Abstract. The versal deformation of Stanley-Reisner schemes associated to equivilar triangulations of the torus is studied. The deformation space is defined by binomials and there is a toric smoothing component which I describe in terms of cones and lattices. Connections to moduli of abelian surfaces are considered. The case of the Möbius torus is especially nice and leads to a projective Calabi-Yau 3-fold with Euler number 6.

Introduction

Versal deformation spaces in algebraic geometry tend to be either smooth, i.e. the object to deform is unobstructed, or much too complicated to compute. In general the equations defining formal versal base spaces are not polynomials. The purpose of this paper is to present an exception. In [AC10] we showed that triangulated surface manifolds with regular edge graph of degree 6, give Stanley-Reisner schemes with nicely presented formal versal deformations spaces defined by polynomials, in fact binomials. Such a surface is either a torus or a Klein bottle.

In this paper, the torus case is studied. Triangulations of (or more generally maps on) surfaces with regular edge graph are called equivilar. In the torus case they were only recently classified and counted by Brehm and Kühnel in [BK08]. The Stanley-Reisner schemes of such tori are all smoothable and they smooth to abelian surfaces.

To avoid non-algebraic abelian surfaces I will work with the functor Def_{X,L} where X is a scheme and L is an invertible sheaf. Since a projective Stanley-Reisner scheme X comes equipped with a very ample line bundle I define Def_{X}^{a} = Def_{(X,\mathcal{O}_{X}(1))} and it is the versal formal element of this functor I consider.

Let T be an equivler triangulated torus and X the Stanley-Reisner scheme. Since the equations defining Def_{X}^{a} are binomial, we may use the results and techniques of Eisenbud and Sturmfels in [ES96] to realize the smoothing components as toric varieties. The main computational part, Section 4, is about the cones and lattices that determine these toric varieties (or their normalizations). I then apply these results to statements about the deformations of X in Section 5.

There should be a connection between the results in this paper and moduli of polarized abelian surfaces. In Section 6 I describe a Heisenberg group H_{T} associated to T. There turns out to be a smooth 3 dimensional subspace \mathcal{M} \subset \text{Def}_{X}^{a} containing all isomorphism classes.
of smoothings of $X$. Moreover, the fibers are exactly the $H_T$ invariant deformations of $X$. There is a finite group acting on $\mathcal{M}$ inducing isomorphisms on the fibers and the quotient space $\mathcal{M}$ can be easily described in toric geometric terms.

To understand the connection to moduli one should extend the results in this paper to the non-polyhedral equivelar maps on the torus. These should correspond to moduli where the polarization class is not represented by a very ample line bundle. This is at the moment work in progress.

In principle one can write equations for abelian surfaces in $\mathbb{P}^{n-1}$ as perturbations of the Stanley Reisner ideal of $T$. I include some details about this ideal in Section 3. An application can be found in Section 7.

The last section deals in detail with the vertex minimal triangulation of the torus, sometimes called the M"obius torus, with 7 vertices. Here the toric geometry of $\text{Def}_X$ is extremely nice and leads to a Calabi-Yau 3-fold with Euler number 6. In this case it is also possible to find all the components of $\text{Def}_X$ and the generic non-smoothable fibers.

It is convenient to work over the ground field $\mathbb{C}$. I use the notation $\text{Def}_X$ for both the functor and the versal base space. Throughout this paper $G^* := \text{Hom}_\mathbb{Z}(G, \mathbb{C}^*)$ is the character group of $G$.

1. Preliminaries

1.1. Equivelar triangulations of the torus. I start by defining the main combinatorial object in this paper. For details and proofs see [BK08]. A map on a surface is called equivelar if there are numbers $p$ and $q$ such that every vertex is $q$-valent and every facet contains exactly $p$ vertices. On a torus we can only have $(p, q)$ equal $(6, 3), (3, 6)$ or $(4, 4)$.

We will consider triangulated tori, i.e. the case $p = 3, q = 6$. In this paper we need honest triangulations and assume the map is polyhedral. This means that the intersection of two triangles is a common face (i.e. empty, vertex or edge).

Every triangulated torus (also the non-polyhedral) are obtained as a quotient of the regular tessellation of the plane by equilateral triangles ($\{3, 6\}$). Denote this tessellation by $\{3, 6\}$. We will need to make this explicit and will refer to the following as the standard description.

We may describe $\{3, 6\}$ as an explicit triangulation of $\mathbb{R}^2$, i.e. we always assume a chosen origin 0 and coordinates $(x, y)$. We may assume the vertices of $\{3, 6\}$ form the rank 2 lattice spanned by $(1, 0)$ and $\frac{1}{2}(1, \sqrt{3})$. Denote this lattice by $T$. We may think of $T$ as the translation subgroup of $\text{Aut}(\{3, 6\})$. Now let $\Gamma \subseteq T$ be a sublattice of finite index and set $T = \{3, 6\}/\Gamma$. Then $T$ is a (not necessarily polyhedral) equivelar triangulated torus. Such a triangulation is called chiral if the $\mathbb{Z}_6$ rotation on $\{3, 6\}$ descends to $T$. Chiral maps on the torus were studied and classified in [Cox49].

We may assume that $\Gamma$ is generated by $a(1, 0)$ and $b(1, 0) + c\frac{1}{2}(1, \sqrt{3})$ for integers $a, b, c$ with $ac \neq 0$, i.e. it is the image of

$$
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}
$$

in the above basis for $T$. In [BK08] Proposition 2 it is shown that two such matrices, $M_1$ and $M_2$, represent isomorphic triangulated tori if and only if $M_2 = PM_1Q$, with $Q \in \text{GL}_2(\mathbb{Z})$ and $P$ in the $D_6$ subgroup generated by the rotation $\rho$ and reflection $\sigma$,

$$
\rho = \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}, \quad \sigma = \begin{pmatrix}
-1 & -1 \\
0 & 1
\end{pmatrix}.
$$
1.2. Deformations of Stanley-Reisner schemes. I refer to [AC10] and the references there for definitions, details and proofs about deformations of Stanley-Reisner schemes. As a general reference for deformation theory see [Ser06].

Let \([n]\) be the set \(\{0, \ldots, n-1\}\) and let \(\Delta_{n-1} := 2^{[n]}\) be the full simplex. Let \(P = \mathbb{C}[x_0, \ldots, x_{n-1}]\) be the polynomial ring in \(n\) variables. If \(a = \{i_1, \ldots, i_k\} \in \Delta_{n-1}\), we write \(x_a \in P\) for the square free monomial \(x_{i_1} \cdots x_{i_k}\). A simplicial complex \(\mathcal{K} \subset \Delta_{n-1}\) gives rise to an ideal

\[I_{\mathcal{K}} := \langle x_p \mid p \in \Delta_{n-1} \setminus \mathcal{K} \rangle \subseteq P.\]

The Stanley-Reisner ring is then \(A_{\mathcal{K}} = P/I_{\mathcal{K}}\). We refer to [Sta96] for more on Stanley-Reisner rings. The corresponding projective Stanley-Reisner scheme is \(\mathbb{P}(\mathcal{K}) = \text{Proj} A_{\mathcal{K}} \subseteq \mathbb{P}^{n-1}\). Note that \(\mathbb{P}(\mathcal{K})\) comes with a very ample line bundle \(\mathcal{O}_{\mathbb{P}(\mathcal{K})}(1)\).

The scheme \(\mathbb{P}(\mathcal{K})\) looks like the geometric realization of \(\mathcal{K}\). It is a union of irreducible components \(X_F = \mathbb{P}^{\dim F}, F\) a facet of \(\mathcal{K}\), intersecting as in \(\mathcal{K}\). There is also a natural open affine cover described in terms of Stanley-Reisner rings. Recall that the link of a face is

\[\text{lk}(f, \mathcal{K}) = \{g \in \mathcal{K} : g \cap f = \emptyset \text{ and } g \cup f \in \mathcal{K}\}.\]

If \(f \in 2^{[n]}\), let \(D_+(x_f) \subseteq \mathbb{P}(\mathcal{K})\) be the chart corresponding to homogeneous localization of \(A_{\mathcal{K}}\) by the powers of \(x_f\). Then \(D_+(x_f)\) is empty unless \(f \in \mathcal{K}\) and if \(f \in \mathcal{K}\) then

\[D_+(x_f) = \mathbb{A}(\text{lk}(f, \mathcal{K})) \times (\mathbb{C}^*)^{\dim f}\]

where \(\mathbb{A}(\mathcal{K})\) denotes \(\text{Spec} A_{\mathcal{K}}\).

The cohomology of the structure sheaf is given by \(H^p(\mathbb{P}(\mathcal{K}), \mathcal{O}_{\mathbb{P}(\mathcal{K})}) \simeq H^p(K; \mathbb{C})\) (Hochster, see [AC10 Theorem 2.2]). If \(\mathcal{K}\) is an orientable combinatorial manifold then the canonical sheaf is trivial ([BE91a Theorem 6.1]). Thus a smoothing of such a \(\mathbb{P}(\mathcal{K})\) would yield smooth schemes with trivial canonical bundle and structure sheaf cohomology equaling \(H^p(K; \mathbb{C})\). In particular if \(\mathcal{K}\) comes from a triangulation of a 2 dimensional torus then a smoothing of \(\mathbb{P}(\mathcal{K})\) is an abelian surface.

In the surface case deformations of Stanley-Reisner schemes may include non-algebraic schemes. It is therefore convenient to work with the functor \(\text{Def}(X,L)\) where \(X\) is a scheme.
and $L$ is an invertible sheaf. (See [Ser06, 3.3.3] and [AC10, 3].) We defined $\text{Def}^a_{\mathbb{P}(K)}$ as $\text{Def}_{\mathbb{P}(K)}(\mathcal{O}_{\mathbb{P}(K)}(1))$. If $K$ is a combinatorial manifold without boundary then

$$\text{Def}^a_{\mathbb{P}(K)}(\mathbb{C}[e]) \simeq H^0(\mathbb{P}(K), T^1_{\mathbb{P}(K)}) \simeq T^1_{\lambda_{K,0}}$$

and $H^0(\mathbb{P}(K), T^2_{\mathbb{P}(K)})$ contains all obstructions for $\text{Def}^a_{\mathbb{P}(K)}$ ([AC10, Theorem 6.1]). For certain surfaces the versal base space for $\text{Def}^a_{\mathbb{P}(K)}$ may be computed and as we shall see this is particularly nice for equivariant triangulated tori.

It follows from the results in [AC10] that if $K$ is a combinatorial manifold without boundary and all vertices have valency greater than or equal 5, then $T^1_{\lambda_{K,0}}$ is the $\mathbb{C}$ vector space on the edges of $K$. Since $K$ is a manifold, the link of an edge must be two vertices, i.e. $\text{lk}(\{p,q\}) = \{\{i\}, \{j\}\}$. If $\varphi_{p,q} \in T^1_{\lambda_{K,0}}$ is the basis element corresponding to $\{p,q\}$ and $x_m$ is in the Stanley-Reisner ideal, then

$$\varphi_{p,q}(x_m) = \frac{x_m x_p x_q}{x_i x_j} \quad \text{if } \{i, j\} \subseteq m$$

$$\text{otherwise.}$$

There is a natural $(n-1)$-dimensional torus action on $\text{Proj}(A_K)$ where $[\lambda_0, \ldots, \lambda_{n-1}] \in (\mathbb{C}^*)^n/\mathbb{C}^*$ takes $x_i \in A_K$ to $\lambda_i x_i$. Note that the induced action on a $\varphi_{p,q} \in T^1_{\lambda_{K,0}}$ as above is

$$\varphi_{p,q} \mapsto \frac{\lambda_p \lambda_q}{\lambda_i \lambda_j} \varphi_{p,q}.$$ 

If $t_{p,q}$ is the corresponding coordinate function on the versal base space then the action is the contragredient, i.e. $t_{p,q} \mapsto (\lambda_i \lambda_j / \lambda_p \lambda_q) t_{p,q}$.

1.3. Binomial ideals. In [ES96] Eisenbud and Sturmfels prove, among other things, that every binomial ideal has a primary decomposition all of whose primary components are binomial. I review here some of the results and notions from that paper. (See also [DMM10].)

If $w = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, write $t^w = \prod t_i^{a_i}$ for a monomial in $P = k[t_1, \ldots, t_n]$, $k$ for the time being is any algebraically closed field. A binomial is a polynomial with at most two terms, $at^v - bt^w$ with $a, b \in k$. A binomial ideal is an ideal of $P$ generated by binomials.

For an integer vector $v$, let $v_+$ and $v_-$, both with non-negative coordinates, be the positive and negative part of $v$, i.e. $v = v_+ - v_-$. In general define, for a sublattice $L \subseteq \mathbb{Z}^n$ the lattice ideal of $L$ by

$$I_L = \langle t^{v_+} - t^{v_-} : v \in L \rangle \subseteq k[t_1, \ldots, t_n].$$

More generally for any character $\rho \in \text{Hom}_\mathbb{Z}(L, \mathbb{C}^*)$, define

$$I_{L,\rho} = \langle t^{v_+} - \rho(v)t^{v_-} : v \in L \rangle.$$ 

If $\rho'$ is an extension of $\rho$ to $\mathbb{Z}^n$, then the automorphism $t_i \mapsto \rho'(\varepsilon_i) t_i$ induces an isomorphism $I_L \simeq I_{L,\rho'}$.

Define the saturation of $L$ in $\mathbb{Z}^n$ as the lattice

$$\text{Sat} L = \{v \in \mathbb{Z}^n : dv \in L \text{ for some } d \in \mathbb{Z}\}.$$ 

Note $\text{Sat} L/L$ is finite. The lattice $L$ is saturated in $\mathbb{Z}^n$ if $\text{Sat} L = L$. The lattice ideal is a prime ideal if and only if $L$ is saturated ([ES96, Theorem 2.1]). In fact [ES96, Corollary 2.3] states that

$$I_L = \bigcap_{\rho \in (\text{Sat} L/L)^*} I_{\text{Sat} L,\rho}$$

is a minimal primary decomposition.
Let $I$ be a binomial ideal. Let $\mathcal{Z} \subseteq \{1, \ldots, n\}$ and let $p_\mathcal{Z} = \langle t_i : i \notin \mathcal{Z} \rangle$ and $\mathbb{Z}^\mathcal{Z} \subset \mathbb{Z}^n$ the sublattice spanned by the standard basis elements $e_i, i \in \mathcal{Z}$. The result we need from [ES96] is the following. (It is not stated in the following form in that paper and in fact much stronger results are proven there.)

**Theorem 1.1.** If characteristic $k$ is 0, the associated primes of the binomial ideal $I$ are all of the form $I_{\text{Sat}} \mathcal{L}_{\mathcal{Z}, \rho} + p_\mathcal{Z}$ for some sublattice $\mathcal{L}_{\mathcal{Z}} \subseteq \mathbb{Z}^\mathcal{Z}$.

For simplicity let us assume $I$ is generated by pure binomials of the form $t^v - t^w$. Define the *exponent vector* of $t^v - t^w$ to be $v - w \in \mathbb{Z}^m$. Let $L \subset \mathbb{Z}^m$ be the sublattice spanned by the exponent vectors of the generators of $I$. It follows from the above and [ES96] Theorem 6.1 that the $I_{\text{Sat}} \mathcal{L}_{\mathcal{L}, \rho}$ will be minimal prime ideals for $I$. It follows from the theorem that all other associated primes must contain some variable $t_i$.

### 1.4. Gorenstein, reflexive and Cayley cones

I recall some notions originally introduced in [BB97] in connection with mirror symmetry. A general reference is [BN08]. If $M \simeq \mathbb{Z}^n$ then set as usual $N = \text{Hom}(M, \mathbb{Z})$. A rational finite polyhedral cone $\sigma \subseteq M_\mathbb{R} = M \otimes \mathbb{R}$ is called *Gorenstein* if there exits $n_\sigma \in N$ with $\langle v, n_\sigma \rangle = 1$ for all primitive generators $v \in M$ of rays of $\sigma$. This means that the affine toric variety $X_\sigma$ is Gorenstein.

The cone $\sigma$ is called *reflexive* if the dual cone $\sigma^\vee$ is also Gorenstein. Let $m_{\sigma^\vee} \in M$ be the determining lattice point. The number $r = \langle m_{\sigma^\vee}, n_\sigma \rangle$ is the *index* of the reflexive cone $\sigma$.

A polytope in $M_\mathbb{R} = M \otimes \mathbb{R}$ is called a lattice polytope if its set of vertices is in $M$. Let $\Delta_1, \ldots, \Delta_r \subseteq L_\mathbb{R}$ be lattice polytopes in a rank $d$ lattice $L$. Let $M = L \oplus \mathbb{Z}^r$, where $\{e_1, \ldots, e_r\}$ is the standard basis for $\mathbb{Z}^r$. The cone

$$\sigma = \{(\lambda_1, \ldots, \lambda_r, \lambda_1 x_1 + \cdots + \lambda_r x_r) \in M_\mathbb{R} : \lambda_i \in \mathbb{R}_{\geq 0}, x_i \in \Delta_i, i = 1, \ldots, r\}$$

is called the *Cayley cone* associated to $\Delta_1, \ldots, \Delta_r$. It is a Gorenstein cone with $n_\sigma = \epsilon_1^* + \cdots + \epsilon_r^*$. A reflexive Gorenstein cone of index $r$ is *completely split* if it is the Cayley cone associated to $r$ lattice polytopes.

### 1.5. Moduli and Heisenberg groups

There is a large amount of literature on moduli of polarized abelian varieties starting with [Mum66]. I mention here only some articles where the surface case is studied in detail: [HS94], [GP98], [GP01], [MS01], [MR03] and [Mar04]. In particular the last three are about the $(1, 7)$ case and I will comment on them in Section 7.

Heisenberg groups are an important ingredient in the construction of these moduli spaces. The following construction is based on [Mum66]. Since we are dealing with surfaces I describe only the 2 dimensional case.

Let $\delta = (d_1, d_2)$ be a list of elementary divisors, i.e. $d_i$ are positive integers and $d_1 | d_2$. Set $K(\delta) = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2}$ with character group $K(\delta)^* = \mu_{d_1} \times \mu_{d_2}$. (Here $\mu_d = \mathbb{Z}_d^*$ are the $d'$th roots of unity.) Define the abstract finite *Heisenberg group* $H_\delta$ as the extension

$$1 \rightarrow \mu_{d_2} \rightarrow H_\delta \rightarrow K(\delta) \oplus K(\delta)^* \rightarrow 0$$

where multiplication in $\mu_{d_2} \oplus K(\delta) \oplus K(\delta)^*$ is defined by

$$(\omega, \tau, \sigma) \cdot (\omega', \tau', \sigma') = (\omega \cdot \omega', \sigma' \cdot \tau, \sigma \cdot \sigma')$$

If $n = d_1 d_2 = |K(\delta)|$ then $H_\delta$ has a unique $n$-dimensional irreducible representation $V(\delta)$ in which the center $\mu_{d_2} \subset \mathbb{C}^*$ acts by its natural character ([Mum66] Proposition 3)). One may realize $V(\delta)$ as the vector space of $\mathbb{C}$ valued functions $f$ on $K(\delta)$. Then the action is defined by

$$((\omega, \tau, \sigma) \cdot f)(\tau') = \omega \cdot \sigma(\tau') \cdot f(\tau + \tau').$$
The representation \( V(\delta) \) is known as the \textit{Schrödinger representation} of the Heisenberg group.

2. Overview

Let \( T \) be an equivelar triangulation of the torus with \( n \) vertices and \( X \subset \mathbb{P}^{n-1} \) the projective Stanley-Reisner scheme associated to \( T \). Let \( \Gamma \) be the sublattice of \( \mathbb{T} \) defining \( T \); i.e. \( T = \{3, 6\}/\Gamma \) and \( G = \mathbb{T}/\Gamma \).

There are three for us important elements of \( T \) and I will call them and their images in \( G \) the \textit{principal translations}. In the standard description (see Section [1.1]) they are

\[
\tau_1 = (1, 0) \quad \tau_2 = \frac{1}{2}(-1, \sqrt{3}) \quad \tau_3 = \frac{1}{2}(-1, -\sqrt{3}).
\]

There is the relation \( \tau_1 + \tau_2 + \tau_3 = 0 \). In \( \mathbb{T} \) and therefore also in \( G \), any pair of them generate the group.

The following proposition is our central observation and allows us a natural identification of the elements of \( G \) and \( \text{vert}_T \), a fact we will use throughout. Recall that given a group \( G \) with a finite set of generators \( S \) one may construct a directed graph, called the \textit{Cayley graph}, where the vertices are the elements of \( G \). The edges are all ordered pairs \( (g, sg) \) for some \( s \in S \). If one assigns a color to each element of \( S \) then the graph is colored by giving the edge \( (g, sg) \) the color of \( s \). I will refer to the underlying undirected graph as a Cayley graph as well.

**Proposition 2.1.** The edge graph of \( T \) is the Cayley graph of \( G \) with respect to the principal translations. In particular the action of \( G \) on \( \text{vert}_T \) is simply transitive and the set of edges of \( T \) is partitioned by the three \( G \) orbits of cardinality \( n \):

\[
\{ \{p, \tau_k(p)\} : p \in \text{vert}_T \}
\]

for \( k = 1, 2, 3 \).

**Proof.** The edge graph of \( \{3, 6\} \) is clearly the Cayley graph of \( \mathbb{T} \) with respect to the principal translations. \( \square \)

An edge in \( T \) is of \textit{type} \( k \) if it is of the form \( \{p, \tau_k(p)\} \). This is the natural coloring of the Cayley graph. The link of \( p \) in \( T \) will be the cycle

\[
(\tau_1(p), -\tau_3(p), \tau_2(p), -\tau_1(p), \tau_3(p), -\tau_2(p)).
\]

This shows that \( \tau_i \neq \pm \tau_j \) if \( T \) is polyhedral.

**Notation.** When describing the interaction between the principal translations it will be useful to have the following convention for the indices. If \( k \) is an element in \( \{1, 2, 3\} \) then I will use the indices \( i, j \) to represent the remaining two elements of \( \{1, 2, 3\} \setminus \{k\} \). I will refer to this as the \( ijk \)-convention.

Recall from Section [1.2] that the tangent space of \( \text{Def}_X^a \) has basis \( \varphi_{p,q}, \{p,q\} \in T \). In our new notation the corresponding perturbation is

\[
\varphi_{p,\tau_k(p)}(x_m) = \begin{cases} x_m x_p x_{\tau_k(p)} & \text{if } \{-\tau_i(p), -\tau_j(p)\} \subseteq m \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( t_{p,q} \) be the dual basis of coordinate functions on \( \mathbb{C}^{3n} \).
For each \( p \in \text{vert} T \) construct the matrix
\[
(2.1) \quad \begin{bmatrix}
    t_{p,\tau_1(p)} & t_{p,\tau_2(p)} & t_{p,\tau_3(p)} \\
    t_{p,-\tau_1(p)} & t_{p,-\tau_2(p)} & t_{p,-\tau_3(p)}
\end{bmatrix}
\]
and take the \( 2 \times 2 \) minors. This yields \( 3n \) quadratic binomials. Let \( I \) to be the ideal generated by them.

**Theorem 2.2** \((\text{AC} \ 10 \  \text{Theorem 6.10})\). The ideal \( I \) defines a versal base space in \((\mathbb{C}^{3n},0)\) for \( \text{Def}^a_X \).

I will misuse notation and also refer to this space as \( \text{Def}^a_X \).

We will employ the results on binomial ideals reviewed in Section 1.3. Let \( L \subset \mathbb{Z}^{3n} \) be the sublattice spanned by the exponent vectors of the generators of \( I \). Index the standard basis of \( \mathbb{Z}^{3n} \), \( \varepsilon_{p,q} \), by the edges of \( T \). The lattice \( L \) is spanned by the \( 3n \) vectors
\[
\begin{align*}
    f_{3,p} &= \varepsilon_{p,\tau_1(p)} + \varepsilon_{p,-\tau_2(p)} - \varepsilon_{p,\tau_3(p)} - \varepsilon_{p,-\tau_1(p)} \\
    f_{2,p} &= \varepsilon_{p,\tau_3(p)} + \varepsilon_{p,-\tau_1(p)} - \varepsilon_{p,\tau_1(p)} - \varepsilon_{p,-\tau_3(p)} \\
    f_{1,p} &= \varepsilon_{p,\tau_2(p)} + \varepsilon_{p,-\tau_3(p)} - \varepsilon_{p,\tau_3(p)} - \varepsilon_{p,-\tau_2(p)} .
\end{align*}
\]
There are relations \( f_{1,p} + f_{2,p} + f_{3,p} = 0 \) for each \( p \) and \( \sum_{p \in \text{vert} T} f_{k,p} = 0 \) for each \( k \). One checks that indeed these are the generating relations and therefore \( \text{rank} L = 2(n-1) \).

The \( \text{I}_{\text{Sat} L,\rho} \) for \( \rho \in (\text{Sat} L/L)^* \) will be minimal prime ideals for \( I \). This means that each \( \rho \in (\text{Sat} L/L)^* \) determines a component of the versal base space. I denote these by \( S_\rho \) and call them the main components of \( \text{Def}^a_X \).

Set \( S \) to be the component for the trivial \( \rho \), that is \( S \) is defined by the toric lattice ideal of \( \text{Sat} L \). In the recent literature it is become normal to include non-normal varieties in the term toric varieties. I will also do this, thus \( S \) is the germ of an affine toric variety. It is in general not normal. Since \( \text{rank} \text{Sat} L = 2(n-1) \), \( \dim S = n + 2 \).

There are isomorphisms \( \rho' : S \simeq S_\rho \) as described in Section 1.3. By their construction, the \( \rho' \) are automorphisms of the polynomial ring restricting to the identity on \( I_L \) and therefore also on \( I \). Thus \( S \simeq S_\rho \) comes from an automorphism of \( \text{Def}^a_X \). This implies that the families over the two components are also isomorphic. So from the point of view of deformations it is enough to study \( S \).

As a toric variety the normalization of \( S \), call it \( \widetilde{S} \), may be described by a rank \( n+2 \) lattice \( M \) and a cone \( \sigma^\vee \subset M_\mathbb{R} \). To find \( M \) and \( \sigma^\vee \) we need to find an integral \( m \times 3n \) matrix \( A \) (for some \( m \)) such that \( \ker A = \text{Sat} L \). Then set \( M = \text{im} A \) and set \( \sigma^\vee \) to be the positive hull of the columns of \( A \) in \( M_\mathbb{R} \). (See e.g. \cite[10]{PT}.) Note it is enough to find \( A \) with rank \( A = n + 2 \) and \( L \subset \ker A \), since then \( \text{rank} L = \text{rank} \ker A \) and \( \ker A \) is obviously saturated, so \( \ker A = \text{Sat} L \). Let \( S \) be the subsemigroup of \( \mathbb{Z}^{m_0}_+ \) generated by the columns of \( A \), thus \( S \) and \( \widetilde{S} \) are the germs at 0 of \( \text{Spec} \mathbb{C}[S] \) and \( \mathbb{C}[\sigma^\vee \cap M] \).

There are two obvious torus actions on \( \text{Def}^a_X \) and the weights of these actions will give us \( A \). First consider the natural action described in Section 1.2 induced by automorphisms of \( X \). Let \( w_{pq} \in \{(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n : \sum a_i = 0\} \) be the weights of this torus action on the basis \( \varphi_{pq} \). With the \( ijk \)-convention
\[
w_{p,\tau_k}(p) = e_p + e_{\tau_k(p)} - e_{-\tau_i(p)} - e_{-\tau_j(p)}
\]
for \( p \in \text{vert} T, k = 1, 2, 3 \), where \( e_p \) are the standard basis for \( \mathbb{Z}^n \).
The coloring of the Cayley graph and the structure of $I$ give us another torus action, not seen on $X$. Clearly the minors of
\[
\begin{bmatrix}
\lambda_1 t_{p,\tau_3}(p) & \lambda_2 t_{p,\tau_3}(p) & \lambda_3 t_{p,\tau_3}(p) \\
\lambda_1 t_{p,-\tau_3}(p) & \lambda_2 t_{p,-\tau_3}(p) & \lambda_3 t_{p,-\tau_3}(p)
\end{bmatrix}
\]
also generate $I$. Thus the $\mathbb{C}^*^3$ action, $(\lambda_1, \lambda_2, \lambda_3) \cdot t_{p,\tau_3}(p) = \lambda_k t_{p,\tau_3}(p)$ preserves $I$.

This leads to the following definition. Write the standard basis for $\mathbb{Z}^3 \oplus \mathbb{Z}^n$ as $\epsilon_1, \epsilon_2, \epsilon_3$ and $e_p, e \in \text{vert } T$. Let $A$ be the $(n + 3) \times 3n$ matrix with columns
\[
(2.3) \quad A_{p,\tau_3}(p) = \epsilon_k + e_p + e_{\tau_3}(p) - e_{-\tau_3}(p)
\]
for $p \in \text{vert } T, k = 1, 2, 3$. By construction $I$ is homogeneous with respect to the multigrading with degree $t_{p,\tau_3}(p) = A_{p,\tau_3}(p)$. It follows that $L \subseteq \ker A$.

For each $p \in \text{vert } T$, the corresponding row in $A$ has a nice description. In columns indexed by the 6 edges having $p$ as vertex, there is a $+1$. In the 6 columns corresponding to edges in $\text{lk}(p)$ we have $-1$. The other entries are 0. Using this one checks that $\text{rank } A = n + 2$.

Let $M = \text{im } A$ and set $M'$ to be the lattice $\mathbb{Z}^3 \oplus \{(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n : \sum a_i = 0\}$, the target of $A$. For any lattice $M$ let $T_M = M^*$ be the corresponding torus. Consider the finite character group $(M'/M)^*$. By standard toric variety theory, see e.g. [Ful93 2.2], $T_M = T_{M'/(M'/M)^*}$ and
\[
\text{Spec } \mathbb{C}[M \cap \sigma^\vee] = \text{Spec } \mathbb{C}[M' \cap \sigma^\vee](M'/M)^*.
\]
Thus the normalizations of the main components will all be isomorphic to
\[
\tilde{S} = (\text{Spec } \mathbb{C}[M \cap \sigma^\vee], 0) = (\text{Spec } \mathbb{C}[M' \cap \sigma^\vee](M'/M)^*, 0).
\]
Our goal is to describe these combinatorial objects to get an as explicit as possible description of the main components.

3. THE STANLEY-REISNER IDEAL OF $T$

In this section let $I = I_T$ be the Stanley-Reisner ideal of the equivelar triangulated torus. The $f$-vector of $T$ is $(n, 3n, 2n)$, so one may compute the Hilbert polynomial of $A_T$, following [Sta96], as
\[
h_{A_T}(z) = n \begin{pmatrix} z - 1 \\ 0 \end{pmatrix} + 3n \begin{pmatrix} z - 1 \\ 1 \end{pmatrix} + 2n \begin{pmatrix} z - 1 \\ 2 \end{pmatrix} = nz^2.
\]
This agrees with the Hilbert function except in degree 0. In particular one computes that
\[
\dim I_2 = \binom{n+1}{2} - 4n = \frac{1}{2}n(n - 7).
\]
The minimum number of cubic generators on the other hand will depend upon the combinatorics of $T$.

To compute the number of cubic generators consider first for every edge $\{p, q\}$ the number
\[
l_{p,q} = |\text{vert } (\text{lk}(\{p\}, T) \cap \text{lk}(\{q\}, T)) \setminus \text{vert } \text{lk}(\{p, q\}, T)|
\]
which can be 0, 1, 2 or 3. By symmetry $l_{p,q}$ will depend only on the type of $\{p, q\}$, so let $k$, $k = 1, 2, 3$, be this common value.

**Lemma 3.1.** The minimum number of cubic generators of $I_T$ is \( \frac{1}{3}n(l_1 + l_2 + l_3) \).
Proof. A cubic monomial generator in $I_T$ corresponds to a set $\{p, q, r\}$ of vertices which is a non-face, but for which every subset is an edge. This means exactly that $\{p, q\} \in T$ and $\{r\} \in \text{lk}(\{p\}) \cap \text{lk}(\{q\}) \setminus \text{lk}(\{p, q\})$. In the sum $\sum_{\{p, q\} \in T} l_{p, q}$ we have counted a given such $\{p, q, r\}$ 3 times.

The following lemma follows from a simple check.

Lemma 3.2. With the $ijk$-convention, $l_k$ is non-zero if and only if $\tau_k = 2\tau_i$ or $\tau_k = 2\tau_j$ or $3\tau_k = 0$.

Proposition 3.3. The ideal $I_T$ is generated by quadratic and cubic monomials. The minimum number of quadratic generators is $\frac{1}{2}n(n - 7)$.

Up to isomorphism, the $T$ which need cubic generators for $I_T$ have one of the following standard presentations:

$\begin{pmatrix} n & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{n} \end{pmatrix}$, $\begin{pmatrix} 3 & 1 \\ 0 & \frac{n}{3} \end{pmatrix}$, $\begin{pmatrix} 3 & 2 \\ 0 & \frac{2}{n} \end{pmatrix}$.

If $T$ is presented by $\begin{pmatrix} n & 2 \\ 0 & 1 \end{pmatrix}$, then the minimum

$$\# \text{cubic generators} = \begin{cases} 21 & \text{if } n = 7 \\ 16 & \text{if } n = 8 \\ n & \text{if } n \geq 9. \end{cases}$$

In the three other cases the minimum

$$\# \text{cubic generators} = \begin{cases} 9 & \text{if } n = 9 \\ \frac{1}{2}n & \text{if } n \geq 10. \end{cases}$$

Proof. The only Stanley-Reisner ideal of a 2-dimensional combinatorial manifold that needs quartic generators is the boundary of the tetrahedron.

By Lemma 3.1 and Lemma 3.2, there will no cubic generators unless some $\tau_k = 2\tau_i$ or some $3\tau_k = 0$. In the first case, after a $D_6$ movement (see Section 1.1) we may assume $\tau_3 = 2\tau_1$. In the standard presentation this means that $(2, 1) \in \Gamma$. One checks using e.g. [BK08, Proposition 3] that, for each $n$, there is only one isomorphism class with this property and that it is represented by the first matrix in the list. In the second case we may assume $3\tau_1 = 0$ and again check possibilities.

To get the number just count the $l_k$ in each case and use Lemma 3.1. \qed

4. Analysis

4.1. The group $G$ and its principal translations. Before proceeding with our analysis of the components I state some facts about $G = \mathbb{T}/\Gamma$ and the $\tau_k$. The proof of the first lemma is an exercise in elementary abelian group theory.

Lemma 4.1. There are the following relationships involving the principal translations.

(i) The quotient group $G/\langle \tau_k \rangle$ is cyclic and the classes of $\tau_i$ and $\tau_j$ are both generators.
In particular $|\tau_i||\tau_j|/n$ is an integer for all $i \neq j$ in $\{1, 2, 3\}$.

(ii) Let $[g]_i \subseteq G$ be the coset of $\langle \tau_i \rangle$ containing $g$. For any $g, h \in G$,
$$|[g]_i \cap [h]_j| = \frac{|\tau_i||\tau_j|}{n}.$$
(iii) The number
\[ \gcd \left( \frac{n}{|\tau_i|}, \frac{n}{|\tau_j|} \right) = \frac{n}{\text{lcm}(|\tau_i|, |\tau_j|)} \]
is the same for all \( i \neq j \).

I will vary between two presentations of \( G \). First there is the standard presentation which is the presentation in Section 1.1. Here \( G \) is a quotient of \( \mathbb{T} \cong \mathbb{Z}^2 \) with basis \( \tau_1, -\tau_3 \) by the image of
\[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}.
\]

Then we may take the symmetric presentation where we think of \( G \) as the quotient of \( \mathbb{Z}^3 \) with basis \( \tau_1, \tau_2, \tau_3 \) by the image of a matrix
\[
R = \begin{pmatrix}
1 & \alpha_{1,1} & \alpha_{2,1} \\
1 & \alpha_{1,2} & \alpha_{2,2} \\
1 & \alpha_{1,3} & \alpha_{2,3}
\end{pmatrix}.
\]

Of course the standard presentation is the symmetric with
\[
R = \begin{pmatrix}
1 & a & b \\
1 & 0 & 0 \\
1 & 0 & -c
\end{pmatrix}.
\]

**Proposition 4.2.** With the symmetric presentation and the \( ijk \)-convention the orders of the principal translations are
\[
|\tau_k| = \frac{n}{\gcd(\alpha_{1,i} - \alpha_{1,j}, \alpha_{2,i} - \alpha_{2,j})}.
\]

**Proof.** The determinant of \( R \) is \( n \). Computing it three different ways one sees that \( \gcd(\alpha_{1,i} - \alpha_{1,j}, \alpha_{2,i} - \alpha_{2,j}) | n \) for all \( i \neq j \). Now \( |\tau_k| \) is the least positive \( m \) with \( m \varepsilon_k \in \text{im} R \). Using Cramers rule, this is the least positive \( m \) with \( m(\alpha_{1,i} - \alpha_{1,j}) \equiv m(\alpha_{2,i} - \alpha_{2,j}) \equiv 0 \pmod{n} \). The result now follows. \( \square \)

Here is the abstract structure of \( G \).

**Proposition 4.3.** If
\[
d = \gcd \left( \frac{n}{|\tau_i|}, \frac{n}{|\tau_j|} \right)
\]
for \( i \neq j \), then the elementary divisors on \( G \) are \( (d, n/d) \). In particular
\[
G \cong \mathbb{Z}_d \times \mathbb{Z}_{n/d}
\]
and \( G \) is cyclic if and only if \( d = 1 \).

**Proof.** These invariants may be computed from the standard presentation. We have \( n = ac \), \( n/|\tau_1| = c \), \( n/|\tau_2| = \gcd(a, b + c) \) and \( n/|\tau_3| = \gcd(a, b) \) (see Proposition 4.2). Thus the \( d \) in the statement equals \( \gcd(a, b, c) \) as it should. \( \square \)
4.2. Sat $L$ and the number of main components. Let $B$ be the $3n \times 3n$ matrix with columns the exponent vectors $[2.2]$ of $I$. As explained in Section 2 Sat $L = \ker A$. Thus there is a commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^{3n} & \longrightarrow & \text{Coker } B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Sat } L & \longrightarrow & \mathbb{Z}^{3n} & \overset{A}{\longrightarrow} & M & \longrightarrow & 0 \\
\end{array}
$$

and the Snake Lemma yields an exact sequence

$$0 \rightarrow \text{Sat } L/L \rightarrow \text{Coker } B \overset{A}{\rightarrow} M \rightarrow 0. $$

Let $d$ be as in Proposition 4.3.

Proposition 4.4. There is an isomorphism $\text{Coker } B \simeq F \oplus \mathbb{Z}_d$ where $F$ is free of rank $n + 2$. In particular

$$\text{Sat } L/L \simeq \mathbb{Z}_d. $$

Proof. We see from the $f_{k,p}$ described in (2.2) that $(\text{Coker } B)^*$ is the set of $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^{*n} \times \mathbb{C}^{*n} \times \mathbb{C}^{*n}$ satisfying

$$\frac{\lambda_{1,p}}{\lambda_{1,\tau_1(p)}} = \frac{\lambda_{2,p}}{\lambda_{2,\tau_2(p)}} = \frac{\lambda_{3,p}}{\lambda_{3,\tau_3(p)}} \quad \text{for all } p \in \text{vert } T. $$

(4.1)

Let $\pi : (\text{Coker } B)^* \rightarrow \mathbb{C}^{*n}$ be the projection on the third factor. Clearly $\ker \pi$ equals

$$\{(\lambda_1, \lambda_2) : \lambda_{1,p} = \lambda_{1,\tau_1(p)}, \lambda_{2,p} = \lambda_{2,\tau_2(p)}, \forall p \in \text{vert } T\} = (\mathbb{C}^*)^{n_{\tau_1}} \times (\mathbb{C}^*)^{n_{\tau_2}}. $$

To prove the statement I will now show that $\text{im } \pi \simeq (\mathbb{C}^*)^{n+2} \bigcap \mathbb{C}^{*n} \times (\mathbb{Z}_d)^*$. Let $\lambda_3 \in \text{im } \pi$. Choose some $p$ and let $O_p^1$ be the $\tau_1$ orbit of $p$. From (4.1) we get

$$\prod_{q \in O_p^1} \frac{\lambda_{3,q}}{\lambda_{3,\tau_3(q)}} = \prod_{p \in O_p^1} \frac{\lambda_{1,q}}{\lambda_{1,\tau_1(q)}} = 1. $$

On the other hand, given $\lambda_3$ satisfying this relation, choose an arbitrary value for $\lambda_{1,p}$ and set

$$\lambda_{1,\tau_1(p)} = \lambda_{1,p} \prod_{k=0}^{r-1} \frac{\lambda_{3,(\tau_3+k\tau_1)(p)}}{\lambda_{3,k\tau_1(p)}}$$

to solve (4.1). The same is of course true for $\tau_2$ orbits.

Thus $\text{im } \pi$ is the set of $\lambda \in \mathbb{C}^{*n}$ with

$$\prod_{p \in O} \frac{\lambda_p}{\lambda_{\tau_3(p)}} = 1 \quad \text{for all } \tau_1 \text{ and } \tau_2 \text{ orbits } O. $$

(4.2)

If $P_i$ are the orbit partitions of vert $T$ by $\tau_i$, then $\tau_3$ acts transitively on $P_1$ and $P_2$ (Lemma 4.1). Thus condition (4.2) translates to

$$\prod_{p \in O} \lambda_p = \prod_{q \in O'} \lambda_q \quad \text{for all } O, O' \in P_1 \text{ and all } O, O' \in P_2. $$

For $O \in P_i$ let this common value be $\mu_i = \prod_{p \in O} \lambda_p, i = 1, 2$. There is a homomorphism $\phi : \text{im } \pi \rightarrow \mathbb{C}^{*2}$, $\lambda \mapsto (\mu_1, \mu_2)$. Clearly $\ker \phi$ is the set of $\lambda \in \mathbb{C}^{*n}$ with $\prod_{p \in O} \lambda_p = 1$ for all $O \in P_1$ and all $O \in P_2$. 
Lemma 4.8. Choose an origin where \((\lambda \text{ for } p)\) after these choices, a solution for (4.3) with \(\tau\) and \(q\).

For Definition 4.6. Define these equations for \(\tau\) by Lemma 4.1. Thus \(\ker \phi \simeq (\mathbb{C}^*)^{n+1-|\tau_1|-|\tau_2|}\).

There is one relation between the \(\mu_i\) namely

\[
\prod_{p \in \text{vert } T} \lambda_p = \mu_1^{n_{|\tau_1|}} = \mu_2^{n_{|\tau_2|}}.
\]

But \(d = \gcd(n/|\tau_1|, n/|\tau_2|)\) so \(\im \phi \simeq \mathbb{C}^* \times (\mathbb{Z}_d)^*\). Adding this up gives the result. \(\square\)

In terms of the structure of \(\text{Def}^n_X\) one has

**Corollary 4.5.** If \(d\) is the first elementary divisor of \(G\) then the number of main components in \(\text{Def}^n_X\) is \(d\).

4.3. The group \((M'/M)^*\). I will compute \((M'/M)^*\). This is of interest in itself, but will also be important for out study of \(S\) in Section 6.

From the weight matrix \(A (2.3)\) we see that \((M'/M)^* \subset \mathbb{C}^{*3} \times (\mathbb{C}^{*n}/\mathbb{C}^*)\) is defined by

\[
(4.3) \quad \mu_k \frac{\lambda_p \lambda_{\tau_i(p)}}{\lambda_{-\tau_i(p)} \lambda_{-\tau_j(p)}} = 1
\]

where \((\mu_1, \mu_2, \mu_3) \in \mathbb{C}^{*3}\) and \(\lambda \in \mathbb{C}^{*n}/\mathbb{C}^*\) has coordinates indexed by \(\text{vert } T\). I will first solve these equations for \(\tau_k \in \mathbb{T}\) and \(p \in \text{vert } \{3, 6\}\) and then see what happens in the quotient. For this purpose we need the following quadratic parabolic function.

**Definition 4.6.** Define \(q : \mathbb{Z}^2 \to \mathbb{Z}\) by

\[
q(x, y) = \frac{1}{2}(x^2 + y^2 - 2xy - x - y) = \frac{1}{2}((x-y)^2 - (x+y)).
\]

**Lemma 4.7.** For \(q\) there are the equalities

\[
q(mx, my) = mq(x, y) + \frac{1}{2} m(m - 1) (x - y)^2
\]

and

\[
q(x + z, y + w) = q(x, y) + q(z, w) + (x - y)(z - w).
\]

**Lemma 4.8.** Choose an origin \(0\) in \(\text{vert } \{3, 6\}\) and \(\lambda_0, \lambda_k, \mu_k \in \mathbb{C}^*, k = 1, 2, 3\) satisfying

\[
(4.4) \quad \lambda_0^3 = \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3}.
\]

After these choices, a solution for (4.3) with \(\tau_k \in \mathbb{T}\) and \(p \in \text{vert } \{3, 6\}\) is unique and given for \(p = \alpha \tau_1(0) + \beta \tau_2(0) + \gamma \tau_3(0)\) by

\[
(4.5) \quad \lambda_p = \lambda_0^{1-\alpha-\beta-\gamma} \lambda_1^{\alpha} \lambda_2^{\beta} \lambda_3^{\gamma} \mu_1^{q(\alpha, \gamma)} \mu_2^{q(\alpha, \beta)} \mu_3^{q(\alpha, \beta)}.
\]
Proof. We first check that the (4.5) is well defined. Assume \( p = \alpha \tau_1(0) + \beta \tau_2(0) + \gamma \tau_3(0) = a \tau_1(0) + b \tau_2(0) + c \tau_3(0) \), then \( \alpha - a = \beta - b = \gamma - c = \delta \) for some \( \delta \). Now \( q(x + \delta, y + \delta) = q(x, y) - \delta \) so the right hand side in (4.5) is changed by multiplication with

\[
\left( \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0^{i} \mu_1 \mu_2 \mu_3} \right)^{\delta}
\]

which equals 1 by condition (4.4).

For the uniqueness consider a vertex p and its link with vertices \( q_1, \ldots, q_6 \). One checks directly that if the \( \mu_k, \lambda_p \) and 3 of the \( \lambda_q \) are given, then the relations (4.3) determine the other 3 \( \lambda_q \). Now beginning in the origin and working outward we see that all \( \lambda_p \) are determined by \( \lambda_0 \) and \( \lambda_{q_k(0)}, k = 1, 2, 3 \).

Finally we must check that these \( \lambda_p \) actually are solutions. We do this only for \( k = 1 \). After plugging (4.5) into (4.3) and some obvious cancellations we arrive at

\[
\frac{\lambda_1 \lambda_2 \lambda_3 \mu_2 q(\alpha + 1, \gamma) \mu_3^{q(\alpha + 1, \beta)}}{\lambda_0^{i} \mu_1 \mu_2 \mu_3^{q(\beta - 1, \gamma)}} = \frac{\lambda_3^{q(\beta - 1, \gamma)} \mu_1^{q(\beta, \gamma - 1)} \mu_2^{q(\alpha, \gamma - 1)} \mu_3^{q(\alpha, \beta - 1)}}{\lambda_0^{i} \mu_1 \mu_2 \mu_3^{q(\alpha + 1, \gamma)}}.
\]

Now \( q(x + 1, y) = q(x, y) + x - y \) and \( q(x - 1, y) = q(x, y) + y - x + 1 \), so this reduces further to

\[
\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_0^{i} \mu_1 \mu_2 \mu_3} = 1.
\]

\( \square \)

Now take the quotient by \( \Gamma \). With the notation of the symmetric presentation let \( \alpha_{t,1} \tau_1 + \alpha_{t,2} \tau_2 + \alpha_{t,3} \tau_3, t = 1, 2 \), be generators of \( \Gamma \) and set \( r_t = \alpha_{t,1} \tau_1(0) + \alpha_{t,2} \tau_2(0) + \alpha_{t,3} \tau_3(0) \).

Lemma 4.9. The character group \( (M'/M)^{\ast} \subset \mathbb{C}^{*3} \times (\mathbb{C}^{*n} / \mathbb{C}^{*}) \) consists of the solutions (4.5) under the condition (4.4) with

\[
\lambda_0 = \lambda_{r_1} = \lambda_{r_2} = 1
\]

and

\[
\mu_1^{A(\alpha_{t,2} - \alpha_{t,3})} \mu_2^{B(\alpha_{t,3} - \alpha_{t,1})} \mu_3^{C(\alpha_{t,1} - \alpha_{t,2})} = 1
\]

for \( t = 1, 2 \) and all integers \( A, B, C \) with \( A + B + C = 0 \).

Proof. Setting \( \lambda_0 = 1 \) corresponds to the second factor being \( \mathbb{C}^{*n} / \mathbb{C}^{*} \). The first condition is clearly necessary.

For the solution to be valid modulo \( \Gamma \) we must have \( \lambda_{m_1 r_1 + m_2 r_2 + p} = \lambda_p \) for all integers \( m_i \) and \( p \in \text{vert}(3, 6) \). Consider first

\[
\lambda_{m_1 r_1 + m_2 r_2} = \lambda_{r_1}^{m_1} \lambda_{r_2}^{m_2}
\]

\[
\cdot \left( \frac{\mu_1^{(\alpha_{t,2} - \alpha_{t,3})} \mu_2^{(\alpha_{t,3} - \alpha_{t,1})} \mu_3^{(\alpha_{t,1} - \alpha_{t,2})}}{\mu_1^{(\alpha_{t,2} - \alpha_{t,3})} \mu_2^{(\alpha_{t,3} - \alpha_{t,1})} \mu_3^{(\alpha_{t,1} - \alpha_{t,2})}} \right)^{m_1 m_2}
\]

\[
\cdot \prod_{t=1}^{2} \left( \frac{\mu_1^{(\alpha_{t,2} - \alpha_{t,3})} \mu_2^{(\alpha_{t,3} - \alpha_{t,1})} \mu_3^{(\alpha_{t,1} - \alpha_{t,2})}}{\mu_1^{(\alpha_{t,2} - \alpha_{t,3})} \mu_2^{(\alpha_{t,3} - \alpha_{t,1})} \mu_3^{(\alpha_{t,1} - \alpha_{t,2})}} \right)^{\frac{1}{2} m_t (m_t - 1)}
\]

by Lemma 4.7 The two conditions in the statement imply that this expression equals 1.

Now, in general, if \( r = \alpha \tau_1(0) + \beta \tau_2(0) + \gamma \tau_3(0) \) and \( p = a \tau_1(0) + b \tau_2(0) + c \tau_3(0) \) then

\[
(4.6) \quad \lambda_{r+p} = \lambda_r \lambda_p \mu_1^{(\beta - \gamma)(b - c)} \mu_2^{(\gamma - \alpha)(c - a)} \mu_3^{(a - \beta)(a - b)}
\]

by Lemma 4.7
Proposition 4.13. It is a Cayley cone by \cite[Proposition 2.3]{BN08}.

Proof. Thus is a column of $A$ and it defines the 1 dimensional face spanned by the column $\sigma_k$. In particular $|M'/M| = n^2d$.

Proof. Consider the projection on the first factor of $\mathbb{C}^* \times \mathbb{C}^{n-1}$ restricted to $(M'/M)^*$. I claim the kernel is $G^*$. Indeed, if we set $\mu_i = 1$ in the conditions of Lemma \ref{lem:kernel} we are left with

$$\lambda_1 \lambda_2 \lambda_3 = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} = 1.$$ 

Thus to prove the statement we must show that the image of the projection is the character group of $\mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_{n/d}$.

Consider the relations among the $\mu_i$ described in Lemma \ref{lem:kernel}. There are 4 generating relations corresponding to $(A,B,C) = (1,-1,0)$ and $(0,-1,1)$. We may use the standard presentation of $G$ to compute them. They are

$$\mu_2^a = \mu_2^a \mu_3^a = \mu_1 \mu_2^{b+c} = \mu_2^{b+c} \mu_3^b = 1.$$ 

One may compute the gcd of minors and find that the elementary divisors of

$$\begin{pmatrix} 0 & 0 & c & 0 \\ a & 0 & b+c & b+c \\ 0 & a & 0 & b \end{pmatrix}$$

are $(d, d, n/d)$.

4.4. The cone $\sigma^\vee$. Let $N' \subseteq N$ be the dual lattices of $M \subseteq M'$ and $\sigma$ the dual cone of $\sigma^\vee$. Recall that $\sigma^\vee$ is the positive hull of the columns of $A$ in $M_R$ and that the columns of $A$ are

$$A_{p,\tau_k(p)} = \epsilon_k + \epsilon_p + \epsilon_{\tau_k(p)} - \epsilon_{-\tau_i(p)} - \epsilon_{-\tau_j(p)}$$

for $k = 1, 2, 3$ and $p \in \text{vert} T$. We will need the easily checked lemma.

Lemma 4.11. If $i \not= j$ and $O$ is a $\tau_i$ orbit in $\text{vert} T$, then $\sum_{q \in O} A_{q,\tau_j(q)} = |\tau_i|\epsilon_j$.

The matrix $A$ has the nice property that the columns generate the rays of $\sigma^\vee$.

Lemma 4.12. Each column of $A$ is a primitive generator in $M$ for a ray of $\sigma^\vee$, thus $\sigma^\vee$ has $3n$ rays.

Proof. For each edge $\{p, \tau_k(p)\}$ of $T$ let $u_{p,\tau_k(p)} \in N'$ be $2(\epsilon_1 + \epsilon_2 + \epsilon_3) - (\epsilon_p^* + \epsilon_{\tau_k(p)}^*)$. If $A_{q,\tau_i(q)}$ is a column of $A$, then $0 \leq \langle A_{q,\tau_i(q)}, u_{p,\tau_k(p)} \rangle \leq 4$ and equals 0 if and only if $p = q$ and $k = l$. Thus $u_{p,\tau_k(p)} \in \sigma$ and it defines the 1 dimensional face spanned by the column $A_{p,\tau_k(p)}$.

Proposition 4.13. The cone $\sigma^\vee \subseteq M_R$ is a Gorenstein cone. It is a Cayley cone associated to 3 lattice polytopes.

Proof. If $n_{\sigma^\vee} = \epsilon_1^* + \epsilon_2^* + \epsilon_3^*$, then clearly $\langle n_{\sigma^\vee}, A_{q,\tau_i(q)} \rangle = 1$ so $\sigma^\vee$ is Gorenstein by Lemma \ref{lem:Gorenstein} and it is a Cayley cone by \cite[Proposition 2.3]{BN08}. 


Remark. It seems a difficult but interesting combinatorial problem to determine the type of polytopes these three are. They vary with the combinatorics of $T$. For example they are in general not $n-1$ dimensional though their Minkowski sum is. If they are $n-1$ dimensional then they must be simplices since they have $n$ vertices. This is the case when the corresponding $\tau_i$ has order $n$.

In fact $\sigma^\vee$ has a finer Cayley structure. By a Cayley structure on $\sigma^\vee$ I mean some set of lattice polytopes $\Delta_1, \ldots, \Delta_r$ such that is $\sigma^\vee$ is the Cayley cone associated to $\Delta_1, \ldots, \Delta_r$. First partition each of the sets of type $k$ columns $\{A_{p, \tau_k}(p) : p \in \text{vert } T\}$ into its $\tau_k$ orbits. This partitions the set of all $3n$ columns into $r$ cells where

$$r = \frac{n}{|\tau_1|} + \frac{n}{|\tau_2|} + \frac{n}{|\tau_3|}.$$ 

Index these cells $o_1, \ldots, o_r$ and view $\mathbb{Z}^r$ as the free abelian group on the $o_i$. Let $\beta_i$ be the standard basis element of $\mathbb{R}^r$ corresponding to $o_i$. The orbit $o_i$ is of type $k$ if it is a $\tau_k$ orbit of type $k$ columns.

Now define the vectors

$$(4.8) \quad m_p = \sum_{k=1}^{3} (e_{\tau_k}(p) - e_{-\tau_k}(p)), \quad p \in \text{vert } T.$$ 

Since

$$A_{-\tau_k}(p,p) = e_k + e_p + e_{-\tau_k}(p) - e_{-(\tau_i - \tau_k)}(p) - e_{-(\tau_j - \tau_k)}(p) = e_k + e_p + e_{-\tau_k}(p) - e_{\tau_j}(p) - e_{\tau_i}(p)$$

we have $m_p = A_{p, \tau_k}(p) - A_{-\tau_k}(p,p)$ for all $k = 1, 2, 3$. Thus $m_p \in M$, so define $M'' \subset M$ to be the sublattice spanned by the $m_p$.

Let $\tilde{\Delta}$ be the support of the Gorenstein cone $\sigma^\vee$, i.e. the polytope $\{x \in \sigma^\vee : \langle n_{\sigma^\vee}, x \rangle = 1\}$.

**Theorem 4.14.** There is an exact sequence

$$0 \to M'' \to M \to \mathbb{Z}^r \to 0$$

where the last map takes $A_{p, \tau_k}(p) \mapsto \beta_i$ if $A_{p, \tau_k}(p) \in o_i$. This projection maps $\tilde{\Delta}$ surjectively on the convex hull of $\{\beta_1, \ldots, \beta_r\}$ and therefore determines a Cayley structure of length $r$ on $\sigma^\vee$.

**Proof.** We must show that the application $A_{p, \tau_k}(p) \mapsto \beta_i$ gives us a well-defined morphism $M \to \mathbb{Z}^r$. This would follow from the following claim:

$$\sum_{k=1}^{3} \sum_{p \in \text{vert } T} \alpha_{k,p} A_{p, \tau_k}(p) = 0 \implies \sum_{A_{p, \tau_k}(p) \in o_i} \alpha_{k,p} = 0, \quad i = 1, \ldots, r.$$
Assume $\sum \sum \alpha_{k,p} A_{p,\tau_k(p)} = 0$ and that $\alpha_i$ is of type $k$. We have

$$0 = \sum_{m=0}^{\tau_k-1} m \tau_k \left( \sum_{l=1}^{3} \sum_{p \in \text{vert } T} \alpha_{l,p} A_{p,\tau_l(p)} \right)$$

$$= \sum_{l=1}^{3} \sum_{m=0}^{\tau_k-1} \alpha_{l,p} \sum_{m=0}^{\tau_k-1} A_{m \tau_k(p), (m \tau_k + \tau_l)(p)}$$

$$= \sum_{i=1}^{n/|\tau_k|} \sum_{\alpha_{i,p} \in \alpha_i} \sum_{A_{p,\tau_k(p)} \in \alpha_i} A_{p,\tau_k(p)}$$

so we must show that the $\{\sum_{A_{p,\tau_k(p)} \in \alpha_i} A_{p,\tau_k(p)} : i = 1, \ldots, n/|\tau_k|\}$ is linearly independent.

Let $G_k$ be the subgroup of $G$ generated by $\tau_k$ acting on $\mathbb{Z}^{n+1}$, with basis $\epsilon_k$ and $e_p, p \in \text{vert } T$, with $\tau_k(\epsilon_k) = \epsilon_k$ and $\tau_k(e_p) = e_{\tau_k(p)}$. Let $[p]$ denote the $G_k$ orbit of $p$ in $\text{vert } T$. The invariant sublattice $(\mathbb{Z}^{n+1})^{G_k}$ has rank $n/|\tau_k|$ and is spanned by $\epsilon_k$ and $\beta_p = \sum_{q \in [p]} e_q$. (If $n = |\tau_k|$ then of course $\beta_p = 0$ and $(\mathbb{Z}^{n+1})^{G_k}$ is spanned by $\epsilon_k$.)

Each $\sum_{A_{p,\tau_k(p)} \in \alpha_i} A_{p,\tau_k(p)} \in (\mathbb{Z}^{n+1})^{G_k}$. If $A_{p,\tau_k(p)} \in \alpha_i$ then one computes

$$\sum_{A_{q,\tau_k(q)} \in \alpha_i} A_{q,\tau_k(q)} = |\tau_k| \epsilon_k + 2 \beta_p - \beta_{[-\tau_i(p)]} - \beta_{[-\tau_j(p)]}.$$ (4.9)

Now both $G_i$ and $G_j$ act transitively on the set of $G_k$ orbits of $\text{vert } T$ by $\tau_i([p]) = [\tau_i(p)]$ and similarly for $G_j$ (see Lemma 4.1.1). So, after choosing some $p_0 \in \text{vert } T$ and setting $\tau_i$ to be the class of $\tau_i$ in $G/G_k$, index the basis by $\beta_p = \beta_m$ if $[p] = m \tau_i([p_0])$. Moreover $[-\tau_j(p)] = [(\tau_k + \tau_i)(p)] = [\tau_i(p)]$. Thus, with new indices, the vectors in (4.9) become

$$|\tau_k| \epsilon_k - \beta_{m-1} + 2 \beta_m - \beta_{m+1}, \quad m = 0, \ldots, n/|\tau_k| - 1$$

(indexed cyclicly) and this is a linearly independent set.

Since $m_p = A_{p,\tau_k(p)} - A_{-\tau_k(p),p}$, for all $k$, they generate the kernel of $M \to \mathbb{Z}^r$. The statement about convex hulls follows from the description of the map. The statement about Cayley structures is again [BNOS Proposition 2.3].

**Remark.** In [BNOS] we are told how to find the $r$ polytopes making up the Cayley structure. The support $\Delta$ is the convex hull of the columns of $A$. Choose some element in each $\alpha_i$ and call it $E_i$ and a basis $E_{r+1}, \ldots, E_{n+2}$ for $M''$. Thus $\{E_1, \ldots, E_{n+2}\}$ is a basis for $M$. Let $E_i^*$ be the dual basis and set for $i = 1, \ldots, r$

$$\tilde{\Delta}_i = \{x \in \Delta : \langle x, E_i^* \rangle = 0 \text{ for } j \in \{1, \ldots, r\} \setminus \{i\}\}.$$  

Write $\tilde{\Delta}_i = \Delta_i \times E_i$ where $\Delta_i$ is a lattice polytope in $M''_R$. The cone $\sigma^\vee$ is the Cayley cone associated to $\Delta_1, \ldots, \Delta_r$.  


5. Deformations

We may pull back the family over $S$ to the normalization $\tilde{S}$, which is finite and generically injective over $S$. Thus if we are only interested in which fibers occur, then we may as well work on $\tilde{S}$.

Let $R$ be the local ring of $\text{Def}_X^a$. In the proof of Theorem 2.2 in [AC10] we constructed a local formal model of the versal family over $\text{Def}_X^a$. That is a collection $U_p$, $p \in \text{vert } T$, of affine schemes and deformations $U_p \to \text{Def}_X^a$ of $U_p$ such that over $R_n = R/\mathfrak{m}_n$, the $U_p \times_{\text{Def}_X^a} \text{Spec } R_n$ could be glued to form a formal versal deformation $X_n \to \text{Spec } R_n$. Thus if $X \to \text{Def}_X^a$ is a formally versal deformation, then

$$(\mathcal{X} \times_{\text{Def}_X^a} \text{Spec } R_n)|_{U_p} \simeq U_p \times_{\text{Def}_X^a} \text{Spec } R_n$$

as formal deformations of $U_p$.

We may therefore apply the following application of Artin approximation.

**Theorem 5.1.** Let $R$ be a local $k$-algebra and assume $X \to \text{Spec } R$ and $Y \to \text{Spec } R$ are two deformations of $X_0$ with isomorphic associated formal deformations. Then $X \setminus X_0$ is smooth near $X_0$ if and only if $Y \setminus X_0$ is smooth near $X_0$.

**Proof.** Let $x \in X_0$. By assumption $\hat{O}_{X,x} \simeq \hat{O}_{Y,x}$, thus by the variant of Artin approximation theorem in [Art69, Corollary 2.6], $X$ and $Y$ are locally isomorphic for the étale topology near $x$.

We know the $U_p$ in detail - see [AC10] Proof of 6.10]. Label the coordinates of $\mathbb{P}^{n-1}$ by $x_p$, $p \in \text{vert } T$. On $U_p$ denote the 6 coordinates by $y_{p,\pm k} = x_{p,\pm \tau_k(p)}/x_p$, $k = 1, 2, 3$. Then $U_p$ is defined by the ideal generated by the 9 equations

$$y_{p,\pm i}y_{p,\mp j} + t_{p,\pm \tau_k(p)}y_{p,\pm k} \quad k = 1, 2, 3$$
$$y_{p,k}y_{p,-k} - t_{p,-\tau_k(p)}t_{p,\tau_k(p)} \quad k = 1, 2, 3.$$  

(5.1)

Recall that $t_{p,-\tau_k(p)}t_{p,\tau_k(p)} = t_{p,\tau_k(p)}t_{p,-\tau_k(p)}$ in $\text{Def}_X^a$; so the last equation makes sense.

Note that if the coordinates of $\text{Def}_X^a$ corresponding to edges of the same type are equated, $t_{p,\tau_k(p)} = t_{q,\tau_k(q)}$ for all $p, q \in \text{vert } T$, the minors of the matrices (2.1) vanish. This defines a smooth 3-dimensional subspace $\mathcal{M}$ of $\text{Def}_X^a$. Recall from Section 1.2 that the action of $G$ on $\text{Def}_X^a$ is the same as the action on the edges of $T$, i.e. $g \cdot t_{pq} = t_{g(p),g(q)}$. It follows immediately that $\mathcal{M} = (\text{Def}_X^a)^G$.

In toric terms we may describe $\mathcal{M}$ this way. Consider the projection on the first factor $p_1 : M' \to \mathbb{Z}^3$. The restriction to $\mathcal{M}$ is surjective and the induced map $M_\mathbb{R} \to \mathbb{R}^3$ maps $\sigma^\vee$ and $S$ onto the positive octant. Thus we have a closed embedding of $\mathbb{C}^3 = \text{Spec } \mathbb{C}[\mathbb{Z}_{\geq 0}^3]$ into both $\text{Spec } \mathbb{C}[M \cap \sigma^\vee]$ and $\text{Spec } \mathbb{C}[S]$. It follows that $\mathcal{M}$ lies in the toric component $\tilde{S}$. The inclusion $\mathcal{M} \subset S$ is the surjection $\mathbb{C}[S] \to \mathbb{C}[\mathbb{Z}_{\geq 0}^3]$ induced by the projection $p_1 : M \to \mathbb{Z}^3$.

Let $K$ be the image of $\mathcal{M}$ under the projection on the second factor

$$M' \to \{(a_0, \ldots, a_n) \in \mathbb{Z}^n : \sum a_i = 0\}.$$

This yields an inclusion $T_K \subseteq T_M$ which corresponds to the natural $(\mathbb{C}^*)^n/\mathbb{C}^*$ action on $\text{Def}_X^a$ induced by the action on $X$ (see Section 1.2). Now $T_M$ and therefore $T_K$ are subspaces of both $S$ and $\tilde{S}$. We will need the following easily proven lemma.

**Lemma 5.2.** Every point in $T_M$ is in a $T_K$ orbit of a point in $T_M \cap \mathcal{M}$.

In terms of deformations
Lemma 5.3. The fibers over a $T_K$ orbit in $S$ (or $\widetilde{S}$) are isomorphic.

Proof. This probably follows from general principles since the action of $T_K$ is induced by exp of the Lie algebra action of $H^0(X, \Theta_X)$ on $T_X^*$. One sees this directly through noting that $T_K$ acts as automorphisms on $U_p$ compatible with the formal gluing (see [AC10] Proof of 6.10).

□

Theorem 5.4. The main components are the only smoothing components of $\text{Def}_X^a$ and the discriminant of $\widetilde{S}$ is $\widetilde{S} \setminus T_M$.

Proof. By Theorem [1] on a non-main component some $t_{p,\tau_k(p)} = 0$. After looking at the equations (5.1) of the local formal model we conclude that $U_p$ will be singular, in fact reducible. But then by Theorem [5.1] $\mathcal{X}$ cannot contain a smooth fiber over this component. By standard toric geometry the same argument applies to fibers over $\widetilde{S} \setminus T_M$.

It remains to show that the fibers over $T_M$ are smooth. Consider a one parameter subfamily over $C \subset M \subset S$. We may assume $C$ is given by $t_{p,\tau_1(p)} = at, t_{p,\tau_2(p)} = bt, t_{p,\tau_3(p)} = ct$ for some $a, b, c \in \mathbb{C}$. Plug this into the equations (5.1) and one sees that if $abc \neq 0$ the charts over this curve have an isolated singularity at 0. Thus if $abc \neq 0$, Theorem [5.1] and generic smoothness imply that $\mathcal{X}|_C$ is a smoothing. Since it is a one-parameter smoothing the nearby fibers will all be smooth. Thus $T_M \cap M$ has only smooth fibers, but then by Lemma [5.2] and Lemma [5.3] the same is true for $T_M$.

6. Moduli

I will construct the Heisenberg group $H_{(d,n/d)}$ from $G$. After choosing an origin in $\{3, 6\}$ there is a one to one correspondence $G \rightarrow \text{vert } T$ given by $\tau \mapsto \tau(0)$. As before label the coordinates of $\mathbb{P}^{n-1}$ by $x_p, p \in \text{vert } T$.

The group $G \subset \text{Aut } T$ acts on the coordinate functions by $\tau(x_p) = x_{-\tau(p)}$ and $G^*$ acts by $\sigma(x_p) = \sigma(\tau_p)^{-1}x_p$. Taken together this defines an inclusion $G \oplus G^* \hookrightarrow \text{PGL}_n(\mathbb{C})$. Now construct a Heisenberg group $H_T \simeq H_{(d,n/d)}$ with Schrödinger representation as in Section 1.5

Lemma 6.1. A point in $\text{Def}_X^a$ is $H_T$ invariant if and only if it is $G$ invariant.

Proof. The induced action of $G^*$ on $\text{Def}_X^a$ is trivial. In fact the proof of Proposition [4.10] shows that $G^*$ is the kernel of $((\mathbb{C}^*)^{n-1} \rightarrow \text{GL}(\text{Def}_X^a(\mathbb{C}[e]))$.

Consider the $\mathbb{Z}^3 \subset M'$ spanned by $\epsilon_1, \epsilon_2$ and $\epsilon_3$ and set $\tilde{M} = \mathbb{Z}^3 \cap M$. There is an exact sequence $0 \rightarrow M \rightarrow M \rightarrow K \rightarrow 0$. The intersection $\sigma^\vee \cap \tilde{M}_\mathbb{R}$ is the positive octant $\mathbb{R}^3_{\geq 0}$ since

$$
\epsilon_k = \frac{1}{n} \sum_{p \in \text{vert } T} A_{p,\tau_k(p)} \in \sigma^\vee.
$$

Let $\tilde{M} = \text{Spec } \mathbb{C}[\tilde{M} \cap \mathbb{Z}^3_{\geq 0}]$ be the corresponding 3-dimensional toric variety. If $\tilde{G} = \mathbb{Z}^3/\tilde{M}$ then $\tilde{M} = \mathbb{C}^3/\tilde{G}^*$. We have already seen $\tilde{G}$ in Proposition [4.10] and know that as abstract group it is isomorphic to $G \times \mathbb{Z}_d$.

I state the following lemma for lack of reference, the proof is straightforward.

Lemma 6.2. Let the cone $\sigma$ and the lattice $N$ determine the affine toric variety $U_\sigma$. Assume the composition of lattice maps $N' \hookrightarrow N \rightarrow N''$ is injective, induces an isomorphism $\sigma' = \sigma \cap N'_\mathbb{R} \simeq \sigma'' = \text{im } \sigma \subset N''_\mathbb{R}$ and rank $N' = \text{rank } N''$. If $K = \ker [N \rightarrow N'']$ then $\ker [T_{N'} \rightarrow T_{N''}] = T_{N'} \cap T_K$ and it is the stabilizer of $U_{\sigma'} \subset U_\sigma$ in $T_K$. 

Theorem 6.3. The composition
\[ \mathbb{C}[^{\bar{M}} \cap \mathbb{R}^3_{\geq 0}] \hookrightarrow \mathbb{C}[M \cap \sigma^\vee] \twoheadrightarrow \mathbb{C}[\mathbb{Z}^3_{\geq 0}] \]
where the last map is induced by \( p_1 : M \to \mathbb{Z}^3 \), is injective and realizes \( \bar{M} \) as \( M/\bar{G}^* \). This identity associates an isomorphism class of \( H_T \) invariant smooth abelian surfaces to each point of the torus \( T_{\bar{M}} \subset \mathcal{M} \).

Proof. The injectivity follows from the injectivity of the composition \( \bar{M} \subseteq M \twoheadrightarrow \mathbb{Z}^3 \). Dualizing this composition we arrive in the situation of Lemma 6.2 with \( N' \) dual to \( \mathbb{Z}^3 \) and \( N'' \) dual to \( \bar{M} \). From toric geometry it follows that \( G^* \simeq \ker[T_{N'} \to T_{N''}] \). Thus by Lemma 6.2, \( G^* \) is isomorphic to the stabilizer subgroup of \( M \) in \( T_K \). The result now follows from the identification \( M = (\text{Def}_{\mathcal{X}})^G \), Lemma 5.3, Lemma 6.1 and the second statement in Theorem 5.4. \( \square \)

Remark. Note that \( \bar{M} \) is the normalization of \( \text{Spec} \mathbb{C}[S \cap \mathbb{Z}^3] \). Theorem 6.3 seems to imply that we see the moduli space for abelian surfaces with level-structure of type \((d, n/d)\) as an open subset of one of these spaces. We may at least think of them as representing the germ at a “deepest” boundary point. This type of claim presupposes an analysis of degenerate abelian surfaces arising from the non-polyhedral equivelar maps on the torus, which is at the moment work in progress.

With the standard description \( \bar{G} \) is the cokernel of the matrix (4.7) in the proof of Proposition 4.10. In each particular case it is straightforward to describe the action of \( \bar{G} \) and thus the singularities of \( \mathcal{M} \). Here are two examples.

Example 6.4. Consider the case where \( G \) is cyclic and one of the \( \tau_i \) generate \( G \). We may assume that this is \( \tau_1 \), so the standard presentation is given by
\[ \begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix} \]
where \( 2 \leq b \leq n - 2 \). (Not all of these are polyhedral.) One computes that the dual lattice,
\[ \bar{N} = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{n} (b(b + 1), -b, b + 1) \]
so the action of \( \bar{G} \) is generated by \( \text{diag}(\zeta_n^{b(b+1)}, \zeta_n^{-b}, \zeta_n^{b+1}) \) where \( \zeta_n \) is a primitive \( n \)'th root of unity. This will yield an isolated quotient singularity if and only if \( n \) is coprime to both \( b \) and \( b + 1 \). This is true if and only if all three \( \tau_k \) generate \( G \).

The quotient singularity will be Gorenstein if and only if \( 1 + b + b^2 \equiv 0 \mod n \), which again is equivalent to \( T \) being chiral. Indeed, if
\[ \rho = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \]
generates the 6-fold rotational symmetry in the standard description, then \( T \) is chiral if and only if \( \Gamma \), as translation subgroup of \( \text{Aut}(\{3, 6\}) \), is fixed by conjugacy with \( \rho \). This is again if and only if \( \rho(\Gamma) \subseteq \Gamma \), when we now view \( \Gamma \) as a sublattice of \( \mathbb{Z}^2 \). The latter is equivalent to
\[ \begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n & b \\ 0 & 1 \end{pmatrix} \]
being an integral matrix. The condition for this is exactly \( 1 + b + b^2 \equiv 0 \mod n \).
Example 6.5. Consider next the case where \( G \) is presented by
\[
\begin{pmatrix}
a & 0 \\
0 & c \\
\end{pmatrix}.
\]
This is polyhedral if \( a, c \geq 3 \). In this case \( d = \gcd(a, c) \) and one easily computes that \( \bar{M} \) is the image of
\[
\begin{pmatrix}
c & 0 & 0 \\
0 & d & 0 \\
0 & 0 & a \\
\end{pmatrix}.
\]
Thus \( \bar{M} = \text{Spec} \mathbb{C}[x^c, y^d, z^a] \simeq \mathbb{C}^3 \).

7. The vertex minimal triangulation \( T_7 \)

From the Euler formula \( v - e + f = 0 \) and the fact that \( 3f = 2e \) for surface manifolds, one concludes that a triangulated torus must have at least 7 vertices. There is exactly one such triangulation and it is equivelar. It is sometimes called the Möbius torus, since he gave the first description in 1861. In 1949 Császár gave the first polyhedral realization of the triangulation in 3-space. See e.g. [BE91b] and the references therein.

7.1. Invariants and Stanley-Reisner scheme. Call this triangulation \( T_7 \) - it is drawn in Figure 2. The group \( G \) is \( \mathbb{Z}_7 \) and the standard presentation is given by
\[
\begin{pmatrix}
7 & 2 \\
0 & 1 \\
\end{pmatrix}
\]
so the pair of divisors is \((1, 7)\). It is chiral and the automorphism group is the Frobenius group \( F_{12} = \mathbb{Z}_7 \rtimes \mathbb{Z}_6 \). The relations among the \( \tau_k \) are \( \tau_2 = 4\tau_1 \) and \( \tau_3 = 2\tau_1 \).

It is the only polyhedral triangulation with complete edge graph, i.e. the edge graph is the complete graph \( K_7 \). The Stanley-Reisner ideal is generated by 21 cubic monomials (see Section 3). The Stanley-Reisner scheme consists of 14 planes.
It appears in the lists of degenerations in both [MR05] and [Mar04], in fact Marini calls it the most special degeneration. It also appears in a central but hidden way in [MS01].

I am grateful to the referee for suggesting the following. In [MS01] Manolache and Schreyer prove that the moduli space of polarized abelian surfaces of type $(1, 7)$ is birational to a Fano threefold commonly known as $V_{22}$. They give an explicit rational parametrization via a triple projection from a special point $p_e$ in $V_{22}$. In their description $p_e$ corresponds to a special cubic curve in $\mathbb{P}^3$, namely three lines through a point, but by [Mar04, Proposition 5.16] this corresponds again exactly to our Stanley-Reisner scheme. Thus the chart in [MS01] and our $M_1$ described later in Section 7.4, are in a sense complimentary.

7.2. The smoothing component, a reflexive cone and a Calabi-Yau threefold. Since $d = 1$, $\text{Def}_X^\tau$ has one main component, the unique smoothing component $S$.

**Theorem 7.1.** The cone $\sigma^\vee$ for $T_7$ is a 9 dimensional completely split reflexive Gorenstein cone of index 3. It is the Cayley cone over three 6-dimensional lattice simplices.

**Proof.** The result follows from implementing our general results from Section 4.4. I have done the computations in Maple using the package Convex ([Fra09]). Choose an origin in vert $T$. Since $T_7$ is chiral, $M$ is Gorenstein, so $\epsilon_1 + \epsilon_2 + \epsilon_3 \in M$. In fact

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = A_{p, \tau_1(p)} + A_{\tau_3(p), (\tau_2 + \tau_3)(p)} + A_{-\tau_2(p), (\tau_3 - \tau_2)(p)}.$$  

As a basis for $M$ choose

$$E_1 = A_{0, \tau_1(0)}, E_2 = A_{\tau_3(0), (\tau_2 + \tau_3)(0)}, E_3 = A_{-\tau_2(0), (\tau_3 - \tau_2)(0)}$$

$$E_{3+i} = m_{i+1}(\tau_1(0))$$

for $i = 1, \ldots, 6$, where the $m_{i\theta}$ are as in (4.8). Thus in this basis $E_i^\sigma = \epsilon_i^\tau$ for $i = 1, 2, 3$.

After expressing the columns of $A$ in this basis, i.e. after turning $M$ into the standard $\mathbb{Z}^9$, one computes that $\sigma^\vee$ is the Cayley cone over the 3 simplices in $\mathbb{R}^6$ described in Table 1. Now plug this into the computer program and find that $\sigma$, the dual cone, has 24 rays, three of them are of course spanned by $\epsilon_1^\tau, \epsilon_2^\tau, \epsilon_3^\tau$. Moreover $m_\sigma := \epsilon_1 + \epsilon_2 + \epsilon_3 \in M$ yields the Gorenstein property on $\sigma$. This proves the theorem.

**Corollary 7.2.** The smoothing component $S$ is the germ of a normal Gorenstein affine toric variety.

**Proof.** Since $\sigma$ is Gorenstein, the point $m_\sigma := \epsilon_1 + \epsilon_2 + \epsilon_3 \in M$ generates the interior of $\sigma^\vee$, in the sense that $\text{int} \sigma \cap M = m_\sigma + \sigma \cap M$. But in this example $m_\sigma$ is in the semigroup generated by the columns of $A$, so the semigroup ring is normal.

The notion of reflexive cones was introduced by Batyrev and Borisov to study mirror symmetry for complete intersection Calabi Yau manifolds in toric varieties. In our example, since

\[
\Delta_1 : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Delta_2 : \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 \end{bmatrix} \quad \Delta_3 : \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}
\]

**Table 1.** The simplices are the convex hull of the columns.
\( \sigma^Y \) is reflexive, the Minkowski sum \( \Delta = \Delta_1 + \Delta_2 + \Delta_3 \) will be a reflexive 6 dimensional polytope. This polytope determines a 6 dimensional (singular) Fano variety \( Y \). The Minkowski decomposition gives us 3 divisors \( E_i \) on \( Y \). If we cut \( Y \) with a general section of each of the \( \mathcal{O}_Y(E_i) \) the result is a singular Calabi-Yau 3-fold and one can now take a crepant resolution to arrive at a Calabi-Yau 3-manifold.

Maxmilian Kreuzer ran the polytope through the computer program PALP (see [KS04]) with the following result.

**Proposition 7.3.** The Calabi Yau 3-manifold arising from the reflexive cone for \( T_7 \) has Hodge numbers \( h^{1,1} = 15 \) and \( h^{1,2} = 12 \). In particular the Euler number is 6.

More may be said about \( \Delta \) but I have not been able to find a good description of it. The \( f \)-vector is \((112, 427, 630, 441, 147, 21, 1)\) and the facets are probably all isomorphic. They all have at least the same \( f \)-vector \((38, 111, 125, 64, 14, 1)\). From the PALP computation one learns that it has 204 = 29 × 7 + 1 lattice points. It might be easier to describe the dual reflexive polytope since it has 21 vertices and 22 (!) lattice points.

Note that if \( Z \) is the total space of the vector bundle \( \bigoplus_{i=1}^3 \mathcal{O}_Y(E_i) \), toric geometry tells us that there is a birational toric morphism \( f : (Z, Y) \to (S, 0) \), where we identify \( Y \) as the zero section.

**Remark.** Since the cones associated to all equivelar triangulated tori are Cayley cones, one could ask if more of them are reflexive. I have not been able to prove, but do conjecture that \( T_7 \) is the only polyhedral triangulation leading to a reflexive cone. There are other non-polyhedral examples.

### 7.3. The non-smoothing components and tilings of the torus

The versal base space in this case is defined by 21 binomials in 21 variables. The numbers are small enough for us to be able to give the full component structure. This was done by delicate use of the ideal quotient command in Macaulay 2 ([GS]).

**Proposition 7.4.** The versal base space \( \text{Def}_{\overline{\mathbb{P}(T_7)}}^0 \) is reduced. It is the union of 29 irreducible components. The 28 non-smoothing components are isomorphic to the germ of the product of the affine cone over \( \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \) and \( \mathbb{C}^3 \) or \( \mathbb{C}^4 \).

When one has the ideal of the component it is in this case easy to find the generic fiber. This will be an interesting scheme since it is a “generic” non-smoothable degenerate abelian surface, i.e. it cannot appear in degenerations of smooth abelian surfaces.

The 28 components come in 4 \( G \) orbits, but 3 of these (the 8 dimensional ones) have isomorphic generic fibers. In Figure 3 I have drawn two tilings of the torus, \( P_1 \) with 1 hexagon (in black), 3 quadrangles and 2 triangles and \( P_2 \) with 1 hexagon and 4 quadrangles. We can associate an embedded rational projective surface to each polygon. A hexagon corresponds to a Del Pezzo surface of degree 6 in \( \mathbb{P}^6 \), a quadrangle corresponds to \( \mathbb{P}^1 \times \mathbb{P}^1 \) embedded via the Segre embedding in \( \mathbb{P}^3 \) and a triangle corresponds to \( \mathbb{P}^2 \). Now take the union of them in \( \mathbb{P}^6 \), intersecting as in \( P \), to make the degenerate abelian surface \( X_P \).

**Proposition 7.5.** The generic fiber over a component of dimension 7 is \( X_{P_1} \) and over a component of dimension 8 it is \( X_{P_2} \).

**Remark.** It is probably better to think of the tilings above as periodic polygonal tilings of the plane with vertices contained in the lattice of vertices of \{3, 6\}, see Figure 4. Note also that \( \Gamma \) is the full translation group of the tiling. All such tilings can be constructed by erasing...
\[\text{Figure 3. The two rigid tilings of the torus.}\]

\[\text{Figure 4. Tilings of the plane that cover the tilings in Figure 3}\]

\(\Gamma\) orbits of edges in \(\{3, 6\}\). I believe that in general the generic fiber over a non-smoothing component may be described by erasing the edges corresponding to deformation parameters that do not vanish on the whole component. It is tempting to conjecture that the components of the non-smoothable fiber are the projective toric varieties associated to the \(\{3, 6\}\) lattice polygons in the tiling.

\textbf{7.4. The space} \(\mathcal{M}\). Since the smoothing component \(S\) is normal in this case the \(\mathcal{M}\) of Theorem 6.3 equals \(\text{Spec} \mathbb{C}[S \cap \mathbb{Z}^3]\). From Example 6.4 we see that \(\mathcal{M} \simeq \mathbb{C}^3/\mathbb{Z}_7\) where \(\mathbb{Z}_7\)
acts with generator \( \text{diag}(\zeta_7^6, \zeta_7^2, \zeta_7^3) \in \text{SL}_3(\mathbb{C}) \). One may check using the criteria of Reid, Shephard-Barron and Tai (see e.g. [Rei80]) that the singularity at the origin is canonical.

7.5. The equations of a Heisenberg invariant smooth surface. Using this deformation theory we can find equations for Heisenberg invariant abelian surfaces in \( \mathbb{P}^6 \). By Lemma 5.2 we need to find the family over the smooth subspace \( \mathcal{M} \) with three parameters \( u_k = t_{p, \tau_k(p)} \).

Let us first index the vertices by their \( \tau_1 \) orbit, i.e. after fixing a vertex \( \{0\} \), \( \{m\} = \{m\tau_1(0)\} \). Thus in cycle notation \( \tau_1 = (0, 1, 2, 3, 4, 5, 6) \). Then \( \tau_2 \) becomes \( (0, 1, 2, 3, 4, 5, 6) \) and \( \tau_3 \) is \( (0, 2, 4, 6, 1, 3, 5) \). The group \( \text{Aut}(T) = F_{42} \) is generated by \( \tau_1 \) and the rotation \( \rho = (1, 5, 4, 6, 2, 3) \). Note that \( \rho \) acts on \( \mathcal{M} \) as the permutation \( (u_1, u_3, u_2) \).

The 21 cubic monomials generating \( I_X \) are one \( F_{42} \) orbit, but it is convenient to partition them in 3 \( \tau_1 \) orbits since they have the nice form

\[
x_{-\tau_k(p)}x_p x_{\tau_k(p)} \quad k = 1, 2, 3 \quad p \in \text{vert} \ T.
\]

Note again that \( \rho \) permutes these 3 orbits.

The first order deformations are easily found and the first order family is defined by the \( \tau_1 \) orbits of

\[
x_0x_1x_2 + u_2(x_1^2x_5 + x_2x_6^2), x_0x_3x_4 + u_3(x_2^2x_6 + x_1x_3^2), x_0x_2x_5 + u_1(x_2^2x_3 + x_4x_5^2).
\]

Instead of lifting equations and relations to continue apply the symmetry.

The ideal must be Heisenberg invariant. In our case the action of \( G^* \) is generated by \( x_m \mapsto \zeta_7^m x_m \). Thus if \( x_{m_1} x_{m_2} x_{m_3} \) is a term in the perturbation of \( x_{-\tau_k(m)} x_m x_{\tau_k(m)} \) we must have

\[
m_1 + m_2 + m_3 \equiv -\tau_k(m) + m + \tau_k(m) \mod 7.
\]

Moreover \( \rho^3 = (1, 6)(2, 5)(3, 4) \) fixes each \( x_{-\tau_k(m)} x_m x_{\tau_k(m)} \), so terms in \( \rho^3 \) orbits in the perturbation must have the same coefficient. Finally we may exclude terms that are in \( I_X \). The upshot is that the family is defined by the orbits of

\[
x_0x_1x_2 + u_2(x_1^2x_5 + x_2x_6^2) + \psi_1(x_1x_2x_3 + x_3x_5x_6)
+ \varphi_1x_0^3 + \xi_1(x_1x_3^2 + x_2^2x_6) + \upsilon_1(x_2^2x_3 + x_4x_5^2)
\]

\[
x_0x_3x_4 + u_3(x_2^2x_6 + x_1x_3^2) + \psi_2(x_1x_2x_4 + x_3x_5x_6)
+ \varphi_2x_0^3 + \xi_2(x_4x_5^2 + x_3^2x_3) + \upsilon_2(x_2^2x_5 + x_2x_6^2)
\]

\[
x_0x_2x_5 + u_1(x_2^2x_3 + x_4x_5^2) + \psi_3(x_1x_2x_4 + x_3x_5x_6)
+ \varphi_3x_0^3 + \xi_3(x_2x_5^2 + x_1^2x_5) + \upsilon_3(x_2^2x_6 + x_1x_3^2)
\]

where the \( \xi_i, \upsilon_i, \varphi_i, \psi_i \) are power series in \( u_1, u_2, u_3 \). Thus the task becomes to find similar expressions for lifted relations and then solving functional equations to make the family flat.

I describe here the answer only for the one parameter deformation \( s = u_1 = u_2 = u_3 \). In this case also \( \psi_1 = \psi_2 = \psi_3 \) etc. so denote the common function by \( \psi \). Let \( f(s) \) be a power series solution for the equation

\[
s^6f(s)^4 - s^4(s + 1)f(s)^3 - (s + 1)^2(s - 1)f(s) + (s + 1)^2 = 0.
\]

**Proposition 7.6.** The family defined by the orbits of equations (7.1) form a flat one parameter smoothing if \( s = u_1 = u_2 = u_3 \) and

\[
\varphi = -\psi = \frac{s^2(s^4f(s)^3 - s - 1)}{1 + s}, \quad \xi = s^2f(s), \quad \upsilon = \frac{s^4f(s)^2}{1 + s}.
\]
Because of the symmetry only 2 relations need to be lifted. I computed the liftings using Maple.

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