RINGEL DUALITY FOR EXTENDED ZIGZAG SCHUR ALGEBRA

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Abstract. Extended zigzag Schur algebras are quasi-hereditary algebras which are conjecturally Morita equivalent to RoCK blocks of classical Schur algebras. We prove that extended zigzag Schur algebras are Ringel self-dual.

1. Introduction

Let $S(m, r)$ be a classical Schur algebra over the ground field $\mathbb{F}$ of characteristic $p > 0$, see [10]. A fundamental fact going back to [3, 11] is that the algebra $S(m, r)$ is (based) quasi-hereditary. The blocks of $S(m, r)$ are classified in [6 (2.12)], [7].

From now on let us assume for simplicity that $r \leq m$. Then the blocks of $S(m, r)$ and of the symmetric group algebra $\mathbb{F}S_r$ are parametrized by pairs $(\rho, d)$ where $\rho$ is a $p$-core and $d$ is a non-negative integer such that $|\rho| + pd = r$. Let $B_{\rho,d}$ denote the corresponding block of $S(m, r)$ and $\bar{B}_{\rho,d}$ denote the corresponding block of $\mathbb{F}S_r$.

A special role in representation theory of $\mathbb{F}S_r$ is played by the so-called RoCK blocks going back to [1, 18]. These are the blocks $\bar{B}_{\rho,d}$ with $\rho$ satisfying certain combinatorial genericity condition with respect to $d$. The corresponding block $B_{\rho,d}$ of the Schur algebra $S(m, r)$ is then also called RoCK. The RoCK blocks of symmetric groups are important because they admit a nice ‘local description.’ Namely, by [9], we have that $\bar{B}_{\rho,d}$ is Morita equivalent to the zigzag Schur algebra $T^Z(d, d)$ defined by Turner [20], see also [8]. In view of [2], this yields a ‘local description’ of all blocks of symmetric groups up to derived equivalence.

On the other hand, an arbitrary RoCK block $B_{\rho,d}$ of $S(m, r)$ is conjecturally Morita equivalent to the extended zigzag Schur algebra $T^Z(d, d)$, see [14] Conjecture 7.60. As a first evidence for this conjecture, it is proved in [14] Theorem 1] that $T^Z(d, d)$ is quasi-hereditary. In this paper we obtain further evidence for this conjecture in terms of Ringel duality.

In fact, Donkin [4 (3.7), (3.11)], [7 §5(2)], [5 §4.1] proves that $S(m, r)$ is Ringel self-dual. It follows from the results of Donkin that the Ringel dual of the block $B_{\rho,d}$ is the block $B_{\rho',d}$. On the other hand, if $B_{\rho,d}$ is RoCK then so is $B_{\rho',d}$, so we expect that the extended zigzag Schur algebra $T^Z(d, d)$ must be Ringel self-dual. This is what we prove in this paper:

Main Theorem. Let $d \leq n$. Then the extended zigzag Schur algebra $T^Z(n, d)$ is Ringel self-dual.

Here $Z$ stands for the extended zigzag algebra corresponding to the quiver with $p$ vertices, see §5.1. In fact, $T^Z(n, d)$ is a special case of the generalized Schur algebras $T^A_a(n, d)$ introduced in [13], with $A = Z$ and $a$ being the degree zero component of the graded algebra $Z$. It is well known that $Z$ is quasi-hereditary and Ringel self-dual. So our Main Theorem is a special case of the following conjecture.

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Conjecture. Let $A$ be a based quasi-hereditary algebra and $d \leq n$. If $A'$ is a Ringel dual of $A$, then a Ringel dual of $T^A_a(n, d)$ is of the form $T^A_{a'}(n, d)$ for some canonical choice of $a'$.

The paper is organized as follows. Section 2 is preliminaries. In particular, §2.3 details necessary facts on based quasi-hereditary algebras and §2.4 is on the combinatorics of partitions and tableaux. Section 3 describes the construction of $T^A(n, d)$ and important results about its (co)standard modules. In Section 4 we define the modified divided power $\sigma$ and tableaux. Section 5 describes the construction of $I$ and we prove the Main Theorem. In particular, in §5.2 we describe a full tilting module for $T^Z(n, d)$; and in §5.3 we compute its endomorphism algebra.

2. Preliminaries

2.1. General notation. For $n \in \mathbb{Z}_{>0}$, we denote $[n] := \{1, 2, \ldots, n\}$. Throughout the paper, $I$ denotes a non-empty finite partially ordered set. We always identify $I$ with the set $\{0, 1, \ldots, \ell\}$ for $\ell = |I| - 1$, so that the standard total order on integers refines the partial order on $I$. For a set $S$, we often write elements of $S^d$ as words $s = s_1 \cdots s_d$ with $s_1, \ldots, s_d \in S$.

The symmetric group $\mathfrak{S}_d$ acts on the right on $S^d$ by place permutations:

$$(s_1 \cdots s_d) \sigma = s_{\sigma 1} \cdots s_{\sigma d}.$$  

For $s = s_1 \cdots s_d \in S^d$ we have the stabilizer $\mathfrak{S}_s := \{ \sigma \in \mathfrak{S}_d \mid s \sigma = s \}$. For $s, t \in S^d$, we write $s \sim t$ if $s \sigma = t$ for some $\sigma \in \mathfrak{S}_d$. If $S_1, \ldots, S_m$ are sets, then $\mathfrak{S}_d$ acts on $S_1^d \times \cdots \times S_m^d$ diagonally. We write $(s_1, \ldots, s_m) \sim (t_1, \ldots, t_m)$ if $(s_1, \ldots, s_m) \sigma = (t_1, \ldots, t_m)$ for some $\sigma \in \mathfrak{S}_d$. If $U \subseteq S_1^d \times \cdots \times S_m^d$ is a $\mathfrak{S}_d$-invariant subset, we denote by $U/\mathfrak{S}_d$ a complete set of the $\mathfrak{S}_d$-orbit representatives in $U$ and we identify $U/\mathfrak{S}_d$ with the set of $\mathfrak{S}_d$-orbits on $U$.

An (arbitrary) ground field is denoted by $\mathbb{F}$. Often we will also need to work over a characteristic 0 principal ideal domain $R$ such that $\mathbb{F}$ is a $R$-module, so that we can change scalars from $R$ to $\mathbb{F}$ (in all examples of interest to us, one can use $R = \mathbb{Z}$). When considering $R$-supermodules (or $\mathbb{F}$-superspaces) below, we always consider $R$ (and $\mathbb{F}$) as concentrated in degree 0.

We use $k$ to denote $\mathbb{F}$ or $R$ and use it whenever the nature of the ground ring is not important. On the other hand, when it is important to emphasize whether we are working over $R$ or $\mathbb{F}$, we will use lower indices; for example for an $R$-algebra $A_R$ and an $A_R$-module $V_R$, after extending scalars we have $V_\mathbb{F} := \mathbb{F} \otimes_R V_R$ is a module over $A_\mathbb{F} := \mathbb{F} \otimes_R A_R$.

2.2. Superalgebras and supermodules. Let $V = \bigoplus_{\varepsilon \in \mathbb{Z}/2} V_\varepsilon$ be a $k$-supermodule. If $v \in V_\varepsilon \setminus \{0\}$ for $\varepsilon \in \mathbb{Z}/2$, we say $v$ is homogeneous, we write $|v| = \varepsilon$, and we refer to $\varepsilon$ as the parity of $v$. If $S \subseteq V$, we denote $S_0 := S \cap V_0$ and $S_1 := S \cap V_1$. If $W$ is another $k$-supermodule, the set of all $k$-linear homomorphisms $\text{Hom}_k(V, W)$ is a $k$-supermodule such that for $\varepsilon \in \mathbb{Z}/2$, we have

$$\text{Hom}_k(V, W)_\varepsilon = \{ f \in \text{Hom}_k(V, W) \mid |f(v)| = |v| + \varepsilon \text{ for all homogeneous } v \}.$$  

The group $\mathfrak{S}_d$ acts on $V^\otimes d$ on the right by automorphisms, such that for all homogeneous $v_1, \ldots, v_d \in V$ and $\sigma \in \mathfrak{S}_d$, we have

$$(v_1 \otimes \cdots \otimes v_d)^\sigma = (-1)^{\langle \sigma; v \rangle} v_{\sigma 1} \otimes \cdots \otimes v_{\sigma d},$$  

where, setting $v := v_1 \cdots v_d \in V^d$, we have put:

$$\langle \sigma; v \rangle := \sharp \{(k, l) \in [d]^2 \mid k < l, \; \sigma^{-1}k > \sigma^{-1}l, \; |v_k| = |v_l| = 1\}.$$  


We consider the \(d\)th divided power \(\Gamma^d V\), which by definition is the subspace of invariants
\[
\Gamma^d V := \{ w \in V^{\otimes d} \mid w^\sigma = w \text{ for all } \sigma \in \mathfrak{S}_d \}. \tag{2.3}
\]
Let \(0 \leq c \leq d\). Given \(w_1 \in V^{\otimes c}\) and \(w_2 \in V^{\otimes (d-c)}\), we define
\[
w_1 \ast w_2 := \sum_{\sigma} (w_1 \otimes w_2)^\sigma \in V^{\otimes d}, \tag{2.4}
\]
where the sum is over all shortest coset representatives \(\sigma\) for \((\mathfrak{S}_c \times \mathfrak{S}_{d-c}) \setminus \mathfrak{S}_d\).

Let \(V\) and \(W\) be \(k\)-supermodules, \(d \in \mathbb{Z}_{\geq 0}\), and let \(v = v_1 \cdot \cdot \cdot v_d \in V^d\), \(w = w_1 \cdot \cdot \cdot w_d \in W^d\) be \(d\)-tuples of homogeneous elements. We denote
\[
\langle v, w \rangle := \#\{(k, l) \in [d]^2 \mid k > l, \ |v_k| = |w_l| = \bar{1}\}. \tag{2.5}
\]

Let \(A\) be a (unital) \(k\)-superalgebra and \(V, W\) be \(A\)-supermodules. A homogeneous \(A\)-superalgebra homomorphism \(f : V \rightarrow W\) is a homogeneous \(k\)-linear map \(f : V \rightarrow W\) satisfying \(f(av) = (-1)^{|f||a|}af(v)\) for all (homogeneous) \(a, v\). For \(\varepsilon \in \mathbb{Z}/2\), let \(\text{Hom}_A(V, W)_\varepsilon\) be the set of all homogeneous \(A\)-supermodule homomorphisms of parity \(\varepsilon\), and let
\[
\text{Hom}_A(V, W) := \text{Hom}_A(V, W)_0 \oplus \text{Hom}_A(V, W)_1.
\]
We refer to the elements of \(\text{Hom}_A(V, W)\) as the \(A\)-supermodule homomorphisms from \(V\) to \(W\). We denote by \(A\)-mod the category of all finitely generated (left) \(A\)-supermodules and all \(A\)-supermodule homomorphisms. We denote by ‘\(\cong\)’ an isomorphism in this category and by ‘\(\sim\)’ an even isomorphism in this category.

We have the parity change functor \(\Pi\) on \(A\)-mod: for \(V \in A\)-mod we have \(IV \in A\)-mod with \((IV)_\varepsilon = V_{\varepsilon+1}\) for all \(\varepsilon \in \mathbb{Z}/2\) and the new action \(a \cdot v = (-1)^{|a||v|}av\) for \(a \in A, v \in V\). We have \(V \cong IV\) via the identity map.

Suppose there is an even superalgebra anti-involution \(\tau : A \rightarrow A\). In particular, \(\tau(ab) = (-1)^{|a||b|}\tau(b)\tau(a)\) for all \(a, b \in A\). Then \(\tau\) is an isomorphism \(A \cong A^{\text{op}}\), where the multiplication in \(A^{\text{op}}\) is defined as \(a \cdot b := (-1)^{|a||b|}ba\). If \(V \in A\)-mod then the \(\tau\)-dual \(V^\tau \in A\)-mod is the dual \(V^*\) as a \(k\)-superalgebra considered as a left \(A\)-supermodule via \((af)(v) := (-1)^{|a||f|}f(\tau(a)v)\) for \(a \in A, f \in V^*, v \in V\).

As usual, the tensor product \(A^{\otimes d}\) is a superalgebra with respect to
\[
(a_1 \otimes \cdots \otimes a_d)(b_1 \otimes \cdots b_d) = (-1)^{|a||b|}a_1b_1 \otimes \cdots \otimes a_db_d,
\]
where we have put \(a := a_1 \cdots a_d, b := b_1 \cdots b_d\) (here and below, in expressions like this, we assume that all elements are homogeneous). If \(V\) is an \(A\)-supermodule then \(V^{\otimes d}\) is a supermodule over \(A^{\otimes d}\) with respect to
\[
(a_1 \otimes \cdots \otimes a_d)(v_1 \otimes \cdots v_d) = (-1)^{|a||v|}a_1v_1 \otimes \cdots \otimes a_dv_d,
\]
where we have again put \(a := a_1 \cdots a_d, v := v_1 \cdots v_d\).

The divided power \(\Gamma^d A\) is a subsuperalgebra of \(A^{\otimes d}\). If \(V\) is an \(A\)-supermodule then
\[
((a_1 \otimes \cdots \otimes a_d)(v_1 \otimes \cdots v_d))^\sigma = (a_1 \otimes \cdots \otimes a_d)^\sigma (v_1 \otimes \cdots \otimes v_d)^\sigma
\]
for all \(a_1, \ldots, a_d \in A, v_1, \ldots, v_d \in V, \sigma \in \mathfrak{S}_d\). So \(\Gamma^d V\) is a subsuperalgebra of the restriction of \(V^{\otimes d}\) to \(\Gamma^d A\). Thus we will always consider \(\Gamma^d V\) as a \(\Gamma^d A\)-superalgebra.
2.3. Based quasi-hereditary algebras. The main reference here is \cite{12}. Let $A$ be a $\mathbb{k}$-superalgebra.

**Definition 2.6.** \cite{12} Let $I$ be a finite partially ordered set and let $X = \bigsqcup_{i \in I} X(i)$ and $Y = \bigsqcup_{i \in I} Y(i)$ be finite sets of homogeneous elements of $A$ with distinguished elements $e_i \in X(i) \cap Y(i)$ for each $i \in I$. For each $i \in I$, we set

$$A^{>i} := \text{span}\{xy \mid j > i, x \in X(j), y \in Y(j)\}.$$  

We say that $I, X, Y$ is heredity data if the following axioms hold:

(a) $B := \{xy \mid i \in I, x \in X(i), y \in Y(i)\}$ is a basis of $A$;

(b) For all $i \in I$, $x \in X(i)$, $y \in Y(i)$ and $a \in A$, we have

$$ax \equiv \sum_{x \in X(i)} l_x^a(a)x' \pmod{A^{>i}} \quad \text{and} \quad ya \equiv \sum_{y' \in Y(i)} r_{y'}^y(a)y' \pmod{A^{>i}}$$

for some $l_x^a(a), r_{y'}^y(a) \in \mathbb{k}$;

(c) For all $i, j \in I$ and $x \in X(i)$, $y \in Y(i)$ we have

$$xe_i = x, \quad e_ix = \delta_{x,e_i}x, \quad e iy = y, \quad ye_i = \delta_{y,e_i}y,$$

$$e_jx = x \text{ or } 0, \quad ye_j = y \text{ or } 0.$$  

If $A$ is endowed with heredity data $I, X, Y$, we call $A$ based quasi-hereditary, and refer to $B$ as a heredity basis of $A$. By (c), $e_i^2 = e_i$ for all $i \in I$, so from now on we call $\{e_i \mid i \in I\}$ the standard idempotents of the heredity data. We set

$$B_{\mathbb{a}} := \{xy \mid i \in I, x \in X(i)_0, y \in Y(i)_0\}, \quad (2.7)$$  

$$B_{\mathbb{c}} := \{xy \mid i \in I, x \in X(i)_1, y \in Y(i)_1\}, \quad (2.8)$$

so that

$$B_0 = B_{\mathbb{a}} \sqcup B_{\mathbb{c}}. \quad (2.9)$$

The heredity data $I, X, Y$ of $A$ is called conforming if $B_{\mathbb{a}}$ spans a unital subalgebra of $A$.

Let $A$ be a based quasi-hereditary superalgebra with heredity data $I, X, Y$ (not necessarily conforming). By \cite{12} Lemma 3.3, $A$ is quasi-hereditary in the sense of Cline, Parshall and Scott and $A$-mod is a highest weight category (see \cite{3} Theorem 3.6). The corresponding standard and costandard modules are defined as follows. Let $i \in I$. Note that $A^{>i}$ the ideal of $A$ generated by $\{e_j \mid j > i\}$, and denote $\tilde{A} := A/A^{>i}, \tilde{a} := a + A^{>i} \in \tilde{A}$ for $a \in A$. The standard $A$-module of highest weight $i$ is defined by $\Delta(i) := \tilde{A}e_i$, which is a free $\mathbb{k}$-module with basis $\{v_x := \tilde{x} \mid x \in X(i)\}$, see \cite{12} §2.3. We also have the right standard $A$-module $\Delta^\text{op}(i) := \tilde{e_i}\tilde{A}$, and by symmetry every result we have about $\Delta(i)$ has its right analogue for $\Delta^\text{op}(i)$, for example $\Delta^\text{op}(i)$ is a free $\mathbb{k}$-module with basis $\{w_y := \tilde{y} \mid y \in Y(i)\}$. The costandard $A$-module of highest weight $i$ is defined by $\nabla(i) := \Delta^\text{op}(i)^* \text{ with } (af)(v) = (-1)^{|a||f|+|a||v|}(va)$ for $a \in A, f \in \Delta(i)^*, v \in \Delta(i)$.

Let $V \in A$-mod. A standard filtration of $V$ is an $A$-supermodule filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_l = V$ such that for every $r = 1, \ldots, l$, we have $W_r/W_{r-1} \cong \Delta(i_r)$ for some $i_r \in I$. We refer to $\Delta(i_1), \ldots, \Delta(i_l)$ as the factors of the filtration, and to $\Delta(i_1)$ (resp. $\Delta(i_l)$) as the bottom (resp. top) factor. If $\mathbb{k} = \mathbb{F}$, $i \in I$ and $V$ has a standard filtration as above, then $\#\{1 \leq r \leq l \mid W_r/W_{r-1} \cong \Delta(i)\}$ does not depend on the choice of the standard filtration and is denoted $(V : \Delta(i))$. In fact, by \cite{5} Proposition A2.2, we have

$$(V : \Delta(i)) = \dim \text{Hom}_A(V, \nabla(i)). \quad (2.10)$$
A costandard filtration of $V$ is an $A$-supermodule filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_l = V$ such that for every $r = 1, \ldots, l$, we have $W_r/W_{r-1} \cong \nabla(i_r)$ for some $i_r \in I$.

Let $T \in A$-mod. We say that $T$ is a tilting supermodule if it has standard and costandard filtrations. We refer to [19] §4 for the integral version of the tilting theory. In particular, by [19] Propositions 4.26, 4.27, for every $i \in I$ there exists a (unique up to isomorphism) indecomposable tilting supermodule $T(i)$ such that $\Delta(i) \subseteq T(i)$ and $T(i)/\Delta(i)$ has a standard filtration with factors of the form $\Delta(j)$ for $j < i$; moreover, for every tilting supermodule $T$ we have $T \cong \bigoplus_{i \in I} T(i)^{m_i}$. In this case $T$ is called a full tilting supermodule if $m_i > 0$ for all $i \in I$. If $T$ is full tilting, the superalgebra $A' := \text{End}_A(T)^{\text{op}}$ is called a Ringel dual of $A$.

The algebra $A'$ is defined uniquely up to Morita superequivalence and is quasi-hereditary, see [19] Proposition 4.26).

2.4. Multipartitions and tableaux. For a partition $\lambda$, we have the conjugate partition $\lambda'$, see [16] p.2. For partitions $\lambda, \mu, \nu$, we denote by $c_{\mu,\nu}^{\lambda}$ the corresponding Littlewood-Richardson coefficient, see [16] §I.9. The Young diagram of $\lambda$ is $[\lambda] := \{(r, s) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid s \leq r\}$. We refer to $(r, s) \in [\lambda]$ as the nodes of $\lambda$.

Let $n \in \mathbb{Z}_{\geq 0}$. We denote $\Lambda(n) := \mathbb{Z}_{\geq 0}$ and interpret it as the set of compositions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $n$ non-negative parts. For such $\lambda$, we set $|\lambda| := \lambda_1 + \cdots + \lambda_n$. The partitions with at most $n$ parts are identified with

$$\Lambda_+(n) := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n) \mid \lambda_1 \geq \cdots \geq \lambda_n\}.$$

For $d \in \mathbb{Z}_{\geq 0}$, we let

$$\Lambda(n, d) := \{\lambda \in \Lambda(n) \mid |\lambda| = d\} \quad \text{and} \quad \Lambda_+(n, d) := \{\lambda \in \Lambda_+(n) \mid |\lambda| = d\}.$$

For $\lambda, \mu \in \Lambda(n)$, we define $\lambda + \mu := (\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n)$.

Recall that $I$ denotes a finite poset. We will consider the set of $I$-multipartitions

$$\Lambda^I(n) := \Lambda(n)^I = \{\lambda = (\lambda(i))_{i \in I} \mid \lambda(i) \in \Lambda(n) \text{ for all } i \in I\}. \quad (2.11)$$

For $\lambda, \mu \in \Lambda^I(n)$ we define $\lambda + \mu$ to be $\nu \in \Lambda^I(n)$ with $\nu(i) = \lambda(i) + \mu(i)$ for all $i \in I$. For $d \in \mathbb{Z}_{\geq 0}$, we have the sets of $I$-multipartitions and $I$-multipartitions of $d$:

$$\Lambda^I(n, d) := \{\lambda \in \Lambda^I(n) \mid \sum_{i \in I} |\lambda(i)| = d\},$$

$$\Lambda_+^I(n, d) := \{\lambda \in \Lambda^I(n, d) \mid \lambda(i) \in \Lambda_+(n) \text{ for all } i \in I\}.$$

For $\lambda \in \Lambda_+^I(n, d)$, we define $[\lambda] := \bigsqcup_{i \in I} [\lambda(i)]$ and $\|\lambda\| := (|\lambda(i)|)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$.

Via our identification $I = \{0, \ldots, \ell\}$, for $\lambda = (\lambda(i))_{i \in I} \in \Lambda^I(n)$, we also write $\lambda = (\lambda^{(0)}, \lambda^{(\ell)})$. For $i \in I$, and $\lambda \in \Lambda(n, d)$, define

$$\iota_i(\lambda) := (0, \ldots, 0, \lambda, 0, \ldots, 0) \in \Lambda^I(n, d), \quad (2.12)$$

with $\lambda$ in the $i$th position. We will slightly abuse this notation writing $\iota_i(d)$ for $\iota_i((d, 0, \ldots, 0))$ and $\iota_i(1,d)$ for $\iota_i((1, \ldots, 1, 0, \ldots, 0))$.

Let $\leq$ be the partial order on $I$. We have a partial order $\leq_I$ on the set $\mathbb{Z}_{\geq 0}^I$ with $(a_i)_{i \in I} \leq_I (b_i)_{i \in I}$ if and only if $\sum_{j \geq i} a_j \leq \sum_{j \geq i} b_j$ for all $i \in I$. Let $\preceq$ be the usual dominance partial order on $\Lambda(n, d)$, i.e. $\lambda \preceq \mu$ if and only if $\sum_{s=1}^n \lambda_r \leq \sum_{s=1}^n \mu_r$ for all $s = 1, \ldots, n$. We define a partial order $\preceq_I$ on $\Lambda^I(n, d)$ via: $\lambda \preceq_I \mu$ whenever $\|\lambda\| \preceq_I \|\mu\|$, or $\|\lambda\| = \|\mu\|$ and $\lambda(i) \preceq_I \mu(i)$ for all $i \in I$.

Let $I, X, Y$ be a heredity data on a $k$-superalgebra $A$ as in [2.3] We introduce colored alphabets $\mathcal{A}_X := \{n\} \times X$ and $\mathcal{A}_{X(i)} := \{n\} \times X(i)$. An element $(i, x) \in \mathcal{A}_X$ is often written
as \( l^x \). If \( L = l^x \in \mathcal{X} \), we denote \( \text{color}(L) := x \). For all \( i \in I \), we fix arbitrary total orders \( \prec \) on the sets \( \mathcal{X}_{(i)} \) such that whenever \( r < s \) (in the standard order on \([n]\)), \( r^x < s^x \) for all \( x \in X(i) \).

Let \( \lambda = (\lambda(0), \ldots, \lambda(l)) \in \Lambda_+^d(n, d) \). Let \( N_1, \ldots, N_d \) be the nodes of \( [\lambda] = [\lambda(0)] \sqcup \cdots \sqcup [\lambda(l)] \) listed along the rows of \( [\lambda(0)] \) from left to right starting from the first row and going down, then along the rows of \( [\lambda(1)] \) from left to right starting with the first row and going down, etc. A function \( T : [\lambda] \to \mathcal{X} \) is called a standard \( X \)-colored \( \lambda \)-tableau if the following conditions hold:

- \( T([\lambda(i)]) \subseteq \mathcal{X}_{(i)} \) for all \( i \in I \);
- if \( r < s \) and \( N_r, N_s \) are in the same row of \([\lambda(i)]\), then \( T(N_r) \leq T(N_s) \) with \( T(N_r) = T(N_s) \) allowed only if \( \text{color}(T(N_r)) \in X(i)_0 \);
- if \( r < s \) and \( N_r, N_s \) are in the same column of \([\lambda(i)]\), then \( T(N_r) \leq T(N_s) \) with \( T(N_r) = T(N_s) \) allowed only if \( \text{color}(T(N_r)) \in X(i)_1 \);

We denote by \( \text{Std}^X(\lambda) \) the set of all standard \( X \)-colored \( \lambda \)-tableaux. For \( T \in \text{Std}^X(\lambda) \), letting \( T(N_r) = l_r^x \in \mathcal{X} \) for \( r = 1, \ldots, d \), we denote \( l^T := l_1 \cdots l_d \) and \( x^T := x_1 \cdots x_d \). For \( \mu \in \Lambda^d(n, d) \), we say that \( T \) has left weight \( \mu \) if there exist \( i_1, \ldots, i_d \in I \) such that \( e_{i_1} x_1 = x_1, \ldots, e_{i_d} x_d = x_d \) and \( \mu = \sum_{c=1}^d \nu_{e_c}(\varepsilon_{c_c}) \). We have the set of all standard \( \lambda \)-tableaux of left weight \( \mu \):

\[
\text{Std}^X(\lambda, \mu) := \{ T \in \text{Std}^X(\lambda) \mid T \text{ has left weight } \mu \}. \tag{2.13}
\]

### 3. Modified divided powers

#### 3.1. Calibrated supermodules and their modified divided powers

Let \( V_R = V_{R,0} \oplus V_{R,1} \) be a free \( R \)-supermodule of finite rank. We say that \( V_R \) is calibrated if we are given a decomposition \( V_{R,0} = V_{R,a} \oplus V_{R,c} \) into two free \( R \)-modules.

Let \( V_R \) be a calibrated \( R \)-supermodule. We choose bases \( B^V_0, B^V_1, B^V_\chi \) of \( V_{R,0}, V_{R,a}, V_{R,1} \), respectively. Thus \( B^V_0 := B^V_0 \sqcup B^V_\chi \) is a basis of \( V_{R,0} \) and \( B^V := B^V_0 \sqcup B^V_1 \sqcup B^V_\chi \) is a basis of \( V_R \). Fix an arbitrary total order \( \prec \) on \( B^V \). Let \( b \in (B^V)^d \). We define

\[
\langle b \rangle := \# \{ (k, l) \in [d]^2 \mid k < l, b_k, b_l \in B^V, b_k > b_l \}. \tag{3.1}
\]

For any \( b \in B^V \), set

\[
[b : b] := \# \{ k \in [d] \mid b_k = b \} \tag{3.2}
\]

and let

\[
[b]_\chi^d := \prod_{b \in B^V_\chi} [b : b]^! \tag{3.3}
\]

Define \( \text{Seq}(B^V, d) \) to be the set of all \( \chi \)-tuples \( b = b_1 \cdots b_d \in (B^V)^d \) such that \( b_k = b_l \) for some \( 1 \leq k \neq l \leq d \) only if \( b_k \in B^V_\chi \). Then \( \text{Seq}(B^V, d) \subseteq (B^V)^d \) is a \( \mathcal{G}_d \)-invariant subset, so we can choose a corresponding set \( \text{Seq}(B^V, d)/\mathcal{G}_d \) of \( \mathcal{G}_d \)-orbit representatives and identify it with the set of all \( \mathcal{G}_d \)-orbits on \( \text{Seq}(B^V, d) \), cf. §2.2.

Recall the divided power \( R \)-supermodule \( \Gamma^d V_R \) from §2.2. For \( b = b_1 \cdots b_d \in \text{Seq}(B^V, d) \), we have elements

\[
x_b := \sum_{b' = b_1' \cdots b_d'} (-1)^{b_1 + \langle b \rangle} b_1' \otimes \cdots \otimes b_d' \in \Gamma^d V_R \quad \text{and} \quad y_b := [b]_\chi^d x_b \in \Gamma^d V_R.
\]
Define the modified divided power
\[ \hat{\Gamma}^d V_R := \text{span}_R \{ y_b \mid b \in \text{Seq}(B^V, d) \} \subseteq \Gamma^d V_R. \]

Note that \( \{ x_b \mid b \in \text{Seq}(B^V, d)/\mathcal{S}_d \} \) is a basis of \( \Gamma^d V_R \) and \( \{ y_b \mid b \in \text{Seq}(B^V, d)/\mathcal{S}_d \} \) is a basis of \( \hat{\Gamma}^d V_R \), cf. \[8\] (3.9).

The \( n = 1 \) case of \[13\] Proposition 4.11 yields:

**Lemma 3.4.** \( \hat{\Gamma}^d V_R \) depends only on \( V_{R,a} \), and not on \( V_{R,c} \) or choice of basis \( B^V \).

If \( V_R, W_R \) are calibrated \( R \)-supermodules, then \( V_R \oplus W_R \) is also a calibrated \( R \)-supermodule with \( (V_R \oplus W_R)_a := V_{R,a} \oplus W_{R,a} \) and \( (V_R \oplus W_R)_c := V_{R,c} \oplus W_{R,c} \). Moreover, recalling the star product from \[2.4\].

**Lemma 3.5.** We have an isomorphism of \( R \)-supermodules
\[ \bigoplus_{d_1 + d_2 = d} (\hat{\Gamma}^{d_1} V_R) \otimes (\hat{\Gamma}^{d_2} W_R) \cong \hat{\Gamma}^d (V_R \oplus W_R), \ y \otimes y' \mapsto y \star y'. \]

**Proof.** Note that \( y_{b^1} \star y_{b^2} = y_{b^1 \cdot b^2} \) for all \( b^1 \in \text{Seq}(B^V, d_1) \), \( b^2 \in \text{Seq}(B^W, d_2) \) and compare bases. \( \square \)

### 3.2. Bilinear form on \( \hat{\Gamma}^d V \)

Let \( V_R \) be a calibrated \( R \)-supermodule. Suppose in addition that we are given a \((R\text{-valued})\) even supersymmetric (or superantisymmetric), non-degenerate bilinear form \( (\cdot, \cdot) \) on \( V_R \), such that \( (V_{R,a}, V_{R,a}) = 0 \) and the \( R \)-complement \( V_{R,c} \) of \( V_{R,a} \) in \( V_{R,0} \) can be chosen so that the restriction of \( (\cdot, \cdot) \) to \( V_{R,a} \times V_{R,c} \) is a perfect pairing. Until the end of this subsection, we always assume that the complement \( V_{R,c} \) has this property.

Since the form is non-degenerate we may select bases \( B^V_{a} = \{ a_1, \ldots, a_r \}, B^V_{c} = \{ c_1, \ldots, c_r \} \) and \( B^V_1 \) for \( V_{R,a}, V_{R,c} \), and \( V_{R,1} \) respectively such that \( (a_i, c_j) = \delta_{i,j} \) for all \( i, j \in [r] \). Set \( B^V = B^V_a \cup B^V_c \cup B^V_1 \). Let \( (B^V)^* = \{ b^* \mid b \in B^V \} \) be the dual basis with respect to \( (\cdot, \cdot) \). Note that \( c_i^* = a_i \) for all \( i \), but it is not necessarily true that \( a_i^* = c_i \). We take \( V'_{R,c} \) to be the \( R \)-span of \( a_1^*, \ldots, a_r^* \), so that \( V_{R,c} \) is another \( R \)-complement of \( V_{R,a} \) in \( V_{R,0} \). We now have \( (B^V)^* = (B^V_a)^* \cup (B^V_c)^* \cup (B^V_1)^* \) where
\[ (B^V_a)^* := B^V_a, \ (B^V_c)^* := \{ a_1^*, \ldots, a_r^* \}, \ (B^V_1)^* = \{ b^* \mid b \in B^V_1 \}. \]

By Lemma 3.3 \( \hat{\Gamma}^d V_R \) is independent of the choice of a complement \( V_{R,c} \) and of the choice of a corresponding basis. So we now have two \( R \)-bases of \( \hat{\Gamma}^d V_R \):
\[ \{ y_b \mid b \in \text{Seq}(B^V, d)/\mathcal{S}_d \} \quad \text{and} \quad \{ y_b \mid b \in \text{Seq}((B^V)^*, d)/\mathcal{S}_d \}. \]

The form \((\cdot, \cdot)\) extends to the form \((\cdot, \cdot)_d\) on \( V_{R,d}^d \) such that
\[ (v_1 \cdots \otimes v_d, w_1 \cdots \otimes w_d)_d = (-1)^{(w, w)} (v_1, w_1) \cdots (v_d, w_d). \quad (3.6) \]
for all \( v = v_1 \cdots v_d, w = w_1 \cdots w_d \in V^d \). Since the form is even, we have for any \( \sigma \in \mathcal{S}_d \):
\[ ((v_1 \otimes \cdots \otimes v_d)^\sigma, (w_1 \otimes \cdots \otimes w_d)^\sigma)_d = (v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d)_d. \quad (3.7) \]
Moreover, for \( b, b' \in (B^V)^d \), we have
\[ (b_1^* \cdots \otimes b_d^*)_d = (-1)^{(b', b)} \delta_{b, b'}. \quad (3.8) \]

**Lemma 3.9.** Let \( b, b' \in \text{Seq}(B^V, d) \). Then \( b^* := b_1^* \cdots b_d^* \in \text{Seq}((B^V)^*, d) \), and \( (y_{b'}, y_{b^*})_d = \pm d! \delta_{b \sim b'} \).
Proof. By \((3.8)\), \((y_b, y_{b^*})_d \neq 0\) only if \(b \sim b'\). So we may assume that \(b' = b\) and that the stabilizer \(\mathfrak{S}_b = \mathfrak{S}_{b^*}\) is a standard parabolic subgroup. As no odd element repeats in \(b\), we have

\[
|\mathfrak{S}_b| = \left( \prod_{b \in B^V} [b : b!] \right) \left( \prod_{b \in B^V} [b^* : b^*]! \right) = \left( \prod_{b \in (B^V)_{c_0}} [b^* : b^*]! \right) \left( \prod_{b \in B^V} [b : b!] \right).
\]

So, using \((3.7)\) and \((3.8)\), we have that \((y_b, y_{b^*})_d\) equals

\[
\left( \left( \prod_{b \in B^V} [b : b!] \right) \left( \prod_{b \in (B^V)_{c_0}} [b^* : b^*]! \right) \right)_{d} \sum_{\sigma \in \mathfrak{S}_d/\mathfrak{S}_b} \sigma \left( \prod_{b \in (B^V)_{c_0}} (b_1 \otimes \cdots \otimes b_d) \right) \left( \prod_{b \in B^V} (b^* \otimes \cdots \otimes b^*_d) \right)_{d} \sum_{\sigma \in \mathfrak{S}_d/\mathfrak{S}_b} \sigma \left( \prod_{b \in B^V} (b_1 \otimes \cdots \otimes b_d) \right) \left( \prod_{b \in (B^V)_{c_0}} (b^* \otimes \cdots \otimes b^*_d) \right)_{d}
\]

\[
= \left( \prod_{b \in B^V} [b : b!] \right) \left( \prod_{b \in (B^V)_{c_0}} [b^* : b^*]! \right) |\mathfrak{S}_d : \mathfrak{S}_b| (b_1 \otimes \cdots \otimes b_d, b^*_1 \otimes \cdots \otimes b^*_d)_{d}
\]

\[
= \pm d!
\]

which completes the proof. \(\square\)

In view of the lemma, we have \((z, w)_d\) is divisible by \(d\) for all \(z, w \in \tilde{\Gamma}^d V_R\). So we can define a new form on \(\tilde{\Gamma}^d V_R\) by setting

\[
(z, w)_\sim := \frac{1}{d!} (z, w)_d \tag{3.10}
\]

for all \(z, w \in \tilde{\Gamma}^d V_R\). The following is now clear from the lemma:

**Proposition 3.11.** The bilinear form \((\cdot, \cdot)_\sim\) on \(\tilde{\Gamma}^d V_R\) is even and non-degenerate. Moreover, it is supersymmetric or superantisymmetric depending on the parity of \(d\) and whether \((\cdot, \cdot)\) is supersymmetric or superantisymmetric.

**3.3. Modified divided powers of algebras.** Let \(A_R = A_{R,0} \oplus A_{R,1}\) be an \(R\)-superalgebra such that \(A_{R,0}\) and \(A_{R,1}\) are free of finite rank as \(R\)-modules. We say that \(A_R\) is a *calibrated superalgebra* if we are given a free \(R\)-module decomposition \(A_{R,0} = a_R \oplus c_R\) such that \(a_R\) is a unital subalgebra of \(A_R\). Choose bases \(B^A_0, B^A_1, B^A_{c_1}\) of \(a_R, c_R, c_{R,1}\), respectively, and set \(B^A = B^A_0 \cup B^A_1 \cup B^A_{c_1}\). Our main examples come from based quasihereditary algebras over \(R\) with conforming heredity data as in \([23]\). In that case we would take \(a_R\) to be the \(R\)-span of \(B_a\), cf. \([27]\).

Note that a calibrated \(R\)-superalgebra is in particular a calibrated \(R\)-supermodule as in \(\S 3.1\) so we have modified divided power \(\Gamma^d A_R\) and elements \(x_b \in \Gamma^d A_R, y_b \in \tilde{\Gamma}^d A_R\) for \(b \in \text{Seq}(B^A, d)\). But to differentiate between algebras and module, when working with algebras, it will be convenient to use another notation:

\[
\xi^b := x_b \quad \text{and} \quad \eta^b := y_b. \tag{3.12}
\]

Recall from \(\S 2.2\) that \(\Gamma^d A_R\) is a subsuperalgebra of \(A^d_{\tilde{\Gamma}}\). Moreover, \(\tilde{\Gamma}^d A_R\) is a (unital) subsuperalgebra of \(\Gamma^d A_R\), see \([13]\) Proposition 3.12], with basis

\[
\{\eta^b \mid b \in \text{Seq}(B^V, d)/\mathfrak{S}_d\}. \tag{3.13}
\]
Recall from [8] §4.1, that $\bigoplus_{d \geq 0} A^\otimes_R$ is a bisuperalgebra with the coproduct $\nabla$ defined so that

$$\nabla : A^\otimes_R - \bigoplus_{c=0}^d A^\otimes_R \otimes A^\otimes_R \otimes A^\otimes_R,$$

$$a_1 \otimes \cdots \otimes a_d \mapsto \sum_{c=0}^d (a_1 \otimes \cdots \otimes a_c) \otimes (a_{c+1} \otimes \cdots \otimes a_d).$$

Moreover, by the $n = 1$ case of [13] Corollary 3.24, $\bigoplus_{d \geq 0} \tilde{\Gamma}^d A_R$ is a sub-bisuperalgebra of $\bigoplus_{d \geq 0} A^\otimes_R$.

There is also another bisuperalgebra structure on $\bigoplus_{d \geq 0} A^\otimes_R$ using the star product $*$ of [2.3]. In fact:

**Lemma 3.14.** [13] Corollary 4.4] $\bigoplus_{d \geq 0} \tilde{\Gamma}^d A_R$ is a sub-bisuperalgebra of $\bigoplus_{d \geq 0} A^\otimes_R$ with respect to the coproduct $\nabla$ and the product $*$

### 3.4. $\tilde{\Gamma}^d V$ as a module over $\tilde{\Gamma}^d A_R$.

Let $A_R = a_R \oplus c_R \oplus A_R \bar{1}$ be a calibrated superalgebra as in the previous subsection. Let $V_R = V_{R, a} \oplus V_{R, c} \oplus V_{R, 1}$ be a calibrated $R$-supermodule with the corresponding basis $B^V = B^a \cup B^c \cup B^1$ as in §3.1 and assume in addition that $V_R$ is an $A_R$-supermodule. We say that $V_R$ is a calibrated $A_R$-supermodule if $a_R V_{R, a} \subseteq V_{R, a}$.

Recall from [2.2] that $\Gamma^d V_R$ is naturally a $\Gamma^d A_R$-supermodule. So upon restriction to the subalgebra $\Gamma^d A_R \subseteq \Gamma^d A_R$, $\tilde{\Gamma}^d V_R$ is a $\tilde{\Gamma}^d A_R$-supermodule. In Lemma 3.16 we will show that $\tilde{\Gamma}^d V_R \subseteq \Gamma^d V_R$ is a $\tilde{\Gamma}^d A_R$-subsupermodule if $V_R$ is a calibrated $A_R$-supermodule.

For $a \in A_R$ and $b, c \in B^V$, we define the structure constants $\kappa_{a,c}^b \in R$ from $ac = \sum_{b \in B^V} \kappa_{a,c}^b b$. For $a \in \text{Seq}(B^A, d)$ and $b, c \in \text{Seq}(B^V, d)$, we also set $\kappa_{a,c}^{b} := \kappa_{a_1,c_1}^{b_1} \cdots \kappa_{a_d,c_d}^{b_d}$.

We want to describe the structure constants $f_{a,c}^{b}$ defined from

$$\xi^a_{x_{c}} = \sum_{b \in \text{Seq}(B^V, d) \setminus \mathcal{S}_d} f_{a,c}^{b} x_{b}.$$  

Recall the stabilizer $\mathcal{S}_a$ from §2.1. The following lemma is an analogue of [13] Corollary 3.7:

**Lemma 3.15.** Let $a \in \text{Seq}(B^A, d)$ and $b, c \in \text{Seq}(B^V, d)$. Let $X$ be the set of all pairs $(a', c') \in \text{Seq}(B^A, d) \times \text{Seq}(B^V, d)$ such that $a' \sim a$, $c' \sim c$ and $|a'_k| + |c'_k| = |b_k|$ for all $k = 1, \ldots, d$. Then

$$f_{a,c}^{b} = \sum_{(a', c') \in X \setminus \mathcal{S}_b} (-1)^{(a') + (c') + (c') + (a', c')} [\mathcal{S}_b : (\mathcal{S}_b \cap \mathcal{S}_{a'} \cap \mathcal{S}_{c'})^a_{a', c'}].$$

**Proof.** Clearly, we have

$$f_{a,c}^{b} = \sum (-1)^{(a') + (c') + (c') + (a', c')} \kappa_{a', c'}^{b},$$

the sum being over all $(a', c') \in \text{Seq}(B^A, d) \times \text{Seq}(B^V, d)$ such that $a' \sim a$ and $c' \sim c$, cf. [8] (3.14). It remains to note that $\kappa_{a', c'}^{b} = 0$ unless $(a', c') \in X$, and for $(a', c'), (a'', c'') \in X$ in the same $\mathcal{S}_b$-orbit the corresponding summands are equal to each other, cf. the proof of [13] Corollary 3.7.

**Lemma 3.16.** If $V_R$ is a calibrated $A_R$-supermodule, then $\tilde{\Gamma}^d V_R \subseteq \Gamma^d V_R$ is a $\tilde{\Gamma}^d A_R$-submodule.
Proof. For each $a \in \text{Seq}(B^A, d)$ and $c \in \text{Seq}(B^V, d)$ we have (working over the field of quotients of $R$):
\[
\eta^a y_c = ([a]_c^1 [c]_c^1) (\xi_c^1 x_c) = \sum_{b \in \text{Seq}(B^V, d) / \mathcal{S}_d} [a]_c^1 [c]_c^1 f_{a,c}^b x_b = \sum_{b \in \text{Seq}(B^V, d) / \mathcal{S}_d} [a]_c^1 [c]_c^1 f_{a,c}^b \frac{[b]_c^1}{[b]_c^1} y_b.
\]
So in view of Lemma 3.15, it suffices to prove that for fixed $a \in \text{Seq}(B^A, d)$, $b, c \in \text{Seq}(B^V, d)$ and $(a', c') \in X$ satisfying $\kappa_{a', c'}^b \neq 0$, the integer
\[
M_{a,c}^b := [a]_c^1 [c]_c^1 (\mathcal{S}_b : (\mathcal{S}_b \cap \mathcal{S}_a \cap \mathcal{S}_c'))
\]
is divisible by $[b]_c^1$. Here, as in Lemma 3.15, $X$ consists of all pairs $(a', c') \in \text{Seq}(B^A, d) \times \text{Seq}(B^V, d)$ such that $a' \sim a$, $c' \sim c$, and $|a'_k| + |c'_k| = |b_k|$ for all $k = 1, \ldots, d$.

For $a \in B^A$ and $b, c \in B^V$, let
\[
m_{a,c}^b = \# \{k \in [d] | a'_k = a, c'_k = c, b_k = b\}.
\]
Then, recalling the notation (3.2), we have
\[
|\mathcal{S}_b \cap \mathcal{S}_a \cap \mathcal{S}_c'| = \prod_{a \in B^A, b, c \in B^V} m_{a,c}^b!, \quad [a : a] = [a' : a] = \sum_{b, c \in B^V} m_{a,c}^b,
\]
\[
[c : c] = [c' : c] = \sum_{a \in B^A, b \in B^V} m_{a,c}^b,
\]
\[
[b : b] = \sum_{a \in B^A, c \in B^V} m_{a,c}^b.
\]

In particular, for all $b, c \in B^V$ and $a \in B^A$, we have integers
\[
z_b := \frac{[b : b]!}{\prod_{a \in B^A, b, c \in B^V} m_{a,c}^b!}, \quad Z_c := \frac{[c : c]!}{\prod_{a \in B^A, b \in B^V} m_{a,c}^b!}, \quad Z_a := \frac{[a : a]!}{\prod_{b \in B^V, c \in B^V} m_{a,c}^b!},
\]
Denoting $C = \prod_{b \in B^V} z_b$, we have
\[
|\mathcal{S}_b : \mathcal{S}_b \cap \mathcal{S}_a \cap \mathcal{S}_c'| = \prod_{a \in B^A, b, c \in B^V} \frac{[b : b]!}{m_{a,c}^b!} = \prod_{b \in B^V} z_b = C \prod_{b \in B^V} z_b.
\]
Let $b \in B^V$. If $a \in B^A_1$ or $c \in B^V_1$, then $m_{a,c}^b \leq 1$ because there are no repeated odd elements in tuples in $\text{Seq}(B^A, d)$ or $\text{Seq}(B^V, d)$. Also observe that if $a \in B^A_1$ and $c \in B^V_1$, then $ac \in V_{R,a}$ by assumption, so, since $b \in B^V_1$, we have $\kappa_{a,c}^b = 0$, hence $m_{a,c}^b = 0$. So
\[
z_b = \frac{[b : b]!}{\left(\prod_{a \in B^A_1, b, c \in B^V_1} m_{a,c}^b!\right) \left(\prod_{a \in B^A_1, b, c \in B^V_1} m_{a,c}^b!\right)}.
\]
Thus we have
\[
M_{a,c}^b = \left( \prod_{a \in B^d} [a : a]! \right) \left( \prod_{c \in B^d} [c : c]! \right) \cdot C \left( \prod_{b \in B^d} \frac{[b : b]!}{\prod_{a \in B^d, c \in B^d} m_{a,c}^{b}!} \right) \left( \prod_{c \in B^d} \frac{[c : c]!}{\prod_{a \in B^d, b \in B^d} m_{a,b}^{c}!} \right) \left( b : b \right)
\]
\[
= C \left( \prod_{a \in B^d} Z_a \right) \left( \prod_{c \in B^d} Z_c \right) \left[ b : b \right]^{d_{c,d}},
\]
which completes the proof. \(\square\)

3.5. More on \(\hat{\Gamma}^d_{A_R}\)-module \(\hat{\Gamma}^d_{V_R}\). Throughout the subsection \(A_R\) is a calibrated superalgebra and \(V_R\) is a calibrated \(A_R\)-supermodule.

Let \(e \in a_R\) be an idempotent such that \(be = b = 0\) for all \(b \in B^d\), cf. [13] §5. Let \(B^d_A = \{ b \in B^A \mid be = b \}\), \(B^d_\alpha = \{ b \in B^A_\alpha \mid be = b \}\) and \(B^d_\epsilon = \{ b \in B^A_\epsilon \mid be = b \}\). We denote by \(\eta^e_d = e^{\otimes d} \in \hat{\Gamma}^d_{A_R}\). In the special case where \(V_R = A_R e\), we always take \(V_{R,a} := a_R e\) with basis \(B^d_A\), \(V_{R,\alpha} := e^{\otimes d} R\) with basis \(B^d_{\alpha}\). In this case we can describe the \(\hat{\Gamma}^d_{A_R}\)-module \(\hat{\Gamma}^d_{V_R}\) explicitly as follows:

**Lemma 3.17.** Let \(e \in a_R\) be an idempotent such that \(be = b = 0\) for all \(b \in B^d\). Then \(\hat{\Gamma}^d(A_R e) \cong (\hat{\Gamma}^d A_R)\eta^e_d\).

**Proof.** Note that \(\hat{\Gamma}^d(A_R e)\) has basis \(\{ y_b \mid b \in \text{Seq}(B^A e, d)/\mathcal{S}_d \}\) and \((\hat{\Gamma}^d A_R)\eta^e_d\) has basis \(\{ \eta^b \mid b \in \text{Seq}(B^A e, d)/\mathcal{S}_d \}\). There is a \(\hat{\Gamma}^d A_R\)-module map \(\varphi : (\hat{\Gamma}^d A_R)\eta^e_d \to \hat{\Gamma}^d(A_R e)\) with \(\varphi(\eta^b) = y_b\) for all \(b \in \text{Seq}(B^A e, d)/\mathcal{S}_d\). \(\square\)

Suppose there is an even superalgebra anti-involution \(\tau : A_R \to A_R\). We make the additional assumption that \(\tau(a) = a\), in which case \(\tau^{\otimes d}\) restricts to a superalgebra anti-involution \(\tau_d^e\) on \(\hat{\Gamma}^d_{A_R}\), see [13] (4.12).

Given \(W \in \hat{\Gamma}^d_{A_R}\text{-mod},\) its \(\tau_d\)-dual \(W^{\tau_d}\) is defined as \(W^{*} := \text{Hom}_{\mathcal{R}}(W, R)\) with the action \((xf)(w) = (-1)^{|x||f|} f(\tau_d(x)w)\) for all \(f \in W^*, w \in W, x \in \hat{\Gamma}^d_{A_R}\). Note that \(W \simeq W^{\tau_d}\) if and only if there is a non-degenerate \(\tau_d\)-contravariant form \(\langle \cdot, \cdot \rangle\) on \(W\), where \(\tau_d\)-contravariance means \(\langle xv, w \rangle = (-1)^{|x||v|} \langle v, \tau_d(x)w \rangle\) for all \(x \in \hat{\Gamma}^d A\) and \(v, w \in W\).

**Lemma 3.18.** Let \(\langle \cdot, \cdot \rangle\) be an even supersymmetric or superantisymmetric, non-degenerate bilinear form on \(V_R\), such that \((V_{R,a}, V_{R,a}) = 0\) and the \(R\)-complement \(V_{R,\epsilon}\) of \(V_{R,a}\) in \(V_{R,0}\) can be chosen so that the restriction of \((\cdot, \cdot)\) to \((V_{R,a} \times V_{R,\epsilon})\) is a perfect pairing. Then \((\cdot, \cdot)_{d}\) is a non-degenerate \(\tau_d\)-contravariant form on \(\hat{\Gamma}^d_{A_R}\text{-mod,}\) \(\hat{\Gamma}^d_{V_R}\).

**Proof.** For \(a = a_1 \cdots a_d \in A^d\) and \(v = v_1 \cdots v_d \in V^d\), we set \(a \cdot v := (a_1 v_1) \cdots (a_d v_d) \in V^d\). Then \((-1)^{a \cdot v} = (-1)^{(a_1 v_1) + \cdots + (a_d v_d)}\). Using this, it is easy to establish that \((\cdot, \cdot)_{d}\) is a \(\tau^{\otimes d}\)-contravariant form on \(V^{\otimes d}\), cf. (36). The lemma now follows from Proposition 3.11. \(\square\)

Recall the coproduct from from §3.3 for supermodules \(X \in \hat{\Gamma}^d_{A_R}\text{-mod}\) and \(Y \in \hat{\Gamma}^{e}\text{-mod}\), we have a structure of a \(\hat{\Gamma}^{d+e}_{A_R}\text{-module}\) on \(X \otimes Y\). Recalling Lemma 3.14 and the isomorphism from Lemma 3.5, we now obtain:
Lemma 3.19. Let $V_R$ and $W_R$ be calibrated $A_R$-supermodules. Then we have an isomorphism of $\bar{\Gamma}^d A_R$-modules

$$\bigoplus_{d_1+d_2=d} (\bar{\Gamma}^{d_1} V_R) \otimes (\bar{\Gamma}^{d_2} W_R) \overset{\sim}{\longrightarrow} \bar{\Gamma}^d (V_R \oplus W_R), \quad y \otimes y' \mapsto y \ast y'.$$

4. Generalized Schur algebras

Throughout the section we fix $n \in \mathbb{Z}_{>0}$ and a calibrated superalgebra $A_R$ over $R$ as in $\S 3.3$. In particular we have fixed $a_R, c_R, A_R, 1$ with bases $B_0, B, B_1,$ and $B = B_0 \sqcup B_1 \sqcup B_1$ is a basis of $A_R$.

4.1. The algebra $T^A(n, d)$. We consider the matrix superalgebra $M_n(A_R)$ with $M_n(A_R)_0 = M_n(A_R)_0$ and $M_n(A_R)_1 = M_n(A_R)_1$. For $b \in B$, we denote by $\bar{\eta}^b \in M_n(A_R)$ so that

$$B^{M_n(A)} := \{ \bar{\eta}^b \mid b \in B, 1 \leq r, s \leq n \}$$

is a basis of $M_n(A_R)$.

This superalgebra is calibrated via $M_n(A_R)_a := M_n(a_R)$ and $M_n(A_R)_c := M_n(c_R)$. So for $d \in \mathbb{Z}_{>0}$, we may define

$$T^A(n, d)_R := \bar{\Gamma}^d M_n(A_R).$$

The theory developed in Section $3$ applies with $A_R$ replaced by $M_n(A_R)$. To facilitate the transition from $A_R$ to $M_n(A_R)$, it is convenient to adopt a special notation. First of all, taking into account that the basis of $M_n(A_R)$ is of the form $\xi^r$, we have element $\bar{\eta}^x$ from $\S 3.12$ labeled by $d$-tuples $x = \xi_{r_1, s_1} \cdots \xi_{r_d, s_d}$ of basis elements from $\S 1.1$ such that $(b_1, r_1, s_1) = (b_j, r_j, s_j)$ for $i \neq j$ only if $b_j$ is even. It is convenient to denote such $\bar{\eta}^x$ rather by $\eta^{b, s}_{r, s}$ where $b = b_1 \cdots b_d$, $r := r_1 \cdots r_d$ and $s := s_1 \cdots s_d$. Thus,

$$\eta^{b, s}_{r, s} := [b, r, s]_l^! \sum_{(c, t, u) \prec (b, r, s)} (-1)^{(b, r, s) + (c, t, u)} \xi^c_{t_1, u_1} \otimes \cdots \otimes \xi^{c_d}_{t_d, u_d} \in T^A(n, d)_R,$$

where

$$\langle b, r, s \rangle := \left\{ (k, l) \in [d]^2 \mid k < l, \ b_k, b_l \in B_1, \ (b_k, r_k, s_k) > (b_l, r_l, s_l) \right\}$$

is the analogue of $\S 3.1$ (where ‘$>$’ is a fixed total order on $B \times [n] \times [n]$) and

$$[b, r, s]_l^! := \prod_{b \in B_0, r, s \in [n]} \left\{ k \in [d] \mid (b_k, r_k, s_k) = (b, r, s) \right\} !$$

is the analogue of $\S 3.3$.

Define $\text{Tri}^B(n, d)$ to be the set of all triples

$$(b, r, s) = (b_1 \cdots b_d, r_1 \cdots r_d, s_1 \cdots s_d) \in B^d \times [n]^d \times [n]^d$$

such that for all $1 \leq k \neq l \leq d$ we have $(b_k, r_k, s_k) = (b_l, r_l, s_l)$ only if $b_k \in B_0$. To connect with the theory developed in $\S 3$ we identify $\text{Tri}^B(n, d)$ with $\text{Seq}(B^{M_n(A)}_d)$ via the map

$$\text{Tri}^B(n, d) \overset{\sim}{\longrightarrow} \text{Seq}(B^{M_n(A)}_d), \quad (b, r, s) \mapsto \xi^{b_1}_{r_1, s_1} \cdots \xi^{b_d}_{r_d, s_d}.$$
Applying the theory developed in [233] replacing \( A_R \) with \( M_n(A_R) \), we also have a co-product \( \nabla \) on \( \bigoplus_{d \geq 0} T^A(n, d) \), which allows us to consider the tensor product \( V \otimes W \) of \( V \in T^A(n, d)_{R} \)-mod and \( W \in T^A(n, e)_{R} \)-mod as a supermodule over \( T^A(n, d + e) \).

Extending scalars from \( R \) to \( \mathbb{F} \), we now define the \( \mathbb{F} \)-superalgebra

\[
T^A(n, d) := \mathbb{F} \otimes_R T^A(n, d)_R.
\]

We denote \( 1_{\mathbb{F}} \otimes \eta^b_{r, s} \in T^A(n, d) \), again by \( \eta^b_{r, s} \). The map \( \text{id}_V \otimes \nabla \) again by \( \nabla \), etc. In particular, given \( V \in T^A(n, d) \)-mod and \( W \in T^A(n, e) \)-mod, we consider \( V \otimes W \) as a \( T^A(n, d + e) \)-superspace module by \( \nabla \).

### 4.2. Quasi-hereditary structure on \( T^A(n, d) \).

Throughout the section, we assume that \( d \leq n \).

Let \( A_R \) be a based quasi-hereditary superalgebra with conforming heredity data \( I, X, Y \), see [233]. In particular, \( A_R \) comes with the standard idempotents \( \{ e_i \mid i \in I \} \) in \( a_R \).

For \( \lambda \in \Lambda(n, d) \), set \( I^\lambda := 1^\lambda \cdot \cdots \cdot n^\lambda \in [n]^d \). For an idempotent \( e \in a_R \), we have an idempotent \( \eta^e := \eta^e_{\lambda^I} \in T^A(n, d) \). For all \( \lambda = (\lambda(0), \ldots, \lambda(d)) \in \Lambda^I(n, d) \), we have the orthogonal idempotents

\[
\eta^\lambda := \eta^e_{\lambda(0)} \ast \cdots \ast \eta^e_{\lambda(d)} \subseteq T^A(n, d).
\]

Given a \( T^A(n, d) \)-superspace module \( V \) we refer to the vectors of \( \eta^\lambda V \) as vectors of weight \( \lambda \).

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n) \), define the monomial \( z^\lambda := z_1^{\lambda_1} \cdots z_n^{\lambda_n} \in \mathbb{Z}[z_1, \ldots, z_n] \). For \( \lambda \in \Lambda^I(n) \), we now set

\[
z^\lambda := z^{\lambda(0)} \otimes z^{\lambda(1)} \otimes \cdots \otimes z^{\lambda(d)} \in \mathbb{Z}[z_1, \ldots, z_n]^{\otimes I}.
\]

Following [13] §5A], see especially [13] Lemma 5.9, for a \( T^A(n, d) \)-superspace module \( V \), we define its formal character

\[
\text{ch} V := \sum_{\mu \in \Lambda^I(n, d)} (\dim \eta^\mu V) z^\mu \in \mathbb{Z}[z_1, \ldots, z_n]^{\otimes I}.
\]

If \( \sum_{i \in I} e_i = 1_A \), then \( 1_{T^A(n, d)} = \sum_{\lambda \in \Lambda^I(n, d)} \eta^\lambda \), but we do not need to assume this. So in general we might have \( \sum_{\mu \in \Lambda^I(n, d)} \eta^\mu V \subseteq V \) for \( V \in T^A(n, d) \)-mod. (Another fact that is not going to be used directly is that \( \text{ch} V \) is a symmetric function.)

**Lemma 4.4.** [13] Lemma 5.10] If \( V \in T^A(n, d) \)-mod and \( W \in T^A(n, e) \)-mod, then \( \text{ch}(V \otimes W) = \text{ch}(V) \text{ch}(W) \).

By [14] Theorem 6.6], \( T^A(n, d) \) is a based quasi-hereditary algebra with standard idempotents \( \{ \eta^\lambda \mid \lambda \in \Lambda^I_+(n, d) \} \), see [14] §6. The corresponding simple, standard and costandard modules are denoted

\[
\{L(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\}, \quad \{\Delta(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\}, \quad \{\nabla(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\},
\]

respectively.

Recalling the \( X \)-colored standard tableaux from [233] for \( \lambda \in \Lambda^I_+(n, d) \), the standard module \( \Delta(\lambda) \) has basis

\[
\{v_T \mid T \in \text{Std}^X(\lambda)\}.
\]

Moreover, if \( T \in \text{Std}^X(\lambda, \mu) \) for some \( \mu \in \Lambda^I(n, d) \), see [233], then

\[
v_T \in \eta^\mu \Delta(\lambda).
\]

**Lemma 4.7.** [15] Lemma 3.23] The formal characters \( \{\text{ch} \Delta(\lambda) \mid \lambda \in \Lambda^I_+(n, d)\} \) are linearly independent.
Recalling the notation (2.12), we have:

**Lemma 4.8.** \[\text{Theorem 6.17(i)}\] Let \(\lambda = (\lambda^{(j)})_{j \in I} \in \Lambda^I_+(n,d)\). Then

\[
\Delta(\lambda) \simeq \bigotimes_{i \in I} \Delta(\epsilon_i(\lambda^{(i)})�)
\]

For \(\lambda = (\lambda^{(j)})_{j \in I} \in \Lambda^I_+(n,d)\), \(\mu = (\mu^{(j)})_{j \in I} \in \Lambda^I_+(n,e)\) and \(\nu = (\nu^{(j)})_{j \in I} \in \Lambda^I_+(n,d + e)\) we define

\[
c_{\lambda,\mu}^{\nu} := \prod_{j \in I} c_{\lambda^{(j)},\mu^{(j)}}^{\nu^{(j)}}.
\] (4.9)

We will crucially use the following

**Theorem 4.10.** \[\text{Main Theorem, Corollary 3.30}\] Let \(\lambda \in \Lambda^I_+(n,d)\) and \(\mu \in \Lambda^I_+(n,e)\). If \(d + e \leq n\) then \(\Delta(\lambda) \otimes \Delta(\mu)\) (resp. \(\nabla(\lambda) \otimes \nabla(\mu)\)) has a standard (resp. costandard) filtration, and

\[
\text{ch}[(\Delta(\lambda) \otimes \Delta(\mu))] = \sum_{\nu \in \Lambda^I_+(n,d + e)} c_{\lambda,\mu}^{\nu} \text{ch}[(\nu)].
\]

An immediate corollary of Theorem 4.10 is

**Corollary 4.11.** Let \(T \in T^A(n,d)\)-mod and \(T' \in T^A(n,e)\)-mod be tilting supermodules. If \(d + e \leq n\) then the tensor product \(T \otimes T'\) is a tilting supermodule over \(T^A(n,d + e)\).

### 4.3. Modified divided powers of modules over \(M_n(A_R)\)

In this subsection we work over \(R\).

Let \(A_R = a_R \oplus c_R \oplus A_{R,1}\) be a calibrated superalgebra as in 3.3 with basis \(B^A = B^A_a \cup B^A_c \cup B^A_1\), and \(V_R = V_{R,a} \oplus V_{R,c} \oplus V_{R,1}\) be a calibrated \(A_R\)-supermodule as in 3.4. Recall that by definition \(T^A(n,d)_R\) is \(\bar{T}^d M_n(A_R)\) for the calibration on \(M_n(A_R)\) induced by that on \(A_R\).

The \(R\)-supermodule of column vectors \(\text{Col}_n(V_R) = V_R^{\otimes n}\) is a left supermodule over \(M_n(A_R)\) in a natural way. In fact, it is a calibrated \(M_n(A_R)\)-supermodule with \(\text{Col}_n(V_{R,a}) := \text{Col}_n(V_{R,a})\) and \(\text{Col}_n(V_{R,c}) := \text{Col}_n(V_{R,c})\). Then by Lemma 3.16 we have \(\bar{T}^d \text{Col}_n(V_R)\) is a left supermodule over \(T^A(n,d)_R = \bar{T}^d M_n(A_R)\).

Extending scalars to \(F\) we get the left module

\[
\bar{T}^d \text{Col}_n(V) := F \otimes_R \bar{T}^d \text{Col}_n(V_R)
\]
over \(T^A(n,d) = F \otimes_R T^A(n,d)_R\).

**Lemma 4.12.** Let \(e \in a_R\) be an idempotent such that \(b = b\) or \(0\) for all \(b \in B^A\). Then \(\bar{T}^d(\text{Col}_n(A_R e)) \simeq M_n(A_R e)_{1,1}^{d,1}\) as \(M_n(A_R)\)-supermodules. The result now follows by applying Lemma 3.17.

**Proof.** First notice that \(\text{Col}_n(A_R e) \simeq M_n(A_R)_{1,1}^{e} \otimes M_n(A_R)_{1,1}^{d,1}\) as \(M_n(A_R)\)-supermodules. The result now follows by applying Lemma 3.17. \(\square\)

We will also need the right module versions of the results of this subsection. If \(V_R\) is a calibrated right \(A_R\)-supermodule, we consider the row vectors \(\text{Row}_n(V_R) = V_R^{\otimes n}\) as a calibrated right \(M_n(A_R)\)-supermodule. Then we obtain a right \(T^A(n,d)_R\)-supermodule structure on \(\bar{T}^d \text{Row}_n(V_R)\) and a right \(T^A(n,d)\)-module \(\bar{T}^d \text{Row}_n(V) := F \otimes_R \bar{T}^d \text{Row}_n(V_R)\). The right module analogue of Lemma 4.12 is then clear.
5. The extended zigzag Schur algebra

5.1. The extended zigzag algebra. Fix \( \ell \geq 1 \) and set
\[
I := \{0, 1, \ldots, \ell\}, \quad J := \{0, 1, \ldots, \ell - 1\}.
\]
Let \( \Gamma \) be the quiver with vertex set \( I \) and arrows \( \{a_{j,j+1}, a_{j+1,j} \mid j \in J\} \) as in the picture:

The extended zigzag algebra \( Z \) is the path algebra \( k\Gamma \) modulo the following relations:
(i) All paths of length three or greater are zero.
(ii) All paths of length two that are not cycles are zero.
(iii) All length-two cycles based at the same vertex are equivalent.
(iv) \( a_{\ell,\ell-1}a_{\ell-1,\ell} = 0 \).

Length zero paths yield the standard idempotents \( \{e_i \mid i \in I\} \) with \( e_i a_{ij} e_j = a_{ij} \) for all admissible \( i, j \). The algebra \( Z \) is graded by the path length: \( Z = \mathbb{Z}^0 \oplus \mathbb{Z}^1 \oplus \mathbb{Z}^2 \). We consider \( Z \) as a superalgebra with \( \mathbb{Z} \) and, up to isomorphism, \( \mathbb{Z}^0 = \mathbb{Z}^0 \oplus \mathbb{Z}^2 \) and \( \mathbb{Z}^1 = \mathbb{Z}^1 \). Define \( c_j := a_{j,j+1}a_{j+1,j} \) for all \( j \in J \).

The superalgebra \( Z \) has an anti-involution \( \tau \) with
\[
\tau(e_i) = e_i, \quad \tau(a_{ij}) = a_{ji}, \quad \tau(c_j) = -c_j.
\]

We consider the total order on \( I \) given by \( 0 < 1 < \cdots < \ell \). For \( j \in I \), we set
\[
X(j) := \begin{cases} \{e_i, a_{i-1,i} \} & \text{if } i > 0, \\ \{e_0\} & \text{if } i = 0 \end{cases} \quad Y(j) := \begin{cases} \{e_i, a_{i,i-1} \} & \text{if } i > 0, \\ \{e_0\} & \text{if } i = 0 \end{cases}
\]
(5.1)

With respect to this data we have:

**Lemma 5.2.** \([2]\) Lemma 4.14 The graded superalgebra \( Z \) is based quasi-hereditary with conforming heredity data \( I, X, Y \). For the corresponding heredity basis \( B \) we have \( B_a = \{e_i \mid i \in I\} \), \( B_0 = \{c_j \mid j \in J\} \), \( B_1 = \{a_{j,j+1}, a_{j+1,j} \mid j \in J\} \).

For \( i \in I \), let \( L(i) = k \cdot v_i \) with \( |v_i| = 0 \) and the action \( e_i v_i = v_i, b v_i = 0 \) for all \( b \in B \setminus \{e_i\} \). This makes \( L(i) \) a \( Z \)-supermodule, and, up to isomorphism, \( \{L(i) \mid i \in I\} \) is a complete set of irreducible \( Z \)-supermodules. Note that \( L(i) \simeq L(i) \).

The standard modules \( \Delta(i) \) have similarly explicit description: \( \Delta(0) \simeq L(0) \), and for \( i > 0 \), \( \Delta(i) \) has basis \( \{v_i, w_i\} \) with \( |v_i| = 0, |w_i| = 1 \) and the only non-trivial action is: \( e_i v_i = v_i, e_{i-1} w_i = w_i, a_{i-1,j} v_i = w_i \). For the costandard modules we have \( \nabla(0) \simeq \Delta(0)^t \simeq L(0) \), and for \( i > 0 \), \( \nabla(i) \simeq \Delta(i)^t \) has basis \( \{v_i^t, w_i^t\} \) with \( |v_i^t| = 0, |w_i^t| = 1 \) and the only non-trivial action is: \( e_i v_i^t = v_i^t, e_{i-1} w_i^t = w_i^t, a_{i-1,i} w_i^t = -v_i^t \).

The indecomposable tilting supermodules over \( Z \) are as follows: \( T(0) \simeq L(0) \) and \( T(i) = \Pi Z e_{i-1} \) for \( i > 0 \). It will actually be more convenient for us to work with the tilting modules \( \Pi I T(i) \). Thus \( \Pi I T(0) = \Pi L(0) \) is 1-dimensional with basis \( \{v_0\} \) where \( v_0 \) is odd, \( \Pi I T(1) = Z e_0 \) has basis \( \{e_0, a_{1,0}, c_0\} \), and, for \( i > 1 \), \( \Pi I T(i) = Z e_{i-1} \) has basis \( \{e_{i-1}, a_{i-2,i-1}, a_{i,i-1}, c_{i-1}\} \). For \( i > 0 \), \( \Pi I T(i) \) has a standard filtration \( \Pi I \Delta(i) \simeq \Delta(i-1) \) (this means \( \Pi I \Delta(i) \subseteq T(i) \) and \( T(i)/\Pi I \Delta(i) \simeq \Delta(i-1) \)) and a costandard filtration \( \nabla(i-1)/\Pi I \nabla(i) \).

We have a full tilting module
\[
T := \bigoplus_{i \in I} \Pi I T(i) = \Pi I L(0) \oplus \bigoplus_{i=0}^{\ell-1} Z e_i.
\]
and the Ringel dual algebra $Z' = \text{End}_Z(T)^{\text{op}}$. In fact, $Z$ is Ringel self-dual, i.e. there is an isomorphism of superalgebras $Z' \cong Z$ which we now proceed to describe.

For any $i \in I$, let $\iota_i : \Pi T(i) \to T$ and $\pi_i : T \to \Pi T(i)$ be the natural embedding and projection. We have the right multiplication maps $\rho_{c_i} : Z e_i \to Z e_i, v \mapsto v c_i$ and $\rho_{a_{ij}} : Z e_i \to Z e_j, v \mapsto (-1)^{|v|} v a_{ij}$. Let $f : \Pi T(0) = \Pi L(0) \leftrightarrow \Pi T(1) = Z e_0$ be the embedding given by $v_0 \mapsto c_0$, and let $g : \Pi T(1) = Z e_0 \to \Pi T(0) = \Pi L(0)$ be the surjection such that $e_0 \mapsto v_0$. Note that $f$ and $g$ are odd. Define the following elements of $Z'$:

- $e'_i := \pi_{\ell-i}$ for all $i \in I$;
- $c'_i := \iota_{\ell-i} \circ \rho_{c_{\ell-i-1}} \circ \pi_{\ell-i}$ for all $i \in J$;
- $a'_{i+1,i} := \iota_{\ell-i} \circ \rho_{a_{\ell-i-2,\ell-i-1}} \circ \pi_{\ell-i-1}$ and $a'_{i,i+1} := \iota_{\ell-i-1} \circ \rho_{a_{\ell-i-1,\ell-i-2}} \circ \pi_{\ell-i}$, for all $i = 0, \ldots, \ell - 2$; 
- $a'_{i,\ell-1} := \iota_1 \circ f \circ \pi_0$ and $a'_{\ell-1,i} := t_0 \circ g \circ \pi_1$.

**Lemma 5.3.** Mapping $e_i \mapsto e'_i$, $a_{i,j} \mapsto a'_{i,j}$, $c_i \mapsto c'_i$ is an isomorphism of superalgebras $Z \cong Z'$. In particular, $Z$ is Ringel self-dual.

**Proof.** Using the fact that dim $\text{Hom}_Z(\Delta(i), \nabla(j)) = 0$ if $i > j$, and is equal to 1 if $i = j$, it is easy to see that

$$\{e'_i \mid i \in I\} \sqcup \{a'_{i,j+1}, a'_{j+1,i}, c'_j \mid j \in J\}$$

is a basis for $Z'$. Hence the given map is a linear isomorphism.

To check that the map is an algebra homomorphism, one proceeds by cases. For example, for $i \in J$, in $Z' = \text{End}_Z(T)^{\text{op}}$ we have

$$a'_{i+1,i} \cdot a'_{i,i+1} = -a'_{i+1,i} \circ a'_{i,i+1}$$

$$= -\iota_{\ell-i} \circ \rho_{a_{\ell-i-2,\ell-i-1}} \circ \pi_{\ell-i-1} \circ \iota_{\ell-i-1} \circ \rho_{a_{\ell-i-1,\ell-i-2}} \circ \pi_{\ell-i}$$

$$= -\iota_{\ell-i} \circ \rho_{a_{\ell-i-2,\ell-i-1}} \circ \rho_{a_{\ell-i-1,\ell-i-2}} \circ \pi_{\ell-i}$$

$$= \iota_{\ell-i} \circ \rho_{a_{\ell-i-1,\ell-i-2}} \circ \pi_{\ell-i}$$

$$= c'_i.$$

The other cases are checked similarly. \qed

We use the isomorphism $Z \cong Z'$ of Lemma [5.3] to transport the heredity data $I, X, Y$ from $Z$ onto a heredity data $I', X', Y'$ for $Z'$ so that $I' = I$ with the same order, and

$$X'(i) := \begin{cases} \{e'_i, a'_{i-1,i} \} & \text{if } i > 0, \\ \{e'_0 \} & \text{if } i = 0. \end{cases} \quad Y'(i) := \begin{cases} \{e'_i, a'_{i,i-1} \} & \text{if } i > 0, \\ \{e'_0 \} & \text{if } i = 0. \end{cases}$$

(Usually, one gets the opposite partial order on $I$ in this place but we have built in a relabeling $i \mapsto \ell - i$ into the construction). With this hereditary data, we have the right modules $L'(i), \Delta'(i), Y'(i)$ and $T'(i)$. For example, $T'(0) \cong L'(0)$ (with $e'_0$ acting as identity and all the other standard generators acting as 0) and $T'(i) \cong \Pi e'_{i-1} Z'$ for $i > 0$. A routine check shows that, as a right $Z'$-module, $T$ decomposes as follows:

$$T = \bigoplus_{i \in I} \Pi T'(i),$$

where the summands are defined explicitly as follows:

- $\Pi T'(0) = k \cdot a_{\ell-1} \subseteq Z e_{\ell-1} = \Pi T(\ell) \subseteq T$;
- $\Pi T'(\ell) = \text{span}_k \langle v_0, e_0, c_0, a_{0,1} \rangle \subseteq \Pi T(0) \oplus \Pi T(1) \oplus \Pi T(2) \subseteq T$ (dropping $a_{0,1}$ if $\ell = 1$);
• for $i \neq 0, \ell$, we set
  \[
  \Pi T'(i) = \text{span}_k(e_{\ell-i}, a_{\ell-i, \ell-i-1}, a_{\ell-i, \ell-i+1}, c_{\ell-i})
  \subseteq \Pi T(\ell - i + 1) \oplus \Pi T(\ell - i) \oplus \Pi T(\ell - i + 2) \subseteq T
  \] (dropping $a_{\ell-i, \ell-i+1}$ if $i = 1$).

In this last paragraph we suppose that $k = R$ and recall the theory of §3.3.4. We have the subalgebra $a_R := \text{span}(B_a) = \text{span}_R(e_i \mid i \in I) \subseteq Z_{R,0}$ and the analogous subalgebra $a'_R = \text{span}_R(e'_i \mid i \in I) \subseteq Z'_{R,0}$. We have $R$-module decomposition $T_{R,0} = T_{R,a} \oplus T_{R,c}$ where $T_{R,a} = \text{span}_R(e_j \mid j \in J)$ and $T_{R,c} = \text{span}_R(c_j \mid j \in J)$. Then it is clear from the explicit construction above that

\[
\Pi T \cdot (T_{R,a}) \cdot a'_R \subseteq T_{R,a}.
\]

For any $i \in I$, $\Pi T(i)_{R,a} := \Pi T(i) \cap T_{R,a}$, $\Pi T(i)_{R,c} := \Pi T(i) \cap T_{R,c}$, $\Pi T'(i)_{R,a} := \Pi T'(i) \cap T_{R,a}$, $\Pi T'(i)_{R,c} := \Pi T'(i) \cap T_{R,c}$. These define calibrations on each $\Pi T(i)_{R}$ and $T_{R}$ as left $Z_{R}$-supermodules, as well as on each $\Pi T'(i)_{R}$ and $T_{R}$ as right $Z'_{R}$-supermodules.

5.2. A full tilting module for $TZ(n, d)$. From now on until the end of the paper we fix $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{>0}$ such that $d \leq n$.

To construct a full tilting module for quasihereditary superalgebra $TZ(n, d)$, we need to first work integrally. Recall from the previous subsection that $T_{R} = (Z_{R}, Z'_{R})$-bimodule.

Recall the constructions of §4.3. In particular, we have the left $M_n(Z_{R})$-module structure on $\text{Col}_n(T_R)$ and a right $M_n(Z'_{R})$-module structure on $\text{Row}_n(T_R)$. Setting as in §4.3 $\text{Col}_n(T_R)_a := \text{Col}_n(T_{R,a})$ and $\text{Row}_n(T_R)_a := \text{Row}_n(T_{R,a})$, we have by (5.4) that $M_n(a) \text{Col}_n(T_R)_a \subseteq \text{Col}_n(T_R)_a$ and $\text{Row}_n(T_R)_a M_n(a) \subseteq \text{Row}_n(T_R)_a$. So by Lemma 3.16 (and its right module analogue), $\hat{T}_d \text{Col}_n(T_R)$ is a left module over $TZ(n, d)_R$ and $\hat{T}_d \text{Row}_n(T_R)$ is a right module over $TZ(n, d)_R$. Similarly, for every $i \in I$, we have left $TZ(n, d)_R$-modules $\hat{T}_d \text{Col}_n(\Pi T(i)_R)$ and right $TZ(n, d)_R$-modules $\text{Row}_n(\Pi T'(i)_R)$. Extending scalars, we have a left supermodule

\[
\mathcal{J}_d := F \otimes_R \hat{T}_d \text{Col}_n(\Pi T(i)_R)
\]

over $TZ(n, d) = F \otimes_R TZ(n, d)_R$.

Recall from §4.2 that for $d \leq n$, the algebra $TZ(n, d)$ is quasi-hereditary with respect to the poset $\Lambda^f(n, d)$ with partial order $\leq_f$, so it has its own standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda^f(n, d)\}$, costandard modules $\{\nabla(\lambda) \mid \lambda \in \Lambda^f(n, d)\}$ and indecomposable tilting modules $\{T(\lambda) \mid \lambda \in \Lambda^f(n, d)\}$. Moreover, by §14 Proposition 6.20, the anti-involution $\tau$ on $Z$ extends to the anti-involution

\[
\tau_{n,d} : TZ(n, d) \to TZ(n, d), \quad \eta_{\mu} \mapsto \eta_{s_{\mu}}^{\tau(b)}(b)
\]

where for a $d$-tuple $b = b_1 \cdots b_d \in \mathbb{B}^d$ we denote $\tau(b) := \tau(b_1) \cdots \tau(b_d)$. As in §15 (2.14), we then have for all $\lambda \in \Lambda^f(n, d)$$: \Delta(\lambda)^{\tau_{n,d}} \simeq \nabla(\lambda)$.

Since $\tau_{n,d}(\eta_{\mu}) = \eta_{\mu}$ for all $\mu \in \Lambda^f(n, d)$, we deduce:

**Lemma 5.6.** For all $\lambda \in \Lambda^f(n, d)$, we have $\text{ch} \Delta(\lambda) = \text{ch} \nabla(\lambda)$.

**Proposition 5.7.** For all $i \in I$, the left $TZ(n, d)$-module $\mathcal{J}_d^i$ is tilting and has a unique maximal weight $\iota_i(1^d)$ (with respect to $\leq_f$).
Proof. Suppose first that \( i = 0 \). Since \( \omega_0(1^d) \) is minimal in \( \Lambda^+_f(n,d) \) it follows that \( T(\omega_0(1^d)) \simeq \Delta(\omega_0(1^d)) \simeq \Delta(\omega_0(1^d)) \). Using the assumption \( d \leq n \), we note that \( \omega_0(1^d) \) is the only weight of \( \mathcal{F}_0^d \) that lies in \( \Lambda^+_f(n,d) \), and the corresponding weight multiplicity is 1, so

\[
\mathcal{F}_0^d \cong L(\omega_0(1^d)) \simeq T(\omega_0(1^d)) \simeq \Delta(\omega_0(1^d)).
\] (5.8)

Let now \( i > 0 \). Since \( \Pi \Pi(i)_R = \mathbb{Z}_R e_{i-1} \), we have \( \text{Col}_n(\Pi \Pi(i)_R) \simeq M_n(\mathbb{Z}_R) \xi_{i-1}^{1} \), and so by Lemma [3.12] and extension of scalars, we have

\[
\mathcal{F}_i^d \simeq T^Z(n,d)\eta_{i,d^1,d}^{e_{i-1}} = T^Z(n,d)\eta_{i-1}(d).
\] (5.9)

In particular, \( \mathcal{F}_i^d \) is projective, and thus has a standard filtration. To prove that \( \mathcal{F}_i^d \) also has a costandard filtration, it suffices to show that it is \( \tau_{n,d} \)-self-dual, or equivalently possesses a non-degenerate \( \tau_{n,d} \)-contravariant bilinear form.

To construct this form we work over \( R \). Recall that for \( i > 1 \), we have \( \Pi \Pi(i)_R = \mathbb{Z}_R e_{i-1} \) has basis \( B^{\Pi \Pi(i)} := \{ e_{i-1}, c_{i-1}, a_{i-2,i-1}, a_{i,i-1} \} \). Consider the bilinear form \((\cdot,\cdot)\) on \( \Pi \Pi(i)_R \) such that

\[
(e_{i-1}, c_{i-1}) = -(c_{i-1}, e_{i-1}) = (a_{i-2,i-1}, a_{i-2,i-1}) = (a_{i,i-1}, a_{i,i-1}) = 1,
\]

and all the other pairings of basis elements are 0. Note that this form is even, non-degenerate, \( \tau \)-contravariant, and superantisymmetric. Extending this form in the obvious way to \( \text{Col}_n(\Pi \Pi(i)_R) \) results in a non-degenerate, superantisymmetric, \( \tau_{n,1} \)-contravariant form again denoted \((\cdot,\cdot)\). Note that this form satisfies the assumptions of Lemma \([3.18]\). Applying Lemma \([3.18]\) we obtain an even, non-degenerate, \( \tau_{n,d} \)-contravariant form on \( \Gamma^d \text{Col}_n(\Pi \Pi(i)_R) \). Extending scalars, we deduce that, \( \mathcal{F}_i^d \) is \( \tau_{n,d} \)-self-dual.

Denote by \( v^b_r \in \text{Col}_n(\Pi \Pi(i)_R) \) the column vector with \( b \in \Pi \Pi(i)_R \) in the \( r \)th position and 0s elsewhere. To see that the unique maximal weight of \( \mathcal{F}_i^d \) is \( \omega_i(1^d) \), observe that the unique maximal weight vector of \( \Pi \Pi(i)_R \) is \( a_{i,i-1} \), which is odd, and so the vector \( v^a_{i,i-1} \cdots v^a_{i,i-1} \in \Gamma^d \text{Col}_n(\Pi \Pi(i)_R) \) has weight \( \omega_i(1^d) \) and all other weight vectors appearing in \( \Gamma^d \text{Col}_n(\Pi \Pi(i)_R) \) have smaller weight.

The case \( i = 1 \) is similar to the case \( i > 1 \) but \( \Pi \Pi(1)_R = \mathbb{Z}_R e_0 \) has basis \( B^{\Pi \Pi(1)} := \{ e_0, c_0, a_{1,0} \} \), and we use the form such that

\[
(e_0, c_0) = -(c_0, e_0) = (a_{1,0}, a_{1,0}) = 1
\]

are the only non-trivial pairings of basis elements.

Recalling that \( T_R \) is a \((\mathbb{Z}_R, T_R')\)-bisupermodule, we can now consider \( M_n(T_R) \) as an \((M_n(\mathbb{Z}_R), M_n(\mathbb{Z}_R'))\)-bisupermodule in the obvious way. Take \( M_n(T_R)_a := M_n(T_R,a) \) and \( M_n(T_R)_c := M_n(T_R,c) \). In view of Lemma \([3.16]\) (and its right module analogue), \( \Gamma^d M_n(T_R) \) is a \((T^Z(n,d)_R, T^Z(n,d)_R')\)-bisupermodule. We now extend the scalars from \( R \) to \( \mathbb{F} \) to get the \((T^Z(n,d), T^Z(n,d))\)-bisupermodule

\[
\mathcal{F} := \mathbb{F} \otimes_R \hat{\Gamma}^d M_n(T_R).
\]

For each composition \( \mu \in \Lambda(n,d) \) define \( \mathcal{F}_\mu^i := \mathcal{F}_\mu^{\mu_1} \otimes \cdots \otimes \mathcal{F}_\mu^{\mu_n} \). Furthermore, for each multicomposition \( \mu = (\mu^{(0)}, \ldots, \mu^{(l)}) \in \Lambda^l(n,d) \) define \( \mathcal{F}_\mu^i := \mathcal{F}_0^{(0)} \otimes \cdots \otimes \mathcal{F}_l^{(0)} \).

Since \( T = \bigoplus_{i \in I} \Pi \Pi(i)_R \), as left supermodules over \( M_n(\mathbb{Z}_R) \), we have

\[
M_n(T_R) \simeq \text{Col}_n(T_R)_{\oplus n} \simeq \bigoplus_{i \in I} \text{Col}_n(\Pi \Pi(i)_R)_{\oplus n}.
\]
Now, using Lemmas 3.19 and extending scalars, we have as left $T^Z(n,d)$-supermodules:

$$\mathcal{T} \simeq \bigoplus_{\mu \in \Lambda^I(n,d)} \mathcal{T}^\mu.$$  \hspace{1cm} (5.10)

For $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^I_+(n,d)$, we define the conjugate multipartition

$$\lambda' := ((\lambda^{(0)})', \ldots, (\lambda^{(\ell)})') \in \Lambda^I_+(n,d).$$  \hspace{1cm} (5.11)

**Proposition 5.12.** As a left $T^Z(n,d)$-supermodule, $\mathcal{T}$ is a full tilting supermodule.

**Proof.** Note that each $T^\mu$ is tilting by Proposition 5.7 and Corollary 4.11. So $\mathcal{T}$ is tilting by (5.10). To show that $\mathcal{T}$ is full tilting, it suffices for each $\lambda \in \Lambda^I_+(n,d)$ to find a summand $T^\mu$ in (5.10) which has maximal weight $\lambda$. Fix $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell)}) \in \Lambda^I_+(n,d)$ and take $\mu = \lambda'$. By Proposition 5.7 again, $T^\mu$ has unique maximal weight $\iota_i(1^s)$ for each $s \in \mathbb{Z}_{>0}$. So

$$\sum_{i=0}^{\ell} \sum_{r=1}^{n} \iota_i(1^{(i)}) = \sum_{i=0}^{\ell} \iota_i(\lambda^{(i)}) = \lambda$$

is the unique maximal weight of $\mathcal{T}^\mu$. \hfill \Box

**Corollary 5.13.** As a left $T^Z(n,d)$-supermodule and as a right $T^Z(n,d)$-supermodule, $\mathcal{T}$ is faithful.

**Proof.** As a left $T^Z(n,d)$-supermodule, $\mathcal{T}$ is faithful since it is full tilting by Proposition 5.12 (a full tilting supermodule is faithful for example by \cite{17}, Lemma 6). The second statement follows similarly from the right module analogue of that proposition. \hfill \Box

5.3. $T^Z(n,d)$ is Ringel self-dual. In view of Corollary 5.13 we have an embedding of $T^Z(n,d)$ into $\text{End}_{T^Z(n,d)}(\mathcal{T})^{\text{op}}$. To prove that this embedding is an isomorphism, we now compute the dimension of $\text{End}_{T^Z(n,d)}(\mathcal{T})$.

Recalling (2.13), for $\lambda \in \Lambda^I_+(n,d)$ and $\mu \in \Lambda^I(n,d)$, let

$$k_{\lambda,\mu} := |\text{Std}^X(\lambda, \mu)|.$$  

By (4.5), (4.6) and Lemma 5.6 we have

$$k_{\lambda,\mu} = \dim \eta_{\mu} \Delta(\lambda) = \dim \eta_{\mu} \nabla(\lambda).$$  \hspace{1cm} (5.14)

Let $i \in I$. If $i \neq 0$, we define

$$\beta_i(d,s) = \iota_{i-1}((s)) + \iota_i((1^d-s)) \in \Lambda^I_+(n,d)$$

for all $0 \leq s \leq d$. We also define

$$\beta_0(d,0) := \iota_0((1^d)).$$

We define by $\Xi_{d,i}$ to be the set of all $\beta_i(d,s)$'s, i.e.

$$\Xi_{d,i} := \begin{cases} \{ \beta_i(d,s) \mid 0 \leq s \leq d \} & \text{if } i \neq 0, \\ \{ \beta_0(d,0) \} & \text{if } i = 0. \end{cases}$$

**Lemma 5.15.** Let $\beta \in \Lambda^I_+(n,d)$ and $i \in I$. Then

$$(\mathcal{T}_i^d : \Delta(\beta)) = \begin{cases} 1 & \text{if } \beta \in \Xi_{d,i}, \\ 0 & \text{otherwise}. \end{cases}$$
Proof. By (5.8), we have $\mathcal{T}_0^d \cong \Delta(\mu_0((1^d)))$, so we may assume that $i \neq 0$. Then by (5.9), we have $\mathcal{T}_i^d \cong T^Z(n, d)\eta_{k_i-1}(d)$. Now, using (2.10) and (5.14), we get
\[
(\mathcal{T}_i^d : \Delta(\beta)) = \dim \text{Hom}_{T^Z(n, d)}(\mathcal{T}_i^d, \nabla(\beta))
= \dim \text{Hom}_{T^Z(n, d)}(T^Z(n, d)\eta_{k_i-1}(d), \nabla(\beta))
= \dim \eta_{k_i-1}(d)\nabla(\beta)
= k_{\beta, \eta_{k_i-1}(d)}.
\]
It remains to observe that $k_{\beta, \eta_{k_i-1}(d)} = 1$ if $\beta = \beta_i(d, s)$ for some $0 \leq s \leq d$ and $k_{\beta, \eta_{k_i-1}(d)} = 0$ otherwise.

Let $0 \leq r \leq d$. Recalling (4.9), our next goal is to compute the Littlewood-Richardson coefficient $c_{\alpha, \beta}^\lambda$ for all $\lambda \in \Lambda_+^I(n, d)$, $\alpha \in \Lambda_+^I(n, d-r)$ and $\beta \in \Xi_{r,i}$. Let $i \in I$ and $\beta = \beta_i(r, s) \in \Xi_{r,i}$, in particular, $0 \leq s \leq r$, and $s = 0$ if $i = 0$. We define $\Omega_{\beta}^\lambda$ to be the set of all $\alpha = (\alpha^{(0)}, \ldots, \alpha^{(t)}) \in \Lambda_+^I(n, d-r)$ such that $\alpha^{(j)} = \lambda^{(j)}$ for all $j \notin \{i-1, i\}$, $[\alpha^{(i-1)}]$ is obtained from $[\lambda^{(i-1)}]$ by removing $s$ boxes from distinct columns, and $[\alpha^{(i)}]$ is obtained from $[\lambda^{(i)}]$ by removing $r-s$ boxes from distinct rows (if $i = 0$, then the condition on $[\alpha^{(i-1)}]$ should be dropped).

Lemma 5.16. Let $0 \leq r \leq d$, $i \in I$, $\lambda \in \Lambda_+^I(n, d)$, $\alpha \in \Lambda_+^I(n, d-r)$ and $\beta \in \Xi_{r,i}$. Then
\[
c_{\alpha, \beta}^\lambda = \begin{cases} 1 & \text{if } \alpha \in \Omega_{\beta}^\lambda \\ 0 & \text{otherwise.} \end{cases}
\]

Proof. The result is an immediate consequence of the Littlewood-Richardson rule. 

For each $\mu \in \Lambda^I(n, d)$, define $\overrightarrow{\mu} = (\overrightarrow{\mu}^{(0)}, \ldots, \overrightarrow{\mu}^{(t)}) \in \Lambda^I(n, d)$ by setting $\overrightarrow{\mu}^{(i)} := \mu^{(i-1)}$ for all $i \in I$. Recall (5.11).

Proposition 5.17. Let $\lambda \in \Lambda_+^I(n, d)$ and $\mu \in \Lambda^I(n, d)$. Then $(\mathcal{T}_\mu : \Delta(\lambda)) = k_{\lambda, \overrightarrow{\mu}}^\mu$.

Proof. We proceed by induction on the number of non-zero parts of $\mu$. To start the induction, we suppose that $\mu$ has only one row, in which case $\mathcal{T}_\mu \cong \mathcal{T}_i^d$ for some $i$, and the result follows from Lemma 5.15. So we may assume that $\mu$ has at least two rows.

Let now $i$ be maximal such that $\mu^{(i)} \neq \emptyset$ and pick the largest $t$ such that $\mu^{(i)}_t \neq 0$. Denote $r := \mu^{(i)}_t$. Let $\nu^{(i)}$ be $\mu^{(i)}$ with last non-zero row removed:
\[
\nu^{(i)} := (\mu^{(i)}_1, \ldots, \mu^{(i)}_t, 0, \ldots, 0) \in \Lambda(n, \mu^{(i)}_t - r),
\]
and $\nu^{(j)} := \mu^{(j)}$ for all $j \neq i$. Set
\[
\nu := (\nu^{(0)}, \ldots, \nu^{(t)}) \in \Lambda^I(n, d-r).
\]
Then $\mathcal{T}_\mu \cong \mathcal{T}_\nu \otimes \mathcal{T}_r$.

By (5.1), we have $X(0) = \{e_0\}$ and $X(i) = \{e_i, a_{i-1,i}\}$ for $i \neq 0$. Recalling §2.3 for $i \neq 0$, we put the total order on $\mathcal{A}_X^{(i)}$ given by:
\[
1^{e_i} < \ldots < n^{e_i} < 1^{a_{i-1,i}} < \ldots < n^{a_{i-1,i}}.
\]
And we endow $\mathcal{A}_X^{(0)}$ with the order $1^{e_0} < \ldots < n^{e_0}$. 


By [15], Corollary (4.13), inductive hypothesis, Lemma \[5.15\] and Lemma \[5.16\] we have
\[
(\mathcal{T}^\mu : \Delta(\lambda)) = (\mathcal{T}^\nu \otimes \mathcal{T}^r : \Delta(\lambda))
\]
\[
= \sum_{\alpha \in \Lambda^i_{(n,d-r)}} \sum_{\beta \in \Lambda^i_{(n,r)}} c^\lambda_{\alpha,\beta}(\mathcal{T}^\nu : \Delta(\alpha))(\mathcal{T}^r : \Delta(\beta))
\]
\[
= \sum_{\alpha \in \Lambda^i_{(n,d-r)}} \sum_{\beta \in \Xi_{r,i}} k_{\alpha,\beta}^\mu,\nu
\]
\[
= \sum_{\alpha \in \Omega^\lambda_{\beta}} \sum_{\beta \in \Xi_{r,i}} |\text{Std}^X(\alpha', \beta')|.
\]

Since \(k_{\alpha',\beta'} = |\text{Std}^X(\hat{\lambda}', \hat{\mu})|\) it remains to prove that there is a bijection
\[
\bigsqcup_{\beta \in \Xi_{r,i}} \bigsqcup_{\alpha \in \Omega^\lambda_{\beta}} \text{Std}^X(\alpha', \beta') \sim \text{Std}^X(\hat{\lambda}', \hat{\mu}).
\]

Let \(\beta \in \Xi_{r,i}, \alpha \in \Omega^\lambda_{\beta}\) and \(T \in \text{Std}^X(\alpha', \beta')\). By definition, \(\beta\) is of the form \(\beta_i(r,s)\). Moreover, the Young diagram \([\alpha]\) is obtained by removing \(s\) nodes from distinct columns of the \((i-1)\)st component of \([\lambda^{(i-1)}]\) and \(r-s\) nodes from distinct rows of the \(i\)th component of \([\lambda^{(i)}]\) of \([\lambda]\). Therefore \([\hat{\alpha}]\) is obtained by removing \(s\) nodes \(N_1, \ldots, N_s\) from distinct rows of the \((\ell-i+1)\)st component of \([\hat{\lambda}]\) and \(r-s\) nodes \(M_1, \ldots, M_{r-s}\) from distinct columns of the \((\ell-i)\)th component of \([\hat{\lambda}]\). Now extend \(T\) to the tableau \(\hat{T} \in \text{Std}^X(\hat{\lambda}', \hat{\mu})\) by setting
\[
\hat{T}(N_1) = \cdots = \hat{T}(N_s) = t^{\ell-i,s-i+1} \quad \text{and} \quad \hat{T}(M_1) = \cdots = \hat{T}(M_{r-s}) = t^{s-i}.
\]
The tableaux \(T\) is indeed standard since, by maximality of \(i\) and \(t\), we have \(T(N) < t^{\ell-i,s-i+1}\) for all \(N\) in the \((\ell-i+1)\)st component of \([\hat{\lambda}]\) and \(T(N) < t^{s-i}\) for all \(N\) in the \((\ell-i)\)th component of \([\hat{\lambda}]\). Thus, \(T \mapsto \hat{T}\) is clearly injective. To see that it is surjective, it suffices to show that for any \(S \in \text{Std}^X(\hat{\lambda}', \hat{\mu})\) there exists \(\beta \in \Xi_{r,i}\) and \(\alpha \in \Omega^\lambda_{\beta}\) with
\[
[\hat{\lambda}'] \setminus \{N \mid S(N) \in \{t^{\ell-i,s-i+1}, t^{s-i}\}\} = [\hat{\alpha}].
\]

Indeed, there are exactly \(\mu^{(\ell-i)}_t = \mu^{(i)}_s\) nodes \(N\) in the Young diagram \([\hat{\lambda}]\) such that \(S(N) \in \{t^{\ell-i,s-i+1}, t^{s-i}\}\). So for some \(0 \leq s \leq r\), we can write
\[
\{N \in [\hat{\lambda}'] \mid S(N) = t^{\ell-i,s-i+1}\} = \{N_1, \ldots, N_s\},
\]
\[
\{N \in [\hat{\lambda}'] \mid S(N) = t^{s-i}\} = \{M_1, \ldots, M_{r-s}\}.
\]
By maximality of \(i\) and \(t\), we have that the nodes \(N_1, \ldots, N_s\) are in the ends of distinct rows of the \((\ell-i+1)\)st component of \([\hat{\lambda}]\) and the nodes \(M_1, \ldots, M_{r-s}\) are in the ends of distinct columns of the \((\ell-i)\)th component of \([\hat{\lambda}]\). It remains to note that removing these nodes produces a shape \([\hat{\alpha}]\) for \(\alpha \in \Omega^\lambda_{\beta}(r,s)\).

**Theorem 5.18.** Let \(d \leq n\). We have \(\text{End}_{\mathcal{T}^Z(n,d)}(\mathcal{T})^\text{op} \cong T^Z(n,d)\). In particular, \(T^Z(n,d)\) is Ringel self-dual.
Proof. In view of Lemma 5.3, we have that $T^Z(n, d) \cong T^{Z'}(n, d)$. In particular, the second statement of the theorem follows from the first one.

By Corollary 5.13, $T^Z(n, d)$ embeds into $\text{End}_{T^Z(n, d)}(\mathcal{F})^{\text{op}}$. So it suffices to show that $\dim \text{End}_{T^Z(n, d)}(\mathcal{F}) = \dim T^Z(n, d)$.

In view of (5.5), we have that each $\mathcal{F}_\mu$ is $\tau_{n,d}$-self-dual and $(\mathcal{F}_\mu : \Delta(\lambda)) = (\mathcal{F}_\mu : \nabla(\lambda))$ for all $\lambda \in \Lambda_+^{(n,d)}$. We now have:

$$\dim \text{End}_{T^Z(n, d)}(\mathcal{F}) = \sum_{\mu, \nu \in \Lambda_+^{(n,d)}} \dim \text{Hom}_{T^Z(n, d)}(\mathcal{F}_\mu, \mathcal{F}_\nu)$$

$$= \sum_{\lambda \in \Lambda_+^{(n,d)}} \sum_{\mu, \nu \in \Lambda_+^{(n,d)}} (\mathcal{F}_\mu : \Delta(\lambda))(\mathcal{F}_\nu : \nabla(\lambda))$$

$$= \sum_{\lambda \in \Lambda_+^{(n,d)}} \sum_{\mu, \nu \in \Lambda_+^{(n,d)}} (\mathcal{F}_\mu : \Delta(\lambda))(\mathcal{F}_\nu : \Delta(\lambda))$$

$$= \sum_{\lambda \in \Lambda_+^{(n,d)}} \sum_{\mu, \nu \in \Lambda_+^{(n,d)}} k_{\lambda,\mu} k_{\lambda,\nu}$$

where we have used (5.10) for the first equality, Proposition A2.2 for the second equality, Proposition 5.17 for the fourth equality and [14] Theorem 5.17 for the last equality. □

References

[1] J. Chuang and R. Kessar, Symmetric groups, wreath products, Morita equivalences, and Brauer’s abelian defect group conjecture. Bull. London Math. Soc. 34 (2002), 174–184.

[2] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Ann. Math. 167 (2008), 245–298.

[3] E. Cline, B. Parshall and L. Scott, Finite-dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988), 85–99.

[4] S. Donkin, On tilting modules for algebraic groups, Math. Z. 212 (1993), 39–60.

[5] S. Donkin, The $q$-Schur Algebra, CUP, Cambridge, 1998.

[6] S. Donkin, On Schur Algebras and Related Algebras II, Journal of Algebra 111 (1987), 354-364.

[7] S. Donkin, On Schur Algebras and Related Algebras IV, Journal of Algebra 168 (1994), 400-429.

[8] A. Evseev and A. Kleshchev, Turner doubles and generalized Schur algebras, Adv. Math. 317 (2017), 665–717.

[9] A. Evseev and A. Kleshchev, Blocks of symmetric groups, semisimple KLR algebras and zigzag Schur-Weyl duality, Ann. of Math. 188 (2018), 453–512.

[10] J.A. Green, Polynomial representations of $GL_n$, 2nd edition, Springer-Verlag, Berlin, 2007.

[11] J.A. Green, Combinatorics and the Schur algebra, J. Pure Appl. Algebra 88 (1993), 89–106.

[12] A. Kleshchev and R. Muth, Based quasi-hereditary algebras, J. Algebra 558 (2020), 504–522.

[13] A. Kleshchev and R. Muth, Generalized Schur algebras, Algebra & Number Theory 14 (2020), 501–544.

[14] A. Kleshchev and R. Muth, Schurifying quasi-hereditary algebras, to appear in Proc. Lond. Math. Soc; arXiv:1810.02849.

[15] A. Kleshchev and I. Weinschelbaum, Good filtrations for generalized Schur algebras, to appear in Transform. Groups; arXiv:2202.08866.

[16] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, OUP, 1995.

[17] C.M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208(1991), 209–223.
[18] R. Rouquier, Repréresentations et catégories dérivées, Rapport d’habilitation, Université de Paris VII, 1998.

[19] R. Rouquier, q-Schur algebras and complex reflection groups, Mosc. Math. J. 8 (2008), 119–158.

[20] W. Turner, Rock blocks, Mem. Amer. Math. Soc. 202 (2009), no. 947., American Mathematical Society, Providence, Rhode Island, 2009.

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