Distinguishing Multi-Partite States
by Local Measurements

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Abstract: We analyze the distinguishability norm on the states of a multi-partite system,
defined by local measurements. Concretely, we show that the norm associated to
a tensor product of sufficiently symmetric measurements is essentially equivalent to
a multi-partite generalisation of the non-commutative $\ell_2$-norm (aka Hilbert-Schmidt
norm): in comparing the two, the constants of domination depend only on the number
of parties but not on the Hilbert spaces dimensions.

We discuss implications of this result on the corresponding norms for the class of all
measurements implementable by local operations and classical communication (LOCC),
and in particular on the leading order optimality of multi-party data hiding schemes.

1. Distinguishability Norms

The task of distinguishing quantum states from accessible experimental data is at the
heart of quantum information theory, appearing right at its historical beginnings – see
[10, 11, and 15] for general reference. Indeed, the special case on which we are focus-
sing in this paper, the discrimination of two states, is the generalisation of hypothesis
testing in classical statistics. There, the optimal discrimination between two hypotheses,
modelled as (for simplicity: discrete) probability distributions $P_0$ and $P_1$, with prior
probabilities $q$ and $1 - q$, respectively, is given by the maximum likelihood rule [7]. The
minimum error probability is thus given by

$$\Pr\{\text{error}\} = \frac{1}{2} \left( 1 - \|qP_0 - (1 - q)P_1\|_1 \right),$$

with the usual $\ell_1$-norm $\|\Delta\|_1 = \sum_{x \in \mathcal{X}} |\Delta_x|$. In this paper, we shall denote by the same symbol its non-commutative generalisation
$\|\Delta\|_1 = \text{Tr} |\Delta|$, i.e. the sum of the singular values of $\Delta$, also known as trace norm.

Owing to the particular role played by measurement in quantum mechanics, however,
any restriction on the set of available measurements leads to a specific norm on density
operators: any decision in the discrimination task must be based on measurement results. Specifically, let the two hypotheses be two quantum states (density operators) $\rho_0$ and $\rho_1$ on some Hilbert space $\mathcal{H}$, with prior probabilities $q$ and $1 - q$, respectively. A generic measurement $M$, i.e. a positive operator valued measure (POVM, aka partition of unity), is given by positive semidefinite operators $M_x \geq 0$, s.t. $\sum_{x \in \mathcal{X}} M_x = \mathbb{1}$.

(In this paper, POVMs will generally be discrete and Hilbert spaces will always be of finite dimension. With suitable adaptations to the proofs, however, our results carry over to general POVMs and infinite dimension.) The Born rule for measurements postulates that the state $\rho_i$ generates a distribution $P_i$ on the outputs of the measurement:

$$P_i(x) = \text{Tr} \rho_i M_x,$$

and hence the minimum error probability in any decision based on $i$ is

$$\Pr\{\text{error}\} = \frac{1}{2} \left( 1 - \| q \rho_0 - (1 - q) \rho_1 \|_1 \right)$$

$$= \frac{1}{2} \left( 1 - \sum_{x \in \mathcal{X}} |\text{Tr}(q \rho_0 - (1 - q) \rho_1) M_x| \right)$$

$$= \frac{1}{2} \left( 1 - \| q \rho_0 - (1 - q) \rho_1 \|_M \right).$$

Observe that $\| \Delta \|_M = \sum_{x \in \mathcal{X}} |\text{Tr} \Delta M_x|$ is a seminorm: it is non-negative, homogeneous and obeys the triangle inequality. However, it may vanish on $\Delta \neq 0$. This is excluded if the measurement $M$ is informationally complete, meaning that the operators $M_x$ span all the operators over the Hilbert space: $\text{span}\{M_x : x \in \mathcal{X}\} = \mathcal{B}(\mathcal{H})$.

If not one but a whole set $\mathcal{M}$ of measurements is given, from which the experimenter may choose, we have an equally natural (semi-)norm

$$\| \Delta \|_\mathcal{M} = \sup_{M \in \mathcal{M}} \| \Delta \|_M,$$

in terms of which the minimum error probability is expressed as $\frac{1}{2} \left( 1 - \| q \rho_0 - (1 - q) \rho_1 \|_\mathcal{M} \right)$. These norms, under certain restrictions of interest on the measurement, will be the object of study in the present paper, and in particular their comparison with the trace norm, which by a classic observation of Holevo [11] and Helstrom [10] equals the distinguishability norm under the set of all possible measurements:

$$\| \Delta \|_{\text{ALL}} = \sup_{M \text{ any POVM}} \| \Delta \|_M = \| \Delta \|_1 = \text{Tr} |\Delta|.$$