Equilibrium Behaviors in Reputation Games

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Abstract

We examine a patient player’s behavior when he can build a reputation in front of a sequence of myopic opponents. With positive probability, the patient player is a commitment type who mechanically plays his Stackelberg action in every period. We characterize the patient player’s action frequencies in equilibrium. Our results clarify the extent to which reputation effects can refine the patient player’s equilibrium behavior.

Keywords: reputation, equilibrium behavior, equilibrium refinement, Wald’s identity

JEL Codes: D82, D83

1 Introduction

Economists have long recognized that individuals, firms, and governments can benefit from good reputations. As shown in the seminal work of Fudenberg and Levine (1989), a patient player can guarantee himself a high payoff when facing a sequence of myopic opponents who believe that the patient player might be committed to play a particular action. Their result can be viewed as an equilibrium refinement, which in many games of interest, selects the patient player’s highest equilibrium payoff in repeated complete information games.

This paper studies the effects of reputations on the patient player’s behavior, which have been underexplored in the reputation literature. Different from existing works on behavior that restrict attention to particular equilibria or games with particular stage-game payoffs, we identify tight bounds on the patient player’s action frequencies that apply to all equilibria under general stage-game payoffs. The motivation of our approach is to derive predictions that can be tested by researchers who do not know which equilibrium players coordinate on. Our findings also clarify the extent to which reputation effects can refine the patient player’s behavior.

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1This includes for example, Bar-Isaac (2003), Phelan (2006), and Liu (2011). There is a separate strand of works that focus on games with particular stage-game payoff functions and examine players’ behaviors in finite horizon reputation games, such as Kreps and Wilson (1982), Milgrom and Roberts (1982), Barro (1986), and Schmidt (1993). By contrast, we study players’ behaviors in infinite horizon games under more permissive conditions on stage-game payoffs.
We analyze repeated games between a patient player and an infinite sequence of myopic opponents, arriving one in each period and each plays the game only once. The patient player is privately informed about his type, which is either a strategic type who maximizes his discounted average payoff, or a commitment type who mechanically plays his optimal pure commitment action (i.e., his Stackelberg action) in every period. The myopic players learn about the patient player’s type by observing all the actions taken in the past.

We examine the extent to which the option to imitate the commitment type can motivate the patient player to play his Stackelberg action. Theorem 1 characterizes tight bounds on the discounted frequencies (or frequencies) with which the strategic-type patient player plays his Stackelberg action in equilibrium. We show that the maximal frequency equals one and the minimal frequency equals the value of the following linear program that can be computed in polynomial time: Choose a distribution over action profiles in order to minimize the probability of the Stackelberg action subject to two constraints. First, each action profile in the support of this distribution satisfies the myopic player’s incentive constraint. Second, the patient player’s expected payoff from this distribution is no less than his Stackelberg payoff. Intuitively, the patient player can approximately attain his Stackelberg payoff by imitating the commitment type. In order to provide him an incentive not to play his Stackelberg action, his continuation value after separating from the commitment type must be at least his Stackelberg payoff.

The substantial part of our proof is to construct equilibria that approximately attain this lower bound when the patient player’s discount factor is close to one. In order to calculate the patient player’s action frequencies, we establish a discounted version of the Wald’s identity (Lemma A.5) that is of separate technical interest.

Theorem 2 builds upon Theorem 1 and identifies a sufficient condition under which a distribution over the patient player’s actions corresponds to his action frequency in some equilibrium of the reputation game. When the patient player’s Stackelberg payoff coincides with his highest equilibrium payoff in the repeated complete information game (such as the entry deterrence game and the product choice game), our sufficient condition is also necessary, in which case for every distribution over action profiles from which the patient player obtains his Stackelberg payoff, there exists an equilibrium of reputation game in which the patient player’s action frequencies coincide with this distribution. Our result implies that reputation effects cannot refine the patient player’s behavior beyond the fact that his equilibrium payoff is weakly greater than his Stackelberg payoff.

Our results are robust when there are multiple pure-strategy commitment types. We exclude mixed-strategy commitment types for two reasons. First, as shown in Fudenberg and Levine (1992), the patient player’s guaranteed payoff in reputation games with mixed-strategy commitment types can be strictly greater than his highest equilibrium payoff in the repeated complete information game. This goes against our interpretation that the presence of commitment type is an equilibrium refinement. Second, commitment types that play the same mixed action in every period are hard to rationalize using rational types that have reasonable stage-game payoffs.
Related Literature: Our paper contributes to the reputation literature by examining the patient player’s behavior. Our research question contrasts to the ones in Fudenberg and Levine (1989, 1992) that focus exclusively on players’ payoffs. Our result clarifies the extent to which reputation effects can refine the patient player’s behavior in repeated complete information games studied by Fudenberg, Kreps, and Maskin (1990).

Existing works that study players’ behaviors in reputation games focus on finite-horizon games or restrict attention to particular equilibria or particular payoff structures. For example, Kreps and Wilson (1982) and Milgrom and Roberts (1982) characterize sequential equilibria in finite horizon entry deterrence games. Schmidt (1993) characterizes Markov equilibria in repeated bargaining games. Phelan (2006), Ekmekci (2011), Liu (2011), and Liu and Skrzypacz (2014) restrict attention to games with monotone-supermodular payoffs or $2 \times 2$ games and characterize the patient player’s behavior in some particular equilibria. By contrast, we examine infinite horizon reputation games and characterize tight bounds on the patient player’s action frequencies that apply to all equilibria. Our results apply as long as the patient player’s Stackelberg payoff is strictly greater than his minmax payoff and the Stackelberg outcome is not a Nash equilibrium of the stage game. By contrast, the uninformed players can flexibly choose their actions in our model, and the reversibility of actions is crucial for our constructive proof. Pei (2020) provides sufficient conditions under which the patient player’s on-path behavior is the same in all equilibria of a reputation game. In contrast to our model that restricts attention to private value environments but allows for general stage-game payoffs, his result requires nontrivial interdependent values and restricts attention to games with monotone-supermodular payoffs.

Cripps et al. (2004) show that when the monitoring structure has full support, the myopic players eventually learn the patient player’s type and play converges to an equilibrium of the repeated complete information game. However, their results do not characterize the speed of convergence or players’ behaviors in finite time, and therefore, do not imply what are the discounted average frequencies with which the patient player plays each of his actions. By contrast, we focus on games with perfect monitoring and examine the patient player’s discounted action frequency. Our measure of the patient player’s behavior is continuous at infinity, and therefore, can be tested by researchers who can only observe players’ behaviors in a finite number of periods.

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3Our result requires an additional requirement, that the patient player has a unique Stackelberg action and the myopic players have a strict best reply against this Stackelberg action. This is satisfied for generic stage-game payoffs given that the stage game is finite.
2 Model

Time is discrete, indexed by \( t = 0, 1, 2, \ldots \). A patient player 1 with discount factor \( \delta \in (0, 1) \) interacts with an infinite sequence of myopic player 2s, arriving one in each period and each plays the game only once. In period \( t \), players simultaneously choose their actions \( (a_t, b_t) \in A \times B \), and receive stage-game payoffs \( u_1(a_t, b_t) \) and \( u_2(a_t, b_t) \). We assume that \( A \) and \( B \) are finite sets, with \( |A| \geq 2 \) and \( |B| \geq 2 \).

We introduce two assumptions on the payoff structure. Let \( BR_1 : \Delta(B) \rightarrow 2^A \setminus \{\emptyset\} \) and \( BR_2 : \Delta(A) \rightarrow 2^B \setminus \{\emptyset\} \) be player 1’s and player 2’s best reply correspondences in the stage-game. The set of player 1’s (pure) Stackelberg actions is \( \text{arg max}_{a \in A} \{\min_{b \in BR_2(a)} u_1(a, b)\} \).

**Assumption 1.** Player 1 has a unique Stackelberg action, and player 2 has a unique best reply against player 1’s Stackelberg action.

Since \( A \) and \( B \) are finite sets, Assumption 1 is satisfied for generic \( u_1 \) and \( u_2 \), for example, when each player has a strict best reply against each of his opponent’s pure actions. Let \( a^* \) be player 1’s Stackelberg action, and let \( b^* \) be player 2’s unique best reply against \( a^* \). We call \( u_1(a^*, b^*) \) player 1’s (pure) Stackelberg payoff.

Let

\[
\mathcal{B}^* \equiv \{\beta \in \Delta(B) | \exists \alpha \in \Delta(A) \text{ s.t. } \text{supp}(\beta) \subset \text{BR}_2(\alpha)\} \subset \Delta(B),
\]

(2.1)
which is the set of player 2’s mixed actions that best reply against some \( \alpha \in \Delta(A) \). Since player 2s are myopic, they will never take actions that do not belong to \( \mathcal{B}^* \). As a result, player 1’s minmax payoff is:

\[
\Sigma_1 \equiv \min_{\beta \in \mathcal{B}^*} \max_{a \in A} u_1(a, \beta).
\]

(2.2)

**Assumption 2.** \( u_1(a^*, b^*) \) is not a Nash equilibrium in the stage game and \( u_1(a^*, b^*) > \Sigma_1 \).

Assumption 2 requires player 1 to have a strict incentive to deviate from his Stackelberg action in the one-shot game, and can strictly benefit from committing to his Stackelberg action. This is satisfied (i) in the entry deterrence games of Kreps and Wilson (1982) and Milgrom and Roberts (1982), in which the incumbent’s Stackelberg action is to fight potential entrants, despite its stage-game payoff is strictly higher when it accommodates entry; (ii) in the product choice game of Mailath and Samuelson (2001, 2006), in which the seller’s Stackelberg action is to supply high quality, despite it can save cost by undercutting quality; (iii) in the monetary and fiscal policy games of Barro (1986) and Phelan (2006), in which the government’s Stackelberg action is to set low inflation rates or low tax rates, although it is tempted to raise inflation in order to boost economic activities or to raise taxes in order to increase tax revenue. Our assumption rules out coordination games such
as the battle of sexes in which \((a^*, b^*)\) is a Nash equilibrium of the stage game, and zero-sum games such as matching pennies and rock-paper-scissors.

Player 1 has private information about his type \(\omega\), which is perfectly persistent. Let \(\omega \in \{\omega^x, \omega^c\}\) in which \(\omega^x\) stands for a commitment type who mechanically plays \(a^x\) in every period, and \(\omega^c\) stands for a strategic type who can flexibly choose his actions in order to maximize his discounted average payoff \(\sum_{t=0}^{+\infty} (1 - \delta)^t u_1(a_t, b_t)\).

Player 2s’ prior belief attaches probability \(\pi \in (0, 1)\) to the commitment type.

Players’ past actions can be perfectly monitored. A typical public history is denoted by \(h' = \{a_s, b_s, \xi_s\}_{s=0}^{t-1}\), which consists of all actions taken in the past and the realizations of public randomization devices \(\xi_s \in [0, 1]\). Let \(\mathcal{H}^t\) be the set of \(h'\) and let \(\mathcal{H} \equiv \cup_{t \in \mathbb{N}} \mathcal{H}^t\). Strategic type player 1’s strategy is \(\sigma_1 : \mathcal{H} \to \Delta(A)\). Player 2s’ strategy is \(\sigma_2 : \mathcal{H} \to \Delta(B)\). Let \(\Sigma_1\) and \(\Sigma_2\) be the set of player 1’s and player 2s’ strategies, respectively.

The solution concept is (Bayes) Nash equilibrium. Let \(\text{NE}(\delta, \pi) \subset \Sigma_1 \times \Sigma_2\) be the set of Nash equilibria under parameter configuration \((\delta, \pi)\). Since the stage game is finite and the game is continuous at infinity, \(\text{NE}(\delta, \pi)\) is non-empty for every \(\delta\) and \(\pi\).

**Existing Result on Equilibrium Payoffs:** The reputation result in Fudenberg and Levine (1989) implies that a patient player 1 can secure his Stackelberg payoff in all equilibria. Formally, for every \(\pi > 0\) and \(\varepsilon > 0\), there exists \(\delta \in (0, 1)\) such that for every \(\delta > \delta\), we have:

\[
\inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} \mathbb{E}_{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{+\infty} (1 - \delta)^t u_1(a_t, b_t) \right] \geq u_1(a^*, b^*) - \varepsilon, \tag{2.3}
\]

where \(\mathbb{E}_{(\sigma_1, \sigma_2)}[\cdot]\) is the expectation operator when player 1 uses \(\sigma_1\) and player 2 uses \(\sigma_2\).

Inequality \((2.3)\) unveils the significant effects of reputations on the patient player’s payoff. Fudenberg and Levine (1989) view this reputation result as a refinement, which selects among the plethora of equilibria in repeated complete information games. According to the folk theorem result in Fudenberg, Kreps, and Maskin (1990), the patient player can attain any payoff between \(v_1\) and

\[
v_1 \equiv \max_{\text{supp}(\beta) \subset \text{BR}_2(\alpha)} \min_{\text{supp}(\alpha)} u_1(a, \beta) \tag{2.4}
\]

in a repeated complete information game against a sequence of myopic opponents. One can verify that \(v_1 \geq \)

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4Our result generalizes to models with multiple commitment types, as long as all commitment types play pure strategies. We exclude mixed-strategy commitment types since we focus on the refinement role of reputations. As shown in Fudenberg and Levine (1992), the patient player’s lowest equilibrium payoff in a reputation game with mixed-strategy commitment types can be strictly greater than his highest equilibrium payoff in the repeated complete information game, which goes against our interpretation.

5Establishing the common properties of all Nash equilibria is a common practice in the reputation literature. Despite we focus on Nash equilibria, the equilibria we construct can survive standard refinements and are not driven by suboptimal behaviors or unreasonable beliefs off the equilibrium path. For example, they are also Perfect Bayesian equilibria and sequential equilibria.
$u_1(a^*, b^*)$, which implies that introducing a commitment type that mechanically plays the Stackelberg action in every period selects equilibria in which player 1’s payoff is between $u_1(a^*, b^*)$ and $v_1$. In games with monotone-supermodular payoffs introduced in Definition 1 which include the product choice game and entry deterrence game as special cases, $v_1 = u_1(a^*, b^*)$, in which case the reputation model selects equilibria in which the patient player attains his highest equilibrium payoff.

3 Results

We examine the extent to which the option to build a reputation can encourage the patient player to play his Stackelberg action, as well as how reputation effects can refine his behavior. We focus on the discounted frequencies of the patient player’s actions. Formally, if the strategic type patient player’s strategy is $\sigma_1$ and player 2’s strategy is $\sigma_2$, then the discounted frequency of action $a \in A$ is:

$$G^{(\sigma_1, \sigma_2)}(a) \equiv \mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t \mathbb{I} \{a_t = a\} \right].$$

(3.1)

In Section 3.1 we characterize the range of discounted frequencies with which the patient player plays $a^*$ in equilibrium, i.e., the values of

$$\inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} G^{(\sigma_1, \sigma_2)}(a^*) \quad \text{and} \quad \sup_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} G^{(\sigma_1, \sigma_2)}(a^*)$$

(3.2)

when $\delta$ is close to 1. In Section 3.2 we characterize the set of action frequencies that can arise in equilibrium.

3.1 Frequency of the Stackelberg Action

When $\delta$ is above some cutoff, there exists an equilibrium $(\sigma_1, \sigma_2)$ in which $G^{(\sigma_1, \sigma_2)}(a^*) = 1$. For example, $\sigma_1(h') = a^*$ and $\sigma_2(h') = b^*$ at every on-path history $h'$. Once player 1 plays an action other than $a^*$, his continuation value equals his minmax payoff $v_1$. Such a punishment is feasible since player 1 separates from the commitment type after any deviation, and according to Fudenberg, Kreps, and Maskin (1990), there exists an equilibrium of the repeated complete information game in which player 1’s payoff is $v_1$. This grim trigger punishment provides the patient player an incentive to play $a^*$ given that $u_1(a^*, b^*) > v_1$.

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6By contrast, reputation results in models with mixed-strategy commitment types cannot be viewed as refinements, since the patient player’s guaranteed payoff in the reputation game can be strictly higher than $v_1$ (for example, in the product choice game).
Theorem 1 characterizes a tight lower bound on the frequency with which the patient player plays \( a^* \). Let
\[
\Gamma \equiv \left\{ (\alpha, b) \in \Delta(A) \times B \big| b \in \text{BR}_2(\alpha) \right\},
\]
which consists of pairs of \((\alpha, b)\) such that \( b \) best replies against \( \alpha \). Let
\[
F^*(u_1, u_2) \equiv \min_{(\alpha, \beta, b_1, b_2, q) \in \Delta(A) \times \Delta(A) \times B \times B \times [0, 1]} \left\{ q\alpha_1(a^*) + (1 - q)\alpha_2(a^*) \right\},
\]
subject to \((\alpha_1, b_1), (\alpha_2, b_2) \in \Gamma\), and
\[
qu_1(\alpha_1, b_1) + (1 - q)u_1(\alpha_2, b_2) \geq u_1(a^*, b^*),
\]
in which \( \alpha_i(a) \) is the probability of action \( a \in A \) in distribution \( \alpha_i \in \Delta(A) \) for every \( i \in \{1, 2\} \). One can verify that \( F^*(u_1, u_2) < 1 \) for every \( u_1 \) and \( u_2 \) that satisfy Assumption 2.

**Theorem 1.** Suppose \( u_1 \) and \( u_2 \) satisfy Assumptions 1 and 2. For every \( \pi \in (0, 1) \),
\[
\lim_{\delta \to 1} \inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} G^{(\sigma_1, \sigma_2)}(a^*) = F^*(u_1, u_2).
\]

Since players have access to a public randomization device, Theorem 1 implies that the frequency with which the patient player plays his Stackelberg action can be any number between \( F^*(u_1, u_2) \) and 1. When players’ stage-game payoffs satisfy Assumption 2 the value of \( F^*(u_1, u_2) \) is strictly less than 1, which implies that there exist equilibria in which an arbitrarily patient player plays his Stackelberg action with frequency bounded away from one despite having the option to build a reputation.

For some intuition behind \( F^*(u_1, u_2) \), notice that the presence of commitment type implies that the patient player can guarantee payoff approximately \( u_1(a^*, b^*) \) by imitating the commitment type. Therefore, he has an incentive to play actions other than \( a^* \) only if his continuation value after separating from the commitment type is no less than \( u_1(a^*, b^*) \). This explains the necessity of constraint (3.5).

The substantial part of our result is to establish the existence of equilibria in which the patient player plays \( a^* \) with frequency approximately \( F^*(u_1, u_2) \). This is nontrivial given that player 1’s mixed actions cannot be perfectly monitored, which implies that player 1 needs to be indifferent when he is supposed to mix. However, player 1’s indifference conditions are not in our linear program. This raises the concern that these indifference conditions may introduce additional constraints on players’ action frequencies. In addition, imperfect monitoring of mixed actions also implies that the patient player’s payoff from either \((\alpha_1, b_1)\) or \((\alpha_2, b_2)\) cannot be
attained in any equilibrium of the repeated game.

In order to illustrate the subtleties, consider the following game between a firm (row player) that chooses its effort level and a sequence of consumers (column player), each of them chooses whether to trust the firm or not:

|   | $T$ | $N$ |
|---|-----|-----|
| $H$ | 1,2 | −3,0 |
| $M$ | 2,1 | −2,0 |
| $L$ | 4,−2 | 0,0 |

Player 1’s Stackelberg action is $M$. His Stackelberg payoff is 2, which is also his highest equilibrium payoff in the repeated complete information game. Action profile $(\frac{1}{2}H + \frac{1}{2}L, T)$ belongs to $\Gamma$, from which player 1’s expected payoff is 2.5. According to (3.4) and (3.5), the value of $F^*$ is 0. Theorem 1 implies that there exist equilibria in which the patient player plays $M$ with frequency arbitrarily close to 0.

However, the conclusion that the patient player plays $M$ with frequency approximately zero seems to be at odds with the result that he can secure payoff approximately 2 in all equilibria. To see this, suppose there exists a history $h'$ at which the strategic-type patient player plays $M$ with zero probability at every $h' \succeq h'$, i.e., only $H$ and $L$ are played in the continuation game. The folk theorem result in Fudenberg, Kreps, and Maskin (1990) suggests that player 1’s continuation value at $h'$ is no more than 1, which is strictly lower than his Stackelberg payoff 2. This implies that at every history where the strategic type plays actions other than $M$ with positive probability, he needs to play $M$ in unboundedly number of periods after separating from the commitment type in order to receive a continuation payoff close to 2. This requirement seems to be in conflict with the one that the frequency of $M$ can be arbitrarily close to 0.

Our proof, which is in Appendix A, constructs equilibria in which the probability with which the patient player plays $M$ infinitely often is strictly positive, but can be arbitrarily close to zero. We explain the main ideas using the above example. Play starts from a phase in which player 2 plays $T$ and player 1 first mixes between $M$ and $L$, and then mixes between $H$ and $L$. If player 1 has played $L$ too frequently in the past, then the continuation play enters an absorbing phase after which player 1 never plays $M$ and his continuation value is at most 1. If player 1 has played $H$ or $M$ too frequently in the past, then the continuation play consists only of outcome $(M, T)$, from which player 1’s continuation value is 2. The calendar time at which play enters the absorbing phase as well as player 1’s continuation value at the absorbing phase depends on the history of player 1’s actions as well as his discount factor, which are constructed to provide him incentives to play his equilibrium mixed actions before play enters the absorbing phase.

An important step of our proof is to verify that the discounted frequency of action $M$ is indeed close to 0.
as $\delta$ approaches unity. We establish a discounted version of the Wald’s identity (Lemma A.5) that bounds the discounted average frequency of $M$ from above, which might be of separate technical interest.

**Compute $F^*(u_1, u_2)$:** We explain how to efficiently compute the value of $F^*(u_1, u_2)$, which is a key step toward applying Theorem 1. First, we show that $F^*(u_1, u_2)$ can be computed in polynomial time.

**Proposition 1.** The program that defines $F^*(u_1, u_2)$ can be computed in polynomial time.

The proof is in Appendix B. Intuitively, this is because for every $(b_1, b_2) \in B \times B$, one can rewrite the constrained optimization problem into one with $2|A| + 1$ variables, each taking value from 0 to 1, and $2|B| + 1$ constraints. This can be computed in polynomial time given that $A$ and $B$ are finite sets.

Next, we show that the constrained optimization problem can be further simplified when players’ stage-game payoff functions are monotone-supermodular, which fits into a number of applications such as entry deterrence, business transactions, and fiscal policies.

**Definition 1.** Players’ payoffs are monotone-supermodular if $A$ and $B$ are totally ordered sets such that $u_1(a, b)$ is strictly decreasing in $a$, and $u_2(a, b)$ has strictly increasing differences in $a$ and $b$.

Let $a$ be the lowest element in $A$ and let $b \in B$ be player 2’s best reply against $a$. If player 2 has multiple best replies against $a$, then pick the one that maximizes player 1’s payoff. Let

$$\Gamma^* \equiv \left\{ (\alpha, b) \in \Gamma | \text{BR}_2(\alpha) \geq 2 \text{ and } b \in \arg\max_{b' \in \text{BR}_2(\alpha)} u_1(\alpha, b') \right\},$$

(3.7)

which is a subset of $\Gamma$ that consists of strategy profiles $(\alpha, b)$ in which player 2’s incentive constraint is binding.

One can verify that under generic $(u_1, u_2)$, $\Gamma^*$ is a finite set.

**Proposition 2.** If players’ payoffs are monotone-supermodular, then:

$$F^*(u_1, u_2) = \min_{(\alpha_1, \alpha_2, b_1, b_2, q) \in \Delta(A) \times \Delta(A) \times B \times B \times [0, 1]} \left\{ q\alpha_1(a^*) + (1 - q)\alpha_2(a^*) \right\},$$

subject to $(\alpha_1, b_1), (\alpha_2, b_2) \in \Gamma^* \cup \{(a, b)\}$, and $qu_1(\alpha_1, b_1) + (1 - q)u_1(\alpha_2, b_2) \geq u_1(a^*, b^*)$.

Proposition 2 implies that in games with monotone-supermodular stage-game payoffs, it is without loss of generality to choose $(\alpha_1, b_1)$ and $(\alpha_2, b_2)$ from a finite subset of $\Gamma$, which consists of strategy profiles in which either player 2’s incentive constraint is binding or player 1’s lowest-cost action is played with probability.

7This definition resembles the one in Liu and Pei (2020) and Pei (2020) except that there is no state that affects players’ payoffs, and furthermore, we do not require $u_1$ to be strictly increasing in $b$. 
1. To understand why, suppose toward a contradiction that \{((\alpha_1, b_1), (\alpha_2, b_2), q) \} solves (3.4), player 2 has a strict incentive to play \( b_1 \) against \( \alpha_1 \), and \( \alpha_1 \) does not attach probability 1 to player 1’s lowest action. Then one can modify the distribution by slightly increasing the probability of \( a \) and decreasing the probability of other actions, after which player 1’s expected payoff strictly increases and the probability of action \( a^* \) strictly decreases. This contradicts the presumption that \{((\alpha_1, b_1), (\alpha_2, b_2), q) \} attains the minimum.

We illustrate how to apply our result using the product choice game in Mailath and Samuelson (2006). Players’ stage-game payoffs are given by:

\[
\begin{array}{ccc}
\text{ } & T & N \\
H & 1, 1 - \gamma^* & -b_N, 0 \\
L & 1 + b_T, -\gamma^* & 0, 0 \\
\end{array}
\]

in which \( b_T > 0 \) is player 1’s cost of playing \( H \) when player 2 plays \( T \), \( b_N \geq 0 \) is his cost of playing \( H \) when player 2 plays \( N \), and \( \gamma^* \in (0, 1) \) is the minimal probability with which player 1 needs to play \( H \) in order to induce player 2 to play \( T \). One can verify that this example satisfies Assumptions 1 and 2, and players’ payoffs are monotone-supermodular once player 1’s actions are ranked according to \( H \succ L \), and player 2’s actions are ranked according to \( T \succ N \).

According to (3.7), \( \Gamma^* \) contains only one action profile \((\gamma^*H + (1 - \gamma^*)L, T)\). Proposition 2 implies that:

\[
F^*(u_1, u_2) = \min_{q \in [0, 1]} q\gamma^*, \quad \text{subject to} \quad qu_1(\gamma^*H + (1 - \gamma^*)L, T) + (1 - q)u_1(L, N) \geq u_1(H, T),
\]

from which we obtain that \( q \geq \frac{1}{(1 - \gamma^*)b_T + 1} \) and therefore, \( F^*(u_1, u_2) = \frac{\gamma^*}{(1 - \gamma^*)b_T + 1} \).

3.2 Equilibrium Action Frequencies

Let

\[
\mathcal{A} \equiv \{ \alpha^* \in \Delta(A) \mid \exists q \in \Delta(\Gamma) \text{ such that } \alpha^* = \int_a \alpha dq \text{ and } \int_{(\alpha, b)} u_1(\alpha, b) dq = u_1(a^*, b^*) \}
\]

be the set of distributions over player 1’s actions such that for every \( \alpha^* \) that belongs to this set, one can find a distribution over action profiles \( q \in \Delta(\Gamma) \) from which player 1’s expected payoff equals his Stackelberg payoff and the marginal distribution over his actions coincides with \( \alpha^* \).

\textbf{Theorem 2.} Suppose \( u_1 \) and \( u_2 \) satisfy Assumptions 7 and 2. For every \( \alpha^* \in \mathcal{A} \) and \( \epsilon > 0 \), there exists
\(\delta \in (0,1)\) such that for every \(\delta > \delta\), there exists \((\sigma_1, \sigma_2) \in NE(\delta, \pi)\) such that:

\[
|G(\sigma_1, \sigma_2)(a) - \alpha^*(a)| < \varepsilon \text{ for every } a \in A. \tag{3.9}
\]

Suppose \(u_1\) and \(u_2\) satisfy Assumptions [1] and [2] and \(u_1(a^*, b^*) = \overline{v}_1\). For every \(\alpha' \notin A\), there exist \(\eta > 0\) and \(\delta \in (0,1)\) such that for every \(\delta > \delta\) and every \((\sigma_1, \sigma_2) \in NE(\delta, \pi)\), we have:

\[
|G(\sigma_1, \sigma_2)(a) - \alpha^*(a)| > \eta \text{ for some } a \in A. \tag{3.10}
\]

The proof is in Appendix A. According to Theorem 2, every action distribution that belongs to \(A\) coincides with the patient player’s action frequency in some equilibria of the reputation game. In games where \(u_1(a^*, b^*) = \overline{v}_1\), i.e., when reputation effects select the patient player’s highest equilibrium payoff in the repeated complete information game, an action distribution is the patient player’s discounted action frequency in some equilibrium if and only if it belongs to \(A\).

In terms of refining the patient player’s equilibrium behaviors for repeated complete information games, our result implies that when \(u_1(a^*, b^*) = \overline{v}_1\), reputation effects cannot refine the patient player’s behavior beyond the fact that his equilibrium payoff being weakly greater than his Stackelberg payoff.

4 Conclusion

We examine the effects of reputation on a patient informed player’s equilibrium behavior, which contrasts to the existing literature that focuses on the patient player’s equilibrium payoff. Our analysis focuses on the patient player’s discounted action frequencies and characterize tight bounds that apply to all equilibria in a broad class of games.

Our results imply that first, in games where the optimal commitment outcome is not a stage-game Nash equilibrium, the long-lived player may play his optimal commitment action with frequency bounded away from one no matter how patient he is. Second, when the patient player’s optimal commitment payoff coincides with his highest equilibrium payoff in the repeated complete information game, reputation effects cannot further refine the patient player’s behavior beyond that fact that his equilibrium payoff is at least his optimal commitment payoff.
A Proof of Theorems 1 and 2

Our proof consists of two parts. In Appendix [A.1] we show that in every equilibrium, the discounted average frequency with which a patient player plays $a^*$ cannot be strictly lower than $F^*(u_1, u_2)$. In another word,

$$\liminf_{\delta \to 1} \inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} G^{(\sigma_1, \sigma_2)}(a^*) \geq F^*(u_1, u_2).$$

(A.1)

We then establish the second statement of Theorem 2 that in games where $\overline{\nu}_1 = u_1(a^*, b^*)$, any action distribution that does not belong to $\mathcal{A}$ cannot be the patient player’s action frequencies in any equilibrium.

In Appendix [A.2] we provide a constructive proof to the sufficiency part of Theorem 1 and the first statement of Theorem 2. We construct a class of equilibria $\{(\sigma_1^\delta, \sigma_2^\delta)\}_{\delta \in (0,1)}$ in which $G^{(\sigma_1^\delta, \sigma_2^\delta)}(a^*)$ converges to $F^*(u_1, u_2)$ when $\delta$ goes to 1, which implies that:

$$\limsup_{\delta \to 1} \inf_{(\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)} G^{(\sigma_1, \sigma_2)}(a^*) \leq F^*(u_1, u_2).$$

(A.2)

More generally, for every $\alpha^* \in \mathcal{A}$, we construct a sequence of equilibria such that $G^{(\sigma_1^\delta, \sigma_2^\delta)}(a)$ converges to $\alpha^*(a)$ for every $a \in A$.

A.1 Part I: Necessity

Let $\Delta(\Gamma)$ be the set of probability distributions on $\Gamma$ whose support has countable number of elements. Let $F(u_1, u_2, \epsilon)$ be the value of the following constrained optimization problem:

$$F(u_1, u_2, \epsilon) \equiv \inf_{p \in \Delta(\Gamma)} \int \alpha(a^*)dp(\alpha, b),$$

subject to

$$\int u_1(\alpha, b)dp(\alpha, b) \geq u_1(a^*, b^*) - \epsilon.$$

Our proof of the necessity part of Theorem 1 consists of three lemmas, which together imply inequality (A.1).

Lemma A.1. For every $\pi > 0$ and $\epsilon > 0$, there exists $\overline{\delta} \in (0, 1)$ such that for every $\delta > \overline{\delta}$,

$$G^{(\sigma_1, \sigma_2)}(a^*) \geq F(u_1, u_2, \epsilon) - (1 - \overline{\delta}) \text{ for every } (\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi).$$

(A.5)

Lemma A.2. For every $u_1$ and $u_2$ that satisfy Assumptions 1 and 2 $\lim_{\epsilon \to 0} F(u_1, u_2, \epsilon) = F(u_1, u_2, 0)$.

Lemma A.3. For every $u_1$ and $u_2$ that satisfy Assumptions 1 and 2 $F^*(u_1, u_2) = F(u_1, u_2, 0)$.
Proof of Lemma A.1: The reputation result in Fudenberg and Levine (1989) implies that for every \( \pi > 0 \) and \( \varepsilon > 0 \), there exists \( \delta \in (0, 1) \) such that for every \( \delta > \delta \).

\[
\mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta) \delta^t u_1(a_t, b_t) \right] \geq u_1(a^*, b^*) - \varepsilon/2 \quad \text{for every } (\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi).
\]  

(A.6)

For given \((\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)\), let \( \mathcal{H}^* \) be a set of on-path histories such that \( h^* \in \mathcal{H}^* \) if and only if

- \( a^* \) was played from period 0 to \( t - 1 \), and \( \sigma_1(h^*) \) assigns positive probability to actions other than \( a^* \).

By construction, for every \( h^* \in \mathcal{H}^* \), player 2’s posterior belief at \( h^* \) assigns probability at least \( \pi \) to the commitment type, and therefore, player 1’s continuation value at \( h^* \) is at least \( u_1(a^*, b^*) - \varepsilon/2 \). Let \( M \equiv \max_{(a,b) \in A \times B} u_1(a,b) \). For every \( a \in \text{supp}(\sigma_1(h^*)) \setminus \{a^*\} \) and \( b \in \text{supp}(\sigma_2(h^*)) \), player 1’s continuation value at \( (h^*, a, b) \), denoted by \( v(h^*, a, b) \), satisfies:

\[
v(h^*, a, b) \geq \frac{1}{\delta} \left( u_1(a^*, b^*) - \frac{\varepsilon}{2} - (1 - \delta)M \right).
\]

The right-hand-side is strictly greater than \( u_1(a^*, b^*) - \varepsilon \) when \( \delta \) is close enough to 1. For every on-path history \( h^* \) such that \( h^* \succeq (h^*, a, b) \), player 2 attaches probability 1 to the rational type at \( h^* \), and therefore, \( \sigma_2(h^*) \) best replies against \( \sigma_1(h^*) \). Therefore, \( (\sigma_1(h^*), b) \in \Gamma \) for every \( b \in \text{supp}(\sigma_2(h^*)) \). Let \( p_{(h^*, a, b)} \in \Delta(\Gamma) \) be a probability measure on \( \Gamma \) such that for every \((\alpha, b) \in \Gamma\),

\[
p_{(h^*, a, b)}(\alpha, b) \equiv \mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{s=t+1}^{\infty} (1 - \delta) \delta^{s-t-1} 1\{\sigma_1(h^s) = \alpha\} \sigma_2(b) \right](h^*, a, b).
\]

(A.7)

By construction, \( p_{(h^*, a, b)} \) has a countable number of elements in its support, and player 1’s continuation value at \( (h^*, a, b) \), denoted by \( v(h^*, a, b) \), satisfies

\[
v(h^*, a, b) = \int u_1(\alpha, b) dp_{(h^*, a, b)}(\alpha, b) \geq u_1(a^*, b^*) - \varepsilon.
\]

(A.8)

The definition of \( F(u_1, u_2, \varepsilon) \) in (A.3) and (A.4) suggests that:

\[
G(h^*, a^*) \equiv \mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{s=t+1}^{\infty} (1 - \delta) \delta^{s-t-1} 1\{a_s = a^*\} \right](h^*, a, b) \geq F(u_1, u_2, \varepsilon).
\]

(A.9)

Next, we compute a lower bound on \( G(h^*, a^*) \). Let \( \tilde{\mathcal{H}} \) be the set of on-path histories \( h^t \equiv (h^{t-1}, a_t, b_t) \) such that \( t \geq 1 \), \( h^{t-1} \in \mathcal{H}^* \), and \( a_{t-1} \neq a^* \). Let \( p^{(\sigma_1, \sigma_2)}(h^t) \) be the ex ante probability of history \( h^t \) under the probability measure induced by \((\sigma_1, \sigma_2)\). By definition, \( 1 - \sum_{h^t \in \tilde{\mathcal{H}}} p^{(\sigma_1, \sigma_2)}(h^t) \) is the ex ante probability with
which player 1 plays \( a^* \) in every period conditional on him being the rational type. Therefore,

\[
G^{(\sigma_1, \sigma_2)}(a^*) = \left( 1 - \sum_{h' \in \mathcal{H}} p^{(\sigma_1, \sigma_2)}(h') + \sum_{h' \in \mathcal{H}} p^{(\sigma_1, \sigma_2)}(h') \left( 1 - \delta^{h-1} + \delta^h X^{(h)}(a^*) \right) \right.
\]

\[
\geq -(1 - \delta) + \left( 1 - \sum_{h' \in \mathcal{H}} p^{(\sigma_1, \sigma_2)}(h') \right) + \sum_{h' \in \mathcal{H}} p^{(\sigma_1, \sigma_2)}(h') \left( 1 - \delta^h + \delta^h X^{(h)}(a^*) \right) 
\]

\[
\geq F(u_{1}, u_{2}, \varepsilon) - (1 - \delta) \geq F(u_{1}, u_{2}, \varepsilon) - (1 - \delta) \quad (A.10)
\]

**Proof of Lemma A.2:** By definition, the value of \( F(u_{1}, u_{2}, \varepsilon) \) is a decreasing function of \( \varepsilon \) and is bounded by \([0, 1] \). Therefore, \( \lim_{\varepsilon \downarrow 0} F(u_{1}, u_{2}, \varepsilon) \) exists and moreover, \( \lim_{\varepsilon \downarrow 0} F(u_{1}, u_{2}, \varepsilon) \leq F(u_{1}, u_{2}, 0) \).

Next, we show that \( \lim_{\varepsilon \downarrow 0} F(u_{1}, u_{2}, \varepsilon) \geq F(u_{1}, u_{2}, 0) \). The optimization problem that defines \( F(u_{1}, u_{2}, \varepsilon) \) implies that for every \( \varepsilon > 0 \), there exists \( p_{\varepsilon} \in \Delta(\Gamma) \) that has countable number of elements in its support such that \( \int \alpha(a^*) d p_{\varepsilon}(\alpha, b) \leq F(u_{1}, u_{2}, \varepsilon) + \varepsilon \) and \( \int u_{1}(\alpha, b) d p_{\varepsilon}(\alpha, b) \geq u_{1}(a^*, b^*) - \varepsilon \).

According to Assumption \( \square \) there exists \( a' \in A \) such that \( u_{1}(a', b^*) > u_{1}(a^*, b^*) \). According to Assumption \( \blacksquare \) \( b^* \) is player 2’s strict best reply against \( a^* \). This implies the existence of \( \alpha^* \in \Delta(A) \) such that \( \alpha^*(a^*) \neq 1, b^* \in \text{BR}_2(\alpha^*) \), and \( u_{1}(\alpha^*, b^*) > u_{1}(a^*, b^*) \). Let \( \rho \equiv u_{1}(\alpha^*, b^*) - u_{1}(a^*, b^*) \). Since the support of \( p_{\varepsilon} \) is countable, there exists \( \alpha_{\varepsilon}^* \in \Delta(A) \) such that \( \alpha_{\varepsilon}^*(a^*) \neq 1, b^* \in \text{BR}_2(\alpha_{\varepsilon}^*) \), \( u_{1}(\alpha_{\varepsilon}^*, b^*) - u_{1}(a^*, b^*) \geq \frac{\rho}{2} \), and \( (\alpha_{\varepsilon}^*, b^*) \) does not belong to the support of \( p_{\varepsilon} \). We construct probability measure \( p_{\varepsilon}' \in \Delta(\Gamma) \) according to:

- \( p_{\varepsilon}'(\alpha_{\varepsilon}^*, b^*) \equiv \frac{2\varepsilon}{\rho + 2\varepsilon} \)
- \( p_{\varepsilon}'(\alpha, b) \equiv \frac{\rho}{\rho + 2\varepsilon} p_{\varepsilon}(\alpha, b) \) for every \( (\alpha, b) \) that belongs to the support of \( p_{\varepsilon} \).

By construction, \( \int u_{1}(\alpha, b) d p_{\varepsilon}'(\alpha, b) \geq u_{1}(a^*, b^*) \), and therefore,

\[
\frac{2\varepsilon}{\rho + 2\varepsilon} + \frac{\rho}{\rho + 2\varepsilon} \left( F(u_{1}, u_{2}, \varepsilon) + \varepsilon \right) \geq \int \alpha(a^*) d p_{\varepsilon}'(\alpha, b) \geq F(u_{1}, u_{2}, 0). \quad (A.11)
\]

This implies that

\[
\lim_{\varepsilon \downarrow 0} \left\{ \frac{2\varepsilon}{\rho + 2\varepsilon} + \frac{\rho}{\rho + 2\varepsilon} \left( F(u_{1}, u_{2}, \varepsilon) + \varepsilon \right) \right\} = \lim_{\varepsilon \downarrow 0} F(u_{1}, u_{2}, \varepsilon) \geq F(u_{1}, u_{2}, 0).
\]

**Proof of Lemma A.3:** The inequality that \( F^*(u_{1}, u_{2}) \geq F(u_{1}, u_{2}, 0) \) is implied by the definitions of \( F^*(u_{1}, u_{2}) \) and \( F(u_{1}, u_{2}, 0) \). In what follows, we show that \( F^*(u_{1}, u_{2}) \leq F(u_{1}, u_{2}, 0) \). For every \( \eta > 0 \), there exists \( p_{\eta} \in \Delta(\Gamma) \) that has countable number of elements in its support such that \( \int \alpha(a^*) d p_{\eta}(\alpha, b) \leq F(u_{1}, u_{2}, 0) + \eta \) and \( \int u_{1}(\alpha, b) d p_{\eta}(\alpha, b) \geq u_{1}(a^*, b^*) \). Let \( \Gamma_{\eta} \) be a countable subset of \( \Gamma \) that contains the support of \( p_{\eta} \). Consider
the following minimization problem:

\[ F_\eta \equiv \min_{p \in \Delta(\Gamma_\eta), (\alpha, b) \in \Gamma_\eta} \sum_{(\alpha, b) \in \Gamma_\eta} p(\alpha, b)\alpha(a^*), \]  

(A.12)

subject to

\[ \sum_{(\alpha, b) \in \Gamma_\eta} p(\alpha, b)u_1(\alpha, b) \geq u_1(a^*, b^*). \]  

(A.13)

By construction, \( F_\eta \leq \int \alpha(a^*)d\eta(\alpha, b) \leq F(u_1, u_2, 0) + \eta \). We show that \( F_\eta \) can be attained via a distribution that contains at most two elements in its support. The Lagrangian of the minimization problem is:

\[ \sum_{(\alpha, b) \in \Gamma_\eta} p(\alpha, b)\alpha(a^*) + \lambda \left( \sum_{(\alpha, b) \in \Gamma_\eta} p(\alpha, b)u_1(\alpha, b) - u_1(a^*, b^*) \right), \]  

(A.14)

where \( \lambda \) is the Lagrange multiplier. If constraint (A.13) is not binding, then the minimum is zero and is attained by a degenerate distribution. If constraint (A.14) is binding, then for every pair of elements \((\alpha, b)\) and \((\alpha', b')\) in the support of the minimand \( p_\eta^* \in \Delta(\Gamma_\eta) \),

\[ \alpha(a^*) + \lambda u_1(\alpha, b) = \alpha'(a^*) + \lambda u_1(\alpha', b). \]  

(A.15)

Label the elements in the support of \( p_\eta^* \) as \( \{(\alpha_i, b_i)\}_{i=1}^{\infty} \). Equation (A.15) implies that for every \( \alpha_i(a^*) \neq \alpha_j(a^*) \),

\[ \frac{u_1(\alpha_i, b) - u_1(\alpha_j, b)}{\alpha_i(a^*) - \alpha_j(a^*)} = -\frac{1}{\lambda}. \]  

(A.16)

Let

\[ \overline{u}_1 \equiv \sup_{(\alpha, b) \in \{(\alpha_i, b_i)\}_{i=1}^{\infty}} u_1(\alpha, b), \quad \underline{u}_1 \equiv \inf_{(\alpha, b) \in \{(\alpha_i, b_i)\}_{i=1}^{\infty}} u_1(\alpha, b), \]

\[ \overline{\alpha} \equiv \sup_{(\alpha, b) \in \{(\alpha_i, b_i)\}_{i=1}^{\infty}} \alpha(a^*), \quad \underline{\alpha} \equiv \inf_{(\alpha, b) \in \{(\alpha_i, b_i)\}_{i=1}^{\infty}} \alpha(a^*). \]

Equation (A.16) implies that

\[ \frac{\overline{u} - \underline{u}}{\overline{\alpha} - \underline{\alpha}} = -\frac{1}{\lambda}. \]

Let \( \gamma \in (0, 1) \) be such that \( \gamma \overline{u}_1 + (1 - \gamma)\underline{u}_1 = u_1(a^*, b^*) \). According to (A.16), we have \( \gamma \overline{\alpha} + (1 - \gamma)\underline{\alpha} = F_\eta \).

Since \( \Delta(A) \times B \) is compact, there exist \((\overline{\alpha}, \overline{b})\) and \((\underline{\alpha}, \underline{b})\) which are limit points of set \( \{(\alpha_i, b_i)\}_{i=1}^{\infty} \) such that \( u_1(\overline{\alpha}, \overline{b}) = \overline{u}_1, \overline{\alpha}(a^*) = \overline{\alpha}, u_1(\underline{\alpha}, \underline{b}) = \underline{u}_1, \) and \( \underline{\alpha}(a^*) = \underline{\alpha} \). Since player 2’s best reply correspondence is upper-hemi-continuous, \((\overline{\alpha}, \overline{b}), (\underline{\alpha}, \underline{b}) \in \Gamma \). Our analysis above suggests that there exists a distribution on \( \Gamma_\eta \cup \{(\overline{\alpha}, \overline{b}), (\underline{\alpha}, \underline{b}) \} \) with at most two elements in its support that satisfies constraint (A.4) and the value of the
objective function \((A.3)\) is at most \(F(u_1, u_2, 0) + \eta\).

Take a decreasing sequence of positive real numbers \(\{\eta_n\}_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} \eta_n = 0\). For every \(n \in \mathbb{N}\), there exists \(p_n \in \Delta(\Gamma)\) with at most two elements in its support that satisfies constraint \((A.4)\) and the value of the objective function is at most \(F(u_1, u_2, 0) + \eta_n\). Since \((\Delta(A_1) \times B)^2\) is compact, there exists a converging subsequence \(\{p_{n_k}\}_{k \in \mathbb{N}}\) such that its limit \(p^*\) has at most two elements in its support, satisfies constraint \((A.4)\), and the value of the objective function is at most \(F(u_1, u_2, 0)\). This implies that \(F^*(u_1, u_2) \leq F(u_1, u_2, 0)\).

**Proof of Statement 2 of Theorem 2:** Since \(\forall 1 = u_1(a^*, b^*)\), for every \(\varepsilon > 0\), there exists \(\delta \in (0, 1)\) such that player 1’s payoff in every equilibrium where \(\delta > \delta\) is no more than \(u_1(a^*, b^*) + \varepsilon\). Let

\[
\mathcal{A}^\varepsilon \equiv \left\{ \alpha^* \in \Delta(A) \mid \exists q \in \Delta(\Gamma) \text{ such that } \alpha^* = \int_{\alpha} \alpha dq \text{ and } \left| \int_{(\alpha, b)} u_1(\alpha, b) dq - u_1(a^*, b^*) \right| \leq \varepsilon \right\}. \quad (A.17)
\]

Lemma \(A.1\) implies that for every \(\alpha' \notin \mathcal{A}^\varepsilon\), there exist \(\eta > 0\) and \(\delta \in (0, 1)\) such that for every \(\delta > \delta\) and every \((\sigma_1, \sigma_2) \in \text{NE}(\delta, \pi)\), we have:

\[
\left| G^{(\sigma_1, \sigma_2)}(a) - \alpha^*(a) \right| > \eta \text{ for some } a \in A. \quad (A.18)
\]

The conclusion of Theorem 2 is obtained since \(\lim_{\varepsilon \to 0} \mathcal{A}^\varepsilon = \mathcal{A}^\emptyset\).

**A.2 Part II: Sufficiency**

First, we argue that in order to establish \((A.2)\), it is without loss of generality to focus on solutions of the constrained optimization problem under which constraint \((3.5)\) is binding. Suppose the constrained minimum of \((3.4)\) is attained by \(\{(\alpha_1, b_1), (\alpha_2, b_2), q\}\) in which \(q u_1(\alpha_1, b_1) + (1 - q)u_1(\alpha_2, b_2) > u_1(a^*, b^*)\). Since \(a^*\) is player 1’s unique Stackelberg action, for every \(a' \neq a^*\), there exists \(b' \in \text{BR}_2(a')\) such that \(u_1(a', b') < u_1(a^*, b^*)\). Let \(r \in [0, 1]\) be defined via:

\[
ru_1(a', b') + (1 - r) u_1(\alpha_1, b_1) + (1 - q) u_1(\alpha_2, b_2) = u_1(a^*, b^*).
\]

Consider an alternative distribution \(q' \in \Delta(\Gamma)\) that attaches probability \(r\) to \((a', b')\), probability \((1 - r)q\) to \((\alpha_1, b_1)\), and probability \(rq\) to \((\alpha_2, b_2)\). The probability of \(a^*\) is weakly lower under \(q'\) compared to that under \(q\), and constraint \((3.5)\) is binding. According to Lemma \(A.3\) there exists a distribution over action profiles supported on two elements under which constraint \((3.5)\) binds and minimizes \((3.4)\).

The above argument implies that the sufficient part of Theorem 1 is implied by the first statement of Theo-
Lemma A.4. For every $\alpha^* \in \mathcal{A}$ and $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that for every $\delta > \delta$, there exists $(\sigma_1, \sigma_2) \in NE(\delta, \pi)$ such that:

$$\left| \mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta) \delta^t \mathbf{1} \{ a_t = a \} \right] - \alpha^*(a) \right| < \varepsilon$$

for every $a \in A$. \hfill (A.19)

Our constructive proof of Lemma [A.4] hinges on the following concentration inequality, which can be viewed as a discounted version of the Wald’s identity. In the next lemma, we focus on the case that $Z_t$ takes positive value with positive probability. Note that the other case is trivial since the sum is always negative.

Lemma A.5. For every $\delta \in (0, 1)$, $c \geq 0$, and sequence of i.i.d. random variables $Z_t$ with finite support and mean $\mu < 0$, and $Z_t$ takes positive value with positive probability, we have:

$$\Pr \left[ \bigcup_{n=1}^{\infty} \left\{ \sum_{t=1}^{n} \delta^t Z_t \geq c \right\} \right] \leq \exp(-r^* \cdot c)$$

where $r^* > 0$ is the smallest positive real number such that $\mathbb{E}_{z \sim Z_1} \left[ \exp(r^* z) \right] = 1$.

Proof. Let $\gamma_{Z,t}(r) = \ln \mathbb{E}_{z \sim Z_t} \left[ \exp(r \delta^t) \right]$, and let

$$q_{Z,t}(z) = p_{Z}(z) \exp(r \delta^t - \gamma_{Z,t}(r)),$$

where $p_{Z}(z)$ is the probability mass function of random variable $Z$. One can verify that $q$ is a well-defined probability measure. For a sequence of random variables $Z^n \equiv \{Z_1, \ldots, Z_n\}$, we have

$$q_{Z^n, r}(z_1, \ldots, z_n) = p_{Z^n}(z_1, \ldots, z_n) \exp \left( \sum_{t=1}^{n} r \delta^t - \sum_{t=1}^{n} \gamma_{Z,t}(r) \right).$$

Let $s_n = \sum_{t=1}^{n} z_t \delta^t$, we have

$$q_{S^n, r}(s_n) = p_{S^n}(s_n) \exp \left( rs_n - \sum_{t=1}^{n} \gamma_{Z,t}(r) \right).$$

Since $q_{S^n, r}$ is a probability measure, we have

$$\mathbb{E} \left[ \exp \left( rs_n - \sum_{t=1}^{n} \gamma_{Z,t}(r) \right) \right] = 1. \hfill (A.20)$$
Let $\gamma(r) \equiv E_{\gamma \sim Z_t}[\exp(r\gamma)]$, we have $\gamma(0) = 1$ and $\gamma'(0) = E_{\gamma \sim Z_t}[\gamma] < 0$. Since $r^* > 0$ is the smallest positive real number such that $E_{\gamma \sim Z_t}[\exp(r^*\gamma)] = 1$, we have $\gamma(r) \leq 1$ for any $0 \leq r \leq r^*$. Since random variables $Z_t$ are i.i.d., we have

$$
\gamma_{Z_t}(r^*) = \ln E_{\gamma \sim Z_t}[\exp(r^*\gamma)] = \ln E_{\gamma \sim Z_t}[\exp(r^*\Delta)] \leq 0
$$

for every $t \geq 1$. By substituting $r = r^*$ in inequality (A.20), we have $E[\exp(r^*s)] \leq 1$.

Let $J$ be the stopping time that the sum $s_J$ first exceeds the threshold $c$, we have

$$
\Pr[s_J \geq c] \cdot E[\exp(r^*s_J)|s_J \geq c] \leq 1,
$$

which implies that

$$
\Pr \left[ \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{n} \Delta Z_t \geq c \right\} \right] = \Pr[s_J \geq c] \leq \exp(-r^* \cdot c).
$$

Back to the proof of Lemma A.4. According to Assumption 2, there exists $\alpha' \in \Delta(A)$ such that $\alpha'(a^*) \neq 1$, $b^* \in BR_2(\alpha')$, and $u_1(\alpha', b^*) > u_1(a^*, b^*)$. Let $a' \neq a^*$ such that $p \equiv \alpha^*(a') > 0$ and $u_1(a', b^*) > u_1(a^*, b^*)$, and let $b' = BR_2(a')$. Since $a^*$ is player 1’s Stackelberg action and $a' \neq a^*$, we have $b' \neq b^*$.

For given $\alpha^* \in \mathcal{A}$, let $q \in \Delta(\Gamma)$ be such that $\alpha^* = \int_{\mathcal{A}} \alpha dq$ and $\int_{\alpha,b} u_1(\alpha, b) dq = u_1(a^*, b^*)$, and let $\varepsilon_1 > 0$ be a small positive real number. Let $Z_1 = u_1(a^*, b^*) - u_1(a, b)$ be a random variable that

- equals $u_1(a^*, b^*) - u_1(a^*, b^*)$ with probability $\varepsilon_1 \alpha'(a^*)$,
- equals $u_1(a^*, b^*) - u_1(a', b^*)$ with probability $\varepsilon_1 \alpha'(a')$,
- with probability $1 - \varepsilon_1$, equals $u_1(a^*, b^*) - u_1(a, b)$ where $(a, b)$ is distributed according to $q$.

One can verify that $Z_1$ has finite support and $E[Z_1] < 0$. Let $r_1^+ > 0$ be the smallest real number such that $E_{\gamma \sim Z_1}[\exp(r_1^+ \gamma)] = 1$.

Similarly, let $Z_2 = u_1(a, b) - \varepsilon_1$ be the random variable that:

- equals $u_1(a^*, b^*) - \varepsilon_1$ with probability $\varepsilon_1 \alpha'(a^*)$,
- equals $u_1(a', b^*) - \varepsilon_1$ with probability $\varepsilon_1 \alpha'(a')$,
- with probability $1 - \varepsilon_1$, equals $u_1(a, b) - \varepsilon_1$ where $(a, b)$ is distributed according to $q$.

Let $r_2^+ > 0$ be the smallest real number such that $E_{\gamma \sim Z_2}[\exp(r_2^+ \gamma)] = 1$. Let $T_1 = \left[ \frac{\gamma_{Z_1}}{u_1(a^*, b^*) - u_1(a, b)} \right]$ where $c \in \mathbb{R}_+$ is such that $\exp(-\min\{r_1^+, r_2^+\} \cdot c) \leq \varepsilon_1$. Note that when $\delta = 1$, the inequality holds with equality, which is the celebrated Wald’s identity established in \textit{Wald} (1944).

\textit{Here} we consider the case that the random variable $Z_2$ takes positive value with positive probability. As will become clearer in the analysis, the case when $Z_t$ only has non-positive support is trivial. We made the same assumption for $Z_2$ as well.
In what follows, we construct an equilibrium when \( \delta > \overline{\delta} \) with
\[
\bar{\delta} = \max \left\{ \frac{\ln(1-\varepsilon^3_1)}{\ln T_1}, 1-\varepsilon^2_1 \right\}. \tag{A.21}
\]

- Play starts from a preparation phase in which player 1 plays \( \alpha' \) and player 2 plays \( b^* \). This phase continues as long as action profile \( (a^*, b^*) \) was played in all previous periods.

- If \( (a', b^*) \) has been played before, then the following strategies are repeatedly played for infinitely many times, and we refer to each repetition as a stage.

1. In the beginning of each stage, both players follow strategy profile \( (\alpha', b^*) \) for \( T_1 \) periods.
2. If action profile \( (a', b^*) \) is observed for fewer than \( T_1 \) periods, then jump to Step 4.
3. If action profile \( (a', b^*) \) is observed for \( T_1 \) periods, play enters a random walk phase. Let \( \bar{T}_2 \equiv \left\lceil \frac{\ln(1-\varepsilon_1)}{\ln \delta} \right\rceil \). For every integer \( t \in [1, \bar{T}_2] \), if player 1’s discounted payoff in the random walk phase is at least \( (1-\delta^t)u_1(a^*, b^*) - c(1-\delta) \) and at most \( (1-\delta^t)(\varepsilon_1 u_1(\alpha', b^*) + (1-\varepsilon_1) E_{(\alpha, b)} q[u_1(\alpha, b)] + \varepsilon_1) + c(1-\delta) \), both players follow action profile \( (\alpha', b^*) \) with probability \( \varepsilon_1 \) and follow the distribution over action profiles \( q \) with probability \( 1-\varepsilon_1 \), dictated by the realization of the public randomization device. Otherwise, the random walk phase stops and let \( T_2 \leq \bar{T}_2 \) be the stopping time.
4. Both players follow action profile \( (a', b') \) in current stage until time \( T \) such that the discounted payoff of agent 1 in current stage is \( (1-\delta^T)u_1(a^*, b^*) \). When \( T \) satisfying the requirement is not an integer, let \( \xi_T \equiv T - \lfloor T \rfloor \), players use the public randomization device \( \xi \sim U[0,1] \) to dictate the continuation play. Both players follow action profile \( (a', b') \) if \( \xi \leq \xi_T \). Otherwise, play enters the next stage.

- At every off-path history, player 2s have ruled out the possibility that player 1 is the commitment type, and the continuation play delivers player 1 his minmax payoff. This is feasible given the folk theorem result in Fudenberg, Kreps, and Maskin (1990).

In the above construction, the discounted payoff for player 1 in each stage equals \( (1-\delta^T)u_1(a^*, b^*) \), in which \( T \in \mathbb{N} \) is the number of time periods in the stage. This implies that the strategic type has an incentive to play the mixed action in the beginning of the game to separate from the commitment type. In addition, one can verify that player 1 has no incentive to make any off-path deviations, since his expected continuation value at every on-path history is strictly greater than \( \nu_1 \) when \( \delta \) is sufficiently close to 1.
In what follows, we show that inequality (A.19) holds, which is sufficient to imply the desired conclusion. Let $\mathcal{E}_1$ be the event that player 1’s discounted payoff in the random walk phase is less than $(1 - \delta^t)u_1(a^*, b^*) - c(1 - \delta)$. Let $\mathcal{E}_2$ be the event that player 1’s discounted payoff in the random walk phase is more than $(1 - \delta^t)(\varepsilon_1 u_1(a^*, b^*) + (1 - \varepsilon_1)E_{(a, b) \sim q}[u_1(\alpha, b)] + \varepsilon_1) + c(1 - \delta)$. First, the probability that event $\mathcal{E}_1$ happens is bounded from above by the probability that $\sum_{t=1}^{T} \delta^t z_{1t}$ is greater than $c$ for some $n \geq 1$ where $z_{1t} \sim Z_1$ for all $t$. According to Lemma A.5, the latter probability is bounded from above by $\exp(-r_1^* \cdot c) \leq \varepsilon_1$, which implies that $\Pr[\mathcal{E}_1] \leq \varepsilon_1$. Similarly, we have $\Pr[\mathcal{E}_2] \leq \varepsilon_1$. Let $\mathcal{E}_3$ be the event that action profile $(a^*, b^*)$ is observed for $T_1$ periods, and by definition we have $\Pr[\mathcal{E}_3] = p^{T_1}$.

We first show that $G^{\sigma_1, \alpha_1}(a) \leq \alpha^*(a) + \varepsilon$ for every $a \in A$. Let $G$ denote the discounted number of times action $a$ is chosen from the beginning of each stage. By construction, we have

$$G \leq (1 - \delta^{T_1}) + (1 - p^{T_1} \cdot (1 - 2\varepsilon_1)) \cdot \delta^{T_1} G + (1 - 2\varepsilon_1) \cdot p^{T_1} \delta^{T_1+T_2} G + p^{T_1} \delta^{T_1} (1 - \delta^{T_2}) (\varepsilon_1 + (1 - \varepsilon_1)\alpha^*(a))$$

$$\Rightarrow G \leq \frac{1 - \delta^{T_1} + p^{T_1} \delta^{T_1} (1 - \delta^{T_2}) (\varepsilon_1 + (1 - \varepsilon_1)\alpha^*(a))}{(1 - 2\varepsilon_1)(1 - \delta^{T_2}) \delta^{T_1} p^{T_1} + (1 - \delta^{T_1})} \leq \frac{\alpha^*(a) + \varepsilon_1}{1 - 2\varepsilon_1}.$$  

To interpret the above inequalities, the first term in the first inequality is the upper bound on the discounted number of times action $a$ is chosen from time 1 to $T_1$; the second term is the upper bound on the discounted number of times action $a$ is chosen in future stages conditional on event $(\mathcal{E}_1 \cup \mathcal{E}_2)$ happens; the third term is the upper bound on the discounted number of times action $a$ is chosen in future stages conditional on event $\neg (\mathcal{E}_1 \cup \mathcal{E}_2)$ happens; and the last term is the upper bound on the discounted number of times action $a$ is chosen in the random walk phase. The second inequality holds by rearranging the terms. By setting the parameter $\varepsilon_1 < p^{T_1}$, the last inequality holds since $1 - \delta^{T_1} \leq \varepsilon_1^{T_1}$ and $1 - \delta^{T_2} \approx \varepsilon_1$. Therefore, we have

$$\mathbb{E}^{(\sigma_1, \alpha_1)} \left[ \sum_{t=0}^{\infty} (1 - \delta^t) \delta^t 1\{a_t = a\} \right] \leq \sum_{t=0}^{\infty} p(1 - p)^t (1 - \delta^t + \delta^t G) \leq \frac{(1 - p)(1 - \delta)}{1 - (1 - p)\delta} G + \frac{\alpha^*(a) + \varepsilon_1}{(1 - (1 - p)\delta)(1 - 2\varepsilon_1)} \leq \alpha^*(a) + \varepsilon,$$

where the last inequality holds for sufficiently small $\varepsilon_1 \ll \varepsilon$.

Next we show that $G^{\sigma_1, \alpha_1}(a) \geq \alpha^*(a) - \varepsilon$ for any action $a$. First we provides upper bounds for the stopping
time $T$ in different events. When event $\mathcal{E}_2 \cap \mathcal{E}_3$ happens, the stopping time $T$ satisfies

\[
(1 - \delta^{T_1 + T_2})M + \delta^{T_1}(1 - \delta^T)u_1(a', b') + (1 - \delta T)u_1(a', b') 
\geq (1 - \delta^{T_1 + T_2})u_1(a', b') - \delta^{T_1 + T_2}(u_1(\alpha', b') - (1 - \delta^{T_1 + T_2})M) 
\geq \delta^{T_1 + T_2} - \frac{(1 - \delta^{T_1 + T_2})M}{u_1(a', b') - u_1(\alpha', b')} \geq \delta^{T_1 + T_2} - \frac{(1 - \delta^{T_1 + T_2})M}{u_1(a', b') - u_1(\alpha', b')} \]

(A.22)

When event $(-\mathcal{E}_2) \cap \mathcal{E}_3$ happens, the stopping time $T$ satisfies

\[
(1 - \delta^{T_1})M + \delta^{T_1}(1 - \delta^T)u_1(\alpha', b') + (1 - \delta T)u_1(a', b') 
\geq (1 - \delta^{T_1})u_1(a', b') - \delta^{T_1}(u_1(\alpha', b') - (1 - \delta^{T_1})M) - c(1 - \delta) + \delta^{T_1 + T_2}(1 - \delta^{T_1 - T_2}u_1(a', b')) 
\geq \delta^{T_1 + T_2} - \frac{\delta^{T_1 + T_2}(u_1(a', b') - (1 - \delta^{T_1})M - c(1 - \delta) - \delta^{T_1}(1 - \delta^{T_2})(\varepsilon_1 u_1(\alpha', b') + \varepsilon_1)}{u_1(a', b') - u_1(\alpha', b')} \geq \delta^{T_1 + T_2} - \frac{\varepsilon_1^2(1 - \delta^{T_1 + T_2})(2\bar{M} + c)}{u_1(a', b') - u_1(\alpha', b')} \]

(A.23)

Finally when event $-\mathcal{E}_3$ happens, the stopping time $T$ satisfies

\[
(1 - \delta^{T_1})M + \delta^{T_1}(1 - \delta^{T_1 - T_1})u_1(a', b') \geq (1 - \delta^{T_1})u_1(a', b') 
\Rightarrow \delta^{T_1} \geq \frac{u_1(a', b') - \delta^{T_1}u_1(\alpha', b') - (1 - \delta^{T_1})M}{u_1(a', b') - u_1(\alpha', b')} 
\geq \delta^{T_1} - \frac{(1 - \delta^{T_1})M}{u_1(a', b') - u_1(\alpha', b')} \]

(A.24)

Let $G$ denote the discounted number of times action $a$ is chosen from the beginning of each stage. By construction, we have

\[
G \geq (1 - p^{T_1})(\delta^{T_1} - \frac{(1 - \delta^{T_1})M}{u_1(a', b') - u_1(\alpha', b')})G + p^{T_1}(1 - \varepsilon_1)(\delta^{T_1 + T_2} - \frac{\varepsilon_1^2(1 - \delta^{T_1 + T_2})(2\bar{M} + c)}{u_1(a', b') - u_1(\alpha', b')}G) + p^{T_1} \varepsilon_1(\delta^{T_1 + T_2} - \frac{(1 - \delta^{T_1 + T_2})M}{u_1(a', b') - u_1(\alpha', b')})G + p^{T_1} \delta^{T_1}(1 - \delta^{T_2})(1 - \varepsilon_1)\alpha^*(a) 
\Rightarrow G \geq \frac{p^{T_1} \delta^{T_1}(1 - \delta^{T_2})(1 - \varepsilon_1)\alpha^*(a) + \alpha^*(a)(1 - \varepsilon_1)}{1 + O(\varepsilon_1)} \geq \frac{\alpha^*(a)(1 - \varepsilon_1)}{1 + O(\varepsilon_1)}. 
\]

The first term in the first inequality is the lower bound on the discounted number of times action $a$ is chosen in future stages conditional on event $-\mathcal{E}_3$ happens; the second term is the lower bound on the discounted number of times action $a$ is chosen in future stages conditional on event $\mathcal{E}_3 \cap (-\mathcal{E}_2)$ happens; the third term is the
lower bound on the discounted number of times action \(a\) is chosen in future stages conditional on event \(\mathcal{E}_3 \cap \mathcal{E}_2\) happens; and the last term is the lower bound on the discounted number of times action \(a\) is chosen in random walk phase. Finally, we have

\[
\mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t 1\{a_t = a\} \right] \geq \sum_{t=0}^{\infty} p(1-p)^t \delta^t G = \frac{\alpha^*(a)(1 - \epsilon_1)}{(1 - (1-p)\delta)(1 + O(\epsilon_1))} \geq \alpha^*(a) - \epsilon.
\]

where the last inequality holds for sufficiently small \(\epsilon_1 \ll \epsilon\). Combining these bounds, we have

\[
|\mathbb{E}^{(\sigma_1, \sigma_2)} \left[ \sum_{t=0}^{\infty} (1 - \delta)^t 1\{a_t = a\} \right] - \alpha^*(a)| \leq \epsilon.
\]

for every action \(a \in A\).

### B Proof of Proposition 1

Let

\[
B^* \equiv \left\{ b \in B \left| \text{there exists } \alpha \in \Delta(A) \text{ such that } b \in BR_2(\alpha) \right. \right\}. \quad \text{(B.1)}
\]

For every \((b_1, b_2) \in B^* \times B^*\), consider the following linear program with \(2|A| + 1\) choice variables and \(2|B^*| + 1\) constraints:

\[
F^{**}(u_1, u_2, b_1, b_2) \equiv \min_{\{a_1(a)\}_{a \in A}, \{a_2(a)\}_{a \in A}, q} \left\{ q \cdot a_1(a^*) + (1 - q) a_2(a^*) \right\}
\]

s.t. \(q \sum_{a \in A} a_1(a) \cdot u_1(a, b_1) + (1 - q) \sum_{a \in A} a_2(a) \cdot u_1(a, b_2) \geq u_1(a^*, b^*) \)

\[
\sum_{a \in A} a_1(a) \cdot u_1(a, b_1) \geq \sum_{a \in A} a_1(a) \cdot u_1(a, b), \quad \forall b \in B
\]

\[
\sum_{a \in A} a_2(a) \cdot u_1(a, b_2) \geq \sum_{a \in A} a_2(a) \cdot u_1(a, b), \quad \forall b \in B
\]

This implies that \(F^{**}(u_1, u_2, b_1, b_2)\) can be solved in time polynomial in \(|A|\) and \(|B^*|\). The program that defines \(F^*(u_1, u_2)\) can be solved by taking the maximum of \(F^{**}(u_1, u_2, b_1, b_2)\) while varying \((b_1, b_2) \in B^* \times B^*\). This can also be computed in polynomial time since there are at most \(|B^*|^2\) pairs of \((b_1, b_2)\).
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