SPANNING TREES WITH MANY LEAVES: NEW LOWER BOUNDS IN TERMS OF
THE NUMBER OF VERTICES OF DEGREE 3 AND AT LEAST 4

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It is proved that every connected graph with \( s \) vertices of degree 3 and \( t \) vertices of degree at least 4 has a spanning tree with \( \frac{2}{3}t + \frac{1}{3}s + \alpha \) leaves, where \( \alpha \geq \frac{t}{3} \). Moreover, \( \alpha \geq 2 \) for all graphs, except for three exclusions. All exclusions are regular graphs of degree 4, and they are explicitly described in the paper. An infinite series of graphs containing only vertices of degrees 3 and 4, for which the maximal number of leaves in a spanning tree is equal to \( \frac{2}{3}t + \frac{1}{3}s + 2 \), is presented. Therefore it is proved that the bound is tight. Bibliography: 12 titles.

1. INTRODUCTION. BASIC NOTATION

We consider undirected graphs without loops and multiple edges. We use standard notation. For a graph \( G \) we denote the set of its vertices by \( V(G) \) and the set of its edges by \( E(G) \). We use the notation \( v(G) \) and \( e(G) \) for the number of vertices and edges of \( G \), respectively.

We denote the degree of a vertex \( x \) in the graph \( G \) by \( d_G(x) \). For any set of vertices \( W \subset V(G) \) we denote by \( d_{G,W}(x) \) the number of vertices of the set \( W \) that are adjacent to \( x \) in the graph \( G \). As usual, we denote the minimal vertex degree of the graph \( G \) by \( \delta(G) \).

Let \( N_G(x) \) denote the neighborhood of a vertex \( x \in V(G) \) (i.e., the set of all vertices adjacent to \( x \)). For a set \( W \subset V(G) \), we denote by \( N_G(W) \) the neighborhood of \( W \) (i.e., the set of all vertices of \( V(G) \setminus W \) that are adjacent to at least one vertex of the set \( W \) in \( G \)).

For any edge \( e \in E(G) \), we denote by \( G \cdot e \) the graph in which the ends of the edge \( e = xy \) are contracted into one vertex, which is incident to all vertices in \( G \) to at least one of the vertices \( x \) and \( y \). We say that the graph \( G \cdot e \) is obtained from \( G \) by contracting the edge \( e \).

For a set of vertices \( R \subset V(G) \) we denote by \( G - R \) the graph obtained from \( G \) upon deleting all vertices of the set \( R \) and all edges incident to deleted vertices.

If \( a, b \in V(G) \) and \( ab \notin V(G) \), then we denote by \( G + ab \) the graph obtained from \( G \) upon adding the edge \( ab \).

We call a set of vertices \( R \subset V(G) \) a cutset if the graph \( G - R \) is disconnected.

Definition 1. For any connected graph \( G \), we denote by \( u(G) \) the maximal number of leaves in a spanning tree of the graph \( G \).

Remark 1. Obviously, if \( F \) is a tree, then \( u(F) \) is the number of its leaves.

Several papers concerning lower bounds for \( u(G) \) have been published. One can see details of the history of this question in [12]. We shall recall only results directly related to our paper.

In 1981, Linial stated a conjecture:

\[
u(G) \geq \frac{d - 2}{d + 1} v(G) + c \quad \text{for } \delta(G) \geq d \geq 3,
\]

where the constant \( c > 0 \) depends only on \( d \). The ground for this conjecture is the following: for every \( d \geq 3 \), one can easily construct an infinite series of graphs with minimal degree \( d \), for which \( \frac{u(G)}{v(G)} \) tends to \( \frac{d - 2}{d + 1} \).

It follows from papers [4, 6, 7] that for \( d \) large enough, Linial’s conjecture is not valid. However, we are interested in the case of small \( d \).

In 1991 Kleitman and West [2] proved that \( u(G) \geq \frac{1}{4} v(G) + 2 \) as \( \delta(G) \geq 3 \) and \( u(G) \geq \frac{2}{5} v(G) + \frac{8}{5} \) as \( \delta(G) \geq 4 \).

In 1996, Griggs and Wu [3] proved once again the statement for \( \delta(G) \geq 4 \) and showed that \( u(G) \geq \frac{1}{5} v(G) + 2 \) as \( \delta(G) \geq 5 \). Hence, Linial’s conjecture holds for \( d = 3, d = 4 \), and \( d = 5 \); for \( d > 5 \), the question remains open.

In [2], a stronger Linial’s conjecture was mentioned:

\[
u(G) \geq \sum_{x \in V(G)} \frac{d_G(x) - 2}{d_G(x) + 1}
\]

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for a connected graph $G$ with $\delta(G) \geq 2$. Clearly, this conjecture is not true, since the weak Linial's conjecture fails for large degrees. We present an infinite series of connected graphs all vertices of which have degrees 3 and 4, disproving this conjecture. Therefore, the strong conjecture fails not only for huge degrees, coming to us from probabilistic methods, but even for degrees 3 and 4.

However, the strong Linial's conjecture inspires attempts of obtaining a lower bound on $u(G)$, in which the contribution of each vertex depends on its degree. But what must the contribution of a vertex of degree $d$ be?

For a connected graph with $v_3$ vertices of degree 3 and $v_4$ vertices of degree at least 4, N. V. Gravin [10] proved that $u(G) \geq \frac{2}{5} \cdot v_4 + \frac{2}{5} \cdot v_3$. In that paper, vertices of degrees 1 and 2 are allowed in the graph. There is no doubt that the constant $\frac{2}{5}$ is optimal, but the constant $\frac{2}{5}$ can be replaced by a greater one, as is shown in our main theorem.

**Theorem 1.** Let $G$ be a connected graph with at least two vertices, let $s$ be the number of vertices of degree 3 and $t$ be the number of vertices of degree at least 4 in $G$. Then $u(G) = \frac{2}{5} s + \frac{1}{5} t + \alpha$, where $\alpha \geq \frac{2}{5}$. Moreover, $\alpha \geq 2$ for all graphs, except for three graphs: $C_6^2$, $C_8^2$ (squares of cycles on 6 and 8 vertices), and $G_8$, a regular graph of degree 4 on 8 vertices, shown in Fig. 1.

![Graphs-exclusions](image)

Fig. 1. Graphs-exclusions.

Note that all three constants of this bound are optimal. There exist infinite series of examples for which this bound is attained. We present series of such graphs containing only vertices of degree 3 and 4.

The proof of this theorem would be much shorter if we exclude from the theorem the last statement. Our interest in finding an exact additive constant is inspired by the desire of obtaining a tight bound that is not a tip effect. The bound with $\alpha = \frac{2}{5}$ is attained only for the graph $C_6^2$! However, there are different infinite series of examples for $\alpha = 2$, and one can see this additive constant in lower bounds for graphs with minimal degrees 3 or 5. Possibly, for this reason Kleitman and West [2] have conjectured that there are only two connected graphs with $\delta(G) \geq 4$ for which the bound $u(G) \geq \frac{2}{5} v(G) + 2$ fails: $C_6^2$ and $C_8^2$. Moreover, it is proved in [2] that for graphs with minimal degree 4 an exclusion must be a 4-regular graph and each its edge must belong to a triangle.

Kleitman and West in their conjecture about graphs-exclusions did not find only the graph $G_8$ on eight vertices (see Fig. 1). However, with the method from [2] one cannot even prove that the set of graphs-exclusions is finite. We shall prove in Theorem 1 a similar statement about graphs-exclusions even for a more general problem.

2. **Proof of Theorem 1**

Let us introduce necessary notation.

**Definition 2.** Let $H$ be an arbitrary graph. We denote by $S(H)$ the set of all vertices of degree 3 of the graph $H$, and by $T(H)$ the set of all vertices of degree at least 4 of the graph $H$.

Let $x \in V(H)$. We set that the cost $c_H(x)$ of the vertex $x$ in the graph $H$ is

$$c_H(x) = \begin{cases} 
\frac{2}{5} & \text{for } x \in T(H), \\
\frac{1}{5} & \text{for } x \in S(H), \\
0 & \text{for } x \notin T(H) \cup S(H).
\end{cases}$$

The cost of the graph $H$ is

$$c(H) = \frac{2}{5} |T(H)| + \frac{1}{5} |S(H)| = \sum_{x \in V(H)} c_H(x).$$
For any set of vertices \( U \subset V(H) \), we define the cost of this set in the graph \( H \) as \( c_H(U) = \sum_{x \in U} c_H(x) \). For any tree \( F \) that is a subgraph of the graph \( H \), we define its cost in the graph \( H \) as \( c_H(F) = c_H(V(F)) \).

For any spanning tree \( F \) of the graph \( H \), we introduce the notation \( \alpha(F) = u(F) - c(H) \). Let \( \alpha(H) \) be the maximum of \( \alpha(F) \) over all spanning trees \( F \) of the graph \( H \).

**Remark 2.** It follows directly from the definition that \( u(G) = c(G) + \alpha(G) \). Hence we want to prove that \( \alpha(G) \geq 2 \) for almost all connected graphs \( G \).

As usual, in constructing the desired spanning tree for a graph \( G \) we assume that the theorem has been proved for all smaller graphs.

**2.1. Reduction rules.** Let us describe two reduction rules.

**R1.** Let \( x \in V(G) \), \( d_G(x) = 2 \), \( N_G(x) = \{a, b\} \), and the vertices \( a \) and \( b \) be not adjacent.
We reduce the graph \( G \) to \( G' = G - x + ab \). Obviously, \( c(G') = c(G) \).

**R2.** Let \( a_1, a_2 \in S \) be adjacent vertices and \( N_G(a_1) \cap N_G(a_2) = \emptyset \).
We reduce the graph \( G \) to \( G' = G - a_1a_2 \). Let \( a \) be the vertex obtained by gluing the vertices \( a_1 \) and \( a_2 \). Clearly, \( d_G'(a) = 4 \), then \( c(G') = c(G) \).

In both cases one can easily transform a spanning tree \( F' \) of the graph \( G' \) to a spanning tree \( F \) of the graph \( G \) with \( u(F) \geq u(F') \) and, therefore, \( \alpha(F) \geq \alpha(F') \). Thus it is easy to see that \( \alpha(G) \geq \alpha(G') \).

**Remark 3.** Later we may assume that the graph considered satisfies the following conditions:

1° any vertex of degree 2 forms a triangle with two vertices of its neighborhood;

2° for any two adjacent vertices of degree 3, their neighborhoods have a nonempty intersection.

**2.2. Dead vertices method. General description.** To prove the theorem, we shall construct the desired spanning tree, using the method of dead vertices, as in papers [2, 3].

**Definition 3.** Let a tree \( F \) be a subgraph of a connected graph \( G \).
We say that a leaf \( x \) of the tree \( F \) is dead if \( N_G(x) \subset V(F) \) and alive otherwise. We denote by \( b(F) \) the number of dead leaves of the tree \( F \).

We set \( \alpha'(F) = \frac{13}{15} u(F) + \frac{2}{15} b(F) - c_G(F) \).

We construct a spanning tree in \( G \) successively, adding the vertices in several steps. Let \( S = S(G) \) and \( T = T(G) \).

Let us describe in detail a step of our algorithm (we call this step \( A \)). Let we have a tree \( F \) before the step \( A \) (of course, \( F \) is a subgraph of the graph \( G \)).

We denote by \( \Delta u \) and \( \Delta b \) the increase of the number of leaves and dead leaves in the tree \( F \), respectively, at the step \( A \), by \( \Delta t \) and \( \Delta s \) are denoted the number of vertices added at this step to the tree \( F \) from \( T \) and \( S \), respectively.

The profit of the step \( A \) is the quantity
\[
p(A) = \frac{13}{15} \Delta u + \frac{2}{15} \Delta b - \frac{2}{5} \Delta t - \frac{1}{5} \Delta s.
\]

Let \( F_1 \) be the tree obtained after the step \( A \). Clearly, \( \alpha'(F_1) = \alpha'(F) + p(A) \). We shall perform only steps with nonnegative profits.

**Remark 4.** 1) It is easy to see that the dead leaves remain dead during all next steps of the construction. When the algorithm terminates, we obtain a spanning tree all of the leaves of which are dead.

2) Since all leaves of a spanning tree \( F \) are dead, we have \( \alpha'(F) = \alpha(F) \).
First we describe all possible steps, and after that we consider the beginning of the construction and estimate \(a(T)\) for the constructed tree \(T\).

We denote by \(W\) the set of all vertices that do not belong to the tree \(F\).

The vertices of the set \(W\) that are adjacent to at least one vertex of the set \(V(F)\) are called vertices of level 1. The vertices of the set \(W\) that do not belong to level 1 and are adjacent to at least one vertex of level 1 are called vertices of level 2.

For each vertex \(x \in W\), we denote by \(P(x)\) the set of all adjacent to \(x\) vertices of the set \(V(F)\).

2.3. A step of the algorithm. We shall try to perform the next step of the algorithm in the following way. We shall pass to the next variant of the step only when all previous variants are impossible. We shall not mention this during the description of steps.

After every complete step (i.e., after a step with nonnegative profit), we will count the parameters of this step: \(\Delta u\), \(\Delta b\), and the profit of the performed step. All these parameters will be necessary to us in the last section of the paper. The number of vertices added at the step to the tree is not a parameter of this step!

We begin with a step, which in fact is not a step, but will help us in the description of other steps.

**Z0. A leaf \(v\) of the tree \(T\), counted as alive, turns out to be dead.**

We do not transform the tree at this step. We take into account information about \(v\) and obtain

\[
\Delta u = 0, \quad \Delta b = 1, \quad p(Z0) = \frac{2}{15}
\]

**Remark 5.** During the description of steps, we consider as alive all leaves of the tree \(F\) that are not said to be dead. Adding an extra dead vertex will be recorded as a step \(Z0\).

2.3.1. Steps of the type A. Let us begin with some easy steps. In the first four variants, new leaves are added to the tree.

**A1.** There is a nonpendant vertex \(x\) of the tree \(F\) adjacent to \(y \in W\).

Then we adjoin \(y\) to \(x\). Since \(c_G(x) \leq \frac{2}{5}\), we have

\[
\Delta u = 1, \quad \Delta b = 0, \quad p(A1) \geq 13 \cdot \frac{2}{15} = \frac{13}{15}
\]

**A2.** There is a vertex \(x \in V(F)\) with \(d_{G,W}(x) \geq 2\).

Then we adjoin to the tree two adjacent to \(x\) vertices of the set \(W\). The cost of two added vertices is no more than \(2 \cdot \frac{2}{5}\), and we obtain

\[
\Delta u = 1, \quad \Delta b = 0, \quad p(A2) \geq 13 \cdot \frac{2}{15} - 2 \cdot \frac{2}{5} = \frac{1}{15}
\]

**A3.** There is a vertex \(x\) of level 1 such that \(d_{G,W}(x) \geq 3\).

First we adjoin to the tree \(F\) the vertex \(x\), and after that we adjoin three vertices of the set \(W\) adjacent to \(x\). The cost of four added vertices is no more than \(4 \cdot \frac{2}{5}\), and we obtain

\[
\Delta u = 2, \quad \Delta b = 0, \quad p(A3) \geq 13 \cdot \frac{2}{15} - 4 \cdot \frac{2}{5} = \frac{2}{15}
\]

![Fig. 3. Steps of type A.](image)

| X | F | F |
|---|---|---|
| A2 | A3 | A4 |
Remark 6. Henceforth we assume that the nonpendant vertices of the tree $F$ are not adjacent to vertices of the set $W$, any leaf of the tree $F$ is adjacent to no more than one vertex of the set $W$, and, finally, any vertex of level 1 is adjacent to no more than two vertices of the set $W$.

In particular, if $x \in T$ is a vertex of level 1, then $|P(x)| \geq 2$ and, after adjoining the vertex $x$ to the tree, at least one vertex of the set $P(x)$ becomes a dead leaf of the obtained tree.

A4. There exists a vertex $x \in S$ of level 1, adjacent to exactly one vertex of the set $W$, namely, to a vertex $y \in T$ of level 2.

First we adjoin the vertices $x, y$ to the tree $F$. Since $y$ is not adjacent to the tree $F$, there are three vertices of the set $W$ that are adjacent to $y$ and are different from $x$. We adjoin these three vertices to the tree. The cost of five added vertices is no more than $\frac{3}{5} + 4 \cdot \frac{2}{5}$, and we obtain

$$\Delta u = 2, \quad \Delta b = 1, \quad p(A4) \geq 2 \cdot \frac{13}{15} + \frac{2}{5} - \frac{1}{5} - 4 \cdot \frac{2}{5} = \frac{1}{15}.$$ 

2.3.2. Steps of the type $M$ and $N$. Further on we consider a more complicated case.

M. There is a vertex $x \in T$ of level 1 such that $d_{G,W}(x) = 2$.

We adjoin the vertex $x$ to the tree. Note that $c_G(x) = \frac{2}{5}$. The vertex $x$ is adjacent to at least two leaves of the tree $F$ (see Fig. 4), whence at least one of these leaves becomes dead. Then we adjoin to the tree two vertices $y_1, y_2 \in W$ adjacent to $x$ (in fact, it is the step A2). Taking the above into account, we obtain

$$\Delta u = 1, \quad \Delta b = 1, \quad p(M) \geq \frac{2}{15} - \frac{2}{5} + p(A2) \geq -\frac{3}{15}.$$ 

N. There is a vertex $x \in S$ of level 1 such that $d_{G,W}(x) = 2$.

We adjoin to the tree the vertex $x$ and two vertices $y_1, y_2 \in W$ adjacent to $x$. Since $c_G(x) = \frac{1}{5}$, similarly to the previous case we obtain

$$\Delta u = 1, \quad \Delta b = 0, \quad p(N) = -\frac{1}{5} + p(A2) \geq -\frac{2}{15}.$$ 

We do not consider that the steps $M$ and $N$ are complete. We have added to the tree three vertices $x, y_1,$ and $y_2$. However, let $F$ be still the tree constructed after the previous complete step. Upon performing the step $M$ or $N$ we have:

- each leaf of the tree $F$ is adjacent to no more than one vertex of the set $W$;
- each vertex of level 1 is adjacent to no more than two vertices of the set $W$.

We aim at performing a step with a profit at least $\frac{2}{5}$.

Steps that we describe below in this section do not include vertices added at the step $M$ or $N$, performed before. Counting the parameters of these steps, we do not take into account vertices added at the preceding step $M$ or $N$.

Let us continue the case analysis.

1. $y_1, y_2 \notin T$.

These vertices cost cheaper than it was calculated above. Hence the profit increases by at least $\frac{2}{5}$, and we obtain

$$\Delta u = \Delta b = 0, \quad p(1) \geq \frac{6}{15}.$$ 

We set $W_1 = W \setminus \{x, y_1, y_2\}$.

If only one of the vertices $y_1$ and $y_2$ belongs to the set $T$, we assume that $y_1 \in T$.

If $y_1, y_2 \in T$, we assume that $d_{G,W_1}(y_1) \geq d_{G,W_1}(y_2)$.

Fig. 4. Steps $M, N, 2$, and $3$. 

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2. \(d_{G,W_1}(y_1) \geq 3\).

We adjoin to the tree three vertices of the set \(W_1\) adjacent to \(y_1\) (in fact, we perform the steps \(A2\) and \(A1\)). We obtain

\[
\Delta u = 2, \quad \Delta b = 0, \quad p(2) = p(A1) + p(A2) \geq \frac{8}{15}.
\]

3. \(d_{G,W_1}(y_1) \leq 1\).

The vertex \(y_1\) is adjacent to no more than three vertices of the set \(W_1\): they are \(x\) and, possibly, \(y_2\) and one vertex of the set \(W_1\). Since \(d_{G}(y_1) \geq 4\), \(y_1\) is adjacent to the tree \(F\), i.e., \(y_1\) is a vertex of level 1. By Remark 6, we have \(|P(y_1)| \geq 2\), whence the number of dead leaves increases by at least 2. Consider two cases.

3.1. If \(y_2 \in T\), then, by the choice of the vertex \(y_1\), we have \(d_{G,W_1}(y_2) \leq 1\). Similarly to the above-said for the vertex \(y_1\), we have two additional dead leaves adjacent to \(y_2\). In this case

\[
\Delta u = 0, \quad \Delta b = 4, \quad p(3.1) = 4 \cdot \frac{2}{15} = \frac{8}{15}.
\]

3.2. If \(y_2 \not\in T\), then the vertex \(y_2\) costs cheaper than we have calculated; hence the profit increases by \(\frac{1}{5}\) and we have

\[
\Delta u = 0, \quad \Delta b = 2, \quad p(3.2) \geq \frac{1}{5} + 2 \cdot \frac{2}{15} = \frac{7}{15}.
\]

4. \(d_{G,W_1}(y_1) = 2\).

Let \(z_1\) and \(z_2\) be two adjacent to \(y_1\) vertices of the set \(W_1\). We adjoin \(z_1\) and \(z_2\) to the tree (it is the step \(A2\)) and obtain

\[
\Delta u = 1, \quad \Delta b = 0, \quad p(4) = p(A2) \geq \frac{1}{15}.
\]

Since that is not enough, we continue the case analysis.

4.1. Among \(y_2, z_1,\) and \(z_2\), there is a vertex adjacent to the tree \(F\).

For example, let \(z_1\) be adjacent to the tree \(F\), i.e., \(z_1\) is a vertex of level 1. For other vertices, arguments are quite similar.

4.1.1. If \(z_1 \in T\), then, by Remark 6, the vertex \(z_1\) must be adjacent to at least two leaves of the tree \(F\), whence the number of dead leaves increases by two, and we obtain

\[
\Delta u = 1, \quad \Delta b = 2, \quad p(4.1.1) \geq p(4) + 2 \cdot \frac{2}{15} \geq \frac{5}{15}.
\]

4.1.2. If \(z_1 \not\in T\), then the vertex \(z_1\) adds one dead leaf and the cost of \(z_1\) decreases by at least \(\frac{1}{5}\). Hence we obtain

\[
\Delta u = 1, \quad \Delta b = 1, \quad p(4.1.2) \geq p(4) + \frac{1}{5} + \frac{2}{15} \geq \frac{6}{15}.
\]

4.2. Among \(y_2, z_1,\) and \(z_2\), there is a vertex not from the set \(T\).

This vertex increases the profit by at least \(\frac{1}{5}\), and we obtain

\[
\Delta u = 1, \quad \Delta b = 0, \quad p(4.2) \geq p(4) + \frac{1}{5} \geq \frac{4}{15}.
\]

Fig. 5. Steps 4, 4.1.1, 4.1.2, and 4.3.

4.3. \(N_G(y_2) = \{x,y_1,z_1,z_2\}\).

In this case, the vertex \(y_2\) is a dead leaf, and we obtain

\[
\Delta u = 1, \quad \Delta b = 1, \quad p(4.3) \geq p(4) + \frac{2}{15} = \frac{3}{15}.
\]
Remark 7. Let us summarize the analyzed cases. In the remaining cases of the steps $M$ and $N$, the vertices $y_1$, $y_2$, $z_1$, and $z_2$ belong to the set $T$ and to the level 2. Moreover, $d_{G,W_1}(y_1) = d_{G,W_1}(y_2) = 2$; hence, the vertices $y_1$ and $y_2$ are adjacent and $d_G(y_1) = d_G(y_2) = 4$.

The vertex $y_2$ cannot be adjacent to both vertices $z_1$ and $z_2$. Without loss of generality, we assume that $y_2$ is not adjacent to $z_1$. Then the vertex $z_1$ is adjacent to at least two vertices of the set $W_2 = W \setminus \{x, y_1, y_2, z_1, z_2\}$.

4.4. $d_G(W_2(z_1)) \geq 3$.

We adjoin to the tree three vertices of the set $W_2$ adjacent to $z_1$ (in fact, we perform the step $A2$ and the step $A1$). We obtain

$$\Delta u = 3, \quad \Delta b = 0, \quad p(4.4) \geq p(4) + p(A2) + p(A1) \geq \frac{9}{15}.$$

4.5. $d_G(W_2(z_1)) = 2$.

Denote by $p_1$ and $p_2$ two adjacent to $z_1$ vertices of the set $W_2$ and adjoin these two vertices to the tree (see Fig. 6). We have performed the step $A2$ and obtain

$$p(4.5) \geq p(4) + p(A2) \geq \frac{2}{15}.$$ 

This is not enough to us, and we continue the case analysis.

4.5.1. Among $p_1$ and $p_2$, there is a vertex of the set $T$ that is adjacent to the tree $F$.

Let it be $p_1$. By Remark 6, the vertex $p_1$ is adjacent to at least two leaves of the tree $F$, whence the number of dead leaves increases by at least 2, and we obtain

$$\Delta u = 2, \quad \Delta b = 2, \quad p(4.5.1) \geq p(4.5) + 2 \cdot \frac{2}{15} \geq \frac{6}{15}.$$

Fig. 6. Steps 4.4, 4.5, and 4.5.1.

Remark 8. In what follows, the vertices $y_1$, $y_2$, $z_1$, $z_2$, $p_1$, and $p_2$ are not adjacent to the tree $F$.

4.5.2. Among $p_1$ and $p_2$, there is a vertex not from the set $T$.

In this case, the profit increases by at least $\frac{1}{5}$, whence we obtain

$$\Delta u = 2, \quad \Delta b = 0, \quad p(4.5.2) \geq p(4.5) + \frac{3}{15} \geq \frac{5}{15}.$$

4.5.3. Among $y_2$, $z_2$, $p_1$, and $p_2$, there is a vertex that is not adjacent to any vertex of the set $W_3 = W \setminus \{x, y_1, y_2, z_1, z_2, p_1, p_2\}$.

This vertex is an extra dead leaf of the constructed tree, whence the profit increases by at least $\frac{1}{5}$, and we obtain

$$\Delta u = 2, \quad \Delta b = 1, \quad p(4.5.3) \geq p(4.5) + \frac{2}{15} \geq \frac{4}{15}.$$

Remark 9. Therefore, all vertices $y_1$, $y_2$, $z_1$, $z_2$, $p_1$, $p_2$ belong to the set $T$ and are not adjacent to the tree $F$. Each of the vertices $y_2$, $z_2$, $p_1$, $p_2$ is adjacent to at least one vertex of the set $W_3$. 

753
\[ 4.5.4. \text{If } d_{G,W_3}(p_1) \geq 2 \text{ or } d_{G,W_3}(p_2) \geq 2. \]

Without loss of generality, we assume that \( d_{G,W_3}(p_1) \geq 2 \). We adjoin to the tree two vertices \( q_1, q_2 \in W_3 \) adjacent to \( p_1 \) (i.e., we perform the step \( A2 \)). We obtain

\[ \Delta u = 3, \quad \Delta b = 0, \quad p(4.5.4) \geq p(4.5) + p(A2) \geq \frac{3}{15} \]

\[ 4.5.5. \quad \text{Then } d_{G,W_3}(p_1) = d_{G,W_3}(p_2) = 1. \]

Then the vertex \( p_1 \) is adjacent to at least two of the vertices \( p_2, y_2, z_2 \), and the vertex \( p_2 \) is adjacent to at least two of the vertices \( p_1, y_2, z_2 \). Let us recall that, by Remark 7, the vertices \( y_1 \) and \( y_2 \) are adjacent and \( d_{G}(y_1) = d_{G}(y_2) = 4 \). Since \( y_2 \) is adjacent to at least one vertex from \( W_3 \), \( y_2 \) cannot be adjacent to both vertices \( p_1 \) and \( p_2 \). Without loss of generality, we assume that \( y_2 \) is not adjacent to \( p_1 \). Then \( d_{G}(p_1) = 4 \), and the vertex \( p_1 \) must be adjacent to both vertices \( p_2 \) and \( z_2 \) (see Fig. 7a).

Note that the vertex \( z_2 \) cannot be adjacent to \( p_2 \). (Otherwise \( z_2 \) would be adjacent to three vertices of the set \( W \setminus \{x, y_1, y_2, z_1, z_2\} \): they are \( p_1, p_2 \), and a vertex from \( W_3 \). In this case, we adjoin these three vertices to \( z_2 \), i.e., perform a step 4.4 with \( z_2 \) instead of \( z_1 \).) Thus the vertex \( p_2 \) is adjacent to \( y_2 \) and \( d_{G}(p_2) = 4 \) (see Fig. 7b).

Moreover, \( y_2 \) is adjacent to exactly two vertices of the set \( W_1 = W \setminus \{x, y_1, y_2\} \) by Remark 7. Since \( y_2 \) is adjacent to \( p_2 \) and to a vertex from \( W_3 \), it is adjacent to neither \( z_1 \) nor \( z_2 \). Hence \( z_1 \) is adjacent to \( z_2 \) and \( d_{G}(z_1) = d_{G}(z_2) = 4 \) (see Fig. 7c).

\[ y_1 \quad y_2 \quad p_1 \quad p_2 \quad z_1 \quad z_2 \]

\[ \text{Fig. 7. Step 4.5.5.} \]

Denote by \( r \) the only vertex of the set \( W_3 \) that is adjacent to \( y_2 \). Now we can apply to \( y_2 \) the same argument as was applied above to \( y_1 \), and we get that two vertices \( r \) and \( p_2 \) adjacent to \( y_2 \) are adjacent to each other and, in addition, \( d_{G}(r) = 4 \) (see Fig. 7d).

Continuing this argument for the vertex \( p_2 \) and two vertices \( p_1 \) and \( z_1 \), adjacent to \( p_2 \), we make sure that one of the vertices \( p_1 \) and \( z_1 \) must be adjacent to \( r \). Since \( z_1 \) cannot be adjacent to \( r \), the vertices \( p_1 \) and \( r \) are adjacent.

Now it is clear (see Fig. 7d) that \( z_2 \) is adjacent to exactly two vertices of the set \( W_2 \): these vertices are \( p_1 \) and a vertex \( r' \in W_3 \). In this case, one can repeat the above argument for the vertex \( z_2 \) instead of \( z_1 \) and conclude that \( p_1 \) is adjacent to \( r' \). Hence, \( r = r' \) and \( z_2 \) is adjacent to \( r \). We obtain the configuration shown in Fig. 7e.

We add to the tree the vertex \( r \) (adjoint it to any vertex adjacent to \( r \)). Note that no added vertex is adjacent to a vertex outside the constructed tree. Let us calculate the parameters of this step from the very beginning: \( \Delta t = 5 \).

\[ \Delta u = 2, \quad \Delta b = 4, \quad p(4.5.5) \geq 2 \cdot \frac{13}{15} + 4 \cdot \frac{2}{15} - 5 \cdot \frac{2}{5} = \frac{4}{15}. \]

**Remark 10.**

1) We have proved that after each of the steps \( M \) and \( N \) we can perform a step with profit at least \( \frac{3}{15} \). We shall always perform after steps \( M \) and \( N \) one of the steps introduced above to obtain the resulting step with nonnegative profit. The notation \( M4.2 \) will mean the step consisting of \( M \) and \( 4.2 \) after it. Similarly for other steps. We call such a step an \( M \)-step if the first step was \( M \) and an \( N \)-step if the first step was \( N \). We call all these steps \( MN \)-steps.

The profit of the steps \( M4.3 \) and \( M4.5.4 \) can be equal to zero, for all other \( MN \)-steps the profit is at least \( \frac{1}{15} \).

Any \( MN \)-step, except for \( M4.5.5 \) and \( N4.5.5 \), cannot be the last step of the algorithm, since at this step we add at least one alive leaf.

2) Let us summarize the analyzed cases. In the remaining cases any vertex of level 1 is adjacent to at most one vertex of the set \( W \) (otherwise we can perform the step \( A3 \) or one of the \( MN \)-steps).
2.3.3. *Steps of the type Z.* In the next cases, the number of leaves of the constructed tree does not vary, but the number of dead leaves increases.

**Z1.** There exists a vertex of level 1 that is not adjacent to \(W\).
Let it be a vertex \(w\). Then \(N_G(w) = P(w)\). We add the vertex \(w\) to the tree. The vertex \(w\) and all vertices of the set \(N_G(w)\), except for one, become dead leaves of the new tree. Therefore, \(\Delta b = d_G(w)\).

**Z1.1.** \(w \in T\).
In this case, the number of dead leaves increases by \(d_G(w) \geq 4\). We consider that \(\Delta b = 4\), and if really \(d_G(w) > 4\) we record this as \(d_G(w) - 4\) additional steps \(Z0\). Thus for the step \(Z1.1\) we have
\[
\Delta u = 0, \quad \Delta b = 4, \quad p(Z1.1) = 4 \cdot \frac{2}{15} - \frac{2}{5} = \frac{2}{15}.
\]

**Z1.2.** \(w \in S\).
In this case, the parameters of the step are
\[
\Delta u = 0, \quad \Delta b = 3, \quad p(Z1.2) = 3 \cdot \frac{2}{15} - \frac{1}{5} = \frac{3}{15}.
\]

**Z1.3.** \(w \notin S \cup T\).
In this case, we have \(p(Z1.3) = \Delta b \cdot \frac{2}{15}\). We will not consider this step further on, since the parameters of this step are the same as the parameters of \(\Delta b\) consecutive steps \(Z0\). (Let us recall that the number of added vertices is not a parameter of a step.)

**Z2.** There are two adjacent vertices \(v\) and \(w\) of level 1.
Let \(v, w\) be these vertices. By Remark 10, other vertices adjacent to \(\{v, w\}\) are leaves of the tree \(F\). Clearly, \(d_G(v) \geq 2\) and \(d_G(w) \geq 2\). Let \(d_G(v) = 2\) and \(N_G(v) = \{x, w\}\). Then \(x\) is adjacent to \(w\) (otherwise we would apply the reduction rule \(R1\)). Hence we have \(d_G(v, x) \geq 2\) for a leaf \(x\) of the tree \(F\), which contradicts Remark 6.

Thus, we have \(v, w \in S \cup T\). The case \(v, w \in S\) is impossible (in this case, we would apply the reduction rule \(R2\)). We add the vertices \(v\) and \(w\) to the tree. In the tree obtained, the vertices \(v, w\) and all vertices of \(P(v) \cup P(w)\), except for two vertices, are dead leaves. Therefore, \(\Delta b = d_G(w) + d_G(v) - 2\). As in the step \(Z1.1\), further on we shall write minimal possible \(\Delta b\) and use, if necessary, steps \(Z0\).

**Z2.1.** If one of the vertices \(v, w\) belongs to \(S\) and the other belongs to \(T\), then
\[
\Delta u = 0, \quad \Delta b = 5, \quad p(Z2.1) = 5 \cdot \frac{2}{15} - \frac{1}{5} - \frac{2}{5} = \frac{1}{15}.
\]

**Z2.2.** If \(v, w \in T\), then
\[
\Delta u = 0, \quad \Delta b = 6, \quad p(Z2.2) = 6 \cdot \frac{2}{15} - \frac{2}{5} = 0.
\]

**Z3.** There is a vertex \(w\) of level 1 adjacent to a vertex \(v \in W \setminus (S \cup T)\).
We add the vertices \(w\) and \(v\) to the tree. If \(d_G(v) = 2\), then either we can apply the reduction rule \(R1\) or \(v\) is adjacent to a vertex of \(P(w)\). Hence, \(d_G(v) = 1\) and in the tree obtained the vertex \(v\) and all vertices of \(P(w)\), except for one, are dead leaves. We have \(\Delta b = d_G(w) - 1\).

**Z3.1.** If \(w \in S\), we obtain
\[
\Delta u = 0, \quad \Delta b = 2, \quad p(Z3.1) = 2 \cdot \frac{2}{15} - \frac{1}{5} = \frac{1}{15}.
\]

**Z3.2.** If \(w \in T\), we obtain
\[
\Delta u = 0, \quad \Delta b = 3, \quad p(Z3.2) = 3 \cdot \frac{2}{15} - \frac{2}{5} = 0.
\]

**Lemma 1.** Let \(w\) be a vertex of level 1. Then \(w \in T\); moreover, the vertex \(w\) is adjacent to a vertex \(v \in S \cup T\) of level 2 and to at least three leaves of the tree \(F\).

**Proof.** By Remark 10, we have \(d_G(w) \leq 1\). Since we cannot perform the step \(Z1\), we have \(d_G(w) = 1\), i.e., \(w\) is adjacent to a vertex \(v \in W\).

Since we cannot perform the step \(Z2\), \(v\) is a vertex of level 2. Since we cannot perform the step \(Z3\), \(v \in T \cup S\).

Since we cannot apply the reduction rule \(R1\), \(v \in T \cup S\).

Finally, we prove that \(w \in T\). Assume the contrary, let \(w \in S\). If \(v \in T\), we can perform a step \(A4\). In the case \(v \in S\) we can apply the reduction rule \(R2\). In both cases we have a contradiction.

Thus, \(w \in T\). Since \(d_G(w) = 1\), the vertex \(w\) is adjacent to at least three leaves of the tree \(F\). \(\square\)
Z4. There exists a vertex in $W$, i.e., $F$ is not a spanning tree.

Let $w_1, \ldots, w_n$ are all vertices of level 1. By Lemma 1, each of these vertices is adjacent with at least three leaves of the tree $F$. Thus there are at least $3n$ alive leaves in the tree $F$, i.e., $u(F) - b(F) \geq 3n$ and

$$u(F) = c_G(F) + \alpha'(F) + \frac{2}{15}(u(F) - b(F)) \geq c_G(F) + \alpha'(F) + \frac{2n}{5}.$$  

(1)

We delete all edges connecting $w_1, \ldots, w_n$ with the tree $F$. As a result, the graph $G$ will be split into $G_1 = G(V(F))$ and $G_2 = G(W)$. Since $c_G(w_i) - c_{G_2}(w_i) = \frac{2}{5}$, we have

$$c(G_2) = c_G(W) - n \cdot \frac{2}{5}.$$  

(2)

Note that the graph $G_2$ can be disconnected, but each of its connected components contains at least 4 vertices and there is a pendant vertex among them (one of the vertices $w_1, \ldots, w_n$). Hence, every connected component of $G_2$ is not a graph-exclusion and contains less vertices than the graph $G$. Thus we can apply the statement of our theorem to any connected component $H$ of the graph $G_2$ and construct a spanning tree $F_H$ in $H$ with $u(F_H) \geq c(H) + 2$. Therefore we can construct a forest $F'$ in the graph $G_2$, which consists of $k$ spanning trees of connected components of $G_2$. Clearly,

$$u(F') \geq c(G_2) + 2k.$$  

(3)

We adjoin each of the $k$ connected components of the forest $F'$ to the tree $F$ and obtain the spanning tree $T$ of the graph $G$. Let us estimate $u(T)$ with the help of inequalities (1), (3), and (2):

$$u(T) = u(F) + u(F') - 2k \geq c_G(F) + \alpha'(F) + \frac{2n}{5} + c(G_2)$$

$$= c_G(V(F)) + c_G(W) + \alpha'(F) = c(G) + \alpha'(F).$$

Thus, in this case we have $\alpha(G) \geq \alpha(T) \geq \alpha'(F)$.

2.4. Beginning of the construction and estimation of $\alpha$. We shall begin the construction of the spanning tree with a base tree $F'$ such that $\alpha'(F')$ is rather large.

We consider several cases. We shall pass to the next variant of the base only when all previous variants are impossible. We begin with the cases where one can easily construct a base tree $F'$ with $\alpha'(F') \geq 2$, thus completing the proof of the theorem.

B1. There are two adjacent vertices $a, a' \in T$ with $N_G(a) \cap N_G(a') = \emptyset$.

We begin with the base tree $F'$ in which the vertices $a$ and $a'$ are adjacent to each other and to all vertices from $N_G(a) \cap N_G(a')$. Clearly, $u(F') = u \geq 6$, $c_G(F') \leq \frac{13u}{15}$, and $\alpha'(F') \geq \frac{13u}{15} - c_G(F') \geq \frac{14u - 13}{15} \geq 2$.

B2. There is a vertex $a \in T$ adjacent to a vertex of degree no more than 2.

Let $v \in N_G(a)$, $d_G(v) \leq 2$. We begin with the base tree $F'$, in which the vertex $a$ is adjacent to all vertices from $N_G(a)$. In the case $d_G(v) = 1$, it is clear that $v$ is a dead leaf of $F'$. Let $d_G(v) = 2$. Since we cannot apply the reduction rule R1, the vertices $a, v$ and a vertex from $N_G(a)$ form a triangle. Hence, in this case the vertex $v$ is a dead leaf of $F'$ as well.

Therefore, $u(F') = d_G(a) = u \geq 4$, $b(F') \geq 1$, $c_G(F') \leq \frac{2u}{5}$, and

$$\alpha'(F') \geq \frac{13u}{15} + \frac{2}{15} - c_G(F') \geq \frac{7u + 2}{15} \geq 2.$$  

In what follows, we shall consider the base trees $F'$ with $\alpha'(F') < 2$. To provide $\alpha(G) \geq 2$, we draw our attention to the end of the construction.
Lemma 2. Assume that the graph $G$ does not contain configurations described in the cases B1 and B2 and one has constructed a spanning tree with the help of our algorithm. Then the following statements hold:

1) If the step Z4 was performed, then there exists a base tree $F'$ with $\alpha'(F') \geq 2$.

2) If the step Z4 was not performed, then the last step of the algorithm does not add new alive leaves and has profit at least $\frac{1}{15}$.

Proof. 1) Let us turn to the step Z4 and to the graph $G_2$ cut from the tree $F$ (see Fig. 8). We consider a connected component $G'$ of the graph $G_2$. Without loss of generality, we assume that $w_1, \ldots, w_k \in V(G')$, and $w_{k+1}, \ldots, w_n \notin V(G')$. Since $G'$ is a smaller connected graph with pendant vertices, one can construct in $G'$ a spanning tree $T'$ with $\alpha(T') = u(T') - c_{G'}(T') \geq 2$.

Regard the tree $T'$ as a subgraph of the graph $G$. Unfortunately, each of the vertices $w_1, \ldots, w_k$ costs $\frac{2}{5}$ in $G$ (while it costs 0 in $G'$), whence $c_{G}(T') = c_{G'}(T') + \frac{2k}{5}$. In addition, the vertices $w_1, \ldots, w_k$ are alive leaves of $T'$ in the graph $G$ (all other leaves of the tree $T'$ are, clearly, dead); therefore in the graph $G$ we have $u(T') - b(T') = k$ and
\[
\alpha'(T') = u(T') - c_{G}(T') - \frac{2}{5}(u(T') - b(T')) = u(T') - c_{G'}(T') - \frac{8k}{15} = 2 - \frac{8k}{15}.
\]

Let us recall details of the step Z4, and for all $i \in \{1, \ldots, k\}$ consider three vertices $x_i, x_i', x_i'' \in V(F)$ adjacent to $w_i$. All $3k$ vertices in such triples are different and do not belong to the tree $T'$. For each $i \in \{1, \ldots, k\}$, we adjoin to $w_i$ the three vertices $x_i, x_i', x_i''$, that is, we perform the step A2 and the step A1 in turn $k$ times. The sum of profits is equal to $k - \frac{2k}{5}$, and as a result we obtain the base tree $F'$ with $\alpha'(F') \geq 2$.

2) Consider the last step. No new alive leaves were added at this step. Looking over the parameters of the steps, one can draw a conclusion that only one of the steps Z0, Z1.1, Z1.2, Z2.1, Z2.2, Z3.1, Z3.2, N4.5.5, and M4.5.5 may be last. The step Z2.2 is impossible, since it needs the configuration from the case B1. The step Z3.2 is impossible, since it needs the configuration from the case B2. Each of the other steps has profit at least $\frac{1}{15}$.

Thus, henceforth it is enough to prove that at the steps which add new alive leaves, a tree $F$ with $\alpha'(F) \geq \frac{20}{15}$ will be constructed. We continue the case analysis.

B3. There is a vertex $a$ of degree at least 5.

We begin with the base tree $F'$ in which the vertex $a$ is adjacent to all vertices from $V(G(a))$. Clearly,
\[
u(F') = d_{G}(a) = u \geq 5, \quad c_{G}(F') = \frac{2}{5}(u + 1), \quad \text{and} \quad \alpha'(F') \geq \frac{13}{15}u - c_{G}(F') \geq \frac{7u - 6}{15} \geq \frac{29}{15},
\]
which is enough.

B4. There is a vertex $x \in S$ adjacent to a vertex of degree no more than 2.

Let $v \in N_{G}(x)$, $d_{G}(v) \leq 2$, $N_{G}(x) = \{v, y_1, y_2\}$. We begin with the base tree $F'$ in which the vertex $x$ is adjacent to all vertices from $V(G(x))$. Similarly to the case B2, the vertex $v$ is a dead leaf of $F'$. Hence, $u(F') = 3$, $b(F') \geq 1$, and
\[
\alpha'(F') \geq \frac{13}{15} \cdot 3 + \frac{2}{15} - c_{G}(F') = \frac{41}{15} - c_{G}(F').
\]

If at least one of the vertices $y_1$ and $y_2$ does not belong to $T$, then $c_{G}(F') \leq 2 \cdot \frac{1}{5} + \frac{2}{5} = \frac{1}{5}$ and $\alpha'(F') \geq \frac{20}{15}$. By Lemma 2, we have $\alpha(G) \geq 2$.

Consider the case $y_1, y_2 \in T$. Then both $y_1$ and $y_2$ are alive leaves, $c_{G}(F') = \frac{1}{5} + 2 \cdot \frac{2}{5} = 1$, and $\alpha'(F') = \frac{20}{15}$. The construction is not complete, since we need the additional profit $\frac{1}{15}$.

In the analysis of the cases $M$ and $N$, we have considered a similar problem of deficiency of the profit $\frac{1}{15}$. We repeat the same argument, perform an $M_{N}$-step, which is possible in our configuration, and obtain a tree $F^*$ with $\alpha'(F^*) \geq \frac{20}{15}$. Moreover, we have $\alpha'(F^*) < 2$ only for the steps $M4.3$ and $M4.5.4$, but in these cases the constructed trees have alive leaves. Hence, the construction is not complete and, by Lemma 2, the last step will give us the profit at least $\frac{1}{15}$ and provide $\alpha(G) \geq 2$.

Remark 11. In the items B2 and B4, we considered all cases where the graph contains a vertex of degree no more than 2. In the item B3, the case where the graph contains a vertex of degree more than 4 was considered. Hence in the sequel we consider only graphs with vertex degrees equal to 3 or 4.

In Table 1, we introduce the parameters of all possible steps. The profits of all steps are multiplied by 15.
We have taken into account that the steps $Z2.2$, $Z3.1$, $Z3.2$, and $Z4$ are impossible. (For the steps $Z2.2$, $Z3.2$, and $Z4$, see details in Lemma 2 and its proof. The step $Z3.1$ is impossible, since the vertex degrees of our graph are at least 3.)

It is not convenient to deal with such a great number of steps. The next lemma will significantly decrease the number of possible steps.

**Lemma 3.** Assume that one have constructed a spanning tree in the graph $G$ with the help of our algorithm. If one of the $MN$-steps mentioned below was performed, then $\alpha(G) \geq 2$.

1) One of the steps $N4.2$, $N4.3$, $N4.4$, $N4.5.2$, $N4.5.3$, and $N4.5.5$. One of the steps $N1$, $N2$, and $N4.5.4$, at which all added vertices, except for $x$, were not adjacent to the tree $F$.

2) One of the steps $M4.1.1$, $M4.5.5$, and $Z0$. One of the steps $M4.2$, $M4.3$, $M4.4$, $M4.5.2$, $M4.5.3$, and $M4.5.5$. One of the steps $M1$, $M2$, $M4.5.4$, at which all added vertices, except for $x$, were not adjacent to the tree $F$.

**Proof.** Let $F$ be a tree constructed before the mentioned $MN$-step. Recall details of $MN$-steps: at this step we adjoin to $F$ a tree $F_0$ the root of which is a vertex $x \in S \cup T$ of level 1. Let $p$ be the profit of this step. Note that at all mentioned steps, the vertex $x$ is adjacent to exactly two vertices of the set $W = V(G) \setminus V(F)$, and all vertices of the adjoined tree $F_0$, except for $x$, are not adjacent to $V(F)$. To make sure of these facts, it suffices to look over details of the mentioned steps and to take into account the conditions of the lemma.

1) For $N$-steps we have $x \in S$ and $p \geq \frac{1}{5}$. Let us recall the calculation of the profit of this step. The vertex $x$ is adjacent to an alive leaf $a$ of the tree $F$. The vertex $a$ is not a leaf of the tree, obtained after adjoining $F_0$ to $F$; owing to this, the profit was decreased by $\frac{1}{10}$. New dead leaves did not appear among the vertices of the tree $F$; therefore, new leaves and new dead leaves of the obtained tree are leaves and dead leaves of the tree $F_0$, counted with the same coefficients as in the calculation of $\alpha'(F_0)$. Therefore, $\alpha'(F_0) = p + \frac{1}{10} \geq \frac{1}{10}$.

Clearly, $N_C(a) \cap N_C(x) = \varnothing$, whence, by Remark 3, we have $a \in T$. Thus, $a$ is adjacent with three vertices $b_1, b_2, b_3 \in V(F)$, and these vertices do not belong to the tree $F_0$. We adjoin to $F_0$ the vertices $a, b_1, b_2, b_3$ (see Fig. 9.1) and obtain, as a result, a new base tree $F_1$. The performed operation gives us the profit $3 \cdot \frac{13}{10} - 4 \cdot \frac{1}{5} = 1$, whence $\alpha'(F_1) \geq \frac{13}{10} + p \geq 2$ and $\alpha(G) \geq 2$.

| Step   | $\Delta u - \Delta b$ | 15-profit |
|--------|------------------------|------------|
| $A1$   | 1                      | 7          |
| $A2$, $A4$, $M4.2$, $N4.3$, $M4.5.3$ | 1          | 1          |
| $A3$, $N4.2$, $M4.5.2$, $N4.5.3$ | 2          | 2          |
| $M1$, $M4.1.2$, $M4.5.1$, $N4.1.1$ | 0          | 3          |
| $N1$, $N4.1.2$, $N4.5.1$ | 1          | 4          |
| $M2$   | 2                      | 5          |
| $N2$, $M4.4$ | 3          | 6          |
| $M3.1$ | -4                     | 5          |
| $N3.1$ | -3                     | 6          |
| $M3.2$ | -2                     | 4          |
| $N3.2$ | -1                     | 5          |
| $M4.1.1$, $N4.5.5$, $Z0$ | -1         | 2          |
| $M4.3$ | 0                      | 0          |
| $N4.4$ | 4                      | 7          |
| $N4.5.2$ | 3          | 3          |
| $M4.5.4$ | 3          | 0          |
| $N4.5.4$ | 4          | 1          |
| $M4.5.5$ | -2         | 1          |
| $Z1.1$ | -4                     | 2          |
| $Z1.2$ | -3                     | 3          |
| $Z2.1$ | -5                     | 1          |
2) The general algorithm of constructing a base tree for $M$-steps.

In this case, $x \in T$. We recall the calculation of the profit of this step. The vertex $x$ is adjacent to two alive leaves $a_1, a_2$ of the tree $F$. One of the vertices $a_1$ and $a_2$ is not a leaf of the tree obtained upon adjoining $F_0$ to $F$ (owing to this, the profit was decreased by $\frac{6}{15}$), the other becomes dead leaf (in view of this, the profit was increased by $\frac{9}{15}$). All other new leaves and new dead leaves of the obtained tree are leaves and dead leaves of the tree $F_0$ calculated with the same coefficients as in the calculation of $\alpha'(F_0)$. Therefore, $\alpha'(F_0) = p + \frac{11}{15}$.

We adjoin to $F_0$ the vertices $a_1, a_2 \in V(F)$ and, as a result, obtain a new base tree $F_1$. The performed operation gives the profit $2 \cdot (\frac{11}{15} - \frac{3}{15}) = \frac{8}{15}$, whence $\alpha'(F_1) \geq \frac{8}{15} + p$. If $a_1, a_2 \not\in T$, then the profit increases by at least $2 \cdot \frac{3}{15}$, and we obtain $\alpha'(F_1) \geq \frac{11}{15}$, which is enough.

Let $a_1 \in T$; then $d_C(a_1) = 4$. Note that $a_1$ is a leaf of the tree $F$, and thus it is not adjacent to vertices of the set $W$, except for $x$; therefore it is adjacent to at least two vertices different from $a_2$ of the tree $F$. These vertices do not belong to the tree $F_1$; we adjoin them to the tree and obtain a new tree $F_2$. If we have added more than two vertices, then we have a profit at least $p(A2) + p(A1) = \frac{8}{15}$ (adjoining two vertices is a step $A2$, adjoining the third vertex is a step $A1$) and $\alpha'(F_2) > 2$, which is enough.

The only remaining case is the case where we have added exactly two vertices; let them be $b_1$ and $b_2$. Note that in this case, the vertices $a_1$ and $a_2$ are adjacent. We have performed a step $A2$ with profit at least $\frac{1}{15}$; thus $\alpha'(F_2) \geq \frac{26}{15} + p$. If $p \geq \frac{4}{15}$, we have $\alpha'(F_2) \geq \frac{30}{15}$, which is enough by Lemma 2. For steps with $p \leq \frac{3}{15}$, we consider two cases: $a_2$ is a dead leaf (see Fig. 9(2a)) and an alive leaf of the tree $F_2$, respectively.

![Fig. 9. Construction of a base tree.](image)

**a.** The vertex $a_2$ is a dead leaf of the tree $F_2$.

This increases the profit by $\frac{2}{15}$ and provides $\alpha'(F_2) \geq \frac{28}{15} + \frac{p}{15}$. Both $b_1$ and $b_2$ are alive leaves of the tree $F_2$; otherwise, the profit increases by at least $\frac{6}{15}$, and we have $\alpha'(F_2) \geq 2$. If $p \geq \frac{4}{15}$, by Lemma 2 we have $\alpha(G) \geq 2$. The only remaining steps are $M4.5.4$ and $M4.3$ with profit 0.

**a1. Step M4.5.4.**

Consider the further construction of a spanning tree with the help of our algorithm. The tree $F_2$ has exactly 7 alive leaves, whence, in the process of construction, at least 7 alive leaves become dead. Let us look at Table 1: any step decreasing the number of alive leaves has profit at least $\frac{1}{15}$. Moreover, we can decrease the number of alive leaves with profit exactly $\frac{1}{15}$ only by 2 or by 5, i.e., less than by 7. Thus for killing of 7 alive leaves, we obtain a profit at least $\frac{2}{15}$ and provide $\alpha(G) \geq 2$.

**a2. Step M4.3.**

Consider the further construction of a spanning tree with the help of our algorithm. The tree $F_2$ has exactly 4 alive leaves, whence, in the process of construction, at least 7 alive leaves become dead. We need an additional profit at least $\frac{2}{15}$; the killing of alive leaves always gives a profit at least $\frac{1}{15}$. The only possible number of alive leaves, the killing of which will provide a profit less than $\frac{2}{15}$ is 5. But to kill 5 alive leaves we must first increase their number by exactly 1, and this operation provides a profit at least $\frac{1}{15}$. In any case, we obtain $\alpha(G) \geq 2$.

**b.** The vertex $a_2$ is an alive leaf of the tree $F_2$.

Since $a_2$ is adjacent to $a_1$, in this case at least one of the vertices $b_1$ and $b_2$ (let it be $b_1$) is not adjacent to $a_2$. If $b_1 \not\in T$, then, owing to this, the profit increases by $\frac{1}{15}$, and we obtain $\alpha'(F_2) \geq \frac{24}{15}$. By Lemma 2, this is enough for $\alpha(G) \geq 2$.

Let $b_1 \in T$. The vertices of the tree $F_0$ are not adjacent to $b_1 \in V(F)$, whence $b_1$ can be adjacent to at most two vertices of $V(F_2)$; the vertex $a_1$ and, maybe, the vertex $b_2$. Therefore, $b_1$ is adjacent to at least two vertices that do not belong to the tree $F_2$. We adjoin all these vertices to the tree $F_2$ and obtain a tree $F_3$.  

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If we have added more than two vertices, then we have a profit at least \( p(A2) + p(A1) \geq \frac{8}{15} \) and \( \alpha'(F_3) > 2 \). Hence, we have added exactly two vertices; let them be \( c_1 \) and \( c_2 \) (see Fig. 9(2b)). We assume that the leaves \( b_2 \), \( c_1 \), \( c_2 \) of the tree \( F_3 \) are alive. If any of them is a dead leaf, it will be calculated at the end of the construction with the help of step \( 20 \).

We have performed the step \( A2 \), obtained the profit \( \frac{1}{15} \) and \( \alpha'(F_3) \geq \frac{27}{15} + p \). If \( p \geq \frac{2}{15} \), then we have \( \alpha' \geq \frac{29}{15} \), which, by Lemma 2, provides \( \alpha(G) \geq 2 \). Only the steps with profit less than \( \frac{2}{15} \) remain: \( M4.5.4, M4.3 \) (with profit 0) and \( M4.2, M4.5.3, M4.5.5 \) (with profit \( \frac{1}{15} \)).

**b1. Step M4.5.4.**
In this case, there are exactly 9 alive leaves in the tree \( F_3 \) and \( \alpha'(F_3) \geq \frac{27}{15} \). Consider the further construction of a spanning tree with the help of our algorithm. At one step, we can kill 1, 2, 3, 4, or 5 alive leaves (see Table 1). The killing of any number of alive leaves, except for 2 and 5, at one step yields a profit at least \( \frac{2}{15} \). Hence, the only number of killed alive leaves, which is at least 9 and does not provide a profit at least \( \frac{2}{15} \) is 10 (one can kill 10 alive leaves with profit \( \frac{2}{15} \)). But to kill 10 alive leaves, we must first increase their number by exactly 1, and this operation provides a profit at least \( \frac{1}{15} \). In any case, we obtain \( \alpha(G) \geq 2 \).

**Remark 12.** Now the steps \( M4.5.4 \) and \( N4.5.4 \) in Lemma 3 are completely analyzed.

Assume that one has constructed a spanning tree \( T \) with \( \alpha(T) < 2 \) in the graph \( G \) with the help of our algorithm and it was performed one of the steps \( M4.5.4 \) and \( N4.5.4 \). Let \( F \) be a tree constructed before this step. Then at least one of the vertices adjoined at the considered step must be adjacent to the tree \( F \). Let us recall details of this step: a leaf adjacent to the tree \( F \) must be among two leaves adjacent to \( p_1 \) at the end of the step (otherwise we would perform one of the previous steps, see Fig. 6 and the description of the steps). We denote this leaf by \( q \).

If \( q \in T \) (in this case, we denote the steps by \( M4.5.4.1 \) and \( N4.5.4.1 \)), then \( q \) must be adjacent to two leaves of the tree \( F \) (by Remark 6), which increases the number of dead leaves by two. Thus, we obtain

\[
p(M4.5.4.1) \geq p(M4.5.4) + 2 \cdot \frac{2}{15} \geq \frac{4}{15}, \quad \Delta u = 4, \quad \Delta b = 3,
p(N4.5.4.1) \geq p(N4.5.4) + 2 \cdot \frac{2}{15} \geq \frac{5}{15}, \quad \Delta u = 4, \quad \Delta b = 2,
\]

If \( q \notin T \) (in this case, we denote the steps by \( M4.5.4.2 \) and \( N4.5.4.2 \)), then we have one extra dead leaf. In addition, the profit increases by at least \( \frac{1}{5} \), since the cost of the vertex \( q \) decreases. In this case,

\[
p(M4.5.4.2) \geq p(M4.5.4) + 2 \cdot \frac{1}{5} \geq \frac{5}{15}, \quad \Delta u = 4, \quad \Delta b = 2,
p(N4.5.4.2) \geq p(N4.5.4) + 2 \cdot \frac{1}{5} \geq \frac{6}{15}, \quad \Delta u = 4, \quad \Delta b = 1.
\]

Now we can claim that any step that increases the number of alive leaves has a profit at least \( \frac{1}{15} \). We shall take into account this property in subsequent arguments.

Let us continue the proof of Lemma 3.

**b2. Step M4.3.**
In this case, there are exactly 6 alive leaves in the tree \( F_3 \) and \( \alpha'(F_3) \geq \frac{27}{15} \). Consider the further construction of a spanning tree with the help of our algorithm. The killing of any number of more than 5 alive leaves gives a profit at least \( \frac{2}{15} \). Moreover, the killing of exactly 6 alive leaves yields a profit at least \( \frac{3}{15} \). Any step that increases the number of alive leaves has a profit at least \( \frac{1}{15} \). In any case, we obtain \( \alpha(G) \geq 2 \).

**b3. Steps M4.2, M4.5.3, M4.5.5.**
In these cases, we have \( \alpha'(F_3) \geq \frac{28}{15} \). In the tree \( F_3 \), we have exactly 4 alive leaves for the step \( M4.5.5 \) and exactly 7 alive leaves in the other cases. Hence we must kill at least 4 alive leaves. The only number of alive leaves the killing of which will not provide the profit \( \frac{2}{15} \) is 5 (profit \( \frac{1}{15} \)). This is possible only for the step \( M4.5.5 \), if we increase the number of alive vertices exactly by 1, but this operation has a profit at least \( \frac{1}{15} \). In any case, we obtain \( \alpha(G) \geq 2 \). \( \square \)

Now there are significant transformations in our table of steps. We exclude steps that provide \( \alpha(G) \geq 2 \) by Lemma 3 and take into account Remark 12. All remaining steps and their parameters are introduced in Table 2.
| Step   | $\Delta u - \Delta b$ | 15-profit |
|--------|------------------------|-----------|
| $A1$   | 1                      | 7         |
| $A2, A4$ | 1                      | 1         |
| $A3$   | 2                      | 2         |
| $M1, M4.1.2, M4.5.1, N4.1.1$ | 0          | 3         |
| $N1, N4.1.2, M4.5.1, M4.5.4.1$ | 1          | 4         |
| $M2, N4.5.4.1, M4.5.4.2$ | 2          | 5         |
| $N2, N4.5.4.2$ | 3          | 6         |
| $M3.1$ | -4                     | 5         |
| $N3.1$ | -3                     | 6         |
| $M3.2$ | -2                     | 4         |
| $N3.2$ | -1                     | 5         |
| $M4.1.1, Z0$ | -1          | 2         |
| $Z1.1$ | -4                     | 2         |
| $Z1.2$ | -3                     | 3         |
| $Z2.1$ | -5                     | 1         |

**Remark 13.** 1) Looking over Table 2, one can easily conclude that
- any step gives a profit at least $\frac{1}{15}$;
- a step or several steps increasing the number of alive leaves by 2 or by 5 gives a profit at least $\frac{2}{15}$;
- any step that preserves the number of alive leaves yields a profit at least $\frac{3}{15}$.

2) The construction of a spanning tree by our algorithm is complete if and only if all leaves of the tree that we have constructed are dead. Hence it is easy to see that the last step of the construction has negative parameter $\Delta u - \Delta b$.

Let us continue the case analysis in the construction of a base tree.

**B5.** *There are two adjacent vertices $a \in T$ and $b \in S$ such that $N_G(a) \cap N_G(b) = \emptyset$.*

We begin with the base tree $F'$ in which the vertices $a$ and $b$ are adjacent to each other and to all vertices from $N_G(a) \cap N_G(b)$. Clearly, $u(F') = 5$, $c_G(F') \leq \frac{1}{5} + 6 \cdot \frac{2}{5} = \frac{13}{5}$, and $\alpha'(F') \geq \frac{5}{13} \cdot c_G(F') \geq \frac{26}{15}$.

If any leaf of the tree $F'$ does not belong to $T$, then its cost decreases by $\frac{1}{5}$, whence $\alpha'(F')$ increases by $\frac{1}{5}$. In this case, we have $\alpha'(F') \geq \frac{26}{15}$, and, by Lemma 2, this is enough.

Therefore, the remaining case is the case where all the leaves of the tree $F'$ belong to $T$, i.e., have degree 4. Let us construct a spanning tree by our algorithm. Consider two cases.

**B5.1.** *In the process of constructing the number of alive leaves was increased.*

At the beginning, this number is equal to 5. The killing of any number of alive leaves, more than 5, gives a profit at least $\frac{2}{15}$. Moreover, the profit may be equal to $\frac{2}{15}$ only for 7 or 10 alive leaves. For another number of alive leaves, we obtain at least $\frac{2}{15}$ for the killing and, in addition, at least $\frac{1}{15}$ for the increase of the number of alive leaves, which provides $\alpha(G) \geq 2$.

Assume that the number of alive leaves was increased to 7 or to 10 (i.e., it was increased by 2 or by 5). By Remark 13, for this increase we have a profit at least $\frac{2}{15}$. After that we have additional $\frac{2}{15}$ for killing of alive leaves, which provides $\alpha(G) \geq 2$.

**B5.2.** *In the process of constructing, the number of alive leaves was not increased.*

Assume that a step that preserves the number of alive leaves was performed and we obtained the tree $F_1$. Clearly, this step cannot be the last step of the construction of a spanning tree. By Remark 13, such a step cannot be the last step and has a profit at least $\frac{2}{15}$. Hence, $\alpha'(F_1) \geq \frac{2}{15}$, and, by Lemma 2, this is enough for $\alpha(G) \geq 2$.

Consider the remaining case where all performed steps have decreased the number of alive leaves. We must kill 5 alive leaves of the tree $F'$. Any way of doing this, except for the step $Z2.1$, provides a profit at least $\frac{4}{15}$ and $\alpha(G) \geq 2$ (see Table 2).

Assume that the step $Z2.1$ was performed at which two adjacent vertices $a'$ of degree 4 and $b'$ of degree 3 were added. In this case our graph consists of 9 vertices: it contains two copies of the tree $F'$; with centers $a, b$ and with centers $a', b'$, and with five common leaves. Let these leaves be $x_1, x_2, x_3 \in N_G(a)$ and $y_1, y_2 \in N_G(b)$.
Since \( d_G(x_1) = d_G(x_2) = d_G(x_3) = d_G(y_1) = d_G(y_2) = 4 \), it follows that \( G(\{x_1, x_2, x_3, y_1, y_2\}) \) is a regular graph of degree 2, i.e., a cycle on five vertices. Hence, there exist two independent edges connecting \( \{x_1, x_2, x_3\} \) with \( \{y_1, y_2\} \). Let these edges be \( x_1y_1 \) and \( x_2y_2 \).

Without loss of generality, we assume that \( x_1 \) and \( y_2 \) are nonneighboring vertices of the 5-vertex cycle

\[
G(\{x_1, x_2, x_3, y_1, y_2\}).
\]

Then \( a, b, x_2, x_3, y_1 \in N_G(x_1) \cup N_G(y_2) \); moreover, one of the vertices \( x_2, x_3, y_1 \) belongs to \( N_G(x_1) \cap N_G(y_2) \).

If \( a' \in N_G(x_1) \cap N_G(y_2) \), then we construct a spanning tree in the following way. We connect \( a' \) with \( x_1 \) and \( y_2 \) and adjoin all other vertices to these three (see Fig. 10a). Similarly in the case \( b' \in N_G(x_1) \cap N_G(y_2) \).

The remaining case is the case where one of the vertices \( a' \) and \( b' \) is adjacent to \( x_1 \), and the other is adjacent to \( y_2 \). Then we connect \( x_1 \) and \( y_2 \) with a vertex from \( N_G(x_1) \cap N_G(y_2) \) (it is proved above that such a vertex exists) and adjoin to \( x_1 \) and \( y_2 \) all other vertices (see Fig. 10b). As a result, in both cases we obtain a spanning tree of the graph \( G \) with 6 leaves. Note that \( 6 > \frac{2}{3} \cdot 7 + \frac{1}{3} \cdot 2 + 2 \), i.e., the theorem is proved in this case.

**Remark 14.** 1) In the remaining cases, the steps Z2.1 and A4 are not performed, since these steps need the configuration considered in the case B3.

2) Further on we assume that any two adjacent vertices \( a, b \in V(G) \) have a common neighbor. Assume the contrary; let \( N_G(a) \cap N_G(b) = \emptyset \). The case \( a, b \in S \) is impossible, since we would apply the reduction rule R2. If \( a, b \in T \), then the graph \( G \) contains the configuration considered in the case B1. If one of the vertices \( a \) and \( b \) belongs to \( S \) and the other lies in \( T \), then the graph contains the configuration considered in the case B3.

**B6. The graph does not contain a vertex of degree 4.**

This means that \( G \) is a regular graph of degree 3. In [2], it is proved that \( u(G) \geq s - \frac{1}{3} + 2 \) for such graphs, whence our theorem follows.

**Lemma 4.** If \( \alpha(G) < 2 \), then after each of the steps M1, N1, M2, N2, M3.1, N3.1, M3.2, N3.2, M4.1.1, N4.1.1, M4.1.2, N4.1.2, M4.5.1, N4.5.1, M4.5.4.1, N4.5.4.1, M4.5.4.2, and N4.5.4.2, an additional (with respect to the parameters of these steps) dead leaf must appear.

**Proof.** Let us recall details of MN-steps. Let \( F \) be a tree before the step, \( W = V(G) \setminus V(F) \). We have adjoined to the tree \( F \) a subtree (denote it by \( F_0 \), see Fig. 11) with a root \( x \in W \) and obtained the tree \( F_1 \) after this step. All vertices of \( W \) adjacent to \( V(F) \) are called vertices of level 1.

Consider any step from our lemma. At this step, one of the leaves added to the tree \( F \), say, \( v \) was adjacent to \( F \) (for the steps M1, N1, M2, N2, M4.5.1, N4.5.4.1, M4.5.4.2, N4.5.4.2, this follows from Lemma 3, for the steps M3.1, N3.1, M3.2, N3.2 this was shown just after the description of the step 3, and for all other steps this follows from their description).

Note that \( v \neq x \) (in all MN-steps the vertex \( x \) is not a leaf). Let us consider the ancestor \( w \) of the leaf \( v \) in the added tree \( F_0 \). By Remark 14, we have \( N_G(v) \cap N_G(w) \neq \emptyset \). Clearly, \( v, w \in W \).

Let \( a \) be a common neighbor of \( v \) and \( w \). Clearly, \( a \not\in V(F) \), since a vertex of the tree \( F \) cannot be adjacent to two vertices of the set \( W \), by Remark 6. Since \( w \) is a nonpendant vertex in the tree obtained, by the construction all vertices adjacent to \( w \) belong to \( V(F) \cup V(F_0) \) (one can verify this fact looking over details of the steps). Hence, \( a \in V(F_0) \). Therefore, \( v \) has two adjacent vertices \( w, a \), which belong to the tree \( F_1 \) constructed after the step. The vertex \( v \in W \) is adjacent to \( V(F) \), whence it belongs to level 1. Then, by Remark 6, the vertex \( v \) cannot be adjacent to more than two vertices from \( W \); thus, the vertex \( v \) is a dead leaf of the tree \( F_1 \). Clearly, it was not calculated in the parameters of the step. \( \square \)
| Step       | $\Delta u - \Delta b$ | 15-profit |
|------------|------------------------|-----------|
| A1         | 1                      | 7         |
| A2         | 1                      | 1         |
| A3         | 2                      | 2         |
| M1, M4.1.2, M4.5.1, N4.1.1 | -1                   | 5         |
| N1, N4.1.2, N4.5.1, M4.5.4.1 | 0                   | 6         |
| M2, N4.5.4.1, M4.5.4.2 | 1                   | 7         |
| M3.1       | -5                     | 7         |
| N3.1       | -4                     | 8         |
| M3.2       | -3                     | 6         |
| N3.2       | -2                     | 7         |
| M4.1.1     | -2                     | 4         |
| Z0         | -1                     | 2         |
| Z1.1       | -4                     | 2         |
| Z1.2       | -3                     | 3         |

Before the last and most complicated case, we rewrite the table of parameters of the steps. We add dead leaves and update the profit for all steps of Lemma 4. In addition, we exclude the steps Z2.1 and A4, which are impossible by Remark 14. Updated parameters of the steps are presented in Table 3.

**B7.** The graph does not satisfy the condition of any previous case.

Then there exists a vertex $a$ of degree 4 in our graph. We connect $a$ with its 4 neighbors and obtain the tree $F'$ with 4 leaves $v_1, v_2, v_3, v_4$ and $\alpha'(F') \geq 4 \cdot \frac{13}{15} - 5 \cdot \frac{2}{5} = \frac{22}{15}$. Let us continue the construction of a spanning tree by our algorithm and consider all performed steps, which have increased the number of alive leaves. We calculate the sum of increases of the number of alive leaves over all these steps and denote this sum by $\ell$. Then the steps that decrease the number of alive leaves must kill $\ell + 4$ alive leaves. Consider several cases.

**B7.1.** $\ell \geq 2$.

It is easy to see from Table 3 that adding $\ell$ alive leaves provides a profit at least $\frac{\ell}{15}$. For $\ell = 2$, we must kill 6 alive leaves. For this we obtain a profit at least $\frac{6}{15}$ (see Table 3), which provides $\alpha(G) \geq 2$.

For $\ell = 3$, we must kill 7 alive leaves. For this we obtain a profit at least $\frac{5}{15}$ (see Table 3), which provides $\alpha(G) \geq 2$.

For $\ell \geq 4$, we must kill at least 8 alive leaves. For this we obtain a profit at least $\frac{4}{15}$ (see Table 3), which also provides $\alpha(G) \geq 2$.

**B7.2.** $\ell = 0$.

By Remark 13, the last step has negative parameter $\Delta u - \Delta b$. It is easy to see from Table 3 that such a step has profit at least $\frac{\ell}{15}$. If a step that preserves the number of alive leaves was performed, it has profit at least $\frac{6}{15}$, which is enough for $\alpha(G) \geq 2$.

Thus, only steps decreasing the number of alive leaves were performed. It is easy to see from Table 3 that there are two ways of killing 4 alive leaves and not providing the profit $\frac{8}{15}$ (and $\alpha(G) \geq 2$):

![Fig. 11. Additional dead leaf.](image-url)
• the step Z0 together with the step Z1.2 (the sum of profits \( \frac{5}{15} \));
• the step Z1.1 (profit \( \frac{1}{15} \)).

Consider these two cases.

**B7.2.1. The step Z0 and the step Z1.2 were performed.**

We have profit \( \frac{1}{15} \), whence \( \alpha(G) \geq \frac{1}{15} \). If at least one leaf of the tree \( F' \) does not belong to \( T \), then the profit increases by \( \frac{1}{15} \), which provides \( \alpha(G) \geq 2 \). Thus, all leaves of \( F' \) belong to \( T \) and have degree 4 in the graph \( G \).

Then there are exactly 6 vertices in the graph \( G \): five vertices of degree 4 and one vertex of degree 3 (added at the step Z1.2). Clearly, this is impossible.

**B7.2.2. The step Z1.1 was performed.**

We have profit \( \frac{1}{15} \), whence \( \alpha(G) \geq \frac{1}{15} = \frac{8}{5} \). The vertex added at the step Z1.1 has degree 4. If at least two of the vertices \( v_1, v_2, v_3, v_4 \) do not belong to \( T \), the profit increases by \( \frac{1}{15} \), which provides \( \alpha(G) \geq 2 \). If exactly one of these vertices does not belong to \( T \), then the graph \( G \) contains five vertices of degree 4 and one vertex of degree 3, which is impossible. Therefore, the only variant for \( \alpha(G) < 2 \) in our case is a regular graph of degree 4 on 6 vertices. Clearly, such a graph is unique — it is \( C_6^4 \), and this graph is really an exclusion (\( \alpha(C_6^4) = \frac{8}{5} \)).

**B7.3. \( \ell = 1 \).**

That is, we have performed exactly one step, increasing the number of alive leaves, and this number was increased exactly by 1. It is easy to see from Table 3 that it is either the step A2, or we have obtained a profit at least \( \frac{7}{15} \)
and have constructed a tree \( F_{1} \) with \( \alpha'(F_{1}) \geq \frac{5}{15} \). In the last case, by Lemma 2, we have \( \alpha(G) \geq 2 \) and complete the proof.

Thus, we have done one step A2 with profit \( \frac{1}{15} \) and have obtained the tree \( F_{1} \) with \( \alpha'(F_{1}) \geq \frac{5}{15} \). As above, if we have performed a step that preserved the number of alive leaves, then \( \alpha(G) \geq 2 \). Hence, the step A2 is the only step, except for steps that decrease the number of alive leaves. It is easy to see from Table 3 that there is the only way of killing 5 alive leaves and not providing the profit \( \frac{7}{15} \) (and \( \alpha(G) \geq 2 \)): the step Z0 together with the step Z1.1 (the sum of profits \( \frac{1}{15} \), we add a vertex of degree 4). These steps provide \( \alpha(G) \geq \frac{10}{15} \).

If at least one of the vertices \( v_1, v_2, v_3, v_4 \) or the two vertices added at the step A2 has degree 3, then the profit increases by \( \frac{1}{15} \), which provides \( \alpha(G) \geq 2 \).

Let us consider the last case where \( G \) is a regular graph of degree 4 on 8 vertices. By Remark 14, each edge of \( G \) belongs to a triangle. Let us verify that there are two such graphs up to isomorphism.

If \( G \) is a vertex 4-connected graph, we make use of paper [5] — it is proved there that \( G \) is either a square of a cycle or an edge graph of the 4-cycle-connected cubic graph. The second case is impossible, since the number of vertices of such an edge graph must be divisible by 3. The first case yields the graph \( C_8^4 \), which is really an exclusion.

Let \( G \) have a cutset \( R \) that consists of less than 4 vertices. It is easy to see that if a set of \( 4 - \ell \) vertices separates a connected component \( H \) in a 4-regular graph, then \( v(H) \geq k + 1 \). Hence, \( |R| \geq 2 \).

If \( |R| = 2 \), then there is only the possibility: the cutset \( R \) must split the graph into exactly two components, each component contains three vertices, and each of these 6 vertices must be adjacent to each of the two vertices of the set \( R \). But then the vertices of the set \( R \) have degree 6, a contradiction.

Let \( |R| = 3 \), \( R = \{ r_1, r_2, r_3 \} \). Then one of the connected components has two vertices (let these vertices be \( a_1, a_2 \)) and the other component has three vertices (\( b_1, b_2, b_3 \)). Clearly, \( a_1 \) and \( a_2 \) are adjacent and each of them is adjacent to each of the vertices \( r_1, r_2, r_3 \) (otherwise \( d_c(a_i) < 4 \)). Hence, each of the vertices \( r_1, r_2, r_3 \) is adjacent to no more than two of the vertices \( b_1, b_2, b_3 \); therefore, the sum of vertex degrees of the graph \( G(\{b_1, b_2, b_3\}) \) is at least 6, i.e., this graph is complete. Consequently, each of the vertices \( b_1, b_2, b_3 \) is adjacent to exactly two of the vertices \( r_1, r_2, r_3 \) and each of the vertices \( r_1, r_2, r_3 \) is adjacent to exactly two of the vertices \( b_1, b_2, b_3 \), i.e., the vertices \( r_1, r_2, r_3 \) are pairwise nonadjacent. Now it is clear that there is only one such graph up to isomorphism, i.e., the graph \( G_8 \), shown in Fig. 1.

2.5. Reduction and counterexamples. Let us prove that if the reduction rule R1 or R2 was applied to a graph \( G \), then the graph \( G \) is not an exclusion.

As we know, the application of the reduction rules R1 and R2 does not decrease \( \alpha(G) \). Hence, it is enough to verify that \( \alpha(G) \geq 2 \) for a graph \( G \), which can be transformed to \( C_6^4, C_8^4 \), or \( G_8 \) by applying the reduction rule R1 or R2. Consider 6 cases.

1. The graph \( G \) can be transformed to \( C_6^4 \) by applying the reduction rule R1.
Let $G$ can be transformed to the square of the cycle $a_1 a_2 a_3 a_4 a_5 a_6$ by deleting a vertex $w$ of degree 2 and adding an edge connecting two vertices of $N_G(w)$. Without loss of generality, we may consider two cases: the added edge is $a_1 a_2$ or $a_1 a_3$. In both cases, it is easy to construct a spanning tree with 5 leaves; see Figs. 12a and 12b. Hence, $u(G) \geq 5 > 6 \cdot \frac{2}{5} + 2$ and $\alpha(G) > 2$.

![Fig. 12. Reduction: case of $C_6^2$.](image)

2. The graph $G$ can be transformed to $C_4^2$ by applying the reduction rule R2.

Let $G$ can be transformed to the square of the cycle $a_1 a_2 a_3 a_4 a_5 a_6$ by contracting an edge $vw$, where $d_G(v) = d_G(w) = 3$. Let the vertex $a_1$ be the result of gluing $v$ and $w$. Then $v$ and $w$ together are adjacent in $G$ to $a_2, a_3, a_5, a_6$. Without loss of generality, we assume that $w$ is adjacent to $a_2$ in the graph $G$. If $a_3$ is adjacent in $G$ to $v$, then we construct a spanning tree of the graph $G$ with 5 leaves as in Fig. 12c. If $a_3$ is adjacent in $G$ to $w$, then $a_5$ is adjacent in $G$ with $v$. In this case, a spanning tree of the graph $G$ with 5 leaves is shown in Fig. 12d. Hence, $u(G) \geq 5 > 6 \cdot \frac{2}{5} + 2$ and $\alpha(G) > 2$.

![Fig. 13. Reduction: case of $C_6^2$.](image)

3. The graph $G$ can be transformed to $C_4^2$ by applying the reduction rule R1.

Let $G$ can be transformed to the square of the cycle $a_1 a_2 \ldots a_8$ by deleting a vertex $w$ of degree 2 and adding an edge connecting two vertices of $N_G(w)$. Without loss of generality, we may consider two cases: the added edge is $a_1 a_2$ or $a_1 a_3$. In both cases, it is easy to construct a spanning tree with 6 leaves; see Figs. 13a and 13b. Hence, $u(G) \geq 6 > 8 \cdot \frac{2}{5} + 2$ and $\alpha(G) > 2$.

4. The graph $G$ can be transformed to $C_6^2$ by applying the reduction rule R2.

Let $G$ can be transformed to the square of the cycle $a_1 a_2 \ldots a_8$ by contracting an edge $vw$, where $d_G(v) = d_G(w) = 3$. Let the vertex $a_1$ be the result of gluing $v$ and $w$. Then $v$ and $w$ together are adjacent in $G$ to $a_2, a_3, a_5, a_7$. Without loss of generality we assume that $w$ is adjacent to $a_2$ in the graph $G$. If $a_3$ is adjacent in $G$ to $v$, then we construct a spanning tree of the graph $G$ with 6 leaves as in Fig. 13c. If $a_3$ is adjacent in $G$ to $w$, then $a_7$ is adjacent in $G$ to $v$. In this case, a spanning tree of the graph $G$ with 6 leaves is shown in Fig. 13d. Hence, $u(G) \geq 6$ and $\alpha(G) > 2$.

5. The graph $G$ can be transformed to $G_8$ by applying the reduction rule R1.

Let $G$ can be transformed to the graph $G_8$ by deleting a vertex $w$ of degree 2 and adding an edge connecting two vertices of $N_G(w)$. We use for $G_8$ the same notation as in Fig. 1b.

By the symmetry of the graph $G_8$, it is enough to consider four cases: the added edge is $a_1 a_2$ (a spanning tree with 6 leaves is shown in Fig. 14a), $b_1 b_3$ (Fig. 14b), $r_3 b_3$ (Fig. 14c), and $r_3 a_1$ (Fig. 14d). Hence, in any case $u(G) \geq 6$ and $\alpha(G) > 2$.

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6. The graph $G$ can be transformed to $G_8$ by applying the reduction rule R2.

Let $G$ be transformed to the graph $G_8$ by contracting an edge $vw$, where $d_G(v) = d_G(w) = 3$. We use for $G_8$ the notation of the previous case. By the symmetry of the graph $G_8$, it is enough to consider three cases: the vertices $v$ and $w$ of the graph $G$ can be contracted into one of the vertices $a_1, b_1, r_1$.

If it is $a_1$, then $N_G(\{v, w\}) = \{a_2, r_1, r_2, r_3\}$. Consider the tree $F_1$ shown in Fig. 15a. It has three nonpandant vertices $a_2, r_2, r_3$ and each of the vertices $w$ and $v$ is adjacent in the graph $G$ to one of them. Thus, $F_1$ can be transformed to a spanning tree of the graph $G$ with 6 leaves.

If $v$ and $w$ are contracted into the vertex $b_1$, then $N_G(\{w, v\}) = \{b_2, b_1, r_1, r_2\}$. Consider the tree $F_2$ shown in Fig. 15b. It has three nonpandant vertices $b_2, b_3, r_2$ and each of the vertices $w$ and $v$ is adjacent in the graph $G$ to one of them. Thus, $F_2$ can be transformed to a spanning tree of the graph $G$ with 6 leaves.

If $v$ and $w$ are contracted into the vertex $r_1$, then $N_G(\{w, v\}) = \{a_1, a_2, b_1, b_2\}$. By symmetry, it is enough to consider two cases:

- $a_1, a_2 \in N_G(w), b_1, b_2 \in N_G(v)$;
- $a_1, b_2 \in N_G(w), a_2, b_1 \in N_G(v)$.

In both cases, we construct a spanning tree of the graph $G$ with 6 leaves, as in Fig. 15c.

Thus, in any case we have $\mu(G) \geq 6$ and $\alpha(G) > 2$.

Now the proof of Theorem 1 is complete.

Fig. 14. Reduction: case of the graph $G_8$ and rule R1.

Fig. 15. Reduction: case of the graph $G_8$ and rule R2.

Fig. 16. Extremal examples.
3. Extremal examples

There are a lot of infinite series of graphs $G$ containing $s > 0$ vertices of degree 3 and $t > 0$ vertices of degree more than 3 such that $u(G) = \frac{3}{2}t + \frac{1}{3}s + 2$. We introduce series of graphs all the vertices of which have degrees 3 and 4. Thus, these graphs are also counterexamples to the strong Linial’s conjecture (see the Introduction).

Let us begin the construction of our graphs. Let $D_i$ be a graph on the vertex set $x_i, y_i, z_i, v_i, a_i, b_i$, where the vertices $x_i, y_i, z_i, v_i$ are pairwise adjacent, the vertex $a_i$ is adjacent to $x_i$ and $y_i$, and the vertex $b_i$ is adjacent to $z_i$ and $v_i$. We make a cycle of such graphs $D_1, \ldots, D_n$ (where $n > 1$) and connect $a_i, a_{i+1}$ with $b_i$ (we set $n + 1 = 1$). The obtained graph is denoted by $H_n$ (see Fig. 16). Clearly, $c(H_n) = 2n \cdot \frac{1}{3} + 4n \cdot \frac{2}{5} = 2n$.

Note that the set of leaves of any spanning tree $T$ of the graph $H_n$ is not a cutset in $H_n$. In view of this, it is easy to see that $u(H_n) = 2n + 2 = c(H_n) + 2$.

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