Bloch Oscillations in the Optical Waveguide Array.

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Abstract

The multiple scattering formalism is proposed describing the guided modes in the optical waveguide array within the framework of macroscopic electrodynamics. It is shown that, under sufficiently general assumptions, our approach justifies the phenomenological model used widely to treat various physical phenomena in the optical micro- and nano-structures. It is found that the theory developed in this paper describes the real experiments in which the Bloch oscillations are observed. Surprisingly, not only qualitative but also reasonably quantitative agreement is found.

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Various artificial materials such as metamaterials [1], photonic crystals [2, 3], and waveguide arrays [4] are considered as promising structures to manipulate light effectively. It is their spatial periodicity that connects these artificial materials and conventional crystals (metal, semiconductors, dielectrics). For this reason, the Bloch functions inherent in the Schrödinger equation with a periodic potential should be typical for the solutions of the Maxwell equations with the periodical dependence of the refractive index. Hence, many physical effects inherent in solid state physics should have their optical counterparts [4].

For the Schrödinger equation, the electron Bloch functions describe the propagating state. It is known that a superimposition of a disorder or an external electrical field may result in the spatial localization of an electron in solids. In the first case, the electron experiences either Anderson localization or dynamic localization. In the second case, an electron experiences the Bloch oscillations and Zener tunneling. In this paper, we investigate the optical analog of the electron Bloch oscillations.

In the standard the first principle solid state physics the description of the phenomena is based on the analysis of the corresponding Schrödinger equation. Within the quasi-classical description, the Bloch oscillations are known as a finite-oscillation motion of an electron in the periodical potential when the external dc electric field is applied. However, such finite motion can be understood setting on the exact solution of the Schrödinger equation as follows. A particle moving in a periodical potential possesses the infinite Bloch state which belongs to the continuous energy spectrum. The application of the external dc field changes the energy spectrum drastically. The electron states become spatially finite, belonging to the discrete energy spectrum called a Wannier-Stark ladder. The wave function for each state of the ladder manifests a localized state, while a wave packet of these functions describes the Bloch oscillations. In general, the period of these oscillations is much larger than either the scattering time of electron with impurities or the Zener tunneling time. For this reason, the Bloch oscillations are never observed in the conventional crystal. In fact, the first observation of the Wannier-Stark ladder [5] and the Bloch oscillations [6] has become possible using the semiconductor superlattices in which the shorter Bloch oscillations period was attained. Also the Bloch oscillations have experimentally been observed for cold atoms and the Bose-Einstein condensates in optical lattices [7].

The optical waveguide array (OWA) considered in this paper enables us a visualization of the
Bloch oscillations in the spatial domain as an oscillatory light beam path. Each waveguide is a homogeneous one and serves as an attractive atomic potential in the crystals. The dc field is mimicked by the monotonic change of the refractive index of the waveguides as one passes from one waveguide to another. The effect is reached, in particular, by applying the temperature gradient across the thermo-optical material [8–10], by a suitable change of the waveguide geometrics [11], or by a circularly-curving the waveguides [12–14].

To describe the optical Bloch oscillations in the array of parallel waveguides (see. Fig. 1), a very viable phenomenological model was proposed in [15]. Along with the optical Bloch oscillations, this model was used to investigate some other various physical effects in the optical structures such as nonlinear Bloch oscillations, the Bloch oscillations in the waveguide arrays with the second-order coupling, the Bloch-Zener oscillations in optical waveguide ladders and binary superlattices, the gradon localization [4, 9, 16–20]. Within this model, a set of modal amplitudes \( a_j(z) \) are introduced which describe a behavior of the effective light amplitude along the \( j \)-th waveguide. According to Ref. [15], these amplitudes obey the system of coupled equations

\[
\left(i \frac{\partial}{\partial z} + \alpha \cdot j\right) a_j + \gamma (a_{j+1} + a_{j-1}) = 0.
\]

(1)

Here \( z \) is the direction along the waveguide axis, the index \( j = 0, \pm 1, \ldots \) determines the position of the waveguide in the array. The first term in Eq. (1) describes the propagation of light beam along the isolated waveguide, the parameter \( \alpha \) being responsible for the refractive index ramp. The second term describes the influence of the nearest neighbor waveguides on the light propagation, the parameter \( \gamma \) being responsible for this influence. The phenomenological constants \( \alpha \) and \( \gamma \) entering Eq. (1) remain unknown and are obtained only as a result of comparison with an experiment.

To our knowledge, equations like (1) have never been derived within the macroscopic electrodynamics approach. This approach assumes that the array of the infinite homogeneous waveguides is considered and the refractive index of each waveguide is known.

A solution to the corresponding Maxwell equations describes a distribution of the electromagnetic field in the whole space. For the array to possess the guided properties, the field should mainly be concentrated inside the array. Thus, we are interested only in those solutions of the Maxwell equations for the waveguide array which vanish as \( |y| \rightarrow \infty \) (see. Fig. 1). These are evanescent modes. It stands to reason, that the solution should possess a finite amplitude inside the array. The goal of the paper is to determine correctly the amplitudes \( a_j(z) \), connecting them with a superposition of these guided mode solutions of the Maxwell equations. A direct derivation
Figure 1: The OWA. The solid line manifests the path of the light beam inherent in the Bloch oscillations.

of Eq. (1) from the Maxwell equations allows us to obtain the constants $\alpha$ and $\gamma$ in terms of the workpiece geometrics and refractive indices of the waveguides and to establish the validity range of the model described by Eq. (1).

Like in quantum mechanics, for the waveguide array under consideration two formulations of the problem are possible. The first one is the problem of scattering of electromagnetic waves by the array. Such problem is investigated in Refs. [21–24]. The approach used in these papers is based on the exact solution for the scattering of the electromagnetic wave by a single infinite cylinder [25]. The solution for the scattering mode does not vanish as it escapes from the array. The second task is similar to that of the bound states in quantum mechanics. It is related with the guided modes inherent in the array. For a single waveguide, the solution is found in [26]. Finding the solution for the waveguide array is just the goal of the paper. The approach we propose is similar to the multisphere Mie scattering formalism developed in our previous papers [27–30] to describe the high-quality guided modes in the arrays of spherical particles.

The paper is organized as follows. First, we derive the system of equations, which describes the guided modes in the array of parallel dielectric waveguides. For this purpose, we expand the electromagnetic field in the vector cylinder harmonics. The system of equations obtained is a formally exact one for the guided mode propagating in the array and consists of the infinite number of equations. To make the problem solvable, we truncate it using the nearest neighbor approximation and zero-harmonic approach. Under these simplifications, one obtains that the amplitudes $a_j(z)$ obey Eq. (1). Finally, we apply the results obtained to treat the real experiments
II. DERIVATION OF THE EQUATION FOR THE GUIDED MODES

Let us consider the array of $N$ parallel cylindrical dielectric waveguides (see Fig. 1). The axes of the waveguides are in the $xz$-plane and are parallel to the $z$-axis. The array is equidistant, $a$ being the distance between the axes of the nearest waveguides. In this paper, the array of infinite parallel cylinder waveguides is considered. All the waveguides are assumed to possess the same radius $R_j = R$ but different refractive indices $n_j$. It is assumed that the contrast between the nearest waveguides is $n_j - n_{j-1} = \text{const}$. 

Suppose that a guided mode with a frequency $\omega$ is excited within the array. Because of the translation invariance in the $z$ direction, all the components of the electromagnetic field describing the guided mode depend on the coordinate $z$ as $e^{i\beta z}$, $\beta$ being a propagation constant (the wave vector component in the $z$-direction) of the guided mode. Thus, all the components of the guided mode are proportional to the factor $e^{-i\omega t} e^{i\beta z}$. Let us consider the guided mode inside and outside of the array.

Since the guided mode possesses a finite value, the electromagnetic field inside of the $j$-th waveguide may be represented in the form

$$
\tilde{E}_j(r) = e^{-i\omega t} e^{i\beta z} \sum_{m=0,\pm1...} e^{im\phi_j} \left( c_{jm} \tilde{N}_{\omega_j \beta m}(\rho_j) - d_{jm} \tilde{M}_{\omega_j \beta m}(\rho_j) \right),
$$

$$
\tilde{H}_j(r) = e^{-i\omega t} e^{i\beta z} n_j \sum_{m=0,\pm1...} e^{im\phi_j} \left( c_{jm} \tilde{M}_{\omega_j \beta m}(\rho_j) + d_{jm} \tilde{N}_{\omega_j \beta m}(\rho_j) \right), \quad \rho_j < R.
$$

Here $r = (x, y, z) = (\rho, z), \rho_j = |\rho - \mathbf{a} j|$, $\phi_j$ is the polar angle of the vector $\rho - \mathbf{a} j$ (see Fig. 2), $\omega_j = n_j \omega$. The vector cylinder harmonics $\tilde{M}_{\omega_j \beta m}(\rho_j)$ and $\tilde{N}_{\omega_j \beta m}(\rho_j)$ are defined as follows

$$
\tilde{N}_{\omega_j \beta m}(\rho_j) = e_r \frac{i\beta}{\kappa_j} J'_m(\kappa_j \rho_j) - e_\phi \frac{m\beta}{\kappa_j^2 \rho_j} J_m(\kappa_j \rho_j) + e_z J_m(\kappa_j \rho_j),
$$

$$
\tilde{M}_{\omega_j \beta m}(\rho_j) = e_r \frac{m\omega_j}{\kappa_j^2 \rho_j} J_m(\kappa_j \rho_j) + e_\phi \frac{i\omega_j}{\kappa_j} J'_m(\kappa_j \rho_j),
$$

where $\kappa_j = \sqrt{\omega_j^2 - \beta^2}$, $J_m(\kappa_j \rho_j)$ is the Bessel function, and the prime means the derivative with respect to the argument $\kappa_j \rho_j$. The functions $\tilde{N}$ and $\tilde{M}$ are orthogonal. Thus, the guided mode
inside the $j$-th rod which possesses the frequency $\omega$, is determined by the propagation constant $\beta$ and by the set of the partial amplitudes $c_{jm}, d_{jm}$.

Let us turn to the electromagnetic field for the same guided mode outside of the array. Each waveguide of the array contributes to this field. The contribution induced by the $j$-th waveguide and vanishing at $y \to \pm \infty$ may be represented in the form

$$
E_j(r) = e^{-i\omega t} e^{i\beta z} \sum_m e^{i m \phi_j} \left( a_{jm} N_{\omega' \beta m}(\rho_j) - b_{jm} M_{\omega' \beta m}(\rho_j) \right),$

$$
H_j(r) = e^{-i\omega t} e^{i\beta z} n_0 \sum_m e^{i m \phi_j} \left( a_{jm} M_{\omega' \beta m}(\rho_j) + b_{jm} N_{\omega' \beta m}(\rho_j) \right), \quad \rho_j > R. \tag{5}
$$

Here another kind of the vector cylinder harmonics is introduced

$$
N_{\omega' \beta m}(\rho_j) = e_r \frac{i \beta}{\zeta'} H'_m(\zeta' \rho_j) - e_\phi \frac{m \beta}{\zeta'^2 \rho_j} H_m(\zeta' \rho_j) + e_z H_m(\zeta' \rho_j), \tag{6}
$$

$$
M_{\omega' \beta m}(\rho_j) = e_r \frac{m \omega'}{\zeta'^2 \rho_j} H_m(\zeta' \rho_j) + e_\phi \frac{i \omega'}{\zeta'} H'_m(\zeta' \rho_j), \tag{7}
$$

where $H_m(\zeta' \rho_j)$ is the Hankel function of the first kind, $\omega' = n^0 \omega$, $\zeta' = \sqrt{\omega'^2 - \beta^2}$, and $n^0$ is the refractive index of the environment. The functions $N$ and $M$ are orthogonal. Thus, the contribution of the $j$-th waveguide into the guided mode field outside the array is defined both by the propagation constant $\beta$ and by the set of the partial amplitudes $a_{jm}, b_{jm}$. Thus, the total field outside the array is
\[
E(r) = \sum_{j=1}^{N} E_j(r), \quad H(r) = \sum_{j=1}^{N} H_j(r).
\] (8)

Note that, for \( \beta = 0 \), Eq. (2) and Eq. (5) transform into the corresponding expressions in Ref. [25], however different notations are used there. Below, the factor \( e^{-i\omega t} e^{i\beta z} \) is omitted, for brevity.

To derive the set of equations which determines the partial amplitudes \( a_{jm}, b_{jm}, c_{jm}, d_{jm} \), one should take into account that the fields \( \tilde{E}_j(R_j), \tilde{H}_j(R_j) \), described by Eq. (2) and the field \( E(R_j), H(R_j) \), described by Eq. (8), are connected by the boundary conditions on the surface of each waveguide of the array; here \( R_j \) be the radius-vector of a point on the surface of the \( j \)-th waveguide. These fields are connected by the six boundary conditions. However, only four of them are independent. It is convenient to choose the four ones which connect the \( \phi \)- and the \( z \)-components of the field. Thus, if the permeability of the waveguide material and the environment is unity, one has

\[
\begin{align*}
(E(R_j))_\phi &= \left( \tilde{E}_j(R_j) \right)_\phi, & (H(R_j))_\phi &= \left( \tilde{H}_j(R_j) \right)_\phi, \\
(E(R_j))_z &= \left( \tilde{E}_j(R_j) \right)_z, & (H(R_j))_z &= \left( \tilde{H}_j(R_j) \right)_z, \quad j = 1, 2, \ldots, N.
\end{align*}
\] (9)

Based on Eq. (9), one can obtain the uniform system of the linear equations with respect to the variables \( a_{jm}, b_{jm}, c_{jm}, d_{jm} \). As is shown in Appendix, the system of equations is decoupled. The amplitudes \( a_{jm}, b_{jm} \) obey the system of equations

\[
\hat{S}^{-1}_{jm} \begin{pmatrix} a_{jm} \\ b_{jm} \end{pmatrix} - \sum_{l \neq j}^{N} \sum_{n=-\infty}^{\infty} U^{lj}_{nm} \begin{pmatrix} a_{ln} \\ b_{ln} \end{pmatrix} = 0.
\] (10)

while the amplitudes \( c_{jm}, d_{jm} \) are expressed in terms of the amplitudes \( a_{jm}, b_{jm} \) (see (A11) of Appendix). The explicit expressions for the \( 2 \times 2 \) matrixes \( \hat{S}^{-1}_{jm} \) and \( U^{lj}_{nm} \) are presented in Appendix. The uniform linear system of equations (10) has a nontrivial solution if its principal determinant, dependent on \( \hat{S}^{-1}_{jm}(\omega, \beta) \) and \( U^{lj}_{nm}(\omega, \beta) \), vanishes. This condition determines the dispersion curve \( \beta(\omega) \) implicitly.

The physical interpretation of Eq. (10) is the following. If the interaction between the waveguides \( U^{lj}_{nm} \) is neglected, each waveguide of the array behaves as an isolated. In this case a nontrivial solution for Eq. (10) exists if \( \det \hat{S}^{-1}_{jm} = 0 \) at least for one pair of the parameters \( (j, m) \). This condition determines the set of the propagation constants \( \beta_{jm}^{(0)} \) as a function of the frequency \( \omega \), which gives rise to a guided mode characterized by orbital number \( m \) and connected with the isolated waveguide \( j \) [26]. If all the isolated waveguides are identical, each one has the same propagation
constant $\beta_{jm}^{(0)} = \beta_m^{(0)}$. In this case, the guided modes with the frequency $\omega$ and the orbital number $m$ are $N$-fold degenerated guided modes. Taking into account the interaction $U_{nm}$ results in the formation of the $N$ hybridized modes, each one characterizing with certain propagation constant $\beta_{km}$, $1 \leq k \leq N$. However, the $N$-fold degeneration in the propagation constant remains, since all the hybridized modes possess the same frequency $\omega$. The values of the propagation constants $\beta_{km}$ belong to a certain band centered around the value $\beta_m^{(0)}$. If the array is equidistant, each of these hybridized modes is characterized by one of $N$ quasi-wave vectors $k_x$ which belong to the Brillouin band $(-\pi/a, \pi/a)$. Thus, for a given frequency, the guided modes possess a certain dependence $\beta(k_x)$ which determines the isofrequency curve. Naturally, if the refractive indices of the waveguides differ or the array is not equidistant, this feature of the guided modes does not hold for.

However, if the distance between the waveguides or their refractive index fluctuates weakly, one may still have the $N$-fold degenerated guided mode degenerated in the propagation constant. In the next section, we consider the case of the equidistant array for which $(n_j - n_{j+1}) = \text{const} \ll n_j$. In this case the propagation constants for the isolated waveguides obey the relation $\beta_j^{(0)} = \beta_0^{(0)} + \alpha \cdot j$, $\alpha \ll \beta_0$.

III. THE NEAREST NEIGHBOR AND THE ZERO-HARMONIC APPROXIMATION

Even if the number of the waveguides in the array is finite, the system of equations (10) is infinite since the number of different harmonics $m$ remains infinite. Below we consider the simplest approximation to these equations, namely, the harmonics with $m = 0$ alone contribute to the guided modes. Then, in the nearest neighbor approximation Eq. (10) takes the explicit form

\[
\frac{a_j}{a_j(\beta)} - \left( U_{j+1}^{j+1} a_{j+1} + U_j^{j-1} a_{j-1} \right) = 0, \tag{11}
\]

\[
\frac{b_j}{b_j(\beta)} - \left( U_{j+1}^{j+1} b_{j+1} + U_j^{j-1} b_{j-1} \right) = 0,
\]

where $a_j = a_{j0}$, $b_j = b_{j0}$, $U_j^{j+1} = U_{0,0}^{j+1}$,

\[
\frac{1}{a_j(\beta)} = \frac{\varepsilon^0 \varepsilon \varepsilon J_0(\varepsilon_j R) H_0'(\varepsilon R) - \varepsilon \varepsilon J_0'(\varepsilon_j R) H_0(\varepsilon R)}{\varepsilon \varepsilon J_0'(\varepsilon_j R) H_0(\varepsilon_j R) - \varepsilon \varepsilon J_0(\varepsilon_j R) H_0'(\varepsilon_j R)}, \tag{12}
\]

\[
\frac{1}{b_j(\beta)} = \frac{\varepsilon \varepsilon J_0(\varepsilon_j R) H_0'(\varepsilon R) - \varepsilon \varepsilon J_0'(\varepsilon_j R) H_0(\varepsilon_j R)}{\varepsilon \varepsilon J_0'(\varepsilon_j R) H_0(\varepsilon_j R) - \varepsilon \varepsilon J_0(\varepsilon_j R) H_0'(\varepsilon_j R)}.
\]
and \( \varepsilon^0 = (n^0)^2, \varepsilon_j = n_j^2 \). The poles of \( \pi_j(\beta) \) and \( \tilde{\beta}_j(\beta) \) determine the guided modes for isolated waveguides.

First of all, let us note that for \( m = 0 \) the system of equations (11) decouples into two ones which describe the \( a- \) and the \( b- \) modes. Let us consider, for example, the features of the modes \( (b_j \equiv 0) \). In this case, we deal with the uniform linear system of equations with respect to the \( N \) variables \( a_j \). Then, a nontrivial solution of Eq. (11) exists if

\[
\det \left| \frac{1}{\pi_j(\beta)} \delta_{ij} + U_i^j (\delta_{i,j-1} + \delta_{i,j+1}) \right| = 0.
\]  (13)

Let the number of the waveguides \( N \) be finite and \( N \gg 1 \). Then, Eq. (13) determines several different \( \beta_k = \beta_k(\omega) \). It is easy to see that, if \( (n_j - n_{j+1}) = \text{const} \ll n_j \), there are \( N \) solutions of Eq. (13). Note that each propagation constant \( \beta_k \) determines the guided mode with the same frequency \( \omega \). Let \( a_j(\beta_k) \) be the normalized solution of Eq. (11) and \( \sum_j |a_j(\beta_k)|^2 = 1 \). Then, the monochromatic guided mode, in the general case, is a linear superposition of the modes with the different \( \beta_k \):

\[
E(t, r) = e^{-i\omega t} \sum_k C_k e^{i\beta_k z} \sum_{j=1}^N a_j(\beta_k) N_{\omega, \beta_k, 0}(\rho_j),
\]

\[
H(t, r) = e^{-i\omega t} n^0 \sum_k C_k e^{i\beta_k z} \sum_{j=1}^N a_j(\beta_k) M_{\omega, \beta_k, 0}(\rho_j).
\]

The factors \( C_k \) determine the linear superposition.

Let us introduce the modal amplitude

\[
a_j(z) = \sum_k C_k e^{i\beta_k z} a_j(\beta_k).
\]  (14)

Since the functions \( N_{\omega, \beta_k, 0}(\rho_j), M_{\omega, \beta_k, 0}(\rho_j) \) vanish rapidly as \( \rho_j \) increases, the field near the \( j \)-th waveguide is mainly determined by the partial amplitudes \( a_j(\beta_k) \). For this reason, the modal amplitude \( a_j(z) \) represents the behavior of the guided modes properly. The coefficients \( C_k \) are obtained from the boundary condition at \( z = 0 \):

\[
a_j(0) = \sum_k C_k a_j(\beta_k),
\]  (15)

\( a_j(0) \) being given. The number of the different coefficients \( C_k \) coincides with the total number of the waveguides in the array \( N \), the number of equations in (15). In what follows, we assume the Gaussian form for the modal amplitude behavior at \( z = 0 \), i.e.
\[ a_j(0) = e^{-\frac{(j-j_0)^2}{\sigma^2} + ik_0a_j}. \] (16)

This means that the external source approximately illuminates the ends of the waveguides with the numbers \( j_0 - \sigma < j < j_0 + \sigma \) and the phase difference between the amplitudes taken at the ends of the nearest waveguides is \( k_0a \).

Thus, to find the guided mode for the array under consideration, one should perform the sequence of operations, namely: using Eq. (13) calculate numerically the set of propagating constants \( \beta_k \); using Eq. (11), calculate the amplitudes \( a_j(\beta_k) \); specify the distribution of the mode amplitude in the cross-section \( z = 0 \), determined by the parameters \( \sigma, j_0, k_0 \) (see Eq. (16)); using Eq. (15), calculate the coefficients \( C_k \). This completely determines the function \( a_j(z) \).

IV. JUSTIFICATION AND DERIVATION OF THE PHENOMENOLOGICAL MODEL

It is demonstrated in the previous section how to calculate the modal amplitude \( a_j(z) \) which describes the guided modes. Let us show that, under sufficiently general assumption, these amplitudes obey phenomenological equation (1). Suppose that the refractive index is \( n_j = n_0 + j \cdot \delta n \), \( \delta n \ll n_0 \). Then, the solutions of the equation \( \frac{1}{a_j(\beta)} = 0 \) for different \( j \) which determine the propagating constants \( \beta_j(0) \) may be represented in the following form:

\[ \beta_j(0) = \beta_{0j} + \alpha \cdot j, \quad \alpha \ll \beta_{0j}. \] (17)

Here \( \alpha \) is the parameter which determines the ramp in \( \beta_j(0) \). If the coupling \( U_j^{\pm 1} \) is weak enough, the set of different solutions of Eq. (13) \( \beta_k \) obeys the condition \( \left| \left[ (\beta_k - \beta_j(0)) / \beta_j(0) \right] \right| \ll 1 \). Then, since

\[ \frac{1}{a_j(\beta_j(0))} = 0, \] (18)

one has

\[ \frac{1}{a_j(\beta_k)} \approx \frac{\partial}{\partial \beta} \frac{1}{a_j(\beta)} \Bigg|_{\beta=\beta_j(0)} \cdot \left( \beta_k - \beta_j(0) \right). \] (19)

Within the same accuracy one can assume that the parameter

\[ \gamma_j(\beta_k) = U_j^{\pm 1}(\beta_k) / \left( \frac{\partial}{\partial \beta} \frac{1}{a_j(\beta)} \right|_{\beta=\beta_j(0)} = \gamma. \] (20)
depends weakly both on the number \( j \) and on the value of the parameter \( \beta_k \). (The correctness of \((19)\) and \((20)\) can easily be verified for any specific physical parameters describing the waveguide array). Then, Eq. \((11)\) goes over to the following one

\[
\left( \beta - \beta_j^{(0)} \right) a_j - \gamma \left( a_{j-1} + a_{j+1} \right) = 0. \tag{21}
\]

This is a uniform system of linear equations with respect to the variable \( a_j \). Let \( \tilde{\beta}_k \) be set of the different values resulting in a nontrivial solution of Eq. \((21)\). As one expects, the number of \( \tilde{\beta}_k \) is \( N \), the number of the waveguides in the array. They are distributed within the interval \( \left| \tilde{\beta}_k - \beta_0^{(0)} \right| \leq \gamma \). Thus, the approximate equation Eq. \((21)\) is valid if

\[
\alpha \ll \gamma \ll \beta_0^{(0)}. \tag{22}
\]

Let us introduce the modal amplitude

\[
\tilde{a}_j (z) = \sum_k \tilde{C}_k e^{i\tilde{\beta}_k z} a_j(\tilde{\beta}_k).
\]

Like modal amplitude \((14)\), the amplitude \( \tilde{a}_j (z) \) represents the monochromatic guided mode properly. It is easy to verify that these amplitudes satisfy the equation

\[
\left( i \frac{\partial}{\partial z} + \beta_j^{(0)} \right) \tilde{a}_j (z) + \gamma \left( \tilde{a}_{j+1} (z) + \tilde{a}_{j-1} (z) \right) = 0. \tag{23}
\]

Under assumption \((17)\) one obtains

\[
\left( i \frac{\partial}{\partial z} + \beta_0^{(0)} + \alpha \cdot j \right) \tilde{a}_j (z) + \gamma \left( \tilde{a}_{j+1} (z) + \tilde{a}_{j-1} (z) \right) = 0. \tag{24}
\]

One can remove the constant \( \beta_0^{(0)} \) from the last equation by means of the phase calibration of the modal amplitudes \( \tilde{a}_j (z) \). Then, the equation obtained coincides with Eq. \((11)\).

The solution of Eq. \((24)\) can analytically be obtained in the case \( \sigma \gg 1 \) (see boundary condition \((16)\)). The solution of Eq. \((24)\) takes on the form

\[
\tilde{a}_j (z) \approx e^{i\alpha (j-j_0) z + i\phi(z)} e^{-\frac{(j-j_0-\delta_j(z))^2}{2\sigma^2} + i k_0 (j-j_0-\delta_j(z))}, \tag{25}
\]

where

\[
\delta_j(z) = \frac{2\gamma}{\alpha} \left( \cos k_0 - \cos (k_0 + \alpha z) \right), \tag{26}
\]

and
\[ \phi(z) = -i \frac{2\gamma}{\alpha} \left[ \left( \sin k_0 - \sin(k_0 + \alpha z) \right) - k_0 \left( \cos k_0 - \cos(k_0 + \alpha z) \right) \right]. \] (27)

The behavior of the factor \( e^{-\frac{(j - j_0 - \delta j(z))^2}{\sigma^2}} \) is the most interesting feature of the solution obtained. For \( z = 0 \), the function \( \delta j(z) \) vanishes and the amplitude \( \tilde{a}_j(0) \) has a noticeable value for the waveguides with the numbers \( j_0 - \sigma \leq j \leq j_0 + \sigma \). However, for \( z > 0 \), \( \delta j(z) \neq 0 \) and the factor \( e^{-\frac{(j - j_0 - \delta j(z))^2}{\sigma^2}} \) describes the shift of the numbers of the waveguides where \( \tilde{a}_j(z) \) possesses a noticeable value. The oscillating dependence of the factor \( \delta j(z) \) manifests the Bloch oscillations. The function \( \tilde{a}_j(z) \) obtained describes the distribution of the intensity in the space \( j, z \). This distribution possesses the maximal value along the sinusoidal trajectory with the period \( 2\pi/\alpha \), while the amplitude excursion of the excitation is \( 2\gamma/\alpha \).

Below we apply the results obtained to the experiments described in Refs. [8, 9].

V. APPLICATION TO THE EXPERIMENT

To describe the guided modes for the array under consideration, on one hand, one must turn to Eq. (11). To obtain the solution of this equation, one should first numerically solve Eq. (13) to find the set \( \beta_k \). Given the parameters \( k_0 \) and \( \sigma \), one obtains the amplitude \( a_j(z) \) (see Eq. (14)). On the other hand, the optical properties of the waveguide array under consideration can be described by the amplitude \( \tilde{a}_j(z) \) (see Eq. (25)). Thus the guided modes may be described either by the function \( a_j(z) \) or by the function \( \tilde{a}_j(z) \). Let us apply the results obtained in the two previous sections to the experiment described in Refs. [8, 9]. In these papers it has been revealed that the guided modes can propagate as Bloch oscillations.

Let us show that the theory proposed agrees with the experiments and numerical simulations in Refs. [8, 9]. In these papers, the wavelength of the laser source is \( \lambda = 633 \text{ nm} \). The experiments are performed for the homogeneous array of the waveguides in an inorganic-organic polymer (the refractive index \( n_{co} = 1.554 \)) on the glass wafers (the refractive index \( n_{sub} = 1.457 \)) with polymer cladding (the refractive index \( n_{cl} = 1.550 \)). Each waveguide has a cross-section of \( 3.5 \times 3.5 \mu m^2 \). The uniform separation of the adjacent waveguides is \( 8.5 \mu m \), the length of waveguides in the array is \( L = 4.5 \text{ cm} \). The uniform array is laterally detuned by taking an advantage of the thermo-optical effect in the polymer (the thermo-optical coefficient \( n_{th} = 10^{-4} \text{ K}^{-1} \)). By the simultaneous heating and cooling of the opposite sides, a lateral temperature gradient is established, leading to a linear variation of the propagation constants of the individual waveguides. The number of the waveguides
in the array \( N = 75 \). The maximal total temperature drop is \( \Delta T = 25 \text{ K} \). This drop results in the maximal value \( \alpha = 250 \text{ m}^{-1} \) within the experiment conditions.

We simulate the optical waveguide array studied in [8, 9] by the array of the evenly spaced identical cylinder waveguides as shown in Fig. 1. The optical and geometrical parameters of the array we consider are close to the parameters of the experiments. We assume that the waveguide radius \( R = 1.975 \mu \text{m} \). This results in the waveguide cross-section close to that in the experiment. The separation between the waveguides \( a = 3R = 5.925 \mu \text{m} \), what approximately corresponds with the experiments. In the lack of the temperature gradient, the refractive index of the waveguides is \( n = 1.554 \). It is assumed that the refractive index of the environment is \( n^0 \approx 0.99 \cdot n \approx 1.538 \), being certain average between \( n_{\text{sub}} \) and \( n_{\text{cl}} \). If the total temperature gradient \( \Delta T \) and the number of the waveguides \( N \) are given, the refractive index \( n_j = n^0 + n_{\text{th}} \cdot \frac{\Delta T}{N} \cdot j \), \(-N/2 \leq j \leq N/2\). These data are enough to determine the parameters entering Eqs. (11), (24). To obtain the solutions of these equations, one should first solve Eq. (13) to find the set \( \beta_k \). As the initial approximation to solve Eq. (13), one can take \( \beta = \beta_0^{(0)} \) which is the solution of Eq. (18) for the isolated waveguide. For the values of the parameters given above, one obtains \( \beta_0^{(0)} = 1.535 \times 10^7 \text{ m}^{-1} \), and using Eq. (20) one obtains \( \gamma = 198 \text{ m}^{-1} \).

To compare the results obtained above with the numerical simulation and the experimental findings performed in Refs. [8, 9], we assume that \( k_0 = 0 \) in Eq. (16) and the temperature gradient between the nearest waveguides \( \delta T = \Delta T/N = 5 \cdot 10^{-2} \text{K} \). Such temperature gradient gives \( \alpha = 44 \text{ m}^{-1} \), agreeing with Refs. [8, 9].

To demonstrate the existence of the Bloch oscillations in the waveguide array, one assumes that \( \sigma = 4 \) in (16). Then, the function \( a_j(z) \) describes these oscillations as shown in Fig. 3. As mentioned above, both the amplitude \( a_j(z) \) (solution of Eq. (14)) and the amplitude \( \tilde{a}_j(z) \) (see Eq. (24)) can describe the effect. In order to compare these results, the function \( a_j(z) \) in Fig. 3 is represented completely by the areas of the different brightness. However, the function \( \tilde{a}_j(z) \) is represented only partially by the bold dots where it has the maximal value.

The result given in Fig. 3 agrees with the numerical simulation in [8, 9].

To compare the results obtained with the experiment, one should investigate the output intensity distribution as a function of the temperature gradient \( \Delta T \). Shown Fig. 4 is the result of the simulation for the Bloch oscillations, respectively for the parameters corresponding to [8, 9]. Since the parameter \( \alpha \) is a single-valued function of \( \Delta T \), Fig. 4 manifests the output intensity distribution as a function of the ramp in the propagation constant \( \alpha \).
Figure 3: The Bloch oscillations for the waveguide array. The brighter is the pixel in the figure, the larger is the light intensity, described by the function $a_j(z)$. The black dotted lines correspond to the position of maximal value for the function $\tilde{a}_j(z)$ (see Eq. (25)).

Figure 4: The output intensity of the Bloch oscillations as function of the temperature gradient $\Delta T$. The brighter is the pixel in the figure the larger is the light intensity, described by function $a_j(L)$. The black dotted line corresponds to the position of the maximal value for the function $\tilde{a}_j(L)$ (see Eq. (25)).
VI. CONCLUSION

In this paper based on the macroscopic electrodynamics approach and the multiple scattering formalism the system of equations is derived describing the guided modes in the cylinder waveguide array. This system contains the infinite number of equations, being a formally exact one. The system of equations can be truncated if one uses the nearest neighbor approximation and zero harmonic approximation. In this case the system reduces to the phenomenological description widely employed for recent decades. Our approach allows us to calculate the unknown parameters which determine the phenomenological equation. So far, these parameters were extracted only as a result of comparison with the experiment. It is found that the theory developed in this paper describes the real experiments [8, 9] in which the Bloch oscillations are observed. Surprisingly, not only qualitative but also reasonable quantitative agreement is found.

Recently it has been communicated that the cylinder waveguide array considered in this paper can be fabricated by means of the direct inscription of photonic band-gap waveguides into bulk optical glass. Our theory may be applied to these systems to attain the accurate description [32].
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Appendix

To use of the system of equations (9), let us represent the field $E(R_j), H(R_j)$ in the form

$$
E(R_j) = E_j(R_j) + \sum_{l \neq j}^N E_l(R_j), \tag{A1}
$$

$$
H(R_j) = H_j(R_j) + \sum_{l \neq j}^N H_l(R_j).
$$

Each of the fields $E_l(R_j), H_l(R_j)$ is expressed in terms of the functions $N(R_j - la)$ and $M(R_j - la)$ (see Eqs. (5)), i.e., these functions are defined with respect to the different reference systems. Let us represent these functions for $l \neq j$ as linear expansions in the functions $\tilde{N}(R_j - ja)$ and $\tilde{M}(R_j - ja)$, i.e. the vector cylinder harmonics taken at the same reference system.

To do this let us employ the Graph theorem [31] given by the formula

$$
H_n(\rho_l) e^{in\phi_l} = \sum_{m=-\infty}^{+\infty} H_{n-m}(a|l - j|) J_m(\rho_j) e^{im\phi_j}, \quad \phi_j < \phi_l, \quad l \neq j.
$$

One can generalize this formula for arbitrary relation between the angles $\phi_j$ and $\phi_l$. As a result, one obtains

$$
H_n(\rho_l) e^{in\phi_l} = \sum_{m=-\infty}^{+\infty} H_{n-m}(a|l - j|) J_m(\rho_j) e^{im\phi_j} [\text{sign}(j - l)]^{n-m}, \quad l \neq j. \tag{A2}
$$

Using Eq. (A2), one can show that

$$
e^{in\phi_l} \begin{pmatrix}
N_{\omega' \beta n} (\rho_l) \\
M_{\omega' \beta n} (\rho_l)
\end{pmatrix} = \sum_{m=-\infty}^{+\infty} U_{nm}^{lj} e^{im\phi_j} \begin{pmatrix}
\tilde{N}_{\omega' \beta m} (\rho_j) \\
\tilde{M}_{\omega' \beta m} (\rho_j)
\end{pmatrix}, \tag{A3}
$$

$$
U_{nm}^{lj} = H_{n-m}(\rho|l - j|) [\text{sign} (j - l)]^{n-m}, \quad l \neq j.
$$

Taking these relations into account, one obtains
\[E_l(r) = \sum_{m=-\infty}^{\infty} e^{im\phi_j} \left( \sum_{n=-\infty}^{\infty} U_{nm}^{lj} a_{ln} \right) \tilde{N}_{\omega'\beta m}(\rho_j) - \]
\[- \sum_{m=-\infty}^{\infty} e^{im\phi_j} \left( \sum_{n=-\infty}^{\infty} U_{nm}^{lj} b_{ln} \right) \tilde{M}_{\omega'\beta m}(\rho_j) \]
\[H_l(r) = n^0 \sum_{m=-\infty}^{\infty} e^{im\phi_j} \left( \sum_{n=-\infty}^{\infty} U_{nm}^{lj} a_{ln} \right) \tilde{M}_{\omega'\beta m}(\rho_j) + \]
\[+ n^0 \sum_{m=-\infty}^{\infty} e^{im\phi_j} \left( \sum_{n=-\infty}^{\infty} U_{nm}^{lj} b_{ln} \right) \tilde{N}_{\omega'\beta m}(\rho_j), \quad \rho_j \geq R, \quad l \neq j \]

(A4)

Let us replace \( r \) with \( R_j \) in Eqs. (A4), (2), (5) and substitute them into the system of equations (9). As a result, one obtains the uniform system of linear equations for the partial amplitudes \( a_{jm}, b_{jm}, c_{jm}, d_{jm} \):

\[a_{jm} \left( N_{\omega'\beta m}\phi \right) - b_{jm} \left( M_{\omega'\beta m}\phi \right) + \left( \sum_{l \neq j, n} U_{nm}^{lj} a_{ln} \right) \left( \tilde{N}_{\omega'\beta m}\phi \right) - \]
\[- \left( \sum_{l \neq j, n} U_{nm}^{lj} b_{ln} \right) \left( \tilde{M}_{\omega'\beta m}\phi \right) = c_{jm} \left( \tilde{N}_{\omega_j\beta m}\phi \right) - d_{jm} \left( \tilde{M}_{\omega_j\beta m}\phi \right) \]

(A5)

\[a_{jm} n^0 \left( M_{\omega'\beta m}\phi \right) + b_{jm} n^0 \left( N_{\omega'\beta m}\phi \right) + \left( \sum_{l \neq j, n} U_{nm}^{lj} a_{ln} \right) n^0 \left( \tilde{M}_{\omega'\beta m}\phi \right) + \]
\[+ \left( \sum_{l \neq j, n} U_{nm}^{lj} b_{ln} \right) n^0 \left( \tilde{N}_{\omega'\beta m}\phi \right) = c_{jm} n_j \left( \tilde{M}_{\omega_j\beta m}\phi \right) + d_{jm} n_j \left( \tilde{N}_{\omega_j\beta m}\phi \right) \]

(A6)

\[a_{jm} \left( N_{\omega'\beta m} \right)_z + \left( \sum_{l \neq j, n} U_{nm}^{lj} a_{ln} \right) \left( \tilde{N}_{\omega'\beta m} \right)_z = c_{jm} \left( \tilde{N}_{\omega_j\beta m} \right)_z \]

(A7)

\[b_{jm} n^0 \left( N_{\omega'\beta m} \right)_z + \left( \sum_{l \neq j, n} U_{nm}^{lj} b_{ln} \right) n^0 \left( \tilde{N}_{\omega'\beta m} \right)_z = d_{jm} n_j \left( \tilde{N}_{\omega_j\beta m} \right)_z \]

(A8)

Here, for brevity, the argument of the functions \( N_{\omega'\beta m}(R), M_{\omega'\beta m}(R), \tilde{N}_{\omega'\beta m}(R), \tilde{M}_{\omega'\beta m}(R) \) is omitted. Here \( R \) is the radius of the rod, \( n_j \) is the refractive index of the \( j \)-th rod, and \( n^0 \) is the refractive index of the medium. In Eqs (A7) and (A8), we have taken in account that \( \left( \tilde{M}_{\omega'\beta m} \right)_z(\rho) = 0 \).
The system of equations (A5) - (A8) may be represented in the form:

\[
\begin{pmatrix}
\hat{M}_{11} & \hat{M}_{12} \\
\hat{M}_{21} & \hat{M}_{22}
\end{pmatrix}
\begin{pmatrix}
a_{jm} \\
b_{jm} \\
c_{jm} \\
d_{jm}
\end{pmatrix} = \begin{pmatrix}
\hat{N}_1 \\
\hat{N}_2
\end{pmatrix} = \begin{pmatrix}
\sum_{l \neq j, n} U_{ln}^{jl} a_{ln} \\
\sum_{l \neq j, n} U_{ln}^{jl} b_{ln}
\end{pmatrix}. \tag{A9}
\]

Here the matrix \(\hat{M}_{11}, \hat{M}_{12}, \hat{M}_{21}, \hat{M}_{22}\) are defined as follows:

\[
\hat{M}_{11} = \begin{pmatrix}
\left( N_{\omega^\prime \beta m} \right)_{\phi} & -\left( M_{\omega^\prime \beta m} \right)_{\phi} \\
n^0 \left( M_{\omega^\prime \beta m} \right)_{\phi} & n^0 \left( N_{\omega^\prime \beta m} \right)_{\phi}
\end{pmatrix}, \quad \hat{M}_{21} = \begin{pmatrix}
\left( N_{\omega^\prime \beta m} \right)_{z} & 0 \\
0 & n^0 \left( N_{\omega^\prime \beta m} \right)_{z}
\end{pmatrix},
\]

\[
\hat{M}_{12} = \begin{pmatrix}
-\left( \tilde{N}_{\omega^\prime \beta m} \right)_{\phi} & \left( \tilde{M}_{\omega^\prime \beta m} \right)_{\phi} \\
-n_j \left( \tilde{M}_{\omega^\prime \beta m} \right)_{\phi} & -n_j \left( \tilde{N}_{\omega^\prime \beta m} \right)_{\phi}
\end{pmatrix}, \quad \hat{M}_{22} = \begin{pmatrix}
-\left( \tilde{N}_{\omega^\prime \beta m} \right)_{z} & 0 \\
0 & -n_j \left( \tilde{N}_{\omega^\prime \beta m} \right)_{z}
\end{pmatrix},
\]

\[
\hat{N}_1 = \begin{pmatrix}
-\left( \tilde{N}_{\omega^\prime \beta m} \right)_{\phi} & \left( \tilde{M}_{\omega^\prime \beta m} \right)_{\phi} \\
n \left( \tilde{M}_{\omega^\prime \beta m} \right)_{\phi} & -n \left( \tilde{N}_{\omega^\prime \beta m} \right)_{\phi}
\end{pmatrix}, \quad \hat{N}_2 = \begin{pmatrix}
-\left( \tilde{N}_{\omega^\prime \beta m} \right)_{z} & 0 \\
0 & -n \left( \tilde{N}_{\omega^\prime \beta m} \right)_{z}
\end{pmatrix}.
\]

Then, the system of equations (A9) is transformed into the form:

\[
\hat{S}^{-1}_{jm} \begin{pmatrix}
\hat{S}_{jm} = \begin{pmatrix}
\hat{M}_{12}^{-1} \hat{M}_{11} - \hat{M}_{22}^{-1} \hat{M}_{21}
\end{pmatrix}^{-1} \begin{pmatrix}
\hat{M}_{12}^{-1} \hat{N}_1 - \hat{M}_{22}^{-1} \hat{N}_2
\end{pmatrix}, \tag{A12}
\end{pmatrix}
\]

\[
\hat{T}_{mj} = -\begin{pmatrix}
\hat{N}_1 \hat{M}_{12} - \hat{N}_2 \hat{M}_{22}
\end{pmatrix}^{-1} \begin{pmatrix}
\hat{N}_1 \hat{M}_{11} - \hat{N}_2 \hat{M}_{21}
\end{pmatrix}. \tag{A13}
\]

Thus, finding the solution of Eq. (A5) - (A8) reduces to solving Eq. (A10) with respect to \(a_{jm}, b_{jm}\), and next calculation of the amplitudes \(c_{jm}, d_{jm}\) using Eq. (A11).

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