Some Two-Loop Corrections to the Finite Temperature Effective Potential in the Electroweak Theory*

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Abstract

Perturbation theory at finite temperature suffers from well-known infrared problems. In the standard model, as a result, one cannot calculate the effective potential for arbitrarily small values of $\phi$, the Higgs expectation value. Because the Higgs field is now known not to be extremely light, it is necessary to determine whether perturbation theory is a reliable guide to properties of the weak phase transition. In this note, we evaluate the most singular contributions to the potential at two loops as well as the leading strong interaction contributions. Above the critical temperature, the strong interaction corrections are reasonably small, while the weak corrections are about 10%, even for rather small values of the Higgs field. At the critical temperature, the weak corrections have a more substantial effect, rendering the transition significantly more first order, but not significantly changing the upper bound on the Higgs mass required for baryogenesis.

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1. Introduction

The possibility that the observed baryon asymmetry may have been created at the electroweak phase transition has renewed interest in understanding the details of this transition. The early studies of Kirzhnitz and Linde [1] and others indicated that in the minimal standard model, if \( m_H \ll M_W \) (\( m_H \) and \( M_W \) are the Higgs and \( W \) mass, respectively) this transition is first order, becoming more weakly so as the Higgs mass increases. This result was based on a study of the leading terms in the finite temperature potential. More recently, questions were raised about this analysis; it was argued that infrared problems might render the phase transition more [2] or less [3] weakly first order. In particular, it was argued that the potential, at two-loop order, should contain linear terms in the scalar field, \( \phi \) (though of differing signs). More careful study of the perturbation expansion, however, shows that these linear terms cancel at this order; the phase transition remains first order, though slightly less so [4].

Still, the finite temperature theory is full of infrared problems. As a result, it is not possible to answer all interesting questions in perturbation theory, even if the coupling is weak. The problems are easily illustrated in the minimal standard model, with a single Higgs doublet, \( \phi \). At high temperatures, the theory becomes essentially three-dimensional. For massless particles, the Feynman diagrams of such a theory exhibit power-law divergences. In the standard model, the gauge boson mass acts as an infrared cutoff. This means that for small \( \phi \), the potential cannot be reliably calculated. The results of ref. [4] imply that perturbation theory is good provided \( M_W(\phi) \gg g^2T \); \( g \) here denotes the gauge boson coupling.\(^\dagger\) Had the analysis of refs. [2] and [3] been correct, one would have had to impose the stronger condition \( M_W(\phi) \gg gT \). We are not really interested in the behavior of the theory for arbitrary values of the fields, however. One question which we would like to address in perturbation theory is the nature of the phase transition. For this, one needs to study the behavior of the potential near the minimum. As we will review below, for weak coupling in this theory the minimum of the potential occurs for \( g\phi \propto g^4T/\lambda \), where \( \lambda \) is the quartic Higgs coupling. Thus for sufficiently small \( \lambda \), the perturbation expansion should be reliable. Indeed, the expansion becomes an expansion in powers of \( g^2, \lambda, \) and \( \lambda/g^2 \). If perturbation theory is to be a useful guide to the behavior of this system, all of these quantities must be small.

On the other hand, we already know from LEP that the Higgs particle cannot be extremely light, and so the quartic coupling cannot be arbitrarily small. Even in extensions of the standard model, designed to produce a more strongly first

\(^\dagger\) We will set the Weinberg angle to zero in most of our discussion, since it is not germane to the issues at hand; it is easily restored where appropriate.
order transition [5] there are likely to be important constraints on the smallness of various couplings. Moreover, at finite temperature, one does not necessarily obtain the factors of $4\pi$ familiar in zero temperature calculations, and corrections are potentially large. It is thus natural to ask whether perturbation theory is a good guide in the experimentally allowed regions of parameter space.

In this note, we investigate some of these questions at two-loop order in the minimal standard model. It is known that in this model one cannot produce an asymmetry, given the present bound on the Higgs mass [4,6]. We choose the minimal model to illustrate our points because of its simplicity; our analysis is readily extended (often trivially extended) to other theories in which the phase transition is likely to be more strongly first order. A complete evaluation of all two-loop diagrams is quite involved. However, it turns out that there are certain leading contributions which can be easily computed. The simplest to describe are the QCD corrections. These are potentially important for top quark loops, due to the large value of the top quark mass and of the strong gauge coupling. They can be obtained by suitably adapting existing QED computations. In addition, there are certain purely electroweak effects which are enhanced by powers of logarithms of the scalar field. We have already remarked that in ref. [4] it was shown that certain leading infrared effects cancel. It turns out that the subleading terms come with logarithms of the gauge boson mass over the temperature. We will compute a set of terms in the potential proportional to $g^4\phi^2 \log(g\phi/2T)$. We will see that, away from the critical temperature, $T_c$, for the phase transition, both the QCD and pure electroweak effects give corrections of order a few per cent to the potential. Thus perturbation theory appears to be in quite good shape. Near $T_c$, however, the situation is more delicate; the weak interaction corrections are in fact quite large, here, and seem to make the transition more first order.

2. Review of Earlier Results

The zero temperature potential, taking into account one-loop corrections, is given by [7,8]

$$ V_0 = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + 2Bv_o^2\phi^2 - \frac{3}{2}B\phi^4 + B\phi^4 \ln \left(\frac{\phi^2}{v_o^2}\right). $$  \hspace{1cm} (2.1)

Here

$$ B = \frac{3}{64\pi^2v_o^4}(2m_W^4 + m_Z^4 - 4m_t^4), $$  \hspace{1cm} (2.2)

$v_o = 246$ GeV is the value of the scalar field at the minimum of $V_0$, $\lambda = \mu^2/v_o^2$. 


\( m_H^2 = 2 \mu^2 \). At a finite temperature, one should add to this expression the term

\[
V_T = \frac{T^4}{2\pi^2} \left[ 6I_- (y_W) + 3I_- (y_Z) - 12I_+ (y_t) \right],
\]

where \( y_i = M_i \phi / v_o T \), and

\[
I_\mp (y) = \pm \int_0^\infty dx \ x^2 \ln \left( 1 \mp e^{-x^2+y^2} \right).
\]

In the high temperature limit it is sufficient to use an approximate expression for \( V(\phi, T) \) [1,5]

\[
V(\phi, T) = D(T^2 - T_o^2) \phi^2 - ET \phi^3 + \frac{\lambda T}{4} \phi^4.
\]

Here

\[
D = \frac{1}{8v_o^2} (2m_W^2 + m_Z^2 + 2m_t^2), \\
E = \frac{1}{4\pi v_o^3} (2m_W^3 + m_Z^3) \sim 10^{-2}, \\
T_o^2 = \frac{1}{2D} (\mu^2 - 4Bv_o^2) = \frac{1}{4D} (m_H^2 - 8Bv_o^2), \\
\lambda_T = \lambda - \frac{3}{16\pi^2 v_o^4} \left( 2m_W^4 \ln \frac{m_W^2}{a_B T^2} + m_Z^4 \ln \frac{m_Z^2}{a_B T^2} - 4m_t^4 \ln \frac{m_t^2}{a_F T^2} \right),
\]

where \( \ln a_B = 2 \ln 4 \pi - 2\gamma \simeq 3.91, \ln a_F = 2 \ln \pi - 2\gamma \simeq 1.14 \). As explained in refs.[4] and [9], the constant \( E \) is reduced by higher order corrections by a factor of 2/3.

This potential leads to a phase transition which is at least weakly first order, basically as a consequence of the term cubic in \( \phi \). At very high temperatures, \( V \) has a unique minimum at \( \phi = 0 \). As the temperature is decreased, a second minimum appears at a temperature

\[
T_1^2 = \frac{T_o^2}{1 - 9E^2 / 8\lambda T_1 D}.
\]

At a temperature \( T_c \), this new minimum becomes degenerate with the minimum
at the origin; $T_c$ and the corresponding minimum of the potential $\phi_c$ are given by

$$T_c^2 = \frac{T_o^2}{1 - E^2/\lambda T_c} \quad \phi_c = \frac{2ET_c}{\lambda T_c}. \quad (2.11)$$

Finally, at $T_o$, the minimum at the origin disappears, and the potential has a unique minimum.

3. QCD Corrections

Because they are conceptually simplest, we consider, first, the strong corrections to the potential. The leading terms arise from a gluonic correction to a top quark loop (fig. 1.) Apart from a trivial group theory factor, this calculation is identical to the calculation of the two-loop free energy in QED [13,14].

If one examines the Feynman diagram containing the top quark (electron in QED) they are completely free of infrared problems even as the quark (electron) mass tends to zero. This is because the discrete Matsubara frequencies for fermions are all non-zero. However, the results of the QED calculation [13,14] diverge logarithmically with the mass $[m^2 \ln(m_e)]$. The reason is not difficult to understand. The QED calculation is performed by subtracting the electron mass on shell. The required counterterm itself diverges logarithmically. In QCD, we want to subtract the mass at some off-shell point; in the present case, we are interested in the mass (or, more precisely, $g_t$), at a scale of order $T$, the temperature. Renormalizing off mass shell in this way, there are no logarithms.

To evaluate the top quark contribution, we begin with the QED expression (for zero chemical potential) [13,14]

$$V = \frac{1}{3} e^2 T^2 \int \frac{d^3p}{(2\pi)^3} \frac{n_F(p)}{E_p} + e^2 \int \frac{d^3p}{2\pi^3} \int \frac{d^3q}{2\pi^3} \frac{1}{E_p E_q}$$

$$\times 2 \left[ 1 + \frac{m^2}{(E_p - E_q)^2 - (\vec{p} - \vec{q})^2} + \frac{m^2}{(E_p + E_q)^2 - (\vec{p} - \vec{q})^2} \right] n_F(p)n_F(q) \quad (3.1)$$

where

$$E^2 = \vec{p}^2 + m^2 \quad n_F(p) = \frac{1}{e^{\beta E_p} + 1}.$$

We will content ourselves with extracting the term of order $m^2$. The terms
involving \( m^2 \) explicitly in eq. (3.1) must be handled with some care (we thank Peter Arnold for pointing this out to us). It is necessary to perform the angular integrals first before taking the \( m \to 0 \) limit. The integration is then straightforward, apart from the integral

\[
\int dx dy \ln \left( \frac{(x-y)^2}{(x+y)^2 (1 + e^x)(1 + e^y)} \right) = \frac{\pi^2}{3} (\ln(2) - 1) \quad (3.2)
\]

(We thank Howard Haber for evaluation of this integral.) The remaining integrals factorize, and are readily evaluated using standard tricks (e.g., see appendix A of ref. [14]). In particular, the integral

\[
f_3(y) = \frac{1}{2} \int_0^\infty \frac{x^2 dx}{(x^2 + y^2)^{1/2}} \frac{1}{e \sqrt{x^2 + y^2} + 1} \quad (3.3)
\]

is given by

\[
f_3 = \frac{\pi^2}{24} + \frac{y^2}{8} (\ln y + \gamma - \ln \pi - \frac{1}{2}) \quad (3.4)
\]

As we have noted above, this does not yet give us the result we want; we need to add a finite counterterm to change the result from an on-mass-shell subtraction to an off-shell one. We choose to use the \( \overline{\text{MS}} \) scheme. A straightforward calculation gives that in the one-loop result, one must replace \( m \) by

\[
m \to m \left\{ 1 - \frac{3e^2}{16\pi^2} \left[ \ln \left( \frac{m^2}{\mu^2} \right) - \frac{4}{3} \right] \right\} \quad (3.5)
\]

To order \( m^2 \), the lowest order result is \( m^2/T^2 \). Putting everything together then gives for the fermionic contribution

\[
V_f = -\frac{7\pi^2 T^4}{60} + \frac{m^2 T^2}{4} + \frac{5g^2 T^4}{72} + \frac{g^2 m^2 T^2}{4\pi^2} \left[ -\frac{1}{2} - \ln(\pi) + \gamma + \frac{2}{3} \ln(2) \right] \quad (3.6)
\]

The corrections both to the leading \( T^4 \) term and to the quadratic term are about 8%.
Fig. 2. Two-loop diagrams which potentially give linear terms in the potential.

Fig. 3. The two diagrams of fig. 2 can be viewed as a correction to the one-loop gauge boson propagator.

4. Pure Electroweak Corrections

We now turn to the purely electroweak corrections. It is important, first, to understand the nature of the perturbation expansion at high temperatures. In general, if one examines the Feynman diagrams for the potential, one sees that they diverge in the infrared as the gauge boson masses (∼gφ/2) tend to zero. The divergences are cut off by the gauge boson masses themselves. For example, at one-loop, the coefficient of the quartic term in the potential is linearly divergent as the mass tends to zero; this is the origin of the cubic term in the potential. At higher loops one encounters more and more severe divergences. From the start, then, it is clear that perturbation theory cannot be good for arbitrarily small values of φ. One can hope, however, that if λ is small enough, the expectation value of the scalar field will be large enough that perturbation theory will be valid near the minimum of the potential. Note that the one-loop potential gives for the location of the minimum at Tc a result which goes as g3/λ. The potential at the minimum is of order g12/λ3.

The analysis of refs. [2] and [3] suggested that at two-loop order the coefficient of the quadratic term in the potential is linearly divergent. Extended to higher orders, this analysis would suggest that the loop expansion parameter is g2/M2W; for φ ∝ g3/λ, this gives λ2/g6 at the minimum. In ref. [4], it was shown that there are no such linear terms; a simple extension of the arguments given there shows that the expansion parameter is g2/MW, or λ/g2 at the minimum.

Actually, while it was shown in ref. [4] that the leading divergences cancel, the subleading terms come with logarithms of the gauge boson mass, i.e., with ln(φ2). To understand how this comes about, it is helpful to describe the results of ref. [4] in a slightly different fashion than explained in that paper. We assume that the Higgs mass is small and neglect Higgs self-interactions. We work, as there, in Coulomb gauge; the theory in the infrared is then like a three-dimensional theory in Landau gauge. As explained there, for small φ one can ignore the contributions of the longitudinal modes. The diagrams involving transverse gauge boson loops of fig. 2 are each potentially quite singular in the infrared. Indeed, while each diagram is formally of order φ2, power-counting suggests that the integrals separately behave as 1/φ, and that these diagrams should thus give linear terms in φ (this is the
source of the linear terms in refs. [2] and [3]). However, the leading divergences cancel between the two diagrams. This is easily understood; the diagrams can be organized as in fig. 3. In other words, one can view the combination of diagrams as a propagator correction to the one-loop gauge boson contribution to the free energy. To verify this requires care with the combinatorics; as explained in ref. [4], it is simplest to verify statements of this kind for tadpoles, rather than directly for the free energy. In any case, it is well-known that the three-dimensional gauge boson propagator vanishes at zero momentum as a consequence of gauge invariance. A straightforward calculation\textsuperscript{*} gives

\begin{equation}
\Pi_{ij} = \frac{11g^2T}{32\sqrt{q^2}}(q^2\delta_{ij} - q_iq_j).
\end{equation}

Because the polarization tensor vanishes linearly with the momentum, the integral behaves for small \( \phi \) as \( \ln(\phi) \). A simple calculation gives

\begin{equation}
\delta V_1 = -\frac{33g^4T^2\phi^2}{256\pi^2} \ln(g\phi/2T),
\end{equation}

(note that, as is conventional in the finite temperature theory, we are again normalizing \( \phi \) as a real field here).

There are other corrections at two loop order which depend on \( \log(\phi) \). These come from the diagrams in fig. 4, which contain scalar fields in intermediate states. These are somewhat more subtle to treat, since one must perform a resummation of some type in order to determine the effective scalar propagator. However, without probing this issue too deeply, it seems likely that any resummed scalar mass will be smaller than the gauge boson mass. For example, even well above the phase transition, the effective scalar mass in the one loop potential is somewhat smaller than the gauge boson mass; near the transition, it is much smaller. So to get an estimate of these effects, we will simply neglect it. In this case, the diagrams are straightforward to evaluate.\textsuperscript{†}

\begin{equation}
\delta V_2 = \frac{9g^4T^2\phi^2}{512\pi^2} \ln(g\phi/2T),
\end{equation}

\textsuperscript{*} This calculation can be found in many textbooks; see, e.g., refs. [16-17]. However, note that the formula the first reference contains a typographical error; we have verified that the formula in the second is correct.

\textsuperscript{†} In evaluating the first diagram, one approach is to route the loop momenta, \( \vec{l} \) and \( \vec{k} \) so that the scalar propagator depends on \( \vec{k} - \vec{l} \). This propagator can then be expanded in a (double) series of Legendre polynomials. The momentum integrals become trivial, and the sums give \( \zeta \)-functions.
while from the second we have

$$\delta V_3 = \frac{3g^4T^2\phi^2}{256\pi^2}\ln(g\phi/2T).$$  \hfill (4.4)

So overall, the shift is

$$\delta V_1 = -\frac{51g^4T^2\phi^2}{512\pi^2}\ln(g\phi/2T).$$  \hfill (4.5)

Away from the critical temperature, this correction is not too large. If the logarithm is of order 2 (corresponding to typical $\phi$'s at the transition), we only obtain a 10% correction. However, near the phase transition, the effect of this term is enhanced, because the quadratic term is so small. For example, keeping only the one-loop correction, the coefficient of the quadratic term at $T_c$ is of order $10^{-2}T_c^2$. This is smaller than the size of the logarithmic term. So the effect of the logarithmic correction is potentially dramatic. In order to evaluate its significance, note first that the various two-loop corrections we have considered alter the value of the critical temperature. Taking $m_H = 35$ GeV, we have first found this new temperature, $\tilde{T}_c$. In fig. 5, we have presented two curves: the one-loop result (with the correction of ref. [4]) at $T_c$, and the two-loop result, at $\tilde{T}_c$. The two-loop correction is seen to render the transition more first order; at the peak, the correction is almost a factor of 3. One can see in fig. 5 that the location of the minimum changes by about 20%. This weakens somewhat (by about 4 or 5 GeV) the bounds mentioned earlier on the Higgs mass.

It is reasonable to ask about the gauge-dependence of this result. This is a subtle question which we will not explore here. We expect, however, that for physically meaningful questions (such as the sphaleron rate at the minimum) these results should be gauge-independent.
5. Conclusions

From this work, it appears that perturbation theory at finite temperature in the standard model is not in terribly bad shape, even for Higgs fields which are not extremely light. Away from the critical temperature, perturbative corrections are quite small. Near $T_c$, however, the situation is somewhat more subtle. Because the leading quadratic terms in the potential vanish at this point, subleading terms which are not precisely quadratic (i.e., have logarithmic modifications) are of great importance. At two-loop order, in the minimal standard model, they render the transition significantly more first order, but only slightly modify the limits on the Higgs mass.

It is natural to ask about the effects of still higher order corrections. Examination of some particular graphs leads us to suspect that the picture presented here will not be drastically altered. Still, given the size of the corrections which we have found, non-perturbative calculations would be of great interest. In particular, we have not carefully investigated the problem of corrections to the potential loops with Higgs fields only, which might be quite important at the transition.
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