Algebraic topological techniques for elliptic problems involving fractional Laplacian

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Abstract. We prove the existence of infinitely many solutions to an elliptic problem by borrowing the techniques from algebraic topology. The solution(s) thus obtained will also be proved to be bounded.

1. Introduction

We propose to study the following singular problem with a mixed operator:

\[ a(-\Delta)^s u + b(-\Delta)u = \lambda |u|^{-\gamma-1} u + \mu |u|^{2^*_s-2} u, \quad \text{in } \Omega, \]
\[ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \]

where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded domain, \( a, b \geq 0, 0 < s < 1 < 2 < 2^*_s < 2^* \), \( \lambda > 0, \mu \in \mathbb{R}, \gamma \in (0, 1) \), and

\[ (-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} dy, \quad \forall \ x \in \Omega, \]

is the fractional Laplacian. We refer the operator \( "a(-\Delta)^s + b(-\Delta)" \) as a mixed operator since it possesses both local as well as nonlocal features, and thus we refer the problems of kind (1.1) as “nonlocal-local” elliptic problems. For a detailed study on this operator refer [6, 13]. The mixed operators with different order are nowadays gaining popularity in applied sciences, in theoretical studies and also in real world applications. The development of the literature includes viscosity solution methods [4], Cahn–Hilliard equations [9], Aubry–Mather theory [11], phase transitions [8],
probability and stochastics [29], fractional damping effects [12], decay estimates for parabolic equations [14], population dynamics [13], Bernstein-type regularity results [7].

We begin by considering the following two spaces $X$ and $Y$ which are the closures of $C^\infty_c(\Omega_1)$ in $H^s(\Omega_1)$ and $H^1(\Omega_1)$, respectively (refer Sect. 2 for these notations). Let $Q = \mathbb{R}^N_2 \setminus ((\mathbb{R}^N_1 \setminus \Omega) \times (\mathbb{R}^N_1 \setminus \Omega))$ and define the spaces as follows:

$$X = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy < \infty \right\}$$

$$Y = \left\{ u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega) \right\}.$$

The spaces $X$, $Y$ are Banach spaces with respect to the following norms:

$$\|u\|_X = \|u\|_2 + \left( \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}} \quad (1.2)$$

and

$$\|u\|_Y = \|u\|_2 + \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \quad (1.3)$$

respectively. The norm $\cdot$ defined in (1.2) is the Gagliardo norm. Note that here $\|u\|_\alpha = (\int_\Omega |u|^\alpha \, dx)^{\frac{1}{\alpha}}$ for $1 < \alpha < \infty$.

The study of the nonlocal-local elliptic problem in (1.1), firstly directed our attention to fix a function space in which the solution(s) will be seeked for. Define the space $Z = \{u : a|u|_{s,2}^2 + b\|u\|_{1,2} < \infty\}$ equipped with the following norm:

$$\|u\| = \left( a|u|_{s,2}^2 + b\|u\|_{1,2}^2 \right)^{\frac{1}{2}}. \quad (1.4)$$

It is easy to see that the space $Z$ is a Banach space and also reflexive with respect to the norm $\| \cdot \|$, defined in (1.4).

The use of algebraic topological techniques to study problems having a singular nonlinearity is a rarity in the literature. Thus, the problem discussed here is new as the consideration of a singularity with a critical exponent with $\mu \in \mathbb{R}$ handled with Morse theoretic approach is not found anywhere in the literature to our knowledge. The question about the existence and multiplicity of positive weak solutions to problem (1.1) with $\mu > 0$ and with either $a = 0$ or $b = 0$ has been answered in [16–18,21,22,31] and the references therein. The authors of these works followed different tools such as variational method, concentration compactness method, Nehari manifold method etc. but none of them used the techniques from algebraic topology to study (1.1) with $a, b > 0$ and $\mu \in \mathbb{R}$.

With the help of algebraic topological techniques, the existence and multiplicity results for the following problem have been established by many researchers under
different growth conditions on the reaction term $f$ (either subcritical or critical growth conditions):

$$Lu = f(x, u), \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega. \quad (1.5)$$

Problems of type (1.5) were treated recently by Iannizzotto et al. [22,23] (finite multiplicity with $L$ being the fractional $p$-Laplacian), Ferrara et al. [15] (at least one non-trivial solution with $L$ being a fractional integro-differential operator). The paper by Papageorgiou and Rădulescu [32], dealt with a nonlinear Robin problem and proved the multiplicity by producing three nontrivial solutions. The techniques thus differed from problem-to-problem addressed.

The double phase problems of type (1.5) with $L$ being a $(p, q)$-Laplacian or a fractional $(p, q)$-Laplacian have been widely studied by many authors with different techniques. For instance, when $L = (-\Delta_p - \Delta_q)$ with $p, q > 1$, Gongbao and Gao [20], Yin and Yang [36] used variational method, Liang et al. [27] used Morse theoretical technique for the case $q = 2 \neq p$, and Marano et al. [28], Mugnai and Papageorgiou [30] used the variational method with Morse theory and truncation comparison techniques. When $L = ((-\Delta_p)^s + (-\Delta_q)^s)$ with $p, q > 1$ and $s \in (0, 1)$, the problem (1.5) has been discussed in [1,5,19,25] using variational methods, and in [10] using the Morse theory. With a combination of variational technique and Morse technique, Papageorgiou et al. [34] established the existence (for $(p - 1)$-superlinear case) and multiplicity (for $(p - 1)$-linear resonant case) for Robin problems with $(p, q)$-Laplacian. For more details on double phase problems one can refer the recent piece of works by Bahrouni et al. [2,3] and the bibliography therein.

Motivated by the former works, in this article, we study the singular problem (1.1) using variational techniques and algebraic topological methods, specifically the Morse theory and the critical groups (refer Sect. 2). We establish the existence of infinitely many solutions to (1.1) in Sect. 3 followed by two subsections. Sections 3.1 and 3.2 deal with the existence part and the multiplicity part, respectively. In the “Appendix”, we will establish the boundedness result of these weak solutions. The books by Perera et al. [35] and Papageorgiou et al. [33] are strongly recommended for a better understanding on the usage of Morse theory in the study of various elliptic PDEs.

2. Mathematical preliminaries

A quintessential condition which the functional requires to satisfy is the the Palais-Smale condition (denoted by $(PS)$-condition) which is as follows.

**Definition 2.1.** Let $X$ be a Banach space, and $I : X \to \mathbb{R}$ be a $C^1(X, \mathbb{R})$ functional. Given $c \in \mathbb{R}$ we say that the functional $I$ satisfies the Palais-Smale condition (or the $(PS)_c$-condition) at level $c$ if any bounded sequence $(u_n) \subset X$ such that $I(u_n) \to c$, and $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence in $X$.

Below are the Sobolev embedding results that will be used throughout the article.
**Theorem 2.2.** ([26]) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $0 < s < 1$, and $2s < N$. Further, assume that $r \leq 2s^* = \frac{2N}{N-2s}$. Then, there exists $C = C(r, s, N, \Omega) > 0$ such that

$$\|u\|_{L^r(\Omega)} \leq C\|u\|_{X}, \forall u \in X.$$ 

Moreover, this embedding is continuous for any $r \in [1, 2s^*]$, and compact for any $r \in [1, 2s^*)$. The above embedding holds also for $Z$.

**Theorem 2.3.** ([26]) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $N > 2$. Then, for every $\bar{r} \leq 2s^* = \frac{2N}{N-2}$ there exists $\bar{C} = \bar{C}(r, N, \Omega) > 0$ such that

$$\|u\|_{L^{\bar{r}}(\Omega)} \leq \bar{C}\|u\|_{Y}, \forall u \in Y.$$ 

Moreover, this embedding is continuous for any $r \in [1, 2s^*]$, and compact for any $r \in [1, 2s^*)$. The above embedding holds also for $Z$.

We now present the fundamental tool that will be used to work with, namely the homology theory [33], which will be followed by the definition of deformation (see [33]).

**Definition 2.4.** A “homology theory” on a family of pairs of spaces $(X, A)$ consists of:

1. A sequence $\{H_k(X, A)\}_{k \in \mathbb{N}_0}$ of abelian groups known as “homology group” for the pair $(X, A)$ (note that for the pair $(X, \phi)$, we write $H_k(X)$).
2. To every map of pairs $\varphi : (X, A) \rightarrow (Y, B)$ is associated a homomorphism $\varphi^* : H_k(X, A) \rightarrow H_k(Y, B)$ for all $k \in \mathbb{N}_0$.
3. To every $k \in \mathbb{N}_0$ and every pair $(X, A)$ is associated a homomorphism $\partial : H_k(X, A) \rightarrow H_{k-1}(A)$ for all $k \in \mathbb{N}_0$.

Here, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. These objects satisfy the following axioms:

(A1) If $\varphi = id_X$, then $\varphi_* = id|H_k(X, A)$.

(A2) If $\varphi : (X, A) \rightarrow (Y, B)$, and $\psi : (Y, B) \rightarrow (Z, C)$ are maps of pairs, then

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$ 

(A3) If $\varphi : (X, A) \rightarrow (Y, B)$ is a map of pairs, then $\partial \circ \varphi_* = (\varphi|_A)_* \circ \partial$.

(A4) If $i : A \rightarrow X$ and $j : (X, \phi) \rightarrow (X, A)$ are inclusion maps, then the following sequence is exact

$$\ldots \rightarrow H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \rightarrow \ldots$$

Recall that a chain $\ldots \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} C_{k-2}(X) \xrightarrow{\partial_{k-2}} \ldots$ is said to be exact if $im(\partial_{k+1}) = ker(\partial_k)$ for each $k \in \mathbb{N}_0$.

(A5) If $\varphi, \psi : (X, A) \rightarrow (Y, B)$ are homotopic maps of pairs, then $\varphi_* = \psi_*$. 

(A6) (Excision): If $U \subseteq X$ is an open set with $\overline{U} \subseteq \text{int}(A)$, and $i : (X \setminus U, A \setminus U) \rightarrow (X, A)$ is the inclusion map, then $i_* : H_k(X \setminus U, A \setminus U) \rightarrow H_k(X, A)$ is an isomorphism.

(A7) If $X = \{\ast\}$, then $H_k(\{\ast\}) = 0$ for all $k \in \mathbb{N}$. 

Definition 2.5. A continuous map $F : X \times [0, 1] \to X$ is a deformation retraction of a space $X$ onto a subspace $A$ if, for every $x \in X$ and $a \in A$, $F(x, 0) = x$, $F(x, 1) \in A$, and $F(a, 1) = a$.

An important result in Morse theory is stated below.

Theorem 2.6. Let $I \in C^2(X)$ satisfy the Palais–Smale condition, and let $a$ be a regular value of $I$. Then, $H_*(X, I^a) \neq 0$, implies that $K_I \cap I^a \neq \emptyset$ where

$$K_I = \{ u \in X : I'(u) = 0 \},$$

and

$$I^a = \{ u \in X : I(u) \leq a \}.$$

Remark 2.7. Another notation which will be used in the article is

$$K_{I,D} = \{ u \in X : I(u) \in D \}$$

where $D$ is a connected subset of $\mathbb{R}$.

Remark 2.8. Prior to applying the Morse lemma we recall that for a Morse function the following holds:

1. $H_*(I^c, I^c \setminus \text{Crit}(I, c)) = \bigoplus_j H_*(I^c \cap N^j, (I^c \setminus \{x^j\}) \cap N^j)$.

2. $H_k(I^c \cap N, I^c \setminus \{x\} \cap N) = \begin{cases} \mathbb{R}, & k = m(x) \\ 0, & \text{otherwise} \end{cases}$

where $m(x)$ is a Morse index of $x$, and $x$ is a critical point of $f$.

3. Further

$$H_k(I^a, I^b) = \bigoplus_{i \in m(x_i) = k} \mathbb{Z} = \mathbb{Z}^{m_k(a,b)}$$

where $m_k(a, b) = n(\{i : m(x_i) = k, x_i \in K_{I,(a,b)}\})$. Here, $n(S)$ is the number of elements in the set $S$.

4. Morse relation

$$\sum_{u \in K_{I,\{a,b\}}} \sum_{k \geq 0} \dim C_k(I, u)t^k = \sum_{k \geq 0} \dim H_k(I^a, I^b)t^k + (1 + t)Q_t$$

for all $t \in \mathbb{R}$.

In this paper, we use the notion of local $(m, n)$-linking $(m, n \in \mathbb{N})$ (see Definition 2.3, [34]) to prove the existence of solution.

Definition 2.9. Let $W$ be a Banach space, $I \in C^1(W, \mathbb{R})$, and $0$ an isolated critical point of $I$ with $I(0) = 0$. Further, assume that $m, n \in \mathbb{N}$. We say that $I$ has a “local $(m, n)$-linking” near the origin if there exist a neighborhood $U$ of $0$, $E_0 \neq \emptyset$, $E \subseteq U$, and $D \subseteq W$ such that $0 \not\in E_0 \subseteq E$, $E_0 \cap D = \emptyset$, and
1. 0 is the only critical point of $I$ in $I^0 \cap U$, where $I^0 = \{ u \in W : I(u) \leq 0 \}$.
2. $\text{Dim } \text{im}(i^*) - \text{Dim } \text{im}(j^*) \geq n$, where $i^* : H_{m-1}(E_0) \rightarrow H_{m-1}(W \setminus D)$ and $j^* : H_{m-1}(E_0) \rightarrow H_{m-1}(E)$ are the homomorphisms induced by the inclusion maps $i : E_0 \rightarrow W \setminus D$ and $j : E_0 \rightarrow E$.
3. $I|_E \leq 0 \leq I|_{U \cap D \setminus \{0\}}$.

3. Proof of the main results

Let $F(x, t) = \int_0^t f(s, x)ds$ be the primitive of $f(x, t) = \lambda |t|^{-\gamma-1}t + \mu |t|^{2^*_s-2}t$. We say $u \in Z$ to be a weak solution of (1.1) if for every $\varphi \in Z$, we have

$$a \int_Q \frac{(u(x) - u(y))}{|x-y|^{N+2s}}(\varphi(x) - \varphi(y))dxdy + b \int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega \lambda |u|^{-\gamma-1}u \varphi(x)dx + \int_\Omega \mu |u|^{2^*_s-2}u \varphi(x)dx.$$  

Clearly, a weak solution to problem (1.1) is a critical point of the corresponding energy functional

$$I(u) = a \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}dxdy + b \int_\Omega |\nabla u|^2 dx - \int_\Omega F(x, u)dx. \quad (3.1)$$

However, it is easy to see that the functional $I$ is not $C^1(Z)$ due to the presence of the singular term. Therefore, instead of working with the original functional $I$, we will use a cut-off functional, $\tilde{I}$.

3.1. Existence of solution using a local (1, 1) linking at 0

We first define the functional $\tilde{I}$. Let us consider the following cut-off function.

$$\xi(t) = \begin{cases} 
1, & \text{if } |t| \leq l \\
\xi \text{ is decreasing, if } l \leq t \leq 2l \\
0, & \text{if } |t| \geq 2l.
\end{cases}$$

Let us now consider the following cut-off problem:

$$a(-\Delta)^su - b\Delta u = \lambda \frac{u}{|u|^{\gamma+1}} + \tilde{g}(u) \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \quad (3.2)$$

where,

$$\tilde{g}(u) = \mu |u|^{2^*_s-2}u \xi(\|u\|).$$
Let $\tilde{F}(x, t) = \int_0^t \tilde{f}(s, x) ds$ be the primitive of $\tilde{f}(x, t) = \lambda |t|^{-\gamma - 1} t + \tilde{g}(t)$. We say $\tilde{u} \in Z$ to be a weak solution of (3.2) if for every $\varphi \in Z$, we have

$$a \iint_Q \frac{(\tilde{u}(x) - \tilde{u}(y))}{|x - y|^{N + 2s}} (\varphi(x) - \varphi(y)) dxdy + b \int_\Omega \nabla \tilde{u} \cdot \nabla \varphi dx$$

$$= \int_\Omega \lambda |\tilde{u}|^{-\gamma - 1} \tilde{u} \varphi(x) dx + \int_\Omega \mu \xi(\|\tilde{u}\|) |\tilde{u}|^{2s^* - 2} \tilde{u} \varphi(x) dx.$$

Consequently, a weak solution to problem (3.2) is a critical point of the corresponding energy functional

$$\bar{I}(u) = \frac{a}{2} \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy + \frac{b}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega \bar{F}(x, u) dx. \quad (3.3)$$

**Remark 3.1.** One can easily see that if $u$ is a weak solution to (3.2) with $\|u\| \leq l$, then $u$ is also a weak solution to (1.1).

**Remark 3.2.** It is easy to observe that $I(u) = I(|u|)$. Hence, if a solution exists it has to be nonnegative. Furthermore, due to the presence of the singular term the solution is forced to be positive a.e. in $\Omega$.

We first verify that that functional $\bar{I}$ satisfies the (PS)- condition. Let us denote

$$S = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2_X}{\|u\|^2_{L^{2s*}(\Omega)}}$$

which is the best Sobolev constant in the Sobolev embedding (Theorem 2.2).

**Theorem 3.3.** The functional $\bar{I}$ satisfies the $(PS)_c$-condition for

$$c < c_* = \left( \frac{1}{2} - \frac{1}{2s^*} \right) \mu_{2s^* - 2s}^{2s^* - 2} S^{\frac{2s^*}{2s^* - 2}} - \left( \frac{1}{2} - \frac{1}{2s^*} \right)^{-\frac{\gamma + 1}{1 - \gamma}} \left[ \frac{2s^* - 1 + \gamma}{\Omega} \right]^{\frac{2s^* - 1 + \gamma}{2s^* - 2s}} \frac{\lambda}{S^{\frac{1}{2s^*}}} \left( 1 - \frac{\lambda}{S^{\frac{1}{2s^*}}} \right)^{\frac{1}{1 - \gamma}},$$

where $c_* > 0$ for sufficiently small $\lambda, |\mu| > 0$.

**Proof.** We see that although the functional has been modified with a cut-off, yet the functional is not $C^1$ which is not in the premise of the Palais-Smale condition. However, we will devise a scheme to tackle this situation by picking the sequence in a way that the singularity at 0 gets avoided since the functional is $C^1$ in $Z \setminus \{0\}$.

Suppose $(u_n) \subset Z$ is an eventually zero sequence, then apparently it converges to 0, and we discard the sequence immediately. Suppose, $(u_n) \subset Z$ is a sequence with infinitely many terms of the sequence equal to 0, then we will hand-pick a subsequence of $(u_n)$ with all non-zero terms. Thus, without loss of generality we will let $(u_n)$ such that $u_n \neq 0$ for every $n \in \mathbb{N}$. Let this sequence $(u_n)$ be such that

$$\bar{I}(u_n) \to c, \quad \text{and} \quad \bar{I}'(u_n) \to 0 \quad (3.5)$$
as \( n \to \infty \). Therefore, let us consider a sequence \( (u_n) \subset \mathbb{Z} \) such that \( \bar{I}(u_n) \to c \) for some \( c \in \mathbb{R} \), and \( \bar{I}'(u_n) \to 0 \) as \( n \to \infty \). It is not difficult to see that the subsequence is bounded in \( \mathbb{Z} \). This has the following consequences:

\[
\begin{align*}
  u_n & \rightharpoonup u_0 \text{ in } \mathbb{Z}; \|u_n\| \to M, \\
  u_n & \to u_0 \text{ in } L^r(\Omega) \text{ for any } 1 \le r < 2^*_s, \\
  u_n & \to u_0 \neq 0 \text{ a.e. in } \Omega,
\end{align*}
\]

as \( n \to \infty \). Now, consider

\[
o(1) = \langle \bar{I}'(u_n), u_n - u \rangle 
= (\|u_n\|^2 - \|u\|^2) - \kappa \int_{\Omega} |u_n|^{-\gamma - 1} u_n (u_n - u) dx \\
- \mu \int_{\Omega} \xi(\|u_n\|)|u_n|^{2^*_s - 2} u_n (u_n - u) dx \\
= (\|u_n\|^2 - \|u\|^2) - \mu \xi(M) \int_{\Omega} |u_n|^{2^*_s} dx + \mu \xi(M) \int_{\Omega} |u|^{2^*_s} dx + o(1) \\
= \|u_n - u\|^2 - \mu \xi(M) \int_{\Omega} |u_n - u|^{2^*_s} dx + o(1).
\]

We obtain

\[
\lim_{n \to \infty} \|u_n - u\|^2 = \mu \xi(M) \lim_{n \to \infty} \int_{\Omega} |u_n - u|^{2^*_s} dx = \mu \xi(M) N^{2^*_s}.
\]

If \( \mu \le 0 \), we produce a contradiction from (3.8), and we guarantee that \( u_n \to u \) strongly in \( \mathbb{Z} \). Therefore, we now proceed for \( \mu > 0 \).

Further, if \( N = 0 \), then we obtain \( u_n \to u \) in \( \mathbb{Z} \) as \( n \to \infty \), since \( M > 0 \). Therefore, we will show that \( N = 0 \). On the contrary let us assume that \( N > 0 \). Thus, we get

\[
0 \le \lim_{n \to \infty} \|u_n - u\|^2 = \mu \xi(M) N^{2^*_s}.
\]

Next, from (3.6), (3.9), and (3.4) we have

\[
SN^2 \le \mu \xi(M) N^{2^*_s} \le \mu N^{2^*_s}
\]

\[
M^2 - \|u\|^2 = \mu \xi(M) N^{2^*_s}.
\]

It also follows from (3.10) that

\[
N \ge \left( \frac{S}{\mu} \right)^{\frac{1}{2^*_s - 2}},
\]

and

\[
M^2 \ge SN^2 \ge \mu \frac{N^{2^*_s}}{2^*_s - 2} S(S^{2^*_s - 2})
= \mu \frac{N^{2^*_s}}{2^*_s - 2} S^{\frac{2^*_s}{2^*_s - 2}}.
\]
Consider
\[ c + o(1) = \tilde{I}(u_n) - \frac{1}{2s} (\tilde{I}'(u_n), u_n) \]
\[ = a \left( \frac{1}{2} - \frac{1}{2s} \right) [u_n]_{s,2}^2 + b \left( \frac{1}{2} - \frac{1}{2s} \right) \| \nabla u_n \|_2^2 \]
\[ + \lambda \left( \frac{1}{2} - \frac{1}{1 - \gamma} \right) \int_{\Omega} |u_n|^{1-\gamma} \, dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{2s} \right) \| u_n \|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u_n|^{1-\gamma} \, dx. \]
(3.13)

Therefore, by the Brezis–Lieb theorem, the Young’s inequality, and the results derived in this theorem, we pass the limit \( n \to \infty \) in (3.13) to obtain the following:

\[ c \geq \left( \frac{1}{2} - \frac{1}{2s} \right) (M^2 + \| u \|^2) - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} \, dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{2s} \right) (M^2 + \| u \|^2) - |\Omega|^{\frac{2s}{2s-1+\gamma}} S^{-\frac{1-\gamma}{1-\gamma}} \frac{\lambda}{1 - \gamma} \| u \|^{1-\gamma} \]
\[ \geq \left( \frac{1}{2} - \frac{1}{2s} \right) (M^2 - \left( \frac{1}{2} - \frac{1}{2s} \right) \frac{\lambda}{1 - \gamma} \left( \frac{2s}{2s-1+\gamma} S^{-\frac{1-\gamma}{1-\gamma}} \| u \|^{1-\gamma} \right) \frac{2}{1+\gamma} \]
\[ = c_*. \]
(3.14)

for sufficiently small \( \lambda > 0 \) and \( |\mu| > 0 \). This is a contradiction. \( \square \)

**Theorem 3.4.** The functional \( \tilde{I} \) has a local \((1, 1)\) - linking at the origin.

**Proof.** We define \( V = \mathbb{R} \). Clearly \( V \) is a one dimensional vector subspace of \( Z \). We now choose \( \nu \in (0, 1) \) small enough so that \( K_{\tilde{I}} \cap B_{\nu}(0) = \{0\} \) where \( B_{\nu}(0) = \{ u \in Z : \| u \| < \nu \} \), and \( K_{\tilde{I}} = \{ u \in Z : \tilde{I}'(u) = 0 \} \). Define

\[ E = V \cap \overline{B_{\nu}(0)} \]

for small enough \( \nu \in (0, 1) \). Note that any two norms are equivalent over \( E \). Therefore, using this to our advantage, for a sufficiently small \( \nu > 0 \) there exists \( \delta > 0 \), we have

\[ \| u \| \leq \nu \Rightarrow \| u \|_{\infty} \leq \delta \text{ for all } u \in E. \]
Therefore,

\[
\bar{I}(u) = \frac{a}{2} \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{b}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} \, dx
\]

\[
- \frac{\mu}{2s} \int_{\Omega} \xi(\|u\|)|u|^2_s \, dx
\]

\[
= \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} \, dx - \frac{\mu}{2s} \int_{\Omega} \xi(\|u\|)|u|^2_s \, dx
\]

\[
\leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} C \|u\|^{1-\gamma} - \frac{\mu}{2s} C' \|u\|^{2s} \leq 0
\]

(3.15)

where equivalence of norms followed from the set \(E\) being in a finite dimensional space \(V\). Further, define

\[
D' = \left\{ u \in Z : \|u\|^2 \geq \frac{4\lambda}{1 - \gamma} \|u\|^{1-\gamma}_1 \right\}.
\]

Thus, for any \(u \in D'\),

\[
\bar{I}(u) = \frac{a}{2} \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{b}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} \, dx
\]

\[
- \frac{\mu}{2s} \int_{\Omega} \xi(\|u\|)|u|^2_s \, dx
\]

\[
= \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} \, dx - \frac{\mu}{2s} \int_{\Omega} |u|^2_s \, dx
\]

\[
\geq \frac{1}{4} \|u\|^2 - \frac{|\mu|}{2s} C \|u\|^{2s} > 0
\]

(3.16)

holds for small enough \(\|u\| < \nu\). This holds for all \(D = D' \cap (B_{\nu}(0) \setminus \{0\})\). Define

\[E_0 = V \cap \partial B_{\nu}(0).\]

Clearly \(E_0 \cap D = \emptyset\). Therefore, we arrive at the following

\[I|_E \leq 0 < I|_D.\]

Further, define

\[h : [0, 1] \times Z \setminus D' \to Z \setminus D'\]

as

\[h(t, u) = (1 - t)u + t \cdot \nu \cdot \frac{\nu}{\|v\|}\]
where \( v = \alpha \in V \). Clearly \( 0 \in D' \) which makes the definition of \( h(\cdot, \cdot) \) valid. Observe that

\[
    h(0, u) = u, \text{ in } Z \setminus D' \\
    h(1, u) = v \frac{u}{\|u\|}, \text{ in } V \cap \partial B_v(0) = E_0.
\]

Thus, \( E_0 \) is a retract of \( Z \setminus D' \). Therefore,

\[
i^* : H_0(E_0) \to H_0(Z \setminus D)
\]

is an isomorphism. Note that \( E_0 = \{v, -v\} \) for some \( v \neq 0 \). Therefore, from \( \dim H_0(E_0) = 2 \) since \( H_0(E_0) = \mathbb{R} \oplus \mathbb{R} \). Thus, \( \dim im(i^*) = 2 \).

Further, \( E \) is an interval \([-\alpha, \alpha]\) which is contractible to a point. Thus, \( H_0(E) = \mathbb{R} \).

Hence, if \( j^* : H_0(E_0) \to H_0(E) \), then \( \dim im(i^*) - \dim im(j^*) = 2 - 1 = 1 \). Thus, the hypothesis of the Definition 2.9 is satisfied, and hence there exists a local \((1, 1)\) linking at 0.

\[\Box\]

**Proposition 3.5.** \( C_k(\bar{I}, \infty) = 0 \) for all \( k \in \mathbb{N} \).

**Proof.** Let \( t > 0 \), and consider

\[
    \frac{d}{dt} \bar{I}(tu) = \langle \bar{I}'(tu), u \rangle
    = \frac{1}{t} \langle \bar{I}'(tu), tu \rangle
    = \frac{1}{t} \left[ a \int \int_Q |tu(x) - tu(y)|^2 |x - y|^{N+2s} \, dxdy + b \int \Omega |\nabla tu|^2 \, dx \right]

\[
    - \int \Omega f(x, tu) \, tdx
    = \frac{1}{t} \left[ a \int \int_Q |tu(x) - tu(y)|^2 |x - y|^{N+2s} \, dxdy + b \int \Omega |\nabla tu|^2 \, dx \right]

\[
    - \lambda \int \Omega |tu|^{1-\gamma} \, dx - \mu \int \Omega \xi(\|tu\|) |tu|^{2*} \, dx
\]

\[\text{(3.18)}\]

Observe that for large \( t > 0 \) we have \( \bar{I}(tu) \geq \underline{J}_0 > c \), for some \( c > 0 \), since \( \bar{I}(tu) \to \infty \) as \( t \to \infty \). Therefore,

\[
    \frac{d}{dt} \bar{I}(tu) > 0
\]

for sufficiently large \( t > 0 \). Thus, there exists a unique \( h(u) > 0 \) such that \( \bar{I}(h(u)u) = \underline{J}_0 \). This actually implies that \( h \in C(\partial B_1) \). On extending \( h(\cdot) \) to \( Z \setminus \{0\} \) by defining

\[
    \tilde{h}(u) = \frac{1}{\|u\|} h \left( \frac{u}{\|u\|} \right)
\]
for all $u \in \mathbb{Z}\setminus\{0\}$, we obtain $\tilde{h} \in \mathbb{Z}\setminus\{0\}$, and $\tilde{I}(\tilde{h}(u)u) = \mathbb{J}_0$. Also if $\tilde{I}(u) = \mathbb{J}_0$ then $\tilde{h}(u) = 1$. Therefore, we define

$$
\hat{h}(u) = \begin{cases} 
1, & \text{if } \tilde{I}(u) \geq \mathbb{J}_0 \\
\tilde{h}(u), & \text{if } \tilde{I}(u) \leq \mathbb{J}_0 
\end{cases}
$$

which makes $\hat{h}$ continuous. Further define

$$
g(t, u) = (1 - s)u + s\hat{h}(u)u
$$

for all $(t, u) \in [0, 1] \times (\mathbb{Z}\setminus\{0\})$. Thus, we have

$$
g(0, u) = u, \ g(1, u) = \tilde{h}(u)u \in \tilde{I}^{\mathbb{J}_0},
$$

and

$$
g(t, .)|_{\tilde{I}^{\mathbb{J}_0}} = id|_{\tilde{I}^{\mathbb{J}_0}}
$$

for all $t \in [0, 1]$. What we conclude from here is that $\tilde{I}^{\mathbb{J}_0}$ is a deformation of $\mathbb{Z}\setminus\{0\}$. By standard definition of a homotopy, it is easy to see that $\partial B_1 = \{u \in \tilde{Z} : \|u\| = 1\}$ is a deformation of $\mathbb{Z}\setminus\{0\}$. Thus, on choosing $\mathbb{J}_0$ sufficiently negative we get

$$
C_k(\tilde{I}, \infty) = H_k(\tilde{Z}, \partial B_1) = 0
$$

for all $k \in \mathbb{N}$.

**Remark 3.6.** We use a variant of the result - *If $X$ is a Banach space, $\tilde{I} \in C^1(X)$, $0 \in K_f$ is isolated, and $\tilde{I}$ has a local $(m, n)$-linking near the origin, then rank $C_m(\tilde{I}, 0) \geq n$*. Since we obtained a $(1, 1)$ linking we conclude that rank of $C_1(\tilde{I}, 0) \geq 1$. In our case $0$ is not a critical point of $\tilde{I}$, however one can still construct $C_1(\tilde{I}, 0)$ owing to not only the fact that $\tilde{I}$ is well-defined at $0$ but also having a ‘sharp’ isolated non regularity there.

**Theorem 3.7.** There exists a solution, say $u_0$, to the problem (3.2).

**Proof.** As seen in the Remark 3.6 we have that rank $C_1(\tilde{I}, 0) \geq 1$. In tandem with this and Proposition 3.5, we are in a position to apply the Proposition 6.2.42 (refer Theorem 4.1 in “Appendix”) of [33] that guarantees the existence of a $u_0 \in \mathbb{Z}$ such that

$$
u_0 \in K_{\tilde{I}}\setminus\{0\},
$$

which further implies that $u_0 \in \mathbb{Z} \cap L^\infty(\Omega)$ is a solution of (3.2). For the boundedness of $u_0$, please refer the Theorem 4.2 in the “Appendix”. Thus, the existence of a nontrivial, bounded solution to (3.2) has been established. □
3.2. Multiplicity results

**Theorem 3.8.** The problem (3.2) has at least two nontrivial solutions in $Z \cap L^\infty(\Omega)$.

**Proof.** From the Theorem 3.7 it can be assumed that there is one nontrivial solution to (3.2). Let us further assume that there is exactly one nontrivial solution to (3.2).

We at first show that $H_k(Z, \bar{I}^{-a})$ for all $k \geq 0$. Pick $a \in \{v \in Z : \|v\| = 1\} = \partial B^\infty$, where $B^\infty = \{v \in Z : \|v\| \leq 1\}$. We make use of the Eq. (3.16) which explains that for small enough $t > 0$ we have $\bar{I}(tu) > 0$. The functional $\bar{I}$ being $C^1$ in $Z \backslash \{0\}$ indicates that $\bar{I}'(tu) > 0$ for this small $t > 0$. Also from (3.19) and in combination with the fact that the functional is superlinear, we have that for large enough $t > 0$, $\bar{I}'(tu) < 0$. Thus, there exists a unique $t(u)$ such that $\bar{I}'(t(u))u = 0$ since due to our assumption that there exists exactly one nontrivial solution. We can thus say that there exists a $C^1$-function $T : Z \backslash\{0\} \rightarrow \mathbb{R}^+$ defined by $u \mapsto t(u)$.

We now define a standard deformation retract $\eta$ of $Z \backslash B_r(0)$ into $\bar{I}^{-a}$ as follows (refer Definition 2.5).

$$
\eta(s, u) = \begin{cases} 
(1-s)u + sT \left( \frac{u}{\|u\|} \right) \frac{u}{\|u\|}, & \|u\| \geq v, \bar{I}(u) \geq -a \\
u, & \bar{I}(u) \leq -a.
\end{cases}
$$

It is not difficult to see that $\eta$ is a $C^1$ function over $[0, 1] \times Z \backslash B_r(0)$. On using the map $\delta(s, u) = \frac{u}{\|u\|}$, for $u \in Z \backslash B_r(0)$ we claim that $H_k(Z, Z \backslash B_r(0)) = H_k(B^\infty, S^\infty)$ for all $k \geq 0$. This is because, $H_k(B^\infty, S^\infty) \cong H_k(\ast, 0)$. From elementary computation of homology groups with two 0-dimensional simplices it is easy to see that $H_k(\ast, 0) = \{0\}$ for each $k \geq 0$. A result in [35] tells us that

$$C_n(\bar{I}, 0) = \begin{cases} \mathbb{R}, & \text{if } m(0) = N \\
0, & \text{otherwise}.
\end{cases}$$

Clearly, $m(0) \geq 2$. Therefore, from the Morse relation in the Remark 2.8–4 and the result above taken from [35], we have for $a > 0$

$$
\sum_{u \in K_{I,[-a,\infty)}} \sum_{k \geq 0} \dim C_k(\bar{I}, u) t^k = t^{m(0)} + p(t) \tag{3.20}
$$

where $m(0)$ is the Morse index of 0, and $p(t)$ contains the rest of the powers of $t$ corresponding to the other critical points, if any. On further using the the Morse relation we obtain

$$t^{m(0)} + p(t) = (1 + t) Q_t. \tag{3.21}$$

This is because the $H_k$s are all trivial groups. Hence, $Q_t$ either contains $t^{m(0)}$ or $t^{m(0)-1}$ or both. Thus, there exists at least two nontrivial $u \in K_{I,[-a,\infty)}$ with $2 \leq m(0) \leq N + 1$.

**Theorem 3.9.** Let $\Omega$ be as above. Then, $\bar{I}$ has infinitely many critical points in $Z$. 

\[\square\]
Proof. We appeal to the Morse theory again from which we obtain the following.

$$\sum_{u \in K_{I, (-a, \infty)}} \sum_{k \geq 0} \dim C_k(I, u)t^k = t^{m(0)} + 2 \sum_{k \geq 0} \alpha_k t^k. \quad (3.22)$$

The factor 2 is due to the fact that if $u$ is a critical point then $(-u)$ is also a critical point. $\alpha_k$'s are nonnegative integers. As in the proof of the Theorem 3.8, we have $H_k(Z, I^{\infty}_{-a}) = 0$. Therefore, we have the following identity over $\mathbb{R}$.

$$t^{m(0)} + 2 \sum_{k \geq 0} \alpha_k t^k = (1 + t)Q_t. \quad (3.23)$$

In particular for $t = 1$ we have $1 + 2A = 2B$, where the series on the left of the identity in (3.23) for $t = 1$ is denoted by $A$, and the term on the right of the same for $t = 1$ is denoted by $B$. This is possible only when the sum is infinite as finite sum leads to a contradiction that there exists an odd and an even number that agree. Thus, there exists infinitely many solutions.

The following is the main existence result for the problem (1.1).

**Theorem 3.10.** The problem (1.1) admits infinitely many nontrivial solutions in $Z \cap L^{\infty}(\Omega)$.

**Proof.** According to Theorems 3.7–3.9, the problem (3.2) has infinitely many solutions in $Z \cap L^{\infty}(\Omega)$. With a suitable choice of $l$ and by Remark 3.2, we conclude our proof.

4. Appendix

**Theorem 4.1.** (Theorem 6.2.42 of Papagiorgiou et al. [33]) If $X$ is a Banach space, $I \in C^1(X)$, $I$ satisfies the Ce-condition, $K_I$ is finite with $0 \in K_I$, and for some $k \in \mathbb{N}$ we have $C_k(I, 0) \neq 0$, $C_k(I, \infty) = 0$, then there exists a $u \in K_I$ such that $\bar{u} < 0$, $C_{k-1}(I, u) \neq 0$ or $I(u) > 0$, and $C_{k+1}(I, u) \neq 0$.

**Theorem 4.2.** Any solution to (3.2) is in $L^{\infty}(\Omega)$.

**Proof.** The argument sketched here is a standard one, and hence we shall only show that an improvement in integrability is possible up to $L^{\infty}$ assuming an integrability of certain order, say $p > 1$. The boundedness follows from a bootstrap argument. Without loss of generality we can consider the set $\Omega' = \{x \in \Omega : u(x) > 1\}$, and thus by the positivity of a fixed solution (refer Remark 3.2), say, $u$ we have...
$u = u^+ > 0 \text{ a.e. in } \Omega$. Let $u \in L^p(\Omega)$ for $p > 1$. Let $a = \max\{\lambda, |\mu|\}$. On testing with $u^p$ to obtain the following:

$$C \|u^{p+1}\|_{\beta^*}^2 \leq \|u^{p+1}\| \leq \left(\lambda \int_{\Omega'} |u|^{p-\gamma} dx + \mu \int_{\Omega'} |u|^{2^*-1+p} dx \right) \frac{(p+1)^2}{4p} \leq a \left(\int_{\Omega'} |u|^p (1 + |u|^{2^*-1}) dx \right) \frac{(p+1)^2}{4p}; \text{ since in } \Omega' \text{ we have } u > 1 \leq 2a \left(\int_{\Omega'} |u|^p |u|^{2^*} dx \right) \frac{(p+1)^2}{4p} \leq 2aC''\|u\|_{\beta^*}^{2^*} \|u^p\|_t; \text{ since by using Hölder’s inequality.}$$

(4.1)

Here, $t = \frac{\beta^*}{\beta^* - 2^*}$ for some $\beta^* > 1$. Thus, we also have

$$C'\|u^p\|_{\beta^*}^2 \leq C'\|u^{p+1}\|_{\beta^*}^2 \leq \|u^{p+1}\|_{\beta^*}^2. \quad (4.2)$$

From the story so far, we know the following

$$C'\|u^p\|_{\beta^*}^2 \leq 2aC''\|u\|_{\beta^*}^{2^*} \|u^p\|_t. \quad (4.3)$$

For the fixed $\beta^* > 1$, we set $\eta = \frac{\beta^*}{\tau t} > 1$ for a suitable choice of $t$, and $\tau = tp$ to get

$$\|u\|_{\eta t} \leq C^\frac{1}{\tau} \|u\|_t; \text{ where } C = 2aC''\|u\|_{\beta^*}^{2^*} \text{ is a fixed quantity for a fixed solution } u. \quad (4.4)$$

Let us now iterate with $\tau_0 = t$, $\tau_{n+1} = \eta \tau_n = \eta^{n+1}t$. After $n$ iterations, the inequality (4.4) yields

$$\|u\|_{\tau_{n+1}} \leq C \sum_{i=0}^{n} \frac{\tau_i}{\tau_i^{1/\tau}} \prod_{i=0}^{n} \left( \frac{\tau_i}{t} \right)^{1/\tau} \|u\|_t. \quad (4.5)$$

By using $\eta > 1$ and the method of iteration, i.e. $\tau_0 = t$, $\tau_{n+1} = \eta \tau_n = \eta^{n+1}t$, we have

$$\sum_{i=0}^{\infty} \frac{t}{\tau_i} = \sum_{i=0}^{\infty} \frac{1}{\eta^i} = \frac{\eta}{\eta - 1},$$

and

$$\prod_{i=0}^{\infty} \left( \frac{\tau_i}{t} \right)^{1/\tau_i} = \eta^{\frac{t^2}{(\eta - 1)t^2}}.$$

Hence, on passing the limit $n \to \infty$ in (4.5), we end up getting

$$\|u\|_\infty \leq C \frac{\eta}{\tau^\eta} \eta^{\frac{\eta^2}{(\eta - 1)^2}} \|u\|_t. \quad (4.6)$$

Thus, $u \in L^\infty(\Omega)$. \qed
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