On the Riemann–Hilbert problem of a generalized derivative nonlinear Schrödinger equation

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Abstract

In this work, we present a unified transformation method directly by using the inverse scattering method for a generalized derivative nonlinear Schrödinger (DNLS) equation. By establishing a matrix Riemann–Hilbert problem and reconstructing potential function \(q(x, t)\) from eigenfunctions \(\{G_j(x, t, \eta)\}\) in the inverse problem, the initial-boundary value problems for the generalized DNLS equation on the half-line are discussed. Moreover, we also obtain that the spectral functions \(f(\eta), s(\eta), F(\eta), S(\eta)\) are not independent of each other, but meet an important global relation. As applications, the generalized DNLS equation can be reduced to the Kaup–Newell equation and Chen–Lee–Liu equation on the half-line.

Keywords: Riemann–Hilbert problem, generalized derivative nonlinear Schrödinger equation, initial-boundary value problems, unified transformation method

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1967, Gardner \textit{et al} \cite{Gardner1967} proposed the famous inverse scattering method (ISM) when studying the fast decay initial value problem of the Korteweg–de Vries equation, which is a powerful tool for solving the initial value problem of nonlinear integrable systems. However, because the ISM was only used to discuss the initial value problem of nonlinear integrable equations and the limitation of the initial value conditions is suitable for infinity, how to extend ISM to the initial-boundary value problems (IBVPs) of nonlinear integrable systems is a major challenge for soliton theory research. In 1997, Fokas \cite{Fokas1997} extended the ISM and proposed a unified transformation method (UTM) to analyze the IBVPs of partial differential equations \cite{Fokas2008}. In 2008, Lenells \cite{Lenells2008} used UTM to analyze the IBVPs of the following derivative nonlinear Schrödinger (DNLS) equation \cite{Degasperis1993, Degasperis1997, Degasperis2000, Procesi1999}:

\[ iq_t + q_{xx} = i|q|^2q_x, \tag{1.1} \]

Equation (1.1) has an important application in plasma physics, which is a model for Alfvén waves propagating parallel to the ambient magnetic field \cite{Frenkel1966, Frenkel1967}. Since then, more and more mathematical physicists have paid attention to the UTM to study the IBVPs of integrable equations \cite{Pelinovsky2011, Lenells2013, Xu2012}. In 2012, Lenells extended UTM to integrable systems related to high-matrix spectral \cite{Lenells2014}, and used UTM to analyze the IBVPs of the Degasperis–Procesi equation \cite{Lenells2013, Lenells2014}. In 2013, Xu and Fan discussed the IBVPs of the Sasa–Satsuma equation through UTM \cite{Xu2014}, and gave the proof of the existence and uniqueness of the solution of the IBVPs of the integrable equation with higher-order matrix spectrum through analyzing a three-wave equation \cite{Xu2015}. Subsequently, more and more scholars have studied the IBVPs of integrable equations with higher-order matrix spectral \cite{Yang2016, Yang2017, Yang2018}. Particularly, the soliton solutions and the long-time asymptotic behavior for the integrable models can be solved by constructing a Riemann–Hilbert (RH) problem. Such as, Wang and Wang investigated the long-time asymptotic behavior of the Kundu–Eckhaus equation \cite{Wang2017}. Yang and Chen obtained the high-order soliton matrix form solution of the Sasa–Satsuma equation \cite{Yang2016}. Ma analyzed multicomponent AKNS integrable hierarchies \cite{Ma2017}, etc.
In 1987, Clarkson and Cosgrove [31] proposed a generalized derivative NLS (GDNLs) equation in the form of
\[ iq_t = q_{xx} + i\alpha|q|^2q_x + i\beta q^2q_x + \kappa |q|^4q, \alpha = \beta, \]  
(1.2)
where \( q \) is the amplitude of the complex field envelope. The equation (1.2) has several applications in optical fibers, nonlinear optics, weakly nonlinear dispersion water waves, quantum field theory, and plasma physics [32, 33], etc. As an example, equation (1.2) can be used to simulate single-mode propagation in the optical fibers, which enjoys traveling and stationary kink envelope solutions of monotonic and oscillatory type. However, it is well known that equation (1.2) has Painlevé property only if \( \kappa = \frac{1}{2}(2\beta - \alpha) \) holds. At this time, equation (1.2) is reduced to an integrable GDNLs model as follows
\[ iq_t = q_{xx} + i\alpha|q|^2q_x + i\beta q^2q_x + \frac{1}{4}\beta(2\beta - \alpha)|q|^4q, \alpha = \beta. \]  
(1.3)
Given \( \alpha = 2\beta \neq 0 \), the equation (1.3) becomes to the DNLS-I (Kad–Newell) equation (1.1), and if \( \alpha = 0, \beta = 0 \), the equation (1.3) becomes to the DNLS-II (Chen–Lee–Liou) equation
\[ iq_t = q_{xx} + i\alpha|q|^2q_x, \]  
(1.4)
whose IBVPs on the half-line has been solved [34]. Recently, the conservation laws of equation (1.3) have been discussed [35]. However, as far as we know, the IBVPs of equation (1.3) have not been analyzed. So we will utilize UTM to study the IBVPs of equation (1.3) on the half-line domain \( \Gamma = \{(x, t): 0 < x < \infty, 0 < t < T\} \) here. Similar to DNLS equation [18] on the interval, the IBVPs of equation (1.3) on the interval will be studied in our future paper.

The design structure of this paper is as follows. In section 2, we give spectral analysis of the Lax pair of equation (1.3). In section 3, some key functions \( f(\eta), s(\eta), F(\eta), S(\eta) \) are further analyzed. In section 4, the RH problem is proposed. Finally, some conclusions and discussions are given in section 5.

2. The spectral analysis
The GDNLs equation (1.3) enjoys a Lax pair as follows [35]
\[ \Phi_t = U(x, t, \eta)\Phi, \]  
(2.1a)
\[ \Phi_t = V(x, t, \eta)\Phi, \]  
(2.1b)
where \( \Phi = (\Phi_1, \Phi_2)^T \) is the vector eigenfunction, the \( 2 \times 2 \) matrices \( U(x, t, \eta), V(x, \eta) \) are given by the following form
\[ U(x, t, \eta) = \begin{pmatrix} \frac{-i\eta^2}{\alpha - \beta} & \frac{i}{4}(\alpha - 2\beta)|q|^2 \xi \\ \eta q & -\frac{i\eta^2}{\alpha - \beta} + \frac{i}{4}(\alpha - 2\beta)|q|^2 \end{pmatrix}, \]  
(2.2a)
\[ V(x, t, \eta) = \begin{pmatrix} V_{11} & \frac{2}{\alpha - \beta}\eta \xi + \frac{\alpha}{\tau}|q|^2 - \tilde{q} \xi \\ -V_{11} & \frac{2}{\alpha - \beta}\eta \xi + \frac{\alpha}{\tau}|q|^2 - i\tilde{q} \xi \end{pmatrix}, \]  
(2.2b)
with
\[ V_{11} = -\frac{2i\eta^4}{(\alpha - \beta)^2} - i\eta^2|q|^2 + \frac{i}{8}(\alpha^2 - \alpha\beta - 2\beta^2)|q|^4 - \frac{2}{\alpha - \beta}(4q_{xx} - \tilde{q}q_x) \]  
(2.3)

2.1. The exact one-form
The equations (2.1a), (2.1b) is equivalent to
\[ \Phi_t + \frac{i}{\alpha - \beta}\eta^2\sigma_3\Phi = M\Phi, \]  
(2.4a)
\[ \Phi_t + \frac{2i}{(\alpha - \beta)^2}\eta^4\sigma_3\Phi = N\Phi, \]  
(2.4b)
where \( \alpha = \beta \), complex number \( \eta \) is a spectral parameter and
\[ M = \eta^2Q - \frac{i}{4}(\alpha - 2\beta)Q^2\sigma_3, \]
\[ N = -[i\eta^2Q^2 + \frac{i}{8}(\alpha^2 - \alpha\beta - 2\beta^2)Q^4 - \frac{2}{\alpha - \beta}(4Q_{xx} - Q_xQ)]\sigma_3 + \frac{2}{\alpha - \beta}\eta^2Q + \frac{i}{2}Q^2 - iQ\sigma_3. \]

One can introduce \( \Psi(x, t, \eta) \) by
\[ \Psi(x, t, \eta) = \Phi(x, t, \eta)e^{[\frac{i}{\alpha - \beta}\eta^2x + \frac{2i}{(\alpha - \beta)^2}\eta^4\tilde{q}^3]\sigma_3}, \]  
(2.5)

hence, equations (2.4a), (2.4b) become to
\[ \Psi_t + \frac{i}{\alpha - \beta}\eta^2[s_3, \Psi] = M\Psi, \]  
(2.6a)
\[ \Psi_t + \frac{2i}{(\alpha - \beta)^2}\eta^4[s_3, \Psi] = N\Psi, \]  
(2.6b)
where \( [s_3, \Psi] = s_3\Psi - \Psi s_3 \). It is easy to see that the above equations give the following full differential
\[ d(e^{[\frac{i}{\alpha - \beta}\eta^2x + \frac{2i}{(\alpha - \beta)^2}\eta^4\tilde{q}^3]s_3}\Psi(x, t, \eta)) = e^{[\frac{i}{\alpha - \beta}\eta^2x + \frac{2i}{(\alpha - \beta)^2}\eta^4\tilde{q}^3]s_3}(Mdx + Ndr)\Phi(x, t, \eta). \]  
(2.7)
One supposes that the following asymptotic expansion
\[ \Psi(x, t, \eta) = D_0 + \frac{D_1}{\eta} + \frac{D_2}{\eta^2} + \frac{D_3}{\eta^3} + O\left(\frac{1}{\eta^4}\right), \quad \eta \rightarrow \infty, \]
(2.8)
is a solution of equations (2.6a), (2.6b). Substituting equation (2.8) into equation (2.6a) and comparing the coefficients for \( \eta^j \), one can get
\[
\begin{align*}
O(\eta^1) : & \quad \frac{2i}{\alpha - \beta}[\sigma_3, D_0] = 0, \\
O(\eta^2) : & \quad \frac{2i}{\alpha - \beta}[\sigma_3, D_1] = 2QD_0, \\
O(\eta^3) : & \quad \frac{2i}{\alpha - \beta}[\sigma_3, D_2] = -iQ^2\sigma_3D_0 + \frac{2}{\alpha - \beta}QD_1, \\
O(\eta^4) : & \quad \frac{2i}{\alpha - \beta}[\sigma_3, D_3] = -iQ^2\sigma_3D_1 + \frac{2}{\alpha - \beta}QD_2, \\
O(\eta^5) : & \quad D_{10} + \frac{2i}{\alpha - \beta}[\sigma_3, D_4] = -iQ^2\sigma_3D_2 + \frac{2}{\alpha - \beta}QD_3, \\
\end{align*}
\]
(2.9)
From \( O(\eta^2) \), one finds that \( D_0 \) enjoys a diagonal matrix form denoted as
\[
D_0 = \begin{pmatrix} D_{011} & 0 \\ 0 & D_{022} \end{pmatrix}
\]
where \( D_{1}^{\text{(odd)}} \) denotes the off-diagonal part of \( D_1 \). At the same time, substituting equation (2.8) into the equation (2.6b) and comparing the coefficient for \( \eta \), we get
\[
\begin{align*}
O(\eta^1) : & \quad \frac{d}{dt}\left[Q^2\sigma_3D_0\right] = 0, \\
O(\eta^2) : & \quad \frac{d}{dt}\left[Q^2\sigma_3D_1\right] = -iQ^2\sigma_3D_0, \\
O(\eta^3) : & \quad \frac{d}{dt}\left[Q^2\sigma_3D_2\right] = -iQ^2\sigma_3D_1 - \frac{2}{\alpha - \beta}QD_1, \\
O(\eta^4) : & \quad \frac{d}{dt}\left[Q^2\sigma_3D_3\right] = -iQ^2\sigma_3D_2 - \frac{2}{\alpha - \beta}QD_2, \\
O(\eta^5) : & \quad \frac{d}{dt}\left[Q^2\sigma_3D_4\right] = -iQ^2\sigma_3D_3 + \frac{2}{\alpha - \beta}QD_3, \\
\end{align*}
\]
(2.10)
Through tedious calculation, one gets
\[
D_{10} = \left[ \frac{i}{8}(\alpha^2 + 2\alpha\beta - \beta^2)|q|^4 + \frac{i}{4}(qq_x - q^2) \right] \sigma_3D_0,
\]
(2.12)
since equations (2.1a), (2.1b) admit the following conservation law
\[
\left( \frac{i}{4}|q|^2 \right)_t = \left[ \frac{i}{8}(\alpha^2 + 2\alpha\beta - \beta^2)|q|^4 + \frac{i}{4}(qq_x - q^2) \right]_t,
\]
the equations (2.10) and (2.12) for \( D_0 \) are consistent, then, one defines
\[
D_0(t, x) = e^{i \int_{x_0}^{x_0, s} \Omega(\xi, t)d\xi},
\]
(2.13)
where \( \Omega \) is the closed one-form and given by
\[
\Omega(x, t) = \Omega_1dx + \Omega_2dt = \frac{1}{2}Q|q|^2dx + \frac{1}{2}(\alpha^2 + 2\alpha\beta - \beta^2)|q|^4
\]
\[
- \frac{i}{4}(qq_x - q^2)dt.
\]
(2.14)
Since the integration of equation (2.13) is independent of the integration path and \( \Omega \) is independent of \( \eta \), one can introduce a key function \( G(x, t, \eta) \) by
\[
\Psi(x, t, \eta) = e^{i \int_{x_0, \eta_0}^{x, \eta} G(x, t, \eta)D_0(x, t)},
\]
(2.15)
then, equation (2.7) is equal to
\[
d\left( e^{-i\left(\frac{\beta}{\alpha - \beta}\right)|q|^2} G(x, t, \eta) \right) = A(x, t, \eta),
\]
(2.16)
where
\[
A(x, t, \eta) = e^{i\left(\frac{\beta}{\alpha - \beta}\right)|q|^2} \int_{x_0}^{x, \eta} B(x, t, \eta)G(x, t, \eta)dx + \int_{x_0, \eta_0}^{x, \eta} N(x, t, \eta)dt
\]
\[
e^{-i\int_{x_0}^{x, \eta} \Omega_1dx + \int_{x_0, \eta_0}^{x, \eta} \Omega_2dt - i\int_{x_0}^{x, \eta} \Omega_3dx}.
\]
(2.17)
It follows from \( M(x, t, \eta), N(x, t, \eta) \) and \( \Omega \) that
\[
M_1(x, t, \eta) = \begin{pmatrix} -i\left(\frac{\beta}{\alpha - \beta}\right)|q|^2 & \eta q e^{2i\int_{x_0}^{x, \eta} \Omega_1} \\ \eta q e^{-2i\int_{x_0}^{x, \eta} \Omega_1} & i\left(\frac{\beta}{\alpha - \beta}\right)|q|^2 \end{pmatrix},
\]
\]
(2.18a)
\[
N_1(x, t, \eta) = \begin{pmatrix} N_{11}^{(1)}(x, t, \eta) & N_{12}^{(1)}(x, t, \eta) \\ N_{21}^{(1)}(x, t, \eta) & -N_{11}^{(1)}(x, t, \eta) \end{pmatrix},
\]
(2.18b)
\[ N_{1}^{(1)}(x, t, \eta) = -i\eta^{2}|q|^{2} - \frac{i}{8}(2\alpha\beta + \beta^{2})|q|^{4} \]
\[ \frac{1}{2}(\alpha - \beta)\eta q - \eta q_{x} - \tilde{g}_{\alpha}q, \]
\[ N_{1}^{(2)}(x, t, \eta) = \left( \frac{2}{\alpha - \beta} \right) \eta^{3}q + \frac{\alpha}{2}|q|^{2} q - \eta q_{x} \right)e^{2i\int_{0}^{\Omega_{0}}}, \]
\[ N_{2}^{(1)}(x, t, \eta) = \left( \frac{2}{\alpha - \beta} \right) \eta^{3}q + \frac{\alpha}{2}|q|^{2} q - \eta q_{x} \right)e^{-2i\int_{0}^{\Omega_{0}},} \]
then equation (2.16) becomes to
\[ G_{x} + \frac{i}{\alpha - \beta}\eta^{2}[\sigma_{3}, G] = M_{G}, \tag{2.19a} \]
\[ G_{x} + \frac{2i}{\alpha - \beta}\eta^{4}[\sigma_{3}, G] = N_{G}. \tag{2.19b} \]

2.2. The Three important functions \([G_{j}(x, t, \eta)]_{3}^{3}\)

For \((x, t) \in \Gamma\), we suppose that \(q(x, t) \in \mathbb{S}\), one defines three eigenfunctions \([G_{j}(x, t, \eta)]_{3}^{3}\) of equations (2.19a), (2.19b) given by
\[ G_{j}(x, t, \eta) = I + \int_{(x, t)}^{(\xi, t)} e^{\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{j}]} A_{j}(\xi, \tau, \eta), \tag{2.20} \]
where \(I = \text{diag}[1, 1]\) is a 2 \(\times\) 2 unit matrix, \(A_{j}(\xi, \tau, \eta)\) is given by equation (2.17), just replacing \(G(\xi, \tau, \eta)\) with \(G_{j}(x, \tau, \eta)\), the integral path \((x, t) \rightarrow (\xi, t)\) is a directed smooth curve and \((x_{1}, t_{1}) = (0, 0), (x_{2}, t_{2}) = (0, T), (x_{3}, t_{3}) = (\infty, t)\). Since the integral of equation (2.20) has nothing to do with the integral path, we select a special integral path parallel to the coordinate axis as shown in figure 1, then we have
\[ G_{1}(x, t, \eta) = I + \int_{0}^{T} e^{-\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{1}]}(M_{G_{1}}(\xi, t, \eta))d\xi \]
\[ + e^{-\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{1}]}(N_{G_{1}}(0, \tau, \eta))d\tau, \tag{2.21a} \]
\[ G_{2}(x, t, \eta) = I + \int_{0}^{T} e^{-\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{2}]}(M_{G_{2}}(\xi, t, \eta))d\xi \]
\[ - e^{-\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{2}]}(N_{G_{2}}(0, \tau, \eta))d\tau, \tag{2.21b} \]
\[ G_{3}(x, t, \eta) = I - \int_{x}^{\infty} e^{-\frac{2i}{\alpha - \beta}\eta^{3}[\sigma_{3}, G_{3}]}(M_{G_{3}}(\xi, t, \eta))d\xi. \tag{2.21c} \]

The first column of equation (2.20) enjoys
\[ \exp \left[ -2i \frac{\eta^{2}}{\alpha - \beta}(x - \xi) + 4i \left( \frac{\eta^{2}}{\alpha - \beta} - \eta^{2} \right)(t - \tau) \right] \tag{2.22a} \]
and the following inequalities
\[ \gamma_{1}: x - \xi \geq 0, t - \tau \geq 0, \tag{2.22b} \]
\[ \gamma_{2}: x - \xi \geq 0, t - \tau \leq 0, \tag{2.22c} \]
are true on curves \([\gamma_{j}]_{3}\), then, the bounded analysis area of eigenfunctions \([G_{j}(x, t, \eta)]_{3}^{3}\) is as follows
\[ [G_{1}](x, t, \eta) : \left\{ \text{Im} \left( \frac{\eta^{2}}{\alpha - \beta} \right) \geq 0 \right\} \cap \left\{ \text{Im} \left( \frac{\eta^{4}}{\alpha - \beta} \right) \geq 0 \right\}, \tag{2.23a} \]
\[ [G_{2}](x, t, \eta) : \left\{ \text{Im} \left( \frac{\eta^{2}}{\alpha - \beta} \right) \geq 0 \right\} \cap \left\{ \text{Im} \left( \frac{\eta^{4}}{\alpha - \beta} \right) \leq 0 \right\}, \tag{2.23b} \]
\[ [G_{3}](x, t, \eta) : \left\{ \text{Im} \left( \frac{\eta^{2}}{\alpha - \beta} \right) \leq 0 \right\} \tag{2.23c} \]
On the other hand, the second column of equation (2.20) contains opposite index terms
\[ \exp \left[ -2i \frac{\eta^{2}}{\alpha - \beta}(x - \xi) - 4i \left( \frac{\eta^{2}}{\alpha - \beta} \right)(t - \tau) \right]. \tag{2.24a} \]
\[ [G_{1}](x, t, \eta) : \left\{ \text{Im} \left( \frac{\eta^{2}}{\alpha - \beta} \right) \leq 0 \right\} \cap \left\{ \text{Im} \left( \frac{\eta^{4}}{\alpha - \beta} \right) \leq 0 \right\}, \tag{2.24b} \]
\[ [G_{2}](x, t, \eta) : \left\{ \text{Im} \left( \frac{\eta^{2}}{\alpha - \beta} \right) \leq 0 \right\} \cap \left\{ \text{Im} \left( \frac{\eta^{4}}{\alpha - \beta} \right) \geq 0 \right\}, \tag{2.24c} \]
Consequently, if we remember that \([G_{j}]_{k}(x, t, \eta), k = 1, 2\) represent \(k\)-column of \([G_{j}(x, t, \eta)]_{3}^{3}\), one can get
for \( \alpha > \beta \),
\[
\begin{align*}
G_1(x, t, \eta) &= ([G_1]^{L_1}(x, t, \eta), [G_1]^{L_2}(x, t, \eta)), \\
G_2(x, t, \eta) &= ([G_2]^{L_1}(x, t, \eta), [G_2]^{L_2}(x, t, \eta)), \\
G_3(x, t, \eta) &= ([G_3]^{L_1}(x, t, \eta), [G_3]^{L_2}(x, t, \eta)), \\
G_4(x, t, \eta) &= ([G_4]^{L_2}(x, t, \eta), [G_4]^{L_2}(x, t, \eta)),
\end{align*}
\tag{2.25}
\]

and for \( \alpha < \beta \),
\[
\begin{align*}
G_1(x, t, \eta) &= ([G_1]^{L_1}(x, t, \eta), [G_1]^{L_2}(x, t, \eta)), \\
G_2(x, t, \eta) &= ([G_2]^{L_1}(x, t, \eta), [G_2]^{L_2}(x, t, \eta)), \\
G_3(x, t, \eta) &= ([G_3]^{L_1}(x, t, \eta), [G_3]^{L_2}(x, t, \eta)), \\
G_4(x, t, \eta) &= ([G_4]^{L_2}(x, t, \eta), [G_4]^{L_2}(x, t, \eta)),
\end{align*}
\tag{2.26}
\]

where \([G_i]^{L_i} \) represents that the bounded analytic region of \([G_i] \) is \( L_i \), \( i = 1, 2, 3, 4 \), and \( L_i \) are shown in Figure 2.

To construct the RH problem of GDNLS equation (1.3), we must define another two important special functions \( \psi(\eta) \) and \( \phi(\eta) \) by
\[
\begin{align*}
G_3(x, t, \eta) &= G_3(x, t, \eta)e^{-\frac{1}{2(\alpha-\beta)}(\psi(\eta))}, \\
G_2(x, t, \eta) &= G_2(x, t, \eta)e^{-\frac{1}{2(\alpha-\beta)}(\psi(\eta))},
\end{align*}
\tag{2.27a}
\tag{2.27b}
\]

upon evaluation at \((x, t) = (0, 0)\) and \((x, t) = (0, T)\), respectively, from equations (2.27a) and (2.27b) we can get
\[
\phi^{-1}(\eta) = e^{-\frac{1}{2(\alpha-\beta)}(\psi(\eta))}, \quad \psi(\eta) = G_3(0, 0, \eta). \tag{2.28}
\]

It follows from (2.27a), (2.27b) and equation (2.28) that
\[
G_2(x, t, \eta) = G_3(x, t, \eta)e^{-\frac{1}{2(\alpha-\beta)}(\psi(\eta))} = \phi^{-1}(\eta), \tag{2.29}
\]

particularly, one also obtains \(G_1(x, t, \eta), G_2(x, t, \eta)\) at \(x = 0\)
\[
\begin{align*}
G_1(0, t, \eta) &= ([G_1]^{L_1}(0, t, \eta), [G_1]^{L_2}(0, t, \eta)), \\
G_2(0, t, \eta) &= ([G_2]^{L_1}(0, t, \eta), [G_2]^{L_2}(0, t, \eta)), \\
G_3(0, t, \eta) &= ([G_3]^{L_1}(0, t, \eta), [G_3]^{L_2}(0, t, \eta)), \\
G_4(0, t, \eta) &= ([G_4]^{L_2}(0, t, \eta), [G_4]^{L_2}(0, t, \eta)),
\end{align*}
\tag{2.30a}
\tag{2.30b}
\]

and \(G_1(x, t, \eta), G_2(x, t, \eta)\) at \(t = 0\)
\[
\begin{align*}
G_1(x, 0, \eta) &= ([G_1]^{L_1}(x, 0, \eta), [G_1]^{L_2}(x, 0, \eta)), \\
G_2(x, 0, \eta) &= ([G_2]^{L_1}(x, 0, \eta), [G_2]^{L_2}(x, 0, \eta)), \\
G_3(x, 0, \eta) &= ([G_3]^{L_1}(x, 0, \eta), [G_3]^{L_2}(x, 0, \eta)), \\
G_4(x, 0, \eta) &= ([G_4]^{L_2}(x, 0, \eta), [G_4]^{L_2}(x, 0, \eta)),
\end{align*}
\tag{2.31a}
\tag{2.31b}
\]

Assume that \(u_0(x) = q(x, t = 0), v_0(t) = q(x = 0, t)\), \(v_1(t) = q_4(x = 0, t)\) are initial condition and boundary conditions of \(q(x, t)\) and \(g_4(x, t)\), then, one get
\[
\begin{align*}
M_1(x, 0, \eta) &= \begin{cases} \\
&
\end{cases}
\tag{2.32a}
\tag{2.32b}
\]

\[
N_1(x, 0, \eta) = \begin{cases} \\
&
\end{cases}
\tag{2.32b}
\]

with
\[ N_{11}(t, \eta) = -i\eta^2 |v_0|^2 - \frac{1}{8} (2\alpha \beta + \beta^2) |v_0|^4 \]
\[ - \frac{\alpha - \beta}{2} (\bar{v}_0 v_1 - \bar{v}_1 v_0), \]
\[ N_{12}(t, \eta) = \left( \frac{2}{\alpha - \beta} \right) \eta^3 v_0 + \frac{\alpha}{2} |v_0|^2 \bar{v}_1 - i\bar{v}_1 \]
\[ \times e^{i \int_0^t \left( \frac{4(\alpha^2 + \alpha \beta - \beta^2)}{2} |v_1|^2 - \frac{4(\alpha \beta - \beta^2)}{2} |v_0|^2 \right) dt}, \]
\[ N_{21}(t, \eta) = \left( \frac{2}{\alpha - \beta} \right) \eta^3 v_0 + \frac{\alpha}{2} |v_0|^2 \bar{v}_1 - i\bar{v}_1 \]
\[ \times e^{-i \int_0^t \left( \frac{4(\alpha^2 + \alpha \beta - \beta^2)}{2} |v_1|^2 - \frac{4(\alpha \beta - \beta^2)}{2} |v_0|^2 \right) dt}. \]

2.3. The other properties of the eigenfunctions

**Proposition 2.1.** The functions
\[ G_j(x, t, \eta) = \{G_j[1](x, t, \eta), G_j[2](x, t, \eta)\}, j = 1, 2, 3, \]
enjoy properties as follows

- \( \det G_j(x, t, \eta) = 1, j = 1, 2, 3, \)
- \( [G_1[1]] \text{ is analytic for } \)
  \[ \eta \in L_1, \text{ and continues to } \bar{L}_1, \alpha > \beta, \]
  \[ \eta \in L_4, \text{ and continues to } \bar{L}_4, \alpha < \beta, \]
- \( [G_1[2]] \text{ is analytic for } \)
  \[ \eta \in L_3, \text{ and continues to } \bar{L}_3, \alpha > \beta, \]
  \[ \eta \in L_2, \text{ and continues to } \bar{L}_2, \alpha < \beta, \]
- \( [G_2[1]] \text{ is analytic for } \)
  \[ \eta \in L_4, \text{ and continues to } \bar{L}_4, \alpha > \beta, \]
  \[ \eta \in L_3, \text{ and continues to } \bar{L}_3, \alpha < \beta, \]
- \( [G_2[2]] \text{ is analytic for } \)
  \[ \eta \in L_1 \cup L_4 \text{ and continues to } L_3 \cup L_4, \alpha > \beta, \]
  \[ \eta \in L_1 \cup L_2 \text{ and continues to } L_4 \cup L_2, \alpha < \beta, \]
- \( \det \psi(\eta), \phi(\eta) \text{ possess the following } 2 \times 2 \)
  \[ \text{matrix from, respectively} \]
\[ \psi(\eta) = \left( \begin{array}{cccc} f(\eta) & s(\eta) \\ \overline{f(\eta)} & \overline{s(\eta)} \end{array} \right), \]
\[ \phi(\eta) = \left( \begin{array}{cccc} F(\eta) & S(\eta) \\ \overline{F(\eta)} & \overline{S(\eta)} \end{array} \right). \] (2.34)

It follows from equations (2.28) and (2.33a), (2.33b) that the following key properties are true

- \( f(\eta) = f(\eta), \quad s(\eta) = s(\eta), \)
- \( F(\eta) = F(\eta), \quad S(\eta) = S(\eta), \)
- \( \det \psi(\eta) = \psi(\eta), \quad \det \phi(\eta) = \phi(\eta) \),
- \( f(\eta) = 1 + O(\eta^{-1}), \quad s(\eta) = O(\eta^{-1}), \)
- \( F(\eta) = 1 + O(\eta^{-1}), \quad S(\eta) = O(\eta^{-1}), \)

2.4. The basic RH problem

To facilitate subsequent calculations, we remember that the following symbolic assumptions
\[ \mu(\eta) = \frac{x^2}{\alpha - \beta} x + \frac{2x^4}{\alpha - \beta} t, \]
\[ h(\eta) = \frac{1}{\eta} \left( \frac{f(\eta)}{F(\eta)} - \frac{s(\eta)}{S(\eta)} \right), \]
\[ g(\eta) = \frac{1}{\eta} \left( \frac{f(\eta)}{F(\eta)} - \frac{s(\eta)}{S(\eta)} \right), \]
\[ \theta(\eta) = \frac{1}{\eta} \left( \frac{f(\eta)h(\eta)}{F(\eta)} \right), \] (2.35)
then, one obtains
\[ \overline{S(\eta)} = f(\eta) g(\eta) + \overline{s(\eta)} h(\eta), \]
\[ h(\eta) = \frac{1}{\eta} \left( \frac{f(\eta)}{F(\eta)} - \frac{s(\eta)}{S(\eta)} \right), \]
\[ h(-\eta) = \frac{1}{\eta} \left( \frac{f(\eta)}{F(\eta)} - \frac{s(\eta)}{S(\eta)} \right), \]
\[ h(\eta) = 1 + O\left( \frac{1}{\eta} \right), \quad g(\eta) = O\left( \frac{1}{\eta} \right), \text{ as } \eta \to \infty, \]
and the \( W(x, t, \eta) \) is defined by

- for \( \alpha > \beta, \)
- \[ W_4(x, t, \eta) = \left( \begin{array}{cccc} G_{14}^{(4,4,0)}(x, t, \eta) \end{array} \right) \]
- \[ W_4(x, t, \eta) = \left( \begin{array}{cccc} G_{14}^{(4,4,0)}(x, t, \eta) \end{array} \right) \]
- \[ W_4(x, t, \eta) = \left( \begin{array}{cccc} G_{14}^{(4,4,0)}(x, t, \eta) \end{array} \right) \] (2.36)
for $\alpha < \beta$, 
\[
W(x, t, \eta) = \begin{cases}
[G_3]^{L_{1,2}}_{n,t}(x, t, \eta), \quad \eta \in L_1, \\
[G_3]^{L_{1,2}}_{n,t}(x, t, \eta), \quad \eta \in L_2, \\
[G_3]^{L_{1,2}}_{n,t}(x, t, \eta), \quad \eta \in L_3, \\
[G_3]^{L_{1,2}}_{n,t}(x, t, \eta), \quad \eta \in L_4.
\end{cases}
\]

These definitions imply that 
\[
\det W(x, t, \eta) = 1, \quad W(x, t, \eta) \to I, \quad \eta \to \infty. \tag{2.38}
\]

In the following, one only gives the case of $\alpha > \beta$ for jump condition and residue relation, and we can discuss the case of $\alpha < \beta$ similarly.

**Theorem 2.3.** For $\alpha > \beta$, set $q(x, t) \in S$, and the function $W(x, t, \eta)$ is given by equation (2.36), then equation (2.36) meets the following jump relation on the curve $L_k, k = 1, \ldots, 4$.

\[
W(x, t, \eta) = W(x, t, \eta)H(x, t, \eta), \quad \eta \in L_k, k = 1, \ldots, 4, \tag{2.39}
\]

where

\[
H(x, t, \eta) = \begin{cases}
H_1(x, t, \eta), \quad \arg \frac{\eta}{\alpha - \beta} = \frac{\pi}{2}, \\
H_2(x, t, \eta) = H_3H_4^{-1}H_1, \quad \arg \frac{\eta}{\alpha - \beta} = \pi, \\
H_3(x, t, \eta), \quad \arg \frac{\eta}{\alpha - \beta} = \frac{3\pi}{2}, \\
H_4(x, t, \eta), \quad \arg \frac{\eta}{\alpha - \beta} = 0,
\end{cases} \tag{2.40}
\]

and

\[
H_1(x, t, \eta) = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \\
H_2(x, t, \eta) = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \\
H_3(x, t, \eta) = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}, \\
H_4(x, t, \eta) = \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]

**Proof.** From equations (2.27a), (2.27b) and (2.34), one finds that

\[
s(\eta)e^{-2\mu(\eta)}[G_1]^{L_1}_{n,t}(x, t, \eta) + f(\eta)[G_1]^{L_2}_{n,t}(x, t, \eta) = [G_3]^{L_{1,2}}_{n,t}(x, t, \eta), \tag{2.41a}
\]

and

\[
F(\eta)[G_1]^{L_1}_{n,t}(x, t, \eta) - S(\eta)e^{2\mu(\eta)}[G_1]^{L_2}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta), \tag{2.42a}
\]

\[
S(\eta)e^{-2\mu(\eta)}[G_1]^{L_1}_{n,t}(x, t, \eta) + F(\eta)[G_1]^{L_2}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta), \tag{2.42b}
\]

\[
S(\eta)e^{-2\mu(\eta)}[G_1]^{L_1}_{n,t}(x, t, \eta) + F(\eta)[G_1]^{L_2}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta), \tag{2.42c}
\]

\[
S(\eta)e^{-2\mu(\eta)}[G_1]^{L_1}_{n,t}(x, t, \eta) + F(\eta)[G_1]^{L_2}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta), \tag{2.42d}
\]

Then, the equations (2.41a), (2.42b) and (2.35) give rise to

\[
\begin{align*}
&h(\eta)[G_3]^{L_{1,2}}_{n,t}(x, t, \eta) - g(\eta)e^{2\mu(\eta)}[G_3]^{L_{1,2}}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta), \\
&g(\eta)e^{-2\mu(\eta)}[G_3]^{L_{1,2}}_{n,t}(x, t, \eta) + h(\eta)[G_3]^{L_{1,2}}_{n,t}(x, t, \eta) = [G_2]^{L_2}_{n,t}(x, t, \eta).
\end{align*}
\]

It follows from the equations (2.36) and (2.39) that

\[
\begin{align*}
&\left[ G_1 \right]^{L_1}_{n,t}(x, t, \eta), \quad \left[ G_3 \right]^{L_{1,2}}_{n,t}(x, t, \eta) \\
&\left[ G_2 \right]^{L_2}_{n,t}(x, t, \eta), \quad \left[ G_3 \right]^{L_{1,2}}_{n,t}(x, t, \eta)
\end{align*}
\]

Therefore, the equations (2.44a)–(2.44d) lead to the jump matrices $\{H(x, t, \eta)\}_{i=1}^4$ defined by equation (2.40).

**Assumption 2.4.** One makes assumptions about the simple zeros of functions $f(\eta)$ and $h(\eta)$ as follows.
• if $\eta$ enjoys $2a$ simple zeros $\{s_i\}^{2a}_{j=1}$, $2a = 2a_1 + 2a_2$. For $\alpha > \beta$, if $\{s_i\}^{2a_1}_{j=1} \in L_1$, then $\{\bar{s}_j\}^{2a_2}_{j=1} \in L_2$. For $\alpha < \beta$, if $\{s_i\}^{2a_2}_{j=1} \in L_2$, then $\{\bar{s}_j\}^{2a_1}_{j=1} \in L_1$.

• if $\bar{f}(\eta)$ enjoys $2b$ simple zeros $\{\bar{s}_j\}^{2b}_{j=1}$, $2b = 2b_1 + 2b_2$. For $\alpha > \beta$, if $\{\bar{s}_j\}^{2b_1}_{j=1} \in L_1$, then $\{s_i\}^{2b_2}_{j=1} \in L_2$. For $\alpha < \beta$, if $\{s_i\}^{2b_2}_{j=1} \in L_2$, then $\{\bar{s}_j\}^{2b_1}_{j=1} \in L_1$.

• The intersection of simple zeros of $h(\eta)$ and $f(\eta)$ is empty.

Proposition 2.5 (The residue conditions). Let $h(\eta) = \frac{dh}{d\eta}$, one enjoys the following residue conditions:

\[
\text{Res}\{W(x, t, \eta)\}, s_j = \begin{cases} 
\frac{1}{s(s_j)f(s_j)}e^{2i\mu(s_j)}[W(x, t, s_j)]_{L_1}, j = 1, \ldots, 2a_1, \\
\frac{1}{s(s_j)f(s_j)}e^{-2i\mu(s_j)}[W(x, t, s_j)]_{L_2}, j = 1, \ldots, 2a_2,
\end{cases}
\]  

\[\text{Res}\{W(x, t, \eta)\}, \bar{s}_j = \begin{cases} 
\frac{S(\bar{s}_j)}{f(\bar{s}_j)}e^{2i\mu(s_j)}[W(x, t, s_j)]_{L_1}, j = 1, \ldots, 2b_1, \\
\frac{S(\bar{s}_j)}{f(\bar{s}_j)}e^{-2i\mu(s_j)}[W(x, t, s_j)]_{L_2}, j = 1, \ldots, 2b_2.
\end{cases}
\]

Proof. One only shows the equation (2.45a). As a result of $W(x, t, \eta) = \left\{\frac{G_1}{f(\eta)}\right\}_{L_2}, \left\{G_3^{(i)}\right\}_{L_2}$, one finds that the zeros $\{s_i\}^{2a}_{j=1}$ of $f(\eta)$ are the poles of $\left\{G_1\right\}_{f(\eta)}$. Then, one gets

\[
\text{Res}\left\{\frac{G_1}{f(\eta)}\right\}_{s_j} = \lim_{\eta \to s_j} \frac{G_1}{f(\eta)} = \frac{G_1(s_j)}{f(s_j)},
\]

\[\text{Res}\left\{\frac{G_1}{f(\eta)}\right\}_{s_j} = \lim_{\eta \to \bar{s}_j} \frac{G_1}{f(\eta)} = \frac{G_1(\bar{s}_j)}{f(\bar{s}_j)},
\]

taking $\eta = s_j$ into the first and second equations of (2.36), we can get

\[
G_1^{(i)}(x, t, s_j) = \frac{1}{s(s_j)}e^{2i\mu(s_j)}[G_3^{(i)}(x, t, s_j)],
\]

together with equations (2.46) and (2.47), one obtains

\[
\text{Res}\left\{\frac{\{G_1\}^{(i)}}{f(\eta)}\right\}_{s_j} = \frac{1}{s(s_j)}e^{2i\mu(s_j)}[G_3^{(i)}](x, t, s_j),
\]

therefore, the equation (2.48) can lead to the equation (2.45a), and the other three equations (2.45b)–(2.45d) can be similarly proved.

2.5. The inverse problem

The inverse problem includes the reconstruction of potential function $q(x, t)$ from spectral functions $\{G_j(x, t, \eta)\}_{L_2}$. It follows from equation (2.10) that $D_{\phi}^{(1)} = \frac{1}{2}(\alpha - \beta)QD_0\alpha$. Since asymptotic expansion in equation (2.8) is a solution of equation (2.7), which implies that

\[
q(x, t) = -\frac{2i}{\alpha - \beta}w(x, t)e^{-2i\int_{a+\eta}^{a}\Omega},
\]

where $G(x, t, \eta)$ is related to $\Psi(x, t, \eta)$ as shown in equation (2.15) and given by

\[
G(x, t, \eta) = I + \frac{w^{(1)}(x, t)}{\eta} + \frac{w^{(2)}(x, t)}{\eta^2} + O\left(\frac{1}{\eta^3}\right), \eta \to \infty.
\]

Meanwhile, $G(x, t, \eta)$ is the solution of equation (2.16) if $w^{(1)}(x, t)$ replaces of $w(x, t)$. It follows from equation (2.49) and its complex conjugate that

\[
\bar{q} - \bar{q} = \frac{4}{(\alpha - \beta)^2}w^2 + \frac{8\alpha}{(\alpha - \beta)^2}w^4.
\]

Then, the one-form $\Omega$ given by equation (2.13) can be expressed by $w(x, t)$

\[
\Omega = \frac{\alpha}{(\alpha - \beta)^2}w^2dx - \left[\frac{\alpha}{(\alpha - \beta)^2}(\pi w_x - w\pi_x) - \frac{6\alpha - 4\beta^2}{(\alpha - \beta)^4}w^4\right]d\tau.
\]

Hence, one can solve the inverse problem according to the following steps successively:

(i) One utilizes any one of the functions $\{G_j(x, t, \eta)\}_{L_2}$ to calculate $w(x, t)$ by $w(x, t) = \lim_{\lambda \to \infty} \{\eta G_j(x, t, \eta)\}_{L_2}$.

(ii) One gets $\Omega(x, t)$ from equation (2.50).

(iii) One computes potential function $q(x, t)$ by equation (2.49).

2.6. The global relation

In this subsection, one gives the spectral functions $f(\eta)$, $s(\eta)$, $F(\eta)$, $S(\eta)$ which are not independent but admit a significant relationship. In fact, at the boundary of the region $(\xi, \tau)$: $0 < \xi < \infty$, $0 < \tau < t$, the integral of the one-form $A(x, t, \eta)$ defined by the equation (2.17) is vanished. Let
Indeed, equation (2.27a) is valid for \( \eta^2 \) in the lower half-plane and the second column of equation (2.24) is valid for \( \eta^2 \) in the upper half-plane, and the expression of \( \phi(t, \eta) \) is

\[
\phi^{-1}(t, \eta) = e^{\frac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}} G_{1}(0, t, \eta). \tag{2.25}
\]

where the first column of equation (2.25) is valid for \( \eta^2 \) in the lower half-plane and the second column of equation (2.25) is valid for \( \eta^2 \) in the upper half-plane, and the expression of \( \phi(t, \eta) \) is

\[
\phi^{-1}(t, \eta) = e^{\frac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}} G_{1}(0, t, \eta). \tag{2.25}
\]

Hence, the (21)-component of equation (2.25) is

\[
f(\eta)S(\eta) - F(\eta)S(\eta) = \frac{\eta}{1 - \alpha \beta} e^{\frac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}} E(\eta), \text{ Im(} \eta^2 \text{) } \geq 0, \tag{2.26}
\]

where \( E(\eta) \) is expressed by

\[
E(\eta) = \int_{0}^{\infty} e^{rac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}}(M_{1}G_{3})(\xi, t, \eta)d\xi. \tag{2.27}
\]

Indeed, equation (2.26) is the so-called global relation.

**3. The functions \( f(\eta) \), \( s(\eta) \), \( R(\eta) \) and \( S(\eta) \)**

**Definition 3.1.** (\( f(\eta) \) and \( s(\eta) \)) Let \( u_0(x) = u(x, 0) \in \mathbb{S} \), one defines the mapping

\[
\gamma : [u_0(x)] \to (f(\eta), s(\eta)),
\]

in terms of

\[
(s(\eta), f(\eta))^T = \begin{cases}
(G_{1}^{2(\alpha\beta)})(x, 0, \eta), & \text{for } \alpha > \beta, \\
(G_{1}^{2(\alpha\beta)})(x, 0, \eta), & \text{for } \alpha < \beta,
\end{cases}
\]

where \( G_{0}(x, 0, \eta) \) is given by

\[
G_{0}(x, 0, \eta) = 1 - \int_{0}^{\infty} e^{\frac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}}(M_{1}G_{3})(\xi, 0, \eta)d\xi,
\]

with \( M_{1}(x, 0, \eta) \) expressed by equation (2.33).

**Proposition 3.2.** The \( f(\eta) \) and \( s(\eta) \) possess the properties as following

(i) \( f(\eta), s(\eta) \) are analytic and bounded for \( \text{Im} \frac{1}{\alpha - \beta} \eta^2 > 0 \) and continuous for \( \text{Im} \frac{1}{\alpha - \beta} \eta^2 \geq 0 \).

(ii) \( f(\eta) = 1 + O\left(\frac{1}{\eta}\right) \), \( s(\eta) = O\left(\frac{1}{\eta}\right) \) as \( \eta \to \infty \).

(iii) \( f(\eta)f(\bar{\eta}) - s(\eta)s(\bar{\eta}) = 1, \eta^2 \in \mathbb{R} \).

(iv) \( f(\eta) = f(\bar{\eta}), s(\eta) = s(\bar{\eta}) \), \( \text{Im} \frac{1}{\alpha - \beta} \eta^2 > 0 \).

(v) The inverse mapping of \( \gamma \) is \( \gamma^{-1} = \mathbb{S}_{1}: (f(\eta), s(\eta)) \to [u_0(x)] \),

which is defined by

\[
u_0(x) = \frac{2i}{\alpha - \beta} \mathbb{W}(x)e^{-2i\eta \frac{1}{\alpha - \beta}} \int_{0}^{\infty} e^{\frac{\eta}{1 - \alpha \beta} \sqrt{t} \text{y} \text{h}}(M_{1}G_{3})(\xi, 0, \eta)d\xi,
\]

where \( \mathbb{W}(x, \eta) \) admits RH problem as follows.

- **W**(1)(x, \( \eta \)) is a section analytic function.
- **W**(1)(x, \( \eta \)) = \( W^{(1)(x)}(\eta) \) \( (H^{(1)(x)}(\eta))^{-1}, \eta \in \mathbb{R} \), and

\[
H^{(1)}(x, \eta) = \left( \frac{1}{\eta}, e^{2i\eta \frac{1}{\alpha - \beta}}, 1 - \sqrt{|\eta|} \right) \tag{3.1}
\]

- **W**(1)(x, \( \eta \)) = \( I + O\left(\frac{1}{\eta}\right) \), \( \eta \to \infty \).
- \( f(\eta) \) possesses 2a simple zeros \( \{z_j\}_j^{2a} \), \( 2a = 2a_1 + 2a_2 \), such that \( \text{Im} \frac{1}{\alpha - \beta} z_j^2 > 0, j = 1, 2, \cdots, 2a_1 \), and \( \text{Im} \frac{1}{\alpha - \beta} z_j^2 < 0, j = 1, 2, \cdots, 2a_2 \).
- The first column of \( W^{(1)}(x, \eta) \) enjoys simple poles at \( \eta = \{z_j\}_1^{2a_1} \). The second column of \( W^{(1)}(x, \eta) \) enjoys
simple poles at \( \eta = (\zeta_j)_1^{2n} \). The relevant residue expression is

\[
\begin{align}
\text{Res}[[W^{(i)}(x, \eta)], \zeta_j] &= \frac{e^{\frac{\pi i}{3}}} {f(\zeta_j) s(\zeta_j)} \left[ W^{(i)}(x, \zeta_j) \right]_2, \quad j = 1, 2, \cdots, 2a \quad (3.2a)
\end{align}
\]

\[
\begin{align}
\text{Res}[[W^{(i)}(x, \eta)], \zeta_j] &= \frac{e^{\frac{\pi i}{3}}} {f(\zeta_j) s(\zeta_j)} \left[ W^{(i)}(x, \zeta_j) \right]_1, \quad j = 1, 2, \cdots, 2a_2 \quad (3.2b)
\end{align}
\]

**Proof.** (i)-(iv) follow from the investigation in section 2.3, and the deduction of (v) can be obtained following [4], where the derivation of \( u_0(x) \) is given in the inverse problem (see section 2.5).

**Definition 3.3.** (\( F(\eta) \) and \( S(\eta) \)) Let \( v_0(t), v_1(t) \in \mathbb{S} \), the mapping

\[
\mathbb{Z}_2 : \{v_0(t), v_1(t)\} \rightarrow \{F(\eta), S(\eta)\},
\]

in terms of

\[
(S(\eta), F(\eta))^T = \begin{cases}
G_1 \eta^{2}(x, 0), \eta, \text{ for } \alpha > \beta, \\
G_2 \eta^{2}(x, 0), \eta, \text{ for } \alpha < \beta,
\end{cases}
\]

where \( G_1(0, t, \eta) \) is given by

\[
G_1(0, t, \eta) = I - \int_{\eta}^{T} \frac{e^{\frac{\pi i}{3}} \eta^j \eta \eta \eta}{(\alpha - \beta)^2} (N_1 G_1)(0, \tau, \eta) d\tau,
\]

and \( N_1(0, t, \eta) \) is expressed by equation (2.32b).

**Proposition 3.4.** The \( F(\eta) \) and \( S(\eta) \) possess the properties as follows

(i) \( F(\eta), S(\eta) \) are analytic and bounded for \( \text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4 \geq 0 \), if \( T = \infty \), the \( F(\eta), S(\eta) \) are defined only for \( \text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4 \geq 0 \).

(ii) \( F(\eta) = 1 + O \left( \frac{1}{\eta} \right) \), \( S(\eta) = O \left( \frac{1}{\eta} \right) \) as \( \eta \rightarrow \infty \),

\[
\text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4 \geq 0.
\]

(iii) \( F(\eta) S(\eta) - S(\eta) F(\eta) = 1, \eta \in \mathbb{C}(\eta^4 \in \mathbb{R}, \text{ if } T = \infty) \).

(iv) \( F(-\eta) = S(\eta), S(-\eta) = -S(\eta), \text{ Im} \frac{2}{(\alpha - \beta)^2} \eta^4 \geq 0 \).

(v) The inverse mapping of \( \mathbb{Z}_2 \) is \( \mathbb{Z}_2 : \{F(\eta), S(\eta)\} \rightarrow \{v_0(t), v_1(t)\} \), which is defined by

\[
\begin{align}
v_0(t) &= -\frac{2i}{(\alpha - \beta)} w^{(i)}(t) e^{\frac{\pi i}{3}} \int_{\eta}^{T} \frac{\eta^j \eta \eta}{(\alpha - \beta)^2} (N_1 G_1)(0, \tau, \eta) d\tau, \\
v_1(t) &= \left[ \frac{2}{(\alpha - \beta)^2} w^{(i)}(t) - w^{(i)}(t) \bar{v}_0(t) w^{(i)}(t) \right] e^{\frac{\pi i}{3}} \int_{\eta}^{T} \frac{\eta^j \eta \eta}{(\alpha - \beta)^2} (N_1 G_1)(0, \tau, \eta) d\tau \\
&- \frac{2i}{(\alpha - \beta)} v(t) w^{(i)}(t) \frac{\eta}{2} v_0(t) \bar{v}_0(t),
\end{align}
\]

where

\[
\begin{align}
\Omega_2(\tau) &= 2(-\alpha^2 + \alpha \beta - \beta^2) w^{(i)}(t)^4 \\
&+ \frac{2\alpha}{(\alpha - \beta)^2} (w^{(i)}(t) w^{(i)}(t) + w^{(i)}(t) w^{(i)}(t)) \\
&- 4\alpha \alpha w^{(i)}(t)^4 - 4\alpha \alpha w^{(i)}(t)^4 \text{ Re}[w^{(i)}(t)] ,
\end{align}
\]

and the functions \( w^{(i)}(t) \), \( i = 1, 2, 3 \) are determined by

\[
W^{(i)}(t, \eta) = I + \frac{w^{(i)}(t)}{\eta} + \frac{w^{(i)}(t)}{\eta^2} + \frac{w^{(i)}(t)}{\eta^3}
\]

\[
+ O \left( \frac{1}{\eta^4} \right), \eta \rightarrow \infty ,
\]

where \( W^{(i)}(t, \eta) \) admits RH problem as follows

- \( W^{(i)}(t, \eta) = \begin{cases}
W^{(i)}(t, \eta), \text{ Im} \frac{2}{(\alpha - \beta)^2} \eta^4 \leq 0,
\\
W^{(i)}(t, \eta), \text{ Im} \frac{2}{(\alpha - \beta)^2} \eta^4 > 0,
\end{cases} \) is a section analytic function.

- \( W^{(i)}(t, \eta) = W_0^{(i)}(t, \eta) \text{H}^{(i)}(t, \eta), \eta^4 \in \mathbb{R}, \) and

\[
H^{(i)}(t, \eta) = \begin{pmatrix}
\frac{1}{\text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4} \\
\frac{-1}{\text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4} \\
\end{pmatrix}. \quad (3.4)
\]

- \( W^{(i)}(t, \eta) = I + \frac{1}{\eta} \) \( \eta \rightarrow \infty .
\]

- \( F(\eta) \) possesses 2k simple zeros \( \{\zeta_j\}_{1}^{2k} \), \( 2k = 2k_1 + 2k_2 \), such that \( \text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4 > 0, j = 1, 2, \cdots, 2k_1 \), and \( \text{Im} \frac{2}{(\alpha - \beta)^2} \eta^4 < 0, j = 1, 2, \cdots, 2k_2 \).

- The first column of \( W^{(i)}(t, \eta) \) enjoys simple poles at \( \eta = (\zeta_j)_1^{2k_1} \), the second column of \( W^{(i)}(t, \eta) \) enjoys simple poles at \( \eta = (\zeta_j)_1^{2k_2} \). The relevant residue expression is

\[
\begin{align}
\text{Res}[[W^{(i)}(t, \eta)], \zeta_j] &= \frac{e^{\frac{\pi i}{3}} \eta^j}{F(\zeta_j) \text{S}(\zeta_j)} [W^{(i)}(t, \zeta_j)], j = 1, 2, \cdots, 2k_1 \quad (3.5a)
\end{align}
\]

\[
\begin{align}
\text{Res}[[W^{(i)}(t, \eta)], \zeta_j] &= \frac{e^{\frac{\pi i}{3}} \eta^j}{F(\zeta_j) \text{S}(\zeta_j)} [W^{(i)}(t, \zeta_j)], j = 1, 2, \cdots, 2k_2 \quad (3.5b)
\end{align}
\]

**Proof.** (i)-(iv) follow from the investigation in section 2.3, and the deduction of (v) can be obtained following [4], where the derivation of \( v_0(t) \) and \( v_1(t) \) are given in appendix.
4. The RH problem

**Theorem 4.1.** Let $u_0(x) \in \mathcal{S}(\mathbb{R}^+)$, the matrix functions $\psi(\eta)$ and $\phi(\eta)$ in terms of $f(\eta)$, $s(\eta)$, $F(\eta)$, $S(\eta)$ are given by equation (2.34), respectively. Assume that the possible simple zeros $\{\zeta_j\}_{j=1}^m$ of function $f(\eta)$ and $\{\zeta_j\}_{j=1}^n$ of function $h(\eta)$ are given by assumption 2.4. Therefore, the matrix-value function $W(x, t, \eta)$ conforms to the following RH problem:

- $W(x, t, \eta)$ is the slice analytic function for $\eta \in L_k$ and continuous to $L_k$, $(k = 1, \ldots, 4)$.
- $W(x, t, \eta)$ jump arises on the curves $\{L_k\}_{k=1}^4$ and admits the jump relation given by theorem 2.3, i.e.

$$W(x, t, \eta) = W_i(x, t, \eta)H(x, t, \eta), \eta \in L_k, \ k = 1, \ldots, 4,$$

- $W(x, t, \eta) = I + O\left(\frac{1}{\eta}\right), \eta \to \infty$.
- $W(x, t, \eta)$ meets the residues conditions given by proposition 2.5.

Hence, the function $W(x, t, \eta)$ is uniquely existing. Then, one can use $W(x, t, \eta)$ to define $q(x, t)$ as

$$q(x, t) = -\frac{2i}{\alpha - \beta}w(x, t)e^{-2i\int_{\eta_0-\Omega}^{\eta_0+\Omega}dt},$$

$$w(x, t) = \lim_{\eta \to \infty}t\{W(x, t, \eta)\}_{21},$$

$$\Omega = \frac{\alpha}{\alpha - \beta^2}|w|^2dx - \left[\frac{\alpha}{\alpha - \beta^2}(\delta_{wyx} - w_{xy}) - \frac{6\alpha\beta - 4\beta^2}{\alpha - \beta^2}|w|^2\right]dt,$$

thus, the function $q(x, t)$ is a solution of the GDNLS equation (1.3). Furthermore, $u(x, 0) = u_d(x)$, $u(0, t) = v_0(t)$, $u_d(0, t) = v_1(t)$.

**Proof.** Indeed, one can manifest the above RH problem following [4].

5. Conclusions and discussions

In this paper, we use UTM to discuss the IBVPs of the generalized DNLS equation (1.3), one can also discuss the equation (1.3) on a finite interval, and analyze the asymptotic behavior of the solution for the equation (1.3) by the Deift–Zhou method [36]. Since the RH problem is equivalent to Gel’fand–Levitan–Marchenko (GLM) theory, one can obtain the soliton solution of the equation (1.3) by solving the GLM equation following [37], which are our future investigation work.

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**Appendix. Recovering $v_0(t)$ and $v_1(t)$**

In this appendix, we will give a proof of equation (3.3), that is, derive $v_0(t)$ and $v_1(t)$ from $W^\eta$. Let $G(x, t, \eta)$ is a solution of equation (2.16). According to equation (2.11), one gets

$$iQ_\eta \sigma_3D_0 = \frac{4i}{\alpha - \beta^2}D_1\sigma_3 - iQ_\eta^2\sigma_3D_2\sigma_3,$$

$$+\frac{2}{\alpha - \beta}Q_\eta D_1\sigma_3 + \frac{\alpha}{2}Q_\eta D_0.$$ (A.1)

where $\Psi(x, t, \eta)$ is the solution of equation (2.7) and enjoys the following form

$$\Psi(x, t, \eta) = D_0 + \frac{D_1}{\eta} + \frac{D_2}{\eta^2} + \frac{D_3}{\eta^3} + O\left(\frac{1}{\eta^4}\right), \eta \to \infty.$$ Since $\Psi(x, t, \eta)$ is defined by equation (2.15) and related to $G(x, t, \eta)$ as follows

$$G(x, t, \eta) = \left(\begin{array}{c} G_{11} \\ G_{21} \\ G_{22} \end{array}\right),$$

then, one gets

$$\Psi(x, t, \eta) = \left(\begin{array}{c} D_0^{11}G_{11} \\ D_0^{12}e^{-2i\int_{\eta_0}^{\eta_0+\Omega} G_{12} \\ D_0^{22}G_{22} \end{array}\right).$$$$\left(\begin{array}{c} D_0^{11}e^{-2i\int_{\eta_0}^{\eta_0+\Omega} G_{11} \\ D_0^{12}e^{-2i\int_{\eta_0}^{\eta_0+\Omega} G_{12} \\ D_0^{22}e^{-2i\int_{\eta_0}^{\eta_0+\Omega} G_{22} \end{array}\right).$$

If seeking

$$G(x, t, \eta) = I + \frac{w^{(1)}}{\eta} + \frac{w^{(2)}}{\eta^2} + \frac{w^{(3)}}{\eta^3} + O\left(\frac{1}{\eta^4}\right), \eta \to \infty,$$

then the (21)-entry of equation (A.1) gives

$$q_\xi = \left[\frac{4}{\alpha - \beta^2}w^{(2)}_{21} - qw^{(2)}_{21}\right]e^{-2i\int_{\eta_0}^{\eta_0+\Omega}} - \frac{2i}{\alpha - \beta}qw^{(2)}_{21} - \frac{\alpha}{2}\bar{q}^2\bar{q}. \quad (A.2)$$

Taking the complex conjugate yields

$$\bar{q}_\xi = \left[\frac{4}{\alpha - \beta^2}\bar{w}^{(2)}_{21} - q\bar{w}^{(2)}_{21}\right]e^{2i\int_{\eta_0}^{\eta_0+\Omega}} + \frac{2i}{\alpha - \beta}\bar{q}w^{(2)}_{21} + \frac{\alpha}{2}\bar{q}\bar{q}^2. \quad (A.3)$$

At the same time, from equation (2.49), one finds

$$q(x, t) = -\frac{2i}{\alpha - \beta}w^{(2)}_{21}e^{-2i\int_{\eta_0}^{\eta_0+\Omega}} - i\bar{q}\bar{q},$$

$$\bar{q}(x, t) = \frac{2i}{\alpha - \beta}w^{(2)}_{21}e^{2i\int_{\eta_0}^{\eta_0+\Omega}}. \quad (A.4)$$
It follows from equations (A.2)–(A.4) that
\[ q_\ell - q_\ell = \frac{8i}{(\alpha - \beta)^2} \mathbb{w}_{21}^{(1)}(w_{21}^{(21)} + w_{21}^{(3)}) \]
\[ - \frac{4i}{\alpha - \beta} q\mathbb{w}_{21}^{(1)}w_{21}^{(21)} - \frac{4i}{\alpha - \beta} qq \text{Re}[w_{11}^{(2)}] - ioq^2q^2, \]
\begin{equation}
(A.5)
\end{equation}
which means that the coefficient \( \Omega_2 = \frac{1}{8}(\alpha^2 + \alpha\beta - \beta^2) \)
\[ |q| - \frac{1}{2} \left( \alpha q_{\ell} - q_{\ell} \right) \] of \( dr \) in the differential form \( \Omega \) defined
in equation (2.14) can be expressed as
\[ \Omega_2 = \frac{1}{8} \left( -\alpha^2 + \alpha\beta - \beta^2 \right) q^2 \]
\[ + \frac{2\alpha}{(\alpha - \beta)^2} \mathbb{w}_{21}^{(1)}w_{21}^{(21)} + w_{21}^{(3)} \]
\[ - \frac{4\alpha}{\alpha - \beta} q\mathbb{w}_{21}^{(1)}w_{21}^{(21)} - \frac{4\alpha}{\alpha - \beta} qq \text{Re}[w_{11}^{(2)}]. \]
\begin{equation}
(A.6)
\end{equation}
Owing to \( q = 4w_{21}^{(1)} \), we calculate equations (A.2), (A.4)–(A.7) at \( x = 0 \) and yield
\[ v_0(t) = -\frac{2i}{\alpha - \beta} w_{11}^{(1)}(t)e^{-2i\int_0^t \mathbb{a}_{21}(r)dr}, \]
\[ v_1(t) = \left[ \frac{4}{(\alpha - \beta)^2} w_{21}^{(21)}(t) - v_0(t) \bar{v}_0(t)w_{21}^{(1)}(t) \right] e^{-2i\int_0^t \mathbb{a}_{21}(r)dr} \]
\[ - \frac{2i}{\alpha - \beta} v_0(t)w_{11}^{(2)}(t) - \frac{i\alpha}{2} v_0(t)^2 \bar{v}_0(t), \]
\begin{equation}
(A.7)
\end{equation}
with
\[ \Omega_2(\tau) = 2(-\alpha^2 + \alpha\beta - \beta^2)w_{21}^{(1)} \]
\[ + \frac{2\alpha}{(\alpha - \beta)^2} w_{21}^{(21)} + w_{21}^{(3)} \]
\[ - \frac{4\alpha}{\alpha - \beta} w_{21}^{(1)} + w_{21}^{(1)} \text{Re}[w_{11}^{(2)}], \]
\begin{equation}
(A.8)
\end{equation}
where the functions \( W^{(j)}(\eta) \), \( j = 1, 2, 3 \) are determined by
\[ W^{(j)}(\eta) = I + \frac{w^{(1)}(t)}{\eta} + \frac{w^{(2)}(t)}{\eta^2} + \frac{w^{(3)}(t)}{\eta^3} \]
\[ + O\left( \frac{1}{\eta^4} \right), \quad \eta \to \infty. \]
\begin{equation}
(A.9)
\end{equation}

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