Attenuation effect and neutrino oscillation tomography

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Attenuation effect is the effect of weakening of contributions to the oscillation signal from remote structures of matter density profile. The effect is a consequence of integration over the neutrino energy within the energy resolution interval. Structures of a density profile situated at distances larger than the attenuation length, $\lambda_{\text{att}}$, are not “seen”. We show that the origins of attenuation are (i) averaging of oscillations in certain layer(s) of matter, (ii) smallness of matter effect: $\epsilon \equiv 2EV/\Delta m^2 \ll 1$, where $V$ is the matter potential, and (iii) specific initial and final states on neutrinos. We elaborate on the graphic description of the attenuation which allows us to compute explicitly the effects in the $\epsilon^2$ order for various density profiles and oscillation channels. The attenuation in the case of partial averaging is described. The effect is crucial for interpretation of oscillation data and for the oscillation tomography of the Earth with low energy (solar, supernova, atmospheric, etc.) neutrinos.

I. INTRODUCTION

The attenuation effect introduced in \cite{1} is the key element for understanding neutrino oscillations in the Earth. It describes weakening of contribution of a remote structure of a matter density profile to the oscillation signal in a detector. The contribution decreases with increase of distance between a structure and a detector because of finite accuracy of reconstruction of the neutrino energy. The latter can be due to finite energy resolution of a detector, or finite width of produced neutrino energy spectrum, or due to kinematics of process when neutrino energy can not be obtained uniquely. The better the energy resolution, the more remote structures can be observed. The attenuation effect applies to the astrophysical (solar, supernova) neutrinos arriving at the surface of the Earth as incoherent fluxes of mass states. It also applies to the terrestrial neutrinos of different origins: the reactor antineutrinos, atmospheric neutrinos of low energies, etc.

The attenuation effect explains why, e.g., Super-Kamiokande can not observe the core of the Earth using the solar neutrinos. The computed curves (see fig. 1 in \cite{2}) do not show increase of the $\nu_e$ regeneration for the zenith angles $|\cos \theta_z| > 0.83$ (core crossing trajectories) in spite of 2 times larger density in the core than in the mantle. The zenith angle dependence of rate of events is flat as it would be in the absence of the core. In the case of Super-Kamiokande, which detects the boron neutrinos with continuous energy spectrum, the attenuation is due to integration over the energy of neutrino since the observed signal is the recoil electron from the $\nu - e$ scattering. As can be seen in fig. 53 of \cite{3} and fig. 2 of \cite{4}, even selection of narrow intervals for the recoil electron energy does not improve the sensitivity to the core. One can observe only small spikes at $|\cos \theta_z| = 0.83$ which are about 30 times smaller than the difference of the night and day signals. The spikes slightly increase in the high recoil energy bins (15 – 16) MeV and (16 – 20) MeV. The reason is that only very high energy part of the neutrino spectrum

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contributes to these bins. Selecting these bins we are effectively narrowing the integration interval over neutrino energies. In contrast, small density jumps close to the surface of the Earth (|cos θ| ~ 0.05 − 0.2) produce much stronger effect.

The core can be seen, in principle, using the beryllium neutrinos with relative width of the line σ_E/E ~ 0.005 [5]. Good energy reconstruction can be achieved in experiments based on the ν-nuclei scattering. As an example, the ν–Ar interactions have been considered [6] and the energy-nadir angle distribution of events during the night time has been computed. It is shown that the nadir angle dependence of the night excess can be interpreted as hierarchical perturbations of the result for constant density profile. The strongest perturbation of the lowest order is produced by the closest to a detector density jumps in the outer mantle. The first order dependence is then perturbed by smaller size effect of the deeper mantle jumps. In turn, this second order approximation is perturbed by the Earth core effect. With improvement of energy resolution of a detector the effect of remote structures increases [5]. The core can be seen with σ_E/E < 0.1.

The attenuation effect has been obtained for the mass-to-flavor transition, e.g. ν_1 → ν_e. Paradoxical result is that for the inverse channel, the flavor-to-mass transition ν_e → ν_1, the situation is opposite: close to a detector structures are attenuated, whereas remote structures can be seen. In the ν_e → ν_e channel the attenuation may or may not be realized depending on features of the profile.

In this paper we further elaborate on physics of the attenuation effect. We clarify the meaning of the attenuation length. Using simple examples we (i) formulate conditions for realization of the effect, (ii) show its specific properties for various density profiles and channels, (iii) prove that it is realized in the lowest order in ε, (iv) consider the cases of partial attenuation. The geometric (graphic) description of the attenuation effect allows us to understand the paradoxical features described above. We show that the attenuation effect is a result of certain averaging of oscillations, smallness of mixing of the neutrino mass states in matter and specific initial and final states of neutrinos. Using the graphic representation we compute effects in all orders in ε. We consider effects in pure flavor channels which can be applied to neutrinos of terrestrial origins: reactor antineutrinos, low energy atmospheric neutrinos, geoneutrinos, neutrinos from π− and μ− decay at rest.

The results obtained here are important for interpretation of data on neutrino oscillation in the Earth and for planning of future experiments aimed at the neutrino oscillation tomography.

The paper is organized as follows: In Sec. II we recall the main points of derivation of the attenuation effect. In Sec. III we clarify meaning of the attenuation length and present graphic description of the effect. The effect in two layers of matter is discussed in Sec. IV. The case of multi-layer medium is explored in Sec. V. In Sec. VI we consider the case of partial averaging. Discussion and conclusions are presented in Sec. VII.

II. ATTENUATION EFFECT

Astrophysical (solar, supernova) neutrinos arrive at the Earth as incoherent fluxes of the mass eigenstates ν_1. In the Earth, each mass state “splits” into eigenstates in matter and oscillates. The mixing angle of the ν_1 and ν_2 mass states in matter, θ', is determined by

\[ \sin 2\theta' = \frac{c_{13}^2 \sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - c_{13}^2)^2 + \sin^2 2\theta_{12}}} = c_{13}^2 \sin 2\theta_{21}^\text{m} \approx \epsilon \sin 2\theta_{21}. \] (1)
Here $\theta_{21}^m$ is the flavor mixing angle in matter, $c_{13} \equiv \cos \theta_{13}$, and

$$\epsilon \equiv \frac{2V_eE}{\Delta m_{21}^2} = 0.03 \left( \frac{E}{10\text{MeV}} \right) \left( \frac{\rho}{2.6\text{g/cm}^3} \right).$$

Thus, the mixing angle of the mass states in matter is suppressed by $\epsilon$. In the Earth for $E < 30$ MeV (solar, supernova neutrinos) oscillations proceed in the low density regime when $\epsilon \ll 1$. The splitting of the eigenvalues of Hamiltonian:

$$\Delta_{21}^m \equiv \frac{\Delta m_{21}^2}{2E} \sqrt{(\cos 2\theta_{12} - c_{13}^2\epsilon)^2 + \sin^2 2\theta_{12}}$$

determines the oscillation length in matter

$$l_m = \frac{2\pi}{\Delta_{21}^m} = l_\nu [1 + \cos 2\theta_{12} c_{13}^2 \epsilon + O(\epsilon^2)],$$

which is close to the vacuum oscillation length $l_\nu = 4\pi E/\Delta m_{21}^2$.

Detector registers the flavor states and for definiteness we will take $\nu_e$. Therefore the relevant transition is $\nu_i \to \nu_e$. Furthermore, at low energies, when matter effect on the 1-3 mixing is negligible, it is enough to find the transition for one mass state, and other can be obtained using unitarity. So, in what follows for definiteness we will focus on the $\nu_1 \to \nu_e$ transition.

Without matter effect the probability equals $P_{1e} = |U_{e1}|^2$, where $U_{e1}$ is the $e1$–element of the mixing matrix in vacuum. Therefore the Earth matter effect is given by the “regeneration factor”

$$f_{\text{reg}} = P_{1e} - |U_{e1}|^2.$$

In the lowest order in $\epsilon$ the factor equals [1, 7]

$$f_{\text{reg}} = C \int_0^L dx \ V_e(x) \sin \phi_{x \to L}^m,$$

where $C \equiv -\frac{1}{2} \sin^2 2\theta_{12} c_{13}^2$ and

$$\phi_{x \to L}(E) \equiv \int_x^L dx \ \Delta_{21}^m(x, E)$$

is the phase acquired from a given point of trajectory $x$ to a detector. $L$ is the total length of trajectory.

The attenuation effect is a consequence of integration of the oscillation probability over the neutrino energy with the neutrino energy reconstruction function $g(E_r, E)$, where $E_r$ and $E$ are the reconstructed and true energies correspondingly. The width of $g(E_r, E)$ is determined by the smallest quantity among (i) a width of neutrino spectrum, (ii) energy resolution of a detector, (iii) an accuracy of the neutrino energy reconstruction determined by kinematics of the process used for a detection. The regeneration factor averaged over the energy equals

$$\bar{f}_{\text{reg}}(E_r) = \int dE \ g(E_r, E)f_{\text{reg}}(E, x),$$

(7)
with $\int dE g(E_r, E) = 1$. Inserting (5) into (7) we obtain

$$\bar{f}_{\text{reg}}(E_r) = C \int_0^L dx \, V(x) F(L - x) \sin \phi_{x \to L}^m(E_r),$$

(8)

where $F(d)$ is the attenuation factor $\Pi$ and $d \equiv L - x$ is the distance from a given structure of the profile to a detector. The factor $F$ is defined by the equality

$$F(L - x) \sin \phi_{x \to L}^m(E_r) = \int dE \, g(E_r, E) \sin \phi_{x \to L}^m(E),$$

(9)

in such a way that for the ideal energy resolution, $g(E_r, E) = \delta(E - E_r)$, one would get $F(L - x) = 1$, i.e., attenuation is absent. Notice that due to presence of sine of the phase in the integral (9), the factor $F(L - x)$ appears as a kind of Fourier transform of the energy resolution function $g(E_r, E)$. For the Gaussian resolution function with width $\sigma_E$,

$$g(E_r, E) = \frac{1}{\sigma_E \sqrt{2\pi}} e^{-\frac{(E_r - E)^2}{2\sigma_E^2}},$$

(10)

we obtain from (8)

$$F(d) \simeq e^{-2 \left( \frac{d}{\lambda_{\text{att}}} \right)^2},$$

where

$$\lambda_{\text{att}} \equiv l_{\nu} \frac{E}{\pi \sigma_E}$$

(11)

is the attenuation length. If $d = \lambda_{\text{att}}$, the factor equals $F(\lambda_{\text{att}}) = 0.135$, and therefore according to (8), a contribution to the oscillation effect of structures with $d > \lambda_{\text{att}}$ is strongly suppressed. As follows from (11), the better the energy resolution of a detector, the more remote structures can be “seen”. Thus, for the relative energy resolution $\sigma_E / E = 0.1$ and $l_{\nu} = 400$ km the attenuation length equals 1470 km and structures of a density profile at $d > 1470$ km can not be observed. For $\sigma_E / E = 0.2$ already structures with $d > 750$ km are strongly attenuated.

The origin of the attenuation can be traced from Eq. (8) where the potential $V(x)$ is integrated with the sine of the phase acquired from coordinate of a structure, $x$, to a detector. Notice that computing number of events in a detector we integrate over energy not just $f_{\text{reg}}$, as in (7), but the product of $f_{\text{reg}}$ with the flux $F$ and cross-section $\sigma$. The product $\sigma F$ depends on energy, but even in this case the results are qualitatively unchanged. If the energy resolution is high, dependence on energy of the product $\sigma F$ can be neglected and the product can be put out of the integral.

The attenuation effect is also realized for the flavor neutrinos of the terrestrial origins (low energy atmospheric neutrinos, neutrinos from pion and muon decay at rest). For these neutrinos loss of coherence can occur in the first layer of matter, e.g., the mantle of the Earth, so that at internal structures the incoherent flux of the eigenstates (close to mass states) arrives. Attenuation is then realized for inner structures.

In what follows we will consider mainly attenuation for the 1-2 mode of oscillations described by the 1-2 sub-system of the complete $3\nu$-system. At low energies the third mass state, $\nu_3$, decouples and dynamics of the $3\nu$ evolution is reduced to the $2\nu$ evolution in the so called propagation basis (see [8] for details), which is related to the flavor basis, in particular, by the 1-3 rotation on the angle $-\theta_{13}$. At this rotation $\nu_e \to \nu'_e$. The matter effect on the 1-3
mixing can be neglected and the remaining $2\nu-$sub-system is characterized by $\theta_{12}$, $\Delta m^2_{21}$ and the potential $c^2_{13}V_e$. So, in what follows we will consider the $2\nu-$transition $\nu_1 \rightarrow \nu'_e$. (We will omit prime keeping in mind that results in the flavor basis can be obtained from the results in the propagation basis by multiplying them by $c_{13}^2$.)

With this, the evolution in the Earth is reduced to the $2\nu$ evolution in the potential $V = c^2_{13}V_e$. For simplicity we omit the subscript $\theta_{12}$ → $\theta$. Then using eq. (1) we find for the mixing of mass states in matter

$$\sin^2 \theta' \approx \frac{\epsilon^2}{4} \frac{\sin^2 2\theta}{(\cos 2\theta - \epsilon)^2 + \sin^2 2\theta} = \frac{1}{4} \epsilon^2 \sin^2 2\theta_m \approx \frac{1}{4} \epsilon^2 \sin^2 2\theta.$$ (12)

Here $c^2_{13}$ is included in the potential and $\epsilon$. We comment on attenuation for the 1-3 mode in Sec. VII.

### III. ATTENUATION EFFECT AND DECOHERENCE

Let us first clarify the meaning of the attenuation length $\lambda_{\text{att}}$. According to (11) the phase acquired by neutrino with energy $E$ over the distance $\lambda_{\text{att}}$ equals

$$\phi(E) = 2\pi \frac{\lambda_{\text{att}}}{l_m} \approx 2\pi \frac{E}{\pi \sigma_E}.$$ Then the difference of phases of neutrinos with the difference of energies $\Delta E$ is

$$\Delta \phi = 2\pi \frac{\Delta E}{\pi \sigma_E}.$$ (13)

For the integration interval, $\Delta E = \pi \sigma_E$ the eq. (13) gives $\Delta \phi = 2\pi$. Therefore integration over the energy resolution leads to averaging of oscillations. So, $\lambda_{\text{att}}$ is the distance (or width of the layer) over which oscillations observed with the energy resolution $\sigma_E$ are averaged.

Let us consider a density profile with some structure, the $s-$layer, e.g. density bump at $x = 0 \div x_s$ and the “decoherence” layer $d$ at $x_s \div (x_s + x_d)$. The bump should have sharp edges, so that the adiabaticity is broken. The decoherence layer has constant or slowly changing density. Suppose a neutrino enters the profile at $x = 0$, while a detector is placed at $x = x_d + x_s$. The distance between the structure and a detector equals $x_d$. (Actually the presence of matter in the $d$ layer is not important.) The densities in $d$ and $s$ are low being of the same order. Recall that the Earth density can be considered as layers with slowly changing density inside the layers and sharp density change on the borders between them [9]. So, our consideration can be immediately applied to this realistic situation.

Suppose $d > \lambda_{\text{att}}$, so that oscillations in the $d-$layer are averaged (or equivalently, coherence of the neutrino state is lost). Let $\nu^{d}_{1m}, \nu^{d}_{2m}$ be the neutrino eigenstates in $d$. Suppose the mass state $\nu_i$ arrives at the $s-$layer and after oscillations in $s$ enters the $d-$layer as $\nu_x$ which can be parametrized as

$$\nu_x = \cos \theta_x \nu^{d}_{1m} + \sin \theta_x \nu^{d}_{2m} e^{-i\phi_x}.$$ (14)

So, the information about the $s-$layer is contained in the angle $\theta_x$ and the phase $\phi_x$. It may happen that some averaging occurred already before arriving at $d$. This can be accounted by
the overall normalization factor of $\nu_x$, $N$, such that $|N|^2 < 1$. We assume also that
\[ \theta_x = B\epsilon, \quad B = O(1). \tag{15} \]
and $\epsilon$ is defined in \[2\]. The phase $\phi_x$ becomes irrelevant due to averaging in the layer $d$.

In terms of the eigenstates $\nu^d_{i m}$ ($i = 1, 2$) the electron neutrino and the mass state $\nu_1$ are given by
\[ \nu_e = \cos \theta_d \nu^d_{1 m} + \sin \theta_d \nu^d_{2 m}, \quad \nu_1 = \cos \theta'_d \nu^d_{1 m} + \sin \theta'_d \nu^d_{2 m}, \tag{16} \]
where $\theta_d$ and $\theta'_d$ are the mixing angles of the flavor states and the mass states in the $d$–layer correspondingly. Then according to (14) the probability to observe $\nu_e$ in a detector equals
\[ P_x = |\langle \nu_e | \nu_x \rangle|^2 = \cos^2 \theta_x \cos^2 \theta_d + \sin^2 \theta_x \sin^2 \theta_d. \tag{17} \]
So, after averaging the information about the structure is encoded in $\theta_x$ only.

In the absence of $s$–layer, the neutrino $\nu_1$ enters immediately the layer $d$ and propagates there. Then instead of (17), we obtain the probability to detect $\nu_e$:
\[ P_1 = |\langle \nu_e | \nu_1 \rangle|^2 = \cos^2 \theta'_d \cos^2 \theta_d + \sin^2 \theta'_d \sin^2 \theta_d. \tag{18} \]
The difference of the probabilities in (17) and (18), which is the measure of effect of the $s$–layer, equals
\[ \Delta P_e \equiv P_x - P_1 = (\sin^2 \theta'_d - \sin^2 \theta_x) \cos 2\theta_d. \tag{19} \]
Since density of the structure is of the order of density in layer $d$, we obtain using (15) and (12)
\[ \Delta P_e = P_x - P_1 \approx (B^2 - 1)\frac{1}{4} \epsilon^2 \sin^2 2\theta_d \cos 2\theta_d \approx (B^2 - 1)\frac{\epsilon^2}{4} \sin^2 2\theta \cos 2\theta. \tag{20} \]
The equality corresponds to the low density case. So, the effect of structure is absent in the first order in $\epsilon$, i.e. attenuated, in agreement with our previous consideration. Its effect appears in the second order in small parameter $\epsilon$.

Averaging eliminates information about the phase, and therefore removes interference, so that small parameters appear being squared.

This result as well as results for more complicated cases can be obtained easily using the graphic representation of oscillations based on analogy of the oscillations and precession of the spin of electron in the magnetic field \[10\] (see Fig. 1 - 8). According to this representation neutrino state is described by the polarization vector in the flavor space:
\[ \mathbf{P} = \frac{1}{2} \bar{\psi} \sigma \psi, \quad \psi^T \equiv (\nu_e, \nu_a). \]
Oscillations are equivalent to precession of the vector $\mathbf{P}$ in the flavor space $(x, y, z)$ around the axis of eigenstates in matter $\mathbf{A}_m$. The axis lies in the $(x, z)$– plane and the angle between the flavor axis $\mathbf{z}$ and $\mathbf{A}_m$ equals $2\theta_m$. The direction of the axis of the mass states in vacuum, $\mathbf{A}_v$, with respect to $\mathbf{z}$ is given by the vacuum mixing angle $2\theta$. We use normalization $|\mathbf{A}_m|^2 = |\mathbf{A}_v|^2 = 1$. The probability to observe $\nu_e$ is given by the projection of $\mathbf{P}$ onto the flavor axis $\mathbf{z}$:
\[ P_e = (\mathbf{P} \cdot \mathbf{z}) + \frac{1}{2}. \]
Let $A_d$ be the axis of eigenstates in the $d$—layer. Loss of coherence (averaging) in $d$ means that neutrino polarization vector $P$ precesses around $A_d$ with decrease of the orthogonal to $A_d$ component. Projection of $P$ on $A_d$ does not change. Thus, the vector $P$ shrinks and eventually coincide with its own projection onto $A_d$:

$$P \rightarrow (P \cdot A_d) A_d.$$ 

Further on we will consider attenuation effect in terms of this graphic representation.

Let $P_x$ be the vector which describes the state $\nu_x$ in the example discussed above. The angle between $P_x$ and the axis $A_d$ equals $2\theta_x$, and $\theta_x$ is defined in (14). Averaging of oscillations in the layer $d$ means that $P_x$ evolves to its projection on $A_d$:

$$P_x \rightarrow \frac{1}{2} \cos 2\theta_x A_d. \quad (21)$$

Similarly, the polarization vector $P_1$, which corresponds to the mass state $\nu_1$, evolves (loosing the coherence) as

$$P_1 \rightarrow \frac{1}{2} \cos 2\theta'_d A_d. \quad (22)$$

Here $2\theta'_d$ is the angle between $P_1$ and $A_d$ introduced in (16). Then the difference of the final vectors in (21) and (22):

$$\frac{1}{2} (\cos 2\theta_x - \cos 2\theta'_d) A_d. \quad (23)$$

Projection of this difference onto the flavor axis $z$ (recall that $(A_d \cdot z) = \cos 2\theta_d$) presents effect of the $s$—layer on the $\nu_e$ survival probability:

$$\Delta P_e \equiv P_x - P_1 = \frac{1}{2} (\cos 2\theta_x - \cos 2\theta'_d) \cos 2\theta_d,$$

which coincides with expression in (19).

IV. TWO LAYERS CASE

Let us consider oscillation effect in the $s$—layer explicitly, assuming first that the densities in $s$ and $d$ are constant. We denote by $A_s$ the axis of eigenstates in the $s$—layer. A density jump on the border between the $s$— and $d$—layers leads to sudden change of the mixing angle in matter, and consequently, to change of direction of the eigenstate axis: $A_s \rightarrow A_d$. The angle between $A_s$ and $A_d$ equals

$$2\Delta\theta_m \equiv 2\theta_s - 2\theta_d = 2\theta'_s - 2\theta'_d. \quad (24)$$

The parameter

$$J_m \equiv \sin 2\Delta\theta_m \sim \epsilon,$$

which we will call the jump factor, quantifies the effect of structure: the effect should be proportional to $J_m$, and if the structure is absent, $\Delta\theta_m = 0$. Notice that $\Delta\theta_m$ is positive, if density in the $s$—layer is larger than that in the $d$—layer, and $\Delta\theta_m < 0$, if the density in $s$ is smaller. For definiteness we will present plots for $\Delta\theta_m > 0$. It is easy to see that formulas we
FIG. 1. Graphic representation of evolution of the neutrino polarization vector in the case of $\nu_1 \rightarrow \nu_e$ transition. Numbers indicate positions of the neutrino vector (red) at the borders of different layers in the order of their crossing. They enumerate vectors $P_1, P_2, \ldots$ (see the text). Position of the vector marked by "0" corresponds to the initial state. The positions of 1' marks the final state $P_{0e}$ (blue) in the absence of the structure. To make effect visible we took rather large $\epsilon$. The left panel: remote structure. The diameter of precession $\sim \epsilon$ and its projection onto the eigenstate axis is given by $\epsilon$. This leads to the attenuation. The right panel: the same as in the left panel but for near structure. The precession diameter equals $\sim \epsilon$, and its projection on the flavor axis is $O(1)$. The attenuation is absent.

will obtain are valid for both cases. If $\Delta \theta_m > 0$, the effect of structure on the $\nu_e$ probability is negative, $\Delta P_e < 0$. It is positive for $\Delta \theta_m < 0$. Oscillations in the $s$–layer may or may not be averaged.

To perform the oscillation tomography (see e.g. [11–15]) one can scan a density profile by changing direction of the neutrino trajectory. This happens, for the solar neutrinos for fixed position of a detector due to the Earth rotation. Let $\eta$ be the nadir angle of neutrino trajectory and $\eta_s$ corresponds to the border of the $s$–layer, so that for $\eta < \eta_s$ neutrino crosses both the $d$– and $s$–layers, whereas for $\eta > \eta_s$ it crosses the $d$– layer only. The length of trajectory in the layers depends on $\eta$. If $x_d > \lambda_{att}$, the oscillations are averaged in the $d$–layer and therefore for $\eta > \eta_s$ the probability $P_{0e}$ does not depend on $\eta$. For $\eta < \eta_s$ one can observe the oscillatory dependence of the probability $P_e$ on $\eta$ induced by the structure, since the length of trajectory in the $s$–layer, $x_s$, changes with $\eta$. Therefore in what follows we will quantify the effect of $s$–layer by the depth of this oscillatory pattern

$$D_e \equiv |\Delta P_{e \max}| = |P_{e \max} - P_{e \min}|,$$

and by difference of the averaged probabilities with, $\bar{P}_e$, and without, $P_{0e}$, the structure:

$$\Delta \bar{P}_e \equiv \bar{P}_e - P_{0e}.$$

In many cases $P_{e \max} = P_{0e}$ and $\Delta \bar{P}_e = 0.5D_e$. Apart from $D_e$ and $\Delta \bar{P}_e$ information about the structure is contained in the period and phase of the oscillatory pattern at $\eta < \eta_s$. 
We will describe the attenuation effect in terms of the diameter of precession in the $s$–layer, $D_s$, and its projection onto one of the eigenstates axes involved. Projection of $D_s$ onto the flavor axis is determined by the flavor mixing angle and therefore does not produce smallness.

Depending on type of density profile and channel of oscillations we obtain the following results.

1. Let us consider the $\nu_1 \to \nu_e$ transition in the profile with a remote structure. Evolution of the neutrino vector is shown in Fig. 1 left. The initial state is $\mathbf{P}(0) = 0.5 \mathbf{A}_v$. In the $s$–layer it precesses around $\mathbf{A}_s$ with the cone angle $2\theta'_s$, which is the angle between $\mathbf{A}_v$ and $\mathbf{A}_s$. Maximal effect in the $s$–layer corresponds to the state $\mathbf{P}_1$ or the precession phase $\pi + 2\pi k$ ($k$ is integer) at the moment when neutrino arrives at the $d$–layer:

$$s – layer: \quad \mathbf{P}(0) \to \mathbf{P}(x_s) = \mathbf{P}_1.$$ 

According to Fig. 1 the angle between $\mathbf{P}_1$ and $\mathbf{A}_d$ equals $2\theta_x = 2\theta'_s + 2\Delta \theta_m$. Notice that $\mathbf{P}_1 = \mathbf{P}_x$ in our consideration in Sec. III.

In the $d$–layer the vector $\mathbf{P}$ precesses around $\mathbf{A}_d$ approaching its projection onto $\mathbf{A}_d$:

$$d – layer: \quad \mathbf{P}_1 \to \mathbf{P}_2 = \frac{1}{2} \cos(2\theta'_d + 2\Delta \theta_m)\mathbf{A}_d.$$ 

Without the structure we have

$$\mathbf{P}(0) \to \mathbf{P}^0(\mathbf{x}_s + \mathbf{x}_d) = \mathbf{P}'_1 = \frac{1}{2} \cos 2\theta'_d \mathbf{A}_d = \frac{1}{2} \cos(2\theta'_s - 2\Delta \theta_m)\mathbf{A}_d.$$ 

Projection of the difference of vectors $[\mathbf{P}_2 - \mathbf{P}'_1]$ onto the flavor axis $\mathbf{z}$ equals

$$D_e = -\Delta \mathbf{P}(\nu_1 \to \nu_e)^{\max} = \cos 2\theta_d \sin 2\theta'_s J_m$$

(25)
giving the depth of the $\nu_e$–oscillations. Since $J_m \sim \sin 2\theta'_s \sim \epsilon$, we obtain $\Delta \mathbf{P}(\nu_1 \to \nu_e)^{\max} \sim \epsilon^2$ in accordance to our consideration above. Attenuation is realized. If oscillations in $s$ are averaged, the effect of the structure is $\Delta \tilde{P}_e = 0.5D_e$. Notice that oscillations correspond to change of the vector $\mathbf{P}$ between positions $\mathbf{P}_2$ and $\mathbf{P}'_1$ in the Fig. 1.

In other terms, the diameter of precession in the $s$–layer equals $D_s = \sin 2\theta'_s$. Its projection onto $\mathbf{A}_d$ (forced by the averaging) is $D \sin 2\Delta \theta_m = \sin 2\theta'_s J_m$, and finally projection onto the flavor axis is given $D_e = \cos 2\theta_d \sin 2\theta'_s J_m$. Thus, the origins of the attenuation are (i) smallness of the diameter of precession $D_s \sim \epsilon$, which is due to the initial mass state $\nu_1$, (ii) projection of $D_s$ onto the eigenstate axis $\mathbf{A}_d$ forced by the averaging in $d$, this produces another smallness $\epsilon$.

Without structure after complete averaging in the $d$–layer we have

$$\tilde{P}_e(\nu_1 \to \nu_e) = \tilde{P}'_1(\nu_e \to \nu_1) = \frac{1}{2}(1 + \cos 2\theta_d \cos 2\theta'_d).$$

(26)

Without averaging maximal value of $P_e(\eta)$ equals $\cos^2 \theta$.

Performing scanning of the profile we will observe at $\eta > \eta_s$ the constant probability $\tilde{P}_e^0$ of (26). For $\eta < \eta_s$ the probability oscillates around the average value $\tilde{P}_e = \tilde{P}_e^0 - 0.5D_e$ with the depth $D_e \sim \epsilon^2$ (25). So that $P_e^{\max}(\eta) = \tilde{P}_e^0$.

2. Let us consider the $\nu_1 \to \nu_e$ transition in the matter profile with structure near a detector
FIG. 2. The same as in Fig. 1 but for the case of $\nu_e \rightarrow \nu_1$ transition. Left panel: remote structure, the attenuation is absent. Right panel: near structure, the attenuation is realized.

(see Fig. 1 right). In the $d$—layer the initial state $\mathbf{P}(0) = 0.5 \mathbf{A}_v$ evolves to its averaged value:

$$d - \text{layer : } \mathbf{P}(0) \rightarrow \mathbf{P}_1 = \frac{1}{2} \cos 2\theta_d' \mathbf{A}_d.$$  \hspace{1cm} (27)

Without structure this gives final position $\mathbf{P}^0 = \mathbf{P}_1'$. Then in the $s$—layer the vector $\mathbf{P}$ precesses around $\mathbf{A}_s$ with the cone angle $2\Delta\theta_m$, see Fig. 1 right. The diameter of precession equals $2|\mathbf{P}(x_d)| \sin 2\Delta\theta_m$. Its projection onto the flavor axis gives the depth of $\nu_e$—oscillations due to the structure:

$$D_e = -\Delta \mathbf{P}(\nu_1 \rightarrow \nu_e)_{\text{max}} = \cos 2\theta_d' \sin 2\theta_s J_m.$$  \hspace{1cm} (28)

Here $2\theta_s$ is the angle between the axis $\mathbf{A}_s$ and $\mathbf{z}$. $\theta_s \approx \theta_{12}$ is the flavor mixing angle in the layer $s$, and it is large. According to (28) $D_e \approx J_m \sin 2\theta_{12} \sim \epsilon$, the effect of structure appears in the lowest order in $\epsilon$, i.e. the attenuation is absent. Change of the averaged probability equals $\Delta \bar{P}_e = -0.5 D_e$.

In contrast to the first case here the projection of $\mathbf{P}$ onto $\mathbf{A}_d$ (induced by averaging) occurs before the oscillations in the $s$—layer. Averaging changes the diameter of the precession in $s$ by factor $\mathcal{O}(1)$, and therefore does not produce additional smallness. The diameter of precession in $s$: $D_s \sim \epsilon$. It should be projected onto the flavor axis immediately which does not produce additional smallness. Thus, the oscillation effect of the close to a detector structures is not suppressed.

In this case, for $\eta < \eta_s$ one will observe oscillations with large depth $D_e \sim \epsilon$ (28) and the average value $\bar{P}_e = \bar{P}_e^0 - 0.5 D_e$. Furthermore, $P_e^{\text{max}} = \bar{P}_e^0$ (26).
3. For comparison let us consider the inverse (although not practical) case of the flavor-to-mass, $\nu_e \rightarrow \nu_1$, transition. Now the initial state, $\nu_e$, is described by $P(0) = 0.5z$. The evolution of $P$ in the matter profile with remote structure is shown in Fig. 2 left. In the $s$–layer $P$ precesses around $A_s$ with large $O(1)$ diameter and evolves to $P_1$ in the case of maximal final effect:

\[
\text{s-layer : } P(0) \rightarrow P(x_s) = P_1.
\]

$P_1$ has the angle $(2\theta_s + 2\Delta \theta_m)$ with respect to $A_d$. In the $d$–layer (precessing around $A_d$) $P$ converges to its projection onto $A_d$ (position 2):

\[
\text{d-layer : } P(x_s) \rightarrow P_2 = \frac{1}{2} \cos(2\theta_s + 2\Delta \theta_m)A_d.
\]

Without the structure we have

\[
P^0(x_s + x_d) = P_1' = \frac{1}{2} \cos 2\theta_d A_d = \frac{1}{2} \cos(2\theta_s - 2\Delta \theta_m)A_d.
\]

Projection of the difference $[P_2 - P_1]$ onto the mass eigenstates axis $A_v$ $[(A_v \cdot A_d) = \cos 2\theta'_d \approx 1]$ gives the difference of probabilities with and without structure: $\Delta P(\nu_e \rightarrow \nu_1) = D_e$ and the latter is given in (28). Thus,

\[
\Delta P(\nu_e \rightarrow \nu_1)_{\text{far}} = \Delta P(\nu_1 \rightarrow \nu_e)_{\text{near}}
\]

This coincidence is the consequence of the $T$–invariance of the physical setup. Namely, the shape of the density profile is time-inverted: $(s - d) \rightarrow (d - s)$, and the initial and final states are permuted $\nu_1 \leftrightarrow \nu_e$.

Thus, $\Delta P(\nu_e \rightarrow \nu_1) \approx \sin 2\theta_{21} J_m \sim \epsilon$, i.e., the effect of structure is not attenuated in spite of its remote position. Here appearance of single factor $\epsilon$ is related to projection of the precession diameter, $O(1)$, onto $A_d$, which is given by $\sin 2\Delta \theta_m$. The observational features are the same as in the case 2. Notice that still averaging leads to suppression: the final depth of oscillations is $O(\epsilon)$ rather than $O(1)$.

4. In contrast to the previous case the near to detector structure is not visible in the $\nu_e \rightarrow \nu_1$ transition, see Fig. 2 right. Now in the $d$–layer

\[
\text{d-layer : } P(0) \rightarrow P_1 = \frac{1}{2} \cos 2\theta_d A_d.
\]

and large initial precession diameter vanishes. In the $s$–layer precession proceeds with small angle $2\Delta \theta_m \sim \epsilon$ around $A_s$, and then the diameter of this precession should be projected onto the axis $A_s$ (given by $\sin 2\theta'_s$) which leads to another $\epsilon$. As a result, we obtain the difference of the probabilities with and without structure as in Eq. (25). Again this is a consequence of the $T$–invariance of the setup.

The origin of attenuation here is (i) reduction of the diameter of precession in $s$: $D_s = \cos 2\theta_d J_m \sim \epsilon$ due to averaging in the $d$–layer, and (ii) projection of $D_s$ onto the mass axis $A_1$ since the final state is $\nu_1$. This gives another $\epsilon$.

So, on the contrary to $\nu_1 \rightarrow \nu_e$, in the $\nu_e \rightarrow \nu_1$ channel the detector “sees” the remote structures, but the closest ones are attenuated. In a sense, the “$\nu_1$–detector” is focused on remote structures, when the initial state is $\nu_1$. 


In general, expressions for the depth of oscillations induced by the structure, $D_e$, have the form of product of the jump factor and the projection factors corresponding to initial and final states. In four cases considered above the probabilities are given by two formulas (25) with and (28) without the attenuation. Both contain the jump factor. They differ by the projection factors in which the flavor and mass mixing angle are permuted: $\theta_d \leftrightarrow \theta'_d$ and $\theta_s \leftrightarrow \theta'_s$. The attenuation is related to the latter – the mixing of the $s$–layer. Sines of these angles enter the diameter of precession, and consequently, appearance of small mass mixing angle $\theta'_s$ gives an additional smallness.

Notice that in the cases of attenuation 1 and 4, two small factors $J_m$ and $\sin 2\theta'_s$ play different roles: in the first case $\sin 2\theta'_s$ determines the diameter of precession, whereas $J_m$ gives the projection onto the axis of eigenstates. In the case 4 - vice versa.

5. Finally, let us consider a remote structure and the $\nu_e \rightarrow \nu_e$ transition. It is similar to the case described in Fig. 2 left, but now the difference of vectors in (29) and (30) should be projected onto the flavor axis which does not produce a smallness:

$$\Delta P(\nu_e \rightarrow \nu_e)^{\text{max}} = -\cos 2\theta_d \sin 2\theta_s J_m.$$  \hspace{1cm} (33)

$\cos 2\theta'_d$ in (27) is substituted here by $\cos 2\theta_d$. So, both projections are given by large flavor mixings and there is no attenuation as in the case 3.

For a near structure (and $\nu_e \rightarrow \nu_e$ mode) we obtain the same result as in (33) due to the $T$–invariance.

Let us consider generalization of the formalism to the case when density in the $d$–layer changes adiabatically. (The same can be done for the $s$–layer). Now the jump factor $J_m$ is determined by difference of the densities immediately before a jump and after a jump. The oscillation phase should be computed by integration (6). The angles $\theta_d$ and $\theta'_d$ in the projection factors should be taken at the outer border the layer $d$ which is not attached to the $s$– layer. Thus, in the case 1 the result is given by eq. (25) with substitution $\theta_d \rightarrow \theta'_d$ in the projection factor, where $\theta'_d$ is the mixing angle at the end of the layer $d$ (i.e., near a detector). In the case 4 the substitution $\theta_d \rightarrow \theta'_d$ should be done. In the second case one should change $\theta'_d \rightarrow \theta'_d$ in Eq. (28), where $\theta'_d$ is the mixing angle in matter in the beginning of layer $d$. In the third case: $\theta'_d \rightarrow \theta'_d$, see Eqs. (31).

V. ATTENUATION IN MULTI-LAYER MEDIUM. TWO JUMPS CASE

Let us consider a matter density profile with three layers: $d_1 – s – d_2$, thus adding another decoherence layer to the profile studied in Sec. IV. Now there are two jumps. We assume that the layers $d_1$ and $d_2$ have the same properties: lengths $x_d$ and densities, and therefore the same eigenstate axis $A_d$ with the direction fixed by $2\theta_d$. The overall profile is symmetric with respect to the center and similar to the Earth density profile, when $d_i$ are identified with the mantle layers, whereas $s$ – with the core. The core appears here as the “structure”. Similar matter profile appears also when neutrino crosses two outer shells of the mantle. We call it as the profile with structure in the middle. As before, we assume that oscillations in $d_1$ and $d_2$ are averaged, while in $s$ – do not.
Let us consider the $\nu_1 \to \nu_e$ oscillations when $\nu_1$ enters $d_1$, see Fig. 3. The initial state is $P(0) = 0.5A_v$. In the first mantle layer $P$ precesses around $A_d$ converging to its projection on $A_d$:

$$d_1 - layer: \quad P(0) \to P_1 = \frac{1}{2} \cos 2\theta'_d A_d.$$  \hspace{1cm} (34)

In the $s$-layer $P$ precesses around $A_s$. Maximal effect corresponds to the state $P_2$ or the precession phase at the end of the layer $\pi + 2\pi k$, where $k$ is integer:

$$s - layer: \quad P(x_d) \to P(x_d + x_s)^{max} = P_2.$$  \hspace{1cm} (35)

The angle between $P_2$ and the axis $A_d$ is $4\Delta \theta_m$, $(A_d \cdot P_2) = \cos 4\Delta \theta_m$. Since the length of the precessing vector in $s$ does not change, we have $|P_2| = |P(x_d)| = 1/2 \cos 2\theta'_d$. Notice that the length is smaller than $1/2$ reflecting departure from pure state due to averaging in $d_1$. $[P(x_d + x_s)$ corresponds to $P_x$ in our original consideration.]

In the third layer due to averaging the vector $P$ evolves to its projection onto axis $A_d$:

$$d_2 - layer: \quad P(x_d + x_s)^{max} \to P_3 = \frac{1}{2} \cos 2\theta'_d \cos 4\Delta \theta_m A_d.$$  \hspace{1cm}

Without $s-$layer we would have $P^0(2x_d + x_p) = P'_1 = P_1$ (34). Then projection of the difference $(P_3 - P'_1)$ onto the flavor axis $z$ gives the depth of oscillations due to the structure:

$$D_c = -\Delta P(\nu_1 \to \nu_e)^{max} = \cos 2\theta'_d \cos 2\theta_d J^2_m.$$  \hspace{1cm} (36)

Again $\Delta P_e \approx \cos 2\theta_1 J^2_m \propto \epsilon^2$, and the attenuation is realized.

The result can be obtained immediately from Fig. 3 as follows. The length of $P$ precessing in the $s-$layer equals $\frac{1}{2} \cos 2\theta'_d$ (34). Then the diameter of precession is $D_s = \cos 2\theta_d \sin 2\Delta \theta_m$.
FIG. 4. The same as in Fig. 3 but for the $\nu_e \rightarrow \nu_e$ transition. The diameter of precession is $\sim \epsilon$, its projection is given by $\epsilon$. The attenuation is present.

the projection of this diameter onto $A_d$ (driven by averaging in $d_2$) is given by $\cos 2\theta_d' \sin^2 2\Delta \theta_m$; finally, its projection onto the flavor axis leads to (36). The change of the average probability (due to structure) equals $\Delta \bar{P}_e = 0.5D_e$.

The origin of attenuation is similar to that in the case 1 of Sec. 4. One $\epsilon$ appears because of smallness of the precession diameter in the $s-$layer. Averaging in $d_1$ gives only small reduction of this diameter. Another $\epsilon$ is a result of projection of this diameter onto axis $A_d$ of the layer $d_2$. This is described by $\sin 2\Delta \theta_m$. Both precession diameter and the projection onto the axis of eigenstates $A_d$ are determined by $J_m$.

This attenuation is realized in the Super-Kamiokande detection of the solar neutrinos. No change of the probability should be seen at $\eta = \eta_s$ in the lowest order in $\epsilon$. In the next order ($\epsilon^2$) for $\eta < \eta_s$ one expects oscillations below $P_{e}^{\text{max}} = \bar{P}_e$ with depth (36) that describes spikes. Due to unitarity the decrease of $P_{1e}$ corresponds to the increase of $P_{2e}$. Since for high energy part of the solar neutrino spectrum neutrinos arrive mainly in the $\nu_2$ mass state, the decrease of $P_{1e}$ means increase of the $\nu_e$ signal.

Let us consider the attenuation for the flavor channel $\nu_e \rightarrow \nu_e$ (see Fig. 4). The result can be obtained immediately from (36). The only difference is that in the first mantle layer the initial (flavor) state is $P(0) = 0.5z$. It evolves to

$$P(0) \rightarrow P_1 = \frac{1}{2} \cos 2\theta_d A_d,$$

so that $\theta_d'$ in Eqs. (34) and (36) should be substituted by $\theta_d$. Therefore, the final difference of
FIG. 5. The same as in Fig. 1 (remote structure, $\nu_1 \rightarrow \nu_e$ transition), but with partial averaging of oscillations in the $d$-layer. Red and blue sections show final precession diameters in the cases of profiles with and without structure correspondingly. The left panel: remote structure. The right panel: near structure. The attenuation is absent in both cases.

the probabilities with and without core equals

$$D_e = -\Delta P(\nu_e \rightarrow \nu_e)^{\max} = \cos^2 2\theta_d J_m^2. \quad (37)$$

$\Delta P(\nu_e \rightarrow \nu_e)^{\max} \sim \epsilon^2$, the structure is attenuated, in contrast to the case of $\nu_e \rightarrow \nu_e$ transition in 2-layers of Sec. III, when suppression was $\sim \epsilon$. The reason for such a difference is that now the state which arrives at the structure (core) is close to the mass state and therefore the oscillation effect in two other layers is similar to that for $\nu_1 \rightarrow \nu_e$ of Sec. IV. So, in the case of complete averaging in $d_1$ the attenuation appears for any initial neutrino state. Here averaging in $d_1$ plays crucial role: the first layer prepares the incoherent system of states close to the mass eigenstates.

In both 3 layer cases the factor $J_m$ appears being squared, which corresponds to the presence of two jumps. The projection factors are given by cosines of the mixing angles and do not produce additional smallness. Now the sign of the effect is fixed and does not depend on the sign of difference of densities. The observational consequences are as in the previous case: for $\eta < \eta_s$ one expects oscillations with the depth (37) below $\bar{P}^d_{\nu_e}$.

Notice that effect of a structure with more than 2 jumps will be still proportional to $J_m^2 \sim \epsilon^2$, although some additional numerical factor can appear. Thus, for 4 jumps of the same size one can find that the maximal total effect is proportional to $4\Delta \theta_m \approx 4J_m^2$, i.e., 4 times larger than in the 2 jumps case due to the parametric enhancement.

When the density in $d$ layer changes adiabatically both $\theta_d$ and $\theta'_d$ in Eq. (36) should be substituted by their values at the surface.

VI. ATTENUATION IN THE CASE OF PARTIAL DECOHERENCE

Partial averaging (decoherence) of oscillations can be described by the factor $\xi \leq 1$ in the interference term of oscillation probability which suppresses the depth of oscillations. For
\( \xi \neq 0 \) the effect of structure also depends on the phase of precession in the \( d \) layer, and we will find maximal possible effect of the structure varying this phase. It can be shown that in the graphic representation the same parameter \( \xi \) describes reduction of the precession diameter: \( D \to \xi D \), projection of \( \mathbf{P} \) onto the precession axis does not change. Notice that \( \xi \) as function of \( x \) depends on the shape of wave packets. For wave packets with the exponential tails we have \( \xi = \exp(-s/\sigma_x) \), where \( s \approx (\Delta m^2/2E^2)t \) is the relative shift of the packets due to difference of the group velocities and \( \sigma_x \) is the width of packet. In this case \( \xi \) obeys multiplicative properties: if \( \xi_1 \) and \( \xi_2 \) are the averaging factors in the layers 1 and 2, the total averaging of oscillations after crossing both layers is given by the product \( \xi^{\text{tot}} = \xi_1 \xi_2 \).

In what follows we will consider effects of \( \xi \neq 0 \) for some cases presented in the previous sections.

1. Let us first study the \( \nu_1 \to \nu_e \) oscillations in the 2 layers profile with remote structure (see Fig. [5] left). In the \( s \)-layer oscillations proceed as the case 1 of Sec. [III]. The phase \( \phi_s \) acquired in this layer determines characteristics of oscillations in the layer \( d \), in contrast to complete averaging case. Maximal effect of oscillations corresponds to the state \( \mathbf{P}_1 \) at the end of \( s \) when the phase equals \( \phi_s = \pi + 2\pi k \). Then precession of \( \mathbf{P} \) in the \( d \)-layer around the axis \( \mathbf{A}_d \) will be with the initial diameter \( D_d = \sin(2\theta' + 2\Delta \theta_m) \). After partial averaging in \( d \) the projection of \( D_d \) onto the flavor axis equals

\[
D_e^{\text{max}}(x_d) = \xi \sin(2\theta'_s + 2\Delta \theta_m) \sin 2\theta_d = \xi \sin(2\theta'_d + 4\Delta \theta_m) \sin 2\theta_d. \tag{38}
\]

Here \( \xi = \xi(x_d) \). Thus, \( D_e^{\text{max}} \sim \epsilon \).

Without the structure the diameter of precession in the beginning would be \( \sin 2\theta'_d \). Partial averaging reduces it down to \( \xi \sin 2\theta'_d \), and projection of the diameter onto the flavor axis gives

\[
D_e^0 = \xi \sin 2\theta'_d \sin 2\theta_d. \tag{39}
\]

So, in the absence of structure the probability \( P_e^0 \) oscillates with the depth \( D_e^0(x_d) \) around average value given in Eq. (26). The oscillation depth decreases with increase of \( x_d \). The probability oscillates below maximal value \( P_e^0 = P_e^{\text{max}} \approx \cos^2 \theta \).

In the presence of structure, the probability \( P_e \) oscillates around nearly the same average value as without the structure (the same in the lowest approximation in \( \epsilon \)), but with bigger depth and the maximal possible depth is given in (38). Now \( P_e^{\text{max}} \) can be even above \( P = \cos^2 \theta \), which is a manifestation of the parametric enhancement of oscillations. This type of the oscillation pattern has been found in [5].

The difference of the depths of oscillations with and without structure equals

\[
D_e^{\text{max}} - D_e^0 = \xi[\sin(2\theta'_d + 4\Delta \theta_m) - \sin 2\theta'_d] \sin 2\theta_d = 2\xi \sin 2\theta_d \cos 2\theta'_s J_m \approx 2\xi \sin 2\theta_d J_m.
\]

That is, the effect of structure is of the order \( \epsilon \xi \). The difference of average probabilities is the same as in Eq. (25): \( \sim \epsilon^2 \), and it does not depend on \( \xi \). Thus, incomplete averaging leads to difference of depths of precession, but does not change averaged values in the lowest order. This is a consequence of the fact that before detection the neutrino vector precesses around the same axis in both cases.

Recall that the depth oscillations in the presence of structure can be smaller than the one without structure if the density in \( s \) is smaller than in \( d \).

2. Let us consider a structure near detector and the \( \nu_1 \to \nu_e \) channel, Fig. [5] right. In the \( d \) layer the polarization vector precesses around \( \mathbf{A}_d \) with the diameter of precession at the end
of the layer

\[ D_d = \xi \sin 2\theta_d'. \]

It can be expressed in terms of \( \bar{\theta}_d' \) – the angle between \( \mathbf{P}(x_d) = \mathbf{P}_1 \) and \( \mathbf{A}_d' \):

\[ D_d = 2|\mathbf{P}_1| \sin 2\bar{\theta}_d', \]

where the length of \( \mathbf{P}_1 \) at the end of layer \( d_1 \) equals

\[ |\mathbf{P}_1| = \frac{1}{2} \cos \frac{2\theta_d'}{2 \cos \theta_d'}. \]  

(41)

The angle \( \bar{\theta}_d \) is determined by the equality

\[ \tan 2\bar{\theta}_d' = \xi \tan \frac{2\theta_d'}{2 \cos \theta_d'}. \]  

(42)

From Eqs. (40), (41) and (42) we obtain the diameter in \( d \)

\[ D_d = \cos 2\theta_d' \tan 2\bar{\theta}_d'. \]  

(43)

The largest final precession depth in the \( s \)–layer is realized the neutrino vector is \( \mathbf{P}_1 \) which corresponds to the phase \( \phi_d = 2\pi k \) at the end of layer \( d \). In this case the angle of precession in \( s \) is \( (2\bar{\theta}_d' + 2\Delta \theta_m) \), and consequently, the diameter of precession in \( s \) equals

\[ D_s = 2|\mathbf{P}_d| \sin(2\bar{\theta}_d' + 2\Delta \theta_m) = \frac{\cos 2\theta_d'}{\cos 2\bar{\theta}_d'} \sin(2\bar{\theta}_d' + 2\Delta \theta_m). \]

Its projection on the flavor axis:

\[ D_{e_{\max}} = \cos \frac{2\theta_d'}{2 \cos \theta_d'} \sin(2\bar{\theta}_d' + 2\Delta \theta_m) \sin 2\theta_s. \]

So, \( D_{e_{\max}} \sim \epsilon \). In the limit of complete averaging, \( \bar{\theta}_d' = 0 \), the above expression coincides with (28). Neglecting the high order corrections it can be rewritten as

\[ D_{e_{\max}} \approx \sin 2\theta_s [\xi \sin 2\theta_d' + J_m]. \]

The average probability equals \( \bar{P} \approx \cos^2 \theta_s \).

Without structure the depth of flavor oscillations (\( z \)-projection of \( D_d \) in (43)) would be

\[ D_0 = \sin 2\theta_d' \cos 2\bar{\theta}_d' \approx \xi \sin 2\theta_d' \sin 2\theta_d'. \]

The average value of the probability: \( \bar{P}_0 = \cos^2 \theta_d \).

The difference of the depths of oscillations with and without structure equals

\[ D_{e_{\max}} - D_0 = \frac{\cos 2\theta_d'}{\cos 2\bar{\theta}_d'} [\sin 2\theta_s \sin(2\bar{\theta}_d' + 2\Delta \theta_m) - \sin 2\theta_d' \sin 2\bar{\theta}_d'] \approx 2 \sin 2\theta_s J_m, \]  

(44)

which does not depend on \( \xi \). So, in the lowest order, partial averaging affects \( D_e \) and \( D_0 \) equally.

The dependence of \( (D_{e_{\max}} - D_0) \) on \( \xi \) appears in the next order in \( \epsilon \) being \( \approx 2\bar{\theta}_d' J_m \sim \xi \epsilon^2 \). If \( \bar{\theta}_d' = 0 \) (complete averaging), we would get from (44) the value \( \Delta P_e = -(D_{e_{\max}} - D_0) \), which
FIG. 6. The same as in Fig. 5 (remote structure) but for the $\nu_e \rightarrow \nu_e$ transition. The attenuation is absent.

coincides with that in [28].

Difference of the averaged probabilities with and without structure is large:

$$\bar{P} - \bar{P}_0 \approx -\sin 2\theta d J_m \sim \epsilon.$$ 

It was no attenuation even in the case of complete averaging in $d$, so that the effect of $s-$layer appeared at the level of $\epsilon$.

If $\phi(x_d) = \pi k$, then the maxima of survival probability with and without the structure are approximately equal: $P_{e}^{\text{max}}(x_d) \approx P_{e}^{\text{max}}(x_d + x_p)$, otherwise the probability with structure is smaller than that without it. Observational effect consists of increase of oscillation depth and decrease of the average probability at $\eta < \eta_s$.

3. Let us consider the $\nu_e \rightarrow \nu_e$ transition and remote structure (Fig. 6). Without $s-$layer the diameter of precession in the $d-$layer equals

$$D^0 = \xi \sin 2\theta_d,$$

and its projection onto the flavor axis is

$$D_e^0 = \xi \sin^2 2\theta_d.$$  \hspace{1cm} (45)

In the limit $\xi = 1$ it coincides with standard oscillation depth.

With structure, precession in the $s-$layer has the diameter $\sin 2\theta_s$. Maximal final depth of oscillations in the $d-$layer corresponds to the vector $P_1$ and the phase $\phi(x_s + x_d) = \pi + 2\pi k$. The angle between $P_1$ and $A_d$ is $(2\theta_s + 2\Delta \theta_m)$, so that the precession diameter in the beginning
of $d-$layer equals
\[
D_d^{\text{max}} = \sin(2\theta_s + 2\Delta\theta_m).
\]
Taking into account averaging and projecting the diameter onto the flavor axis we obtain the depth of oscillations at a detector
\[
D_e^{\text{max}} = \xi \sin 2\theta_d \sin(2\theta_s + 2\Delta\theta_m).
\]
The difference of the depths with and without structure equals according to (45) and (46)
\[
D_e^{\text{max}} - D_0^e = 2\xi \sin 2\theta_d \cos 2\theta_s J_m.
\]
(47)
The difference is the order of $\epsilon$, i.e., attenuation is absent as in the case 5 of Sec. 11.

The position of neutrino vector $P_2$ corresponds to the minimal value of the probability $P_{e\text{min}}$ at the end. The maximal value of the probability in the presence of structure, $P_{e\text{max}}$, is in the position $P'_2$, which is realized when the phase of precession in $s$ is $2\pi k$, that is, the neutrino enters the $d-$layer as $P(0)$. The difference of the maximal and minimal probabilities can be found from the Fig. 6:
\[
P_{e\text{max}} - P_{e\text{min}} = (\xi \sin 2\theta_d \cos 2\theta_s + \cos 2\theta_d \sin 2\theta_s) J_m
\]
(48)
which differs from (47). In the limit $\xi \to 0$ it is reduced to the expression (33) for the complete averaging.

It is straightforward to show that the difference of the average oscillation probabilities is the same as in the case of complete averaging in the layer $d$, see (31). So, here we have oscillations with $O(1)$ depth. The differences of depths of oscillations and average values (with and without structure) are of the order $\epsilon$.

Notice that it was no attenuation even with complete averaging. Incomplete averaging does not change the difference of average probabilities, but produces difference of depths of oscillations (47) of the order $\epsilon$.

4. Let us consider $\nu_1 \rightarrow \nu_e$ transition in the symmetric profile with three layers $(d_1 - s - d_2)$ (see Fig. 7). If $\xi_1$ and $\xi_2$ are the averaging factor in the layers $d_1$ and $d_2$, we obtain the depth of oscillations without structure $(d_1 - d_2)$:
\[
D_e^{\text{max}}(x_d) = \xi_1 \xi_2 \sin 2\theta'_d \sin 2\theta_d,
\]
(49)
which differs from (39) by additional power of $\xi$. The average value of probability is given in (26).

As in the case 2 of this section, we use the angle $\bar{\theta}'_d$ (42) between the polarization vector at the end of layer $d_1$, $P_1$, and the axis $A_d$. Then the precession angle in the $s-$layer is $(2\bar{\theta}'_d + 2\Delta\theta_m)$. The length of $P_1$ is given in (41). Maximal final precession depths corresponds to the phase of oscillations in the $s-$layer $\phi_s = \pi + 2\pi k$ ($k-$ integer) when neutrino state is described by $P_2$. The angle of precession in the layer $d_2$ – the angle between $P_2$ and $A_d$ is $(2\bar{\theta}'_d + 4\Delta\theta_m)$. Consequently, the initial diameter of precession in $d_2$ equals
\[
D = \frac{\cos 2\bar{\theta}'_d}{\cos 2\bar{\theta}_d} \sin(2\bar{\theta}'_d + 4\Delta\theta_m).
\]
Averaging in the layer $d_2$ gives another factor $\xi_2$, and then projection on the flavor axis leads
FIG. 7. The same as in Fig. 3 but with partial averaging in $d_1$ and $d_2$. The attenuation is absent.

to

$$D_e^{max} = \xi_2 \frac{\cos 2\theta_d'}{\cos 2\theta_d} \sin(2\bar{\theta}_d + 4\Delta \theta_m) \sin 2\theta_d.$$ 

The oscillations proceed around the average value

$$\bar{P}_e = \frac{1}{2} + \frac{\cos 2\theta_d'}{2 \cos 2\theta_d} \cos(2\bar{\theta}_d + 4\Delta \theta_m) \cos 2\theta_d.$$ 

The difference of average probabilities with and without structure is very small $\sim \epsilon^2$.

Difference of the oscillation depth with and without structure equals

$$D_e^{max} - D_e^0 = \xi_2 \cos 2\theta_d \left[ \frac{\cos 2\theta_d'}{\cos 2\theta_d} \sin(2\bar{\theta}_d + 4\Delta \theta_m) - \xi_1 \sin 2\theta_d \right].$$ (50)

(In the limit $\Delta \theta_m = 0$, the expression in the brackets vanishes, as it should be.) In the lowest approximation in $\epsilon$ Eq. (50) reduces to

$$D_e^{max} - D_e^0 \approx 2\xi_2 \sin 2\theta_d J_m = O(\xi \epsilon).$$

Thus, the difference of depths is proportional to $\xi_2$, whereas dependence on $\xi_1$ is absent. There is no attenuation: $D_e^{max} - D_e^0 \propto \xi_2 \epsilon$. The difference of averaged probabilities, $\Delta \bar{P}_e \sim \epsilon^2$, is slightly changed from that in Fig. 3 There is no change of average values of the probabilities in the lowest order in $\epsilon$, since oscillations in the last layer occur in both cases (with and without structure) around the same axis.

5. Let us consider the $\nu_e \rightarrow \nu_e$ transition in the case of partial averaging in the symmetric profile with three layers, Fig. 8. The only difference from the previous case is the initial state given now by $P(0) = \frac{1}{2} z$, instead of $P(0) = \frac{1}{2} A_v$. Therefore in formulas obtained for the $\nu_1 \rightarrow \nu_e$ transition $\theta_d'$ should be substituted by $\bar{\theta}_d$. As in the case 2, we introduce the angle $\bar{\theta}_d$.
between the polarization vector $\mathbf{P}_1$ at the end of layer $d_1$ after partial averaging and the axis $\mathbf{A}_d$. It is determined by the equality

$$\tan 2\tilde{\theta}_d = \xi \tan 2\theta_d.$$  \hspace{1cm} (51)

Then the precession angle in the $s$–layer (the angle between $\mathbf{P}_1$ and $\mathbf{A}_s$) is $(2\tilde{\theta}_d + 2\Delta \theta_m)$. The length of the vector equals

$$|\mathbf{P}_1| = \frac{\cos 2\theta_d}{2 \cos 2\theta_d}.$$  

Maximal final depths of oscillations corresponds to the phases of oscillations $\phi_d = 2\pi k$ (position $\mathbf{P}_1$) in the $d_1$–layer and $\phi_s = \pi + 2\pi k'$ (position $\mathbf{P}_2$) in the $s$–layer ($k, k'$ are integers). Under these conditions the parametric enhancement of oscillations occur and the precession angle in the $d_2$–layer, i.e. the angle between $\mathbf{P}_2$ and $\mathbf{A}_d$, becomes $2\tilde{\theta}_d + 4\Delta \theta_m$. Consequently, the initial diameter of precession in $d_2$ equals

$$D_{\text{max}}^e = \frac{\cos 2\theta_d}{\cos 2\theta_d} \sin(2\tilde{\theta}_d + 4\Delta \theta_m).$$

Averaging in the layer $d_2$ gives another factor $\xi_{2e}$, and then projection onto the flavor axis leads to

$$D_{\text{max}}^{e}\xi_{2e} = \frac{\cos 2\theta_d}{\cos 2\theta_d} \sin(2\tilde{\theta}_d + 4\Delta \theta_m) \sin 2\theta_d.$$  \hspace{1cm} (52)
The depth of oscillations without structure equals

\[ D_{e}^{\text{max}}(x_d) = \xi_1 \xi_2 \sin^2 2\theta_d, \]

which again differs from \((49)\) by substitution \(\theta'_d \to \theta_d\). The difference of the depths with \((52)\) and without \((53)\) structure,

\[ D_{e}^{\text{max}} - D_{e}^0 = \xi_2 \sin 2\theta_d \left[ \frac{\cos 2\theta_d}{\cos 2\theta_d} \sin(2\bar{\theta}_d + 4\Delta \theta_m) - \xi_1 \sin 2\theta_d \right], \]

is similar to that in \((50)\) with substitution \(\theta'_d \to \theta_d\) in the parenthesis. It can be rewritten as

\[ D_{e}^{\text{max}} - D_{e}^0 = \xi_2 \sin 2\theta_d \left[ \cos 2\theta_d \cos 2\Delta \theta_m J_m - \xi_1 \sin 2\theta_d J_m^2 \right] \approx \xi_2 \sin 4\theta_d J_m. \tag{54} \]

There is no attenuation and \(D_{e}^{\text{max}} - D_{e}^0 \propto \xi_2 \epsilon\). Attenuation is reproduced if \(\xi_2 = 0\), i.e. in the case of complete averaging in the \(d_2\) layer, then the diameter of precession becomes zero in the lowest order. Dependence on \(\xi_1\) appears in the \(\epsilon^2\) order, that is, attenuated.

The oscillations proceed around the average value

\[ \bar{P}_e = \frac{1}{2} + \frac{\cos^2 2\theta_d}{2 \cos 2\theta_d} \cos(2\bar{\theta}_d + 4\Delta \theta_m). \]

Without the structure we would have \(\bar{P}_e^0 = 0.5(1 + \cos^2 2\theta_d)\). The difference of average probabilities with and without structure equals

\[ \Delta \bar{P}_e = -\cos^2 2\theta_d \left[ \xi_1 \tan 2\theta_d \cos 2\Delta \theta_m J_m + J_m^2 \right]. \tag{55} \]

Now partial averaging leads to \(\Delta D_e \sim \xi_2 \epsilon\) and \(\Delta \bar{P}_e \sim \xi_1 \epsilon\). Thus, the difference of average values depends on \(\xi_1\) but it does not depend on \(\xi_2\). In the limit \(\xi_1 = 0\): \(\Delta \bar{P}_e \propto \epsilon^2\) and the attenuation is recovered. Notice that the differences of depth and average values depends on different \(\xi\): \(\xi_2\) and \(\xi_1\) correspondingly. This can be used for tomography.

The structure in the density profile changes the depth of precession which can be larger or smaller than that without structure depending on phases of oscillations in the \(d_1\) and \(s\)-layers. In the case of smaller density in the \(s\)-layer than in the \(d\) layers, the sign of \(2\Delta \theta_m\), and consequently, the sign of \(J_m\) change. As a result, the difference of precession diameters \((54)\) becomes negative and the difference of average values \((55)\) becomes positive.

For \(\xi \gg \epsilon\), the effect of structure appears in the lowest order in \(\epsilon\) in all the cases (channel, profile), although it can be suppressed by \(\xi\).

VII. DISCUSSION AND CONCLUSIONS

1. Attenuation effect is the effect of loss of sensitivity of the oscillation signal to remote structures of density profile due to finite neutrino energy resolution. We presented the graphic (geometric) description of the effect. We show that the effect is a result of

- small mixing of the mass states in matter;
- incoherence of the neutrino state arriving at a structure;
• averaging of oscillations (loss of coherence) between a structure and a detector.

Contributions to the oscillation effect of structures at distances larger than the attenuation length are suppressed by additional power of $\epsilon$.

The attenuation length is the distance over which oscillations integrated over the energy resolution interval of neutrinos are averaged. In other terms, it is a distance over which the wave packets of the size determined by the energy reconstruction function are separated in space. The attenuation is realized in the lowest order in $\epsilon$. The remote structures produce effects in the $\epsilon^2$ order. The better the relative energy resolution $\sigma_E/E$, the more remote structures can be seen.

The conditions of attenuation are valid for a multi-layer medium. In the case of several different structures the conditions should be applied to each structure independently. Interplay between different structures will show up in the next order in $\epsilon$. Actually, we saw this interplay in the case of two jumps.

2. The effect of remote structure is proportional to the change of the mixing parameter $J_m \equiv \sin 2\Delta \theta_m \sim \epsilon$ and the projection factors. The attenuation is realized if one of the projection factors is $\sim \epsilon$. For the profile with core (two jumps) the jump factor appears as $J_m^2$ in the probability. For more than 2 jumps the effect is still proportional to $J_m^2$ with some additional coefficients.

3. In terms of graphic representation the effect of structure is determined by the diameter of precession and its projection onto the eigenstate axis (which depends on setup and channel of transition). This allows us to understand immediately why in the case of flavor to mass transition $\nu_e \rightarrow \nu_1$ the sensitivity is mainly to remote structures (see [1]). The detector of neutrinos $\nu_1$ is “focused” on structures to which neutrino state $\nu_1$ arrives.

Graphic description allows us to explicitly compute effects in $\epsilon^2$ and higher orders and also obtain results for different positions of a structure and channels of oscillations.

Attenuation is a result of (i) suppression of the precession diameter in the $s$-layer either due to specific initial state (state arriving at the structure) or due to averaging, and (ii) smallness of projection of the diameter onto the eigenstates axis.

4. In the case of partial averaging the attenuation is absent or weak. For all the configurations (channels, profiles) the effect appears in the lowest order in $\epsilon$. In expressions obtained for complete attenuation one factor $\epsilon$ is substituted by $\xi$, and the effect is given by $\xi \epsilon$. So, it may be suppressed, if $\xi$ is small.

5. From the observational point of view, in the case of complete averaging one will see constant $P_e$ at $\eta > \eta_s$ and the oscillatory pattern at $\eta < \eta_s$. In the case of attenuation the depth of oscillations and change of the average probability are of the order $\epsilon^2$. In absence of the attenuation these parameters are of the order $\epsilon$.

6. Similarly, one can consider the attenuation in the 1-3 channel. There are two features here: the vacuum angle is relatively small, so the eigenstate axes are turned closer to the flavor axis. Consequently, in the $2\nu$-case we would get the same formulas as before with just substitution $\theta_{12} \rightarrow \theta_{13}$, and $\epsilon \rightarrow \epsilon_{13} = E V_e/\Delta m^2_{31}$. Low density means here $E < 1$ GeV. The $2\nu$-case can be realized in the region $(0.2 - 1)$ GeV where the 1-2 phase is small (evolution is frozen).

7. On practical side, the operations of integration over the energy (wave function of a detector) and integration of the evolution equation can be permuted. That is, one can first integrate over the neutrino energy obtaining wave packets and then consider the flavor evolution, or first compute the flavor evolution and then perform the energy integration. In the first case it is clear that one can simply neglect effects of remote structures in consideration from the beginning.
The attenuation effect should be taken into account at interpretation of experimental data on neutrino oscillations in the Earth and in planning of future experiments devoted to the Earth oscillation tomography.

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