On quantum $\mathfrak{osp}(1|2\ell)$-Toda chain *

A.A. Gerasimov, D.R. Lebedev and S.V. Oblezin

Abstract. The orthosymplectic superalgebra $\mathfrak{osp}(1|2\ell)$ is the closest analog of standard Lie algebras in the world of super Lie algebras. We demonstrate that the corresponding $\mathfrak{osp}(1|2\ell)$-Toda chain turns out to be an instance of a $BC_\ell$-Toda chain. The underlying reason for this relation is discussed.

1 Introduction

Representation theory is an essential tool in finding explicit solutions of known quantum integrable systems as well as construction of new ones. An important class of finite-dimensional quantum integrable systems allowing representation theory interpretation is provided by Toda chains. It is known that integrable Toda chains are classified by a class of root systems that include root systems of finite dimensional Lie algebras as well as their affine counterparts. For the Toda chains associated with the root systems of finite dimensional Lie algebras the corresponding integrable systems can be solved explicitly by representation theory methods [K1], [GW] (see [STS] for a review). Precisely, the eigenfunctions of the quantum Hamiltonians are given by special matrix elements of principal series representations of the totally split real form of the corresponding Lie group. The resulting functions should be considered as generalized Whittaker functions associated with the corresponding finite-dimensional Lie algebras [K1], [Ha]. These functions allow quite explicit integral representations (see e.g. [GLO1]).

The class of integrable Toda chains is however a bit larger than the class of finite/affine Lie algebras and includes in particular non-reduce root systems $BC_\ell$ combining $B_\ell$ and $C_\ell$ root systems. The corresponding $BC_\ell$-Toda system is an important element of the web of Toda type theories connected by various intertwining relations [GLO2]. Although $BC_\ell$ root system fits naturally in the classification of finite Lie algebra root systems the problem of construction of the Lie algebra type object corresponding to the non-reduced root $BC_\ell$ systems seems not yet obtained a satisfactory resolution. However, one should recall that $BC_\ell$ root systems appear in the Cartan classification of symmetric spaces [H], [L].

*Talk given by the second author at the "Polivanov-90" conference, 16-17 December 2020, Steklov Mathematical Institute of Russian Academy of Science.
BCℓ-Toda chain can be solved via representation theory methods using a generalization of \(C_\ell\)-Whittaker functions (see e.g. [J] for \(\ell = 1\) and a remark in [RS], relevant to \(BC_\ell\) classical Toda system). This unfortunately does not elucidate the question of the interpretation of \(BC_\ell\)-Toda eigenfunctions as standard Whittaker functions for some group-like object. One should add that the integrability of the quantum \(BC_\ell\)-Toda chain for generic coupling constants was proven independently in [S] using Yangian representation theory (aka quantum inverse scattering methods). This however also does not clarify the question of existence of a group-like structure behind \(BC_\ell\) root systems.

In this note we consider quantum Toda chains associated with the super Lie algebras \(osp(1|2\ell)\). This series of super Lie algebras occupy a special place in the world of super Lie algebras. In particular, it is the only instance of simple super Lie algebras for which the corresponding category of finite-dimensional representations is semi-simple and thus allows direct analogs of the standard constructions of representation theory of semisimple Lie algebras [Kac1]. In connection with this fact one should mention that \(osp(1|2\ell)\) is the unique super Lie algebra with finitely-generated center of its universal enveloping algebra. The special properties of \(osp(1|2\ell)\) makes it natural to consider the associated quantum integrable systems.

In this note we demonstrate that \(osp(1|2\ell)\)-Toda chain may be also considered as a Toda chain associated with the \(BC_\ell\) root system. This allows us to solve \(BC_\ell\)-Toda chain by standard representation theory methods i.e. by identifying the corresponding eigenfunctions with \(osp(1|2\ell)\)-Whittaker functions.

The underlying reason for the appearance of \(BC_\ell\) root structure in \(osp(1|2\ell)\)-Toda chain becomes clear by comparing \(BC_\ell\) root data with that of the super Lie algebra \(osp(1|2\ell)\). Actually the only difference is the opposite parity of the maximal commutative subalgebra eigenspaces in the Cartan decomposition corresponding to short roots of non-reduced \(BC_\ell\) root system. This difference however does not affect the expressions for quantum Hamiltonians of the corresponding Toda chain.

Exposition of the paper goes as follows. In Section 2 we provide basic facts on the structure of the orthosymplectic super Lie algebra \(osp(1|2\ell)\). In Section 3 we construct \(osp(1|2\ell)\)-Whittaker functions associated with representations of the super Lie algebra \(osp(1|2\ell)\) and demonstrate that these functions are eigenfunctions of the quadratic Hamiltonian of \(BC_\ell\)-Toda chain for special values of the coupling constants. Finally, in Section 4 we discuss the structure of root system of \(osp(1|2\ell)\) versus \(BC_\ell\) root system and provide an explanation of the apparent identification of quadratic Hamiltonian of \(osp(1|2\ell)\)-Toda chain with that of \(BC_\ell\)-Toda chain.

Acknowledgments: The research of the second (D.R.L.) and third (S.V.O.) authors was supported by RSF grant 16-11-10075.

2 Basic facts on the super Lie algebra \(osp(1|2\ell)\)

We start with the basic definition of a Lie superalgebra structure and then we describe explicitly the algebra \(osp(1|2\ell)\) in detail. This is a standard material that can be found in
standard sources on super algebras e.g. [Kac1], [Kac2].

The notion of Lie superalgebra is a direct generalization of the notion of Lie algebra to the category of vector superspaces. Vector superspace \( V = V_0 \oplus V_1 \) is a \( \mathbb{Z}_2 \)-graded vector space with the parity \( p \) taking values 0 and 1 on \( V_0 \) and \( V_1 \) respectively. The tensor product structure is given by twisting of the standard tensor product structure in the category of vector spaces

\[
 v \otimes w = (-1)^{p(v) \cdot p(w)} w \otimes v, \quad v \in V, \quad w \in W,
\]

for \( v \) and \( w \) are homogeneous elements with respect to the \( \mathbb{Z}_2 \)-grading.

**Definition 2.1** The structure of super Lie algebra on super vector space \( g = g_0 \oplus g_1 \) is given by a bilinear operation \([\cdot, \cdot]\), called the bracket, so that for any homogeneous elements \( X, Y, Z \in g \) the following hold:

\[
 p([X, Y]) = p(X) + p(Y),
\]

\[
 [X, Y] = -(-1)^{p(X) \cdot p(Y)} [Y, X],
\]

\[
 [X, [Y, Z]](-1)^{p(X) \cdot p(Z)} + [Z, [X, Y]](-1)^{p(Y) \cdot p(Z)} + [Y, [Z, X]](-1)^{p(X) \cdot p(Y)} = 0.
\]

We will be interested in a special instance of super Lie algebras, the ortho-symplectic super Lie algebra \( \mathfrak{osp}(1|2\ell) \). To define this algebra let us first introduce the super Lie algebra \( \mathfrak{gl}(1|2\ell) \).

**Definition 2.2** The super Lie algebra \( \mathfrak{gl}(1|2\ell) \) is generated by

\[
 E_{0,i}, \quad E_{i,0}, \quad p(E_{0,i}) = p(E_{i,0}) = 1, \quad 0 \leq i \leq 2\ell,
\]

\[
 E_{kl}, \quad p(E_{kl}) = 0, \quad 1 \leq k, l \leq 2\ell,
\]

subjected to the following relations:

\[
 [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{p(i) \cdot p(l)} \delta_{il}E_{kj}, \quad 0 \leq i, j \leq 2\ell, \quad 0 \leq k, l \leq 2\ell.
\]

The super Lie algebra \( \mathfrak{gl}(1|2\ell) \) may be identified with the super Lie algebra structure \( (\text{End}(V), [, ]) \) on the space \( \text{End}(V) \) of endomorphisms of the superspace

\[
 V = \mathbb{R}^{1|2\ell} = V_0 \oplus V_1, \quad V_0 = \mathbb{R}^{0|2\ell}, \quad V_1 = \mathbb{R}^{1|0},
\]

in the following way. Any zero parity linear endomorphism \( A \in \text{End}(V) \) is given by the matrix of the following shape:

\[
 A = \begin{pmatrix}
 A_{11} & A_{12} \\
 A_{21} & A_{22}
\end{pmatrix}, \quad A_{11} : V_1 \to V_1, \quad A_{12} : V_0 \to V_1, \quad A_{21} : V_1 \to V_0, \quad A_{22} : V_0 \to V_0,
\]
where entries of blocks $A_{11}$, $A_{22}$ are even while the entries of $A_{12}$, $A_{21}$ are odd so that
\[
\text{End}(V)_0 = \left\{ \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \right\}, \quad \text{End}(V)_1 = \left\{ \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\}.
\] (2.9)
The super brackets on End($V$) are defined on homogeneous elements $X, Y \in \text{End}(V)$ as follows
\[
[X, Y] = X \circ Y - (-1)^{p(X)\cdot p(Y)} Y \circ X.
\] (2.10)
The description of $g(1|2\ell)$ given in Definition 2.2 is then obtained via fixing a bases in $V$
\[
\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2\ell}\} \subset \mathbb{R}^{1|2\ell}, \quad p(\varepsilon_0) = 1, \quad p(\varepsilon_k) = 0, \quad 1 \leq k \leq 2\ell.
\] (2.11)
The generators $E_{ij}$ are identified with the elementary matrices in End($V$) with the only non-zero elements in the $i$-th row and the $j$-th column.

**Definition 2.3** The super transposition of a matrix $A \in \text{End}(V)$ is defined by
\[
A^\top = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^\top = \begin{pmatrix} A_{11}^t & -A_{21}^t \\ A_{12}^t & A_{22}^t \end{pmatrix},
\] (2.12)
where $X^t$ if the standard transposition of a matrix $X$.

**Lemma 2.1** Super transposition (2.12) possesses the following properties:
\[
(Av)^t = v^t A^\top, \quad A \in \text{End}(V), \quad v \in V,
\] (2.13)
\[
(A \cdot B)^\top = B^\top \cdot A^\top,
\] (2.14)
\[
(A^\top)^\top = \Pi A \Pi^{-1},
\] (2.15)
where $\Pi$ is the parity operator with the matrix $\begin{pmatrix} -1 & 0 \\ 0 & \text{Id}_{2\ell} \end{pmatrix}$.

**Proof**: Given $v \in V$ let us write it down in the basis $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2\ell}\}$:
\[
v = \xi \varepsilon_0 + \sum_{i=1}^{2\ell} v_i \varepsilon_i,
\] (2.16)
with odd Grassmann coordinate $\xi$, and even coordinates $v_i$. Then we have
\[
(Av)^t_i = a_{i0} \xi + a_{i1} v_1 + \ldots + a_{i,2\ell} v_{2\ell},
\] (2.17)
and on the other hand,
\[
(v^t A^\top)_0 = \xi a_{00} + v_1 a_{0,1} + \ldots + v_n a_{0,n},
\]
\[
(v^t A^\top)_k = -\xi a_{k0} + v_1 a_{k,1} + \ldots + v_n a_{k,n}, \quad 1 \leq k \leq 2\ell.
\] (2.18)
Taking into account that
\[ \xi a_{00} = a_{00} \xi, \quad -\xi a_{k,0} = a_{k,0} \xi, \quad 1 \leq k \leq 2\ell, \quad (2.19) \]
we deduce the first assertion. The second assertion can be verified by straightforward computation. The third assertion follows from the definition: on the one hand, we have
\[ (A^T)^\top = \left( \begin{array}{cc} A_{11}^t & -A_{21}^t \\ A_{12}^t & A_{22}^t \end{array} \right)^\top = \left( \begin{array}{cc} A_{11} & -A_{12} \\ -A_{12} & A_{22} \end{array} \right); \quad (2.20) \]
on the other hand, in the standard basis \((2.11)\) the matrix of parity operator reads
\[ \Pi = \left( \begin{array}{cc} -1 & 0 \\ 0 & \text{Id}_{2\ell} \end{array} \right), \quad (2.21) \]
so the assertion easily follows. \(\square\)

Now \(\mathfrak{osp}(1|2\ell)\) may be defined as a subalgebra of the general linear superalgebra \(\mathfrak{gl}(1|2\ell)\).

Introduce the following involution:
\[ \theta : \mathfrak{gl}(1|2\ell) \rightarrow \mathfrak{gl}(1|2\ell), \quad X \mapsto X^\theta := -J X^\top J^{-1}, \quad (2.22) \]
where
\[ J = \left( \begin{array}{cc} 1 & 0 \\ 0 & -\text{Id}_\ell \\ 0 & \text{Id}_\ell \end{array} \right) \in \text{End}(V_0). \quad (2.23) \]

**Definition 2.4** The orthosymplectic super Lie algebra \(\mathfrak{osp}(1|2\ell)\) is defined as the \(\theta\)-invariant subalgebra of \(\mathfrak{gl}(1|2\ell)\):
\[ \mathfrak{osp}(1|2\ell) = \left\{ X \in \mathfrak{gl}(1|2\ell) : \quad X^\theta = X \right\} \]
\[ = \left\{ X = \left( \begin{array}{cc} y & x \\ -x & A \\ \text{Id}_\ell & B \end{array} \right) : \quad B^t = B, \quad C^t = C \right\} \subset \mathfrak{gl}(1|2\ell). \quad (2.24) \]

According to the classification of simple super Lie algebras [Kac1] one associates the root system \(B_{0,\ell}\) to the super Lie algebra \(\mathfrak{osp}(1|2\ell)\). Let \(\{\epsilon_1, \ldots, \epsilon_\ell\} \subset \mathbb{R}^\ell\) be an orthogonal basis in \(\mathbb{R}^\ell\) with respect to the scalar product \((,\)\). Then simple root system \(\Delta^+(B_{0,\ell})\) of type \(B_{0,\ell}\) consists of even simple positive roots \(\Delta^{(2)}_0\) and odd simple positive roots \(\Delta^{(2)}_1\):
\[ \Delta^{(2)}_0(B_{0,\ell}) = \left\{ \alpha_k = \epsilon_{\ell+1-k} - \epsilon_{\ell+2-k}, \quad 1 < k \leq \ell \right\}, \quad \Delta^{(2)}_1(B_{0,\ell}) = \left\{ \alpha_1 = \epsilon_\ell \right\}, \quad (2.25) \]
indexed by \(I = \{1, \ldots, \ell\}\). The simple co-roots \(\alpha^\vee_i, i \in I\) are defined in a standard way:
\[ \alpha^\vee_i := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \quad i \in I. \]

Note that the set \(\Delta^+(B_{0,\ell})\) of positive roots contains the sub-system of even positive roots of \(C_\ell\) root system with the corresponding set of simple roots:
\[ \Delta^+(C_\ell) = \left\{ 2\alpha_1 = 2\epsilon_\ell, \quad \alpha_k = \epsilon_{\ell+1-k} - \epsilon_{\ell+2-k}, \quad 1 < k \leq \ell \right\} \subset \Delta^+(B_{0,\ell}). \quad (2.26) \]
The Cartan matrix $A = \|A_{ij}\|$ associated with the simple root system (2.25) is defined by the standard formula

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j \in I.$$  \hspace{1cm} (2.27)

Thus the Cartan matrix of $^s\Delta^+(B_{0,\ell})$ coincides with the standard $B_\ell$-type Cartan matrix

$$A = \begin{pmatrix}
2 & -2 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 2 & -1 \\
0 & \ldots & 0 & -1 & 2
\end{pmatrix}.$$  \hspace{1cm} (2.28)

The Cartan decomposition for $\mathfrak{osp}(1|2\ell)(\mathbb{C})$ reads

$$\mathfrak{osp}(1|2\ell)(\mathbb{C}) = \bigoplus_{i \in I} \mathbb{C}h_i \oplus \bigoplus_{\alpha \in \Delta_0^+} (\mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}) \oplus \bigoplus_{\beta \in \Delta_i^+} (\mathbb{C}X_\beta \oplus \mathbb{C}X_{-\beta}),$$

$$\Delta_0^+ = \{2\epsilon_i; \quad \epsilon_i \pm \epsilon_j, \quad i < j, \quad i, j \in I\}, \quad \Delta_i^+ = \{\epsilon_i, \quad i \in I\}.$$  \hspace{1cm} (2.29)

and the Cartan-Weyl relations are the following:

$$[X_{\epsilon_i}, X_{\epsilon_j}] = (1 + \delta_{ij})X_{\epsilon_i + \epsilon_j}, \quad [X_{-\epsilon_i}, X_{-\epsilon_j}] = -(1 + \delta_{ij})X_{-\epsilon_i - \epsilon_j},$$

$$[X_{\epsilon_i}, X_{-\epsilon_i}] = a_{ii}, \quad i \in I;$$

$$[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_j}] = X_{\epsilon_i}, \quad [X_{\epsilon_i - \epsilon_j}, X_{-\epsilon_i}] = -X_{-\epsilon_j},$$

$$[X_{\epsilon_i}, X_{-\epsilon_i - \epsilon_j}] = X_{-\epsilon_j}, \quad [X_{\epsilon_i - \epsilon_j}, X_{\epsilon_i + \epsilon_j}] = X_{\epsilon_j}, \quad i < j;$$

$$[X_\alpha, X_{-\alpha}] = h_{\alpha\gamma} = \sum_{i \in I} (\alpha^{\gamma}, \epsilon_i)a_{ii},$$

$$[h_{\alpha\gamma}, X_\gamma] = \alpha^{\gamma}(\gamma)X_\gamma, \quad \alpha, \gamma \in \Delta^+. $$

The Serre relations on $X_{\alpha_i}, \alpha_i \in ^s\Delta^+(B_{0,\ell})$ have the following form:

$$\text{ad}_{X_{\alpha_i}}^2(X_{\alpha_1}) = 0, \quad \text{ad}_{X_{-\alpha_1}}^2(X_{-\alpha_1}) = 0,$$

$$\text{ad}_{X_{\alpha_i}}^{1-a_{ij}}(X_{\alpha_j}) = 0, \quad \text{ad}_{X_{-\alpha_1}}^{1-a_{ij}}(X_{-\alpha_1}) = 0, \quad i, j \in I.$$  \hspace{1cm} (2.31)

The Cartan-Weyl generators $X_\alpha$ may be represented via matrix embedding (2.24) of $\mathfrak{osp}(1|2\ell)$ as follows:

$$X_{\epsilon_i} = E_{i,0} + E_{0,\ell+i}, \quad X_{-\epsilon_i} = E_{0,i} - E_{\ell+i,0}. $$  \hspace{1cm} (2.32)
\begin{align*}
X_{i,-\ell} &= E_{ij} - E_{2\ell+1-i,2\ell+1-j}, \\
X_{-\ell,i+\ell} &= E_{ji} - E_{2\ell+1-j,2\ell+1-i}, \\
X_{\ell,i+\ell} &= E_{i,\ell+j} + E_{j,\ell+i}, \\
X_{-\ell,i-\ell} &= E_{\ell+j,i} + E_{\ell+i,j}, \quad i < j, \\
X_{2\ell,i} &= E_{i,\ell+i}, \\
X_{-2\ell,i} &= E_{\ell+i,i}, \quad i \in I.
\end{align*}

The Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{osp}(1|2\ell) \) is spanned by
\begin{equation}
\label{eq:2.34}
\mathfrak{h}_i = E_{ii} - E_{\ell+i,\ell+i}, \quad i \in I.
\end{equation}

For a class of super Lie algebras \( \mathfrak{g} \) allowing non-degenerate invariant pairing \(|\) there is a canonical construction of the quadratic Casimir element \( C_2 \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \) of the center of the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) (see e.g. [Kac1]). Let us chose a pair \( \{u_i, i \in I\}, \{u^i, i \in I\} \) of dual bases in the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \), and let \( \{X_\alpha, X^\alpha, \alpha \in \Delta^+\} \) be the Cartan-Weyl generators normalized by \( (X^\alpha|X_\alpha) = 1 \). Then the quadratic Casimir element \( C_2 \) allows for the following presentation:
\begin{equation}
\label{eq:2.35}
C_2 = \sum_{i \in I} u_i u_i + \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} X_\alpha X^\alpha + X^\alpha X_\alpha.
\end{equation}

To define the quadratic Casimir element for \( \mathfrak{osp}(1|2\ell) \) we shall first introduce an invariant non-degenerate invariant pairing. The super Lie algebra \( \mathfrak{osp}(1|2\ell) \) allows the invariant scalar product defined as follows
\begin{equation}
\label{eq:2.36}
(X|Y) := \frac{1}{2} \text{str} \left( \rho_t(X) \circ \rho_t(Y) \right), \quad X, Y \in \mathfrak{osp}(1|2\ell),
\end{equation}
where \( \rho_t : \mathfrak{osp}(1|2\ell) \to \text{End}(\mathbb{C}^{1|2\ell}) \) is the tautological representation of \( \mathfrak{osp}(1|2\ell) \) in \( \mathbb{C}^{1|2\ell} \). The supertrace of \( A \in \text{End}(\mathbb{C}^{1|2\ell}) \) of the shape (2.3) is given by
\begin{equation}
\label{eq:2.37}
\text{str} \left( A \right) = \text{str} \left( \begin{array}{cc} A_{11} & A_{12} \\
A_{21} & A_{22} \end{array} \right) = -\text{tr} \left( A_{11} \right) + \text{tr} \left( A_{22} \right).
\end{equation}

The explicit form of the invariant scalar product (2.36) may be directly derived using the matrix representation (2.24).

**Lemma 2.2** The invariant scalar product (2.36) on super Lie algebra \( \mathfrak{osp}(1|2\ell) \) is as follows
\begin{equation}
\label{eq:2.38}
\begin{aligned}
(h_i|h_i) &= 1, \quad i \in I; \\
(X_\alpha|X_\alpha) &= \frac{2}{(\alpha,\alpha)}, \quad \alpha \in \Delta^+_0; \\
(X_\beta|X_\beta) &= 1, \quad \beta \in \Delta_1^+, \\
\end{aligned}
\end{equation}
with the rest of the products being zero.

**Proof** : Validity of (2.38) may be checked directly. Thus for example we have using (2.34)
\begin{equation}
\label{eq:2.39}
(h_i|h_j) = \frac{1}{2} \text{str} \left( \rho_t(E_{ii} - E_{\ell+i,\ell+i}) \rho_t(E_{jj} - E_{\ell+j,\ell+j}) \right) = \delta_{ij}.
\end{equation}
Similarly, using (2.32) we obtain
\[ (X_\epsilon|X_{-\epsilon_j}) = \frac{1}{2} \text{str} \left( \rho_t(E_{i,0} + E_{0,t+i})\rho_t(E_{0,j} - E_{t+j,0}) \right) = \delta_{ij}. \] (2.40)

Similarly one might check the expressions for remaining products. \(\Box\)

**Proposition 2.1** The following expression provides the quadratic Casimir element (2.35) for \(osp(1|2\ell)\):
\[ C_2 = \sum_{i \in I} \left( h_i^2 - X_\epsilon_i X_{-\epsilon_i} + X_{-\epsilon_i} X_\epsilon_i \right) \]
\[ + \sum_{\alpha \in \Delta_0^+} \frac{(\alpha,\alpha)}{2} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha). \] (2.41)

**Proof:** Using the expressions (2.38) for the invariant pairing we have
\[ h^i = h_i, \quad i \in I; \]
\[ X^\alpha = \frac{(\alpha,\alpha)}{2} X_{-\alpha}, \quad \alpha \in \Delta_0^+; \quad X^\beta = X_{-\beta}, \quad \beta \in \Delta_1^+. \] (2.42)

Substituting (2.42) into (2.35) we arrive at (2.41), and complete the proof. \(\Box\)

In the following it will more convenient to use another set of notations for generators which is adapted to the matrix form (2.24)
\[ y_i = X_\epsilon_i, \quad x_i = X_{-\epsilon_i}; \]
\[ a_{ii} = h_i, \quad b_{ii} = X_{2\epsilon_i}, \quad c_{ii} = X_{-2\epsilon_i}, \quad i \in I; \]
\[ a_{ij} = X_{\epsilon_i-\epsilon_j}, \quad a_{ji} = X_{-\epsilon_i+\epsilon_j}, \quad i < j. \] (2.43)

In addition to (2.30) the even part of \(osp(1|2\ell)\) satisfies the following relations:
\[ [b_{ij}, b_{kl}] = 0, \quad [c_{ij}, c_{kl}] = 0, \quad [b_{ij}, c_{kl}] = \delta_{jk}a_{il}, \]
\[ [a_{ii}, b_{kl}] = (\delta_{ik} - \delta_{il})b_{kl}, \quad [a_{ii}, c_{kl}] = -(\delta_{ik} - \delta_{il})b_{kl}. \] (2.44)

Using these notations the quadratic Casimir element (2.41) may be written as follows:
\[ C_2 = \sum_{i=1}^{\ell} (a_{ii}^2 + x_i y_i - y_i x_i) + 2(c_{ii} b_{ii} + b_{ii} c_{ii}) \]
\[ + \sum_{i<j} (a_{ij} a_{ji} + a_{ji} a_{ij}) + (b_{ij} c_{ij} + c_{ij} b_{ij}). \] (2.45)

From now on we will consider the real form \(osp(1|2\ell)(\mathbb{R})\) of the orthosymplectic super Lie algebra, such that the generators \(a_{ii}, b_{ii}, c_{ii}, i \in I, a_{ij}, a_{ji}, b_{ij}, c_{ij}, i < j\) as well as \(x_i\) and \(y_i\) are defined to be real.
3 The osp\((1\mid 2\ell)\)-Whittaker function

In this section we construct the Whittaker function associated with the super Lie algebra \(\text{osp}(1\mid 2\ell)\). There is a classical approach to the construction of Whittaker functions associated with semisimple Lie algebras \([J], [K1], [K2], [H]\). Below we give a modified version of this construction due to Kazhdan-Kostant (see \([E]\)).

Given a super Lie algebra \(\mathfrak{g}\), let \(\mathcal{U}(\mathfrak{g})\) be the corresponding universal enveloping algebra and let \(\mathcal{Z}(\mathcal{U}(\mathfrak{g})) \subset \mathcal{U}(\mathfrak{g})\) be its center. A \(\mathcal{U}(\mathfrak{g})\)-module \(V\) admits an infinitesimal character \(\chi\) if there is a homomorphism \(\chi : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \rightarrow \mathbb{C}\) such that \(zv = \chi(z)v\) for all \(z \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))\) and \(v \in V\). Given a character \(\chi\) of a nilpotent super Lie subalgebra \(\mathfrak{n} \subset \mathfrak{g}\),

\[
\chi : \mathfrak{n} \rightarrow \mathbb{C}^{[1]} ;
\]

we define a Whittaker vector \(\psi \in V\) by the following relations:

\[
X \cdot \psi = \chi(X) \psi, \quad \forall X \in \mathfrak{n} \subset \mathfrak{g}.
\]

The Whittaker vector \(\psi \in V\) is called cyclic, if it generates \(V\): \(\mathcal{U}(\mathfrak{g}) \cdot \psi = V\). A \(\mathcal{U}(\mathfrak{g})\)-module \(V\) is called a Whittaker module if it contains a cyclic Whittaker vector. A pair of \(\mathcal{U}(\mathfrak{g})\)-modules \(V\) and \(V'\) is called dual if there exists a non-degenerate pairing

\[
\langle , \rangle : V \times V' \rightarrow \mathbb{C}^{[1]},
\]

which is \(\mathbb{C}\)-antilinear in the first variable and \(\mathbb{C}\)-linear in the second one, and such that

\[
\langle X \cdot v', v \rangle = -(-1)^{\rho(v') - \rho(X)} \langle v', X \cdot v \rangle, \quad v \in V, \ v' \in V', \ X \in \mathfrak{g}.
\]

Now we restrict ourselves to the case of the orthosymplectic super Lie algebra \(\text{osp}(1\mid 2\ell)\). Let \(V_\lambda\) be a \(\mathcal{U}(\text{osp}(1\mid 2\ell))\)-module with an infinitesimal central character allowing a vector \(v_\lambda \in V_\lambda\) defined by (see the notations of (2.29), (2.30)):

\[
h_{\alpha^\vee} \cdot v_\lambda = \alpha^\vee(\lambda)v_\lambda, \quad X_\alpha \cdot v_\lambda = 0, \quad \forall X_\alpha \in \mathfrak{n}_+ \subset \text{osp}(1\mid 2\ell), \quad \alpha \in \Delta^+,
\]

where \(\lambda\) is an element of the dual to the Cartan subalgebra \(\mathfrak{h} \subset \text{osp}(1\mid 2\ell)\). Value of the quadratic Casimir element \(C_2\) on \(V_\lambda\) is uniquely determined by (3.5). Indeed let us re-write the Casimir element \(C_2\) from (2.41) as follows:

\[
C_2 = \sum_{i \in I} \left( \alpha_{ii}^2 - a_{ii} + 2X_{-\epsilon_i}X_{\epsilon_i} \right) + \sum_{\alpha \in \Delta_0^+} \frac{(\alpha, \alpha)}{2} \left( h_{\alpha^\vee} + 2X_{-\alpha}X_{\alpha} \right)
\]

\[
= \sum_{i \in I} \left( \alpha_{ii}^2 + 2\rho(\epsilon_i)a_{ii} + 2X_{-\epsilon_i}X_{\epsilon_i} \right) + \sum_{\alpha \in \Delta_0^+} (\alpha, \alpha) X_{-\alpha}X_{\alpha},
\]

where

\[
\rho(q) = \frac{1}{2} \left( \sum_{\alpha \in \Delta_0^+} \alpha(q) - \sum_{\beta \in \Delta_1^+} \beta(q) \right).
\]
Thus using (3.5) \( C_2 \) takes the following value on \( v_\lambda \in \mathcal{V}_\lambda \):
\[
C_2(v_\lambda) = \sum_{i \in I} (a_{ii}^2 + 2\rho(\epsilon_i)a_{ii})(v_\lambda) = (\lambda, \lambda + 2\rho)(v_\lambda).
\]

(3.8)

In the following we consider those \( \mathcal{V}_\lambda \) that allow a structure of the Whittaker modules and also allow integration of the action of the Cartan subalgebra \( \mathfrak{h} \subset \text{osp}(1|2\ell) \) to the action of the corresponding maximal torus \( H \). Precisely let \( \mathcal{V}_\lambda \) and \( \mathcal{V}'_\lambda \) be a dual pair of Whittaker modules cyclically generated by Whittaker vectors \( \psi_R \in \mathcal{V}_\lambda \) and \( \psi_L \in \mathcal{V}'_\lambda \). Explicitly, the Whittaker vectors \( \psi_R \in \mathcal{V}_\lambda \) and \( \psi_L \in \mathcal{V}'_\lambda \) are defined by the following conditions:
\[
X_\alpha \cdot \psi_R = \chi_R(X_\alpha) \psi_R, \quad X_{-\alpha} \cdot \psi_L = \chi_L(X_{-\alpha}) \psi_L, \quad \forall \alpha \in \Delta^+,
\]
where \( \chi_R : n_+ \to \mathbb{C}^{1|1} \) and \( \chi_L : n_- \to \mathbb{C}^{1|1} \) are the characters of the opposite nilpotent super Lie subalgebras \( n_+ \subset \text{osp}(1|2\ell) \):
\[
\chi_R : n_+ = \left( \bigoplus_{\alpha \in \Delta^+_0} \mathbb{C}X_\alpha \oplus \bigoplus_{\beta \in \Delta^+_1} \mathbb{C}X_{\beta} \right) \to \mathbb{C}^{1|1},
\]
\[
\chi_L : n_- = \left( \bigoplus_{\alpha \in \Delta^+_0} \mathbb{C}X_{-\alpha} \oplus \bigoplus_{\beta \in \Delta^+_1} \mathbb{C}X_{-\beta} \right) \to \mathbb{C}^{1|1}.
\]

(3.10)

Lemma 3.1
(i) The function \( \chi_R : n_+ \to \mathbb{C}^{1|1} \) defined by
\[
\chi_R(X_{\epsilon_i}) = i^{3/2} \xi_{\alpha_i}^+ \in i^{3/2}\mathbb{R}^{1|0}, \quad \chi_R(X_{2\epsilon_i}) = i(\xi_{\alpha_i}^+)^2 \in i\mathbb{R}^{0|1},
\]
\[
\chi_R(X_{\epsilon_i+1-k-\epsilon_{i+2-k}}) = i\xi_{\alpha_k}^+ \in i\mathbb{R}^{0|1}, \quad 1 < k \leq \ell,
\]
\[
\chi_R(X_{\epsilon_k}) = \chi_R(X_{\alpha}) = 0, \quad 1 \leq k < \ell, \quad \alpha \in \Delta^+_0 \setminus s\Delta^+_0,
\]
is a character of the super Lie subalgebra \( n_+ \subset \text{osp}(1|2\ell) \).

(ii) Similarly, the function \( \chi_L : n_- \to \mathbb{C}^{1|1} \) defined by
\[
\chi_L(X_{-\epsilon_i}) = i^{3/2} \xi_{\alpha_i}^- \in i^{3/2}\mathbb{R}^{1|0}, \quad \chi_L(X_{-2\epsilon_i}) = i(\xi_{\alpha_i}^-)^2 \in i\mathbb{R}^{0|1},
\]
\[
\chi_L(X_{-\epsilon_i+1-k+\epsilon_{i+2-k}}) = i\xi_{\alpha_k}^- \in i\mathbb{R}^{0|1}, \quad 1 < k \leq \ell,
\]
\[
\chi_L(X_{-\epsilon_k}) = \chi_L(X_{-\alpha}) = 0, \quad 1 \leq k < \ell, \quad \alpha \in \Delta^+_0 \setminus s\Delta^+_0,
\]
is a character of the super subalgebra \( n_- \subset \text{osp}(1|2\ell) \).

Proof: We provide the proof in the case of \( \chi_R \) while the case of \( \chi_L \) can be treated in a similar way. Let us verify that (3.11) defines a character \( \chi_R \) of the super subalgebra \( n_+ \subset \text{osp}(1|2\ell) \) by checking the compatibility of (3.11) with the appropriate Cartan-Weyl relations (2.30):
\[
[X_{\epsilon_i}, X_{\epsilon_j}] = 2X_{\epsilon_i}^2 = 2X_{2\epsilon_i}, \quad [X_{\epsilon_i}, X_{\epsilon_j}] = X_{\epsilon_{i+j}}, \quad i, j \in I,
\]
\[
[X_{\epsilon_i-\epsilon_{i+1}}, X_{\epsilon_{i+1}}] = X_{\epsilon_i}, \quad [X_{\epsilon_i-\epsilon_{i+1}}, [X_{\epsilon_i-\epsilon_{i+1}}, X_{2\epsilon_{i+1}}]] = 2X_{2\epsilon_i}, \quad 1 \leq i < \ell,
\]
\[
[X_{\epsilon_i-\epsilon_j}, X_{2\epsilon_j}] = X_{\epsilon_i+\epsilon_j}, \quad i < j,
\]
\[
[X_{\epsilon_i-\epsilon_j}, X_{\epsilon_j-\epsilon_k}] = X_{\epsilon_i-\epsilon_k}, \quad i < j < k,
\]
(3.13)
and with the Serre relations (2.31):

\[
\text{ad}_{X_{\ell}}^2 (X_{e\ell}) = 0, \quad \text{ad}_{X_{\ell}}^3 (X_{e\ell-1-e\ell}) = \text{ad}_{X_{\ell}}^2 (X_{e\ell-1-e\ell}) (X_{e\ell}) = 0, \\
\text{ad}_{X_{2e\ell}}^2 (X_{e\ell-1-e\ell}) = \text{ad}_{X_{e\ell-1-e\ell}}^2 (X_{2e\ell}) = 0, \\
\text{ad}_{X_{e\ell-1-e\ell}}^2 (X_{e\ell-e\ell+i}) = \text{ad}_{X_{e\ell-e\ell+i+1}}^2 (X_{e\ell-e\ell+i}) = 0, \quad 1 < i < \ell.
\]  

(3.14)

From the defining relations (3.11) we see that \(\chi_R\) takes non-zero values only on the simple root generators \(X_{e\ell}, X_{e_k-e_{k+1}}, 1 \leq k < \ell\) and on the special non-simple root generator \(X_{2e\ell}\). The latter follows from the first relation from (3.13) for \(i = \ell\):

\[
[X_{e\ell}, X_{e\ell}] = 2X_{e\ell}^2 = 2X_{2e\ell}.
\]  

(3.15)

Indeed, given \(X_{e\ell} \cdot \psi_R = i^{3/2} \xi_{\alpha_1}^+ \psi_R\) one readily deduces

\[
2X_{e\ell}^2 \cdot \psi_R = 2X_{e\ell} \cdot (X_{e\ell} \cdot \psi_R) = 2X_{e\ell} \cdot (i^{3/2} \xi_{\alpha_1}^+ \psi_R) = -2i^{3/2} \xi_{\alpha_1}^+(X_{e\ell} \cdot \psi_R) = 2i(\xi_{\alpha_1}^+)^2 \psi_R,
\]

which matches with \(2X_{2e\ell} \cdot \psi_R = 2i(\xi_{\alpha_1}^+)^2 \psi_R\). Similarly, the first relation from (3.13) for \(1 \leq i < \ell\) yields

\[
\chi_R(X_{2e_i}) = \frac{1}{2} \chi_R(X_{e_i})^2 = 0, \quad 1 \leq i < \ell,
\]  

(3.17)

and the other relation in the first line of (3.11) entails

\[
\chi_R(X_{e_i+e_j}) = \chi_R(X_{e_i}X_{e_j} + X_{e_j}X_{e_i}) = 0, \quad i < j.
\]  

(3.18)

The Serre relations imply that \(\dim n_+ = |\Delta^+| = \ell^2 + \ell = \ell(\ell + 1)\). Thus the rest of the defining relations (3.11) are provided by the fact that given \(X_\alpha \in n_+\) and \(X_\beta, X_\gamma \in n_+\), such that \(\alpha = \beta + \gamma\) and not both \(\beta, \gamma\) are odd, we have

\[
\chi_R(X_\alpha) = \chi_R(X_{\beta}X_{\gamma} - X_{\gamma}X_{\beta}) = 0.
\]  

(3.19)

Namely, the last line of (3.13) for each \(i < j < k\) with \(1 < i - k < \ell\) implies

\[
\chi_R(X_{e_i-e_k}) = \chi_R(X_{e_i-e_j}X_{e_j-e_k} - X_{e_j-e_k}X_{e_i-e_j}) = 0.
\]  

(3.20)

Then the second and the third lines of (3.13) for each \(1 \leq i < k \leq \ell\) we have

\[
\chi_R(X_{e_i}) = \chi_R(X_{e_i-e_k}X_{e_k} - X_{e_k}X_{e_i-e_k}) = 0.
\]  

(3.21)

Finally, we check that the remaining relations

\[
[X_{e_i-e_j}, X_{2e_j}] = X_{e_i+e_j}, \quad [X_{e_i-e_j}, X_{2e_j}] = X_{e_i+e_j}, \\
\text{ad}_{X_{e_i-e_j}}^2 (X_{2e_j}) = [X_{e_i-e_j}, [X_{e_i-e_j}, X_{2e_j}]] = X_{2e_i}, \quad i < j,
\]  

(3.22)

are consistent with the defining relations (3.11). This completes our proof. \(\Box\)
Remark 3.1 Our choice of the characters (3.12), (3.11) in Lemma 3.1 is compatible with the notion of a unitary operators in the case of super Hilbert spaces (see [DM]). Note however that in our case we do not require the Hilbert space structure but only an invariant pairing.

Definition 3.1 Let $V_\lambda$ and $V_\lambda'$ be a dual pair of cyclic Whittaker modules with the action of the Casimir element given by (3.8). The $\mathfrak{osp}(1\vert 2\ell)$-Whittaker function is defined by

$$
\Psi_\lambda(e^q) = e^{-\rho} \langle \psi_L, e^{-h_q} \cdot \psi_R \rangle, \quad h_q = \sum_{i \in I} q_i a_{ii},
$$

(3.23)

where $\rho$ is the half-sum of positive even roots minus the half-sum of positive odd roots, given by (3.1).

Proposition 3.1 The $\mathfrak{osp}(1\vert 2\ell)$-Whittaker function (3.23) is a solution to the following eigenvalue problem:

$$
\mathcal{H}^{\mathfrak{osp}(1\vert 2\ell)}_2 \cdot \Psi_\lambda(e^q) = -(\lambda + \rho)^2 \Psi_\lambda(e^q),
$$

(3.24)

$$
\mathcal{H}^{\mathfrak{osp}(1\vert 2\ell)}_2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + 2 \sum_{\alpha \in \Delta^+} \xi^- \xi^+ e^{\alpha} + 4(\xi^- \xi^+) e^{2\alpha}(q),
$$

(3.25)

where $\rho$ is given by (3.7), and $\Delta^+ = \Delta^+(B_0,\ell)$ is defined in (2.25).

Proof: On the one hand, by our construction we read from (3.8):

$$
\langle \psi_L, e^{-h_q} C_2 \psi_R \rangle = (\lambda, \lambda + 2\rho) \langle \psi_L, e^{-h_q} C_2 \psi_R \rangle.
$$

(3.26)

On the other hand, the action of the Casimir element $C_2 \in \mathcal{Z}(\mathcal{U}(\mathfrak{osp}(1\vert 2\ell)))$ is equivalent to action on (3.23) of a certain second-order differential operator. Namely, from (2.45) we take

$$
C_2 = \sum_{i \in I} \left( a_{ii}^2 + 2\rho(\epsilon_i) a_{ii} + 2X_{-\epsilon_i}X_{\epsilon_i} \right) + \sum_{\alpha \in \Delta^+} (\alpha, \alpha) X_{-\alpha}X_{\alpha},
$$

(3.27)

and substituting this into (3.26) we obtain:

$$
\sum_{i \in I} \langle \psi_L, e^{h_q} \left( a_{ii}^2 + 2\rho(\epsilon_i) a_{ii} \right) \psi_R \rangle = \sum_{i \in I} \left\{ \frac{\partial^2}{\partial q_i^2} - 2\rho(\epsilon_i) \frac{\partial}{\partial q_i} \right\} \langle \psi_L, e^{-h_q} \cdot \psi_R \rangle.
$$

(3.28)

Taking into account the defining equations (3.12), (3.11) and the hermitian property (3.4) of $\langle , \rangle$ we find out

$$
2 \sum_{i \in I} \langle \psi_L, e^{-h_q} X_{-\epsilon_i}X_{\epsilon_i} \psi_R \rangle = -2 \sum_{i \in I} e^{q_i}(-1)^{p(X_{-\epsilon_i})} p(\psi_L) \langle X_{-\epsilon_i} \psi_L, e^{-h_q} X_{\epsilon_i} \psi_R \rangle
$$

(3.29)

$$
= -2(-1)^{p(X_{-\epsilon_i})} p(\psi_L) e^{q_i} t^{3/2} \xi^- \psi_L, e^{-h_q} e^{3/2} \xi^+ \psi_R
$$

$$
= -2(-1)^{p(X_{-\epsilon_i})} p(\psi_L) (-1)^{p(\xi^+)} e^{q_i} t^{3/2} \xi^- \psi_L, e^{-h_q} e^{3/2} \xi^+ \psi_R
$$

$$
= -2(-1)^{p(X_{-\epsilon_i})} p(\psi_L) (-1)^{p(\xi^+)} e^{q_i} t^{3/2} \xi^- \psi_L, e^{-h_q} \psi_R
$$

$$
= -2 \xi^- e^{q_i} e^{(q_i)(\psi_L), e^{-h_q} \psi_R}.
$$
Here we use the fact that $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-antilinear in the first variable and it is $\mathbb{C}$-linear in the second variable. In a similar way we derive

$$
\sum_{\alpha \in \Delta^+_0} (\alpha, \alpha) \langle \psi_L, e^{-h_q} X_{-\alpha} X_{\alpha} \psi_R \rangle = - \sum_{\alpha \in \Delta^+_0} (\alpha, \alpha) e^{\alpha(q)} \langle X_{-\alpha} \psi_L, e^{h_q} X_{\alpha} \psi_R \rangle
$$

$$
= -2 \sum_{i=2}^\ell i^{-1} \xi^- \xi^+ e^{\alpha_i(q)} \langle \psi_L, e^{-h_q} \psi_R \rangle
$$

$$
- 4i^{-1}(\xi^- \xi^+)^2 e^{2\alpha_i(q)} \langle \psi_L, e^{-h_q} \psi_R \rangle.
$$

Collecting the contributions above we obtain the following:

$$
\langle \psi_L, e^{-h_q} C_2 \psi_R \rangle = \left\{ \sum_{i \in I} \left( \frac{\partial^2}{\partial q_i^2} - 2\rho(\epsilon_i) \frac{\partial}{\partial q_i} \right) \right\} \langle \psi_L, e^{-h_q} \cdot \psi_R \rangle.
$$

Now we observe that

$$
e^{-\rho(q)} \frac{\partial}{\partial q_i} e^{\rho(q)} = \frac{\partial}{\partial q_i} + \rho(q)' q_i = \frac{\partial}{\partial q_i} + \rho(\epsilon_i),
$$

$$
e^{-\rho(q)} \frac{\partial^2}{\partial q_i^2} e^{\rho(q)} = \frac{\partial^2}{\partial q_i^2} + 2\rho(q)' q_i \frac{\partial}{\partial q_i} + (\rho(q)' q_i)^2
$$

$$
= \frac{\partial^2}{\partial q_i^2} + 2\rho(\epsilon_i) \frac{\partial}{\partial q_i} + \rho(\epsilon_i)^2,
$$

hence we deduce the following:

$$
\sum_{i \in I} e^{-\rho(q)} \left\{ \frac{\partial^2}{\partial q_i^2} - 2\rho(\epsilon_i) \frac{\partial}{\partial q_i} \right\} e^{\rho(q)}
$$

$$
= \sum_{i \in I} \left\{ \frac{\partial^2}{\partial q_i^2} + 2\rho(\epsilon_i) \frac{\partial}{\partial q_i} + \rho(\epsilon_i)^2 - 2\rho(\epsilon_i) \left( \frac{\partial}{\partial q_i} + \rho(\epsilon_i) \right) \right\}
$$

$$
= \sum_{i \in I} \frac{\partial^2}{\partial q_i^2} - \rho^2.
$$

Finally, we collect all the contributions and substitute them into (3.26) to deduce the following:

$$
\left\{ \sum_{i \in I} \frac{\partial^2}{\partial q_i^2} - 4(\xi^- \xi^+)^2 e^{2\alpha_i(q)} - 2 \sum_{\alpha_i \in \Delta^+} \xi^- \xi^+ e^{\alpha_i(q)} - \rho^2 \right\} \cdot \Psi_\lambda(e^\eta)
$$

$$
= (\lambda, \lambda + 2\rho) \Psi_\lambda(e^\eta),
$$

(3.34)
where \( \Delta^+ = \Delta^+(B_{0,t}) \). This easily entails the assertion \((3.24)\). \(\Box\)

**Remark 3.2** In the special case \( \lambda = \mu - \rho \), the eigenvalue equation \((3.24)\) reads

\[
H^{osp(1|2\ell)}_2 \psi_\lambda(e^q) = \mu^2 \psi_\lambda(e^q),
\]

\[
H^{osp(1|2\ell)}_2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + 2 \sum_{\alpha_i \in \Delta^+} \xi^-_i \xi^+_i e^{\alpha_i(q)} + 4 (\xi^-_i \xi^+_i)^2 e^{2\alpha_1(q)}.
\] (3.35)

Let us introduce the corresponding couplings:

\[
g_i^2 = \xi^-_i \xi^+_i, \quad i \in I.
\] (3.36)

**Lemma 3.2** The \(osp(1|2\ell)\)-Whittaker function \((3.23)\) depends on \(\xi^\pm_\alpha\) via \(g_i^2\), \(i \in I\).

**Proof**: The \(osp(1|2\ell)\)-Whittaker function \((3.23)\)

\[
\Psi_\lambda(e^q | \xi^\pm_\alpha) = e^{-\rho(q)} \langle \psi_L, e^{-h_q} | \psi_R \rangle, \quad h_q = \sum_{i \in I} q_i a_{ii},
\] (3.37)

satisfies the following obvious relation: given \(Q = \exp \{ \sum_{i=1}^\ell \theta_i h_i \} \in H\)

\[
\langle \psi_L, Q e^{-h_q} Q^{-1} \psi_R \rangle = \langle \psi_L, e^{-h_q} | \psi_R \rangle.
\] (3.38)

The adjoint action of \(Q\) on the left and right Whittaker vectors \(\psi_R, \psi_L\) \((3.12), (3.11)\) changes them, so that the eigenvalues \(\xi^\pm_\alpha\) of the corresponding \(n_\pm\)-characters are changed as follows:

\[
\xi^\pm_\alpha \rightarrow \xi^\pm_\alpha e^{\pm \theta_\ell}, \quad \xi^\pm_\alpha \rightarrow \xi^\pm_\alpha e^{(\theta_{\ell+1-i} - \theta_{\ell+2-i})}, \quad 1 < i \leq \ell.
\] (3.39)

The invariance of the Whittaker function under this transformation implies that the \(osp(1|2\ell)\)-Whittaker function depends on \(\xi^\pm_\alpha\) only via quadratic combinations \(\xi^\pm_\alpha \xi^-_\alpha\). \(\Box\)

**Lemma 3.3** Let us consider a specialization of the \(osp(1|2\ell)\)-Toda chain by taking arbitrary special values \(g_i^2 = \kappa_i^2 \in \mathbb{R}\) of the couplings. Then by a linear change of variables \(q_i\) one can bring the quadratic Hamiltonian

\[
H^{osp(1|2\ell)}_2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + 2 \sum_{i=2}^{\ell} \kappa_i^2 e^{\alpha_i(q)} + 2 \kappa_1^2 e^{\alpha_1(q)} + 4 \kappa_1^4 e^{2\alpha_1(q)},
\] (3.40)

to the following canonical form:

\[
H^{osp(1|2\ell)}_2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + \sum_{i=2}^{\ell} e^{\alpha_i(q)} + e^{\alpha_1(q)} + e^{2\alpha_1(q)}.
\] (3.41)

**Proof**: Indeed it is easy to check that the following transformation of variables,

\[
q_\ell \rightarrow q_\ell - \ln 2 - \ln \kappa_1^2, \\
q_k \rightarrow q_k - (\ell + 1 - k) \ln 2 - \ln \kappa_k^2 + \ln \kappa_{\ell-k}^2, \quad 1 \leq k < \ell,
\] (3.42)

applied to \((3.25)\) gives \((3.41)\). \(\Box\)
4 On $\mathfrak{osp}(1|2\ell)$ as a Lie algebra of type $BC_\ell$

Let us recall the construction of the Toda chain associated with the general root system. Let $\Delta$ be a rank $\ell$ root system realized as a set of vectors in $V = \mathbb{R}^\ell$. Chose an orthogonal basis $\{\epsilon_i, i \in I\}$ in $V$ and the dual basis $\{\epsilon^i, i \in I\}$ in $V^*$, both indexed by $I = \{1, \ldots, \ell\}$. Then elements $q \in V^*$ allow decomposition $q = \sum_{i=1}^\ell q_i \epsilon^i$. Let $^*\Delta^+$ be as a set of simple positive roots in $\Delta$. The quadratic quantum Hamiltonian of the Toda chain associated with the root system $\Delta$ is given by

$$H_{\Delta^+}^2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + \sum_{\alpha \in ^*\Delta^+} g_\alpha^2 e^{\alpha(q)}$$

(4.1)

with the coupling constants $g_\alpha^2$. Note that the Hamiltonian depends only on the structure of simple positive roots $^*\Delta^+$.

Now let us specialize this expression to the case of $BC_\ell$-root system, the unique non-reduced root system satisfying basic axioms of root systems of finite-dimensional Lie algebras (for a description of $BC_\ell$ root system see e.g. [H], [L]). The set of simple positive roots of the $BC_\ell$ root system is given by

$$^*\Delta^+(BC_\ell) = \{2\epsilon_\ell; \epsilon_\ell, \epsilon_i - \epsilon_{i+1}, 1 \leq i < \ell\}.$$ (4.2)

The Cartan matrix $A = \|A_{ij}\|$ associated with the set of simple positive roots is defined via standard formula

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j \in I.$$ (4.3)

Note that the Cartan matrix corresponding to $BC_\ell$ is degenerate. For example the Cartan matrix for $\ell = 5$ is given by (2.25):

$$A = \begin{pmatrix}
2 & 4 & -2 & 0 & 0 \\
1 & 2 & -1 & 0 & 0 \\
-1 & -2 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ (4.4)

Using the general formula (4.1) for $BC_\ell$-root system we arrive at the following

$$H_{BC_\ell}^2 = -\sum_{i \in I} \frac{\partial^2}{\partial q_i^2} + \sum_{i=1}^{\ell-1} g_i^2 e^{\epsilon_i - \epsilon_{i+1}} + g_\ell^2 e^{2\epsilon_\ell} + g_{\ell+1}^2 e^{2\epsilon_{\ell+1}}.$$ (4.5)

It is clear that specialization of the quadratic Hamiltonian (4.5) of the $BC_\ell$-Toda chain to the case of the coupling constants $g_i^2 = 1, i \in I$ coincides with the quadratic Hamiltonian (3.41) of $\mathfrak{osp}(1|2\ell)$-Toda chain. On the other hand for generic values of $g_i$ in (4.5) it is not possible by linear changes of variables $q_i$ to transform the Hamiltonian (4.5) into the Hamiltonian with $g_i^2 = 1$. Thus $\mathfrak{osp}(1|2\ell)$-Toda chain realized a special class of $BC_\ell$-Toda chains.

There is a question on the underlying reason for this phenomenon. It is easy to see that the simple positive roots (4.2) of $BC_\ell$ and that of $B_{0,\ell}$ (2.25) are closely related. There are
however two differences. First, the short simple root of $\mathfrak{osp}(1|2\ell)$ has odd parity while in $BC_\ell$ root system it is an even root. Second, while in the case of super Lie algebra $\mathfrak{osp}(1|2\ell)$ the corresponding root system includes the roots $\pm 2\epsilon_\ell$, these roots are not simple and thus do not enter the expression for the corresponding Cartan matrix. If however we formally add the root $2\epsilon_\ell$ to the set of positive simple roots then the corresponding Cartan matrix constructed according to (4.3) precisely coincides with the Cartan matrix of $BC_\ell$ root system.

The fact that in the case of $\mathfrak{osp}(1|2\ell)$ the terms of the Cartan decomposition (2.29) corresponding to short roots are odd actually does not manifest itself in the expressions for the Hamiltonians of the corresponding Toda chain. Indeed, according to Lemma 3.2 the eigenvalues $\xi_{\pm \alpha_i}^\pm$ in (3.11), (3.12) enter the expressions for quantum Hamiltonians only via combinations $g_{i}^\pm = \xi_{\alpha_i}^+\xi_{-\alpha_i}^-$. Therefore $B_{0,\ell}$-Toda chains turns out to be a special case of $BC_\ell$-Toda chain. We have checked this explicitly for quadratic Hamiltonian in the previous Section 3.

It is natural to wonder whether we might treat the super Lie algebra $\mathfrak{osp}(1|2\ell)$ as a proper candidate for the Lie algebra structure associated with $BC_\ell$ root system. Such identification has at least one obvious caveat. The root system $BC_\ell$ allows embedding of roots systems $B_\ell$ and $C_\ell$ having isomorphic Weyl groups $W_{B_\ell} \simeq W_{C_\ell}$. It would be natural to expect the same property for the corresponding Lie algebras i.e. a candidate for the Lie algebra associated with $BC_\ell$ should allow an embedding of the Lie algebras $\mathfrak{so}_{2\ell+1}$ and $\mathfrak{sp}_{2\ell}$ associated with the roots systems $B_\ell$ and $C_\ell$ correspondingly. While there indeed exists an embedding $\mathfrak{sp}_{2\ell} \subset \mathfrak{osp}(1|2\ell)$ the super Lie algebra $\mathfrak{osp}(1|2\ell)$ does not allow an embedding of $\mathfrak{so}_{2\ell+1}$.

References

[E] P.I. Etingof, *Whittaker functions on quantum groups and q-deformed Toda operators*, Differential Topology, Infinite-Dimension Lie algebras and Applications, AMS Transl. Ser. 2, vol.194, AMS, Rhode Island, 1999, 9–25.

[DM] P. Deligne, J. Morgan, *Notes on Supersymmetry (following Joseph Bernstein)*, in Quantum Fields and Strings: A Course for Mathematicians I. AMS, IAS, 1999, 41–97.

[GLO1] A. Gerasimov, D. Lebedev, S. Oblezin, *New integral representation of the Whittaker functions for classical Lie groups*, Russian Math.Surveys, 67:1 (2012),192; [arXiv:0705.2886].

[GLO2] A. Gerasimov, D. Lebedev, S. Oblezin, *Quantum Toda chains intertwined*, Algebra i Analiz, 2010, Volume 22, Issue 3, 107–141; [arXiv:0907.0299].

[GW] R. Goodman, N.R. Wallach, *Classical and quantum-mechanical systems of Toda lattice type III*, Commun. Math. Phys., 105(3) (1986) 473–509.

[J] H. Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France, 95, 1967, 243–309.
[Ha] M. Hashizume, *Whittaker functions on semi-simple Lie groups*, Hiroshima Math. J., 12, 1982, 259–293.

[H] S. Helgason, *Differential Geometry, Lie groups, and Symmetric Spaces*, AMS, 2001.

[Kac1] V.G. Kac, *Lie Super algebras*, Adv. in Mathematics 26, 1977, 8–96.

[Kac2] V.G. Kac, *A Sketch of Lie Superalgebra Theory*, Commun. Math. Phys. 53, 1977, 31–64.

[K1] B. Kostant, *Quantization and representation theory*, in Proc. of Symposium on Representations and Lie groups, Oxford 1977, London Math. Soc. Lect. Notes Series, 1979, 34, 287–316.

[K2] B. Kostant, *On Whittaker vectors and representation theory*, Invent.Math. 48(2) (1978) 101–184.

[L] O. Loos, *Symmetric Spaces I, II*, W.A. Benjamen, 1969.

[RS] A.G. Reyman, M.A. Semenov-Tian-Shansky, *Group-Theoretical Methods in the Theory of Finite-Dimensional Integrable systems*, in Integrable systems II. Encyclopedia of Mathematical Sciences, Vol. 16 Dynamical Systems VII., Springer-Verlag, New York, 1994, 116–225.

[STS] M.A. Semenov-Tian-Shansky, *Quantization of Open Toda lattice*, in Integrable systems II. Encyclopedia of Mathematical Sciences, Vol. 16 Dynamical Systems VII., Springer-Verlag, New York, 1994, 226–259.

[S] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J.Phys.A: Math.Gen v.21 (1988) 2375–2389.

**A.A.G.** Laboratory for Quantum Field Theory and Information, Institute for Information Transmission Problems, RAS, 127994, Moscow, Russia; E-mail address: anton.a.gerasimov@gmail.com

**D.R.L.** Laboratory for Quantum Field Theory and Information, Institute for Information Transmission Problems, RAS, 127994, Moscow, Russia; Moscow Center for Continuous Mathematical Education, 119002, Bol. Vlasievsky per. 11, Moscow, Russia; E-mail address: lebedev.dm@gmail.com

**S.V.O.** School of Mathematical Sciences, University of Nottingham, University Park, NG7 2RD, Nottingham, United Kingdom; Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia; E-mail address: oblezin@gmail.com