Time scale defined by the fractal structure of the price fluctuations in foreign exchange markets

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Abstract. In this contribution, a new time scale named C-fluctuation time is defined by price fluctuations observed at a given resolution. The intraday fractal structures and the relations of the three time scales: real time (physical time), tick time and C-fluctuation time, in foreign exchange markets are analyzed. The data set used is trading prices of foreign exchange rates; US dollar (USD)/Japanese yen (JPY), USD/Euro (EUR), and EUR/JPY. The accuracy of the data is one minute and data within a minute are recorded in order of transaction. The series of instantaneous velocity of C-fluctuation time flowing are exponentially distributed for small \( C \) when they are measured by real time and for tiny \( C \) when they are measured by tick time. When the market is volatile, for larger \( C \), the series of instantaneous velocity are exponentially distributed.

1. Introduction
There are at least four candidates for the proper time scale to use for analyzing market prices; physical time (real time), trading time, the number of trade, and cumulated trading volume [3]. Mandelbrot defined a time-scale, called trading time which - compared to clock time - flows slowly during some period and fast during others [9][10]. Let \( t \) be clock time and \( T(t) \) a trading time, typically multifractal time. Fractional Brownian motions in multifractal time is represented as \( X(t) = B_H(T(t)) \) where \( B_H \) denotes fractal Brownian motions. Clark analyzed the price series as a probability process \( X(T(t)) \) where directing process \( T(t) \) is another probability process and proposed to apply cumulated trading volume as the directing process [2]. In this article, another time scale which evolves the price fluctuation is defined and the relations of the three time scales: real time, tick time and newly defined C-fluctuation time, in foreign exchange markets are analyzed. Takayasu et. al. showed that the statistics of trade intervals in foreign exchange data is well characterized by a Poisson process with its mean given by a moving average of trade intervals for about 2 minutes [11]. In this study the intervals between extreme values determined by a resolution scale are investigated. The data set used is trading prices of foreign exchange rates; US dollar (USD)/Japanese yen (JPY), USD/Euro (EUR), and EUR/JPY. The accuracy of the data is one minute and data within a minute are recorded in order of transaction.

2. Scaling of fluctuations
Given a time series \( g(t) \), local maxima determined by a resolution scale \( C \) along price axis are defined as follows [6]. If there exist \( d_1, d_2 > 0 \) such that \( g(s-d_1) < g(s)-C \), \( g(s+d_2) < g(s)-C \),
and \( g \) attains its maximum at \( s \) in \([s - d_1, s + d_2]\), then \( g(t) \) has a C-local maximum at \( s \). The scale \( C \) is the accuracy of measuring the extreme values. Making \( C \) larger filters out more data. The ordinary local maximum is the case \( C = 0 \): 0-local maximum. In Fig. 1, \( g(t) \) has a C-local maximum at \( s_1^C \), and \( C' \)-local maxima at \( s_1^{C'} \) and \( s_3^{C'} \). Likewise, if there exist \( d_1, d_2 > 0 \) such that \( g(s - d_1) > g(s) - C \), \( g(s + d_2) > g(s) - C \), and \( g(s) = \min_{s-d_1<t<s+d_2} g(t) \), then \( g(t) \) has a C-local minimum at \( s \). In Fig. 2, \( g(t) \) has a C-local minimum at \( s_2^C \), and \( C' \)-local minima at \( s_1^{C'} = s_2^C \) and \( s_2^{C'} \). These extreme values are related to the ex-post optimal trading under the transaction cost \( C \) and based on the same idea as a charting technique: Kagi chart [8].

Given a resolution \( C \), the time series \( g(t) \) attains its C-extreme values (C-local maximum or C-local minimum) at \( s_1^C, s_2^C, s_3^C, \cdots \). Let \( R(C) \) denotes the sum of the absolute variation between neighboring C-extreme values:

\[
R(C) = \sum |g(s_{i+1}^C) - g(s_i^C)|. \tag{1}
\]

The fold dimension is defined as follows. If the relation:

\[
R(C)/C \sim C^{-D_f}
\]

holds, we say that the exponent \( D_f \) is the fold dimension of the time series. Since \( R(C) \) is an invariant for time scale stretching, the fold dimension is independent of time scale. Let \( D_b \) denote the local box dimension of a time series, and \( H \) the Hurst exponent [5]. The relation: \( D_b = 2 - H \) holds. The fold dimension is a kind of latent dimension equal to \( 1/H \) in the self-affine function generated by a special case of Iterated Function System [1],[7]. To correspond the fractal dimension of time series, \( 2-1/D_f \) is used. Given resolution \( C \), a time scale: C-fluctuation time can be defined as

\[
\tau_C(t) = \mu \{ s_i^C \left| s_i^C < t \right. \} \tag{3}
\]

where \( \mu() \) denotes counting measure. To calculate the local box dimension of the time series by Higuchi method, the graph of the time series is covered with rectangles whose base are equal in length [4]. Here we cover the graph with rectangles equal in height. Applying C-fluctuation time as the time scale, these rectangles are also equal in length, and therefore the fold dimension is calculated by the procedure same as Higuchi method. The relation between C-fluctuation time and real (or tick) time The time intervals between these neighboring C-extreme values can be measured by real-time (clock time) and tick-time (the order of submission)

3. Results
The data used is foreign exchange rates tick data of US dollar (USD)/Japanese yen (JPY), USD/Euro (EUR), and EUR/JPY provided by CQG Inc. The record is given in the order of transaction. The transaction time is recorded with an accuracy of one minute. The data within a minute are recorded in order of transaction. There are four types of transaction; bid, ask, trade and settle, and the study presented here uses trading price.

Four periods in 2008 are analyzed in this article: July 14-18, August 18-22, September 15-19, and October 27-31. Each period is five business days of one week since Monday until Friday. Fig. 3 shows the distribution of the fluctuations: the frequency of price changes from the last trade of USD/EUR, USD/JPY and EUR/JPY in these four weeks. In the first two periods the market was calm, and in the latter two periods the market was volatile.

Fig. 6 shows double logarithmic plot of \( R(C)/C \) with \( C \) of EUR/USD in the four periods. The fold dimension \( D_f \) can be calculated from the effective slope of this double logarithmic plot. The fold dimension \( D_f \) of each period is calculated with the scaling region of 5-20 pips. To compare the fractal dimension of time series, \( 2-1/D_f \) is shown in Fig. 7. The exchange rates are persistent in Sep. 15-19 and Oct. 27-31, especially in EUR/JPY.
Figure 1. C-local maximum. $g(t)$ has a C-local maximum at $s_1^C$, and $C'$-local maxima at $s_1^{C'} = s_1^C$ and $s_3^{C'}$.

Figure 2. C-local minimum. $g(t)$ has a C-local minimum at $s_2^C$, and $C'$-local minima at $s_4^{C'} = s_2^C$ and $s_2^{C'}$.

Figure 3. The distribution of the fluctuations: the frequency of price changes from the last trade; Jul. 14-18, Aug. 18-22, Sep. 15-19 and Oct. 27-31, 2008. (a)USD/EUR, (b)USD/JPY, (c)EUR/JPY
Figure 4. Double logarithmic plot of $R(C)/C$ with $C$ (EUR/USD); Jul. 14-18, Aug. 18-22, Sep. 15-19 and Oct. 27-31, 2008.

Figure 5. $(2 - 1/D_f)$ for EUR/USD ($\circ$), USD/JPY ($\triangle$) and EUR/JPY ($\Box$) in Jul. 14-18, Aug. 18-22, Sep. 15-19 and Oct. 27-31, 2008. $D_f$ is the fold dimension calculated with scaling region 5-20 pips.

Fig. 6 and 7 show examples of the relation of the three time scales: real time, tick time and C-fluctuation time ($C=7$ pips) in the case of EUR/USD, Oct. 29, 2008. Fig. 6 shows the transformation of the time scale: C-fluctuation time into real time. In Fig. 7 a part of the transformation is from C-fluctuation time into tick time.

Fig. 8-13 show the distribution of the intervals of C-extreme values. In Fig. 8-10 the intervals between C-extreme values are measured by tick time. In Fig. 11-13 the intervals are measured by real time (minutes). Since time is recorded with an accuracy of one minute, the intervals are also measured by real time with an accuracy of one minute. The interval of 0 minute means that the interval is less than 60 seconds and the measuring error is greater than 0 second and less than 60 seconds. In case of the interval of other minutes, the measuring error is greater than -60 seconds and less than 60 seconds. Considering this measuring error, the bin width of 0 minute must be a half of these of other minutes. The figures (a) of Fig. 8-13 are the distributions in Aug. 18-22 and The figures (b) of Fig. 8-13 are the ones in Sep. 15-19. Fig. 8 and 11 show the distribution of EUR/USD, Fig. 9 and 12 show the ones of USD/JPY and Fig. 10 and 13 show the ones of EUR/JPY. The interval between C-extreme values is the increment of C-fluctuation time measured by another time scale such as real time. With small resolution $C$ (1-2 pips), the intervals between C-extreme values are exponentially distributed in tick time. In EUR/JPY the intervals between C-extreme values are exponentially distributed with larger range. When the market is volatile, for larger resolution $C$ (1-4 pips), the intervals between C-extreme values are exponentially distributed. When the market is not volatile, with the resolution $C$ (3-5 pips), the intervals between C-extreme values are exponentially distributed in real time. When the market is volatile, for larger resolution $C$ (5-15 pips), the intervals between C-extreme values are exponentially distributed in real time.

4. Summary
The series of instantaneous velocity of C-fluctuation time flowing, i.e. the intervals between C-extreme values, are exponentially distributed for small $C$ when they are measured by real time and for smaller $C$ (1-2 pips) when they are measured by tick time. When the market is volatile,
Figure 6. transformation of the time scale: C-fluctuation time (C=7 pips) into real time (EUR/USD, Oct. 29, 2008).

Figure 7. transformation of the time scale: C-fluctuation time (C=7 pips) into tick time (EUR/USD, Oct. 29, 2008).

Figure 8. Interval of C-extreme values measured by tick time (EUR/USD, 2008) (a) Aug. 18-22, (b) Sep. 15-19

for larger C (1-4 pips), the intervals between C-extreme values are exponentially distributed. When the market is not volatile, with the resolution C (3-5 pips), the intervals between C-extreme values are exponentially distributed in real time. When the market is volatile, for larger resolution C (5-15 pips), the intervals between C-extreme values are exponentially distributed in real time.

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Figure 9. Interval of C-extreme values measured by tick time (USD/JPY, 2008) (a) Aug.18-22, (b) Sep.15-19

Figure 10. Interval of C-extreme values measured by tick time (EUR/JPY, 2008) (a) Aug.18-22, (b) Sep.15-19

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Figure 11. Interval of C-extreme values measured by real time (EUR/USD, 2008) (a) Aug.18-22, (b) Sep.15-19

Figure 12. Interval of C-extreme values measured by real time (USD/JPY, 2008) (a) Aug.18-22, (b) Sep.15-19

Figure 13. Interval of C-extreme values measured by real time (EUR/JPY, 2008) (a) Aug.18-22, (b) Sep.15-19