Abstract

In this paper we introduce a new combinatorial approach to analyze the trace of large powers of Wigner matrices. Our approach is motivated from the paper by Soshnikov [36]. However the counting approach is different. We start with classical word sentence approach similar to Anderson and Zeitouni [1] and take the motivation from Sinai and Soshnikov [35], Soshnikov [36] and Péké [32] to encode the words to objects similar to Dyck paths. To be precise the map takes a word to a Dyck path with some edges removed from it. Using this new counting we prove edge universality for large Wigner matrices with sub-Gaussian entries. One novelty of this approach is unlike Sinai and Soshnikov [35], Soshnikov [36] and Péké [32] we do not need to assume the entries of the matrices are symmetrically distributed around 0. We hope this method will be applicable to many other scenarios in random matrices.

1 Introduction

Since the groundbreaking discovery of Wigner [43], Wigner matrices have been a topic of key interest in the mathematics and physics communities. Later on these matrices proved to be important for many models in engineering, high dimensional statistics and many other branches. Since the introduction of these matrices many problems regarding the eigenvalue distribution of these matrices have been solved. The results are so vast and diverse that we shall not be able to discuss all of them in this introduction. Here we mention some of them which we find relevant in the context of current paper.

In particular, a Wigner matrix is a $n \times n$ symmetric (hermitian) matrix with real (complex) entries where the entries of the upper diagonal part are i.i.d. with mean 0 and variance $\frac{1}{n}$. One is interested in the eigenvalue distribution of the matrix when the dimension grows to infinity. The study of eigenvalues of Wigner matrices started with characterizing the limiting distribution of the histogram of the eigenvalues. This is done
in the seminal papers of Wigner [43] and Wigner [42]. It is known that this limiting
distribution exists and coined as the famous semicircular distribution. In particular, it
is given by the following density.

\[ f(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4-x^2} & \text{when } |x| \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad (1.1) \]

However after specifying the spectral distribution, there has been a remarkable ad-
vancement in this topic. One of the most important direction is the universality prop-
ties of the Wigner matrices.

In general there are two types of universality properties of Wigner matrices. The
first one is the bulk universality and the second one is the edge universality. In this paper
we shall mostly consider the edge universality. However for the shake of completeness
we briefly describe the important results on the bulk universality of the Wigner matrices.
In general bulk universality concerns about the point process at any fixed point
in the interior of the semicircle distribution. When the entries of the matrices are real
(complex) Gaussians, they are called GOE (GUE). The explicit eigenvalue distributions
in these cases are well known. For the explicit formula in the GOE case one might look
at (4.1). In general, doing calculation with the exact eigenvalue formulas are consider-
ably difficult in the GOE case than the GUE case. Using the exact formulas in the GUE
case in the early days in a series of papers, Wigner, Dyson, Gaudin, and Mehta proved
that under proper rescaling, the joint eigenvalue distribution or the point process at an
interior point of the spectrum is described by the sine kernel. One might look at Mehta
[28], Mehta and Gaudin [30], Dyson [9] and Dyson [10] for some references. In these
papers they also conjectured that this result holds for general Wigner matrices. These
results were proved for a very general class of invariant ensembles by Deift et al. [7],
Deift et al. [8], Bleher and Its [6] and Pastur and Shcherbina [31].

Later on Johansson [25] proved bulk universality for Wigner divisible ensembles.
For general Wigner matrices a new approach was introduced by Erdos, Schlein, Yau,
and others. One might look at [14], [13], [18], [22], [15], [17], [19], [20], [16], [21]
for some references. In another different approach bulk universality was also proved by
Tao and Vu [37]. One might also look at Erdős and Yau [12], Erdős and Yau [11] for a
review on the literature.

Apart from the bulk universality a different type of universality is observed at the
edge of the spectrum. Here we look at the point processes of the eigenvalues near ±2
which is the support of the semicircular distribution. Using the explicit distribution of
the eigenvalues the fluctuation of the largest eigenvalue of the Wigner matrix was first
proved in the seminal works of Tracy and Widom [39], [40]. In particular it is proved
that

\[ \mathbb{P} \left[ n^{-\frac{1}{2}} (\lambda_{1,n} - 2) \leq s \right] \to F_\beta(s) \quad (1.2) \]

where the Tracy-Widom distribution functions \( F_\beta \) can be described by Painleve equa-
tions, and \( \beta = 1, 2, 4 \) corresponds to Orthogonal/Unitary/Symplectic ensemble, respec-
tively. Here \( \lambda_{1,n} \geq \ldots \geq \lambda_{n,n} \) are the eigenvalues of the Wigner matrix \( W \). The joint
distribution of \( k \) largest eigenvalues can be expressed in terms of the Airy kernel, which was shown by Forrester [23]. In general the joint distribution of \((\lambda_{1,n}, \ldots, \lambda_{k,n})\) will also converge after proper rescaling and centering. In particular

\[
P \left[ n^{\frac{2}{3}} (\lambda_{1,n} - 2) \leq s_1, \ldots, n^{\frac{2}{3}} (\lambda_{k,n} - 2) \leq s_k \right] \to F_{\beta,k}(s_1, \ldots, s_k). \tag{1.3}
\]

The \( k \) dimensional distribution \( F_{\beta,k} \) will also be coined as Tracy Widom distribution. Now coming to the universality at the edge, the first result of this kind was obtained by Soshnikov [36]. He assumed that the distributions of the entries of the matrix are sub-Gaussian and symmetric. The method of this paper is combinatorial in nature and is the main inspiration of our paper. Essentially the proof analyzes the trace of a high power of the Wigner matrix. Based on a similar technique and truncation Ruzmaikina [34] proved the universality under the assumption that entries are symmetric and the tail of the entries of the matrix decay at the rate \( x^{-18} \). The universality for the non-symmetric entries was first proved by Tao and Vu [38]. Here one assumes that entries have vanishing third moment and the tail decays exponentially. Finally through a different approach initiated by Erdős, Yau and others the vanishing third moment condition was removed. One might look at [21] and [18]. In these papers the results are obtained through a detailed analysis of the Green’s function of the matrix. Finally a necessary and sufficient condition for the edge universality was obtained in Lee and Yin [27]. This paper proves that the edge universality holds if and only if \( \lim_{s \to \infty} s^{4} P \left[ |x_{1,2}| \geq s \right] \to 0 \). Here \( x_{1,2} = \sqrt{n} W(1,2) \).

Before moving to the next section, we spend a few moments about the novelty of the current paper. As mentioned earlier we use the combinatorial approach initiated by Soshnikov [36]. However unlike Soshnikov [36] we do not need to assume that entries are symmetrically distributed. This is done by a very refined encoding of the contributing words defined in section 7. In particular the encoding in this paper out performs the encoding by Füredi and Komlós [24], Vu [41] and Péché and Soshnikov [33]. To the best of our limited knowledge this is the first paper to establish the edge universality for general non-symmetric entry distribution through combinatorial methods. We also hope the counting strategy introduced in this paper will be useful in many other different scenarios. On the other hand, by the methods in this paper we have been able to prove a combinatorial description of the Tracy Widom law. This method might be useful to characterize the Tracy Widom law when the exact calculation is not available.

2 The model

In this section we introduce the matrix ensembles. Firstly we start with the definition of Wigner matrices.

**Definition 2.1.** We call a matrix \( W = (x_{i,j}/\sqrt{n})_{1 \leq i,j \leq n} \) to be a Wigner matrix if \( x_{i,j} = \bar{x}_{j,i}, \) \( (x_{i,j})_{1 \leq i < j \leq n} \) are i.i.d., \( E[x_{i,j}] = 0 \) and \( E[|x_{i,j}|^2] = 1. \)
In this paper we deal with the real symmetric matrices and for the ease of calculation we scale the whole matrix by a factor 2. With slight abuse of notation we shall also call this matrix a Wigner matrix and denote it by \( W \). Following are the assumptions of the matrices we consider.

**Assumption 2.1.** We consider a matrix \( W \) given by \( W = (x_{i,j}/ \sqrt{n})_{1 \leq i,j \leq n} \) such that the following conditions are satisfied:

(i) \( x_{i,j} = x_{j,i} \) for \( i \leq j \).

(ii) \( \text{Var}(x_{i,j}) = \frac{1}{4} \)

(iii) \( (x_{i,j})_{1 \leq i < j \leq n} \) are i.i.d.

(iv) \( \mathbb{E}[x_{i,j}^2] \leq (\text{const.} k)^k \forall k \in \mathbb{N} \)

Given a Wigner matrix \( W \) of dimension \( n \times n \) we denote its eigenvalues by \( \lambda_{1,n} \geq \ldots \geq \lambda_{n,n} \). It is well known that for a Wigner matrix in Definition 2.1, the measure \( \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i,n}} \) converge weakly to the semicircular law in the almost sure sense. The law is given by density in (1.1). When we scale the entries by a factor 2, the distribution is supported in \( [-1,1] \) and its density is given by

\[
f(x) = \begin{cases} 
\frac{2}{n} \sqrt{1 - x^2} & \text{when } |x| \leq 1 \\
0 & \text{otherwise}
\end{cases} \tag{2.1}
\]

### 3 Powers of generating function of Catalan numbers

Since Dyck paths play a crucial role in this paper, we discuss the some properties of the Dyck paths and the generating function of the Dyck paths.

**Definition 3.1.** (Dyck paths) A Dyck path of length \( 2k \) is a path of a simple symmetric random walk which starts from \( y = 0 \), returns to \( y = 0 \) after \( 2k \) step and it always stays on or above the \( X \) axis.

**Definition 3.2.** (Catalan numbers) The count of all Dyck paths of length \( 2k \) is well known, coined as the \( k \) th Catalan number and is given by

\[
C_k = \begin{cases} 
1 & \text{if } k = 0 \\
\frac{1}{k+1} \binom{2k}{k} & \text{otherwise.}
\end{cases} \tag{3.1}
\]

It is well known that whenever a random variable follows the semicircular distribution given in (1.1), then its \( 2k \) th moment is \( C_k \).
Definition 3.3. The generating function of the Catalan numbers and its powers will be quantities of interest. The generating function of the Catalan numbers is denoted by $C(x)$ and is defined as follows:

$$C(x) = \sum_{k=0}^{\infty} C_k x^k.$$  \hfill (3.2)

The $m$th power of $C(x)$ will be of fundamental interest. Fortunately the an explicit formula for the $m$th power of $C(x)$ is known (see Lang [26] for a reference). This is given as

$$C^m(x) = \sum_{k=0}^{\infty} \frac{m^m (2k + m)^k}{2} x^k.$$  \hfill (3.3)

4 A brief overview of Tracy-Widom law and related stuffs

In this section we give a very brief overview of the point process corresponding to the eigenvalues of a Wigner matrix and the Tracy-Widom law. This part is mostly taken from Soshnikov [36].

We start with the definition of GOE (Gaussian Orthogonal Ensemble)

Definition 4.1. Suppose we have a Wigner matrix $W$ as defined in Assumption 2.1. Then $W$ is called a GOE (Gaussian Orthogonal Ensemble) if $x_{i,j} \sim \text{indep } N(0, \frac{1}{4})$ and $x_{i,i} \sim \text{indep } N(0, \frac{1}{2})$.

Definition 4.2. (Eigenvalue distribution of Wigner matrices:) For GOE the eigenvalue distribution is well known. Suppose $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a GOE matrix and we assume $P_{n,1}(\lambda_1, \ldots, \lambda_n)$ is the eigenvalue distribution. Then

$$dP_{n,1}(\lambda_1, \ldots, \lambda_n) = C \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j| \exp \left\{-n \sum_{i=1}^{n} \lambda_i^2 \right\} \prod_{i=1}^{n} d\lambda_i.$$  \hfill (4.1)

Although not explicitly known, we shall denote the eigenvalue distribution of a general Wigner matrix by $P_{n,\text{Gen}}(\lambda_1, \ldots, \lambda_n)$

Studying the eigenvalues near the support is done by studying the $k$ point correlation function of the eigenvalue distribution. This is what we define next.

Definition 4.3. ($k$ point correlation function) The $k$ point correlation function of the eigenvalue distribution of the GOE is defined as

$$\rho_{n,1,k}(\lambda_1, \ldots, \lambda_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} \frac{d^n P_{n,1}(\lambda_1, \ldots, \lambda_n)}{d\lambda_1 d\lambda_2 \ldots d\lambda_n} d\lambda_{k+1} \ldots d\lambda_n.$$  \hfill (4.2)

Similarly for general Wigner matrices the $k$ point correlation function is defined as

$$\rho_{n,\text{Gen},k}(\lambda_1, \ldots, \lambda_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} \frac{d^n P_{n,\text{Gen}}(\lambda_1, \ldots, \lambda_n)}{d\lambda_1 d\lambda_2 \ldots d\lambda_n} d\lambda_{k+1} \ldots d\lambda_n.$$  \hfill (4.3)
For the precise formulas of $\rho_{n,1,k}(\lambda_1, \ldots, \lambda_k,n)$ one might look at Mehta [29] Chapter 5 and 6. $k$ point correlation functions are particularly useful in calculating the moments of the number of eigenvalues in an interval $I \subset \mathbb{R}$. Let $\nu_{n,I,1}(\text{resp. } \nu_{n,I,\text{Gen}})$ be the number of eigenvalues in $I$. Then the mathematical expectation of $\nu_{n,I,1}(\text{resp. } \nu_{n,I,\text{Gen}})$ is given by the formula

$$E[\nu_{n,I,1}(\text{resp. } \nu_{n,I,\text{Gen}})] = \int_I \rho_{n,1,1}(x)(\text{resp. } \rho_{n,1,\text{Gen}}(x))dx,$$

and in general

$$E[\nu_{n,I,1}(\nu_{n,I,1} - 1) \ldots (\nu_{n,I,1} - k + 1)] = \int I \rho_{n,1,k}(x_1, \ldots, x_k)dx_1 \ldots dx_k. \tag{4.5}$$

In the subsequent part of this section we shall only consider the GOE case since the explicit form of the eigenvalue distribution is known.

In order to find out the scaling limit of the eigenvalues at the edge, we shrink the length of the interval $I$ such that in expectation there are only finitely many eigenvalues in the interval. In order to achieve this for a given point $x$ we take the interval $I_{n,x} = \left[ x - \frac{c_1}{\rho_{n,1,1}(x)}, x + \frac{c_2}{\rho_{n,1,1}(x)} \right]$ so that the integral

$$\int_{I_{n,x}} \rho_{n,1,1}(y)dy = O(1).$$

This allows us to consider the following rescaling of the eigenvalues for any given $x$

$$\lambda_{i,n} := x + y_{i,n} \frac{1}{\rho_{n,1,1}(x)}.$$

we also consider the following rescaled correlation function:

$$R_{n,1,k,x}(y_1, \ldots, y_k) = \frac{1}{\rho_{n,1,1}(x)} \rho_{n,1,k}(\lambda_{1,n}, \ldots, \lambda_{k,n})$$

It is easy to see that if $I_{n,x} = \left[ x - \frac{c_1}{\rho_{n,1,1}(x)}, x + \frac{c_2}{\rho_{n,1,1}(x)} \right]$ is an interval containing $x$ and is of length $O(\frac{1}{\rho_{n,1,1}(x)})$ then

$$E[\nu_{n,I_{n,x,1}}(\nu_{n,I_{n,x,1}} - 1) \ldots (\nu_{n,I_{n,x,1}} - k + 1)] = \int_{[c_1,c_2]^2} R_{n,1,k,x}(y_1, \ldots, y_k)dy_1 \ldots dy_k.$$

To show that $\nu_{n,I_{n,x,1}}$ converges to a limit in distribution as $n \to \infty$, one needs to show that the rescaled $k$-point correlation functions have a limit too. This is well known in the Gaussian case (i.e. $R_{n,1,k,x}(y_1,n, \ldots, y_{k,n}) \to R_{1,k,x}(y_1, \ldots, y_k)$). One might look at Mehta [29] for a reference. Here we shall be interested in $R_{1,k,1}(y_1, \ldots, y_k)$. For the GOE case the function $R_{1,k,1}(y_1, \ldots, y_k)$ is defined in the following way: Let

$$K(y, z) = \frac{\mathcal{A}(y)\mathcal{A}'(z) - \mathcal{A}(z)\mathcal{A}'(y)}{y - z}. \tag{4.6}$$
Here $A_i(x)$ is the Airy function and is given by the solution of the differential equation $f''(y) = y f(y)$ with the asymptotics $f(y) \sim \frac{1}{2 \sqrt{\pi} y^{\frac{1}{2}}} \exp \left\{ -\frac{2}{3} y^{\frac{3}{2}} \right\}$ as $y \to \infty$. Now $R_{1,k,1}$ can be represented as the square root of determinant of a $2k \times 2k$ matrix consisting of $2 \times 2$ blocks $(\xi_i(y, y_j))_{1 \leq i, j \leq k}$. To define $\xi_1$ we introduce a few more notations. Let

$$DK(y, z) = -\frac{d}{dz} K(y, z)$$
$$JK(y, z) = -\int_{\infty}^y K(t, z) dt - \frac{1}{2} \text{sgn}(y - z).$$

Then

$$\xi_1(y, z) = \begin{bmatrix} K(y, z) + \frac{1}{2} A_i(y) \int_{-\infty}^t A_i(t) dt & -\frac{1}{2} A_i(y) A_i(z) + DK(y, z) \\ JK(y, z) + \frac{1}{2} \int_{-\infty}^t A_i(u) du + \frac{1}{2} \int_{-\infty}^z A_i(u) du \int_{-\infty}^z A_i(v) dv & K(z, y) + \frac{1}{2} A_i(z) \int_{-\infty}^y A_i(t) dt \end{bmatrix}$$

and

$$R_{1,k,1}(y_1, \ldots, y_k) = \sqrt{\det \left( \xi_1(y_i, y_j) \right)_{1 \leq i, j \leq k}}.$$

Finally the distribution of the largest entry of this point process can be determined in the following way: Let $\lambda_{\text{max}}$ be the largest entry of the limiting point process. Then

$$P[\lambda_{\text{max}} \leq t] = P[\text{# of points of the point process } \in [t, \infty) = 0]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[t, \infty)^k} R_{1,k,1}(y_1, \ldots, y_k) dy_1 \ldots dy_k. \quad (4.10)$$

This concludes our discussion about the GOE Tracy-Widom distribution.

## 5 Main results

In this section we state our main result. These results can also be proved under weaker assumptions by the approach of Erdős, Yau and others (see [21] and [18]). However as mentioned at the beginning of the paper, the fundamental objective of this paper is to provide a combinatorial method to prove these results and extend the approach in Soshnikov [36] for matrices where entries are sub-Gaussian but possibly have non-symmetric distributions.

We now state the main result of this paper.

**Theorem 5.1.** Suppose $W$ is a Wigner matrix satisfying Assumption 2.1 with eigenvalues $\lambda_{1,n} \geq \ldots \geq \lambda_{n,n}$. Then following are true:

(i) The point process at the edge of the spectrum (i.e. at $x = \pm 1$ according to notations in Section 4) converges weakly to the limiting point process at the edge of the spectrum of GOE ensemble.
(ii) As a consequence, for any fixed $k$ the joint distribution of $(n^{\frac{2}{3}}(\lambda_{1,n} - 1), \ldots, n^{\frac{2}{3}}(\lambda_{k,n} - 1))$ converge weakly to the limiting joint distribution of the top $k$ eigenvalues of the GOE. In particular $n^{\frac{2}{3}}(\lambda_{1,n} - 1)$ weakly converges to the GOE Tracy Widom distribution.

6 Strategies to prove the edge universality

As discussed earlier, our main technique is combinatorial in nature. In this section we briefly give an overview of the strategies to prove the edge universality. At the beginning, we start with the approach by Soshnikov [36]. Here given the matrix $W$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, we write

$$\lambda_{i,n} = 1 + \frac{\theta_{i,n}}{2n^{\frac{2}{3}}}.$$  

if $\lambda_{i,n} > 0$ and

$$\lambda_{i,n} = -1 - \frac{\tau_{i,n}}{2n^{\frac{2}{3}}}$$  

if $\lambda_{i,n} \leq 0$. Now we shall consider a very high power of the matrix and compute it’s trace. In particular, for any $t \in (0, \infty)$ we consider $[tn^{\frac{2}{3}}]$ and consider $\text{Tr}[W^2[tn^{\frac{2}{3}}]]$. Since

$$\text{Tr}[W^k] = \sum_{i=1}^{n} \lambda_{i,n}^k,$$

we have

$$\text{Tr}[W^2[tn^{\frac{2}{3}}]] = \sum_{\lambda_{i,n} > 0} \left(1 + \frac{\theta_{i,n}}{2n^{\frac{2}{3}}}\right)^{2[tn^{\frac{2}{3}}]} + \sum_{\lambda_{i,n} \leq 0} \left(1 + \frac{\tau_{i,n}}{2n^{\frac{2}{3}}}\right)^{2[tn^{\frac{2}{3}}]}.$$  

(6.1)

Now among the terms in (6.1), we can ignore the terms when $0 \leq \lambda_{i,n} \leq \left(1 - \frac{1}{2n^{\frac{2}{3}}}\right)$ and $0 \geq \lambda_{i,n} \geq (-1 + \frac{1}{2n^{\frac{2}{3}}})$. This is due to the fact that both $\left(1 - \frac{1}{2n^{\frac{2}{3}}}\right)^{[tn^{\frac{2}{3}}]} = O(e^{-cn^{\frac{4}{3}}})$ for some fixed constant $c$. As a consequence, the sum corresponding to these terms goes to 0.

Before proceeding further, we introduce two results. These results are the main technical contribution of the paper and their proofs are given in Section 8. Similar results can be found in Soshnikov [36] but with additional assumption that the distributions of the entries of the matrix are symmetric around 0.

**Theorem 6.1.** Consider the Wigner matrix $W$ satisfying Assumption 2.1. Then for any fixed $t \in (0, \infty)$ taking $k = [tn^{\frac{2}{3}}]$, we have the following results

1. $\mathbb{E} \text{Tr}[W^2] = O(1)$ and $\mathbb{E} \text{Tr}[W^{2k+1}] = o(1)$. 
2. If the limit of $\lim_{n \to \infty} \text{E} \left[ \text{Tr} \left[ W^{2k} \right] \right]$ for some $t \in (0, \infty)$ exists, then the limit only depends on the first and second moment of entries.

3. As the limit exists for Gaussian entries, the limit exists and is universal for any Wigner matrix satisfying Assumption 2.1.

**Theorem 6.2.** Consider the Wigner matrix $W$ satisfying Assumption 2.1. Then for any fixed $t_1, \ldots, t_l \in (0, \infty)^l$ taking $k_i = \left[ t_in^{\frac{2}{3}} \right]$, we have the following results

1. 
   \[ \text{E} \left[ \prod_{i=1}^l \left[ \text{Tr} \left[ W^{k_i} \right] - \text{E} \left[ \text{Tr} \left[ W^{k_i} \right] \right] \right] \right] = O(1). \]  
   \hspace{1cm} (6.2)

2. If the limit in (6.2) exists for some $t_1, \ldots, t_l$, then the limit only depends on the first and second moment of entries.

3. As the limit exists for Gaussian entries, the limit exists and is universal for any Wigner matrix satisfying Assumption 2.1.

Now assuming part 1 and 2 of both Theorems 6.1 and 6.2 we describe rest of the proof techniques.

We now divide the sum in (6.1) into a few further cases. In particular we consider

\[ s_1 = \sum_{\lambda_{i,n} \geq 1 + \frac{2}{n^{\frac{2}{3}}}} \lambda_{i,n}^{2\left[ n^{\frac{2}{3}} \right]} \]
\[ s_2 = \sum_{\lambda_{i,n} \leq -1 - \frac{2}{n^{\frac{2}{3}}}} \lambda_{i,n}^{2\left[ n^{\frac{2}{3}} \right]} \]  
\hspace{1cm} (6.3)

We shall at first show that the terms described in (6.3) go to 0 almost surely. We shall show this only for $s_1$ and the argument for $s_2$ is exactly same.

\[ \mathbb{P} \left[ s_1 > 0 \right] \leq \mathbb{P} \left[ \lambda_{1,n} > \left( 1 + \frac{2}{n^{\frac{2}{3}}} \right) \right] \]
\[ \leq \mathbb{P} \left[ \text{Tr}[W^{2\left[ n^{\frac{2}{3}} \right]}] \geq \left( 1 + \frac{2}{n^{\frac{2}{3}}} \right)^{2\left[ n^{\frac{2}{3}} \right]} \right] \]
\[ \leq \text{E} \left[ \text{Tr}[W]^{2\left[ n^{\frac{2}{3}} \right]} \right] \left( 1 + \frac{2}{n^{\frac{2}{3}}} \right)^{2\left[ n^{\frac{2}{3}} \right]} = O \left( e^{-cn^{\frac{1}{3}}} \right). \]  
\hspace{1cm} (6.4)
So by Borel-Cantelli theorem $s_1 = 0$ almost surely. As a consequence,

$$\text{Tr}[W^{2[n^\frac{2}{3}]}] - \sum_{(1-\frac{2}{3n^3}) \leq A_n \leq (1+\frac{2}{3n^3})} \lambda_{i,n}^2 - \sum_{(-1-\frac{2}{3n^3}) \leq A_n \leq (-1+\frac{2}{3n^3})} \lambda_{i,n}^2 \rightarrow 0.$$  \hspace{1cm} (6.5)

Since all the terms of the l.h.s. of (6.5) are always greater than 0 and by Theorem 6.2 we have

$$\limsup_n \mathbb{E} \left[ \text{Tr} \left[ W^{2[n^\frac{2}{3}]} \right]^2 \right] = O(1)$$

we have the expectation of the l.h.s. of (6.5) also goes to 0 by uniform integrability. Now

$$\sum_{(1-\frac{2}{3n^3}) \leq A_n \leq (1+\frac{2}{3n^3})} \lambda_{i,n}^2 + \sum_{(-1-\frac{2}{3n^3}) \leq A_n \leq (-1+\frac{2}{3n^3})} \lambda_{i,n}^2$$

$$= \left( \sum_{\theta_j \leq n^\frac{1}{3}} e^{i\theta_j} + \sum_{\tau_j \leq n^\frac{1}{3}} e^{i\tau_j} \right) \left( 1 + O \left( n^{-\frac{1}{3}} \right) \right)$$

$$\Rightarrow \mathbb{E} \left[ \sum_{(1-\frac{2}{3n^3}) \leq A_n \leq (1+\frac{2}{3n^3})} \lambda_{i,n}^2 + \sum_{(-1-\frac{2}{3n^3}) \leq A_n \leq (-1+\frac{2}{3n^3})} \lambda_{i,n}^2 \right]$$

$$= \left( 1 + O \left( n^{-\frac{1}{3}} \right) \right) \left( \int_{-\infty}^{\infty} e^{iy} R_{n,1,1}(y) dy + \int_{-\infty}^{\infty} e^{iy} R_{n,1,1,-1}(y) dy \right)$$

Similarly one can prove that

$$\mathbb{E} \left[ \text{Tr} \left[ W^{2[n^\frac{2}{3}]+1} \right] \right]$$

$$= \mathbb{E} \left[ \sum_{(1-\frac{2}{3n^3}) \leq A_n \leq (1+\frac{2}{3n^3})} \lambda_{i,n}^{2[n^\frac{2}{3}]+1} - \sum_{(-1-\frac{2}{3n^3}) \leq A_n \leq (-1+\frac{2}{3n^3})} \lambda_{i,n}^{2[n^\frac{2}{3}]+1} \right]$$

$$= \left( 1 + O \left( n^{-\frac{1}{3}} \right) \right) \left( \int_{-\infty}^{\infty} e^{iy} R_{n,1,1,1}(y) dy - \int_{-\infty}^{\infty} e^{iy} R_{n,1,1,-1}(y) dy \right)$$

Hence

$$\frac{1}{2} \mathbb{E} \left[ \text{Tr} \left[ W^{2[n^\frac{2}{3}]} \right] + \text{Tr} \left[ W^{2[n^\frac{2}{3}]+1} \right] \right] = \left( 1 + O \left( n^{-\frac{1}{3}} \right) \right) \left( \int_{-\infty}^{\infty} e^{iy} R_{n,1,1,1}(y) dy \right).$$

(6.8)

This implies that for the Gaussian case the l.h.s. of the last expression of (6.6) exists and equals to

$$\int_{-\infty}^{\infty} e^{iy} R_{1,1,1}(y) dy.$$  \hspace{1cm} (6.9)
This proves the part 3 of Theorem 6.1. 
Same can be said for part 3 of Theorem 6.2. However here we shall get a polynomial of multi-dimensional Laplace transform.
Now coming back to general case, by these two results we get the Laplace transforms of the general correlation functions also converge to the same limit as the Gaussian. As convergence of the Laplace transform for all \( t > 0 \) implies the weak convergence of a measure, we have the correlation functions for the general case converge weakly to the limit of the correlation function of the Gaussian case. This completes the proof of Theorem 5.1.

7 Combinatorial preliminaries

7.1 Introductory definitions

In this subsection we develop some preliminaries about the method of moments and the word sentence approach for random matrices.

To begin with we start with a matrix \( W \) of dimension \( n \times n \). Its \( k \) th moment is given by

\[
\text{Tr}[W^k] = \sum_{i_0,i_1,...,i_{k-1},i_0} W_{i_0,i_1} \cdots W_{i_{k-1},i_0}.
\]

The word sentence method systematically analyzes the tuples \((i_0,\ldots,i_{k-1},i_0)\) for some suitable \( k \). To do this we need some notations and definitions.

In this part we give a very brief introduction to words, sentences and their equivalence classes essential for the combinatorial analysis of random matrices. The definitions are taken from Anderson et al. [2] and Anderson and Zeitouni [1]. For more general information, see [2, Chapter 1] and [1].

**Definition 7.1 (S words).** Given a set \( S \), an \( S \) letter \( s \) is simply an element of \( S \). An \( S \) word \( w \) is a finite sequence of letters \( s_1 \ldots s_k \), at least one letter long. An \( S \) word \( w \) is closed if its first and last letters are the same. In this paper \( S = \{1,\ldots,n\} \) where \( n \) is the number of nodes in the graph.

Two \( S \) words \( w_1, w_2 \) are called equivalent, denoted \( w_1 \sim w_2 \), if there is a bijection on \( S \) that maps one into the other. For any word \( w = s_1 \ldots s_k \), we use \( l(w) = k \) to denote the length of \( w \), define the weight \( wt(w) \) as the number of distinct elements of the set \( s_1,\ldots,s_k \) and the support of \( w \), denoted by \( \text{supp}(w) \), as the set of letters appearing in \( w \). With any word \( w \) we may associate an undirected graph, with \( wt(w) \) vertices and at most \( l(w) - 1 \) edges, as follows.

**Definition 7.2 (Graph associated with a word).** Given a word \( w = s_1 \ldots s_k \), we let \( G_w = (V_w, E_w) \) be the graph with set of vertices \( V_w = \text{supp}(w) \) and (undirected) edges \( E_w = \{s_i, s_{i+1}\}, i = 1,\ldots,k-1 \).
The graph $G_w$ is connected since the word $w$ defines a path connecting all the vertices of $G_w$, which further starts and terminates at the same vertex if the word is closed. We note that equivalent words generate the same graphs $G_w$ (up to graph isomorphism) and the same passage-counts of the edges. Given an equivalence class $w$, we shall sometimes denote $\#E_w$ and $\#V_w$ to be the common number of edges and vertices for graphs associated with all the words in this equivalence class $w$.

**Definition 7.3 (Weak Wigner words).** Any word $w$ will be called a *weak Wigner word* if the following conditions are satisfied:

1. $w$ is closed.
2. $w$ visits every edge in $G_w$ at least twice.

Suppose now that $w$ is a weak Wigner word. If $wt(w) = (l(w) + 1)/2$, then we drop the modifier “weak” and call $w$ a *Wigner word*. (Every single letter word is automatically a Wigner word.) Except for single letter words, each edge in a Wigner word is traversed exactly twice. If $wt(w) = (l(w) - 1)/2$, then we call $w$ a *critical weak Wigner word*.

It is a very well known result in random matrix theory that there is a bijection from the set of the Wigner words of length $2k + 1$ to the set of Dyck paths of length $2k$. We briefly discuss this map when we construct the well behaved words.

### 7.2 Mapping of words to Dyck paths

The fundamental idea of Soshnikov [36] is to map the closed words such that all edges are traversed even number of times to Dyck paths. It is worth noting that given a Dyck path there will be multiple equivalence classes of words. In particular the map is not one to one. The main goal of this section is to understand this map explicitly and extend the ideas to possible cases when the closed words does not have all edges traversed even number of times. To understand the ideas clearly we need the following terminologies.

**Definition 7.4.** (Well behaved words) These are the words that can be naturally mapped to a Dyck path in the following way. We start with a Wigner word and from Anderson et al. [2] we know that the corresponding graph is a tree. We merge some vertices in the tree to incorporate cycles. As an example one might consider the following word: $w = (1, 2, 3, 5, 3, 2, 4, 2, 1)$ and we merge the vertices 5 and 1 and we call the common letter 1. So the transformed word is $w' = (1, 2, 3, 1, 3, 2, 4, 2, 1)$. These words can be mapped to a Dyck path as follows: one start a random walk from 0 and whenever one traverse an edge odd number of time one goes one step up in the random walk and whenever one traverse an edge even number of times one goes one step down in the random walk. For example the random walk values corresponding to the word $(1, 2, 3, 1, 3, 2, 4, 2, 1)$ look like $(0, 1, 2, 3, 2, 1, 2, 1, 0)$.with the additional constraint that the vertex corresponding to value 3 in the random walk is labeled as 1 which is the same as the vertex corresponding to value 0.
Before moving forward, we state an important convention. In several places we need to count the number of times a Dyck path returns to certain level. However from the construction of the Dyck paths it is clear when the Dyck path falls down from a certain level and comes back to the level from below, these points have possibly different labels. So whenever we talk about the Dyck path coming to a specific level we shall always mean that the Dyck path returns to the level before falling down.

Unfortunately not all words are well behaved. Perhaps the simplest example of such word is \((1, 2, 3, 1, 2, 3, 1)\). Observe that if we want to construct a Dyck path just like the well behaved words we shall encounter problems. To make the idea understandable, we start constructing the Dyck path as follows we start from 0(vertex 1) and strictly increase to 3 (which again comes to vertex 1) but once we reach that point there is
no obvious way to continue the Dyck path. This happens because the next edge is \(\{1, 2\}\) which although appeared in the Dyck path previously but not the edge appeared immediately before which is \(\{1, 3\}\).

Now we peek at the stack interpretation of the Dyck path. From this example we see that we encounter problems continuing the Dyck path when an edge is closed in the word which is not the top most edge in the stack.

This leads to the definition of non well behaved words.

**Definition 7.5.** (Non well behaved words) The non well behaved words are defined as follows. We start a Dyck path following the vertex exploration of the word. However there is at least one instant such the Dyck path can not be continued further if we follow the vertex exploration of the graph. In other words the word exploration is such that there is an instant when the word closes an edge which is not at the top of the stack at that time instant.

To elaborate our idea clearly we go back to the example of the word \((1, 2, 3, 1, 2, 3, 1)\). So we start forming a Dyck path in standard way. In particular the random walk goes strictly in the upward direction until it hits 3(with corresponding vertex 1). Now the Dyck path can’t be continued. From this point we start creating the segments. The rest of the word looks like 1, 2, 3, 1. So at the first step it closes the edge \(\{1, 2\}\) which corresponds to the upward step from 0 to 1 in the Dyck path formed till now. To match this edge in our first segment we draw a downward segment from 1 to 0. (Observe that this creates discontinuity in the path. So in the last step of assembling we assemble
the segments in such a way that it corresponds to a Dyck path.) In the next two steps 
we create two more downward segments from 2 to 1 (corresponding to edge \{3, 2\}) 
and from 3 to 2 (corresponding to edge \{3, 1\}). Now in the final step we assemble the 
segments to get a Dyck path. In particular for the example we are concerned, there is 
only one Dyck path which can be assembled from the segments. That is a random walk 
strictly increase from 0 to 3 and then strictly decreased from 3 to 0. 
In the next part we elaborate this idea for general non well behaved words. We in 
particular, give an algorithm (Algorithm 7.1) to encode a general non well behaved 
word.

To formulate Algorithm 7.1 we need to reinterpret the definition of type of an instant 
and open and closed instants in Soshnikov [36].

**Definition 7.6.** *(Type of an instant)* We start with a word (possibly non well behaved) 
and we assume that it is mapped to several segments of a Dyck path. (How these 
segments are formed will be given in Algorithm 7.1. However this doesn’t make in-
consistencies in the definition of the type of an instant. This is true since every thing is 
done in a dynamical way. In particular when we want to define the type of an instant 
it can be defined by the segments created upto that instant.) We call an instant to be 
of type \( k \geq 1 \) if there is an upward step at that instant in the segment and it is the \( k \) 
th appearance of the vertex corresponding to that instant as the right endpoint of an 
upward step.

**Definition 7.7.** *(Open and closed instants)* An instant of type \( > 1 \) is called open if the 
exploration of the word is such that when the instant is encountered, there is at least 
one unmatched edge which is not the immediate edge, on the vertex corresponding to 
that instant. Otherwise an instant is called closed.

To clarify the idea consider the word \((1, 2, 3, 1, 3, 2, 4, 3, 4, 2, 1)\). Firstly observe 
that this word is well behaved. However the idea can be interpreted analogously for 
non well behaved words. The random walk corresponding to this word looks like this: 
it starts from 0 and move in strictly upward direction until it reaches 3 (corresponding 
vertex is 1). Then it goes down 2 steps to reach 1 (corresponding vertex is 2). Now 
it further goes upward 2 steps to reach 3 once again (corresponding vertices are 4 and 
3). Then it strictly decrease to come to 0. Observe that there are two instants of type 2 
in this example. The first one is the second appearance of 1 and the second one is the 
third appearance of 3. Among them the second appearance of 1 is open as there is an 
unmatched edge (i.e. \{1, 2\}) apart from the immediate edge (i.e. \{1, 3\}). On the other 
hand the third appearance of 3 is closed since all the edges incident to 3 apart from the 
immediate edge (i.e. \{4, 3\}) (i.e. edges \{2, 3\} and \{3, 1\}) are matched.
Now we are ready to state Algorithm 7.1.

**Algorithm 7.1.** We now give an algorithm to encode a possibly non well behaved word. The Algorithm is performed in the following four steps.

**Step 1:** Given a word we start forming the Dyck path corresponding to the exploration of the word. We continue until we encounter the first open instant of type 2. A key observation is that the Dyck path can be continued without introducing discontinuity until the first open instant of type 2. Once this instant is encountered the word can do the following three things:

(a) It can close most recently appeared edge.

(b) It can create a new edge.

(c) It can close an open edge incident to the vertex corresponding to the type 2 instant.

**Step 2:** In the first two cases one continues the Dyck path without introducing discontinuity. In the third case we go to the level where the edge appeared as an open edge and create a downward segment.

**Step 3:** In the next step the word can either close an edge or create a new edge. If it creates a new edge, we start a new segment of a Dyck path from the level of the most recent vertex. On the other hand if it closes an edge then we create a downward segment at the level of the closing edge.

**Step 4:** We continue **Step 3** until we encounter another open instant of type $k > 1$ and
we go back to **Step 1** and **Step 2**. The process is terminated once all the edges have been traversed.

Observe that the walk introduced by Algorithm 7.1 is not necessarily continuous. To inspect the structure minutely we introduce a terminology called the skeleton word.

**Definition 7.8.** We start with a word $w$ and use Algorithm 7.1 to get a walk $P(w)$ which is possibly disconnected. Now we collect all the edges such that there exists at least one type $k > 1$ open instant at which the edge is open. We call this collection pre-skeleton edges. The skeleton word $S(w)$ is the minimal word which contains all the traversals of the edges just mentioned.

As an example we again consider the word $(1, 2, 3, 1, 3, 2, 4, 3, 4, 2, 1)$. As mentioned earlier, the type 2 instant for vertex 1 is open. However the type 2 instant for vertex 3 is closed. Hence the skeleton word is $(1, 2, 3, 1, 3, 2, 1)$. The following figure gives an exposition of $S(w)$. 

![Exposition of S(w)](image-url)
One crucial observation is that given a word \( w \), \( S(w) \) is also a closed word. Although the path corresponding to \( S(w) \) in \( P(w) \) might not be continuous. The main scope of the rest of this section is to understand the structure of \( S(w) \) for a word \( w \). How \( w \) is related to \( S(w) \). In order to count all the words we at first fix the skeleton word and enumerate the number of words with this skeleton and finally sum over all the skeleton words.
7.3 The structure of skeleton words and their relationship with the words

We start with a word \( w \) and wait until the arrival of the first type 2 open instant. There are \( r \) many (say) open edges at this instant. We call these edges

\[
(i_0 = (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)).
\]

We at first prove that for any \( i \) \( \alpha_i = \beta_{i-1} \). This is true due to the following reason. Suppose there is \( i \) such that \( \alpha_i \neq \beta_{i-1} \). Firstly observe that the level of \( \alpha_i \) can not be below the level of \( \beta_{i-1} \), this will contradict the fact that \((\alpha_{i-1}, \beta_{i-1})\) is open. Now suppose \( \alpha_i \) is at a level strictly higher that \( \beta_{i-1} \). This also can’t happen as the path until the first type 2 open vertex is continuous hence in order to arrive at the level of \( \alpha_i \) the path has to cross the intermediate levels and there is at least one open edge incident to each of these levels before reaching the level of \( \alpha_i \). This contradicts to the fact that \( i = (i - 1) + 1 \).

Now observe that the path restricted in between any \( \beta_{i-1} \) and \( \alpha_i \) is a Dyck path. So until the first appearance of the type 2 open instant the path looks like this: We start with from the starting vertex and cover Dyck path of some length then it goes one step up to reach \( \beta_1 \). Then it covers another Dyck path of some length and then moves one step up. This continues up to \( r \) steps. Now we arrive at the first type 2 open instant. Then it covers another Dyck path (might be of length 0) and after that the path possibly jumps to close an edge. Now it closes edge for some amount of time and with in each closing instant it covers a dyck path of some length. Then it goes up again for some time to reach the second vertex of type \( k > 1 \). In particular in between any two consecutive edge in the skeleton word there is a Dyck path of some length. Now the skeleton path looks like this: it starts from the initial vertex and goes strictly upward until it hits the first type 2 instant in the skeleton word. Then it closes some edges then again goes strictly up to reach second type \( k > 1 \) instant in the skeleton word, then it closes some edge and so on.

One fact about this representation is, the words corresponding to the intermediate Dyck paths will have empty skeleton words. This introduces an additional constraint. However for the purpose of calculation we shall ignore this constraint as we are only concerned about the upper bound.

7.4 Calculation of number of Skeleton words

Now we give an upper bound to the number of skeleton words. Observe that firstly there is only one choice until the word hits the first open type 2 instant. Up to this instant the walk increases monotonically. Now once the word hits the first type 2 open instant, we have to fix the location of the point where this type 2 instant appear for the first time. Once that is also fixed, in the next step to continue the word there is at most 3 choices. It can close the immediately traversed edge or it can close one of the remaining maximum two open edges incident to the instant of the first appearance of the type 2 instant. Now the word closes some edges for a few steps. Then the walk goes
up for some steps until we hit the second type $k > 1$ vertex in the skeleton word. We fix the number of upward steps, this fix the location of the second type $k > 1$ instant in the skeleton word. Now we have to fix the instant where this vertex appeared as type $k - 1 \geq 1$ instant for the first time. Now the walk again decreases for some steps. Then it goes up to hit the next type $k > 1$ instant and so on. Let $p_i$ denote the length of the $i$ th upward chunk before appearance of $i$ th type $k > 1$ vertex after the $(i - 1)$ th downward chunk, $q_i$ denotes the length of the $i$ downward chunk. Let $r_i$ denote the positions of the first appearances of the type $k \geq 2$ vertices in the pre-skeleton edges. Firstly observe that if a vertex does not appear as a type $k > 1$ instant, then there is only one way to continue the skeleton word. It goes one step up or one step down depending upon the position of that instant as an upward or downward chunk respectively. We at first fix the values of $\{p_i\}_{i=1}^N, \{q_i\}_{i=1}^{N-1}$ and $r_i$’s and find the number of skeleton words with these parameters.

We at first come to the power of $n$ for this skeleton word. Observe that in the $i$ upward chunk, there are $p_{i+1}$ many vertices. Now every upward chunk ends with a type $k > 1$ vertex and also the starting vertex of each upward chunk apart from the first one is also fixed. So this gives us at most $n^{\sum_{i=1}^N (p_i - 1) + 1}$ choices.

Now we look at a vertex (say $i$) which appear as a type $j \geq 2$ instant. In this case there will be $j$ different upward instants where this vertex appears. Let among them there are $\alpha_{i,j}$ instants with no downward chunk. Then at these instants there is only one choice to continue the skeleton word i.e. to go upward. However these instants increase the number of open directions incident to the vertex. Now each one of such instants will increase the number of open directions by 2. As a consequence when we arrive at a type $l \leq j$ instant corresponding to the vertex $i$, the maximum number of choices to continue the skeleton word is bounded by $2\alpha_{i,j} + 1$. Now notice that the instants with downward chunks do not increase the number of open direction incident to a vertex. Since they introduce and close an open direction at that instant. Now along with the upward instants there might be downward instants where the walk arrives at the vertex $i$. At these instants there might be multiple choice for the next step. However observe that whenever the walk arrive at the vertex $i$ going downward we either do not have any choice or the choices might be from the other type $k \geq 2$ vertices. Now while leaving vertex $i$ there is multiple number of choices. Whenever the walk arrive and leave at vertex $i$ while going downward, it closes two open directions. As a consequence, there will be at most $2\alpha_{i,j}$ of these leavings will each have at most $2\alpha_{i,j}$ choices.

Let there be $N$ type $k > 1$ instants. Among them there are $N_j$ instants each of type $j$. Giving $N = \sum_{j=2}^N (j - 1)N_j$. If we fix the values of $\{p_i\}_{i=1}^N, \{q_i\}_{i=1}^{N-1}$ and $\{r_i\}_{i=1}^{\sum_{j=1}^N N_j}$, the number of skeleton words are bounded by

$$
\prod_{j}^{N_j} \prod_{\alpha_{i,j}=1}^{(2\alpha_{i,j} + 1)^{j-1-\alpha_{i,j}}(2\alpha_{i,j})^{\alpha_{i,j}}} n^{\sum_{i=1}^N p_i - N + 1} \\
\leq \prod_{j}^{(3j)^{j-1}N_j} n^{\sum_{i=1}^N p_i - N + 1}.
$$

(7.2)
choices. Now note that instead of $N$ the index of $q_i$ runs up to $N - 1$. This due to the fact that once all $\{p_i\}_{i=1}^N, \{q_i\}_{i=1}^{N-1}$ and $\{r_i\}_{i=1}^{N-1}$ are fixed then $q_N$ has to be fixed. This is due to the fact that the skeleton word is closed. So in the last downward chunk one goes on closing the edges until it hits the starting vertex.

Observe that in the bound (7.2) we have possibly counted many cases which do not correspond to a valid word. We call such cases infeasible words. Not all infeasible word can be described. However we describe a special class of infeasible words which will be useful at a later scenario. We here start with a feasible word and look at the skeleton walk. This skeleton walk corresponds to a Dyck path with some edges removed from it. Now we construct an infeasible in the following way. We start creating the word from the starting point and we follow the word until we arrive at the first instant where we have multiple choice while closing an edge. Now we take any of the choice and continue closing edges until we reach the next instant where face multiple choices while closing an edge and continue. Here we might close some edge which did not arrive yet. This will create the infeasiblities. Observe that traversals in these fashion may not give the complete word since we can run out of choices to continue before all edges are traversed. In this case we start with the left most edge which is not traversed and continue.

We shall see later that these $p_i, q_i$ and $r_i$’s are typically of order $n^{\frac{1}{2}}$ when the length of the word is of order $n^{\frac{3}{2}}$, all the type $k > 1$ open instants are actually type 2 and $N$ is finite with high probability. In this case (7.2) reduces to

$$3^N n^{\sum_{i=1}^N p_r - N + 1}$$

(7.3)

8 Proof of Theorem 6.1

We at first prove a lemma which is needed for proving the universality.

**Lemma 8.1.** Consider the uniform probability measure on $m$ many Dyck paths of length $2k_1, \ldots, 2k_m$ for any given $m$ such that $\sum_{i=1}^m 2k_i = 2k$ a fixed number. We call this probability measure $\mathbb{P}_{D,k,m}$. We now fix $\tau$ many levels $q_1, \ldots, q_\tau$. Let $N_{k_i,m}(q_i)$ denote the number of returns to level $q_i$ in the $i$ th Dyck path before falling down. Then the random variables $N_{k_1,m}(q_1), \ldots, N_{k_\tau,m}(q_\tau)$ can be stochastically dominated by $\tau$ many i.i.d. random variables with common distribution $X$ such that $\mathbb{P}[X \geq l] \leq \frac{l^2}{2^\tau}$.

**Proof.** The proof of this lemma is some what tricky. We divide it in the following steps: **Step 1:** We at first consider the levels of the starting points of the Dyck paths. With slight abuse of notation we call any such level as level 0. Let the corresponding length of the Dyck path be $2k_i$ for some $i$ and we look at the the returns to level 0 before falling down in the $i$ th Dyck path. We prove here that $\mathbb{P}_{D,k,m}[N_{k_i,m}(0) \geq l] \leq \frac{l^2}{2^\tau}$. To prove this result we compare this probability with a similar probability of the simple symmetric random walk of length $2k_i$. At this point we at first fix all the other Dyck
paths and consider the uniform probability measure of the simple symmetric random walk of length $2k_i$. We call this measure $\mathbb{P}_{R,k_i}$. Let $N_{k_i}(t,0)$ be the collection of random walk paths which starts from 0 and returns to 0 at time point $2k_i$ and in between time points 0 to $2k_i$ it returns to 0, exactly $t$ many times. Then

$$\mathbb{P}_{R,k_i} [\text{The random walk returns to 0, } t + 1 \text{ times } \cap x(2k_i) = 0] = \frac{\#N_{k_i}(t,0)}{4^{k_i}} < 1. \quad (8.1)$$

Now we inspect a typical path in $N_{k_i}(t,0)$. Observe that the random walk starts from 0 and in the very next step it can go either above or below 0. Now once the path goes either above 0 or below 0 it stays there until it returns to 0 for the first time. Now in the next step it also goes either 0 or below 0 and stays there until it returns to 0 for the second time. In particular after any return to 0 there are 2 possible ways to choose in the very next step determining whether the random walk goes above or below 0 in the next part of time before returning to 0. If we consider a Dyck path, it always stays above 0. Hence the number of Dyck paths which start from 0 and returns to 0 exactly $t + 1$ number of times is given by $\frac{1}{2} \#N_{k_i}(t,0)$. As a consequence if we consider the uniform measure over all Dyck paths of length exactly $2k_i$ (call it $\mathbb{P}_{k_i}$), then under this measure the probability that a Dyck path comes to 0 for exactly $t$ times is given by:

$$\mathbb{P}_{k_i} [\text{The Dyck path comes to 0 exactly } t + 1 \text{ times}] \leq \frac{\#N_{k_i}(t,0)}{2^i C_{2k_i}} \leq \frac{k^2}{2^i} \leq \frac{k^2}{2^i}. \quad (8.2)$$

Since the number of returns to 0 in the $i$th Dyck path doesn’t depend on the other Dyck paths, the number of paths such that the $i$th Dyck path comes to 0 exactly $t + 1$ many times is bounded by $\frac{k^2}{2^i} \prod_i C_{2k_i}$. Hence the probability

$$\mathbb{P}_{D,k,m} [N_{k_i,m}(0) = t] = \frac{\# \text{of paths having the required property}}{\sum_{k_1,\ldots,k_m = k} \prod_i C_{2k_i}} \leq \frac{k^2}{2^i} \prod_i C_{2k_i} \leq \frac{k^2}{2^i} \prod_i C_{2k_i} \leq \frac{k^2}{2^i}. \quad (8.3)$$

**Step 2:** In this step we prove the remaining of Lemma 8.1. The proof is done by conditioning and induction on the level. In step 1, we have proved an upper bound on the tail of the number of returns to 0 for the $i$th Dyck path. Here we show that the conditional distribution of the number of returns to level 1 given the number of returns to level 0 follows the same upper bound. Since given the lengths of the Dyck paths the number of times one path returns to a level is independent of the number of times another different path returns to another level, if $N_{k_1,m}(q_1)$ and $N_{k_2,m}(q_2)$ belong to different Dyck paths, then the proof can be done by simply looking at the distribution of $N_{k_1,m}(q_1) | N_{k_2,m}(q_2)$. On the other hand, if $N_{k_1,m}(q_1)$ and $N_{k_2,m}(q_2)$ denote the return to two different levels of the same Dyck path, the proof can be completed by the repeated use of the argument we give next.
Now we go by conditioning. In particular we look at \( N_{k_i,m}(1) \mid N_{k_i,m}(0) \). To this end we at first fix the value of \( N_{k_i,m}(0) \) to be \( t_1 \) and assume the random walk returns to 0 at instants \( 2k_{i,1}, 2k_{i,1} + 2k_{i,2}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \). Since any of such paths return to 0 exactly at instants \( 2k_{i,1}, 2k_{i,1} + 2k_{i,2}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \), the total number of paths of this type is given by \( C_{2k_{i,1}} \cdots C_{2k_{i,t_1}} \). Here we have used the fact that the random walk returns to 0 exactly at instants \( 2k_{i,1}, 2k_{i,1} + 2k_{i,2}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \) which implies the random walk goes one step down at instants \( 2k_{i,1}, 2k_{i,1} + 2k_{i,2}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \) and goes one step up at instants \( 1, 2k_{i,1} + 1, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} + 1 \). This explains the quantity \( C_{2k_{i,1}} \cdots C_{2k_{i,t_1}} \). By \( N_{k_i,m}(1) \) we shall denote the number of returns to 1 in the left most chunk. The arguments for the other chunks is exactly the same. Now fix a path satisfying this property and call it \( \omega \). Let the path returns to level 1, \( t_2 \) times in the left most chunk. Observe that the conditional probability of this path given \( N_{k_i,m}(0) = t_1 \) and \( k_{i,1}, \ldots, k_{i,t_1} \) is:

\[
\mathbb{P}_{D,k,m}(\omega \mid N_{k_i,m}(0) = t_1 \cap \text{the returns are at } 2k_{i,1}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \cap \\
\text{The length of the Dyck paths are } k_1, \ldots, k_m] = \frac{1}{C_{2k_{i,1}} \cdots C_{2k_{i,t_1}}} \\
= \frac{1}{C_{2k_{i,1}} C_{2k_{i,2}} \cdots C_{2k_{i,t_1}}} \\
\Rightarrow \mathbb{P}_{D,k,m}(N_{k_i,m}(1) = t_2 \mid N_{k_i,m}(0) = t_1 \cap \text{the returns are at } 2k_{i,1}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \cap \\
\text{The length of the Dyck paths are } k_1, \ldots, k_m] \leq \frac{k_{i,1}^2 C_{2k_{i,1}} \cdots C_{2k_{i,t_1}} C_{2k_{i,1}} \cdots C_{2k_{i,t_1}}}{2^{t_2} C_{2k_{i,1}} \cdots C_{2k_{i,t_1}} C_{2k_{i,1}} \cdots C_{2k_{i,t_1}}} \\
\leq \frac{k_{i,1}^2}{2^{t_2}} \leq \frac{k_i^2}{2^{t_2}} \quad (8.4)
\]

Since the r.h.s. does not depend on \( t_1 \) and the values \( 2k_{i,1}, \ldots, 2k_{i,1} + \ldots + 2k_{i,t_1} \) and \( k_1, \ldots, k_m \), we have the required result.

\( \square \)

We now fix a number of Dyck paths along with their starting points and calculate the number of words corresponding to the Dyck paths. This calculation is quite similar to Pêché [32].

**Proposition 8.1.** Given any number \( m \), we fix \( m \) many Dyck paths \( P_1, \ldots, P_m \) of length \( 2k_1, \ldots, 2k_m \) respectively. We also fix the initial points of the Dyck paths. Then the number of words obeying the Dyck path exploration is asymptotically of the order \( n^{2m-1}k_i \).

**Proof.** We at first start with the path \( P_1 \) and fix its vertices then move to \( P_2 \) and so on. Firstly observe that there can be common vertices among the Dyck paths \( P_1, \ldots, P_m \). We also have to keep track of the instants of type \( k > 1 \). In our calculation we shall
keep track of the vertices appeared in all the previous Dyck paths. In particular, while calculating the words corresponding to \( P_l \) we shall keep track of the vertices appeared in \( P_1, \ldots, P_{l-1} \) and define the type \( k > 1 \) vertices whenever a vertex appeared as the right end point of an upward edge for the \( k \) th time in the joint path tuple \( P_1, \ldots, P_m \).

Let \( (\Gamma_j)_{j=1}^\infty \) be the number of vertices of type \( j \) in the joint path tuple \( P_1, \ldots, P_m \). We at first start with the instants of type 2. There are a total of \( \Gamma_2 \) of them. Firstly we fix their locations. As there are a total of \( \sum_{i=1}^m k_i \) many up ward instants, the type 2 instants will be among them. Hence the locations are given by \( \Gamma_2 \) positions \( j_1 < j_2 < \ldots < j_{\Gamma_2} \), where \( j_i \)'s are within \( \{1, 2, \ldots, \sum_{i=1}^m k_i \} \). Given the locations \( j_1 < \ldots < j_{\Gamma_2} \) we are to choose their values. For the location \( j_i \) there are at most \( (j_1 - 1) \) choices for the value of the first type 2 instant. Similarly for the locations \( j_i \) there are at most \( (j_i - i) \) choices for the value of the \( i \) th type 2 instant. Hence the total number of choices for the type 2 instants are given by

\[
\sum_{1\leq j_1 < \ldots < j_{\Gamma_2} \leq \sum_{i=1}^m k_i} \prod_{i=1}^{\Gamma_2} (j_i - i)
\]

\[
\leq \sum_{1\leq j_1 < \ldots < j_{\Gamma_2} \leq \sum_{i=1}^m k_i} \prod_{i=1}^{\Gamma_2} j_i = \frac{1}{\Gamma_2!} \sum_{1\leq j_1 \neq j_2 \neq \ldots \neq j_{\Gamma_2} \leq \sum_{i=1}^m k_i} \prod_{i=1}^{\Gamma_2} j_i
\]

\[
\leq \frac{1}{\Gamma_2!} \sum_{i \leq j_1 < j_2 \leq \sum_{i=1}^m k_i} \prod_{i=1}^{\Gamma_2} j_i = \frac{1}{\Gamma_2!} \left( \frac{\sum_{i=1}^m k_i}{2} \right)^{\Gamma_2} \leq \frac{1}{\Gamma_2!} \left( \frac{\sum_{i=1}^m k_i}{2} \right)^{\Gamma_2}.
\]

(8.5)

A similar calculation proves that the number of choices for the type \( j \) instant is bounded by

\[
\frac{1}{\Gamma_j!^{(j-1)\Gamma_j}} \left( \sum_{i=1}^m k_i \right)^{\Gamma_j}.
\]

As a consequence, we get the total number of words corresponding to paths \( P_1, \ldots, P_m \) are bounded by

\[
\sum_{i=1}^l \frac{1}{\Gamma_2!} \left( \frac{\sum_{i=1}^m k_i}{2} \right)^{\Gamma_2} \sum_{\Gamma_3, \Gamma_4, \ldots, \Gamma_{j-1}} \prod_{j \geq 2} \frac{1}{(j-1)!^{\Gamma_j}} \left( \sum_{i=1}^m k_i \right)^{\Gamma_j}.
\]

(8.6)

(8.6)

Here we have used the fact that the initial choices of the paths are fixed. This corresponds to the power \( \sum_{i=1}^m k_i \) instead of \( \sum_{i=1}^m k_i + m \). Now we analyse the term \( \prod_{l=1}^{\sum_{j=2}^l (j-1)\Gamma_j} (n-l+1) \) in some details. Observe that

\[
\prod_{l=1}^{\sum_{j=2}^l (j-1)\Gamma_j} (n-l+1)
\]

\[
(n-l+1)
\]

\[
= n^{\sum_{j=2}^l (j-1)\Gamma_j} \prod_{l=1}^{\sum_{j=2}^l (j-1)\Gamma_j} \left( \frac{n-l+1}{n} \right)
\]

(8.7)

\[
= n^{\sum_{j=2}^l (j-1)\Gamma_j} \prod_{l=1}^{\sum_{j=2}^l (j-1)\Gamma_j} \left( 1 - \frac{l-1}{n} \right).
\]

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So the whole thing boils down to analyse $\sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} \log \left(1 - \frac{l-1}{n}\right)$. Now using the fact that $\frac{x}{(1-x)} \leq \log(1 + x) \leq x$

$$\sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} - \frac{l-1}{n} \leq -\frac{l-1}{n} \leq \sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} \log \left(1 - \frac{l-1}{n}\right) \leq \sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} - \frac{l-1}{n}$$

$$\Rightarrow \sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} \log \left(1 - \frac{l-1}{n}\right) = -\left(1 + O\left(\frac{\Sigma_i k_l}{n}\right)\right) \sum_{l=1}^{\Sigma_i k_l - \Sigma_{j \geq 2} (j-1) \Gamma_j} \frac{l-1}{n}$$

$$\leq -\frac{1}{2n} \left(1 + O\left(\frac{\Sigma_i k_l}{n}\right)\right) \left(\sum_i k_l - \sum_{j \geq 2} (j-1) \Gamma_j\right)^2$$

$$\leq -\frac{1}{2n} \left(1 + O\left(\frac{\Sigma_i k_l}{n}\right)\right) \left(\sum_i k_l\right)^2 - 2 \left(\sum_i k_l\right) \left(\sum_{j \geq 2} (j-1) \Gamma_j\right)$$

$$= -\frac{(\sum_i k_l)^2}{2n} + O(1) + \left(1 + O\left(\frac{\Sigma_i k_l}{n}\right)\right) \frac{(\sum_i k_l) \left(\sum_{j \geq 2} (j-1) \Gamma_j\right)}{n}$$

$$\leq -\frac{(\sum_i k_l)^2}{2n} + O(1) + 2 \frac{(\sum_i k_l) \left(\sum_{j \geq 2} (j-1) \Gamma_j\right)}{n}$$

(8.8)

Here in the last line of (8.8) we have used the fact that $\sum_{i=1}^{m} k_i = O(n^{1/2})$. Now exponentiating (8.8) and putting it in (8.6) we have the count of the words are bounded by

$$O(1)n^{\sum_i k_l - \sum_{j \geq 2} (j-1) \Gamma_j} \exp\left(-\frac{(\sum_i k_l)^2}{2n}\right) \exp\left(2 \frac{(\sum_i k_l) \left(\sum_{j \geq 2} (j-1) \Gamma_j\right)}{n}\right)$$

$$\sum_{\Gamma_2}^{1} \left(\frac{(\sum_{i=1}^{m} k_i)^2}{2}\right)^{\Gamma_2} \prod_{j \geq 3} \left(\frac{1}{(j-1)! \Gamma_j!}\right)^{k_j}$$

$$\approx n^{\sum_i k_l} \exp\left(-\frac{(\sum_i k_l)^2}{2n}\right) \prod_{j \geq 3} \left(\frac{1}{(j-1)! \Gamma_j!}\right)^{\sum_i k_j}$$

$$\sum_{\Gamma_2}^{1} \left(\frac{(\sum_i k_l)^2}{2n}\right)^{\Gamma_2} \prod_{j \geq 3} \left(\frac{1}{(j-1)! \Gamma_j!}\right)^{\sum_i k_j}$$

$$\approx n^{\sum_i k_l} \exp\left(-\frac{(\sum_i k_l)^2}{2n}\right) \exp\left(\frac{(\sum_i k_l)^2}{2n}\right) \exp\left(\sum_{j \geq 3} \frac{(\sum_i k_j)^j}{(j-1)! n^{j-1}}\right)$$

(8.9)
It can be showed with some elementary calculation that the term
\[
\exp\left(-\frac{\sum_i k_i^2}{2n}\right) \exp\left(\frac{\sum_i k_i^2}{2n} \frac{2 \sum_j k_j}{n}\right) \exp\left(\sum_{j \geq 3} \frac{\sum_i k_i^j}{(j-1)!n^{j-1}} \right) \asymp 1
\]
whenever \(\sum_i k_i \approx n^{3/2}\). As a consequence, we get the required bound on the word count. This completes the proof. \(\square\)

We now state a proposition which tells that among all the words, the words with every edge traversed exactly twice give a bounded contribution to the trace.

**Proposition 8.2.** Suppose we consider all the words of length \(2k + 1\) where every edge traversed exactly twice. We call this class of words \(W_{2k}\). Then the following is true whenever \(k = \lfloor tn^{3/2}\rfloor\) for some \(t \in (0, \infty)\):

\[
\frac{1}{n^k} \sum_{w \in W_{2k}} E[X_w] = O(1).
\]

**Proof.** At first observe that as every edge traversed exactly twice \(E[X_w] = \left(\frac{1}{3}\right)^k = \frac{1}{2^{2k}}\). Hence (8.11) is equivalent to proving \(#W_{2k} = O\left(2^{2k}\right)\). This is what we prove here.

As we mentioned earlier, the key approach of this paper is to fix a skeleton word and then do the calculation of the number of words having that specific skeleton word and finally take the sum over all the skeleton words.

So at the beginning we fix a skeleton word. Let there be \(N\) type \(k \geq 2\) vertices. We at first fix the values of \(\{p_i\}_i^{N}, \{q_i\}_i^{N-1}, \{r_i\}_i^{N}\) where these parameters were defined in subsection 7.4. As we have all the edges traversed exactly twice, we have \(\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i\).

Now once the skeleton word is fixed, we spend \(\sum_{i=1}^{N} (p_i + q_i) = 2 \sum_{i=1}^{N} p_i = 2m\) many edges. In particular there will be \(2m + 1\) many Dyck paths adjacent to each of these edges of the skeleton word. We call them \(P_1, \ldots, P_{2m+1}\). Let their lengths be \(2k_1, \ldots, 2k_{2m+1}\) respectively. Then we have that

\[
\sum_{i=1}^{2m+1} 2k_i + 2m = 2k.
\]

Since the vertices of the skeleton path are fixed, the end points of these Dyck paths are fixed. By Proposition 8.1, we have the number of words corresponding to these Dyck paths are of the order \(n^{\sum_{i=1}^{2m+1} k_i}\). On the other hand there are at most \(n^{\sum_{i=1}^{N} p_i - N + 1} = n^{m-N+1}\) many choices for the vertices in the skeleton word. So in particular, fixing the skeleton word and the Dyck paths adjacent to each edge in the skeleton word we get the number of words are bounded by \(n^{m-N+1+\sum_{i=1}^{2m+1} k_i} = n^{k-N+1}\).

We now calculate the total number of choices of \((P_1, \ldots, P_{2m+1})\). By (3.3) we have this is exactly equal to \(\frac{2m+1}{2k+1} \binom{2k+1}{k+m+1}\).
Now in the final step we take the sum over the choices of the skeleton words. At first we assume that all \( \{p_i\}_{i=1}^{N}, \{q_i\}_{i=1}^{N-1} \) and \( \{r_j\}_{j=1}^{N_j} \) are positive. We also assume all type \( k \) instants to be actually of type 2. Hence \( N = N_2 = \sum_j N_j \). By subsection 7.4 this is bounded by
\[
3^N n^{\sum_i p_i - N + 1}.
\] (8.13)
Hence in this case we get the all possible words are bounded by
\[
\sum_N \sum_{p_1, \ldots, p_N} \sum_{q_1, \ldots, q_{N-1}} \sum_{r_1, \ldots, r_N} 3^N 2m + 1 \left( \frac{2k + 1}{2k + 1} \right)^{k-N+1} (k + m + 1)^n
\] (8.14)
Firstly observe that
\[
\left( \frac{2k + 1}{k + m + 1} \right) = \frac{(2k)!}{(k - m)! (k + m)!} k + m + 1 \leq 2 \binom{2k}{k+m}
\] (8.15)
Hence we replace the \( \binom{2k+1}{k+m+1} \) in (8.14) by \( 2 \binom{2k}{k+m} \). We now apply Stirling approximation to get that whenever \( m \leq k - 1 \)
\[
\left( \frac{2k}{k+m} \right) \times (1 + o(1)) 2^{2k+1} \frac{\sqrt{k}}{\sqrt{\pi (k^2 - m^2)}} \exp \left( - \sum_{l \geq 2 | \text{ even}} \frac{2m^l}{k^{l-1} l (l-1)} \right).
\] (8.16)
We shall apply (8.16) in (8.14). Further we at first fix \( N \) and sum over all the other indexes. Now our sum in (8.14) is of the order of
\[
2^{2k+1} n^k \sum_N 3^N \sum_{p_1, \ldots, p_N} \sum_{q_1, \ldots, q_{N-1}} \sum_{r_1, \ldots, r_N} \frac{2m + 1}{2k+1} \frac{\sqrt{k}}{\sqrt{\pi (k^2 - m^2)}} \exp \left( - \sum_{l \geq 2 | \text{ even}} \frac{2m^l}{k^{l-1} l (l-1)} \right) \left( \frac{1}{\sqrt{k}} \right)^{3N-3}
\] (8.17)
The key idea is to represent the sum inside the summand \( \sum_{p_1, \ldots, p_N} \sum_{q_1, \ldots, q_{N-1}} \sum_{r_1, \ldots, r_N} \) as a Riemann sum of an integral. To do this we take the mesh size of \( \frac{1}{\sqrt{k}} \) and write \( x_i := \frac{p_i}{\sqrt{k}}, X_j := \sum_{i=1}^j x_i, y_i := \frac{q_i}{\sqrt{k}}, Y_j := \sum_{i=1}^j y_i \) and \( z_i := \frac{r_i}{\sqrt{k}} \). Since the final function only depends on \( m = \sum_{i=1}^N p_i \), we do a trick to change the co-ordinates from \( x_1, \ldots, x_N \) to \( X_1, \ldots, X_N \) and \( y_1, \ldots, y_{N-1} \) to \( Y_1, \ldots, Y_{N-1} \). This transformation is one to one and we have the additional constraint \( X_1 \leq X_2 \leq \ldots \leq X_N \) and similarly \( Y_1 \leq Y_2 \leq \ldots \leq Y_{N-1} \). We also define \( P_1, \ldots, P_N \) and \( Q_1, \ldots, Q_N \) in the analogous way. Now we fix
Now observe that the sum under the summand \( m = \sum_{i=1}^{N} p_i \) and take the sum

\[
\sum_{p_1 \leq p_2 \leq \cdots \leq p_N = m} f(m)
\]

\[
\leq \sum_{p_1 \leq p_2 \leq \cdots \leq p_N = m} f(m)
\]

\[
= \frac{\sqrt{k}^{2N-2}}{\sqrt{k}^{2N-2}} \sum_{p_1 \leq p_2 \leq \cdots \leq p_N = m} f(m)
\]

\[
\leq f(m) \left( \frac{\sqrt{k}}{N-1} \right)^{2N-2} \int_{X_1 \leq Y_1 \leq \cdots \leq X_N \leq \frac{m}{\sqrt{k}}} \int_{Y_1 \leq Y_2 \leq \cdots \leq Y_N = \frac{m}{\sqrt{k}}} dX_i \prod_{i=1}^{N-1} dY_i
\]

\[
= f(m) \left( \frac{\sqrt{k}}{N-1} \right)^{2N-2} \frac{1}{(N-1)!^2} \left( \frac{m}{\sqrt{k}} \right)^{2N-2}
\]

Now we consider the sum over the indexes \( r_i \)'s. We know that for each \( i, 1 \leq r_i \leq m \). So

\[
\sum_{r_1, \ldots, r_N} 1 \leq m^N.
\]

Putting these in (8.17) we reduce our job to bound

\[
2^{2k+4} n^k \sum_{N} \sum_{N \leq m \leq k-1} \left( \frac{\sqrt{k}}{N-1} \right)^{3N-2} \frac{1}{(N-1)!^2} \left( \frac{m}{\sqrt{k}} \right)^{3N-2}
\]

\[
\frac{2m+1}{2k+1} \frac{\sqrt{k}}{\pi(k^2 - m^2)} \exp \left( - \sum_{l \leq 2 \text{ even}} \frac{2m!}{k^{l-1} l(l-1)} \right) \left( \frac{1}{\sqrt{k}} \right)^{3N-3}
\]

\[
\leq 2^{2k+4} n^k \sum_{N} \sum_{N \leq m \leq k-1} \frac{1}{(N-1)!^2} \sqrt{k} \left( \frac{m}{\sqrt{k}} \right)^{3N-2} \left( \frac{2m+1}{2k+1} \frac{\sqrt{k}}{\pi(k^2 - m^2)} \exp \left( - \frac{m^2}{k} \right) \right)
\]

Now observe that the sum under the summand \( \sum_{N \leq m \leq k-1} \) is a Riemann sum of the function

\[
\int_{\frac{N}{\sqrt{k}}}^{\frac{N+1}{\sqrt{k}}} X_N^{3N-2} \left( 2X_N + \frac{1}{\sqrt{k}} \right) \frac{\sqrt{k}}{2k + 1} \frac{\sqrt{k}}{\sqrt{k - X_N^2}} \exp \left( -X_N^2 \right) dX_N
\]

Note that the function in (8.21) is not monotonically decreasing. However the function inside the integrand of (8.21) can be uniformly dominated by a function of the form:

\[
\zeta(X_N) := c_1 \exp \left( -X_N^2 \right) \mathbb{1}_{0 \leq X_N \leq c} + c_2 X_N^{3N-1} \exp \left( -X_N^2 \right) \mathbb{1}_{c \leq X_N}
\]

where \( c \) is a deterministic constant independent of \( N \) and \( k \) and the function \( \zeta(X_N) \) is
monotonically decreasing. As a consequence,

$$\sum_{N \leq m \leq k-1} \sqrt{k} \left( \frac{m}{\sqrt{k}} \right)^{3N-2} \frac{2m + 1}{2k + 1} \sqrt{k} \frac{1}{\sqrt{\pi (k^2 - m^2)}} \exp \left( - \frac{m^2}{k} \right)$$

$$\leq \int_{\frac{k^{-1}}{\sqrt{k}}}^{\frac{k-1}{\sqrt{k}}} \zeta(X_N) dX_N$$

$$\leq c_1 c'_1 + c_2 \int_0^{\infty} X_N^{\xi + 3N-1} \exp \left( -X_N^2 \right) dX_N$$

$$= c_1 c'_1 + \frac{c_2}{2} \int_0^{\infty} \frac{\xi + 3N}{2} \exp(-\varepsilon) d\varepsilon$$

$$= c_1 c'_1 + \frac{c_2}{2} \Gamma \left( \frac{\xi + 3N}{2} \right) \left( \frac{3N}{2} \right)$$

It is easy to see that

$$\sum_{N=1}^{\infty} 3^N \left( c_1 c'_1 + \frac{c_2}{2} \Gamma \left( \frac{\xi + 3N}{2} \right) \right) \frac{1}{(N-1)!^2} \approx 1. \quad (8.24)$$

Now we consider the boundary case \( m = k \). When \( m = k \) we have

$$\lim_{k \to \infty} \sum_{1 \leq N \leq k} 3^N \left( \frac{2k + 1}{2k + 1} \right)^{3N-1} \frac{1}{2^{2k}}$$

$$= \lim_{k \to \infty} \sum_{1 \leq N \leq k} 3^N \left( \frac{1}{(N-1)!^2} \right) \left( \frac{1}{2^{2k}} \right)$$

$$= \lim_{k \to \infty} \sum_{1 \leq N \leq k} N \exp \left( N \log 3 - k \log 4 + \frac{3N - 1}{2} \log k - 2 \log(N)N - N \right)$$

$$\leq \lim_{k \to \infty} \sum_{1 \leq N \leq k} k \exp \left( N \log 3 - k \log 4 + \frac{3N - 1}{2} \log k - 2 \log(N)N - N \right)$$

Maximizing the term in the exponential we get that \( \log N = \frac{3}{2} \log k + c \). Hence among the terms inside the exponential \( \exp(-k \log 4) \) dominates. Now if we take the sum it is \( \text{Poly}(k) \frac{1}{2^k} \to 0 \).

Hence (8.14) is asymptotically of the same order as \( n^k 2^{2k} \).

Now we consider the other case when there is at least one instant of type strictly greater than 2 or there is at least one instant where \( q_i = 0 \). We prove that in these cases we get a negligible contribution.

To start with we need a few notations. Let \( N_j \) be the total number of vertices of type \( j \) and for the \( i \) th of type \( j \) vertex let \( a_{i,j} \) be the number of instants with no downward chunk. It is easy to observe that \( N = \sum_j (j-1)N_j \). As before we fix the values of \( \{p_i\}_{i=1}^{N_j} \), \( \{q_i\}_{i=1}^{N_j-1} \) and \( \{r_i\}_{i=1}^{\sum_j N_j} \). Here we observe that some \( q_i \)'s will be 0. From subsection
we have given \( \{p_i\}_{i=1}^{N}, \{q_i\}_{i=1}^{N-1} \) and \( \{r_i\}_{i=1}^{\sum_{i=1}^{N_j}} \) and \( \{\alpha_{i,j}\} \)'s, the number of skeleton words of this kind is bounded by:

\[
n^{\sum_{i=1}^{N_j} p_i - \sum_{i=1}^{N_j-1} q_i} \prod_j (3j)^{(j-1)N_j}. \tag{8.26}\]

Now arguing as before we get the total number of all possible words are bounded by

\[
\sum_{N_j} \sum_{\alpha_{i,j}} \sum_{p_1, \ldots, p_N} \sum_{q_1, \ldots, q_{N-1}} \sum_{r_i} \sum_{\sum_{i=1}^{N_j} p_i = m} 2m + 1 \left( \frac{2k+1}{2k+1} \right)^{N_k+1} \prod_j (3j)^{(j-1)N_j}.
\]

The rest of the argument is dedicated to bound (8.27). To begin with the number of \( q_i \)'s which are non zero are given by

\[
N - M := N - 1 - \sum_{j=1}^{N_j} \sum_{i=1}^{N_j} \alpha_{i,j} + 1 = N - \sum_{j=1}^{N_j} \sum_{i=1}^{N_j} \alpha_{i,j}. \tag{8.28}\]

Now we need to sum

\[
\sum_{p_1, \ldots, p_N} \sum_{\sum_{i=1}^{N_j} p_i = m} 1. \tag{8.29}\]

This argument is somewhat similar to the calculation we did for type \( j > 2 \) vertices. However we need to be more cautious here due to the additional factors \( \prod_j (3j)^{(j-1)N_j} \). In particular for type \( j \) instants, there will be \( N_j \) many instants repeated \( j-1 \) times each. Now our task is to bound how many ways these positions can be arranged. Observe that we are dealing with \( N_j \) positions repeated \( (j-1) \) times where \( j \) varies. In particular the number of ways to arrange these positions are

\[
\frac{N!}{\prod_j ((j-1)!)^{N_j}}. \tag{8.30}\]

However note that here we made some over counting. This is due to the fact when we take the choices of \( n \) we already account for the ordering of the vertices. In particular \( jN_j \) vertices of type \( j \) will be counted \( N_j! \) many times. So we get the exact count is

\[
\frac{N!}{\prod_j ((j-1)!)^{N_j} \prod_j N_j!}. \tag{8.31}\]
Now the total number of choices of (8.29) is bounded by

\[
\sum_{p_1, \ldots, p_{N-1} \mid \sum_i p_i = m} 1 \leq \frac{1}{(N-1)!^m} \prod_j (j-1)!^{N_j} \prod_j m^{\Sigma_j N_j} \prod_j (j-1)!^{N_j-1}^{m^{\Sigma_j N_j}} \tag{8.32}
\]

On the other hand by exactly same argument of the proof of part 1 of this proposition, we have that

\[
\sum_{q_j \neq 0} 1 \leq \frac{1}{(N-M)!^m} \prod_j (j-1)!^{N_j} \prod_j \Sigma_j^{N_j} m^\alpha j. \tag{8.33}
\]

Let \(N_j = \gamma_j N\). Hence \(\sum_j (j-1)\gamma_j = 1\).
Putting (8.32) and (8.33) in (8.27) we have:

\[
\sum_{N_j} \sum_{\alpha_{i,j}} \sum_{p_{i,j},...,p_{N_j} \neq 0} \frac{2^N 2m + 1}{2k + 1} \left( \frac{2k + 1}{k + m + 1} \right) n^{k-N_j} \prod_j (3j)^{(j-1)N_j} \\
\leq \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions of 0 q values}} \sum_{m} n^{k-N_j} 2^N \frac{2m + 1}{2k + 1} \left( \frac{2k + 1}{k + m + 1} \right) \\
\sum_{N_j} \prod_j \left( N - \sum_{i} \sum_{x_i = 1}^{N_j} \alpha_{i,i} \right) \prod_j (3j)^{(j-1)N_j} \\
\approx n^k 2^{k+\frac{1}{2}} \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions}} \left( \frac{1}{\sqrt{k}} \right) 3^N \frac{2^N}{\sqrt{\pi(k^2 - m^2)}} \exp \left( -\frac{m^2}{k} \right) \frac{2m + 1}{2k + 1} \sum_{N_j} \prod_j \left( N - \sum_{i} \sum_{x_i = 1}^{N_j} \alpha_{i,i} \right) \prod_j (3j)^{(j-1)N_j} \\
\approx n^k 2^{k+\frac{1}{2}} \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions}} \left( \frac{1}{\sqrt{k}} \right) 3^N \frac{2^N}{\sqrt{\pi(k^2 - m^2)}} \exp \left( -\frac{m^2}{k} \right) \frac{2m + 1}{2k + 1} \sum_{N_j} \prod_j \left( N - \sum_{i} \sum_{x_i = 1}^{N_j} \alpha_{i,i} + \xi \right) \prod_j (3j)^{(j-1)N_j} \\
\leq n^k 2^{k+\frac{1}{2}} \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions}} \left( \frac{1}{\sqrt{k}} \right) 3^N \frac{2^N}{\sqrt{\pi(k^2 - m^2)}} \exp \left( -\frac{m^2}{k} \right) \frac{2m + 1}{2k + 1} \sum_{N_j} \prod_j \exp \left( (j-1)N_j \log(3j) \right) \\
\leq n^k 2^{k+\frac{1}{2}} \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions}} \left( \frac{1}{\sqrt{k}} \right) 3^N \frac{2^N}{\sqrt{\pi(k^2 - m^2)}} \exp \left( -\frac{m^2}{k} \right) \frac{2m + 1}{2k + 1} \sum_{N_j} \prod_j \exp \left( (j-1)N_j \log(3j) \right) \\
\approx n^k 2^{k+\frac{1}{2}} \sum_{N_j} \sum_{\alpha_{i,j}} \sum_{\text{positions}} \left( \frac{1}{\sqrt{k}} \right) 3^N \frac{2^N}{\sqrt{\pi(k^2 - m^2)}} \exp \left( -\frac{m^2}{k} \right) \frac{2m + 1}{2k + 1} \sum_{N_j} \prod_j \exp \left( (j-1)N_j \log(3j) \right) \\
(8.34)
\]

This analysis is somewhat cumbersome and tedious. We analyze them term by term.

We at first simply bound \( \frac{\Gamma(N - \frac{M}{2})}{N - M} \leq N^N \approx \exp \left( \frac{M}{2} \log N \right) \).
Next we look at the term

\[
\exp\left\{- \sum_j (N\gamma_j \log(N\gamma_j))\right\}
\]

\[
= \exp\left\{-N \sum_j \gamma_j (\log(N) + \log(\gamma_j))\right\}
\]

\[
= \exp\left\{-N \log N \sum_j \gamma_j - N \sum_j \gamma_j \log(\gamma_j)\right\}
\]

Our first job is to bound the term

\[
\exp\left\{-N \sum_j \gamma_j \log(\gamma_j)\right\}
\]

So we come to the following optimization problem

\[
\max_{\gamma_j} - \sum_j \gamma_j \log(\gamma_j)
\]

subj. to

\[
\sum_j (j-1)\gamma_j = 1
\]

We at first transform the variables to \(z_j = (j-1)\gamma_j\). This reduces the optimization problem to

\[
\max_{z_j} - \sum_j \frac{z_j}{j-1} \log\left(\frac{z_j}{j-1}\right)
\]

subj. to

\[
\sum_j z_j = 1
\]

Now we apply the method of Lagrange multiplier to optimize

\[
f(z_1, \ldots, z_N, \lambda) = - \sum_j \frac{z_j}{j-1} \log\left(\frac{z_j}{j-1}\right) - \lambda \left(\sum_j z_j - 1\right).
\]

Observe that the function \((z_1, \ldots, z_N) \mapsto - \sum_j \frac{z_j}{j-1} \log\left(\frac{z_j}{j-1}\right)\) is a concave function and the set \(\sum_j z_j = 1\) is a convex set. Hence the method of Lagrange multiplier gives the unique maximizer.

Taking the partial derivative of \(f(z_1, \ldots, z_N, \lambda)\) with respect to \(z_j\) and setting it to 0 we have

\[
\frac{\partial f(z_1, \ldots, z_N, \lambda)}{\partial z_j} = - \frac{1}{j-1} \log\left(\frac{z_j}{j-1}\right) - \frac{1}{j-1} - \lambda = 0
\]

\[
\Rightarrow -1 - \lambda(j-1) = \log\left(\frac{z_j}{j-1}\right)
\]

\[
\Rightarrow \exp\left(-1 - \lambda(j-1)\right) = \frac{z_j}{j-1}
\]

\[
\Rightarrow z_j = (j-1) \exp\left(-1 - \lambda(j-1)\right).
\]
Now we apply the constraint to have

$$\sum_{j=2}^{N} (j - 1)^2 \exp (-1 - \lambda(j - 1)) = 1.$$  \hspace{1cm} (8.40)

It is easy to observe that the solution to the above equation is unique and the value $\lambda$ remains uniformly bounded over $N$. As a consequence the value

$$- \sum_{j} \frac{z_j}{(j - 1)} \log \left( \frac{z_j}{(j - 1)} \right) = \sum_{j} \exp (-1 - \lambda(j - 1))(1 + \lambda(j - 1))$$

also remains uniformly bounded. Hence (8.35) is bounded by

$$C'' \exp \left( -N \log N \sum_{j} \gamma_j \right).$$ \hspace{1cm} (8.42)

Now if we have a look at (8.34), then inside the exponential only linear functions of $\gamma_j$’s remain. To make things clear we have a look at the term inside the summand of (8.34).

$$\left( \frac{1}{\sqrt{k}} \right)^{\sum_{j=1}^{N/2} (j - 1)} NC'' \exp \left( \log N \log \left( N \sum_{j} \gamma_j \right) \right) = \sum_{j} \frac{z_j}{(j - 1)} \log \left( \frac{z_j}{(j - 1)} \right)

= \sum_{j} \exp (-1 - \lambda(j - 1))(1 + \lambda(j - 1))$$

implying

$$\exp \left( N \sum_{j} \gamma_j (j \log j - (j - 1) \log(j - 1)) \right) \leq C''''.$$

Now $(j - 1) \log j - (j - 1) \log(j - 1)$ is of the order $\log(j + 1) + 1$. So

$$\sum_{j} \gamma_j ((j - 1) \log j - (j - 1) \log(j - 1)) \leq c$$ \hspace{1cm} (8.44)

implying

$$\exp \left( N \sum_{j} \gamma_j (j \log j - (j - 1) \log(j - 1)) \right) \leq C''''. \hspace{1cm} (8.43)$$
Same can be said about \( \exp \left( N \log 3 \sum j \gamma_j (j - 1) \right) \). Also \( \frac{M}{2} \left( \log N - \log k \right) < 0 \).

As a consequence, the only interesting term is

\[
\exp \left( - \frac{N \log N}{2} \sum j \gamma_j - \frac{\sum j (j - 2) N_j}{2} \log k \right)
= \exp \left( - \frac{N \log N}{2} \sum j \gamma_j - \frac{N \log k}{2} \sum j (j - 2) \gamma_j \right)
\]

(8.45)

Now we come to another optimization problem:

\[
\max -N \log N \sum j \gamma_j - N \log k \sum j \gamma_j (j - 2)
\text{ subj. to }
\sum j (j - 1) \gamma_j = 1
\]

(8.46)

We again transform it to \( z_j = (j - 1) \gamma_j \) to reduce the problem to

\[
\max -N \log N \sum \frac{z_j}{(j - 1)} - N \sum \frac{z_j}{(j - 1)} (j - 2)
\text{ subj. to }
\sum j z_j = 1.
\]

(8.47)

This is equivalent to minimize

\[
\frac{N \log N}{j - 1} + \frac{N \log k (j - 2)}{(j - 1)}
= \frac{N \log N}{j - 1} + N \log k - \frac{N \log k}{j - 1}
= \frac{N}{j - 1} (\log N - \log k) + N \log k.
\]

(8.48)

Since \( j - 1 \geq 1 \) and \( \log k \geq \log N \), we have the minimum value of (8.48) is \( N \log N \).

Now we take the sum over all possible \( N_j \)'s and \( \alpha_{i,j} \)'s. Firstly \( N_j \) are chosen in two steps. At first we fix \( x \) and choose \( x \) many positions from \( \{1, 2 \ldots, N\} \). This can be done in \( \binom{N}{x} \) ways. This chosen set is the collection of \( j \)'s such that \( N_j \neq 0 \). Now we partition \( N \) balls into \( \sum j (j - 1) \) bins such that each bin has nonzero number of balls. This can be done in at most \( \left( \binom{N - 1}{\sum_j (j - 1)} \right) \leq 2^{N-1} \) ways. As a consequence the total number of choices of \( N_j \)'s is bounded by \( 4^N \). Finally we choose \( \alpha_{i,j} \) many points from \( (j - 1) \). These are the positions where the corresponding \( q \) value is 0. So we have

\[
\sum \sum \alpha_{i,j} \text{ positions of 0 q values} \leq \sum (j - 1) \alpha_{i,j} \leq 2^{\sum (j - 1) N_j} = 2^N
\]

(8.49)

Hence the sum in (8.34) is finite. Now we need to prove it actually goes to 0. This is true due to the fact that we simply bound the term \( \exp \left( \frac{M}{2} (\log N - \log k) \right) \) by 1. However
for small $N$ this term goes to zero. To make this clear we have a look at (8.34). We 
have this equation reduces to

$$
\sum_{N} \exp \left( \frac{M}{2} \left( \log N - \log k \right) - \frac{N \log N}{2} + CN \right)
$$

(8.50)

The tail of the above equation also decays like $\exp \left( -cN \log N \right)$ for some finite constant $c$. So if we take $N_0$ large enough such that $\exp \left( -cN \log N \right) \leq \frac{1}{\sqrt{k}}$, then

$$
\sum_{N \leq N_0} \exp \left( \frac{M}{2} \left( \log N - \log k \right) - \frac{N \log N}{2} \right) \leq \left( \frac{1}{\sqrt{k}} \right)^{M(1-\varepsilon)}
$$

for any $\varepsilon > 0$. On the other hand

$$
\sum_{N \leq N_0} \exp \left( \frac{M}{2} \left( \log N - \log k \right) - \frac{N \log N}{2} \right) \leq \left( \frac{1}{\sqrt{k}} \right).
$$

So (8.34) goes to 0 whenever $M \geq 1$. On the other hand if there exists some $j > 2$ such that $\gamma_j > 0$, the same argument can be used to prove that (8.34) goes to 0. This completes the proof.

Now we come to an important proposition for proving the universality.

**Proposition 8.3.** Suppose the entries $x_{i,j}$ satisfy condition (iii) of Assumption 2.1. Let $W_{\geq 3,k}$ denote the class of words of length $k + 1$ where every edge is traversed at least twice and some edge is traversed at least thrice. Then following is true whenever $k = [tn^2]$ for some $t \in (0, \infty)$:

$$
\frac{1}{n^2} \sum_{w \in W_{\geq 3,k}} E[X_w] \rightarrow 0.
$$

(8.51)

The proof Proposition 8.3 is somewhat long and tedious.

We at first fix some notations. Firstly observe that in the skeleton word, after the end of each downward chunk there is an upward chunk of length at least 1. So there will be $N$ such endpoints. We enumerate them from left to right. Along with these points we also add the points which correspond to the first arrival to the level of $N$ endpoints considered here. Let the total number of points in this collection be $\tilde{N}$. Observe that $\tilde{N} \leq 2N$. Now we introduce a partition of $\{1, 2, \ldots, \tilde{N}\}$ in the following way. We place $i$ and $j$ in the same block if the $i$th and $j$th endpoint is in the same level and if we look the word from left to right it does not fall below the level of the $i$th endpoint in between the $i$th and $j$th endpoint. Although, the total number of partitions of $\{1, \ldots, \tilde{N}\}$ is quite large, we shall see the number of feasible partition is only of exponential order. One can see that this partition is a non-crossing partition of $\{1, \ldots, \tilde{N}\}$. It is a well known fact that there is a bijection from the set of all non-crossing partition of $\{1, \ldots, \tilde{N}\}$ to
the Dyck paths of length $2\tilde{N}$. As a consequence the total number of such partitions are bounded by $16^N$. The following argument explicitly constructs this map. Observe that the interpretation of this map will be useful to us to bound certain quantities.

**Lemma 8.2.** For any $\tilde{N}$ let $NC(\tilde{N})$ denotes the set of non-crossing partitions of $\{1, \ldots, \tilde{N}\}$. Then there is a bijection from $NC(\tilde{N})$ to the all set of all Dyck paths of length $2\tilde{N}$.

**Proof.** We start with a non crossing partition of $\tilde{N}$ and look at the block containing 1. The point 1 corresponds to the starting point of the Dyck path and all the other entries correspond to the returns to 0. Observe that here we are not counting the endpoint of the Dyck path which will be specified from the other parameters. From all these points we place an upward edge from left to right. Now we look at the blocks which are just one step above to the current block. Here the first entries of each block denote the entry to the level 1 and all the other entries denote the returns to the level 1 before falling down. We place these points at level 1 and put an upward edge from left to right from these points. We continue in this fashion until we run out of choices. Observe that by performing this procedure, we have specified all the upward edges in the Dyck path. So we fill up the remaining downward edges to to get a Dyck path. Since from each point we construct exactly one upward edge, the length of the Dyck path is $2\tilde{N}$. This way given a non-crossing partition we created a Dyck path. On the other hand given a Dyck path the non-crossing partition is obvious. □

Here we give an example of a non-crossing partition and the corresponding Dyck path. Let the non-crossing partition be $\{1\}, \{2, 6, 7\}, \{3, 5\}, \{4\}, \{8\}$. The corresponding Dyck path is given in the following figure:
Observe that the Dyck path formed in the process will have exactly $\hat{N}$ points marked on it. From these points we choose the points which corresponds to the start of an upward chunk in the filled up path. A naive upper bound to the choice is $2^\hat{N} \leq 4^N$.

In our refined count of the skeleton word we shall fix one such partition and shall call it $\mathcal{P}$.

Now given this partition $\mathcal{P}$, and the parameters $p'_i$'s(to be specified), we at first create a path of length $2m$. This path is the path obtained by filling up the gaps in the skeleton path.

So in the next part we give two algorithms: at first we give an algorithm to construct the filled up path from the parameters $p'_i$'s and then we give an algorithm to form the skeleton path from this filled up path after fixing the additional parameters.

Algorithm 8.1. Given a non-crossing partition $\mathcal{P}$, we fix the Dyck path ($\mathcal{D}$) according to Lemma 8.2. This algorithm consists of the following steps:

**Step 1** We look at the left most upward section of the path $\mathcal{D}$. Let us assume that in this section there are $\kappa$ many points which indicates the start of an upward chunk. In order to form the filled up word we at first move $p'_1, p'_2, \ldots, p'_\kappa$ step upward. This makes the length of the left most upward segment of the filled up path $\left(p'_1 + \ldots + p'_\kappa\right)$.

**Step 2** Now we go $p'_{\kappa+1}$ steps downward to reach a level. This level corresponds to the first level where the Dyck path $\mathcal{D}$ returns more than once after the first upward chunk.

**Step 3** Next the Dyck path $\mathcal{D}$ has to move in the upward direction. Now we consider two cases as follows:

1. The next return to this level is the final return to the level before falling down.
2. The walk returns to this level more than once after the current return.

Now we use the parameter $p'_{\kappa+2}$. In both the cases we move $p'_{\kappa+2}$ step up to reach a level. However for case 2 we need to ensure the return to this level. Hence we need to fix a parameter depending on the future choices of the parameters.

**Step 4:** Now we arrive at a new level. There can be two further cases from here.

i. This level is the starting point of new upward chunk.

ii. The next edge from this level is a downward edge.

For case i we go back to **Step 1** with parameter $p'_{\kappa+3}$.

For case ii, if the next level in the path $\mathcal{D}$ in the downward direction is not a level previously explored, then we go $p'_{\kappa+3}$ step down and move to **Step 3**.

Further if the next level in the path $\mathcal{D}$ in the downward direction is a level previously explored, we go down by a pre-specified number which denotes the difference between the current level and the next level. This explain the case 2. in **Step 3**. We now again go to **Step 3**.

Before moving forward we make an important observation. Firstly observe that from Algorithm 8.1 we get a word with several points on it specified. We call these
points marked points. We shall see later that these points denote the starting points of upward segments in the skeleton word.

In the next part we shall look at type $j \geq 2$ instants which correspond to repetitions of edges and use an endpoint of at least one previous appearance of the edge. These type $j \geq 2$ instants require the skeleton word to return to a certain level at least twice. Here we construct the type $j \geq 2$ instant by going one step up from the endpoint of the corresponding downward chunk. This also requires the vertex corresponding to this type $j \geq 2$ instant to appear immediately after at least one arrival of the walk to the level of the endpoint of downward chunk before falling down.

Now given the filled up word, we want to construct the skeleton word. Our strategy is to fix sufficient parameters (including the filled up words) which specify the skeleton word uniquely. First of all instead of taking any arbitrary permutation of the type $j \geq 2$ instants we at first take a permutation of a subclass of type $j \geq 2$ instants. These instants will be called non-ignored instants. All the other instants will be called ignored instants. Our strategy is to at first take a permutation of the non-ignored instants and given this permutation we give an upper bound on the number of all permutations of the type $j \geq 2$ instants satisfying certain properties. Which instants we take for the initial permutation depends on the other parameters in somewhat complicated way but when we count the permutations we do not take into consideration that the additional constraints are specified. Further when we want to use some properties of the permutation depending on the constraints we shall assume that the properties hold for all the permutation. Mathematically this is formalized in the following way: Suppose we have a set $C$ and we consider elements $x \in C$ having some property $\varrho$. We consider the indicator function $I_\varrho(x)$ to denote whether $x$ has property $\varrho$ or not. Let $f$ be a function of the properties. Then
\[ \sum_{x \in C} f(\varrho) I_\varrho(x) \leq f(\varrho) \#C. \]

Now we consider a fixed level $q$. By the filled up path we know the positions of returns to this level before falling down. Let there be $\Delta_q$ such indexes $i_1 < \ldots < i_{\Delta_q}$. We shall make the ignored instants such that apart from the first appearance of an edge all the instants appearing immediately after a return to the level $q$ are ignored instants.

Now for a given level $q$, suppose the filled up word comes to the level $\Delta_q$ times and among these $\Delta_q$ returns say the $\eta_{i,j,q}$ be the number of times the $i$th vertex of type $j$ appears immediately after a return to $q$. Then the number of choices for such cases are given by
\[ \prod_q \frac{\Delta_q^{\sum_i \eta_{i,j,q}}}{\prod_j \prod_i \eta_{i,j,q}!} \]  

(8.52)

We now fix certain choices which we call the movement choices. These choices determined dynamically while forming the skeleton word. A choice of such kind determines what the skeleton word should do in the next step. Firstly at every instant it determines whether the word should create a new edge or close an existing edge immediately after an instant. If at the next instant the word closes an edge it determines exactly which edge should it close when there are multiple choices for the closing edge.
One should also notice that if the word comes to a level for which the corresponding vertex appear only in the current level, then there is a unique movement choice for that instant.

Fixing all these parameters we shall provide an algorithm to construct the skeleton word.

**Algorithm 8.2.** The algorithm consists of the following steps.

**Step 1** At first we follow the left most upward segment of the filled up path. This is the first upward segment of the skeleton path. Now at this point we use the value of $r_i$’s and the permutation of the non ignored vertices to get the location of the initial point of the first type $j \geq 2$ instant.

**Step 2** At this point there is at most three choices to continue the word:

(a) It can create an upward edge.

(b) It can close the immediately traversed edge.

(c) It can close one of the other two edges incident to the vertex of consideration.

In case (a) we move to **Step 1** and continue until we encounter the second type $j \geq 2$ instant.

**Step 3** In cases (b) and (c) we start closing edges obeying the movement choices. We continue this until we encounter an instant where the movement choice directs the walk to go up. Now there can be two cases at this point. Firstly we can encounter an instant of moving upward while closing an edge left to right. Here we take follow the upward segment adjacent to the current level. Secondly we can encounter an instant of moving upward while closing an edge right to left. In this case we follow the upward segment starting from the current level which is the next available one from left to right. After completion of **Step 3** we move to **Step 1** and move upwards until we encounter the next marked instant.

Before moving further, we now count the number of free parameters in the filled up word. We actually also take into consideration the fact that whenever we have a type $j \geq 2$ instant corresponding to repetitions of edges that uses an endpoint of at least one previous appearance of the edge, the corresponding value of $p_i$ is just 1. By specifying the filled up path we have specified the starting points of each upward segment. So without any further constraints there are $2N - 1$ free parameters. However multiple returns to a level before falling down decreases the number of free parameters. First of all we have seen that suppose the filled up word comes to a level $q$, $\Delta_q$ times, then there is a decrease of $\Delta_q$ free parameters. Now among these free parameters let there $\delta$ many positions corresponding to the ignored vertices. Then the corresponding values of $p_i$’s are just 1. If the non-ignored vertex is the first appearance of that vertex then the number of decrease in free parameters is same as the number of ignored vertices. Otherwise the decrease in free parameters is even smaller. In other words the total decrease in free parameters are greater than or equal to $\Delta_q + \delta \geq 2\delta$. 

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Now along with the repetitions of discussed type, there can be another type of repetitions. Here the repeated edge does not use the level of an endpoint of at least one previous appearance of the edge. We at first choose the permutation of the non-ignored instants in such a way that it covers first traversal of all the edges in the skeleton word. Now we consider the each repeated edge one by one. For every edge traversed more than thrice we shall consider only one of its endpoint as an ignored instant at that step and we shall look for a position to place the endpoint. Special care needs to be taken for edges traversed odd number of times where we also need to specify which endpoint of the edge we are considering as an ignored instant. We shall discuss this later in details.

We at first fix a permutation of the non-ignored instants. Now we look at the repeated edges one by one and start filling up the ignored instants. We assume that for \(i\) th type \(j\) vertex there are \(\eta_{i',j,i,j}\) many instants of the \(i'\) th type \(j'\) vertex appearing as ignored instants. This gives us a choice of

\[
\prod_{j} \prod_{i=1}^{N_j} \frac{j!}{\eta_{i,j,i,j}!}.
\]

(8.53)

Here \(\zeta_{i,j} = \sum_{j} \sum_{i} \eta_{i,j,i,j}\).

Now we come to the reduction of free parameters due to this kind of repetitions. For any edge \(e = \{e_1, e_2\}\) repeated \(r_e\) times, we fix \(2\left[\frac{r_e}{2}\right]\) free parameters. This is due to the fact that the corresponding lengths of upward chunks are 1 and the downward chunks are 0. In particular here also for each such ignored instants we fix two parameters for each such ignored instants.

Next we need an argument before we state a bound for the number of skeleton words keeping the repetitions of edges in mind. This argument is for the edges closing the repetitions. Here we fix an edge \(e\) with endpoints \(\{e_1, e_2\}\). Suppose it is repeated \(r_e\) times. Then the edge \(e\) is traversed \(\left[\frac{r_e}{2}\right]\) times in the down ward direction. Among these downward traversals let \(r_{e,e_1}\) denote the number of times the edge is closed in the order \((e_1, e_2)\) and \(r_{e,e_2}\) be the number of times the edge is closed in the order \((e_2, e_1)\). Clearly \(r_{e,e_1} + r_{e,e_2} = \left[\frac{r_e}{2}\right]\). Now we recall the construction of the skeleton word. While closing edges we considered all possible choices to close an edge. This gives rise to the \((3j)^j!\) factor. However all these choices might not lead to a feasible word. For example when we consider the edge \(e\) which repeated \(r_e\) times, these downward traversals are ordered. In particular the first downward traversal closes the first upward traversal, the second downward closes the second upward traversal and so on. However this ordering is not preserved while we calculate all the possible choices for closing the edges. This reduces the word count. In particular, every feasible word can be mapped to \(\prod_e r_{e,e_1}! r_{e,e_2}!\) many infeasible words in the following way. We start the skeleton word from the starting point and we continue until we hit the first instant where an edge is traversed multiple times is closed. Say this edge is \(e\) and we close the edge in the order \((e_1, e_2)\). Now we choose any one from the \(r_{e,e_1}\) closings of the edge and close that edge. Next we continue until hit the next such instant and so on. Observe that traversals in these fashion may not give the complete word since we can run out of choices to continue before all
edges are traversed. In this case we start with the left most edge which is not traversed and continue. Although these words are not feasible but we have counted them. Now \( r_{e_1}!r_{e_2}! \geq \left( \frac{1}{2} \right) \left( \frac{e}{2} \right)! \). We get the actual feasible word count is less than \( 2^N \prod_e \left( \frac{1}{e} \right)! \) times the count we introduced in (7.2).

Now we look at the edges traversed odd number of times in detail. We start with a basic but fundamental observation. To begin with we fix the filled up path. Suppose an edge is traversed multiple number of times in the skeleton word, then it can be traversed odd number of time only under the following condition. If we look at the filled up word, then in the skeleton word all but the final exploration of the edge is covered. If this is not the case, then the given filled up path has to be modified. Since both the filled up word and the skeleton word are closed and the skeleton word is obtained by removing some edges from the exploration of the filled up word the removed word is a collection of closed words. Further as we just discussed, no removed edge can be traversed more than once, the removed edges are disjoint collection of cycles. Hence number of edges incident to each vertex is even. Now we look at a vertex. Say this vertex is of type \( j \geq 2 \) for some \( j \) and there are \( 2\delta \) many removed edges incident to that vertex. By looking at the argument where bounded the number of choices for closing an edge for each vertex, we see that in this case the number of choices for closing the edges for this vertex is bounded by \((3j)^{j-\delta}\) instead of \((3j)^j\).

Now we prove a few important facts about edges traversed odd number of times. We consider a level \( q \) and assume that the filled up path comes \( \Delta_q \) number of times before falling down. In this case we claim that there can be at most two instants among the \( \Delta_q \) returns to the level \( q \), such that immediate upward edge after a return to level \( q \) is traversed but it is never closed. Further this position is uniquely determined by the movement choices. This is due to the following observation. In the skeleton word suppose we look at the returns to the level \( q \) before falling down. Then among these returns there can be only one return after closing an edge from right to left. All the other returns will be by closing an edge left to right. Now in order to create an upward edge from the level \( q \) the walk has to return to the level \( q \). Further if the return is from left to right the only choice for creating the upward edge is the upward edge is placed right next to the closed edge. As a consequence the only possibility that an immediate upward edge after a return to level \( q \) is traversed but it is never closed is the upward edge right before the walk returns to the level \( q \) from right to left. However in this case observe that the edge corresponding to the last return to \( q \) might never been closed. One might look at the following figure for further insights.

Finally we spend some time on specifying the parameters in counting the skeleton words. By specifying the parameters which determine the skeleton word, we make a function from the space of the parameters to the space of the skeleton words. Now given the filled up word, the permutations of type \( j \) vertices, the movement choice and other parameters the skeleton word is uniquely determined. However in general there is no reason for this function to be one to one. This give us some liberty to describe some relationship between the parameters. In particular, we shall impose some constraints on
the number of non-ignored vertices. First of all if we look at a given level \( l \) and consider the multiple \((\geq 2)\) returns of the filled up word to that level, then the vertices just one step above the level which are connected to the level have to be ignored vertices. Now we come to the other type of edge repetitions. Here we have some options. For example we consider the \( i \) th vertex of type \( j \) and look at the number of edges this vertex appeared as an endpoint of an oddly traversed edge. We have argued that this has to be even. Now among these edges some will be repetitions of the first kind and the others will be repetitions of the second kind. For the first kind we have no choices as we have just discussed. However, for the repetition of the second kind we have some options since the both endpoints of the such edges are symmetric. Here we mark the ignored and non-ignored vertices in such a way that the difference between the number of times the that vertex appears as a non-ignored vertex and the number of times the vertex appears as an ignored vertex is at most 1. Given the skeleton word, this is done in the following simple way. For every vertex of type \( j \geq 2 \) we mark it as ignored or non-ignored. First of all some non-ignored vertices are fixed due to the repetitions of the first kind. As we know that if we consider the edges traversed odd number of times then to each vertex there are even number of such edges incident to it. As a consequence if we look at the graph corresponding to these edges, they will be a disjoint union of several disjoint Euler circuits. Now we fix one such Euler circuit and consider the traversal of the Euler circuit. We now divide the edges in two parts. Firstly the edges for which the edge coming immediately after is an edge corresponding to repetition of first type. The other edges are the edges for which the edge coming immediate after is an edge corresponding to repetition of second type. For the edges in the second part we denote the instant corresponding to the second endpoint of the next edge as an ignored instant. However for edges in first part we do not have any choices for the second endpoint of the next edge.

Now given a permutation of such kind we enumerate the number of times a vertex appears at an ignored instant. At first we consider the edges in the first part. Suppose for the \( i \) th type \( j \) vertex there are \( 2\delta'_{i,j} \) many edges incident to it with the following property. When we consider the Euler circuit, both the edges entering and going out of the vertex belong to the repetition of the first kind. We call this set to be the edges of type \( I \). Similarly define \( 2\delta''_{i,j} \) to be the number of edges where both the edges entering and going out of the vertex belong to the repetition of the second kind. We call this set to be the edges of type \( II \). Now there can be two other types of edges. Firstly there can be edges incident to a vertex such that when the edge enters a vertex according to the Euler circuit it is a repetition of the first kind and the edge going out of the vertex is a repetition of the second kind. We call this set to be the edges of type \( III \). Similarly the exact opposite of it can also happen which will be called type \( IV \). We denote the number of such edges by \( 2\kappa''_{i,j} \) and \( 2\kappa'_{i,j} \) respectively. We now give an upper bound to the number of times the given vertex appears at an ignored instant. Firstly observe that among the edges of type \( I \) we don’t have a choice for which vertex is ignored and which is not ignored. So for edges of this type we put an upper bound of \( 2\delta'_{i,j} \). On the other hand for edges of type \( II \) we put an upper bound of \( \delta'_{i,j} \) instead of \( 2\delta'_{i,j} \). Now for edges
of type $III$ we upper bound it by $k'_{i,j}$. Since here we have the edge coming out of the vertex is a repetition of second kind. So we don’t consider that instant as new ignored instant. On the other hand the edge entering to the vertex can correspond to a new ignored instant for which we do not have any choice. Finally for type $IV$ we bound it by $2k'_{i,j}$ as the edge entering the vertex is a repetition of second kind so we count that instant as a new ignored instant and the edge coming out of the vertex is a repetition of first kind so it is possible that that instant is an ignored instant too. So our upper bound is $2\delta'_{i,j} + 2k'_{i,j} + \delta''_{i,j} + k''_{i,j}$.

We now count the number of permutations of the non-ignored vertices. There are $N - L$ many of them and assume that $N'_j$ of them are of type $j$. So that $\sum_j (j - 1)N'_j = N - L$. Further we assume that for the $i$ th vertex of type $j$, $L_{i,j}$ many of them are non-ignored. So permuting all these gives us a choice of

$$\frac{(N - L)!}{\prod_j N'_j! \prod_j \prod_{i=1}^{N_j} (j - L_{i,j})!}.$$ (8.54)

Compiling these factors and the arguments given before we have the following upper bound to the number of skeleton words.

**Lemma 8.3.** **Fixing the parameters we just discussed, the number of skeleton words is bounded by**

$$n^{\sum_{l=1}^{p_{i-1}} N_{i-1} + N_i} C^N \prod_{\epsilon \leq \frac{1}{2}} \frac{1}{\left\lfloor \frac{1}{2} \right\rfloor !} \frac{(N - L)!}{\prod_j N'_j! \prod_j \prod_{i=1}^{N_j} (j - L_{i,j})!} \prod_q \frac{\Delta_i \eta_i^{\sum_j \epsilon_{i,j,q}}}{\prod_j \prod_i \eta_i \eta_j} \prod_i \prod_j \prod_{r} \eta'_{i,j,p,r}.$$ (2m)^{2N-2L-2+\sum_i N_i} \frac{(2N - 2L - 2)!}{(2N - 2L - 2)!}. (8.55)

**Proof.** The proof of this lemma is essentially compiling the arguments given after the statement of Proposition 8.3 until now. So we omit the details.

Before going into the formal proof of Proposition 8.3, we spend some time about the possible choices of the remaining parameters. Observe that the partition $P(N)$ can be chosen in at most $C^N$ ways. Now given $L$ the parameters $N'_j$s and $N_j$’s also can be chosen in at most $C^N$ ways for some finite $C$. Finally we come to the number of choices for the $\eta_{i,j,q}$’s and $\eta'_{i,j,p,r}$’s. Here situations are a little tricky since some $\eta_{i,j,q}$ and $\eta'_{i,j,p,r}$ can be 0. However here we use the property that first traversal of every edge in the skeleton word is covered by the permutation of the non-ignored instants.

We at first do the calculation for $\eta_{i,j,q}$. Firstly observe that for by the permutations of the non-ignored instants which edges come at level $q$ are determined. Now the remaining numbers can be obtained in at most $2\sum_i \eta_{i,j,q}$ ways. Now observe that $\sum_i \sum_j \eta_{i,j,q} \leq 2N$. So the total number of choices for this kind is at most $4^N$. Now for the other case observe that when we construct the permutation of the non-ignored instants, the $i$ th instant of type $j$ can be endpoint of at most $2j$ many edges. Hence the $\zeta_{i,j}$
is partitioned at most $2j$ many groups. So the choice is bounded by $\binom{c_{i,j}+2j-1}{2j-1} \leq 2^{c_{i,j}+2j}$. Hence taking the product over all possible $i$ and $j$'s we find that this number is bounded by $8^N$.

We now have the enough machineries to prove Proposition 8.3.

**Proof of Proposition 8.3.** The proof is divided into two parts. In the first part we fix the choices in the associated Dyck paths and calculate the quantities for the skeleton word. In the second part we consider the repetitions of edges in the Dyck paths.

Now we go into **Step 1** of the proof.

**Step 1** Here we start with the skeleton word. There are $\left(\sum_{i=1}^{N} p_i + q_i\right) = m + m'$ many edges in the skeleton word. Associated to the endpoints of these edges there will be $m + m' + 1$ many Dyck paths. Let their lengths be $2k_1, \ldots, 2k_{m+m'+1}$ respectively. So that we have $\sum_{i=1}^{m+m'+1} 2k_i + m + m' = k$. We shall show in **step 2** the factor coming from the Dyck paths in (8.51) is $o\left(n^{2m+m'+1}k\right)$ whenever there is at least one edge in the Dyck path traversed at least four times.

Now we calculate the total number of choices of the Dyck paths. By (3.3) this is exactly equal to $\frac{m+m'+1}{k+1} \binom{k+1}{\frac{k+m+m'+2}{2}}$. Since by our assumption the skeleton word and the Dyck paths have disjoint edges for any word $w$, $E[X_w] = E[X_{S(w)}]E[X_{D(w)}]$. Here $S(w)$ is the skeleton word and $X_{D(w)}$ denotes the corresponding random variables in the Dyck paths.

The main goal of this step is to show that

$$\sum_{S(w)} E[X_{S(w)}] \frac{m + m' + 1}{k + 1} \binom{k + 1}{\frac{k+m+m'+2}{2}} = 2^{k-m-m'} o\left(n^{\frac{m+m'}{2}}\right)$$

whenever the words $S(w)$ varies over all skeleton words with $m + m'$ number of edges, every edge traversed at least twice and some edge is traversed at least thrice. Suppose an edge $e$ is traversed $r_e$ number of times in the skeleton word. Then by (iv) of Assumption 2.1 we have

$$E[X_{S(w)}] \leq \frac{1}{2^{m+m'}} C^N \prod_e \left\lceil \frac{r_e}{2} \right\rceil !. \quad (8.57)$$

Now we fix all the required parameters of the skeleton word. Our main calculation tool is (8.55). By (8.55) we have

$$\sum_{S(w)} E[X_{S(w)}] \frac{m + m' + 1}{k + 1} \binom{k + 1}{\frac{k+m+m'+2}{2}} \leq \frac{1}{2^{m+m'}} \sum_{\text{parameters}} n^{m-N+1} C^N \prod_e \left\lceil \frac{r_e}{2} \right\rceil ! \times \prod_j \frac{1}{j!} \prod_{i=1}^{N_j} \frac{1}{(j - L_{i,j})!} \prod_{j} \frac{\Delta_q^{\sum_{i\neq j} \eta_{i,j,q}} \prod_{i=1}^{N_j} \eta_{i,j,q} \prod_{i=1}^{N_j} \prod_{i'} \eta_{i,j',i'}}{\prod_{i} \prod_{j} \prod_{i'} \eta_{i,j',i'}} \times \left(\frac{(2m)^{2N-2L-2+\sum_{i\neq j} N_j}}{(2N - 2L - 2)!} \right) \prod_e \left\lceil \frac{r_e}{2} \right\rceil !. \quad (8.58)$$

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We analyze (8.58) step by step. We at first cancel the $\prod_{e \left[ \frac{L}{2} \right]} !$ from numerator and denominator. Next we cancel the term $\prod_j \prod_{i=1}^{N_j} j^{j-\delta_{i,j}}$ in the numerator with the term $\prod_j \prod_{i=1}^{N_j} (j-L_{i,j})!$ as far as we can and we bound the remaining term by $\prod_j \prod_{i=1}^{N_j} j^{(L_{i,j} - \delta_{i,j})}$. By Lemma 8.1, we know that $\Delta_q$‘s can be dominated by i.i.d. sub-exponential random variables under of uniform measure all simple symmetric random walks of length $2N$. On the other hand the total number of non-crossing partition is bounded by $4^N$. So the term

$$\sum_{P \in NC(N)} \prod_q \Delta_q \eta_{i,q} \leq 4^N \prod_{q} \left( \sum_i \eta_{i,q} \right) !. \quad (8.59)$$

Our next task is to fix the parameters $N, L_{i,j}, N_j, \omega_q, \eta_{i,q}$’s and take the following sum

$$n^{-N+1} \sum_{m \geq N} \frac{(N-L)! (2m)^{N-2L-2+\sum_j N_j}}{\prod_j N_j! (2N-2L-2)!} \prod_{j} \left( \frac{k+1}{k+m+m+2} \right) \quad (8.60)$$

By the same arguments as the arguments given to bound (8.34), we have (8.60) is bounded by

$$\left( \frac{1}{n} \right)^L 2^k \exp \left( \frac{-(N-L) \log(N-L)}{2} \right). \quad (8.61)$$

Observe that $\exp \left( \frac{-(N-L) \log(N-L)}{2} \right)$ is bounded by $C^N \left( \frac{N-L}{2} \right) !$. Now we write

$$\left( \frac{N-L}{2} \right)! = \frac{N+1}{2} \frac{N+2}{2} \cdots \frac{N}{2} \geq \frac{N}{2} \left( \frac{N}{2} \right)^{\frac{N}{2}} \quad (8.62)$$

$$\Rightarrow \exp \left( \frac{-(N-L) \log(N-L)}{2} \right) \leq \exp \left( - \frac{N \log N}{2} + \frac{L \log N}{2} \right).$$

So (8.61) is bounded by $2^k \exp \left( -L \log k - \frac{N \log N}{2} \right)$. We plug this in (8.58) to get the following reduced form:

$$\sum_{\text{parameters}} 2^{k-m-m'} C^N \prod_j \prod_{i=1}^{N_j} j^{L_{i,j} - \delta_{i,j}} \prod_q \left( \sum_{j,i} \eta_{i,j,q} \right) ! \prod_j \prod_{i=1}^{N_j} \prod_{j'} \prod_{j''} \eta_{i,j,j',j''} ! \prod_{j} \prod_{i=1}^{N_j} j^{\hat{\xi}_{i,j}} \exp \left( -L \log k - \frac{N \log N}{2} \right)$$

$$= \sum_{\text{parameters}} 2^{k-m-m'} C^N \prod_j \prod_{i=1}^{N_j} j^{L_{i,j} - \delta_{i,j}} \prod_q \left( \sum_{j,i} \eta_{i,j,q} \right) ! \prod_j \prod_{i=1}^{N_j} \prod_{j'} \prod_{j''} \eta_{i,j,j',j''} ! \exp \left( -L \log k - \frac{N \log N}{2} \right) \quad (8.63)$$

We now focus on the term

$$\prod_j \prod_{i=1}^{N_j} j^{L_{i,j} - \delta_{i,j}} \prod_q \left( \sum_{j,i} \eta_{i,j,q} \right) ! \prod_j \prod_{i=1}^{N_j} \prod_{j'} \prod_{j''} \eta_{i,j,j',j''} !$$
Now recall the discussion we had the edges traversed odd number of times. We divide the parameter $2\delta_{i,j}$ into four parts $2\delta'_{i,j}, 2\delta''_{i,j}, 2\kappa'_{i,j}, 2\kappa''_{i,j}$. Here $2\delta_{i,j} = 2\delta'_{i,j} + 2\delta''_{i,j} + 2\kappa'_{i,j} + 2\kappa''_{i,j}$. Now among $\eta_{i,j,q}$'s and $\eta'_{i,j',j''}$'s some will correspond to the odd number of repetitions. In these cases we replace $\eta_{i,j,q}$ by $(\eta_{i,j,q} - 1)$ and $\eta'_{i,j',j''}$ by $(\eta'_{i,j',j''} - 1)$ in the factorial of the denominator. To denote this mathematically we introduce parameters $\tau_{i,j,q}$ and $\tau'_{i,j',j''}$. This parameter is either 1 or 0 where 1 means the corresponding edge is traversed odd number of times and 0 means the corresponding edge is traversed even number of times.

Now observe that the $i$ th type $j$ vertex appears as the endpoint other than the ignored one for repetitions of second type $\kappa''_{i,j} + \delta''_{i,j}$ times. From the factor $j^{\kappa''_{i,j}}$ we take out $\delta''_{i,j} + \kappa''_{i,j}$ th power of $j$.

As a consequence, our factor reduces to

$$
\frac{j^{\kappa''_{i,j} + \delta''_{i,j}}}{\prod_{j'} \prod_{j''} \left( \eta'_{i,j',j''} - \tau'_{i,j',j''} \right)!} \quad (8.64)
$$

We have proved that for repetitions of the first type at a given level $q$ there can be at most two instants with edges traversed odd number of times. So we take out $(\sum_{ji} \eta_{i,j,q})^2$ from $(\sum_{ji} \eta_{i,j,q})!$ and bound the squared term by $2(\sum_{ji} \eta_{i,j,q})!$. So our factor is lesser than or equal to

$$
2 \sum_{ji} (\sum_{ji} \eta_{i,j,q} - \sum_{ji} \tau_{i,j,q})! \prod_{j} \prod_{j'} (\eta_{i,j,q} - \tau_{i,j,q})! \quad (8.65)
$$

Now we apply the following fact about multinomial coefficient. Suppose we have $\tau$ numbers $\gamma_1, \ldots, \gamma_\tau$ and for each $\tau$, $\gamma_\tau$ is partitioned into $N$ many groups. Let $\gamma_{t,k}$ be the frequency of $k$ th group. Then

$$
\prod_{t=1}^{\tau} \frac{\gamma_t!}{\prod_{k=1}^{N} \gamma_{t,k}!} \leq \frac{(\sum_{i} \gamma_i)!}{\prod_{k=1}^{N} (\sum_{i} \gamma_{t,k})!} \quad (8.66)
$$

Applying this to (8.64) and (8.65) we arrive at the following quantity.

$$
\prod_{j} \prod_{i} \frac{j^{\delta''_{i,j} + \kappa''_{i,j}}}{\prod_{j'} \prod_{j''} (\eta'_{i,j',j''} - \tau'_{i,j',j''})!} \prod_{q} \frac{(\sum_{ji} \eta_{i,j,q} - \sum_{ji} \tau_{i,j,q})!}{\prod_{j} \prod_{j'} (\eta_{i,j,q} - \tau_{i,j,q})!} \quad (8.67)
$$

Here $\varphi_{i,j} = 2\delta''_{i,j} + 2\kappa''_{i,j} + \delta''_{i,j} + \kappa''_{i,j}$.

So now we have the following reduction

$$
\frac{\prod_{j} \prod_{i=1}^{\eta_i} (L_{i,j} - \varphi_{i,j})!}{\prod_{j} \prod_{j'=1}^{N_j} (L_{i,j} - \varphi_{i,j})!} \leq C^N \prod_{j} \prod_{i=1}^{N_j} j^{\delta''_{i,j} + \delta''_{i,j} + \kappa''_{i,j}} = C^N \prod_{j} \prod_{i} j^{\delta_{i,j}}. \quad (8.68)
$$
Now we neglect \( j \leq 1000 \). Since the product \( \prod_{j \leq 1000} \prod_{i=1}^{N_j} \delta_{i,j} \leq \left(1000^{1000}\right)^N \). Now for \( j \geq 1000 \), we have \( \sum_{j \geq 1000} \sum_{i=1}^{N_j} \delta_{i,j} \leq \frac{1000}{999} \sum_{j \geq 1000} (j-1)N_j = \frac{1000}{999} \cdot N \).

Now the rest of the argument is dedicated to bound \( \sum_{i} \sum_{j=1}^{N_j} \delta_{i,j} \). Observe that the \( i \) th type \( j \) vertex has appeared as an endpoint of \( 2\delta_{i,j} \) many edges which are traversed odd number of times. As a consequence

\[
\sum_{j} \sum_{i} \delta_{i,j} = \# \text{ of edges traversed odd number of times}
\]

Now to each such edge \( e \) we assign two type \( j \geq 2 \) instants in the following way. We know that all these edges are traversed at least thrice. Hence each edge appears in the upward direction at least twice. The right endpoints of such edges are type \( j \geq 2 \) instants. Now we only need to consider those edges whose at least one endpoint is type \( j \geq 1000 \). Now say there are \( E_1 \) edges with exactly one vertex is type \( j \geq 1000 \) the other vertex is of type \( 2 \leq j < 1000 \) and \( E_2 \) edges with both vertex is type \( j \geq 1000 \) or one endpoint of type 1. So we need to calculate an upper bound for \( E_2 + \frac{1}{2} E_1 \). Now for edges corresponding to \( E_1 \) instead of two we consider only one instant and we can choose the instant to be at least the second appearance of that instant. As a consequence \( 2E_2 + E_1 \leq N + \sum_{j \geq 1000} N_j \leq \frac{1000}{999} \cdot N \). So \( E_2 + \frac{1}{2} E_1 \leq \frac{1000}{999} \cdot N \).

Plugging this into (8.63) we have the following upper bound to (8.63):

\[
\sum_{\text{parameters}} 2^{k-m-m'} C^N \exp \left\{ \delta \log N - \delta \log k - \frac{N \log N}{2} \right\} \\
\leq \sum_{\text{parameters}} 2^{k-m-m'} C^N \left( \frac{1}{\sqrt{n}} \right)^\delta \exp \left\{ \delta \log N - \frac{\delta}{4} \log N - \frac{N \log N}{2} \right\} \\
\leq \sum_{\text{parameters}} 2^{k-m-m'} C^N \left( \frac{1}{\sqrt{n}} \right)^\delta \exp \left\{ \frac{3\delta}{4} \log N - \frac{N \log N}{2} \right\} \\
\leq \sum_{\text{parameters}} 2^{k-m-m'} C^N \left( \frac{1}{\sqrt{n}} \right)^\delta \exp \left\{ N \log N \left( \frac{3 \times 500}{4 \times 999} - \frac{1}{2} \right) \right\}.
\]

Since \( \frac{1}{2} > \frac{3 \times 500}{4 \times 999} \) this concludes our proof.

**Step 2:** Now we come to the **Step 2** of the proof. Here we consider the repetitions in the Dyck paths. The main idea of the proof is similar to the proof we have just done but some calculations are different. Like **Step 1** here also we consider two types of repetitions. In the first type we consider the repetitions of the edges when they use the level of an endpoint of at least one previous appearance of the edge. In the second type we consider the repetitions when they do not use the level of an endpoint of any previous appearance of the edge.

At this point we fix the skeleton word and the Dyck paths. Let the Dyck paths be \( D_1, \ldots, D_{m+m'+1} \) with their respective lengths \( 2k_1, \ldots, 2k_{m+m'+1} \). We also denote
\( w_1, \ldots, w_{m+m'+1} \) to be all possible words corresponding to the Dyck paths \( D_1, \ldots, D_m, D_{m+m'+1} \).

Our fundamental goal is to prove the following bound.

\[
\sum_{w'_1, \ldots, w'_m} E[X_D(w)] = \frac{1}{2^{k-(m+m')}} o\left(n^{\Sigma_i k_i}\right)
\]  

(8.70)

Here we assume \( \Gamma \) be the total number of type \( j \geq 2 \) instant where for any given \( j \) there are \( \Gamma_j \) many instants of type \( j \). In particular \( \Gamma = \sum_{j \geq 2} (j-1)\Gamma_j \).

Here also we shall at first fix a permutation of the non-ignored instants (they are defined in the same manner as the skeleton word) which along with the type 1 instants specify the first traversal of all the edges. Let \( L_D \) denote the number of ignored instants in this case and \( \Gamma'_j \) be the number of non-ignored instants repeated exactly \( j \) times. Further we assume that the \( i \) th type \( j \) instant appears \( L_{i,j,D} \) times as an ignored instant.

Now the permutations of the non-ignored vertices can be done in

\[
\frac{(\Gamma - L_D)!}{\prod_j \Gamma'_j! \prod_j \prod_{i=1}^{N_j} (j - L_{i,j,D})!}
\]

(8.71)

ways.

Now for each level \( q \) we assume the corresponding Dyck path come to the level \( \Delta_{q,D} \) times. From Lemma 8.1 we know that if we consider the uniform distribution of over the feasible paths, then we know that the \( \Delta_{q,D} \)’s can be dominated by i.i.d. copies of \( X \) where \( P[X \geq t] \leq n^{\theta \frac{1}{2}} \) where \( \theta \) is a fixed constant not depending on any other parameters. In particular

\[
E[\Delta_{q,D}^\eta] \leq \begin{cases} c, & \text{if } \eta \leq 1000 \\ n^\theta \eta!, & \text{otherwise.} \end{cases}
\]

From the choices of the non-ignored vertices which edges are repeated within the \( \Delta_{q,D} \) returns to the level \( q \) are determined. Now like before suppose the \( i \) th type \( j \) instant comes immediately after the return to the level \( q \) before falling is \( \eta_{i,j,q,D} \) times. These positions can be chosen in

\[
\frac{\Delta_{q,D}^\Sigma_{\eta_{i,j,q,D}}}{\prod_j \prod_i \eta_{i,j,q,D}!}
\]

(8.72)

ways.

Now like the calculation for the skeleton words we assume that for \( i \) th type \( j \) instant there are \( \eta'_{i,j,i',j',D} \) many ignored instants corresponding to \( i' \) th type \( j' \) instant. Let \( \xi_{i,j} = \sum_{i,j} \eta_{i,j,i',j',D} \). The total choices for this is bounded by

\[
\frac{j^\xi_{i,j}}{\prod_j \eta_{i,j,i',j',D}!}
\]

(8.73)

Now we come to the expectation of a random variable coming in the product \( X_D(w) \).

Suppose a random variable (corresponding to edge \( e = \{e_1, e_2\} \)) is repeated \( 2r_e \) times in
the product $X_{D(w)}$. Then we divide this repetition in four parts $r_{e,1}, r_{e,2}, r_{e,3}$ and $r_{e,4}$. Here $2r_{e,1}$ denotes the number of times the edge is repeated as first kind where the instants corresponding to the level of return has label $e_1$. Similarly $2r_{e,2}$ denotes the number of times the edge is repeated as second kind where the instants corresponding to the level of return has label $e_2$. Finally $2r_{e,3}$ and $2r_{e,4}$ denotes the number of time the edge is repeated as second kind with the ignored instant having label $e_1$ and $e_2$ respectively. Now the expectation of the random variable is bounded by $r_e!$. It is easy to see that $r_e! \leq C^e \prod_{i=1}^{4} r_{e,i}!$. Now for the repetitions of the first kind several different values of $q$ can correspond to same vertex. So $r_{e,1}$ and $r_{e,2}$ is further partitioned into say $\tau$ many groups. Each corresponds to returns to a single level. Now if we have different levels corresponding to the same value of the label, we divide them into two parts. In the first part we consider where the value of $\sum_{i,j} \eta_{i,j,q,f}^D$ is less than 1000 and where the sum is greater than 1000. We assume $E_{\text{Unif}}$ denotes the uniform measure on all the feasible paths and $\tau'$ be the number of groups where the sum is less than 1000 and $\tau''$ be the number of groups where the sum is greater than 1000. Here we are dealing with the following sum:

$$\sum \sum_{\eta_{i,j,q,f}^D} E_{\text{Unif}} \left[ \prod_{f=1}^{\tau} \frac{\Sigma_{i,j} \eta_{i,j,q,f}^D}{\prod_j \prod_i \eta_{i,j,q,f}^D} \right]$$

$$\leq \sum \sum_{\eta_{i,j,q,f}^D} \left[ \prod_{f=1}^{\tau} \frac{1}{\prod_j \prod_i \eta_{i,j,q,f}^D} \right] \prod_{f=1}^{\tau'} \left[ c^i \sum_{i,j} \eta_{i,j,q,f}^D \leq 1000 + I \sum_{i,j} \eta_{i,j,q,f}^D > 1000 \right] n^{\eta_{i,j,q,f}^D} \eta_{i,j,q,f}^D$$

$$\leq C^{\tau'} \sum \sum_{\eta_{i,j,q,f}^D} n^{\tau''} \frac{\sum_{i,j} \eta_{i,j,q,f}^D \eta_{i,j,q,f}^D}{\prod_j \prod_i \eta_{i,j,q,f}^D}$$

$$\leq C^{\tau'} \sum \sum_{\eta_{i,j,q,f}^D} n^{\tau''} \frac{\sum_{i,j} \eta_{i,j,q,f}^D \eta_{i,j,q,f}^D}{\prod_j \prod_i \eta_{i,j,q,f}^D}$$

Now we give the upper bound to $\sum_{w_1, w_2, \ldots, w_m} E[X_{D(w)}]$. This is done by the modified bound on the words we discussed so far keeping the repetitions of the edges in mind.
In particular,

$$
\sum_{w_1, \ldots, w_{m+m'+1}} E[X_{D(w)}] 
\leq \left(\frac{1}{2}\right)^{k-m-m'} n^{\sum_i k_i - \sum_j (j-1)\Gamma_j} \exp\left(-\frac{(\sum_i k_i)^2}{2n}\right) \sum \frac{1}{\Gamma_{2}'^\alpha} \left(\frac{(\sum_i k_i)^2}{2}\right)^{\Gamma_{2}'^\alpha} 
\sum_{\Gamma_j' | j \geq 3} \prod_j \Gamma_j' \prod_{i=1}^{N_j} (j-L_{i,j,D})! \frac{(\sum_i k_i)^{\Gamma-L_{D}}+\sum_j \Gamma_j'}{(\Gamma-L_D)!} \prod_j \prod_i \left(\eta_{i,j,p,f,D}'\right) \prod_j \prod_i \left(\sum_{j=1}^{r''} \eta_{i,j,q,f,D}'\right)! 
= o\left(n^{\sum_i k_i} \left(\frac{1}{2}\right)^{k-m-m'}\right)
$$

Here the factors \(\left(\frac{1}{2}\right)^{k-m-m'}\) and the factorials in the numerator comes from \(E[X_{D(w)}]\). Also note that we have skipped the details analysis of (8.75) since this is very similar to the previous parts.

Compiling Propositions 8.3, 8.1 and 8.2 we arrive at the proof of Theorem 6.1.

9 Proof of Theorem 6.2

From discussion at the beginning of this paper, we have seen that in order to prove the Tracy Widom distribution at the edge, in addition to bounding the expectation of high value of traces we also need to bound the joint moments. In this section we deal with this problem.

Before going into the proof of Theorem 6.2, we state another important algorithm. This algorithm takes two closed words \(w_1\) and \(w_2\) of lengths \(k_1+1\) and \(k_2+1\) as input such that the words have at least one edge in common and gives a closed word \(w_3\) of length \(k_1+k_2+1\) as an output which has the same edge set as the union of the edges of \(w_1\) and \(w_2\). Variants of this algorithm has appeared in Banerjee [3], Banerjee and Bose [4] and Banerjee and Ma [5].

Algorithm 9.1. We start with two words \(w_1\) and \(w_2\) such that \(w_1\) and \(w_2\) shares an edge. Let \(\{\alpha, \beta\}\) be the first edge in \(w_2\) which is repeated in \(w_1\). We consider the first appearance of \(\{\alpha, \beta\}\) in \(w_2\). Without loss of generality we assume that the first appearance of the edge \(\{\alpha, \beta\}\) appears in the word \(w_2\) in the order \((\alpha, \beta)\). We now consider any appearance (for concreteness say the first) of the edge \(\{\alpha, \beta\}\) in the word \(w_1\). This appearance
\{\alpha, \beta\} can be traversed in \(w_1\) in the order (\(\alpha, \beta\)) or (\(\beta, \alpha\)). Considering these we have the word \(w_2\) looks like
\[
\begin{align*}
    w_2 &= (\alpha_0, \alpha_1, \ldots, \alpha_{p_1}, \alpha, \beta, \ldots, \alpha_{k_1-1}, \alpha_0) \\
    \text{(9.1)}
\end{align*}
\]
and the word \(w_1\) looks like
\[
\begin{align*}
    w_1 &= (\beta_0, \beta_1, \ldots, \beta_{q_1}, \alpha, \beta, \ldots, \beta_{k_2-1}, \beta_0) \\
    \text{(9.2)}
\end{align*}
\]
or
\[
\begin{align*}
    w_1 &= (\beta_0, \beta_1, \ldots, \beta_{q_1}, \alpha, \ldots, \beta_{k_2-1}, \beta_0). \\
    \text{(9.3)}
\end{align*}
\]
Now we output the word \(w_3\) as follows:

1. Suppose \(w_1\) is of the form (9.2), then
\[
\begin{align*}
    w_3 &= (\alpha_0, \alpha_1, \ldots, \alpha_{p_1}, \alpha, \beta, \beta_{q_1+3}, \ldots, \beta_{k_2-1}, \beta_0, \ldots, \beta_{q_1}, \alpha, \beta, \alpha_{p_1+3}, \ldots, \alpha_{k_1-1}, \alpha_0) \\
    \text{(9.4)}
\end{align*}
\]

2. On the other hand when \(w_1\) is of the form (9.3),
\[
\begin{align*}
    w_3 &= (\alpha_0, \alpha_1, \ldots, \alpha_{p_1}, \alpha, \beta, \beta_{q_1+3}, \ldots, \beta_{k_2-1}, \beta_0, \ldots, \beta_{q_1}, \alpha, \beta, \alpha_{p_1+3}, \ldots, \alpha_{k_1-1}, \alpha_0) \\
    \text{(9.5)}
\end{align*}
\]

With Algorithm 9.1 in hand we are now ready to prove Theorem 6.2.

**Proof of Theorem 6.2.** Let for any word \(w = (\alpha_0, \ldots, \alpha_{k-1}, \alpha_0)\), \(X_w\) denote the random variable
\[
X_w := \prod_{j=0}^{k-1} x_{\alpha_j, \alpha_{j+1}}. \\
\text{(9.6)}
\]
Firstly observe that
\[
\begin{align*}
    &\mathbb{E} \left[ \prod_{i=1}^{l} \left[ \text{Tr} \left[ W^{k_i} \right] - \mathbb{E} \left[ \text{Tr} \left[ W^{k_i} \right] \right] \right] \right] \\
    &= \left( \frac{1}{n} \right)^{\sum_{i=1}^{l} k_i} \sum_{w_1 \ldots w_l} \mathbb{E} \prod_{i=1}^{l} (X_{w_i} - \mathbb{E}[X_{w_i}]) \\
    \text{(9.7)}
\end{align*}
\]
where \(w_i\) is of length \(k_i + 1\) for \(1 \leq i \leq l\) respectively. Now given the words \(w_1, \ldots, w_l\) we introduce a partition \(\varsigma\) of \(\{1, \ldots, l\}\) in the following way: We put \(i\) and \(j\) in the same block if \(G_{w_i}\) and \(G_{w_j}\) share an edge. Let \(d(\varsigma)\) be the number of blocks in \(\varsigma\). Given the words \(w_1, \ldots, w_l\) we sequentially embed them in \(d(\varsigma)\) closed words with disjoint edges by applying Algorithm 9.1 sequentially. Now given \(d(\varsigma)\) closed words with disjoint edge set our goal is to construct the words \(w_1 \ldots w_l\). We call these words \(w_1, \ldots, w_d(\varsigma)\). To this end we shall also fix the partition \(\varsigma\) and the order in which Algorithm 9.1 is used. Let for a block \(B_i = j_{1,i}, \ldots, j_{|B_i|,i}\) of the partition \(\varsigma\), the order is given by \((j_{\tau_i(1),i}, \ldots, j_{\tau_i(|B_i|,i)})\). Here \(\tau_i\) is a permutation of \(\{1, \ldots, |B_i|\}\).
Now we start with the word \( w_1 \). From \( w_1 \) we at first extract \( w_{τ_1(|B_1|),1} \) and the word \( w'_{1,1} \) such that we apply Algorithm 9.1 to \( w_{1,1}' \) and \( w_{τ_1(|B_1|),1} \). We proceed in this way \(|B_1|\) times where at each step we replace \( w_1 \) by the first word we obtain after reverting Algorithm 9.1. Then we work with \( B_2, \ldots, B_{d(γ)} \) in the similar fashion.

Now we enter into one of the main parts of the proof. Here we discuss the procedure to invert Algorithm 9.1 to reconstruct the words. First of all given \( w_1 \), in order to reconstruct \( w_{τ_1(|B_1|),1} \) and \( w'_{1,1} \) we need to choose three points on the word \( w_1 \). The first and third point have the same label and they determine where the word \( w_{τ_1(|B_1|),1} \) is cut to implement the algorithm. The second point determines the endpoint of the word \( w'_{1,1} \). Also observe that once the first point is chosen, the third point comes exactly after \( l(w_{1,1}') - 1 \) steps. However the second point can be anywhere in between the first and the third point.

In the next part we prove that the first point can’t be arbitrary. Here we consider the following cases. Firstly it can happen that the edge \( \{α, β\} \) appears exactly once in each of the words \( w'_{1,1} \) and \( w_{τ_1(|B_1|),1} \). In this case we shall prove that the vertex \( α \) is a type \( j \geq 2 \) instant. At this point recall from Algorithm 9.1 that both appearances of the edge \( \{α, β\} \) happen in the order \( (α, β) \) in the word \( w_1 \). Also this edge is the first edge in \( w_{τ_1(|B_1|),1} \) which is repeated in \( w'_{1,1} \). So the level of the first point can not be reached after closing an edge from right to left within \( l(w_{1,1}') - 1 \) steps after first point. Also since the edge \( \{α, β\} \) appears exactly once in both \( w'_{1,1} \) and \( w_{τ_1(|B_1|),1} \), the edge is not closed within \( l(w_{1,1}') - 2 \) steps after first point. So the only way to reach the vertex \( α \) is, another level has the label \( α \). Hence \( α \) is a type \( j \geq 2 \) vertex.

In all the other cases since the edge \( \{α, β\} \) is traversed at least twice in the word \( w'_{1,1} \), at least one of the vertices \( α, β \) is a type \( j \geq 2 \) vertex.

Now we consider two cases depending on the position of the third point.

(i) Firstly the third point is at a different level than the first point.

(ii) Secondly the third point is at the same level as the first point.

We at first consider case (i). Since \( \{α, β\} \) is the first edge in \( w_{τ_1(|B_1|),1} \) which is repeated in \( w'_{1,1} \), if we look at \( l(w_{1,1}') - 1 \) steps after first point, the walk never falls down of the level of the first point. As a consequence, within the \( l(w_{1,1}') - 1 \) steps after first point there is always an open edge incident to the vertex \( α \). This implies if the third point is at a different level than the first point, then the level of the third point at a level of a type \( j \geq 2 \) open instant. Hence in order to choose the first and the third point, one needs to look at the skeleton word, choose two appearances of a type type \( j \geq 2 \) instant and look at the returns to these chosen levels. Now we spend some time on the total number of words in this case. Like usual we at first fix the skeleton word and fix two instants where same vertex is repeated as type \( j \geq 2 \) instants. Let \( m_2 \) be the length of the skeleton word between these two points while \( m_1 + m_2 \) be the total length of the skeleton word. Then the total number of Dyck paths in between the chosen points is given by \( \frac{m_2 + 1}{l(l(w_{1,1}'), l(w_{1,1}'), m_2 + 1)} \). On the other hand the number of Dyck paths in the other part
can be chosen in \( \frac{m_1+1}{l(w'_{1,1})} \left( \frac{m_i}{l(w'_{i,1})} \right) \) ways.

Now we look at case (ii). We further reduce it into the following cases:

(a) The whole word \( w'_{1,1} \) is inside a Dyck path.

In this case we choose a point on the word \( w_{r_{1}(\beta_{1i})} \) and from that point we choose a Dyck path of length \( l(w'_{1,1}) - 1 \). Next just after last point of the Dyck path we create an upward edge with endpoint \( \beta_i \). Considering the first edge in the Dyck path to be \((\alpha, \beta)\) with starting point \( \alpha \), the upward edge after the last point of the Dyck path is a type \( j \geq 2 \) instant. The words in this case is chosen in the following way. We pick a point on the word \( w_{r_{1}(\beta_{1i})} \) and place a Dyck path of length \( l(w'_{1,1}) - 1 \) immediately after that point. So the the choices for the word \( w'_{1,1} \) is simply \( C_{\lfloor w'_{1,1}\rceil - 1} \).

(b) The word \( w'_{1,1} \) spans beyond a Dyck path. Observe that as \( \{\alpha, \beta\} \) is the first edge in \( w_{r_{1}(\beta_{1i})} \) repeated in \( w'_{1,1} \), the first point in this case has to be in the same level as a point on the skeleton word. Here we divide it into further two cases depending on the edge \( \{\alpha, \beta\} \) after the third point belonging or not belonging to a Dyck path.

Case I: Here we assume that the edge \( \{\alpha, \beta\} \) after the third point belongs to the skeleton word. Hence the edge \( \{\alpha, \beta\} \) after the first point belongs to skeleton word. Observe that here the instant immediately after the first point and the instant immediately after the third point are type \( j \geq 2 \). Hence here we arrive at a situation similar to case (i). The only difference is instead of choosing the type \( j \) instants as the first and the third point we choose the points immediately before them. The calculation of the number of words is also the same as case (i).

Case II: Here we assume that the edge \( \{\alpha, \beta\} \) after the third point belongs to a Dyck path. Since the first and the third point is in the same level of a point in the skeleton word, the calculation of the number of words is also same as case (i). However here we have an additional constraint that the instant immediately after the third point is a type \( j \geq 2 \) instant and the third point is at the same level as a point on the skeleton word. So it is equivalent to choose a point on the skeleton word and construct the type \( j \geq 2 \) instant from that level. This will reduce the count. One might look at the detailed explanation below.

We now prove Theorem 6.2. We only prove part 1 and 2 of it. Further for convenience we assume that \( l = 2 \). The case for general \( l \) can be proved by repeated use of the arguments given here. The fundamental idea of the proof is as follows: We shall prove that only case (i) and Case I of part (b) of case (ii) give higher contribution. All the other cases give a negligible contribution. Further in these cases we know that with high probability \( m_1, m_2 \) are of \( O(n^{\frac{2}{3}}) \) while \( l(w'_{1,1}) \) is of \( O(n^{\frac{5}{3}}) \). This makes the value \( m_3+1 \left( \frac{m_i}{l(w'_{i,1})} \right) = 2^{l(w'_{1,1})} O\left( \frac{1}{l(w'_{1,1})} \right) \). The factor \( \left( \frac{1}{l(w'_{1,1})} \right) \) cancels out with the possible choices of the second point which is also of \( O(l(w'_{1,1})) \). As a result we get the total contribution is of \( O(1) \). In the next part we formalize these arguments.
We now consider each case discussed in the proof separately.

(i): This is the most important case and is responsible for the main contributions. We shall use the results in Propositions 8.3 and 8.2. First of all observe that

\[
\sum_{w_1, w_2} \left( \frac{1}{n} \right)^{k_1 + k_2} E \left[ (X_{w_1} - E[X_{w_1}]) (X_{w_2} - E[X_{w_2}]) \right]
\]

\[
\sum_{w_1, w_2} \left( \frac{1}{n} \right)^{k_1 + k_2} \left( E[X_{w_1}X_{w_2}] - E[X_{w_1}]E[X_{w_2}] \right).
\]

(9.8)

Now if \( w_1 \) and \( w_2 \) do not share an edge then \( E[X_{w_1}X_{w_2}] = E[X_{w_1}]E[X_{w_2}] \). So we shall ignore such words. Among the remaining words, we have dealt with the term \( \sum_{w_1, w_2} \left( \frac{1}{n} \right)^{k_1 + k_2} E[X_{w_1}]E[X_{w_2}] \) in Propositions 8.3 and 8.2. So we only need to consider the first term. Also following the notation used in the proof so far, we shall denote \( w_1 \) by \( w_1' \) and \( w_2 \) by \( w_{r_1(|B_1|, 1)} \). We apply Algorithm 9.1 to these words to get \( w_1 \). Let the
parameters be defined analogously for the word \( w_1 \). Then

\[
\sum_{w_{1,1}^Iw_{1,1}^I} \sum_{w_1} \left( \frac{1}{n} \right)^{\frac{1}{2}N(w_1^I)+l(w_1^I)+w_1^I} E[Xw_{1,1}^I Xw_{1,1}^I]
\]

\[
= \sum_{w_1} \left( \frac{1}{n} \right)^{\frac{1}{2}N(w_1^I)+1} |f^{-1}(w_1)| E[Xw_1]
\]

\[
\leq \sum_{w_1} \left( \frac{1}{n} \right)^{\frac{1}{2}N(w_1^I)+1} N^2 l(w_1^I) E[Xw_1]
\]

\[
\leq \left( \frac{1}{n} \right)^{\frac{k-m+m'}{2}} n \sum_{i} k_i \exp \left( -\frac{(\sum k_i)^2}{2n} \right) \sum_{\Gamma_j} \frac{1}{\Gamma_j} \left( \frac{(\sum k_i)^2}{2} \right) \Gamma_j
\]

\[
\leq \left( \frac{1}{n} \right)^{k-m+m'} n \sum_{i} k_i \sum_{\Gamma_j} \prod_{i} \prod_{j \geq 2} \frac{\Gamma_j^{\frac{k_i}{\Gamma_j}} |\prod_{j=1}^{N_j} (j-L_{i,j})|!}{(\Gamma_j-L_D)!} \sum_{\eta_i,j,i'|f_j} \prod_{i' \neq i} \left( \eta_i,j,i'||f_j \right) \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right) \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[
\sum_{\eta_i,j,i'|f_j} \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right) \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[
\leq \left( \frac{1}{n} \right)^{k-m+m'} n \sum_{i} k_i \sum_{\Gamma_j} \prod_{i} \prod_{j \geq 2} \frac{\Gamma_j^{\frac{k_i}{\Gamma_j}} |\prod_{j=1}^{N_j} (j-L_{i,j})|!}{(\Gamma_j-L_D)!} \sum_{\eta_i,j,i'|f_j} \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right) \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[
\left( \frac{1}{n} \right)^{k-m+m'} n \sum_{i} k_i \sum_{\Gamma_j} \prod_{i} \prod_{j \geq 2} \frac{\Gamma_j^{\frac{k_i}{\Gamma_j}} |\prod_{j=1}^{N_j} (j-L_{i,j})|!}{(\Gamma_j-L_D)!} \sum_{\eta_i,j,i'|f_j} \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right) \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[
1 \leq \frac{1}{2} m+m'
\]

\[
\sum_{\text{parameters}} n^{m-N+1} C^N
\]

\[
\frac{1}{e^{\frac{1}{2}} \prod_{j=1}^{N_j} (j-L_{i,j})!} \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right) \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[
(2m+2m')^{2N-2L_i-2+\sum_{j} N_j}
\]

\[
(2m+2m')^{2N-2L_i-2+\sum_{j} N_j}
\]

\[
(2m+2m')^{2N-2L_i-2+\sum_{j} N_j}
\]

\[
N^2 l(w_1^I) m^2 + 1 \left( \frac{l(w_1^I)+m^2+1}{2} \right) m_1 + 1 \left( \frac{l(w_1^I)+m^2+1}{2} \right) \prod_{i} \prod_{j \neq i} \left( \eta_i,j,i'|f_j \right)
\]

\[ (9.9) \] can be handled exactly in the same way we handled Proposition 8.2 and 8.3 to get that \( (9.9) \) is of \( O(1) \). Further if the word \( w \) has at least one edge repeated at least thrice, then the sum goes to 0.

It can be proved that the sums in all the other cases go to 0. Since the equations are somewhat long and are not very informative, for the next cases we shall only point out due to which factors the corresponding sum goes to 0.

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Case (a): Here the word $w'_{1,1}$ is given by a Dyck path of length $l(w'_{1,1})$. This Dyck path can be placed in anywhere of the word. So in order to form $w_1$, we at first form a word of length $l(w_{τ(1B_1),1})$. Now we choose a left endpoint of an upward edge in the word (this choice is necessary since the edge $\{α, β\}$ from the third point is an upward edge and the given upward edge serves for that). Now the Dyck path can be chosen in $C_{\frac{l(w'_{1,1})-1}{2}} = 2^{\frac{l(w'_{1,1})-1}{2}}O\left(\frac{1}{n}\right)$. Now the position of the first point has $O\left(n^\frac{3}{2}\right)$ many choices. However we loose a factor of $n$, since the instant immediately after the third point is a type $j \geq 2$ instant. On the other hand the second point has $O\left(n^\frac{3}{2}\right)$ many choices. Compiling these we have

$$
\sum_{w_1} \left(\frac{1}{n}\right)^{\frac{l(w_{1,1})-1}{2}} |f^{-1}(w_1)| E[X_{w_1}] = O\left(\frac{n^\frac{3}{2}}{n^2}\right) = O\left(\frac{1}{n^\frac{3}{2}}\right) \to 0.
$$

Case (b):

Case I: This case is almost identical to case (i). However the only difference is, since the edge $\{α, β\}$ is repeated at least thrice, we have

$$
\sum_{w_1} \left(\frac{1}{n}\right)^{\frac{l(w_{1,1})-1}{2}} |f^{-1}(w_1)| E[X_{w_1}] \to 0.
$$

We omit the details.

Case II: In this case the word count will be similar to case (i). However the edge $\{α, β\}$ immediately after the third point belongs to a Dyck path. Since the first point is in the same level of a point in the skeleton word, the number of choices of the first point is $O(m_1) = O(n^\frac{3}{2})$. On the other hand since the instant immediately after the third point is a type $j \geq 2$ instant, we lose a factor of $n$. Finally the choice of the second point is of $O\left(n^\frac{3}{2}\right)$ and the factor $\frac{m_2+1}{l(w'_{1,1})} = O\left(\frac{1}{l(w'_{1,1})}\right)$. Compiling these we get

$$
\sum_{w_1} \left(\frac{1}{n}\right)^{\frac{l(w_{1,1})-1}{2}} |f^{-1}(w_1)| E[X_{w_1}] = O\left(\frac{n^\frac{3}{2}+\frac{3}{2}}{n^{2+\frac{3}{2}}}\right) = O\left(\frac{1}{n^{\frac{3}{2}}}\right) \to 0.
$$

This concludes the proof.

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