Combined flavor symmetry violation and lepton number violation in neutrino physics

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Heavy singlet neutrinos admit Majorana masses which are not possible for the Standard Model particles. This suggest new possibilities for generating the masses and mixing angles of light neutrinos. We present a model of neutrino physics which combines the source of lepton number violation with the flavor symmetry responsible for the hierarchy in the charged lepton and quark sector. This is accomplished by giving the scalar field effecting the lepton number violation a nonzero charge under the horizontal flavor symmetry. We find an economical model which is consistent with the measured values of the atmospheric and solar neutrino mass-squares and mixing angles.

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I. INTRODUCTION

The measurement of neutrino oscillations has created opportunities and challenges for model builders. The additional information regarding the masses and mixing angles in the neutrino sector provide valuable targets for models which aspire to explain the experimental data. The fact that the data has a combination of what appear to be large and small mixing angles as well as what might be at least a moderate hierarchy in neutrino masses is not what usually results in a generic way from the most simple models.

Nir and Shadmi[1, 2] made an interesting observation regarding the unique characteristics of the neutrino sector where the right-handed neutrino is a Standard Model gauge singlet. In their framework new possibilities exist for generating mass terms because there is a new scale associated with the fact that the heavy neutrinos can have a Majorana mass that violates lepton number. They imagine a symmetry

\[
G_{SM} \times U(1)_H \times U(1)_L,
\]

where \(G_{SM}\) is the Standard Model gauge group, \(U(1)_H\) is the usual horizontal (or flavor) symmetry of Froggatt-Nielsen\[3\] models, and \(U(1)_L\) is lepton number. They propose (in the usual way) a scalar field \(S_H\) which carries charge \(-1\) under \(U(1)_H\) which generates the mass and mixing angle hierarchies in the charged lepton and quark sectors of the Standard Model. In addition there is a pair of scalar fields \(S_L\) and \(\bar{S}_L\) that are singlets under \(G_{SM} \times U(1)_H\) and have lepton number \(-2\) and \(+2\). The symmetry breaking involving these fields effects lepton number violation and allows for nonzero neutrino masses. They looked at a few examples in the case of two light neutrino generations to illustrate their point in the setting of the Froggatt-Nielsen framework. In particular they showed that neutrino anarchy can result even when the horizontal flavor charges of the charged lepton families are different for each generation.

Related ideas have been explored by Barr and Kyae\[4, 5\] in grand unified models. In their papers the new interactions are introduced into fully specified grand unified theories, but the source of the new contributions to the mass matrices comes from integrating out of the theory heavy vectorlike fermions like those in a Froggatt-Nielsen models.

II. SPECIFICATION OF THE MODEL

In this paper we provide a realistic explanation for the neutrino oscillation data which can be obtained in a very economical way. The fundamental new feature we add to the models examined in Ref. [1, 2] is to give the lepton number violating scalar field a horizontal charge. We refer to this as “combining” the lepton number violation with the flavor symmetry violation. The model we present here falls into a class that implements an \(L_e - L_\mu - L_\tau\) symmetry. This results in a pseudo-Dirac form for the mixing between the electron neutrino and the maximally mixed state from the muon- and tau-neutrino sector. It has been noted[6, 7, 8, 9, 10] that a light neutrino mass matrix of the form

\[
m \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

has two large (in fact maximal) mixing angles, a contribution \(U_{e3} = 0\) and an inverted hierarchy. It also has \(\Delta m^2 = 0\) and \(m^2\) which is very roughly the pattern observed in neutrino oscillations. Our model achieves approximately this form (absent the extra \(\mu-\tau\) symmetry that guarantees the exact bimaximal form) but implements perturbations, consistent with Eq. (1), that are needed to be in agreement with the solar and atmospheric neutrino data.

In Ref. [1] examples were provided that involved two light neutrino generations and that exhibited the variety of behavior that can arise in that case. In this section...
we introduce a full three-generation model. The importance of neutrinos being Majorana plays a crucial role in the properties of the light neutrino mass and mixing angles. The model implements an accidental $L_e - L_u - L_f$ symmetry that is broken perturbatively by the interactions of the Froggatt-Nielsen fields whose vevs break the horizontal symmetry $U(1)_H$.

Following Nir and Shadmi we assume there is a scalar field $S_H$ that is a singlet of $G_{SM} \times U(1)_L$ and carries charge $-1$ under $U(1)_H$. This field is the usual Froggatt-Nielsen scalar whose vacuum expectation value (vev) generates the hierarchy in the charged lepton and quark sectors. We will also assume there is another scalar field $S_H'$ that is a singlet under $G_{SM} \times U(1)_L$ and carries charge $+1$ under $U(1)_H$. The fields $S_H$ and $S_H'$ admit interactions that are incompatible with the $L_e - L_u - L_f$ symmetry and as a consequence generate the necessary perturbations that make the model phenomenologically viable. The lepton number symmetry $U(1)_L$ is broken by two scalar fields $S_{HL}$ and $\bar{S}_{HL}$ which carry charges $-2$ and $+2$ respectively, under $U(1)_L$. We also assign these two fields charges $-1$ and $+1$ respectively, under $U(1)_H$.

It is in this last respect that our model differs from the standard Model singlets. The fields $S_{HL}$ and $\bar{S}_{HL}$ singlets under the horizontal symmetry $U(1)_H$. For this reason we have chosen to denote this lepton number violating field $S_{HL}$ (rather than $S_L$).

The scales in the model are:

1) the electroweak breaking scale determined by the vevs of Higgs doublets, $(\phi_u, d)$;

2) $M_{HL} \equiv \langle S_{HL} \rangle \sim \langle \bar{S}_{HL} \rangle$, the lepton number breaking scale

3) $M_H \equiv \langle S_H \rangle \sim \langle S_H' \rangle$, the horizontal symmetry breaking scale;

4) $M_F$, the mass scale of Froggatt-Nielsen vector-like quarks and leptons.

The Froggatt-Nielsen parameter

$$
\lambda_H \equiv \frac{\langle S_H \rangle}{M_F} = \frac{M_H}{M_F},
$$

is small and is often associated with the value of Cabibbo angle ($\sim 0.2$) since it is used to generate the mass and mixing angle hierarchies in the quark and charged lepton sectors of the Standard Model. The lepton number breaking introduces another parameter

$$
\lambda_L \equiv \frac{(S_H)^2}{\langle S_{HL} \rangle} = \frac{M_H^2}{M_{HL}^2},
$$

The new feature of this type of model is that this parameter need not be small in which case some new features in the generation of neutrino masses and mixing can result. We postulate the existence of the following fields and associated $U(1)_H$ charges

$$
L_{+1}, L_0, L_0, N_{+1}, N_0, \bar{N}_0.
$$

The fields $L$ are the three generations of lepton doublets in the Standard Model, while the fields $N$ are the Standard Model singlets. The fields $L$ and $\bar{N}$ give rise to the usual Dirac mass terms. Vector-like couplings can arise between $N$ and $\bar{N}$ while lepton violating couplings arise between two $N$ fields or two $\bar{N}$ fields. Setting the $U(1)_H$ charges equal in the lepton doublet fields can yield large mixing angles in the neutrino mass matrix and has been called neutrino anarchy \cite{11, 12, 13, 14, 15}. The charges in Eq. 5 guarantee a semi-anarchy \cite{16} in the second and third generations (because the $U(1)_H$ charges are the same), and will also exhibit an inverted hierarchy. This collection of heavy neutrino fields is quite economical. Other fields can be present without changing the features of the light neutrino mass matrix. As mentioned in the introduction a new ingredient we introduce here is to assign a nonzero horizontal charge to the lepton number violating scalar fields $S_{HL}$ and $\bar{S}_{HL}$. First consider the situation where the horizontal symmetry breaking is turned off (i.e. $\lambda_H = 0$). With the charges of the fields chosen as in Eq. 5, the following symmetric mass matrix results:

$$
\begin{pmatrix}
0 & 0 & 0 & \phi_u & 0 & 0 \\
0 & 0 & 0 & 0 & \phi_u & 0 \\
0 & 0 & 0 & 0 & 0 & \phi_u \\
\phi_u & 0 & 0 & M_F & 0 & \bar{S}_{HL} \\
0 & \phi_u & 0 & 0 & M_F & \bar{S}_{HL} \\
0 & 0 & \bar{S}_{HL} & 0 & 0 & M_F \\
0 & \phi_u & 0 & 0 & \bar{S}_{HL} & M_F \\
0 & \phi_u & 0 & 0 & \bar{S}_{HL} & M_F
\end{pmatrix}.
$$

In this matrix the entries $S_{HL}$ and $\bar{S}_{HL}$ represent their vevs, $M_{HL} \equiv \langle S_{HL} \rangle \sim \langle \bar{S}_{HL} \rangle$. We imagine the scale $M_{HL}$ is comparable to the scale $M_F$ so that the superheavy $4 \times 4$ subblock is fully mixed. This implies that the $\lambda_L$ parameter in Eq. 4 should be sufficiently small, $\lambda_L \lesssim \lambda_H$. This is the only condition on $\lambda_L$ and our results can henceforth be described in terms of the small parameter $\lambda_H$ alone. The effect of assigning a nonzero $U(1)_H$ charge to the fields giving rise to lepton number violation, $S_{HL}$ and $\bar{S}_{HL}$, is to move the contributions off the diagonal, and as a consequence generate a pseudo-Dirac structure in the effective light neutrino mass matrix. It should be understood that there are undetermined order one coefficients in front of every nonzero term in the matrix as in all model of this Froggatt-Nielsen type. Some of the entries are related because the matrix is symmetric, but other entries which appear identical are only equivalent up to these coefficients (for example, $M_F$’s in the $4 - 5$ entry and in the $7 - 8$ entry).

Integrating out the heavy sector gives an effective light neutrino mass matrix

$$
\begin{pmatrix}
0 & \sin \theta & \cos \theta \\
\sin \theta & 0 & 0 \\
\cos \theta & 0 & 0
\end{pmatrix}.
$$

The overall scale $m$ is generated as a seesaw-type mass of order $O(\langle \phi_u \rangle^2 / \langle S_{HL} \rangle)$ and the undetermined ratio of order
one coefficients has been expressed as an angle. This matrix is of much interest because it has two large angles and a small one. It can be understood as involving large mixing between the second and third family, together with a pseudo-Dirac structure relating the first family to them. The pseudo-Dirac case is characterized in the two generation case by the matrix
\[
m \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]
and the matrix in Eq. (7) is clearly of this form.

The effective light neutrino mass matrix in Eq. (7) is a well-known mass matrix form that almost gives acceptable values for the neutrino parameters
\[
\begin{align*}
\sin^2 2\theta_A &= \sin^2 2\theta, \\
\sin^2 2\theta_{\odot} &= 1,
\end{align*}
\]
where the A and \( \odot \) subscripts refer to the atmospheric and solar angles respectively. The mass eigenvalues of the matrix are 0, \(-m\), and +\(m\). As noted earlier this structure is an interesting starting point for constructing realistic masses and mixing angles to describe the neutrino oscillation data. The atmospheric neutrino mixing angle is the same as \(\theta\) which can be large. In our formulation this angle is anarchical because the muon and tau lepton doublet horizontal charges are equal to each other. The other large angle is in fact maximal, and may be large for a fundamentally different reason. We will exploit this distinction between the two large angles in our model[18].

The mixing angles observed in neutrino oscillation experiments are the angles in the matrix
\[
U_{\text{PMNS}} = U_L^\dagger U_{\nu},
\]
where \(U_L\) is the matrix that diagonalizes the charged lepton sector, while \(U_{\nu}\) diagonalizes the effective light neutrino mass matrix. It is expected in Froggatt-Nielsen models of the type we are considering that the mixing angles in the charged lepton sector should be small and CKM-like since the electron, muon and tau lepton exhibit a strong hierarchy in masses. Therefore we expect statements about the mixing angles arising from \(U_{\nu}\) to provide the substantial contributions to the mixing angles observed in the experiments in \(U_{\text{PMNS}}\).

The mixing matrix \(U_L\) has one maximal angle, one angle \(\theta\) which is given in terms of (undetermined) order one coefficients in the full mass matrix in Eq. (4). In fact, because of the symmetries of the model, the source of the light neutrino masses arises from a limited number of interactions, and \(\tan \theta\) is simply the ratio of the 3 – 7 and the 2 – 7 entries in the matrix in Eq. (9). (For the special case \(\tan \theta = 1\) one has maximal mixing for the atmospheric neutrinos.) The remaining angle is such that \(U_{e3} = 0\).

With a small perturbation the mass matrix in Eq. (7) can be made roughly consistent with the experimental data which requires a solar mixing angle \(\theta_{\odot} \approx 35^\circ\) somewhat smaller than maximal and a mass splitting between the two nonzero eigenvalues to account for a small \(\Delta m^2\).

Including the effects of the field \(S_H\) (through the coupling \(S_H N_{+1} \bar{N}_0\) which conserves the horizontal charge \(U(1)_H\)) gives a contribution that can be regarded as a perturbation
\[
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 & \phi_u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \phi_u \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_F & S_{HL} \\
0 & 0 & 0 & S_{HL} & 0 & 0 \\
0 & \phi_u & \phi_u & PM_F & S_{HL} & M_F
\end{pmatrix}
\end{align*}
\]
where \(P = O(M_H/M_F) = O(\lambda_H)\). This perturbs the matrix in Eq. (7) to the following form,
\[
m \begin{pmatrix}
z & \sin \theta & \cos \theta \\
\sin \theta & 0 & 0 \\
\cos \theta & 0 & 0
\end{pmatrix},
\]
where the small quantity \(z\) is of order the perturbation, \(P\).

The lepton numbers can be assigned to the fermion fields as in Table I. These are consistent with the \(U(1)_L\) symmetry which ensures lepton number conservation in that all the interactions conserve overall lepton number \(L\).

The underlying \(L_e - L_\mu - L_\tau\) symmetry can be understood from the charges in Eq. (5). Charges can be assigned to the fields \(S_{HL}\) and \(\bar{S}_{HL}\) so that the interactions generating the masses from \(M_{HL}\) and the Dirac masses \(\phi\) all respect this symmetry. The field \(S_H\) whose usual role is to provide the small parameter that gives rise to flavor hierarchies in the Standard Model, here also provides a source for the breaking of the \(L_e - L_\mu - L_\tau\) symmetry. This symmetry arises as an accidental symmetry due to the charge assignments of the fields, and does not necessarily arise for some more fundamental reason. Nevertheless it would be interesting to see if this scheme could arise from a grand unified model.

The \(S_H\) and \(S'_H\) fields are singlets under \(U(1)_L\) while the fields \(S_{HL}\) and \(\bar{S}_{HL}\) are charged. Since the \(S_H\) field is presumably responsible for the hierarchy in the quark and charged lepton sectors, we assign it zero also under the \(L_e - L_\mu - L_\tau\) symmetry[19]. These fields are assigned the \(L_e - L_\mu - L_\tau\) charges according to Table I.

### Table I

| \(L_e\) | \(L_\mu + L_e\) | \(L_e - L_\mu - L_\tau\) | \(L\) | \(H\) |
|---|---|---|---|---|
| 1 | 0 | +1 | +1 | +1 |
| 0 | +1 | -1 | +1 | 0 |
| 0 | +1 | -1 | +1 | 0 |
| +1 | 0 | +1 | +1 | +1 |
| 0 | +1 | 0 | +1 | 0 |
| 0 | -1 | 0 | +1 | 0 |
| 0 | +1 | 0 | -1 | 0 |
| 0 | +1 | 0 | -1 | 0 |
The interaction \( S_H N_+ \bar{N}_0 \) has charge \( L_e - L_\mu - L_\tau = +2 \), and so does not respect the \( L_e - L_\mu - L_\tau \) symmetry and generates mass terms in the full neutrino matrix when \( S_H \) gets a vev. It fills in the \( 1 - 1 \) entry of the light neutrino mass matrix involving the parameter \( z \). If there is a new field \( S'_H \) with \( U(1)_H \) charge +1, then it provides a coupling \( S'_H N_0 \bar{N}_- \) with charge +1, then it provides a source of tension in all models employing a perturbative symmetry involving a single \( \phi \) or \( \phi' \).

To obtain a good fit to the solar and atmospheric neutrino data we will need to include the effects of both types of \( L_e - L_\mu - L_\tau \) symmetry breaking (see the next section). Including both perturbations \( P \) and \( P' \),

\[
\begin{pmatrix}
  0 & 0 & 0 & 0 & \phi_u & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & \phi_u & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \phi_u \\
  0 & 0 & 0 & 0 & M_F & S_{HL} & 0 \\
  \phi_u & 0 & 0 & M_F & 0 & P'M_F & S_{HL} \\
  0 & \phi_u & 0 & S_{HL} & P'M_F & 0 & M_F \\
  0 & \phi_u & \phi_u & P'M_F & S_{HL} & 0 & M_F
\end{pmatrix},
\]

where \( P' = O(\lambda_H) \). The effective light neutrino mass matrix in Eq. (7) becomes,

\[
m \begin{pmatrix}
  0 & 0 & 0 & \sin \theta & 0 & \cos \theta \\
  0 & 0 & 0 & \sin \theta & z' \sin^2 \theta & \sin \theta \cos \theta \\
  0 & 0 & 0 & \cos \theta & z' \sin \theta \cos \theta & 0 \\
  \end{pmatrix},
\]

where the small quantity \( z' \) is of order the perturbation, \( P' \). The relationship between the four terms in the 2–3 subblock of this matrix is enforced by the restricted nature of the coupling of the lepton doublets \( L \) to the heavy neutrino fields \( N \) and \( \bar{N} \).

To obtain a good fit to the solar and atmospheric neutrino data we will need to include the effects of both types of \( L_e - L_\mu - L_\tau \) symmetry breaking (see the next section). Including both perturbations \( P \) and \( P' \),

\[
\begin{pmatrix}
  0 & 0 & 0 & 0 & \phi_u & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & \phi_u & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \phi_u \\
  0 & 0 & 0 & 0 & M_F & S_{HL} & P'M_F \\
  \phi_u & 0 & 0 & M_F & 0 & P'M_F & S_{HL} \\
  0 & \phi_u & 0 & S_{HL} & P'M_F & 0 & M_F \\
  0 & \phi_u & \phi_u & P'M_F & S_{HL} & 0 & M_F
\end{pmatrix},
\]

yields a light neutrino mass matrix that can be phenomenologically acceptable for describing all the data from atmospheric and solar neutrino oscillations,

\[
m \begin{pmatrix}
  z & \sin \theta & \cos \theta \\
  \sin \theta & z' \sin^2 \theta & \sin \theta \cos \theta \\
  \cos \theta & \sin \theta \cos \theta & z' \sin \theta \cos \theta \\
  \end{pmatrix}.
\]

This matrix has one massless neutrino and \( U_{e3} = 0 \).
Finally the neutrinoless double beta decay parameter in the general case of Eq. \((\text{[17]})\) is given by \(\langle m\rangle_{33} = zm\).

In our model this parameter is therefore set to the scale \(\sim \lambda_H \sqrt{\Delta m^2_{21}}\).

The dependence of the solar mixing angle and the ratio of the mass-squares in Eq. \((20)\) indicates that a small fine-tuning \(z \approx -z'\) can give rise to acceptable values, \(1 - \sin^2 2\theta_\odot \approx \lambda_H^2\) and \(R \approx \lambda_H^2\). The terms which break the \(L_e - L_\mu - L_\tau\) symmetry do so in such a way to generate entries in a very specific form in the light neutrino mass matrix. In fact this simple structure gives a prediction for the mixing angle, \(U_{e3} = 0\).

It is easy to adjust the charge assignments to make the perturbations \(P\) and \(P'\) any power of the parameter \(\lambda_H\). We have chosen \(P, P' \sim \lambda_H\) because this gives the best overall fit to the experimental data.

IV. SUMMARY AND CONCLUSIONS

The model we have constructed in this note exhibits the following features:

- The large mixing in the atmospheric neutrino oscillations results by assuming the same horizontal \(U(1)_H\) charge for the second and third flavor generation for the \(SU(2)_L\) lepton doublets of the Standard Model. This is the usual anarchy in the neutrino sector in Froggatt-Nielsen models, and is sometimes called a semi-anarchy since it does not extend to the first generation.

- The large mixing in the solar neutrino oscillations results from the fact that there are two distinct kinds of couplings for the heavy neutrino sector: 1) there are lepton-number violating Majorana masses of order \(M_{HL}\), and 2) there are vector-like mass of order \(M_F\). The resulting large angle in the light neutrino mass matrix becomes the solar neutrino mixing angle and is of the pseudo-Dirac form. The feature of the model which resulted in this pseudo-Dirac mass matrix is the nonzero \(U(1)_H\) charge of the fields \(S_{HL}\) and \(\bar{S}_{HL}\) that give rise to lepton number violation.

- In Froggatt-Nielsen models it is difficult to achieve a light neutrino mass matrix that has two large mixing angles and a small third mixing (\(U_{e3}\)). In the model presented in this paper the source of the two large angles is separated into the distinct mechanisms of the previous bullet items.

- Perturbations are present in the model in terms of the usual Froggatt-Nielsen parameter \(\lambda_H\) that can yield a realistic set of mass-squares and mixing angles. The model has an inverted hierarchy with one massless neutrino, and it requires \(U_{e3} = 0\) barring small contributions from the charged lepton sector. The solar neutrino mixing angle satisfies \(1 - \sin^2 2\theta_\odot \sim P^2, P'^2\). The ratio of mass-squared parameters for the neutrino oscillation experiments satisfy \(R = \Delta m^2_{21}/\Delta m^2_{31} \sim P, P'\) where \(P\) and \(P'\) are perturbations related to the small horizontal symmetry parameter (Cabibbo angle) of \(\lambda_H \sim 0.2\). A modest amount of fine tuning between the two sources of perturbation are needed to suppress the ratio \(R\) to an acceptable value.

In summary we have constructed a model which “combines” lepton number violation with the horizontal symmetry violation. By assigning the lepton number violating field \(S_{HL}\) a horizontal symmetry charge, a pseudo-Dirac structure is imposed on the light neutrino mass matrix. Since the model contains two scales associated with the breaking of lepton number on the one hand and the breaking of the horizontal symmetry on the other, the small Froggatt-Nielsen parameter naturally introduces the needed perturbations to the \(L_e - L_\mu - L_\tau\) symmetry to accommodate the experimental data. This idea may be of more general use in flavor models based on the Froggatt-Nielsen mechanism.

Acknowledgments

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Note added: After this work was submitted to the arXiv, we learned that our model falls into a class of models governed by the scaling ansatz introduced in Ref. \([17]\). Some further description of the phenomenology of these models can be found there.

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[18] Sometimes an additional symmetry is imposed to force \( \sin \theta = \cos \theta = \frac{1}{\sqrt{2}} \) so the the atmospheric neutrino mixing angle is maximal. This gives a bimaximal neutrino mixing scenario, but as we have argued here, the maximality of the solar and atmospheric neutrino angles then arises in different and generically unrelated ways.
[19] Alternatively one can assign individual lepton number charges to the \( S_H \) and \( S_H' \) fields in which case the \( L_e - L_\mu - L_\tau \) symmetry only arises after they acquire vevs. However then the fields do not respect the \( U(1)_L \) symmetry in the interactions with the quark sector.