Gravity with a Linear Action

and

Gravitational Singularities

George Savvidy
Institute of Nuclear and Particle Physics
Demokritos National Research Center
Ag. Paraskevi, Athens, Greece

Abstract

Motivated by quantum mechanical considerations we earlier suggested an alternative action for discretised quantum gravity which has a dimension of length. It is the so called ”linear” action. The proposed action is a ”square root” of the classical area action in gravity and has in front of the action a new constant of dimension one. Here we shall consider the continuous limit of the discretised linear action. We shall demonstrate that in the modified theory of gravity there appear space-time regions of the Schwarzschild radius scale which are unreachable by test particles. These regions are located in the places where standard theory of gravity has singularities. We are confronted here with a drastically new concept that there may exist space-time regions which are excluded from the physical scene, being physically unreachable by test particles or observables. If this concept is accepted, then it seems plausible that the gravitational singularities are excluded from the modified theory.
1 Introduction

Unification of gravity with other fundamental forces within the superstring theory stimulated the interest to the theory of quantum gravity and to physics at Planck scale \[1, 2, 3, 4, 5\]. In particular, string theory predicts modification of the gravitational action at Planck scale with additional high derivative terms. This allows to ask fundamental questions concerning physics at Planck scale referring to these effective actions and, in particular, one can try to understand how they influence the gravitational singularities \[6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22\].

It is appealing to extend this approach to different modifications of classical gravity which follow from the string theory and also to develop an alternative approach which is based on new geometrical principles \[28, 29, 30\]. These geometrical principles allow to extend the notion of the Feynman integral over paths to an integral over space-time manifolds with the property that in cases when a manifold degenerates into a single world line the quantum mechanical amplitude becomes proportional to the length of the world line. This approach to quantum gravity is based on the idea that quantum mechanical amplitudes should be proportional to the "length" of the quantum fluctuations when they are of geometrical origin. In other words, in this limit the action should reduce to the action of the relativistic particle \[34\].

The discretised version of the linear action can be derived from geometrical principles: \(\alpha\) the coincidence of the transition amplitude with the Feynman path amplitude for a manifold which is degenerated to a single world line and \(\beta\) the continuity of the transition amplitudes under the manifold deformations \[28, 29, 30\]. In accordance with \(\alpha\) the quantum mechanical amplitude should be proportional to the length of the space-time manifold and therefore it must be proportional to the linear combination of the lengths of all edges of discretised space-time manifold \(\sum_{<i,j>} \lambda_{ij} \cdot \Theta_{ij}\), where \(\lambda_{ij}\) is the length of the edge between two vertices \(<i>\) and \(<j>\), summation is over all edges \(<i,j>\) and \(\Theta_{ij}\) is unknown angular factor, which can be defined through the continuity principle \(\beta\). The deficit angel \(\Theta_{ij}\) should vanish in cases when the triangulation has flat surfaces, thus \(\sum_{<i,j>} \lambda_{ij} \cdot \sum(2\pi - \sum \beta_{ijk})\), where \(\beta_{ijk}\) are the angles on the cone which appear in the normal section of the edge \(<ij>\) and triangle \(<ijk>\). Combining terms belonging to a given triangle \(<ijk>\) we shall get a sum \(\lambda_{ij} + \lambda_{jk} + \lambda_{ki} = \lambda_{ijk}\) which is equal to the

\[\text{\textsuperscript{*}See also the references [31, 32, 33, 34}.\]
perimeter of the triangle \(<ijk>\) \([28, 29, 30]\):

\[
S_L = \sum_{<ijk>} \lambda_{ijk} \cdot \omega_{ijk}^{(2)}
\]

(1.1)

and where \(\omega_{ijk}^{(2)}\) is the deficit angle associated with the triangle \(<ijk>\).

The linear character of the action requires the existence of a new fundamental coupling constant \(m_P\) of dimension \(1/cm\). It is natural to call these amplitudes “linear” or “goni-hedric” because the definition contains the sum of products of the characteristic lengths and deficit angles. Thus the principles of \textit{linearity} and \textit{continuity} allow to define the linear action which can be considered as a “square root” of classical Regge area action in gravity \([35, 39]\). It is useful to compare the linear action (1.1) with the Regge area action \([35, 39]\). In (1.1) the \textit{perimeter} of a triangle is multiplied by the deficit angle. In the Regge action the \textit{area} of the triangle \(\sigma_{ijk}\) is multiplied by the deficit angle

\[
S_R = \sum_{<ijk>} \sigma_{ijk} \cdot \omega_{ijk}^{(2)}
\]

(1.2)

and represents the discretised version of the standard continuous area action in gravity:

\[
S_A = -\frac{c^3}{16\pi G} \int R \sqrt{-g} d^4x,
\]

(1.3)

where \(G = 6.67 \cdot 10^{-8} \text{ cm}^3 \text{ g sec}^{-2}\) is the gravitational coupling constant. The dimension of the measure \([\sqrt{-g} d^4x]\) is \(cm^4\), the dimension of the scalar curvature \([R]\) is \(1/cm^2\), thus the integral \(\int R \sqrt{-g} d^4x\) has dimension \(cm^2\) and measures the ”area” of the universe. The coupling constant \(\frac{c^3}{16\pi G}\) in front of the integral has therefore the dimension \(\frac{g}{\text{sec}}\).

It is unknown to the author how to derive a continuous limit of the linear action (1.1) in a unique way. In this circumstance we shall try to construct a possible linear action for a smooth space-time universe by using the available geometrical invariants. Any expression which is quadratic in the curvature tensor and includes two derivatives can be a candidate for the linear action. These invariants have the following form:

\[
I_1 = -\frac{1}{180} R_{\mu\nu\lambda\rho\sigma} R_{\mu\nu\lambda\rho\sigma}, \quad I_2 = +\frac{1}{36} R_{\mu\nu\lambda\rho} \Box R_{\mu\nu\lambda\rho},
\]

(1.4)

and we shall consider a linear combination of the above expressions:

\[
S_L = -M c \int \frac{3}{8\pi} (1 - \gamma) \sqrt{I_1 + \gamma I_2} \sqrt{-g} d^4x,
\]

(1.5)

\textsuperscript{†}The general form of the action is presented in the Appendix.
where we introduced the corresponding mass parameter $M$ and the dimensionless parameter $\gamma$. The dimension of the invariant $[\sqrt{I_1 + \gamma I_2}]$ is $1/cm^3$, thus the integral $\int \sqrt{I_1 + \gamma I_2}\sqrt{-g}d^4x$ has the dimensions of $cm$ and measures the "length" of the universe. The expression (1.5) fulfils our basic physical requirement on the action that it should have the dimension of length and should be similar to the action of the relativistic particle

$$S = -Mc\int ds = -Mc^2\int \sqrt{1 - \frac{v^2}{c^2}} dt. \quad (1.6)$$

Both expressions contain the geometrical invariants which are not in general positive definite under the square root operation. In the relativistic particle case (1.6) the expression under the root becomes negative for a particle moving with a velocity which exceeds the velocity of light. In that case the action develops an imaginary part and quantum mechanical suppression of amplitudes prevents a particle from exceeding the velocity of light [36, 37, 38]. A similar mechanism was implemented in the Born-Infeld modification of electrodynamics with the aim to prevent the appearance of infinite electric fields [40].

One can expect that in case of modified gravity with linear action (1.5) there may appear space-time regions which are unreachable by the test particles as in that regions the expression under the root were negative. If these "locked" space-time regions happen to appear and if that space-time regions include singularities, then one can expect that the gravitational singularities are naturally excluded from the theory due to the fundamental principles of quantum mechanics. The question of consistency of the new action principle, if it is the right one, can only be decided by their physical consequences.

In the next sections we shall consider the black hole singularities and the physical effects which are induced by the inclusion of the linear action (1.5). As we shall see, the expression under the root becomes negative in the region which is smaller than the Schwarzschild radius $r_g$ and includes the singularities. For the observer which is far away from the horizon the linear action perturbation induces the advance precession of the perihelion and has a profound influence on the physics near the horizon. We are confronted here with a drastically unusual concept that there may exist space-time regions which are excluded from the physical scene, being physically unreachable by test particles or observables. If one accepts this concept, then it seems plausible that the gravitational singularities are excluded from the modified theory. In this paper we have only taken the first steps to describe the phenomena which are caused by the additional linear term in the gravitational action proposed in [28, 29, 30].
2 Schwarzschild Black Hole Singularities

The modified action which we shall consider is a sum

\[ S = -\frac{c^3}{16\pi G} \int R\sqrt{-g}d^4x - Mc\int \frac{3}{8\pi} (1 - \gamma)\sqrt{I_1 + \gamma I_2} \sqrt{-g}d^4x, \]  

(2.7)

where we introduced a dimensionless parameter \( \gamma \) in order to consider a general linear combination of the invariants. The additional linear term has high derivatives of the metric and the equations of motion which follow from the variation of this action are much more complicated than in the standard case, and we shall consider them in the forthcoming sections. Here we shall consider the perturbation of the Schwarzschild solution which is induced by the additional term in the action and try to understand how it influences the black hole physics and the singularities. The Schwarzschild solution has the form

\[ ds^2 = (1 - \frac{r_g}{r})c^2dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 - r^2d\Omega^2, \]  

(2.8)

where \( g_{00} = 1 - \frac{r_g}{r}, \ g_{11} = -(1 - \frac{r_g}{r})^{-1}, \ g_{22} = -r^2, \ g_{33} = -r^2 \sin^2 \theta, \) and

\[ r_g = \frac{2GM}{c^2}, \quad \sqrt{-g} = r^2 \sin \theta. \]

The nontrivial quadratic curvature invariant in this case has the form

\[ I_0 = \frac{1}{12} R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \left(\frac{r_g}{r^3}\right)^2 \]  

(2.9)

and shows that the singularity located at \( r = 0 \) is actually a curvature singularity. The event horizon is located where the metric component \( g_{rr} \) diverges, that is, at \( r_{\text{horizon}} = r_g \).

The expressions for the two curvature polynomials (1.4) of our interest are\(^1\)

\[ I_1 = \frac{r_g^2(r - r_g)}{r^9}, \quad I_2 = \frac{r_g^3}{r^9}, \]  

(2.10)

and on the Schwarzschild solution the action acquires additional term of the form

\[ S = -Mc^2 \int \frac{3}{2} \varepsilon \sqrt{1 - \varepsilon \frac{r_g}{r}} \frac{r_g}{r^2} dr dt, \]  

(2.11)

where

\[ \varepsilon = 1 - \gamma. \]

\(^1\)It should be stressed that all other invariant polynomials of the same dimensionality can be expressed in terms of \( I_1 \) and \( I_2 \), and they are given in Appendix.
As one can see, the expression under the square root in (2.11) becomes negative at
\[ r < \varepsilon r_g, \quad 0 < \varepsilon \leq 1 \] (2.12)
and defines the region which is unreachable by the test particles. The size of the region depends on the parameter \( \varepsilon \) and is smaller than the gravitational radius \( r_g \) (see Fig. 1).

This result seems to have profound consequences on the gravitational singularity at \( r = 0 \). In a standard interpretation of the singularities, which appear in spherically symmetric gravitational collapse, the singularity at \( r = 0 \) is hidden in the sense that no signal from it can reach infinity. The singularities are not visible for the outside observer, but hidden behind an event horizon. In that interpretation the singularities are still present in the theory. In the suggested scenario it seems possible to eliminate the singularities from the theory based on the fundamental principles of quantum mechanics. The singularities are excluded from the theory on the same level as the motion of particles with a velocity which exceeds the speed of light.

The quantum mechanical amplitude in terms of the path integral has the form
\[
\Psi = \int \exp \left( \frac{i}{\hbar} S \right) \mathcal{D} g_{\mu \nu}(x),
\]
where integration is over all diffeomorphism nonequivalent metrics. For the Schwarzschild massive object which is at rest we can find the expression for the action
\[
S = -M c^2 \int_{\varepsilon r_g}^{\infty} \frac{3}{2} \varepsilon \sqrt{1 - \varepsilon \frac{r_g}{r}} \frac{r_g}{r^2} dr dt = M c^2 t
\] (2.13)
and confirm that it is proportional to the length \( t \) of the space-time trajectory, as it should be for the relativistic particle at rest, so that the corresponding amplitude can be written in the form
\[
\Psi \approx \exp \left( \frac{i}{\hbar} \sum_n M_n c^2 t \right),
\] (2.14)
where the summation is over all bodies in the universe.

The perturbation (2.11) generates a contribution to the distance invariant \( ds \) in (2.8) of the form
\[
\frac{3}{2} \int_r^{\infty} \varepsilon \sqrt{1 - \varepsilon \frac{r_g}{r}} \frac{r_g}{r^2} dr = \left[ 1 - \left( 1 - \varepsilon \frac{r_g}{r} \right)^{3/2} \right] \] (2.15)
and allows to calculate the correction to the purely temporal component of the metric tensor (2.8) caused by the additional term in the action
\[
g_{00} = 1 - \frac{r_g}{r} - \left[ 1 - \left( 1 - \varepsilon \frac{r_g}{r} \right)^{3/2} \right]^2.
\] (2.16)
Figure 1: The graphic of the potential function $g_{00} = 1 - \frac{r_g}{r} - \left[1 - \left(1 - \frac{\varepsilon r_g}{r}\right)^{3/2}\right]^2$. At $r \to \infty$ the $g_{00} \to 1$ and at $r \to \varepsilon r_g$ the $g_{00} \to -\frac{1}{\varepsilon}$.

The equation used to determine gravitational time dilation near a massive body is modified in this case and the proper time between events is defined now by the equation

$$d\tau = \sqrt{g_{00}}dt = \sqrt{1 - \frac{r_g}{r} - \left[1 - \left(1 - \frac{\varepsilon r_g}{r}\right)^{3/2}\right]^2} dt$$

(2.17)

and therefore $d\tau \leq dt$, as in standard gravity. It follows from (2.16) that near the gravitational radius $r \approx r_g$ a purely temporal component of the metric tensor has the form

$$g_{00} \approx 1 - \frac{r_g}{r} - \varepsilon^2 \frac{9}{4} \left(\frac{r_g}{r}\right)^2 + O(\varepsilon^3)$$

(2.18)

and the infinite red shift which appears in the standard case at $r = r_g$ now appears at

$$r \approx r_g \left(1 + \frac{9}{4} \varepsilon^2\right) + O(\varepsilon^4).$$

(2.19)

To define the perturbation of the trajectories of the test particles we shall study the behaviour of the solutions of the Hamilton-Jacobi equation for geodesics, which is modified by the perturbation of the metric:

$$g^{\mu\nu} \frac{\partial A}{\partial x^\mu} \frac{\partial A}{\partial x^\nu} = g^{00} \left(\frac{\partial A}{\partial t}\right)^2 - \frac{1}{g^{00}} \left(\frac{\partial A}{\partial r}\right)^2 - \frac{1}{r^2} \left(\frac{\partial A}{\partial \phi}\right)^2 = m^2 c^2.$$

The solution has the form

$$A = -Et + L\phi + A(r),$$

(2.20)
where $E$ and $L$ are the energy and angular momentum of the test particle and

$$A(r) = \int \left[ \left( g^{00} \frac{E^2}{c^2} - m^2 c^2 - \frac{L^2}{r^2} \right) g^{00} \right]^{1/2} dr. \quad (2.21)$$

In the non-relativistic limit $E = E' + mc^2, E' \ll mc^2$, and in terms of a new coordinate $r(r - r_g) = r'$ we shall get

$$A(r) \approx \int \left[ \left( \frac{E'^2}{c^2} + 2E'm \right) + \frac{1}{r'}\left(4E'mr_g + m^2 c^2 r_g\right) - \frac{1}{r'^2} \left(L^2 - \frac{3}{2} m^2 c^2 r_g^2 (1 + \frac{3}{2} \varepsilon^2) \right) \right]^{1/2} dr'. \quad (2.22)$$

The geodesic trajectories are defined by the equation $\phi + \partial A(r)/\partial L = Const$ and the advance precession of the perihelion $\delta \phi$ expressed in radians per revolution is given by the expression

$$\delta \phi = \frac{3 \pi m^2 c^2 r_g^2}{2L^2} \left(1 + \frac{3}{2} \varepsilon^2 \right) = \frac{6 \pi GM}{c^2 \alpha (1 - e^2)} \left(1 + \frac{3}{2} \varepsilon^2 \right), \quad (2.23)$$

where $\alpha$ is the semi-major axis and $e$ is the orbital eccentricity. As one can see from the above result, the precession is advanced by the additional factor $1 + \frac{3}{2} \varepsilon^2$. The upper bound on the value of $\varepsilon$ can be extracted from the observational data for the advanced precession of the Mercury perihelion, which is $42.98 \pm 0.04$ seconds of arc per century, thus

$$\varepsilon \leq 0.16.$$ 

For the light propagation we shall take $m^2 = 0, E = \omega_0, L = \rho \omega_0/c$ in (2.21):

$$A(r) = \frac{\omega_0}{c} \int \sqrt{(g^{00} - \frac{\rho^2}{r^2}) g^{00}} dr \approx \frac{\omega_0}{c} \int \sqrt{1 + 2 \frac{r_g^2}{r} - \frac{\rho^2}{r^2}} dr + \mathcal{O}(\varepsilon^2 r_g^2 / r^2). \quad (2.24)$$

The trajectory is defined by the equation $\phi + \partial A(r)/\partial \rho = Const$ and in the given approximation the deflection of light ray remains unchanged:

$$\delta \phi = \frac{2 \rho_g}{\rho}, \quad (2.25)$$

where $\rho$ is the distance from the centre of gravity. The deflection angle is not influenced by the perturbation, which is of order $\mathcal{O}(\varepsilon^2 r_g^2 / \rho^2)$, and does not impose a sensible constraint on $\varepsilon$. In the next sections we shall consider perturbation of the Reissner-Nordström and the Kerr solutions.
3 Reissner-Nordström Solution

The Reissner-Nordström solution has the form

\[ ds^2 = (1 - \frac{rg}{r} + \frac{r_g^2}{r^2})c^2 dt^2 - (1 - \frac{rg}{r} + \frac{r_g^2}{r^2})^{-1} dr^2 - r^2 d\Omega^2 , \]  

(3.26)

where

\[ r_g = \frac{2GM}{c^2} , \quad r_Q^2 = \frac{Q^2G}{c^4} , \quad \sqrt{-g} = r^2 \sin \theta. \]

The nontrivial quadratic curvature invariant is

\[ I_0 = \frac{1}{12} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{3r^2r_g^2 - 12rr_g^2r_Q^2 + 14r^4}{3r^8} , \]

(3.27)

and it shows that the singularity is located at \( r = 0 \). The event horizon and internal Cauchy horizon are located where the metric component \( g_{rr} \) diverges:

\[ r_\pm = \frac{1}{2}(rg \pm \sqrt{r_g^2 - 4r_Q^2}). \]

(3.28)

The solutions with \( r_Q > r_g/2 \) represent a naked singularity. As we shall see below, at these charges \( r_Q \) the linear action becomes a complex valued function and prevents the appearance of the naked singularities.

The expression for the two curvature polynomials of our interest are:

\[ I_1 = \frac{(r^2 - rr_g + r_Q^2)(45r^2r_g^2 - 216rr_g^2r_Q^2 + 304r^4_Q)}{45r^{12}} , \]

\[ I_2 = \frac{9r^3r_g^3 + 36r^3r_g^2r_Q^2 - 96r^2r_g^2r_Q^2 - 88r^2r_Q^4 + 264rr_g^2r_Q^4 - 200r_Q^6}{9r^{12}}. \]

(3.29)

It is convenient to introduce the dimensionless quantities:

\[ \hat{r} = \frac{r}{r_g} , \quad \hat{r}_Q = \frac{r_Q}{r_g} , \]

(3.30)

and express the linear action on the Reissner-Nordström solution in the following form:

\[ S = -Me^2 \int \frac{3}{2} \epsilon \sqrt{\left(1 - \frac{b}{\hat{r}} - \frac{c}{\hat{r}^2} + \frac{f}{\hat{r}^3} - \frac{e}{\hat{r}^4}\right)} \frac{1}{\hat{r}^2} d\hat{r} dt , \]

(3.31)

where

\[ b = \epsilon + \hat{r}_Q^2 \frac{4(1 + 5\epsilon)}{5} , \quad c = \hat{r}_Q^2 \frac{(219 - 480\epsilon)}{45} + \hat{r}_Q^4 \frac{8(17 - 55\epsilon)}{45} , \]

\[ f = \hat{r}_Q^4 \frac{8(20 - 33\epsilon)}{9} , \quad e = \hat{r}_Q^6 \frac{8(87 - 125\epsilon)}{45} . \]
If the charge of the black hole is equal to zero, \( \hat{r}_Q = 0 \), then the action \( S \) reduces to the expression \( (2.11) \) on the Schwarzschild solution. For the extremal black hole of the change \( r_Q = r_g/2 \) the fourth order polynomial under the root in \( (3.31) \) is positive for \( r > r_g/2 \), is equal to zero at \( r = r_g/2 \) and is negative for \( r < r_g/2 \). The region which is "locked" for the test particles in this case is defined by \( r = r_g/2 \) and prevents the appearance of the naked singularities.

It is helpful to represent the polynomial under the square root in \( (3.31) \) in the form

\[
\left(1 - \frac{b}{\hat{r}} - \frac{c}{\hat{r}^2} + \frac{f}{\hat{r}^3} - \frac{e}{\hat{r}^4}\right) = (1 - \frac{\hat{r}_1}{\hat{r}})...(1 - \frac{\hat{r}_4}{\hat{r}}), \tag{3.32}
\]

where \( \hat{r}_i, i = 1, ..., 4 \) are the roots of the fourth order polynomial. The largest positive real valued root at which the polynomial turns out to be negative is defined as \( \hat{r}_4 \). Near that radius one can approximate the polynomial as

\[
(1 - \frac{\hat{r}_1}{\hat{r}_4})(1 - \frac{\hat{r}_2}{\hat{r}_4})(1 - \frac{\hat{r}_3}{\hat{r}_4})(1 - \frac{\hat{r}_4}{\hat{r}}) = \Gamma(\hat{r}_1, ..., \hat{r}_4) (1 - \frac{\hat{r}_4}{\hat{r}}). \tag{3.33}
\]

Thus the action will take the form

\[
S \approx -Mc^2 \Gamma \int \frac{3}{2} \varepsilon \sqrt{1 - \frac{\hat{r}_4}{\hat{r}}} \frac{1}{\hat{r}^2} d\hat{r} dt \tag{3.34}
\]

and it has a similar form with the expression which we analysed in the case of the Schwarzschild solution \( (2.11) \). As one can see, the expression under the square root in \( (3.34) \) becomes negative at

\[
r < r_4(r_Q, \varepsilon) \tag{3.35}
\]

and defines the region which is unreachable by the test particles.

The Table I presents the values of the radius \( \hat{r}_4 \) at which the polynomial under the root changes its sign from positive to negative and the action becomes complex as a function of the charge \( \hat{r}_Q \) and the parameter \( \varepsilon \). As it follows from the Table I for the extremal black hole, \( \hat{r}_Q = 1/2 \), the "locked" region has the radius \( \hat{r}_4 = 1/2 \) and increases with the charge \( \hat{r}_Q > 1/2 \), preventing the appearance of naked singularities. At \( \hat{r}_Q = 1 \) the locked region has the radius \( \hat{r}_4 \approx 2.14 \). For \( \hat{r}_Q < 1/2 \) the locked region is smaller than horizon \( \hat{r}_4 < 1/2 \).
Charge Parameter Maximal real solution

| \( \hat{r}_Q \) | \( \varepsilon \) | \( \hat{r}_4 \) |
|---|---|---|
| 1 | 0.1 | 2.14 |
| 1/2 | 0.1 | 0.50 |
| 1/4 | 0.1 | 0.47 |
| 1/8 | 0.1 | 0.29 |
| 1/16 | 0.1 | 0.18 |

Table 1: The table of the solutions \( \hat{r}_4 \) at which the four order polynomial under the root function in (3.31) becomes negative. The value of \( \hat{r}_4 \) is measured in \( r_g \) units (3.30). For the extremal black hole \( \hat{r}_Q = 1/2 \) the "locked" region has the radius \( \hat{r}_4 = 1/2 \) and increases with the black hole charge \( \hat{r}_Q > 1/2 \), thus preventing the appearance of naked singularities. For \( \hat{r}_Q < 1/2 \) the locked region is smaller than horizon \( \hat{r}_4 < 1/2 \), that is \( r_4 < r_g/2 \).

### 4 Kerr Solution

Let us also consider the Kerr metric

\[
ds^2 = (1 - \frac{r_g}{\rho^2})c^2 dt^2 - \frac{\rho^2}{r^2 - r_g r + a^2} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g a^2}{\rho^2} \sin^2 \theta) \sin^2 \theta d\phi^2 + 2 \frac{r_g a}{\rho^2} \sin^2 \theta d\phi dt,
\]

where

\[
g_{00} = 1 - \frac{r_g}{\rho^2}, \quad g_{11} = -\frac{\rho^2}{r^2 - r_g r + a^2}, \quad g_{22} = -\rho^2, \quad g_{33} = -(r^2 + a^2 + \frac{r_g a^2}{\rho^2} \sin^2 \theta) \sin^2 \theta,
\]

\[
g_{03} = g_{30} = \frac{r_g a}{\rho^2} \sin^2 \theta
\]

and

\[
a = \frac{J}{M c}, \quad r_g = \frac{2GM}{c^2}, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \sqrt{-g} = \rho^2 \sin \theta.
\]

The nontrivial quadratic curvature invariant is

\[
I_0 = \frac{1}{12} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{r_g^2(r^2 - a^2 \cos^2 \theta)[(r^2 + a^2 \cos^2 \theta)^2 - 16a^2 r_g^2 \cos^2 \theta]}{(r^2 + a^2 \cos^2 \theta)^6}
\]

and it shows that the singularity located at \( r = 0, \quad \theta = \pi/2 \) is a curvature singularity. The event horizon is defined by the largest root of the equation \( r^2 - r_g r + a^2 = 0 \) where the metric component \( g_{11} \) diverges:

\[
r_{\text{horizon}} = \frac{1}{2} (r_g + \sqrt{r_g^2 - 4a^2}).
\]
For $a > r_g/2$ there are no real valued solutions and there is no event horizon. With no event horizons to hide it from the rest of the universe, the black hole ceases to be a black hole and will instead be a naked singularity. The outer ergosurface is defined by the equation where the purely temporal component $g_{00}$ of the metric changes the sign from positive to negative:

$$ r_{ergosur} = \frac{1}{2} \left( r_g + \sqrt{r_g^2 - 4a^2 \cos^2 \theta} \right). \tag{4.40} $$

These two critical surfaces are tangent to each other at poles $\theta = 0, \pi$ and they exist only when $a < r_g/2$. The space between these two surfaces defines the ergosphere. At maximum value of the angular momentum $a = r_g/2$ these surfaces are defined by the equations

$$ r_{horizon} = \frac{r_g}{2}, \quad r_{ergosur} = \frac{r_g}{2} (1 + \sin \theta). \tag{4.41} $$

Let us now consider the expressions for the curvature polynomials $I_1$ and $I_2$ in the case of Kerr solution

$$ I_1 = \frac{r_g^2 (r^2 - r_g r + a^2 \cos^2 \theta) [r^8 - 28a^2 r^6 \cos^2 \theta + 70a^4 r^4 \cos^4 \theta - 28a^6 r^2 \cos^6 \theta + a^8 \cos^8 \theta]}{(r^2 + a^2 \cos^2 \theta)^9}, \tag{4.42} $$

$$ I_2 = \frac{r_g^3 (r^2 - 3a^2 \cos^2 \theta) (r^6 - 33a^2 r^4 \cos^2 \theta + 27a^4 r^2 \cos^4 \theta - 3a^6 \cos^6 \theta)}{(r^2 + a^2 \cos^2 \theta)^9}. $$

It is convenient to introduce the dimensionless quantities

$$ \hat{r} = \frac{r}{r_g}, \quad \hat{a} = \frac{a}{r_g}, \quad \sigma^2 = \hat{a}^2 \cos^2 \theta \tag{4.43} $$

so that the linear action will takes the form

$$ S = -M c^2 \int \frac{3}{2} \varepsilon \sqrt{f(\hat{r}, \hat{a}, \varepsilon, \theta)} \frac{1}{\hat{r}^2} d\hat{r} dt, \tag{4.44} $$

where

$$ f(\hat{r}, \hat{a}, \varepsilon, \theta) = \left( 1 - \varepsilon \frac{27\sigma^2}{\hat{r}^2} - \frac{8\sigma^2}{\hat{r}^3} + \frac{36\sigma^2 \varepsilon}{\hat{r}^3} + \frac{42\sigma^4}{\hat{r}^4} + \frac{56\sigma^4}{\hat{r}^5} - \frac{126\sigma^4 \varepsilon}{\hat{r}^5} + \frac{42\sigma^6}{\hat{r}^6} - \frac{56\sigma^6 \varepsilon}{\hat{r}^7} + \frac{84\sigma^6 \varepsilon}{\hat{r}^7} - \frac{27\sigma^8}{\hat{r}^8} + \frac{8\sigma^8}{\hat{r}^9} - \frac{9\sigma^8 \varepsilon}{\hat{r}^9} + \frac{\sigma^{10}}{\hat{r}^{10}} \right) \left( 1 + \frac{\sigma^2}{\hat{r}^2} \right)^{-9}. \tag{4.45} $$

The region which is locked for the test particles is defined by the largest real positive root of the polynomial (4.45) at which the polynomial turns out to be negative. It is denoted as $\hat{r}_{10} = \hat{r}_{10}(\hat{a}, \theta, \varepsilon)$. Let us consider the situation with maximal angular momentum $a = r_g/2$ as in (4.41). The roots $\hat{r}_{10}(1/2, \theta, \varepsilon,)$ can be found numerically for different parameters of the Kerr solutions. In the maximal angular momentum case some of the values are:

$$ a = r_g/2, \quad \theta = 0, \pi \quad r_{horizon} = r_g/2, \quad r_{ergosur} = r_g/2, \quad r_{10} = 2.6r_g $$

$$ a = r_g/2, \quad \theta = \pi/2, \quad r_{horizon} = r_g/2, \quad r_{ergosur} = r_g, \quad r_{10} = r_g/2. \tag{4.46} $$
Thus the singularity is unreachable by the test particles. In the case of smaller angular momentum the locked region shrinks and is inside the event horizon.

In the sections 2-4 we considered the perturbation of the exact solutions of the classical gravity. It is a difficult task to find the exact solutions of the equations which follow from the action (1.5) and (2.7). We were unable to find exact solutions of these equations, but in order to have an idea how these equations look we shall derive the corresponding field equations in the most simple case of the linear action in the next section.

5 Equation of Motion

As an example let us consider the simplest form of the linear action:

\[ S_L = -\frac{Mc}{8\pi} \int \sqrt{4R_{\mu\nu}^{\,\lambda\rho} R^{\lambda\rho} - g} d^4x, \]  

(5.47)

where the dimension of the integral is the length and the action measures the "linear size" of the universe. Using the Bianchi identity \( 2R_{\mu\nu}^{\,\lambda\rho} = \partial_\nu R \) one can represent the action in the equivalent form:

\[ S_L = -\frac{Mc}{8\pi} \int \sqrt{g} \partial_\mu R \partial_\nu R \sqrt{-g} d^4x. \]  

(5.48)

The corresponding equations of motion can be obtained by variation of the action with respect to the metric \( \bar{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \) and \( \bar{g}^{\mu\nu} = g^{\mu\nu} - \delta g^{\mu\nu} \). We have

\[ \delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(D_\mu \delta g_{\alpha\nu} + D_\nu \delta g_{\alpha\mu} - D_\alpha \delta g_{\mu\nu}) \]

then \( \delta R_{\beta\gamma\alpha\delta} = D_\lambda \delta \Gamma^\nu_{\beta\alpha} - D_\alpha \delta \Gamma^\nu_{\beta\lambda} \) and

\[ \delta R_{\mu\nu} = \frac{1}{2}(-D^2 \delta g_{\mu\nu} + D_\alpha D_\mu \delta g^\alpha_{\nu} + D_\alpha D_\nu \delta g^\alpha_{\mu} - D_\mu D_\nu \delta g^\alpha_{\alpha}) , \]

thus

\[ \delta R = -R_{\mu\nu} \delta g^{\mu\nu} + D_\mu D_\nu \delta g^{\mu\nu} - D^2 \delta g^\alpha_{\alpha} \]

and \( \sqrt{-g} = \sqrt{-\bar{g}}(1 + \delta g^\alpha_{\alpha}/2) \). Using the above formulas we can calculate the variation of linear action (5.48)

\[ \delta S_L \propto \int d^4x \sqrt{-g} \left( \frac{-\delta g^{\mu\nu} \partial_\mu R \partial_\nu R + 2g^{\mu\nu} \partial_\mu R \partial_\nu \delta R}{2\sqrt{g^\alpha\beta \partial_\alpha R \partial_\beta R}} + \frac{1}{2} \frac{\sqrt{g^{\mu\nu} \partial_\mu R \partial_\nu R} g_{\mu\nu} \delta g^{\mu\nu}}{\sqrt{g^\alpha\beta \partial_\alpha R \partial_\beta R}} \right) = \]

\[ = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left( \left( R_{\mu\nu} - D_\mu D_\nu + g_{\mu\nu} D^2 \right) D_\lambda \left( \frac{\partial^\lambda R}{\sqrt{g^\alpha\beta \partial_\alpha R \partial_\beta R}} \right) + \right. \]

\[ + \left. \frac{1}{2} \left( - \frac{\partial_\mu R \partial_\nu R}{\sqrt{g^\alpha\beta \partial_\alpha R \partial_\beta R}} + g_{\mu\nu} \sqrt{g^\alpha\beta \partial_\alpha R \partial_\beta R} \right) \right] = 0 \]
and find the field equation

\[ \Lambda_{\mu\nu} = \left( R_{\mu\nu} - D_\mu D_\nu + g_{\mu\nu} D^2 \right) D_\alpha \Pi^\alpha + \frac{1}{2} \left( g_{\mu\nu} \Pi^\alpha \partial_\alpha R - \Pi_\mu \partial_\nu R \right) = 0, \]  

(5.49)

where we introduced the operator \( \Pi_\mu \) to represent the equation in a compact form:

\[ \Pi_\mu = \frac{\partial_\mu R}{\sqrt{g^{\alpha\beta} \partial_\alpha R \partial_\beta R}}, \quad \Pi^2 = 1, \quad D_\mu \Pi_\nu \partial_\nu R = 0. \]

The equivalent form of the equation is

\[ \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - D_\mu D_\nu + g_{\mu\nu} D^2 \right) D_\alpha \Pi^\alpha + \frac{1}{2} \left( g_{\mu\nu} D_\alpha (\Pi^\alpha R) - \Pi_\mu \partial_\nu R \right) = 0. \]  

(5.50)

Taking the covariant derivative \( D^\alpha \) of the above equation one can get convinced that it identically vanishes as a consequence of the diffeomorphism invariance of the action (5.47).

These equations contain the high derivative terms and are highly nonlinear. The classical and quantum gravity theories with generic higher curvature terms and corresponding high derivative equations have been considered in [23, 24, 25] and recently in [26, 27].

If one considers the sum

\[ S = -\frac{c^3}{16\pi G} \int R \sqrt{-g} d^4x - \frac{M}{8\pi} \int \sqrt{g^{\mu\nu} \partial_\mu R \partial_\nu R} \sqrt{-g} d^4x, \]  

(5.51)

then the equation will take the form

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{2GM}{c^2} \Lambda_{\mu\nu} = 0 \]  

(5.52)

and because \( 2GM/c^2 = r_g \) we shall have

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + r_g \Lambda_{\mu\nu} = 0. \]  

(5.53)

The equations are highly nonlinear and we were unable to find the exact solutions yet. It is a challenging problem and in future work we will analyse the field equations which follow from the action (2.7). In a forthcoming publication we shall consider a perturbative solution of these equations imposing spherical symmetry [41]. In this paper we have only taken the first steps to describe the phenomena which are caused by the additional linear term in the gravitational action.

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6 Appendix

The general form of the linear action has the form:

$$S_L = -m_p c \int \frac{3}{8\pi} \sqrt{\sum_{i=1}^{3} \eta_i K_i + \sum_{i=1}^{4} \chi_i J_i + \sum_{i=1}^{9} \gamma_i I_i} \sqrt{-gd^4x},$$

where the curvature invariants have the form

$$I_0 = \frac{1}{12} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho},$$
$$I_1 = -\frac{1}{180} R_{\mu\nu\lambda\rho,\sigma} R^{\mu\nu\lambda\rho,\sigma},$$
$$I_2 = +\frac{1}{36} R_{\mu\nu\lambda\rho} \Box R^{\mu\nu\lambda\rho},$$
$$I_3 = -\frac{1}{72} \Box (R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}) = 5I_1 - I_2,$$
$$I_4 = -\frac{1}{90} R_{\mu\nu\lambda\rho,;\alpha} R^{\alpha\nu\lambda\rho,;\mu} = I_1,$$
$$I_5 = -\frac{1}{18} (R^{\alpha\nu;\lambda\rho} R_{\mu;\nu\lambda\rho})_{;\mu;\alpha} = 5I_1 - I_2,$$
$$I_6 = -\frac{1}{18} (R^{\alpha\nu;\lambda\rho} R_{\mu;\nu\lambda\rho})_{;\alpha;\mu} = 5I_1 - I_2,$$
$$I_7 = \frac{1}{18} R_{\alpha\nu\lambda\rho;\mu;\alpha;\mu} = I_2,$$
$$I_8 = R_{\nu\lambda;\mu} R^{\rho\nu\lambda;\rho;\sigma},$$
$$I_9 = R^{\alpha\nu\lambda\rho} R_{\nu\lambda;\mu;\alpha;\mu},$$

$$J_0 = R_{\mu\nu} R^{\mu\nu}, \quad J_1 = R_{\mu\nu;\lambda} R^{\mu\nu;\lambda}, \quad J_2 = R^{\mu\nu} \Box R_{\mu\nu}, \quad J_3 = \Box (R^{\mu\nu} R_{\mu\nu}), \quad J_4 = R_{\mu\rho} ;^{\mu} R^{\rho\sigma;\nu},$$
$$K_0 = R^2, \quad K_1 = R_{;\mu} R^{;\mu}, \quad K_2 = R \Box R, \quad K_3 = \Box R^2.$$

The $\eta_i, \chi_i$ and $\gamma_i$ are free parameters. Some of the invariants can be expressed through others using Bianchi identities.

References

[1] A. Sakharov, Vacuum quantum fluctuations in curved space and the theory of gravitation, Dokl. Akad. Nauk SSSR 177 (1967) 70

[2] H.A. Buchdahl, Non-Linear Lagrangians and Cosmological Theory, Monthly Notices Roy. Astron. Soc. 150 (1970) 1.

[3] A. A. Starobinsky, A New Type of Isotropic Cosmological Models Without Singularity, Phys. Lett. 91B (1980) 99. doi:10.1016/0370-2693(80)90670-X
[4] S. L. Adler, *Einstein Gravity as a Symmetry Breaking Effect in Quantum Field Theory*, Rev. Mod. Phys. **54** (1982) 729. doi:10.1103/RevModPhys.54.729

[5] M. Gasperini and G. Veneziano, *Pre - big bang in string cosmology*, Astropart. Phys. **1** (1993) 317. doi:10.1016/0927-6505(93)90017-8.

[6] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965) 57. doi:10.1103/PhysRevLett.14.57

[7] D. Christodoulou, *Examples of naked singularity formation in the gravitational collapse of a scalar field*, Ann. Math. **140** (1994) 607-653. doi:10.2307/2118619.

[8] S. Hawking, *Occurrence of Singularities in Open Universes*, Phys. Rev. Lett. **15** (1965) 689.

[9] S. Hawking, *The Occurrence of Singularities in Cosmology*, Proc. R. Soc. London **A294** (1966) 511-521.

[10] S. Hawking, *The Occurrence of Singularities in Cosmology. II.*, Proc. R. Soc. London **A295** (1966) 490-493.

[11] S. Hawking, *The Occurrence of Singularities in Cosmology. III. Causality and Singularities*, Proc. R. Soc. London **A300** (1967) 187-201.

[12] S. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. R. Soc. London **A314** (1970) 529.

[13] H. P. Robertson, *Relativistic Cosmology*, Rev. Mod. Phys. **5** (1935) 62.

[14] A. Raychaudhure, *Relativistic Cosmology I.*, Phys. Rev. **98** (1955) 1123.

[15] A. Komar, *Necessity of Singularities in the Solution of the Field Equations of General Relativity*, Phys. Rev. **104** (1956) 544.

[16] M. A. Markov, *Limiting density of matter as a universal law of nature*, JETP Lett. **36** (1982) 265.

[17] Y. I. Anini, *The Limiting Curvature Hypothesis: Towards a Theory of Gravity Without Singularities*, Current Topics in Mathematical Cosmology, Proceedings of the International Seminar held in Potsdam, Germany, 30 March - 4 April, 1998. Edited by M. Rainer and H.-J. Schmidt. World Scientific Press, 1998., p.183.
[18] R. H. Brandenberger and C. Vafa, *Superstrings in the Early Universe*, Nucl. Phys. B 316 (1989) 391. doi:10.1016/0550-3213(89)90037-0

[19] E. Alvarez, *Superstring Cosmology*, Phys. Rev. D 31, 418 (1985) Erratum: [Phys. Rev. D 33, 1206 (1986)]. doi:10.1103/PhysRevD.31.418, 10.1103/PhysRevD.33.1206

[20] R. F. Baierlein, D. H. Sharp and J. A. Wheeler, *Three-Dimensional Geometry as Carrier of Information about Time*, Phys. Rev. 126 (1962) 1864. doi:10.1103/PhysRev.126.1864

[21] R. H. Brandenberger, V. F. Mukhanov and A. Sornborger, *A Cosmological theory without singularities*, Phys. Rev. D 48 (1993) 1629. doi:10.1103/PhysRevD.48.1629

[22] S. Deser and G. W. Gibbons, *Born-Infeld-Einstein actions?*, Class. Quant. Grav. 15 (1998) L35. doi:10.1088/0264-9381/15/5/001.

[23] K. S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, Phys. Rev. D 16 (1977) 953. doi:10.1103/PhysRevD.16.953

[24] K. S. Stelle, *Classical Gravity with Higher Derivatives*, Gen. Rel. Grav. 9 (1978) 353. doi:10.1007/BF00760427

[25] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, Phys. Rev. D 67 (2003) 084009 doi:10.1103/PhysRevD.67.084009 [hep-th/0212292].

[26] A. Kehagias, C. Kounnas, D. Lüst and A. Riotto, *Black hole solutions in $R^2$ gravity*, JHEP 1505 (2015) 143 doi:10.1007/JHEP05(2015)143 [arXiv:1502.04192 [hep-th]].

[27] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst and A. Riotto, *Aspects of Quadratic Gravity*, Fortsch. Phys. 64 (2016) no.2-3, 176 doi:10.1002/prop.201500100 [arXiv:1505.07657 [hep-th]].

[28] G. K. Savvidy and K. G. Savvidy, *Interaction hierarchy: Gonihedric string and quantum gravity*, Mod. Phys. Lett. A 11 (1996) 1379. doi:10.1142/S0217732396001399.

[29] J. Ambjorn, G. K. Savvidy and K. G. Savvidy, *Alternative actions for quantum gravity and the intrinsic rigidity of the space-time*, Nucl. Phys. B 486 (1997) 390. doi:10.1016/S0550-3213(96)00660-8.
[30] G. K. Savvidy, *Quantum gravity with linear action. Intrinsic rigidity of space-time*, Nucl. Phys. Proc. Suppl. **57** (1997) 104. doi:10.1016/S0920-5632(97)00358-7.

[31] J. Ambjorn, B. Durhuus and J. Frohlich, *Diseases of Triangulated Random Surface Models, and Possible Cures*, Nucl. Phys. B **257** (1985) 433. doi:10.1016/0550-3213(85)90356-6.

[32] R. V. Ambartsumian, G. S. Sukiasian, G. K. Savvidi and K. G. Savvidy, *Alternative model of random surfaces*, Phys. Lett. B **275** (1992) 99. doi:10.1016/0370-2693(92)90857-Z.

[33] J. Ambjorn, J. L. Nielsen, J. Rolf and G. K. Savvidy, *Spikes in quantum Regge calculus*, Class. Quant. Grav. **14** (1997) 3225. doi:10.1088/0264-9381/14/12/009.

[34] G. Savvidy, *The gonihedric paradigm extension of the Ising model*, Mod. Phys. Lett. B **29** (2015) no.32, 1550203 doi:10.1142/S0217984915502036.

[35] T. Regge, *General Relativity Without Coordinates*, Nuovo Cim. **19** (1961) 558. doi:10.1007/BF02733251.

[36] W. Pauli, *Relativistic Field Theories of Elementary Particles*, Rev. Mod. Phys. **13** (1941) 203. doi:10.1103/RevModPhys.13.203.

[37] R. P. Feynman, *The Theory of positrons*, Phys. Rev. **76** (1949) 749. doi:10.1103/PhysRev.76.749.

[38] R. P. Feynman, *Space - time approach to quantum electrodynamics*, Phys. Rev. **76** (1949) 769. doi:10.1103/PhysRev.76.769.

[39] J.A.Wheeler, *Regge Calculus and Schwarzschild Geometry* in Relativity, Groups and Topology ed. B. DeWitt and C. DeWitt (New York: Gordon and Breach, 1964) pp 463-501.

[40] M. Born and L. Infeld, *Foundations of the New Field Theory*, Proc. R. Soc. Lond. **A 1934** (1934) 144, doi: 10.1098/rspa.1934.0059.

[41] G. Savvidy (to be published).