MOMENT MAP AND MATRIX INTEGRALS

Gaussian separation of variables

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April 22, 2021

Abstract

We discuss the geometry behind some integrals related to structure constants of the Liouville conformal field theory.

§0. Introduction. Three incarnations of a quadratic map

This note is a followup of [BS]; it is mostly a review of known results. We discuss some geometry lying behind the computations from [ZZ] and [BR], and their p-adic and adelic analogs.

0.1. Moment ternoon. Let $K$ be a field of characteristic $\neq 2$. We will discuss certain quadratic map between two affine spaces

$$\mu : K^6 \rightarrow K^3$$

(0.1.1)

It may be introduced in three ways.

(a) As an exterior multiplication

$$\mu : K^3 \times K^3 \rightarrow \Lambda^2(K^3), \ (x,y) \mapsto x \wedge y.$$  

(0.1a)

(b) As a moment map. Regard $X(K) = K^6$ as the cotangent space to $Y(K) = K^3$; the group $H = SO_3(K)$ acts on $Y(K)$ in an obvious way; this action is Hamiltonian, and $\mu$ is the momentum map, $K^3$ on the right being identified with $\mathfrak{h}^* := \text{Lie}(H)^*$:

$$\mu : T^*Y \rightarrow \mathfrak{h}^*$$

(0.1b)
This is the archetypical moment map, wherefrom its very name has appeared. Its three components are "angular momenta".

(c) As a \textit{quotient map}. Identify $X(K)$ with the space of $2 \times 3$ matrices; the group $G = SL_2(K)$ acts upon $X(K)$ from the left, and $\mu$ may be identified (at least birationally) with the quotient map

$$\mu : X(K) \rightarrow G(K) \backslash X(K) = Y(K), \quad (0.1c)$$

$Y$ being identified with a categorical quotient of $X$, by the Igusa criterion, cf. [I].

0.2. In [BR] the map $\mu$ (for $K = \mathbb{R}$) has been used for a computation of certain triple integral $I(a, b, c; \mathbb{R})$, $a, b, c \in \mathbb{C}$ over $Y(\mathbb{R})$, see (1.1.7) below.

A similar integral for $K = \mathbb{C}$ has appeared previously in [ZZ] (cf. also [Z]). We can take $K$ to be a nonarchimedian local field; the integral $I(a, b, c; \mathbb{Q}_p)$ has been introduced and computed in [BS]. In \textit{op. cit.} a $q$-deformation of $I(a, b, c)$ is discussed as well.

The upshot of the trick from [BR] is that an integral $I(a, b, c; K)$ over $Y(K)$ is represented as a ratio of two Gaussian integrals over $X(K)$ and $Y(K)$.

From our viewpoint it might be considered as an integral over a fiber $X_t := \mu^{-1}(t)$, $t \in Y(K)$, and indeed, in the original definition in [ZZ] $I(a, b, c; \mathbb{C})$ has appeared as an integral over $G(\mathbb{C})$, see §1 below.

0.3. I am much obliged to M.Finkelberg for consultations; among others things he explained to me that 0.1 is a particular case (and a part) of a general superalgebra construction described e.g. in [BFT], see §4 below.

§1. Some geometry behind an integral

1.1. \textbf{Complex and real integrals.} The following integral appears in [ZZ] (4.17)

$$I(\sigma_1, \sigma_2, \sigma_3; \mathbb{C}) = \int_{\mathbb{C}^3} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} (1 + |z_i|^2)^{-2\sigma_i} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} (z_i - z_{i+1})^{-2-2\nu_{i+1}} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} |d^2z_i|, \quad (1.1.1)$$

where $|d^2z| = dx dy$, $z = x + iy$. Here

$$\nu_i := \sigma_i - \sigma_{i+1} - \sigma_{i+2}; \quad (1.1.2)$$

thus

$$-2\sigma_i = \nu_i + \nu_{i+1}. \quad (1.1.3)$$
Its value is
\[ I(\sigma_1, \sigma_2, \sigma_3; \mathbb{C}) = \pi^3 \Gamma\left(\sum_i \sigma_i - 1\right) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma\left(-\nu_i\right) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma\left(2\sigma_i\right) \] (1.1.4)

A real version of (1.1.1) looks as follows:
\[ I(\sigma_1, \sigma_2, \sigma_3; \mathbb{R}) = \int_{\mathbb{R}^3} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} (1 + x_i^2)^{-2\sigma_i} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} |x_i - x_{i+1}|^{-2(1+\nu_i)} dx_1 dx_2 dx_3, \] (1.1.5)

cf. [BS] (a), §6. After a change of variables
\[ x_i = \tan \alpha_i \]
it becomes
\[ I(\sigma_1, \sigma_2, \sigma_3; \mathbb{R}) = \frac{1}{8} \int_{[-\pi,\pi]^3} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} |\sin(\alpha_i - \alpha_{i+1})|^{-2(1+\nu_i)} d\alpha_1 d\alpha_2 d\alpha_3, \] (1.1.6)
in this form it appears in [BR], §5, (19). The computation from loc. cit. gives the following answer:
\[ I(\sigma_1, \sigma_2, \sigma_3; \mathbb{R}) = \pi^{3/2} \Gamma\left(\sum_i \sigma_i - 2\right) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma\left(-1/2 - \nu_i\right) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma\left(2\sigma_i - 1\right) \] (1.1.7)

1.2. Reduction to Gaussian integrals. The authors of [BR] have proposed an
elegant method for computing \( I(\sigma_1, \sigma_2, \sigma_3; \mathbb{R}) \).

Consider the real vector space \( V = \mathbb{R}^3 \) and the multiplication map
\[ \mu : V \times V \rightarrow \Lambda^2 V, \quad (x, y) \mapsto x \wedge y \] (1.2.1)
If we identify the 6-dimensional vector space \( V \times V \) with the space of \( 2 \times 3 \) matrices
and the 3-dimensional vector space \( \Lambda^2 V \) with \( \mathbb{R}^3 \) using a base \( \{e_i \wedge e_{i+1}\} \) then \( \mu \) will be a "moment" map
\[ \mu : X = M_{2,3} \rightarrow Y = \mathbb{R}^3 \] (1.2.3)
which assigns to a matrix \( A \) its three \( 2 \times 2 \) minors,
\[ \mu(A) = (|A_{12}|, |A_{23}|, |A_{31}|). \]

\textit{Structure of the formula}
Let us consider the formula (1.1.7). Denote

\[ A(\mathbb{R}) = \pi^{3/2} \Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 2), \]

\[ B(\mathbb{R}) = \prod_{i=1}^{3} \Gamma(-1/2 - \nu_i), \]

and

\[ C(\mathbb{R}) = \prod_{i=1}^{3} \Gamma(2\sigma_i - 1). \]

Then

\[ I(\mathbb{R}) = A(\mathbb{R}) \cdot \frac{B(\mathbb{R})}{C(\mathbb{R})}. \quad (1.2.4) \]

Bernstein and Reznikov interpret the numerator \( B(\mathbb{R}) \) (resp. the denominator \( C(\mathbb{R}) \)) as a Gaussian integral over \( X \) (resp. \( Y \)); the prefactor \( A(\mathbb{R}) \) appears due, roughly speaking, to the map \( \mu \).

A complex version of the above computation is described in [BS] (b).

1.3. Fibers of the moment map and \( SL_2 \). Let us try to understand the above computation "motivically". Consider various fibers

\[ X_t := \mu^{-1}(t) \subset X, \ t \in Y; \]

they are subvarieties of \( X = M_{2,3}(F) \) where \( F = \mathbb{C}, \mathbb{R}, \mathbb{Q}_p, \ldots \).

Among them there is a distinguished one

\[ Z = X_0 \subset X = M_{2,3}, \]

- the subvariety of matrices of rank 1, \( \dim Z = 4 \). For \( t \neq 0 \) \( \dim X_t = 3 \).

We may say that morally the "motive" \( [X_y] \) is a ratio

\[ [X_t] = \frac{[X]}{[Y]} \quad (1.3.1) \]

and ask if our integral \( I \) may be interpreted as an integral over \( X_t \).

1.3.1. We remark that the group \( G = SL_2(F) \) is acting upon \( X \) by left multiplication and respects the fibers \( X_t \), due to the equality of minors

\[ (gA)_{ij} = gA_{ij}, \ g \in G, \ A \in X. \]
If we pick a point \( x \in X_t \), the map 

\[ \nu_x : G \rightarrow X_t, \quad g \mapsto gx \]

is a birational isomorphism since both guys are 3-dimensional.

Therefore we may guess that probably our integral \( I \) may be interpreted as an integral over \( G \).

**1.3.2.** Note that there is also an action of another group \( H = SO(3) \) on \( X \), along the rows of a matrix.

We can identify \( X \) with \( T^*V \), and then \( \mu \) will be the moment map for this action, if we identify \( Y \) with \( \mathfrak{h} = \text{Lie}(H) \), see [A], Appendix 5.

The fibers \( X_t \subset X \) will be Lagrangian subvarieties.

**1.4.** The above guess turns out to be true. Consider the case \( F = \mathbb{C} \) treated in [ZZ]. Originally \( I(\mathbb{C}) \) is defined in [ZZ] as follows:

\[
I(\sigma_1, \sigma_2, \sigma_3; \mathbb{C}) = \int_{G(\mathbb{C})} \left( |b|^2 + |d|^2 \right)^{-2\sigma_1} \left( |a + b|^2 + |c + d|^2 \right)^{-2\sigma_2} \left( |a|^2 + |c|^2 \right)^{-2\sigma_3} d\mu_{G(\mathbb{C})},
\]

(1.4.1) where

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{C}), \]

and

\[ d\mu_G(g) = 4d^2 da \, db \, dc \, dd \]

is a Haar measure on \( G(\mathbb{C}) \), see op. cit. (4.14), (2.44).

The integral (1.4.1) takes a form (1.1.1) after a change of variables

\[ z_1 = b/d, \quad z_2 = (a + b)/(c + d), \quad z_3 = a/c, \]

(1.4.2) cf. op. cit. (4.15).

It comes out in turn from an integral

\[
J(z_1, z_2, z_3) := \pi^3 \int_{G(\mathbb{C})} \prod_{i=1}^{3} |g \cdot z_i|^{2\sigma_i} d\mu_G(g) =
\]
\[
\prod_{i \in \mathbb{Z}/3\mathbb{Z}} |z_i - z_{i+1}|^{2\nu_i+2} I(\sigma_1, \sigma_2, \sigma_3; \mathbb{C})
\]

(1.4.3)

where

\[|g \cdot z|^2 := |az + b|^2 + |cz + d|^2,\]

cf. op. cit. (4.13). This integral is a "quasiclassical limit" of a three-point correlation function in the Liouville CFT.

§2. \textit{p}-adic and adelic

2.1. \textit{p}-adic. Consider the \textit{p}-adic field \(K = \mathbb{Q}_p\); for \(x \in \mathbb{Q}_p\) we set

\[|x|_p := p^{-\nu_p(x)},\]

let \(d_p x\) denote the Haar measure on \(K\) normalised in such a way that

\[\int_{\mathbb{Z}_p} d_p x = 1.\]

Define a function \(\psi_p : K \rightarrow \mathbb{R}\) by

\[\psi_p(x) = \max\{|x|_p, 1\},\]

(2.1.1)

- it is a \textit{p}-adic analog of \(|z|^2 + 1, \ z \in \mathbb{C}\), cf. [BS], 2.5.

A \textit{p}-adic version of (1.1.1) is

\[I(\sigma_1, \sigma_2, \sigma_3; \mathbb{Q}_p) = \int_{\mathbb{Q}_p^3} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \psi_p(x)^{-2\sigma_i} |x_i - x_{i+1}|^{-1-\nu_i+2} d_p x_1 d_p x_2 d_p x_3,\]

(2.1.2)

cf. [BS] (a), 1.8. Its value has been computed in [B], [BS] (a), Theorem 1.9.

Define a function

\[\Gamma_p(s) := \frac{1}{1 - p^{-s}}, \ s \in \mathbb{C}\]

(2.1.3)

Then

\[I(\sigma_1, \sigma_2, \sigma_3; \mathbb{Q}_p) = \Gamma_p(2)^{-1} \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma_p(-\nu_i) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \Gamma_p(2\sigma_i).\]

(2.1.4)

Note that this result contains some regularisation behind the scene. Namely, the integral (2.1.2) is written as a sum of integrals which converge in different half-spaces of values for parameters \(\sigma_i\); each of these summands is calculated, and is in the obvious way
extended to a meromorphic function on \( \mathbb{C}^3 \). Our functions are similar to the "\( p \)-adic beta function" from [GGPS], Ch. II, 5.5.

**A matrix version**

The same integral is equal to a matrix one

\[
I(\sigma_1, \sigma_2, \sigma_3; \mathbb{Q}_p) = \int_{G(\mathbb{Q}_p)} (|b|_p^2 + |d|_p^2)^{-2\sigma_1} (|a + b|_p^2 + |c + d|_p^2)^{-2\sigma_2} (|a|_p^2 + |c|_p^2)^{-2\sigma_3} d\mu_{G(\mathbb{Q}_p)},
\]

**2.2. Balance.** In *op. cit.* a slightly different \( p \)-adic Gamma is used, namely

\[
\Gamma_p(s) := \frac{1}{1 - p^{-s}}.
\]

It appears naturally in calculation but goes from the final answer since in it we have four Gammas both in the numerator and in the denominator.

This circumstance is lucky for the Euler product (see below).

**2.3. Global: an Euler product.** Let \( P \) denote the set of all rational primes \( p \).

Consider a product

\[
I(\sigma_1, \sigma_2, \sigma_3)_{A} := \prod_p \Gamma_p(2)^{-1} \Gamma_p(\sum_i \sigma_i - 1) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \zeta(\nu_i) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \zeta(2\sigma_i)
\]

This product converges for \((\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3\) such that

\[
D : \Re(\sigma_i) > 1/2, \quad \sum_i \Re(\sigma_i) > 1, \quad \Re(\sigma_i) + \Re(\sigma_{i+1}) - \Re(\sigma_{i+2}) > 1
\]

This domain is nonempty: it contains a subset

\[
\Re(\sigma_1) = \Re(\sigma_2) = \Re(\sigma_3) > 1.
\]

The value obviously is

\[
I(\sigma_1, \sigma_2, \sigma_3)_{A} = \zeta(2)^{-1} \zeta(\sum_i \sigma_i - 1) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \zeta(-\nu_i) \prod_{i \in \mathbb{Z}/3\mathbb{Z}} \zeta(2\sigma_i)
\]
Let $A^f(\mathbb{Q})$ be the ring of finite adèles for $\mathbb{Q}$. It is tempting to conjecture that $I(\sigma_1, \sigma_2, \sigma_3)_A$ is equal to some integral over $A^f(\mathbb{Q})$.

The adelic integral similar to (1.4.1), (2.1.2) does not make sense: one needs some regularization. One of the possible ways would be a ratio of two Gaussian integrals.

§3. $q$-deformations

3.1. A $q$-deformation of the ZZ integral is proposed and calculated in [BR] (a), where it is formulated in the form of an "exotic" Macdonald constant term identity, certain generalization of this identity for the root system $A_2$. In this form it was discovered earlier by W.Morris, [M].

$q$-deformed Liouville triple correlators, in their gauge theory avatars, appear in the physical papers, cf. [CPT] and references therein.

It is not excluded that this identity may be proven by BR Gaussian trick as well, with Jackson integrals replacing the usual ones.

§4. Second moment and supersymmetry

4.1. Orthosymplectic Lie superalgebra. The action of the groups $G = SL_2$ and $H = SO(3)$ on the space $X = \text{Mat}_{2,3}$ commute with each other, in other words, the group

$$\mathfrak{g} = G \times H$$

is acting on $X$. The groups $G$ and $H$ look similar if one recalls that $SL_2 = Sp(2)$; both of them have type $B_1 = C_1 = A_1$.

Consider a vector superspace $V = V_0 \oplus V_1$, dim $V_0 = 3$, dim $V_1 = 2$ equipped with a nondegenerate bilinear form $B$ which is symmetric on $V_0$ and skew-symmetric on $V_1$, such that $V_0$ and $V_1$ are orthogonal with respect to $B$.

The Lie superalgebra $\mathfrak{osp}(3|2)$ is by definition a subalgebra of $\mathfrak{gl}(3|2)$ consisting of endomorphisms respecting $B$.

If $B$ is given by the matrix

$$B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}$$
then $\mathfrak{osp}(3|2)$ consists of matrices of the form

$$
\begin{pmatrix}
0 & -u & -v & x & x' \\
v & a & 0 & y & y' \\
u & 0 & -a & z & z' \\
-x' & -z' & -y' & d & e \\
x & z & y & f & -d
\end{pmatrix},
$$

cf. [Mu], 2.3.

Thus

$$
g_0 = \mathfrak{so}(3) \oplus \mathfrak{sp}(2) = \mathfrak{so}(3) \oplus \mathfrak{sl}_2,
$$

so

$$
dim g_0 = dim g_1 = 6.
$$

We have

$$
g_0 = \text{Lie}(\mathcal{G}),
$$

and on the other hand one can identify

$$
g_1 \cong X
$$

so that the action of $\mathcal{G}$ on $X$ will be the adjoint action.

4.2. The moment ternoon. We have a superbracket map

$$
\mu_{(a)} : g_1 \longrightarrow g_0 = \text{Lie}(G) \oplus \text{Lie}(H), \quad \mu_{(a)}(x) = \frac{1}{2}[x, x] \quad (4.2.1a)
$$

whose first component

$$
\mu_{(a1)} : g_1 \longrightarrow \text{Lie}(G)
$$

is the map (0.1a), at the same time we have a partner

$$
\mu_{(a2)} : g_1 \longrightarrow \text{Lie}(H).
$$

Next, we have the moment map

$$
\mu_{(b)} : g_1 \longrightarrow g_0^* = (\text{Lie}(G))^* \oplus \text{Lie}(H)^*, \quad (4.2.1b)
$$

whose first component

$$
\mu_{(b1)} : g_1 \longrightarrow (\text{Lie}(G))^*
$$

is the map (0.1b),
and finally we have the two quotient maps

\[ \mu_{(c_1)} : g_1 \rightarrow g_1/G \]  

which is the map (0.1c), and its partner

\[ \mu_{(c_2)} : g_1 \rightarrow g_1/H. \]

4.3. For a generalization to an arbitrary \( \mathfrak{osp}(m|2n) \) see e.g. [BFT], 2.1, 2.8.

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