NONABELIAN HARMONIC ANALYSIS AND FUNCTIONAL EQUATIONS ON COMPACT GROUPS

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ABSTRACT. Making use of nonabelian harmonic analysis and representation theory, we solve the functional equation
\[ f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = f_5(x)f_6(y) \]
on arbitrary compact groups. The structure of its general solution is completely described. Consequently, several special cases of the above equation, in particular, the Wilson equation and the d’Alembert long equation, are solved on compact groups.

1. Introduction

Let \( G \) be a group. The d’Alembert equation
\[ f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad (1.1) \]
where \( f : G \to \mathbb{C} \) is the function to determine, has a long history (see [2]). It is easy to check that if \( \varphi \) is a homomorphism from \( G \) into the multiplicative group of nonzero complex numbers, the function \( f(x) = (\varphi(x) + \varphi(x)^{-1})/2 \) is a solution of Eq. (1.1) on \( G \). Such solutions and the zero solution are called classical solutions. Kannappan [13] proved that if \( G \) is abelian, then all solution of Eq. (1.1) are classical. This was generalized to certain nilpotent groups in [6, 7, 11, 15, 16]. On the other hand, Corovei [6] constructed a nonclassical solution of Eq. (1.1) on the quaternion group \( Q_8 \). It was realized later that Corovei’s solution is nothing but the restriction to \( Q_8 \) of the normalized trace function \( \text{tr}/2 \) on \( SU(2) \), which is a nonclassical solution of Eq. (1.1) on \( SU(2) \) (c.f. [1, 22]). Recently, it was proved in [22, 23] that any nonclassical continuous solution of Eq. (1.1) on a connected compact group factors through \( SU(2) \), and that the function \( \text{tr}/2 \) is the only nonclassical continuous solution on \( SU(2) \). This was generalized by Davison to arbitrary compact groups in [8], and further to any topological groups in [9] (with the group \( SU(2) \) replaced by \( SL(2, \mathbb{C}) \)). Hence Eq. (1.1) on topological groups has been completely solved. For more results related to Eq. (1.1), we refer to [1, 5, 10, 18, 19, 20] and the survey [17].
A well-known generalization of the d’Alembert equation is the Wilson equation
\[ f(xy) + f(xy^{-1}) = 2f(x)g(y), \]  
(1.2)
where \( f \) and \( g \) are unknown complex functions on \( G \). It was first considered by Wilson [21] and has also been extensively studied (see [9, 10, 11, 16, 17] and the references therein). It turns out in [9] that Eq. (1.2) is directly related to Eq. (1.1), where solutions of Eq. (1.2) were used to construct the homomorphism \( G \to SL(2, \mathbb{C}) \) mentioned in the previous paragraph. Furthermore, it was shown (see, e.g., [16]) that if \( f \) and \( g \) satisfy Eq. (1.2) and \( f \neq 0 \), then \( g \) is a solution of the d’Alembert long equation
\[ f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y). \]  
(1.3)
The question of solving Eq. (1.3) on arbitrary topological groups was raised in [8]. However, the approaches in [8, 9, 22] do not apply to Eqs. (1.2) and (1.3).

The purpose of this paper is to study the equation
\[ f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = f_5(x)f_6(y), \]  
(1.4)
where \( f_i : G \to \mathbb{C} \) (\( i = 1, \ldots, 6 \)) are unknown functions. It is clear that Eq. (1.4) includes Eqs. (1.1)–(1.3) as special cases. We will find all \( L^2 \)-solutions of Eq. (1.4) on arbitrary compact groups. Consequently, we will solve Eqs. (1.2) and (1.3) on compact groups completely. Here, it is worth mentioning that, under some mild conditions, nonzero \( L^2 \)-solutions of the d’Alembert equation (1.1) (or some of its variant forms) exist only when \( G \) is compact (c.f. [14]).

Our main ingredients are nonabelian harmonic analysis on compact groups and representation theory. Let \( G \) be a compact group. Then the Fourier transform transforms a square integrable function \( f \) on \( G \) into an operator-valued function \( \hat{f} \) on \( \hat{G} \), the unitary dual of \( G \). Applying the Fourier transform to both sides of Eq. (1.4) and taking some representation theory into account, we will convert Eq. (1.4) into a family of matrix equations. We call a tuple of matrices satisfying such matrix equations an admissible (matrix) tuple. There are three types of admissible tuples, i.e., complex, real, and quaternionic types, which correspond to the three types of the representations \( [\pi] \in \hat{G} \), respectively. To determine the admissible tuples is a question of linear algebra. We will find all admissible tuples of each type. Then applying the Fourier inversion formula, we obtain the general solution of Eq. (1.4).

The structure of the general solution of Eq. (1.4) can be compared with that of linear differential equations, where any solution is the sum of a particular solution and a solution of the associated homogeneous differential equation. In our case, the homogeneous equation associated with Eq. (1.4) is
\[ f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = 0. \]  
(1.5)
It is obvious that the solutions of Eq. (1.5) form a closed subspace of \( L^2(G)^4 \), and that the sum of a solution of Eq. (1.4) and a solution of Eq. (1.5) is
also a solution of Eq. (1.4). Some obvious solutions of Eq. (1.5) are provided by central functions. We will determine the orthogonal complement of these obvious solutions in the solution space of Eq. (1.5) by constructing a spanning set using irreducible representations of \( G \) into \( O(1) \), \( O(2) \), and \( SU(2) \). We will also prove that any solution of Eq. (1.4) is the sum of a solution of Eq. (1.5) and a pure normalized solution of Eq. (1.4) (see Section 3 for the definitions), and will determine all pure normalized solutions of Eq. (1.4), which correspond to irreducible representations of \( G \) into \( U(1) \), \( O(2) \), \( SU(2) \), and \( O(3) \). This provides a complete picture of the general solution of Eq. (1.4). These results will be proved in Theorems 5.2–5.5. As applications, we will solve several special cases of Eq. (1.4), including Eqs. (1.2) and (1.3). In particular, we will show that all nontrivial solutions of Eqs. (1.2) and (1.3) factor through \( SU(2) \), and that the general solutions of Eq. (1.3) and Eq. (1.1) are the same.

The paper is organized as follows. Some basic properties of the Fourier transform on compact groups and some facts in representation theory will be briefly reviewed in Section 2. In Section 3 we will give some basic definitions related to Eq. (1.4), introduce the notion of admissible matrix tuples, reveal their relations with Eq. (1.4), and present some examples which are the building blocks of the general solution. Then in Section 4 we will determine all admissible matrix tuples. The main results will be proved in Section 5. The general solutions of several special cases of Eq. (1.4) will be given in Section 6.

We should point out that one could apply our method in this paper to some other types of functional equations on compact groups, and that the method may be also generalized to solve functional equations on non-compact groups admitting Fourier transforms.

Throughout this paper, \( G \) denotes a compact group, \( dx \) the normalized Haar measure on \( G \), and \( L^2(G) \) the Hilbert space of all square integrable functions on \( G \) with respect to \( dx \). By solutions of Eq. (1.4) (or its special cases) on \( G \) we always mean its \( L^2 \)-solutions.

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2. Preliminaries

As mentioned in the introduction, our basic tools in this paper are Fourier analysis on compact groups and some results in representation theory. In this section, we briefly review some fundamental facts in these two subjects that will be used later.

2.1. Fourier analysis. We mainly follow the approach of [12, Chapter 5]. Let \( \hat{G} \) be the unitary dual of the compact group \( G \). For \( [\pi] \in \hat{G} \), we view \( \pi \) as a homomorphism \( \pi : G \rightarrow U(d_\pi) \), where \( d_\pi \) is the dimension of the representation space. Let \( \mathbb{M}(n, \mathbb{C}) \) denote the space of all \( n \times n \) complex
matrices. For \( f \in L^2(G) \), the Fourier transform of \( f \) is defined by
\[
\hat{f}(\pi) = d_\pi \int_G f(x) \pi(x)^{-1} dx \in M(d_\pi, \mathbb{C}), \quad [\pi] \in \hat{G}.
\]
Note that for the sake of convenience, our definition is different from the one in \cite{12} by a factor \( d_\pi \). In our setting, the Fourier inversion formula is
\[
f(x) = \sum_{[\pi] \in \hat{G}} \text{tr}(\hat{f}(\pi)\pi(x)), \quad x \in G.
\]
If \( f \in L^2(G) \) has the form \( f(x) = \text{tr}(A_1 \pi_1(x)) + \cdots + \text{tr}(A_k \pi_k(x)) \), where \([\pi_1], \ldots, [\pi_k] \in \hat{G}\) are distinct and \( A_i \in M(d_\pi, \mathbb{C}) \), then, by the Peter-Weyl Theorem, we have \( \text{supp}(\hat{f}) \subseteq \{[\pi_1], \ldots, [\pi_k]\} \) and \( \hat{f}(\pi_i) = A_i \). Here \( \text{supp}(\hat{f}) = \{[\pi] \in \hat{G} \mid \hat{f}(\pi) \neq 0\} \).

Let \( L^2_c(G) \) be the subspace of central functions in \( L^2(G) \), i.e., \( f \in L^2_c(G) \) if and only if \( f(xy) = f(yx) \) for almost all \( x, y \in G \). Then \( f \in L^2_c(G) \) if and only if \( \hat{f}(\pi) \) is a scalar matrix for every \([\pi] \in \hat{G}\). Let \( L^2_c(G) \perp \) be the orthogonal complement of \( L^2_c(G) \) in \( L^2(G) \). By the Fourier inversion formula, one can show that \( f \in L^2_c(G) \perp \) if and only if \( \text{tr}(\hat{f}(\pi)) = 0 \) for every \([\pi] \in \hat{G}\).

A crucial property of the Fourier transform is that it converts the regular representations of \( G \) into matrix multiplications. As usual, the left and right regular representations of \( G \) in \( L^2(G) \) are defined by
\[
(L_y f)(x) = f(y^{-1}x), \quad (R_y f)(x) = f(xy),
\]
respectively, where \( f \in L^2(G) \) and \( x, y \in G \). Then it is easy to show that
\[
(L_y f)(\pi) = \hat{f}(\pi)\pi(y)^{-1}, \quad (R_y f)(\pi) = \pi(y)\hat{f}(\pi).
\]

2.2. Representation theory. For a positive integer \( n \), let \( I_n \) denote the \( n \times n \) identity matrix, and if \( n \) is even, let \( J_n = \begin{bmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{bmatrix} \). If \( n \) is clear from the context, we will simply denote \( I = I_n \) and \( J = J_n \). Recall that \( \text{Sp}(n) = \{ x \in U(n) \mid xJx^tJ^t = I \} \) if \( n \) is even, where \( A^t \) refers to the transpose of a matrix \( A \). We recall the following definitions.

**Definition 2.1.** Let \( \pi : G \to U(n) \) be an irreducible representation.

1. \( \pi \) is of **complex type** if \([\bar{\pi}] \neq [\pi]\).
2. \( \pi \) is of **real type** if there exists \( x \in U(n) \) such that \( x\pi(G)x^{-1} \subseteq O(n) \).
3. \( \pi \) is of **quaternionic type** if \( n \) is even and there exists \( x \in U(n) \) such that \( x\pi(G)x^{-1} \subseteq \text{Sp}(n) \).

What is really important for us is the equivalence classes of representations. So if \( \pi \) is of real (resp. quaternionic) type, we will always assume that \( \pi(G) \subseteq O(n) \) (resp. \( \pi(G) \subseteq \text{Sp}(n) \)).
Let \( \hat{G}_c \) (resp. \( \hat{G}_r, \hat{G}_q \)) denote the set of (equivalence classes of) irreducible representations of \( G \) of complex (resp. real, quaternionic) type. Then we have the following basic fact.

**Theorem 2.1.** \( \hat{G} \) is the disjoint union of \( \hat{G}_c, \hat{G}_r, \) and \( \hat{G}_q. \)

**Proof (sketched).** For an irreducible representation \( \pi : G \to U(n) \), we consider the representation \( \rho \) of \( G \) in \( \mathbb{M}(n, \mathbb{C}) \) defined by \( \rho(g)(A) = \pi(g)AA^t \). Let \( \mathbb{M}(n, \mathbb{C})^G \) denote the space of matrices \( A \) such that \( \rho(g)(A) = A \) for all \( g \in G \). Then \( [\pi] = [\rho] \) if and only if \( \dim \mathbb{M}(n, \mathbb{C})^G = 1 \). In this case, any nonzero matrix in \( \mathbb{M}(n, \mathbb{C})^G \) is invertible. It is easy to see that \( \mathbb{M}(n, \mathbb{C}) \) is decomposed as the \( G \)-invariant direct sum of the space of symmetric matrices \( \mathbb{M}_{\text{symm}}(n, \mathbb{C}) \) and the space of skew-symmetric matrices \( \mathbb{M}_{\text{skew}}(n, \mathbb{C}) \). Hence \( [\pi] = [\rho] \) if and only if either \( \dim \mathbb{M}_{\text{symm}}(n, \mathbb{C})^G = 1 \) (which means that \( \pi(G) \) lies in a conjugate of \( O(n) \)), or \( \dim \mathbb{M}_{\text{skew}}(n, \mathbb{C})^G = 1 \) (which means that \( n \) is even and \( \pi(G) \) lies in a conjugate of \( Sp(n) \)). Since \( \dim \mathbb{M}(n, \mathbb{C})^G = 1 \), the two cases can not occur simultaneously. For more details, see [3, Section 2.6]. \( \square \)

We define an equivalence relation on \( \hat{G} \) for which the equivalence class of \( [\pi] \) is \( \{[\pi], [\bar{\pi}]\} \) if \( [\pi] \in \hat{G}_c \), and \( \{[\pi]\} \) if \( [\pi] \in \hat{G}_r \) or \( \hat{G}_q \). We denote the equivalence class of \( [\pi] \) with respect to this equivalence relation by \( [[\pi]] \), and the set of all equivalence classes by \( \hat{G} \).

3. **Constructing solutions from admissible tuples**

We first introduce some notions on solutions of Eq. (1.4), and examine their basic properties. For \( g, h \in L^2(G) \), let \( g \otimes h \) be the function on \( G^2 \) defined by \( g \otimes h(x, y) = g(x)h(y) \). As being a solution of Eq. (1.4) is a property about \( f_1, f_2, f_3, f_4 \) and \( f_5 \otimes f_6 \), it is natural to denote a solution as a 5-tuple \( \mathcal{F} = (f_1, f_2, f_3, f_4, f_5 \otimes f_6) \) of functions. But sometimes we will also write the 5-tuple \( \mathcal{F} \) as \( (f_i)_{i=1}^6 \) or simply \( (f_i) \) for convenience.

The corresponding homogeneous equation (1.5) is important for us. Its solutions are 4-tuples of functions \( (f_1, f_2, f_3, f_4) \), and form a closed subspace of \( L^2(G)^4 \) in the usual way. If \( (f_i)_{i=1}^4 \) is a solution of Eq. (1.4) satisfying \( f_5 \otimes f_6 \equiv 0 \), then \( (f_i)_{i=1}^4 \) is a solution of Eq. (1.5). In this case, without loss of generality, we always assume that \( f_5 \equiv f_6 \equiv 0 \). Conversely, if \( (f_i)_{i=1}^4 \) is a solution of Eq. (1.5), then \( (f_1, f_2, f_3, f_4, 0) \) is a solution of Eq. (1.4), where 0 is the zero function on \( G \). We identify \( (f_i)_{i=1}^4 \) with \( (f_1, f_2, f_3, f_4, 0) \), and call such a solution a homogeneous solution of Eq. (1.4). We say that it is the *trivial solution* if furthermore \( f_i \equiv 0 \) for \( 1 \leq i \leq 4 \). If \( \mathcal{F} = (f_i)_{i=1}^6 \) is a solution and \( \mathcal{F}' = (f'_i)_{i=1}^4 \) is a homogeneous solution, then their sum \( \mathcal{F} + \mathcal{F}' = (f_1 + f'_1, f_2 + f'_2, f_3 + f'_3, f_4 + f'_4, f_5 \otimes f_6) \) is also a solution. It is obvious that if \( c_1, c_2 \in L_2^2(G) \), then

\[
\mathcal{F}_{c_1, c_2} = (c_1, -c_1, c_2, -c_2)
\]
is a homogeneous solution. We say that a solution \((f_i)_{i=1}^6\) of Eq. (1.4) is \textit{normalized} if \(f_1 - f_2, f_3 - f_4 \in L_2^2(G)^\perp\). Then any solution of Eq. (1.4) can be uniquely decomposed as a sum \(\mathcal{F} + \mathcal{F}_{c_1,c_2}\), where \(\mathcal{F}\) is normalized and \(\mathcal{F}_{c_1,c_2}\) is given by (3.1). Furthermore, in the Hilbert space of homogeneous solutions of Eq. (1.4), normalized homogeneous solutions form the orthogonal complement of the space of solutions of the form \(\mathcal{F}_{c_1,c_2}\). Finally, we say that a solution \(\mathcal{F} = (f_i)_{i=1}^6\) of Eq. (1.4) is \textit{pure} if \(\bigcup_{i=1}^6 \text{supp}(\hat{f}_i) \subseteq \varpi\) for some \(\varpi \in \hat{G}\). In this case, we say that \(\mathcal{F}\) is \textit{supported on} \(\varpi\).

In Section 5, we will determine all pure normalized solutions of Eq. (1.4), prove that pure normalized homogeneous solutions span the space of normalized homogeneous solutions, and that any solution is the sum of a pure normalized solution and a homogeneous solution.

We will convert Eq. (1.4) into a family of matrix equations. We call solutions of these matrix equations admissible matrix tuples, whose definitions are as follows. For \(A, B, C, D, E, F \in \mathbb{M}(n, \mathbb{C})\), we consider the linear maps \(\Phi_{c,A,B}, \Phi_{r,A,B,C,D}, \Phi_{q,A,B,C,D}\) (if \(n\) is even), and \(\Psi_{E \otimes F}\) from \(\mathbb{M}(n, \mathbb{C})\) into itself defined by

\[
\Phi_{c,A,B}(X) = AX + XB,
\Phi_{r,A,B,C,D}(X) = AX + XB + (CX + XD)^t,
\Phi_{q,A,B,C,D}(X) = AX + XB + J(CX + XD)^t J^t,
\Psi_{E \otimes F}(X) = \text{tr}(EX)F.
\]

It is easy to see that \(\Psi_{E \otimes F}\) depends only on \(E \otimes F \in \mathbb{M}(n, \mathbb{C}) \otimes \mathbb{M}(n, \mathbb{C})\), and that if \(n\) is even we have

\[
\Phi_{r,A,B,C,D}(X) = -\Phi_{q,A,B,C,D}(X) J.
\]

Definition 3.1. Let \(A, B, C, D, E, F \in \mathbb{M}(n, \mathbb{C})\).

1. \((A, B, E \otimes F)\) is an admissible tuple of complex type \((c\text{-admissible tuple abbreviated})\) if \(\text{tr}(A) = \text{tr}(B)\) and \(\Phi_{c,A,B} = \Psi_{E \otimes F}\).
2. \((A, B, C, D, E \otimes F)\) is an admissible tuple of real type \((r\text{-admissible tuple abbreviated})\) if \(\text{tr}(A) = \text{tr}(B)\), \(\text{tr}(C) = \text{tr}(D)\), and \(\Phi_{r,A,B,C,D} = \Psi_{E \otimes F}\).
3. \((A, B, C, D, E \otimes F)\) is an admissible tuple of quaternionic type \((q\text{-admissible tuple abbreviated})\) if \(n\) is even, \(\text{tr}(A) = \text{tr}(B)\), \(\text{tr}(C) = \text{tr}(D)\), and \(\Phi_{q,A,B,C,D} = \Psi_{E \otimes F}\).

We refer to \(n\) as the \textit{order} of the above admissible matrix tuples. An admissible tuple \(\mathcal{T}\) is \textit{homogeneous} if \(E \otimes F = 0\), and is \textit{trivial} if \(A = B = C = D = 0\). It is obvious that trivial admissible tuples are homogeneous. If \(\mathcal{T}\) is homogeneous, we always assume that \(E = F = 0\).

We should mention that the trace conditions in Definition 3.1 are not essential. As we will see later, they are imposed so that admissible tuples correspond to normalized solutions. This will simplify some arguments below.
We will determine all admissible matrix tuples in the next section. In the rest of this section, we explain how to construct pure normalized solutions of Eq. (1.4) from admissible tuples. We also exhibit some examples of admissible tuples, which indeed include all nontrivial ones. The solutions constructed from these examples form the building blocks of the general solution of Eq. (1.4).

We begin with a simple example.

Example 3.1. Let $\varepsilon_1, \delta_1, \varepsilon_2, \delta_2 \in \mathbb{C}$. Then $(\varepsilon_i \delta_j / 2, \varepsilon_i \delta_j / 2, \varepsilon_i \delta_j)$ $(i, j = 1, 2)$ are 1-ordered $c$-admissible tuples. Define the tuple of functions $\mathcal{F}^{U(1)}_{\varepsilon_1, \delta_1, \varepsilon_2, \delta_2} = (f_i)_{i=1}^6$ as

$$
\begin{align*}
  f_1(x) &= f_2(x) = (\varepsilon_1 \delta_1 x + \varepsilon_2 \delta_2 \bar{x}) / 2, \\
  f_3(x) &= f_4(x) = (\varepsilon_1 \delta_2 x + \varepsilon_2 \delta_1 \bar{x}) / 2, \\
  f_5 \otimes f_6(x, y) &= (\varepsilon_1 x + \varepsilon_2 \bar{x})(\delta_1 y + \delta_2 \bar{y}),
\end{align*}
$$

Then it is easy to check that $\mathcal{F}^{U(1)}_{\varepsilon_1, \delta_1, \varepsilon_2, \delta_2}$ is a pure normalized solution of Eq. (1.4) on $U(1)$ supported on $\{[\iota_{U(1)}]\}$, where $\iota_{U(1)}$ is the identity representation of $U(1)$. It is homogeneous if and only if it is the trivial solution.

The general principle of constructing solutions from admissible tuples of real and quaternionic types is as follows. For a closed irreducible subgroup $K$ of $U(n)$ and a matrix $L \in \mathbb{M}(n, \mathbb{C})$, we define the function $f_L$ on $K$ as $f_L(x) = \text{tr}(Lx)$, $x \in K$. Then we have $\text{supp}(\hat{f}_L) \subseteq \{[\iota_K]\}$ and $\hat{f}_L(\iota_K) = L$, where $\iota_K : K \to U(n)$ is the inclusion. For a 5-tuple $\mathcal{T} = (A, B, C, D, E \otimes F)$, where $A, \ldots, F \in \mathbb{M}(n, \mathbb{C})$, we define the 5-tuple of functions

$$
\mathcal{F}^K_\mathcal{T} = (f_A, f_B, f_C, f_D, f_E \otimes f_F).
$$

Clearly, $f_E \otimes f_F$ depends only on $E \otimes F$.

**Proposition 3.1.** We keep the notation as above.

1. If $\mathcal{T}$ is an $n$-ordered $r$-admissible tuple, then $\mathcal{F}^{O(n)}_\mathcal{T}$ is a pure normalized solution of Eq. (1.4) on $O(n)$ supported on $\{[\iota_{O(n)}]\}$. $\mathcal{F}^{O(n)}_\mathcal{T}$ is homogeneous if and only if $\mathcal{T}$ is homogeneous.

2. If $n$ is even and $\mathcal{T}$ is an $n$-ordered $q$-admissible tuple, then $\mathcal{F}^{Sp(n)}_\mathcal{T}$ is a pure normalized solution of Eq. (1.4) on $Sp(n)$ supported on $\{[\iota_{Sp(n)}]\}$. $\mathcal{F}^{Sp(n)}_\mathcal{T}$ is homogeneous if and only if $\mathcal{T}$ is homogeneous.
Proof. (1) Since $\Phi^r_{A,B,C,D} = \Psi_{E \otimes F}$, for all $x, y \in O(n)$ we have
\[
f_A(xy) + f_B(yx) + f_C(xy^{-1}) + f_D(y^{-1}x)
= \text{tr}(Ax) + \text{tr}(Byx) + \text{tr}(Cxy^t) + \text{tr}(Dy^t x)
= \text{tr}(Ax y + x By + x^t C y + D^t x^t y)
= \text{tr}(\Phi^r_{A,B,C,D}(x)y)
= \text{tr}(E x F y)
= f_E(x) f_F(y).
\]
So $\mathcal{F}^{O(n)}_T$ is a solution of Eq. (1.4) on $O(n)$. Obviously it is a pure solution supported on $\{[\lambda]\}$, where $\lambda = \iota_{O(n)}$. Since $\text{tr}(\hat{f}_A(\lambda) - \hat{f}_B(\lambda)) = \text{tr}(A - B) = 0$, we have $f_A - f_B \in L^2_c(O(n))^\perp$. Similarly, $f_C - f_D \in L^2_c(O(n))^\perp$. Thus $\mathcal{F}^{O(n)}_T$ is normalized. It is homogeneous if and only if $f_E \equiv 0$ or $f_F \equiv 0$, which is equivalent to $E \otimes F = \hat{f}_E(\iota) \otimes \hat{f}_F(\iota) = 0$, i.e., $T$ is homogeneous.

(2) Since $\Phi^r_{A,B,C,D} = \Psi_{E \otimes F}$, for all $x, y \in Sp(n)$ we have
\[
f_A(xy) + f_B(yx) + f_C(xy^{-1}) + f_D(y^{-1}x)
= \text{tr}(Ax) + \text{tr}(Byx) + \text{tr}(Cx J y^t) + \text{tr}(D J y^t J^t x)
= \text{tr}(Ax y + x By + J^t x^t C y + J^t D^t x^t J y)
= \text{tr}(\Phi^r_{A,B,C,D}(x)y)
= \text{tr}(E x F y)
= f_E(x) f_F(y).
\]
Hence $\mathcal{F}^{Sp(n)}_T$ is a solution of Eq. (1.4) on $Sp(n)$. The proofs of the other assertions in (2) are similar to those of the corresponding parts in (1) and omitted here. \Halmos

Note that if $\varphi : G \to K$ is a homomorphism and $\mathcal{F}^K = (f_i)$ is a solution of Eq. (1.4) on $K$, then $\mathcal{F}^K \circ \varphi = (f_i \circ \varphi)$ is a solution on $G$. Some relations between $\mathcal{F}^K$ and $\mathcal{F}^K \circ \varphi$ are revealed in the following assertion.

**Proposition 3.2.** Let $\pi : G \to U(n)$ be an irreducible representation of complex (resp. real, quaternionic) type, and let $K = U(n)$ (resp. $O(n)$, $Sp(n)$). If $\mathcal{F}^K = (f_i)$ is a pure solution of Eq. (1.4) on $K$ supported on $[[\iota_K]]$, then $\mathcal{F}^K \circ \pi$ is a pure solution of Eq. (1.4) on $G$ supported on $[[\pi]]$, and $\mathcal{F}^K \circ \pi$ is normalized (resp. homogeneous) if and only if $\mathcal{F}^K$ is normalized (resp. homogeneous).

**Proof.** It suffices to prove that if $f$ is a function on $K$ with supp$((\hat{f}) \subseteq [[\iota_K]])$, then supp$((f \circ \pi)) \subseteq [[\pi]]$, $f \circ \pi \in L^2_c(G)^\perp$ if and only if $f \in L^2_c(K)^\perp$, and $f \circ \pi \equiv 0$ if and only if $f \equiv 0$. Suppose that $\pi$ is of complex type. Then $f$ is of the form $f(x) = \text{tr}(Ax) + \text{tr}(B \tilde{x})$, where $x \in U(n)$, $A, B \in \mathbb{M}(n, \mathbb{C})$. Hence $(f \circ \pi)(y) = \text{tr}(A \pi(y)) + \text{tr}(B \bar{\pi}(y))$, $y \in G$. This implies that $(f \circ \pi)(\pi) = A$,
(f \circ \pi)(\pi) = B, and (f \circ \pi')(\pi') = 0 if [\pi'] \notin [[\pi]]. So \text{supp}((f \circ \pi)) \subseteq [[\pi]]. Moreover, we have
\[ f \circ \pi \in L^2(G) \iff \text{tr}A = \text{tr}B = 0 \iff f \in L^2(K), \]
\[ f \circ \pi \equiv 0 \iff A = B = 0 \iff f \equiv 0. \]
The proofs of the other two cases are similar and left to the reader. \( \square \)

Example 3.2. Any 1-ordered \( r \)-admissible tuple is of the form
\[ T_{a,b} = (a/2, a/2, b/2, b/2, a + b) \]
for some \( a, b \in \mathbb{C} \). It is homogeneous if and only if \( a + b = 0 \). We define the tuple of functions \( \mathcal{G}^{O(1)}_{a,b} = (f_i)_{i=1}^6 \) as
\[
\begin{align*}
\mathcal{G}^{O(1)}_{a,b} : \quad & f_1(x) = f_2(x) = ax/2, \\
& f_3(x) = f_4(x) = bx/2, \quad x, y \in O(1). \\
& f_5 \otimes f_6(x, y) = (a + b)xy,
\end{align*}
\]
Then \( \mathcal{G}^{O(1)}_{a,b} = \mathcal{G}^{U(1)}_{a,b} \). By Proposition 3.1 (1), it is a pure normalized solution of Eq. (1.4) on \( O(1) \) supported on \( \{ [O(1)] \} \). It is homogeneous if and only if \( a + b = 0 \). Note that \( \mathcal{G}^{U(1)}_{a,1,b,0} \) is the restriction of the solution \( \mathcal{G}^{U(1)}_{a,1,1,0} \) on \( U(1) \) (see Example 3.1). But it may occur that \( \mathcal{G}^{U(1)}_{a,1,b,0} \) is non-homogeneous while \( \mathcal{G}^{O(1)}_{a,b} \) is homogeneous. This fact is meaningful when we construct the general solution of Eq. (1.4) on arbitrary compact groups (see Section 5). For later reference, we denote \( \mathcal{G}^{O(1)}_{a} = \mathcal{G}^{O(1)}_{2a,-2a} \). In our notation of homogeneous solutions, \( \mathcal{G}^{O(1)}_{a} \) is the 4-tuple of functions \( (f_i)_{i=1}^4 \) defined as
\[ \mathcal{G}^{O(1)}_{a} : \quad f_1(x) = f_2(x) = -f_3(x) = -f_4(x) = ax, \quad x \in O(1). \]

Now we consider admissible matrix tuples of higher order. Since the bilinear pairing \((X, Y) \mapsto \text{tr}(XY)\) on \( M(n, \mathbb{C}) \) is non-degenerate, for a linear map \( \Gamma : M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}) \), we can define its adjoint \( \Gamma^\dagger \) by \( \text{tr}(\Gamma(X)Y) = \text{tr}(X\Gamma^\dagger(Y)) \) for all \( X, Y \in M(n, \mathbb{C}) \). It is straightforward to check that
\[
\begin{align*}
(\Phi^{c}_{B,A})^\dagger &= \Phi^{c}_{B,A}, \\
(\Phi^{c}_{B,A,C,D})^\dagger &= \Phi^{c}_{B,A,C^*,D^*}, \\
(\Phi^{q}_{B,A,C,D})^\dagger &= \Phi^{q}_{B,A,C^*,D^*,J}, \\
(\Psi_{E \otimes F})^\dagger &= \Psi_{F \otimes E}.
\end{align*}
\]

Lemma 3.3. Let \( A, B \in M(2, \mathbb{C}) \) be such that \( \text{tr}(A) = \text{tr}(B) \).

(1) The tuples
\[ T_{A,B} = (A, B, -A, -B, -JA^* + B^*J) \otimes J, \]
\[ (T_{A,B})^\dagger = (A, B, -B^*, -A^*, -J \otimes (AJ + JB)) \]

(2) Any 1-ordered \( r \)-admissible tuple is of the form
\[ T_{a,b} = (a/2, a/2, b/2, b/2, a + b) \]
for some \( a, b \in \mathbb{C} \). It is homogeneous if and only if \( a + b = 0 \). We define the tuple of functions \( \mathcal{G}_{a,b}^{O(1)} = (f_i)_{i=1}^6 \) as
\[
\begin{align*}
\mathcal{G}_{a,b}^{O(1)} : \quad & f_1(x) = f_2(x) = ax/2, \\
& f_3(x) = f_4(x) = bx/2, \quad x, y \in O(1). \\
& f_5 \otimes f_6(x, y) = (a + b)xy,
\end{align*}
\]
are \( r \)-admissible. They are homogeneous if and only if \( \text{tr}(A) = 0 \) and \( B = A^t \).

(2) The tuples

\[
\mathcal{T}_{A,B}^q = (A, B, A, B, (A + B) \otimes I),
\]

\[
(\mathcal{T}_{A,B}^q)^\dagger = (A, B, JB^tJ^t, JA^tJ^t, I \otimes (A + B))
\]

are \( q \)-admissible. They are homogeneous if and only if \( \text{tr}(A) = 0 \) and \( B = A \).

Proof. We first prove the assertions for \( \mathcal{T}_{A,B}^r \) and \( \mathcal{T}_{A,B}^q \). Since \( Y + JY^tJ^t = \text{tr}(Y)I \) for any \( Y \in \mathbb{M}(2, \mathbb{C}) \), we have

\[
\Phi_{A,B,A,B}^q(X) = AX + XB + J(AX + XB)^tJ^t = \text{tr}(AX + XB)I = \Psi_{(A+B) \otimes I}(X).
\]

(3.7)

So \( \mathcal{T}_{A,B}^q \) is \( q \)-admissible. By (3.2), we have

\[
\Phi_{A,B,-A,-B}^r(X) = -\Phi_{-A,-JB,J,-JB,J}^q(XJ).J
\]

\[
= -\Psi_{(A-JBJ) \otimes I}(XJ).J
\]

\[
= \Psi_{-(JA+BJ) \otimes J}(X).
\]

So \( \mathcal{T}_{A,B}^r \) is \( r \)-admissible.

Now by (3.4)–(3.6), we have

\[
\Phi_{A,B,-B',-A'}^r(X) = (\Phi_{B,A,B,A}^q)^\dagger = (\Psi_{(A+B) \otimes I})^\dagger = \Psi_{I \otimes (A+B)},
\]

(3.8)

\[
\Phi_{A,B,-B',-A'}^r = (\Phi_{A,B,-B,-A}^r)^\dagger = (\Psi_{-(JB+AJ) \otimes J})^\dagger = \Psi_{-J \otimes (AJ+JB)}.
\]

Hence \( (\mathcal{T}_{A,B}^q)^\dagger \) and \( (\mathcal{T}_{A,B}^r)^\dagger \) are admissible tuples of quaternionic and real type, respectively.

The conditions of being homogeneous are easy to prove and left to the reader. \( \square \)

Remark 3.1. The families \( \mathcal{T}_{A,B}^r \) and \( (\mathcal{T}_{A,B}^q)^\dagger \) (resp. \( \mathcal{T}_{A,B}^q \) and \( (\mathcal{T}_{A,B}^r)^\dagger \)) are not mutually exclusive. Indeed, it is easy to check that \( \mathcal{T}_{A,B}^r = (\mathcal{T}_{A,B}^q)^\dagger \) if and only if \( B = A^t \), and \( \mathcal{T}_{A,B}^q = (\mathcal{T}_{A,B}^r)^\dagger \) if and only if \( B = \text{tr}(A)I - A \). In particular, if \( \mathcal{T}_{A,B}^r \) (or, equivalently, \( (\mathcal{T}_{A,B}^q)^\dagger \)) is homogeneous, then \( \mathcal{T}_{A,B}^r = (\mathcal{T}_{A,B}^q)^\dagger \). Similarly, if \( \mathcal{T}_{A,B}^q \) (or \( (\mathcal{T}_{A,B}^r)^\dagger \)) is homogeneous, then \( \mathcal{T}_{A,B}^q = (\mathcal{T}_{A,B}^r)^\dagger \).

Example 3.3. Let \( A, B \in \mathbb{M}(2, \mathbb{C}) \) with \( \text{tr}(A) = \text{tr}(B) \). By Proposition 3.1 (1) and Lemma 3.2 (1), the tuples of functions \( \mathcal{T}_{A,B}^{O(2)} = \mathcal{T}_{A,B}^{O(2)} \) and \( (\mathcal{T}_{A,B}^{O(2)})^\dagger = (\mathcal{T}_{A,B}^{O(2)})^\dagger \) are pure normalized solutions of Eq. (14) on \( O(2) \) supported on
\{[t_{O(2)}]\}. Writing explicitly, we have

\begin{align*}
\mathcal{F}^{O(2)}_{A,B} : & \begin{cases} 
  f_1(x) = -f_3(x) = \text{tr}(Ax), \\
  f_2(x) = -f_4(x) = \text{tr}(Bx), \\
  f_5 \otimes f_6(x,y) = -\text{tr}((JA + BJ)x)\text{tr}(y),
\end{cases} & x, y \in O(2); \\
(\mathcal{F}^{O(2)}_{A,B})^\dagger : & \begin{cases} 
  f_1(x) = -f_4(x^{-1}) = \text{tr}(Ax), \\
  f_2(x) = -f_3(x^{-1}) = \text{tr}(Bx), \\
  f_5 \otimes f_6(x,y) = -\text{tr}(x)\text{tr}((AJ + JB)y),
\end{cases} & x, y \in O(2).
\end{align*}

The solutions \(\mathcal{F}^{O(2)}_{A,B}\) and \((\mathcal{F}^{O(2)}_{A,B})^\dagger\) are homogeneous if and only if \(\text{tr}(A) = 0\) and \(B = A^t\). In this case the two solutions are equal (see Remark 3.1). We denote \(\mathcal{F}^{O(2)}_A = \mathcal{F}^{O(2)}_{A,A^t} = (\mathcal{F}^{O(2)}_{A,B})^\dagger\) if \(\text{tr}(A) = 0\). The functions in \(\mathcal{F}^{O(2)}_A\) are

\begin{align*}
\mathcal{F}^{O(2)}_A : & f_1(x) = f_2(x^{-1}) = -f_3(x) = -f_4(x^{-1}) = \text{tr}(Ax), & x \in O(2).
\end{align*}

**Example 3.4.** Let \(A, B \in \mathbb{M}(2, \mathbb{C})\) with \(\text{tr}(A) = \text{tr}(B)\). By Proposition 3.1 (2), Lemma 3.3 (2) and the fact \(Sp(2) = SU(2)\), the tuples of functions \(\mathcal{F}^{SU(2)}_{A,B} = \mathcal{F}^{SU(2)}_{A,B} = (\mathcal{F}^{SU(2)}_{A,B})^\dagger\) are pure normalized solutions of Eq. (1.4) on \(SU(2)\) supported on \(\{t_{SU(2)}\}\). The functions in these solutions are

\begin{align*}
\mathcal{F}^{SU(2)}_{A,B} : & \begin{cases} 
  f_1(x) = f_3(x) = \text{tr}(Ax), \\
  f_2(x) = f_4(x) = \text{tr}(Bx), \\
  f_5 \otimes f_6(x,y) = \text{tr}((A + B)x)\text{tr}(y),
\end{cases} & x, y \in SU(2); \\
(\mathcal{F}^{SU(2)}_{A,B})^\dagger : & \begin{cases} 
  f_1(x) = \text{tr}(Ax), \\
  f_2(x) = \text{tr}(Bx), \\
  f_3(x) = \text{tr}(A)\text{tr}(x) - f_2(x), \\
  f_4(x) = \text{tr}(A)\text{tr}(x) - f_1(x), \\
  f_5 \otimes f_6(x,y) = \text{tr}(x)\text{tr}((A + B)y),
\end{cases} & x, y \in SU(2).
\end{align*}

These solutions are homogeneous if and only if \(\text{tr}(A) = 0\) and \(B = -A\), and in this case we have \(\mathcal{F}^{SU(2)}_{A,B} = (\mathcal{F}^{SU(2)}_{A,B})^\dagger\). We denote \(\mathcal{F}^{SU(2)}_A = \mathcal{F}^{SU(2)}_{A,A^t} = (\mathcal{F}^{SU(2)}_{A,A^t})^\dagger\) if \(\text{tr}(A) = 0\). Writing explicitly, it is

\begin{align*}
\mathcal{F}^{SU(2)}_A : & f_1(x) = -f_3(x) = f_3(x) = -f_4(x) = \text{tr}(Ax), & x \in SU(2).
\end{align*}

Now we consider 3-ordered \(r\)-admissible tuples. We view elements of \(\mathbb{C}^3\) as column vectors. For \(u, v \in \mathbb{C}^3\), let \(\langle u, v \rangle = u^t v\) be the standard bilinear pairing, and define

\(\tau_{u,v} = uv^t - \frac{1}{2} \langle u, v \rangle I_3 \in \mathbb{M}(3, \mathbb{C})\).
Let $\mathbb{M}_{\text{skew}}(3, \mathbb{C})$ denote the space of $3 \times 3$ skew-symmetric complex matrices. For $u = (u_1, u_2, u_3)^t \in \mathbb{C}^3$, let

$$\sigma_u = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{M}_{\text{skew}}(3, \mathbb{C}).$$

Note that for $w \in \mathbb{C}^3$, $\sigma_u w$ is (the complex analogue of) the cross product $u \times w$ of $u$ and $w$.

**Lemma 3.4.** For any $u, v \in \mathbb{C}^3$, the tuple

$$\mathcal{T}_{u,v} = (\tau_{u,v}, \tau_{v,u}, -\tau_{u,v}, -\tau_{v,u}, \sigma_u \otimes \sigma_v)$$

is $r$-admissible. It is homogeneous if and only if it is the trivial tuple.

**Proof.** Firstly we consider the representations $\rho_1$ and $\rho_2$ of the Lie algebra $\mathfrak{gl}(3, \mathbb{C})$ in $\mathbb{M}_{\text{skew}}(3, \mathbb{C})$ and $\mathbb{C}^3$ defined by

$$\rho_1(A)(Y) = AY + YA^t, \quad \rho_2(A)(w) = (\text{tr}(A)I_3 - A^t)w,$$

respectively, where $A \in \mathfrak{gl}(3, \mathbb{C})$, $Y \in \mathbb{M}_{\text{skew}}(3, \mathbb{C})$, $w \in \mathbb{C}^3$. We claim that the linear isomorphism $\sigma : \mathbb{C}^3 \rightarrow \mathbb{M}_{\text{skew}}(3, \mathbb{C})$ sending $w$ to $\sigma_w$ is an equivalence between $\rho_1$ and $\rho_2$, i.e.,

$$\rho_1(A)(\sigma_w) = \sigma(\rho_2(A)(w)) \quad (3.10)$$

for all $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w \in \mathbb{C}^3$. To prove this, we note (the complex analogue of) the equality for scalar triple products, i.e., for all $w, w_1, w_2 \in \mathbb{C}^3$, we have

$$\langle \sigma_w w_1, w_2 \rangle = \det[w, w_1, w_2],$$

where $[w, w_1, w_2]$ is the $3 \times 3$ matrix specified by column vectors. Now let $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w, w_1, w_2 \in \mathbb{C}^3$. Then we have

$$\langle \rho_1(A)(\sigma_w)w_1, w_2 \rangle = \langle (A\sigma_w + \sigma_w A^t)w_1, w_2 \rangle$$

$$= \langle A\sigma_w w_1, w_2 \rangle + \langle \sigma_w A^t w_1, w_2 \rangle$$

$$= \langle \sigma_w w_1, A^t w_2 \rangle + \langle \sigma_w A^t w_1, w_2 \rangle$$

$$= \det[w, w_1, A^t w_2] + \det[w, A^t w_1, w_2]$$

and

$$\langle \sigma(\rho_2(A)(w))w_1, w_2 \rangle = \det[\rho_2(A)(w), w_1, w_2]$$

$$= \det[(\text{tr}(A)I_3 - A^t)w, w_1, w_2]$$

$$= \text{tr}(A) \det[w, w_1, w_2] - \det[A^t w, w_1, w_2].$$

This proves (3.10) by noting the fact that

$$\det[Aw, w_1, w_2] + \det[w, Aw_1, w_2] + \det[w, w_1, Aw_2] = \text{tr}(A) \det[w, w_1, w_2]$$

for all $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w, w_1, w_2 \in \mathbb{C}^3$. 

Now we notice that
\[
\rho_2(\tau_{u,v})(w) = -\langle u, w \rangle v = \frac{1}{2} \text{tr}(\sigma_u \sigma_w)v,
\]
\[
\tau_{u,v}^t = \tau_{v,u}, \quad \sigma_u^t = -\sigma_u.
\]
From these identities, (3.9), and (3.10), it follows that for all \(X \in M(3, \mathbb{C})\) we have
\[
\Phi_{\tau_{u,v},\tau_{v,u},\tau_{u,v}^t}(X) = \tau_{u,v}X + X \tau_{v,u} - (\tau_{u,v}X + X \tau_{v,u})^t
\]
\[
= \rho_1(\tau_{u,v})(X - X^t) + (X - X^t)^t \tau_{u,v}^t
\]
\[
= -\sigma(\rho_2(\tau_{u,v})(\sigma^{-1}(X - X^t)))
\]
\[
= -\sigma(\langle u, \sigma^{-1}(X - X^t) \rangle v) = -\langle u, \sigma^{-1}(X - X^t) \rangle \sigma(v)
\]
\[
= \frac{1}{2} \text{tr}(\sigma_u(X - X^t))\sigma_v = \text{tr}(\sigma_u X)\sigma_v
\]
\[
= \Psi_{\sigma_u \otimes \sigma_v} (X).
\] (3.11)
This proves that \(\mathcal{J}_{u,v}\) is \(r\)-admissible. If \(\mathcal{J}_{u,v}\) is homogeneous, then \(\sigma_u = 0\) or \(\sigma_v = 0\), which implies that \(u = 0\) or \(v = 0\). Hence it is the trivial tuple. \(\square\)

**Example 3.5.** For \(u, v \in \mathbb{C}^3\), we define the tuple of functions \(\mathcal{F}^{O(3)}_{u,v}\) as \(\mathcal{F}^{O(3)}_{u,v}\).

The functions in \(\mathcal{F}^{O(3)}_{u,v}\) are
\[
\mathcal{F}^{O(3)}_{u,v} : \begin{cases} 
  f_1(x) = f_2(x^{-1}) = -f_3(x) = -f_4(x^{-1}) = \text{tr}(\tau_{u,v}x), & x, y \in O(3). \\
  f_5 \otimes f_6(x, y) = \text{tr}(\sigma_u x)\text{tr}(\sigma_v y),
\end{cases}
\]

Then Proposition 3.1 (1) and Lemma 3.4 imply that \(\mathcal{F}^{O(3)}_{u,v}\) is a pure normalized solution of Eq. (1.4) on \(O(3)\) supported on \(\{[\iota_{O(3)}]\}\). It is homogeneous if and only if it is the trivial solution.

## 4. Determination of admissible tuples

In this section we determine all admissible matrix tuples, which are completely described in the following three propositions. We keep the same notation from Section 3.

**Proposition 4.1.** Let \(\mathcal{J} = (A, B, E \otimes F)\) be an \(n\)-ordered \(c\)-admissible tuple.

1. If \(n = 1\), then \(\mathcal{J} = (a, a, 2a)\) for some \(a \in \mathbb{C}\).

2. If \(n \geq 2\), then \(\mathcal{J}\) is the trivial tuple.

**Proposition 4.2.** Let \(\mathcal{J} = (A, B, C, D, E \otimes F)\) be an \(n\)-ordered \(r\)-admissible tuple.

1. If \(n = 1\), then \(\mathcal{J} = \mathcal{J}_{a,b}\) for some \(a, b \in \mathbb{C}\).

2. If \(n = 2\), then \(\mathcal{J} = \mathcal{J}_{A,B}^r\) or \((\mathcal{J}_{A,B}^r)^\dagger\) for some \(A, B \in M(2, \mathbb{C})\) with \(\text{tr}(A) = \text{tr}(B)\).
(3) If \( n = 3 \), then \( \mathcal{T} = \mathcal{T}_{u,v} \) for some \( u, v \in \mathbb{C}^3 \).

(4) If \( n \geq 4 \), then \( \mathcal{T} \) is the trivial tuple.

**Proposition 4.3.** Let \( n \) be even, and let \( \mathcal{T} = (A, B, C, D, E \otimes F) \) be an \( n \)-ordered \( q \)-admissible tuple.

1. If \( n = 2 \), then \( \mathcal{T} = T_{A,B} \) or \( (T_{A,B})^\dagger \) for some \( A, B \in M(2, \mathbb{C}) \) with \( \text{tr}(A) = \text{tr}(B) \).
2. If \( n \geq 4 \), then \( \mathcal{T} \) is the trivial tuple.

The assertions in Propositions 4.1 (1) and 4.2 (1) are trivial. It remains to prove the others. Since our proofs of 4.3 (2) and 4.2 (2) make use of 4.2 (4) and 4.3 (1), respectively, and the proofs of 4.1 (2) and 4.2 (4) are similar, we proceed the proofs in the following order:

\[ 4.1 \, (2), 4.2 \, (4) \Rightarrow 4.3 \, (2), 4.3 \, (1) \Rightarrow 4.2 \, (2), 4.2 \, (3). \]

**Proof of Proposition 4.1 (2).** Denote \( \Phi = \Phi_{c_{A,B}} \) and \( \mathbb{N}_n = \{1, \ldots, n\} \). Since \( \Phi = \Psi_{E \otimes F} \), we have \( \dim \text{Im}(\Phi) \leq 1 \). So the entries \( \Phi(X)_{ij} (i, j \in \mathbb{N}_n) \) of \( \Phi(X) \), viewed as linear polynomials in the entries \( X_{ij} \) of \( X \), are mutually linearly dependent. We make the convention that if a linear polynomial \( p \) in the variables \( y_1, \ldots, y_m \) is written in the reduced form as \( p(y) = a_1 y_1 + a_2 y_2 + \cdots \), then the terms being omitted do not contain \( y_1 \) and \( y_2 \).

Let \( i, j \in \mathbb{N}_n \), \( i \neq j \). It is easy to see that

\[
\Phi(X)_{ii} = A_{ij} X_{ji} + 0X_{jj} + \cdots ,
\]

\[
\Phi(X)_{ij} = 0X_{ji} + A_{ij} X_{jj} + \cdots .
\]

Since they are linearly dependent, we must have \( A_{ij} = 0 \). So \( A \) is diagonal. Similarly, \( B \) is diagonal. Now we have

\[
\Phi(X)_{rs} = (A_{rr} + B_{ss}) X_{rs} \quad \text{for all} \quad r, s \in \mathbb{N}_n.
\]

Setting \((r, s) = (i, i), (i, j), (j, i), (j, j)\), we get four polynomials. Their mutual linear dependence implies that at most one of the four sums \( A_{ii} + B_{ii}, A_{ii} + B_{jj}, A_{jj} + B_{ii}, A_{jj} + B_{jj} \) is nonzero. This forces that they are all zero. So \( A = -B \in \mathbb{C}I \). But we have \( \text{tr}(A) = \text{tr}(B) \). Hence \( A = B = 0 \). This proves that \( \mathcal{T} \) is the trivial tuple. \( \square \)

We use the similar idea to prove 4.2 (4).

**Proof of Proposition 4.2 (4).** Denote \( \Phi = \Phi_{r_{A,B,C,D}} \). Then \( \dim \text{Im}(\Phi) \leq 1 \) and \( \Phi(X)_{ij} (i, j \in \mathbb{N}_n) \) are mutually linearly dependent. Let \( i, j \in \mathbb{N}_n \) with \( i \neq j \). Since \( n \geq 4 \), there exist \( k, l \in \mathbb{N}_n \) such that \( i, j, k, l \) are distinct. Then we compute

\[
\Phi(X)_{ik} = A_{ij} X_{jk} + 0X_{jl} + \cdots ,
\]

\[
\Phi(X)_{il} = 0X_{jk} + A_{ij} X_{jl} + \cdots .
\]
Since they are linearly dependent, we have $A_{ij} = 0$. So $A$ is diagonal. Similarly, $B, C, D$ are diagonal. Now we have

$$
\Phi(X)_{rs} = (A_{rr} + B_{ss})X_{rs} + (C_{ss} + D_{rr})X_{sr}
$$

for all $r, s \in \mathbb{N}_n$.

Setting $(r, s) = (i, j), (i, l), (k, j), (k, l)$, we get four polynomials. Their mutual linear dependence implies that at most one of $A_{ii} + B_{jj}, A_{ii} + B_{ll}, A_{kk} + B_{jj}, A_{kk} + B_{ll}$ is nonzero. This forces that they are all zero. So $A_{ii} + B_{jj} = 0$ whenever $i \neq j$. This is impossible unless $A = -B \in \mathbb{C}I$. But we have

$$
\text{tr}(A) = \text{tr}(B).
$$

So $A = B = 0$. Similarly, $C = D = 0$. Hence $T$ is trivial. \hfill \Box

We now use Proposition 4.2 (4) to prove Proposition 4.3 (2).

**Proof of Proposition 4.3 (2).** Suppose $n \geq 4$ and $T$ is $q$-admissible. Then it follows from (3.2) that the tuple $(A, -JBJ, -C, JDJ, -(JE) \otimes (FJ))$ is $r$-admissible. By Proposition 4.2 (4), we have

$$
A = -JBJ = -C = JDJ = 0.
$$

So $A = B = C = D = 0$. \hfill \Box

Similarly, due to (3.2), Proposition 4.2 (2) is equivalent to Proposition 4.3 (1). We find that the proof of Proposition 4.3 (1) is easier to write up. So we prove it first. In the following proof, we will constantly use the fact that $Y + JY^tJ^t = \text{tr}(Y)I$ for all $Y \in \mathbb{M}(2, \mathbb{C})$ without any further mention.

**Proof of Proposition 4.3 (1).** Denote $\Phi = \Phi_{A,B,C,D}^q$. Then $\dim \text{Im}(\Phi) \leq 1$ and $\Phi(X)_{ij} (i, j \in \mathbb{N}_2)$ are mutually linearly dependent. We divide the proof into two steps.

**Step (i).** First we assume that $C = -A$ and $D = -B$. We prove that $\text{tr}(A) = 0, B = A$, and $\Phi(X) = 2\text{tr}(X)A$.

In this case, we have

$$
\Phi(X) = AX + XB - J(AX + XB)^tJ^t.
$$

Let $(i, j) = (1, 2)$ or $(2, 1)$. Since

$$
\Phi(X)_{ii} = (A_{ij} - B_{ij})X_{ji} + \cdots,
\Phi(X)_{ij} = 0X_{ji} + 2A_{ij}X_{jj} + 2B_{ij}X_{ii} + \cdots,
\Phi(X)_{ji} = 0X_{ij} + 2A_{jj}X_{ij} + \cdots,
\Phi(X)_{ii} = (A_{ii} + B_{ii})X_{ii} - (A_{jj} + B_{jj})X_{jj},
$$

their linear dependence implies that $A_{ij} = B_{ij}$. Using this, it is easy to compute that

$$
\Phi(X)_{ij} = 2(A_{ii} + B_{jj})X_{ij} + \cdots,
\Phi(X)_{ji} = 0X_{ij} + 2(A_{jj} + B_{ii})X_{ji} + \cdots,
\Phi(X)_{ii} = (A_{ii} + B_{ii})X_{ii} - (A_{jj} + B_{jj})X_{jj}.
$$

We claim that $A_{ii} + B_{jj} = 0$. For otherwise, if $A_{ii} + B_{jj} \neq 0$, then by the mutual linear dependence, we have $A_{jj} + B_{ii} = A_{jj} + B_{ii} = A_{jj} + B_{jj} = 0$, which conflicts with $A_{ii} + B_{jj} \neq 0$. Now if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$.
But \( \text{tr}(A) = \text{tr}(B) \). Hence \( \text{tr}(A) = 0 \) and \( B = A \). This also implies that \( C = D = -A = JA^tJ^t \). By (3.8), we have \( \Phi(X) = 2\text{tr}(X)A \).

**Step (ii).** Now we prove the general case. Since

\[
\Phi(X) - J\Phi(X)^tJ^t = (A - C)X + X(B - D) - J[(A - C)X + X(B - D)]^tJ^t,
\]

\[
\Psi_{E \otimes F}(X) - J\Psi_{E \otimes F}(X)^tJ^t = \text{tr}(EX)(F - JF^tJ^t),
\]

the tuple \((A - C, B - D, C - A, D - B, E \otimes (F - JF^tJ^t))\) is \(q\)-admissible. By Step (i), we have \( \text{tr}(A - C) = 0 \),

\[
B - D = A - C,
\]

and

\[
\Phi(X) - J\Phi(X)^tJ^t = 2\text{tr}(X)(A - C).
\]

This also implies that the traces of \( A, B, C, D \) are the same. On the other hand, we have

\[
\Phi(X) + J\Phi(X)^tJ^t = (A + C)X + X(B + D) + J[(A + C)X + X(B + D)]^tJ^t
\]

\[
= \text{tr}((A + B + C + D)X)I
\]

\[
= 2\text{tr}((A + D)X)I.
\]

Hence

\[
\Phi(X) = \text{tr}(X)(A - C) + \text{tr}((A + D)X)I.
\]  

(4.2)

There are two cases to consider.

**Case (a).** \( A - C \) and \( I \) are linearly dependent. Then \( A - C \) is a scalar matrix. But we have \( \text{tr}(A) = \text{tr}(C) \). Hence \( C = A \). From (4.1), we see that \( D = B \). By (3.7), we have \( \Phi = \Psi_{(A + B) \otimes I} \) and \( T = T^q_{A,B} \).

**Case (b).** \( A - C \) and \( I \) are linearly independent. Since \( \dim \text{Im}(\Phi) \leq 1 \), by (4.2), the dimension of the subspace

\[
\{(\text{tr}(X), \text{tr}((A + D)X)) \mid X \in M(2, \mathbb{C})\}
\]

of \( \mathbb{C}^2 \) is less than or equal to 1. This implies that \( A + D \) is a scalar matrix. By (4.1), \( B + C \) is also a scalar matrix. Hence

\[
D = A + D - A = \frac{1}{2}\text{tr}(A + D)I - A = \text{tr}(A)I - A = JA^tJ^t.
\]

Similarly, we have \( C = JB^tJ^t \). From (3.8), we see that \( \Phi = \Psi_{I \otimes (A + B)} \) and \( T = (T^q_{A,B})^\dagger \).

**Proof of Proposition 4.2 (2).** By (3.2), the tuple \((A, -JB, -C, JD, -(JE) \otimes (FJ))\) is \(q\)-admissible, which must be \( T^q_{A-JBJ} \) or \( (T^q_{A-JBJ})^\dagger \) by Proposition 4.3 (1). This implies that \( T \) is equal to \( T^q_{A,B} \) or \( (T^q_{A,B})^\dagger \).

**Proof of Proposition 4.2 (3).** We will make use of the representations \( \rho_1 \) and \( \rho_2 \) of \( \mathfrak{gl}(3, \mathbb{C}) \) in \( M_{\text{skew}}(3, \mathbb{C}) \) and \( \mathbb{C}^3 \) defined in (3.9).
Proof of Proposition 4.2 (3). Denote $\Phi = \Phi_{A,B,C,D}$. Then $\dim \text{Im}(\Phi) \leq 1$ and $\Phi(X)_{ij}$ ($i, j \in \mathbb{N}_3$) are mutually linearly dependent. We divide the proof into two steps.

Step (i). First we assume that $C = A$, $D = B$. Then

$$\Phi(X) = AX + XB + (AX + XB)^t.$$  

We prove that $A = B = 0$.

Let $i, j \in \mathbb{N}_3$, $i \neq j$. Let $k \in \mathbb{N}_3$ with $\{i, j, k\} = \mathbb{N}_3$. Since

$$\Phi(X)_{ii} = 0X_{jk} + 2A_{ij}X_{ji} + \cdots,$$

$$\Phi(X)_{ik} = A_{ij}X_{jk} + \cdots,$$

their linear dependence implies that $A_{ij} = 0$. So $A$ is diagonal. Similarly, $B$ is diagonal. Now we have six polynomials

$$\Phi(X)_{rr} = 2(A_{rr} + B_{rr})X_{rr},$$

$$\Phi(X)_{rs} = (A_{rr} + B_{ss})X_{rs} + (A_{ss} + B_{rr})X_{sr},$$

whose mutual linear dependence forces that $A = -B \in \mathbb{C}I$. But $\text{tr}(A) = \text{tr}(B)$. So $A = B = 0$.

Step (ii). Now we prove the general case. Since

$$\Phi(X) + \Phi(X)^t = (A + C)X + X(B + D) + [(A + C)X + X(B + D)]^t;$$

$$\Psi_{E \otimes F}(X) + \Psi_{E \otimes F}(X)^t = \text{tr}(EX)(F + F^t),$$

the tuple $(A + C, B + D, A + C, B + D, E \otimes (F + F^t))$ is $r$-admissible. By Step (i), we have $A + C = 0$, $B + D = 0$. So

$$\Phi(X) = AX + XB - (AX + XB)^t.$$  

Let $i, j \in \mathbb{N}_3$, $i \neq j$. Let $k \in \mathbb{N}_3$ with $\{i, j, k\} = \mathbb{N}_3$. Since

$$\Phi(X)_{ij} = (A_{ij} - B_{ji})X_{jj} + \cdots,$$

$$\Phi(X)_{ik} = 0X_{jj} + A_{ij}X_{jk} - B_{ji}X_{kj} + \cdots,$$

their linear dependence implies that

$$B_{ji} = A_{ij}. \quad (4.3)$$

We now prove that

$$A_{ii} - B_{ii} = A_{jj} - B_{jj}. \quad (4.4)$$

If both $\Phi(X)_{ik}$ and $\Phi(X)_{jk}$ are identically zero, from the expressions

$$\Phi(X)_{ik} = (A_{ii} + B_{kk})X_{ik} - (A_{kk} + B_{ii})X_{ki} + \cdots,$$

$$\Phi(X)_{jk} = (A_{jj} + B_{kk})X_{jk} - (A_{kk} + B_{jj})X_{kj} + \cdots,$$
we get $A_{ii} + B_{kk} = A_{jj} + B_{kk} = A_{kk} + B_{ii} = A_{kk} + B_{jj} = 0$, which implies (4.4). If one of $\Phi(X)_{ik}$ and $\Phi(X)_{jk}$, say $\Phi(X)_{ik}$, is not identically zero. Then we have the linearly dependent polynomials

$$
\Phi(X)_{ik} = B_{ik} X_{ij} - A_{kj} X_{ji} + \cdots,
$$

$$
\Phi(X)_{ij} = (A_{ii} + B_{jj}) X_{ij} - (A_{jj} + B_{ii}) X_{ji} + \cdots.
$$

Since $\Phi(X)_{ik} \neq 0$ and $B_{ik} = A_{kj}$, we must have $A_{ii} + B_{jj} = A_{jj} + B_{ii}$, which also implies (4.4). By (4.4), there exists $\alpha \in \mathbb{C}$ such that $B_{ii} = A_{ii} + \alpha$. But $\text{tr}(A) = \text{tr}(B)$. So we have

$$
B_{ii} = A_{ii}. \tag{4.5}
$$

From (4.3) and (4.5), we have $B = A^t$. Thus

$$
\Phi(X) = A(X - X^t) + (X - X^t) A^t.
$$

Denote $\Phi_1 = \Phi|_{\mathbb{M}_{\text{skew}}(3, \mathbb{C})}$. Then

$$
\Phi_1(Y) = 2(AY + YA^t)
$$

for $Y \in \mathbb{M}_{\text{skew}}(3, \mathbb{C})$, and we have dim $\text{Im}(\Phi_1) \leq 1$.

Now we consider the representations $\rho_1$ and $\rho_2$ of $\mathfrak{gl}(3, \mathbb{C})$ in $\mathbb{M}_{\text{skew}}(3, \mathbb{C})$ and $\mathbb{C}^3$ defined in (3.3). Note that $\Phi_1 = 2\rho_1(A)$. From the proof of Lemma 3.4 we know that $\rho_1$ and $\rho_2$ are equivalent. So

$$
\text{rank}(A - \text{tr}(A) I) = \text{rank}(\text{tr}(A) I - A^t) = \dim \text{Im}(\rho_2(A)) = \dim \text{Im}(\rho_1(A)) = \dim \text{Im}(\Phi_1) \leq 1.
$$

Hence there exist $u, v \in \mathbb{C}^3$ such that $A - \text{tr}(A) I = uv^t$, i.e., $A = uv^t - \langle u, v \rangle I/2 = \tau_{u,v}$. Now we have $B = A^t = \tau_{v,u}^t$, $C = -A = -\tau_{u,v}$, and $D = -B = -\tau_{v,u}$. By (3.11), we have $\Phi = \Psi_{\sigma_u \otimes \sigma_v}$. Therefore $\mathcal{F} = \mathcal{F}_{u,v}$. \hfill \Box

5. The main theorems

Using the results about admissible tuples obtained in the previous section, in this section we prove our main theorems (Theorems 5.2–5.5 below). We first prove a lemma, which is crucial for converting Eq. (1.4) to matrix equations.

**Lemma 5.1.** $\mathcal{F} = (f_i)_{i=1}^6$ is a solution of Eq. (1.4) on $G$ if and only if

$$
\text{tr}[(\hat{f}_1(\pi) X + X \hat{f}_2(\pi)) \pi(y) + (\hat{f}_3(\pi) X + X \hat{f}_4(\pi))^t \pi(y)] = \text{tr}(\hat{f}_5(\pi) X f_6(y)) \tag{5.1}
$$

for all $y \in G$, $[\pi] \in \hat{G}$, and $X \in \mathbb{M}(d_\pi, \mathbb{C})$.

**Proof.** Eq. (1.4) can be rewritten as

$$
R_y f_1 + L_{y^{-1}} f_2 + R_{y^{-1}} f_3 + L_y f_4 = f_6(y) f_5.
$$

Taking the Fourier transform, we see that this is equivalent to

$$
\pi(y) \hat{f}_1(\pi) + \hat{f}_2(\pi) \pi(y) + \pi(y)^{-1} \hat{f}_3(\pi) + \hat{f}_4(\pi) \pi(y)^{-1} = f_6(y) \hat{f}_5(\pi)
$$

where $\hat{f}_i = \mathcal{F}_{g, \pi} f_i(\pi)$.
for all $[\pi] \in \hat{G}$. Then the lemma follows from the fact that a matrix $A \in \mathcal{M}(d_\pi, \mathbb{C})$ is equal to 0 if and only if $\text{tr}(AX) = 0$ for all $X \in \mathcal{M}(d_\pi, \mathbb{C})$. □

In our first theorem we determine all pure normalized solutions of Eq. (1.4). We keep the notation from Examples 3.1 and 3.3.

**Theorem 5.2.** Let $[\pi] \in \hat{G}$, and let $\mathcal{F}$ be a nontrivial pure normalized solution of Eq. (1.4) on $G$ supported on $[[\pi]]$. Denote $K = U(d_\pi)$, $O(d_\pi)$ or $Sp(d_\pi)$ according to the type of $\pi$. Then $\mathcal{F} = \mathcal{F}^K \circ \pi$, where $\mathcal{F}^K$ is a solution of Eq. (1.4) on $K$, and the only possibilities of $K$ and $\mathcal{F}^K$ are as follows:

1. $K = U(1)$ and $\mathcal{F}^K = \mathcal{F}^U(1)_{\varepsilon_1, \delta_1, \varepsilon_2, \delta_2}$ for some $\varepsilon_1, \delta_1, \varepsilon_2, \delta_2 \in \mathbb{C}$;
2. $K = O(2), \mathcal{F}^K = \mathcal{F}^{O(2)}_{A,B}$ or $(\mathcal{F}^{O(2)}_{A,B})^\dagger$ for some $A, B \in \mathcal{M}(2, \mathbb{C})$ with $\text{tr}(A) = \text{tr}(B)$;
3. $K = SU(2), \mathcal{F}^K = \mathcal{F}^{SU(2)}_{A,B}$ or $(\mathcal{F}^{SU(2)}_{A,B})^\dagger$ for some $A, B \in \mathcal{M}(2, \mathbb{C})$ with $\text{tr}(A) = \text{tr}(B)$;
4. $K = O(3)$ and $\mathcal{F}^K = \mathcal{F}^{O(3)}_{u,v}$ for some $u, v \in \mathbb{C}^3$.

**Proof.** Since $\mathcal{F}$ is normalized, we have

$$\text{tr}(\hat{f}_1(\pi) - \hat{f}_2(\pi)) = \text{tr}(\hat{f}_3(\pi) - \hat{f}_4(\pi)) = 0.$$  

According to the types of $\pi$ (c.f. Theorem 2.1), there are three cases to consider.

**Case (a).** $\pi$ is of complex type, i.e., $[\pi] \neq [\bar{\pi}]$. Applying Lemma 5.1 to $\pi$ and $\bar{\pi}$, we have

$$\hat{f}_1(\pi)X + X\hat{f}_2(\pi) = \text{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi),$$

$$\left(\hat{f}_3(\pi)X + X\hat{f}_4(\pi)\right)^\dagger = \text{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi),$$

$$\hat{f}_1(\pi)X + X\hat{f}_2(\pi) = \text{tr}(\hat{f}_5(\bar{\pi})X)\hat{f}_6(\bar{\pi}),$$

$$\left(\hat{f}_3(\pi)X + X\hat{f}_4(\pi)\right)^\dagger = \text{tr}(\hat{f}_5(\bar{\pi})X)\hat{f}_6(\bar{\pi})$$

for all $X \in \mathcal{M}(d_\pi, \mathbb{C})$. So the 3-tuples

$$\left(\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_5(\pi) \odot \hat{f}_6(\pi)\right), \quad \left(\hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \odot \hat{f}_6(\pi)^\dagger\right),$$

$$\left(\hat{f}_1(\bar{\pi}), \hat{f}_2(\bar{\pi}), \hat{f}_5(\bar{\pi}) \odot \hat{f}_6(\bar{\pi})\right), \quad \left(\hat{f}_3(\bar{\pi}), \hat{f}_4(\bar{\pi}), \hat{f}_5(\bar{\pi}) \odot \hat{f}_6(\bar{\pi})^\dagger\right)$$

are $c$-admissible. Since $\mathcal{F}$ is nontrivial and supported on $[[\pi]]$, these tuples can not be all trivial. By Proposition 4.1, we have $d_\pi = 1$, i.e., $K = U(1)$. Let $\hat{f}_5(\pi) = \varepsilon_1, \hat{f}_5(\bar{\pi}) = \varepsilon_2, \hat{f}_6(\pi) = \delta_1, \hat{f}_6(\bar{\pi}) = \delta_2$. Then

$$\hat{f}_1(\pi) = \hat{f}_2(\pi) = \varepsilon_1\delta_1/2, \quad \hat{f}_1(\bar{\pi}) = \hat{f}_2(\bar{\pi}) = \varepsilon_2\delta_2/2,$$

$$\hat{f}_3(\pi) = \hat{f}_4(\pi) = \varepsilon_1\delta_2/2, \quad \hat{f}_3(\bar{\pi}) = \hat{f}_4(\bar{\pi}) = \varepsilon_2\delta_1/2.$$  

From Example 3.1 and the Fourier inversion formula, we see that $\mathcal{F} = \mathcal{F}^{U(1)}_{\varepsilon_1, \delta_1, \varepsilon_2, \delta_2} \circ \pi$. 

□
Case (b). \( \pi \) is of real type, i.e., \( \pi(G) \subseteq O(d_\pi) \). Then \( \hat{\pi}(y) = \pi(y) \) for all \( y \in G \). By Lemma 5.1, for all \( X \in \mathbb{M}(d_\pi, \mathbb{C}) \) we have
\[
\hat{f}_1(\pi)X + X\hat{f}_2(\pi) + (\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^t = \text{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi).
\]
So the 5-tuple
\[
\mathcal{T}_r = (\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\pi))
\]
is \( r \)-admissible. Since \( \mathcal{F} \) is nontrivial and supported on \([\pi]\), \( \mathcal{T}_r \) is nontrivial. By Proposition 4.2, we have \( K = O(1) \), \( O(2) \) or \( O(3) \), respectively.

If \( d_\pi = 1 \), then \( \mathcal{T}_r = \mathcal{T}_{a,b} \) and \( \mathcal{F} = \mathcal{F}_{a,b}^{O(1)} \circ \pi \) for some \( a, b \in \mathbb{C} \) (see Example 3.2). As mentioned in Example 3.2, this case can be absorbed into the case of \( K = U(1) \).

If \( d_\pi = 2 \), then \( \mathcal{T}_r = \mathcal{T}_{A,B}^r \) or \((\mathcal{T}_{A,B}^r)^t\), and \( \mathcal{F} = \mathcal{F}_{A,B}^{O(2)} \circ \pi \) or \((\mathcal{F}_{A,B}^{O(2)})^t \circ \pi \) for some \( A, B \in \mathbb{M}(2, \mathbb{C}) \) with \( \text{tr}(A) = \text{tr}(B) \) (see Example 3.3).

If \( d_\pi = 3 \), then \( \mathcal{T}_r = \mathcal{T}_{u,v} \) and \( \mathcal{F} = \mathcal{F}_{u,v}^{O(3)} \circ \pi \) for some \( u, v \in \mathbb{C}^3 \) (see Example 3.4).

Case (c). \( \pi \) is of quaternionic type, i.e., \( d_\pi \) is even and \( \pi(G) \subseteq Sp(d_\pi) \). Then \( \hat{\pi}(y) = J\pi(y)J^t \) for all \( y \in G \). By Lemma 5.1, for all \( X \in \mathbb{M}(d_\pi, \mathbb{C}) \) we have
\[
\hat{f}_1(\pi)X + X\hat{f}_2(\pi) + J(\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^tJ^t = \text{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi).
\]
So the 5-tuple
\[
\mathcal{T}_q = (\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\pi))
\]
is \( q \)-admissible. As before, \( \mathcal{T}_q \) is nontrivial. By Proposition 4.3, we have \( d_\pi = 2 \) and \( \mathcal{T}_q = \mathcal{T}_{A,B}^q \) or \((\mathcal{T}_{A,B}^q)^t\). Hence \( K = Sp(2) = SU(2) \), and \( \mathcal{F} = \mathcal{F}_{A,B}^{SU(2)} \circ \pi \) or \((\mathcal{F}_{A,B}^{SU(2)})^t \circ \pi \) for some \( A, B \in \mathbb{M}(2, \mathbb{C}) \) with \( \text{tr}(A) = \text{tr}(B) \) (see Example 3.4).

Our next theorem gives all pure normalized homogeneous solutions.

**Theorem 5.3.** Under the same conditions as in Theorem 5.2, if moreover \( \mathcal{F} \) is homogeneous, then the only possibilities of \( K \) and \( \mathcal{F}^K \) are as follows:

1. \( K = O(1) \) and \( \mathcal{F}^K = \mathcal{F}_a^{O(1)} \) for some \( a \in \mathbb{C} \);
2. \( K = O(2) \) and \( \mathcal{F}^K = \mathcal{F}_A^{O(2)} \) for some \( A \in \mathbb{M}(2, \mathbb{C}) \) with \( \text{tr}(A) = 0 \);
3. \( K = SU(2) \) and \( \mathcal{F}^K = \mathcal{F}_A^{SU(2)} \) for some \( A \in \mathbb{M}(2, \mathbb{C}) \) with \( \text{tr}(A) = 0 \).

**Proof.** This follows directly from the proof of Theorem 5.2 and the conditions for \( \mathcal{F}^K \) being homogeneous given in Examples 3.4 [3.5].

The next theorem characterizes the space of normalized homogeneous solutions.
Theorem 5.4. The Hilbert space of normalized homogeneous solutions of Eq. (1.4) is spanned by pure normalized homogeneous solutions.

Proof. Let $F = (f_i)_{i=1}^4$ be a normalized homogeneous solution of Eq. (1.4) on $G$. It suffices to prove that $F$ is the sum of some pure normalized homogeneous solutions. For $\varpi \in [\hat{G}]$, let $f_i^\varpi(x) = \sum_{[\pi] \in \varpi} \text{tr}(\hat{f}_i(\pi)x(\pi))$, $1 \leq i \leq 4$.

Then

$$(f_i^\varpi)^\gamma(\pi) = \begin{cases} \hat{f}_i(\pi), & [\pi] \in \varpi; \\ 0, & [\pi] \notin \varpi. \end{cases}$$

By Lemma 5.1, $F^\varpi = (f_i^\varpi)_{i=1}^4$ is a pure normalized homogeneous solution of Eq. (1.4) supported on $\varpi$, and we have $F = \sum_{\varpi \in [\hat{G}]} F^\varpi$. This proves the theorem. 

Finally we prove the theorem on the structure of the general solution of Eq. (1.4) on $G$.

Theorem 5.5. Any solution of Eq. (1.4) on $G$ is of the form

$$F_0 + F_h,$$

where $F_0$ is a pure normalized solution and $F_h$ is a homogeneous solution.

Proof. Let $F = (f_i)_{i=1}^6$ be a solution of Eq. (1.4) on $G$. Applying Lemma 5.1 to $F$ and taking the Fourier transform at the both sides of (5.1), we obtain that $\text{supp}((\text{tr}(f_5(\pi)x)f_6)^\gamma) \subseteq [[\pi]]$ for all $[\pi] \in \hat{G}$ and $X \in M(d_\pi, \mathbb{C})$. So if $[\pi] \in \text{supp}(\hat{f}_6)$, then $\text{supp}(\hat{f}_6) \subseteq [[\pi]]$. Hence there exists $\varpi_0 \in [\hat{G}]$ such that $\text{supp}(\hat{f}_5) \cup \text{supp}(\hat{f}_6) \subseteq \varpi_0$. Let $f_i^{\varpi_0}(x) = \sum_{[\pi] \in \varpi_0} \text{tr}(\hat{f}_i(\pi)x(\pi))$, $1 \leq i \leq 4$.

Then

$$(f_i^{\varpi_0})^\gamma(\pi) = \begin{cases} \hat{f}_i(\pi), & [\pi] \in \varpi_0; \\ 0, & [\pi] \notin \varpi_0. \end{cases}$$

By Lemma 5.1, $F_0 = (f_1^{\varpi_0}, f_2^{\varpi_0}, f_3^{\varpi_0}, f_4^{\varpi_0}, f_5 \otimes f_6)$ is a pure normalized solution supported on $\varpi_0$. So $F_h = (f_i - f_i^{\varpi_0})_{i=1}^4$ is a homogeneous solution of Eq. (1.4) on $G$, and we have $F = F_0 + F_h$. 

Theorems 5.2–5.5 provide a complete picture of the general solution of Eq. (1.4) on the compact group $G$. They also provide a method about how to construct all solutions. For a fixed $G$, we first find all irreducible representations of $G$ into $U(1)$, $O(2)$, $SU(2)$, and $O(3)$. Then using Theorem 5.2 we
find all pure normalized solutions. Theorem 5.3 gives all pure normalized homogeneous solutions. Here we should be careful that representations into $O(1)$ provide nontrivial homogeneous solutions. Theorem 5.4 tells us that pure normalized homogeneous solutions and solutions of the form $F_{c_1,c_2}$ span the space of homogeneous solutions. Thus we determine all homogeneous solutions. Finally by Theorem 5.5, we get the general solution by picking an arbitrary pure normalized solution and take its sum with an arbitrary homogeneous solution. We illustrate this by finding the general solution of Eq. (1.4) on $SU(2)$.

**Example 5.1 (General Solution on $SU(2)$).** It is well known that for each positive integer $d$ there exists exactly one $d$-dimensional irreducible representation of $SU(2)$ (see, e.g., [3]). The 1-dimensional one is the trivial representation. So it is a representation into $O(1)$. The 2-dimensional one is the identity representation. The 3-dimensional one is the adjoint representation $Ad$ in the Lie algebra $su(2)$ of $SU(2)$, which can be viewed as a representation into $O(3)$. As the 1-dimensional representation is into $O(1)$, when applying Theorem 5.2 (1), we can use Example 3.2. Indeed, as the 1-dimensional representation is trivial, the pure normalized solutions obtained from Theorem 5.2 (1) are constant solutions. They are of the form

$$f_1 \equiv f_2 \equiv a/2, \quad f_3 \equiv f_4 \equiv b/2, \quad f_5 \otimes f_6 \equiv a + b \quad (5.2)$$

for some $a, b \in \mathbb{C}$. The pure normalized solutions obtained by applying Theorem 5.2 (3)–(4) to the identity representation and the adjoint representation are $F_{A,B}^{SU(2)}$, $(F_{A,B}^{SU(2)})^\dagger$, and $F_{u,v}^{O(3)} \circ Ad$. Thus we get all pure normalized solutions of Eq. (1.4) on $SU(2)$. Now applying Theorem 5.3 we obtain all pure normalized homogeneous solutions. They are $f_1 \equiv f_2 \equiv -f_3 \equiv -f_4 \equiv \text{const}$ and $F_{A'}^{SU(2)}$. By Theorem 5.4, all homogeneous solutions of Eq. (1.4) on $SU(2)$ are of the form

$$f_1(x) = \text{tr}(A'x) + c_1(x) + \alpha,$$

$$f_2(x) = -\text{tr}(A'x) - c_1(x) + \alpha,$$

$$f_3(x) = \text{tr}(A'x) + c_2(x) - \alpha,$$

$$f_4(x) = -\text{tr}(A'x) - c_2(x) - \alpha,$$

where $A' \in M(2, \mathbb{C})$, $c_1, c_2 \in L^2_c(G)$, $\alpha \in \mathbb{C}$. Finally, by Theorem 5.5, the general solution of Eq. (1.4) on $SU(2)$ is given by $\mathcal{F}_0 + \mathcal{F}_h$, where $\mathcal{F}_0 \in \{5.2, F_{A,B}^{SU(2)}, (F_{A,B}^{SU(2)})^\dagger, F_{u,v}^{O(3)} \circ Ad\}$ and $\mathcal{F}_h$ is given by (5.3).

### 6. Applications

In this section, we consider some functional equations on compact groups which are special cases of Eq. (1.4). In particular, we solve the Wilson equation and the d’Alembert long equation on compact groups. We also recover the general solution of the d’Alembert equation that was obtained in [8, 22].
We first consider the equation
\[ f(xy) + g(xy^{-1}) = h(x)k(y), \] (6.1)
where \( f, g, h, k : G \to \mathbb{C} \) are the unknowns. It is clear that Eq. (6.1) corresponds to the special case of Eq. (1.4) where \( f_2 \equiv f_4 \equiv 0 \). We denote a solution of Eq. (6.1) by \( F = (f, g, h \otimes k) \), and say that it is homogeneous if \( h \otimes k \equiv 0 \).

If \( F = (f, g, h \otimes k) \) is a solution and \( F' = (f', g', 0) \) is a homogeneous solution of Eq. (6.1), then \( F + F' = (f + f', g + g', h \otimes k) \) is also a solution of Eq. (6.1).

We first construct some homogeneous solutions of Eq. (6.1).

**Example 6.1.** Let \( \pi : G \to O(1) \) be a homomorphism, and let \( a \in \mathbb{C} \). We view \( \pi \) as a function on \( G \). Then \( F_{\pi, a} = (a\pi, -a\pi, 0) \) is a homogeneous solution of Eq. (6.1) on \( G \). More generally, if \( \pi_j : G \to O(1) \) are distinct homomorphisms and \( a_j \in \mathbb{C} \) (\( j = 1, 2, \ldots \)), then
\[
\sum_{j \geq 1} F_{\pi_j, a_j} = \left( \sum_{j \geq 1} a_j\pi_j, -\sum_{j \geq 1} a_j\pi_j, 0 \right)
\]
is a homogeneous solution, provided that \( \sum_{j \geq 1} |a_j|^2 < \infty \).

Now we construct some solutions of Eq. (6.1) on \( U(1), O(2), \) and \( SU(2) \).

**Example 6.2.** Let \( G = U(1) \). For \( \varepsilon_1, \delta_1, \varepsilon_2, \delta_2 \in \mathbb{C} \), define
\[
\begin{aligned}
f(x) &= \varepsilon_1\delta_1 x + \varepsilon_2\delta_2 \bar{x}, \\
g(x) &= \varepsilon_1\delta_2 x + \varepsilon_2\delta_1 \bar{x}, \\
h \otimes k(x, y) &= (\varepsilon_1 x + \varepsilon_2 \bar{x})(\delta_1 y + \delta_2 \bar{y}),
\end{aligned}
\]
x, y \in U(1).

It is easy to check that \( (f, g, h \otimes k) \) is a solution of Eq. (6.1) on \( U(1) \).

**Example 6.3.** Let \( G = O(2) \). For \( P \in M(2, \mathbb{C}) \), define
\[
\begin{aligned}
f(x) &= -g(x) = \text{tr}(Px), \\
h \otimes k(x, y) &= -\text{tr}(JPx)\text{tr}(Jy),
\end{aligned}
\]
x, y \in O(2).

Then \( (f, g, h \otimes k) \) is a solution of Eq. (6.1) on \( O(2) \).

**Example 6.4.** Let \( G = SU(2) \). For \( P \in M(2, \mathbb{C}) \), define
\[
\begin{aligned}
f(x) &= g(x) = \text{tr}(Px), \\
h \otimes k(x, y) &= \text{tr}(Px)\text{tr}(y),
\end{aligned}
\]
x, y \in SU(2).

Then \( (f, g, h \otimes k) \) is a solution of Eq. (6.1) on \( SU(2) \).

We leave the verification of the above examples to the reader. The following result claims that the above examples are the building blocks of the general solution of Eq. (6.1) on \( G \).
Theorem 6.1. Any solution of Eq. (5.1) on $G$ is of the form

$$\mathcal{F} \circ \pi + \sum_{j \geq 1} \mathcal{F}_{\pi_j, a_j},$$

where $\pi : G \to K$ is an irreducible representation with $K = U(1), O(2)$ or $SU(2)$, $\mathcal{F}$ is a solution of Eq. (6.1) on $K$ as in Examples 6.2, 6.4 and $\sum_{j \geq 1} \mathcal{F}_{\pi_j, a_j}$ as in Example 6.7.

Proof. Let $(f, g, h \otimes k)$ be a solution of Eq. (6.1). Then $(f, 0, g, 0, h \otimes k)$ is a solution of Eq. (1.4). By Theorems 5.2–5.5, there exist $c_1, c_2 \in L^2(G)$ and irreducible representations $\pi_j : G \to K_j$ ($j \geq 0$) with $[[\pi_j]]$’s distinct, such that

$$(f, 0, g, 0, h \otimes k) = \mathcal{F}_{c_1, c_2} + \sum_{j \geq 0} \mathcal{F}^{K_j} \circ \pi_j,$$  

where $\mathcal{F}^{K_0} = (f_1^{K_0}, f_2^{K_0}, f_3^{K_0}, f_4^{K_0}, f_5^{K_0} \otimes f_1^{K_0})$ is a solution of Eq. (1.4) on $K_0$, $\mathcal{F}^{K_j} = (f_1^{K_j}, f_2^{K_j}, f_3^{K_j}, f_4^{K_j})$ ($j \geq 1$) is a homogeneous solution of Eq. (1.4) on $K_j$, and the only possibilities of $K_j, \pi_j$, and $\mathcal{F}^{K_j}$ are given in Theorems 5.2 and 5.3. Note that this implies

$$c_1 = \sum_{j \geq 0} f_2^{K_j} \circ \pi_j, \quad c_2 = \sum_{j \geq 0} f_4^{K_j} \circ \pi_j$$

and

$$f = \sum_{j \geq 0} (f_1^{K_j} + f_2^{K_j}) \circ \pi_j, \quad g = \sum_{j \geq 0} (f_3^{K_j} + f_4^{K_j}) \circ \pi_j.$$  

Without loss of generality, we may assume that each $\mathcal{F}^{K_j}$ is a nontrivial solution.

We first prove that $K_0 \neq O(3)$. Suppose $K_0 = O(3)$. Then $\mathcal{F}^{K_0} = \mathcal{F}_{u, v}^{O(3)}$ for some $u, v \in \mathbb{C}^3$. Since $\mathcal{F}^{K_j} \circ \pi_j$ is a pure solution of Eq. (1.4) on $G$ supported on $[[\pi_j]]$ for any $j \geq 0$ and $[[\pi_j]]$ are distinct, we have $(f_2^{K_j} \circ \pi_j)^{\dagger}(\pi_0) = 0$ if $j \geq 1$. Hence

$$\hat{c}_1(\pi_0) = \sum_{j \geq 0} (f_2^{K_j} \circ \pi_j)^{\dagger}(\pi_0) = (f_2^{K_0} \circ \pi_0)^{\dagger}(\pi_0) = \tau_{v, u},$$

where $\tau_{v, u}$ is as in Lemma 5.4. Since $c_1$ is a central function, $vu = \hat{c}_1(\pi_0) + \langle u, v \rangle I_3/2$ is a scalar matrix. This implies that $vu = 0$, i.e., $u = 0$ or $v = 0$. Hence $\mathcal{F}^{K_0}$ is the trivial solution, a contradiction.

Now we prove that if $K_j = O(2)$, then $j = 0$ and $\mathcal{F}^{K_0} = \mathcal{F}_{A, B}^{O(2)}$ for some $A \in \mathbb{M}(2, \mathbb{C})$. We know that if $K_j = O(2)$, then $\mathcal{F}^{K_j} = \mathcal{F}_{A, B}^{O(2)}$ or $(\mathcal{F}_{A, B}^{O(2)})^{\dagger}$ for some $A, B \in \mathbb{M}(2, \mathbb{C})$ with $\text{tr}(A) = \text{tr}(B)$, and $B = A^{\dagger}$ with $\text{tr}(A) = 0$ if $j \geq 1$. If $\mathcal{F}^{K_j} = \mathcal{F}_{A, B}^{O(2)}$, similar to the above proof, we obtain that $B = \hat{c}_1(\pi_j)$ is a scalar matrix. So $B = \text{tr}(A)/2$. If $j \geq 1$, then $A = B = 0$, conflicting with the assumption that $\mathcal{F}^{K_j}$ is nontrivial. Hence $j = 0$. If $\mathcal{F}^{K_j} = (\mathcal{F}_{A, B}^{O(2)})^{\dagger}$, then
similarly \( B = \hat{c}_1(\pi_j) \) and \(-A^t = \hat{c}_2(\pi_j)\) are scalar matrices. So \( A = B = \lambda I \) for some \( \lambda \in \mathbb{C} \). By Remark 3.12 this case can be absorbed into the former case. Note that if we set \( P = A + \text{tr}(A)I/2 \), then we have

\[
\begin{aligned}
&f_1^K(x) + f_2^K(x) = -(f_3^K(x) + f_4^K(x)) = \text{tr}(Px), \\
&f_5^K \otimes f_6^K(x, y) = -\text{tr}(JPx)\text{tr}(Jy),
\end{aligned}
\]

A similar argument shows that if \( K_j = SU(2) \), then \( j = 0 \) and \( \mathcal{F}^{K_0} = \mathcal{F}_{A, B}^{SU(2)} \) for some \( A \in \mathbb{M}(2, \mathbb{C}) \). In this case if we set \( P = A + \text{tr}(A)I/2 \), then we have

\[
\begin{aligned}
&f_1^K(x) + f_2^K (x) = f_3^K(x) + f_4^K(x) = \text{tr}(P x), \\
&f_5^K \otimes f_6^K(x, y) = \text{tr}(Px)\text{tr}(y),
\end{aligned}
\]

The above proofs also imply that if \( j \geq 1 \), then \( K_j = O(1) \) and \( \mathcal{F}^{K_j} = \mathcal{F}_{aj}^{O(1)} \) for some \( a_j \in \mathbb{C} \). In this case we have

\[
\begin{aligned}
f_1^K (x) + f_2^K (x) = -(f_3^K(x) + f_4^K(x)) = \lambda x, \quad x \in O(1).
\end{aligned}
\]

Now we know that there are three possibilities for \( K_0 \), i.e., \( K_0 = U(1), O(2), \) or \( SU(2) \). In each case, it is easy to see from (6.3) - (6.6) that

\[
(f, g, h \otimes k) = \mathcal{F} \circ \pi_0 + \sum_j \mathcal{F}_{\pi_j, a_j},
\]

where \( \mathcal{F} \) is a solution of Eq. (6.1) on \( K_0 \) as in Examples 6.2 - 6.4. The proof of the theorem is completed by setting \( K = K_0 \) and \( \pi = \pi_0 \).

Now we consider the special case of Eq. (6.1) where \( f \equiv g \).

**Theorem 6.2.** The general solution of the equation

\[
f(xy) + f(xy^{-1}) = h(x)k(y)
\]

is

\[
\begin{aligned}
f(x) &= \text{tr}(P \pi(x)), \\
h \otimes k(x, y) &= \text{tr}(P \pi(x)\text{tr}(\pi(y))),
\end{aligned}
\]

where \( \pi : G \rightarrow SU(2) \) is a homomorphism and \( P \in \mathbb{M}(2, \mathbb{C}) \).

**Proof.** Clearly, the general solution of Eq. (6.7) corresponds to the solutions of Eq. (6.1) for which \( f \equiv g \). By Theorem 6.1 the functions \( f \) and \( g \) in a solution of Eq. (6.1) has the form

\[
f = f^K \circ \pi + \sum_{j \geq 1} a_j \pi_j, \quad g = g^K \circ \pi - \sum_{j \geq 1} a_j \pi_j,
\]

where \( K = U(1), O(2) \) or \( SU(2) \), \( \pi : G \rightarrow K \) and \( \pi_j : G \rightarrow O(1) \) are distinct irreducible representations, \( f^K \) and \( g^K \) are functions on \( K \) as in Examples 6.2 - 6.4. Applying the Fourier transform, it is easy to see that \( f \equiv g \) if and only
if \( f^K \equiv g^K \) and \( a_j = 0 \). Restricting our attention to nontrivial solutions, we can see that either \( K = U(1) \) and \( \delta_1 = \delta_2 \) (in the notation of Example 6.2), or \( K = SU(2) \). If \( K = SU(2) \) we have reached the conclusion of the theorem. If \( K = U(1) \) and \( \delta_1 = \delta_2 =: \delta \), then the homomorphism \( x \mapsto \text{diag}(\pi(x), \bar{\pi}(x)) \in SU(2) \) and \( P = \text{diag}(\varepsilon_1 \delta, \varepsilon_2 \delta) \) satisfy our requirements. \( \square \)

The following corollaries are straightforward from Theorem 6.2.

**Corollary 6.3.** Any nontrivial solution of the Wilson equation (1.2) is of the form

\[
\begin{align*}
f(x) &= \text{tr}(P\pi(x)), \\
g(x) &= \frac{1}{2}\text{tr}(\pi(x)),
\end{align*}
\]

where \( \pi : G \to SU(2) \) is a homomorphism and \( P \in \mathbb{M}(2, \mathbb{C}) \).

**Corollary 6.4.** Any nontrivial solution of the equation

\[
f(xy) + f(xy^{-1}) = 2g(x)f(y)
\]

is of the form

\[
\begin{align*}
f(x) &= \text{atr}(\pi(x)), \\
g(x) &= \frac{1}{2}\text{tr}(\pi(x)),
\end{align*}
\]

where \( \pi : G \to SU(2) \) is a homomorphism and \( a \in \mathbb{C} \).

**Corollary 6.5.** Any nontrivial solution of the d’Alembert equation (1.1) is of the form

\[
f(x) = \frac{1}{2}\text{tr}(\pi(x)),
\]

where \( \pi : G \to SU(2) \) is a homomorphism.

Indeed, to prove Corollaries 6.3–6.5 it suffices to examine the solutions of Eq. (6.7) satisfying \( h \equiv 2f, k \equiv 2f, \) and \( h \equiv 2k \equiv 2f \), respectively.

Now we apply the results in the previous section to another type of equations.

**Theorem 6.6.** Let \((f, h \otimes k)\) be a solution of the equation

\[
f(xy) + f(xy^{-1}) + f(yx) + f(y^{-1}x) = h(x)k(y).
\]

(6.9)

Then either there exist an irreducible representation \( \pi : G \to O(2) \) and \( a \in \mathbb{C} \) such that

\[
\begin{align*}
f(x) &= \text{atr}(J\pi(x)), \\
h \otimes k(x, y) &= 2\text{atr}(J\pi(x))\text{tr}(\pi(y)),
\end{align*}
\]

or there exist a representation \( \pi : G \to SU(2) \) and \( A \in \mathbb{M}(2, \mathbb{C}) \) such that

\[
\begin{align*}
f(x) &= \text{tr}(A\pi(x)), \\
h \otimes k(x, y) &= 2\text{tr}(A\pi(x))\text{tr}(\pi(y)).
\end{align*}
\]
Proof. It suffices to consider the solutions of Eq. (1.4) satisfying $f_1 \equiv f_2 \equiv f_3 \equiv f_4$. We write the general solution of Eq. (1.4) as the right hand side of (6.2). In particular, we have

$$f_1 = c_1 + \sum_{j \geq 0} f_{1j} \circ \pi_j, \quad f_2 = -c_1 + \sum_{j \geq 0} f_{2j} \circ \pi_j.$$ 

So $f_1 \equiv f_2$ implies that

$$2c_1 + \sum_{j \geq 0} (f_{1j} - f_{2j}) \circ \pi_j = 0.$$ 

But the two summands above belong to $L^2_c(G)$ and $L^2_c(G) \perp$, respectively. So we have $c_1 \equiv 0$. Similarly, we have $c_2 \equiv 0$. By considering the Fourier transform, it is easy to see that $f_{1j} \equiv f_{2j} \equiv f_{3j} \equiv f_{4j}$ for any $j \geq 0$. Now one can verify that $F^K_j$ is trivial if $j \geq 1$, and $K_0$, $\pi_0$, and $F^K_0$ take one of the following forms:

1. $K_0 = U(1)$ and $F^K_0 = F^{U(1)}_{\varepsilon_0,\varepsilon_0,\varepsilon_0,\varepsilon_0}$ for some $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}$;
2. $K_0 = O(2)$ and $F^K_0 = (F_{a,\varepsilon,\varepsilon,\varepsilon})^\dagger$ for some $a \in \mathbb{C}$;
3. $K_0 = SU(2)$ and $F^K_0 = F^{SU(2)}_{A,A}$ for some $A \in \mathbb{M}(2,\mathbb{C})$.

The last two cases obviously satisfy the conclusion of the theorem. For the first case, it suffices to set $\pi(x) = \text{diag}(\pi_0(x), \bar{\pi}_0(x)) \in SU(2)$ and $A = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. □

Similar to Corollaries 6.3–6.5, we have the following corollaries.

Corollary 6.7. Let $(f, g)$ be a nontrivial solution of the equation

$$f(xy) + f(xy^{-1}) + f(yx) + f(y^{-1}x) = 4f(x)g(y).$$  \hspace{1cm} (6.10)

Then either there exist an irreducible representation $\pi : G \to O(2)$ and $a \in \mathbb{C}$ such that

$$\begin{cases} f(x) = \text{atr}(J\pi(x)), \\ g(x) = \frac{1}{2}\text{tr}(\pi(x)), \end{cases}$$

or there exist a representation $\pi : G \to SU(2)$ and $A \in \mathbb{M}(2, \mathbb{C})$ such that

$$\begin{cases} f(x) = \text{tr}(A\pi(x)), \\ g(x) = \frac{1}{2}\text{tr}(\pi(x)). \end{cases}$$

Corollary 6.8. Any nontrivial solution of the equation

$$f(xy) + f(xy^{-1}) + f(yx) + f(y^{-1}x) = 4g(x)f(y)$$ \hspace{1cm} (6.11)

is of the form

$$\begin{cases} f(x) = \text{atr}(\pi(x)), \\ g(x) = \frac{1}{2}\text{tr}(\pi(x)), \end{cases}$$

where $\pi : G \to SU(2)$ is a homomorphism and $a \in \mathbb{C}$. 

Corollary 6.9. Any nontrivial solution of the d’Alembert long equation (1.3) is of the form

\[ f(x) = \frac{1}{2} \text{tr}(\pi(x)), \]

where \( \pi : G \to SU(2) \) is a homomorphism.

From Corollary 6.9 we see that the solutions of the d’Alembert long equation (1.3) and the d’Alembert equation (1.1) are the same. The similar result for step 2 nilpotent groups was proved in [16].

The factorization property of the d’Alembert equation on compact groups was studied in [8, 9, 22, 23]. To conclude this section, we summarize the same property of the above equations as follows.

Corollary 6.10. The following factorization properties hold.

1. All nontrivial solutions of Eqs. (6.7) and (6.11) on a compact group factor through \( SU(2) \).
2. All nontrivial solutions of Eq. (6.9) on a compact group factor through \( O(2) \) or \( SU(2) \).

As a simple consequence, all nontrivial solutions of every special case of Eqs. (6.7) and (6.11), in particular, the Wilson equation and the d’Alembert long equation, factor through \( SU(2) \).

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