Multidimensional quasi-exactly solvable potentials with two known eigenstates

V.M. Tkachuk, T.V. Fityo
Ivan Franko Lviv National University, Chair of Theoretical Physics
12 Drahomanov Str., Lviv UA–79005, Ukraine
E-mail: tkachuk@ktf.franko.lviv.ua

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Abstract

A general approach for constructing multidimensional quasi-exactly solvable (QES) potentials with explicitly known eigenfunctions for two energy levels is proposed. Examples of new QES potentials are presented.

Key words: quasi-exactly solvable potentials, multidimensional potentials.

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1 Introduction

The importance of exactly solvable potentials in quantum mechanics is well known. But even in one-dimensional case the number of such potentials is limited. Therefore, much attention has been given to the quasi-exactly solvable (QES) potentials for which a finite number of energy levels and corresponding wavefunctions are known in explicit form. The first examples of such potentials were given in [1, 2, 3, 4, 5] and subsequently many QES potentials were established [6, 7, 8, 9, 10, 11, 12, 13] (for review see book by Ushveridze [14]).

In multidimensional case many exactly solvable systems admit separation of variables and thus they are reduced to one-dimensional problems [15].
The investigation of multidimensional and many-body systems which are not amenable to separation of variables is important from mathematical and physical points of view.

One of the most remarkable exactly solvable models in $N$-body quantum mechanics is the Calogero model [16, 17]. This model describes one-dimensional $N$-body problem with quadratic and inverse square interacting potential. Soon after these papers Sutherland extended Calogero model to a model where interaction takes place on a circle [18, 19, 20]. The further progress was connected with Olshanetsky-Perelomov integrable systems [21] (see also review [22]). In [23] explicit examples of QES $N$-body problems on a line were introduced for the first time. Later QES $N$-body problems were studied in [24, 25].

Recently, there has been achieved some progress in studying exact solvability of such systems. In [26] a new class of QES many-body Hamiltonians in arbitrary dimension was constructed. The multidimensional Darboux transformation was proposed in [27] and new examples of QES multidimensional matrix Schrödinger operators were presented there. The authors of [28] have developed a systematic procedure for constructing exactly solvable and QES many-body potentials by purely algebraic means. The method of multidimensional supersymmetric (SUSY) quantum mechanics was applied to the investigation of $N$-particle systems and an explicit construction of exactly solvable 3-particle as well as QES $N$-particle problems on a line were presented [29, 30, 31]. Very recently, two new methods based on higher-order SUSY quantum mechanics for the investigation of two-dimensional quantum systems, whose Hamiltonians are not amenable to separation of variables, were presented in [32].

In this paper we develop a simple method for generation of QES potentials with two known eigenstates in arbitrary dimension. For one-dimensional case the problem of constructing QES potential with two known levels was solved completely in the frame of SUSY quantum mechanics [33, 34, 35] or using a simple method in which two wave functions are chosen in such a way that they lead to the same potential [36, 37]. Just this method will be generalized for multidimensional case.

Note that in fact the idea of the papers [33, 34, 35, 36, 37] is based on the inverse method. For the first time this method was used by Ushveridze [14] for construction QES potentials. It rests on the very simple idea: instead of looking for solution of Schrödinger equation for a given potential one should reconstruct the potential starting with appropriately chosen eigenfunctions.
We would like to mention also paper [38] where detailed comparison of supersymmetric approach proposed in [33] and Turbiner-Ushveridze approach [6] based on finite-dimensional representations of $sl(2,R)$ was done.

2 Construction of QES multidimensional potentials

We consider the Schrödinger equation

$$H \psi = E \psi$$ \hspace{1cm} (1)

in $n$-dimensional case with the Hamiltonian

$$H = -\frac{1}{2} \Delta + V(x_1, \cdots x_n),$$ \hspace{1cm} (2)

where $\Delta = (\nabla, \nabla) = \sum \frac{\partial^2}{\partial x_i^2}$; $\nabla$ is the nabla operator in $n$-dimensional space.

The wave function of the ground state with the energy $E_0$ has no zeros and can be written in the following form

$$\psi_0 = e^{-F},$$ \hspace{1cm} (3)

where $F$ is a nonsingular function of $x_1, \ldots x_n$.

Substituting (3) into (1) we express the potential in terms of the wave function $\psi_0$:

$$V = E_0 + \frac{1}{2} \frac{\Delta \psi_0}{\psi_0} = E_0 + \frac{1}{2} \left[ (\nabla F)^2 - \Delta F \right].$$ \hspace{1cm} (4)

Note that this transformation of Schrödinger equation from linear form to non-linear one is well known in literature. See, for instance, review by Turbiner, where this procedure was used for construction of convergent perturbation theory in quantum mechanics [39].

The wave function of the excited state with the energy $E_1$ we write as follows

$$\psi_1 = \phi e^{-F},$$ \hspace{1cm} (5)

where $\phi = \psi_1/\psi_0$. This eigenfunction must lead to the same potential $V$ and two eigenfunctions satisfy the following equation

$$E_0 + \frac{1}{2} \frac{\Delta \psi_0}{\psi_0} = E_1 + \frac{1}{2} \frac{\Delta \psi_1}{\psi_1}.$$ \hspace{1cm} (6)
In terms of $\phi$ and $F$ functions this equation reads

$$2(\nabla F, \nabla \phi) = \Delta \phi + 2\varepsilon \phi,$$  \hspace{1cm} (7)

where

$$\varepsilon = E_1 - E_0.$$  

In one-dimensional case this equation can be easily solved with respect to $F$ [36, 37]. Then choosing different generating functions $\phi$ we obtain different QES one-dimensional potentials with two known eigenstates.

In multidimensional case the solution of equation (7) is a nontrivial task. In this paper we find the solution of (7) with respect to $F$ for some special cases of function $\phi$. Note that equation (7) is a nonuniform linear equation with respect to $F$. Any solution of it can be written in the following form

$$F = f + \tilde{f},$$  \hspace{1cm} (8)

where $f$ is a particular solution of (7) and $\tilde{f}$ is a general solution of the uniform equation

$$(\nabla \tilde{f}, \nabla \phi) = 0.$$  \hspace{1cm} (9)

Direct substitution of function (8) into equation (7) shows that $F$ is indeed a solution of the equation. In order to satisfy square integrability of wavefunctions (3) and (5) we have to choose such $f$ and $\tilde{f}$ that $F \to +\infty$ when $|\vec{x}| \to +\infty$.

Note that in one-dimensional case this equation gives only trivial solution $\tilde{f} = \text{const}$, which does not influence the potential (4). New point of multidimensional case is that uniform equation (9) has many nontrivial solutions $\tilde{f}$. This gives more possibilities for constructing QES potentials in multidimensional case for given $\phi$ in comparison with the one-dimensional case.

**Case 1.** This case corresponds to the following form of $\phi$

$$\phi = \sum_{i=1}^{n} \phi_i(x_i).$$  \hspace{1cm} (10)

Then equation (7) can be rewritten as

$$2\sum_{i=1}^{n} \phi_i \frac{\partial F}{\partial x_i} = \sum_{i=1}^{n} (\phi_i'' + 2\varepsilon \phi_i),$$  \hspace{1cm} (11)
where $\phi'_i = \frac{d\phi_i}{dx_i}$ and $\phi''_i = \frac{d^2\phi_i}{dx_i^2}$.

The particular solution of equation (11) can be found in the form of the function with separated variables

$$f = \sum_{i=1}^{n} f_i(x_i),$$

(12)

where

$$\frac{df_i}{dx_i} = \frac{\phi''_i + 2\varepsilon \phi_i + \lambda_i}{2\phi'_i},$$

(13)

and the constants $\lambda_1, \ldots \lambda_n$ have to satisfy the following condition

$$\sum_{i=1}^{n} \lambda_i = 0.$$  

(14)

The solution of the uniform equation

$$\sum_{i=1}^{n} \phi_i \frac{\partial \tilde{f}}{\partial x_i} = 0$$

(15)

reads

$$\tilde{f} = \tilde{f}(\chi_1(x_1) - \chi_2(x_2), \chi_1(x_1) - \chi_3(x_3), \ldots, \chi_1(x_1) - \chi_n(x_n)),$$

(16)

where $\tilde{f}$ is an arbitrary function of $n-1$ arguments and

$$\chi_i(x) = \int^{x} \frac{dy}{\phi'_i(y)}.$$  

(17)

Finally, the solution of (11) reads

$$F = \sum_{i=1}^{n} \int^{x_i} \frac{\phi''_i(x) + 2\varepsilon \phi_i(x) + \lambda_i}{2\phi'_i(x)} dx + \tilde{f}(\chi_1(x_1) - \chi_2(x_2), \ldots).$$  

(18)

Note that in the case of $\tilde{f} = 0$ the variables of potential $V$ are separated. It is the function $\tilde{f}$ that is responsible for nonseparability of variables in potential $V$.

**Case 2.** Let us choose the generating function $\phi$ in the following form

$$\phi = \prod_{i=1}^{n} \phi_i(x_i).$$  

(19)
Now equation (7) reads
\[ 2 \sum_{i=1}^{n} \frac{\phi_i'}{\phi_i} \partial F = \sum_{i=1}^{n} \frac{\phi_i''}{\phi_i} + 2\varepsilon. \] (20)

The solution is similar to the solution of the previous case. We can write it as
\[ F = \sum_{i=1}^{n} \int x_i \frac{\phi_i''(x)}{2\phi_i'(x)} dx + \tilde{f}(\chi_1(x_1) - \chi_2(x_2), \ldots), \] (21)
where the constants \( \lambda_1, \lambda_2, \ldots \) also satisfy condition (14), \( \tilde{f} \) is an arbitrary function of \( n - 1 \) arguments and
\[ \chi_i(x) = \int x \frac{\phi_i(y)}{\phi_i'(y)} dy. \] (22)

3 Examples

QES potential is given by equation (4), where the function \( F \) is represented by expression (18) in the case 1 and by expression (21) in the case 2. For this QES potential we know two energy levels \( E_0 \) and \( E_1 \) as well as the corresponding wave functions (3), (5). Choosing various functions \( \phi_i \) for the case 1 or the case 2 we obtain different QES potentials. Without loss of generality we choose in all expressions \( E_0 = 0 \) and then \( E_1 = \varepsilon \).

The functions \( \phi_i \) and the parameters \( \lambda_i \) must be chosen in such a way that the function \( f \) is a nonsingular one and \( \psi_0, \psi_1 \) are square integrable functions. Note that in multidimensional case we have also possibility to choose different \( \tilde{f} \) for this purpose.

Example 1. Let us choose
\[ \phi = \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2 \]
which corresponds to the case 1. For this generating function we obtain
\[ \frac{df_i}{dx_i} = \frac{a_i + \lambda_i + a_i \varepsilon x_i^2}{2a_i x_i}. \]
To satisfy the nonsingularity of the functions $f_i$ we have to choose $\lambda_i = -a_i$. Note that $a_i$ satisfy the same condition as $\lambda_i$, namely $\sum a_i = 0$. Then

$$f_i(x_i) = \frac{\varepsilon}{4} x_i^2, \quad \chi_i(x_i) = \frac{1}{a_i} \ln x_i.$$

And finally, we obtain

$$F(x_1, x_2, \ldots) = \frac{\varepsilon}{4} \sum_{i=1}^n x_i^2 + \tilde{f} \left( \frac{x_1^{1/a_1}}{x_2^{1/a_2}}, \frac{x_1^{1/a_1}}{x_3^{1/a_3}}, \ldots \right),$$

here we rewrite $\tilde{f} \left( \frac{1}{a_1} \ln x_1 - \frac{1}{a_2} \ln x_2, \frac{1}{a_1} \ln x_1 - \frac{1}{a_3} \ln x_3, \ldots \right)$ as some new function

$$\tilde{f} \left( \frac{x_1^{1/a_1}}{x_2^{1/a_2}}, \frac{x_1^{1/a_1}}{x_3^{1/a_3}}, \ldots \right).$$

Let us apply these results to the two-dimensional case. Now, $a_1 + a_2 = 0$ and we may choose $a_1 = -a_2 = 1$. Then

$$F(x, y) = \frac{\varepsilon}{4} (x^2 + y^2) + \tilde{f}(xy).$$

QES potential and wave functions with zero and $\varepsilon$ energy levels read

$$V(x, y) = \frac{1}{2} \left[ (\tilde{f}')^2 - \tilde{f}'' + \frac{\varepsilon^2}{4} (x^2 + y^2) + \varepsilon xy \tilde{f}' - \frac{\varepsilon}{2} \right],$$

$$\psi_0(x, y) = c_0 e^{-\frac{\varepsilon}{4} (x^2 + y^2) - \tilde{f}(xy)},$$

$$\psi_1(x, y) = c_1 (x^2 - y^2) e^{-\frac{\varepsilon}{4} (x^2 + y^2) - \tilde{f}(xy)},$$

where $c_0$ and $c_1$ are normalization constants.

Choosing $\tilde{f}$ as some polynomial we reproduce the two-dimensional potentials studied in [40] as interesting examples of bottomless potentials with bound states.

For $\tilde{f} = 0$ we obtain the isotropic harmonic oscillator. Then, in Dirac notation, $\psi_0 = |0, 0\rangle$ is the ground state eigenfunction, and $\psi_1 = \frac{1}{\sqrt{2}} (|2, 0\rangle - |0, 2\rangle)$ corresponds to the second excited energy level.

Note that $\psi_0$ corresponds to the ground state if $\tilde{f}$ is a nonsingular function. For singular $\tilde{f}$ wave function $\psi_0$ may correspond to an excited state. For example, let us choose $\tilde{f} = -\ln(xy)$. Then

$$\psi_0(x, y) = c_0 xy e^{-\frac{\varepsilon}{4} (x^2 + y^2)}$$
corresponds to the second excited energy level.

**Example 2.** This example illustrates the case 2. We choose

\[ \phi_i = x_i, \quad \phi = \prod_{i=1}^{n} x_i. \]

Using (21) we obtain

\[ F = \sum_{i=1}^{n} \left( \frac{2}{n} \varepsilon + \lambda_i \right) \frac{x_i^2}{4} + \tilde{f}(x_1^2 - x_2^2, x_1^2 - x_3^2, \ldots); \]

remember that \( \sum \lambda_i = 0 \).

Let us consider a particular three-dimensional case with \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) and \( \tilde{f} = \alpha(2x^2 - y^2 - z^2)^2 \). Then

\[ V = \left( \frac{\varepsilon^2}{18} - 16\alpha \right) x^2 + \left( \frac{\varepsilon^2}{18} - 4\alpha \right) (y^2 + z^2) + \\
+ 4\alpha(2x^2 - y^2 - z^2)^2 \left( 2\alpha(4x^2 + y^2 + z^2) \frac{\varepsilon}{3} \right) - \frac{1}{2} \varepsilon; \]

\[ \psi_0 = c_0 e^{-\frac{\varepsilon^2 + y^2 + z^2}{6} - \alpha(2x^2 - y^2 - z^2)^2}, \]

\[ \psi_1 = c_1 x y z e^{-\frac{\varepsilon^2 + y^2 + z^2}{6} - \alpha(2x^2 - y^2 - z^2)^2}, \]

where \( c_0 \) and \( c_1 \) are normalization constants.

For \( \alpha = 0 \) we have harmonic oscillator. Then \( \psi_0 = |0, 0, 0, 0\rangle \) is the ground state function, and \( \psi_1 = |1, 1, 1, 1\rangle \) corresponds to the third excited energy level. For \( \alpha \neq 0 \) we obtain nontrivial QES three-dimensional potential, the variables of which cannot be separated.

Here we would like to underline that question about separation of variables in multidimensional Schrödinger equation is an interesting and nontrivial task (see, for instance, [41, 42]). We plan to discuss this problem in a part concerning our approach in a separate paper.

**4 Conclusions**

We developed a simple approach for constructing QES multidimensional potentials with two known energy levels and corresponding wave functions. The
The proposed method is a direct extension of the general approach proposed earlier for one-dimensional case \cite{36,37}. The central point of our approach is equation (7) and the main problem is to solve this equation with respect to the function $F$. In contrast to one-dimensional case, when the corresponding equation can be easily solved, there is a nontrivial task to solve it in multidimensional case. In this paper we find general solutions of equation (7) and construct new multidimensional QES potentials for some special cases of function $\phi$. Finding of other solutions of equation (7) is an interesting problem for further investigations.

Let us stress the following new point which appears in multidimensional case in contrast to one-dimensional case. Namely, in one-dimensional case $\tilde{f} = \text{const}$ and therefore the ratio of the two eigenfunctions $\psi_1/\psi_0 = \phi$ and distance between corresponding energy levels $\varepsilon$ entirely determine the potential energy $V$ \cite{34,36,37}. In multidimensional case the potential energy $V$ is determined by $\phi$, $\tilde{f}$ and $\varepsilon$. Equation (9) allows many nontrivial solutions for $\tilde{f}$. As result we obtain a family of potentials $V$ for fixed $\phi$ and fixed energy gap $\varepsilon$ choosing different solution for $\tilde{f}$. This feature of multidimensional case is explicitly shown in the example 1, where potential $V(x, y)$ for fixed $\phi$ and $\varepsilon$ depends on arbitrary function $\tilde{f}(xy)$. Thus, in multidimensional case there are more possibilities for constructing QES potentials for a given $\phi$ in comparison with the one-dimensional case.

In one-dimensional case all eigenfunctions can be easily ordered using oscillation theorem which states that the $n$-th eigenfunction has $n$ zeros. For multidimensional case it is known that ground state eigenfunction also has no zero and the corresponding energy level is non-degenerated \cite{43}, but there is no theorem which connects zeros and order of excited eigenfunctions. Therefore, one cannot give an exact answer between which states the energy gap is calculated. We can only state that if an eigenfunction has no zeros then this function corresponds to the ground state.

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