The Coin Exchange Problem and the Structure of Cube Tilings

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Abstract

Let \( k_1, \ldots, k_d \) be positive integers and let \( D \) be a subset of \( [k_1] \times \cdots \times [k_d] \), whose complement can be decomposed into disjoint sets of the form \( \{x_1\} \times \cdots \times \{x_{s-1}\} \times [k_s] \times \{x_{s+1}\} \times \cdots \times \{x_d\} \). We conjecture that the number of elements of \( D \) can be represented as a linear combination of the numbers \( k_1, \ldots, k_d \) with non-negative integer coefficients. A connection of this conjecture with the structure of periodic cube tilings is revealed.

For any positive integer \( n \), we denote by \([n]\) the set \( \{1, \ldots, n\} \). We extend this notation to vectors \( \mathbf{k} = (k_1, \ldots, k_d) \) with positive integer coordinates: \([\mathbf{k}] := [k_1] \times \cdots \times [k_d] \). If all \( k_i \) are greater than 1, then \([\mathbf{k}]\) is said to be a (discrete) \( d \)-box. A line in \([\mathbf{k}]\) is any set of the form

\[
\{x_1\} \times \cdots \times \{x_{s-1}\} \times [k_s] \times \{x_{s+1}\} \times \cdots \times \{x_d\},
\]

where \( s \in [d] \), and \( x_i \in [k_i] \). A subset \( D \) of \([\mathbf{k}]\) is said to be complementable by lines if its complement \( [\mathbf{k}] \setminus D \) can be represented as a union of disjoint lines.

A non-negative integer \( n \) is representable by \( \mathbf{k} \) if there are non-negative integers \( n_1, \ldots, n_d \) such that

\[
n = n_1k_1 + \cdots + n dk_d.
\]

In other words, the amount \( n \) can be changed using coins of denominations \( k_1, \ldots, k_d \). As a consequence of this interpretation, the problem of representability is often called the coin exchange problem.

The following conjecture arises from certain problems concerning periodic cube tilings, as we shall explain it later on.

**Conjecture**

For each \( d \)-box \([\mathbf{k}]\), if \( D \subseteq [\mathbf{k}] \) is complementable by lines, then the size \( |D| \) of \( D \) is representable by \( \mathbf{k} \).
It is not difficult to confirm this conjecture for \( k = (m, \ldots, m, n) \), where \( m \) and \( n \) are arbitrary positive integers. If \( d = 3 \), then verification of the conjecture reduces to a strictly numerical problem:

Show that for every positive integers \( 1 < k_1 < k_2 < k_3, \) and \( 1 \leq l_i \leq k_i - 1, \) the number

\[
l_1l_2l_3 + (k_1 - l_1)(k_2 - l_2)(k_3 - l_3)
\]

is representable by \((k_1, k_2, k_3)\).

This problem has been tested for a wide range of data by M. Haluszczak and, independently, by A. Ziejiński, an MSc student of the second author. In particular, it has been tested for all \( 1 < k_1 < k_2 < k_3 \leq 700 \) and all \( l_i, i = 1, 2, 3, \) satisfying the constraints.

We define a cube in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) to be any translate of the unit cube \([0, 1)^d\). Let \( T \) be a subset of \( \mathbb{R}^d \). The family \([0, 1)^d + T := \{[0, 1)^d + t: t \in T\}\) is said to be a cube tiling of \( \mathbb{R}^d \) if for each pair of distinct vectors \( s, t \in T \) the cubes \([0, 1)^d + s\) and \([0, 1)^d + t\) are disjoint and \( \bigcup(0, 1)^d + T = \mathbb{R}^d \). Let \( k := (k_1, \ldots, k_d) \) be a vector with all coordinates that are positive integers. The tiling \([0, 1)^d + T\) is said to be \( k \)-periodic if for every vector of the standard basis \( e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 1) \) one has

\[
T + ke_i = T.
\]

We define the (flat) torus \( \mathbb{T}^d_k \), to be the set \([0, k_1) \times \cdots \times [0, k_d)\) with addition mod \( k\):

\[
x \oplus y := ((x_1 + y_1) \mod k_1), \ldots, (x_d + y_d) \mod k_d). 
\]

We can extend the notion of a cube so that it will apply to flat tori: Cubes in \( \mathbb{T}^d_k \) are the sets of the form \([0, 1)^d \oplus t\), where \( t \in \mathbb{T}^d_k \). It is clear that we can speak about cube tilings of \( \mathbb{T}^d_k \) and that there is a canonical ‘one-to-one’ correspondence between these tilings and the \( k \)-periodic tilings of \( \mathbb{R}^d \).

From now on, \( k \) is assumed to have all coordinates greater than 1.

If \([0, 1)^d \oplus T\) is a cube tiling of \( \mathbb{T}^d_k \) and \( S \subseteq T \), then we say that the packing \([0, 1)^d \oplus S\) is a simple component of the cube tiling if \( S \) is an equivalence class of the relation ‘\( \sim \)' defined on \( T \) as follows:

\[
x \sim y \text{ if and only if } x - y \in \mathbb{Z}^d.
\]

For each \( t \in \mathbb{T}^d_k \), the integer code of \( t \) is defined by

\[
\varepsilon(t) := ([t_1] + 1, \ldots, [t_d] + 1).
\]

Clearly, \( \varepsilon \) maps \( \mathbb{T}^d_k \) into \( [k] \).

One can prove the following rather non-trivial result.

**Theorem 1** If \([0, 1)^d + S\) is a simple component of a cube tiling \([0, 1)^d \oplus T\) of \( \mathbb{T}^d_k \), then \( \varepsilon(S) \subseteq [k] \) is complementable by lines.

If \( D \subseteq [k] \) is complementable by lines, then there is a cube tiling \([0, 1)^d \oplus T\) of \( \mathbb{T}^d_k \) with a simple component \([0, 1)^d + S\) such that \( \varepsilon(S) = D \).
If, in addition, we take into account that by Keller’s theorem (see any of the three papers we refer to), the restriction $\varepsilon|T$ of $\varepsilon$ to $T$ is a bijection for each cube tiling $[0,1]^d \oplus T$, then the conjecture can be rephrased as follows:

If $[0,1]^d + S$ is a simple component of the cube tiling $[0,1]^d + T$ of the torus $\mathbb{T}^d_k$, then the size $|S|$ of $S$ is representable by $k$.

References

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