Large-time Behavior of Magnetohydrodynamics with Temperature-Dependent Heat-Conductivity

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Abstract. For the strong solutions to the equations of a planar magnetohydrodynamic compressible flow with the heat conductivity proportional to a nonnegative power of the temperature, we first prove that both the specific volume and the temperature are proved to be bounded from below and above independently of time. Then, we also show that the global strong solution is nonlinearly exponentially stable as time tends to infinity. This is the first result obtaining the exponential stability behavior of strong solutions to the equations of a planar magnetohydrodynamic compressible flow without any smallness conditions on the data. Our result can be regarded as a natural generalization of the previous ones for the compressible Navier-Stokes system to MHD system with either constant heat-conductivity or nonlinear and temperature-depending heat-conductivity. As a direct consequence, it is shown that the global strong solution to the constant heat-conductivity MHD system whose existence is obtained by Kazhikhov in 1987 is nonlinearly exponentially stable.

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1. Introduction

The governing equations of a planar magnetohydrodynamic compressible flow written in the Lagrange variables read as follows:

\[
\begin{align*}
vt &= ux, \\
u_t + (P + \frac{1}{2}|b|^2)_x &= \left(\frac{ux}{v}\right)_x, \\
w_t - bx &= \left(\frac{w_x}{v}\right)_x, \\
(vb)_t - wx &= \left(\frac{bx}{v}\right)_x, \\
\left(e + \frac{u^2 + |w|^2 + v|b|^2}{2}\right)_t + \left(ux\left(P + \frac{1}{2}|b|^2\right) - w \cdot b\right)_x &= \left(\frac{\theta_x}{v} + \mu \frac{uu_x}{v} + \lambda \frac{w \cdot w_x}{v} + \nu \frac{b \cdot b_x}{v}\right)_x,
\end{align*}
\]

where \( t > 0 \) is time, \( x \in \Omega = (0, 1) \) denotes the Lagrange mass coordinate, and the unknown functions \( v > 0, u, w \in \mathbb{R}^2, b \in \mathbb{R}^2, e > 0, \theta > 0, \) and \( P \) are, respectively, the specific volume of the gas, longitudinal velocity, transverse velocity, transverse magnetic field, internal energy, absolute temperature and pressure. \( \mu \) and \( \lambda \) are the viscosity of the flow, \( \nu \) is the magnetic diffusivity of the magnetic field, and \( \kappa \) is the heat conductivity.
In this paper, we consider a perfect gas for magnetohydrodynamic flow, that is, \( P \) and \( e \) satisfy
\[
P = R \theta / v, \quad e = c_v \theta + \text{const},
\]
where both specific gas constant \( R \) and heat capacity at constant volume \( c_v \) are positive constants. We also assume that \( \mu, \lambda, \) and \( \nu \) are positive constants, and \( \kappa \) satisfies
\[
\kappa = \tilde{\kappa} \theta^\beta,
\]
with constants \( \tilde{\kappa} > 0 \) and \( \beta \geq 0 \).

The system (1.1)–(1.7) is supplemented with initial conditions
\[
(v, u, \theta, b, w)(x, 0) = (v_0, u_0, \theta_0, b_0, w_0)(x), \quad x \in \Omega,
\]
and boundary ones
\[
(u, b, w, \theta_x) \big|_{\partial\Omega} = 0,
\]
where the initial data (1.8) should be compatible with the boundary conditions (1.9).

Magnetohydrodynamics (MHD), concerning the flow of electrically conducting fluids in the presence of magnetic fields, covers a wide range of physical objects from liquid metals to cosmic plasmas [6, 10, 17, 19, 23, 24, 31]. The central point of MHD theory is that conductive fluids can support magnetic fields. The partial differential equations of MHD can in principle be derived from Boltzmann’s equation assuming space and time scales to be larger than all inherent scale-lengths such as the Debye length or the gyro-radii of the charged particles [6, 17, 23, 24, 31]. In fact, one can deduce from the Chapman-Enskog expansion for the first level of approximation in kinetic theory that the viscosity \( \mu \) and heat conductivity \( \kappa \) are functions of temperature alone (see Chapman-Colwig [7]). These dependencies, especially the dependence of viscosity on temperature, brings great difficulties and challenges to mathematical analysis and numerical calculation. Thus, to study this problem, we first consider the case that the viscosity is a positive constant and the heat conductivity proportional to a nonnegative power of the temperature, as shown as in the equation (1.7).

There is huge literature on the studies of the global existence and large time behavior of solutions to the compressible Navier-Stokes system and MHD system. Indeed, for compressible Navier-Stokes system (1.1) (1.2) (1.5) with \( b \equiv w \equiv 0 \), Kazhikhov and Shelukhin [22] first obtained the global existence of solutions for constant coefficients \( (\beta = 0) \) with large initial data. From then on, much effort has been made to generalize this approach to other cases (for \( \beta > 0 \), see [15, 18] and the reference therein). As for MHD system, the are many results concerning the global existence of solutions with large initial data (see [2, 8, 9, 11–14, 16, 21, 33] and the references therein). In particular, Kazhikhov [21] (see also [2]) first for \( \beta = 0 \) and very recently Huang-Shi-Sun [16] for \( \beta > 0 \) proved that

**Lemma 1.1.** (\([16, 21]\)) Let \( \beta \geq 0 \). Suppose that the initial data \((v_0, u_0, \theta_0, b_0, w_0)\) satisfies
\[
(v_0, \theta_0) \in H^1((0, 1)), \quad (u_0, b_0, w_0) \in H^1_0((0, 1)),
\]
and
\[
\inf_{x \in (0,1)} v_0(x) > 0, \quad \inf_{x \in (0,1)} \theta_0(x) > 0.
\]
Then, the initial-boundary-value problem (1.1)–(1.9) has a unique strong solution \((v, u, \theta, b, w)\) such that for each fixed \( T > 0 \),
\[
\begin{aligned}
&v, \theta \in L^\infty(0, T; H^1(0, 1)), \quad u, b, w \in L^\infty(0, T; H^1_0(0, 1)), \\
v_t \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \\
u_t, \theta_t, b_t, w_t, u_{xx}, \theta_{xx}, b_{xx}, w_{xx} \in L^2((0, 1) \times (0, T)),
\end{aligned}
\]
and for each \((x, t) \in [0, 1] \times [0, T]\)
\[
C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C,
\]
where \( C > 0 \) is a constant depending on the data and \( T \).
Concerning the large-time behavior of solutions to the compressible Navier-Stokes system (1.1) (1.2) (1.5) with \( b \equiv w \equiv 0 \), Kazhikhov [20] (see also [1,3–5,25–29,32] among others) first obtained that for the case that \( \beta = 0 \), the strong solution is nonlinearly exponentially stable as time tends to infinity. Very recently, Huang-Shi [15] prove that the same result still holds for the compressible Navier-Stokes system (1.1) (1.2) (1.5) with \( b \equiv w \equiv 0 \) for \( \beta > 0 \). However, for the MHD system (1.1)–(1.9), it seems to us that the known lower and upper bounds of the specific volume \( v \) and the temperature \( \theta \) depend on the time \( T \), see [16,21], so it is impossible to study the large time asymptotic behavior of solutions in the setting in [16,21]. In fact, the main aim of this paper is to prove that the global strong solutions whose existence is guaranteed by Lemma 1.1 are indeed nonlinearly exponentially stable as time tends to infinity for \( \beta \geq 0 \).

We now state our main result as follows.

**Theorem 1.2.** Under the conditions of Lemma 1.1, there exist positive constants \( C \) and \( \eta_0 \) both depending only on the data such that the unique strong solution \((v, u, \theta, b, w)\) of the initial-boundary-value problem (1.1)–(1.9) obtained by Lemma 1.1 satisfies for any \((x, t) \in (0,1) \times (0,\infty)\),

\[
C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C,
\]

and for any \( t > 0 \),

\[
\|(v - v_s, u, \theta - \theta_s, b, w)\|_{H^1(0,1)} \leq Ce^{-\eta_0 t},
\]

with

\[
v_s \triangleq \int_0^1 v_0 dx, \quad \theta_s \triangleq \int_0^1 \left( \theta_0 + \frac{u_0^2 + |w_0|^2 + v_0|b_0|^2}{2c_v} \right) dx.
\]

A few remarks are in order.

**Remark 1.3.** Although there are many results concerning the large-time behavior of strong solutions to the compressible Navier-Stokes system (1.1) (1.2) (1.5) with \( b \equiv w \equiv 0 \) (see [1,3–5,15,20,26–29,32] and the reference therein), it seems to us that it is still open for the large-time behavior of strong solutions to the compressible MHD system without any smallness conditions on the data even for \( \beta = 0 \). Therefore, our result is the first one obtaining the large-time behavior of strong solutions to the equations of a planar magnetohydrodynamic compressible flow without any smallness conditions on the data.

**Remark 1.4.** Our result can be regarded as a natural generalization of previous ones for compressible Navier-Stokes system with either constant heat conductivity [20] or temperature-dependent one [15] to the temperature-dependent heat conductivity MHD system with the constant heat conductivity as a special case.

We now make some comments on the analysis of this paper. The key step to study the large-time behavior of the global strong solutions is to get the time-independent lower and upper bounds of both \( v \) and \( \theta \) (see (2.1), (2.17), (2.41), and (2.85)). Compared with [15,20,29], the main difficulties come from the interaction of the hydrodynamic and electrodynamic effects and the degeneracy and nonlinearity of the heat conductivity. Hence, to overcome these difficulties, some new ideas are needed. The key observations are as follows: First, after modifying the ideas due to [15,20], we obtain an explicit expression of the specific volume \( v \) (see (2.8)) which together with a lower bound of the temperature \( \theta \) (see (2.14)) shows that \( v \) is bounded from below time-independently (see (2.1)). Then, for \( \beta > 0 \), we find that (see (2.18))

\[
\int_0^T \max_{x \in [0,1]} \left( \theta^{1/2}(x, t) - 2^{1/2} \right)^2 dt \leq C,
\]

which gives the uniform upper bound of \( v \). For \( \beta = 0 \), it seems much more difficult to bound \( v \) from above time-independently. We first prove a new estimate that (see (2.31))

\[
\int_0^T \int_0^1 \frac{|w|}{v} dx dt \leq C,
\]

and for any \( t > 0 \),

\[
\|(v - v_s, u, \theta - \theta_s, b, w)\|_{H^1(0,1)} \leq Ce^{-\eta_0 t},
\]

with

\[
v_s \triangleq \int_0^1 v_0 dx, \quad \theta_s \triangleq \int_0^1 \left( \theta_0 + \frac{u_0^2 + |w_0|^2 + v_0|b_0|^2}{2c_v} \right) dx.
\]
which play an important role in the analysis. Then we refine the strategy of Kazhikhov ([21]), that is, we prove that the $L^\infty(0,T;L^2(0,1))$-norm of $(\ln v)_x$ can be bounded time-independently by a log-type inequality (see (2.40)), which together with the Gronwall inequality in turn gives the uniform upper bound of $v$ (see (2.41)). Next, for $\beta > 0$ and for the upper and lower bounds of $\theta$, we modify slightly the ideas due to [15], that is, we prove that the $L^\infty(0,\infty;L^p(0,1))$-norm of $\theta^{-1}$ is bounded (see (2.24)), which yields that the $L^2((0,1)\times(0,\infty))$-norm of $\theta_x$ is bounded provided $\beta > 1$ (see (2.70)). Finally, for $\beta \in [0,1]$, we mainly follow the idea due to [15] to prove that the $L^2((0,1)\times(0,\infty))$-norm of $\theta_x$ can be bounded by the $L^2(0,\infty;L^2(0,1))$-norm of $u_x$, which plays an important role in bounding the $L^2((0,1)\times(0,\infty))$-norm of both $\theta_x$ and $u_{xx}$ (see Lemma 2.7) for $\beta \in [0,1]$. The whole procedure will be carried out in the next section.

2. Proof of Theorem 1.2

Without loss of generality, we assume that $\lambda = \nu = \mu = \tilde{\kappa} = R = c_v = 1$, and that

$$\int_0^1 v_0 dx = 1, \quad \int_0^1 \left( \theta_0 + \frac{u_0^2 + |w_0|^2 + v_0|b_0|^2}{2} \right) dx = 1.$$  \hspace{1cm} (2.1)

We first state the time-independent lower bound of $v$.

**Lemma 2.1.** For $\beta \geq 0$, it holds that for any $(x,t) \in [0,1] \times [0,\infty)$,

$$v(x,t) \geq C_0,$$  \hspace{1cm} (2.2)

where (and in what follows) $C_0$ and $C$ denote some generic positive constants depending only on $\beta, \|v_0, u_0, \theta_0, b_0, w_0\|_{H^1(0,1)}$, $\inf_{x \in [0,1]} v_0(x)$, and $\inf_{x \in [0,1]} \theta_0(x)$.

**Proof.** First, it follows from (1.1), (1.5), and (1.9) that for $t > 0$

$$\int_0^1 v(x,t) dx \equiv 1, \quad \int_0^1 \left( \theta + \frac{u^2 + |w|^2 + v|b|^2}{2} \right)(x,t) dx \equiv 1.$$  \hspace{1cm} (2.3)

Then, denoting

$$\sigma \triangleq \frac{u_x}{v} - \frac{\theta}{v} - \frac{1}{2}|b|^2,$$  \hspace{1cm} (2.4)

we rewrite (1.2) as

$$u_t = \sigma_x.$$  \hspace{1cm} (2.5)

Integrating this with respect to $x$ over $(0,x)$ gives

$$\left( \int_0^x u dy \right)_t = \sigma - \sigma(0,t),$$  \hspace{1cm} (2.6)

which implies

$$v\sigma(0, t) = v\sigma - v \left( \int_0^x u dy \right)_t.$$  \hspace{1cm} (2.7)

Integrating this with respect to $x$ over $(0,1)$, we obtain after using (1.9), (2.2), and (2.3) that

$$\sigma(0, t) = \int_0^1 \left( u_x - \theta - \frac{v}{2}|b|^2 \right) dx - \left( \int_0^1 v \int_0^x u dy dx \right)_t + \int_0^1 u_x \int_0^x u dy dx.$$  \hspace{1cm} (2.8)

Integrating this with respect to $x$ over $(0,1)$, we obtain after using (1.9), (2.2), and (2.3) that

$$\sigma(0, t) = \int_0^1 \left( u_x - \theta - \frac{v}{2}|b|^2 \right) dx - \left( \int_0^1 v \int_0^x u dy dx \right)_t + \int_0^1 u_x \int_0^x u dy dx.$$  \hspace{1cm} (2.9)

Integrating this with respect to $x$ over $(0,1)$, we obtain after using (1.9), (2.2), and (2.3) that

$$\sigma(0, t) = \int_0^1 \left( u_x - \theta - \frac{v}{2}|b|^2 \right) dx - \left( \int_0^1 v \int_0^x u dy dx \right)_t + \int_0^1 u_x \int_0^x u dy dx.$$  \hspace{1cm} (2.10)
Combining (2.4), (1.1), and (2.5) yields
\[ v(x, t) = D(x, t) Y(t) \exp \left\{ \int_0^t \left( \theta + \frac{v}{2} |b|^2 \right) v^{-1} d\tau \right\}, \]  
(2.6)

with
\[ D(x, t) = v_0 \exp \left\{ \int_0^x (u(y, t) - u_0(y)) dy \right\} \times \exp \left\{ - \int_0^1 \int_0^x u dy dx + \int_0^1 \int_0^x v_0 dy dx \right\}, \]  
(2.7)

and
\[ Y(t) = \exp \left\{ - \int_0^t \int_0^1 \left( u^2 + \frac{v}{2} |b|^2 + \theta \right) dxd\tau \right\}. \]  

Using (2.6), direct computation gives
\[ v(x, t) = D(x, t) Y(t) \left\{ 1 + \int_0^t \int_0^1 \left( \theta + \frac{v}{2} |b|^2 \right)(x, \tau) \frac{D(x, \tau) Y(\tau)}{D(x, \tau) Y(\tau)} d\tau \right\}. \]  
(2.8)

Next, using (1.1)–(1.4), we rewrite the energy equation (1.5) as
\[ \theta_t + \frac{\theta}{v} u_x = \left( \frac{\theta^2}{v} \frac{\partial}{\partial x} \right) u_x + \frac{u_x^2 + |w|^2 + |b|^2}{v} \theta - \ln \theta. \]  
(2.9)

Multiplying (1.1), (1.2), (1.3), (1.4), and (2.9) by \(1 - v^{-1}, u, w, b, \) and \(1 - \theta^{-1}\) respectively, adding them altogether and integrating the result over \((0,1) \times (0, t),\) we obtain the following energy-type inequality that for all \(t > 0,\)
\[ \int_0^1 \left( \frac{u^2 + |w|^2 + v |b|^2}{2} + (v - \ln v) + (\theta - \ln \theta) \right) dx \]  
\[ + \int_0^t V(s) ds \leq e_0, \]  
(2.10)

where
\[ V(t) \triangleq \int_0^1 \left( \frac{\theta^2}{v \theta^2} + \frac{u_x^2 + |w|^2 + |b|^2}{v \theta} \right)(x, t) dx. \]

and
\[ e_0 \triangleq \int_0^1 \left( \frac{u_0^2 + |w_0|^2 + v_0 |b_0|^2}{2} + (v_0 - \ln v_0) + (\theta_0 - \ln \theta_0) \right) dx. \]

Next, applying Jensen’s inequality to the convex function \(\theta - \ln \theta\) leads to
\[ \int_0^1 \theta dx - \ln \int_0^1 \theta dx \leq \int_0^1 (\theta - \ln \theta) dx, \]

which together with (2.10) and (2.2) gives
\[ \bar{\theta}(t) \triangleq \int_0^1 \theta(x, t) dx \in [\alpha_1, 1], \]  
(2.11)

where \(0 < \alpha_1 < \alpha_2\) are two roots of
\[ x - \ln x = e_0. \]
Next, both (2.2) and Cauchy’s inequality imply
\[
\left| \int_0^1 \int_0^x v \, u \, dy \, dx \right| \leq \int_0^1 \left| \int_0^x u \, dy \right| \, dx \\
\leq \int_0^1 v \left( \int_0^1 u^2 \, dy \right)^{1/2} \, dx \\
\leq C,
\]
which combined with (2.7) shows
\[
C^{-1} \leq D(x, t) \leq C. \tag{2.12}
\]
Moreover, one deduces from (2.2) that
\[
\alpha_1 \leq \int_0^1 \left( u^2 + v |b|^2 + \theta \right) \, dx \leq 2,
\]
which yields that for any \(0 \leq \tau < t < \infty\),
\[
e^{-2t} \leq Y(t) \leq 1, \quad e^{-2(t-\tau)} \leq \frac{Y(t)}{Y(\tau)} \leq e^{-\alpha_1(t-\tau)}. \tag{2.13}
\]
Next, denoting \(f_+ \triangleq \max\{f, 0\}\), we have
\[
\left( \theta^{\beta+1}(t) - \theta^{\beta+1}(x, t) \right)_+ \leq \int_0^1 \left| \frac{\partial_x}{\theta^2 v} \left( \theta^{\beta+1}(t) - \theta^{\beta+1}(x, t) \right) \right| \, dx
\leq C \left( \int_0^1 \frac{\theta \theta^2 x \, dx}{\theta^2 v} \right)^{1/2} \left( \int_0^1 1_{(\theta \leq \bar{\theta})} \theta v \, dx \right)^{1/2}
\leq CV^{1/2}(t),
\]
which implies that for \(t > 0\),
\[
\min_{x \in [0, 1]} \theta(x, t) \geq \frac{\alpha_1}{4} - CV(t). \tag{2.14}
\]
Combining (2.8) with (2.12)–(2.14) yields
\[
v(x, t) \geq C^{-1} \int_0^t e^{-2(t-\tau)} \min_{x \in [0, 1]} \theta(x, \tau) \, d\tau
\geq C^{-1} \int_0^t e^{-2(t-\tau)} \left( \frac{\alpha_1}{4} - CV(\tau) \right) \, d\tau
\geq \frac{C^{-1} \alpha_1}{8} - \frac{C^{-1} \alpha_1}{8} e^{-2t} - C \int_0^t e^{-2(t-\tau)} V(\tau) \, d\tau. \tag{2.15}
\]
Since
\[
\int_0^t e^{-2(t-\tau)} V(\tau) \, d\tau = \int_0^{t/2} e^{-2(t-\tau)} V(\tau) \, d\tau + \int_{t/2}^t e^{-2(t-\tau)} V(\tau) \, d\tau
\leq e^{-t} \int_0^\infty V(\tau) \, d\tau + \int_{t/2}^t V(\tau) \, d\tau \to 0, \text{ as } t \to \infty,
\]
it follows from (2.15) that there exists some \(\bar{T} > 0\) such that
\[
v(x, t) \geq \frac{C^{-1} \alpha_1}{16} \tag{2.16}
\]
for all \((x, t) \in [0, 1] \times [\bar{T}, +\infty)\).
Finally, using (2.8), (2.12), and (2.13), we get
\[ v(x, t) \geq C^{-1} e^{-2T}, \]
for all \((x, t) \in [0, 1] \times [0, \tilde{T}]\), which together with (2.16) implies
\[ v(x, t) \geq C_0 \triangleq \min \left\{ \frac{C^{-1} \alpha_1}{16}, C^{-1} e^{-2\tilde{T}} \right\}, \]
for all \((x, t) \in [0, 1] \times [0, +\infty)\). We finish the proof of Lemma 2.1.

\[ \square \]

To obtain the upper bound of \(v\), we set
\[ M_v(t) \triangleq 1 + \max_{x \in [0, 1]} v(x, t). \]
Then we have the following time-independent upper bound of \(v\) for \(\beta > 0\).

**Lemma 2.2.** For \(\beta > 0\), there exists a positive constant \(C\) such that for all \((x, t) \in [0, 1] \times [0, +\infty)\),
\[ v(x, t) \leq C. \tag{2.17} \]

**Proof.** First, we claim that
\[ \int_0^T \max_{x \in [0, 1]} \left( \theta^\frac{\beta}{2} (x, t) - 2^\frac{\beta}{2} \right)^2 + dt \leq C. \tag{2.18} \]
Indeed, on the one hand, for \(\beta \in [1, \infty)\), we have
\[ \int_0^T \max_{x \in [0, 1]} \left( \theta^\frac{\beta}{2} (x, t) - 2^\frac{\beta}{2} \right)^2 + dt \leq C \int_0^T \max_{x \in [0, 1]} \left( \theta^\beta (x, t) - 2^\beta \right)^2 + dt \]
\[ \leq C \int_0^T \left( \int_0^1 \left| \partial_x \left( \theta^\beta (x, t) - 2^\beta \right) \right| dx \right)^2 dt \]
\[ \leq C \int_0^T \int_0^1 \frac{\theta^\beta \theta_x^2}{\theta^2 v} dx \int_0^1 v dx dt \]
\[ \leq C. \tag{2.19} \]
On the other hand, for \(\beta \in (0, 1)\) and \(\eta \triangleq (2-\beta)/4 \in (0, \frac{1}{2})\), integrating (2.9) multiplied by \((\theta^n - 4^n)_+, \theta^{n-1}\) over \((0, 1) \times (0, T)\), we get
\[ \frac{\beta}{2} \int_0^T \int_{(\theta > 4)(t)} \frac{\theta^{-1+\beta/2} \theta_x^2}{v} dx dt \]
\[ + \int_0^T \int_0^1 \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v} (\theta^n - 4^n)_+ \theta^{n-1} dx dt \]
\[ = \frac{1}{2\eta} \int_0^1 \left( (\theta^n - 4^n)_+^2 - (\theta^n - 4^n)_+^2 \right) dx + 4^n (1 - \eta) \int_0^T \int_{(\theta > 4)(t)} \frac{\theta^\beta \theta_x^2}{v \theta^2 - \eta} dx dt \]
\[ \leq C + \frac{\beta}{4} \int_0^T \int_{(\theta > 4)(t)} \frac{\theta^{-1+\beta/2} \theta_x^2}{v} dx dt + \frac{1}{2} \int_0^T \int_0^1 \frac{u_x^2}{v} (\theta^n - 4^n)_+ \theta^{n-1} dx dt, \tag{2.20} \]
where in the last inequality we have used
\[
\int_0^T \int_0^1 \frac{1}{v} (\theta^n - 4^n) + \theta^{n+1} dx dt
\leq C \int_0^T \max_{x \in [0,1]} (\theta^{1/2}(x,t) - 2^{1/2})^2 \int_0^1 \theta^{2n} dx dt
\leq \varepsilon \int_0^T \int_{(\theta > \delta)(t)} \frac{\theta^{-1+\beta/2}\theta^2}{v} dx dt + C(\varepsilon),
\]
due to
\[
\int_0^T \max_{x \in [0,1]} (\theta^{1/2}(x,t) - 2^{1/2})^2 dt
\leq C \int_0^T \int_0^1 \frac{\theta^2}{\partial v} dx \int_0^1 v dx dt
\leq \varepsilon \int_0^T \int_0^1 \frac{\theta^{-1+\beta/2}\theta^2}{v} dx dt + C(\varepsilon) \int_0^T \int_0^1 \frac{\theta^{-2+\beta/2}\theta^2}{v} dx dt
\leq \varepsilon \int_0^T \int_{(\theta > \delta)(t)} \frac{\theta^{-1+\beta/2}\theta^2}{v} dx dt + C(\varepsilon).
\]
Combining (2.20) with (2.21) gives (2.18) for \(\beta \in (0,1)\) which together with (2.19) proves (2.18).

Next, it follows from (2.2) and (2.11) that
\[
\left| \theta^{1/2}(x,t) - \tilde{\theta}^{1/2}(t) \right| \leq C \left| \theta^{\alpha_1/2}(x,t) - \tilde{\theta}^{\alpha_1/2}(t) \right|
\leq C \left( \int_0^1 \frac{\theta^2}{\partial v} dx \right)^{1/2} \left( \int_0^1 \theta v dx \right)^{1/2}
\leq CV^{1/2}(t) M_v^{1/2}(t),
\]
which together with (2.11) shows
\[
\theta(x,t) \leq C + CV(t) M_v(t),
\]
for all \((x,t) \in [0,1] \times [0, \infty)\).

Finally, standard calculations give
\[
\max_{x \in [0,1]} |b|^2(x,t) \leq C \int_0^1 |b \cdot b_x| dx
\leq C \int_0^1 \frac{|b_x|^2}{v\theta} dx + C \int_0^1 v\theta |b|^2 dx
\leq CV(t) + C \max_{x \in [0,1]} (\theta^{1/2}(x,t) - 2^{1/2})^2 + C,
\]
which together with (2.8), (2.12), (2.13), and (2.23) leads to
\[
v(x,t) \leq C + C \int_0^t e^{-\alpha_1(t-\tau)} \max_{x \in [0,1]} (\theta + v |b|)^2(x,\tau) d\tau
\leq C + C \int_0^t e^{-\alpha_1(t-\tau)} (1 + V(\tau)) M_v(\tau) d\tau
+ C \int_0^t \max_{x \in [0,1]} (\theta^{1/2}(x,\tau) - 2^{1/2})^2 M_v(\tau) d\tau.
\]
We thus obtain (2.17) from this, (2.10), (2.18), and the Gronwall inequality. The proof of Lemma 2.2 is finished. \(\square\)
For $\beta > 0$, to obtain the time-independent lower bound of the temperature, we need the following uniform (with respect to time) estimate on the $L^\infty(0, T; L^p)$-norm of $\theta^{-1}$.

**Lemma 2.3.** For $\beta > 0$ and any $p > 0$, there exists some positive constant $C(p)$ such that

$$
\sup_{0 \leq t \leq T} \int_0^1 \theta^{-p} \, dx + \int_0^T \int_0^1 \theta^{\beta - 1} \theta^2 \, dx \, dt \leq C(p).
$$

(2.24)

**Proof.** It suffices to prove (2.24) for $p \neq 1$ since it holds for $p = 1$ due to (2.10). Multiplying (2.9) by $1/\theta^p$ and integration by parts gives

$$
\frac{1}{p - 1} \left( \int_0^1 \theta^{1-p} \, dx \right) + p \int_0^1 \frac{\theta^2 \theta_x^2}{v \theta^p} \, dx + \int_0^1 \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v \theta^p} \, dx
$$

$$
= \int_0^1 \left( \theta^{1-p} - 1 \right) u_x \, dx + \int_0^1 \frac{u_x}{v} \, dx
$$

$$
\leq C(p) \int_0^1 \left( \theta^{1/2} - 1 \right) \left( \int_0^1 \theta^{1-p} \, dx \right)^{1/2} \left( \int_0^1 \frac{u_x^2}{v \theta^p} \, dx \right)^{1/2}
$$

(2.25)

$$
+ C(p) \max_{x \in [0, 1]} \left( \theta^{1/2} - 1 \right) \int_0^1 |u_x| \, dx + \left( \int_0^1 \ln v \, dx \right)
$$

$$
\leq C(p) \max_{x \in [0, 1]} \left( \theta^{1/2} - 1 \right)^2 \left( 1 + \int_0^1 \theta^{1-p} \, dx \right) + \frac{1}{2} \int_0^1 \frac{u_x^2}{v \theta^p} \, dx
$$

$$
+ C(p) \left( \int_0^1 |u_x| \, dx \right)^2 + \left( \int_0^1 \ln v \, dx \right).
$$

Next, by (2.2), (2.11), direct computation shows that for $\beta \geq 0$ and any real number $q$,

$$
|1 - \theta^q| \leq C(q)|1 - \theta|
$$

$$
\leq C(q) \int_0^1 \left( u^2 + |w|^2 + v|b|^2 \right) \, dx
$$

$$
\leq C \int_0^1 \left( |u_x| + |w_x| + |b_x| \right) \, dx
$$

(2.26)

$$
\leq C \left( \int_0^1 \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v \theta} \, dx \right)^{1/2} \left( \int_0^1 v \theta \, dx \right)^{1/2}
$$

$$
\leq CV^{1/2}(t) M_\theta^{1/2}(t),
$$

which together with (2.22), (2.17), and (2.10) yields that for $\beta > 0$,

$$
\int_0^T \max_{x \in [0, 1]} \left( \theta^{1/2} - 1 \right)^2 \, dt \leq C.
$$

(2.27)

Finally, noticing that for $p \in [0, 1]$,

$$
\int_0^1 \theta^{1-p} \, dx \leq \int_0^1 \theta \, dx + 1 \leq C,
$$

and that both (2.10) and (2.2) imply that for $\beta \geq 0$,

$$
\sup_{0 \leq t < \infty} \int_0^1 |\ln v| \, dx \leq C,
$$

(2.28)
after using (2.26), (2.10), (2.27), and the Gronwall inequality, we obtain (2.24) from (2.25) and finish the proof of Lemma 2.3.

□

Lemma 2.4. For \( \beta \geq 0 \), there exists a positive constant \( C \) such that for all \( T > 0 \),

\[
\sup_{0 \leq t \leq T} \int_0^1 \int_0^1 \left( (1 + \theta)v_x^2 + u_x^2 + |b_x|^2 + |w_x|^2 \right) \, dx \, dt \leq C. \tag{2.29}
\]

Proof. Case 1 (\( \beta = 0 \)). First, multiplying (1.3) by \( w \) and integrating the resulting equality over \((0,1)\) yields

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 |w|^2 \, dx + \int_0^1 \frac{|w_x|^2}{v} \, dx \leq C \int_0^1 |w_x||b| \, dx
\]

\[
\leq C \int_0^1 |w_x||b| (\ln \theta - \ln \bar{\theta})_+ \, dx + C \int_0^1 |w_x||b| (1 - \ln \theta)_+ \, dx
\]

\[
\leq C \left( \int_0^1 \frac{|w_x|^2}{v} \, dx \right)^{1/2} \left( \int_0^1 v|b|^2 \, dx \right)^{1/2} \max_{x \in [0,1]} (\ln \theta - \ln \bar{\theta})_+
\]

\[
+ C \left( \int_0^1 \frac{|w_x|^2}{v\theta} \, dx \right)^{1/2} \max_{x \in [0,1]} (|b|(1 - \ln \theta)_+)
\]

\[
\leq \frac{1}{2} \int_0^1 \frac{|w_x|^2}{v} \, dx + CV(t), \tag{2.30}
\]

where in the last inequality we have used the following two simple facts:

\[
\max_{x \in [0,1]} (\ln \theta - \ln \bar{\theta})_+ \leq C \int_0^1 |(\ln \theta - \ln \bar{\theta})_+| \, dx
\]

\[
\leq C \left( \int_0^1 v \, dx \int_0^1 \frac{\theta^2}{v\theta^2} \, dx \right)^{1/2}
\]

\[
\leq CV^{1/2}(t),
\]

and

\[
\max_{x \in [0,1]} (|b|(1 - \ln \theta)_+)
\]

\[
\leq C \int_0^1 |(b(1 - \ln \theta)_+)| \, dx
\]

\[
\leq C \int_0^1 |b| \theta^{-1/2} \, dx + C \int_0^1 |\theta x| \, dx
\]

\[
\leq C \left( \int_0^1 \frac{b^2}{v\theta} \, dx \int_0^1 v \, dx \right)^{1/2} + C \left( \int_0^1 v|b|^2 \, dx \int_0^1 \frac{\theta^2}{v\theta^2} \, dx \right)^{1/2}
\]

\[
\leq CV^{1/2}(t).
\]

Denoting

\[
\tilde{V}(t) \triangleq \int_0^1 \frac{|w_x|^2}{v} \, dx + V(t),
\]

we obtain from (2.30) and (2.10) that

\[
\int_0^T \tilde{V}(t) \, dt \leq C. \tag{2.31}
\]
Next, using (1.1), we rewrite (1.2) as

\[
(ln v)_{xt} = u_t + \left( \frac{\theta}{v} \right)_x + b \cdot b_x.
\]  

Adding (2.32) multiplied by \((ln v)_x\) to (1.4) by \(\nu b\), and integrating the resulting equality over \((0, 1) \times (0, T)\), one has

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( (ln v)^2 + v^2 |b|^2 \right) dx + \int_0^1 \left( |b_x|^2 + \frac{\theta}{v} (ln v)^2 \right) dx \\
= \frac{d}{dt} \int_0^1 u (ln v)_x dx + \int_0^1 \frac{u_x^2}{v} dx + \int_0^1 \frac{\theta_x (ln v)_x}{v} dx + \int_0^1 v w_x \cdot b dx.
\]

Then, on the one hand, for the last term on the righthand side of (2.33), we have by (2.30),

\[
\left| \int_0^1 v w_x \cdot b dx \right| \leq CM_v(t) \int_0^1 |w_x| \|b\| dx
\leq CM_v(t) V(t).
\]

On the other hand, for the third term on the righthand side of (2.33), integrating by part gives

\[
\int_0^1 \frac{\theta_x (ln v)_x}{v} dx = - \int_0^1 \ln v \frac{\theta_x}{C_0} \left( \frac{\theta_x}{v} \right)_x dx \\
= - \int_0^1 \ln v \left( \theta_t + \frac{\theta}{v} u_x - \frac{u_x^2}{v} + \frac{|w_x|^2 + |b_x|^2}{v} \right) dx
\]

\[
= - \left( \int_0^1 \frac{\theta}{v} \ln v dx \right)_t + \int_0^1 \frac{\theta u_x}{v} dx \\
- \int_0^1 \frac{\theta u_x}{v} dx + \int_0^1 \frac{u_x^2}{v} + \frac{|w_x|^2 + |b_x|^2}{v} \ln v \frac{v}{C_0} dx.
\]

For the second term on the righthand side of (2.35), we have

\[
\int_0^1 \frac{\theta}{v} u_x dx = \int_0^1 \frac{\theta - 1}{v} u_x dx + \int_0^1 \frac{u_x}{v} dx \\
\leq \varepsilon \int_0^1 \frac{u_x}{v} dx + C(\varepsilon) \int_0^1 (\theta - 1)^2 dx + \left( \int_0^1 \ln v dx \right)_t \\
\leq \varepsilon \int_0^1 \frac{u_x}{v} dx + C(\varepsilon)V(t) M_v(t) + \left( \int_0^1 \ln v dx \right)_t,
\]

where in the last inequality we have used

\[
\int_0^1 (\theta - 1)^2 dx \leq C \max_{x \in [0, 1]} \left( \theta^{1/2} - 1 \right)^2 \\
\leq C \max_{x \in [0, 1]} \left( \theta^{1/2} - \theta^{1/2} \right)^2 + C(1 - \theta^2) \\
\leq C \int_0^1 |\theta_x|^2 \theta^{-1/2} dx + CV(t) M_v(t) \\
\leq C \int_0^1 \frac{\theta x}{\theta^2 v} dx \int_0^1 v \theta dx + CV(t) M_v(t) \\
\leq CV(t) M_v(t)
\]
due to (2.26). Similarly, for the third term on the righthand side of (2.35), we have
\[
- \int_0^1 \frac{\theta}{v} u_x \ln \frac{v}{C_0} \, dx \\
\leq C \left( \int_0^1 \frac{u_x^2}{v} \, dx + V(t)M_v(t) \right) \ln M_v(t) - \frac{1}{2} \left( \int_0^1 \ln \frac{v}{C_0} \, dx \right),
\]
(2.38)
due to (2.36). Putting (2.36) and (2.38) into (2.35) gives
\[
\int_0^1 \frac{\theta}{v} (\ln v)_x \, dx \leq \frac{d}{dt} \int_0^1 \left( -\theta \ln \frac{v}{C_0} + \ln v - \frac{1}{2} \ln^2 \frac{v}{C_0} \right) \, dx \\
+ C \left( \int_0^1 \frac{u_x^2 + |b_x|^2}{v} \, dx + \bar{V}(t)M_v(t) \right) \ln M_v(t),
\]
which together with (2.33) and (2.34) leads to
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \ln v \right)_x^2 + |v b|^2 + 2\theta \ln \frac{v}{C_0} + \ln^2 \frac{v}{C_0} - 2 \ln v - 2u(\ln v)_x \, dx \\
+ \int_0^1 \left( |b_x|^2 + \frac{\theta}{v} (\ln v)_x^2 \right) \, dx \\
\leq C \left( \int_0^1 \frac{u_x^2 + |b_x|^2}{v} \, dx + \bar{V}(t)M_v(t) \right) \ln M_v(t),
\]
(2.39)
Then, integrating (2.9) over (0,1) × (0,t), we have by (2.36) and (2.28)
\[
\int_0^t \int_0^1 \frac{u_x^2 + |b_x|^2}{v} \, dx ds \leq C + C \int_0^t \bar{V}(s)M_v(s) \, ds,
\]
which combined with (2.39) yields
\[
\sup_{0 \leq s \leq t} \int_0^1 (\ln v)_x^2 \, dx + \int_0^t \int_0^1 \left( |b_x|^2 + \frac{\theta}{v} (\ln v)_x^2 \right) \, dx ds \\
\leq C \ln \sup_{0 \leq s \leq t} M_v(s) + C \int_0^t \bar{V}(s)M_v(s) \, ds \ln \sup_{0 \leq s \leq t} M_v(s) \\
\leq C(\varepsilon) \left( 2 + \int_0^t \bar{V}(s)M_v(s) \, ds \right) \ln \left( 2 + \int_0^t \bar{V}(s)M_v(s) \, ds \right) \\
+ C(\varepsilon) + \varepsilon \sup_{0 \leq s \leq t} M_v(s),
\]
(2.40)
where in the second inequality we have used
\[
fg \leq e^f - f - 1 + (1 + g) \ln (1 + g) - g, \text{ for any } f, g \geq 0,
\]
with
\[
f = \frac{1}{2} \ln \sup_{0 \leq s \leq t} M_v(s), \quad g = 2C \int_0^t \bar{V}(s)M_v(s) \, ds.
\]
Then, direct computation shows
\[
v - 1 \leq C \left( \int_0^1 v^2 \, dx \right)^{1/2} \left( \int_0^1 \frac{v^2}{v_x^2} \, dx \right)^{1/2} \\
\leq CM_v^{1/2}(t) \left( \int_0^1 \frac{v_x^2}{v^2} \, dx \right)^{1/2},
\]
which gives
\[ M_v(t) \leq C + C \int_0^1 (\ln v)^2 dx. \]
Combining this, (2.40), (2.31), and the Gronwall inequality shows that for any \((x, t) \in [0, 1] \times [0, +\infty),\)
\[ v(x, t) \leq C, \quad (2.41) \]
which together with (2.40) and (2.31) implies
\[ \sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx + \int_0^1 \int_0^T (\theta v_x^2 + u_x^2 + |b_x|^2 + |w_x|^2) dx dt \leq C. \quad (2.42) \]
Finally, direct computation shows
\[ \int_0^1 v_x^2 dx = \int_0^1 v_x^2 (1 - \theta) dx + \int_0^1 v_x^2 \theta dx \]
\[ \leq \int_0^1 v_x^2 (1 - \theta) dx + \int_0^1 v_x^2 \theta dx \quad (2.43) \]
\[ \leq \frac{1}{2} \int_0^1 v_x^2 dx + C \max_{x \in [0, 1]} (\theta^{1/2} - 1)^2 + \int_0^1 v_x^2 \theta dx, \]
where in the last inequality we have used (2.42). Combining this, (2.42), (2.37), and (2.41) gives (2.29) for \(\beta = 0.\)

**Case 2** \((\beta > 0).\) First, we rewrite the momentum equation (1.2) as
\[ (u - \frac{v_x}{v})_t = - \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right)_x. \]
Multiplying the above equation by \(u - \frac{v_x}{v}\) and integrating the resultant equality yields that for any \(t \in (0, T)\)
\[ \frac{1}{2} \int_0^t \int_0^1 \left( u - \frac{v_x}{v} \right)^2 dx dt - \frac{1}{2} \int_0^1 \left( u - \frac{v_x}{v} \right)(x, 0) dx \]
\[ = \int_0^t \int_0^1 \left( \theta v_x - \theta \frac{x}{v} - b \cdot b_x \right) \left( u - \frac{v_x}{v} \right) dx dt \]
\[ = - \int_0^t \int_0^1 \frac{\theta v_x^2}{v^3} dx dt + \int_0^t \int_0^1 \frac{u v_x}{v^2} dx dt \]
\[ = - \int_0^t \int_0^1 \frac{\theta x}{v} \left( u - \frac{v_x}{v} \right) dx dt - \int_0^t \int_0^1 b \cdot b_x \left( u - \frac{v_x}{v} \right) dx dt \]
\[ = - \int_0^t \int_0^1 \frac{\theta v_x^2}{v^3} dx dt + \sum_{i=1}^3 I_i. \quad (2.44) \]
Each \(I_i (i = 1, 2, 3)\) can be estimated as follows:

First, Cauchy’s inequality gives
\[ |I_1| \leq \frac{1}{4} \int_0^t \int_0^1 \frac{\theta v_x^2}{v^3} dx dt + C \int_0^T \int_0^1 u_x^2 \theta dx dt \]
\[ \leq \frac{1}{4} \int_0^t \int_0^1 \frac{\theta v_x^2}{v^3} dx dt + C, \quad (2.45) \]
where we have used
\[ \int_0^T \int_0^1 u^2 \theta dxdt \leq C \int_0^T \int_0^1 u^2((\theta^{1/2} - 1)^2 + 1) dxdt \]
\[ \leq C \int_0^T \left( \int_0^1 u^2 dx + \max_{x \in [0,1]} \left( \theta^{1/2} - 1 \right)^2 \right) dt \]
\[ \leq C, \tag{2.46} \]
due to (2.26), (2.10), and (2.27).

Next, using (2.46), (2.10), and (2.24) with \( p = \beta \), we get
\[ |I_2| \leq C \int_0^T \int_0^1 u^2 \theta dxdt + C \int_0^T \int_0^1 \theta^{-1} \theta_x^2 dxdt + \frac{1}{2} \int_0^T \int_0^1 \frac{\theta v^2}{v^3} dxdt \]
\[ \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\theta v^2}{v^3} dxdt. \tag{2.47} \]

Finally, integrating (2.9) over \((0, 1) \times (0, T)\), we have by (2.28)
\[ \int_0^T \int_0^1 \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v} dxdt \]
\[ = \int_0^1 \theta dx - \int_0^1 \theta_0 dx + \int_0^T \int_0^1 \theta - \frac{1}{v} u_x dx + \int_0^1 \ln v dx - \int_0^1 \ln v_0 dx \]
\[ \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{u_x^2}{v} dxdt + C \int_0^T \max_{x \in [0,1]} (\theta^{1/2} - 1)^2 dt, \]
which together with (2.27) and (2.17) gives
\[ \int_0^T \int_0^1 (u_x^2 + |w_x|^2 + |b_x|^2) dxdt \leq C. \tag{2.48} \]

Combining this with Cauchy’s inequality leads to
\[ |I_3| \leq C \int_0^T \int_0^1 \left( |b_x|^2 + |b|^2 \left( u - \frac{v_x}{v} \right)^2 \right) dxdt \]
\[ \leq C + C \int_0^T V(t) \int_0^1 \left( u - \frac{v_x}{v} \right)^2 dxdt, \tag{2.49} \]
due to
\[ \max_{0 \leq x \leq 1} |b|^2 \leq C \int_0^1 \frac{|b|^2}{v^\theta dx} \int_0^1 v^\theta dx \leq CV(t). \]

Then, putting (2.45), (2.47), and (2.49) into (2.44), we obtain after using the Gronwall inequality and (2.10) that
\[ \sup_{0 \leq t \leq T} \int_0^1 \left( u - \frac{v_x}{v} \right)^2 dx + \int_0^T \int_0^1 \frac{\theta v_x^2}{v^3} dxdt \leq C, \]
which together with (2.10) and (2.17) gives
\[ \sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx + \int_0^T \int_0^1 v_x^2 \theta dxdt \leq C. \tag{2.50} \]

Finally, since (2.43) still holds for \( \beta > 0 \) due to (2.50), we have
\[ \int_0^T \int_0^1 v_x^2 dxdt \leq C, \]
which together with (2.50) and (2.48) proves (2.29) for \( \beta > 0 \). The proof of Lemma 2.4 is finished. \( \square \)
Lemma 2.5. For $\beta \geq 0$, there is a positive constant $C$ such that for all $T > 0$,
\begin{equation}
\sup_{0 \leq t \leq T} \int_0^1 (|b|^2 + |w|^2) \, dx + \int_0^T \int_0^1 (|b|^2 + |b_{xx}|^2 + |w|^2 + |w_{xx}|^2) \, dx \, dt \leq C. \tag{2.51}
\end{equation}

Proof. First, rewriting (1.3) as
\begin{equation}
w_t = \frac{w_{xx}}{v} - \frac{w_x v_x}{v^2} + b_x, \tag{2.52}
\end{equation}
multiplying (2.52) by $w_{xx}$, and integrating the resulting equality over $(0, 1) \times (0, T)$, we obtain after using (1.9), (2.29), and Cauchy’s inequality that
\begin{align}
\frac{1}{2} \int_0^1 |w|^2 \, dx + \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} \, dx \, dt + C \int_0^T \int_0^1 (|b|^2 + |w|^2 |v_x|^2) \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} \, dx \, dt + C \int_0^T \max_{x \in [0, 1]} |w_x|^2 \, dt. \tag{2.53}
\end{align}

Noticing that for any $f \in \{f_0^1 f \, dx = 0\} \cup \{f(0) = 0\}$,
\begin{equation}
\max_{x \in [0, 1]} f^2 \leq 2 \left(\int_0^1 f^2 \, dx\right)^{1/2} \left(\int_0^1 f_x^2 \, dx\right)^{1/2}, \tag{2.54}
\end{equation}
we get for any $\varepsilon > 0$,
\begin{align}
\int_0^T \max_{x \in [0, 1]} |w_x|^2 \, dt &\leq C(\varepsilon) \int_0^T \int_0^1 |w_x|^2 \, dx \, dt + \varepsilon \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} \, dx \, dt \\
&\leq C(\varepsilon) + \varepsilon \int_0^T \int_0^1 \frac{|w_{xx}|^2}{v} \, dx \, dt, \tag{2.55}
\end{align}
which combined with (2.53) leads to
\begin{equation}
\sup_{0 \leq t \leq T} \int_0^1 |w|^2 \, dx + \int_0^T \int_0^1 |w_{xx}|^2 \, dx \, dt \leq C. \tag{2.56}
\end{equation}

Combining this, (2.52), (2.55), and (2.29) gives
\begin{align}
\int_0^T \int_0^1 |w_t|^2 \, dx \, dt &\leq C \int_0^T \int_0^1 (|b|^2 + |w_{xx}|^2 + |v_x|^2 |w_x|^2) \, dx \, dt \\
&\leq C + C \int_0^T \max_{x \in [0, 1]} |w_x|^2 \, dt \\
&\leq C. \tag{2.57}
\end{align}

Next, rewriting (1.4) as
\begin{equation}
b_t = \frac{w_x}{v} + \frac{b_{xx}}{v^2} - \frac{b_x v_x}{v^3} - \frac{b u_x}{v}, \tag{2.58}
\end{equation}
multiplying (2.58) by $b_{xx}$ and integrating the result over $(0.1) \times (0, T)$, we deduce from (2.29), (2.10), (2.54), and Cauchy’s inequality that

\[
\frac{1}{2} \int_0^1 |b_x|^2 \, dx + \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dxdt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dxdt + C \int_0^T \int_0^1 \left( |b_x|^2 v_x^2 + u_x^2 |b|^2 + |w_x|^2 \right) \, dxdt \\
\leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dxdt + C \int_0^T \max_{x \in [0,1]} |b_x|^2 dt + \max_{(x,t) \in [0,1] \times [0,T]} |b|^2 \\
\leq C + \frac{3}{4} \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dxdt + C \int_0^T \int_0^1 |b_x|^2 \, dxdt \\
+ C \sup_{0 \leq t \leq T} \int_0^1 |b|^2 \, dx + \frac{1}{4} \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx \\
\leq C + \frac{3}{4} \int_0^T \int_0^1 \frac{|b_{xx}|^2}{v^2} \, dxdt + \frac{1}{4} \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx,
\]

which implies

\[
\sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx + \int_0^T \int_0^1 |b_{xx}|^2 \, dxdt \leq C.
\]  

(2.59)

Hence,

\[
\max_{(x,t) \in [0,1] \times [0,T]} |b|^2 \leq C + C \sup_{0 \leq t \leq T} \int_0^1 |b_x|^2 \, dx \leq C.
\]  

(2.60)

Finally, it follows from (2.58), (2.59), (2.29), and (2.60) that

\[
\int_0^T \int_0^1 |b_t|^2 \, dxdt \leq C \int_0^T \int_0^1 \left( |b_{xx}|^2 + |b_x|^2 v_x^2 + |w_x|^2 + |b|^2 u_x^2 \right) \, dxdt \\
\leq C + C \int_0^T \left( \max_{x \in [0,1]} |b_x|^2 + \int_0^1 u_x^2 \, dx \right) \, dt \\
\leq C + C \int_0^T \int_0^1 \left( |b_x|^2 + |b_{xx}|^2 \right) \, dxdt \\
\leq C.
\]

Combining this, (2.56), (2.57), and (2.59) gives (2.51) and finishes the proof of Lemma 2.5. \hfill \square

For further uses, we need the following estimate on the $L^2((0,1) \times (0, T))$-norm of $\theta_x$ for $\beta \in [0, 1]$.

**Lemma 2.6.** If $\beta \in [0, 1]$, there exists a positive constant $C$ such that for all $T > 0$,

\[
\int_0^T \int_0^1 \theta_x^2 \, dxdt \leq C + C \int_0^T \left( \int_0^1 u_x^2 \, dx \right)^2 \, dt.
\]  

(2.61)
Proof. First, multiplying (2.9) by $\theta^{1-\frac{\alpha}{2}}$ and integration by parts gives

\[
\frac{2}{4-\beta} \left( \int_0^1 \theta^{2-\frac{\alpha}{2}} \frac{dx}{t} \right)_t + \frac{(2-\beta)}{2} \int_0^1 \frac{\theta^{\frac{\alpha}{2}} \theta^2}{v} dx
= \int_0^1 \left( \frac{\theta^{2-\frac{\alpha}{2}}}{v} - \theta^{2-\frac{\alpha}{2}} \right) u_x dx + \left( 1 - \theta^{2-\frac{\alpha}{2}} \right) \int_0^1 v_x dx
- \int_0^1 \frac{u_x}{v} dx + \int_0^1 \frac{1}{v} \left( u_x^2 + |b_x|^2 + |w_x|^2 \right) dx
\leq C \int_0^1 \left| \theta^{2-\frac{\alpha}{2}} - \bar{\theta}^{2-\frac{\alpha}{2}} \right| |u_x| dx + CV(t) \left( \int_0^1 \ln v dx \right)_t
+ C \int_0^1 \theta^{1-\frac{\alpha}{2}} (u_x^2 + |b_x|^2 + |w_x|^2) dx,
\]

where in the last inequality we have used (2.26). Direct calculations yield that for any $\delta > 0$

\[
\int_0^1 \left| \theta^{2-\frac{\alpha}{2}} - \bar{\theta}^{2-\frac{\alpha}{2}} \right| |u_x| dx
\leq C \max_{x \in [0,1]} \left| \theta^{1-\frac{\alpha}{2}} - \bar{\theta}^{1-\frac{\alpha}{2}} \right| \left( \int_0^1 \left( \theta^{2-\frac{\alpha}{2}} + 1 \right) dx \right)^{1/2} \left( \int_0^1 u_x^2 dx \right)^{1/2}
\leq \delta \left( \int_0^1 \theta^{-\frac{\alpha}{2}} |u_x| dx \right)^2 + C(\delta) \int_0^1 \left( \theta^{2-\frac{\alpha}{2}} + 1 \right) dx \int_0^1 u_x^2 dx
\leq C \delta \int_0^1 \theta^{3/2} u_x^2 dx + C(\delta) V(t) + C(\delta) \int_0^1 \left( \theta^{2-\frac{\alpha}{2}} + 1 \right) dx \int_0^1 u_x^2 dx,
\]

and that for any $\delta > 0$

\[
\int_0^1 \theta^{1-\frac{\alpha}{2}} (u_x^2 + |b_x|^2 + |w_x|^2) dx
\leq C \left( \max_{x \in [0,1]} \left| \theta^{1-\frac{\alpha}{2}} - \bar{\theta}^{1-\frac{\alpha}{2}} \right| + 1 \right) \int_0^1 (u_x^2 + |b_x|^2 + |w_x|^2) dx
\leq C \left( \int_0^1 \theta^{-\frac{\alpha}{2}} |u_x| dx + 1 \right) \int_0^1 (u_x^2 + |b_x|^2 + |w_x|^2) dx
\leq \delta \int_0^1 \left( \theta^{-\frac{\alpha}{2}} + \theta \frac{\bar{\theta}}{2} \right) \theta_x^2 dx + C(\delta) \left( \int_0^1 u_x^2 dx \right)^2
+ C \int_0^1 (u_x^2 + |b_x|^2 + |w_x|^2) dx.
\]

Putting (2.63) and (2.64) into (2.62), choosing $\delta$ suitably small, and using (2.10), (2.29), (2.51), and the Gronwall inequality, one obtains

\[
\int_0^1 \theta^{2-\beta/2} dx + \int_0^T \int_0^1 \theta^{3/2} \theta_x^2 dx dt \leq C + C \int_0^T \left( \int_0^1 u_x^2 dx \right)^2 dt,
\]
which together with (2.10) implies
\[
\int_0^T \int_0^1 \theta_x^2 dx dt \leq C \int_0^T \int_0^1 \left( \theta^{\beta-2} + \theta^{\beta/2} \right) \theta_x^2 dx dt \\
\leq C + C \int_0^T \int_0^1 \theta^{\beta/2} \theta_x^2 dx dt \\
\leq C + C \int_0^T \left( \int_0^1 u_x^2 dx \right)^2 dt.
\]
This gives (2.61) and finishes the proof of Lemma 2.6. \qed

**Lemma 2.7.** For $\beta \geq 0$, there is a positive constant $C$ such that for all $T > 0$,
\[
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 dx + \int_0^T \int_0^1 \left( u_t^2 + u_{xx}^2 + \theta_x^2 \right) dx dt \leq C. \tag{2.65}
\]

**Proof.** First, rewriting (1.2) as
\[
\delta > \text{(2.51) and (2.29).}
\]
Due to (2.69) and (2.29), we have
\[
\int_0^T \int_0^1 u_x^2 dx dt \leq C \int_0^T \int_0^1 \left( u_t^2 + u_{xx}^2 + \theta_x^2 \right) dx dt.
\]
where in the last inequality we have used (2.54). Putting (2.68) into (2.67) and choosing $\delta$ suitably small yields
\[
\int_0^1 \left( \theta_x^2 + \theta^2 v_x^2 + |b|^2 |b_x|^2 + u_x^2 v_x^2 \right) dx \\
\leq C \left( \max_{x \in [0,1]} u_x^2 + \max_{x \in [0,1]} (\theta - \tilde{\theta})^2 + 1 \right) \int_0^1 u_x^2 dx + C \int_0^1 (|b_x|^2 + \theta^2) dx \\
\leq C \max_{x \in [0,1]} u_x^2 + C \max_{x \in [0,1]} (\theta - \tilde{\theta})^2 + C \int_0^1 (v_x^2 + |b|) dx + C \int_0^1 \theta_x^2 dx \\
\leq \delta \int_0^1 u_{xx} dx + C(\delta) \int_0^1 u_x^2 dx + C \int_0^1 (v_x^2 + |b|) dx + C \int_0^1 \theta_x^2 dx,
\]
where in the last inequality we have used (2.54). Putting (2.68) into (2.67) and choosing $\delta$ suitably small yields
\[
\int_0^1 u_x^2 dx + \int_0^T \int_0^1 u_{xx} dx dt \leq C + C \int_0^T \int_0^1 \theta_x^2 dx dt, \tag{2.69}
\]
due to (2.51) and (2.29).

Next, on the one hand, if $\beta > 1$, choosing $p = \beta - 1$ in (2.24) shows
\[
\int_0^T \int_0^1 \theta_x^2 dx dt \leq C, \tag{2.70}
\]
which along with (2.69) gives
\[
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 dx + \int_0^T \int_0^1 u_{xx} dx dt + \int_0^T \int_0^1 \theta_x^2 dx dt \leq C. \tag{2.71}
\]
On the other hand, if $\beta \in [0,1]$, it follows from (2.69), (2.61), (2.29), and the Gronwall inequality that (2.71) still holds.
Finally, it follows from (2.66), (2.71), (2.68), and (2.29) that

$$\int_0^T \int_0^1 u_t^2 dx dt \leq C,$$

which together with (2.71) gives (2.65) and finishes the proof of Lemma 2.7.

\[\square\]

**Lemma 2.8.** For $\beta \geq 0$, there exists a positive constant $C$ such that for all $T > 0$,

$$\sup_{0 \leq t \leq T} \int_0^1 \theta_t^2 dx + \int_0^T \int_0^1 (\theta_t^2 + \theta_{xx}^2) dx dt \leq C. \tag{2.72}$$

**Proof.** First, multiplying (2.9) by $\theta$ and integrating the result over $(0, 1)$ yields

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \theta^2 dx + \int_0^1 \frac{\theta^2 \theta_x^2}{v} dx$$

$$= - \int_0^1 \frac{\theta^2 - 1}{v} u_x dx - \int_0^1 \frac{u_x}{v} dx + \int_0^1 \left( u_x^2 + |w_x|^2 + |b_x|^2 \right) \frac{\theta}{v} dx \tag{2.73}$$

$$\leq C \max_{x \in [0, 1]} \left( u_x^2 + |w_x|^2 + |b_x|^2 + (\theta - 1)^2 \right) - \frac{d}{dt} \int_0^1 \ln v dx,$$

where we have used (2.2). It follows from (2.26), (2.65), and (2.29) that

$$\int_0^T \max_{x \in [0, 1]} (\theta - 1)^2 dt \leq C \int_0^T \max_{x \in [0, 1]} (\theta - \bar{\theta})^2 dt + C \int_0^T (1 - \bar{\theta})^2 dt$$

$$\leq C \int_0^T \int_0^1 \theta_t^2 dx dt + C \int_0^T V(t) dt \leq C,$$

which together with (2.73), (2.51), and (2.65) gives

$$\sup_{0 \leq t \leq T} \int_0^1 \theta^2 dx + \int_0^T \int_0^1 \theta \theta_t^2 dx dt \leq C. \tag{2.74}$$

Next, noticing that integration by parts leads to

$$\int_0^1 \theta^3 \theta_t \left( \frac{\theta^2 \theta_x}{v} \right) dx = - \int_0^1 \theta \theta_x \left( \frac{\theta^2 \theta_x}{v} \right)_t dx$$

$$= - \int_0^1 \theta^3 \theta_x \left( \theta^2 \theta_x \right)_t dx$$

$$= - \frac{1}{2} \int_0^1 \left( \theta^2 \theta_x \right)_t^2 dx$$

$$= - \frac{1}{2} \left( \int_0^1 \frac{(\theta^2 \theta_x)^2}{v} dx \right)_t - \frac{1}{2} \int_0^1 \frac{(\theta^2 \theta_x)^2 u_x}{v^2} dx,$$
multiplying (2.9) by $\theta^3 \theta_t$ and integrating the resultant equality over $(0,1)$, we have
\[
\int_0^1 \theta^3 \theta_t^2 \, dx + \frac{1}{2} \left( \int_0^1 \left( \theta^3 \theta_x \right)^2 \, dx \right)_t \]
\[= -\frac{1}{2} \int_0^1 \left( \theta^3 \theta_x \right)^2 \, dx + \int_0^1 \theta^3 \theta_t \left( -\theta u_x + u_x^2 + |w_x|^2 + |b_x|^2 \right) \, dx \]
\[\leq C \max_{x \in [0,1]} (|u_x| \theta^{3/2}) \int_0^1 \theta^{3/2} \theta_x^2 \, dx + \frac{1}{2} \int_0^1 \theta^3 \theta_t^2 \, dx \]
\[+ C \int_0^1 \theta^3 \theta_x^2 \, dx + C \int_0^1 \theta^3 (u_x^4 + |w_x|^4 + |b_x|^4) \, dx \]
\[\leq C \int_0^1 \theta^3 \theta_x^2 \, dx + C \left( 1 + \int_0^1 \theta^3 \theta_x^2 \, dx \right) \max_{x \in [0,1]} \left( u_x^2 + u_x^4 + |w_x|^4 + |b_x|^4 \right), \tag{2.75} \]
due to
\[
\max_{x \in [0,1]} (\theta^{3+1} - \bar{\theta}^{3+1})^2 \leq C \int_0^1 \theta^3 \theta_x^2 \, dx. \tag{2.76} \]

Next, combining (2.54) and (2.65) gives
\[
\int_0^T \max_{x \in [0,1]} u_x^4 \, dt \leq C \int_0^T \int_0^1 u_x^2 \, dx \int_0^1 u_{xx} \, dx \, dt \]
\[\leq C \int_0^T \int_0^1 u_{xx} \, dx \, dt \leq C. \tag{2.77} \]

Using (2.51) and applying similar arguments to $b$ and $w$ implies
\[
\int_0^T \max_{x \in [0,1]} |b_x|^4 \, dt \leq C, \quad \int_0^T \max_{x \in [0,1]} |w_x|^4 \, dt \leq C. \tag{2.78} \]

We deduce from (2.75), (2.74), (2.77), (2.78), and the Gronwall inequality that
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \theta^3 \theta_x \right)^2 \, dx + \int_0^T \int_0^1 \theta^3 \theta_x^2 \, dt \leq C, \tag{2.79} \]
which together with (2.76) in particular gives
\[
\max_{(x,t) \in [0,1] \times [0,T]} \theta(x,t) \leq C. \tag{2.80} \]

Next, it follows from (2.76), (2.80), and (2.65) that
\[
\int_0^T \int_0^1 \left( \theta^{3+1} - \bar{\theta}^{3+1} \right)^2 \, dx \, dt \leq C \int_0^T \int_0^1 \theta_{xx}^2 \, dx \, dt \leq C, \tag{2.81} \]
which together with (2.80), (2.79), and (2.9) shows
\[
\int_0^T \int_0^1 \left( \theta^{3+1} - \bar{\theta}^{3+1} \right)^2 \, dx \, dt \leq C \int_0^T \int_0^1 \theta_{xx}^2 \, dx \, dt \leq C, \tag{2.82} \]
\[
\leq C + C \int_0^T \int_0^1 \left( u_x^2 + |b_x|^2 + |w_x|^2 \right) \, dx \, dt \leq C. \]
Thus, both \(2.81\) and \(2.82\) lead to
\[
\lim_{t \to \infty} \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx = 0,
\]
which together with \(2.79\) gives
\[
\max_{x \in [0,1]} (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^4 \leq C \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \int_0^1 \theta^{2\beta} \theta_x^2 \, dx
\]
\[
\leq C \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \, dx \to 0 \text{ as } t \to \infty.
\]

(2.83)

It thus follows from \(2.83\) and \(2.11\) that there exists some \(T_0 > 0\) such that
\[
\theta(x, t) \geq \alpha_1/2,
\]
for all \((x, t) \in [0, 1] \times [T_0, \infty)\). Moreover, it follows from \([16,21]\) that there exists some constant \(C \geq 2/\alpha_1\) such that
\[
\theta(x, t) \geq C^{-1},
\]
for all \((x, t) \in [0, 1] \times [0, T_0]\). Combining this, \(2.84\), and \(2.80\) yields that for all \((x, t) \in [0, 1] \times [0, \infty)\),
\[
C^{-1} \leq \theta(x, t) \leq C,
\]
which together with \(2.79\) leads to
\[
\sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 \, dx + \int_0^T \int_0^1 \theta_x^2 \, dx \, dt \leq C.
\]

(2.86)

Finally, it follows from \(2.9\) that
\[
\frac{\theta^2 \theta_{xx}}{v} = -\frac{\beta\theta - 1}{\theta^2} + \frac{\theta^2 \theta_x v_x}{\theta^2} - \frac{u_x^2 + |b_x|^2 + |w_x|^2}{v} + \frac{\theta u_x}{v} + \theta_t,
\]
which together with \(2.85\), \(2.29\), \(2.77\), \(2.78\), \(2.65\), \(2.86\), and \(2.54\) yields
\[
\int_0^T \int_0^1 \theta_x^2 \, dx \, dt \leq C \int_0^T \int_0^1 \left( \theta_x^2 + \theta_x v_x + u_x^4 + |b_x|^4 + |w_x|^4 + u_x^2 + \theta_t^2 \right) \, dx \, dt
\]
\[
\leq C + C \int_0^T \max_{x \in [0,1]} \theta_x^2 \, dt
\]
\[
\leq C + \frac{1}{2} \int_0^T \int_0^1 \theta_x^2 \, dx \, dt.
\]

Combining this with \(2.86\) proves \(2.72\) and finishes the proof of Lemma 2.8. □

Finally, we have the following nonlinearly exponential stability of the strong solutions.

**Lemma 2.9.** There exist some positive constants \(C\) and \(\eta_0\) both depending only on \(\beta, \|(v_0, u_0, \theta_0, b_0, w_0)\|_{H^1(0,1)}, \inf_{x \in [0,1]} v_0(x),\) and \(\inf_{x \in [0,1]} \theta_0(x)\) such that
\[
\| (v - 1, u, \theta - 1, b, w)(\cdot, t) \|_{H^1(0,1)} \leq C e^{-\eta_0 t}.
\]

(2.87)

**Proof.** Noticing that all the constants \(C\) in Lemmas 2.4, 2.5, 2.7, and 2.8 are independent of \(T\), we have
\[
\int_0^\infty \left| \frac{d}{dt} \|(v_x, u_x, \theta_x, b_x, w_x)(\cdot, t)\|_{L^2(0,1)}^2 \right| \, dt \leq C,
\]
where we have used
\[
\int_0^1 u_x u_{xx} \, dx = -\int_0^1 u_t u_{xx} \, dx.
\]
It thus follows from (2.88), (2.29), and (2.72) that
\[
\lim_{t \to \infty} \| (v_x, u_x, \theta_x, b_x, w_x)(\cdot, t) \|_{L^2(0,1)} = 0,
\]
which in particular implies
\[
\lim_{t \to \infty} \| (v - 1, u, \theta - 1, b, w)(\cdot, t) \|_{H^1(0,1)} = 0.
\] (2.89)

With (2.89) at hand, the proof of (2.87) is standard (c.f. [30]). □

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