State-dependent graviton noise in the equation of geodesic deviation

Z. Haba  
Institute of Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Plac Maxa Borna 9, Poland

We consider an equation of the geodesic deviation appearing in the problem of gravitational wave detection in an environment of gravitons. We investigate a state-dependent graviton noise (as discussed in a recent paper by Parikh, Wilczek and Zahariade) from the point of view of the Feynman integral and stochastic differential equations. The evolution of the density matrix in an environment of gravitons is obtained. We express the time evolution by a solution of a stochastic geodesic deviation equation with a noise dependent on the quantum state of the gravitational field.

I. INTRODUCTION

The study of the effect of an environment, which is not directly observable, on the motion of macroscopic bodies, began with the phenomenon of Brownian motion. The mathematical theory of Brownian motion allowed to confirm the presence of invisible molecules (see [1]). In a recent paper Parikh, Wilczek and Zahariade (PWZ) [2] suggest that in a similar way we can detect the environment of gravitons in spite of the negative conclusions of Dyson [3]. According to PWZ the noise resulting from the ground state or coherent state of gravitons is weak and practically undetectable. However, the high temperature states and squeezed states of the graviton can lead to a substantial increase of noise which can be detected in the gravitational wave detection experiments. The authors [2] derive their results by an investigation of the geodesic deviation equation in the environment of the quantized gravitational field. They apply the influence functional method [4] in order to transform the evolution of the transition probability of macroscopic bodies into an expectation value with respect to the noise disturbing the geodesic deviation equation. Studies on the effect of gravitons on the motion of other particles appeared earlier [5][6][7][8][9][10], but these papers did not tackle directly the geodesic deviation equation. The quantum noise from the environment has been investigated in other fields of physics [11][12]. It can be detected in quantum optics [13] and solid state physics.

The evolution of the density matrix in an environment of unobservable particles is usually treated by means of the influence functional [4] which starts from the Feynman integral in order to transform the evolution of the quantum state of the gravitational field into an expectation value with respect to the Brownian motion developed in [14][15]. In such a case the dependence of the evolution of the density matrix on the state of the gravitational field is exhibited explicitly.

The plan of the paper is the following. In sec.2 we derive in a novel way the transformation of the Feynman integral in quantum mechanics into an expectation value over a state-dependent noise. In sec.3 we show how this transformation applies to quantum field theory. In sec.4 we discuss in detail perturbations by noise resulting from the time-dependent Gaussian states. In sec.5 we apply the method to the BPZ model (as introduced in [2]) in the one mode approximation. In sec.6 the influence functional method is in an arbitrary Gaussian state. We apply a method of transforming the Feynman integral into an expectation value with respect to the Brownian motion developed in [14][15]. In such a case the dependence of the evolution of the density matrix on the state of the gravitational field is exhibited explicitly.

II. STATE-DEPENDENT TRANSFORMATION OF THE FEYNMAN INTEGRAL

Let us consider first a simple model of the Schrödinger equation of quantum mechanics in one dimension with the time-dependent potential $V_t$:

$$i\hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_t \right) \psi. \quad (1)$$

The solution $\psi_t$ with the initial (boundary) condition $\psi_0$ at $t = 0$ can be expressed by the Feynman integral

$$\psi_t(x) = \int Dq \exp \left( \frac{i}{\hbar} \int_0^t \left( \frac{1}{2m} \frac{dq_s}{ds}^2 - V_{t-s}(q(s)) \right) ds \right) \psi_0(q_t(x)), \quad (2)$$

where $q_0(x) = x.$
Assume that the initial condition is a product of the wave functions

$$
\psi_0 = \psi_0^\varphi \chi_0.
$$

We define a stochastic process $q_s$ depending on $\psi_0^\varphi$ and defined as a functional of the Brownian motion $b_s$. The stochastic process $q_s$ is defined as a solution of the stochastic equation

$$
dq_s = \alpha ds + adb_s.
$$

Here, the Brownian motion is defined as the Gaussian process with the covariance

$$
E[b_t b_s] = \min(t, s).
$$

We show that with a proper choice of $\alpha$ and $\alpha$ the solution of the Schrödinger equation (1) can be expressed as

$$
\psi_t(x) = \psi_t^\varphi(x) E\left[ \chi_0(q_t(x)) \right],
$$

where $q_t(x)$ is the solution of eq.(4) with the initial condition $q_0(x) = x$. The expectation value $E[...]$ is with respect to the Brownian motion $b_t$.

The idea is that if in eq.(2) $\psi_0(q_t) = \psi_0^\varphi(q_t) \chi_0(q_t)$ then we can represent $\psi_0^\varphi(q_t) = \exp \ln \left( \psi_0^\varphi(q_t) \right)$ as an integral

$$
\ln \psi_0^\varphi(q_t(x)) = \ln \psi_0^\varphi(x) + \int_0^t d \ln \psi_{t-s}^\varphi(q_s(x)).
$$

With a proper choice of $\alpha$ and $\alpha$ in eq.(4) the potential term in eq.(2) will cancel. More precisely we shall have

$$
\frac{i}{\hbar} \int_0^t \left( \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 - V_{t-s}(q(s)) \right) + \ln(\psi_0^\varphi(q_t(x))) = -\frac{1}{\hbar} \int_0^t \left( \frac{dq}{dt} \right)^2 + \ln \psi_t^\varphi(x).
$$

So that the integration over $q$ in the Feynman formula (1) will be replaced by an average over Brownian paths $b_t$.

It is crucial for the derivation that functional of the Brownian motion satisfy a modified differential formula (Ito formula [16]) following from the non-differentiability of the Brownian paths $b_t$,

$$
df_s = \nabla f_s \circ dq + \partial_s f_s ds = \nabla f_s dq + \frac{1}{2} \partial^2 f_s \left( ds \right)^2 + \partial_s f_s ds,
$$

where $\circ$ denotes the Stratonovič differential (the differential $db$ without circle is called Ito differential [16]) and on the rhs of (9) we insert $< \left( ds \right)^2 >= ds$ (we denote $E[...]$ by $< ... >$ as a shorthand). If the stochastic equation (4) is satisfied and if $f_s$ is a function of $q$

$$
df_s = \nabla f_s \circ dq + \partial_s f_s ds = \nabla f_s dq + \frac{1}{2} a^2 \nabla^2 f_s ds + \partial_s f_s ds.
$$

We insert $f_s = \ln \psi_t^\varphi$. Then

$$
\frac{d \ln \psi_{t-s}^\varphi(q_s)}{ds} = \nabla \ln \psi_{t-s}^\varphi dq_s + \frac{1}{2} a^2 \nabla^2 \psi_{t-s}^\varphi ds + \partial_s \psi_{t-s}^\varphi ds.
$$

We have

$$
\nabla^2 \ln \psi_{t-s}^\varphi = (\nabla^2 \ln \psi_{t-s}^\varphi)(\psi_{t-s}^\varphi)^{-1} - (\psi_{t-s}^\varphi)^{-2} (\nabla \psi_{t-s}^\varphi)^2.
$$

We use the fact that $\psi_{t-s}^\varphi$ satisfies the Schrödinger equation (1). Then, in equation (10) we can express $\nabla^2 \psi_{t-s}$ by $V_{t-s}$ and $\partial_s \psi_{t-s}^\varphi$. We can check after an insertion of eqs.(7) and (10) in eq.(8) that eq.(8) is satisfied if $a^2 = \frac{i\hbar}{m}$ and $\alpha = \frac{i\hbar}{m} \nabla \ln \psi_{t-s}^\varphi$. In such a case in the exponential of the Feynman integral there remains solely $-\frac{1}{2} \int ds \left( \frac{dq}{dt} \right)^2$.

The stochastic equation (4) reads

$$
dq_s = \i\hbar \nabla \ln \psi_{t-s}^\varphi dq_s + \sqrt{\frac{i\hbar}{m}} db_s.
$$

We can repeat the procedure for the Feynman integral solving Schrödinger equation with a final boundary condition $\psi_{\tau} = \psi_\tau$

$$
\psi_t(x) = \int dq \exp \left( \frac{i}{\hbar} \int_0^t \left( \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 - V_s(q(s)) \right) ds \right) \psi_q(x, t),
$$

where $q(x, t) = x$. Let $\psi_\tau = \psi_\tau^\varphi \chi_\tau$ where $\psi_\tau^\varphi$ is the solution of eq.(1). As before we use the identity

$$
\ln \psi_\tau^\varphi(q_t(x, t)) = \int_t^\tau d \ln \psi_\tau^\varphi(q_s(x, t)) + \ln \psi_t^\varphi(x).
$$

We perform the differentiation $d \ln \psi$ as in eq.(10). We check that if $q_s$ satisfies the equation

$$
dq_s = -\frac{i\hbar}{m} \nabla \ln \psi_s^\varphi(q_s) ds + \sqrt{-\frac{i\hbar}{m}} db_s,
$$

then the Feynman integration is reduced to the Brownian average

$$
\psi_t(x) = \psi_t^\varphi(x) E\left[ \chi_\tau(q_t(x, t)) \right].
$$

The imaginary time version of our formulas (at least the ones when $\psi^\varphi$ is the ground state) belongs to the standard theory of stochastic differential equations (see [17] for eq.(11) and [18] for eq.(14)). The derivation of the Feynman integral has been discussed earlier in a different form in [14][15]. There are some mathematical subtleties concerning analytic properties of functions of the stochastic processes as our formalism is an analytic continuation of the one well known for the imaginary time.

The formalism can be extended to arbitrary number of dimensions and to Lagrangians of the form

$$
L = \frac{1}{2g} \frac{dx^l}{dt} \frac{dx^k}{dt}. \label{16}
$$

Then the Schrödinger equation (1) holds with the Hamiltonian

$$
H = \frac{1}{2} g^{-2} p_l g^{-2} p^l + V, \label{17}
$$
where \( g = \text{det}\{g_{jk}\}, g^{jk} \) are the matrix elements of the inverse matrix and
\[
p_j = g_{jk} \frac{dx^k}{ds}
\]
(18)
is the momentum in classical mechanics. In quantum mechanics
\[
p_j = -i\hbar \frac{\partial}{\partial x^j}
\]
(19)
The corresponding stochastic equation (an analog of eq.(11)) reads
\[
dq^j = \frac{i\hbar}{m} g^{jk} \partial_k \ln(\psi^g_{t-s})ds + \sqrt{\frac{i\hbar}{m}} e_k \circ db^k,
\]
(20)
where \( e_k \circ e^k = g^{jk} \). As a simple example we could take the harmonic oscillator with the ground state solution
\[
\psi_g(x) = \exp(-\frac{m\omega}{2\hbar}x^2).
\]
(21)
The stochastic equation (11) reads
\[
dq = -i\omega qdt + \sqrt{\frac{i\hbar}{m}} db.
\]
(22)
A simple calculation gives
\[
\int dx |\psi^g_t(x)|^2 E[q_t(x)q_{t'}(x)] = \frac{\hbar}{2m\omega} \exp(-i\omega|t - t'|).
\]
(23)
The rhs of eq.(23) is the expectation value in the ground state of the time-ordered products of Heisenberg picture position operators of the harmonic oscillator. In this sense the stochastic process \( q_t \) has a physical meaning as its correlation functions coincide with quantum expectation values.

III. THE RELATIVISTIC QUANTUM FIELD THEORY

In the canonical field theory with the Hamiltonian
\[
\mathcal{H} = \frac{1}{2} \int dx \left( \Pi^2 + (\omega \Phi)^2 \right) + \int dx V(\Phi),
\]
(24)
where
\[
\omega = \sqrt{-\Delta + m^2}
\]
and
\[
[\Phi(x), \Pi(y)] = i\hbar \delta(x - y)
\]
consider the Schrödinger equation
\[
-i\hbar \partial_t \psi = \mathcal{H} \psi
\]
(25)
and its imaginary time version
\[
-\hbar \partial_t \psi = \mathcal{H} \psi.
\]
(26)
Let
\[
\psi_0 = \psi_g \chi_0,
\]
(27)
where \( \psi_g \) is the ground state. Then, \( \chi \) satisfies the equation
\[
\hbar \partial_t \chi = \frac{1}{2} \int dx \left( \Pi^2 - (\Pi \ln \psi_g)\Pi \right) \chi,
\]
(28)
where
\[
\Pi(x) = -i\hbar \frac{\delta}{\delta \Phi(x)}
\]
(29)
Eq.(28) is a diffusion equation in infinite dimensional spaces. An approach to Euclidean field theory based on this equation has been developed in [19]. It follows that the solution of eq.(28) can be expressed as
\[
\chi_t(\Phi) = E\left[ \chi_0 \left( \Phi_t(\Phi) \right) \right],
\]
(30)
where \( \Phi_t(\Phi) \) is the solution of the stochastic equation
\[
d\Phi_t(x) = \hbar \frac{\delta}{\delta \Phi(x)} \ln \psi_g dt + \sqrt{\hbar} dW_t(x)
\]
(31)
with the initial condition \( \Phi \). \( E[...] \) denotes an expectation value with respect to the Wiener process (Brownian motion) defined by the covariance
\[
E\left[ W_t(x)W_s(y) \right] = \min(t, s) \delta(x - y).
\]
(32)
The correlation functions of the Euclidean field in the ground state \( \psi_g \) can be expressed by the correlation functions of the stochastic process \( \Phi_t \).

Let us consider the simplest example: the free field. Then, the ground state is
\[
\psi_g = \exp\left( -\frac{1}{2\hbar} \Phi \omega \Phi \right).
\]
(33)
Eq.(31) reads
\[
d\Phi = -\omega \Phi dt + \sqrt{\hbar} dW
\]
(34)
with the solution (with the initial condition \( \Phi \) at \( t_0 \))
\[
\Phi_t = \exp(-\omega(t-t_0)\Phi + \sqrt{\hbar} \int_{t_0}^t \exp(-\omega(t-s))dW_s).
\]
(35)
We can calculate
\[
\int D\Phi \exp\left( -\frac{1}{2\hbar} \Phi \omega \Phi \right) E[\Phi_t(\Phi)\Phi_{t'}(\Phi')] = \hbar \left( \frac{1}{2\hbar} \exp(-|t - t'|\omega) \right)(x, x').
\]
(36)
In eq.(36) the rhs denotes the kernel of the operator.
If the ground state has some analyticity property then we can extend the time in eqs. (25)-(36) \( t \rightarrow \text{it} \) from imaginary time to the real time \([14][15]\). An analytic continuation to real time of eq. (34) reads

\[
d\Phi_t = -i\omega \Phi_t dt + \sqrt{i\hbar} dW. \tag{37}
\]

The solution is

\[
\Phi_t = \exp(-i\omega(t-t_0))\Phi + \sqrt{i\hbar} \int_{t_0}^{t} \exp(-i\omega(t-s))dW_s. \tag{38}
\]

The solution of the Schrödinger equation (25) has the same form (6) with \( \Phi_t(\Phi) \) of eq. (38) replacing \( q_t(x) \). We can see that

\[
\begin{align*}
\left< \psi_g, T \left( \exp(\int dt \Phi_t^\dagger(f_t)\psi_g) \right) \right> \\
= \int d\Phi \exp(-\frac{i}{\hbar}\Phi \omega \Phi) E \left[ \exp(\int dtdx f_t(\Phi_t(\Phi))) \right] \\
= \exp \left( \frac{\hbar}{2} \int dt d\Gamma_t \left(f_t, 2\omega^{-1} \exp(-i\omega|t-t'|)f_t' \right) \right),
\end{align*}
\tag{39}
\]

where \( \Phi_t^\dagger(f) = \int dx \Phi_t^\dagger(x)f(x) \) is the quantum field and \( T \) denotes the time-ordered product.

We can transform the complex equation (37) into two real equations defining

\[
\Phi = \phi_1 + i\phi_2 \tag{40}
\]

and

\[
\phi_+ = \phi_1 + \phi_2, \tag{41}
\]

\[
\phi_- = \phi_1 - \phi_2. \tag{42}
\]

Then it follows from eq. (37) that

\[
\phi_+ = \omega^{-1}\phi_- \tag{43}
\]

and \( \phi_- \) satisfies the random wave equation

\[
\partial^2_t \phi_- = -\omega^2\phi_- + \sqrt{2\hbar}\omega\partial_t W, \tag{44}
\]

which has the solution

\[
\phi_-(t) = \cos(\omega t)\phi_-(0) + \omega^{-1}\sin(\omega t)\partial_t \phi_-(0) \\
+ \sqrt{2\hbar} \int_0^t \sin(\omega(t-s))dW_s, \tag{45}
\]

where \( \partial_t \phi_-(0) = \omega \phi_+(0) \).

IV. TIME-DEPENDENT REFERENCE STATE

There are time-dependent solutions in the Gaussian form of the Schrödinger equation for the free field theory

\[
\psi^\theta_t(\Phi) = A(t) \exp \left( \frac{i}{2\hbar}(\Phi \Gamma_t \Phi + 2J_t \Phi) \right). \tag{46}
\]

In the time-dependent case eq. (11) in quantum field theory takes the form [17]

\[
d\Phi_t = -\Gamma_{t-} \Phi_t ds - J_{t-} ds + \sqrt{i\hbar} dW_t. \tag{47}
\]

Let \( \Phi_s(\Phi) \) be the solution of eq. (47) with the initial condition \( \Phi \) then the solution of the Schrödinger equation is

\[
\psi_t = \psi_t^\theta E \left[ \chi(\Phi_t(\Phi)) \right]. \tag{48}
\]

With the time-dependent interaction \( V_t \) in eq. (24) the Feynman formula reads (where \( \psi_t^\theta \) is the solution of free field theory)

\[
\psi_t = \psi_t^\theta E \left[ \exp \left( -\frac{i}{\hbar} \int_0^t V_{t-s}(\Phi_s) ds \chi(\Phi_t(\Phi)) \right) \right]. \tag{49}
\]

If we impose the final boundary condition at \( t = \tau \) on the solutions \( \Phi_t(\Phi, \tau) \) of the stochastic equation

\[
d\Phi_t = \Gamma_t \Phi_t dt + J_t dt + \sqrt{-i\hbar} dW_t \tag{50}
\]

then we can express the solution of the Schrödinger equation with the final condition \( \chi_\tau \) at \( t = \tau \) as [18]

\[
\psi_t(\Phi) = \psi_t^\theta(\Phi) E \left[ \chi_\tau(\Phi_\tau(t, \Phi)) \right]. \tag{51}
\]

Let us decompose eq. (50) \( (J = 0) \) into the real and imaginary parts

\[
\Phi = \phi_1 + i\phi_2 \tag{52}
\]

\[
\Gamma = \omega_1 + i\omega_2, \tag{53}
\]

Then

\[
\partial_t \phi_+ = \omega_1 \phi_+ + \omega_2 \phi_- \tag{54}
\]

\[
\partial_t \phi_- = \omega_1 \phi_- - \omega_2 \phi_+ + \sqrt{2\hbar}\partial_t W. \tag{55}
\]

Expressing \( \phi_- \) by \( \phi_+ \) from eq. (54) and inserting it in eq. (55) we obtain an equation (if \( \omega_2 \neq 0 \))

\[
\partial_t \left( \omega_2^{-1}(\partial_t \phi_+ - \omega_1 \phi_-) \right) = \omega_1 \omega_2^{-1}(\partial_t \phi_+ - \omega_1 \phi_-) - \omega_2 \phi_+ + \sqrt{2\hbar}\partial_t W.
\]

Hence, \( \phi_+ \) satisfies the wave equation with friction (we further develop the theory of stochastic wave equations for the quantum theory of fields in expanding universes in [21] as an extension of Starobinsky stochastic inflation [20]).

As an application to the particle motion in a (quantized) gravitational wave let us consider the Hamiltonian of a harmonic oscillator

\[
H = -\frac{\hbar^2}{2} \nabla^2 + \frac{1}{2}\omega^2 x^2. \tag{56}
\]
Let us look for a time-dependent solution of the Schrödinger equation in the form
\[ \psi^g = A(t) \exp \left( \frac{i}{\hbar} (x \Gamma t + 2 J_t x) \right). \] (57)

Then, \( \psi^g \) is a solution of the Schrödinger equation (1) if
\[ i \hbar \partial_t \ln A = - \frac{i \hbar}{2} \Gamma + \frac{1}{2} J_t^2, \] (58)
and
\[ - \partial_t J = \Gamma J \] (59)

It can be seen that the Riccatti equation (60) is equivalent to
\[ \frac{d^2 u}{dt^2} + \omega^2 u = 0, \] (61)

where
\[ u = \exp \left( \int^t \Gamma \right). \] (62)

Then, \( J = Cu^{-1} \) with a certain constant \( C \). The general solution of eq.(61) is
\[ u = \sigma \cos(\omega t) + \delta \sin(\omega t), \] (63)

where \( \sigma, \delta \) are complex numbers.

Eq.(47) \((J = 0)\) reads
\[ dq_s = - \partial_t \ln u_{t-s} q_s ds + \sqrt{\hbar} db_s \]
\[ = - \omega ( - \sigma \sin(\omega(t-s)) + \delta \cos(\omega(t-s))) \]
\[ (\sigma \cos(\omega(t-s)) + \delta \sin(\omega(t-s)))^{-1} q_s ds + \sqrt{\hbar} db_s \] (64)

with the solution
\[ q_s(q) = u^{-1}_{s-t} q + \sqrt{\hbar} u_{t-s} \int^s_{0} u_{s-r}^{-1} db_r. \] (65)

We have
\[ (\partial_s^2 + \omega^2) < q_s > = 0 \]
and
\[ E[(q_s - < q_s >)(q_{s'} - < q_{s'} >)] \]
\[ = i \hbar u_{t-s} u_{t-s'} \int^{\min(s, s')}_{0} u_{s-r}^{-2} ds \]
\[ = i \hbar \omega^{-2} u_{t-s} u_{t-s'} \]
\[ (\sigma^2 + \delta^2)^{-1} \left( \Gamma(t) - \Gamma(t - \min(s, s')) \right). \] (66)

If \( \frac{\partial}{\partial t} = i \sigma \) then \( \Gamma(0) = i \omega \) and the solution \( \psi_t \) starts from a real Gaussian function. The case \( \delta = i \sigma \) corresponds to the ground state (21) \((m = 1)\). It can be shown that the formula (66) is continuous with respect to the limit \( \delta \rightarrow i \sigma \).

V. PARTICLE INTERACTING WITH GRAVITONS: PWZ MODEL IN ONE MODE APPROXIMATION

Parikh,Wilczek and Zahariade [2] consider two masses \( M \) and \( m_0 \) interacting with gravitational field \( q_\omega \). In a one mode approximation the Lagrangian describing the geodesic deviation of the \( m_0 \) mass \((\text{in the free falling frame})\) is \( [2][22] \)
\[ \mathcal{L} = \frac{1}{2} ((\frac{dq_\omega}{ds})^2 - \omega^2 q_\omega^2) + \frac{1}{2} m_0 (\frac{\delta}{ds})^2 - m_0 \lambda \frac{dq_\omega}{ds} \frac{dq_\omega}{ds} \]
\[ = \frac{1}{2} g_{jk} \frac{dq_j}{ds} \frac{dq_k}{ds} - \frac{1}{2} \omega^2 q_\omega^2, \] (67)

where \( \lambda = \sqrt{8 \pi G}, x = (q_\omega, \xi), \xi \) is the geodesic deviation.

\[ (g_{jk}) = \left[ \begin{array}{cc} 1 & -m_0 \lambda \xi \ 
- m_0 \lambda \xi & \frac{m_0 \lambda \xi}{m_0} \end{array} \right]. \] (68)

The Lagrange equation is
\[ \frac{d^2 \xi}{ds^2} = m_0 \lambda \frac{dq_\omega}{ds} \xi. \]

The Hamiltonian is determined by eq.(17) with
\[ [g^{jk}] = (m_0 - g^2 \xi^2)^{-1} \left[ \begin{array}{cc} m_0 & m_0 \lambda \xi \\
- m_0 \lambda \xi & 1 \end{array} \right]. \] (69)

We could quantize the model (67) with the explicit quantum version (17) of the Hamiltonian. However, the averaging over gravitons is simple only in the influence functional approach.

In general, the wave function evolution in a potential \( V \) and in the gravitational field \( g_{\mu\nu} \) is
\[ \psi(x, q) = \int Dx Dg \exp \left( \frac{i}{\hbar} \int \mathcal{L}(g) \right) \exp \left( \frac{i}{\hbar} \int_0^t (\frac{m_0}{2} g_{\mu\nu} \frac{dq_\mu}{ds} \frac{dq_\nu}{ds} - V(x) \right) \psi(x_t(x)), \] (70)

where \( \mathcal{L}(g) \) is the gravitational Lagrangian and \( q \) are the initial values of the metric. The density matrix in an environment of gravitons described by variables \( q \) is
\[ \rho(t, x; t', x') = \int Dq \psi(t, x, q) \psi^*(t, x', q). \] (71)

In the PWZ model (67), where the gravitational field is described by one mode \( q_\omega \), when we apply the transformation (8) to \( q \) with the ground state (21) \((m = 1)\) then from eq.(49) we obtain
\[ \rho_{t}(\xi, \xi') = \int dq \exp (- \frac{i}{\hbar} q^2) \]
\[ \left( \frac{i}{\hbar} \int_0^t (\frac{m_0}{2} \frac{dq^*_\omega}{ds} \frac{dq^*_\omega}{ds} - \frac{m_0}{2} \frac{dq^*_\omega}{ds} \frac{dq^*_\omega}{ds} - V(\xi) + V(\xi')) \right) \]
\[ \exp \left( \frac{i m_0}{\hbar} \int_0^t (q_{t-s} (\frac{dq^*_\omega}{ds} \frac{dq^*_\omega}{ds} - i q_{t-s} (\frac{dq^*_\omega}{ds} \frac{dq^*_\omega}{ds}) ) \right) \]
\[ \chi_{\omega}(\xi, q_\omega(q)) \chi_{\omega}(\xi', q_\omega(q)^*) \right] \equiv K_t \rho_0, \] (72)

where \( q^* \) is another realization of the process \( q \) (and the complex conjugation of this realization), we have denoted the evolution kernel of the density matrix by \( K_t \).
The calculation of the expectation value (72) gives (we use the solution of eq.(22) and assume that the initial condition \( \chi_0 \) does not depend on \( q \))

\[
K_1 \simeq \int dq D\xi D\xi' \exp(-\frac{i}{\hbar} \int_0^t \frac{d^2}{ds^2} ds - \frac{m_0}{\hbar} \int_0^t \frac{d^2}{ds^2} ds - V(\xi) + V(\xi'))
\]

\[
-\frac{-\lambda}{\hbar} \int_0^t (q f_s - q f_s') ds - q \exp(-i\omega(t-s)) f_s - q \exp(i\omega(t-s)) f_s'
\]

\[
\exp \left( -\frac{\lambda_2^2}{\hbar^2} \int_0^t dsds' \left( E[(q_{t-s} - q_{t-s}')(q_{t-s}' - q_{t-s}')] f_s f_s' + E[(q_{t-s} - q_{t-s}')(q_{t-s}' - q_{t-s}'')] f_s f_s' \right) \right),
\]

where \( f = \frac{m_0}{\hbar} \frac{d^2}{ds^2}, f' = \frac{m_0}{\hbar} \frac{d^2}{ds^2} \). In eq.(73) we have (from eq.(22))

\[
\frac{\lambda}{\hbar} \int_0^t dsds' \left( \exp(-i\omega|s-s'|) + \exp(-i\omega(2t-s-s')) \right).
\]

If the oscillator is in a time-dependent state then we should insert the solution (63) in the Feynman formula (72). Hence, instead of eq.(73) we have

\[
\frac{\lambda}{\hbar} \int_0^t dsds' \left( E[(q_{t-s} - q_{t-s}')(q_{t-s}' - q_{t-s}')] f_s f_s' + E[(q_{t-s} - q_{t-s}')(q_{t-s}' - q_{t-s}'')] f_s f_s' \right),
\]

where

\[
E[(q_{t-s} - q_{t-s}')(q_{t-s}' - q_{t-s}')] = i\hbar u_s u_s' \int_{\min(t-s,t-s')} d\tau u(t-\tau)^{-2} \]

\[
= i\hbar \omega^2 u_s u_s' (\sigma^2 + \delta^2)^{-1} \]

\[
(\Gamma(t) - \Gamma(\max(s,s'))) \] (75)

After calculation of the \( q \) integral in eqs.(73) and (75) we obtain a quadratic functional of \( f_s \) and \( f_s' \) in the exponential. In the simplest case (73) of the ground state of the oscillator we obtain

\[
K_1 \simeq \int D\xi D\xi' \exp \left( \frac{i}{\hbar} \int_0^t \frac{d^2}{ds^2} ds - \frac{m_0}{\hbar} \int_0^t \frac{d^2}{ds^2} ds - V(\xi) + V(\xi') \right)
\]

\[
\exp \left( -\frac{\lambda_2^2}{\hbar^2} \int_0^t dsds' \left( f_s f_s' \exp(-i\omega|s-s'|) + f_s f_s' \exp(i\omega|s-s'|) - 2f_s f_s' \exp(i\omega|s-s'|) \right) \right).
\]

We write

\[
X = \frac{1}{2}(\xi + \xi'),
\]

\[
y = \xi - \xi'.
\]

We expand the exponential in eq.(77) in \( y \). Then, the terms independent of \( y \) cancel and there remains (up to the terms quadratic in \( y \))

\[
\rho_t \simeq \int D\xi D\xi' \exp \left( \frac{i}{\hbar} \int_0^t ds \left( (m_0 \frac{d^2}{ds^2} - V'(X)) \right) \frac{\delta^2}{2\hbar^2} \int_0^t dsds' \left( \left( -i\sin(\omega(s-s')) \left( \frac{d^2}{ds^2} \right) + \frac{d^2}{ds^2} + \frac{d^2}{ds^2} \right) \cos(\omega(s-s')) \right) \right) \rho_0(X).
\]

The term linear in \( y \) gives a modification of the equation of motion of the \( \xi \) coordinate whereas the term quadratic in \( y \) is a noise acting upon the particle [7].

In the expression (75) of the time-dependent reference state we obtain

\[
\rho_t \simeq \exp \left( -\frac{\lambda_2^2}{\hbar^2} \frac{i}{\hbar} \int_0^t dsds' \left( L \left( (u_s - u_s')^2 - u_s u_s' (\sigma^2 + \delta^2)^{-1} (\Gamma(t) - \Gamma(\max(s,s'))) \right) + u_s u_s' (\sigma^2 + \delta^2)^{-1} (\Gamma(t) - \Gamma^2(\max(s,s'))) \right) \rho_0(X).
\]

We expand (80) in \( y \) again. Let us consider the general expression appearing after an expansion in \( y \) till the second order terms in eqs.(79)-(80)

\[
\int D\xi D\xi' \exp \left( \frac{i}{\hbar} \int_0^t ds \left( (m_0 \frac{d^2}{ds^2} + V'(X)) + L(X) + \frac{1}{2\hbar} M y \right) \right) \rho_0(X)
\]

\[
\equiv \int D\xi D\xi' \exp \left( \frac{i}{\hbar} \int_0^t ds \left( (L + \frac{1}{2\hbar} M y) \right) \right) \rho_0(X)
\]

\[
= \int D\xi D\xi' \exp \left( -\frac{i}{\hbar} \int_0^t ds \left( yL + \frac{1}{2\hbar} M y \right) \right) \rho_0(X)
\]

\[
= \int D\xi D\xi' \left( -\frac{i}{\hbar} \int_0^t ds \left( y - i\hbar M^{-1} L \right) \right) \rho_0(X),
\]

where by \( L \) we denoted a functional of \( X \), \( M \) is an operator and by \( L \) we denoted the term proportional to \( y \). If we introduce \( Y = M^{-\frac{1}{2}}L \) then \( Y \) becomes a Gaussian variable with the white noise distribution which can be represented as \( \partial W \). The factor depending on \( y \) is integrated out contributing just a constant. The calculation of \( \rho_t \) is reduced to an average over solutions of the stochastic equation

\[
m_0 \frac{d^2}{ds^2} + V'(X) + L(X) = M^{-\frac{1}{2}}\partial_s W.
\]

In the next section dealing with a thermal state of the gravitational field we approximate \( L \) and \( M \) by local functions of \( X \).

VI. INFINITE NUMBER OF MODES: THERMAL STATE OF GRAVITONS

We are to generalize the results of sec.5 to an infinite number of modes of the gravitational field. We have studied earlier [7] an analogous model of a particle geodesic motion in an environment of quantum gravitational waves in a thermal state. There are minor
changes from the setting of Parikh, Wilczek and Zahariade [2] where the effect of gravitons on geodesic deviation is considered. With the infinite number of gravitational wave modes in the model (67) of sec.5 we shall have \( \omega = |k| \) (we set the velocity of light \( c = 1 \)), and we sum up the terms with \( g_\omega \) over \( k \) with the measure \( dk \). Now the Lagrangian (67) is [22]
\[
\mathcal{L} = \frac{1}{8} \int dk h^*_\alpha(k,s)(\frac{d}{ds})^2 h_\alpha(k,s) + \frac{1}{2} m_0 \frac{d^2 \chi}{ds^2} - \frac{1}{4} m_0 \lambda \frac{d^2 h^*_\alpha}{ds^2} \xi_t \xi_t - \frac{1}{4} m_0 \lambda (2\pi)^{-\frac{1}{2}} \int dk \frac{d^2 h^*_\alpha}{ds^2} (k) \xi_t \xi_t \cdot \nabla
\]
where the metric perturbation \( h_{\alpha t} \) of the Minkowski metric \( g_{\mu \nu} \) \((g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu})\) is in the transverse-traceless gauge, \( k \) is the wave vector and the geodesic deviation \( \xi \) in a gravitational transverse-traceless gauge has only the spatial components \( \xi_r \). We apply the decomposition of \( h_{\alpha t} = h^\delta_{\alpha t} + h^\gamma_{\alpha t} \) into the classical wave solution \( h^\delta_{\alpha t} \) and the quantized \((\text{graviton})\) part \( h^\gamma_{\alpha t} \). \( h^\gamma_{\alpha t} \) is decomposed in the amplitudes \( h_\alpha \) (where \( \alpha = +, - \), in the linear polarization \() \) by means of the polarization tensors \( e^\gamma_r \) \[23][24]\n\[
h_{\alpha t}(x,t) = h^\gamma_{\alpha t}(x,t) + (2\pi)^{-\frac{1}{2}} \int dk (h^\delta_{\alpha t} e_\gamma^r \exp(-i\mathbf{k}\mathbf{x}) + h^\gamma_{\alpha t} e_\gamma^r \exp(i\mathbf{k}\mathbf{x})).
\]
The \( h^\delta_{\alpha t} \) are quantized at finite temperature \( T \) with the Gibbs distribution \( \exp(-\beta H) \), \( \beta^{-1} = k_B T \) where \( k_B \) is the Boltzmann constant. The classical part of the wave-particle interaction can be considered as a time-dependent external potential \( V(\xi) \) with
\[
V = \frac{m_0}{4} \frac{d^2 h^\gamma_{\alpha t}}{ds^2} \xi^k \xi^\alpha.
\]
Denote
\[
f^\gamma_r = \frac{m_0}{2} \frac{d^2}{ds^2} \xi^r \xi^t
\]
(84)
\[
f^\delta_r = \frac{m_0}{2} \frac{d^2}{ds^2} \xi^r \xi^t.
\]
(85)

Then, as in[7] (there are some misprints of signs in [7]; for the derivation of the thermal formula see [25],sec.18, see also [5][8][10];the gravitational case is analogous to the electromagnetic one treated in [26]) we obtain for the density matrix evolution kernel
\[
K(t;\xi';\xi) = \int D\xi D\xi' \exp\left(\frac{im_0}{2\hbar} \int_0^T ds \left( \frac{d\xi}{ds} \frac{d\xi'}{ds} - \frac{d\xi}{ds} \frac{d\xi}{ds} \right) \right) \exp\left(\frac{2}{T} \int_0^T ds \int_0^T ds' \left( \left( f^{r l} - f^{r l} \right) C_{r l; mn} (f^{mn} + f^{mn}) - \left( f^{r l} - f^{r l} \right) A_{r l; mn} (f^{mn} - f^{mn}) \right) \right).
\]
(86)

In (86) we have a decomposition of the finite temperature transverse-traceless graviton propagator \( D \) into the real and imaginary parts \( D = A + iC \)
\[
A_{r l; mn} (x-x', s-s') = 2\beta \hbar (2\pi)^{-3} \int \frac{dk}{k} \Lambda_{r l; mn} \cos(k(x-x')) \cos(k(s-s')) \coth\left(\frac{\hbar k}{2T}\right),
\]
(87)
\[
C_{r l; mn} (x-x', s-s') = 2\hbar (2\pi)^{-3} \int \frac{dk}{k} \Lambda_{r l; mn} \cos(k(x-x')) \sin(k(s-s')).
\]
(88)

where
\[
2\Lambda_{r l; mn} = 2\epsilon_{0} \epsilon_{mn} = ((\delta_{tm} - k^{-2} k_m k_n) (\delta_{jn} - k^{-2} k_j k_n)
+ (\delta_{jm} - k^{-2} k_j k_n) (\delta_{tn} - k^{-2} k_t k_n)
- \frac{2}{5} (\delta_{ij} - k^{-2} k_i k_j) (\delta_{kn} - k^{-2} k_j k_n)).
\]

We neglect the dependence on \( x \) in eqs.(87)-(88). Then the angular average over \( k k^{-1} \) gives
\[
\frac{1}{4\pi} < A_{ij; mn} >= \frac{1}{5} (\delta_{ij} \delta_{jm} + \delta_{im} \delta_{jm} - \frac{2}{15} \delta_{ij} \delta_{im} \delta_{jm}).
\]
(89)

When we neglect the \( x \) dependence of the propagators (assuming \( \xi \) is small in comparison with the gravitational wave length) and average over the angles then the \( k \)-integral \( dk \approx \alpha \pi d k k^2 \) in the high temperature limit of \( A_{r l; mn} \) in eq.(87) gives \( \delta(s-s') \). In \( C_{r l; mn} \) (88) we write (as in [7]) \( \sin(k(s-s')) = -k^{-1} \partial_\alpha \cos(k(s-s')) \). Then, integrating over \( k \) we obtain \( \partial_\alpha \delta(s-s') \). In such a case the evolution kernel of the density matrix reads
\[
K_\xi(t;\xi';\xi) = \int D\xi D\xi' \exp\left(\frac{im_0}{2\hbar} \int_0^T ds \left( \frac{d\xi}{ds} \frac{d\xi'}{ds} - \frac{d\xi}{ds} \frac{d\xi}{ds} \right) \right) \exp\left(\frac{2}{T} \int_0^T ds \int_0^T ds' \left( \left( q^{r l} - q^{r l} \right) \partial_\alpha (q^{r l} + q^{r l})
- \frac{w}{2\pi^2} (q^{r l} - q^{r l}) (q^{r l} - q^{r l}) \right) \right),
\]
where
\[
q^{r l} = \frac{1}{2} \frac{d^2}{ds^2} (\xi^t \xi^l - \frac{1}{3} \delta^{r l} \xi^r \xi^l)
\]
(91)
results from a resummation of \( f^{r l} \) with \( < A_{ij; rt} > \)
\[
\gamma = \frac{8\pi G m_0}{10\pi}
\]
(92)
\[
w = \frac{2\gamma}{\beta}
\]
(93)

We expand eq.(90) around \( X \) (78) (now with three spatial indices)
\[
q^{r l} - q^{r l} = \frac{d^2}{ds^2} (X^r y^l + X^l y^r - \frac{2}{3} \delta^{r l} X^3 y^l)
\]
In the exponential (90) the term linear in \( y \) becomes
\[
y_n \left( - \frac{d^2 X_n}{ds^2} + \frac{\alpha}{2} \frac{d^2 h^\gamma_{\alpha t}}{ds^2} X_r
+ \frac{8\pi G m_0}{10\pi} X_k \frac{d^2 x_r}{ds^2} (\frac{1}{3} X_r X_r \delta_{nl} - X_n X_l) \right)
\]
(94)

The term quadratic in \( y \) is the noise term. The calculation of the evolution kernel is reduced to an expectation value over the solutions of the stochastic equation
\[
- \frac{d^2 X_n}{ds^2} + \frac{\alpha}{2} \frac{d^2 h^\gamma_{\alpha t}}{ds^2} X_r
+ \frac{8\pi G m_0}{10\pi} X_k \frac{d^2 x_r}{ds^2} (\frac{1}{3} X_r X_r \delta_{nl} - X_n X_l)
= m_0^{-1} \sqrt{w(M)} \frac{d^2 X_n}{ds^2} \partial_\alpha W_r.
\]
(95)
As explained in the derivation of eq.(81) the term quadratic in $y$ defines the operator $M$. From eq.(90) we obtain that $M$ is an operator defined by the bilinear form (on the rhs of eq.(95) we have the square root of this matrix)

$$y^r M^{rt} y^t = 2 \int ds \left( \frac{\partial^2}{\partial x^r} (X^j y^j) \frac{\partial^2}{\partial x^t} (X^j y^j) + \frac{\partial^2}{\partial x^r} (X^j y^j) \frac{\partial^2}{\partial x^t} (X^j y^j) - \frac{2}{3} \frac{\partial^2}{\partial x^r} (X^j y^j) \frac{\partial^2}{\partial x^t} (X^j y^j) \right) \tag{96}$$

We can see that the noise on the rhs of eq.(95) has the factor $\sqrt{T}$ which can be large at high temperature.

**VII. GENERAL GAUSSIAN STATE OF THE GRAVITON**

We generalize the results of sec.5 on averaging over the oscillator modes to infinite number of modes of the gravitational field. The gravitational perturbation in the transverse-traceless gauge is decomposed into the amplitudes $h^\alpha$ by means of the polarization tensors $e_{il}^\alpha, \alpha = +, -, \ h_{rl} = e_{il}^0 h_\alpha$. The gravitational Hamiltonian $H = H_+ + H_-$ has the same form as for two independent scalar fields $\Phi \rightarrow h_\alpha$ (eq.(24) with $m = V = 0$). In such a case we have two independent equations (60) for $\Gamma_\alpha$ and two independent $u_\alpha$. The quantum interaction in eq.(67) in multimode case takes the form

$$\frac{\Lambda_{mn;rl}}{(2\pi)^{\frac{d}{2}}} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \kappa^k \kappa^l = \frac{\Lambda_{mn}}{(2\pi)^{\frac{d}{2}}} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} f^\alpha,$$

where $f_\alpha = e_{rl}^\alpha f^{rl}$.

We consider a solution of the Schrödinger equation (25) in the Gaussian form (if we did not split the gravitational field into $h^w$ and $h^q$ then the classical part would be obtained from the term $Jh$ in eq.(57))

$$\psi_\alpha(h) = \exp \left( i \frac{\hbar}{2} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \Gamma_\alpha k^2 h_\alpha \right). \tag{97}$$

In eq.(97) $(\Gamma^\alpha - \Gamma^{\alpha*})^{-1}$ has a physical meaning as the squeezing of the amplitude $h^\alpha$ in the uncertainty relations (squeezed states are produced during inflation[28][29]). In general, $\Gamma^+$ and $\Gamma^\times$ are independent solutions of eq.(60). We restrict our discussion to the case $\Gamma^+ = \Gamma^\times \equiv \Gamma$ (and $u_+ = u_\times$). Then, the gravitational field resulting from the state (97) is covariant under rotations. In the expectation values (77) and (80) we shall have sums of the form

$$f_\alpha f_\alpha \rightarrow \Lambda_{mn;rl} f^{mn} f^{rl} \tag{98}$$

which after averaging over $kk^{-1}$ will be expressed by $q^{rl}$ (91) in the way analogous to the thermal case of sec.6

$$(\Lambda_{mn;rl} f^{mn} f^{rl}) \rightarrow <\Lambda_{mn;rl}> f^{mn} f^{rl}. \tag{99}$$

In detail

$$K_t \simeq \exp \left( -i \frac{\lambda^2 m_0^2}{4 \hbar} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \left( f^{rl}(u_t^{-1} u_+ f_+ - u_t^{-1} u_- f_-) ds \right)^2 \right. \tag{100}$$

$$-i \frac{\lambda^2 m_0^2}{4 \hbar} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \left( u_+ u_- (\sigma^2 + \delta^2)^{-1} \right. \tag{101}$$

which was local in time (for a small $\beta$) owing to the $k^{-1}$ factor coming from $\coth(\frac{\beta}{2\hbar})$. We do not have such a factor here. Nevertheless, we can see that the non-local final noise can be large owing to the squeezing factor $(\Gamma - \Gamma^*)^{-1}$ in eq.(99). Explicitly, this term is

$$\exp \left( -i \frac{\lambda^2 m_0^2}{4 \hbar} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \left( u_+ u_- (\sigma^2 + \delta^2)^{-1} \int ds \int ds' \left( u_t^{-1} u_+ u_- f_+ f_- + u_t^{-1} u_+ u_- f_+ f_- \right) \right) \right). \tag{102}$$

The exponential in (99) is of the form

$$\exp(i\alpha_q q + i\alpha_q q' + i\alpha_q q''). \tag{103}$$

When $\frac{\lambda}{2\hbar} = i$ and $\Gamma = \hbar k$ then in the integral (99) we obtain (this is an infinite mode version of eq.(77))

$$K_t \simeq \exp \left( -i \frac{\lambda^2 m_0^2}{4 \hbar} \int \frac{dk}{(2\pi)^{\frac{d}{2}}} \left( \exp(-ik|s - s'|) q_\alpha (s) q_\alpha (s') \right. \right) \tag{104}$$

where $q_\alpha (s) = -k^{-1} \partial_s (\phi(s))$ we integrate over $k$ obtaining $\partial_s (s - s')$ in a similar way as in the thermal case arriving at the phase factor

$$\exp \left( -i \frac{\lambda}{2\hbar} \int_0^t (q_\alpha (s) q_\alpha (s) + q_\alpha (s) q_\alpha (s')) \right). \tag{105}$$

We obtain the same rhs of the stochastic equation as in the thermal state, but the noise resulting from eq.(102) is different than the one of eq.(95) (non-local in time and non-Markovian).

In eq.(99) the exponential of the evolution kernel for the density matrix is of the form

$$K_t \simeq \int d\xi d\bar{\xi} \exp \left( i \int \frac{d\xi}{2\hbar} \int_0^t ds (m_0 \frac{d\xi^r}{d^r} \frac{d\xi^r}{d^r} - m_0 \frac{d\xi^r}{d^r} \frac{d\xi^r}{d^r} + V(\xi) - V(\bar{\xi}) + i \int ds d\bar{\xi} (q^{rl} - q^{rl}) C(q^{rl} + q^{rl}) + (q^{rl} + q^{rl}) A(q^{rl} + q^{rl}) + (q^{rl} - q^{rl}) B(q^{rl} - q^{rl}) \right) \tag{106}$$

for which after averaging over $kk^{-1}$ will be expressed by $q^{rl}$ (91) in the way analogous to the thermal case of sec.6.
If the exponential (99)-(101) is written in the form (104) then $\alpha_1 = A + B + C, \alpha_2 = A + B - C, \alpha_3 = 2(A - B - C)$.

The functions $A, B, C$ can be read from eqs. (99)-(100). Expanding in $y$ we obtain the non-Markovian stochastic equation as derived in eq. (81). In eq. (104) the $C$ term is proportional to $y$ whereas the $A$ and $B$ terms are quadratic in $y$. So the $C$ term gives the modification of the equation for the geodesic deviation whereas the $A, B$ terms contribute to the noise.

VIII. SUMMARY

We have calculated the average of the density matrix over the gravitational field in a thermal state and in a general Gaussian state. The result shows that for gravitons in high temperature or in a highly squeezed state the modification of the geodesic deviation equation applied in a gravitational wave detection can have large noise amplitude. In general, the perturbation of the geodesic deviation equation is non-local and non-Markovian. Our results on the stochastic geodesic deviation equation show some differences in comparison to PWZ[2] which may come from approximations used by those authors. The formula for the noise (although quite complicated) can be useful in order to distinguish the contribution of the graviton noise from other sources of noise in the gravitational wave detection. The noise could be detectable if gravitational waves come from an inflationary stage of universe evolution (squeezing) or from a merge of hot neutron stars.

IX. APPENDIX: HIGH TEMPERATURE (CLASSICAL LIMIT)

We can treat the system in an environment of thermal gravitons by means of the same procedure as applied in[27]. The equation for the gravitational field resulting from the Lagrangian (82) is

$$\frac{d^2 h_{\nu}^{\mu}(k)}{dt^2} + k^2 h_{\nu}^{\mu}(k) = (2\pi)^{-\frac{3}{2}} \lambda f^{\nu\mu}. \quad (105)$$

We write the solution of eq. (105) (assuming that when $t \leq t_0$, where $t_0$ is the initial time, $h_{\nu}^{\mu}$ behaves as free wave) in the form (the transverse-traceless gauge)

$$h_{\nu}^{\mu}(k) = h_{\nu}^{\mu}(k) + h_{\nu}^{\mu}(k) + h_{\nu}^{\mu}(k)$$

$$= h_{\nu}^{\mu}(k) + e_{\nu}^{\mu}(\delta h_{0}^{\mu}(k) + k^{-1}\Pi_{0}^{\mu}(\sin(kt)))$$

$$+ (2\pi)^{-\frac{3}{2}} \Lambda_{\nu\mu} \int_{t_0}^{t} k^{-3} \sin(k(t - t')) f_{m,n}(t') dt' , \quad (106)$$

here

$$h_{\nu}^{\mu}(t, x) = (2\pi)^{-\frac{3}{2}} \int dK \exp(ikx) h_{\nu}^{\mu}(k)$$

is the gravitational wave, $h^{\mu\nu}$ describes the thermal modes distributed with the Gibbs equilibrium measure, $h_{0}^{\mu}$ and $\Pi_{0}^{\mu}$ are random initial conditions. $h_{\nu}^{\mu}$ is the gravitational field created by the motion of $m_{n}$. $h_{\nu}^{w}$ and $h_{\nu}^{h}$ satisfy the homogeneous equation

$$\frac{d^2 h_{\nu}^{h}(k)}{dt^2} + k^2 h_{\nu}^{h}(k) = 0.$$
where we assumed that at $t_0$ the first and the second derivatives of $f_{mn}$ are zero. If we assume that in (114) $f_{mn}(t_0) = 0$ then we can write eq.(108) in the form

\[
\frac{d^2 \xi}{dt^2} = \frac{\lambda}{2} \frac{\partial^2 f_{rl}}{\partial t^2} \xi - \frac{\lambda}{2} (\delta_{rm} \delta_{ln} - \frac{1}{3} \delta_{rl} \delta_{mn}) \xi \partial^3 f_{mn} + N_{rl}(t) \xi.
\]  

(116)

Eq.(116) is the classical limit of eq.(95).