Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential

Dedicated to Professor Yoshio Tsutsumi on his 60th birthday

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Abstract

We study strong instability of standing waves $e^{i\omega t}\phi_\omega(x)$ for nonlinear Schrödinger equations with $L^2$-supercritical nonlinearity and a harmonic potential, where $\phi_\omega$ is a ground state of the corresponding stationary problem. We prove that $e^{i\omega t}\phi_\omega(x)$ is strongly unstable if $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1}\leq 0$, where $E$ is the energy and $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$ is the $L^2$-invariant scaling.

1 Introduction

This paper is concerned with the instability of standing waves $e^{i\omega t}\phi_\omega(x)$ for the nonlinear Schrödinger equation with a harmonic potential

$$i\partial_t u = -\Delta u + |x|^2u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 1$ and $1 < p < 2^* - 1$. Here, $2^*$ is defined by $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.

It is known that for any $\omega \in (-N, \infty)$, there exists a unique positive solution (ground state) $\phi_\omega(x)$ of the stationary problem

$$-\Delta \phi + |x|^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N$$

in the energy space

$$X := \{v \in H^1(\mathbb{R}^N) : |x|v \in L^2(\mathbb{R}^N)\}.$$
Note that the condition $\omega > -N$ appears naturally in the existence of positive solutions for (1.2), because the first eigenvalue of $-\Delta + |x|^2$ is $N$. For the uniqueness of positive solutions for (1.2), see [7, 8, 9, 16].

The Cauchy problem for (1.1) is locally well-posed in the energy space $X$ (see [3, §9.2] and [12]). That is, for any $u_0 \in X$ there exist $T_{\text{max}} = T_{\text{max}}(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T_{\text{max}}), X)$ of (1.1) with initial condition $u(0) = u_0$ such that either $T_{\text{max}} = \infty$ (global existence) or $T_{\text{max}} < \infty$ and

$$\lim_{t \to T_{\text{max}}} \|u(t)\|_X = \infty$$

(finite time blowup). Moreover, the solution $u(t)$ satisfies the conservations of charge and energy

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad E(u(t)) = E(u_0)$$

for all $t \in [0, T_{\text{max}})$, where the energy $E$ is defined by

$$E(v) = \frac{1}{2} \|\nabla v\|_{L^2}^2 + \frac{1}{2} \|xv\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}.$$  (1.4)

Here we give the definitions of stability and instability of standing waves.

**Definition 1.** We say that the standing wave solution $e^{i\omega t} \phi_\omega$ of (1.1) is **stable** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in X$ and $\|u_0 - \phi_\omega\|_X < \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists globally and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta} \phi_\omega\|_X < \varepsilon.$$  

Otherwise, $e^{i\omega t} \phi_\omega$ is said to be **unstable**.

**Definition 2.** We say that $e^{i\omega t} \phi_\omega$ is **strongly unstable** if for any $\varepsilon > 0$ there exists $u_0 \in X$ such that $\|u_0 - \phi_\omega\|_X < \varepsilon$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.

Before we state our main result, we recall some known results on the stability and instability of standing waves $e^{i\omega t} \phi_\omega$ for (1.1). When $\omega$ is sufficiently close to $-N$, the standing wave solution $e^{i\omega t} \phi_\omega$ of (1.1) is stable for any $p \in (1, 2^* - 1)$ (see [5]). On the other hand, when $\omega$ is sufficiently large, the standing wave solution $e^{i\omega t} \phi_\omega$ of (1.1) is stable if $1 < p \leq 1 + 4/N$ (see [5, 4]), and unstable if $1 + 4/N < p < 2^* - 1$ (see [6]). More precisely, it is proved in [6] that $e^{i\omega t} \phi_\omega$ is unstable if $\partial_\lambda^2 E(\phi_\lambda^\omega)|_{\lambda = 1} < 0$, where $v^\lambda(x) = \lambda^{N/2} v(\lambda x)$ is the $L^2$-invariant scaling (see also [13]).

However, the strong instability of $e^{i\omega t} \phi_\omega$ has been unknown for (1.1), although there are some results on blowup (see, e.g., [2, 19, 17]).

Now we state our main result in this paper.
Theorem 1. Let $N \geq 1$, $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and let $\phi_\omega$ be the positive solution of (1.2). Assume that $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} \leq 0$. Then, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is strongly unstable.

We remark that by the scaling $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$ for $\lambda > 0$, we have $\|v^\lambda\|_{L^2}^2 = \|v\|_{L^2}^2$ and

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\lambda^{-2}}{2}\|xv\|_{L^2}^2 - \frac{\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$  \hspace{1cm} (1.5)

Here and hereafter, we put $\alpha := \frac{N}{2}(p-1) > 2$.

Moreover, we define $S_\omega(v) = E(v) + \frac{\omega}{2}\|v\|_{L^2}^2$ for $v \in X$. Then, $\phi_\omega$ satisfies $S'_\omega(\phi_\omega) = 0$, and

$$0 = \partial_\lambda S_\omega(\phi^\lambda_\omega)|_{\lambda=1} = \|\nabla \phi_\omega\|_{L^2}^2 - \|x\phi_\omega\|_{L^2}^2 - \frac{\alpha}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1},$$

$$\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = \|\nabla \phi_\omega\|_{L^2}^2 + 3\|x\phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}$$

$$= 4\|x\phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-2)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}.$$  \hspace{1cm} (1.6)

Thus, the condition $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} \leq 0$ is equivalent to

$$\frac{\|x\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} \leq \frac{\alpha(\alpha-2)}{4(p+1)}.$$  \hspace{1cm} (1.7)

Furthermore, it is proved in Section 2 of [6] that

$$\lim_{\omega \to \infty} \frac{\|x\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} = 0.$$  \hspace{1cm} (1.8)

Therefore, as a corollary of Theorem 1, we have the following.

Corollary 2. Let $N \geq 1$, $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and let $\phi_\omega$ be the positive solution of (1.2). Then, there exists $\omega_0 \in (-N, \infty)$ depending only on $N$ and $p$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is strongly unstable for all $\omega \in (\omega_0, \infty)$.  \hspace{1cm} (1.9)
We give the proof of Theorem 1 in Sections 2 and 3. In Section 2, we introduce a subset \( B_\omega \) of \( X \), and Theorem 1 is reduced to Theorem 3. In Section 3, we give the proof of Theorem 3. The key to the proof of Theorem 3 is Lemma 4. The proof of Lemma 4 relies heavily on the specialty of harmonic potential, and it is not applicable to other types of nonlinear Schrödinger equations. For example, consider the nonlinear Schrödinger equation with a delta potential in one space dimension

\[
i \partial_t u = -\partial_x^2 u - \gamma \delta(x) u - |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},
\]

where \( \gamma > 0 \), \( \delta(x) \) is the delta measure at the origin, \( 5 < p < \infty \), \( \omega > \gamma^2/4 \), and \( \phi_\omega \) is a unique positive solution of the corresponding stationary problem.

It is proved in [15] that if \( E_\gamma(\phi_\omega) > 0 \), then \( e^{i\omega t} \phi_\omega \) is strongly unstable for (1.6) (see also [11, 14] for related results), where the energy for (1.6) is defined by

\[
E_\gamma(v) = \frac{1}{2} \| \partial_x v \|_{L^2}^2 - \frac{\gamma}{2} |v(0)|^2 - \frac{1}{p+1} \| v \|_{L^{p+1}}^{p+1}, \quad v \in H^1(\mathbb{R}).
\]

Here, we remark that \( E_\gamma(\phi_\omega) > 0 \) implies \( \partial^2_\lambda E_\gamma(\phi_\omega) \big|_{\lambda=1} < 0 \) for this case. The problem for (1.6) is completely different from that for (1.1), and the proof of Lemma 4 in this paper is not applicable to (1.6). So, it is still an open problem whether the standing wave solution \( e^{i\omega t} \phi_\omega \) of (1.6) is strongly unstable or not for the case where \( E_\gamma(\phi_\omega) \leq 0 \) and \( \partial^2_\lambda E_\gamma(\phi_\omega) \big|_{\lambda=1} < 0 \) (see [15, §4] for more remarks).

# 2 Proof of Theorem 1

Throughout this section, we assume that \( 1 + 4/N < p < 2^* - 1 \), \( \omega > -N \), and \( \phi_\omega \) is the positive solution of (1.2). We put \( \alpha = N(p-1)/2 > 2 \).

The proofs of blowup and strong instability of standing waves for nonlinear Schrödinger equations rely on the virial identity (see, e.g., [1, 3, 10, 18]). Let \( u(t) \) be the solution of (1.1) with \( u(0) = u_0 \in X \). Then, the function \( t \mapsto \| xu(t) \|_{L^2}^2 \) is in \( C^2[0, T_{\text{max}}) \), and satisfies

\[
\frac{d^2}{dt^2} \| xu(t) \|_{L^2}^2 = 16 P(u(t)) \quad (2.1)
\]

for all \( t \in [0, T_{\text{max}}) \), where

\[
P(v) = \frac{1}{2} \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| xv \|_{L^2}^2 - \frac{\alpha}{2(p+1)} \| v \|_{L^{p+1}}^{p+1}. \quad (2.2)
\]
Moreover, we define

\[ R(v) = \|\nabla v\|^2_{L^2} + 3\|xv\|^2_{L^2} - \frac{\alpha(\alpha - 1)}{p + 1}\|v\|^{p+1}_{L^{p+1}}. \]  

(2.3)

Note that by (1.5), we have

\[ P(v^\lambda) = \frac{1}{2}\lambda\partial_{\lambda} E(v^\lambda), \quad R(v^\lambda) = \lambda^2 \partial_{\lambda}^2 E(v^\lambda) \]  

(2.4)

for \( \lambda > 0 \). We also define

\[ A_\omega = \{ v \in X : E(v) < E(\phi_\omega), \| v \|^2_{L^2} = \| \phi_\omega \|^2_{L^2}, \| v \|^p_{L^p} > \| \phi_\omega \|^p_{L^p} \} \]

\[ B_\omega = \{ v \in A_\omega : P(v) < 0 \}. \]

**Lemma 1.** Assume that \( R(\phi_\omega) \leq 0 \). Then, \( \phi_\lambda^\omega \in B_\omega \) for all \( \lambda > 1 \).

**Proof.** First, we have \( \| \phi_\lambda^\omega \|^2_{L^2} = \| \phi_\omega \|^2_{L^2} \) and \( \| \phi_\lambda^\omega \|^{p+1}_{L^{p+1}} = \lambda^\alpha \| \phi_\omega \|^{p+1}_{L^{p+1}} > \| \phi_\omega \|^{p+1}_{L^{p+1}} \) for \( \lambda > 1 \).

Next, by drawing the graphs of the functions \( \lambda \mapsto P(\phi_\lambda^\omega) \) and \( \lambda \mapsto E(\phi_\lambda^\omega) \) using (1.3), (2.4), \( P(\phi_\omega) = 0 \) and \( R(\phi_\omega) \leq 0 \), we see that \( P(\phi_\lambda^\omega) < P(\phi_\omega) = 0 \) and \( E(\phi_\lambda^\omega) < E(\phi_\omega) \) for \( \lambda > 1 \). This completes the proof. \( \square \)

Since \( \phi_\lambda^\omega \to \phi_\omega \) in \( X \) as \( \lambda \to 1 \), Theorem 1 follows from Lemma 1 and the following Theorem 3.

**Theorem 3.** Let \( N \geq 1, 1 + 4/N < p < 2^*-1, \omega > -N \), and assume that \( R(\phi_\omega) \leq 0 \). If \( u_0 \in B_\omega \), then the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) blows up in finite time.

We give the proof of Theorem 3 in the next section.

### 3 Proof of Theorem 3

Throughout this section, we assume that \( 1 + 4/N < p < 2^*-1, \omega > -N \), and \( \phi_\omega \) is the positive solution of (1.2).

**Lemma 2.** If \( v \in X \) satisfies \( \| v \|^2_{L^2} = \| \phi_\omega \|^2_{L^2} \) and \( \| v \|^p_{L^p} = \| \phi_\omega \|^p_{L^p} \), then \( E(\phi_\omega) \leq E(v) \).
Proof. It is well known that

\[ S_{\omega}(\phi_{\omega}) = \inf \left\{ S_{\omega}(v) : v \in X, \|v\|_{L^{p+1}}^{p+1} = \|\phi_{\omega}\|_{L^{p+1}}^{p+1} \right\} \]

(see, e.g., Lemma 3.1 of [6]), where \( S_{\omega}(v) = E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 \).

Thus, if \( v \in X \) satisfies \( \|v\|_{L^2}^2 = \|\phi_{\omega}\|_{L^2}^2 \) and \( \|v\|_{L^{p+1}}^{p+1} = \|\phi_{\omega}\|_{L^{p+1}}^{p+1} \), then

\[ E(\phi_{\omega}) = S_{\omega}(\phi_{\omega}) - \frac{\omega}{2}\|\phi_{\omega}\|_{L^2}^2 \leq S_{\omega}(v) - \frac{\omega}{2}\|v\|_{L^2}^2 = E(v). \]

This completes the proof.

Lemma 3. The set \( A_{\omega} \) is invariant under the flow of (1.1). That is, if \( u_0 \in A_{\omega} \), then the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) satisfies \( u(t) \in A_{\omega} \) for all \( t \in [0, T_{\text{max}}) \).

Proof. This follows from the conservation laws (1.3) and Lemma 2.

By (1.4), (2.2) and (2.3), we have

\[ E(v) - P(v) = \|xv\|_{L^2}^2 + \frac{\alpha - 2}{2(p+1)}\|v\|_{L^{p+1}}^{p+1}, \quad (3.1) \]

\[ R(v) - 2P(v) = 4\|xv\|_{L^2}^2 - \frac{\alpha(\alpha - 2)}{p+1}\|v\|_{L^{p+1}}^{p+1}. \quad (3.2) \]

The following lemma is the key to the proof of Theorem 3.

Lemma 4. Assume that \( R(\phi_{\omega}) \leq 0 \). If \( v \in X \) satisfies

\[ P(v) \leq 0, \quad \|v\|_{L^2}^2 = \|\phi_{\omega}\|_{L^2}^2, \quad \|v\|_{L^{p+1}}^{p+1} > \|\phi_{\omega}\|_{L^{p+1}}^{p+1}, \quad (3.3) \]

then \( E(\phi_{\omega}) \leq E(v) - P(v) \).

Proof. Let \( v \in X \) satisfy (3.3).

If \( \|xv\|_{L^2}^2 \geq \|x\phi_{\omega}\|_{L^2}^2 \), then it follows from \( P(\phi_{\omega}) = 0 \) and (3.1) that

\[ E(\phi_{\omega}) = \|x\phi_{\omega}\|_{L^2}^2 + \frac{\alpha - 2}{2(p+1)}\|\phi_{\omega}\|_{L^{p+1}}^{p+1} \leq \|xv\|_{L^2}^2 + \frac{\alpha - 2}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} = E(v) - P(v). \]

So, in what follows, we assume that

\[ \|xv\|_{L^2}^2 \leq \|x\phi_{\omega}\|_{L^2}^2. \quad (3.4) \]
Let
\[ \lambda_0 = \left( \frac{\| \phi_\omega \|^p_{L^{p+1}}}{\| v \|^p_{L^{p+1}}} \right)^{1/\alpha}. \]  
(3.5)

Then, \( 0 < \lambda_0 < 1, \| v^{\lambda_0} \|^p_{L^{p+1}} = \| \phi_\omega \|^p_{L^{p+1}}, \| v^{\lambda_0} \|^2_{L^2} = \| \phi_\omega \|^2_{L^2}, \) and it follows from Lemma 2 that
\[ E(\phi_\omega) \leq E(v^{\lambda_0}). \]  
(3.6)

Next, we define
\[ f(\lambda) := E(v^\lambda) - \lambda^2 P(v) = \frac{\lambda^2 + \lambda^{-2}}{2} \| xv \|^2_{L^2} + \frac{\alpha \lambda^2 - 2 \lambda^\alpha}{2(p + 1)} \| v \|^p_{L^{p+1}} \]
for \( \lambda > 0. \) Then, \( f(\lambda_0) \leq f(1) \) if and only if
\[ (p + 1)(\lambda_0^2 + \lambda_0^{-2} - 2) \| xv \|^2_{L^2} \leq (2\lambda_0^\alpha - \alpha \lambda_0^2 + \alpha - 2) \| v \|^p_{L^{p+1}}. \]  
(3.7)

Here, since \( P(\phi_\omega) = 0 \) and \( R(\phi_\omega) \leq 0, \) it follows from (3.2) that
\[ 4(p + 1)\| x\phi_\omega \|^2_{L^2} \leq \alpha(\alpha - 2)\| \phi_\omega \|^p_{L^{p+1}}. \]  
(3.8)

Thus, by (3.4), (3.8) and (3.5), we have
\[ 4(p + 1)(\lambda_0^2 + \lambda_0^{-2} - 2) \| xv \|^2_{L^2} = 4(p + 1)(\lambda_0 - \lambda_0^{-1})^2 \| xv \|^2_{L^2} \]
\[ \leq 4(p + 1)(\lambda_0 - \lambda_0^{-1})^2 \| x\phi_\omega \|^2_{L^2} \]
\[ \leq \alpha(\alpha - 2)(\lambda_0 - \lambda_0^{-1})^2 \| \phi_\omega \|^p_{L^{p+1}} = \alpha(\alpha - 2)(\lambda_0 - \lambda_0^{-1})^2 \lambda_0^2 \| v \|^p_{L^{p+1}}. \]

Therefore, (3.7) holds if
\[ \alpha(\alpha - 2)(\lambda_0 - \lambda_0^{-1})^2 \lambda_0^2 \leq 4(2 \lambda_0^\alpha - \alpha \lambda_0^2 + \alpha - 2). \]  
(3.9)

Here, we put \( \beta = \alpha/2 \) and define
\[ g(s) := s^\beta - 1 - \beta(s - 1) - \frac{\beta(\beta - 1)}{2}(s - 1)^2 s^{\beta - 1} \]
for \( s > 0. \) Then, (3.9) is equivalent to \( g(\lambda_0^2) \geq 0. \) By the Taylor expansion of \( s^\beta \) at \( s = 1, \) we have
\[ g(\lambda_0^2) = \frac{\beta(\beta - 1)}{2}(\lambda_0^2 - 1)^2 \{ \xi^{\beta - 2} - (\lambda_0^2)^{\beta - 1} \} \]
for some \( \xi \in (\lambda_0^2, 1). \) Since \( \beta > 1 \) and \( \lambda_0^2 < \xi < 1, \) we have
\[ (\lambda_0^2)^{\beta - 1} \leq \xi^{\beta - 1} \leq \xi^{\beta - 2}. \]
and obtain \( g(\lambda_0^2) \geq 0 \). Thus, we have (3.7) and \( f(\lambda_0) \leq f(1) \).

Finally, since \( P(v) \leq 0 \), it follows from (3.6) that

\[
E(\phi_\omega) \leq E(v^{\lambda_0}) \leq E(v^{\lambda_0}) - \lambda_0^2 P(v) = f(\lambda_0) \leq f(1) = E(v) - P(v).
\]

This completes the proof.

Lemma 5. Assume that \( R(\phi_\omega) \leq 0 \). Then, the set \( B_\omega \) is invariant under the flow of (1.1).

Proof. Let \( u_0 \in B_\omega \) and \( u(t) \) be the solution of (1.1) with \( u(0) = u_0 \). Since \( \mathcal{A}_\omega \) is invariant under the flow of (1.1), we have only to show that \( P(u(t)) < 0 \) for all \( t \in [0, T_{\max}) \).

Suppose that there exists \( t_1 \in (0, T_{\max}) \) such that \( P(u(t_1)) \geq 0 \). Then, by the continuity of the function \( t \mapsto P(u(t)) \), there exists \( t_0 \in (0, t_1] \) such that \( P(u(t_0)) = 0 \). Moreover, by Lemma 5 we have \( \|u(t_0)\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2 \) and \( \|u(t_0)\|_{L^{p+1}}^{p+1} > \|\phi_\omega\|_{L^{p+1}}^{p+1} \). Thus, by Lemma 4 we have

\[
E(\phi_\omega) \leq E(u(t_0)) - P(u(t_0)) = E(u(t_0)).
\]

On the other hand, since it follows from Lemma 5 that \( E(u(t_0)) < E(\phi_\omega) \), this is a contradiction.

Therefore, \( P(u(t)) < 0 \) for all \( t \in [0, T_{\max}) \).

Now we give the proof of Theorem 3.

Proof of Theorem 3. Let \( u_0 \in B_\omega \) and let \( u(t) \) be the solution of (1.1) with \( u(0) = u_0 \). Then, by Lemma 5 \( u(t) \in B_\omega \) for all \( t \in [0, T_{\max}) \).

Moreover, by the virial identity (2.1), Lemma 4 and the conservation of energy (1.3), we have

\[
\frac{1}{16} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = P(u(t)) \leq E(u(t)) - E(\phi_\omega) = E(u_0) - E(\phi_\omega) < 0
\]

for all \( t \in [0, T_{\max}) \), which implies \( T_{\max} < \infty \). This completes the proof.

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