ON THE EXISTENCE OF PARA-KENMOTSU MANIFOLDS

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Abstract. This note provides a quite obvious observation that the condition (2.7), which is a part of the original definition of the so-called para-Kenmotsu manifolds [9], does not make sense, and thus this concept is void. So, it is proved that the para-Kenmotsu manifolds does not exist under the condition mentioned above.

1. Introduction

The notion of almost para-contact structure was introduced by Sato [4, 5]. This structure is an analogue of the almost contact structure [2, 11] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd dimensional but an almost para-contact manifold could be even dimensional as well.

After that, T. Adati and K. Matsumoto [1] defined and studied para-Sasakian and SP-Sasakian manifolds which are regarded as a special kind of almost contact Riemannian manifolds. Before Sato, Kenmotsu [3] defined a class of almost contact Riemannian manifolds.

Later, in 1995, Sinha and Sai Prasad [9] have defined a class of almost para contact metric manifolds namely para-Kenmotsu (p-Kenmotsu) and special para Kenmotsu (sp-Kenmotsu) manifolds. Unfortunately without example which to prove the existence of this type of manifolds.

Para-Kenmotsu manifolds in this sense, are recently developed in several works, for example as [7, 8, 10].

This note provides a quite obvious observation that the formula (2.7), which is a part of the original definition of the so-called para-Kenmotsu manifolds [9], does not make sense, and thus this concept is void.

Firstly, let’s give a brief information for para-Kenmotsu manifolds.
2. Preliminaries

Let $M$ be an $n$-dimensional differentiable manifold equipped with structure tensors $(\phi, \xi, \eta)$ where $\phi$ is a tensor of type $(1,1)$, $\xi$ is a vector field, $\eta$ is a 1-form such that:

\begin{align}
\eta(\xi) &= 1, \\
\phi^2 X &= X - \eta(X)\xi,
\end{align}

for all $X$ vector field on $M$. Then $M$ is called an almost para-contact manifold.

Let $g$ be the Riemannian metric in an $n$-dimensional almost para-contact manifold $M$ such that

\begin{align}
\eta(X) &= g(\xi, X), \\
\phi \xi &= 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align}

for all vector fields $X$ and $Y$ on $M$. Then the manifold $M$ is said to admit an almost para-contact Riemannian structure $(\phi, \xi, \eta, g)$ and the manifold is called an almost para-contact Riemannian manifold.

A manifold $M$ with Riemannian metric $g$ admitting a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and 1-form $\eta$ satisfying equations (2.1), (2.2), (2.5) along with

\begin{align}
d\eta &= 0, \\
\nabla_X \xi &= \phi^2 X = X - \eta(X)\xi, \\
(\nabla_X \nabla_Y \eta) Z &= (-g(X, Z) + \eta(X)\eta(Z))\eta(Y) + (-g(X, Y) + \eta(X)\eta(Y))\eta(Z),
\end{align}

is called a para-Kenmotsu manifold or briefly P-Kenmotsu manifold [9], where $\nabla$ denote the Levi-Civita connection with respect to $g$.

In the following, we will give three reasons why the condition (2.7) is never satisfied.

3. Existence of Para-Kenmotsu Manifolds

Our objection to the condition (2.7) lies in the following three serious problems:

**First problem:**

In the basic condition (2.7), putting

$$T(X,Y)Z = (-g(X,Z) + \eta(X)\eta(Z))\eta(Y) + (-g(X,Y) + \eta(X)\eta(Y))\eta(Z).$$

Then, for all $f$ function on $M$, we have

$$\left(\nabla_X \nabla_Y \eta\right) Z = \left(\nabla_X (f \nabla_Y \eta)\right) Z = X(f) (\nabla_Y \eta) Z + f (\nabla_X \nabla_Y \eta) Z.$$
But in the other side we have
\[
T(X, fY)Z = ( - g(X, Z) + \eta(X) \eta(Z))\eta(fY) + ( - g(X, fY) + \eta(X) \eta(fY))\eta(Z)
\]
\[
= f \left( ( - g(X, Z) + \eta(X) \eta(Z))\eta(Y) + ( - g(X, Y) + \eta(X) \eta(Y))\eta(Z) \right)
\]
\[
= fT(X, Y)Z,
\]
i.e.
\[
(\nabla_X \nabla_{fY} \eta)Z = fT(X, Y)Z + X(f) (\nabla_Y \eta)Z.
\]

This means that one side of the condition (2.7) is \(C^\infty(M)\)-linear in the second variable \(Y\) the other is not.

**Second problem:**

In the basic condition (2.7), since \(d\eta = 0\) i.e for all \(Y\) and \(Z\) vector fields on \(M\)
\[
(\nabla_Y \eta)Z = (\nabla_Z \eta)Y
\]
Then, we have
\[
(\nabla_X \nabla_Y \eta)Z = \nabla_X \left( \nabla_Y \eta \right)Z - \nabla_Y \eta \left( \nabla_X Z \right)
\]
\[
= \nabla_X \left( \nabla_Z \eta \right)Y - \nabla_Y \eta \left( \nabla_X Z \right)
\]
\[
\neq (\nabla_X \nabla_Z \eta)Y,
\]
and in the other side we have
\[
T(X, Y)Z = ( - g(X, Z) + \eta(X) \eta(Z))\eta(Y) + ( - g(X, Y) + \eta(X) \eta(Y))\eta(Z)
\]
\[
= T(X, Z)Y,
\]
This means that one side of the condition (2.7) admits the commutativity between \(Y\) and \(Z\) the other is not.

**Third problem:**

From (2.6), we have
\[
2d\eta(X, Y) = 0 \iff (\nabla_X \eta)Y = (\nabla_Y \eta)X
\]
\[
\iff g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X),
\]
by setting \(Y = \xi\), we find
\[
\nabla_\xi \xi = 0.
\]
(3.1)
Therefore, from (2.7), we have
\[
(\nabla_X \nabla_\xi \eta)Z = -g(X, Z) + \eta(X) \eta(Z),
\]
(3.2)
and in the other side we have
\[
(\nabla_X \nabla_\xi \eta) Z = \nabla_X \left( (\nabla_\xi \eta) Z \right) - (\nabla_\xi \eta) (\nabla_X Z) = Xg(\nabla_\xi \xi, Z) - g(\nabla_\xi \xi, \nabla_X Z) = 0.
\]
(3.3)
From (3.2) and (3.3) shows the contradiction.

This clearly shows that condition (2.7) is never fulfills and therefore the definition of para-Kenmotsu structure as well as the theorem 3.1 in [6] must be reconsidered.

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