Selfishness need not be bad: a general proof

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Abstract

This article studies the user behavior in non-atomic congestion games. We consider non-atomic congestion games with continuous and non-decreasing functions and investigate the limit of the price of anarchy when the total user volume approaches infinity. We deepen the knowledge on asymptotically well designed games [28], limit games [28], scalability [28] and gaugeability [7] that were recently used in the limit analyses of the price of anarchy for non-atomic congestion games. We develop a unified framework and derive new techniques that allow a general limit analysis of the price of anarchy. With these new techniques, we are able to prove a global convergence on the price of anarchy for non-atomic congestion games with arbitrary polynomial
price functions and arbitrary user volume vector sequences, see Theo-
rem 2. This means that non-atomic congestion games with polynomial
price functions are asymptotically well designed. Moreover, we show
that these new techniques are very flexible and robust and apply also
to non-atomic congestion games with price functions of other types.
In particular, we prove that non-atomic congestion games with reg-
ularly varying price functions are also asymptotically well designed,
provided that the price functions are slightly restricted, see Theorem
3 and Theorem 4. Our proofs are direct and very elementary with-
out using any heavy machinery. They only use basic properties of
Nash equilibrium and system optimum profiles, simple facts about
the asymptotic notation $O(\cdot), \Omega(\cdot)$, etc, and induction. Our results
greatly generalize recent results from [8], [6], [7] and [28]. In particu-
lar, our results further support the view of [28] with a general proof
that selfishness need not be bad for non-atomic congestion games.

1 Introduction

1.1 Motivation

Nowadays, traffic congestion has almost become a daily annoyance to every
citizen in large cities of China. According to the newest data from AMap [1]
in 2017, more than 26% of cities in China experienced traffic congestion in
rush hours, 55% of cities experienced low speed, and only 19% of cities did
not suffer from traffic congestion.

Traffic congestion does not only considerably enlarge travel latency, but
also causes serious economic loss. We take the capital city of China, Beijing,
as an example. The average economic loss caused by congestion in 2017 was
about 4,013 RMB per person, see [2], which accounts for 3.1% of the annual
GDP of Beijing in that year. Note that the annual GDP growth of Beijing
was only about 6.8% in 2017. This means that traffic congestion has almost
destroyed one third of the potential economic growth of Beijing.

To alleviate problems caused by traffic congestion, the government of
China has actively implemented a series of traffic management measures in
some large cities in recent years, including the even and odd license plate
number rule, license plate lotteries, encouraging public transportation and
others. These measures definitely prevent further deterioration of traffic, but
not yet completely cure congestion.

Road traffic conditions are a direct result from simultaneous travel of cit-
izens in a particular area. Given road conditions, the routing behavior of travelers almost determines how the traffic develops. Thus, to comprehensively cure congestion, a preliminary step is to well understand the routing behavior of travelers. In particular, we need to find out the extent to which the autonomous routing behavior of travelers contributes to congestion. This motivates the present article.

1.2 The static model

To that end, one needs to model road traffic appropriately. A popular static model for road traffic is the so-called non-atomic congestion game (NCG), see [18] or [13]. NCGs are non-cooperative games of perfect information. In an NCG, users (players) are collected into $K$ different groups according to some measurement on their similarities, for a fixed integer $K \in \mathbb{N}_+$. Associated with each group $k \in \mathcal{K} := \{1, \ldots, K\}$ is a finite non-empty set $\mathcal{S}_k$ containing all strategies only available to users from group $k$. Every user engaged in the game chooses a strategy $s \in \mathcal{S} := \bigcup_{k \in \mathcal{K}} \mathcal{S}_k$ that he will follow, and every chosen strategy $s \in \mathcal{S}$ consumes $r(a, s)$ units of resource $a$ for each $a \in A$. Here, $A$ is a finite non-empty set containing all available resources, and $r(a, s)$ is a fixed non-negative constant denoting the consumed (or demanded) volume of resource $a$ by strategy $s$ for each $a \in A$ and each $s \in \mathcal{S}$. The eventual price of a resource $a \in A$ depends only on its consumed volume. Given a vector $d = (d_k)_{k \in \mathcal{K}}$ of user volumes, a feasible strategy profile $f$ is an assignment that assigns to each of the $d_k$ users from group $k \in \mathcal{K}$ a feasible strategy $s \in \mathcal{S}_k$ for each group $k \in \mathcal{K}$. See [28] or Section 2 for details.

Obviously, NCGs model road traffic on a macroscopic level. Then, resources $a \in A$ will be arcs (streets) of the underlying road network, a group $k \in \mathcal{K}$ will be a travel origin-destination (OD) pair, and a strategy $s \in \mathcal{S}_k$ will be a path from the $k$-th origin to the $k$-th destination, for each $k \in \mathcal{K}$. The constant $r(a, s)$ is just an indicator function of the membership relation “$a \in s$” for each arc $a \in A$ and each path $s \in \mathcal{S}$. A feasible strategy profile $f$ is then a feasible traffic (path) assignment \cite{10} for all the $T(d) := \sum_{k \in \mathcal{K}} d_k$ travelers. As our results hold on a more general level, we will not stick to these terminologies of road traffic in the sequel, but still be able to understand the routing behavior of travelers.

In an NCG, the price of a resource $a \in A$ is often expressed as a non-negative, non-decreasing and continuous function $\tau_a(\cdot)$ of its demanded volume, see, e.g., [19], [20], [22], [13], [23]. Popular price functions are polynomials. For instance, latency functions $\tau_a(\cdot)$ in road traffic are conventionally
assumed to be Bureau of Public Road (BPR) functions \[14\], which are polynomials of degree 4. In our study, we will follow this fashion, and emphasize on polynomial price functions \(\tau_a(\cdot)\) and others that are related to polynomials, e.g., regularly varying functions \[3\]. However, the polynomials we will consider are general, i.e., they are allowed to have different degrees. We do not consider strategies that are completely free, i.e., \(\sum_{a \in A} r(a, s) \cdot \tau_a(x) \equiv 0\) for all \(x \geq 0\), for some group \(k \in K\) and some strategy \(s \in S_k\). This is rather reasonable in congestion game, since users with choices of free strategies are actually outside the underlying game!

1.3 Selfish user behavior

NCGs are non-cooperative, and so users are considered to be selfish. They would like to use strategies minimizing their own cost. For instance, travelers would like to follow a quickest path, so as to reduce their travel latency. In general, the cost of a user is just the cost of the strategy adopted by that user. Given a vector \(d = (d_k)_{k \in K}\) of user volumes and a feasible profile \(f\), the cost of a strategy \(s \in S\) equals \(\sum_{a \in A} r(a, s) \cdot \tau_a(f_a)\), where \(f_a\) denotes the total consumed volume of resource \(a\) w.r.t. profile \(f\), for each \(a \in A\). Obviously, for road traffic, the cost of a strategy \(s \in S\) is just the total travel time (latency) along path \(s\).

The selfish behavior of users will eventually lead the underlying game into a so-called Wardrop equilibrium (WE) \[27\], in which every user follows a cheapest strategy he could follow given the choices of all other users, see Section 2 for details. Under our assumption of continuous and non-decreasing price functions \(\tau_a(\cdot)\), a WE is actually a pure Nash equilibrium (NE) \[18\], in which all users will loyally adhere to the choices they have done, since no unilateral change in strategy can introduce any extra profit. Therefore, the game will enter a steady state if no external force interferes. An NE (equivalently, a WE) is thus a macro model of user (selfish) behavior. The average cost of users in an NE profile therefore reflects the cost users need to pay in practice. Note that under our setting of continuous and non-decreasing price functions \(\tau_a(\cdot)\), all NE profiles will have the same cost, see, e.g., \[26\].

A question of great interest in NCGs is whether user (selfish) behavior is harmful. This actually concerns whether the selfish behavior of users will damage social welfare, i.e., increases the average cost of users engaged in the game. If this is the case, then we may need to employ some external forces or measures to break up the equilibrium induced by the selfish behavior, so as to get closer to social welfare. For road traffic, possible external measures
could be some road guidance policies like congestion pricing, see, e.g., [4], [11], [5], [17] and [12].

The price of anarchy (PoA), a concept stemming from [16], is a popular measure for the “deficiency” of selfish user behavior. Given a vector \( d = (d_k)_{k \in K} \) of user volumes, a feasible strategy profile \( f^* \) is said to be at system optimum (SO) if it minimizes the average cost of users engaged in the game. The value of the PoA for non-decreasing price functions \( \tau_a(\cdot) \) equals the ratio of the average cost of users in an NE profile over that in an SO profile. Obviously, the larger the value of PoA, the more selfish user behavior demages the social welfare.

The value of the PoA does not only reflect the extent to which the selfish user behavior ruins social welfare, but also the potential benefit we could get if all users were appropriately guided. For road traffic, the value of the PoA thus shows the extent to which the average latency can be reduced in principle, if all travelers use the “right” paths. Therefore, for our purpose, we need a close inspection of the value of the PoA.

1.4 The state of the art

Traditionally, selfish user behavior is considered to be harmful, see, e.g., the studies in [19], [20], [21], [24], [22], [9], [25], and [23]. These studies investigated an upper bound of the PoA for several classes of price functions. They demonstrated that the worst-case upper bound can be very large. A famous example motivating these studies is Pigou’s game, see, e.g., [13] or Figure 1. In Pigou’s game, there is only one user group with two strategies of price functions \( x^\beta \) and 1, respectively, for some constant \( \beta \geq 0 \). The PoA of this game equals \( T/(T - (\beta + 1)^{-1/\beta} + (\beta + 1)^{-1}) \), where \( T \geq 1 \) is the volume of users engaged in the game. Obviously, considering all possible \( \beta \),

\begin{align*}
\text{Figure 1: Pigou’s game}
\end{align*}
the worst-case upper bound of the PoA is infinity, since the PoA tends to ∞ as β → ∞, if T = 1.

Worst-case upper bounds of the PoA are actually not a valid evidence to show that selfish user behavior is bad, especially not for a large total volume of users. For instance, the PoA of Pigou’s game actually tends to 1, as T approaches infinity, for each fixed β ≥ 0. This means that selfish user behavior may well guarantee social welfare if the total user volume is large. Generally, the total travel demand in rush hours is usually very large. Therefore, to comprehend the value of the PoA is still needed, especially for the case of heavy traffic, i.e., the case when the total travel demand T(d) = \sum_{k\in K} d_k is large.

Recently, several studies have been done in this direction, see [8], [6], [7], and [28]. Colini et al. [8] considered two special cases: (i) games with a single OD pair, and (ii) games with a single OD pair with parallel feasible paths. For case (i), they proved that if one of the feasible paths has a latency function that is bounded by a constant from above, then the PoA will converge to 1 as the total demand T(d) tends to infinity. For case (ii), they proved that if the latency functions τ_a(·) are regularly varying [3], then the PoA will converge to 1 as the total demand T(d) tends to infinity.

Colini et al. [7] continued the study of [8]. They investigated more general cases, i.e., games with multiple OD pairs. Using the notion of regular variation [3], they proposed the concept of gaugeable games that are defined only for particular user volume vector sequences \{d^{(n)}\}_{n\in \mathbb{N}} fulfilling the condition that \( \lim_{n\to \infty} \sum_{k\in K_{\text{light}}} d_{k}^{(n)} T(d^{(n)}) > 0 \) for each group k \in K, see [7] or Subsection 3.1 for details. With the technique of regular variation [3], they proved that the PoA of gaugeable games converges to 1 as the total demand T(d) = \sum_{k\in K} d_k tends to infinity. However, this convergence result only holds for particular user volume vector sequences, due to the sequence-specific nature of gaugeable games. See Subsection 3.1 for details about the convergence results of gaugeable games.

Colini et al. [6] further continued the study of [7]. They applied the technique of gaugeability [7] to NCGs with polynomial price functions. Due to the sequence-specific nature of gaugeability [7], they assumed a user volume vector sequence \{d^{(n)}\}_{n\in \mathbb{N}} with \( \lim_{n\to \infty} \frac{d_{k}^{(n)}}{T(d^{(n)})} > 0 \) for each group k \in K. They proved that if all τ_a(·) are polynomials, then \( \lim_{n\to \infty} \text{PoA}(d^{(n)}) = 1 \), provided that \( \lim_{n\to \infty} T(d^{(n)}) = \infty \). Moreover, Colini et al. [6] further extended the study of [7] to light traffic, i.e., T(d) → 0.

All the results of [8], [6] and [7] are derived with the technique of regular
A different study was done by Wu et al. They assumed arbitrary user volume vector sequence \( \{d^{(n)}\}_{n \in \mathbb{N}} \) and aimed to explore properties of asymptotic well designed games, i.e., games in which the PoA tends to 1, as the total volume \( T(d) \) approaches infinity. They proposed the concept of scalable games, and proved that all scalable games are asymptotically well designed. They also proved that gaugeable games are special cases of scalable games w.r.t. the particular user volume vector sequences assumed by gaugeability. Moreover, they provided examples that are scalable, but not gaugeable.

In addition, Wu et al. made a detailed study of the case that all \( \tau_a(\cdot) \) are BPR-functions. They proved for this particular case that an SO profile is an \( \epsilon \)-approximate NE profile \(^{19}\) for a small \( \epsilon > 0 \), and the PoA equals \( 1 + O(T(d)^{-\gamma}) \), where \( \gamma \) is the common degree of the BPR-functions. This proves a conjecture proposed by O’Hare et al. \(^{15}\). They also proved for this particular case that the cost of both, an NE-profile and an SO-profile, can be asymptotically approximated by \( L(d) \cdot T(d)^{\gamma} \), where \( L(d) \) is a computable constant that only depends on distribution \( d := \left( d_k/T(d) \right)_{k \in K} \) of users among groups, when the total volume \( T(d) \) is large enough. However, Wu et al. still failed to show that NCGs with general polynomials are asymptotically well designed.

In summary, \(^{8},^{6},^{7}\) and \(^{28}\) definitely show that selfish behavior need not be bad for the case of a large \( T(d) \) under certain conditions. However, one important question has remained open, namely, whether NCGs with arbitrary polynomial price functions \( \tau_a(\cdot) \) are asymptotically well designed. Although \(^{6}\) and \(^{7}\) have preliminary results towards this question, their results only partially answer this question due to the sequence-specific nature of their study.

Polynomial functions are so popular because they usually serve as a first prototype to understand quantitative relations between variables. Price functions \( \tau_a(\cdot) \) are key components of an NCG and model the quantitative relations between resource prices and demanded volumes. NCGs with polynomial price functions \( \tau_a(\cdot) \) are thus of great importance in practice. The open question about the PoA for polynomials thus concerns properties of the selfish user behavior in such games for a large user volume \( T(d) \) and is thus of great interest to our purpose of understanding heavy traffic.
1.5 Our main result

We will continue the study of \cite{28}. However, to better understand the state of the art, we will first discuss the notions of scalability \cite{28} and gaugeability \cite{7}, and give a detailed description of the known results from \cite{28}, \cite{6} and \cite{7}, see Subsection 3.1.

We then apply the concepts of scalability and limit games stemming from \cite{28}. We first prove that if the limit game exists, then an NCG is asymptotically well designed if and only if it is scalable, see Theorem 1. This deepens the knowledge about scalability and asymptotically well designed games. For an even deeper understanding, we adapt some algebraic ideas to our analysis and consider decompositions of NCGs. We prove that the class of asymptotically well designed games is actually closed under direct sums, see Corollary 2. This demonstrates in a certain sense the extent of the notion of asymptotically designed games.

To obtain a general proof for the convergence of the PoA, we develop a new technique called \textit{asymptotic decomposition}. This technique generalizes the idea of direct sums, and is designed for handling general NCGs in the limit analysis. We are able to demonstrate its power by applying it to NCGs with arbitrary polynomial price functions and NCGs with regularly varying price functions, see Theorem 2, Theorem 3 and Theorem 4.

With the asymptotic decomposition, we are able to prove that NCGs with arbitrary polynomial price functions are asymptotically well designed, see Theorem 2. This completely solves the convergence of the PoA of NCGs with polynomial price functions, and thus the aforementioned open question. This result greatly extends the findings of \cite{28}, \cite{8}, \cite{7} and \cite{6} for road traffic, and deepens the understanding that selfishness need not be bad, and might be the best choice in a bad environment. Moreover, this result also indicates that selfish routing is actually not the main cause of congestion, when the total travel demand $T(d)$ is large. In particular, if the total travel demand stays high, then we cannot significantly reduce the average travel latency by any road guidance policies.

Theorem 2 also brings some insight into free market economics. In market economics, resources correspond to factors of production, groups correspond to sets of suppliers manufacturing a particular type of product, and resource prices $\tau_a(\cdot)$ are the purchasing prices of those production factors. In a free market system, the prices $\tau_a(\cdot)$ of production factors are completely determined by the demanded volumes, and are often assumed to be polynomial functions. Theorem 2 then shows that given the demand of each product
type, the free market will autonomously minimize the average manufacturing cost, when the total number of suppliers is large.

Asymptotic decomposition also applies to NCGs with price functions of other types. To demonstrate this, we also applied this technique to NCGs with regularly varying price functions. The result shows that these NCGs are in general also asymptotically well designed, see Theorem 3. In particular, with this technique, we are able to remove the sequence limitation for gaugeable games and generalize the main result Theorem 4.4 in [7] for gaugeability, see Theorem 4.

Theorem 2, Theorem 3 and Theorem 4 definitely demonstrate the power of asymptotic decomposition. They assume an arbitrary user volume vector sequence, and thus the results hold globally. In particular, together they constitute a general proof that selfishness need not be bad for NCGs. Their proofs are direct and very elementary without using any heavy machinery, and only use some basic properties of Nash equilibrium and system optimum profiles, simple facts about asymptotic notation $O(\cdot), \Omega(\cdot)$, etc, and a simple induction over the group set $\mathcal{K}$.

1.6 The structure of the paper

The remainder of the article is arranged as follows: Section 2 gives a detailed description of the NCG model that we will study. Section 3 gives a detailed description on known results and then reports our results. Section 4 gives a brief summary of the whole article. To improve readability, we move the elementary but long proofs of our results to the Appendix.

2 The Model

In our study, we follow the formulation of NCGs in [28]. This formulation is slightly different from the traditional model commonly used in the literature, see, e.g., [13]. Traditionally, a strategy is assumed to be a subset of resources. In our study, we employ a constant $r(a, s) \geq 0$ to reflect the relation between a resource $a \in A$ and a strategy $s \in S$. This slightly generalizes our results.

An NCG is represented by a tuple

$$\Gamma = (\mathcal{K}, A, S, (r(a, s))_{a \in A, s \in S}, (\tau_a)_{a \in A}, d),$$

where:
• $\mathcal{K}$ is a finite non-empty set of groups. We assume, w.l.o.g., that $\mathcal{K} = \{1, \ldots, K\}$, i.e., there are $K$ groups of users.

• $A$ is a finite non-empty set of resources that will be demanded by users engaged in the game.

• $\mathcal{S} = \bigcup_{k \in \mathcal{K}} \mathcal{S}_k$ is a finite non-empty set of available strategies. Herein, each $\mathcal{S}_k$ contains all strategies available to users in group $k$ for each $k \in \mathcal{K}$. We assume that $\mathcal{S}_k \cap \mathcal{S}_{k'} = \emptyset$, provided that $k \neq k'$, for each $k, k' \in \mathcal{K}$.

• $r(a, s) \geq 0$ denotes the demanded volume of resource $a$ by a user adopting strategy $s$, for each $a \in A$ and each $s \in \mathcal{S}$.

• $\tau_a : [0, +\infty) \mapsto [0, +\infty)$ denotes the price function of resource $a$ for each $a \in A$. We assume that each $\tau_a(x)$ is a continuous function that is non-negative and non-decreasing for all $x \geq 0$, for all $a \in A$.

• $d = (d_k)_{k \in \mathcal{K}}$ is a non-negative user volume vector, where each component $d_k \geq 0$ represents the volume of users belonging to group $k \in \mathcal{K}$.

In our study, we assume further that for each group $k \in \mathcal{K}$ and each strategy $s \in \mathcal{S}_k$,

$$\sum_{a \in \mathcal{S}_k} r(a, s) \cdot \tau_a(x) \neq 0. \quad (1)$$

Note that (1) is a reasonable assumption. Otherwise, there would be a group $k \in \mathcal{K}$, whose users can consume resources without paying any price. This actually conflicts with the spirit of congestion games in practice.

In an NCG, users usually want to adopt strategies that minimize their own cost. However, the cost of a user does not only depend on his/her choice, but also on the choices of other users, i.e., the cost of a user is eventually determined by the strategy profile formed by all users engaged in the game. Herein, a feasible strategy profile $f$ can be represented by a vector $f = (f_s)_{s \in \mathcal{S}}$, where:

p1) Each component $f_s \geq 0$ represents the total volume of users adopting strategy $s$, for each strategy $s \in \mathcal{S}$.

p2) $\sum_{s \in \mathcal{S}_k} f_s = d_k$, for each group $k \in \mathcal{K}$, which indicates that every user must choose a strategy to follow.
Consider a feasible strategy profile \( f = (f_s)_{s \in S} \). The demanded volume (or consumed volume) of each resource \( a \in A \) w.r.t. profile \( f \), denoted by \( f_a \), can be computed as

\[
f_a = \sum_{s \in S} r(a, s) \cdot f_s.
\]

Thus, the price of a resource \( a \in A \) w.r.t. profile \( f \), denoted by \( \tau_a(f_a) \), equals

\[
\tau_a(f_a) = \sum_{s \in S} r(a, s) \cdot \tau_a(f_s).
\]

Then, the cost of a strategy \( s \in S \) w.r.t. profile \( f \), denoted by \( \tau_s(f) \), equals

\[
\tau_s(f) = \sum_{a \in A} r(a, s) \cdot \tau_a(f_a).
\]

The average cost of users w.r.t. profile \( f \) equals

\[
C(f) := \frac{1}{T(d)} \cdot \sum_{s \in S} f_s \cdot \tau_s(f) = \frac{1}{T(d)} \cdot \sum_{a \in A} f_a \cdot \tau_a(f_a),
\]

where \( T(d) = \sum_{k \in K} d_k \) denotes the total volume of users in the game.

The selfishness of users will autonomously lead their choices to eventually form a feasible profile \( \tilde{f} = (\tilde{f}_s)_{s \in S} \) satisfying that

\[
\forall k \in K \forall s, s' \in S_k \left( \tilde{f}_s > 0 \implies \tau_s(f) \leq \tau_s'(\tilde{f}) \right),
\]

i.e., every user chooses a “cheapest” strategy he/she could follow w.r.t. profile \( \tilde{f} \). Such profiles are called Wardrop equilibria (WE, [27]). Under our assumption on the price functions \( \tau_a(\cdot) \), Wardrop equilibria coincide with the pure Nash equilibria (NE). A feasible strategy profile \( f = (f_s)_{s \in S} \) is said to be at NE, if

\[
\forall k \in K \forall s, s' \in S_k \left( f_s > 0 \implies \left( \forall \epsilon (f_s > \epsilon > 0) \implies \tau_s(f) \leq \tau_{s'}(f^{1+\epsilon}) \right) \right),
\]

where \( f^{1+\epsilon} = (f^{1+\epsilon}_s)_{s' \in S} \) is a feasible strategy profile that moves \( \epsilon \) users from strategy \( s \) to strategy \( s' \), i.e., for each strategy \( s'' \in S \),

\[
f_s^{1+\epsilon} = \begin{cases} f_s' & \text{if } s'' \notin \{s, s'\}, \\ f_s' - \epsilon & \text{if } s'' = s, \\ f_s' + \epsilon & \text{if } s'' = s'. \end{cases}
\]

In the sequel, we shall always put a tilde above a strategy profile, if the strategy profile is an NE profile (or, equivalently a Wardrop equilibrium).
An NE profile $\tilde{f}$ is a macro model for the selfish behavior of users in practice. Under our assumption on price functions $\tau_a(\cdot)$, all NE profiles $\tilde{f}$ have the same average cost, see e.g. [26] for a proof. Obviously, these profiles are user “optimal” (2), and stable (3) to some extent.

Besides NE profiles, system optimum (SO) profiles are also of great interest, for the sake of achieving social welfare. Formally, a feasible strategy profile $f^* = (f^*_s)_{s \in S}$ is an SO profile if it solves the following program:

$$\begin{align*}
\min & \quad C(f) \\
\text{s.t.} & \quad \sum_{s \in S_k} f_s = d_k, \forall k \in K, \\
& \quad f_s \geq 0, \forall s \in S.
\end{align*}$$

(4)

In the sequel, we shall always use a star in the superscript of a feasible profile to indicate that it is an SO profile.

In general, an NE profile need not be a solution to the program (4). The PoA is a popular index to show the extent to which the selfish user behavior destroys social welfare in practice. It is a concept stemming from [16], and can be defined as follows

$$\text{PoA} := \frac{C(\tilde{f})}{C(f^*)} = \frac{\sum_{s \in S} \tilde{f}_s \cdot \tau_s(\tilde{f})}{\sum_{s \in S} f^*_s \cdot \tau_s(f^*)},$$

(5)

where $\tilde{f}$ is an NE profile, and $f^*$ is an SO profile.

As mentioned, we will investigate the limit of the PoA when the total volume $T(d) = \sum_{k \in K} d_k$ approaches infinity. Therefore, to avoid ambiguity, we shall denote by PoA($d$) the corresponding PoA for user volume vector $d = (d_k)_{k \in K}$ in the sequel.

3 Selfishness need not be bad: a general discussion

In this Section, we consider the limit of the PoA under our assumption of continuous, non-decreasing and non-negative price functions $\tau_a(\cdot)$. In particular, we will emphasize on the polynomial functions and regularly varying functions that have been recently studied in [28], [8], [6] and [7]. To better understand our result, we first introduce some relevant concepts and results from [28] and [7].
3.1 Scalability and gaugeability

NCGs are static models for decision-making behavior of selfish users (players) in systems with limited resources. Designing an NCG such that the selfish choices of users autonomously optimize social welfare is in general not easy, see e.g. [20]. However, such games exist, see [28].

Definition 1 (See [28]). An NCG $\Gamma$ is said to be a well designed game (WDG), if $\text{PoA}(d) = 1$ for each given user volume vector $d = (d_k)_{k \in K}$ with total user volume $T(d) > 0$.

Obviously, selfish behavior of users in a WDG should be strongly favored, as it leads the underlying system into a steady state with minimum average cost. Readers may refer to [28] for examples of WDGs.

As mentioned, WDGs are often too restrictive for designing them in practice. Nevertheless, an important goal in NCGs concerns how to effectively allocate limited resources to a large volume of users. Therefore, an alternative choice to designing a WDG is to design an NCG that will approximate a WDG when the total user volume $T(d)$ becomes large. This inspires the concept of an asymptotically well designed game (AWDG) [28].

Definition 2 (see [28]). An NCG $\Gamma$ is said to be an AWDG, if the $\text{PoA}(d)$ converges to 1 as $T(d)$ approaches infinity. For later use, we also denote the class of all AWDGs as AWDG.

Scalable games introduced by Wu et al. [28] are examples of AWDGs. These games require the existence of a well designed limit game for each sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$ of user volume vectors with

$$\lim_{n \to \infty} T(d^{(n)}) = \sum_{k \in K} d_k^{(n)} = \infty,$$

where $d_k^{(n)}$ is the $k$-th component of $d^{(n)} = (d_k^{(n)})_{k \in K}$ and denote the user volume in group $k$ for each $k \in K$ and each $n \in \mathbb{N}$.

Definition 3. Given a sequence $\{d^{(n)} = (d_k^{(n)})_{k \in K}\}_{n \in \mathbb{N}}$ of user volume vectors with $\lim_{n \to \infty} T(d^{(n)}) = \lim_{n \to \infty} \sum_{k \in K} d_k^{(n)} = \infty$, an NCG

$$\Gamma^\infty = (K^\infty, A, S^\infty, (r(a,s))_{a \in A, s \in S^\infty}, (\tau_a^\infty)_{a \in A}, d)$$

is called a limit of the NCG

$$\Gamma = (K, A, S, (r(a,s))_{a \in A, s \in S}, (\tau_a)_{a \in A}, d)$$
w.r.t. the user volume vector sequence \(\{d^{(n)}\}_{n \in \mathbb{N}}\), if there exists an infinite subsequence \(\{n_i\}_{i \in \mathbb{N}}\) such that:

L1) For each \(k \in \mathcal{K}\),
\[
\lim_{i \to \infty} \frac{d^{(n_i)}_k}{T(d^{(n_i)})} = d_k,
\]
where \(d_k \in [0, 1]\) is the limit volume of group \(k\).

L2) There exists a sequence \(\{g_i\}_{i \in \mathbb{N}}\) of positive scaling factors, such that
\[
\lim_{i \to \infty} \frac{\tau_a(T(d^{(n_i)})x)}{g_i} = \tau^\infty_a(x)
\]
for all \(x > 0\), where \(\tau^\infty_a(\cdot)\) is the limit price of resource \(a\), for each \(a \in A\).

L3) Each limit price function \(\tau^\infty_a(\cdot)\) is either a continuous and non-decreasing function, or \(\tau^\infty_a(x) \equiv \infty\) for all \(x > 0\), for each \(a \in A\). And for each group \(k \in \mathcal{K}\),

L3.1) either group \(k\) is negligible w.r.t. scaling factors \(\{g_i\}_{i \in \mathbb{N}}\), i.e.,
\[
\lim_{i \to \infty} \sum_{s \in S_k} f^{(n_i)}_s \cdot \frac{\tau_a(f^{(n_i)})}{T(d^{(n_i)}) \cdot g_i} = 0,
\]
where each \(f^{(n_i)}\) is an arbitrary feasible strategy profile of \(\Gamma\) w.r.t. user volume vector \(d^{(n_i)}\) for each \(i \in \mathbb{N}\),

L3.2) or there exists a strategy \(s \in S_k\) that is tight w.r.t. scaling factors \(\{g_i\}_{i \in \mathbb{N}}\), i.e., \(\tau^\infty_a(x) \not\equiv \infty\) for \(x > 0\), for each resource \(a \in A\) with \(r(a,s) > 0\).

L4) Put
\[
S^\infty := \{s \in S : s \text{ is tight w.r.t. } \{g_i\}_{i \in \mathbb{N}}\},
\]
\[
\mathcal{K}^\infty := \{k \in \mathcal{K} : k \text{ is not negligible, or } S_k \cap S^\infty \neq \emptyset\}.
\]

The cost of NE profiles of the limit game \(\Gamma^\infty\) is positive w.r.t. the limit user volume vector \((d_k)_{k \in \mathcal{K}^\infty}\).

**Definition 4** (See also [28]). An NCG \(\Gamma\) is called a scalable game if, for each user volume sequence \(\{d^{(n)}\}_{n \in \mathbb{N}}\) with total volume \(T(d^{(n)}) \to \infty\) as \(n \to \infty\), there is a well designed game \(\Gamma^\infty\) that is a limit of \(\Gamma\) w.r.t. the user volume sequence \(\{d^{(n)}\}_{n \in \mathbb{N}}\).
Conditions L3) and L4) of Definition 3 are imposed to guarantee that the cost of NE profiles in the limit game $\Gamma^\infty$ will be neither unbounded, nor vanishing w.r.t. the scaling factors $\{g_i\}_{i \in \mathbb{N}}$. Therefore, the scaling factors $\{g_i\}_{i \in \mathbb{N}}$ should be carefully chosen with reference to the sequence $\{T(d^{(n)})\}_{n \in \mathbb{N}}$ of user volume vectors and price functions $\tau_a(\cdot)$, so as to fulfill these conditions.

Note that limit games only consider tight strategies $s \in S^\infty$ and “non-negligible” groups $k \in K^\infty$. This is actually reasonable, since users will asymptotically adopt only tight strategies w.r.t. both, NE profiles and SO profiles, see the proofs of Lemma 1 and Theorem 1 in the Appendix.

Condition L2) of Definition 3 can be further relaxed. Note that for each resource $a \in A$, only those $x > 0$ make sense that are possible consumed volumes of resource $a$ w.r.t. the limit $\Gamma^\infty$. We can therefore require only that the limit price functions $\tau_a^\infty(x)$ exist for $x \in I_a \cap (0, \infty)$, where $I_a$ is a non-empty set containing all the possible consumed volumes of resource $a$ w.r.t. the limit game $\Gamma^\infty$, for each $a \in A$. See [28] for details.

Note that it is possible that there are several limit games for a given user volume sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$, see the following example.

Example 1. Consider the NCG $\Gamma$ shown in Figure 2. The game has two groups (OD pairs), each of which has two strategies (two parallel paths). The price functions are listed above the corresponding paths. Let $\{d^{(n)} = (d_1^{(n)}, d_2^{(n)})\}_{n \in \mathbb{N}}$ be a user volume vector sequence such that

$$d_1^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}, \end{cases} \quad d_2^{(n)} = \begin{cases} n & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

where $d_1^{(n)}$ denotes user volume of the upper group, and $d_2^{(n)}$ denotes user volume of the lower group, for each $n \in \mathbb{N}$. Obviously, w.r.t. subsequence
\( \{2i\}_{i \in \mathbb{N}} \) and scaling factor sequence \( \{g_i = i\}_{i \in \mathbb{N}} \), the lower group is negligible since \( d_2^{(2i)} \equiv 0 \) for all \( i \in \mathbb{N} \). Moreover, limit price functions

\[
\lim_{i \to \infty} \frac{2 \cdot (2i \cdot x) + 1}{2i} = 2x \quad \text{and} \quad \lim_{i \to \infty} \frac{3 \cdot (2i \cdot x) + 1}{2i} = 3x
\]

exist and are continuous and non-decreasing for all \( x > 0 \).

Furthermore, the NCG game \( \Gamma_1^\infty \) consisting of these two limit price functions and the upper group has positive average cost for NE profiles w.r.t. the limit user volume vector \( (d_i) \), where \( d_1 = 1 \). Thus, \( \Gamma_1^\infty \) is a limit game of \( \Gamma \) w.r.t. the given user volume vector sequence \( \{d^{(n)}\}_{n \in \mathbb{N}} \).

Similarly, considering the subsequence \( \{2i + 1\}_{i \in \mathbb{N}} \) and scaling factor sequence \( \{g_i = (2i + 1)^2\}_{i \in \mathbb{N}} \), we can define another limit game \( \Gamma_2^\infty \) consisting of the lower group and two price functions \( 4x, 5x \), respectively.

Obviously, these two limit games \( \Gamma_1^\infty \) and \( \Gamma_2^\infty \) are different. However, both of them are limits of the given NCG \( \Gamma \) w.r.t. the user volume sequence \( \{d^{(n)}\}_{n \in \mathbb{N}} \).

Wu et al. \cite{28} proved that for an NCG \( \Gamma \) and a given user volume vector \( \{d^{(n)}\}_{n \in \mathbb{N}} \) with total volume \( T(d^{(n)}) \to \infty \) as \( n \to \infty \), if \( \Gamma^\infty \) is the limit of \( \Gamma \) for an infinite subsequence \( \{n_i\}_{i \in \mathbb{N}} \) and a scaling factor sequence \( \{g_i\}_{i \in \mathbb{N}} \), then the average cost of NE profiles normalized by the scaling factor sequence \( \{g_i\}_{i \in \mathbb{N}} \) converges to the total cost of NE profiles of \( \Gamma^\infty \).

**Lemma 1.** Consider an NCG

\[
\Gamma = (K, A, S, (r(a, s))_{a \in A, s \in S}, (\tau_a)_{a \in A}, d),
\]

in which all price functions \( \tau_a(\cdot) \) are non-negative, non-decreasing and continuous. Let \( \{d^{(n)}\}_{n \in \mathbb{N}} \) be an arbitrary sequence of user volume vectors such that the total volume \( T(d^{(n)}) = \sum_{k \in K} d_k^{(n)} \to \infty \) as \( n \to \infty \), where each \( d^{(n)} = (d_k^{(n)})_{k \in K} \) for each \( n \in \mathbb{N} \). Let \( f^{(n)} = (f_s^{(n)})_{s \in S} \) be an NE profile of \( \Gamma \) for the user volume vector \( d^{(n)} \), for each \( n \in \mathbb{N} \). If \( \Gamma \) has a limit

\[
\Gamma^\infty = (K^\infty, A, S^\infty, (r(a, s))_{a \in A, s \in S^\infty}, (\tau_a^\infty)_{a \in A}, d)
\]

for the user volume vector sequence \( \{d^{(n)}\}_{n \in \mathbb{N}} \), then there exists an infinite subsequence \( \{n_i\}_{i \in \mathbb{N}} \) and a sequence \( \{g_i\}_{i \in \mathbb{N}} \) of positive numbers such that

\[
\lim_{i \to \infty} \frac{C(f^{(n_i)})}{g_i} = \sum_{s \in S^\infty} \hat{f}_s^\infty \cdot \tau_s^\infty(\hat{f}^\infty),
\]

where \( \hat{f}^\infty = (\hat{f}_s)_{s \in S^\infty} \) is some NE profile of the limit game \( \Gamma^\infty \).
Proof. See the proof of Theorem 3.2 in [28], or the appendix for an alternative proof.

Using Lemma 1, Wu et al. [28] then proved that all scalable games are asymptotically well designed. In that proof, the condition that the limit game is well designed plays a pivotal role, which actually implies that the average costs of NE profiles is asymptotically not larger than the average costs of SO profiles. We summarize this in Lemma 2.

Lemma 2. Every scalable game is an AWDG.

Proof. See the proof of Theorem 3.2 in [28] for details.

Theorem 1 below continues the study of Wu et al. [28]. It states that scalable games and AWDG's actually coincide when they have a limit game. Moreover, we showed in the proof of Theorem 1 that users will asymptotically adopt only tight strategies w.r.t. SO profiles, and Lemma 1 also applies to SO profiles. Therefore, it is reasonable to consider only tight strategies in limit games.

Theorem 1. Consider an NCG $\Gamma$ and a user volume vector $\{d^{(n)}\}_{n \in \mathbb{N}}$ with the total volume $T(d^{(n)}) \to \infty$ as $n \to \infty$. If $\Gamma$ has a limit game $\Gamma^\infty$ w.r.t. the given user volume vector, and the limit game $\Gamma^\infty$ is not well designed, then $\Gamma$ is not an AWDG.

Proof. See the Appendix.

Specific for polynomial price functions, Theorem 3.2 from Wu et al. [28] directly yields that NCGs with polynomial price functions $\tau_a(\cdot)$ of the same degree are asymptotically well designed. In that case, we can take scaling factors $g_n = T(d^{(n)})^\gamma$ for each given user volume vector sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$ with $T(d^{(n)}) \to \infty$ as $n \to \infty$, where $\gamma \geq 0$ is the common degree of all polynomials. Then, the corresponding limit game is well designed.

To show this for general polynomial price functions $\tau_a(\cdot)$ with different degrees, we will use some helpful notations from [6]. For each resource $a \in A$, let $\rho_a(\cdot)$ be the degree of polynomial $\tau_a(\cdot)$. Put $\rho_s = \max\{\rho_a : r(a, s) > 0, a \in A\}$ and $\rho_k = \min\{\rho_s : s \in S_k\}$ for each $s \in S_k$ and $k \in K$. If $\rho_k = \rho_l$ for all $k, l \in K$, then the above argument shows that the underlying NCG is scalable, and therefore asymptotically well designed. We summarize this in Corollary [1].
**Corollary 1.** Consider an NCG $\Gamma$ with polynomial price functions $\tau_a(\cdot)$. If $\rho_k = \rho_l$ for all $k, l \in K$, then $\Gamma$ is asymptotically well designed.

**Proof.** Let $\{d^{(n)}\}_{n \in \mathbb{N}}$ be an arbitrary user volume vector such that $T(d^{(n)}) \to \infty$ as $n \to \infty$. Let $\rho = \rho_k$ for some $k \in K$. Put $g_n = T(d^{(n)})^\rho$ for each $n \in \mathbb{N}$. By assumption (1), one can then easily show that the limit of $\Gamma$ w.r.t. sequence $\{n\}_{n \in \mathbb{N}}$ and the scaling factor sequence $\{g_n\}_{n \in \mathbb{N}}$ is well designed. \( \square \)

The result of Wu et al. [28] does not directly apply if $\rho_k \neq \rho_l$ for some $l, k \in K$. The reason is that, in this case, there need not exist a unified limit game for all groups for some user volume vector sequences $\{d^{(n)}\}_{n \in \mathbb{N}}$. We thus need additional arguments in this case and leave the proof of this case to Subsection 3.3.

To better understand the current state of the art, we now introduce some relevant results from [6] and [7]. They employed an alternative technical path to prove the convergence of the PoA. They introduced the so-called gaugeable games, which consider only particular sequences of user volume vectors. Gaugeability is a concept based on the notion of regular variation [3]. A non-negative function $g(\cdot)$ is said to be regularly varying, if the limit

$$
\lim_{t \to \infty} \frac{g(t \cdot x)}{g(t)} = q(x) \in (0, \infty)
$$

exists for all $x > 0$.

**Definition 5** (See also [7]). An NCG $\Gamma$ is said to be gaugeable for a user volume vector $\{d^{(n)}\}$, if there exists a regularly varying function $g(\cdot)$ such that:

\begin{itemize}
  \item [G1)] The limit
  $$
  \lim_{n \to \infty} \frac{\tau_a(x)}{g(x)} =: q_a \in [0, \infty]
  $$
  exists for all resource $a \in A$.

  \item [G2)] For each group $k \in K$, there exists a strategy $s \in S_k$ such that
  $$
  q_a < \infty, \quad \text{for all resource } a \in A \text{ with } r(a, s) > 0.
  $$

  \item [G3)] The lower limit
  $$
  \lim_{n \to \infty} \sum_{k \in K_{\text{tight}}} \frac{T_k(d^{(n)})}{T(d^{(n)})} > 0,
  $$
\end{itemize}

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where $\mathcal{K}_{\text{tight}}$ is the set of all tight groups, and a group $k$ is said to be tight in such a case if

\[
0 < \min_{s \in S_k} \max\{q_a : r(a, s) > 0, a \in A\} < \infty.
\]

With the technique of regular variation [3], Colini et al. [7] proved that for each NCG $\Gamma$, if $\Gamma$ is gaugeable w.r.t. a user volume vector $\{d(n)\}_{n \in \mathbb{N}}$, then $\text{PoA}(d(n)) \to 1$ as $n \to \infty$, see Theorem 4.4 in [7]. Here, we recall that $\text{PoA}(d(n))$ denotes the price of anarchy w.r.t. user volume vector $d(n)$, for each $n \in \mathbb{N}$.

In fact, if $\Gamma$ is gaugeable w.r.t. a user volume vector $\{d(n)\}_{n \in \mathbb{N}}$, then $\Gamma$ has a well designed limit w.r.t. that user volume vector sequence. Let $g(\cdot)$ be the required regularly varying function in Definition 5. By $G_1$), the limit price functions

\[
\tau_a^\infty(x) = \lim_{n \to \infty} \frac{\tau_a(T(d(n))x)}{g_n} = q_a \cdot x^\rho
\]

exist, where the scaling factors $g_n = g(T(d(n)))$, and $\rho \geq 0$ is the regular variation index of $g$ in Karamata’s Characterization Theorem [3]. Moreover, by $G_2$) and $G_3$), one can easily check that the limit game consisting of groups $\mathcal{K}^\infty = \mathcal{K}$ and price functions $\tau_a^\infty$ is well designed. See Wu et al. [28] for details. Therefore, gaugeable games are special cases of scalable games. To explicitly show this, we define the “sequential counterpart” of scalable games in a natural way.

**Definition 6.** Consider an NCG $\Gamma$ and some user volume vector $\{d(n)\}_{n \in \mathbb{N}}$ with $T(d(n)) \to \infty$ as $n \to \infty$. $\Gamma$ is said to be scalable w.r.t. the sequence $\{d(n)\}_{n \in \mathbb{N}}$, if for each infinite subsequence $\{n_i\}_{i \in \mathbb{N}}$, $\Gamma$ has a well designed limit w.r.t. the subsequence $\{d(n_i)\}_{i \in \mathbb{N}}$.

Obviously, an NCG $\Gamma$ is scalable if and only if $\Gamma$ is scalable w.r.t. each user volume sequence $\{d(n)\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} T(d(n)) = \infty$. However, if $\Gamma$ is only scalable w.r.t. some user volume sequence $\{d(n)\}_{n \in \mathbb{N}}$, then $\Gamma$ need not to be globally scalable. Nevertheless, Lemma 3 below states that this restricted notion of scalability already generalizes the gaugeability of [7].

**Lemma 3.** Consider an NCG $\Gamma$ and some user volume vector $\{d(n)\}_{n \in \mathbb{N}}$ with $T(d(n)) \to \infty$ as $n \to \infty$. If $\Gamma$ is gaugeable w.r.t. the sequence $\{d(n)\}_{n \in \mathbb{N}}$, then $\Gamma$ is also scalable w.r.t. that user volume vector sequence.

**Proof.** See the proof of Corollary 3.1 in Wu et al. [28] for details. \qed
The difference between scalable games and gaugeable games are therefore obvious. Scalable games consider arbitrary user volume vector sequence, while gaugeable games consider particular user volume vector sequence satisfying \((7)\). Thus, the convergence result of the PoA for scalable games is global, while that for gaugeable games holds only locally.

Actually, scalability is more general than gaugeability even for a specific user volume vector sequence \(\{d(n)\}_{n \in \mathbb{N}}\). Gaugeability requires that there exists a uniform regularly varying function \(g(\cdot)\) for the whole sequence that fulfills conditions \(G1\)-\(G3\). As shown above, this results in the same well designed limit game for every subsequence of \(\{d(n)\}_{n \in \mathbb{N}}\). However, scalability allows different subsequences of \(\{d(n)\}_{n \in \mathbb{N}}\) to have different well designed limit games. The NCG \(\Gamma\) in Example 1 has two limit games w.r.t. the given user volume vector sequence, and both of them are well designed, see Wu et al. \[28\] for details. We summarize this in Lemma 4.

**Lemma 4.** There exists an NCG \(\Gamma\) and a user volume vector sequence \(\{d(n)\}_{n \in \mathbb{N}}\) with \(T(d(n)) \to \infty\) as \(n \to \infty\), such that \(\Gamma\) is scalable w.r.t. \(\{d(n)\}_{n \in \mathbb{N}}\), but not gaugeable w.r.t. \(\{d(n)\}_{n \in \mathbb{N}}\).

**Proof.** See the proof of Theorem 3.4 in Wu et al. \[28\], or Example 1. \(\square\)

Now, let us return to the discussion of NCGs with arbitrary polynomial price functions \(\tau_a(\cdot)\). By considering a particular user volume vector sequence \(\{d(n)\}_{n \in \mathbb{N}},\) Colini et al. \[6\] proved that \(\text{PoA}(d(n)) \to 1\), as \(n \to \infty\), see also Corollary 4.7 in \[7\]. They assumed that for each \(k \in \mathcal{K}\),

\[
\lim_{n \to \infty} \frac{d_k(n)}{T(d(n))} > 0.
\]

(8)

Obviously, \((8)\) fulfills \((7)\). Let \(g(x) = T(d(n))^{\rho}\), where \(\rho = \max\{\rho_k : k \in \mathcal{K}\}\). They actually proved that if \(\{d(n)\}_{n \in \mathbb{N}}\) satisfies \((8)\), then the underlying NCG is gaugeable w.r.t. \(\{d(n)\}_{n \in \mathbb{N}}\) and regularly varying function \(g(\cdot)\). We summarize this in Lemma 5.

**Lemma 5.** Consider an NCG \(\Gamma\) with polynomial price functions \(\tau_a(\cdot)\), and a user volume vector sequence \(\{d(n)\}_{n \in \mathbb{N}}\) such that \(T(d(n)) \to \infty\) as \(n \to \infty\), and satisfies \((8)\). Then, \(\Gamma\) is scalable w.r.t. \(\{d(n)\}_{n \in \mathbb{N}},\) i.e., \(\text{PoA}(d(n)) \to 1\) as \(n \to \infty\).

**Proof.** The proof follows immediately from Lemma 3. \(\square\)
Although Lemma 5 shows an inspiring result for general polynomial price functions, we can not directly conclude that NCGs with polynomial price functions are asymptotically well designed, due to the sequence-specific nature of Lemma 5. To show that NCGs with polynomial price functions are asymptotically well designed, we still need a more sophisticated analysis. Motivated by Lemma 3, Lemma 4, and Example 1, we will use the idea of scalability for a general proof.

3.2 AWDG is closed under direct sum

A direct application of scalability does not lead to a general proof. To see this, consider again Example 1, but now with user volume vector sequence $d^n = (d_1^n = n^2, d_2^n = n)$. In this case, the two groups are mutually non-negligible, and we cannot find a suitable scaling factor sequence that results in a well designed limit game. However, a closer inspection shows that either of the two groups has its own well designed limit game w.r.t. the given user volume sequence. This inspires us to consider the two groups separately.

**Definition 7.** An NCG $\Gamma$ is said to have mutually disjoint groups (MDGs), if

$$\sum_{k \in \mathcal{K}} \mathbb{1}_{\{x > 0\}}(r(a, s_k)) \leq 1, \quad (9)$$

for each $a \in A$, for each $K$-dimensional strategy vector $(s_1, \ldots, s_K) \in \prod_{k \in \mathcal{K}} S_k$, where $\mathbb{1}_{\{x > 0\}}(\cdot)$ is the indicator function of set $\{x : x > 0\}$.

Condition (9) in Definition 7 implies that users from different groups cannot share resources. Therefore, users from different groups of an NCG with MDGs will not affect each other when they determine strategies to follow. This means that each group in an NCG with MDGs forms an independent subgame. Let

$$\Gamma = (\mathcal{K}, A, S, (r(a, s))_{a \in A, s \in S}, (\tau_a)_{a \in A}, d)$$

be an NCG with MDGs. For each group $k \in \mathcal{K}$, we denote by

$$\Gamma|_k = (\{k\}, A, S, (r(a, s))_{a \in A, s \in S_k}, (\tau_a)_{a \in A}, (d_k))$$

the $k$-marginal of $\Gamma$, and denote by $f|_k = (f_s)_{s \in S_k}$ the $k$-marginal of a feasible strategy $f = (f_s)_{s \in S}$.  

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Lemma 6. Consider an NCG $\Gamma$ with MDGs. Then, $f^* = (f^*_s)_{s \in S}$ is an SO profile of $\Gamma$ if and only if the $k$-th marginal profile $f^*|_k = (f^*_s)_{s \in S_k}$ is an SO profile of the $k$-marginal game $\Gamma|_k$, for each $k \in K$. This holds similarly for NE profiles.

Proof. Trivial. \qed

With Lemma 6 and by applying scalability to each of the mutually independent marginals, we can easily derive that NCGs with mutually disjoint and scalable marginals are asymptotically well designed.

Lemma 7. Consider an NCG $\Gamma$ with MDGs. If all the marginals are scalable, then $\Gamma$ is asymptotically well designed.

Proof. This is easy by observing the trivial facts that

$$\text{PoA}(d) = \frac{C(\hat{f})}{C(f^*)} = \frac{\sum_{k \in K} \sum_{s \in S_k} \hat{f}_s \cdot \tau_s(\hat{f})}{\sum_{k \in K} \sum_{s \in S_k} f^*_s \cdot \tau_s(f^*)}$$

and that each marginal is scalable, and thus the marginal PoA

$$\frac{\sum_{s \in S_k} \hat{f}_s \cdot \tau_s(\hat{f})}{\sum_{k \in K} \sum_{s \in S_k} f^*_s \cdot \tau_s(f^*)},$$

tends to 1, as $d_k \to \infty$, for each $k \in K$. Herein we recall assumption (1) that there are no free strategies, and that

$$\tau_s(\hat{f}) = \tau_s(\hat{f}|_k) \quad \text{and} \quad \tau_s(f^*) = \tau_s(f^*_|_k),$$

if $s \in S_k$, for each $k \in K$. \qed

Combining Corollary 1 and Lemma 7, we obtain immediately that NCGs with MDGs and polynomial price functions $\tau_a(\cdot)$ are asymptotically well designed.

A direct extension of Lemma 7 considers the direct sum of asymptotically well designed games. Let

$$\Gamma_l = \left( K_l, A_l, S_l, (r(a,s))_{a \in A_l, s \in S_l}, (\tau_a)_{a \in A_l}, d(l) \right)$$

be an NCG, for $l = 1, \ldots, m$, such that $A_1, \ldots, A_m$ are mutually disjoint. Then, we call the game

$$\left( \bigcup_{l=1}^m K_l, \bigcup_{l=1}^m A_l, \bigcup_{l=1}^m S_l, (r(a,s))_{a \in \bigcup_{l=1}^m A_l, s \in \bigcup_{l=1}^m S_l}, (\tau_a)_{a \in \bigcup_{l=1}^m A_l}, \bigcup_{l=1}^m d(l) \right)$$

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the direct sum of $\Gamma_1, \ldots, \Gamma_m$, denoted by $\oplus_{l=1}^m \Gamma_l$, where $\bigcup_{l=1}^m d(l)$ means the concatenation of vectors $d(1), \ldots, d(m)$. Obviously, direct sums of asymptotically well designed games are again asymptotically well designed.

**Corollary 2.** The class AWDG is closed under direct sums.

**Proof.** Trivial. \qed

Corollary 2 suggests a possible approach to check whether an NCG is asymptotically well designed. One can try to decompose the underlying NCG into a direct sum of several independent marginals, and then check the scalability of each marginal. Here, we allow compound marginals, i.e., each marginal can contain more than one group. The independence between them then means that users from different marginals do not affect each other when they determine strategies to follow.

However, in general, it could be difficult to find such a decomposition, since different groups might compete for the same resources. The above discussion has already shown that if collisions are heavy or slight, then it is easy to check whether the underlying NCG is asymptotically well designed. In fact, if all groups heavily collide on resources, i.e., every resource can be used by all groups, then the game is not decomposable and Lemma 2 applies, see e.g., Corollary 1. On the other hand, if groups do not collide on resources, i.e., if they can be partitioned into several mutually disjoint classes w.r.t. the use of resources, then the game is decomposable and Corollary 2 may apply.

The above two cases are, somehow, regular. Below, we consider the case of irregular collisions, which might be the general case in practice.

### 3.3 Asymptotic decomposition: a general proof for polynomial price functions

In general, it may be difficult to directly apply the idea of scalability, since there need not exist a unified well designed game for all groups for some user volume vector sequence. Moreover, it may also be difficult to directly decompose the game, due to irregular collisions of groups on resources. If this is the case, then one may consider an asymptotic decomposition. The idea is similar to direct sums. However, one needs to additionally deal with the problems caused by the irregular collisions.

An asymptotic decomposition is based on a suitable partition of the group set $K$. Consider an arbitrary sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$ of user volume vectors such
that $T(d^{(n)}) \to \infty$ as $n \to \infty$. The decomposition aims to eventu ally partition $\mathcal{K}$ into $\mathcal{K}_0, \ldots, \mathcal{K}_t$, for some integer $t \geq 0$, such that $\mathcal{K} = \bigcup_{m=0}^t \mathcal{K}_m$, and

$$\lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^n \mathcal{K}_u} \sum_{s \in \mathcal{S}_k} f^{(n)}_s \cdot \tau_{s}(\hat{f}^{(n)})}{\sum_{k \in \bigcup_{u=0}^n \mathcal{K}_u} \sum_{s \in \mathcal{S}_k} f^{(n)}_s \cdot \tau_{s}(f^{(n)})} = 1,$$  \tag{10}

for each $m = 0, \ldots, t$, where $f^{(n)}_s$, $\hat{f}^{(n)}_s$ are SO and NE profiles w.r.t. $d^{(n)}$, respectively, for each $n \in \mathbb{N}$. Obviously, if such partition can indeed be constructed, then the underlying NCG is well designed, due to the arbitrary choice of $\{d^{(n)}\}_{n \in \mathbb{N}}$.

One can try to construct the partition inductively. In the beginning of each inductive step $l = 0, \ldots, t$, we assume that we have already constructed classes $\mathcal{K}_0, \ldots, \mathcal{K}_{l-1}$, such that $\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \ldots \subseteq \mathcal{K}_{l-1} \subseteq \mathcal{K}$, and (10) holds for step $m = l - 1$, where we employ the convention that $\mathcal{K}_{-1} = \emptyset$, $0 = 1$, and “$\mathcal{K}_0, \mathcal{K}_{-1}$” means “$\emptyset$”. The objective of step $l$ then is to construct class $\mathcal{K}_l \subseteq \mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u$ such that (10) holds again for $m = l$.

To construct $\mathcal{K}_l$, one can inspect the remaining groups more closely, and pick those groups $k \in \mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u$ that have a non-vanishing limit proportion in the remaining total user volume $T_l(d^{(n)}) := T(d^{(n)}) - \sum_{k \in \bigcup_{u=0}^{l-1} \mathcal{K}_u} d^{(n)}_k$, since these groups are most significant in the limit. To show (10) for $m = l$, one needs to argue that these groups are either asymptotically independent of the groups that have been considered before, or negligible compared to them.

To that end, one needs to suitably estimate the costs of users w.r.t. NE profiles $f^{(n)}_s$ and SO profiles $f^{(n)}_s$, respectively. By comparing the cost of users from groups $k \in \mathcal{K}_l$ with those from groups $k' \in \bigcup_{u=0}^{l-1} \mathcal{K}_u$, one can then learn whether groups $k \in \mathcal{K}_l$ are negligible. If they are negligible, then (10) holds trivially for $m = l$. Otherwise, groups $k \in \mathcal{K}_l$ will be asymptotically independent of groups $k' \in \bigcup_{u=0}^{l-1} \mathcal{K}_u$, since the cost of users from groups $k' \in \bigcup_{u=0}^{l-1} \mathcal{K}_u$ will be negligible compared with the cost of users from groups $k \in \mathcal{K}_l$. If this is the case, one can then check the scalability of groups from $\mathcal{K}_l$ under the condition that users from groups $k' \in \bigcup_{u=0}^{l-1} \mathcal{K}_u$ adopt strategies that they used in NE profiles $\hat{f}^{(n)}_s$ and SO profiles $f^{(n)}_s$, respectively. Moreover, if these groups are scalable, then (10) follows immediately for $m = l$.

This procedure can tactically avoid the impact of possible irregular collisions in the limit analysis by comparing the costs of users from different classes $\mathcal{K}_l$. If the above partition can be constructed, then the underlying game decomposes into several asymptotically independent subgames corresponding to the partition $\mathcal{K}_0, \ldots, \mathcal{K}_t$. Although these subgames share re-
source set $A$, they are asymptotically independent, since the choices of users from one subgame will be asymptotically independent of those from other subgames.

Asymptotic decomposition can be successfully applied to NCGs with arbitrary polynomial price functions, which directly implies that NCGs with arbitrary polynomial price functions are asymptotically well designed. Theorem 2 summarizes this result. We move the detailed decomposition procedure to the Appendix.

**Theorem 2.** Consider an NCG $\Gamma$ with polynomial price functions $\tau_a(\cdot)$ such that each $\tau_a(x)$ is non-negative and non-decreasing for all $x \geq 0$ for $a \in A$. Then, $\Gamma$ is asymptotically decomposable, and thus asymptotically well designed.

**Proof.** See the appendix.

Different from the direct sum in Subsection 3.2, an SO profile need not be locally a system optimum w.r.t. some “marginals” corresponding to the partition in the asymptotic decomposition. This introduces extra difficulties in the application of scalability. The proof of Theorem 2 overcomes them by considering SO profiles as NE profiles w.r.t. price functions $c_a(x) := x\tau'_a(x) + \tau_a(x)$, where each $\tau'_a(\cdot)$ is the derivative function of $\tau_a(\cdot)$. Note that NE profiles are still at (pure) Nash equilibrium w.r.t. each marginal under the condition that users from other marginals adhere to the strategies they used in corresponding NE profiles.

The proof of Theorem 2 is based on three basic properties of polynomial functions. The first is that polynomial functions can be asymptotically sorted according to their degrees, which forms a base for the cost comparison and the construction of scaling factors at each inductive step. The second is that the price functions $c_a(x) = x\tau'_a(x) + \tau_a(x)$ are comparable with the price functions $\tau_a(x)$, i.e., $\lim_{x \to \infty} \frac{c_a(x)}{\tau_a(x)} = q_a$ for some constant $q_a \in (0, \infty)$, when all $\tau_a(\cdot)$ are polynomials. This plays a pivotal role when we check scalability for marginals in the inductive steps. The last properly is the relatively clear structure of polynomial functions, from which we can obtain suitable scaling factors $g_a^{(l)}$ at each inductive step $l$.

The proof of Theorem 2 is very elementary. It does not involve any advanced techniques, but only mathematical induction, basic calculus, and a very crude ranking of non-negative functions. Therefore, it should be widely readable.
Theorem 2 fully settles the convergence of the PoA for arbitrary polynomial price functions. This greatly extends the partial results from [6], [7] and [28] for polynomial price functions. Due to the popularity of polynomial functions, Theorem 2 may apply to a more general context other than road traffic, e.g. the scenario of free market mentioned in the Introduction.

3.4 A further extension: a general proof for regularly varying price functions

The idea of asymptotic decomposition may apply also to NCGs with price functions of other types. Polynomial functions are special cases of regularly varying functions. This subsection aims to apply the asymptotic decomposition to this more general notion.

By Karamata’s Characterization Theorem [3], a regularly varying function $\tau(\cdot)$ can be written as

$$\tau(x) = x^\rho \cdot Q(x),$$

where $\rho \in \mathbb{R}$ is called the regular variation index of $\tau(\cdot)$ and $Q(x)$ is a slowly varying function, i.e., for each $x > 0$,

$$\lim_{t \to \infty} \frac{Q(tx)}{Q(t)} = 1.$$ 

The class of regularly varying functions is very extensive and includes many popular analytic functions, e.g., all affine functions, polynomials, logarithms, and others.

Although regularly varying functions are more extensive than polynomial functions, they actually have similar properties. Lemma 8 summarizes these properties.

**Lemma 8.** Consider a regularly varying function $\tau(\cdot)$ with index $\rho \geq 0$. Then, the following statements hold.

a) For each $\epsilon > 0$,

$$\lim_{x \to \infty} \frac{\tau(x)}{x^{\rho+\epsilon}} = 0,$$

and

$$\lim_{x \to \infty} \frac{\tau(x)}{x^{\rho-\epsilon}} = \infty.$$ 

b) For each non-negative function $g(\cdot)$, if

$$\lim_{x \to \infty} \frac{g(x)}{\tau(x)} = q \in (0, \infty)$$

for some constant $q$, then $g(\cdot)$ is also regularly varying with index $\rho$. 

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c) For each regularly varying function $g(\cdot)$ with index $\rho'$, the quotient

$$\frac{g(x)}{\tau(x)}$$

is again regularly varying, but with index $\rho' - \rho$. Therefore, the quotient of two slowly varying non-zero functions is again slowly varying.

Proof. See the Appendix, or [3].

Lemma 8 a) and Lemma 8 c) indicate a partial ordering on the class of regularly varying functions, i.e., regularly varying functions of different indices can be sorted according to their indices. However, two regularly varying functions of the same index are generally incomparable. Therefore, we cannot directly reuse the simple ordering of polynomial functions when we apply the asymptotic decomposition to NCGs with regularly varying functions. Lemma 8 b) will be implicitly used in our discussion. It guarantees that the auxiliary price functions $c_a(x) = x\tau'_a(x) + \tau_a(x)$ are again regularly varying and have the same indices as the price functions $\tau_a(x)$.

Due to the generality of regularly varying functions, we cannot have a uniform argument. To simplify the discussion, we shall consider regularly varying functions with particular properties in this Subsection, e.g., convex and differentiable regularly varying functions.

Lemma 9. Consider a regularly varying function $\tau(\cdot)$ that is non-decreasing, non-negative, convex and differentiable on $[0, \infty)$. Then, the regular variation index of $\rho$ is non-negative, and

$$\lim_{x \to \infty} \frac{x \cdot \tau'(x)}{\tau(x)} = \rho \geq 0.$$  \hspace{1cm} (11)

Proof. See the Appendix.

Lemma 8 b) and Lemma 9 yield immediately that the auxiliary price functions $c_a(x) = x\tau'_a(x) + \tau_a(x)$ are again regularly varying, provided that the price functions $\tau_a(x)$ are regularly varying and convex. Moreover, the marginal games in an asymptotic decomposition will be asymptotically well designed in this case, since (11) holds. We summarize this result in Theorem 3.
Theorem 3. Consider an NCG with price functions $\tau_a(\cdot)$ satisfying all the conditions of Lemma\[7\]. If the price functions are mutually comparable, i.e., for each $a, b \in A$, the limit

$$\lim_{x \to \infty} \frac{\tau_a(x)}{\tau_b(x)} = q_{a,b} \in [0, \infty]$$

exists for some constant $q_{a,b}$, then $\Gamma$ is asymptotically well designed.

Proof. See the Appendix. \(\square\)

Theorem 3 generalizes Corollary 4.8 in [7]. Colini et al. [7] showed that NCGs with regularly varying and mutually comparable price functions $\tau_a(\cdot)$ are gaugeable w.r.t. each user volume vector sequence $\{d(n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} T(d(n)) = \infty$ and $\lim_{n \to \infty} \frac{d(n)}{T(d(n))} > 0$ for each $k \in K$. Obviously, this is only a partial result, due to the sequence-specific condition on $\{d(n)\}_{n \in \mathbb{N}}$ in Colini et al. [7]. With Lemma 3, such games are obviously scalable w.r.t. these specific user volume sequences. With the asymptotic decomposition, Theorem 3 now successfully removes the sequence limitation and gives a global convergence of the PoA for such games, when the price functions $\tau_a(\cdot)$ are convex.

The condition that all $\tau_a(\cdot)$ are mutually comparable implies a suitable ordering on the $\tau_a(\cdot)$, see the proof of Theorem 3 for details on the ordering. By Lemma 8(b), this ordering also carries over to the auxiliary price functions $c_a(\cdot)$. Hence, the proof of Theorem 3 directly defines the ordering on the resources $a \in A$. With this ordering, we can then compare the cost and construct scaling factors at each inductive step.

Convexity is only needed to guarantee [11]. The proof of Theorem 3 is still valid if we use [11] instead of convexity. Therefore, Theorem 3 can be further extended. For instance, if we substitute convexity by concavity in Theorem 3 then the conclusion still holds.

Moreover, we can even weaken the condition that all price functions $\tau_a(\cdot)$ are regularly varying and mutually comparable. Actually, similar arguments may still apply when only some of the price functions $\tau_a(\cdot)$ are regularly varying and mutually comparable. If this is the case, we need that for each subset $K' \subseteq K$, there exists a regularly varying function $g(\cdot)$ such that:

$G1'$ For each $a \in A$, the limit

$$\lim_{x \to \infty} \frac{\tau_a(x)}{g(x)} = q_a \in [0, \infty].$$

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For each $k \in K'$, there exists an strategy $s \in S_k$ such that
\[ \max\{q_a : a \in A \text{ and } r(a, s) > 0\} < \infty. \]

There exists a group $k \in K'$ such that
\[ \min_{s \in S_k} \max\{q_a : a \in A \text{ and } r(a, s) > 0\} \in (0, \infty). \]

By additionally assuming condition (11), one can again obtain a proof for this case by a slight modification of the proof of Theorem 3. Theorem 4 below summarizes this result.

**Theorem 4.** Consider an NCG $\Gamma$ with non-decreasing, non-negative and differentiable price functions $\tau_a(\cdot)$. If (11) holds for each $a \in A$, and if for each non-empty subset $K' \subset K$ there exists a regularly varying function $g(\cdot)$ fulfilling $G1')-G3')$, then $\Gamma$ is asymptotically decomposable, and thus asymptotically well designed.

**Proof.** See the appendix. \qed

Conditions $G1')-G3')$ correspond to $G1)-G3)$ in Definition 5 for gaugeability. However, they are more flexible and general than gaugeability. They apply to arbitrary user volume sequences, while gaugeability only applies to user volume sequences fulfilling condition (7). Hence, Theorem 4 is not only an extension of Theorem 2, but also of the main result about gaugeability (Theorem 4.4) in Colini et al. [7].

The proof of Theorem 4 does not need an ordering of the price functions $\tau_a(\cdot)$. Conditions $G1')-G3')$ already imply the existence of suitable scaling factors at each inductive step in the asymptotic decomposition. With these scaling factors, we can then compare the cost of marginals.

Therefore, condition (11) and the existence of suitable scaling factors $g^{(l)}_n$ at each inductive step are pivotal when we apply the asymptotic decomposition. The ordering of price functions is only needed for constructing scaling factors and comparing the cost of marginals at each inductive step. If the existence of suitable scaling factors at each inductive step can be guaranteed in advance, then we do not need such an ordering of the price functions. Note that then the cost can be compared by comparing the scaling factors.

Actually, condition (11) may not be so restrictive for regularly varying functions in practice. By Karamata's Representation Theorem for slowly
varying functions $\tau(\cdot)$ can be written as
\[ \tau(x) = x^\rho \cdot \exp(\eta(x) + \int_b^x \frac{\epsilon(t)}{t} dt), \]
where $\lim_{x \to \infty} \eta(x) = \kappa \in \mathbb{R}$, $\lim_{x \to \infty} \epsilon(x) = 0$ and $b \geq 0$ is a constant dependent of function $\tau(\cdot)$. If $\tau(x)$ is differentiable and non-decreasing, then we can assume w.l.o.g. that the bounded function $\eta(x)$ is differentiable and $\epsilon(x)$ is continuous. Then,
\[ \lim_{x \to \infty} \frac{x\tau'(x)}{\tau(x)} = \rho + \lim_{x \to \infty} x\eta'(x), \]
which is a non-negative constant when the limit $\lim_{x \to \infty} x\eta'(x)$ exists. Note that the limit $\lim_{x \to \infty} x\eta'(x)$ exists if $\eta(x)$ converges to $\kappa$ eventually in a relatively steady way. Such $\eta(\cdot)$ may possess some regular properties, e.g., convexity, concavity, monotonicity, and others.

Note that there are differentiable, non-decreasing and regularly varying functions $\tau(x)$ that do not fulfill (11). For instance, consider $\tau(x) = x^3 \cdot \exp\left(1 - \frac{1}{x} \sin(x)\right)$ for large enough $x$. Here, $\eta(x) = 1 - \frac{\sin x}{x}$ and the limit $\lim_{x \to \infty} x\eta'(x)$ does not exist, since $x\eta'(x) = \frac{1}{x} \sin x - \cos x$ diverges as $x \to \infty$. Although such price functions are meaningful in theory, they may not be of interest in practice because of their irregular properties.

Theorem 2, Theorem 3 and Theorem 4 further demonstrate the power of scalability stemming from [28]. They together deepen our knowledge on user behavior in NCGs. In particular, they positively support the view of [28] that selfish user behavior need not be bad.

4 Conclusion

We have unified recent results from [8], [6], [7] and [28] on the convergence of the PoA for NCGs. We have reformulated the concept of limit games that are implicitly used in [28]. With the concept of limit games, we were able to bring new insight into AWDGs, see, e.g., Theorem 4 and Corollary 2. We have deepened the knowledge of scalability introduced by Wu et al. [28]. We have introduced the technique of asymptotic decomposition that allows us to analyze the convergence of the PoA for NCGs with general price functions.

With this new technique, we were able to show that NCGs with arbitrary polynomial price functions $\tau_n(\cdot)$ are asymptotically well designed, see Theorem 2. This completes the results from [8], [6], [7] and [28] for NCGs with
polynomial price functions. Moreover, we were able to apply the asymptotic decomposition to NCGs with regularly varying price function, and prove that these NCGs are also asymptotically well designed, see Theorem 3 and Theorem 4. Our results definitely demonstrate the power of scalability, and positively support the view of [28] on user behavior in NCGs.

The profit goals of users in an NCG are in general inconsistent with the profit goal of the underlying central authority. Both of them want to minimize cost. But the difference is that users only locally minimize their own cost, while the central authority wants to globally minimize social cost. Our results show that the local minimization will lead to a glocal minimization when the volume of users becomes large. Thus, selfishness need not be bad in general.

Future work in this direction could consider the saturation point of an NCG, i.e., a threshold value for user volumes, beyond which NE profiles will almost be SO profiles. This could be very interesting in game theory. NCGs are open games, i.e., users can freely join such games. When the user volume reaches its saturation, what the users need to do is to perform as selfish as possible, since this is the best choice in a bad environment.

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Appendix

Proof of Lemma 1

Proof of Lemma 1. Consider an NCG

$$
\Gamma = \left( \mathcal{K}, A, \mathcal{S}, (r(a, s))_{a \in A, s \in S}, (\tau^a)_{a \in A}, d \right),
$$
and an arbitrary user volume vector sequence \( \{d^{(n)}\}_{n \in \mathbb{N}} \) such that the total volume \( T(d^{(n)}) \to \infty \) as \( n \to \infty \). Let \( \tilde{f}^{(n)} \) be an NE profile w.r.t. user volume vector \( d^{(n)} = (d^{(n)}_k)_{k \in \mathcal{K}}, \) for each \( n \in \mathbb{N} \). We assume that \( \Gamma \) has a limit
\[
\Gamma^\infty = \left( \mathcal{K}^\infty, A, \mathcal{S}^\infty, (r(a,s))_{a \in A, s \in \mathcal{S}^\infty}, (\tau^\infty_a)_{a \in A, d} \right)
\]
w.r.t. \( \{d^{(n)}\}_{n \in \mathbb{N}} \).

By Definition 3, we obtain that there is an infinite subsequence \( \{n_i\}_{i \in \mathbb{N}} \) and a sequence \( \{g_i\}_{i \in \mathbb{N}} \) of positive scaling factors, s.t., conditions L1-L4 hold. Note that we can further assume that:

- The limit
  \[
  \lim_{i \to \infty} \frac{\tilde{f}^{(n_i)}}{T(d^{(n_i)})} = \tilde{f}^\infty_s \in [0,1]
  \]
  exists, for some constant \( \tilde{f}^\infty_s \), for each \( s \in \mathcal{S} \).

Otherwise, we can take an infinite subsequence \( \{n_{i_j}\}_{j \in \mathbb{N}} \) fulfilling the above condition. Let
\[
\tilde{f}_a^\infty := \sum_{s \in \mathcal{S}} r(a,s) \cdot \tilde{f}^\infty_s
\]
for each \( a \in A \). We aim to prove Lemma 1 with the subsequence \( \{n_i\}_{i \in \mathbb{N}} \), scaling factor sequence \( \{g_i\}_{i \in \mathbb{N}} \), and \( \tilde{f}^\infty := (\tilde{f}^\infty_s)_{s \in \mathcal{S}^\infty} \).

To that end, we need some auxiliary facts. Fact 1 below indicates that prices of tight strategies are always well “preserved” in the limit w.r.t. any sequence of feasible strategy profiles. Therefore, for each tight strategy \( s \in \mathcal{S}^\infty \), we obtain that
\[
\lim_{i \to \infty} \tau_s(\tilde{f}^{(n_i)}) = \sum_{a \in A} r(a,s) \cdot \tau_a(\tilde{f}_a^\infty) < \infty.
\]

**Fact 1.** Consider a tight strategy \( s \in \mathcal{S}^\infty \). Let \( f^{(n_i)} \) be a feasible strategy profile w.r.t. user volume vector \( d^{(n_i)} \), for each \( i \in \mathbb{N} \). If
\[
\lim_{i \to \infty} \frac{f^{(n_i)}}{T(d^{(n_i)})} = \mu'_{s'},
\]
for some constant \( \mu'_{s'} \in [0,1] \), for each \( s' \in \mathcal{S}, \) then
\[
\tau^\infty_s(\mu) = \sum_{a \in A} r(a,s) \cdot \tau^\infty_a(\mu_a) = \lim_{i \to \infty} \frac{\sum_{a \in A} r(a,s) \cdot \tau_a(f^{(n_i)}_a)}{g_i},
\]
where \( \mu_a = \sum_{s' \in \mathcal{S}} r(a,s') \mu'_{s'} \) for each \( a \in A \).
**Proof of Fact 2.** Note that for each \( a \in A, \)

\[
\mu_a = \sum_{s' \in S} r(a, s') \mu_{s'} = \sum_{s' \in S} r(a, s') \cdot \lim_{i \to \infty} \frac{f_{a}^{(n_i)}}{T(d(n_i))} = \lim_{i \to \infty} \sum_{s' \in S} r(a, s') \cdot \frac{f_{a}^{(n_i)}}{T(d(n_i))} \in [0, \infty).
\]

Therefore, for an arbitrarily fixed \( \epsilon > 0, \)

\[
\max \{0, \mu_a - \epsilon\} \leq \frac{f_{a}^{(n_i)}}{T(d(n_i))} \leq \mu_a + \epsilon,
\]

for each resource \( a \in A, \) for \( i \) large enough. As a result, for \( i \) large enough, we obtain for each \( a \in A \) that

\[
\tau_a \left( T(d(n_i)) \cdot \max \{0, \mu_a - \epsilon\} \right) \leq \tau_a \left( f_{a}^{(n_i)} \right) \leq \tau_a \left( T(d(n_i)) \cdot (\mu_a + \epsilon) \right), \tag{12}
\]

since each price function \( \tau_a(\cdot) \) is non-decreasing.

Since \( s \) is tight, by L3.2), we then obtain for each resource \( a \in A \) with \( r(a, s) > 0 \) that \( \tau_a^{\infty}(\cdot) \) is a continuous and non-decreasing function, and that

\[
\tau_a^{\infty}(\mu_a + \epsilon) = \lim_{i \to \infty} \frac{\tau_a \left( T(d(n_i)) \cdot (\mu_a + \epsilon) \right)}{g_i} \geq \lim_{i \to \infty} \frac{\tau_a \left( f_{a}^{(n_i)} \right)}{g_i} \geq \lim_{i \to \infty} \frac{\tau_a \left( f_{a}^{(n_i)} \right)}{g_i} \geq \tau_a \left( \max \{0, \mu_a - \epsilon\} \right) = \lim_{i \to \infty} \frac{\tau_a \left( T(d(n_i)) \cdot \max \{0, \mu_a - \epsilon\} \right)}{g_i}.
\]

By \( \text{[12]} \), the continuity of \( \tau_a^{\infty}(\cdot), \) and the arbitrary choice of \( \epsilon, \) we then obtain for each \( a \in A \) with \( r(a, s) > 0 \) that

\[
\tau_a^{\infty}(\mu_a) = \lim_{i \to \infty} \frac{\tau_a \left( f_{a}^{(n_i)} \right)}{g_i},
\]

which, in turn, implies Fact 1. \( \square \)

Fact 2 below considers the limit prices of non-tight strategies w.r.t. NE profiles. It indicates that we can completely ignore those non-tight strategies in the limit analysis. By Fact 2, we obtain that

\[
f_{a}^{\infty} = \sum_{s \in S} r(a, s) \cdot f_{s}^{\infty} = \sum_{s \in S^\infty} r(a, s) \cdot \tilde{f}_{s}^{\infty},
\]

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and that the limit volume
\[ d_k = \sum_{s \in S_k \cap S^\infty} \tilde{f}_s^\infty \]
for each group \( k \in K^\infty \), and that
\[ d_k = \sum_{s \in S_k} \tilde{f}_s^\infty = 0, \]
for each group \( k \in K \setminus K^\infty \) (since such groups do not have tight strategies). Therefore, \( \tilde{f}^\infty = (\tilde{f}_s^\infty)_{s \in S^\infty} \) is a strategy profile of the limit game \( \Gamma^\infty \) w.r.t. user volume vector \( (d_k)_{k \in K} \) and \( K^\infty \).

**Fact 2.** Consider a non-tight strategy \( s \in S \setminus S^\infty \). Then, we obtain that:

1) \( \tilde{f}_s^\infty = 0 \).
2) \( \lim_{i \to \infty} \frac{j_i(n_i) \cdot \tau_s(\tilde{f}(n_i))}{T(d(n_i)) \cdot g_i} = 0 \).

*Proof of Fact 2.* Before we start the proof, let us recall a basic property of non-tight strategies. By L2) and L3) of Definition 3, we obtain that
\[ \lim_{i \to \infty} \tau_s(\tilde{f}(n_i)) g_i = \tau^\infty_a(x) \equiv \infty, \forall x > 0, \]
for some \( a \in A \) with \( r(a, s) > 0 \), since \( s \) is non-tight.

For 1) : Let \( k \in K \) be the group such that \( s \in S_k \). We prove 1) by contradiction. We suppose that \( \tilde{f}_s^\infty > 0 \). Then, by L2) of Definition 3 with an argument similar to that for Lemma 1, we obtain that
\[ \lim_{i \to \infty} \frac{\tau_s(\tilde{f}(n_i))}{g_i} = \infty, \quad \text{and} \quad \lim_{i \to \infty} \frac{j_i(n_i) \cdot \tau_s(\tilde{f}(n_i))}{T(d(n_i)) \cdot g_i} = \infty, \]
since \( \tilde{f}_s^\infty > 0 \). Thus, by L3.1) of Definition 3, group \( k \) is non-negligible. Hence, group \( k \) must have tight strategies. Let \( s' \in S_k \cap S^\infty \) be a tight strategy. By Fact 1, we obtain that
\[ \lim_{i \to \infty} \frac{\tau_{s'}(\tilde{f}(n_i))}{g_i} < \infty = \lim_{i \to \infty} \frac{\tau_s(\tilde{f}(n_i))}{g_i}. \]
Hence, for \( i \) large enough,
\[ \tau_{s'}(\tilde{f}(n_i)) < \tau_s(\tilde{f}(n_i)). \]
This, in turn, implies that $f^{(n_i)}_s \equiv 0$ for $i$ large enough, due to the user optimality (2) of NE profiles. This contradicts with the assumption that $f^\infty_s > 0$.

Therefore, $f^\infty_s = 0$ must hold.

For 2): We prove 2) again by contradiction. We assume that

$$\lim_{i \to \infty} f^{(n_i)}_s \cdot \tau_s(f^{(n_i)}) > 0.$$  \hfill (13)

By L3.1 of Definition 3, we obtain immediately that $s$ must be a strategy from some non-negligible group. We assume, w.l.o.g., that

$$\lim_{i \to \infty} f^{(n_i)}_s \cdot \tau_s(f^{(n_i)}) = \lim_{i \to \infty} f^{(n_i)}_s \cdot \tau_s(f^{(n_i)}) \cdot T(d^{(n_i)}) \cdot g_i = 0.$$  \hfill (13)

Otherwise, one can take an infinite subsequence of $\{n_i\}_{i \in \mathbb{N}}$ fulfilling the above condition.

By 1), we can obtain that

$$\lim_{i \to \infty} f^{(n_i)}_s \cdot \tau_s(f^{(n_i)}) \cdot T(d^{(n_i)}) \cdot g_i = 0,$$

and thus

$$\lim_{i \to \infty} \frac{\tau_s(f^{(n_i)}) \cdot T(d^{(n_i)}) \cdot g_i}{g_i} = \infty.$$  \hfill (13)

Then, with an argument argument similar as that in the proof of 1), one can prove again that $f^{(n_i)}_s \equiv 0$ for $i$ large enough. This yields that

$$\frac{f^{(n_i)}_s \cdot \tau_s(f^{(n_i)})}{T(d^{(n_i)}) \cdot g_i} \equiv 0,$$

for $i$ large enough. This obviously contradicts with the assumption (13). Hence, 2) must hold.

By Fact 1 and Fact 2 we obtain immediately that

$$\lim_{i \to \infty} \frac{C(f^{(n_i)})}{g_i} = \lim_{i \to \infty} \sum_{s \in S^\infty} \frac{f^{(n_i)}_s \cdot \tau_s(f^{(n_i)})}{T(d^{(n_i)}) \cdot g_i} = \sum_{s \in S^\infty} \tilde{f}^\infty_s \cdot \tau_s^\infty(\tilde{f}^\infty).$$  \hfill (13)

The fact that $\tilde{f}^\infty$ is an NE profile of $\Gamma^\infty$ follows immediately from that limit preserves numerical ordering “$\geq$”.

Combining all of the above, the proof of Lemma 1 is completed.
Proof of Theorem 1

Proof of Theorem 1. Consider an NCG
\[ \Gamma = \left( \mathcal{K}, A, \mathcal{S}, (r(a, s))_{a \in A, s \in \mathcal{S}}, (\tau)_{a \in A} \right), \]
and an arbitrary user volume vector sequence \( \{d(n)\}_{n \in \mathbb{N}} \) such that the total volume \( T(d(n)) \to \infty \) as \( n \to \infty \).

We assume that the game \( \Gamma^\infty = \left( \mathcal{K}^\infty, A, \mathcal{S}^\infty, (r(a, s))_{a \in A, s \in \mathcal{S}^\infty}, (\tau_s^\infty)_{a \in A} \right) \)
is the limit game of \( \Gamma \) w.r.t. a subsequence \( \{n_i\}_{i \in \mathbb{N}} \) and a scaling factor sequence \( \{g_i\}_{i \in \mathbb{N}} \), where all the components of \( \Gamma^\infty \) are defined as in Definition 3.

We suppose that \( \Gamma^\infty \) is not well designed. We aim to show in this case that the PoA\( (d(n_i)) \) does not converge to 1 as \( i \to \infty \), which, in turn, implies that the game \( \Gamma \) is not asymptotically well designed.

Let \( \{\tilde{f}^{(n_i)}\}_{i \in \mathbb{N}} \) and \( \{f^*(n_i)\}_{i \in \mathbb{N}} \) be an NE profile sequence and an SO profile sequence w.r.t. \( \{d(n_i)\}_{i \in \mathbb{N}} \), respectively. We assume, w.l.o.g., that
- the limits
  \[ \lim_{i \to \infty} \frac{\tilde{f}^{(n_i)}_s}{T(d(n_i))} = \tilde{f}^\infty_s \in [0, 1], \quad \text{and} \quad \lim_{i \to \infty} \frac{f^*(n_i)_s}{T(d(n_i))} = f^{*\infty}_s \in [0, 1] \]
exist for some constants \( \tilde{f}^\infty_s, f^{*\infty}_s \), for each \( s \in \mathcal{S} \). Otherwise, we can take an infinite subsequence of \( \{n_i\}_{i \in \mathbb{N}} \) fulfilling this condition.

By Fact 1 and Fact 2 in the proof of Lemma 1 we obtain immediately that
\[ \lim_{i \to \infty} \frac{C(\tilde{f}^{(n_i)})}{g_i} = \sum_{s \in \mathcal{S}^\infty} \tilde{f}^\infty_s \cdot \tau_s^\infty(\tilde{f}^\infty) \in (0, \infty), \]
\[ \lim_{i \to \infty} \frac{\sum_{s \in \mathcal{S}^\infty} f_s^*(n_i) \cdot \tau_s(f^*(n_i))}{T(d(n_i)) \cdot g_i} = \sum_{s \in \mathcal{S}^\infty} f^{*\infty}_s \cdot \tau_s^\infty(f^{*\infty}) \in [0, \infty), \]
and that \( \tilde{f}^\infty = (\tilde{f}^\infty_s)_{s \in \mathcal{S}^\infty} \) is an NE profile of \( \Gamma^\infty \).

We now aim to show that Fact 2 applies also to SO profiles.

By the fact that each \( f^*(n_i) \) is an SO profile, we then obtain immediately that
\[ \lim_{i \to \infty} \frac{C(f^*(n_i))}{g_i} \leq \lim_{i \to \infty} \frac{C(\tilde{f}^{(n_i)})}{g_i} < \infty, \]
which, in turn, implies that for each non-tight strategy \( s \in \mathcal{S} \setminus \mathcal{S}^\infty \), \( f_s^* = 0 \). Otherwise, if \( f_s^* > 0 \), then by L3 of Definition 3, \( \lim_{i \to \infty} \frac{C(f_s(n_i))}{g_i} = \infty \).

Therefore, \( f^* = (f_s^*)_{s \in \mathcal{S}^\infty} \) is also a feasible strategy profile of \( \Gamma^\infty \).

We now aim to prove for each non-tight strategy \( s \in \mathcal{S} \setminus \mathcal{S}^\infty \) that
\[
\lim_{i \to \infty} \frac{f_s(n_i) \cdot \tau_s(f_s(n_i))}{T(d(n_i)) \cdot g_i} = 0.
\]

Similarly, we prove this by contradiction. We assume, w.l.o.g., that there is exactly one non-tight strategy, i.e., \( |\mathcal{S} \setminus \mathcal{S}^\infty| = 1 \). Let \( s \) denote this unique non-tight strategy. For the case of more non-tight strategies, an almost identical argument will apply.

We assume now, w.l.o.g., that
\[
\lim_{i \to \infty} \frac{f_s(n_i) \cdot \tau_s(f_s(n_i))}{T(d(n_i)) \cdot g_i} > 0.
\]

Then, by L3.1) of Definition 3, \( s \) must be a strategy from some non-negligible group \( k \in \mathcal{K} \). By L3.2) of Definition 3, there must exist a tight strategy \( s' \in \mathcal{S}_k \). To derive a contradiction to (14), we now construct some artificial feasible strategy profiles. For each \( i \in \mathbb{N} \), we put
\[
h_s^{(n_i)} = \begin{cases} 
  f_s^{(n_i)} & \text{if } s'' \notin \{s, s'\}, \\
  f_s^{(n_i)} + f_s^{(n_i)} & \text{if } s'' = s', \\
  0 & \text{if } s'' = s.
\end{cases}
\]

For the case of more than one non-tight strategies, one can similarly move the users adopting non-tight strategies to tight strategies. However, the explicit definition of profiles \( h_s^{(n_i)} = (h_s^{(n_i)})_{s'' \in \mathcal{S}} \) will become very complicated.

Obviously, for each \( s'' \in \mathcal{S} \),
\[
\lim_{i \to \infty} \frac{h_s^{(n_i)}}{T(d(n_i))} = f_s^*:\]

since \( f_s^* = 0 \). Moreover, by Fact 1 of Lemma 1, we obtain that
\[
\lim_{i \to \infty} \frac{\sum_{s'' \in \mathcal{S}^\infty} h_s^{(n_i)} \cdot \tau_{s''}(h_s^{(n_i)})}{T(d(n_i))g_i} = \lim_{i \to \infty} \frac{\sum_{s'' \in \mathcal{S}^\infty} f_s^{(n_i)} \cdot \tau_{s''}(f_s^{(n_i)})}{T(d(n_i))g_i} = \sum_{s'' \in \mathcal{S}^\infty} f_s^{*} \cdot \tau_{s''}(f_s^{*}).
\]
Note that in profiles $h^{(n_1)}$, only tight strategies are used. Thus, we obtain that
\[
\lim_{i \to \infty} \frac{C(h^{(n_1)})}{g_i} = \lim_{i \to \infty} \frac{\sum_{s'' \in S^\infty} f_{s''}^{(n_1)} \cdot \tau_{s''}(f^{(n_1)})}{T(d^{(n_1)}) \cdot g_i}.
\]
With (14), we obtain further that
\[
\lim_{i \to \infty} \frac{C(f^{(n_1)})}{g_i} > \lim_{i \to \infty} \frac{\sum_{s'' \in S^\infty} f_{s''}^{(n_1)} \cdot \tau_{s''}(f^{(n_1)})}{T(d^{(n_1)}) \cdot g_i} = \lim_{i \to \infty} \frac{C(h^{(n_1)})}{g_i}.
\]
This yields that
\[
C(h^{(n_1)}) < C(f^{(n_1)})
\]
for $i$ large enough. This contradicts with the fact that profiles $f^{(n_1)}$ are all system optimum.

Therefore, Fact 2 also applies to SO profiles. Thus we obtain that
\[
\lim_{i \to \infty} \frac{C(f^{(n_1)})}{g_i} = \sum_{s \in S^\infty} f_{s}^{*,\infty} \cdot \tau_s(f^{*,\infty}).
\]
We now aim to show that $f^{*,\infty}$ is an SO profile of $\Gamma^\infty$. Let $f = (f_s)_{s \in S^\infty}$ be an arbitrary feasible strategy profile of $\Gamma^\infty$. To show that $f^{*,\infty}$ is an SO profile of $\Gamma^\infty$, we only need to show that its cost is not larger than the cost of $f$, due to the arbitrary choice of $f$. Note that there must exist a sequence \(\{f^{(n_1)}\}_{i \in \mathbb{N}}\) of feasible profiles such that
\[
f_s = \lim_{i \to \infty} \frac{f_s^{(n_i)}}{T(d^{(n_i)})}
\]
for each tight strategy $s \in S^\infty$. Actually, we can put for each $k \in K^\infty$ with $d_k^{(n_i)} > 0$ and $s \in S_k$ that
\[
f_s^{(n_i)} = \frac{d_s^{(n_i)} \cdot f_s}{d_k^{(n_i)}} \quad \forall i \in \mathbb{N},
\]
and put for each other $k \in K$ and $s \in S_k$ that
\[
f_s^{(n_i)} = \frac{d_s^{(n_i)}}{|S_k|} \quad \forall i \in \mathbb{N}.
\]
Since each $f^{(n_1)}$ is system optimal, we obtain that
\[
\frac{C(f^{(n_1)})}{g_i} \leq \frac{C(f^{(n_1)})}{g_i} \quad \forall i \in \mathbb{N}.
\]
Letting $i \to \infty$, this yields by Fact 1, $L^3$ and the definition of $K^\infty$ that

$$
\lim_{i \to \infty} \frac{C(f^{(n_i)})}{g_i} = \sum_{s \in S^\infty} f^{s,\infty} \cdot \tau^\infty_s (f^{*,\infty}) \leq \sum_{s \in S^\infty} f_s \cdot \tau^\infty_s (f) = \lim_{i \to \infty} \frac{C(f^{(n_i)})}{g_i},
$$

since $f$ defines only on tight strategies, and since each group $k \in K\setminus K^\infty$ is negligible without tight strategy and thus must have zero limit volume, i.e., $d_k = 0$. Hence, $f^{*,\infty}$ is an SO profile of $\Gamma^\infty$.

Since $\Gamma^\infty$ is not well designed, we thus obtain that

$$
\sum_{s \in S^\infty} f^{s,\infty}_s \cdot \tau^\infty_s (f^{*,\infty}) < \sum_{s \in S^\infty} \tilde{f}^{\infty}_s \cdot \tau^\infty_s (\tilde{f}^{\infty}).
$$

This implies that

$$
\lim_{i \to \infty} \text{PoA}(d^{(n_i)}) = \lim_{i \to \infty} \frac{C(\tilde{f}^{(n_i)})}{C(f^{(n_i)})} = \frac{\sum_{s \in S^\infty} \tilde{f}^{\infty}_s \cdot \tau^\infty_s (\tilde{f}^{\infty})}{\sum_{s \in S^\infty} f^{s,\infty}_s \cdot \tau^\infty_s (f^{*,\infty})} > 1.
$$

Therefore, $\Gamma$ is not well designed. \qed

**Proof of Theorem 2**

**Proof of Theorem 2.** We prove Theorem 2 by applying the idea of asymptotic decomposition. To well demonstrate the idea, we will give a very detailed proof. The proof will be direct and elementary without using a heavy machinery. It only uses the definition and connection between NE profiles and SO profiles, simple facts about the asymptotic notations $O(\cdot), \Omega(\cdot), \Theta(\cdot), o(\cdot)$ and $\omega(\cdot)$, and a suitable induction along the user groups.

Let $\{d^{(n)}\}_{n \in \mathbb{N}}$ be an arbitrary sequence of user volume vectors such that:

- Each $d^{(n)} = (d^{(n)}_1, \ldots, d^{(n)}_K)$ is a vector, where the $k$-th component $d^{(n)}_k$ represents the user volume of the $k$-th group for $k = 1, \ldots, K$, for each $n \in \mathbb{N}$.
- The total user volume $T(d^{(n)}) = \sum_{k=1}^{K} d^{(n)}_k \to +\infty$ as $n \to +\infty$.

To prove the Theorem 2, we only need to show that $\lim_{n \to \infty} \text{PoA}(d^{(n)}) = 1$, due to the arbitrary choice of the user volume vector sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$.

Note that

$$
\lim_{n \to \infty} \text{PoA}(d^{(n)}) = 1 \quad (15)
$$
follows immediately from the fact that

$$\lim_{n \to \infty} \text{PoA}(d^{(n)}) = 1,$$

since it holds trivially that

$$\lim_{n \to \infty} \text{PoA}(d^{(n)}) \geq 1.$$ 

Therefore, we assume, w.l.o.g., that the limit

$$\lim_{n \to \infty} \text{PoA}(d^{(n)}) \in [1, +\infty] \quad (16)$$

exists. Otherwise, we can take an infinite subsequence \(\{n_i\}_{i \in \mathbb{N}}\) such that the limit

$$\lim_{i \to \infty} \text{PoA}(d^{(n_i)}) = \lim_{n \to \infty} \text{PoA}(d^{(n)}) \in [1, +\infty]$$

exists, and we can then restrict our discussion to this subsequence.

With assumption (16), (15) follows immediately from the existence of an infinite subsequence \(\{n_i\}_{i \in \mathbb{N}}, s.t.,\)

$$\lim_{i \to \infty} \text{PoA}(d^{(n_i)}) = 1.$$ 

So, in the application of asymptotic decomposition, we can take a series of nested infinite subsequences of the sequence \(\{n\}_{n \in \mathbb{N}}\). To simplify notation, we will not explicitly use the terminology of subsequences, but assume that the user volume vector sequence \(\{d^{(n)}\}_{n \in \mathbb{N}}\) itself fulfills some required properties, if corresponding subsequences do exist.

We now introduce some notations stemming from [6]. Let \(\rho_a\) be the degree of the polynomial \(\tau_a(\cdot)\) for each resource \(a \in A\). Accordingly, we can define the degree of a strategy \(s \in S_k\) as

$$\rho_s := \max\{\rho_a : r(a, s) > 0 \text{ and } a \in A\},$$

and the degree of a group \(k \in \{1, \ldots, K\}\) as

$$\rho_k := \min\{\rho_s : s \in S_k\}.$$ 

Although these notations are trivial, they help us construct a suitable ordering on the resource set \(A\), and accordingly on the sets \(S\) and \(K\). We can compare two resources \(a, b \in A\) through their degrees \(\rho_a, \rho_b\). This will be very helpful when we construct the scaling factors at each inductive step in the asymptotic decomposition.
Moreover, we need to compare cost at each inductive step during asymptotic decomposition. This requires basic knowledge on the asymptotic notation. Let $h(x)$ be a non-negative real-valued function. The big $O$ notation $O(h(x))$ denotes the class of all non-negative real-valued functions $q(x)$ such that $\lim_{x \to \infty} \frac{q(x)}{h(x)} < +\infty$, and the small $o$ notation $o(h(x))$ denotes the class of all non-negative real-valued functions $q(x)$ such that $\lim_{x \to \infty} \frac{q(x)}{h(x)} = 0$.

Similarly, $\Theta(h(x))$ denotes the class of all non-negative real-valued functions $q(x)$ such that $\lim_{x \to \infty} \frac{q(x)}{h(x)} > 0$, and $\omega(h(x))$ denotes the class of all non-negative real-valued functions $q(x)$ such that $\lim_{x \to \infty} \frac{q(x)}{h(x)} = +\infty$. We put $\Theta(h(x)) = O(h(x)) \cap \Omega(h(x)).$ In addition, we write $h(x) \approx q(x)$ if $\lim_{x \to \infty} \frac{h(x)}{q(x)} = 1$.

Let $f^*(n)$, $\tilde{f}^*(n)$ be an SO profile and an NE profile, respectively, w.r.t. user volume vector $d^{(n)} = (d_k^{(n)})_{k \in \mathcal{K}}$, for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$\text{PoA}(d^{(n)}) = \frac{C(\tilde{f}^{(n)})}{C(f^*(n))} = \frac{\sum_{k \in \mathcal{K}} \sum_{s \in \mathcal{S}_k} \tilde{f}_s^{(n)} \cdot \tau_s(\tilde{f}^{(n)})}{\sum_{k \in \mathcal{K}} \sum_{s \in \mathcal{S}_k} f_s^{(n)} \cdot \tau_s(f^*(n))}.$$  \quad (17)

Let $C_k(\tilde{f}^{(n)}) := \sum_{s \in \mathcal{S}_k} \tilde{f}_s^{(n)} \cdot \tau_s(\tilde{f}^{(n)})$ and $C_k(f^*(n)) := \sum_{s \in \mathcal{S}_k} f_s^{(n)} \cdot \tau_s(f^*(n))$ denote the total cost of users from group $k \in \mathcal{K}$ w.r.t. NE profile $\tilde{f}^{(n)}$ and SO profile $f^*(n)$, respectively, for all $k \in \mathcal{K}$ and each $n \in \mathbb{N}$. By \cite{17}, we obtain for each $n \in \mathbb{N}$ that

$$\text{PoA}(d^{(n)}) = \frac{\sum_{k \in \mathcal{K}} C_k(\tilde{f}^{(n)})}{\sum_{k \in \mathcal{K}} C_k(f^*(n))}.$$ \quad (18)

We are now ready to start the asymptotic decomposition. We aim to inductively partition the set $\mathcal{K}$ into mutually disjoint non-empty subsets $\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_t$ for some integer $t \geq 0$, and prove at each step $m = 0, \ldots, t$ that

$$\lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^{m} \mathcal{K}_u} C_k(\tilde{f}^{(n)})}{\sum_{k \in \bigcup_{u=0}^{m} \mathcal{K}_u} C_k(f^*(n))} = 1.$$ \quad (19)

This procedure will not only form a proof for Theorem \cite{2} but also asymptotically decompose the underlying game.

**Step $m = 0$** : construct $\mathcal{K}_0$, and prove \cite{19} for $m = 0$.

Before we formally start step $m = 0$, we first introduce a trivial but useful fact about NE profiles.. Note that for each $n \in \mathbb{N}$ and each $k \in \mathcal{K}$, by the
user optimality \(2\), users from group \(k\) have the same cost \(\tilde{L}_k^{(n)}\) w.r.t. the NE profile \(\tilde{f}^{(n)}\). Then, we obtain that

\[
C(\tilde{f}^{(n)}) = \frac{1}{T(d^{(n)})} \sum_{k \in \mathbb{N}} \tilde{\lambda}_k^{(n)} \cdot d_k^{(n)}.
\]

By \([19]\), each SO profile \(f^{s(n)}\) is actually an NE profile w.r.t. to the auxiliary price functions \(c_a(x) := (x \cdot \tau_a(x))' = x \cdot \tau'_a(x) + \tau_a(x)\), for each \(n \in \mathbb{N}\). Obviously, each \(c_a(\cdot)\) is again a polynomial of the same degree as \(\tau_a(\cdot)\), for each \(a \in A\). Let \(\tilde{L}_k^{(n)}\) be the cost of users from group \(k \in \mathcal{K}\) w.r.t. price functions \(c_a(\cdot)\) and the corresponding NE profile \(f^{s(n)}\), for each \(n \in \mathbb{N}\). Since \(\tau_a(x) \in \Theta(c_a(x))\) for each \(a \in A\), the cost of each user from group \(k \in \mathcal{K}\) is then in \(\Theta(L_k^{(n)})\) w.r.t. SO profiles \(f^{s(n)}\) and price functions \(\tau_a(\cdot)\).

As a key component at each step of the asymptotic decomposition, we are now to estimate \(\tilde{L}_k^{(n)}, L_k^{(n)}\) for each \(k \in \mathcal{K}\). This will be the base for the cost comparison. Claim \([7]\) below states that the degree \(\rho_k\) of a group \(k \in \mathcal{K}\) reflects the magnitude of the cost of its users in both, NE and SO profiles.

**Claim 1.** \(\tilde{L}_k^{(n)}, L_k^{(n)} \in O(T(d^{(n)})^{\rho_k}),\) for each group \(k \in \mathcal{K}\).

**Proof of Claim \([7]\).** Let \(k \in \{1, \ldots, K\}\) be an arbitrarily fixed user group.

By the definition of \(\rho_k\), there must exist some strategy \(s_0 \in S_k\), such that \(\rho_{s_0} = \rho_k = \min\{\rho_s : s \in S_k\}\). Let \(f^{(n)}\) be an arbitrary feasible strategy profile w.r.t. user volume vector \(d^{(n)}\), for each \(n \in \mathbb{N}\). Then \(\tau_{s_0}(f^{(n)}) \in O(T(d^{(n)})^{\rho_k})\), since there are at most \(T(d^{(n)})\) users adopting strategy \(s_0\) and the degree of \(s_0\) is \(\rho_k\). Therefore, \(\tau_{s_0}(f^{s(n)}), \tau_{s_0}(\tilde{f}^{(n)}) \in O(T(d^{(n)})^{\rho_k})\). By the user optimality \(2\) of NE profiles, we thus obtain that

\[
\tilde{L}_k^{(n)} \leq \tau_{s_0}(\tilde{f}^{(n)}) \in O(T(d^{(n)})^{\rho_k}).
\]

Recall that \(c_{s_0}(f^{s(n)}) \in \Theta(\tau_{s_0}(f^{s(n)}))\), and \(f^{s(n)}\) is an SO profile w.r.t. price functions \(c_a(\cdot)\), for each \(n \in \mathbb{N}\). Therefore, again by \(2\), we obtain that

\[
L_k^{(n)} \leq c_{s_0}(f^{s(n)}) \in O(T(d^{(n)})^{\rho_k}).
\]

\(\Box\)

We can now formally start step \(m = 0\). To facilitate our discussion, we assume, w.l.o.g., that
Moreover, for each \( k \) and thus for each \( a \), \( f_s \) states that the average cost of users from groups \( k \) \( f \) satisfies

\[
\alpha \leq \rho \leq \alpha \:
\]

For each \( \alpha \), \( f \) is feasible for the user volume vector \( d \). 

Note also that both \( f_{s^{(0,\infty)}} \), \( f_{s^{(0,\infty)}} \) and there exist some constants \( f_{s^{(0,\infty)}} \), \( f_{s^{(0,\infty)}} \) \( 0 \), \( 1 \), \( f \) \( K \) \( K \) such that

\[
\alpha \leq \rho \leq \alpha \:
\]

We now define \( \alpha \). Let \( \alpha \) := \( \max \{ \rho : d_k^{(0,\infty)} > 0, k \in K \} \) and \( \alpha \) := \( \{ k \in K : d_k^{(0,\infty)} > 0 \ \text{or} \ \rho_k \leq \alpha \} \). Note that \( \alpha \) \( 0 \), \( \alpha \) \( K \) \( K \) such that

\[
\alpha \leq \rho \leq \alpha \:
\]

Note also that both \( f_{s^{(0,\infty)}} \) \( K \) \( K \) and \( f_{s^{(0,\infty)}} \) \( K \) \( K \) are feasible strategies for the user volume vector \( d^{(0,\infty)} := (d_k^{(0,\infty)})_{k \in K} \), i.e.,

\[
d_k^{(0,\infty)} = \sum_{s \in s_k} f_{s^{(0,\infty)}} = \sum_{s \in s_k} f_{s^{(0,\infty)}} \ \forall k \in K.
\]

Moreover, for each \( k \in K \) \( K \) and \( s \in s_k \),

\[
f_{s^{(0,\infty)}} = 0, \ \text{and} \ \ f_{s^{(0,\infty)}} = 0,
\]

and thus for each \( a \in A \),

\[
f_{a^{(0,\infty)}} = \sum_{s \in s_k} r(a, s) \cdot f_{s^{(0,\infty)}} \ \text{and} \ f_{a^{(0,\infty)}} = \sum_{s \in s_k} r(a, s) \cdot f_{s^{(0,\infty)}}.
\]

By Claim \( 1 \), \( \hat{L}_k^{(n)} \), \( L_k^{(n)} \) \( K \) \( 0 \) \( d^{(0,\infty)} \) \( K \).

It remains at step 0 to prove \( 19 \) for \( m = 0 \). Claim \( 2 \) asserts this. It states that the average cost of users from groups \( k \in K \) w.r.t. NE profiles \( f^{(n)} \) will be asymptotically equal to that of those users w.r.t. SO profiles \( f^{(n)} \), no matter which strategies the users from the other groups \( k \in K \)
The proof of Claim 2 is inspired by the proof of the main result Theorem 3.2 in [28]. However, here, we need an additional argument for SO profiles $f^*(n)$, since they now need not be SO profiles of the marginal game consisting of groups $k \in K_0$. The idea to handle this is to consider profiles $f^*(n)$ as NE profiles of the corresponding game with auxiliary price functions $c_a(\cdot)$. Interestingly, the two corresponding marginal games converge to limit games sharing NE profiles, since

$$\lim_{x \to \infty} \frac{c_a(x)}{\tau_a(x)} = 1 + \rho_a$$

for each $a \in A$. A trivial fact hidden in the proof is that users from groups $k \in K \setminus K_0$ do not affect the limit behavior of users from $K_0$, since they only account for a negligible limit proportion in the whole user volume. This makes it possible to independently consider groups $k \in K_0$ in the limit analysis.

**Claim 2.**

$$\lim_{n \to \infty} \sum_{k \in K_0} C_k(f^*(n)) = 1.$$

Moreover,

$$\sum_{k \in K_0} C_k(f^*(n)), \sum_{k \in K_0} C_k(f^*(n)) \in \Theta(T(d(n))^{\alpha_0+1}).$$

**Proof of Claim 2.** The main step of this proof is to show that the marginal game consisting of all groups in $K_0$ will “converge” to a limit game with user volume vector $d^{(0,\infty)} = (d_k^{(0,\infty)})_{k \in K_0}$, and $(f_s^{(0,\infty)})_{s \in S_k,k \in K_0}$ and $(f_s^{*(0,\infty)})_{s \in S_k,k \in K_0}$ are both NE profiles of the limit game. To show this, we employ a similar argument as in the proof of Theorem 3.2 in [28].

Let $g_n := T_0(d(n))^{\alpha_0}$ be a scaling factor for each $n \in \mathbb{N}$. Note that for each resource $a \in A$ and each $x > 0$,

$$\lim_{n \to \infty} \frac{\tau_a(T_0(d(n))x)}{g_n} =: \tau_a^{(0,\infty)}(x) = \begin{cases} 0, & \text{if } \rho_a < \alpha_0, \\ b_a \cdot x^{\alpha_0}, & \text{if } \rho_a = \alpha_0, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\lim_{n \to \infty} \frac{c_a(T_0(d(n))x)}{g_n} =: c_a^{(0,\infty)}(x) = \begin{cases} 0, & \text{if } \rho_a < \alpha_0, \\ b_a \cdot (\alpha_0 + 1) \cdot x^{\alpha_0}, & \text{if } \rho_a = \alpha_0, \\ \infty, & \text{otherwise,} \end{cases}$$

where $b_a > 0$ is the coefficient of term $x^{\alpha_0}$ in the polynomial $\tau_a(\cdot)$ if $\tau_a(\cdot)$ has degree $\alpha_0$, for all $a \in A$. 

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Let us define the two “limit” games
\[
\Gamma_{\tau}^{(0,\infty)} := (\mathcal{K}_0, A, \bigcup_{k \in \mathcal{K}_0} S_k, (r(a, s))_{a \in A, s \in S_k}, (\tau_{a}^{(0,\infty)})_{a \in A}, d^{(0,\infty)})
\]
and
\[
\Gamma_{c}^{(0,\infty)} := (\mathcal{K}_0, A, \bigcup_{k \in \mathcal{K}_0} S_k, (r(a, s))_{a \in A, s \in S_k}, (c_{a}^{(0,\infty)})_{a \in A}, d^{(0,\infty)}).
\]
Obviously, \(\Gamma_{\tau}^{(0,\infty)}\) and \(\Gamma_{c}^{(0,\infty)}\) have the same NE profiles and SO profiles, since
\[
c_{a}(x)^{(0,\infty)} = (\alpha_0 + 1) \cdot \tau_{a}(x)^{(0,\infty)}
\]
for all \(a \in A\).

We now aim to show that \((\tilde{f}_{s}^{(0,\infty)})_{s \in S_k, k \in \mathcal{K}_0}\) is an NE profile of the game \(\Gamma_{\tau}^{(0,\infty)}\). Let us arbitrarily fix some \(k \in \mathcal{K}_0\) and two strategies \(s, s' \in S_k\) with \(\tilde{f}_{s}^{(0,\infty)} > 0\). This implies by Fact 2 that \(s\) is tight. By L3) of Definition 3, we obtain that \(\tau_{s}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}) < \infty\). Thus, if \(\tau_{s'}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}) = \infty\), then
\[
\tau_{s}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}) \leq \tau_{s'}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}),
\]
(22)
We now assume that \(\tau_{s'}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}) < \infty\). Then, by L3) of Definition 3, \(s'\) is also tight. We will prove that (22) still holds in this case, which, in turn, implies that \((\tilde{f}_{s}^{(0,\infty)})_{s \in S_k, k \in \mathcal{K}_0}\) is an NE profile of the game \(\Gamma_{\tau}^{(0,\infty)}\), due to the arbitrary choice of \(s, s'\).

We recall that
\[
\lim_{n \to \infty} \frac{\tilde{f}_{s}^{(n)}}{T_0(d^{(n)})} = \tilde{f}_{s}^{(0,\infty)} > 0.
\]
Thus, we obtain for large enough \(n\) that
\[
\frac{\tilde{f}_{s}^{(n)}}{T_0(d^{(n)})} > 0,
\]
which implies that \(\tilde{f}_{s}^{(n)} > 0\) for large enough \(n\). Since each \(\tilde{f}_{s}^{(n)}\) is an NE profile for each \(n \in \mathbb{N}\), we further obtain by the user optimality (2) that
\[
\tau_{s}(\tilde{f}_{s}^{(n)}) \leq \tau_{s'}(\tilde{f}_{s}^{(n)}),
\]
for large enough \(n\). Hence, by Fact 1 in the proof of Lemma 1, we obtain that
\[
\tau_{s}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}) = \lim_{n \to \infty} \frac{\tau_{s}(\tilde{f}_{s}^{(n)})}{g_n} \leq \lim_{n \to \infty} \frac{\tau_{s'}(\tilde{f}_{s}^{(n)})}{g_n} = \tau_{s'}^{(0,\infty)}(\tilde{f}_{s}^{(0,\infty)}),
\]
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since both $s$ and $s'$ are tight.

Hence, $(\tilde{f}_s^{(0,\infty)})_{s \in S_k, k \in K_0}$ is an NE profile of the game $\Gamma^{(0,\infty)}_\tau$. Similarly, we can prove that $(f_s^{*,(0,\infty)})_{s \in S_k, k \in K_0}$ is an NE profile of the game $\Gamma^{(0,\infty)}_c$, which, in turn, implies that $(f_s^{*,(0,\infty)})_{s \in S_k, k \in K_0}$ is also an NE profile of the game $\Gamma^{(0,\infty)}_\tau$. Therefore, $(\tilde{f}_s^{(0,\infty)})_{s \in S_k, k \in K_0}$ and $(f_s^{*,(0,\infty)})_{s \in S_k, k \in K_0}$ have equal cost w.r.t. the game $\Gamma^{(0,\infty)}_\tau$. Hence,

$$\sum_{k \in K_0} \sum_{s \in S_k : \rho_s \leq \alpha_0} f_s^{(0,\infty)} \cdot \sum_{a \in A} r(a, s) \cdot \tau_a^{(0,\infty)}(\tilde{f}_a^{(0,\infty)}) = 1,$$

where we observe the fact that for $k \in K_0$, each strategy $q \in S_k$ with degree $\rho_q > \alpha_0$ will be non-tight, since there exists a strategy $p \in S_k$ with $\rho_p = \rho_k \leq \alpha_0$. Moreover, both the numerator and denominator in (23) are positive and finite, i.e., in $\Theta(1)$, since there exists a $k \in K_0$ such that $d_k^{(0,\infty)} > 0$ and $\rho_k = \alpha_0$.

By Fact 1 and Fact 2 in the proof of Lemma 1, we obtain immediately that

$$\lim_{n \to \infty} \frac{\sum_{k \in K_0} C_k(\tilde{f}_s^{(n)})}{T(d^{(n)}) \cdot g_n} = \sum_{k \in K_0} \sum_{s \in S_k : \rho_s \leq \alpha_0} f_s^{(0,\infty)} \cdot \sum_{a \in A} r(a, s) \cdot \tau_a^{(0,\infty)}(\tilde{f}_a^{(0,\infty)}) \in \Theta(1),$$

and

$$\lim_{n \to \infty} \frac{\sum_{k \in K_0} C_k(f_s^{*(n)})}{T(d^{(n)}) \cdot g_n} = \sum_{k \in K_0} \sum_{s \in S_k : \rho_s \leq \alpha_0} f_s^{*,(0,\infty)} \cdot \sum_{a \in A} r(a, s) \cdot \tau_a^{(0,\infty)}(f_a^{*,(0,\infty)}) \in \Theta(1).$$

Here (25) is obtained from the fact that profiles $f_s^{*(n)}$ are NE profiles of the corresponding game with price functions $c_s(\cdot)$, and that

$$\lim_{n \to \infty} \frac{c_s(T(d^{(n)})z)}{g_n} = c_s^{(0,\infty)}(z) = (1 + \rho_a)\tau_a^{(0,\infty)}(z) = (1 + \rho_a) \cdot \lim_{n \to \infty} \frac{\tau_a(T(d^{(n)})z)}{g_n}.$$

Claim 2 then follows from (23), (24) and (25).
In the proof of Claim \(2\) and the scaling factors \(g_n\) play pivotal roles as the equation \(26\) does not hold without them.

If \(\mathcal{K}_0 = \mathcal{K}\), then we have already finished the decomposition and completed the proof of Theorem \(2\) by Claim \(2\). We assume now that \(\mathcal{K}\setminus \mathcal{K}_0 \neq \emptyset\). In this case, we need to further partition \(\mathcal{K}\setminus \mathcal{K}_0\) and proceed to step \(m = 1\).

**Step \(m = 1\): Construct \(\mathcal{K}_1\), and prove (19) for \(m = 1\).**

To facilitate our discussion, we will first define some notations and propose further assumptions on the fixed user volume sequence \(\{d^{(n)}\}_{n \in \mathbb{N}}\).

For each \(n \in \mathbb{N}\) and each \(a \in A\), we denote by

\[
\tau_a^{(1,n)}(x) := \tau_a(x + f_a^{(n)}(\mathcal{K}_0)) \quad \text{and} \quad c_a^{(1,n)}(x) := c_a(x + f_a^{(n)}(\mathcal{K}_0))
\]

the price function of resource \(a\) under the condition that the users from groups \(k \in \mathcal{K}_0\) stick to strategies they used in NE profiles \(\tilde{f}^{(n)}\) and SO profiles \(f^s^{(n)}\), respectively. Here,

\[
\tilde{f}_a^{(n)}(\mathcal{K}_0) := \sum_{k \in \mathcal{K}_0, s \in S_k} \sum_{a \in A} r(a, s) \tilde{f}_s(n) \in O(T_0(d^{(n)}))
\]

and

\[
f_a^s(n)(\mathcal{K}_0) := \sum_{k \in \mathcal{K}_0, s \in S_k} \sum_{a \in A} r(a, s) f_s^{(n)} \in O(T_0(d^{(n)}))
\]

denote the volumes of resource \(a\) consumed by users from groups \(k \in \mathcal{K}_0\) w.r.t. NE profile \(\tilde{f}^{(n)}\) and SO profile \(f^s^{(n)}\), respectively, for each \(a \in A\) and each \(n \in \mathbb{N}\).

Let

\[f^s(n)(\mathcal{K}\setminus \mathcal{K}_0) := (f^s_k(n))_{k \in \mathcal{K}\setminus \mathcal{K}_0; s \in S_k} \text{ and } \tilde{f}^{(n)}(\mathcal{K}\setminus \mathcal{K}_0) := (\tilde{f}_s(n))_{k \in \mathcal{K}\setminus \mathcal{K}_0; s \in S_k}\]

be the “marginal” profiles consisting of users from the remaining groups \(\mathcal{K}\setminus \mathcal{K}_0\) w.r.t. the SO profile \(f^s^{(n)}\) and NE profile \(\tilde{f}^{(n)}\), respectively, for each \(n \in \mathbb{N}\). Obviously,

\[
\tau_a^{(1,n)}(\tilde{f}^{(n)}(\mathcal{K}\setminus \mathcal{K}_0)) = \tau_a(\tilde{f}^{(n)}) \quad \text{and} \quad c_a^{(1,n)}(f^s(n)(\mathcal{K}\setminus \mathcal{K}_0)) = c_a(f^s(n))
\]

for each \(a \in A\) and each \(n \in \mathbb{N}\).

Let

\[
\Gamma_\tau^{(1,n)} := (\mathcal{K}\setminus \mathcal{K}_0, A, \bigcup_{k \in \mathcal{K}\setminus \mathcal{K}_0} \mathcal{S}_k, (r(a, s))_{a \in A, s \in S_k, k \in \mathcal{K}\setminus \mathcal{K}_0}, (\tau_a^{(1,n)})_{a \in A}, (d_k^{(n)})_{k \in \mathcal{K}\setminus \mathcal{K}_0})
\]

and

\[
\Gamma_c^{(1,n)} := (\mathcal{K}\setminus \mathcal{K}_0, A, \bigcup_{k \in \mathcal{K}\setminus \mathcal{K}_0} \mathcal{S}_k, (r(a, s))_{a \in A, s \in S_k, k \in \mathcal{K}\setminus \mathcal{K}_0}, (c_a^{(1,n)})_{a \in A}, (d_k^{(n)})_{k \in \mathcal{K}\setminus \mathcal{K}_0})
\]

for each \(n \in \mathbb{N}\) and each \(a \in A\).
be the corresponding marginal games under the condition that users from groups in \( \mathcal{K}_0 \) stick to the strategies they used in NE profile \( \hat{f}^{(n)} \) and SO profile \( f^*(n) \), respectively, for each \( n \in \mathbb{N} \). Obviously, profiles \( \tilde{f}^{(n)}(\mathcal{K}\setminus\mathcal{K}_0) \) and \( f^*(n)(\mathcal{K}\setminus\mathcal{K}_0) \) are NE profiles of the two marginal games \( \Gamma^{(1,n)}_\tau \) and \( \Gamma^{(1,n)}_c \), respectively, w.r.t. the user volume vector \( (d_k^{(n)})_{k\in\mathcal{K}\setminus\mathcal{K}_0} \), for each \( n \in \mathbb{N} \).

At step \( m = 1 \), we shall employ an argument similar to that for step \( m = 0 \). However, we shall now consider the marginal profiles and the marginal games consisting of groups from \( \mathcal{K}\setminus\mathcal{K}_0 \).

Let \( T_1(d^{(n)}) = \sum_{k\in\mathcal{K}\setminus\mathcal{K}_0} d_k^{(n)} \) be the total volume of users from groups in \( \mathcal{K}\setminus\mathcal{K}_0 \), for each \( n \in \mathbb{N} \). Then, \( T_1(d^{(n)}) \in o(T_0(d^{(n)})) \). Then, it follows trivially for the price functions \( \tau_{a}^{(1,n)}(\cdot) \) and \( c_{a}^{(1,n)}(\cdot) \) that, for each \( x \geq 0 \),

\[
\tau_{a}^{(1,n)}(T_1(d^{(n)})x) \in \Theta\left( \max \{ \tau_{a}(T_1(d^{(n)})x), \tau_{a}(\hat{f}^{(n)}(\mathcal{K}_0)) \} \right) \tag{27}
\]

and

\[
c_{a}^{(1,n)}(T_1(d^{(n)})x) \in \Theta\left( \max \{ c_{a}(T_1(d^{(n)})x), c_{a}(f^*(n)(\mathcal{K}_0)) \} \right), \tag{28}
\]

for each \( a \in A \), since both \( \tau_{a}(\cdot) \) and \( c_{a}(\cdot) \) are asymptotically non-decreasing polynomials for all \( a \in A \).

Similarly, we assume, w.l.o.g., that

- the limit \( \lim_{n \to \infty} \frac{d_k^{(n)}}{T_1(d^{(n)})} =: d_k^{(1,\infty)} \) exists for some constant \( d_k^{(1,\infty)} \in [0,1] \), for each \( k \in \mathcal{K}\setminus\mathcal{K}_0 \),

- the limits

\[
\lim_{n \to \infty} \frac{\hat{f}^{(n)}}{T_1(d^{(n)})} =: \hat{f}_s^{(1,\infty)} \quad \text{and} \quad \lim_{n \to \infty} \frac{f^*(n)}{T_1(d^{(n)})} =: f_s^{*(1,\infty)}
\]

exist for some constants \( \hat{f}_s^{(1,\infty)}, f_s^{*(1,\infty)} \in [0,1] \) for each \( k \in \mathcal{K}\setminus\mathcal{K}_0 \) and each \( s \in \mathcal{S}_k \).

Otherwise, we can again take an infinite subsequence \( \{n_i\}_{i\in\mathbb{N}} \) fulfilling these two conditions. Again, let

\[
\hat{f}_a^{(1,\infty)} := \sum_{k\in\mathcal{K}\setminus\mathcal{K}_0} \sum_{s\in\mathcal{S}_k} r(a,s) \cdot \hat{f}_s^{(1,\infty)} = \sum_{k\in\mathcal{K}\setminus\mathcal{K}_0} \sum_{s\in\mathcal{S}_k} r(a,s) \cdot \hat{f}_s^{(1,\infty)}
\]

and

\[
f_a^{*(1,\infty)} := \sum_{k\in\mathcal{K}\setminus\mathcal{K}_0} \sum_{s\in\mathcal{S}_k} r(a,s) \cdot f_s^{*(1,\infty)} = \sum_{k\in\mathcal{K}\setminus\mathcal{K}_0} \sum_{s\in\mathcal{S}_k} r(a,s) \cdot f_s^{*(1,\infty)}
\]
for each resource $a \in A$, where $\mathcal{K}_1$ is defined below.

Now we are ready to partition $\mathcal{K}\setminus \mathcal{K}_0$. We define $\alpha_1 := \max\{\rho_k : k \in \mathcal{K}\setminus \mathcal{K}_0, d_k^{1,\infty} > 0\} > \alpha_0$, and $\mathcal{K}_1 := \{k \in \mathcal{K}\setminus \mathcal{K}_0 : \rho_k \leq \alpha_1\}$. Obviously,

$$\sum_{k \in \mathcal{K}_1} \frac{d_k^{(n)}}{T_1(d^{(n)})} \to \sum_{k \in \mathcal{K}_1} d_k^{1,\infty} = 1 \quad \text{as} \quad n \to \infty,$$

and there exists $k \in \mathcal{K}_1$ such that $\rho_k = \alpha_1$ and $d_k^{(n)} \in \Theta(T_1(d^{(n)}))$.

To show (19), we need a tighter bound of $\tilde{L}_k^{(n)}$ and $L_k^{(n)}$ for each $k \in \mathcal{K}_1$. Note that for each $k \in \mathcal{K}_1$, $\tilde{L}_k^{(n)}$ is still the cost of users from group $k$ w.r.t. the (marginal) NE profile $\tilde{f}^{(n)}(\mathcal{K}\setminus \mathcal{K}_0)$ and the (marginal) game $\Gamma^{1,n}$. Similarly, $L_k^{(n)}$ is still the cost of users from group $k$ w.r.t. the (marginal) NE profile $f^{(n)}(\mathcal{K}\setminus \mathcal{K}_0)$ and the (marginal) game $\Gamma^{1,n}$, for each $n \in \mathbb{N}$.

Similar to the Claim 2 at step $m = 0$, Claim 3 estimates the cost of users w.r.t. the two marginal games and corresponding NE profiles.

**Claim 3.** For each group $k \in \mathcal{K}_1$,

$$\tilde{L}_k^{(n)}, L_k^{(n)} \in O\left(\max\{g_a^{(0)}, g_a^{(1)}\}\right),$$

where $g_a^{(0)} = T_0(d^{(n)})^{\alpha_0}$ is the scaling factor at step $m = 0$, and $g_a^{(1)} = T_1(d^{(n)})^{\alpha_1}$ will be the scaling factor used at step $m = 1$, for each $n \in \mathbb{N}$.

**Proof of Claim 3**. We first prove that for each resource $a \in A$,

$$\tau_a(\tilde{f}_a^{(n)}(\mathcal{K}_0)), \tau_a(f_a^{(n)}(\mathcal{K}_0)) \in O(g_a^{(0)}). \quad (29)$$

We only prove this for NE profiles $\tilde{f}^{(n)}$. An almost identical argument applies to the SO profiles $f^{(n)}$.

Consider an arbitrarily fixed $a \in A$. If $j_a^{(n)}(\mathcal{K}_0) = \sum_{s \in S_{k'}, k' \in \mathcal{K}_0} r(a, s) \cdot \tilde{f}_s^{(n)} > 0$, then $\tilde{f}_s^{(n)} > 0$ for some $s \in S_{k'}$ with $r(a, s) > 0$ and for some $k' \in \mathcal{K}_0$. By Claim 1, if $\tilde{f}_s^{(n)} > 0$, i.e., $s$ is used by some users from group $k' \in \mathcal{K}_0$, then

$$\tau_s(\tilde{f}^{(n)}) = \tilde{L}_{k'}^{(n)} \in O(T(d^{(n)})^{\alpha_0}) = O(g_a^{(0)}),$$

which in turn implies that

$$\tau_a(\tilde{f}_a^{(n)}(\mathcal{K}_0)) \in O(g_a^{(0)}).$$
since
\[ \tau_a(\tilde{f}^{(n)}(K_0)) \leq \tau_a(\tilde{f}^{(n)}). \]

If \( \tilde{f}^{(n)}(K_0) = 0 \), then
\[ \tau_a(\tilde{f}^{(n)}(K_0)) = \eta_a \in \Theta(1), \]
for some constant \( \eta_a \geq 0 \). Thus we obtain that
\[ \tau_a(\tilde{f}^{(n)}(K_0)) \in O(g^{(n)}). \]

A similar result holds for SO profiles \( f^{*} \).

We are now ready to finish the proof of Claim 3. We first prove this for NE profiles \( \tilde{f}^{(n)}(K \setminus K_0) \). By (29), (27) and (28), we obtain for each \( a \in A \) that
\[
\tau_a^{(1,n)}(\tilde{f}^{(n)}(K \setminus K_0)) = \tau_a^{(1,n)}(T_1(d^{(n)})\tilde{f}^{(n)}(K \setminus K_0) T_1(d^{(n)}))
\in O\left(\max\{g^{(0)}, T_1(d^{(n)})\rho_a\}\right).
\]
where we observe that \( \frac{\tilde{f}^{(n)}(K \setminus K_0)}{T_1(d^{(n)})} \in O(1) \),
since
\[
\lim_{n \to \infty} \frac{\tilde{f}^{(n)}(K \setminus K_0)}{T_1(d^{(n)})} = \tilde{f}^{(1,\infty)}_a \in O(1).
\]

Let \( k \in K_1 \) be an arbitrarily fixed group. For each strategy \( s \in S_k \), we obtain by the above discussion that the price of \( s \) w.r.t. the NE profiles \( \tilde{f}^{(n)}(K \setminus K_0) \) and the marginal games \( \Gamma^{(1,n)}_{\tau} \) are in
\[
O\left(\max\{g^{(0)}_n, T_1(d^{(n)})\rho_s\}\right).
\]
By the user optimality (2) of NE profiles, we then obtain that the cost \( \tilde{L}^{(n)}_k \) of users in each group \( k \in K_1 \) is in
\[
O\left(\max\{g^{(0)}_n, T_1(d^{(n)})\rho_s\}\right) \subseteq O\left(\max\{g^{(0)}_n, g^{(1)}_n\}\right),
\]
since \( \rho_s \geq \rho_k \) and \( \rho_k \leq \alpha_1 \) for each \( s \in S_k \) and \( k \in K_1 \).

An almost identical argument carries over to the NE profiles \( f^{*}(n)(K \setminus K_0) \) w.r.t. the marginal games \( \Gamma^{(1,n)}_c \). This completes the proof. \( \square \)
With Claim 3, we can now prove (19) for \( m = 1 \). To this end, we make the further assumption that

\[
\lim_{n \to \infty} \frac{g_n^{(1)}}{g_n^{(0)}} = \lim_{n \to \infty} \frac{T_1(d^{(n)})^{\alpha_1}}{T_0(d^{(n)})^{\alpha_0}} = \beta_0,
\]

for some constant \( \beta_0 \in [0, \infty) \). Otherwise, one can take a subsequence fulfilling this condition.

**Claim 4.**

\[
\lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(\tilde{f}(n))}{\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(f^*(n))} = 1.
\]

Moreover,

\[
\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(\tilde{f}(n)), \quad \sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(f^*(n)) \in \Theta \left( \max\{T_0(d^{(n)})^{\alpha_0+1}, T_1(d^{(n)})^{\alpha_1+1}\} \right)
\]

\[
= \Theta \left( \max\{T_0(d^{(n)}) \cdot g_n^{(0)}, T_1(d^{(n)}) \cdot g_n^{(1)}\} \right).
\]

**Proof of Claim 4.** We separate the discussion into two cases.

**Case 1:** \( \beta_0 < \infty \) In this case, the total cost of the groups from \( K_1 \) are negligible w.r.t. the total cost of the groups from \( K_0 \). By Claim 2, see (24) and (25), we obtain for large enough \( n \) that

\[
\sum_{k \in K_0} C_k(\tilde{f}(n)) \approx \sum_{k \in K_0} C_k(f^*(n)) \in \Theta \left( T_0(d^{(n)}) \cdot g_n^{(0)} \right).
\]

With Claim 3 we obtain that

\[
\sum_{k \in K_1} C_k(\tilde{f}(n)) = \sum_{k \in K_1} d_k^{(n)} \cdot \tilde{L}_k^{(n)} \in O \left( T_1(d^{(n)}) \cdot g_n^{(0)} \right),
\]

since \( g_n^{(1)} \in O(g_n^{(0)}) \) in this case.

Note that

\[
\tau_a(f^*(n)) \in \Theta \left( c_a(f^*(n)) \right) \quad \text{and} \quad c_a(f^*(n)) = c_a^{(1,n)}(f^*(n)(K \setminus K_0))
\]

for each \( a \in A \). Thus, again by Claim 3 we obtain

\[
\sum_{k \in K_1} C_k(f^*(n)) \in \Theta \left( \sum_{k \in K_1} d_k^{(n)} \cdot L_k^{(n)} \right) \leq O \left( T_1(d^{(n)}) \cdot g_n^{(0)} \right).
\]
Since $T_1(d^{(n)}) \in o(T_0(d^{(n)}))$, we obtain that
\[ \sum_{k \in K_1} C_k(\tilde{f}^{(n)}) \in o\left( \sum_{k \in K_0} C_k(\tilde{f}^{(n)}) \right) \quad \text{and} \quad \sum_{k \in K_1} C_k(f_*^{(n)}) \in o\left( \sum_{k \in K_0} C_k(f_*^{(n)}) \right). \]
This gives
\[ \lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^1 \mathcal{K}_u} C_k(\tilde{f}^{(n)})}{\sum_{k \in \bigcup_{u=0}^1 \mathcal{K}_u} C_k(f_*^{(n)})} = \lim_{n \to \infty} \frac{\sum_{k \in K_0} C_k(\tilde{f}^{(n)})}{\sum_{k \in K_0} C_k(f_*^{(n)})} = 1. \]

With Claim 2, we obtain that
\[ \sum_{k \in \bigcup_{u=0}^1 \mathcal{K}_u} C_k(\tilde{f}^{(n)}), \quad \sum_{k \in \bigcup_{u=0}^1 \mathcal{K}_u} C_k(f_*^{(n)}) \in \Theta\left( \max\{ T_0(d^{(n)}), T_1(d^{(n)}) \} \right), \]
since $g_n^{(1)} \in O(g_n^{(0)})$ and $T_1(d^{(n)}) \in o(T_0(d^{(n)}))$.

(Case 2: $\beta_0 = \infty$) In this case, we can use a similar argument as in the proof of Claim 2 to show that
\[ \lim_{n \to \infty} \sum_{k \in K_1} C_k(\tilde{f}^{(n)}) = 1. \]
To this end, we need to look more closely into the price functions $\tau^{(1,n)}_a(\cdot)$ and $\epsilon^{(1,n)}_a(\cdot)$.

Let $a \in A$ be an arbitrarily fixed resource. We aim to show that the price of $a$ is asymptotically determined only by users from groups $k \in K_1$ under the assumption that $\beta_0 = \infty$ and the scaling factor is $g_n^{(1)} = T_1(d^{(n)})^{\alpha_1}$ for each $n \in \mathbb{N}$.

If $\rho_a < \alpha_1$, then we obtain by (27) and (28) that the limit
\[ \lim_{n \to \infty} \frac{\tau^{(1,n)}_a(T_1(d^{(n)})x)}{g_n^{(1)}} = 0, \quad (30) \]
for each $x > 0$, since $g_n^{(0)} \in o(g_n^{(1)})$ and
\[ \tau^{(1,n)}_a(T_1(d^{(n)})x) \in O\left( \max\{ g_n^{(0)}, T_1(d^{(n)})^{\rho_a} \} \right) \subseteq o(g_n^{(1)}). \]

If $\rho_a \geq \alpha_1$, then we obtain by the definition of $\tau^{(1,n)}_a(\cdot)$ and (29) that the limit
\[ \lim_{n \to \infty} \frac{\tau^{(1,n)}_a(T_1(d^{(n)})x)}{g_n^{(1)}} = \lim_{n \to \infty} \frac{\tau_a(T_1(d^{(n)})x)}{T_1(d^{(n)})^{\alpha_1}}, \quad (31) \]
exists for all $x \geq 0$. Here we observe that
\[ \tau_a(\tilde{f}^{(n)}(K_0)) \in O(g_n^{(0)}) \subseteq o(g_n^{(1)}). \]

Similarly, we obtain that for each $x > 0$ and each $a \in A$ with $\rho_a \geq \alpha_1$, the limit
\[ \lim_{n \to \infty} \frac{c_a^{(1,n)}(T_1(d^{(n)})x)}{g_n^{(1)}} = \lim_{n \to \infty} \frac{c_a(T_1(d^{(n)})x)}{g_n^{(1)}} \]
exists. Then, (30) and (31) yield for each $x > 0$ and $a \in A$ that
\[ \lim_{n \to \infty} \frac{\tau_a^{(1,n)}(T_1(d^{(n)})x)}{g_n^{(1)}} =: \tau_a^{(1,\infty)}(x) = \begin{cases} 0, & \text{if } \rho_a < \alpha_1, \\ b_a \cdot (\alpha_1 + 1)x^{\alpha_1}, & \text{if } \rho_a = \alpha_1, \\ \infty, & \text{if } \rho_a > \alpha_1, \end{cases} \]
\[ \lim_{n \to \infty} \frac{c_a^{(1,n)}(T_1(d^{(n)})x)}{g_n^{(1)}} =: c_a^{(1,\infty)}(x) = \begin{cases} 0, & \text{if } \rho_a < \alpha_1, \\ b_a \cdot (\alpha_1 + 1)x^{\alpha_1}, & \text{if } \rho_a = \alpha_1, \\ \infty, & \text{if } \rho_a > \alpha_1, \end{cases} \tag{32} \]
where, again, $b_a > 0$ is the coefficient of the term $x^{\alpha_1}$ of polynomial $\tau_a(\cdot)$, if $\tau_a(\cdot)$ does have degree $\alpha_1$, for each $a \in A$. Notice that (32) actually indicates that users from groups $k \in \mathcal{K}\setminus \bigcup_{u=0}^t \mathcal{K}_u$ can be ignored when we discuss the limit behavior of users from groups $k \in \mathcal{K}_1$, since their total volume is negligible compared to that of users from groups $k \in \mathcal{K}_1$.

By (32), with an argument similar to that for Claim 2, we can show that
\[ f^{s(1,\infty)} := (f^{s(1,\infty)})_{k \in \mathcal{K}_1:s \in S_k} \quad \text{and} \quad f^{(1,\infty)} := (f^{(1,\infty)})_{k \in \mathcal{K}_1:s \in S_k} \]
are NE profiles w.r.t. the limit marginal games
\[ \Gamma^{(1,\infty)} := (\mathcal{K}_1\cup \bigcup_{k \in \mathcal{K}_1} S_k, (r(a, s))_{a \in A, s \in S_k, k \in \mathcal{K}_1}, (\tau_a^{(1,\infty)})_{a \in A, (d_k^{(1,\infty)})_{k \in \mathcal{K}_1}}) \]
and
\[ \Gamma^{(c,1,\infty)} := (\mathcal{K}_1\cup \bigcup_{k \in \mathcal{K}_1} S_k, (r(a, s))_{a \in A, s \in S_k, k \in \mathcal{K}_1}, (c_a^{(1,\infty)})_{a \in A, (d_k^{(1,\infty)})_{k \in \mathcal{K}_1}}), \]
respectively.

Moreover, we can show that
\[ \lim_{n \to \infty} \frac{\sum_{k \in \mathcal{K}_1} C_k(\tilde{f}^{(n)})}{T_1(d^{(1)})g_n^{(1)}} = \sum_{k \in \mathcal{K}_1} \sum_{s \in S_k: \rho_s \leq \alpha_1} \sum_{a \in A} r(a, s) \tau_a^{(1,\infty)}(\tilde{f}_a^{(1,\infty)}) \in \Theta(1) \]
and
\[
\lim_{n \to \infty} \frac{\sum_{k \in K_1} C_k(f^*(n))}{T_1(d) \cdot g^{(1)}_n} = \sum_{k \in K_1} \sum_{s \in S_k : p_s \leq \alpha_1} f^s_{s}(1,\infty) \sum_{a \in A} r(a, s) \tau^{(1,\infty)}_a(f^s_{a}(1,\infty)) \in \Theta(1).
\]

Note that there exists at least one group \( k \in K_1 \), such that \( d^{(1,\infty)}_k > 0 \) and \( \rho_k = \alpha_1 \). Therefore,
\[
\sum_{k \in K_1} \sum_{s \in S_k : p_s \leq \alpha_1} \tilde{f}^s_{s}(1,\infty) \cdot \sum_{a \in A} r(a, s) \tau^{(1,\infty)}_a(f^s_{a}(1,\infty)) = 1,
\]
since the two games \( \Gamma^{(1,\infty)}_r \) and \( \Gamma^{(1,\infty)}_c \) have the same NE profiles, the numerator is the cost of the NE profile \( \tilde{f}^s_{s}(1,\infty) \), and the denominator is the cost of the NE profile \( f^s_{s}(1,\infty) \).

As a result, we obtain that
\[
\lim_{n \to \infty} \frac{\sum_{k \in K_1} C_k(\tilde{f}^{(n)})}{\sum_{k \in K_1} C_k(f^*(n))} = 1,
\]
which, in turn, implies by Claim 2 that
\[
\lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(\tilde{f}^{(n)})}{\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(f^*(n))} = 1.
\]
This completes the proof of Claim 4.

Consequently,
\[
\sum_{k \in K_1} C_k(\tilde{f}^{(n)}) \approx \sum_{k \in K_1} C_k(f^*(n)) \in \Theta\left(T_1(d^{(n)}) \cdot g^{(1)}_n\right).
\]

With Claim 2 we then further obtain that
\[
\sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(\tilde{f}^{(n)}), \sum_{k \in \bigcup_{u=0}^{1} K_u} C_k(f^*(n)) \in \Theta\left(\max\{T_0(d^{(n)})g^{(0)}_n, T_1(d^{(n)})g^{(1)}_n\}\right).
\]

If \( K = K_0 \cup K_1 \), then we have already finished the decomposition and completed the whole proof with Claim 4. Otherwise, we can continue with an argument similar to those at steps \( m = 0, 1 \). Note that this procedure will
eventually terminate, since the number $|\mathcal{K}| = K$ of groups is finite. We now outline the general inductive step $m = l$ for some integer $l = 0, \ldots, K$.

**Step $m = l$: construct $\mathcal{K}_l$, and prove (19) for $m = l$.**

We assume that we have partitioned $\mathcal{K}$ into $\mathcal{K}_0, \ldots, \mathcal{K}_{l-1}, \mathcal{K}\setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u$ for some integer $l = 0, \ldots, K$, where we use a convention that $\bigcup_{u=0}^{l-1} \mathcal{K}_u = \emptyset$, and $\mathcal{K}_{l-1} = \emptyset$. Moreover, we make the following inductive assumptions.

**IA1.** The limit
\[
\lim_{n \to \infty} \frac{T_{u+1}(d^{[n]}_{i_0})}{T_{u}(d^{[n]}_{i_0})} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\sum_{k \in \mathcal{K}_{i-1}} d^{[n]}_k}{T_{i-1}} = 1,
\]
for each $u = 0, 1, \ldots, l-2$, where each $T_u(d^{[n]}) = \sum_{k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u} d^{[n]}_k$, for each $n \in \mathbb{N}$ for $u = 0, \ldots, l-1$.

**IA2.** Define
\[
\alpha_u := \max \{ \rho_k : k \in \mathcal{K}_u \}
\]
for each $u = 0, 1, \ldots, l-1$.

**IA3.** For each $k \in \bigcup_{u=0}^{l-1} \mathcal{K}_u$,
\[
\hat{f}^{(n)}_k, L^{(n)}_k \in O\left( \max \left\{ g_0, \ldots, g_{l-1} \right\} \right),
\]
where each $g^{(u)}_n = T_u(d^{[n]})^{\alpha_u}$, for $u = 0, \ldots, l-1$.

**IA4.** For each $u = 0, \ldots, l-1$,
\[
\lim_{n \to \infty} \sum_{k \in \bigcup_{i=0}^{n_0} \mathcal{K}_i} C_k \left( \hat{f}^{(n)}_k \right) = 1,
\]
and
\[
\sum_{k \in \bigcup_{i=0}^{n_0} \mathcal{K}_i} C_k \left( f^{*\langle n \rangle} \right), \quad \sum_{k \in \bigcup_{i=0}^{n_0} \mathcal{K}_i} C_k \left( \hat{f}^{(n)} \right) \in \Theta\left( \max \{ T_0(d^{[n]})g^{(0)}_n, \ldots, T_u(d^{[n]})g^{(u)}_n \} \right).
\]

Note that we only need to check the validity of IA3-IA4 at each step. IA1 follows immediately from the definition of total user volumes $T_u(d^{[n]})$ of the marginals, and IA2 defines the “degree” of each $\mathcal{K}_u$ for $u = 0, \ldots, l-1$.

Again, if $\bigcup_{u=0}^{l-1} \mathcal{K}_u = \mathcal{K}$, then we have already finished the decomposition and completed the proof. Otherwise, we can apply an argument similar to
those above. Due to the heavy similarity, we now only list the key components at this general step \( m = l \), but omit the detailed proof.

Similarly we need the following assumptions:

- The limit
  \[
  \lim_{n \to \infty} \frac{d_k^{(n)}}{T_l(d^{(n)})} =: d_k^{(l, \infty)} \in [0, 1]
  \]
  exists for each \( k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1}\mathcal{K}_u \), where \( T_l(d^{(n)}) = \sum_{k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1}\mathcal{K}_u} d_k^{(n)} \).

- The limit
  \[
  \lim_{n \to \infty} \tilde{f}_s^{(n)} =: f_s^{(l, \infty)} \quad \text{and} \quad \lim_{n \to \infty} f_s^{*(n)} =: f_s^{*(l, \infty)}
  \]
  exist for some constants \( \tilde{f}_s^{(l, \infty)}, f_s^{*(l, \infty)} \in [0, 1] \) for each \( s \in \mathcal{S}_k \), and each \( k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1}\mathcal{K}_u \).

Otherwise, we can again take an infinite subsequence to fulfill these assumptions.

Similarly, we define

\[
\alpha_l := \max \left\{ \rho_k : k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1}\mathcal{K}_u, d_k^{(l, \infty)} > 0 \right\},
\]
and put

\[
\mathcal{K}_l := \left\{ k \in \mathcal{K}\setminus \bigcup_{u=0}^{l-1}\mathcal{K}_u : \rho_k \leq \alpha_l \right\}.
\]

Obviously, \( \mathcal{K}_l, \alpha_l \) and \( T_l(d^{(n)}) \) together validate \( IA1-IA2 \) for step \( m = l \). Moreover, there exists \( k \in \mathcal{K}_l \) such that \( d_k^{(l, \infty)} > 0 \) and \( \rho_k = \alpha_l \).

Similarly, we put

\[
\tau_a^{(l,n)}(x) := \tau_a \left( x + \tilde{f}_s^{(n)} \left( \bigcup_{u=0}^{l-1}\mathcal{K}_u \right) \right)
\]
and

\[
c_a^{(l,n)}(x) := c_a \left( x + f_s^{*(n)} \left( \bigcup_{u=0}^{l-1}\mathcal{K}_u \right) \right),
\]
be the price functions under the condition that users from groups in \( \bigcup_{u=0}^{l-1} \mathcal{K}_u \)
stick to the strategies they used in NE profiles \( \tilde{f}^{(n)} \) and SO profiles \( f^{(n)} \),
respectively, for each \( n \in \mathbb{N} \) and each \( a \in A \). Here

\[
\tilde{f}^{(n)}(\bigcup_{u=0}^{l-1} \mathcal{K}_u) := \sum_{k \in \mathcal{F}_{u}^{(n)}} \sum_{s \in S_k} r(a, s) \cdot \tilde{f}^{(n)}
\]

and

\[
f^{(n)}(\bigcup_{u=0}^{l-1} \mathcal{K}_u) := \sum_{k \in \mathcal{F}_{u}^{(n)}} \sum_{s \in S_k} r(a, s) \cdot f^{(n)}
\]

are the volumes of resource \( a \) consumed by users from groups \( \in \bigcup_{u=0}^{l-1} \mathcal{K}_u \)
w.r.t. the profiles \( \tilde{f}^{(n)} \) and \( f^{(n)} \), respectively, for each \( n \in \mathbb{N} \) and each \( a \in A \).

Obviously,

\[
\tau_a(\tilde{f}^{(n)}) = \tau_a(\tilde{f}^{(n)}(\mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u)) \quad \text{and} \quad c_a(f^{(n)}) = c_a(f^{(n)}(\mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u))
\]

for each \( n \in \mathbb{N} \), and each resource \( a \in A \). Here

\[
\tilde{f}^{(n)}(\mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u) := \tilde{f}^{(n)} - \tilde{f}^{(n)}(\bigcup_{u=0}^{l-1} \mathcal{K}_u) = \sum_{k \in \mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u} \sum_{s \in S_k} r(a, s) \cdot \tilde{f}_s
\]

denotes the volume of resource \( a \) consumed by users from groups \( k \in \mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u \)
w.r.t. NE profile \( \tilde{f}^{(n)} \). This yields similarly for \( f^{(n)}(\mathcal{K} \setminus \bigcup_{u=0}^{l-1} \mathcal{K}_u) \).

By the inductive assumption IA3 and a similar argument to the proof of Claim 3 we obtain for each \( a \in A \) that

\[
\tau_a(\tilde{f}^{(n)}(\bigcup_{u=0}^{l-1} \mathcal{K}_u)) \in O\left( \max\{ g_n^{(0)}, \ldots, g_n^{(l-1)} \} \right)
\]

and

\[
c_a(f^{(n)}(\bigcup_{u=0}^{l-1} \mathcal{K}_u)) \in O\left( \max\{ g_n^{(0)}, \ldots, g_n^{(l-1)} \} \right).
\]

Since both \( \tau_a(\cdot) \) and \( c_a(\cdot) \) are asymptotically non-decreasing, we thus obtain that

\[
c_a^{(1,n)}(T_l(d^{(n)}), x), \tau_a^{(1,n)}(T_l(d^{(n)}), x) \in O\left( \max \left\{ T_l(d^{(n)})^{\rho_n}, g_n^{(0)}, \ldots, g_n^{(l-1)} \right\} \right)
\]

for each \( a \in A \) and each \( x \geq 0 \). An argument similar to that for Claim 3 gives:
Claim 5. For each \( k \in \mathcal{K}_i \),
\[
\tilde{L}_k^{(n)}, L_k^{(n)} \in O\left( \max\{g_n^{(0)}, \ldots, g_n^{(l)}\}\right),
\]
where \( g_n^{(l)} = T_l(d_n)^{\alpha_l} \).

Claim 5 validates the inductive assumption \( IA_3 \) for step \( m = l \).

To validate the inductive assumption \( IA_4 \) for step \( m = l \), we assume that the limit
\[
\lim_{n \to \infty} \frac{g_l^{(n)}}{\max\{g_n^{(0)}, \ldots, g_n^{(l-1)}\}} = \beta_{l-1} \in [0, \infty]
\]
exists for some constant \( \beta_{l-1} \). In the case that \( \beta_{l-1} < \infty \), we obtain that \( g_n^{(l)} \in O\left( \max\{g_n^{(0)}, \ldots, g_n^{(l-1)}\}\right) \), which implies that groups \( \in \mathcal{K}_i \) are negligible w.r.t. groups \( \in \bigcup_{u=0}^{l-1} \mathcal{K}_u \), since \( T_l(d_n) \in o(T_{l-1}(d_n)) \). Hence, if \( \beta_{l-1} < \infty \), then \( IA_4 \) is valid for step \( m = l \).

In the case that \( \beta_{l-1} = \infty \), we obtain that \( g_n^{(l)} \in \omega\left( \max\{g_n^{(0)}, \ldots, g_n^{(l-1)}\}\right) \), which implies that the behavior of users from groups \( \mathcal{K}_i \) is asymptotically independent of users from groups \( \bigcup_{u=0}^{l-1} \mathcal{K}_u \). Then, an argument similar to that for Claim 4 applies. This yields Claim 6 below, and validates \( IA_4 \) for step \( m = l \).

Claim 6. \[
\lim_{n \to \infty} \frac{\sum_{k \in \bigcup_{u=0}^l \mathcal{K}_i} C_k\left(\tilde{f}^{(n)}\right)}{\sum_{k \in \bigcup_{u=0}^l \mathcal{K}_i} C_k\left(f^{*^{(n)}}\right)} = 1,
\]
and
\[
\sum_{k \in \bigcup_{u=0}^l \mathcal{K}_i} C_k\left(f^{*^{(n)}}\right), \sum_{k \in \bigcup_{u=0}^l \mathcal{K}_i} C_k\left(\tilde{f}^{(n)}\right) \in \Theta\left( \max\{T_0(d_n)^{g_n^{(0)}}, \ldots, T_l(d_n)^{g_n^{(l)}}\}\right).
\]

All above together validate the inductive assumptions \( IA_1-IA_4 \) for step \( m = l \). So the induction completes, and Theorem 2 is proved.

Proof of Lemma 8

Proof of Lemma 8. A proof of Lemma 8 may already exist in [3]. However, we cannot directly access [3]. Our knowledge on regular variation is actually indirectly obtained from Wikipedia on the page https://en.wikipedia.org/wiki/Slowly_varying_function.
Therefore, we supply a detailed proof to ensure completeness of this paper.

**Proof of a)**: Let $\epsilon > 0$ be an arbitrarily fixed constant. Using Karamata’s Characterization Theorem and Representation Theorem, see, e.g., [3], we can write

$$\tau(x) = x^\rho \cdot e^{\eta(x) + \int_1^x \frac{\xi(t)}{t} \, dt},$$

where

- $\eta(x)$ is a real-valued measurable function such that $\lim_{x \to \infty} \eta(x) = p$ for some constant $p \geq 0$,
- $b \geq 0$ is a constant, and $\xi(x)$ is a real-valued measurable function such that $\lim_{x \to \infty} \xi(x) = 0$.

Thus, we obtain that

$$\lim_{x \to \infty} \frac{\tau(x)}{x^\rho + \epsilon} = \lim_{x \to \infty} e^{-\epsilon \ln x + \eta(x) + \int_1^x \frac{\xi(t)}{t} \, dt} = \lim_{x \to \infty} e^{-\int_1^x \frac{\xi(t)}{t} \, dt + p + \int_1^x \frac{\xi(t)}{t} \, dt} = \lim_{x \to \infty} e^{-\int_1^x \frac{\xi(t)}{t} \, dt} = 0,$$

where we observe that

$$\lim_{x \to \infty} \int_1^x \frac{\epsilon - \xi(t)}{t} \, dt = \infty,$$

since $\xi(t) \to 0$ as $t \to \infty$, and $\epsilon > 0$.

Similarly, one can prove that

$$\lim_{x \to \infty} \frac{\tau(x)}{x^\rho - \epsilon} = \infty.$$

**Proof of b)**: For each $x > 0$,

$$\lim_{t \to \infty} \frac{g(tx)}{g(t)} = \lim_{t \to \infty} \frac{g(tx)}{g(t)} \cdot \lim_{t \to \infty} \frac{\tau(tx)}{\tau(t)} = x^\rho.$$

**Proof of c)**: For each $x > 0$,

$$\lim_{t \to \infty} \frac{\frac{g(tx)}{\tau(tx)}}{\frac{g(t)}{\tau(t)}} = \lim_{t \to \infty} \frac{g(tx)}{g(t)} \cdot \lim_{t \to \infty} \frac{\tau(tx)}{\tau(t)} = x^{\rho'} \cdot x^{-\rho} = x^{\rho' - \rho} \in (0, \infty).$$

□

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Proof of Lemma 9

Proof of Lemma 9. We assume that $\tau(x)$ is nondecreasing, non-negative, convex, differentiable and regularly varying with index $\rho \in \mathbb{R}$.

We first show that $\rho \geq 0$. Since $\tau(\cdot)$ is convex,

$$\tau((1 + \eta)x) = \tau((1 - \eta) \cdot x + \eta \cdot 2x) \leq (1 - \eta)\tau(x) + \eta\tau(2x)$$

for each $x \geq 0$ and each $\eta \in [0, 1]$. Therefore,

$$\frac{\tau((1 + \eta)x)}{\tau(x)} \leq 1 - \eta + \eta \frac{\tau(2x)}{\tau(x)}$$

for each $x$ with $\tau(x) > 0$. Letting $x \to \infty$, we obtain from the regular variation of $\tau(\cdot)$ that

$$(1 + \eta)^\rho \leq 1 - \eta + \eta \cdot 2^\rho \quad (33)$$

for each $\eta \in [0, 1]$, which, in turn, implies that $\rho \geq 0$. Otherwise, if $\rho < 0$, then

$$\frac{\partial}{\partial \eta} \left(1 - \eta + \eta \cdot 2^\rho - (1 + \eta)^\rho\right) = 2^\rho - 1 - \rho(1 + \eta)^{\rho-1} < 0,$$

when

$$0 \leq \eta < \left(\frac{\rho}{2^\rho - 1}\right)^\frac{1}{\rho} - 1 \in (0, 1).$$

Therefore, if $\rho < 0$, then we obtain for these $\eta$ that

$$1 - \eta + \eta \cdot 2^\rho - (1 + \eta)^\rho < 1 - 0 + 0 \cdot 2^\rho - (1 + 0)^\rho = 0,$$

which contradicts (33).

So, convexity of $\tau(\cdot)$ implies that $\rho \geq 0$.

Convexity and differentiability of $\tau(\cdot)$ further imply for each $t > 0$ and $x > 0$ that

$$\frac{1}{t} x \cdot \tau'(x) \leq \int_x^{(1 + \frac{1}{t})x} \tau'(u)du = \tau\left((1 + \frac{1}{t})x\right) - \tau(x).$$

Therefore,

$$\frac{x \cdot \tau'(x)}{t \tau(x)} \leq \left(\frac{\tau((1 + \frac{1}{t})x)}{\tau(x)} - 1\right)$$
for each $t > 0$ and $x > 0$. Letting $x \to \infty$, the regular variation of $\tau(x)$ yields

$$\lim_{x \to \infty} \frac{x\tau'(x)}{\tau(x)} \leq t \left( (1 + \frac{1}{t})^\rho - 1 \right)$$

for each $t > 0$. Note that $\rho \geq 0$ implies that

$$\lim_{t \to \infty} t \left( (1 + \frac{1}{t})^\rho - 1 \right) = \lim_{z \to 0} \frac{(1 + z)^\rho - 1}{z} = \rho.$$

So, altogether, if $\tau(\cdot)$ is convex and differentiable, then

$$\lim_{x \to \infty} \frac{x\tau'(x)}{\tau(x)} \leq \rho.$$

Similarly, using again the convexity and differentiability of $\tau$, we obtain for each $t > 1$ and each $x > 0$ that

$$\frac{1}{t} x\tau'(x) \geq \int_{(1 - \frac{1}{t}) x}^{x} \tau'(u) du = \tau(x) - \tau \left( (1 - \frac{1}{t}) x \right).$$

An almost identical argument to the above then yields

$$\lim_{x \to \infty} \frac{x\tau'(x)}{\tau(x)} \geq \lim_{z \to 0} \frac{1 - (1 - z)^\rho}{z} = \rho.$$

Altogether, we obtain that

$$\lim_{x \to \infty} \frac{x\tau'(x)}{\tau(x)} = \rho \geq 0,$$

which completes the proof. \qed

**Proof of Theorem 3**

Proof of Theorem 3. The proof is similar to that for Theorem 2. To save space, we only sketch the main idea and give details for some crucial points.

Consider an arbitrary sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$ of user volume vectors such that $\lim_{n \to \infty} T(d^{(n)}) = \infty$, and consider an NE profile $\bar{f}^{(n)}$ and an SO profile $f^{\star(n)}$ for each $n \in \mathbb{N}$. We want to apply the asymptotic decomposition to this sequence. The Theorem will follow directly from the arbitrary choice of the sequence $\{d^{(n)}\}_{n \in \mathbb{N}}$.
To construct suitable scaling factors at each inductive step, we now aim to define a suitable ordering $\preceq$ on the set $A$ of resources. Note that the degrees of the polynomial price functions serve as such an ordering in the proof of Theorem 2. Here, the $\tau_a(\cdot)$ are generally no longer polynomials, thus such an ordering is not so obvious.

Let $\rho_a$ denote the regular variation index of $\tau_a(\cdot)$ for each $a \in A$. Note that the “degrees” $\rho_a$ now cannot be used directly as such an ordering on the set $A$. We instead use the mutual comparability of the price functions to define a suitable ordering $\preceq$ on the set $A$. We put

$$a \preceq b \iff \lim_{x \to \infty} \tau_a(x) / \tau_b(x) = q_{a,b} < \infty.$$ 

Obviously, by Lemma 8, $\rho_a < \rho_b$ implies that $a \preceq b$. But the inverse need not be true.

This ordering on $A$ also carries over to the case that the price functions are $c_a(x) = x\tau'_a(x) + \tau_a(x)$. By Lemma 9 and Lemma 8 and the convexity of each $\tau_a(\cdot)$, we obtain immediately for each $a \in A$ that $c_a(\cdot)$ is also regular varying with the same index $\rho_a$, and

$$\lim_{x \to \infty} c_a(x) / \tau_a(x) = \rho_a + 1.$$ 

For each $s \in S$, let $\bar{a}_s$ be a maximum element of $\{a \in A : r(a, s) > 0\}$ w.r.t. the ordering $\preceq$. Then, we obtain by the mutual comparability that

$$q_{\bar{a}_s, \bar{a}'_s} = \lim_{x \to \infty} \tau_{\bar{a}_s}(x) / \tau_{\bar{a}'_s}(x) \in (0, \infty),$$ 

where $\bar{a}'_s$ is another maximum element of $\{a \in A : r(a, s) > 0\}$ w.r.t. the ordering $\preceq$. This means that these maximum elements are mutually equivalent w.r.t. the ordering $\preceq$. Hence, by Lemma 8, two maximum elements of $\{a \in A : r(a, s) > 0\}$ have the same regular variation index. Again, the inverse need not be true.

With the ordering $\preceq$, we can now easily identify the cheapest strategy $s_k^*$ for each group $k \in K$. Obviously, $s_k^*$ is a strategy $s \in S_k$ such that $\bar{a}_s \preceq \bar{a}_{s'}$ for each $s' \in S_k$.

The ordering $\preceq$ on resource set $A$ are crucial for the construction of the scaling factors of the marginal games at each inductive step in the asymptotic decomposition.

step $m = 0$: Construct $K_0$ and the scaling factors $g_n^{(0)}$
Let us put $T_0(d^{(n)}) = T(d^{(n)})$ for each $n \in \mathbb{N}$.

Similar to the proof of Theorem 2 we assume w.l.o.g. that the limits
\[
\lim_{n \to \infty} \frac{d_k^{(n)}}{T_0(d^{(n)})} = d_k^{(0,\infty)}, \quad f_s^{(0,\infty)} = \lim_{n \to \infty} \frac{f_s^{(n)}}{T_0(d^{(n)})}, \quad \text{and} \quad f_s^{*(0,\infty)} = \lim_{n \to \infty} \frac{f_s^{*(n)}}{T_0(d^{(n)})}
\]
exist for each $k \in K$ and $s \in S$. We put $\alpha_0 = \max_{\leq} \{ \bar{a}_{s_k} : k \in K, d_k^{(0,\infty)} > 0 \} \in A$, where the maximization is w.r.t. the ordering $\leq$. Note that if there are multiple maxima, then we can pick arbitrary one of them. Moreover, we put $K_0 = \{ k \in K : \bar{a}_{s_k} \leq \alpha_0 \}$, and $g_n^{(0)} = \tau_{a_0}(T_0(d^{(n)}))$ for each $n \in \mathbb{N}$.

With the user optimality (2), we can easily obtain for each $k \in K_0$ that
\[
\tilde{L}_k^{(n)} \cdot L_k^{*(n)} \in O(g_n^{(0)}) = O(\tau_{a_0}(T_0(d^{(n)}))),
\]
since there exists an strategy $s \in S_k$ such that $\bar{a}_s \leq \alpha_0$ for each $k \in K_0$, and the maximum volume of users adopting a strategy is in $O(T_0(d^{(n)}))$.

With the regular variation of the price functions $\tau_a(\cdot)$ and $c_a(\cdot)$, we obtain for each $x > 0$ that
\[
\lim_{n \to \infty} \frac{\tau_a(T(d^{(n)})) x}{g_n^{(0)}} = \lim_{n \to \infty} \frac{\tau_a(T(d^{(n)})) x}{\tau_{a_0}(T_0(d^{(n)}))} = q_{a,a_0} \cdot x^{a_0},
\]
and
\[
\lim_{n \to \infty} \frac{c_a(T(d^{(n)})) x}{g_n^{(0)}} = \lim_{n \to \infty} \frac{c_a(T(d^{(n)})) x}{\tau_{a_0}(T_0(d^{(n)}))} = (1 + \rho_{a_0}) q_{a,a_0} \cdot x^{a_0}.
\]
Therefore, we can then obtain from an argument similar to that for Claim 2 that
\[
\sum_{k \in K_0} C_k(\tilde{f}^{(n)}) \approx \sum_{k \in K_0} C_k(f_s^{*(n)}) \in \Theta(T_0(d^{(n)})) g_n^{(0)}.
\]
Here, we observe that there exists at least one group $k \in K_0$ such that $\bar{a}_{s_k} = \alpha_0$ and $d_k^{(0,\infty)} > 0$.

\textbf{step $m = l$ : Construct $K_l$ and scaling factors $g_n^{(l)}$}

Similar to the proof of Theorem 2 we now make inductive assumptions that we have constructed $K_0, \ldots, K_{l-1}$ and $g_n^{(0)}, \ldots, g_n^{(l-1)}$ such that for each $k \in \bigcup_{u=0}^{l-1} K_u$
\[
\tilde{L}_k^{(n)} \cdot L_k^{*(n)} \in O\left(\max\{g_n^{(0)}, \ldots, g_n^{(l-1)}\}\right),
\]

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and
\[
\sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f^{(n)}) \approx \sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f^{* (n)})
\]
\[
\in \Theta \left( \max \{ T_0(d^{(n)})g^{(0)}_n, \ldots, T_{l-1}(d^{(n)})g^{(l-1)}_n \} \right).
\]

Moreover, we assume for each \( u = 0, \ldots, l - 2 \) that
\[
\lim_{n \to \infty} \frac{T_{u+1}(d^{(n)})}{T_u(d^{(n)})} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\sum_{k \in K \setminus \bigcup_{u=0}^{l-1} K_u} d^{(n)}_k}{T_{l-1}(d^{(n)})} = 0.
\]

To construct \( K_l \) and \( g^{(l)}_n \), we make a further assumption that the limits
\[
\lim_{n \to \infty} \frac{d^{(n)}_k}{T_l(d^{(n)})} = d^{(l, \infty)}_k, \quad \lim_{n \to \infty} \frac{f^{(n)}_s}{T_l(d^{(n)})} = f^{(l, \infty)}_s, \quad \text{and} \quad \lim_{n \to \infty} \frac{f^{* (n)}_s}{T_l(d^{(n)})} = f^{* (l, \infty)}
\]
exist for each \( s \in S_k \) and \( k \in K \setminus \bigcup_{u=0}^{l-1} K_u \), where \( T_l(d^{(n)}) = \sum_{k \in K \setminus \bigcup_{u=0}^{l-1} K_u} d^{(n)}_k \) for each \( n \in \mathbb{N} \).

Let \( \alpha_l = \max \{ \tilde{a}_{s_i^*} : k \in K \setminus \bigcup_{u=0}^{l-1} K_u, d^{(l, \infty)}_k > 0 \} \) and \( K_{l/2} = \{ k \in K \setminus \bigcup_{u=0}^{l-1} K_u : \tilde{a}_{s_i^*} \leq \alpha_l \} \). Then, we put \( g^{(l)}_n = \tau_{\alpha_l}(T_l(d^{(n)})) \).

Similarly, we can obtain for each \( k \in K_l \) that
\[
\tilde{L}_k^{(n)}, L_k^{(n)} \in O \left( \max \{ g^{(0)}_n, \ldots, g^{(l)}_n \} \right),
\]
since both \( \tau_a(\cdot) \) and \( c_a(\cdot) \) are non-decreasing, and thus for each \( a \in A \) and each \( x > 0 \)
\[
\tau_a^{(l, n)}(T_l(d^{(n)}))x = \tau_a(T_l(d^{(n)}))x + \tilde{f}_a^{(n)} \bigcup_{u=0}^{l-1} K_u)
\]
\[
\in O \left( \max \{ \tau_a(T_l(d^{(n)})), \tau_a(\tilde{f}_a^{(n)} \bigcup_{u=0}^{l-1} K_u) \} \right)
\]
and
\[
c_a^{(l, n)}(T_l(d^{(n)}))x = c_a(T_l(d^{(n)}))x + f_a^{* (n)} \bigcup_{u=0}^{l-1} K_u)
\]
\[
\in O \left( \max \{ c_a(T_l(d^{(n)})), c_a(f_a^{* (n)} \bigcup_{u=0}^{l-1} K_u) \} \right).
\]
So, if \( g_n(l) \in O(\max\{g_n(0), \ldots, g_n(l-1)\}) \), then all groups in \( K_l \) are negligible w.r.t. groups \( \bigcup_{u=0}^{l-1} K_u \). Otherwise, \( g_n(l) \in \omega(\max\{g_n(0), \ldots, g_n(l-1)\}) \). If this is the case, we can then independently consider \( K_l \) with an argument similar to that for Theorem 2. This will yield
\[
\sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f(n)) \approx \sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f^*(n)) \\
\in \Theta(\max\{T_0(d(n))g_n(0), \ldots, T_{l-1}(d(n))g_n(l-1), T_l(d(n))g_n(l)\}),
\]
which completes the induction and finishes the proof. \( \square \)

**Proof of Theorem 4**

*Proof of Theorem 4.* This proof is very similar to those for Theorem 2 and Theorem 3. We thus only sketch the main idea.

Note that we will not need a suitable ordering on resources in this proof. Conditions \( G1'-G3' \) have already guaranteed the existence of a suitable scaling factor sequence \( \{g_n(l)\}_{n \in \mathbb{N}} \) at each inductive step.

We assume that we are now at an inductive step \( m = l \), and we have already shown that for each \( k \in \bigcup_{u=0}^{l-1} K_u \)
\[
\hat{L}_k^{(n)}, L_k^{*(n)} \in O(\max\{g_n(0), \ldots, g_n(l-1)\}),
\]
and
\[
\sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f(n)) \approx \sum_{k \in \bigcup_{u=0}^{l-1} K_u} C_k(f^*(n)) \\
\in \Theta(\max\{T_0(d(n))g_n(0), \ldots, T_{l-1}(d(n))g_n(l-1)\}),
\]
where all the notations are the same as those for the proofs of Theorem 2 and Theorem 3. Moreover, we make the further inductive assumptions that
\[
\lim_{n \to \infty} \frac{T_{u+1}(d(n))}{T_u(d(n))} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\sum_{k \in K \setminus \bigcup_{u=0}^{l-1} K_u} d_k^{(l,u)}}{T_{l-1}(d(n))} = 0
\]
for each \( u = 0, \ldots, l - 2 \). If \( K = \bigcup_{u=0}^{l-1} K_u \), then we terminate. Otherwise, we continue as follows.

We define \( K_l = \{k \in K \setminus \bigcup_{u=0}^{l-1} K_u : d_k^{(l,\infty)} > 0\} \), where all \( d_k^{(l,\infty)} \) are again defined similarly as in the proofs of Theorem 2 and Theorem 3. The
condition of Theorem 4 implies that there is a regularly varying function \( g(\cdot) \) for \( K_l \) such that \( G'_{l-1} \) hold. We put \( g_{i_l}^{(l)} = g(T_i(d^{(n)})) \), where we put again \( T_i(d^{(n)}) = \sum_{k \in K_i} d_k^{(n)} \) for each \( n \in \mathbb{N} \).

Then, by the user optimality (2) and conditions \( G'_{l-1} \), we can similarly obtain for each \( k \in K_l \) that

\[
\tilde{L}_k^{(n)} = L_k^{(n)} \in O\left( \max\{g_n^{(0)}, \ldots, g_n^{(l)}\}\right),
\]

since \( \tau_a(x) \) is non-decreasing, \([11]\) and thus \( c_a(x) = x\tau_a'(x) + \tau_a(x) \) are asymptotically non-decreasing for each \( a \in A \). By comparing \( g_n^{(l)} \) with \( \max\{g_n^{(0)}, \ldots, g_n^{(l-1)}\} \), we can again obtain with conditions \([11] \) and \( G3' \) that

\[
\sum_{k \in \bigcup_{u=0}^l K_u} C_{k}(f^{(n)}) \approx \sum_{k \in \bigcup_{u=0}^l K_u} C_{k}(f^{*^{(n)})}
\]

\[
\in \Theta\left( \max\{T_0(d^{(n)})g_n^{(0)}, \ldots, T_{l-1}(d^{(n)})g_n^{(l-1)}, T_l(d^{(n)})g_n^{(l)}\}\right),
\]

which validates the inductive assumption at step \( m = l \).

Therefore, the game is asymptotic decomposable and asymptotically well designed. \( \square \)

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