Tunable nonlinear band gaps in a sandwich-like meta-plate

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Abstract This paper aims to explore the working mechanism of sandwich-like meta-plates by periodically attaching nonlinear mass-beam-spring resonators for low-frequency wave attenuation. The nonlinear MBS resonator consists of a mass, a cantilever beam, and a spring to provide negative stiffness in the transverse vibration of the resonator; this stiffness is tunable by changing the parameters of the spring. Considering the nonlinear stiffness of the resonator, the energy method is applied to obtain the dispersion relation of the sandwich-like meta-plate, and the band-gap bounds related to the amplitude of the resonator are derived by dispersion analysis. For a finite-sized sandwich-like meta-plate with a fully free boundary condition subjected to external excitations, its dynamic equation is established by the Galerkin method. The frequency response analysis of the meta-plate is carried out by numerical simulation, with the band-gap range obtained in good agreement with that of the theoretical one. Results show that the band-gap range of the present meta-plate is tunable by designing the structural parameters of the MBS resonator. Furthermore, by analyzing the vibration suppression of the finite-sized meta-plate, it is observed that the nonlinearity of the resonators can widen the wave attenuation range of the meta-plate.

Keywords Meta-plate · Nonlinear band gaps · Wave transmittance · MBS resonator

1 Introduction

Over the past decades, phononic crystals (a multiphase dielectric material ordered with symmetries resembling those of atomic-scale crystalline materials) have attracted considerable attention due to their scattering mechanism (Bragg scattering) [1–5]. Bragg scattering is a phenomenon whereby wave scattering at the periodic lattice interface creates frequency bands that stop the propagation of phonons/photons. On this basis, acoustic metamaterials, which exhibit unique dynamic behaviors at long wavelengths, were proposed and became a research hotspot because of their range of potential applications in acoustic absorption [6, 7] and acoustic/elastic cloaking [8, 9], and as band gaps [10, 11]. Generally, acoustic metamaterials with locally resonant elements are considered a smart design of composite structures to obtain subwavelength band gaps [12–14].

Compared to phononic crystals devised to achieve wave manipulation by Bragg scattering, locally resonant metamaterials comprised of unit cells with subwavelength dimensions can guide and control wave propagation by the locally resonant properties [13, 15–17], which do not rely on the design

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periodicity. In locally resonant metamaterials, the band gaps that produce low-frequency wave attenuation are mainly related to the resonant frequency of the resonators, which is influenced by the inner attached mass and the restoring force [18]. The wave transmission properties of metamaterials are commonly independent of the constituent materials, but dependent on the coupling characteristics between the artificial microstructure units and elastic waves [10, 19, 20]. Therefore, the generation and adjustability of locally resonant band gaps are attracting increasing interest. Furthermore, various methods, including the PWE method [21, 22], the harmonic balance method [23, 24], and the finite element method [21, 25, 26], have been applied to calculate the band gap and wave attenuation of locally resonant metamaterials. In addition, numerical simulation [15, 16, 21, 27] and experiments [10, 20, 28] have been utilized to reveal the wave propagation characteristics of locally resonant metamaterials.

For easy clarification of the problem, linear local resonators [21, 23, 29–32] are always applied to analyze the band-gap properties of structures. Nevertheless, in such a linear design, the resulting band-gap range is limited by the requirement of lightweight and compact designs and application to large-amplitude excitation. Due to the added mass constraints, it is especially difficult for linear meta-plates to achieve both a low frequency and a wide band gap for vibration suppression, and the linear resonators increase the number of resonance peaks in the passband of the finite structure. These factors limit the application of locally resonant meta-plates in the new devices and new structures.

On the other hand, to realize greater application of metamaterials, the nonlinearity of the structure must be analyzed. Nonlinear metamaterials are artificially periodic structures with nonlinear dynamic properties [33, 34], such as nonlinear resonance, internal resonance, modal coupling, and chaos, which are expected to provide more control mechanisms for the nonlinear elastic wave. Currently, metamaterial nonlinearity attracts increasing attention [35–40]. In the context of nonlinear metamaterials, several investigations follow the premise of elastic wave propagation analysis in nonlinear media using the methods of transfer matrix [41, 42], perturbation analysis [43–45], and multiple scales [46–48]. Narisetti et al. [49] investigated wave propagation in one-dimensional nonlinear lattice materials. Some studies have also focused on the design of nonlinear resonators in elastic metamaterials. Wang et al. [50] designed a metamaterial plate with high-static–low-dynamic stiffness resonators and explored the nonlinear frequency response of the meta-plate under excitation with different amplitudes. Fang et al. [51] presented the chaotic band gaps of nonlinear acoustic metamaterials with two nonlinearly coupled resonances, achieving ultra-low- and ultrabroadband wave suppressions. Campana et al. [52] described the wave propagation in a periodic structure with local Duffing nonlinear resonators. Emerson and Manimala [53] utilized the tunability of the amplitude-dependent response of a nonlinear local resonator to manipulate the direction and bandwidth of wave propagation in a metamaterial. Based on the chaotic vibrations of bistable resonators under large-amplitude excitation, Xia et al. [54] investigated the nonlinear wideband attenuation in a metamaterial beam, whose validity was also demonstrated by an experiment. These investigations shed light on the design of nonlinear metamaterials to adjust the band-gap range. However, to the best of our knowledge, the mathematical formulations and numerical simulations on the wave propagation characteristics of a sandwich-like meta-plate with periodically attached nonlinear resonators are rare.

Our previous work [55] investigated the wave propagation properties of a meta-plate with linear resonators (the mass-beam type). The present work aims to study the influence of a nonlinear resonator (the mass-beam-spring type) on the band-gap characteristics of a sandwich-like meta-plate. The stiffness of the MBS resonator is tunable by changing the compressibility of the spring at the equilibrium position. Considering the nonlinear stiffness of the resonator and using the energy method, the band-gap bounds related to the amplitude of the resonator is derived by dispersion analysis. The dynamic equation of the sandwich-like meta-plate under external excitations is obtained by the Galerkin method. The wave propagation properties of the finite-sized meta-plate, whose band-gap range is compared with that of the theoretical one, are investigated by finite element simulation and the Galerkin method. The present research is helpful for the design of a meta-plate for low-frequency vibration control.
2 Mathematical formulations

2.1 Design concept

Figure 1a shows the schematic diagram of a sandwich-like meta-plate with nonlinear resonators, where the two faceplates are connected by several vertical rods and the nonlinear MBS resonators are attached periodically on the vertical rods. The unit cell of the metamaterial plate in Fig. 1b shows that the nonlinear resonator is made of a cantilever beam, a mass and a spring, which can provide negative stiffness in the transverse vibration of the resonators. When the resonator is in a static equilibrium state, corresponding to zero external force, the spring and cantilever beam sit in the same horizontal plane, as shown in Fig. 1c. It is assumed that the mass can only oscillate in the vertical direction. Figure 1d shows the deformation of the MBS resonator under restoring force, which is given as follows:

\[ f_r = \frac{3E_b I_b}{l^3} w_r - k \left( \frac{a_0}{1-\eta} - \sqrt{a_0^2 + w_r^2} \right) \frac{w_r}{\sqrt{a_0^2 + w_r^2}} \]  

where \( l, E_b \) and \( I_b \) are the length, elastic modulus, and moment of inertia of the cantilever beam, respectively. \( w_r \) is the transverse displacement of the mass. \( a_0 \) is the length of the spring at \( w_r = 0 \). \( k \) is the stiffness of the spring. \( \eta (0 \leq \eta < 0.6) \) is the pre-compression ratio of the spring, which is defined as the proportion of the compressed length \( (L-a) \) of the spring at \( w_r = 0 \) to the original length \( L \) of the spring. It should be pointed out that when \( \eta < 0 \), the spring will always be in tension, this case is not considered in this article. The stiffness of the nonlinear resonator is derived by differentiating Eq. (1) with respect to \( w_r \) as below:

\[ k_r = P \left( 1 - \mu \left( \frac{1}{1-\eta} \left( 1 + (w_r/a_0)^2 \right)^{3/2} - 1 \right) \right) \]  

(2)

where \( P = 3E_b I_b l^3 \) is the cantilever stiffness, and \( \mu = k/P \) is the ratio of the spring stiffness to the cantilever stiffness. Using the Taylor series at zero equilibrium, Eq. (2) can be presented as:

\[ K_r = \beta_0 + \beta_1 w_r^2 + \sigma \]  

(3)

where

\[ \beta_0 = P \left( 1 - \mu \left( \frac{1}{1-\eta} - 1 \right) \right) \]  

(4)

\[ \beta_1 = \frac{3\mu P}{2(1-\eta)a_0^2} \]  

(5)

Fig. 1 a Schematic diagram of a sandwich-like meta-plate with periodically attached MBS resonators. b The unit cell of the meta-plate. c The physical prototype of an MBS resonator in a static equilibrium state. d Deformation of the MBS resonator under a restoring force
2.2 Dispersion relation of an infinite-sized meta-plate with MBS resonators

In this section, a unit cell, shown in Fig. 2, is used to analyze the dispersion relation of an infinite-sized sandwich-like meta-plate with a nonlinear resonator. For the MBS resonator, the deformation of the cantilever beam is assumed linearly elastic and the weights of the spring and the cantilever beam are neglected. Based on Newton’s second law, the governing equation of the nonlinear resonator is expressed as:

\[
m \ddot{w}_r = K_r (w_r - w_0) = (\beta_0 + \beta_1 w_r^2) (w_r - w_0)
\]

where \( w_0 \) is the transverse displacement of the faceplate at the position of the vertical rod, corresponding to the coordinate origin, and \( w_r \) is the resonator’s displacement.

For free vibration of the nonlinear resonator without counting the effect of the resonator on the faceplates, Eq. (6) can be rewritten as follows:

\[
\ddot{w}_r + \omega_n^2 (w_r + \varphi w_r^3) = 0
\]

where \( \omega_n^2 = \beta_0/m \) and \( \varphi = \beta_1/\beta_0 \). Based on the Harmonic Balance Method (HBM) [33], the amplitude-related natural frequency \( \omega_n \) for a single-degree-of-freedom nonlinear resonator is as follows:

\[
\omega_n^2 = \omega_0^2 \left( 1 + \varphi \left( \frac{3}{4} r_1^2 + \frac{3}{4} r_1 r_3 + \frac{3}{2} r_3^2 \right) \right)
\]

where \( r_1 \) is the amplitude of the fundamental harmonic and \( r_3 \) is the amplitude of the third harmonic. The relationship between \( r_1 \) and \( r_3 \) is \( r_1^2 \varphi - 21r_1^2 r_3 \varphi - 27r_1 r_3^2 \varphi - 51r_3^3 \varphi - 32r_3 = 0 \).

As for the two thin faceplates, the strain–displacement relations based on the classical plate theory are expressed as follows:

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix}^T = 
\begin{pmatrix}
-\frac{\partial^2 w}{\partial x^2} & -\frac{\partial^2 w}{\partial y^2} & -2\varepsilon \frac{\partial^2 w}{\partial x \partial y}
\end{pmatrix}^T
\]

where \( w \) is the transverse displacement of the faceplates.

The constitutive relations of faceplates are presented as follows:

\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} = 
\begin{pmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{pmatrix} \begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix}
\]

where

\[
Q_{11} = Q_{22} = \frac{E}{1 - v^2}, Q_{12} = Q_{21} = \frac{vE}{1 - v^2},
Q_{66} = \frac{E}{2(1 + v)}
\]

\( E \) and \( v \) are Young’s modulus and Poisson’s ratio of the faceplate, respectively. Then the elastic energy \( U \) and the kinetic energy \( T \) of the faceplates can be defined as:

\[
U = \frac{1}{2} \int_V (\varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \tau_{xy} \gamma_{xy}) dV
\]

\[
T = \frac{1}{2} \int_V \rho w^2 dV
\]

where \( \rho \) and \( V \) are the density and volume of the faceplates, respectively. At the vertical rod position, the non-conservative work \( W \) is given by:

\[
W = K_r (w_r - w_0) w_0 = (\beta_0 + \beta_1 w_r^2) (w_r - w_0) w_0
\]
When elastic waves are propagated in the infinite-sized meta-plate composed of the above periodic unit cell, \( w, w_0, \) and \( w_r \) are defined in the form of the exponential function as:

\[
w = q e^{i(2\pi x + \beta y - \omega t)}, \quad w_0 = q e^{-i\omega t}, \quad w_r = re^{-i\omega t}
\]  

(15)

where \( q \) is the displacement amplitude of the meta-plate and \( \omega \) is the wave frequency. The wavenumbers along the \( x \) and \( y \) axes are given by \( \alpha = 2\pi/\lambda_1 \) and \( \beta = 2\pi/\lambda_2 \), where \( \lambda_1 \) and \( \lambda_2 \) are the corresponding wavelengths.

The following is Hamilton’s principle:

\[
\int_{t_0}^{t_1} \delta(T - U)dt + \int_{t_0}^{t_1} \delta Wdt = 0
\]

(16)

Substituting Eqs. (12)–(14) for Eq. (16), then substituting Eq. (15) for Eqs. (6) and (16), the following results are obtained:

\[
\left[ \frac{8 \sin(\alpha/2) \sin(\beta/2)}{\pi \beta} \left( \rho \omega^2 - \frac{h^3}{12} Q_{11} (\alpha^2 + \beta^2)^2 \right) - \frac{K_m}{m} \right] \{ q \} = 0
\]

(17)

Equation (17) shows that the non-zero solutions of \( q \) and \( r \) exist only if the determinant of the matrix is zero:

\[
\frac{8 \sin(\alpha/2) \sin(\beta/2)}{\pi \beta} \left( \rho \omega^2 - \frac{h^3}{12} Q_{11} (\alpha^2 + \beta^2)^2 \right) - \frac{K_m}{m} = 0
\]

(18)

where \( K_m/\lambda \) is taken as \( \omega^2 \). Substituting Eq. (8) for Eq. (18), the dispersion equation of the infinite-sized meta-plate with nonlinear resonators is then expressed as:

\[
\frac{8 \sin(\alpha/2) \sin(\beta/2)}{\pi \beta} \left( \rho \omega^2 - \frac{h^3}{12} Q_{11} (\alpha^2 + \beta^2)^2 \right) \left( \omega^2 - \omega_0^2 \left( 1 + \phi \left( \frac{3}{4} r_1^2 + \frac{3}{4} r_1 r_3 + \frac{3}{2} r_3^2 \right) \right) \right) - \omega_0^2 \left( 1 + \phi \left( \frac{3}{4} r_1^2 + \frac{3}{4} r_1 r_3 + \frac{3}{2} r_3^2 \right) \right) m \omega^2 = 0
\]

(19)

The wave frequency \( \omega \) related to \( \alpha \) and \( \beta \) is obtained by solving Eq. (19). The lower bound of the band gap corresponds to wave frequency \( \omega \) with \( \alpha \) and \( \beta \to \infty \) and the upper bound of the band gap corresponds to wave frequency \( \omega \) with \( \alpha \) and \( \beta \to 0 \). When the wave frequency of the band-gap bounds is divided by \( 2\pi \), the band gap is presented as:

\[
\left( \frac{\omega_0}{2\pi} \right) \sqrt{1 + \phi \left( \frac{3}{4} r_1^2 + \frac{3}{4} r_1 r_3 + \frac{3}{2} r_3^2 \right)} \left( 1 + \frac{\omega}{\omega_0} \right)
\]

(20)

where \( \omega_0 \) is the natural frequency corresponding to the linear stiffness and \( \gamma = m/2ab\rho \) is the ratio of the mass of the resonator to the mass of the faceplates. Equations (3)–(5) and (20) show that the lower bound and width of the band gap can be modified by adjusting the parameters \( \eta, \mu, \) and \( \gamma \). Moreover, it should be noted that the band-gap bound of the infinite-sized meta-plate with nonlinear resonators is related to the resonator amplitude; as the amplitude increases, the lower bound of the band gap moves to a higher frequency.

In fact, the higher the harmonic frequency, the smaller the amplitude. To simplify the expression (Eq. (20)) of the nonlinear band gap, only the amplitude \( r_1 \) of the fundamental harmonic is retained as the preliminary approximation, and the accuracy of the “first-order” approximate solution is also verified in Sect. 3.2, as shown in Fig. 6. Accordingly, Eq. (20) can be rewritten as:

\[
\left( \frac{\omega_0}{2\pi} \right) \sqrt{1 + \phi r_1^2} \left( 1 + \frac{\omega}{\omega_0} \right)
\]

(21)

2.3 Dynamic model of the finite-sized meta-plate with MBS resonators

According to the Galerkin method, the transverse displacement of the faceplates at any point \( (x, y) \) is given by:

\[
w(x, y, t) = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}(x, y)p_{ij}(t) = \xi^T(x, y)p(t)
\]

(22)

where \( m \) and \( n \) are the order of the eigenfunction, \( p(t) \) is a generalized coordinate vector, and \( \xi(x, y) \) is the displacement shape function which is assumed to satisfy the fully free boundary condition [50] and is given by
\[ \tilde{\zeta}_{ij}(x,y) = \Phi_i(x) \Psi_j(y) \]  
(23)

where

\[
\Phi_i(x) = \sin \gamma_{ix} x + \sinh \gamma_{ix} L_x 
+ \cos \gamma_{ix} L_x - \cos \gamma_{ix} L_x 
(\cos \gamma_{ix} x + \cosh \gamma_{ix} x) 
\]
(24)

\[
\Psi_j(y) = \sin \gamma_{jy} y + \sinh \gamma_{jy} y 
+ \cos \gamma_{jy} y - \cosh \gamma_{jy} y 
(\cos \gamma_{jy} y + \cosh \gamma_{jy} y) 
\]
(25)

The eigenvalues \( \gamma_{ix} \) and \( \gamma_{jy} \) can be obtained by solving the following eigen-equations:

\[
\begin{cases}
\cos \gamma_{ix} L_x \cosh \gamma_{ix} L_x = 1, & i = 1, 2, 3 \ldots \\
\cos \gamma_{jy} L_y \cosh \gamma_{jy} L_y = 1, & j = 1, 2, 3 \ldots 
\end{cases}
\]
(26)

where \( L_x \) and \( L_y \) denote the length and width of the meta-plate. To obtain the governing equations of the meta-plate subjected to external excitation \( f_q(t) \), Eqs. (22)–(25) are substituted into Eq. (16), the outcome of which is combined with Eq. (6) to obtain the following detailed expressions:

\[
\begin{bmatrix} M_{ij} \end{bmatrix} p(t) + K p(t) = \xi^T (x_0, y_0) f_0(t) + \sum_{r=1}^{8} \xi^T (x_r, y_r) f_r(x_r, y_r, t) + \sum_{r=1}^{8} \xi^T (x_r, y_r) f_r(x_r, y_r, t) + m \ddot{\omega}_r(t) + f_r(x_r, y_r, t) = 0
\]
(27)

where \( M \) and \( K \) are the mass matrix and stiffness matrix, respectively.

The restoring force \( f_r(x_r, y_r, t) \) caused by the resonator at point \((x_r, y_r)\) is given by:

\[
f_r(x_r, y_r, t) = \frac{3 \mu A h}{2} \omega_r (t) - \xi^T (x_r, y_r) p(t) 
- k \left( \frac{1}{(1 - \eta) \left( 1 + \left( \frac{\omega_r(t) - \xi^T (x_r, y_r) p(t)}{\omega_0} \right)^2 \right)^{\frac{\eta}{\gamma}}} - 1 \right)
\]
(28)

Considering the damping of the resonator and the modal damping of the faceplates, Eq. (27) can be rewritten as

\[
\begin{bmatrix} M_{ij} \end{bmatrix} p(t) + C p(t) + K p(t) = \xi^T (x_0, y_0) f_0(t) + \sum_{r=1}^{8} \xi^T (x_r, y_r) f_r(x_r, y_r, t) 
+ \sum_{r=1}^{8} \xi^T (x_r, y_r) f_r(x_r, y_r, t) 
+ m \ddot{\omega}_r(t) + f_r(x_r, y_r, t) + f_r(x_r, y_r, t) = 0
\]
(29)

By introducing structural damping, the transient response can quickly converge in the numerical simulation. \( C \) and \( f_d(x_r, y_r, t) \) are given by:

\[
C = 2 \zeta K M, \quad f_d(x_r, y_r, t) = 2 \zeta_0 \sqrt{m \beta_0} \dot{w}_r(t) 
- \xi^T (x_r, y_r) p(t)
\]
(30)

where \( \zeta \) is the modal damping ratio and \( \zeta_0 \) is the damping ratio of the resonator. Note that the generalized coordinate \( p(t) \) can be obtained by solving Eq. (29) using the fourth-order Runge–Kutta method embedded in \texttt{ode45}. Then substituting \( p(t) \) into Eq. (22) yields the steady-state response of the meta-plate under an external force \( f_q \).

### 3 Results and discussion

#### 3.1 Verification of the band-gap properties

In this section, the numerical simulations based on the finite element method (FEM) and the Galerkin method are carried out to demonstrate the validity of the theoretical band gap. The material properties and geometrical parameters of the sandwich-like meta-plate are listed in Tables 1 and 2. The numerical simulation model of the sandwich-like meta-plate with \( 8 \times 8 \) unit cells is presented in Fig. 3. It is assumed that a harmonic excitation \( f_q = q_0 \sin \omega t \), with a bandwidth from 1 to 300 Hz, acts on the origin of the meta-plate \((P_s \text{ in Fig. 3)} \), whose response is adopted as the input signal, and the time history of displacement for the test point \( P_t \) in Fig. 3 is chosen as the output signal. The wave attenuation property is estimated by the wave transmittance in decibel (dB), which is defined as

\[
T_r = 20 \log \frac{|q_{out}|}{|q_{in}|}
\]
(31)

In the numerical simulation based on the FEM, COMSOL Multiphysics® 5.5 is used to perform the wave transmittance analysis of the finite-sized meta-plate, with opposite boundary completely clamped support and others free. Note that \( q_{in} \) and \( q_{out} \) are the input acceleration amplitude and the output acceleration amplitude, respectively. The wave transmission curves are shown by the solid red lines in Fig. 4. As for the numerical simulation based on the Galerkin method, \( q_{in} \) and \( q_{out} \) in Eq. (31) represent the input and output root-mean-squared (RMS) displacement of the response obtained by solving Eq. (29) using the Runge–Kutta method at a certain frequency. Taking
into account the damping and nonlinear stiffness of the resonator, the solid blue lines in Fig. 4 represent the transmission spectra, where the harmonic excitation amplitude $q_0$ is taken as 1 N.

It can be seen in Fig. 4 that a sharp drop and an increase appear in the frequency region with the negative transmittance in dB, and the corresponding frequency range represents a band gap. To
demonstrate the validity of the present theoretical band gaps, the band-gap ranges (gray shaded area) obtained by dispersion analysis (Eq. (21)) are compared in Fig. 4. Note that the amplitude $r_1$ is taken as 0.001 m in Eq. (21). Clearly, the attenuation ranges of the measured transmission spectra are in excellent agreement with the theoretical band gaps predicted by Eq. (21). Consequently, the theoretical band gaps of the meta-plate can be demonstrated by the wave transmittance using the FEM and the Galerkin method.

3.2 Dispersion properties

In the proposed infinite-sized meta-plate with a nonlinear resonator, a unit cell, shown in Fig. 2, is used to analyze the dispersion relation. It is noted from Eq. (19) that the wave frequency $\omega$ is related to each pair of $\alpha$ and $\beta$. The material properties and geometrical parameters of the unit cell for dispersion analysis are given in Tables 1 and 2.

Figure 5a and c shows the 3D plots of the dispersion surfaces in terms of the wave frequency $\omega$ as the wave numbers $\alpha$ and $\beta$ for the linear system ($\beta_1 = 0$). According to Eq. (19), two real and positive values of $\omega$ can be obtained for each pair of $\alpha$ ($0 < \alpha < 50$) and $\beta$ ($0 < \beta < 50$), corresponding with the two dispersion surfaces in Fig. 5a. Meanwhile, Fig. 5b and d presents the frequency value corresponding to four edges (M-Γ-Π-X-M) of the dispersion surfaces. It is clearly observed from Fig. 5b that the band gap among the two dispersion curves is 63.4–126.1 Hz, which means that the structural vibration is suppressed in this frequency range, where the pre-compression ratio $\eta$ is 0.2. However, for the sandwich plate without resonators, there is only one dispersion curve, as shown in Fig. 5d. The reason is that when the harmonic wave propagates in the plain sandwich plate, the faceplate is self-dispersive as the wave speed changes.

To verify the accuracy of the nonlinear band gap (Eq. (21)) that retains only the fundamental harmonic as the preliminary approximation, Fig. 6 compares the nonlinear band gap containing only the fundamental harmonic amplitude ($r_1 = 0.006$ m, $r_3 = 0$) and the nonlinear band gap containing the fundamental and third harmonic amplitudes ($r_1 = 0.006$ m, $r_3 = 7.93 \times 10^{-5}$ m), and their results are almost identical. In this case, the accuracy of the “first-order” approximate solution of the harmonic balance method can be guaranteed. Hence, the simplified band-gap expression (Eq. (21)) will be used to analyze the band-gap adjustment in the subsequent discussion.

Meanwhile, it is noted from Eq. (3) that the stiffness of the resonator is composed of a linear term and a nonlinear term. Equation (21) shows that the dispersion solutions of the nonlinear meta-plate depend on the resonator amplitude $r_1$. The linear ($\beta_1 = 0$) and nonlinear ($\beta_1 \neq 0$) dispersion curves are shown in Fig. 7. The effect of the resonator amplitude $r_1$ on the bound frequencies of band gaps is analyzed in Fig. 8. For the nonlinear meta-plate with MBS resonators, as the resonator amplitude increases, the band-gap range shifts to a high frequency, which means that the nonlinearity of the resonator can realize band-gap adjustment of the meta-plate.

3.3 Band-gap adjustment by structural parameters

In Sect. 3.2, a unit cell of the infinite-sized meta-plate is analyzed to predict the dispersion relation. In this section, for the finite-sized meta-plate with periodically attached MBS resonators, the effect of the structural parameters on the wave transmittance is investigated. Considering the nonlinear stiffness of the MBS resonator, the governing equations of the sandwich-like meta-plate subjected to external excitation are derived by the Galerkin method in Sect. 2.3. For the sake of consistency, the material properties and geometrical parameters of Sect. 3.2 are used in the following investigations on finite-sized meta-plate. The four edges of the finite-sized meta-plate are fully free according to the shape function expressions in Eqs. (24) and (25). The numerical simulation model of the meta-plate with 8 × 8 unit cells is the same as the model in Sect. 3.1, as shown in Fig. 3.
Firstly, the effect of structural parameters on the nonlinear stiffness of the MBS resonator is analyzed. Based on Eq. (2) and using \( kr/P \), the dimensionless stiffness expression of the MBS resonator is presented as follows:

\[
kr = \frac{1}{C_0 l + \frac{1}{g}} + \frac{1}{C_0 g} \left( \frac{r}{C_1 l} \right)^2 = \frac{2}{C_0 l} \left( \frac{3}{2} \right)
\]

where \( kr = kr/P \) and \( w_r = w_r/l \).

The relationship between \( kr, \mu \), and \( \eta \) at the equilibrium position \( w_r = 0 \) is presented in Fig. 9. It is noted in Fig. 9b that the curve \( L_1 \), corresponding to \( kr = 0 \), divides the projection of \( kr \) on the \( xy \)-plane into two areas \( S_1 (kr \geq 0) \) and \( S_2 (kr \leq 0) \). To ensure the effectiveness of the nonlinear resonator stiffness \( (kr \geq 0), \) the values of \( \mu \) and \( \eta \) should be taken in the \( S_1 \) area. Moreover, the effects of the pre-compression ratio of the spring and the non-dimensional transverse displacement of the mass on the non-dimensional stiffness of the nonlinear resonator are shown in Fig. 10. From the 3D plot, one can find that the
stiffness of the resonator decreases as the pre-compression ratio \( \eta \) increases at \( \tilde{w}_r = 0 \), with the minimum stiffness obtained at the equilibrium position. Figure 11 shows the non-dimensional stiffness of the nonlinear resonator with a different ratio \( l \). It can be observed that when the value of the ratio \( \mu \) is 0, corresponding to a sandwich-like meta-plate with a mass-beam linear resonator [55], the stiffness of the nonlinear resonator remains constant with change in the pre-compression ratio \( \eta \) and the transverse displacement \( \tilde{w}_r \). However, in Fig. 11a, as \( \mu \) increases to 1.5, the stiffness of the resonator can be conveniently tuned to any value from zero to the stiffness of the cantilever by changing the pre-compression ratio \( \eta \) at \( \tilde{w}_r = 0 \), which means that the novel sandwich-like meta-plate with MBS resonators can generate a tunable low-frequency band gap wider than that of the meta-plate with a mass-beam linear resonator. It is also noted from Fig. 11b that the smaller the value of \( \mu \), the weaker the nonlinear property of the MBS resonator. These results are helpful for the design of the proposed nonlinear resonator in the numerical simulations.

Based on the nonlinear stiffness analysis of the MBS resonator, the effects of \( \eta \) and \( \mu \) on the wave transmission (as shown in Figs. 12 and 13) obtained by the Galerkin method (\( q_0 = 1 \) N) and on the band gap (as shown in Fig. 14) obtained by the dispersion analysis (\( \beta_1 \neq 0, \ r_1 = 0.001 \) m) are investigated. Note that the ratio \( \mu \) is taken as 1.323 in Fig. 12 and the pre-compression ratio \( \eta \) is taken as 0.3 in Fig. 13. It is observed from Figs. 12 and 14a that as the pre-compression ratio \( \eta \) increases, the width of the band gap becomes narrow and the lower bound of the band gap moves from a high-frequency region to a low-frequency region. Moreover, the wave attenuation range shifts to a low-frequency range when the ratio \( \mu \) increases, as shown in Figs. 13 and 14b. The above band-gap change is the same as the change in the resonator stiffness with \( \mu \) and \( \eta \), as shown in Fig. 9b. As is well known, the band gap of a locally resonant metamaterial is determined by the stiffness of its resonator. Therefore, for the nonlinear sandwich-like meta-plate with MBS resonators, the tunable feature of
the band gaps can be achieved by the design of the structural parameters of the MBS resonators.

3.4 Effect of external excitation on wave transmittance

As mentioned in Sect. 3.3, the wave transmittances of the meta-plate with MBS resonators are illustrated in Figs. 12 and 13; the corresponding low excitation amplitude $q_0$ is 1 N. It is also noted from Fig. 11a that the nonlinearity of the resonator stiffness is related to the displacement response of the resonator. Therefore, the effect of external excitation on the wave transmittance will be analyzed in this section. The material properties and geometrical parameters are the same as those of the model in Sect. 3.1.

Figure 15 shows the wave transmittance curves of the meta-plate with nonlinear resonators under excitation with different amplitudes, where the pre-compression ratio $\eta$ of Eq. (21) is taken as the maximum value of the RMS displacement of the resonator in the band-gap range by solving Eq. (29). It can be found from Fig. 15 that with increase in the excitation amplitude, the wave attenuation range evidently shifts from a low-frequency region to a high-frequency one. This trend is consistent with the change in band gaps given by Eq. (21) as the resonator amplitude
increases. Figure 16 shows the RMS displacements of the faceplate and resonators under large-amplitude excitation $(q_0 = 10 \text{ N})$, where $\omega = 40 \text{ Hz}$ and $\omega = 140 \text{ Hz}$ are in the passband range, and $\omega = 80 \text{ Hz}$ is in the band-gap range. It is observed from Fig. 16c and d that the vibration of the faceplate and resonators mainly occurs near the excitation point, while in other areas the faceplate vibration is negligible. This can be ascribed to the fact that the resonator vibrates in resonance with the incoming excitation within the band gap. Meanwhile, it is an interesting phenomenon that even though the RMS displacements of the resonators and faceplate in the low-frequency region are of the same order of magnitude, the resonators have little effect on the vibration suppression of the faceplate, as shown in Fig. 16a and b. The reason is that when the excitation frequency is in the low-frequency passband region, the resonators cannot provide a reaction force to suppress the faceplate vibration due to the zero phase difference between the displacement responses of the resonator and faceplate. Moreover, many peaks are observed in the band-gap range for the case of large-amplitude excitation, as shown by the purple dotted line in Fig. 15a. The
reason is that large-amplitude excitation results in complicated dynamic behaviors, such as coupled vibration and chaos, which can weaken wave attenuation at certain frequencies. Therefore, when the excitation frequency is taken as a constant value within the wave attenuation range, such as $\omega = 75$ Hz, the transient responses of the meta-plate under different excitation amplitudes are as shown in Fig. 17. Compared with the transient responses under a small-excitation amplitude, it is interesting to note that large-amplitude excitation significantly diminishes the wave attenuation efficiency.

To further demonstrate the effect of nonlinearity on the dynamic behaviors of a meta-plate, the bifurcation diagrams of the displacement amplitude versus the excitation amplitude at points $P_s$ and $P_r$ are shown in Fig. 18. Note that the harmonic excitation frequency is taken as 60 Hz, which is within the band-gap ranges. It is observed from Fig. 18 that as the excitation amplitude increases from 0 to 20 N, a route to chaos is presented, corresponding to the change in the displacement amplitude of the meta-plate from the gradient growth to interwell oscillations. As for the meta-plate in this article, when the excitation amplitude $q_0$ is greater than 6.9 N, the dynamic behaviors change into chaos. Comparing the displacement amplitudes of the ‘excitation source’ (Fig. 18a) and ‘test point’ (Fig. 18b), it is obviously seen that the bifurcation phenomenon becomes weak at point $P_r$. The reason is that the sandwich-like meta-plate can control wave propagation through its locally resonant properties.
The phase portraits and the corresponding Poincare sections under different excitation amplitudes are shown in Fig. 19. For small-amplitude excitation, the phase portrait is an elliptic orbit (Fig. 19a, d, e) or a line-like flat curve (Fig. 19b), and the corresponding Poincare sections are clustered near a point due to the periodicity of the dynamic behavior. However, as the excitation amplitude $q_0$ increases to 20 N, the phase portraits of the meta-plate exhibit a cluttered state and the Poincare sections become dispersed, as shown in Fig. 19c, f, which mean that the dynamic behavior of the meta-plate is chaotic. In addition, the transverse displacement responses at points $P_s$ (Fig. 19a–c) and $P_r$ (Fig. 19d–f) are compared to reveal the wave attenuation properties of the meta-plate with MBS resonators. Especially for small-amplitude excitation ($q_0 = 1$ N and $q_0 = 5$ N), there are two orders of magnitude difference in the transverse displacements of points $P_s$ and $P_r$. However, when large-amplitude excitation ($q_0 = 20$ N) is applied to point $P_s$, the corresponding responses are chaotic (Fig. 19c, f), the displacement response is of the same order of magnitude, and its attenuation is not obvious. This interesting phenomenon is helpful for wave manipulation.

4 Conclusion

In this paper, the concept of a sandwich-like meta-plate for wave manipulation is considered while accounting for the nonlinearity of the meta-plate’s mass-beam-spring (MBS) resonator. Based on Hamilton’s principle and the harmonic balance method, the expression of the band-gap bounds related to the amplitude of the resonator is derived by dispersion analysis. A coupled mathematical model is established to analyze the nonlinear dynamic behavior of a finite-sized meta-plate periodically attached to MBS-damper systems acting as local resonators. Some
conclusions can be drawn as follows. The attenuation ranges of the transmission spectra obtained by the FEM and the Galerkin method agree with the band gaps predicted by the dispersion analysis. By investigating the dispersion relation of an infinite-sized meta-plate, it is noted that with increase in the resonator amplitude, the nonlinear theoretical band gap shifts to a high-frequency region. The stiffness of the MBS resonator can be adjusted to a low value by increasing pre-compression of the springs at the equilibrium position, and thus ultra-low-frequency band gaps can be obtained through the design of a meta-plate with MBS resonators. Nevertheless, for large-amplitude excitation, which corresponds to the nonlinearity of resonators, the frequency within the wave attenuation range is evidently higher, and the wave attenuation becomes worse in the band-gap range compared with the wave attenuation in the band-gap range of the linear finite-sized meta-plate. Moreover, the nonlinearity of MBS resonators widens the band gap of sandwich-like meta-plates.

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**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest concerning the publication of this manuscript.

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