WELL-POSEDNESS AND SCATTERING FOR A SYSTEM OF QUADRATIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS WITH LOW REGULARITY INITIAL DATA

HIROYUKI HIRAYAMA
Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya, 464-8602, Japan

Abstract. In the present paper, we consider the Cauchy problem of a system of quadratic derivative nonlinear Schrödinger equations which was introduced by M. Colin and T. Colin (2004) as a model of laser-plasma interaction. The local existence of the solution of the system in the Sobolev space $H^s$ for $s > d/2 + 3$ is proved by M. Colin and T. Colin. We prove the well-posedness of the system with low regularity initial data. For some cases, we also prove the well-posedness and the scattering at the scaling critical regularity by using $U^2$ space and $V^2$ space which are applied to prove the well-posedness and the scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch (2009).

1. Introduction

We consider the Cauchy problem of the system of Schrödinger equations:

\[
\begin{align*}
(i\partial_t + \alpha \Delta)u &= -(\nabla \cdot w)v, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
(i\partial_t + \beta \Delta)v &= -(\nabla \cdot \bar{w})u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
(i\partial_t + \gamma \Delta)w &= \nabla(u \cdot \nabla v), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \\
(u(0, x), v(0, x), w(0, x)) &= (u_0(x), v_0(x), w_0(x)), \quad x \in \mathbb{R}^d
\end{align*}
\]

where $\alpha, \beta, \gamma \in \mathbb{R}\{0\}$ and the unknown functions $u, v, w$ are $d$-dimensional complex vector valued. The system (1.1) was introduced by Colin and Colin in [6] as a model of laser-plasma interaction. (1.1) is invariant under the following scaling transformation:

\[
A_\lambda(t, x) = \lambda^{-1}A(\lambda^{-2}t, \lambda^{-1}x) \quad (A = (u, v, w)), \quad (1.2)
\]

and the scaling critical regularity is $s_c = d/2 - 1$. The aim of this paper is to prove the well-posedness and the scattering of (1.1) in the scaling critical Sobolev space.

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First, we introduce some known results for related problems. The system (1.1) has quadratic nonlinear terms which contains a derivative. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata ([23]) proved that a necessary condition for the $L^2$ well-posedness of the problem:

$$
\begin{cases}
  i\partial_t u - \Delta u = b_1(x)\nabla u, & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^d
\end{cases}
$$

is the uniform bound

$$
\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}, R > 0} \left| \text{Re} \int_0^R b_1(x + r\omega) \cdot \omega \, dr \right| < \infty.
$$

Furthermore, Christ ([5]) proved that the flow map of the Cauchy problem:

$$
\begin{cases}
  i\partial_t u - \partial_x^2 u = u\partial_x u, & t \in \mathbb{R}, \ x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}
\end{cases}
$$

(1.3)

is not continuous on $H^s$ for any $s \in \mathbb{R}$. While, there are positive results for the Cauchy problem:

$$
\begin{cases}
  i\partial_t u - \Delta u = \pi(\nabla \cdot \pi), & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^d
\end{cases}
$$

(1.4)

Grünerk (12) proved that (1.4) is globally well-posed in $L^2$ for $d = 1$ and locally well-posed in $H^s$ for $d \geq 2$ and $s > s_c (= d/2 - 1)$. For more general problem:

$$
\begin{cases}
  i\partial_t u - \Delta u = P(u, \nabla u, \nabla \pi), & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^d
\end{cases}
$$

(1.5)

there are many positive results for the well-posedness in the weighted Sobolev space ([1], [2], [3], [4], [21], [27]). Kenig, Ponce and Vega ([21]) also obtained that (1.5) is locally well-posed in $H^s$ (without weight) for large enough $s$ when $P$ has no quadratic terms.

The Benjamin–Ono equation:

$$
\partial_t u + H\partial_x^2 u = u\partial_x u, \ (t, x) \in \mathbb{R} \times \mathbb{R}
$$

(1.6)

is also related to the quadratic derivative nonlinear Schrödinger equation. It is known that the flow map of (1.6) is not uniformly continuous on $H^s$ for $s > 0$ ([22]). But the Benjamin–Ono equation has better structure than the equation (1.3). Actually, Tao ([28]) proved that (1.6) is globally well-posed in $H^1$ by using the gauge transform. Furthermore, Ionescu and Kenig ([18]) proved that (1.6) is
globally well-posed in $H^s_r$ for $s \geq 0$, where $H^s_r$ is the Banach space of the all real valued function $f \in H^s$.

Next, we introduce some known results for systems of quadratic nonlinear derivative Schrödinger equations. Ikeda, Katayama and Sunagawa ([19]) considered with null form nonlinearity and obtained the small data global existence and the scattering in the weighted Sobolev space for the dimension $d \geq 2$ under the condition $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) = 0$. While, Ozawa and Sunagawa ([25]) gave the examples of the quadratic derivative nonlinearity which causes the small data blow up for a system of Schrödinger equations. As the known result for (1.1), we introduce the work by Colin and Colin ([6]). They proved that the local existence of the solution of (1.1) for $s > d/2 + 3$. There are also some known results for a system of Schrödinger equations with no derivative nonlinearity ([7], [8], [9], [15], [16]). Our results are an extension of the results by Colin and Colin ([6]) and Grünrock ([12]).

Now, we give the main results in the present paper. For a Banach space $H$ and $r > 0$, we define $B_r(H) := \{ f \in H \mid \|f\|_H \leq r \}$. Furthermore, we put $\theta := \alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma)$ and $\kappa := (\alpha - \beta)(\alpha - \gamma)(\beta + \gamma)$. Note that if $\alpha, \beta, \gamma \in \mathbb{R}\{0\}$ and $\theta \geq 0$, then $\kappa \neq 0$.

**Theorem 1.1.**

(i) We assume that $\alpha, \beta, \gamma \in \mathbb{R}\{0\}$ satisfy $\kappa \neq 0$ if $d \geq 4$, and $\theta > 0$ if $d = 2, 3$. Then (1.1) is globally well-posed for small data in $\dot{H}^{s_c}$. More precisely, there exists $r > 0$ such that for all initial data $(u_0, v_0, w_0) \in B_r(\dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c})$, there exists a solution

$$(u, v, w) \in \dot{X}^{s_c}([0, \infty)) \subset C([0, \infty); \dot{H}^{s_c})$$

of the system (1.1) on $(0, \infty)$. Such solution is unique in $\dot{X}^{s_c}([0, \infty))$ which is a closed subset of $\dot{X}^{s_c}([0, \infty))$ (see (6.11) and (6.12)). Moreover, the flow map

$$S_+ : B_r(\dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c}) \ni (u_0, v_0, w_0) \mapsto (u, v, w) \in \dot{X}^{s_c}([0, \infty))$$

is Lipschitz continuous.

(ii) The statement in (i) remains valid if we replace the space $\dot{H}^{s_c}, \dot{X}^{s_c}([0, \infty))$ and $\dot{X}^{s_c}([0, \infty))$ by $H^s, X^s([0, \infty))$ and $X^s([0, \infty))$ for $s \geq s_c$.

**Remark 1.1.** Due to the time reversibility of the system (1.1), the above theorems also hold in corresponding intervals $(−\infty, 0)$. We denote the flow map with respect to $(-\infty, 0)$ by $S_-$. 

**Corollary 1.2.**

(i) We assume that $\alpha, \beta, \gamma \in \mathbb{R}\{0\}$ satisfy $\kappa \neq 0$ if $d \geq 4$, and $\theta > 0$ if $d = 2, 3$. 

Let $r > 0$ be as in Theorem 1.4. For every $(u_0, v_0, w_0) \in B_r(\dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c})$, there exists $(u_\pm, v_\pm, w_\pm) \in \dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c}$ such that

$$S_\pm(u_0, v_0, w_0) - (e^{it_\alpha \Delta} u_\pm, e^{it_\beta \Delta} v_\pm, e^{it_\gamma \Delta} w_\pm) \to 0$$

in $\dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c}$ as $t \to \pm \infty$.

(ii) The statement in (i) remains valid if we replace the space $\dot{H}^{s_c}$ by $H^s$ for $s \geq s_c$.

**Theorem 1.3.** Let $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$.

(i) Let $d \geq 4$. We assume $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $s > s_c$. Then (1.1) is locally well-posed in $H^s$. More precisely, for any $r > 0$ and for all initial data $(u_0, v_0, w_0) \in B_r(H^s \times H^s \times H^s)$, there exist $T = T(r) > 0$ and a solution

$$(u, v, w) \in X^s([0, T]) \subset C([0, T]; H^s)$$

of the system (1.1) on $(0, T]$. Such solution is unique in $X^s([0, T])$ which is a closed subset of $X^s([0, T])$. Moreover, the flow map

$$S_+: B_r(H^s \times H^s \times H^s) \ni (u_0, v_0, w_0) \mapsto (u, v, w) \in X^s([0, T])$$

is Lipschitz continuous.

(ii) Let $d = 2, 3$. We assume $s > s_c$ if $\theta > 0$, $s \geq 1$ if $\theta \leq 0$ and $\kappa \neq 0$, and $s > 1$ if $\alpha = \beta$. Then the statement in (i) remains valid.

(iii) Let $d = 1$. We assume $s \geq 0$ if $\theta > 0$, $s \geq 1$ if $\theta = 0$, and $s \geq 1/2$ if $\theta < 0$ and $(\alpha - \gamma)(\beta + \gamma) \neq 0$. Then the statement in (i) remains valid.

**Remark 1.2.** For the case $d = 1, 1 > s \geq 1/2, \theta < 0$ and $(\alpha - \gamma)(\beta + \gamma) \neq 0$, we prove the well-posedness as $X^s([0, T]) = X^s_{\alpha, b}([0, T]) \times X^s_{\beta, b}([0, T]) \times X^s_{\gamma, b}([0, T])$, where $X^s_{\alpha, b}$ denotes the standard Bourgain space which is the completion of the Schwarz space with respect to the norm $||u||_{X^s_{\alpha, b}} := ||\xi||^s (\tau + \sigma \xi^2) \tilde{u}||_{L^2_{\xi}}$ (see Appendix A).

System (1.11) has the following conservation quantities (see Proposition 7.1): 

$$M(u, v, w) := 2||u||_{L^2}^2 + ||v||_{L^2}^2 + ||w||_{L^2}^2,$$

$$H(u, v, w) := \alpha||\nabla u||_{L^2}^2 + \beta||\nabla v||_{L^2}^2 + \gamma||\nabla w||_{L^2}^2 + 2\text{Re}(w, \nabla (u \cdot \nabla))_{L^2}.$$

By using the conservation law for $M$ and $H$, we obtain the following result.

**Theorem 1.4.**

(i) Let $d = 1$ and assume that $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ satisfy $\theta > 0$. For every $(u_0, v_0, w_0) \in L^2 \times L^2 \times L^2$, we can extend the local $L^2$ solution of Theorem 1.3 globally in time.

(ii) We assume that $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ have the same sign and satisfy $\kappa \neq 0$ if $d = 2, 3$ and $(\alpha - \gamma)(\beta + \gamma) \neq 0$ if $d = 1$. There exists $r > 0$ such that for every
\((u_0, v_0, w_0) \in B_r(H^1 \times H^1 \times H^1)\), we can extend the local \(H^1\) solution of Theorem 1.3 globally in time.

While, we obtain the negative result as follows.

**Theorem 1.5.** Let \(d \geq 1\) and \(\alpha, \beta, \gamma \in \mathbb{R}\setminus\{0\}\). We assume \(s \in \mathbb{R}\) if \((\alpha-\gamma)(\beta+\gamma) = 0\), \(s < 1\) if \(\theta = 0\), and \(s < 1/2\) if \(\theta < 0\). Then the flow map of (1.1) is not \(C^2\) in \(H^s\).

Furthermore, for the equation (1.4), we obtain the following result.

**Theorem 1.6.** Let \(d \geq 2\). Then, the equation (1.4) is globally well-posed for small data in \(\dot{H}^{s_c}\) (resp. \(H^s\) for \(s \geq s_c\)) and the solution converges to a free solution in \(\dot{H}^{s_c}\) (resp. \(H^s\) for \(s \geq s_c\)) asymptotically in time.

**Remark 1.3.** The results by Grünrock ([12]) are not contained the critical case \(s = s_c\) and global property of the solution. In this sense, Theorem 1.6 is the extension of the results by Grünrock ([12]).

The main tools of our results are \(U^p\) space and \(V^p\) space which are applied to prove the well-posedness and scattering for KP-II equation at the scaling critical regularity by Hadac, Herr and Koch ([13], [14]). After their work, \(U^p\) space and \(V^p\) space are used to prove the well-posedness of the 3D periodic quintic nonlinear Schrödinger equation at the scaling critical regularity by Herr, Tataru and Tzvetkov ([17]) and to prove the well-posedness and the scattering of the quadratic Klein-Gordon system at the scaling critical regularity by Schottdorf ([26]).

**Notation.** We denote the spatial Fourier transform by \(\hat{\cdot}\) or \(\mathcal{F}_x\), the Fourier transform in time by \(\mathcal{F}_t\) and the Fourier transform in all variables by \(\hat{\cdot}\) or \(\mathcal{F}_{tx}\). For \(\sigma \in \mathbb{R}\), the free evolution \(e^{i\sigma \Delta}\) on \(L^2\) is given as a Fourier multiplier

\[
\mathcal{F}_x[e^{it\sigma \Delta} f](\xi) = e^{-it\sigma|\xi|^2} \hat{f}(\xi).
\]

We will use \(A \lesssim B\) to denote an estimate of the form \(A \leq CB\) for some constant \(C\) and write \(A \sim B\) to mean \(A \lesssim B\) and \(B \lesssim A\). We will use the convention that capital letters denote dyadic numbers, e.g. \(N = 2^n\) for \(n \in \mathbb{Z}\) and for a dyadic summation we write \(\sum_N a_N := \sum_{n \in \mathbb{Z}} a_{2^n}\) and \(\sum_{N \geq M} a_N := \sum_{n \in \mathbb{Z}, 2^n \geq M} a_{2^n}\) for brevity. Let \(\chi \in C_0^\infty((-2, 2))\) be an even, non-negative function such that \(\chi(t) = 1\) for \(|t| \leq 1\). We define \(\psi(t) := \chi(t) - \chi(2t)\) and \(\psi_N(t) := \psi(N^{-1}t)\). Then, \(\sum_N \psi_N(t) = 1\) whenever \(t \neq 0\). We define frequency and modulation projections

\[
\mathcal{F}_N u(\xi) := \psi_N(\xi) \hat{u}(\xi), \quad \mathcal{F}_0 u(\xi) := \psi_0(\xi) \hat{u}(\xi), \quad \mathcal{Q}_M^\pm u(\tau, \xi) := \psi_M(\tau + \sigma|\xi|^2) \hat{u}(\tau, \xi),
\]
where $\psi_0 := 1 - \sum_{N \geq 1} \psi_N$. Furthermore, we define $Q_{\geq M} := \sum_{N \geq M} Q_N$ and $Q_{< M} := Id - Q_{\geq M}$.

The rest of this paper is planned as follows. In Section 2, we will give the definition and properties of the $U^p$ space and $V^p$ space. In Sections 3, 4 and 5, we will give the bilinear and trilinear estimates which will be used to prove the well-posedness. In Section 6, we will give the proof of the well-posedness and the scattering (Theorems 1.1, 1.3, 1.6 and Corollary 1.2). In Section 7, we will give the a priori estimates and show Theorem 1.4. In Section 8, we will give the proof of $C^2$-ill-posedness (Theorem 1.5). In Appendix A, we will give the proof of the bilinear estimates for the standard 1-dimensional Bourgain norm under the condition $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) \neq 0$.

2. $U^p$, $V^p$ spaces and their properties

In this section, we define the $U^p$ space and the $V^p$ space, and introduce the properties of these spaces which are proved by Hadac, Herr and Koch ([13], [14]).

We define the set of finite partitions $Z$ as

$$Z := \{ \{t_k\}_{k=0}^K | K \in \mathbb{N}, -\infty < t_0 < t_1 < \cdots < t_K \leq \infty \}$$

and if $t_K = \infty$, we put $v(t_K) := 0$ for all functions $v : \mathbb{R} \to L^2$.

**Definition 1.** Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in Z$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} ||\phi_k||_{L^2}^p = 1$ we call the function $a : \mathbb{R} \to L^2$ given by

$$a(t) = \sum_{k=1}^K 1_{[t_{k-1}, t_k)}(t)\phi_{k-1}$$

a "$U^p$-atom". Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j | a_j : U^p \text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^\infty |\lambda_j| < \infty \right\}$$

with the norm

$$||u||_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| | u = \sum_{j=1}^\infty \lambda_j a_j, a_j : U^p \text{-atom}, \lambda_j \in \mathbb{C} \right\}.$$

**Definition 2.** Let $1 \leq p < \infty$. We define the space of the bounded $p$-variation

$$V^p := \{ v : \mathbb{R} \to L^2 | ||v||_{V^p} < \infty \}$$

with the norm

$$||v||_{V^p} := \sup_{\{t_k\}_{k=0}^K \in Z} \left( \sum_{k=1}^K ||v(t_k) - v(t_{k-1})||_{L^2}^p \right)^{1/p}. \quad (2.1)$$
Likewise, let $V^{p}_{rc}$ denote the closed subspace of all right-continuous functions $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$, endowed with the same norm (2.1).

**Proposition 2.1.** Let $1 \leq p < q < \infty$.

(i) $U^p$, $V^p$ and $V^p_{rc}$ are Banach spaces.

(ii) Every $u \in U^p$ is right-continuous as $u : \mathbb{R} \to L^2$.

(iii) For every $u \in U^p$, $\lim_{t \to -\infty} u(t) = 0$ and $\lim_{t \to \infty} u(t)$ exists in $L^2$.

(iv) For every $v \in V^p$, $\lim_{t \to -\infty} v(t)$ and $\lim_{t \to \infty} v(t)$ exist in $L^2$.

(v) The embeddings $U^p \hookrightarrow V^p_{rc} \hookrightarrow U^q \hookrightarrow L^\infty_t (\mathbb{R}; L^2_x (\mathbb{R}^d))$ are continuous.

For Proposition 2.1 and its proof, see Propositions 2.2, 2.4 and Corollary 2.6 in [13].

**Theorem 2.2.** Let $1 < p < \infty$ and $1/p + 1/p' = 1$. If $u \in V^1_{rc}$ be absolutely continuous on every compact intervals, then

$$
\|u\|_{U^p} = \sup_{v \in V^p \setminus \{0\}, \|v\|_{V^p} = 1} \left\| \int_{-\infty}^{\infty} (u'(t), v(t))_{L^2(\mathbb{R})} dt \right\|.
$$

For Theorem 2.2 and its proof, see Theorem 2.8 Proposition 2.10 and Remark 2.11 in [13].

**Definition 3.** Let $1 \leq p < \infty$. For $\sigma \in \mathbb{R}$, we define

$$
U^p_{\sigma} := \{ u : \mathbb{R} \to L^2 | e^{-it\sigma \Delta} u \in U^p \}
$$

with the norm $\|u\|_{U^p_{\sigma}} := \|e^{-it\sigma \Delta} u\|_{U^p}$,

$$
V^p_{\sigma} := \{ v : \mathbb{R} \to L^2 | e^{-it\sigma \Delta} v \in V^p \}
$$

with the norm $\|v\|_{V^p_{\sigma}} := \|e^{-it\sigma \Delta} v\|_{V^p}$ and similarly the closed subspace $V^p_{-rc,\sigma}$.

**Remark 2.1.** We note that $\|\overline{u}\|_{U^p_{\sigma}} = \|u\|_{U^p_{-\sigma}}$ and $\|\overline{v}\|_{V^p_{\sigma}} = \|v\|_{V^p_{-\sigma}}$.

**Proposition 2.3.** Let $1 < p < \infty$. We have

$$
\|Q^\sigma_M u\|_{L^p_t L^2_x} \lesssim M^{-1/p} \|u\|_{V^p_{\sigma}}, \quad \|Q^\sigma_{\geq M} u\|_{L^p_t L^2_x} \lesssim M^{-1/p} \|u\|_{V^p_{\sigma}}, \quad (2.2)
$$

$$
\|Q^\sigma_{< M} u\|_{V^p_{\sigma}} \lesssim \|u\|_{V^p_{\sigma}}, \quad \|Q^\sigma_{\geq M} u\|_{V^p_{\sigma}} \lesssim \|u\|_{V^p_{\sigma}}, \quad (2.3)
$$

$$
\|Q^\sigma_{< M} u\|_{V^p_{\sigma}} \lesssim \|u\|_{V^p_{\sigma}}, \quad \|Q^\sigma_{\geq M} u\|_{V^p_{\sigma}} \lesssim \|u\|_{V^p_{\sigma}}. \quad (2.4)
$$

For Proposition 2.3 and its proof, see Corollary 2.18 in [13]. (2.2) for $p \neq 2$ can be proved by the same way.
Proposition 2.4. Let
\[ T_0 : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d) \]
be a \( m \)-linear operator and \( I \subset \mathbb{R} \) be an interval. Assume that for some \( 1 \leq p, q < \infty \)
\[ ||T_0(e^{it\sigma_1 \Delta} \phi_1, \cdots, e^{it\sigma_m \Delta} \phi_m)||_{L^p(I;L^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^m ||\phi_i||_{L^2(\mathbb{R}^d)}. \]
Then, there exists \( T : U^p_{r_1} \times \cdots \times U^p_{r_m} \to L^p_I(I;L^q(\mathbb{R}^d)) \) satisfying
\[ ||T(u_1, \cdots, u_m)||_{L^p(I;L^q(\mathbb{R}^d))} \lesssim \prod_{i=1}^m ||u_i||_{U^p_{r_i}} \]
such that \( T(u_1, \cdots, u_m)(t)(x) = T_0(u_1(t), \cdots, u_m(t))(x) \) a.e.

For Proposition 2.4 and its proof, see Proposition 2.19 in [13].

Proposition 2.5 (Strichartz estimate). Let \( \sigma \in \mathbb{R} \setminus \{0\} \) and \( (p, q) \) be an admissible pair of exponents for the Schrödinger equation, i.e. \( 2 \leq q \leq 2d/(d-2) \) \((2 \leq q < \infty \) if \( d = 2 \), \( 2 \leq q \leq \infty \) if \( d = 1 \)), \( 2/p = d(1/2 - 1/q) \). Then, we have
\[ ||e^{it\sigma \Delta} \varphi||_{L^p_tL^q_x} \lesssim ||\varphi||_{L^2_x} \]
for any \( \varphi \in L^2(\mathbb{R}^d) \).

By Proposition 2.4 and 2.5, we have following:

Corollary 2.6. Let \( \sigma \in \mathbb{R} \setminus \{0\} \) and \( (p, q) \) be an admissible pair of exponents for the Schrödinger equation, i.e. \( 2 \leq q \leq 2d/(d-2) \) \((2 \leq q < \infty \) if \( d = 2 \), \( 2 \leq q \leq \infty \) if \( d = 1 \)), \( 2/p = d(1/2 - 1/q) \). Then, we have
\[ ||u||_{L^p_tL^q_x} \lesssim ||u||_{V^p_{\sigma}}, \quad u \in U^p_{\sigma}, \quad (2.5) \]
\[ ||u||_{L^p_tL^q_x} \lesssim ||u||_{V^p_{\bar{\sigma}}}, \quad u \in V^p_{\bar{\sigma}}, \quad (1 \leq \bar{\sigma} < p). \quad (2.6) \]

Proposition 2.7. Let \( q > 1 \), \( E \) be a Banach space and \( T : U^q_g \to E \) be a bounded, linear operator with \( ||Tu||_E \leq C_q||u||_{U^q_g} \) for all \( u \in U^q_g \). In addition, assume that for some \( 1 \leq p < q \) there exists \( C_p \in (0,C_q] \) such that the estimate \( ||Tu||_E \leq C_p||u||_{U^p_g} \) holds true for all \( u \in U^p_g \). Then, \( T \) satisfies the estimate
\[ ||Tu||_E \lesssim C_p \left( 1 + \ln \frac{C_q}{C_p} \right)||u||_{V^p_{g}}, \quad u \in V^p_{g,r(\sigma)}, \]
where implicit constant depends only on \( p \) and \( q \).

For Proposition 2.7 and its proof, see Proposition 2.20 in [13].
3. Bilinear Strichartz estimates

In this section, implicit constants in \( \ll \) actually depend on \( \sigma_1, \sigma_2 \).

**Lemma 3.1.** Let \( d \in \mathbb{N} \), \( s_c = d/2 - 1 \), \( b > 1/2 \) and \( \sigma_1, \sigma_2 \in \mathbb{R} \setminus \{0\} \). For any dyadic numbers \( L, H \in \mathbb{Z}^2 \) with \( L \ll H \), we have

\[
|\langle (P_H u_1)(P_L u_2) \rangle_{L^2_x} \leq L^{s_c} \left( \frac{L}{H} \right)^{1/2} |\langle P_H u_1 \rangle_{X^{0,b}_x} \| P_L u_2 \|_{X^{0,b}_x},
\]

(3.1)

where \( \|u\|_{X^{0,b}_x} := \| (\tau + \sigma|\xi|^2)^b \hat{u} \|_{L^2_{x \xi}} \).

**Proof.** For the case \( d = 2 \) and \((\sigma_1, \sigma_2) = (1, \pm 1)\), the estimate (3.1) is proved by Colliander, Delort, Kenig, and Staffilani (10), Lemma 1. The proof for general case as following is similar to their argument.

We put \( g_1(\tau_1, \xi_1) := \langle \tau_1 + \sigma_1 |\xi_1|^2 \rangle^b \tilde{P}_H \hat{u}_1(\tau_1, \xi_1), g_2(\tau_2, \xi_2) := \langle \tau_2 + \sigma_2 |\xi_2|^2 \rangle^b \tilde{P}_L \hat{u}_2(\tau_2, \xi_2) \) and \( A_N := \{ \xi \in \mathbb{R}^d \mid |\xi| \leq 2N \} \) for a dyadic number \( N \) by the Plancherel’s theorem and the duality argument, it is enough to prove the estimate

\[
I := \left| \int_{\mathbb{R}} \int_{\mathbb{R}} A_L \int_{\mathbb{R}} f(\tau_1 + \tau_2, \xi_1 + \xi_2) \frac{g_1(\tau_1, \xi_1)}{(\tau_1 + \sigma_1 |\xi_1|^2)^b} \frac{g_2(\tau_2, \xi_2)}{(\tau_2 + \sigma_2 |\xi_2|^2)^b} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right| 
\leq \frac{L^{(d-1)/2}}{H^{1/2}} \|f\|_{L^2_{x \xi}} \|g_1\|_{L^2_{x \xi}} \|g_2\|_{L^2_{x \xi}}
\]

(3.2)

for \( f \in L^2_{x \xi} \). We change the variables \((\tau_1, \tau_2) \leftrightarrow (\theta_1, \theta_2)\) as \( \theta_i = \tau_i + \sigma_i |\xi_i|^2 \) \((i = 1, 2)\) and put

\[
F(\theta_1, \theta_2, \xi_1, \xi_2) := f(\theta_1 + \theta_2 - \sigma_1 |\xi_1|^2 - \sigma_2 |\xi_2|^2, \xi_1 + \xi_2),
\]

\[
G_i(\theta_i, \xi_i) := g_i(\theta_i - \sigma_i |\xi_i|^2, \xi_i), \quad (i = 1, 2).
\]

Then, we have

\[
I \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(\theta_1)^b(\theta_2)^b} \left( \int_{A_L} \int_{A_H} |F(\theta_1, \theta_2, \xi_1, \xi_2)G_1(\theta_1, \xi_1)G_2(\theta_2, \xi_2)| d\xi_1 d\xi_2 \right) d\theta_1 d\theta_2 
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(\theta_1)^b(\theta_2)^b} \left( \int_{A_L} \int_{A_H} |F(\theta_1, \theta_2, \xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{1/2} \|G_1(\theta_1, \cdot)\|_{L^2_{x \xi}} \|G_2(\theta_2, \cdot)\|_{L^2_{x \xi}} d\theta_1 d\theta_2
\]

by the Cauchy-Schwarz inequality. For \( 1 \leq j \leq d \), we put

\[
A_{jH} := \{ \xi_1 = (\xi_1^{(1)}, \cdots, \xi_1^{(d)}) \in \mathbb{R}^d \mid H/2 \leq |\xi_1| \leq 2H, \ |\xi_1^{(j)}| \geq H/(2\sqrt{d}) \}
\]

and

\[
K_j(\theta_1, \theta_2) := \int_{A_L} \int_{A_{jH}} |F(\theta_1, \theta_2, \xi_1, \xi_2)|^2 d\xi_1 d\xi_2.
\]

We consider only the estimate for \( K_1 \). The estimates for other \( K_j \) are obtained by the same way.
Assume $d \geq 2$. By changing the variables $(\xi_1, \xi_2) = (\xi_1^{(1)}, \ldots, \xi_1^{(d)}, \xi_2^{(1)}, \ldots, \xi_2^{(d)}) \mapsto (\mu, \nu, \eta)$ as

$$
\begin{cases}
\mu = \theta_1 + \theta_2 - \sigma_1|\xi_1|^2 - \sigma_2|\xi_2|^2 \in \mathbb{R}, \\
\nu = \xi_1 + \xi_2 \in \mathbb{R}^d, \\
\eta = (\xi_2^{(2)} \cdot \cdots \cdot \xi_2^{(d)}) \in \mathbb{R}^{d-1},
\end{cases}
$$

(3.3)

we have

$$d\mu d\nu d\eta = 2|\sigma_1\xi_1^{(1)} - \sigma_2\xi_2^{(1)}|d\xi_1 d\xi_2$$

and

$$F(\theta_1, \theta_2, \xi_1, \xi_2) = f(\mu, \nu).$$

We note that $|\sigma_1\xi_1^{(1)} - \sigma_2\xi_2^{(1)}| \sim H$ for any $(\xi_1, \xi_2) \in A_H^1 \times A_L$ with $L \ll H$. Furthermore, $\xi_2 \in A_L$ implies that $\eta \in [-2L, 2L]^{d-1}$. Therefore, we obtain

$$K_1(\theta_1, \theta_2) \lesssim \frac{1}{H} \int_{[-2L, 2L]^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |f(\mu, \nu)|^2 d\mu d\nu d\eta \sim \frac{L^{d-1}}{H} ||f||^{2}_{L^2_{q_2}}.$$

As a result, we have

$$I \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(\theta_1)^b(\theta_2)^b} \left( \sum_{j=1}^{d} K_j(\theta_1, \theta_2) \right)^{1/2} ||G_1(\theta_1, \cdot)||_{L^2_{q_1}} ||G_2(\theta_2, \cdot)||_{L^2_{q_2}} \, d\theta_1 d\theta_2$$

$$\lesssim \frac{L^{(d-1)/2}}{H^{1/2}} ||f||_{L^2_{q_2}} ||g_1||_{L^2_{q_1}} ||g_2||_{L^2_{q_2}}$$

by the Cauchy-Schwarz inequality and changing the variables $(\theta_1, \theta_2) \mapsto (\tau_1, \tau_2)$ as $\theta_i = \tau_i + \sigma_i|\xi_i|^2$ ($i = 1, 2$).

For $d = 1$, we obtain the same result by changing the variables $(\xi_1, \xi_2) \mapsto (\mu, \nu)$ as $\mu = \theta_1 + \theta_2 - \sigma_1|\xi_1|^2 - \sigma_2|\xi_2|^2$, $\nu = \xi_1 + \xi_2$ instead of (3.3). □

**Corollary 3.2.** Let $d \in \mathbb{N}$, $s_c = d/2 - 1$ and $\sigma_1, \sigma_2 \in \mathbb{R}\setminus\{0\}$.

(i) If $d \geq 2$, then for any dyadic numbers $L, H \in 2\mathbb{Z}$ with $L \ll H$, we have

$$|||P_Hu_1)(P_L^2u_2)|||_{L^2_{[2]}(\mathbb{R})} \lesssim L^{s_c} \left( \frac{L}{H} \right)^{1/2} ||P_Hu_1||_{v_{\sigma_1}^2} ||P_Lu_2||_{v_{\sigma_2}^2},$$

(3.4)

$$|||P_Hu_1)(P_Lu_2)|||_{L^2_{[2]}(\mathbb{R})} \lesssim L^{s_c} \left( \frac{L}{H} \right)^{1/2} \left( 1 + \ln \frac{H}{L} \right)^2 ||P_Hu_1||_{v_{\sigma_1}^2} ||P_Lu_2||_{v_{\sigma_2}^2}. $$

(3.5)

(ii) If $d = 1$, then for any dyadic numbers $L, H \in 2\mathbb{Z}$ with $L \ll H$, we have

$$|||P_Hu_1)(P_Lu_2)|||_{L^2([0,1] \times \mathbb{R})} \lesssim \frac{1}{H^{1/2}} ||P_Hu_1||_{v_{\sigma_1}^2} ||P_Lu_2||_{v_{\sigma_2}^2},$$

(3.6)

$$|||P_Hu_1)(P_Lu_2)|||_{L^2([0,1] \times \mathbb{R})} \lesssim \min \left\{ L^{1/6}, \frac{(1 + \ln H)^2}{H^{1/2}} \right\} ||P_Hu_1||_{v_{\sigma_1}^2} ||P_Lu_2||_{v_{\sigma_2}^2}. $$

(3.7)
Proof. To obtain (3.4) and (3.6), we use the argument of the proof of Corollary 2.21 (27) in [13]. Let \( \phi_1, \phi_2 \in L^2(\mathbb{R}^d) \) and define \( \phi_j(x) := \phi_j(\lambda x) \) (\( j = 1, 2 \)) for \( \lambda \in \mathbb{R} \).

By using the rescaling \((t, x) \mapsto (\lambda^2 t, \lambda x)\), we have
\[
\|P_H(e^{it\sigma_1 \Delta} \phi_1)P_L(e^{it\sigma_2 \Delta} \phi_2)\|_{L^2([-T, T] \times \mathbb{R}^d)} = \lambda^{s_c+2}\|P_{\lambda H}(e^{it\sigma_1 \Delta} \phi_1^\lambda)P_{\lambda L}(e^{it\sigma_2 \Delta} \phi_2^\lambda)\|_{L^2([-\lambda^{-2} T, \lambda^{-2} T] \times \mathbb{R}^d)}.
\]

Therefore by putting \( \lambda = \sqrt{T} \) and Lemma 3.1, we have
\[
\|P_H(e^{it\sigma_1 \Delta} \phi_1)P_L(e^{it\sigma_2 \Delta} \phi_2)\|_{L^2([-T, T] \times \mathbb{R}^d)} \lesssim \sqrt{T}^{2(s_c+1)} L^{s_c} \left( \frac{L}{H} \right)^{1/2} \|P_{\sqrt{T}H} \phi_1^{\sqrt{T}}\|_{L^2_T} \|P_{\sqrt{T}L} \phi_2^{\sqrt{T}}\|_{L^2_T} = L^{s_c} \left( \frac{L}{H} \right)^{1/2} \|P_H \phi_1\|_{L^2_T} \|P_L \phi_2\|_{L^2_T}.
\]

Let \( T \to \infty \), then we obtain
\[
\|P_H(e^{it\sigma_1 \Delta} \phi_1)P_L(e^{it\sigma_2 \Delta} \phi_2)\|_{L^2_{tx}} \lesssim L^{s_c} \left( \frac{L}{H} \right)^{1/2} \|P_H \phi_1\|_{L^2_T} \|P_L \phi_2\|_{L^2_T}
\]
and (3.4), (3.6) follow from proposition 2.4.

To obtain (3.5) and (3.7), we first prove the \( U^4 \) estimate for \( d \geq 2 \) and \( U^8 \) estimate for \( d = 1 \). Assume \( d \geq 2 \). By the Cauchy-Schwarz inequality, the Sobolev embedding
\[
\dot{W}^{s_c, 2d/(d-1)}(\mathbb{R}^d) \hookrightarrow L^{2d}(\mathbb{R}^d)
\]
and (2.5), we have
\[
\|(P_H u_1)(P_L u_2)\|_{L^2_{tx}} \lesssim L^{s_c} \|P_H u_1\|_{L^4_T L^{2d/(d-1)}_x} \|P_L u_2\|_{L^4_T L^{2d/(d-1)}_x}
\]
(3.8)
for any dyadic numbers \( L, H \in 2^Z \). While if \( d = 1 \), then by the Hölder’s inequality and (2.5), we have
\[
\|(P_H u_1)(P_L u_2)\|_{L^2([0,1] \times \mathbb{R})} \lesssim \|1_{[0,1]}\|_{L^4_T} \|P_H u_1\|_{L^4_T L^4_1} \|P_L u_2\|_{L^8_T L^4_1}
\]
(3.9)
for any dyadic numbers \( L, H \in 2^Z \). We use the interpolation between (3.4) and (3.8) via Proposition 2.7. Then, we get (3.5) by the same argument of the proof of Corollary 2.21 (28) in [13]. The estimate (3.7) follows from
\[
\|(P_H u_1)(P_L u_2)\|_{L^2([0,1] \times \mathbb{R})} \leq \|1_{[0,1]}\|_{L^6_T} L^{1/6} \|P_H u_1\|_{L^6_T L^6_1} \|P_L u_2\|_{L^6_T L^6_1}
\]
(3.10)
and the interpolation between (3.6) and (3.9), where we used the Hölder’s inequality, the Sobolev embedding \( \dot{W}^{1/6,3}(\mathbb{R}) \hookrightarrow L^6(\mathbb{R}) \) and (2.6) to obtain (3.10). \( \square \)
4. Time global estimates for $d \geq 2$

In this and next section, implicit constants in $\ll$ actually depend on $\sigma_1, \sigma_2, \sigma_3$.

**Lemma 4.1.** Let $d \in \mathbb{N}$. We assume that $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$ and $(\tau_1, \xi_1), (\tau_2, \xi_2), (\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^d$ satisfy $\tau_1 + \tau_2 + \tau_3 = 0, \xi_1 + \xi_2 + \xi_3 = 0$.

(i) If there exist $1 \leq i, j \leq 3$ such that $|\xi_i| \ll |\xi_j|$, then we have

$$\max_{1 \leq j \leq 3} |\tau_j + \sigma_j|_{\xi_j}^2 \gtrsim \max_{1 \leq j \leq 3} |\xi_j|^2. \quad (4.1)$$

(ii) If $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, then we have (4.1).

**Proof.** By the triangle inequality and the completing the square, we have

$$M_0 := \max_{1 \leq j \leq 3} |\tau_j + \sigma_j|_{\xi_j}^2$$

$$\gtrsim |\sigma_1|_{\xi_1}^2 + |\sigma_2|_{\xi_2}^2 + |\sigma_3|_{\xi_3}^2$$

$$= |(\sigma_1 + \sigma_3)|_{\xi_1}^2 + 2\sigma_3 \xi_1 \cdot \xi_2 + (\sigma_2 + \sigma_3)|_{\xi_2}^2$$

$$= |\sigma_1 + \sigma_3| - \frac{\sigma_3}{\sigma_1 + \sigma_3} \xi_2^2 + \frac{\sigma_1 \sigma_2 \sigma_3}{(\sigma_1 + \sigma_3)^2} \left( \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3} \right) |\xi_2|^2. \quad (4.2)$$

We first prove (i). By the symmetry, we can assume $|\xi_1| \sim |\xi_3| \gtrsim |\xi_2|$. If $|\xi_1| \gg |\xi_2|$, then we have $M_0 \gtrsim |\xi_1|^2 \sim \max_{1 \leq j \leq 3} |\xi_j|^2$ by (4.2). Next, we prove (ii). By the symmetry, we can assume $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3|$. If $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, then we have $M_0 \gtrsim |\xi_2|^2 \sim \max_{1 \leq j \leq 3} |\xi_j|^2$ by (4.2). \qed

In the following Propositions and Corollaries in this and next section, we assume $P_{N_1} u_1 \in V_{-rc, \sigma_1}^2, P_{N_2} u_2 \in V_{-rc, \sigma_2}^2$ and $P_{N_3} u_3 \in V_{-rc, \sigma_3}^2$ for each $N_1, N_2, N_3 \in 2^\mathbb{Z}$. Propositions 4.2, 4.3 and its proofs are based on Proposition 3.1 in [13].

**Proposition 4.2.** Let $d \geq 2$, $s_c = d/2 - 1$, $0 < T \leq \infty$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$. For any dyadic numbers $N_2, N_3 \in 2^\mathbb{Z}$ with $N_2 \sim N_3$, we have

$$\left| \sum_{N_1 \ll N_2} N_{\text{max}} \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right|$$

$$\lesssim \left( \sum_{N_1 \ll N_2} N_{1}^{2s_c} \|P_{N_1} u_1\|_{V_{s_c}^2}^2 \right)^{1/2} \|P_{N_2} u_2\|_{V_{s_c}^2} \|P_{N_3} u_3\|_{V_{s_c}^2}, \quad (4.3)$$

where $N_{\text{max}} := \max_{1 \leq j \leq 3} N_j$. 


Proof. We define \( f_{j,N,T} := 1_{[0,T]}P_{N_j}u_j \) \((j = 1, 2, 3)\). For sufficiently large constant \( C \), we put \( M := C^{-1}N^2_{\text{max}} \) and decompose \( Id = Q_{\leq M}^{\sigma_j} + Q_{\geq M}^{\sigma_j} \) \((j = 1, 2, 3)\). We divide the integrals on the left-hand side of (4.3) into eight piece of the form

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^{\sigma_1} f_{1,N_1,T})(Q_2^{\sigma_2} f_{2,N_2,T})(Q_3^{\sigma_3} f_{3,N_3,T}) dx dt
\]

with \( Q_j^{\sigma_j} \in \{Q_{\leq M}^{\sigma_j}, Q_{\geq M}^{\sigma_j}\} \) \((j = 1, 2, 3)\). By the Plancherel’s theorem, we have

\[
(4.4) = c \int_{\tau_1 + \tau_2 + \tau_3 = 0} \int_{\xi_1 + \xi_2 + \xi_3 = 0} \mathcal{F}[Q_1^{\sigma_1} f_{1,N_1,T}](\tau_1, \xi_1) \mathcal{F}[Q_2^{\sigma_2} f_{2,N_2,T}](\tau_2, \xi_2) \mathcal{F}[Q_3^{\sigma_3} f_{3,N_3,T}](\tau_3, \xi_3),
\]

where \( c \) is a constant. Therefore, Lemma 4.1 (i) implies that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_{\leq M}^{\sigma_1} f_{1,N_1,T})(Q_{\leq M}^{\sigma_2} f_{2,N_2,T})(Q_{\leq M}^{\sigma_3} f_{3,N_3,T}) dx dt = 0
\]

when \( N_1 \ll N_2 \). So, let us now consider the case that \( Q_j^{\sigma_j} = Q_{\geq M}^{\sigma_j} \) for some \( 1 \leq j \leq 3 \).

First, we consider the case \( Q_1^{\sigma_1} = Q_{\geq M}^{\sigma_1} \). By the Hölder’s inequality and the Sobolev embedding \( \hat{H}^{s_1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \), we have

\[
\left| \sum_{N_1 \ll N_2} N_{\text{max}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1} f_{1,N_1,T})(Q_{\geq M}^{\sigma_2} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3} f_{3,N_3,T}) dx dt \right| \leq \left\| \sum_{N_1 \ll N_2} N_{\text{max}} |\nabla|^{s_1} Q_{\geq M}^{\sigma_1} f_{1,N_1,T} \right\|_{L^2_t L^2_x} \left\| Q_{\geq M}^{\sigma_2} f_{2,N_2,T} \right\|_{L^1_t L^{2d/(d-1)}_x} \left\| Q_{\geq M}^{\sigma_3} f_{3,N_3,T} \right\|_{L^1_t L^{2d/(d-1)}_x}.
\]

Furthermore, by the \( L^2 \) orthogonality and (2.2) with \( p = 2 \), we have

\[
\left\| \sum_{N_1 \ll N_2} N_{\text{max}} |\nabla|^{s_1} Q_{\geq M}^{\sigma_1} f_{1,N_1,T} \right\|_{L^2_t L^2_x} \lesssim \left( \sum_{N_1 \ll N_2} N_{\text{max}}^2 N_{\text{max}}^{2s_1} M^{-1} \| f_{1,N_1,T} \|_{V_{s_1}^2}^2 \right)^{1/2}.
\]

While by (2.6) and (2.3), we have

\[
\| Q_{\geq M}^{\sigma_2} f_{2,N_2,T} \|_{L^1_t L^{2d/(d-1)}_x} \lesssim \| f_{2,N_2,T} \|_{V_{s_2}^2}, \quad \| Q_{\geq M}^{\sigma_3} f_{3,N_3,T} \|_{L^1_t L^{2d/(d-1)}_x} \lesssim \| f_{3,N_3,T} \|_{V_{s_3}^2}.
\]

Therefore, we obtain

\[
\left\| \sum_{N_1 \ll N_2} N_{\text{max}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1} f_{1,N_1,T})(Q_{\geq M}^{\sigma_2} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3} f_{3,N_3,T}) dx dt \right\| \lesssim \left( \sum_{N_1 \ll N_2} N_{\text{max}}^2 \| P_{N_1} u_1 \|_{V_{s_1}^2}^2 \right)^{1/2} \| P_{N_2} u_2 \|_{V_{s_2}^2} \| P_{N_3} u_3 \|_{V_{s_3}^2},
\]

since \( M \sim N_{\text{max}}^2 \) and \( \| 1_{[0,T]} f \|_{V_{\sigma}^2} \lesssim \| f \|_{V_{\sigma}^2} \) for any \( \sigma \in \mathbb{R} \) and any \( T \in (0, \infty) \).
Next, we consider the case $Q_3^{σ_3} = Q_2^{σ_1}$. By the Cauchy-Schwarz inequality, we have

$$\left| \sum_{N_1 \ll N_2} N_{\text{max}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^{σ_1} f_{1,N_1,T})(Q_2^{σ_2} f_{2,N_2,T})(Q_3^{σ_3} f_{3,N_3,T}) \, dx \, dt \right|$$

$$\leq \sum_{N_1 \ll N_2} N_{\text{max}} \|(Q_1^{σ_1} f_{1,N_1,T})(Q_2^{σ_2} f_{2,N_2,T})\|_{L^2_{tx}} \|(Q_3^{σ_3} f_{3,N_3,T})\|_{L^2_{tx}}.$$

Furthermore, by (2.2) with $p = 2$, we have

$$\|(Q_3^{σ_3} f_{3,N_3,T})\|_{L^2_{tx}} \lesssim M^{-1/2}\|f_{3,N_3,T}\|_{V^2_{σ_3}}. \quad (4.6)$$

While by (3.5), (2.3) and the Cauchy-Schwarz inequality for the dyadic sum, we have

$$\sum_{N_1 \ll N_2} \|(Q_1^{σ_1} f_{1,N_1,T})(Q_2^{σ_2} f_{2,N_2,T})\|_{L^2_{tx}}$$

$$\lesssim \sum_{N_1 \ll N_2} N^{2c} \left( \frac{N_1}{N_2} \right)^{1/4} \|Q_1^{σ_1} f_{1,N_1,T}\|_{V^2_{σ_1}} \|Q_2^{σ_2} f_{2,N_2,T}\|_{V^2_{σ_2}}$$

$$\lesssim \left( \sum_{N_1 \ll N_2} N^{2c} \|f_{1,N_1,T}\|_{V^2_{σ_1}}^2 \right)^{1/2} \|f_{2,N_2,T}\|_{V^2_{σ_2}}. \quad (4.7)$$

Therefore, we obtain

$$\left| \sum_{N_1 \ll N_2} N_{\text{max}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^{σ_1} f_{1,N_1,T})(Q_2^{σ_2} f_{2,N_2,T})(Q_3^{σ_3} f_{3,N_3,T}) \, dx \, dt \right|$$

$$\lesssim \left( \sum_{N_1 \ll N_2} N^{2c} \|P_{N_1} u_1\|_{V^2_{σ_1}}^2 \right)^{1/2} \|P_{N_2} u_2\|_{V^2_{σ_2}} \|P_{N_3} u_3\|_{V^2_{σ_3}},$$

since $M \sim N^2_{\text{max}}$ and $\|1_{[0,T]} f\|_{V^2_{σ}} \lesssim \|f\|_{V^2_{σ}}$ for any $σ \in \mathbb{R}$ and any $T \in (0, \infty]$.

For the case $Q_2^{σ_2} = Q_3^{σ_3}$ is proved in exactly same way as the case $Q_3^{σ_3} = Q_{≥M}$.

**Proposition 4.3.** Let $d \geq 2$, $s_c = d/2 - 1$, $s \geq 0$, $0 < T \leq \infty$ and $σ_1$, $σ_2$, $σ_3 \in \mathbb{R} \setminus \{0\}$ satisfy $(σ_1 + σ_2)(σ_2 + σ_3)(σ_3 + σ_1) \neq 0$. For any dyadic numbers $N_1$, $N_2 \in 2^\mathbb{Z}$ with $N_1 \sim N_2$, we have

$$\left( \sum_{N_1 \ll N_2} N^{2c} \sup_{\|u_3\|_{V^2_{σ_3}} = 1} \left( \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dxdt \right)^2 \right)^{1/2} \quad (4.8)$$

$$\lesssim N^{2c} \|P_{N_1} u_1\|_{V^2_{σ_1}} \|P_{N_2} u_2\|_{V^2_{σ_2}} \|P_{N_3} u_3\|_{V^2_{σ_3}},$$

where $N_{\text{max}} := \max_{1 \leq j \leq 3} N_j$. 
Proof. We define $f_{j,N_{j},T} := 1_{[0,T]}P_{N_{j}}u_{j}$ $(j = 1, 2, 3)$. For sufficiently large constant $C$, we put $M := C^{-1}N_{\text{max}}^{2}$ and decompose $Id = Q_{<M}^{j} + Q_{\geq M}^{j}$ $(j = 1, 2, 3)$. We divide the integrals on the left-hand side of (4.8) into eight pieces of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{d}} (Q_{1}^{j}f_{1,N_{1},T})(Q_{2}^{j}f_{2,N_{2},T})(Q_{3}^{j}f_{3,N_{3},T})dxdt$$

(4.9)

with $Q_{j}^{j} \in \{Q_{\geq M}^{j}, Q_{<M}^{j}\}$ $(j = 1, 2, 3)$. By the same argument of the proof of Proposition 4.2, we consider only the case that $Q_{j}^{j} = Q_{\geq M}^{j}$ for some $1 \leq j \leq 3$.

First, we consider the case $Q_{1}^{1} = Q_{\geq M}^{1}$. By the Cauchy-Schwarz inequality, we have

$$\left|\int_{\mathbb{R}} \int_{\mathbb{R}^{d}} (Q_{\geq M}^{1}f_{1,N_{1},T})(Q_{2}^{j}f_{2,N_{2},T})(Q_{3}^{j}f_{3,N_{3},T})dxdt\right|$$

$$\leq ||Q_{\geq M}^{1}f_{1,N_{1},T}||_{L_{x}^{2}} ||(Q_{2}^{j}f_{2,N_{2},T})(Q_{3}^{j}f_{3,N_{3},T})||_{L_{x}^{2}}.$$

Furthermore by (2.2) with $p = 2$, we have

$$||Q_{\geq M}^{1}f_{1,N_{1},T}||_{L_{x}^{2}} \lesssim M^{-1/2}||f_{1,N_{1},T}||_{v_{1}^{2}}.$$  
(4.10)

While by (3.5) and (2.3), we have

$$||(Q_{2}^{j}f_{2,N_{2},T})(Q_{3}^{j}f_{3,N_{3},T})||_{L_{x}^{2}} \lesssim N_{3}^{2s} \left(\frac{N_{3}}{N_{2}}\right)^{1/4} ||f_{2,N_{2},T}||_{v_{2}^{2}} ||f_{3,N_{3},T}||_{v_{3}^{2}}$$  
(4.11)

when $N_{3} \ll N_{2}$. Therefore, we obtain

$$\sum_{N_{3} \ll N_{2}} N_{3}^{2s} \sup_{\|u_{j}\|_{v_{j}^{2}} = 1} \left|\int_{\mathbb{R}} \int_{\mathbb{R}^{d}} (Q_{\geq M}^{1}f_{1,N_{1},T})(Q_{2}^{j}f_{2,N_{2},T})(Q_{3}^{j}f_{3,N_{3},T})dxdt\right|^{2}$$

$$\lesssim N_{1}^{2s} \|P_{N_{1}}u_{1}\|_{v_{1}^{2}}^{2} N_{2}^{2s} \|P_{N_{2}}u_{2}\|_{v_{2}^{2}}^{2}$$

by $M \sim N_{\text{max}}^{2}$, $N_{1} \sim N_{2}$ and $\|1_{[0,T]}f\|_{v_{\sigma}^{2}} \lesssim \|f\|_{v_{\sigma}^{2}}$ for any $\sigma \in \mathbb{R}$ and any $T \in (0, \infty]$.

Next, we consider the case $Q_{3}^{3} = Q_{\geq M}^{3}$. We define $\tilde{P}_{N_{3}} = P_{N_{3}/2} + P_{N_{3}} + P_{2N_{3}}$. By the Cauchy-Schwarz inequality, we have

$$\left|\int_{\mathbb{R}} \int_{\mathbb{R}^{d}} (Q_{1}^{1}f_{1,N_{1},T})(Q_{2}^{j}f_{2,N_{2},T})(Q_{\geq M}^{3}f_{3,N_{3},T})dxdt\right|$$

$$\lesssim ||\tilde{P}_{N_{3}}((Q_{1}^{1}f_{1,N_{1},T})(Q_{2}^{j}f_{2,N_{2},T}))||_{L_{x}^{2}} ||Q_{\geq M}^{3}f_{3,N_{3},T}||_{L_{x}^{2}}$$

since $P_{N_{3}} = \tilde{P}_{N_{3}}P_{N_{3}}$. Furthermore, by (2.2) with $p = 2$, we have

$$||Q_{\geq M}^{3}f_{3,N_{3},T}||_{L_{x}^{2}} \lesssim M^{-1/2}||f_{3,N_{3},T}||_{v_{3}^{2}}.$$  
(4.12)
Therefore, we obtain
\[
\sum_{N_3 \ll N_2} N_3^{2s} \sup_{\|u_3\|_{V_{s,3}^2} = 1} \left| N_{\max} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^s f_{1,N_1,T})(Q_2^s f_{2,N_2,T})(Q_3^s_{\geq M} f_{3,N_3,T}) dx dt \right|^2 
\]
\[
\lesssim \sum_{N_3 \ll N_2} N_3^{2s} N_3^{2s} M^{-1||} \tilde{P}_{N_3}((Q_1^s f_{1,N_1,T})(Q_2^s f_{2,N_2,T})) ||^2_{L_{t,x}^2} \tag{4.13}
\]
\[
\lesssim N_3^{2s} \| (Q_1^s f_{1,N_1,T})(Q_2^s f_{2,N_2,T}) \|^2_{L_{t,x}^2} 
\]
\[
\lesssim N_3^{2s} \| P_{N_1} u_1 \|^2_{L_{x,1}^2} \| P_{N_2} u_2 \|^2_{L_{x,1}^2} \| P_{N_3} u_3 \|^2_{L_{x,1}^2},
\]
by $M \sim N_3^{2s}$, $N_1 \sim N_2$, $L^2$-orthogonality, \((3.8)\), the embedding $V_{1,rc}^2 \hookrightarrow U^4$, \((2.3)\) and \(\|1_{[0,T]} f\|_{V_{2}^s} \lesssim \| f \|_{V_{2}^s}\) for any $\sigma \in \mathbb{R}$ and any $T \in (0, \infty]$.

For the case $Q_2^s = Q_{\geq M}^s$ is proved in exactly same way as the case $Q_1^s = Q_{\geq M}^s$. □

**Proposition 4.4.** Let $s_c = d/2 - 1$ and $\sigma < T \leq \infty$.

(i) Let $d \geq 4$. For any $\sigma_1$, $\sigma_2$, $\sigma_3 \in \mathbb{R}\setminus\{0\}$ and any dyadic numbers $N_1$, $N_2$, $N_3 \in 2\mathbb{Z}$ , we have
\[
N_{\max} \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \lesssim N_{\max}^{s_c} \| P_{N_1} u_1 \|_{V_{s_1}^2} \| P_{N_2} u_2 \|_{V_{s_1}^2} \| P_{N_3} u_3 \|_{V_{s_1}^2},
\]
where $N_{\max} := \max_{1 \leq j \leq 3} N_j$.

(ii) Let $d = 2$, $3$ and $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. For any dyadic numbers $N_1$, $N_2$, $N_3 \in 2\mathbb{Z}$, we have \((4.14)\).

**Proof.** First, we consider the case $d \geq 4$. By the Hölder’s inequality, the Sobolev embedding $\tilde{W}^{s_c-1,6d/(3d-4)}(\mathbb{R}^d) \hookrightarrow L^{3d/4}(\mathbb{R}^d)$ and \((2.6)\), we have
\[
\text{(L.H.S of (4.14))} \lesssim N_{\max} \| P_{N_1} u_1 \|_{L^{6d/(3d-4)}_{t,x} L^{6d/(3d-4)}_{t,x}} \| P_{N_2} u_2 \|_{L^{6d/(3d-4)}_{t,x} L^{6d/(3d-4)}_{t,x}} \| \nabla |^{s_c-1} P_{N_3} u_3 \|_{L^{6d/(3d-4)}_{t,x} L^{6d/(3d-4)}_{t,x}} 
\]
\[
\lesssim N_{\max}^{s_c} \| P_{N_1} u_1 \|_{V_{s_1}^2} \| P_{N_2} u_2 \|_{V_{s_1}^2} \| P_{N_3} u_3 \|_{V_{s_1}^2}.
\]

Next, we consider the case $d = 2$, $3$ and $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. We define $f_{j,N_j,T} := 1_{[0,T]} P_{N_j} u_j$ ($j = 1, 2, 3$). For sufficiently large constant $C$, we put $M := C^{-1} N_{\max}^{2s}$ and decompose $Id = Q_{\leq M}^j + Q_{\geq M}^j$ ($j = 1, 2, 3$). We divide the integral on the left-hand side of \((4.14)\) into eight piece of the form
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^s f_{1,N_1,T})(Q_2^s f_{2,N_2,T})(Q_3^s_{\geq M} f_{3,N_3,T}) dx dt \tag{4.15}
\]
with $Q_j^s \in \{Q_{\leq M}^j, Q_{\geq M}^j\}$ ($j = 1, 2, 3$). Since $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$, Lemma \((4.1)\)(ii) implies that
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} (Q_1^s f_{1,N_1,T})(Q_2^s f_{2,N_2,T})(Q_3^s_{\leq M} f_{3,N_3,T}) dx dt = 0
\]
for any $N_1, N_2, N_3 \in 2^\mathbb{Z}$. So, let us now consider the case that $Q_{\sigma_j}^{\alpha_j} = Q_{\geq M}^{\alpha_j}$ for some $1 \leq j \leq 3$. We consider only for the case $Q_1^{\alpha_1} = Q_{\geq M}^{\alpha_1}$ since for the other cases is same manner.

By the Cauchy-Schwarz inequality, we have
\[
\left| \int_\mathbb{R} \int_\mathbb{R}^d (Q_{\geq M}^{\alpha_1} f_{1,N_1,T})(Q_{\geq M}^{\alpha_2} f_{2,N_2,T})(Q_{\geq M}^{\alpha_3} f_{3,N_3,T}) dx dt \right| \leq \left| \|Q_{\geq M}^{\alpha_1} f_{1,N_1,T}\|_{L_{t,x}^2} \right| \left( \|Q_{\geq M}^{\alpha_2} f_{2,N_2,T}\|_{L_{t,x}^2} \right) \left( \|Q_{\geq M}^{\alpha_3} f_{3,N_3,T}\|_{L_{t,x}^2} \right).
\]
Furthermore by (2.2) with $p = 2$, we have
\[
\|Q_{\geq M}^{\alpha_1} f_{1,N_1,T}\|_{L_{t,x}^2} \lesssim M^{-1/2} \|f_{1,N_1,T}\|_{V_2^{\alpha_1}}.
\]
(4.16)

While by (3.3), the embedding $V^2_{-\infty} \hookrightarrow U^4$ and (2.3), we have
\[
\|\left( Q_{\geq M}^{\alpha_2} f_{2,N_2,T}\right)(Q_{\geq M}^{\alpha_3} f_{3,N_3,T}) \|_{L_{t,x}^2} \lesssim N_{\max} \|f_{2,N_2,T}\|_{V_2^{\alpha_2}} \|f_{3,N_3,T}\|_{V_2^{\alpha_3}}.
\]
(4.17)
Therefore, we obtain
\[
N_{\max} \left| \int_\mathbb{R} \int_\mathbb{R}^d (Q_{\geq M}^{\alpha_1} f_{1,N_1,T})(Q_{\geq M}^{\alpha_2} f_{2,N_2,T})(Q_{\geq M}^{\alpha_3} f_{3,N_3,T}) dx dt \right| \lesssim N_{\max} \|P_{N_1} u_1\|_{V_2^{\alpha_1}} \|P_{N_2} u_2\|_{V_2^{\alpha_2}} \|P_{N_3} u_3\|_{V_2^{\alpha_3}},
\]
since $M \sim N_{\max}^2$ and $\|1_{[0,T]} f\|_{V_2^\infty} \lesssim \|f\|_{V_2^\infty}$ for any $\sigma \in \mathbb{R}$ and any $T \in (0, \infty]$. \hfill \Box

Proposition 4.3 and Proposition 4.4 imply the following:

**Corollary 4.5.** Let $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$ if $d \geq 4$, and $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$ if $d = 2, 3$. Then the estimate (4.8) holds if we replace $\sum_{N_3 < N_2}$ by $\sum_{N_3 \leq N_2}$.

5. **Time local estimates**

**Proposition 5.1.** Let $s > s_c$ ($= d/2 - 1$), $0 < T < \infty$ if $d \geq 2$ and $s \geq 0$, $T = 1$ if $d = 1$. We assume $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$. For any dyadic numbers $N_2, N_3 \in 2^\mathbb{Z}$ with $N_2 \sim N_3$, we have
\[
\left| \sum_{N_1 < N_2} N_{\max} \int_0^T \int_\mathbb{R}^d (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \lesssim T^{\delta} \left( \sum_{N_1 < N_2} (N_1 + N_2) \|P_{N_1} u_1\|_{V_2^{\alpha_1}}^2 \right)^{1/2} \|P_{N_2} u_2\|_{V_2^{\alpha_2}} \|P_{N_3} u_3\|_{V_2^{\alpha_3}},
\]
for some $\delta > 0$, where $N_{\max} := \max_{1 \leq j \leq 3} N_j$. 

Proof. First, we assume \(d \geq 2\). We choose \(\delta > 0\) satisfying \(\delta < (s-s_c)/2\) and \(\delta \ll 1\). In the proof of proposition 4.2 for L.H.S of (4.5), we use the Sobolev embedding \(\dot{H}^{s_c+2\delta} \hookrightarrow L^{d/(1-2\delta)}\) instead of \(\dot{H}^{s_c} \hookrightarrow L^{d}\). Then we have

\[
\left| \sum_{N_1 \ll N_2} \mathcal{N}_{\max} \int \int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1} f_{1,N_1,T})(Q_{\geq M}^{\sigma_2} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3} f_{3,N_3,T})dxdt \right|
\leq \sum_{N_1 \ll N_2} \mathcal{N}_{\max} |\nabla|^{s_c+2\delta} Q_{\geq M}^{\sigma_1} f_{1,N_1,T} |_{L_t^2 L_x^p} \|Q_{\geq M}^{\sigma_2} f_{2,N_2,T}\|_{L_t^p L_x^q} \|Q_{\geq M}^{\sigma_3} f_{3,N_3,T}\|_{L_t^p L_x^q}
\leq T^{\delta} \sum_{N_1 \ll N_2} \mathcal{N}_{\max} |\nabla|^s Q_{\geq M}^{\sigma_1} f_{1,N_1,T} |_{L_t^2 L_x^p} \|Q_{\geq M}^{\sigma_2} f_{2,N_2,T}\|_{L_t^p L_x^q} \|Q_{\geq M}^{\sigma_3} f_{3,N_3,T}\|_{L_t^p L_x^q}
\]

with \((p, q) = (4/(1-2\delta), 2d/(d-1+2\delta))\) which is the admissible pair of the Strichartz estimate. Furthermore for L.H.S of (4.6), we use the Hölder’s inequality and (2.2) with \(p = 2/(1-2\delta)\) instead of \(p = 2\). Then we have

\[
\|Q_{\geq M}^{\sigma_3} f_{3,N_3,T}\|_{L_t^2 L_x^q} \leq \|1_{[0,T]}\|_{L_t^{1/\delta}} \|Q_{\geq M}^{\sigma_3} f_{3,N_3,T}\|_{L_t^{2/(1-2\delta)} L_x^q} \lesssim T^{\delta/(1-2\delta)/2} \|f_{3,N_3,T}\|_{V_{2,q}^3}.
\]

For the other part, by the same way of the proof of proposition 4.2 we obtain (5.1).

Next, we assume \(d = 1\). In the proof of proposition 4.2 for L.H.S of (4.5), we use the Hölder’s inequality as follows:

\[
\left| \sum_{N_1 \ll N_2} \mathcal{N}_{\max} \int \int_{\mathbb{R}^d} (Q_{\geq M}^{\sigma_1} f_{1,N_1,T})(Q_{\geq M}^{\sigma_2} f_{2,N_2,T})(Q_{\geq M}^{\sigma_3} f_{3,N_3,T})dxdt \right|
\leq \sum_{N_1 \ll N_2} \mathcal{N}_{\max} Q_{\geq M}^{\sigma_1} f_{1,N_1,T} |_{L_t^2 L_x^p} \|Q_{\geq M}^{\sigma_2} f_{2,N_2,T}\|_{L_t^p L_x^q} \|Q_{\geq M}^{\sigma_3} f_{3,N_3,T}\|_{L_t^p L_x^q}.
\]

We note that (8, 4) is the admissible pair of the Strichartz estimate for \(d = 1\). Furthermore for the first inequality in (1.7), we use (3.7) instead of (3.5). For the other part, by the same way of the proof of proposition 4.2 we obtain (5.1) with \(T = 1\). \(\square\)

**Proposition 5.2.** Let \(s > s_c = (d-2)/2\), \(0 < T < \infty\) if \(d \geq 2\) and \(s \geq 0\), \(T = 1\) if \(d = 1\). We assume \(\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}\) satisfy \((\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0\). For any dyadic numbers \(N_1, N_2 \in 2^\mathbb{Z}\) with \(N_1 \sim N_2\), we have

\[
\left( \sum_{N_3 \ll N_2} \mathcal{N}_{\max} \sup_{\|u_3\|_{V_{2,q}^3}} \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3)dxdt \right)^{1/2} \lesssim T^{\delta} (N_1 \vee 1)^s \|P_{N_1} u_1\|_{V_{2,q}^3} (N_2 \vee 1)^s \|P_{N_2} u_2\|_{V_{2,q}^3}.
\]

for some \(\delta > 0\), where \(\mathcal{N}_{\max} := \max_{1 \leq j \leq 3} N_j\).
Proof. First, we assume $d \geq 2$. We choose $\delta > 0$ satisfying $\delta < (s - s_c)/2$ and $\delta \ll 1$. In the proof of proposition 4.3 for L.H.S of (4.10) and (4.12), we use the Hölder’s inequality and (2.22) with $p = 2/(1 - 2\delta)$ instead of $p = 2$. Then we have

$$
\| Q_{\geq M}^3 f_{1,N_1,T} \|_{L^2_t L^2} \leq \| 1_{[0,T]} \|_{L^{1/4}_t} \| Q_{\geq M}^3 f_{1,N_1,T} \|_{L^{2/(1 - 2\delta)}_t L^2_x} \lesssim T^\delta M^{-(1 - 2\delta)/2} \| f_{1,N_1,T} \|_{V^3_{2\delta}},
$$

$$
\| Q_{\geq M}^3 f_{5,N_1,T} \|_{L^2_t L^2} \leq \| 1_{[0,T]} \|_{L^{1/4}_t} \| Q_{\geq M}^3 f_{5,N_3,T} \|_{L^{2/(1 - 2\delta)}_t L^2_x} \lesssim T^\delta M^{-(1 - 2\delta)/2} \| f_{5,N_3,T} \|_{V^3_{2\delta}}.
$$

For the other part, by the same way of the proof of proposition 4.3, we obtain (5.2). Next, we assume $d = 1$. In the proof of proposition 4.3 for L.H.S of (4.11), we use (3.7) instead of (3.5) and for the third inequality in (4.13), we use (3.9) and $V^2_{2\delta} \hookrightarrow U^4$ instead of (3.8) and $V^2_{2\delta} \hookrightarrow U^4$. For the other part, by the same way of the proof of proposition 4.3, we obtain (5.2) with $T = 1$.

**Proposition 5.3.**

(i) Let $d \geq 4$, $s > s_c$ and $0 < T < \infty$. For any $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$, any dyadic numbers $N_1, N_2, N_3 \in 2^\mathbb{Z}$ and $1 \leq j \leq 3$, we have

$$
\left| N_j \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \lesssim T^\delta (N_j \vee 1)^s \| P_{N_1} u_1 \|_{V^3_{2\delta}} \| P_{N_2} u_2 \|_{V^3_{2\delta}} \| P_{N_3} u_3 \|_{V^3_{2\delta}},
$$

for some $\delta > 0$.

(ii) Let $d = 1, 2, 3$, $s \geq 1$, $0 < T < \infty$. For any $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$, any dyadic numbers $N_1, N_2, N_3 \in 2^\mathbb{Z}$ and $1 \leq j \leq 3$, we have (5.3).

(iii) Let $s > s_c$, $0 < T < \infty$ if $d = 2, 3$ and $s \geq 0$, $T = 1$ if $d = 1$. We assume $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\setminus\{0\}$ satisfy $\sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0$. For any dyadic numbers $N_1, N_2, N_3 \in 2^\mathbb{Z}$ with $N_1 \sim N_2 \sim N_3$ and $1 \leq j \leq 3$, we have (5.3).

**Proof.** By symmetry, it is enough to prove for $j = 3$. We choose $\delta > 0$ satisfying $\delta < (s - s_c)/2$ and $\delta \ll 1$.

First, we consider the case $d \geq 4$. By the Hölder’s inequality and the Sobolev embedding $\dot{W}^{s_c + 2\delta - 1, 6d/(3d - 4 + 12\delta)}(\mathbb{R}^d) \hookrightarrow L^{3d/4}(\mathbb{R}^d)$, we have

$$
\left| \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \lesssim \| 1_{[0,T]} \|_{L^{1/4}_t} \| P_{N_1} u_1 \|_{L^3_t L^6} \| P_{N_2} u_2 \|_{L^3_t L^6} \| \nabla \|_{s_c + 2\delta - 1} \| P_{N_3} u_3 \|_{L^3_t L^6},
$$

with $(p, q) = (3/(1 - 3\delta), 6d/(3d - 4 + 12\delta))$ which is the admissible pair of the Strichartz estimate. Therefore we obtain (5.3) by (2.6).
Second, we consider the case \( d = 1, 2, 3 \) and \( \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\} \) are arbitrary. By the Hölder’s inequality and (2.6), we have

\[
\left| \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \\
\lesssim \|1_{[0,T]}\|_{L^{4/(4-d)}} \|P_{N_1} u_1\|_{L^{12/d} L^2} \|P_{N_2} u_2\|_{L^{12/d} L^2} \|P_{N_3} u_3\|_{L^{12/d} L^2} \\
\lesssim T^{1-d/4} \|P_{N_1} u_1\|_{V_{2^1}} \|P_{N_2} u_2\|_{V_{2^2}} \|P_{N_3} u_3\|_{V_{2^3}}
\]

and obtain (5.3) as \( \delta = 1 - d/4 \) for \( s \geq 1 \).

Third, we consider the case \( d = 2, 3 \) and \( \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0 \). In the proof of proposition 4.4 for L.H.S of (4.16), we use the Hölder’s inequality and (2.2) with \( p = 2/(1 - 2\delta) \) instead of \( p = 2 \). Then we have

\[
\|Q_{\geq M}^3 f_{1,N_1,T}\|_{L^2_{t,x}} \leq \|1_{[0,T]}\|_{L^{1/4}} \|Q_{\geq M}^3 f_{1,N_1,T}\|_{L^{2/(1-2\delta)} L^2_t} \lesssim T^\delta M^{-(1-2\delta)/2} \|f_{1,N_1,T}\|_{V_{2^1}}.
\]

For the other part, by the same way of the proof of proposition 4.4, we obtain (5.3).

Finally, we consider the case \( d = 1 \) and \( \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0 \). In the proof of proposition 4.4 for L.H.S of (4.17), we use (3.9) and \( V_{-rc}^2 \hookrightarrow U^8 \) instead of (3.8) and \( V_{-rc}^2 \hookrightarrow U^4 \). For the other part, by the same way of the proof of proposition 4.4, we obtain (5.3) with \( T = 1 \).

\( \square \)

Proposition 5.2 and Proposition 5.3 imply the following:

**Corollary 5.4.** Let \( 0 < T < \infty \) if \( d \geq 2 \) and \( T = 1 \) if \( d = 1 \). We assume \( \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\} \) satisfy \( (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0 \).

(i) Let \( s > s_c \) if \( d \geq 4 \), and \( s \geq 1 \) if \( d = 1, 2, 3 \). Then the estimate (5.2) holds if we replace \( \sum_{N_3 \ll N_2} \) by \( \sum_{N_3 \ll N_2} \).

(ii) Let \( s > s_c \) if \( d = 2, 3 \) and \( s \geq 0 \) if \( d = 1 \). We assume \( \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\} \) satisfy \( \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) > 0 \). Then the estimate (5.2) holds if we replace \( \sum_{N_3 \ll N_2} \) by \( \sum_{N_3 \ll N_2} \).

Let \( (i, j, k) \) is one of the permutation of \( (1, 2, 3) \). If \( \sigma_i + \sigma_j = 0 \), then Proposition 4.1 (i) fails only for the case \( |\xi_k| < |\xi_i| \sim |\xi_j| \). We obtain following estimates for the case \( |\xi_k| \ll |\xi_i| \sim |\xi_j| \).

**Corollary 5.5.** Let \( s > s_c \) if \( d \geq 4 \), and \( s > 1 \) if \( d = 2, 3 \).

(i) We assume \( \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\} \) satisfy \( \sigma_2 + \sigma_3 = 0 \) and \( (\sigma_1 + \sigma_2)(\sigma_3 + \sigma_1) \neq 0 \). Then for any \( 0 < T < \infty \), and any dyadic numbers \( N_2, N_3 \in 2^\mathbb{Z} \) with \( N_2 \sim N_3 \), we
have

\[ \left| \sum_{N_1 \ll N_2} N_1 \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \]

\[ \lesssim T^8 \left( \sum_{N_1 \ll N_2} (N_1 \lor 1)^2 \| P_{N_1} u_1 \|_{V^2_{1/3}}^2 \right) \left\| P_{N_2} u_2 \right\|_{V^2_{3/5}} \left\| P_{N_3} u_3 \right\|_{V^2_{3/5}} \]

(ii) We assume \( \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\} \) satisfy \( \sigma_1 + \sigma_2 = 0 \) and \( (\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0 \). Then for any \( 0 < T < \infty \), and any dyadic numbers \( N_1, N_2 \in 2^Z \) with \( N_1 \ll N_2 \), we have

\[ \left( \sum_{N_3 \leq N_2} N_{3}^{2s} \sup_{\| v_3 \|_{V^2_{3/5}} = 1} \left| N_3 \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right|^2 \right)^{1/2} \]

\[ \lesssim T^8 (N_1 \lor 1)^s \left\| P_{N_1} u_1 \right\|_{V^2_{1/3}} (N_2 \lor 1)^s \left\| P_{N_2} u_2 \right\|_{V^2_{3/5}} \]

for some \( \delta > 0 \).

Proof. By the Hölder’s inequality, \( V^2_{r,c} \hookrightarrow L^\infty(\mathbb{R}; L^2) \) and (3.5), we have

\[ \left| N_1 \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right| \]

\[ \leq N_1 \| 1_{[0,T]} \|_{L^\infty_r} \| (P_{N_1} u_1)(P_{N_2} u_2) \|_{L^2_x} \| P_{N_3} u_3 \|_{L^\infty_r L^2_x} \]

\[ \lesssim T^{1/2} N_1^{s_c + 1} \left( \frac{N_1}{N_2} \right)^{1/2} \| P_{N_1} u_1 \|_{V^2_{1/3}} \| P_{N_2} u_2 \|_{V^2_{3/5}} \| P_{N_3} u_3 \|_{V^2_{3/5}} \]

for \( N_1 \ll N_2 \). We use (5.6) for the summation for \( N_1 < 1 \) and use (5.3) with \( j = 1 \) for the summation for \( 1 \leq N_1 \ll N_2 \). Then, we obtain (5.4) by the Cauchy-Schwarz inequality for the dyadic sum.

The estimate (5.5) is obtained by using (5.3) with \( j = 3 \).

6. PROOF OF THE WELL-POSEDNESS AND THE SCATTERING

In this section, we prove Theorems 1.1, 1.3, 1.6 and Corollary 1.2. To begin with, we define the function spaces which spaces will be used to construct the solution.

Definition 4. Let \( s, \sigma \in \mathbb{R} \).

(i) We define \( \dot{Z}^s_\sigma := \{ u \in C(\mathbb{R}; \dot{H}^s(\mathbb{R}^d)) \cap U^2_{\sigma} \mid \| u \|_{\dot{Z}^s_\sigma} < \infty \} \) with the norm

\[ \| u \|_{\dot{Z}^s_\sigma} := \left( \sum_{N} N^{2s} \| P_{N} u \|_{U^2_{\sigma}}^2 \right)^{1/2} \]

(ii) We define \( Z^s_\sigma := \{ u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_{\sigma} \mid \| u \|_{Z^s_\sigma} < \infty \} \) with the norm

\[ \| u \|_{Z^s_\sigma} := \| u \|_{\dot{Z}^s_\sigma} + \| u \|_{\dot{Z}^s_\sigma} \]
Also a Banach space (see Remark 2.23 in [13]).

\[ \|u\|_{\tilde{Y}_\sigma} := \left( \sum_{N} N^{2s} \| P_N u \|_{Y_\sigma}^2 \right)^{1/2} \]

(iv) We define \( Y_\sigma := \{ u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V_{r,c}^2 \mid \| u \|_{Y_\sigma} < \infty \} \) with the norm

\[ \|u\|_{Y_\sigma} := \|u\|_{\tilde{Y}_\sigma} + \|\dot{u}\|_{\tilde{Y}_\sigma}. \]

**Remark 6.1.** Let \( E \) be a Banach space of continuous functions \( f : \mathbb{R} \to H \), for some Hilbert space \( H \). We also consider the corresponding restriction space to the interval \( I \subset \mathbb{R} \) by

\[ E(I) = \{ u \in C(I, H) \mid \exists v \in E \text{ s.t. } v(t) = u(t), \ t \in I \} \]

endowed with the norm \( \|u\|_{E(I)} = \inf \{ \|v\|_E \mid v(t) = u(t), \ t \in I \} \). Obviously, \( E(I) \) is also a Banach space (see Remark 2.23 in [13]).

We define the map \( \Phi(u, v, w) = (\Phi_{T,\alpha,u_0}^{(1)}(w, v), \Phi_{T,\beta,v_0}^{(1)}(\overline{w}, v), \Phi_{T,\gamma,w_0}^{(2)}(u, \overline{v})) \) as

\[ \Phi_{T,\sigma,\varphi}^{(1)}(f, g)(t) := e^{i t \sigma \Delta} \varphi - I_{T,\sigma}^{(1)}(f, g)(t), \]

\[ \Phi_{T,\sigma,\varphi}^{(2)}(f, g)(t) := e^{i t \sigma \Delta} \varphi + I_{T,\sigma}^{(2)}(f, g)(t), \]

where

\[ I_{T,\sigma}^{(1)}(f, g)(t) := \int_{0}^{t} 1_{[0,T]}(t') e^{i(t-t') \sigma \Delta} (\nabla \cdot f(t')) g(t') dt', \]

\[ I_{T,\sigma}^{(2)}(f, g)(t) := \int_{0}^{t} 1_{[0,T]}(t') e^{i(t-t') \sigma \Delta} \nabla (f(t') \cdot g(t')) dt'. \]

To prove the existence of the solution of (1.1), we prove that \( \Phi \) is a contraction map on a closed subset of \( \tilde{Z}_\alpha^s([0, T]) \times \tilde{Z}_\beta^s([0, T]) \times \tilde{Z}_\gamma^s([0, T]) \) or \( Z_\alpha^s([0, T]) \times Z_\beta^s([0, T]) \times Z_\gamma^s([0, T]) \). Key estimates are the followings:

**Proposition 6.1.** We assume that \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\} \) satisfy the condition in Theorem 1.1. Then for \( s_c = d/2 - 1 \) and any \( 0 < T \leq \infty \), we have

\[ \|I_{T,\alpha}^{(1)}(w, v)\|_{\tilde{Z}_\alpha^s} \lesssim \|w\|_{\tilde{Y}_\alpha^s} \|v\|_{\tilde{Y}_\beta^s}, \]  
\[ \|I_{T,\beta}^{(1)}(\overline{w}, u)\|_{\tilde{Z}_\beta^s} \lesssim \|w\|_{\tilde{Y}_\gamma^s} \|u\|_{\tilde{Y}_\beta^s}, \]  
\[ \|I_{T,\gamma}^{(2)}(u, \overline{v})\|_{\tilde{Z}_\gamma^s} \lesssim \|u\|_{\tilde{Y}_\alpha^s} \|v\|_{\tilde{Y}_\beta^s}. \]

**Proof.** We prove only (6.3) since (6.1) and (6.2) are proved by the same way. We show the estimate

\[ \|I_{T,\gamma}^{(2)}(u, \overline{v})\|_{\tilde{Z}_\gamma^s} \lesssim \|u\|_{\tilde{Y}_\alpha^s} \|v\|_{\tilde{Y}_\beta^s} + \|u\|_{\tilde{Y}_\alpha^s} \|v\|_{\tilde{Y}_\beta^s}. \]
for \( s \geq 0 \). (6.3) follows from (6.4) as \( s = s_c \). We put \((u_1, u_2) := (u, \overline{v})\) and \((\sigma_1, \sigma_2, \sigma_3) := (\alpha, -\beta, -\gamma)\). To obtain (6.4), we use the argument of the proof of Theorem 3.2 in [13]. We define

\[
J_1 := \left\| \sum_{N_2} \sum_{N_1 \ll N_2} I_{T, -\sigma_3}^{(2)} (P_{N_1} u_1, P_{N_2} u_2) \right\|_{\dot{Z}_{-\sigma_3}^s},
\]

\[
J_2 := \left\| \sum_{N_2} \sum_{N_1 \sim N_2} I_{T, -\sigma_3}^{(2)} (P_{N_1} u_1, P_{N_2} u_2) \right\|_{\dot{Z}_{-\sigma_3}^s},
\]

\[
J_3 := \left\| \sum_{N_1} \sum_{N_2 \ll N_1} I_{T, -\sigma_3}^{(2)} (P_{N_1} u_1, P_{N_2} u_2) \right\|_{\dot{Z}_{-\sigma_3}^s},
\]

where implicit constants in \( \ll \) actually depend on \( \sigma_1, \sigma_2, \sigma_3 \).

First, we prove the estimate for \( J_1 \). By Theorem 2.2, we have

\[
J_1 \leq \left\{ \sum_{N_3} N_3^{2s} \left( \sum_{N_2 \sim N_3} \left\| e^{it\sigma_3 \Delta} P_{N_3} \sum_{N_1 \ll N_2} I_{T, -\sigma_3}^{(2)} (P_{N_1} u_1, P_{N_2} u_2) \right\|_{U^2} \right)^2 \right\}^{1/2}
\]

\[
= \left\{ \sum_{N_3} N_3^{2s} \left( \sum_{N_2 \sim N_3} \sup_{\| u_3 \|_{V^2_{\sigma_3}} = 1} \left\| \sum_{N_1 \ll N_2} N_3 \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right\|_{U^2} \right)^2 \right\}^{1/2}.
\]

Therefore by Proposition 4.2 we have

\[
J_1 \lesssim \left\{ \sum_{N_3} N_3^{2s} \left( \sum_{N_2 \sim N_3} \sup_{\| u_1 \|_{V^2_{\sigma_3}} = 1} \left( \sum_{N_1 \ll N_2} N_1^{2\sigma_c} \| P_{N_1} u_1 \|_{V^2_{\sigma_1}}^2 \right)^{1/2} \| P_{N_2} u_2 \|_{V^2_{\sigma_2}} \| P_{N_3} u_3 \|_{V^2_{\sigma_3}}^2 \right)^2 \right\}^{1/2}
\]

\[
\lesssim \left( \sum_{N_1} N_1^{2\sigma_1} \| P_{N_1} u_1 \|_{V^2_{\sigma_1}}^2 \right)^{1/2} \left( \sum_{N_2} N_2^{2\sigma_2} \| P_{N_2} u_2 \|_{V^2_{\sigma_2}}^2 \right)^{1/2}
\]

\[
= \| u_1 \|_{Y^c_{\sigma_1}} \| u_2 \|_{Y^c_{\sigma_2}}.
\]

Second, we prove the estimate for \( J_2 \). By Theorem 2.2, we have

\[
J_2 \leq \sum_{N_2} \sum_{N_1 \sim N_2} \left( \sum_{N_3 \leq N_2} N_3^{2\sigma_1} \| e^{it\sigma_3 \Delta} P_{N_3} I_{T, -\sigma_3}^{(2)} (P_{N_1} u_1, P_{N_2} u_2) \|_{U^2}^2 \right)^{1/2}
\]

\[
= \sum_{N_2} \sum_{N_1 \sim N_2} \left( \sum_{N_3 \leq N_2} N_3^{2\sigma_1} \sup_{\| u_3 \|_{V^2_{\sigma_3}} = 1} \left\| \int_0^T \int_{\mathbb{R}^d} (P_{N_1} u_1)(P_{N_2} u_2)(P_{N_3} u_3) dx dt \right\|_{U^2}^2 \right)^{1/2}.
\]
Therefore by Corollary 1.1 and Cauchy-Schwarz inequality for dyadic sum, we have
\[ J_2 \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} N_2^{s_c} \| P_{N_1} u_1 \|_{V_{\alpha_1}} N_2^{s} \| P_{N_2} u_2 \|_{V_{\alpha_2}} \]
\[ \lesssim \left( \sum_{N_1} N_1^{2s_c} \| P_{N_1} u_1 \|^2_{V_{\alpha_1}} \right)^{1/2} \left( \sum_{N_2} N_2^{2s} \| P_{N_2} u_2 \|^2_{V_{\alpha_2}} \right)^{1/2} \]
\[ = \| u_1 \|_{\dot{Y}_{\alpha_1}^{s_c}} \| u_2 \|_{\dot{Y}_{\alpha_2}^{s_c}}. \]

Finally, we prove the estimate for \( J_3 \). By the same manner as for \( J_1 \), we have
\[ J_3 \lesssim \| u_1 \|_{\dot{Y}_{\alpha_1}^{s_c}} \| u_2 \|_{\dot{Y}_{\alpha_2}^{s_c}}. \]

Therefore, we obtain (6.4) since \( \| u_1 \|_{\dot{Y}_{\alpha_1}^{s_c}} = \| u \|_{\dot{Y}^{s_c}_\alpha} \) and \( \| u_2 \|_{\dot{Y}_{\alpha_2}^{s_c}} = \| v \|_{\dot{Y}^{s_c}_\beta} \).

**Corollary 6.2.** We assume that \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\} \) satisfy the condition in Theorem 1.1 Then for \( s \geq s_c \) \((= d/2 - 1)\) and any \( 0 < T \leq \infty \), we have
\[ ||I_{T, \alpha}^{(1)}(w, v)||_{Z_{\sigma}} \lesssim ||w||_{Y^s} ||v||_{Y^{s}_\beta}, \] (6.5)
\[ ||I_{T, \beta}^{(1)}(\bar{w}, u)||_{Z_{\sigma}} \lesssim ||w||_{Y^s} ||u||_{Y^{s}_\alpha}, \] (6.6)
\[ ||I_{T, \gamma}^{(2)}(u, \bar{v})||_{Z_{\sigma}} \lesssim ||u||_{Y^{s}_\alpha} ||v||_{Y^{s}_\beta}. \] (6.7)

**Proof.** We prove only (6.7) since (6.5) and (6.6) are proved by the same way. By (6.6), we have
\[ ||I_{T, \gamma}^{(2)}(u, \bar{v})||_{Z_{\sigma}} = ||I_{T, \gamma}^{(2)}(u, \bar{v})||_{Z_{\sigma}} + ||I_{T, \gamma}^{(2)}(u, \bar{v})||_{Z_{\sigma}} \]
\[ \lesssim ||u||_{Y^{s}_\alpha} ||v||_{Y^{s}_\beta} + ||u||_{Y^{s}_\beta} ||v||_{Y^{s}_\alpha} + ||u||_{Y^{s}_\alpha} ||v||_{Y^{s}_\beta} + ||u||_{Y^{s}_\beta} ||v||_{Y^{s}_\alpha}. \]
We decompose \( u = P_0 u + (I d - P_0) u \) and \( v = P_0 v + (I d - P_0) v \). Since
\[ ||P_0 u||_{Y^{s}_\alpha} \lesssim ||P_0 u||_{Y^{s}_\alpha}, \] ||(I d - P_0) u||_{Y^{s}_\beta} \lesssim ||(I d - P_0) u||_{Y^{s}_\beta}, \]
\[ ||P_0 v||_{Y^{s}_\beta} \lesssim ||P_0 v||_{Y^{s}_\beta}, \] ||(I d - P_0) v||_{Y^{s}_\beta} \lesssim ||(I d - P_0) v||_{Y^{s}_\beta}\]
for \( s \geq s_c \), we obtain (6.7). \( \square \)

**Proposition 6.3.**
(i) Let \( d \geq 2 \). We assume that \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\} \) and \( s \in \mathbb{R} \) satisfy the condition in Theorem 1.3 Then there exists \( \delta > 0 \) such that for any \( 0 < T < \infty \), we have
\[ ||I_{T, \alpha}^{(1)}(w, v)||_{Z_{\sigma}} \lesssim T^{\delta} \||w||_{Z_{\sigma}} ||v||_{Z_{\sigma}}, \] (6.8)
\[ ||I_{T, \beta}^{(1)}(\bar{w}, u)||_{Z_{\sigma}} \lesssim T^{\delta} \||w||_{Z_{\sigma}} ||u||_{Z_{\sigma}}, \] (6.9)
\[ ||I_{T, \gamma}^{(2)}(u, \bar{v})||_{Z_{\sigma}} \lesssim T^{\delta} \||u||_{Z_{\sigma}} ||v||_{Z_{\sigma}}. \] (6.10)
(ii) Let \( d = 1 \). We assume that \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\} \) and \( s \in \mathbb{R} \) satisfy the condition in Theorem 1.3 except the case \( 1 > s \geq 1/2, \theta < 0 \) and \( (\alpha - \gamma)(\beta + \gamma) \neq 0 \). Then we have \((6.8) - (6.10)\) with \( T = 1 \).

**Proof.** We obtain \((6.8) - (6.10)\) by using Proposition 6.1 and Corollary 5.5 if \( \alpha \neq \beta \), using Corollary 5.5 if \( d \geq 2 \) and \( \alpha = \beta \) instead of Proposition 4.2 and Corollary 4.5 in the proof of Proposition 6.1. \( \square \)

**Proof of Theorem 1.1.** We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. For \( s \in \mathbb{R} \) and interval \( I \subset \mathbb{R} \), we define

\[
\dot{X}^s(I) := \dot{Z}^s_\alpha(I) \times \dot{Z}^s_\beta(I) \times \dot{Z}^s_\gamma(I).
\]  

Furthermore for \( r > 0 \), we define

\[
\dot{X}^s_r(I) := \left\{ (u, v, w) \in \dot{X}^s(I) \bigm| ||u||_{\dot{Z}^s_\alpha(I)}, ||v||_{\dot{Z}^s_\beta(I)}, ||w||_{\dot{Z}^s_\gamma(I)} \leq 2r \right\}
\]  

which is a closed subset of \( \dot{X}^s(I) \). Let \( (u_0, v_0, w_0) \in B_r(\dot{H}^s \times \dot{H}^s \times \dot{H}^s) \) be given. For \( (u, v, w) \in \dot{X}^s_r([0, \infty)) \), we have

\[
||\Phi^{(1)}_{T, t, \alpha, u_0}(w, v)||_{\dot{Z}^s_{\alpha}([0, \infty))} \leq ||u_0||_{\dot{H}^s} + Cr ||w||_{\dot{Z}^s_{\alpha}([0, \infty))} ||v||_{\dot{Z}^s_{\alpha}([0, \infty))} \leq r(1 + 4Cr),
\]

\[
||\Phi^{(1)}_{T, t, \beta, v_0}(w, u)||_{\dot{Z}^s_{\beta}([0, \infty))} \leq ||v_0||_{\dot{H}^s} + Cr ||w||_{\dot{Z}^s_{\beta}([0, \infty))} ||u||_{\dot{Z}^s_{\beta}([0, \infty))} \leq r(1 + 4Cr),
\]

\[
||\Phi^{(2)}_{T, t, \gamma, w_0}(u, v)||_{\dot{Z}^s_{\gamma}([0, \infty))} \leq ||w_0||_{\dot{H}^s} + Cr ||u||_{\dot{Z}^s_{\gamma}([0, \infty))} ||v||_{\dot{Z}^s_{\gamma}([0, \infty))} \leq r(1 + 4Cr)
\]  

and

\[
||\Phi^{(1)}_{T, t, \alpha, u_0}(w_1, v_1) - \Phi^{(1)}_{T, t, \alpha, u_0}(w_2, v_2)||_{\dot{Z}^s_{\alpha}([0, \infty))} \leq 2Cr \left( ||w_1 - w_2||_{\dot{Z}^s_{\alpha}([0, \infty))} + ||v_1 - v_2||_{\dot{Z}^s_{\alpha}([0, \infty))} \right),
\]

\[
||\Phi^{(1)}_{T, t, \beta, v_0}(w_1, u_1) - \Phi^{(1)}_{T, t, \beta, v_0}(w_2, u_2)||_{\dot{Z}^s_{\beta}([0, \infty))} \leq 2Cr \left( ||w_1 - w_2||_{\dot{Z}^s_{\beta}([0, \infty))} + ||u_1 - u_2||_{\dot{Z}^s_{\beta}([0, \infty))} \right),
\]

\[
||\Phi^{(2)}_{T, t, \gamma, w_0}(u_1, v_1) - \Phi^{(1)}_{T, t, \gamma, w_0}(u_2, v_2)||_{\dot{Z}^s_{\gamma}([0, \infty))} \leq 2Cr \left( ||u_1 - u_2||_{\dot{Z}^s_{\gamma}([0, \infty))} + ||v_1 - v_2||_{\dot{Z}^s_{\gamma}([0, \infty))} \right)
\]  

by Proposition 6.1

\[
||e^{i\sigma t}\varphi||_{\dot{Z}^s_{\gamma}([0, \infty))} \leq ||1([0, \infty])e^{i\sigma t}\varphi||_{\dot{Z}^s_{\gamma}([0, \infty))} \leq ||\varphi||_{\dot{H}^s},
\]

where \( C \) is an implicit constant in \((6.11) - (6.3)\). Therefore if we choose \( r \) satisfying

\[
r < (4C)^{-1},
\]

then \( \Phi \) is a contraction map on \( \dot{X}^s_r([0, \infty)) \). This implies the existence of the solution of the system \((1.1)\) and the uniqueness in the ball \( \dot{X}^s_r([0, \infty)) \). The Lipschitz continuously of the flow map is also proved by similar argument. \( \square \)

Theorem 1.3 except the case \( d = 1, 1 > s \geq 1/2, \theta < 0 \) and \( (\alpha - \gamma)(\beta + \gamma) \neq 0 \) is proved by the same way for the proof of Theorem 1.1.
Remark 6.2. For \( d = 1 \) and \( s > s_c \) (in particular \( s \geq 0 \)), we can assume the \( H^s \)-norm of the initial data is small enough by the scaling (1.2) with large \( \lambda \) since \( s_c < 0 \).

**Proof of Corollary 1.2.** We prove only the homogeneous case. The inhomogeneous case is also proved by the same way. By Proposition 6.1, the global solution \((u, v, w) \in \dot{X}^{s_c}([0, \infty))\) of (1.1) which was constructed in Theorem 1.1 satisfies
\[
N^{s_c}(e^{-i\alpha \Delta} P_N I^{(1)}_{\infty, a}(w, v), e^{-i\beta \Delta} P_N I^{(2)}_{\infty, \beta}(u, w), e^{-i\gamma \Delta} P_N I^{(2)}_{\infty, \gamma}(u, v)) \in V^2 \times V^2 \times V^2
\]
for each \( N \in 2\mathbb{Z} \). This implies that
\[
(u, v, w) := \lim_{t \to \infty} (u_0 - e^{-i\alpha \Delta} I^{(1)}_{\infty, a}(w, v), v_0 - e^{-i\beta \Delta} I^{(1)}_{\infty, \beta}(u, w), w_0 + e^{-i\gamma \Delta} I^{(2)}_{\infty, \gamma}(u, v))
\]
exists in \( \dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c} \) by Proposition 2.1 (4). Then we obtain
\[
(u, v, w) - (e^{i\alpha \Delta} u_+, e^{i\beta \Delta} v_+, e^{i\gamma \Delta} w_+) \to 0
\]
in \( \dot{H}^{s_c} \times \dot{H}^{s_c} \times \dot{H}^{s_c} \) as \( t \to \infty \). \( \square \)

Theorem 1.6 is proved by using the estimate (6.1) and (6.5) for \((\alpha, \beta, \gamma) = (-1, 1, 1)\).

### 7. A PRIORI ESTIMATES

In this section, we prove Theorem 1.4. We define
\[
M(u, v, w) := 2||u||^2_{L^2} + ||v||^2_{L^2} + ||w||^2_{L^2}
\]
\[
H(u, v, w) := \alpha ||\nabla u||^2_{L^2} + \beta ||\nabla v||^2_{L^2} + \gamma ||\nabla w||^2_{L^2} + 2\text{Re}(w, \nabla (u \cdot \overline{v})),
\]
and put \( M_0 := M(u_0, v_0, w_0), H_0 := H(u_0, v_0, w_0) \).

**Proposition 7.1.** For the smooth solution \((u, v, w)\) of the system (1.1), we have
\[
M(u, v, w) = M_0, \quad H(u, v, w) = H_0
\]

**Proof.** For the system
\[
(i \partial_t + \alpha \Delta) u = - (\nabla \cdot w)v \quad (7.1)
\]
\[
(i \partial_t + \beta \Delta) v = - (\nabla \cdot w)u \quad (7.2)
\]
\[
(i \partial_t + \gamma \Delta) w = \nabla (u \cdot \overline{w}) \quad (7.3)
\]

We have the conservation law for \( M \) by calculating
\[
\text{Im} \int_{\mathbb{R}^d} \{-2u \times \overline{(7.1)} + (\overline{v} \times (7.2)) + (\overline{w} \times (7.3))\} dx
\]
and for $H$ by calculating
\[
\text{Re} \int_{\mathbb{R}^d} \{ (\partial_t u \times (7.1)) + (\partial_t \overline{v} \times (7.2)) + (\partial_t \overline{w} \times (7.3)) \} \, dx.
\]

\[\Box\]

The following a priori estimates imply Theorem 1.4

**Proposition 7.2.** We assume $\alpha$, $\beta$ and $\gamma$ have the same sign and put
\[
\rho_{\text{max}} := \max\{ |\alpha|, |\beta|, |\gamma| \}, \quad \rho_{\text{min}} := \min\{ |\alpha|, |\beta|, |\gamma| \}.
\]
(i) Let $d = 1, 2$. For the data $(u_0, v_0, w_0) \in H^1 \times H^1 \times H^1$ satisfying
\[
M_0^{1-d/4} \ll \rho_{\text{min}},
\]
there exists $C > 0$ such that for the solution $(u, v, w) \in (C([0, T]; H^1))^3$ of (1.1), the following estimate holds:
\[
\sup_{0 \leq t \leq T} \left( \| \nabla u(t) \|^2_{L^2_x} + \| \nabla v(t) \|^2_{L^2_x} + \| \nabla w(t) \|^2_{L^2_x} \right) \leq \frac{H_0 + C M_0^{1-d/4}}{\rho_{\text{min}} - C M_0^{1-d/4}}.
\]
(ii) Let $d = 3$. If the data $(u_0, v_0, w_0) \in H^1 \times H^1 \times H^1$ satisfies
\[
\| \nabla u_0 \|^2_{L^2_x} + \| \nabla v_0 \|^2_{L^2_x} + \| \nabla w_0 \|^2_{L^2_x} < \epsilon^2 / \rho_{\text{max}}
\]
for some $\epsilon$ with $0 < \epsilon \ll 1$, then for the solution $(u, v, w) \in (C([0, T]; H^1))^3$ of (1.1), the following estimate holds:
\[
\sup_{0 \leq t \leq T} \left( \| \nabla u(t) \|^2_{L^2_x} + \| \nabla v(t) \|^2_{L^2_x} + \| \nabla w(t) \|^2_{L^2_x} \right) < 3 \epsilon^2 / \rho_{\text{min}}.
\]

**Proof.** We put
\[
F = F(t) := \| \nabla u(t) \|^2_{L^2_x} + \| \nabla v(t) \|^2_{L^2_x} + \| \nabla w(t) \|^2_{L^2_x}.
\]
Since $\alpha$, $\beta$ and $\gamma$ are same sign, we have
\[
F \leq \frac{1}{\rho_{\text{min}}} (H(u, v, w) + 2|((\nabla \cdot w), (u \cdot \overline{v}))_{L^2_x}|).
\]
By the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality we have
\[
|((\nabla \cdot w), (u \cdot \overline{v}))_{L^2_x}| \leq \| \nabla \cdot w \|_{L^2_x} \| u \|_{L^2_x} \| v \|_{L^2_x} \lesssim \| \nabla \cdot w \|_{L^2_x} \| u \|_{L^2_x}^{1-d/4} \| \nabla u \|_{L^2_x}^{d/4} \| v \|_{L^2_x}^{1-d/4} \lesssim M(u, v, w)^{1-d/4} F^{(d+2)/4}
\]
for $d \leq 4$. Therefore, by using Proposition 7.1 we obtain
\[
F \leq \frac{1}{\rho_{\text{min}}} \left( H_0 + C M_0^{1-d/4} F^{(d+2)/4} \right)
\]

(7.8)
for some constant $C > 0$. For $d \leq 2$ we have $F^{(d+2)/4} \leq 1 + F$ because of $(d+2)/4 \leq 1$. Therefore if (7.4) holds, then the estimate (7.5) follows from (7.8).

By the same argument as above, we obtain

$$H_0 \leq \rho_{\text{max}} F(0) + 2 |((\nabla \cdot w(0)), (u(0) \cdot v(0)))_{L^2_x}| \leq \rho_{\text{max}} F(0) + CM_0^{1-d/4}F(0)^{(d+2)/4}$$

for some constant $C > 0$ and $d \leq 4$. Therefore if (7.6) holds for some $\epsilon$ with $0 < \epsilon \ll 1$, we have

$$H_0 < \epsilon^2 (1 + CM_0^{1-d/4} \rho_{\text{max}}^{-(d+2)/4} \epsilon^{(d-2)/2}).$$

By choosing $\epsilon$ sufficiently small, we have $H_0 < 2\epsilon^2$ for $d = 3$ (and also $d = 4$). Therefore the estimate

$$F \leq \frac{1}{\rho_{\text{min}}}(2\epsilon^2 + CM_0^{1-d/4}F^{(d+2)/4})$$

follows from (7.8). If there exists $t_0 \in [0,T]$ such that $F(t_0) < 4\epsilon^2/\rho_{\text{min}}$ for sufficiently small $\epsilon$, then we have $F(t_0) < 3\epsilon^2/\rho_{\text{min}}$ by (7.9). Since $F(0) < \epsilon^2/\rho_{\text{min}} < 4\epsilon^2/\rho_{\text{min}}$ and $F(t)$ is continuous with respect to $t$, we obtain (7.7). \qed

8. $C^2$-ill-posedness

In this section, we prove Theorem 1.5. We rewrite Theorem 1.5 as follows:

**Theorem 8.1.** Let $d \geq 1$, $0 < T \ll 1$ and $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$. We assume $s \in \mathbb{R}$ if $(\alpha - \gamma)(\beta + \gamma) = 0$, $s < 1$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) = 0$, and $s < 1/2$ if $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) < 0$. Then for every $C > 0$ there exist $f, g \in H^s(\mathbb{R}^d)$ such that

$$\sup_{0 \leq t \leq T} \left\| \int_0^t e^{i(t-t')\gamma \Delta} \nabla ((e^{it'\alpha \Delta} f)(e^{it'\beta \Delta} g)) dt' \right\|_{H^s} \geq C \| f \|_{H^s} \| g \|_{H^s}. \quad (8.1)$$

**Proof.** We prove only for $d = 1$. For $d \geq 2$, it is enough to replace $D_1, D_2$ and $D$ by $D_1 \times [0,1]^{d-1}, D_2 \times [0,1]^{d-1}$ and $D \times [1/2,1]^{d-1}$ in the following argument. We use the argument of the proof of Theorem 1 in [24]. For the sets $D_1, D_2 \subset \mathbb{R}$, we define the functions $f, g \in H^s(\mathbb{R})$ as

$$\hat{f}(\xi) = 1_{D_1}(\xi), \quad \hat{g}(\xi) = 1_{D_2}(\xi).$$

First, we consider the case $(\alpha - \gamma)(\beta + \gamma) = 0$. We assume $\alpha - \gamma = 0$. (For the case $\beta + \gamma = 0$ is proved by similar argument.) We put $M := -(\beta + \gamma)/2\gamma$, then we have

$$\alpha |\xi_1|^2 - \beta |\xi - \xi_1|^2 - \gamma |\xi|^2 = 2\gamma \{\xi_1 - M(\xi - \xi_1)\}(\xi - \xi_1).$$
For $N \gg 1$, we define the sets $D_1, D_2$ and $D \subset \mathbb{R}$ as

$$D_1 := [N, N + N^{-1}], \quad D_2 := [N^{-1}, 2N^{-1}], \quad D := [N + 3N^{-1}/2, N + 2N^{-1}]$$

Then, we have

$$\|f\|_{H^s} \sim N^{s-1/2}, \quad \|g\|_{H^s} \sim N^{-1/2}, \quad |(\hat{f} \ast \hat{g})(\xi)| \gtrsim N^{-1} 1_D(\xi)$$

and

$$\int_0^t e^{-it'(\alpha|\xi_1|^2 - \beta|\xi - \xi_1|^2 - \gamma|\xi|^2)} dt' \sim t$$

for any $\xi \in D_1$ satisfying $\xi - \xi_1 \in D_2$ and $0 \leq t \ll 1$. This implies

$$\sup_{0 \leq t \leq T} \left\| \int_0^t e^{i(t-t')\gamma D} \nabla ((e^{i(t-t')f})(e^{i(t-t')g})) dt' \right\|_{H^s} \gtrsim N^{s+1}$$

Therefore we obtain (8.1) because $s - 1/2 > s - 1$ for any $s \in \mathbb{R}$.

Second, we consider the case $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) = 0$. We put $M := \gamma/(\alpha - \gamma)$, then $M \neq -1$ since $\alpha \neq 0$ and we have

$$\alpha|\xi_1|^2 - \beta|\xi - \xi_1|^2 - \gamma|\xi|^2 = (\alpha - \gamma)|\xi_1 - M(\xi - \xi_1)|^2.$$
Because $M_+ \neq M_-$, at least one of $M_+$ and $M_-$ is not equal to $-1$. We can assume $M_+ \neq -1$ without loss of generality. For $N \gg 1$, we define the sets $D_1$, $D_2$ and $D \subset \mathbb{R}$ as

$$D_1 := [N, N + N^{-1}], \quad D_2 := [N/M_+, N/M_+ + N^{-1}/|M_+|],$$

$$D := [(1 + 1/M_+)N + N^{-1}/2, (1 + 1/M_+)N + N^{-1}].$$

Then, we have

$$||f||_{H^s} \sim N^{s-1/2}, \quad ||g||_{H^s} \sim N^{s-1/2}, \quad |(\hat{f} \ast \hat{g})(\xi)| \gtrsim N^{-1}1_D(\xi)$$

and

$$\int_0^t e^{-it'\langle \xi \rangle^2 - \beta |\xi| - \gamma |\xi|^2} d\xi \sim t$$

for any $\xi \in D_1$ satisfying $\xi - \xi_1 \in D_2$ and $0 \leq t \ll 1$. This implies

$$\sup_{0 \leq t \leq T} \left\| \int_0^t e^{i(t-t')\gamma} \nabla((e^{it\alpha \Delta} f)(e^{it\beta \Delta} g)) dt \right\|_{H^s} \gtrsim N^{s-1/2}.$$ 

Therefore we obtain (8.1) because $s - 1/2 > 2s - 1$ for any $s < 1/2$. \hfill \Box

**Appendix A. Bilinear estimates for 1D Bourgain norm**

In this section, we give the bilinear estimates for the standard 1-dimensional Bourgain norm under the condition $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) \neq 0$. Which estimates imply the well-posedness of (1.1) for $1 > s \geq 1/2$ as the solution $(u, v, w)$ be in the Bourgain space $X^s([0, T]) \times X^s([0, T]) \times X^s([0, T])$.

**Lemma A.1.** Let $\sigma_1$, $\sigma_2$, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0$ and $(\tau_1, \xi_1)$, $(\tau_2, \xi_2)$, $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}$ satisfy $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$. If there exist $1 \leq i, j \leq 3$ such that $|\xi_i| \ll |\xi_j|$, then we have

$$\max_{1 \leq j \leq 3} |\tau_j + \sigma_j \xi_j^2| \gtrsim \xi_3^2.$$ 

**Proof.** For the case $\sigma_1 + \sigma_2 \neq 0$, proof was complete in Lemma 4.1. We assume $\sigma_1 + \sigma_2 = 0$. Then we have

$$M_0 := \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau_2 + \sigma_2 \xi_2^2|, |\tau_3 + \sigma_3 \xi_3^2|\}$$

$$\gtrsim |\sigma_1 \xi_1^2 + \sigma_2 \xi_2^2 + \sigma_3 \xi_3^2|$$

$$= |\xi_3||(\sigma_1 + \sigma_3)\xi_3 + 2\sigma_1 \xi_2|$$

$$= |\xi_3||(\sigma_2 + \sigma_3)\xi_3 + 2\sigma_2 \xi_1|$$

by the triangle inequality. Therefore if $|\xi_i| \ll |\xi_j|$ for some $1 \leq i, j \leq 3$, then we have $M_0 \gtrsim \xi_3^2$. \hfill \Box
Lemma A.2. We assume $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\{0\}$ satisfy $\theta := \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \neq 0$. For any $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$ with $|\xi| \geq 1$ and $b > 1/2$, we have

$$\int_{R} \int_{R} \frac{d\tau_1 d\xi_1}{(\tau + \sigma_1 \xi_1^2)^{2b}(\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2^{2b})} \lesssim \langle (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 \rangle^{-1/2}$$

and

$$\int_{|\xi_1| > |\xi - \xi_1| \text{ or } |\xi_1| < |\xi - \xi_1|} \int_{R} \frac{d\tau_1 d\xi_1}{(\tau + \sigma_1 \xi_1^2)^{2b}(\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2^{2b})} \lesssim \langle \xi \rangle^{-1},$$

where implicit constants in $\ll$ actually depend on $\sigma_1, \sigma_2$.

Proof. We put $I(\tau, \xi) := (\text{L.H.S of (A.1)})$. By Lemma 2.3, (2.8) in [20], we have

$$I(\tau, \xi) \lesssim \int_{R} \frac{d\xi_1}{(\sigma_1 \xi_1^2 + \sigma_2 (\xi - \xi_1)^2 + \sigma_3 \xi^2 + (\tau - \sigma_3 \xi^2)^2)^{2b}}.$$ We change the variable $\xi_1 \mapsto \mu$ as $\mu = \sigma_1 \xi_1^2 + \sigma_2 (\xi - \xi_1)^2 + \sigma_3 \xi^2$, then we have

$$d\mu = 2|\sigma_1 \xi_1 - \sigma_2 (\xi - \xi_1)| d\xi_1 \sim |(\sigma_1 + \sigma_2) \mu - \theta \xi^2|^{1/2} d\xi_1.$$ Therefore if $\sigma_1 + \sigma_2 = 0$, we obtain

$$I(\tau, \xi) \lesssim \frac{1}{|\xi|} \int_{R} \frac{d\mu}{(\mu + (\tau - \sigma_3 \xi^2)^2)^{2b}} \lesssim \langle \xi \rangle^{-1}$$

for $b > 1/2$ since $\theta \neq 0$ and $|\xi| \geq 1$. While if $\sigma_1 + \sigma_2 \neq 0$, we obtain

$$I(\tau, \xi) \lesssim \int_{R} \frac{d\mu}{(\mu + (\tau - \sigma_3 \xi^2)^2)^{2b} |(\sigma_1 + \sigma_2) \mu - \theta \xi^2|^{1/2}} \lesssim \langle (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 \rangle^{-1/2}$$

for $b > 1/2$ by Lemma 2.3, (2.9) in [20]. The estimate (A.2) follows from

$$d\mu = 2|\sigma_1 \xi_1 - \sigma_2 (\xi - \xi_1)| d\xi_1 \sim \max\{|\xi_1|, |\xi - \xi_1|\} d\xi_1 \sim |\xi| d\xi_1$$

when $|\xi_1| \gg |\xi - \xi_1|$ or $|\xi_1| \ll |\xi - \xi_1|$.

Proposition A.3. We assume $d = 1$ and $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\{0\}$ satisfy $(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) \neq 0$ and $\theta := \sigma_1 \sigma_2 \sigma_3 (1/\sigma_1 + 1/\sigma_2 + 1/\sigma_3) \neq 0$. Then for $3/4 \geq b > 1/2$ and $1 > s \geq 1/2$, we have

$$\| (\partial_x u_3) u_2 \|_{X^{s, b}_{\sigma_3, -1}} \lesssim \| u_3 \|_{X^{s, b}_{\sigma_3}} \| u_2 \|_{X^{s, b}_{\sigma_2}},$$

(A.3)

$$\| \partial_x (u_1 u_2) \|_{X^{s, b}_{\sigma_3}} \lesssim \| u_1 \|_{X^{s, b}_{\sigma_1}} \| u_2 \|_{X^{s, b}_{\sigma_2}},$$

(A.4)

where

$$\| u \|_{X^{s, b}_{\sigma}} := \| \langle \xi \rangle^s (\tau + \sigma \xi^2)^b u \|_{L^2_x \xi}.$$
Proof. We prove only \( \text{(A.4)} \) since the proof of \( \text{(A.3)} \) is similar. By the Cauchy-Schwarz inequality, we have

\[
\|\partial_\tau (u_1 u_2)\|_{X^{s, b-1}} \lesssim \|I\|_{L^\infty_{\tau, \xi}} \|u_1\|_{X^{s, b}} \|u_2\|_{X^{s, b}},
\]

where

\[
I(\tau, \xi) := \left( \frac{\langle \xi \rangle^2 |\xi|^2}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s} \langle \tau - \tau_1 + \sigma_2(\xi - \xi_1)^2 \rangle^{2b}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}(\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2)^{2b}} d\tau_1 d\xi_1 \right)^{1/2}.
\]

It is enough to prove \( I(\tau, \xi) \lesssim 1 \) for \( |\xi| \geq 1 \). For fixed \( (\tau, \xi) \in \mathbb{R} \times \mathbb{R} \), we divide \( \mathbb{R} \times \mathbb{R} \) into three regions \( S_1, S_2, S_3 \) as

\[
S_1 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | |\xi| \ll |\xi_1|\}
\]

\[
S_2 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | |\xi| \gtrsim |\xi_1|, \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2|\} \gtrsim \xi^2\}
\]

\[
S_3 := \{(\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{R} | |\xi| \gtrsim |\xi_1|, \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2|\} \ll \xi^2\}
\]

First, we consider the region \( S_1 \). For any \( (\tau_1, \xi_1) \in S_1 \), we have

\[
\langle \xi \rangle^{2s} |\xi|^2 \langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s} \lesssim \langle \xi \rangle^{2-2s}
\]

because \( |\xi| \ll |\xi_1| \sim |\xi - \xi_1| \). Therefore, we have

\[
I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{2-2s}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}((\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2)^{1/2}} \right)^{1/2}
\]

for \( b > 1/2 \) by \( \text{(A.1)} \). Because \( \theta \neq 0 \),

\[
\xi^2 = \frac{1}{\theta} \left\{ (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2 - (\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) \right\}.
\]

Therefore we obtain

\[
I(\tau, \xi) \lesssim \left( \frac{1}{(\tau - \sigma_3 \xi^2)^{2s - (2b - 1)}((\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2)^{1/2}} \right)^{1/2}
\]

\[
\lesssim 1
\]

for \( 3/4 \geq b > 1/2 \) and \( 1 > s \geq 1/2 \).

Second, we consider the region \( S_2 \). We assume \( |\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2| \gtrsim \xi^2 \) \((\gtrsim |\xi|^{1/3}|\xi_1|^{1/3} + |\xi - \xi_1|^{1/3}) \) since for the case \( |\tau_1 + \sigma_1 \xi_1^2| \gtrsim \xi^2 \) is same argument. Then, we have

\[
I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{2s}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s} \langle \tau - \tau_1 + \sigma_2(\xi - \xi_1)^2 \rangle^{2b}}{(\tau - \tau_1 + \sigma_2(\xi - \xi_1)^2)^{2b}} d\tau_1 d\xi_1 \right)^{1/2}.
\]

Because

\[
\int_{\mathbb{R}} \frac{d\tau_1}{(\tau_1 + \sigma_1 \xi_1^2)^{2b}} \lesssim 1
\]
for $b > 1/2$, we obtain
\[ I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{2s}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \int_{\mathbb{R}} \langle \xi_1 \rangle^{2s + 2b - 1} d\xi_1 \right)^{1/2} \]
\[ \lesssim \left( \frac{1}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \right)^{1/2} \]
\[ \lesssim 1 \]

for $1 \geq b > 1/2$ and $s \geq 1/2$ by Lemma 2.3, (2.8) in [20].

Finally, we consider the region $S_3$. To begin with, we consider the case $|\tau - \sigma_3 \xi^2| \gtrsim |\xi|^2$. Then we have
\[ \frac{\langle \xi \rangle}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{-2s} \lesssim \begin{cases} \langle \xi \rangle^{4b - 2 - 2s} & \text{if } |\xi_1| \sim |\xi - \xi_1| \\ \langle \xi \rangle^{4b - 2} & \text{if } |\xi_1| \gg |\xi - \xi_1| \text{ or } |\xi_1| \ll |\xi - \xi_1| \end{cases} \]

since $|\xi| \sim \max\{|\xi_1|, |\xi - \xi_1|\}$ for any $(\tau, \xi) \in S_3$. Therefore we obtain
\[ I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{4b - 2 - 2s}}{(\sigma_1 + \sigma_2)(\tau - \sigma_3 \xi^2) + \theta \xi^2} \right)^{1/2} \lesssim 1 \]

for $3/4 \geq b > 1/2$ and $s \geq 1/2$ by (A.1) and (A.2). Next, we consider the case $|\tau - \sigma_3 \xi^2| \ll |\xi|^2$. Because $(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) \neq 0$ and
\[ \max\{|\tau_1 + \sigma_1 \xi_1^2|, |\tau - \tau_1 + \sigma_2 (\xi - \xi_1)^2|, |\tau - \sigma_3 \xi^2|\} \ll |\xi|^2, \]

we have $|\xi| \sim |\xi - \xi_1| \sim |\xi_1|$ by Lemma A.1. Therefore, we have
\[ I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{2 - 2s}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \right)^{1/2} \]

for $b > 1/2$ by (A.1). Because $\theta \neq 0$ and $|\tau - \sigma_3 \xi^2| \ll |\xi|^2$, we have
\[ I(\tau, \xi) \lesssim \left( \frac{\langle \xi \rangle^{1 - 2s}}{(\tau - \sigma_3 \xi^2)^{2(1-b)}} \right)^{1/2} \lesssim 1 \]

for $1 \geq b > 1/2$ and $s \geq 1/2$.

**Corollary A.4.** We assume $d = 1$ and $\alpha, \beta, \gamma \in \mathbb{R}\setminus\{0\}$ satisfy $(\alpha - \gamma)(\beta + \gamma) \neq 0$ and $\alpha \beta \gamma(1/\alpha - 1/\beta - 1/\gamma) \neq 0$. Then for $3/4 \geq b > 1/2$ and $1 > s \geq 1/2$, we have
\[ ||(\partial_x w)v||_{X^{s,b}_a} \lesssim ||w||_{X^{s,b}_a} ||v||_{X^{s,b}_a}, \quad \text{(A.5)} \]
\[ ||(\partial_x \overline{w})u||_{X^{s,b}_a} \lesssim ||w||_{X^{s,b}_a} ||u||_{X^{s,b}_a}, \quad \text{(A.6)} \]
\[ ||\partial_x (u \overline{v})||_{X^{s,b}_a} \lesssim ||u||_{X^{s,b}_a} ||v||_{X^{s,b}_a}. \quad \text{(A.7)} \]
Proof. (A.5) follows from (A.3) with \((u_2, u_3) = (v, w)\) and \((\sigma_1, \sigma_2, \sigma_3) = (-\alpha, \beta, \gamma)\).

(A.6) follows from (A.3) with \((u_2, u_3) = (u, \overline{w})\) and \((\sigma_1, \sigma_2, \sigma_3) = (\alpha, -\beta, -\gamma)\). (A.7) follows from (A.4) with \((u_1, u_2) = (u, v)\) and \((\sigma_1, \sigma_2, \sigma_3) = (\alpha, -\beta, -\gamma)\). □

Theorem 1.3 (iii) under the condition \(1 > s \geq 1/2, \theta = \alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) < 0\) and \((\alpha - \gamma)(\beta + \gamma) \neq 0\) follows from Lemma 2.1 in [11] and Corollary A.4.

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(H. Hirayama)

E-mail address, H. Hirayama: m08035f@math.nagoya-u.ac.jp