Intrinsic Diophantine approximation on circles and spheres

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Abstract
We study Lagrange spectra arising from intrinsic Diophantine approximation of circles and spheres. More precisely, we consider three circles embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$ and three spheres embedded in $\mathbb{R}^3$ or $\mathbb{R}^4$. We present a unified framework to connect the Lagrange spectra of these six spaces with the spectra of $\mathbb{R}$ and $\mathbb{C}$. Thanks to prior work of Asmus L. Schmidt on the spectra of $\mathbb{R}$ and $\mathbb{C}$, we obtain as a corollary, for each of the six spectra, the smallest accumulation point and the initial discrete part leading up to it completely.

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\section{INTRODUCTION}

In the classical Diophantine approximation, we study approximations of irrational numbers by rational numbers. For any irrational number $\alpha$, there exists an $L \geq 1$ such that the inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{Lq^2}$$

admits infinitely many integer solutions $p$ and $q$ that are relatively prime. The \textit{Lagrange number} $L(\alpha)$ of $\alpha$ is defined to be the supremum of such $L$'s. Equivalently, we have

$$L(\alpha) = \limsup_{p/q \in \mathbb{Q}} \left( q^2 |\alpha - \frac{p}{q}| \right)^{-1}.$$
The Lagrange spectrum $\mathcal{L}$ is the set of Lagrange numbers (that are finite) of all irrational numbers. Many properties of $\mathcal{L}$ are known. For example, $\sqrt{5}$ is the smallest value of $\mathcal{L}$, called the Hurwitz constant, and 3 is the smallest limit point of $\mathcal{L}$ in $\mathbb{R}$. See [1, 3, 5] for details.

On the other hand, structures of Lagrange spectra arising from intrinsic Diophantine approximation are less known. After introducing a few notations, we briefly review some existing results on intrinsic Diophantine approximation, focusing on the case of unit spheres $S^n$ in $\mathbb{R}^{n+1}$. Let $(X, d)$ be a complete metric space. Assume that $Z$ is a countable dense subset of $X$. In addition, we assume that there is a height function $H_t_Z : Z \to \mathbb{R}_{\geq 0}$. Given the data $(X, d, Z, H_t_Z)$, we define the Lagrange number $L(X, d, Z, H_t_Z)(P)$ of $P \in X \setminus Z$ to be

$$L(X, d, Z, H_t_Z)(P) = \limsup_{Z \in Z} \frac{1}{H_t_Z(Z) d(P, Z)}.$$

Also, we define the Lagrange spectrum to be

$$\mathcal{L}(X, d, Z, H_t_Z) = \{L(X, d, Z, H_t_Z)(P) \mid P \in X \setminus Z, \ L(X, d, Z, H_t_Z)(P) > 0\}.$$ 

When the choices of $d$ and $H_t_Z$ are clear from the context, we will write $L(X, Z)(P)$ instead of $L(X, d, Z, H_t_Z)(P)$. Furthermore, we may simply write $L(P)$ whenever the data $(X, d, Z, H_t_Z)$ are implicitly understood and thus there is no danger of confusion. Similarly, we will write $\mathcal{L}(X, Z)$ when the context makes clear the choices of $d$ and $H_t_Z$.

Using these notations, if we let $X = \mathbb{R}$, equipped with the usual Euclidean distance $d$, and $Z = \mathbb{Q}$ with the height function on $\mathbb{Q}$ being $H_t_\mathbb{Q}(p/q) = |q|^2$, the case $(X, Z) = (\mathbb{R}, \mathbb{Q})$ corresponds to the classical Lagrange spectrum studied by Markoff in [11, 12].

Particularly relevant to the present article is the intrinsic Diophantine approximation of $n$-spheres $(X, Z) = (S^n, S^n \cap Q^{n+1})$. Here,

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

is the unit $n$-sphere in $\mathbb{R}^{n+1}$. The distance $d$ on $S^n$ is the Euclidean distance inherited from the ambient space $\mathbb{R}^{n+1}$. Also, we define the height function on $S^n \cap Q^{n+1}$ to be

$$H_{S^n \cap Q^{n+1}}(p/q) = |q|$$

with $p \in Z^{n+1}$ primitive, meaning that all coefficients of $p$ have no common divisor $> 1$.

With respect to these data $(S^n, d, S^n \cap Q^{n+1}, H_{S^n \cap Q^{n+1}})$, it appears that Kopetzky was the first person to determine the initial discrete part of the spectrum $\mathcal{L}(S^1, S^1 \cap Q^2)$ of the 1-sphere. He obtained his results in [9, 10] by relying on prior results [14, 15] of Asmus L. Schmidt. In [4], the authors of the present paper investigated $\mathcal{L}(S^1, S^1 \cap Q^2)$ independently of Kopetzky’s results and determined the structure of the initial discrete part of $\mathcal{L}(S^1, S^1 \cap Q^2)$ more explicitly by using a different set of tools.

For $n \geq 2$, much less is known about $\mathcal{L}(S^n, S^n \cap Q^{n+1})$. Moshchevitin [13] found that the minimum of $\mathcal{L}(S^2, d, S^n \cap Q^3, H_{S^n \cap Q^3})$ is greater than or equal to $\frac{\sqrt{3}}{2}$. In [8], Kleinbock and Merrill made an important contribution to the study of implicit Diophantine approximation of the $n$-sphere for general $n \geq 2$. See also [6]. We should note here that Kleinbock and Merrill use the sup norm $d_{\sup}(x, y) := \sup_{i=0}^n |x_i - y_i|$ on $S^n$, not the Euclidean distance $d$, while they use the same
TABLE 1 Definitions of the six spaces \((\mathcal{X}, \mathcal{Z})\). In all cases, the distance \(d\) on \(\mathcal{X}\) is the Euclidean distance in the ambient spaces \(\mathbb{R}^n\) of \(\mathcal{X}\). And \(\text{Ht}_{\mathcal{Z}}\) is given by \(\text{Ht}_{\mathcal{Z}}(p/q) = |q|\) with primitive \(p \in \mathbb{Z}^l\).

| \(\mathcal{X}\)                  | \(\mathcal{Z}\)                  |
|--------------------------------|---------------------------------|
| \(\mathcal{S}_1^I\) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} | \(\mathcal{S}_1^I \cap \mathbb{Q}^2\) |
| \(\mathcal{S}_1^II\) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 2\} | \(\mathcal{S}_1^II \cap \mathbb{Q}^2\) |
| \(\mathcal{S}_1^III\) = \{(x_0, x_1, x_2) \in \mathcal{W} \mid x_0^2 + x_1^2 + x_2^2 = 1\} | \(\mathcal{S}_1^III \cap \mathbb{Q}^3\) |
| \(\mathcal{S}_2^I\) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} | \(\mathcal{S}_2^I \cap \mathbb{Q}^3\) |
| \(\mathcal{S}_2^II\) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 2\} | \(\mathcal{S}_2^II \cap \mathbb{Q}^3\) |
| \(\mathcal{S}_2^III\) = \{(x_0, x_1, x_2, x_3) \in \mathcal{W} \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\} | \(\mathcal{S}_2^III \cap \mathbb{Q}^4\) |

height function as in (1). One of their results in [8] says that, for each \(n \geq 1\), there exists \(c_n > 0\) such that

\[
\mathcal{L}(\mathcal{S}_n, d_{\text{sup}}, \mathcal{S}_n \cap \mathbb{Q}^{n+1}, \text{Ht}_{\mathcal{S}_n \cap \mathbb{Q}^{n+1}}) \subseteq (c_n, \infty).
\]

When we combine [14] with Corollary 1.4, our result shows that the minimum of \(\mathcal{L}(S^2, d, S^2 \cap \mathbb{Q}^3, \text{Ht}_{S^2 \cap \mathbb{Q}^3})\) is \(\frac{\sqrt{3}}{2}\), which improves Moshchevitin’s lower bound \(\frac{1}{2} \sqrt{\frac{\pi}{3}}\). Note that the case \(\mathcal{X} = \mathcal{S}_2^I\) in Corollary 1.4 is the same as \((S^2, S^2 \cap \mathbb{Q}^3)\) above. Additionally, it is easy to see that, for any \(x, y \in S^n\),

\[
\frac{1}{\sqrt{n+1}} \leq d(x, y) \leq d_{\text{sup}}(x, y) \leq d(x, y).
\]

From this, we see that

\[
L(S^n, d, S^n \cap \mathbb{Q}^{n+1}, \text{Ht}_{S^n \cap \mathbb{Q}^{n+1}})(P) \leq L(S^n, d_{\text{sup}}, S^n \cap \mathbb{Q}^{n+1}, \text{Ht}_{S^n \cap \mathbb{Q}^{n+1}})(P) \leq \sqrt{n+1} \cdot L(S^n, d, S^n \cap \mathbb{Q}^{n+1}, \text{Ht}_{S^n \cap \mathbb{Q}^{n+1}})(P) \tag{2}
\]

for \(P \in S^n \setminus \mathbb{Q}^{n+1}\). So we conclude

\[
\sqrt{\frac{3}{2}} \leq \inf \mathcal{L}(S^2, d_{\text{sup}}, S^2 \cap \mathbb{Q}^3, \text{Ht}_{S^2 \cap \mathbb{Q}^3}) \leq \frac{3}{\sqrt{2}}. \tag{3}
\]

The plan for the paper is as follows. We aim to study Lagrange spectra of certain spaces \(\mathcal{X}\) listed in Table 1, all of which are homeomorphic to \(S^1\) or \(S^2\). Note that \(\mathcal{S}_1^I = S^1\) and \(\mathcal{S}_2^I = S^2\) in the above discussions. We will construct maps from either \(\mathbb{R}\) or \(\mathbb{C}\) to \(\mathcal{X}\) via stereographic projections. To be precise, we introduce the six spaces \((\mathcal{X}, \mathcal{Z})\) that are listed in Table 1.

In each of the six cases, the distance function \(d\) on \(\mathcal{X}\) will be the usual Euclidean distance inherited from its ambient Euclidean space \(\mathbb{R}^{n+1}\) or, in the case of \(\mathcal{S}_3^\mathcal{III}\), \(\mathbb{R}^{n+2}\). Also, we define the
set $\mathcal{Z}$ of rational points to be $\mathcal{Z} = \mathcal{X} \cap \mathbb{Q}^l$, where $l = n + 1$ or $n + 2$. The height function on $\mathcal{Z}$ is defined in the same way as in (1). For each of the three cases $\mathcal{X} = S_1^1, S_1^2, S_1^3$, we will construct a map $\Phi : \mathbb{R} \longrightarrow \mathcal{X}$. When $\mathcal{X} = S_1^2, S_2^2, S_1^3$, the map will be $\Phi : \mathbb{C} \longrightarrow \mathcal{X}$. Then we present a common set of conditions (Φ-i) and (Φ-ii) (see Lemma 2.2) for all the above maps $\Phi$ to satisfy. In §3, we define the height functions on certain sets $K$ of rational points of $\mathbb{R}$ and of $\mathbb{C}$. When the conditions (Φ-i) and (Φ-ii) are satisfied, the Lagrange numbers of $\xi$ and $\Phi(\xi)$ are the same, up to a fixed constant factor. In §4, we give the definitions of $\Phi$ for each of the six cases and prove that (Φ-i) and (Φ-ii) are satisfied for these six maps.

We note here that $(C, K)$, even when $K = \mathbb{Q} (\sqrt{-1})$, is not the same as $(\mathbb{R}^2, \mathbb{Q}^2)$ and therefore our results do not establish any relation between the Lagrange spectrum of $S^2$ and that of $\mathbb{R}^2$. For this reason, the authors do not know if one can use a similar method to study $(S^n, S^n \cap \mathbb{Q}^{n+1})$ for $n \geq 3$.

Now, we state our main theorem for $S^1$, that is, for the cases $\mathcal{X} = S_1^1, S_1^2, S_1^3$.

**Theorem 1.1.** Consider $(\mathbb{R}, K)$ where $K = \sqrt{2} \mathbb{Q}$ for the statements (a) and (b) and $K = \mathbb{Q}$ for (c). The distance $d$ on $\mathbb{R}$ is the usual Euclidean distance and the definitions of height functions on $K$ are given in §3.

(a) There exists a continuous bijection $\Phi_1^1 : \mathbb{R} \longrightarrow S_1^1 \setminus \{n\}$ for some $n \in S_1^1 \cap \mathbb{Q}^2$ such that $\Phi_1^1$ maps $\sqrt{2} \mathbb{Q}$ onto $S_1^1 \cap \mathbb{Q}^2 \setminus \{n\}$ and

$$L_{(\mathbb{R}, \sqrt{2} \mathbb{Q})}(\xi) = \sqrt{2} L_{(S_1^1, S_1^1 \cap \mathbb{Q}^2)}(\Phi(\xi))$$

for every $\xi \in \mathbb{R} \setminus \sqrt{2} \mathbb{Q}$.

(b) There exists a continuous bijection $\Phi_1^2 : \mathbb{R} \longrightarrow S_1^2 \setminus \{n\}$ for some $n \in S_1^2 \cap \mathbb{Q}^2$ such that $\Phi_1^2$ maps $\sqrt{2} \mathbb{Q}$ onto $S_1^2 \cap \mathbb{Q}^2 \setminus \{n\}$ and

$$L_{(\mathbb{R}, \sqrt{2} \mathbb{Q})}(\xi) = 2 L_{(S_1^2, S_1^2 \cap \mathbb{Q}^2)}(\Phi(\xi))$$

for every $\xi \in \mathbb{R} \setminus \sqrt{2} \mathbb{Q}$.

(c) There exists a continuous bijection $\Phi_1^3 : \mathbb{R} \longrightarrow S_1^3 \setminus \{n\}$ for some $n \in S_1^3 \cap \mathbb{Q}^3$ such that $\Phi_1^3$ maps $\mathbb{Q}$ onto $S_1^3 \cap \mathbb{Q}^3 \setminus \{n\}$ and

$$L_{(\mathbb{R}, \mathbb{Q})}(\xi) = \sqrt{2} L_{(S_1^3, S_1^3 \cap \mathbb{Q}^3)}(\Phi(\xi))$$

for every $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

**Corollary 1.2.** For $S^1$, we have

$$\mathcal{L}(S_1^1, S_1^1 \cap \mathbb{Q}^2) = \frac{1}{\sqrt{2}} \mathcal{L}(\mathbb{R}, \sqrt{2} \mathbb{Q}),$$

$$\mathcal{L}(S_1^2, S_1^2 \cap \mathbb{Q}^2) = \frac{1}{2} \mathcal{L}(\mathbb{R}, \sqrt{2} \mathbb{Q}),$$
\[ L(S_{\text{III}}^1, S_{\text{III}}^1 \cap Q^2) = \frac{1}{\sqrt{2}} L(\mathbb{R}, Q). \]

The usefulness of Corollary 1.2 comes from the fact that the spectra \( L(\mathbb{R}, \sqrt{2}Q) \) and \( L(\mathbb{R}, Q) \) have been studied by many authors and we already know quite a bit about their structure. For example, it is possible to relate Schmidt’s prior result in [16] with the space \( (\mathbb{R}, \sqrt{2}Q) \). See [7] for more discussion on this. As a result, we deduce that the smallest accumulation point of \( L(S_{\text{I}}^1, S_{\text{I}}^1 \cap Q^2) \) is 2 and

\[ L(S_{\text{I}}^1, S_{\text{I}}^1 \cap Q^2) \cap (0, 2) = \left\{ \sqrt{4 - \frac{1}{x^2}} \middle| x = 1, 5, 11, 29, \ldots \right\} \cup \left\{ \sqrt{4 - \frac{2}{y^2}} \middle| y = 1, 3, 11, 17, \ldots \right\}. \]

Here, \( x \) and \( y \) arise from an integer solution \((x, y_1, y_2)\) (with \( y = y_1 \) or \( y = y_2 \)) satisfying the Diophantine equation

\[ 2x^2 + y^2 + y_2^2 = 4xy_1y_2. \quad (4) \]

This gives a complete description of the initial discrete part of \( L(S_{\text{I}}^1, S_{\text{I}}^1 \cap Q^2) \). One can make similar statements for \( L(S_{\text{II}}^1, S_{\text{II}}^1 \cap Q^2) \).

Next, we present our results for \( S^2 \), that is, for the cases \( \lambda' = S_{\text{I}}^2, S_{\text{II}}^2, S_{\text{III}}^2 \).

**Theorem 1.3.** Consider \((\mathbb{C}, K)\) where

\[
K = \begin{cases} 
\mathbb{Q}(\sqrt{-1}) & \text{for (a),} \\
\mathbb{Q}(\sqrt{-2}) & \text{for (b),} \\
\mathbb{Q}(\sqrt{-3}) & \text{for (c).}
\end{cases}
\]

The distance \( d \) on \( \mathbb{C} \) is the usual Euclidean distance and the definitions of height functions on \( K \) are given in §3.

(a) There exists a continuous bijection

\[
\Phi_{\text{I}}^2 : \mathbb{C} \rightarrow S_{\text{I}}^2 \setminus \{n\}
\]

for some \( n \in S_{\text{I}}^2 \cap Q^3 \) such that \( \Phi_{\text{I}}^2 \) maps \( \mathbb{Q}(\sqrt{-1}) \) onto \( S_{\text{I}}^2 \cap Q^3 \setminus \{n\} \) and

\[
L_{(\mathbb{C}, \mathbb{Q}(\sqrt{-1}))}(\xi) = \sqrt{2} L_{(S_{\text{I}}^2, S_{\text{I}}^2 \cap Q^3)}(\Phi(\xi))
\]

for every \( \xi \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-1}). \)

(b) There exists a continuous bijection

\[
\Phi_{\text{II}}^2 : \mathbb{C} \rightarrow S_{\text{II}}^2 \setminus \{n\}
\]

for some \( n \in S_{\text{II}}^2 \cap Q^3 \) such that \( \Phi_{\text{II}}^2 \) maps \( \mathbb{Q}(\sqrt{-2}) \) onto \( S_{\text{II}}^2 \cap Q^3 \setminus \{n\} \) and

\[
L_{(\mathbb{C}, \mathbb{Q}(\sqrt{-2}))}(\xi) = \sqrt{2} L_{(S_{\text{II}}^2, S_{\text{II}}^2 \cap Q^3)}(\Phi(\xi))
\]

for every \( \xi \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-2}). \)
There exists a continuous bijection

\[ \Phi_{III}^2 : \mathbb{C} \to \mathbb{S}^2_{III} \setminus \{\mathbf{n}\} \]

for some \( \mathbf{n} \in \mathbb{S}^2_{III} \cap \mathbb{Q}^4 \) such that \( \Phi_{III}^2 \) maps \( \mathbb{Q}(\sqrt{-3}) \) onto \( \mathbb{S}^2_{III} \cap \mathbb{Q}^4 \setminus \{\mathbf{n}\} \) and

\[ L_{(\mathbb{C}, \mathbb{Q}(\sqrt{-3}))}(\xi) = \sqrt{2} L_{(\mathbb{S}^2_{III}, \mathbb{S}^2_{III} \cap \mathbb{Q}^4)}(\Phi(\xi)) \]

for every \( \xi \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-3}) \).

Corollary 1.4. For \( S^2 \), we have

\[ \mathcal{L}(S^2_1, S^2_1 \cap \mathbb{Q}^3) = \frac{1}{\sqrt{2}} \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-1})) \],

\[ \mathcal{L}(S^2_{II}, S^2_{II} \cap \mathbb{Q}^3) = \frac{1}{\sqrt{2}} \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-2})) \],

\[ \mathcal{L}(S^2_{III}, S^2_{III} \cap \mathbb{Q}^4) = \frac{1}{\sqrt{2}} \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-3})) \].

As before, this corollary allows us to use the known theorems of Schmidt [14, 17, 18] about the spectra \( \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-1})), \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-2})), \) and \( \mathcal{L}(\mathbb{C}, \mathbb{Q}(\sqrt{-3})) \), and to obtain complete descriptions of the initial discrete parts of the three spectra of \( S^2 \) in the corollary. For instance, the smallest limit point of \( \mathcal{L}(S^2_1, S^2_1 \cap \mathbb{Q}^3) \) is \( \sqrt{2} \) and its initial discrete part is

\[ \mathcal{L}(S^2_1, S^2_1 \cap \mathbb{Q}^3) \cap (0, \sqrt{2}) = \left\{ \sqrt{2 - \frac{1}{2x^2}} | x = 1, 5, 11, 29, \ldots \right\} \cup \left\{ \sqrt{\frac{3}{10}} \sqrt{41} \right\} \].

Here, \( x \) is the first integer in the triple \( (x, y_1, y_2) \) which satisfies the Diophantine equation (4). As for \( \mathcal{L}(S^2_{II}, S^2_{II} \cap \mathbb{Q}^3) \), the minimum value (so-called Hurwitz’s bound) is 1 and its smallest limit point is (cf. [18])

\[ \left( \frac{4(82 662 667 + 11 577 720 \sqrt{47})}{405 186 721} \right)^{1/2} \].

Also, the minimum value of \( \mathcal{L}(S^2_{III}, S^2_{III} \cap \mathbb{Q}^4) \) is \((13/4)^{1/4}\) and its smallest limit point is (cf. [17])

\[ \left( \frac{14 + 8 \sqrt{3}}{13} \right)^{1/2} \].

In our last theorem (Theorem 1.5), we show that the six maps \( \Phi \) in Theorems 1.1 and 1.3 can be extended to hyperbolic spaces (either a hyperbolic plane or a hyperbolic 3-space) as isometries. To explain, let \( \mathbf{B} \) be the open ball whose boundary is \( \mathbf{S} \). Also we write \( \mathbb{H}^2 = \{(x, t) \in \mathbb{R}^2 | t > 0\} \) and \( \mathbb{H}^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R} | t > 0\} \), the hyperbolic plane and the hyperbolic 3-space, respectively, having \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) or \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) as its boundary. In §6, we will construct a map \( \hat{\Phi} \) (for each of the six cases) from either \( \mathbb{H}^2 \cup \hat{\mathbb{R}} \) or \( \mathbb{H}^3 \cup \hat{\mathbb{C}} \) to \( \mathbf{B} \), which coincides with \( \Phi \) on the boundary of its domain. Furthermore, we will show that \( \hat{\Phi} \) satisfies the following properties:
The radii $R$ and dilation factors $C$ for each of the six cases of $\mathcal{X}$.

| $\mathcal{X}$ | $R$ | $C$ | $\mathcal{X}$ | $R$ | $C$ |
|----------------|-----|-----|----------------|-----|-----|
| $S_1^I$       | 1   | $\sqrt{2}$ | $S_3^I$       | 1   | $\sqrt{2}$ |
| $S_2^I$       | $\sqrt{2}$ | 2   | $S_4^I$       | $\sqrt{2}$ | $\sqrt{2}$ |
| $S_3^I$       | $\sqrt{2/3}$ | $\sqrt{2}$ | $S_5^I$       | $\sqrt{3/2}$ | $\sqrt{2}$ |

**Theorem 1.5.** The map $\Phi$ is an isometry from either $\mathbb{H}^2 \cup \hat{\mathbb{R}}$ or $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ to $\mathcal{B}$, such that the restriction of $\Phi$ to the boundary of its domain is the same as $\Phi$. In addition, for the radii $R$ of $S$ and dilation factors $C$ for $\Phi$ defined in §2 (see Table 2), $\Phi$ maps a horosphere (in $\mathbb{H}^2$ or $\mathbb{H}^3$) based at a rational point $z \in K$ (cf. (8)) with radius $1/(2Ht_K(z))$ to a horosphere in $\mathcal{B}$ based at $\Phi(z)$ with radius

$$\rho = \frac{R}{1 + (2R/C) \cdot Ht_{S \cap \mathbb{Q}^l}(\Phi(z))}. \tag{5}$$

Furthermore, any two horospheres based at two distinct rational points $z$ and $z'$ in $S$ are either tangent or disjoint.

Geometric meanings of the dilation factors $C$ will become clearer once we introduce all related objects in §2. To be brief, $C$ is a dilation factor by which either $\mathbb{R}$ or $\mathbb{C}$ is enlarged, before it is mapped onto $S$. Also, note that $C$ is the ratio between two Lagrange numbers in Theorems 1.1 and 1.3. See Lemma 2.2, especially the condition (\(\Phi\)-i). The horosphere (in $\mathbb{H}^2$ or $\mathbb{H}^3$) based at a rational point $z \in K$ with radius $1/(2Ht_K(z))$ is called the Ford horosphere. By Theorem 1.5, the geometric picture of the Ford horosphere can be applied to the Diophantine approximation on $S$.

## 2 SETUP AND SOME LEMMAS

Let $n$ be a positive integer and let $l \geq n + 1$. We fix an $(n + 1)$-dimensional plane $W$ inside of $\mathbb{R}^l$, which is defined over $\mathbb{Q}$. This plane $W$, which will play the role of an ambient space, is equipped with the Euclidean distance $d(x, y) = |x - y|$ inherited from $\mathbb{R}^l$.

Later in this paper, we will let $n = 1$ for $S^1$ and $n = 2$ for $S^2$. Furthermore, for $S_1^n$ and $S_2^n$, we will let $l = n + 1$ and $W = \mathbb{R}^{n+1}$. For $S_3^{n+1}$, we will let $l = n + 2$ and $W$ be the $(n + 1)$-dimensional hyperplane in $\mathbb{R}^{n+2}$ given by

$$W = \{(x_0, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} \mid x_0 + \cdots + x_{n+1} = 1\}.$$  

We fix the data $(S, n, P)$, satisfying the following conditions:

- $S$ is an $n$-dimensional sphere (inside of $W$) centered at a point $c \in W$ with radius $R > 0$ possessing a rational point $n$, that is, $n \in S \cap \mathbb{Q}^l$.
- $P$ is an $n$-dimensional plane (inside of $W$) not containing $n$ and perpendicular to $n - c$ (with respect to the usual dot product in $\mathbb{R}^l$). We denote by $D$ the (shortest) distance between $n$ and $P$.
- The product $RD$ is a rational number.

Additionally, we let $\tilde{S} \subset W$ be the sphere centered at $n$ with radius $\sqrt{2RD}$ and write $\tilde{W} = W \cup \{\infty\}$. Then we define $\Psi : \tilde{W} \longrightarrow \tilde{W}$ to be the reflection in $\tilde{S}$ of $\tilde{W}$ (cf. [2, §3.1]) (Figure 1).
From a general formula of reflections (see for instance [2, (3.1.1)]), we have

\[
\Psi(x) = n + \frac{2RD(x - n)}{|x - n|^2}.
\]  

(6)

The restriction of \( \Psi \) to \( S \) gives a stereographic projection of \( S \) at \( n \). More precisely, the point \( \Psi(x) \) coincides with the (unique) point of \( P \) such that the three points \( x, \Psi(x), n \) lie on the same line. On the other hand, when \( \Psi \) is restricted to \( P \), it induces a one-to-one correspondence \( \Psi : P \rightarrow S \setminus \{n\} \). Recall that \( RD \) is assumed to be a rational number and consequently \( \Psi \) is defined over \( \mathbb{Q} \) (see (6)). Therefore, \( \Psi \) also maps \( P \cap \mathbb{Q}^l \) bijectively onto \( (S \setminus \{n\}) \cap \mathbb{Q}^l \). The following lemma is a straightforward consequence of (6) so we will omit the proof.

**Lemma 2.1** (the chordal metric on \( S \)). For \( x, y \in P \),

\[
|\Psi(x) - \Psi(y)| = \frac{2RD|x - y|}{|x - n||y - n|}.
\]

Let \( F \) be

\[
F = \begin{cases} 
\mathbb{R} & \text{if } n = 1, \\
\mathbb{C} & \text{if } n = 2.
\end{cases}
\]

(7)

Also, let \( K \) be a countable dense subset of \( F \), which is equipped with a height function \( Ht_K \). Later, we will choose \( K \) to be

\[
K = \begin{cases} 
\sqrt{2}\mathbb{Q} & \text{for } \mathbb{S}_I^1, \\
\sqrt{2}\mathbb{Q} & \text{for } \mathbb{S}_II^1, \\
\mathbb{Q} & \text{for } \mathbb{S}_III^1, \\
\mathbb{Q}(\sqrt{-1}) & \text{for } \mathbb{S}_I^2, \\
\mathbb{Q}(\sqrt{-2}) & \text{for } \mathbb{S}_II^2, \\
\mathbb{Q}(\sqrt{-3}) & \text{for } \mathbb{S}_III^2.
\end{cases}
\]

(8)

Suppose that there is a continuous bijection

\[
\varphi : F \rightarrow P
\]

(9)
such that $\varphi(K) = P \cap Q^l$. Finally, we define

$$\Phi = \Psi \circ \varphi : F \to S \setminus \{n\}. \quad (10)$$

**Lemma 2.2** (Main lemma). Suppose that the map $\Phi$ satisfies the two conditions:

(\Phi-i) There exists a positive constant $C > 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| = C|x_1 - x_2|$$

for all $x_1, x_2 \in F$.

(\Phi-ii) For any $z \in K$,

$$\frac{Ht_{S \cap Q^l}(\Phi(z))}{Ht_{K}(z)} = \frac{|\varphi(z) - n|^2}{2RD}$$

Then

$$L_{(F,K)}(\xi) = C \cdot L_{(S,S \cap Q^l)}(\Phi(\xi))$$

for every $\xi \in F \setminus K$.

**Proof.** Let $\xi \in F \setminus K$ and $z \in K$. From Lemma 2.1, we have

$$|\Phi(\xi) - \Phi(z)| = \frac{2RD|\varphi(\xi) - \varphi(z)|}{|\varphi(\xi) - n||\varphi(z) - n|}. $$

Combine this with (\Phi-i) and (\Phi-ii) to obtain

$$\frac{Ht_{S \cap Q^l}(\Phi(z))|\Phi(\xi) - \Phi(z)|}{Ht_{K}(z)|\xi - z|} = \frac{Ht_{S \cap Q^l}(\Phi(z))|\varphi(\xi) - \varphi(z)|}{Ht_{K}(z)|\xi - z|} \cdot \frac{2RD}{|\varphi(\xi) - n||\varphi(z) - n|} = C \frac{|\varphi(z) - n|}{|\varphi(\xi) - n|}. $$

This proves the lemma. \hfill \Box

### 3 SUMMARY OF HEIGHT FUNCTIONS

#### 3.1 The height function on $(\mathbb{R}, K)$ when $S = S^I_{1}, S^I_{II}, S^I_{III}$

Recall that the domain of our map $\Phi$ in this case is $F = \mathbb{R}$. We choose the set $K$ of its rational points to be

$$K = \begin{cases} \sqrt{2}Q & \text{if } S = S^I_{1}, \\ \sqrt{2}Q & \text{if } S = S^I_{II}, \\ Q & \text{if } S = S^I_{III}. \end{cases}$$
For the last case $K = \mathbb{Q}$, the height function on $\mathbb{Q}$ is the usual one, that is, $Ht_q(\frac{p}{q}) = q^2$ for a reduced fraction $\frac{p}{q}$. When $K = \sqrt{2}\mathbb{Q}$, we define the height function on $\sqrt{2}\mathbb{Q}$ as follows. Let $r \in \sqrt{2}\mathbb{Q}$. Write $r = \sqrt{2} \cdot p/q_1$ with coprime integers $p$ and $q_1$. If $q_1$ is odd, then we let $q = q_1$. If $q_1$ is even, then ($p$ must be odd and) we let $q = q_1/2$. As a result, we can express $r$ as

$$r = \begin{cases} \frac{\sqrt{2}p}{q} & \text{with } q \text{ odd, or} \\ \frac{p}{\sqrt{2}q} & \text{with } p \text{ odd} \end{cases}$$

for coprime integers $p$ and $q$ in a unique way (up to the signs of $p$ and $q$). We define

$$Ht_{\sqrt{2}\mathbb{Q}}(r) = \begin{cases} Ht_{\sqrt{2}\mathbb{Q}}\left(\frac{\sqrt{2}p}{q}\right) = q^2, & \text{if } r = \frac{\sqrt{2}p}{q} \text{ with } q \text{ odd,} \\ Ht_{\sqrt{2}\mathbb{Q}}\left(\frac{p}{\sqrt{2}q}\right) = 2q^2, & \text{if } r = \frac{p}{\sqrt{2}q} \text{ with } p \text{ odd}. \end{cases}$$

(12)

To put it in another way, suppose that $x$ is the numerator and $y$ is the denominator in the fraction in (11), so that $r = x/y$ in both cases. The height of $r$ is then

$$Ht_{\sqrt{2}\mathbb{Q}}(r) = Ht_{\sqrt{2}\mathbb{Q}}\left(\frac{x}{y}\right) = y^2.$$

(13)

Note that $x^2$, $y^2$ and $\sqrt{2}xy$ are all integers satisfying

$$\gcd\left(x^2, y^2, \sqrt{2}xy\right) = 1.$$  

(14)

For more detailed discussions on the Diophantine approximation in $(\mathbb{R}, \sqrt{2}\mathbb{Q})$, consult [7].

### 3.2 The height function on $(\mathbb{C}, K)$ when $S = S^2_1, S^2_{II}, S^2_{III}$

The domain of $\Phi$ in this case is $F = \mathbb{C}$ and the set $K$ of its rational points is chosen to be

$$K = \begin{cases} \mathbb{Q}(\sqrt{-1}) & \text{if } S = S^2_1, \\ \mathbb{Q}(\sqrt{-2}) & \text{if } S = S^2_{II}, \\ \mathbb{Q}(\sqrt{-3}) & \text{if } S = S^2_{III}. \end{cases}$$

In this case, we denote by $\mathcal{O}_K$ the ring of integers of $K$ and write each $r \in K$ as $r = \alpha/\beta$ where $\alpha$ and $\beta$ are elements of $\mathcal{O}_K$ such that the ideal generated by $\alpha$ and $\beta$ is the entire $\mathcal{O}_K$. Then we define

$$Ht_K(z) = Ht_K\left(\frac{\alpha}{\beta}\right) = \beta\beta^\sigma = |\beta|^2.$$  

(15)

Here, $\beta^\sigma$ is the complex conjugate of $\beta$ in $\mathbb{C}$. Since the class number of $\mathcal{O}_K$ is one for the three choices of $K$ above, the expression $z = \alpha/\beta$ is unique up to a unit of $\mathcal{O}_K$ and therefore $Ht_K(z)$ is well defined.
3.3 | The height function on \((S, S \cap \mathbb{Q}^l)\)

In all six cases of \(S\), the rational points of \(S\) are defined by \(S \cap \mathbb{Q}^l\) (where \(l = n + 1 \) or \(n + 2\)). We utilize the standard height function on \(\mathbb{Q}^l\). That is, for each \(r \in S \cap \mathbb{Q}^l\), we write \(r = p/q\) with \(p = (p_1, \ldots, p_l) \in \mathbb{Z}^l\) and \(q \in \mathbb{Z}\) such that
\[
gcd(p_1, \ldots, p_l, q) = 1
\]
and define
\[
H_{S \cap \mathbb{Q}^l}(r) = q.
\]
Note that the integers \(p_1, \ldots, p_l, q\) satisfy
\[
p_1^2 + \cdots + p_l^2 = kq^2
\]
for \(k = 1\) or \(k = 2\). Therefore the gcd condition (16) is equivalent to \(gcd(p_1, \ldots, p_l) = 1\). In other words, the definition (17) is equivalent to the one used by Kleinbock and Merrill (1).

4 | PROOF OF THEOREM 1.1; THE CASE OF \(S^1\)

4.1 | The case \(S = S^1\)

Let \(W = \mathbb{R}^2\) and define
\[
S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}, \\
n = (0, 1), \\
P = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}.
\]
This yields
\[
R = D = 1,
\]
so that the conditions in §2 are satisfied. Also, the formula (6) becomes
\[
\Psi(x_1, x_2) = \frac{(2x_1, x_1^2 + x_2^2 - 1)}{x_1^2 + x_2^2 - 2x_2 + 1}.
\]
Next, we define \(\varphi : \mathbb{R} \rightarrow P\) to be
\[
\varphi(t) = (\sqrt{2}t - 1, 0).
\]
Note that \(\varphi\) maps \(\sqrt{2}\mathbb{Q}\) to \(P \cap \mathbb{Q}^2\) bijectively. Also, this gives
\[
|\varphi(x_1) - \varphi(x_2)| = \sqrt{2}|x_1 - x_2|.
\]
so \(\varphi\) satisfies the condition (\(\Phi\)-i) in Lemma 2.2 with \(C = \sqrt{2}\).
To verify the condition \((\Phi\text{-}ii)\), we compute \(H \sqrt{2} \mathbb{Q}(r)\) and \(H_S \cap \mathbb{Q}^2(\Phi(r))\) for \(r \in \sqrt{2} \mathbb{Q}\). Using the notations in §3.1, we write \(r = x/y\), so that \(H \sqrt{2} \mathbb{Q}(r) = y^2\) (cf. (13)). To compute \(H_{S \cap \mathbb{Q}^2}(\Phi(r))\), we use (19) to get
\[
\Phi(r) = \frac{1}{r^2 - \sqrt{2}r + 1} \left( \sqrt{2}r - 1, r^2 - \sqrt{2}r \right)
= \left( \frac{\sqrt{2}xy - y^2, x^2 - \sqrt{2}xy}{x^2 - \sqrt{2}xy + y^2} \right).
\]
We verify the gcd condition (16) for \(\Phi(r)\) using (14):
\[
gcd \left( \sqrt{2}xy - y^2, x^2 - \sqrt{2}xy, x^2 - \sqrt{2}xy + y^2 \right) = \gcd \left( x^2, y^2, \sqrt{2}xy \right) = 1.
\]
This gives
\[
H_{S \cap \mathbb{Q}^2}(\Phi(r)) = x^2 - \sqrt{2}xy + y^2. \quad (21)
\]
So,
\[
\frac{H_{S \cap \mathbb{Q}^2}(\Phi(r))}{H \sqrt{2} \mathbb{Q}(r)} = \frac{x^2 - \sqrt{2}xy + y^2}{y^2} = r^2 - \sqrt{2}r + 1.
\]
On the other hand,
\[
\frac{|\varphi(r) - n|^2}{2RD} = \frac{|(\sqrt{2}r - 1, -1)|^2}{2} = r^2 - \sqrt{2}r + 1. \quad (22)
\]
This proves \((\Phi\text{-}ii)\) in this case.

4.2 The case \(S = S^1_{II}\)

As in §4.1, we let \(W = \mathbb{R}^2\). We define
\[
S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 2\},
n = (1, 1),
P = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}.
\]
This yields
\[
R = \sqrt{2}, \quad D = \sqrt{2} \quad (23)
\]
and
\[
\Psi(x_1, x_2) = \frac{(x_1^2 + x_2^2 + 2x_1 - 2x_2 - 2, x_1^2 + x_2^2 - 2x_1 + 2x_2 - 2)}{x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2}. \quad (24)
\]
In this case, we define \( \varphi : \mathbb{R} \rightarrow \mathbb{P} \) to be
\[
\varphi(t) = \left( \sqrt{2t} - 1, 1 - \sqrt{2t} \right).
\tag{25}
\]
Then we get
\[
|\varphi(t_1) - \varphi(t_2)|^2 = \left| (\sqrt{2}(t_1 - t_2), \sqrt{2}(t_2 - t_1)) \right|^2 = 4(t_1 - t_2)^2,
\]
which proves (Φ-i) with \( C = 2 \).

For (Φ-ii), let \( r \in \sqrt{2}\mathbb{Q} \) and write \( r = x/y \) as before, so that \( \text{Ht}_{\sqrt{2}\mathbb{Q}}(r) = y^2 \). To compute \( \text{Ht}_{\sqrt{2}\mathbb{Q}^2}(\Phi(r)) \) in this case, we use (24) and (25) to get
\[
\Phi(r) = \Psi(\varphi(r)) = \Psi \left( \sqrt{2}r - 1, 1 - \sqrt{2}r \right)
= \frac{(r^2 - 1, r^2 - 2\sqrt{2}r + 1)}{r^2 - \sqrt{2}r + 1} = \frac{(x^2 - y^2, x^2 - 2\sqrt{2}xy + y^2)}{x^2 - \sqrt{2}xy + y^2}.
\]
The gcd condition (16) is verified by (14), that is,
\[
\gcd \left( x^2 - y^2, x^2 - 2\sqrt{2}xy + y^2, x^2 - \sqrt{2}xy + y^2 \right) = \gcd \left( x^2, y^2, \sqrt{2}xy \right) = 1.
\]
Therefore we have
\[
\text{Ht}_{\sqrt{2}\mathbb{Q}^2}(\Phi(r)) = x^2 - \sqrt{2}xy + y^2.
\tag{26}
\]
On the other hand,
\[
\frac{|\varphi(r) - n|^2}{2RD} = \frac{(\sqrt{2}r - 2)^2 + 2r^2}{4} = r^2 - \sqrt{2}r + 1.
\tag{27}
\]
Combining this with (26), we obtain (Φ-ii) in this case.

### 4.3 The case \( S = S_{\text{III}}^1 \)

In this case, we let the ambient space \( W \) be
\[
W = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = 1\}
\]
and define
\[
\begin{align*}
S &= \{(x_0, x_1, x_2) \in W \mid x_0^2 + x_1^2 + x_2^2 = 1\}, \\
\mathbf{n} &= (0, 0, 1), \\
\mathbb{P} &= \{(x_0, x_1, 0) \in W \mid x_0 + x_1 = 1\}.
\end{align*}
\]
This gives
\[ R = \sqrt{\frac{2}{3}}, \quad D = \sqrt{\frac{3}{2}} \tag{28} \]
and
\[ \Psi(x_0, x_1, x_2) = \frac{(2x_0, 2x_1, x_0^2 + x_1^2 + x_2^2 - 1)}{x_0^2 + x_1^2 + x_2^2 - 2x_2 + 1}. \tag{29} \]
Note that \( \Psi(x) \in W \) for \( x \in W \). In this case, We define \( \varphi : \mathbb{R} \to \mathbb{P} \) to be
\[ \varphi(t) = (t, 1 - t, 0). \tag{30} \]
The property \((\Phi-i)\) is easily seen to be satisfied with \( C = \sqrt{2} \) because
\[ |\varphi(t_1) - \varphi(t_2)|^2 = 2(t_1 - t_2)^2. \]
For each \( r \in \mathbb{Q} \), we write \( r = \frac{p}{q} \) as a reduced fraction, so that \( \text{Ht}_{\mathbb{Q}}(r) = q^2 \). Also, To compute \( \text{Ht}_{S \cap \mathbb{Q}}(\Phi(r)) \), we see that
\[ \Phi(r) = \Psi(r, 1 - r, 0) = \frac{(r, 1 - r, r^2 - r)}{r^2 - r + 1} = \frac{(pq, q^2 - pq, p^2 - pq)}{p^2 + q^2 - pq}. \]
Since
\[ \gcd(pq, q^2 - pq, p^2 - pq, p^2 + q^2 - pq) = \gcd(p^2, q^2, pq) = 1, \]
the \( \gcd \) condition \((16)\) is satisfied in this case and we have
\[ \text{Ht}_{S \cap \mathbb{Q}}(\Phi(r)) = p^2 + q^2 - pq. \tag{31} \]
Finally,
\[ \frac{|\varphi(r) - n|^2}{2RD} = \frac{r^2 + (1 - r)^2 + (-1)^2}{2} = \frac{p^2 - pq + q^2}{q^2}. \tag{32} \]
Therefore, \((31)\) and \((32)\) give \((\Phi-\text{ii})\) in this case.

5 | PROOF OF THEOREM 1.3; THE CASE OF \( S^2 \)

To prove Theorem 1.3, we will construct a map \( \varphi : \mathbb{C} \to \mathbb{P} \) for each of the cases \( S = S_1^2, S_\text{II}_2, S_\text{III}_2 \).
As explained in §1, the set of rational points \( K \) of \( \mathbb{C} \) will be chosen to be
\[ K = \begin{cases} \mathbb{Q}(\sqrt{-1}) & \text{for the case } S = S_1^2, \\ \mathbb{Q}(\sqrt{-2}) & \text{for the case } S = S_\text{II}_2, \\ \mathbb{Q}(\sqrt{-3}) & \text{for the case } S = S_\text{III}_2. \end{cases} \]
It will be convenient to choose an integral basis over \( \mathbb{Z} \) for the ring \( \mathcal{O}_K \) of integers of \( K \) as \( \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega_K \) where

\[
\omega_K = \begin{cases} 
  i = \sqrt{-1} & \text{for the case } S = S_1^2, \\
  \sqrt{-2} & \text{for the case } S = S_2^2, \\
  (-1 + \sqrt{-3})/2 & \text{for the case } S = S_3^2.
\end{cases}
\]

**Lemma 5.1.** The notations are as above. Let \( r \) be a nonzero element \( K \) and write \( r = \alpha/\beta \) where \( \alpha \) and \( \beta \) are elements of \( \mathcal{O}_K \), which generate \( \mathcal{O}_K \). Let \( a, b, c \) be the (rational) integers given by

\[
a + b\omega_K = \alpha\beta^\sigma \quad \text{and} \quad c = |\beta|^2 = \beta\beta^\sigma.
\]

Here, \( \sigma \) is the nontrivial automorphism of \( K/\mathbb{Q} \) (the complex conjugation). Then we have

(a) \( r = (a + b\omega_K)/c \);
(b) \( \text{Ht}_K(r) = c \);
(c) \( c \) divides \( |a + b\omega_K|^2 \) (in \( \mathbb{Z} \));
(d) \( |\alpha|^2 = |a + b\omega_K|^2/c \) and
(e) \( \text{gcd}(|\alpha|^2, |\beta|^2, a, b) = 1 \).

**Proof.** By the definition of \( a, b, c \), the statements (a) and (b) in the lemma are trivially true. Also, we have

\[
\frac{|a + b\omega_K|^2}{c} = \frac{\alpha\beta^\sigma \cdot \alpha^\sigma \beta}{\beta\beta^\sigma} = \alpha\alpha^\sigma,
\]

which proves (c) and (d). It remains to prove (e).

Suppose that there exists a (rational) prime \( p \) dividing \( |\alpha|^2 \) and \( |\beta|^2 \) in \( \mathbb{Z} \). Then \( \alpha \) must be an element of a prime ideal, say, \( \mathfrak{p}_\alpha \) of \( \mathcal{O}_K \) lying above \( p \). Similarly, \( \beta \) is an element of, say, \( \mathfrak{p}_\beta \) lying above \( p \). Since \( \alpha \) and \( \beta \) are assumed to generate \( \mathcal{O}_K \) it follows that \( \alpha \not\in \mathfrak{p}_\beta \) and \( \beta \not\in \mathfrak{p}_\alpha \). In particular, \( \mathfrak{p}_\alpha \neq \mathfrak{p}_\beta \). Since they are both prime ideals of \( \mathcal{O}_K \) above \( p \), we must have \( \mathfrak{p}_\alpha = \mathfrak{p}_\beta^\sigma \) (and the prime \( p \) must split in \( \mathcal{O}_K \)). Consequently, \( \beta^\sigma \not\in \mathfrak{p}_\alpha^\sigma = \mathfrak{p}_\beta \).

Now, suppose that \( p \) additionally divides \( a \) and \( b \). Then \( a + b\omega_K \) is in \( p\mathcal{O}_K = \mathfrak{p}_\alpha \mathfrak{p}_\beta \subset \mathfrak{p}_\beta \). However, we already proved that \( \alpha \not\in \mathfrak{p}_\beta \) and \( \beta^\sigma \not\in \mathfrak{p}_\beta \), so that \( a + b\omega_K = \alpha\beta^\sigma \not\in \mathfrak{p}_\beta \). This is a contradiction. So, \( p \) cannot divide both \( a \) and \( b \) and we obtain \( \text{gcd}(|\alpha|^2, |\beta|^2, a, b) = 1 \). \( \square \)

5.1 The case \( S = S_1^2 \)

The ambient space in this case is \( W = \mathbb{R}^3 \) and we define

\[
S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\},
\]
\[
n = (0, 0, 1),
\]
\[
P = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}\}.
\]

This gives

\[
R = D = 1 \quad (33)
\]
and
\[ \Psi(x_1, x_2, x_3) = \frac{(2x_1, 2x_2, x_1^2 + x_2^2 + x_3^2 - 1)}{x_1^2 + x_2^2 + x_3^2 - 2x_3 + 1}. \]  
(34)

We define \( \varphi : \mathbb{C} \rightarrow \mathbb{P} \) to be
\[ \varphi(u + iv) = (u - v, u + v - 1, 0). \]  
(35)

To prove (\( \Phi-i \)) in this case, we let \( z_1 = u_1 + v_1i \) and \( z_2 = u_2 + v_2i \). Then
\[ |\varphi(z_1) - \varphi(z_2)|^2 = |((u_1 - u_2) - (v_1 - v_2), (u_1 - u_2) + (v_1 - v_2), 0)|^2 \]
\[ = 2|z_1 - z_2|^2. \]

So this proves (\( \Phi-i \)) with \( C = \sqrt{2} \).

For (\( \Phi-ii \)), let \( r \in K \) and write \( r = \alpha/\beta = (a + bi)/c \) using the notations in Lemma 5.1. To find \( Ht_{\mathbb{S} \cap \mathbb{Q}^3}(\Phi(r)) \), we compute \( \Phi(r) \) using Lemma 5.1:
\[ \Phi(r) = \Psi\left( \varphi\left( \frac{a + bi}{c} \right) \right) = \Psi\left( \frac{a}{c} - \frac{b}{c}, \frac{a}{c} + \frac{b}{c} - 1, 0 \right) \]
\[ = \frac{((a - b)c, (a + b - c)c, a^2 + b^2 - (a + b)c)}{a^2 + b^2 + c^2 - (a + b)c} \]
\[ = \left( \frac{a - b, a + b - |\beta|^2, |\alpha|^2 - a - b}{|\alpha|^2 + |\beta|^2 - a - b} \right). \]

And
\[ \gcd(a - b, a + b - |\beta|^2, |\alpha|^2 - a - b, |\alpha|^2 + |\beta|^2 - a - b) = \gcd(|\alpha|^2, |\beta|^2, a - b, a + b). \]  
(36)

Note that prime 2 does not split in \( \mathbb{Z}[i] \), so 2 cannot divide \( \gcd(|\alpha|^2, |\beta|^2) \) (see the proof of Lemma 5.1). Therefore, we see that the \( \gcd \) condition (16) for \( \Phi(r) \) is satisfied by Lemma 5.1 because
\[ \gcd(|\alpha|^2, |\beta|^2, a - b, a + b) = \gcd(|\alpha|^2, |\beta|^2, a, b) = 1. \]  
(37)

Hence we obtain
\[ Ht_{\mathbb{S} \cap \mathbb{Q}^3}(\Phi(r)) = |\alpha|^2 + |\beta|^2 - a - b. \]

On the other hand,
\[ \left| \varphi(r) - n \right|^2 = \frac{1}{2RD} = \frac{\left| \left( \frac{a}{c} - \frac{b}{c}, \frac{a}{c} + \frac{b}{c} - 1, -1 \right) \right|^2}{2} \]
\[ = \frac{|\alpha|^2 + |\beta|^2 - a - b}{c^2} \]
\[ = \frac{Ht_{\mathbb{S} \cap \mathbb{Q}^3}(\Phi(r))}{Ht_K(r)}, \]

which gives (\( \Phi-ii \)) in this case.
5.2 The case $S = S^2_\Pi$

As in the case of $S^2_1$, the ambient space is again $W = \mathbb{R}^3$. We define

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 2\},$$

$$n = (0, 1, 1),$$

$$P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 + x_3 = 1\},$$

which gives

$$R = \sqrt{2}, \quad D = 1/\sqrt{2}$$

and

$$\Psi(x_1, x_2, x_3) = \frac{(2x_1, x_1^2 + x_2^2 + x_3^2 - 2x_3, x_1^2 + x_2^2 + x_3^2 - 2x_2)}{x_1^2 + x_2^2 + x_3^2 - 2x_2 - 2x_3 + 2}.$$  \hspace{0.5cm} (39)

In this case, we define $\varphi : \mathbb{C} \longrightarrow P$ to be

$$\varphi(u + v\sqrt{-2}) = (1 - 2v, u, 1 - u).$$  \hspace{0.5cm} (40)

Writing $z_1 = u_1 + \sqrt{-2}v_1$ and $z_2 = u_2 + \sqrt{-2}v_2$, we get

$$|\varphi(z_1) - \varphi(z_2)|^2 = |(-2(u_1 - u_2), u_1 - u_2, -(u_1 - u_2))|^2$$

$$= 2((u_1 - u_2)^2 + 2(v_1 - v_2)^2) = 2|z_1 - z_2|^2,$$

which proves (Φ-i) with $C = \sqrt{2}$.

For (Φ-ii), we proceed similarly using Lemma 5.1. Let $r = \alpha/\beta = (a + b\sqrt{-2})/c \in K = \mathbb{Q}(\sqrt{-2})$. Then

$$\Phi(r) = \Psi\left(\varphi\left(\frac{a + b\sqrt{-2}}{c}\right)\right) = \Psi\left(1 - \frac{2b}{c}, \frac{a}{c}, 1 - \frac{a}{c}\right)$$

$$= \frac{((c-2b)c, a^2 + 2b^2 - 2bc, a^2 + 2b^2 + c^2 - 2(a + b)c)}{a^2 + 2b^2 + c^2 - (a + 2b)c}$$

$$= \frac{(|\beta|^2 - 2b, |\alpha|^2 - 2b, |\alpha|^2 + |\beta|^2 - 2(a + b))}{|\alpha|^2 + |\beta|^2 - (a + 2b)}.$$  \hspace{0.5cm} (41)

To check the gcd condition (16) for $\Phi(r)$, we note

$$\gcd(|\beta|^2 - 2b, |\alpha|^2 - 2b, |\alpha|^2 + |\beta|^2 - 2(a + b), |\alpha|^2 + |\beta|^2 - (a + 2b))$$

$$= \gcd(|\alpha|^2, |\beta|^2, a, 2b).$$  \hspace{0.5cm} (41)
Again, prime 2 does not split in $\mathbb{Z}[\sqrt{-2}]$, so 2 cannot divide $\gcd(|\alpha|^2, |\beta|^2)$. So we deduce from Lemma 5.1 that

$$\gcd(|\alpha|^2, |\beta|^2, a, 2b) = \gcd(|\alpha|^2, |\beta|^2, a, b) = 1.$$  

(42)

So we obtain

$$\text{Ht}_{\mathcal{S}\cap\mathbb{Q}^3}(\Phi(r)) = |\alpha|^2 + |\beta|^2 - (a + 2b).$$

Also,

$$\frac{|\varphi(r) - n|^2}{2RD} = \frac{|(1 - \frac{2b}{c}, \frac{a}{c} - 1, -\frac{a}{c})|^2}{2} = \frac{a^2 + 2b^2 + c^2 - c(a + 2b)}{c^2}$$

$$= \frac{|\alpha|^2 + |\beta|^2 - (a + 2b)}{c}$$

$$= \frac{\text{Ht}_{\mathcal{S}\cap\mathbb{Q}^3}(\Phi(r))}{\text{Ht}_{\mathcal{K}}(r)},$$

which gives (Φ-ii) in this case.

5.3  The case $\mathcal{S} = S^2_{\text{III}}$

In this case, the ambient space $W$ is defined to be

$$W = \{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_0 + x_1 + x_2 + x_3 = 1 \}.$$

Also, we define

$$\begin{align*}
\mathcal{S} &= \{ (x_0, x_1, x_2, x_3) \in W \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \}, \\
\mathbf{n} &= (0, 0, 0, 1), \\
\mathcal{P} &= \{ (x_0, x_1, x_2, 0) \in W \mid x_0 + x_1 + x_2 = 1 \}.
\end{align*}$$

This yields

$$R = \sqrt{3}/2, \quad D = 2/\sqrt{3}$$

(43)

and

$$\Psi(x_0, x_1, x_2, x_3) = \frac{(2x_0, 2x_1, 2x_2, x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1)}{x_0^2 + x_1^2 + x_2^2 + x_3^2 - 2x_3 + 1}.$$  

(44)

We define $\varphi : \mathbb{C} \longrightarrow \mathcal{P}$ to be

$$\varphi(u + v\omega_K) = (1 - u, u - v, v, 0).$$  

(45)
Recall that in this case \( K = \mathbb{Q}(\sqrt{-3}) \) and \( \omega_K = (-1 + \sqrt{-3})/2 \). It is a simple exercise to show that

\[
|\varphi(z_1) - \varphi(z_2)|^2 = 2|z_1 - z_2|^2,
\]

which proves (\( \Phi \)-i) with \( C = \sqrt{2} \).

For (\( \Phi \)-ii), we write \( r = \alpha/\beta = (a + b\omega_K)/c \) and

\[
\Phi(r) = \Psi\left(\varphi\left(\frac{a + b\omega_K}{c}\right)\right) = \Psi\left(1 - \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 0\right)
\]

\[
= \left(\frac{(c - a)c, (a - b)c, bc, a^2 - ab + b^2 - ac}{a^2 - ab + b^2 + c^2 - ac}\right)
\]

\[
= \left(\frac{|\beta|^2 - a, a - b, b, |\alpha|^2 - a}{|\alpha|^2 + |\beta|^2 - a}\right).
\]

The gcd condition (16) for \( \Phi(r) \) is again satisfied by Lemma 5.1 because

\[
gcd(|\beta|^2 - a, a - b, b, |\alpha|^2 - a, |\alpha|^2 + |\beta|^2 - a) = gcd(|\alpha|^2, |\beta|^2, a, b) = 1.
\]

So,

\[
H_{tS}^{n+1}(\Phi(r)) = |\alpha|^2 + |\beta|^2 - a. \quad (46)
\]

We have

\[
\frac{|\varphi(r) - n|^2}{2RD} = \frac{a^2 - ab + b^2 + c^2 - ac}{c^2}
\]

\[
= \frac{|\alpha|^2 + |\beta|^2 - a}{c}
\]

\[
= \frac{H_{tS}^{n+1}(\Phi(r))}{H_{tK}(r)}.
\]

This proves (\( \Phi \)-ii) in this case.

6 PROOF OF THEOREM 1.5

We continue to use the notations introduced in the beginning of \( \S 2 \). Recall from (7) that \( F \) is by definition either \( \mathbb{R} \) or \( \mathbb{C} \), depending on \( n = 1 \) or \( n = 2 \). As in the introduction, we define \( H^{n+1} \) to be

\[
H^{n+1} = \{(z, s) \in F \times \mathbb{R} \mid z \in F, s > 0\},
\]

so that \( H^{n+1} \) is either the hyperbolic surface (when \( n = 1 \)) or the hyperbolic 3-space (when \( n = 2 \)). In particular, we will identify \( F \) with the set \( F \times \{0\} \). Then \( \tilde{F} = F \cup \{\infty\} \) is the boundary of \( H^{n+1} \) and therefore \( H^{n+1} = \tilde{H}^{n+1} \cup \tilde{F} \).
Now we define the maps $\bar{\varphi}$ and $\bar{\Psi}$, extending $\varphi$ and $\Psi$ (see Figure 2). Recall that $\varphi : \mathbb{F} \rightarrow \mathbb{P}$ is a map (cf. (9)) satisfying the conditions $(\Phi\text{-i})$ and $(\Phi\text{-ii})$ in Lemma 2.2. For $(z, s) \in \mathbb{H}^{n+1}$, we define $\bar{\varphi} : \mathbb{H}^{n+1} \rightarrow \mathbb{W}$ to be

$$\bar{\varphi}(z, s) = \varphi(z) + Cs \frac{R D}{|p - n|}.$$ 

Here, $C$ is the same constant appearing in $(\Phi\text{-i})$ of Lemma 2.2.

To define $\bar{\Psi}$, we let $\mathcal{H}$ be the connected component of $\mathcal{W} \setminus \mathcal{P}$ containing $n$. Denote by $\text{Ref}_p$ and $\text{Ref}_s$ the reflections of $\mathcal{W}$ in $\mathcal{P}$ and in $\mathcal{S}$, respectively. Then we define

$$\bar{\Psi} = \text{Ref}_s \circ \text{Ref}_p.$$  

Then it is clear that $\bar{\Psi}$ maps $\bar{\mathcal{H}}$ onto $\bar{\mathcal{B}}$ and it is a hyperbolic isometry between them. Also, it is easy to see that $\bar{\varphi}$ and $\bar{\Psi}$ extend $\varphi$ and $\Psi$, as indicated in Figure 2.

**Lemma 6.1.** The notations are the same as in §2. Additionally, let $r$ be the radius of a reflected horosphere by $\text{Ref}_p$ based at $p$ tangent to $\mathcal{P}$, which is on the opposite side to $n$ with respect to $\mathcal{P}$ (see Figure 3). Then the radius $\rho$ of the horosphere based at $\Psi(p)$ tangent to $\mathcal{S}$ is given by the formula

$$\rho = \frac{2r RD}{|p - n|^2 + 2r D}.$$
Proof. The distance between \( \mathbf{n} \) and the center of the horosphere (in the opposite side of \( \mathbf{n} \)) based at \( \mathbf{p} \in \mathbf{P} \) with radius \( r \) is

\[
\sqrt{(D + r)^2 - D^2 + |\mathbf{p} - \mathbf{n}|^2}.
\]

Thus the distances from \( \mathbf{n} \) to the closest and farthest points on the horosphere are

\[
\sqrt{(D + r)^2 - D^2 + |\mathbf{p} - \mathbf{n}|^2} \pm r.
\]

Recall that the radius of \( \mathbf{S} \) is \( \sqrt{2RD} \). From a general property of the reflection \( \Psi \), we see that the diameter of the image of the horosphere under \( \Psi \) has diameter

\[
\frac{2RD}{\sqrt{(D + r)^2 - D^2} - |\mathbf{p} - \mathbf{n}| - r} - \frac{2RD}{\sqrt{(D + r)^2 - D^2} + |\mathbf{p} - \mathbf{n}| + r} = \frac{4rRD}{(D + r)^2 - D^2 + |\mathbf{p} - \mathbf{n}|^2 - r^2}.
\]

Simplifying this expression, we obtain the formula for \( \rho \), which completes the proof. \( \square \)

Using Lemma 6.1, it is now straightforward to prove the formula (5) for \( \rho \) in Theorem 1.5. Suppose that there is a Ford horosphere in \( \mathbb{H}^{n+1} \) based at \( z \in K \) with radius \( 1/(2 \text{Ht}_K(z)) \). From (Φ-i), we see that this horosphere is mapped by the extended map \( \overline{\varphi} \) to a horosphere with radius \( C/(2 \text{Ht}_K(z)) \) based at \( \varphi(z) \). Then, after we further apply Ref \( \mathbf{P} \), we are in the situation described in Lemma 6.1 with \( \mathbf{p} = \varphi(z) \) and \( r = C/(2 \text{Ht}_K(z)) \). We use Lemma 6.1 with this and simplify using the condition (Φ-ii) to obtain (5).

It remains to show the last assertion in Theorem 1.5, namely, two horospheres on \( \mathbb{F} \) based at \( z, z' \in K \) are either disjoint or tangent. Let \( z, z' \) be two distinct points of \( K \) and write \( z = \alpha/\beta \) and \( z' = \alpha'/\beta' \) as before. Since the horospheres based at \( z, z' \) have radius \( 1/(2|\beta|^2), 1/(2|\beta'|^2) \), the square of the distance between the centers of the horospheres based at \( z \) and \( z' \) is

\[
\left( \frac{1}{2|\beta|^2} - \frac{1}{2|\beta'|^2} \right)^2 + \left| \frac{\alpha}{\beta} - \frac{\alpha'}{\beta'} \right|^2 = \left( \frac{1}{2|\beta|^2} + \frac{1}{2|\beta'|^2} \right)^2 + \frac{|\alpha \beta' - \alpha' \beta|^2 - 1}{|\beta|^2 |\beta'|^2} \geq \left( \frac{1}{2|\beta|^2} + \frac{1}{2|\beta'|^2} \right)^2.
\]

Here, we use the fact that \( |\alpha \beta' - \alpha' \beta|^2 \) is at least 1. For the case of \( n = 1 \), see Figures 4–6. This completes the proof of Theorem 1.5.

The horosphere based at \( \Phi(z) \) with (Euclidean) radius \( \rho = R/(1 + 2(R/C) \text{Ht}_z(z)) \) is centered at

\[
\mathbf{c} + \left( 1 - \frac{\rho}{R} \right)(\mathbf{z} - \mathbf{c}) = \mathbf{c} + \frac{2R \text{Ht}_z(z)(\mathbf{z} - \mathbf{c})}{C + 2R \text{Ht}_z(z)},
\]
where \( c \) is the center of the sphere \( S \). Hence, the square of the distance between the centers of the horospheres based at \( z \) and \( z' \) is

\[
\left| \frac{2Rh(z - c)}{C + 2Rh} - \frac{2Rh(z' - c)}{C + 2Rh'} \right|^2 = \left( \frac{2R^2h}{C + 2Rh} \right)^2 + \left( \frac{2R^2h'}{C + 2Rh'} \right)^2 - \frac{8R^2hh'(z - c) \cdot (z' - c)}{(C + 2Rh)(C + 2Rh')}. \]

Here, we used \( h = H_t(z) \) and \( h' = H_t(z') \) to lighten the notations. On the other hand, the square of the sum of radii of the horospheres based at \( z \) and \( z' \) is

\[
(r_z + r_{z'})^2 = \left( \frac{RC}{C + 2Rh} + \frac{RC}{C + 2Rh'} \right)^2
= \left( \frac{2R^2h}{C + 2Rh} \right)^2 + \left( \frac{2R^2h'}{C + 2Rh'} \right)^2 - \frac{8R^2hh'\left( R^2 - C^2/(2hh') \right)}{(C + 2Rh)(C + 2Rh')}.
\]
Therefore, we conclude that

\[(z - c) \cdot (z' - c) \leq R^2 - \frac{C^2}{2Ht_S(z)Ht_S(z')}.
\] (48)

Moreover, two horospheres based at \(z\) and \(z'\) are tangent if and only if the equality in (48) holds true.

For the rest of the paper, we give an explicit description of (48) in terms of the (Euclidean) coordinates in their ambient spaces for each of the six spaces we consider.

### 6.1 The case \(S = S^1_1\)

A rational point \(z = (\frac{a}{c}, \frac{b}{c})\) on \(S^1_1\) has height \(Ht(z) = c\), the Ford horocycle based at \(z\) has radius \(1/(1 + \sqrt{2}c)\). For \(z = (\frac{a}{c}, \frac{b}{c})\) and \(z' = (\frac{a'}{c'}, \frac{b'}{c'})\), the inequality (48) implies that

\[(z - c) \cdot (z' - c) = \frac{aa' + bb'}{cc'} \leq 1 - \frac{1}{cc'}.
\]
Horocycles for the circle $S_{III}$. The radius of the horocycle at $(a_d, b_d, c_d)$ is $\sqrt{2/(\sqrt{3} + 2d)}$ on $S_{III}$ (above). The projected horocycle at $\Phi^{-1}(a_c, b_c)$ on $\mathbb{R}$ has radius $1/(2(a + b))$ (below).

Therefore, we have $aa' + bb' - cc' \leq -1$ and two horocycles based at $z = (a, b, c)$ and $z' = (a', b', c')$ are tangent if and only if $aa' + bb' - cc' = -1$. In Figure 4, we give pictures of horospheres based at the following rational points on $S_{I}$ and corresponding points on $\mathbb{R}$:

\[
\begin{align*}
(\frac{-1}{1}, 0_1) &= \Phi(0), & (0_1, -\frac{1}{1}) &= \Phi\left(\frac{1}{\sqrt{2}}\right), & (\frac{3}{5}, -\frac{4}{5}) &= \Phi\left(\frac{2\sqrt{2}}{3}\right), \\
(\frac{4}{5}, -\frac{3}{5}) &= \Phi\left(\frac{3}{2\sqrt{2}}\right), & (\frac{1}{1}, 0_1) &= \Phi\left(\sqrt{2}\right), & (\frac{12}{13}, 6_{13}) &= \Phi\left(\frac{5}{2\sqrt{2}}\right), \\
(\frac{15}{17}, \frac{8}{17}) &= \Phi\left(\frac{4\sqrt{2}}{3}\right), & (\frac{4}{5}, \frac{3}{5}) &= \Phi\left(\frac{3}{\sqrt{2}}\right), & (\frac{21}{29}, 20_{29}) &= \Phi\left(\frac{5\sqrt{2}}{3}\right), \\
(\frac{20}{29}, \frac{21}{29}) &= \Phi\left(\frac{7}{2\sqrt{2}}\right), & (\frac{3}{5}, \frac{4}{5}) &= \Phi\left(2\sqrt{2}\right), & (\frac{0}{1}, 1_1) &= \Phi(\infty).
\end{align*}
\]

Note that the height of $\Phi^{-1}(\frac{a}{c}, \frac{b}{c})$ is $c - b$. 
6.2 | The case $S = S_{II}^1$

A rational point $z = \left( \frac{a}{c}, \frac{b}{c} \right)$ on $S_{II}^1$ has height $Ht(z) = c$, the Ford horocycle based at $z$ has radius $\sqrt{2}/(1 + \sqrt{2}c)$. For $z = \left( \frac{a}{c}, \frac{b}{c} \right)$ and $z' = \left( \frac{a'}{c'}, \frac{b'}{c'} \right)$, the inequality (48) implies that

$$(z - c) \cdot (z' - c) = \frac{aa' + bb'}{cc'} \leq 2 - \frac{2}{cc'}.$$ 

Therefore, we have $aa' + bb' - 2cc' \leq -2$ and two horocycles based at $z = \left( \frac{a}{c}, \frac{b}{c} \right)$ and $z' = \left( \frac{a'}{c'}, \frac{b'}{c'} \right)$ are tangent if and only if $aa' + bb' - cc' = -2$. In Figure 5, we give pictures of horospheres based at the following rational points on $S_{II}^1$ and corresponding points on $\mathbb{R}$:

$$\begin{align*}
\left( -\frac{1}{1}, \frac{1}{1} \right) &= \Phi(0), \\
\left( -\frac{1}{1}, \frac{1}{1} \right) &= \Phi\left( \frac{1}{\sqrt{2}} \right), \\
\left( -\frac{1}{5}, \frac{7}{5} \right) &= \Phi\left( \frac{2\sqrt{2}}{3} \right), \\
\left( \frac{1}{5}, \frac{2}{5} \right) &= \Phi\left( \frac{3}{2\sqrt{2}} \right), \\
\left( \frac{1}{5}, \frac{2}{5} \right) &= \Phi\left( \frac{\sqrt{2}}{2} \right), \\
\left( \frac{23}{17}, -\frac{7}{17} \right) &= \Phi\left( \frac{4\sqrt{2}}{3} \right), \\
\left( \frac{7}{5}, -\frac{1}{5} \right) &= \Phi\left( \frac{3}{\sqrt{2}} \right), \\
\left( \frac{41}{29}, -\frac{1}{29} \right) &= \Phi\left( \frac{5\sqrt{2}}{3} \right), \\
\left( \frac{7}{5}, \frac{1}{5} \right) &= \Phi\left( 2\sqrt{2} \right), \\
\left( \frac{1}{1}, \frac{1}{1} \right) &= \Phi(\infty).
\end{align*}$$

Note that the height of $\Phi^{-1}(\frac{a}{c}, \frac{b}{c})$ is $c - (a + b)/2$.

6.3 | The case $S = S_{III}^1$

A rational point $z = \left( \frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c} \right)$ on $S_{III}^1$ has height $Ht(z) = a + b + c$, the Ford horocycle based at $z$ has radius

$$\rho = \frac{\sqrt{2}}{\sqrt{3} + 2(a + b + c)}.$$ 

Since $c = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$, for $z = \left( \frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c} \right)$ and $z' = \left( \frac{a'}{a'+b'+c'}, \frac{b'}{a'+b'+c'}, \frac{c'}{a'+b'+c'} \right)$ we have

$$(z - c) \cdot (z' - c) = z \cdot z' - z \cdot c - z' \cdot c + c \cdot c$$

$$= \frac{aa' + bb' + cc'}{(a + b + c)(a' + b' + c')} - \frac{1}{3} - \frac{1}{3} + \frac{1}{3}$$

$$= \frac{(a + b + c)(a' + b' + c')}{(a + b + c)(a' + b' + c')} - \frac{ab' + ac' + ba' + bc' + ca' + cb'}{(a + b + c)(a' + b' + c')} - \frac{1}{3}$$

$$= \frac{2}{3} - \frac{ab' + ac' + ba' + bc' + ca' + cb'}{(a + b + c)(a' + b' + c')}.$$
The inequality (48) implies that 
\[ ab' + ac' + ba' + bc' + ca' + cb' \geq 1. \]

So we conclude that two horocycles based at \( z = (\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}) \) and \( z' = (\frac{-a'}{a'+b'+c'}, \frac{-b'}{a'+b'+c'}, \frac{c'}{a'+b'+c'}) \) are tangent if and only if \( ab' + ac' + ba' + bc' + ca' + cb' = 1 \).

In Figure 6, we give pictures of horospheres based at the following rational points on \( S^1_{III} \) and corresponding points on \( R^3_3 \):

\[
\begin{align*}
(\frac{-2}{7}, \frac{3}{7}, \frac{6}{7}) &= \Phi(-2), \\
(\frac{0}{1}, \frac{1}{1}, \frac{0}{1}) &= \Phi(0), \\
(\frac{6}{7}, \frac{3}{7}, \frac{-2}{7}) &= \Phi(\frac{2}{3}), \\
(\frac{0}{1}, \frac{0}{1}, \frac{1}{1}) &= \Phi(\infty).
\end{align*}
\]

Note that the height of \( \Phi^{-1}(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}) \) is \( a + b \).

### 6.4 The case \( S = S^2_I \)

A rational point \( z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) on \( S^2_I \) has height \( Ht(z) = d \), the Ford horosphere based at \( z \) has radius \( 1/(1 + \sqrt{2}d) \). For \( z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) and \( z' = (\frac{a'}{d'}, \frac{b'}{d'}, \frac{c'}{d'}) \), the inequality (48) implies that

\[
(z - c) \cdot (z' - c) = \frac{aa' + bb' + cc'}{dd'} \leq 1 - \frac{1}{dd'}.
\]

Therefore, we have \( aa' + bb' + cc' - dd' \leq -1 \) and two horospheres based at \( z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) and \( z' = (\frac{a'}{d'}, \frac{b'}{d'}, \frac{c'}{d'}) \) are tangent if and only if \( aa' + bb' + cc' - dd' = -1 \). In Figure 7, we give diagrams of horospheres based at six rational points on \( S^2_I \) and corresponding points on \( C^3_3 \):

\[
\begin{align*}
(\frac{1}{1}, \frac{0}{1}, \frac{0}{1}) &= \Phi(1), \\
(\frac{0}{1}, \frac{1}{1}, \frac{0}{1}) &= \Phi(1+i), \\
(\frac{0}{1}, \frac{0}{1}, \frac{1}{1}) &= \Phi(\infty), \\
(\frac{-1}{1}, \frac{0}{1}, \frac{0}{1}) &= \Phi(i), \\
(\frac{0}{1}, \frac{-1}{1}, \frac{0}{1}) &= \Phi(0), \\
(\frac{0}{1}, \frac{0}{1}, \frac{-1}{1}) &= \Phi\left(\frac{1}{1-i}\right).
\end{align*}
\]

Note that the height of \( \Phi^{-1}(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) is \( d - c \).

### 6.5 The case \( S = S^2_{II} \)

A rational point \( z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) on \( S^2_{II} \) has height \( Ht(z) = d \), the Ford horosphere based at \( z \) has radius \( \sqrt{2}/(1 + 2d) \). For \( z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) \) and \( z' = (\frac{a'}{d'}, \frac{b'}{d'}, \frac{c'}{d'}) \), the inequality (48) implies that

\[
(z - c) \cdot (z' - c) = \frac{aa' + bb' + cc'}{dd'} \leq 2 - \frac{1}{dd'}.
\]
These graphs show how horospheres at $S$ or $C$ are tangent to one another. When two vertices are connected, the corresponding horosphere are tangent to one another. The radius of horospheres based at $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ and $\Phi^{-1}(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ are $\frac{1}{1 + \sqrt{2c}}$ (above) and $\frac{1}{2(d-c)}$ (below), respectively.

Therefore, we have $aa' + bb' + cc' - 2dd' \leq -1$ and two horospheres based at $z = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ and $z' = (\frac{a'}{d'}, \frac{b'}{d'}, \frac{c'}{d'})$ are tangent if and only if $aa' + bb' + cc' - 2dd' = -1$. In Figure 8, we give diagrams of horospheres based at twelve rational points on $S_\Pi$ and corresponding points on $C$:

$$
\begin{align*}
\left(\frac{1}{1}, \frac{1}{1}, \frac{0}{1}\right) &= \Phi(1), & \left(\frac{1}{1}, \frac{-1}{1}, \frac{0}{1}\right) &= \Phi\left(\frac{1}{1-\omega}\right), & \left(\frac{-1}{1}, \frac{1}{1}, \frac{0}{1}\right) &= \Phi(1 + \omega), \\
\left(\frac{-1}{1}, \frac{-1}{1}, \frac{0}{1}\right) &= \Phi\left(\frac{-1+\omega}{1+\omega}\right), & \left(\frac{1}{1}, \frac{0}{1}, \frac{1}{1}\right) &= \Phi(0), & \left(\frac{1}{1}, \frac{0}{1}, \frac{-1}{1}\right) &= \Phi\left(\frac{\omega}{1+\omega}\right), \\
\left(\frac{-1}{1}, \frac{0}{1}, \frac{1}{1}\right) &= \Phi(\omega), & \left(\frac{-1}{1}, \frac{1}{1}, \frac{-1}{1}\right) &= \Phi\left(\frac{2}{1-\omega}\right), & \left(\frac{0}{1}, \frac{1}{1}, \frac{1}{1}\right) &= \Phi(\infty), \\
\left(\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}\right) &= \Phi\left(\frac{-1+\omega}{\omega}\right), & \left(\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}\right) &= \Phi\left(\frac{-1}{\omega}\right), & \left(\frac{0}{1}, \frac{-1}{1}, \frac{-1}{1}\right) &= \Phi\left(\frac{1+\omega}{2}\right).
\end{align*}
$$

Note that the height of $\Phi^{-1}(\frac{a}{d'}, \frac{b}{d'}, \frac{c}{d'})$ is $2d - b - c$. 


These graphs show how horospheres at $S$ or $C$ are tangent to one another. When two vertices are connected, the corresponding horosphere are tangent to one another. The radius of horospheres based at $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ and $\Phi^{-1}(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ are $\sqrt{\frac{2}{1+2c}}$ (above) and $\frac{1}{2(2d-b-c)}$ (below), respectively.

6.6 The case $S = S^2_{III}$

A rational point $z = (\frac{a}{a+b+c+d}, \frac{b}{a+b+c+d}, \frac{c}{a+b+c+d}, \frac{d}{a+b+c+d})$ on $S^2_{III}$ has height $Ht(z) = a + b + c + d$, the Ford horosphere based at $z$ has radius

$$\rho = \frac{\sqrt{3}}{2 + \sqrt{6(a + b + c + d)}}.$$
Since \( \mathbf{c} = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \), for two rational points \( \mathbf{z} \) and \( \mathbf{z}' \) we have

\[
(\mathbf{z} - \mathbf{c}) \cdot (\mathbf{z}' - \mathbf{c}) = \mathbf{z} \cdot \mathbf{z}' - \mathbf{z} \cdot \mathbf{c} - \mathbf{z}' \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}
\]

\[
= \frac{aa' + bb' + cc' + dd'}{(a + b + c + d)(a' + b' + c' + d')} = \frac{1}{4} - \frac{1}{4} + \frac{1}{4}
\]

\[
= \frac{(a + b + c + d)(a' + b' + c' + d')}{(a + b + c + d)(a' + b' + c' + d')} = \frac{1}{4}
\]
\[
- \frac{ab' + ac' + ad' + ba' + bc' + bd' + ca' + cb' + cd' + da' + db' + dc'}{(a + b + c + d)(a' + b' + c' + d')}
= \frac{3}{4} - \frac{ab' + ac' + ad' + ba' + bc' + bd' + ca' + cb' + cd' + da' + db' + dc'}{(a + b + c + d)(a' + b' + c' + d')}
\]

The inequality (48) implies that

\[ab' + ac' + ad' + ba' + bc' + bd' + ca' + cb' + cd' + da' + db' + dc' \geq 1.\]

Two horospheres based at \(z\) and \(z'\) are tangent if and only if

\[ab' + ac' + ad' + ba' + bc' + bd' + ca' + cb' + cd' + da' + db' + dc' = 1.\]

In Figure 9, we give diagrams of horospheres based at eight rational points on \(\mathbb{S}^2_{\text{III}}\) and corresponding points on \(\mathbb{C}\):

\[
\begin{align*}
\left(\frac{1}{2}, \frac{1}{2}, 1, 1\right) &= \Phi(0), \\
\left(0, \frac{1}{1}, 0, \frac{1}{1}\right) &= \Phi(1), \\
\left(0, 0, \frac{1}{1}, \frac{1}{1}\right) &= \Phi(1 + \omega), \\
\left(0, 0, 0, \frac{1}{1}\right) &= \Phi(\infty), \\
\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) &= \Phi(2 + \omega), \\
\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) &= \Phi(\omega), \\
\left(\frac{1}{2}, 2, -\frac{1}{2}, \frac{1}{2}\right) &= \Phi\left(\frac{1}{1-\omega}\right) \\
\left(1, 2, 2, -\frac{1}{2}\right) &= \Phi(\omega).
\end{align*}
\]

Note that the height of \(\Phi^{-1}(\frac{a}{a+b+c+d}, \frac{b}{a+b+c+d}, \frac{c}{a+b+c+d}, \frac{d}{a+b+c+d})\) is \(a + b + c\).

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