Inclusive weak decays of heavy hadrons with power suppressed terms at NLO

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Abstract

Within the heavy quark expansion techniques for the heavy hadron weak decays we analytically compute the coefficient of the power suppressed dimension five chromo-magnetic operator at next-to-leading order of QCD perturbation theory with the full dependence on the final state quark mass. We present explicit expressions for the total width of inclusive semileptonic decays including the power suppressed terms and for a few moments of decay differential distributions. One of the important phenomenological applications of our results is precision analysis of the decays of bottom mesons to charmed final states and extraction of the numerical value for the CKM matrix entry $|V_{cb}|$. 

1 Introduction

Presently the Standard Model of fundamental interactions is being thoroughly tested experimentally at colliders, but no definite signs of New Physics have been detected beyond the framework of the Standard Model. Neither new particles have been explicitly seen nor any significant deviations from the Standard Model values in the loop sensitive Wilson coefficients for flavor changing observables have been determined in high precision data (as a review, see e.g. [1]). Thus, the Standard Model has successfully passed all tests in the areas where it is certainly valid as a low-energy effective theory.

However, there is definitely life beyond SM. Some new phenomena – like neutrino masses and mixing - can be readily incorporated in a rather straightforward manner to extensions of SM. The other new effects – like dark matter – are of cosmological nature and related to still poorly understood realm of gravity and, strictly speaking, are outside the physics of the standard model domain. Nevertheless it seems certain that the scale of the traditionally expected extensions of the standard model – like supersymmetry or extra dimensions – has definitely moved from few TeV region to a higher one in energies that can make it unreachable at accelerators in foreseeing future, e.g. [2]. Since the New Physics scale moved higher the direct observation of new physics phenomena will not probably be explicit even at new machines (still one should wait for the results of the 14 TeV run of LHC!). In case that nothing will be seen the new phenomena beyond the standard model (if any at all!) can only be identified through detecting slight discrepancies between theoretical predictions within the SM and precision measurements at low energy with available tools.

Accurate theoretical predictions within the SM are of crucial importance in such a scenario. For these predictions to be reliable one first needs the precise numerical values for the key parameters of the SM itself. The least precisely quantitatively known sector of the SM is a quark flavor one where the quark Yukawa couplings to the Higgs field are not well known numerically. In the standard model they translate into the mixing angles between generations gathered in the CKM matrix and the vacuum expectation value of the Higgs field. The latter can be determined from the leptonic sector. Note that the flavor sector is also a most promising place in investigating the Higgs mechanism that is definitely of an effective origin and probably will be modified in future as the presence of a fundamental scalar in the “final” theory does not look convincing. All in all the flavor physics of quarks is the promising place to search for new physics and should be thoroughly studied (see, e.g. [3, 4]).

While the quark weak decays are mediated by the charged weak currents at tree level, which are believed not to have sizable contributions of possible new physics, their study is of importance
for precise determination of the numerical values of the CKM matrix elements. However, obtaining solid theoretical predictions for processes with quarks at the fundamental level requires the use of genuinely nonperturbative computational methods like QCD lattice calculations since eventually one has to make prediction for the experimental quantities that include hadrons and cannot be described in perturbation theory of QCD due to confinement. This is principal part of the problem but there is also a pure technical part. Even if the direct computation in terms of quarks would be relevant to the world of hadrons that partly can be made possible by choosing proper observables one will still face the problem of computational complexity of the calculation with sufficient accuracy that requires a rather large order of perturbation theory. The example is the description of the process $b \to s\gamma$.

Taking just the parton level of computation for hadronic processes one makes the technical part equivalent to that of the leptonic calculations where the benchmark level for the technical part of the computation is the evaluation of the muon lifetime. The muon decay is a source for the determination of the Fermi constant $G_F$ with high accuracy from a leptonic sector. First radiative corrections have been computed long time ago [5, 6]. To match the precision of the present experimental data for muon lifetime, the theoretical calculations have to be performed with very high accuracy. In this case the calculations are feasible, since the purely leptonic decays are well described within perturbation theory and the expansion parameter $\alpha \approx 1/137$ is small. The latest theoretical result includes the second order (NNLO) radiative corrections in the fine structure constant expansion [7]

$$\Gamma(\mu \to \nu_\mu e\bar{\nu}_e) = \frac{G_F^2 m_\mu^5}{192\pi^3} \left\{ 1 + \left( \frac{25}{8} - \frac{\pi^2}{2} \right) \frac{\alpha_r}{\pi} + (6.743 + \Delta\Gamma^{\text{had}}) \left( \frac{\alpha_r}{\pi} \right)^2 \right\}. \tag{1}$$

Here $m_\mu$ is the muon mass. The numerical value for the electron mass $m_e$ is set to zero everywhere but in the expression for the expansion parameter

$$\alpha_r = \alpha + \frac{2\alpha^2}{3\pi} \ln \frac{m_\mu}{m_e}.$$

The expressions with account for nonvanishing electron mass are known. The quantity $\Delta\Gamma^{\text{had}} = -0.042 \pm 0.002$ is the hadronic contribution that is known with uncertainty of about 5%. It cannot be computed from first principles for light quarks and is obtained by integrating the experimental data for the photon vacuum polarization. Note that the similar situation emerges with precision analysis one of the key leptonic observable – the muon anomalous magnetic moment $g - 2$. At present the hadronic contributions related to light quarks give the main uncertainty of theoretical prediction (e.g. [8, 9]). It is a general feature that quark sector influences even pure leptonic processes if the required accuracy is high enough, e.g. [10].
Eq. (1) results in an $O(1\text{ppm})$ accuracy of the theoretical expression for the lifetime that is competitive for precision comparison with modern experimental data. As for the quark sector is concerned there is a good set of data for $s \to u$ weak transitions that corresponds to $K \to \pi e\bar{\nu}_e$ decays at the hadron level, but it is hopeless to compute the related rate theoretically at present because of strong infrared problems in theoretical treatment of reactions with light hadrons.

For heavy hadrons the theoretical treatment of the decays is however possible because the large mass of the heavy quark constitutes a perturbative scale that is much larger than $\Lambda_{QCD}$. The leading logarithmic effects related to that scale have been discussed long ago [11]. Later there have been created a framework for the possibility for an expansion in powers of $\Lambda_{QCD}/m_Q$ where $m_Q$ is the quark mass and $\Lambda_{QCD} \sim 500 \text{ MeV}$ is a typical hadronic scale [12, 13, 14]. Top quarks do not form mesons due to their short lifetime, charmed mesons are probably not heavy enough, rendering the convergence in the inverse mass marginal, but the case of bottom-meson decays is certainly tractable in this way and thus has been intensively studied. The technique is applicable to $b \to u$ and $b \to c$ transition and both to semileptonic and purely hadronic inclusive decays. For definiteness, we will stick to semileptonic $b \to c$ decays.

In the present paper we analytically compute the coefficient of a power suppressed dimension five chromo-magnetic operator at next-to-leading order of QCD perturbation theory with the full dependence on the final state quark mass. The results of the analogous computation in the massless limit for the final state quark have been presented earlier in ref. [15]. Here we present explicit expressions for the total width of inclusive semileptonic decays and few moments of differential distributions with full dependence on the final state quark mass. One of the important phenomenological applications of our results is precision analysis of the decays of bottom mesons to charmed final states and an extraction of the numerical value for the CKM matrix entry $|V_{cb}|$.

The paper is organized as follows. In the next section we give a general representation for the decay width of a heavy hadron in a form suitable for computation in QCD. In Sect. 3 we give necessary basics of Heavy Quark Effective Theory (HQET) that is a working tool for the present calculation. In Sect. 4 we write down the Heavy Quark Expansion (HQE) for the decay rate. The actual computation and results are described in Sect. 5. In Appendices we give the explicit expressions for our master integrals and some long analytical expressions for the coefficients of HQE.
2 QCD representation for the decay rate

It is difficult to compute an hadronic decay rate since the underlying theory of strong interactions – QCD – is formulated in terms of quarks and the hadrons only appear in the strong coupling regime as bound states. Therefore one can use either numerical calculation on the lattice or find special observables for which perturbation theory calculation is feasible in some form. Such observables are inclusive ones since the sum over hadronic states can be related to the sum over the quark-gluon states using unitarity of the theory. In case the initial state is treatable in perturbation theory, i.e. it is a leptonic one as in $e^+e^-$-annihilation into hadrons or hadronic $\tau$-lepton decays then the results can be uniquely obtained in perturbation theory. In cases when the initial state is hadronic, i.e. it is non-treatable in perturbation theory, one uses a factorization idea – to separate scales and compute the short distance effects in perturbation theory while long distance properties are coded in hadronic matrix elements. The famous example of the latter approach is the analysis of deep inelastic scattering of leptons on hadrons. The analogue of deep inelastic scattering in heavy quark physics is inclusive decays of heavy hadrons. One can use either fully hadronic (non-leptonic) decays or semileptonic ones. The number of experimental observables in inclusive hadronic decays is however limited to basically the total rate of the process. In semileptonic decays the presence of leptons in the final states gives more kinematical flexibility still retaining the rigorous theoretical description of the process.

The low-energy effective Lagrangian $\mathcal{L}_{\text{eff}}$ for semileptonic $b \to c$ transitions is a Fermi four-fermion one

$$\mathcal{L}_{\text{eff}} = 2\sqrt{2}G_F V_{cb} (\bar{b}_L \gamma_\mu c_L)(\bar{\nu}_L \gamma^\mu \ell_L) + \text{h.c.}$$  \hspace{1cm} (2)

with left-handed fermion fields. The numerical value for Fermi constant $G_F$ is determined from pure leptonic weak processes and known with high precision. The mixing angle $V_{cb}$ is the main interest in decay measurements with hadronic initial states [16]. The precision analysis of such processes is important both for the flavor sector and Higgs mechanism investigations in search for new physics.

Using unitarity of the $S$-matrix the inclusive decay rate $B \to X_c \ell \bar{\nu}_\ell$ is obtained from taking the absorptive part of the forward matrix element of the transition operator $\mathcal{T}$ [17] that is the second order term of the perturbation theory expansion in the interaction Lagrangian $\mathcal{L}_{\text{eff}}$,

$$\mathcal{T} = i \int dx \, T \{\mathcal{L}_{\text{eff}}(x)\mathcal{L}_{\text{eff}}(0)\} .$$  \hspace{1cm} (3)

Note that the transition operator $\mathcal{T}$ is a non-local functional of the particle fields and is given by the integral over all possible scales. There is no much hope to handle such an operator in QCD that includes all scales as well and no large parameter is available in case of the two-point
correlator in Eq. (3). However, one can hope that some transitions or matrix elements are still short distance dominated even if it is not a universal feature of the correlator given in Eq. (3) itself and may depend on external states. For light hadrons (like kaon) it is definitely not the case and taking matrix elements cannot help in isolating a short distance dominant part of the correlator in (3). Heavy hadrons have an additional simplification that makes the computation of some matrix elements possible in perturbation theory by separating the scales involved.

The idea is that when taking a matrix element over a heavy hadron containing a heavy quark with mass \( m_Q \gg \Lambda_{QCD} \) the correlator does acquire a large internal scale, \( m_Q \), that enables scale separation. For actual separation of scales one applies the operator product expansion (OPE) techniques. These ideas are formalized through the notion of effective theories. Within the heavy hadron with momentum \( p_H \) and mass \( M_H \) the large part of the momentum is due to a pure kinematical contribution of the heavy quark \( p_H = m_Q v + \Delta \) with \( v = p_H/M_H \) being the velocity of the hadron and \( \Delta \) is related to the light degrees of freedom and interactions between them and the heavy quark. One can already extract the factor related to the large quark part of the momentum explicitly at the level of field variables when afterwards the matrix element over a heavy hadron is taken. The heavy quark field can be separated into the fast oscillating phase and a slow changing field \( h_v(x) \) with a typical momentum of order \( \Delta \sim \Lambda_{QCD} \)

\[
Q(x) \sim e^{-i(m_Q v)x} h_v(x).
\] (4)

The velocity \( v = p_H/M_H \) is finite in the limit of infinitely heavy quarks \( m_Q \gg \Lambda_{QCD} \). This program is realized within the effective theory for heavy quarks. In order to make the dependence of the decay width on the heavy quark mass \( m_Q \) explicit and to build up an expansion in \( \Lambda_{QCD}/m_Q \), one matches a time-ordered product of full QCD operators in entering to the transition operator \( T \) onto an expansion in terms of Heavy Quark Effective Theory (HQET) [18, 19]. Presently the Heavy Quark Expansion in inclusive semileptonic \( b \rightarrow c \) transitions provides a level of theoretical precision in the prediction of the total inclusive rate for \( B \rightarrow X_c \ell \bar{\nu}_\ell \) within two percent. The structure of the HQE is given by [20]

\[
\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell) = \Gamma^0|V_{cb}|^2 \left[ a_0(1 + \frac{\mu^2}{2m_b^2}) + a_2 \frac{\mu^2}{2m_b^2} + a_3 \frac{\bar{\rho}_3}{m_b^3} + a_4 \frac{\bar{\rho}_4}{m_b^4} + \mathcal{O}\left(\frac{\Lambda_{QCD}^4}{m_b^4}\right)\right]
\]

where \( \Gamma^0 = G_F^2 m_b^5/(192\pi^3) \) and \( m_b \) is the \( b \)-quark mass. The precise definition and the proper choice of the most suitable mass parameter for the heavy quark field is extensively discussed in the literature. The power suppressed terms are given by the forward matrix elements of the local operators of growing dimensionality in HQET over the heavy hadron state. Their numerical values
are determined by the corresponding power of the QCD infrared parameter of $\Lambda_{QCD}$. These are nonperturbative quantities either to be computed within some non-perturbative techniques such as lattice QCD or to be fitted to experimental data. The kinetic energy parameter $\mu_\pi^2$ is given by the nonrelativistic kinetic energy operator of the heavy quark within the heavy hadron. The chromo-magnetic parameter $\mu_G^2$ is given by the matrix element of the magnetic dipole operator. These two operators give the leading power suppressed contribution and were intensively studied. The higher order power suppressed terms are becoming important at present as the experimental data improves. The parameter $\bar{\rho}_3$ describes the contribution of dimension six operators that are Darwin term and spin-orbit interaction. The general parameter $\bar{\rho}_4$ is a contribution of a rather large number of dimension seven operators $[21]$. The coefficients $a_i$ are functions of the quark and lepton masses and have a perturbative expansion in the strong coupling constant $\alpha_s(m_b)$. The leading term coefficient $a_0$ is known analytically to $O(\alpha_s^2)$ precision in the massless limit of the final state quark $[22]$. At this order the mass corrections have been analytically accounted for the total width as an expansion in final fermion mass in ref. $[23]$ and for the differential distribution numerically in $[24]$. The coefficient of the kinetic energy parameter is linked to the coefficient $a_0$ by Lorentz invariance, see the explicit analysis in $[25]$. The NLO correction to the coefficient of the chromo-magnetic parameter $a_2$ has been investigated recently in $[26]$ where the differential distribution has been computed and the total decay rate has been then obtained by a process of numerical integration over the phase space. The $\alpha_s$ correction to the chromo-magnetic parameter coefficient $a_2$ has been analytically computed in ref. $[15]$ in the massless limit. Here we give the result with full mass dependence in analytical form. Our calculation of the coefficient $a_2$ is in fact a matching computation between QCD and HQET. For this reason we present some facts about HQET relevant for our discussion in the next section.

3 Basics of HQET

A heavy quark near its mass-shell is described by a field $h_v(x)$ which is a remnant of the whole QCD fermion field $Q(x)$. In fact, it effectively contains only large components of the Dirac bi-spinor that describe the quark and not the antiquark. One achieves the separation of the components by using the projector $P_+ = (1+\gamma)/2$ where $v$ is the external velocity that determines the remnant fields $h_v(x)$ and the whole construction of HQET. Note that obtaining HQET as the effective theory from QCD is very close in spirit to the well known procedure of obtaining the nonrelativistic limit of QCD or, earlier, QED. The field variables and Lagrangians are just the same in both nonrelativistic QCD and
HQET. The quark velocity \( v \) is fixed in the presence of the heavy hadron by its momentum. Usually the common choice for the velocity is \( v = p_H/M_H \). The behavior of time and space components of the formal Lorentz four-tensors differs in HQET. It is useful to split a four-vector \( p^\mu \) in longitudinal and transverse parts, namely \( p^\mu = v^\mu(vp) + p^\mu_\perp \). The covariant derivative of QCD is \( \pi^\mu = i\partial^\mu + g_s A^\mu \) with the splitting \( \pi^\mu = v^\mu(v\pi) + \pi^\mu_\perp \).

The quantity \( h_v \) is the heavy-quark field entering the HQET Lagrangian [18, 19]. The effective Lagrangian of HQET can be obtained in a concise form at tree level by integrating out the \( P_- \) part of the heavy quark field \( H_v \), \( H_v = P_- H_v \), with the result

\[
\mathcal{L}_{HQET} = \bar{h}_v(\pi v)h_v + \bar{h}_v\pi_\perp\frac{1}{2m + \pi v}\pi_\perp h_v. \tag{5}
\]

Here the first term is just the residual energy of the quark while the second one describes the effects of the removed (integrated out) antiquark. It is non-local that is the price for integrating the antiquark out. In the limit \( m \gg \pi v \) one can expand the second term in a series in the inverse large mass and obtain a local Lagrangian up to a given order in the mass expansion

\[
\mathcal{L}_{HQET} = \bar{h}_v(\pi v)h_v + \bar{h}_v\pi_\perp\frac{1}{2m + \pi v}\pi_\perp h_v - \bar{h}_v\pi_\perp\frac{\pi v}{4m^2}\pi_\perp h_v. \tag{6}
\]

It is inconvenient to have time derivatives in a term that is formally a correction since then the fields \( h_v \) are not correctly canonically normalized. Therefore the redefinition of the fields is used to remove time derivatives

\[
h_v \to \left(1 + \frac{\pi_\perp\pi_\perp}{8m^2}\right) h_v
\]

and get the Lagrangian for the new modes \( h_v \) (for which we retain the same notation though) in the form

\[
\mathcal{L}_{HQET} = \mathcal{O}_v + \frac{1}{2m}(\mathcal{O}_\pi + C_{mag}(\mu)\mathcal{O}_G) + \frac{1}{2m^2}\mathcal{O}_3 + \mathcal{O}\left(\frac{\Lambda^3_{QCD}}{m^3}\right) \tag{8}
\]

with

\[
C_{mag}(\mu) = 1 + \frac{\alpha_s(\mu)}{2\pi}\left\{C_F + C_A\left(1 + \ln\frac{\mu}{m_b}\right)\right\} \tag{9}
\]

being the coefficient of chromo-magnetic operator \( \mathcal{O}_G \) including the QCD radiative correction of the order \( \alpha_s \) [27]. For new modes \( h_v \), the terms of the order \( O(1/m_b^2) \) in the Lagrangian contain no time derivative [19, 28]. Here we introduced the notation used below. The quantity \( \mathcal{O}_v = \bar{h}_v\pi v h_v \) is the leading power energy operator that is independent of the heavy quark mass and spin and gives the famous spin-flavor symmetry of HQET. The quantity \( \mathcal{O}_\pi = \bar{h}_v\pi_\perp h_v \) is a kinetic energy operator and \( \mathcal{O}_G = \bar{h}_v\sigma_{\mu\nu}[\pi^\mu_\perp,\pi^\nu_\perp]/2h_v \) is a chromo-magnetic operator. They constitute classical subleading power operators. Higher terms are given by the operator \( \mathcal{O}_3 = \bar{h}_v[\pi^\mu_\perp, [\pi^\mu_\perp, \pi v]]h_v \) that can further be converted into a linear combination of the Darwin \( \mathcal{O}_D = \bar{h}_v[\pi^\mu_\perp, [\pi^\mu_\perp, \pi v]]h_v \) and spin-orbit term
\( \mathcal{O}_{\text{SL}} = \bar{h}_v \sigma_{\mu\nu} [\pi^\mu_{\perp}, [\pi^\nu_{\perp}, \pi_v]] h_v, \mathcal{O}_3 = c_D \mathcal{O}_D + c_{\text{SL}} \mathcal{O}_{\text{SL}} \) with coefficients \( c_D, c_{\text{SL}} \) known at the next-to-leading order of perturbative expansion in the strong coupling constant. The discussion of order \( 1/m_Q^2 \) terms in the HQET Lagrangian is relevant for our computation because of the necessity to precisely fix the definition of the fields \( h_v \) entering the heavy quark expansion.

4 HQE for the width correlator

For further convenience we introduce a normalized transition operator \( \tilde{T} \) through the relation

\[
\text{Im} T = \Gamma^0 |V_{cb}|^2 \tilde{T}.
\] (10)

With the use of heavy quark effective theory the heavy quark expansion is simply a matching from QCD to HQET

\[
\tilde{T} = C_0 \mathcal{O}_0 + C_v \mathcal{O}_v \frac{\mathcal{O}_v}{m_b} + C_{\pi} \mathcal{O}_{\pi} \frac{\mathcal{O}_{\pi}}{2m_b^2} + C_G \mathcal{O}_G \frac{\mathcal{O}_G}{2m_b^2}.
\] (11)

The local operators \( \mathcal{O}_i \) in the expansion (11) are ordered by their dimensionality \( \mathcal{O}_0 = \bar{h}_v h_v, \mathcal{O}_v = \bar{h}_v v \pi h_v, \mathcal{O}_{\pi} = \bar{h}_v \pi^2_\perp h_v, \mathcal{O}_G = \bar{h}_v \frac{1}{2} [\pi^\mu_\perp, \pi^\nu_\perp] h_v \). The coefficients of these operators are obtained by matching the relevant matrix elements between QCD and HQET. Note that after taking a matrix element over the hadronic state (like the \( B \)-meson) one can use equations of motion for HQET fields \( h_v \) to eliminate the operator \( \mathcal{O}_v \). By the same token there is an operator \( \mathcal{O}_5 = \bar{h}_v (v \pi)^2 h_v \) that is of higher order in the large mass expansion after going on shell using equations of motion of HQET. Thus, the expansion (11) is a matching relation from QCD to HQET with proper operators up to dimension five with the corresponding coefficient functions. The coefficients are independent of external states and one can take them at will. We take a heavy quark on shell and gluons as external states for matching to QCD.

Note that one can use the full QCD fields for the heavy quark expansion expansion as well. However the choice of the proper basis of operators is not so straightforward as in HQET. Still it is convenient to choose the local operator \( \bar{b} \gamma_\mu b \) defined in full QCD as a leading term of heavy quark expansion [29]. Indeed, the current \( \bar{b} \gamma_\mu b \) is conserved and its forward matrix element with hadronic states is absolutely normalized. For implementing this setup one needs an expansion (matching) of a full QCD local operator \( \bar{b} \gamma_\mu b \) in HQE through HQET operators. The expansion reads

\[
\bar{b} \gamma_\mu b = \mathcal{O}_0 - \mathcal{C}_\pi \frac{\mathcal{O}_\pi}{2m_b^2} + \mathcal{C}_G \frac{\mathcal{O}_G}{2m_b^2} + O(\Lambda_{QCD}^3/m_b^2)
\] (12)

up to necessary order in the strong coupling \( \alpha_s \). The coefficient of the leading power operator \( \mathcal{O}_0 \) has no radiative corrections and the kinetic operator has the coefficient related to the leading one due to Lorentz (reparameterization) invariance.
Substituting the expansion (12) into eq. (11) one obtains after using the equation of motion for the operator $O_v$ in the forward matrix elements

$$\tilde{T} = C_0 \left\{ \bar{b}y^b - \frac{Q_\pi}{2m_b^2} \right\} + \left\{ -C_v C_{\text{mag}}(\mu) + C_G - \tilde{C}_G C_0 \right\} \frac{O_G}{2m_b^2}. \quad (13)$$

Note that for phenomenological applications the numerical value for the chromo-magnetic moment parameter $\mu_G^2$, related to the forward matrix element of the operator $O_G$, is usually taken from the mass splitting between the pseudoscalar and vector ground-state mesons. The mass difference of bottom mesons $m_{B^*}^2 - m_B^2 = \Delta m_B^2 = 0.49 \text{ GeV}^2$ is given by

$$\frac{1}{2M_B} C_{\text{mag}}(\mu) \langle B(p_B) | O_G(\mu) | B(p_B) \rangle = \frac{3}{4} \Delta m_B^2 \quad (14)$$

(up to higher order $1/m_Q$ corrections) where we use the relativistic normalization of states. Therefore the coefficient in front of the renormalization group invariant combination $C_{\text{mag}}(\mu) O_G(\mu)$ can be useful. In such normalization one gets after taking the forward matrix element of the expansion in Eq. (13) the representation

$$\Gamma(B \to X_c \bar{\nu}_\ell \ell) = \Gamma^0 |V_{cb}|^2 \left\{ C_0 \left( 1 + \frac{\mu^2}{2m_b^2} \right) + \left( -C_v + \frac{C_G - \tilde{C}_G C_0}{C_{\text{mag}}} \right) \frac{3\Delta m_B^2}{8m_b^2} \right\}. \quad (15)$$

5 Description of the calculation and results

5.1 Generalities and techniques

The matching procedure consists in computing matrix elements with partonic states (on-shell quarks and gluons) at both sides of the expansion (11). The coefficient function $C_0$ of the dimension three operator $\bar{h}_v h_v$ determines the total width of the heavy quark and at the same time the leading contribution to the width of a bottom hadron with HQE technique. At NLO the calculation of the transition operator $\tilde{T}$ in (3) requires to consider three-loop diagrams with external heavy quark lines on shell. The leading order result is well known and requires the calculation of the two-loop Feynman integrals of the simplest topology – the sunset type ones [30]. At the NLO level one needs the on-shell three-loop integrals with massive lines due to the massive $c$-quark. In Fig. 1 we show some typical three-loop diagrams both for the partonic part and power corrections of the decay rate.

The computation has been performed in dimensional regularization used for both ultraviolet and infrared singularities. We used the systems of symbolic manipulations REDUCE [31] and Mathematica [32] with original codes written for the calculation. The package FeynCalc [33] is used...
Figure 1: Perturbation theory diagrams for the matching computation at NLO level, (left) - partonic type, right - power correction type: insertion of an external gluon

for manipulating Dirac matrices and four vectors under Mathematica. The reduction to master integrals has been done within the integration by parts technique \[34]. The original codes have been used for most of the diagrams and then the program LiteRed \[35] has been used for checking and further application to complicated diagrams. The master integrals have been computed directly and then checked with the program HypExp \[36]. The renormalization is performed on-shell by the multiplication of the bare (direct from diagrams) results by the on-shell renormalization constant \( Z_{2OS} \)

\[
Z_{2OS} = 1 - C_F \frac{\alpha_s(\mu)}{4\pi} \left( \frac{3}{\epsilon} + 3 \ln \left( \frac{\mu^2}{m_b^2} \right) + 4 \right). \tag{16}
\]

It is convenient to fix the normalization point to the \( b \)-quark mass \( \mu = m_b \) in the practical computation. The \( \mu \)-dependence can be easily restored from the knowledge of anomalous dimensions.

We present and discuss the obtained results below.

5.2 The leading power coefficient \( C_0 \): partonic width

By using the described methods we reproduce the known result for the heavy quark width which is given by the contribution of the leading operator \( \mathcal{O}_0 \). The coefficient \( C_0 \) is

\[
C_0 = C_0^{LO} + C_F \frac{\alpha_s}{\pi} C_0^{NLO} \tag{17}
\]

where the LO contribution reads

\[
C_0^{LO} = 1 - 8r - 12r^2 \ln(r) + 8r^3 - r^4 \tag{18}
\]
and the NLO contribution reads

\[
C_0^{NLO} = (1 - r^2) \left\{ \left( \frac{25}{8} - \frac{239}{6} r + \frac{25}{8} r^2 \right) + \left( -\frac{17}{6} + \frac{32}{3} r - \frac{17}{6} r^2 \right) \ln(1 - r) \right\} \\
+ \left( -10 - 45r + \frac{2}{3} r^2 - \frac{17}{6} r^3 \right) r \ln(r) + \left( -18 - \frac{r^2}{2} \right) r^2 \ln^2(r) \\
+ (2 + 60r^2 + 2r^4) \ln(1 - r) \ln(r) + (1 + 16r^2 + r^4) (3\text{Li}_2(r) - \pi^2/2) \\
+ 16 r^{3/2} (1 + r) \left( \pi^2 - 4 \left\{ \text{Li}_2(\sqrt{r}) - \text{Li}_2(-\sqrt{r}) \right\} + 2 \ln(r) \ln \frac{1 + \sqrt{r}}{1 - \sqrt{r}} \right) 
\]

(19)

with \( r = m_c^2/m_b^2 \). Here \( \text{Li}_2(r) \) is polylogarithm, \( \text{Li}_2(r) = \sum n r^n/n^2 \). The combination

\[
r^{1/2} \left( \pi^2 - 4 \left\{ \text{Li}_2(\sqrt{r}) - \text{Li}_2(-\sqrt{r}) \right\} + 2 \ln(r) \ln \frac{1 + \sqrt{r}}{1 - \sqrt{r}} \right)
\]

(20)
is a part of one master integral in the computation and it always appears in this form. It contains a specific odd contribution \( r^{1/2}\pi^2 \) while the rest is in fact formally even in \( m_c \). The analytical expression at NLO in Eq. (19) has been first given by Nir [37].

The behavior near the border of the decay phase space (\( r \sim 1 \)) of the NLO correction

\[
C_0^{NLO}(r \to 1) = \frac{3}{10} (1 - r)^5 + O((1 - r)^6)
\]

(21)
is similar to that of the LO which is

\[
C_0^{LO}(r \to 1) = \frac{2}{5} (1 - r)^5 + O((1 - r)^6).
\]

(22)

A typical feature of the result at next-to-leading order is the presence of odd powers of the charm quark mass like \( r^{3/2} \). Of course, it does not mean that there is a symmetry \( m_c \to -m_c \). At the small mass limit \( r \to 0 \) only the simplest term of such structure \( \pi^2 r^{3/2} \) survives with a rather large coefficient.

We define the bottom quark mass being a pole one because it is convenient for computing the relevant matrix elements in QCD with on-shell quark states. The definition of charmed quark mass can be either the pole scheme or \( \overline{\text{MS}} \)-scheme one. The relation between the two definitions up to necessary order is

\[
m_c^{\text{pol}} = m_c^{\overline{\text{MS}}} (\mu) \left( 1 + C_F \frac{\alpha_s}{4\pi} \left( 3 \ln \frac{\mu^2}{m_c^2} + 4 \right) \right)
\]

(23)
The numerical value for the charmed quark mass is best known in the \( \overline{\text{MS}} \)-scheme [38, 39]. It is rather small and cannot be be perturbatively cast into the pole mass scheme with any reliable control over uncertainties due to convergence of perturbation series expansion [40]. The numerical value for the bottom quark mass has been discussed in the literature for a long time and many
estimates are available. Also there is an extensive discussion which particular scheme of defining the quark mass parameter which is the most suitable for this particular observable \[41, 42\].

In Fig. 2 we give the plot of the coefficient $C_0^{\text{LO}}(r)$ and also the normalized next-to-leading coefficient $\hat{C}_0^{\text{NLO}}(r)$ in the pole mass scheme for $m_c$.

In Fig. 3 we give the plot of the mass dependence of the coefficient $C_0^{\text{NLO}}(r)$ in different mass schemes for $m_c$.

In the small mass limit for the charmed quark one finds

$$C_0^{\text{NLO}}(r)|_{r \to 0} = \left( \frac{25}{8} - \frac{\pi^2}{2} \right) + 2r(6 \ln(r) + 17) + 16\pi^2 r^{3/2} + O(r^2 \ln^2 r). \quad (24)$$

We have computed the results for the coefficient $C_0$ in massless limit, $C_0(0)$, independently that serves partly as a check of our full mass calculation.

The relative magnitude of the NLO contribution at a typical value of mass ratio $r = 0.07$ is

$$C_0(0.07) = 0.60 - 0.78 C_F \frac{\alpha_s}{\pi} = 0.6(1 - C_F \frac{\alpha_s}{\pi} 1.31) \quad (25)$$

while in massless limit it is

$$C_0(0) = 1 - C_F \frac{\alpha_s}{\pi} 1.8. \quad (26)$$

The numerical value for the bottom quark mass $m_b$ is important for phenomenological applications and discussed in the literature (see, e.g. \[11\]). The dependence on the charm quark mass is essential
Figure 3: Mass dependence of the coefficient $C_0^{NLO}(r)$ in pole and $\overline{\text{MS}}$ schemes with $\mu = m_b$ and $\mu = m_c$.

but still it follows mainly the pattern of that at leading order. This similarity supports the idea of ref. [15] that the computation in massless limit can be useful for physical applications as the normalization and the extrapolation with the leading order massive result can be a reasonable approximation for the mass dependence at NLO. We will see how it works or does not work for other coefficients later.

5.3 The $vD$-operator coefficient $C_v$

Here we present the result for the coefficient $C_v$ which is an auxiliary quantity in our approach since the operator is reexpressed through the other contributions at the level of matrix elements. The coefficient $C_v$ is singled out by taking the matrix element between $b$-quarks on shell and one gluon with vanishing momentum and longitudinal polarization, i.e. the gluon field is chosen on the form $A_\mu = v_\mu(vA)$. Here $A_\mu$ is a matrix in color space $A_\mu = A_\mu^a t^a$. The result for the coefficient $C_v$

$$C_v = C_v^{LO} + C_F \frac{\alpha_s}{\pi} C_v^{NLO}$$  \hspace{1cm} (27)

reads

$$C_v^{LO} = 5 - 24r - 12r^2 \ln(r) + 24r^2 - 8r^3 + 3r^4.$$  

In Fig. 4 we plot the charmed quark mass dependence of $C_v$. 

Figure 4: Mass dependence of the coefficient $C_v(r)$

For the NLO part $C_v^{NLO}$ we give explicitly only the expression for the small $r$ expansion. The structure of the whole contribution is very similar to that of $C_v^{NLO}$. The expression is rather long and given in Appendix B. The small mass expansion reads

$$C_v^{NLO} = \left( -\frac{25}{24} - \frac{\pi^2}{2} \right) + 48r - 8\pi^2 r^{3/2} + O(r^2 \ln^2 r).$$

(28)

The leading term of the expression coincides with the independent computation in the massless limit done in [15]

$$C_v|_{r=0} = 5 + C_F \frac{\alpha_s}{\pi} \left\{ -\frac{25}{24} - \frac{\pi^2}{2} \right\}. \quad (29)$$

As for the mass dependence of the coefficient $C_v$, for the typical value of $r = 0.07$ one finds

$$C_v(0.07) = 3.6 - 3.8 C_F \frac{\alpha_s}{\pi} = 3.6 \left( 1 - C_F \frac{\alpha_s}{\pi} 1.1 \right)$$

(30)

while in the massless limit one has

$$C_v(0) = 5(1 - C_F \frac{\alpha_s}{\pi} 1.2).$$

(31)

One sees again a rather reasonable accuracy for the mass dependence extrapolation at NLO.

The coefficient $C_v$ has no $C_A$ color structure, it contains only the $C_F$ Casimir invariant. This property matches the possibility to compute this coefficient using a small momentum expansion near the quark mass shell, $p = m v + k$. Still, an explicit cancellation of the contribution proportional to the color structure $C_A$ and cancellation of poles with the same renormalization constant $Z_2^{QS}$ shown in Eq. (16) is a powerful check of the final result.
The large $m_c$ behavior at the border of phase space is

$$C^{NLO}_v (r \to 1) = -3(1-r)^4 + O((1-r)^5)$$  \hspace{1cm} (32)

and

$$C^{LO}_v (r \to 1) = 4(1-r)^4 + O((1-r)^5).$$  \hspace{1cm} (33)

5.4 The coefficient $C^r_G - C_0 \tilde{C}_G \equiv C^r_G$:

chromo-magnetic operator

For the chromo-magnetic operator coefficient we directly compute the difference between contributions to the width correlator in Eq. (11) and the local $\bar{b}b$ operator in Eq. (12) multiplied by the leading power coefficient $C_0(r)$, $C^r_G = C_G - C_0 \tilde{C}_G$. We write this coefficient as leading order term and radiative correction in the form

$$C^r_G (r) = C^{r,LO}_G (r) + \frac{\alpha_s}{\pi} \left\{ C_A C^{r,NLO,A}_G (r) + C_F C^{r,NLO,F}_G (r) \right\}$$  \hspace{1cm} (34)

where the NLO coefficient is separated into two color structures with $C_A$ and $C_F$ color group invariants. In Fig. 5 we present the plot of the mass dependence for the coefficient of the chromo-magnetic operator for QCD with $C_A = 3$ and $C_F = 4/3$. One sees that the mass dependence of

![Figure 5: Mass dependence of the coefficient $C^r_G = C_G - C_0 \tilde{C}_G$ with $\mu = m_b$](image)

$C^r_G$ at NLO is much sharper than in previous cases. This is unexpected and makes the conjecture
about a uniform phase space suppression for the coefficients less accurate. The explicit leading order expression reads

$$C_G^{r, \text{LO}} = 2 - 16r - 24r^2 \ln(r) + 16r^3 - 2r^4 = 2C_0^{\text{LO}}. \tag{35}$$

The NLO coefficients with full mass dependence are too long, whereas the expanded results are

$$C_G^{r, \text{NLO}, A} = -\frac{8\pi^2 \sqrt{r}}{3} + r \left( \ln^2(r) - 25 \ln(r) + 2\frac{\pi^2}{3} - 25 \right) - \frac{\pi^2}{9} + \frac{49}{18},$$

$$C_G^{r, \text{NLO}, F} = \frac{32\pi^2 \sqrt{r}}{3} + r \left( -4 \ln^2(r) + 68 \ln(r) - 4\pi^2 + 21 \right) - \frac{7\pi^2}{9} - \frac{47}{36}. \tag{36}$$

At the border of phase space we obtain

$$C_G^{r, \text{NLO}}(r \to 1) = C_F (1 - r)^4 + \frac{1}{5} [2C_F - 3C_A] (1 - r)^5 + O((1 - r)^6) \tag{37}$$

and

$$C_G^{r, \text{LO}}(r \to 1) = \frac{4}{5}(1 - r)^5 + O((1 - r)^6). \tag{38}$$

In the massless limit the $C_G^r$ coefficient is given by

$$C_G^r(0) = 2 + \frac{\alpha_s}{\pi} \left\{ C_A \left( \frac{49}{18} - \frac{\pi^2}{9} \right) + C_F \left( -\frac{47}{36} - \frac{7\pi^2}{9} \right) \right\}. \tag{39}$$

This result has been independently determined by the direct computation using the technology developed for the massless case.

### 5.5 Coefficient $C_{\mu_G^2} = -C_v + C_G^r/C_{\text{mag}}$:

**the matrix element of $C_{\text{mag}} \mathcal{O}_G$**

This coefficient is the final result after the use of equations of motion. We prefer to give the coefficient in front of the renormalization group invariant combination that enters the HQET Lagrangian. This combination also determines the mass splitting in the ground state multiplets due to spin orientation.

Thus, the final coefficient of the matrix element of the chromo-magnetic operator with account of equation of motion after taking hadronic matrix elements reads

$$C_{\mu_G^2}(r) = -C_v(r) + \frac{C_G^r(r)}{C_{\text{mag}}(\mu)}. \tag{40}$$

This is a coefficient in front of the matrix element of the renormalization invariant combination $C_{\text{mag}}(\mu) \mathcal{O}_G(\mu)$.

In Fig. 6 we plot the mass dependence of this final coefficient.
Figure 6: The mass dependence of the coefficient $C_{\bar{\mu}G}(r)$ at LO and NLO in the pole mass scheme. Color blind: $C_A = 3$, $C_F = 4/3$. Left panel – the whole phase space, right panel – zoomed image of the small mass region $0 < r < 0.1$.

Writing again the decomposition of the whole coefficient in $\alpha_s$ order

$$C_{\bar{\mu}G}(r) = C_{\bar{\mu}G}^{LO} + \frac{\alpha_s}{\pi} \left\{ C_A C_{\bar{\mu}G}^{NLO,A} + C_F C_{\bar{\mu}G}^{NLO,F} \right\} \quad (41)$$

we obtain at the leading order the well known result

$$C_{\bar{\mu}G}^{LO} = -3 + 8r - 24r^2 - 12r^2 \log(r) + 24r^3 - 5r^4. \quad (42)$$

The whole expressions are given in Appendix C. Here we present the new result at NLO as a small $r$ expansion only

$$C_{\bar{\mu}G}^{NLO,A} = \frac{2\pi^2 r}{3} - 17r - \frac{8\pi^2 \sqrt{r}}{3} + r \log^2(r) - 25r \log(r) - \frac{\pi^2}{9} + \frac{31}{18},$$

$$C_{\bar{\mu}G}^{NLO,F} = -4\pi^2 r - 19r + \frac{32\pi^2 \sqrt{r}}{3} - 4r \log^2(r) + 68r \log(r) - \frac{5\pi^2}{18} - \frac{91}{72}. \quad (43)$$

The color blind expansion for QCD ($C_A = 3$, $C_F = 4/3$) reads

$$C_{\bar{\mu}G}^{NLO} = \left( \frac{94}{27} - \frac{19\pi^2}{27} \right) + \frac{56\pi^2 \sqrt{r}}{9}$$

$$+ \frac{1}{3} r \left( -7 \ln^2(r) + 47 \ln(r) - 10\pi^2 - 229 \right) + \frac{280}{27} \pi^2 r^{3/2}$$

$$+ \frac{1}{81} r^2 \left( -1251 \ln^2(r) - 1917 \ln(r) - 216\pi^2 - 5750 \right) + O(r^{5/2}). \quad (44)$$

The very large contribution of the $\sqrt{r}$ term leads to a very fast change of the coefficient $C_{\bar{\mu}G}^{NLO}$ from its massless limit value with an increase of the charm quark mass. Numerically one finds

$$C_{\bar{\mu}G}^{NLO} = -3.46 + 61.41 \sqrt{r} + r \left( -2.3 \ln^2(r) + 15.7 \ln(r) - 109.2 \right) + O(r^{3/2}). \quad (45)$$
In the massless limit the new result is

\[ C_{\mu G} = -3 + \frac{\alpha_s}{\pi} \left\{ C_A \left( \frac{31}{18} - \frac{\pi^2}{9} \right) - C_F \left( \frac{91}{72} + \frac{5\pi^2}{18} \right) \right\}. \]

Note that the \( C_F \) part of this coefficient differs from the result given in ref. \[15\]. The difference is given by \( 2C_0^{NLO} \) and it emerged because in \[15\] only the leading order of \( C_0 \) coefficient was used for subtracting the contribution of the local \( \bar{b}b \) operator.

The \( \mu \) dependence of the prefactor of \( \mathcal{O}_G \) in Eq. (13) matches the leading order anomalous dimension of chromo-magnetic operator \[27\], such that \( C_{\mu G} \) is \( \mu \) independent.

The end-of-spectrum behavior reads

\[ C_{\mu G}^{NLO}(r \to 1) = 4C_F(1-r)^4 + \left[ \frac{3}{10}C_F - C_A \right](1-r)^5 + O((1-r)^6) \] (46)

for NLO and

\[ C_{\mu G}^{LO}(r \to 1) = -4(1-r)^4 + O((1-r)^6) \] (47)

for the leading order contribution.

The mass parameter of the heavy quark \( m_b \) is chosen to be the pole mass which is a proper formal parameter for perturbative computations in HQET (see discussion in \[20\]). After having obtained the results of perturbation theory computation for the coefficients of HQE, one is free to change this parameter to any other \[42\].

6 Discussion of the results

6.1 The total width

The radiative corrections are of reasonable magnitude and are well under control for the numerical values of the coupling constant for \( \mu \sim 2-4 \) GeV (for the numerical value see, e.g. \[43\]). This provides a clean application of the results to phenomenology. The final quark mass dependence is remarkable. It is very fast for small \( m_c \) therefore the decays into light quarks \( u \) for bottom mesons and \( d \) for charmed mesons should be treated with care.

The coefficients of HQE have been also calculated in ref. \[44\] where the analytical computation has been performed for the hadronic tensor and the final integration over the phase space has been done numerically. Such a setup has advantages for direct comparison with experimental data since the experimental cuts in the phase space can be readily introduced.
We can make a literal comparison with the results of [44] for the total width. Our result in the format of ref. [44] is

\[ \Gamma = \Gamma_0^m \left( (1 - 1.7776^{\alpha_s / \pi})(1 - \frac{\mu^2}{2m_b^2}) - (1.9449 + 2.4235^{\alpha_s / \pi})\frac{\mu^2_G}{m_b^2} \right) \] (48)

for \( r = 0.0625 \) that literally coincides with the results of ref. [44].

For phenomenological applications and comparison with experiment within our approach one can compute moments of the differential distribution (see, e.g. [45]). It is straightforward to compute almost any moment in the invariant lepton pair mass, lepton pair energy or invariant mass of the hadronic system. We present few such moments below.

### 6.2 Moments of differential distribution

Note that our computation is organized such that it allows for computation of certain moments of differential distribution. We can build up moments over the leptonic pair invariant mass squared \( q^2 \), \( (q = p_\ell + p_\nu) \) and the partonic invariant mass squared \( (p - q)^2 \), \( p \) is the momentum of the bottom quark and \( p = m_bv \). It is possible because we have the leptonic part and the partonic parts separately in an intermediate representation of computed diagrams – one can compute the moments in \( q^2 \) or/and in \( (p - q)^2 \). The total lepton energy moments (the moments in the variable \( pq \)) are just the linear combinations of those two sets. We present the analytical results for few moments at small \( r \) expansion for brevity. The analytical expression for the total width is given for further comparison with the moments. It reads

\[
\frac{\Gamma}{\Gamma_0} = 1 - 8r + C_F \frac{\alpha_s}{4\pi} \left( \frac{25}{2} - 2\pi^2 - 8r(6\ln(r) + 17) \right) + \\
\frac{\bar{\mu}^2}{2m_b^2} \left\{ -3 + 8r + \frac{\alpha_s}{4\pi} \times \\
\left( C_A \left( \frac{2\pi^2r}{3} - 17r - \frac{8\pi^2\sqrt{r}}{3} + r \ln^2(r) - 25r \ln(r) - \frac{\pi^2}{9} + \frac{31}{18} \right) \\
+ C_F \left( -4\pi^2r - 19r + \frac{32\pi^2\sqrt{r}}{3} - 4r \ln^2(r) + 68r \ln(r) - \frac{5\pi^2}{18} - \frac{91}{72} \right) \right\} \right].
\] (49)

The normalized \( q^2 \) moments of the total width with \( C_{\bar{\mu}^2} \) coefficient are given below. For convenience they are normalized to unity at leading order of power, small mass, and perturbative expansions. The normalization can be obtained independently. Indeed, the \( x = q^2/m_b^2 \) distribution in massless limit at LO is given by

\[
\frac{1}{\Gamma_0} \frac{d\Gamma}{dx} = 2(1 - x)(1 + 2x).
\] (50)
The normalization factors for the moments $n = 1 - 3$ are then $N(M_n^q) = \{3/10, 2/15, 1/14\}$. For example,

$$N(M_1^q) = \int_0^1 2(1-x)(1+2x)dx = 3/10. \tag{51}$$

First moment $(q^2/m_b^2)$ is

$$M_1^q = 1 - 15r + C_F \frac{\alpha_s}{4\pi} \left( 13 - 2\pi^2 - r \left( 90 \ln(r) + \frac{10\pi^2}{9} + 355 \right) \right) +$$

$$\frac{\bar{\mu}_G^2}{2m_b^2} \left( -\frac{25}{3} + \frac{\alpha_s}{\pi} \left( C_A \left( -\frac{80\pi^2\sqrt{r}}{9} - \frac{25\pi^2}{27} + \frac{260}{27} \right) + C_F \left( \frac{320\pi^2\sqrt{r}}{9} + \frac{65\pi^2}{54} - \frac{763}{36} \right) \right) \right).$$

Second moment $(q^2/m_b^2)^2$ is

$$M_2^q = 1 - 24r + C_F \frac{\alpha_s}{4\pi} \left( \frac{604}{45} - 2\pi^2 + r \left( -144 \ln(r) - 2\pi^2 - \frac{6813}{10} \right) \right) +$$

$$\frac{\bar{\mu}_G^2}{2m_b^2} \left( -15 + \frac{\alpha_s}{\pi} \left( C_A \left( -20\pi^2\sqrt{r} - \frac{17\pi^2}{12} + \frac{541}{40} \right) + C_F \left( 80\pi^2\sqrt{r} + \frac{7\pi^2}{3} - 41 \right) \right) \right). \tag{52}$$

Third moment $(q^2/m_b^2)^3$ is

$$M_3^q = 1 - 35r + C_F \frac{\alpha_s}{4\pi} \left( \frac{1243}{90} - 2\pi^2 + r \left( -210 \ln(r) - \frac{14\pi^2}{5} - \frac{20195}{18} \right) \right) +$$

$$\frac{\bar{\mu}_G^2}{2m_b^2} \left( -23 + \frac{\alpha_s}{\pi} \left( C_A \left( -\frac{112\pi^2\sqrt{r}}{3} - \frac{35\pi^2}{18} + \frac{27217}{1620} \right) + C_F \left( \frac{448\pi^2\sqrt{r}}{3} + \frac{67\pi^2}{18} - \frac{1088429}{16200} \right) \right) \right).$$

The $q^2$ moments are very stable and hardly change with $n$ besides the total normalization. Usually one argues that radiative corrections should increase or decreases depending on the momentum flow through the diagram – we see no simple explanation for the change of radiative corrections.

The moments in partonic variable $(p - q)^2 - m_c^2$ are defined through the relation

$$M_n^H = \int \frac{(p - q)^n}{m_b^{2n}} \frac{d\Gamma}{\Gamma_0} \tag{53}$$

and have been considered in [45]. They are given below for $n = 1 - 3$ analytically within small $r$ expansion.

First moment $((p - q)^2 - m_c^2)^1$ is

$$M_1^H = C_F \frac{\alpha_s}{\pi} \left( \frac{71r}{24} + \frac{3}{2} r \log(r) + \frac{91}{600} \right) +$$

$$\frac{\bar{\mu}_G^2}{2m_b^2} \left( \frac{\alpha_s}{\pi} \left( C_A \left( -\frac{611r}{108} - \frac{22}{9} r \log(r) - \frac{29}{180} \right) \right) + C_F \left( -\frac{73\pi^2 r}{36} + \frac{457r}{108} - \frac{67\pi^2 r}{36} - \frac{77}{45} \right) \right) - \frac{3r}{2} + \frac{1}{2}. \tag{54}$$
Second moment \(((p - q)^2 - m_c^2)^2\)

\[
M_2^H = C_F \frac{\alpha_s}{\pi} \left( \frac{5}{432} - \frac{137r}{600} \right) + \frac{\bar{\mu}_G^2}{2m_b^2} \frac{\alpha_s}{\pi} \left( C_A \left( r \left( \frac{\ln r}{18} + \frac{163}{1080} \right) + \frac{1}{72} \right) + C_F \left( r \left( \frac{25 \ln r}{18} + \frac{3703}{1080} \right) + \frac{347}{3600} \right) \right). \tag{55}
\]

Third moment \(((p - q)^2 - m_c^2)^3\) is

\[
M_3^H = C_F \frac{\alpha_s}{\pi} \left( \frac{377}{176400} - \frac{119r}{3600} \right) + \frac{\bar{\mu}_G^2}{2m_b^2} \frac{\alpha_s}{\pi} \left( C_A \frac{43}{16200} + C_F \frac{11537}{1587600} \right). \tag{56}
\]

This set is such that moments vanish at leading order. Therefore one cannot discuss the relative magnitude of radiative corrections. Our results for \(n = 1 - 2\) coincide with those of ref. [45].

We also compute the relevant moments numerically with full mass dependence for a typical value of the mass ratio. For the \(q^2\) moments up to third order we obtain with \(r = 0.0625\)

\[
M_0^q = (1 - 1.7776\frac{\alpha_s}{\pi}) - 3.8898(1 - 0.9206\frac{\alpha_s}{\pi}) \frac{\bar{\mu}_G^2}{2m_b^2}, \\
M_1^q = (1 - 1.6500\frac{\alpha_s}{\pi}) - 8.9901(1 - 0.6834\frac{\alpha_s}{\pi}) \frac{\bar{\mu}_G^2}{2m_b^2}, \\
M_2^q = (1 - 1.5575\frac{\alpha_s}{\pi}) - 14.394(1 - 0.5578\frac{\alpha_s}{\pi}) \frac{\bar{\mu}_G^2}{2m_b^2}, \\
M_3^q = (1 - 1.4847\frac{\alpha_s}{\pi}) - 19.997(1 - 0.4666\frac{\alpha_s}{\pi}) \frac{\bar{\mu}_G^2}{2m_b^2}. \tag{56}
\]

Numerically for partonic moments in \(H = (p - q)^2 - m_c^2\) with \(r = 0.0625\) one obtains

\[
M_1^H = 0.0569\frac{\alpha_s}{\pi} + 0.397(1 - 2.304\frac{\alpha_s}{\pi}) \frac{\bar{\mu}_G^2}{2m_b^2}, \\
M_2^H = 0.00575\frac{\alpha_s}{\pi} + 0.0554 \frac{\alpha_s}{\pi} \frac{\bar{\mu}_G^2}{2m_b^2}, \tag{57}
\]

\[
M_3^H = 0.00114\frac{\alpha_s}{\pi} + 0.00694 \frac{\alpha_s}{\pi} \frac{\bar{\mu}_G^2}{2m_b^2},
\]

where \(\bar{\mu}_G^2 = C_\text{mag}(\mu)\mu_G^2(\mu)\).

It is also possible to compute the moments of the lepton energy spectrum that is of interest from the experimental point of view. However, here a few more technical problems arise. On the one hand the whole set up of the analytical calculation has to be modified, since leptonic tensor has to be taken as a differential distribution rather than fully integrated over the lepton phase space. On the other hand there is the question of how to deal with \(\gamma_5\) in dimensional regularization. For the cases we discussed here we always have a situation when there is an even (in fact two) number of \(\gamma_5\)-matrices within the trace over Dirac matrices both in leptonic and hadronic parts, so
we simply and consistently use anticommuting $\gamma_5$. However, in the calculation of the moments of the charged-lepton energy one has also consider an odd number of $\gamma_5$-matrices in the traces, which causes an additional complication of the calculation. Nevertheless, with the technology developed here, these problems can be tackled and we plan to present a calculation of lepton-energy moments in a separate publication.

6.3 Phenomelogical outlook

This paper has been devoted to the description of the technical aspects of the calculation of the perturbative QCD corrections for subleading powers in the $1/m$ expansion. Aside from more theoretical consideration, such as the discussion of the mass dependence of the various terms of the heavy quark expansion, such a calculation has a variety of phenomenological applications, of which the most prominent one is its application to inclusive semileptonic $b \to c$ transitions. These decays are currently believed to be the most precise method to determine the CKM matrix element $V_{cb}$. In this method, $V_{cb}$ is extracted from the heavy quark expansion for the total rate, while the heavy quark expansion parameters $\mu_\pi$, $\rho_D$ etc. are extracted from the moments of the differential rates. Based on this methodology, the theoretical uncertainty in $V_{cb}$ has been reduced to a level below 1%, while the total uncertainty (including the experimental as well as the uncertainty in the extraction of the heavy quark expansion parameters) is at the level of 2%. The current extractions of $V_{cb}$ do not yet include the $\alpha_s \mu_G^2$ contributions, which are parametrically the largest missing pieces in the analysis.

From the experimental side, the lepton energies cannot be measured to arbitrarily low values. Thus either an extrapolation is necessary or one has to include a cut into the theoretical predictions. Since an extrapolation involves a model dependence, it is more favorable to include a lepton-energy cut into the theoretical prediction.

However, unlike in the numerical study of [26] [46], such a cut cannot be implemented in an analytical calculation, at least not exactly. Thus in order to make phenomenological use of the analytical calculation one needs to take into account effects of such a cut, which needs further study. We plan to return to this in a separate publication.

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Appendices

A Master integrals

Here we present the results for master integrals entering our calculation in dimensional regularization with $D = 4 - 2\varepsilon$. The general integral

$$S(a, b; m^2, p^2) = \frac{1}{i\pi^{D/2}} \int \frac{d^Dk}{(m^2 - k^2)^a(-(p-k)^2)^b}$$

(58)

develops a cut at $p^2 = s > m^2$ with a discontinuity $\rho(a, b; m^2, s)$ that is

$$\rho(a, b; m^2, s) = \frac{1}{2\pi i} \left( S(a, b; m^2, s + i0) - S(a, b; m^2, s - i0) \right).$$

(59)

It is a spectrum of a general sunset diagram

$$\rho(a, b; m^2, s) = \frac{\Gamma(D/2 - b)}{\Gamma(a)\Gamma(b)\Gamma(D - a - 2b + 1)} z^{D - a - 2b} s^{D/2 - a - b} {}_2F_1(D/2 - b, 1 - b; D - a - 2b + 1; z)$$

(60)

with $z = 1 - m^2/s$. Here $\, {}_2F_1(a, b; c; z)$ is a hypergeometric function. In our case $m = m_c$ and $s = m^2_b$.

A.1 Master integrals at LO: two loop

At LO there are two master integrals. In both cases it is a two loop sunset with one heavy ($m_c$) line. The internal massive line can be a normal one or doubled which is denoted by a dot on it, see Fig. 7.

![Figure 7: Two-loop master integral. A dotted line indicates one additional power of the propagator.](image)

The closed form for a master integral with a normal line is

$$M_{00} = S(1, 1; 0, -1)\rho(1, 2 - D/2; m_c^2, m_b^2)$$

(61)
where \( S(1,1;0,-1) \) is a scalar massless loop that is expressible through \( \Gamma \)-functions

\[
S(a,b;0,-1) = \frac{\Gamma(a+b-D/2)\Gamma(D/2-a)\Gamma(D/2-b)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)}
\]

(62)

We usually set \( m_b = 1 \) in the computation. The \( \varepsilon \)-expansion of this integral can be obtained with the program HypExp or independently. At the leading order of \( \varepsilon \)-expansion one has

\[
M_{00} = \frac{1}{2} + m^2 \ln \left( m^2 \right) - \frac{m^4}{2} + O(\varepsilon)
\]

(63)

with \( m = m_c \) and \( m_b = 1 \).

The second master integral (dotted) belongs to the same class of sunsets and can be obtained as a derivative in \( m_c \)

\[
M_{01} = -\frac{d}{dm_c^2} M_{00}.
\]

(64)

The closed form for this dotted leading order master integral is

\[
M_{01} = S(1,1;0,-1) \rho(2,2-D/2; m_c^2, m_b^2).
\]

(65)

### A.2 Master integrals at NLO: three loop

At NLO there are master integrals that are factorizable, of sunset-type, and nontrivial.

#### A.2.1 Factorizable integrals

The factorized master integrals contain a closed massive loop that can be either of charmed quark or bottom quark.

\[
M_{11} = T_0(m_c) M_{00}
\]

(66)

with \( T_0(m) \) being a massive tadpole

\[
T_0(m) = m^{D-2} \Gamma(1-D/2)
\]

(67)
and the other one
\[ M_{12} = T_0(m_c)M_{01}. \] (68)

The master integrals with a \( b \)-quark tadpole are
\[ M_{41} = T_0(m_b)M_{00} \] (69)
and the other one
\[ M_{42} = T(m_b)M_{01}. \] (70)

In actual computation the bottom quark mass is set to unity.

A.2.2 Sunset-type integrals

Non-factorizable but still simple master integrals \( M_{21}, M_{22} \) are of the sunset type.

The normal one is given by the basic integral
\[ M_{21} = S(1,1; 0, -1)S(1, 2-D/2; 0, -1)\rho(1, 3 - D, m_c^2, m_b^2) \] (71)

The dotted one is its derivative in loop (charmed quark) mass
\[ M_{22n} = -\frac{d}{dm_c^2}M_{21} \] (72)

which is again a three loop sunset
\[ M_{21} = S(1, 1; 0, -1)S(1, 2-D/2; 0, -1)\rho(2, 3 - D, m_c^2, m_b^2). \] (73)

A.2.3 Nontrivial master integrals

There two nontrivial master integrals that can be chosen in a variety of ways. We define the first nontrivial master integral \( N_p \) as a sum of left (dotted at bottom line) and right (dotted on charm line) diagrams in Fig. ?? . In words, this can be expressed as \( N_p = dot.m_b + dot.m_c. \)
We managed to compute the $\varepsilon$-expansion of $N_p$ up to the necessary order. It reads

$$N_p = N_p^{LO} + \varepsilon N_p^{NLO}$$

with

$$N_p^{LO} = -2(1 - r) - (1 + r) \log(r)$$

and

$$N_p^{NLO} = 4\sqrt{r} \left( 4\text{Li}_2\left(-\sqrt{r}\right) - \text{Li}_2\left(\sqrt{r}\right) + \pi^2 + 2 \ln(r) \ln\frac{\sqrt{r} + 1}{1 - \sqrt{r}} \right)
$$

$$+ \left(\frac{1}{2} - \frac{r}{2}\right) \ln^2(r) - 3(r + 1) \ln(r)
$$

$$+ 4(r + 1) \ln(1 - r) \ln(r) + 8(1 - r) \ln(1 - r) + 14(r - 1) .$$

These are master integrals entering partonic contribution for the total width and $C_v$ coefficient.

At NLO one more master integral appears to be necessary for the $C_G$ coefficient. It is represented by the difference of left and right diagrams in Fig. 10. In words, this can be expressed as $N_p = \text{dot.m}_b - \text{dot.m}_c$. We need only the leading term of its $\varepsilon$-expansion that reads

$$N_m = (1 - r) \left( -4 \text{Li}_2(r) + \frac{2\pi^2}{3} - 4 \ln(1 - r) \ln(r) + 2 \ln(r) \right) - 2r \ln^2(r) .$$

### A.3 Master integrals in massless case

We have calculated all quantities in the massless limit independently. The reduction procedure and master integrals have been obtained independently as well. In massless case master integrals can be found in a concise form. These master integrals are represented by Feynman diagrams given in Fig. 11.

At leading order there is one master integral

$$M^0_{00} = S(1, 1; 0, -1)S(1, 2 - D/2; 0, -1) \frac{\sin(2\pi\varepsilon)}{\pi} .$$

At NLO in massless case there are three master integrals:
a) factorizable integral
\[ M_{i1}^0 = T_0(m_b)M_{00}^0; \] (79)

b) sunset integral
\[ M_{21}^0 = S(1, 1; 0, -1)S(1, 2 - D/2; 0, -1)S(1, 3 - D; 0, -1)\frac{\sin(3\pi\varepsilon)}{\pi}; \] (80)

c) complicated integral
\[ N^0 = -S(1, 1; 0, -1)\frac{\Gamma(1 - \varepsilon)^2}{\Gamma(2 - \varepsilon)\Gamma(3 - 3\varepsilon)}\mathbf{4} F_2\{\varepsilon, 1 - \varepsilon, 1; \{3 - 3\varepsilon, 2 - \varepsilon\}, 1\}. \] (81)

**B C\textsubscript{v} coefficient at NLO with full mass dependence**

The expression for the coefficient \( C_v \) is
\[
C_v^{NLO} = (3\text{Li}_2(r) - \frac{1}{2}\pi^2)(1 - 16r^2 - 3r^4) - \frac{1}{24}(1 - r)(25 - 1011r - 1487r^2 + 189r^3)
+ \frac{1}{6}r(12 + 450r + 4r^2 + 45r^3)\ln(r) - \frac{1}{6}(1 - r)(11 + 11r + 83r^2 - 45r^3)\ln(1 - r)
+ \frac{3}{2}r^2(4 + r^2)\ln^2(r) + 2(1 - 30r^2 - 3r^4)\ln(1 - r)\ln(r)
+ 8r^{3/2}(1 + 3r)\left(4\text{Li}_{2}^- - \pi^2 - 2\ln\left(\frac{1 + \sqrt{r}}{1 - \sqrt{r}}\right)\ln(r)\right)
\] (82)

where \( \text{Li}_2^- = \text{Li}_2(\sqrt{r}) - \text{Li}_2(-\sqrt{r}) \).
Here we give results for the $C^{NLO}_{\mu_G}$ coefficient. At NLO we give both color structures.

The $C_A$ color structure coefficient of $\alpha_s/\pi$ reads

$$C^{NLO,C_A}_{\mu_G} = \frac{1}{108} (1 - r) \left( 156 - 4081r - 354r^2 - 405r^3 \right) + \frac{1}{9} (6\text{Li}_2(r) - \pi^2) (1 - 6r + 24r^2 - 11r^3) - \frac{(1 - r)}{54r} (15 + 20r - 196r^2 - 292r^3 - 27r^4) \ln(1 - r) - \frac{1}{54} r \left( 786 + 972r + 131r^2 - 27r^3 \right) \ln(r) - \frac{2}{9} (1 + 9r - 93r^2 + 19r^3) \ln(1 - r) \ln(r) + \frac{1}{9} r (9 - 33r + 5r^2) \ln(r)^2 + \frac{8}{3} r^{1/2} (1 - \frac{11}{3} r) \left( 4\text{Li}_2 - \pi^2 - 2 \ln(r) \ln \left( \frac{1 + \sqrt{r}}{1 - \sqrt{r}} \right) \right). \tag{83}$$

The $C_F$ color structure is

$$C^{NLO,C_F}_{\mu_G} = -\frac{1}{216} (1 - r) \left( 321 - 13747r + 5421r^2 - 3807r^3 \right) + \frac{1}{18} (6\text{Li}_2(r) - \pi^2) (5 + 72r - 72r^2 - 88r^3 + 45r^4) - \frac{(1 - r)}{54r} (12 - 19r + 917r^2 - 1795r^3 + 585r^4) \ln(1 - r) + \frac{1}{54} r \left( 1500 - 330r + 2668r^2 - 585r^3 \right) \ln(r) + \frac{2}{9} (11 + 54r - 48r^2 - 94r^3 + 45r^4) \ln(1 - r) \ln(r) - \frac{1}{18} r \left( 72 + 60r - 112r^2 + 45r^3 \right) \ln(r)^2 + \frac{32}{3} (1 - \frac{4}{3} r) r^{1/2} \left( 4\text{Li}_2 - \pi^2 - 2 \ln(r) \ln \left( \frac{1 + \sqrt{r}}{1 - \sqrt{r}} \right) \right). \tag{84}$$

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