Generalized comonotonicity and new axiomatizations of Sugeno integrals on bounded distributive lattice

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Abstract

Two new generalizations of the relation of comonotonicity of lattice-valued vectors are introduced and discussed. These new relations coincide on distributive lattices and they share several properties with the comonotonicity for the real-valued vectors (which need not hold for $L$-valued vectors comonotonicity, in general). Based on these newly introduced generalized types of comonotonicity of $L$-valued vectors, several new axiomatizations of $L$-valued Sugeno integrals are introduced. One of them brings a substantial decrease of computational complexity when checking an aggregation function to be a Sugeno integral.

Keywords: bounded distributive lattice, comonotonicity, information fusion, Sugeno integral

1. Introduction

Sugeno \cite{19} has introduced an integral (called $F$-integral, or fuzzy integral in the original source) for fusion of information obtained in a fuzzy set characterized by its membership function. This integral was acting on $[0, 1]$ due to the range of membership functions of fuzzy sets, and it was built by means of...
the basic fuzzy connectives min and max, following the introduction of fuzzy sets due to Zadeh [20]. Recall that the Sugeno integral is formally introduced as the Lebesgue and Choquet integrals, replacing the standard arithmetic operations + and · on the real unit interval [0,1] by the lattice operations max and min on the bounded chain [0,1]. Similarly as the fuzzy sets with [0,1]-valued membership functions were generalized by Goguen [5] into \( L \)-valued fuzzy sets, where \( L \) stands for a bounded distributive lattice, also the Sugeno integral can be generalized to act on \( L \). Formally, even non-distributive lattices \( L \) could be considered, however, then some ambiguities may occur. For example, the values of considered functionals with fixed values in boolean vectors (formally, the underlying fuzzy measure or capacity) need not be unique when two forms of formulas for the Sugeno integral, see Definition 2.1 are considered. Among several deep studies of \( L \)-valued Sugeno integrals we recall \([10, 12]\) and especially \([1]\), where several equivalent axiomatic characterizations appear. Note that papers \([10]\) and \([12]\) study the Sugeno integrals from the different point of views as it is intended in this paper. In a recent paper \([10]\) a new characterizing property of Sugeno integrals, based on the preservation of certain equivalence relations (the so-called compatibility), has been presented. In \([12]\) it was shown that Sugeno integrals form a subclass of weighted lattice polynomial functions, which can be characterized by an important median based decomposition formula.

Let us mention that various types of integrals have many applications within the theory of aggregation functions. To illustrate that the study of Sugeno integrals from various points of view is still very active area of aggregation functions, we refer the reader to recent papers \([3, 4, 8, 14, 17]\). More specifically, in \([8, 4]\) a new promising approach via so-called clone theory to a study of aggregation functions on bounded lattices has been started. We expect to apply these results for better understanding of properties of Sugeno integrals on distributive lattices. Moreover, following the spirit of papers \([3, 14, 17]\) dealing with certain non-additive measures, we expect to introduce Sugeno integrals on non-distributive lattices.

The axiomatic approach to Sugeno integrals was studied in several papers,
including [13, 15, 16] and it is based on the notion of comonotonicity of real functions (of real vectors in the case of discrete Sugeno integrals). However, in the case of lattice-valued vectors, the relation of comonotonicity has some undesirable properties. For example, there can exist a vector \( x \) such that it is not comonotone with itself (i.e., \( x \) and \( x \) are not comonotone), neither it is comonotone with an arbitrary constant vector \( c \), i.e., \( x \) and \( c \) are not comonotone. Thus a generalization of the comonotonicity relation for \( L \)-valued vectors avoiding the above mentioned defects is a challenging problem.

The aim of this contribution is the introduction of generalized comonotonicity relations with properties similar to the real-valued vectors comonotonicity and a subsequent development of the Sugeno integral theory on bounded distributive lattices as an important tool in \( L \)-valued information fusion. We introduce two new types of comonotonicity, named generalized comonotonicity and dual generalized comonotonicity here, and we apply them to get new axiomatic characterizations of \( L \)-valued Sugeno integrals.

The organization of the paper is as follows: In Section 2 we recall the basic definitions and notions concerning the discrete Sugeno integrals. The notions of generalized comonotonicity and dual generalized comonotonicity are introduced and studied in Section 3. New axiomatic characterizations of \( L \)-valued Sugeno integrals are discussed in Section 4.

2. Preliminaries

Let \( L \) be a bounded distributive lattice. Through the paper we denote by 0 and 1 the bottom and the top element of \( L \), respectively. Recall that a function \( f: L^n \to L, n \geq 1 \) being an integer, is called an aggregation function (on \( L \)) whenever it is monotone and satisfies two boundary conditions, \( f(0, \ldots ,0) = 0 \) and \( f(1, \ldots ,1) = 1 \).

For any integer \( n \geq 1 \) we set \([n] = \{1, \ldots ,n\}\). Recall that given an \( L \)-valued capacity \( m: 2^{[n]} \to L \), i.e., a set function satisfying \( m(X) \leq m(Y) \) for \( X \subseteq Y \subseteq [n] \) and \( m(\emptyset) = 0, m([n]) = 1 \), there are two equivalent expressions
for the Sugeno integral.

**Definition 2.1.** Let $n \geq 1$ be a positive integer and $m : 2^{[n]} \to L$ be an $L$-valued capacity. The Sugeno integral with respect to the capacity $m$ is defined by

$$Su_m(x) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} x_i) = \bigwedge_{I \subseteq [n]} (m([n] \setminus I) \lor \bigvee_{i \in I} x_i),$$

where $x = (x_1, \ldots, x_n) \in L^n$.

Observe, that for any $L$-valued capacity $m$, the Sugeno integral $Su_m$ is an aggregation function on $L$.

Two vectors $x = (x_1, \ldots, x_n) \in L^n$ and $y = (y_1, \ldots, y_n) \in L^n$ are said to be **comonotone** if $x_i \leq x_j$ and $y_i \leq y_j$ or $x_i \geq x_j$ and $y_i \geq y_j$ for all pairs $i, j \in \{1, \ldots, n\}$. Equivalently, $x, y \in L^n$ are comonotone if and only if there is a permutation $\sigma$ of the set $[n]$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(i)} \leq \cdots \leq x_{\sigma(n)}$ and $y_{\sigma(1)} \leq \cdots \leq y_{\sigma(i)} \leq \cdots \leq y_{\sigma(n)}$.

For $c \in L$, denote by $c = (c, \ldots, c)$ the constant vector. A function $f : L^n \to L$ is said to be

- **inf-homogeneous** if, for every $x \in L^n$ and every $c \in L$, $f$ satisfies $f(c \land x) = c \land f(x)$
- **sup-homogeneous** if, for every $x \in L^n$ and every $c \in L$, $f$ satisfies $f(c \lor x) = c \lor f(x)$
- **comonotone supremal** if $f(x \lor y) = f(x) \lor f(y)$ for every pair of comonotone vectors $x, y \in L^n$
- **comonotone infimal** if $f(x \land y) = f(x) \land f(y)$ for every pair of comonotone vectors $x, y \in L^n$.

Let us note that if $L$ is a chain (e.g., the real line), inf(sup)-homogeneity and comonotone supremality (infimality) are commonly referred to as min(max)-homogeneity and comonotone maxitivity (minitivity) respectively.

We conclude this section with recalling the following well-known characterization of the discrete Sugeno integrals on bounded chains, see e.g. [1, 6].
Proposition 2.2. Let $L$ be a bounded chain and $f : L^n \to L$, $n \geq 1$, be an aggregation function. The following conditions are equivalent:

(i) $f$ is a discrete Sugeno integral.

(ii) $f$ is comonotone maxitive and min-homogeneous.

(iii) $f$ is comonotone minitive and max-homogeneous.

Let us note that the crucial step in the proof relies on finding an appropriate permutation $\sigma$ of the set $[n]$, such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$. Obviously, this is, in general, no longer possible provided $L$ contains incomparable elements.

3. Generalized and dually generalized comonotonicity

In this section we introduce the notions of generalized and dually generalized comonotonicity and study their basic properties.

3.1. Definition and intuition

The notion of comonotonicity of vectors can be introduced for any poset, and, in particular, for any bounded distributive lattice. However, then some genuine properties of comonotonicity of real vectors are lost. For example, for real valued vectors (functions), for any vector $x$ and any constant vector $c$ the couple $x, c$ is comonotone. Similarly, the couple $x, x$ is comonotone for any real valued vector, but not for $L$-valued vectors once $L$ is not a chain. Of course, these problems are caused by a possible incomparability of some elements of $L$. To eliminate the above mentioned defects, we introduce a new concept of generalized comonotonicity and its dual counterpart.

Definition 3.1. Let $L$ be a lattice. Given two $n$-ary vectors $x$ and $y$, we call them generalized comonotone (g-comonotone, for short) if for every pair $i, j \in \{1, \ldots, n\}$ we have

$$(x_i \vee y_i) \wedge (x_j \vee y_j) = (x_i \wedge x_j) \vee (y_i \wedge y_j).$$

(2)
Two $n$-ary vectors $\mathbf{x}$ and $\mathbf{y}$ are called \textit{dually generalized comonotone} if for every pair $i, j \in \{1, \ldots, n\}$ we have

$$
(x_i \land y_i) \lor (x_j \land y_j) = (x_i \lor x_j) \land (y_i \lor y_j).
$$

(3)

We call an $n$-ary aggregation function $f$ on $L$ \textit{g-comonotone supremal}, if

$$
f(\mathbf{x} \lor \mathbf{y}) = f(\mathbf{x}) \lor f(\mathbf{y}),
$$

and dually, $f$ is said to be \textit{g-comonotone infimal}, if

$$
f(\mathbf{x} \land \mathbf{y}) = f(\mathbf{x}) \land f(\mathbf{y})
$$

for any pair of generalized comonotone vectors $\mathbf{x}, \mathbf{y} \in L^n$. The similar notions can be applied in the case of dually generalized comonotone vectors.

Let us remark that the identities (2) and (3) are already known in the literature as so-called interchange identities. Interchange identity have its origin in category theory where it is related to a characterization of natural transformations of functors, we refer the reader to a classic book [11]. Not going into details, similar identities relating the Lie and the Jordan products are also deeply studied in computer algebra.

Formally, let $\bullet$ and $\circ$ be two binary operations on a set. Then the following identity

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$$

is called the interchange identity (compare also the commuting of aggregation functions discussed in [18]).

It can be easily seen that putting $\circ = \lor$ and $\bullet = \land$ we obtain the identity (2), and similarly, $\circ = \land$ and $\bullet = \lor$ yields the dual identity (3).

Regarding $\lor$ and $\land$ as vertical and horizontal compositions respectively, the identity (2) expresses the equivalence of two decompositions of a $2 \times 2$ array:

$$
(x_i \lor y_i) \land (x_j \lor y_j) \equiv \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} = \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} \equiv (x_i \land x_j) \lor (y_i \land y_j).
$$

Hence, two vectors $\mathbf{x}, \mathbf{y} \in L^n$ are g-comonotone if and only if for any choice $i, j$ of indexes the above interchange identity is fulfilled.
To give more intuition concerning g-comonotonicity, Figure 1 schematically illustrates configuration in a lattice when the inputs \(x_i, x_j, y_i, y_j\) are pairwise incomparable elements.

3.2. Properties

We start an investigation of the introduced notions with the following simple lemma, relating comparability and classical comonotonicity with g-comonocity and dual g-comonocity respectively.

**Lemma 3.2.** Let \(L\) be a lattice. If \(x, y \in L^n\) are comonotone or comparable, then they are g-comonotone as well as dually g-comonotone.

**Proof.** Let \(x, y \in L^n\) be comonotone vectors and \(i, j \in \{1, \ldots, n\}\) be any two indexes. Without loss of generality, assume that \(x_i \leq x_j\) and \(y_i \leq y_j\). Then it is easily seen that

\[
(x_i \lor y_i) \land (x_j \lor y_j) = (x_i \lor y_i) = (x_i \land x_j) \lor (y_i \land y_j),
\]

proving that \(x\) and \(y\) are g-comonotone. Similarly, one can show that \(x\) and \(y\) are also dually g-comonotone.

Further, let \(x\) and \(y\) be two comparable vectors. Assume that \(x \leq y\). Then for any two indexes \(i, j \in \{1, \ldots, n\}\) we have \(x_i \leq y_i\) and \(x_j \leq y_j\). From this we
Figure 2: The sets $A(x)$, $B(x)$ and $C(x)$ for a given vector $x \in [0,1]^2$.

obtain

$$(x_i \lor y_i) \land (x_j \lor y_j) = y_i \land y_j = (x_i \land x_j) \lor (y_i \land y_j).$$

The dual g-comonotonicity can be proved analogously.

The previous lemma shows that the notions of g-comonotonicity and dual g-comonotonicity generalize that of comonotonicity and comparability. Moreover, for a constant vector $c$, a vector $x \in L^n$ is comonotone with $c$ only if the set $\{x_1, \ldots, x_n\}$ forms a chain in $L$. Similarly, if $c \in L \setminus \{0,1\}$, there are vectors which are not comparable with $c$. However, for any vector $x \in L^n$, $x$ and $c$ are g-comonotone (dually g-comonotone). Indeed, substituting $y_i = y_j = c$ in (2), then applying distributivity of $L$, we obtain the equality

$$(x_i \lor c) \land (x_j \lor c) = (x_i \land x_j) \lor (c \land c)$$

for any $i, j \in \{1, \ldots, n\}$. Similarly, the dual g-comonotonicity for such a pair of vectors can be verified.

**Example 3.3.** Consider the product $L = [0,1] \times [0,1]$. For a fixed element $x = (x_1, x_2) \in [0,1]^2$, consider the sets of all vectors $y$ such that $x$ and $y$ are comonotone, comparable and g-comonotone, respectively. Particularly, we put $A(x) = \{y \mid x, y \text{ comonotone}\}$, $B(x) = \{y \mid x, y \text{ comparable}\}$ and $C(x) = \{y \mid x, y \text{ g-comonotone}\}$. For a point $x = (x_1, x_2) \in [0,1]^2$ with $x_1 > x_2$, these sets are depicted in Figure 2. As the figure indicates, for this particular point $x$, the
set $C(x)$ is the union of the sets $A(x)$ and $B(x)$.

We show that this is valid in general, i.e., $C(x) = A(x) \cup B(x)$ for all $x \in [0,1]^2$. To observe this, let $x = (x_1, x_2)$ be such that $x_1 > x_2$ and $y = (y_1, y_2)$ be arbitrary. Assume that $y \notin A(x) \cup B(x)$. Then necessarily $y_1 < y_2$ (since $x$, $y$ are not comonotone), and $y_1 < x_1, y_2 > x_2$ as $x$, $y$ are incomparable.

However, from this we obtain

$$(x_1 \lor y_1) \land (x_2 \lor y_2) = x_1 \land y_2 > x_2 \lor y_1 = (x_1 \land x_2) \lor (y_1 \land y_2),$$

i.e., $x$, $y$ are not g-comonotone. Hence $C(x) \subseteq A(x) \cup B(x)$ and this inclusion can be also proved for $x$ satisfying $x_1 < x_2$. Note that $A(x) = [0,1]^2$ provided $x = (x, x)$ for some $x \in [0,1]$. Since the opposite inclusion $C(x) \supseteq A(x) \cup B(x)$ follows from Lemma 3.2, we obtain $C(x) = A(x) \cup B(x)$ for all $x \in [0,1]^2$.

Let us remark that the above equality $C(x) = A(x) \cup B(x)$ does not hold in higher dimensions.

**Example 3.4.** Consider $L = [0,1]^3$ and the vectors $x = (0.6, 0.3, 0.5)$ and $y = (0.7, 0.2, 0.9)$. It can be easily seen that they are incomparable as well as they are not comonotone. On the other hand, they fulfill equalities

$$(x_1 \lor y_1) \land (x_2 \lor y_2) = 0.7 \land 0.3 = 0.3 \lor 0.2 = (x_1 \land x_2) \lor (y_1 \land y_2),$$

$$(x_1 \lor y_1) \land (x_3 \lor y_3) = 0.7 \land 0.9 = 0.7 \lor 0.5 = (x_1 \land x_3) \lor (y_1 \land y_3),$$

$$(x_2 \lor y_2) \land (x_3 \lor y_3) = 0.3 \land 0.9 = 0.3 \lor 0.2 = (x_2 \land x_3) \lor (y_2 \land y_3),$$

showing that they are g-comonotone.

Observe that the generalized comonotonicity in $[0,1]^n$, $n \geq 2$ can be characterized as follows: by definition, two vectors $x$ and $y$ are g-comonotone if and only if for any $i, j \in \{1, \ldots, n\}$ the pairs $(x_i, x_j)$ and $(y_i, y_j)$ satisfy (2). However, given fixed $i, j \in \{1, \ldots, n\}$, according to Example 3.3 the equation (2) is valid if and only if $(y_i, y_j) \notin A(x_i, x_j) \cup B(x_i, x_j)$, i.e., when the pairs $(x_i, x_j)$ and $(y_i, y_j)$ are comparable or $(x_i, x_j)$ and $(y_i, y_j)$ are comonotone.
Coming back to our example, note that the vectors \( (x_1, x_2) \) and \( (y_1, y_2) \) are comonotone, the vectors \( (x_1, x_3) \) and \( (y_1, y_3) \) are comparable, while the vectors \( (x_2, x_3) \) and \( (y_2, y_3) \) are comonotone.

For an arbitrary lattice \( L \), the notions of g-comonotonicity and dual g-comonotonicity need not be equivalent. However, in what follows we show that this will be the case when considering the distributive lattices.

**Theorem 3.5.** On any distributive lattice \( L \), generalized comonotonicity is self-dual, i.e., it is equivalent to dual generalized comonotonicity.

**Proof.** Assume that \( x, y \in L^n \) are g-comonotone and \( \{i, j\} \subseteq \{1, \ldots, n\} \) be a pair of indexes. Then

\[
(x_i \lor y_i) \land (x_j \lor y_j) = (x_i \land x_j) \lor (y_i \land y_j).
\]

Applying distributivity of \( L \), we obtain

\[
(x_i \land x_j) \lor (y_i \land y_j) \lor (x_i \land y_j) \lor (y_i \land y_j) = (x_i \land x_j) \lor (y_i \land y_j),
\]

which is equivalent to

\[
(x_i \land y_j) \lor (y_i \land x_j) \leq (x_i \land x_j) \lor (y_i \land y_j).
\]

Note that the equivalence follows from the fact that in any lattice \( M \), for \( a, b \in M \) we have \( a \lor b = b \) if and only if \( a \leq b \). In our case \( a = (x_i \land y_j) \lor (y_i \land x_j) \) and \( b = (x_i \land x_j) \lor (y_i \land y_j) \).

Now, applying distributivity once more and the above inequality for the right-hand side of the dual g-comonotonicity \( \text{(3)} \) we obtain \( (x_i \lor x_j) \land (y_i \lor y_j) = (x_i \land y_i) \lor (x_i \land y_j) \lor (x_j \land y_i) \lor (x_j \land y_j) \leq (x_i \land y_i) \lor (x_i \land x_j) \lor (y_i \land y_j) \lor (x_j \land y_j) \).

Clearly, \( (x_i \land y_i) \lor (x_i \land x_j) = x_i \land (y_i \lor x_j) \leq x_i, (y_i \land y_j) \lor (x_j \land y_j) = (y_i \lor x_j) \land y_j \leq y_j \), hence \( (x_i \lor x_j) \land (y_i \lor y_j) \leq x_i \lor y_j \).

Similarly, \( (x_i \lor y_i) \lor (y_i \land y_j) \leq y_i \) and \( (x_i \land x_j) \lor (x_j \land y_j) \leq x_j \), i.e. \( (x_i \lor x_j) \land (y_i \lor y_j) \leq x_j \lor y_i \) and we have

\[
(x_i \lor x_j) \land (y_i \lor y_j) \leq (x_j \lor y_i) \land (x_i \lor y_j)
\]
or equivalently

\[(x_i \lor x_j) \land (y_i \lor y_j) \land (x_j \lor y_i) \land (x_i \lor y_j) = (x_i \lor x_j) \land (y_i \lor y_j).\]

By distributivity of \(L\), this is equivalent to

\[(x_i \land y_i) \lor (x_j \land y_j) = (x_i \lor x_j) \land (y_i \lor y_j),\]

which shows that \(x\) and \(y\) are dually g-comonotone.

The converse implication can be done in a similar way by using dual arguments. \(\square\)

Before we prove an important lemma concerning g-comonotonicity, or equivalently dual g-comonotonicity in distributive lattices, we recall the following well-known fact, cf. [7].

**Remark 3.6.** Let \(L\) be a distributive lattice and let \((\lambda_{i,0})_{i \in I}, (\lambda_{i,1})_{i \in I}, I \neq \emptyset\) finite, be two families of elements of \(L\). Then

\[\bigwedge_{i \in I} (\lambda_{i,0} \lor \lambda_{i,1}) = \bigvee_{\varphi \in \{0,1\}^I} \bigwedge_{i \in I} \lambda_{i,\varphi(i)}\]  

(4)

and dually

\[\bigvee_{i \in I} (\lambda_{i,0} \land \lambda_{i,1}) = \bigwedge_{\varphi \in \{0,1\}^I} \bigvee_{i \in I} \lambda_{i,\varphi(i)},\]  

(5)

where \(\{0,1\}^I = \{\varphi \mid \varphi : I \to \{0,1\}\}\) denotes the set of all functions with domain \(I\) and values in \(\{0,1\}\).

**Lemma 3.7.** Let \(L\) be a distributive lattice and \(x, y \in L^n\) be two \(n\)-ary vectors. Then \(x\) and \(y\) are g-comonotone if and only if

\[\bigwedge_{i \in I} (x_i \lor y_i) = \bigwedge_{i \in I} x_i \lor \bigwedge_{i \in I} y_i\]  

(6)

for any non-empty subset \(I \subseteq \{1, \ldots, n\}\).

Similarly, \(x\) and \(y\) are dually g-comonotone if and only if

\[\bigvee_{i \in I} (x_i \land y_i) = \bigvee_{i \in I} x_i \land \bigvee_{i \in I} y_i\]  

(7)

for any non-empty subset \(I \subseteq \{1, \ldots, n\}\).
The proof is a series of logical steps. The second one can be proved using the dual arguments. Obviously, \( (\text{II}) \) applied to a two-element subset \( I = \{i, j\} \) yields \( (\text{I}) \), i.e., two vectors \( \mathbf{x}, \mathbf{y} \) are g-comonotone, provided they satisfy \( (\text{I}) \).

Conversely, assume that \( \mathbf{x} \) and \( \mathbf{y} \) are g-comonotone. Then evidently \( (\text{II}) \) holds for \( I = \emptyset \) as well as for any one or two-element subset \( I \subseteq \{1, \ldots, n\} \).

Thus, assume further that \( I \subseteq \{1, \ldots, n\} \) is an arbitrary subset, \( |I| = m \) where \( 3 \leq m \leq n \) and that \( (\text{II}) \) is valid for any subset \( J \subseteq \{1, \ldots, n\} \) with \( |J| = m - 1 \).

Then with respect to the induction hypothesis, we obtain

\[
\bigwedge_{i \in I}(x_i \lor y_i) = \bigwedge_{i \in I} \bigwedge_{j \in I \setminus \{i\}}(x_j \lor y_j) = \bigvee_{j \in I \setminus \{i\}}\bigwedge_{i \in I}(x_j \lor y_j).
\]

For \( i \in I \) put \( \lambda_{i,0} = \bigwedge_{j \in I \setminus \{i\}} x_j \) and \( \lambda_{i,1} = \bigwedge_{j \in I \setminus \{i\}} y_j \). According to \( (\text{II}) \) we have

\[
\bigwedge_{i \in I}(x_i \lor y_i) = (\lambda_{i,0} \lor \lambda_{i,1}) = \bigvee_{\varphi \in \{0,1\}^I} \bigwedge_{i \in I} \lambda_{i,\varphi(i)}.
\]

For the constant functions \( \varphi_0, \varphi_1 : I \to \{0,1\} \) such that \( \varphi_0(i) = 0 \) and \( \varphi_1(i) = 1 \) for all \( i \in I \) we have

\[
\bigwedge_{i \in I} \lambda_{i,\varphi_0(i)} = \bigwedge_{i \in I} \bigwedge_{j \in I \setminus \{i\}} x_j = \bigwedge_{i \in I} x_i \quad \text{and} \quad \bigwedge_{i \in I} \lambda_{i,\varphi_1(i)} = \bigwedge_{i \in I} \bigwedge_{j \in I \setminus \{i\}} y_j = \bigwedge_{i \in I} y_i.
\]

If \( \varphi : I \to \{0,1\} \) is non-constant, then \( \varphi^{-1}(0) \) or \( \varphi^{-1}(1) \) contains at least two elements since \( |I| \geq 3 \). Suppose that \( \{i_1, i_2\} \subseteq \varphi^{-1}(0) \). Then

\[
\bigwedge_{i \in I} \lambda_{i,\varphi(i)} \leq \lambda_{i_1,\varphi(i_1)} \land \lambda_{i_2,\varphi(i_2)} = \lambda_{i_1,0} \land \lambda_{i_2,0} = \bigwedge_{j \in I \setminus \{i_1\}} x_j \land \bigwedge_{j \in I \setminus \{i_2\}} x_j = \bigwedge_{i \in I} x_i.
\]

Similarly,

\[
\bigwedge_{i \in I} \lambda_{i,\varphi(i)} \leq \bigwedge_{i \in I} y_i,
\]

provided \( \{i_1, i_2\} \subseteq \varphi^{-1}(1) \). Consequently, we obtain

\[
\bigwedge_{i \in I}(x_i \lor y_i) = \bigvee_{\varphi \in \{0,1\}^I} \bigwedge_{i \in I} \lambda_{i,\varphi(i)} = \bigwedge_{i \in I} x_i \lor \bigwedge_{i \in I} y_i \lor \bigvee_{\varphi \in \{0,1\}^I \setminus \varphi_0, \varphi_1} \bigwedge_{i \in I} \lambda_{i,\varphi(i)} = \bigwedge_{i \in I} x_i \lor \bigwedge_{i \in I} y_i,
\]

which shows that \( (\text{II}) \) also holds for the subset \( I \). □
As a consequence of the previous two assertions we obtain the following theorem.

**Theorem 3.8.** Let $L$ be a distributive lattice and $x, y \in L^n$ be two $n$-ary vectors. The following conditions are equivalent:

(i) The vectors $x$ and $y$ are $g$-comonotone.

(ii) The vectors $x$ and $y$ are dually $g$-comonotone.

(iii) $\bigwedge_{i \in I} (x_i \lor y_i) = \bigwedge_{i \in I} x_i \lor \bigwedge_{i \in I} y_i$ holds for each subset $I \subseteq \{1, \ldots, n\}$.

(iv) $\bigvee_{i \in I} (x_i \land y_i) = \bigvee_{i \in I} x_i \land \bigvee_{i \in I} y_i$ holds for each subset $I \subseteq \{1, \ldots, n\}$.

**Proof.** According to Theorem 3.5 the conditions (i) and (ii) are equivalent, while (i) if and only if (iii) and (ii) if and only if (iv) follow from Lemma 3.7. □

4. Alternative characterizations of Sugeno integrals on bounded distributive lattices

The goal of this section is to study some of the characteristic properties of $L$-valued Sugeno integrals, mainly with respect to the defined concept of $g$-comonotonicity.

**Lemma 4.1.** Let $L$ be a bounded distributive lattice and $m: 2^n \to L$ be an $L$-valued capacity. Then the discrete Sugeno integral $S_m$ is inf-homogeneous and $g$-comonotone supremal as well as sup-homogeneous and $g$-comonotone infimal.

**Proof.** Let $m: 2^n \to L$ be a capacity given on $[n]$. Recall (see Definition 2.1), that the corresponding Sugeno integral is defined for all $x = (x_1, \ldots, x_n) \in L^n$ by

$$S_m(x) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} x_i).$$

For an element $c \in L$ and the constant vector $c = (c, \ldots, c)$ we obtain

$$S_m(c \land x) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} (c \land x_i)) = \bigvee_{I \subseteq [n]} (c \land m(I) \land \bigwedge_{i \in I} x_i).$$
Since $m(\emptyset) = 0$, distributivity of $L$ yields

$$\text{Su}_m(c \land x) = c \land \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} x_i) = c \land \text{Su}_m(x),$$

i.e., $\text{Su}_m$ is inf-homogeneous.

Further, to show that $\text{Su}_m$ is g-comonotone supremal, let $x, y \in L^n$ be two generalized comonotone vectors. Then due to (iii) of Theorem 3.8 we have

$$\text{Su}_m(x \lor y) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} (x_i \lor y_i)) = \bigvee_{I \subseteq [n]} \left( m(I) \land \left( \bigwedge_{i \in I} x_i \lor \bigwedge_{i \in I} y_i \right) \right),$$

which is by distributivity equal to

$$\bigvee_{I \subseteq [n]} \left( (m(I) \land \bigwedge_{i \in I} x_i) \lor (m(I) \land \bigwedge_{i \in I} y_i) \right) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} x_i) \lor \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} y_i).$$

Hence $\text{Su}_m(x \lor y) = \text{Su}_m(x) \lor \text{Su}_m(y)$, proving that the Sugeno integral $\text{Su}_m$ is g-comonotone supremal.

Using the dual expression for the Sugeno integral, one can show in the same way that $\text{Su}_m$ is also sup-homogeneous and g-comonotone infimal. \qed

**Remark 4.2.** According to Lemma 3.2 as comonotone vectors are also g-comonotone, we obtain that any discrete Sugeno integral on a bounded distributive lattice is comonotone supremal and comonotone infimal. Obviously, g-comonotone supremality of a function $f$ represents the stronger condition than comonotone supremality in general, since the equality $f(x \lor y) = f(x) \lor f(y)$ is required for more pairs of vectors in the former case.

Before we prove that inf-homogeneity together with comonotone supremality characterize discrete Sugeno integrals, we recall one result from [1].

**Proposition 4.3.** An aggregation function $f$ on a bounded distributive lattice $L$ is a discrete Sugeno integral if and only if $f$ is inf-homogeneous and sup-homogeneous.

With respect to this result a natural question can be raised whether comonotone supremality implies sup-homogeneity. If this is the case, then the proposed
characterization would be just a simple consequence of the result from [1]. In what follows we will discuss this question. First, we prove the following simple, but important lemma.

**Lemma 4.4.** Let \( L \) be a bounded lattice. If \( f : L^n \to L \) is an inf-homogeneous or sup-homogeneous aggregation function, then \( f \) is idempotent.

**Proof.** Assume that \( f \) is inf-homogeneous. Then

\[
f(x, \ldots, x) = x \land f(1, \ldots, 1) = x \land 1 = x.
\]

Dually, if \( f \) is sup-homogeneous, we obtain

\[
f(x, \ldots, x) = x \lor f(0, \ldots, 0) = x \lor 0 = x,
\]

hence in both cases the function \( f \) is idempotent. \( \Box \)

In what follows we give an example of aggregation function which is comonotone supremal but fails to be sup-homogeneous.

**Example 4.5.** Let \( L \) be a bounded distributive lattice with at least three elements and \( n \geq 1 \) an integer. Consider the constant aggregation function \( h : L^n \to L \) given by \( h(0, \ldots, 0) = 0 \) and \( h(x) = 1 \) otherwise. It is easily seen that \( h \) is a \( \lor \)-homomorphism, i.e., \( h(x \lor y) = h(x) \lor h(y) \) for all \( x, y \in L^n \), thus it is comonotone supremal. On the other hand, \( h \) is not idempotent, therefore it cannot be sup-homogeneous. Note that if \( L \) is a two element chain, then each aggregation function on \( L \) is inf-homogeneous as well as sup-homogeneous. Similarly, one can find an aggregation function which is comonotone infimal but fails to be inf-homogeneous.

Let us note that the previous consideration, i.e., the usage of non-surjective \( \lor \)-homomorphism, can be modified to obtain a relatively rich class of aggregation functions which are comonotone supremal but not sup-homogeneous. Let \( L \) be a bounded distributive lattice, \(|L| \geq 3\). Given an integer \( n \geq 1 \), consider a function \( h : L^n \to L \) which is comonotone supremal, e.g., \( Su_m \) represents such
function. Further, let \( g: L \to L \) be a non-surjective \( \lor \)-homomorphism preserving the bottom and the top element of \( L \) respectively. Then the composition \( h \circ g: L^n \to L \) is also a comonotone supremal aggregation function, which fails to be idempotent (it is not surjective). With respect to Lemma 4.4 the function \( h \circ g \) cannot be sup-homogeneous.

The following statement is a simple corollary of Proposition 4.3.

**Corollary 4.6.** Let \( L \) be a distributive lattice and \( f: L^n \to L \) be an inf-homogeneous and \( g \)-comonotone supremal aggregation function. Then \( f \) is a Sugeno integral.

**Proof.** If \( f \) is inf-homogeneous, then it is idempotent. Since every constant vector \( c = (c, \ldots, c) \), \( c \in L \), is \( g \)-comonotone with any \( x \in L^n \), from \( g \)-comonotone supremality of \( f \) it follows that \( f(c \lor x) = f(c) \lor f(x) = c \lor f(x) \), i.e., \( f \) is sup-homogeneous. Consequently, Proposition 4.3 yields that \( f \) is a Sugeno integral. \( \square \)

**Remark 4.7.** Note, that using the dual arguments, it can be similarly shown that if \( f \) is a sup-homogeneous and \( g \)-comonotone infimal aggregation function, then \( f \) is a Sugeno integral.

In the next theorem we introduce an important simplification of Proposition 4.3. Following the notation introduced in [2], we say that \( f: L^n \to L \) is Boolean inf-homogeneous (resp. Boolean sup-homogeneous) if

\[
f(c \land x) = c \land f(x) \quad \left( f(c \lor x) = c \lor f(x) \right)
\]

for all \( x \in \{0,1\}^n \subseteq L^n \) and for all constant vectors \( c = (c, \ldots, c), c \in L \).

**Lemma 4.8.** If a function \( f: L^n \to L \) is Boolean inf-homogeneous and Boolean sup-homogeneous, then it is idempotent.

**Proof.** For any \( c \in L \) we obtain:

\[
c \leq f(0) \lor c = f(0 \lor c) = f(c) = f(1 \land c) = f(1) \land c \leq c.
\]
**Theorem 4.9.** Let $L$ be a distributive lattice and $f : L^n \to L$ be an aggregation function on $L$. The function $f$ is a discrete Sugeno integral on $L$ if and only if it is Boolean sup-homogeneous and Boolean inf-homogeneous.

**Proof.** If $f$ is a Sugeno integral on $L$, then according to Proposition 4.3, $f$ is inf and sup-homogeneous. Consequently it is clear that the ordinary inf(sup)-homogeneity implies the Boolean inf(sup)-homogeneity.

Conversely, assume that $f$ is Boolean inf and sup-homogeneous. For $I \subseteq [n]$, put $m(I) = f(1_I)$. Obviously $m : 2^{[n]} \to L$ is an $L$-valued capacity on the set $[n]$, since $f$ is non-decreasing and fulfills the boundary conditions.

For all $x \in L^n$ we show that $Su_m(x) \leq f(x)$ as well as $f(x) \leq Su_m(x)$. Given a subset $I \subseteq [n]$, let $u_I = (\bigwedge_{i \in I} x_i, \ldots, \bigwedge_{i \in I} x_i)$ be the constant vector. Then $u_I \land 1_I \leq x$. Indeed, $(u_I \land 1_I)_i = 0 \leq x_i$ if $i \notin I$, while $(u_I \land 1_I)_i = \bigwedge_{j \in I} x_j \leq x_i$ provided $i \in I$. As $f$ is idempotent (Lemma 4.4) and Boolean inf-homogeneous, we obtain

$$\bigwedge_{i \in I} x_i \land m(I) = \bigwedge_{i \in I} x_i \land f(1_I) = f(u_I \land 1_I) \leq f(x_1, \ldots, x_n). \tag{8}$$

Consequently, since (8) holds for each subset $I \subseteq [n]$, we have

$$Su_m(x) = \bigvee_{I \subseteq [n]} (m(I) \land \bigwedge_{i \in I} x_i) \leq f(x).$$

Conversely, for a subset $I \subseteq [n]$, let $v_I = (\bigvee_{i \in I} x_i, \ldots, \bigvee_{i \in I} x_i)$ be the constant vector. Then $x \leq v_I \lor 1_{[n] \setminus I}$ and similarly as in the previous case we obtain

$$f(x_1, \ldots, x_n) \leq f(v_I \lor 1_{[n] \setminus I}) = f(v_I) \lor f(1_{[n] \setminus I}) = \bigvee_{i \in I} x_i \lor m([n] \setminus I). \tag{9}$$

Consequently, from (9) using the dual expression for a Sugeno integral, we obtain

$$f(x) \leq \bigwedge_{I \subseteq [n]} (m([n] \setminus I) \lor \bigvee_{i \in I} x_i) = Su_m(x).$$

\[\square\]
Let us remark that Lemma 4.8 and Theorem 4.9 were proved for bounded chains in [2].

**Lemma 4.10.** Let $L$ be a bounded distributive lattice and $f : L^n \to L$ be an aggregation function. If $f$ is comonotone supremal, then $f$ is Boolean sup-homogeneous. Dually, if $f$ is comonotone infimal, then it is Boolean inf-homogeneous.

**Proof.** Let $c = (c, \ldots, c), c \in L$ and $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ be arbitrary vectors. Since the set $\{x_1, \ldots, x_n\} \subseteq \{0,1\}$ forms a chain in $L$, it follows that $c$ and $x$ are comonotone. Hence the comonotone supremality (comonotone infimality) of $f$ implies that $f$ is Boolean sup-homogeneous (Boolean inf-homogeneous). \qed

Applying the previous lemma and Theorem 4.9 we obtain the following corollaries.

**Corollary 4.11.** If an aggregation function is comonotone supremal and comonotone infimal, then it is a Sugeno integral.

**Corollary 4.12.** If an aggregation function is inf-homogeneous and comonotone supremal or sup-homogeneous and comonotone infimal, then it is a Sugeno integral.

Summarizing our results we obtain the following seven equivalent axiomatic characterizations of the discrete $L$-valued Sugeno integrals:

**Theorem 4.13.** Let $L$ be a bounded distributive lattice and let $f : L^n \to L$ be an $n$-ary aggregation function on $L$. The following conditions are equivalent:

(i) $f$ is a Sugeno integral on $L$.
(ii) $f$ is inf-homogeneous and $g$-comonotone supremal.
(iii) $f$ is sup-homogeneous and $g$-comonotone infimal.
(iv) $f$ is inf-homogeneous and comonotone supremal.
(v) $f$ is sup-homogeneous and comonotone infimal.
(vi) $f$ is comonotone supremal and comonotone infimal.
(vii) \( f \) is \( g \)-comonotone supremal and \( g \)-comonotone infimal.
(viii) \( f \) is Boolean sup-homogeneous and Boolean inf-homogeneous.

**Proof.** The implications (i) implies (ii) as well as (i) implies (iii) are due to Lemma 4.1. The converse implications follow from Corollary 4.6 and Remark 4.7.

The implications (i) implies (iv) and (i) implies (v) are due to Lemma 4.1 and Remark 4.2, while Corollary 4.12 yields the converse implications.

The equivalence (i) iff (vi) follows from Lemma 4.1, Remark 4.2 and Corollary 4.11.

The implication (i) implies (vii) follows from Lemma 4.1. The converse implication follows from Corollary 4.11, since if an aggregation function is \( g \)-comonotone supremal(infimal) it is also comonotone supremal(infimal), cf. Remark 4.2.

Finally, (i) iff (viii) is stated in Theorem 4.9. □

Let us note that from the computational point of view, to verify any of the two conditions in (vii) one must check \(|L| \cdot 2^n\) pairs of vectors, while to formally check inf(sup)-homogeneity one must consider \(|L| \cdot |L|^n\) pairs of vectors.

5. Concluding remarks

We have discussed new axiomatic characterizations of Sugeno integrals on bounded distributive lattices. Some of them are based on new notions of generalized comonotonicity and dual generalized comonotonicity of \( L \)-valued vectors, and they show new stronger properties satisfied by the Sugeno integrals. On the other hand, some other generalizes and simplifies an axiomatization given in [1], bringing as a by-product a drastic reduction of computational complexity of verification an aggregation function to be a Sugeno integral. For example, for a bounded distributive lattice \( L \) with cardinality \( |L| = k > 2 \), the reduction factor is \((k/2)^n\). We expect applications of our results in information fusion and multicriteria decision support when the \( L \)-valued scales are considered, in particular in the case of linguistic scales different from chains.
Concerning the further research in this area, we aim to study the Sugeno integrals on particular lattices, such as the ordinal or horizontal sums of lattices. Here we expect, among others, to solve a challenging problem of ordinal (horizontal) sums of $L$-valued Sugeno integrals.

**Acknowledgements**

The first author was supported by the international project Austrian Science Fund (FWF)-Grant Agency of the Czech Republic (GAČR) number 15-34697L; the second author by the Slovak VEGA Grant 1/0420/15; the third author by the project of Palacký University Olomouc IGA PrF2015010 and by the Slovak VEGA Grant 2/0044/16.

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