On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity

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Abstract

We associate to any irreducible germ $S$ of complex quasi-ordinary hypersurface an analytically invariant semigroup. We deduce a direct proof (without passing through their embedded topological invariance) of the analytical invariance of the normalized characteristic exponents. These exponents generalize the generic Newton-Puiseux exponents of plane curves. Incidentally, we give a toric description of the normalization morphism of the germ $S$.

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1 Introduction

In this paper we generalize to arbitrary dimensions results obtained first for surfaces in [22] and published in [23].

A classical way to study an irreducible germ $C$ of complex analytical plane curve is to introduce its Newton-Puiseux series in some coordinate system $(X,Y)$, which allows to define its so-called characteristic (Newton-Puiseux) exponents. If the coordinates are generic - which means that the $Y$-axis and the embedded reduced tangent cone of the curve are transversal - the characteristic exponents do not depend on them and their collection is a complete invariant of the embedded topological type of the curve.

Another way to study the germ $C$ is to associate to it a semigroup $\Gamma(C)$. We recall two ways of doing this. Both of them yield isomorphic (abstract) semigroups:

1) Take the values by the canonical valuation of the elements of the local algebra $A$ of $C$. In other words, take the orders of vanishing of $\nu^*(h)$ at the base point of $C$, where $\nu : C \to C$ is the normalization morphism of $C$ and $h$ varies through $A$. They form a sub-semigroup of $(\mathbb{N}, +)$ which obviously depends only on the analytical type of the germ $C$.

2) Take the orders of the series $h(\xi)$, where $h$ varies in $C\{X,Y\}$ and $\xi$ is a Newton-Puiseux series of $C$ in the coordinates $(X,Y)$. They form a sub-semigroup of $(\mathbb{Q}_+, +)$ which can be expressed in terms of the characteristic exponents of $C$.

Seen as an abstract semigroup, $\Gamma(C)$ is also a complete invariant of the embedded topological type of $C$. For the preceding claims, see [25] and [24]. In [21], we noticed that from the isomorphism of the two semigroups one can deduce the analytical invariance of the (generic) characteristic exponents of $C$. 
Here we extend this idea to a class of higher-dimensional hypersurface germs, the so-called quasi-ordinary ones, for which a generalization of the characteristic exponents can be defined. If \( S \) is an irreducible germ of hypersurface of dimension \( d \), such exponents are associated to any quasi-ordinary projection:

\[
\psi : S \to \mathbb{C}^d,
\]

which is by definition a finite morphism whose discriminant locus is contained in a hypersurface with normal crossings.

Lipman generalized the notion of generic Newton-Puiseux exponents of plane curves, by defining the normalized characteristic exponents of \( S \) (see section 2). The irreducible quasi-ordinary germ of hypersurface \( S \) being given, there is always a quasi-ordinary projection which has moreover normalized characteristic exponents.

It is a natural question to study the degree of invariance (analytic or topological) of the normalized characteristic exponents. In [9], Gau proves their embedded topological invariance when \( S \) is a germ of surface. Then, Gau and Lipman [10], [17] generalize this result to arbitrary dimensions.

In [11], [12], González Pérez generalizes the second of the constructions presented above of the semigroup of a plane curve to the case of an irreducible quasi-ordinary hypersurface germ \( S \). He starts from a fixed quasi-ordinary polynomial \( f \) which defines \( S \) (see definition 2.2). Instead of the order of a series in one variable, he uses the set of vertices of the Newton polyhedron of a series in various variables. He defines the semigroup \( \Gamma(f) \) as the set of vertices of the Newton polyhedra of the fractional series \( h(\xi) \), where \( h \) varies in \( \mathbb{C}\{X_1,\ldots,X_d\}\{Y\} - (f) \) and \( \xi \) is a fractional series representing a root of \( f \) (see section 2). The same semigroup is obtained if one considers only functions \( h \) such that \( h(\xi) \) has a dominating exponent (see section 8). Using the embedded topological invariance of the normalized characteristic exponents, he shows that up to isomorphism, this semigroup does not depend on the quasi-ordinary polynomial \( f \). Moreover, it is a complete invariant of the embedded topological type of the germ, and a fortiori it is an analytical invariant of the germ.

Here we avoid using Lipman-Gau’s results. Instead, we generalize the first definition given in the introduction.

If \( f \) is a quasi-ordinary polynomial defining \( S \), we remark that one can take as the root \( \xi \) of \( f \) the restriction \( Y \mid_S \), as \( f(Y) \mid_S = 0 \). With this choice of root, we have the equality:

\[
\psi^*(h(\xi)) = h \mid_S.
\]

This remark is the starting point of our method of construction of an intrinsic semigroup using the elements of the algebra \( \mathcal{A} \).

In dimension \( d \geq 2 \), there is no canonical valuation associated to \( S \). We generalize definition 1 of the introduction by constructing a canonical morphism \( \theta : (\mathbb{R}, P) \to (S, 0) \) whose source is a smooth space, and a canonical divisor with normal crossings \( \mathcal{H} \) on \( \mathbb{R} \) at \( P \). Then we restrict to those functions \( h \in \mathcal{A} \) such that the components of the divisor \( (\theta^*(h)) \) are either components of \( \mathcal{H} \) or do not contain the intersection of its components (we say then that \( \theta^*(h) \) has a dominating exponent with respect to \( \mathcal{H} \) at \( P \), this exponent being the tuple formed by the orders of vanishing of \( \theta^*(h) \) along the components of \( \mathcal{H} \)). The set of these dominating exponents obviously forms a semigroup of rank not greater than the number of components of \( \mathcal{H} \) at \( P \). We denote it by \( \Gamma'_{P}(S) \).
To construct the morphism $\theta$, the idea is to use the structure of the couple $(S, \text{Sing}(S))$. For $d \geq 2$, one cannot take simply as morphism $\theta$ a normalization $\nu : \overline{S} \to S$, as $\overline{S}$ is not necessarily smooth. So, in order to get a smooth source, we compose $\nu$ with a finite morphism $\mu : \mathcal{R} \to \overline{S}$ (see section 11), that we call an orbifold mapping. We construct the hypersurface with normal crossings by looking at the preimage $(\nu \circ \mu)^{-1}(\text{Sing}(S))$. This preimage is not necessarily a hypersurface, as $\text{Sing}(S)$ may have components of codimension 2 (see section 7). Therefore, we look only at its components of codimension 1 in $S$. Let $s$ be their number.

If $s = d$, we take $\theta := \nu \circ \mu$, $P := \theta^{-1}(0)$ and $H := \theta^{-1}(\text{Sing}(S))$. Then $\theta^*(h |_S)$ has a d.e. with respect to $H$ at $P$, whenever $h(\xi)$ has a d.e. The relation (1) allows to construct a morphism $\Phi_P : \Gamma(f) \to \Gamma'_P(S)$. In this case, our main theorem says that $\Phi_P$ realizes an isomorphism of semigroups (theorem 9.1).

If $s < d$, the situation is more complex. Then (see section 10), we construct a third morphism $\eta : \mathcal{R} \to \mathcal{R}$ as a composition of blowing-ups of smooth centers, determined canonically by the structure of $(\mathcal{R}, (\nu \circ \mu)^{-1}(\text{Sing}(S)))$. We do this third step in order to get more components of codimension 1 of the preimage of $\text{Sing}(S)$ passing through a same point. We arrive at germs $\overline{H}_P$ having $c'$ components, where $c'$ denotes the reduced equisingular dimension of $S$, defined in section 12.

In order to get an isomorphism between $\Gamma'_P(S)$ and a semigroup generalizing construction 2 for plane curves, we are obliged to modify González Pérez' definition in order to get first of all an equality of ranks. We define (see section 12) the reduced semigroup $\Gamma'(f)$ of $f$ by taking the vertices of the Newton polyhedron of $h(\xi)$, where $h(\xi)$ is now expressed as a fractional series in the first $c'$ variables $X_1, \ldots, X_{c'}$. It is a semigroup of rank $c'$.

We define then a finite set $P \subset H$ such that for any $P \in P$, the situation is analogous to the one explained before in the case $s = d$. Namely, the function $\theta^*(h |_S)$ has a d.e. with respect to $H$ at $P$, whenever $h(\xi)$ has a d.e. The relation (1) allows to construct a morphism $\Phi_P : \Gamma'(f) \to \Gamma'_P(S)$ (see section 13).

Our main theorem is:

**Theorem 13.2** The morphism $\Phi_P$ realizes an isomorphism between the semigroups $\Gamma'(f)$ and $\Gamma'_P(S)$.

The important point for the proof is that the morphisms $\nu, \mu, \eta$ have toric representatives. In particular, we think our toric construction of a normalization morphism (see section 5) has independent interest. The needed background of toric geometry is recalled in section 4.

As $\Gamma'(f)$ depends on $f$ but not on $P$ and $\Gamma'_P(S)$ just the other way round, we see that up to isomorphism, the semigroup obtained like this is an analytic invariant of $S$. In section 15 we deduce from this fact:

**Corollary 13.5** The normalized characteristic exponents are analytical invariants of the germ $S$.

Other proofs of Corollary 13.5 were given in the case of surfaces by Lipman [15], [16] and Luengo Velasco [18], [19]. Their method was to compare the normalized characteristic exponents with a canonical resolution procedure by blowing-ups of smooth centers. Such a canonical procedure by blowing-ups of smooth centers in a way deducible only from the characteristic exponents is
presently unavailable in higher dimensions. In the case of surfaces, an embedded resolution of this kind was obtained by Ban and McEwan in [4].

At a first reading, one should understand the proof of theorem 9.1. The needed material is presented in the sections which precede it. In the sections 10, 11 we present the modifications needed to prove the general case. We end the paper by a comparison with the case 23 of dimension 2.

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2 The quasi-ordinary hypersurface germs and their characteristic exponents

In this section we introduce the basic objects of study of this article.

If $P$ is a point on a complex space $\mathcal{V}$, we denote by $\mathcal{O}_{\mathcal{V}, P}$ the local algebra of $\mathcal{V}$ at $P$. If $\mathcal{H}$ is a hypersurface on an analytical manifold $\mathcal{V}$ and $P$ is a point of $\mathcal{H}$, we say that $\mathcal{H}$ has normal crossings at $P$ if there are local coordinates for $\mathcal{V}$ at $P$ such that the germ $\mathcal{H}_P$ of $\mathcal{H}$ at $P$ is the union of some hypersurfaces of coordinates. If $\hat{\phi}$ is an analytic function defined on a complex space, we denote by $\mathcal{Z}(\phi)$ its (reduced) zero-locus.

In the sequel we will denote with the same letter a germ and a sufficiently small representative of it. It will be deduced from the context if one deals with one or the other notion.

If $a \in \mathbb{R}$, we denote by $[a]$ its integral part. If $F \in K[Y]$ is a polynomial with coefficients in a field $K$, we denote by $d_Y(F)$ its degree and by $R(F)$ the set of its roots in an algebraic closure of the basis field, taken with multiplicities. If $E$ is a finite set, we denote by $|E|$ its cardinal. If $a, b \in \mathbb{Z}$ with $a \leq b$, we denote $[[a, b]] := \{a, a + 1, \ldots, b\}$.

Let $d \geq 1$ be an integer. Define the algebra of fractional series $\mathbb{C}[[X]] := \lim_{N \to \infty} \mathbb{C}\{X_1^{\frac{1}{N}}, \ldots, X_d^{\frac{1}{N}}\}$, where $X := (X_1, \ldots, X_d)$. If $m = (m_1, \ldots, m_d) \in \mathbb{Q}_+^d$, we denote $X^m := X_1^{m_1} \cdots X_d^{m_d}$. If $\eta \in \mathbb{C}\{X\}$ can be written $\eta = X^m u(X)$, with $m \in \mathbb{Q}_+^d$ and $u \in \mathbb{C}\{X\}$, $u(0, \ldots, 0) \neq 0$, we say that $\eta$ has a dominating exponent (abbreviated "d.e.") this exponent being denoted by $v_X(\eta) := m$.

If $\eta \in \mathbb{C}[X]$, we define its Newton polyhedron $N_X(\eta)$ to be the convex hull in $\mathbb{R}^d$ of the set $\text{Supp}_X(\eta) + \mathbb{R}_+^d$, where $\text{Supp}_X(\eta)$ denotes the support of $\eta$ written as a series in the variables $X$. If $\eta$ has a d.e., then $N_X(\eta) = \{v_X(\eta)\} + \mathbb{R}_+^d$, which shows that the Newton polyhedron is a generalization of the dominating exponent.

Definition 2.1 Let $\mathcal{A}$ be a reduced equi-dimensional local complex-analytical algebra of dimension $d$ and $(\mathcal{S}, 0)$ a germ of complex space such that $\mathcal{O}_{\mathcal{S}, 0} \simeq \mathcal{A}$. The algebra $\mathcal{A}$ and the germ $(\mathcal{S}, 0)$ are called quasi-ordinary if there exists a finite morphism $\psi$ from $(\mathcal{S}, 0)$ to a smooth space of the same dimension, whose discriminant locus is contained in a hypersurface with normal crossings. Such a morphism $\psi$ is also called quasi-ordinary.
All germs of curves are quasi-ordinary with respect to any finite morphism whose target is a smooth curve.

Quasi-ordinary germs appear naturally in the Jung method of resolution of the singularities of a germ by embedded resolution of the discriminant locus of a finite morphism from the germ to a smooth space of same dimension (see the original article Jung [13] and a more recent presentation by Laufer [14] for the case of dimension 2). Zariski [20] gave an alternative method of resolution of singularities of surfaces that need a study of quasi-ordinary germs. Quasi-ordinary hypersurface germs were first systematically studied by Lipman [15] when $d = 2$, see also the survey [16]. This study was extended to any $d \geq 2$ in Lipman [17].

In the special case in which $A$ is of embedding dimension $d + 1$, one can find local coordinates $X$ on the target space of $\psi$ such that the discriminant locus of $\psi$ is contained in $Z(X_1 \cdots X_d)$ and an element $Y$ in the maximal ideal of $A$ such that $(\psi, Y)$ embeds $(S, 0)$ in $C^d \times C$. So $\psi$ appears as a map:

$$\psi : S \to C^d.$$  

By the Weierstrass preparation theorem, the image of $S$ by $(\psi, Y)$, identified in the sequel with $S$, is defined by a unitary polynomial $f \in C\{X\}[Y]$. The discriminant locus of $\psi$ is defined by the discriminant $\Delta_Y(f)$ of $f$, which has therefore a d.e.  

**Definition 2.2** Let $f \in C\{X\}[Y]$ be unitary. If $\Delta_Y(f)$ has a d.e., we say that $f$ is **quasi-ordinary**. If $A \simeq C\{X\}[Y]/(f)$, with $f$ quasi-ordinary, we say that $f$ is a **qo-defining polynomial** of $S$ and of the algebra $A$.

The following theorem, called "of Jung-Abhyankar" (see [1], [17]), generalizes the theorem of Newton-Puiseux for plane curves.

**Theorem 2.3** If $f \in C\{X\}[Y]$ is quasi-ordinary, then the set $R(f)$ of roots of $f$ embeds canonically in the algebra $\hat{C}\{X\}$.

In the sequel, we consider $R(f)$ as a subset of $\hat{C}\{X\}$. Moreover, we suppose that $f$ is irreducible. Then all the differences of roots of $f$ have d.e., which are totally ordered for the componentwise order (see [15], [17]). Denote them by $A_1 < \cdots < A_G$, $A_i = (A_{i1}, \ldots, A_{id})$, $\forall i \in \{1, \ldots, G\}$.

**Definition 2.4** We say that the vectors $A_1, \ldots, A_G \in \mathbb{Q}_+^d$ are the **characteristic exponents** and the monomials $X^{A_1}, \ldots, X^{A_G}$ are the **characteristic monomials** of $f$.

After possibly permuting the variables $X_1, \ldots, X_d$, we can ensure that:

$$(A_1, \ldots, A_G) \geq_{\text{lex}} \cdots \geq_{\text{lex}} (A^1, \ldots, A^G). \quad (2)$$

Here $\geq_{\text{lex}}$ denotes the lexicographic ordering. In what follows, we suppose that this condition is always verified.

**Definition 2.5** We say that $f$ is a **normalized** qo-defining polynomial of $S$ if (2) is verified and either $A^1_1 \neq 0$ or $A^1_1 > 1$.  

5
Lipman [15] proved that any irreducible quasi-ordinary germ of hypersurface
has normalized qo-defining polynomials (see also [16, 17] and [11]).

Following Lipman [17], we define inductively the abelian groups \( M = M_0 := \mathbb{Z}^d \), \( M_i := M_{i-1} + \mathbb{Z}A_i \), \( \forall i \in \{1,...,G\} \) and the successive indices \( N_i := \text{card}(M_i/M_{i-1}) \), \( \forall i \in \{1,...,G\} \). Following González Pérez [11, 12], we define the vectors \( \overline{A}_1,...,\overline{A}_G \in \mathbb{Q}_+^d \) as:

\[
\overline{A}_1 := A_1, \quad \overline{A}_i := N_{i-1}\overline{A}_{i-1} + A_i - A_{i-1}, \quad \forall i \in \{2,...,G\}, \quad \overline{A}_{G+1} := \infty. \quad (3)
\]

It can be easily seen that \( M_G \) is also generated by \( M_0 \) and \( \overline{A}_1,...,\overline{A}_G \). Moreover, one has a canonical way of writing the elements of \( M_G \) (see [11, 12]):

**Lemma 2.6** Every element of \( M_G \) can be written in a unique way as a sum \( A+i_0A_1+\cdots+i_{G-1}A_G \), where \( A \in M_0 \) and \( 0 \leq i_k \leq N_{k+1}-1, \forall k \in \{0,...,G-1\} \).

**Proof:** As \( N_k = \text{card}(M_k/M_{k-1}) \), we deduce that \( N_k\overline{A}_k \in M_{k-1}, \forall k \in \{1,...,G\} \). From this we deduce immediately the existence of an expression verifying the asked property.

In order to prove the uniqueness, notice first that, if \( i \in \{1,...,G\} \), one has: \( N_i = \min\{k \in \mathbb{N}^*, kA_i \in M_{i-1}\} = \min\{k \in \mathbb{N}^*, k\overline{A}_i \in M_{i-1}\} \). Then, suppose by contradiction that \( \exists (i_0,...,i_{G-1}) \neq (j_0,...,j_{G-1}) \) and \( A,B \in M \) so that \( A+i_0A_1+\cdots+i_{G-1}A_G = B+j_0A_1+\cdots+j_{G-1}A_G \). Define \( p := \min\{k \in \{0,...,G-1\}, i_k = j_k, \forall k \geq k\} \). Then \( p \geq 0 \) and \( (i_p-j_p)\overline{A}_{p+1} = (B-A) + \sum_{k=p+1}^{G-1}(j_k-i_k)\overline{A}_{k+1} \in M_p \). But \( 0 < |i_p-j_p| \leq N_{p+1}-1 \), which contradicts the previous remark. \( \square \)

3 The singular locus of the germ

The characteristic exponents allow to describe precisely the singular locus \( \text{Sing}(S) \) of \( S \). Before stating this description, we introduce some notations taken from [17].

If \( I \subset \{1,...,d\} \), let \( D_I \) be the linear subspace of \( \mathbb{C}^d \) defined by \( \{X_i = 0, \forall i \in I\} \). Its codimension is \( |I| \). Denote \( Z_I := S \cap \psi^{-1}(D_I) \). In [17], Lipman shows that the spaces \( Z_I \) are irreducible. For simplicity, we denote \( Z_i := Z_{\{i\}}, \forall i \in \{1,...,d\} \).

**Definition 3.1** The minimal number \( c \in \{1,...,d\} \) with the property that \( A_i^k = 0, \forall i \in \{1,...,G\}, \forall k \in \{c+1,...,d\} \) is called the equisingular dimension of the quasi-ordinary projection \( \psi \).

Recalling that the characteristic exponents verify condition (2), we see that \( c \) represents the number of variables appearing with non-zero exponents among the monomials \( X^{A_1},...,X^{A_G} \). The name is motivated by the fact that \( \psi \) is then an equisingular deformation of a \( c \)-dimensional quasi-ordinary germ, but not of a smaller-dimensional germ (see Ban [3]).

The following theorem is a reformulation of theorem 7.3 in Lipman [17]:

**Theorem 3.2** The irreducible components of \( \text{Sing}(S) \) are of the form \( Z_I \), with \( I \subset \{1,...,c\} \) and \( |I| \in \{1,2\} \). Moreover:
1) If \( i \in \{1,\ldots,c\} \), then \( Z_i \) is a component of \( \text{Sing}(S) \) if and only if one has not simultaneously \( A_k^i = 0, \forall k \in \{1,\ldots,G-1\} \) and \( A_G^i = \frac{1}{\chi_i} \).

2) If \( \{i,j\} \subset \{1,\ldots,c\} \) with \( i \neq j \), then \( Z_{\{i,j\}} \) is a component of \( \text{Sing}(S) \) if and only if neither \( Z_i \) nor \( Z_j \) are components of \( \text{Sing}(S) \).

3) If \( \{i,j\} \subset \{1,\ldots,c\} \) with \( i \neq j \) and \( Z_{\{i,j\}} \) is a component of \( \text{Sing}(S) \), then the germ of \( S \) at any point \( P \) of \( Z_{\{i,j\}} \cup \cup_{k \notin \{i,j\}} Z_k \) is isomorphic to the subgerm of \( (\mathbb{C}^d+1,0) \) defined by the equation \( Y \cdot N = T_1T_2 \).

Let \( s \in \{0,\ldots,c\} \) be such that \( Z_i \subset \text{Sing}(S) \) for \( i \in \{1,\ldots,s\} \) and \( Z_i \not\subset \text{Sing}(S) \) either. Our ordering convention \( 2 \) and the previous theorem imply that there is always such an \( s \), which is equal to the number of components of \( \text{Sing}(S) \) having codimension \( 1 \) in \( S \). Then \( Z_{\{i,j\}} \) is a component of \( \text{Sing}(S) \) if and only if \( i \neq j \) and \( i,j \in \{s+1,\ldots,c\} \). We get:

\[
\text{Sing}(S) = \bigcup_{1 \leq i \leq s} Z_i \cup \bigcup_{j \neq i} Z_{\{i,j\}}
\]

(4)

**Examples** (where \( G = 1 \) and \( c = 3 \)):

1) If \( S = \{(X,Y), Y^N - X_1X_2X_3 = 0\} \), with \( N > 1 \), then \( s = 0 \) and \( \text{Sing}(S) = Z_{\{1,2\}} \cup Z_{\{2,3\}} \).

2) If \( S = \{(X,Y), Y^N - X_1^2X_2X_3 = 0\} \), with \( N > 1 \), \( B^1 > 1 \), then \( s = 1 \) and \( \text{Sing}(S) = Z_1 \cup Z_{\{2,3\}} \).

3) If \( S = \{(X,Y), Y^N - X_1^2X_2^2X_3 = 0\} \), with \( N > 1 \), \( B^1 > 1 \), \( B^2 > 1 \), then \( s = 2 \) and \( \text{Sing}(S) = Z_1 \cup Z_2 \).

4) If \( S = \{(X,Y), Y^N - X_1^2X_2^2X_3^3 = 0\} \), with \( N > 1 \), \( B^1 > 1 \), \( B^2 > 1 \), \( B^3 > 1 \), \( \text{gcd}(N,B^1,B^2,B^3) = 1 \), then \( s = 3 \) and \( \text{Sing}(S) = Z_1 \cup Z_2 \cup Z_3 \).

### 4 A reminder of toric geometry

For the constructions of sections \( 5, 6, 10 \) we need some elementary results of toric geometry. For details, one can consult Fulton’s \( 5 \) or Oda’s \( 20 \) book.

A lattice \( W \) is a finitely generated free abelian group. An element \( v \neq 0 \) of \( W \) is called primitive if it cannot be written \( v = av' \) with \( a \in \mathbb{N}^* - \{1\} \), \( v' \in W \). The dual lattice \( M \) of \( W \) is by definition \( \text{Hom}(W,\mathbb{Z}) \). If \( \phi : \overline{W} \to W \) is a morphism of lattices and \( K \) denotes a field, denote by \( W_K \) the \( K \)-vector space generated by \( W \) and by \( \phi_K \) the corresponding morphism of vector spaces.

The elements of \( M \) should be thought about here as exponents of monomials and those of \( W \) as weights of those monomials.

Let \( \sigma \) be a strictly convex rational polyhedral (abbreviated “s.c.r.p.”) cone in \( W_{\mathbb{R}} \) and \( \overline{\sigma} := \{ u \in M_{\mathbb{R}}, (u,v) \geq 0, \forall v \in \sigma \} \) its dual cone inside \( M_{\mathbb{R}} \). This dual cone is again s.c.r.p., by Gordan’s lemma. The affine toric variety \( Z(W,\sigma) \) of weight lattice \( W \) and cone \( \sigma \) is by definition:

\[
Z(W,\sigma) := \text{Spec} \mathbb{C}^{[\sigma \cap M]}
\]

where \( \mathbb{C}^{[\sigma \cap M]} \) denotes the \( \mathbb{C} \)-algebra of the additive semigroup \( \sigma \cap M \). Denote by \( \chi^m \) the monomial of \( \mathbb{C}^{[\sigma \cap M]} \) which has the exponent \( m \).
If \( v_1, \ldots, v_k \in \mathcal{W} \), we denote by \( \mathbf{R}_+(v_1, \ldots, v_k) \) the cone generated by them. A s.c.r.p. cone is called regular if it is generated by a subset of a basis of the lattice \( \mathcal{W} \). The variety \( \mathcal{Z}(\mathcal{W}, \sigma) \) is smooth if and only if the cone \( \sigma \) is regular.

If \( \phi : \overline{\mathcal{W}} \to \mathcal{W} \) is a morphism and \( \overline{\sigma} \subset \overline{\mathcal{W}}_R \), \( \sigma \subset \mathcal{W}_R \) are s.c.r.p. cones such that \( \phi \mathbf{R}_+(\overline{\sigma}) \subset \sigma \), then there is a canonical induced toric morphism \( \phi_* : \mathcal{Z}(\overline{\mathcal{W}}, \overline{\sigma}) \to \mathcal{Z}(\mathcal{W}, \sigma) \).

If \( \phi = \text{id}\mathcal{W} \), then \( \phi_* \) is an embedding. As a particular case of this, by taking \( \phi = \text{id}\mathcal{W} \) and \( \overline{\sigma} = \{0\} \), we obtain a canonical embedding of the complex torus \( T_{\mathcal{W}} := \mathcal{Z}(\mathcal{W}, \{0\}) \simeq (\mathbb{C}^*)^{\dim \mathcal{W}} \) in any affine toric variety \( \mathcal{Z}(\mathcal{W}, \sigma) \). Moreover, there is a canonical action of the torus \( T_{\mathcal{W}} \) on \( \mathcal{Z}(\mathcal{W}, \sigma) \), such that \( \mathcal{Z}(\mathcal{W}, \{0\}) \) is the only open orbit. With respect to these actions, the preceding morphisms are equivariant.

If \( \phi \) is an inclusion of finite index and \( \phi \mathbf{R}_+(\overline{\sigma}) = \sigma \), then \( \phi_* \) is a finite map. More precisely, we have the following proposition (corollary 1.16 of [20]):

**Proposition 4.1** If \( \phi : \overline{\mathcal{W}} \to \mathcal{W} \) presents \( \overline{\mathcal{W}} \) as a submodule of finite index of \( \mathcal{W} \), then \( \phi_* : \mathcal{Z}(\overline{\mathcal{W}}, \overline{\sigma}) \to \mathcal{Z}(\mathcal{W}, \sigma) \) coincides with the projection for the quotient of \( \mathcal{Z}(\overline{\mathcal{W}}, \overline{\sigma}) \) with respect to the natural action of the finite group \( \mathcal{W}/\phi(\overline{\mathcal{W}}) \).

The natural action alluded to comes from the following action of \( \mathcal{W} \) on the monomials of \( \mathbf{C}[\overline{\sigma}] \):

\[
v_\chi e^{-2\pi i (\overline{\chi}, \chi)} \overline{\chi}.
\]

(5)

Let us express the morphism \( \phi_* \) using coordinates in the case in which \( \dim \mathcal{W}_R = \dim \overline{\mathcal{W}}_R = d \) and \( \sigma, \overline{\sigma} \) are regular cones of the maximal dimension. Let \( v_1, \ldots, v_d \in \mathcal{W} \) and \( \overline{v}_1, \ldots, \overline{v}_d \in \overline{\mathcal{W}} \) be the unique primitive elements situated on the edges of \( \sigma \), respectively \( \overline{\sigma} \). Write \( \phi(\overline{v}_j) := \sum_{i=1}^d \alpha_i^j v_i, \forall j \in \{1, \ldots, d\} \). The hypothesis \( \phi(\overline{\sigma}) \subset \sigma \) implies that \( \alpha_i^j \in \mathbb{N}, \forall (i, j) \in \{1, \ldots, d\}^2 \). Let \( u_1, \ldots, u_d \in \mathcal{M} \) and \( \overline{u}_1, \ldots, \overline{u}_d \in \overline{\mathcal{M}} \) be the dual bases of \( v_1, \ldots, v_d \), respectively \( \overline{v}_1, \ldots, \overline{v}_d \). The adjoint morphism \( \hat{\phi} : \mathcal{M} \to \overline{\mathcal{M}} \) verifies then \( \hat{\phi}(u_i) = \sum_{j=1}^d \alpha_i^j \overline{u}_j, \forall i \in \{1, \ldots, d\} \). The monomials \( U_i := \chi^u_i, 1 \leq i \leq d \) and \( \overline{U}_i := \overline{\chi}^\overline{u}_i, 1 \leq i \leq d \) are free generators of the group algebras \( \mathbf{C}[\sigma \cap \mathcal{M}] \), respectively \( \mathbf{C}[\overline{\sigma} \cap \overline{\mathcal{M}}] \). Then:

**Lemma 4.2** The morphism \( \phi^* : \mathbf{C}[\sigma \cap \mathcal{M}] \to \mathbf{C}[\overline{\sigma} \cap \overline{\mathcal{M}}] \) can be expressed as:

\[
\begin{cases}
U_1 = \overline{U}_1^{u_1} \cdots \overline{U}_d^{u_d} \\
\vdots \\
U_d = \overline{U}_1^{u_1} \cdots \overline{U}_d^{u_d}
\end{cases}
\]

This shows that with respect to the coordinates \( U, \overline{U} \) of the two algebras, the morphism \( \phi_* \) is monomial. Let us look at its effect on the Newton polyhedron of a fractional series in the coordinates \( U \). If \( \eta \in \mathbf{C}\{U\}, \eta = \sum_{m \in \text{Supp}_U(\eta)} c_m U^m \), where \( \text{Supp}_U(\eta) \subset M_{\mathbf{Q}} \), one has \( \phi^*(\eta) = \sum_{m \in \text{Supp}_U(\eta)} c_m \phi^* m \). So, \( \text{Supp}_U(\phi^*(\eta)) \subset \hat{\phi}(\text{Supp}_U(\eta)) \). If \( m, m' \in M \) and \( m' \in \{m\} + \overline{\sigma} \), then from the condition \( \phi^*(\sigma) \subset \overline{\sigma} \) we get immediately that \( \phi(m') \in \{\phi(m)\} + \overline{\sigma} \). We deduce:

**Lemma 4.3** The vertices of \( \mathcal{N}_{\mathcal{W}}(\phi^*(\eta)) \) are images of vertices of \( \mathcal{N}_U(\eta) \).
Define a fan $\Sigma$ in $W_\mathbb{R}$ to be a finite collection of s.c.r.p. cones, such that for any $\sigma \in \Sigma$, all the faces are also in $\Sigma$, and the intersection of any two elements of $\Sigma$ is also in $\Sigma$. For example, to any s.c.r.p. cone $\sigma$ is associated canonically a fan, the set of the faces of $\sigma$. The support $|\Sigma|$ of the fan $\Sigma$ is by definition the union of the cones composing it.

Using the affine toric morphisms defined before, the affine toric varieties $Z(W, \sigma)$, for $\sigma \in \Sigma$, can be equivariantly glued in a new variety $Z(W, \Sigma)$, called the toric variety of weight lattice $W$ and fan $\Sigma$. It is always normal. It is smooth if and only if the fan $\Sigma$ is regular, i.e. its constituting cones are all regular.

The orbits of the action of $T_W$ on $Z(W, \Sigma)$ are in 1-to-1 correspondence with the cones of $\Sigma$. Denote by $O_{\sigma}$ the orbit corresponding to $\sigma \in \Sigma$ and by $V_{\sigma}$ its closure. One has the equality $\dim V_{\sigma} + \dim \sigma = \dim W_\mathbb{R}$.

The notion of toric morphism can be extended to this more general setting, starting from a morphism $\phi : \overline{W} \to W$ verifying $\forall \sigma \in \Sigma$, $\exists \tau_0 \in \Sigma$ such that $\phi_R(\sigma) \subset \tau$. Then one obtains an associated (equivariant) toric morphism $\phi_* : Z(W, \Sigma) \to Z(W, \Sigma)$. A particular case of this construction is obtained if one considers a subdivision of the fan $\Sigma$, i.e. a second fan $\Sigma'$ in $W_\mathbb{R}$ such that $|\Sigma'| = |\Sigma|$ and $\forall \sigma \in \Sigma$, $\exists \tau \in \Sigma$ with $\sigma \subset \tau$. The associated toric morphism is then proper and birational.

Let us specialize this even more. Consider a regular fan $\Sigma \subset W_\mathbb{R}$ and let $\sigma_0$ be one of its cones, of dimension $e \geq 2$. Let $V_1, ..., V_e$ be the primitive elements of $W$ situated on the edges of $\sigma_0$ and $V_{\sigma_0} := V_1 + \cdots + V_e$. Denote $\sigma_0 := R_\lambda(V_0, ..., V_{j-1}, V_{j+1}, ..., V_e)$, $\forall j \in \{1, ..., e\}$. Each $\sigma \in \Sigma$ with $\sigma_0 \subset \sigma$ can be written uniquely as $\sigma = \sigma_0 + \tau$, with $\tau \in \Sigma$ and $\sigma_0 \cap \tau = \{0\}$. Denote then $\sigma^j := \sigma_0^j + \tau$, $\forall j \in \{1, ..., e\}$.

**Definition 4.4** The star-subdivision $\Sigma^*(\sigma_0)$ of $\Sigma$ with respect to $\sigma_0$ is the fan: $(\Sigma - \{\sigma \in \Sigma, \sigma_0 \subset \sigma\}) \cup \{\text{the faces of } \sigma^j, \sigma \in \Sigma, \sigma_0 \subset \sigma, 1 \leq j \leq e\}$.

The particular case in which $\Sigma$ is the fan composed of the faces of $\sigma_0$, we write $\Sigma^*(\sigma_0)$ instead of $\Sigma^*(\sigma_0)$.

The importance of this particular subdivision comes from the following proposition (see [22]), which shows that the associated toric morphism is intrinsic from an analytical viewpoint:

**Proposition 4.5** The equivariant morphism obtained by passing from $\Sigma$ to the star-subdivision $\Sigma^*(\sigma_0)$ is isomorphic to the blow-up of $Z(W, \Sigma)$ along $V_{\sigma_0}$.

## 5 A toric normalization of the germ

In this section we show how one can obtain using toric geometry a canonical normalization morphism of the germ $(\mathcal{S}, 0)$ in the chosen coordinate system $X$. These are results obtained in [22].

We denote $W = \mathbb{Z}^d$ and let $\sigma_0$ be the cone generated by the canonical basis of $\mathbb{Z}^d$. It is a regular cone. We identify the space $C^d$ of coordinates $X$ with $Z(W, \sigma_0)$. Denote by $u_1, ..., u_d$ the primitive elements of $M = \text{Hom}(W, \mathbb{Z})$ situated on the edges of $\sigma_0$, such that $X_i = \chi^{u_i}$, $\forall i \in \{1, ..., d\}$.

Coming back to the quasi-ordinary morphism $\psi$, denote $D := D_1 \cup \cdots \cup D_d$ and $Z := Z_1 \cup \cdots \cup Z_d$. Then $S - Z \xrightarrow{\psi} C^d - D$ is an unramified covering. As the
fundamental group $\pi_1(C^d - D)$ is abelian, this covering is galoisian, its Galois group being $W/W(\psi)$. Here $W(\psi)$ denotes the following subsemigroup of $W$:

$$W(\psi) := \psi_*\pi_1(S - Z) \hookrightarrow \pi_1(C^d - D) = W.$$

Apart from $C^d = Z(W, \sigma_0)$, consider also the affine $d$-dimensional toric variety $Z(W(\psi), \sigma_0)$ and the canonical finite toric morphism:

$$\gamma_{W:W(\psi)} : (Z(W(\psi), \sigma_0), 0) \rightarrow (Z(W, \sigma_0), 0),$$

where in both cases we denote by $0$ the point which is the unique closed orbit of the corresponding toric variety.

By proposition [11], $Z(W, \sigma_0) \cong Z(W(\psi), \sigma_0)/(W/W(\psi))$. The restriction

$$\gamma_{W:W(\psi)} : Z(W(\psi), \sigma_0) - \gamma_{W:W(\psi)}^{-1}(D) \rightarrow Z(W, \sigma_0) - D$$

is an unramified covering with automorphism group precisely $W/W(\psi)$, which shows that one can complete a commutative diagram:

$$\begin{array}{ccc}
Z(W(\psi), \sigma_0) - \gamma_{W:W(\psi)}^{-1}(D) & \xrightarrow{\nu} & S - D \\
\gamma_{W:W(\psi)} & \downarrow & \psi \\
C^d & \xrightarrow{\psi} & \\
\end{array}$$

The morphism $\nu$ can be extended to a continuous morphism from $(Z(W(\psi), \sigma_0), 0)$ to $(S, 0)$. As the variety $(Z(W(\psi), \sigma_0), 0)$ is normal, by the Riemann extension theorem, the morphism $\nu$ is in fact everywhere analytic, which shows that it is a normalization of $(S, 0)$. Noticing that we had not used the fact that $S$ is a hypersurface, we get:

**Theorem 5.1** For any morphism $\psi : S \rightarrow C^d$, unramified over $(C^*)^d$ and such that $S$ is irreducible, one has the following commutative diagram, in which $\nu$ is a normalization morphism:

$$\begin{array}{ccc}
(Z(W(\psi), \sigma_0), 0) & \xrightarrow{\nu} & (S, 0) \\
\gamma_{W:W(\psi)} & \downarrow & \psi \\
(C^d, 0) & \xrightarrow{\psi} & \\
\end{array}$$

In the special case in which $S$ is a hypersurface germ, we can express the lattice $W(\psi)$ using the characteristic exponents of $\psi$. In order to do this, let us introduce the dual lattices $W_k$ of the lattices $M_k$ defined in section 2. One has the inclusions: $M = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_G$, $W = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_G$.

Introduce also the sequence of fields extensions (see [17] and [11]): $\text{Frac}(C\{X\}) = L = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_G$, where: $L_k := L(X^{A_1}, \ldots, X^{A_\varnothing_0})$, $\forall k \in \{0, \ldots, G\}$. One has the following lemma, proved in [11] (see also [22]):

**Lemma 5.2** 1) For every $i, j \in \{0, \ldots, G\}$, $i < j$, the fields extension $L_j : L_i$ is galoisian and $\text{Gal}(L_j : L_i) \cong W_j/W_i$.

2) If $\xi \in R(f)$ then $\text{Frac}(A) = L(\xi) = L_G$.

3) If $N = \text{d}_Y(f)$, then $N = N_1 \cdots N_G$. 
The action of the group \( W/W(\psi) = \text{Gal}(L(\xi) : L) \) on the field \( L(\xi) \) can be canonically lifted as an action of the group \( W \) by multiplication with roots of the unity on the monomials of \( L(\xi) \) (compare with the relation (5)):

\[
v.X^u := e^{2\pi i \langle v, u \rangle} X^u, \quad \forall v \in W.
\]

Here \( u \) varies through the set of exponents of the monomials in \( L(\xi) \), i.e. through the lattice \( M_G \). So \( W(\psi) = \{ v \in W, \langle v, u \rangle \in \mathbb{Z}, \forall u \in M_G \} = W \cap \text{Hom}(M_G, \mathbb{Z}) = W_G \). We got like this:

**Proposition 5.3** Let \( f \in \mathbb{C}\{X\}[Y] \) an irreducible quasi-ordinary polynomial and \( \psi \) be the associated quasi-ordinary projection. Then \( W(\psi) = W_G \).

Using this identification, theorem 5.1 becomes:

**Corollary 5.4** If \( f \) is an irreducible quasi-ordinary polynomial defining the germ \( S \), then one has the following commutative diagram, in which \( \nu \) is a normalization morphism:

\[
\begin{array}{ccc}
(Z(W_G, \sigma_0), 0) & \xrightarrow{\nu} & (S, 0) \\
\gamma_{W:W_G} \downarrow & & \downarrow \psi \\
(C^d, 0) & &
\end{array}
\]

An algebraic proof of this result was obtained by González Pérez in [11]. In the case of surfaces, the normalization of an irreducible quasi-ordinary germ has a Jung-Hirzebruch singularity (see Barth-Peters-Van de Ven [5]).

## 6 The canonical orbifold map

In this section we show that there is a canonical finite morphism \( \mu \) whose target is the normalization \( \overline{\mathcal{S}} \) of \( S \), its source being a smooth germ \( \mathcal{R} \). It is a particular case of the orbifold maps defined by Deligne and Mostow in [7], and also of a construction described by Prill in [24] using Grauert-Remmert’s existence theorem (see Bell-Narasimhan [6] for a presentation of this last theorem).

We denote again by \( 0 \) the base point \( \nu^{-1}(0) \) of \( \overline{\mathcal{S}} \). Consider the extrinsic isomorphism \( \overline{\mathcal{S}} \cong Z(W_G, \sigma_0) \) of the previous section. Let \( \hat{W} \) be the sublattice of \( W_G \) generated by the smallest non-zero elements of \( W_G \) situated on the edges of \( \sigma_0 \). Then \( \sigma_0 \) is regular with respect to \( \hat{W} \), and so \( Z(\hat{W}, \sigma_0) \) is smooth. Consider the toric map:

\[
\mu : Z(\hat{W}, \sigma_0) \to Z(W_G, \sigma_0)
\]

obtained by changing the lattice. Denote by \( \check{u}_1, \ldots, \check{u}_d \) the primitive elements of \( M = \text{Hom}(\hat{W}, \mathbb{Z}) \) situated on the edges of \( \sigma_0 \), such that the image of \( u_i \) is proportional with \( u_i \) in \( M_R \). Introduce \( \check{U}_i := \chi^i \check{u}_i, \forall i \in \{1, \ldots, d\} \). Then, the composition \( \nu \circ \mu : Z(W_G, \sigma_0) \to Z(W_G, \sigma_0) \) is given in the coordinates \((\check{U}_i)_{1 \leq i \leq d}, (X_i)_{1 \leq i \leq d}\) by equations of the form \( X_i = \check{U}_i^{m_i}, m_i \in \mathbb{N}^*, \forall i \in \{1, \ldots, d\} \). Denote also by \( \check{\nu}_1, \ldots, \check{\nu}_d \) the dual basis of \( \check{u}_1, \ldots, \check{u}_d \).

By proposition 4.1, the morphism \( \mu \) is the quotient map of \( Z(\hat{W}, \sigma_0) \) by the natural action of the finite group \( W_G/\hat{W} \). Moreover, it can be easily seen
that $W_G/W$ does not contain complex reflections. This shows that the locus $F_\mu$ of the fixed points of the elements of $W_G/W$ distinct from the identity has codimension at least 2 in $Z(W, \sigma_0)$. Moreover, $\mu^{-1}(\text{Sing}(Z(W_G, \sigma_0))) \subset F_\mu$.

As $Z(W, \sigma_0)$ is smooth, the complement $Z(W, \sigma_0) - \mu^{-1}(\text{Sing}(Z(W_G, \sigma_0)))$ is simply connected, and so the restriction of $\mu$ over the smooth part of $Z(W_G, \sigma_0)$ is the universal covering map. This shows its uniqueness, by the same arguments as in [24] or [7]. More precisely, we have the following result, which is a particular case of proposition 14.3 of [7], proved there algebraically:

**Lemma 6.1** For $i = 1, 2$, let $\mu_i : (R_i, P_i) \to (S, 0)$ be two finite maps unramified in codimension 1, with smooth sources $R_i$. Then there exists an isomorphism $(R_1, P_1) \cong (R_2, P_2)$ making the following diagram commutative:

\[
\begin{array}{ccc}
(R_1, P_1) & \xrightarrow{\mu_1} & (R_2, P_2) \\
\downarrow & & \downarrow \\
(S, 0) & & (S, 0)
\end{array}
\]

We denote $R := Z(W, \sigma_0)$. The preceding lemma shows that the morphism $\mu : R \to S$ is independent of the particular isomorphism $S \cong Z(W_G, \sigma_0)$ under consideration.

**Definition 6.2** We call the mapping $\mu$ described before the canonical orbifold mapping associated to $S$.

The vocabulary is motivated by the fact that $\mu$ is locally the quotient map of the action of a finite group on $R$.

Denote $P_0 := (\nu \circ \mu)^{-1}(0)$. So we have constructed a canonical finite map of germs of analytical spaces $\nu \circ \mu : (R, P_0) \to (S, 0)$.

### 7 Expansions according to semiroots

Semiroots were introduced for arbitrary irreducible quasi-ordinary polynomials by González Pérez in [11]. For their use in the study of plane curves, see [21]. Here we need them as an essential tool in the proofs of theorem 13.2.

**Definition 7.1** Let us fix $\xi \in R(f)$. Take any $k \in \{0, \ldots, G\}$. A unitary polynomial $f_k \in C[X][Y]$ is called a $k$-semiroot of $f$ if $f_k$ is of degree $N_1 \cdots N_k$, and $f_k$ has a d.e. with $v(f_k(\xi)) = N_{k+1}$. A $(G + 1)$-tuple $(f_0, \ldots, f_G)$ such that $\forall k \in \{0, \ldots, G\}$, $f_k$ is a $k$-semiroot of $f$ is called a complete system of semiroots for $f$.

These objects are independent of the choice of $\xi$.

Let $(f_0, \ldots, f_G)$ be a complete system of semiroots for $f$ (which always exists, for example the minimal polynomials of suitable truncations of $\xi$ or the characteristic approximate roots of $f$, see [11]). Generalizing immediately Abhyankar [2] (see also [21]), we have:
Lemma 7.2 Any $h \in \mathbb{C}\{X\}[Y]$ can be uniquely written as a finite sum $h = \sum c_{i_0 \cdots i_G} f_{G}^{i_0} \cdots f_{G}^{i_G}$, with $c_{i_0 \cdots i_G} \in \mathbb{C}\{X\}$, the $(G+1)$-tuples $(i_0, \ldots, i_G) \in \mathbb{N}^{G+1}$ verifying $0 \leq i_k \leq N_{k+1} - 1$, $\forall k \in \{0, \ldots, G-1\}$ and $i_G \leq \left\lfloor \frac{d_Y(h)}{d_Y(f_G)} \right\rfloor$.

Proof: Make the euclidean division of $h$ by $f_G$ and of the successive quotients by $f_G$, till one obtains a quotient of degree $< d_Y(f_G)$. Then one gets the $f_G$-adic expansion of $h$ by $f_G$, which has the form $h = \sum c_{i_0 \cdots i_G} f_{G}^{i_0} \cdots f_{G}^{i_G}$, where $i_G \leq \left\lfloor \frac{d_Y(h)}{d_Y(f_G)} \right\rfloor$. Then iterate this, making at each step the $f_{k-1}$-adic expansions of the coefficients $c_{i_0 \cdots i_G}$.

The unicity comes from the remark that the $Y$-degrees of the terms $c_{i_0 \cdots i_G} f_{G}^{i_0} \cdots f_{G}^{i_G}$ are pairwise distinct. To see it, suppose by contradiction that $\exists (i_0, \ldots, i_G) \neq (j_0, \ldots, j_G)$ and $d_Y(c_{i_0 \cdots i_G} f_{G}^{i_0} \cdots f_{G}^{i_G}) = d_Y(c_{j_0 \cdots j_G} f_{G}^{j_0} \cdots f_{G}^{j_G})$. Then $\exists p \in \{0, \ldots, G\}$ so that $i_k = j_k, \forall k > p$ and $i_p \neq j_p$. Suppose for example that $i_p > j_p$. Then $(i_p - j_p) N_1 \cdots N_p = \sum_{k=0}^{p-1} (j_k - i_k) N_1 \cdots N_k \leq \sum_{k=0}^{p-1} (N_{k+1} - 1) N_1 \cdots N_k = N_1 \cdots N_p - 1$, and so $i_p - j_p < 1$, which is a contradiction. □

Definition 7.3 The preceding equality is called the $(f_0, \ldots, f_G)$-adic expansion of $h$. The finite set $\{ (i_0, \ldots, i_G), c_{i_0 \cdots i_G} \neq 0 \}$ is called the $(f_0, \ldots, f_G)$-adic support of $h$, denoted $\text{Supp}_{(f_0, \ldots, f_G)}(h)$.

The following lemma, which generalizes the properties of Abhyankar’s expansions in terms of semiroots in the plane branch case (see 2 and 21), is a simple consequence of lemma 24.

Lemma 7.4 If $h = \sum c_{i_0 \cdots i_G} f_{G}^{i_0} \cdots f_{G}^{i_G}$ is the $(f_0, \ldots, f_G)$-adic expansion of $h \in \mathbb{C}\{X\}[Y]$, then for every $\xi \in R(f)$, the sets of vertices of the Newton polyhedra $N_X(c_{i_0 \cdots i_G}(f_0(\xi))^{i_0} \cdots (f_G(\xi))^{i_G})$ are pairwise disjoint, when $(i_0, \ldots, i_G)$ varies through the $(f_0, \ldots, f_G)$-adic support of $h$.

In the sequel, the previous lemma will be important combined with the following one:

Lemma 7.5 If $h_1, \ldots, h_p \in \mathbb{C}\{X\}$ and the sets of vertices of the Newton polyhedra $N_X(h_1), \ldots, N_X(h_p)$ are pairwise disjoint, then $N_X(h_1 + \cdots + h_p)$ is the convex hull of the union $N_X(h_1) \cup \cdots \cup N_X(h_p)$. In particular, each vertex of $N_X(h_1 + \cdots + h_p)$ is a vertex of one of the polyhedra $N_X(h_1), \ldots, N_X(h_p)$.

Proof: Denote by $\text{Conv}(E)$ the convex hull of a set $E$. If $h := h_1 + \cdots + h_p$, one has always the inclusion: $N_X(h) \subset \text{Conv}(N_X(h_1) \cup \cdots \cup N_X(h_p))$, as each monomial of $h$ is a monomial of one of the $h_i$.

Conversely, each vertex of $\text{Conv}(N_X(h_1) \cup \cdots \cup N_X(h_p))$ is a vertex of one of the $N_X(h_i)$. The hypothesis of the lemma implies that it is necessarily also a vertex of $N_X(h)$, which proves the converse inclusion. □

8 Various definitions of semigroups

A difficulty for extending the second definition of the semigroup of a plane irreducible curve is that in dimension $> 1$, fractional series may have no dominating exponent.
In [11], [12], González Pérez considers a more general notion, by using instead of the dominating exponents, the Newton polyhedra \( N_X(h(\xi)) \) for varying \( h \in \mathbb{C}\{X\}[Y] - (f) \) and the set of their vertices:

\[
\Gamma_X(f) := \{ A \in \mathbb{Q}^d_+ \mid A \text{ is a vertex of } N_X(h(\xi)), \ h \in \mathbb{C}\{X\}[Y] - (f) \}.
\]

Here \( \xi \) denotes again an arbitrary root of \( f \).

One sees immediately the independence of the set \( \Gamma_X(f) \) from the choice of \( \xi \). In [11], [12], González Pérez proves:

**Proposition 8.1** One has the equality of sets:

\[
\Gamma_X(f) = N^d + N\overline{A}_1 + \cdots + N\overline{A}_G.
\]

**Proof:** This is an immediate consequence of lemmas 7.4 and 7.5. \( \square \)

So, the set \( \Gamma_X(f) \) has the structure of semigroup for the addition, a fact which was not a priori clear from the definition.

Now we modify the previous definition, by considering only those functions which have a dominating exponent, as we find it convenient for stating theorem 13.2. We introduce the following subset of \( \mathbb{Q}^d_+ \):

\[
\Gamma_D(f) := \{ v_X(h(\xi)), \ h \in \mathbb{C}\{X\}[Y] - (f), \ h(\xi) \text{ has a d.e.} \}.
\]

This time it is clear that it is a semigroup. In fact one obtains the same semigroup as before, again an immediate consequence of lemmas 7.4 and 7.5:

**Proposition 8.2** One has the equality of semigroups:

\[
\Gamma_D(f) = N^d + N\overline{A}_1 + \cdots + N\overline{A}_G.
\]

This motivates the following definition:

**Definition 8.3** The semigroup \( \Gamma_X(f) = \Gamma_D(f) \) is called the semigroup of \( A \) with respect to \( f \), denoted \( \Gamma(f) \).

Let us introduce now some notions needed to give in sections 9 and 13 another definition of a semigroup associated to \( f \), generalizing the first one of the introduction.

Let \( V \) be a complex analytical manifold of dimension \( n \) and \( H \) a hypersurface of \( V \). Let \( P \in H \) be a point such that the germ of \( H \) at \( P \) has normal crossings.

Let \( (H_1, P), \ldots, (H_r, P) \) be the irreducible components of \( (H, P) \), where \( 1 \leq r \leq n \). From now on, their ordering is supposed to be fixed.

**Definition 8.4** The \( r \)-codimensional submanifold \( C(H, P) := (H_1, P) \cap \cdots \cap (H_r, P) \) of \( V \) is called the center of \( H \) at \( P \).

Let \((x_1, \ldots, x_n)\) be local coordinates of \( V \) at \( P \), such that \( H_i = Z(x_i), \forall i \in \{1, \ldots, r\} \). We say in this case that they are adapted to \( H \) at \( P \). Take \( h \in \mathcal{O}_V, P \).

Write \( h = x_1^{m_1} \cdots x_r^{m_r} u, \) with \( u \in \mathcal{O}_V, P \) not divisible by any of \( x_1, \ldots, x_r \). As \( m_i \) is the multiplicity of \( H_i \) as an irreducible component of the principal divisor \( (h) \), we have:
Lemma 8.5 The r-tuple \((m_1,\ldots,m_r)\) depends only on the ordering of the components of the germ \((\mathcal{H},P)\), and not on the choice of the adapted coordinates \(x_1,\ldots,x_n\).

Let us define a subclass of elements of \(\mathcal{O}_{\mathcal{V},P}\), distinguished by their relation with \(\mathcal{H}\):

**Definition 8.6** We say that \(h \in \mathcal{O}_{\mathcal{V},P}\) has a dominating exponent (abbreviated "d.e.") with respect to the germ \((\mathcal{H},P)\) if \(C(\mathcal{H},P)\) is not included in the closure of \(Z(h) - \mathcal{H}\). The r-tuple \((m_1,\ldots,m_r) \in \mathbb{N}^r\), written \(v_{\mathcal{H},P}(h)\), is called the dominating exponent of \(h\) with respect to \((\mathcal{H},P)\).

Lemma 8.5 shows that the dominating exponent is independent of the chosen adapted coordinates, once we have chosen an ordering of the components of \(\mathcal{H}\) at \(P\). In coordinates \((x_1,\ldots,x_n)\) adapted to \(\mathcal{H}\) at \(P\), we see that \(h\) has a d.e. with respect to \((\mathcal{H},P)\) if and only if the series \(h \in C\{x_1,\ldots,x_n\}\) has a d.e. with respect to \((x_1,\ldots,x_r)\), in the sense of section 11, i.e. if and only if \(u(x_1,\ldots,x_r,0,\ldots,0) \neq 0\).

If \(\mathcal{B}\) is a subalgebra of \(\mathcal{O}_{\mathcal{V},P}\), let us introduce the following subsemigroup of the multiplicative semigroup \((\mathcal{B},\cdot)\):

\[E_{\mathcal{H},P}(\mathcal{B}) := \{ h \in \mathcal{B} - \{0\}, h \text{ has a d.e. with respect to } (\mathcal{H},P) \} \]

This allows us to define:

**Definition 8.7** The semigroup of \(\mathcal{B}\) with respect to \((\mathcal{H},P)\) is the following subsemigroup of \((\mathbb{N}^r,+)\):

\[\Gamma_{\mathcal{H},P}(\mathcal{B}) := \{ v_{\mathcal{H},P}(h), h \in E_{\mathcal{H},P}(\mathcal{B}) - \{0\} \} \]

This semigroup consists simply of the dominating exponents with respect to \((\mathcal{H},P)\) of the elements of \(\mathcal{B}\) which have d.e.

9 The simplest case of the main theorem

Let us suppose in this section that \(s = d\). Then \(\text{Sing}(\mathcal{S}) = \bigcup_{1 \leq i \leq d} Z_i\) (see section 3).

Let \(\nu : \mathcal{S} \rightarrow \mathcal{S}\) be the normalization morphism of \(\mathcal{S}\) and \(\mu : \mathcal{R} \rightarrow \mathcal{S}\) be the orbifold map of \(\mathcal{S}\). Denote \(\theta := \nu \circ \mu : \mathcal{R} \rightarrow \mathcal{S}\) and let \(\theta^*\) be the corresponding morphism of sheaves of local algebras. Define \(\overline{\mathcal{H}} := \theta^{-1}(\text{Sing}(\mathcal{S}))\). Then, using the toric presentations of the morphisms \(\nu\) and \(\mu\) given in the sections 5 and 6, we see that \(\overline{\mathcal{H}}\) is a divisor with normal crossings whose center at \(P := \theta^{-1}(0)\) is 0-dimensional, reduced to \(P\) itself. Denote:

\[\Gamma_P(\mathcal{S}) := \Gamma_{\overline{\mathcal{H}},P}(\theta^*(A)_P)\]

This sub-semigroup of \((\mathbb{N}^d,+)\) is obviously an analytical invariant of \(\mathcal{S}\). The following theorem, which is the main one of this article specialized to the case \(s = d\), shows that this semigroup is isomorphic to \(\Gamma(f)\). The general case is stated in theorem 13.2.
Theorem 9.1 Let $f$ be a quasi-ordinary defining polynomial of $S$. Suppose that $s = d$. If $h$ varies through $C\{X\}|Y|$ such that $h(\xi)$ has a d.e., then $\theta^*(h|_S)$ has a d.e. with respect to $\overline{\mathcal{H}}$ at $P := \theta^{-1}(0)$ and one obtains a well-defined mapping:

$$
\Phi_P : \Gamma(f) \rightarrow \Gamma^P(S) \\
v_X(h(\xi)) \rightarrow v_{\mathcal{C}_P}^* (\theta^*(h|_S)_P)
$$

which realizes an isomorphism of semigroups.

Proof: Denote $\overline{\psi} := \psi \circ \theta : \mathcal{R} \rightarrow \mathbb{C}^d$.

As the image of $Y \in C\{X\}|Y|$ in $\mathcal{A}$ verifies the equation $f(X,Y) = 0$, one sees that $Y|_S$ can be thought as an element of $R(f)$. Denoting it by $\xi$, one has the equality $1$: $\overline{\psi}(h(\xi)) = \theta^*(h|_S)$, $\forall h \in C\{X\}|Y$.

Taking the toric representatives constructed before of the morphisms $\mu$ and $\nu$, the point $P$ is the orbit of dimension 0 and one can choose canonical toric coordinates adapted to $\overline{\mathcal{H}}$ at $P$. With such coordinates, the morphism $\overline{\psi}$ is monomial, and using formula $11$, we see that $\Phi_P$ is injective.

In order to prove its surjectivity, we must show that if $h \in C\{X\}|Y$ is such that $\theta^*(h|_S) \in E_{\mathcal{C},P}(\theta^*(A))$, then one can find another element $h' \in C\{X\}|Y$ such that $h(\xi)$ has a d.e. and $v^{\mathcal{C}}_{\mathcal{C},P}(\theta^*(h'|_S)_P) = v^{\mathcal{C}}_{\mathcal{C},P}(\theta^*(h|_S)_P)$. As $f|_S = 0$, we can suppose that $\operatorname{deg}(h) < \operatorname{deg}(f)$, after possibly making the euclidian division of $h$ by $f$. We consider then a complete system $(f_0, ..., f_G)$ of semiroots of $f$ and the $(f_0, ..., f_G)$-adic expansion of $h$, which by our hypothesis is of the form $h = \sum c_{i_0 \cdots i_G-1} f_0^{i_0} \cdots f_G^{i_G-1}$. Using lemma $16$, we see that there exists a tuple $(i_0, ..., i_{G-1}, 0) \in \operatorname{Supp}(f_0, ..., f_G)(h)$ such that $v^{\mathcal{C},P}(\theta^*(h|_S)_P) = v^{\mathcal{C},P}(\theta^*(c_{i_0 \cdots i_{G-1}} f_0^{i_0} \cdots f_G^{i_G-1}|_S)_P) = v^{\mathcal{C},P}(\theta^*(X^m f_0^{i_0} \cdots f_G^{i_G-1}|_S)_P)$, where $m$ is one of the vertices of the Newton polyhedron $\mathcal{N}_X(c_{i_0 \cdots i_{G-1}})$. The term $X^m f_0^{i_0} \cdots f_G^{i_G-1}$ has a d.e, which proves that $\Phi_P$ is surjective. $\square$

10 A canonical sequence of blowing-ups

In this section we consider $(\nu \circ \mu)^{-1}(\text{Sing}(S))$ as a subspace of $\mathcal{R}$ and we construct from it a canonical map $\eta : \overline{\mathcal{R}} \rightarrow \mathcal{R}$ obtained as a composition of blowing-ups of smooth centers. This construction will be used in order to replace the morphism $\theta$ of theorem $6$ by $\nu \circ \mu \circ \eta$ (see section $13$).

Define:

$$
\begin{align*}
\quad c' & := \begin{cases} 
\quad c - 2, & \text{if } s = c - 2, \\
\quad c, & \text{if } s \neq c - 2
\end{cases} \\
\quad D' & := D_1 \cup \cdots \cup D_c \\
\quad Z' & := \bigcup_{1 \leq i \leq s} Z_i, \text{ if } s \in \{c - 2, c - 1\}, \\
\quad \cup \bigcup_{1 \leq i \leq s} Z_i \cup Z_{[s+1, c]}, \text{ if } s \notin \{c - 2, c - 1\}
\end{align*}
$$

We say that $c'$ is the reduced equisingular dimension of $\psi$ (see also definition $20$). When $s \in \{c - 1, c\}$, the space $Z'$ is precisely the union of the components of $\text{Sing}(S)$ which have codimension 1 in $S$. As $Z_{[s+1, c]} = \bigcap_{i \notin \{s+1, c\}} Z(i,i)$, using formula $11$ we see that $Z' \hookrightarrow \text{Sing}(S)$ depends only on the analytical structure of $S$. 

16
When constructing \( \eta \), we consider several cases, according to the values of \( s \), the number of components of Sing(\( S \)) of codimension 1 in \( S \), and of the equisingular dimension \( c \):

1) **Suppose that** \( s \leq c - 3 \).

If \( I \subset \{1, \ldots, d\} \), denote \( U_I := (\nu \circ \mu)^{-1}(Z_I) \). Using the toric construction of the composition \( \nu \circ \mu \) presented in the last two sections, one sees that in the canonical toric coordinates \( \tilde{U}_1, \ldots, \tilde{U}_d \) of \( R \), one has \( U_I = \bigcap_{i \in I} Z(\tilde{U}_i) \). Formula (\ref{eq:3}) shows that:

\[
(\nu \circ \mu)^{-1}(\text{Sing}(S)) = \bigcup_{1 \leq i \leq s} U_i \bigcup_{j \in \{s+1, \ldots, c\}} \bigcup_{j \neq l} U_{(j,l)}.
\]

**Consider first the case in which** \( c = d \) **and** \( s = 0 \). Then Sing(\( S \)) has only components of codimension 2 in \( S \): Sing(\( S \)) = \( \bigcap_{I \subset \{1, \ldots, d\}, |I| = 2} Z_I \).

Then the axis of the coordinates \( \tilde{U}_1, \ldots, \tilde{U}_d \) can be obtained analytically from \((\nu \circ \mu)^{-1}(\text{Sing}(S)) \). Indeed, if \( L \) is the axis of the \( \tilde{U}_i \), one has \( L^i = \bigcap_{i \in I \neq l} U_i \).

Let \( \pi_1 : R_1 \to R \) be the blow-up of \( R \) at \( P_0 = (\nu \circ \mu)^{-1}(0) \). Denote by \( H_1 := \pi_1^{-1}(P_0) \) the exceptional divisor of \( \pi_1 \) and by \( P_1^i \) the point where the strict transform of \( L^i \) meets the exceptional divisor \( H_1 \).

Let \( \pi_2 : R_2 \to R_1 \) be the blow-up of \( R_1 \) at all the points \( P_1^i \), for \( i \in \{1, \ldots, d\} \). Denote by \( H_2 := \pi_2^{-1}(P_1^i) \) the components of the exceptional divisor of \( \pi_2 \), by \( H_{1,2} \) the strict transform of \( H_1 \) by \( \pi_2 \). Denote by \( L^{(i,j)} \) the line of the \((c-1)\)-dimensional projective space \( H_1 \) joining \( P_1^i \) and \( P_1^j \). Define \( P_2^{i,j} \) to be the point where the strict transform of \( L^{(i,j)} \) meets the divisor \( H_2^{(i,j)} \).

More generally, suppose that \( R_1 \) is already constructed, with \( d \geq k \geq 2 \). Let \( \pi_k : R_k \to R_{k-1} \) be the blow-up of \( R_{k-1} \) at all the points \( P_{k-1}^{i_1, \ldots, i_{k-1}} \), for \( i_1, \ldots, i_{k-1} \in \{1, \ldots, d\} \) pairwise distinct. The components of the exceptional divisor of \( \pi_k \) are \( H_{k-1}^{i_1, \ldots, i_{k-1}} := \pi_k^{-1}(P_{k-1}^{i_1, \ldots, i_{k-1}}) \). For \( l \in \{1, \ldots, k-1\} \) and \( j_1, \ldots, j_{k-1} \in \{1, \ldots, d\} \) pairwise distinct, the strict transform by \( \pi_k \) of \( H_{k-1}^{j_1, \ldots, j_{k-1}} \) is denoted \( H_{k-1, l}^{l,j_1, \ldots, j_{k-1}} \). Denote by \( L^{i_1, \ldots, i_{k-2}, j_{k-1}, i_k} \) the line joining the points \( P_{k-1}^{i_1, \ldots, i_{k-2}, j_{k-1}} \) and \( P_{k-1}^{i_1, \ldots, i_{k-2}, i_k} \) of the \((d-1)\)-dimensional projective space \( H_{k-1}^{i_1, \ldots, i_{k-2}} \). Define \( P_{k}^{i_1, \ldots, i_k} \) to be the point where the strict transform of the line \( L^{i_1, \ldots, i_{k-2}, j_{k-1}, i_k} \) meets the exceptional divisor \( H_{k}^{i_1, \ldots, i_{k-1}} \).

Denote \( \overline{R} := R_d \) and \( \eta := \pi_1 \circ \pi_2 \circ \cdots \circ \pi_d : R_d \to R \). Let:

\[
\overline{H} := \eta^{-1}(P_0) = \bigcup_{k, i_1, \ldots, i_{k-1} \in \{1, \ldots, d\}} H_{k,d}^{i_1, \ldots, i_{k-1}}
\]

be the exceptional divisor of \( \eta \). It is by construction a divisor with normal crossings. The minimal dimension of the centers (see definition \ref{def:centers}) of \( \overline{H} \) is 0 and is attained precisely at the points \( P_{d}^{i_1, \ldots, i_d} \), where \((i_1, \ldots, i_d) \) varies among the \( d! \) permutations of \( \{1, \ldots, d\} \). Denote this set by \( \overline{P} \).

The morphism \( \eta \) is isomorphic with a toric morphism obtained from \( Z(W, \sigma_0) \) by a sequence of star-subdivisions with respect to cones of dimension \( d \). One begins by taking the star-subdivision \( \sigma_0^0 \) of \( \sigma_0 \) with respect to itself. This gives
the morphism $\pi_1$. Then one star-subdivides each cone $\sigma_j$, $1 \leq j \leq d$ (see the notations preceding before definition 1.3), and gets $\pi_2$. At each one of the next steps, one subdivides only part of the cones of the maximal dimension. The point is that at each step it is possible to number canonically the edges of each cone from 1 to $d$ (the numbering depends on the cone looked upon). Then one subdivides only cones of the form $(\cdot \cdot \cdot ((\sigma_j^*)^i_2) \cdot \cdots )^i_k$, with $i_1, \ldots , i_k$ pairwise distinct.

If $\Sigma$ denotes the fan obtained by composing all these subdivisions, the morphism $\eta$ is isomorphic with the toric morphism $\mathcal{Z}(\tilde{W}, \tilde{\Sigma}) \to \mathcal{Z}(\tilde{W}, \sigma_0)$ and $\overline{\mathcal{R}}$ is the union of the orbits of dimension 0 of the action of $T_{\tilde{W}}$ on $\mathcal{Z}(\tilde{W}, \sigma_0)$.

Consider then the case in which $c = d$ and $0 < s \leq d - 3$.

The codimension 1 part of $(\nu \circ \mu)^{-1}(\text{Sing}(S))$ is, by formula (6), the space $\bigcup_{1 \leq i \leq s} \mathcal{U}_i$. Its center at the point $P_0$ is the space $\bigcap_{1 \leq i \leq s} \mathcal{U}_i \mu = \mathcal{U}_i[1,s]$, which in the toric setting is the subspace of the coordinates $\tilde{L}$ inside $\nu$ which are blown-up. Using the fibers of the normalization morphism with respect to cones of dimension $(c - s)$ isomorphic to a toric morphism obtained by a sequence of star-subdivisions with respect to cones of dimension $(d - s)$.

Indeed, one has simply to add $(d - c)$ to the dimensions of all the varieties which are blown-up. Using the fibers of the normalization morphism $\nu : \overline{\mathcal{R}} \to R$, one has at its disposal at each step of blowing-up, canonical families of projective spaces in which to join points. This gives the canonical smooth varieties whose strict transforms meet the exceptional divisors at the new centers of blowing-up.

At the end, one obtains again a map $\eta : \overline{\mathcal{R}} \to R$ having as exceptional divisor $\overline{\mathcal{H}} := \eta^{-1}(\mathcal{U}_i[1,s]) \to \overline{\mathcal{R}}$. The minimal dimension of its centers is now $(s + d - c)$. Among the points at which the dimension of the center is $(s + d - c)$, a discrete set $\overline{\mathcal{P}}$ of $(c - s)!$ elements is analytically determined, as formed by the points which are moreover situated on the fiber $\eta^{-1}(P_0)$. We define $\overline{\mathcal{H}} := \eta^{-1}(\mathcal{U}_i[1,s]) \cup \bigcup_{1 \leq i \leq s} \mathcal{U}_i$. Then $\overline{\mathcal{H}}$ is again a divisor with normal crossings and at the points of $\overline{\mathcal{P}}$, the dimension of its center is $d - c$. Now $\eta$ is isomorphic to a toric morphism obtained by a sequence of star-subdivisions with respect to cones of dimension $(c - s)$. 


2) Suppose that \( s = c - 1 \). Then we define \( \eta : \mathcal{R} \to \mathcal{R} \) to be the blow-up of \( \mathcal{U}_{[i,c]} \) and \( \mathcal{H} := \eta^{-1}(\bigcup_{1 \leq i \leq s} \mathcal{U}_i) \). The strict transform of \( \mathcal{U}_{[1,s]} = \bigcap_{1 \leq i \leq s} \mathcal{U}_i \) cuts the fiber \( \eta^{-1}(P_0) \) in a unique point \( P_1 \). We define \( \mathcal{P} := \{ P_1 \} \).

3) Suppose that \( s \in \{ c, c - 2 \} \). Then we do not modify \( \mathcal{R} \). So, \( \eta = \text{id}_\mathcal{R} \) and \( \mathcal{H} = \mathcal{R} \). We define then \( \mathcal{H} := \bigcup_{1 \leq i \leq s} \mathcal{U}_i \) and \( \mathcal{P} := \{ P_0 \} \).

As all the previous constructions are analytically canonical presentations of toric constructions, we have obtained the following proposition:

**Proposition 10.1** If \( \mathcal{R}, \eta, \mathcal{H}, \mathcal{P} \) are constructed as before, there is a fan \( \Sigma \) obtained from \( \sigma_0 \) by a sequence of star-subdivisions with respect to cones of dimension \( d - c' + s \), such that \( \eta \) is isomorphic to the restriction of the toric map \( \eta_T : \mathcal{Z}(\mathcal{W}, \Sigma) \to \mathcal{Z}(\mathcal{W}, \sigma_0) \) to a neighborhood of \( \eta_T^{-1}(0) \). Moreover, the components of \( \mathcal{P} \) correspond by this isomorphism to orbit closures of codimension 1, the points of \( \mathcal{P} \) to orbits of dimension 0 and at each point of \( \mathcal{P} \), the hypersurface \( \mathcal{H} \) has exactly \( c' \) components.

## 11 Reduced Newton polyhedra

The notions developed in these sections are needed in the next one in order to define the reduced semigroup of an irreducible quasi-ordinary polynomial.

Let \( \check{X} \) denote a subset with \( d \) elements of the variables \( X \).

If \( \eta \in \mathcal{C}(\check{X}) \) can be written \( \eta = (\check{X})^w u(\check{X}) \), with \( \check{m} \in \mathcal{Q}_d^d \), and \( u \in \mathcal{C}(\check{X}) \) verifying \( u(\check{X}, 0) \neq 0 \), we say that \( \eta \) has a dominating exponent with respect to \( \check{X} \), exponent denoted by \( \nu_{\check{X}}(\eta) := \check{m} \). Here \( u(\check{X}, 0) \) denotes the series obtained by anulating all the variables \( X_i \) which are not in \( \check{X} \).

In fact one can also extend to this setting the notion of Newton polyhedron.

If \( \eta \in \mathcal{C}(\check{X}) \), we define the *reduced Newton polyhedron* \( \mathcal{N}_{\check{X}}(\eta) \) of \( \eta \) with respect to \( \check{X} \), to be the convex hull in \( \mathbb{R}^d \) of the set \( \text{Supp}_{\check{X}}(\eta) + \mathbb{R}_d^d \), where \( \text{Supp}_{\check{X}}(\eta) \) denotes the support of \( \eta \) written as a series in the variables \( \check{X} \), with coefficients in the algebra \( \mathcal{C}(\check{X} - \check{X}) \).

One can obtain the reduced Newton polyhedron \( \mathcal{N}_{\check{X}}(\eta) \) from the knowledge of the usual Newton polyhedron \( \mathcal{N}_{\check{X}}(\eta) \). In order to see it, let \( M = \mathbb{Z}^d \) be the lattice of exponents of the monomials in \( \mathcal{C}(\check{X}) \) and let \( \check{M} \simeq \mathbb{Z}^d \) be the sublattice of exponents of the monomials in \( \mathcal{C}(\check{X}) \). Denote by \( \check{p} : M \to \check{M} \) the canonical projection of \( M \) on \( \check{M} \).

**Lemma 11.1** One has the equality: \( \mathcal{N}_{\check{X}}(\eta) = \check{p}_\mathbb{R}(\mathcal{N}_{\check{X}}(\eta)) \).

**Proof:** We can suppose without loss of generality that \( \check{X} \) is composed of the first \( d \) variables \( X_1, \ldots, X_d \).

Let \( \check{A} \in \check{M}_\mathbb{R} \) be a vertex of \( \mathcal{N}_{\check{X}}(\eta) \). Then there is a monomial of \( \eta \) of the form \( \check{X}^{\check{A}_1} X_{d+1}^{\check{A}_{d+1}} \cdots X_{d-d}^{\check{A}_{d-d}} \), and so \( \check{A} = \check{A}_1, \ldots, \check{A}_{d-d} \in \mathcal{N}_{\check{X}}(\eta) \Rightarrow \check{A} \in \check{p}_\mathbb{R}(\mathcal{N}_{\check{X}}(\eta)) \).

This implies that \( \mathcal{N}_{\check{X}}(\eta) \subset \check{p}_\mathbb{R}(\mathcal{N}_{\check{X}}(\eta)) \).

Take now a vertex \( \check{A} \) of \( \check{p}_\mathbb{R}(\mathcal{N}_{\check{X}}(\eta)) \). It is immediate to see that it is the projection of a vertex of \( \mathcal{N}_{\check{X}}(\eta) \). Choose one such vertex by \( \check{A}, \check{A}_1, \ldots, \check{A}_{d-d} \in \mathcal{N}_{\check{X}}(\eta) \Rightarrow \check{A} \in \mathcal{N}_{\check{X}}(\eta) \Rightarrow \check{A} \in \mathcal{N}_{\check{X}}(\eta) \).

This implies the reverse inclusion \( \check{p}_\mathbb{R}(\mathcal{N}_{\check{X}}(\eta)) \subset \mathcal{N}_{\check{X}}(\eta) \). \( \square \)
Even if we do not need it later in this work, we indicate here a generalization of the dominating exponent, as we think that it has independent interest. This generalization gives an intrinsic meaning to the notion of reduced Newton polyhedron.

Consider a germ \((\mathcal{H}, P)\) of hypersurface with normal crossings on a smooth variety \(\mathcal{V}\). We suppose that the \(r\) components \(\mathcal{H}_1, \ldots, \mathcal{H}_r\) of \(\mathcal{H}\) at \(P\) are taken in a fixed order.

**Proposition 11.2** Let \(x = (x_1, \ldots, x_n)\) be a system of local coordinates of \(\mathcal{V}\) at \(P\), such that \(Z(x_i)_P = (\mathcal{H}_i)_P, \forall i \in \{1, \ldots, r\}\). Denote \(\hat{x} := (x_1, \ldots, x_r)\). If \(h \in \mathcal{O}_{\mathcal{V}, P}\), then its reduced Newton polyhedron \(\mathcal{N}_g(h)\) depends only on the pair \((h, \mathcal{H})\) and of the ordering of the components of \(\mathcal{H}\) at \(P\).

**Proof:** Let \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) be two coordinate systems of \(\mathcal{V}\), adapted to \(\mathcal{H}\) at \(P\). Then, \(\forall k \in \{1, \ldots, r\}, x_k = y_k u_k\), with \(u_k \in \mathcal{O}_{\mathcal{V}, P}\). If \(h \in \mathcal{O}_{\mathcal{V}, P}\) and \(h = \sum \text{Supp}_{i}(h) c_{i_1, \ldots, i_r} x_1^{i_1} \cdots x_r^{i_r}\) is the expression of \(h\) as a series in the variables \(x_1, \ldots, x_r\) with coefficients in \(\mathbb{C}\{x_{r+1}, \ldots, x_n\}\), one deduces that \(h = \sum \text{Supp}_{\hat{y}}(h) c_{i_1, \ldots, i_r} u_1^{i_1} \cdots u_r^{i_r} y_1 \cdots y_r\).

Notice that \(c_{i_1, \ldots, i_r} u_1^{i_1} \cdots u_r^{i_r}\), seen as an element of \(\mathbb{C}\{y_{r+1}, \ldots, y_n\}\), is not necessarily an element of \(\mathbb{C}\{y_1, \ldots, y_n\}\).

Let \(m \in \mathbb{N}^r\) be a vertex of \(\mathcal{N}_g(h)\) (where \(g := (y_1, \ldots, y_r)\)). It is the exponent of a \((y_1, \ldots, y_r)\)-monomial appearing in the development of one of the terms \((c_{i_1, \ldots, i_r} u_1^{i_1} \cdots u_r^{i_r}) y_1^{i_1} \cdots y_r^{i_r}\) introduced before. So, for this exponent \((i_1, \ldots, i_r)\), we have \(m \in \{(i_1, \ldots, i_r)\} + \mathbb{R}^r_+\). But \((i_1, \ldots, i_r) \in \text{Supp}_{\hat{y}}(h) \Rightarrow m \in \mathcal{N}_{\hat{y}}(h)\). As this is true for any vertex \(m\) of \(\mathcal{N}_{\hat{y}}(h)\), it implies that \(\mathcal{N}_{\hat{y}}(h) \subset \mathcal{N}_{\hat{y}}(h)\). Permuting now the roles of \(x\) and \(y\), we get the desired equality \(\mathcal{N}_{\hat{x}}(h) = \mathcal{N}_{\hat{y}}(h)\). □

We call the invariant Newton polyhedron of the previous proposition the Newton polyhedron of \(h\) with respect to \((\mathcal{H}, P)\) and we denote it by \(\mathcal{N}(\mathcal{H}, P)(h)\).

The function \(h\) has a d.e. with respect to \((\mathcal{H}, P)\) (see definition 8.6) if and only if this polyhedron has only one vertex, which is then equal to \(v(\mathcal{H}, P)(h)\). So, this notion generalizes that of dominating exponent.

### 12 The reduced semigroup

Recall that the reduced equisingular dimension \(c'\) of \(S\) was defined in section 10.

If \(E\) is a set and \(V \subset E^{d}\), we denote by \(V'\) the \(c'\)-tuple of the first \(c'\)-coordinates of \(V\). Now we particularise the constructions of section 11 to the case where \(\hat{X} = X'\).

The following lemma is an immediate consequence of theorem 8.2. We use the notation \((a)^j := (a, \ldots, a) \in \mathbb{R}^j\).

**Lemma 12.1** For \(i \in \{1, \ldots, G - 1\}\), one has \(A_i = A'_i \oplus (0)^{d-c'}\), and:

\[
A_G = \begin{cases} 
A'_G \oplus (0)^{d-c'}, & \text{if } s \neq c - 2 \\
A'_G \oplus \left(\frac{1}{N_c}\right)^2 \oplus (0)^{d-c}, & \text{if } s = c - 2 
\end{cases}
\]

The relations 3 remain true if one considers the vectors \(A'_i\) and \(\overline{A}_i\) instead of the vectors \(A_i\) and \(A_i\).
We imitate now the construction of the semigroups $\Gamma_X(f)$ and $\Gamma_D(f)$, by introducing the following subsets of $Q_+^\prime$:

\[
\Gamma_D'(f) := \{ v_X(h(\xi)), h \in C[XY] - (f), h(\xi) \text{ has a d.e. with respect to } X' \},
\]

\[
\Gamma_X'(f) := \{ A' \in Q_+^\prime, A' \text{ is a vertex of } N_X(h(\xi)), h \in C[XY] - (f) \}.
\]

For the same reasons as before, they are additive semigroups.

We have the following proposition, to be compared with propositions S.1 and S.2.

**Proposition 12.2** One has the equality of semigroups:

\[
\Gamma_X'(f) = \Gamma_D'(f) = N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_G.
\]

**Proof:** Take any $A \in N^c$ and $(i_0, \ldots, i_{G-1}) \in N^G$. As $v_X((X')^A(f_0(\xi)))^{i_0} \cdots (f_{G-1}(\xi))^i_{G-1} = A + i_0\overrightarrow{A}_1 + \cdots + i_{G-1}\overrightarrow{A}_G$, one gets the inclusions $N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_G \subset \Gamma_D'(f) \subset \Gamma_X'(f)$.

Suppose now that $A' \in \Gamma_X'(f)$. Then there is an element $h \in C[XY] - (f)$ such that $A'$ is a vertex of $N_X(h(\xi))$. By lemma 14.1 there is a vertex of $N_X(h(\xi))$ of the form $(A', j_1, \ldots, j_{G-1})$. But, by proposition 5.1 this vertex is an element of $N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_G$, which implies that $A'$ is an element of $N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_G$. 

This motivates the following definition:

**Definition 12.3** The semigroup $\Gamma_X'(f) = \Gamma_D'(f)$ is called the reduced semigroup of $A$ with respect to $f$, denoted $\Gamma'(f)$.

If $(\Gamma, +)$ is a finitely generated abelian semigroup without torsion, let $\hat{\Gamma}$ be the lattice generated by $\Gamma$. Denote by $\sigma(\Gamma)$ the convex cone generated by $\Gamma$ in $\hat{\Gamma}_R$.

Particularize this to the reduced semigroup of $f$. As $\Gamma'(f) \subset Q_+^\prime$, the cone $\sigma(\Gamma'(f))$ is strictly convex. Let $u^1, \ldots, u^c$ be the smallest non-zero elements of $\Gamma'(f)$ situated on the edges of $\sigma(\Gamma'(f))$. The following lemma shows that almost all the vectors $\overrightarrow{A}_1, \ldots, \overrightarrow{A}_G$ of $Q_+^\prime$ are determined by the isomorphism type of $\Gamma'(f)$.

**Lemma 12.4** For any $j \geq 1$, if $\alpha_1, \ldots, \alpha_{j-1}$ are already defined and verify $\Gamma'(f) \neq Nu^1 + \cdots + Nu^c + Na_1 + \cdots + Na_{j-1}$, there exists a unique smallest element $\alpha_j$ of $\Gamma'(f)$ not contained in the semigroup $Nu^1 + \cdots + Nu^c + Na_1 + \cdots + Na_{j-1}$. Define $g \in N$ by $\Gamma'(f) = Nu^1 + \cdots + Nu^c + Na_1 + \cdots + Na_g$ and $e \in \{0, 1\}$ to be either 0 if $s \neq c - 2$ or 1 if $s = c - 2$. Then $g \in \{G - e, G\}$ and, after possibly permuting $u^1, \ldots, u^c$, the components of $\alpha_1, \ldots, \alpha_{G-e}$ written in the basis $u^1, \ldots, u^c$ coincide with the vectors $\overrightarrow{A}_1, \ldots, \overrightarrow{A}_{G-e}$ of $Q_+^\prime$ associated to any normalized go-defining polynomial of $S$.

**Proof:** By lemma 12.1, $\forall k \in \{1, \ldots, G-e\}$, one has $N_k = \min\{j \in N^e, j\overrightarrow{A}_k \in Z^c + Z\overrightarrow{A}_1 + \cdots + Z\overrightarrow{A}_{k-1}\}$. As $N_k > 1$, we get $\overrightarrow{A}_k \notin N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_{k-1}$. But if $\overrightarrow{A} \in \Gamma'(f)$ and $\overrightarrow{A} \notin N^c + N\overrightarrow{A}_1 + \cdots + N\overrightarrow{A}_{k-1}$, then $\overrightarrow{A} = \overrightarrow{A}_0 + \sum_{j=1}^{G} i_j\overrightarrow{A}_j$, with
$A'_0 \in \mathbb{N}^c$ and at least one of $i_k, \ldots, i_G$ is non-zero. This implies that $\mathbf{T}' \geq \mathbf{T}'_k$, and so $\mathbf{T}'_k$ is the unique smallest element of $\Gamma'(f)$ not contained in the semigroup $\mathbb{N}^c + \mathbb{N} \mathbf{T}'_1 + \cdots + \mathbb{N} \mathbf{T}'_{k-1}$. As $(u^1, \ldots, u^c)$ form obviously a permutation of the canonical generators of the semigroup $(\mathbb{N}^c, +)$, this proves the lemma. \hfill $\Box$

This lemma will be used in the passage from the analytical invariance of the reduced semigroup to the analytical invariance of the normalized characteristic exponents (Corollary 13.5).

13 The main results

In the sequel we suppose that the germ $(\mathcal{S}, 0)$ is irreducible, quasi-ordinary of dimension $d \geq 2$ and embedding dimension $d+1$. Moreover, we suppose that $0$ is not smooth on $\mathcal{S}$. Let $f$ be a qo-defining polynomial of $\mathcal{S}$. With our hypothesis, $f$ is irreducible.

Define $\nu : \mathcal{S} \to \mathcal{S}$ to be the normalization morphism of $\mathcal{S}$, studied in section 5. Let $\mu : \mathcal{R} \to \mathcal{S}$ be the orbifold map of $\mathcal{S}$, introduced in section 6. Let also $\eta : \mathcal{R} \to \mathcal{R}$ be the canonical modification of $\mathcal{R}$ defined in section 10.

Denote $\theta := \nu \circ \mu \circ \eta : \mathcal{R} \to \mathcal{S}$. Let $\theta^*$ be the corresponding morphism of sheaves of local algebras. By construction, the morphism $\theta$ depends only on the analytical type of $\mathcal{S}$ and not on any particular qo-defining polynomial. Comparing the construction done in the previous section and the definition of $\mathcal{Z}'$ given in section 10, we get:

$$\mathcal{H} = \theta^{-1}(\mathcal{Z}')$$.

Let $P$ be a point of $\mathcal{S}$. By proposition 10.1, the hypersurface $\mathcal{H}$ has $c'$ components at $P$. The following definition generalizes the first one presented in section 1 for plane curves:

**Definition 13.1** The reduced semigroup of $\mathcal{S}$ with respect to $P$, denoted $\Gamma'_p(\mathcal{S})$, is the following subsemigroup of $(\mathbb{N}^c, +)$: $\Gamma'_p(\mathcal{S}) := \Gamma_{\mathcal{H},P}(\theta^*(A)_P)$.

The semigroup $\Gamma'_p(\mathcal{S})$ depends obviously only on the choice of the point $P$. Theorem 13.2 shows that in fact, up to isomorphism, this semigroup is independent of $P$, and so it is an analytical invariant of $\mathcal{S}$:

**Theorem 13.2** Let $f$ be a quasi-ordinary defining polynomial of $\mathcal{S}$. For every point $P \in \mathcal{H}$, the image of the composition $E'(f) \to A \to \mathcal{O}_{\mathcal{H},P}$ of the restriction mapping and of $\theta^*$ is contained in the set $E_{\mathcal{H},P}(\theta^*(A)_P)$. It induces a well-defined mapping $\Phi_P$ which realises an isomorphism of semigroups:

$$\Phi_P : \Gamma'(f) \to \Gamma'_p(\mathcal{S}) \quad \nu_{\mathcal{H},P}(h(\xi)) \to \nu_{\mathcal{H},P}(\theta^*(h|_{\mathcal{H}})_P)$$.

Here $h$ varies through $E'(f)$.

We prove this theorem in the next section. It obviously generalizes the theorem 9.1.

As the left-hand semigroup does not depend on the choice of the point $P \in \mathcal{H}$, and the right-hand one does not depend on the choice of qo-defining polynomial $f$, we get:
Corollary 13.3 As an abstract semigroup, \( \Gamma'(f) \) does not depend on the chosen defining polynomial \( f \) of \( S \). We call it the reduced semigroup of \( S \), denoted \( \Gamma'(S) \).

As a by-product of the proof of theorem 13.2, we get a way to associate to some elements of \( A \) a value in the semigroup \( \Gamma'(S) \):

Corollary 13.4 Let \( f \) be a qo-defining polynomial of the germ \( S \) and \( \xi \in R(f) \). If \( h \in \mathcal{E}'(f) \), then the dominating exponent \( v_X(h(\xi)) \), seen as an element of the abstract semigroup \( \Gamma'(S) \), depends only on the image \( h|_S \in A \), and not on the choice of \( f \).

Our main application of theorem 13.2 is the following one:

Corollary 13.5 The characteristic exponents of a normalized qo-defining polynomial \( f \) of the germ \( S \) are analytical invariants of \( S \) (we recall that we suppose condition (2) is verified).

We postpone the proof to section 15.

Remark: From Corollary 13.5 one can deduce also the analytical invariance of the semigroup \( \Gamma(f) \), as was done in [11] using the inversion formulæ, expressing arbitrary characteristic exponents in terms of normalized ones.

14 Proof of the theorem 13.2

Recall from section 3 that \( s \) denotes the number of components of \( \text{Sing}(S) \) which have codimension 1 in \( S \).

Denote \( \overline{\psi} := \psi \circ \theta : \mathcal{R} \to \mathbb{C}^d \). We have the following commutative diagram:

\[
\begin{array}{cccccc}
P & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{H} = \theta^{-1}(Z') & \longrightarrow & Z' & \longrightarrow & \text{Sing}(S) \\
\downarrow & & \downarrow & & \theta & & \downarrow & & \\
\mathcal{R} & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}^d \\
\eta & & \mu & & \psi & & \psi & & \leftarrow D' \\
\end{array}
\]

As in section 3 take \( \xi := Y|_S \). One uses again formula 11.

One can choose as representative of \( \overline{\psi} \) a localisation to an open set of a toric morphism. Indeed, following the sections 5, 6 and 10 one can realize representatives of \( \nu, \mu, \eta \) as toric morphisms \( \nu : Z(W_G, \sigma_0) \to Z(W, \sigma_0) \), \( \mu : Z(W, \sigma_0) \to Z(W_G, \sigma_0) \), \( \eta : Z(W, \hat{\sigma}) \to Z(W, \sigma_0) \). With such representatives of the morphisms, the point \( P \) is an orbit of dimension 0 and \( \mathcal{H} \) is a union of closures of the orbits of codimension 1. Let \( T := (T_1, \ldots, T_d) \) be the toric coordinates of \( \mathcal{R} \) centered at \( P \). They are adapted to \( \mathcal{H} \) at \( P \). So, \( \mathcal{H}P = Z(T_1) \cup \cdots \cup Z(T_d) \).

With such coordinates, the morphism \( \overline{\psi} \) is monomial.

Let \( \tau_1, \ldots, \tau_d \in \hat{M} \) be such that \( T_i = \chi^\tau_i, \forall i \in \{1, \ldots, d\} \). Denote by \( T' \) the set of coordinates \( T_1, \ldots, T_d \) and by \( p' \) the projection of \( M \) on the sublattice \( \hat{M}' \).
generated by \( \tau_1, \ldots, \tau_r \). If \( \phi : W \rightarrow W \) is the morphism obtained by composing the changes of lattices of sections, we see that \( p' \circ \phi : M \rightarrow M' \) depends only on the restriction of \( p' \circ \phi \) to \( M' \) and that the restriction \( p' \circ \phi : M' \rightarrow M' \) is an isomorphism. Using formula (11), we see that \( \Phi_P \) is injective.

In order to prove its surjectivity, we must show that if \( h \in \mathbb{C}\{X\}\{Y\} \) verifies \( \theta^*(h \mid S) \in E_P(\theta^*(A) \mid P) \), then one can find another element \( \tilde{h} \in E'(f) \) such that \( v_{E_P}(\theta^*(h \mid S) \mid P) = v_{E_P}(\theta^*(\tilde{h} \mid S) \mid P) \). As \( f \mid S = 0 \), we can suppose that \( \deg(h) < \deg(f) \), after possibly making the euclidian division of \( h \) by \( f \). We consider then a complete system \( (f_0, \ldots, f_G) \) of semiroots of \( f \) and the \((f_0, \ldots, f_G)\)-adic expansion of \( h \), which by our hypothesis is of the form \( h = \sum c_{i_0\ldots i_{G-1}} f_0^{i_0} \cdots f_G^{i_{G-1}} \).

Formula (11) implies \( v_{E_P}(\theta^*(\tilde{h} \mid S) \mid P) = v_{E_P}(\tilde{\psi}(h(\xi))) \). By lemma 13.8 this vector is the image by \( p' \) of a vertex \( \mu \) of \( N_{\tilde{X}}(\tilde{\psi}(h(\xi))) \). By lemma 13.8 \( \mu = \phi(m) \), where \( m \) is a vertex of \( N_{\tilde{X}}(\tilde{\psi}(h(\xi))) \). Now, lemma 14.3 shows that \( m \) is a vertex of one of the polyhedra \( N_{\tilde{X}}(c_{i_0\ldots i_{G-1}} f_0^{i_0} \cdots f_G^{i_{G-1}}) \). In particular, it is of the form \( v_{N}((X^A f_0^{i_0} \cdots f_G^{i_{G-1}}) \mid S) \). But \( X^A f_0^{i_0} \cdots f_G^{i_{G-1}} \in E'(f) \), showing that we can take \( \tilde{h} := X^A f_0^{i_0} \cdots f_G^{i_{G-1}} \). This proves that \( \Phi_P \) is surjective. \( \square \)

15 Proof of the corollary 13.5

We need first some classical results about germs of plane curves and about the equisingularity of plane sections of a germ of hypersurface.

One has the following classical theorem, attributed sometimes to M.Noether (see proposition 6.5 in [21]):

**Proposition 15.1** Let \( x, y \) be two indeterminates. Let \( f, g \in \mathbb{C}\{[x]\}\{y\} \) be irreducible unitary polynomials. Denote by \( (f, g) \) their intersection number and by:

\[
K(f, g) := \max\{v_x(\xi - \eta), \xi \in R(f), \eta \in R(g)\}
\]

the exponent of coincidence of \( f \) and \( g \), where their roots are seen as Newton-Puiseux series. If \( A_1, \ldots, A_G \) are the characteristic exponents of \( f \), that \( A_{G+1} := +\infty \) and \( k \in \{0, \ldots, G\} \) is the smallest integer so that \( K(f, g) < A_{k+1} \), then:

\[
\frac{(f, g)}{dy(f)dy(g)} = \frac{\mathcal{A}_k}{N_1 \cdots N_{k-1}} + \frac{K(f, g) - A_k}{N_1 \cdots N_k}.
\]

The following proposition is an easy consequence of the also classical inversion formulas for plane germs (see proposition 4.3 in [21]):

**Proposition 15.2** Let \( C \) be a non-regular germ of irreducible plane curve and \( a_1 > 1 \) be its first characteristic exponent in generic coordinates. Then the possible values of the first characteristic exponent in various coordinate systems span the set \( \{a_2, a_2^{-1}\} \cup \{\frac{1}{a_1}, m \leq \lceil a_1 \rceil\} \). Moreover, if the first characteristic exponents of \( C \) in two coordinate systems coincide, then the whole sequence of such exponents coincide.
The following proposition is also classical (see [27, 28]):

**Proposition 15.3** Let \( C \) and \( C' \) be two germs of reduced plane curves. The following conditions are equivalent:

1) There is a 1-to-1 correspondence \( C_i \leftrightarrow C'_i \) between the components of \( C \) and \( C' \) such that \( \forall i, C_i \) and \( C'_i \) have the same sequence of characteristic Newton-Puiseux exponents and \( \forall i \neq j \), one has the equality \((C_i, C_j) = (C'_i, C'_j)\) of intersection numbers.

2) The germs \( C \) and \( C' \) have isomorphic processes of embedded resolution by blowing-ups.

3) The dual graphs of their total transforms by the minimal embedded resolution morphisms are isomorphic.

This motivates the following definition:

**Definition 15.4** The germs \( C \) and \( C' \) are said to be equisingular if the equivalent conditions of the previous proposition are verified.

Let \((\mathcal{V}, 0)\) be a germ of irreducible complex analytical space of dimension \( d \) and embedding dimension \( d + 1 \). Let \( \mathcal{H} \) be an irreducible germ of hypersurface on \( \mathcal{V} \). Consider an embedding \( E : (\mathcal{V}, 0) \to (\mathbb{C}^{d+1}, 0) \) and denote by \( \psi_E \) the projection on the first \( d \) coordinates. Suppose that \( \psi_E \) is finite. Let \( P \) be a smooth point of \( \psi_E(\mathcal{H}) \), and let \( L \) be a smooth germ of curve in \( \mathbb{C}^d \times \{0\} \), transversal to \( \psi_E(\mathcal{H}) \) at \( P \). Let \( Q \) be any point of \( \psi_E^{-1}(P) \cap \mathcal{V} \). Zariski [27] proves the following:

**Theorem 15.5** The equisingularity type of the germ of plane curve \((\psi_E^{-1}(L) \cap \mathcal{V}, Q)\) depends only on \((\mathcal{V}, \mathcal{H}, 0)\) and not on the choices of \( E, P, Q, L \).

For more informations about the notion of equisingularity, one can consult Teissier [25].

Let us pass now to the proof of the corollary.

Suppose that \( f \) is a normalized \( q_0 \)-defining polynomial of \( S \). Lemma [12.4] shows that the characteristic exponents \( A_1, \ldots, A_G \) of \( f \) are known, once \( A'_1, \ldots, A'_G \) and \( N_G \) are known. The same lemma shows that the knowledge of these last quantities is equivalent with the one of \( A'_1, \ldots, A'_{G-1}, A'_G \) and \( N_G \). By lemma [12.4] the abstract semigroup \( \Gamma'(S) \) allows to determine the vectors \( A'_1, \ldots, A'_{G-1} \) of \( \mathbb{Q}^G_+ \). The same lemma shows that \( \forall i \in \{1, \ldots, G-1\} \), the number \( N_i \) is the index of \( \mathbb{Z}^{G-1} + \mathbb{Z} A'_1 + \cdots + \mathbb{Z} A'_{i-1} \) in \( \mathbb{Z}^{G-1} + \mathbb{Z} A'_1 + \cdots + \mathbb{Z} A'_G \), which proves its analytical invariance. Moreover, the same is true for \( A_G' \) and \( N_G \), excepted when \( s = c - 2 \). So, the corollary is proved for \( s \neq c - 2 \).

**Suppose now that** \( s = c - 2 \). We will show that in addition to \( A'_1, \ldots, A'_{G-1} \), one can also determine \( A'_G \) and \( N_G \) from the analytical type of \((S, 0)\). In order to do it, we shall use no more the analytical invariance of the semigroup \( \Gamma'(f) \). Instead, we shall use the results presented at the beginning of the section.

In what follows, if \( P \) is a point of \( \mathbb{C}^d \) and \( I \subset [[1, d]] \), we denote by \( D^P_I \) the \( \mid I \mid \)-codimensional affine subspace of \( \mathbb{C}^d \) passing through \( P \) and parallel to \( D_I \).

By theorem [3.2] the variety \( Z_{(c-1,c)} \) is analytically distinguished as the only component of \( \text{Sing}(S) \) of codimension 2 in \( S \). We can obtain \( N_G \) from
the analytical structure of $\mathcal{S}$, as the least number of blowing-ups of the strict transforms of $Z_{(c-1,c]}$, needed to desingularize $\mathcal{S}$ at their generic points. This is an immediate consequence of theorem 15.2 point 3).

It remains to show that $A^*_G$ is analytically determined by $\mathcal{S}$. We do it by showing that $A^*_G \in \mathbb{Q}_+$ is analytically determined for any $i \in \{1, \ldots, c-2\}$.

1) **Consider the case where $c = 3$ and $G = 1$.** Then $A_1 = (A^1_1, \frac{1}{N^1_1}, \frac{1}{M^1_1}) \oplus (0)^{d-3}$, with $B^1_1 := N^1_1 A^1_1 \in \mathbb{N}^* - \{1\}$ (see theorem 15.2). Take $P \in D_1 \setminus (D_2 \cup D_3)$. Denote $C^P := \psi^{-1}(D^P_{[2,d]}) \cap \mathcal{S}$. The intersection $\psi^{-1}(P) \cap \mathcal{S}$ is reduced to one point $Q$. By theorem 15.2, the equisingularity type of $(C^P, Q)$ is an analytical invariant of $(\mathcal{S}, 0)$. The germ $(C^P, Q)$ has gcd$(N^1_1, B^1_1)$ irreducible components, and so this number is also analytically determined by $(\mathcal{S}, 0)$.

If $\gcd(N^1_1, B^1_1) = 1$, then $\frac{B^1_1}{N^1_1}$ is the unique Newton-Puiseux characteristic exponent of $C^P \to \psi^{-1}(D^P_{[2,d]})$ with respect to the coordinates $(X_1, Y)$ of the plane $\psi^{-1}(D^P_{[2,d]})$. As $A^1_1 \neq \frac{1}{N^1_1}$, neither $A^1_1$ nor $\frac{1}{A^1_1}$ is an integer. By proposition 15.2 we deduce that $A^1_1$ is determined as the unique first characteristic exponent of $(C^P, Q)$, for varying coordinate systems, having $N^1_1$ as the denominator of its irreducible form.

If $\gcd(N^1_1, B^1_1) > 1$, then take two irreducible components of $C^P$ at $Q$. By proposition 15.1, their intersection number at $Q$ is $\frac{N^1_1}{\gcd(N^1_1, B^1_1)} \cdot \frac{B^1_1}{\gcd(N^1_1, B^1_1)}$. As $N^1_1$ and gcd$(N^1_1, B^1_1)$ are already determined, this determines $B^1_1$, and consequently $A^1_1 = \frac{B^1_1}{N^1_1}$.

2) **Consider the case where $c = 3$ and $G > 1$.** Then the characteristic monomials of $f$ are $X^A_{11}, \ldots, X^A_{12}, X^A_{21} X^A_{22} X^A_{11}, X^A_{21} X^A_{31} X^A_{12}$. We take the same notations as in the previous case. If $A^i_G$ is a characteristic exponent of the components of the germ $(C^P, Q)$ in the coordinates $(X_1, Y)$, then proposition 15.2 shows that it is determined by the equisingularity type of $(C^P, Q)$, as $A^i_G$ is not a characteristic exponent of the components of $(C^P, Q)$, then there are at least two such components. We look at the intersection number $I$ of any two of them. By proposition 15.1 we have $I = \frac{1}{N^1_1 \cdots N^i_{G-1}} = N^{i+1}_{G-1} - A^{i+1}_{G-1} + A^{i+1}_G$, which determines $A^{i+1}_G$ from the knowledge of $A^1_G, \ldots, A^{i-1}_{G-1}$, $I$. Indeed, lemma 12.1 shows that $N^1_G, \ldots, N^{i-1}_{G-1}$ can be deduced from $A^1, \ldots, A^{i-1}_{G-1}$.

3) **Consider the case where $c \geq 4$ and $G \geq 1$.** Then the characteristic monomials of $f$ are $X^A_{11} \cdots X^A_{12}, \ldots, X^A_{i-2} \cdots X^A_{i-1} X^A_{i}, X^A_{i+1} \cdots X^A_{i} X^A_{i+2} \cdots X^A_{i+1} X^A_{i+2}$. We want to show that for all $i \in \{1, \ldots, c-2\}$, $A^*_G \in \mathbb{Q}_+$ is determined by the analytical structure of $(\mathcal{S}, 0)$.

Let $P(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_d)$ be a point of $D_i \setminus \cup_{j \neq i} D_j$. Define $C^P := \psi^{-1}(D^P_{[1,d]-\{i\}}) \cap \mathcal{S}$. It is a curve embedded in the plane $\psi^{-1}(D^P_{[1,d]-\{i\}})$. Localize it at any point $Q \in \psi^{-1}(P) \cap \mathcal{S}$ and look at the equisingularity type of the germ $(C^P, Q)$. In particular, at the characteristic exponents of the irreducible components and at the intersection numbers of the pairs of components, as in the previous two cases. Each component has the same characteristic exponents as the curve with Newton-Puiseux series $X^A_{i+1} \cdots X^A_{i} X^A_{i+2} \cdots X^A_{i+1} X^A_{i+2}$. These characteristic exponents form a (possibly strict) subset of $\{A^1_G, \ldots, A^*_G\}$. We subdivide this case in two subcases, analogous with the cases 1), respectively 2) treated before.
Suppose that \( A_1, ..., A_{G-1} \in \mathbb{N} \). Then \( B^1_G := N_G A^1_G \in \mathbb{N} \) and we determine \( A^1_G \) as in case 1), using the fact that \( s = c - 2 \) implies \( B^1_G > 1 \).

Suppose that at least one of \( A_1, ..., A_{G-1} \) is in \( \mathbb{Q} - \mathbb{N} \). Then, as \( A_1, ..., A_{G-1} \) are known, we can determine the characteristic exponents among them. Then we determine \( A^1_G \) as in case 2).

\[ \square \]

16 Comparison with the 2-dimensional case

Let us now compare this work with the paper \[23\], in which we had obtained the analytical invariance of the semigroup and of the normalized characteristic exponents in the case of quasi-ordinary surfaces.

In \[23\] the strategy of proof was the same. The morphism \( \theta : \overline{R} \rightarrow S \) was obtained also as a composition \( \theta = \nu \circ \mu \circ \eta \), with \( \eta : \overline{S} \rightarrow S \) the normalization morphism of \( S \). The proof of the isomorphism of the semigroups was basically the same as here.

The main difference is that we defined \( \mu \) to be the minimal resolution of \( \overline{S} \). The last morphism \( \eta \) was either the identity or the blow-up of a point. We took as hypersurface \( \overline{H} \) the full preimage \( \theta^{-1}(\text{Sing}(S)) \) of the singular locus of \( S \) and as set \( \overline{P} \) the union of its singular points. With the exception of the case when \( S \) was analytically isomorphic with the germ of a quadratic cone at its vertex, we showed that \( \overline{P} \neq \emptyset \). At the points of \( \overline{P} \), the curve \( \overline{H} \) had two local components, and so it was not needed to consider reduced semigroups, as done in the present paper.

The obstruction to extend that method to higher dimensions was that in general one has no more unicity of the minimal resolution of \( \overline{S} \). Even the existence of some canonical non-minimal resolution would have been enough, if it could be obtained by a toric morphism once \( \overline{S} \) was presented as a germ of toric variety in the way explained in section 5.

The solution of our problem of extension to higher dimensions came once we looked for a morphism \( \mu \) without asking it to be birational. Indeed, the attributes of \( \mu \) that were important for us were that its source was smooth and that it could be presented as a toric morphism once \( \overline{S} \) was presented as a germ of toric variety in the way explained in section 5. We could then realize this by considering the orbifold map introduced in section 6.

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28
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