LOW TEMPERATURE DOMINANCE OF PION-LIKE EXCITATIONS IN THE MASSIVE GROSS-NEVEU MODEL AT ORDER 1/N∗

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Abstract

We perform a $1/N$-expansion of the partition function of the massive Gross-Neveu model in 1+1 dimensions. The procedure allows for the inclusion of the contribution of scalar and pseudoscalar composites (of order $1/N$) to the equation of state. The naive expectation that the bosonic fluctuations correct significantly the mean field approximation at low temperatures is confirmed by our calculations. Actually the relevant degrees of freedom of hadronic matter at low temperatures are found to be pion-like excitations, rather than the fundamental constituents.

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1. Introduction

The equilibrium thermodynamics of strong-interacting matter has been studied, up to now, by quite different approaches. Apart from lattice simulations, most of the other studies rely on phenomenological models, which try to capture some relevant physical features of the theory [1, 2, 3]. The absence of a common starting point in the literature is due to the non-perturbative character of low energy QCD, and to the vicinity of the expected phase transition, at temperatures of the order of $\Lambda_{QCD}$. Thus, in employing analytical methods to study the transition, one is faced by this major difficulty of QCD, that of passing from the hadronic regime to that of quarks and gluons, based on the same fundamental lagrangian.

For very low temperatures, important results have been obtained on the assumption that the thermodynamics of hadronic matter in this regime is that of a gas of weakly interacting pions [4], since pions are the lowest mass excitations and are thus expected to dominate the partition function. On the other hand, at very high temperatures, we expect a gas of almost free quarks and gluons. Results obtained in these limits, which are both far from the temperatures of deconfinement and chiral symmetry restoration, cannot help to explore the nature of the transition itself.

For such an exploration much of the effort done so far is based on the mean field approximation to the effective potential at finite temperature for four-fermion models or other low-energy effective models of QCD. Unfortunately, the mean field approximation neglects fluctuations, and it misses completely, in such a way, the role of bosonic contributions to the thermodynamical potential. These however should be, as already said, the relevant ones at very low temperatures.

Recently there have been attempts to recover some information missing in the mean field approximation to the effective potential at finite temperature for four-fermion models or other low-energy effective models of QCD. Unfortunately, the mean field approximation neglects fluctuations, and it misses completely, in such a way, the role of bosonic contributions to the thermodynamical potential. These however should be, as already said, the relevant ones at very low temperatures.

For the moment being these attempts have been made for the Nambu-Jona Lasinio (NJL) model [7].

Here we present an application, based on a general scheme which allows for the $1/N$ expansion of the effective action in four-fermion models [8], to the Gross-Neveu model in 1+1 dimensions [9]. We include in the Lagrangian a current fermion mass to avoid infrared divergencies, and include the one pion-like composite states and the scalar composite. To verify whether their contribution to the free energy is dominant at low temperatures, we have to evaluate the effective potential at least at order $1/N$. In the imaginary time
formalism, we have to sum up over discrete Matsubara frequencies twice: the first time for fermions, the second one for bosons.

The effective potential at this order of the expansion appears as a sum of various terms. The first is the zeroth order, mean field term. It has a zero temperature part and a finite temperature part, obtained by summing over fermionic energies.

A second term is a pure temperature term, obtained by summing over both fermion and boson energies. It contains a Bose distribution function and it is expected to dominate in the expression for the pressure at low temperatures. In the following we show this to be indeed the case, and that in particular the largest contribution comes from the integration over almost on shell external pion momenta (close to the pion propagator pole).

The last term is a mixed one, a zero temperature term with respect to the sum over Bose frequencies, depending on temperature through the bosonic propagator, obtained by summing over Fermi frequencies.

This part of the effective potential presents non trivial aspects which are interesting as far as the model itself is concerned, but its detailed study would go beyond the scope of the present work. Actually we are primarily interested in showing how, by applying a general formula for four fermion theories \[8\], one can evaluate the pressure at order \(1/N\), and exhibit the role of pions at low temperatures. Since the mixed term is subleading for very low temperatures and small quark masses (see sect.5), it does not appear in our final results.

In the following paragraph we give a review of the \(1/N\) expansion of the partition function, for the case of the Gross-Neveu model in 1+1 dimensions. In sect.3 and sect.4 we extend the formalism to finite temperature, and calculate the pressure at order \(1/N\). In sect.5 we show the temperature behaviour of the pressure for the pure bosonic term discussed above and compare it to the mean field term. The results are summarized in the conclusions.

Finally we summarize some useful calculations in the appendices, namely the analytic properties of the inverse bosonic propagator (App.A), and the method employed for summing over the bosonic frequencies (App.B).

2. \(1/N\) expansion

We start from the lagrangian of the massive Gross-Neveu model in 1+1 dimensions \[9\]

\[
\mathcal{L} = \bar{\psi} \left( i\hat{\partial} - M \right) \psi + \frac{g^2}{2} \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right]
\]  

(2.1)
where $\psi$ is a multiplet of $N$ degenerate fermion fields, and $M$ is a bare fermion mass which explicitly breaks the chiral invariance.

The $1/N$ expansion can be carried out as follows [8]. According to standard methods the generating functional can be rewritten by introducing a scalar field $\sigma$ and a pseudoscalar field $\pi$ [9]. At vanishing sources, we have

$$Z = \int \mathcal{D}(\bar{\psi}\psi)\mathcal{D}(\sigma)\mathcal{D}(\pi) e^{i \int d^2x \left[ \bar{\psi}(i\hat{\partial} - M + g\sigma + ig\gamma_5\pi)\psi - \frac{1}{2}(\sigma^2 + \pi^2) \right]} \quad (2.2)$$

By carrying out the integral over the fermion fields, we obtain an effective action for scalar and pseudoscalar fields

$$Z = \int \mathcal{D}(\sigma)\mathcal{D}(\pi) e^{iS_B} \equiv e^{iW} \quad (2.3)$$

Now we shift the $\sigma$-field, $g\sigma \rightarrow g\sigma + M$, and write

$$S_B = \int d^2x \left[ -1 \frac{1}{2}(\sigma^2 + \pi^2)(1 + \delta Z) - \frac{M\sigma}{g} \right] - i \log \det(i\hat{\partial} + g\sigma + ig\gamma_5\pi) \quad (2.4)$$

where $\delta Z$ is a counterterm which renormalizes the fermion loop [9].

Finally we redefine the parameters of the model as follows

$$\lambda \equiv Ng^2; \quad \alpha \equiv \frac{M}{Ng^2}; \quad \phi \equiv (g\sigma, g\pi); \quad a \equiv (1, i\gamma_5) \quad (2.5)$$

Thus, the lagrangian for the $\phi$-field is given by

$$S_B = \int d^2x \left[ -\frac{N}{2\lambda} \phi^2(1 + \delta Z) - N\alpha \phi_1 \right] - iN\text{Tr} \log(i\hat{\partial} + \phi \cdot a) \equiv \int d^2x \mathcal{L}_B(\phi) \quad (2.6)$$

where we have used the identity $\log \det = \text{tr} \log$, and carried out the trace over fermion indices.

Up to now we have simply integrated over the fermion fields, passing from a description in terms of elementary fermionic constituents to a description in terms of bosonic composites. At this point we are ready to expand the generating functional $Z$ in eq.(2.3) in series of $1/N$. For this we follow the method of ref. [10], that we briefly review (notice that we are now performing an $1/N$ expansion and not the formal $\hbar$-expansion of the original reference).

Let us couple an external constant source $J$ to the field $\phi(x)$, and expand $\phi(x)$ around its vacuum expectation value $\bar{\phi}$

$$\phi(x) = \phi'(x) + \bar{\phi} \quad (2.7)$$

$$\bar{\phi}[J] = \langle 0^+ | \phi(x) | 0^- \rangle_J = \frac{\delta W}{\delta J} \quad (2.8)$$
Since $J$ is constant, from the Lorentz invariance of the vacuum, it follows that $\bar{\varphi}$ is constant too. By supposing $\bar{\varphi}[J]$ to be invertible, we can define the effective action $\Gamma[\bar{\varphi}]$ as the Legendre transform of $W[J]$

$$\Gamma[\bar{\varphi}] = W[J[\bar{\varphi}]] - \int d^2x J[\bar{\varphi}] \equiv -\mathcal{V}(\bar{\varphi}) \int d^2x$$

(2.9)

From the definitions it follows that, at $J = 0$, $\varphi$ must satisfy the stationary equation

$$\frac{\delta \mathcal{V}(\varphi)}{\delta \varphi} \bigg|_{\varphi = \bar{\varphi}} = 0$$

(2.10)

At this point $W[J]$ can be separated into two terms

$$W[J] \equiv W_0[J] + W_1[J]$$

(2.11)

with $W_0[J]$ given by

$$W_0[J] = \int d^2x \left( \mathcal{L}_B(\bar{\varphi}) + J\bar{\varphi} \right)$$

(2.12)

and $W_1[J]$ solution of the integral equation [10]

$$W_1[J] \equiv -i \log \int \mathcal{D}(\phi') \exp \left[ i \int d^2x \left( \mathcal{L}_B^{(2)}(\phi', \bar{\varphi}) - \phi' \frac{\delta W_1}{\delta \bar{\varphi}} \right) \right]$$

(2.13)

The lagrangian $\mathcal{L}_B^{(2)}(\phi', \bar{\varphi})$ is $\mathcal{L}_B(\phi' + \bar{\varphi})$ minus the constant and linear terms in $\phi'$, and can be obtained from eq. (2.6). We have

$$S_B = \int d^2x \left[ -\frac{N}{2\lambda} (\phi' + \bar{\varphi})^2 (1 + \delta Z) - N\alpha(\phi_1 + \bar{\varphi}_1) \right] - iN\text{Tr} \log(i\hat{\varphi} + \bar{\varphi} \cdot a + \phi' \cdot a)$$

(2.14)

Furthermore, by defining

$$(i\hat{\varphi} + \bar{\varphi} \cdot a)G(x_1 - x_2) = \delta^2(x_1 - x_2)$$

(2.15)

it follows

$$\text{Tr} \log(i\hat{\varphi} + \bar{\varphi} \cdot a + \phi' \cdot a) = \text{Tr} \log(i\hat{\varphi} + \bar{\varphi} \cdot a) + \text{Tr} \log (1 + G\varphi \cdot a)$$

(2.16)

Therefore, the standard mean field fermionic term in eq. (2.12), at $J = 0$, is

$$W_0 = \int d^2x \left[ -\frac{N}{2\lambda} \bar{\varphi}^2 (1 + \delta Z) - N\alpha\bar{\varphi}_1 \right] - iN\text{Tr} \log(i\hat{\varphi} + \bar{\varphi} \cdot a)$$

$$= \int d^2x \left[ -\frac{N}{2\lambda} \bar{\varphi}^2 (1 + \delta Z) - N\alpha\bar{\varphi}_1 + iN \int \frac{d^2p}{(2\pi)^2} \log(\bar{\varphi}^2 - p^2) \right]$$

(2.17)

This is the only term which survives in the $N \to +\infty$ limit, since $W_1$ is of order $1/N$ with respect to $W_0$. 
The evaluation of \( W_1 \) requires to expand \( S_B \) in eq.(2.14) in powers of \( \phi' \), starting from the quadratic terms. By rescaling the fields, \( \phi' \rightarrow \phi'/\sqrt{N} \), and by ordering the terms in the exponent of eq.(2.13) in powers of \( 1/N \), it turns out that the leading term is just the quadratic one.

Thus, by keeping only the leading term of the \( 1/N \) expansion, the field-dependent part of \( W_1 \) can be obtained by

\[
e^{iW_1} = \int D(\phi') \exp \left\{ -\frac{i}{2\lambda} \int d^2x \phi'^2(1 + \delta Z) - \frac{1}{2} \int d^2x_1d^2x_2 \text{Tr} \left[ G(x_1 - x_2)\phi'(x_1) \cdot aG(x_2 - x_1)\phi'(x_2) \cdot a \right] \right\} \quad (2.18)
\]

By going over to the Fourier space, we can diagonalize the exponent in the previous equation, thus obtaining a simple gaussian integral.

We finally have

\[
W_1 = \sum_{j=\sigma,\pi} \frac{i}{2} \int d^2x \int \frac{d^2p}{(2\pi)^2} \log \left[ \frac{i}{2\lambda} D_{0,j}^{-1}(p) \right] \quad (2.19)
\]

where \( iD_{0,j}^{-1}(p) \) is the zero temperature inverse bosonic propagator, whose explicit form is given in Appendix A.

At this point it is straightforward to verify that, apart from the linear term in \( \bar{\varphi}_1 \), both \( W_0 \) and \( W_1 \) depend on the chirally invariant combination \( \bar{\varphi}^2 = g^2(\bar{\sigma}^2 + \bar{\pi}^2) \), which is the solution of eq.(2.10). From this condition it follows that the pseudoscalar component vanishes, i.e.

\[
\bar{\varphi} = (g\bar{\sigma}, 0) \quad (2.20)
\]

3. Finite temperature formalism

The finite temperature extension is straightforward. For imaginary times we have

\[
\int d^2x \rightarrow -i\beta \int dx \equiv -i\beta V ; \quad \int \frac{d^2p}{(2\pi)^2} \rightarrow \frac{i}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{dp}{2\pi} \quad (3.1)
\]

while the finite temperature effective potential, obtained as the generalization to finite temperatures of eq.(2.9), is just the free energy density \( F \)

\[
y \rightarrow -\frac{i}{\beta V} W = -\frac{\log Z}{\beta V} \equiv F \quad (3.2)
\]

According to the \( 1/N \) expansion discussed in the previous paragraph, we can write

\[
F = F^F + \sum_{j=\sigma,\pi} F_j^B \quad (3.3)
\]
where $\mathcal{F}^F$ is the purely fermionic standard mean field term coming from the one-loop calculation ($\omega_n = (2n+1)\pi/\beta$)

$$\mathcal{F}^F = \frac{N}{2\lambda} \varphi^2 (1 + \delta Z) + N\alpha \varphi - \frac{N}{\beta} \sum_n \int \frac{dp}{2\pi} \log \left( \varphi^2 + p^2 + \omega_n^2 \right)$$

(3.4)

and $\mathcal{F}^B$ is the contribution of bosonic fluctuations ($\nu_n = 2n\pi/\beta$)

$$\mathcal{F}^B_j = \frac{1}{2\beta} \sum_n \int \frac{dp}{2\pi} \log \left[ \frac{i}{2\lambda} D_j^{-1}(i\nu_n, p) \right]$$

(3.5)

The expression of the finite temperature inverse bosonic propagator $iD^{-1}$ and its analytic properties are given in Appendix A.

Expression (3.4) and (3.3) put in eq.(3.3) give the equation of state, when evaluated at the solution of the gap equation (2.10) at finite temperature

$$\left. \frac{\partial \mathcal{F}(\varphi, T)}{\partial \varphi} \right|_{\varphi = \bar{\varphi}(T)} = 0$$

(3.6)

which determines the evolution of the fermion condensate with temperature.

By following a standard renormalization procedure [9] and evaluating the sums in the last term of eq.(3.4) [11], the fermionic term $\mathcal{F}^F$ can be written as

$$\mathcal{F}^F(\varphi, T) = N \left[ \frac{\varphi^2}{4\pi} \left( \log \frac{\varphi^2}{m_0^2} - 1 \right) + \alpha \varphi - \frac{2}{\pi \beta} \int_0^{+\infty} dp \log \left( 1 + e^{-\beta \sqrt{p^2 + \varphi^2}} \right) \right]$$

(3.7)

where $m_0$ is the dynamical fermion mass generated at $\alpha = 0, T = 0$.

The sums that appear in the bosonic part of the free energy, eq.(3.4), can be transformed into integrals by standard methods [12]. In this procedure we must pay attention to the fact that the finite temperature inverse bosonic propagator $iD^{-1}(\omega, p)$ has a cut across the origin in the complex $\omega$-plane (for $|\omega| < p$), besides the cut extending from the continuum threshold to $+\infty$ (that is present even at $T = 0$) [3, 17]. A detailed derivation is shown in Appendix B, from which it follows that the bosonic term can be cast in the form ($n_B(\omega) = (e^{\beta \omega} - 1)^{-1}$)

$$\mathcal{F}^B_j = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_0^{+\infty} d\omega \log \left[ iD_j^{-1}(i\omega) \right]$$

$$+ \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_0^{+\infty} d\omega \ n_B(\omega) \log \left[ \frac{iD_j^{-1}(\omega + i\epsilon, p)}{iD_j^{-1}(\omega - i\epsilon, p)} \right]$$

(3.8)

apart from infinities independent of $\beta$ and $\varphi$. 
Furthermore, since the function \( g(z) = iD^{-1}(z, p) \) satisfies the Schwarz reflection principle, \( g^*(z) = g(z^*) \), by reintroducing the explicit dependence on \( \varphi \) and \( T \), we can write

\[
\mathcal{F}_B^j (\varphi, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_{0}^{+\infty} d\omega \log \left[ iD_j^{-1}(i\omega, p; \varphi, T) \right] + \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_{0}^{+\infty} d\omega n_B(\omega) \left[ \arg \left( iD_j^{-1}(\omega + i\epsilon, p; \varphi, T) \right) - \pi \right]
\]  

(3.9)

where \( \arg(f) \) is the argument \( \theta \in [0, 2\pi) \) of the complex number \( f \equiv |f| \exp(i\theta) \).

4. **Free energy density and pressure at order 1/\(N\)**

According to the 1/\(N\) expansion discussed in the previous paragraph, \( \mathcal{F} \) can be written as the sum of the purely fermionic mean field term \( \mathcal{F}^F \equiv N\mathcal{F}_0 \), and of the contribution of the bosonic fluctuations \( \sum_{j=\sigma,\pi} \mathcal{F}_B^j \equiv \mathcal{F}_1 \)

\[
\mathcal{F}(\bar{\varphi}(T), T) = N\mathcal{F}_0(\bar{\varphi}(T), T) + \mathcal{F}_1(\bar{\varphi}(T), T)
\]  

(4.1)

where \( \bar{\varphi}(T) \) is the solution of the gap equation (3.6). Therefore, in order to get a true 1/\(N\) expansion of the free energy density in eq.(4.1), we need also to expand \( \bar{\varphi}(T) \) in series of 1/\(N\)

\[
\bar{\varphi}(T) = \bar{\varphi}_0(T) + \frac{1}{N}\bar{\varphi}_1(T) + \cdots
\]  

(4.2)

By requiring eq.(3.6) to be satisfied at each order in 1/\(N\), we obtain that \( \bar{\varphi}_0(T) \) is the solution of the mean field gap equation

\[
\frac{\partial \mathcal{F}_0(\varphi, T)}{\partial \varphi} \bigg|_{\bar{\varphi}_0(T)} = 0
\]  

(4.3)

whereas \( \bar{\varphi}_1(T) \) is given by

\[
\bar{\varphi}_1(T) = -\frac{\partial \mathcal{F}_1(\varphi, T)}{\partial \varphi} \bigg|_{\bar{\varphi}_0(T)} \cdot \left( \frac{\partial^2 \mathcal{F}_0(\varphi, T)}{\partial \varphi^2} \bigg|_{\bar{\varphi}_0(T)} \right)^{-1}
\]  

(4.4)

Finally, by inserting eq.(4.2) into eq.(4.1), we have

\[
\mathcal{F}(\bar{\varphi}(T), T) = N\mathcal{F}_0(\bar{\varphi}_0(T), T) + \frac{\partial \mathcal{F}_0}{\partial \varphi} \bigg|_{\bar{\varphi}_0(T)} \bar{\varphi}_1(T) + \mathcal{F}_1(\bar{\varphi}_0(T), T) + \cdots
\]  

\[
= N\mathcal{F}_0(\bar{\varphi}_0(T), T) + \mathcal{F}_1(\bar{\varphi}_0(T), T) + \cdots
\]  

(4.5)

Let us now consider the pressure per fermion species, at order 1/\(N\)

\[
\mathcal{P}(T) = -\mathcal{F}(\bar{\varphi}_0(T), T)/N
\]

\[
= \mathcal{P}_0(\bar{\varphi}_0(T), T) + \frac{1}{N}\mathcal{P}_1(\bar{\varphi}_0(T), T) + \cdots
\]  

(4.6)
According to the previous paragraph the mean field pressure is given by
\[ P_0(\bar{\varphi}_0, T) = -\frac{\bar{\varphi}_0^2}{4\pi} \left( \log \frac{\bar{\varphi}_0^2}{m_0^2} - 1 \right) - \alpha \bar{\varphi}_0 + \frac{2}{\pi \beta} \int_0^{+\infty} dp \log \left( 1 + e^{-\beta \sqrt{p^2 + \bar{\varphi}_0^2}} \right) \] (4.7)

whereas the contribution at order 1/\(N\) can be separated into two terms
\[ P_1(\bar{\varphi}_0, T) = -\sum_{j=\sigma,\pi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \int_0^{+\infty} d\omega \log \left[ iD_j^{-1}(i\omega, p; \bar{\varphi}_0, T) \right] \] (4.8)
\[ -\sum_{j=\sigma,\pi} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} dp \int_0^{+\infty} d\omega \ n_B(\omega) \left[ \arg \left( iD_j^{-1}(\omega + i\epsilon, p; \bar{\varphi}_0, T) \right) - \pi \right] \]

In the next section we show the results for the temperature behaviour of the pressure, expanded as in eq.(4.6), within the approximation of low temperatures and small current quark masses.

5. Results

Let us first comment on the second term in eq.(4.8), which is the one expected to give the dominant bosonic behaviour at low temperatures. Actually such a term contains the Bose distribution function \(n_B(\omega)\), and, if \(iD^{-1}\) were the inverse free bosonic propagator, it would correspond to the standard pressure for a free boson gas [5], and as such it would contain the typical temperature behaviour of a gas of spinless particles.

Even in the case of the inverse propagator having a more complex structure, the major contribution to the pressure at low temperatures should come from the integration over the region containing the poles corresponding to the composite bosons. In particular, since the pion corresponds to the lowest bosonic mass excitation because of its pseudo-goldstones nature, it characterizes the thermodynamics at low temperatures, where the leading behaviours are of the form \(\sim \exp(-m/T)\), for the particles of mass \(m\).

Notice that in the present work we are including an explicit symmetry breaking, avoiding infrared divergencies related to the low dimensionality of the model. If one could go continuously to the massless case, one would recover the power law behavior in \(T\) of the thermodynamical quantities in the pion sector, evidencing the role of the pions as Goldstone bosons associated to the spontaneous breaking of the chiral symmetry.

In conclusion the pion term in the second sum of eq.(4.8) is expected to dominate the total pressure in eq.(4.6) for low temperatures and small values of \(\alpha = M/Ng^2\), which is responsible for the explicit breaking of the chiral symmetry. In Figs.1-3 we show that this is indeed the case, in the present model.
We notice that the Gross-Neveu model is peculiar since in presence of a bare fermion mass no bound states are present in the scalar sector (see Appendix A). However this particular feature of the model should not affect its physical interest to mimic some aspects of QCD at low temperatures, where the contribution of the scalar pole is depressed since \( m_\sigma \gg m_\pi \).

Let us now consider the first term in the r.h.s. of eq.(4.8). Even this term has a temperature dependence, this time not through the Bose distribution function, but through the Fermi distribution functions appearing in the bosonic propagator at finite \( T \), obtained by summing over fermions frequencies.

This term can be separated in a zero temperature part

\[
P_1^0 \equiv - \sum_{j=\sigma,\pi} \frac{1}{2\pi} \int_0^{+\infty} dp \int dp \int d\omega \ b \ \log \left[ iD^{-1}_{0,j}(i\omega, p; \varphi_0) \right]
\]

and a finite temperature part

\[
P_1^\beta \equiv - \sum_{j=\sigma,\pi} \frac{1}{2\pi} \int_0^{+\infty} dp \int_0^{+\infty} d\omega \ b \ \log \left[ 1 + \frac{D^{-1}_{0,j}}{D^{-1}_{\beta,j}} \right]
\]

with an analogous separation for the inverse bosonic propagator (see eq.(A.3))

\[
D^{-1}_j \equiv D^{-1}_{0,j} + D^{-1}_{\beta,j}
\]

\( P_1^0 \) is minus the zero temperature effective potential of order \( 1/N \) (evaluated at the minimum \( \varphi_0 \)). It is UV divergent and needs to be renormalized. The renormalization procedure for the massless Gross-Neveu model has been studied in literature [13, 14, 15], although an explicitly invariant expression for the renormalized effective potential at this order has not been given\(^1\).

However this term (once renormalized) is subleading, at low temperatures, with respect to pion pole. This can be verified by performing a low temperature expansion, and taking into account that \( P_1^0 \) depends on temperature only through \( \varphi_0(T) \), which has the following behaviour for low temperatures

\[
\varphi_0(T) = \varphi_0(0) + \delta \varphi_0(T) + \ldots
\]

with

\[
\delta \varphi_0(T) \sim \sqrt{T} e^{-\frac{\varphi_0(0)}{T}} \quad (5.5)
\]

\(^1\)We remind that since a bare fermion mass is present, there are no infrared divergencies in the pion sector, that otherwise would be present [13, 10].
Thus, as $T$ goes to zero, $\mathcal{P}_1^0$ is exponentially depressed with respect to the pressure term coming from the integration around the pion pole, since $\bar{\varphi}_0(0) > m_\pi$.

Let us now consider $\mathcal{P}_1^\beta$. We have verified numerically that at low temperature (and small enough values of $\alpha$) also this term is subleading with respect to the pion pole term. This can be in part understood from the fact that its temperature dependence comes from the Fermi distribution functions, that, again, have an exponentially decreasing behavior at low temperatures, characterized by $\bar{\varphi}_0(0)$.

In addition both $\mathcal{P}_1^0$ and $\mathcal{P}_1^\beta$ exhibit an imaginary part below a certain value of the quark condensate. This problem has been already noticed in literature for the zero temperature effective potential [13, 15], and we will not go further in its discussion. A complete calculation goes beyond the scope of this paper, since the problems are specific to the Gross-Neveu model, whereas the dominance of the pion pole is surely a general feature of QCD.

We now discuss the behaviour of the pure temperature pion pressure term discussed above (see second line of eq.(4.8))

$$\mathcal{P}_1^\pi(\bar{\varphi}_0, T) \equiv -\lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_{0}^{+\infty} d\omega \, n_B(\omega) \left[ \arg \left( iD_\pi^{-1}(\omega + i\epsilon, p; \bar{\varphi}_0, T) \right) - \pi \right]$$

(5.6)

and we compare it to the mean field term, eq.(4.7). The results are shown in Figs.1-2, where we plot various terms of the pressure (divided by the square temperature) vs $T/m_0$ for different values of the chiral symmetry breaking parameter $\alpha$, and of the number $N$ of fermion species.

According to the previous paragraph, and by subtracting the bag constant $\mathcal{P}_0(\bar{\varphi}_0(0), 0)$, the expression for the pressure at order $1/N$ that we consider is

$$\mathcal{P}(T) \equiv \mathcal{P}_0(\bar{\varphi}_0, T) - \mathcal{P}_0(\bar{\varphi}_0(0), 0) + \frac{1}{N} \mathcal{P}_1^\pi(\bar{\varphi}_0, T)$$

(5.7)

Furthermore, to better identify the role of the pion pole, we have divided the integrations that appear in eq.(5.6) into two regions: (i) $p < \omega < \sqrt{p^2 + 4\bar{\varphi}_0^2}$ and (ii) $(\omega^2 - p^2)(\omega^2 - p^2 - 4\bar{\varphi}_0^2) > 0$. The region (i) is the one which contains the pion pole, and it is from here that the pressure takes its major contribution at low temperatures, always for small values of $\alpha$.

In Fig.1(a) we show, for $\alpha = 0.01m_0$, the contribution to the pressure of the pion term $\mathcal{P}_1^\pi$ in eq.(5.6), decomposed as follows: contribution of the pion pole (continuous) and that coming from the integration over the region (ii) (dashed). In Fig.2(a) are shown the same quantities of Fig.1(a), for $\alpha = 0.001m_0$.

As anticipated in the previous paragraphs, we see that at low temperatures the pseudoscalar bound state (provided the current quark mass is small enough) dominates over
the mean field pressure, reflecting the fact that the pion is the lowest mass state by virtue of its nature of pseudogoldstone. Its contribution grows more and more as we decrease the bare fermion mass. In the massless case we would recover a power law behaviour in the limit $T \to 0$, instead of the exponential one shown in Figs. 1-2 (i.e. the curves of $P/T^2$ and the one of $P_1^\pi/T^2$ containing the contribution of the pion pole would tend to a constant value and not to zero).

In Fig. 1(b) we plot $P/T^2$ (see eq.(5.7)) for $N = 3$ (short-dashed), $N = 10$ (medium-dashed), and in the mean field case $N = +\infty$ (continuum line). In this figure $\alpha = 0.01m_0$. The analog results for $\alpha = 0.001m_0$ are shown in Figs. 2(b).

Finally, in Fig. 3, we compare $P_1^\pi$ to the sigma contribution coming from the second term of eq.(4.8) for $\alpha = 0.01m_0$ (a) and $\alpha = 0.001m_0$ (b). Although, as already noticed, the sigma term is peculiar in this model (since there is no scalar bound state) we show this figures to make manifest that its contribution is negligible at low temperatures, whereas, close above the critical temperature $T_c = 0.57m_0$, the scalar and pseudoscalar fields have to be practically degenerate, since the chiral symmetry is almost restored (there is no real restoration due to the explicit symmetry breaking term).

6. Conclusions

We have applied a general formula, valid for four-fermion theories, to calculate the finite temperature effective potential at order $1/N$ in the massive Gross-Neveu model in $D = 1+1$. We have shown that the pion-like composite dominates the thermodynamics at low temperatures, correcting significantly the mean field approximation. We have shown in particular that the leading contribution at low temperatures comes from integration close to the pion pole. This feature, which is expected to be also general to QCD and related models, emerges clearly from our calculations.

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Appendix

A. Analytic structure of the bosonic propagator

The zero temperature inverse propagator in eq.(2.19) is given by (we omit the index $j$)

$$D_0^{-1} (\omega, p; \varphi) = i [(1 + \delta Z) + i \Pi(\omega, p)]$$

$$= i \left[ (1 + \delta Z) - 2i\lambda \int \frac{d^2q}{(2\pi)^2} \frac{q_\mu (q_\mu + p_\mu) \pm \varphi^2}{((q_\mu + p_\mu)(q_\mu + p_\mu) - \varphi^2)(q_\mu q_\mu - \varphi^2)} \right]$$

(A.1)

where $\pm$ refer to the scalar and pseudoscalar self energy $\Pi(\omega, p)$ respectively, and $p_\mu = (\omega, p)$. By renormalizing in a standard way \([9]\), we have

$$\frac{2\pi}{\lambda} i D_0^{-1} (\omega, p; \varphi) = - \log \frac{\varphi^2}{m_0^2}$$

$$+ \frac{(\omega^2 - p^2 - \epsilon_M^2)}{\sqrt{(\omega^2 - p^2)(\omega^2 - p^2 - 4\varphi^2)}} \log \frac{\sqrt{(\omega^2 - p^2)(\omega^2 - p^2 - 4\varphi^2)} - (\omega^2 - p^2)}{\sqrt{(\omega^2 - p^2)(\omega^2 - p^2 - 4\varphi^2)} + (\omega^2 - p^2)}$$

(A.2)

At finite temperature the inverse propagator $D^{-1}$ appearing in eq.(3.5) can be calculated by usual methods \([12]\)

$$D^{-1}(i\nu_n, p; \varphi, T) = D_0^{-1}(i\nu_n, p; \varphi) + 2i\lambda \int \frac{d\xi}{2\pi} \left\{ \frac{n_F(E_\xi)}{E_\xi} \left[ \frac{1}{(E_q - i\nu_n)^2 - E_{q+p}^2} + \frac{1}{(E_q + i\nu_n)^2 - E_{q+p}^2} \right] \right\}$$

(A.3)

where

$$\nu_n = 2n\pi/\beta; \quad n_F(E) = \frac{1}{e^{\beta E} + 1}; \quad E_q = \sqrt{q^2 + \varphi^2}$$

(A.4)

Eq.(A.3) can be cast in a more compact form

$$\frac{2\pi}{\lambda} i D^{-1}(i\nu_n, p; \varphi, T) = \frac{2\pi}{\lambda} i D_0^{-1}(i\nu_n, p; \varphi) - 4 \int_0^{+\infty} \frac{d\xi}{2\pi} \left\{ \frac{n_F(E_\xi)}{E_\xi} \sum_{\xi_1, \xi_2 = \pm 1} \frac{1}{(E_q + \xi_1 i\nu_n)^2 - E_{q+\xi_2 p}^2} \right\}$$

(A.5)

In order to evaluate the second term of eq.(A.8) we take the inverse bosonic propagator as a function of $\omega + i\epsilon$ instead of $i\nu_n$, namely $i D^{-1}(\omega + i\epsilon, p; \varphi, T)$. Then, the real and imaginary parts of $i D^{-1}$ can be evaluated, in the limit $\epsilon \to 0$, by using

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = PV \frac{1}{x} - i\pi \delta(x)$$

(A.6)
Defining $\Delta \equiv (\omega^2 - p^2 - 4\varphi^2)/(\omega^2 - p^2)$, we have

$$
\text{Re} \left[ \frac{2\pi}{\lambda} iD^{-1}_\sigma \right] = \begin{cases} 
- \log \frac{\omega^2}{m_0^2} - 2\sqrt{-\Delta} \arctan \left( \frac{1}{\sqrt{-\Delta}} \right) - I^\beta + \Delta I^\beta_1(\omega, p) & \Delta < 0 \\
- \log \frac{\omega^2}{m_0^2} + \sqrt{\Delta} \log \left| \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1} \right| - I^\beta + \Delta I^\beta_1(\omega, p) & \Delta > 0
\end{cases}
$$

(A.7)

$$
\text{Im} \left[ \frac{2\pi}{\lambda} iD^{-1}_\sigma \right] = \begin{cases} 
0 & \Delta < 0 \\
\pi \sqrt{\Delta} \frac{\sinh \left( \frac{\beta \omega}{2} \right)}{\cosh \left( \frac{\beta \omega}{2} \right) + \cosh \left( \frac{\beta p \sqrt{\Delta}}{2} \right)} & \Delta > 0
\end{cases}
$$

(A.8)

$$
\text{Re} \left[ \frac{2\pi}{\lambda} iD^{-1}_\pi \right] = \begin{cases} 
- \log \frac{\omega^2}{m_0^2} + \frac{2}{\sqrt{-\Delta}} \arctan \left( \frac{1}{\sqrt{-\Delta}} \right) - I^\beta + I^\beta_1(\omega, p) & \Delta < 0 \\
- \log \frac{\omega^2}{m_0^2} + \frac{1}{\sqrt{\Delta}} \log \left| \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1} \right| - I^\beta + I^\beta_1(\omega, p) & \Delta > 0
\end{cases}
$$

(A.9)

$$
\text{Im} \left[ \frac{2\pi}{\lambda} iD^{-1}_\pi \right] = \begin{cases} 
0 & \Delta < 0 \\
\pi \frac{\sinh \left( \frac{\beta \omega}{2} \right)}{\sqrt{\Delta} \cosh \left( \frac{\beta \omega}{2} \right) + \cosh \left( \frac{\beta p \sqrt{\Delta}}{2} \right)} & \Delta > 0
\end{cases}
$$

(A.10)

where we have defined the following integrals

$$
I^\beta = 4 \int_0^{+\infty} dq \frac{n_F(E_q)}{E_q} \quad (A.11)
$$

$$
I^\beta_1(\omega, p) = 4(\omega^2 - p^2)^2 \cdot \int_0^{+\infty} dq \frac{n_F(E_q)}{E_q} \frac{(\omega^2 - p^2)^2 - 4\omega^2 E_q^2 - 4p^2 q^2}{((\omega^2 - p^2)^2 - 4\omega^2 E_q^2 + 4p^2 q^2)^2 - 16p^2 q^2(\omega^2 - p^2)^2} \quad (A.12)
$$

When $\Delta > 0$ (and $\omega^2 > 0$) the last integral can be rewritten as follows

$$
I^\beta_1(\omega, p) = \frac{1}{4\omega p \sqrt{\Delta}} \text{PV} \int_0^{+\infty} dq \frac{n_F(E_q)}{E_q} \cdot \left( \frac{1}{q^2 - q^2_+} - \frac{1}{q^2 - q^2_-} \right) \quad (A.13)
$$
with \( \bar{q}_\pm = \left( p \pm \omega \sqrt{\Delta} \right) / 2 \)

Let us now summarize the main features of \( iD^{-1}(z,p) \), with \( z = \omega + i\epsilon \), which follow from eqs. (A.7)-(A.10)

a) at \( T = 0 \) the inverse propagator \( iD_0^{-1} \) has a cut along the real \( z \)-axis, extending from the continuum threshold \( |\omega| = \sqrt{p^2 + 4\varphi^2} \) to \( +\infty \)

b) for \( T \neq 0 \) a cut is present even for \( |\omega| < p \). This is due to the fact that at finite temperature “scattering” processes are possible between virtual scalar/pseudoscalar particles with space like momenta \( (\omega^2 - p^2 < 0) \) and quarks of the medium [5, 17]

c) in the pseudoscalar sector \( iD^{-1} \) has a zero, at least for low temperatures, for \( \omega^2 - p^2 \sim \alpha m_0 \), corresponding to the pion bound state [18]

d) no scalar bound states are present for \( \alpha \neq 0 \), both at zero [19] and finite temperature [18]

Finally, we give the expression for the inverse propagator evaluated at imaginary energies, which appears in the first term eq.(4.8) \( (\Delta_E \equiv (\omega^2 + p^2 + 4\varphi^2)/(\omega^2 + p^2)) \)

\[
\frac{\lambda}{2\pi} iD^{-1}_\sigma(i\omega, p; \varphi, T) = -\log \frac{\varphi^2}{m_0^2} + \sqrt{\Delta_E} \log \left| \frac{\sqrt{\Delta_E} - 1}{\sqrt{\Delta_E} + 1} \right| - I^\beta + \Delta_E I_1^\beta(i\omega, p) \tag{A.14}
\]

\[
\frac{\lambda}{2\pi} iD^{-1}_\pi(i\omega, p; \varphi, T) = -\log \frac{\varphi^2}{m_0^2} + \frac{1}{\sqrt{\Delta_E}} \log \left| \frac{\sqrt{\Delta_E} - 1}{\sqrt{\Delta_E} + 1} \right| - I^\beta + I_1^\beta(i\omega, p) \tag{A.15}
\]

for the scalars and pseudoscalars respectively.

**B. Sum over bosonic frequencies**

The bosonic sums in eq.(3.3) can be transformed into integrals by means of a standard procedure (for a review of finite temperature methods see [12]). Here we have to pay attention to the fact that, at finite temperature, the inverse bosonic propagator \( iD^{-1}(z,p) \) (\( \text{Re} z = \omega \)) has a cut extending along the real \( z \)-axis for \( -p < \omega < p \).

Let us start by considering the following sum

\[
\frac{1}{\beta} \sum_n f(i\nu_n) ; \quad \nu_n = 2n\pi/\beta \tag{B.1}
\]

where we suppose that \( f(z) \) has no singularities on the complex plane apart from cuts along the real axis. Notice that in our case \( iD^{-1}(z,p) \), although it has a cut for \( -p < \omega < p \), is well defined in \( z = 0 \), since the discontinuity across the cut goes to zero as \( \omega \) goes to zero.
The first step is then to use the residue theorem to trade the sum for an integral over a circuit around the imaginary axis (see Fig.4). Since the integral along the whole contour \( \Gamma \) in Fig.4(a) gives not only the sum of the residues in \( z = i\nu_n \), but also the discontinuity across the cut, we have to subtract it, namely subtract the contribution of the contour \( C = C_l + C_r \) (horizontal lines).

Thus we can write

\[
\frac{1}{\beta} \sum_n f(i\nu_n) = \frac{1}{2\pi i} \left[ \int_\Gamma dz f(z)n_B(z) - \int_C dz f(z)n_B(z) \right]
\]

\[
= \lim_{\delta, \epsilon \to 0} \frac{1}{2\pi i} \left[ \int_{-i\epsilon + \delta}^{-i\epsilon - \delta} dz f(z)n_B(z) + \int_{i\epsilon + \delta}^{i\epsilon - \delta} dz f(z)n_B(z) \right]
\]

\[
+ \lim_{\delta, \epsilon \to 0} \frac{1}{2\pi i} \left[ \int_{-i\infty - \delta}^{-i\infty + \delta} dz f(z)n_B(z) + \int_{i\infty - \delta}^{i\infty + \delta} dz f(z)n_B(z) \right]
\]

where \( n_B(z) \) is the Bose distribution function

\[
n_B(z) = \frac{1}{\exp(\beta z) - 1}
\]

which has poles in \( z = i\nu_n \), with residue equal to \( 1/\beta \).

The third and fourth integrals of eq. (B.2) can be transformed by a change of variable, \( z \to -z \). Thus we obtain

\[
\frac{1}{\beta} \sum_n f(i\nu_n) = \lim_{\delta, \epsilon \to 0} \frac{1}{2\pi i} \int_{-i\infty + \delta}^{-i\infty - \delta} dz n_B(z) \left[ f(z) + f(-z) \right]
\]

\[
+ \lim_{\delta, \epsilon \to 0} \frac{1}{2\pi i} \int_{i\infty - \delta}^{i\infty + \delta} dz n_B(z) \left[ f(z) + f(-z) \right] + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz f(z)
\]

(B.4)

where we have taken into account the fact that no singularities are encountered in performing the integral of \( f(z) \) on the imaginary axis (last term of eq. (B.4)).

Finally, provided \( f(z) \exp(-\beta|z|) \) vanishes sufficiently fast at infinity and \( f(z) \) has no singularities on the complex plane (apart from the already mentioned cuts along the real axis), we can rotate the first integrals in eq. (B.4) along the circuit shown in Fig.4(b), and write

\[
\frac{1}{\beta} \sum_n f(i\nu_n) = \lim_{\epsilon \to 0} \frac{1}{\pi i} \int_{-\infty}^{+\infty} d\omega n_B(\omega) \left[ f(\omega + i\epsilon) - f(\omega - i\epsilon) \right] + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega f(i\omega)
\]

(B.5)

where we have also supposed \( f(z) = f(-z) \), and the limit \( \delta \to 0 \) has been performed.
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Figures captions

Fig.1  (a) Plot of the contribution to the pressure of the pion term $P_1^\pi$ in eq.(5.6), decomposed as follows: contribution of the pion pole (continuous) and that coming from the integration over the region $(\omega^2 - p^2)(\omega^2 - p^2 - 4\varphi^2_0) > 0$ (dashed). These curves are obtained for $\alpha = 0.01m_0$.

(b) Plot of $P/T^2$ (see eq.5.7) for $N = 3$ (short-dashed), $N = 10$ (medium-dashed), and in the mean field case $N = +\infty$ (continuous line). In this figure $\alpha = 0.01m_0$.

Fig.2  (a) Plot of the contribution to the pressure of the pion term $P_1^\pi$ in eq.(5.6), decomposed as follows: contribution of the pion pole (continuous) and that coming from the integration over the region $(\omega^2 - p^2)(\omega^2 - p^2 - 4\varphi^2_0) > 0$ (dashed). These curves are obtained for $\alpha = 0.001m_0$.

(b) Plot of $P/T^2$ (see eq.5.7) for $N = 3$ (short-dashed), $N = 10$ (medium-dashed), and in the mean field case $N = +\infty$ (continuous line). In this figure $\alpha = 0.001m_0$.

Fig.3  (a) Plot of the contribution to the pressure $P_1$ in eq.(4.8) of the pion (continuous) and of the sigma (dashed). These curves are obtained for $\alpha = 0.01m_0$.

(b) Plot of the contribution to the pressure $P_1$ in eq.(4.8) of the pion (continuous) and of the sigma (dashed). These curves are obtained for $\alpha = 0.001m_0$.

Fig.4  (a) Contour integration used in eq.(B.2). $\Gamma$ labels the whole contour, while $C = C_l + C_r$ is the horizontal contour circumscribing the cut across the origin.

(b) Contour integration used in eq.(B.4). $\Gamma'$ labels the vertical contour along right side of imaginary axis, which is transformed in the horizontal contours $C_r$ and $C'_r$ circumscribing the cuts along the real positive axis.
Fig. 1
Fig. 2
Fig. 3