Superspace BRST/BV Operators of Superfield Gauge Theories

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Abstract: We consider the superspace BRST and BV description of 4\(D\), \(\mathcal{N} = 1\) super-Maxwell theory and its non-abelian generalization Super Yang–Mills. By fermionizing the superspace gauge transformation of the gauge superfields, we define the nilpotent superspace BRST symmetry transformation (\(s\)). After introducing an appropriate set of anti-superfields and defining the superspace antibracket, we use it to construct the BV-BRST nilpotent differential operator (\(s\)) in terms of superspace covariant derivatives. The anti-superfield independent terms of \(s\) provide a superspace generalization of the Koszul–Tate resolution (\(\delta\)). In the linearized limit, the set of superspace differential operators that appear in \(s\) satisfy a nonlinear algebra which can be used to construct a BRST charge \(Q\), without requiring pure spinor variables. \(Q\) acts on the Hilbert space of superfield states, and its cohomology generates the expected superspace equations of motion.

Keywords: BRST; BV; superspace; Super Yang–Mills

1. Introduction

The Becchi–Rouet–Stora–Tutin (BRST) formalism offers a modern approach of describing gauge theories. The power of this technique has been demonstrated repeatedly from the formulation of field theories that correspond to given first quantized systems all the way to string field theory and the study of string interactions.

Initially, BRST symmetry [1–4] was introduced as a method of quantizing gauge field theories. In general, the quantization procedure of gauge theories is not straightforward, and requires the involvement of ghost fields. The two basic characteristics of gauge theories that lead to the introduction of ghosts are (a) gauge redundancy and (b) the tensorial nature of gauge fields. Specifically, one must eliminate the non-dynamical degrees of freedom of gauge fields and eliminate the states with non-positive definite norm which correspond to the time components (we use the mostly plus convention) of gauge fields. The first was completely understood by the Fadeev–Popov procedure from the view point of a path-integral integration measure effect. Both these issues are addressed in the BRST quantization procedure which systematically generates not only the Fadeev–Popov ghosts—which give rise to appropriate gauge fixing conditions—but also additional propagating ghosts (non-minimal sector) that cancel the negative norm states.

This approach was later extended to the Batalin–Vilkovisky (BV) [5–7] formalism, which promotes the above BRST symmetry to a fundamental principal of the theory, which is automatically incorporated by the introduction of additional fields, the antifields. The antifields have the interpretation of sources of BRST transformations and were initially introduced as a technique to capture the renormalization of the composite operators which emerged in the BRST transformations of interacting gauge theories. However, antifields...
play a crucial role, whether one studies quantum or classical aspects of the theory. They allow the definition of a symplectic structure in the space of fields and antifields, the antibracket. The antibracket can be understood as a covariant analog of the Hamiltonian Poisson bracket, and as a consequence of various Hamiltonian concepts (e.g., canonical transformations), it can be introduced and applied \[8–10\]. This makes the field–antifield formalism a very powerful tool, which could be used in cases where the usual Fadeev–Popov method fails. An important class of such theories is the one with open symmetry algebras, meaning that the commutator of two symmetries generates trivial symmetries. In general, supersymmetric theories fall into this class, unless one considers theories with off-shell supersymmetry, like the ones that have superspace descriptions (see the relevant discussion in \[11\]).

Moreover, the antibracket formalism gained popularity among string theorists, when it was applied to the bosonic open string field theory \[12–15\], and later, to closed string field theory \[16\]. This proved to be very useful for the study of string interactions and various string vacua containing non-perturbative objects like D-branes \[17\]. The BRST approach to string field theory originated in the work of Siegel \[18\], where a nilpotent, fermionic BRST operator is constructed which commutes with observables, acts on the state space of the theory, and its cohomology defines the physical spectrum. This state space BRST operator is a reflection of the BRST symmetry of the target space fields, as generated from the antibracket in the Lagrangian path integral formulation. Interestingly, similar state space BRST operators have been constructed to describe higher spin gauge fields \[19–34\]. This is another indication of the close relationship between higher spins and string theory.

Finally, for maximally supersymmetric theories, it has been shown that a covariant quantization can be achieved by using pure spinor variables \[35\], and the spectrum of these theories is captured by the cohomology of a pure spinor BRST operator \[36\]. The free action was constructed \[37\] in terms of a pure spinor superfield \(\lambda\) \(\Psi(x, \theta, \lambda)\) and the pure spinor BRST operator \(Q = \lambda^a \mathcal{D}_a\). However, these descriptions were not manifestly maximally supersymmetric, and only worked for linearized theories. Later, both issues were addressed by the introduction of additional –non-minimal– pure spinors that modified the BRST charge operator \[38\], and by adopting the BV formalism, and solving the master equation \[39,40\].

The purpose of this paper is to explore the BRST symmetry, BV and BFV (Batalin–Fradkin–Vilkovisky) formulations of supersymmetric gauge theories (see \[41,42\] for earlier considerations) from the superspace point of view, without the use of pure spinor variables. Similarly to string theory applications, we construct a superspace BRST operator acting on the (target) space of superfields, and a corresponding BRST operator that acts on a Hilbert space of superfield states.

Following the BRST procedure, for a given supersymmetric gauge theory with a superspace description, we introduce a nilpotent, BRST symmetry operator \(s\) \(s^2 = 0\), by fermionizing the superspace gauge transformations of the theory via the replacement of all gauge parameter superfields, with the corresponding ghost superfields of opposite statistics. The superspace Lagrangian is deformed by the addition of an appropriate \(s\)-exact term, which includes additional (non-minimal sector) ghost superfields, like the Nakanishi–Lautrup ghost superfield and its BRST-doublet partner.

In the BV description, for every superfield and ghost superfield, we introduce a conjugate variable, the anti-superfield. In the superfield/anti-superfield space, we define the superspace antibracket and use it to construct a nilpotent, superspace differential operator \(s\) \(s^2 = 0\). This is a superspace BRST differential operator which, acting on (anti-) superfields, can be decomposed into two pieces: \(s = \gamma + \delta\). The \(\gamma\) part is the now renamed superspace BRST symmetry, and its action on superfields coincides with \(s\). The second part \(\delta\) is the superspace generalization of the Koszul–Tate resolution \[43\], which implements the superspace equations of motion. As expected, both \(\gamma\) and \(\delta\) are nilpotent and anticommute.

Finally, getting inspiration from string field theory, we define a Hilbert space of states, where the arbitrary state vector \(|\Psi\rangle\) can be expanded on some basis \(|\phi_s\rangle\) with superfield...
coefficients $\chi_s (|\Psi\rangle = \sum_s |\phi_s\rangle \chi_s)$. In this Hilbert space, we define a nilpotent BRST operator $Q (Q^2 = 0)$, such that its action on the Hilbert space vectors coincides with the action of $s$ on the superfield coefficients. This is done in two steps: (1) we identify the set of differential operators that appear in $s$ and (2) find their algebraic properties in order to construct a nilpotent Hilbert space BRST operator via the Fradkin–Fradkina algorithm [44]. The cohomology of $Q$ must correctly generate the superspace equations of motion and gauge transformations for the superfields of the theory, and thus produce the expected physical spectrum.

We would like to emphasize that, although the existence of such BRST charge is proven for general gauge theories [45–48], the explicit construction of it can be a nontrivial task. For supersymmetric theories with superspace descriptions, the only available methodology for constructing the BRST charge is based on pure spinors. In the pure spinor approach, the nilpotence of the BRST charge is based on the pure spinor constraint, and not on the algebraic properties of the characteristic set of differential operators that participate in the description of the theory. Our methodology can offer an alternative method for constructing such nilpotent charges. This could be beneficial for the construction of a manifestly supersymmetric BRST charge that describes higher spin supermultiplets.

The proposed procedure can be applied to any gauge theory in superspace. In this paper, we demonstrate it for 4D, $\mathcal{N} = 1$ super-Yang–Mills (SYM) and its linearized limit, the vector supermultiplet. We explicitly construct the superspace BRST symmetry operator $s$ and extract the superspace Koszul–Tate complex $\delta$. For the linearized theory, the set of superspace differential operators that appear in $s$ satisfy a nonlinear algebra. Nevertheless, using known corrections [49,50] of the Fradkin–Fradkina algorithm, we find and explicit expression for $Q$ in terms of supersymmetric covariant derivatives, without the need to introduce pure spinors. The cohomology of this superspace BRST charge correctly generates the gauge symmetry and superspace equations of motion for the vector supermultiplet.

Before discussing all of the above, we first remind the reader of the BRST, BV, and BFV descriptions in the context of a simple example: Maxwell theory. This corresponds to the bosonic sector of the linearized SYM, and it is the simplest example where the non-supersymmetric shadows ($s, s', \gamma, \delta, Q$) of the quantities discussed above can be introduced and easily constructed. Moreover, it is explicitly shown that there is a Hilbert space of field states, and a nilpotent charge $Q$, such that the action of the BRST operator $s$, generated by the antibracket, on fields and antifields components of the field states, is equal to the action of $Q$ on the basis vectors of the Hilbert state $\{|e_i\rangle\}$: $s|\psi\rangle = Q|\psi\rangle$. The BRST charge $Q$ constructed in this way is identical to the Batalin–Fradkin–Vilkovisky (BFV) nilpotent charge constructed from the algebra of the physical constraints that define the on-shell propagating degrees of freedom of the theory.

2. Review of BRST, BV, and BFV Descriptions of Maxwell Theory

In the Lagrangian path integral approach, the BRST method is based on the fact that, if the theory under consideration is a gauge theory, then there is a nilpotent, odd BRST symmetry ($\delta$), and one can deform the Lagrangian of the theory by the addition of an $\delta$-exact term, without affecting the path integral

$$\mathcal{L} \rightarrow \mathcal{L} + \delta \Omega .$$

(1)

The deformed action remains invariant

$$\delta \mathcal{L} + \delta^2 \Omega = 0$$

(2)

because both terms independently vanish. The second term vanishes because $\delta$ must be nilpotent ($\delta^2 = 0$), and the first term vanishes because we choose the BRST transformation of the fields that appear in the starting Lagrangian $\mathcal{L}$ to have exactly the same form as their corresponding gauge transformations. Therefore, the Bianchi identities that guarantee the gauge invariance of $\mathcal{L}$ will also guarantee $\delta$-invariance. The tension between $\delta$ with
odd symmetry and the same structure as the gauge transformations is resolved by the replacement of every gauge parameter by a ghost field of opposite statistics. This way, one can immediately define the BRST symmetry transformation of any gauge field. This is usually referred to as the fermionization of the gauge transformation. For the case of Maxwell’s theory, we obtain:

\[ \delta A_m = \partial_m \lambda \rightarrow sA_m = \partial_m c \]

(3)

where \( c \) is a ghost field. Moreover, because of the nilpotence of this BRST transformation, one determines the BRST transformation of the ghost field:

\[ s^2 A_m = \partial_m (sc) = 0 \Rightarrow sc = 0 . \]

(4)

Because the deformation (1) of the Lagrangian is \( \delta \)-exact, the theory is independent of any particular choice of \( \Omega \). In the context of quantizing the theory, one selects this deformation, such that it generates an appropriate gauge fixing condition by integrating a ghost field, as expected from the Fadeev–Popov procedure. For Maxwell’s theory, usually one chooses the Landau–Fermi gauge, or more generally, the Feynman gauge:

\[ \mathcal{L} \rightarrow \mathcal{L} + \rho (\partial^m A_m + \frac{\xi}{2} \rho) \]

(5)

where \( \rho \) is a bosonic ghost field, and \( \xi \) is the usual Feynman gauge parameter. However, this deformation fails to satisfy the consistency condition of being \( \delta \)-closed:

\[ \delta [ \rho (\partial^m A_m + \frac{\xi}{2} \rho) ] = \delta (\rho) (\partial^m A_m + \xi \rho) + \rho \Box c \neq 0 . \]

(6)

Even if we choose \( \delta \rho = 0 \), there is no way to cancel the second term; hence, in order for the deformation to be consistent, we must add additional terms. It is straightforward to check that the complete answer is:

\[ \delta \Omega = \rho (\partial^m A_m + \frac{\xi}{2} \rho) - \beta \Box c \]

(7)

where \( \beta \) is an additional fermionic ghost field. By assigning to \( \rho \) and \( \beta \) the following \( \delta \) transformations

\[ \delta \rho = 0 , \; \delta \beta = \rho \]

(8)

one can check that the right-hand side not only is \( \delta \)-closed, but also \( \delta \) exactly like the left-hand side, with \( \Omega \) taking the form

\[ \Omega = \beta (\partial^m A_m + \frac{\xi}{2} \rho) . \]

(9)

The fermionic ghost fields \( \beta \) and \( c \) correspond to the Fadeev–Popov ghosts, and the bosonic ghost field \( \rho \) is the Nakanishi–Lautrup field. Moreover, the \( (\beta , \rho) \) pair defines the non-minimal sector of the theory, which must have two properties. First, it must allow the construction of an action for the fermionic ghost \( c \) (accomplished by \( \beta \)). Second, \( \beta \) cannot be BRST-isolated (\( \delta \)-closed), because it may introduce new gauge invariant quantities and affect the physical sector of the theory (accomplished by \( \rho \)). Transformations (8) have a special structure, which identifies the non-minimal sector ghosts \( \beta \) and \( \rho \) as a BRST-doublet. BRST-doublets play a very crucial role in BRST constructions, because they do not contribute non-trivial pieces in the cohomology of the theory. In the case of string theory or the (super)particle, similar BRST-doublets are introduced.

In the BV formalism, one goes a step further, and for every field/ghost, introduces a “conjugate” variable, the antifield. At the moment, antifields are considered as classical sources that generate \( \delta \) transformations of the corresponding fields or ghosts, and thus appear in the action in the following manner: Antifield \( \delta (\text{Field}) \). Therefore, the antifield of a specific field/ghost carries appropriate quantum numbers/indices, in order for these
terms to be well defined additions to the deformed Lagrangian. For Maxwell theory, we obtain

\[ L_{BV} = \frac{1}{2} A^m (\Box A_m - \partial_m \partial^n A_n) + \rho (\partial^m A_m + \frac{i}{2} \beta) - \beta \Box c + A^m m \partial_m c + \beta^* \rho \] (10)

where \( A^m, c^*, \rho^*, \beta^* \) are the antifields of \( A_m, c, \rho, \beta \), respectively. In this case, \( c^* \) and \( \rho^* \) drop out of the Lagrangian, because their corresponding ghost fields are \( s \)-closed. Furthermore, in order for these additional terms to be BRST-invariant:

\[ s[\text{Antifield } \Phi_i] = s[\text{Field } \Phi_i] \Rightarrow s[\text{Antifield } \Phi_i] = 0 \] (11)

we choose all antifields to be \( s \)-closed.

For a general gauge theory, the variation of the BV action (\( S \)) under the \( s \)-BRST transformations takes the form:

\[ \delta S = \int \left\{ \delta \Phi^i \frac{\delta S}{\delta \Phi^i} + s \Phi^*_i \frac{\delta S}{\delta \Phi^*_i} \right\} \] (12)

where \( \Phi^i \) is symbolically the set of gauge fields and ghosts that participate in the BRST description of the gauge theory, as discussed above, and \( \Phi^*_i \) are their corresponding antifields. However, because the antifields are \( s \)-closed and appear in the action such that \( \frac{\delta S}{\delta \Phi^*_i} = s \Phi^i \), one arrives at the expression:

\[ \delta S = \int \frac{\delta S \delta S}{\delta \Phi_i^* \delta \Phi^i} \] (13)

This motivates the definition of a binary bracket, called the antibracket

\[ (F, G) = \int \left\{ \frac{\delta F}{\delta \Phi_i^*} \frac{\delta G}{\delta \Phi^i} + \frac{\delta G}{\delta \Phi_i^*} \frac{\delta F}{\delta \Phi^i} \right\} \] (14)

Using the antibracket, the \( s \)-invariance of the action \( S \) takes the simple form

\[ (S, S) = 0 \] (15)

which is known as the classical master equation. It is easy to check that the fields and antifields are conjugate variables with respect to the antibracket, but most importantly, because the action obeys the classical master equation, one can define a nilpotent differential operator \( s (s^2 = 0) \) that acts in the space of fields and antifields

\[ sF \equiv (S, F) \] (16)

for any field or antifield \( F \). The \( s \) transformations of the fields and antifields of Maxwell theory are:

\[ sA_m = \partial_m c , \quad sA^*_m = \Box A^m - \partial^m \partial_n A^n - \partial^m \rho \] (17a)

\[ s\rho = 0 , \quad s\rho^* = \partial^m A_m + \rho + \beta^* \] (17b)

\[ s\beta = \rho , \quad s\beta^* = -\Box c \] (17c)

\[ sc = 0 , \quad sc^* = \Box \beta + \partial_m A^m \] (17d)

These transformations are nilpotent as expected, and they have very interesting structures. The \( s \)-transformation of any field matches their \( s \) transformation exactly, whereas the \( s \)-transformation of the antifields knows about the equations of motion of the corresponding fields.
In general, the differential operator $s$ can be decomposed into two operators: $s = \gamma + \delta$. Operator $\gamma$ is the newly renamed BRST symmetry transformation, and is defined as follows:

$$\gamma \Phi^i \equiv \frac{\delta S}{\delta \Phi^i} = \beta \Phi^i, \quad \gamma \Phi^*_i \equiv \frac{\delta}{\delta \Phi^*_i} \left[ \Phi^*_i \beta (\Phi^i) \right]. \quad (18)$$

For fields, $\gamma$ coincides with the $\beta$ transformation, but for antifields, it has additional terms originating from the variation of the antifield terms in the action. The $\delta$ part is called the Koszul–Tate resolution (see [43,51]) and it implements the equations of motion. It is defined as follows:

$$\delta \Phi^i \equiv 0, \quad \delta \Phi^*_i \equiv \frac{\delta}{\delta \Phi^*_i} \left[ S - \Phi^*_i \beta (\Phi^i) \right]. \quad (19)$$

Adding Equations (18) and (19) automatically gives $s$, as defined in (16). Additionally, $\gamma$ and $\delta$ are nilpotent and anticommute: $\gamma^2 = 0, \delta^2 = 0, \gamma \delta + \delta \gamma = 0$.

For Maxwell’s theory, the action of $\gamma$ and $\delta$ operators on the fields and antifields is:

$$\gamma A_m = \partial_m c, \quad \gamma A^*_m = 0, \quad \delta A_m = 0, \quad \delta A^*_m = \Box A^m - \partial \partial_A^m A^m - \partial^m \partial A^m \quad (20a)$$

$$\gamma \rho = 0, \quad \gamma \rho^* = \beta^*, \quad \delta \rho = 0, \quad \delta \rho^* = \partial_A^m A^m + \rho \quad (20b)$$

$$\gamma \beta = \rho, \quad \gamma \beta^* = 0, \quad \delta \beta = 0, \quad \delta \beta^* = -\Box c \quad (20c)$$

$$\gamma c = 0, \quad \gamma c^* = \partial_m A^*_m, \quad \delta c = 0, \quad \delta c^* = \Box \beta \quad (20d)$$

Having constructed the BRST operator $s$ on the space of fields and antifields, we can define a corresponding BRST charge operator $Q$, which acts on an appropriately defined Hilbert space of states. For every pair of field $\Phi^i$ and antifield $\Phi^*_i$, we introduce a corresponding pair of state vectors $|e(\Phi^i)\rangle$ and $|e(\Phi^*_i)\rangle$, and we consider the vector space which is spanned by the states $\{|e(\Phi^i)\rangle, |e(\Phi^*_i)\rangle\}$. The most general state is written as

$$|\psi\rangle = \sum_i \chi_i |e(\Phi^i)\rangle + \sum_i \nu^i |e(\Phi^*_i)\rangle \quad (21)$$

where $\chi_i$ and $\nu^i$ are field components of $|\psi\rangle$. Furthermore, for each of the basis vectors, we assign a ghost number value, which is defined as being opposite to the ghost number of the corresponding field or antifield. Specifically, if the field theoretic ghost number value of the field $\Phi^i$ is $gh(\Phi^i) = g^i$, then we define the vector space ghost number value of the corresponding vectors to be

$$gh(|e(\Phi^i)\rangle) \equiv -g^i, \quad gh(|e(\Phi^*_i)\rangle) \equiv 1 + g^i \quad (22)$$

For Maxwell theory, we have the following set of states:

$$gh^{\#} = 0: \{ |e(A)\rangle, |e(\rho)\rangle, |e(\beta^*)\rangle \}$$

$$gh^{\#} = 1: \{ |e(A^*)\rangle, |e(\rho^*)\rangle, |e(\beta)\rangle \}$$

$$gh^{\#} = -1: \{ |e(c)\rangle \}$$

$$gh^{\#} = 2: \{ |e(c^*)\rangle \}$$
This structure of states can be captured by a Hilbert space with a vacuum state $|\omega\rangle$ and three fermionic creation operators $\eta, \zeta, \pi$ with ghost values $gh(\eta) = gh(\zeta) = -gh(\pi) = 1$ and $gh(|\omega\rangle) = 0$

$$
gh# = 0 : \{ |\omega\rangle, \eta \pi_\xi |\omega\rangle, \zeta \pi_\xi |\omega\rangle \}
$$

$$
gh# = 1 : \{ \eta |\omega\rangle, \zeta |\omega\rangle, \eta \zeta \pi_\xi |\omega\rangle \}
$$

$$
gh# = -1 : \{ \pi_\xi |\omega\rangle \}
$$

$$
gh# = 2 : \{ \eta \zeta |\omega\rangle \}
$$

On this basis, the most general state is a linear combination of the above states

$$
|\psi\rangle = |\psi(\text{-}1)\rangle + |\psi(0)\rangle + |\psi(1)\rangle + |\psi(2)\rangle
$$

where

$$
|\psi(\text{-}1)\rangle = \pi_\xi |u\rangle, \quad |u\rangle = u |\omega\rangle
$$

$$
|\psi(0)\rangle = |w_1\rangle + \eta \pi_\xi |w_2\rangle + \zeta \pi_\xi |w_3\rangle, \quad |w_i\rangle = w_i |\omega\rangle
$$

$$
|\psi(1)\rangle = \eta |z_1\rangle + \zeta |z_2\rangle + \eta \zeta |z_3\rangle, \quad |z_i\rangle = z_i |\omega\rangle
$$

$$
|\psi(2)\rangle = \eta \zeta |v\rangle, \quad |v\rangle = v |\omega\rangle
$$

and the coefficients $u, w_i, z_i, v$ are elements of the field–antifield space. Therefore, the BRST operator $s$ can act on the state $|\psi\rangle$ by acting on these coefficients. On the other hand, we can define a corresponding BRST operator $Q$ that acts on the states of this Hilbert space, which is equivalent to the action of $s$ (17)

$$
Q |\psi\rangle = s |\psi\rangle.
$$

$Q$ must be odd, have a ghost value of one, and must be constructed out of the oscillators of the Hilbert space $\eta, \zeta, \pi$ and their conjugate $\pi_\eta, \pi_\zeta, \pi$: $\{ \eta, \pi_\eta \} = 1$, $\{ \zeta, \pi_\zeta \} = 1$, $\{ \xi, \pi_\xi \} = 1$. The most general ansatz for $Q$ is

$$
Q = \eta \Lambda^{(0)} + \zeta \Lambda^{(1)} + \xi \Lambda^{(-1)}
$$

$$
+ \eta \zeta \pi_\eta B_1 + \eta \xi \pi_\xi B_2 + \eta \zeta \pi_\zeta B_3
$$

$$
+ \eta \xi \pi_\eta \Gamma_1 + \eta \xi \pi_\xi \Gamma_2 + \eta \xi \pi_\zeta \Gamma_3
$$

$$
+ \zeta \xi \pi_\eta \Delta_1 + \zeta \pi_\xi \Delta_2 + \zeta \pi_\zeta \Delta_3
$$

$$
+ \eta \xi \pi_\eta \pi_\xi K_1 + \eta \xi \zeta \pi_\eta \pi_\zeta K_2 + \eta \zeta \pi_\xi \pi_\zeta K_3
$$

Equation (25) can be expanded according to (23)

$$
Q |\psi(\text{-}1)\rangle + Q |\psi(0)\rangle + Q |\psi(1)\rangle + Q |\psi(2)\rangle = s |\psi(\text{-}1)\rangle + s |\psi(0)\rangle + s |\psi(1)\rangle + s |\psi(2)\rangle
$$

and by matching the Hilbert space ghost number (and the field–antifield space ghost number) of the two sides, we obtain the following set of equations

$$
Q |\psi(\text{-}1)\rangle = s |\psi(0)\rangle, \quad (28a)
$$

$$
Q |\psi(0)\rangle = s |\psi(1)\rangle, \quad (28b)
$$

$$
Q |\psi(1)\rangle = s |\psi(2)\rangle, \quad (28c)
$$

$$
Q |\psi(2)\rangle = 0 = s |\psi(\text{-}1)\rangle. \quad (28d)
$$
These equations can be solved to fix all the coefficients of $|\psi\rangle$ and determine all the field operators in $Q$. We find:

$$u = c, \quad |u\rangle = |c\rangle = c|\omega\rangle,$$
$$v = c^*, \quad |v\rangle = |c^*\rangle = c^*|\omega\rangle,$$
$$w_1 = A^m a^+_m, \quad |w_1\rangle = |A\rangle = A^m a^+_m|\omega\rangle, \quad \rho = A^m a^+_m - \partial^m a^+_m \rho^*, \quad |z_1\rangle = -|A\rangle - \Lambda(-1)|\rho^*\rangle,$$
$$w_2 = -\beta^*, \quad |w_2\rangle = -|\beta^*\rangle = -\beta^*|\omega\rangle, \quad \rho = -\rho^* + (1 + \partial^m a^+_m) \beta, \quad |z_2\rangle = -|\rho^*\rangle + (1 + \Lambda(-1))|\beta\rangle,$$
$$w_3 = \rho, \quad |w_3\rangle = |\rho\rangle = \rho|\omega\rangle, \quad \rho = -\rho^* + (1 + \partial^m a^+_m) \beta, \quad |z_3\rangle = -|\rho\rangle + (1 + \Lambda(-1))|\beta\rangle,$$

where $a^+_m$ and $a_m$ is a pair of bosonic creation and annihilation operators $[a_m, a^+_n] = \eta_{mn}$ and

$$Q = \eta \Box + \zeta \partial^m a_m + \zeta \partial^m a^+_m - \zeta \xi \tau_7.$$

This $Q$ operator is nilpotent by construction, and it is identical to the BFV-BRST charge constructed from the algebra of operators $\Lambda(0) = \Box$, $\Lambda(1) = \partial^m a_m$, $\Lambda(-1) = \partial^m a^+_m$:

$$[\Lambda(0), \Lambda(1)] = 0, \quad [\Lambda(0), \Lambda(-1)] = 0, \quad [\Lambda(1), \Lambda(-1)] = \Lambda(0).$$

The general physical state (zero ghost number) cohomology of $Q$ is given by:

$$|\phi\rangle = |A\rangle + \eta \tau_2 |B\rangle + \zeta \tau_5 |C\rangle$$

with a transformation:

$$\delta |\phi\rangle = Q |L\rangle, \quad |L\rangle = \tau_5 |\Lambda\rangle$$

$$\Rightarrow \begin{cases} 
\delta |A\rangle = \Lambda(-1)|\lambda\rangle, \\
\delta |B\rangle = \Lambda(0)|\lambda\rangle, \\
\delta |C\rangle = \Lambda(1)|\lambda\rangle.
\end{cases}$$

and equations of motion:

$$Q |\phi\rangle = 0 \Rightarrow \begin{cases} 
\Lambda(0)|A\rangle - \Lambda(-1)\Lambda(1)|A\rangle + \Lambda(-1)\Lambda(-1)|C\rangle = 0, \\
\Lambda(1)|A\rangle - \Lambda(-1)|D\rangle - |B\rangle = 0, \\
\Lambda(0)|D\rangle - \Lambda(1)\Lambda(3)|A\rangle + \Lambda(1)\Lambda(-1)|D\rangle = 0.
\end{cases}$$

For the case $|\phi\rangle = |\psi(0)\rangle$ and $|L\rangle = |\psi(-1)\rangle$, one recovers the Maxwell theory equations and transformations.

3. Superspace BRST Description of Super-Maxwell Theory

In this section, we apply the above concepts in superspace aiming towards a BRST-BV/BFV description of the vector supermultiplet, which is the supersymmetric extension of Maxwell theory. The superspace description of the theory is given in terms of a real scalar superfield $V(x, \theta, \bar{\theta})$ with the following superspace action principle

$$S = \frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} \, V \bar{D}^\gamma \bar{D}_\gamma V$$

which is invariant under the gauge transformation $\delta V = \bar{D}^\gamma L + D_\gamma L$. The operators $D_\alpha$ and $\bar{D}_\dot{\alpha}$ are the supersymmetric covariant derivatives, and the off-shell spectrum of the theory is a Maxwell spin 1 gauge field, a decoupled auxiliary real scalar field and a spin 1/2 fermion.

Similar to the discussion in the previous section, we promote the gauge symmetry of the theory to a nilpotent, superspace BRST symmetry $\delta$. The action of $\delta$ on the gauge
superfield \( V \) is found by fermionizing the original gauge transformation, and replace the gauge parameter superfields with ghost superfields:

\[
\mathcal{A} V = \mathcal{D}^2 \epsilon + \mathcal{D}^2 \bar{\epsilon}.
\]

Using the nilpotence of \( \mathcal{A} \), we also find that

\[
0 = \mathcal{A}^2 V = \mathcal{D}^2 (\mathcal{A} \epsilon) + \mathcal{D}^2 (\mathcal{A} \bar{\epsilon}) \Rightarrow \mathcal{A} \epsilon = 0, \mathcal{A} \bar{\epsilon} = 0
\]

An important comment is that, unlike Maxwell theory, the differential operators that appear in the gauge transformation of the gauge superfield have a non-zero kernel and are based on the algebra of the supersymmetric covariant derivatives, we can introduce ghosts for ghosts, ghosts for ghosts for ghosts, etc., ad infinitum \( \mathcal{A} \epsilon = 0 + \mathcal{D}^4 \bar{\epsilon} \), \( \mathcal{A} \bar{\epsilon} = 0 + \mathcal{D}^5 \mathcal{D}^2 \epsilon )\ldots \). This is similar to \( \bar{\epsilon} \). For the purpose of this paper, these contributions are not required, and we will not consider them.

Next, we deform the Lagrangian by adding an appropriate \( \mathcal{A} \)-exact term, \( \mathcal{A} \Omega \). Motivated by the quantization procedure for this theory, this deformation will include a gauge fixing condition accompanied by the corresponding Lagrange multiplier ghost superfield. The usual superspace gauge fixing conditions for gauge superfield \( V \) are \( \mathcal{D}^2 V = 0 \) and \( \mathcal{D}^2 \bar{V} = 0 \), which correspond to the superspace extension of the Landau–Fermi gauge in Maxwell theory. Therefore, we consider the deformation:

\[
\mathcal{A} \Omega = \rho (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \bar{\rho} (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \ldots
\]

where \( \rho \) and \( \bar{\rho} \) are Nakanishi–Lautrup superfield ghosts, \( \xi \) is the Feynman gauge parameter, and the dots represent additional terms that we have in order to make the right-hand side of the equation \( \mathcal{A} \)-exact. Using the nilpotence of \( \mathcal{A} \), we find the consistence condition

\[
0 = (\mathcal{A} \rho) (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + (\mathcal{A} \bar{\rho}) (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \rho \mathcal{D}^2 \mathcal{D}^2 \epsilon + \bar{\rho} \mathcal{D}^2 \mathcal{D}^2 \bar{\epsilon} + \mathcal{A} (\ldots)
\]

which determines the missing terms to be

\[
\mathcal{A} \Omega = \rho (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \bar{\rho} (\mathcal{D}^2 V + \frac{\xi}{2} \rho) - \beta \mathcal{D}^2 \mathcal{D}^2 \epsilon - \bar{\beta} \mathcal{D}^2 \mathcal{D}^2 \bar{\epsilon}.
\]

The remaining \( \mathcal{A} \) transformations are

\[
\mathcal{A} \beta = \rho, \mathcal{A} \rho = 0, \mathcal{A} \bar{\beta} = \bar{\rho}, \mathcal{A} \bar{\rho} = 0
\]

and \( \beta \) is a fermionic ghost superfield which forms a BRST-doublet with \( \rho \). This is similar to \( \bar{\beta} \) and \( \bar{\rho} \). This is expected because there are two gauge parameter superfields \( (\bar{L}, L) \), and therefore, by fermionizing them, we obtain two fermionic ghost superfields \( (\epsilon, \bar{\epsilon}) \), and for each one of them, we must introduce a BRST-doublet, as discussed in the previous section. The non-minimal sector of this theory is the two BRST-doublets \((\beta, \rho)\) and \((\bar{\beta}, \bar{\rho})\). Furthermore, using the above \( \mathcal{A} \) transformations, one can check that the right-hand side of (39) is \( \mathcal{A} \)-exact, and solves for \( \Omega \)

\[
\Omega = \beta (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \bar{\beta} (\mathcal{D}^2 V + \frac{\xi}{2} \rho).
\]

This deformation is further extended by the BV anti-superfields. For each (ghost) superfield, we introduce an anti-superfield with opposite statistics and appropriate superspace ghost number, and add to the action a [Anti-superfield \( \mathcal{A} \) (Superfield)] term. The superspace BV Lagrangian takes the form:

\[
\mathcal{L}_{BV} = \frac{1}{2} \mathcal{V} \mathcal{D}^2 \mathcal{D}_V V + \rho (\mathcal{D}^2 V + \frac{\xi}{2} \rho) + \bar{\rho} (\mathcal{D}^2 V + \frac{\xi}{2} \rho) - \beta \mathcal{D}^2 \mathcal{D}^2 \epsilon - \bar{\beta} \mathcal{D}^2 \mathcal{D}^2 \bar{\epsilon} + V^* (\mathcal{D}^2 \epsilon + \mathcal{D}^2 \bar{\epsilon}) + \beta^* \rho + \bar{\beta}^* \bar{\rho}
\]
where $V^*$, $\beta^*$, $\bar{\beta}^*$, $\rho^*$, $\bar{\rho}^*$, $c^*$, $\bar{c}^*$ are the anti-superfields corresponding to $V$, $\beta$, $\bar{\beta}$, $\rho$, $\bar{\rho}$, $c$, $\bar{c}$, respectively. The ghosts $\rho$, $\bar{\rho}$, $c$, $\bar{c}$ are $\mathcal{s}$-closed, thus their conjugate anti-superfields drop out of the BV superspace action. In order to maintain the $\mathcal{s}$ BRST symmetry of the BV action, we assign, to all the anti-superfields, trivial $\mathcal{s}$ transformations ($\mathcal{s}$-closed):

$$\mathcal{s}V^* = \mathcal{s}\beta^* = \mathcal{s}\bar{\beta}^* = \mathcal{s}\rho^* = \mathcal{s}\bar{\rho}^* = \mathcal{s}c^* = \mathcal{s}\bar{c}^* = 0.$$  \hfill (42)

Now, we can define a full superspace antibracket $(.,.):

$$(F,G) \equiv \int d^4x\ d^2\theta\ d\bar{\theta}\{\frac{\delta F}{\delta \Phi}, \frac{\delta G}{\delta \Phi}\}$$

$$= \int d^4x\ d^2\theta\ d\bar{\theta}\left\{\frac{\delta F}{\delta \Phi} \frac{\delta G}{\delta \Phi^*} + \frac{\delta G}{\delta \Phi} \frac{\delta F}{\delta \Phi^*} \right\}$$

$$+ \frac{\delta F}{\delta \Phi^*} \frac{\delta G}{\delta \Phi} + \frac{\delta G}{\delta \Phi^*} \frac{\delta F}{\delta \Phi} + \frac{\delta F}{\delta \Phi^*} \frac{\delta F}{\delta \Phi^*} + \frac{\delta G}{\delta \Phi^*} \frac{\delta G}{\delta \Phi^*} \right\}$$

$$= \int d^4x\ d^2\theta\ d\bar{\theta}\left\{\frac{\delta F}{\delta \Phi} \frac{\delta G}{\delta \Phi^*} + \frac{\delta G}{\delta \Phi} \frac{\delta F}{\delta \Phi^*} \right\}$$

Using this bracket, the $\mathcal{s}$ invariance of the BV superspace action takes the form of a superspace classical master equation

$$(S,S) = 2\mathcal{s}S = 0.$$  \hfill (44)

As discussed previously, by populating the first slot of the antibracket with the superspace BV action $S$, we define a nilpotent superspace BRST differential operator $\mathcal{s}$ in the space of superfields and anti-superfields

$$\mathcal{s}F \equiv (S,F).$$  \hfill (45)

The action of $\mathcal{s}$ on the superfields and anti-superfields of the vector supermultiplet is:

$$\mathcal{s}V = D^2c + D^2\bar{c}, \quad \mathcal{s}V^* = D^2\bar{D}^2D\gamma V + D^2\rho + \bar{D}^2\bar{\rho},$$

$$\mathcal{s}\rho = 0, \quad \mathcal{s}\rho^* = \beta^* + D^2\bar{\gamma} + \beta\bar{\rho},$$

$$\mathcal{s}\bar{\rho} = 0, \quad \mathcal{s}\bar{\rho}^* = \bar{\beta}^* + D^2\gamma + \bar{\beta}\rho,$$

$$\mathcal{s}\beta = \rho, \quad \mathcal{s}\beta^* = -D^2\bar{D}^2c,$$

$$\mathcal{s}\bar{\beta} = \bar{\rho}, \quad \mathcal{s}\bar{\beta}^* = -\bar{D}^2\bar{D}^2\bar{c},$$

$$\mathcal{s}c = 0, \quad \mathcal{s}c^* = -D^2V^* + D^2\bar{D}^2c,$$

$$\mathcal{s}\bar{c} = 0, \quad \mathcal{s}\bar{c}^* = -D^2\bar{V}^* + D^2\bar{D}^{2}\bar{c}.$$  \hfill (46a-g)

As expected, the action of $\mathcal{s}$ on superfields is identical to their $\mathcal{s}$ transformations, while $\mathcal{s}$ on anti-superfields, it gives the equations of motion for the corresponding superfields. By splitting these equations of motion into two pieces, the piece coming from the anti-superfield terms of the action and the rest, we can decompose $\mathcal{s}$ into two nilpotent and anticommuting operators $\gamma$ and $\delta$: $\mathcal{s} = \gamma + \delta$. The $\gamma$ part is the superspace BRST symmetry operator, and $\delta$ is the superspace Koszul–Tate resolution differential:
Using the above, it is straightforward to verify that $\gamma^2 = 0$, $\delta^2 = 0$, $\{\gamma, \delta\} = 0$.

The last step is to use the $s$ transformations, in order to define a nilpotent BRST charge operator in a Hilbert space of states. In the previous section, we demonstrated explicitly how this can be done for the Maxwell theory. The result of this procedure can be summarized in four steps. First, we take, as an input, the list of operators that appear in the nilpotent differential operator defined by the antibracket. Second, if some of these operators carry free indices, then we introduce pairs of creation and annihilation oscillators, with appropriate statistics to dress them and absorb these free indices. Third, we calculate the algebra of all these dressed operators and apply the Fradkin and Fradkina algorithm \cite{44} for the construction of a corresponding Hilbert space, nilpotent BRST charge operator. Finally, we select an appropriate vacuum state, in order to define the cohomology of the BRST charge in this Hilbert space.

In this case, according to Equations (46a)–(46g), we must consider the following list of superspace operators

$$\{ \mathcal{D}^\gamma \mathcal{D}^2 \mathcal{D}_\gamma, \mathcal{D}^2, \mathcal{D}^3 \mathcal{D}^2, \mathcal{D}^3 \mathcal{D}^2 \mathcal{D} \}$$

None of these operators carry free indices, therefore we do not require the introduction of any additional oscillators besides the ghost oscillators of the Fradkin–Fradkina process. The algebra of these operators is as follows:

$$[\mathcal{D}^\gamma \mathcal{D}^2 \mathcal{D}_\gamma, \mathcal{D}^2] = 0, \quad [\mathcal{D}^\gamma \mathcal{D}^2 \mathcal{D}_\gamma, \mathcal{D}^3 \mathcal{D}^2] = 0, \quad [\mathcal{D}^\gamma \mathcal{D}^3 \mathcal{D}_\gamma, \mathcal{D}^2 \mathcal{D}^3] = 0, \quad [\mathcal{D}^\gamma \mathcal{D}^3 \mathcal{D}_\gamma, \mathcal{D}^2 \mathcal{D}^2 \mathcal{D}^2] = 0,$$

$$[\mathcal{D}^2, \mathcal{D}^3 \mathcal{D}^2] = \mathcal{D}^2 \mathcal{D}^2 - \mathcal{D}^2 \mathcal{D}^2, \quad [\mathcal{D}^2, \mathcal{D}^2 \mathcal{D}^2] = -\Box \mathcal{D}^2, \quad [\mathcal{D}^2, \mathcal{D}^3 \mathcal{D}^2] = \Box \mathcal{D}^2,$$

$$[\mathcal{D}^2, \mathcal{D}^3 \mathcal{D}^2 \mathcal{D}] = 0.$$

Notice that the d’Alembertian operator emerges in some of the commutators. In superspace, this is not an independent operator, because it can be expressed as a linear combination of other operators. We follow the conventions of Superspace:

$$\Box = \mathcal{D}^2 \mathcal{D}^2 + \mathcal{D}^2 \mathcal{D}^2 - \mathcal{D}^\gamma \mathcal{D}^2 \mathcal{D}_\gamma.$$

This makes the above algebra non-linear. The construction of the BRST operators of non-linear algebras has been found in \cite{49,50}. Using these results, we find that the most general BRST charge that we can write is:

$$Q = \eta \mathcal{D}^\gamma \mathcal{D}^2 \mathcal{D}_\gamma + \xi \mathcal{D}^2 + \xi \mathcal{D}^2 \mathcal{D}^2 + \xi \mathcal{D}^2 \mathcal{D}^2 \mathcal{D} + \kappa \Box - \xi \mathcal{D}^2 \mathcal{D} \mathcal{D}_\gamma + \xi \mathcal{D}^2 \mathcal{D}_\gamma \mathcal{D} \mathcal{D}.$$  

(51)
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where $\kappa = -\xi^2 \pi_\xi - \xi^2 \pi_\xi - \xi^2 \pi_\xi - \xi^2 \pi_\xi$ and $\eta, \xi, \xi$, $\xi, \xi$ are fermionic ghost oscillators with conjugate oscillators $\pi_\eta, \pi_\xi, \pi_\xi, \pi_\xi$. The $D^2 D^2$ terms in the above expression can be decomposed into an anticommutator and a commutator

$$D^2 D^2 = \frac{1}{2} \{D^2, D^2\} + \frac{1}{2} [D^2, D^2], \quad D^2 D^2 = \frac{1}{2} \{D^2, D^2\} - \frac{1}{2} [D^2, D^2]$$

and the BRST charge takes the form

$$Q = \eta D^\gamma D^2 D_\gamma + \xi D^2 + \xi D^2 + \rho [D^2, D^2] + 2(\xi \rho \pi_\xi - \xi \rho \pi_\xi) \Box - \xi \pi_\rho + \beta \{D^2, D^2\}$$

where $\beta = \frac{1}{2}(\xi + \xi)$, $\rho = \frac{1}{2}(\xi - \xi)$ and $\pi_\rho = \pi_\xi - \pi_\xi$ is $\rho$'s conjugate oscillator $\{\rho, \pi_\rho\} = 1$. Because of (50), the $\{D^2, D^2\}$ term is not an independent generator, and its effect is already captured by the $D^\gamma D^2 D_\gamma$ and $\Box$ terms. Therefore, we can ignore this term (choose $\beta = 0$) and focus on the cohomology of the following BRST charge

$$Q = \eta D^\gamma D^2 D_\gamma + \xi D^2 + \xi D^2 + \rho [D^2, D^2] + 2(\xi \rho \pi_\xi - \xi \rho \pi_\xi) \Box - \xi \pi_\rho$$

acting in the reduced Hilbert space generated out of the fermionic oscillators $\eta, \xi, \xi, \rho$ and their conjugates. In this Hilbert space, we select the vacuum $|0\rangle$ such that it is annihilated by the oscillators $\pi_\eta, \xi, \xi, \rho$

$$\pi_\eta|0\rangle = \xi|0\rangle = \xi|0\rangle = \rho|0\rangle = 0.$$ (55)

Thus, the most general state of this Hilbert space takes the form

$$|\Phi\rangle = \sum_{a,b,c,d} \eta^a \pi_\rho^b \pi_\xi^c \pi_\xi^d |\Phi_{a,b,c,d}\rangle$$

where $a,b,c,d = 0, 1, |\Phi_{a,b,c,d}\rangle = \Phi_{a,b,c,d}|0\rangle$ and $\Phi_{a,b,c,d}$ is a superfield coefficient. The Hilbert space ghost number of this state is:

$$gh(|\Phi\rangle) = a - b - c - d.$$ (57)

The zero ghost state (physical state) is

$$|\Psi\rangle = |V\rangle + \eta \pi_\rho |A\rangle + \eta \pi_\xi |B\rangle + \eta \pi_\xi |B\rangle$$

and we can also construct three gauge parameter states with ghost values $-1$ ($|\Lambda_{-1}\rangle$), $-2$ ($|\Lambda_{-2}\rangle$) and -3 ($|\Lambda_{-3}\rangle$):

$$|\Lambda_{-1}\rangle = \pi_\rho |\lambda\rangle + \pi_\xi |L\rangle + \pi_\xi |L\rangle + \eta \pi_\rho \pi_\xi |\omega_1\rangle + \eta \pi_\rho \pi_\xi |\omega_2\rangle + \eta \pi_\xi \pi_\xi |\omega_3\rangle,$$ (59a)

$$|\Lambda_{-2}\rangle = \pi_\rho \pi_\xi |w_1\rangle + \pi_\rho \pi_\xi |w_2\rangle + \pi_\xi \pi_\xi |w_3\rangle + \eta \pi_\rho \pi_\xi \pi_\xi |w_4\rangle,$$ (59b)

$$|\Lambda_{-3}\rangle = \pi_\rho \pi_\xi |\pi_\xi \xi\rangle.$$ (59c)

The transformation laws of the superfield coefficients at each ghost level are derived from the action of the BRST charge (54) on the previous level state. Hence, we obtain:

$$\delta Q |\Lambda_{-2}\rangle = Q |\Lambda_{-3}\rangle \Rightarrow \begin{cases} \delta |w_1\rangle = D^2 |z\rangle, \\ \delta |w_2\rangle = -D^2 |z\rangle, \\ \delta |w_3\rangle = [D^2, D^2] |z\rangle, \\ \delta |w_4\rangle = D^\gamma D^2 D_\gamma |z\rangle \end{cases}$$

(60)
\[ \delta \langle \Lambda_{-1} \rangle = Q \langle \Lambda_{-2} \rangle \Rightarrow \begin{cases} \delta \langle \lambda \rangle = -D^2|w_1\rangle - \bar{D}^2|w_2\rangle + |w_3\rangle, \\ \delta \langle L \rangle = -D^2|w_3\rangle + [D^2, \bar{D}^2]|w_1\rangle + 2\Box|w_1\rangle, \\ \delta \langle L \rangle = D^2|w_3\rangle + [D^2, \bar{D}^2]|w_2\rangle - 2\Box|w_2\rangle, \\ \delta \langle \omega_1 \rangle = D^2\bar{D}^2\bar{D}_7|w_1\rangle - \bar{D}^2|w_4\rangle, \\ \delta \langle \omega_2 \rangle = D^2\bar{D}^2\bar{D}_7|w_2\rangle + D^2|w_4\rangle, \\ \delta \langle \omega_3 \rangle = D^2\bar{D}^2\bar{D}_7|w_3\rangle - [D^2, \bar{D}^2]|w_2\rangle. \end{cases} \] (61)

\[ \delta \langle \Psi \rangle = Q \langle \Lambda_{-1} \rangle \Rightarrow \begin{cases} \delta \langle V \rangle = D^2|L\rangle + \bar{D}^2|\bar{L}\rangle + [D^2, \bar{D}^2]|\lambda\rangle, \\ \delta \langle A \rangle = D^2\bar{D}^2\bar{D}_7|\lambda\rangle + D^2|\omega_1\rangle + \bar{D}^2|\omega_2\rangle - |\omega_3\rangle, \\ \delta \langle B \rangle = D^2\bar{D}^2\bar{D}_7|L\rangle + \bar{D}^2|\omega_3\rangle - [D^2, \bar{D}^2]|\omega_1\rangle - 2\Box|\omega_1\rangle, \\ \delta \langle \bar{B} \rangle = D^2\bar{D}^2\bar{D}_7|\bar{L}\rangle - D^2|\omega_3\rangle - [D^2, \bar{D}^2]|\omega_2\rangle + 2\Box|\omega_2\rangle. \end{cases} \] (62)

The equation of motion for the physical state is:

\[ Q \langle \Psi \rangle = 0 \Rightarrow D^2\bar{D}^2\bar{D}_7|V\rangle - D^2|B\rangle - \bar{D}^2|\Gamma\rangle - [D^2, \bar{D}^2]|A\rangle = 0. \] (63)

Using the gauge freedom, the states \( |A\rangle \) and \( |\lambda\rangle \) can be eliminated, and we obtain

\[ D^2\bar{D}^2\bar{D}_7|V\rangle = |\Phi\rangle + |\bar{\Phi}\rangle \] (64)

where \( |\Phi\rangle \) is the chiral state \( \langle \Phi| = \bar{D}^2|\bar{B}\rangle, \bar{D}_7|\Phi\rangle = 0 \) and \( |\bar{\Phi}\rangle \) is the antichiral state \( \langle \bar{\Phi}| = D^2|B\rangle, D_7|\Phi\rangle = 0 \). The transformation of the states \( |V\rangle, \langle \Phi|, \langle \bar{\Phi}| \) are

\[ \delta |V\rangle = D^2|\bar{L}\rangle + \bar{D}^2|L\rangle, \delta |\Phi\rangle = 0, \delta |\bar{\Phi}\rangle = 0 \] (65)

and as an integrability condition to (64), the chiral and antichiral states satisfy their expected equations of motion

\[ D^2|\Phi\rangle = 0, B^2|\bar{\Phi}\rangle = 0. \] (66)

This includes a consistent sector of the theory, where the chiral and antichiral state vanish (\( |\Phi\rangle = 0 = |\bar{\Phi}\rangle \)). This sector correspond to the free, massless vector supermultiplet given by the state \( |V\rangle \), which satisfies the equation of motion

\[ D^2|V\rangle = 0, \delta |V\rangle = D^2|L\rangle + \bar{D}^2|\bar{L}\rangle. \] (67)

as expected from the superspace action (34).

4. Superspace BRST Description of Super Yang Mills

The full 4D, \( N = 1 \) SYM theory, which is the non-abelian extension of the vector multiplet, is known to be described by the superspace action

\[ S = -\frac{1}{4} Tr \int d^4x \, d^4\theta \, (e^{-V} D^\gamma e^V) \bar{D}^2(e^{-V} D_\gamma e^V) + h.c. \] (68)

where \( V = V^I T_I \) and \( T_I \) are the generators of an internal symmetry group. The action is invariant under the gauge transformation

\[ e^{\delta V} = e^{i\tilde{\Lambda}} e^{V - i\Lambda} \Rightarrow \delta V = -\frac{i}{2} L_V [\Lambda + \tilde{\Lambda} + \coth(\frac{1}{2} L_V) (\Lambda - \tilde{\Lambda})] \] (69)

\[ = D^2L + D^2L - \frac{1}{2} [V, D^2L - D^2L] + O(V^2) \]

where \( i\Lambda = D^2L \), \( i\Lambda = -D^2L \) and \( L = L^I T_I \) and \( \tilde{L} = L^I T_I \).
The procedure discussed in the previous section can also be applied to this theory. The superspace $\mathcal{s}$ BRST transformation is obtained by fermionizing the above gauge transformation
\begin{equation}
\mathcal{s}V = D^2c + \bar{D}^2\bar{c} - \frac{1}{2}[V, D^2c - \bar{D}^2\bar{c}] + \ldots \tag{70}
\end{equation}
where $c = e^T_1T_1$ and $\bar{c} = e^\dagger_1\bar{T}_1$. The nilpotence of $\mathcal{s}$ on $V$ fixes the action of $\mathcal{s}$ on the ghosts $c$ and $\bar{c}$:
\begin{equation}
\mathcal{s}^2V = 0 \implies \mathcal{s}c = -\frac{1}{2}\{c, D^2c\}, \quad \mathcal{s}\bar{c} = \frac{1}{2}\{\bar{c}, \bar{D}^2\bar{c}\} \tag{71}
\end{equation}
Using these transformations, the $\mathcal{s}$-exact deformation of the SYM Lagrangian is
\begin{equation}
\mathcal{s}\Omega = Tr \left\{ \rho(D^2V + \frac{2}{5}\bar{\rho}) + \bar{\rho}(D^2V + \frac{2}{5}\rho) 
- \beta D^2\left\{ -\frac{i}{2}L_V[i\bar{D}^2c - iD^2\bar{c} + \coth(\frac{i}{2}L_V)(i\bar{D}^2c + iD^2\bar{c})]\right\} \right\} - \bar{\beta} D^2\left\{ -\frac{i}{2}L_V[i\bar{D}^2c - iD^2\bar{c} + \coth(\frac{i}{2}L_V)(i\bar{D}^2c + iD^2\bar{c})]\right\} \tag{72}
\end{equation}
where $(\rho, \beta)$ and $(\bar{\rho}, \bar{\beta})$ are the two algebra valued BRST doublets: $\mathcal{s}\beta = \rho$, $\mathcal{s}\bar{\rho} = 0$, $\mathcal{s}\bar{\beta} = \bar{\rho}$, and $\mathcal{s}\beta = 0$.

By adding the corresponding anti-superfields, the superspace BV Lagrangian for SYM takes the form
\begin{equation}
\mathcal{L}_{BV} = Tr \left\{ -\frac{1}{4}(e^{-V}D^\gamma e^V)\bar{D}^2(e^{-V}D_\gamma e^V) - \frac{1}{4}(e^{-V}D^\gamma e^V)D^2(e^{-V}D_\gamma e^V) 
+ \rho(D^2V + \frac{2}{5}\bar{\rho}) + \bar{\rho}(D^2V + \frac{2}{5}\rho) 
- \beta D^2\left\{ -\frac{i}{2}L_V[i\bar{D}^2c - iD^2\bar{c} + \coth(\frac{i}{2}L_V)(i\bar{D}^2c + iD^2\bar{c})]\right\} \right\} - \bar{\beta} D^2\left\{ -\frac{i}{2}L_V[i\bar{D}^2c - iD^2\bar{c} + \coth(\frac{i}{2}L_V)(i\bar{D}^2c + iD^2\bar{c})]\right\} \tag{73}
\end{equation}
After expanding around the free theory of the previous section, and taking the trace, the BV Lagrangian becomes
\begin{equation}
\mathcal{L}_{BV} = \frac{1}{2}V^I D^I \bar{D}^2 V_I + \frac{1}{2}V^I D^I (D_\gamma V^K) f_{KAI} + \frac{1}{4} V^I \bar{D}^2 (D_\gamma V^K) f_{KAI} + \rho(D^2V + \frac{2}{5}\bar{\rho}) + \bar{\rho}(D^2V + \frac{2}{5}\rho) 
- \beta D^2\mathcal{D}^I \bar{D}^2 \mathcal{D}_I + \frac{1}{2} \beta^I \bar{D}^2 \left( V^K (D^2 c^A - \bar{D}^2 \bar{c}^A) \right) f_{KAI} 
- \bar{\beta} D^2 \bar{D}^2 \bar{c}_I + \frac{1}{2} \bar{\beta}^I \bar{D}^2 \left( V^K (D^2 c^A - \bar{D}^2 \bar{c}^A) \right) f_{KAI} 
+ V^{*I} D^2 \mathcal{D}^I + V^{*I} D^2 \mathcal{D}_I + \frac{1}{2} V^{*I} V^K (D^2 c^A - \bar{D}^2 \bar{c}^A) f_{KAI} 
- \frac{1}{2} e^{*I} e^K \bar{D}^2 c^A f_{KAI} + \frac{1}{2} e^{*I} e^K \bar{D}^2 \bar{c}^A f_{KAI} + \beta^{*I} \rho_I + \bar{\beta}^{*I} \bar{\rho}_I 
+ \ldots
\end{equation}
where $f_{IJ}^K$ are the structure constants of the internal Lie algebra $[T_I, T_J] = f_{IJ}^K T_K$. The algebra indices are lowered using the Cartan-Killing metric $g_{IJ} = Tr(T_I T_J)$. The constants $f_{IJK} = f_{IJ}^L g_{LK}$ have the properties $f_{IJK} = -f_{IKJ} = -f_{IJK}$.

Using the superspace antibracket, we find the action of $\mathcal{s}$ on the superfields and anti-superfields of SYM:
\begin{align}
\gamma V^I &= D^2c^I + D^2\xi^I - \frac{1}{2} V^K (D^2\xi^A - D^2c^A) f_{KAI}^I, \\
\gamma V^* I &= D^\gamma D^2 D^\gamma V I + \frac{1}{4} \left[ D^\gamma D^2 (D^\gamma V^K V^A) + D^\gamma D^2 (D^\gamma V^K D^\gamma V^A) - D^\gamma D^2 V^K D^\gamma V^A \right] f_{KAI} \\
&\quad + \frac{1}{4} \left[ D^\gamma D^2 (D^\gamma V^K V^A) + D^\gamma D^2 (D^\gamma D^\gamma V^K V^A) - D^\gamma D^2 D^\gamma V^K D^\gamma V^A \right] f_{KAI} + D^2 \rho I + D^2 \bar{\rho} I \\
\gamma \bar{\rho}^I &= 0, \quad \gamma \bar{\rho}^* I = \bar{\beta}^* I + D^2 V I + \xi \bar{\rho} I, \\
\gamma \bar{\beta}^I &= 0, \quad \gamma \bar{\beta}^* I = \bar{\beta}^* I + D^2 V I + \xi \bar{\rho} I, \\
\gamma \bar{\beta}^* I &= \bar{\beta}^* I = -D^2 D^2 c I + \frac{1}{2} D^2 \left( V^K (D^2 c^A - D^2 c^A) \right) f_{KAI}, \\
\gamma \bar{\beta} I &= \bar{\beta} I = -D^2 D^2 \xi I + \frac{1}{2} D^2 \left( V^K (D^2 c^A - D^2 c^A) \right) f_{KAI}, \\
\gamma \bar{c}^I &= -\frac{1}{2} \bar{c}^K D^2 c^A f_{KAI}^I, \\
\gamma \bar{c}^* I &= D^2 \bar{D}^2 \bar{\beta} I - D^2 V^* I - \frac{1}{2} D^2 \left( D^2 \bar{\beta}^K + D^2 \bar{\beta}^K - V^* K \right) V^A \right] f_{AIK} \\
&\quad + \frac{1}{2} \left( e^{*K} D^2 c^A + D^2 (e^{*K} c^A) \right) f_{IAK},
\end{align}

It is straightforward to decompose these transformations to their $\gamma$ and $\delta$ components as defined in (18) and (19). The action of the $\gamma$-BRST symmetry transformation on the SYM superfields and anti-superfields is:

\begin{align}
\gamma V^I &= D^2c^I + D^2\xi^I - \frac{1}{2} V^K (D^2\xi^A - D^2c^A) f_{KAI}^I, \\
\gamma V^* I &= -\frac{1}{2} V^* K (D^2\xi^A - D^2c^A) f_{IKA}, \\
\gamma \rho^I &= 0, \quad \gamma \rho^* I = \beta^* I, \\
\gamma \bar{\rho}^I &= 0, \quad \gamma \bar{\rho}^* I = \bar{\beta}^* I, \\
\gamma \beta^I &= \rho I, \quad \gamma \beta^* I = 0, \\
\gamma \bar{\beta}^I &= \bar{\rho} I, \quad \gamma \bar{\beta}^* I = 0, \\
\gamma \bar{c}^I &= -\frac{1}{2} c^K D^2 c^A f_{KAI}^I, \\
\gamma \bar{c}^* I &= -D^2 V^* I - \frac{1}{2} D^2 \left( V^* K V^A \right) f_{AIK} - \frac{1}{2} \left( e^{*K} D^2 c^A + D^2 (e^{*K} c^A) \right) f_{IAK}, \\
\gamma \bar{\rho}^I &= \frac{1}{2} e^K D^2 c^A f_{KAI}^I, \\
\gamma \bar{\beta}^I &= -\frac{1}{2} e^K D^2 c^A f_{KAI}^I, \\
\gamma \bar{\beta}^* I &= D^2 V^* I + \frac{1}{2} D^2 \left( V^* K V^A \right) f_{AIK} + \frac{1}{2} \left( e^{*K} D^2 c^A + D^2 (e^{*K} c^A) \right) f_{IAK}
\end{align}

and the action of the Koszul–Tate complex $\delta$ on the SYM anti-superfields is:
\[
\begin{align}
\delta V^* &= D^\gamma \bar{D}^2 D_\gamma V_I + \frac{1}{4} \left[ D^\gamma \bar{D}^2 (D_\gamma V^K V^\Lambda) + D^\gamma (\bar{D}^2 D_\gamma V^K V^\Lambda) - \bar{D}^2 D^\gamma V^K D_\gamma V^\Lambda \right] f_{KAI} \\
&+ \frac{1}{4} \left[ D^\gamma D^2 (D_\gamma V^K V^\Lambda) + D^\gamma (\bar{D}^2 D_\gamma V^K V^\Lambda) - \bar{D}^2 D^\gamma V^K D_\gamma V^\Lambda \right] f_{KAI} + D^2 \rho_I + \bar{D}^2 \bar{\rho}_I \\
\delta \rho^* &= D^2 V_I + \xi \bar{\rho}_I, \quad \delta \bar{\rho}^* = \bar{D}^2 V_I + \xi \rho_I, \\
\delta \beta^* &= -D^2 D^2 c_I + \frac{1}{2} D^2 (K(D^2 c^\Lambda - \bar{D}^2 \bar{c}^\Lambda)) f_{KAI}, \\
\delta \bar{\beta}^* &= -\bar{D}^2 D^2 \bar{c}_I + \frac{1}{2} \bar{D}^2 (K(D^2 c^\Lambda - \bar{D}^2 \bar{c}^\Lambda)) f_{KAI}, \\
\delta c^* &= D^2 D^2 \beta_I + \frac{1}{2} D^2 \left( (D^2 \beta^K + \bar{D}^2 \bar{\beta}^K) V^\Lambda \right) f_{IAK}, \\
\delta \bar{c}^* &= D^2 D^2 \bar{\beta}_I - \frac{1}{2} D^2 \left( (D^2 \beta^K + \bar{D}^2 \bar{\beta}^K) V^\Lambda \right) f_{IAK}
\end{align}
\]

Unlike the free theory example of Section 3, the Hilbert space description of SYM cannot be captured by a sole BRST charge \( Q \). The nonlinear terms require the introduction of a new kind of product “\(*\)” in the superfield state space. This product will assign to a pair of superfield states \((|A\rangle, |B\rangle)\) a new state \(|A\rangle \ast |B\rangle\). Hence, for interacting theories, the Lagrangian BRST symmetry operator \( s \) corresponds to the doublet \((Q, \ast)\) acting in an appropriately defined Hilbert space of superfield states \( s(|\Psi\rangle) = Q(|\Psi\rangle) + |\Psi\rangle \ast |\Psi\rangle\). This is similar to string field theory, where there is a \( \ast\)-product, and the physical spectrum of the theory is described by the equation \( Q\Psi + \Psi \ast \Psi = 0 \).

5. Summary

It has been demonstrated repeatedly that knowing the BRST description of a gauge theory is a very powerful tool. In this paper, we explore the BRST description of supersymmetric gauge theories, while maintaining supersymmetry manifest. For a given superspace formulated gauge theory, we follow the usual BRST procedure and construct a nilpotent BRST symmetry operator \( s \) by fermionizing the original gauge symmetry. For every gauge parameter superfield, we introduce a Fadeev–Popov ghost superfield, and two additional ghost superfields which play the role of a BRST doublet. The superspace action is deformed by an appropriate \( s\)-exact term which generates gauge fixing conditions and removes negative norm states from the physical spectrum of the theory. This deformation is further extended by the addition of anti-superfield terms, according to the BV procedure. Using a superspace anti-bracket, we define the nilpotent BRST-BV differential operator \( (s) \) in superspace, and identify its decomposition to the superspace BRST symmetry transformation \( (\gamma) \) and the superspace Koszul–Tate complex \( (\delta) \). We apply this procedure to \( 4D, \mathcal{N} = 1 \) super-Maxwell theory, and its non-abelian extension, super-Yang–Mills. For both theories, we derive explicit expressions for all these nilpotent operators in terms of the superspace covariant derivatives.

Moreover, for the linearized theory, we explore its BRST-BFV description in terms of a nilpotent BRST charge operator \( (Q) \) acting on the Hilbert space of superfield states. Superspace BRST charges have been constructed previously for maximally supersymmetric theories by introducing pure spinors. The nilpotence of these BRST charges is automatic due to the pure spinor constraint. In this paper, we follow another approach, and construct a Hilbert space BRST charge based on the algebra of the appropriate set of superspace differential operators. Specifically, we consider the set of linearly independent differential operators that appear in \( s \). Due to the algebraic properties of the supersymmetric covariant derivative, the algebra of these operators is nonlinear. Nevertheless, the Fradkin–Fradkina algorithm can be appropriately modified and applied, in order to construct a manifestly supersymmetric nilpotent BRST charge without requiring pure spinor variables. By choos-
ing an appropriate vacuum state, we show that its physical state cohomology generates the correct superspace equations. For the interacting theory, the physical spectrum of the theory cannot be reproduced solely in terms of a BRST charge, but an additional structure must be introduced, in order to capture the nonlinear terms that correspond to the gluing of superfield states in Hilbert space.

There are several interesting future directions that we want to investigate. First of all, this methodology must be applied to gauge theories with tensorial prepotential superfields, such as supergravity and higher spin supermultiplets, in order to explore the structure of the nilpotent operators for these theories. It is very interesting that for non-supersymmetric Maxwell theory, the physical state cohomology \((33), (32)\) is also valid for higher spin gauge theories, by simply allowing the expansion of the states in terms of bosonic oscillators that carry spacetime indices. This happens because the gauge invariance of higher spin theories (Bianchi identities) relies on exactly the same algebra as Maxwell theory. However, for supersymmetric higher spin theories, this is not true; hence, we must apply the same procedure to theories described by higher rank gauge superfields. Secondly, we must explore the properties of the additional product rule \((\ast)\) structure, between superfield states required by the BRST-BFV description of interacting gauge theories. String interactions in string field theory offer an example of such a structure.

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