SHADOWING NEAR NONHYPERBOLIC FIXED POINTS

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(Communicated by Lan Wen)

ABSTRACT. We use Lyapunov type functions to find conditions of finite shadowing in a neighborhood of a nonhyperbolic fixed point of a one-dimensional or two-dimensional homeomorphism or diffeomorphism. A new concept of shadowing in which we control the size of one-step errors is introduced in the case of a nonisolated fixed point.

1. Introduction. The shadowing property of dynamical systems (diffeomorphisms or flows) is now well-studied (see, for example, the monographs [1, 2] and the recent survey [3]). This property means that, near approximate trajectories (so-called pseudotrajectories), there exist exact trajectories of the system.

In this paper, we are interested in shadowing near fixed points.

Mostly, standard methods allow one to show that a diffeomorphism has the shadowing property near a hyperbolic fixed point, and this property is Lipschitz, see [1].

One can mention several papers which contain methods of proving the shadowing property for systems with nonhyperbolic behavior (see, for example, [4, 5]).

Our approach is based on the method of Lyapunov type functions. First, Lyapunov type functions were used in the study of shadowing and topological stability by Lewowicz in [6]. We apply results on Lyapunov type functions obtained in our paper [7].

Let us state the problem of shadowing in general.

Let \( f \) be a homeomorphism of a metric space \((M, \text{dist})\).

In this paper, we define a finite \(d\)-pseudotrajectory of \( f \) as a set of points \( \{p_k \in M : 0 \leq k \leq m\} \) such that

\[
\text{dist}(f(p_k), p_{k+1}) \leq d, \quad 0 \leq k \leq m - 1.
\]

2010 Mathematics Subject Classification. Primary: 37C50.

Key words and phrases. Dynamical system, shadowing, fixed point, Lyapunov function.

Supported by the Russian Foundation for Basic Research (project 12-01-00275); the first author is also supported by the Chebyshev Laboratory, Faculty of Mathematics and Mechanics, St. Petersburg State University, under grant 11.G34.31.0026 of the Government of Russian Federation.
In the study of shadowing in noninvariant sets (such as neighborhoods of fixed points), the concept of finite shadowing is natural.

We say that \( f \) has the finite shadowing property in a set \( K \subset M \) if, for any \( \varepsilon > 0 \), there exists a \( d > 0 \) such that if \( \{p_k \in K : 0 \leq k \leq m\} \) is a \( d \)-pseudotrajectory of \( f \), then there exists a point \( r \) such that

\[
\text{dist}(f^k(r), p_k) \leq \varepsilon, \quad 0 \leq k \leq m.
\] (1)

Let us emphasize that in the above definition, \( d \) depends on \( K \) and \( \varepsilon \) but not on \( m \).

The structure of the paper is as follows. In Sec. 2, we treat the one-dimensional case. We prove a simple general statement (Theorem 2.1) and show that, in some cases, the dependence of \( d \) on \( \varepsilon \) can be clarified. Section 3 is devoted to the method of Lyapunov type functions developed in [7]. In Sec. 4, we give general conditions of finite shadowing in the two-dimensional case and then treat in detail an important example of a diffeomorphism of the form

\[
f(x, y) = (x - x^{2n+1} + X(x, y), y + y^{2m+1} + Y(x, y)),
\] (2)

where \( n, m \) are natural numbers and \( X, Y \) are smooth functions that vanish at the origin together with their Jacobi matrices.

Finally, Sec. 5 is devoted to shadowing near a nonisolated fixed point. We study a simple (but nontrivial) example of the diffeomorphism

\[
f(x, y) = \left( \frac{x}{2}, y(1 + x^2) \right),
\] (3)

for which the origin is a nonisolated fixed point (any point \((0, y)\) is a fixed one).

Of course, such a system does not have the usual shadowing property. In this case, we work with a new concept of shadowing in which we control the size of one-step errors.

Our methods can be applied to dynamical systems with phase space of arbitrary dimension; in this paper, we restrict the consideration to one-dimensional and two-dimensional systems to clarify the presentation of the main ideas.

\section*{2. One-dimensional case.} First we consider the problem of shadowing near a nonhyperbolic fixed point of a homeomorphism in the simplest, one-dimensional, case.

Let \( f \) be a homeomorphism of a neighborhood \( U \) of a fixed point \( 0 \in \mathbb{R} \) to its image.

We consider in detail the case where \( f \) is nonhyperbolically expanding in a neighborhood of a fixed point; the case of nonhyperbolic contraction near a fixed point is treated similarly.

We impose simplest possible conditions on \( f \); in a sense, precisely this topological form of conditions is generalized by our conditions in the two-dimensional case.

\textbf{Condition 1.} There exist numbers \( a, A > 0 \) such that if \( |x| \leq A \) and \( 0 < v < a \), then

\[
f(x + v) - f(x) > v, \quad f(x - v) - f(x) < -v.
\] (4)

Denote by \( B(r, A) \) the closed \( r \)-neighborhood of a set \( A \).

\textbf{Theorem 2.1.} If condition 1 is satisfied, then \( f \) has the finite shadowing property in the set \( B = B(A, 0) \).
Proof. Fix $\varepsilon > 0$; we assume that $\varepsilon \leq a$.

Condition 1 implies that if $x \in B$, then
$$B(\varepsilon, f(x)) \subset \text{Int} f(B(\varepsilon, x)).$$

The compactness of the set $B$ and the continuity of $f$ imply that there exists a $d > 0$ such that
$$B(d, B(\varepsilon, f(x))) \subset f(B(\varepsilon, x))$$
for $x \in B$.

Let $\{p_k \in B : 0 \leq k \leq m\}$ be a $d$-pseudotrajectory of $f$; denote $C_k = B(\varepsilon, p_k)$. We claim that
$$C_{k+1} \subset f(C_k), \quad k = 0, \ldots, m - 1. \tag{6}$$

Indeed, let $f(C_k) = [s, t]$ and let $y \in C_{k+1}$. Inclusion (5) with $x = p_k$ implies that
$$f(p_k) + \varepsilon + d \leq t.$$

Now it follows from the inequalities $|y - p_{k+1}| \leq \varepsilon$ and $|f(p_k) - p_{k+1}| < d$ that
$$|y - f(p_k)| < \varepsilon + d;$$

hence,
$$y < f(p_k) + \varepsilon + d \leq t.$$

Similarly one shows that $y > s$, which proves (6).

It follows from inclusions (6) that the set
$$C = \cap_{k=0}^{m} f^{-k}(C_k)$$
is not empty. Clearly, for any point $r \in C$, inequalities (1) hold. \hfill \Box

In Theorem 2.1, we can say nothing about the dependence of $d$ on $\varepsilon$. For a particular example of a diffeomorphism with a nonhyperbolic fixed point considered below, such a dependence can be clarified.

Example 1. Let $f(x) = x + x^{2n+1} + X(x)$, where $n$ is a natural number. Take $\varepsilon > 0$ and let
$$S(x, \varepsilon) := \frac{(x + \varepsilon)^{2n+1} - x^{2n+1}}{\varepsilon}.$$

Then $S(x, \varepsilon)$ is a polynomial in $x$ of even degree with positive leading coefficient. Since the derivative of $S(x, \varepsilon)$ in $x$ has a unique zero $x = -\varepsilon/2$, the inequality
$$S(x, \varepsilon) \geq S\left(-\frac{\varepsilon}{2}, \varepsilon\right) = \frac{\varepsilon^{2n}}{2^{2n}}$$
holds. Thus, the form
$$S(x, \varepsilon) - \frac{\varepsilon^{2n}}{1 + 2^{2n}}$$
of degree $2n$ is positive definite, and there exists a positive number $\alpha = \alpha(n)$ such that
$$S(x, \varepsilon) - \frac{\varepsilon^{2n}}{1 + 2^{2n}} \geq \alpha(x^{2n} + \varepsilon^{2n}). \tag{7}$$

Assume that
$$\frac{|X(x + v) - X(x)|}{|v|} = o(x^{2n} + v^{2n}), \quad x, v \to 0. \tag{8}$$

Then there exist $A, a > 0$ such that if $|x| \leq A$ and $0 < \varepsilon < a$, then
$$|X(x + \varepsilon) - X(x)| \leq \frac{\alpha \varepsilon}{2}(x^{2n} + \varepsilon^{2n}). \tag{9}$$
Let
\[ d = \frac{\varepsilon^{2n+1}}{1 + 2^{2n}}. \tag{10} \]

If \( \{p_k\} \) is a \( d \)-pseudotrajectory with \( |p_k| \leq A \), then it follows from (7) and (9) that

\[
f(p_k + \varepsilon) - f(p_k) = \varepsilon + \varepsilon S(p_k, \varepsilon) + X(p_k + \varepsilon) - X(p_k) \geq \varepsilon + \frac{\alpha \varepsilon}{2} (p_k^{2n} + \varepsilon^{2n}) + d > \varepsilon + d.
\]

This relation and a similar relation for \( f(p_k - \varepsilon) - f(p_k) \) mean that an analog of inclusion (5) holds for any \( p_k \).

Repeating the proof of Theorem 1, we conclude that \( f \) has the finite shadowing property in the set \( B(A, 0) \).

Note that, for example, condition (8) is valid if \( X(x) = x^{2n+2} \).

Our reasoning also shows that if \( X(x) \equiv 0 \), then \( f \) has the finite shadowing property in the whole line \( \mathbb{R} \) with the same dependence of \( d \) on \( \varepsilon \) given by (10).

**Remark 1.** In [8], S. Tikhomirov used a different approach to show that the diffeomorphism \( f(x) = x + x^3 \) has the shadowing property with \( d = \varepsilon \varepsilon^3 \).

### 3. Lyapunov functions and shadowing.

As was mentioned in the Introduction, we consider in detail the problem of finite shadowing for a homeomorphism \( f \) of the plane \( \mathbb{R}^2 \) in two cases: Case I (the origin is an isolated nonhyperbolic fixed point) and Case NI (the \( y \)-axis consists of fixed points).

We apply the approach based on pairs of Lyapunov type functions developed in the paper [7].

Let us formulate the sufficient conditions of finite shadowing obtained in [7] in a form modified to fit our purposes in this paper.

Let \( K_0 = \mathbb{R}^2 \) in Case I, and let

\[ K = \{(x, y) : 0 < |x| < 1\} \]

in Case NI.

We assume that there exist two continuous nonnegative functions \( V \) and \( W \) defined on \( K_0 \times K_0 \) such that \( V(p, p) = W(p, p) = 0 \) for any \( p \in K_0 \) and the conditions (C1)-(C9) stated below are satisfied.

We formulate our conditions not directly in terms of the functions \( W \) and \( V \) but in terms of some geometric objects defined via these functions.

Fix a positive number \( \delta \) and a point \( p \in K_0 \) and let

\[ P(\delta, p) = \{q \in K_0 : V(q, p) \leq \delta, W(q, p) \leq \delta\}, \]
\[ Q(\delta, p) = \{q \in P(\delta, p) : V(q, p) = \delta\}, \quad R(\delta, p) = \{q \in P(\delta, p) : W(q, p) = \delta\}, \]
\[ T(\delta, p) = \{q \in P(\delta, p) : V(q, p) = 0\}. \]

Set
\[ \text{Int}^0 P(\delta, p) = \{q \in P(\delta, p) : V(q, p) < \delta, W(q, p) < \delta\}, \]
\[ \partial^0 P(\delta, p) = Q(\delta, p) \cup R(\delta, p), \]
\[ \text{Int}^0 Q(\delta, p) = \{q \in P(\delta, p) : V(q, p) = \delta, W(q, p) < \delta\}. \]

Let \( K \) be a subset of \( K_0 \) (in our basic examples, \( K \) is a small closed neighborhood of the origin in Case I and \( K \) is a neighborhood of the origin in \( K_0 \) in Case NI).
Conditions (C1) – (C4) contain our assumptions on the geometry of the sets introduced above. In conditions (C2) – (C4), \( p \in K \), and \( \delta, \Delta \) are arbitrary positive numbers such that \( \delta < \Delta \).

(C1) For any \( \varepsilon > 0 \) there exists a \( \Delta_0 = \Delta_0(\varepsilon) > 0 \) such that \( P(\Delta_0, p) \subset B(\varepsilon, p) \) for \( p \in K \).

(C2) \( Q(\delta, p) \) is not a retract of \( P(\delta, p) \);

(C3) \( Q(\delta, p) \) is a retract of \( P(\delta, p) \setminus T(\delta, p) \);

(C4) there exists a retraction \( \sigma : P(\Delta, p) \to P(\delta, p) \) such that if \( V(q, p) \neq 0 \), then \( V(\sigma(q), p) \neq 0 \).

In the next group of conditions, we state our assumptions on the behavior of the introduced objects and their images under the homeomorphism \( f \).

Let \( p, q \in K \) and let \( 0 < \delta < \Delta \). We say that condition \( W(\delta, \Delta, p, q) \) is satisfied if

(C5) \( f(P(\delta, p)) \subset \text{Int}^0 P(\Delta, q) \), \( f^{-1}(P(\delta, q)) \subset \text{Int}^0 P(\Delta, p) \);

(C6) \( f(T(\delta, p)) \subset \text{Int}^0 P(\delta, q) \);

(C7) \( f(T(\Delta, p)) \cap Q(\delta, q) = \emptyset \);

(C8) \( f(P(\delta, p)) \cap \partial^0 P(\delta, q) \subset \text{Int}^0 Q(\delta, q) \);

(C9) \( f(Q(\delta, p)) \cap P(\delta, q) = \emptyset \).

The same reasoning as in [7] proves the following statement.

**Proposition 1.** Assume that conditions (C2) – (C4) hold. Let \( p_0, \ldots, p_m \in K \). If \( 0 < \delta < \Delta \) and condition \( W(\delta, \Delta, p_k, p_{k+1}) \) is satisfied for any \( k = 0, \ldots, m - 1 \), then there is a point \( r \in P(\delta, p_0) \) such that

\[ f^k(r) \in P(\delta, p_k), \quad k = 1, \ldots, m. \]  \hfill (11)

Thus, to show that \( f \) has the finite shadowing property in a neighborhood \( K \) of the origin, it is enough to find functions \( V \) and \( W \) that satisfy conditions (C1) – (C4) and to show that for any \( \Delta > 0 \) there exists a \( \delta \in (0, \Delta) \) with the following property: There exists a \( d > 0 \) such that if \( p, q \in K \) and \( |q - f(p)| \leq d \), then condition \( W(\delta, \Delta, p, q) \) is satisfied.

Indeed, take any \( \varepsilon > 0 \), find a corresponding \( \Delta_0 \) (see condition (C1)), then take suitable \( \Delta < \Delta_0 \) and \( \delta \), and finally find a \( d > 0 \) having the above property. Then,
if \( p_0, \ldots, p_m \in K \) is a \( d \)-pseudotrajectory of \( f \), this pseudotrajectory is \( \varepsilon \)-shadowed by any point \( r \) that satisfies inclusions (11).

We realize this scheme in the next section considering Case I. In Case NI, \( f \) does not have the usual shadowing property, and we have to modify the concept of shadowing (see Sec. 5).

4. Two-dimensional case. Isolated fixed point. Now we consider a two-dimensional homeomorphism

\[ f(x, y) = (g(x, y), h(x, y)) \]

having a fixed point at the origin and assume that \( f \) is contracting in the direction of variable \( x \) and expanding in the direction of variable \( y \) (and both the contraction and expansion are not assumed to be hyperbolic).

Let \( p = (p_x, p_y) \) be the coordinate representation of a point \( p \in \mathbb{R}^2 \).

In the case considered, we introduce two functions,

\[ V(p, q) = |p_y - q_y| \quad \text{and} \quad W(p, q) = |p_x - q_x|. \]

For such functions, conditions (C1) – (C4) are obviously satisfied for any \( p \in \mathbb{R}^2 \) and any \( 0 < \delta < \Delta \).

We first formulate general conditions of finite shadowing for an arbitrary compact subset \( K \) of the plane and then apply them to our first basic example.

**Condition 2.** For any \( \Delta_0 > 0 \) there exist \( \delta, \Delta > 0 \) such that \( \delta < \Delta < \Delta_0 \) and if \( p \in K \), then condition (C5) with \( q = f(p) \) is satisfied,

\[ |g(p_x + v, p_y + w) - g(p_x, p_y)| < \delta \quad \text{for} \quad (v, w) \in H(\delta), \]

where

\[ H(\delta) = \{|v| \leq \delta, \ w = 0\} \cup \{|v| = \delta, \ |w| \leq \delta\}, \]

\[ |h(p_x + v, p_y) - h(p_x, p_y)| < \delta \quad \text{for} \quad 0 \leq |v| \leq \Delta, \]

and

\[ |h(p_x + v, p_y + w) - h(p_x, p_y)| > \delta \quad \text{for} \quad 0 \leq |v| \leq \delta, \ |w| = \delta. \]

**Theorem 4.1.** If \( K \) is a compact subset of the plane and condition 2 is satisfied, then \( f \) has the finite shadowing property in the set \( K \).
Proof. First we show that condition 2 implies that condition $W(\delta, \Delta, p, f(p))$ is satisfied for any $p \in K$.

Since

$$T(\delta, p) = \{ r : |r_x - p_x| \leq \delta, r_y = p_y \},$$

conditions (12) with $w = 0$ and (13) imply condition (C6).

Similarly, condition (13) implies condition (C7).

Since condition (14) holds,

$$f(Q(\delta, p)) \cap P(\delta, f(p)) = \emptyset.$$  \hfill (15)

Inequalities (12) for $|v| = \delta$, $0 \leq |w| \leq \delta$ combined with (15) show that the image of the boundary of the set $P(\delta, p)$ under the homeomorphism $f$ does not intersect the set $R(\delta, f(p))$.

Set

$$S = \{(g(p) + a, h(p) + b) \mid |a| \geq \delta, |b| \leq \delta \}.$$  

It is obviously that $R(\delta, f(p)) \subset S$. From (12), (13) and (14) it follows that $f(\partial_0 P(\delta, p)) \cap S = \emptyset$. Because of connectedness of $f(P(\delta, p))$ it follows also that $f(P(\delta, p)) \cap S = \emptyset$ and condition (C8) is proved.

Finally, condition (C9) follows from condition (14).

Thus, we have shown that conditions (12) – (14) imply that condition $W(\delta, \Delta, p, f(p))$ is satisfied for any $p \in K$.

Finally, we note that since $K$ is compact and $f$, $V$, and $W$ are continuous, the form of conditions (C5) – (C9) implies that there exists a $d > 0$ depending only on $\delta$ and $\Delta$ such that if $p \in K$ and $|q - f(p)| < d$, then condition $W(\delta, \Delta, p, q)$ is satisfied.

Now Theorem 4.1 is a corollary of the proposition stated in the previous section.

Example 2. Consider a diffeomorphism (2) in which $n, m$ are natural numbers and $X, Y$ are smooth functions that vanish at the origin together with their Jacobi matrices.

First we fix a small closed neighborhood $K$ of the origin (in what follows, we make it as small as our future conditions require).

Let us assume that if $p \in K$, then

$$(2n + 1)p^{2n}x - \frac{\partial X}{\partial x}(p) - \nu \frac{\partial X}{\partial y}(p) > 0, \quad p_x \neq 0, |\nu| \leq 1,$$  \hfill (16)

and

$$(2m + 1)p^{2m}y + \frac{\partial Y}{\partial y}(p) + \nu \frac{\partial Y}{\partial x}(p) > 0, \quad p_y \neq 0, |\nu| \leq 1.$$  \hfill (17)

We take the same functions $V$ and $W$ as above in this section.

It follows from the form of $f$ that for any $\alpha > 0$ we can find a neighborhood $K$ such that

$$\left\| \frac{\partial f}{\partial (x, y)}(p) \right\|, \left\| \frac{\partial f^{-1}}{\partial (x, y)}(p) \right\| \leq 1 + \alpha, \quad p \in K.$$  \hfill (18)

Thus, if $\alpha > 0$ is given, then, for $\delta$ small enough and $K$ properly chosen, condition (C5) is satisfied with $\Delta = (1 + \alpha)\delta$.

We take $\alpha < 1$ (in which case we may take $\Delta = 2\delta$ in condition (C5)) and assume that $K$ is so small that

$$\left| \frac{\partial Y}{\partial x}(p) \right| \leq \frac{1}{4}, \quad p \in K.$$
If $p \in K$ and $|v| \leq \Delta = 2\delta$, then

$$|h(p_x + v, p_y) - h(p_x, p_y)| = |Y(p_x + v, p_y) - Y(p_x, p_y)| \leq \frac{1}{4}|v| \leq \frac{\delta}{2};$$

thus, condition (13) is satisfied.

Now let us check condition (14). If $0 \leq |v| \leq \delta$ and $|w| = \delta$, then $v = \nu w$ for some $|\nu| \leq 1$.

Assume, for definiteness, that $w > 0$ and estimate, using condition (17):

$$h(p_x + \nu w, p_y + w) - h(p_x, p_y) = \int_0^w \frac{d}{dt} h(p_x + \nu t, p_y + t) dt =$$

$$= \int_0^w \nu \frac{\partial h}{\partial x}(p_x + \nu t, p_y + t) + \frac{\partial h}{\partial y}(p_x + \nu t, p_y + t) dt =$$

$$+ 1 + (2m + 1)(p_y + t)^{2m}) dt > w \geq \delta,$$

where we take into account that $p_y + t$ is not identically zero. This proves condition (14) (the case $w < 0$ is treated similarly).

To prove condition (12), we consider the case where $|v| = \delta$, $|w| \leq d$; the case where $w = 0$ is treated similarly (note that in both cases, $w = \nu v$ with $|\nu| \leq 1$).

We assume, for definiteness, that $v = \delta$, represent $w = \nu v$ with $|\nu| \leq 1$, and estimate, using condition (16):

$$g(p_x + \delta, p_y + \nu\delta) - g(p_x, p_y) = \int_0^{\delta} \frac{d}{dt} g(p_x + t, p_y + \nu t) dt =$$

$$= \int_0^{\delta} [1 - (2n + 1)(p_x + t)^{2n} + \frac{\partial X}{\partial x}(p_x + t, p_y + \nu t) +$$

$$+ \nu \frac{\partial X}{\partial y}(p_x + t, p_y + \nu t)] dt < \delta.$$

The case $v = -\delta$ is treated similarly.

Let us consider as a “test perturbation” a monomial $Y(x, y) = ax^k y^l$ in (2), where $a \in \mathbb{R}$, $k \geq 0$, and $l \geq 1$. Let $(x, y)$ be a point in a neighborhood $K$ of the origin with $y \neq 0$.

Taking $\nu = 0$ in (17) and dividing the result by $y^{2m}$, we get the following necessary condition:

$$alx^k y^{l-2m-1} > -(2m + 1).$$

This condition is obviously satisfied in a small $K$ if $l \geq 2m + 1$ (and if $|a|$ is small in the case where $l = 2m + 1$ and $k = 0$).

If $l < 2m + 1$, then the necessary condition looks as follows: $a > 0$, $k$ is even, and $l$ is odd.

Let us write condition (17) in $K$ in the form

$$(2m + 1)g^{2m} + alx^k y^{l-1} > |a k x^{k-1} y^l|. \quad (19)$$

If $l \geq 2m + 1$ (and if $|a|$ is small in the case where $l = 2m + 1$ and $k = 0$), condition (17) is obviously satisfied in a small neighborhood of the origin.

In the case where $l \leq 2m$, we get one more necessary condition: $k + l \geq 2m + 1$. Indeed, since $a > 0$, $k$ is even, and $l$ is odd, it is enough to consider (17) for $x, y \geq 0$. 


Now let us write (17) in the form
\[
a x^{k-1} y^{l-1} (ky - lx) < (2m + 1) y^{2m}.
\]
If \(k + l < 2m + 1\), set \(x = \frac{kh}{2r}\) and \(y = b\) with small \(b > 0\). We get an inequality of the form
\[
0 < \text{const} < b^{2m - k - l + 1}
\]
which cannot be satisfied for all small \(b\).

Elementary calculations show that if \(k + l < 2m + 1\), \(a > 0\), \(k\) is even, \(l\) is odd, and \(k + l \geq 2m + 1\), then condition (19) is satisfied in a small neighborhood of the origin.

We can obtain similar conditions if \(X(x, y)\) in (2) is also a monomial.

Our methods allow us to estimate the dependence of \(d\) on \(\varepsilon\) in the finite shadowing property for the considered case.

For example, if \(X(x, y) = a_1 x^{k_1} y^{l_1}\) and \(Y(x, y) = a_2 x^{k_2} y^{l_2}\) in (2) with \(k_1 > 2n + 1\) and \(l_2 > 2m + 1\), then the same reasoning as above shows that there exists a neighborhood \(K\) of the origin and a small number \(c > 0\) such that if a \(\delta > 0\) is given and \(\{p_k\}\) is a finite set of points in \(K\) with
\[
|f(p_k) - p_{k+1}| \leq c\delta^p,
\]
where \(p = \max(2n+1, 2m+1)\), then conditions \(W(\delta, 2\delta, p_{k}, p_{k+1})\) are satisfied. This means that \(f\) has in \(K\) the finite shadowing property with the following dependence of \(d\) on \(\varepsilon\): \(d = c\varepsilon^p\).

5. Two-dimensional case. Nonisolated fixed point. In this section, we consider a model example of the diffeomorphism (3) for which the origin is a nonisolated fixed point (any point \((0, y)\) is a fixed one).

Of course, \(f\) does not have the shadowing property.

Nevertheless, we show that \(f\) has an analog of the finite shadowing property if we consider pseudotrajectories \(\{p_k\}\) with \((p_k)_x \neq 0\) and allow the “errors”
\[
|f(p_k) - p_{k+1}|
\]
to depend on \((p_k)_x\). The errors must be smaller for smaller values of \(|(p_k)_x|\). Such an approach (in the case of a nontransverse homoclinic point) has been suggested by S. Tikhomirov.

We restrict our consideration to the case of a diffeomorphism \(f\) of a very simple form (3) to simplify presentation (as the reader will see, even this case is not completely trivial); of course, our reasoning can be applied in more general situations.

Note that
\[
f^{-1}(x, y) = \left(2x, -\frac{y}{1 + 4x^2}\right).
\]

Thus, we consider a finite pseudotrajectory \(p_0, \ldots, p_m\) and assume that \((p_k)_x \neq 0\) and
\[
|f(p_k) - p_{k+1}| \leq d(p_k)_x^2, \quad k = 0, \ldots, m - 1,
\]
for some \(d > 0\).

Our main result is as follows. Recall that in our case,
\[
K_0 = \{(x, y) : 0 < |x| < 1\}.
\]
Theorem 5.1. There exists a neighborhood $K$ of the origin and a number $c > 0$ such that, for any $\varepsilon > 0$ and any pseudotrajectory $p_0, \ldots, p_m$ in $K \cap K_0$ that satisfies conditions (21) with $d = c\varepsilon$ there exists a point $r$ for which inequalities (1) are satisfied.

Proof. As above, in our proof we use the approach based on Lyapunov functions, but now one of the functions is modified. We take $V(q,p) = |p_y - q_y|$ and $W(q,p) = \frac{|p_x - q_x|}{|p_x|((1 - |p_x|)}$.

Clearly, these functions are nonnegative and continuous on $K_0 \times K_0$ and they vanish if their arguments coincide.

It is obvious that conditions (C1) – (C4) are satisfied (and we can take $\Delta_0(\varepsilon) = \varepsilon/2$ in condition (C1)).

In the following proof, we take $\delta = \Delta N = \Delta_0 N = \varepsilon/2N$ (the constant $N$ is chosen below) and $d = c\delta$ in condition (21).

First we take $c = 1$ and then make $c$ smaller preserving the same notation $c$; the same is done with the neighborhood $K$.

Our main goal is to check condition $W(\delta, \Delta, p_k, p_{k+1})$ for consecutive points $p_k, p_{k+1}$ of the pseudotrajectory considered for properly chosen $\delta$ and $\Delta$; after that, we apply Proposition 1 stated in Sec. 3.

To make the presentation shorter, let $p_k = p = (x, y)$ and $p_{k+1} = q = (x', y')$.

Thus, we may assume that $|x|, |x'|, |y|, |y'|$ are as small as we need.

First we claim that there exists a number $N > 0$ such that if $K$ is a small neighborhood of the origin, $p, q \in K$, and $\delta < 1$, then inclusions (C5) hold with $\Delta = N\delta$.

A point $(x + v, y + w)$ belongs to $P(\delta, p)$ if and only if $|v| \leq \delta|x|(1 - |x|)$ and $|w| \leq \delta$. Similar inequalities define the set $P(\delta, q)$.

Thus, to prove our claim, we have to show that there exists a number $N$ such that if $|v| \leq \delta|x|(1 - |x|)$ and $|w| \leq \delta$, then

$$|\frac{x + v}{2} - x'| < N\delta |x'| (1 - |x'|)$$

and

$$|(y + w)(1 + (x + v)^2) - y'| < N\delta.$$ 

In addition, we have to show that if $|v| \leq \delta|x'| (1 - |x'|)$ and $|w| \leq \delta$, then

$$|2(x' + v) - x| < N\delta |x| (1 - |x|)$$

and

$$\left| \frac{y' + w}{1 + 4(x' + v)^2} - y \right| < N\delta$$

(see the formula (20) for $f^{-1}$).

Let us prove our statement on the existence of a number $N$ for which the third of the above inequalities holds (the remaining inequalities are established using a similar reasoning).

We may assume that $|x| < 1/4$ and $\delta < 1$. Then it follows from (21) that

$$|x' - x/2| < dx^2 < \delta|x|/4 < |x|/4.$$
which implies that $|x'| < |x|$ and $|2x' - x| < \delta|x|/2$. In addition,

$$|v| \leq \delta|x'|(1 - |x'|) < \delta|x'| < \delta|x|$$

and

$$\delta|x|(1 - |x|) > \delta|x|/2.$$  

Combining these inequalities, we see that if $N = 3$, then

$$N\delta|x|(1 - |x|) > 3\delta|x|/2 > |2x' - x| + |v|,$$

as required.

In what follows, we take $\Delta = N\delta$ with a fixed $N$.

Now we check conditions (C6) and (C7). To simplify presentation, we assume that $x > 0$ (and $x$ is as small as we need).

We note that

$$f(T(\delta, p)) = \left\{ \left( \frac{x + v}{2}, y(1 + (x + v)^2) \right) : |v| \leq \delta x(1 - x) \right\}.$$  

Thus, the projection of $f(T(\delta, p))$ to the $x$ axis is the segment $[A^-, A^+]$, where

$$A^- = \frac{x}{2} - \frac{\delta x}{2}(1 - x) \quad \text{and} \quad A^+ = \frac{x}{2} + \frac{\delta x}{2}(1 - x).$$

At the same time, if $x' = x/2 + u$, then the projection of $P(\delta, q)$ to the $x$ axis is the segment $[B^+(u), B^-(u)]$, where

$$B^-(u) = \frac{x}{2} + u - \frac{\delta(x + 2u)}{2} \left( 1 - \frac{x}{2} - u \right),$$  

$$B^+(u) = \frac{x}{2} + u + \frac{\delta(x + 2u)}{2} \left( 1 - \frac{x}{2} - u \right),$$

and $|u| \leq dx^2$ (in the above formulas, we note that if $d$ and $x$ are small, then $x + 2u > 0$).

Since

$$B^+(0) = A^+ + \frac{\delta x^2}{4} \quad \text{and} \quad B^-(0) = A^- - \frac{\delta x^2}{4},$$

it is easy to understand that there exists a $c > 0$ (independent of $x$ and $\delta$) such that if $d \leq c\delta$ in (21), then

$$[A^+, A^-] \subset (B^-, B^+).$$

(23)

To complete the proof of condition (C6) and prove condition (C7), we note that

$$f(T(\Delta, p)) = \left\{ \left( \frac{x + v}{2}, y(1 + (x + v)^2) \right) : |v| \leq N\delta x(1 - x) \right\}.$$  

If $y' = y(1 + x^2) + u$, then the $y$ coordinate of any point of the set $Q(\delta, q)$ is either $y' - \delta$ or $y' + \delta$.

Let us represent

$$y' + \delta - y(1 + (x + v)^2) = \delta - y(2xv + v^2) + u.$$  

Since $|v| \leq N\delta x(1 - x)$, we have the estimates

$$|xv| \leq N\delta x^2(1 - x), \quad v^2 \leq N^2\delta^2 x^2(1 - x)^2.$$  

If the neighborhood $K$ is small (so that $|x|$ and $|y|$ are small enough), we conclude from the inequality $|u| \leq c\delta x^2$ (with $c$ fixed above) that

$$y' + \delta - y(1 + (x + v)^2) > 0, \quad |v| \leq N\delta x(1 - x),$$

which completes the proof of condition (C6) and proves condition (C7), because

$$\frac{x + v}{2} \leq \frac{x}{2} + \frac{\delta x}{2}(1 - x) = A^-.$$
for small $\delta$, i.e., $f(T(\Delta, p))$ does not intersect the “upper” component of $Q(\delta, q)$. A similar reasoning is applicable to the “lower” component. This proves condition (C7) (and, combined with inclusion (23), condition (C6)).

It remains to check conditions (C8) and (C9). Let us start with condition (C9). Consider a point $r = (x + v, y + \delta)$ of the “upper” component of $Q(\delta, p)$. In this case, $|v| \leq \delta x(1 - x)$. The $y$ component of the point $f(r)$ is

$$(y + \delta)(1 + (x + v)^2).$$

If $y' = y(1 + x^2) + u$, then the projection of the set $P(\delta, q)$ to the $y$ axis is the segment $D = [y(1 + x^2) + u - \delta, y(1 + x^2) + u + \delta]$.

Let us represent

$$(y + \delta)(1 + (x + v)^2) - y(1 + x^2) - u - \delta =$$

$$= \delta(x + v)^2 + y(2xv + v^2) - u.$$

The estimates

$$\delta(x + v)^2 \geq \delta x^2(1 - \delta)^2$$

and

$$|y(2xv + v^2)| \leq |y|(2\delta x^2 + \delta^2 x^2) \leq \delta x^2/2$$

(which is valid if $|y|$ and $\delta$ do not exceed a small value independent of $x$) imply that there exists a constant $c$ (independent of $x$, $y$, and $\delta$) such that if $u \leq c\delta x^2$, then the projection of the point $f(r)$ to the $y$ axis does not belong to the segment $D$.

Applying a similar reasoning to points $r = (x + v, y - \delta)$, we complete the proof of condition (C9). Estimates of a similar form prove condition (C8).

To complete the proof of Theorem 5.1, we take into account the equality $d = c\delta$ and relations (22).

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Received xxxx 20xx; revised xxxx 20xx.

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