Kronecker’s Double Series and Exact Asymptotic Expansion for Free Models of Statistical Mechanics on Torus

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Abstract

For the free models of statistical mechanics on torus, exact asymptotic expansions of the free energy, the internal energy and the specific heat in the vicinity of the critical point are found. It is shown that there is direct relation between the terms of the expansion and the Kronecker’s double series. The latter can be expressed in terms of the elliptic $\theta$-functions in all orders of the asymptotic expansion.

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I. INTRODUCTION

It is well known that the singularities in thermodynamic functions associated with a critical point occur only in the thermodynamic limit when dimension $L$ of the system under consideration tends to infinity. In such a limit, the critical fluctuations are correlated over a distance of the order of correlation length $\xi_{\text{bulk}}$ that may be defined as the length scale governing the exponential decay of correlation functions. Besides these two fundamental lengths, $L$ and $\xi_{\text{bulk}}$, there is also the microscopic length of interactions $a$. Thermodynamic quantities thus may in principle depend on the dimensionless ratios $\xi_{\text{bulk}}/L$ and $a/L$. The Finite-Size Scaling (FSS) hypothesis [1] assumes that in the scaling interval, for temperatures so close to the critical point that $a \ll \xi_{\text{bulk}} \sim L$, the microscopic length drops out and the behaviour of any thermodynamic quantity can be described in terms of the universal scaling function of the scaling variable $t = \xi_{\text{bulk}}/L$. However, non-universal corrections to FSS do exist. These sometimes can be viewed as asymptotic series in powers of $a/L$.

Two-dimensional models of statistical mechanics have long served as a proving ground in attempts to understand critical behavior and to test the general ideas of FSS. Very few of them have been solved exactly [2]. Notably, the so-called free models that can be treated in terms of non-interacting quasi-particles on the lattice. The Ising model [3] (which is equivalent to the model of free lattice fermions) and the Gaussian model (which can be viewed as the model of free lattice bosons) are the most prominent examples.

For all free models, exact asymptotic expansion of the free energy on an infinite cylinder of circumference $L$ can easily be obtained by direct application of the Euler-Maclaurin summation formula [4,5]. However, derivation of such an expansion on a torus of area $S$ and aspect ratio $\rho$ is much more difficult problem. For the Ising model on torus, such an expansion has first been studied by Ferdinand and Fisher [6]. Starting with the explicit expression for the partition function [7], they have calculated two leading terms, $f_{\text{bulk}}$ and $f_0(\rho)$, of the expansion

$$F_{T=T_c}(\rho, S) = f_{\text{bulk}} S + f_0(\rho) + \sum_{p=1}^{\infty} f_p(\rho) S^{-p}$$

Their calculations have recently been pushed forward to get two next sub-leading terms [8,9].

It is the purpose of this paper to derive all terms of the exact asymptotic expansion of the logarithm of the partition function on torus for a class of free exactly solvable models of statistical mechanics. Our approach is based on an intimate relation between the terms of the asymptotic expansion and the so-called Kronecker’s double series. Besides the aesthetic appeal of the exact expansion, there is also physical motivation to study non-universal corrections to FSS. The problem is that in numerical simulations of lattice models one usually studies relatively small lattices. Therefore, to compare the results of high precision numerical simulations to the theoretical predictions one cannot neglect sub-leading corrections to FSS [10]. Non-universal terms in the asymptotic expansion also provide important information about the structure of irrelevant operators in conformal field theory [11].

2
II. FREE MODELS OF STATISTICAL MECHANICS

In this section, we formulate three basic models of statistical mechanics: Ising model, dimer model and Gaussian model. These models are often referred to as "free" since they were shown to be equivalent to free fermions or free bosons on the lattice. Partition functions of all these models can be written in terms of the only object — the partition function with twisted boundary conditions $Z_{\alpha,\beta}(\mu)$. Exact asymptotic expansion of the later will be our main objective in the subsequent section.

A. Ising model

The Ising model is usually formulated as follows. Consider a planar square lattice of size $M \times N$ with periodic boundary conditions, i.e. torus. To each site $(m, n)$ of the torus a spin variable is ascribed, $s_{m,n}$, with two possible values: +1 or −1. Two nearest neighbor spins, say $s_{m,n}$ and $s_{m,n+1}$ contribute a term $-J s_{m,n} s_{m,n+1}$ to the Hamiltonian, where $J$ is some fixed energy. Therefore, the Hamiltonian is simply the sum of all such terms, one for each edge of the lattice

$$H(s) = -J \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} (s_{m,n} s_{m,n+1} + s_{m,n} s_{m,n+1})$$

(2)

The partition function of the Ising model is given by the sum over all spin configurations on the lattice

$$Z_{\text{Ising}}(J) = \sum_{\{s\}} e^{-H(s)}$$

It is convenient to set up another parameterizations of the interaction constant $J$ in terms of the mass variable $\mu = \ln \sqrt{sh 2J}$. Critical point corresponds to the massless case $\mu = 0$.

An explicit expression for the partition function of the Ising model on $M \times N$ torus, which was given originally by Kaufmann [12], can be written as

$$Z_{\text{Ising}}(\mu) = \frac{1}{2} \left(\sqrt{2 e^\mu}\right)^{MN} \left\{Z_{\frac{1}{2},\frac{1}{2}}(\mu) + Z_{0,\frac{1}{2}}(\mu) + Z_{\frac{1}{2},0}(\mu) + Z_{0,0}(\mu)\right\}$$

(3)

where we have introduced the partition function with twisted boundary conditions

$$Z_{\alpha,\beta}^2(\mu) = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} 4 \left[ \sin^2 \left( \frac{\pi(n+\alpha)}{N} \right) + \sin^2 \left( \frac{\pi(m+\beta)}{M} \right) + 2 \sinh^2 \mu \right]$$

Here $\alpha = 0$ corresponds to the periodic boundary conditions for the underlying free fermion in the $N$-direction while $\alpha = \frac{1}{2}$ stands for anti-periodic boundary conditions. Similarly $\beta$ controls boundary conditions in $M$-direction. With the help of the identity [13]

$$4 \left| \sinh (M \omega + i \pi \beta) \right|^2 = 4 \left[ \sinh^2 M \omega + \sin^2 \pi \beta \right] = \prod_{m=0}^{M-1} 4 \left[ \sinh^2 \omega + \sin^2 \left( \frac{\pi(m+\beta)}{M} \right) \right]$$

the partition function with twisted boundary conditions $Z_{\alpha,\beta}$ can be transformed into simpler form.
\[ Z_{\alpha,\beta}(\mu) = \prod_{n=0}^{N-1} 2 \left| \text{sh} \left[ M \omega_{\mu} \left( \frac{\pi(n+\alpha)}{N} + i\pi\beta \right) \right] \right| \]  

(4)

where lattice dispersion relation has appeared

\[ \omega_{\mu}(k) = \text{arcsinh} \sqrt{\sin^2 k + 2 \text{sh}^2 \mu} \]  

(5)

This is nothing but the functional relation between energy \( \omega_{\mu} \) and momentum \( k \) of a free quasi-particle on the planar square lattice.

**B. Dimer model**

A ”dimer” is a two-atom molecule. When drawn on a lattice it covers two adjacent sites of the lattice and the bond that joins them. The ”dimer problem” is to determine the number of ways of covering of a given lattice with dimers, so that all sites are occupied and no two dimers overlap. If we consider planar square lattice of size \( 2M \times 2N \) wrapped on a torus then the number of dimers must be \( 2MN \) and the number of distinct dense coverings of the lattice (the partition function) has been calculated by Kasteleyn and Fisher \[ \text{[14,15]} \]. This can be expressed in terms of the same partition function with twisted boundary conditions Eq.(4) as

\[ Z_{\text{Dimer}} = \frac{1}{2} \left\{ Z_{1,1}^{2,1}(0) + Z_{0,1}^{2,1}(0) + Z_{1,0}^{2,0}(0) - Z_{0,0}^{2,0}(0) \right\} \]  

(6)

First two leading terms of the asymptotic expansion of this partition function has been obtained by Ferdinand \[ \text{[16]} \]. Let us also mention that dimers are always in the critical point and have no phase transition.

**C. Gaussian model**

Let us now turn to a boson analog of Ising model, which is often referred to as Gaussian model. Again, we consider planar square lattice of size \( N \times M \) wrapped on a torus. To each site \((m,n)\) of the lattice we assign a continuous variable \( x_{m,n} \). The Hamiltonian of the model is

\[ H(x) = -J \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left( x_{m,n} x_{m+1,n} + x_{m,n} x_{m,n+1} - 2x_{m,n}^2 \right) \]  

(7)

The partition function of the model can be written as

\[ Z(J) = \int_{\mathbb{R}^{MN}} e^{-H(x)} \, d\sigma(x) \]

If the measure \( d\sigma(x) \) in the phase space \( \mathbb{R}^{MN} \) is chosen to be Gaussian

\[ d\sigma_{\text{Gauss}}(x) = \pi^{-MN/2} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} e^{-x_{m,n}^2} \, dx_{m,n} \]
the integration can be done explicitly and the partition function of the free boson model can be written in terms of the partition function with twisted boundary conditions Eq. (1) and parameterization $J^{-1} = 4 \cosh^2 \mu$ as

$$Z_{\text{Gauss}}(\mu) = \left(\sqrt{2} \cosh \mu\right)^{MN} \left[ Z_{0,0}(\mu) \right]^{-1}$$

(8)

This model exhibit phase transition at the point $\mu_c = 0$ where the partition function is divergent. This is due to the presence of so-called zero mode, i.e. due to the symmetry transformation $x_{mn} \rightarrow x_{mn} + \text{const}$, which leave the Hamiltonian (7) invariant. Correlation functions of disorder operator in this model have been studied by Sato, Miwa and Jimbo [17].

The reason why this model is often considered as boson analog of the Ising model is that, one can choose another measure in the phase space, which makes this model equivalent to the Ising model considered above

$$d\sigma_{\text{Ising}}(x) = 2^{-MN} \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} \left[ \delta(x_{mn} - 1) + \delta(x_{mn} + 1) \right] dx_{mn}$$

where $\delta$'s are Dirac $\delta$-functions. With such a definition the variables $x_{mn}$ can actually take only two values: +1 or −1, so that $x_{mn}^2 = 1$. In this case integration can be replaced by summation over discrete values of $x_{mn} = \pm 1$ and the Hamiltonian (7) coincides with the Hamiltonian of the Ising model (3) up to a constant.

III. ASYMPTOTIC EXPANSION IN TERMS OF KRONECKER’S DOUBLE SERIES

In the previous section it was shown that partition functions of three basic free models of statistical mechanics can be expressed in terms of the only object: the partition function with twisted boundary condition $Z_{\alpha,\beta}(\mu)$. In this section we shall obtain exact asymptotic $1/S$-expansion of the partition function near the critical point. For reader’s convenience, all the technical details of our calculations and the definitions of the special functions are summarized in the appendices attached to the paper.

First of all, let us mention the symmetry properties of the partition function $Z_{\alpha,\beta}(\mu)$. From its definition (4) one can easily verify that it is even and periodic with respect to its arguments $\alpha$ and $\beta$

$$Z_{\alpha,\beta}(\mu) = Z_{\alpha,-\beta}(\mu) = Z_{-\alpha,\beta}(\mu)$$

$$Z_{\alpha,\beta}(\mu) = Z_{1+\alpha,\beta}(\mu) = Z_{\alpha,1+\beta}(\mu)$$

These imply that twist angles $\alpha$ and $\beta$ can be taken from the interval $[0, 1]$. Then, one can note that for all twists $(\alpha, \beta) \neq (0, 0)$ the partition function $Z_{\alpha,\beta}(\mu)$ is even with respect to its mass argument $\mu$. Hence, near the critical point $(\mu = 0)$ we have

$$Z_{\alpha,\beta}(\mu) = Z_{\alpha,\beta}(0) + \frac{\mu^2}{2!} Z_{\alpha,\beta}''(0) + \ldots \quad (\alpha, \beta) \neq (0, 0)$$

(9)
The only exception is the point where both $\alpha$ and $\beta$ are equal to zero. This case has to be treated separately since at this point the partition function turns to zero. As a result, we have

$$Z_{0,0}(\mu) = \mu Z'_{0,0}(0) + \frac{\mu^3}{3!} Z''_{0,0}(0) + \ldots \quad (\alpha, \beta) = (0, 0)$$

(10)

In what follows notation $Z_{\alpha,\beta}(\mu)$ will imply $(\alpha, \beta) \neq (0, 0)$.

A. Asymptotic Expansion of $Z_{\alpha,\beta}(0)$

Considering the logarithm of the partition function with twisted boundary conditions, Eq.(11), we note, that it can be transformed as

$$\ln Z_{\alpha,\beta}(0) = M \sum_{n=0}^{N-1} \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + \sum_{n=0}^{N-1} \ln \left| 1 - e^{-2 \left[ M \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i \pi \beta \right] } \right|$$

(11)

The second sum here vanishes in the formal limit $M \to \infty$ when the torus turns into infinitely long cylinder of circumference $N$. Therefore, the first sum gives the logarithm of the partition function with twisted angle $\alpha$ on that cylinder. Its asymptotic expansion can be found with the help of the Euler-Maclaurin summation formula (Appendix A)

$$M \sum_{n=0}^{N-1} \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) = \frac{S}{\pi} \int_0^\pi \omega_0(x) \, dx - \pi \lambda \rho B_2^\alpha - 2 \pi \rho \sum_{p=1}^{\infty} \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\lambda_{2p}}{(2p)!} \frac{B_{2p+2}^\alpha}{2p+2}$$

(12)

Where $\int_0^\pi \omega_0(x) \, dx = 2\gamma$, $\gamma = 0.915965...$ is Catalan’s constant and $B_p^\alpha$ are so-called Bernoulli polynomials. We have also used the symmetry property, $\omega_0(k) = \omega_0(\pi - k)$, of the lattice dispersion relation (11) and its Taylor expansion

$$\omega_0(k) = k \left( \lambda + \sum_{p=1}^{\infty} \frac{\lambda_{2p}}{(2p)!} k^{2p} \right)$$

(13)

where $\lambda = 1$, $\lambda_2 = -2/3$, $\lambda_4 = 4$, etc. In what follows, we shall not use the special values of these coefficients assuming the possibility for generalizations.

The second term in Eq.(11) we may transform as

$$\sum_{n=0}^{N-1} \ln \left| 1 - e^{-2 \left[ M \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i \pi \beta \right] } \right| =$$

$$= \Re \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \sum_{n=0}^{\left[ N/2 \right] - 1} e^{-2m \left[ M \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i \pi \beta \right] } + \sum_{n=0}^{N - \left[ N/2 \right] - 1} e^{-2m \left[ M \omega_0 \left( \frac{\pi(n+1+\alpha)}{N} \right) + i \pi \beta \right] } \right\}$$

(14)

The argument of the first exponent can be expanded in powers of $1/S$ if we replace the lattice dispersion relation $\omega_0(x)$ with its Taylor expansion (13)

$$\exp \left\{ -2\pi m \left[ \lambda \rho(n + \alpha) + i \beta \right] - 2\pi m \rho \sum_{p=1}^{\infty} \frac{\lambda_{2p}}{(2p)!} \left( \frac{\pi^2 \rho}{S} \right)^p (n + \alpha)^{2p+1} \right\}$$
Taking into account the relation between moments and cumulants (Appendix B), we obtain asymptotic expansion of the first exponent itself in powers of $1/S$

$$e^{-2m \left[ M \omega_0 \left( \frac{n+\alpha}{N} \right) + i\pi \beta \right]} = e^{-2\pi m \left[ \lambda \rho(n+\alpha) + i\beta \right]}$$

$$- 2\pi m \rho \sum_{p=1}^{\infty} \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p}}{(2p)!} (n + \alpha)^{2p+1} e^{-2\pi m \left[ \lambda \rho(n+\alpha) + i\beta \right]}$$

The differential operators $\Lambda_{2p}$ that have appeared here can be expressed via coefficients $\lambda_{2p}$ of the expansion of the lattice dispersion relation as (Appendix C), or Kronecker’s double series, $K^{\alpha,\beta}\left( \tau \right)$ (Appendix D). Namely, with the help of the identities (C2) and (D1) we obtain

$$\Lambda_2 = \lambda_2$$

$$\Lambda_4 = \lambda_4 + 3\lambda_2^2 \frac{\partial}{\partial \lambda}$$

$$\Lambda_6 = \lambda_6 + 15\lambda_4\lambda_2\frac{\partial}{\partial \lambda} + 15\lambda_3^2 \frac{\partial^2}{\partial \lambda^2}$$

$$\vdots$$

$$\Lambda_p = \sum_{r=1}^{p} \sum \left( \frac{\lambda_{p_1}}{p_1!} \right) \cdots \left( \frac{\lambda_{p_r}}{p_r!} \right) k_1! \cdots k_r! \frac{\partial^k}{\partial \lambda^k}$$

Here summation is over all positive numbers $\{k_1 \ldots k_r\}$ and different positive numbers $\{p_1, \ldots, p_r\}$ such that $p_1k_1 + \ldots + p_rk_r = p$ and $k = k_1 + \ldots + k_r - 1$.

The expansion for the second exponent in Eq.(14) can be obtained along the same lines by substitution: $\alpha \rightarrow 1 - \alpha$. Plugging the expansion of both of the exponents back into Eq.(14) we obtain

$$\sum_{n=0}^{N-1} \left[ 1 - e^{-2\left[ M \omega_0 \left( \frac{n+\alpha}{N} \right) + i\pi \beta \right]} \right] =$$

$$= - \Re \sum_{m=1}^{\infty} \frac{1}{m} \left[ \sum_{n=0}^{[\frac{N}{2}] - 1} e^{-\pi n \left[ \lambda \rho(n+\alpha) + i\beta \right]} + \sum_{n=0}^{N - [\frac{N}{2}] - 1} e^{-\pi n \left[ \lambda \rho(n+\alpha - 1) + i\beta \right]} \right]$$

$$+ 2\pi \rho \sum_{p=1}^{\infty} \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p}}{(2p)!} \Re \sum_{m=1}^{\infty} \left[ \sum_{n=0}^{[\frac{N}{2}] - 1} (n + \alpha)^{2p+1} e^{-2\pi n \left[ \lambda \rho(n+\alpha) + i\beta \right]} \right]$$

$$+ \sum_{n=0}^{N - [\frac{N}{2}] - 1} (n + 1 - \alpha)^{2p+1} e^{-2\pi n \left[ \lambda \rho(n+\alpha - 1) + i\beta \right]} \right]$$

In all these series, summation over $n$ can be extended to infinity. The resulting errors are exponentially small and do not affect our asymptotic expansion in any finite power of $1/S$.

The key point of our analysis is the observation that all the series that have appeared in such an expansion can be obtained by resummation of either elliptic theta function, $\theta_{\alpha,\beta}(\tau)$ (Appendix E), or Kronecker’s double series, $K^{\alpha,\beta}\left( \tau \right)$ (Appendix D). Namely, with the help of the identities (E2) and (E1) we obtain

$$\sum_{n=0}^{N-1} \ln \left[ 1 - e^{-2\left[ M \omega_0 \left( \frac{n+\alpha}{N} \right) + i\pi \beta \right]} \right] = \ln \left[ \frac{\theta_{\alpha,\beta}(i\lambda \rho)}{\eta(i\lambda \rho)} \right] + \pi \lambda \rho B_{2}^{\alpha}$$

$$- 2\pi \rho \sum_{p=1}^{\infty} \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p}}{(2p)!} \Re K_{2p+2}^{\alpha,\beta}(i\lambda \rho) - B_{2p+2}^{\alpha}$$

$$- 2\pi \rho \sum_{p=1}^{\infty} \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p}}{(2p)!} \Re K_{2p+2}^{\alpha,\beta}(i\lambda \rho) - B_{2p+2}^{\alpha} \quad (15)$$
Substituting Eqs. (12) and (15) into Eq. (11) we finally obtain exact asymptotic expansion of the logarithm of the partition function with twisted boundary conditions in terms of the Kronecker’s double series

$$
\ln Z_{\alpha,\beta}(0) = \frac{S}{\pi} \int_0^\pi \omega_0(x) \, dx + \ln \left| \frac{\theta_{\alpha,\beta}(i\lambda \rho)}{\eta(i\lambda \rho)} \right| \\
-2\pi \rho \sum_{p=1}^\infty \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p} \Re K_{2p+2}^{\alpha,\beta}(i\lambda \rho)}{(2p)!} 
$$

(16)

Note, that Bernoulli polynomials $B_p^\alpha$ have finally dropped out from the asymptotic expansion on torus. This actually means that Kronecker’s double series can be considered as elliptic generalizations of Bernoulli polynomials.

B. Asymptotic Expansion of $Z'_{0,0}(0)$

As it has already been mentioned, we have to treat the case $(\alpha, \beta) = (0, 0)$ separately. Taking the derivative of Eq. (4) with respect to mass variable $\mu$ and then considering limit $\mu \rightarrow 0$ we obtain

$$
Z'_{0,0}(0) = 2\sqrt{2} M \prod_{n=1}^{N-1} \left| 2 \sin \left( \frac{\pi}{N} \right) \right|
$$

Asymptotic expansion of this expression can be found along the same lines as above. In terms of the Kronecker’s double series, the expansion can be written as

$$
\ln Z'_{0,0}(0) = \frac{S}{\pi} \int_0^\pi \omega_0(x) \, dx + \frac{1}{2} \ln 8\rho S + 2 \ln \left| \eta(i\lambda \rho) \right| \\
-2\pi \rho \sum_{p=1}^\infty \left( \frac{\pi^2 \rho}{S} \right)^p \frac{\Lambda_{2p} \Re K_{2p+2}^{0,0}(i\lambda \rho)}{(2p)!} 
$$

(17)

C. Asymptotic Expansion of $Z''_{\alpha,\beta}(0)$

The analysis of the $Z''_{\alpha,\beta}(0)$ is a little more involved. Taking the second derivative of Eq. (11) with respect to mass variable $\mu$ and then considering limit $\mu \rightarrow 0$ we obtain

$$
\frac{Z''_{\alpha,\beta}(0)}{Z_{\alpha,\beta}(0)} = \Re M \sum_{n=0}^{N-1} \omega''_0 \left( \frac{\pi(n+\alpha)}{N} \right) \cth \left[ M \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i\pi \beta \right] \\
= M \sum_{n=0}^{N-1} \omega''_0 \left( \frac{\pi(n+\alpha)}{N} \right) + 2 \Re M \sum_{n=0}^{N-1} \sum_{m=1}^{\infty} \omega''_0 \left( \frac{\pi(n+\alpha)}{N} \right) e^{-2m[M \omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i\pi \beta]} 
$$

(18)

where $\omega''_0(x)$ is the second derivative of $\omega_\mu(x)$ with respect to $\mu$ at criticality

$$
\omega''_0(x) = \frac{2}{\sin x \sqrt{1 + \sin^2 x}}
$$
Using Taylor’s theorem, the asymptotic expansion of the $\omega_0''(x)$ can be written in the following form

$$\omega_0''(x) = \frac{\kappa}{x} \left[1 + \sum_{p=1}^{\infty} \frac{\kappa_{2p}}{(2p)!} x^{2p}\right]$$

where $\kappa = 2$, $\kappa_2 = -2/3$, $\kappa_4 = 172/15$, etc. Again, in what follows, we shall not use the special values of these coefficients assuming the possibility for generalizations.

The first sum in Eq. (18) we may transform as

$$M \sum_{n=0}^{N-1} \frac{\omega_0''(\pi(n+\alpha)/N)}{n} = M \sum_{n=0}^{N-1} f\left(\pi(n+\alpha)/N\right) + \kappa S \sum_{n=0}^{N-1} \left(\frac{1}{n + \alpha} + \frac{1}{n + 1 - \alpha}\right)$$

where we have introduce the function $f(x) = \omega_0''(x) - \kappa/x - \kappa/(\pi - x)$. This function and all its derivatives are integrable over the interval (0, $\pi$). Thus, for the first term in Eq. (19) we may use again the Euler-Maclaurin summation formula (Appendix A), and after a little algebra we obtain

$$M \sum_{n=0}^{N-1} \frac{\omega_0''(\pi(n+\alpha)/N)}{n} = \frac{S}{\pi} \int_0^\pi f(x) \, dx - \pi \rho \kappa \sum_{p=1}^{\infty} \left(\frac{\pi^2 \rho}{S}\right)^{p-1} \frac{\kappa_{2p} B_2^\alpha}{p(2p)!} + \kappa S \pi \sum_{p=1}^{\infty} \frac{B_2^\alpha}{p} \frac{1}{N^{2p}}$$

where $\int_0^\pi f(x) \, dx = 2 \ln 2 - 4 \ln \pi$. The second sum in Eq. (19) can be written in terms of the digamma function $\psi(x)$.

$$\sum_{n=0}^{N-1} \left(\frac{1}{n + \alpha} + \frac{1}{n + 1 - \alpha}\right) = [\psi(N + \alpha) + \psi(N + 1 - \alpha) - \psi(\alpha) - \psi(1 - \alpha)]$$

The asymptotic expansion of the digamma function $\psi(x)$ is given by (see Appendix E)

$$\psi(N + \alpha) = \ln N - \sum_{p=1}^{\infty} (-1)^p \frac{B_p^\alpha}{p} \frac{1}{N^p}$$

Using the symmetry properties of the Bernoulli polynomials $B_p^\alpha$, namely, $B_{2p}^{1-\alpha} = B_{2p}^\alpha$ and $B_{2p+1}^{1-\alpha} = -B_{2p+1}^\alpha$, the Eq. (21) can be rewritten as

$$\sum_{n=0}^{N-1} \left(\frac{1}{n + \alpha} + \frac{1}{n + 1 - \alpha}\right) = 2 \ln N - \sum_{p=1}^{\infty} \frac{B_{2p}^\alpha}{p} \frac{1}{N^{2p}} - \psi(\alpha) - \psi(1 - \alpha)$$

Plugging Eqs. (20) and (23) back in Eq. (19) we have finally obtain

$$M \sum_{n=0}^{N-1} \frac{\omega_0''(\pi(n+\alpha)/N)}{n} = \frac{2\kappa S}{\pi} \left[\frac{1}{2\kappa} \int_0^\pi f(x) \, dx + \ln N - \frac{\psi(\alpha) + \psi(1 - \alpha)}{2}\right]$$

$$- \pi \rho \kappa \sum_{p=1}^{\infty} \left(\frac{\pi^2 \rho}{S}\right)^{p-1} \frac{\kappa_{2p} B_2^\alpha}{p(2p)!}$$
Let us now consider the second sum in Eq. (18). Note that function \( \omega''_0(x) \) can be represented as

\[
\omega''_0(x) = \frac{\kappa}{x} \exp \left\{ \sum_{p=1}^{\infty} \frac{\varepsilon_{2p}}{(2p)!} x^{2p} \right\}
\]

where coefficients \( \varepsilon_{2p} \) and \( \kappa_{2p} \) are related to each other through relation between moments and cumulants (Appendix B). Following along the same lines as in the section (3.1), the second sum in Eq. (18) can be written as

\[
2 \text{Re} M \sum_{n=0}^{N-1} \sum_{m=1}^{\infty} \omega''_0 \left( \frac{\pi(n+\alpha)}{N} \right) e^{-2\pi m \left[ M\omega_0 \left( \frac{\pi(n+\alpha)}{N} \right) + i\pi\beta \right]} =
\]

\[
= \frac{2\kappa S}{\pi} \text{Re} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{n+\alpha} e^{-2\pi m \left[ \rho(n+\alpha)+i\pi\beta \right]} + \frac{1}{n+1-\alpha} e^{-2\pi m \left[ \rho(n+1-\alpha)+i\pi\beta \right]} \right\}
\]

\[
+ \kappa\pi\rho\Omega_2 \text{Re} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ (n+\alpha) e^{-2\pi m \left[ \lambda\rho(n+\alpha)+i\pi\beta \right]} + (n+1-\alpha) e^{-2\pi m \left[ \lambda\rho(n+1-\alpha)+i\pi\beta \right]} \right\}
\]

\[
- \kappa\pi\rho \sum_{p=2}^{\infty} \frac{\Omega_{2p}}{p(2p)!} \left( \frac{\pi^2\rho}{S} \right)^{p-1} \text{Re} K_{2p}^{\alpha,\beta} (i\lambda\rho) + \kappa\pi\rho \sum_{p=2}^{\infty} \frac{\kappa_{2p} B_{2p}^{\alpha}}{p(2p)!} \left( \frac{\pi^2\rho}{S} \right)^{p-1}
\]

The differential operators \( \Omega_{2p} \) that have appeared here can be expressed via coefficients \( \omega_{2p} = \varepsilon_{2p} + \lambda_{2p} \frac{\partial}{\partial \alpha} \) as

\[
\Omega_2 = \omega_2 \\
\Omega_4 = \omega_4 + 3\omega_2^2 \\
: 
\]

Let us introduce the function \( R_{\alpha,\beta}(\rho) \)

\[
R_{\alpha,\beta}(\rho) = -\frac{\psi(\alpha) + \psi(1-\alpha)}{2} + \text{Re} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{n+\alpha} e^{-2\pi m \left[ \rho(n+\alpha)+i\pi\beta \right]} + \frac{1}{n+1-\alpha} e^{-2\pi m \left[ \rho(n+1-\alpha)+i\pi\beta \right]} \right\}
\]

which first derivative with respect to \( \rho \) is given by (see Appendix C, Eq. (C4))

\[
\frac{\partial}{\partial \rho} R_{\alpha,\beta}(\rho) = -2 \frac{\partial}{\partial \rho} \ln |\theta_{\alpha,\beta}(i\rho)| + \frac{1}{2\pi} \left( \frac{\partial}{\partial \rho} \ln |\theta_{\alpha,\beta}(i\rho)| \right)^2 \]

For the cases \((\alpha, \beta) = (0, 1/2), (1/2, 0), (1/2, 1/2)\) the second term in above equation is equal to zero, and for \( R_{\alpha,\beta}(\rho) \) we obtain

\[
R_{\alpha,\beta}(\rho) = -2 \ln |\theta_{\alpha,\beta}(i\rho)| + C_E + 2 \ln 2
\]

where \( C_E \) is the Euler constant.

With the help of the identity (C3) the Eq. (24) can be rewritten as
\[ 2 \text{Re} M \sum_{n=0}^{N-1} \sum_{m=1}^{\infty} \omega_0^{(n+m)} e^{-2m[M\omega_0(\frac{n+m}{N}) + i\pi \beta]} = \]
\[ = \frac{2\kappa S}{\pi} \left[ R_{\alpha,\beta}(\rho) + \frac{\psi(\alpha) + \psi(1-\alpha)}{2} \right] + \frac{\kappa}{2} \left( \kappa_2 \rho \frac{\partial}{\partial \rho} + \lambda_2 \rho^2 \frac{\partial^2}{\partial \rho^2} \right) \ln \left| \frac{\theta_{\alpha,\beta}(i\rho)}{\eta(i\rho)} \right| \]
\[ - \kappa \pi \rho \sum_{p=2}^{\infty} \frac{\Omega_{2p}}{p(2p)!} \left( \frac{\pi^2 \rho}{S} \right)^{p-1} \text{Re} K_{2p}^{\alpha,\beta}(i\lambda \rho) \]

Substituting Eqs. (24) and (30) into Eq. (18) we finally obtain exact asymptotic expansion of the \( Z_{\alpha,\beta}''(0) \)

\[
\frac{Z_{\alpha,\beta}''(0)}{Z_{\alpha,\beta}(0)} = \frac{2\kappa S}{\pi} \left[ \frac{1}{2\kappa} \int_0^\pi f(x) \, dx + \ln N + R_{\alpha,\beta}(\rho) \right] + \frac{\kappa}{2} \left( \kappa_2 \rho \frac{\partial}{\partial \rho} + \lambda_2 \rho^2 \frac{\partial^2}{\partial \rho^2} \right) \ln \left| \frac{\theta_{\alpha,\beta}(i\rho)}{\eta(i\rho)} \right| \]
\[ - \kappa \pi \rho \sum_{p=2}^{\infty} \frac{\Omega_{2p}}{p(2p)!} \left( \frac{\pi^2 \rho}{S} \right)^{p-1} \text{Re} K_{2p}^{\alpha,\beta}(i\lambda \rho) \] (31)

D. Asymptotic Expansion of \( Z_{0,0}''(0) \)

Let us now consider the case \((\alpha, \beta) = (0, 0)\). Eq. (10) implies immediately that

\[
\lim_{\mu \to 0} \frac{Z_{0,0}''(\mu)}{Z_{0,0}(\mu)} = \frac{Z_{0,0}''(0)}{Z_{0,0}(0)} \] (32)

Taking the third derivative of Eq. (4) with respect to mass variable \(\mu\) and then considering limit \(\mu \to 0\) we obtain

\[
\frac{Z_{0,0}'''(0)}{Z_{0,0}'(0)} = 2M^2 - 1 + 3M \sum_{n=1}^{N-1} \omega_0^{(n)} \left( \frac{\pi n}{N} \right) \text{cth} \left[ M\omega_0 \left( \frac{\pi n}{N} \right) \right] \] (33)

Asymptotic expansion of the \( Z_{0,0}'(0) \) can be found along the same lines as above. In terms of the Kronecker’s double series, the expansion can be written as

\[
\frac{Z_{0,0}'''(0)}{Z_{0,0}'(0)} = \frac{6\kappa S}{\pi} \left[ \frac{1}{2\kappa} \int_0^\pi f(x) \, dx + \ln N + C_E - \ln \eta(i\rho) \right] + 3\kappa \left( \kappa_2 \rho \frac{\partial}{\partial \rho} + \lambda_2 \rho^2 \frac{\partial^2}{\partial \rho^2} \right) \ln \eta(i\rho) - 1 \]
\[ - 3\kappa \pi \rho \sum_{p=2}^{\infty} \frac{\Omega_{2p}}{p(2p)!} \left( \frac{\pi^2 \rho}{S} \right)^{p-1} K_{2p}^{0,0}(i\lambda \rho) \] (34)

Expansions (13), (17), (31) and (34) are the main results of the paper. Kronecker’s double series \( K_{\mu}^{\alpha,\beta} \) with \(\alpha\) and \(\beta\) taking values 0 and 1/2 can all be expressed in terms of the elliptic \(\theta\)-functions only (Appendix F).
IV. ASYMPTOTIC EXPANSION OF THE FREE ENERGY,
THE INTERNAL ENERGY AND THE SPECIFIC HEAT

After reaching this point, one can easily write down all the terms of the exact asymptotic expansion \( F = -\ln Z \), at the critical point for all three models under consideration. For the Ising model and the Gaussian model we have found that the exact asymptotic expansion of the internal energy, \( U = -\partial \ln Z/J \), and the specific heat, \( C = \partial^2 \ln Z/J^2 \), at the critical point can be written in the following form

\[
U = u_{\text{bulk}} S + \sum_{p=0}^{\infty} u_p(\rho) S^{-p+\frac{1}{2}}
\]

\[
C = c_{\text{bulk}} S + \sum_{p=0}^{\infty} c_{2p}(\rho) S^{-p+\frac{1}{2}} + \sum_{p=0}^{\infty} c_{2p+1}(\rho) S^{-p}
\]

1. The asymptotic expansion of the free energy of the Ising model is

\[
f_{\text{bulk}} = -\ln \sqrt{2} - \frac{2\gamma}{\pi}
\]

\[
f_0(\rho) = -\ln \frac{\theta_2 + \theta_3 + \theta_4}{2\eta}
\]

\[
f_1(\rho) = -\frac{\pi^3 \rho^4}{180} \left[ \frac{3(\theta_2 + \theta_3 + \theta_4)}{\theta_2 + \theta_3 + \theta_4} + \theta_2 \theta_3 \theta_4 \right] \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
f_2(\rho) = -\frac{\pi^6 \rho^5}{18432} \left[ \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right] \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
- \frac{31 \pi^5 \rho^4}{24192} \left( \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right) \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
- \frac{1512}{72576} \left( \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right) \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
- \frac{4536}{4608} \left( \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right) \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
u_{\text{bulk}} = -\sqrt{2}
\]

\[
u_0(\rho) = -2\sqrt{\rho} \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4}
\]

\[
u_1(\rho) = \frac{\pi^3 \rho^{5/2}}{48} \left[ \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right] \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2
\]

\[
u_2(\rho) = \frac{\pi^6 \rho^{4/2}}{4608} \left[ \frac{3 \theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right] \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^3
\]
\[
\begin{align*}
&\frac{\pi^6 \rho^{9/2}}{9216} \theta_2 \theta_3 \theta_4 \left[ 23(\theta_2^{17} + \theta_3^{17} + \theta_4^{17}) + 8\theta_2^4 \theta_4^4 (5\theta_2^9 + 5\theta_3^9 - 8\theta_3^9) + 2\theta_2^2 \theta_3^2 \theta_4^2 \theta_2^2 \theta_4^2 \right] \\
&\quad - \frac{\pi^5 \rho^{7/2}}{192} \theta_2 \theta_3 \theta_4 \left[ \theta_2^9 (\theta_2^3 - \theta_4^3) + \theta_2^3 \theta_3^3 (\theta_2^3 + \theta_4^3) - \theta_2^3 (\theta_3^3 + \theta_4^3) \right] \left( 1 + \frac{\pi \rho}{2} \theta_3^4 + 4\rho \frac{\partial}{\partial \rho} \ln \theta_2 \right) \\
&\quad : \\
&c_{\text{bulk}}(\rho) = \frac{8}{\pi} \left( \ln \sqrt{\frac{S}{\rho}} + \ln \frac{2^{5/2}}{\pi} + C_E - \frac{\pi}{4} \right) - \frac{4}{\pi} \left( \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \right)^2 - \frac{16 \sum_{i=2}^4 \ln \theta_i}{\pi} \\
&c_0(\rho) = -2\sqrt{2}\sqrt{\rho} \frac{\theta_2 \theta_3 \theta_4}{\theta_2 + \theta_3 + \theta_4} \\
&c_1(\rho) = \frac{\pi^2 \rho^2}{6} \left( \theta_2^3 - \theta_3^3 \right) \theta_2 \theta_3 \ln \frac{\theta_2}{\theta_3} + \left( \theta_2^3 - \theta_3^3 \right) \theta_4 \theta_3 \ln \frac{\theta_4}{\theta_3} + \left( \theta_2^3 - \theta_3^3 \right) \theta_3 \theta_4 \ln \frac{\theta_3}{\theta_4} \\
&\quad + \frac{\pi^2 \rho^2}{9} \left( \theta_2^3 \theta_3^3 \theta_4^3 \right) \left( \theta_2^3 - \theta_3^3 - \theta_4^3 \right) \frac{1}{\theta_2 + \theta_3 + \theta_4} \\
&\quad + \frac{\pi \rho}{9} \theta_2^3 - \theta_3^3 + \theta_3 (\theta_2^3 - \theta_4^3) - 2\theta_2 \theta_4 (\theta_2^3 - \theta_4^3) \left( 1 + 4\rho \frac{\partial}{\partial \rho} \ln \theta_2 \right) \\
&c_2(\rho) = \frac{\pi^3 \rho^{5/2}}{24\sqrt{2}} \frac{\theta_2 \theta_3 \theta_4 (\theta_2^3 + \theta_3^3 + \theta_4^3)}{(\theta_2 + \theta_3 + \theta_4)^2} \\
&\quad : \\
\end{align*}
\]

We have also used the following relations between derivatives of the elliptic functions

\[\frac{\partial}{\partial \rho} \ln \theta_3 = \frac{\pi}{4} \theta_3^4 + \frac{\partial}{\partial \rho} \ln \theta_2 \quad \text{and} \quad \frac{\partial}{\partial \rho} \ln \theta_4 = \frac{\pi}{4} \theta_4^4 + \frac{\partial}{\partial \rho} \ln \theta_2\]

Note, that with the help of the identities

\[\frac{\partial}{\partial \rho} \ln \theta_2 = -\frac{1}{2} \theta_2^2 E, \quad \text{and} \quad \frac{\partial E}{\partial \rho} = \frac{\pi^2}{4} \theta_3^4 \theta_4^4 - \frac{\pi}{2} \theta_4^4 E\]

one can express all derivatives of the elliptic functions in terms of the elliptic functions \( \theta_2, \theta_3, \theta_4 \) and the elliptic integral of the second kind \( E \)

2. Similar expansion of the free energy of the dimer model is

\[f_{\text{bulk}} = -\frac{\gamma}{\pi} \]

\[f_0(\rho) = -\ln \frac{\theta_2^2 + \theta_3^2 + \theta_4^2}{2\eta^2} \]

\[f_1(\rho) = -\frac{\pi^3 \rho^2}{90} \left( \frac{\theta_2^{10} + \theta_3^{10} + \theta_4^{10}}{\theta_2^2 + \theta_3^2 + \theta_4^2} \right) + \frac{\theta_2^2 \theta_3^2 \theta_4^2 \theta_2^2 \theta_3^2 \theta_4^2}{\theta_2^2 + \theta_3^2 + \theta_4^2} \left[ \theta_2^2 \theta_3^2 - \theta_2^2 \theta_4^2 - \theta_3^2 \theta_4^2 \right] \]

\[f_2(\rho) = -\frac{\pi^6 \rho^4}{16200} \left( \frac{\theta_2^2 \left( \theta_2^2 \theta_3^2 \theta_4^2 \right)^2 + \theta_2^2 \left( \theta_2^2 \theta_3^2 \theta_4^2 \right)^2 + \theta_2^2 \left( \theta_2^2 \theta_3^2 \theta_4^2 \right)^2}{\theta_2^2 + \theta_3^2 + \theta_4^2} \right) \]
energy and the specific heat for a class of free exactly solvable models of statistical mechanics.

In this paper, we have derived exact asymptotic expansion of the partition function with twisted boundary conditions at the critical point, Eqs. (16) and (17). As an application of this result, we have obtained exact asymptotic expansion of the free energy, the internal energy and the specific heat of the Gaussian model after subtraction of the zero modes, \( \ln \mu \sqrt{8S}, \sqrt{2S/\mu} \text{ and } S(1/\mu + 1/\mu^2) \), respectively, can be written as

\[
\begin{align*}
\mathcal{F}_{\text{bulk}} &= \frac{2\gamma}{\pi} - \ln \sqrt{2} \\
\mathcal{F}_0(\rho) &= \ln \sqrt{\rho} \eta^2 \\
\mathcal{F}_1(\rho) &= \frac{\pi^3 \rho^2}{180} (\theta_2^4 - \theta_2^4 \theta_3^4 - \theta_3^4 \theta_1^4) \\
\mathcal{F}_2(\rho) &= \frac{\pi^6 \rho^4}{1512} (\theta_2^4 - 2\theta_3^4 \theta_4^4) + \frac{\pi^5 \rho^3}{1512} (\theta_4^4 + \theta_1^4)(\theta_3^4 + \theta_2^4)(\theta_2^4 - \theta_1^4) \left( 1 + 4\rho \frac{\partial}{\partial \rho} \ln \theta_2 \right) \\
&\vdots
\end{align*}
\]

For internal energy there are no finite-size correction terms (\( u_{\text{bulk}} = 0, u_p = 0 \) for all \( p \)).

\[
\begin{align*}
\mathcal{C}_{\text{bulk}}(\rho) &= \frac{8}{\pi} \left( \ln \sqrt{\frac{S}{\rho}} + \ln \frac{\sqrt{2}}{\pi} + C_E - \frac{\pi}{4} + \ln \eta \right) \\
\mathcal{C}_0(\rho) &= 0 \\
\mathcal{C}_1(\rho) &= \frac{2}{3} \left( 1 + 4\rho \frac{\partial}{\partial \rho} \ln \eta + 4\rho^2 \frac{\partial^2}{\partial \rho^2} \ln \eta \right) \\
\mathcal{C}_2(\rho) &= 0 \\
&\vdots
\end{align*}
\]

V. SUMMARY

In this paper, we have derived exact asymptotic expansion of the partition function with twisted boundary conditions at the critical point, Eqs. (16) and (17). As an application of this result, we have obtained exact asymptotic expansion of the free energy, the internal energy and the specific heat for a class of free exactly solvable models of statistical mechanics.
on torus. Moreover, the partition function of the dimer model on the Klein bottle can also be expressed via the partition function with twisted boundary conditions, namely $Z_{\text{Klein}} = Z_{1/4,1/2}(0)$ \[E\]. Exact asymptotic expansion of the latter can immediately be written down with the help of our general formula \[G\]. An interesting question is: whether this is also the case for other free models? This, however, is the problem for the future.

**APPENDIX A: EULER-MACLAURIN SUMMATION FORMULA.**

Suppose that $F(x)$ together with its first $2m$ derivatives is continuous within the interval $(a,b)$. Then the general Euler-Maclaurin summation formula states \[J\]

$$
\sum_{n=0}^{N-1} F(a + nh + \alpha h) = \frac{1}{h} \int_{a}^{b} F(\tau) \, d\tau + \sum_{k=1}^{p} \frac{h^{k-1}}{k!} B_k(\alpha) \left( F^{(k-1)}(b) - F^{(k-1)}(a) \right) - R_p(\alpha)
$$

(A1)

where $p \leq 2m$, $0 \leq \alpha \leq 1$, $h = (b - a)/N$ and reminder term $R_p(\alpha)$ is given by

$$
R_p(\alpha) = \frac{h^{p}}{p!} \int_{0}^{1} \hat{B}_p(\alpha - \tau) \left\{ \sum_{n=0}^{N-1} F^{(p)}(a + nh + \tau h) \right\} \, d\tau
$$

(A2)

$\hat{B}_p(\alpha)$ are so called periodic Bernoulli functions which are defined as follows

$$
\hat{B}_p(\alpha) = -\frac{p!}{(-2\pi i)^p} \sum_{n \neq 0} \frac{e^{-2\pi i n\alpha}}{n^p}
$$

(A3)

These functions have singularities at the integer values of $\alpha$ and on the interval $\alpha \in [0, 1]$ they coincide with so-called Bernoulli polynomials

- $B_1(\alpha) = \alpha - \frac{1}{2}$
- $B_2(\alpha) = \alpha^2 - \alpha + \frac{1}{6}$
- $B_3(\alpha) = \alpha^3 - \frac{3}{2}\alpha^2 + \frac{1}{2}\alpha$
- $B_4(\alpha) = \alpha^4 - 2\alpha^3 + \alpha^2 - \frac{1}{30}$

The generating function for the Bernoulli polynomials is

$$
\frac{\lambda e^{\lambda \alpha}}{e^\lambda - 1} = 1 + \sum_{p=1}^{\infty} \frac{\lambda^p}{p!} B_p(\alpha)
$$

Fourier transform of the generating function gives the important identity

$$
\frac{\lambda e^{2\pi i n\alpha}}{e^{2\pi i} - 1} = -\sum_{n=0}^{\infty} e^{2\pi i (n+\alpha)} = \frac{1}{2\pi i} \sum_{n=-\infty}^{+\infty} e^{-2\pi i n\alpha} \frac{e^{-2\pi i \alpha}}{z + n}
$$

(A4)

It is well known that Euler-Maclaurin summation formula is closely related to (generally speaking divergent) asymptotic series. For further discussion of the properties of these series the interested reader is referred to the book of Hardy \[K].
In this paper we are mainly interested in sums of the form
\[ \frac{1}{N} \sum_{n=0}^{N-1} f \left( \frac{\pi(n + \alpha)}{N} \right) \]  
\hfill (A5)

The asymptotic expansion of the sum (A5) in the limit \( N \to \infty \) can be obtained from Eq. (A4) by setting \( a = 0, b = \pi \). If we assume that all the derivatives of \( f(x) \) are integrable over the interval \((0, \pi)\), i.e., the integral in Eq. (A4) is finite, we can formally extend the sum in Eq. (A4) to \( k = \infty \) and drop the reminder term \( R_p(\alpha) \). In this case we can write the asymptotic expansion of the sum (A5) as follows
\[ \frac{1}{N} \sum_{n=0}^{N-1} f \left( \frac{\pi(n + \alpha)}{N} \right) = \frac{1}{\pi} \int_0^\pi f(\tau) \, d\tau + \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{\pi}{N} \right)^k \frac{B_k(\alpha)}{k!} \left( f^{(k-1)}(\pi) - f^{(k-1)}(0) \right) \]  
\hfill (A6)

Note, that we also use abbreviated notation for Bernoulli polynomials: \( B_p(\alpha) = B_p^\alpha \).

APPENDIX B: RELATION BETWEEN MOMENTS AND CUMULANTS

Moments \( Z_k \) and cumulants \( F_k \) which enters the expansion of exponent 
\[ \exp \left\{ \sum_{k=1}^{\infty} \frac{x^k}{k!} F_k \right\} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} Z_k \]
are related to each other as [21]
\[ Z_1 = F_1 \]
\[ Z_2 = F_2 + F_1^2 \]
\[ Z_3 = F_3 + 3F_1F_2 + F_1^3 \]
\[ Z_4 = F_4 + 4F_1F_3 + 3F_2^2 + 6F_2F_1^2 + F_1^4 \]
\[ \vdots \]
\[ Z_k = \sum_{i_1=1}^{k} \sum_{i_2=1}^{k_i} \ldots \sum_{i_r=1}^{k_i \ldots i_r} \left( \frac{F_{k_1}}{k_1!} \right)^{i_1} \ldots \left( \frac{F_{k_r}}{k_r!} \right)^{i_r} \frac{k!}{i_1! \ldots i_r!} \]
where summation is over all positive numbers \( \{i_1 \ldots i_r\} \) and different positive numbers \( \{k_1, \ldots, k_r\} \) such that \( k_1i_1 + \ldots + k_ri_r = k \).

APPENDIX C: ELLIPTIC THETA FUNCTIONS

We adopt the following definition of the elliptic \( \theta \)-functions:
\[ \theta_{\alpha, \beta}(z, \tau) = \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \tau \left( n + \frac{1}{2} - \alpha \right)^2 + 2\pi i \left( n + \frac{1}{2} - \alpha \right) \left( z + \frac{1}{2} - \beta \right) \right\} \]
\[ = \eta(\tau) \exp \left\{ \pi i \tau (\alpha^2 - \alpha + \frac{1}{6}) + 2\pi i (\frac{1}{2} - \alpha)(z + \frac{1}{2} - \beta) \right\} \]
\[ \times \prod_{n=0}^{\infty} \left[ 1 - e^{2\pi i \tau (n + \alpha) - 2\pi i (z - \beta)} \right] \left[ 1 - e^{2\pi i \tau (n + 1 - \alpha) + 2\pi i (z - \beta)} \right] \]
These should be compared with the notations of Mumford [22].

The elliptic \( \theta \)-functions satisfies the heat equation

\[
\frac{\partial}{\partial \tau} \theta_{\alpha,\beta}(z, \tau) = \frac{1}{4\pi i} \frac{\partial^2}{\partial z^2} \theta_{\alpha,\beta}(z, \tau) \tag{C1}
\]

Relation to standard notations is

\[
\begin{align*}
\theta_{0,0}(z, \tau) &= \theta_1(z, \tau) \\
\theta_{0,\frac{1}{2}}(z, \tau) &= \theta_2(z, \tau) \\
\theta_{\frac{1}{2},0}(z, \tau) &= \theta_4(z, \tau) \\
\theta_{\frac{1}{2},\frac{1}{2}}(z, \tau) &= \theta_3(z, \tau)
\end{align*}
\]

The Dedekind \( \eta \)-function is usually defined as

\[
\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} \left[ 1 - e^{2\pi i n \tau} \right]
\]

Considering the functions \( \theta_{\alpha,\beta}(\tau) = \theta_{\alpha,\beta}(0, \tau) \) and \( \eta(\tau) \) of pure imaginary aspect ratio, \( \tau = i\rho \), we obtain the identity

\[
\ln \left| \frac{\theta_{\alpha,\beta}(i\rho)}{\eta(i\rho)} \right| + \pi \rho B_2^\alpha = \sum_{n=0}^{\infty} \ln \left| 1 - e^{-2\pi \rho(n+\alpha)} - 2\pi i \beta \right|
\]

\[
+ \sum_{n=0}^{\infty} \ln \left| 1 - e^{-2\pi \rho(n+1-\alpha)} - 2\pi i \beta \right| \tag{C2}
\]

Taking the derivative of Eq. (C2) with respect to \( \rho \) we can obtain the following useful identity

\[
\text{Re} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[ (n+\alpha) e^{-2\pi m \rho(n+\alpha)+i\beta} + (n+1-\alpha) e^{-2\pi m \rho(n+1-\alpha)+i\beta} \right] =
\]

\[
\frac{B_2^\alpha}{2} + \frac{1}{2\pi} \frac{\partial}{\partial \rho} \ln \left| \frac{\theta_{\alpha,\beta}(i\rho)}{\eta(i\rho)} \right| \tag{C3}
\]

Taking the second derivative of \( \ln |\theta_{\alpha,\beta}(z, i\rho)| \) with respect to \( z \) at \( z = 0 \) and using the heat equation Eq. (C1) we obtain

\[
\frac{\partial}{\partial \rho} \left( \text{Re} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{n+\alpha} e^{-2\pi m \rho(n+\alpha)+i\beta} + \frac{1}{n+1-\alpha} e^{-2\pi m \rho(n+1-\alpha)+i\beta} \right\} \right) =
\]

\[
-2 \frac{\partial}{\partial \rho} \ln |\theta_{\alpha,\beta}(i\rho)| + \frac{1}{2\pi} \left( \frac{\partial}{\partial z} \ln |\theta_{\alpha,\beta}(i\rho)| \right)^2 \tag{C4}
\]
APPENDIX D: KRONECKER’S DOUBLE SERIES

Kronecker’s double series can be defined as

\[ K_{p}^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^{p}} \sum_{n,n \in \mathbb{Z} \atop (m,n) \neq (0,0)} \frac{e^{-2\pi i (n\alpha + m\beta)}}{(n + \tau m)^{p}} \]

In this form, however, they cannot be directly applied to our analysis. We need to cast them in a different form. To this end, let us separate from the double series a subseries with \( m = 0 \)

\[ K_{p}^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^{p}} \sum_{n \neq 0} \frac{e^{-2\pi i n\alpha}}{n^{p}} - \frac{p!}{(-2\pi i)^{p}} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi i (n\alpha + m\beta)}}{(n + \tau m)^{p}} \]

Here the first sum gives nothing but Fourier representation of Bernoulli polynomials Eq.(A3). The second sum can be rearranged with the help of the identity

\[ \frac{p!}{(-2\pi i)^{p}} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi i n\alpha}}{(z + n)^{p}} = p \sum_{n=0}^{\infty} (n + \alpha)^{p-1} e^{2\pi i(z+n+\alpha)} \]

which can easily be derived from Eq.(A4) differentiating it \( p \) times. The final result of our resummation of double Kronecker sum is

\[ K_{p}^{\alpha,\beta}(\tau) = B_{\alpha}^{p} - p \sum_{m \neq 0} \sum_{n=0}^{\infty} (n + \alpha)^{p-1} e^{2\pi i m (\tau(n+\alpha) - \beta)} \]

Considering real part of the Kronecker sums with pure imaginary aspect ratio, \( \tau = i\rho \), we can further rearrange this expression to get summation only over positive \( m \geq 1 \)

\[ B_{2p}^{\alpha} - \text{Re} \, K_{2p}^{\alpha,\beta}(i\rho) = \frac{(2p)!}{(-4\pi i)^{p}} \text{Re} \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi i (n\alpha + m\beta)}}{(n + i\rho m)^{2p}} \]

\[ = 2p \text{Re} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (n + \alpha)^{2p-1} e^{-2\pi m (\rho(n+\alpha) + i\beta)} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (n + 1 - \alpha)^{2p-1} e^{-2\pi m (\rho(n+1-\alpha) + i\beta)} \right\} \]  

(D1)

APPENDIX E: ASYMPTOTIC EXPANSION OF THE DIGAMMA FUNCTION

Let us start with well known expansion of the digamma function \( \psi(N) \)

\[ \psi(x) = \ln x - \frac{1}{2x} - \sum_{p=1}^{\infty} \frac{B_{2p}}{2p} \frac{1}{x^{2p}} \]

\[ = \ln x - \sum_{p=1}^{\infty} (-1)^{p} \frac{B_{p}}{p} \frac{1}{x^{p}} \]  

(E1)

Plugging in the above expansion \( x = N + \alpha \) and expand the resulting factors \( \ln (1 + \alpha/N) \), \((1 + \alpha/N)^{-p}\) in powers of \( N^{-1} \) we obtain
ψ(N + α) = ln N − ∞ \sum_{p=1}^{∞} (-1)^p \frac{α^p}{pN^p} − \sum_{p=1}^{∞} \sum_{k=0}^{∞} (-1)^{k+p} B_p \frac{(p + k - 1)!}{k!p!} \frac{α^k}{N^{p+k}}

= ln N − ∞ \sum_{p=1}^{∞} (-1)^p \frac{α^p}{pN^p} − \sum_{l=1}^{∞} \sum_{p=1}^{l} (-1)^l B_p \frac{(l - 1)!}{(l - p)!p!} \frac{α^{l-p}}{N^l}

= ln N − ∞ \sum_{l=1}^{∞} \sum_{p=0}^{l} (-1)^l B_p \frac{(l - 1)!}{(l - p)!p!} \frac{α^{l-p}}{N^l}

(E2)

Using the relation between Bernoulli polynomials $B_p^α$ and Bernoulli numbers $B_p$

$B_p^α = \sum_{p=0}^{l} B_p \frac{l!}{(l - p)!p!} α^{l-p}$

(E3)

we finally obtain the Eq. (23)

ψ(N + α) = ln N − ∞ \sum_{p=1}^{∞} (-1)^p \frac{B_p^α}{p} \frac{1}{N^p}$

(E4)

APPENDIX F: REDUCTION OF KRONECKER’S DOUBLE SERIES TO THETA FUNCTIONS

Let us consider Laurent expansion of the Weierstrass function

ψ(z) = \frac{1}{z^2} + \sum_{(n,m)≠(0,0)} \left[ \frac{1}{(z - n - τm)^2} - \frac{1}{(n + τm)^2} \right]

= \frac{1}{z^2} + \sum_{p=2}^{∞} a_p(τ) z^{2p-2}

The coefficients $a_p(τ)$ of the expansion can all be written in terms of the elliptic θ-functions with the help of the recursion relation [24]

$a_p = \frac{3}{(p - 3)(2p + 1)} (a_2 a_{p-2} + a_3 a_{p-3} + ... + a_{p-2} a_2)$

where first terms of the sequence are

$a_2 = \frac{π^4}{192} (\theta_2^4 \theta_3^4 - \theta_2^3 \theta_4^4 + \theta_3^3 \theta_4^4)

a_3 = \frac{π^6}{192} (\theta_2^4 + \theta_3^4) (\theta_2^4 - \theta_3^4) (\theta_3^4 + \theta_4^4)

a_4 = \frac{1}{3} a_2^2

a_5 = \frac{3}{11} (a_2 a_3)

a_6 = \frac{1}{39} (2a_2^2 + 3a_3^2)

...
Kronecker functions $K_{2p}^{0,0}(\tau)$ are related directly to the coefficients $a_p(\tau)$

$$K_{2p}^{0,0}(\tau) = -\frac{(2p)!}{(-4\pi^2)^p(2p - 1)} a_p(\tau)$$

Kronecker functions $K_{2p}^{\alpha,\beta}(\tau)$ with $\alpha$ and $\beta$ taking values 0 or 1/2 can in their turn be related to the function $K_{2p}^{0,0}(\tau)$ by means of simple resummation of Kronecker’s double series

$$K_p^{0,\frac{1}{2}}(\tau) = 2K_p^{0,0}(2\tau) - K_p^{0,0}(\tau)$$
$$K_p^{1,0}(\tau) = 2^{1-p}K_p^{0,0}(\tau/2) - K_p^{0,0}(\tau)$$
$$K_p^{\frac{1}{2},\frac{1}{2}}(\tau) = (1 + 2^{2-p})K_p^{0,0}(\tau) - 2^{1-p}K_p^{0,0}(\tau/2) - 2K_p^{0,0}(2\tau)$$

Thus, Kronecker functions $K_{2p}^{\alpha,\beta}(\tau)$ with $\alpha$ and $\beta$ taking values 0 or 1/2 can all be expressed in terms of the elliptic $\theta$-functions only. For practical calculations the following identities are also helpful

$$2\theta_2^2(2\tau) = \theta_2^2 - \theta_4^2 \quad \theta_2^2(\tau/2) = 2\theta_2\theta_3$$
$$2\theta_3^2(2\tau) = \theta_3^2 + \theta_4^2 \quad \theta_3^2(\tau/2) = \theta_2^2 + \theta_3^2$$
$$2\theta_4^2(2\tau) = 2\theta_3\theta_4 \quad \theta_4^2(\tau/2) = \theta_3^2 - \theta_2^2$$

From the general formulas above we can easily write down all the Kronecker functions that have appeared in our asymptotic expansions

$$K_4^{0,0}(\tau) = \frac{1}{30}(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4 - \theta_3^4\theta_4^4)$$
$$K_4^{0,\frac{1}{2}}(\tau) = \frac{1}{30}(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4)$$
$$K_4^{1,0}(\tau) = \frac{1}{30}(\frac{7}{8}\theta_2^8 - \theta_2^4\theta_3^4)$$
$$K_4^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{1}{30}(\frac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4)$$

$$K_6^{0,0}(\tau) = \frac{1}{84}(\theta_2^4 + \theta_3^4)(\theta_2^4 - \theta_3^4)(\theta_2^4 + \theta_4^4)$$
$$K_6^{0,\frac{1}{2}}(\tau) = \frac{1}{84}(\theta_2^4 + \theta_4^4)(\frac{31}{16}\theta_2^8 + \theta_3^4\theta_4^4)$$
$$K_6^{1,0}(\tau) = -\frac{1}{84}(\theta_2^4 + \theta_3^4)(\frac{31}{16}\theta_2^8 + \theta_2^4\theta_3^4)$$
$$K_6^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{1}{84}(\theta_2^4 - \theta_3^4)(\frac{31}{16}\theta_2^8 - \theta_2^4\theta_3^4)$$

Note that when $\rho \to \infty$ we have limits $\theta_2 \to 0$, $\theta_4 \to 1$, $\theta_3 \to 1$ and the Kronecker’s function reduce to the Bernoulli polynomials.

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