Mixed symmetry gauge fields in a flat background

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Abstract

We present a list of all inequivalent bosonic covariant free particle wave equations in a flat spacetime of arbitrary dimension. The wave functions are assumed to have a finite number of components. We relate these wave equations to equivalent versions obtained following other approaches.

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1 Elementary particles as irreducible representations of the Poincaré group

Wigner showed that the rules of quantum mechanics, combined with the principle of special relativity, imply that the classification of all possible wave equations describing the evolution of the states of a free relativistic particle moving in the Minkowski space $\mathbb{M}^4$ is equivalent to the classification of all unitary irreducible representations (UIRs) of the Poincaré group$^1$ $ISO(3, 1) \equiv \mathbb{R}^4 \rtimes SO(3, 1)$ [1].

The classification of all linear relativistic wave equations in Minkowski spacetime will be referred to as Wigner’s programme. It was completed in 1939 when, using the method of induced representations, Wigner showed that the UIRs of $ISO(3, 1)$ are characterized by two real numbers: the square of the momentum $p^2$, and the spin $S$ [1]. Physical considerations further impose$^2$ $p^2 = -m^2 \leq 0$ (no tachyon) and $2S \in \mathbb{N}$ (discrete spin).

Subsequently, for every UIR of $ISO(3, 1)$, a linear partial differential equation (PDE) is given, the solutions $\phi$ of which transform according to that representation. The map $\phi : \mathbb{R}^4 \rightarrow V$ stands for the particle wave function, where the vector space $V$ (over $\mathbb{C}$) denotes the representation space for the little group $\ell_4$ of proper Lorentz transformations that preserve the particle’s four-momentum $p_\mu$. Every UIR of the Poincaré group is determined by a UIR of the little group (acting on the spin part of the wave function) [1].

$^1$To deal with double-valued representations, i.e. fermions, we should actually consider the double covering $\mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ of $ISO(3, 1)$.

$^2$We use the “mostly plus” signature $(-, +, \cdots, +)$ for the metric $\eta_{\mu\nu}$.
In the process of second quantization, the wave function $\phi$ is interpreted as a classical field which, in turn, is itself quantized. It is thus of prime importance to derive the above-mentioned wave equations from a variational principle. In order to easily control the Poincaré invariance when introducing interaction terms, it is convenient to start with a free covariant Lagrangian. Thus, the determination of a corresponding covariant Lagrangian for each free particle wave equation in $\mathbb{M}^4$ constitutes the second step in the canonical field quantization scheme. The latter problem will be referred to as Fierz-Pauli’s programme. This programme was initiated in 1939 \[2\] and was completed in the seventies by Singh and Hagen for the massive case ($p^2 < 0$) \[3\] and by Fang and Fronsdal for the massless case ($p^2 = 0$) \[4, 5\].

2 Higher-spin gauge theories

In all fundamental field theories known to date, Nature seems to have limited herself to spins $S \leq 2$, although in principle nothing prevents us from theoretically investigating higher-spin (i.e. $S > 2$) elementary fields since, from a group theoretic point of view, the Lorentz group $SO(3, 1)$ allows representations with any integer (or half-integer) spin. Incidentally, Fronsdal’s programme consists in introducing consistent interactions among massless higher-spin fields \[4\]. This problem was stated in 1978 but still remains an open mathematical question of field theory. Numerous preliminary results have recently been obtained (see \[6\] and references therein) which reveal surprising properties of higher-spin gauge fields.

Fierz-Pauli’s programme is a very old problem. Its generalization to arbitrary space-time dimension $D$ constitutes the first step towards the introduction of consistent interactions among arbitrary higher-spin fields. In the 80’s, string field theory brought interest in this direction \[7, 8\]. Fierz-Pauli’s programme generalization is currently under investigation \[9, 10, 11, 12\].

The aim of this talk is restricted to the presentation of an exhaustive list of inequivalent covariant wave equations for free relativistic particles moving in a flat background $\mathbb{M}^D$, which will be referred to as Bargmann-Wigner’s theorem (since it was achieved for $D = 4$ by those authors \[13\]). This theorem is itself preliminary to Fierz-Pauli’s programme completion\(^3\).

To start with, Wigner’s programme is easily generalized to the Poincaré group $ISO(D−1, 1)$ (see, e.g. \[14\]):

**Lemma** (Wigner’s programme)

Let $\ell_D \subset SO(D−1, 1)$ be a little group corresponding to $p^2 \leq 0$ and $p^\mu \neq 0$. Any UIR of $\ell_D$ with representation space $V$ provides a UIR of the Poincaré group $ISO(D−1, 1)$ the representation space of which is the Hilbert space $\mathcal{H}$ (with $L^2$ norm) of positive energy

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\(^3\)Actually this programme has been completed in the $OSp(1, 1 \mid 2)$ formalism \[7\]. We are grateful to W. Siegel for calling this fact to our attention.
solutions $\phi : \mathbb{R}^D \rightarrow V$ of the wave equation\(^4\)

$$\tag{1} \Box \phi - m^2 \phi = 0 .$$

If $p^\mu \neq 0$, then the little groups $\ell_D$ are isomorphic to $SO(D - 1)$ for $p^2 < 0$, $ISO(D - 2)$ for $p^2 = 0$, and $SO(D - 2, 1)$ for $p^2 > 0$. A natural requirement is that the field $\phi$ should possess a finite number of components, i.e. $\dim(V) < \infty$. This removes the unphysical tachyonic representations with $p^2 > 0$ because $SO(D - 2, 1)$ is non-compact. The UIRs of $ISO(D - 2)$ are induced from those of $SO(D - 2)$ and, since we want a finite-dimensional representation, the non-compact subgroup $\mathbb{R}^{D-2}$ must act trivially on the wave functions. Moreover, in order to ensure parity invariance, we are led to consider finite-dimensional irreducible representations (irreps) of the orthogonal groups $O(D - \ell)$ with $\ell = 1$ when $p^2 < 0$, and $\ell = 2$ when $p^2 = 0$. Therefore, the Hilbert space $\mathcal{H}$ for a massive particle in $\mathbb{M}^D$ is isomorphic to the one obtained by a dimensional reduction of a massless particle in $\mathbb{M}^{D+1}$. By construction, Fierz-Pauli’s programme for finite-component fields can thus be restricted, without loss of generality, to the massless case (see [15] for completely symmetric fields).

Finite-dimensional irreps of $O(n)$ are characterized by Young diagrams. For the sake of simplicity, the following discussion will be limited from now on to single-valued (i.e. tensor) representations of the orthogonal groups. The space of multilinear applications from $\mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n$ to $\mathbb{C}$ is denoted by $T(\mathbb{R}^n)$. We further denote by $V^G_Y$ the vector space of tensors in $T(\mathbb{R}^n)$ which are irreducible under $G \subset GL(D, \mathbb{R})$ and whose symmetry properties are associated with the Young diagram $Y$.

We are interested in fields $\phi$ which have representation space $V = V^O(D - \ell)_Y$. The case $D = 4$ is very particular in the sense that each tensor irrep of $O(2)$ and $O(3)$ is equivalent to a completely symmetric tensor irrep (pictured by a one-row Young diagram with $S$ columns for a spin $S$ particle). This significant simplification enabled the completion of Fierz-Pauli’s programme in $\mathbb{M}^4$. When $D > 4$, more complicated Young diagrams (corresponding to “mixed symmetry” tensor fields) are generated, the analysis of which requires appropriate mathematical tools.

3 Bargmann-Wigner’s theorem

Unfortunately, by construction the wave equation\(^1\) is only covariant under the “little group” $O(D - \ell)$, and not under the Lorentz group $O(D - 1, 1)$. Consequently, more work is required in order to obtain a version of Bargmann-Wigner’s theorem for $\mathbb{M}^D$. The usual technique consists in considering a new wave equation for a tensor field $\phi : \mathbb{R}^D \rightarrow V^GL(D, \mathbb{R})_Y$ which is irreducible under the general linear group in $D$ dimensions, thereby ensuring Lorentz covariance. The solution space of \(^1\) is denoted by $\Phi_Y$. The dimension of the representation space $V^GL(D, \mathbb{R})_Y$ is much bigger than the dimension of $V^O(D - \ell)_Y$, which generally implies that extra non-physical degrees of freedom have been added. In other words, the Hilbert space $\mathcal{H}_Y$ (see the lemma of section \(^2\)) is a strict subspace of $\Phi_Y$. Furthermore, the scalar product of $\Phi_Y$ is not positive definite.

\(^4\)Boundary conditions and regularity requirements should be specified when solving PDEs. In the lemma, we assume that the “ket” $\phi \in L^2(\mathbb{R}^D) \otimes V$. This choice is convenient because $(a)$ it provides an obvious norm for $\mathcal{H}$, $(b)$ it selects solutions such that $| \phi(x) | \nrightarrow 0$, and $(c)$ if we consider $\phi$ as a temperate distribution (since the “bra” $\phi \in \mathcal{S}'(\mathbb{R}^D \otimes V)$ then we are always allowed to convert linear PDEs into algebraic equations by going to the momentum representation.
3.1 Massive particle

The massive case is easy to deal with since it is only necessary to remove the longitudinal components of the corresponding wave functions to obtain irreps of $O(D-1)$ ($\ell = 1$). To the mass-shell equation (1), we must add (i) the transversality condition

$$\partial \cdot \phi = 0$$

and (ii) the tracelessness of the field $\phi$. On-shell, the field $\phi$ is thus irreducible under the group $O(D-1,1)$: it takes values in the representation space $V^{O(D-1,1)}_{Y} \subset T(\mathbb{R}^{D})$. The covariant equations (1) and (2) for a traceless field $\phi : \mathbb{R}^{D} \to V^{O(D-1,1)}_{Y}$ take us back to the representation space $H_{Y}$.

3.2 Massless particle

In the massless case, the situation is a bit more cumbersome. In order to have irreps of $O(D-2)$ ($\ell = 2$), it is necessary to remove the components of the corresponding wave functions lying along the light-cone directions.

A remedy is to introduce redundancies in the solution space $\Phi_{Y}$ by resorting to gauge symmetries. In mathematical terms, one quotients $\Phi_{Y}$ by the gauge orbits, which leads to the original Hilbert space $H_{Y}$ of physical states (one completely fixes the gauge). This class of relativistic wave equation is essential because it should enable the realization of Fierz-Pauli’s programme for arbitrary $D$. In the “metric-like” [4, 9, 8] and “frame-like” [16] approaches, off-shell trace conditions are further imposed on the gauge field and the gauge parameters (in order to avoid the use of auxiliary fields). By relaxing the orthodox requirement of locality, Francia and Sagnotti were recently able to forego these trace conditions [17] (for mixed symmetry fields, see [10, 11, 18]).

Their field equations were elegantly formulated in terms of the curvature tensor introduced by de Wit and Freedman [19] in 1980. This tensor is invariant under gauge transformations with unconstrained gauge parameters. It was already used in 1965 by Weinberg in his analysis of massless higher-spin fields in $\mathbb{M}^{4}$ [20] (also see the inspiring pedagogical review on higher-spin fields in the book [21]).

To formulate the theorem we make use of a specific choice of conventions where sets of antisymmetrized indices are privileged. More precisely, we define a multiform “of spin $S$” as a field $K : \mathbb{R}^{D} \to \Lambda_{[S]}(\mathbb{R}^{D})$ which takes value in the algebra $\Lambda_{[S]}(\mathbb{R}^{D}) \equiv \otimes^{S} \Lambda(\mathbb{R}^{D})$ of polynomials in the generators $d_{i}x^{\mu}$ ($i = 1, 2, \ldots, S$; $\mu = 0, 1, \ldots, D - 1$). For fixed $i$, the $d_{i}x^{\mu}$’s generate the exterior algebra $\Lambda(\mathbb{R}^{D})$. The nilpotent operators $d_{i} \equiv d_{i}x^{\mu}\partial_{\mu}$ generalize the usual exterior differential $d$ of the de Rham complex $\Omega(\mathbb{R}^{D})$. With the help of the Minkowski metric, we define the Hodge operators $*_{i}$ as well as the codifferentials $\delta_{i} \equiv *_{i}d_{i} *_{i}$. A multiform $K$ of spin $S$ is said to be harmonic if it is closed ($d_{i}K = 0$) and coclosed ($\delta_{i}K = 0$) for all $i \in \{1, \ldots, S\}$.

Proposition (Bargmann-Wigner’s theorem)

Let $\overline{Y}$ be a Young diagram with at least two rows of equal length $S$ and let $Y$ be the Young diagram obtained by removing the first row of $\overline{Y}$. Any tensor irrep of $O(D-1,1)$ with representation space $V^{O(D-1,1)}_{\overline{Y}}$ provides a massless UIR of the group $IO(D-1,1)$ associated with the Young diagram $Y$, the representation space of which is the space of harmonic irreducible multiforms $K : \mathbb{R}^{D} \to V^{O(D-1,1)}_{\overline{Y}}$. 

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of spin $S$. This latter space is isomorphic to the Hilbert space $\mathcal{H}_Y$ of physical states $\phi : \mathbb{R}^D \to V_Y^{O(D-2)}$ that are solutions of (7).

The proof of the Bargmann-Wigner theorem is straightforward in both cases ($\ell = 1, 2$). The corresponding set of differential equations is equivalent to a set of algebraic conditions on the components of the Fourier transform of the corresponding tensor field. An explicit check shows that these algebraic conditions constrain the tensor components to belong to the appropriate space $V_Y^{O(D-\ell)}$. Details are given in [22].

The gauge-invariant curvature tensor for completely symmetric fields [19, 20] is generalized for a mixed symmetry gauge field $\phi : \mathbb{R}^D \to V_Y^{GL(D,\mathbb{R})}$ as follows [10, 11, 23]

$$K \equiv d_1 d_2 \ldots d_S \phi : \mathbb{R}^D \to V_Y^{GL(D,\mathbb{R})}. \tag{3}$$

In [10, 23], field equations were proposed for gauge fields irreducible under $GL(D, \mathbb{R})$. This was motivated by a systematic generalization of the work [24]. On-shell, the curvature tensor $K$ was taken to be traceless and harmonic, which is a simple generalization of the Maxwell equations ($S = 1$), the linearized Einstein equation ($S = 2$) and the Bargmann-Wigner equations\(^5\) of [13, 20, 25] for completely symmetric gauge fields of arbitrary spin $S$. The field equations of [10, 23] should be equivalent to the ones proposed in [7]. Thanks to our proposition, the on-shell fieldstrength $K : \mathbb{R}^D \to V_Y^{O(D-1,1)}$ provides a UIR of the Poincaré group corresponding to the gauge field $\phi : \mathbb{R}^D \to V_Y^{O(D-2)}$ in the light-cone gauge. Indeed, gauge fields in the light-cone gauge are essentially fieldstrengths [7, 21].

The local wave equations that we provide contain many derivatives of the gauge field because they are built out of the curvature tensor (3), but nevertheless can simply be brought back to a second order form upon partial gauge-fixing [18]. Our proposition proves that an apparently ill-behaved higher-derivative field equation can in fact be the correct one (leaving aside the subtle issue of a well-behaved realization of Fierz-Pauli’s programme). Following the method sketched in [18], we checked that the generalized Poincaré lemma of [10] relates the previous higher-derivative field equations to the local second-order field equations of [8] for any mixed symmetry field. This procedure introduces a supplementary non-local term which enables to abandon the trace conditions on the gauge parameters of the local approaches (in perfect agreement with the results of [17] for completely symmetric fields). These non-local second-order field equations are equivalent to the ones of [11]. This proves the complete generality of the procedure sketched in [18].

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\(^5\)Indeed, Bargmann and Wigner used fields which transform according to the $[(S,0) \oplus (0,S)]$ representations of $SL(2, \mathbb{C})$. In other words, they considered fieldstrengths [21]. To compare with the works [25], one must consider their field equations (i) in the limit where the cosmological constant goes to zero and (ii) in the gauge where the (frame-like) gauge field is equal to the (metric-like) completely symmetric field.
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