Receding Horizon Control in Deep Structured Teams: A Provably Tractable Large-Scale Approach with Application to Swarm Robotics

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Abstract—In this paper, a deep structured tracking problem is introduced for a large number of decision-makers. The problem is formulated as a linear quadratic deep structured team, where the decision-makers wish to track a global target cooperatively while considering their local targets. For the unconstrained setup, the gauge transformation technique is used to decompose the resultant optimization problem in order to obtain a low-dimensional optimal control strategy in terms of the local and global Riccati equations. For the constrained case, however, the feasible set is not necessarily decomposable by the gauge transformation. To overcome this hurdle, we propose a family of local and global receding horizon control problems, where a carefully constructed linear combination of their solutions provides a feasible solution for the original constrained problem. The salient property of the above solutions is that they are tractable with respect to the number of decision-makers and can be implemented in a distributed manner. In addition, the main results are generalized to cases with multiple sub-populations and multiple features, including leader-follower setup, cohesive cost function and soft structural constraint. Furthermore, a class of cyber-physical attacks is proposed in terms of perturbed influence factors. A numerical example is presented to demonstrate the efficacy of the results.

I. INTRODUCTION

Swarm tracking arises in many engineering applications such as robotics, smart grids and economics, where a group of decision-makers wish to track a target collectively. To solve the swarm tracking problem, one common practice is to propose a strategy based on the consensus algorithms, where the decision-makers are guaranteed to reach the target after a sufficiently large horizon [1]–[4]. Alternatively, one can define a cost function consisting of the tracking cost (penalizing the distance between every agent and the target) and the formation cost (penalizing the relative distances between the agents). Given a differentiable parametrized strategy, gradient decent methods can be utilized to search for a locally optimal solution [5]. On the other hand, it is difficult to find a scalable solution for large-scale swarms, in practice. This is because there is often a set of state and action constraints, leading to a non-trivial feasible set, such that any naive solution suffers from the curse of dimensionality with respect to the number of decision-makers.

To address the above shortcoming, we introduce deep structured tracking problem wherein a large number of decision-makers wish to track a global target while taking into account their local targets. The idea of deep structured tracking stems from a newly emergent class of large-scale decentralized control systems called deep structured teams [6]–[14].

In deep structured teams/games, decision-makers are coupled through a set of linear regressions of the states and actions of the decision-makers, which is similar in spirit to the coupling of neurons in feed-forward deep neural networks (DNN). For example, it is shown in this paper that a feed-forward DNN with rectified linear unit activation function may be viewed as a special case of deep structured teams, where neurons are agents with affine dynamics and affine constraints, and layers are time steps. In general, a key step to obtain a low-dimensional solution for the linear quadratic deep structured model is to decompose the optimization problem by a gauge transformation, initially proposed in [15] and showcased in risk-sensitive model [7], decentralized estimation [12], reinforcement learning [8], [10], [11], nonzero-sum game [9], minmax optimization [16], leader-follower tracking [17], [18], and mean-field teams [19], [20].

To consider state and action constraints, we use receding horizon control in this article as a popular industrial methodology, also known as model predictive control, rolling horizon planning, dynamic matrix control and dynamic linear programming [21]–[24]. In particular, we propose a family of two low-dimensional receding horizon control problems, where a carefully constructed linear combination of their solutions provides a feasible solution. In addition, we generalize our main results to include multiple sub-populations, multiple features and cyber-physical attacks. In contrast to the consensus-based algorithms, our approach is a control-based algorithm that is scalable with respect to the number of agents; see Subsection IV-B.2 for similarities and differences between consensus and (optimal) control algorithms.

The remainder of the paper is organized as follows. In Section II the problem is formulated and in Section III the main results are obtained. In Sections IV and V the main results are extended to multiple sub-populations, multiple features and cyber-physical attacks. A numerical example is presented in Section VI to verify the obtained theoretical results. In Section VII some conclusions are drawn.

II. PROBLEM FORMULATION

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ refer to the sets of real and natural numbers, respectively. Given any $n \in \mathbb{N}$, $\mathbb{N}_n$ is the finite set $\{1, 2, \ldots, n\}$. For any vector $x$ and square matrix
In this paper, the dynamics $\phi$ is a deterministic affine function, $I_i$ is the information set of agent $i$, and strategy $\pi$ is computed by a set of local and global Riccati equations and quadratic programmings for the unconstrained and constrained cases, respectively.

Fig. 1. The interaction (coupling) between agents in deep structured teams is similar in spirit to that of neurons in a deep feed-forward neural network.

Fig. 2. A feed-forward DNN with Rectified Linear Unit (ReLU) activation function may be viewed as a special case of deep structured teams, where neurons are agents, and layers are time steps. In particular, the dynamics of agents is a single integrator, which is an affine function, with affine constraints, wherein $W_i$ and $b_i$ represent the weight matrix and bias vector, respectively. An alternative formulation of DNN with ReLU function is where $A_i = 0$ and $x_{t+1} = u_t$, $u_t \geq 0$, $u_t = W_{i+1}[x_t, x_t^2, \ldots, x_t^n]' + b_i$.

where $m_i := \frac{1}{n} \alpha_i M$ and $M := \frac{1}{n} \sum_{i=1}^n m_i$. An important special case of the center of mass is where $\{\alpha_i = \frac{m_i}{m} \geq 0, \forall i \in \mathbb{N}_n\}$ is a convex combination of scalars.

At time $t \in \mathbb{N}$, the state of agent $i \in \mathbb{N}_n$ evolves as:

$$x_{t+1}^i = A_t x_t^i + B_t u_t^i,$$

where $A_t$ and $B_t$ are matrices with appropriate dimensions.

A. Cost function

Let $r_t^i \in \mathbb{R}^{d_r}$ denote the local reference of agent $i$ at time $t \in \mathbb{N}_T$ indicating the center of its safe zone and $s_t \in \mathbb{R}^{d_s}$ denote the global reference of the swarm determining the desired trajectory of the center of agents. To this end, we define a cost function with a common penalty function penalizing the mismatch between the center of swarm $\bar{x}_t^\alpha$ and the global reference $s_t$. More precisely, for any $i \in \mathbb{N}_n$ and $t \in \mathbb{N}_T$, the cost of agent $i \in \mathbb{N}_n$ is defined as:

$$c_t^i = \gamma_i (\|x_t^i - r_t^i\|_Q + \|u_t^i\|_{R_t}) + \|\bar{x}_t^\alpha - s_t\|_{Q_t} + \|u_t^\alpha\|_{R_t},$$

where $\gamma_i > 0$ denotes the importance of the local cost of agent $i$ and matrices $Q_t$, $R_t$, $Q_{t'}$, and $R_t$ are symmetric with appropriate dimensions. The first term forces agent $i$ to be close to its safe zone whose center is given by $r_t^i$ and the second one considers energy consumption of agent $i$. The third term incentivizes the center of the swarm to track the global target whereas the forth term (which can be set to zero, i.e. $R_t = 0$) smooths the trajectory of the center of the swarm by preferring small values for the deep action $u_t^\alpha$.

B. Problem statement

The agents are interested to collaborate to minimize a common cost function defined as:

$$J_n := \frac{1}{n} \sum_{t=1}^T \sum_{i=1}^n c_t^i(x_t^1, \ldots, x_t^n, u_t^1, \ldots, u_t^n).$$
Remark 1. Note that our main results hold for any setup in which the per-step cost in (4) can be represented as a summation of local cost functions (in terms of local states and local actions) and global cost functions (in terms of deep states and deep actions). Below, we present two such cases.

- Any weighted cross-terms in $c_i$ can be formulated as:
  $$\frac{1}{n} \sum_{i=1}^{n} \alpha_i (x_i^{\alpha})^T Q_i \bar{x}_i^{\alpha} = \|\bar{x}_i^{\alpha}\|_{Q_i}.$$

- Any weighted tracking cost can be expressed as:
  $$\frac{1}{n} \sum_{i=1}^{n} \alpha_i \|x_i^{\alpha} - F_i \bar{x}_i^{\alpha}\|_{Q_i} = \frac{1}{n} \sum_{i=1}^{n} \alpha_i \|x_i^{\alpha}\|_{Q_i} + \|\bar{x}_i^{\alpha}\|_{Q_i},$$

where $Q_i := (I - F_i)^T Q (I - F_i) - Q_i$.

Problem 1 (Optimal control). Find a scalable optimal strategy such that the team cost in (4) is minimized, i.e.,

$$J_n^* := \min_{u_1^{T}, u_2^{T}, \ldots, u_n^{T}} \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} c_i^{T}(x_i^{1}, \ldots, x_i^{n}, u_i^{1}, \ldots, u_i^{n}),$$

subject to:

$$x_i^{t+1} = A_t x_i^{t} + B_t u_i^{t}, \quad \forall i \in \mathbb{N}_n, \forall t \in \mathbb{N}_T.$$

Problem 2 (Receding horizon control (RHC)). Develop a scalable RHC for the following constrained optimization:

$$\min_{u_1^{T}, u_2^{T}, \ldots, u_n^{T}} \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} c_i^{T}(x_i^{1}, \ldots, x_i^{n}, u_i^{1}, \ldots, u_i^{n}),$$

subject to:

$$x_i^{t+1} = A_t x_i^{t} + B_t u_i^{t}, \quad \forall i \in \mathbb{N}_n, \forall t \in \mathbb{N}_T,$$
$$a \leq x_i^{t} \leq b, \quad a, b \in \mathbb{R}^{d_x},$$
$$c \leq u_i^{t} \leq d, \quad c, d \in \mathbb{R}^{d_u},$$
$$\bar{a} \leq \bar{x}_i^{t} \leq \bar{b}, \quad \bar{a}, \bar{b} \in \mathbb{R}^{d_x},$$
$$\bar{c} \leq \bar{u}_i^{t} \leq \bar{d}, \quad \bar{c}, \bar{d} \in \mathbb{R}^{d_u}.$$

For the special case of non-negative influence factors (e.g., convex combination), the effective lower and upper bounds imposed on the deep state and deep action in Problem 2 are $\max(\bar{a}, a)$, $\max(\bar{c}, c)$, $\min(\bar{a} b, b)$ and $\min(\bar{a} d, d)$.

C. Main challenges and contributions

The first challenge is the curse of dimensionality with respect to the number of agents, where the augmented matrices are fully dense. To overcome this challenge, we use a gauge transformation (i.e., a change of coordinates) to decompose the optimization problem in order to obtain a low-dimensional solution in terms of two scale-free Riccati equations. We show that the centralized solution can be implemented in a distributed manner wherein every agent needs access to only the deep state (rather than the entire joint state). The second challenge is that the feasible set of the constrained optimization problem (Problem 2) is not fully decomposable by the gauge transformation, which means that the solution of Problem 1 is not directly applicable in this case. To this end, we propose two scale-free RHCs under mild conditions for every agent. We show that a carefully constructed linear combination of the solutions of the proposed RHCs provides a feasible solution for Problem 2.

III. MAIN RESULTS FOR PROBLEMS 1 AND 2

In this section, we present the main results for Problems 1 and 2. Prior to delving into theoretical results, we define two types of tracking as follows.

Definition 2 (Strong and weak swarm tracking). When the center of swarm reduces to the center of mass, the swarm tracking is called strong; otherwise, it is called weak.

Proposition 1. Suppose that the tracking is weak and the center of swarm is not at the origin. Then, there is at least one agent that does not converge to the center of swarm.

Proof. The proof follows from contradiction. Suppose all agents converge to the center of swarm at some time $t \in \mathbb{N}_n$, i.e. $x_i^t = \bar{x}_i^t$, $\forall i \in \mathbb{N}_n$. From (1), one has $\frac{1}{n} \sum_{t=1}^{n} \alpha_i \bar{x}_i^t = \bar{x}_i^t$, which holds if and only if $\bar{x}_i^t = 0$.

In general, weak tracking arises in various situations wherein the center of swarm is not properly balanced. This unbalanced property may be caused by an external force (e.g., cyber-physical attack) or by the designer (e.g., when agents wish to monitor a target without getting close to it).

A. Gauge transformation and Riccati equation

The first step to solve a linear quadratic deep structured team is to use a gauge transformation, initially introduced in [15] and showcased in [8], to define auxiliary variables as the deviation of the local variables from deep (weighted) variables. We use the following gauge transformation:

$$\Delta x_i^t := x_i^t - \frac{\alpha_i}{\gamma_i} \bar{x}_i^t, \quad \Delta u_i^t := u_i^t - \frac{\alpha_i}{\gamma_i} \bar{u}_i^t, \quad \Delta u_i^t := r_i^t - \frac{\alpha_i}{\gamma_i} \bar{r}_i^t,$$

where $\bar{r}_i^t := \frac{1}{n} \sum_{i=1}^{n} \alpha_i r_i^t$. From (3), one has

$$\Delta x_i^{t+1} = A_t \Delta x_i^t + B_t \Delta u_i^t,$$

and

$$\bar{x}_i^{t+1} = A_t \bar{x}_i^t + B_t \bar{u}_i^t.$$

Let $\mu := \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i^2}{\gamma_i}$.

Lemma 1. The per-step cost function in equation (4) at any time $t \in \mathbb{N}$ can be written as:

$$\frac{1}{n} \sum_{i=1}^{n} \gamma_i (\|\Delta x_i^t - \Delta r_i^t\|_{Q_i} + \|\Delta u_i^t\|_{R_i}) + \|\bar{x}_i^t - s_t\|_{\tilde{Q}_t} + (2 - \mu) \|\bar{x}_i^t - \bar{r}_i^t\|_{Q_i} + \|\bar{u}_i^t\|_{R_i} + (2 - \mu) \|\bar{u}_i^t\|_{R_i}.$$

Proof. The proof follows directly from (3), the gauge transformation (5) and the fact that

$$\frac{1}{n} \sum_{i=1}^{n} \alpha_i (\Delta x_i^t - \Delta r_i^t)^T Q_i (\bar{x}_i^t - \bar{r}_i^t) = (1 - \mu) \|\bar{x}_i^t - \bar{r}_i^t\|_{Q_i},$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \alpha_i (\Delta u_i^t)^T R_i (\bar{u}_i^t) = (1 - \mu) \|\bar{u}_i^t\|_{R_i}.$$

■
Denote $Q_t := (2 - \mu)Q_t + \tilde{Q}_t$, and $R_t := (2 - \mu)R_t + \tilde{R}_t$.

**Assumption 1.** At any time $t \in \mathbb{N}$, $Q_t$ and $Q_t$ are positive semi-definite and $R_t$ and $R_t$ are positive definite.

Define local and global Riccati equations as follows:

$$
P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (B_t^\top P_{t+1} B_t + R_t)^{-1} \times B_t^\top P_{t+1} A_t, \forall t \in \mathbb{N}_{T-1},
$$

$$
P_t = Q_t + A_t^\top P_{t+1} A_t - A_t^\top P_{t+1} B_t (B_t^\top P_{t+1} B_t + R_t)^{-1} \times B_t^\top P_{t+1} A_t, \forall t \in \mathbb{N}_{T-1},
$$

with $P_T = Q_T$ and $P_T = Q_T$.

**Theorem 1.** Let Assumption 1 hold. The optimal strategy of agent $i \in \mathbb{N}_n$ at any time $t \in \mathbb{N}_{T-1}$ is given by:

$$
u_t^a := \theta_t^a x_t^a + \frac{\alpha_t}{\gamma_t} (\theta_t^a - \theta_t^a) \bar{v}_t^a + \tilde{L}_t \bar{v}_t^{a+1},$$

where gain matrices $\{\theta_t^a, \theta_t^a, L_t, \bar{L}_t\}$ and correction signals $\{\{v_t^a := \tilde{v}_t^{a+1}, \bar{v}_t\}$ are obtained from the solution of the local and global Riccati equations (8) for any $t \in \mathbb{N}_{T-1}$.

$$
\begin{align*}
\theta_t^a &:= - (B_t^\top P_{t+1} B_t + R_t)^{-1} B_t^\top P_{t+1} A_t, \\
L_t &:= (B_t^\top P_{t+1} B_t + R_t)^{-1} B_t^\top P_{t+1} A_t, \\
\tilde{L}_t &:= (B_t^\top P_{t+1} B_t + R_t)^{-1} B_t^\top P_{t+1} A_t, \\
\bar{L}_t &:= (B_t^\top P_{t+1} B_t + R_t)^{-1} B_t^\top P_{t+1} A_t,
\end{align*}
$$

and

$$
\begin{align*}
\bar{v}_t &:= (A_t + B_t \theta_t^a)^\top \bar{v}_t^{a+1} + Q_t \Delta_t^{a+1}, & i \in \mathbb{N}_n, \\
\bar{v}_t &:= Q_T \Delta_T^{a+1}, & i \in \mathbb{N}_n, \\
v_t^a &:= (A_t + B_t \theta_t^a)^\top \bar{v}_t^{a+1} + (2 - \mu) Q_t \bar{v}_t^{a+1} + \tilde{Q}_t s_t, \\
\bar{v}_t &:= (2 - \mu) Q_T \bar{v}_t^{a+1} + Q_T s_T.
\end{align*}
$$

**Proof.** The proof follows from equations (6) and (7) and Lemma 1 where the optimization in Problem 1 can be decomposed to $n + 1$ smaller optimizations. More precisely, there is a local linear quadratic regulator (LQR) for every $i \in \mathbb{N}_n$ with state and action $\{\Delta_t, \Delta_t\}$ and tracking signal $\{\Delta_t\}$. Since $\gamma_i > 0$, it does not affect the optimization problem. Therefore, one has for any $i \in \mathbb{N}_n$: $\Delta_t^a = \theta_t^a \Delta_t^a + L_t \bar{v}_t^{a+1}$ and $s_t$, where $\bar{v}_t^a = \tilde{\bar{v}}_t^a + L_t \bar{v}_t^{a+1}$. From gauge transformation (5), it results that:

$$
u_t^a = \Delta_t^a \bar{v}_t^a + \frac{\alpha_t}{\gamma_t} \bar{v}_t^a, & i \in \mathbb{N}_n.
$$

**B. Receding horizon control**

A naive way to solve the centralized RHC in Problem 2 leads to a large-scale optimization problem that is intractable with respect to the number of agents. In addition, the centralized RHC does not necessarily decompose into scalable problems after the gauge transformation. This is in contrast to the unconstrained model wherein the centralized solution coincides with two scalable optimal control problems. To overcome this hurdle, we propose two scalable RHC problems whose feasible sets are a subset of the feasible set of the centralized RHC problem. In particular, to distinguish

between the state and action of the proposed RHC problems and those of the original Problem 2, we use notations $y$ and $u$ instead of $x$ and $u$, respectively. Define one local and one global RHC problem as follows.

**Problem 3 (Local RHC).** For any agent $i \in \mathbb{N}_n$, and horizon $H \in \mathbb{N}$, find a solution for the following minimization:

$$
\min_{\Delta_t, \Delta_t} \sum_{t=1}^{t+H} \|\Delta_t \|_{Q_t} + \|\Delta_t \|_{R_t},
$$

s.t. $\Delta_t^a = A_t \Delta_t + B_t \Delta_t^a$, $\tau \in \{t, \ldots, t + H - 1\}$,

$$
\bar{a}_t \leq \Delta_t^a \leq \bar{b}_t, \quad \bar{a}_t, \bar{b}_t \in \mathbb{R}^{d_x},
$$

$$
\bar{c}_t \leq \Delta_t^a \leq \bar{d}_t, \quad \bar{c}_t, \bar{d}_t \in \mathbb{R}^{d_u}.
$$

**Problem 4 (Global RHC).** Given any prediction horizon $H \in \mathbb{N}$, find a solution for the following minimization:

$$
\min_{\Delta_t, \Delta_t} \sum_{t=1}^{t+H} \|\Delta_t^a + \|\Delta_t^a - s_t \|_{Q_t} + \|\Delta_t^a \|_{R_t},
$$

s.t. $\Delta_t^a = A_t \Delta_t + B_t \Delta_t^a$, $\tau \in \{t, \ldots, t + H - 1\}$,

$$
\bar{a}_t \leq \Delta_t^a \leq \bar{b}_t, \quad \bar{a}_t, \bar{b}_t \in \mathbb{R}^{d_x},
$$

$$
\bar{c}_t \leq \Delta_t^a \leq \bar{d}_t, \quad \bar{c}_t, \bar{d}_t \in \mathbb{R}^{d_u}.
$$

**Remarked 2.** At any time $t \in \mathbb{N}_T$, one can solve the above open-loop control problems by quadratic programming. Notice that the feasible set of the proposed RHC Problems 3 and 4 is not necessary equal to that of the RHC Problem 2.

Now, we introduce a family of bounds for Problems 3 and 4 such that their solution is valid for Problem 2.

**Assumption 2.** Let $\alpha_t \in (0, 1], \alpha_t \geq \gamma_t, \forall t \in \mathbb{N}_n$. Let also $a, b, \bar{a} > 0$, $a, \bar{a} < 0$, $d, \bar{d} > 0$, and $d, \bar{d} < 0$.

**Theorem 2.** Let Assumptions 1 and 2 hold. For any $\lambda \in (0, 1)$ and $i \in \mathbb{N}_n$, suppose that the boundaries of the local and global RHC Problems 3 and 4 are given by:

$$
\bar{a}_t := \frac{\lambda}{1 - \lambda} \bar{a}_t, \quad \bar{b}_t := \frac{\lambda}{1 - \lambda} \bar{b}_t,
$$

$$
\bar{c}_t := \frac{\lambda}{1 - \lambda} \bar{c}_t, \quad \bar{d}_t := \frac{1 - \lambda}{1 - \lambda} \bar{d}_t,
$$

$$
\bar{a} := (1 - \lambda) \max(\bar{a} \bar{a}, \bar{a}), \quad \bar{b} := (1 - \lambda) \min(\bar{a} \bar{b}, \bar{a}),
$$

$$
\bar{c} := (1 - \lambda) \max(\bar{a} \bar{c}, \bar{a}), \quad \bar{d} := (1 - \lambda) \min(\bar{a} \bar{d}, \bar{a}).
$$

Then, at any time $t \in \mathbb{N}_T$, the following linear combination:

$$
u_t^a = \Delta_t^a \bar{v}_t^a + \frac{\alpha_t}{\gamma_t} \bar{v}_t^a, \quad i \in \mathbb{N}_n,
$$

is a feasible solution for Problem 2.

**Proof.** In the first step, we show that the above limits construct a non-empty set. To avoid repetition, we only prove the results for those constraints imposed on state spaces because similar arguments hold for action spaces. Since $1 - \lambda > 0, 0 < \bar{a} \leq 1, \max(\bar{a} \bar{a}, \bar{a}) < 0$, and $\min(\bar{a} \bar{b}, \bar{a}) > 0$, one can conclude that for every $i \in \mathbb{N}_n$,

$$
a < 0 < b \quad \text{and} \quad \bar{a}_t := \frac{\lambda}{1 - \lambda} a < \bar{a}_t < \frac{\lambda}{1 - \lambda} b = \bar{b}_t.$$
In the second step, we show that the above limits present a feasible set for Problem 2. By definition, for any $i \in \mathbb{N}_n$ at time $t \in \mathbb{N}$:

$$\tilde{a}_i \leq \Delta y^i_t \leq \tilde{b}_i \text{ and } \tilde{a}_i \leq \tilde{y}^i_t \leq b_i.$$  

(9)

Therefore, one arrives at:

$$a_i + \frac{\alpha_i}{\gamma_i} a \leq y^i_t := \Delta y^i_t + \frac{\alpha_i}{\gamma_i} \tilde{y}^i_t \leq \tilde{b}_i + \frac{\alpha_i}{\gamma_i} b_i.$$  

(10)

where the left-hand side of (10) is lower-bounded by

$$(\frac{\lambda}{1-\lambda}) a + \frac{\alpha_i}{\gamma_i} a \geq (\frac{\lambda}{1-\lambda} + 1) a = \max(\tilde{a}a, \tilde{a}) \geq \tilde{a} a \geq a$$

and the right-hand side of (10) is upper-bounded by

$$(\frac{\lambda}{1-\lambda}) b + \frac{\alpha_i}{\gamma_i} b \leq (\frac{\lambda}{1-\lambda} + 1) b = \min(\tilde{a} b, \tilde{b}) \leq \tilde{a} b \leq b.$$  

As a result, one can conclude that the lower and upper bounds on local states in Problem 2 are satisfied, i.e., $a \leq y^i_t \leq b$. In addition, it is straightforward to show that the lower and upper bounds on the deep state in Problem 2 is satisfied, where

$$\tilde{a} \leq \max(\tilde{a}a, \tilde{a}) \leq \frac{\lambda}{1-\lambda} \tilde{a} + a \leq \frac{1}{n} \sum_{i=1}^{n} \alpha_i y^i_t =$$

$$\frac{1}{n} \sum_{i=1}^{n} \alpha_i \Delta y^i_t + \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 \frac{\gamma_i}{\gamma_i} \tilde{y}^i_t \leq \frac{\lambda}{1-\lambda} b + \tilde{b} \leq \min(\tilde{a} b, \tilde{b}).$$

Thus, solution of Problems 3-4 is feasible for Problem 2. □

**Remark 3.** Consider a special case when influence factors are a convex combination, i.e., $\alpha_i \in (0, 1], \forall i \in \mathbb{N}_n$, and $\tilde{\alpha} = 1$ such that $\alpha_i \leq \gamma_i$, $\lambda = \frac{1}{2}$, $a = \tilde{a} = -b = -\tilde{b}$ and $c = \bar{c} = -d = -\bar{d}$. From Theorem 2 one can show that the following bounds provide a feasible solution: $\tilde{a}_i = -\tilde{b}_i = \tilde{a} = -\tilde{b} = \frac{1}{2} \tilde{a}$ and $\bar{c}_i = \bar{d}_i = \bar{c} = \bar{d} = \frac{1}{2} c, \forall i \in \mathbb{N}_n$.

In contrast to Theorem 2 that only holds for positive factors, we present a new theorem with more conservative bounds including both negative and positive factors. Define $m_x := \min(b, \tilde{b})$ if $\min(b, \tilde{b}) + \max(a, \tilde{a}) < 0$ and $m_x := -\max(a, \tilde{a})$ if $\min(b, \tilde{b}) + \max(a, \tilde{a}) > 0$. Similarly, define $m_a := \min(d, \tilde{d})$ if $\min(d, \tilde{d}) + \max(c, \bar{c}) < 0$ and $m_a := -\max(c, \bar{c})$ if $\min(d, \tilde{d}) + \max(c, \bar{c}) > 0$.

**Assumption 3.** Let $\alpha_i \in [-1, 1]$ and $\alpha_i \leq \gamma_i$, $\forall i \in \mathbb{N}_n$. Let also $b, \tilde{b} > 0, a, \tilde{a} > 0$ and $d, \tilde{d} < 0$.

**Theorem 3.** Let Assumptions 2 and 3 hold. For any $\lambda \in (0, 1)$ and $i \in \mathbb{N}_n$, suppose that the boundaries of the local and global RHC Problems 3 and 4 are given by:

$$-\tilde{a}_i := \tilde{b}_i := \lambda m_x, \quad -\bar{c}_i := \bar{d}_i := \lambda m_a,$$

$$-a := b := (1-\lambda) m_x, \quad -c := d := (1-\lambda) m_a.$$  

Then, at any time $t \in \mathbb{N}_T$, the following linear combination:

$$u^i_t = \Delta u^i_t + \frac{\alpha_i}{\gamma_i} \tilde{r}^i_t, \quad i \in \mathbb{N}_n,$$

is a feasible solution for Problem 2.

**Proof.** The proof proceeds along the same lines as the proof of Theorem 2. In the first step, we show that the above limits construct a non-empty set, i.e., from the definition of $m_x$,

$$a \leq -m_x, \tilde{a} \leq -m_x, m_x \leq b, m_x < \tilde{b},$$

where for any $i \in \mathbb{N}_n$, $\tilde{a}_i = -\lambda m_x < 0, \tilde{b}_i = \lambda b, \tilde{a} = -\tilde{b}$ and $\tilde{a} = -\tilde{b}$. The left-hand side of inequality (11) is lower-bounded as follows:

$$a \leq -m_x = -\lambda m_x + (1-\lambda) m_x \leq \tilde{a} + \frac{\alpha_i}{\gamma_i} b_i,$$

and its right-hand side is upper bounded as:

$$b \geq m_x = \lambda m_x + (1-\lambda) m_x \geq \tilde{b} + \frac{\alpha_i}{\gamma_i} b_i.$$  

Thus, one has $a \leq \Delta y^i_t + \frac{\alpha_i}{\gamma_i} \tilde{y}^i_t \leq b$. From Theorem 3 one can show that the following bounds provide a feasible solution: $\tilde{a}_i = -\tilde{b}_i = a = \tilde{b} = -\tilde{b} = \frac{1}{2} \tilde{b}$ and $c = \bar{c} = -d = -\bar{d}$. From Theorem 2 one can show that the following bounds provide a feasible solution: $\tilde{a}_i = -\tilde{b}_i = \tilde{a} = -\tilde{b} = \frac{1}{2} \tilde{a}$ and $c = \bar{c} = -d = -\bar{d} = \frac{1}{2} c, \forall i \in \mathbb{N}_n$.

**Remark 4.** When the optimal solution in Theorem 1 lies in the feasible set of the proposed distributed RHC, the RHC solution for a sufficiently large prediction horizon $H = T$ can be explicitly obtained by Ricatti equations 5.

**C. Distributed and decentralized implementations.**

The obtained LQR and RHC solutions can be implemented in a distributed manner, where each agent solves two low-dimensional Ricatti equations and quadratic programings, respectively, and compute its action based on local (private) information $\{x^i_t, r^i_t, \alpha_i, \gamma_i\}$ and global (public) information $\{\tilde{x}^i_t, \tilde{r}^i_t, \tilde{s}_i, \mu\}$.

1. **Stochastic model & certainty equivalence:** Suppose that the dynamics (2) have additive noises such that $x^i_{t+1} = A_i x^i_t + B_i u^i_t + w^i_t, \forall i \in \mathbb{N}_n$, where $\{w^i_t\}_{1:T}$ is an independent stochastic process. This generalization does not affect the solution of Problem 1 because of the certainty equivalence theorem. In a such case, there is no loss of optimality in replacing the noises with their expectations. For Problem 2, however, certainty equivalence theorem does not hold. Nonetheless, one can use the certainty equivalence approximation (where the noise is replaced by its expectation) to convert the stochastic dynamics to deterministic ones and establish recursive feasibility [25]-[27]. When it comes to
TABLE I
DIFFERENCES BETWEEN CONSENSUS AND OPTIMAL CONTROL

| Consensus and distributed averaging | Optimal control |
|------------------------------------|-----------------|
| **Objective**                      | Agents allocate resources efficiently during horizon $T$, (not necessarily large $T$), where one may have $x^i_{T} \neq x^j_{T} \neq x^i_{T}$, $\forall i,j$. |
| **Model**                          | Agents reach identical value after a sufficiently large horizon i.e. $x^i_{\infty} = x^j_{\infty} = x^i_{\infty}$, $\forall i,j$. |
| **Information**                    | Dissemination of information is via many local interactions (not suitable for costly communications). |
| **Solution approach**              | Dissemination of information is via one-shot cloud-based server (not suitable for hard constrained communication graph). |

A distributed implementation of the stochastic model, each agent at every time $t$ only requires to observe the deep state $x^i_t \in \mathbb{R}^{d_x}$ (whose size is independent of the number of agents unlike the centralized joint state $(x^1_t, \ldots, x^n_t)$).

2) **Two-time-scale distributed (consensus-based) solution:** Suppose that the agents’ communication graph does not allow the immediate observation of the deep state. In this case, one can use a two-time scale distributed optimization strategy. At each time instant $t$, agents run a consensus algorithm to compute the deep state after a sufficiently large number of iterations. Such a two-time-scale distributed implementation is practical in many control applications, especially where the communication (information) process is significantly faster than the physical (control) process.

3) **Fully decentralized information in asymptotic model:** It is not always feasible to share the deep state among agents especially when the number of agents is very large. In such a case, the deep state can be predicted (rather than communicated) by the infinite-population approximation because the dynamics of the infinite-population model is deterministic due to the strong law of large numbers. See for example [12], [18], where the predicted case (sub-optimal solution) converges to the communicated case (optimal solution) at the rate $1/n$. This leads to a fully decentralized control structure, where each agent needs to observe only its local information.

IV. MULTIPLE SUB-POPULATIONS AND FEATURES

So far, we have assumed that the matrices in dynamics (2) and cost function (3) are identical for all agents (i.e., one population) and the agents are coupled through one set of factors (i.e., one feature). In this section, we briefly discuss the generalization of our main results to cases with multiple sub-populations and multiple features.

A. Multiple sub-populations

Consider a population consisting of $S \in \mathbb{N}$ sub-populations, where matrices in dynamics and cost functions of agents in each sub-population are identical. In such a case, the unconstrained optimization problem gets decomposed into $S + 1$ smaller LQR problems [7]. Similarly, one can propose $S + 1$ distributed RHC problems. See [17], [18] for an example with two sub-populations where one sub-population contains one leader and another sub-population a large number of followers.

B. Multiple features

Consider a population where agents are coupled through $f \in \mathbb{N}$ sets of influence factors (features). For example, any directed weighted graph can be decomposed by singular value decomposition and any undirected weighted graph by spectral decomposition, respectively, where features represent the singular vectors and eigenvectors. In such a case, every feature may be viewed as a virtual sub-population; hence, the unconstrained optimization problem decomposes into $f + 1$ LQR problems [7]. Analogously, one can construct $f + 1$ distributed RHC problems. In what follows, we present two problems with more than one set of features.

1) **Cohesive cost function:** It is possible to add a cost function to (3) in order to incorporate the cohesiveness of the swarm where the team cost becomes $\beta \hat{c}_t + (1 - \beta)c^L_t$, $\beta \in [0,1]$ and the cohesive cost $c^L_t$ is a quadratic function of the relative distances, i.e.

$$c^L_t := x^T_t L x_t = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} (x^i_t - x^k_t)^T L^{i,k} (x^i_t - x^k_t),$$

where $L$ is not necessarily a symmetric matrix. However, a special case of the cohesive cost function is Laplacian, i.e.

$$c^L_t = x^T_t L x_t := \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} A(i,k) (x^i_t - x^k_t)^T (x^i_t - x^k_t) = \sum_{(i,k) \in \mathcal{E}} (x^i_t - x^k_t)^T (x^i_t - x^k_t),$$

where $A$ and $L$ are the adjacency and Laplacian (symmetric) matrices of an undirected weighted graph and $\mathcal{E}$ is the edge set. The same analogy holds for a more general cost function

$$c^L_t := x^T_t \text{diag} (\alpha_1, \ldots, \alpha_n)^{-1} L x_t,$$

resulting in consensus to a weighted average [1]–[3], [28]. In addition, one can use decomposition methods such as

$$x^T_t L x_t := \sum_{j=1}^{n} \sigma_j x^T_j U_j V^T_j x_t, \quad (\text{singular value decomposition})$$

$$x^T_t L x_t := \sum_{j=1}^{n} \lambda_j^2 x^T_j V^T_j V_j x_t, \quad (\text{spectral decomposition})$$

to decompose (12) and restrict attention to a few dominant features associated with the largest singular values and eigenvalues, respectively. Although consensus and optimal control are two different problems (see Table I for a few...
differences), they are related in some sense. In particular, the consensus problem may be formulated as a linear time-invariant system with integrators and an infinite-horizon time-average quadratic cost function. In such a case, the consensus strategy makes all the relative distances as well as any tracking distance from the consensus value go to zero [29]. On the other hand, the optimal control strategy may be viewed as a solution to the problem of finding the best topology for the communication graph with quadratic similarity index; see [17, Corollary 1], for example.

2) Soft structural constraint: It is possible to add a soft-constraint term to \(3\) in order to take into account the structure of the control strategy, where the hard constraint \(u_i = Hz_t\) is replaced by the quadratic soft constraint \((u_i - Hz_t)^\top(u_i - Hz_t)\). Analogously, one can consider the structure of dynamics, where \(x_{t+1} = Sx_t\) is replaced by \((x_{t+1} - Sx_t)^\top(x_{t+1} - Sx_t)\). Therefore, one can use the singular value decomposition and spectral theorem to generate dominant features associated with the above quadratic cost functions, similar to those in the cohesive cost function.

V. CYBER-PHYSICAL ATTACKS

In this section, we propose a new class of cyber-physical attacks formulated as perturbed influence factors. Let \(z_i \in \mathbb{R}\) be the status of the attack associated with agent \(i \in \mathbb{N}_n\). Let \(\alpha_i := \alpha_i(z_i)\) denote the attack factor, which is a function of \(z_i\). Depending on the attack function, we can define different types of attacks. Below, we mention a few cases.

- **Denial of service.** This is when \(z_i = 0\), if agent \(i\) is attacked, and \(z_i = 1\), if not attacked, where \(\alpha_i = \alpha_i(z_i)\).
- **Leader attack.** This is when one (leading) agent is targeted, i.e., the one with the largest influence factor.

We can also define various defence mechanisms as follows.

- **Isolated mechanism.** In this case, agent \(i\) is dispensable; hence, it gets isolated by choosing a relatively small value (i.e., close to zero) for factor \(\alpha_i\). Subsequently, agent \(i\) has a negligible effect in the center of the swarm and will be ignored by the swarm.
- **Protected mechanism.** In this scenario, agent \(i\) is important; hence, it gets protected by other agents via making its factor \(\alpha_i\) considerably larger. As a result, agent \(i\) would have a significant effect in the center of the swarm. In particular, the larger \(\alpha_i\), the closer the center of the swarm is to agent \(i\). This mechanism is helpful for situations in which other agents must cover the attacked agent by moving to its vicinity.

**Remark 5.** In practice, one can extend the above setup to time-varying attacks by developing a two-time-scale framework, where attacks are occurred in the slower scale and the RHC (or MPC) is deployed in the faster scale.

VI. SIMULATIONS

**Example 1.** Consider a group of robots that are interested to move towards a target collectively. Let the influence factors \(\alpha_i \geq 0, i \in \mathbb{N}_n\), construct a center of mass i.e. \(\bar{\alpha} = 1\); hence, tracking is strong according to Definition 2.

In our simulations, control horizon is \(T = 100\) and number of robots is \(n = 100\). Let the dynamics of the robots be linearised such that \(A = B = (\text{diag}(1, 1))\), and their team cost function be defined as follows:

\[
\frac{1}{n} \sum_{i=1}^{T} \sum_{i=1}^{n} \alpha_i(\|x_i^t - \bar{x}_i^0\|_Q + \|u_i^t\|_R + \|\bar{x}_i^0 - s\|_Q),
\]

where \(Q = \text{diag}(5, 50), \bar{Q} = \text{diag}(1, 1)\) and \(R = \text{diag}(100, 100)\). In addition, we consider a case in which one robot is physically attacked and other \((n - 1)\) robots follow a protected mechanism to cover it, as described in Section V. Let \(z_i = 1\) denote that agent \(i\) is attacked and \(z_i = 0\) denote that it is not, \(i \in \mathbb{N}_n\). In this case, the perturbed influence factor of robot \(i \in \mathbb{N}_n\) can be defined as:

\[
\bar{\alpha}_i := n \rho z_i + \frac{n}{n-1}(1 - \rho)(1 - z_i),
\]

where \(\frac{1}{n} \sum_{i=1}^{n} \bar{\alpha}_i = 1\) and \(\rho \in [0, 1]\) determines the level of protection. The larger \(\rho\), the closer the center of mass is to the attacked (targeted) robot, providing more protection.
The results of our simulations are depicted in Figures 3 and 4, where we display only 30 out of 100 robots to ease the exposition. In these figures, the blue dotted line is the trajectory of the center of mass and the red dashed line is that of the attacked robot. In particular, it is shown in Figure 4 that the robots can collectively reach the target \( s = (2, 2) \) in the normal case (where influence factors are homogeneous \( \alpha_i = 1 \)) as well as the attacked case (where the perturbed influence factors are calculated for \( \rho = 0.9 \)). Furthermore, we consider a similar setting wherein control signals are bounded such that \( |x_i^a| < 0.2, \forall i \). To solve the resultant problem, we use quadratic programming to find a solution for the proposed local and global RHCs, where \( H = 10 \) and \( \lambda = 0.5 \). It is demonstrated in Figure 4 that the robots can collaboratively reach the target while respecting their control constraints.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

We introduced deep structured tracking for a large number of decision-makers, where the interaction between them is modelled by influence factors. The influence factors can represent physical features and constraints (e.g., the center of swarm) as well as non-physical ones (e.g., adhesive behaviour of the swarm). For the unconstrained and constrained cases, two low-dimensional solutions were proposed. On the one hand, the unconstrained solution was shown to be optimal, obtained by solving two scale-free Riccati equations, where its extension to the infinite-horizon cost function is straightforward. On the other hand, the constrained solution took affine constraints into account, where establishing its stability is difficult due to the time-varying nature of the solution. In addition, the main results were generalized to multiple sub-populations and multiple features.

There are several possible future directions. For example, one can consider (a) different forms of dynamics (e.g. aerial and ground vehicles) and cyber-physical attacks (e.g. time-varying attacks with two-time-scale framework, denial of service, or minmax optimization with adversarial player); (b) output feedback, \( H_2 \) and \( H_\infty \) control algorithms; (c) a more general model with non-symmetric weighting matrices as well as non-quadratic and non-convex cost functions using interior-point methods; (d) constrained reinforcement learning and data-driven approaches, and (e) the investigation of the optimal feasible set (e.g., best \( \lambda \) in Theorems 2 and 3).

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