Brownian motion and thermal capacity

Davar Khoshnevisan Yimin Xiao
University of Utah Michigan State University

September 5, 2012

Abstract

Let $W$ denote $d$-dimensional Brownian motion. We find an explicit formula for the essential supremum of Hausdorff dimension of $W(E) \cap F$, where $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$ are arbitrary nonrandom compact sets. Our formula is related intimately to the thermal capacity of Watson (1978). We prove also that when $d \geq 2$, our formula can be described in terms of the Hausdorff dimension of $E \times F$, where $E \times F$ is viewed as a subspace of space-time.

Keywords: Brownian motion, thermal capacity, Euclidean and space-time Hausdorff dimension.

AMS 2000 subject classification: Primary 60J65, 60G17; Secondary 28A78, 28A80, 60G15, 60J45.

1 Introduction

Let $W := \{W(t)\}_{t \geq 0}$ denote standard $d$-dimensional Brownian motion where $d \geq 1$. The principal aim of this paper is to describe the Hausdorff dimension $\dim_H(W(E) \cap F)$ of the random intersection set $W(E) \cap F$, where $E$ and $F$ are compact subsets of $(0, \infty)$ and $\mathbb{R}^d$, respectively. This endeavor solves what appears to be an old problem in the folklore of Brownian motion; see Mörters and Peres [17, p. 289].

In general, the Hausdorff dimension of $W(E) \cap F$ is a random variable, and hence we seek only to compute the $L^\infty(P)$-norm of that Hausdorff dimension. The following example—due to Gregory Lawler—highlights the preceding assertion: Consider $d = 1$, and set $E := \{1\} \cup [2, 3]$ and $F := [1, 2]$. Also consider the two events

$$A_1 := \{1 \leq W(1) \leq 2, W([2, 3]) \cap [1, 2] = \emptyset\},$$
$$A_2 := \{W(1) \notin [1, 2], W([2, 3]) \subset [1, 2]\}.$$  

1.1

*Research supported in part by NSF grant DMS-1006903.
Evidently $A_1$ and $A_2$ are disjoint; and each has positive probability. However, \( \dim_{\mu}(W(E) \cap F) = 0 \) on $A_1$, whereas \( \dim_{\mu}(W(E) \cap F) = 1 \) on $A_2$. Therefore, \( \dim_{\mu}(W(E) \cap F) \) is nonconstant, as asserted.

Our first result describes our contribution in the case that $d \geq 2$. In order to describe that contribution let us define $\rho$ to be the \textit{parabolic metric} on “space-time” $\mathbb{R}_+ \times \mathbb{R}^d$; that is,

\[
\rho((s, x); (t, y)) := \max \left( |t - s|^{1/2}, \|x - y\| \right).
\]

(1.2)

The metric space $\mathbf{S} := (\mathbb{R}_+ \times \mathbb{R}^d, \rho)$ is also called \textit{space time}, and Hausdorff dimension of the compact set $E \times F$—viewed as a set in $\mathbf{S}$—is denoted by \( \dim_{\mu}(E \times F; \rho) \). That is, \( \dim_{\mu}(E \times F; \rho) \) is the supremum of $s \geq 0$ for which

\[
\lim_{\epsilon \to 0} \inf \left( \sum_{j=1}^{\infty} |\rho\text{-diam}(E_j \times F_j)|^s \right) < \infty,
\]

(1.3)

where the infimum is taken over all closed covers \( \{E_j \times F_j\}_{j=1}^{\infty} \) of $E \times F$ with \( \rho\text{-diam}(E_j \times F_j) < \epsilon \), and “\( \rho\text{-diam}(\Lambda) \)” denotes the diameter of the space-time set $\Lambda$, as measured by the metric $\rho$.

**Theorem 1.1.** If $d \geq 2$, then

\[
\|\dim_{\mu}(W(E) \cap F)\|_{L^\infty(P)} = \dim_{\mu}(E \times F; \rho) - d,
\]

(1.4)

where “\( \dim_{\mu} A < 0 \)” means “$A = \emptyset$.” Display (1.4) \textit{continues to hold for $d = 1$, provided that “=” is replaced by “$\leq$.”}

The following example shows that (1.4) does not always hold for $d = 1$: Consider $E := [0, 1]$ and $F := \{0\}$. Then, a computation on the side shows that \( \dim_{\mu}(W(E) \cap F) = 0 \) a.s., whereas \( \dim_{\mu}(E \times F; \rho) - d = 1 \).

On the other hand, Proposition 1.2 below shows that if $|F| > 0$, where $| \cdot |$ denotes the Lebesgue measure, then $W(E) \cap F$ shares the properties of the image set $W(E)$.

**Proposition 1.2.** If $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and $|F| > 0$, then

\[
\|\dim_{\mu}(W(E) \cap F)\|_{L^\infty(P)} = \min \{d, 2 \dim_{\mu} E\}.
\]

(1.5)

If \( \dim_{\mu} E > 1/2 \) and $d = 1$, then $P\{W(E) \cap F \geq 0\} > 0$.

When $F \subset \mathbb{R}^d$ satisfies $|F| > 0$, it can be shown that \( \dim_{\mu}(E \times F; \rho) = 2 \dim_{\mu} E + d \). Hence (1.5) coincides with (1.4) when $d \geq 2$. Proposition 1.2 is proved by showing that, when $|F| > 0$, there exists an explicit “smooth” random measure on $W(E) \cap F$. Thus, the remaining case, and this is the most interesting case, is when $F$ has Lebesgue measure 0. The following result gives a suitable [though quite complicated] formula that is valid for all dimensions, including $d = 1$. 2
Theorem 1.3. If $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and $|F| = 0$, then

$$\|\dim (W(E) \cap F)\|_{L^\infty(P)} = \sup \left\{ \gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} E_\gamma(\mu) < \infty \right\},$$

(1.6)

where $\mathcal{P}_d(E \times F)$ denotes the collection of all probability measures $\mu$ on $E \times F$ that are “diffuse” in the sense that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$E_\gamma(\mu) := \int\int \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma} \mu(ds \, dx) \mu(dt \, dy).$$

(1.7)

Theorems 1.1 and 1.3 are the main results of this paper. But it seems natural that we also say a few words about when $W(E) \cap F$ is nonvoid with positive probability, simply because when $P\{W(E) \cap F = \emptyset\} = 1$ there is no point in computing the Hausdorff dimension of $W(E) \cap F$!

It is a well-known folklore fact that $W(E)$ intersects $F$ with positive probability if and only if $E \times F$ has positive thermal capacity in the sense of Watson [23, 24]. (For a simpler description see Proposition 1.4 below.) This folklore fact can be proved by combining the results of Doob [4] on parabolic potential theory; specifically, one applies the analytic theory of [4, Chapter XVII] in the context of space-time Brownian motion as in [4, §13, pp. 700–702]. When combined with Theorem 3 of Taylor and Watson [22], this folklore fact tells us the following: If

$$\dim(E \times F ; \varrho) > d,$$

(1.8)

then $W(E) \cap F$ is nonvoid with positive probability; but if $\dim(E \times F ; \varrho) < d$ then $W(E) \cap F = \emptyset$ almost surely. Kaufman and Wu [11] contain related results. And our Theorem 1.1 states that the essential supremum of the Hausdorff dimension of $W(E) \cap F$ is the slack in the Taylor–Watson condition (1.8) for the nontriviality of $W(E) \cap F$.

The proof of Theorem 1.3 yields a simpler interpretation of the assertion that $E \times F$ has positive thermal capacity, and relates one of the energy forms that appear in Theorem 1.3, namely $E_0$, to the present context. For the sake of completeness, we state that interpretation next in the form of Proposition 1.4.

Proposition 1.4. Suppose $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and has Lebesgue measure 0. Then $P\{W(E) \cap F \neq \emptyset\} > 0$ if and only if there exists a probability measure $\mu \in \mathcal{P}_d(E \times F)$ such that $E_0(\mu) < \infty$.

Theorems 1.1 and 1.3 both proceed by checking to see whether or not $W(E) \cap F$ [and a close variant of it] intersect a sufficiently-thin random set. This so-called “codimension idea” was initiated by S.J. Taylor [21] and has been used in other situations as well [6, 15, 19]. A more detailed account of the history of stochastic codimension can be found in the recent book of Mörters and Peres [17, p. 287].
The broad utility of this method—using fractal percolation sets as the [thin] testing random sets—was further illustrated by Yuval Peres [18].

Throughout this paper we adopt the following notation: For all integers \( k \geq 1 \) and for every \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \), \( \|x\| \) and \( |x| \) respectively define the \( \ell^2 \) and \( \ell^1 \) norms of \( x \). That is,

\[
\|x\| := (x_1^2 + \cdots + x_k^2)^{1/2} \quad \text{and} \quad |x| := |x_1| + \cdots + |x_k|.
\]

(1.9)

The rest of the paper is organized as follows. Proposition 1.2 is proved in Section 2. Then, in Sections 5 and 3, Theorems 1.1 and 1.3 are proved in reverse order, since the latter is significantly harder to prove. The main ingredient for proving Theorem 1.3 is Theorem 3.1 whose proof is given in Section 4. Proposition 1.4 is proved in Section 4.5.

2 Proof of Proposition 1.2

The upper bound in (1.5) follows from the well-known fact that \( \dim_H W(E) = \min\{d, 2 \dim_H E\} \) almost surely. In order to establish the lower bound in (1.5), we first construct a random measure \( \nu \) on \( W(E) \cap F \), and then appeal to a capacity argument. The details follow.

Choose and fix a constant \( \gamma \) such that

\[
0 < \gamma < \min\{d, 2 \dim_H E\}.
\]

(2.1)

According to Frostman’s theorem, there exists a Borel probability measure \( \sigma \) on \( E \) such that

\[
\int \int \frac{\sigma(ds) \sigma(dt)}{|s - t|^{\gamma/2}} < \infty.
\]

(2.2)

For every integer \( n \geq 1 \), we define a random measure \( \mu_n \) on \( E \times F \) via

\[
\int f \, d\mu_n := (2\pi n)^{d/2} \int_{E \times F} f(s, x) \exp \left( -\frac{n \|W(s) - x\|^2}{2} \right) \sigma(ds) \, dx,
\]

(2.3)

for every Borel measurable function \( f : E \times F \to \mathbb{R}_+ \). Equivalently,

\[
\int f \, d\mu_n = \int_{E \times F} \sigma(ds) \, dx \int_\mathbb{R}^d d\xi \exp \left( i\langle \xi, W(s) - x \rangle - \frac{\|\xi\|^2}{2n} \right),
\]

(2.4)

thanks to the characteristic function of a Gaussian vector.

Let \( \nu_n \) be the image measure of \( \mu_n \) under the random mapping \( g : E \times F \to \mathbb{R}_d \) defined by \( g(s, x) := W(s) \). That is, \( \int \phi \, d\nu_n := \int (\phi \circ g) \, d\mu_n \) for all Borel-measurable functions \( \phi : \mathbb{R}_d \to \mathbb{R}_+ \). It follows from (2.3) that, if \( \{\nu_n\}_{n=1}^\infty \) has a subsequence which converges weakly to \( \nu \), then \( \nu \) is supported on \( W(E) \cap F \). This \( \nu \) will be the desired random measure on \( W(E) \cap F \). Thus, we plan to prove that: (i) \( \{\nu_n\}_{n=1}^\infty \) indeed has a subsequence which converges weakly; and
(ii) Use this particular ν to show that P{dimₙ(W(E) ∩ F) ≥ γ} > 0. This will demonstrate (1.5).

In order to carry out (i) and (ii), it suffices to verify that there exist positive and finite constants c₁, c₂, and c₃ such that

\[ E(\|\nu_n\|) ≥ c₁, \quad E(\|\nu_n\|^2) ≤ c₂, \quad (2.5) \]

and

\[ E \int \frac{\nu_n(dx) \nu_n(dy)}{\|x - y\|^\gamma} ≤ c₃, \quad (2.6) \]

simultaneously for all n ≥ 1, where \(\|\nu_n\| := \nu_n(\mathcal{R})\) denote the total mass of \(\nu_n\).

The rest hinges on a well-known capacity argument that is explicitly hashed out in [8, pp. 204-206]; see also [13, pp. 75-76].

It follows from (2.4) and Fubini’s theorem that

\[ E(\|\nu_n\|) = \frac{1}{(2\pi(s + n - 1))^{d/2}} \exp \left(-\frac{\|x\|^2}{2(s + n - 1)}\right) \sigma(ds) \ dx. \quad (2.7) \]

Since \(E \subset (0, \infty)\) is compact, we have \(\inf E ≥ \delta\) for some constant \(\delta > 0\). Hence, (2.7) implies that \(\inf_{n \geq 1} E(\|\nu_n\|) ≥ c₁\) for some constant \(c₁ > 0\), and this verifies the first inequality in (2.5). For the second inequality, we use (2.3) to see that

\[ \|\nu_n\| = \|\mu_n\| = (2\pi n)^{d/2} \int E \exp \left(-\frac{n\|W(s) - x\|^2}{2}\right) \sigma(ds) \ dx. \quad (2.8) \]

We may replace \(F\) by all of \(\mathcal{R}^d\) in order to find that \(\|\nu_n\| ≤ (2\pi)^d\) a.s.; whence follows the second inequality in (2.5). Similarly, we prove (2.6) by writing

\[ \int \frac{\nu_n(dx) \nu_n(dy)}{\|x - y\|^\gamma} = \int_{(E \times F)^2} \frac{\sigma(ds) \sigma(dt) dx \ dy}{\|W(t) - W(s)\|^\gamma} \times (2\pi n)^d \exp \left(-\frac{n\|W(s) - x\|^2 - n\|W(t) - y\|^2}{2}\right). \]

We may replace \(F\) by \(\mathcal{R}^d\), use the scaling property of \(W\) and the fact that \(\gamma < d\) in order to see that

\[ E \int \frac{\nu_n(dx) \nu_n(dy)}{\|x - y\|^\gamma} ≤ c \int \frac{\sigma(ds) \sigma(dt)}{|s - t|^{\gamma/2}} \quad \text{a.s.} \]

Therefore, (2.6) follows from (2.2).

Finally, we prove the last statement in Proposition 1.2. Since \(\dimₙ E > 1/2\), Frostman’s theorem assures us that there exists a Borel probability measure \(\sigma\) on \(E\) such that (2.2) holds with \(\gamma = 1\). We construct a sequence of random measures \(\{\nu_n\}_{n=1}^\infty\) as before, and extract a subsequence that converges weakly to a random Borel measure \(\nu\) on \(W(E) \cap F\) such that \(P\{\|\nu\| > 0\} > 0\).
Let \( \hat{\nu} \) denote the Fourier transform of \( \nu \). In accord with Plancherel's theorem, a sufficient condition for \( P\{|W(E) \cap F| > 0\} > 0 \) is that \( \hat{\nu} \in L^2(\mathbb{R}) \). We apply Fatou’s lemma to reduce our problem to the following:

\[
\sup_{n \geq 1} \mathbb{E} \int_{-\infty}^{\infty} |\hat{\nu}_n(\theta)|^2 \, d\theta < \infty. \tag{2.9}
\]

By (2.4) and Fubini’s theorem,

\[
\mathbb{E} \int_{-\infty}^{\infty} |\hat{\nu}_n(\theta)|^2 \, d\theta = \int_{-\infty}^{\infty} d\theta \mathbb{E} \int_{\mathbb{R}^2} \mu_n(ds \, dx) \mu_n(dt \, dy) e^{i\theta(W(s)-W(t))} \\
= \int_{-\infty}^{\infty} d\theta \int_{(E \times F)^2} \sigma(ds) \sigma(dt) \, dx \, dy \int_{\mathbb{R}^2} d\xi \, d\eta \exp \left( -i(\xi x + \eta y) - \frac{\xi^2 + \eta^2}{2n} \right) E \left( e^{i[(\xi + \eta)W(s) + (\eta - \theta)W(t)]} \right). \tag{2.10}
\]

When \( 0 < s < t \), this last expectation can be written as

\[
\exp \left( -\frac{s}{2} (\xi + \eta)^2 - \frac{t-s}{2} (\eta - \theta)^2 \right). \tag{2.11}
\]

By plugging this into (2.10), we can write the triple integral in \([d\theta \, d\xi \, d\eta]\) of (2.10) as

\[
\int_{\mathbb{R}^2} e^{-i(\xi x + \eta y)} \exp \left( -\frac{\xi^2 + \eta^2}{2n} - \frac{s}{2} (\xi + \eta)^2 \right) \, d\xi \, d\eta \\
\times \int_{-\infty}^{\infty} \exp \left( -\frac{t-s}{2} (\eta - \theta)^2 \right) \, d\theta \tag{2.12}
\]

where \( p(x, y) \) denotes the joint density function of a bivariate normal distribution with mean vector 0 and covariance matrix \( \Gamma^{-1} \), where

\[
\Gamma := \begin{pmatrix}
  s + n^{-1} & s \\
  s & s + n^{-1}
\end{pmatrix}. \tag{2.13}
\]

We plug (2.12) into (2.10), replace \( F \) by \( \mathbb{R}^d \) to integrate \([dx \, dy]\) in order to find that

\[
\sup_{n \geq 1} \mathbb{E} \int_{-\infty}^{\infty} |\hat{\nu}_n(\theta)|^2 \, d\theta \leq \text{const} \cdot \iint \sigma(ds) \sigma(dt) \frac{1}{|s-t|^{1/2}} < \infty. \tag{2.14}
\]

This yields (2.9) and finishes the proof of Proposition 1.2.
Here and throughout, \( B_x(\epsilon) := \{ y \in \mathbb{R}^d : \| x - y \| \leq \epsilon \} \) denotes the radius-\( \epsilon \) ball about \( x \in \mathbb{R}^d \). Also, define \( \nu_d \) to be the volume of \( B_0(1) \); that is,
\[
\nu_d := \frac{2 \cdot \pi^{d/2}}{d \Gamma(d/2)}.
\]
(3.2)

Recall that \( \{ W(t) \}_{t \geq 0} \) denotes a standard Brownian motion in \( \mathbb{R}^d \), and consider the following “parabolic Green function”: For all \( t > 0 \) and \( x \in \mathbb{R}^d \):
\[
p_t(x) := e^{-\|x\|^2/(2t)} \cdot \frac{1}{(2\pi t)^{d/2}} \cdot 1_{(0,\infty)}(t).
\]
(3.3)

The seemingly-innocuous indicator function plays an important role in the sequel; this form of the heat kernel appears earlier in Watson [24, 23] and Doob [4, (4.1), p. 266].

As indicated in the Introduction, our proof of Theorem 1.3 is based on the co-dimension argument to check whether or not \( W(E) \cap F \) intersect a sufficiently-thin “testing” random set. One example of such testing sets could be the range of a stable Lévy process \( X = \{ X(t) \}_{t \geq 0} \) in \( \mathbb{R}^d \) with index \( \alpha \in (0, 2] \). However, this choice will only work for \( d \leq 3 \) due to the fact that, for any Borel set \( G \subset \mathbb{R}^d \) with \( \dim_H G < d - \alpha \), \( X(\mathbb{R}^+) \cap G = \emptyset \) a.s.

To avoid this restriction and for future applications, we will use the range of an \( N \)-parameter additive stable Lévy process with index \( \alpha \) as the testing set for proving Theorem 1.3.

Let \( X^{(1)}, \ldots, X^{(N)} \) be \( N \) isotropic stable processes with common stability index \( \alpha \in (0, 2] \). We assume that the \( X^{(j)} \)'s are totally independent from one another, as well as from the process \( W \), and all take their values in \( \mathbb{R}^d \). We assume also that \( X^{(1)}, \ldots, X^{(N)} \) have right-continuous sample paths with left-limits. This assumption can be—and will be—made without incurring any real loss in generality. Finally, our normalization of the processes \( X^{(1)}, \ldots, X^{(N)} \) is described as follows:
\[
E \left[ \exp \left( i \langle \xi, X^{(k)}(1) \rangle \right) \right] = e^{-\|\xi\|^\alpha/2} \quad \text{for all} \quad 1 \leq k \leq N \quad \text{and} \quad \xi \in \mathbb{R}^d.
\]
(3.4)

Define the corresponding additive stable process \( X_\alpha := \{ X_\alpha(t) \}_{t \in \mathbb{R}_+^N} \) as
\[
X_\alpha(t) := \sum_{k=1}^N X^{(k)}(t_k) \quad \text{for all} \quad t := (t_1, \ldots, t_N) \in \mathbb{R}_+^N.
\]
(3.5)

Also, define \( C_\gamma \) to be the capacity corresponding to the energy form (1.7). That is, for all compact sets \( U \subset \mathbb{R}_+^N \times \mathbb{R}^d \) and \( \gamma \geq 0 \),
\[
C_\gamma(U) := \left[ \inf_{\mu \in \mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu) \right]^{-1}.
\]
(3.6)
Theorem 3.1. If \( d > \alpha N \) and \( F \subset \mathbb{R}^d \) has Lebesgue measure 0, then

\[
P \left\{ W(E) \cap \overline{X_{\alpha}(\mathbb{R}^N)} \cap F \neq \emptyset \right\} > 0 \iff C_{d-\alpha N}(E \times F) > 0.
\] (3.7)

Here and in the sequel, \( \overline{A} \) denotes the closure of \( A \).

Remark 3.2. It can be proved that the same result continues to hold even if we remove the closure sign. We will not delve into this here because we do not need the said refinement. \( \Box \)

We can now apply Theorem 3.1 to prove Theorem 1.3. Theorem 3.1 will be established subsequently.

Proof of Theorem 3.1. Suppose \( \alpha \in (0, 2] \) and \( N \in \mathbb{Z}_+ \) are chosen such that \( d > \alpha N \). If \( X_{\alpha} \) denotes an \( N \)-parameter additive stable process \( \mathbb{R}^d \) whose index is \( \alpha \in (0, 2] \), then [12, Example 2, p. 436]

\[
\text{codim } \overline{X_{\alpha}(\mathbb{R}^N)} = d - \alpha N.
\] (3.8)

This means that the closure of \( X_{\alpha}(\mathbb{R}^N) \) will intersect any nonrandom Borel set \( G \subset \mathbb{R}^d \) with \( \text{dim}_H(G) > d - \alpha N \), with positive probability; whereas the closure of \( X_{\alpha}(\mathbb{R}^N) \) does not intersect any \( G \) with \( \text{dim}_H(G) < d - \alpha N \), almost surely.

Define

\[
\Delta := \sup \left\{ \gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}
\] (3.9)

with the convention that \( \sup \emptyset = 0 \).

If \( \Delta > 0 \) and \( d - \alpha N < \Delta \), then \( C_{d-\alpha N}(E \times F) > 0 \). It follows from Theorem 3.1 and (3.8) that

\[
P \{ \text{dim}_H(W(E) \cap F) \geq d - \alpha N \} > 0.
\] (3.10)

Because \( d - \alpha N \in (0, \Delta) \) is arbitrary, we have \( \| \text{dim}_H(W(E) \cap F) \|_{L^\infty(P)} \geq \Delta \).

Similarly, Theorem 3.1 and (3.8) imply that

\[
d - \alpha N > \Delta \Rightarrow \| \text{dim}_H(W(E) \cap F) \|_{L^\infty(P)} \leq d - \alpha N \quad \text{almost surely.}
\] (3.11)

Hence \( \| \text{dim}_H(W(E) \cap F) \|_{L^\infty(P)} \leq \Delta \) whenever \( \Delta \geq 0 \). This proves the theorem. \( \Box \)

4 Proof of Theorem 3.1

Our proof of Theorem 3.1 is divided into separate parts. We begin by developing a requisite result in harmonic analysis. Then, we develop some facts about additive Lévy processes. After that, we prove Theorem 3.1 in two separate parts.

8
4.1 Isoperimetry

Recall that a function \( \kappa : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}^+ \) is tempered if it is measurable and
\[
\int_{\mathbb{R}^n} \frac{\kappa(x)}{(1 + \|x\|)^m} \, dx < \infty \quad \text{for some } m \geq 0.
\] (4.1)

A function \( \kappa : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}^+ \) is said to be positive definite if it is tempered and for all rapidly-decreasing test functions \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \),
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x) \kappa(x-y) \phi(y) \, dx \, dy \geq 0.
\] (4.2)

We make heavy use of the following result of Foondun and Khoshnevisan [5, Corollary 3.7]; for a weaker version see [14, Theorem 5.2].

**Lemma 4.1.** If \( \kappa : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}^+ \) is positive definite and lower semicontinuous, then for all finite Borel measures \( \mu \) on \( \mathbb{R}^n \),
\[
\iint \kappa(x-y) \mu(dx) \mu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\kappa}(|\hat{\mu}|) \, d\xi.
\] (4.3)

Here and in the sequel, \( \hat{g} \) denotes the Fourier transform of a function (or a measure) \( g \).

Lemma 4.1 implies two “isoperimetric inequalities” that are stated below as Propositions 4.2 and 4.4. Recall that a finite Borel measure \( \nu \) on \( \mathbb{R}^d \) is said to be positive definite if \( \hat{\nu}(|\hat{\sigma}|) \geq 0 \) for all \( \xi \in \mathbb{R}^d \).

**Proposition 4.2.** Suppose \( \kappa : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) is a lower semicontinuous positive-definite function such that \( \kappa(x) = \infty \) iff \( x = 0 \). Suppose \( \nu \) and \( \sigma \) are two positive definite probability measures on \( \mathbb{R}^d \) that satisfy the following:

1. \( \kappa \) and \( \kappa \ast \nu \) are uniformly continuous on every compact subset of \( \{ x \in \mathbb{R}^d : \|x\| \geq \eta \} \) for each \( \eta > 0 \); and

2. \( (\tau, x) \mapsto (p_\tau \ast \sigma)(x) \) is uniformly continuous on every compact subset of \( \{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d : t \wedge \|x\| \geq \eta \} \) for each \( \eta > 0 \).

Then, for all finite Borel measures \( \mu \) on \( \mathbb{R}_+ \times \mathbb{R}^d \),
\[
\iint (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu(dt \, dx) \mu(ds \, dy) \leq \iint p_{|t-s|}(x-y) \kappa(x-y) \mu(dt \, dx) \mu(ds \, dy).
\] (4.4)

**Remark 4.3.** The very same proof shows the following slight enhancement: Suppose \( \kappa \) and \( \nu \) are the same as in Proposition 4.2. If \( \sigma_1 \) and \( \sigma_2 \) share the
Therefore, for fixed 
\( t > s \) all 
\( \xi \) for all 
\( \xi \) and 
\( \kappa \). For all 
\( \eta > 0 \) define 
\( \nu \). Products of positive-definite functions are positive definite themselves. Therefore, for fixed 
\( t > s \), Lemma 4.1 applies, and tells us that for all Borel probability measures \( \rho \) on \( \mathbb{R}^d \), and for all nonnegative \( t \neq s \),

\[
\iint (p_{t-s} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \rho(dx) \rho(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} d\zeta \ e^{-(t-s)\|\xi\|^2/2} \sigma(\xi) \varsigma(\zeta) \rho(\xi - \zeta) |\rho(\xi)|^2.
\]

Because the preceding is valid also when \( \sigma = \nu = \delta_0 \), and since \( 0 \leq \hat{\sigma}(\xi), \hat{\nu}(\xi) \leq 1 \) for all \( \xi \in \mathbb{R}^d \), it follows that for all nonnegative \( t \neq s \),

\[
\iint (p_{t-s} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \rho(dx) \rho(dy) \leq \iint p_{t-s}(x-y)\kappa(x-y) \rho(dx) \rho(dy).
\]

This inequality continues to hold when \( \rho \) is a finite Borel measure on \( \mathbb{R}^d \), by scaling. Thus, thanks to Tonelli’s theorem, the proposition is valid whenever \( \mu(dx) = \lambda(dt)\rho(dx) \) for two finite Borel measures \( \lambda \) and \( \rho \), respectively defined on \( \mathbb{R}_+ \) and \( \mathbb{R}^d \).

Now let us consider an compactly-supported finite measure \( \mu \) on \( \mathbb{R}_+ \times \mathbb{R}^d \). For all \( \eta > 0 \) define

\[
\mathcal{G}(\eta) := \{(t, s, x, y) \in (\mathbb{R}_+)^2 \times (\mathbb{R}^d)^2 : |t - s| \wedge \|x - y\| \geq \eta\}.
\]

It suffices to prove that for all \( \eta > 0 \),

\[
\iint_{\mathcal{G}(\eta)} (p_{t-s} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu(dx) \mu(dy) \leq \iint_{\mathcal{G}(\eta)} p_{t-s}(x-y)\kappa(x-y) \mu(dx) \mu(dy).
\]

\( \square \)
This is so, because \( \kappa(0) = \infty \) and (4.6) readily tell us that \( \mu \times \mu \) does not charge
\[
\{ (t, s, x, y) \in (\mathbb{R}_+)^2 \times (\mathbb{R}^d)^2 : x = y \};
\]
and therefore,
\[
\lim_{\eta \downarrow 0} \iint_{\mathcal{G}(\eta)} (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu(\text{d}t \, \text{d}x) \mu(\text{d}s \, \text{d}y)
\]
\[
= \iint_{\mathbb{R} \setminus \{x \neq y\}} (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu(\text{d}t \, \text{d}x) \mu(\text{d}s \, \text{d}y)
\]
\[
= \iint (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu(\text{d}t \, \text{d}x) \mu(\text{d}s \, \text{d}y).
\]

And similarly,
\[
\lim_{\eta \downarrow 0} \iint_{\mathcal{G}(\eta)} p_{|t-s|}(x-y)\kappa(x-y) \mu(\text{d}t \, \text{d}x) \mu(\text{d}s \, \text{d}y)
\]
\[
= \iint p_{|t-s|}(x-y)\kappa(x-y) \mu(\text{d}t \, \text{d}x) \mu(\text{d}s \, \text{d}y).
\]

And the proposition follows, subject to (4.10).

Next we verify (4.10) to finish the proof.

One can check directly that \( \mathcal{G}(\eta) \cap \text{supp}(\mu \times \mu) \) is compact, and both mappings \((t, s, x, y) \mapsto (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \) and \((t, s, x, y) \mapsto p_{|t-s|}(x-y)\kappa(x-y) \) are uniformly continuous on \( \mathcal{G}(\eta) \cap \text{supp}(\mu \times \mu) \).

We can find finite Borel measures \( \{\lambda_j\}_{j=1}^{\infty} \) on \( \mathbb{R}_+ \) and \( \{\rho_j\}_{j=1}^{\infty} \) on \( \mathbb{R}^d \) such that \( \mu \) is the weak limit of \( \mu_N := \sum_{j=1}^{\infty} (\lambda_j \times \rho_j) \) as \( N \to \infty \). The already-proved portion of this proposition implies that for all \( \eta > 0 \) and \( N \geq 1 \),
\[
\iint_{\mathcal{G}(\eta)} (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(x-y) \mu_N(\text{d}t \, \text{d}x) \mu_N(\text{d}s \, \text{d}y)
\]
\[
\leq \iint_{\mathcal{G}(\eta)} p_{|t-s|}(x-y)\kappa(x-y) \mu_N(\text{d}t \, \text{d}x) \mu_N(\text{d}s \, \text{d}y).
\]

Let \( N \uparrow \infty \) to deduce (4.10) and hence the proposition. \( \square \)

**Proposition 4.4.** Suppose \( \kappa : \mathbb{R} \to \mathbb{R}_+ \) is a lower semicontinuous positive-definite function such that \( \kappa(x) = \infty \iff x = 0 \). Suppose \( \nu \) and \( \sigma \) are two positive definite probability measures, respectively on \( \mathbb{R} \) and \( \mathbb{R}^d \), that satisfy the following:

1. \( \kappa \) and \( \kappa \ast \nu \) are uniformly continuous on every compact subset of \( \{ x \in \mathbb{R} : \| x \| \geq \eta \} \) for each \( \eta > 0 \); and
2. \((\tau, x) \mapsto (p_{|\tau|} \ast \sigma)(x) \) is uniformly continuous on every compact subset of \( \{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d : \| x \| \geq \eta \} \) for each \( \eta > 0 \).
Then, for all finite Borel measures $\mu$ on $\mathbb{R}_+ \times \mathbb{R}^d$,

\[
\iint (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(s-t) \mu(dt \, dx) \mu(ds \, dy) \leq \iint p_{|t-s|}(x-y)\kappa(s-t) \mu(dt \, dx) \mu(ds \, dy) .
\]

(4.15)

Proof. It suffices to prove the proposition in the case that

\[
\mu(ds \, dx) = \lambda(ds)\rho(dx),
\]

(4.16)

for finite Borel measures $\lambda$ and $\rho$, respectively on $\mathbb{R}_+$ and $\mathbb{R}^d$. See, for instance, the argument beginning with (4.9) in the proof of Proposition 4.2. We shall extend the definition $\lambda$ so that it is a finite Borel measure on all of $\mathbb{R}$ in the usual way: If $A \subset \mathbb{R}$ is Borel measurable, then $\lambda(A) := \lambda(A \cap \mathbb{R}_+)$. This slight abuse in notation should not cause any confusion in the sequel.

Tonelli’s theorem and Lemma 4.1 together imply that in the case that (4.16) holds,

\[
\iint (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(s-t) \mu(dt \, dx) \mu(ds \, dy) = \iint \lambda(dt) \lambda(ds) (\kappa \ast \nu)(s-t) e^{-|t-s|\|\xi\|^2/2} \]

(4.17)

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\kappa}(\xi)\hat{\nu}(\xi)|^2 d\xi \iint \lambda(dt) \lambda(ds) (\kappa \ast \nu)(s-t) e^{-|t-s|\|\xi\|^2/2} \]

\[
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\kappa}(\xi)|^2 d\xi \iint \lambda(dt) \lambda(ds) (\kappa \ast \nu)(s-t) e^{-|t-s|\|\xi\|^2/2} .
\]

The map $\tau \mapsto \exp\{ -|\tau| \cdot \|\xi\|^2/2 \}$ is positive definite on $\mathbb{R}$ for every fixed $\xi \in \mathbb{R}^d$; in fact, its inverse Fourier transform is a [scaled] Cauchy density function, which we refer to as $\vartheta_{\xi}$. Therefore, in accord with Lemma 4.1,

\[
\iint (\kappa \ast \nu)(s-t) e^{-|t-s|\|\xi\|^2/2} \lambda(dt) \lambda(ds) = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\lambda}(\tau)|^2 (\hat{\kappa} \ast \vartheta_{\xi})(\tau) d\tau \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\lambda}(\tau)|^2 (\hat{\kappa} \ast \vartheta_{\xi})(\tau) d\tau \]

(4.18)

\[
= \iint \kappa(s-t) e^{-|t-s|\|\xi\|^2/2} \lambda(dt) \lambda(ds) .
\]

The last line follows from the first identity, since we can consider $\nu = \delta_0$ as a possibility. Therefore, it follows from (4.17) and (4.18) that

\[
\iint (p_{|t-s|} \ast \sigma)(x-y)(\kappa \ast \nu)(s-t) \mu(dt \, dx) \mu(ds \, dy) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\kappa}(\xi)|^2 d\xi \iint \lambda(dt) \lambda(ds) \kappa(s-t) e^{-|t-s|\|\xi\|^2/2} \]

\[
= \iint \lambda(dt) \lambda(ds) \kappa(s-t) \iint \rho(dx) \rho(dy) p_{|t-s|}(x-y) ;
\]

12
the last line follows from the first identity in (4.17) by considering the special
case that \( \nu = \delta_0 \) and \( \sigma = \delta_0 \). This proves the proposition in the case that \( \mu \) has
the form (4.16), and the result follows. \( \Box \)

### 4.2 Additive stable processes

In this subsection we develop a “resolvent density” estimate for the additive
stable process \( X_\alpha \).

First of all, note that the characteristic function \( \xi \mapsto \mathbb{E} \exp(i\langle \xi, X_\alpha(t) \rangle) \)
of \( X_\alpha(t) \) is absolutely integrable for every \( t \in \mathbb{R}^d_+ \setminus \{ 0 \} \). Consequently, the
inversion formula applies and tells us that we can always choose the following
as the probability density function of \( X_\alpha(t) \):

\[
g_t(x) := g_t(\alpha; x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i \langle x, \xi \rangle - |t| \cdot \|\xi\|^{\alpha/2}} \, d\xi. \tag{4.19}
\]

**Lemma 4.5.** Choose and fix some \( a, b \in (0, \infty)^N \) such that \( a_j \leq b_j \) for all \( 1 \leq j \leq N \). Define

\[
[a, b] := \{ s \in \mathbb{R}^N_+ : a_j \leq s_j \leq b_j \text{ for all } 1 \leq j \leq N \}. \tag{4.20}
\]

Then, for all \( M > 0 \) there exists a constant \( A_0 \in (1, \infty) \) —depending only on the parameters \( d, N, M, \alpha, \min_{1 \leq j \leq N} a_j, \) and \( \max_{1 \leq j \leq N} b_j \)—such that for all \( x \in [-M, M]^d \),

\[
A_0^{-1} \leq \int_{[a, b]} g_t(x) \, dt \leq A_0. \tag{4.21}
\]

**Proof.** Let \( \vec{1} := (1, \ldots, 1) \in \mathbb{R}^N \). Then, we may also observe the scaling relation,

\[
g_t(x) = |t|^{-d/\alpha} g_{\vec{1}} \left( \frac{x}{|t|^{1/\alpha}} \right), \tag{4.22}
\]

together with the fact \( g_{\vec{1}} \) is an isotropic stable-\( \alpha \) density function on \( \mathbb{R}^d \). The upper bound in (4.21) follows from (4.22) and the boundedness of \( g_{\vec{1}}(z) \).

On the other hand, the lower bound in (4.21) follows from (4.22), the well-known fact that \( g_{\vec{1}}(z) \) is continuous and strictly positive everywhere, and the following standard estimate: For all \( R > 0 \) there exists \( C(R) \in (1, \infty) \) and \( c(R) \in (0, 1) \) such that

\[
c(R) \|z\|^{d+\alpha} \leq g_{\vec{1}}(z) \leq \frac{C(R)}{\|z\|^{d+\alpha}} \text{ for all } z \in \mathbb{R}^d \text{ with } \|z\| \geq R. \tag{4.23}
\]

See [12, Proposition 3.3.1, p. 380], where this is proved for \( R = 2 \). The slightly more general case where \( R > 0 \) is arbitrary is proved in exactly the same manner. \( \Box \)

**Proposition 4.6.** Choose and fix some \( b \in (0, \infty)^N \) and define \([0, b]\) as in
Lemma 4.5, and assume \( d > \alpha N \). Then, for all \( M > 0 \) there exists a constant
\(A_1 \in (1, \infty)\) —depending only on \(d, N, M, \alpha, \min_{1 \leq j \leq N} b_j, \) and \(\max_{1 \leq j \leq N} b_j\) —such that for all \(x \in [-M, M]^d,\)

\[
\frac{1}{A_1 \|x\|^{d-\alpha N}} \leq \int_{[0, b]} g_t(x) \, dt \leq \frac{A_1}{\|x\|^{d-\alpha N}}. \tag{4.24}
\]

**Proof.** Since

\[
\int_{[0, b]} g_t(x) \, dt \leq e|b| \int_{\mathbb{R}^N} e^{-|t|} g_t(x) \, dt, \tag{4.25}
\]

the proof of Proposition 4.1.1 of [12, p. 420] shows that the upper bound in (4.24) holds for all \(x \in \mathbb{R}^d.\)

For the lower bound, we first recall the notation \(\vec{1} := (1, \ldots, 1) \in \mathbb{R}^N,\) and then apply (4.22) and then (4.23) in order to find that

\[
\int_{[0, b]} g_t(x) \, dt = \int_{[0, b]} |t|^{-d/\alpha} g_t \left( \frac{x}{|t|^{1/\alpha}} \right) \, dt \\
\geq \frac{c(1)}{\|x\|^{d-\alpha}} \cdot \int_{t \in [0, b]: |t|^{1/\alpha} \leq \|x\|} |t| \, dt. \tag{4.26}
\]

Clearly, there exists \(R_0 > 0\) sufficiently small such that whenever \(\|x\| \leq R_0,\)

\[
\int_{t \in [0, b]: |t|^{1/\alpha} \leq \|x\|} |t| \, dt \geq \text{const} \cdot \|x\|^\alpha(N+1), \tag{4.27}
\]

and the result follows. On the other hand, if \(\|x\| > R_0,\) then the preceding display still holds uniformly for all \(x \in [-M, M]^d.\) This proves the proposition. \(\square\)

We mention also the following; it is an immediate consequence of Proposition 4.6 and the scaling relation (4.22).

**Lemma 4.7.** Choose and fix some \(b \in (0, \infty)^N\) and define \([0, b]\) as in Lemma 4.5. Then there exists a constant \(A_2 \in (1, \infty)\) —depending only on \(d, N, \alpha, \min_{1 \leq j \leq N} b_j, \) and \(\max_{1 \leq j \leq N} b_j\) —such that for all \(x \in \mathbb{R}^d,\)

\[
\int_{[0, 2b]} g_t(x) \, dt \leq A_2 \int_{[0, b]} g_t(x) \, dt. \tag{4.28}
\]

**Proof.** Let \(M > 1\) be a constant. If \(x \in [-M, M]^d,\) then (4.28) follows from Proposition 4.6. And if \(\|x\| \geq M,\) then, (4.28) holds because of (4.22) and (4.23), together with the well-known fact that \(g_1\) is continuous and strictly positive everywhere; compare with the first line in (4.26). \(\square\)
4.3 First part of the proof

Our goal, in this first half, is to prove the following:

\[
\mathcal{C}_{d-\alpha N}(E \times F) > 0 \quad \Rightarrow \quad P\left\{W(E) \cap X\alpha(R_+^N) \cap F \neq \emptyset\right\} > 0. \tag{4.29}
\]

Because \(E \subset (0, \infty)\) and \(F \subset \mathbb{R}^d\) are assumed to be compact, there exists a \(q \in (1, \infty)\) such that

\[
E \subseteq [q^{-1}, q] \quad \text{and} \quad F \subseteq [-q, q]^d. \tag{4.30}
\]

We will use \(q\) for this purpose unwaiveringly.

Define

\[
f_\epsilon(x) := \frac{1}{\nu_d \epsilon^d} 1_{B_0(\epsilon)}(x) \quad \text{and} \quad \phi_\epsilon(x) := (f_\epsilon * f_\epsilon)(x). \tag{4.31}
\]

For every \(\mu \in \mathcal{P}_d(E \times F)\) and \(\epsilon > 0\) we define a variable \(Z_\epsilon(\mu)\) by

\[
Z_\epsilon(\mu) := \int_{[1,2]^N} du \int_{E \times F} \mu(ds \, dx) \phi_\epsilon(W(s) - x)\phi_\epsilon(X_\alpha(u) - x). \tag{4.32}
\]

**Lemma 4.8.** There exists a constant \(a \in (0, \infty)\) such that

\[
\inf_{\mu \in \mathcal{P}(E \times F)} \inf_{\epsilon \in (0,1)} E[Z_\epsilon(\mu)] \geq a. \tag{4.33}
\]

**Proof.** Thanks to the triangle inequality, whenever \(u \in B_0(\epsilon/2)\) and \(v \in B_0(\epsilon/2)\), then we have also \(u - v \in B_0(\epsilon)\) and \(v \in B_0(\epsilon)\). Therefore, for all \(u \in \mathbb{R}^d\) and \(\epsilon > 0\),

\[
\phi_\epsilon(u) = \frac{1}{\nu_d \epsilon^d} \int_{\mathbb{R}^d} 1_{B_0(\epsilon)}(u - v)1_{B_0(\epsilon)}(v) \, dv \\
\geq \frac{1}{\nu_d \epsilon^d} 1_{B_0(\epsilon/2)}(u) \int_{\mathbb{R}^d} 1_{B_0(\epsilon/2)}(v) \, dv \geq 2^{-d} f_{\epsilon/2}(u). \tag{4.34}
\]

Because \(f_{\epsilon/2}\) is a probability density, and since \(\epsilon \in (0, 1)\), the preceding implies that for all \(u \in [1,2]^N\) and \(x \in \mathbb{R}^d\),

\[
(\phi_\epsilon * g_u)(x) = \int_{\mathbb{R}^d} \phi_\epsilon(u) g_u(x - u) \, du \\
\geq 2^{-d} \int_{\mathbb{R}^d} f_{\epsilon/2}(u) g_u(x - u) \, du \geq 2^{-d} \inf_{\|x - z\| \leq 1/2} g_u(z). \tag{4.35}
\]

Since \(F \subseteq [-q, q]^d\), Lemma 4.5 tells us that

\[
a_0 := \inf_{u \in [1,2]^N} \inf_{x \in F} \inf_{\epsilon \in (0,1)} (\phi_\epsilon * g_u)(x) > 0. \tag{4.36}
\]
And therefore, for all $\epsilon > 0$ and $\mu \in \mathcal{P}(E \times F)$,

$$
E[Z_\epsilon(\mu)] = \int_{E \times F} \mu(\text{d}x) \ (\phi_\epsilon \ast p_s)(x) \int_{[1,2]^N} \text{d}u \ (\phi_\epsilon \ast g_u)(x) \tag{4.37}
$$

$$
\geq a_0 \int_{E \times F} \ (\phi_\epsilon \ast p_s)(x) \mu(\text{d}x) \geq a_0 \inf_{s \in [1/4,q]} \inf_{\epsilon \in (0,1)} \inf_{x \in F} (\phi_\epsilon \ast p_s)(x),
$$

which is clearly positive.

\begin{proof}

First of all, let us note the following complement to (4.34):

$$
\phi_\epsilon(z) \leq 2^d f_{2\epsilon}(z) \text{ for all } \epsilon > 0 \text{ and } z \in \mathbb{R}^d.
$$

Define, for the sake of notational simplicity,

$$
Q_\epsilon(t,x;s,y) := \phi_\epsilon(W(t) - x)\phi_\epsilon(W(s) - y). \tag{4.40}
$$

Next we apply the Markov property to find that for all $(t,x)$ and $(s,y)$ in $E \times F$ such that $s < t$, and all $\epsilon > 0$,

$$
E[Q_\epsilon(t,x;s,y)] = E[\phi_\epsilon(W(s) - y)\phi_\epsilon(\tilde{W}(t-s) + W(s) - x)], \tag{4.41}
$$

where $\tilde{W}$ is a Brownian motion independent of $W$. An application of (4.39) yields

$$
E[Q_\epsilon(t,x;s,y)] \leq 4^d E[f_{2\epsilon}(W(s) - y)f_{2\epsilon}(\tilde{W}(t-s) + W(s) - x)] \tag{4.42}
$$

$$
\leq 8^d E[f_{2\epsilon}(W(s) - y)f_{4\epsilon}(\tilde{W}(t-s) - x + y)],
$$

thanks to the triangle inequality. Consequently, we may apply independence and (4.34) to find that

$$
E[Q_\epsilon(t,x;s,y)] \leq 8^d E[f_{2\epsilon}(W(s) - y)] E[f_{4\epsilon}(W(t-s) - x + y)]
$$

$$
\leq 32^d E[\phi_{4\epsilon}(W(s) - y)] E[\phi_{8\epsilon}(W(t-s) - x + y)]
$$

$$
= 32^d (\phi_{4\epsilon} \ast p_s)(y) \cdot (\phi_{8\epsilon} \ast p_{t-s})(x - y).
$$

Since $s \in E$, it follows that $s \geq 1/q$ and hence $\sup_{x \in \mathbb{R}^d} p_s(z) \leq p_{1/q}(0)$. Thus,

$$
E[\phi_\epsilon(W(t) - x)\phi_\epsilon(W(s) - y)] \leq 32^d p_{1/q}(0) \cdot (\phi_{8\epsilon} \ast p_{t-s})(x - y). \tag{4.44}
$$

By symmetry, the following holds for all $(t,x),(s,y) \in E \times F$ and $\epsilon > 0$:

$$
E[\phi_\epsilon(W(t) - x)\phi_\epsilon(W(s) - y)] \leq 32^d p_{1/q}(0) \cdot (\phi_{8\epsilon} \ast p_{t-s})(x - y). \tag{4.45}
$$

\end{proof}
Similarly, we have the following for all \((u, x), (v, y) \in [1, 2]^N \times F\) and \(\epsilon > 0\):

\[
E \left[ \phi_\epsilon (X_\alpha (u) - x) \phi_\epsilon (X_\alpha (v) - y) \right] \leq 32^d K \cdot (\phi_{8 \epsilon} * g_{u - v})(x - y),
\]

where \(K := g_0(1/q, \ldots, 1/q)(0) < \infty\) by (4.19), and the definition of \(g_t(z)\) has been extended to all \(t \in \mathbb{R}^N \setminus \{0\}\) by symmetry, viz.,

\[
g_t(z) := |t|^{-d/\alpha} g_1 \left( \frac{z}{|t|^{1/\alpha}} \right) \quad \text{for all} \quad z \in \mathbb{R}^d \quad \text{and} \quad t \in \mathbb{R}^N \setminus \{0\},
\]

where we recall \(1 := (1, \ldots, 1) \in \mathbb{R}^N\). It follows easily from the preceding that \(E(|Z_\epsilon(\mu)|^2)\) is bounded from above by a constant multiple of

\[
\int \int (\phi_{8 \epsilon} * p_{|t-s|})(x - y) \left( \int_{[1,2]^N} (\phi_{8 \epsilon} * g_{u - v})(x - y) \, du \, dv \right)
\]

\[
\times \mu( dt \, dx) \, \mu( ds \, dy),
\]

uniformly for all \(\epsilon > 0\). Define

\[
\kappa(z) := \int_{[0,1]^N} g_u(z) \, du \quad \text{for all} \quad z \in \mathbb{R}^d.
\]

Then we have shown that, uniformly for every \(\epsilon > 0\),

\[
E \left( |Z_\epsilon(\mu)|^2 \right) \leq \text{const} \cdot \int \int (\phi_{8 \epsilon} * p_{|t-s|})(x - y) (\phi_{8 \epsilon} * \kappa)(x - y) \, \mu( dt \, dx) \, \mu( ds \, dy).
\]

It follows easily from (4.19) that the conditions of Proposition 4.2 are met for 

\[
\sigma(dx) := v(dx) := \phi_{8 \epsilon}(x) \, dx,
\]

and therefore that proposition yields the following bound: Uniformly for all \(\epsilon > 0\),

\[
E \left( |Z_\epsilon(\mu)|^2 \right) \leq \text{const} \cdot \int \int p_{|t-s|}(x - y) \kappa(x - y) \, \mu( dt \, dx) \, \mu( ds \, dy).
\]

According to Proposition 4.6, \(\kappa(z) \leq \text{const} \|z\|^{d-\alpha N} \) uniformly for all \(z \in \{x - y : x, y \in F\}\), and the proof is thus completed. \(\square\)

Now we establish (4.29).

**Proof of Theorem 3.1: First half.** If \(C_{d-\alpha N}(E \times F) > 0\), then there exists \(\mu_0 \in P_d(E \times F)\) such that \(\mathcal{E}_{d-\alpha N}(\mu_0) < \infty\), by definition. We apply the Paley–Zygmund inequality [12, p. 72] to Lemma 4.8 and Proposition 4.9, with \(\mu\) replaced by \(\mu_0\), to find that for all \(\epsilon > 0\),

\[
P\{Z_\epsilon(\mu_0) > 0\} \geq \frac{[EZ_\epsilon(\mu_0)]^2}{E \left( |Z_\epsilon(\mu_0)|^2 \right)} \geq \frac{a^2/b}{\mathcal{E}_{d-\alpha N}(\mu_0)}.
\]
If $Z_\varepsilon(\mu_0)(\omega) > 0$ for some $\omega$ in the underlying sample space, then it follows from (4.32) and (4.34) that
\[
\inf_{s \in E} \inf_{x \in F} \inf_{u \in [1,2]^N} \max (\|W(s) - x\|, \|X_\alpha(u) - x\|)(\omega) \leq \epsilon, \tag{4.53}
\]
for the very same $\omega$. As the right-most term in (4.52) is independent of $\epsilon > 0$, the preceding establishes (4.29); i.e., the first half of the proof of Theorem 3.1. ⊓⊔

### 4.4 Second part of the proof

For the second half of our proof we aim to prove that
\[
P \left\{ W(E) \cap X_\alpha([a,b]^N) \cap F \neq \emptyset \right\} > 0 \Rightarrow C_{d-\alpha N}(E \times F) > 0, \tag{4.54}
\]
for all positive real numbers $a < b$. This would complete our derivation of Theorem 3.1. In order to simplify the exposition, we make some reductions. Since $F$ has Lebesgue measure 0, we may and will assume that $E$ has no isolated points. Furthermore, we will take $[a,b]^N = [1,3/2]^N$ and prove the following slightly-weaker statement.
\[
P \left\{ W(E) \cap X_\alpha \left([1,3/2]^N\right) \cap F \neq \emptyset \right\} > 0 \Rightarrow C_{d-\alpha N}(E \times F) > 0. \tag{4.55}
\]
This is so, because $X^{(1)}, \ldots, X^{(N)}$ are right-continuous random functions that possess left-limits. However, we omit the details of this more-or-less routine argument.

Henceforth, we assume that the displayed probability in (4.55) is positive. Let $\partial$ be a point that is not in $\mathbb{R}_+ \times \mathbb{R}^N_{\geq}$, and we define an $E \times [1,3/2]^N \cup \{\partial\}$-valued random variable $T = (S, U)$ as follows.

1. If there is no $(s, u) \in E \times [1,3/2]^N$ such that $W(s) = X_\alpha(u) \in F$, then $T = (S, U) := \partial$.

2. If there exists $(s, u) \in E \times [1,3/2]^N$ such that $W(s) = X_\alpha(u) \in F$, then we define $T = (S, U)$ inductively. Let $S$ denote the first time in $E$ when $W$ hits $X_\alpha([1,3/2]^N) \cap F$, namely,
\[
S := \inf \left\{ s \in E : W(s) \in X_\alpha \left([1,3/2]^N\right) \cap F \right\}. \tag{4.56}
\]

Then, we define inductively,
\[
U_1 := \inf \left\{ u_1 \in [1,3/2] : X_\alpha(u_1, u_2, \ldots, u_N) = W(S) \right\},
\]
\[
U_2 := \inf \left\{ u_2 \in [1,3/2] : X_\alpha(U_1, u_2, \ldots, u_N) = W(S) \right\},
\]
\[
\vdots
\]
\[
U_N := \inf \{ u_N \in [1,3/2] : X_\alpha(U_1, \ldots, U_{N-1}, u_N) = W(S) \}. \tag{4.57}
\]
Because the sample functions of $X_\alpha$ are right-continuous coordinatewise, we can deduce that $W(S) = X_\alpha(U)$ on $\{(S, U) \neq \partial\}$.

Now for every two Borel sets $G_1 \subseteq E$ and $G_2 \subseteq F$ we define

$$
\mu(G_1 \times G_2) := P\{S \in G_1, X_\alpha(U) \in G_2 \mid T \neq \partial\}.
$$

(4.58)

Since $P\{T \neq \partial\} > 0$, it follows that $\mu$ is a bona fide probability measure on $E \times F$. Moreover, $\mu \in \mathcal{P}_d(E \times F)$, since for every $t > 0$,

$$
\mu(\{t\} \times F) = P\{S = t, X_\alpha(U) \in F \mid T \neq \partial\} \leq \frac{P\{W(t) \in F\}}{P\{T \neq \partial\}} = 0.
$$

(4.59)

For every $\epsilon > 0$, we define $Z_\epsilon(\mu)$ by (4.32), but insist on one [important] change. Namely, now, we use the Gaussian mollifier,

$$
\phi_\epsilon(z) := \frac{1}{(2\pi\epsilon^2)^{d/2}} \exp\left(-\frac{\|z\|^2}{2\epsilon^2}\right),
$$

(4.60)

in place of $f_\epsilon \ast f_\epsilon$. [The change in the notation is used only in this portion of the present proof.]

Thanks to the proof of Lemma 4.8,

$$
\inf_{\epsilon \in (0, 1)} E[Z_\epsilon(\mu)] > 0.
$$

(4.61)

Similarly to the proof of (4.50) (e.g., up to a constant factor, the inequalities (4.45) and (4.46) still hold), we have

$$
\sup_{\epsilon \in (0, 1)} E\left(|Z_\epsilon(\mu)|^2\right) \leq \text{const} \cdot \iint (\phi_{8\epsilon} \ast p_{|t-s|})(x-y)(\phi_{8\epsilon} \ast \kappa)(x-y) \mu(ds \, dx) \mu(dt \, dy),
$$

(4.62)

where $\kappa$ is defined by (4.49). Define

$$
\tilde{\kappa}(z) := \int_{[0,1/2]^\infty} g_{\epsilon}(z) \, dt \quad \text{for all } z \in \mathbb{R}^d.
$$

(4.63)

Thanks to Lemma 4.7,

$$
\sup_{\epsilon, \eta \in (0, 1)} E\left(|Z_{\epsilon}(\mu)|^2\right) \leq \text{const} \cdot \iint (\phi_{8\epsilon} \ast p_{|t-s|})(x-y)(\phi_{8\epsilon} \ast \kappa)(x-y) \mu(ds \, dx) \mu(dt \, dy).
$$

(4.64)

Now we are ready to explain why we had to change the definition of $\phi_\epsilon$ from $f_\epsilon \ast f_\epsilon$ to the present Gaussian ones: In the present Gaussian case, both subscripts of “$8\epsilon$” can be replaced by “$\epsilon$” at no extra cost; see (4.65) below. Here is the reason why:
First of all, note that \( \phi_\epsilon \) is still positive definite; in fact, \( \phi_\epsilon(\xi) = e^{-\epsilon^2 \|\xi\|^2 / 2} > 0 \) for all \( \xi \in \mathbb{R}^d \). Next—and this is important—we can observe that \( \phi_\epsilon \leq \hat{\phi}_\delta \) whenever \( 0 < \delta < \epsilon \). And hence, the following holds, thanks to Remark 4.3:

\[
\mathop{\sup}_{\epsilon, \eta \in (0,1)} E \left( |Z_\epsilon(\mu)|^2 \right) \leq \text{const} \cdot \int (\phi_\epsilon * p_{|t-s|})(x-y)(\phi_\epsilon * \hat{\kappa})(x-y) \mu(ds \, dx) \mu(dt \, dy).
\]

This proves the assertion that “\( 8\epsilon \) can be replaced by \( \epsilon \).”

Now define a partial order \( \prec \) on \( \mathbb{R}^N \) as follows: \( u \prec v \) if and only if \( u_i \leq v_i \) for all \( i = 1, \ldots, N \). Let \( X_\alpha \) denote the \( \sigma \)-algebra generated by the collection \( \{X_\alpha(u)\}_{u \prec v} \). Also define \( G := \{G_t\}_{t \geq 0} \) to be the usual augmented filtration of the Brownian motion \( W \).

According to Theorem 2.3.1 of [12, p. 205], \( \{X_\alpha\} \) is a commuting \( N \)-parameter filtration [12, p. 233]. Hence, so is the \( (N+1) \)-parameter filtration

\[
\mathcal{F} := \{F_{s,u}; \ s \geq 0, \ u \in \mathbb{R}_+^N \},
\]

where \( F_{s,u} := G_s \times X_\alpha u \).

Now, for any fixed \( (s,u) \in \mathbb{R}^n \times [1, 3/2]^N \),

\[
E[Z_\epsilon(\mu) \mid \mathcal{F}_{s,u}] \geq \int_{V(u)} \mu(d \mathcal{T}_c(t,x,v)),
\]

where

\[
V(u) := \{v \in [1, 2]^N : u_j \leq v_j \text{ for all } 1 \leq j \leq N\},
\]

and

\[
T_c(t,x,v) := E[Z_\epsilon(W(t) - x) \mid G_s, X_\alpha(v) - x].
\]

Thanks to independence, and the respective Markov properties of the processes \( W, X^{(1)}, \ldots, X^{(N)} \),

\[
T_c(t,x,v) = E[\phi_\epsilon(W(t) - x) \mid G_s] \cdot E[\phi_\epsilon(X_\alpha(v) - x) \mid X_\alpha] = (\phi_\epsilon * p_{t-s})(x - W(s)) \cdot (\phi_\epsilon * p_{v-u})(x - X_\alpha(u)).
\]

Therefore, the definition (4.63) of \( \hat{\kappa} \) and the triangle inequality together reveal that with probability one,

\[
E[Z_\epsilon(\mu) \mid \mathcal{F}_{s,u}] \geq 1_{\{(S,U) \neq \emptyset\}}(\omega) \times \int_{E \times F} (\phi_\epsilon * p_{t-s})(x - W(s))(\phi_\epsilon * \hat{\kappa})(x - X_\alpha(u)) \mu(dt \, dx).
\]

This inequality is valid almost surely, simultaneously for all \( s \) in a dense countable subset of \( E \) (which will be assumed as a subset of \( \mathbb{Q}_+ \) for simplicity of notation) and all \( u \in [1, 3/2]^N \cap \mathbb{Q}_+^N \).
Select points with rational coordinates that converge, coordinatewise from the above, to \((S(\omega), U(\omega))\). In this way we find that
\[
\sup_{s \in E, u \in F} \mathbb{E} \left[ Z_\epsilon(\mu) \left| \mathcal{F}_{s,u} \right. \right] \geq 1_{\{(S,U) \neq \partial\}}(\omega) 
\times \int_{E \times F} (\phi_\epsilon * p_{t-s})(x - W(S)(\phi_\epsilon * \tilde{\kappa})(x - X_\alpha(U)) \mu(dt \, dx).
\]
(4.72)

This is valid \(\omega\) by \(\omega\). We square both sides of (4.72) and then apply expectations to both sides in order to obtain the following:
\[
\mathbb{E} \left\{ \left( \sup_{(s,u) \in Q^N} \mathbb{E} \left[ Z_\epsilon(\mu) \left| \mathcal{F}_{s,u} \right. \right] \right)^2 \right\} \geq \mathbb{P} \left\{ (S,U) \neq \partial \right\}
\times \mathbb{E} \left[ \left( \int_{E \times F} \Psi_\epsilon(t,x) \mu(dt \, dx) \right)^2 \right| (S,U) \neq \partial \right],
\]
(4.73)
where
\[
\Psi_\epsilon(t,x) := (\phi_\epsilon * p_{t-s})(x - W(S)(\phi_\epsilon * \tilde{\kappa})(x - X_\alpha(U)).
\]

According to (4.58), and because \(W(S) = X_\alpha(U)\) on \(\{(S,U) \neq \partial\}\), the conditional expectation in (4.73) is equal to the following:
\[
\int \left( \int_{E \times F} (\phi_\epsilon * p_{t-s})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y) \mu(dt \, dx) \right)^2 \mu(ds \, dy).
\]
(4.74)

In view of the Cauchy–Schwarz inequality, the quantity in (4.74) is at least
\[
\left( \int \int_{E \times F} (\phi_\epsilon * p_{t-s})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y) \mu(dt \, dx) \mu(ds \, dy) \right)^2,
\]
which is, in turn, greater than or equal to
\[
\frac{1}{4} \left( \int \int (\phi_\epsilon * p_{t-s})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y) \mu(dt \, dx) \mu(ds \, dy) \right)^2,
\]
(4.75)
by symmetry.

The preceding estimates from below the conditional expectation in (4.73). And this yields a bound on the right-hand side of (4.73). We can also obtain a good estimate for the left-hand side of (4.73). Indeed, the \((N + 1)\)-parameter filtration \(\mathcal{F}\) is commuting; therefore, according to Cairoli’s strong \((2,2)\) inequality [12, Theorem 2.3.2, p. 235],
\[
\mathbb{E} \left\{ \left( \sup_{(s,u) \in Q^N} \mathbb{E} \left[ Z_\epsilon(\mu) \left| \mathcal{F}_{s,u} \right. \right] \right)^2 \right\} \leq 4^{N+1} \mathbb{E} \left( |Z_\epsilon(\mu)|^2 \right),
\]
(4.76)
and this is in turn at most a constant times the final quantity in (4.75); compare with (4.65). In this way, we are led to the following bound:

\[
P \{ (S, U) \neq \varnothing \} \leq \text{const} \cdot \left[ \iint (\phi_\epsilon * p_{t-s})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y) \mu(dt \, dx) \mu(ds \, dy) \right]^{-1}.
\]

Since the implied constant is independent of \( \epsilon \), we can let \( \epsilon \downarrow 0 \). As the integrand is lower semicontinuous, we obtain the following from simple real-variables considerations:

\[
P \{ (S, U) \neq \varnothing \} \leq \text{const} \cdot \left[ \iint p_{t-s}(x-y)\tilde{\kappa}(x-y) \mu(dx) \mu(dy) \right]^{-1}.
\]

By Proposition 4.6, the term in the reciprocated brackets is equivalent to the energy \( \mathcal{E}_{d-\alpha N}(\mu) \) of \( \mu \), and because \( \mu \) is a probability measure on \( E \times F \), we obtain the following:

\[
P \{ (S, U) \neq \varnothing \} \leq \text{const} \cdot C_{d-\alpha N}(E \times F).
\]

This yields (4.55), and hence Theorem 3.1. \( \square \)

### 4.5 Proof of Proposition 1.4

The method for proving Theorem 3.1 can be modified to prove Proposition 1.4.

**Proof of Proposition 1.4 (Sketch).** The proof for the sufficiency follows a similar line as in Section 4.3; we merely exclude all appearances of \( X_\alpha(u) \), and keep careful track of the incurred changes. This argument is based on a second-moment argument and is standard. Hence we only give a brief sketch for the proof of the more interesting necessity.

Assume that \( P \{ W(E) \cap F \neq \varnothing \} > 0 \) and let \( \Delta \) be a point that is not in \( \mathbb{R}_+ \). Define \( \tau := \inf \{ s \in E : W(s) \in F \} \) on \( \{ W(E) \cap F \neq \varnothing \} \), where \( \inf \varnothing := \Delta \) [in this instance].

Let \( \mu \) be the probability measure on \( E \times F \) defined by

\[
\mu(G_1 \times G_2) := P \{ \tau \in G_1, W(\tau) \in G_2 \mid \tau \neq \Delta \}.
\]

Since \( F \) has Lebesgue measure 0, we have \( \mu \in \mathcal{P}_d(E \times F) \). The rest of the proof is similar to the argument of Section 4.4, but is considerably simpler. Therefore, we omit the many remaining details. \( \square \)

### 5 Proof of Theorem 1.1

Choose and fix an \( \alpha \in (0, 1) \), and define \( X_\alpha \) to be a symmetric stable process in \( \mathbb{R} \) with index \( \alpha \). That is, \( X_\alpha \) is the same process as \( X_\alpha \) specialized to \( N = d = 1 \).
As before, we denote the transition probabilities of $X_\alpha$ by
\[ g_t(x) := \frac{\mathbb{P}\{X_\alpha(t) \in dx\}}{dx} = \frac{1}{\pi} \int_0^\infty \cos(\xi|x|)e^{-\xi^2/2} \, d\xi. \] (5.1)

We define $v$ to be the corresponding 1-potential density. That is,
\[ v(x) := \int_0^\infty g_t(x)e^{-t} \, dt. \] (5.2)

It is known that for all $m > 0$ there exists $c_m = c_{m,\alpha} > 1$ such that
\[ c_m^{-1}|x|^{\alpha-1} \leq v(x) \leq c_m|x|^{\alpha-1} \quad \text{if } |x| \leq m; \] (5.3)
see [12, Lemma 3.4.1, p. 383]. Since $\alpha \in (0,1)$, the preceding remains valid even when $x = 0$, as long as we recall that $1/0 := \infty$.

The following forms the first step toward our proof of Theorem 1.1.

**Lemma 5.1.** Suppose there exists a $\mu \in \mathcal{P}(E \times F)$, the collections of all probability measures on $E \times F$, such that $I_{\beta+2(1-\alpha)}(\mu)$ is finite, where for $\beta > 0$,
\[ I_\beta(\mu) := \int \int \frac{e^{-\|x-y\|^2/(2(t-s))}}{|t-s|^{\beta/2}} 1_{\{s \neq t\}} \, \mu(ds \, dx) \mu(dt \, dy). \] (5.4)

Then, the random set $E \cap W^{-1}(F)$ intersects the closure of $X_\alpha(\mathbb{R}_+)$ with positive probability.

**Remark 5.2.** It is possible, but significantly harder, to prove that the sufficient condition of Lemma 5.1 is also necessary. We will omit the proof of that theorem, since we will not need it.

**Proof.** The proof is similar in spirit to that of Proposition 1.2. For all fixed $\epsilon > 0$ and probability measures $\mu$ on $(0,\infty) \times \mathbb{R}^d$, we define the following parabolic version of (4.32), using the same notation for $\phi_\epsilon := f_\epsilon * f_\epsilon$, etc.:
\[ Y_\epsilon(\mu) := \int_0^\infty e^{-t} \, dt \int \mu(ds \, dx) \, \phi_\epsilon(W(s) - x)\phi_\epsilon(X_\alpha(t) - s). \] (5.5)

Just as we did in Lemma 4.8, we can find a constant $c \in (0, \infty)$—depending only on the geometry of $E$ and $F$—such that uniformly for all $\mu \in \mathcal{P}(E \times F)$ and $\epsilon \in (0,1),$
\[ \mathbb{E}[Y_\epsilon(\mu)] = \int_0^\infty e^{-t} \, dt \int \mu(ds \, dx) \, (\phi_\epsilon * p_\epsilon)(x)(\phi_\epsilon * g_\epsilon)(s) \geq c; \] (5.6)
but now we apply (5.3) in place of Lemma 4.5.

And we proceed, just as we did in Proposition 4.9, and prove that
\[ \mathbb{E}\left(Y_\epsilon(\mu)^2\right) \leq \text{const} \cdot I_{\beta+2(1-\alpha)}(\mu). \] (5.7)

The only differences between the proof of (5.7) and that of Proposition 4.9 are the following:
- Here we appeal to Proposition 4.4, whereas in Proposition 4.9 we made use of Proposition 4.2; and

- We apply (5.3) in place of both Proposition 4.6 and Lemma 4.7. Otherwise, the details of the two computations are essentially exactly the same.

Lemma 5.1 follows from another application of the Paley–Zygmund lemma [12, p. 72] to (5.6) and (5.7); the Paley–Zygmund lemma is used in a similar way as in the proof of the first half of Theorem 3.1. We omit the details, since this is a standard second-moment computation.

Next, we present measure-theoretic conditions that are respectively sufficient and necessary for $I_{d+2(1-\alpha)}(\mu)$ to be finite for some Borel space-time probability measure $\mu$ on $E \times F$.

**Lemma 5.3.** We always have

$$\dim_H(E \times F : \varrho) \leq \sup \left\{ \beta > 0 : \inf_{\mu \in \mathcal{P}(E \times F)} I_\beta(\mu) < \infty \right\}.$$  \hfill (5.8)

**Proof.** For all space-time probability measures $\mu$, and $\tau > 0$ define the space-time $\tau$-dimensional Bessel–Riesz energy of $\mu$ as

$$\Upsilon_\tau(\mu : \varrho) := \int \int \mu(ds \, dx) \mu(dt \, dy) \left[ \varrho((s, x) ; (t, y)) \right]^\tau.$$  \hfill (5.9)

A suitable formulation of Frostman’s theorem [22] implies that

$$\dim_H(E \times F ; \varrho) = \sup \{ \tau > 0 : \Upsilon_\tau(\mu ; \varrho) < \infty \}.$$  \hfill (5.10)

We can consider separately the cases that $\|x - t\|^2 \leq |s - t|$ and $\|x - y\|^2 > |s - t|$, and hence deduce that

$$e^{-\|x-y\|^2/(2|s-t|)} \leq \min \left( \frac{c}{\|x-y\|^{2\beta}}, \frac{1}{|s-t|^\beta} \right),$$  \hfill (5.11)

where $c := \sup_{z>1} z^{2\beta} e^{-z/2}$ is finite. Consequently, $I_{2\beta}(\mu) \leq c' \Upsilon_{2\beta}(\mu ; \varrho)$, with $c' := \max(c, 1)$, and (5.8) follows from (5.10).

**Lemma 5.4.** With probability one,

$$\dim_H(E \cap W^{-1}(F)) \leq \dim_H(E \times F : \varrho) - d/2.$$  \hfill (5.12)

**Proof.** Choose and fix some $r > 0$. Let $T(r)$ denote the collection of all intervals of the form $[t-r^2, t+r^2]$ that are in $[1/q, q]$. Also, let $S(r)$ denote the collection of all closed Euclidean balls of radius $r$ that are contained in $[-q, q]^d$. Recall that $X_\alpha$ is a symmetric stable process of index $\alpha \in (0, 1)$ that is independent of $W$. It is well known that uniformly for all $r \in (0, 1)$,

$$\sup_{I \in T(r)} P \{ X_\alpha([0, 1]) \cap I \neq \emptyset \} \leq \text{const} \cdot r^{2(1-\alpha)}.$$  \hfill (5.13)
see [12, Lemma 1.4.3., p. 355], for example. It is just as simple to prove that
the following holds uniformly for all \( r \in (0, 1) \):
\[
\sup_{I \in \mathcal{T}(r)} \sup_{J \in \mathcal{S}(r)} P \{ W(I) \cap J \neq \emptyset \} \leq \text{const} \cdot r^d. \tag{5.14}
\]

[Indeed, conditional on \( \{ W(I) \cap J \neq \emptyset \} \), the random variable \( W(t) \) comes to
within \( r \) of \( J \) with a minimum positive probability, where \( t \) denotes the smallest
point in \( J \).] Because \( W(I) \cap J \neq \emptyset \) if and only if \( W^{-1}(J) \cap I \neq \emptyset \), it follows
that uniformly for all \( r \in (0, 1) \),
\[
\sup_{I \in \mathcal{T}(r)} \sup_{J \in \mathcal{S}(r)} P \{ W^{-1}(J) \cap I \cap X_\alpha([0, 1]) \neq \emptyset \} \leq \text{const} \cdot r^{d+2(1-\alpha)}. \tag{5.15}
\]

Define
\[
\mathcal{R} := \bigcup_{r \in (0, 1)} \{ I \times J : I \in \mathcal{T}(r) \text{ and } J \in \mathcal{S}(r) \}. \tag{5.16}
\]

Thus, \( \mathcal{R} \) denotes the collection of all “space-time parabolic rectangles” whose \( \varrho \)-diameter lies in the interval \((0, 1)\).

Suppose \( d + 2(1 - \alpha) > \dim_h(E \times F; \varrho) \). By the definition of Hausdorff
dimension, and a Vitali-type covering argument—see Mattila [16, Theorem 2.8,
p. 34]—for all \( \epsilon > 0 \) we can find a countable collection \( \{ E_j \times F_j \}_{j=1}^\infty \) of elements
of \( \mathcal{R} \) such that: (i) \( \bigcup_{j=1}^\infty (E_j \times F_j) \) contains \( E \times F \); (ii) The \( \varrho \)-diameter of \( E_j \times F_j \) is positive and less than one [strictly] for all \( j \geq 1 \); and (iii) \( \sum_{j=1}^\infty \varrho \text{-diam}(E_j \times F_j)^{d+2(1-\alpha)} \leq \epsilon \). Thanks to (5.15),
\[
P \{ W^{-1}(F) \cap E \cap X_\alpha([0, 1]) \neq \emptyset \} \leq \sum_{j=1}^\infty P \{ W^{-1}(F_j) \cap E_j \cap X_\alpha([0, 1]) \neq \emptyset \}
\leq \text{const} \cdot \sum_{j=1}^\infty \varrho \text{-diam}(E_j \times F_j)^{d+2(1-\alpha)} \leq \text{const} \cdot \epsilon. \tag{5.17}
\]

Since neither the implied constant nor the left-most term depend on the value of \( \epsilon \), the preceding shows that \( W^{-1}(F) \cap E \cap X_\alpha([0, 1]) \) is empty almost surely.

Now let us recall half of McKean’s theorem [12, Example 2, p. 436]: If \( \dim_h(A) > 1 - \alpha \), then \( X_\alpha([0, 1]) \cap A \) is nonvoid with positive probability. We apply McKeans theorem, conditionally, with \( A := W^{-1}(F) \cap E \) to find that if \( d + 2(1 - \alpha) > \dim_h(E \times F; \varrho) \), then
\[
\dim_h(W^{-1}(F) \cap E) \leq 1 - \alpha \quad \text{almost surely.} \tag{5.18}
\]

The preceding is valid almost surely, simultaneously for all rational values of \( 1 - \alpha \) that are strictly between one and \( \frac{1}{2}(\dim_h(E \times F; \varrho) - d) \). Thus, the result follows. \( \square \)

**Proof of Theorem 1.1.** By the modulus of continuity of Brownian motion, there exists a null set off which \( \dim_h(W(A) \leq 2 \dim_h A \), simultaneously for all Borel
sets $A \subseteq \mathbb{R}_+$ that might—or might not—depend on the Brownian path itself. Since $W(E \cap W^{-1}(F)) = W(E) \cap F$, Lemma 5.4 implies that

$$\dim_{\mu}(W(E) \cap F) \leq \dim_{\mu}(E \times F; \varrho) - d \quad \text{almost surely.} \quad (5.19)$$

For the remainder of the proof we assume that $d \geq 2$, and propose to prove that

$$\|\dim_{\mu}(W(E) \cap F)\|_{L^\infty(P)} \geq \dim_{\mu}(E \times F; \varrho) - d. \quad (5.20)$$

Henceforth, we assume without loss of generality that $\dim_{\mu}(E \times F; \varrho) > d$; (5.21) for there is nothing left to prove otherwise. In accord with the theory of Taylor and Watson [22], (5.21) implies that $P\{W(E) \cap F \neq \emptyset\} > 0$.

According to Kauffman’s uniform-dimension theorem [9], the Hausdorff dimension of $W(E) \cap F$ is almost surely equal to twice the Hausdorff dimension of $E \cap W^{-1}(F)$. Therefore, it suffices to prove the following in the case that $d \geq 2$:

$$\|\dim_{\mu}(E \cap W^{-1}(F))\|_{L^\infty(P)} \geq \frac{\dim_{\mu}(E \times F; \varrho) - d}{2}, \quad (5.22)$$

as long as the right-hand side is positive. If $\alpha \in (0, 1)$ satisfies

$$1 - \alpha < \frac{\dim_{\mu}(E \times F; \varrho) - d}{2}, \quad (5.23)$$

then Lemma 5.3 implies that $I_{d+2(1-\alpha)}(\mu) < \infty$ for some $\mu \in \mathcal{P}(E \times F)$. Thanks to Lemma 5.1, $E \cap W^{-1}(F) \cap \overline{X_\alpha([0,1])} \neq \emptyset$ with positive probability. Consequently,

$$P\{\dim_{\mu}(E \cap W^{-1}(F)) \geq 1 - \alpha\} > 0, \quad (5.24)$$

because the second half of McKean’s theorem implies that if $\dim_{\mu}(A) < 1 - \alpha$, then $\overline{X_\alpha(\mathbb{R}_+)} \cap A = \emptyset$ almost surely. Since (5.24) holds for all $\alpha \in (0, 1)$ that satisfy (5.23), (5.22) follows. This completes the proof.

**Remark 5.5.** Let us mention the following byproduct of our proof of Theorem 1.1: For every $d \geq 1$,

$$\|\dim_{\mu}(E \cap W^{-1}(F))\|_{L^\infty(P)} = \frac{\dim_{\mu}(E \times F; \varrho) - d}{2}. \quad (5.25)$$

When $d = 1$, this was found first by Kaufman [10], who used other arguments [for the harder half]. See Hawkes [7] for similar results in case $W$ is replaced by a stable subordinator of index $\alpha \in (0, 1)$.

We conclude this paper with some problems that continue to elude us.
Open Problems. Theorems 1.1 and 1.3 together imply that when \( d \geq 2 \) and \( F \subset \mathbb{R}^d \) has Lebesgue measure 0,

\[
\sup \left\{ \gamma > 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\} = \dim_H(E \times F; \rho) - d.
\]  
(5.26)

The preceding is a kind of “parabolic Frostman theorem.” And we saw in the introduction that (5.26) is in general false when \( d = 1 \). We would like to better understand why the one-dimensional case is so different from the case \( d \geq 2 \). Thus, we are led naturally to a number of questions, three of which we state below:

P1. Equation (5.26) is, by itself, a theorem of geometric measure theory. Therefore, we ask, “Is there a direct proof of (5.26) that does not involve random processes, broadly speaking, and Kaufman’s uniform-dimension theorem [9], in particular”?

P2. When \( d \geq 2 \), (5.26) gives an interpretation of the capacity form on the left-hand side of (5.26) in terms of the geometric object on the right-hand side. Can we understand the left-hand side of (5.26) geometrically in the case that \( d = 1 \)?

P3. The following interesting question is due to an anonymous referee: Are there quantitative relationships between a rough hitting-type probability of the form \( P\{\dim(W(E) \cap F) > \gamma\} \) and the new capacity form of Benjamini et al [1] [see also [17, Theorem 8.24]]? We suspect the answer is “yes,” but do not have a proof.

Acknowledgements. Many hearty thanks are due to Professors Gregory Lawler and Yuval Peres. The former showed us the counterexample in the Introduction, and the latter introduced us to the problem that is being considered here.

We thank the anonymous referee for pointing out several mistakes in an earlier formulation of Theorem 1.3 in a previous draft of this manuscript.

References

[1] Benjamini, Itai, Robin Pemantle, and Yuval Peres, Martin capacity for Markov chains, *Ann. Probab.* **23** (1995) 1332–1346.

[2] Blumenthal, R. M. and R. K. Getoor, Dual processes and potential theory, In: *Proc. Twelfth Biennial Sem. Canad. Math. Congr. on Time Series and Stochastic Processes; Convexity and Combinatorics* (Vancouver, B.C., 1969) pp. 137–156, Canad. Math. Congr., Montreal, Que., 1970.

[3] Blumenthal, R. M. and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York-London, 1968.

[4] Doob, Joseph L., *Classical Potential Theory and Its Probabilistic Counterpart*, Springer Verlag, Berlin, 2002.

[5] Foondun, Mohammad and Davar Khoshnevisan, Lévy processes, the stochastic heat equation with spatially-colored forcing, and intermittence, *preprint* (2009).

27
[6] Hawkes, John, Trees generated by a simple branching process, *J. London Math. Soc. (2)* 24 (1981) no. 2, 373–384.
[7] Hawkes, John, Measures of Hausdorff type and stable processes, *Mathematika* 25 (1978) 202–212.
[8] Kahane, J.-P., *Some Random Series of Functions*, 2nd ed. Cambridge Univ. Press, Cambridge, 1985.
[9] Kaufman, Robert, Une propriété métrique du mouvement brownien, *C. R. Acad. Sci. Paris* 268 (1969) 727–728.
[10] Kaufman, Robert, Measures of Hausdorff-type, and Brownian motion, *Mathematika* 19 (1972) 115–119.
[11] Kaufman, Robert and Jan Wei Wu, Parabolic potential theory, *J. Differential Equations* 43(2) (1982) 204–234.
[12] Khoshnevisan, Davar, *Multiparameter Processes*, Springer Verlag, New York, 2002.
[13] Khoshnevisan, Davar and Yimin Xiao, Level sets of additive Lévy processes, *Ann. Probab.* 30 (2002) 62–100.
[14] Khoshnevisan, Davar and Yimin Xiao, Harmonic analysis of additive Lévy processes, *Probab. Theory and Related Fields* 145 (2009) 459–515.
[15] Lyons, Russell, Random walks and percolation on trees, *Ann. Probab.* 18 (1990) no. 3, 931–958.
[16] Mattila, Pertti, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.
[17] Mörters, Peter and Yuval Peres, *Brownian Motion*, Cambridge University Press, Cambridge, 2010.
[18] Peres, Yuval, *Probability on Trees*, In: Lectures on Probability Theory and Statistics (Saint-Flour, 1997), 193–280, Lecture Notes in Math. 1717, Springer, Berlin, 1999.
[19] Peres, Yuval, Remarks on intersection-equivalence and capacity-equivalence, *Ann. Inst. H. Poincaré: Phys. Théor.* 64 (1996) no. 3, 339–347.
[20] Orey, Steven, Polar sets for processes with stationary independent increments, In: *Markov Processes and Potential Theory* (Proc. Sympos. Math. Res. Center, Madison, Wis., 1967) pp. 117–126, Wiley, New York, 1967.
[21] Taylor, S. J., Multiple points for the sample paths of the symmetric stable process, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 5 (1966) 247–264.
[22] Taylor, S. J. and N. A. Watson, A Hausdorff measure classification of polar sets for the heat equation, *Math. Proc. Cambridge Philos. Soc.* 97(2) (1985) 325–344.
[23] Watson, N. A., Thermal capacity, *Proc. London Math. Soc. (3)* 37(2) (1978) 342–362.
[24] Watson, N. A., Green functions, potentials, and the Dirichlet problem for the heat equation, *Proc. London Math. Soc. (3)* 33(2) (1976) 251–298 [corrigendum: *Proc. London Math. Soc. (3)* 37(1) (1978) 32–34.]

Davar Khoshnevisan  
Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090  
Email: davar@math.utah.edu
URL: http://www.math.utah.edu/~davar

Yimin Xiao
Department of Statistics and Probability, A-413 Wells Hall, Michigan State University,
East Lansing, MI 48824
Email: xiao@stt.msu.edu
URL: http://www.stt.msu.edu/~xiaoyimi