ON SPACE-TIME PERIODIC SOLUTIONS OF THE
ONE-DIMENSIONAL HEAT EQUATION

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Abstract. We look for solutions $u(x, t)$ of the one-dimensional heat equation
$u_t = u_{xx}$ which are space-time periodic, i.e. they satisfy the property
$u(x + a, t + b) = u(x, t)$
for all $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$, and derive their Fourier series expansions.
Here $a \geq 0$, $b \geq 0$ are two constants with $a^2 + b^2 > 0$. For general equation
of the form $u_t = u_{xx} + Au_x + Bu$, where $A$, $B$ are two constants, we also
have similar results. Moreover, we show that non-constant bounded periodic
solution can occur only when $B > 0$ and is given by a linear combination of
$\cos\left(\sqrt{B}(x + At)\right)$ and $\sin\left(\sqrt{B}(x + At)\right)$.

1. Introduction; periodic solutions of the heat equation. The goal of this
paper is to look for periodic solutions of some simple parabolic linear equations. In
Section 2 we first focus on the one-dimensional heat equation

$$u_t = u_{xx}, \quad (x, t) \in (-\infty, \infty) \times (-\infty, \infty). \quad (1)$$

Clearly it has infinitely many solutions defined on the whole space $(x, t) \in \mathbb{R}^2$.
Among them, some have special property. For example, the solutions $u(x, t) = e^{-t} \cos x, e^{-t} \sin x$, are periodic in space with period $2\pi$, i.e. they satisfy

$$u(x + 2\pi, t) = u(x, t), \quad \forall (x, t) \in \mathbb{R}^2. \quad (2)$$

Another less obvious solutions are $u(x, t) = e^{\pm x/\sqrt{2}} \cos \left(t \pm x/\sqrt{2}\right), \quad e^{\pm x/\sqrt{2}} \sin \left(t \pm x/\sqrt{2}\right)$, which are periodic in time with period $2\pi$, i.e. they satisfy the heat equation and

$$u(x, t + 2\pi) = u(x, t), \quad \forall (x, t) \in \mathbb{R}^2. \quad (3)$$

A natural question is: is there any other solution of the heat equation (1) which satisfies the property

$$u(x + a, t + b) = u(x, t), \quad \forall (x, t) \in \mathbb{R}^2 \quad (4)$$
for some constants $a \geq 0$, $b \geq 0$ with $a^2 + b^2 > 0$? The answer is affirmative and for
$u(x,t)$ satisfying (1) and (4), its Fourier series expansion is derived. See Theorem 2.10.

In Section 3 we consider the more general linear equation
\[ u_t(x,t) = u_{xx}(x,t) + Au(x,t) + Bu(x,t), \]  
where $A$, $B$ are two constants. We conclude results similar to the heat equation.

An interesting result is that non-constant bounded periodic solutions $u(x,t)$ of
equation (5) can occur only when $B > 0$ and are given by linear combinations of
$\cos(\sqrt{B}(x + At))$ and $\sin(\sqrt{B}(x + At))$, $(x,t) \in \mathbb{R}^2$. See Theorem 3.8.

To go on, we first define the following:

**Definition 1.1.** Let $a \geq 0, b \geq 0$ be two constants with $a^2 + b^2 > 0$. If $u(x,t)$
satisfies (4) with $a > 0$, $b > 0$, we say it is **space-time periodic with period**
$(a,b) \in \mathbb{R}^2$. If $a > 0, b = 0$, then it is **space-periodic with period** $a$; and if $a = 0, b > 0$,
it is **time-periodic with period** $b$.

**Remark 1.2.** If a function $u(x,t)$ is both space-periodic with period $a$ and
time-periodic with period $b$, then it is also space-time periodic with period $(a,b) \in \mathbb{R}^2$.

However, the converse is not true. For example, the function $u(x,t) = x-t$ is space-time periodic with period $(a,a) \in \mathbb{R}^2$ for any $a > 0$, but it is neither space-periodic
nor time-periodic.

**Remark 1.3.** The following observation is important and easy to verify: Assume
that $v$ is the **parabolic rescaling** of another function $u$, i.e. $v(x,t) = u(\lambda x, \lambda^2 t)$ for some
number $\lambda \neq 0, 1$. Then $u$ is a solution of the heat equation if and only if
$v$ is a solution of the heat equation. Moreover, $u$ is space-time periodic with period
$(a,b) \in \mathbb{R}^2$ if and only if $v$ is space-time periodic with period $(a/\lambda, b/\lambda^2) \in \mathbb{R}^2$. Note
that the two vectors $(a,b)$, $(a/\lambda, b/\lambda^2)$ are independent unless $a = 0$ (time-periodic) or $b = 0$ (space-periodic).

To end this introduction section, we would like to point out that, due to our
limited knowledge on the literature, somehow we cannot find any paper discussing
topic related to space-time periodic solutions of the heat equation. However, in our
opinion, this topic is interesting on its own and should have some applications and
connections with other fields. In Section 3.2 of the paper, we see that if $u(x,t)$ is a
traveling wave solution of the heat equation (or of the more general equation (74)
in Section 3), then it is also a space-time periodic solution (but not vice versa).

Therefore, properties of space-time periodic solutions can be applied to traveling
wave solutions. One can view this as a simple example of application and it also
provides motivation for the study of this type of solutions.

Finally, for readers who would like to acquire the basic knowledge of the one-
dimensional heat equation, we recommend the two excellent books by Cannon [1]
and Widder [2].

2. **Periodic solutions of the heat equation.**

2.1. **Space-periodic solutions of the heat equation.** It is known that for a
space-periodic solution $u(x,t)$ of the heat equation with period $2\pi$, its Fourier series
expansion (expansion with respect to space variable) is given by
\[ u(x,t) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} e^{-n^2t} (a_n(0) \cos nx + b_n(0) \sin nx), \quad \forall (x,t) \in \mathbb{R}^2, \]  
where $A$, $B$ are two constants. We conclude results similar to the heat equation.

An interesting result is that non-constant bounded periodic solutions $u(x,t)$ of
equation (5) can occur only when $B > 0$ and are given by linear combinations of
$\cos(\sqrt{B}(x + At))$ and $\sin(\sqrt{B}(x + At))$, $(x,t) \in \mathbb{R}^2$. See Theorem 3.8.

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\[ u(x,t) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} e^{-n^2t} (a_n(0) \cos nx + b_n(0) \sin nx), \quad \forall (x,t) \in \mathbb{R}^2, \]  
where $A$, $B$ are two constants. We conclude results similar to the heat equation.

An interesting result is that non-constant bounded periodic solutions $u(x,t)$ of
equation (5) can occur only when $B > 0$ and are given by linear combinations of
$\cos(\sqrt{B}(x + At))$ and $\sin(\sqrt{B}(x + At))$, $(x,t) \in \mathbb{R}^2$. See Theorem 3.8.
Remark 2.2. Lemma 2.1 is also a consequence of equation but still linear. For example, let equation and is space-periodic with period 2π. Letting m → ∞, we obtain v(x, t) ≡ const. and so is u(x, t).

Remark 2.2. Lemma 2.1 is also a consequence of Liouville theorem for heat equation, which says that if u(x, t) is a bounded solution to the heat equation $\Delta u = 0$ in $\mathbb{R}^{n+1}$, then it must be a constant function. Note that any function u(x, t) which is both space-periodic and time-periodic must be a bounded function on $\mathbb{R}^2$. This is not true if u(x, t) is only space-periodic (or only time-periodic, or only space-time periodic).

Remark 2.3. Lemma 2.1 (or Liouville theorem) fails if the equation is not heat equation but still linear. For example, let u(x, t) = sin(x + t), (x, t) ∈ $\mathbb{R}^2$. Then it is a bounded function and is space-periodic (with period 2π), time-periodic (with period 2π), and also space-time periodic with period (π, π). It satisfies the equation

$$u_t (x, t) = u_{xx} (x, t) + u_x (x, t) + u(x, t), \quad (x, t) \in \mathbb{R}^2.$$  

However, it is not a constant function. See Lemma 3.6 also.

2.2. Time-periodic solutions of the heat equation. As for a time-periodic solution u(x, t) of the heat equation with period 2π, its Fourier series expansion (expansion with respect to time variable) has the form

$$u(x, t) = \frac{a_0 (x)}{2} + \sum_{n=1}^{\infty} \left( a_n (x) \cos nt + b_n (x) \sin nt \right),$$  

where

$$
\begin{align*}
a_n (0) &= \frac{1}{\pi} \int_0^{2\pi} u(x, 0) \cos nx \, dx, \quad n = 0, 1, 2, 3, \\
b_n (0) &= \frac{1}{\pi} \int_0^{2\pi} u(x, 0) \sin nx \, dx, \quad n = 1, 2, 3, \ldots.
\end{align*}
$$

Note that (6) implies $u(x, t) \to a_0 (0)/2$ exponentially for all $x \in (-\infty, \infty)$ as $t \to \infty$. In this situation, the solution u(x, t) is uniquely determined by its initial data u(x, 0) over a 2π interval, say $x \in [0, 2\pi]$.

Note that each n-th term in the summation of (6) is a solution of the heat equation and is space-periodic with period 2π/n. Up to a parabolic rescaling $(x, t) \leftrightarrow (\lambda x, \lambda^2 t)$, $\lambda = 1/n$, it is a linear combination of $e^{-t} \cos x$, $e^{-t} \sin x$. Thus, up to parabolic scaling, any 2π space-periodic solution $u(x, t)$ of the heat equation is an infinite superposition of the three solution functions

$$1, \quad e^{-t} \cos x, \quad e^{-t} \sin x. \quad (7)$$

We may call them the generating functions for the 2π space-periodic solutions of the heat equation.

With the help of Fourier series expansion (6), we can easily prove the following:

**Lemma 2.1.** Assume that $u(x, t)$ is a solution of the heat equation on $\mathbb{R}^2$ and is both space-periodic with period $a > 0$ and time-periodic with period $b > 0$, then it must be a constant function.

**Proof.** Let $v(x, t) = u(\lambda x, \lambda^2 t)$, where $\lambda = a/(2\pi)$. Then $v(x, t)$ is a solution of the heat equation which is space-periodic with period $2\pi > 0$ and time-periodic with period $b/\lambda^2 > 0$. By (6), we know that

$$\lim_{t \to \infty} v(x, t) = \frac{1}{2\pi} \int_0^{2\pi} v(x, 0) \, dx, \quad \forall x \in (-\infty, \infty).$$

Since it is also time-periodic with period $b/\lambda^2$, we have $v(x, t) = v(x, t + mb/\lambda^2)$ for all $m \in \mathbb{N}$ and all $(x, t) \in \mathbb{R}^2$. Letting $m \to \infty$, we obtain $v(x, t) \equiv \text{const.}$ and so is $u(x, t)$. \qed
uniquely determined if we know the initial condition. This, we can find the general real solutions of the system (12). The solution can be given by

\[
\begin{align*}
    a_n(x) &= \frac{1}{\pi} \int_0^{2\pi} u(x,t) \cos nt dt, \\
    b_n(x) &= \frac{1}{\pi} \int_0^{2\pi} u(x,t) \sin nt dt,
\end{align*}
\]

(10)

As we have, we get

\[
\sum_{n=1}^{\infty} n (-a_n(x) \sin nt + b_n(x) \cos nt) = \frac{a_0''(x)}{2} + \sum_{n=1}^{\infty} (a_n''(x) \cos nt + b_n''(x) \sin nt)
\]

and so we need to solve the system of ODE:

\[
\frac{a_0''(x)}{2} = 0, \quad \begin{cases} a_n''(x) = nb_n(x), \\
    b_n''(x) = -na_n(x), \quad n = 1, 2, 3, \ldots
\end{cases}
\]

(11)

We first have \(a_0(x) = cx + d\) for some real constants \(c, d\).

Let \(\tilde{a}_n(x) = a_n'(x)\) and \(\tilde{b}_n(x) = b_n'(x)\). The system (11) can be written as the first system for \((a_n, \tilde{a}_n, b_n, \tilde{b}_n)\):

\[
\begin{pmatrix} a_n'(x) \\ \tilde{a}_n'(x) \\ b_n'(x) \\ \tilde{b}_n'(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & 1 \\ -n & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_n(x) \\ \tilde{a}_n(x) \\ b_n(x) \\ \tilde{b}_n(x) \end{pmatrix}.
\]

(12)

The eigenvalues \(\lambda\) of the matrix in (12) satisfy the equation \(\lambda^4 + n^2 = 0\) and are given by \(\lambda = \pm \sqrt{n} \alpha, \pm \sqrt{n} \tilde{\alpha}\), where \(\alpha = 1/\sqrt{2} + i (1/\sqrt{2})\), \(\tilde{\alpha} = 1/\sqrt{2} - i (1/\sqrt{2})\). By this, we can find the general real solutions of the system (12). The solution can be uniquely determined if we know the initial condition \((a_n(0), \tilde{a}_n(0), b_n(0), \tilde{b}_n(0))\).

By (10) and \(\tilde{a}_n(x) = a_n'(x), \quad \tilde{b}_n(x) = b_n'(x)\), it suffices to know the values of \(u(0,t)\) and \(u_x(0,t)\) for \(t \in [0,2\pi]\).

From the above discussion, we can obtain the following main result for the time-periodic case:

**Lemma 2.4.** Assume that \(u(x,t)\) is a time-periodic solution of the heat equation with period \(2\pi\) and we know the values of \(u(0,t)\) and \(u_x(0,t)\) for \(t \in [0,2\pi]\). Then one can determine \(u(x,t)\) on \(\mathbb{R}^2\) uniquely. Moreover, \(u(x,t)\) has the following Fourier series expansion

\[
u(x,t) = \frac{cx + d}{2} + \sum_{n=1}^{\infty} \left( \gamma_1^{(n)} e^{\sqrt{\frac{n}{2}} x} \cos \left( \sqrt{\frac{n}{2}} x + nt \right) + \gamma_2^{(n)} e^{\sqrt{\frac{n}{2}} x} \sin \left( \sqrt{\frac{n}{2}} x + nt \right) + \gamma_3^{(n)} e^{-\sqrt{\frac{n}{2}} x} \cos \left( \sqrt{\frac{n}{2}} x - nt \right) + \gamma_4^{(n)} e^{-\sqrt{\frac{n}{2}} x} \sin \left( \sqrt{\frac{n}{2}} x - nt \right) \right),
\]

(13)

where the coefficients \(c, d, \gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}\) are uniquely determined by \(u(0,t)\) and \(u_x(0,t)\) for \(t \in [0,2\pi]\).

**Proof.** Since we know the eigenvalues of the matrix in (12), the general real solution for \(a_n(x)\) is given by

\[
a_n(x) = \gamma_1^{(n)} e^{\sqrt{\frac{n}{2}} x} \cos \sqrt{\frac{n}{2}} x + \gamma_2^{(n)} e^{\sqrt{\frac{n}{2}} x} \sin \sqrt{\frac{n}{2}} x + \gamma_3^{(n)} e^{-\sqrt{\frac{n}{2}} x} \cos \sqrt{\frac{n}{2}} x
\]
for arbitrary real constants $\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}$. By the first identity of (11), i.e. $b_n(x) = a''_n(x)/n$ and $a_n(x) = a'_n(x), \hat{b}_n(x) = b'_n(x)$, one can determine $b_n(x), \hat{a}_n(x), a_n(x)$. All of them are linear combinations of the four functions in (14), with coefficients certain suitable linear combinations of $\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}$.

Since we know $(a_n(0), \hat{a}_n(0), b_n(0), \hat{b}_n(0))$ and $a_0(0), a'_0(0)$, the six constants $c, d, \gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}$ can be determined uniquely.

Now we have

$$a_n(x) \cos nt + b_n(x) \sin nt = \begin{cases} \gamma_1^{(n)} e^{\sqrt{\frac{n}{2}} x} \cos \left(\sqrt{\frac{n}{2}} x + \frac{\gamma_2^{(n)} e^{\sqrt{\frac{n}{2}} x}}{\sqrt{\frac{n}{2}}} \right), \\ \gamma_3^{(n)} e^{-\sqrt{\frac{n}{2}} x} \cos \left(\sqrt{\frac{n}{2}} x + \frac{\gamma_4^{(n)} e^{-\sqrt{\frac{n}{2}} x}}{\sqrt{\frac{n}{2}}} \right) \end{cases} \cos nt$$

and the above can be simplified as

$$a_n(x) \cos nt + b_n(x) \sin nt = \begin{cases} \gamma_1^{(n)} e^{\sqrt{\frac{n}{2}} x} \cos \left(\sqrt{\frac{n}{2}} x + nt \right) + \gamma_2^{(n)} e^{\sqrt{\frac{n}{2}} x} \sin \left(\sqrt{\frac{n}{2}} x + nt \right), \\ \gamma_3^{(n)} e^{-\sqrt{\frac{n}{2}} x} \cos \left(\sqrt{\frac{n}{2}} x - nt \right) + \gamma_4^{(n)} e^{-\sqrt{\frac{n}{2}} x} \sin \left(\sqrt{\frac{n}{2}} x - nt \right). \end{cases}$$

From it, we get the series (13).

**Remark 2.5.** Note that each $n$-th term in the summation of (13) is a solution of the heat equation and is time-periodic with period $2\pi/n$. Up to parabolic rescaling $(x, t) \leftrightarrow (\lambda x, \lambda^2 t), \lambda = 1/\sqrt{n}$, it is a linear combination of

$$e^{\sqrt{\frac{n}{2}} x} \cos \left(\frac{x}{\sqrt{2}} \right), \quad e^{\sqrt{\frac{n}{2}} x} \sin \left(\frac{x}{\sqrt{2}} \right), \quad e^{-\sqrt{\frac{n}{2}} x} \cos \left(\frac{x}{\sqrt{2}} \right), \quad e^{-\sqrt{\frac{n}{2}} x} \sin \left(\frac{x}{\sqrt{2}} \right),$$

where each function in (15) is time-periodic with period $2\pi$. Thus, up to parabolic scaling, any $2\pi$ time-periodic solution $u(x, t)$ of the heat equation is an infinite superposition of the six solution functions:

$$1, \quad x, \quad e^{\pm \sqrt{\frac{n}{2}} x} \cos \left(\frac{x}{\sqrt{2}} \right), \quad e^{\pm \sqrt{\frac{n}{2}} x} \sin \left(\frac{x}{\sqrt{2}} \right).$$

Similar to (7), we call them the generating functions for the $2\pi$ time-periodic solutions of the heat equation.

Until now, we have known that for a space-periodic solution $u(x, t)$ of the heat equation with period $2\pi$, it is uniquely determined by $u(x, 0)$ for $x \in [0, 2\pi]$ (initial condition); and for a time-periodic solution $u(x, t)$ of the heat equation with period $2\pi$, it is uniquely determined by $u(0, t)$ and $u_x(0, t)$ for $t \in [0, 2\pi]$ (boundary condition). As we shall see in Lemma 2.6 below, these two conditions are actually equivalent. In fact, both conditions can appear in some simple situation. For example, let us try to find space-time polynomial solutions of the heat equation. If we look for a polynomial solution of the form

$$u(x, t) = p_0(x) + p_1(x) t + p_2(x) t^2 + p_3(x) t^3 + p_4(x) t^4 + \cdots \text{(finite terms only)},$$
where each \( p_i(x) \) is a polynomial in \( x \in (-\infty, \infty) \) and substitute (17) into the heat equation, we get the identity

\[
p_k(x) = \frac{1}{k!}p_0^{(2k)}(x) \quad \text{for all } k \in \mathbb{N}.
\]  

(18)

Since \( p_0(x) \) is a polynomial with finite degree, the above process will stop at some \( k \in \mathbb{N} \). Moreover, we see that all \( p_k(x) \), \( k \geq 1 \), are uniquely determined by \( p_0(x) = u(x,0) \), \( x \in (-\infty, \infty) \), which is the initial condition of the solution. On the other hand, if we look for a polynomial solution of the form

\[
u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}p_0^{(2k)}(x)\]

(19)

where each \( p_i(t) \) is a polynomial in \( t \in (-\infty, \infty) \) and substitute (19) into the heat equation, we get the identities

\[
p_k(t) = \begin{cases} 
p_0^{(m)}(t), & k = 2m \text{ is even} \\
p_1^{(m)}(t), & k = 2m + 1 \text{ is odd.}
\end{cases}
\]

In this case we see that all \( p_k(t) \), \( k \geq 2 \), are determined by the following boundary condition \( (x = 0) \):

\[
u(0,t) = p_0(t) \quad \text{and} \quad u_x(0,t) = p_1(t), \quad t \in (-\infty, \infty).
\]

(20)

We now prove that the two conditions are actually equivalent. Let \( u(x,t) \) be a solution of the heat equation on \( \mathbb{R} \) given by

\[
u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) \, dy, \quad x \in \mathbb{R}, \quad t > 0
\]

(21)

and \( v(x,t) \) be another solution of the heat equation on \( \mathbb{R} \) given by

\[
u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y) \, dy, \quad x \in \mathbb{R}, \quad t > 0,
\]

(22)

where both \( f(y) \) and \( g(y) \) are continuous functions on \( \mathbb{R} \) which grow at most exponentially. We have:

**Lemma 2.6.** Let \( u(x,t) \) and \( v(x,t) \) be given by (21) and (22) respectively. If \( u(x,t) \) and \( v(x,t) \) satisfy the same boundary condition, given by

\[
u(0,t) = v(0,t) \quad \text{and} \quad u_x(0,t) = v_x(0,t), \quad \forall \ t \in (0, \infty).
\]

(23)

Then we must have

\[
f(y) = g(y), \quad \forall \ y \in (-\infty, \infty).
\]

(24)

As a consequence, we have \( u(x,t) = v(x,t) \) for all \( x \in \mathbb{R}, \ t \in (0, \infty) \).

**Proof.** Let \( h(y) = f(y) - g(y) \). It is a continuous function on \( \mathbb{R} \) which grows at most exponentially. The condition (23) implies

\[
\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} h(y) \, dy = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} yh(y) \, dy = 0, \quad \forall \ t \in (0, \infty).
\]

(25)

We claim that \( h(y) \equiv 0 \). Since the two identities in (25) are true for all \( t \), one can differentiate the first identity with respect to \( t \) successively and get

\[
\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}} y^{2n} h(y) \, dy = 0, \quad \forall \ n = 0, 1, 2, 3, \ldots, \quad \forall \ t \in (0, \infty).
\]

(26)
Similarly, if we do the same on the second identity of (25), we get
\[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} y^{2n+1} h(y) \, dy = 0, \quad \forall \ n = 0, 1, 2, 3, ..., \quad \forall \ t \in (0, \infty). \tag{27} \]
Thus we conclude
\[ \int_{-\infty}^{\infty} \left( e^{-\frac{x^2}{4t}} h(y) \right) y^n \, dy = 0, \quad \forall \ n = 0, 1, 2, 3, ..., \quad \forall \ t \in (0, \infty), \tag{28} \]
which is the same as
\[ \int_{-\infty}^{\infty} \left( e^{-y^2} \right)^s h(y) y^n \, dy = 0, \quad \forall \ s \in (0, \infty), \quad \forall \ n = 0, 1, 2, 3, .... \tag{29} \]
The only function \( h(y) \) which can satisfy (29) is \( h(y) \equiv 0 \). The proof is done. \( \square \)

2.3. Space-time periodic solutions of the heat equation. In this section, we assume that \( u(x, t) \) is a solution of the heat equation which is space-time periodic with period \((a, b) \in \mathbb{R}^2, a > 0, b > 0\). We will perform a change of variables to convert it into time-periodic case or space-periodic case. By this, we can derive its Fourier series expansion.

2.3.1. Convert to time-periodic case. Consider the change of variables \((x, t) \leftrightarrow (\sigma, t)\), where
\[ \sigma = \frac{bx - at}{b} = x - \frac{a}{b} t, \quad u(x, t) \leftrightarrow v(\sigma, t). \tag{30} \]
Chain rule implies \( u_t = (-a/b)v_\sigma + v_t, \ u_x = v_\sigma, \ u_{xx} = v_{\sigma\sigma}. \) Therefore, if we have \( u_t = u_{xx} \), then \( v(\sigma, t) \) satisfies
\[ v_t = v_{\sigma\sigma} + \frac{a}{b} v_\sigma. \tag{31} \]
Moreover, if \( u(x, t) \) satisfies (4), \( v(\sigma, t) \) will satisfy
\[ v(\sigma, t + b) = u(x + a, t + b) = u(x, t) = v(\sigma, t), \quad \forall \ (\sigma, t) \in (-\infty, \infty) \times (-\infty, \infty), \tag{32} \]
i.e. \( v(\sigma, t) \) is time-periodic with period \( b > 0 \), satisfying equation (31), which is no longer a heat equation.

Instead of (9), the Fourier series for \( v(\sigma, t) \) now takes the form
\[ v(\sigma, t) = \frac{a_0}{2} (\sigma) + \sum_{n=1}^{\infty} \left( a_n (\sigma) \cos \left( \frac{2n\pi}{b} t \right) + b_n (\sigma) \sin \left( \frac{2n\pi}{b} t \right) \right), \quad \sigma = x - \frac{a}{b} t, \tag{33} \]
where
\[ \begin{cases} a_n (\sigma) = \frac{2}{b} \int_{0}^{b} v(\sigma, t) \cos \left( \frac{2n\pi}{b} t \right) \, dt, & n = 0, 1, 2, 3, ... \\ b_n (\sigma) = \frac{2}{b} \int_{0}^{b} v(\sigma, t) \sin \left( \frac{2n\pi}{b} t \right) \, dt, & n = 1, 2, 3, .... \end{cases} \tag{34} \]
By (31), and comparing the coefficients, we obtain the following system of second order ODE for the coefficients functions:
\[ ba''_0 (\sigma) + aa'_0 (\sigma) = 0, \quad \begin{cases} ba''_n (\sigma) = -aa'_n (\sigma) + 2n\pi b_n (\sigma), \\ bb''_n (\sigma) = -ab'_n (\sigma) - 2n\pi a_n (\sigma), \end{cases} \quad n = 1, 2, 3, .... \tag{35} \]
To determine the solution uniquely for each \( n \in \mathbb{N} \), we need to know the values of \( a_0 (0), \ a'_0 (0), \ a_n (0), \ a'_n (0), \ b_n (0), \ b'_n (0) \). \( \tag{36} \)
where we note that \( \sigma = 0 \) corresponds to the line \( bx - at = 0 \) and by (34) we need to know the values of \( v (0, t) = u (at/b, t) \) and \( v_x (0, t) = u_x (at/b, t) \) along the segment \( bx - at = 0 \), \( 0 < t < b \) (same as \( 0 < x < a \)).

**Remark 2.7.** In the special case when \( a = 0 \), \( b = 2\pi \), we get \( \sigma = x \), and the system (35) becomes the previous (11).

**Remark 2.8.** In terms of complex number, if we let \( z_n (\sigma) = a_n (\sigma) + ib_n (\sigma) \), then the system (35) can be written as the following single complex ODE:

\[
 b z_n'' (\sigma) + a z_n' (\sigma) + (2n\pi i) z_n (\sigma) = 0, \quad n = 1, 2, 3, \ldots \quad (37)
\]

Similarly to (12), we now obtain the 4 \times 4 system:

\[
\begin{pmatrix}
 a_n' (\sigma) \\
 \bar{a}_n' (\sigma) \\
 b_n' (\sigma) \\
 \bar{b}_n' (\sigma)
\end{pmatrix} = \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 0 & -a & 2n\pi & 0 \\
 0 & 0 & 0 & 1 \\
 -2n\pi & 0 & 0 & -a
\end{pmatrix} \begin{pmatrix}
 a_n (\sigma) \\
 \bar{a}_n (\sigma) \\
 b_n (\sigma) \\
 \bar{b}_n (\sigma)
\end{pmatrix}. \quad (38)
\]

The eigenvalues \( \lambda \) of the coefficient matrix in (38) satisfy the characteristic equation

\[
 \left( \lambda + \frac{a}{b} \right)^2 + \left( \frac{2n\pi}{b} \right)^2 = 0, \quad n = 1, 2, 3, \ldots
\]

which gives \( \lambda^2 + (a/b) \lambda = \pm (2n\pi/b) i \).

If we let \( \lambda = p + iq \), \( p, q \in \mathbb{R} \), then for the equation \( \lambda^2 + (a/b) \lambda = (2n\pi/b) i \), we get the system

\[
 p^2 - q^2 + \frac{a}{b} p = 0, \quad \left( 2p + \frac{a}{b} \right) q = \frac{2n\pi}{b},
\]

which gives two solutions for \( p, q \), given by

\[
(p, q) = \left( -\frac{1}{2b} (a - \Gamma), \frac{1}{2b} \Lambda \right), \quad \left( -\frac{1}{2b} (a + \Gamma), \frac{1}{2b} \Lambda \right), \quad (39)
\]

where\(^2\)

\[
\Gamma = \frac{\sqrt{2}}{2} \sqrt{a^4 + 64\pi^2 b^2 n^2 + a^2}, \quad \Lambda = \frac{\sqrt{2}}{2} \sqrt{a^4 + 64\pi^2 b^2 n^2 - a^2}. \quad (40)
\]

On the other hand, for the equation \( \lambda^2 + (a/b) \lambda = -(2n\pi/b) i \), we get the solutions

\[
(p, q) = \left( -\frac{1}{2b} (a - \Gamma), -\frac{1}{2b} \Lambda \right), \quad \left( -\frac{1}{2b} (a + \Gamma), \frac{1}{2b} \Lambda \right). \quad (41)
\]

Thus the four eigenvalues of the coefficient matrix in (38) are given by

\[
\lambda = \frac{1}{2b} \left[ -(a + \Gamma) \pm i\Lambda \right], \quad \frac{1}{2b} \left[ -(a - \Gamma) \pm i\Lambda \right]. \quad (42)
\]

By the identity \( p^2 - q^2 + \frac{a}{b} p = 0 \), \( \Gamma \) and \( \Lambda \) satisfy the equations:

\[
\begin{cases}
 \left[ \frac{1}{2b} (a + \Gamma) \right]^2 - \left( \frac{1}{2b} \Lambda \right)^2 - \frac{a}{b} \left[ \frac{1}{2b} (a + \Gamma) \right] = 0 \\
 \left[ \frac{1}{2b} (a - \Gamma) \right]^2 - \left( \frac{1}{2b} \Lambda \right)^2 - \frac{a}{b} \left[ \frac{1}{2b} (a - \Gamma) \right] = 0.
\end{cases} \quad (43)
\]

Other useful identities for \( \Gamma \), \( \Lambda \) are

\[
\Gamma^2 + \Lambda^2 = \sqrt{a^4 + 64\pi^2 b^2 n^2}, \quad \Gamma^2 - \Lambda^2 = a^2, \quad \Gamma\Lambda = 4n\pi b. \quad (44)
\]

\(^2\)Note that the numbers \( \Gamma \) and \( \Lambda \) here both depend on \( n \).
Thus the general real solution for $a_n (\sigma)$ is given by

$$a_n (\sigma) = \begin{cases} 
\gamma_1^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_2^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) \\
+ \gamma_3^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_4^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) 
\end{cases} \quad (45)$$

for arbitrary real constants $\gamma_1^{(n)}$, $\gamma_2^{(n)}$, $\gamma_3^{(n)}$, $\gamma_4^{(n)}$.

As for $b_n (\sigma)$, one can determine it by the first equation of (35), i.e.

$$b_n (\sigma) = \frac{1}{2n\pi} \left[ ba''_n (\sigma) + aa'_n (\sigma) \right]. \quad (46)$$

By the help of (43) and (44) and after simplification, we can get

$$b_n (\sigma) = \begin{cases} 
- \gamma_2^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_1^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) \\
+ \gamma_4^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) - \gamma_3^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) 
\end{cases} \quad (47)$$

As for $a'_0 (\sigma)$, it satisfies the equation $ba''_0 (\sigma) + aa'_0 (\sigma) = 0$ with general solution given by

$$a_0 (\sigma) = ce^{-\frac{\alpha}{2} \sigma} + d, \quad \text{where} \quad \sigma = x - \frac{a}{b} t$$

for arbitrary constants $c$, $d$. Thus the Fourier series (33) takes the form

$$u (x, t) = v (\sigma, t) = \frac{ce^{-\frac{\alpha}{2} \sigma} + d}{2} + \sum_{n=1}^{\infty} \left[ \begin{array}{c} 
\gamma_1^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_2^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) \\
+ \gamma_3^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_4^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) \\
- \gamma_2^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) + \gamma_1^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) \\
+ \gamma_4^{(n)} e^{-\frac{\alpha + \beta}{2\sigma} \sigma} \cos \left( \frac{\Lambda n}{2 \sigma} \right) - \gamma_3^{(n)} e^{-\frac{\alpha - \beta}{2\sigma} \sigma} \sin \left( \frac{\Lambda n}{2 \sigma} \right) 
\end{array} \right] \cos \left( \frac{2\pi n}{b} t \right). \quad (48)$$

**Remark 2.9.** One can check that when $a = 0$, $b = 2\pi$, the above is reduced to (after a rearrangement of constants) (13).

Similar to the discussions in the case when $a = 0$, $b = 2\pi$, one can simplify the eight terms in the summation of (48) into four terms. As a result, we conclude the following:

**Theorem 2.10.** Let $a > 0$, $b > 0$ be two given numbers and let $p (t)$, $q (t)$ be two given periodic functions (with period $b > 0$) defined on $t \in (-\infty, \infty)$. Then there is a unique solution $u (x, t)$ of the heat equation $u_t = u_{xx}$ satisfying

$$u \left( \frac{a}{b}, t \right) = p (t), \quad u_x \left( \frac{a}{b}, t \right) = q (t), \quad \forall \ t \in [0, b] \quad (49)$$

and the space-time periodic condition

$$u (x + a, t + b) = u (x, t), \quad \forall \ (x, t) \in \mathbb{R}^2. \quad (50)$$
Moreover, \( u(x,t) \) has the following Fourier series expansion

\[
\begin{align*}
\quad u(x,t) &= v(\sigma,t) = ce^{-\frac{\sigma}{b}t} + \frac{d}{2} \\
&\quad + \sum_{n=1}^{\infty} \left( \gamma_1^{(n)} e^{-\frac{\sigma a}{b}t} \cos \left( \frac{n\pi}{b} \sigma - \frac{2n\pi}{b} t \right) + \gamma_2^{(n)} e^{-\frac{\sigma a}{b}t} \sin \left( \frac{n\pi}{b} \sigma - \frac{2n\pi}{b} t \right) \\
&\quad + \gamma_3^{(n)} e^{-\frac{\sigma a}{b}t} \cos \left( \frac{n\pi}{b} \sigma + \frac{2n\pi}{b} t \right) + \gamma_4^{(n)} e^{-\frac{\sigma a}{b}t} \sin \left( \frac{n\pi}{b} \sigma + \frac{2n\pi}{b} t \right) \right),
\end{align*}
\]

where

\[
\begin{align*}
\sigma &= x - \frac{a}{b} t, \quad \Gamma = \frac{\sqrt{2}}{2} \sqrt{\frac{1}{a^4 + 64\pi^2 b^2 n^2 + a^2}}, \quad \Lambda = \frac{\sqrt{2}}{2} \sqrt{\frac{1}{a^4 + 64\pi^2 b^2 n^2 - a^2}}.
\end{align*}
\]

Here the coefficients \( c, d, \gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)} \) in (51) are uniquely determined by the condition (49).

**Remark 2.11.** The series (51) looks much more complicated than (6) and (13). Same as before, we note that each \( n \)-th term in the summation of (51) is itself a space-time periodic solution of the heat equation with period \((a/n, b/n)\) (note that the direction of the vector \((a/n, b/n)\) is the same as the direction of the vector \((a,b)\)). A natural question here is whether there are generating functions for the series (51). Due to Remark 1.3, our understanding is that, unlike the generating functions in (7) and (16), there are no generating functions for space-time periodic solutions of the heat equation with period \((a,b)\), \( a > 0, b > 0 \).

### 2.3.2 Convert to space-periodic case.

Instead of (30), now we consider the change of variables: \((x,t) \longleftrightarrow (x,\tau)\), where

\[
\tau = \frac{at - bx}{a} = t - \frac{b}{a} x, \quad (x,t) \longleftrightarrow (x,\tau).
\]

Chain rule implies

\[
\begin{align*}
\quad u_t &= w_\tau, \quad u_x = w_x - \frac{b}{a} w_\tau, \quad u_{xx} = w_{xx} - \frac{2b}{a} w_{x\tau} + \frac{b^2}{a^2} w_{\tau\tau}.
\end{align*}
\]

Therefore, if \( u(x,t) \) satisfies the heat equation \( u_t = u_{xx} \), \( w(x,\tau) \) satisfies

\[
\begin{align*}
\quad w_\tau &= w_{xx} - \frac{2b}{a} w_{x\tau} + \frac{b^2}{a^2} w_{\tau\tau}.
\end{align*}
\]

Moreover, we have

\[
\begin{align*}
\quad w(x+a,\tau) &= u(x+a,t+b) = u(x,t) = w(x,\tau), \quad \forall (x,\tau) \in (-\infty,\infty) \times (-\infty,\infty),
\end{align*}
\]

i.e. \( w(x,\tau) \) is now space-periodic with period \( a > 0 \), satisfying equation (53), which is unfortunately much more complicated than the heat equation.

As \( w(x,\tau) \) is space-periodic with period \( a \), one can use Fourier series to expand it as

\[
\begin{align*}
\quad w(x,\tau) &= \frac{a_0(\tau)}{2} + \sum_{n=1}^{\infty} \left( a_n(\tau) \cos \left( \frac{2n\pi}{a} x \right) + b_n(\tau) \sin \left( \frac{2n\pi}{a} x \right) \right), \quad \tau = t - \frac{b}{a} x,
\end{align*}
\]

where

\[
\begin{align*}
\quad \left\{ \begin{array}{ll}
\quad a_n(\tau) &= \frac{2}{a} \int_0^a w(x,\tau) \cos \left( \frac{2n\pi}{a} x \right) \, dx, & n = 0, 1, 2, 3, \\
\quad b_n(\tau) &= \frac{2}{a} \int_0^a w(x,\tau) \sin \left( \frac{2n\pi}{a} x \right) \, dx, & n = 1, 2, 3, \ldots
\end{array} \right.
\]

(55)
From (54) and (53), we get
\[ w_\tau = \frac{a_\beta(\tau)}{2} + \sum_{n=1}^{\infty} \left( a_n(\tau) \cos \left( \frac{2n\pi}{\alpha} x \right) + b_n(\tau) \sin \left( \frac{2n\pi}{\alpha} x \right) \right) \]
and
\[ \begin{cases} w_{xx} = -\sum_{n=1}^{\infty} \left( \frac{2n\pi}{\alpha} \right)^2 \left( a_n(\tau) \cos \left( \frac{2n\pi}{\alpha} x \right) + b_n(\tau) \sin \left( \frac{2n\pi}{\alpha} x \right) \right) \\ \frac{-2b}{a} w_\tau = -\frac{2b}{a} \sum_{n=1}^{\infty} \left( \frac{2n\pi}{\alpha} \right) \left( b_n'(\tau) \cos \left( \frac{2n\pi}{\alpha} x \right) - a_n'(\tau) \sin \left( \frac{2n\pi}{\alpha} x \right) \right) \\ \frac{b^2}{\alpha^2} w_{\tau\tau} = \frac{b^2}{\alpha^2} \frac{a_n''(\tau)}{2} + \frac{b^2}{\alpha^2} \sum_{n=1}^{\infty} \left( a_n''(\tau) \cos \left( \frac{2n\pi}{\alpha} x \right) + b_n''(\tau) \sin \left( \frac{2n\pi}{\alpha} x \right) \right) \end{cases} \]
Comparing the coefficients, we need to solve \( b^2 a_n''(\tau) = a^2 a_n'(\tau) \) and the system of second order ODE for the coefficients functions:
\[ \begin{cases} b^2 a_n''(\tau) = a^2 a_n'(\tau) + (4n\pi b) b_n'(\tau) + 4n^2 \pi^2 a_n(\tau), \\ b^2 b_n''(\tau) = a^2 b_n'(\tau) - (4n\pi b) a_n'(\tau) + 4n^2 \pi^2 b_n(\tau), \quad n = 1, 2, 3, \ldots \end{cases} \]  
(56)
To determine the solution uniquely, we need to know the values of
\[ a_0(0), \quad a_0'(0), \quad a_n(0), \quad a_n'(0), \quad b_n(0), \quad b_n'(0), \]
where we note that \( \tau = 0 \) corresponds to the line \( bx - at = 0 \) and by (55) we need to know the values of \( w(0,0) = u(x,bx/a) \) and \( w_\tau(0,0) = u_t(x,bx/a) \) along the segment \( bx - at = 0, \quad 0 < x < a \) (same as \( 0 < t < b \)).

Remark 2.12. In the special case when \( a = 2\pi, \quad b = 0 \), we get \( \tau = t \), and the system (56) becomes a first order system:
\[ \begin{align*} a_0'(\tau) &= 0, \\ a_n'(\tau) &= -n^2 a_n(\tau), \\ b_n'(\tau) &= -n^2 b_n(\tau), \quad n = 1, 2, 3, \ldots, \end{align*} \]
which gives the familiar expansion (6).

Remark 2.13. In terms of complex number, if we let \( z_n(\tau) = a_n(\tau) + ib_n(\tau) \), then the system (56) can be written as the following single complex ODE:
\[ b^2 z''_n(\tau) + (4n\pi bi - a^2) z'_n(\tau) - 4n^2 \pi^2 z_n(\tau) = 0, \quad n = 1, 2, 3, \ldots \]  
(58)

By the above discussion, we can temporarily conclude the following:

Lemma 2.14. Let \( a > 0, \quad b > 0 \) be two given numbers and let \( p(x), \quad q(x) \) be two given periodic functions (with period \( a > 0 \)) defined on \( x \in (-\infty, \infty) \). Then there is a unique solution \( u(x,t) \) of the heat equation \( u_t = u_{xx} \) satisfying
\[ u \left( x, \frac{b}{a} t \right) = p(x), \quad u_t \left( x, \frac{b}{a} t \right) = q(x), \quad \forall \ x \in [0,a] \]  
(59)
and the space-time periodic condition
\[ u(x + a, t + b) = u(x, t), \quad \forall \ (x,t) \in \mathbb{R}^2. \]  
(60)

The Fourier series expansion of \( u(x,t) \) is, however, much more difficult to derive. We will discuss it just briefly in the following. Letting \( \tilde{a}_n(\tau) = a_n'(\tau), \quad \tilde{b}_n(\tau) = b_n'(\tau) \), the system (56) can be written as the first system for \( (a_n, \tilde{a}_n, b_n, \tilde{b}_n) \):
\[
\begin{pmatrix}
  a_n'(\tau) \\
  \tilde{a}_n'(\tau) \\
  b_n'(\tau) \\
  \tilde{b}_n'(\tau)
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  \frac{4n^2 \pi^2}{b^2} & \frac{a^2}{b^2} & 0 & 0 \\
  0 & 0 & 0 & \frac{4n\pi}{b} \\
  0 & -\frac{4n\pi}{b} & \frac{4n^2 \pi^2}{b^2} & \frac{a^2}{b^2}
\end{pmatrix}
\begin{pmatrix}
  a_n(\tau) \\
  \tilde{a}_n(\tau) \\
  b_n(\tau) \\
  \tilde{b}_n(\tau)
\end{pmatrix}
\]  
(61)
We also have \( a_0(\tau) = ce^{(a^2/\pi^2)\tau + d} \) for some real constants \( c, d \).

The coefficient matrix \( M \) in (61) has the following characteristic equation

\[
\left[ \lambda \left( \frac{a^2}{b^2} - \lambda \right) + \frac{4n^2\pi^2}{b^2} \right]^2 + \left( \lambda \frac{4n\pi}{b} \right)^2 = 0,
\]

which gives

\[
\lambda \left( \frac{a^2}{b^2} - \lambda \right) + \frac{4n^2\pi^2}{b^2} = \pm \left( \lambda \frac{4n\pi}{b} \right) i. \tag{62}
\]

If we let \( \lambda = p + iq, \ p, q \in \mathbb{R} \), then for the equation \( \lambda (a^2/b^2 - \lambda) + 4n^2\pi^2/b^2 = (\lambda 4n\pi/b) i \), we get the system

\[
(p - \beta)^2 = (q + \delta)^2 + \beta^2, \quad (p - \beta) q = -p\delta,
\]

where \( \beta = a^2/(2b^2) \), \( \delta = 2n\pi/b \). One can check that the two solutions for \( p, q \) in (63) are given by

\[
(p, q) = \left( \beta - \frac{\beta^2 + \Psi^2}{\beta\delta}, -\delta + \Psi \right), \quad \left( \beta + \frac{\beta^2 + \Psi^2}{\beta\delta}, -\delta - \Psi \right), \tag{64}
\]

where

\[
\Psi = \frac{\sqrt{2}}{2} \sqrt{\beta\sqrt{\beta^2 + 4\delta^2} - \beta^2}, \quad \beta = \frac{a^2}{2b^2}, \quad \delta = \frac{2n\pi}{b}. \tag{65}
\]

On the other hand, for the equation \( \lambda (a^2/b^2 - \lambda) + 4n^2\pi^2/b^2 = -(\lambda 4n\pi/b) i \), we get the system (63) with \( \delta \) replaced by \(-\delta\) and the two solutions in this case are

\[
(p, q) = \left( \beta + \frac{\beta^2 + \Psi^2}{\beta\delta}, \delta + \Psi \right), \quad \left( \beta - \frac{\beta^2 + \Psi^2}{\beta\delta}, \delta - \Psi \right). \tag{66}
\]

Thus the four eigenvalues of the coefficients matrix \( M \) in (61) are given by

\[
\lambda = \left( \beta + \frac{\beta^2 + \Psi^2}{\beta\delta} \right) \pm i (\delta + \Psi), \quad \left( \beta - \frac{\beta^2 + \Psi^2}{\beta\delta} \right) \pm i (\delta - \Psi), \tag{67}
\]

where \( \Psi \) is given by (65). Moreover, if we substitute \( \beta = a^2/(2b^2) \) and \( \delta = 2n\pi/b \) into \( \Psi \), we find

\[
\Psi = \frac{\sqrt{2}}{2} \sqrt{\beta\sqrt{\beta^2 + 4\delta^2} - \beta^2} = \frac{a}{2b^2} \Lambda, \tag{68}
\]

where \( \Lambda \) is in (40). In particular, we have

\[
\beta^2 + \Psi^2 = \frac{a^4}{4b^4} + \left( \frac{a^2}{2b^2} \right)^2 = \frac{a^4}{4b^4} + \frac{a^2}{4b^4} \left( \frac{1}{2} \sqrt{a^4 + 64\pi^2b^2n^2} - a^2 \right) = \frac{a^2}{4b^4} \Gamma^2,
\]

which, together with the identity \( \Gamma \Lambda = 4n\pi b \), gives

\[
\beta + \frac{\beta^2 + \Psi^2}{\beta\delta} \Psi = \frac{a}{2b^2} (a + \Gamma), \quad \beta - \frac{\beta^2 + \Psi^2}{\beta\delta} \Psi = \frac{a}{2b^2} (a - \Gamma). \tag{69}
\]

Thus the four eigenvalues in (67) can be written as

\[
\lambda = \frac{a}{2b^2} (a + \Gamma) \pm i \left( \frac{2n\pi}{b} + \frac{a}{2b^2} \Lambda \right), \quad \frac{a}{2b^2} (a - \Gamma) \pm i \left( \frac{2n\pi}{b} - \frac{a}{2b^2} \Lambda \right), \tag{70}
\]

where \( \Gamma, \Lambda \) are from (40).
By (70), the general real solution for \(a_n(\tau)\) is given by

\[
a_n(\tau) = \left\{ \begin{array}{ll}
+\frac{\gamma_1^{(n)}}{1!} e^{\frac{2\pi}{b^2}(a+\Gamma)\tau} \cos \left((\frac{2\pi}{b} + \frac{a}{2b^2} \Lambda) \tau\right) + \frac{\gamma_2^{(n)}}{2!} e^{\frac{2\pi}{b^2}(a-\Gamma)\tau} \sin \left((\frac{2\pi}{b} + \frac{a}{2b^2} \Lambda) \tau\right) \\
+\frac{\gamma_3^{(n)}}{3!} e^{\frac{2\pi}{b^2}(a+\Gamma)\tau} \cos \left((\frac{2\pi}{b} - \frac{a}{2b^2} \Lambda) \tau\right) + \frac{\gamma_4^{(n)}}{4!} e^{\frac{2\pi}{b^2}(a-\Gamma)\tau} \sin \left((\frac{2\pi}{b} - \frac{a}{2b^2} \Lambda) \tau\right)
\end{array} \right.
\]  

for arbitrary real constants \(\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}\). If we know the values of \(u(x, bx/a)\) and \(u_t(x, bx/a)\) for \(x \in (0, a)\) (see (59)), then one can determine the constants \(\gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_3^{(n)}, \gamma_4^{(n)}\) uniquely. As for \(b_n(\tau)\), by the first equation of the system (56), we have

\[
b'_n(\tau) = \frac{1}{4n\pi b} \left[ b^2a''_n(\tau) - a^2a'_n(\tau) - 4n^2\pi^2a_n(\tau) \right],
\]

which gives

\[
b''_n(\tau) = \frac{1}{4n\pi b} \left[ b^2a'''_n(\tau) - a^2a''_n(\tau) - 4n^2\pi^2a'_n(\tau) \right]
\]

and the second equation of (56) gives

\[
b_n(\tau) = \frac{1}{4n^2\pi^2} \left[ b^2b'_n(\tau) - a^2b'_n(\tau) + (4n\pi b) a'_n(\tau) \right]
\]

\[
= \frac{1}{4n^2\pi^2} \left\{ \frac{b^2}{4n\pi b} a''_n(\tau) - a^2a'_n(\tau) - 4n^2\pi^2a_n(\tau) \right\}
\]

\[
= \frac{1}{4n^2\pi^2} \left\{ \frac{b^4}{4n^2\pi b} a''''_n(\tau) - a^2b^2a'_n(\tau) - 4n^2\pi^2a''_n(\tau) \right\} + \left( \frac{3\pi b}{4n\pi b} + \frac{a^4}{4n^2 \pi b} \right) a''_n(\tau) + \frac{a^2}{b} n\pi a_n(\tau)
\]  

(72)

By (72), one can also determine the coefficients for \(b_n(\tau)\) uniquely.

Since we already know the functions \(a_0(\tau), a_n(\tau), b_n(\tau)\), one can determine the Fourier series expansion for \(u(x, t)\) in (54). Letting \(\tau = t - bx/a\), the Fourier series expansion for \(u(x, t)\) can be determined. Computational details are left to the readers.

Finally, one can easily verify the following identities:

\[
\left\{ \begin{array}{ll}
\frac{a}{2b^2} (a + \Gamma) \tau = -\frac{1}{2b} (a + \Gamma) \sigma, & \frac{a}{2b^2} (a - \Gamma) \tau = -\frac{1}{2b} (a - \Gamma) \sigma \\
\left( \frac{2\pi}{b} + \frac{a}{2b^2} \Lambda \right) \tau = -\left( \frac{1}{2b} + \frac{2\pi}{a} \right) \sigma, & \left( \frac{2\pi}{b} - \frac{a}{2b^2} \Lambda \right) \tau = \left( \frac{1}{2b} - \frac{2\pi}{a} \right) \sigma
\end{array} \right.
\]

(73)

where \(\tau = t - bx/a\), \(\sigma = x - at/b\). For a space-time periodic solution \(u(x, t)\) of the heat equation with period \((a, b)\), (73) provides a relation between the expansion of \(u(x, t)\) with respect to time and the expansion of \(u(x, t)\) with respect to space.

3. Some remarks on the space-time periodic solutions of a general linear equation. Consider a linear equation of the form

\[
u_t = u_{xx} + Au_x + Bu, \quad u = u(x, t),
\]

where \(A, B\) are two real constants. If we let

\[
r(x, t) = e^{\frac{2\pi}{b^2}(B - \frac{A^2}{4})t} u(x, t),
\]

(75)

we see that \(u\) satisfies (74) if and only if \(r\) satisfies the heat equation \(r_t = r_{xx}\). Therefore, if \(r(x, t)\) is space-time periodic with period \((B - A^2/4, A/2) \in \mathbb{R}^2\), so is \(u(x, t)\), and vice versa. However, for period different from \((B - A^2/4, A/2)\), there seems to be no useful relation at all.

Since equation (74) is linear, the method of Fourier series expansion is still valid. For example, if \(u(x, t)\) is space-time periodic with period \((a, b), a > 0, b >
0, then we can perform the change of variables as in (30) and obtain the equation for \( v(\sigma,t) \), which is now time-periodic with period \( b \) and satisfies

\[
v_{t} = v_{\sigma\sigma} + \left(A + \frac{a}{b}\right)v_{\sigma} + Bv.
\]  

(76)

Now we can expand \( v(\sigma,t) \) in Fourier series with respect to time and get a system of second order ODE for the coefficients functions \( a_{0}(\sigma) \), \( a_{n}(\sigma) \), \( b_{n}(\sigma) \). Then the remaining procedure is exactly the same as that in the case \( u_{t} = u_{xxz} \), except that now the system of ODE is more difficult to solve. The conclusion is the same if we perform the change of variables as in (52) to convert \( u(x,t) \) into a space-periodic function \( w(x,\tau) \).

3.1. Complex exponential function method. As equation (74) is linear, by the help of complex exponential function, we can obtain specific space, time, and space-time periodic real solutions.

The following result is obvious. For convenience of reference, we state it as a lemma:

**Lemma 3.1.** Let \( a \geq 0, b \geq 0 \) be two given constants with \( a^{2} + b^{2} > 0 \). Assume that \( u(x,t) \) is a complex function with the form

\[
u(x,t) = e^{(z^{2}+Az+B)t+zx}, \quad (x,t) \in \mathbb{R}^{2},
\]  

(77)

where \( z \in \mathbb{C} \) is a complex number satisfying the equation

\[
b(z^{2}+Az+B) + az = 2\pi ki
\]  

(78)

for some integer \( k \in \mathbb{Z} \), then \( u(x,t) \) is a complex solution of (74) on \( (x,t) \in \mathbb{R}^{2} \) and is space-time periodic with period \( (a,b) \in \mathbb{R}^{2} \). In particular, the real part (or imaginary part) of \( u(x,t) \) is a real solution of (74) and is space-time periodic with period \( (a,b) \in \mathbb{R}^{2} \).

**Remark 3.2.** Lemma 3.1 includes the cases \( a = 0, b > 0 \) and \( a > 0, b = 0 \). Also, for \( b > 0, a \geq 0 \), and \( z \) satisfying (78), one can write the complex solution \( u(x,t) \) as

\[
u(x,t) = e^{(z^{2}+Az+B)t+zx} = e^{z(x-\frac{a}{2}t)} \cdot e^{i\left(\frac{2\pi k}{a}t\right)}, \quad (x,t) \in \mathbb{R}^{2}
\]  

(79)

and for \( b = 0, a > 0 \), and \( z \) satisfying (78) (now \( z = 2\pi ki/a \)), one can write the complex solution \( u(x,t) \) as

\[
u(x,t) = e^{(z^{2}+Az+B)t+zx} = e^{\left(B^{-\frac{a^{2}+z^{2}}{a^{2}}}t+\frac{2\pi k}{a}(x+At)\right)}.
\]  

(80)

In the following we take \( k = 1 \) in (78) and use Lemma 3.1 to derive three periodic solutions of (74). For arbitrary \( k \in \mathbb{Z} \), the derivation is similar.

**Example 3.3.** Take \( k = 1, a > 0, b = 0 \) in (78). We get the solution \( z = 2\pi i/a \). Hence by (80) we get

\[
u(x,t) = e^{\left(B^{-\frac{a^{2}+z^{2}}{a^{2}}}t+\frac{2\pi k}{a}(x+At)\right)}
\]  

\[
= e^{\left(B^{-\frac{a^{2}+z^{2}}{a^{2}}}t\right)} \left[ \cos \left(\frac{2\pi}{a}(x+At)\right) + i \sin \left(\frac{2\pi}{a}(x+At)\right) \right].
\]

The real and imaginary part of the above function are space-periodic solutions of (74) on \( \mathbb{R}^{2} \) with period \( a > 0 \).
Example 3.4. Take $k = 1, \ a = 0, \ b > 0$ in (78). We get the equation

$$z^2 + Az + B = \frac{2\pi}{b}i,$$  \hspace{1cm} (81)

which can be written as

$$\left(z + \frac{A}{2}\right)^2 = \left(\frac{A^2}{4} - B\right) + \frac{2\pi}{b}i$$

and if we let $z + A/2 = p + iq, \ p, \ q \in \mathbb{R}$, we get the system

$$p^2 - q^2 = \frac{A^2}{4} - B, \ pq = \frac{\pi}{b}.$$  \hspace{1cm} (82)

One can verify that the two real solutions for $p, q$ are given by

$$(p,q) = \pm (\Delta, \Theta),$$

where

$$\Delta = \frac{\sqrt{2}}{4} \sqrt{\left(A^2 - 4B\right)^2 + \left(\frac{8\pi}{b}\right)^2 + (A^2 - 4B)} > 0$$

and

$$\Theta = \frac{\sqrt{2}}{4} \sqrt{\left(A^2 - 4B\right)^2 + \left(\frac{8\pi}{b}\right)^2 - (A^2 - 4B)} > 0.$$  

Thus we get the two complex solutions for equation (81), given by

$$z(x,t) = e^{(z^2 + Az + B)t + zx} = e^{(-\frac{A}{2} + \Delta)x + i\left(\frac{2\pi}{b}t + \Theta x\right)}.$$  

Thus the functions

$$w(x,t) = e^{(-\frac{A}{2} + \Delta)x} \cos\left(\frac{2\pi}{b}t + \Theta x\right), \ v(x,t) = e^{(-\frac{A}{2} + \Delta)x} \sin\left(\frac{2\pi}{b}t + \Theta x\right).$$  \hspace{1cm} (83)

are two time-periodic solutions of (74) on $\mathbb{R}^2$ with period $b > 0$. Another two solutions are given by

$$\tilde{w}(x,t) = e^{(-\frac{A}{2} - \Delta)x} \cos\left(\frac{2\pi}{b}t - \Theta x\right), \ \tilde{v}(x,t) = e^{(-\frac{A}{2} - \Delta)x} \sin\left(\frac{2\pi}{b}t - \Theta x\right).$$  \hspace{1cm} (84)

Example 3.5. Take $k = 1, \ a > 0, \ b > 0$ in (78). We get the equation

$$z^2 + \left(A + \frac{a}{b}\right)z + B = \frac{2\pi}{b}i,$$  \hspace{1cm} (85)

which is the same as (81) except that we replace $A$ by $A + a/b$. Thus the two complex solutions of (85) are given by $z = (- (A + a/b)/2) \pm (\Delta_0 + i\Theta_0)$, where now

$$\Phi = \frac{\sqrt{2}}{4} \sqrt{\left[\left(A + \frac{a}{b}\right)^2 - 4B\right]^2 + \left(\frac{8\pi}{b}\right)^2 + \left[\left(A + \frac{a}{b}\right)^2 - 4B\right]}$$

and

$$\Psi = \frac{\sqrt{2}}{4} \sqrt{\left[\left(A + \frac{a}{b}\right)^2 - 4B\right]^2 + \left(\frac{8\pi}{b}\right)^2 - \left[\left(A + \frac{a}{b}\right)^2 - 4B\right]}.$$
If we take the + sign in \( z \), by (85), we have
\[
z^2 + Az + B = \frac{2\pi}{b} - \frac{a}{b} z = \frac{1}{2} \left( A + \frac{a}{b} \right) - \frac{a}{b} \Psi + i \left( \frac{2\pi}{b} - \frac{a}{b} \right)
\]
and
\[
(z^2 + Az + B) t + zx = \left[ \frac{1}{2} \left( A + \frac{a}{b} \right) - \frac{a}{b} \Phi + i \left( \frac{2\pi}{b} - \frac{a}{b} \right) \right] t + \left[ -\frac{1}{2} \left( A + \frac{a}{b} \right) + \Phi + i \Psi \right] x
\]
\[
= \left[ -\frac{1}{2} \left( A + \frac{a}{b} \right) + \Phi \right] \left( x - \frac{a}{b} t \right) + i \left[ \frac{2\pi}{b} t + \Psi \left( x - \frac{a}{b} t \right) \right]
\]
\[
:= Z_1(x, t) + iZ_2(x, t).
\]

By the above we get two space-time periodic solutions of (74) on \( \mathbb{R}^2 \) with period \( (a, b) \), \( a > 0, \ b > 0 \):
\[
w(x, t) = e^{Z_1(x, t)} \cos (Z_2(x, t)), \quad v(x, t) = e^{Z_1(x, t)} \sin (Z_2(x, t)). \tag{86}
\]

Another two solutions are given by
\[
\bar{w}(x, t) = e^{Z_3(x, t)} \cos (Z_4(x, t)), \quad \bar{v}(x, t) = e^{Z_3(x, t)} \sin (Z_4(x, t)), \tag{87}
\]
where
\[
Z_3(x, t) = \left[ -\frac{1}{2} \left( A + \frac{a}{b} \right) - \Phi \right] \left( x - \frac{a}{b} t \right), \quad Z_4(x, t) = \frac{2\pi}{b} t - \Psi \left( x - \frac{a}{b} t \right).
\]

An important question for equation (74) is the existence of non-constant bounded periodic solutions of (74) on \( \mathbb{R}^2 \). These periodic solutions are interesting since we can capture them due to their boundedness. We have seen one such example in Remark 2.3. For the heat equation \( u_t = u_{xx} \), any bounded solution must be a constant due to Liouville theorem. However, unlike the heat equation, for equation (74) with \( B > 0 \), we do have non-constant bounded periodic solutions on \( \mathbb{R}^2 \). More precisely, we have:

**Lemma 3.6.** (a) If equation (74) has a complex solution \( u(x, t) \) of the form (77) and is non-constant, bounded, and periodic (either space-periodic or time-periodic or space-time periodic), then we must have \( B > 0 \). (b) Conversely, assume \( B > 0 \). If \( a \geq 0, \ b \geq 0 \) (with \( a^2 + b^2 > 0 \)) and \( k \in \mathbb{Z} \) satisfy the identity
\[
(a + bA) \sqrt{B} = 2k\pi \tag{88}
\]
then
\[
u(x, t) = \cos \left( \sqrt{B} (x + At) \right), \quad \sin \left( \sqrt{B} (x + At) \right), \quad (x, t) \in \mathbb{R}^2. \tag{89}
\]
are non-constant bounded space-time periodic solutions of (74) on \( \mathbb{R}^2 \) with period \( (a, b) \).

**Remark 3.7.** The conclusion of (b) in Lemma 3.6 includes the case \( a > 0, \ b = 0 \) and the case \( a = 0, \ b > 0 \). Therefore, the functions in (89) are simultaneously space-periodic, time-periodic, and space-time periodic. Also note that the period \( (a, b) \) does not appear in the functions of (89).

**Proof.** The assertion (b) is obvious. For \( B > 0 \), it is easy to check that functions in (89) are solutions of (74) on \( \mathbb{R}^2 \) and are non-constant bounded space-time periodic with period \( (a, b) \) if (88) is satisfied.
For (a), if we are in the case \(b > 0\), \(a \geq 0\), then by (79) the number \(z\) satisfying (78) must be pure imaginary, i.e. \(z = iq\) for some \(q \in \mathbb{R}\), otherwise the term \(e^{z(x-(a/b)t)}\) in (79) cannot be bounded and so is \(u(x,t)\). The case \(q = 0\) cannot happen since for \(z = 0\), \(u(x,t)\) becomes
\[
u(x,t) = e^{(z^2+Az+B)t+zx} = e^{Bt},
\]
which is either unbounded (if \(B \neq 0\)) or constant (if \(B = 0\)). Thus we have \(z = iq\), \(q \neq 0\), and since it satisfies (78), we get \(B = q^2 > 0\). If we are in the case \(b = 0\), \(a > 0\), then by (80) we must have \(B = 4\pi^2k^2/a^2 \geq 0\) and if \(B = 0\), we get \(k = 0\) and by (80) again \(u(x,t)\) will be a constant function. Thus we have \(B > 0\).

The proof is done.

Motivated by Lemma 3.6, we can state the following general result, which does not refer to the form (77) (and its real or imaginary part).

**Theorem 3.8.** Non-constant bounded periodic (either space-periodic or time-periodic or space-time periodic) real solutions \(u(x,t)\) of equation (74) occur only when \(B > 0\) and are given by (89) and their linear combinations.

**Remark 3.9.** By (89), we conclude that a non-constant bounded periodic solution of (74) is simultaneously space-periodic, time-periodic, and space-time periodic.

**Proof.** We divide the proof into three cases. The idea is to use Fourier series again.

**Space-periodic case.** Assume \(u(x,t)\) is a non-constant bounded solution of (74) which is space-periodic with period \(a > 0\). By Fourier series expansion, we have
\[
u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} \left( a_n(t) \cos \left( \frac{2\pi n x}{a} \right) + b_n(t) \sin \left( \frac{2\pi n x}{a} \right) \right), \quad \forall (x,t) \in \mathbb{R}^2
\]
(90)
where
\[
\begin{align*}
&\left\{ \begin{array}{l}
\forall n = 0, 1, 2, 3, \ldots \quad a_n(t) = \frac{2}{a} \int_0^a u(x,t) \cos \left( \frac{2\pi n x}{a} \right) dx, \\
\forall n = 1, 2, 3, \ldots \quad b_n(t) = \frac{2}{a} \int_0^a u(x,t) \sin \left( \frac{2\pi n x}{a} \right) dx,
\end{array} \right.
\end{align*}
\]
(91)
As \(u(x,t)\) is bounded, the functions \(a_n(t)\), \(b_n(t)\) are all bounded for any \(n\). By equation (74), the coefficient functions satisfy the system of ODE:
\[
\begin{align*}
a_n'(t) &= Ba_n(t), \\
b_n'(t) &= -A^2 \frac{2\pi}{a} a_n(t) + (B - \left( \frac{2\pi}{a} \right)^2) b_n(t), \quad n = 1, 2, 3, \ldots
\end{align*}
\]
(92)
In terms of complex number, we get (let \(z_n(t) = a_n(t) + ib_n(t)\), which is a bounded complex function)
\[
z_n'(t) = \left[ B - \left( \frac{2\pi}{a} \right)^2 + i \left( -A^2 \frac{2\pi}{a} \right) \right] z_n(t), \quad t \in (-\infty, \infty), \quad n = 1, 2, 3, \ldots
\]
(93)
Since \(a_0(t)\), \(a_n(t)\), \(b_n(t)\) are bounded functions of time for all \(n = 1, 2, 3, \ldots\), if \(B \neq 0\) and \(B \neq \left( \frac{2k\pi}{a} \right)^2\) for some \(k \in \mathbb{N}\), we must have \(a_0(t) \equiv 0\) and \(z_k(t) \equiv 0\), i.e. the terms \(a_0(t)/2\) and \(a_k(t) \cos (\cdots) + b_k(t) \sin (\cdots)\) disappear in the Fourier series expansion (90). In particular, if \(B < 0\), there is no non-constant bounded space-periodic solution of (74) with any period \(a > 0\).
If $B = 0$, only the term $a_0(t)$ can survive in the Fourier series expansion (90) and we get $a_0(t) \equiv \text{const.}$ and again there is no non-constant bounded space-periodic solution with any period $a > 0$.

If $B > 0$, then one can choose suitable period $a > 0$ and some $k \in \mathbb{N}$ satisfying the identity

$$\frac{2k\pi}{a} = \sqrt{B},$$

then it is possible to have non-constant bounded space-periodic solution with this particular period $a > 0$. In such a case, we have $a_0(t) \equiv 0$ and $z_n(t) \equiv 0$ for all $n \neq k$. Now the Fourier series expansion of $u(x,t)$ is reduced to

$$u(x,t) = a_k(t) \cos \left( \sqrt{B}x \right) + b_k(t) \sin \left( \sqrt{B}x \right), \quad \sqrt{B} = \frac{2k\pi}{a},$$

where by (92) and (94), we have

$$a_k(t) = c_1 \cos \left( A\sqrt{B}t \right) + c_2 \sin \left( A\sqrt{B}t \right), \quad b_k(t) = c_2 \cos \left( A\sqrt{B}t \right) - c_1 \sin \left( A\sqrt{B}t \right)$$

for some real constants $c_1, c_2$. With this, the function $u(x,t)$ in (95) is space-periodic with period $a > 0$, non-constant, and bounded. After simplification, we get

$$u(x,t) = c_1 \cos \left( \sqrt{B}(x + At) \right) + c_2 \sin \left( \sqrt{B}(x + At) \right), \quad (x,t) \in \mathbb{R}^2, \quad c_1, c_2 \in \mathbb{R},$$

which is given by (89).

**Time-periodic case.** Assume $u(x,t)$ is a non-constant bounded solution of (74) which is time-periodic with period $b > 0$. By Fourier series expansion, we have

$$u(x,t) = \frac{a_0(x)}{2} + \sum_{n=1}^{\infty} \left( a_n(x) \cos \left( \frac{2n\pi t}{b} \right) + b_n(x) \sin \left( \frac{2n\pi t}{b} \right) \right),$$

where

$$a_n(x) = \frac{2}{b} \int_0^b u(x,t) \cos \left( \frac{2n\pi t}{b} \right) dt, \quad b_n(x) = \frac{2}{b} \int_0^b u(x,t) \sin \left( \frac{2n\pi t}{b} \right) dt,$$

for all $n = 1, 2, 3, ...$

Now the functions $a_0(x), a_n(x), b_n(x)$ are all bounded for all $n = 1, 2, 3, ...$ and by equation (74), they satisfy the system of ODE:

$$a''_n(x) + Aa'_n(x) + Ba_0(x) - \frac{2n\pi}{b}b_n(x) = 0$$

$$b''_n(x) + Ab'_n(x) + Bb_0(x) + \frac{2n\pi}{b}a_n(x) = 0.$$

In terms of complex number, we get (let $z_n(x) = a_n(x) + ib_n(x)$, which is a bounded complex function)

$$z''_n(x) + Az'_n(x) + \left( B + i\frac{2n\pi}{b} \right) z_n(x) = 0, \quad x \in (-\infty, \infty), \quad n = 1, 2, 3, ...$$

Note that, for fixed $n \in \mathbb{N}$, if the two complex roots $r_1, r_2$ of the characteristic equation

$$r^2 + Ar + \left( B + i\frac{2n\pi}{b} \right) = 0, \quad B \in \mathbb{R}, \quad b > 0, \quad n \in \mathbb{N}$$

are both not pure imaginary, then we must have $z_n(x) \equiv 0$ due to the boundedness of $z_n(x)$. Hence the terms $a_n(x) \cos (\cdot \cdot) + b_n(x) \sin (\cdot \cdot)$ disappear in the Fourier
series expansion (97). When $B < 0$, there is no pure imaginary root of the characteristic equation (101) for any $n \in \mathbb{N}$ and the only bounded solution $a_0 (x)$ for the ODE $a_0'' + Aa_0' + Ba_0 = 0$ is $a_0 (x) \equiv 0$. Thus for $B < 0$, there is no non-constant bounded time-periodic solution of (74) with any period $b > 0$.

When $B = 0$, there is no pure imaginary root of the characteristic equation (101) for any $n \in \mathbb{N}$ and the only bounded solution $a_0 (x)$ for the ODE $a_0'' + Aa_0' = 0$ is $a_0 (x) \equiv const$. Again, there is no non-constant bounded time-periodic solution of (74) with any period $b > 0$.

When $B > 0$ and $A \neq 0$, one can choose suitable period $b > 0$ and some $k \in \mathbb{N}$ satisfying the identity

$$\frac{2k\pi}{b} = \begin{cases} A\sqrt{B}, & \text{if } A > 0, \\
-A\sqrt{B}, & \text{if } A < 0. \end{cases}$$

(102)

In such a case, it is possible to have non-constant bounded time-periodic solution with this particular period $b > 0$. Now we can decompose the characteristic equation (101) as

$$r^2 + Ar + \left(B + i\frac{2k\pi}{b}\right) = \begin{cases} \left(r + i\sqrt{B}\right) \left(r - i\sqrt{B} + \frac{1}{\sqrt{B}} \frac{2k\pi}{b}\right), & \text{if } A > 0 \\
\left(r - i\sqrt{B}\right) \left(r + i\sqrt{B} - \frac{1}{\sqrt{B}} \frac{2k\pi}{b}\right), & \text{if } A < 0, \end{cases}$$

(103)

which has a pure imaginary root $r = i\sqrt{B}$ (or $-i\sqrt{B}$). Thus, for $n = k$, the only bounded solution $z_k (x) = a_k (x) + ib_k (x)$ of the complex ODE (100) is given by

$$z_k (x) = a_k (x) + ib_k (x) = \begin{cases} (c_1 + ic_2) e^{-i\sqrt{B}x}, & \text{if } A > 0 \\
(c_1 + ic_2) e^{i\sqrt{B}x}, & \text{if } A < 0 \end{cases}$$

(104)

for some real constants $c_1, c_2$. To be complete, we still have to look at the equation $a'_0 (x) + Aa_0 (x) + Ba_0 (x) = 0$. One can easily see that for $A \neq 0$ and $B > 0$, the only bounded solution for $a_0 (x)$ is $a_0 (x) \equiv 0$.

At this moment, we can conclude that when $B > 0$, $A \neq 0$, there exist non-constant bounded time-periodic solution $u (x,t)$ if (102) is satisfied for $b > 0$ and for some $k \in \mathbb{N}$. In such a case, the solution $u (x,t)$ is time-periodic with period $b > 0$ and its Fourier series expansion is reduced to

$$u (x,t) = a_k (x) \cos \left(\frac{2k\pi}{b} t\right) + b_k (x) \sin \left(\frac{2k\pi}{b} t\right).$$

(105)

This is because the characteristic equation of the complex ODE (100) has no pure imaginary solution at all if $n \neq k$ (note that now $b$, $k$ satisfy (102)) and so $z_n (x) \equiv 0$ for $n \neq k$. By (102) and (104), (105) can be reduced to the form (96).

Finally, when $B > 0$, $A = 0$, there is no pure imaginary root for the characteristic equation (101) for any $n \in \mathbb{N}$. However, the equation for $a_0 (x)$ becomes $a_0'' (x) + Ba_0 (x) = 0$ and, it has non-constant bounded time-periodic solutions $a_0 (x) = \cos \left(\sqrt{B}x\right)$ or $\sin \left(\sqrt{B}x\right)$. Again, they are reduced to the form (96).

**Space-time periodic case.** Assume $u (x,t)$ is a non-constant bounded solution of (74) which is space-time periodic with period $(a,b)$, $a > 0$, $b > 0$. This case can be reduced to the previous case. We do the change of variables (30) and the new function $v (\sigma,t)$ satisfies equation (76). As the function $v (\sigma,t)$ is now non-constant,
bounded, and time-periodic with period \( b > 0 \), the above discussion implies that this can happen only when \( B > 0 \) and there are some \( k \in \mathbb{N} \) and \( a > 0 \), \( b > 0 \) satisfying
\[
\frac{2k\pi}{b} = \begin{cases} 
(A + \frac{a}{b}) \sqrt{B}, & \text{if } A + \frac{a}{b} > 0, \\
-(A + \frac{a}{b}) \sqrt{B}, & \text{if } A + \frac{a}{b} < 0.
\end{cases}
\tag{106}
\]

Moreover, by (96) the solution \( v(\sigma, t) \) has the form
\[
v(\sigma, t) = c_1 \cos \left( \sqrt{B} \left( \sigma + \left( A + \frac{a}{b} \right) t \right) \right) + c_2 \sin \left( \sqrt{B} \left( \sigma + \left( A + \frac{a}{b} \right) t \right) \right).
\]

Back to \( u(x, t) \), we get (note that \( u(x, t) = v(\sigma, t) \), where \( \sigma = x - (a/b)t \))
\[
u(x, t) = c_1 \cos \left( \sqrt{B} (x + At) \right) + c_2 \sin \left( \sqrt{B} (x + At) \right)
\tag{107}
\]
and by (106), we have \( u(x + a, t + b) = u(x, t) \) for all \((x, t) \in \mathbb{R}^2 \). Thus we conclude that non-constant bounded space-time periodic solutions of equation (74) occur only when \( B > 0 \) and are given by (107). It has period \((a, b) \), \( a > 0 \), \( b > 0 \), if \( a \), \( b \) satisfy the condition (106) for some \( k \in \mathbb{N} \).

By the above three cases, the proof of Theorem 3.8 is done. \( \square \)

3.2. Comparing traveling wave solutions and periodic solutions of the equation (74). Let \( c \neq 0 \) be a constant. A solution \( u(x, t) \) of (74) of the form \( u(x, t) = h(x - ct) \), \((x, t) \in \mathbb{R}^2 \), for some function \( h(\theta) \) defined on \( \mathbb{R} \) is called a traveling wave solution. Without loss of generality, we may assume \( c > 0 \). A traveling wave solution \( u(x, t) = h(x - ct) \) can be viewed as the traveling of the graph \( y = h(x) \), \( x \in (-\infty, \infty) \), along the positive \( x \)-direction with speed \( c \), hence the name.

If \( u(x, t) = h(x - ct) \) is a traveling wave solution, then it is space-time periodic with period \((a, b) \) as long as \( a > 0 \), \( b > 0 \) satisfy the identity \( a = cb \). However, not every space-time periodic solution of (74) is a traveling wave solution. For example, the space-time periodic solutions (51) of the heat equation are not traveling wave solutions. Similarly, the space-time periodic solutions (86) of (74) are not traveling wave solutions. On the other hand, however, the non-constant bounded space-time periodic solutions of (74) in (89) are traveling wave solutions.

We can look for traveling wave solutions of (74) by plugging \( u(x, t) = h(x - ct) \) into (74) to get the ODE for \( h(\theta) \):
\[
h''(\theta) + (A + c) h'(\theta) + Bh(\theta) = 0, \quad \theta \in (-\infty, \infty).
\]
If \( (A + c)^2 - 4B > 0 \), its general solution is given by \( h(\theta) = c_1 e^{r_+ \theta} + c_2 e^{r_- \theta} \), where \( r_{\pm} \) are the two distinct real roots of the characteristic equation of the above ODE, and if \( (A + c)^2 - 4B = 0 \), its general solution is given by \( h(\theta) = c_1 e^{r\theta} + c_2 \theta e^{r\theta} \), \( r = -(A + c)/2 \), and if \( (A + c)^2 - 4B < 0 \), its general solution is given by \( h(\theta) = c_1 e^{r\theta} \cos (\lambda \theta) + c_2 e^{r\theta} \sin (\lambda \theta) \), \( \lambda = 2^{-1} \sqrt{4B - (A + c)^2} \), where \( c_1 \), \( c_2 \) are arbitrary constants.

By the above, we conclude the following:

**Corollary 3.10.** Non-constant bounded traveling wave solutions \( u(x, t) \) of equation (74) occur only when \( B > 0 \). They have traveling speed \( c = -A \) and are given by (89) and their linear combinations.
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