Comprehensive Analysis of ZNN Models for Computing Complex-Valued Time-Dependent Matrix Inverse

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ABSTRACT Computing complex-valued time-dependent matrix inverse is an important procedure in statistics and control theory. Zeroing neural network (ZNN) becomes an effective method for solving time-dependent issues. This paper proposes a complex-valued synthetical ZNN (CVSZNN) model for computing complex-valued time-dependent matrix inverse within different noise environments. Two important property is synthetically dissected in the construction process of CVSZNN model. First, the finite-time convergence, to ensure the efficiency of CVSZNN model for solving complex-valued time-dependent issue. Secondly, the anti-noise property, to guarantee the robustness of CVSZNN model in presence of noises. Two important aspects above have been validated by theoretical analysis and numerical experiments. Furthermore, the impact of parameters values is investigated based on the experimental results. As compared with the existing ZNN models for computing complex-valued time-dependent matrix inverse, the superiority of the proposed CVSZNN model is displayed fully.

INDEX TERMS Zeroing neural network, time-dependent, finite-time convergence, complex-valued, anti-noise, matrix inverse.

I. INTRODUCTION

The computing of matrix inverse, as a fundamental step of many issues, appears in mathematics, statistics, control theory, and application can also be found in machine learning [1], robotics [2], and pattern recognition [3]. Because of its indispensable roles, lots of algorithms have been investigated to improve the speed and accuracy of the solution to the inverse matrix in general. Most of the algorithms can be classified into iterative algorithms and neural networks. The iterative algorithms is a typical serial algorithm [4]–[6]. For instance, the accurate solution can be obtained with operations of finite sequence [4]. Wu proposed iterative algorithms to solve matrix equations [6]. The iterative algorithms should be operated within every sampling period. Because of the sampling rate becomes too high within each sampling period, the iterative algorithms will fail to complete calculation when they are applied to solve time-dependent issues. In this case, researchers have to search various parallel methods to facilitate the processing.

Recurrent neural network (RNN) is a typical parallel method. With the high-speed parallel and distributed hardware processing characters, RNN becomes an effective approach to solving various issues (including the matrix inversion) [7]–[13]. For example, Chen et al. developed RNNs for solving linear matrix equation, the solution errors can converge to zero as the time goes on [10]. However, for the time-dependent case, the solution errors of RNNs are always instability because of neglecting the velocity compensation of time-dependent coefficients. In order to eliminate the instability of the solution errors, Zhang et al. proposed a novel zeroing neural network (ZNN) for solving time-dependent issues [14]–[17], which is a great progress in this field. The superior property of ZNN has been verified in [14]–[21]. Wang et al. researched and developed ZNN models for solving complex-valued problem [21], [22]. The traditional design formula of ZNN is presented as follows:

\[ \dot{E}(t) = -\gamma E(t), \]  

(1)
where $E(t)$ denotes the error monitoring function, the parameter $\gamma > 0$. The formula (1) has been proved to be stable and effective to guarantee $E(t)$ vanish to zero with exponential convergence in noise-free environment [14], [23].

Owing to time is precious in solving time-depended issues, many efforts have been devoted to finite-time convergence researches in neural network field. Literature [24] reveals a fact that the convergence speed can be accelerated by embedding appropriate nonlinear activation function [25]–[28]. Li designed a sign-bi-power function accelerate the convergence of ZNN in finite time [25]. Xiao proposed an nonlinear neural network to solve the time-depended matrix equations in finite time [28]. The design formula of finite-time convergence ZNN (FCZNN) is presented as

$$\dot{E}(t) = -\gamma \Psi(E(t)), \quad (2)$$

where $\gamma > 0$, $\Psi(\cdot)$ denotes a nonlinear activation function.

It worth pointing out, the traditional ZNN design formula (1) is sensitive to noise. However, noise is ineluctable [29]. To overcome the drawback of (1), an integration-enhanced ZNN (IEZNN) is proposed in [30]. The design formula of IEZNN is provided as

$$\dot{E}(t) = -\gamma E(t) - \lambda \int_0^t E(\tau) d\tau, \quad (3)$$

where $\gamma > 0$, $\lambda > 0$. Due to the anti-noise property of (3), the application of ZNN models are extended in practical world [31], [32].

Two conclusions can be drawn from the aforementioned discussion. First, the convergence speed of the ZNN model can be accelerated within finite time by embedding a well-designed nonlinear activation function. Second, the anti-noise property of the ZNN model can be enhanced by adding an integral term. Note that the aforementioned ZNN models are efficient for real-valued matrix, few literatures have been involved in complex-valued domain, not to mention the computing of the time-depended complex inverse. However, complex-valued matrices may appear in many situations, such as frequency domain identification processes on real time, or the input signals contain the magnitude and phase information [33], [34]. The goal of our work is to establish a complex-valued synthetical ZNN (CVSZNN) model with noise resistance and finite-time convergence for computing complex-valued time-depended matrix inverse.

The remainder of this paper consists of the following sections. Problem formulation and the conventional ZNN (i.e., FCZNN and IEZNN) models are introduced in Section II. In Section III, the novel design formula is presented, in addition, the CYSZNN model is constructed according to the formula. In Section IV, the finite-time convergence and anti-noise property of the CYSZNN model are analyzed in theory. Numerical examples are performed in Section V with the FCZNN, IEZNN and CVSZNN model. Section VI concludes this work.

The main contributions of this work can be summarized as below.

- A novel unified formula that includes the integral term and nonlinear activation function is first proposed in the complex-valued domain.
- The CVSZNN model with finite-time convergence and noise resistance is first established and investigated for solving the complex-valued time-depended matrix inverse.
- The superior properties of the CVSZNN model are derived in theory.
- The results of numerical experiments demonstrate that the proposed CVSZNN model superior than existing models for computing complex-valued time-depended matrix inverse.

II. PROBLEM FORMULATION AND PREVIOUS ZNN MODELS

In this section, the problem formulation is presented for further investigation. Then, the FCZNN and IEZNN are submitted for comparison.

A. PROBLEM FORMULATION

In this paper, we study the complex-valued time-dependent matrix inverse as following form:

$$C(t)Y(t) = I, \quad 0 < t < +\infty \quad (4)$$

where $C(t) \in \mathbb{C}^{n \times n}$ is a given nonsingular time-dependent matrix, $I$ is an identity matrix, $Y(t) \in \mathbb{C}^{n \times n}$ is an unknown time-dependent matrix to be obtained.

For the solution of the time-depended complex-valued inverse, the error-monitoring function is defined as follows:

$$E(t) = C(t)Y(t) - I. \quad (5)$$

B. PREVIOUS ZNN MODELS

For comparison, the corresponding FCZNN model is given as [15]:

$$\dot{E}(t) = -\xi \Phi(E(t)). \quad (6)$$

where $\xi > 0$ is a scaling parameter for measuring the convergence rate of the FCZNN model, and $E(t)$ is the corresponding error-monitoring function. Thereinto, $\Phi(\cdot)$ denotes a complex-valued nonlinear activation-function, which is an odd and monotonically increasing function. $\Phi(\cdot)$ operates the real part and imaginary part of complex separately. $\Phi(\cdot)$ is defined as [25]

$$\Phi(P + iQ) = \phi(P) + i\phi(Q), \quad (7)$$

where $P$, $Q \in \mathbb{R}^{n \times n}$, $i$ denotes the imaginary part, $\phi(\cdot)$ is an odd and monotonically increasing function. Actually, the nonlinear function $\phi(\cdot)$ is well-defined can improve the convergence rate of ZNN model. In [25], sign-bi-power function $\phi(\cdot)$ is proven that it is superior than existing nonlinear activation functions. Therefore, we adopt sign-bi-power function $\phi(\cdot)$ to nudge the ZNN models converge to the theoretical solution in finite-time. The sign-bi-power function $\phi(\cdot)$ is...
defined as
\[ \phi(x) = \frac{S'(x) + S'(x)}{2}, \]  
where \( x \in \mathbb{R}, 0 < \gamma < 1 \), and \( S(\cdot) \) is designed as follows:
\[ S'(x) = \begin{cases} |x|^\gamma, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -|x|^\gamma, & \text{if } x < 0. \end{cases} \]

Substituting (5) into (6), the FCZNN model is obtained as
\[ C(t)\dot{Y}(t) = -\hat{C}(t)Y(t) - \xi \Phi(C(t)Y(t) - I). \]  
In [25], the FCZNN model (9) can effectively solve the complex-valued time-dependent issue in noiseless situation. However, the FCZNN model (9) cannot work well in noise situation. This is due to the design flaw of the traditional design formula (6).

In view of the noise interference, an improved design formula [30] is designed as follows:
\[ \dot{E}(t) = -\xi E(t) - \eta \int_0^t E(s) ds, \]  
where the scaling parameters \( \xi > 0 \), \( \eta > 0 \) and an integral item are added. Replacing (5) into (10), we can obtain the IEZNN model as
\[ C(t)\dot{Y}(t) = -\hat{C}(t)Y(t) - \xi \Phi(C(t)Y(t) - I) 
- \eta \int_0^t (C(s)Y(s) - I) ds, \]  
which has a better anti-noise performance in solving time-dependent problems. However, it need a wide time range to obtain the solution.

III. DESIGN OF CVSZNN MODEL
In accordance with the previous discussion, a unified design formula is proposed to deal with defects of the FCZNN (9) and IEZNN (11) models in this section. In addition, by adopting the unified design formula and embedding nonlinear activation function, the CVSZNN model is presented for solving time-dependent inverse in noisy situation.

First, we design the unified formula as follows:
\[ \dot{E}(t) = -\xi \Phi_1(E(t)) - \eta \Phi_2(E(t)) + \xi \int_0^t \Phi_1(E(s)) ds, \]  
where \( \xi > 0 \) and \( \eta > 0 \) are the scale factors, \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) denote complex-valued matrix mappings: \( \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n} \). In this paper, \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) is designed same as \( \Phi(\cdot) \) in (7), and we still adopt sign-bi-power function embedded in \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \). According to the unified design formula (12) and the error-monitoring function (5), the CVSZNN model can be obtained as follows:
\[ C(t)\dot{Y}(t) = -\hat{C}(t)Y(t) - \xi \Phi_1(C(t)Y(t) - I) 
- \eta \Phi_2(C(t)Y(t) - I) 
+ \xi \int_0^t \Phi_1(C(s)Y(s) - I) ds, \]  
In the next section, we will prove the finite-time convergence and the anti-noise property of the CVSZNN model. Besides, in order to discuss the anti-noise property of the CVSZNN model (13) for computing complex-valued time-dependent matrix inverse, the CVSZNN model (13), with noise polluted, can be rewritten as
\[ C(t)\dot{Y}(t) = -\hat{C}(t)Y(t) - \xi \Phi_1(C(t)Y(t) - I) 
- \eta \Phi_2(C(t)Y(t) - I) 
+ \xi \int_0^t \Phi_1(C(s)Y(s) - I) ds + \Lambda(t), \]  
where \( \Lambda(t) \in \mathbb{C}^{n \times n} \) is the matrix-form noise, which is divided into constant noise and time-depended noise.

Remark 1: The convergence of neural network systems (9), (11) and (13) is adjusted by parameters \( \xi \) and \( \eta \). And \( \gamma \) is the parameter of sign-bi-power function (8). A large value of \( \xi \) and \( \eta \) can accelerate the convergence. Therefore, in applications, parameters \( \xi \) and \( \eta \) should be set as large as hardware permits.

Remark 2: Firstly, the CVSZNN model adopts a nonlinear activation function, compared with the IEZNN model, the CVSZNN model owns superior performance in noise immunity and convergence. Then, using CVSZNN model to solve problems has more advantages. Secondly, it is worth noting that the proposed complex dynamic system contains complex weights and thresholds as well as complex inputs and output signals. This is clearly different from the approach which separates a complex problem into real and imaginary parts and then uses a real recurrent neural network for the solution. Therefore, the proposed complex dynamic system is more economical for parallel processing (in terms of circuit implementation).

IV. THEORETICAL DERIVATIONS
The corresponding property of FCZNN (9) and IEZNN (11) have been proven in [25] and [30], respectively. This section proves the global convergence of CVSZNN model (13) for solving complex-valued time-dependent matrix inverse. Further more, this section verifies the finite-time convergence and anti-noise performance of CVSZNN model (13).

Theorem 1: The state matrix \( Y(t) \) synthesized by CVSZNN model (13) globally converges to theoretical solution \( C^{-1}(t) \) of (4).

Proof: Owing to the design formula
\[ \dot{E}(t) = -\xi \Phi_1(E(t)) - \eta \Phi_2(E(t)) + \xi \int_0^t \Phi_1(E(s)) ds, \]  
element-wise, we have
\[ \dot{e}_{ab}(t) = -\xi \Phi_1(e_{ab}(t)) - \eta \Phi_2(e_{ab}(t)) + \xi \int_0^t \Phi_1(e_{ab}(s)) ds, \]  
where \( e_{ab}(t) \in \mathbb{C} \) denotes the \( ab \)th item of the complex matrix \( E(t) \). Based on the definition of \( \Phi(\cdot) \) in (7), we have
\[ \Phi_1(e_{ab}(t)) = \phi_1(p_{ab}(t)) + i\phi_1(q_{ab}(t)), \]  
\[ \Phi_2(e_{ab}(t)) = \phi_2(p_{ab}(t)) + i\phi_2(q_{ab}(t)), \]
where \( p_{ab}(t) \in \mathbb{R} \) and \( q_{ab}(t) \in \mathbb{R} \) denote the real and imaginary parts of \( \varepsilon_{ab}(t) \), respectively. According to (15), we obtain two equations in real number domain as

\[
\dot{p}_{ab}(t) = -\xi \phi_1(p_{ab}(t)) - \eta \phi_2(p_{ab}(t)) + \xi \int_0^t \phi_1(p_{ab}(s)) \, ds,
\]

\[\tag{16a}\]

\[
\dot{q}_{ab}(t) = -\xi \phi_1(q_{ab}(t)) - \eta \phi_2(q_{ab}(t)) + \xi \int_0^t \phi_1(q_{ab}(s)) \, ds,
\]

\[\tag{16b}\]

For the variable \( p_{ab}(t) \), we set

\[\omega_{ab}(t) = p_{ab}(t) + \xi \int_0^t \phi_1(p_{ab}(s)) \, ds, \quad \forall a, b \in \{1, 2, \ldots, n\}, \tag{17}\]

we have

\[\dot{\omega}_{ab}(t) = \dot{p}_{ab}(t) + \xi \phi_1(p_{ab}(t)). \tag{18}\]

Then, according to Eqs. (16a), (17) and (18), we obtain

\[\dot{\omega}_{ab}(t) = -\eta \phi_2(\omega_{ab}(t)), \tag{19}\]

which is the traditional design formula (6). Obviously, \( \omega_{ab}(t) \) converges to 0 in finite-time with the nonlinear activation function \( \phi_2(\cdot) \). Then, we construct a Lyapunov function candidate

\[V_{ab}(t) = (\theta p_{ab}^2(t) + \omega_{ab}^2(t))/2 (\text{with} \theta > 0) \text{ for the abth item (16)a).}\]

Evidently, \( V_{ab}(t) \) is positive definite, its time derivative

\[\dot{V}_{ab}(t) = \dot{\omega}_{ab}(t) \dot{p}_{ab}(t) + \omega_{ab}(t) \dot{\omega}_{ab}(t) = \dot{\omega}_{ab}(t) [\dot{p}_{ab}(t) + \omega_{ab}(t)] - \eta \omega_{ab}(t) \phi_2(\omega_{ab}(t)) = -\eta \omega_{ab}(t) \phi_2(\omega_{ab}(t)). \tag{20}\]

With \( \dot{V}_{ab}(t) \leq 0 \) holding true, we can conclude \( V_{ab}(t) \leq V_{ab}(0) \), then we have

\[\frac{\theta p_{ab}^2(t)}{2} \leq V_{ab}(0), \quad \text{and} \quad \frac{\theta \omega_{ab}^2(t)}{2} \leq V_{ab}(0), \tag{21}\]

and also we have

\[|p_{ab}(t)| \leq \sqrt{2V_{ab}(0)/\theta}, \quad \text{and} \quad |\omega_{ab}(t)| \leq \sqrt{2V_{ab}(0)}. \tag{22}\]

Further, we obtain

\[U_1 = \{|p_{ab}(t)| \leq \sqrt{2V_{ab}(0)/\theta}, \quad p_{ab}(t) \in \mathbb{R}\}, \tag{23}\]

\[U_2 = \{|\omega_{ab}(t)| \leq \sqrt{2V_{ab}(0)}, \quad \omega_{ab}(t) \in \mathbb{R}\}. \tag{24}\]

According to the mean value theorem, we have

\[\phi_2(\omega_{ab}(t)) - \phi_2(0) = (\omega_{ab}(t) - 0) \frac{\partial \phi_2(\omega_{ab}(t))}{\partial \omega_{ab}}|_{\omega_{ab}(t) \in U_2}. \tag{25}\]

Owing to function feature of \( \phi_2(\cdot) \), we can determined that \( \phi_2(0) = 0 \), and \( \partial \phi_2(\omega_{ab}(t))/\partial \omega_{ab} > 0 \). We let

\[D_0 = \text{sub}\{\partial \phi_2(\omega_{ab}(t))/\partial \omega_{ab}\}_{\omega_{ab}(t) \in U_2} > 0, \tag{26}\]

and obtain

\[|\phi_2(\omega_{ab}(t))| \leq D_0 |\omega_{ab}(t)|. \tag{27}\]

Therefore, we determine that

\[|p_{ab}(t)| \phi_2(\omega_{ab}(t)) \leq \|p_{ab}(t)\| |\phi_2(\omega_{ab}(t))| \leq D_0 |p_{ab}(t)| |\omega_{ab}(t)|. \tag{28}\]

By replacing (21) into (20), we obtain

\[\dot{V}_{ab}(t) = -\theta \xi p_{ab}(t) \phi_1(p_{ab}(t)) - \theta \eta p_{ab}(t) \phi_2(\omega_{ab}(t)) - \eta \omega_{ab}(t) \phi_2(\omega_{ab}(t)) \leq -\theta \xi D_1 p_{ab}^2(t) - \theta \eta D_0 |p_{ab}(t)| |\omega_{ab}(t)| - \eta D_0 \omega_{ab}^2(t) = -\theta \left(\sqrt{\frac{\xi}{\theta}} D_1 |p_{ab}(t)| - \frac{\eta D_0}{2} |\omega_{ab}(t)|^2 \right)^2 - \eta \left(\frac{D_2}{\theta} - \frac{\eta^2 D_0^2}{4 \xi D_1} \right) \omega_{ab}^2(t). \tag{29}\]

Similarly, \( D_1 = \inf\{\|\partial \phi_1(p_{ab}(t))/\partial p_{ab}\|_{p_{ab}(t) \in U_1} \geq 0 \), and \( D_2 = \inf\{\|\partial \phi_2(\omega_{ab}(t))/\partial \omega_{ab}\|_{\omega_{ab}(t) \in U_2} > 0 \) can be obtained by the mean value theorem. If

\[\theta \in (0, \frac{4 \xi D_1 U_1^2}{\eta D_0^2} \text{ and } \frac{\eta D_0}{\theta} - \frac{\eta^2 D_0^2}{4 \xi D_1} \geq 0, \tag{30}\]

then, \( \dot{V}_{ab}(t) \leq 0 \), which means, \( V_{ab}(t) \) is negative definite. By the Lyapunov stability theory, \( p_{ab}(t) \) globally converges to 0. Similarly, it can be proven that \( q_{ab}(t) \) globally converges to 0. In summary, \( \varepsilon_{ab}(t) \) globally converges to 0. That is, starting from any initial state \( Y(0) \), the state matrix \( Y(t) \) globally converges to the initial solution \( C^{-1}(0) \). The proof is completed. \( \square \)

**Theorem 2:** The state matrix \( Y(t) \) synthesized by CVSZNN model (13) converges to the theoretical solution \( C^{-1}(0) \) of (4) in finite time:

\[t_{up} = \frac{(\xi + \eta) \delta_0^{1-\gamma}}{\xi \eta (1 - \gamma)}, \tag{31}\]

where \( \delta_0 = \max_{j,k,l,m} |p_{jk}(0)|, |q_{lm}(0)| \), with \( p_{jk}(0) \) being the real part of the jkth element of \( \mathbb{E}(0) \), and \( q_{lm}(0) \) being the imaginary part of the lmth element of \( \mathbb{E}(0) \), \( \xi, \eta, \gamma \) are defined in (12) and (8).

**Proof:** Based on Eq. (15) mentioned in above theorem, by selecting

\[\sigma_{ab}(t) = \varepsilon_{ab}(t) + \xi \int_0^t \Phi_1(\varepsilon_{ab}(s)) \, ds, \quad \forall a, b \in \{1, 2, \ldots, n\}, \tag{32}\]

we have

\[\dot{\sigma}_{ab}(t) = \varepsilon_{ab}(t) + \xi \Phi_1(\varepsilon_{ab}(t)). \tag{33}\]

Then, according to Eqs. (15), (23) and (24), we obtain

\[\dot{\sigma}_{ab}(t) = -\eta \Phi_2(\sigma_{ab}(t)), \tag{34}\]
which is the traditional design formula (6), where $\Phi_2(\cdot) = \Phi(\cdot)$. According to the definition of $\Phi(\cdot)$, (25) yields

$$
\dot{p}_{ab}(t) = -\eta\phi_2(p_{ab}(t)) = -\eta\phi_2(q_{ab}(t)),
$$

(26a)

$$
\dot{q}_{ab}(t) = -\eta\phi_2(q_{ab}(t)) = -\eta\phi_2(q_{ab}(t)),
$$

(26b)

where $p_{ab}(t) \in \mathbb{R}$ and $q_{ab}(t) \in \mathbb{R}$ denote the real and imaginary parts of $\sigma_{ab}(t)$, respectively. We can determine that $\sigma_{ab}(0) = \epsilon_{ab}(0)$ with $t = 0$. Then, we define Lyapunov function candidates $\mu_{pab}(t) = p_{ab}^2(t)$ and $\mu_{qab}(t) = q_{ab}^2(t)$. Based on the comparison lemma [35], we have $\mu_{pab} \leq \mu^* = \mu_{qab} \leq \mu_*$ where $\mu^*$ denotes the Lyapunov function associated with the maximum initial elements in absolute value for the real or the imaginary parts, i.e., $\mu^* = \delta^2(t)$, $\delta(t) = -\eta\phi(\delta(t)), \delta(0) = \delta_0 = \max_{j,k,l,m}\{|p_{jk}(0)|, |q_{lm}(0)|\}$. As $\mu_{pab} \leq \mu^*$ and $\mu_{qab} \leq \mu_*$, both $\mu_{pab}$ and $\mu_{qab}$ achieve zero when $\mu^*$ achieves zero, which means the upper bound of the convergence time is the time for $\mu^*$ reaches zero. The time derivative of $\mu^*$ is

$$
\dot{\mu}^* = -2\eta\delta(t)\phi(\delta(t)) = -\eta\delta(t)(S'(\delta(t)) + S(\delta(t)))
$$

$$
\leq -2\eta\delta(t)^{\gamma+1} = -2\eta\mu^*^{\frac{\gamma+1}{\gamma}}.
$$

We solve the differential inequality $\dot{\mu}^* \leq -2\eta\mu^*^{\frac{\gamma+1}{\gamma}}$. Then, we obtain

$$
(\mu^*(t))^{\frac{1}{\gamma}} \leq \left\{\begin{array}{ll}
|\delta(0)|^{1-\gamma} - \eta(1-\gamma), & \text{if } 0 \leq t \leq \frac{|\delta(0)|^{1-\gamma}}{\eta(1-\gamma)} \frac{1}{\gamma} \ , \\
0, & \text{if } t > \frac{|\delta(0)|^{1-\gamma}}{\eta(1-\gamma)} \frac{1}{\gamma} ,
\end{array}\right.
$$

therefore, after a certain point $|\delta(0)|^{1-\gamma}/\eta(1-\gamma)$, $\mu^*$ converges to zero. For $t > |\delta(0)|^{1-\gamma}/\eta(1-\gamma)$, $\sigma_{ab}(t)$ decreases to zero because of $\mu_{pab}(t) = p_{ab}^2(t)$ and $\mu_{qab}(t) = q_{ab}^2(t)$ and $\sigma_{ab}(0) = \epsilon_{ab}(0)$. Then, we conclude that $\sigma_{ab}(t)$ converges to zero when $t \in (\delta_0^{1-\gamma}/\eta(1-\gamma), +\infty)$. That is, the upper bound of convergence time $t_1$ of $\sigma_{ab}(t)$ is obtained as

$$
t_1 \leq \frac{\delta_0^{1-\gamma}}{\xi(1-\gamma)}.
$$

The result above means that $\sigma_{ab}(t)$ converges to the equilibrium point, and $\sigma_{ab}(t)$ equals 0 after a finite period. Based on above result and Eq. (24), we can conclude that $\epsilon_{ab}(t) = -\xi\Phi(\epsilon_{ab}(t)) (\Phi(1) = \Phi(\cdot))$. Similarly, the upper bound of convergence time $t_2$ is obtained as

$$
t_2 < \frac{\delta_0^{1-\gamma}}{\xi(1-\gamma)}.
$$

Eventually, we conclude that CVSZN model (13) can converge to zero with a finite time, and the upper bound of convergence time

$$
t_{up} = t_1 + t_2 < \frac{(\xi + \eta)\delta_0^{1-\gamma}}{\xi\eta(1-\gamma)}.
$$

The proof completes. □
function is set as $\gamma$ generality, in this paper, the parameter in nonlinear activation dependent matrix inverse in noise situations. Without loss of generality, in this paper, the parameter in nonlinear activation function is set as $\gamma = 0.2$. 

Example 1: The complex-valued time-dependent matrix $C(t)$ is considered as follow:

$$C(t) = \begin{bmatrix} i \sin(2t) & -i \cos(2t) \\ i \cos(2t) & i \sin(2t) \end{bmatrix} \in \mathbb{C}^{2 \times 2}. \quad (31)$$

For verifying the accuracy of the proposed CVSZNN model (13), the theoretical inverse of (31) can be gotten as

$$C^{-1}(t) = \begin{bmatrix} -i \sin(2t) & -i \cos(2t) \\ i \cos(2t) & -i \sin(2t) \end{bmatrix} \in \mathbb{C}^{2 \times 2}. \quad (32)$$

First, the experimental results generated by the proposed finite-time convergent CVSZNN model (13) for computing inverse of complex-valued time-dependent matrix (31) in noiseless situation are shown in Figures. 1-2. In Figure. 1, starting from a randomly initial state $Y(0) \in \mathbb{C}^{2 \times 2}$, state matrix $Y(t) \in \mathbb{C}^{2 \times 2}$ generated by CVSZNN model (13) with $\xi = \eta = 4$ converges to the time-depended theoretical inverse (32) precisely and rapidly. In Fig. 3, the residual error $\|\mathcal{E}(t)\|_F = \|C(t)Y(t) - I\|_F$ synthesized by CVSZNN model (13) decreases directly to zero within finite time $0.3$ s, which is less than the upper bound $t_{up} = (\xi + \eta)\theta(1-\gamma)/\xi \eta(1-\gamma) \approx 0.625$ s. Furthermore, the Figure. 2 shows the neural state trajectories of CVSZNN solution and the theoretical solution for imagine part. Then, the results verify the global and finite time convergence performance proven in Theorem 1 and 2.

For comparative purposes, the FCZNN model (9), the IEZNN model (3) and the proposed CVSZNN model (13) are adopted to compute inverse of the complex-valued time-depended matrix (31). we set $\xi = \eta = 6$ and assume such three models are polluted by matrix-form constant noise $\Lambda(t)$, each element of $\Lambda(t)$ is set to be $9 + 12i$. the corresponding numerical experimental results are displayed in Figures. 4-6. In Figure. 4(a), Figure. 5(a), Figure. 6(a), the blue curve denotes the state trajectories synthesized by three ZNN model, the red curve denotes the theoretical inverse $C^{-1}(t)$, Figure. 4(a) plots that the state matrix synthesized by FCZNN model (9) does not converge the theoretical inverse $C^{-1}(t)$ when polluted by the constant noise $\Lambda(t)$. In Figure. 4(b), the residual error $\|\mathcal{E}(t)\|_F = \|C(t)Y(t) - I\|_F$ synthesized by FCZNN model (9) cannot decrease to zero and remains at nearly $3$. From which, FCZNN model (9) cannot compute the complex-valued time-depended matrix inverse effectively because of the constant noise. Figure. 5(a) shows the anti-noise property of IEZNN model (3). In Figure. 5(b),

V. SIMULATION AND VERIFICATION

In the Section IV, the convergence and robustness property of the proposed CVSZNN model (13) for computing complex-valued time-dependent matrix inverse are analysed. This section, numerical experiment results are conducted to validate the efficacy and superiority of finite-time convergent CVSZNN model (13) for solving complex-valued time-dependent matrix inverse in noise situations. Without loss of generality, in this paper, the parameter in nonlinear activation function is set as $\gamma = 0.2$.

Example 1: The complex-valued time-dependent matrix $C(t)$ is considered as follow:

$$C(t) = \begin{bmatrix} i \sin(2t) & -i \cos(2t) \\ i \cos(2t) & i \sin(2t) \end{bmatrix} \in \mathbb{C}^{2 \times 2}. \quad (31)$$

For verifying the accuracy of the proposed CVSZNN model (13), the theoretical inverse of (31) can be gotten as

$$C^{-1}(t) = \begin{bmatrix} -i \sin(2t) & -i \cos(2t) \\ i \cos(2t) & -i \sin(2t) \end{bmatrix} \in \mathbb{C}^{2 \times 2}. \quad (32)$$

First, the experimental results generated by the proposed finite-time convergent CVSZNN model (13) for computing inverse of complex-valued time-dependent matrix (31) in noiseless situation are shown in Figures. 1-2. In Figure. 1, starting from a randomly initial state $Y(0) \in \mathbb{C}^{2 \times 2}$, state matrix $Y(t) \in \mathbb{C}^{2 \times 2}$ generated by CVSZNN model (13) with $\xi = \eta = 4$ converges to the time-depended theoretical inverse (32) precisely and rapidly. In Fig. 3, the residual error $\|\mathcal{E}(t)\|_F = \|C(t)Y(t) - I\|_F$ synthesized by CVSZNN model (13) decreases directly to zero within finite time $0.3$ s, which is less than the upper bound $t_{up} = (\xi + \eta)\theta(1-\gamma)/\xi \eta(1-\gamma) \approx 0.625$ s. Furthermore, the Figure. 2 shows the neural state trajectories of CVSZNN solution and the theoretical solution for imagine part. Then, the results verify the global and finite time convergence performance proven in Theorem 1 and 2.

For comparative purposes, the FCZNN model (9), the IEZNN model (3) and the proposed CVSZNN model (13) are adopted to compute inverse of the complex-valued time-depended matrix (31). we set $\xi = \eta = 6$ and assume such three models are polluted by matrix-form constant noise $\Lambda(t)$, each element of $\Lambda(t)$ is set to be $9 + 12i$. the corresponding numerical experimental results are displayed in Figures. 4-6. In Figure. 4(a), Figure. 5(a), Figure. 6(a), the blue curve denotes the state trajectories synthesized by three ZNN model, the red curve denotes the theoretical inverse $C^{-1}(t)$, Figure. 4(a) plots that the state matrix synthesized by FCZNN model (9) does not converge the theoretical inverse $C^{-1}(t)$ when polluted by the constant noise $\Lambda(t)$. In Figure. 4(b), the residual error $\|\mathcal{E}(t)\|_F = \|C(t)Y(t) - I\|_F$ synthesized by FCZNN model (9) cannot decrease to zero and remains at nearly $3$. From which, FCZNN model (9) cannot compute the complex-valued time-depended matrix inverse effectively because of the constant noise. Figure. 5(a) shows the anti-noise property of IEZNN model (3). In Figure. 5(b),
FIGURE 4. Adopting the FCZNN model (2) with $\xi = 6$ to solving complex-valued time-depended inverse of (31) in the presence of constant noise $9 + 12i$. (a) State trajectories of FCZNN model (2). (b) Residual errors $\|E(t)\|_F$ synthesized by FCZNN model (2).

FIGURE 5. Adopting the IEZNN model (11) with $\xi = \eta = 6$ to solving complex-valued time-depended inverse of (31) in the presence of constant noise $9 + 12i$. (a) State trajectories of IEZNN model (11). (b) Residual errors $\|E(t)\|_F$ synthesized by IEZNN model (11).

the residual error $\|E(t)\|_F$ of IEZNN model (3) vanish to zero within $5 \, \text{s}$. In Figure 6(a), the state matrix synthesized by CVSZNN model (13) converges accurately and quickly to the theoretical inverse $C^{-1}(t)$. Figure 6(b) demonstrates that the residual error $\|E(t)\|_F$ of the proposed CVSZNN model (13) vanish to zero at a finite time $0.4 \, \text{s}$. In the constant noise situation, the convergence speed of CVSZNN model (13) is approximately 12 times faster than IEZNN model (3). From which, the superiority of CVSZNN model (13) is verified in constant situation.

Different noises are further investigated on aforementioned three models with $\xi = \eta = 10$, we analyze the differences of constant noise $\Lambda(t) = [6 + 8i]^{2 \times 2}$ and time-depended noise $\Lambda(t) = [2t + 2ti]^{2 \times 2}$, the corresponding simulation results are depicted in Figure 7. Figure 7(a) with the constant noise $\Lambda(t) = [6 + 8i]^{2 \times 2}$, it can be concluded that the residual error $\|E(t)\|_F$ of CVSZNN model (13) converges fleetly to zero within $0.2 \, \text{s}$, while the one of IEZNN model (3) vanish to zero within $6 \, \text{s}$, which is 30 times lower than the proposed CVSZNN model (13). On the contrary, the residual error $\|E(t)\|_F$ of FCZNN model (9) does not decrease to zero and remains in a settled level. Figure 7(b) with the time-depended linear noise $\Lambda(t) = [2t + 2ti]^{2 \times 2}$, it indicates that the residual error $\|E(t)\|_F$ of CVSZNN model (13) decreases rapidly to zero within finite time $0.1 \, \text{s}$, while the ones of IEZNN model (3) and FCZNN model (9) cannot converges to zero, the residual error $\|E(t)\|_F$ of IEZNN model (3) remains in a stable level, that of FCZNN model (9) increases as time goes on. Therefore, the results verify Theorem 3 and the advantage of the presented CVSZNN model (13) for computing the complex-valued time-depended matrix inverse in different noises.
FIGURE 6. Adopting the CVSZNN model (13) with $\xi = \eta = 6$ to solving complex-valued time-dependent inverse of (31) in the presence of constant noise $9 + 12i$. (a) State trajectories of CVSZNN model (13). (b) Residual errors $\|E(t)\|_F$ synthesized by CVSZNN model (13).

Example 2: A more complicated complex-valued time-dependent matrix is considered for further investigation.

$$C(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) & \cdots & c_{1n}(t) \\ c_{21}(t) & c_{22}(t) & \cdots & c_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}(t) & c_{n2}(t) & \cdots & c_{nn}(t) \end{bmatrix} \in \mathbb{C}^{n \times n}. \quad (33)$$

with $c_{ab}(t)$ denotes the $ab$th element of $C(t)$. Thereinto,

$$c_{ab}(t) = \begin{cases} e^{it}, & a = b, \\ b + e^{-it}, & a > b, \\ a + e^{-it}, & a < b. \end{cases}$$

Because of the complexity of complex-valued matrix (with 16 elements), it is difficult to obtain the theoretical inverse. Then, we only demonstrate the residual error $\|E(t)\|_F$ synthesized by FCZNN model (9), the IEZNN model (3) and the proposed CVSZNN model (13) with $\xi = \eta = 4$ in constant noise and time-dependent noise. The corresponding results are displayed in Figure 8. In Figure 8(a), with the constant noise $\Lambda(t) = [3 + 4i]^{4 \times 4}$, the residual errors of the IEZNN (3) and the novel CVSZNN model (13) decrease to zero with 4 s and 0.4 s, while the residual errors of FCZNN model (9) still cannot vanish toward zero and remains stable at approximately 2.6 s. The residual errors of three model with the time-dependent noise $\Lambda(t) = [t + ti]^{4 \times 4}$ are presented in Figure 8(b). In this situation, the residual errors of FCZNN model (9) and IEZNN model (3) do not converge to zero, however, the residual error of the CVSZNN model (13) converges quickly to zero. The stability, robustness and finite-time convergence of CVSZNN model (13) are further verified.

Now, let we investigate the effect of different values of parameters $\xi$ and $\eta$ on the convergence speed of CVSZNN.
model (13). Here, we still consider the constant noise and linear noise, the corresponding experimental results are shown in Figure 9. Figure 9(a) depicts the residual error of CVSZNN model (13) in the presence of constant noise $2 + 2i$ with different values of parameters, the red solid curve denotes $\xi = \eta = 10$, the residual error decreases to 0 at 0.17 s, the blue dash-dotted curve denotes $\xi = \eta = 100$, the residual error decreases to 0 at 0.02 s, which is nearly 9 times faster than that with $\xi = \eta = 10$. The convergence speed of CVSZNN model (13) in the presence of linear noise $2t + ti$ with different values of parameters is displayed in Figure 9(b), from which, the convergence speed is also further accelerated by setting larger values of $\xi$ and $\eta$. Therefore, we conclude that the parameters in CVSZNN model (13) play a noise-resistant and accelerated role in computing complex-valued time-dependent matrix inverse.

VI. CONCLUSION

In this paper, based on a novel unified design formula, we first propose a complex-valued synthetical ZNN (CVSZNN) model for computing complex-valued time-dependent matrix inverse in different noise situation. The noise resistance and finite-time convergence of CVSZNN model have been investigated in theory. In the numerical experiments, by comparing the simulation results of FCZNN model and IEZNN model, the efficiency and superiority of CVSZNN model have been demonstrated. Furthermore, the effect of CVSZNN model with different parameters values has been investigated. In summary, the noise resistance and finite-time convergence of the proposed CVSZNN model are all illustrated for computing complex-valued time-dependent matrix inverse within different additive noises. Designing the corresponding discrete-time ZNN model is the future research direction. The application scope of this work can be promoted.
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