A Blass-Sagan bijection on Eulerian equivalence classes

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Abstract. Following the treatment of Blass and Sagan, we present an algorithmic bijection between the Eulerian equivalence classes of totally cyclic orientations and the spanning trees without internal activity edges for a given graph.

Key words. orientations, acyclic orientations, totally cyclic orientations, Tutte polynomials, cut equivalence, Eulerian equivalence, Eulerian-cut equivalence, external activity, internal activity, directed cut, directed cycle

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1. Introduction

To generalize the chromatic polynomials of graphs, Tutte [22] introduced the dichromatic polynomials in two variables which we know as Tutte polynomials. Without much additional effort, one can define Tutte polynomials for arbitrary matroids. Ardila [12] also defined the Tutte polynomials on hyperplane arrangements. Many interesting invariants of graphs and matroids can be computed directly from these polynomials [3, 6, 9]. It is worth mentioning that the Tutte polynomials play an important role in statistical mechanics, where the partition functions are just simple variants of these polynomials; the Jones polynomials and Kauffman polynomials in knot theory are also closely related to them; see [5]. To find other new interpretations for specializations of Tutte polynomials has interested many mathematicians [7, 8, 19, 20, 24], etc. In this paper we concentrate on the evaluation of the Tutte polynomial at several special points in terms of equivalence classes of orientations on graphs.

The first remarkable result on the connection between acyclic orientations of graphs and the Tutte polynomial is due to Stanley [20], who gave the interpretation of the chromatic polynomial at negative integers. Then it was generalized by Chen [7] to interpret the integral and modular tension polynomials of Kochol [17] at nonnegative integers, where acyclic orientations and their cut equivalence classes are used to describe the decomposition of these polynomials. Green and Zaslavsky [16] proved that the number of acyclic orientations with a unique source at a given vertex is the special value of the Tutte polynomial at (1, 0), and a fascinating result in [7] is that this value also counts the number of cut equivalence classes of acyclic orientations.
Dual to Stanley’s result, the number of totally cyclic orientations also can be given by the Tutte polynomial \[23\]. Utilizing the theory of Ehrhart polynomials as in \[7\], Chen and Stanley \[8\] studied the integral and modular flow polynomials, where they gave a similar decomposition as the tension polynomials in terms of totally cyclic orientations and their Eulerian equivalence classes. Dually, the number of Eulerian equivalence classes of totally cyclic orientations is equal to the special value of the Tutte polynomial at \((0, 1)\).

Using the convolution formula due to Kook, Reiner and Stanton \[18\], we recover the result of Stanley in \[21\] which states that the value of the Tutte polynomial at \((2, 1)\) enumerates in-sequences of orientations, i.e., the Eulerian equivalence classes of orientations; by duality the value at \((1, 2)\) enumerates the cut equivalence classes of all orientations. Another result from the convolution formula is the interpretation of the value of the Tutte polynomial at \((1, 1)\) in terms of Eulerian-cut equivalence classes of orientations. Gioan independently \[14\] obtained the same result on interpretations of the Tutte polynomial at \((0, 1), (1, 0), (1, 2), (2, 1)\) and \((1, 1)\), where the cycle-cocycle systems are used instead of Eulerian-cut equivalence classes.

As Tutte originally defined, a fundamental property of the Tutte polynomial is that it has a spanning tree expansion. Therefore, specializations of the Tutte polynomial inherit the interpretations in terms of spanning trees. A natural question arises: to find the bijections between the set of some equivalence classes of orientations and spanning trees with special property. The related work has been done. Blass and Sagan \[4\] constructed an algorithmic bijection between the set of acyclic orientations and the broken circuit complex. This algorithm was modified by Gebhard and Sagan \[13\] to give a bijection between the set of acyclic orientations with a unique sink at a given vertex and the set of spanning trees without external activity edges. Gioan \[14\] gave a bijection between the set of cut equivalence classes of acyclic orientations and the set of of acyclic orientations with a unique sink at a given vertex. The combination of the above two bijections leads to a bijection between cut equivalence classes of acyclic orientations and spanning trees without external activity edges. Gioan and Vergnas \[15\] also established the activity preserving bijections between spanning trees and orientations.

The main task of this paper is to give a Blass-Sagan bijection between Eulerian equivalence classes of totally cyclic orientations and spanning trees without internal activity edges. As each cut equivalence class of acyclic orientations has an acyclic orientation with a unique sink at a given vertex, our bijection would be helpful to find the corresponding representative element for each Eulerian equivalence class of totally cyclic orientations.

2. Definitions and notations

Let \(G = (V, E)\) be a graph with vertex set \(V\) and edge set \(E\), in which multiple edges and loops are allowed. Given \(e \in E\), let \(G - e = (V, E \setminus \{e\})\). Thus \(G - e\) is obtained from \(G\) by deleting the edge \(e\). Let \(G/e\) be the multigraph obtained from \(G\) by contracting the edge \(e\). Throughout this paper the graphs are assumed to be always connected.

Now let us define the Tutte polynomial \(T_G(x, y)\) for a graph \(G\) recursively. First, let \(T_{E_n}(x, y) = 1\), where \(E_n\) is the empty \(n\)-graph for \(n \geq 1\). In general, we have

\[
T_G(x, y) = \begin{cases} 
  xT_{G/e}(x, y) & \text{if } e \text{ is a bridge,} \\
  yT_{G-e}(x, y) & \text{if } e \text{ is a loop,} \\
  T_{G-e}(x, y) + T_{G/e}(x, y) & \text{if } e \text{ is neither a bridge nor a loop.}
\end{cases}
\]
The Eulerian equivalence classes

As we remarked at the beginning, the original definition of \( T_G(x, y) \) is in terms of spanning trees of \( G \). We adopt the notions of [3] in the following. For a connected graph \( G = (V, E) \), a tree \( F = (V', E') \) is a spanning tree of \( G \) if \( V' = V \) and \( E' \subset E \). If \( G \) is not connected, the spanning trees of all components form a spanning forest of \( G \). Now let us impose an order on the edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \), with \( e_i \) preceding \( e_j \) if \( i < j \). Fix a spanning forest \( F \) of \( G \). For each edge \( e_i \) in \( F \), we call \( U_F(e_i) = \{e_j \in E(G) : (F - e_i) + e_j \text{ is a spanning forest}\} \) the cut defined by \( e_i \). If \( e_i \) is the smallest edge of the cut it defines, we call \( e_i \) an internally active edge of \( F \). Similarly, for each edge \( e_j \) not in \( F \), we call \( Z_F(e_j) = \{e_i \in E(G) : e_i \text{ is an edge on the unique cycle of } F + e_j\} \) the cycle defined by \( e_j \). If \( e_j \) is the smallest edge of the cycle it defines, we call \( e_j \) an externally active edge. We say that a spanning forest has internal activity \( i \) and external activity \( j \) if there are precisely \( i \) internally active edges and precisely \( j \) externally active edges, denoted by an \((i, j)\)-forest. Tutte originally defined

\[
T_G(x, y) = \sum_{i,j} t_{ij} x^i y^j, \tag{1}
\]

where \( t_{ij} \) is the number of \((i, j)\)-forests.

Recall that a cut of \( G \) is a partition \([S, T]\) of the vertex set \( V \) such that the removal of \([S, T]\), the set of all edges between \( S \) and \( T \), disconnects the graph \( G \). For a digraph \((G, \varepsilon)\), where \( \varepsilon \) is an orientation of \( G \), we denote by \((S, T)_\varepsilon\) the set of all edges going from \( S \) to \( T \), and by \((T, S)_\varepsilon\) the set of all edges going from \( T \) to \( S \). A bond is a minimal cut. A bond \([S, T]\) is called directed relative to \( \varepsilon \) if \((S, T)_\varepsilon = \emptyset \) or \((T, S)_\varepsilon = \emptyset \). A cut is called directed if it can be decomposed into a disjoint union of directed bonds. Let \( \mathcal{O}(G) \) denote the set of all orientations of \( G \), \( \mathcal{AO}(G) \) the set of all orientations without directed cycles, and \( \mathcal{BO}(G) \) the set of all orientations without directed cuts.

Given an orientation \( \varepsilon \) of \( G \), a directed edge \( e = (u, v) \) is called cut flippable if there are no directed paths either from \( u \) to \( v \) or from \( v \) to \( u \) in \( G - \varepsilon \). An directed edge \( e \) relative to \( \varepsilon \) is called cycle flippable if there are directed paths both from \( u \) to \( v \) and from \( v \) to \( u \) in \( G - \varepsilon \).

We call two orientations \( \varepsilon_1 \) and \( \varepsilon_2 \) cut-equivalent, denoted by \( \varepsilon_1 \sim_c \varepsilon_2 \), if the spanning subgraph induced by the edge set \( \{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\} \) is a directed cut with respect to \( \varepsilon_1 \) or \( \varepsilon_2 \). It is easy to see that \( \sim_c \) is an equivalence relation on \( \mathcal{O}(G) \), and it also induces an equivalence relation on \( \mathcal{AO}(G) \).

Similarly, we define the Eulerian equivalence relations as follows. We call two orientations \( \varepsilon_1 \) and \( \varepsilon_2 \) Eulerian equivalent, denoted by \( \varepsilon_1 \sim_e \varepsilon_2 \), if the spanning subgraph induced by the edge set \( \{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\} \) is a directed Eulerian graph with respect to \( \varepsilon_1 \) or \( \varepsilon_2 \), i.e., the in-degree is equal to the out-degree at each vertex. It is easy to see that \( \sim_e \) is an equivalence relation on \( \mathcal{O}(G) \), and it induces an equivalence relation on \( \mathcal{BO}(G) \).

We also need the concept of Eulerian-Cut equivalence over orientations. Two orientations \( \varepsilon_1 \) and \( \varepsilon_2 \) are called to be Eulerian-cut equivalent, denoted by \( \varepsilon_1 \sim_{ec} \varepsilon_2 \), if the spanning subgraph induced by the edge set \( \{e \in E(G) \mid \varepsilon_1(e) \neq \varepsilon_2(e)\} \) is a disjoint union of a directed Eulerian graph and a directed cut with respect to \( \varepsilon_1 \) or \( \varepsilon_2 \). The relation \( \sim_{ec} \) is also an equivalence relation on \( \mathcal{O}(G) \).

By definitions, the two orientations (B-1) and (B-2) in Fig. 4 are cut equivalent, (B-2) and (B-3) are Eulerian equivalent, while (B-1) and (B-3) are Eulerian-cut equivalent.
Fig. 1. Equivalence relations among three orientations.

3. Eulerian equivalence classes

Using the theory of Ehrhart polynomials, Chen and Stanley obtained the following nice result, which is independently discovered by Gioan [14].

**Theorem 1.** [8, Theorem 1.2] For any graph $G$, let $\alpha(G)$ denote the number of Eulerian equivalence classes of $\mathcal{BO}(G)$. Then

$$\alpha(G) = T_G(0, 1).$$ (2)

In the following we will present two proofs of the above theorem. The first proof is purely inductive according to the inductive definition of Tutte polynomials, and the second one is an algorithmic bijection similar to the modified Blass-Sagan algorithm [13].

3.1. The inductive proof

For any fixed edge $e = (u, v)$, it is clear that there always exists an orientation $\varepsilon$ in each Eulerian equivalence class of $\mathcal{BO}(G)$ such that the edge $e$ is directed from $u$ to $v$ with respect to $\varepsilon$. Notice that the edge $e$ has the same cycle flippable property in each Eulerian equivalence class, i.e., for any two equivalent totally cyclic orientations $\varepsilon$ and $\varepsilon'$ with $\varepsilon(e) = \varepsilon'(e)$, then $e$ is cycle flippable relative to $\varepsilon$ if and only if it is cycle flippable relative to $\varepsilon'$. Therefore, in each equivalence class we can choose an orientation with $e$ directed from $u$ to $v$ as a representative element.

**Proof of Theorem**. We shall deduce the assertion from the following four properties of the function $\alpha(G)$.

(i) If $G = E_n$, then $\alpha(G) = 1$.
(ii) If $e$ is a loop, then $\alpha(G) = \alpha(G - e)$.
(iii) If $e$ is a bridge, then $G$ has no totally cyclic orientations so $\alpha(G) = 0$.
(iv) Finally, suppose that $e$ is neither a bridge nor a loop. Consider an equivalence class of $\mathcal{BO}(G)$, and the orientation $\varepsilon$ is its representative element. If $e$ is cycle flippable relative to $\varepsilon$, then all orientations equivalent to $\varepsilon$ give an equivalence class of $\mathcal{BO}(G - e)$; otherwise, they give an equivalence class of $\mathcal{BO}(G/e)$. Also, all appropriate equivalence classes of $\mathcal{BO}(G - e)$ and $\mathcal{BO}(G/e)$ arise in this way. Therefore, in this case we have

$$\alpha(G) = \alpha(G - e) + \alpha(G/e).$$

Since $\alpha(G)$ and $T_G(0, 1)$ satisfy the same boundary conditions and recurrence relations, the desired result immediately follows.
3.2. The bijective proof

From Equation (1) we see that the value $T_G(0,1)$ counts the number of spanning trees without internal activity edges. To prove Theorem 1, it suffices to establish a bijection between these spanning trees of $G$ and Eulerian equivalence classes of $BO(G)$.

Fix an orientation $\varepsilon$ of $G$ (not necessarily totally cyclic or acyclic), which we will refer to as the normal orientation. Fix the total order imposed on the edges which defines the internal and external activity. We say that an orientation $\varepsilon'$ is reduced if for each edge $e \in E(G)$ either $\varepsilon'(e) = \varepsilon''(e)$ or there exists no directed cycle containing $e$ with other edges smaller than $e$.

For any oriented arc $e = \vec{uv}$, we denote the oppositely oriented arc by $e' = \vec{vu}$. To unorient an arc $e$ for an orientation $\varepsilon$ of $G$, it means that we will just add the oppositely oriented arc $e'$. Given a graph with unoriented edges, let $G'$ be the contraction of $G$, which is the graph where all unoriented edges have been contracted. The orientation of $G'$ is inherited from the original graph $G$. We say that $G$ is reduced if its contraction $G'$ is reduced with respect to the inherited normal orientation. For any two orientations $\varepsilon_1$ and $\varepsilon_2$ of $G$ with unoriented edges, we say that they are Eulerian equivalent if the two inherited orientations of the contraction $G'$ are Eulerian equivalent.

Lemma 1. For the normal orientation $\varepsilon$ and the total order on edges fixed as above, there exists one and only one reduced orientation in each Eulerian equivalence class of $BO(G)$.

Proof. Given an Eulerian equivalence class, we first show that there exists at least one reduced orientation. Start with one arbitrary totally cyclic orientation, say $\varepsilon_0$. If $\varepsilon_0$ is reduced, then we are done. Otherwise, find the largest edge, say $e_0$, which doesn’t satisfy the reduced property. It means that $\varepsilon(e_0) \neq \varepsilon_0(e_0)$ and there exists one directed cycle which contains $e_0$ and all other edges on the cycle are smaller than $e_0$. By reversing the orientation of this cycle, we obtain another Eulerian equivalent orientation $\varepsilon_1$ with all edges larger than or equal to $e_0$ satisfying the reduced property. Iterating the above process, we will get one orientation equivalent to $\varepsilon_0$, with all its edges satisfying the reduced property.

Now we show that the reduced orientation is unique in the Eulerian equivalence class. Suppose there are two reduced equivalent orientations $\varepsilon'$ and $\varepsilon''$. Consider the spanning subgraph induced by the edge set $\{e \in E(G) \mid \varepsilon'(e) \neq \varepsilon''(e)\}$. If not empty, then it must contain a directed cycle with respect to $\varepsilon'$ or $\varepsilon''$. Therefore, the largest edge on this cycle satisfies the reduced property only for one of two orientations $\varepsilon'$ and $\varepsilon''$. This is a contradiction.

As shown above, from an arbitrary orientation $\varepsilon'$ we can obtain the reduced orientation in each Eulerian equivalence class with the iterated process. For convenience we call it the normalization of $\varepsilon'$.

In the following we will construct an algorithm which maps each reduced totally cyclic orientation to a spanning tree without internal activity edges. Due to the above lemma, we obtain the desired bijection. With the total order imposed on the edge set, each oriented edge is sequentially examined and is either deleted or unoriented using the following algorithm:

1. Input a graph $(G, \varepsilon)$, where $\varepsilon$ is an orientation of $G$ with some unoriented edges.
2. Let $(G', \varepsilon')$ be the contraction of $(G, \varepsilon)$ with all unoriented edges having been contracted. If $\varepsilon'$ is not reduced, then we take the reduced representation $\varepsilon''$ in its Eulerian equivalence class.
(S3) Consider the largest edge $e$ of $G'$. If $e$ is a loop or cycle flippable with respect to $\varepsilon''$, then we delete $e$ from $G'$. Otherwise, we unorient $e$ in $G'$. Reset $G$ to be the graph recovered from $G'$ by adding back all unoriented edges. Reset $\varepsilon$ to be the orientation of $G$ obtained from $\varepsilon''$, i.e., for all oriented edge $e'$ we have $\varepsilon''(e') = \varepsilon(e')$. If $G$ contains at least one oriented edge with respect to $\varepsilon$, then go to Step (S2). Otherwise, go to Step (S4).

(S4) Output the graph $G$.

For an example of how the above algorithm works, see Figure 2, where $I$ denotes the unorientation, $II$ denotes the deletion, and $III$ denotes the normalization.

To show that this algorithm actually does produce a bijection, we shall first introduce a sequence of sets, $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q$, such that $\mathcal{O}_0$ is the set of all reduced totally cyclic orientations of $G$, and $\mathcal{O}_q$ (where $q = |E(G)|$) is the set of all spanning trees of $G$ without internal activity edges. We will show that the $k$-th step of the algorithm gives a bijection, $f_k : \mathcal{O}_{k-1} \rightarrow \mathcal{O}_k$, where $\mathcal{O}_k$ is defined as the set of all orientations $\varepsilon$ of spanning subgraphs of $G$ satisfying the following conditions:

(a) Each of the first $k$ largest edges of $G$ is either present in $\varepsilon$ (as an unoriented edge) or absent from $\varepsilon$, but each of the remaining $q - k$ edges is present in $\varepsilon$ in exactly one orientation, and there does not exist a cycle only consisting of unoriented edges.
(b) $\varepsilon$ is totally cyclic.
(c) $\varepsilon$ is reduced.
(d) For each unoriented edge $e$ in the subgraph, if $e$ is a bridge which separates the subgraph into two components $C_1$ and $C_2$, there exists at least one edge strictly smaller than $e$ in the edge cut $E_G|_{C_1, C_2}$.

From the above conditions, we see that $O_0$ is indeed the set of all reduced representations of the totally cyclic orientations of $G$, and $O_q$ is indeed the set of all spanning trees without internal activity edges.

**Lemma 2.** $f_k$ maps $O_{k-1}$ into $O_k$.

**Proof.** It suffices to verify that properties (a)–(d) listed previously are still satisfied after the algorithm is applied at the $k$-th stage.

(a) If the $k$-th largest edge $e$ is cycle flippable then the algorithm will delete it; otherwise, the algorithm will unorient it. Therefore, it will not create a new cycle consisting of only unoriented edges.
(b) Clearly, to unorient an edge and delete the cycle flippable edge will not destroy the totally cyclic property.
(c) This is ensured by Step (S2) of the algorithm.
(d) Suppose that there exists some unoriented edge $e$ as a bridge in the subgraph such that $e$ is the smallest edge in the edge cut $E_G|_{C_1, C_2}$. Therefore, in the process of the algorithm all edges of $E_G|_{C_1, C_2}$ except $e$ will be deleted, i.e., all these oriented edges are cycle flippable. Now consider the second smallest edge $e_0$ of $E_G|_{C_1, C_2}$. Clearly, $e_0$ must not be cycle flippable, and the algorithm will unorient it. This is a contradiction.

To prove that $f_k$ is bijective, we first give the following two lemmas:

**Lemma 3.** Given an orientation $\varepsilon \in O_{k-1}$, let $e$ be the largest oriented edge of the underlying graph $G$. Let $G' = G - e$, and $\varepsilon'$ be the orientation of $G'$ inherited from $\varepsilon$. If $\varepsilon$ is reduced and $e$ is cycle flippable in $\varepsilon$, then $\varepsilon'$ is reduced. Moreover, $f_k(\varepsilon(G)) = \varepsilon'(G')$.

**Proof.** Suppose that $\varepsilon'$ is not reduced. There must exist one edge $e'$ which is smaller than $e$ and doesn’t satisfy the reduced property in $G'$. Clearly, $e'$ also doesn’t satisfy the reduced property for the orientation $\varepsilon$ in $G$, which is contrary to the fact that $\varepsilon$ is reduced.

**Lemma 4.** Given any two distinct reduced totally cyclic orientations $\varepsilon_1$ and $\varepsilon_2$ of $G$, suppose that the largest oriented edge $e$ is neither cycle flippable with respect to $\varepsilon_1$ nor $\varepsilon_2$. Let $\varepsilon'_1$ (resp. $\varepsilon'_2$) be the orientation of $G$ obtained from $\varepsilon_1$ (resp. $\varepsilon_2$) by unorienting the edge $e$. Then $\varepsilon'_1$ and $\varepsilon'_2$ are not Eulerian equivalent as orientations of the contraction graph $G/e$.

**Proof.** Since $\varepsilon_1, \varepsilon_2$ are reduced and $e$ is the largest edge in $G$, we must have $\varepsilon_1(e) = \varepsilon_2(e)$. Suppose that $\varepsilon'_1$ and $\varepsilon'_2$ are Eulerian equivalent, then the edge set $\{e' \in E(G/e) \mid \varepsilon'_1(e') \neq \varepsilon'_2(e')\}$ can be taken as a disjoint union $\bigcup_i C_i$, where each $C_i$ is a directed cycle in $G/e$ with respect to $\varepsilon'_1$ or $\varepsilon'_2$. The set $\{e' \in E(G/e) \mid \varepsilon'_1(e') \neq \varepsilon'_2(e')\}$ can not be empty, otherwise we will have $\varepsilon_1 \sim_e \varepsilon_2$, contradicting the fact that they are distinct reduced orientations. If for each $i$ the edges in $G$ corresponding to the edges of $C_i$ also form a cycle, then we also have $\varepsilon_1 \sim_e \varepsilon_2$. Otherwise, suppose for some $i$ the edges in $G$ corresponding to the
edges of $C_i$ do not form a cycle, but together with the edge $e$ they will form a cycle. If $C_i$ and $e$ form a directed cycle with respect to $\varepsilon_1$ (resp. $\varepsilon_2$), then $e$ will be cycle flippable with respect to $\varepsilon_2$ (resp. $\varepsilon_1$), which is again a contradiction.

\begin{proof}
First we prove that $f_k$ is one to one. Suppose $\varepsilon_1$ and $\varepsilon_2$ are two distinct elements of $O_{k-1}$ which are both mapped to $\varepsilon \in O_k$ by the algorithm. Since the algorithm only affects the $k$-th large edge, we note that in both $\varepsilon_1$ and $\varepsilon_2$, the cases are same for the first $k-1$ large edges of $G$. We note that $\varepsilon$ was not obtained from $\varepsilon_1$ and $\varepsilon_2$ by deletion. Otherwise, $\varepsilon_1$ and $\varepsilon_2$ will be the same due to Lemma 3. Thus we only need consider the case that $\varepsilon$ was obtained from $\varepsilon_1$ and $\varepsilon_2$ by unorienting the $k$-th edge and applying the normalization. By Lemma 4, this is also impossible.

Then we prove that that $f_k$ maps $O_{k-1}$ onto $O_k$. For any $\varepsilon \in O_k$ such that the $k$-th edge $e$ of $G$ is absent in the underlying spanning subgraph, we just add the edge $e$ in the subgraph and normally orient it. Denote the orientation of this new diagram by $\varepsilon'$. Since $\varepsilon$ is totally cyclic and the underlying graph is connected, $\varepsilon'$ is still totally cyclic. Notice that $e$ is the largest oriented edge with respect to $\varepsilon'$. Therefore, $\varepsilon'$ is also reduced and the directed edge $e$ is cycle flippable. It means that $\varepsilon' \in O_{k-1}$, and the $k$-th stage of the algorithm will map $\varepsilon'$ to $\varepsilon$.

For any $\varepsilon \in O_k$ such that the $k$-th edge $e$ of $G$ is unoriented in the underlying spanning subgraph, we construct one orientation $\varepsilon' \in O_{k-1}$ as follows.

1. Choose an orientation of $e$ such that the new orientation is totally cyclic. Note that such an orientation always exists.
2. Normalize the new orientation. If the directed edge $e$ is not cycle flippable, then return the orientation. Otherwise, go to (3).
3. Reorient the edge $e$ oppositely, then reorient the directed cycle containing $e$ oppositely, and go to (2).

Let $\varepsilon'$ be the returned orientation. Clearly, $e$ is not cycle flippable with respect to $\varepsilon'$, and $\varepsilon' \in O_{k-1}$. The $k$-th stage of the algorithm will map $\varepsilon'$ to $\varepsilon$.

\end{proof}

\begin{remark}
In fact, the acyclic orientations with only one given source (or sink) can be considered as the representative elements of cut equivalence classes of acyclic orientations. But for Eulerian equivalence classes of totally cyclic orientations, the dual representative elements are not known. E. Gioan mentioned to use the degree sequences to characterize the Eulerian equivalence classes. In this paper our reduced orientations are also representative elements of Eulerian equivalence classes, but they depend on the total order on the edge set and the fixed normal orientation.

\end{remark}

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