Properties of non-BPS $SU(3)$ monopoles

YA. SHNIR

Institute of Physics, University of Oldenburg
D-26111, Oldenburg, Germany

Abstract

This paper is concerned with magnetic monopole solutions of $SU(3)$ Yang-Mills-Higgs system beyond the Bogomol’nyi-Prasad-Sommerfield limit. The different $SU(2)$ embeddings, which correspond to the fundamental monopoles, as well the embedding along composite root are studied. The interaction of two different fundamental monopoles is considered. Dissolution of a single fundamental non-BPS $SU(3)$ monopole in the limit of the minimal symmetry breaking is analysed.

PACS numbers: 11.15.-q, 11.27.+d, 14.80.Hv
1 Introduction

One of the most interesting direction of the modern field theory is related with study of magnetic monopoles in spontaneously broken gauge theories. It was shown almost immediately after discovering of the celebrated 't Hooft-Polyakov monopole solution of the simple $SU(2)$ Yang-Mills-Higgs theory [1, 2], that existence of such monopole field configurations is a generic prediction of grand unified theories.

More complicated is the question of existence of multimonopole solutions with topological charge $n > 1$. Here most progress have been made in the original $SU(2)$ theory, where, in the limit of vanishing scalar potential (so called Bogomolnyi-Prasad-Sommerfield (BPS) limit), the monopoles satisfy the first order equations. The latter are just 3-dimensional reduction of the integrable self-duality equations. However there is no analytical solution of the Yang-Mills-Higgs model which would correspond to a system of two separated 't Hooft-Polyakov monopoles in a general case of non-vanishing Higgs potential.

There are arguments that such a system corresponds to a saddle point of the energy functional of the Yang–Mills–Higgs system [3, 4], not a minimum as in the single-monopole case. Therefore, this configuration in not static and we have to take into account the effect of the interaction between the monopoles which was analysed in [5, 6, 7].

It would be rather difficult to find an exact description of the time evolution of such a system in the general case. Considering the $SU(2)$ Yang-Mill-Higgs system, Taubes proved that the magnetic dipole solution could exists [4]. The detailed consideration of this axially symmetric $SU(2)$ monopole-antimonopole pair configuration was given in [8, 10]. Recently, more general static equilibrium solutions have been constructed, representing chains, where $m$ monopoles and antimonopoles alternate along the symmetry axis [11].

Clearly, first step beyond the simplest $SU(2)$ gauge theory is to consider an extended $SU(3)$ model. The properties of the corresponding monopoles, which generalise the spherically symmetric 't Hooft–Polyakov solution, were discussed in [12, 13]. This solution is actually a simple embedding of the $SU(2)$ monopole into corresponding Cartan subalgebra of the $SU(3)$ model. However the pattern of symmetry breaking of latter theory is different. One have to separate two different situations: the minimal symmetry breaking $SU(3) \rightarrow U(2)$ and the maximal symmetry breaking: $SU(3) \rightarrow U(1) \times U(1)$. Moreover, the 2-monopole solution arises in this picture on a very natural way as a configuration which corresponds to the composite root of the Cartan-Weyl basis.

Detailed analyse of the BPS monopole solutions in the gauge theory with a large gauge group by E. Weinberg [15, 16, 17] unexpectedly shows that some of these solutions correspond to massless monopoles, so called cloud. The moduli space approximation allows to study the interaction energy of well separated BPS monopoles and the formation of the non-Abelian cloud [18, 19]. However there is very little information about properties of non-BPS $SU(3)$ multi-monopoles, their interaction and behaviour in the massless limit since the powerful Nahm formalism cannot be applied in that case.

If we restrict our discussion to the system of two monopoles, we could expect that, alongside with a straightforward embedding of $SU(2)$ spherically symmetric monopole configuration along a given positive simple root of the Cartan-Weyl basis [9, 23], there
are some other solutions. The latter in the BPS limit would correspond to the distinct fundamental monopoles which are embedded along composite roots.

Clearly, the character of the interaction between these monopoles depends from their relative orientation in the group space and it has to be quite different from the interaction of the monopoles, which are embedded along the same simple root of the Cartan-Weyl basis. Furthermore, if the symmetry is broken minimally, there should be a counterpart of the massless BPS monopole whose existence was recently proved from the Hahm formalism [18, 19].

In the present note we discuss the spectrum of the spherically symmetric non-BPS monopole solutions of the $SU(3)$ Yang-Mill-Higgs theory and analyse the properties of the solution with minimal symmetry breaking.

2 Cartan-Weyl basis of $SU(3)$ group

We consider the $SU(3)$ Yang-Mills-Higgs Lagrangian density

$$-L = \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} D_\mu \Phi D^\mu \Phi + V(\Phi)$$

where $F_{\mu\nu} = F_{\mu\nu}^a T^a$, $\Phi = \Phi^a T^a$ and the symmetry breaking Higgs potential is

$$V(\Phi) = \lambda (|\Phi|^2 - 1)^2.$$  

where we used the $su(3)$ norm $|\Phi|^2 = \Phi^a \Phi^a$.

The $su(3)$ Lie algebra is given by a set of traceless matrices $T^a = \lambda^a / 2$ where $\lambda^a$ are the standard Gell-Mann matrices, that is we use the normalisation $\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab}$. The structure constants are $f^{abc} = \frac{1}{4} \text{Tr} [\lambda^a, \lambda^b] \lambda^c$ and in the adjoint representation $(T^a)_{bc} = f^{abc}$.

Discussion of monopoles in a gauge theory of higher rank is closely connected with notion of the Cartan-Weyl basis [15, 16, 17]. Let us briefly review the basic elements of this approach.

The diagonal, or Cartan subalgebra of $SU(3)$ is given by two generators

$$H_1 \equiv T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad H_2 \equiv T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which are composed into the vector $\vec{H} = (H_1, H_2)$. The Cartan-Weyl basis of the $SU(3)$ group can be constructed by addition to the commuting elements $\vec{H}$ two raising and lowering generators $E_{+\vec{\beta}}$, each for one of two simple roots $\vec{\beta} = (\vec{\beta}_1, \vec{\beta}_2)$:

$$[H_i, E_{\vec{\beta}}] = \beta_i E_{\vec{\beta}}; \quad [E_{\vec{\beta}}, E_{-\vec{\beta}}] = 2 \vec{\beta} \cdot \vec{H}$$

Most general $SU(3)$-symmetry breaking scalar potential can be written in a different form [14, 23]. To set a correspondence with the related discussion of the BPS monopoles we shall take into account the pattern of the symmetry breaking on a different way.
We take the basis of simple roots as (cf. figure 1)

\[ \vec{\beta}_1 = (1, 0); \quad \vec{\beta}_2 = (-1/2, \sqrt{3}/2) \]  

(5)

Third positive root is given by the composition of the first two roots \( \vec{\beta}_3 = \vec{\beta}_1 + \vec{\beta}_2 = (1/2, \sqrt{3}/2) \). Note that all these roots have a unit length, that is our choice corresponds to the self-dual basis: \( \vec{\beta}_i^* = \vec{\beta}_i \).

For any given root \( \vec{\beta}_i \) the generators \( \vec{\beta} \cdot \vec{H}, E_{\pm \vec{\beta}} \) form an \( su(2) \) algebra. Let us write these generators explicitly in the above defined basis of the simple roots. For \( \vec{\beta}_1 \) we have

\[
T_{(1)}^3 = \vec{\beta}_1 \vec{H} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} ; \\
E_{\vec{\beta}_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad E_{-\vec{\beta}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .
\]

(6)

For the second simple root \( \vec{\beta}_2 \) we have

\[
T_{(2)}^3 = \vec{\beta}_2 \vec{H} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} ; \\
E_{\vec{\beta}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ; \quad E_{-\vec{\beta}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .
\]

(7)

The generators of the \( su(2) \) subalgebra which correspond to the third composite root are given by the set of matrices

\[
T_{(3)}^3 = \vec{\beta}_3 \vec{H} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \\
E_{\vec{\beta}_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad E_{-\vec{\beta}_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .
\]

(8)

Clearly, the set of matrices \( T_{(k)}^a, k = 1, 2, 3 \), which includes \( T_{(k)}^3 \) of Esq (6), (7) and (8), and

\[
T_{(k)}^1 = \frac{1}{2} (E_{\vec{\beta}_k} + E_{-\vec{\beta}_k}) , \quad T_{(k)}^2 = \frac{1}{2i} (E_{\vec{\beta}_k} - E_{-\vec{\beta}_k})
\]

(9)

satisfy the commutation relations of the \( su(2) \) algebras associated with the simple roots \( \vec{\beta}_1, \vec{\beta}_2 \) and \( \vec{\beta}_3 \) respectively.

In other words, the basis of the simple roots \( \beta_1, \beta_2 \) corresponds to two different ways to embed the \( SU(2) \) subgroup into \( SU(3) \). Upper left and lower right \( 2 \times 2 \) blocks are
corresponding to the subgroups generated by the simple roots $\beta_1$ and $\beta_2$ respectively. The third composite root $\vec{\beta}_3$ generates the $SU(2)$ subgroup which lies in the corner elements of the $3 \times 3$ matrices of the $SU(3)$. Note there is also so-called maximal embedding, which is given by the set of matrices

$$
\tilde{T}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \tilde{T}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \\
\tilde{T}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

These matrices up to an unitary transformation are equivalent to the vector representation of $SU(2)$. The very detailed analyse of the corresponding solutions is presented in [23]. We shall not consider the maximal embedding in this note.

3 $SU(3)$ two-monopole configurations

The asymptotic value of the scalar field in some fixed direction (e.g. positive $z$-axis) can be chosen to lie in the Cartan subalgebra

$$
\Phi_0 = \vec{h} \cdot \vec{H},
$$

where $\vec{h} = (h_1, h_2)$ is 2-component vector in the space of Cartan subalgebra. Clearly, that is a generalisation of the $SU(2)$ boundary condition $\Phi_0 = \sigma_3/2$.

Orientation of the Higgs field in the $SU(3)$ root space corresponds to the pattern of the symmetry breaking. If the Higgs vector $\vec{h}$ is orthogonal to none from the simple roots $\vec{\beta}_i$, the symmetry is maximally broken to the maximal Abelian torus $U(1) \times U(1)$. If inner product of $\vec{h}$ and either of the simple roots is vanishing, there are two choices of the basis of simple roots with positive inner product with $\vec{h}$ which are related by Weyl reflections. If, for example $\vec{h}$ is orthogonal to $\vec{\beta}_2$, we can choose between two possibilities: $(\vec{\beta}_1, \vec{\beta}_2)$ and $(\vec{\beta}_1 + \vec{\beta}_2, -\vec{\beta}_2)$. This is the case of the minimal symmetry breaking $SU(3) \rightarrow U(2)$.

Furthermore, the magnetic charge of the $SU(3)$ monopole is also defined as a vector in the root space, that is

$$
g = \vec{g} \cdot \vec{H}
$$

where the vector magnetic charge lies on the dual root lattice\(^2\) [20]

$$
\vec{g} = \sum_{i=1}^{r} n_i \vec{\beta}_i^* = \vec{g}_1 + \vec{g}_2 = n_1 \vec{\beta}_1 + n_2 \vec{\beta}_2
$$

\(^2\)Recall that in our self-dual basis $\vec{\beta}_i^* = \vec{\beta}_i$. 

\[4\]
where \( n_1 \) and \( n_2 \) are integer and \( \vec{g}_1, \vec{g}_2 \) are the magnetic charges associated with the corresponding simple roots. Thus,

\[
g = \vec{g} \cdot \vec{H} = \left(n_1 - \frac{n_2}{2}\right) H_1 + \frac{\sqrt{3}}{2} n_2 H_2
\]  

(12)

An SU(3) spherically symmetrical monopole configuration can be constructed by a simple embedding \([9, 21, 23]\). The recipe is obvious: we have to choose one of the simple roots having a positive inner product with the scalar field, for example, \( \vec{\beta}_1 \), and embed the ‘t Hooft-Polyakov solution into the corresponding SU(2) subgroup. For example, embedding into left upper corner SU(2) subgroup defines the \( \beta_1 \)-monopole which is characterised by the vector charge \( \vec{g} = (1, 0) \) while the embedding into lower right corner SU(2) subgroup defines the \( \beta_2 \)-monopole with the vector charge \( \vec{g} = (0, 1) \). Similarly, one can embed the SU(2) axially symmetric monopole-antimonopole saddle point configuration of \([8, 10]\) which yields the state \( \vec{g} = (0, 0) \) or two SU(2) monopoles configuration of \([3]\) which, depending from the root we choose, yields the states \( \vec{g} = (2, 0) \) or \( \vec{g} = (0, 2) \) respectively.

Embedding of the spherically symmetric SU(2) monopole along composite root \( \vec{\beta}_3 \) gives a \((1, 1)\) monopole with the magnetic charge

\[
g = \vec{g} \cdot \vec{H} = \frac{1}{2} H_1 + \frac{\sqrt{3}}{2} H_2
\]

The analysis based on the index theorem shows \([16]\), that this configuration is a simple superposition of two other fundamental solutions and could be continuously deformed into solution which describes two well separated single \( \beta_1 \) and \( \beta_2 \) monopoles.

It is known that the character of interaction between the SU(3) BPS monopoles depends from the type of the embedding \([20]\). This is also correct for non-BPS extension. Indeed, then there is only long-range electromagnetic field which mediates the interaction between two widely separated non-BPS monopoles, that is they are considered as classical point-like particles with magnetic charges \( g_i = \vec{g}_i \cdot \vec{H} = \vec{\beta}_i \cdot \vec{H} \). For a non-zero scalar coupling \( \lambda \) the contribution of the scalar field is exponentially suppressed. The energy of the electromagnetic interaction then originates from the kinetic term of the gauge field \( \frac{1}{2} \text{Tr} F_{\mu \nu} F^{\mu \nu} \) in the Lagrangian \([11]\). Therefore an additional factor \( \text{Tr}[\vec{\beta}_1 \cdot \vec{H})(\vec{\beta}_2 \cdot \vec{H})] = (\vec{\beta}_1 \cdot \vec{\beta}_2) \) appears in the formula for the energy of electromagnetic interaction. In the case under consideration \( (\vec{\beta}_1 \cdot \vec{\beta}_2) = -\frac{1}{2} \) while \( (\vec{\beta}_1 \cdot \vec{\beta}_1) = 1 \). This corresponds to an attraction of two different fundamental SU(3) monopoles and repulsion of two monopoles of the same SU(2) subalgebra due to non-trivial group structure. The energy of interaction between the \( \beta_1 \) and \( \beta_2 \) monopoles then is:

\[
V_{\text{int}} = -\frac{(r_1 r_2)}{r_1 + r_2}.
\]

We can check this conclusion by making use of an analogy with the classical electrodynamics of point-like charges. Let us suppose that both monopoles are located on the \( z \)-axis at the points \((0, 0, \pm R)\).

The electromagnetic field of that configuration can be calculated in the Abelian gauge where the gauge field become additive \([23]\). If the monopoles are embedded along the same
simple root, say $\beta_1$, we can write the components of the gauge field as

$$A_r = A_\theta = 0; \quad A_\phi = (1 + \cos \theta_1) \frac{\sigma_3^{(1)}}{2} + (1 + \cos \theta_2) \frac{\sigma_3^{(2)}}{2}$$

(13)

Simple calculation yields the components of the electromagnetic field strength tensor

$$F_{r\theta} = 0; \quad F_{r\phi} = r R \sin^2 \theta \left( \frac{1}{r_1^3} \frac{\sigma_3^{(1)}}{2} - \frac{1}{r_2^3} \frac{\sigma_3^{(1)}}{2} \right);$$

$$F_{\theta\phi} = -r^2 \sin \theta \left( \frac{r - R \cos \theta \sigma_3^{(1)}}{r_3^3} \frac{\sigma_3^{(1)}}{2} + \frac{r + R \cos \theta \sigma_3^{(1)}}{r_3^3} \frac{\sigma_3^{(1)}}{2} \right),$$

(14)

where $r_1, r_2$ are the distances of the point $r$ to the points at which monopoles are placed. The field energy becomes

$$E = \text{Tr} \left( \frac{1}{r^2 \sin^2 \theta} F_{r\phi}^2 + \frac{1}{r^4 \sin^2 \theta} F_{\theta\phi}^2 \right) = \frac{1}{2} \left[ \left( \frac{r_1}{r_3^3} \right)^2 + \left( \frac{r_2}{r_3^3} \right)^2 + \frac{2(r_1 r_2)}{r_1^3 r_2^3} \right]$$

(15)

that is the potential energy of the electromagnetic interaction of two $\beta_1$ monopoles is repulsive. However, for a $\beta_3$ configuration with vector charge $\vec{q} = (1, 1)$ the components of the gauge fields are

$$A_r = A_\theta = 0; \quad A_\phi = (1 + \cos \theta_1) \frac{\sigma_3^{(1)}}{2} + (1 + \cos \theta_2) \frac{\sigma_3^{(2)}}{2}$$

(16)

and, because $\text{Tr} \sigma_3^{(1)} \sigma_3^{(2)} = -1$, the field energy is

$$E = \frac{1}{2} \left[ \left( \frac{r_1}{r_3^3} \right)^2 + \left( \frac{r_2}{r_3^3} \right)^2 - \frac{(r_1 r_2)}{r_1^3 r_2^3} \right]$$

(17)

that is $\beta_1$ and $\beta_2$ monopoles attract each other with a half-force comparing to the case of the repulsion of two $\beta_1$ monopoles.

Note that for a system of two well separated $SU(3)$ dyons the energy of the classic long-range electromagnetic interaction also includes the electric part. The electric charges of the $\beta_1$ and $\beta_2$ dyons are defined with respect to different $U(1)$ subgroups, that is alongside the magnetic charges they are vectors in the root space: $Q_i = q_i (\vec{\beta}_i \cdot \vec{H})$. Thus, the potential of interaction of two identical dyons remains proportional to the inner product $(\vec{\beta}_i \cdot \vec{\beta}_j)$ as in the case of purely magnetically charged configuration.

### 3.1 Spherically symmetric $SU(3)$ monopole configuration

In this simplified consideration above we neglected both the structure of the monopole core and the contribution of the short-range massive scalar field. Such an approximation
to the low energy dynamics of the $SU(3)$ monopoles have been applied in the moduli space approach [26]. However the mechanism of the interaction becomes more complicated if the symmetry is broken minimally. Then one of the fundamental monopoles is losing its identity as a located field configuration. If this monopole would be isolated it would spread out and disappear. But as its core overlaps with the second massive monopole, it ceases to expand [27, 28, 18, 19].

To analyze the behavior of two distinct fundamental monopole systems in that limit we study the spherically symmetric $SU(3)$ monopoles in a more consistent way. In the BPS limit our numerical results can be compared with the consideration of the paper [19] where the Nahm formalism was used to calculate the monopole energy density.

For each simple root $\vec{\beta}_i$, which defines an $SU(2)$ subgroup with corresponding generators $T^a_{(i)}$, we can define an embedded $SU(2)$ monopole as [21]

$$A_n = A^a_n T^a_{(i)}; \quad \Phi = \Phi^a T^a_{(i)} + \phi(h), \quad \text{where} \quad \phi(h) = \left( \vec{h} - \vec{\beta}_i (\vec{h} \cdot \vec{\beta}_i) \right) \vec{H}$$

(18)

The additional invariant term $\phi(h)$ is added to the Higgs field to satisfy the boundary conditions on the spatial asymptotic. In our basis of the simple roots we can write

$$\vec{\beta}_1: \quad \phi(h) = \frac{h_2}{2\sqrt{3}} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{array} \right);$$

$$\vec{\beta}_2: \quad \phi(h) = \frac{1}{4} \left( h_1 + \frac{h_2}{\sqrt{3}} \right) \left( \begin{array}{ccc} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right);$$

$$\vec{\beta}_3: \quad \phi(h) = \frac{1}{4} \left( h_1 - \frac{h_2}{\sqrt{3}} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

(19)

Clearly, the embedding (18) is very convenient to obtain spherically symmetric monopoles [17]. It is also helpful to examine the fields and low-energy dynamics of the charge two BPS monopoles [18]. Depending on the boundary conditions and pattern of the symmetry, some other ansätze can be implemented to investigate static monopole solutions, as, for example, the harmonic map ansatz [22] which was used to construct non-Bogomol’nyi $SU(N)$ BPS monopoles.

In our consideration we shall consider ansätze for the Higgs field of a spherically symmetric $\vec{\beta}_i$ monopole configuration. Depending on the way of the $SU(2)$-embedding, it can be taken\(^3\) as a generalization of the the embedding (18)

$$\vec{\beta}_i: \quad \Phi(r) = \Phi_1(r) \tau^{(i)}_r + \frac{\sqrt{3}}{2} \Phi_2(r) D^{(i)};$$

$$A_r = 0; \quad A_\theta = [1 - K(r)] \tau^{(i)}_\theta; \quad A_\phi = -\sin \theta [1 - K(r)] \tau^{(i)}_\phi$$

(20)

\(^3\)The first of these ansätze (in a different basis of the simple roots) was already used in [23, 24].
where \( i = 1, 2, 3 \) and we make use of the \( su(2) \) matrices \( \tau^{(i)}_{\alpha} = \left( T^{a}_{(i)\alpha} \right) \), \( \tau^{(i)}_{\theta} = \left( T^{a}_{(i)\theta} \right) \) and \( \tau^{(i)}_{\phi} = \left( T^{a}_{(i)\phi} \right) \). The diagonal matrices \( D^{(i)} \), which define the embedding along corresponding simple root, are simple \( SU(3) \) hypercharge

\[
D^{(1)} \equiv Y = \frac{2}{\sqrt{3}}H_2 = \frac{1}{3}\text{diag}(1, 1, -2),
\]

the \( SU(3) \) electric charge operator

\[
D^{(2)} \equiv Q = T^3 + \frac{Y}{2} = H_1 + \frac{H_2}{\sqrt{3}} = \frac{1}{3}\text{diag}(2, -1, -1)
\]

and its conjugated

\[
D^{(3)} \equiv \bar{Q} = T^3 - \frac{Y}{2} = \frac{1}{3}\text{diag}(1, -2, 1).
\]

The normalization of the ansatz (20) corresponds to the \( su(3) \)-norm of the Higgs field \( |\Phi|^2 = \Phi_1^2 + \Phi_2^2 \) for any embedding.

Inserting the ansatz (20) into the Lagrangian density (11) yields

\[
-\mathcal{L} = \text{Tr} \left\{ \frac{1}{r^2} F_{r\theta}^2 + \frac{1}{r^2 \sin^2 \theta} F_{r\phi}^2 + \frac{1}{r^4 \sin^2 \theta} F_{\theta\phi}^2 \right\} + \text{Tr} \left\{ (D_r \Phi)^2 + \frac{1}{r^2} (D_\theta \Phi)^2 + \frac{1}{r^2 \sin \theta} (D_\phi \Phi)^2 \right\} + \lambda (|\Phi|^2 - 1)^2
\]

\[
= \frac{1}{2r^2} \left\{ 2(r\partial_r K)^2 + (1 - K^2)^2 \right\} + \frac{1}{2r^2} \left\{ (r\partial_r \Phi_1)^2 + (r\partial_r \Phi_2)^2 + 2K^2 \Phi_1^2 \right\} + \lambda (\Phi_1^2 + \Phi_2^2 - 1)^2,
\]

Straightforward variation of the Lagrangian \( \mathcal{L} = \int d^4xL \) with respect to the gauge field profile functions \( K(r) \) and the scalar field functions \( \Phi_1(r), \Phi_2(r) \) gives the system of the coupled non-linear differential equations of second order:

\[
0 = \partial_r^2 K - \frac{K(K^2 - 1)}{r^2} - \Phi_1^2 K = 0; \quad (22)
\]

\[
0 = 2\Phi_1 K^2 + 4\lambda r^2 \Phi_1 (\Phi_1^2 + \Phi_2^2 - 1) - r^2 \partial_r^2 \Phi_1 - 2r \partial_r \Phi_1;
\]

\[
0 = 4\lambda r^2 \Phi_2 (\Phi_1^2 + \Phi_2^2 - 1) - r^2 \partial_r^2 \Phi_2 - 2r \partial_r \Phi_2
\]

Clearly, these equations are identical for any \( SU(2) \) embedding. However the boundary conditions we have to impose on the Higgs field, depend from the type of the embedding.

Let us consider the behavior of the scalar field of the configurations (20) along positive direction of the \( z \)-axis. We obtain

\[
\bar{\beta}_1 : \quad \Phi(r, \theta) \bigg|_{\theta=0} = \Phi_1 H_1 + \Phi_2 H_2 = (\tilde{h} \cdot \bar{H});
\]

\[
\bar{\beta}_2 : \quad \Phi(r, \theta) \bigg|_{\theta=0} = \frac{1}{2} \left[ (\sqrt{3}\Phi_2 - \Phi_1) H_1 + (\sqrt{3}\Phi_1 + \Phi_2) H_2 \right] = (\tilde{h} \cdot \bar{H});
\]

\[
\bar{\beta}_3 : \quad \Phi(r, \theta) \bigg|_{\theta=0} = \frac{1}{2} \left[ (\sqrt{3}\Phi_2 + \Phi_1) H_1 + (\sqrt{3}\Phi_1 - \Phi_2) H_2 \right] = (\tilde{h} \cdot \bar{H}).
\]
That yields the components of the vector $\vec{h}$ which determines the nature of the symmetry breaking.

The boundary conditions we can impose on configurations, which minimise the action (11) are of different types. First, the Higgs potential vanishes on the spacial asymptotic, that is as $r \to \infty$

$$|\Phi|^2 = \Phi_1^2 + \Phi_2^2 = 1$$

Second, the inner product of the vector $\vec{h}$ with all roots have to be non-negative for any embedding. That yields

$$\vec{\beta}_1: \quad (\vec{\beta}_1 \cdot \vec{h}) = \Phi_1 \geq 0; \quad (\vec{\beta}_2 \cdot \vec{h}) = -\frac{\Phi_1}{2} + \frac{\sqrt{3}}{2} \Phi_2 \geq 0; \quad (\vec{\beta}_3 \cdot \vec{h}) = \frac{\Phi_1}{2} + \frac{\sqrt{3}}{2} \Phi_2 \geq 0$$

$$\vec{\beta}_2: \quad (\vec{\beta}_1 \cdot \vec{h}) = -\frac{\Phi_1}{2} + \frac{\sqrt{3}}{2} \Phi_2 \geq 0; \quad (\vec{\beta}_2 \cdot \vec{h}) = \Phi_1 \geq 0; \quad (\vec{\beta}_3 \cdot \vec{h}) = \frac{\Phi_1}{2} - \frac{\sqrt{3}}{2} \Phi_2 \geq 0$$

$$\vec{\beta}_3: \quad (\vec{\beta}_1 \cdot \vec{h}) = \frac{\Phi_1}{2} + \frac{\sqrt{3}}{2} \Phi_2 \geq 0; \quad (\vec{\beta}_2 \cdot \vec{h}) = \frac{\Phi_1}{2} - \frac{\sqrt{3}}{2} \Phi_2 \geq 0; \quad (\vec{\beta}_3 \cdot \vec{h}) = \Phi_1 \geq 0. \quad (23)$$

Thirdly, the covariant derivatives of the Higgs field have to vanish at spacial infinity, that is

$$D_r \Phi = r \partial_r \Phi_1 \tau^{(i)} + \frac{\sqrt{3}}{2} \partial_r \Phi_2 D^{(i)} = 0;$$
$$D_\theta \Phi = (K - 1) \Phi_1 \tau^{(i)} = 0;$$
$$D_\phi \Phi = \sin \theta (K - 1) K \Phi_1 \tau^{(i)} = 0 \quad (24)$$

And finally, the solution has to be regular at the origin. The condition on the short distance behavior implies

$$K(r) \to 1; \quad \Phi_1(r) \to 0; \quad \partial_r \Phi_2(r) \to 0,$$

as $r \to 0$. The energy density also goes to 0 in that limit.

\section{Composite monopole solution and various limits of the symmetry breaking}

We are interested in the investigation of the properties of the configuration, which correspond to the embedding along the composite root $\vec{\beta}_3$. The physical meaning of the third of the ansätze for the scalar field (20) becomes more clear if we note that on the spacial asymptotic this configuration really corresponds to the Higgs field of two distinct fundamental monopoles, $(1, 0)$ and $(0, 1)$. Indeed, outside of the cores of these monopoles in the Abelian gauge the scalar field can be written as superposition:

$$\Phi(r \to \infty) = v_1 T^3_{(1)} + v_2 T^3_{(2)} = \frac{1}{2} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 - v_1 & 0 \\ 0 & 0 & -v_2 \end{pmatrix}$$
where the Higgs field of the $\beta_1$ and $\beta_2$ monopoles is taking the vacuum values $v_1, v_2$ respectively.

Rotation of this configuration by the matrices of the $SU(2)$ subgroup which is defined by the third composite root $\vec{\beta}_3$

$$U = \begin{pmatrix} \cos \frac{\theta}{2} & 0 & \sin \frac{\theta}{2} e^{-i\phi} \\ 0 & 1 & 0 \\ -\sin \frac{\theta}{2} e^{i\phi} & 0 & \cos \frac{\theta}{2} \end{pmatrix}$$
yields

$$U^{-1} \Phi U = \frac{1}{2} [v_1 + v_2] \tau^{(3)} + \frac{3}{4} [v_1 - v_2] \tilde{Q}$$

Up to the obvious reparametrization of the shape functions of the scalar field

$$\Phi_1 \rightarrow \frac{1}{2} (F_1(r) + F_2(r)); \quad \Phi_2 \rightarrow \frac{\sqrt{3}}{2} (F_1(r) - F_2(r))$$

where the functions $F_1, F_2$ have the vacuum expectation values $v_1, v_2$ respectively, the configuration (26) precisely corresponds to the third of the ansätze (20). Note that because the $su(3)$-norm of the scalar field is set to be unity, the vacuum values must satisfy the condition

$$v_1^2 + v_2^2 - v_1 v_2 = 1$$

Moreover, the reparametrization (26) allows to write the scalar field of the $\beta_3$ monopole along positive direction of the z-axis as

$$\vec{\beta}_3 : \Phi(r \rightarrow \infty, \theta) \bigg|_{\theta=0} = \left( v_1 - \frac{v_2}{2} \right) H_1 + \frac{\sqrt{3}}{2} v_2 H_2 = (v_1 \vec{\beta}_1 + v_2 \vec{\beta}_2) \cdot \vec{H} = (\vec{h} \cdot \vec{H})$$

Thus, the asymptotic values $v_1$ and $v_2$ are the coefficients of the expansion of the vector $\vec{h}$ in the basis of the simple roots and on the spacial asymptotic the fields $F_1(\vec{\beta}_1 \cdot \vec{H})$ and $F_2(\vec{\beta}_2 \cdot \vec{H})$ can be identified with the Higgs fields of the first and second fundamental monopole respectively.

The solution of the equations (22) becomes very simple in the BPS limit. Then the third equation is decoupled and its solution, which is regular at the origin, is just a constant $\Phi_2 = C, C \in [0; 1]$. The shape functions of the scalar and gauge field are well known rescaled Bogomol’nyi solutions

$$K(r') = \frac{r'}{\sinh r'}; \quad \Phi_1(r') = \sqrt{1 - C^2} \coth r' - \frac{1}{r'}, \quad \text{where} \quad r' = r \sqrt{1 - C^2}$$

with a long-range field $\Phi_1$.

Now we may treat the configuration, which corresponds to the minimal $SU(3)$ symmetry breaking, as a special case of maximal symmetry breaking, namely we can start from an arbitrary orientation of the vector $\vec{h}$ which is compatible with the boundary conditions (28).
Embedding along the composite simple root $\vec{\beta}_3$ gives two fundamental monopoles, which in the case of the maximal symmetry breaking, are charged with respect to different $U(1)$ subgroups and are on top of each other. The configuration with a minimal energy corresponds to the boundary condition $(\Phi_1)_{\text{vac}} = 1, (\Phi_2)_{\text{vac}} = 0$. We can interpret it, by making use of Eq. (26), as two identical monopoles of the same mass. This degeneration is lifted as the value of the constant solution $\Phi_2 = C$ increases, the vector of the Higgs field $\vec{h}$ smoothly rotates in the root space and the boundary conditions begin to vary.

According to the parametrization (26) increasing of the constant $C$ results in the splitting of the vacuum values of the scalar fields of the first and second fundamental monopoles, the $\beta_1$-monopole is getting heavier than the $\beta_2$-monopole. Note that the shape function of the vector field remains almost unaffected by this variation of the boundary conditions as shown in Fig 7.

One would expect that in the limiting case of the minimal symmetry breaking the $\beta_2$ monopole is getting massless, that is in that limit the vacuum value of the field $F_2$ should vanish, $v_2 \to 0$. However two monopoles are overlapped and the presence of the massive monopole changes the situation. Indeed, the symmetry outside the core of the $\beta_1$-monopole is broken down to $U(1)$ which also changes the pattern of the symmetry breaking by the scalar field of the second monopole. One can see that the vector $\vec{h}$ becomes orthogonal to the simple root $\vec{\beta}_2$ when $(\Phi_1)_{\text{vac}} = \frac{3}{2}, (\Phi_2)_{\text{vac}} = 0$, or $v_2 = \frac{\sqrt{3}}{2}$. Going back to the Eq. (27) we can see that in this case on the spatial asymptotic the scalar field along $z$ axis is

$$\Phi(r \to \infty, \theta)\bigg|_{\theta=0} = \frac{3}{4} v_1 \bar{D}(2)$$

where $D(2) = H_1 + \frac{1}{\sqrt{3}} H_2$. Thus, the symmetry is still maximally broken and both monopoles are massive.

Equation (27) indicates that the symmetry is minimally broken if the vector $\vec{h}$ becomes orthogonal to the simple root $\vec{\beta}_1$ and $v_1 = \frac{\sqrt{3}}{2}$. Then the eigenvalues of the scalar field are the same as $H_2$, that is the unbroken symmetry group is really $U(2)$. However for the third composite root such a situation corresponds to the negative value of the inner product $(\vec{h} \cdot \vec{\beta}_1)$ and it has to be excluded. Thus, the maximal vacuum value of the second component of the Higgs field of $\beta_3$-monopole is $(\Phi_2)_{\text{vac}} = \frac{1}{2}$. This is a border value which, according to (23), separates the composite $\beta_3$-monopole from a single fundamental $\beta_i$-monopole, for which $(\vec{h} \cdot \vec{\beta}_i) \geq 0$ if $(\Phi_2)_{\text{vac}} \geq \frac{1}{2}$. In that case the pattern of symmetry breaking becomes more simple. Futher increasement of the vacuum value $(\Phi_2)_{\text{vac}}$ smoothly moves the configuration to the limit $\Phi_1 \to 0$. Then the vector $\vec{h}$ becomes orthogonal to one of the simple roots that is the gauge symmetry is broken minimally and this is the case of a "massless" monopole.

In a general case of non-zero scalar coupling $\lambda$ the system of equations (22) may be solved numerically. This calculation have been done in [23] for the all range of possible boundary conditions. Here we only make another interpretation of the results, which is related with separation of the single fundamental monopole from a composite one.

As in the paper [23], we performed the calculation using COLSYS package. The profile
function of the gauge and scalar field are plotted for several values of the vacuum expectation value \((\Phi_2)_{\text{vac}} = C\) in Figures 2 and 7. In the following table, we summarize our results of the evaluation of the mass of the configuration in units of \(4\pi\) as a function of the boundary conditions on the vacuum value \((\Phi_2)_{\text{vac}} = C\) for values \(\lambda = 0\) (BPS monopole), and \(\lambda = 1, 10\):

| Embedding          | \(C\) | \(M\) \((\lambda=0)\) | \(M\) \((\lambda=1)\) | \(M\) \((\lambda=10)\) |
|-------------------|-------|-------------------------|-------------------------|-------------------------|
| Single \(\beta_1\)-monopole | 1.0   | 0.0                     | 0.0                     | 0.0                     |
|                   | 0.9999| 0.011                   | 0.011                   | 0.011                   |
|                   | 0.99  | 0.042                   | 0.042                   | 0.042                   |
|                   | 0.99  | 0.138                   | 0.139                   | 0.140                   |
|                   | 0.95  | 0.309                   | 0.314                   | 0.317                   |
|                   | 0.90  | 0.433                   | 0.456                   | 0.472                   |
|                   | 0.80  | 0.597                   | 0.634                   | 0.682                   |
|                   | 0.70  | 0.711                   | 0.777                   | 0.820                   |
|                   | 0.60  | 0.797                   | 0.897                   | 0.971                   |
|                   | 0.50  | 0.863                   | 1.001                   | 1.034                   |
| Composite \(\beta_2\)-monopole | 0.50  | 0.863                   | 1.001                   | 1.034                   |
|                   | 0.40  | 0.913                   | 1.090                   | 1.121                   |
|                   | 0.30  | 0.951                   | 1.166                   | 1.217                   |
|                   | 0.20  | 0.976                   | 1.229                   | 1.306                   |
|                   | 0.10  | 0.992                   | 1.273                   | 1.390                   |
|                   | 0.05  | 0.996                   | 1.284                   | 1.423                   |
|                   | 0.0   | 1.000                   | 1.291                   | 1.467                   |

The calculations show that the energy of interaction of two distinct fundamental non-BPS monopoles of the same mass \((\Phi_2)_{\text{vac}} = 0\), which are on top of each other, is very strong and this configuration seems to be an absolute minimum of the energy functional.

One may see that in the another limiting case \((\Phi_2)_{\text{vac}} = 1/2\) the mass degeneration between a single fundamental monopole and composite monopole is preserved at finite scalar coupling. Note that a single fundamental monopole becomes massless if the profile function of the vector field is constant everywhere: \(K = 1\). This is a case of the minimal symmetry breaking.

**Minimal symmetry breaking solution**

Let us consider behavior of a single fundamental monopole solution as the vacuum expectation value \((\Phi_2)_{\text{vac}}\) approaches the limit \(C = 1\). Figure shows that the monopole is spreading out in space as it was expected. First, we note that in this limit the potential of the scalar field \(V(\Phi)\) vanishes everywhere for any value of \(\lambda\), thus it is a BPS-like configuration. The ‘hedgehog’ component \(\Phi_1\) vanishes while the second component remains
a constant: $\Phi_2 = H_2$, thus this configuration becomes topologically trivial and can be continuously deformed into trivial solution $K = 1$.

Indeed, the energy, which corresponds to (21), in the limit of the minimal symmetry breaking becomes simple

$$E = 4\pi \int dr \left\{ (\partial_r K)^2 + \frac{(1 - K^2)^2}{2r^2} \right\}$$

Moreover, the covariant derivatives of the Higgs field (24) vanish everywhere while the non-Abelian magnetic field becomes

$$B_r = \frac{1 - K^2}{r^2} \tau_r^{(i)}; \quad B_\theta = \frac{\partial_r K}{r} \tau_\theta^{(i)}; \quad B_\phi = -\frac{\partial_r K}{r} \tau_\phi^{(i)}$$

that is the radial derivative $\partial_r K$ is not vanishing as $r \to \infty$. It would define the magnitude of the angular components $B_\theta \sim B_\phi \sim r^{-1}$ which would appear beside the radial Coulomb field $B_r \sim r^{-2}$. However the topological charge $g = \text{Tr} \int d^2SB_\Phi = 0$ since $\Phi = H_2$.

If we identify the term $\frac{(1 - K^2)^2}{2r^2}$ as a new scalar “potential” of the field $K(r)$, the energy functional (29) gets some similarity with the one-dimensional $\phi^4$ theory. However the kink solution of the latter model interpolate between two-fold degenerated vacua $\phi_{\text{vac}} = \pm 1$, while the potential of the former model vanishes on the spacial asymptotic and the profile function $K(r)$ would interpolate between $K(0) = 1$ and $K(r \to \infty) \to 0$ and such a configuration is instable. Indeed, let us consider the small spherically symetric fluctuations of the fields about the configuration with minimal symmetry breaking:

$$K(r) \to K(r) + a(r); \quad \Phi_1(r) \to \chi(r); \quad \Phi_2(r) \to 1 + \eta(r)$$

Then expansion of the energy in terms of these variations yelds the operator of second derivatives

$$D^{(2)} = \begin{pmatrix}
-\partial_r^2 + \frac{3K^2 - 1}{r^2} & 0 & 0 \\
0 & -\frac{1}{2} \partial_r^2 + \frac{K^2}{r^2} & 0 \\
0 & 0 & -\frac{1}{2} \partial_r^2 + 4\lambda
\end{pmatrix}$$

which is diagonal and has eigenfunctions

$$D^{(2)} \begin{pmatrix}
a \\
r\chi \\
r\eta
\end{pmatrix} = -\omega^2 \begin{pmatrix}
a \\
r\chi \\
r\eta
\end{pmatrix}$$

Each negative eigenvalue $-\omega^2$ generates an exponentially growing instability of the configuration. One may easily see that this equation has a number of negative modes in the region $r \to \infty$.

Thus, in the limiting case of the minimal symmetry breaking a single isolated fundamental monopole is dissolving into the topologically trivial sector and only a constant component of the scalar field $\Phi_2 = H_2$ survives.
Acknowledgements

I am very grateful to Jutta Kunz for invaluable help and support. All the numerical calculations were performed with her help. I would like to acknowledge the hospitality at the Abdus Salam International Center for Theoretical Physics where this work was completed (ICTP Preprint, IC/2003/180).

References

[1] G. ‘t Hooft, Nucl. Phys. B79 (1974) 276.
[2] A.M.Polyakov, Pis’ma JETP 20 (1974) 430.
[3] R.S. Ward, Comm. Math. Phys., 79 (1981) 317.
[4] C.H. Taubes, Comm. Math. Phys., 86 (1982) 257.
[5] J.N. Goldberg, Pong Soo Jang, Soo Young Park and K. Wali, Phys. Rev., D18, 542 (1978).
[6] B. Kleihaus, J. Kunz and D.H. Tchrakian, Mod. Phys. Lett., A13 (1998) 2523.
[7] V.G. Kiselev and Ya.M. Shnir, Phys. Rev. D57 (1997) 5174.
[8] B. Rüiber, Eine axialsymmetrische magnetische Dipollösung der Yang-Mills-Higgs-Gleichungen, MS Thesis, University of Bonn (1985).
[9] E. Corrigan, D.I. Olive, D.B.Fairlie and J. Nuyts, Nucl. Phys. B106 (1976) 475.
[10] B. Kleihaus and J. Kunz, Phys. Rev., D61 (2000) 025003.
[11] B. Kleihaus, J. Kunz and Ya. Shnir, Phys. Lett.. B570 (2003) 237.
[12] A. Chakrabarti, Ann. Inst. H. Poincaré, 23 (1975) 235.
[13] W.J. Marciano and H. Pagels, Phys. Rev., D12 (1975) 1093.
[14] J. Burzlaff, Phys. Rev., D23 (1981) 1329.
[15] E.J. Weinberg, Phys. Rev., D20 (1979) 936.
[16] E.J. Weinberg, Nucl. Phys., B167 (1980) 500.
[17] E.J. Weinberg, Nucl. Phys., B203 (1982) 445.
[18] P. Irwin, Phys. Rev., D56 (1997) 5200.
[19] Changhai Lu, Phys. Rev., D58 (1998) 125010.
[20] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys., B125 (1977) 1.

[21] F.A. Bais, Phys. Rev., D18 (1978) 1206.

[22] Th. Ioannidou and P. Sutcliffe, Phys. Rev., D60 (1999) 105009.

[23] J. Kunz and D. Masak, Phys. Lett., B196 (1987) 513.

[24] Y. Brihaye and B. Piette, Phys. Rev., D64 (2001) 084010.

[25] J. Arafune, P.G.O. Freund and C.J.Goebel, Journal of Math. Phys., 16 (1975) 433.

[26] K. Lee, E.J. Weinberg and Piljin Yi, Phys. Rev., D54 (1996) 1633; J.P. Gauntlett and D. Lowe, Nucl. Phys. B472 (1996) 194.

[27] A.S. Dancer, Nonlinearity, 5 (1992) 1355.

[28] A.S. Dancer, Comm. Math. Phys., 158 (1993) 545.
Figure 1: $SU(3)$ simple root basis.

Figure 2: Structure functions of the Higgs field component $\Phi_1(r)$ of the single fundamental monopole for different boundary conditions on the second component $\Phi_2(r)$ ($\lambda = 1$).
Figure 3: Structure function of the Higgs field component $\Phi_2(r)$ of the single fundamental monopole with different boundary conditions ($\lambda = 1$).

Figure 4: Structure function of the gauge field $K(r)$ of the single fundamental monopole for different boundary conditions on the second component of the Higgs field $\Phi_2$. The monopole is spreading out as $\Phi_2(r \to \infty)$ is approaching the limit $C = 1$ which corresponds to the minimal symmetry breaking.
Figure 5: Structure function of the Higgs field component $\Phi_1(r)$ of the composite monopole for different boundary conditions on the second component $\Phi_2(r)$ ($\lambda = 1$).

Figure 6: Structure functions of the Higgs field component $\Phi_2(r)$ of the composite monopole for different boundary conditions ($\lambda = 1$).
Figure 7: Structure functions of the gauge field $K(r)$ of the composite monopole for different boundary conditions on the second component of the Higgs field $\Phi_2$ ($\lambda = 1$).