The Vaidya metric: expected and unexpected traits of evaporating black holes

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The ingoing Vaidya metric is introduced as a model for a non-rotating uncharged black hole emitting Hawking radiation. This metric is expected to capture the physics of the spacetime for radial coordinates up to a small multiple (\(\geq 1\)) of the Schwarzschild radius. For larger radii, it will give an excellent approximation to the spacetime geometry in the case of astrophysical black holes (\(M \geq M_\odot\)), except at extremely large distances from the horizon (exceeding the cosmic particle horizon). In the classroom, the model may serve as a first exploration of non-stationary gravitational fields. Several interesting predictions are developed. First, particles dropped early enough before complete evaporation of the black hole cross its horizon as easily as with an eternal black hole. Second, the Schwarzschild radius takes on the properties of an apparent horizon, and the true event horizon of the black hole is inside of it, because light can escape from the shrinking apparent horizon.

In 2005, Aste and Trautmann considered a toy model, which is expected to capture the physics of the spacetime for radial coordinates up to a small multiple (\(\geq 1\)) of the Schwarzschild radius. For larger radii, it will give an excellent approximation to the spacetime geometry in the case of astrophysical black holes (\(M \geq M_\odot\)), except at extremely large distances from the horizon (exceeding the cosmic particle horizon). In the classroom, the model may serve as a first exploration of non-stationary gravitational fields. Several interesting predictions are developed. First, particles dropped early enough before complete evaporation of the black hole cross its horizon as easily as with an eternal black hole. Second, the Schwarzschild radius takes on the properties of an apparent horizon, and the true event horizon of the black hole is inside of it, because light can escape from the shrinking apparent horizon.

Third, a particle released from rest close enough to the apparent horizon is strongly repelled and may escape to infinity. An interpretation is given, demonstrating that such a particle would be able to compete, for a short time, in a race with a photon.

Keywords: Black holes, Hawking radiation, evolving horizons

I. INTRODUCTION

Schwarzschild black holes\(^1,2\) are among the most popular examples of spacetimes studied in general relativity courses. Their metric is simple, enables a straightforward discussion of the four classical tests of general relativity, and allows one to explore some strong-field effects as well as the concept of an event horizon.

The metric is stationary, so effects of time-dependent gravitational fields are absent. Beyond that, the interesting phenomenon of gravitational repulsion of radially infalling particles does not arise in the Schwarzschild metric. It happens for charged spherically symmetric black holes, described by the Reissner-Nordström metric\(^3\) but only well inside the event horizon. Some non-stationary metrics turn out to exhibit it in more accessible places.

A simple time-dependent metric is the Vaidya metric\(^4\) describing the exterior of a spherically symmetric mass distribution that absorbs or emits null dust, i.e., massless particles. While spectacular time-dependent effects may be discussed in studying the properties of gravitational waves\(^5\) the Vaidya metric is not devoid of some interesting ones either.

Note that the Vaidya metric is not purely of academic interest. It arises in a natural way in phenomenological models of a black hole evaporating by Hawking radiation, an object that is not fully describable by a stationary metric. Since Vaidya metrics are models of the spacetime resulting from Hawking radiation, not of the radiation itself (which is mostly electromagnetic in nature), Hawking radiation will be hardly touched upon in the main part of this article. Rather, we will give a discussion of its essential features in Sec. \(\secnum{11}\) preceding the presentation of our model and its results.

In 2005, Aste and Trautmann considered a toy model\(^6\) in which they made the mass term appearing in the Schwarzschild metric dependent on the global time. With this metric, they found that the black hole evaporates under an infalling particle, before the latter can cross the horizon. It was pointed out\(^7\) that this model is not physically meaningful; the Schwarzschild time coordinate inside the event horizon is unrelated to the time coordinate outside due to the fact that it is not continuous across the horizon (diverging to infinity there).

An improvement was suggested in Ref. \(\ref{7}\), where Gullstrand-Painlevé (GP) coordinates\(^8\) were taken as a starting point to introduce a time-dependent mass parameter. These coordinates are continuous across the event horizon of the Schwarzschild spacetime, so making the mass time-dependent at least has a well-defined meaning. In this model, an observer falling towards an evaporating black hole will ordinarily cross the Schwarzschild radius without any problem (except close to the time of final dissolution of the hole) and then hit the central singularity in finite proper time. Clearly, this is still a toy model with a number of unrealistic features. For example, since the mass parameter depends on the time coordinate only, it will change instantaneously for all observers sitting at different radial positions. However, energy is radiated away at a finite velocity. Therefore, two coordinate stationary observers with different radii should experience, at the same time, the attraction of different masses inside their respective radial shells.

This problem may be solved by using Eddington-Finkelstein (EF) coordinates\(^9,10\) instead. Making the mass dependent on the EF time leads to a Vaidya metric, in which the mass parameter is constant not on a spacelike but on a null surface. Because this means that the radiated energy moves at the speed of light, compatibility with causality is achieved, and the emitted particles must be massless, such as photons. The construction of such a metric is still a far cry from producing a dynamical spacetime that would describe an evaporating black hole in a quantum mechanically consistent way. We do not know the time dependence of Hawking radiation at times close to the presumed final disappearance of the black hole, where quantum effects should become strong, and we cannot calculate the back reaction between the Hawking radiation flux and the evolving metric.
non-perturbatively. Even a semiclassical self-consistent solution does not seem to be available, in which the metric would be a classical field arising from the quantum mechanical expectation value $\langle T_{\mu\nu} \rangle$ of the stress-energy tensor.\textsuperscript{12} Nevertheless, the simplest guess at what the semiclassical time-dependent metric of an evaporating black hole should look like, subject to the condition of a minimum of realism regarding energy transport, would be a Vaidya metric, leaving one with the freedom to specify largely arbitrary time-dependent mass functions.\textsuperscript{13}

As will be discussed below, if the dynamical horizon of the description is to correspond to a future horizon, the metric must be an ingoing Vaidya metric. Such a metric with a mass parameter that decreases as a function of time corresponds to a black hole losing mass by in-falling negative-energy null dust. This feature, while acceptable near the horizon (as will be seen in the next section), is not very realistic far from it, where Hawking radiation clearly is outgoing and consists of positive-energy particles. Then, to add more realism to the model, the ingoing Vaidya metric near the horizon should be complemented by an outgoing one at larger radial coordinates, with the surface joining the two metrics corresponding to a pair creation location.\textsuperscript{14} A few years ago, a model for an evaporating black hole without any spacetime singularities was presented that employs these features.\textsuperscript{15}

Since our purpose is to present a model that is useful to a pair creation location, we consider, in Sec. III, the maximally extended Schwarzschild metric near the horizon should be complemented by an outgoing one at larger radial coordinates, with the surface joining the two metrics corresponding to a pair creation location.\textsuperscript{14} A few years ago, a model for an evaporating black hole without any spacetime singularities was presented that employs these features.\textsuperscript{15}

The remainder of the paper is organized as follows. After the explanatory Sec. II on Hawking radiation, we consider, in Sec. III the maximally extended Schwarzschild metric in a diagram using Kruskal-Szekeres (KS) coordinates.\textsuperscript{15,16} Discussing which regions of the full spacetime are covered by ingoing and outgoing EF coordinates, we decide which of the two sets is the best starting point in constructing a Vaidya metric. Section IV gives the equations of motion for a test particle falling radially into the model black hole and simplifies them to a form that can be compared with the Newtonian limit. These equations are solved numerically in Sec. V. The existence of repulsive effects of the time-dependent gravitational field is demonstrated analytically. Finally, some conclusions are given in Sec. VI.

II. HAWKING RADIATION

Hawking provided a physical mechanism for Bekenstein’s idea that the area of the event horizon of a black hole corresponds to its entropy and its surface gravity to its temperature (each with the appropriate proportionality constant). In two papers exploring this quantum mechanical mechanism,\textsuperscript{17,18} he presented a calculation based on quantum field theoretical considerations, according to which a field operator associated with a vacuum state in Minkowski spacetime (at past null infinity $\mathcal{I}^-$) will, after the spacetime has become curved due to the formation of a black hole, be associated with a state of non-zero particle content near the event horizon. Some of these particles, mostly photons, escape to future null infinity $\mathcal{I}^+$, and the calculation proceeds by following the field at $\mathcal{I}^+$ backwards in time through the interaction with the collapsing star to $\mathcal{I}^-$, using analytic continuation techniques to evaluate the spectral properties of the so-created radiation, which turns out to be thermal with temperature and entropy given by the Bekenstein expressions. The field calculation may be visualized in terms of a wave picture, with the properties of the ingoing wave coming from $\mathcal{I}^-$ modified by its interaction with the event horizon. Detailed considerations involve negative frequency components of the wave falling into the black hole and positive frequency components being amplified, allowing part of them to escape to $\mathcal{I}^+$.

It should be noted that Hawking himself provided an alternative picture of the process already in the extended version of his first explanation of the effect.\textsuperscript{19} It focuses on the particle content of the quantum fields involved rather than on the decomposition of the field into waves with amplitudes given by annihilation and creation operators. In this picture, Hawking described a negative energy flux inward across the horizon, coming from the surrounding region in which, by virtue of quantum fluctuations, pairs of virtual particles appear and disappear. One of these typically has positive and the other negative energy (so as to satisfy energy conservation in the long run). He then stated: The negative particle is in a region which is classically forbidden but it can tunnel through the event horizon to the region inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exist as a real particle with a timelike momentum vector even though its energy relative to infinity as measured by the time translation Killing vector is negative. The other particle of the pair, having a positive energy, can escape to infinity where it constitutes a part of the thermal emission described above. [...] Instead of thinking of negative energy particles tunnelling through the horizon in the positive sense of time one could regard them as positive energy particles crossing the horizon on past-directed world-lines and then being scattered on to future-directed world-lines by the gravitational field. Although Hawking emphasized that this is a heuristic picture, not to be taken too literally, it may be more appealing than the picture of positive-frequency and negative-frequency wave components, where the energy balance is not readily visible with the overall wave consisting predominantly of positive-frequency components.

Of course, when talking of particles here, we mean quantum particles without committing ourselves to their manifestation in a localized or wavy manner. In fact, the particles of Hawking radiation are not very point-like. The wavelength of the maximum of the spectrum of Hawking radiation for a solar-mass black hole is about 47 km at infinity, whereas the Schwarzschild radius is a mere 3 km. So these photons would be detectable with
antennas rather than with photomultipliers, i.e., they would be considered waves rather than particles. Moreover, they are so poorly localized that it is impossible to indicate a single value for the redshift they have undergone since their creation near the black hole horizon (because the redshift is significantly different for different parts of the wave). A wave emerging at about two Schwarzschild radii from the center of the geometry will have only a roughly 40% higher frequency at that position than at infinity, meaning that its most probable wavelength would still be more than 30 km. Photons with such wavelengths might easily tunnel distances on the order of 10 km, i.e., three Schwarzschild radii. Hence, in adopting the particle picture, we should not imagine the radiation to come from the surface constituted by the horizon. Its location of creation may be washed out with an uncertainty on the order of the Schwarzschild radius. In fact, the “quantum atmosphere”, in which Hawking radiation is created, has recently been estimated, from a 1 + 1 dimensional calculation, to extend in radial coordinate from 1.5 to 2 Schwarzschild radii.

Newer calculations of Hawking radiation have been presented which are much closer in spirit to the picture of tunneling quantum particles than Hawking’s original one. One advantage of calculations that are manifestly based on a tunneling process is that it is not necessary to consider the complete collapse geometry before formation of the black hole. While the details of the collapse do not influence the final result, the picture usually drawn, having null trajectories in a Penrose diagram that connect $\mathcal{I}^−$ with $\mathcal{I}^+$ (and pile up near the horizon) may favor misconceptions such as the idea that all of the Hawking radiation comes from the time before the surface of the collapsing star crosses the horizon which would mean that the black hole remains incipient forever, i.e., that it does not actually form. A nice compact discussion of features of Hawking radiation may be found in Ref. 2.

Experimental verification of Hawking radiation from a typical black hole with a mass exceeding that of our sun is virtually impossible, because the effect is so tiny. It has been suggested that sonic black holes (in which a fluid takes on supersonic speeds so that there should be Hawking-like sound emission from the “sonic horizon”) might allow experimental access to the phenomenon. Indeed, success in detecting Hawking radiation from a Bose-Einstein condensate as a black-hole analog was reported recently.

Returning to real black holes, an object falling towards the event horizon seems to get slower and freeze there due to the fact that photons take longer and longer to escape from the vicinity of the horizon. What is more, it turns out that the object will take infinite Schwarzschild time to actually reach the horizon, so the slowing-down would, it appears, not just be an optical illusion. That a black hole dissolves via Hawking radiation in finite time then leads to an apparent paradox: If it takes infinite time for an observer to fall into an eternal black hole with fixed event horizon, falling into an evaporating black hole should also take infinite time. After all, the attracting mass decreases and the horizon recedes, so if anything, one should expect an infaller to take longer for the evaporating black hole than for the static one. But then the black hole must disappear, before the observer can fall into it. This of course begs the question how an event horizon could form in the first place, because the surface of the star will, on collapse, behave similarly to the infaller. On the other hand, observers can fall into eternal black holes within a finite interval of their proper time and Hawking radiation is a weak effect for stellar-mass black holes, which suggests that crossing of the event horizon by an observer should happen essentially in the same way as for a static black hole. But it is a contradiction for an observer to both fall and not fall into a black hole. This paradox has been resolved in Ref. 7 modeling the metric of an evaporating black hole appropriately, by use of a time coordinate that does not diverge at the horizon and is resolved in this paper the same way (but using a different time coordinate).

In such a model, there must be a region of negative energy density outside the horizon, as transpires from the quote reproduced above from Hawking’s 1975 paper. In classical general relativity, negative energy density is considered forbidden and solutions requiring it are viewed as unrealistic, but in the presence of quantum mechanics, they cannot be discarded automatically. Our considerations in the following will be based on a classical model, but negative energy density is admitted in order to mimic the quantum effect of negative-energy virtual particles.

III. THE SCHWARZSCHILD SPACETIME IN TERMS OF EDDINGTON-FINKELSTEIN COORDINATES

In this section, we illustrate the idea that different coordinate systems may cover different parts of the spacetime manifold. The necessity to capture certain features (such as a future event horizon) in the description may then restrict the choice of possible coordinates.

The maximally extended Schwarzschild solution, corresponding to an eternal spherically symmetric black hole — and a bit more — is describable via KS coordinates. The resulting diagram given in Fig. 1 will also be useful in exhibiting the difference between the two EF coordinate systems.

The metric is

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \left(dT^2 - dX^2\right) - r^2 d\Omega^2,$$ (1)

where $r_s = 2GM/c^2$ defines the Schwarzschild radius ($M$ is the black hole mass, $G$ Newton’s gravitational constant, $c$ the vacuum speed of light) and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ abbreviates the line element on a unit sphere. The radial coordinate $r$ of the Schwarzschild metric is expressible via $X$ and $T$ with the help of $X^2 - T^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s}$.

The diagram of Fig. 1 may be read as giving a two-dimensional section of the spacetime at fixed $\theta$ and $\varphi$. One of its interesting features is that radial light rays
are parallel to either of the two bisectors of the pair of coordinate axes, so it is easy to discuss the exchange of light signals of observers in the geometry. Event horizons are given in Fig. 1 by the future and past light cones of the origin. The Schwarzschild time coordinate is timelike only in the regions I and III ( \( X > |T| \) and \( X < -|T| \), respectively) but spacelike in the regions II and IV ( \( T > |X| \) and \( T < -|X| \), respectively, with \( T^2 - X^2 < 1 \)), because the absolute value of the slope of any straight line through the origin is bigger than \( 1 \) in these two regions. Correspondingly, the coordinate \( r \) behaves like a spatial coordinate in regions I and III, but like a time coordinate in regions II and IV.

Customarily, region I ( \( r > r_s, -\infty < t < \infty \) ) is considered as describing the outside of a Schwarzschild black hole. The complete set of Schwarzschild coordinates, including \( r < r_s, -\infty < t < \infty \), then either describes regions I and II or regions I and IV. That Schwarzschild coordinates become singular at \( r = r_s \) may be seen from the fact that all points with \( r = r_s \) (except \( (X,T) = (0,0) \)) also have either \( t = \infty \) or \( t = -\infty \), so the pair \((r,t) = (r_s,\infty)\) actually describes an infinite set of events instead of a single one (and so does \((r,t) = (r_s,-\infty)\)). The patches of Schwarzschild coordinates separated by \( r = r_s \) are not continuously connected (\( t \) becomes infinite between them), therefore the ambiguity whether \( r < r_s \) corresponds to region II or IV.

In practice, identification is always made with region II, because the horizon between regions IV and I, sometimes called antihorizon, is permeable from IV to I only and thus corresponds to a white hole (into which nothing can fall, but from which stuff may be ejected). The vacuum solution, described by the metric, is not applicable inside a star, so observers in region I will see an illusionary horizon (the surface of the collapsing star) instead of an antihorizon. Once a black hole has formed, the vacuum part of the metric will resemble that of an eternal black hole with only regions I and II explorable.

A pair of photons, one escaping one infalling, is indicated in the figure, the first drawn as a solid arrow; the second as a dashed arrow. This is to illustrate the point that for an observer \( A \), whose approximate position is marked in the figure, Hawking radiation seems to come from the illusionary horizon because he will not see the infalling photon and the trajectory of the escaping one is parallel to the horizon in the KS diagram. However, an observer \( B \) who is closer to the illusionary horizon will not see that particular photon, because it does not really come from the surface of the collapsing star. For a shrinking black hole, observer \( A \) will see radiation that was emitted at a slightly higher temperature than that seen by \( B \) and the difference in photon content was created in the spacetime interval between the two observers, which typically will mean that it was created after the observation made at \( B \)'s spacetime position but before the observation by \( A \). An observer falling freely through the horizon should not detect any Hawking radiation at all at the crossing point, if the equivalence principle continues to hold at the horizon, as suggested by Ref. [34].

Let us now discuss EF coordinates. One way to obtain them from Schwarzschild coordinates is to first introduce the tortoise coordinate

\[
r^* = r + r_s \ln\left|\frac{r}{r_s} - 1\right|,
\]

which approaches \(-\infty\) as \( r \to r_s \), and then to introduce, as new time coordinate, either

\[
v = t + \frac{r^*}{c},
\]

yielding the ingoing EF metric with coordinates \( v, r, \vartheta, \) and \( \varphi \) or

\[
u = t - \frac{r^*}{c},
\]

which gives the outgoing EF metric with coordinates \( u, r, \vartheta, \) and \( \varphi \). It is then easy to determine the coordinate transformations leading from the KS metric to the EF metrics. These are

\[
X = \frac{1}{2}e^{cv/2r^*} + \frac{1}{2}\left(\frac{r}{r_s} - 1\right)e^{r/r_s - cv/2r_s},
\]

\[
T = \frac{1}{2}e^{cv/2r^*} - \frac{1}{2}\left(\frac{r}{r_s} - 1\right)e^{r/r_s - cv/2r_s},
\]

and

\[
X = \frac{1}{2}e^{-cv/2r^*} + \frac{1}{2}\left(\frac{r}{r_s} - 1\right)e^{r/r_s + cv/2r_s},
\]

\[
T = \frac{1}{2}e^{-cv/2r^*} - \frac{1}{2}\left(\frac{r}{r_s} - 1\right)e^{r/r_s - cv/2r_s}.
\]
respectively. We note that all values $-\infty < v < \infty$ and $0 < r < \infty$ are admissible in the transformation \((6)\). Moreover, for $r > r_s$, we have $X > 0$, which excludes region III, and for $r < r_s$, we have $T > 0$, which excludes region IV. Therefore, ingoing EF coordinates cover regions I and II continuously. This is depicted in Fig. 1 by the region with solid hatch lines. Since constant $v$ implies that $X + T = \text{const.}$ (as follows immediately from \((6)\)), the hatch lines also correspond to lines of constant coordinate $v$. So $v$ is a null coordinate rather than a timelike one; at a fixed position $r$, it may nonetheless nicely serve as a time coordinate.

Similar considerations for equations \((6)\) show that outgoing EF coordinates cover regions I and IV continuously; they describe a white-hole metric. The dashed hatching depicts their region of validity and dashed lines correspond to lines of constant coordinate $u$.

It is then clear that if we wish to describe an evaporating black hole (with a future horizon), our starting point should not be outgoing but ingoing EF coordinates. In terms of these, the Schwarzschild line element reads

\[
d s^2 = (1 - \frac{r_s}{r}) c^2 d\tilde{v}^2 - 2c d\tilde{v} d\tilde{r} - r^2 d\Omega^2. \tag{7}
\]

This metric is nonsingular at $r = r_s$, in spite of the fact that the prefactor of $d\tilde{v}^2$ vanishes there. The nondiagonal term $c d\tilde{v} d\tilde{r}$ ensures that none of the eigenvalues of the metric become zero at $r = r_s$.

\section{IV. Ingoing Vaidya Metric and Equations of Motion}

A model for an evaporating black hole is obtained by allowing the Schwarzschild radius to become $v$-dependent, which produces an ingoing Vaidya metric:

\[
d s^2 = \left(1 - \frac{r_s(v)}{r}\right) c^2 dv^2 - 2c dv dr - r^2 d\Omega^2. \tag{8}
\]

Our goal is then to study the equations of motions of particles moving in this metric, in order to become conversant with the geodesic motion in this non-stationary setting.

The Vaidya metric is not a vacuum solution. However, only one of the covariant components of the Ricci tensor is non-zero: $R_{uv} = r_s'(v)c/r^2 = 2GM'(v)/r^2 c$. The scalar curvature vanishes. Hence, the stress-energy tensor also has only one nonvanishing element in the coordinates used in \((8)\), which is given by $T_{uv} = M'(v)c^3/4\pi r^2$. Note that this is negative, if the mass $M$ is a decreasing function of $v$.

Consider the line of constant Schwarzschild time $t = t^*$ drawn in Fig. 1. Moving upward to the right on this line means increasing $v$, as $t$ remains constant but $r$ increases (Eq. \((4)\)). This means that observers at larger $r$ and fixed $t = t^*$ see smaller masses inside a sphere of radius $r$, as $M(v)$ diminishes with increasing $v$. This must be so for the ingoing Vaidya metric, because with increasing $r$, more and more negative-energy null dust is between the observer at $r$ and the horizon.

However, this is not what is to be expected far from the horizon for a true evaporating black hole. Hawking radiation is outgoing and has positive energy density, so the mass inside a sphere of radius $r$ should increase with $r$ at constant $t$. Note that this would be the case if the mass depended on the coordinate $u$ of outgoing EF coordinates. The value of $u$ is infinite at $X = T$ (see Eq. \((6)\)) and decreases along the line $t = t^*$ as one moves to the right (Eq. \((11)\)). Therefore, far from the horizon, the outgoing Vaidya metric is more compatible with Hawking radiation than the ingoing one. But near the horizon, the retarded Vaidya time $u$ becomes singular, so for a description including the horizon and its interior, it is not an option, whereas the ingoing Vaidya metric works fine and its negative energy density agrees with quantum mechanical considerations. We do not switch between the two metrics (at some prescribed surface) as was done in Refs. \[15\] and \[16\], because the model would become unnecessarily complicated. The effects to be discussed here will be (slightly) modified quantitatively by such a model improvement but not qualitatively. A detailed discussion of the quality of the approximation is given after Eq. \((19)\), at the end of this section.

To specify the model completely, we will assume

\[
    r_s(v) = \begin{cases} 
    k'(v_l - v)^{1/3} & \text{for } v \leq v_l \\
    0 & \text{for } v > v_l. 
    \end{cases} \tag{9}
\]

This time dependence arises from the relationship for Hawking radiation emitted by a macroscopic black hole and as seen by a distant observer. The temperature of a black hole is inversely proportional to its mass,\[20\]

\[
    T_{BH}(M) = \frac{\hbar c^3}{8\pi G k_B M}, \tag{10}
\]

where Planck’s and Boltzmann’s constants appear in standard notations. The thermal radiation of a black body at this temperature is proportional to $T_{BH}$ but also to the surface area of the black hole, which goes as $r_s^2 \propto M^2 \propto T_{BH}^{-2}$, so the total power output is proportional to $T_{BH}^4 \propto M^{-2}$,

\[
    P_{BH}(M) = \frac{\hbar c^6}{15360\pi G^2 M^{-2}}, \tag{11}
\]

from which we obtain a differential equation for the mass

\[
    \frac{dM}{dv} = - \frac{\hbar c^4}{15360\pi G^2 M^2}, \tag{12}
\]

which is solved by $M^3(v) = \tilde{k}(v_l - v)$, where

\[
    v_l = 5120M_0^5\frac{\pi G^2}{\hbar c^4}. \tag{13}
\]

is the lifetime of the black hole. $M_0$ is its initial mass, and $\tilde{k} = M_0/v_l$. The quantity $k'$ is then just $2Gk^{1/3}/c^2$ (so
that \( r_s(0) = 2GM_0/c^2 \). In Ref. 12 it is argued that the dependency \( [9] \), based on a fixed-background calculation, cannot hold down to mass zero, as this would lead to an infinite flux of radiated particles. At least near the end of evaporation, the functional law must then be modified, an effect that we have studied but which has little impact on the results given here, so we skip its discussion.

The equations of motion for a test particle falling freely in the Vaidya spacetime may be obtained from the Lagrangian

\[
L = \left(1 - \frac{r_s(v)}{r}\right) c^2 \dot{v}^2 - 2 c \dot{v} \dot{r} - \frac{r_s(r)}{2r} c^2 \dot{v}^2 ,
\]

(14)
in which a dot denotes a derivative with respect to the proper time \( \tau \) of the particle. The Euler-Lagrange equations read

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 .
\]

(15)

Writing them out for \( q = \varphi \) and \( q = \vartheta \) we find that they are solved by \( \varphi = \text{const.} \) and \( \vartheta = \text{const.} \). Therefore, purely radial geodesics exist and to find these, we may consider an effective Lagrangian, obtained from Eq. (14) by dropping the last two terms. The equations for the two other coordinates then are \( (r_{s,v} \equiv dr_s/dv = r_s(v)):\)

\[
\left(1 - \frac{r_s}{r}\right) c^2 \dot{v}^2 + \frac{r_s}{r^2 c^2} \dot{v} \dot{r} - \frac{r_{s,v}}{2r} c^2 \dot{v}^2 = 0 ,
\]

(16)

\[
\ddot{v} + \frac{r_s}{2r^2} c^2 \dot{v}^2 = 0 ,
\]

(17)

and the definition of the effective Lagrangian will be useful for simplifications of the equations of motion:

\[
\left(\frac{ds}{d\tau}\right)^2 = c^2 = \left(1 - \frac{r_s(v)}{r}\right) c^2 \dot{v}^2 - 2 c \dot{v} \dot{r} .
\]

(18)

This expresses that the four-speed of a massive particle is equal to \( c \). First, we use (17) in (16) to eliminate \( \ddot{v} \) and then we replace the newly arisen term \( \left(1 - \frac{r_s(v)}{r}\right) c^2 \dot{v}^2 \) with the help of (18) (by \( c^2 + 2c \dot{v} \dot{r} \)), which leads to a cancellation of all mixed terms containing the product \( \dot{v} \dot{r} \). The result is

\[
\ddot{r} = -\frac{r_s(v)}{2r^2} c^2 - \frac{r_{s,v}}{2r} c^2 \dot{v}^2 .
\]

(19)

Equations (16) and (17), together with the side condition (18) on the four-velocity constitute the equations to be solved for radial fall towards the evaporating black hole. They are equivalent to a set of three first-order equations. In fact, it can be shown that Eq. (17) is a consequence of Eqs. (16) and (18) (at least for \( r \) values greater than \( r_s(v) \)). A unique solution requires initial conditions for \( r \) (usually \( > r_s \)), \( \dot{r} \) and \( v \).

In the limit \( r_s = \text{const.} \), i.e., for an eternal black hole, the second term on the r.h.s. of Eq. (19) is zero and inserting the definition of the Schwarzschild radius, we obtain \( \ddot{r} = -GM/r^2 \), which has the same form as Newton’s law for the free fall of a particle in the gravitational field of a point mass (or a spherically symmetric mass distribution with radius smaller than \( r \)). Since the time derivatives are with respect to proper time rather than absolute Newtonian time, the dynamics will look Newtonian only at sufficiently small velocities, as long as the difference between the rate of proper time of the particle and that of an external observer does not become visible.

If we now take a temporal variation of the mass term \( r_s \) into account, then we have both contributions, the first being a generalized Newtonian law with time-dependent mass term \( (r_s(v) \sim M(v)) \) and the standard \( 1/r^2 \) dependence, whereas the second term contains the time derivative of the mass and decays as \( 1/r \). For diminishing mass, the second term is positive, so it is a repulsive contribution to the acceleration of the test particle.

In a Newtonian universe, we would expect only the first term to be present (with the advanced time in the argument of \( r_s \) replaced by absolute time). This should be true even when the mass varies with time, because Newtonian gravity spreads instantaneously. Hence, what we see in general relativity instead is an effect of gravity traveling at a finite velocity.

Let us estimate the relative sizes of the second and first terms. For \( v < v_1 \), the ratio between their magnitudes is

\[
Q = \frac{|r''_s(v)| r^2}{r_s(v)c^2} = \frac{r^2 \dot{v}^2}{3c(v - v)} \approx \frac{r^2 \dot{v}^2}{3cv}. \]

(20)

The last approximation is valid for black holes having a mass bigger than that of the sun and up to times \( v \) well exceeding the current lifetime of the universe \( (13.8 \times 10^9 \text{y}) \), because we have \( v_1 > 2.1 \times 10^{87} \text{y} = 6.6 \times 10^{74} \text{s} \). From Eq. (15), we obtain the estimates \( \dot{v} < 1/\sqrt{1 - \frac{r}{r_s}} \) for negative \( \dot{r} \) and \( \dot{v} < (1 + \sqrt{2})/(1 - \frac{r}{r_s}) \) for positive \( \dot{r} \). Hence, if we require the radius \( r \) to be larger than \( 1.01r_s \), we have \( \dot{v}^2 < 6/(1 - \frac{r}{r_s})^2 < 10^5 \). Setting an upper limit for the radial coordinate by requiring \( r < 10^{11}\text{ly} \approx 10^{27}\text{m} \), which is larger than the particle horizon of the currently favored model of the universe, we find that \( Q < 1.7 \times 10^{-52} \) in the specified temporal and spatial range.

On time scales of the current lifetime of the universe, \( r_s \) is constant for non-rotating astrophysical black holes \( (M \geq M_\odot) \), the mass of the sun), as long as nothing falls on them. The reason is that the power output of their Hawking radiation is extremely small (below \( 10^{48} \text{W} \)), leading to a negligible amount of radiated energy in comparison with their mass energy for at least \( 10^{48} \text{y} \). Obviously, such a black hole must be very well described by the Schwarzschild geometry, except possibly (very) close to the horizon. Our estimate shows that the ingoing Vaidya metric is, on the radial and time scales discussed, indistinguishable from the Schwarzschild metric (written in EF coordinates), as would be the outgoing Vaidya metric with comparable parameters. Therefore, radial particle trajectories in the spacetime of a non-rotating astrophysical black hole will be described to an extremely good approximation by Eq. (19) for \( r > 1.01r_s \) and time scales of up to a few billion years.
Closer to the horizon, the static approximation may be insufficient, because the second term of Eq. (19) can become large. But there, the ingoing Vaidya metric should be good, because it faithfully reproduces the negative energy density near the horizon. According to a fairly rigorous calculation in 1 + 1 dimensions, the sum of the outgoing and ingoing fluxes in the radial rest frame, which is proportional to $T_{uu} + T_{vv}$, becomes zero at $r \approx 1.6r_s$ (and is negative for smaller $r$), so the Vaidya metric with its negative energy density should be good up to about this radial coordinate, and this should be true independent of the mass of the black hole. This is also consistent with Ref. [21]. Hence, for black holes of realistic size, approximating the metric via the ingoing Vaidya metric will be viable for all interesting regions of spacetime. Moreover, it has the advantage of analytic accessibility that is lost, if instead we patch two or more metrics together in order to gain a tiny bit of accuracy at larger $r$.

In order to render the time-dependent Schwarzschild radius visible in our figures in the next section, we have to consider black holes with much smaller masses than will be found in astrophysics. For these, we expect the ingoing Vaidya metric to be a good approximation up to $r = 1.6r_s$ and a decent approximation out to a few more Schwarzschild radii, so answers obtained from it should be qualitatively correct. Results will become qualitatively incorrect, once the second term of Eq. (19) becomes bigger than the first one at large $r$, due to the fact that the second term falls off with a weaker power of $r$ than the first. The dominant second term will then have the wrong sign, because it would be attractive for an outgoing Vaidya metric.

V. NUMERICAL SOLUTION AND ANALYTIC DISCUSSION

To gain understanding about the dynamics in this metric, the equations of motion should be integrated numerically for exemplary situations. They are simple enough to make this a nice practical exercise.

In a number of cases, we solved the geodesic equations beyond the evaporation time, where the derivative of $r_s(v)$ has a singularity $\propto (v_t - v)^{-2/3}$. This poses an obstacle to direct numerical integration. Solvers for ordinary differential equations with time step control will not normally cross that point, in attempting to resolve the singularity. Therefore, a numerical approach avoiding the appearance of diverging quantities was developed. Its details will not be presented here, in order to keep the paper concise.

In Fig. 2, the trajectories of a particle released from rest at different heights and approaching the Schwarzschild radius well before evaporation are presented. The parameters are the same as in Fig. 1 of Ref. [7] and the outcome is similar. The particle falls into the black hole in all cases.

Our results are given in terms of nondimensional variables, with $\tilde{r} \equiv r_s(0)$ chosen as the length unit and the speed of light set equal to 1. Then the time unit is also fixed (at the value $\tilde{v} = r_s(0)/c$). By choosing a nondimensional value for the time $v_t$ of evaporation, defined by $r_s(v_t) = 0$, we therefore fix a physical quantity, which is the initial mass of the black hole. To see this, consider

$$\frac{v_t}{\tilde{v}} = \frac{cv_t}{r_s(0)} = \frac{2560M_0^2\pi G}{h c} = \frac{2560\pi M_0^2}{m_p^2},$$

where $m_p = \sqrt{\hbar c/G} = 2.176 \times 10^{-8}$ kg is the Planck mass. For the values $v_t/\tilde{v} = 100, 10$, and 5 used in our figures, we have $M_0 = 0.112 m_p$, $0.0352 m_p$, and $0.0249 m_p$, respectively, extremely small masses indeed. Black holes with such a small mass cannot arise from direct gravitational collapse. They might be the result of a density fluctuation in some violent subatomic process, of a kind that could possibly have arisen very shortly after the big bang. These black holes would, however, have long since vanished. Yet, it is conceivable that some primordial black holes starting out with masses around $1.8 \times 10^{11}$ kg, approaching the end of their life today, might have current masses close to a Planck mass.

The reason for considering such unrealistic cases here is that for a black hole of about one solar mass, the nondimensional $v_t$ would be on the order of $10^8$, and the Schwarzschild radius would look constant in our figures. To see a particle actually survive the evaporation of the black hole, we must tune its starting distance and time so that it closes in on the horizon position only near the point of complete evaporation (or later).

Before looking at this kind of behavior, let us briefly clarify whether the Schwarzschild radius $r_s(v)$ is still an event horizon, when $r_s$ is decreasing as a function of time. The equation of motion for an outgoing radial light ray follows from Eq. (5) by setting $d\Omega = 0$, $ds = 0$, and $dv \neq 0$:

$$\frac{dr}{dv} = \frac{1}{2} \hat{c} \left(1 - \frac{r_s(v)}{r}\right).$$
This has been integrated in Fig. 3 for a few initial values $r = r_0$ close to $r_s(0)$.

A light ray sent from the Schwarzschild radius $r_s(0)$ will obviously escape, having zero coordinate velocity at $v = 0$ and positive velocity at any later time. Interestingly, some rays starting their journey with $r < r_s(0)$ manage to cross the shrinking Schwarzschild radius, which therefore is not a true event horizon anymore, but rather an apparent horizon. An apparent horizon is a limiting trapped null surface; i.e., outward-directed light rays emitted outside the apparent horizon will move outward, while outward-directed light rays emitted inside it will move inward.

Escaping light rays from inside start falling inward but do so more slowly than the horizon shrinks, so they are passed by the latter and afterwards move outward. Light sent from sufficiently far inside this apparent horizon will, however, fall into the singularity, so there still exists an event horizon, separating events from which null infinity can never be reached from those that may send signals to null infinity. That horizon, however, does not exist indefinitely; it is gone after the time $v_1$. Moreover, if quantum mechanics leads to avoidance of the singularity as is generally believed, possibly replacing it with a fuzzy region of fluctuating spacetime that contains quantum fields at high energy density, whatever was caught in that high-energy density region would get out again (albeit transformed into photons or other elementary particles), once the region dissolved, and escape to future null or timelike infinity. But then there would be no event horizon by definition.

Let us next consider a situation, where it is possible for a particle to miss the singularity in the black hole. This is depicted in Fig. 3. Particles starting from a radius slightly exceeding $3.2 r_s(0)$ will not cross the apparent horizon. The inward velocity of both the particle starting at $3.5 r_s(0)$ and that starting at $4 r_s(0)$ decreases a little just before complete evaporation, which is obviously due to the repulsive term in \( \dot{v} \).

The model can even produce overall repulsion after an initial phase of inward falling. However, since for this to happen, the second term of Eq. (19) must become larger than the first one, due to $r$ getting large, the ingoing Vaidya metric will not be a good model for an evaporating black hole anymore, so we will not consider this case here.

Let us instead determine the range of $r_0$ values, for which the force on the particle is repulsive already in the initial phase of its fall. Since we release it from rest, we may assume that $\dot{r} \approx 0$. Using (18), we then express $\dot{v}$ by functions of $r$ and $v$ alone and find from Eq. (19)

$$\dot{v} = - \frac{r_s(v)c^2}{2r^2} + \frac{r_s'(v)c}{2(r_s - r)}. \quad (23)$$

Setting $\ddot{v} = 0$, we obtain a quadratic equation for $r$ with the roots

$$r_{01} = \frac{r_s c}{2r_{s,v}} + \sqrt{\frac{r_{s,c}^2}{4r_{s,v}^2} + \frac{r_{s,c}^2}{r_{s,v}}} \approx \frac{r_s c}{r_{s,v}} = \frac{3cv_1}{c}, \quad (24)$$

$$r_{02} = \frac{r_s c}{2r_{s,v}} - \sqrt{\frac{r_{s,c}^2}{4r_{s,v}^2} + \frac{r_{s,c}^2}{r_{s,v}}} \approx r_s \left(1 + \frac{r_s}{3cv_1}\right). \quad (25)$$

Repulsion dominates for $r_0 > r_{01}$ and $r_0 < r_{02}$. The first result is irrelevant for stellar-mass evaporating black holes, because it is exorbitantly large, bigger than $6.3 \times 10^{57}$ly, well beyond $10^{50}$ times the particle horizon of the universe.

On the other hand, the second result tells us that there is repulsion in the ingoing Vaidya metric with decreasing mass slightly outside, but near, the Schwarzschild radius. And this close to the horizon, the ingoing Vaidya metric can be safely assumed to yield a decent description!

We first verify the validity of the analytic prediction by numerical simulation. Fig. 4 gives the trajectories, for $v_1/\dot{v} = 10$, of a particle starting at a radius that exceeds $r_s(0)$ by just one percent and of one starting only one thousandth of $r_s(0)$ above the apparent horizon. Eq. (25) predicts initial repulsion for $r_s < r_0 < 1.033 r_s$ here.

The first particle starts moving away from the apparent horizon, being repelled more strongly than attracted,
but as its distance from $r_s(v)$ increases, it is pulled back, turns around, and finally falls into the singularity. However, the second particle, starting significantly closer to the apparent horizon, experiences such a strong repulsion that it actually escapes from the black hole. In order to gain some understanding, we have also plotted the trajectory of an outgoing light ray starting together with the second particle. What becomes immediately clear is that the two trajectories are pretty close to each other, so the particle reaches a speed that is close to the speed of light. It is then easy to form a meaningful mental picture by remembering the river model of a black hole, proposed by Hamilton and Lisle. According to this idea, space about a black hole may be viewed as flowing inward across the event horizon, where it reaches the speed of light. This explains nicely why nothing can cross the event horizon from the inside out – motion relative to space cannot exceed the speed of light. With the Vaidya metric, inflowing space reaches the speed of light at the time-dependent apparent horizon rather than at the event horizon, but otherwise the situation is similar. Positioning a particle at rest slightly above the apparent horizon actually means giving it a very high outward speed with respect to the inflowing space. The closer to the apparent horizon this is done, the closer the outward speed is to the speed of light. A particle extremely close to the apparent horizon should therefore move like a photon. Since outgoing light will always escape when emitted outside the horizon, a massive particle close enough to it will do the same.

Note that it is impossible to start a particle from rest inside the apparent horizon, as its velocity with respect to the inflowing space would have to exceed $c$. However, it is possible to give a particle slightly inside, and close to the apparent horizon an initial condition corresponding to a sufficiently small inward velocity so that the shrinking horizon will catch up with it, allowing it to escape to infinity. Indeed, we find in simulations (not shown) that a particle which is started slightly inside the initial Schwarzschild radius with a coordinate velocity not significantly below $u_0 = -c\sqrt{r_s(0)/r_0} - 1$ manages to cross the apparent horizon and to escape to future timelike infinity.

We expect this result to be robust with regard to a modification of the metric outside of a neighborhood of $r_s$. Any outside metric patched to the ingoing Vaidya metric must not add an additional horizon. Light that reaches the outer metric will therefore escape to future infinity. A particle accelerated to relativistic velocities by the inner metric will then generally also escape to infinity on entering the outer metric.

VI. CONCLUSIONS

The Vaidya metric is an appealing tool to present certain effects of non-stationary gravitational fields in classroom, having applications to black-hole evaporation. Simple analytical results demonstrate interesting physical effects, in particular gravitational repulsion in the presence of a shrinking apparent horizon. The numerical studies presented here are not too advanced, and can be assigned to students as exercises.

Use of the ingoing Vaidya metric with a decreasing mass term is motivated by its utility in modeling a spacetime geometry with a shrinking future (apparent) horizon. We have argued that this kind of description is reasonable near and outside the apparent horizon, because negative energy density is expected to arise there due to quantum effects and the Vaidya metric considered describes an inflow of negative energy carried by null dust. If we take Hawking’s view that this is the same as an outflow of positive-energy particles along past-directed geodesics, then we can say that what the ingoing Vaidya metric fails to account for is the final scattering of these quanta off the gravitational field that makes their path future-directed again. This would certainly be bad, if the purpose was to describe these particles, i.e., Hawking radiation itself. However, we were mainly interested in physical effects of the modifications of the metric brought about by Hawking radiation. For most of the time a black hole exists, Hawking radiation is a small effect. The metric at some distance from the horizon will essentially be the Schwarzschild metric; i.e., $r_s$ changes so slowly that the second term on the right-hand-side of Eq. (19) is negligible and it also does not really matter whether the argument of the first term is the advanced time $v$, the retarded time $u$ or even the Schwarzschild time $t$.

This is no longer true near the end of the black hole’s life, because then Hawking radiation becomes a strong effect and even may dominate the metric. We must then model it more accurately in the whole spacetime. It is therefore concluded that results (not discussed here) such as net repulsion after full evaporation are an artifact of the ingoing Vaidya metric in most cases. This effect is not likely to occur for a black hole at the end of its life.

FIG. 5. Trajectories of particles released from rest at $r_0 = 1.01 r_s(0)$ and $r_0 = 1.001 r_s(0)$. The second particle escapes to infinity. The path of a light ray starting from the same $r_0$ as the second particle is given for comparison.
A more sophisticated approach, properly reproducing the (future-directed) outflow of positive energy far from the horizon would be to keep the ingoing Vaidya metric for radii up to a small multiple of $r_+$ and use an outgoing Vaidya metric for larger $r$. There would be matching conditions for the places where the two metrics meet. Realizing this approach might be a bit too involved for the classroom, and it would not provide any quantitative gains for the description of realistic astrophysical black holes.

The properties of evaporating black holes obtained within our model and discussed here should be robust against model variations aiming at a generally more quantitative description. Two of these are pretty intuitive. The first is that, for particles falling towards the black hole sufficiently long before evaporation, there is no problem crossing the horizon. Hawking radiation does not prevent them from falling in. This would be expected from a comparison of the time scales on which Hawking radiation becomes relevant with those for an infaller to cross the horizon, if a time coordinate is used that does not become singular at the horizon. Second, the nature of the horizon changes. The event horizon of the stationary limit of the Vaidya metric (which is the corresponding EF metric) turns into an apparent horizon. That is, it remains a trapped null surface, but light may eventually escape from it due to its shrinking in time. Contrary to apparent horizons of growing black holes (the normal case), this apparent horizon is outside of the event horizon, obviously because the standard condition of positive energy density everywhere is violated.

Moreover, it is expected that a counterintuitive aspect found here will prevail in a generally more quantitative description, viz. the strong repulsion experienced by a particle released from rest just above the horizon. Such a particle may escape to infinity and this effect, predicted from the Vaidya metric (but also from a time-dependent generalization of the GP metric) is likely to be a qualitatively correct result. Once the particle has been accelerated to nearly the speed of light in a thin shell above the horizon, it will also escape in a model, where the outer part of the metric has been changed into something more realistic. Because this outer metric will not contain a second horizon, any relativistic particle should not have problems escaping, just as a light ray would.

Arguably, the effect is not so counterintuitive after all, because of the presence of negative energy density. Repulsion in the Reissner-Nordström metric is explicable because of negative pressure (which is stress) exerted by the electric field. In the equations of motion, an electrostatic potential term appears, representing negative energy. This then suggests that it is the negative energy density in both the Reissner-Nordström solution and the Vaidya metric that is responsible for the appearance of repulsion. A difference, however, is that the electrostatic energy term in the Reissner-Nordström geodesic equations exceeds the mass energy of the black hole inside the inner horizon. By contrast, a volume integral of the negative energy density of the Vaidya metric, constructed to represent an evaporating black hole, will be much smaller than the remaining mass energy of the black hole for all times except very briefly before complete evaporation. Moreover, the repulsion will happen only for particles almost coordinate stationary near the horizon. An infalling particle having already acquired a substantial inward velocity at the horizon will not experience repulsion ($\dot{r}$ remains finite at $r = r_+$ in (13) for $\dot{r} < 0$). In the Reissner-Nordström case, the repulsion is velocity independent. These considerations suggest that the presence of negative energy density alone does not explain the repulsion effect in the case considered here, although it may be a necessary condition for radial repulsion.

On the other hand, there is no need to invoke energy considerations at all to understand repulsion the way we presented it in discussing Fig. 5. If the apparent horizon is receding, no matter what the reason, then a coordinate stationary particle close enough to it will have essentially the speed of outgoing light relative to the inflowing space in the river model of a black hole. Such a relativistic particle will be ejected and is likely to escape from the black hole. In this view, the repulsion is not due to a repelling force but rather an inertial effect. By contrast, a particle that has a sufficiently large inward velocity slightly outside the horizon is much slower than light with respect to the inflowing space and thus will not be able to avoid being captured.

Finally, the system considered here seems to be the simplest one exhibiting radial gravitational repulsion outside a horizon. In principle, our prediction of particles being ejected violently from near the horizon could be tested in a sonic black-hole analog set up with a receding horizon.

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constant value of that coordinate is spacelike (timelike).

The single event \((X,T) = (0,0)\) has an infinity of time coordinates \(t\), just as the pole of a sphere has an infinity of \(\varphi\) values in spherical coordinates.

A. J. S. Hamilton, “Hawking radiation inside a Schwarzschild black hole,” Gen. Rel. Grav. 50, 50 (48pp) (2018)

The representation is symbolical, because pair creation should, as noted before, not be considered a pointlike event.

However, since \(A\) is at a larger \(r\) coordinate than \(B\), the radiation he sees will be more strongly redshifted than that seen by \(B\).

D. Singleton and S. Wilburn, “Hawking Radiation, Unruh Radiation, and the Equivalence Principle,” Phys. Rev. Lett. 107, 081102 (2011)

A freely falling observer is in a local inertial system, so local experiments do not allow her to detect the black hole horizon on crossing it. If the horizon is a condition for Hawking radiation in an almost stationary spacetime, then a freely falling observer cannot see radiation from that horizon, once she reaches it. Radiation from a distant horizon is not forbidden by the equivalence principle, as demonstrated by the simple fact that an observer at infinity does observe Hawking radiation. Ref. [4] gives support to the validity of the equivalence principle at the horizon only for certain accelerated observers, by showing that Hawking radiation and Unruh radiation have the same temperature for them. Note that the calculation of a non-zero Hawking temperature [4] for a freely falling observer passing the horizon is insufficient to prove observability of the radiation. What has to be proven in addition is that its intensity (or its gray-body factor) is non-zero.

T. Opatrný and L. Richterek, “Black hole heat engine,” Am. J. Phys. 80, 66–71 (2012)

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Interested readers can find them in an earlier version of this article, deposited on arXiv: 2103.08340v2 [gr-qc].

J. Piesnack, “Freier Fall von Testteilchen im Gravitationsfeld verdampfender Schwarzer Löcher,” Bachelor Thesis, Otto-von-Guericke Universität Magdeburg (2020)

The approximation \(r_s \ll r\) may be directly applied in [23] to obtain the approximate result for \(r_{01}\). It also implies \(r_s \ll c\), which has to be used in the equation for \(r_{02}\), not satisfying \(r_s \lesssim r_{02}\).

A. J. S. Hamilton and J. P. Lisle, “The river model of black holes,” Am. J. Phys. 76, 519–532 (2008)

This is true for classical objects. For quantum tunneling, the speed of light is not an insurmountable barrier.

This holds for other metrics with a receding apparent horizon as well.2.