Sensor and Regional Gradient Detectability of Distributed Systems

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Abstract. In this paper the Regional Gradient Detectability (RG – Detectability) based on Sensors and Actuators Structures (SAS). So, we reflect a class of Distributed Parameter Systems (DPSs) in Sobolev Space (SS) such that the dynamics of the system is ruled by Strongly Continuous SemiGroup (SCSS). More accurately, we discover numerous outcomes connected with diverse types of structures with a view to accomplish that Regional Gradient Observability (RG – Observability) notion. Therefore, we establish that, the existence of Gradient Detectable System (GD – System) is not realized in general case, but it perhaps Regionally Gradient Detectable System (RG – Detectable System).

Keywords: RG – Detectability, Diffusion System, RGS – Sensors, RG – Observers.

1 Introduction

Analysis of DPSs has established a lot consideration in the literatures with several ([1-3] and references therein). The study of this notion through the structure of the sensors and the actuators was developed by El Jai and Pritchard in ref.s [4]. So that, the conception of Regional Analysis was presented by El Jai, Zerrik and Al-Saphory et al. in [5-7] and it involves in considering the comportment of systems not in all the Domain but only a certain part ω of the Domain. Thus, the work is induced by many concrete-real problems, in thermic, mechanic, environment...[7-8]. So, Detectability concept was an important consideration in DPSs analysis, because it is possible to reconstruct an specific observer for the reflected system. An extension of this notion, is that of Regional Detectability which was explored recently by Al – Saphory and El Jai et al. [6, 9-10], and was concentrated on the orientation state in a specified part ω of R. The objective of this study is to introduce and analysis the rigorous results devoted to the RG – Detectability notion. Moreover, we give a demonstration that there is a linked between this concept and the number of GS-Sensors their positions and the geometrical domains of the systems. The main goal behind presenting this concept is that to contract with some physical problems related to the model.
represented by (Figure 1) below, one is interested in detecting the state in the ℜ zone rather than in the entire space ℜ [8].

**Figure 1.** The real detected model of region ℜ with fixed and mobile sensors. This problem is situated in a class of so-called ℜ-G-Detectability and ℜ-G-Observability problem developed by Al-Saphory and El-Jai et al. as in [5-7, 9-21, 23]. This method can be modified to detect the gradient of the state in ℜ of this model depending on structures sensors and actuators as in [8]. The outline of this research is structured as following: The next section devoted to the Formulation Problem and Preliminaries. The third section, devotes to the introduction of ℜ-G-Detectability. Thus, we examine the relation between this notion and ℜ-G-Observability and sensors structures.

2 Formulation Problem and Preliminaries

Let the following assumptions be given:

- ℜ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial \mathbb{R}$.
- $\mathbb{R}$ be a nonempty given subregion of $\mathbb{R}$.
- We denote $\mathcal{Q} = \mathbb{R} \times [0, \infty[, \Phi = \mathbb{R} \times [0, \infty[.$
- The space $\mathcal{X}, \mathcal{U} \text{ and } \mathcal{Z}$ be separable SS such that $\mathcal{X}$ is the space of states, $\mathcal{U}$ the space inputs and $\mathcal{Z}$ the space of measurements, with $\mathcal{X} = H^1(\mathbb{R}), \mathcal{U} = L^2(0, \infty, R^p)$ is the control space and $\mathcal{Z} = L^2(0, \infty, R^q)$ where $p$ and $q$ hold for the number of actuators and sensors [1]. The reflected is given by subsequent DPs

\[
\begin{align*}
\frac{\partial x}{\partial t}(\zeta, t) &= Ax(\zeta, t) + Bu(t) & \mathcal{Q} \\
x(\zeta, 0) &= x_0(\zeta) & \mathbb{R} \\
x(\eta, t) &= 0 & \Phi \\
\end{align*}
\]  

(1)

augmented with the output function

\[
z(., t) = Cx(., t)
\]

(2)

where $A$ is a linear differential operator of second order type, which is a SCSS $(S_A(t))_{t \geq 0}$ on the SS $H^1(\mathbb{R})$. The operator $B \in L(R^p, \mathcal{X})$ and $C \in L(H^1(\omega), R^q)$ depend on the SAS [4,11]. Thus, the mathematical model illustrates in Figure 2 is further more general than the specific case represented in (Figure 1).
Thus, by using [4] the system (1) has a solution of unique type described by

$$x(\xi, t) = S_A(t) x_0(\xi) + \int_0^t S_A(t - \tau) B u(\tau) d\tau$$  \hspace{1cm} (3)

The sensors definitely of zone type, symbolized by $\mathcal{D}_i \subset \mathbb{R}$ represent the support (or $\mathcal{D}_i \subset \partial \mathbb{R}$) and $f \in L^2(\mathcal{D}_i)$, represent the distribution with $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$, where $1 \leq i \neq j \leq q$. Then, the measurements (2) formulated by

$$z(\xi, t) = < x(\xi, t), f_i(\cdot) >_{L^2(\mathcal{D}_i)}$$  \hspace{1cm} (4)

The sensors definitely of pointwise type, symbolized by $(b_i, \delta_{b_i})$, such that $b_i \in \mathbb{R}$, then,

$$z(\xi, t) = x(b_i, t).$$  \hspace{1cm} (5)

Let

$$K: x \in X \to Kx = CS_A(\cdot) x \in Z$$  \hspace{1cm} (6)

characterizes the observation operator, then

$$z(\xi, t) = K(t) x_0(\cdot) .$$  \hspace{1cm} (7)

We indicate by $K^*: Z \to X$ the related adjoint of $K$ formed via

$$K^*z^*(\cdot, t) = \int_0^t S_A(s) C^*z^*(\cdot, s) ds$$  \hspace{1cm} (8)

If $\mathbb{R} \subset \mathbb{R}$, we consider the function

$$\chi_{\mathbb{R}}: (L^2(\mathbb{R}))^n \to (L^2(\mathbb{R}))^n$$  \hspace{1cm} (9)

and where $x |_{\mathbb{R}} (\xi, t)$ is the restriction of the state $x (\xi, t)$ to $\mathbb{R}$. Let $\nabla$ be the operator is given by

$$\nabla: H^1(\mathbb{R}) \to (L^2(\mathbb{R}))^n$$  \hspace{1cm} (10)

$$x \to \nabla x = (\frac{\partial x}{\partial \xi}, \ldots, \frac{\partial x}{\partial \xi})$$
So $\nabla^*$ symbolized the related operator of adjoint type of $\nabla$. It is interested to reconsider some notion reformulated in the next statements [12-13]:

- Systems which is linked to (1)-(2) are said to be Exactly (resp. Weakly) Regionally Gradient Observable in $\mathbb{R}$ (or ERG-Observable (resp. WRG-Observable)) if:
  \[ \text{Im}\chi_\mathcal{R} \nabla K^* = (L^2(\mathbb{R}))^n \text{ (resp. } \text{Im}\chi_\mathcal{R} \nabla K'(\cdot) = (L^2(\mathbb{R}))^n) \]

- System is ERG — observable (resp. WRG-Observable, the it is ERG-Observable (resp. WRG-Observable) [12].
- Systems ERG-Observable (resp. WRG-Observable), they are $\mathcal{E} \mathcal{R}^1 \mathcal{G}$-Observable (resp. $\mathcal{W} \mathcal{R}^1 \mathcal{G}$-Observable) for all $\mathbb{R}^1 \subset \mathbb{R}$.
- The couples $(\mathcal{D}_v, f_1)_{\text{loc}}$ are Regional Gradient Strategic Sensors (RGS-Sensors) if the systems (1)-(2) is WRG-Observable. Thus, in the situation of boundary domain, we mention to ref. [14].

**3. Regional Exponential Gradient Detectability**

The Gradient Detectability is in some sense a dual notion of Gradient Stabilizability [2-4]. In Elfaij and Pritchard [4] this notion was considered and analyzed in the whole domain $\mathbb{R}$. In this section, we shall extend these results to the regional case by considering $\mathbb{R}$ as region of $\mathbb{R}$. RGS-Detectability characterization needs some assumptions which are related to the Gradient Stability.

**3.1. Definitions and Characterizations**

As well-known in [1-4] the problem of Gradient Stability is one of the most important in the analysis of Control Systems. Thus, for this purpose we announce some definitions and characterizations derived from [9-10] in the subsequent:

- The Semigroup $(S_\mathcal{A}(t))_{t \geq 0}$ is said to be Gradient Stable (G-Stable) on $(L^2(\mathbb{R}))^n$, if $\forall x_0 \in H^1(\mathbb{R})$ then, $x(t)$ in system (1) verifies the condition,
  \[ \lim_{t \to \infty} x(t) = 0. \]  
  \[ \text{(11)} \]

- System (2.1) so-called G-Stable, if $\mathcal{A}$ produces a SCSG which is G-Stable on $(L^2(\mathbb{R}))^n$. Thus, the system (2.1) is G-Stable, if and only if, $M_\mathcal{G}, \alpha_\mathcal{G} > 0$, such that
  \[ \|\nabla S_\mathcal{A}(\cdot)\|_{(L^2(\mathbb{R}))^n} \leq M_\mathcal{G}e^{-\alpha_\mathcal{G}t}, \forall t \geq 0 \]

If $(S_\mathcal{A}(t))_{t \geq 0}$ is G-Stable, then $\forall x_0 \in H^1(\mathbb{R})$, we have,
\[ \lim_{t \to \infty} \|\nabla x(\cdot,t)\|_{(L^2(\mathbb{R}))^n} = \lim_{t \to \infty} \|\nabla S_\mathcal{A}(\cdot) x_0\|_{(L^2(\mathbb{R}))^n} = 0 \]  
  \[ \text{(12)} \]

- Systems (1)-(2) so-called G-Detectable, if $\exists H: R^q \to (L^2(\mathbb{R}))^n$, $\exists (\mathcal{A} - HC)$ produces $(S_H(t))_{t \geq 0}$ which is G-Stable on $(L^2(\mathbb{R}))^n$.
- System is G-Detectable, then it is conceivable to rebuild an G-Observer for the corresponding system [15-16].
• Reflect the system
\[
\begin{aligned}
\frac{\partial u}{\partial t}(\xi, t) &= Ax(\xi, t) + Bu(t) + H_{EG}(x(\xi, t) - Cy(\xi, t)) \quad \Omega \\
y(\xi, 0) &= y_0(\xi) \\
y(\eta, t) &= 0 \\
\end{aligned}
\]
(13)

Then \( y(\xi, t) \) gives the estimations for the state gradient \( x(\xi, t) \) because the equation
\[
e(\xi, t) = x(\xi, t) - y(\xi, t)
\]
and verified
\[
\frac{\partial e}{\partial t}(\xi, t) = (A - H_{EG}C)e(\xi, t)
\]
such that
\[
e(\xi, 0) = x(\xi, t) - y_0(\xi, t).
\]
So, the system is \( G\)-Detectable, it is probable to select \( H_G \) which identify the form
\[
\lim_{t \to \infty} \|e(\xi, t)\|_{(L^2(\mathbb{R}))^n} = 0.
\]

**Remark 3.1:** In this research, it is necessary that, the relation (12) to be true for the considered region \( \mathcal{R} \). therefore,
\[
\lim_{t \to \infty} \|\nabla x(\xi, t)\|_{(L^2(\mathbb{R}))^n} = \lim_{t \to \infty} \|\nabla S(\xi)\|_{(L^2(\mathcal{R}))^n} = 0
\]
we may study in the next this as \( G\)-Stability.

**Definition 3.2:** System so-called \( G\)-Stable, if \( A \) produces a SCSG which is \( G\)-Stable on \( (L^2(\mathbb{R}))^n \).

**Definition 3.3:** Systems so-called \( G\)-Detectable if \( \exists H_{RG}: \mathbb{R}^q \to (L^2(\mathbb{R}))^n \) such that (\( A - H_{RG}C \)) produces a SCSG \( (S_{RG}(t))_{t \geq 0} \) which is \( G\)-Stable.

**Remark 3.4:**
I- System stays \( G\)-Detectable, remains \( G\)-Detectable.

II- System stays \( G\)-Detectable, remains \( G\)-Detectable, for all \( \mathbb{R}^1 \subseteq \mathbb{R} \) however the reverse is not true as in the next outcome.

**3.2. Example of Counter type**

Contemplate the system motivated by the pointwise control \((\bar{b}, \delta_E)\)
\[
\begin{aligned}
\frac{\partial x}{\partial t}(\zeta, t) &= \gamma_1 \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \gamma_2 x(\zeta, t) + \delta(\zeta - \bar{b})u(t) \quad ]0, 1[, \ t > 0 \\
x(\zeta, 0) &= x_0(\zeta) \\
x(\zeta, 0, t) &= x(a, t) = 0 \\
\end{aligned}
\]
where \( \gamma_1 > 0, \gamma_2 > 0, \mathbb{R} = ]0, 1[ \) and \( \bar{b} \in \mathbb{R} \) is the position of the control \((\bar{b}, \delta_E)\). The measurement may be found by the use of sensor of internal pointwise type. Then,
\[ z(t) = \int_0^t x(\xi, t) \delta(\xi - b)d\xi. \]  

where \( b \in \mathbb{R} \) is the position of the sensor \((b, \delta_b)\) (Figure 3).

**Figure 3.** Show of sensor and actuator for internal pointwise type.

\[ \mathcal{A} = (\gamma_1 \frac{\partial^2}{\partial \xi^2} + \gamma_2) \text{ creates a } SCSG \ \{S_{\mathcal{A}}(t)\}_{t \geq 0} \text{ on } X = H^1(\mathbb{R}). \]  

If \( b \in \mathbb{Q} \cap [0, 1] \) (where \( \mathbb{Q} \) is the rational numbers), then \((b, \delta_b)\) is not GS-Sensor for the system of (15) and therefore the systems (15)-(16) is not \( G\)-Detectable in \( \mathbb{R} \) [4]. We deliberate the region \( \mathfrak{R} = ]a, \beta[ \cup [0, 1] \) if \( \frac{b-a}{\beta-a} \notin \mathbb{Q} \cap [0, 1] \), then the \((b, \delta_b)\) is \( \mathfrak{RGS}\)-Sensor for (15) [13] and hence (15)-(16) is \( \mathfrak{RG}\)-Detectable System [9].

On the other hand, one can simply deduced the next outcomes.

**Corollary 3.5:** \( EG\)-Observable system is \( G\)-Detectable.

**Corollary 3.6:** Systems (2.1)-(2.2) \( \mathfrak{RG}\)-Observable, then it is \( \mathfrak{RG}\)-Detectable.

Now, supposing that

\[ x_0 = 0 \Rightarrow S_{\mathcal{A}}(t)x_0 = 0, \]

then, \( \exists \gamma > 0 \), which satisfies this inequality

\[ \|X_\omega \nabla x\|_{(L^2(\mathbb{R}))^n} \leq \gamma \|CS_{\mathcal{A}}(\cdot)x_0\|_{L^2(0, \infty, \mathbb{R}^n)}, \quad \forall x_0 \in (L^2(\mathbb{R}))^n \]

### 3.3. \( \mathfrak{RG}\)-Detectability and Sensor

Let \((\varphi_{m_j})\) be the eigenfunctions of complete set type of \( \mathcal{A} \) with \( \lambda_m \) the corresponding eigenvalues and \( m_m \) represents the multiplicity. In addition, the function sets \((\psi_{m_j})\) described by \( \psi_{m_j}(\xi) = \chi_\mathfrak{R} \nabla \varphi_{m_j}(\xi) \) in \((L^2(\mathbb{R}))^n \) and suppose that the system (1) s. Thus, the sufficient condition of \( \mathfrak{RG}\)-detectability is given by the following main result.

**Theorem 3.7:** Suppose that there are \( q \) zone sensors \((\mathcal{D}_i, f_i)_{1 \leq i \leq q} \) and. The systems (2.1)-(2.2) is \( \mathfrak{RG}\)-Detectable if and only if:

1. \( q \geq m \)
2. \( \text{rank } G_m = m_m, \forall m, m = 1, \ldots, J \) with
$G_m = (G_m)_{ij} = \begin{cases} 
\psi_{m_j}(b_i), f_i(.) > \ell^2(\mathcal{G}_i) & \text{for zone sensors} \\
\psi_{m_j}(b_i) & \text{for pointwise sensors} \\
< - \frac{\partial \psi_{m_j}}{\partial \varphi}, f_i(.) > _{L^2(\Gamma_j)} & \text{for boundary zone sensors} 
\end{cases}$

and the spectrum of $\mathcal{A}$ holds $j$ eigenvalues with non-negative real parts, such that sup $m_m = m < \infty$, and $j = 1, \ldots, m_m$

**Proof:** Let $x(\xi, t) = [x_1(\xi, t), x_2(\xi, t)]^T$, where $x(\xi, t)$ is the state component of the unstable part of the system (1), is given by

\[
\begin{aligned}
\frac{\partial^2 x_1(\xi, t)}{\partial t} &= \mathcal{A}_1 x_1(\xi, t) + PBu(t) & \xi \\
x_1(\xi, 0) &= x_{01}(\xi) & \mathbb{R} \\
x_1(\eta, t) &= 0 & \Phi
\end{aligned}
\tag{17}
\]

where $x_2(\xi, t)$ is the state part of the stable formulated via

\[
\begin{aligned}
\frac{\partial^2 x_2(\xi, t)}{\partial t} &= \mathcal{A}_2 x_2(\xi, t) + (I - P)Bu(t) & \xi \\
x_2(\xi, 0) &= x_{02}(\xi) & \mathbb{R} \\
x_2(\eta, t) &= 0 & \Phi
\end{aligned}
\tag{18}
\]

The hypotheses (2) of theorem 3.7, permits that $(D, f_i)_{1\leq i\leq q}$ of $\mathcal{RGS}$-Sensors for the unstable subsystem of (1), and then, the subsystem (17) is $\mathcal{RGS}$-Observable [12]. Meanwhile the equation (17) is of limited dimensional, then it is $\mathcal{RGS}$-Detectable. Therefore it is $\mathcal{RGS}$-Detectable, and hence

\[
\|x_1(\cdot, t)\|_{(L^2(\mathbb{R}))^n} \leq M_{\mathcal{RGS}}^1 e^{-a_{\mathcal{RGS}}^2(t)} \|Px_0(.)\|_{(L^2(\mathbb{R}))^n}.
\]

then, $\mathcal{A}_2$ produce a SCSG of stable type on $(L^2(\mathbb{R}))^n$. So that, $\exists M_{\mathcal{RGS}}^2, a_{\mathcal{RGS}}^2 > 0$ holds the next inequality

\[
\|x_2(\cdot, t)\|_{(L^2(\mathbb{R}))^n} \leq M_{\mathcal{RGS}}^2 e^{-a_{\mathcal{RGS}}^2(t)} \|P\|_{(L^2(\mathbb{R}))^n} x_0(.)\|_{(L^2(\mathbb{R}))^n} + \int_0^t M_{\mathcal{RGS}}^2 e^{-a_{\mathcal{RGS}}^2(t)} \|P\|_{(L^2(\mathbb{R}))^n} x_0(.)\|_{(L^2(\mathbb{R}))^n} \|u(t)\| dt
\]

Consequently, $x(\xi, t) \to 0$ when $t \to \infty$. As a final point, the system (1)-(2) is $\mathcal{RGS}$-Detectable.

Conversely, if (1)-(2) is $\mathcal{RGS}$-Detectable System, then, the equation (17) is $\mathcal{RGS}$-Detectable System. If a system is $\mathcal{RGS}$-Observable, that is,

\[
[K\nabla x_0(\cdot, t) = 0 \Rightarrow x_0(\cdot, t) = 0] \text{ ([12])}.
\]

On behalf of $x^*(\cdot, t) \in (L^2(\mathbb{R}))^n$, we have

\[
K\nabla x_0^*(\cdot, t) = (\sum_{m=1}^j e^{\lambda_m t} < \psi_{m_j}(.), x_0^*\nabla x^*(\cdot, t) >_{(L^2(\mathbb{R}))^n} < \psi_{m_j}(.), f_j(\cdot, t) >_{L^2(\mathbb{R})} \leq q_j)
\]
If the rank $G_m x_m \neq m_m$ for $m, m = 1, ..., f$, there exists $x^*(., t) \in (L^2(\mathbb{R}))^n$, such that $Kv x^* x^*(., t) = 0$, this leads

$$\sum_{m=1}^{f} < \psi_{m}(.), x^*(., t) > (L^2(\mathbb{R}))^n < \psi_{m}(.), f_{j}(., t) > (L^2(\mathbb{R}))^n 1 \leq k \leq q$$

The component of the state $x_m$ is described via

$$x_m(., t) = [ < \psi_{1}(.), x^*(., t) > (L^2(\mathbb{R}))^n < \psi_{j_m}(.), x^*(., t) > (L^2(\mathbb{R}))]^T \neq 0$$

therefore

$$G_m x_m = 0 \text{ for } m, m = 1, ..., f.$$  

So, the equation (17) not remains $WRG$-observable system and hence $(D, f_1)_{1 \leq k \leq q}$ not stays $RGS$-Sensor. Finally (1)-(2) is not $RGS$-Deteatable system [14] and then, the rank $G_m \neq m_m, \forall m, m = 1, ..., f$. □

4. Application to Diffusion Systems

Deliberate $DPs$s of Diffusion type with $\mathcal{A} = \Delta$ well – defined on $\mathbb{R}_k$ wherever $1 \leq k \leq 2$. Contemplate $\mathbb{R}_1 = ]0, a_1[ \times ]0, a_2[$. So that we symbolize $\mathcal{Q}_1 = \mathbb{R}_1 \times ]0, \infty[, \mathcal{Z}_1 = \mathbb{R}_1 \times ]0, \infty[ \text{ and } \mathcal{Z}_2 = \mathbb{R}_2 \times ]0, \infty[ \text{ with boundaries } \Phi_1 = \partial \mathcal{Q}_1 \times ]0, \infty[ \text{ and } \Phi_2 = \partial \mathcal{Z}_2 \times ]0, \infty[$.

4.1. Zone Sensor

Consider the following problems.

4.1.1. Figure 4 case

Reflect the following form

$$\begin{align*}
\frac{\partial x}{\partial t}(\zeta, t) &= \Delta x(\zeta, t) + x_{bg}(\zeta) u(t) & \mathcal{Q}_1 \\
x(\zeta, 0) &= x_0(\zeta) & \mathbb{R}_1 \\
x(0, t) &= x(a, t) = 0 & \mathcal{Z}_1
\end{align*}$$

(19)

where $\mathcal{D} \in \mathbb{R}_1$ is the position of the control, $g \in L^2(\mathcal{D})$ and (19) characterize the heat conduction system [12]. Assume that $\mathbb{R}_1 = ]a, \beta[ \subset ]0, a[ \text{, } a > 0$ be the interested region (Figure 4).
Figure 4. Show of sensor and actuator for internal zone type in one dimension.

The measurements maybe get by $(D, f)$ where $D \subset \mathbb{R}$ is the sensor location, the $f \in L^2(D)$. The output information is characterized via the form

$$z(t) = \int_D x(\zeta, t) f(\zeta) d\zeta$$

So, we have

$$\varphi_m(\zeta) = \frac{2}{\sqrt{\beta-\alpha}} \sin m \frac{(\zeta-\alpha)}{(\beta-\alpha)} \quad \text{and} \quad \lambda_m = -\gamma \frac{m^2\pi^2}{(\beta-\alpha)^2}$$

Here, by supposing that $\frac{a^2}{\beta^2} \notin \mathbb{Q}$, then $\lambda_m = \lambda_1$, and a unique sensor is guaranteed the $\mathbb{R}_1GS$-Sensor [11].

**Corollary 4.1:** If the information support is given by $D = [\zeta_0 - l, \zeta_0 + l] \subset ]0, a[\), so $f$ is of type symmetric about $\zeta_0$, then (91)-(20) is $\mathbb{R}_1G$-Detectable systems satisfies this condition $\frac{m(\zeta_0-\alpha)}{(\beta-\alpha)} \notin \mathbb{Q}$ for all $m, m = 1, \ldots, f$.

4.1.2. **Figure 5 case**

We deliberate the system (4.1)-(4.2) is specified by

$$\begin{align*}
\frac{\partial x}{\partial t}(\zeta_1, \zeta_2, t) &= \Delta x(\zeta_1, \zeta_2, t) + \chi_D g(\zeta_1, \zeta_2) u(t) & \Omega_2 \\
x(\zeta_1, \zeta_2, 0) &= x_0(\zeta_1, \zeta) & \mathbb{R}_2 \\
x(\eta_1, \eta_2, t) &= 0 & \Phi_2 \\
z(t) &= \int_D x(\zeta_1, \zeta_2, t) f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 & \Xi_2
\end{align*}$$

where $D, \overline{D} \subset \mathbb{R}_2$ or $(D, \overline{D} = \Gamma \subset \partial \mathbb{R}_2)$ are the positions of the sensor and actuator as (Figure 5).
Figure 5. Show of sensor and actuator for internal and boundary zone type in two dimensions.

Suppose that \( \mathbb{R}_2 = [a_1, \beta_1 [\times] a_2, \beta_2 [\subset] o, a_1 [\times] o, a_2 [\subset] o \) and \( a_1 > o \) and \( a_2 > o \). For this case consider the following functions:

\[
\varphi_{nm}(\zeta_1, \zeta_2) = \frac{2}{\sqrt{(\beta_1 - a_1)(\beta_2 - a_2)}} \sin n \pi \frac{\zeta_1 - a_1}{\beta_1 - a_1} \sin m \pi \frac{\zeta_2 - a_2}{\beta_2 - a_2}
\]

as well as with

\[
\lambda_{nm} = -\left( \frac{n^2}{(\beta_1 - a_1)^2} + \frac{m^2}{(\beta_2 - a_2)^2} \right) \pi^2.
\]

If \( (\beta_1 - a_1)^2 \notin \mathbb{Q} \), hence \( \lambda_{nm} = 1 \), and then a unique sensor is guaranteed \( \mathbb{R}_2 GS-Sensor \) (see [13]).

Assume that measurement support is rectangular through \( D = [\zeta_{01} - l_1, \zeta_{01} + l_1] \times [\zeta_{02} - l_2, \zeta_{02} + l_2] \in \mathbb{R}_2 \). If \( f_1 \) is symmetric about \( \zeta_1 = \zeta_{01} \) and \( f_2 \) is symmetric about \( \zeta_2 = \zeta_{02} \), then we obtained the subsequent outcome. \( \zeta_{01}, \zeta_{02} \)

**Corollary 4.2:** If (4.5) is \( \mathbb{R}_2 G-Detectable System \) if \( \frac{n}{(\zeta_{01} - a_1)} \notin \mathbb{Q} \) and \( \frac{m}{(\zeta_{02} - a_2)} \notin \mathbb{Q} \) for all \( n, m = 1, ..., f \).

In the situation of zone sensor type boundary, we have

\[
\begin{align*}
\frac{dx}{dt}(\zeta_1, \zeta_2, \tau) &= \Delta x(\zeta_1, \zeta_2, \tau) + \chi_D g(\zeta_1, \zeta_2) u(\tau) \\
\frac{dx}{dt}(\eta_1, \eta_2, \tau) &= 0 \\
x(\zeta_1, \zeta_2, 0) &= x_0(\zeta_1, \zeta_2) \\
\varphi(\tau) &= \int_{\Gamma} f(\zeta_1, \zeta_2, \tau) d\zeta_1 d\zeta_2.
\end{align*}
\]

with \( \Gamma \subset \partial \mathbb{R} \) as well as \( f \in L^2(\Gamma) \). In the case \( (D, f) \) is situated on the boundary type, so that \( \Gamma = [\eta_{01} - l, \eta_{01} + l] \times \{a_2\} \). Thus obtain the following results:

**Corollary 4.3.:**
I- Case One Side Type: Let $\Gamma = [\eta_0 - l, \eta_0 + l] \times \{a_2\} \subset \partial \mathbb{R}$ and $f$ be of symmetric type about the point $\eta = \eta_1$, so (25) is $\mathbb{R}_2G$-Detectable System if $\frac{m(\eta_0 - a_1)}{(\beta_1 - a_1)} \notin \mathbb{Q}$ for all $m, n = 1, \ldots, J$.

II- Case Two Side Type: Let $\Gamma = [\eta_0 - l_1, a_1] \times \{a_2\} \cup \{a_1\} \times [\eta_0, \eta_0 + l_2] \subset \partial \mathbb{R}$ and $f|_{\Gamma}$ be of symmetric type about the point $\eta_1 = \eta_0$, and also $f|_{\Gamma}$ be of symmetric type about the point $\eta_2 = \eta_0$, so, the (25) is $\mathbb{R}_2G$-Detectable System if $\frac{n(\eta_0 - a_1)}{(\beta_1 - a_1)}$ as well as with $\frac{m(\eta_0 - a_1)}{(\beta_2 - a_2)} \notin \mathbb{Q}$ for all $n, m = 1, \ldots, J$.

4.2. Pointwise Sensor

In this subsection, we consider the following cases:

4.2.1. Figure 6 case

Here, the system (19)-(20) is given by $(\bar{b}, \delta_{\bar{b}})$

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial x}{\partial t}(\zeta, t) = \Delta x(\zeta, t) + \delta(\zeta - \bar{b})u(t) \\
x(\zeta, 0) = x_0(\zeta) \\
x(0, t) = x(a, t) = 0 \\
z(t) = \int_{0}^{t} x(\zeta, t) \delta(\zeta - \bar{b})d\zeta
\end{array} \right. \\
Q_1, R_1, \Phi_1, \Xi_1
\end{aligned}
$$

(26)

where $\bar{b}, b \in \mathbb{R}_1$ are the positions of actuator and sensor (see Figure 6).

**Figure 6.** Show of sensor and actuator for internal pointwise type in one dimension.

The measurement is obtained via $(b, \delta_b)$ situated on $\partial \mathbb{R}_1$. Then the system (26) represented via

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial x}{\partial t}(\zeta, t) = \Delta x(\zeta, t) + \delta(\zeta - b)u(t) \\
x(\zeta, 0) = x_0(\zeta) \\
\frac{\partial x}{\partial \nu}(0, t) = x(a, t) = 0 \\
z(t) = \int_{0}^{b} x(\zeta, t) \delta(\zeta - b)d\zeta
\end{array} \right. \\
Q_1, R_1, \Phi_1, \Xi_1
\end{aligned}
$$

(27)

Thun, we have the following result.

**Corollary 4.4:**
I- Case of Internal Type. Let \( b \in ]0, a[ \) be a position sensor, so, (26) is \( \mathcal{R}_1 \)-Detectable System if \( \frac{m(b-a)}{(b-a)} \notin \mathbb{Q}, \forall m, m = 1, ..., J \).

II- Case of Boundary Type. Let \( b \in \{0, a\} \), be a position sensor, so, the (27) is \( \mathcal{R}_1 \)-Detectable System if \( \frac{m(b-a)}{(b-a)} \notin \mathbb{Q}, \forall m, m = 1, ..., J \).

4.2.2. **Figure 7 case**

Contemplate the system

\[
\begin{aligned}
&\frac{dx}{dt}(\zeta_1, \zeta_2, t) = \Delta x(\zeta_1, \zeta_2) + \delta(\zeta_1 - \overline{b}_1, \zeta_2 - \overline{b}_2)u(t) \\
&x(\zeta_1, \zeta_2, 0) = x_0(\zeta_1, \zeta_2) \\
&\frac{dx}{dv}(\eta_1, \eta_2, t) = 0 \\
&z(t) = \int_{\Omega_2} x(\zeta_1, \zeta_2, t) \delta(\zeta_1 - b_1, \zeta_2 - b_2) d\zeta_1 d\zeta_2
\end{aligned}
\]

where \( \overline{b} = (\overline{b}_1, \overline{b}_2) \in \mathbb{R}_2 \) and \( b = (b_1, b_2) \in \partial \mathbb{R}_2 \) as (Figure 7).

**Figure 7.** Show of sensor and actuator for internal and boundary pointwise type in two dimension.

So, the (28) is represented via

\[
\begin{aligned}
&\frac{dx}{dt}(\zeta_1, \zeta_2, t) = \Delta x(\zeta_1, \zeta_2, t) + \delta(\zeta_1 - \overline{b}_1, \zeta_2 - \overline{b}_2)u(t) \\
&x(\zeta_1, \zeta_2, 0) = x_0(\zeta_1, \zeta_2) \\
&\frac{dx}{dv}(\eta_1, \eta_2, t) = 0 \\
&z(t) = \int_{\partial \Omega_2} x(\zeta_1, \zeta_2, t) \delta(\zeta_1 - b_1, \zeta_2 - b_2) d\zeta_1 d\zeta_2
\end{aligned}
\]

Thus, we obtain.

**Corollary 4.5**

I- Case of Internal Type. Let \( \frac{n(b_1-a_1)}{(b_1-a_1)} \) and \( \frac{m(b_2-a_2)}{(b_2-a_2)} \notin \mathbb{Q}, \forall n, m = 1, ..., J \), so (28) is \( \mathcal{R}_2 \)-Detectable System.
II- Case of Internal filament Type. Let \((\sigma, \delta)\) such that \(\sigma = \text{Im}(\gamma)\) is of symmetric type about \(b = (b_1,b_2)\). So, (29) is \(\mathfrak{R}_2G\)-Detectable System if \(\frac{n(b_1-a_1)}{(\beta_1-a_1)}\) and \(\frac{m(b_2-a_2)}{(\beta_2-a_2)} \notin \mathbb{Q}, \forall \ n, m = 1, \ldots, J.\)

III- Case of Boundary Type. Let \(\frac{n(b_1-a_1)}{(\beta_1-a_1)}\) and \(\frac{m(b_2-a_2)}{(\beta_2-a_2)} \notin \mathbb{Q}, \forall \ n, m = 1, \ldots, J,\) So, (29) is \(\mathfrak{R}_2G\)-Detectable System.

6. Conclusion

In the paper we have examined the being of the sufficient condition of \(\mathfrak{R}G\)-Detectable System. Thus, we have discussed and analyzed the \(\mathfrak{R}G\)-Detection problem for a class \(\mathcal{DPS}_c\) in \(H^1(\mathbb{R})\) with \(\mathfrak{GS}\)-Sensors and regional \(\mathfrak{R}G\)-Observer. Finally, Numerous interesting consequences relating to select sensors and actuators are discovered and exemplified in precise locations.

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