(Anti)symmetric multivariate exponential functions
and corresponding Fourier transforms

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Abstract
We define and study symmetrized and antisymmetrized multivariate
exponential functions. They are defined as determinants and antideterminants
of matrices whose entries are exponential functions of one variable. These
functions are eigenfunctions of the Laplace operator on the corresponding
fundamental domains satisfying certain boundary conditions. To symmetric
and antisymmetric multivariate exponential functions there correspond Fourier
transforms. There are three types of such Fourier transforms: expansions into
the corresponding Fourier series, integral Fourier transforms and multivariate
finite Fourier transforms. Eigenfunctions of the integral Fourier transforms are
found.

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1. Introduction

In mathematical and theoretical physics, very often we deal with functions on the Euclidean
space $E_n$ which are symmetric or antisymmetric with respect to the permutation (symmetric)
group $S_n$. For example, such functions describe collections of identical particles. Symmetric
and antisymmetric solutions appear in the theory of integrable systems. Characters of finite-
dimensional representations of semisimple Lie algebras are symmetric functions. Moreover,
according to the Weyl formula for these characters, each such character is a ratio of
antisymmetric functions.

The aim of this paper is to describe and study symmetrized and antisymmetrized
multivariate exponential functions and the corresponding Fourier transforms. Antisymmetric
multivariate exponential functions (we denote them by $E^{-\lambda}(x)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $x = (x_1, x_2, \ldots, x_n)$) are determinants of $n \times n$ matrices, whose entries are the usual exponential
functions of one variable, $E^{-\lambda}(x) = \det(e^{2\pi i \lambda_j x_i})_{i,j=1}^n$. The symmetric multivariate exponential
functions $E_1^+(x)$ are antiderivatives of the same $n \times n$ matrices (for a definition of antiderivatives see below).

As in the case of the exponential functions of one variable, we may consider three types of antisymmetric and symmetric multivariate exponential functions.

(a) Functions $E_m^-(x)$ and $E_m^+(x)$ with $m = (m_1, m_2, \ldots, m_n)$, $m_i \in \mathbb{Z}$; they determine Fourier series expansions in multivariate symmetric and antisymmetric exponential functions.

(b) Functions $E_\lambda^-(x)$ and $E_\lambda^+(x)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R}$; these functions determine integral multivariate Fourier transforms.

(c) Functions $E_n^-(x)$ and $E_n^+(x)$, where $n = (x_1, x_2, \ldots, x_n)$ take a finite set of values; they determine multivariate finite Fourier transforms.

Functions (b) are antisymmetric (symmetric) with respect to the elements of the symmetric group $S_n$. (Anti)symmetries of functions (a) are described by a wider group, since exponential functions $e^{2\pi i m x}$, $m \in \mathbb{Z}$, of one variable are invariant with respect to shifts $x \rightarrow x + k, k \in \mathbb{Z}$. These (anti)symmetries are described by the elements of the affine symmetric group $S_n^{\text{aff}}$ which is a product of the group $S_n$ and the group $T_n$, consisting of shifts in the space $E_n$ by vectors $r = (r_1, r_2, \ldots, r_n), r_j \in \mathbb{Z}$. A fundamental domain $F(S_n^{\text{aff}})$ of the group $S_n^{\text{aff}}$ is a certain bounded subset of $\mathbb{R}^n$ (see subsection 2.2).

The functions $E_1^+(x)$ give solutions of the Neumann boundary value problem on a closure of the fundamental domain $F(S_n)$. The functions $E_\lambda^-(x)$ are solutions of the Laplace equation $\Delta f = \mu f$ on the domain $F(S_n)$ vanishing on the boundary $\partial F(S_n)$ of $F(S_n)$.

Functions on the fundamental domain $F(S_n^{\text{aff}})$ can be expanded into series in the functions (a). These expansions are an analogue of the usual Fourier series for functions of one variable. Functions (b) determine an (anti)symmetrized Fourier integral transform on the fundamental domain $F(S_n)$ of the symmetric group $S_n$. This domain consists of points $x \in E_n$, such that $x_1 > x_2 > \cdots > x_n$.

Functions (c) are used to determine (anti)symmetric finite (that is, on a finite set) Fourier transforms. These finite Fourier transforms are given on grids consisting of points in the fundamental domain $F(S_n^{\text{aff}})$.

Symmetric and antisymmetric exponential functions are closely related to symmetric and antisymmetric orbit functions defined in [1, 2] and studied in detail in [3, 4]. In fact, symmetric and antisymmetric exponential functions are connected with orbit functions corresponding to the Coxeter–Dynkin diagram $A_n$. Discrete orbit function transforms, corresponding to Coxeter–Dynkin diagrams of low order, were studied in detail and it was shown that they are very useful for applications [5–13].

The exposition of the theory of orbit functions in [3, 4] strongly depends on the theory of Weyl groups, properties of root systems, etc. In this paper we avoid this dependence. We use only the permutation (symmetric) group and properties of determinants and antiderivatives. It is well known that a determinant $\det(a_{ij})_{i,j=1}^n$ of the $n \times n$ matrix $(a_{ij})_{i,j=1}^n$ is defined as

$$
\det(a_{ij})_{i,j=1}^n = \sum_{w \in S_n} (\det w)a_{1,w(1)}a_{2,w(2)}\cdots a_{n,w(n)} = \sum_{w \in S_n} (\det w)a_{w(1),1}a_{w(2),2}\cdots a_{w(n),n},
$$

where $S_n$ is the symmetric group of $n$ symbols $1, 2, \ldots, n$, the set $(w(1), w(2), \ldots, w(n))$ means the set $w(1, 2, \ldots, n)$ and $\det w$ denotes a determinant of the transform $w$, that is, $\det w = 1$ if $w$ is an even permutation and $\det w = -1$ otherwise. Along with a determinant,
we shall use an antideterminant \( \det^+ \) of the matrix \( (a_{ij})_{i,j=1}^n \) which is defined as the sum of all terms, entering to the expression for the corresponding determinant, taken with the sign +,

\[
\det^+ (a_{ij})_{i,j=1}^n = \sum_{w \in S_n} a_{1,w(1)}a_{2,w(2)} \cdots a_{n,w(n)} = \sum_{w \in S_n} a_{w(1),1}a_{w(2),2} \cdots a_{w(n),n}.
\]  

(2)

Symmetrized and antisymmetrized multivariate polynomials were studied by several authors (see, for example, [14, 15]). In this paper, we investigate symmetric and antisymmetric multivariate exponential functions.

2. Symmetric and antisymmetric multivariate exponential functions

A symmetric multivariate exponential function of \( x = (x_1, x_2, \ldots, x_n) \) is defined as the function

\[
E^+_\lambda(x) \equiv E^+_\lambda(\lambda_1, \lambda_2, \ldots, \lambda_n)(x) = \det^+ (e^{2\pi i \lambda_i x_i})_{i,j=1}^n
= \det^+ \left( \begin{array}{cccc}
e^{2\pi i \lambda_1 x_1} & e^{2\pi i \lambda_2 x_2} & \cdots & e^{2\pi i \lambda_n x_n} \\
e^{2\pi i \lambda_1 x_2} & e^{2\pi i \lambda_2 x_1} & \cdots & e^{2\pi i \lambda_n x_n} \\
\vdots & \vdots & \ddots & \vdots \\
e^{2\pi i \lambda_1 x_n} & e^{2\pi i \lambda_2 x_n} & \cdots & e^{2\pi i \lambda_n x_1} \end{array} \right)
= \sum_{w \in S_n} e^{2\pi i \lambda w(x)} = \sum_{w \in S_n} e^{2\pi i \lambda w(x)},
\]

(3)

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a set of real numbers, which determines the function \( E^+_\lambda(x) \) and \( (\lambda, x) \) denotes the scalar product in the \( n \)-dimensional Euclidean space \( E_n \), \( \langle \lambda, x \rangle = \sum_{i=1}^n \lambda_i x_i \).

When \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are integers, we denote this set of numbers as \( m \equiv (m_1, m_2, \ldots, m_n) \),

\[
E^+_{m_1,m_2,\ldots,m_n}(x) = \det^+ (e^{2\pi i m_i x_i})_{i,j=1}^n.
\]

(4)

It is seen from expression (2) for an antideterminant \( \det^+ \) that the symmetric exponential functions \( E^+_\lambda(x) \) satisfy the relation

\[
E^+_\lambda(x_1 + a, x_2 + a, \ldots, x_n + a) = e^{2\pi i (\lambda_1 + \lambda_2 + \cdots + \lambda_n) a} E^+_\lambda(x).
\]

(5)

This means that it is enough to consider the function \( E^+_\lambda(x) \) on the hyperplane \( x_1 + x_2 + \cdots + x_n = b \), where \( b \) is a fixed number (we denote this hyperplane by \( \mathcal{H}_b \)). A transition from one hyperplane \( \mathcal{H}_b \) to another \( \mathcal{H}_c \) is fulfilled by multiplication by a usual exponential function \( e^{2\pi i \lambda (c-b)} \), where \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n \),

\[
E^+_\lambda(x)|_{x \in \mathcal{H}_b} = e^{2\pi i (\lambda_1 + \lambda_2 + \cdots + \lambda_n) (c-b)} E^+_\lambda(x)|_{x \in \mathcal{H}_c}.
\]

(6)

It is useful to consider the functions \( E^+_\lambda(x) \) on the hyperplane \( \mathcal{H}_0 \). For \( x \in \mathcal{H}_0 \) we have the relation

\[
E^+_\lambda(x)|_{x \in \mathcal{H}_0} = E^+_\lambda(x).
\]

(7)

It is seen from expression (2) for an antideterminant \( \det^+ \) that its expression does not change under permutations of rows or under permutations of columns. This means that for any permutation \( w \in S_n \) we have

\[
E^+_\lambda(wx) = E^+_\lambda(x), \quad E^+_\lambda(wx) = E^+_\lambda(x).
\]

Therefore, it is enough to consider only symmetric exponential functions \( E^+_\lambda(x) \) with \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), such that

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]
Such \( \lambda \) are called dominant. The set of all dominant \( \lambda \) is denoted by \( D^+_\ast \). Below, considering symmetric exponential functions \( E^+_\ast(x) \), we assume that \( \lambda \in D^+_\ast \).

Antisymmetric multivariate exponential functions of \( x = (x_1, x_2, \ldots, x_n) \) are defined as the functions

\[
E^-_{\lambda}(x) = E^-_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x) := \det(e^{2\pi i \langle \lambda, x \rangle})_{j=1}^n
\]

\[
= \det \begin{pmatrix}
e^{2\pi i \lambda_1 x_1} & e^{2\pi i \lambda_2 x_2} & \cdots & e^{2\pi i \lambda_n x_n} \\
e^{2\pi i \lambda_2 x_1} & e^{2\pi i \lambda_3 x_2} & \cdots & e^{2\pi i \lambda_n x_n} \\
\vdots & \vdots & \ddots & \vdots \\
e^{2\pi i \lambda_n x_1} & e^{2\pi i \lambda_1 x_2} & \cdots & e^{2\pi i \lambda_{n-1} x_n}
\end{pmatrix}
\]

\[
= \sum_{w \in S_n} (\det w) e^{2\pi i \langle \lambda, w x \rangle} = \sum_{w \in S_n} (\det w) e^{2\pi i (\lambda, w x)}, \quad (8)
\]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) is a set of real numbers. When \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are integers, we denote them as \( m = (m_1, m_2, \ldots, m_n) \),

\[ E_m(x) = \det(e^{2\pi i m_i x_j})_{j=1}^n. \quad (9) \]

It is seen from properties of determinants that the functions \( E^-_{\lambda}(x) \) satisfy the relation

\[
E^-_{\lambda}(x_1 + a, x_2 + a, \ldots, x_n + a) = e^{2\pi i (\lambda_1 + \lambda_2 + \cdots + \lambda_n) a} E^-_{\lambda}(x), \quad (10)
\]

that is, it is enough to consider functions \( E^-_{\lambda}(x) \) on some hyperplane \( H_0 \). As in the case of symmetric exponential functions, a transition from one hyperplane \( H_0 \) to another \( H \) for the function \( E^-_{\lambda} \) is fulfilled by means of multiplication by a usual exponential function \( e^{2\pi i \langle \lambda, (c-b) \rangle} \), where \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n \). For points \( x \) of the hyperplane \( H_0 \) we have the relation

\[
E^-_{\lambda_1 + \lambda_2 + \cdots + \lambda_n}(x) = E^-_{\lambda}(x). \quad (11)
\]

It follows from properties of determinants that \( E^-_{\lambda}(x) = 0 \) if \( \lambda \) has at least two coinciding numbers or if \( x \) has at least two coinciding coordinates. For any permutation \( w \in S_n \) we receive

\[
E^-_{\lambda}(w x) = (\det w) E^-_{\lambda}(x), \quad E^-_{\lambda}(w x) = (\det w) E^-_{\lambda}(x). \quad (12)
\]

This means that it is enough to consider the antisymmetric exponential functions \( E^-_{\lambda}(x) \) for \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), such that

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_n.
\]

Such \( \lambda \) are called strictly dominant. The set of these \( \lambda \) is denoted by \( D^+_{\ast} \).

3. Affine symmetric group and fundamental domains

We have seen that the functions \( E^+_\ast(x) \) are symmetric with respect to the permutation group \( S_n \), that is, \( E^+_\ast(w x) = E^+_\ast(x) \), \( w \in S_n \). The symmetric exponential functions \( E_m^+ \) with integral \( m = (m_1, m_2, \ldots, m_n) \) admit additional symmetries related to the periodicity of the exponential functions \( e^{2\pi i m y} \), \( r \in \mathbb{Z}, y \in \mathbb{R} \). These symmetries are described by the discrete group of shifts in the space \( E_n \) by vectors

\[ r_1 e_1 + r_2 e_2 + \cdots + r_n e_n, \quad r_i \in \mathbb{Z}, \]

where \( e_1, e_2, \ldots, e_n \) are the unit vectors in directions of the corresponding axes. We denote this group by \( T_n \). Permutations of \( S_n \) and shifts of \( T_n \) generate a group which is denoted as \( S_n^\text{aff} \) and is called the affine symmetric group. The group \( S_n^\text{aff} \) is a semidirect product of its subgroups \( S_n \) and \( T_n \),

\[
S_n^\text{aff} = S_n \times T_n, \quad \text{where} \quad T_n \text{ is an invariant subgroup, that is, } w t w^{-1} \in T_n \text{ for } w \in S_n \text{ and } t \in T_n.
\]
An open simply connected set $F \subset \mathbb{R}^n$ is called a fundamental domain for the group $S_n^{aff}$ (for the group $S_n$) if it does not contain equivalent points (that is, points $x$ and $x'$, such that $x' = wx$, where $w$ belongs to $S_n^{aff}$ or $S_n$, respectively) and if its closure contains at least one point from each $S_n^{aff}$-orbit (from each $S_n$-orbit). Recall that a $S_n^{aff}$-orbit of a point $x \in \mathbb{R}^n$ is the set of points $wx, w \in S_n^{aff}$.

It is evident that the set $D^*_n$ of all points $x = (x_1, x_2, \ldots, x_n)$, such that $x_1 > x_2 > \cdots > x_n$

constitutes a fundamental domain for the group $S_n$ (we denote it as $F(S_n)$). The set of points $x = (x_1, x_2, \ldots, x_n) \in D^*_n$, such that

$1 > x_1 > x_2 > \cdots > x_n > 0$

constitutes a fundamental domain for the affine group $S_n^{aff}$ (we denote it as $F(S_n^{aff})$).

As we have seen, the functions $E^+_m(x)$ are symmetric with respect to the permutation group $S_n$. This means that it is enough to consider the functions $E^+_m(x)$ only on the closure of the fundamental domain $F(S_n)$. Values of $E^+_m$ on other points are obtained by using symmetry.

The symmetricity of functions $E^+_m$ with integral $m = (m_1, m_2, \ldots, m_n)$ with respect to the affine symmetric group $S_n^{aff}$,

$E^+_m(wx + r) = E^+_m(x), \quad w \in S_n, \quad r \in T_n,$

(13)

means that we may consider $E^+_m(x)$ only on the closure of the fundamental domain $F(S_n^{aff})$, that is, on the set of points $x$, such that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. Values of $E^+_m(x)$ on other points are obtained by using relation (13).

The exponential functions $E^-_m(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$ also admit additional symmetries related to the periodicity of the usual exponential functions $e^{2\pi i w}, r \in \mathbb{Z}, y \in \mathbb{R}$. These symmetries are described by the affine symmetric group $S_n^{aff}$. We have

$E^-_m(wx + r) = (\det w)E^-_m(x), \quad w \in S_n, \quad r \in T_n,$

(14)

that is, it is enough to consider the functions $E^-_m(x)$ only on the closure of the fundamental domain $F(S_n^{aff})$. Values of $E^-_m(x)$ on other points are obtained by using relation (14).

The functions $E^+_{\lambda^-}(x), \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i \in \mathbb{R}$, are antisymmetric with respect to the symmetric group $S_n$,

$E^+_{\lambda^-}(wx) = (\det w)E^+_{\lambda^-}(x), \quad w \in S_n.$

For this reason, we may consider $E^+_{\lambda^-}$ only on the fundamental domain $F(S_n)$.

4. Properties

Symmetricity and antisymmetricity of symmetric and antisymmetric multivariate exponential functions are the main properties of these functions. However, they possess many other interesting properties.

4.1. Behaviour on boundary

The symmetric and antisymmetric functions $E^+_{\lambda}(x)$ and $E^-_{\lambda}(x)$ are finite sums of exponential functions. Therefore, they are continuous functions of $x_1, x_2, \ldots, x_n$ and have continuous derivatives of all orders in $\mathbb{R}^n$.

The closure $\overline{F(S_n)}$ of the fundamental domain $F(S_n)$ without points of $F(S_n)$ is called a boundary of the fundamental domain $F(S_n)$ and is denoted by $\partial F(S_n)$. A point $x = (x_1, x_2, \ldots, x_n) \in \overline{F(S_n)}$ belongs to $\partial F(S_n)$ if and only if at least two coordinates $x_i, x_j$
in $x$ coincide. It is clear that the boundary $\partial F(S_n)$ is composed of points of $F(S_n)$ belonging to the hyperplanes given by the equations

$$x_i = x_j, \quad i, j = 1, 2, \ldots, n, \quad i \neq j.$$  

Similarly, the boundary $\partial F(S_n)$ of the fundamental domain $F(S_n)$ consists of points of $F(S_n)$ which do not belong to $F(S_n)$. A point $x = (x_1, x_2, \ldots, x_n) \in \partial F(S_n)$ belongs to $\partial F(S_n)$ if and only if at least two coordinates $x_i, x_j$ in $x$ coincide or if one of the conditions $x_1 = 1, x_n = 0$ is fulfilled.

It follows from the properties of determinants that the function $E^+_\lambda(x)$ vanishes on the boundary $\partial F(S_n)$.

$$E^+_\lambda(x) = 0, \quad \text{for} \quad x \in \partial F(S_n).$$

This relation is true for $E^+_m(x), m_i \in \mathbb{Z}$. In this case, we also have $E^+_m(x) = 0$ for points $x \in \partial F(S_n)$, such that $x_1 = x_n = 1$.

For the symmetric multivariate functions $E^+_\lambda(x)$ we have

$$\frac{\partial E^+_\lambda(x)}{\partial n} = 0 \quad \text{for} \quad x \in \partial F(S_n),$$

where $n$ is the normal to the boundary $\partial F(S_n)$.

### 4.2. Complex conjugation

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a strictly dominant element, that is, $\lambda_1 > \lambda_2 \ldots > \lambda_n$. We have

$$E^{-\lambda}(x) = \sum_{w \in S_n} (\det w) e^{2\pi i (w\lambda)_1 x_1 + \cdots + (w\lambda)_n x_n},$$  

where $(w\lambda)_1, (w\lambda)_2, \ldots, (w\lambda)_n$ are the coordinates of the point $w\lambda$.

The element $- (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is strictly dominant if the element $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is strictly dominant. In the group $S_n$ there exists an element $w_0$, such that

$$w_0(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1).$$

It is easy to calculate that the set $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is obtained from $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by $n(n - 1)/2$ permutations of two neighbouring numbers. Clearly, $\det w_0 = 1$ if this number is even and $\det w_0 = -1$ otherwise. Thus,

$$\det w_0 = 1 \quad \text{for} \quad n = 4k \quad \text{and} \quad n = 4k + 1,$$

$$\det w_0 = -1 \quad \text{for} \quad n = 4k - 2 \quad \text{and} \quad n = 4k - 1,$$

where $k$ is an integer. It follows from here that in the expressions for the exponential functions $E_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x)$ and $E^{-\lambda}(x)$ there are summands

$$e^{2\pi i (w\lambda)_1 x_1 + \cdots + (w\lambda)_n x_n} \quad \text{and} \quad e^{-2\pi i (w\lambda)_1 x_1 + \cdots + (w\lambda)_n x_n},$$

respectively, which are complex conjugate to each other. Moreover, the first expression is contained with the sign $(\det w_0)$ in $E_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x)$, that is, expressions (16) are contained in $E_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x)$ and $E^{-\lambda}(x)$ with the same sign for $n = 4k, 4k + 1$ and with opposite signs for $n = 4k - 2, 4k - 1, k \in \mathbb{Z}$.

Similarly, in expressions (15) for the function $E^{-\lambda}(x)$ and for the function $E^{-\lambda}(x)$ all other summands are (up to a sign, which depends on a value of $n$) pairwise complex conjugate. Therefore,

$$E^{-\lambda}(x) = E^{-\lambda}(x)$$
for \( n = 4k, 4k + 1 \) and

\[
E^{-}_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x) = -E^{-}_{(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)}(x)
\]  

(18)

for \( n = 4k - 2, 4k - 1 \).

According to (17) and (18), if

\[
(\lambda_1, \lambda_2, \ldots, \lambda_n) = -(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1),
\]

(19)

then the function \( E^{-}_{\lambda}(x) \) is real for \( n = 4k, 4k + 1 \) and pure imaginary for \( n = 4k - 2, 4k - 1 \).

Moreover, the right-hand side of (15) for this case consists of pairs of terms which give sine or cosine functions. It is representable as a sum of cosines of angles if

\[ n = 4k, 4k + 1 \] and as

\[ a sum of sines of angles multiplied by \( i = \sqrt{-1} \) if \[ n = 4k - 2, 4k - 1 \].

It is proved similarly that for the symmetric exponential functions \( E^{+}_{\lambda}(x) \) and \( E^{+}_{\mu_{\lambda}}(x) \) we have the following relation:

\[
E^{+}_{(\lambda_1, \lambda_2, \ldots, \lambda_n)}(x) = E^{-}_{(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)}(x).
\]

4.3. Scaling symmetry

For \( c \in \mathbb{R} \), let \( c\lambda = (c\lambda_1, c\lambda_2, \ldots, c\lambda_n) \). Then

\[
E^{-}_{c\lambda}(x) = \sum_{w \in W} (\det w) e^{2\pi i |cw\lambda, x|} = \sum_{w \in W} (\det w) e^{2\pi i |w\lambda, cx|} = E^{-}_{\lambda}(cx).
\]

The equality \( E^{-}_{c\lambda}(x) = E^{-}_{\lambda}(cx) \) expresses the scaling symmetry of exponential functions \( E^{-}_{\lambda}(x) \). The scaling symmetry is true also for symmetric exponential functions, \( E^{+}_{\lambda}(x) = E^{+}_{\lambda}(cx) \).

4.4. Duality

Due to invariance of the scalar product \( \langle \cdot, \cdot \rangle \) with respect to the symmetric group \( S_n \), \( \langle w\mu, wy \rangle = \langle \mu, y \rangle \), for \( x = (x_1, x_2, \ldots, x_n) \), \( x_i \neq x_j, i \neq j \), we have

\[
E^{-}_{\lambda}(x) = \sum_{w \in W} (\det w) e^{2\pi i |\lambda, w^{-1}x|} = \sum_{w \in W} (\det w) e^{2\pi i |x, w\lambda|} = E^{-}_{\lambda}(x).
\]

This relation expresses the duality of antisymmetric orbit functions. The duality is true also for the symmetric exponential functions, \( E^{+}_{\lambda}(x) = E^{+}_{\lambda}(\lambda) \).

4.5. Orthogonality on the fundamental domain \( F(S_n^{\text{aff}}) \)

The antisymmetric exponential functions \( E^{-}_{m} \) with \( m = (m_1, m_2, \ldots, m_n) \in D^{+}_n, m_j \in \mathbb{Z} \), are orthogonal on \( F(S_n^{\text{aff}}) \) with respect to the Euclidean measure,

\[
|F(S_n^{\text{aff}})|^{-1} \int_{F(S_n^{\text{aff}})} E^{-}_{m}(x) E^{-}_{m'}(x) \, dx = |S_n| \delta_{mm'},
\]

(20)

where the overbar means complex conjugation, \( |S_n| \) means a number of elements in the set \( S_n \) and \( |F(S_n^{\text{aff}})| \) is an area of the fundamental domain \( F(S_n^{\text{aff}}) \). This relation follows from the equality

\[
\int_{\mathbb{T}} E^{-}_{m}(x) E^{-}_{m'}(x) \, dx = |S_n| \delta_{mm'}
\]
where $T$ is the torus in $E_n$ consisting of points $x = (x_1, x_2, \ldots, x_n), 0 \leq x_i < 1$, which is a consequence of orthogonality of the exponential functions $e^{2\pi i j(x)}$ (entering into the definition of $E_n(x)$) for different sets $\mu$.

If to assume that an area of $T$ is equal to 1, $|T| = 1$, then $|F(S_m^\aff)| = |S_m|^{-1}$ and formula (20) takes the form

$$\int_{F(S_m^\aff)} E_m^+(x)E_m^-(x) \, dx = \delta_{mm'}.$$  \hspace{1cm} (21)

In expression (3) for symmetric exponential functions there can be coinciding summands. For this reason, for the symmetric exponential functions $E_m^+$ with $m = (m_1, m_2, \ldots, m_n) \in D_+, m_j \in \mathbb{Z}$, relation (21) is replaced by

$$\int_{F(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx = |S_m|\delta_{mm'},$$  \hspace{1cm} (22)

where $|S_m|$ is a number of elements in the subgroup $S_m$ of $S_n$ consisting of elements $w \in S_n$, such that $wm = m$.

4.6. Orthogonality of symmetric and antisymmetric exponential functions.

Let $w_i$ $(i = 1, 2, \ldots, n-1)$ be the permutation of coordinates $x_i$ and $x_{i+1}$. We create the domain $F^{\ext}(S_m^\aff) = F(S_m^\aff) \cup w_i F(S_m^\aff)$, where $F(S_m^\aff)$ is the fundamental domain for the affine group $S_n^\aff$. Since for $m = (m_1, m_2, \ldots, m_n) \in D_+, m_j \in \mathbb{Z}$, we have $E_m^+(wix) = E_m^+(x)$ and $E_m^-(wix) = -E_m^-(x)$, then

$$\int_{F^{\ext}(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx = 0.$$  \hspace{1cm} (23)

Indeed, due to symmetry and antisymmetry of symmetric and antisymmetric exponential functions, respectively, we have

$$\int_{F^{\ext}(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx = \int_{F(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx + \int_{w_iF(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx$$

$$= \int_{F(S_m^\aff)} E_m^+(x)E_m^+(x) \, dx + \int_{F(S_m^\aff)} E_m^+(x)(-E_m^-(x)) \, dx = 0.$$

Relation (23) is a generalization of the orthogonality of the functions sine and cosine on the interval $(0, 2\pi)$.

5. Special cases

The special case of symmetric and antisymmetric exponential functions at $\lambda = \frac{1}{2}(n-1, n-3, \ldots, -n+3, -n+1) \equiv \rho$ is of great interest since it is met in the representation theory. The antisymmetric exponential function $E_\rho^-(x)$ is given by the formula

$$E_\rho^-(x) = (2\pi)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} \sin \pi (x_i - x_j).$$  \hspace{1cm} (24)

It follows if to represent $\sin \pi (x_i - x_j)$ in terms of exponential functions, then to fulfil multiplication of these functions and to compare with expression (8) for $E_\rho^+(x)$.

Let us set $(m_1, m_2, \ldots, m_n) = (n-1, n-2, \ldots, 1, 0) \equiv \rho'$. The antisymmetric exponential function $E_\rho'(x)$ can be written in the form of the Vandermonde determinant,

$$E_\rho'(x) = \det(e^{2\pi i x_j} e^{2\pi i x_k})_{i,j=1} = \prod_{k<l}(e^{2\pi i x_k} - e^{-2\pi i x_l}).$$  \hspace{1cm} (25)
The last equality follows from the expression for the Vandermonde determinant. Since \( \rho' = \rho + \frac{n-1}{2} \), expressions (24) and (25) are connected by the relation
\[
E^{-\rho'}(x) = e^{\pi i |x|/(n-1)} E^{-\rho}(x),
\]
where \(|x| = \sum_{i=1}^n x_i\).

It is easy to see that the function \( E^{-\rho}(x) \) does not vanish on intrinsic points of the fundamental domain \( F(\text{aff}^n) \).

The symmetric counterpart \( E^+_{\rho}(x) \) of formula (24) for the antisymmetric exponential function \( E^{-\rho}(x) \) has the form
\[
E^+_{\rho}(x) = 2^{n(n-1)/2} \prod_{1 \leq i < j \leq n} \cos \pi (x_i - x_j). \tag{26}
\]

6. Solutions of the Laplace equation

The Laplace operator on the \( n \)-dimensional Euclidean space \( E_n \) in the orthogonal coordinates \( x = (x_1, x_2, \ldots, x_n) \) has the form
\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.
\]

We take any summand in the expression for symmetric or antisymmetric multivariate exponential function and act upon it by the operator \( \Delta \). We get
\[
\Delta e^{\pi i (w(\lambda_1)x_1 + \cdots + w(\lambda_n)x_n)} = -4\pi^2 \langle \lambda, \lambda \rangle e^{\pi i (w(\lambda_1)x_1 + \cdots + w(\lambda_n)x_n)},
\]
where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) determines \( E^{-\lambda}(x) \). Since this action of \( \Delta \) does not depend on a summand from the expression for symmetric or antisymmetric exponential function, we have
\[
\Delta E^{-\lambda}(x) = -4\pi^2 \langle \lambda, \lambda \rangle E^{-\lambda}(x), \quad \Delta E^{+\lambda}(x) = -4\pi^2 \langle \lambda, \lambda \rangle E^{+\lambda}(x). \tag{27}
\]

Formula (27) can be generalized in the following way. Let \( \sigma_k(y_1, y_2, \ldots, y_n) \) be the \( k \)th elementary symmetric polynomial of degree \( k \), that is,
\[
\sigma_k(y_1, y_2, \ldots, y_n) = \sum_{1 \leq k_1 < k_2 < \cdots < k_k \leq n} y_{k_1} y_{k_2} \cdots y_{k_k}.
\]

Then for \( k = 1, 2, \ldots, n \) we have
\[
\sigma_k \left( \frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2} \right) E^{\pm}_{\lambda}(x) = (-4\pi^2)^k \sigma_k (\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2) E^{\pm}_{\lambda}(x). \tag{28}
\]

Note that \( n \) differential equations (28) are algebraically independent.

Thus, the antisymmetric exponential functions \( E^{-\lambda}(x) \) are eigenfunctions of the operators \( \sigma_k \left( \frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2} \right) \), \( k = 1, 2, \ldots, n \), on the fundamental domain \( F(S_n) \) of the symmetric group \( S_n \) satisfying the boundary condition
\[
E^{-\lambda}(x) = 0 \quad \text{for} \quad x \in \partial F(S_n). \tag{29}
\]

Similarly, the symmetric exponential functions \( E^{+\lambda}(x) \) are eigenfunctions of the operators \( \sigma_k \left( \frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2} \right) \), \( k = 1, 2, \ldots, n \), on the fundamental domain \( F(S_n) \) satisfying the boundary condition
\[
\frac{\partial E^{+\lambda}(x)}{\partial n} = 0 \quad \text{for} \quad x \in \partial F(S_n),
\]
where \( n \) is the normal to the boundary \( \partial F(S_n) \). That is, they are the solutions of the Neumann boundary value problem for the domain \( F(S_n) \).
7. Expansions in (anti)symmetric exponential functions on $F(S_n^{\text{aff}})$

Symmetric and antisymmetric exponential functions determine symmetric and antisymmetric multivariate Fourier transforms which generalize the usual Fourier transform.

As in the case of exponential functions of one variable, (anti)symmetric exponential functions determine three types of Fourier transforms:

(a) Fourier transforms related to the functions $E_m^f(x)$ with $m = (m_1, m_2, \ldots, m_n)$, $m_j \in \mathbb{Z}$ (Fourier series);
(b) Fourier transforms related to $E_m^f(x)$ with $\lambda \in D_+$;
(c) symmetrized and antisymmetrized multivariate finite Fourier transforms.

In this section, we consider expansions in (anti)symmetric exponential functions on the fundamental domain $F(S_n^{\text{aff}})$. Let $f(x)$ be a symmetric (with respect to the affine symmetric group $S_n^{\text{aff}}$) continuous function on the $n$-dimensional Euclidean space $E_n$ which has continuous derivatives. We may consider this function on the torus $T$ which is a closure of the union of the sets $wF(S_n^{\text{aff}})$, $w \in S_n$. The function $f(x)$, as a function on $T$, can be expanded in the exponential functions $e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} \ldots e^{2\pi i m_n x_n}$, $m_i \in \mathbb{Z}$. We have

$$f(x) = \sum_{m_i \in \mathbb{Z}} c_m e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} \ldots e^{2\pi i m_n x_n}, \quad (30)$$

where $m = (m_1, m_2, \ldots, m_n)$. Due to the symmetry $f(wx) = f(x)$, for any $w \in S_n$, we have

$$f(wx) = \sum_{m_i \in \mathbb{Z}} c_m e^{2\pi i m_1 x_1} \ldots e^{2\pi i m_n x_n} = \sum_{m_i \in \mathbb{Z}} c_m e^{2\pi i m_1 x_1} \ldots e^{2\pi i m_n x_n}$$

$$= \sum_{m_i \in \mathbb{Z}} c_{wm} e^{2\pi i m_1 x_1} \ldots e^{2\pi i m_n x_n} = f(x) = \sum_{m_i \in \mathbb{Z}} c_m e^{2\pi i m_1 x_1} \ldots e^{2\pi i m_n x_n}.$$

Therefore, the coefficients $c_m$ satisfy the conditions $c_{wm} = c_m$, $w \in S_n$. Collecting in (30) exponential functions at the same $c_{wm}$, $w \in S_n$, we obtain the expansion

$$f(x) = \sum_{m \in P_+} c_mE_m^f(x), \quad (31)$$

where $P_+ = D_+ \cap \mathbb{Z}^n$. Thus, any symmetric (with respect to $S_n$) continuous function $f$ on $T$ which has continuous derivatives (that is, any continuous function on $F(S_n^{\text{aff}})$ with continuous derivatives) can be expanded in the symmetric exponential functions $E_m^f(x)$, $m \in P_+$.

By the orthogonality relation (22), the coefficients $c_m$ in expansion (31) are determined by the formula

$$c_m = |S_m|^{-1} \int_{F(S_n^{\text{aff}})} f(x)E_m^f(x) \, dx, \quad (32)$$

where, as before, $|S_m|$ is a number of elements in the subgroup $S_m$ of $S_n$ consisting of $w \in S_n$, such that $wm = m$. Moreover, the Plancherel formula

$$\sum_{m \in P_+} |c_m|^2 = |S_m|^{-1} \int_{F(S_n^{\text{aff}})} |f(x)|^2 \, dx \quad (33)$$

holds, which means that the Hilbert spaces with the appropriate scalar products are isometric.

Formula (32) is the symmetrized Fourier transform of the function $f(x)$. Formula (31) gives an inverse transform. Formulas (31) and (32) give the symmetric multivariate Fourier transforms corresponding to the symmetric exponential functions $E_m^f(x)$, $m \in P_+$. 
Analogous transforms hold for the antisymmetric exponential functions \( E_m^-(x) \), \( m \in P^* \equiv D^*_+ \cap \mathbb{Z}^n \). Let \( f(x) \) be an antisymmetric (with respect to the symmetric group \( S_n \)) continuous function on the \( n \)-dimensional torus \( T \), which has continuous derivatives. We may consider this function as a function on \( F(S_n^{aff}) \). Then we have the expansion

\[
f(x) = \sum_{m \in P^*} c_m E_m^+(x), \quad \text{where} \quad c_m = \int_{F(S_n^{aff})} f(x) \overline{E_m^-(x)} \, dx. \tag{34}
\]

Moreover, the Plancherel formula holds

\[
\sum_{m \in P^*} |c_m|^2 = \int_{F(S_n^{aff})} |f(x)|^2 \, dx. \tag{35}
\]

Let \( \mathcal{L}^2(F(S_n^{aff})) \) denote the Hilbert space of functions on the fundamental domain \( F(S_n^{aff}) \), which behaves on the boundary \( \partial F(S_n^{aff}) \) of the fundamental domain \( F(S_n^{aff}) \) in the same way as the functions \( E_m^-(x) \) do. Let

\[
(f_1, f_2) = \int_{F(S_n^{aff})} f_1(x) \overline{f_2(x)} \, dx
\]

be a scalar product in this space. Formulae (34) and (35) show that the set of exponential functions \( E_m^+(x), m \in P^* \), forms an orthogonal basis of \( \mathcal{L}^2(F(S_n^{aff})) \).

Let \( F^{ext}(S_n^{aff}) = F(S_n^{aff}) \cup F(w_1 S_n^{aff}) \) denote the set from section 4. Then we can extend the symmetric and antisymmetric Fourier transforms to the functions from the Hilbert space \( \mathcal{L}^2(F^{ext}(S_n^{aff})) \) with the scalar product

\[
(f_1, f_2) = \int_{F^{ext}(S_n^{aff})} f_1(x) \overline{f_2(x)} \, dx.
\]

This transform is of the form

\[
f(x) = \sum_{m \in P^*} c_m E_m^+(x) + \sum_{m \in P^*} c_m^* E_m^-(x), \tag{36}
\]

where

\[
c_m = |S_n|^{-1} \int_{F(S_n^{aff})} f(x) \overline{E_m^-(x)} \, dx, \quad c_m^* = \int_{F(S_n^{aff})} f(x) \overline{E_m^+(x)} \, dx. \tag{37}
\]

The corresponding Plancherel formula holds. The functions \( E_m^+(x), m \in P^* \) and \( E_m^-(x), m \in P^* \), form a complete orthogonal basis of the Hilbert space \( \mathcal{L}^2(F^{ext}(S_n^{aff})) \).

8. Multivariate Fourier transforms on the fundamental domain \( F(S_n) \)

Expansions (31) and (34) of functions on the fundamental domain \( F(S_n^{aff}) \) are respectively expansions in the symmetric and antisymmetric exponential functions \( E_m^+(x) \) and \( E_m^-(x) \) with integral \( m = (m_1, m_2, \ldots, m_n) \). The exponential functions \( E_\lambda^+(x) \) and \( E_\lambda^-(x) \) with \( \lambda \) lying in the fundamental domain \( F(S_n) \) (and not obligatory integral) are not invariant (anti-invariant) with respect to the corresponding affine symmetric group \( S_n^{aff} \). They are invariant (anti-invariant) only with respect to the permutation group \( S_n \). A fundamental domain of \( S_n \) coincides with the set \( D^*_+ \) consisting of the points \( x \), such that \( x_1 > x_2 > \cdots > x_n \). For this reason, the functions \( E_\lambda^+(x), \lambda \in D^*_+ \), and \( E_\lambda^-(x), \lambda \in D^*_+ \), determine Fourier transforms on \( D^*_+ \).

We began with the usual Fourier transforms on \( \mathbb{R}^n \):

\[
\hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle \lambda, x \rangle} \, dx, \tag{38}
\]
\[ f(x) = \int_{\mathbb{R}^n} \tilde{f}(\lambda) e^{-2\pi i \langle \lambda, x \rangle} d\lambda. \] (39)

Let the function \( f(x) \) be anti-invariant with respect to the symmetric group \( S_n \), that is, 
\( f(wx) = (\det w)f(x), \, w \in S_n \). It is easy to check that \( \tilde{f}(\lambda) \) is also anti-invariant with respect to the group \( S_n \). Replace \( \lambda \) by \( w\lambda \) in (38), \( w \in S_n \), multiply both sides by \( \det w \), and sum these both sides over \( w \in S_n \). Then instead of (38) we obtain
\[ \tilde{f}(\lambda) = \int_{D_+} f(x) E_{\lambda}^+(x) d\lambda, \quad \lambda \in D_+, \] (40)

where we have taken into account that \( f(x) \) is anti-invariant with respect to \( S_n \).

Similarly, starting from (39), we obtain the inverse formula
\[ f(x) = \int_{D_+} \tilde{f}(\lambda) E_{\lambda}^+(x) d\lambda. \] (41)

For the transforms (40) and (41) the Plancherel formula
\[ \int_{D_+} |f(x)|^2 d\lambda = \int_{D_+} |\tilde{f}(\lambda)|^2 d\lambda \]
holds. Formulae (40) and (41) determine the antisymmetric multivariate Fourier transform on the domain \( F(S_n) \).

Similarly, starting from formulae (38) and (39) we receive the symmetric multivariate Fourier transform on the domain \( F(S_n) \):
\[ \tilde{f}(\lambda) = \int_{D_+} f(x) E_{\lambda}^-(x) d\lambda, \quad f(x) = \int_{D_+} \tilde{f}(\lambda) E_{\lambda}^-(x) d\lambda. \] (42)

The corresponding Plancherel formula holds.

9. Finite Fourier transform

Along with the integral Fourier transform in one variable there exists a discrete Fourier transform in one variable running over a finite set. Similarly, it is possible to introduce finite multivariate antisymmetric and symmetric Fourier transforms, based on antisymmetric and symmetric exponential functions. We first consider the finite Fourier transform in one variable, which will be used below. In the following sections, we expose a general antisymmetric and symmetric Fourier transforms (under exposition we shall use the methods developed in [16]).

Let us fix a positive integer \( N \) and consider the numbers
\[ e_{mn} := N^{-1/2} \exp(2\pi i mn/N), \quad m, n = 1, 2, \ldots, N. \] (43)

The matrix \( (e_{mn})_{m,n=1}^N \) is unitary, that is,
\[ \sum_k e_{mk} e_{nk} = \delta_{mn}, \quad \sum_k e_{km} e_{kn} = \delta_{mn}. \] (44)

Indeed, according to the formula for a sum of a geometric progression we have
\[ t^a + t^{a+1} + \ldots + t^{ar} = (1 - t)^{-1}(1 - t^{a+1}), \quad t \neq 1, \]
\[ t^a + t^{a+1} + \ldots + t^{ar} = r + 1, \quad t = 1. \]

Setting \( t = \exp(2\pi i (m - n)/N), \, a = 1 \) and \( r = N - 1 \), we prove (44).

Let \( f(n) \) be a function of \( n \in \{1, 2, \ldots, N\} \). We may consider the transform
\[ \sum_{n=1}^N f(n) e_{mn} \equiv N^{-1/2} \sum_{n=1}^N f(n) \exp(2\pi i mn/N) = \tilde{f}(m). \] (45)
Then due to unitarity of the matrix \((em_n)_{m,n=1}^N\), we express \(f(n)\) as a linear combination of conjugates of the functions (43):

\[
f(n) = N^{-1/2} \sum_{m=1}^N \tilde{f}(m) \exp(-2\pi imn/N).
\]

(46)

The function \(\tilde{f}(m)\) is a finite Fourier transform of \(f(n)\). This transform is a linear map. Formula (46) gives an inverse transform. The Plancherel formula

\[
\sum_{m=1}^N |\tilde{f}(m)|^2 = \sum_{n=1}^N |f(n)|^2
\]

holds for transforms (45) and (46). This means that the finite Fourier transform conserves the norm introduced in the space of functions on \(\{1, 2, \ldots, N\}\).

10. Antisymmetric multivariate discrete Fourier transforms

We use the discrete exponential function (43),

\[
e_m(s) := N^{-1/2} \exp(2\pi ims), \quad s \in F_N \equiv \left\{ \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1 \right\}, \quad m \in \mathbb{Z}^{\geq 0},
\]

(47)

and make a multivariate discrete exponential function by taking a product of \(n\) copies of these functions,

\[
e_m(s) := e_{m_1}(s_1)e_{m_2}(s_2) \cdots e_{m_n}(s_n)
\]

\[
= N^{-n/2} \exp(2\pi im_1 s_1) \exp(2\pi im_2 s_2) \cdots \exp(2\pi im_n s_n),
\]

(48)

where \(s = (s_1, s_2, \ldots, s_n) \in F_N^n\) and \(m = (m_1, m_2, \ldots, m_n) \in (\mathbb{Z}^{\geq 0})^n\). Now we take these multivariate functions for integers \(m_i\), such that \(m_1 > m_2 > \cdots > m_n \geq 0\) and make an antisymmetrization. As a result, we obtain a finite version of the antisymmetric exponential functions (8),

\[
E_m^-(s) := |S_n|^{-1/2} \det(e_{m_j}(s_j))_{j,m=1}^n = |S_n|^{-1/2} N^{-n/2} E_m^-(s),
\]

(49)

where, as before, \(|S_n|\) is the order of the symmetric group \(S_n\).

The \(n\)-tuples \(s\) in (49) run over \(F_N^n = F_N \times \cdots \times F_N\) \((n\) times). We denote by \(\hat{F}_N^n\) the subset of \(F_N^n\) consisting of \(s \in F_N^n\), such that

\[
s_1 > s_2 > \cdots > s_n.
\]

The set \(\hat{F}_N^n\) is a finite subset of the fundamental domain \(F(S_n^\text{aff})\) of the group \(S_n^\text{aff}\).

Note that acting by permutations \(w \in S_n\) upon \(\hat{F}_N^n\) we obtain the whole set \(F_N^n\) without those points which are invariant under some nontrivial permutation \(w \in S_n\). Clearly, the function (49) vanishes on the last points.

Since the discrete exponential functions \(e_m(s)\) satisfy the equality \(e_m(s) = e_{m+N s}\), we do not need to consider them for all values \(m \in \mathbb{Z}^{\geq 0}\). It is enough to consider them for \(m \in \{1, 2, \ldots, N\}\). By \(\hat{D}_N^n\) we denote the set of integer \(n\)-tuples \(m = (m_1, m_2, \ldots, m_n)\), such that

\[
N \geq m_1 > m_2 > \cdots > m_n > 0.
\]

We need a scalar product in the space of linear combinations of the functions (48). It is natural to give it by the formula

\[
\langle e_m(s), e_{m'}(s) \rangle = \prod_{i=1}^n \langle e_{m_i}(s_i), e_{m'_i}(s_i) \rangle = \prod_{i=1}^n \sum_{s_i \in F_N} e_{m_i}(s_i) e_{m'_i}(s_i) = \delta_{mm'},
\]

(50)

where \(m_i, m'_i \in \{1, 2, \ldots, N\}\). Here we used relation (44).
Proposition 1. For \( m, m' \in \hat{D}_N^n \) the discrete functions (49) satisfy the orthogonality relation
\[
\langle \tilde{E}^{-m}(s), \tilde{E}^{-m'}(s) \rangle = |S_n| \sum_{s \in \hat{F}_N^n} \tilde{E}^{-m}(s) \tilde{E}^{-m'}(s) = \delta_{mm'},
\]
where the scalar product is determined by formula (50).

Proof. Since \( m_1 > m_2 > \cdots > m_n > 0 \) and \( m'_1 > m'_2 > \cdots > m'_n > 0 \), then due to the definition of the scalar product we have
\[
\langle \tilde{E}^{-m}(s), \tilde{E}^{-m'}(s) \rangle = |S_n| \sum_{s \in \hat{F}_N^n} \tilde{E}^{-m}(s) \tilde{E}^{-m'}(s) = |S_n|^{-1} \sum_{w \in S_n} \prod_{i=1}^n \sum_{s_i \in \hat{F}_N^n} e_{m_w(s_i)}(s_i) \tilde{E}^{-m'(s_i)}(s_i) = \delta_{mm'},
\]
where \((m_w(1), m_w(2), \ldots, m_w(n))\) is obtained from \((m_1, m_2, \ldots, m_n)\) by action by the permutation \(w \in S_n\). Since functions \( \tilde{E}^{-m}(s) \) are antisymmetric with respect to \( S_n \), then
\[
\sum_{s \in \hat{F}_N^n} \tilde{E}^{-m}(s) \tilde{E}^{-m'}(s) = |S_n| \sum_{s \in \hat{F}_N^n} \tilde{E}^{-m}(s) \tilde{E}^{-m'}(s).
\]
This proves the proposition. \(\square\)

Let \( f \) be a function on \( \hat{F}_N^n \) (or an antisymmetric function on \( F_N^n \)). Then it can be expanded in the functions (49) as
\[
f(s) = \sum_{m \in \hat{D}_N^n} a_m \tilde{E}^{-m}(s),
\]
where the coefficients \( a_m \) are determined by the formula
\[
a_m = |S_n| \sum_{m \in \hat{D}_N^n} f(s) \tilde{E}^{-m}(s).
\]
We have taken into account the facts that numbers of elements in \( \hat{D}_N^n \) and \( \hat{F}_N^n \) are the same and that the discrete functions (49) are orthogonal with respect to the scalar product (51). We call expansions (53) and (54) the antisymmetric multivariate discrete Fourier transforms. These expansions can be written in terms of the exponential function
\[
E^{-m}(s) = \det(2\pi i m_i s_j)_{i,j=1}^n,
\]
where \( e_m(s) \) are given by (47).

11. Symmetric multivariate discrete Fourier transforms

Let us give a symmetric multivariate discrete Fourier transform. For this we take the multivariate exponential functions (48) for integers \( m_i \), such that
\[
N \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 1
\]
and make a symmetrization. We obtain a finite version of the symmetric exponential functions (3),
\[
\tilde{E}^+(s) := |S_n|^{-1/2} \det^*(e_m(s_j))_{i,j=1}^n = |S_n|^{-1/2} N^{-n/2} E^+(s),
\]
where the discrete functions \( e_m(s) \) are given by (47).
The $n$-tuples $s$ in (56) run over $F^m_N \equiv F_N \times \cdots \times F_N$ ($n$ times). We denote by $\hat{F}^m_N$ the subset of $F^m_N$ consisting of $s = (s_1, s_2, \ldots, s_n) \in F^m_N$, such that

$$s_1 \geq s_2 \geq \cdots \geq s_n.$$  

The set $\hat{F}^m_N$ is a finite subset of the closure of the fundamental domain $F(s_{\text{aff}})$.

Note that acting by permutations $w \in S_n$ upon $\hat{F}^m_N$ we obtain the whole set $\hat{F}^m_N$, where each point, having some coordinates $m_i$ coinciding, is repeated several times. Namely, a point $s$ is contained $|S_n|$ times in $\{w \hat{F}^m_N; w \in S_n\}$, where $S_n$ is the subgroup of $S_n$ consisting of elements $w \in S_n$, such that $w s = s$.

By $\hat{D}^m_N$ we denote the set of integer $n$-tuples $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, such that

$$N \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 1.$$  

**Proposition 2.** For $\mathbf{m}, \mathbf{m}' \in \hat{D}^m_N$ the discrete functions (56) satisfy the orthogonality relation

$$\langle \hat{E}^+_{\mathbf{m}}(s), \hat{E}^+_{\mathbf{m}'}(s) \rangle = |S_n| \sum_{s \in \hat{F}^m_N} |S_n|^{-1} \hat{E}^+_{\mathbf{m}}(s) \hat{E}^+_{\mathbf{m}'}(s) = |S_n| \delta_{\mathbf{m}\mathbf{m}'}.$$  

**Proof.** This proposition is proved in the same way as Proposition 1, but we have to take into account the difference between $\hat{F}^m_M$ and $\hat{F}^m_M$. Due to the definition of the scalar product we have

$$\langle \hat{E}^+_{\mathbf{m}}(s), \hat{E}^+_{\mathbf{m}'}(s) \rangle = \sum_{s \in \hat{F}^m_N} \hat{E}^+_{\mathbf{m}}(s) \hat{E}^+_{\mathbf{m}'}(s)$$

$$= |S_n|^{-1} |S_m| \sum_{w \in S_n} \prod_{i=1}^n \sum_{s \in F_N} e_{m_{wi}}(s_i) e_{m'_{wi}}(s_i) = |S_n| \delta_{\mathbf{m}\mathbf{m}'}.$$  

Here we have taken into account that there appear additional summands (with respect to (52)) because some summands on the right-hand side of (56) can coincide.

Since the functions $\hat{E}^+_{\mathbf{m}}(s)$ are symmetric with respect to $S_n$, then

$$\sum_{s \in \hat{F}^m_N} \hat{E}^+_{\mathbf{m}}(s) \hat{E}^+_{\mathbf{m}'}(s) = |S_n| \sum_{s \in \hat{F}^m_N} |S_n|^{-1} \hat{E}^+_{\mathbf{m}}(s) \hat{E}^+_{\mathbf{m}'}(s),$$

where we have taken into account that under an action by $S_n$ upon $\hat{F}^m_N$ a point $s$ appears $|S_n|$ times in $\hat{F}^m_N$. This proves the proposition.  

Let $f$ be a function on $\hat{F}^m_N$ (or a symmetric function on $F^m_N$). Then it can be expanded in functions (56) as

$$f(s) = \sum_{\mathbf{m} \in \hat{D}^m_N} a_{\mathbf{m}} \hat{E}^+_{\mathbf{m}}(s).$$  

(58)

The coefficients $a_{\mathbf{m}}$ are determined by the formula

$$a_{\mathbf{m}} = |S_n| |S_m|^{-1} \sum_{s \in \hat{F}^m_N} |S_n|^{-1} f(s) \hat{E}^+_{\mathbf{m}}(s).$$  

(59)

Expansions (58) and (59) follow from the facts that numbers of elements in $\hat{D}^m_N$ and $\hat{F}^m_N$ are the same and from the orthogonality relation (57). We call expansions (58) and (59) the symmetric multivariate discrete Fourier transforms.
12. Eigenfunctions of (anti)symmetric Fourier transforms

Let $H_n(x)$, $n = 0, 1, 2, \ldots$, be the well-known Hermite polynomials of one variable. They satisfy the relation

$$\int_{-\infty}^{\infty} e^{2\pi i px} e^{-\pi p^2} H_m(\sqrt{2\pi} p) dp = i^{-m} e^{-\pi x^2} H_m(\sqrt{2\pi} x)$$  \hspace{1cm} (60)

(see, for example, subsection 12.2.4 in [17]).

We create polynomials of many variables

$$H_m(x) = H_{m_1, m_2, \ldots, m_n}(x_1, x_2, \ldots, x_n) := H_{m_1}(x_1) H_{m_2}(x_2) \cdots H_{m_n}(x_n).$$  \hspace{1cm} (61)

The functions $e^{-|x|^2/2} H_m(x)$, $m = 0, 1, 2, \ldots, i = 1, 2, \ldots, n$, (62) where $|x|$ is the length of the vector $x$, form an orthogonal basis of the Hilbert space $L^2(\mathbb{R}^n)$ with the scalar product $(f, g) := \int_{\mathbb{R}^n} f(x) g(x) dx$, where $dx = dx_1 dx_2 \cdots dx_n$.

We make symmetrization and antisymmetrization of the functions

$$\gamma_m(x) := e^{-|x|^2} H_m(\sqrt{2\pi} x)$$

(obtained from (62) by replacing $x$ by $\sqrt{2\pi} x$) by means of the symmetric and antisymmetric multivariate exponential functions:

$$\int_{\mathbb{R}^n} E_+^\gamma(x) e^{-\pi |x|^2} H_m(\sqrt{2\pi} x) dx = i^{-m} e^{-\pi |x|^2} H_{m}^{\text{sym}}(\sqrt{2\pi} \lambda),$$  \hspace{1cm} (63)

$$\int_{\mathbb{R}^n} E_-^\gamma(x) e^{-\pi |x|^2} H_m(\sqrt{2\pi} x) dx = i^{-m} e^{-\pi |x|^2} H_{m}^{\text{anti}}(\sqrt{2\pi} \lambda).$$  \hspace{1cm} (64)

It is easy to see that the polynomials $H_m^{\text{sym}}$ and $H_m^{\text{anti}}$ indeed are symmetric and antisymmetric, respectively, with respect to the group $S_n$,

$$H_m^{\text{sym}}(w\lambda) = H_m^{\text{sym}}(\lambda), \quad H_m^{\text{anti}}(w\lambda) = (\det w) H_m^{\text{anti}}(\lambda), \quad w \in S_n.$$

For this reason, we may consider $H_m^{\text{sym}}(\lambda)$ for values of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $H_m^{\text{anti}}(\lambda)$ for values of $\lambda$, such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. The polynomials $H_m^{\text{sym}}$ are of the form

$$H_m^{\text{sym}}(\lambda) = \det(\theta(m_1, \lambda_1))_{i,j=1}^n$$  \hspace{1cm} (65)

and the polynomials $H_m^{\text{anti}}$ of the form

$$H_m^{\text{anti}}(\lambda) = \det(\theta(m_1, \lambda_1))_{i,j=1}^n.$$  \hspace{1cm} (66)

Moreover, $H_m^{\text{anti}}(\lambda) = 0$ if $m_i = m_{i+1}$ for some $i = 1, 2, \ldots, n - 1$. For this reason, we may consider the polynomials $H_m^{\text{sym}}(\lambda)$ for integer $n$-tuples $m$, such that $m_1 \geq m_2 \geq \cdots \geq m_n$ and the polynomials $H_m^{\text{anti}}(\lambda)$ for integer $n$-tuples $m$, such that $m_1 > m_2 > \cdots > m_n$.

Let us apply the symmetric Fourier transform (42) (we denote it as $\hat{\gamma}$) to the symmetric functions (65). Taking into account formula (63) we obtain

$$\hat{\gamma}(e^{-\pi |x|^2} H_m^{\text{sym}}(\sqrt{2\pi} x)) = \frac{1}{|S_n|} \int_{\mathbb{R}^n} E_+^\gamma(x) e^{-\pi |x|^2} H_m^{\text{sym}}(\sqrt{2\pi} x) dx$$

$$= i^{-m} e^{-\pi |x|^2} H_{m}^{\text{sym}}(\sqrt{2\pi} \lambda),$$

that is, the functions $e^{-\pi |x|^2} H_m^{\text{sym}}(\sqrt{2\pi} x)$ are eigenfunctions of the symmetric Fourier transform $\hat{\gamma}$. Since these functions for $m_i = 0, 1, 2, \ldots, i = 1, 2, \ldots, n, m_1 \geq m_2 \geq \cdots \geq m_n$, form
an orthogonal basis of the Hilbert space $L^2_0(\mathbb{R}^n)$ of functions from $L^2(\mathbb{R}^n)$ symmetric with respect to $S_n$ (that is, of the Hilbert space $L^2(D_n)$), then they constitute a complete set of eigenfunctions of this transform. Thus, this transform has only four eigenvalues $i, -i, 1, -1$ in $L^2_0(\mathbb{R}^n)$. This means that we have $\tilde{\mathbf{S}}^3 = 1$.

Now we apply the antisymmetric Fourier transform (40) (we denote it as $\tilde{\mathbf{F}}$) to the antisymmetric function $e^{-\pi |x|^2} H^\text{anti} m (\sqrt{2\pi} x)$. Taking into account formula (64) we obtain
\[
\tilde{\mathbf{F}}\left(e^{-\pi |x|^2} H^\text{anti} m (\sqrt{2\pi} x)\right) = \frac{1}{|S_n|} \int_{\mathbb{R}^n} E^\text{-} (x) e^{-\pi |x|^2} H^\text{anti} m (\sqrt{2\pi} x) \, dx
\]
that is, the functions $e^{-\pi |x|^2} H^\text{anti} m (\sqrt{2\pi} x)$ are eigenfunctions of the transform $\tilde{\mathbf{F}}$. Since these functions for $m_1 = 0, 1, 2, \ldots; i = 1, 2, \ldots, n, m_1 > m_2 > \cdots > m_n \geq 0$, form an orthogonal basis of the Hilbert space $L^2_-(\mathbb{R}^n)$ of functions from $L^2(\mathbb{R}^n)$ antisymmetric with respect to $W$, then they constitute a complete set of eigenfunctions of this transform. Thus, this transform has only four eigenvalues $i, -i, 1, -1$. This means that, as in the previous case, we have $\tilde{\mathbf{S}}^3 = 1$.

It is possible to find eigenfunctions of the symmetric and antisymmetric finite Fourier transforms. It is done by using the results in [18, 19]. The method and results will be exposed in one of the forthcoming papers on this subject.

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References

[1] Patera J 2004 Proc. Inst. Math. Natl. Acad. Sci. Ukraine 30 1152–60
[2] Patera J 2005 SIGMA 1 paper 025
[3] Klimyk A U and Patera J 2006 SIGMA 2 paper 06
[4] Klimyk A U and Patera J 2007 SIGMA 3 paper 023
[5] Atoyan A and Patera J 2004 J. Math. Phys. 45 2468–91
[6] Patera J and Zaratsyan A 2005 J. Math. Phys. 46 053514
[7] Patera J and Zaratsyan A 2005 J. Math. Phys. 46 113506
[8] Kashuba I and Patera J 2007 J. Phys. A: Math. Theor. 40 1751–74
[9] Atoyan A and Patera J 2005 CRM Proc. Lect. Notes 39 1–15
[10] Atoyan A, Patera J, Sahakian V and Akhperjanian A 2005 Astropart. Phys. 23 79–95
[11] Patera J, Zaratsyan A and Zhu H-M 2006 SPIE Electron. Imaging 6064A-06 S1
[12] Germain M, Patera J and Zaratsyan A 2006 SPIE Electron. Imaging 6064A-03 S2
[13] Germain M, Patera J and Allard Y 2006 Proc. SPIE 6065 387–95
[14] Karlin S and McGregor J 1962 Bull. Am. Math. Soc. 68 204–9
[15] Berens H, Schmidt H and Xu Y 1995 Arch. Math. 64 26–32
[16] Moody R V and Patera J 2006 SIGMA 2 paper 76
[17] Vilenkin N Ja and Klimyk A U 1993 Representations of Lie Groups and Special Functions vol 2 (Dordrecht: Kluwer)
[18] Mehta M L 1987 J. Math. Phys. 28 781–5
[19] Atakishiyev N 2006 Int. J. Mod. Phys. A 21 4993–5006