THE MAGNETIC LIOUVILLE EQUATION AS A SEMI-CLASSICAL LIMIT

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Abstract

The Liouville equation with non-constant magnetic field is obtained as a limit in the Planck constant $\hbar$ of the von Neumann equation with the same magnetic field. The convergence is with respect to an appropriate semi-classical pseudo distance, and consequently with respect to the Monge-Kantorovich distance. Uniform estimates both in $\epsilon$ and $\hbar$ are proved for the specific 2D case of a magnetic vector potential of the form $\frac{1}{\epsilon} x \perp$. As an application, an observation inequality for the von Neumann equation with a magnetic vector potential is obtained. These results are a magnetic variant of the works [F. Golse and T. Paul. Arch. Ration. Mech. Anal., 223 (1): 57–94 (2017)] and [F. Golse and T. Paul. Math. Models Methods Appl. Sci., World Scientific Publishing, In press, 32 (5) (2022)] respectively.

1 Introduction

We derive rigorously the Liouville equation with non-constant magnetic field from the von Neumann equation with the same magnetic field. The Liouville equation with magnetic vector potential $A(x)$, electric potential $V$ and initial data $f^\text{in}$ reads

$$
\begin{align*}
\partial_t f + \left\{ \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x), f \right\} &= 0, \\
|f|_{t=0} &= f^\text{in},
\end{align*}
$$

(1)

where the unknown $f := f(t, x, \xi)$ is a time dependent ($t \in [0, \tau] \subset [0, \infty)$) probability density on $\mathbb{R}^d \times \mathbb{R}^d$, and $\{\cdot, \cdot\}$ are the Poisson brackets defined by

$$
\{f, g\} = \nabla_\xi f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g.
$$

The quadratic term $\frac{1}{2} |x|^2$ can be interpreted as an confinement potential and its inclusion is a technicality, in the sense that its utility is not apparent at the level of the formal calculations but at the level of the spectral theory required in order to put the calculations on rigorous grounds. The von Neumann equation with magnetic vector potential $A(x)$, electric potential $V$ and initial data $R^\text{in}_\hbar$ reads

$$
\begin{align*}
\partial_t R_\hbar + \frac{i}{\hbar} \left[ \frac{1}{2} |\hbar \nabla_y + A(y)|^2 + \frac{1}{2} |y|^2 + V(y), R_\hbar \right] &= 0, \\
R|_{t=0} &= R^\text{in}_\hbar.
\end{align*}
$$

(2)

Here $\hbar > 0$ is the Planck constant, the unknown $R_\hbar$ is a time dependent density operator on $\mathcal{H} := L^2(\mathbb{R}^d)$ and $[T, S]$ designates the commutator of operators $T$ and $S$ defined by

$$
[T, S] = TS - ST.
$$

By a density operator on a Hilbert space $\mathcal{H}$ we mean a trace class operator $R$ on $\mathcal{H}$ such that

$$
R = R^* \geq 0, \text{trace}_{\mathcal{H}}(R) = 1,
$$

and the set of all such operators is denoted $\mathcal{D}(\mathcal{H})$. 

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The derivation of the Liouville equation from the von Neumann equation in the absence of a magnetic field was investigated in [14]—in particular see Theorem 2.5. In fact, the authors consider there the slightly more involved case of the Vlasov and Hartree equations, which are nonlinear versions of equations \(^{(1)}\) and \(^{(2)}\) respectively. The authors in [14] successfully derive Vlasov from Hartree (or Liouville from von Neumann for our purposes) with respect to the Monge-Kantorovich distance of exponent 2 in the limit as \(\hbar \to 0\). When no magnetic field is included and the electric potential \(V\) is sufficiently well behaved in terms of regularity and decay, the existing results on quantum, semi-classical and classical mean field limits are fairly satisfactory. The case of singular potentials (such as the Coulomb potential) is more challenging: in [23] the Vlasov equation is obtained as a semi-classical limit of the Hartree equation, with Coulomb potential. However the method introduced in [23] does not yield quantitative estimates and is obtained along a sub-sequence of \(\hbar\). In [2] it is shown that the Wigner function of the solution of Hartree’s equation is \(L^{2}\)-close to its weak limit, that is, the solution of Vlasov, but this convergence, which is with respect to a semi-classical analogue of the Monge-Kantorovich distance, is obtained provided the electric potential is sufficiently regular and decaying and the Sobolev norm of the initial datas satisfies appropriate growth rates with respect to \(\hbar\). More recently, the problem of establishing a quantitative rate of convergence for the semi-classical limit with respect to in the presence of singular interactions (Coulomb singularity included) has been dealt in the work [8]– with respect to the Monge-Kantorovich distance and in [10]– with respect to the Schatten norms. Interestingly, the convergence obtained in [19] is even global in time. Other related developments include [8] and [9] which consider a semi-classical mean field limit for Boson systems, and [5], [11], [20] and [26], which consider a semi-classical mean field limit for Fermionic systems. Less is known about how the inclusion of a magnetic field influences the semi-classical asymptotic (even when the electric potential is sufficiently nice). The work [1] proves weak convergence of the Wigner transform associated to the (magnetic) von Neumann equation to the solution of Liouville equation \(^{(1)}\), whereas in the present work the convergence is with respect to the Monge-Kantorovich distance of exponent 2 and is moreover quantitative. In the context of semi-classical mean-field limits, we mention the work [24] which derives the Hartree equation as a mean-field limit of the \(N\)-body Schrödinger equation with Coulomb potential and a magnetic field by employing methods of second quantization. However, this convergence is not uniform in the Planck constant—uniform convergence with respect to \(\hbar\) remains open for electric potentials with Coulomb singularity, both in the presence and absence of a magnetic field. Another work which is closely related to combined semi-classical mean field limit of the quantum many body problem with magnetic field is [25].

This paper can be viewed as an additional step towards a deeper understanding of magnetic semi-classical limits. The regularity assumptions that we impose on the electric potential are identical to the conditions in [14]. Extending the result to electric potentials with Coulomb singularity is left for future investigation. We adapt the methods introduced in [14] in order to obtain Monge-Kantorovich convergence when including a Lipschitz continuous magnetic vector potential \(A(x)\) with Lipschitz gradient. In Section \(^{(1)}\) the same problem will be investigated, but with a 2D magnetic vector potential which carries the specific form \(A(x) = \frac{1}{\epsilon} x^\perp\) (here \(x^\perp := (−x_2, x_1)\), and in this case the limit will be shown to be uniform both in \(\epsilon\) and \(\hbar\) which are both viewed as small parameters. In Section \(^{(2)}\) we will apply these results in order to obtain a quantitative observation type inequality (necessary background to be reviewed in the sequel) for the von Neumann equation. Both results are of semi-classical type since the distance considered compares a quantum object (a solution of the Hartree or von Neumann equation) with a classical object (solution of the Vlasov or Liouville equation).

Let us now give a rough outline of the idea of the proof in [14], which in turn will clarify the contribution of this work in comparison to the existing literature. For simplicity, let us for the moment omit the quadratic term \(\frac{1}{2} |y|^{2}\). Our starting point is the definition of a semi-classical pseudo-distance as described above. To properly motivate this definition, we recall the definition of the Monge-Kantorovich distance at the classical level. Given \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d)\), a coupling of \(\mu\) and \(\nu\) is a probability measure \(\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\) such that

\[
\pi_1 = \mu, \quad \pi_2 = \nu,
\]

where \(\pi_1, \pi_2\) are the first and second marginals of \(\pi\) respectively. The set of all couplings of \(\mu\) and \(\nu\) is denoted by \(\Pi(\mu, \nu)\). We denote by \(\mathcal{P}_p(\mathbb{R}^d)\) the set of Borel probability measures with finite moments of order \(p\) (\(1 \leq p < \infty\)), i.e.

\[
\mathcal{P}_p(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|^p \, d\mu(x) < \infty \right\}.
\]
For each $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ the Monge-Kantorovich distance of exponent $p$ of $\mu$ and $\nu$ is defined by

$$\text{dist}_{MK,p}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx dy) \right)^{\frac{1}{p}}.$$

We wish now to modify the above definition in a way that would allow to compare probability densities (which in practice will be solution to the Vlasov or Liouville equations) and density operators (which in practice will be solution to the Hartree or von Neumann equations). Thus, a definition of a coupling between these two objects is sought after. As a general rule, when moving from classical to quantum, integrals of functions are replaced by traces of the corresponding operator, and therefore we are lead to

**Definition 1.1.** (Definition 2.1 in [14]) Let $f(x, \xi)$ be a a probability density on $\mathbb{R}^d \times \mathbb{R}^d$. Let $R$ be a density operator on $\mathfrak{H}$. A coupling of $f$ and $R$ is a measurable function $Q : \mathbb{R}^d \times \mathbb{R}^d \to \mathfrak{L}(\mathfrak{H})$ such that

$$Q(x, \xi) = Q(x, \xi)^* \geq 0 \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$$

and

(i). $\text{trace}(Q(x, \xi)) = f(x, \xi)$ for a.e. $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$

and

(ii). $\int_{\mathbb{R}^d \times \mathbb{R}^d} Q(x, \xi) dxd\xi = R$.

The set of all couplings of $f$ and $R$ is denoted by $\mathcal{C}(f, R)$, and is nonempty as witnessed by the operator valued function $(x, \xi) \mapsto f(x, \xi)R$. Note that the integrand on the left hand side of (ii) is an element of the separable Banach space $L^1(\mathfrak{H})$ (because of (i)), and in accordance the integral should be interpreted as a Bochner integral. We also denote by $\mathcal{D}^2(\mathfrak{H})$ the space of density operators with finite second quantum moments, i.e. all $R \in \mathcal{D}(\mathfrak{H})$ such that

$$\text{trace} \left( \sqrt{R} (-\hbar^2 \Delta + |y|^2 \sqrt{R}) \right) < \infty.$$

Equipped with the right notion of a coupling enables to mimic the definition of the Monge-Kantorovich distance as follows.

**Definition 1.2.** For each probability density $f = f(x, \xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ and each $R \in \mathcal{D}^2(\mathfrak{H})$ set

$$E_h(f, R) := \inf_{Q \in \mathcal{C}(f, R) \cap \mathcal{D}^2(\mathfrak{H})} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \sqrt{Q(x, \xi)} c_h(x, \xi) \sqrt{Q(x, \xi)} \right) dxd\xi \right)^{\frac{1}{2}},$$

where $c_h$ is a function of $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ with values in the set of unbounded operators on $\mathfrak{H}$, called the cost function, and defined by

$$c_h(x, \xi) := \frac{1}{2} |x - y|^2 + \frac{1}{2} |\xi + i\hbar \nabla_y|^2 = \frac{1}{2} \sum_{k=1}^{d} (x_k - y_k)^2 + \frac{1}{2} \sum_{k=1}^{d} (\xi_k + i\hbar \partial_y \xi_k)^2.$$

The differential operator $c_h(x, \xi)$ is a semi-classical version of the cost function from optimal transport, and is viewed as an operator indexed by $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ acting on the space $L^2(\mathbb{R}^d)$. The infimum is restricted to couplings in $\mathcal{D}^2(\mathfrak{H})$ in order to avoid ambiguity in the definition of the trace. Consider the time dependent quantity

$$\mathcal{E}_h(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \sqrt{Q_h(t, x, \xi)} c_h(x, \xi) \sqrt{Q_h(t, x, \xi)} \right) dxd\xi$$

where $Q_h(t, x, \xi)$ is the solution to the Cauchy problem

$$\begin{cases}
\partial_t Q_h + \left\{ \frac{1}{2} |\xi + A(x)|^2 + V(x), Q_h \right\} + \frac{i}{\hbar} \left\{ \frac{1}{2} |\xi + A(y)|^2 + V(y), Q_h \right\} = 0, \\
Q_h(0, x, \xi) = Q_h^{(n)},
\end{cases}$$

$$Q_h(0, x, \xi) = Q_h^{(n)},$$
for $Q_h^{\text{in}} \in C(f^{\text{in}}, R^{\text{in}})$. When no magnetic field is included, i.e. $A \equiv 0$, the argument in [14] rests upon establishing a Gronwall estimate for $\mathcal{E}_h$. We can formally differentiate $\mathcal{E}_h$ in time and use the cyclic property of the trace (and specifically the identity $\text{trace}(T[S,R]) = -\text{trace}(R[T,S])$) to get

$$
\mathcal{E}_h(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( Q_h(t,x,\xi) \left\{ \frac{1}{2} |\xi|^2 + V(x), c_h(x,\xi) \right\} \right) dxd\xi
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( Q_h(t,x,\xi) \left[ -\frac{\hbar^2}{2} \Delta_y + V(y), c_h(x,\xi) \right] \right) dxd\xi.
$$

The core of the proof in [14] leading to the desired Gronwall estimate for $\mathcal{E}_h(t)$ is in obtaining the following operator inequality

$$
\left\{ \frac{1}{2} |\xi|^2 + V(x), c_h(x,\xi) \right\} + \frac{i}{\hbar} \left[ \frac{\hbar^2}{2} \Delta y + V(y), c_h(x,\xi) \right] \leq \Lambda c_h(x,\xi),
$$

where $\Lambda$ is some constant which can be chosen to be independent of $\hbar$. In general, the Poisson brackets of a polynomial (in the variables $(x,\xi)$) with a second order differential operator is a second order differential operator and the commutator of second order differential operators is a third order differential operator. Since a third order differential operator cannot be controlled by a second order differential operator, this indicates that an “abstract nonsense” argument cannot lead to the inequality (3). Indeed, the key observation in [14] is that for the special operators of interest there is a “cancellation phenomena” leading to the inequality (3). However, when a non-constant magnetic field is included this cancellation fails, and there is no reason to expect that the operator

$$
\left\{ \frac{1}{2} |\xi + A(x)|^2 + V(x), c_h(x,\xi) \right\} + \frac{i}{\hbar} \left[ \frac{1}{2} |i\hbar \nabla_y + A(y)|^2 + V(y), c_h(x,\xi) \right]
$$

can be controlled by the cost function $c_h(x,\xi)$. The idea that we wish to convey here is that considering a magnetic cost function defined by

$$
\bar{c}_h(x,\xi) := \frac{1}{2} |x - y|^2 + \frac{1}{2} |\xi + A(x) - A(y) + i\hbar \nabla y|^2
$$

enables us to overcome this obstacle, since the functional associated to it

$$
\bar{\mathcal{E}}_h(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_S \left( \sqrt{Q_h(t,x,\xi)\bar{c}_h(x,\xi)} \sqrt{Q_h(t,x,\xi)} \right) dxd\xi
$$

can be easily shown to be dominated by $\mathcal{E}_h$. We should remark though that this equivalence between $\mathcal{E}_h$ and $\bar{\mathcal{E}}_h$ is not uniform with respect to the Lipschitz constant of $A$, and is therefore not suitable for the case where $A(x) = \frac{1}{\epsilon} e^{-\frac{x^2}{2}}$ and $\epsilon$ is taken to 0. This regime is addressed in section [4].

The next section is aimed at fixing the notations, recalling necessary background and stating the main results. The main results are then proved in Section 3 and Section 4. How to deduce Monge-Kantorovich convergence from the evolution estimates of the latter sections is explained in Section 5. Section 6 is an application of the main results to observation inequalities. Finally, Section 7 further elaborates on some spectral theory subtleties which are created due to the magnetic field, thereby putting the conceptual argument on rigorous grounds.

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The magnetic vector potential $A \in C^\infty(\mathbb{R}^d)$ will always be taken to be:

(A). If $d = 2$: $A(x) = \frac{1}{2}x^\perp$.

(A'). If $d \geq 3$: Lipschitz with Lipschitz gradient, i.e. $|A(x) - A(y)| \leq K |x - y|$ and $|\nabla A(x) - \nabla A(y)| \leq K' |x - y|$ for some constant $K, K' > 0$. In addition assume without loss of generality $A(0) = 0$.

As we will see in Section (7), $D^2_A(\mathfrak{H}) = D^2(\mathfrak{H})$ whenever $A$ is sublinear (for $d \geq 3$) or $A(x) = \frac{1}{2}x^\perp$ ($d = 2$). The distinction between 2D and arbitrary dimension $d$ should not be regarded as an essential point because according to Remark 2.11 in [17], it is possible to formulate a general statement unifying all dimensions $d \geq 2$, but we were unable to locate a reference with a full proof of this. In the sequel, the electric potential $V$ is assumed to be real-valued function verifying the following condition

$$V(x) = V(-x), \quad V \in C^{1,1}(\mathbb{R}^d).$$

The Poisson brackets of functions $f(x, \xi), g(x, \xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ (which may be operator valued) are defined by

$$\{f, g\} := \nabla_x f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g.$$ 

As customary, $[\cdot, \cdot]$ is the commutator defined by

$$[T, S] := TS - ST,$$

and $\vee$ is the anti-commutator defined by

$$T \vee S := TS + ST.$$

We will see that $E_h(f, R)$ (see Definition 1.2) can be bounded from below (and above in case $R$ carries the form of a special operator called Toeplitz operator) in terms of the Monge-Kantorovich distance. In Section 5 we will explain how $E_h$ is related to the Husimi transform and the Monge-Kantorovich distance $\text{dist}_{MK,2}$, and therefore we now turn our attention to briefly review some elementary definitions and facts from the theory of Husimi transforms and their relatives. Let $z = (p, q) \in \mathbb{R}^d \times \mathbb{R}^d$. For each $\hbar$ we consider the complex valued $L^2$-function on $\mathbb{R}^d$ (called the coherent state) defined by

$$\langle z, \hbar \rangle \langle x \rangle := (\pi \hbar)^{-\frac{d}{2}} e^{-\frac{(x-q)^2}{2\hbar}} e^{ip \cdot x}.$$ 

We denote by $|z, \hbar\rangle \langle z, \hbar| : \mathfrak{H} \to \mathfrak{H}$ the orthogonal projection on the line $\mathbb{C} |z, \hbar\rangle$ in $\mathfrak{H}$. For each finite or positive Borel measure $\mu$ on $\mathbb{R}^d \times \mathbb{R}^d$ we define the Toeplitz operator $\text{OP}^T_h(\mu) : \mathfrak{H} \to \mathfrak{H}$ at scale $\hbar$ with symbol $\mu$ by the formula

$$\text{OP}^T_h(\mu) := \frac{1}{(2\pi \hbar)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z, \hbar\rangle \langle z, \hbar| \mu(dz).$$

The operator $|z, \hbar\rangle \langle z, \hbar|$ is trace class (as a rank 1 self-adjoint projection) and the integral should be interpreted as a Bochner integral, while the operator $\text{OP}^T_h(\mu)$ is possibly unbounded. We also recall the definition of the Wigner and Husimi transforms.
**Definition 2.1.** For $R$ an unbounded operator on $\mathfrak{H}$ with integral kernel $r \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$, the *Wigner transform of scale* $\hbar$ of $R$ is the distribution on $\mathbb{R}^d \times \mathbb{R}^d$ defined by the formula

$$W_{\hbar}[R](x, \xi) := \frac{1}{(2\pi)^d} \mathcal{F}_2 (r \circ j_{\hbar})(x, \xi),$$

where $j_{\hbar}(y, \eta) := (y + \frac{i}{\hbar} \eta, y - \frac{i}{\hbar} \eta)$ and $\mathcal{F}_2$ stands for the Fourier transform with respect to the second variable. The *Husimi transform of scale* $\hbar$ is the function on $\mathbb{R}^d \times \mathbb{R}^d$ defined by the formula

$$\tilde{W}_{\hbar}[R](x, \xi) := e^{\frac{\hbar}{2} \Delta - i \hbar y \cdot \xi} W_{\hbar}[R](x, \xi).$$

**Remark 2.2.** Denoting by $G^d_a$ the centered Gaussian density on $\mathbb{R}^d$ with covariance matrix $aI$ we can equivalently write

$$\tilde{W}_{\hbar}[R](x, \xi) = G^{2d}_a \ast (W_{\hbar}[R])(x, \xi).$$

If $\mu$ is a finite or positive Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$ then

$$W_{\hbar}[\text{OP}_T^\hbar(\mu)] = \frac{1}{(2\pi \hbar)^{d/2}} G^{2d}_a \ast \mu$$

(see formula (51) in [13]).

The following formula relates the Toeplitz operator to the Husimi transform

**Theorem 2.3.** ([13], Formula (54)). If $R \in D(\mathcal{S})$ and $\mu$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ then $\text{OP}_T^\hbar(\mu)$ is trace class and

$$\text{trace} \left( \text{OP}_T^\hbar(\mu) R \right) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{W}_{\hbar}[R](x, \xi) \mu(dx, d\xi). \quad (6)$$

An additional functional analytic tool which we freely use is Kato’s theory of the self-adjointness of perturbed self-adjoint unbounded operators.\(^2\) We denote by $\mathcal{H}$ the quantum Hamiltonian defined by

$$\mathcal{H} := \frac{1}{2} \left(-i \hbar \nabla_y + A(y)\right)^2 + \frac{1}{2} |y|^2 + V(y) := \frac{1}{2} \mathcal{H}_0 + V(y).$$

The operator $\mathcal{H}_0$ is called the *quantum kinetic energy* and plays the role of the perturbed operator while the potential $V$ plays the role of the perturbation. The reason why $\mathcal{H}_0$ is an essentially self-adjoint operator on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is explained Section [7]. By Kato’s theorem (see e.g. Theorem 6.4 in [28]), in order to assert that $\mathcal{H}_0$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ it is sufficient to show that $V$ (viewed as a multiplication operator) is relatively bounded, with relative bound $< 1$, with respect to $\mathcal{H}_0$—which is trivially true since $V$ is bounded. By Stone’s theorem, that $\mathcal{H}$ is essentially self-adjoint implies that the operator $U(t) := e^{rac{i t \mathcal{H}}{\hbar}}$ is unitary. The same considerations permit to view $c_\hbar^\lambda(x, \xi), \tilde{c}_\hbar^\lambda(x, \xi)$ as essentially self-adjoint operators with domains $D(c_\hbar^\lambda(x, \xi)), D(\tilde{c}_\hbar^\lambda(x, \xi))$ respectively such that

$$D \left(-\frac{\hbar^2}{2} \Delta + \frac{1}{2} |y|^2 \right) \subset D(c_\hbar^\lambda(x, \xi))$$

and

$$D \left(\frac{1}{2} \left(-i \hbar \nabla_y + A(y)\right)^2 + \frac{1}{2} |y|^2 \right) \subset D(\tilde{c}_\hbar^\lambda(x, \xi)).$$

See Section [7] for more useful information about the spectral theory of the magnetic/non-magnetic harmonic oscillator. At this stage, it is worth mentioning that the problem of finding a unique self-adjoint extension for magnetic Hamiltonians has been investigated in [3], where the authors prove a Kato type theorem under significantly weaker assumptions on $V$ and $A$. We can now state our main theorems.

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\(^2\)The power of Kato’s theory is reflected mainly in its capacity of handling singular perturbations such as the Coulomb potential. Since the electric potential is assumed to be fairly regular, Kato’s theory is not strictly necessary here.
Theorem 2.4. Let $V$ satisfy (7) and $V + \frac{1}{2}|x|^2 \geq 0$. Set $L := \text{Lip}(\nabla V)$. Let $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field satisfying (9) with constants $d, K$ and $K'$. Let $f^{\text{in}}$ be a probability density on $\mathbb{R}^d \times \mathbb{R}^d$ with finite second moments, i.e.
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x|^2 + |\xi|^2\right) f^{\text{in}}(x, \xi) dx d\xi < \infty.
\]
Assume in addition there is some $\rho_0 > 0$ such that
\[
\text{supp}(f^{\text{in}}) \subset \left\{(x, \xi) \mid \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \leq \rho_0^2\right\}.
\]
Let $f(t, \cdot)$ be the solution of the Liouville equation (7) with initial data $f^{\text{in}}$. Let $R_0^{\text{in}} = \text{OP}^T((2\pi\hbar)^d \mu^{\text{in}})$ where $\mu^{\text{in}}$ is a Borel probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ with finite second moments, and let $R(t)$ be the solution to the von Neumann equation (3) with initial data $R_0^{\text{in}}$. Then for all $t \in [0, \tau]$
\[
\text{dist}_{MK,2} \left(f(t, \cdot), \hat{W}_h[R(t)]\right)^2 \leq \beta(K)e^{\alpha(L, K', \rho_0)t} \left(\text{dist}_{MK,2}(f^{\text{in}}, \mu^{\text{in}})^2 + \frac{d\hbar}{2}\right) + \frac{d\hbar}{2}
\]
where
\[
\alpha(L, K', \rho_0) := 2 + L^2 + 4\rho_0 \max(2K'^2, 1)
\]
and
\[
\beta(K) := \max(2, 1 + 2K'^2)^2.
\]
Remark 2.5. As we will see, the initial finite second moments assumption on $f^{\text{in}}$ and the assumption $R_0^{\text{in}} \in D^2(\mathcal{F})$ are propagated in time. Consequently (see appendix B in [13] and especially formulas (54) and (48)), $\hat{W}_h[R(t)]$ has finite second moments and therefore can indeed be compared with $f(t, \cdot)$ through $\text{dist}_{MK,2}$. This remark is also relevant for Theorem 2.6 below.

Our second result concerns a similar question, but with magnetic vector potential which corresponds to planar rotation by $\frac{\pi}{2}$. Namely, we take $A(x) := \frac{1}{2}x^x$ and we are interested in the limit both as $\epsilon \rightarrow 0$, $\hbar \rightarrow 0$.

Theorem 2.6. Let $A(x) = \frac{1}{2}x^x$ and let $V$ satisfy (7). Set $L := \text{Lip}(\nabla V)$. Let $f^{\text{in}}_e$ be a probability density on $\mathbb{R}^2 \times \mathbb{R}^2$ with finite second moments. Let $f_e(t, \cdot)$ be the solution of the Liouville equation (7) with initial data $f^{\text{in}}_e$. Let $R_0^{\text{in}} = \text{OP}^T((2\pi\hbar)^2 \mu^{\text{in}})$ where $\mu^{\text{in}}$ is a Borel probability measure on $\mathbb{R}^2 \times \mathbb{R}^2$ with finite second moments, and let $R_e(t)$ be the solution to the von Neumann equation (2) with initial data $R_0^{\text{in}}$. Then for all $t \in [0, \tau]$
\[
\text{dist}_{MK,2} \left(f_e(t, \cdot), \hat{W}_h[R_e(t)]\right)^2 \leq e^{Ct} \left(\text{dist}_{MK,2}(f^{\text{in}}_e, \mu^{\text{in}})^2 + \frac{1}{2}(1 + \epsilon^2)\hbar\right) + \frac{1}{2}(1 + \epsilon^2)\hbar
\]
with $C := \max(2, \epsilon^2(1 + L^2))$.

Remark 2.7. Explicitly, the solution of equation (2) can be written as $R(t) = U^*(t)R^{\text{in}}U(t)$, whereas the solution of equation (1) is obtained as the push-forward under the flow of Newton’s second order system of ODEs (whose existence and uniqueness is guaranteed by the Cauchy-Lipschitz theorem). Namely, if $\Phi_t(x, \xi) = (X(t), \Xi(t))$ is the flow of the system
\[
\begin{aligned}
\frac{d}{dt} X(t) &= \Xi(t) + A(X(t)) \\
\frac{d}{dt} \Xi(t) &= -\nabla A(X(t))(\Xi + A(X(t)) - X(t) - \nabla V(X(t)) - \Xi(0) = \xi,
\end{aligned}
\]
then $f(t, x, \xi) = f^{\text{in}}(\Phi_{-t}(x, \xi))$. In the special case where $A(x) = \frac{1}{2}x^x$, the system (7) becomes
\[
\begin{aligned}
\frac{d}{dt} X(t) &= \Xi(t) + \frac{1}{2} X(t) \\
\frac{d}{dt} \Xi(t) &= \frac{1}{2} \Xi(t) - \frac{1}{2} X(t) - X(t) - \nabla V(X(t)) - \Xi(0) = \xi
\end{aligned}
\]
Remark 2.8. Both statements of Theorem 2.4 and 2.6 are valid (with some appropriate modifications) for the Hartree and Vlasov equations. In this case, the estimates for the terms arising from the electric potential are slightly more involved due to the convolutional non-linearity. See Theorem 2.5 in [14] for a guidance on how this is to be done.
3 Arbitrary Non-constant Magnetic Field

In this section we consider the question of deriving the Liouville equation from the von Neumann equation in the presence of a magnetic vector potential verifying the assumption \( (\mathbf{A}) \). Recall that \( f, R_h \) are the solutions to equations (1) and (2) with magnetic vector potential \( A(x) \) and with initial data \( f^{in}, R^{in} \), respectively. Let \( Q^{in} \in \mathcal{C}(f^{in}, R^{in}) \cap D^2(\mathfrak{S}) \) and let \( Q = Q(t, x, \xi) \) be defined by

\[
Q(t, x, \xi) := U(t)^* Q^{in}(\Phi_{-t}(x, \xi)) U(t).
\]

Alternatively \( Q(t, x, \xi) \) is the unique solution of the Cauchy problem for the semi-classical coupling equation

\[
\begin{align*}
\partial_t Q_h + \left\{ \frac{1}{2} |\xi + A(x)|^2 + V(x), Q_h \right\} + \frac{i}{\hbar} \left[ \frac{1}{2} |\xi - i\hbar \nabla_y + A(y)|^2 + V(y), Q_h \right] &= 0, \\
Q_h(0, x, \xi) &= Q^{in}_h \in \mathcal{C}(f^{in}, R^{in}) \cap D^2(\mathfrak{S}).
\end{align*}
\]

Eventually we wish to obtain an evolution estimate on the time dependent quantity

\[
\mathcal{E}_h^\lambda(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\mathfrak{S} \left( \sqrt{Q(t, x, \xi)} c_h^\lambda(x, \xi) \sqrt{Q(t, x, \xi)} \right) dxd\xi \geq 0,
\]

where

\[
c_h^\lambda(x, \xi) := \frac{1}{2} \lambda^2 |x - y|^2 + \frac{1}{2} |\xi + i\hbar \nabla_y|^2.
\]

We insert a parameter \( \lambda > 0 \) in the cost function in order to optimize some constants which will show up in Section 9. However, for the purpose of establishing Theorem 2.4 this is unnecessary, and so the reader is advised to follow the forthcoming calculation with \( \lambda = 1 \). We will define an auxiliary functional \( \tilde{\mathcal{E}}^\lambda_h(t) \) by

\[
\tilde{\mathcal{E}}^\lambda_h(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\mathfrak{S} \left( \sqrt{Q(t, x, \xi)} \tilde{c}_h^\lambda(x, \xi) \sqrt{Q(t, x, \xi)} \right) dxd\xi \geq 0,
\]

where

\[
\tilde{c}_h^\lambda(x, \xi) := \frac{\lambda^2}{2} |x - y|^2 + \frac{1}{2} |\xi + A(x) - A(y) + i\hbar \nabla_y|^2.
\]

Recall that \( Q \) propagates in time the coupling property

**Lemma 3.1.** (Lemma 4.2 in [14]) With the same notations and assumptions of Theorem 2.4, if \( Q^{in} \in \mathcal{C}(f^{in}, R^{in}) \) then \( Q(t) \in \mathcal{C}(f(t), R(t)) \) for all \( t \geq 0 \).

3.1 Classical/Quantum Finite Second Moments are Propagated in Time

We proceed by proving that equations (1) and (2) propagate in time finite second moments, i.e that for all \( t \in [0, \tau] \), \( f(t) \) has finite second moments and \( R(t) \in D^2(\mathfrak{S}) \), provided this is true for \( t = 0 \). As we will see, both of these observations justify the finiteness of \( \mathcal{E}_h^\lambda(t) \) and \( \tilde{\mathcal{E}}^\lambda_h(t) \) for all times.

**Lemma 3.2.** Let \( f^{in} \) be a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) f^{in}(x, \xi) dxd\xi < \infty.
\]

Let \( f \) be a solution of the Cauchy problem (3) with initial data \( f^{in} \). There is a constant \( C > 0 \) such that for all \( t \in [0, \tau] \)

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) f(t, x, \xi) dxd\xi \leq C \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) f^{in}(x, \xi) dxd\xi + \|V\|_\infty \right).
\]
Proof. The proof is a straightforward modification of Lemma 4.1 in [14]. Let \( X(t) \) and \( \Xi(t) \) be the solutions of equation (\ref{eq:lemma4.1}). By conservation of energy for the Liouville equation we have

\[
\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \frac{1}{2} |\Xi(t) + A(X(t))|^2 + \frac{1}{2} |X(t)|^2 + V(X(t)) \right) f_{\text{in}}(x, \xi) dx d\xi = 0,
\]

which entails

\[
\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\Xi(t) + A(X(t))|^2 + |X(t)|^2 \right) f_{\text{in}}(x, \xi) dx d\xi
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\xi + A(x)|^2 + |x|^2 \right) f_{\text{in}}(x, \xi) dx d\xi + \int_{\mathbb{R}^d \times \mathbb{R}^d} (V(x) - V(X(t))) f_{\text{in}}(x, \xi) dx d\xi.
\]

Clearly, for all \( t \geq 0 \)

\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (V(x) - V(X(t))) f_{\text{in}}(x, \xi) dx d\xi \right| \leq 2 \| V \|_{\infty},
\]

and therefore

\[
\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\Xi(t) + A(X(t))|^2 + |X(t)|^2 \right) f_{\text{in}}(x, \xi) dx d\xi
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\xi + A(x)|^2 + |x|^2 \right) f_{\text{in}}(x, \xi) dx d\xi + 2 \| V \|_{\infty}
\]

\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\xi|^2 + \left( \frac{2K^2 + 1}{2} \right) |x|^2 \right) f_{\text{in}}(x, \xi) dx d\xi + 2 \| V \|_{\infty}.
\]

Hence

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\xi + A(x)|^2 + |x|^2 \right) f(t, x, \xi) dx d\xi
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |X(t)|^2 + |\Xi(t) + A(X(t))|^2 \right) f_{\text{in}}(x, \xi) dx d\xi
\]

\[
\leq C \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) f_{\text{in}}(x, \xi) dx d\xi + \| V \|_{\infty} \right),
\]

which concludes the proof.

We recall the following observation, which will be freely used in the sequel.

Lemma 3.3. (Lemma 2.3 in [16]) If \( S = S^* \geq 0 \) is unbounded with domain \( D(S) \subset \mathcal{H} \) and \( T = T^* \geq 0 \) is trace class with eigenvectors \( (e_k)_{k \geq 1} \subset D(S) \) and eigenvalues \( (\alpha_k)_{k \geq 1} \) respectively, then \( \mathcal{L}(\mathcal{H}) \ni \sqrt{TS\sqrt{T}} = \left( \sqrt{TS\sqrt{T}} \right)^* \geq 0 \) and

\[
\text{trace} \left( \sqrt{TS\sqrt{T}} \right) = \sum_{k=1}^{\infty} \alpha_k \langle e_k, Se_k \rangle.
\]

Lemma 3.4. Let \( R(t) \) be the solution to the Cauchy problem (\ref{eq:lemma3.4}) with initial data \( R_{\text{in}} \in D^2(\mathcal{H}) \). Then \( R(t) \in D^2(\mathcal{H}) \) for all \( t \in [0, \tau] \).
**Proof.** By the Hilbert-Schmidt theorem, let \((e_k)_{k \geq 1}\) be a complete system of eigenvectors of \(R^{in}\). Because \(\mathcal{H} = V = \mathcal{H}_0\) we have

\[
\langle e_k, \sqrt{R(t)} \mathcal{H}_0 \sqrt{R(t)} e_k \rangle = \langle e_k, U(t)^* \sqrt{R^{in}} U(t) \mathcal{H}_0 U(t)^* \sqrt{R^{in}} U(t) e_k \rangle \\
= \langle e_k, U(t)^* \sqrt{R^{in}} \mathcal{H} \sqrt{R^{in}} U(t) e_k \rangle - \langle e_k, U(t)^* \sqrt{R^{in}} V \sqrt{R^{in}} U(t) e_k \rangle \\
= \langle U(t) e_k, \sqrt{R^{in}} \mathcal{H} \sqrt{R^{in}} U(t) e_k \rangle - \langle U(t) e_k, \sqrt{R^{in}} V \sqrt{R^{in}} U(t) e_k \rangle.
\]

Since \(U^*(t) e_k \in D(\mathcal{H}_0) = D\left(-\frac{1}{2} \hbar^2 \Delta + \frac{1}{2} |y|^2 \right)\) (see Corollary 7.5), the trace of \(\sqrt{R(t)} \mathcal{H}_0 \sqrt{R(t)}\) is well defined and

\[
\text{trace} \left( \sqrt{R(t)} \mathcal{H}_0 \sqrt{R(t)} \right) = \sum_{k=1}^{\infty} \langle e_k, \sqrt{R(t)} \mathcal{H}_0 \sqrt{R(t)} e_k \rangle \leq \sum_{k=1}^{\infty} \langle U(t) e_k, \sqrt{R^{in}} \mathcal{H} \sqrt{R^{in}} U(t) e_k \rangle + \|V\|_\infty \]

\[
= \text{trace} \left( \sqrt{R^{in}} \mathcal{H} \sqrt{R^{in}} \right) + \|V\|_\infty < \infty,
\]

as desired.

\(\blacksquare\)

**Remark 3.5.** The assumption that \(\mu^{in}\) is a Borel probability measure on \(\mathbb{R}^d \times \mathbb{R}^d\) with finite second moments implies that the Toeplitz operator \(OP^{T}_\hbar ((2\pi \hbar)^d \mu^{in})\) belongs to the space \(D^2(\mathcal{F})\)– see Proposition 2.3 in [15]. This fact will also be restated explicitly in Section 5.

We now provide a proof of a magnetic version of Lemma 2.2 in [15] which acts as a justification to the fact that the functionals \(\mathcal{E}^{\hbar}_0(t)\) and \(\mathcal{E}^{\hbar}_1(t)\) are well defined for all \(t \in [0, \tau]\). Again, the argument here is almost identical to the one proposed in [15]. As a preliminary we recall the following observation

**Proposition 3.6.** (Lemma 2.1 in [14]) Let \(T \in \mathcal{L}(\mathcal{F})\) satisfy \(T^* \geq 0\) and let \(S\) be an unbounded operator such that \(S = S^* \geq 0\). Then

\[
\text{trace} \left( \sqrt{T} S \sqrt{T} \right) = \text{trace} \left( \sqrt{ST} \sqrt{S} \right) \in [0, \infty].
\]

**Lemma 3.7.** Let \(R^{in} \in D^2(\mathcal{F})\) and let \(f^{in}\) be a probability density on \(\mathbb{R}^d \times \mathbb{R}^d\) with finite second moments. Suppose \(Q^{in}(x, \xi) \in \mathcal{C}(f^{in}, R^{in}) \cap D^2(\mathcal{F})\). Then

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t,x,\xi) c^{\hbar}_0(x,\xi) \sqrt{Q(t,x,\xi)}} \right) dxd\xi \\
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) f^{in}(x, \xi) dxd\xi + \text{trace} \left( \sqrt{R(t)} \left( |-i\hbar \nabla_y + A(y)|^2 + \lambda^2 |y|^2 \right) \sqrt{R(t)} \right) < \infty.
\]
Proof. Note the operator inequality
\[ c_h^λ(x, ξ) ≤ \lambda^2 |x|^2 + |ξ + A(x)|^2 + \lambda^2 |y|^2 + |−ih∇_y + A(y)|^2. \]

Since \( Φ_{−1} : \mathbb{R}^d × \mathbb{R}^d → \mathbb{R}^d × \mathbb{R}^d \) is a diffeomorphism, we see that \( Q^i_{in}(Φ_{−1}(x, ξ)) ∈ \mathcal{D}^2(Ω) \), which explains why
the trace of \( \sqrt{Q(t, x, ξ)}c_h^λ(x, ξ) \sqrt{Q(t, x, ξ)} \) is well defined. Therefore
\[
\int_{\mathbb{R}^d × \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, ξ)}c_h^λ(x, ξ) \sqrt{Q(t, x, ξ)} \right) dx dξ
\]
\[
≤ \int_{\mathbb{R}^d × \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, ξ)} \left( \lambda^2 |x|^2 + |ξ + A(x)|^2 \right) \sqrt{Q(t, x, ξ)} \right) dx dξ
\]
\[
+ \int_{\mathbb{R}^d × \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, ξ)} \left( \lambda^2 |y|^2 + |−ih∇_y + A(y)|^2 \right) \sqrt{Q(t, x, ξ)} \right) dx dξ.
\]
The first integral is
\[
\int_{\mathbb{R}^d × \mathbb{R}^d} \left( \lambda^2 |x|^2 + |ξ + A(x)|^2 \right) \text{trace} (Q(t, x, ξ)) dx dξ ≤ \int_{\mathbb{R}^d × \mathbb{R}^d} \left( \lambda^2 |x|^2 + |ξ + A(x)|^2 \right) f(t, x, ξ) dx dξ.
\]
Owing to Proposition 3.6, the second integral is recast as
\[
\int_{\mathbb{R}^d × \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, ξ)} \left( \lambda^2 |y|^2 + |−ih∇_y + A(y)|^2 \right) \sqrt{Q(t, x, ξ)} \right) dx dξ
\]
\[
= \int_{\mathbb{R}^d × \mathbb{R}^d} \text{trace} \left( \sqrt{\lambda^2 |y|^2 + |−ih∇_y + A(y)|^2} Q(t, x, ξ) \sqrt{\lambda^2 |y|^2 + |−ih∇_y + A(y)|^2} \right) dx dξ
\]
\[
= \text{trace} \left( \sqrt{\lambda^2 |y|^2 + |−ih∇_y + A(y)|^2} R(t) \sqrt{\lambda^2 |y|^2 + |−ih∇_y + A(y)|^2} \right)
\]
\[
= \text{trace} \left( \sqrt{R(t)} \left( \lambda^2 |y|^2 + |−ih∇_y + A(y)|^2 \right) \sqrt{R(t)} \right).
\]

\[\square\]

3.2 The Gronwall Estimate

We shall first obtain a Gronwall estimate on \( \tilde{E}_h^λ(t) \). Once this is achieved, the desired estimate for \( E_h^λ(t) \) would follow easily, as implied by the following simple

**Lemma 3.8.** Let the assumptions of Theorem (2.4) hold. Let \( λ > 0 \) and let \( E_h^λ(t) \) and \( \tilde{E}_h^λ(t) \) be as defined in (3) and (11). It holds that
\[
\frac{1}{\max(2, 2λ^2 + 2K^2)} \tilde{E}_h^λ(t) \leq E_h^λ(t) ≤ \max \left( 2, \frac{λ^2 + 2K^2}{λ^2} \right) \tilde{E}_h^λ(t).
\]

**Proof.** By the triangle inequality
\[
\tilde{c}_h^λ(x, ξ) = \frac{λ}{2} |x − y|^2 + \frac{1}{2} |ξ + A(x) − A(y) + ih∇_y|^2
\]
\[
≤ \frac{λ}{2} |x − y|^2 + |ξ + ih∇_y|^2 + |A(x) − A(y)|^2
\]
\[
≤ \frac{λ^2 + 2K^2}{2} |x − y|^2 + |ξ + ih∇_y|^2 \leq \max \left( 2, \frac{λ^2 + 2K^2}{λ^2} \right) c_h^λ(x, ξ),
\]

\[\square\]
while

\[ c_h^\lambda(x, \xi) = \frac{\lambda^2}{2} |x - y|^2 + \frac{1}{2} |\xi + A(x) - A(y) + i\hbar \nabla_y - (A(x) - A(y))|^2 \]

\[ \leq \frac{\lambda^2}{2} |x - y|^2 + K^2 |x - y|^2 + |\xi + A(x) - A(y) + i\hbar \nabla_y|^2 \]

\[ = \frac{\lambda^2 + 2K^2}{2} |x - y|^2 + |\xi + A(x) - A(y) + i\hbar \nabla_y|^2 \]

\[ \leq \max \left( 2, \frac{\lambda^2 + 2K^2}{\lambda^2} \right) \tilde{c}_h^\lambda(t, x, \xi). \]

\[ \square \]

We are now ready to prove the following intermediate inequality, which is the core inequality leading to the estimate announced in Theorem 2.4.

**Theorem 3.9.** With the same notations and assumptions of Theorem 2.4,\(^5\)

\[ E_h^\lambda(f(t), R_h(t))^2 \leq \beta(K, \lambda)e^{\alpha(L', K', \lambda, \rho_0)t}E_h^{\text{fin}}(f^{\text{fin}}, R_h^{\text{fin}}) \]

where

\[ \beta(K, \lambda) := \max \left( 2, \frac{\lambda^2 + 2K^2}{\lambda^2} \right)^2 \]

and

\[ \alpha(L', K', \lambda, \rho_0) := 1 + \max \left( 1, \frac{1 + L^2}{\lambda^2} \right) + 4\rho_0 \max \left( \frac{2K^2}{\lambda^2}, 1 \right). \]

The proof below should be regarded as a formal proof. Subtleties such as differentiability in time of \( \tilde{E}_h^\lambda(t) \) can be addressed using the eigenfunction expansion method demonstrated in [15]—the same method is also used to justify rigorously the estimate reported in Theorem 2.4 and is therefore not repeated here. The verification that the method of Section 4 (which, as remarked, borrows from [15]) can be adapted to the forthcoming calculations is standard.

**Proof of Theorem 2.4.** **Step 0.** By Lemma 3.1 and Lemma 3.8, for each \( t \geq 0 \)

\[ \tilde{E}_h^\lambda(t) \geq \frac{1}{\max(2, \frac{\lambda^2 + 2K^2}{\lambda^2})}E_h^\lambda(f(t), R_h(t))^2. \]

Set

\[ I_1 := \frac{i}{\hbar} \left[ \frac{1}{2} |\nabla A(y)|^2, c_h^\lambda(x, \xi) \right] + \left\{ \frac{1}{2} |\xi + A(x)|^2, \tilde{c}_h^\lambda(x, \xi) \right\} \]

and

\[ I_2 := \frac{i}{\hbar} \left[ \frac{1}{2} |y|^2 + V(y), c_h^\lambda(x, \xi) \right] + \left\{ \frac{1}{2} |x|^2 + V(x), \tilde{c}_h^\lambda(x, \xi) \right\}. \]

We compute

\[ \frac{d}{dt} \text{trace} \left( c_h^\lambda(x, \xi)Q(t, x, \xi) \right) = \text{trace} \left( \tilde{c}_h^\lambda(x, \xi) \partial_t Q(t, x, \xi) \right) \]

\[ = -\text{trace} \left( c_h^\lambda(x, \xi) \left\{ \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right\} \right) \]

\[ -\text{trace} \left( \tilde{c}_h^\lambda(x, \xi) \frac{i}{\hbar} \left[ \frac{1}{2} |\nabla A(y)|^2 + \frac{1}{2} |y|^2 + V(y) \right] \right). \]
\[
\text{and}
\]
\[
= \text{trace} \left( \left\{ \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x), \tilde{c}^\lambda_h(x, \xi) \right\} Q(t, x, \xi) \right)
\]
\[
+ \text{trace} \left( \left\{ \frac{i}{\hbar} \left[ \frac{1}{2} - i\hbar \nabla_y + A(y) \right]^2 + \frac{1}{2} |y|^2 + V(y), \tilde{c}^\lambda_h(x, \xi) \right\} Q(t, x, \xi) \right)
\]
\[
= \text{trace} \left( I_1 Q(t, x, \xi) \right) + \text{trace} \left( I_2 Q(t, x, \xi) \right).
\]

**Step 1. Estimating** \(\text{trace} \left( I_1 Q(t, x, \xi) \right)\). To make the equations a bit lighter, set

\[
\Pi_k := -i\hbar \partial_k + A_k(y)
\]

and

\[
D_k = D_k(x, \xi) := \xi_k + A_k(x) - \Pi_k.
\]

In the following calculations Einstein summation is in free use.

\[
\frac{i}{\hbar} \left[ \frac{1}{2} - i\hbar \nabla_y + A(y) \right]^2, \tilde{c}^\lambda_h(x, \xi)
\]
\[
= \frac{1}{2} \frac{i}{\hbar} \Pi_k \vee \left[ \Pi_k, \lambda^2 \frac{1}{2} (x_l - y_l)^2 + \frac{1}{2} D^2_l \right]
\]
\[
= \frac{\lambda^2}{4} \frac{i}{\hbar} \Pi_k \vee ((x_l - y_l) \vee [\Pi_k, (x_l - y_l)])
\]
\[
+ \frac{1}{\hbar} \Pi_k \vee \left( D_l \vee \frac{1}{2} [\Pi_k, D_l] \right)
\]
\[
= -\frac{\lambda^2}{2} \Pi_k \vee ((x_l - y_l) \partial_{y_k}(y_i)) + \frac{1}{4} \Pi_k \vee (D_l \vee a_{kl}(y)),
\]

(12)

where in the last equation we abbreviated \(a_{kl}(y) := \partial_y A_k(y) - \partial_{y_k} A_l(y)\). In addition

\[
\left\{ \frac{1}{2} |\xi + A(x)|^2, \tilde{c}^\lambda_h(x, \xi) \right\} = \left\{ \frac{1}{2} (\xi_k + A_k(x))^2, \frac{\lambda^2}{2} (x_l - y_l)^2 + \frac{1}{2} D^2_l \right\}
\]
\[
= \lambda^2 (\xi_k + A_k(x))(x_k - y_k) - \frac{1}{4} a_{kl}(x) \vee ((\xi_k + A_k(x)) \vee D_l).
\]

(13)

Adding up the first terms of equations (12) and (13) gives

\[
\text{trace} \left( \left( \frac{\lambda^2}{2} (\xi_k + A_k(x)) \vee (x_k - y_k) - \frac{\lambda^2}{2} \Pi_k \vee ((x_l - y_l) \partial_{y_k}(y_i)) \right) Q(t, x, \xi) \right)
\]
\[
= \text{trace} \left( \frac{\lambda^2}{2} D_k \vee (x_k - y_k) Q(t, x, \xi) \right)
\]
\[
\leq \text{trace} \left( \left( \frac{1}{2} D^2_k + \frac{\lambda^2}{2} (x_k - y_k)^2 \right) Q(t, x, \xi) \right)
\]

(14)

Adding up the second terms of equations (12) and (13) gives

\[
\frac{1}{4} \Pi_k \vee (D_l \vee a_{kl}(y)) - \frac{1}{4} (\xi_k + A_k(x)) \vee (D_l \vee a_{kl}(x))
\]
\[
= -\frac{1}{4} (\xi_k + A_k(x)) \vee (D_l \vee (a_{kl}(x) - a_{kl}(y)))
\]

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\[-\frac{1}{4}(\xi_k + A_k(x)) \vee (D_t \vee a_{kl}(y)) + \frac{1}{4} \Pi_k \vee (D_t \vee a_{kl}(y)) = -\frac{1}{4}(\xi_k + A_k(x)) \vee (D_t \vee (a_{kl}(x) - a_{kl}(y))) - \frac{1}{4} D_k \vee (D_t \vee a_{kl}(y)).\]

We claim that the second line is identically 0, i.e.

**Claim 3.10.** \[\sum_{k,l} D_k \vee (D_l \vee a_{kl}) = 0.\]

The proof of Claim (3.10) is postponed to the end of this section. We arrive at

\[
\text{trace } (I_1 Q(t,x,\xi)) \leq \text{trace } (\tilde{c}_h^\lambda(\Phi_t(x,\xi))Q(t,x,\xi)) + \frac{1}{4} \text{trace } ((\xi_k + A_k(x)) \vee (D_t \vee (a_{kl}(y) - a_{kl}(x))) Q(t,x,\xi)).
\]

**Step 2. Estimating I_2.** We compute

\[
i \hbar \left[ V(y), \tilde{c}_h^\lambda(x,\xi) \right] = i \hbar \left[ V(y), \frac{\lambda^2}{2} |x - y|^2 + \frac{1}{2} D_k^2 \right] = i \hbar D_k \vee [V(y), D_k] = i \hbar D_k \vee [V(y), i\hbar \partial_{y_k}] = \frac{1}{2} D_k \vee (\partial_{y_k} V(y)).
\]

The same calculation with V replaced by \(\frac{1}{2} |y|^2\) gives

\[
i \hbar \left[ \frac{1}{2} |y|^2, \tilde{c}_h^\lambda(x,\xi) \right] = \frac{1}{2} D_l \vee y_l.
\]

The Poisson brackets are recast as

\[
\left\{ \frac{1}{2} |x|^2 + V(x), \tilde{c}_h^\lambda(x,\xi) \right\} = -(x_k + \partial_{x_k} V(x)) D_k.
\]

Thus

\[
I_2 = \frac{1}{2} D_k \vee (\partial_{y_k} V(y)) - \partial_{x_k} V(x) D_k + \frac{1}{2} D_l \vee (y_k \partial_{y_l} y_k) - \frac{1}{2} D_l \vee x_l = \frac{1}{2} D_k \vee (\partial_{y_k} V(y) - \partial_{x_k} V(x)) + \frac{1}{2} D_k \vee (y_l - x_l).
\]

Hence

\[
\text{trace } (I_2 Q(t,x,\xi)) \leq \text{trace } \left( \left( \frac{1}{2} |\xi + i\hbar \nabla_y + A(x) - A(y)|^2 + \frac{1}{2} L^2 |x - y|^2 \right) Q(t,x,\xi) \right)
\]
Proof. Let \( \rho \) be precisely the place where we use the assumption that the support of \( \frac{1}{\lambda^2} \). The final ingredient needed in order to complete the estimate is mastering the term on the second line, which is
\( d \times R \times R + 1 \times 0 \int \int \frac{1}{1 + \frac{1 + L^2}{\lambda^2}} \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) ds

\[ + \int_0^t \left( 1 + \max \left( \frac{1 + L^2}{\lambda^2} \right) \right) \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) ds. \]

Integrating on \( R^d \times R^d \) yields
\[ \int \left( \frac{1}{\lambda^2} \right) \left( \frac{1}{\lambda^2} \right) \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) dx d\xi \]
\[ \leq \int \text{trace}(\tilde{c}_h(x, \xi)Q^\text{in}(x, \xi)) \) dx d\xi \]

\[ + \frac{1}{4} \int_0^t \int \left( (\xi_k + A_k(x)) \right) \left( (\xi_k + A_k(x)) \right) Q(t, x, \xi) \) dx d\xi ds \]

\[ + \int_0^t \left( 1 + \max \left( \frac{1 + L^2}{\lambda^2} \right) \right) \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) dx d\xi ds. \] (17)

The final ingredient needed in order to complete the estimate is mastering the term on the second line, which is precisely the place where we use the assumption that the support of \( f^\text{in} \) is “not too large”.

Claim 3.11. The following estimate holds
\[ \frac{1}{4} \int_0^t \int \frac{1}{\lambda^2} \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) dx d\xi ds \]

\[ \leq 4\rho_0 \max \left( \frac{2K^2}{\lambda^2}, 1 \right) \int \text{trace}(\tilde{c}_h(x, \xi)Q(t, x, \xi)) \) dx d\xi. \]

Proof. Let \( \chi \in C_0^\infty(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1, \chi(r) \equiv 1 \) for \( |r| > 2\rho_0^2 \) and \( \chi(r) \equiv 0 \) for \( |r| < \rho_0^2 \). Observe that
\[ \frac{d}{dt} \int \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) f(t, x, \xi) dx d\xi \]
\[ = - \int \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) f(t, x, \xi) dx d\xi \]
\[ \begin{align*}
\int_{\mathbb{R}^d \times \mathbb{R}^d} & \left\{ \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x), \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) \right\} f(t, x, \xi) dxd\xi = 0, \\
\end{align*} \]

and as a result

\[ \begin{align*}
\int_{\mathbb{R}^d \times \mathbb{R}^d} & \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) f(t, x, \xi) dxd\xi \\
= & \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) f^{in}(x, \xi) dxd\xi = 0.
\end{align*} \]

Since \( Q(t, x, \xi) \) is a coupling of \( f(t, x, \xi) \) and \( R(t) \), it follows that

\[ \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) \text{trace} \left( Q(t, x, \xi) \right) \equiv 0, \]

so that

\[ \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) Q(t, x, \xi) \equiv 0. \]

We thus conclude

\[ \begin{align*}
\frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} & \text{trace} \left( (\xi_k + A_k(x)) \lor (D_l \lor (a_{kl}(y) - a_{kl}(x))) Q(t, x, \xi) \right) dxd\xi \\
= & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( 1 - \chi \left( \frac{1}{2} |\xi + A(x)|^2 + \frac{1}{2} |x|^2 + V(x) \right) \right) (\xi_k + A_k(x)) \\
& \times \text{trace} \left( D_l \lor (a_{kl}(y) - a_{kl}(x)) Q(t, x, \xi) \right) dxd\xi \\
\leq & 2\rho_0 \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \text{trace} \left( D_l \lor (a_{kl}(y) - a_{kl}(x)) Q(t, x, \xi) \right) \right| dxd\xi \\
\leq & 2\rho_0 \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \left( D_l^2 + |a_{kl}(y) - a_{kl}(x)|^2 \right) Q(t, x, \xi) \right) dxd\xi \\
\leq & 2\rho_0 \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \left( D_l^2 + 2K^2 |x - y|^2 \right) Q(t, x, \xi) \right) dxd\xi \\
\leq & 4\rho_0 \max \left( \frac{2K^2}{\lambda^2}, 1 \right) \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \tilde{c}_h^\lambda(x, \xi) Q(t, x, \xi) \right) dxd\xi.
\end{align*} \]

The combination of inequality (17) and Claim (3.11) yields

\[ \begin{align*}
\int_{\mathbb{R}^d \times \mathbb{R}^d} & \text{trace} \left( \tilde{c}_h^\lambda(x, \xi) Q(t, x, \xi) \right) dxd\xi \\
\leq & \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \tilde{c}_h^\lambda(x, \xi) Q^{in}(x, \xi) \right) dxd\xi.
\end{align*} \]
Gronwall inequality as applied to inequality (18) implies
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \tilde{c}_h^\lambda(x,\xi)Q(t,x,\xi) \right) \, dx \, d\xi \leq e^{(1+\max(1,\frac{1+L^2}{\lambda^2})+4\rho_0 \max\left(\frac{2K^2\lambda^2}{\lambda^2},1\right)) \int_0^t \text{trace} \left( \tilde{c}_h^\lambda(x,\xi)Q(s,x,\xi) \right) \, dx \, d\xi}. \tag{18}
\]

The application of Lemma (3.8) produces
\[
\frac{1}{\max\left(2,\frac{\lambda^2+2K^2}{\lambda^2}\right)} E_h^\lambda(f(t),R_h(t))^2 \leq \frac{1}{\max\left(2,\frac{\lambda^2+2K^2}{\lambda^2}\right)} E_h^\lambda(t) \leq \tilde{E}_h^\lambda(t) \leq \tilde{E}_h^\lambda(0) e^{\alpha(K',L,\lambda,\rho_0)t} E_h^\lambda(0).
\]

Minimizing the right hand side of the above inequality as \(Q^{\text{in}} \in C(f^{\text{in}},R_h^{\text{in}}) \cap D^2(\mathfrak{h})\) yields
\[
E_h^\lambda(f(t),R_h(t))^2 \leq \beta(K,\lambda) e^{\alpha(K',L,\lambda,\rho_0)t} E_h^\lambda(f^{\text{in}},R_h^{\text{in}})^2,
\]

as claimed.

\[\square\]

In order to finish the proof we are left to justify Claim (3.10)

**Proof. Step 1. Calculation of \([D_k,D_l]\).** We expand
\[
D_kD_l = (\xi_k + A_k(x) - A_k(y) + ih\partial_{y_k})(\xi_l + A_l(x) - A_l(y) + ih\partial_{y_l})
\]
\[
= (\xi_k + A_k(x))(\xi_l + A_l(x)) + (\xi_k + A_k(x))(-A_l(y) + ih\partial_{y_l}) + (-A_k(y) + ih\partial_{y_k})(\xi_l + A_l(x))
\]
\[
- h^2 \partial_{y_k} A_l - A_l ih \partial_{y_k} + A_k ih \partial_{y_l} + A_k A_l.
\]

From this identity we derive the relation
\[
D_kD_l - D_l D_k = ih\partial_{y_l} A_k - ih\partial_{y_k} A_l,
\]
so that
\[
a_{kl}(y) = \frac{1}{ih}(D_kD_l - D_l D_k).
\]

**Step 2. Calculation of \(D_l \vee a_{kl}\).** By step 1
\[
\begin{align*}
\rho(D_l \vee a_{kl}) &= D_l \vee (D_kD_l - D_l D_k) \\
&= D_l(D_kD_l) + (D_kD_l)D_l - D_l(D_l D_k) - (D_l D_k)D_l \\
&= (D_lD_k)D_l + (D_kD_l)D_l - D_l(D_l D_k) - (D_l D_k)D_l \\
&= (D_kD_l)D_l - D_l(D_l D_k).
\end{align*}
\]
We consider the time dependent quantity \( f \). Let \( \Theta_0 \).

Theorem 4.1. Intermediate inequality justifies that we use the solution to the von Neumann equation (2) with initial data \( f \).

In contrast with the previous section we consider here a magnetic cost function with an \( \epsilon \) weight in front of the quantum part. We prove here the following intermediate inequality.

**Theorem 4.1.** Let \( A(x) = \frac{1}{\epsilon} x^1 \) and let \( V \) satisfy (??). Let \( f^{in}_\epsilon \) be a probability density on \( \mathbb{R}^2 \times \mathbb{R}^2 \) such that

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( |x|^2 + |\xi|^2 \right) f^{in}_\epsilon(x,\xi) dx d\xi < \infty.
\]

Let \( f_\epsilon \) be the solution of the Liouville equation (1) with initial data \( f^{in}_\epsilon \). Let \( R^{in}_{h,\epsilon} \in D^2(\mathfrak{f}) \) and let \( R_{\epsilon,h} \) be the solution to the von Neumann equation (2) with initial data \( R^{in}_{\epsilon,h} \). Then

\[
E_{h,\epsilon}(f_\epsilon(t), R_{\epsilon,h}(t)) \leq e^{\max(2,\epsilon^2(1+L^2))t} \cdot E_{h,\epsilon}(f^{in}_\epsilon, R^{in}_{\epsilon,h}).
\]

For each \( Q^{in}(x,\xi) \in C(f^{in}_\epsilon, R^{in}_{\epsilon,h}) \cap D^2(\mathfrak{f}) \) let \( Q = Q_{h,\epsilon}(t, x, \xi) \) be defined by

\[
Q(t, x, \xi) := U^*(t)Q^{in}(\Phi_{-t}(x, \xi))U(t).
\]

We consider the time dependent quantity

\[
\mathcal{E}_{h,\epsilon}(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr} \left( \sqrt{Q(t, x, \xi)} \left( \frac{1}{2} |x - y|^2 + \frac{1}{2} \epsilon^2 |\xi + i\hbar \nabla_y|^2 \right) \right) \sqrt{Q(t, x, \xi)} \ dx d\xi \geq 0.
\]

By Lemma [3.4] and Lemma [3.2] \( R(t) \in D^2(\mathfrak{f}) \) and \( f(t) \) has finite second moments, which in view of Lemma [3.7] justifies that \( \mathcal{E}_{h,\epsilon}(t) < \infty \).

**Proof of theorem [2.4].** **Step 0. Smooth Approximation.** By Lemma [3.1]

\[
\mathcal{E}_{h,\epsilon}(t) \geq E_{h,\epsilon}(f_\epsilon(t), R_{\epsilon,h}(t))^2.
\]

In addition
For a.e. \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\) let \(e_1(x, \xi), \ldots, e_k(x, \xi), \ldots\) be a \(\mathcal{H}\)-complete orthonormal system of eigenvectors of \(Q^{\text{in}}(x, \xi)\) with eigenvalues \(\mu_1(x, \xi), \ldots, \mu_k(x, \xi), \ldots\) respectively. Then

\[
\text{trace} \left( U^*(t) \sqrt{Q^{\text{in}}(x, \xi)} U(t) c_{h, e} (\Phi_t(x, \xi)) U(t) \sqrt{Q^{\text{in}}(x, \xi)} U(t) \right) = \sum_{k=1}^{\infty} \mu_k \left( U^*(t) e_k, c_{h, e} (\Phi_t(x, \xi)) U^*(t) e_k \right).
\]

For each \(\varphi \in C_0^\infty(\mathbb{R}^d)\) the map

\[
t \mapsto \langle U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) U^*(t) \varphi \rangle
\]

is \(\text{Lip}([0, \tau])\) (see Lemma 4.2 for more details) and one computes its time derivative as follows

\[
\frac{d}{dt} \langle U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) U^*(t) \varphi \rangle = \left\langle -\frac{i}{\hbar} \mathcal{H} U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) U^*(t) \varphi \right\rangle
\]

\[
- \left\langle U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) \frac{i}{\hbar} \mathcal{H} U^*(t) \varphi \right\rangle + \left\langle U^*(t) \varphi, \frac{d}{dt} (c_{h, e} (\Phi_t(x, \xi))) U^*(t) \varphi \right\rangle
\]

\[
= \frac{i}{\hbar} \left\langle \mathcal{H} U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) U^*(t) \varphi \right\rangle - \frac{i}{\hbar} \left\langle U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) \mathcal{H} U^*(t) \varphi \right\rangle + \left\langle U^*(t) \varphi, (X(t) - y) \cdot (\Xi + A(X(t)) + \frac{c^2}{2} (\Xi + i\hbar \nabla_y) \vee (-\nabla A(X(t))(\Xi + A(X(t))) - X(t) - \nabla V(X(t))) U^*(t) \varphi \right\rangle
\]

The first two terms are paired together to yield a commutator while the last term is recognized as a Poisson bracket:

\[
\frac{d}{dt} \langle U^*(t) \varphi, c_{h, e} (\Phi_t(x, \xi)) U^*(t) \varphi \rangle
\]

\[
= \frac{i}{\hbar} \left\langle U^*(t) \varphi, [\mathcal{H}, c_{h, e} (\Phi_t(x, \xi))] U^*(t) \varphi \right\rangle + \left\langle U^*(t) \varphi, \left\{ \frac{1}{2} |\Xi + A(X)|^2 + \frac{1}{2} |X|^2 + V(X), \frac{1}{2} |X - y|^2 + \frac{c^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} (\Phi_t(x, \xi)) U^*(t) \varphi \right\rangle.
\]

Here the Poisson bracket is with respect to \(X, \Xi\). We omit the time variable, since it will be invisible in the forthcoming calculations.
\[
\begin{aligned}
&+ \left\{ \frac{1}{2} |\Xi + A(X)|^2 + \frac{1}{2} |X|^2 + V(X), \frac{1}{2} |X - y|^2 + \frac{\epsilon^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} (\Phi_t(x, \xi)) \\
&= \left\{ \frac{1}{2} |\Xi + A(X)|^2, \frac{1}{2} |X - y|^2 + \frac{\epsilon^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} \\
&+ \frac{i}{\hbar} \left\{ \frac{1}{2} |i\hbar \nabla_y + A(y)|^2, \frac{1}{2} |X - y|^2 + \frac{\epsilon^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} \\
&+ \left\{ V(X) + \frac{1}{2} |X|^2, \frac{1}{2} |X - y|^2 + \frac{\epsilon^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} \\
&+ \frac{i}{\hbar} \left\{ V(y) + \frac{1}{2} |y|^2, \frac{1}{2} |X - y|^2 + \frac{\epsilon^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\}.
\end{aligned}
\]

Denote by \( \delta_1(t) \) the sum of the first two terms and by \( \delta_2(t) \) the sum of the last two terms. We proceed through the following steps.

**Step 1. Vanishing of \( \delta_1(t) \).** The vanishing of \( \delta_1(t) \) reflects the main novelty of this section, as it is the main reason for the fact that the final estimate is uniform in \( \epsilon \). The estimate of \( \delta_2(t) \) would follow by an argument similar (and in fact simpler) to the one in [14]. Recall that for brevity we denote

\[
\Pi_k := -i\hbar \partial_k + A_k(y)
\]

and

\[
D_k = D_k(X, \Xi) := \Xi_k + A_k(X) - \Pi_k.
\]

As usual, Einstein summation is freely used. We expand

\[
\frac{i}{\hbar} \left\{ \frac{1}{2} |i\hbar \nabla_y + A(y)|^2, \frac{1}{2} |X - y|^2 \right\}
\]

\[
= \frac{1}{4} \frac{i}{\hbar} \Pi_k \lor ((X_t - y_t) \lor [\Pi_k, (X_t - y_t)])
\]

\[
= \frac{1}{4} \frac{i}{\hbar} \Pi_k \lor ((X_t - y_t) \lor (i\hbar \partial_k y_t))
\]

\[
= -\frac{1}{4} \Pi_k \lor ((X_t - y_t) \lor (\partial_k y_t)) = -\frac{1}{2} \Pi_k \lor (X_t - y_t),
\]

where the second equation is because the commutator of two multiplication operators is 0. Furthermore

\[
\frac{i}{\hbar} \left\{ \frac{1}{2} (-i\hbar \partial_{y_k} + A_k(y))^2, \frac{\epsilon^2}{2} (\Xi_l + i\hbar \partial_{y_l})^2 \right\}
\]

\[
= \frac{\epsilon^2}{4} \frac{i}{\hbar} \left[ \Pi_k^2, (\Xi_l + i\hbar \partial_{y_l})^2 \right]
\]

\[
= \frac{\epsilon^2}{4} \frac{i}{\hbar} \Pi_k \lor ((\Xi_l + i\hbar \partial_{y_l}) \lor [\Pi_k, (\Xi_l + i\hbar \partial_{y_l})])
\]

\[
= \frac{\epsilon^2}{4} \frac{i}{\hbar} \Pi_k \lor ((\Xi_l + i\hbar \partial_{y_l}) \lor (-i\hbar \partial_{y_l} A_k(y)))
\]

\[
= \frac{\epsilon^2}{4} \Pi_k \lor ((\Xi_l + i\hbar \partial_{y_l}) \lor (\partial_{y_l} A_k(y)))
\]

\[
= \frac{\epsilon^2}{2} \Pi_k \lor ((\Xi_l + i\hbar \partial_{y_l})(\nabla A)_{l_k}),
\]

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so that
\[
\frac{i}{\hbar} \left[ \frac{1}{2} | -i \hbar \nabla_y + A(y) |^2, \frac{1}{2} | X - y |^2 + \frac{\epsilon^2}{2} | \Xi + i \hbar \nabla_y |^2 \right]
\]
\[= -\frac{1}{2} \Pi_k \vee (X_k - y_k) + \frac{\epsilon^2}{2} \Pi_k \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A)_k). \tag{22} \]

As for the Poisson brackets, we compute that
\[
\{ (\Xi_k + A_k(X))^2, \epsilon^2 (\Xi_k + i \hbar \partial_{y_k})^2 \}
\]
\[= \epsilon^2 (\Xi_k + A_k(X)) \vee \{ \Xi_k + A_k(X), (\Xi + i \hbar \partial_{y_k})^2 \}
\]
\[= -\epsilon^2 (\Xi_k + A_k(X)) \vee \{ (\Xi + i \hbar \partial_{y_k})^2, \Xi_k + A_k(X) \}
\]
\[= -\epsilon^2 (\Xi_k + A_k(X)) \vee (\Xi + i \hbar \partial_{y_k}) \vee \partial_{X_k} A_k), \tag{23} \]

since
\[
\{ \Xi + i \hbar \partial_{y_k}, \Xi_k + A_k(X) \} = \{ \Xi, A_k(X) \} = \partial_{X_k} A_k.
\]

In addition
\[
\left\{ \frac{1}{2} | \Xi + A(X) |^2, \frac{1}{2} | X - y |^2 \right\} = (\Xi + A(X)) \cdot (X - y). \tag{24} \]

So, gathering equations (22), (23) and (24) we get
\[
(\Xi + A(X)) \cdot (X - y) - \frac{1}{4} \epsilon^2 (\Xi_k + A_k(X)) \vee (\Xi + i \hbar \partial_{y_k}) \vee \partial_{X_k} A_k)
\]
\[-\frac{1}{2} \Pi_k \vee (X_k - y_k) + \frac{\epsilon^2}{2} \Pi_k \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A)_k)
\]
\[= \frac{1}{2} D_k \vee (X_k - y_k) - \frac{\epsilon^2}{2} D_k \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A)_k)
\]
\[= \frac{1}{2} (i \hbar \partial_{y_k} + \Xi_k) \vee (X_k - y_k) - \frac{\epsilon^2}{2} D_k \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A)_k),
\]

where the last equation is because \( A(y) - A(X) \perp X - y \). Since \( \nabla A = \frac{1}{\epsilon} J \) where \( J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) we see that
\[
\frac{1}{2} (i \hbar \nabla_y + \Xi) \vee (X - y) + \frac{1}{2} \epsilon^2 (A(y) - A(X)) \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A))
\]
\[= \frac{1}{2} (i \hbar \nabla_y + \Xi) \vee (X - y) + \frac{1}{2} (y - X)^\perp \vee (\xi + i \hbar \nabla_y)^\perp = 0.
\]

Moreover
\[
(i \hbar \partial_{y_k} + \Xi_k) \vee ((\Xi + i \hbar \nabla_y) \cdot (\nabla A))_k = 0,
\]

which shows that \( \delta_1(t) = 0. \)

**Step 2. Controlling \( \delta_2(t) \).** We have
\[
\left\{ V(X), \frac{1}{2} | X - y |^2 + \frac{\epsilon^2}{2} | \Xi + i \hbar \nabla_y |^2 \right\}
\]
which by Gronwall implies

\[ \left\{ V(X), \frac{1}{2} r^2 |\Xi + i\hbar \nabla_y|^2 \right\} = -r^2 \nabla V(X) \cdot (\Xi + i\hbar \nabla_y), \]

and

\[
i \frac{i}{\hbar} \left[ V(y), \frac{1}{2} |X - y|^2 + \frac{1}{2} r^2 |\Xi + i\hbar \nabla_y|^2 \right] = \frac{i r^2}{2\hbar} \left[ V(y), |\Xi + i\hbar \nabla_y|^2 \right]
\]

\[
= \frac{i r^2}{2\hbar} \left[ V(y), (\Xi_k + i\hbar \partial_{y_k})^2 \right] = \frac{i r^2}{2\hbar} (\Xi_k + i\hbar \partial_{y_k}) \vee |V(y), \Xi_k + i\hbar \partial_{y_k}|
\]

\[
= -\frac{i r^2}{2\hbar} (\Xi_k + i\hbar \partial_{y_k}) \vee i\hbar \partial_{y_k} V = \frac{r^2}{2} (\Xi_k + i\hbar \partial_{y_k}) \vee \partial_{y_k} V.
\]

Therefore we may write

\[
\left\{ V(X), \frac{1}{2} |X - y|^2 + \frac{r^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} + i \frac{r}{\hbar} \left[ V(y), \frac{1}{2} |X - y|^2 + \frac{r^2}{2} |\Xi + i\hbar \nabla_y|^2 \right] = -r^2 \partial_X V(X) (\Xi_k + i\hbar \partial_{y_k}) + \frac{r^2}{2} (\Xi_k + i\hbar \partial_{y_k}) \vee \partial_{y_k} V(y)
\]

\[
= \frac{r^2}{2} (\partial_{y_k} V(y) - \partial_X V(X)) \vee (\Xi_k + i\hbar \partial_{y_k}).
\]

Furthermore

\[
\left\{ \frac{1}{2} |X|^2, \frac{1}{2} |X - y|^2 + \frac{r^2}{2} |\Xi + i\hbar \nabla_y|^2 \right\} + i \frac{r}{\hbar} \left[ \frac{1}{2} |y|^2, \frac{1}{2} |X - y|^2 + \frac{r^2}{2} |\Xi + i\hbar \nabla_y|^2 \right] = -\frac{r^2}{2} X_k \vee (\Xi_k + i\hbar \partial_{y_k}) + \frac{r^2}{2} (\Xi_k + i\hbar \partial_{y_k}) \vee y_k = \frac{r^2}{2} (\Xi_k + i\hbar \partial_{y_k}) \vee (X_k - y_k).
\]

Hence

\[
\langle U^*(t) \varphi, \delta_2(t) U^*(t) \varphi \rangle
\]

\[
= \left\langle U^*(t) \varphi, \frac{r^2}{2} (\partial_{y_k} V(y) - \partial_X V(X)) \vee (\Xi_k + i\hbar \partial_{y_k}) U^*(t) \varphi \right\rangle
\]

\[
+ \left\langle U^*(t) \varphi, \frac{r^2}{2} (X_k - y_k) \vee (\Xi_k + i\hbar \partial_{y_k}) U^*(t) \varphi \right\rangle
\]

\[
\leq \frac{r^2}{2} \left\langle U^*(t) \varphi, (L^2 |X - y|^2 + (\Xi_k + i\hbar \partial_{y_k})^2) U^*(t) \varphi \right\rangle
\]

\[
+ \frac{r^2}{2} \left\langle U^*(t) \varphi, (|X - y|^2 + (\Xi_k + i\hbar \partial_{y_k})^2) U^*(t) \varphi \right\rangle
\]

\[
\leq \max(2, \alpha^2(1 + L^2)) \langle U^*(t) \varphi, c_{\hbar, \alpha}(\Phi_1(x, \xi)) U^*(t) \varphi \rangle . \tag{25}
\]

**Step 3. Gronwall estimate.** Combining step 1 with inequality (25) gives

\[
\langle U^*(t) \varphi, c_{\hbar, \alpha}(\Phi_1(x, \xi)) U^*(t) \varphi \rangle
\]

\[
\leq \langle \varphi, c_{\hbar, \alpha}(x, \xi) \varphi \rangle + \max(2, \alpha^2(1 + L^2)) \int_0^t \langle U^*(s) \varphi, c_{\hbar, \alpha}(\Phi_1(s, \xi)) U^*(s) \varphi \rangle ds,
\]

which by Gronwall implies
\[ \langle U^*(t) \varphi, c_{h, \epsilon}(\Phi_t(x, \xi)) \rangle_U \leq \langle \varphi, c_{h, \epsilon}(x, \xi) \rangle e^{\max(2, c^2(1+L^2))t}. \]

By density of \( C_0^\infty(\mathbb{R}^2) \) in the domain of \( c_{h, \epsilon}(x, \xi) \) this implies (see Lemma 4.3 for more details)

\[ \langle U^*(t) e_k, c_{h, \epsilon}(\Phi_t(x, \xi)) \rangle_U \leq \langle e_k, c_{h, \epsilon}(x, \xi) \rangle e^{\max(2, c^2(1+L^2))t}. \]

Multiplying both sides by \( \mu_k \geq 0 \) and summing over \( k \) yields (for a.e. \( (x, \xi) \))

\[ \sum_{k=1}^\infty \mu_k \langle U^*(t) e_k, c_{h, \epsilon}(\Phi_t(x, \xi)) \rangle_U \leq \sum_{k=1}^\infty \mu_k \langle e_k, c_{h, \epsilon}(x, \xi) \rangle e^{\max(2, c^2(1+L^2))t}. \]

Integrating both sides on \( \mathbb{R}^2 \times \mathbb{R}^2 \) produces

\[ E_{h, \epsilon}(f_\epsilon(t), R_{h, \epsilon}(t))^2 \leq E_{h, \epsilon}(t) \leq E_{h, \epsilon}(0)e^{\max(2, c^2(1+L^2))t}. \]

Minimizing the right hand side of the above inequality as \( Q^{in} \in C(J^{in}, R^{in}_{h, \epsilon}) \cap D^2(\mathcal{F}) \) yields

\[ E_{h, \epsilon}(f_\epsilon(t), R_{h, \epsilon}(t))^2 \leq e^{\max(2, c^2(1+L^2))t} E_{h, \epsilon}(f_\epsilon^{in}, R_{h, \epsilon}^{in})^2, \]

as claimed.

\[ \square \]

The following lemmata elaborate on the approximation procedure used in the proof of Theorems 2.6 and 2.7.

With the same notation as before we have

**Lemma 4.2.** Let \( \varphi \in C_0^\infty(\mathbb{R}^d) \). The map \( t \mapsto \langle U^*(t) \varphi, c_{h, \epsilon}(\Phi_t(x, \xi))U^*(t) \varphi \rangle \) belongs to \( \text{Lip}([0, \tau]) \).

**Proof.** Set

\[ c^1(t, x, \xi) := \lambda |X(t) - y| \]

and

\[ c^2(t, x, \xi) := \Xi(t) + i\hbar \nabla y. \]

Then

\[ \left| \langle U^*(t) \varphi, c_{h, \epsilon}^\lambda(\Phi_t(x, \xi))U^*(t) \varphi \rangle - \langle U^*(s) \varphi, c_{h, \epsilon}^\lambda(\Phi_s(x, \xi))U^*(s) \varphi \rangle \right| \]

\[ \leq \frac{1}{2} \left| \langle c^1(t, x, \xi)U^*(t) \varphi, c^1(t, x, \xi)U^*(t) \varphi \rangle - \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(s, x, \xi)U^*(s) \varphi \rangle \right| \]

\[ + \frac{1}{2} \left| \langle c^2(t, x, \xi)U^*(t) \varphi, c^2(t, x, \xi)U^*(t) \varphi \rangle - \langle c^2(s, x, \xi)U^*(s) \varphi, c^2(s, x, \xi)U^*(s) \varphi \rangle \right|. \]

By elementary properties of semigroups (see e.g. Section 7.4.1 in [12]) there is a constant \( M > 0 \) such that

\[ \left| \langle c^1(t, x, \xi)U^*(t) \varphi, c^1(t, x, \xi)U^*(t) \varphi \rangle - \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(s, x, \xi)U^*(s) \varphi \rangle \right| \]

\[ \leq \left| \langle c^1(t, x, \xi)U^*(t) \varphi, c^1(t, x, \xi)U^*(t) \varphi \rangle - \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(t, x, \xi)U^*(s) \varphi \rangle \right| \]

\[ + \left| \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(t, x, \xi)U^*(t) \varphi \rangle - \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(t, x, \xi)U^*(s) \varphi \rangle \right| \]

\[ + \left| \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(t, x, \xi)U^*(t) \varphi \rangle - \langle c^1(s, x, \xi)U^*(s) \varphi, c^1(t, x, \xi)U^*(s) \varphi \rangle \right| \]
By Corollary 7.5 in Section 7 we have

$$D_{\mathcal{H}} = D(\mathcal{O})$$

(recall the notation $\mathcal{O} := -\frac{\hbar^2}{2} \Delta + \frac{1}{2} |y|^2$) so that

$$U^*(t) \varphi \in C^1([0, \tau]; D(\mathcal{O})).$$

In particular, the mixed norms $\|c^1(t, x, \xi)U^*(t)\varphi\|_{L^\infty([0, \tau]; \mathcal{H})}$, $\|c^2(t, x, \xi)U^*(t)\varphi\|_{L^\infty([0, \tau]; \mathcal{H})}$ are finite, which establishes the statement.

$$\Box$$

**Lemma 4.3.** Fix $k \in \mathbb{N}$ and $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Let $\{\varphi_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \to e_k$ in $D(\mathcal{H})$. Then

$$\langle U^*(t) \varphi_n, c^1_n(x, \xi)U^*(t)\varphi_n \rangle \to_{n \to \infty} \langle U^*(t)e_k, c^1_n(x, \xi)U^*(t)e_k \rangle.$$  

**Proof.** One has

$$\|\langle U^*(t)\varphi_n, c^1_n(x, \xi)U^*(t)\varphi_n \rangle - \langle U^*(t)e_k, c^1_n(x, \xi)U^*(t)e_k \rangle\|$$

$$\leq \|\langle U^*(t)\varphi_n, c^1_n(x, \xi)U^*(t)(\varphi_n - e_k) \rangle\|$$

$$+ \|\langle U^*(t)(\varphi_n - e_k), c^1_n(x, \xi)U^*(t)e_k \rangle\|.$$  

First

$$\|\langle U^*(t)\varphi_n, c^1_n(x, \xi)U^*(t)(\varphi_n - e_k) \rangle\|^2$$

$$\leq \langle U^*(t)\varphi_n, c^1_n(x, \xi)U^*(t)\varphi_n \rangle \langle U^*(t)(\varphi_n - e_k), c^1_n(x, \xi)U^*(t)(\varphi_n - e_k) \rangle$$

$$\leq \langle U^*(t)\varphi_n, (\lambda^2 |x|^2 + |\xi|^2 + \lambda^2 |y|^2 - \hbar^2 \Delta_y)U^*(t)\varphi_n \rangle \times$$

$$\langle U^*(t)(\varphi_n - e_k), (\lambda^2 |x|^2 + |\xi|^2 + \lambda^2 |y|^2 - \hbar^2 \Delta_y)U^*(t)(\varphi_n - e_k) \rangle.$$  

(26)

By Corollary 7.5 in Section 7 we have

$$\|\langle U^*(t)(\varphi_n - e_k), (\lambda^2 |x|^2 + |\xi|^2 + \lambda^2 |y|^2 - \hbar^2 \Delta_y)U^*(t)(\varphi_n - e_k) \rangle\|$$

$$\leq (1 + \lambda^2 |x|^2 + |\xi|^2) \|\varphi_n - e_k\|_2^2 + C(\lambda) \|\mathcal{H}_0U^*(t)(\varphi_n - e_k)\|_2^2$$

$$= (1 + \lambda^2 |x|^2 + |\xi|^2) \|\varphi_n - e_k\|_2^2 + C(\lambda) \|(\mathcal{H} - V)U^*(t)(\varphi_n - e_k)\|_2^2$$

$$\leq (1 + \lambda^2 |x|^2 + |\xi|^2 + 2 \|V\|_\infty^2 + 2C(\lambda)) \|\varphi_n - e_k\|_2^2 + \|\mathcal{H}(\varphi_n - e_k)\|_2^2.$$
which shows that the first factor in the right hand of (26) is bounded (uniformly in \( R \)). If we insert a parameter \( As \) will become clear in Section 6, it will be of use to consider a cost function with a weighted classical part. Namely,

\[
\text{5 Monge-Kantorovich Convergence}
\]

Similarly, we have

\[\| \nabla f \|_2^2 \leq (1 + \lambda^2 |x|^2 + |\xi|^2 + 2 \| V \|_\infty^2 + 2C(\lambda)) \| \varphi_n - e_k \|^2 + \| \mathcal{K} \| \| \varphi_n \|^2\]

which shows that the first factor in the right hand of (26) is bounded (uniformly in \( n \)) and so the right hand side is

\[\lesssim (1 + \lambda^2 |x|^2 + |\xi|^2 + 2 \| V \|_\infty^2 + 2C(\lambda)) \| \varphi_n - e_k \|^2 + \| \mathcal{K} (\varphi_n - e_k) \|_2^2 \to 0.\]

\[\Box\]

5 Monge-Kantorovich Convergence

As will become clear in Section 6, it will be of use to consider a cost function with a weighted classical part. Namely, we insert a parameter \( \lambda > 0 \) in the cost function in order to optimize some constants which will show up in Section 6. Define

\[c_\lambda^\rho(x, \xi) := \frac{1}{2} \lambda^2 |x - y|^2 + \frac{1}{2} |\xi + i\hbar \nabla y|^2.
\]

Accordingly, define

\[E^\lambda_h(\rho, R) := \inf_{Q \in \mathcal{C}(\rho, R) \cap \mathcal{D}^2(S)} \left( \int \text{trace}(\sqrt{Q(x, \xi)} c_\lambda^\rho(x, \xi) \sqrt{Q(x, \xi)}) dx d\xi \right)^{\frac{1}{2}}.
\]

In the following lemma we gather lower and upper bounds on \( E^\lambda_h(\rho, R_h) \) in terms of the Monge-Kantorovich distance.

**Lemma 5.1.** (Proposition 2.3 in [15] and Section 3 in [14]) Let \( f \) be a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) and let \( \mu \) be a Borel probability measure on \( \mathbb{R}^d \times \mathbb{R}^d \) with

\[\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x|^2 + |\xi|^2 \right) \mu(x, \xi) dx d\xi < \infty.
\]

1. If \( R_h = \text{OP}_h^{\mathcal{T}}((2\pi \hbar)^d \mu) \) then \( R_h \in \mathcal{D}^2(S) \) and one has the following bounds

\[E^\lambda_h(f, R_h)^2 \leq \text{max}(1, \lambda^2) \text{dist}_{MK,2}(f, \mu)^2 + \frac{1}{4}(\lambda^2 + 1) \hbar.
\]

2. If \( R \in \mathcal{D}^2(S) \) then

\[E^\lambda_h(f, R)^2 \geq \text{dist}_{MK,2}(f, \widetilde{W}_h[R])^2 - \frac{1}{4}(\lambda^2 + 1) \hbar.
\]

**Proof of Theorem 2.4** By Theorem 3.4, Lemma 5.1 and Lemma 3.1 we have

\[\text{dist}_{MK,2}(f(t), \widetilde{W}_h[R(t)])^2 - \frac{1}{4}(1 + \lambda^2) \hbar \leq E^\lambda_h(f(t), R(t))^2
\]

\[\leq \beta(K, \lambda) c^{\alpha(L, K', \lambda, \rho_0) t} E_h(f^\infty, R^\infty_h)^2
\]

\[\leq \beta(K, \lambda) c^{\alpha(L, K', \lambda, \rho_0) t} \left( \text{max}(1, \lambda^2) \text{dist}_{MK,2}(f^\infty, \mu)^2 + \frac{1}{4}(\lambda^2 + 1) \hbar \right),
\]

which for \( \lambda = 1 \) gives

\[\text{dist}_{MK,2}(f(t), \widetilde{W}_h[R(t)])^2 - \frac{1}{2} \hbar \leq \beta(K) c^{\alpha(L, K', \rho_0) t} \left( \text{dist}_{MK,2}(f^\infty, \mu)^2 + \frac{1}{2} \hbar \right).
\]
Proof of Theorem 2.6. By Theorem 4.1, Lemma 5.1, and Lemma 3.4, we have
\[
\begin{align*}
\text{dist}_{MK,2}(f_e(t), \text{W}_h[R(t)])^2 - \frac{1}{2}(1 + \epsilon^2)\hbar &\leq E_{h,e}(f_e(t), R_{e,h}(t))^2 \\
&\leq e^{\max(2, \epsilon^2(1+L^2))t} E_{h,e}(f_e^{\text{fin}}, R_{e,h}^{\text{fin}})^2 \\
&\leq e^{\max(2, \epsilon^2(1+L^2))t} \left( \text{dist}_{MK,2}(f_e^{\text{fin}}, \mu)^2 + \left( \frac{1 + \epsilon^2}{2} \right) \hbar \right).
\end{align*}
\]
\hfill \square

6 Observation Inequality

As an application of the method demonstrated in Section 5, we can investigate the problem of observation type inequalities for the von Neumann equation. Following the tradition, let us introduce the notion of an observation inequality for the linear Schrödinger equation, whose Cauchy problem is
\[
\begin{align*}
i\hbar \partial_t \psi &= -\frac{1}{2}\hbar^2 \Delta \psi + V(y)\psi, \quad \psi(0, x) = \psi^{\text{in}}. \tag{28}
\end{align*}
\]
As before we assume that \( V \in C^{1,1}(\mathbb{R}^d) \). An observation inequality for equation (28) is an inequality of the form
\[
\|\psi^{\text{in}}\|_2^2 \leq C_{\text{OBS}} \int_0^T \int_\Omega |\psi(t, x)|^2 dx dt,
\]
for some \( T > 0, \Omega \subseteq \mathbb{R}^d \) an open set, \( C_{\text{OBS}} > 0 \) some constant and all initial data \( \psi^{\text{in}} \) which satisfies some constraints related to \( \Omega \). Note that the conservation of the \( L^2 \) norm forces \( C_{\text{OBS}} \geq \frac{1}{\pi} \) (for \( \|\psi^{\text{in}}\| \neq 0 \)). Observability inequalities were first introduced in [21], as a dual notion of controllability. A more modern exposition on the subject can be found in [21] in the context of the linear and nonlinear Schrödinger equation. We now introduce a geometric condition due to Bardos-Lebeau-Rauch [4], which is known to imply observability for equation (28). Recall that the assumption \( V \in C^{1,1}(\mathbb{R}^d) \) implies that Newton’s system of ODEs (17) has a unique flow \((X(t), \Xi(t))\), and we denote \( \Phi_t(x, \xi) = (X(t, x, \xi), \Xi(t, x, \xi)) \).

Definition 6.1. Let \( K \subset \mathbb{R}^d \times \mathbb{R}^d \) be compact, \( \Omega \subset \mathbb{R}^d \) open and \( T > 0 \). The triplet \((K, \Omega, T)\) is said to satisfy the \textit{Bardos-Lebeau-Rauch geometric condition} (henceforth \( \text{GC} \)) if for each \( (x, \xi) \in K \) there is some \( t = t(x, \xi) \in (0, T) \) such that \( X(t, x, \xi) \in \Omega \).

Denote by \( \Omega_\delta \) the \( \delta \)-neighborhood of \( \Omega \), i.e.
\[
\Omega_\delta := \left\{ x \in \mathbb{R}^d \mid \text{dist}(x, \Omega) < \delta \right\}.
\]
Adapting the methods of [15] (especially Theorem 4.1), we formulate and prove a quantitative observation inequality for the von Neumann equation (2), by utilizing the propagation estimate of Section 5 as well as the optimal transport theory of Section 3. The proof of the forthcoming theorem is almost identical to the proof in [15], and is included for the purpose of clarifying the link between Theorem (2.4) and observability, which is not obvious—indeed the optimal transport approach demonstrated in [15] (which is the same approach used here) is not a standard tool in some of the earlier literature on the subject.

Theorem 6.2. Let \( V \) satisfy (4). Let \( K \subset \mathbb{R}^d \times \mathbb{R}^d \) be compact, \( \Omega \subset \mathbb{R}^d \) open and \( T > 0 \). Suppose that the triplet \((K, \Omega, T)\) satisfies \( \text{GC} \). Let \( \chi \in \text{Lip}(\mathbb{R}^d) \) with \( \chi(x) > 0 \) for all \( x \in \Omega \). For each \( t \geq 0 \) set
\[
R(t) = U^*(t)R^{\text{fin}}U(t), \quad f(t, X, \Xi) = f^{\text{fin}}(\Phi_{-t}(X, \Xi)),
\]
where \( R_{\text{in}} = \text{OP}_{\hbar}^{T}\left((2\pi \hbar)^d f^{\text{in}}\right) \) and \( f^{\text{in}} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) is supported on \( K \) with \( \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x|^2 + |\xi|^2\right) f^{\text{in}}(x,\xi) dx d\xi < \infty \). Then for all \( \lambda > 0 \)

\[
\int_0^T \text{trace} \left( \chi R(t) \right) dt \geq \inf_{(x,\xi) \in K} \int_0^T \chi(X(t, x, \xi)) dt - \frac{\operatorname{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} \sqrt{(\lambda^2 + 1) \hbar} \frac{2}{\alpha(L, K', \lambda, \rho, 0)} \left(e^{a L K^0 K, \lambda, \rho, 0} - 1\right).
\]

Consequently,

\[
\int_0^T \text{trace} \left(1_{\Omega_{t}} R(t) \right) dt \geq C_{\text{OBS}} = C_{\text{OBS}} \text{trace}(R_{\text{in}})
\]

for all \( \delta > c \), where

\[
C_{\text{OBS}} = C_{\text{OBS}}(K, \Omega, T, \delta, h, K, K', L, \rho_0) > 0
\]

and

\[
c = c(K, \Omega, T, L, h, K, K', L, \rho_0) > 0
\]

admit explicit formulas.

**Proof. Step 1.** We compute that

\[
\text{trace} \left( \chi R(t) \right) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x)f(t, x, \xi) dx d\xi
\]

\[
= \text{trace} \left( \chi \int \int_{\mathbb{R}^d \times \mathbb{R}^d} Q(t, x, \xi) dx d\xi\right) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) \text{trace} \left(Q(t, x, \xi)\right) dx d\xi
\]

\[
= \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left((\chi - \chi(x)) Q(t, x, \xi)\right) dx d\xi.
\]

Therefore

\[
\left|\text{trace} \left( \chi R(t) \right) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x)f(t, x, \xi) dx d\xi\right|
\]

\[
\leq \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left(\sqrt{Q(t, x, \xi)} |\chi - \chi(x)| \sqrt{Q(t, x, \xi)}\right) dx d\xi
\]

\[
\leq \operatorname{Lip}(\chi) \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left(\sqrt{Q(t, x, \xi)} |x - y| \sqrt{Q(t, x, \xi)}\right) dx d\xi
\]

\[
\leq \operatorname{Lip}(\chi) \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left(\sqrt{Q(t, x, \xi)} \left(\eta |x - y|^2 + \frac{1}{\eta}\right) \sqrt{Q(t, x, \xi)}\right) dx d\xi
\]

\[
= \operatorname{Lip}(\chi) \left(\eta \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left(\sqrt{Q(t, x, \xi)} |x - y|^2 \sqrt{Q(t, x, \xi)}\right) dx d\xi + \frac{1}{\eta}\right).
\]
Now we minimize the right hand side with respect to \( \eta \) by taking
\[
\eta := \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, \xi)} |x - y|^2 \sqrt{Q(t, x, \xi)} \right) dx d\xi \right)^{\frac{1}{2}},
\]
which produces
\[
\left| \text{trace} \left( \chi_R(t) \right) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dx d\xi \right| \leq 2\text{Lip}(\chi) \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace} \left( \sqrt{Q(t, x, \xi)} |x - y|^2 \sqrt{Q(t, x, \xi)} \right) dx d\xi \right)^{\frac{1}{2}}
\leq \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \sqrt{\mathcal{E}_h^\chi(t)} \sqrt{\beta(K, \lambda)} e^{\frac{\alpha(L, K', \lambda, \rho_0)}{2}} \sqrt{\mathcal{E}_h^\chi(0)} \right)^{\frac{1}{2}}
\leq \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \sqrt{\mathcal{E}_h^\chi(t)} \leq \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} e^{\frac{\alpha(L, K', \lambda, \rho_0)}{2}} \sqrt{\mathcal{E}_h^\chi(0)},
\]
where the last inequality is due to step 3 in Section 3. Minimizing inequality (29) on all \( Q^{in} \in C(f^{in}, R^{in}) \) yields
\[
\left| \text{trace} \left( \chi_R(t) \right) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dx d\xi \right| \leq \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} e^{\frac{\alpha(L, K', \lambda, \rho_0)}{2}} \mathcal{E}_h^\chi(f^{in}, R^{in}).
\]

**Step 2.** We have
\[
\int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dx d\xi = \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f^{in}(\Phi_{-t}(x, \xi)) dx d\xi
\]
\[
= \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi(X(t, x, \xi)) f^{in}(x, \xi) dx d\xi.
\]
Therefore inequality (30) implies
\[
\int_0^T \text{trace}(\chi R(t)) dt \geq \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_0^T \chi(X(t, x, \xi)) dt \right) f^{in}(x, \xi) dx d\xi - \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} e^{\frac{\alpha(L, K', \lambda, \rho_0)}{2}} \mathcal{E}_h^\chi(f^{in}, R^{in}) \int_0^T e^{\frac{\alpha(L, K', \lambda, \rho_0) t}{2}} dt
\]
\[
= \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_0^T \chi(X(t, x, \xi)) dt \right) f^{in}(x, \xi) dx d\xi - \frac{2\sqrt{2} \text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} e^{\frac{\alpha(L, K', \lambda, \rho_0)}{2}} \mathcal{E}_h^\chi(f^{in}, R^{in}) \frac{2(e^{\frac{\alpha(L, K', \lambda, \rho_0) T}{2}} - 1)}{\alpha(L, K', \lambda, \rho_0)}
\]
\[ \geq \inf_{(x, \xi) \in K} \int_0^T \chi(X(t, x, \xi)) dt - \frac{2\sqrt{2}\text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} \mathbb{E}_h^\lambda(f^{\text{in}}, R^n)^2(e^{\frac{\alpha(L, K', \lambda, \rho_0)T}{\lambda^2}} - 1) \]

\[ \geq \inf_{(x, \xi) \in K} \int_0^T \chi(X(t, x, \xi)) dt - \frac{2\sqrt{2}\text{Lip}(\chi)}{\lambda} \sqrt{\beta(K, \lambda)} \sqrt{(\lambda^2 + 1)} dt \frac{2(e^{\frac{\alpha(L, K', \lambda, \rho_0)T}{\lambda^2}} - 1)}{\alpha(L, K', \lambda, \rho_0)} \] (31)

where the last inequality is due to Lemma \ref{lem:5.1}.

**Step 3.** We explain how to remove the cutoff \( \chi \) in step 2, which is the second statement of Theorem \ref{thm:6.2}.

Since \( \Omega \) is open the indicator \( 1_\Omega \) is lower semicontinuous. By condition (GC), for each \( (x, \xi) \in K \) there is some \( t(x, \xi) \in (0, T) \) such that \( 1_\Omega(X(t(x, \xi), x, \xi)) = 1 \). For each \( (x, \xi) \in K \) consider the set

\[ S(x, \xi) := \left\{ t \in (0, T) \mid 1_\Omega(X(t, x, \xi)) > \frac{1}{2} \right\}. \]

Evidently \( S(x, \xi) \) is open. Therefore there is some \( \tau(x, \xi) \) such that \( [t(x, \xi) - \tau(x, \xi), t(x, \xi) + \tau(x, \xi)] \subset S(x, \xi) \) which means

\[ \int_0^T 1_\Omega(X(t, x, \xi), x, \xi)) dt \geq \int_{[t(x, \xi) - \tau(x, \xi), t(x, \xi) + \tau(x, \xi)]} 1_\Omega(X(t(x, \xi), x, \xi)) dt \geq \tau(x, \xi) > 0. \] (32)

Furthermore, by Fatou’s lemma the function

\[ (x, \xi) \mapsto \int_0^T 1_\Omega(X(t(x, \xi), x, \xi)) dt \]

is lower semicontinuous on \( K \), and positive on \( K \) because of \ref{lem:3.1}. As \( K \) is compact, set

\[ C(T, K, \Omega) := \inf_{(x, \xi) \in K} \int_0^T 1_\Omega(X(t(x, \xi), x, \xi)) dt > 0, \]

and put

\[ \chi_\delta(x) = \left(1 - \frac{\text{dist}(x, \Omega)}{\delta} \right)_+. \]

Note that \( \chi_\delta \in \text{Lip}(\mathbb{R}^d) \) and \( \text{Lip}(\chi_\delta) = \frac{1}{\delta} \). Clearly \( 1_\Omega \leq \chi_\delta \leq 1_{\Omega_\delta} \) and consequently

\[ \int_0^T \text{trace} (\chi_\delta R(t)) dt \leq \int_0^T \text{trace} (1_{\Omega_\delta} R(t)) dt. \]

Thus, in view of inequality \ref{eq:31} applied for \( \chi_\delta \) we get

\[ \int_0^T \text{trace} (1_{\Omega_\delta} R(t)) dt \geq C(T, K, \Omega) - \frac{\sqrt{2}}{\delta^2} \sqrt{\beta(K, \lambda)} \sqrt{(\lambda^2 + 1)} dt \frac{2(e^{\frac{\alpha(L, K', \lambda, \rho_0)T}{\lambda^2}} - 1)}{\alpha(L, K', \lambda, \rho_0)}. \]

In order to maximize the right hand side of the last inequality put

\[ C^*(T, K, K', L, d, h) := \inf_{\lambda > 0} \frac{1}{\lambda} \sqrt{\beta(K, \lambda)} \sqrt{(\lambda^2 + 1)} dt \frac{2(e^{\frac{\alpha(L, K', \lambda, \rho_0)T}{\lambda^2}} - 1)}{\alpha(L, K', \lambda, \rho_0)}. \]

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If we pick $\delta > 0$ so that

$$C_{OBS} := C(T, K, \Omega) - \frac{1}{\delta} C^*(T, K', L, d, \hbar) > 0,$$

then we finally get

$$\int_0^T \text{trace } (1_{\Omega_t} R(t)) \, dt \geq C_{OBS} \text{trace}(R^\text{in}).$$

\[\square\]

7 On the Domain of the Magnetic/Non-Magnetic Harmonic Oscillator

Roughly speaking, Section 3 and Section 4 focused on how the presence of a magnetic field influences the formal calculations leading to the evolution inequality for $E_{\hbar}$. Interestingly, the presence of a magnetic field also influences the spectral theory which is required in order to put these formal calculations on rigorous grounds. In the proof of both of the main Theorems 2.4 and 2.6 we relied on the fact that the domain of the magnetic harmonic oscillator identifies with the domain of the (non-magnetic) harmonic oscillator. This is quite apparent in the case where $A$ is bounded, but more subtle for sublinear $A$, which is the case of interest. The purpose of this section is to review the relevant literature, as well as state some bounds which are presumably not new, nevertheless do not appear explicitly enough in the literature. We start by elaborating on the essential self-adjointness of the cost function. Recall the notations

$$\mathcal{H}_0 := \frac{1}{2} | -i\hbar \nabla_x + A(y) |^2 + \frac{1}{2} | y |^2$$

and

$$K := \frac{1}{2} | -i\hbar \nabla_y + A(y) |^2$$

and

$$\Pi_k := -i\hbar \partial_k + A_k(y).$$

The following theorem is a restatement of Theorem 1.1 in [27] for the specific settings of interest

**Theorem 7.1.** Let $A \in \text{Lip}_{\text{loc}}(\mathbb{R}^d)$ and $W \in L^2_{\text{loc}}(\mathbb{R}^d)$ with $W \geq 0$. Then $\frac{1}{2} | -i\hbar \nabla_y + A(y) |^2 + \frac{1}{2} W(y)$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}^d)$. The domain of $\frac{1}{2} | -i\hbar \nabla_y + A(y) |^2 + \frac{1}{2} W(y)$ is given by

$$D = \left\{ \varphi \in L^2(\mathbb{R}^d) \left| \frac{1}{2} | -i\hbar \nabla_y + A(y) |^2 + \frac{1}{2} W(y) \in L^2(\mathbb{R}^d) \right. \right\}.$$

An immediate consequence is that both operators

$$\tilde{c}_k^\lambda(x, \xi) = \frac{\lambda^2}{2} | x - y |^2 + \frac{1}{2} | \xi + A(x) - A(y) + i\hbar \nabla_y |^2$$

and

$$c_k^\lambda(x, \xi) = \frac{\lambda^2}{2} | x - y |^2 + \frac{1}{2} | \xi + i\hbar \nabla_y |^2$$

are essentially self-adjoint on $C^\infty_0(\mathbb{R}^d)$ with domains

$$D (\tilde{c}_k^\lambda(x, \xi)) = \left\{ \varphi \in \mathcal{S} | \tilde{c}_k^\lambda(x, \xi) \varphi \in \mathcal{S} \right\}$$

and

$$D (c_k^\lambda(x, \xi)) = \left\{ \varphi \in \mathcal{S} | c_k^\lambda(x, \xi) \varphi \in \mathcal{S} \right\}.$$
respectively. In addition, it is an exercise to check (see e.g. Footnote 3 in [7]) that the domain of the harmonic oscillator \( O := -\frac{\hbar^2}{2} \Delta + \frac{1}{2} |y|^2 \) is characterized as
\[
D(O) = H^2(\mathbb{R}^d) \cap \{ \varphi \in \mathcal{S} \mid |y|^2 \varphi \in \mathcal{S} \}.
\]
The characterization of \( D(\mathcal{H}_0) \) is a somewhat more challenging task. We proceed by explaining how to compare \( D(\mathcal{H}_0) \) and \( D(O) \), eventually we wish to show they are the same. Let us confine ourselves to the case \( d \geq 3 \) (see Remark 7.7 for the case \( d = 2 \)). We start with

**Lemma 7.2.** Let \( A \) satisfy (A') with constant \( K \). For each \( \varphi \in C_0^\infty(\mathbb{R}^d) \) one has the following estimates
\[
\int_{\mathbb{R}^d} |x|^2 |\varphi|^2 (x) dx \leq \| \mathcal{H}_0 \varphi \|_2^2 + \| \varphi \|_2^2
\]
and
\[
\int_{\mathbb{R}^d} \hbar^2 |\nabla \varphi|^2 (x) dx \leq \max(4, K^2) \left( \| \mathcal{H}_0 \varphi \|_2^2 + \| \varphi \|_2^2 \right).
\]

**Proof.** One has
\[
\int_{\mathbb{R}^d} |x|^2 |\varphi|^2 (x) dx \leq \langle K \varphi, \varphi \rangle + \int_{\mathbb{R}^d} |x|^2 |\varphi|^2 (x) dx
\]
\[
= \int_{\mathbb{R}^d} \nabla \varphi (x) \left( K + |x|^2 \right) \varphi(x) dx
\]
\[
\leq 2 \| \mathcal{H}_0 \varphi \|_2 \| \varphi \|_2 \leq \| \mathcal{H}_0 \varphi \|_2^2 + \| \varphi \|_2^2,
\]
which is the first inequality. The second inequality is implied from inequality (33) as follows
\[
\int_{\mathbb{R}^d} \hbar^2 |\partial_k \varphi|^2 (x) dx = \|(\Pi_k - A_k(x)) \varphi \|_2^2
\]
\[
\leq 2 \| \Pi_k \varphi \|_2^2 + K^2 \int_{\mathbb{R}^d} |x|^2 |\varphi|^2 (x) dx
\]
\[
= 4 \int_{\mathbb{R}^d} \nabla \varphi (x) \Pi_k \varphi(x) dx + K^2 \int_{\mathbb{R}^d} |x|^2 |\varphi|^2 (x) dx
\]
\[
\leq \max(8, 2K^2) \int_{\mathbb{R}^d} \nabla \varphi (x) \left( \frac{1}{2} \Pi_k^2 + \frac{1}{2} |x|^2 \right) \varphi(x) dx
\]
\[
\leq \max(4, K^2) \left( \| \mathcal{H}_0 \varphi \|_2^2 + \| \varphi \|_2^2 \right),
\]
which is the second inequality.

The following “magnetic maximal inequality” is a specification of Theorems 2.10 and 4.1 in [17] to the scenario considered in the present work.
Lemma 7.3. Let $A$ satisfy $(h')$. There is a constant $C > 0$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ it holds that
\[
||K\varphi||_2^2 \leq C \left( ||\mathcal{H}_0\varphi||_2^2 + ||\varphi||_2^2 \right)
\]
and
\[
|||y|^2 \varphi||_2^2 \leq C \left( ||\mathcal{H}_0\varphi||_2^2 + ||\varphi||_2^2 \right).
\]

Next we obtain an estimate for a weighted $L^2$ norm of the gradient in terms of the norm attached to $\mathcal{H}_0$.

Lemma 7.4. Let $A$ satisfy $(h')$. There is a constant $C > 0$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ it holds that
\[
\int_{\mathbb{R}^d} h^2 |x|^2 |\nabla \varphi|^2 (x) dx \leq C \left( ||\mathcal{H}_0\varphi||_2^2 + ||\varphi||_2^2 \right).
\]

Proof. We manipulate the integral
\[
\int_{\mathbb{R}^d} |x|^2 \overline{\varphi}(x) \mathcal{H}_0 \varphi(x) dx
\]
as follows:
\[
\int_{\mathbb{R}^d} |x|^2 \overline{\varphi}(x) \mathcal{H}_0 \varphi(x) dx = \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \overline{\varphi}(x) \Pi_k^2 \varphi(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2 (x) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} \Pi_k \left( |x|^2 \varphi \right) (x) \Pi_k \varphi(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2 (x) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} h^2 |x|^2 |\partial_{x_k} \varphi|^2 (x) dx + \frac{1}{2} \int_{\mathbb{R}^d} i h\partial_{x_k} \overline{\varphi} A_k(x) \varphi(x) dx + \int_{\mathbb{R}^d} (i h x_k \overline{\varphi})(-i h \partial_{x_k} \varphi)(x) dx
\]
\[
+ \int_{\mathbb{R}^d} i h x_k |\varphi|^2 (x) A_k(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |A_k|^2 |x|^2 |\varphi|^2 (x) dx
\]
\[
- \frac{1}{2} A_k(x) |x|^2 \overline{\varphi}(x) h \partial_{x_k} \varphi(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2 (x) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} h^2 |x|^2 |\partial_{x_k} \varphi|^2 (x) dx + \Re \left( \int_{\mathbb{R}^d} |x|^2 i h \partial_{x_k} \overline{\varphi} A_k(x) \varphi(x) dx \right) + \int_{\mathbb{R}^d} h^2 x_k \overline{\varphi}(x) h \partial_{x_k} \varphi(x) dx
\]
\[
+ \int_{\mathbb{R}^d} i h x_k |\varphi|^2 (x) A_k(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |A_k|^2 |x|^2 |\varphi|^2 (x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2 (x) dx
\]
We equate the real part of the left hand side with the real part of the right hand side in order to find
\[
\Re \left( \int_{\mathbb{R}^d} |x|^2 \overline{\varphi}(x) \mathcal{H}_0 \varphi(x) dx \right)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} h^2 |x|^2 |\partial_{x_k} \varphi|^2 (x) dx + \Re \left( \int_{\mathbb{R}^d} |x|^2 i h \partial_{x_k} \overline{\varphi} A_k(x) \varphi(x) dx \right) + \Re \left( \int_{\mathbb{R}^d} h^2 x_k \overline{\varphi}(x) h \partial_{x_k} \varphi(x) dx \right)
\]

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+ \frac{1}{2} \int_{\mathbb{R}^d} |A_k|^2 |x|^2 \varphi^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2(x) dx.  \tag{34}

By Young’s inequality and the assumption on $A$ we can bound from below the second and third terms in the right hand side of equation (34):

$$
\Re \left( \int_{\mathbb{R}^d} |x|^2 i\hbar \partial_{x_k} \overline{\varphi} A_k(x) \varphi(x) dx \right)
\geq \frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx - \int_{\mathbb{R}^d} |x|^2 |A_k(x)|^2 |\varphi|^2(x) dx
\geq -\frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx,
$$

and

$$
\Re \left( \int_{\mathbb{R}^d} \hbar^2 x_k \partial_{x_k} \varphi(x) dx \right)
\geq -\frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx. \tag{36}
$$

Inequalities (35) and (36) together with identity (34) imply the following inequality

$$
\frac{1}{2} \left\| |x|^2 \varphi \right\|_2^2 + \frac{1}{2} \left\| \mathcal{H}_0 \varphi \right\|_2^2
\geq \frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx
- \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx
+ \frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |A_k|^2 |x|^2 |\varphi|^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} |x|^4 |\varphi|^2(x) dx
\geq \frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx
- \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\varphi|^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx,
$$

which is recast as

$$
\frac{1}{4} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx \leq \frac{1}{2} \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left( \frac{1}{2} + K^2 \right) \int_{\mathbb{R}^d} |x|^4 |\varphi|^2(x) dx
+ \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |x|^2 |\varphi|^2(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} \hbar^2 |\partial_{x_k} \varphi|^2(x) dx.
$$

Invoking Lemma [7.2] and Lemma [7.2] we get

$$
\int_{\mathbb{R}^d} \hbar^2 |x|^2 |\partial_{x_k} \varphi|^2(x) dx
$$
\[
\begin{align*}
&\leq \frac{1}{2} \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left( \frac{1}{2} + K^2 \right) C \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right) \\
&\quad + \frac{\hbar^2}{2} \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right) + \max \left( 2, \frac{K^2}{2} \right) \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right) \\
&\quad \leq C \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right),
\end{align*}
\]
as asserted.

As a corollary we find that the norm attached to the magnetic harmonic oscillator is equivalent to the norm attached to the (non-magnetic) harmonic oscillator.

**Corollary 7.5.** There is a constant \( C > 0 \) such that for all \( \varphi \in C_0^\infty(\mathbb{R}^d) \) it holds that

\[
\frac{1}{C} \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right) \leq \left\| \mathcal{O} \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \leq C \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right).
\]

**Proof.** For each \( \varphi \in C_0^\infty(\mathbb{R}^d) \)

\[
\left\| \Pi_k \varphi \right\|_2 = \left\langle \Pi_k \varphi, \Pi_k \varphi \right\rangle \leq \frac{\left\| \varphi \right\|_2^2 + \left\| \Pi_k \varphi \right\|_2^2}{2}.
\]

Consequently (here we use again Einstein summation)

\[
\left\| \mathcal{O} \varphi \right\|_2^2 = \left\| -\frac{\hbar^2}{2} \Delta \varphi + \frac{1}{2} |y|^2 \varphi \right\|_2^2
\]

\[
= \left\| \left( \frac{1}{2}(-i\hbar \partial_{y_k} + A^k(y) - A^k(y))^2 + \frac{1}{2} |y|^2 \right) \varphi \right\|_2^2
\]

\[
= \left\| \left( \frac{1}{2} \Pi_k^2 - \frac{1}{2} \Pi_k \varphi y_k + \frac{1}{2} |A_k(y)|^2 + \frac{1}{2} |y|^2 \right) \varphi \right\|_2^2
\]

\[
\leq \left( \left\| K \varphi \right\|_2 + \frac{3K^2 + 1}{2} \left\| \left| y \right|^2 \varphi \right\|_2 + \frac{\hbar^2}{2} K \left\| \varphi \right\|_2 + \frac{1}{2} \left\| A \cdot \hbar \nabla \varphi \right\|_2 \right)^2
\]

\[
\leq C \left( \left\| \mathcal{H}_0 \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right),
\]

where both of the last inequalities follow from Lemma [23] and Lemma [24]. The lower bound for \( \left\| \mathcal{O} \varphi \right\|_2^2 \) is proved in a similar manner:

\[
\left\| \mathcal{H}_0 \varphi \right\|_2^2 = \left\| \left( \frac{1}{2} \Pi_k^2 + \frac{1}{2} |y|^2 \right) \varphi \right\|_2^2
\]

\[
\leq \left\| \left( -\hbar^2 \Delta + |y|^2 \right) \varphi \right\|_2^2 + \left\| A^k(y) \varphi \right\|_2^2 + \left\| (-i\hbar \partial_{y_k} A^k) \varphi \right\|_2^2 + \left\| A \cdot i\hbar \nabla \varphi \right\|_2^2
\]

\[
\leq \left\| \left( -\hbar^2 \Delta + |y|^2 \right) \varphi \right\|_2^2 + K^2 \left\| \left| y \right|^2 \varphi \right\|_2^2 + \hbar^2 K^2 \left\| \varphi \right\|_2^2 + 3 \int_{\mathbb{R}^3} \hbar^2 |x|^2 \left| \nabla \varphi \right|^2 (x) dx
\]

\[
\leq \left\| \left( -\hbar^2 \Delta + |y|^2 \right) \varphi \right\|_2^2 + C \left( \left\| \left( -\hbar^2 \Delta + |y|^2 \right) \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right)
\]

\[
+ \hbar^2 K^2 \left\| \varphi \right\|_2^2 + C \left( \left\| \left( -\hbar^2 \Delta + |y|^2 \right) \varphi \right\|_2^2 + \left\| \varphi \right\|_2^2 \right).
\]
Finally, in light of the above discussion, the following conclusion is immediate

**Corollary 7.6.** The following inclusions hold

\[
    D(\mathcal{E}) \subset D\left(\mathcal{E}_0(x,\xi)\right)
\]

and

\[
    D(\mathcal{H}) \subset D\left(\mathcal{H}_0(x,\xi)\right).
\]

**Remark 7.7.** According to Remark 2.11 in [17] the case \(d = 2\) does not introduce any particular difficulties, and the result remains true up to some minor modifications. However, we were are unable to locate a reference which includes a full treatment for the case \(d = 2\), and therefore we decided to formulate Theorem 2.4 in dimension \(\geq 3\), although likely this can be avoided with some more effort. Anyhow, the case of \(d = 2\) with constant magnetic field of the type \((\mathbf{A})\) – which is the case of interest for Section 4 – has been already handled in Lemma 6.6 in [6].

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