THE COVERING RADIi OF THE 2-TRANSITIVE UNITARY, SUZUKI, AND REE GROUPS

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Abstract. We study the covering radii of 2-transitive permutation groups of Lie rank one, giving bounds and links to finite geometry.

Key words: covering radius; 2-transitive permutation group; unitary group; Suzuki group; Ree group

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1. Introduction

Let \( n \geq 2 \). Define the Hamming distance \( d_n \) on the symmetric group \( S_n \) by letting

\[ d_n(g, h) = n - |\text{fix}(gh^{-1})| \]

for any \( g, h \in S_n \), where \( \text{fix}(x) \) denotes the set of fixed points for \( x \in S_n \). Equipped with Hamming distance \( S_n \) is then a metric space. As usual, the distance of a point \( v \) from a subset \( C \) in \( S_n \) is

\[ d_n(v, C) := \min\{d_n(v, c) \mid c \in C\}, \]

and the covering radius of \( C \) is

\[ \text{cr}_n(C) := \max\{d_n(v, C) \mid v \in S_n\}. \]

Covering radii of subgroups of \( S_n \) were first studied in a seminal paper of Cameron and Wanless [1], where an upper bound was given in terms of the transitivity:

Proposition 1.1. ([1, Proposition 16]) If \( G \) is a \( t \)-transitive permutation group of degree \( n \), then \( \text{cr}_n(G) \leq n - t \).

In light of this result, the authors of [1] were then interested in the covering radius of 2-transitive permutation groups. For example, they showed that \( \text{cr}_{q+1}(\text{PSL}(2, q)) = q - \gcd(2, q) \) for all \( q \), and determined \( \text{cr}_{q+1}(\text{AGL}(2, q)) \) and \( \text{cr}_{q+1}(\text{PGL}(2, q)) \) for \( q \not\equiv 1 \pmod{6} \). For \( q \equiv 1 \pmod{6} \), a determination of \( \text{cr}_{q+1}(\text{AGL}(2, q)) \) and \( \text{cr}_{q+1}(\text{PGL}(2, q)) \) was accomplished in [4] and [5], respectively. In determining the covering radii of these groups, the key was to find a large lower bound on the covering radius, which turns out to be ad hoc for each group. To shed light on the covering radii of 2-transitive permutation groups, especially on establishing their lower bounds, we apply a general idea of “field automorphism” construction to obtain lower bounds of the other 2-transitive permutation groups that are simple groups of (twisted) Lie rank one: \( \text{PSU}(3, q) \), \( \text{Sz}(q) \) and \( \text{Ree}(q) \). This leads to the following main results of this paper, whose proof is at the end of Sections 2, 3 and 4 respectively.

Theorem 1.2. Let \( q = p^f \) with prime \( p \). Then

\[ q^3 - p \leq \text{cr}_{q^3+1}(\text{PGU}(3, q)) \leq \text{cr}_{q^3+1}(\text{PSU}(3, q)) \leq q^3 - \gcd(2, q). \]

In particular, if \( q \) is even then

\[ \text{cr}_{q^3+1}(\text{PGU}(3, q)) = \text{cr}_{q^3+1}(\text{PSU}(3, q)) = q^3 - 2. \]

Moreover, for the field automorphism \( h \) defined in [2], \( d_{q^3+1}(h, \text{PGU}(3, q)) = q^3 - p \).
Theorem 1.3. Let $q = 2^{2m+1}$. Then $q^2 - 4 \leq cr_{q^2+1}(Sz(q)) \leq q^2 - 2$. Moreover, for the field automorphism $h$ defined in (8), $d_{q^2+1}(h, Sz(q)) = q^2 - 4$.

Theorem 1.4. Let $q = 3^{2m+1}$. Then $q^3 - 27 \leq cr_{q^3+1}(Ree(q)) \leq q^3 - 1$. Moreover, for the field automorphism $h$ defined in (10), $d_{q^3+1}(h, Ree(q)) = q^3 - 27$.

Cameron and Wanless [1] mentioned a geometric interpretation for their work, namely a link between the covering radius of PGL(2, q) and the classical Minkowski planes. However they did not give many details. In Section 5 we explain this link in detail and then establish a similar link between the covering radius of PGU(3, q) and a certain finite geometry $U_{2,2}$. We also pose the problem of characterising the classical ovoids of $U_{2,2}$ via incidence geometry.

2. UNITARY GROUPS

Let $q = p^f$ with $p$ prime, and $V$ be a 3-dimensional vector space over $\mathbb{F}_{q^2}$ equipped with the unitary form
\begin{equation}
(1) \quad x_1y_3^2 + x_2y_2^2 + x_3y_1^2
\end{equation}
for any vectors $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ in $V$. Denote by $\Omega$ the set of totally isotropic one-dimensional subspaces of $V$ and $\varphi$ the quotient map of GU($V$) modulo its center. Then $|\Omega| = q^3 + 1$ and GU($V$)$^\varphi = PGU(3, q)$ is a permutation group on $\Omega$. For any $(\langle x_1, x_2, x_3 \rangle) \in V$, let
\begin{equation}
(2) \quad \langle (x_1, x_2, x_3) \rangle^h = \langle (x_1^p, x_2^p, x_3^p) \rangle.
\end{equation}
Then $h$ is a permutation of $V$, and we have PGU(3, q) $\rtimes \langle h \rangle = P\Gamma U(3, q)$.

Lemma 2.1. For each $\gamma \in \mathbb{F}_{q^2}$ and $\delta \in \mathbb{F}_{q^2}$ such that $\delta^{q+1} = 1$, the system of equations
\begin{equation}
(3) \quad \begin{cases}
  a^{p-1} = \gamma^{q+1} \\
  b^p = b\gamma\delta \\
  a + a^q + b\delta^{q+1} = 0
\end{cases}
\end{equation}
has at most $p - 1$ solutions $(a, b) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$.

Proof. We divide it into three cases by the value of $\gamma^{(q^2-1)/(p-1)}$.

Case 1. $\gamma^{(q^2-1)/(p-1)} = -1$. Let $(a, b) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ be a solution of (3). Then by the first line of (3) we have
\[a^q + a = a ((a^{p-1})^{(q-1)/(p-1)} + 1)\]
\[= a ((\gamma^{q+1})^{(q-1)/(p-1)} + 1) = a \left(\gamma^{(q^2-1)/(p-1)} + 1\right) = 0,
\]
which together with the third line of (3) yields $b = 0$. Moreover, the first line of (3) has at most $p - 1$ solutions for $a$. Hence there are at most $p - 1$ such pairs $(a, b)$.

Case 2. $\gamma^{(q^2-1)/(p-1)} = 1$ and $\gamma^{(q^2-1)/(p-1)} \neq -1$. Note that this condition indicates $p > 2$. Let $(a, b) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ be a solution of (3). Then by the first line and third line of (3) we have $a \neq 0$ (since $\gamma \neq 0$) and
\begin{equation}
(4) \quad 2a = a \left(\gamma^{(q^2-1)/(p-1)} + 1\right) = a ((\gamma^{q+1})^{(q-1)/(p-1)} + 1) = a ((a^{p-1})^{(q-1)/(p-1)} + 1) = a^q + a = -b\delta^{q+1}.
\end{equation}
In particular, $b \neq 0$. It then follows from the second line of (3) that $b^{p-1} = \gamma\delta$, which has at most $p - 1$ solutions for $b$. Thus, since $a$ is determined by $b$ as (4) shows, we conclude that there are at most $p - 1$ such $(a, b)$. 

Case 3. $\gamma^{(q^2 - 1)/(p-1)} \neq \pm 1$. In this case, any solution $(a, b) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2$ of (3) would satisfy (using the first and third lines of (3))

$$b^q + 1 = -a(a^q + 1) = -a \left( (a^{p-1})^{(q-1)/(p-1)} + 1 \right) = -a \left( (\gamma^{q+1})^{(q-1)/(p-1)} + 1 \right) = -a \left( (\gamma^{(q-1)/(p-1)} + 1 \right) \neq 0$$

and hence (using the second line of (3))

$$1 = b^{q^2 - 1} = (b^{p-1})^{(q^2 - 1)/(p-1)} = (\gamma^{q+1})^{(q-1)/(p-1)} \neq (\gamma^{q+1})^{(q-1)/(p-1)}$$

contrary to the condition $\gamma^{q+1} \neq 1$. This completes the proof. \hfill $\square$

Lemma 2.2. For each $g \in \text{PGU}(3, q)$ we have $|\text{fix}(gh^{-1})| \leq p + 1$, where $h$ is defined as in (2).

Proof. Suppose $g$ is an element of $\text{PGU}(3, q)$ such that $gh^{-1}$ fixes at least 2 points of $\Omega$, say $\langle u \rangle$ and $\langle v \rangle$. Note that $\langle e_1 \rangle$ and $\langle e_2 \rangle$ lie in $\Omega$ by (1), where $e_1 = (1, 0, 0)$ and $e_2 = (0, 0, 1)$. Since $\text{PGU}(3, q)$ is 2-transitive on $\Omega$, there exists $x \in \text{PGU}(3, q)$ such that $\langle u \rangle^x = \langle e_1 \rangle$ and $\langle v \rangle^x = \langle e_2 \rangle$. Accordingly, $x^{-1}gh^{-1}x$ fixes $\langle e_1 \rangle$ and $\langle e_2 \rangle$.

Let $X = \langle \text{PGU}(3, q), h \rangle = \text{PGU}(3, q) \rtimes \langle h \rangle = \text{PGL}(3, q)$ and

$$Y = \left\{ \begin{pmatrix} \gamma^{q+1} & \gamma^q \delta \\ 0 & 1 \end{pmatrix} \middle| \gamma, \delta \in \mathbb{F}_q^\times, \delta^{q+1} = 1 \right\}.$$

Observe that $Y$ preserves the form in (1). We have $Y^\varphi \leq \text{PGU}(3, q)$ and $Y^\varphi \rtimes \langle h \rangle \leq X_{e_1,e_2}$. Define a map

$$\theta : (\gamma, \delta) \mapsto \begin{pmatrix} \gamma^{q+1} \\ \gamma^q \delta \\ 1 \end{pmatrix}^\varphi$$

from $\mathbb{F}_q^\times \times \{ \delta \in \mathbb{F}_q^\times | \delta^{q+1} = 1 \}$ to $Y^\varphi$. Then $\theta$ is a group epimorphism, and

$$|\text{ker}(\theta)| = |\{ \gamma, \delta \in \mathbb{F}_q^\times | \gamma^{q+1} = 1, \delta = \gamma^{-q} \delta^{q+1} = 1 \}|$$

$$= \{ \gamma \in \mathbb{F}_q^\times | \gamma^{q+1} = 1 \} = q + 1.$$

Consequently, $|Y^\varphi| = (q^2 - 1)(q + 1)/|\text{ker}(\theta)| = q^2 - 1$. Since $X$ is 2-transitive on $\Omega$, it follows that

$$|X_{e_1,e_2}| = \frac{|X|}{|\Omega|(|\Omega| - 1)} = \frac{|\text{PGL}(3, q)|}{(q^3 - 1)q^3} = 2(q^2 - 1)f = |Y^\varphi \rtimes \langle h \rangle|.$$

This implies that $X_{e_1,e_2} = Y^\varphi \rtimes \langle h \rangle$.

Now as $x^{-1}gh^{-1}x \in X_{e_1,e_2}$, there exists an integer $i$ such that

$$x^{-1}gh^{-1}x = yh^i$$

for some $y \in Y^\varphi$. Since $\text{PGL}(3, q) = \text{PGU}(3, q) \rtimes \langle h \rangle$, we deduce that $h^i = h^{-1}$, taking both sides of the above equality modulo $\text{PGU}(3, q)$. Hence $y^{-1} = x^{-1}gh^{-1}x$ has the same number of fixed points as $gh^{-1}$. Write

$$y = \begin{pmatrix} \gamma^{q+1} \\ \gamma^q \delta \\ 1 \end{pmatrix}^\varphi.$$
with \( \gamma \in \mathbb{F}_q^\times \), \( \delta \in \mathbb{F}_q^\times \) and \( \delta^{q+1} = 1 \). To complete the proof, we show that \( |\text{fix}(yh^{-1}) \setminus \{\langle e_1 \rangle, \langle e_3 \rangle\}| \leq p - 1 \).

For any fixed point \( \alpha = \langle (\alpha_1, \alpha_2, \alpha_3) \rangle \) of \( yh^{-1} \) other than \( \langle e_1 \rangle \) and \( \langle e_3 \rangle \), we have

\[
\alpha_1\alpha_3^q + \alpha_2^{q+1} + \alpha_3\alpha_1^q = 0
\]

by \([1]\). If \( \alpha_1 = 0 \) or \( \alpha_3 = 0 \), then \((5)\) would imply \( \alpha_2 = 0 \), contrary to the assumption that \( \alpha \neq \langle e_1 \rangle \) or \( \langle e_3 \rangle \). Consequently, \( \alpha = \langle (a, b, 1) \rangle \) for some \( (a, b) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2 \) with \( a \neq 0 \).

It follows from \([1]\) that

\[
a + a^q + b^{q+1} = 0,
\]

and

\[
\langle (a\gamma^{q+1}, b\gamma^q\delta, 1) \rangle = \langle (a^p, b^p, 1) \rangle
\]

since \( \alpha^q = \alpha^h \). Note that \((6)\) is equivalent to

\[
\begin{cases}
a^{p-1} = \gamma^{q+1} \\
b^p = b^{q^2}
\end{cases}
\]

as \( a \neq 0 \). We then conclude by Lemma \([2,1]\) that there are at most \( p-1 \) such \( \alpha \), completing the proof. \( \square \)

**Lemma 2.3.** \( d_{q^3+1}(h, \PGU(3, q)) = q^3 - p \), where \( h \) is defined as in \((2)\).

**Proof.** By Lemma \([2,2]\) \( d_{q^3+1}(h, g) \geq q^3 + 1 - |\text{fix}(gh^{-1})| \geq q^3 - p \) for each \( g \in \PGU(3, q) \). Hence \( d_{q^3+1}(h, \PGU(3, q)) \geq q^3 - p \). To complete the proof, we only need to show the existence of \( g \in \PGU(3, q) \) such that \( d_{q^3+1}(h, g) \leq q^3 - p \), or equivalently, the existence of \( g \in \PGU(3, q) \) such that \( |\text{fix}(hg^{-1})| \geq p+1 \). Let \( u = \langle (1, 0, 0) \rangle \in \Omega \) and \( v = \langle (0, 0, 1) \rangle \in \Omega \). Note that any element of \( \Omega \) other than \( u \) and \( v \) has form \( \langle (a, b, 1) \rangle \) for some \( (a, b) \in \mathbb{F}_q^2 \times \mathbb{F}_q^2 \) with \( a \neq 0 \) and \( a + a^q + b^{q+1} = 0 \).

First assume that \( p = 2 \). Then \( u \), \( v \) and \( \langle (1, 0, 1) \rangle \) are all elements of \( \Omega \) fixed by \( h \). Hence \( |\text{fix}(h)| \geq 3 \), as desired.

Next assume that \( p > 2 \). Let \( \omega \) be a generator of \( \mathbb{F}_q^\times \), and

\[
g = \begin{pmatrix}
\omega^{(q+1)(p-1)/2} & \omega^{q(p-1)/2} \\
0 & 1
\end{pmatrix} \in \PGU(3, q).
\]

Then it is straightforward to verify that \( u \), \( v \) and

\[
\langle (\omega^{i(q^2-1)/(p-1)+(q+1)/2}, 0, 1) \rangle
\]

with \( 1 \leq i \leq p-1 \) are all elements of \( \Omega \) fixed by \( hg^{-1} \). Hence \( |\text{fix}(hg^{-1})| \geq p+1 \), as desired. This completes the proof. \( \square \)

**Proof of Theorem \([1,2]\)** Since \( \PSU(3, q) \) is a 2-transitive subgroup of \( S_{q^{3+1}} \), we deduce from \([1]\) Proposition 16] that \( \text{cr}_{q^3+1}(\PSU(3, q)) \leq q^3 - 1 \). Moreover, equality cannot be attained if \( q \) is even according to \([1]\) Theorem 21]. Hence

\[
\text{cr}_{q^3+1}(\PSU(3, q)) \leq q^3 - \gcd(2, q).
\]

Let \( h \) be as defined in \((2)\). Then

\[
\text{cr}(\PSU(3, q)) \geq \text{cr}(\PGU(3, q)) \geq d_{q^3+1}(h, \PGU(3, q)).
\]

This together with Lemma \([2,3]\) and \((7)\) verifies Theorem \([1,2]\).
Let \( q = 2^{2m+1}, \ell = 2^{m+1} \), and
\[
\Omega = \{(a, b, c) \in \mathbb{F}_q^3 \mid \begin{array}{l}
abla = ab + \alpha^{\ell+2} + b^\ell
\end{array}\} \cup \{\infty\}.
\]
Clearly, \(|\Omega| = q^2 + 1\). Let
\[
\begin{align*}
\infty^h &= \infty & \text{and} & & (a, b, c)^h &= (a^2, b^2, c^2) \\
\end{align*}
\]
Then \( h \) is a permutation of \( \Omega \), and \( X := Sz(q) \rtimes \langle h \rangle \) is a permutation group on \( \Omega \). We shall prove that \( \delta_{q^2+1}(h, Sz(q)) \geq q^2 - 4 \), which gives a lower bound for the covering radius of \( Sz(q) \).

**Lemma 3.1.** For each \( \kappa \in \mathbb{F}_q^\times \), the system of equations
\[
\begin{align*}
\kappa a &= a^2 \\
\kappa^{\ell+1} b &= b^2 \\
\kappa^{\ell+2} c &= c^2 \\
c &= ab + a^{\ell+2} + b^\ell
\end{align*}
\]
has exactly four solutions \( (a, b, c) \in \mathbb{F}_q^3 \).

**Proof.** From the first three lines of (9) we deduce that \( a = 0 \) or \( \kappa = 0 \) or \( \kappa^{\ell+1} \) and \( c = 0 \) or \( \kappa^{\ell+2} \). This gives eight candidates for \( (a, b, c) \). Verifying the fourth line of (9) for these eight candidates we conclude that
\[
(0, 0, 0), \quad (0, \kappa^{\ell+1}, \kappa^{\ell+2}), \quad (\kappa, 0, \kappa^{\ell+2}), \quad (\kappa, \kappa^{\ell+1}, \kappa^{\ell+2})
\]
are all solutions of (9) and are the only ones. \( \square \)

**Lemma 3.2.** \( \delta_{q^2+1}(h, Sz(q)) = q^2 - 4 \), where \( h \) is defined as in (5).

**Proof.** Suppose \( g \) is an element of \( Sz(q) \) such that \( gh^{-1} \) fixes at least two points of \( \Omega \), say \( u \) and \( v \). Let \( o = (0, 0, 0) \in \Omega \). Since \( Sz(q) \) is 2-transitive on \( \Omega \), there exists \( x \in Sz(q) \) such that \( u^x = \infty \) and \( v^x = o \). Accordingly, \( x^{-1}gh^{-1}x \) fixes \( \infty \) and \( o \).

For any \( \kappa \in \mathbb{F}_q^\times \) denote \( y_\kappa \) the permutation of \( \Omega \) defined by
\[
\infty^{y_\kappa} = \infty \quad \text{and} \quad (a, b, c)^{y_\kappa} = (\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c),
\]
and let \( Y = \{y_\kappa \mid \kappa \in \mathbb{F}_q^\times\} \). Then \( Y \) form a subgroup of \( X_{\infty o} \) [Page 250]. Since \( X \) is 2-transitive on \( \Omega \), it follows that
\[
|X_{\infty o}| = \frac{|X|}{|\Omega|(|\Omega| - 1)} = \frac{|Sz(q)|(2m + 1)}{(q^2 + 1)q^2} = (q - 1)(2m + 1) = |Y \rtimes \langle h \rangle|.
\]
This implies that \( X_{\infty o} = Y \rtimes \langle h \rangle \).

Now as \( x^{-1}gh^{-1}x \in X_{\infty o} \), there exist an integer \( i \) and \( \kappa \in \mathbb{F}_q^\times \) such that
\[
x^{-1}gh^{-1}x = y_\kappa h^i.
\]
Since \( X = Sz(q) \rtimes \langle h \rangle \), we deduce that \( h^i = h^{-1} \), taking both sides of the above equality modulo \( Sz(q) \). Hence \( y_\kappa h^{-1} = x^{-1}gh^{-1}x \) has the same number of fixed points as \( gh^{-1} \).

For any fixed point \( (a, b, c) \) of \( y_\kappa h^{-1} \) other than \( \infty \), we have
\[
c = ab + a^{\ell+2} + b^\ell
\]
since \( (a, b, c) \in \Omega \), and
\[
\begin{align*}
\kappa a &= a^2 \\
\kappa^{\ell+1} b &= b^2 \\
\kappa^{\ell+2} c &= c^2
\end{align*}
\]
since $\alpha^y = \alpha^h$. It then follows from Lemma 3.1 that $|\text{fix}(y, h^{-1}) \setminus \{\infty\}| \leq 4$, and so $|\text{fix}(gh^{-1})| = |\text{fix}(y, h^{-1})| \leq 5$. This implies that $|\text{fix}(gh^{-1})| \leq 5$ for any $g \in \text{Sz}(q)$, which yields

$$d_{q^2+1}(h, \text{Sz}(q)) \geq (q^2 + 1) - 5 = q^2 - 4.$$  

Moreover, Lemma 3.1 implies that

$$d_{q^2+1}(h, y_\kappa) = q^2 + 1 - |\text{fix}(y_\kappa, h^{-1})| = q^2 + 1 - (1 + 4) = q^2 - 4$$

for any $\kappa \in \mathbb{F}_q^\times$. We then conclude that $d_{q^2+1}(h, \text{Sz}(q)) = q^2 - 4$, as the lemma asserts. \hfill \Box

**Proof of Theorem 1.3.** Since $\text{Sz}(q)$ is a 2-transitive subgroup of $S_{q^2+1}$, we deduce from [11 Proposition 16] that $\text{cr}_{q^2+1}(\text{Sz}(q)) \leq q^2 - 1$. Moreover, the equality cannot be attained according to [11 Theorem 21]. Hence

$$\text{cr}_{q^2+1}(\text{Sz}(q)) \leq q^2 - 2.$$  

This together with Lemma 3.2 leads to Theorem 1.3.

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**4. Ree groups**

Let $q = 3^{2m+1}$, $\ell = 3^{m+1}$, and

$$\Omega = \{(a, b, c, \lambda_1(a, b, c), \lambda_2(a, b, c), \lambda_3(a, b, c)) \mid (a, b, c) \in \mathbb{F}_3^3 \cup \{\infty\},$$

where

$$\lambda_1(a, b, c) = a^2b - ac + b^\ell - a^{\ell+3},$$

$$\lambda_2(a, b, c) = a^\ell b - c^\ell + ab^2 + bc - a^{2\ell+3},$$

$$\lambda_3(a, b, c) = ac^\ell - a^{\ell+1}b^\ell + a^{\ell+3}b + a^2b^2 - b^{\ell+1} - c^2 + a^{2\ell+4}.$$  

Clearly, $|\Omega| = q^3 + 1$. Let $\infty^h = \infty$ and

$$\begin{align*}
(a, b, c, \lambda_1(a, b, c), \lambda_2(a, b, c), \lambda_3(a, b, c))^h = & (a^3, b^3, c^3, \lambda_1(a^3, b^3, c^3), \lambda_2(a^3, b^3, c^3), \lambda_3(a^3, b^3, c^3)) \\
= & (\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c, \lambda_1(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c), \lambda_2(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c), \lambda_3(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c)),
\end{align*}$$

for any $(a, b, c) \in \mathbb{F}_q^3$. Then $h$ is a permutation of $\Omega$, and $X := \text{Ree}(q) \rtimes \langle h \rangle$ is a permutation group on $\Omega$. We shall prove that $d_{q^2+1}(h, \text{Ree}(q)) \geq q^3 - 27$, which gives a lower bound for the covering radius of $\text{Ree}(q)$.

**Lemma 4.1.** $d_{q^2+1}(h, \text{Ree}(q)) = q^3 - 27$, where $h$ is defined as in (10).

**Proof.** Suppose $g$ is an element of $\text{Ree}(q)$ such that $gh^{-1}$ fixes at least two points of $\Omega$, say $u$ and $v$. Let $o = (0, 0, 0) \in \Omega$. Since $\text{Ree}(q)$ is 2-transitive on $\Omega$, there exists $x \in \text{Ree}(q)$ such that $x^u = \infty$ and $x^v = o$. Accordingly, $x^{-1}gh^{-1}x$ fixes $\infty$ and $o$.

For any $\kappa \in \mathbb{F}_q^\times$ denote by $y_\kappa$ the permutation of $\Omega$ defined by $\infty^{y_\kappa} = \infty$ and

$$(a, b, c, \lambda_1(a, b, c), \lambda_2(a, b, c), \lambda_3(a, b, c))^{y_\kappa} = (\lambda_1(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c), \lambda_2(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c), \lambda_3(\kappa a, \kappa^{\ell+1} b, \kappa^{\ell+2} c)),$$

and let $Y = \{y_\kappa \mid \kappa \in \mathbb{F}_q^\times\}$. Then $Y$ forms a subgroup of $X_{\infty}$ [2 Page 251]. Since $X$ is 2-transitive on $\Omega$, it follows that

$$|X_{\infty}| = \frac{|X|}{|\Omega|(|\Omega| - 1)} = \frac{|\text{Ree}(q)|(2m + 1)}{(q^3 + 1)q^3} = (q - 1)(2m + 1) = |Y \rtimes \langle h \rangle|.$$  

This implies that $X_{\infty} = Y \rtimes \langle h \rangle$.

Now as $x^{-1}gh^{-1}x \in X_{\infty}$, there exist an integer $i$ and $\kappa \in \mathbb{F}_q^\times$ such that

$$x^{-1}gh^{-1}x = y_\kappa h^i.$$
Since \( X = \text{Ree}(q) \times \langle h \rangle \), we deduce that \( h^i = h^{-1} \), taking both sides of the above equality modulo \( \text{Ree}(q) \). Hence \( y_i h^{-1} = x_i g h^{-1} \) has the same number of fixed points as \( g h^{-1} \). For any fixed point \((a, b, c, \lambda)\) modulo \( \text{Ree}(q) \), we have

\[
\begin{aligned}
\kappa a &= a^3 \\
\kappa^{4+1} b &= b^3 \\
\kappa^{4+2} c &= c^3
\end{aligned}
\]

since \( \alpha^\kappa = \alpha^h \). As each line of (11) has most three solutions, (11) has at most 27 solutions \((a, b, c) \in \mathbb{F}_q^3\). It follows that \( |\text{fix}(y_i h^{-1}) \setminus \{\infty\}| \leq 27 \), and so \( |\text{fix}(g h^{-1})| = |\text{fix}(y_i h^{-1})| \leq 28 \). This implies that \( |\text{fix}(g h^{-1})| \leq 28 \) for any \( g \in \text{Ree}(q) \), which yields

\[
d_{q+1}(h, \text{Ree}(q)) \geq (q^3 + 1) - 28 = q^3 - 27.
\]

Moreover, (11) has exactly 27 solutions \((a, b, c) \in \mathbb{F}_q^3\) when \( \kappa = 1 \). Hence \( y_i h^{-1} \) fixes exactly 28 points. This implies that

\[
\text{cr}_{q+1}(h, y_i) = (q^3 + 1) - 28 = q^3 - 27.
\]

We then conclude that \( d_{q+1}(h, \text{Ree}(q)) = q^3 - 27 \), as the lemma asserts. \( \square \)

**Proof of Theorem 4.4.** Since \( \text{Ree}(q) \) is a 2-transitive subgroup of \( S_{q+1} \), we deduce from [1, Proposition 16] that

\[
\text{cr}_{q+1}(\text{Ree}(q)) \leq q^3 - 1.
\]

This together with Lemma 4.1 leads to Theorem 4.4.

5. Geometric Interpretations for the Covering Radii of \( \text{PGL}(2, q) \) and \( \text{PGU}(3, q) \)

Let us first describe in detail how the covering radius problem for \( \text{PGL}(2, q) \) relates to certain configurations in an associated Minkowski plane that was alluded to by Cameron and Wanless [1]. Consider the projective line \( \text{PG}(1, q) \) for which \( \text{PGL}(2, q) \) has its natural sharply 3-transitive action of degree \( q + 1 \). Consider the Segre variety \( S_{m,n}(q) \) lying in \( \text{PG}((m + 1)/(n + 1) - 1, q) \) obtained by taking the simple tensors of pairs of homogeneous coordinates of points of \( \text{PG}(m, q) \) and \( \text{PG}(n, q) \). So for example, \( S_{1,1}(q) \) is the hyperbolic quadric \( \mathbb{Q}^+(3, q) \) in projective 3-space, and \( S_{1,2}(q) \) is the Segre threefold in \( \text{PG}(5, q) \). We refer to [3, Chapter 4] for more on finite Segre varieties and their properties. Consider \( S_{1,1}(q) \). The Segre map in this instance is given by:

\[
\text{PG}(1, q) \times \text{PG}(1, q) \to S_{1,1}
\]

\[
((X_1, X_2), (X'_1, X'_2)) \mapsto (X_1, X_2) \otimes (X'_1, X'_2) = \begin{bmatrix} X_1 X'_1 & X_1 X'_2 \\ X_2 X'_1 & X_2 X'_2 \end{bmatrix}.
\]

We can identify the output with a row vector \((X_1 X'_1, X_1 X'_2, X_2 X'_1, X_2 X'_2)\); the homogeneous coordinates for a point of \( \text{PG}(3, q) \). Moreover, each element of \( S_{1,1} \) is a zero of the following quadratic form on \( \mathbb{F}_q^4 \):

\[
Q(X) = X_1 X_4 - X_2 X_3, \quad \text{for} \ X = (X_1, X_2, X_3, X_4) \in \mathbb{F}_q^4.
\]

The quadratic form \( Q \) gives rise to a hyperbolic quadric \( \mathbb{Q}^+(3, q) \) whose points are precisely those of \( S_{1,1} \). Consider a permutation \( \pi \in \text{Sym}(\text{PG}(1, q)) \). Then we can identify \( \pi \) with a set of ordered pairs in \( \text{PG}(1, q) \times \text{PG}(1, q) \):

\[
\pi \leftrightarrow \{(X, X^\pi) : X \in \text{PG}(1, q)\}.
\]

We can then apply the Segre map above to obtain

\[
\pi \to \text{Graph}(\pi) := \{X \otimes X^\pi : X \in \text{PG}(1, q)\}.
\]
Geometrically, Graph$(\pi)$ is an ovoid of $S_{1,1}$ because two points $X \otimes X^\pi$ and $Y \otimes Y^\pi$ are orthogonal under the associated bilinear form if and only if $X = Y$. In other words, Graph$(\pi)$ is a set of $q + 1$ points, no two orthogonal in $S_{1,1}$. Now we can easily distinguish the permutations that lie in PGL$(2, q)$. Recall that elements of PGL$(2, q)$ are Möbius transformations of the form

$$
\mu_{a,b,c,d} := t \mapsto \frac{at + b}{ct + d}, \quad a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0.
$$

So

$$
\text{Graph}(\mu_{a,b,c,d}) = \{(ct + d, at + b, ct^2 + dt, at^2 + bt) : t \in \mathbb{F}_q \cup \{\infty\}\},
$$

which is a conic section of $Q^+(3, q)$. The converse is also true. So in the bijection above, we have that PGL$(2, q)$ is orthogonal to $\pi \otimes \pi$ amongst all ovoids of $S_{1,1}$. Therefore, we have

$$
\text{cr}_{q+1}(\text{PGL}(2, q)) = \max_{O \in \text{ooids of } Q^+(3, q)} \left( \min_{C \in \text{conics of } Q^+(3, q)} |O \cap C| \right).
$$

Alternatively, we could use the language of Benz planes. The classical Minkowski plane $\mathcal{M}(q)$ consists of three types of objects: points, lines, and circles. The points of $\mathcal{M}(q)$ are just the points of $Q^+(3, q)$, the lines are the generators of $Q^+(3, q)$, and the circles are the non-tangent plane sections (i.e., conics) of $Q^+(3, q)$. Incidence is inherited from the ambient projective space. So to compute $\text{cr}_{q+1}(\text{PGL}(2, q))$, we need to consider sets of points of $\mathcal{M}(q)$ that meet every line once and every circle in at most $s$ points, and we seek to minimise $s$.

We will consider a special set of points lying in the Segre variety $S_{2,2}(q^2) \subseteq \text{PG}(8, q^2)$. Suppose we have an Hermitian form $\beta$ from $\mathbb{F}_q^3$ to $\mathbb{F}_q^2$. Define $\beta \otimes \beta$ to be the form on $\mathbb{F}_q^3$ defined by

$$(\beta \otimes \beta)(u_1 \otimes v_1, u_2 \otimes v_2) := \beta(u_1, u_2)\beta(v_1, v_2)$$

and extend linearly. This form is Hermitian. The set of totally isotropic points of $\beta \otimes \beta$ is a Hermitian variety which we will denote by $H(8, q^2)$.

Now consider the totally isotropic points of $\beta$; an Hermitian curve $H(2, q^2)$ of $q^3 + 1$ points of $\text{PG}(2, q^2)$. This variety gives us the natural 2-transitive action of PGU$(3, q)$. This time, we apply the Segre map to pairs of elements of the Hermitian curve:

$$
H(2, q^2) \times H(2, q^2) \rightarrow S_{2,2}
$$

$$(u, v) \mapsto u \otimes v.
$$

This map is injective, but not surjective. There is no natural name for its image, but we will call it $U_{2,2}$.

**Lemma 5.1.** Let $(P_1, P_2) \in H(2, q^2) \times H(2, q^2)$. Then the set of points orthogonal with or equal to $P_1 \otimes P_2$ (in $U_{2,2}$) is

$$
\{P_1 \otimes X : X \in H(2, q^2)\} \cup \{X \otimes P_2 : X \in H(2, q^2)\}.
$$

**Proof.** Suppose $X \otimes Y$ is orthogonal to $P_1 \otimes P_2$ with respect to $\beta \otimes \beta$. Then

$$
\beta(X, P_1)\beta(Y, P_2) = (\beta \otimes \beta)(X \otimes Y, P_1 \otimes P_2) = 0.
$$

1In terms of tensors, the bilinear form can be written as $B(u_1 \otimes u_2, v_1 \otimes v_2) = \beta(u_1, v_1)\beta(u_2, v_2)$, where $\beta$ is defined by $\beta(x, y) = x_1y_2 - x_2y_1$, and then extend $B$ linearly. So $0 = B(X \otimes X^\pi, Y \otimes Y^\pi) = \beta(X, Y)\beta(X^\pi, Y^\pi)$ implies $X = Y$ or $X^\pi = Y^\pi$, which are equivalent as $\pi$ is a bijection.

2In fact, the (hyper)plane of intersection has dual homogeneous coordinates $[b, -d, a, -c]$ and it is not difficult to calculate that this plane is non-degenerate with respect to the form $Q$. 


However, no two distinct points of the Hermitian curve are orthogonal, and so we must have either \( X = P_1 \) or \( Y = P_2 \).

Now consider a permutation \( f \in \text{Sym}(H(2, q^2)) \). Let \( \text{Graph}(f) \) be
\[
\{ X \otimes X^f : X \in H(2, q^2) \}
\]
as a subset of \( \mathcal{U}_{2,2} \). Then:

**Proposition 5.2.** For each \( f \in \text{Sym}(H(2, q^2)) \), the set \( \text{Graph}(f) \) is a set of \( q^3 + 1 \) totally isotropic points of \( H(8, q^2) \), no two distinct elements orthogonal in \( H(8, q^2) \).

**Proof.** Suppose two elements \( P \otimes P^f \) and \( Q \otimes Q^f \) are orthogonal with respect to the form \( \beta \otimes \beta \). Then
\[
\beta(P, Q)\beta(P^f, Q^f) = (\beta \otimes \beta)(P \otimes P^f, Q \otimes Q^f) = 0.
\]
and hence \( P = Q \).

The converse is also true.

**Proposition 5.3.** Let \( \{ P_i \otimes Q_i : i = 0, \ldots, q^3 \} \) be a set of points of \( \mathcal{U}_{2,2} \), no two distinct elements orthogonal. Then the map \( P_i \to Q_i \) is a bijection.

**Proof.** Suppose \( Q_i = Q_j \) for some \( i, j \in \{ 0, \ldots, q^3 \} \). Then
\[
(\beta \otimes \beta)(P_i \otimes Q_i, P_j \otimes Q_j) = \beta(P_i, P_j)\beta(Q_i, Q_j) = 0
\]
and hence \( i = j \). Therefore, the map \( P_i \to Q_i \) is a bijection.

So we define an *ovoid* of \( \mathcal{U}_{2,2} \) to be a set of \( q^3 + 1 \) elements, that are pairwise non-orthogonal. For example, the Frobenius automorphism \( h \) that we used in the proof of Theorem 1.2 gives rise to the ovoid \( \{ P \otimes P^h : P \in H(2, q^2) \} \) of \( \mathcal{U}_{2,2} \).

**Proposition 5.4.** For each \( f \in \text{PGU}(3, q) \), the set \( \text{Graph}(f) \) spans a projective 5-space of \( \text{PG}(8, q^2) \), and its perp with respect to \( H(8, q^2) \) is a plane \( \pi_f \) which does not intersect \( S_{2,2} \). Moreover, \( \pi_f \) is totally isotropic when \( q \) is even, and non-degenerate when \( q \) is odd.

**Proof.** The diagonal \( D := \{ P \otimes P : P \in \text{PG}(2, q^2) \} \) pertains to the projective points arising from the symmetric tensors of the vector space \( V := \mathbb{F}_q^3 \). The dimension of the symmetric square of \( \mathbb{F}_q^3 \) is \( \binom{3}{2} = 6 \), and therefore, \( D \) spans a projective 5-space. Moreover, as the Hermitian curve spans \( \text{PG}(2, q^2) \), we also know that \( \text{Graph}(1) \) spans the same 5-space, where 1 denotes the trivial element of \( \text{PGU}(3, q) \). Now \( \text{PGU}(3, q) \) acts on \( \{ \text{Graph}(f) : f \in \text{PGU}(3, q) \} \) in the following way:
\[
\text{Graph}(f)^g = \text{Graph}(fg).
\]

Note that since \( \text{PGU}(3, q) \) acts transitively on itself by right multiplication, we see that \( \text{PGU}(3, q) \) also acts transitively on \( \{ \text{Graph}(f) : f \in \text{PGU}(3, q) \} \). So every \( \text{Graph}(f) \) will span a space projectively equivalent to the space spanned by \( \text{Graph}(1) \).

Now let us consider the perp \( \pi \) of \( \text{Graph}(1) \). Take the alternating tensors \( A \) and construct projective points:
\[
A = \{ X \otimes Y - Y \otimes X : X, Y \in \text{PG}(2, q^2) \}.
\]
Then for all \( P, X, Y \in \text{PG}(2, q^2) \) we have
\[
(\beta \otimes \beta)(X \otimes Y - Y \otimes X, P \otimes P) = (\beta \otimes \beta)(X \otimes Y, P \otimes P) - (\beta \otimes \beta)(Y \otimes X, P \otimes P) = \beta(X, P)\beta(Y, P) - \beta(Y, P)\beta(X, P) = 0.
\]
Therefore, \( A \subseteq \pi \). However, we know that \( A \) has (algebraic) dimension \( \binom{3}{2} = 3 \) and hence \( A = \pi \). If \( q \) is even, then \( A \) is a subspace of \( D \), and hence totally isotropic. Otherwise, \( A \cap D \) is trivial and \( A \) is non-degenerate.

In analogy with the \( \text{PGL}(2, q) \) example from before, we will call \( \text{Graph}(f) \), where \( f \in \text{PGL}(3, q) \), a \textit{classical ovoid} of \( U_{2,2} \). So in the injection above, we have that \( \text{PGU}(3, q) \) is distinguished amongst all permutations of \( H(2, q^2) \) by taking classical ovoids amongst all ovoids of \( U_{2,2} \).

**Problem 5.5.** Find an incidence geometry characterisation of classical ovoids of \( U_{2,2} \).

We would then have a meaningful way to determine the following:

\[
\text{cr}_{q^3+1}(\text{PGU}(3, q)) = \max_{O \in \text{ovoids of } U_{2,2}} \left( \min_{C \in \text{classical ovoids of } U_{2,2}} |O \cap C| \right).
\]

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**REFERENCES**

[1] P. J. Cameron and I. M. Wanless, Covering radius for sets of permutations, *Discrete Math.*, 293 (2005), no. 1-3, 91–109.

[2] J. D. Dixon and B. Mortimer, *Permutation groups*, Springer-Verlag, New York, 1996.

[3] J. W. P. Hirschfeld and J. A. Thas, *General Galois geometries*, Springer Monographs in Mathematics, London, 2016.

[4] I. M. Wanless and X. Zhang, Transversals of Latin squares and covering radius of sets of permutations, *European J. Combin.*, 34 (2013), 1130–1143

[5] B. Xia, The covering radius of \( \text{PGL}_2(q) \), *Discrete Math.*, 340 (2017), no. 10, 2469–2471.

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