Abstract

We show that a recently developed method for generating bounds for the discrete energy states of the non-hermitian $-ix^3$ potential (Handy 2001) is applicable to complex rotated versions of the Hamiltonian. This has important implications for extension of the method in the analysis of resonant states, Regge poles, and general bound states in the complex plane (Bender and Boettcher (1998)).
I. INTRODUCTION

In a recent work, Handy (2001) presented a novel quantization formalism for generating converging bounds to the (complex) discrete spectra of non-hermitian, one dimensional, potentials. The first part of this analysis makes use of the fact that the modulus squared of the wavefunction, \( S(x) \equiv |\Psi(x)|^2 \), for \( x \in \mathbb{R} \), and any (complex) potential function, \( V(x) \), satisfies a fourth order, linear differential equation:

\[
- \frac{1}{V_I - E_I} S^{(4)} - \left( \frac{1}{V_I - E_I} \right)' S^{(3)} + 4 \left( \frac{V_R - E_R}{V_I - E_I} \right) S^{(2)} + \left( 4 \left( \frac{V_R'}{V_I - E_I} \right)' + 2 \left( \frac{V_R'}{V_I - E_I} \right) \right) S^{(1)} + \left( 4 (V_I - E_I) + 2 \left( \frac{V_R'}{V_I - E_I} \right) \right) S = 0, \tag{1}
\]

where \( S^{(n)}(x) \equiv \partial^n_n S(x) \), and \( E = E_R + iE_I \), etc.

The bounded \((L^2)\) and nonnegative solutions of this differential equation uniquely correspond to the physical states. For the case of rational fraction potentials, there is an associated (recursive) moment equation, whose solutions are parameterized by the energy, \( E \). Constraining these through the appropriate Moment Problem conditions for nonnegative functions (Shohat and Tamarkin (1963)), generates rapidly converging bounds on the discrete state energies. This entire procedure is referred to as the Eigenvalue Moment Method (EMM) and was originally developed by Handy and Bessis (1985) and Handy et al (1988a,b). An important theoretical and algorithmic component is the use of linear programming (Chvatal (1983)).

The EMM procedure is not solely a numerical implementation of some very important mathematical theorems. It is possible to implement it algebraically, deriving bounding formulas for the eigenenergies (Handy and Bessis (1985)). Furthermore, it does not depend on the hermitian or non-hermitian structure of the Hamiltonian. What is important is that the desired solutions be the unique nonnegative and bounded solutions of the differential system being investigated. In this sense, it provides an important example of how positivity can be used as a quantization procedure.

Many important problems can be analyzed through the application of EMM on Eq.(1),
and its generalization as outlined in Sec. II. This includes generating bounds to Regge poles (which first motivated Handy’s formalism), to be presented in a forthcoming work (Handy and Msezane (2001)). Other problems include the calculation of resonant states as well as bound states in the complex plane as discussed by Bender and Boettcher (1998), and elaborated upon by others cited in the more recent work of Mezincescu (2001).

However, the realization of these depends on understanding the complex rotation extension of Handy’s original formulation. Indeed, such concerns introduce a new class of EMM problems not encountered before, and thereby motivate the present work.

Our basic objective is to extend the $S(x)$-EMM formulation to complex rotated transformations of the Hamiltonian $p^2 - ix^3$. We are particularly interested in the effectiveness of such an analysis. Indeed, once the correct moment equation is developed, one finds that the EMM approach works very well in this case, and yields the anticipated result, as detailed in Sec. III.

In the following sections we present a more general derivation of the $S(x)$ equation, and apply the resulting formalism to the rotated $-ix^3$ Hamiltonian.
II. GENERALIZED S-EQUATION

Consider the (normalized) Schrodinger equation
\[-\partial_x^2 \Psi(x) + V(x)\Psi(x) = E\Psi(x),\]
\hspace{1cm} (2)
for complex energy, \(E\), and complex potential, \(V(x)\). Assume that the (complex) bound state, \(\Psi(x)\), lies within the complex-\(x\) plane, along some infinite contour, \(C\). Let \(x(\xi)\) define a differentiable map from a subset of the real axis to the entire complex contour:
\[x(\xi) : \xi \in \mathbb{R} \rightarrow C.\]
\hspace{1cm} (3)
The transformed Schrodinger equation is
\[-\left(D(\xi)\partial_\xi\right)^2 \Psi(\xi) + V(\xi)\Psi(\xi) = E\Psi(\xi),\]
\hspace{1cm} (4)
where \(D(\xi) \equiv (\partial_x x)^{-1}\), and \(V(\xi) \equiv V(x(\xi))\). Alternatively, we may rewrite the above as
\[H_\xi \Psi(\xi) \equiv A(\xi)\partial_\xi^2 \Psi(\xi) + B(\xi)\partial_\xi \Psi(\xi) + C(\xi)\Psi(\xi) = 0,\]
\hspace{1cm} (5)
where \(A(\xi) \equiv -(D(\xi))^2\), \(B(\xi) = -\frac{1}{2}\partial_\xi (D(\xi))^2\), and \(C(\xi) = V(\xi) - E\). In the previous work by Handy (2001), he showed how such an expression, without the \(\partial_\xi \Psi\) term, leads to a fourth order equation for \(S(\xi) = |\Psi(\xi)|^2\). We broaden this derivation to include the above general case.

Define the following four functions:
\[\Sigma_1(\xi) \equiv \Psi^*(\xi)H_\xi \Psi(\xi) + c.c. = 0,\]
\hspace{1cm} (6)
\[\Sigma_2(\xi) \equiv \Psi'^*(\xi)H_\xi \Psi(\xi) + c.c. = 0,\]
\hspace{1cm} (7)
\[\Delta_1(\xi) \equiv \Psi^*(\xi)H_\xi \Psi(\xi) - c.c. = 0,\]
\hspace{1cm} (8)
and
\[\Delta_2(\xi) \equiv \Psi'^*(\xi)H_\xi \Psi(\xi) - c.c. = 0.\]
\hspace{1cm} (9)
Define

\[ S(\xi) = \Psi^*(\xi)\Psi(\xi), \]  
(10)

\[ P(\xi) = \Psi'^*(\xi)\Psi'(\xi), \]  
(11)

\[ J(\xi) = \frac{\Psi(\xi)\partial_\xi \Psi^*(\xi) - \Psi^*(\xi)\partial_\xi \Psi(\xi)}{2i}, \]  
(12)

and

\[ T(\xi) = \frac{\partial_\xi \Psi(\xi)\partial^2_\xi \Psi^*(\xi) - \partial_\xi \Psi^*(\xi)\partial^2_\xi \Psi(\xi)}{2i}. \]  
(13)

These correspond to important physical quantities. The first two are nonnegative functions corresponding to the probability density and the “momentum density”, while \( J(\xi) \) is the probability flux.

We then have (i.e. \( A = A_R + iA_I \), etc.):

\[ \Sigma_1(\xi) = (S''(\xi) - 2P(\xi))A_R(\xi) + S'(\xi)B_R(\xi) + 2S(\xi)C_R(\xi) + 2(B_I(\xi) + A_I(\xi)\partial_\xi)J(\xi) = 0, \]  
(14)

\[ \frac{\Delta_1(\xi)}{i} = (S''(\xi) - 2P(\xi))A_I(\xi) + S'(\xi)B_I(\xi) + 2S(\xi)C_I(\xi) - 2(B_R(\xi) + A_R(\xi)\partial_\xi))J(\xi) = 0, \]  
(15)

\[ \Sigma_2(\xi) = P'(\xi)A_R(\xi) + 2T(\xi)A_I(\xi) + 2P(\xi)B_R(\xi) + S'(\xi)C_R(\xi) - 2J(\xi)C_I(\xi) = 0, \]  
(16)

and

\[ \frac{\Delta_2(\xi)}{i} = P'(\xi)A_I(\xi) - 2T(\xi)A_R(\xi) + 2P(\xi)B_I(\xi) + S'(\xi)C_I(\xi) + 2J(\xi)C_R(\xi) = 0. \]  
(17)

For the case of the ordinary (real potential) Schrödinger equation where \( x = \xi \), and \( A_I = 0, B = 0, \) and \( C_I = 0 \), we can combine Eq.(14) (divided by \( A_R \)) and Eq.(16) to generate a third order differential equation for \( S \) (Handy (1987a,b), Handy et al (1988c)).
If $A_I = 0$ and $B = 0$, the case considered by Handy (2001), then Eq.(14) (divided by $A_R$) yields an equation for $P$. Differentiating this relation, and inserting Eq. (16) defines a third order equation in $S$, which also depends on $J$. We can eliminate $J$ by differentiating (after dividing out $\frac{C_I}{A_R}$) one more time, and substituting Eq.(15). The resulting expression corresponds to that in Eq.(1), assuming $A_R = 1$.

If $A_I = 0$ and $C_I \neq 0$, or $A_R = 0$ and $C_I \neq 0$, we can determine $J$ from Eq.(16) or Eq.(17), respectively, and proceed with the analysis discussed below.

If both $A_{R,I} \neq 0$, multiplying Eq.(16) and Eq.(17) by $A_R(\xi)$ and $A_I(\xi)$, respectively, and adding, yields an equation for $J$:

$$J(\xi) = \frac{P'(\xi)|A(\xi)|^2 + 2P(\xi)Re(A(\xi)B^*(\xi)) + S'(\xi)Re(A(\xi)C^*(\xi))}{-2Im(A(\xi)C^*(\xi))}. \tag{18}$$

We can now substitute this expression, $J(\xi; P, P', S')$, into Eq.(14) and Eq.(15) yielding two coupled differential equations for $P$ and $S$. Instead of the formalism presented by Handy (2001), we can work with these two coupled equations, since they involve two nonnegative configurations (for the physical solutions). However, depending on the nature of the function coefficients (i.e. $A(\xi), B(\xi), C(\xi)$) it may be expedient to reduce these to one fourth order, linear differential equation for $S$. This is particularly the case corresponding to that discussed by Handy (2001).

For completeness, we note that the analytic extension of $S(\xi)$ onto the complex-$\xi$ plane (which is of no immediate concern to us) corresponds to

$$S(\xi) = \Psi^*(\xi^*)\Psi(\xi). \tag{19}$$

That is, if we assume that $\Psi(\xi)$ is analytic at the origin (for simplicity), then $\Psi(\xi) = \sum_{j=0}^{\infty} c_j \xi^j$ and $\Psi^*(\xi^*) = \sum_{j=0}^{\infty} c_j^* \xi^j$ are convergent power series, defining an analytic product which is nonnegative along the real $\xi$-axis.
III. THE COMPLEX ROTATED $P^2 + (IX)^3$ HAMILTONIAN

Consider the non-hermitian potential problem

$$-\Psi''(x) - ix^3\Psi(x) = E\Psi(x),$$

first considered in the work by Bender and Boettcher (1998). From their analysis, it is known that bound states exist along the real axis. They satisfy

$$E = \frac{\int dx P(x) - i \int dx x^3 S(x)}{\int dx S(x)}.$$  \hfill (21)

If the physical solution conforms with the $\mathcal{PT}$ invariant nature of the Hamiltonian,

$$\Psi^*(-x) = \Psi(x),$$  \hfill (22)

then $S(-x) = S(x)$, and the discrete state energy is real. Exhaustive investigations by many individuals supports this. These include the works by (in addition to those cited) Bender et al (1999), Bender et al (2000), Bender and Wang (2001), Caliceti (2000), Delabaere and Pham (1999), Delabaere and Trinh (2000), Levai and Znojil (2000), Mezincescu (2000,2001), Shin (2000), and Znojil (2000). The analysis of Handy (2001) has also been extended to the case of complex $E$, also supporting that the discrete states of the $-ix^3$ potential do not violate $\mathcal{PT}$ symmetry (Handy (2001)). In accordance with all of this, we proceed assuming that the discrete spectrum is real.

Let $x \equiv e^{i\theta}|x|$. Adopting Bender and Boettcher’s notation, it is known that the bound states decrease exponentially fast in any asymptotic direction within the ‘wedges’ defined by

$$|\theta - \theta_{L,R}| < \frac{\pi}{5},$$  \hfill (23)

where $\theta_L = -\pi + \frac{\pi}{10}$, and $\theta_R = -\frac{\pi}{10}$. The anti-Stokes lines (along which the asymptotic decrease is fastest) is centered at $\theta_{L,R}$.

We first considered the general transformation
for $0 < \xi < \infty$, and $\theta$ lying within the boundedness wedge(s). Such transformations would be relevant not only for the $-ix^3$ potential, but all the other non-Hermitian potentials considered by Bender and Boettcher. However, the generation of a moment equation, following the formalism in Section II, made this approach inefficient, due to the large order of the associated moment equation relation.

Instead of this approach, the $-ix^3$ problem affords a particularly simpler formulation, since the wedges intersect the entire real axis. This is not the case for all the other potentials (i.e. $(-ix)^n$, $n \geq 4$) considered by them. The generalization of the present approach can be made to these cases as well; however, we defer this to another communication, since it requires a completely different linear programming formalism.

Consistent with the formalism in Sec. II, if we take

$$x = e^{i\theta} \xi,$$  \hspace{1cm} (25)

where $\xi \in (-\infty, +\infty)$, then the bound state solution $\Psi(e^{i\theta} \xi)$ will be asymptotically zero (i.e. at $\xi = \pm \infty$) for

$$-\frac{\pi}{10} < \theta < \frac{\pi}{10}. \hspace{1cm} (26)$$

Clearly, this includes the real axis. However, for fixed $\theta \neq 0$, the closer $e^{i\theta} \xi$ is to the anti-Stokes line along the positive real-$\xi$ axis (i.e. $\theta \rightarrow \theta_R$) the farther is $e^{i\theta} \xi$ from its respective anti-Stokes line, for $\xi < 0$ (and conversely). Only along the real axis ($\theta = 0$), will the exponential decay be equally balanced between both the positive and negative real axis.

The work by Handy(2001) introduces a very accurate bounding technique which was used to generate the first five low lying states. Our immediate interest is in testing the same EMM formalism with respect to complex rotations of the system.
A. Moment Equation for the Complex Rotated $-ix^3$ Potential

As noted, we will be solving the Schrodinger equation along the complex-$x$ infinite contour defined by $x = e^{i\theta}\xi$, for $-\infty < \xi < +\infty$. The resulting differential equation is

$$-\partial_\xi^2\Psi(\xi) - ie^{i5\theta}\xi^3\Psi(\xi) = Ee^{i2\theta}\Psi(\xi).$$

(27)

We will assume $E$ to be real. The resulting, fourth order differential equation for $S(\xi)$ is then

$$\frac{1}{\Lambda(\xi)}S^{(4)}(\xi) - \frac{3c_5\xi^2}{\Lambda(\xi)^2}S^{(3)}(\xi) + \frac{4c_2E - 4\xi^3s_5}{\Lambda(\xi)}S^{(2)}(\xi) - \frac{6\xi^2(2c_2c_5E + c_5s_5\xi^3 + 3Es_2s_5)}{\Lambda(\xi)^2}S^{(1)}(\xi)$$

$$= \frac{4c_2^3\xi^9 + 12c_5^2s_2E\xi^6 + 12c_5s_2E^2\xi^3 + 4E^3s_2^3 - 6c_5s_5\xi^4 + 12Es_2s_5}{\Lambda(\xi)^2}S^{(0)}(\xi) = 0,$$

(28)

where

$$\Lambda(\xi) = c_5\xi^3 + Es_2,$$

(29)

$c_n \equiv \cos(n\theta)$, $s_n \equiv \sin(n\theta)$, and $S^{(n)}(\xi) \equiv \partial_\xi^n S(\xi)$.

We know that all of the solutions to this equation are regular in the entire complex $\xi$ plane, despite the singular function coefficient, $\frac{1}{\Lambda}$ (Handy(2001)). There is a real singularity at $\xi = b$, defined by

$$b = -\left(\frac{Es_2}{c_5}\right)^{\frac{1}{3}} \in \mathbb{R}.$$

(30)

In terms of the implicit $\theta$ dependence, $\lim_{\theta \to 0} b(\theta) = 0$.

In all of the previous applications of the EMM approach, we have never considered problems with function coefficients singular within the physical domain. The types of singularities encountered have either been close to the physical domain, or at the boundaries.

Despite the fact that the singularities in question do not affect the regular nature of the solutions, they are of concern within the EMM formalism, as discussed in the recent work by Handy, Trallero-Giner, and Rodriguez (HTR, 2001). Intuitively, let us represent the above fourth order equation by $O_\xi S(\xi)$. If one is to generate a moment equation by multiplying
both sides of this differential equation by $\Lambda(\xi)^2$, one must make sure that the resulting moment equation does not admit solutions to the more general problem of $O_\xi S(\xi) = D(\xi - b)$, where $D(\xi - b)$ is a distribution-like expression supported at $b$. Such expressions disappear when multiplied by $\Lambda(\xi)$ (with a zero at $b$). If one is not careful, the naive moment equation may not correspond to the desired problem, $D = 0$. That is, an inappropriate specification of the moment equation would not produce any converging bounds. Such concerns are generally inconsequential, within the EMM framework, if it is known that the desired solution also has a zero at $b$ (HTR, 2001). For the present problem, this is not the case, and a suitable modification of the standard EMM formalism is required.

In order to address this problem, one should first perform a translation change of variables:

$$\chi = \xi - b.$$  

(31)

The ensuing fourth order differential equation will then involve function coefficients whose singularity is no more singular than $\frac{1}{\chi^2}$. Specifically, let $\Lambda(\chi) = \chi \Upsilon(\chi)$, where $\Upsilon(\chi) = c_5(3b^2 + 3b\chi + \chi^2) \neq 0$. Upon multiplying both sides of the translated fourth order equation ($O_\chi S(\chi) = 0$) by $\Upsilon(\chi)$, one obtains an expression of the form

$$\frac{C_4(\chi)}{\chi} S^{(4)}(\chi) + \frac{C_3(\chi)}{\chi^2} S^{(3)}(\chi) + \frac{C_2(\chi)}{\chi} S^{(2)}(\chi) + \frac{C_1(\chi)}{\chi^2} S^{(1)}(\chi) + \frac{C_0(\chi)}{\chi^2} S(\chi) = 0,$$  

(32)

where all the $C_n(\chi)$ are polynomials in $\chi$.

Let us now multiply by $\chi^\rho$, where $\rho \geq 0$. Upon doing so, through the operational procedure corresponding to ‘integration by parts’, we can rewrite the expression

$$\chi^\rho \Upsilon(\chi) O_\chi S(\chi) = \left( \sum_{j=0}^{4} U_j(\chi^{\rho+9}, \ldots, \chi^{\rho-2}; \rho) \right) S(\chi) + \partial_\chi \left( T(\chi; S(\chi), \ldots, S^{(4)}(\chi)) \right) = 0,$$  

(33)

involving the total derivative expression noted, and functions of the power expressions, $\chi^q$, where the exponent may be negative.

Since the physical solution must decay exponentially, along the asymptotic directions ($\chi \rightarrow \pm \infty$), we can now integrate along a contour that sits on top of the real axis, except near the origin where it deviates around it. Denote this contour by $C$. Define the moments
\[ \mu_\rho \equiv \int_C d\chi \chi^\rho S(\chi). \]  

If \( \rho \geq 0 \), because of the analyticity of the solution, these moments correspond to the Hamburger moments taken along the real axis: \( \mu_\rho = \int_{-\infty}^{+\infty} d\chi \chi^\rho \Psi(\chi) \), for \( \rho \geq 0 \).

The moments will satisfy the moment equation (i.e. \( p = \rho - 2 \geq -2 \))

\[ \mu_{p-3} \left[ -\frac{3Es_2}{b} p(4 - 4p - p^2 + p^3) \right] + \mu_{p-2} \left[ -\frac{3Es_2}{b^2} p(-4 - p + 4p^2 + p^3) \right] \]
\[ + \mu_{p-1} \left[ -\frac{Es_2}{b^3} p(p + 2)(6 + 12b^2c_2E + 7p + p^2 - 12b^5s_5) \right] \]
\[ + \mu_p \left[ \frac{6Es_2}{b^4} (-2c_2E(4 + 5p + p^2) + s_5b^3(14 + 25p + 8p^2)) \right] \]
\[ + \mu_{p+1} \left[ \frac{2Es_2}{b^5} (-2c_2E(12 + 8p + p^2) + b^3s_5(168 + 169p + 38p^2)) \right] \]
\[ + \mu_{p+2} \left[ \frac{12Es_2}{b^6} (42 + 30p + 5p^2) \right] + \mu_{p+3} \left[ \frac{6Es_2}{b^7} (18E^2s_2^2 + bs_5(56 + 31p + 4p^2)) \right] \]
\[ + \mu_{p+4} \left[ \frac{2Es_2}{b^8} (162E^2s_2^2 + bs_5(42 + 19p + 2p^2)) \right] + \mu_{p+5} \left[ \frac{432E^3s_2^3}{b^9} \right] + \mu_{p+6} \left[ \frac{324E^3s_2^3}{b^{10}} \right] \]
\[ + \mu_{p+7} \left[ \frac{144E^3s_2^3}{b^{11}} \right] + \mu_{p+8} \left[ \frac{36E^3s_2^3}{b^{12}} \right] + \mu_{p+9} \left[ \frac{4E^3s_2^3}{b^{13}} \right] = 0, \]  

(35)

where \( p \geq -2 \).

We note that for \( p \geq 0 \), the first nine Hamburger moments, \( \{\mu_0, \ldots, \mu_8\} \) generate all of the other moments (\( \mu_p, p \geq 9 \)) through the linear relation

\[ \mu_p = \sum_{\ell=0}^8 \tilde{M}_{p,\ell} \mu_\ell, \]  

(36)

where the coefficients \( \tilde{M}_{p,\ell} \) satisfy the moment equation (Eq.(35), for \( p \geq 0 \)) with respect to the \( p \)-index. In addition, they satisfy the initialization conditions \( \tilde{M}_{\ell_1,\ell_2} = \delta_{\ell_1,\ell_2} \), for \( 0 \leq \ell_{1,2} \leq 8 \).

If the moment equation is restricted to \( p \geq 0 \), then the associated differential equation would be of the form \( \mathcal{O}_\chi S(\chi) = \mathcal{D}(\chi) \), with \( \mathcal{D} \) an unknown distribution supported at the origin (as discussed before). By also including \( p = -1, -2 \), we are imposing \( \mathcal{D} = 0 \).
For these two values of the \( p \) index, the moment equation becomes \((p = -1)\)

\[
\begin{align*}
\mu_0 \left[ \frac{2Es_2}{b^3}(-10c_2E + 37b^3s_5) \right] + \mu_1 \left[ \frac{204Es_2s_5}{b^3} \right] + \mu_2 \left[ \frac{6Es_2}{b^3}(18E^2s_2^2 + 29bs_5) \right] \\
+ \mu_3 \left[ \frac{432E^3s_2^3}{b^6} \right] + \mu_4 \left[ \frac{324E^3s_2^3}{b^6} \right] + \mu_5 \left[ \frac{144E^3s_2^3}{b^6} \right] \\
+ \mu_7 \left[ \frac{36E^3s_2^3}{b^8} \right] + \mu_8 \left[ \frac{4E^3s_2^3}{b^9} \right] = \Sigma(\mu_{-1}, \mu_{-2}, \mu_{-4}),
\end{align*}
\]

where

\[
\Sigma(\mu_{-1}, \mu_{-2}, \mu_{-4}) = \left(18bEs_2s_5\mu_{-1} - \frac{Es_2}{b^3}(12b^2c_2E - 12b^5s_5)\mu_{-2} - 18Es_2 \right) \mu_{-4},
\]

and \((p = -2)\)

\[
\begin{align*}
\mu_0 \left[ \frac{24Es_2s_5}{b} \right] + \mu_1 \left[ \frac{6Es_2}{b^3}(18E^2s_2^2 + 10bs_5) \right] + \mu_2 \left[ \frac{2Es_2}{b^4}(162E^2s_2^2 + 12bs_5) \right] + \mu_3 \left[ \frac{432E^3s_2^3}{b^6} \right] \\
+ \mu_4 \left[ \frac{324E^3s_2^3}{b^6} \right] + \mu_5 \left[ \frac{144E^3s_2^3}{b^6} \right] + \mu_6 \left[ \frac{36E^3s_2^3}{b^8} \right] + \mu_7 \left[ \frac{4E^3s_2^3}{b^9} \right] = \frac{2}{b} \Sigma(\mu_{-1}, \mu_{-2}, \mu_{-4}).
\end{align*}
\]

Therefore, we can use both relations to constrain the \( \mu_8 \) moment in terms of the moments \( \{\mu_0, \mu_1, \ldots, \mu_7\} \). These in turn can be used with the moment equation (Eq.(35)), for \( p \geq 0 \), to generate all of the other Hamburger moments. Thus, we have

\[
\mu_8 = \sum_{\ell=0}^{7} \tilde{M}_{8,\ell} \mu_\ell,
\]

defined by subtracting Eq.(39) from Eq.(37) (after dividing the former by \( \frac{2}{b} \)).

Incorporating this within the expression in Eq.(36),

\[
\mu_p = \sum_{\ell=0}^{7} \tilde{M}_{p,\ell} \mu_\ell + \tilde{M}_{p,8} \sum_{\ell=0}^{7} \tilde{M}_{8,\ell} \mu_\ell,
\]

yields

\[
\mu_p = \sum_{\ell=0}^{7} M_{p,\ell} \mu_\ell,
\]

where

\[
M_{p,\ell} = \begin{cases} 
\tilde{M}_{8,\ell}, & p = 8 \\
\tilde{M}_{p,\ell} + \tilde{M}_{p,8} \tilde{M}_{8,\ell}, & p \geq 9,
\end{cases}
\]

and \( M_{\ell_1,\ell_2} = \delta_{\ell_1,\ell_2}, \) for \( 0 \leq \ell_1, \ell_2 \leq 7 \). This is the moment generating relation required in implementing EMM.
An additional important point is specifying the choice of normalization. Since we are only dealing with Hamburger moments, some of the odd order moments may be negative. However, the even order moments are positive, and by working with the first 8 of them

\[ u_\ell \equiv \mu_{2\ell}, \quad (44) \]

we can impose the normalization

\[ \sum_{\ell=0}^{7} u_\ell = 1. \quad (45) \]

Working with the positive quantities, \( u_\ell \), makes the implementation of the linear programming algorithm more efficient (since one is working within a bounded polytope corresponding to the unit hypercube, \([0, 1]^7\)).

Thus, we take

\[ u_\ell = \mu_{2\ell} = \sum_{\ell_0}^{7} M_{2\ell, \ell_0} \mu_{\ell_0}, \quad (46) \]

for \( 0 \leq \ell \leq 7 \). The inverse of this relation is denoted by

\[ \mu_\ell = \sum_{\ell_0=0}^{7} N_{\ell, \ell_0} u_{\ell_0}, \quad (47) \]

and substituted in Eq.(42) producing

\[ \mu_p = \sum_{\ell, \ell_0=0}^{7} M_{p, \ell} N_{\ell, \ell_0} u_{\ell_0}, \quad (48) \]

or

\[ \mu_p = \sum_{\ell=0}^{7} \mathcal{M}_{p, \ell} u_\ell, \quad (49) \]

for \( p \geq 0 \), where \( \mathcal{M}_{p, \ell} = \sum_{\ell_0=0}^{7} M_{p, \ell} N_{\ell_0, \ell}. \)

Finally, the normalization condition is incorporated \( (u_0 = 1 - \sum_{\ell=1}^{7} u_\ell) \), producing

\[ \mu_p = \sum_{\ell=0}^{7} \mathcal{\hat{M}}_{p, \ell} \hat{u}_\ell, \quad (50) \]

where
\[ \hat{\mathcal{M}}_{p,\ell} = \begin{cases} \mathcal{M}_0, & p = 0 \\ \mathcal{M}_{p,\ell} - \mathcal{M}_0, & p \geq 1 \end{cases}, \quad (51) \]

and

\[ \hat{u}_\ell = \begin{cases} 1, & \ell = 0 \\ u_\ell, & 1 \leq \ell \leq 7 \end{cases}. \quad (52) \]

**B. Numerical Results**

The numerical implementation of the EMM procedure is given in Table I. We only investigate the lowest energy state. The parameter \( P_{\text{max}} \) denotes the maximum moment order generated, \( \{\mu_p | 0 \leq p \leq P_{\text{max}}\} \). We see that as the rotation angle increases, the tightness of the bounds decreases, as expected. Furthermore, for \( \theta > \frac{\pi}{10} \), no EMM solution is generated, consistent with the fact that the bound state becomes exponentially unbounded, in the corresponding direction in the complex \( x \) plane.
TABLES

TABLE I. Bounds for the Ground State Energy of the $-i\alpha^3$ Potential Using Eq. ( ) ($P_{\text{max}} = 40$)

| $\theta$ | $E_{L,0}$   | $E_{U,0}$   |
|----------|-------------|-------------|
| 0*       | 1.15619     | 1.15645     |
| .01      | 1.15617     | 1.15652     |
| .05      | 1.15510     | 1.15761     |
| .10      | 1.14201     | 1.17298     |
| .15      | 1.0         | 1.3         |
| .20      | .5          | 8.0         |

*Using Stieltjes EMM formulation by Handy(2001)
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