Nonparametric Estimation of Linear Multiplier in SDEs driven by General Gaussian Processes

B.L.S. Prakasa Rao

CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India

Abstract: We investigate the asymptotic properties of a kernel-type nonparametric estimator of the linear multiplier in models governed by a stochastic differential equation driven by a general Gaussian process.

1 Introduction

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modeling in a wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence (cf. Prakasa Rao (2010)).

Our aim in this paper is to investigate the asymptotic properties of a non-parametric kernel-type estimator for estimating the linear multiplier in a stochastic differential equation driven by a Gaussian process. Nonparametric estimation of a linear multiplier for processes driven by $\alpha$-stable noise is investigated in Prakasa Rao (2021). Asymptotic properties of the minimum $L_1$-norm estimator of the drift parameter of an Ornstein-Uhlenbeck process driven by general Gaussian processes are investigated in Prakasa Rao (2022). El Machkouri et al. (2015), Chen and Zhou (2020) and Lu (2022) study parameter estimation for an Ornstein-Uhlenbeck process driven by a general Gaussian process.

1 Mathematics Subject Classification: Primary 60G22, 62 M 09.

Keywords and phrases: Kernel method; Linear multiplier; Nonparametric estimation; Stochastic differential equation; Trend coefficient; Gaussian Process.
2 Preliminaries

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a stochastic basis satisfying the usual conditions and the processes discussed in the following are \((\mathcal{F}_t)\)-adapted. Further the natural filtration of a process is understood as the \(P\)-completion of the filtration generated by this process. We consider a centered Gaussian process \(G \equiv \{G_t, 0 \leq t \leq T\}\).

Let us consider a stochastic process \(\{X_t, t \in [0,1]\}\) defined by the stochastic differential equation

\[
\frac{dX_t}{dt} = \theta(t)X_tdt + \varepsilon\, dG_t, \, X_0 = x_0, \, 0 \leq t \leq 1,
\]

where the function \(\theta(t)\) is an unknown. We assume that the Gaussian process \(G(.)\) satisfies the condition \(E[\sup_{0 \leq s \leq T} |G(s)|] \) is finite and it has Holder continuous paths of positive order. We assume that integration with respect to the Gaussian process \(G\) is defined as a Young integral (cf. Nourdin (2012)). This class of Gaussian processes \(G\) includes fractional Brownian motion, sub-fractional Brownian motion and bifractional Brownian motion (cf. Mishura and Zili (2018)) under some conditions.

We now consider the problem of estimation of the function \(\theta(t)\) based on the observation of process \(X = \{X_t, 0 \leq t \leq 1\}\) and study its asymptotic properties as \(\varepsilon \to 0\).

Let \(x = \{x_t, 0 \leq t \leq 1\}\) be the solution of the ordinary differential equation

\[
\frac{dx_t}{dt} = \theta(t)x_t, x_0, 0 \leq t \leq 1.
\]

Observe that

\[x_t = x_0 \exp(\int_0^t \theta(s)ds)\].

We assume that (A1) the function \(\theta(t)\) is bounded over the interval \([0,T]\) by a constant \(L\).

**Lemma 2.1:** Let \(X_t\) and \(x_t\) be the solutions of the equation (2.1) and (2.2) respectively. Then, with probability one,

\[
(a) |X_t - x_t| \leq e^{LT} \varepsilon \sup_{0 \leq s \leq t} |G_s|
\]

and

\[
(b) \sup_{0 \leq t \leq T} E|X_t - x_t| \leq e^{LT} \varepsilon E[\sup_{0 \leq t \leq T} |G(t)|].
\]
Proof of (a) : Let \( u_t = |X_t - x_t| \). Then

\[
(2.5) \quad u_t \leq \int_0^t |\theta(v)(X_v - x_v)| \, dv + \epsilon |G_t| \\
\leq L \int_0^t u_v \, dv + \epsilon \sup_{0 \leq s \leq t} |G_s|.
\]

Applying the Gronwall’s lemma (cf. Lemma 1.12, Kutoyants (1994), p.26), it follows that

\[
(2.6) \quad u_t \leq \epsilon e^{LT} \sup_{0 \leq s \leq t} |G_s|.
\]

Proof of (b) : From (2.3), we have,

\[
(2.7) \quad E|X_t - x_t| \leq e^{Lt} \epsilon E\left[ \sup_{0 \leq s \leq t} |G_s| \right].
\]

Hence

\[
(2.8) \quad \sup_{0 \leq t \leq T} E|X_t - x_t| \leq \epsilon e^{LT} E\left[ \sup_{0 \leq t \leq T} |G(t)| \right].
\]

3 Estimation of the Drift function

Let \( \Theta_0(L) \) denote the class of all functions \( \theta(.) \) with the same bound \( L \). Let \( \Theta_k(L) \) denote the class of all functions \( \theta(.) \) which are uniformly bounded by the same constant \( L \) and which are \( k \)-times differentiable satisfying the condition

\[
|\theta^{(k)}(x) - \theta^{(k)}(y)| \leq L'|x - y|, \ x, y \in \mathbb{R}
\]

for some constant \( L' > 0 \). Here \( g^{(k)}(x) \) denotes the \( k \)-th derivative of \( g(.) \) at \( x \) for \( k \geq 0 \). If \( k = 0 \), we interpret the function \( g^{(0)}(.) \) as the function \( g(.) \).

Let \( K(u) \) be a bounded function with finite support \([A, B]\) with \( A < 0 < B \) satisfying the condition

(\( A_2 \)) \( K(u) = 0 \) for \( u < A \) and \( u > B \); and \( \int_A^B K(u) \, du = 1 \).

Boundedness of the function \( K(\cdot) \) with finite support \([A, B]\) implies that

\[
\int_{-\infty}^{\infty} |K(u)|^2 \, du < \infty; \quad \int_{-\infty}^{\infty} |u^j K(u)| \, du < \infty, \ j \geq 0.
\]

We define a kernel type estimator \( \hat{\theta}_t \) of the function \( \theta(t) \) by the relation

\[
(3.1) \quad \hat{\theta}_t X_t = \frac{1}{\varphi_\epsilon} \int_0^T K\left( \frac{\tau - t}{\varphi_\epsilon} \right) dX_\tau
\]
where the normalizing function $\varphi_\epsilon \to 0$ as $\epsilon \to 0$. Let $E_\theta(.)$ denote the expectation when the function $\theta(.)$ is the linear multiplier.

**Theorem 3.1:** Suppose that the linear multiplier $\theta(.) \in \Theta_0(L)$ and the function $\varphi_\epsilon \to 0$ as $\epsilon \to 0$. Further suppose that the conditions $(A_1), (A_2)$ hold. Then, for any $0 < c \leq d < T$, the estimator $\hat{\theta}_t$ satisfies the property

$$\lim_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_0(L)} \sup_{c \leq t \leq d} E_\theta(\|\hat{\theta}_t X_t - \theta(t)x_t\|) = 0.$$

In addition to the conditions $(A_1), (A_2)$, suppose the following condition holds.

$(A_3) \int_{-\infty}^\infty u^j K(u)du = 0$ for $j = 1, 2, ..., k.$

**Theorem 3.2:** Suppose that the function $\theta(.) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{k+1}$. Suppose the conditions $(A_1) - (A_3)$ hold. Then

$$\lim_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_\theta(\|\hat{\theta}_t X_t - \theta(t)x_t\|)\epsilon^{-1} < \infty.$$

**Theorem 3.3:** Suppose that the function $\theta(.) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{1/(k+1)}$. Further suppose that the conditions $(A_1) - (A_3)$ hold. Then the asymptotic distribution of

$$\varphi_\epsilon^{-(k+1)}(\hat{\theta}_t X_t - \theta(t)x_t),$$

as $\epsilon \to 0$ is the distribution of a Gaussian random variable with mean

$$m = \frac{J^{(k+1)}(x_t)}{(k+1)!} \int_{-\infty}^\infty K(u)u^{k+1}du$$

and variance $R(t,t)$ as $\epsilon \to 0$ where $J(t) = \theta(t)x(t)$.

4 Proofs of Theorems 3.1-3.3:

**Proof of Theorem 3.1:** From the equation (3.1), we have

$$E_\theta[\|\hat{\theta}_t X_t - \theta(t)x_t\|] = E_\theta[\frac{1}{\varphi_\epsilon} \int_0^T K\left(\frac{\tau-t}{\varphi_\epsilon}\right) (\theta(\tau)x_\tau - \theta(\tau)x_\tau) d\tau$$

$$+ \frac{1}{\varphi_\epsilon} \int_0^T K\left(\frac{\tau-t}{\varphi_\epsilon}\right) \theta(\tau)x_\tau d\tau - \theta(t)x_t + \frac{\epsilon}{\varphi_\epsilon} \int_0^T K\left(\frac{\tau-t}{\varphi_\epsilon}\right) dG_{\tau}].$$
\[ \begin{align*}
\leq & \ E_\theta \left[ \frac{1}{\varphi_\epsilon} \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) (\theta(\tau)X_\tau - \theta(\tau)x_\tau) d\tau \right] \\
+ & \ E_\theta \left[ \frac{1}{\varphi_\epsilon} \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) \theta(\tau)x_\tau d\tau - \theta(t)x_t \right] \\
+ & \frac{\epsilon}{\varphi_\epsilon} E_\theta \left[ | \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) dG_\tau | \right] \\
= & \ I_1 + I_2 + I_3 \ (\text{say}).
\end{align*} \]

Apply the change of variables \( u = (\tau - t)\varphi_\epsilon^{-1}, v = (\tau' - t)\varphi_\epsilon^{-1} \) and let \( \epsilon_1 = \min(\epsilon', \epsilon'') \), where \( \epsilon' = \sup\{ \epsilon : \varphi_\epsilon \leq -\frac{\epsilon}{\sqrt{T}} \} \) and \( \epsilon'' = \sup\{ \epsilon : \varphi_\epsilon \leq -\frac{T - d}{\varphi_\epsilon} \} \). Then, for \( 0 < \epsilon < \epsilon_1 \),

(4. 2) \[ I_3 = \frac{\epsilon}{\varphi_\epsilon} E_\theta \left[ \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) dG_\tau \right] \]

\[ = \frac{\epsilon}{\varphi_\epsilon} \left[ \int_0^T \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) K \left( \frac{\tau' - t}{\varphi_\epsilon} \right) R(\tau, \tau') d\tau d\tau' \right]^{1/2} \]

\[ = \frac{\epsilon}{\varphi_\epsilon} \varphi_\epsilon \left[ \int_0^\infty \int_0^\infty |K(u)K(v)R(t + u\varphi_\epsilon, t + v\varphi_\epsilon)| du dv \right]^{1/2} \]

\[ \leq C_1 \epsilon (\text{by using (A2)}) \]

for some positive constant \( C_1 \) by observing that

\[ \int_\infty \int_\infty |K(u)K(v)R(t + u\varphi_\epsilon, t + v\varphi_\epsilon)| du dv \]

tends to \( R(t, t) \int_\infty \int_\infty K(u)K(v) du dv = R(t, t) \) by the condition \( (A_1) \) as \( \epsilon \to 0 \) by Bochner’ theorem (cf. Prakasa Rao (1983)) as \( \epsilon \to 0 \). Since \( \epsilon \to 0 \), it follows that \( I_1 \) tends to zero.

Furthermore

(4. 3) \[ I_2 = E_\theta \left[ \frac{1}{\varphi_\epsilon} \int_0^T K \left( \frac{\tau - t}{\varphi_\epsilon} \right) \theta(\tau)x_\tau d\tau - \theta(t)x_t \right] \]

\[ \leq E_\theta \left[ \int_\infty |K(u)(\theta(t + \varphi_\epsilon u)x_t + \varphi_\epsilon u - \theta(t)x_t)| du \right] \]

\[ \leq L \left[ \int_\infty |K(u)| \varphi_\epsilon du \right] \]

\[ \leq C_2 \varphi_\epsilon \]
for some positive constant $C_2$. Hence $I_2$ tends to zero as $\epsilon \to 0$. Furthermore note that

\begin{align}
(4.4) \quad I_1 &= E_\theta \left[ \frac{1}{\varphi_\epsilon} \int_0^T K \left( \frac{t-\tau}{\varphi_\epsilon} \right) (\theta(\tau)X_\tau - \theta(\tau)x_\tau) d\tau \right] \\
&= E_\theta \left[ \left| \int_{-\infty}^{\infty} K(u) (\theta(t+\varphi_\epsilon u)X_{t+\varphi_\epsilon u} - \theta(t+\varphi_\epsilon u)x_{t+\varphi_\epsilon u}) \right| du \right] \\
&\leq LE_\theta \left( \int_{-\infty}^{\infty} |K(u)| \sup_{0 \leq t+\varphi_\epsilon u \leq T} (|X_{t+\varphi_\epsilon u} - x_{t+\varphi_\epsilon u}|) \right) du \quad \text{(by using the condition } (A_1)) \\
&\leq LE_\theta \left( \int_{-\infty}^{\infty} |K(u)| e^{LT} \right) E \left[ \sup_{0 \leq s \leq T} |G_s| \right] \\
&\leq C_3 \epsilon \quad \text{(by using } (2.4)\text{)}
\end{align}

for some positive constant $C_4$ depending on $T$ and $L$. Hence $I_3$ tends to zero as $\epsilon \to 0$. Theorem 3.1 is now proved by using the equations (4.1) to (4.4).

**Remarks:** From the proof presented above, it is possible to choose the functions $c_\epsilon$ and $d_\epsilon$ such that $c_\epsilon \to 0, d_\epsilon \to T$ and satisfy the conditions

\[
\frac{c_\epsilon}{\varphi_\epsilon} \geq -A, \quad \frac{T - d_\epsilon}{\varphi_\epsilon} \geq B
\]

(for instance, choose $c_\epsilon = -A \varphi_\epsilon$ and $d_\epsilon = T - B \varphi_\epsilon$). Then the estimator $\hat{\theta}_t$ satisfies the property that

\begin{equation}
(4.5) \quad \lim_{\epsilon \to 0} \sup_{\theta(t) \in \Theta(L)} \sup_{c_\epsilon \leq t \leq d_\epsilon} E_\theta(\hat{\theta}_t X_t - \theta(t)x_t) = 0.
\end{equation}

**Proof of Theorem 3.2:** Let $J(t) = \theta(t)x_t$. By the Taylor’s formula, for any $u \in R$,

\[
J(y) = J(u) + \sum_{r=1}^{k} J^{(r)}(u) \frac{(y-u)^r}{r!} + [J^{(k)}(z) - J^{(k)}(u)] \frac{(y-u)^k}{k!}
\]

for some $z$ such that $|z - u| \leq |y - u|$. Using this expansion, the equation (4.2) and the conditions in the expression $I_2$ defined in the proof of Theorem 3.1, it follows that

\[
I_2 \leq \left[ \int_{-\infty}^{\infty} K(u) (J(t + \varphi_\epsilon u) - J(t)) \right] du
\]
Proof of Theorem 3.3:

This completes the proof of Theorem 3.2.

\[ I_2 \leq C_4 L \left[ \int_{-\infty}^{\infty} |K(u)u^{k+1}| \varphi_{\epsilon}^{k+1}(k!)^{-1} du \right] \]

for some \( z_u \) such that \( |x_t - z_u| \leq |x_{t+\varphi_u t} - x_t| \leq C|\varphi_u u| \) for some positive constant \( C \). Hence

\[
(4. 6) \quad I_2 \leq C_5 (k!)^{-1} \int_{-\infty}^{\infty} |K(u)u^{k+1}| du \]

for some positive constant \( C_6 \) depending on \( A, B, T \) and \( L \). Combining the relations (4.2), (4.4) and (4.6), we get that there exists a positive constant \( C_7 \) depending on \( T, L, A, B \) such that

\[
sup_{c \leq t \leq d} E_\theta |\tilde{\theta}_t x_t - \theta(t)x_t| \leq C_7 (\epsilon + \varphi_{\epsilon}^{k+1} + \epsilon).
\]

Choosing \( \varphi_{\epsilon} = \frac{1}{\epsilon^{k+1}} \), we get that

\[
\limsup_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_\theta |\tilde{\theta}_t x_t - \theta(t)x_t|^{\epsilon^{-1}} < \infty.
\]

This completes the proof of Theorem 3.2.

Proof of Theorem 3.3:

From the equation (3.1), we obtain that

\[
(4. 7) \quad \tilde{\theta}_t x_t - \theta(t)x_t = \left[ \frac{1}{\varphi_{\epsilon}} \int_0^T K \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) \theta(t)x_{\tau} d\tau \right.
\]

\[
\qquad \quad + \frac{1}{\varphi_{\epsilon}} \int_0^T K \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) \theta(t)x_{\tau} d\tau - \theta(t)x_t + \frac{\epsilon}{\varphi_{\epsilon}} \int_0^T K \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) dG_{\tau} \]

\[
\qquad \quad = \left[ \int_{-\infty}^{\infty} K(u) \left( \theta(t+\varphi_u u)x_{t+\varphi_u u} - \theta(t+\varphi_u u)x_{t+\varphi_u u} \right) du \right.
\]

\[
\qquad \quad + \int_{-\infty}^{\infty} K(u) \left( \theta(t+\varphi_u u)x_{t+\varphi_u u} - \theta(t)x_t \right) du \]

\[
\qquad \quad + \frac{\epsilon}{\varphi_{\epsilon}} \int_0^T K \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) dG_{\tau} \right].
\]
Let \( J(t) = \theta(t)x_t \). By the Taylor’s formula, for any \( u \in R \),

\[
J(y) = J(u) + \sum_{r=1}^{k+1} \frac{J^{(r)}(u)(y - u)^j}{j!} + \left[ J^{(k+1)}(z) - J^{(k+1)}(x) \right] \frac{(y - u)^{k+1}}{(k + 1)!}
\]

for some \( z \) such that \(|z - u| \leq |y - u|\). Let

\[
m = \frac{J^{(k+1)}(x_t)}{(k + 1)!} \int_{-\infty}^{\infty} K(u)u^{k+1} du
\]

and

\[
R_1(t) = \varphi^{-k-1}(k) \int_0^T K \left( \frac{\tau - t}{\varphi} \right) (\theta(\tau)x_{\tau} - \theta(\tau)x_t) d\tau.
\]

By arguments similar to those given in (4.3) for obtaining upper bounds, it follows that

\[
E|R_1(t)| \leq C\varphi^{-k} \epsilon.
\]

Let

\[
R_2(t) = \varphi^{-k-1}(k) \int_0^T K \left( \frac{\tau - t}{\varphi} \right) (\theta(\tau)x_{\tau} - \theta(\tau)t) d\tau.
\]

Observe that

\[
R_2(t) = m + o(1)
\]

by an application of the Taylor’s expansion under the condition \((A_3)\).

Furthermore

\[
(4.8) \varphi^{-k-1}(k)(\hat{\theta}_t x_t - \theta(t)x_t) = \epsilon \varphi^{-k-2}(k) \int_0^T K \left( \frac{\tau - t}{\varphi} \right) dG_\epsilon + R_2(t) + R_1(t)
\]

\[
= \epsilon \varphi^{-k-2}(k) \int_0^T K \left( \frac{\tau - t}{\varphi} \right) dZ_\epsilon + m + o(1) + O_p(\varphi^{-k} \epsilon).
\]

Let \( \varphi \) be chosen so that \((\varphi)^{-1} = \epsilon \varphi^{-k-2}(k)\). Such a choice is \( \varphi = \epsilon^v \) where \( v = (k + 1)^{-1} \).

We will now study the asymptotic behaviour of the random variable

\[
W_\epsilon = (\varphi)^{-1} \int_0^T K \left( \frac{\tau - t}{\varphi} \right) dZ_\epsilon
\]

as \( \epsilon \to 0 \). Note that

\[
(4.9) \varphi^{-k-1}(k)(\hat{\theta}_t x_t - \theta(t)x_t) = W_\epsilon + m + o(1)
\]

Note that \( W_\epsilon \) is Gaussian with mean zero and variance \( R(t, t) + o(1) \) as \( \epsilon \to 0 \).

\[
m = \frac{J^{(k+1)}(x_t)}{(k + 1)!} \int_{-\infty}^{\infty} K(u)u^{k+1} du
\]
where \( J(t) = \theta(t)x_t \). Hence

\[
\varphi_\epsilon^{-(k+1)}(\hat{\theta}_t X_t - \theta(t)x_t)
\]

is Gaussian with mean

\[
m = \frac{J^{(k+1)}(x_t)}{(k+1)!} \int_{-\infty}^{\infty} K(u) u^{k+1} du
\]

and variance \( R(t, t) \) as \( \epsilon \to 0 \).

Note that the results given above deal with asymptotic properties of the estimator for the function

\[
J(t) = \theta(t)x_t = \theta(t)x_0 \exp(\int_0^t \theta(s) \, ds).
\]

We will now present another method for the estimation of the linear multiplier \( \theta(t) \).

## 5 Estimation of the Multiplier \( \theta(.) \)

Let \( \Theta \rho(L_\gamma) \) be a class of functions uniformly bounded and \( k \)-times continuously differentiable for some integer \( k \geq 1 \) with the \( k \)-th derivative satisfying the Holder condition of the order \( \gamma \in (0, 1) \):

\[
|\theta^{(k)}(t) - \theta^{(k)}(s)| \leq L_\gamma |t - s|^{\gamma}, \rho = k + \gamma.
\]

From the Lemma 3.1, it follows that

\[
|X_t - x_t| \leq \epsilon e^{Lt} \sup_{0 \leq s \leq T} |G_s|.
\]

Let

\[
A_t = \{ \omega : \inf_{0 \leq s \leq t} X_s(\omega) \geq \frac{1}{2} x_0 e^{-Lt} \}
\]

and let \( A = A_T \). Define the process \( Y \) with the differential

\[
dY_t = \theta(t)I(A_t)dt + \epsilon X_t^{-1}I(A_t) \, dG_t, 0 \leq t \leq T.
\]

We will now construct an estimator of the function \( \theta(.) \) based on the observation of the process \( Y \) over the interval \([0, T]\). Define the estimator

\[
\tilde{\theta}(t) = I(A) \frac{1}{\varphi_\epsilon} \int_0^T K(\frac{t-s}{\varphi_\epsilon}) \, dY_s
\]
where the kernel function $K(.)$ satisfies the conditions $(A1) - (A3)$. Observe that

$$E|\tilde{\theta}(t) - \theta(t)| = E|I(A)\frac{1}{\varphi_\epsilon} \int_0^T K\left(\frac{t - s}{\varphi_\epsilon}\right)(\theta(s) - \theta(t))ds + I(A^c)\theta(t) + I(A)\frac{\epsilon}{\varphi_\epsilon} \int_0^T K\left(\frac{t - s}{\varphi_\epsilon}\right)X_s^{-1}dG_s|$$

$$\leq E|I(A)\int_R K(u)[\theta(t + \omega_\epsilon) - \theta(t)]du + |\theta(t)|P(A^c) + \frac{\epsilon}{\varphi_\epsilon} E|I(A)\int_0^T K\left(\frac{t - s}{\varphi_\epsilon}\right)X_s^{-1}dG_s|$$

$$= I_1 + I_2 + I_3. \ (say)$$

Applying the Taylor’s theorem and using the fact that the function $\theta(t) \in \Theta_\rho(L_\gamma)$, it follows that

$$I_1 \leq \frac{L_\gamma}{(k + 1)!} \varphi_\epsilon^\rho \int_R |K(u)\rho^0|du.$$

Note that, by Lemma 3.1,

$$P(A^c) = P\left(\inf_{0 \leq t \leq T} X_t < \frac{1}{2}x_0e^{-LT}\right)$$

$$\leq P\left(\inf_{0 \leq t \leq T} |X_t - x_t| + \inf_{0 \leq t \leq T} x_t < \frac{1}{2}x_0e^{-LT}\right)$$

$$\leq P\left(\inf_{0 \leq t \leq T} |X_t - x_t| < \frac{1}{2}x_0e^{-LT}\right)$$

$$\leq P\left(\sup_{0 \leq t \leq T} |X_t - x_t| > \frac{1}{2}x_0e^{-LT}\right)$$

$$\leq P(\epsilon e^{LT}\sup_{0 \leq t \leq T} |G_t| > \frac{1}{2}x_0e^{-LT})$$

$$= P\left(\sup_{0 \leq t \leq T} |G_t| > \frac{x_0}{2\epsilon}e^{-2LT}\right)$$

$$\leq \left(\frac{x_0}{2\epsilon}e^{-2LT}\right)^{-1}E\sup_{0 \leq t \leq T} |G_t|$$

$$\leq D\epsilon$$

for some positive constant $D$ by the assumption on the Gaussian process $G$. The upper bound obtained above and the fact that $|\theta(s)| \leq L, 0 \leq s \leq T$ leads an upper bound for the term $I_2$. We have used the inequality

$$x_t = x_0 \exp(\int_0^t \theta(s)ds) \geq x_0e^{-Lt}$$

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in the computations given above. Furthermore

\[
\begin{align*}
|E[I(A) \int_0^T K(\frac{t-s}{\varphi_\epsilon})X_s^{-1}dG_s] & = |E[\int_0^T K(\frac{t-s}{\varphi_\epsilon})X_s^{-1}I(A_s)dG_s] \\
& \leq C_9E[|\int_0^T |K(\frac{t-s}{\varphi_\epsilon})X_s^{-1}I(A_s)|dG_s| \\
& \leq C_{10}\epsilon^{LT}(\int_0^T |G(\frac{t-s}{\varphi_\epsilon})|dG_s) \\
& \leq C_{11}\varphi_\epsilon(R(t,t) + o(1))
\end{align*}
\]

for some positive constant \(C_{11}\) which leads to an upper bound on the term \(I_3\). Combining the above estimates, it follows that

\[
E|\tilde{\theta}(t) - \theta(t)| \leq C(\varphi_\epsilon^\rho + \epsilon + \epsilon)
\]

for some positive constant \(C\). Choosing \(\varphi_\epsilon = \epsilon^{1/\rho}\), we obtain that

\[
E|\tilde{\theta}(t) - \theta(t)| \leq C\epsilon
\]

for some positive constants \(C\). Hence we obtain the following result implying the uniform consistency of the estimator \(\tilde{\theta}(t)\) as an estimator of \(\theta(t)\) as \(\epsilon \to 0\).

**Theorem 5.1:** Let \(\theta \in \Theta_\rho(L)\) and \(\varphi_\epsilon = \epsilon^{1/\rho}\). Suppose the conditions \((A_1) - (A_3)\) hold. Then, for any interval \([c,d] \subset [0,T]\),

\[
\limsup_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_\rho(L)} \sup_{c \leq t \leq d} E|\tilde{\theta}(t) - \theta(t)|\epsilon^{-1} < \infty.
\]

**Funding:** This work was supported under the scheme “INSA Senior Scientist” by the Indian National Science Academy while the author was at the CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad 500046, India.

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CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India.

e-mail: blsprao@gmail.com