1 Introduction

Despite extension groups between modules over an algebra are very easy to define and taught nowadays in every standard course in homological algebra, it is still to be very difficult to compute them explicitly for a given pair of modules. One of such problems is a computation of extension groups between Weyl modules over the Schur algebra $S(n, r)$. It was shown in the joint work [4] of the author with Ana Paula Santana that this problem is closely related to the construction of a minimal projective resolution of the trivial module $\mathbb{K}$ over Kostant form $\mathfrak{U}_\mathbb{K}(sl^+_n)$ of the universal enveloping algebra of the Lie algebra $sl^+_n$.

In this paper we compute the first three steps of a minimal projective resolution of $\mathbb{K}$ for $n = 3$. For this we use the Anick resolution constructed in [1]. Our result depends on the knowledge of a Gröbner basis for $\mathfrak{U}_\mathbb{K}(sl^+_3)$.

In the Section 2 we recall the definition of Gröbner basis and in the Section 6 the construction of the Anick’s resolution. Then we proceed with the definition of $\mathfrak{U}_\mathbb{K}(sl^+_n)$ in Section 3. The Sections 4, 5, 7 contain new results. In particular, we describe the first three steps of the minimal projective resolution for trivial module over $\mathfrak{U}_\mathbb{K}(sl^+_3)$.

2 Gröbner basis

Let $X$ be a set. We denote by $X^*$ the set of all words with letters in $X$. Then $X^*$ is a free monoid generated by $X$ with the multiplication given by concatenation.

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of words and the unity $e$ given by the empty word. There is a partial order $\prec$ on $X^*$ given by the incursion of words. Note that $\prec$ is the coarsest partial order on $X^*$ such that $X^*$ is an ordered monoid with $e$ the least element of $X^*$. A monoidal order on $X^*$ is a total order that refines $\prec$.

Let $\mathbb{K}$ be a field. We denote by $\mathbb{K}\langle X^* \rangle$ a vector space spanned by $X^*$. A vector space $\mathbb{K}\langle X^* \rangle$ is a free associative algebra generated by $X$. We will call the elements of $X^*$ monomials, and the elements of $\mathbb{K}\langle X^* \rangle$ polynomials. Define the support of $p \in \mathbb{K}\langle X^* \rangle$ to be the set of element in $X^*$ with non-zero coefficients in $p$. If $\leq$ is a monoidal order on $X^*$ then we define the leading monomial $\text{lm}(p)$ of $p \in \mathbb{K}\langle X^* \rangle$ to be the maximal element of support of $p$ with respect $\leq$. Define the leading term $\text{lt}(p)$ of $p$ to be the leading monomial of $p$ with coefficient it enters in $p$. A monoidal order $\leq$ on $X^*$ can be extended to a partial order $\leq$ on $\mathbb{K}\langle X^* \rangle$ by the rule

$$p \leq q \iff \text{lm}(p) < \text{lm}(q)$$

$$\text{lt}(p) = \text{lt}(q) \text{ and } p - \text{lt}(p) \leq q - \text{lt}(q).$$

Note that in the case $\text{lm}(p) = \text{lm}(q)$ but $\text{lt}(p) \neq \text{lt}(q)$ the polynomials $p$ and $q$ are incompatible.

The pair $(m, f)$, where $m$ is a monomial and $f$ an element of $\mathbb{K}\langle X^* \rangle$, is called a rewriting rule if $m > f$. Note that every element $p \in \mathbb{K}\langle X^* \rangle$ gives a rewriting rule $r(p) = (\text{lm}(p), f)$ where $f = (p - \text{lt}(p))/\lambda$ and $\lambda$ is the leading coefficient of $p$. We will say that $h$ is a result of application of $(m, f)$ to $g$ if there is $m' \in \text{supp}(g)$ such that $m' = uvw$ for some $u, v \in X^*$, and $h = g - \lambda m' + \lambda uf v$, where $\lambda$ is the coefficient of $m$ in $g$. We will write in this situation $g \rightarrow_r h$. If $r = r(p)$ for some $p \in \mathbb{K}\langle X^* \rangle$ then we write $g \rightarrow_r h$ instead of $g \rightarrow_{r(p)} h$. Let $S$ be a collection of rewriting rules or polynomials. Then $g \rightarrow^*_S h$ denotes that there is $r \in S$ such that $g \rightarrow_r h$. Formally, $\rightarrow^*_S$ is a set relation on $\mathbb{K}\langle X^* \rangle$. We denote by $\rightarrow^*_S$ the reflexive and transitive closure of $\rightarrow_S$. An element $g$ of $\mathbb{K}\langle X^* \rangle$ is called non-reducible with respect to the set of rewriting rules or polynomials $S$ if $g$ is a minimal element of $\mathbb{K}\langle X^* \rangle$ with respect to $\rightarrow^*_S$.

**Definition 1.** Let $A$ be an algebra over a field $\mathbb{K}$ and $X = \{a_i | i \in I\}$ a set of generators of $A$. Denote by $\pi$ the canonical projection from $\mathbb{K}\langle X^* \rangle$ to $A$. We say that a subset $S$ of $\ker(\pi)$ is a Gröbner basis of $\ker(\pi)$ if $\pi$ restricted on the vector space of non-reducible elements with respect $\{r(p) | p \in S\}$ is an isomorphism of $\mathbb{K}$-vector spaces. A Gröbner basis $S$ is called reduced if elements $p \in S$ are non-reducible with respect to $S \setminus \{p\}$.

Suppose that $\leq$ is an artinian monoidal order on $X^*$, that is every descending chain in $X^*$ stabilizes. Let $f \in \mathbb{K}\langle X^* \rangle$. If $f$ is reducible with respect to a Gröbner basis then there is $f_1$ such that $f \rightarrow_S f_1$. By definition of Gröbner basis $f_1 < f$ with respect to the induced ordering on $\mathbb{K}\langle X^* \rangle$. If $f_1$ is reducible we can find $f_2$ such that $f_1 \rightarrow_S f_1, f_1 > f_2$ and so on. Thus we get a descending sequence $f > f_1 > f_2 > \ldots$. As we assumed that the ordering $\leq$ is artinian this sequence have to break. Thus there is $f'$ that is non-reducible with respect to $S$ and $f \rightarrow_S f'$. We call $f'$ the normal form of $f$ with respect to $S$ and denote
it by $NF(f, S)$. Note that the use of the article “the” is justified by the fact that $f'$ is unique. In fact suppose there are $f'$ and $f''$ such that $f \rightarrow_S f'$ and $f \rightarrow_S f''$. Then $f' - f'' = (f' - f) + (f - f'') \in \ker(\pi)$ is an element of the kernel of the natural projection $\pi: \mathbb{K}(X^*) \to A$. Moreover, all monomials in $f' - f''$ are non-reducible with respect to $S$. Since the images of non-reducible monomials with respect to $S$ give a basis of $A$ under the map $\pi$ it immediately follows that $f' - f'' = 0$.

The notion of Gröbner basis is closely connected with the notion of critical pairs. We say that two monomials $m_1, m_2 \in X^*$ overlaps if there are $u, v, w \in X^*$ such that $m_1 = uw$ and $m_2 = vw$. Note that two given monomials can have different overlappings. To make things more convenient we define an overlapping as a triple $(m, m_1, m_2)$, such that there are $u, v \in X^*$ such that $m = m_1 v$ and $m = m_2 w$.

**Definition 2.** A critical pair is a triple $(w, r_1, r_2)$, where $w$ is a word and $r_1 = (m_1, f_1)$, $r_2 = (m_2, f_2)$ are rewriting rules such that there are $u, v \in X^*$ with the property

$$w = um_1 = m_2 v \text{ or } w = um_1 v = m_2.$$ 

A word $w$ is called the tip of the critical pair $(w, r_1, r_2)$.

Let $(w, r_1, r_2)$ be a critical pair with $r_1, r_2 \in S$ and $u, v \in X^*$ such that $w = um_1 = m_2 v$ (or $w = um_1 v = m_2$). It is called reducible if $uf_1 - f_2 v \rightarrow_S 0$ (respectively $uf_1 v - f_2 \rightarrow_S 0$). The set of rewriting rules $S$ is called complete if all critical pairs $(w, r_1, r_2)$ with $r_1, r_2 \in S$ are reducible.

**Theorem 1.** Suppose $\preceq$ is artinian monoidal ordering on $X^*$. A subset $S$ of $\mathbb{K}(X^*)$ is a Gröbner basis of a two-sided ideal $I \subseteq \mathbb{K}(X^*)$ if and only if the set of rewriting rules $\{ r(p) \mid p \in S \}$ is complete.

We shall need the following proposition

**Proposition 1.** Suppose $R$ is a complete rewriting system in variables $X$ and $Y$ is a subset of $X$. We denote by $R(Y)$ the subset of $R$ that consist from all the rules $(m, p)$ such that $m \in Y^*$. If for all $(m, p) \in R(Y)$ we have $p \in \mathbb{K}(Y^*)$ then $R(Y)$ is a complete rewriting system.

**Proof.** Suppose $f \in \mathbb{K}(Y^*)$ and $f \rightarrow_R g$ then $f \rightarrow_{(m, p)} g$ for some $(m, p) \in R$. Since $m \preceq m'$ for some $m' \in \text{supp}(f)$ and $m' \in Y^*$ we get that $(m, p) \in R(Y)$. By assumption of the proposition we get $p \in \mathbb{K}(Y^*)$. Therefore $g \in \mathbb{K}(Y^*)$ and $f \rightarrow_{R(Y)} g$. Now by repetition we get that $f \in \mathbb{K}(Y^*)$ and $f \rightarrow_R g$ implies that $f \rightarrow_{R(Y)} g$.

Suppose that $(w, r_1, r_2)$ is an overlap of two rules from $R(Y)$ and $u, v \in Y^*$ are such that $w = m_1 v = um_2 \ (w = um_1 v = m_2)$. Then $p_1 v - up_2 \in \mathbb{K}(Y^*)$ \(up_1 v - p_2 \in \mathbb{K}(Y^*)\) and $p_1 v - up_2 \rightarrow_R 0 \ (up_1 v - p_2 \rightarrow_R 0)$, since $R$ is complete. But then $p_1 v - up_2 \rightarrow_{R(Y)} 0 \ (up_1 v - p_2 \rightarrow_{R(Y)} 0)$, which shows that $R(Y)$ is complete. \qed
3 Konstant form of universal enveloping algebra

Denote by $\mathfrak{sl}_3^+$ the Lie algebra of upper triangular nilpotent $3 \times 3$ matrices. Let $\mathfrak{U}_3^+$ be its universal enveloping algebra over $\mathbb{C}$. We shall consider $\mathfrak{sl}_3^+$ with the standard basis

$$e_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

They also generate $\mathfrak{U}_3^+$ as an associative algebra. It follows from the Poincare-Birkhoff-Witt Theorem, that the set

$$B = \left \{ e^{k_\alpha}_\alpha e^{k_{\alpha+\beta}}_{\alpha+\beta} e^{k_\beta}_{\beta} \mid k_\alpha, k_{\alpha+\beta}, k_\beta \in \mathbb{N} \right \}$$  

is a $\mathbb{C}$-basis of $\mathfrak{U}_3^+$. For $\omega \in \{\alpha, \alpha+\beta, \beta\}$, denote by $e^{(k)}_\omega$ the element $\frac{1}{k!}e^{k}_\omega$ of the algebra $\mathfrak{U}_n$. We define $\mathfrak{U}_3^+(\mathbb{Z})$ to be the $\mathbb{Z}$-sublattice of $\mathfrak{U}_3^+(\mathbb{C})$ generated by the set

$$B = \left \{ e^{(k_\alpha)}_\alpha e^{(k_{\alpha+\beta})}_{\alpha+\beta} e^{(k_\beta)}_{\beta} \mid k_\alpha, k_{\alpha+\beta}, k_\beta \in \mathbb{N} \right \}.$$  

Proposition 2. The set $\mathfrak{U}_3^+(\mathbb{Z})$ is a subring of $\mathfrak{U}_3^+(\mathbb{C})$. In other words, $\mathfrak{U}_3^+(\mathbb{Z})$ is a $\mathbb{Z}$-algebra. It is called the Kostant form of the universal enveloping algebra $\mathfrak{U}_3^+(\mathbb{C})$ over $\mathbb{Z}$.

Proof. For a proof see [2, Lemma 2 after Proposition 3] and [2, Remark 3] thereafter.

Definition 3. For any field $K$, the algebra $\mathfrak{U}_3^+(K) := K \otimes_{\mathbb{Z}} \mathfrak{U}_3^+(\mathbb{Z})$ is called Kostant form of the algebra $\mathfrak{U}_3^+(\mathbb{C})$ over $K$.

Let $S$ be a free monoid generated by $\alpha$ and $\beta$. We define $S$-grading on $\mathfrak{U}_3^+(\mathbb{C})$ by

$$\deg(e_\alpha) := \alpha \quad \deg(e_{\alpha+\beta}) := \alpha + \beta \quad \deg(e_\beta) := \beta.$$  

This grading extends to the grading of $\mathfrak{U}_3^+(\mathbb{C})$ by

$$\deg \left ( e^{k_\alpha}_\alpha e^{k_{\alpha+\beta}}_{\alpha+\beta} e^{k_\beta}_{\beta} \right ) := (k_\alpha + k_{\alpha+\beta}) \alpha + (k_{\alpha+\beta} + k_\beta) \beta.$$  

It induces a grading on the algebras $\mathfrak{U}_3^+(\mathbb{Z})$ and $\mathfrak{U}_3^+(K)$, for an arbitrary field $K$, such that

$$\deg \left ( e^{(k_\alpha)}_\alpha e^{(k_{\alpha+\beta})}_{\alpha+\beta} e^{(k_\beta)}_{\beta} \right ) := (k_\alpha + k_{\alpha+\beta}) \alpha + (k_{\alpha+\beta} + k_\beta) \beta.$$  

We also define the norm $N$ on $S$ by

$$N(k_\alpha \alpha + k_\beta \beta) := k_\alpha + k_\beta$$  

and will denote the composition of $\deg$ with $N$ by $\text{Deg}$. 

4
4 Big Gröbner basis

In this section we describe a Gröbner basis of the algebra $\mathcal{U}_\mathbb{K}$ with respect to the generating set $X = \{ e_{(k_{\alpha})}, e_{(k_{\alpha+\beta})}, e_{(k_{\beta})} \mid k_{\alpha}, k_{\alpha+\beta}, k_{\beta} \in \mathbb{N} \}$. Let $Y = \{ e_{\alpha}, e_{\alpha+\beta}, e_{\beta} \}$. We order $Y^*$ by deg-lexicographical ordering induced by the ordering

$$e_{\alpha} < e_{\alpha+\beta} < e_{\beta}$$

on $Y$. We have the map $\phi: X^* \to Y^*$ of free monoids induced by

$$e_{\alpha}^{(k)} \mapsto e_{\alpha}^{k}, \quad e_{\alpha+\beta}^{(k)} \mapsto e_{\alpha+\beta}^{k}, \quad e_{\beta}^{(k)} \mapsto e_{\beta}^{k}.$$

We define the ordering $\preceq$ on $X^*$ as follows. If $\phi(u) < \phi(v)$, then $u \preceq v$. If $\phi(u) = \phi(v)$ and the length of $u$ is less then the length of $v$, then $u \preceq v$. If $\phi(u) = \phi(v)$ and both words $u$ and $v \in X^*$ have the same length, then we compare them lexicographically with respect to the ordering

$$e_{\alpha} < e_{\alpha+\beta} < e_{\beta} < e_{(2)}^{(2)} < e_{(2)}^{(2)} < \cdots < e_{(k)}^{(k)} < e_{(k)}^{(k)} < \cdots$$

on $X$. Since deg-lexicographical ordering on $Y^*$ is terminating and every fiber of $\phi$ is finite, it follows that also the ordering $\preceq$ on $X^*$ is terminating. It is also easy to see that $\preceq$ is monomial. In fact, let $u, v, w \in X^*$. Then $\phi(u) < \phi(v)$ implies $\phi(uw) < \phi(vw)$; if $\phi(u) = \phi(v)$ and the length of $u$ is less then the length of $v$, then $\phi(uw) = \phi(vw)$ and the length of $uw$ is less then the length of $vw$; if $\phi(u) = \phi(v)$, $u$ and $v$ have the same length, and $u < v$ with respect to the lexicographical ordering, then $\phi(uw) = \phi(vw)$, $uw$ and $vw$ have the same length, and $uw$ is less then $vw$ with respect to the lexicographical ordering. Thus $u \preceq v$ implies $uw \preceq vw$. The stability with respect to the left multiplication is verified analogous.

**Theorem 2.** Let $X$ and the ordering on $X$ be as above. Then the following set of rewriting rules is complete:

1. $$e_{(k)}^{(l)} \alpha \mapsto \begin{pmatrix} k + l \\ k \end{pmatrix} e_{(k+l)}^{(l)}$$

2. $$e_{\alpha+\beta}^{(k)} \alpha \mapsto \begin{pmatrix} k + l \\ k \end{pmatrix} e_{\alpha+\beta}^{(k+l)}$$

3. $$e_{\beta}^{(k)} \alpha \mapsto \begin{pmatrix} k + l \\ k \end{pmatrix} e_{\beta}^{(k+l)}$$

4. $$e_{\alpha+\beta}^{(k)} \alpha \mapsto \begin{pmatrix} k + l \\ k \end{pmatrix} e_{\alpha+\beta}^{(k+l)}$$

5. $$e_{\beta}^{(k)} \alpha \mapsto \sum_{j=0}^{\min(k,l)} (-1)^j e_{\alpha}^{(j)} e_{\alpha+\beta}^{(k-j)}$$

6. $$e_{\beta}^{(k)} \alpha \mapsto e_{\alpha+\beta}^{(k)} e_{\beta}^{(k)}$$

where $k, l \in \mathbb{N}$. 

5
Proof. It is clear that the set

$$B = \left\{ e^{(k_a)}_\alpha e^{(k_{a+\beta})}_{\alpha+\beta} e^{(k_\beta)}_\beta \mid k_\alpha, k_{a+\beta}, k_\beta \in \mathbb{N} \right\}$$

is the set of non-reducible words with respect to the given rewriting system. By definition, the natural image of $B$ in $U^+_{\mathbb{K}}(\mathbb{K})$ is a basis of $U^+_{\mathbb{K}}(\mathbb{K})$. Therefore, it is enough to check that for every rule the left hand side and the right hand side are equal in $U^+_{\mathbb{K}}(\mathbb{K})$. This is obvious for (1), (2), (3), (4), (6). Thus we have only to check the claim for (5). We have to prove the equality

$$e^{(k)}_\beta e^{(l)}_\alpha = \sum_{j=0}^{\min(k,l)} (-1)^j e^{(l-j)}_\alpha e^{(j)}_{\alpha+\beta} e^{(k-j)}_\beta$$

in $U^+_{\mathbb{K}}(\mathbb{K})$. Clearly it is enough to prove the same equality in $U^+_{\mathbb{Z}}(\mathbb{Z})$ and, therefore in $U^+_{\mathbb{C}}(\mathbb{C})$. We will do this by induction on the minimum of $k$ and $l$. The case $\min(k,l) = 1$ splits into two cases $k = 1$ and $l = 1$. The case $k = 1$, we prove by induction on $l$. For $k = l = 1$ we have

$$e_\beta e_\alpha = e_\alpha e_\beta - e_{\alpha+\beta}.$$ 

Suppose we have proved equality for $k = 1$ and $l \leq l_0$. Let us check it for $l = l_0 + 1$.

$$e_\beta e^{(l)}_\alpha = \frac{1}{l} e_\beta e^{(l-1)}_\alpha e_\alpha \quad \text{induction assumption}$$

$$= \frac{1}{l} \left( e^{(l-1)}_\alpha e_\beta - e^{(l-2)}_\alpha e_{\alpha+\beta} \right) e_\alpha$$

$$= \frac{1}{l} \left( e^{(l-1)}_\alpha e_\beta e_\alpha - e^{(l-1)}_\alpha e_{\alpha+\beta} - e^{(l-2)}_\alpha e_\alpha e_{\alpha+\beta} \right)$$

$$= e^{(l)}_\alpha e_\beta - \frac{1}{l} (1 + l - 1) e^{(l-1)}_\alpha e_{\alpha+\beta}$$

$$= e^{(l)}_\alpha e_\beta - e^{(l-1)}_\alpha e_{\alpha+\beta}.$$

Now we prove the equality in the case $l = 1$ and $k \geq 2$. Suppose it is proved for all $k \leq k_0$. Let us show it for $k = k_0 + 1$. We have

$$e^{(k)}_\beta e^{(k-1)}_\alpha e_\alpha = \frac{1}{k} e_\beta e^{(k-1)}_\alpha e^2_\alpha$$

$$= \frac{1}{k} \left( e_\beta e^{(k-1)}_\alpha e_\alpha - e_\beta e^{(k-2)}_{\alpha+\beta} e^{(k-1)}_\beta \right)$$

$$= \frac{1}{k} \left( e_\alpha e_\beta e^{(k-1)}_\alpha - e_{\alpha+\beta} e^{(k-2)}_\beta e^{(k-1)}_\alpha \right)$$

$$= e_\alpha e^{(k)}_\beta - e_{\alpha+\beta} e^{(k-1)}_\beta.$$ 

Suppose we have prove equality for all $k$ and $l$ such that $\min(k,r) \leq m_0$. Let us prove it for $\min(k,r) = m_0 + 1$. There are two cases $k = m_0 + 1$ and $l = m_0 + 1$. 

\[\text{6}\]
Then is given by the formula
\[
e^{(k)}_\beta e^{(l)}_\alpha = \frac{1}{k} e_\beta e^{(k-1)} e^{(l)}_\alpha
\]

\[
= \frac{1}{k} \sum_{s=0}^{k-1} (-1)^s e_\beta e^{(l-s)} e^{(k-1-s)} e_\alpha + e^{(s)} e^{(k-1-s)} e_\alpha + e^{(l-s-1)} e_\alpha + e^{(s)} e^{(k-1-s)} e_\alpha
\]

\[
= \frac{1}{k} \sum_{s=0}^{k-1} (-1)^s \left( e^{(l-s)} e^{(s)} e^{(k-1-s)} e_\alpha + e^{(l-s-1)} e_\alpha + e^{(s)} e^{(k-1-s)} e_\alpha + e^{(l-s)} e^{(s)} e^{(k-1-s)} e_\alpha \right)
\]

\[
= \frac{1}{k} \sum_{s=0}^{k} ((-1)^s (k - s) - (-1)^{s-1} s) e^{(l-s)} e^{(s)} e^{(k)} = \sum_{s=0}^{k} (-1)^s e^{(l-s)} e^{(s)} e^{(k)}.
\]

Corollary 1. Let \( p \) be a characteristic of the field \( \mathbb{K} \) and \( m \geq 0 \). Then the linear span \( \Lambda^m_+(\mathbb{K}) \) of the set
\[
B' = \left\{ e^{(k_\alpha)}_\alpha e^{(k_\beta+\gamma)} e^{(k_\beta)}_\beta \left| k_\alpha, k_\alpha+\beta, k_\beta \leq p^m - 1 \right. \right\}
\]
is a subalgebra of \( \Lambda^m_+(\mathbb{K}) \).

Proof. We claim that \( \Lambda^m_+(\mathbb{K}) \) is the subalgebra \( A \) of \( \Lambda^m_+(\mathbb{K}) \) generated by the set
\[
X' = \left\{ e^{(k)}_\alpha, e^{(k)}_{\alpha+\beta}, e^{(k)}_\beta \left| k \leq p^m - 1 \right. \right\}.
\]

It is enough to show that the set \( B' \) is a basis of \( A \). Let \( R \) be rewriting system defined in Theorem 2. We claim that \( R(X') \) is complete. To prove this we apply Proposition 1. It is obvious for the rules (3), (4), and (5), that if the left hand side is an element of \( (X')^* \), then all the monomials on the right hand side are also elements of \( (X')^* \). Moreover, if \( k + l \leq p^m - 1 \) then the same is true for the rewriting rules (1), (2), and (5). Suppose \( k, l \leq p^m - 1 \) and \( k + l \geq p^m \). Then \( \binom{k+l}{k} = 0 \) in \( \mathbb{K} \). In fact, the degree of \( p \) in the prime decomposition of \( n! \) is given by the formula
\[
\sum_{j=0}^{\infty} \left[ \frac{n}{p^j} \right].
\]

Therefore, the degree of \( p \) in the prime decomposition of \( \binom{k+l}{k} \) is
\[
\sum_{j=0}^{\infty} \left( \left\lfloor \frac{k+l}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{l}{p^j} \right\rfloor \right) = \left\lfloor \frac{k+l}{p^m} \right\rfloor + \sum_{j=0}^{l-1} \left( \left\lfloor \frac{k+l}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{l}{p^j} \right\rfloor \right) \geq \left\lfloor \frac{k+l}{p^m} \right\rfloor = 1 > 0.
\]

Therefore, for the rules (1), (2), (5) and \( k + l \geq p^m \), we get
\[
e^{(k)}_\alpha e^{(l)}_\alpha \to 0 \quad e^{(k)}_{\alpha+\beta} e^{(l)}_{\alpha+\beta} \to 0 \quad e^{(k)}_\beta e^{(l)}_\beta \to 0.
\]
This shows that $R(X')$ is complete. Now, it is obvious that $B'$ is the set of non-reducible monomials in the alphabet $X'$ with respect to the rewriting system $R(X')$. This shows that $B'$ is a basis of the algebra $A'$.

\[\square\]

5 Small Gröbner basis

The Gröbner basis obtained in the previous section is not convenient for the construction of minimal projective resolution of $\mathbb{K}$, since the Anick resolution is much closer to the minimal resolution, if the chosen generating set is minimal.

Denote $e^{(p^k)}_\alpha$ by $a_k$ and $e^{(p^k)}_\beta$ by $b_k$.

**Theorem 3.** For any $m \in \mathbb{N}_0$ the set $Z_m := \{a_l, b_l \mid l \leq m - 1\}$ generates the algebra $\mathfrak{U}_m^m(\mathbb{K})$. And, therefore, the set $Z := \{a_l, b_l \mid l \in \mathbb{N}_0\}$ generates $\mathfrak{U}_3^m(\mathbb{K})$.

**Proof.** We know that $U^m_3(\mathbb{K})$ is generated by the elements $e^{(k)}_\alpha$, $e^{(k)}_{\alpha+\beta}$, $e^{(k)}_\beta$, $k \leq p^m - 1$. Thus it is enough to show that these elements can be written as linear combination of monomials in $a_l, b_l, l \leq m - 1$.

Suppose $k = k_0 p^{m-1} + k_{-1} p^{m-2} + \cdots + k_0$ with $0 \leq k_s \leq p - 1$. Then it follows from the Lucas’ theorem [3 (137)] and (1), (2), (3), that

\[
e^{(k)}_\alpha = \prod_{s=0}^{m-1} e^{(k_s p^s)}_\alpha, \quad e^{(k)}_{\alpha+\beta} = \prod_{s=0}^{m-1} e^{(k_s p^s)}_{\alpha+\beta}, \quad e^{(k)}_\beta = \prod_{s=0}^{m-1} e^{(k_s p^s)}_\beta.
\]

Now, for any $0 \leq k_s \leq p - 1$ the integer $k_s!$ is invertible in $\mathbb{K}$. Therefore

\[
e^{(k_s p^s)}_\alpha = \frac{1}{k_s!} \left( e^{(p^s)}_\alpha \right)^{k_s}, \quad e^{(k_s p^s)}_{\alpha+\beta} = \frac{1}{k_s!} \left( e^{(p^s)}_{\alpha+\beta} \right)^{k_s}, \quad e^{(k_s p^s)}_\beta = \frac{1}{k_s!} \left( e^{(p^s)}_\beta \right)^{k_s}.
\]

Thus the algebra $\mathfrak{U}_m^m(\mathbb{K})$ is generated by the elements $a_l, b_l$, and $e^{(p^l)}_{\alpha+\beta}, l \leq m - 1$.

From (5), it follows that

\[
e^{(p^l)}_{\alpha+\beta} = (-1)^{p^l} \left( e^{(p^l)}_\beta e^{(p^l)}_\alpha - \sum_{j=0}^{p^l-1} (-1)^j e^{(p^l-j)}_\alpha e^{(p^l-j)}_{\alpha+\beta} e^{(p^l-j)}_\beta \right).
\]

From this equality by recursion on $l$, it follows that $e^{(p^l)}_{\alpha+\beta}$ can be written as a linear combination of monomials in $a_s, b_s$ with $s \leq l$.

We will consider Deg-lexicographical order on $Z_m$ that corresponds to the ordering

\[a_0 < b_0 < a_1 < b_1 < \cdots < a_m < b_m.\]

on $Z_m$. To establish the Gröbner basis of $\mathfrak{U}_m^m(\mathbb{K})$ for the generating set $Z_m$ with respect to the above ordering, we prove some equalities between the elements $a_k, b_k, k \in \mathbb{N}_0$.

**Proposition 3.** For any $k$ we have $a_k^p = b_k^p = 0$. 

8
Proposition 5. For any \( a_k \) and \( b_k \) of Proposition 4.

Proof. We know that \( a_k \) is an element of the subalgebra \( \mathbb{U}_3^{k+1}(\mathbb{K}) \), and that \( a_k \) is a linear multiple of \( a_{k+1} \). Since \( a_{k+1} \notin \mathbb{U}_3^{k+1}(\mathbb{K}) \), it follows that the coefficient of multiplication is zero, and therefore \( a_k = 0 \). The claim \( b_k = 0 \) is proved in the same way.

Proposition 4. For any \( l \) and \( k \) elements \( a_l \) and \( a_k \) commutes. Similarly \( b_l \) and \( b_k \).

Proof. Obvious.

Proposition 5. For any \( l > k \) we have
\[
a_l b_k - b_k a_l + (-1)^{l-k} a_l^{p-1} b_k a_k a_{k+1} \cdots a_l^{p-1} = 0 \quad (7)
b_l a_k - a_k b_l + (-1)^{l-k} b_l a_k b_{k+1} \cdots b_l^{p-1} = 0 \quad (8)
\]
in \( \mathbb{U}_3^{l+1}(\mathbb{K}) \).

Proof. First we note that \( a_k^{p-1} = -e_{\alpha}((p-1)p^k) \). In fact, \( a_k^{p-1} = (p-1)!e_{\alpha}((p-1)p^k) \).

Now \( (p-1)! \) is the product of all elements in \( \mathbb{P}_p^* \). The elements of \( \mathbb{P}_p^* \setminus \{1, -1\} \) can be grouped in pairs \( \{\lambda, \lambda^{-1}\} \) with \( \lambda \neq \lambda^{-1} \). Therefore the product \( (p-1)! \) equals to \( 1 \cdot -1 = -1 \).

By (5), we get
\[
b_k a_l = e_{\beta}^{(p^k)} e_{\alpha}^{(p^l)} = e_{\alpha}^{(p^l)} e_{\beta}^{(p^k)} + \sum_{j=1}^{p^k} (-1)^j e_{\alpha}^{(p^l-j)} e_{\alpha+\beta}^{(p^k-j)} e_{\beta}^{(p^k-j)}
\]
\[= a_l b_k + \sum_{j=1}^{p^k} (-1)^j e_{\alpha}^{(p^l-j)} e_{\alpha+\beta}^{(p^k-j)} e_{\beta}^{(p^k-j)}.
\]

For \( 1 \leq p^k \), we have
\[
p^l - j = (p-1)p^{l-1} + (p-1)p^{l-2} + \cdots + (p-1)p^k + (p^k - j), \quad (9)
\]
where \( p^k - j \leq p^k - 1 \). Therefore from (11) and the Lucas’ theorem, it follows that \( e_{\alpha}^{(p^l-j)} e_{\alpha+\beta}^{(p^k-j)} = e_{\alpha}^{((p-1)p^k)} e_{\alpha+\beta}^{(p^k-j)} \cdot (p^{p-1}+p^k-j) \). Moreover, \( (p^l)_{(p^l)} = 0 \) and \( e_{\alpha}^{((p-1)p^k)} e_{\alpha+\beta}^{(p^k+1-p^k)} = 0 \). Therefore
\[
\sum_{j=1}^{p^k} (-1)^j e_{\alpha}^{(p^l-j)} e_{\alpha+\beta}^{(p^k-j)} = e_{\alpha}^{((p-1)p^k)} \sum_{j=0}^{p^k} (-1)^j e_{\alpha}^{(p^l-p^k+1+1)} e_{\alpha+\beta}^{(p-j)} e_{\beta}^{(p^k-j)}
\]
\[= e_{\alpha}^{((p-1)p^k)} \left( e_{\beta}^{(p^k)} e_{\alpha}^{(p^l-p^k+1)} \right),
\]
where in the last step we used (2). Now
\[
p^l - p^{k+1} + p^k = (p-1)p^{l-1} + \cdots + (p-1)p^{k+1} + p^k.
\]
Therefore by the Lucas’ theorem and (11)
\[ e_{\alpha}^{(p^k-p^{k+1}+p^j)} = e_{\alpha}^{(p^k)}e_{\alpha}^{((p-1)p^{k+1})} \ldots e_{\alpha}^{((p-1)p^j-1)} \]
\[ = (-1)^{j-k-1} a_k a_{k+1}^{p-1} \ldots a_{j-1}^{p-1}. \]

Finally we get
\[ b_k a_l = a_l b_k + (-1)^{j-k-1} a_k a_{k+1}^{p-1} \ldots a_{j-1}^{p-1}. \]

Now we prove (8). We have by (5)
\[ e_{\alpha}^{(p^j-1)(p^{j-1})} = e_{\alpha}^{(p^j)}e_{\alpha}^{(p^{j-1})} + \sum_{j=1}^{p^k} (-1)^j e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k)} e_{\beta}^{(p^j-k)}. \]

From (10), the Lucas’ theorem and (8), we get
\[ e_{\beta}^{(p^j-k)} = e_{\beta}^{(p^j)}e_{\beta}^{((p-1)p^j-1)} \]

Taking into the account that \( b_{s}^{p-1} = -e_{\beta}^{((p-1)p^s)}, \) for all \( s \in \mathbb{N}_0, \) and
\[ e_{\beta}^{(p^j-k+1+p^s)} e_{\beta}^{(p^s+1+p^s)} = 0, \]

we get
\[ \sum_{j=1}^{p^k} (-1)^j e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k)} e_{\beta}^{(p^j-k)} = \left( \sum_{j=0}^{p^k} e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k)} e_{\beta}^{(p^j-k)} \right) (-1)^{j-k-1} b_k b_{k+1}^{p-1} \ldots b_{j-1}^{p-1} \]
\[ = (-1)^{j-k-1} b_k a_k b_{k+1}^{p-1} \ldots b_{j-1}^{p-1} \]
and (8) follows.

**Proposition 6.** For all \( k \in \mathbb{N}_0, \) we have \((b_k a_k)^p = (a_k b_k)^p.\)

**Proof.** We have
\[ (b_k a_k)^p = \left( a_k b_k + \sum_{j=1}^{p^k-1} e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k)} e_{\beta}^{(p^j-k)} + (-1)^p e_{\alpha+\beta}^{(p^k)} \right)^p. \]

Let \( 1 \leq j \leq p^k - 1 \) and \( 1 \leq s \leq p^k - 1. \) Then
\[ e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k)} e_{\beta}^{(p^j-k)} = \min(p^j-k, p^j-k) \]
\[ = \sum_{r=0}^{\min(p^j-k, p^j-k)} (-1)^r e_{\alpha}^{(p^j-k)} e_{\alpha+p}^{(p^j-k-r)} e_{\beta}^{(p^j-k-r)} e_{\beta}^{(p^j-k-s)} e_{\beta}^{(p^j-k-s)}. \]
Every monom on the right hand side of the last formula has the form
\[ e_{(k_1)} e_{(k_2)} e_{(k_3)} e_{(k_4)} e_{(k_5)} e_{(k_6)} e_{(k_7)} \]
with \( 0 \leq k_i \leq p^k - 1 \) for all \( 1 \leq i \leq 7 \). In particular, every such monom is an element of \( \Omega_5^k(K) \). Moreover, we have equations
\[
\begin{align*}
    k_1 + k_3 &= p^k \\
    k_2 + k_4 + k_5 &= p^k \\
    k_3 + k_4 + k_6 &= p^k \\
    k_5 &= p^k.
\end{align*}
\]
Thus \( k_1 + k_2 + 2(k_3 + k_4 + k_5) + k_6 + k_7 = 4p^k \). Therefore at least one of the following inequalities holds
\[
\begin{align*}
    k_1 + k_2 &\geq p^k \\
    k_3 + k_4 + k_5 &\geq p^k \\
    k_6 + k_7 &\geq p^k.
\end{align*}
\]
Since the elements with divided power greater or equal then \( p^k \) do not lie in \( \Omega_5^k(K) \), we get that one of the products
\[ e_{(k_1)} e_{(k_2)} e_{(k_3)} e_{(k_4)} e_{(k_5)} e_{(k_6)} e_{(k_7)} \]
is zero. This shows that
\[ e_{(p^k - j)} e_{(j)} e_{(p^k - j)} e_{(j)} e_{(p^k - s)} e_{(s)} e_{(p^k - s)} = 0. \]
In particular, any two elements in the sum \( \sum_{j=1}^{p^k - 1} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} e_{(j)} e_{(p^k - s)} e_{(s)} e_{(p^k - s)} \) commute.

Now consider for \( 1 \leq j \leq p^k - 1 \) the product
\[
\begin{align*}
    e_{(p^k)} e_{(p^k)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} &= e_{(p^k)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} \\
    &\quad + \sum_{r=1}^{p^k - j} (-1)^r e_{(p^k)} e_{(p^k - j - r)} e_{(j)} e_{(p^k - r)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)} e_{(j)} e_{(p^k - j)}.
\end{align*}
\]
Every monom on the right hand side for \( 1 \leq r \leq p^k - 1 \) can be written in the form
\[ e_{(p^k)} e_{(k_1)} e_{(k_2)} e_{(k_3)} e_{(k_4)} e_{(k_5)} e_{(k_6)} e_{(k_7)}, \]
where \( 1 \leq k_i \leq p^k - 1 \) for \( 2 \leq i \leq 5 \). Moreover, we have
\[
\begin{align*}
    k_2 + k_3 &= p^k \\
    k_3 + k_5 &= p^k.
\end{align*}
\]
which implies \(k_2 + k_3 + k_4 + k_5 = 2p^k\) and, therefore, \(k_2 + k_3 \geq p^k\) or \(k_4 + k_5 \geq p^k\). By the same consideration as above, we get that \(e_{\alpha+\beta}^{(k_2)} e_{\alpha+\beta}^{(k_3)} = 0\) or \(e_{\beta}^{(k_4)} e_{\beta}^{(k_5)} = 0\), and thus
\[
e_{\alpha}^{(p^k)} e_{\alpha}^{(k_2)} e_{\alpha+\beta} e_{\alpha}^{(k_3)} e_{\beta}^{(k_4)} e_{\beta}^{(k_5)} = 0.
\]
Therefore
\[
e_{\alpha}^{(p^k)} e_{\beta}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} = e_{\alpha}^{(p^k)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} .
\]
Similarly, it can be shown that for \(1 \leq j \leq p^k - 1\)
\[
e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} .
\]
Thus \(e_{\alpha}^{(p^k)} e_{\beta}^{(p^k)}\) commute with every summand of
\[
\sum_{j=1}^{p^k - 1} (-1)^j e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} .
\]
It is obvious, that \(e_{\alpha+\beta}^{(p^k)}\) commutes with \(e_{\alpha}^{(p^k)} e_{\beta}^{(p^k)}\) and with every summand of
\[
\sum_{j=1}^{p^k - 1} (-1)^j e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} .
\]
Therefore
\[
\left( a_k b_k + \sum_{j=1}^{p^k - 1} e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} + (-1)^{p^k} e_{\alpha+\beta}^{(p^k)} \right)^p = \\
= (a_k b_k)^p + \sum_{j=1}^{p^k - 1} (-1)^{p^k} \left( e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(p^k - j)} \right)^p + \left( e_{\alpha+\beta}^{(p^k)} \right)^p = (a_k b_k)^p .
\]
\[\square\]

**Proposition 7.** If \(\text{char} K \geq 3\), then for any \(k \in \mathbb{N}_0\), we have
\[
b_k^2 a_k - 2b_k a_k b_k + a_k b_k^2 = 0 \\
b_k a_k^2 - 2a_k b_k a_k + a_k^2 b_k = 0.
\]

**Proof.** We have
\[
b_k^2 a_k = \binom{2p^k}{p^k} e_{\beta}^{(p^k)} e_{\alpha}^{(p^k)} = 2 \sum_{j=0}^{p^k} e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(2p^k - j)} = 2 e_{\alpha}^{(p^k)} e_{\beta}^{(2p^k)} + 2 \sum_{j=1}^{p^k} e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(2p^k - j)}
\]
\[
a_k b_k^2 + 2 \sum_{j=1}^{p^k} e_{\alpha}^{(p^k - j)} e_{\alpha+\beta} e_{\beta}^{(2p^k - j)}
\]

12
On the other hand

\[ b_k a_k b_k = e^{(p^k)} (p^k) e^{(p^k)} = \sum_{j=0}^{p^k} e^{(p^k-j)} e^{(p^k-j)} e^{(p^k)}. \]

For \( j = 0 \), we have \( e^{(p^k-j)} (p^k) = b_k^2 \). On the other hand, for \( 1 \leq j \leq p^k \), we have \( 2p^k = p^k + (p^k - j) \), and \( 0 \leq p^k - j \leq p^k - 1 \). Therefore, from the Lucas' theorem it follows that \( (2p^k-j) = 1 \), and \( e^{(p^k-j)} e^{(p^k)} = e^{(2p^k-j)} \). Thus

\[ b_k a_k b_k = a_k b_k^2 + \sum_{j=1}^{p^k} e^{(p^k-j)} e^{(p^k-j)} e^{(2p^k-j)} \]

and

\[ b_k^2 a_k - 2b_k a_k b_k = a_k b_k^2 \]

The second equality follows from the first one, after noticing that \( e^{(k)} \rightarrow e^{(k)} \), \( e^{(k)} \rightarrow e^{(k)} \), \( e^{(k)} \rightarrow (-1)^k e^{(k)} \) can be prolonged to an automorphism of \( \mathfrak{U}^n_m \).

Denote by \( \pi_m \) the natural projection \( \mathbb{K} \langle Z_m^* \rangle \rightarrow \mathfrak{U}^n_m \).

**Proposition 8.** Suppose \( \text{char} \mathbb{K} = p \). The following set \( G_m \) of elements in \( \mathbb{K} \langle X_m^* \rangle \)

\[
\begin{align*}
a b_k - b_k a_l + (-1)^{l-k} a_k^{p-1} a_k a_k^{p-1} \ldots a_k^{p-1}, & \quad 0 \leq k < l \leq m - 1 \\
b_l a_k - a_k b_l - (-1)^{l-k} b_k a_k b_{k+1} \ldots b_{l-1}, & \quad 0 \leq k < l \leq m - 1 \\
a_1 a_k + a_k a_l, & \quad 0 \leq k < l \leq m - 1 \\
b_l b_k + b_k b_l, & \quad 0 \leq k < l \leq m - 1 \\
(b_k a_k)^p - (a_k b_k)^p, & \quad 0 \leq k \leq m - 1 \\
a_k^p, & \quad 0 \leq k \leq m - 1 \\
b_k^p, & \quad 0 \leq k \leq m - 1 \\
b_k^2 a_k - 2b_k a_k b_k + a_k b_k^2, & \quad 0 \leq k \leq m - 1, \ p \geq 3, \\
b_k a_k^2 - 2a_k b_k a_k + a_k b_k^2, & \quad 0 \leq k \leq m - 1, \ p \geq 3
\end{align*}
\]

is a reduced Gröbner basis of \( \text{ker}(\pi_m) \).

**Proof.** If follows from Propositions \( 3, 4, 5, 6 \) that \( G_m \) is a subset of \( \text{ker}(\pi) \). Thus it is enough to show that the images of non-reducible monomials in \( X_m^* \) give a basis of \( \mathfrak{U}^n_m \). Since the images of non-reducible monomials in \( X_m^* \) generate \( \mathfrak{U}^n_m \) as a vector space and \( \mathfrak{U}^n_m \) is finite dimensional, it is enough to show that the number of non-reducible monomials in \( X_m^* \) with respect to \( G_m \) is less or equal to the dimension of \( \mathfrak{U}^n_m \). From Corollary it follows that the dimension of \( \mathfrak{U}^n_m \) is \( (p^m)^3 = p^{3m} \).
Let $t$ be a monomial non-reducible with respect to $G_m$. Since $t$ does not contain submonomials $a_ib_k, \ ia_ka_k, \ ib_k$ for $0 \leq k < l \leq m-1$, the indices of variables in $t$ weakly increase from the left to right. We denote by $t_k$ a submonomial of $t$ that consists from the all variables with index $k$. Then $t = t_0t_1 \ldots t_{m-1}$.

For $0 \leq k \leq m-1$, the monomial $t_k$ has the form

$$a_k^{i_1}b_k^{s_1} \ldots a_k^{i_n}b_k^{s_n},$$

where $n \in \mathbb{N}$, $r_i$ and $s_i$ are non-zero natural numbers, except probably of $r_1$ and $s_n$. Since $t_k$ does not contain submonomials $b_k^2a_k$ and $b_k^2a_k^2$, we see that $s_1 = r_2 = \cdots = s_{n-1} = r_n = 1$. Thus $t_k$ has the form $a_k^{i_1}(b_kak_k)^{n-1} b_k^{s_n}$. As $t_k$ does not contain subwords $a_k^{i_1}$, $(b_kak_k)^{n-1}$, and $b_k^{s_n}$, we get that $r_1 \leq p-1$, $n-1 \leq p-1$, and $s_n \leq p-1$. Thus the number of different possibilities for $t_k$ does not exceed $p^3$, and the number of different possibilities for $t$ does not exceed $(p^3)^m = p^{3m}$. \(\square\)

**Corollary 2.** The set $G$ is a reduced Gröbner basis of $\ker(\pi)$, where $\pi$ is the natural projection $\mathbb{K}(\langle X^* \rangle) \to \mathbb{U}_1^m(\mathbb{K})$.

**Proof.** It is clear that $G \subset \ker(\pi)$. Denote by $R$ the rewriting system $\{ r(p) \mid p \in G \}$. It is enough to show that any critical pair $(w, r_1, r_2)$, with $r_1, r_2 \in R$ is reducible. For a given critical pair $(w, r_1, r_2)$ there is an $m \geq 0$, such that all monomials in $w, r_1, r_2$ lie in $Z_m$. By Proposition the set $G_m$ is a Gröbner basis, therefore any critical pair $(w, r_1, r_2)$ with $w \in Z_m^*, \ r_1, r_2 \in R_m = \{ r(p) \mid p \in G_m \}$ is reducible. \(\square\)

### 6 Anick resolution

The Anick resolution was introduced in [?]. Let $A$ be an algebra over a field $\mathbb{K}$ and $\varepsilon : A \to \mathbb{K}$ a homomorphism of algebras. Let $X = a_1, \ldots$ be a set of generators of $A$ and $G \subset \mathbb{K}(\langle X^* \rangle)$ a reduced Gröbner basis with respect to a monomial ordering $\leq$ on $X^*$. For this set of data Anick constructed a free resolution of $\mathbb{K}$ over $A$, which is nowadays called Anick resolution. We will describe only the first four steps of Anick’s construction under additional assumption that $\varepsilon(x) = 0$ for all $x \in X$.

First we define sets $T_k$, $k = -1, 0, 1, 2$, that will serve as bases of $A$-free modules $P_k$. Denote by $T_{-1}$ the set $\{e\}$ with one element $e$ and by $T_0$ the set $X$. The set $T_1$ is the set of all leading monomials in $G$. Denote by $\bar{T}_2$ the set of all possible overlaps of elements of $T_1$. Every element of $\bar{T}_2$ is a triple $(w, r_1, r_2)$. We say that an overlap $(w, r_1, r_2)$ is minimal if there is no overlap $(w', r_1', r_2')$ such that $w'$ is a subword of $w$. Note that if an overlap $(w, r_1, r_2)$ is minimal then the rules $r_1$ and $r_2$ are uniquely determined by $w$. In fact, suppose that $(w, r_1, r_2) \in \bar{T}_2$. Then $w = m_1v = m'_1v'$. But this means that $m_1$ is a subword of $m'_1$ or $m'_1$ is a subword of $m_1$. Since $G$ is a reduced Gröbner basis it follows that $r_1 = r_1'$. Similarly $r_2 = r_2'$. 

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14
Define $T_2$ to be the set of all words $w$ in $X^*$ such that there exists a minimal overlap $(w, r_1, r_2)$. Denote for $k = -1, 0, 1, 2$ by $P_k$ the $A$-linear span of $T_k$. Let $M$ be the set of all non-reducible monomials with respect to $G$. Then for $k = -1, 0, 1, 2$ the set

$$N_k = \{m.t \mid m \in M, \ t \in T_k\}$$

is the basis of $P_k$ over $K$.

The sets $N_k$ have a full ordering induced by the ordering $\leq$ on $X^*$ via the map $m.t \mapsto mt$. We define maps $\delta_n: P_n \to P_{n-1}$ and $j_n: P_{n-1} \to P_n$ as follows

$$\delta_0(m.x) := NF(mx,G).e$$
$$j_0(u.x.e) := u.x$$
$$\delta_1(m.t) := NF(mt',G).x, \text{ where } t = t'.x.$$ 

Now let $m \in M$ and $x \in X$. Suppose there are $u, v \in M$ such that $m = uv$ and $vx \in T_1$. Then we define $j_1(m.x) = u.vx$. Otherwise we let $j_1(m.x) = 0$. Note that $j_1$ is well-defined as $m = uv = u'v'$ would imply that $v \preceq v'$ or $v' \preceq v$ and therefore $vx \preceq v'x$ or $v'x \preceq v'x$. But since $G$ is reduced Gröbner basis any two different elements of $T_1$ are incompatible with respect to $\preceq$ (in other words $T_1$ is an anti-chain in the Anick’s terminology).

Let $w \in T_2$ be such that $w = m_1v = um_2$ with $m_1, m_2 \in T_1$. Define $\delta_2(m.u) = NF(mu,G).m_2$.

Suppose $t \in T_1$ and $m \in M$. If $m = uv$ for some $u, v \in M$ such that $vt \in T_2$ then we define $j_2(m.t) = u.vt$. Note that if such $u$ and $v$ exist then they are unique as $G$ is a reduced Gröbner basis. If there is no $u$ and $v$ with the above property then we let $j_2(m.t) = 0$.

Note, that if $A$ is an $S$-graded algebra, where $S$ is a monoid, then the maps $j_n$ and $\delta_n$ are homomogeneous with respect to the induced grading on the modules $P_k$.

Now we define homomorphisms of left $A$-modules $d_n: P_n \to P_{n-1}$ and homomorphisms of $K$-vector spaces $i_n: \ker(d_{n-1}) \to P_n$ for $n = 0, 1, 2$ by induction.

Since $d_n$ is a homomorphism of free $A$-modules it is enough to define $d_n$ on the basis elements $t$, where $t \in T_n$. On the other hand $i_n$ is a homomorphism of $K$ vector spaces, moreover we do not have any convenient basis for $\ker(d_{n-1})$. We will define $i_n$ by induction on the leading term of $f \in \ker(d_{n-1})$.

$$d_0(t) := \delta_0(t)$$
$$i_0(m.e) := j_0(m.e)$$
$$d_{n+1}(t) := \delta_{n+1}(t) - i_n d_n(\delta_{n+1}(t))$$
$$i_n(f) := j_n(lt(f)) + i_n(f - d_n(j_n(lt(f))))$$

Note that it is not obvious that $d_n$ and $i_n$ are well-defined. This a part of the claim of Proposition [2]. The following proposition is proved in [?]. Note that Anick [?] constructed modules $P_n$ and maps $d_n$ for all $n \in \mathbb{N}$.
Proposition 9. The sequence of left $A$-modules

$$P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{\varepsilon} \mathbb{K} \rightarrow 0$$

is an exact complex. If $A$ is an $S$-graded algebra, then all the maps above are homogeneous. Moreover, the highest term in the expansion $d_n (.)$ is $\delta_n (.)$ for every $n$ and $t \in T_n$.

Let $j < m$. We will consider the algebra $\Omega^{km}_1 (\mathbb{K})$ with the generating set $X_{jm}$ and the Gröbner basis $G_{jm}$. The sets $T_1$ and $T_2$ in this case are given by

$$T_1 = \{ a_l b_k, b_l a_k, a_l a_k, b_l b_k \mid j \leq k \leq l \leq m - 1 \}$$

$$\cup \{ a^p_k, b^p_k, (b_k a_k)^p \mid j \leq k \leq m - 1 \}$$

$$\cup \{ b^2_k a_k, b_k a^2_k \mid j \leq k \leq m - 1, p \geq 3 \}$$

$$T_2 = \{ a_l a_k, a_l a_k, a_l a_k, a_l b_k, a_l b_k, b_l a_k, b_l b_k \mid j \leq k < l \leq m - 1 \}$$

$$\cup \{ a^p_k, a^p_l, j \leq k \leq l \leq m - 1 \}$$

$$\cup \{ b^p_k a_k, b^p_l a_k, b^p_k b_k, b^p_l b_k, (b_l a_k)^p a_k, (b_l a_k)^p b_k \mid j \leq k \leq l \leq m - 1 \}$$

$$\cup \{ b^2_k a_k, b^2_l a_k, b^2_k b_k, b^2_l b_k, a_k, b_k, a^2_k, b^2_k \mid j \leq k \leq m - 1, p \geq 3 \}$$

In the tables below we list the values of $\deg w$ for the elements of $T_1$ and $T_2$.

| $w \in T_1$ | $\deg (w)$ |
|-------------|-------------|
| $a_l b_k$   | $p^l \alpha + p^k \beta$ |
| $b_l a_k$   | $p^k \alpha + p^k \beta$ |
| $a_l a_k$   | $(p^l + p^k) \alpha$ |
| $b_l b_k$   | $(p^l + p^k) \beta$ |

| $w \in T_2$ | $\deg (w)$ |
|-------------|-------------|
| $a_l a_k$   | $(p^l + p^k + p^k) \alpha$ |
| $a_l b_k$   | $(p^l + p^k) \alpha + p^l \beta$ |
| $b_l b_k$   | $p^l \alpha + (p^l + p^k) \beta$ |

For $j \leq k < l \leq m - 1$

| $w \in T_1$ | $\deg (w)$ |
|-------------|-------------|
| $a_l a_k$   | $(p^l + p^k) \alpha + p^k \beta$ |
| $a_l b_k$   | $(p^l + p^k) \alpha + p^l \beta$ |
| $b_l a_k$   | $p^l \alpha + (p^l + p^k) \beta$ |
| $b_l b_k$   | $(p^l + p^k + p^k) \beta$ |

| $w \in T_2$ | $\deg (w)$ |
|-------------|-------------|
| $a_l a_k$   | $(p^l + p^k + p^k) \alpha$ |
| $a_l b_k$   | $(p^l + p^k) \alpha + p^l \beta$ |
| $b_l a_k$   | $p^l \alpha + (p^l + p^k) \beta$ |
| $b_l b_k$   | $(p^l + p^k + p^k) \beta$ |
For \( j \leq k < l \leq m - 1 \)

\[
\begin{array}{c|c|c}
    w \in T_2 & \deg(w) & w \in T_2 \\
    \hline
    a_1a_k^l & (p^l + p^{l+1})\alpha & a_1a_k^l & (p^l + p^k)\alpha \\
    a_1b_k^p & p^l\alpha + p^{l+1}\beta & a_1b_k^p & (p^l + p^k)\alpha + p^l\beta \\
    a_1(b_ka_k)^p & (p^l + p^{k+1})\alpha + p^l\beta & b_1b_k^p & (p^l + p^k)\alpha + p^l\beta \\
    b_1a_k^p & p^{k+1}\alpha + p^l\beta & b_1a_k^p & (p^l + p^k)\alpha + p^l\beta \\
    b_1(b_ka_k)^p & p^{k+1}\alpha + (p^l + p^k)\beta & b_1(b_ka_k)^p & (p^l + p^k)\alpha + (p^l + p^k)\beta \\
\end{array}
\]

For \( j \leq k < l \leq m - 1 \)

\[
\begin{array}{c|c|c}
    w \in T_2 & \deg(w) & w \in T_2 \\
    \hline
    a_k^{p+1} & (p^{k+1} + p^k)\alpha & b_k(b_ka_k)^p & p^{k+1}\alpha + (p^l + p^k)\beta \\
    b_k^{p+1} & (p^{k+1} + p^k)\beta & (b_ka_k)^{p+1} & (p^{k+1} + p^k)\beta \\
\end{array}
\]

For \( p \geq 3 \)

\[
\begin{array}{c|c|c|c}
    w \in T_2 & \deg(w) & w \in T_2 & \deg(w) \\
    \hline
    a_1b_ka_k & (p^l + p^k)\alpha + 2p^l\beta & b_1a_ka_k & (p^l + p^k)\alpha + 2p^l\beta \\
    a_1b_ka_k^2 & (p^l + 2p^k)\alpha + p^k\beta & b_1a_ka_k & (p^l + 2p^k)\beta \\
    b_1b_ka_k & p^l\alpha + (2p^k + p^l)\beta & b_1a_ka_k & (2p^l + p^k)\alpha + p^l\beta \\
    b_1b_ka_k^2 & 2p^l\alpha + (p^l + p^k)\beta & b_1b_ka_k^2 & 2p^l\alpha + (p^l + p^k)\beta \\
    b_1b_ka_k^3 & 2p^l\alpha + 2p^k\beta & b_1b_ka_k^3 & 2p^l\alpha + 2p^k\beta \\
\end{array}
\]

Analizing these tables we get

**Proposition 10.** The set \( W \) of pairs \((w_1, w_2) \in T_1 \times T_2\) such that \( \deg(w_1) = \deg(w_2) \) is given by

\[
\left\{ \begin{array}{l}
    (a_1b_ka_k, a_1b_ka_k^p), (a_1b_ka_k, b_1a_k, a_1a_k^p), (b_1a_k, b_1b_ka_k^p), \\
    ((b_ka_k)^p, b_1a_k, b_1b_ka_k^p), (a_1b_ka_k, a_1a_k^p), (a_1a_k, a_1a_k^p), \\
    (b_1a_1b_ka_k, b_1b_ka_k^p), (a_1a_1b_ka_k, a_1a_1a_ka_k^p), (b_1b_ka_k, b_1b_ka_k^p), \\
    (b_ka_k, a_1b_ka_k^2), (b_ka_k, b_1a_1b_ka_k^2), (a_1a_1a_1a_ka_k^2), (b_1b_ka_k, b_1b_ka_k^2) \\
\end{array} \right\}
\]

\[
\cup \left\{ (a_ka_k^2, a_ka_k^2), (b_ka_k^2, b_ka_k^2), (a_1a_1a_1b_1a_1(b_1a_1a_1a_1a_1)^2), (b_1b_1b_1b_1(b_1b_1a_1a_1a_1a_1a_1)^2) \right\} \mid p = 2 \}
\]

### 7 Minimal resolution

Let \( \Gamma \) be a monoid, and \( A \) an \( \Gamma \)-graded algebra over a field \( K \). For \( \gamma \in \Gamma \) we denote by \( A[\gamma] \) the left \( \Gamma \)-graded \( A \)-module defined by

\[
A[\gamma]_\alpha := \bigoplus_{\beta \in \Gamma: \beta \gamma = \alpha} A_\beta.
\]

The modules \( A[\gamma] \) are projective modules in the category of \( S \)-graded \( A \)-modules, and every projective module is a direct summand of a direct sum of modules of the form \( A[\gamma] \) (cf. [3] Propositions 4.1, 5.1).
Let $M$ be a $\Gamma$-graded $A$-module. A submodule $N$ of $M$ is called small if for any submodule $T$ of $M$ such that $T + N = M$, we have $T = M$. A projective cover of a module $M$ is a projective object $P$ together with an epimorphism $\psi: P \to M$ such that the kernel of $\psi$ is a small subobject of $P$. It is well known fact that if projective cover exists for the module $M$, the it is unique (cf. [3, Theorem 5.1]). It is proved in Proposition 5.2 of [3], that if $\Gamma$ is an artinian ordered monoid such that the neutral element $0$ of $\Gamma$ is the minimal element, and $A_0$ is a perfect ring, then every $\Gamma$-graded $A$-module possess a projective cover. Note that the conditions of this criterion hold for $\Upsilon_3^+(K)$ considered as an $S$-graded algebra, where $S$ is the free commutative monoid generated by $\alpha$ and $\beta$. In fact, $S$ is artinian with respect to the lexicographical ordering, and $0$ is the minimal element with respect to this ordering. Moreover, the 0-th component of $\Upsilon_3^+(K)$ is isomorphic to $K$ and thus is perfect.

A projective resolution

$$\cdots \to P_k \xrightarrow{d_k} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} M \to 0$$

of a $\Gamma$-graded $A$-module $M$ is called minimal if the kernel of $d_j$ is a small subobject of $P_j$, $j \geq -1$, or, equivalently, if the image of $d_i$ is a small subobject of $P_{i+1}$ for $i \geq 0$. From the uniqueness of the projective cover, it follows that the minimal resolution is unique up to isomorphism if it exists. Moreover, if every $\Gamma$-graded $A$-module possess a projective cover, then the existence of a minimal projective resolution for every $\Gamma$-graded $A$-module can be proved by induction.

Proposition 11. Let $(s_i)_{i \in I}$ be a family of elements in $S$. Denote by $N_i$ the direct sum $\bigoplus_{s \in S, s \neq 0} \Upsilon_3^+(K)[s_i]$. Then $N \subset \bigoplus_{i \in I} \Upsilon_3^+(K)[s_i]$ is a small submodule if and only if $M \subset \bigoplus_{i \in I} N_i$.

We will denote the submodule $\bigoplus_{i \in I} N_i$ of $P := \bigoplus_{i \in I} \Upsilon_3^+(K)[s_i]$, with $N_i$ defined as above, by $\text{Rad}(P)$.

Corollary 3. The Anick resolution of $\Upsilon_3^+(K)$ with respect to the Gröbner basis $G$ gives two first steps of the minimal resolution for the trivial module $K$.

Proof. We have to check that the images of $d_1$ and $d_0$ are small subobjects of $P_0$ and $P_{-1}$, respectively. For $d_0$, this is obvious, as $d_0(x) = x.c$ for any $x \in X$, and $x.c \in \Upsilon_3^+(K)|_{\text{deg}(x)}$, $\text{deg}(x) \neq 0$.

Now, let $w \in T_1$. We write $w$ in the form $u.x$, where $x \in X$. Then

$$d_1(w) = u.x - i(NF(w, G)).$$

Since, $NF(w, G)$ does not contain the terms of length one, we see that $i(NF(w, G))$ is an element of $\bigoplus_{x \in X} \bigoplus_{y \neq 0} \Upsilon_3^+(K)[\text{deg}(y)]$.

Unfortunately, the third step of the Anick resolution, in our situation, does not give the third step of the minimal resolution, since $P_1$ contains redundant summands. To find out which summands of $P_1$ should be skipped, we will analyze the differential $d_2: P_2 \to P_1$. 

18
First of all, note that if \( w \in T_2 \) is such that \( \deg(w) \) is different from the degrees of elements in \( T_1 \), then \( d_2(w) \in \text{Rad}(P_1) \).

Thus, the only elements \( w \in T_2 \), for which it can happen that \( d_2(w) \not\in \text{Rad}(P_1) \), are given by the second components of the pairs in \( W \) defined in Proposition 10.

Now, suppose \( w \in T_2 \) and \( u_1, \ldots, u_k \in T_1 \) is the full set of elements in \( T_1 \) such that \( \deg(w) = \deg(u) \). Then \( d_2(w) \) is an element of \( \text{Rad}(P_1) \) if and only if coefficients of \( u_1, \ldots, u_k \) in the expansion of \( d_2(w) \) are zero.

**Lemma 1.** Suppose \( w \in T_2 \) and \( u \in T_1 \) are such that \( \deg(w) = \deg(u) \) and \( u < w \). Then the coefficient of \( u \) in the expansion of \( d_2(w) \) is zero.

**Proof.** Recall that the \( K \)-basis of \( P_1 \) is given by \( B := \{ v', v'' \mid NF(v', G) = v', v'' \in T_1 \} \). This basis is ordered via the map \( B \to X^*, v'.v'' \to v'.v'' \). Now, write \( w \) in the form \( v'.v'' \) with \( v'' \in T_2 \) and \( v' \) non-reducible word in \( X^* \). Then the maximal element in the expansion of \( d_2(w) \) is \( \delta_2(w) = v'.v'' \). Since \( w < u \), we have \( v'.v'' < u \). Therefore, the coefficient of \( u \) in the expansion of \( d_2(w) \) is zero. \( \square \)

**Corollary 4.** The images of \( a_k b_k^{p-1} \), \( b_k a_k^p \) for \( l \geq k + 2 \), \( a_k a_k^l \), \( b_k b_k a_k \), \( b_k a_k^l \) for \( l \geq k + 1 \), \( a_k a_k^{p+1} \), \( b_k^{p+1} \) for any \( k \), and \( a_k a_k^2 \), \( b_k b_k^{p-1} \), \( \alpha_k \), \( \alpha_k (\alpha_k a_k-1)^2 \), \( \alpha_k (\alpha_k a_k-1)^2 \) for \( p = 2 \) under \( d_2 \) are the elements \( \text{Rad}(P_1) \).

Note that for the elements \( b_k b_k - a_k b_k \) and \( a_k b_k \) of \( T_2 \) there are two elements in \( T_1 \) of the same degree, namely \( a_k b_k \) and \( b_k a_k \). Since \( a_k b_k > a_k b_k \), it follows from Lemma 1 that the coefficient of \( a_k b_k \) in the expansion of \( d_2(a_k b_k) \) is zero.

**Proposition 12.** The coefficient of \( a_k b_k \) in the expansion of \( d_2(a_k b_k) \) is zero. The coefficients of \( b_k a_k \) in the expansion of \( d_2(a_k b_k) \) and in the expansion of \( d_2(a_k b_k) \) equal \(-1\).

**Proof.** We have

\[
d_2(a_k b_k) = a_k b_k - i (a_k d_1(b_k)).
\]

It is easy to check that \( d_1(b_k) = b_k - a_k \). To compute \( i (a_k b_k - a_k b_k) \), we have to find the leading term in the normal form of \( a_k b_k - a_k b_k \).

Note that the elements \( a_k b_k \) and \( b_k a_k \) lie in the subalgebra \( U_{3}^{k+1} (K) \) of \( U_{k}^{k+1} (K) \), and the homogeneous component of degree \( p^{k+1} \alpha + (p - 1)p^2 \beta \) of \( U_{k+1}^{k+1} (K) \) has the basis

\[
a_k b_k - \left\{ a_k^{p-1} (b_k a_k)^j b_k^{p-1} \bigg| 0 \leq j \leq p - 1 \right\}.
\]

We get from Proposition 10 and the above remark

\[
a_k b_k = b_k a_k + b_k b_k - a_k b_k = \cdots = b_k^{p-1} a_k + \sum_{j=1}^{p} \lambda_j a_k^{p-j} (b_k a_k)^j b_k^{p-j}.
\]
for some coefficients \( \lambda_j \in \mathbb{K} \). We see that \( b_k^{p-1}a_{k+1} \) is the maximal term in the expansion of \( a_{k+1}b_k^{p-1} \). Since \( j \left( b_k^{p-1}a_{k+1}b_k \right) = b_k^{p-1}a_{k+1}b_k \) we get

\[
d_2(a_{k+1}b_k^p) = a_{k+1}b_k^p - b_k^{p-1}a_{k+1}b_k + i \left( b_k^{p-1}a_{k+1}b_k - b_k^{p-1}d_1(a_{k+1}b_k) + \sum_{j=1}^{p} \lambda_j a_k^{p-j}(b_k a_k)^j b_k^{p-1-j}b_k \right).
\]

We have

\[
d_1(a_{k+1}b_k) = a_{k+1}b_k - b_k.a_{k+1} + a_k^{p-1}b_k.a_k.
\]

Therefore

\[
b_k^{p-1}a_{k+1}b_k - b_k^{p-1}d_1(a_{k+1}b_k) = b_k^p.a_{k+1} - b_k^{p-1}a_k^{p-1}b_k.a_k = -b_k^{p-1}a_k^{p-1}b_k.a_k.
\]

Since \( b_k \) and \( a_k \) are the elements of \( U_{3,k}^{k,k}(\mathbb{K}) \), the element \( b_k^{p-1}a_k^{p-1} \) is a linear combination of the elements in \( \left\{ a_k^{p-1-j}(b_k a_k)^j b_k^{p-1-j} \mid 0 \leq j \leq p - 1 \right\} \). Note that \((b_k a_k)^p \) is maximal among them.

**Lemma 2.** The coefficient of \((b_k a_k)^p \) in the normal expansion of \( b_k^{p-1}a_k^{p-1} \) is 1.

**Proof.** Since \( U_{3,k}^{k,k}(\mathbb{K}) \) is isomorphic to \( U_{0,0}^{k,k}(\mathbb{K}) \) via \( F_k \), and \( F_k(a_0) = a_k, F_k(b_0) = b_k \), it is enough to prove that the coefficient of \((ba)^p \) in the normal expansion of \( b_k^{p-1}a_k^{p-1} \) is 1, where \( a := a_0 \) and \( b := b_0 \).

We have by (19)

\[
e^{(p-1)}_{\alpha}e^{(p-1)}_{\beta} = \sum_{j=0}^{p} (-1)^j e^{(p-1-j)}_{\alpha}e^{(j)}_{\alpha+\beta}e^{(p-1-j)}_{\beta}.
\]

For \( 0 \leq j \leq p-1, e^{(p-1-j)}_{\alpha}e^{(j)}_{\alpha+\beta}e^{(p-1-j)}_{\beta} \) is a non-zero multiple of \( a^{p-1-j}(ab-ba)^j b^{p-1-j} \).

Consider \( a^{p-1-j}(ab-ba)^j b^{p-1-j} \) as an element of the free algebra \( \mathcal{F} \) generated by \( a \) and \( b \). Then every term in \( a^{p-1-j}(ab-ba)^j b^{p-1-j} \) is less than \((ba)^p \). Since during the Gröbner reduction process the terms can only decrease, we see that the coefficient of \((ba)^p \) in the normal expansion of \( a^{p-1-j}(ab-ba)^j b^{p-1-j} \) is zero.

Recall that \( (p-1)! = -1 \text{(mod } p) \). Thus \( e^{(p-1)}_{\alpha+\beta} = -e^{(p-1)}_{\alpha+\beta} = -(ab-ba)^p \).

Again, if we consider \((ab-ba)^p \) as an element of \( \mathcal{F} \), the all the terms in \((ab-ba)^p \) are less than \((ba)^p \) except \((-1)^{p-1}(ba)^{p-1} \). Thus in the normal expansion of \((ab-ba)^p \) the term \((ba)^p \) enters with the coefficient \((-1)^{p-1} \).

Since the coefficient of \( e^{(p-1)}_{\alpha+\beta} \) in (19) is \((-1)^p \), we see that the coefficient of \((ba)^p \) in the normal expansion of

\[
b_k^{p-1}a_k^{p-1} = \left((-e^{(p-1)}_{\beta}) (-e^{(p-1)}_{\alpha}) \right) = e^{(p-1)}_{\beta}e^{(p-1)}_{\alpha}
\]

20
Since \((b_k a_k)^{p-1} b_k a_k\) is greater then \(a_k^{p-j} (b_k a_k)^j b_k^{p-1-j} b_k\) for any \(1 \leq j \leq p\), we see that \((b_k a_k)^{p-1} a_k b_k\) is the maximal term in the normal expansion of

\[
b_k^{p-1} a_{k+1} b_k - b_k^{p-1} d_1 (a_{k+1} b_k) + \sum_{j=1}^{p} \lambda_j a_k^{p-j} (b_k a_k)^j b_k^{p-1-j} b_k.
\]

Since \(j \left( (b_k a_k)^{p-1} b_k a_k \right) = (b_k a_k)^p\), we get

\[
d_2 (a_{k+1} b_k^p) = a_{k+1} b_k^p - b_k^{p-1} a_{k+1} b_k - (b_k a_k)^p - i \text{ (smaller terms).}
\]

Now we consider \(d_2 (b_{k+1} a_k^p)\). We have

\[
d_2 (b_{k+1} a_k^p) = b_{k+1} a_k^p - i \left( b_{k+1} d_1 (a_k^p) \right).
\]

It is easy to check that \(d_1 (a_k^p) = a_k^{p-1} a_k\). Thus

\[
d_2 (b_{k+1} a_k^p) = b_{k+1} a_k^p - i \left( b_{k+1} a_k^{p-1} a_k \right).
\]

Now \(b_{k+1} a_k^{p-1}\) is an element of \(U_3^{k,k+1} (\mathbb{K})\) of degree \((p - 1) p^k + p^{k+1} \beta\), and thus is a linear combination of the elements

\[
V := \left\{ a_k^{p-1-j} (b_k a_k)^j b_k^{p-j} \mid 0 \leq j \leq p - 1 \right\} \cup a_k^{p-1} b_{k+1}.
\]

Note that for any \(v \in V\), we have \(va_k < a_{k+1} b_{k+1}\). Therefore, \(a_{k+1} b_{k+1}\) enters with the coefficient zero in the expansion of \(d_2 (b_{k+1} a_k^p)\).

Now \((b_k a_k)^{p-1} b_k\) is the maximal element of \(V\).

**Lemma 3.** The coefficient of \((b_k a_k)^{p-1} b_k\) in the normal expansion of \(b_{k+1} a_k^{p-1}\) is 1.

**Proof.** Since \(U_3^{k,k+1} (\mathbb{K})\) is isomorphic to \(U_3^{0,1} (\mathbb{K})\) via \(F_k\), it is enough to check that the coefficient of \((b_0 a_0)^{p-1} b_0\) in the normal expansion of \(b_1 a_0^{p-1}\) is 1.

We have by \((5)\)

\[
e_{\beta}^{(p)} e_{\alpha}^{(p-1)} = \sum_{j=1}^{p-1} (-1)^j e_{\alpha}^{(p-j-1)} e_{\alpha+\beta}^{(j)} e_{\beta}^{(p-j)}.
\]

For \(1 \leq j \leq p-2\), \(e_{\alpha}^{(p-1-j)} e_{\alpha+\beta}^{(j)} e_{\beta}^{(p-j)}\) is a non-zero multiple of

\[
f := a_0^{p-j-1} (a_0 b_0 - b_0 a_0)^j b_0^{p-j}.
\]

Now the coefficient of \((b_0 a_0)^{p-1} b_0\) in \(f\) is zero since all terms of \(f\) considered as an element of the free algebra generated by \(a_0\) and \(b_0\) are less then \((b_0 a_0)^{p-1} b_0\).
For \( j = 0 \), we have \( e_{\alpha}^{(p-1)} e_{\beta} = -a_0^{p-1} b_1 \) and \( a_0^{p-1} b_1 < (b_0 a_0)^{p-1} b_0 \). For \( j = p - 1 \), we get

\[
e_{\alpha}^{(p-1)} e_{\beta} = -(a_0 b_0 - b_0 a_0)^{p-1} b_0.
\]

By the considerations as in Lemma 2, we see that the coefficient of \( (b_0 a_0)^{p-1} b_0 \) in \( (a_0 b_0 - b_0 a_0)^{p-1} b_0 \) is \(-1\)\(^{p-1}\). Therefore the coefficient of \( (b_0 a_0)^{p-1} b_0 \) in

\[
b_1 a_0^{p-1} = -e_{\beta}^{(p)} e_{\alpha}^{(p-1)}
\]

is \(-(-1)^{p-1} (-1)^{p-1}\) = 1.

Since \( j \left( (b_k a_k)^{p-1} a_k b_k \right) = (b_k a_k)^p \), we get

\[
d_2 \left( (b_k a_k)^p \right) = b_{k+1} a_k - (b_k a_k)^p + i \text{ (smaller terms)}.
\]

**Corollary 5.** The elements \( d_2 \left( (b_{k+1} a_k)^p \right) + (b_k a_k)^p \) and \( d_2 \left( (a_{k+1} b_k)^p \right) + (b_k a_k)^p \) belong to \( \text{Rad} (P_1) \).

Now we modify the Anick resolution in order to make it minimal up to the third step. We define \( T'_1 = \{ w \in T_1 \mid w \neq (b_k a_k)^p \} \) and \( T'_2 = \{ w \in T_2 \mid w \neq a_{k+1} b_k^p \} \). Denote by \( P'_1 \) and \( P'_2 \) the \( A \)-linear spans of \( T'_1 \) and \( T'_2 \), respectively. We define \( d'_1 : P'_1 \to P_0 \) to be the restriction of \( d_1 \) to \( P'_1 \). To define \( d'_2 : P'_2 \to P'_1 \) we proceed as follows. For every \( w \in T'_2 \), the expression \( d_2 \left( (b_k a_k)^p \right) \) can be written as \( A \)-linear combination

\[
\sum_{w \in T'_2} r_u u + \sum_k r_k (b_k a_k)^p.
\]

Define \( d'_2 (w) \) by the expression

\[
\sum_{w \in T'_2} r_u u + \sum_k r_k (d_2 \left( (a_{k+1} b_k)^p \right) + (b_k a_k)^p).
\]

Then it is obvious that the image of \( d'_2 \) is a subset of \( \text{Rad} (P'_1) \).

**Proposition 13.** The complex

\[
P'_2 \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{\varepsilon} K \to 0
\]

is exact at the term \( P'_1 \).

**Proof.** Denote by \( P''_2 \) and \( P''_1 \) the submodules of \( P_2 \) and \( P_1 \) generated by the elements of the form \( (a_{k+1} b_k^p) \) and \( (a_{k+1} b_k^p) \). It is obvious that \( P''_2 \) is a free submodule. Since \( (b_k a_k)^p \) has coefficient \(-1\) in the expansion of \( d_2 \left( (a_{k+1} b_k^p) \right) \), it is easy to show by induction on \( k \), that \( P''_1 \) is a free submodule of \( P_1 \). Moreover, it is obvious that the restriction of \( d_2 \) on \( P''_2 \) induces an isomorphism between
Define the $A$-homomorphism $\phi_2: P_2 \to P_2'$ by $\phi_2 (.w) = .w$ if $w \in T_2'$ and $\phi_2 (.w) = 0$ if $w \notin T_2'$. Define the $A$-homomorphism $\phi_1: P_1 \to P_1'$ by $\phi_1 (.w) = .w$ if $w \in T_1'$ and $\phi_1 (.w) = .w + d_2 (.a_{k+1}b_k)$.

Then the diagram

\[
\begin{array}{ccc}
P_2'' & \cong & P_1'' \\
\downarrow & & \downarrow \\
P_2 & \xrightarrow{d_2} & P_1 \\
\downarrow & & \downarrow \\
P_2' & \cong & P_1'
\end{array}
\]

is commutative, and all its columns are exact. Therefore the corresponding vertical spectral sequence collapse at stage 1. The stage 1 of the horizontal spectral sequence has the form

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\Ker (d_2) & 0 & \Coker (d_1) \\
\Ker (d_2') & H_1' & \Coker (d_1')
\end{array}
\]

It is easy to see that there is no high differentials that starts or terminate at $H_1'$. Thus $H_1'$ is zero and $P_1'$ is exact at the term $P_1'$.

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