Convergent discrete Laplace-Beltrami operators over surfaces

Jyh-Yang Wu †, Mei-Hsiu Chi ‡ and Sheng-Gwo Chen ¶

Abstract

The convergence problem of the Laplace-Beltrami operators plays an essential role in the convergence analysis of the numerical simulations of some important geometric partial differential equations which involve the operator. In this note we present a new effective and convergent algorithm to compute discrete Laplace-Beltrami operators acting on functions over surfaces. We prove a convergence theorem for our discretization. To our knowledge, this is the first convergent algorithm of discrete Laplace-Beltrami operators over surfaces for functions on general surfaces. Our algorithm is conceptually simple and easy to compute. Indeed, the convergence rate of our new algorithm of discrete Laplace-Beltrami operators over surfaces is $O(r)$ where $r$ represents the size of the mesh of discretization of the surface.

Keywords: Local tangential polygon; discrete Laplace-Beltrami operators; Configuration Equation

1 Introduction

Let $\Sigma$ be a smooth surface in the 3D space. The Laplace-Beltrami (LB) operator is a natural generalization of the classical Laplacian $\Delta$ from the Euclidean space to $\Sigma$. It is well-known that the LB operator is closed related to the mean curvature normal by the relation $\Delta_{\Sigma}(p) = 2H(p)$. The LB operator plays important role not just in the study of geometric properties of $\Sigma$, but also in the investigation of physical problems, like heat flow and wave equations, on $\Sigma$. Moreover, the LB operator has recently many applications in a variety of different areas, such as surface processing [6, 13], signal processing [14, 15, 16, 18] and geometric partial differential equations [11, 11, 12]. Since the objective underlying surfaces to be considered are usually represented as discrete meshes in these applications, there are tremendous needs in practice to discretize the LB operators.

Even though the computation of the LB operators is important for many applications, there does not exist a simple "convergent" discrete approximation
of the LB operators for general surfaces. In this paper we shall present a new effective and convergent algorithm to compute discrete Laplace-Beltrami operators acting on functions over surfaces. In fact, we shall prove the following convergence theorem.

Main Theorem. Given a smooth function $h$ on a regular surface $\Sigma$ and a triangular surface mesh $S = (V, F)$ with mesh size $r$, one has

$$\Delta\Sigma h(v) = \Delta_A h(v) + O(r)$$ (1)

where the discrete LB operator $\Delta_A h(v)$ is given in Equation (2). We shall give a mathematical proof of this convergence result. To our knowledge, this is the first convergent algorithm of discrete Laplace-Beltrami operators over surfaces for functions on general surfaces. The idea of our algorithm can be divided into two parts: First, we shall introduce a notation of the local tangential polygon and lift functions and vectors on a triangular mesh, obtained from the discretization of the surface under consideration, to the local tangential polygon, and second, we shall give a new method to define the discrete Laplace-Beltrami (LB) operator acting on functions on a 2D polygon. Our algorithm is conceptually simple and easy to compute. The convergence rate of our new algorithm of discrete Laplace-Beltrami operators over surfaces is $O(r)$ where $r$ represents the size of the mesh of discretization of the surface. We also present our numerical results to support this in section 4.

2 Laplace-Beltrami operator and its discretizations

Let $\Sigma$ be a regular surface in the 3D Euclidean space $\mathbb{R}^3$. Consider a local parameterization $h : U \rightarrow \Sigma$ with $h(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)) \in \Sigma$, $(u_1, u_2) \in U \subset \mathbb{R}^2$. For the details, we refer to do Carmo [8, 9]. Then the Laplace-Beltrami operator $\Delta\Sigma$ applying to a $C^2$ function $f$ on $\Sigma$ is given by

$$\Delta\Sigma f = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u_i} \left[ g^{ij} \sqrt{g} \frac{\partial f}{\partial u_j} \right]$$ (2)

where $(g^{ij})$ is the inverse of the matrix $(g_{ij})$ with $g = \det(g_{ij})$ and

$$g_{ij} = \langle \frac{\partial h}{\partial u_i}, \frac{\partial h}{\partial u_j} \rangle .$$ (3)

Consider a triangular discretization $S = (V, F)$ of the surface $\Sigma$, where $V = \{v_i | 1 \leq i \leq n_V \}$ is the list of vertices and $F = \{T_k | 1 \leq k \leq n_F \}$ is the list of triangles. Let $v$ be a vertex in $V$ and $N(v)$ the index of one-ring neighbors of the vertex $v$. Next we recall several discretizations of $\Delta f$ for a $C^2$ function $f$ on as follows. For more discussions, see also Xu [19, 20].

2.1 Taubin’s et al. Discretization

Taubin considered in [16] the following form of discretization of $\Delta f$:

$$\Delta f(v) = \sum_{i \in N(v)} \omega_i (f(v_i) - f(v))$$ (4)
where the weights $\omega_i$ are nonnegative numbers with $\sum_{i \in N(v)} \omega_i = 1$. There are several choices for the weights $\omega_i$. An obvious choice is the uniform weights $\omega_i = \frac{1}{|N(v)|}$ where $|N(v)|$ is the cardinality of the set $N(v)$. A general way to determine the weights $\omega_i$ is to use the following formulation:

$$\omega_i = \frac{\phi(v, v_i)}{\sum_{k \in N(v)} \phi(v, v_k)}$$

(5)

with a nonnegative function $\phi(v, v_i)$. Fujiwara takes $\phi(v, v_i) = \frac{1}{\|v_i - v\|}$. Desbrun’s et al. [7] defines the weights $\omega_i$ as

$$\omega_i = \frac{\cot \alpha_i + \cot \beta_i}{\sum_{k \in N(v)} \cot \alpha_k + \cot \beta_k}$$

(6)

where $\alpha_i$ and $\beta_i$ are the triangles as shown Figure 1.

**Figure 1:** The angles $\alpha_i$ and $\beta_i$.

It is obvious that the discretization (5) of $\Delta f$ can not be a correct approximation of $\Delta f$ since it approaches zero as the size of the surface mesh goes to zero.

### 2.2 Mayer’s et al. Discretization

For a $C^2$ function $f$ on $\Sigma$, Green’s formula gives

$$\int_{D(z, \epsilon)} \Delta f(x) dx = \int_{\partial D(z, \epsilon)} \partial_n f(s) ds$$

(7)

where $D(z, \epsilon)$ is a small disk at a point $z$ on the surface $\Sigma$, and $n$ is the intrinsic outer normal of the boundary of the disk. Mayer discretized (7) at $v$ over the triangular surface mesh $S$ and obtained the following approximation

$$\Delta f(v) = \frac{1}{A(v)} \sum_{i \in N(v)} \frac{\|v_k - v_i\| + \|v_m - v_i\|}{2\|v - v_i\|} (f(v_i) - f(v))$$

(8)

where $A(v)$ is the sum of areas of triangles around $v$, and $k, m \in N(v) \cap N(v_i)$. It can be checked directly that the formula (8) is derived from (7) by approximating $\int_{D(z, \epsilon)} \Delta f(x) dx$, $\partial_n f(s)$ and $ds$ with $\Delta f(v)A(v)$, $\frac{(v_i - f(v))}{\|v_i - v\|}$ and $\frac{\|v_k - v_i\| + \|v_m - v_i\|}{2\|v - v_i\|}$, respectively. Therefore, the discretization in (8) is an approximation of $\Delta f$ at $v$. 

3
2.3 Desbrun’s et al. Discretization

It is well-known in the theory of differential geometry that the mean curvature normal satisfies the following formula:

$$\lim_{\text{diam}(A) \to 0} \frac{3\nabla A}{2A} = -H(p)$$  \hspace{1cm} (9)

where $A$ is the area of a small region around the point $p$, and $\nabla$ is the gradient with respect to the $(x, y, z)$ coordinates of $p$. From Equation (9), Desbrun et al. got the following approximation:

$$\Delta f(v) = \frac{3}{A(v)} \sum_{i \in N(v)} \frac{\cot \alpha_i + \cot \beta_i}{2} |f(v_i) - f(v)|.$$  \hspace{1cm} (10)

where $N(v)$ is the index set of 1-ring neighboring vertices of vertex $v$, $\alpha_i$ and $\beta_i$ are as in (6) and $A(v)$ is the sum of areas of triangles around $v$.

2.4 Xu’s Discretization

In 2004, Xu presented two discrete Laplace-Beltrami methods in a triangular mesh from Green’s formula and the quadratic fitting. Following Equation (7), xu introduced his discretization

$$\Delta f(v) = \frac{1}{2A(p)} \sum_{i \in N(v)} <\nabla f(p) + \nabla f(v_i), \nu_i> \|v_i - v_i+\|$$.  \hspace{1cm} (11)

where $\nabla f(v)$ is the gradient of $f$ at $v$, $A(v)$ is the sum of area of the triangles that contain $v$ and $\nu_i$ is the unit outward normal of the edge $v_i v_{i+}$. Xu also used the biquadratic fitting of the surface data and function data to calculate the approximate LB operator. He introduced complexity weights of the equation (4). This kind of weights can be found in [20].

The convergence problem of these discrete LB operators over triangular surface meshes has been investigated by Liu, Xu and Zhang in [10, 19, 20]. None of the above mentioned discretizations of the LB operators has ever been proved to be convergent for general surfaces and functions. The Desbrun et al.’s discretization (10) has been investigated under some very restricted conditions. It is shown in [10, 19, 20] that the discretization (10) converges to the LB operator under the conditions that the valence of the vertex $v$ is 6 and $v = F(q_i)$, $v_i = F(q_{i+})$ for a smooth parametric surface $F$ and the relations $q_{i+3} + q_i = 2q_i$, $i = 1, 2, 3$ hold, where $q_i$, $i = 1, 2, \cdots, 6$ are one-ring neighboring vertices of $q_i$ in the 2D domain. See [10, 19, 20] for more details.

3 A new convergent discrete algorithm for LB operators

In this section we will describe a simple and effective method to define the discrete LB operator on functions on a triangular mesh. The primary ideas were developed in Chen, Chi and Wu [4, 5] where we try to estimate discrete partial derivatives of functions on 2D scattered data points. Indeed, the method
that we use to develop our algorithm is divided into two main steps: First, we lift the 1-neighborhood points to the tangent space and obtain a local tangential polygon. Second, we use some geometric ideas to lift functions to the tangent space. We call this a local tangential lifting (LTL) method. Then we present a new algorithm to compute their Laplacians in the 2D tangent space. This means that the LTL process allows us to reduce 2D curved surface problems to 2D Euclidean problems. As one will see later, our approach of discretization is quite different from the discretizations discussed in section 2.

Consider a triangular surface mesh \( S = (V, F) \), where \( V = \{v_i | 1 \leq i \leq n_V\} \) is the list of vertices and \( F = \{f_k | 1 \leq k \leq n_F\} \) is the list of triangles.

### 3.1 The local tangential lifting (LTL) method

To describe the local tangential lifting (LTL) method, we introduce the approximating tangent plane \( T_S A(v) \) and the local tangential polygon \( P_A(v) \) at the vertex \( v \) of as follows:

1. The normal vector \( N_A(v) \) at the vertex \( v \) in \( S \) is given by
   \[
   N_A(v) = \frac{\sum_{T \in T(v)} \omega_T N_T}{\| \sum_{T \in T(v)} \omega_T N_T \|} \quad (12)
   \]
   where \( T(v) \) is the set of triangles that contain the vertex \( v \), \( N_T \) is the unit normal to a triangle face \( T \) and the centroid weight is given in \([2,3]\) by
   \[
   \omega_T = \frac{1}{\| G_T - v \|^2} \quad (13)
   \]
   where \( G_T \) is the centroid of the triangle face \( T \) determined by
   \[
   G_T = \frac{v + v_i + v_j}{3} \quad (14)
   \]
   Note that the letter \( A \) in the notation \( N_A(v) \) stands for the word "Approximation".

2. The approximating tangent plane \( T_S A(v) \) of \( S \) at \( v \) is now determined by
   \[
   T_S A(v) = \{w \in \mathbb{R}^3 | w \perp N_A(v)\}.
   \]

3. The local tangential polygon \( P_A(v) \) of \( v \) in \( T_S A(v) \) is formed by the vertices \( \bar{v}_i \) which is the lifting vertex of \( v_i \) adjacent to \( v \) in \( V \):
   \[
   \bar{v}_i = (v_i - v) - < v_i - v, N_A(v) > N_A(v) \quad (15)
   \]
   as in figure 2

4. We can choose an orthonormal basis \( e_1, e_2 \) for the tangent plane \( T_S A(v) \) of \( S \) at \( v \) and obtain an orthonormal coordinates \((x, y)\) for vectors \( w \in T_S A(v) \) by \( w = x e_1 + y e_2 \). We set \( \bar{v}_i = x_i e_1 + y_i e_2 \) with respect to the orthonormal basis \( e_1, e_2 \).
Next we explain how to lift locally a function defined on \( V \) to the local tangential polygon \( P_A(v) \). Consider a function \( h \) on \( V \). We will lift locally the function \( h \) to a function of two variables, denoted by \( \bar{h} \), on the vertices \( \bar{v}_i \) in \( P_A(v) \) by simply setting
\[
\bar{h}(x_i, y_i) = h(v_i) \quad (16)
\]
and \( \bar{h}(\vec{0}) = h(v) \) where \( \vec{0} \) is the origin of \( TS_A(v) \). Then one can extend the function \( \bar{h} \) to be a piecewise linear function on the whole polygon \( P_A(v) \) in a natural and obvious way.

### 3.2 A new discrete 2D Laplacian algorithm and configuration equation

In this section we present a new discrete 2D algorithm for Laplacians acting on functions on the 2D domains in the \( x - y \) plane. Given a \( C^2 \) function \( f \) on a domain \( \Omega \) in the \( x - y \) plane with the origin \( (0, 0) \in \Omega \), Taylor’s expansion for two variables \( x \) and \( y \) gives
\[
f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{x^2}{2} f_{xx}(0, 0) + xy f_{xy}(0, 0) + \frac{y^2}{2} f_{yy}(0, 0) + O(r^3)
\]
when \( r = x^2 + y^2 \) is small.

Consider a family of neighboring points \( (x_i, y_i) \in \Omega \), \( i = 1, 2, \ldots, n \), of the origin \( (0, 0) \). Take some constants \( \alpha_i \), \( i = 1, 2, \ldots, n \), with \( \sum_{i=1}^{n} \alpha_i^2 = 1 \). Then one has
\[
\sum_{i=1}^{n} \alpha_i (f(x_i, y_i) - f(0, 0)) = \left( \sum_{i=1}^{n} \alpha_i x_i \right) f_x(0, 0) + \left( \sum_{i=1}^{n} \alpha_i y_i \right) f_y(0, 0) + \frac{1}{2} \left( \sum_{i=1}^{n} \alpha_i x_i^2 \right) f_{xx}(0, 0) + \frac{1}{2} \left( \sum_{i=1}^{n} \alpha_i y_i^2 \right) f_{yy}(0, 0) + O(r^3)
\]
We choose the constants \( \alpha_i \), \( i = 1, 2, \ldots, n \), so that they satisfy the following equations:
\[
\sum_{i=1}^{n} \alpha_i x_i = 0, \quad \text{(I)}
\]
\[
\sum_{i=1}^{n} \alpha_i y_i = 0, \quad \text{(II)}
\]
\[
\sum_{i=1}^{n} \alpha_i x_i y_i = 0, \quad \text{(III)}
\]
and
\[
\sum_{i=1}^{n} \alpha_i x_i^2 = \sum_{i=1}^{n} \alpha_i y_i^2
\]
or equivalently
\[
\sum_{i=1}^{n} \alpha_i (x_i^2 - y_i^2) = 0, \quad \text{(IV)}
\]

One can rewrite these equations in a matrix form and obtain the following configuration equation:
\[
\begin{pmatrix}
  x_1, & x_2, & \cdots, & x_n \\
  y_1, & y_2, & \cdots, & y_n \\
  x_1 y_1, & x_2 y_2, & \cdots, & x_n y_n \\
  x_1^2 - y_1^2, & x_2^2 - y_2^2, & \cdots, & x_n^2 - y_n^2
\end{pmatrix}
\begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_n
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\quad (19)
\]

The solutions \( \alpha_i \) of this equation allow us to obtain a formula for the Laplacian \( \Delta f(0,0) \):
\[
\Delta f(0,0) = f_{xx}(0,0) + f_{yy}(0,0)
\]
\[
= \frac{2}{n} \sum_{i=1}^{n} \alpha_i (f(x_i, y_i) - f(0,0)) \]
\[
\frac{\sum_{i=1}^{n} \alpha_i x_i^2}{\sum_{i=1}^{n} \alpha_i y_i^2} + O(r) \quad (20)
\]

For the reason of symmetry, Equation (20) also gives
\[
\Delta f(0,0) = \frac{4}{n} \sum_{i=1}^{n} \alpha_i (f(x_i, y_i) - f(0,0)) \]
\[
\frac{\sum_{i=1}^{n} \alpha_i x_i^2 + y_i^2}{\sum_{i=1}^{n} \alpha_i x_i^2 + y_i^2} + O(r) \quad (21)
\]

since we have \( \sum_{i=1}^{n} \alpha_i x_i^2 = \sum_{i=1}^{n} \alpha_i y_i^2 \). It is worth to point out that Equation (21) is an generalization of the well-known 5-point Laplacian formula. In the 5-point Laplacian case, we have the origin \((0,0)\) along with 4 neighboring points \((s,0)\), \((0,s)\), \((-s,0)\) and \((0,-s)\) for sufficiently small positive number \(s\). From Equation (19), one can find a solution \( \alpha_i = \frac{1}{2} \) for \( i = 1, 2, 3, 4 \), in this case.
3.3 A new discrete approximation for LB operators over surfaces

Now we can come back to handle the local lifting function \( \tilde{h} \) and propose a new discrete approximation for the LB operator over a triangular surface mesh \( S \). From Equation \( (21) \), we can define a discrete LB operator \( \Delta_A \) for the function \( h \) at the vertex \( v \) by

\[
\Delta_A h(v) = \frac{4 \sum_{i=1}^{n} \alpha_i (h(v_i) - h(v))}{\sum_{i=1}^{n} \alpha_i (x_i^2 + y_i^2)}
\]

(22)

where the constants \( \alpha_i, i = 1, 2, \cdots, n \) satisfy the configuration equation \( (19) \). Note again we have \( \tilde{v}_i = (v_i - v) - <v_i - v, N_A(v)> N_A(v) \) and \( \tilde{v}_i = x_i e_1 + y_i e_2 \). Indeed, this definition of the discrete LB operator \( \Delta_A \) is independent of the choice of the orthonormal basis \( e_1, e_2 \). It depends on the choice of the constants \( \alpha_i \). To obtain a unique solution \( \alpha_i \) for \( i = 1, 2, \cdots, n \), with \( \sum_{i=1}^{n} \alpha_i^2 = 1 \), one can simply choose 5 closest neighboring points \( \tilde{v}_i \) of the origin in \( P_A(v) \). In case that the local polygon has less than 5 vertices, one can also lift the 2 or 3-ring of neighboring vertices of \( v \) in \( V \). In this way, we can call \( \Delta_A h(v) \) a 6-point Laplacian formula.

3.4 A convergence theorem for the discrete LB operators

In this section we will prove that the discrete LB operator \( \Delta_A h \) for a smooth function \( h \) on a regular surface \( \Sigma \) is a convergent \( O(r) \) approximation of the true LB operator \( \Delta_{\Sigma} h \). To show this, let \( \Sigma \) be a smooth regular surface in the 3D Euclidean space \( \mathbb{R}^3 \) and \( p \in \Sigma \). Consider the exponential map \( \exp_p : T\Sigma(p) \to \Sigma \) from the tangent plane \( T\Sigma(p) \) of \( \Sigma \) at the point \( p \) into the surface \( \Sigma \). See do Carmo \cite{8,9} for discussions about the properties of the exponential map \( \exp_p \). One of the well-known properties of the exponential map \( \exp_p \) is that it is a local diffeomorphism around the origin \( \tilde{0} \in T\Sigma(p) \). In other words, if \( W \) is a sufficiently small open domain around the origin \( \tilde{0} \), \( \exp_p : W \to D \) is a diffeomorphism where \( D = \exp(W) \) is an open domain around \( p \). In particular, the inverse of \( \exp_p \) exists on \( D \).

Given a smooth function \( h \) on the regular surface \( \Sigma \), we can lift \( h \) via the exponential map \( \exp_p \) locally to obtain a smooth function \( \hat{h} \) defined on \( W \in T\Sigma(p) \) by setting

\[
\hat{h}(w) = h(\exp_p(w))
\]

(23)

for \( w \in W \). Fix an orthonormal basis \( \tilde{e}_1, \tilde{e}_2 \) for the tangent space \( T\Sigma(p) \). This gives us a coordinate system on \( T\Sigma(p) \). Namely, for \( w \in W \) we have \( w = x\tilde{e}_1 + y\tilde{e}_2 \) for two constants \( x \) and \( y \). Without ambiguity, we can identify the vector \( w \in W \) with the vector \( (x, y) \) with respect to the orthonormal basis \( \tilde{e}_1, \tilde{e}_2 \). In this way, the function \( \hat{h} \) can also give us a smooth function \( \hat{h} \) of two variables \( x \) and \( y \) by defining

\[
\hat{h}(x, y) = \hat{h}(w)
\]

(24)

for \( w = x\tilde{e}_1 + y\tilde{e}_2 \). Using these notations, we will prove
Lemma 1. One has
\[ \Delta \Sigma h(p) = \Delta \hat{h}(0) = \Delta \hat{h}(0, 0) \quad (25) \]

Proof:
It is well-known that the LB operator \( \Delta \Sigma h(p) \) acting on a smooth function \( h \) at a point \( p \) can be computed from the second derivatives of \( h \) along any two perpendicular geodesics with unit speed. See do Carmo [3] for details. Indeed, we consider the following two perpendicular geodesics with unit speed in \( \Sigma \) by using the orthonormal vectors \( \hat{e}_1, \hat{e}_2 \):

\[ c_i(t) = \exp_p(t\hat{e}_i), \quad i = 1, 2 \quad (26) \]

with \( c_i(0) = p \) and \( \frac{dc_i}{dt}(0) = \hat{e}_i \). One has

\[ \Delta \Sigma h(p) = \frac{d^2}{dt^2} h(c_1(t))|_{t=0} + \frac{d^2}{dt^2} h(c_2(t))|_{t=0} = \frac{d^2}{dt^2}\hat{h}(t\hat{e}_1)|_{t=0} + \frac{d^2}{dt^2}\hat{h}(t\hat{e}_2)|_{t=0} = \Delta \hat{h}(0) \quad (27) \]

\[ = \frac{\partial^2}{\partial x^2}\hat{h}(0, 0) + \frac{\partial^2}{\partial y^2}\hat{h}(0, 0) = \hat{\Delta} \hat{h}(0, 0) \]

Next we consider a triangular surface mesh \( S = \{V, F\} \) for the regular surface \( \Sigma \), where \( V = \{v_i|1 \leq i \leq n_V\} \) is the list of vertices and \( F = \{T_k|1 \leq k \leq n_F\} \) is the list of triangles and the mesh size is less than \( r \). Fix a vertex \( v \) in \( V \). For each face \( T \in F \) containing \( v \), we have

\[ N_\Sigma(v) = N_T + O(r^2) \quad (28) \]

where \( N_\Sigma(v) \) is the unit normal vector of the true tangent plane \( T_\Sigma(v) \) of \( \Sigma \) at \( v \) and \( N_T \) is the unit normal vector of the face \( T \). Since the approximating normal vector \( N_A(v) \), defined in section 3.1 is a weighted sum of these neighboring face normals \( N_T \), we have

Lemma 2. One has
\[ N_\Sigma(v) = N_A v + O(r^2) \quad (29) \]

Due to this lemma, the orthonormal basis \( \hat{e}_1, \hat{e}_2 \) for the tangent plane \( T_\Sigma(v) \) will give us an orthonormal basis \( e_1, e_2 \) for the approximating tangent space \( T_{\Sigma_A}(v) = \{w \in \mathbb{R}^3|w \bot N_A(v)\} \) by the Gram-Schmidt process in linear algebra:

\[ e_1 = \frac{\hat{e}_1 - \langle \hat{e}_1, N_A(v) \rangle N_A(v)}{\|\hat{e}_1 - \langle \hat{e}_1, N_A(v) \rangle N_A(v)\|}, \]

and

\[ e_2 = \frac{\hat{e}_2 - \langle \hat{e}_2, N_A(v) \rangle N_A(v) - \langle \hat{e}_2, e_1 \rangle e_1}{\|\hat{e}_2 - \langle \hat{e}_2, N_A(v) \rangle N_A(v) - \langle \hat{e}_2, e_1 \rangle e_1\|}. \]

Logically speaking, one can first choose an orthonormal basis \( e_1, e_2 \) for the approximating tangent space \( T_{\Sigma_A}(v) \) and then apply the Gram-Schmidt process to obtain an orthonormal basis \( \hat{e}_1, \hat{e}_2 \) for the tangent plane \( T_\Sigma(v) \). In either way, we always have by Lemma 2 the following relations.
Lemma 3. One has
\[ \tilde{e}_i = e_i + O(r^2), \quad i = 1, 2 \]  \hspace{1cm} (30)

Consider a neighboring vertex \( v_i \) of \( v \) in \( V \). For \( r \) small enough, we can use the inverse of the exponential map \( \exp_p \) to lift the vertex \( v_i \) up to the tangent plane \( T\Sigma(v) \) and obtain
\[ \tilde{v}_i = \exp_{v_i}^{-1}(v_i) \in T\Sigma(v) \]
and
\[ \tilde{v}_i = x_i \tilde{e}_1 + y_i \tilde{e}_2 \]
for some constants. As discussed in section 3.1, we can also lift the vertex \( v_i \) up to the approximating tangent plane \( T\Sigma_A(v) \) and get
\[ \bar{v}_i = (v_i - v) - <v_i - v, N_A(v)> N_A(v) \]
and
\[ \bar{v}_i = x_i e_1 + y_i e_2 \]
for some constants \( x_i, y_i \). Then Lemmas 2 and 3 yield

Lemma 4. One has
\[
\begin{align*}
\tilde{x}_i &= x_i + O(r^2) \\
\tilde{y}_i &= y_i + O(r^2)
\end{align*}
\]  \hspace{1cm} (31)

Using these relations, one can solve the configuration equation (19) for \( (\tilde{x}_i, \tilde{y}_i) \) and \( (x_i, y_i) \) respectively and obtain their corresponding solutions \( \tilde{\alpha}_i \) and \( \alpha_i \) with the relation
\[ \tilde{\alpha}_i = \alpha_i + O(r^2) \]  \hspace{1cm} (32)
Note that the lifting function \( \tilde{h} \) is a smooth function of two variables \( x \) and \( y \). Equation (21) in section 3.2 now gives an approximation of the Laplacian \( \Delta \tilde{h}(0, 0) \):
\[ \Delta \tilde{h}(0, 0) = \frac{4 \sum_{i=1}^{n} \tilde{\alpha}_i (\tilde{h}(x_i, y_i) - \tilde{h}(0, 0))}{\sum_{i=1}^{n} \tilde{\alpha}_i (\tilde{x}_i^2 + \tilde{y}_i^2)} + O(r) \]  \hspace{1cm} (33)

The relations (24), (29) and (30) imply
\[ \Delta \tilde{h}(0, 0) = \frac{4 \sum_{i=1}^{n} \alpha_i (h(v_i) - h(0))}{\sum_{i=1}^{n} \alpha_i (x_i^2 + y_i^2)} + O(r). \]  \hspace{1cm} (34)

This along with Lemma 4 proves the following convergence theorem.

Main Theorem. Given a smooth function \( h \) on a regular surface \( \Sigma \) and a triangular surface mesh \( S = (V, F) \) with mesh size \( r \), one has
\[ \Delta \Sigma h(v) = \Delta_A h(v) + O(r) \]  \hspace{1cm} (35)
where the discrete LB operator \( \Delta_A h(v) \) is defined by Equation (23).
Remark 1. The discussions in this section also indicate that as long as we have $O(r)$-convergent algorithms to estimate gradients, Laplacians and other intrinsic derivatives of 2D smooth functions, the LTL method and methods in section 3.4 will allow us to develop corresponding discrete convergent algorithms over 3D surfaces. It is possible to obtain a $O(r^2)$ algorithm by extending Taylor’s expansion \cite{12} to the third order and improving the configuration equation \cite{12}.

4 Numerical simulations

In this section, we shall compare two convergent Laplace-Beltrami methods: Xu’s method (see \cite{20}) and our proposed method. We take four functions,

$$F_1(x, y) = \sqrt{4 - (x - 0.5)^2 - (y - 0.5)^2}.$$  
$$F_2(x, y) = \tanh(9x - 9y).$$  
$$F_3(x, y) = \frac{1.25 + \cos(5.4y)}{6 + (5x - 1)^2}.$$  
$$F_4(x, y) = \exp\left(-\frac{81}{16}(x - 0.5)^2 + (y - 0.5)^2\right).$$

over $xy$-plane as three dimensional surfaces.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{triangulation.png}
\caption{The triangulation of the domain.}
\end{figure}

The exact and approximated mean curvatures are computed as some selected interior domain points $(x_i, y_j)$ with $x_i, y_j \not\in \{0, 1\}$. The domain (a) is a three directional triangular partition, the domain (b) is a four directional triangular partition and the domain (c) is a unstructured triangular partition. To observe the convergence property, these domains are recursively subdivided by the bisection linear subdivisions. Hence, $h = \frac{h_0}{2^i}, i = 1, 2, \cdots$, where $h_0 = \sqrt{0.2}, \frac{0.2}{\sqrt{2}}, 0.23$ are the maximal value of edge lengths of the triangulations (a), (b) and (c), respectively.

The maximal errors of these simulations are shown in the Table 1. Table 2 shows the time costs for the computations in the domain (c) with $h = \frac{1}{2^i}$. Obviously, our proposed method is more accurate and faster than Xu’s method. Furthermore, the convergent rate of our method is also better than Xu’s method.
Table 1: The maximal errors of Laplacian

|               | Xu’s method    | Our method    |
|---------------|----------------|---------------|
| Domain (a)    |                |               |
| $F_1$         | $2.34E-03 \cdot h^2$ | $2.36E-05 \cdot h^4$ |
| $F_2$         | $1.81E+01 \cdot h^2$ | $3.15E-02 \cdot h^3$ |
| $F_3$         | $9.15E-01 \cdot h^2$ | $1.58E-02 \cdot h^3$ |
| $F_4$         | $1.50E+01 \cdot h^2$ | $1.68E+00 \cdot h^3$ |
| Domain (b)    |                |               |
| $F_1$         | $4.24E-03 \cdot h^2$ | $2.64E-05 \cdot h^2$ |
| $F_2$         | $1.89E+01 \cdot h^2$ | $5.13E-01 \cdot h^2$ |
| $F_3$         | $1.47E+01 \cdot h^2$ | $4.29E-01 \cdot h^2$ |
| $F_4$         | $3.51E+01 \cdot h^2$ | $7.84E-01 \cdot h^2$ |
| Domain (c)    |                |               |
| $F_1$         | $1.49E-01$      | $1.53E-03 \cdot h^3$ |
| $F_2$         | $1.94E+00 \cdot h^3$ | $1.59E-01 \cdot h^2$ |
| $F_3$         | $9.22E-01$      | $3.00E-01 \cdot h$ |
| $F_4$         | $5.41E-01 \cdot h$ | $1.36E+01 \cdot h^2$ |

Table 2: Time costs for the computations of domain (c)

|               | Xu’s method | Our method |
|---------------|-------------|------------|
| (seconds)     |             |            |
| $F_1$         | 0.024       | 0.010      |
| $F_2$         | 0.026       | 0.014      |
| $F_3$         | 0.026       | 0.015      |
| $F_4$         | 0.025       | 0.012      |

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