Calculations relating to some special Harmonic numbers

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Abstract
We report on the results of a computer search for primes $p$ which divide an Harmonic number $H_{\lfloor p/N \rfloor}$ with small $N > 1$.

Keywords: Harmonic numbers, Fermat quotient, Fermat’s Last Theorem

1 Introduction

Before its complete proof by Andrew Wiles, a major result for the first case of Fermat’s last theorem (FLT), that is, the assertion of the impossibility of $x^p + y^p = z^p$ in integers $x, y, z$, none of which is divisible by a prime $p > 2$, was the 1909 theorem of Wieferich [25] that the exponent $p$ must satisfy the congruence

$$q_p(2) := 2^{p-1} - 1 \equiv 0 \pmod{p},$$

where $q_p(2)$ is known as the Fermat quotient of $p$, base 2. This celebrated result was to be generalized in many directions. One of these was the extension of the congruence to bases other than 2, the first such step being the proof of an analogous theorem for the base 3 by Mirimanoff [17] in 1910. Another grew from the recognition that some of these criteria could be framed in terms of certain special Harmonic numbers of the form

$$H_{\lfloor p/N \rfloor} := \sum_{j=1}^{\lfloor p/N \rfloor} \frac{1}{j} \quad (1)$$

for $N > 1$, and with $\lfloor \cdot \rfloor$ denoting the greatest-integer function. There are nice historical overviews of these developments in [19] and [18] (pp. 155–59), which go into greater detail than we attempt here.

At the time when Wieferich’s and Mirimanoff’s results appeared, it was already known that three of these Harmonic numbers had close connections to the Fermat quotient, and satisfied the following congruences (all modulo $p$):
\[
H_{\lfloor p/2 \rfloor} \equiv -2 \cdot q_p(2) \tag{2}
\]
\[
H_{\lfloor p/3 \rfloor} \equiv -\frac{3}{2} \cdot q_p(3) \tag{3}
\]
\[
H_{\lfloor p/4 \rfloor} \equiv -3 \cdot q_p(2). \tag{4}
\]
All these results are due to Glaisher; those for \( N = 2 \) and 4 will be found in ([11], pp. 21-22, 23), and that for \( N = 3 \) in ([10], p. 50). Although it was not the next Harmonic number criterion published, it will be convenient to dispense next with the case \( N = 6 \). The apparatus needed to evaluate this Harmonic number appears in a 1905 paper of Lerch ([16], p. 476, equations 14 and 15), but the implications of Lerch’s result seem to have been long overlooked, and were only developed in 1938 by Emma Lehmer ([15], pp. 356ff), who gave the first row of the following congruence modulo \( p \):

\[
H_{\lfloor p/6 \rfloor} \equiv -2 \cdot q_p(2) - \frac{3}{2} \cdot q_p(3) \tag{5}
\]
\[
\equiv -\frac{1}{2} \cdot q_p(432).
\]
The second row shown is obtained from Lehmer’s result by applying in reverse the logarithmic and factorization rules for the Fermat quotient given by Eisenstein [7], and this has been done both to indicate a possible computational simplification, and to emphasize the connection with studies of the vanishing of the Fermat quotient for composite bases, notably the ongoing work of Richard Fischer [9]. The fact that the vanishing of \( H_{\lfloor p/6 \rfloor} \mod p \) is a necessary condition for the failure of the first case of FLT for the exponent \( p \) is an immediate consequence of the theorems of Wieferich and Mirimanoff.

The four congruences above exhaust the cases of \( H_{\lfloor p/N \rfloor} \) that can be evaluated solely in terms of Fermat quotients. Historically, the case \( N = 5 \) also has its origins in this era, and was in fact introduced before that of \( N = 6 \). In 1914, Vandiver [24] proved that the vanishing of both \( H_{\lfloor p/5 \rfloor} \) and \( q_p(5) \) are necessary conditions for \( p \) to be an exception to the first case of FLT. Unlike the four cases already considered, here the connection of the Harmonic number with FLT was discovered before any evaluation of it (beyond the definitional one) was known. It was almost eighty years later that the connection between these results would become apparent. The ingredients needed for the evaluation of this Harmonic number were presented in a 1991 paper by Williams ([26], p. 440), and nearly simultaneously by Z.H. Sun ([22], pt. 3, Theorems 3.1 and 3.2); and though they do not write out the formula explicitly, it is clearly implied to be

\[
H_{\lfloor p/5 \rfloor} \equiv -\frac{5}{4} \cdot q_p(5) - \frac{5}{4} \cdot F_{p-(\frac{\bar{5}}{p})} / p \pmod{p}, \tag{6}
\]
where \( F_{p-(\frac{\bar{5}}{p})} / p \) is the Fibonacci quotient (OEIS A092330), with \( F \) a Fibonacci number and \((\frac{\bar{5}}{p})\) a Jacobi symbol. In light of Vandiver’s theorems on \( H_{\lfloor p/5 \rfloor} \) and
This result immediately established that the vanishing of the Fibonacci quotient mod $p$ was yet another criterion for the failure of the first case of FLT for the exponent $p$. The fact was announced almost immediately in the celebrated paper by the Sun brothers [23], which gave fresh impetus to an already vast literature on the Fibonacci quotient; and in its honor the primes $p$ which divide their Fibonacci quotient were named Wall-Sun-Sun-Primes. These remain hypothetical, as not a single instance has been found despite tests to high limits (see “Wall-Sun-Sun Prime Search” at [http://www.primegrid.com](http://www.primegrid.com)).

The remaining known formulae for the type of special Harmonic numbers in which we are interested are of much more recent origin. In some cases they were discovered simultaneously, or nearly so, by more than one researcher, and we hope we have not done injustice to any of the participants. Apart from a few published formulae which are apparently in error or underdetermined, we have the following congruences (all modulo $p$) which are undoubtably correct:

\[
H_{[p/8]} \equiv -4 \cdot q_p(2) - 2 \cdot U_{p-(\frac{5}{p})}(2,-1)/p
\]  

\[
H_{[p/10]} \equiv -2 \cdot q_p(2) - \frac{5}{4} \cdot q_p(5) - \frac{15}{4} \cdot F_{p-(\frac{5}{p})}/p
\]  

\[
H_{[p/12]} \equiv -3 \cdot q_p(2) - \frac{3}{2} \cdot q_p(3) - 3 \cdot \left(\frac{3}{p}\right) \cdot U_{p-(\frac{5}{p})}(4,1)/p
\]  

\[
H_{[p/24]} \equiv -4 \cdot q_p(2) - \frac{3}{2} \cdot q_p(3) - 4 \cdot U_{p-(\frac{5}{p})}(2,-1)/p
\] 

\[
- 3 \cdot \left(\frac{3}{p}\right) \cdot U_{p-(\frac{5}{p})}(4,1)/p - 6 \cdot \left(\frac{6}{p}\right) \cdot U_{p-(\frac{5}{p})}(10,1)/p
\]

where the $\left(\frac{\cdot}{p}\right)$ are Jacobi symbols, and

- $F_{p-(\frac{5}{p})}/p$ is as before the Fibonacci quotient (OEIS A092330)
- $U_{p-(\frac{5}{p})}(2,-1)/p$ is the Pell quotient (OEIS A000129)
- $U_{p-(\frac{5}{p})}(4,1)/p$ is a quotient derived from the Lucas sequence 1, 4, 15, 56, 209, ... (OEIS A001353)
- $U_{p-(\frac{5}{p})}(10,1)/p$ is a quotient derived from the Lucas sequence 1, 10, 99, 980, 9701, ... (OEIS A004189)

- $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$
- $\left(\frac{3}{p}\right) = (-1)^{(p+1)/6}$
- $\left(\frac{6}{p}\right) = (-1)^{(p+5)/12}$. 

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The result for $N = 8$ is derived from the 1991 paper by Williams ([26], p. 440), with an equivalent result also appearing in Sun ([22], pt. 3, Theorem 3.3). To the best of our knowledge, at the time of the discovery of this formula in 1991 the vanishing of $H_{\lfloor p/8 \rfloor}$ mod $p$ was not recognized as a condition for the failure of the first case of FLT for the exponent $p$, and this only became evident with the appearance of the 1995 paper of Dilcher and Skula [3] discussed below.

The result for $N = 10$ is due to a 1992 paper by Z.H. Sun ([22], pt. 3, Theorem 3.1). Since the vanishing of all the individual components was by then known to be a necessary condition for the failure of the first case of FLT for the exponent $p$, the same thing automatically became true for $H_{\lfloor p/10 \rfloor}$.

The result for $N = 12$ is also derived from the 1991 paper of Williams ([26], p. 440), and that for $N = 24$ is from a 2011 paper of Kuzumaki & Urbanowicz ([14], p. 139).

In their landmark joint paper of 1995, Dilcher and Skula [3] proved among other things that the vanishing mod $p$ of $H_{\lfloor p/N \rfloor}$ was a necessary criterion for the failure of the first case of FLT for the exponent $p$, for all $N$ from 2 to 46. This result gives retrospective interest to the evaluations of $H_{\lfloor p/8 \rfloor}$ and $H_{\lfloor p/12 \rfloor}$, and furnishes the main motivation for the present study.

1.1 The Divisibility of Harmonic Numbers

It may be helpful to distinguish the purpose of the present study with that of the more general problem of the divisibility of Harmonic numbers. Eswarathasan and Levine [8], noting that all primes divide the Harmonic numbers of indices $p - 1$, $p(p - 1)$, and $p^2 - 1$, defined harmonic primes as primes that divide only those Harmonic numbers, and anharmonic primes as those that divide additional Harmonic numbers. From the fact that we consider only cases [1] where $p | H_{\lfloor p/N \rfloor}$ with $N > 1$, it will be evident that any such $p$ is anharmonic, and that we are seeking a subset of primes that divide Harmonic numbers of relatively small index; for example, from Table 4 we see that $N = 2$ is solved by $p = 1093$, implying that $1093 | H_{546}$, while $N = 46$ is solved by $p = 11731$, implying that $11731 | H_{255}$. In constrast, the work of Boyd [1] and Rogers [20] entails, in part, finding Harmonic numbers divisible by relatively small primes, such as the case $11 | H_{3546471722268916272}$. The overlap between our results and theirs is minor.

While the formulae (2) through (10) are framed in terms of $p$, they may conversely be seen as conditions involving the divisibility of Harmonic numbers. For example, with $N = 2$ (the Wieferich primes), we are asking for cases where the numerator of $H_n$ is divisible by $2n + 1$. The still unsolved case $N = 5$ asks whether it is possible for $H_n$ to be divisible by $5n + 1$, $5n + 2$, $5n + 3$, or $5n + 4$.

1.2 Computational Considerations

In the cases where such formulae as (2) through (10) exist for the kinds of special Harmonic numbers with which we are concerned, it would be difficult to
overemphasize their superiority for computational purposes over naïve application of the definition of the Harmonic number as a sum of reciprocals (1). As we noted in [4], the limits attainable by the two methods differ by several orders of magnitude.

The nine cases of $H_{⌊p/N⌋}$ represented by formulae have all been tested to high limits (Table 2) without finding solutions for $H_{⌊p/5⌋}$ and $H_{⌊p/12⌋}$, and this raises the question of whether they ever vanish mod $p$. These two cases have resisted tests to limits that are at least four orders of magnitude greater than the least zeros (wherever one is known) of $H_{⌊p/N⌋}$ with $N ≤ 46$. These limits are at least two orders of magnitude greater than the least zeros (whenever one is known) of all the Fermat quotients $q_p(b)$ with base $b ≤ 100$, than all three known zeros of the Pell quotient (see OEIS A238736 for its zeros), and than the lesser of the known zeros of the Lucas quotient $U_{p−(3/p)}(4,1)/p$ (see OEIS A238490 for its zeros). But if there are mathematical reasons for the observed non-vanishing of $H_{⌊p/5⌋}$ and $H_{⌊p/12⌋}$, they remain elusive.

On the other hand, we are not inclined to attach as much significance to the fact that there are still no known zeros of $H_{⌊p/N⌋}$ for $N = 17, 18, 20, 29, 31, 43$, as these cases require brute-force processing and have still not been particularly well tested. Explicit computation of Harmonic numbers, even using modular arithmetic, is an unappealing scenario; as noted by Schwindt, for fixed $N$ it has algorithmic complexity of order $p \log p$. Where such computation is unavoidable, it can be somewhat accelerated by employing the elementary rearrangement

$$H_n = H_{⌊n/4⌋} + \frac{3}{2} \cdot \sum_{j=⌊n/4⌋+1}^{⌊n/2⌋} \frac{1}{j} + \sum_{k=⌊n/2⌋+1}^{n} \frac{1}{k}, \quad (11)$$

which reduces the number of terms by about $\frac{1}{4}$. The term $H_{⌊n/4⌋}$ could likewise be decomposed in a similar manner, though for the sake of simplicity this was not done in the present study; and more sophisticated rearrangements are possible, where all the even terms are successively generated from the odd terms. But such devices can bring about only modest improvements in runtime.

As to the solution of the remaining cases with $N = 17, 18, 20, 29, 31, 43$, there are indeed heuristic reasons for doubting the efficacy of the present means of testing. If one considers the sequence of least divisors $p$ of Harmonic numbers $H_{⌊p/N⌋}$ for $N = 2$ through 46 (see Table 5), arranged in ascending order of $p$ and displayed on a logarithmic scale (see Figure 1), it is readily seen that the points tend to describe a line of positive curvature.

Thus, within the limits studied, it has been found that the sequence of least divisors $p$ grows at a superexponential rate. This portion of our data represents about a year of processing time on a typical desktop computer with a 3.20 GHz processor. Perhaps a few more such $p$ can be found using the present methods and equipment, but it does not seem reasonable to expect that all six will be (assuming they exist).
Figure 1: Least divisors \( p < 161,600,000 \) of Harmonic numbers \( H_{\lfloor p/N \rfloor} \) for \( N = 2 \) through 46, arranged in ascending order of \( p \); horizontal axis = order of discovery; vertical axis = log \( p \).

2 Previous Calculations

The following account does not aim to be complete, but we hope that it includes all significant contributions to the problem. In referring to earlier work, for the sake of brevity we focus mainly on the results of successful searches. In [4], we noted that a statement in pp. 389–390 of [3], implying that \( H_{\lfloor p/N \rfloor} \equiv 0 \mod p \) has a solution \( p < 2,000 \) for every \( N \) between 2 and 46 other than 5, is incorrect (at least if we require \( p > N \) to ensure that \( H_{\lfloor p/N \rfloor} \) is not a vacuous sum).

The divisors \( p \) of \( H_{\lfloor p/2 \rfloor} \) and \( H_{\lfloor p/4 \rfloor} \) are the Wieferich primes (OEIS A001220). The first of these, 1093, was found in 1913 by Meissner, and the second, 3511, in 1922 by Beeger. The Wieferich primes have inspired some of the most intensive numerical searches ever conducted, but no further instances have been found. The current search record appears to be \( p < 140,000,000,000,000,000,000 \), held by PrimeGrid (see “Wieferich Prime Search” at www.primegrid.com).

The divisors \( p \) of \( H_{\lfloor p/3 \rfloor} \) are the Mirimanoff primes (OEIS A014127). The first of these, 11, was announced by Jacobi in 1828 [12], the second, 1006003, by Kloss in 1965 [13]. No further instances have been found. The current search record appears to be \( p < 970,453,984,500,000 \), held by Dorais and Klyve [6].

We have found no computational literature on \( H_{\lfloor p/5 \rfloor} \), though inevitably, reports of negative results are more difficult to locate. One wonders why such results were not reported by Schwindt [21], since he could have obtained them as a byproduct of his study of the interval \( H_{\lfloor p/5 \rfloor} - H_{\lfloor p/6 \rfloor} \). In any case, we have tested \( H_{\lfloor p/5 \rfloor} \) to a much greater limit than that attainable by Schwindt’s method, without finding any instance where it vanishes mod \( p \).

The divisors of \( H_{\lfloor p/6 \rfloor} \) (OEIS A238201, but ignoring solutions with \( p < 6 \)) were investigated in 1983 by Schwindt [21], whose study employed a rather naive method and only reached \( p < 600,000 \), finding the single nontrivial case \( p = 61 \).
We know of no further computations relating to these numbers until they were obtained in another guise as the zeros of \( q_p(432) \) by Richard Fischer [9], whose ongoing test, which had reached \( p < 50,998,592,653,399 \) as of 19 May 2014, has found two further solutions. All three solutions found by these writers were rediscovered in the course of the present study, and are included in Tables 3 and 4 below.

3 The Present Calculations

The present study, which supersedes the corresponding section of [5], does not attempt to extend existing calculations in the cases \( N = 2, 3, 4, 6 \), and relies on known values for those cases in the tables below. Although Cikánek [2] also gives similar conditions for the first case of FLT for \( N \) up to 94 (with certain qualifications), it was decided to focus computing resources on the original range considered by Dilcher and Skula.

In [4], we reported that there are no solutions of (1) with \( p < 3,750,000 \) for \( N = 5, 12, 13, 17, 18, 20, 29, 30, 31, 43 \). Since then, the search has been extended more than forty-fold to 161,600,000, and solutions have been found for \( N = 13 \) and \( N = 30 \) (see Table 4 below), leaving unsettled only eight cases out of 45. For values obtainable by the formulae (2) through (10), the work was performed in PARI, which is excellently suited for efficient modular arithmetic involving recurrence sequences, and it has now been run almost continuously on a typical desktop computer for about one year. The following limits have been attained:

| \( N \) | limit of \( p \) |
|---|---|
| 5 | 3,799,000,000,000 |
| 8 | 960,900,000,000 |
| 10 | 960,900,000,000 |
| 12 | 3,622,000,000,000 |
| 24 | 960,900,000,000 |

For those values of \( N \) necessitating a brute-force search, the work was performed concurrently on a separate machine, and as previously noted it also represents about a year of processing time. After testing several programming languages, the best runtimes were again found to be achieved in PARI, which is very efficient at calculating modular inverses. To expedite the computations, values of \( H_{\lfloor p/N \rfloor} \) were obtained either by addition or by subtraction from the nearest neighbor that could be calculated by formula. Thus, \( H_{\lfloor p/7 \rfloor} \) was obtained from \( H_{\lfloor p/8 \rfloor} \), \( H_{\lfloor p/9 \rfloor} \) from \( H_{\lfloor p/10 \rfloor} \), \( H_{\lfloor p/N \rfloor} \) with \( N = 11, 13, 14, 15, 16 \) from \( H_{\lfloor p/12 \rfloor} \), and \( H_{\lfloor p/N \rfloor} \) for \( 17 \geq N \geq 46 \) from \( H_{\lfloor p/24 \rfloor} \). These calculations were separated into five concurrent runs; hence the limits so far attained vary, as follows:
Table 2: Test limits on divisors $p$ of Harmonic numbers $H_{[p/N]}$ for other $N$

| $N$  | limit of $p$               |
|------|--------------------------|
| 7    | 225,700,000              |
| 9    | 236,400,000              |
| 11, 13, 14, 15, 16 | 161,600,000 |
| 17–23| 223,700,000              |
| 25–46| 201,600,000              |

Solutions are still unknown for $N = 5, 12, 17, 18, 20, 29, 31, 43$, and so the work continues.

The results of the searches follow. First (Table 3), we report separately those results for which the search-limits are given in Table 1 above, since these have particular interest and in some cases coincide with OEIS sequences. Then (Table 4) we report all results combined, including those in Table 3. We believe that the zero $p = 31251349243$ for $N = 24$ is the largest presently known for any value of $N < 46$.

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Table 3: Divisors $p$ of Harmonic numbers $H_{[p/N]}$ for which a formula exists

| $N$ | $p$                                                                 | OEIS reference   |
|-----|----------------------------------------------------------------------|------------------|
| 2   | 1093, 3511                                                            | A001220          |
| 3   | 11, 1006003                                                           | A014127          |
| 4   | 1093, 3511                                                            | A001220          |
| 5   | —                                                                     |                  |
| 6   | 61, 1680023, 7308036881                                               | A238201          |
| 8   | 269, 8573, 1300709, 11740973, 241078561                                 |                  |
| 10  | 227, 17539, 4750159                                                   |                  |
| 12  | —                                                                     |                  |
| 24  | 137, 577, 247421, 307639, 366019, 5262591617, 31251349243              |                  |
Table 4: Divisors $p$ of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for all $N$, 2 through 46

| $N$ | $p$ | $N$ | $p$ |
|-----|-----|-----|-----|
| 2   | 1093, 3511 | 25  | 137 |
| 3   | 11, 1006003 | 26  | 137, 67939 |
| 4   | 1093, 3511 | 27  | 137, 23669 |
| 5   | — | 28  | 20101 |
| 6   | 61, 1680023, 7308036881 | 29  | — |
| 7   | 652913 | 30  | 27089407 |
| 8   | 269, 8573, 1300709, 11740973, 241078561 | 31  | — |
| 9   | 677, 6691 | 32  | 761 |
| 10  | 227, 17539, 4750159 | 33  | 761 |
| 11  | 246277, 1156457 | 34  | 1553 |
| 12  | — | 35  | 4139, 4481, 4598569 |
| 13  | 43214711 | 36  | 1297 |
| 14  | 2267, 6898819 | 37  | 1439, 26833 |
| 15  | 134227 | 38  | 2473, 3527, 4047089 |
| 16  | 38723, 38993, 4292543 | 39  | 407893 |
| 17  | — | 40  | 509, 177553 |
| 18  | — | 41  | 509, 151883 |
| 19  | 521, 911 | 42  | 509, 190657 |
| 20  | — | 43  | — |
| 21  | 1423, 5693, 5782639, 212084723 | 44  | 6967, 27361 |
| 22  | 2843 | 45  | 609221 |
| 23  | 137, 264391 | 46  | 11731 |
| 24  | 137, 577, 247421, 307639, 366019, 5262591617, 31251349243 | — | — |
Table 5: Least divisors $p < 161,600,000$ of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for $N = 2$ through 46, arranged in ascending order of $p$

| $p$   | $N$  |
|-------|------|
| 11    | 3    |
| 61    | 6    |
| 137   | 23   |
| 137   | 24   |
| 137   | 25   |
| 137   | 26   |
| 137   | 27   |
| 227   | 10   |
| 269   | 8    |
| 509   | 40   |
| 509   | 41   |
| 509   | 42   |
| 521   | 19   |
| 677   | 9    |
| 761   | 32   |
| 761   | 33   |
| 1093  | 2    |
| 1093  | 4    |
| 1297  | 36   |
| 1423  | 21   |
| 1439  | 37   |
| 1553  | 34   |
| 2267  | 14   |
| 2473  | 38   |
| 2843  | 22   |
| 4139  | 35   |
| 6967  | 44   |
| 11731 | 46   |
| 20101 | 28   |
| 38723 | 16   |
| 134227| 15   |
| 246277| 11   |
| 407893| 39   |
| 609221| 45   |
| 652913| 7    |
| 27089407| 30 |
| 43214711| 13 |
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