On the Structure of Stationary and Axisymmetric Metrics

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Abstract
We study the structure of stationary and axisymmetric metrics solving the vacuum Einstein equations of General Relativity in four and higher dimensions, building on recent work in hep-th/0408141. We write the Einstein equations in a new form that naturally identifies the sources for such metrics. The sources live in a one-dimensional subspace and the entire metric is uniquely determined by them. We study in detail the structure of stationary and axisymmetric metrics in four dimensions, and consider as an example the sources of the Kerr black hole.
1 Introduction

In recent years there have been a great deal of attention towards research in black holes in higher-dimensional space-times. For four-dimensional asymptotically flat space-times the Uniqueness Theorems states that only one type of black holes exists for a given set of asymptotic charges. In particular, for four-dimensional gravity without matter, the Kerr black hole \[1\] is the unique black hole solution for a given mass and angular momentum \[2, 3, 4, 5\]. Contrary to this, we have learned that for more than four dimensions there are generically many available phases of black holes.\(^1\) For Kaluza-Klein spaces, i.e. Minkowski-space times a circle, the phase structure is very rich with interesting phase transitions between different kinds of black holes, and in some cases even an uncountable number of different phases are available \[9, 10\]. For five-dimensional asymptotically flat space-times

\(^1\)Notice though the uniqueness theorems of \[6, 7, 8\] for higher-dimensional static black holes in flat space-times.
without matter we have in addition to the Myers-Perry rotating black hole solution [11] also the recently discovered rotating black ring solution of Emparan and Reall [12].

To understand the phase structure of black holes in higher dimensions it is important to search for new black hole solutions. But the non-linearity of General Relativity makes it in general very hard to find new solutions. However, a class of metrics for which the Einstein equations simplify considerably are the stationary and axisymmetric metrics. These are $D$-dimensional metrics possessing $D-2$ mutually commuting Killing vector fields. Among black hole metrics of this type are the Kerr rotating black hole in four dimensions [1], the Myers-Perry rotating black hole in five dimensions [11] and the rotating black ring [12].

In [14] a canonical form for stationary and axisymmetric metrics were found for which the Einstein equations effectively reduce to a differential equation on an axisymmetric $D-2$ by $D-2$ matrix field living in three dimensional flat space. The results of [14] generalizes those of Papapetrou [15, 16] for stationary and axisymmetric metrics in four dimensions. Moreover, the results of [14] also generalizes the work of [17] on the Weyl-type metrics, which are $D$-dimensional metrics with $D-2$ mutually commuting and orthogonal Killing vector fields. Ref. [18] is the higher-dimensional generalization of the analysis of Weyl [17] on static and axisymmetric metrics in four dimensions.

In [14] it is furthermore shown that any stationary and axisymmetric solution have a certain structure of its sources, called the rod-structure. The rod-structure is connected to the three dimensional flat space that the reduced Einstein equations can be seen to live in. In this three-dimensional flat space, the sources for the metric consist of thin rods lying along a certain axis of the space. To each of these rods can be attached a direction in the $D-2$ dimensional vector space that the Killing vector fields are spanning. The analysis of stationary and axisymmetric solutions in terms of its rod-structure generalizes the analysis of [18] for Weyl-type metrics, which again generalizes the analysis of four-dimensional Weyl-type metrics (see for example [21] and references therein).

In this paper we find a new formulation of the reduced form of the Einstein equations found in [14]. In the new formulation the sources for a given stationary and axisymmetric solution are naturally identified. These sources consist of two $D-2$ by $D-2$ matrix valued functions living on an axis in the above-mentioned three-dimensional flat space that the reduced version of the Einstein equations lives in. It is argued that these sources uniquely determines the solution. One can therefore say that the problem of finding a stationary and axisymmetric solutions is reduced to a one-dimensional problem, i.e. the problem of finding the appropriate sources for the solution.

We examine the general properties of the sources, which are intimately connected to the rod-structure of a solution. As an important part of this, we describe how to continue
the sources across rod endpoints. It was conjectured in [14] that a given rod-structure corresponds to a unique stationary and axisymmetric metric. The idea being that the rod-structure for a solution contains all information about the solution. However, we do not find in this paper sufficient constraints to fix the sources in terms of the rod-structure. We leave this as a problem for future research.

For completeness we note here that there have been several recent interesting developments in the study of stationary and axisymmetric solutions in higher dimensions. In [22, 23] a new tool to categorize stationary and axisymmetric solutions by drawing the so-called Weyl Card diagrams have been developed. Moreover, solution generating techniques for five-dimensional solutions have been explored in [24, 25, 26]. In particular, in [26] it is shown using [14] that Einstein equations are integrable for stationary and axisymmetric metrics, and this is subsequently employed in rederiving the Myers-Perry five-dimensional rotating black hole.

In Section 2 we review the results of [14] that we build on in this paper. In Section 3 we find the new formulation for the Einstein equations, and we identify the sources for the stationary and axisymmetric metrics. We furthermore examine the properties of these sources. In Section 4 we prove in the four-dimensional case that stationary and axisymmetric solutions are uniquely determined by their sources. We also comment on the higher-dimensional cases. In Section 5 we find a way to relate the behavior of the sources on each side of a rod endpoint. We prove that we in principle can make a full determination of the sources on one side of a rod endpoint by knowing the sources on the other side of the endpoint. In Section 6 we consider the asymptotic behavior of the sources for asymptotically flat space-times. In Section 7 we consider the example of the Kerr black hole, find its matrix-valued potential and its sources, and we consider the behavior of the sources near the endpoints of the rods. In Section 8 we present our conclusions.

2 Review of stationary and axisymmetric solutions

We review in the following some of the most important results of [14] for use in this paper. As mentioned in the Introduction the analysis and results of [14] builds on and generalizes results of Refs. [17, 15, 16, 18]. See the Introduction or [14] for details on this.

2.1 Canonical metric and equations of motion

We consider in this section $D$-dimensional stationary and axisymmetric solutions of the vacuum Einstein equations. These are $D$-dimensional Ricci-flat manifolds with $D-2$ commuting linearly independent Killing vector fields $V_{(i)}$, $i = 1, ..., D-2$. For such solutions we can find coordinates $x^i$, $i = 1, ..., D-2$, along with $r$ and $z$, so that

$$V_{(i)} = \frac{\partial}{\partial x^i},$$

(2.1)
for $i = 1, \ldots, D - 2$, and such that the metric is of the canonical form
\begin{equation}
\text{ds}^2 = \sum_{i,j=1}^{D-2} G_{ij} \text{d}x^i \text{d}x^j + e^{2\nu} (\text{d}r^2 + \text{d}z^2),
\end{equation}
with
\begin{equation}
r = \sqrt{|\det(G_{ij})|}.
\end{equation}
For a given $r$ and $z$ we can view $G_{ij}$ as a $D - 2$ by $D - 2$ real symmetric matrix with $G^{ij}$ as its inverse. With this, the vacuum Einstein equations $R_{\mu\nu} = 0$ for the metric (2.2) with the constraint (2.3) can be written as
\begin{equation}
G^{-1} \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) G = (G^{-1} \partial_r G)^2 + (G^{-1} \partial_z G)^2,
\end{equation}
\begin{equation}
\partial_r \nu = -\frac{1}{2r} + \frac{r}{8} \text{Tr} \left( (G^{-1} \partial_r G)^2 - (G^{-1} \partial_z G)^2 \right),
\partial_z \nu = \frac{r}{4} \text{Tr} \left( G^{-1} \partial_r G G^{-1} \partial_z G \right).
\end{equation}
From this one sees that one can find stationary and axisymmetric solutions of the vacuum Einstein equations by finding a solution $G(r, z)$ of the equation (2.4). Given $G(r, z)$ one can then subsequently always find a solution $\nu(r, z)$ to (2.5) since these equations are integrable.

We can make a further formal rewriting of (2.4) by recognizing that the derivatives respects the symmetries of a flat three-dimensional Euclidean space with metric
\begin{equation}
\text{ds}^2_3 = \text{d}r^2 + r^2 \text{d}\gamma^2 + \text{d}z^2.
\end{equation}
Here $\gamma$ is an angular coordinate of period $2\pi$. Therefore, if we define $\nabla$ to be the gradient in three-dimensional flat Euclidean space, we can write (2.4) as
\begin{equation}
G^{-1} \nabla^2 G = (G^{-1} \nabla G)^2.
\end{equation}
For use in the paper we elaborate here a bit more on the precise meaning of the formula (2.7). Using the auxiliary coordinate $\gamma$ we can define the Cartesian coordinates $\sigma_1$ and $\sigma_2$ as
\begin{equation}
\sigma_1 = r \cos \gamma, \quad \sigma_2 = r \sin \gamma.
\end{equation}
Clearly, we have then that $(\sigma_1, \sigma_2, z)$ are Cartesian coordinates for the flat three-dimensional Euclidean space defined in (2.6) since the metric in these coordinates is
\begin{equation}
\text{ds}^2_3 = d\sigma_1^2 + d\sigma_2^2 + dz^2.
\end{equation}
We write the Cartesian components of a three-dimensional vector as $\vec{w} = (w_1, w_2, w_z)$, and the gradient used in (2.7) is then given by
\begin{equation}
\nabla = (\partial_1, \partial_2, \partial_z),
\end{equation}
where $\partial_1 = \partial / \partial \sigma_1$ and $\partial_2 = \partial / \partial \sigma_2$.

\footnote{Two further assumptions are required as premises in the derivation (see [14] for explanation): (i) The tensor $V_{(i)}^{(\mu_1)} V_{(i)}^{(\mu_2)} \cdots V_{(D-2)}^{(\mu_{D-2})} D^\rho V_{(i)}^{(\rho)}$ vanishes at least one point of the manifold for a given $i = 1, \ldots, D - 2$. (ii) $\det(G_{ij})$ is non-constant on the manifold.}

\footnote{It is important to remark that $\gamma$ is not an actual physical variable for the solution (2.2), but rather an auxiliary coordinate that is useful for understanding the structure of Eqs. (2.4).}
2.2 Rod-structure of a solution

If we consider the condition (2.3) we see that \( \det(G(0, z)) = 0 \). Therefore, the dimension of the kernel of \( G(0, z) \) must necessarily be greater than or equal to one for any \( z \), i.e. \( \dim(\ker(G(0, z))) \geq 1 \).

In [14] we learned that in order to avoid curvature singularities it is a necessary condition on a solution that \( \dim(\ker(G(0, z))) = 1 \), except for isolated values of \( z \). We therefore restrict ourselves to solutions where this applies. Naming the isolated \( z \)-values as \( a_1, ..., a_N \), we see that the \( z \)-axis splits up into the \( N + 1 \) intervals \( [-\infty, a_1], [a_1, a_2], ..., [a_{N-1}, a_N], [a_N, \infty] \). For a given stationary and axisymmetric solution we thus have that the \( z \)-axis at \( r = 0 \) is divided into these intervals, called rods.

Consider now a specific rod \([z_1, z_2]\). From [14] we know that we can find a vector

\[ v = v^i \frac{\partial}{\partial x^i}, \]

such that

\[ G(0, z)v = 0, \]

for \( i = 1, ..., D - 2 \) and \( z \in [z_1, z_2] \). This vector \( v \) is called the direction of the rod \([z_1, z_2]\).

We note that if \( G_{ij}v^iv^j/r^2 \) is negative (positive) for \( r \to 0 \) we say the rod \([z_1, z_2]\) is time-like (space-like). For a space-like rod \([z_1, z_2]\) we clearly have a potential conical singularity for \( r \to 0 \) when \( z \in [z_1, z_2] \). However, if \( \eta \) is a coordinate made as a linear combination of \( x^i, i = 1, ..., D - 2 \), with

\[ \frac{\partial}{\partial \eta} = v^i \frac{\partial}{\partial x^i}, \]

then we can cure the conical singularity at the rod by requiring the coordinate \( \eta \) to have the period

\[ \Delta \eta = 2\pi \lim_{r \to 0} \frac{\sqrt{r^2v^i v^j}}{G_{ij}v^i v^j}, \]

for \( z \in [z_1, z_2] \).

3 New formulation and identification of sources

In this section we rewrite the vacuum Einstein equations in a way that gives a natural identification of the sources for a given stationary and axisymmetric solution. The purpose of identifying such sources is that one can hope to reduce the problem of finding new solutions to the problem of finding the sources for new solutions. We comment on the general philosophy behind the introduction of these sources in the end of this section.
3.1 New formulation

We begin by defining the field $\vec{C}(r, z)$ by

$$\vec{C} = G^{-1} \vec{\nabla} G \ .$$  \hspace{1cm} (3.1)

Thus, $C_r = G^{-1} \partial_r G$ and $C_z = G^{-1} \partial_z G$. With this definition, the equation (2.7) for $G(r, z)$ is equivalent to the equation $\vec{\nabla} \cdot \vec{C} = 0$ for $r > 0$. However, for $r = 0$ we can have sources for $\vec{C}$. Therefore, we require $\vec{C}(r, z)$ to obey the equation

$$\vec{\nabla} \cdot \vec{C} = 4\pi \delta^2(\sigma) \rho(z) ,$$  \hspace{1cm} (3.2)

i.e. that $r^{-1} \partial_r(r C_r) + \partial_z C_z = 4\pi \delta^2(\sigma) \rho(z)$. Here $\delta^2(\sigma) \equiv \delta(\sigma_1) \delta(\sigma_2)$, where $\sigma_1$ and $\sigma_2$ are the Cartesian coordinates defined in (2.8). Thus, the delta-function $\delta^2(\sigma)$ expresses that we have sources for $\vec{C}(r, z)$ at $r = 0$. The $D^{-2}$ by $D^{-2}$ matrix-valued function $\rho(z)$ in (3.2) then parameterizes the sources for $\vec{C}(r, z)$ at $r = 0$.

Interestingly, the equation (3.2) is now a linear first order differential equation which is equivalent to the non-linear second order equation (2.7). However, the price of introducing $\vec{C}$ is that $\vec{C}$ in addition should obey the non-linear equation

$$\vec{\nabla} \times \vec{C} + \vec{C} \times \vec{C} = 0 \ ,$$  \hspace{1cm} (3.3)

i.e. that $\partial_r C_z + \partial_z C_r + [C_r, C_z] = 0$. The equation (3.3) is found from $\vec{\nabla} \times (G^{-1} \vec{\nabla} G) = \vec{\nabla} G^{-1} \times \vec{\nabla} G = -G^{-1} \vec{\nabla} GG^{-1} \times \vec{\nabla} G = -(G^{-1} \vec{\nabla} G) \times (G^{-1} \vec{\nabla} G)$. Note that there are no source terms to the equation (3.3) so it also holds for $r = 0$. In conclusion, we have exchanged the non-linear equation second order equation (2.7) with the two first order equations (3.2) and (3.3).

We see from Eq. (3.2) that $C_{ij}(r, z)$ resembles an electric field with $\rho_{ij}(z)$ being the charge density at $r = 0$. However, the $\vec{C} \times \vec{C}$ term of Eq. (3.3) introduces terms that mix the different components in such a way that $\vec{C}(r, z)$ cannot be obtained with linear superposition of fields.

The constraint (2.3) on the determinant of $G$ is equivalent to demanding

$$\text{Tr}(\vec{C}) = 2\vec{\nabla} \log r \ ,$$  \hspace{1cm} (3.4)

i.e. that $\text{Tr}(C_r) = 2/r$ and $\text{Tr}(C_z) = 0$. Therefore, if we take the trace of (3.2) we get, using (3.4) and $\vec{\nabla}^2 \log r = 2\pi \delta^2(\sigma)$, the important identity

$$\text{Tr}(\rho) = 1 \ ,$$  \hspace{1cm} (3.5)

which holds for all $z$. Eq. (3.5) is a generalization of the requirement on the Generalized Weyl solutions of [18] that the total rod density is constant for all $z$. As shown in [14], we

\footnote{Note our conventions for the cross product is such that the epsilon symbol has $\epsilon^{12z} = 1$ which in $(r, \gamma, z)$ coordinates means that $\epsilon^{rz} = 1$.}
get from this and demanding absence of singularities for \( r \to 0 \) that there is precisely one rod present for any given value of \( z \) (except for isolated values of \( z \)).

Note that we can express the equations \( 2.26 \) for \( \nu(r, z) \) in terms of \( \vec{C}(r, z) \) as
\[
\partial_r \nu = -\frac{1}{2r} + \frac{r}{8} \text{Tr}(C_r^2 - C_z^2), \quad \partial_z \nu = \frac{r}{4} \text{Tr}(C_r C_z).
\] (3.6)

These equations ensure that we can find \( \nu(r, z) \) from \( \vec{C}(r, z) \) alone.

It is interesting to consider how many different \( G(r, z) \) that corresponds to the same \( \vec{C}(r, z) \). The answer is provided by the following lemma:

**Lemma 3.1** Let \( G_1(r, z) \) and \( G_2(r, z) \) be given such that \( G_1^{-1} \vec{\nabla} G_1 = G_2^{-1} \vec{\nabla} G_2 \), with \( G_1 \) and \( G_2 \) being invertible for \( r \neq 0 \). Then we have that \( G_2 G_1^{-1} \) is a constant matrix.

**Proof:** Define \( M = G_2 G_1^{-1} \). Then
\[
G_2^{-1} \vec{\nabla} G_2 = (MG_1)^{-1} \vec{\nabla} (MG_1) = G_1^{-1} M^{-1}((\vec{\nabla} M) G_1 + M \vec{\nabla} G_1) = G_1^{-1} M^{-1}(\vec{\nabla} M) G_1 + G_1^{-1} \vec{\nabla} G_1.
\]

We see from this that \( \vec{\nabla} M = 0 \) for \( r \neq 0 \). This means that \( M \) is a constant matrix. \( \square \)

The above lemma means that given \( \vec{C}(r, z) \) we can determine the corresponding \( G(r, z) \) up to a constant matrix, hence if we for example require certain asymptotic boundary conditions on \( G(r, z) \) then we can determine \( G(r, z) \) uniquely for a given \( \vec{C}(r, z) \).

### 3.2 Defining a potential for \( \vec{C} \)

We define the \( D - 2 \) by \( D - 2 \) matrix-valued function \( A(r, z) \) by
\[
C_r = -\frac{1}{r} \partial_z A, \quad C_z = \frac{1}{r} \partial_r A.
\] (3.7)

This definition is meaningful for \( r > 0 \) since \( r^{-1} \partial_r (r C_r) + \partial_z C_z = 0 \) implies the integrability condition \( \partial_r \partial_z A = \partial_z \partial_r A \). Hence, for a given \( \vec{C} \) we can find \( A \) such that \( 3.7 \) is fulfilled for \( r > 0 \). The value of \( A \) for \( r = 0 \) is then found by demanding continuity of \( A \) at \( r = 0 \).

Clearly, \( A(r, z) \) is a potential for \( \vec{C}(r, z) \). If we define the field \( \vec{A}(r, z) \) by \( A_r(r, z) = A_z(r, z) = 0 \) and \( A_\gamma(r, z) = A(r, z) \) we can furthermore write that \( \vec{C} = \vec{\nabla} \times \vec{A} \). Therefore, the potential \( A(r, z) \) can be seen as a component of the vector potential \( \vec{A}(r, z) \).\(^8\)

Note also that the definition of \( A \) from \( \vec{C} \) is ambiguous since if one uses the vector potential \( \vec{A}' = \vec{A} + \vec{\nabla} \alpha \) for a given \( \alpha(r, \gamma, z) \) one gets the same \( \vec{C} \). However, demanding \( \partial_\gamma A' = A'_z = 0 \) and that \( \partial_\gamma A' = 0 \) we get that the definition \( 3.7 \) for \( A(r, z) \) is only ambiguous up to an additive constant.

We can now use Gauss’ law for \( \vec{C} \) to derive an important identity. Consider the cylindrical volume \( V = \{ r \leq r_0, \; z_1 \leq z \leq z_2 \} \). Then we have
\[
\frac{1}{2\pi} \int_V \vec{\nabla} \cdot \vec{C} dV = \frac{1}{2\pi} \int_{\partial V} \vec{n} \cdot \vec{C} dS\]
\[= -\int_{r=0}^{r_0} \partial_r A|_{z=z_2} dr + \int_{r=0}^{r_0} \partial_r A|_{z=z_2} dr - \int_{z=z_1}^{z_2} \partial_z A|_{r=r_0} dz = A(0, z_1) - A(0, z_2).
\] (3.8)

\(^8\)Note that in Cartesian coordinates \( \vec{A} = (A_1, A_2, A_3) = (A \partial_1, A \partial_2, A \partial_3, 0) \).
Using (3.2) we then get\(^9\)
\[
\rho(z) = -\frac{1}{2} \partial_z A(0, z) .
\] (3.9)

Therefore, we get from (3.7) that
\[
\rho(z) = \frac{1}{2} \lim_{r \to 0} r C_r .
\] (3.10)

We see then that \(2\rho = \lim_{r \to 0} r G^{-1} \partial_r G\) which means that one can compute \(\rho\) from the metric directly.

We have thus solved (3.2) by introducing the potential \(A(r, z)\) via (3.7) and demanding (3.9). However, we still have (3.3) which for \(A(r, z)\) becomes
\[
r \partial_r \left( \frac{1}{r} \partial_r A \right) + \partial_z^2 A + \frac{1}{r} [\partial_r A, \partial_z A] = 0 .
\] (3.11)

Thus, (2.7) for \(G(r, z)\) have now been translated into (3.11) for \(A(r, z)\). The constraint (2.3), which is equivalent to (3.4), translates for the potential \(A(r, z)\) to
\[
\text{Tr}(A) = -2z ,
\] (3.12)

up to an additive constant.

### 3.3 Behavior of a solution near \(r = 0\)

Equation (3.10) shows that \(\rho(z)\) characterizes the behavior of \(C_r(r, z)\) for \(r \to 0\). However, it will be clear in the following that we in addition need to take into account the behavior of \(C_z(r, z)\) for \(r \to 0\). We define therefore the \(D - 2\) by \(D - 2\) matrix-valued function \(A(z)\) by
\[
A(z) = C_z(0, z) .
\] (3.13)

We comment on the finiteness of \(A(z)\) below. From (3.4) it is easily seen that
\[
\text{Tr}(A) = 0 .
\] (3.14)

Moreover, from considering the \(r \to 0\) limit of (3.3), we get
\[
\rho' = [\rho, A] .
\] (3.15)

Therefore, if one knows \(A(z)\) it is possible to find \(\rho(z)\), at least up to an additive constant matrix.

Consider now a rod \([z_1, z_2]\), as defined in Section 2.2. Without loss of generality we consider the rod \([z_1, z_2]\) to have direction \(v = (1, 0, ..., 0)\). We consider what happens for general \(v\) below. Using the analysis of the metric near a rod in [13] we see then that for

\(^9\)The identity (3.9) can also be derived using the Cartesian coordinates (2.8). Here one uses \([\partial_1, \partial_2] \gamma = 2\pi \delta^2(\sigma)\). Note that just as for the Dirac monopole in electrodynamics one encounters the Dirac string at \(r = 0\). The integral version of the derivation (3.8) instead avoids this issue.
$r \to 0$ and $z \in ]z_1, z_2[$ we have that $G_{11} = O(r^2)$, $G_{1i} = G_{i1} = O(r^2)$ and $G_{ij} = O(1)$ with $i, j = 2, ..., D - 2$, while for the inverse we have that $G^{11} = O(r^{-2})$, $G^{1i} = G^{i1} = O(1)$ and $G^{ij} = O(1)$ with $i, j = 2, ..., D - 2$. Moreover, writing $G_{11} \simeq \pm e^{\lambda(z)}r^2$, we have that $G^{11} \simeq \pm e^{-\lambda(z)}r^{-2}$ for $r \to 0$ and $z \in ]z_1, z_2[$. From (3.11) this gives then for $r \to 0$ and $z \in ]z_1, z_2[$ that $C_{r,11} \simeq 1/r$, $C_{r,1i} = O(r^{-1})$, $C_{r,i1} = O(r)$ and $C_{r,ij} = O(r)$ with $i, j = 2, ..., D - 2$, while $C_{z,11} \simeq \lambda'(z)$, $C_{z,1i} = O(1)$, $C_{z,i1} = O(r^2)$ and $C_{z,ij} = O(1)$ with $i, j = 2, ..., D - 2$.

Using then (3.10) and (3.13) we conclude that both $\rho(z)$ and $\Lambda(z)$ are well-defined and smooth for $z \in ]z_1, z_2[$. Moreover, for $\rho(z)$ we get that $\rho_{11} = 1$ and that $\rho_{ij} = 0$ with $i = 2, ..., D - 2$ and $j = 1, ..., D - 2$ for $z \in ]z_1, z_2[$, while for $\Lambda(z)$ we get that $\Lambda_{11} = \lambda'(z)$ and $\Lambda_{ij} = 0$ with $i = 2, ..., D - 2$ for $z \in ]z_1, z_2[$. Note that we can infer from this that $\text{Tr}(\rho \Lambda) = \lambda'$. This is used below.

Consider now instead a given rod $[z_1, z_2]$ of direction $v$, with $v$ being arbitrary. Clearly, we can find $\lambda(z)$ such that $v^T G v = \pm \lambda(z) r^2 v^T v$. Using now the above analysis along with Appendix A we can conclude for $\rho(z)$ with $z \in ]z_1, z_2[$ that

$$\rho v = v, \quad \text{Tr}(\rho) = 1, \quad \rho^2 = \rho, \quad (3.16)$$

and that there exists linearly independent vectors $w_1, ..., w_{D-3}$ independent of $z$ such that

$$\rho^T w_i = 0, \quad i = 1, ..., D - 3. \quad (3.17)$$

We see from (3.16) that $v$ is an eigenvector for $\rho$ with eigenvalue one. Concerning the properties involving $\Lambda$, we have for $z \in ]z_1, z_2[$ that

$$\Lambda v = \lambda' v, \quad \text{Tr}(\rho \Lambda) = \lambda'. \quad (3.18)$$

Thus, $v$ is also an eigenvector for $\Lambda(z)$. If we consider $\nu(r, z)$ for $r \to 0$, we see from (2.8) that $\partial_r \nu \simeq -(1 - \text{Tr}(\rho^2))/(2r)$. Using now (3.16) we see that $\text{Tr}(\rho^2) = 1$ and hence that $\nu(0, z)$ is well-defined for $z \in ]z_1, z_2[$. If we instead consider $\partial_z \nu$ for $r \to 0$ we get from (2.5) and (3.18) that

$$\partial_z \nu(0, z) = \frac{1}{2} \text{Tr}(\rho \Lambda) = \frac{1}{2} \lambda'(z), \quad (3.19)$$

for $z \in ]z_1, z_2[$. Integrating this, we get $2\nu(0, z) = \lambda(z) + \text{constant}$. This ensures that for space-like rods we can remove the potential conical singularity by choosing a suitable periodicity of the corresponding coordinate, while for time-like rods it ensures the temperature of the corresponding horizon is constant on the horizon. We see furthermore from (3.19) that we can compute $\nu(0, z)$ for all $z$ by knowing $\rho(z)$ and $\Lambda(z)$, and demanding continuity of $\nu(0, z)$.

**General philosophy**

The general philosophy of introducing the sources $\rho(z)$ and $\Lambda(z)$ is that all information about the solution is stored in these two matrix-valued function. This is shown explicitly
in the four-dimensional case in Section 4. Thus, we have reduced the relevant physics to these two matrix-valued functions living on a one-dimensional subspace. In particular, all the physical quantities like the mass, angular momenta, temperature, entropy, quadrupole moment, etc. can be obtained from \( \rho(z) \) and \( \Lambda(z) \) directly. We show this explicitly for the most important asymptotic quantities in Section 6.

The idea is moreover that the sources \( \rho(z) \) and \( \Lambda(z) \) should be uniquely determined by the rod-structure, i.e. by specifying the division of the \( z \)-axis into separate intervals, along with specifying the direction of these rods in the linear vector space of Killing vector fields. This would be a generalization of the solution generating technique for Weyl solutions \( [17, 18] \). In this paper we do not go all the way to determine \( \rho(z) \) and \( \Lambda(z) \) from the rod-structure, but in addition to the steps taken in this section we take a further important step in this direction in Section 5.

4 On uniqueness of \( A(r, z) \) given \( \rho(z) \) and \( \Lambda(z) \)

The purpose of this section is to argue that stationary and axisymmetric solutions of the vacuum Einstein equations are uniquely determined by the sources \( \rho(z) \) and \( \Lambda(z) \) introduced in Section 3. We address this here by arguing that the potential \( A(r, z) \) introduced in Section 3 can be uniquely determined by \( \rho(z) \) and \( \Lambda(z) \).

We prove in this section in detail that \( A(r, z) \) is uniquely determined by the two matrix-valued functions \( \rho(z) \) and \( \Lambda(z) \) in four dimensions. This means that we can reduce the question of finding a solution for \( A(r, z) \) to finding the two functions \( \rho(z) \) and \( \Lambda(z) \) that lives on a one-dimensional subspace. We comment on the proof for arbitrary dimensions in the end.

The general idea is to expand \( A(r, z) \) around \( r = 0 \). The expansion is

\[
A(r, z) = \sum_{n=0}^{\infty} A_n(z) r^{2n}. \tag{4.1}
\]

Since the behavior of \( A(r, z) \) is singular around rod endpoints we assume \( z_1 < z < z_2 \) where \([z_1, z_2] \) is one of the rods of the solution. We get then from (3.11)

\[
A''_0 = 2[A'_0, A_1], \quad A_2 + \frac{1}{2}[A_2, A'_0] = -\frac{1}{8} A''_1 - \frac{1}{4}[A_1, A'_1], \tag{4.2}
\]

\[
A_{n+1} + \frac{1}{2n}[A_{n+1}, A'_0] = -\frac{1}{4n(n+1)} A''_n - \sum_{k=1}^{n} \frac{n-l+1}{2n(n+1)} [A_{n-k+1}, A'_k]. \tag{4.3}
\]

Note that \( A'_0 = -2\rho \) and \( A_1 = \frac{1}{8} \Lambda \). Therefore, the basic idea in the following is to show that we can find all \( A_n \) for \( n \geq 2 \) from \( A'_0 \) and \( A_1 \).

We now restrict ourselves to four dimensions. We can assume without loss of generality that we have a rod \([z_1, z_2] \) in the direction

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.4}
\]
Define
\[ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]
(4.5)

These three matrices have the properties
\[ \alpha^2 = \alpha, \quad \beta^2 = 0, \quad \gamma^2 = 0, \quad \alpha \beta = \beta \alpha = 0, \quad \alpha \gamma = 0, \quad \gamma \alpha = \gamma, \quad \beta \gamma = \alpha, \quad \gamma \beta = I - \alpha, \quad [\alpha, \beta] = \beta, \quad [\alpha, \gamma] = -\gamma, \quad [\beta, \gamma] = 2\alpha - I. \]
(4.6)

From the properties of \( \rho \) of Section 3.3 we see that it should have the form \( \rho(z) = \alpha + h(z)\beta \)
where \( h(z) \) is a function. Moreover, the most general form of \( \Lambda \) is \( \Lambda(z) = \lambda'(z)(2\alpha - I) + (h'(z) + 2h(z)\lambda'(z))\beta \) where \( \lambda(z) \) is defined in Section 3.3. Note that \( A_0 = -2\alpha - 2h\beta \).

Since \( \text{Tr}(A_n) = 0 \) for \( n \geq 1 \) we can write
\[ A_n = a_n(2\alpha - I) + b_n\beta + c_n\gamma, \]
(4.7)
for \( n \geq 1 \) and \( z_1 < z < z_2 \). Note that \( a_1 = \frac{1}{2}q', \quad b_1 = \frac{1}{2}h' + hq' \) and \( c_1 = 0 \). From (4.3) we get then for \( n \geq 1 \)
\[ 4n(n+1)a_{n+1} + 4(n+1)h c_{n+1} + a_n'' + 2 \sum_{k=1}^{n} (n-k+1)(b_{n-k+1}c_k - c_{n-k+1}b_k) = 0, \]
\[ 4(n+1)^2b_{n+1} - 8(n+1)h a_{n+1} + b_n'' + 4 \sum_{k=1}^{n} (n-k+1)(a_{n-k+1}b_k - b_{n-k+1}a_k) = 0, \]
\[ 4(n-1)(n+1)c_{n+1} + c_n'' + 4 \sum_{k=1}^{n} (n-k+1)(c_{n-k+1}a_k - a_{n-k+1}c_k) = 0. \]
(4.8)

For \( n \geq 2 \) we see that these three equations uniquely determine \( a_{n+1}, b_{n+1} \) and \( c_{n+1} \) from \( a_k, b_k \) and \( c_k \) for \( k \leq n \). However, for \( n = 1 \) these equations do not constrain \( c_2(z) \). Therefore \( c_2(z) \) can be an arbitrary function, according to these equations.

In order to fix \( c_2(z) \) we should impose that we want \( A(r, z) \) to be a potential for \( \tilde{C}(r, z) \)
such that \( \tilde{C} = G^{-1}\tilde{V}G \) with \( G^T = G \). Then we know from Appendix B that \( h(z) \) and \( \lambda'(z) \) can be chosen freely. On the other hand, from the expansion of \( G(r, z) \) in Appendix B \( h(z) \) and \( \lambda'(z) \) are also the only free functions. Therefore, \( c_2(z) \) must be determined in terms of \( h \) and \( \lambda' \). Indeed from (3.12) we get
\[ c_2(z) = -\frac{1}{4}e^{2\lambda}h'. \]
(4.9)

So, given \( h(z) \) and \( \lambda'(z) \), we must choose \( c_2(z) \) as (4.9) in order for \( A(r, z) \) to be a potential for which a symmetric \( G(r, z) \) exists. Said in another way, imposing (3.11) on \( A(r, z) \) ensures the existence of a \( G(r, z) \) such that \( \tilde{C} = G^{-1}\tilde{V}G \), but imposing in addition that \( G^T = G \) means that we also have to fix \( c_2(z) \).

We can formulate the above as a theorem:

**Theorem 4.1** Let a rod \([z_1, z_2]\) be given. Given \( \rho(z) \) and \( \Lambda(z) \) for \( z_1 < z < z_2 \) there exists precisely one \( A(r, z) \) that solves (3.11) and that corresponds to a symmetric \( G(r, z) \), for \( z_1 < z < z_2 \).
To show that this theorem works in arbitrary dimensions one should generalize the
procedure above. One can of course without loss of generality choose a rod of direction
(1, 0, ..., 0) generalizing the four dimensional case. The proof basically rest on having
that $A_{n+1}$ for a given $n$ is determined by Eq. (4.3). If we take into account the general
properties of $\rho$ from Section 3.3 we see that this is only problematic for $n = 1$. For $n = 1$
we have that only the top entry of the first column of the matrix $A_2 - [A_2, \rho]$ is non-zero,
no matter what $A_2$ is. Therefore, given $\rho$ and $\Lambda$ we can choose $D - 3$ additional arbitrary
functions and still obey (3.11), generalizing the arbitrariness of $c_2(z)$ for four dimensions.
We expect these $D - 3$ free functions to be determined by the $G^T = G$ condition, just as
in four dimensions.

5 Smoothness conditions at rod endpoints

In this section we take a further step in understanding how to obtain the sources $\rho(z)$ and
$\Lambda(z)$ from specifying the rod-structure for a solution. What we address in this section is
how to relate the sources on both sides of a rod endpoint. Using this one can find the
sources on one side of a rod endpoint from knowing the sources on the other side of the
rod endpoint.

We can address the question of how to connect the sources across rod endpoints in a
mathematical way as follows. We have seen that $\rho(z)$ and $\Lambda(z)$ are smooth matrix valued
functions away from endpoints of rods. But, this is not true at the rod endpoints. This
is directly related to the fact that the potential $A(r, z)$ is not smooth near rod endpoints.
However, we want the solution as a whole to be regular, thus we need a criteria for when
a solution is well-behaved near a rod endpoint. This is the subject of this section.

Our basic requirement for smoothness of a solution is:

• Let $z_*$ be given. Define the coordinates

$$p = z - z_* , \quad q = \sqrt{r^2 + (z - z_*)^2}.$$  \hspace{1cm} (5.1)

We then require of a solution that $A(p, q)$ should be smooth in a neighborhood of
$(p, q) = (0, 0)$.

Away from the endpoints, this reduce to usual requirement of smoothness of $A(r, z)$ near
$r = 0$, which already is contained in the previous sections. However, if $z_*$ is an endpoint,
 i.e. if we suppose that we are considering a solution with two different rods $[z_1, z_*]$ and
$[z_*, z_2]$ so that $z = z_*$ divides the two rods, then the above requirement becomes non-trivial.

In the following we use the above smoothness requirement to analyze the behavior of
$\rho(z)$ and $\Lambda(z)$ across endpoints. Let now $z_*$ be given, and consider the $(p, q)$ coordinates
as defined in (5.1). We define

$$T^{(m, n)} = \partial_p^m \partial_q^n A|_{(p, q) = (0, 0)}.$$  \hspace{1cm} (5.2)
Theorem 5.1

Eqns. (5.5)-(5.7) relates the behavior of $T$ if we consider $\rho$, $q$, and $z$ to their behavior for $z \to z^+$. This means that knowing $\rho(z)$ and $\Lambda(z)$ for $z > z_*^+$ we can get information on $\rho(z)$ and $\Lambda(z)$ for $z < z_*$. The question is now whether one can fully determine $\rho(z)$ and $\Lambda(z)$ for $z < z_*$. This is in fact possible though one needs to use the extra constraints on $T(p,q)$ coming from expanding

$$
\rho = \frac{1}{2} \sum_{k=0}^{\infty} \frac{p_k}{k!} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \left( -\text{sgn}(p)^l T^{(k-l+1,l)} - \text{sgn}(p)^l T^{(k-l,l+1)} \right),
$$

(5.3)

$$
p \Lambda = \sum_{k=0}^{\infty} \frac{p_k}{k!} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \text{sgn}(p)^l T^{(k-l,l+1)}.
$$

(5.4)

If we consider $T^{(0,1)}$ we see from (5.3) and (5.4) that

$$
\lim_{z \to z_*^+} \rho(z) = \lim_{z \to z_*^+} \rho(z) = \lim_{z \to z_*^+} (z - z_*) \Lambda(z) = - \lim_{z \to z_*} (z - z_*) \Lambda(z).
$$

(5.5)

Considering instead $T^{(1,0)}$ and $T^{(2,0)} - T^{(0,2)}$, we get

$$
\lim_{z \to z_*^+} \left[ 2 \rho(z) + (z - z_*) \Lambda(z) \right] = \lim_{z \to z_*^-} \left[ 2 \rho(z) + (z - z_*) \Lambda(z) \right],
$$

(5.6)

$$
\lim_{z \to z_*^+} \left[ \rho'(z) + ((z - z_*) \Lambda(z))' \right] = \lim_{z \to z_*^-} \left[ \rho'(z) + ((z - z_*) \Lambda(z))' \right].
$$

(5.7)

Eqs. (5.5)-(5.7) relates the behavior of $\rho(z)$ and $\Lambda(z)$ for $z \to z^+$ to their behavior for $z \to z^-$. This means that knowing $\rho(z)$ and $\Lambda(z)$ for $z > z_*$ we can get information on $\rho(z)$ and $\Lambda(z)$ for $z < z_*$. The question is now whether one can fully determine $\rho(z)$ and $\Lambda(z)$ for $z < z_*$. This is in fact possible though one needs to use the extra constraints on $T(p,q)$ coming from expanding

$$
\partial_p^2 A + \partial_q^2 A + \frac{2p}{q} \partial_p \partial_q A = \frac{1}{q} [\partial_p A, \partial_q A],
$$

(5.8)

around $(p,q) = (0,0)$. Eq. (5.8) is Eq. 3.11 in $(p,q)$ coordinates. The first few terms of this expansion give

$$
[T^{(1,0)}, T^{(0,1)}] = 0, \quad 2T^{(1,1)} = [T^{(2,0)}, T^{(0,1)}] + [T^{(1,0)}, T^{(1,1)}],
$$

(5.9)

$$
T^{(2,0)} + T^{(0,2)} = [T^{(1,0)}, T^{(1,2)}] + [T^{(1,1)}, T^{(0,1)}].
$$

That we can connect $\rho(z)$ and $\Lambda(z)$ across a rod endpoint is formulated in the following theorem:

**Theorem 5.1** Let two rods $[z_1, z_*]$ and $[z_*, z_2]$ be given. Assume that we know $\rho(z)$ and $\Lambda(z)$ for $z_1 < z < z_*$. Then we can determine $\rho(z)$ and $\Lambda(z)$ for $z_* < z < z_2$ uniquely by the smoothness condition on $A(p,q)$ formulated above with $(p,q)$ given in (5.8).

We can prove this as follows. Since we know $\rho(z)$ and $\Lambda(z)$ for $z_1 < z < z_*$ we can determine $A(r,z)$ for $z_1 < z < z_*$ using Theorem 4.1. Employing the coordinate transformation

\[10\] We assume here the validity of Theorem 4.1 for all dimensions, although we strictly speaking only have proven Theorem 4.1 in detail in four dimensions.
we can then find \( A(p,q) \) for \( p < 0 \) and \( q \) in a certain range. Since \( A(p,q) \) is assumed to be smooth in \((p,q) = (0,0)\) we see that all the derivatives of \( A(p,q) \) at \((p,q) = (0,0)\) are the same whether one approaches from \( p < 0 \) or \( p > 0 \). Thus, we can determine \( T^{(m,n)} \) as defined by (5.2) from our knowledge of \( \rho(z) \) and \( \Lambda(z) \) for \( z_1 < z < z_s \). Using now (5.3) and (5.4) we can find the values of the derivatives of \( \rho(z) \) and \( (z-z_s)\Lambda(z) \) as \( z \to z^+_s \) to all orders. This then uniquely determines \( \rho(z) \) and \( \Lambda(z) \) for \( z_s < z < z_2 \).

6 Asymptotically flat space-times

In this section we consider solutions that asymptote to four- and five-dimensional Minkowski spaces. We work in units where Newton’s constant is set to one.

An important purpose of this section is to show that one can find the asymptotic quantities like mass and angular momenta directly from the sources \( \rho(z) \) and \( \Lambda(z) \). This is an alternative way of viewing the statement that all the information about a solution is contained in the sources \( \rho(z) \) and \( \Lambda(z) \).

Four-dimensional asymptotically flat space-times

From [14] we know the asymptotic behavior of four dimensional solutions asymptoting to Minkowski space. Using this, we get the following asymptotic behavior for \( A(r,z) \)

\[
A_{11} = -\frac{2Mz}{\sqrt{r^2 + z^2}} , \quad A_{12} = \frac{2Jz(3r^2 + 2z^2)}{(r^2 + z^2)^{3/2}} , \quad A_{21} = -\frac{2Jz}{(r^2 + z^2)^{3/2}} ,
\]

in the asymptotic region \( \sqrt{r^2 + z^2} \to \infty \) with \( z/\sqrt{r^2 + z^2} \) finite. Here \( M \) is the mass and \( J \) is the angular momentum. Combining this with Eq. (3.8) and Eq. (3.2) we get

\[
\int_{-\infty}^{\infty} dz \rho_{11}(z) = 2M , \quad \int_{-\infty}^{\infty} dz \rho_{12}(z) = -4J , \quad \int_{-\infty}^{\infty} dz \rho_{21}(z) = 0 .
\]

Using Eq. (8.10) we get that \( \rho(z) \) and \( \Lambda(z) \) asymptotically behave as

\[
\rho(z) = \begin{pmatrix} 0 & 0 \\ -\frac{2J}{|z|^3} & 1 \end{pmatrix} , \quad \Lambda(z) = \text{sgn}(z) \begin{pmatrix} \frac{2M}{z^2} & 0 \\ \frac{6J}{z^4} & -\frac{2M}{z^2} \end{pmatrix} ,
\]

for \( z \to \pm \infty \). Note that the entries with zeroes are zero to all orders in \( 1/z \) as can be seen from the properties of \( \rho \) and \( \Lambda \) found in Section 3.3.
Finally, we see from Eq. (6.4) using Eq. (3.10) that to leading order Eq. (6.4) together with Eq. (3.8) and Eq. (3.2) we get

\[ z = 2M \sqrt{r^2 + z^2} + z + \frac{2(M + \eta)}{3\pi} \frac{z}{\sqrt{r^2 + z^2}}, \]

in the asymptotic region \( \sqrt{r^2 + z^2} \to \infty \) with \( z/\sqrt{r^2 + z^2} \) finite. Here \( M \) is the mass and \( J_1 \) and \( J_2 \) are the angular momenta. See [14] for comments on \( \eta \) and \( \zeta \). Using now Eq. (6.4) together with Eq. (6.8) and Eq. (6.2) we get

\[
\begin{align*}
\int_{-\infty}^{\infty} dz \rho_{11}(z) &= \frac{4M}{3\pi}, \\
\int_{-\infty}^{\infty} dz \rho_{12}(z) &= -\frac{4J_1}{\pi}, \\
\int_{-\infty}^{\infty} dz \rho_{13}(z) &= -\frac{4J_2}{\pi}, \\
\int_{-\infty}^{\infty} dz \rho_{21}(z) &= \int_{-\infty}^{\infty} dz \rho_{23}(z) = \int_{-\infty}^{\infty} dz \rho_{31}(z) = \int_{-\infty}^{\infty} dz \rho_{32}(z) = 0.
\end{align*}
\]

Finally, we see from Eq. (6.4) using Eq. (3.10) that to leading order

\[
\begin{align*}
\rho(z) &= \begin{pmatrix} 0 & 0 & 0 \\ -J_1/z^2 & 1 & 2\zeta/z^2 \\ 0 & 0 & 0 \end{pmatrix}, \\
\Lambda(z) &= \begin{pmatrix} 0 & 0 & 0 \\ -1/z & 0 & 1/z \\ 0 & 0 & 1/z \end{pmatrix} + \begin{pmatrix} 4M/3\pi z^2 & 0 & 8J_2/\pi z^2 \\ 3J_1/z^3 & -2(M + \eta)/3\pi z^2 & 8\zeta/z^2 \\ J_2/z^3 & 0 & -2(M - \eta)/3\pi z^2 \end{pmatrix},
\end{align*}
\]

for \( z \to \infty \) and

\[
\begin{align*}
\rho(z) &= \begin{pmatrix} 0 & 0 & 0 \\ -J_2/z^2 & 2\zeta/z^2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
\Lambda(z) &= \begin{pmatrix} 0 & 0 & 0 \\ 1/z & 0 & 1/z \\ 0 & 0 & 1/z \end{pmatrix} + \begin{pmatrix} -4M/3\pi z^2 & 8J_1/\pi z^2 & 0 \\ J_1/z^3 & 2(M + \eta)/3\pi z^2 & 0 \\ 3J_2/z^3 & -8\zeta/z^2 & 2(M - \eta)/3\pi z^2 \end{pmatrix},
\end{align*}
\]

for \( z \to -\infty \).

### 7 Example: The Kerr black hole

To illustrate the methods of this paper to analyze stationary and axisymmetric solutions, we consider in detail the Kerr black hole [1].
The Kerr metric is formulated most easily in the prolate spherical coordinates \((x, y)\) defined as
\[
 r = \alpha \sqrt{(x^2 - 1)(1 - y^2)} , \quad z = \alpha xy , \quad x \geq 1 , \quad -1 \leq y \leq 1 , \quad (7.1)
\]
where \(\alpha > 0\) is a number. The metric is then given as
\[
 G = \begin{pmatrix} -X & -X\tilde{A} \\ -X\tilde{A} & X^{-1}(r^2 - \tilde{A}^2x^2) \end{pmatrix} , \quad (7.2)
\]
\[
 X = \frac{x^2 \cos^2 \lambda + y^2 \sin^2 \lambda - 1}{(1 + x \cos \lambda)^2 + y^2 \sin^2 \lambda} , \quad \tilde{A} = \frac{2\alpha \tan \lambda (1 - y^2)(1 + x \cos \lambda)}{x^2 \cos^2 \lambda + y^2 \sin^2 \lambda - 1} . \quad (7.3)
\]
We now find the potential \(A\) for the Kerr metric. For this we record that in prolate spherical coordinates \(\vec{C} = \vec{\nabla} \times \tilde{A}\) becomes
\[
 C_x = -\frac{1}{\alpha(x^2 - 1)} \partial_y A , \quad C_y = \frac{1}{\alpha(1 - y^2)} \partial_x A . \quad (7.4)
\]
Using this we get the following potential for the Kerr black hole
\[
 A_{11} = -\frac{2\alpha y((1 + x \cos \lambda)^2 + \sin^2 \lambda)}{\cos \lambda((1 + x \cos \lambda)^2 + y^2 \sin^2 \lambda)} , \quad A_{21} = -\frac{2y \sin \lambda}{(1 + x \cos \lambda)^2 + y^2 \sin^2 \lambda} , \quad (7.5)
\]
\[
 A_{12} = 6\alpha^2 y \left(1 - \frac{y^2}{3}\right) \frac{\sin \lambda}{\cos^2 \lambda} + \frac{2\alpha^2 y(1 - y^2)^2 \sin \lambda \tan \lambda}{(1 + x \cos \lambda)^2 + y^2 \sin^2 \lambda} .
\]
From [14] we have that the rod-structure of the Kerr black hole consist of a space-like rod \([-\infty, -\alpha]\) in the direction \((0, 1)\), a time-like rod \([-\alpha, \alpha]\) in the direction \((1, \Omega)\) and a space-like rod \([\alpha, \infty]\) in the direction \((0, 1)\). Here \(\Omega\) is given as
\[
 \Omega = \frac{\sin \lambda \cos \lambda}{2\alpha(1 + \cos \lambda)} . \quad (7.6)
\]
We now find the source distributions \(\rho(z)\) and \(\Lambda(z)\) for the Kerr metric. For \(z < -\alpha\) we have
\[
 \rho = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} , \quad h(z) = -\frac{2\sin \lambda \cos \lambda (1 - \frac{z}{\alpha} \cos \lambda)}{\alpha \left((1 - \frac{z}{\alpha} \cos \lambda)^2 + \sin^2 \lambda\right)^2} , \quad \Lambda = \begin{pmatrix} -\lambda' & 0 \\ h' + 2h\lambda' & \lambda' \end{pmatrix} , \quad \lambda'(z) = \frac{2 \left((1 - \frac{z}{\alpha} \cos \lambda)^2 - \sin^2 \lambda\right)}{\alpha \cos \lambda \left(1 - \frac{z^2}{\alpha^2}\right) \left((1 - \frac{z}{\alpha} \cos \lambda)^2 + \sin^2 \lambda\right)} . \quad (7.7)
\]
This corresponds to the \([-\infty, -\alpha]\) rod. For \(z > \alpha\) we have
\[
 \rho = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} , \quad h(z) = -\frac{2\sin \lambda \cos \lambda (1 + \frac{z}{\alpha} \cos \lambda)}{\alpha \left((1 + \frac{z}{\alpha} \cos \lambda)^2 + \sin^2 \lambda\right)^2} , \quad \Lambda = \begin{pmatrix} -\lambda' & 0 \\ h' + 2h\lambda' & \lambda' \end{pmatrix} , \quad \lambda'(z) = \frac{2 \left((1 + \frac{z}{\alpha} \cos \lambda)^2 - \sin^2 \lambda\right)}{\alpha \cos \lambda \left(1 - \frac{z^2}{\alpha^2}\right) \left((1 + \frac{z}{\alpha} \cos \lambda)^2 + \sin^2 \lambda\right)} . \quad (7.8)
\]
This corresponds to the \([\alpha, \infty]\) rod. Finally for \(-\alpha < z < \alpha\) we have

\[
\rho = \left( \frac{1 - \Omega h}{\Omega - \Omega^2 h}, \frac{h(z)}{\Omega h} \right), \quad h(z) = \frac{1}{\Omega} \left( \frac{\sin \lambda}{\Omega^2 \alpha} \frac{(1 + \cos \lambda)^2 - \frac{z^2}{\alpha^2} \sin^2 \lambda}{(1 + \cos \lambda)^2 + \frac{z^2}{\alpha^2} \sin^2 \lambda} \right),
\]

\[
\Lambda = \left( \frac{-\Omega h' + (1 - 2\Omega h)\lambda'}{\Omega h'} + 2h' \lambda', \frac{-\Omega^2 h' + 2\Omega(1 - \Omega h)\lambda'}{\Omega h'} - (1 - 2\Omega h)\lambda' \right),
\]

\[
\lambda'(z) = \frac{4(1 + \cos \lambda)z}{\alpha^2 \left(1 - \frac{z^2}{\alpha^2}\right) \left[(1 + \cos \lambda)^2 + \frac{z^2}{\alpha^2} \sin^2 \lambda\right]}. \quad (7.9)
\]

corresponding to the \([-\alpha, \alpha]\) rod.

One can now easily check that \(\rho(z)\) and \(\Lambda(z)\) given above for the Kerr black hole obeys the properties found in Section 3.3. It is also interesting to consider the smoothness conditions of Section 5 on the endpoints of the rods. Consider the endpoint \(z = \alpha\). Using (7.9) we compute

\[
\lim_{z \to \alpha^-} \rho = \begin{pmatrix} 1 \\ -\frac{z}{\alpha} \end{pmatrix}, \quad \lim_{z \to \alpha^-} (z - \alpha)\Lambda = \begin{pmatrix} -1 \\ -2\Omega \end{pmatrix},
\]

\[
\lim_{z \to \alpha^-} (\rho' + ((z - \alpha)\Lambda)') = \frac{1 - 2\cos \lambda}{2\alpha} \begin{pmatrix} 1 \\ -2\Omega \end{pmatrix}. \quad (7.10)
\]

Using (7.9) we compute

\[
\lim_{z \to \alpha^+} \rho = \begin{pmatrix} 0 \\ -\Omega \end{pmatrix}, \quad \lim_{z \to \alpha^+} (z - \alpha)\Lambda = \begin{pmatrix} 1 \\ 2\Omega \end{pmatrix},
\]

\[
\lim_{z \to \alpha^+} (\rho' + ((z - \alpha)\Lambda)') = \frac{1 - 2\cos \lambda}{2\alpha} \begin{pmatrix} 1 \\ 2\Omega \end{pmatrix}. \quad (7.11)
\]

From (7.10) and (7.11) one can now easily check explicitly that (5.5)-(5.7) are obeyed. Similarly one can check the smoothness conditions involving higher derivatives of \(\rho\) and \(\Lambda\).

### 8 Conclusions

In this paper we have made a new formulation of the Einstein equations for stationary and axisymmetric metrics. This was done in Section 4 by introducing the field \(\vec{C} = G^{-1}\vec{\nabla}\vec{G}\) and its vector potential \(\vec{A}\) given by \(\vec{C} = \vec{\nabla} \times \vec{A}\). This enabled us to naturally identify the sources \(\rho(z)\) and \(\Lambda(z)\) for a given solution. As argued in Section 4 a solution should be completely determined once \(\rho(z)\) and \(\Lambda(z)\) are given. Hence, we have reduced the problem of finding stationary and axisymmetric solution to the problem of specifying the sources \(\rho(z)\) and \(\Lambda(z)\). This means that we have reduced the problem of finding solutions
to a one-dimensional problem, since $\rho(z)$ and $\Lambda(z)$ are matrix-valued functions of just one variable.

That all information about a solution is contained in the sources $\rho(z)$ and $\Lambda(z)$ is also confirmed by the fact that one can extract all physical quantities from these. In particular, in Section 6 we consider how to find the most important asymptotically measurable quantities for four- and five-dimensional asymptotically flat solutions.

The remaining problem is to find an effective method to determine $\rho(z)$ and $\Lambda(z)$ solely from information about the rod-structure of the solution. A step towards this is provided in Section 3 in which we analyze the constraints on $\rho(z)$ and $\Lambda(z)$ coming from the $r \to 0$ limit of the Einstein equations, and from imposing a particular rod-structure. These constraints significantly reduce the freedom of choice for $\rho(z)$ and $\Lambda(z)$, in particular for the four-dimensional case the left-over freedom is in terms of a single function of $z$. A further step towards restricting $\rho(z)$ and $\Lambda(z)$ when crossing a rod endpoint. However, we do not find a sufficient amount of restrictions on $\rho(z)$ and $\Lambda(z)$ to determine them completely from the given rod-structure. We leave therefore this as an open problem for future research.

In Section 7 we applied our considerations to the example of the Kerr black hole. We found the potential $A(r, z)$ for the Kerr black hole and from that the sources $\rho(z)$ and $\Lambda(z)$. Moreover, we checked successfully that the conditions of Section 5 on the specific sources when crossing rod endpoints are obeyed.

In conclusion, we have accomplished several steps towards the goal of this paper, which is to find a general method of finding stationary and axisymmetric metrics given a specific rod-structure. However, we are still lacking a complete determination of the sources. Despite this, it is clear from the uniqueness theorems in four dimensions that one for example should be able to determine completely the sources of the Kerr black hole. On the other hand, we do not now of any direct construction of the Kerr black hole. Indeed, if one determines $\rho(z)$ and $\Lambda(z)$ directly from the rod-structure of the Kerr black hole it will be the first example of such a direct construction. One can therefore say that by devising a method to find the sources solely from the rod-structure one would be obtaining a generalization of the uniqueness theorems in four dimensions to stationary and axisymmetric solutions in higher dimensions.

Clearly, the problem of finding the sources from the rod-structure is interesting also with respect to the search for new black hole solutions, in particular the five-dimensional asymptotically flat black ring with two angular momenta, generalizing [12], and non-zero angular momenta solutions with Kaluza-Klein bubbles and event horizons [19].
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A Rigid transformations

Consider a coordinate transformation
\[ \tilde{x}^i = \sum_j M_{ij} x^j. \]  
(A.1)

Here \( M \) can be any (constant) invertible matrix. We see then that
\[ \tilde{G} = M^{-T} G M^{-1}, \quad \tilde{C} = M \tilde{C} M^{-1}, \quad \tilde{A} = M A M^{-1}. \]  
(A.2)

Moreover, we have
\[ v^i = \sum_j M_{ij} v^j, \quad \frac{\partial}{\partial \tilde{x}^i} = \sum_j (M^{-T})_{ij} \frac{\partial}{\partial x^j}. \]  
(A.3)

With a slight abuse of notation we write the first equation of (A.3) as
\[ \tilde{v} = M v, \]  
(A.4)
i.e. in this equation we read \( v \) and \( \tilde{v} \) as vectors in \( \mathbb{R}^{D-2} \) instead of vector fields in the \((D-2)\)-dimensional linear space spanned by the Killing vectors (clearly \( v^i \partial/\partial x^i \) is not affected by the transformation (A.1)). Thus, the point of the above is that by a transformation (A.1) we can transform a given non-zero vector \( v \) into any non-zero vector that we want. We just need to take into account (A.2).

B Analysis of EOMs in four dimensions

In this appendix we consider the expansion of the EOMs (2.4) for \( G \) around \( r = 0 \) in four dimensions. In doing this, we show that a solution of the EOMs is completely determined by two functions. We use this in Section 4.

We parameterize \( G \) as
\[ G = \begin{pmatrix} -X & -X \tilde{A} \\ -X \tilde{A} X^{-1} (r^2 - \tilde{A}^2 X^2) & \end{pmatrix}. \]  
(B.1)

The EOMs (2.4) in terms of \( X \) and \( \tilde{A} \) are
\[ X \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) X = (\partial_r X)^2 + (\partial_z X)^2 - \frac{X^4}{r^2} \left( (\partial_r \tilde{A})^2 + (\partial_z \tilde{A})^2 \right), \]  
(B.2)
\[ \partial_r \left( \frac{X^2}{r} \partial_r \tilde{A} \right) + \partial_z \left( \frac{X^2}{r} \partial_z \tilde{A} \right) = 0. \]  
(B.3)
We consider now the behavior of a solution for \( r \) small (i.e. near \( r = 0 \)) and for \( z \) away from the endpoints of any rod. Then we can expand \( X \) and \( \tilde{A} \) as

\[
X(r, z) = \sum_{n=0}^{\infty} f_n(z) r^{2n}, \quad \tilde{A}(r, z) = \sum_{n=0}^{\infty} g_n(z) r^{2n}.
\] (B.4)

Considering then the leading part of \((B.2)\) we get that either \( f_0(z) \) is zero, or \( g_0(z) \) is constant.

We consider first the case in which \( g_0(z) \) is constant. We write

\[
f_0(z) = e^{-\lambda(z)}, \quad g_0(z) = -a, \quad g_1(z) = -h(z)e^{2\lambda(z)}.
\] (B.5)

The first few terms in \( X \) and \( \tilde{A} \) then takes the form

\[
X = e^{-\lambda(z)} + r^2 f_1(z) + r^4 f_2(z) + \mathcal{O}(r^6),
\]

\[
\tilde{A} = -a - r^2 h(z) e^{2\lambda(z)} + r^4 g_2(z) + r^6 g_3(z) + \mathcal{O}(r^8).
\] (B.6)

One can now find \( f_1(z) \) and \( g_2(z) \) from the EOMs \((B.2)-(B.3)\):

\[
f_1 = \frac{1}{4} e^{-\lambda} \lambda'' - e^{\lambda} h^2, \quad g_2 = \frac{1}{8} e^{2\lambda} (h'' + 4h\lambda'' + 2h'\lambda') - e^{4\lambda} h^3.
\] (B.7)

By going order by order one can similar find \( f_2(z) \) and \( g_3(z) \), and then \( f_3(z) \) and \( g_4(z) \), and so on. Therefore, once \( h(z) \) and \( \lambda(z) \) are given, \( X(r, z) \) and \( \tilde{A}(r, z) \) are uniquely determined. We can now find \( \rho \) and \( \Lambda \):

\[
\rho = \begin{pmatrix} ah & a(1-ah) \\ h & 1-ah \end{pmatrix}, \quad \Lambda = \begin{pmatrix} a(h' + 2h\lambda') - \lambda' & -a^2(h' + 2h\lambda') + 2a\lambda' \\ h' + 2h\lambda' & -a(h' + 2h\lambda') + \lambda' \end{pmatrix}.
\] (B.8)

One sees from this that if one knows both \( \rho \) and \( \Lambda \) then \( X(r, z) \) and \( \tilde{A}(r, z) \) are uniquely determined. (Notice though the constant part of \( \lambda(z) \). However, that is not essential since that can be scaled to whatever one wants).

We consider now the case for which \( f_0(z) = 0 \). We write

\[
f_0(z) = 0, \quad f_1(z) = e^{\lambda(z)}, \quad g_0(z) = h(z).
\] (B.9)

We then get

\[
\rho = \begin{pmatrix} 1 & h \\ 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda' h' + 2h\lambda' \\ 0 & -\lambda' \end{pmatrix}.
\] (B.10)

Writing

\[
C_2(r, z) = \Lambda(z) + 4r^2 A_2(z) + \mathcal{O}(r^4).
\] (B.11)

we also compute that

\[
A_{2,21} = -\frac{1}{4} e^{2\lambda} h'.
\] (B.12)

This is used in Section \( \square \). One can solve for \( f_2(z) \) and \( g_1(z) \) from the EOMs \((B.2)-(B.3)\):

\[
f_2(z) = -\frac{1}{4} e^{\lambda} \lambda'', \quad g_1(z) = \frac{1}{8} h'' - \frac{1}{4} h'\lambda' .
\] (B.13)

By going order by order one can similar find \( f_3(z) \) and \( g_2(z) \), and then \( f_4(z) \) and \( g_3(z) \), and so on. Therefore, once \( h(z) \) and \( \lambda(z) \) are given, \( X(r, z) \) and \( \tilde{A}(r, z) \) are uniquely determined.
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