The Parity Bit in Quantum Cryptography

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Abstract

An \(n\)-bit string is encoded as a sequence of non-orthogonal quantum states. The parity bit of that \(n\)-bit string is described by one of two density matrices, \(\rho_0^{(n)}\) and \(\rho_1^{(n)}\), both in a Hilbert space of dimension \(2^n\). In order to derive the parity bit the receiver must distinguish between the two density matrices, e.g., in terms of optimal mutual information. In this paper we find the measurement which provides the optimal mutual information about the parity bit and calculate that information. We prove that this information decreases exponentially with the length of the string in the case where the single bit states are almost fully overlapping. We believe this result will be useful in proving the ultimate security of quantum cryptography in the presence of noise.
I. INTRODUCTION

A major question in quantum information theory \([1,4]\) is “how well can two quantum states, or more generally, two density matrices \(\rho_0\) and \(\rho_1\), be distinguished?” In terms of a communication scheme this question is translated to an identification task: A sender (Alice) sends a bit \(b = i\) \((i = 0; 1)\) to the receiver (Bob) by sending the quantum state \(\rho_i\), and the receiver does his best to identify the value of that bit, i.e. the quantum state. The Two-dimensional Hilbert space \(\mathcal{H}_2\) is usually used to implement such a binary channel, so the transmitted signals can be polarization states of photons, spin-states of spin-half particles etc. The transmitted states may be pure states or density matrices, and need not be orthogonal. Usually, the mutual information \(I\) is used to describe distinguishability, such that \(I = 0\) means indistinguishable, and \(I = 1\) (for a binary channel) means perfect distinguishability. The ensemble of signals is agreed on in advance, and the main aim of Alice and Bob is to optimize the average mutual information over the different possible measurements at the receiving end. For a (simple) example, two orthogonal pure states transmitted through an error-free channel are perfectly distinguishable; The optimal mutual information \((I = 1)\) is obtained if Bob measures in an appropriate basis. Finding the optimal mutual information is still an open question for most ensembles. Some cases with known analytic solutions are the case of two pure states and the case of two density matrices in two dimensions with equal determinants \([3,4]\). There are no known analytic solutions for two non-trivial density matrices in dimensions higher than two. In this paper we find a solvable case which has also practical implications.

Let a single bit be presented by one of two quantum states, \(\rho_0\) and \(\rho_1\), in \(\mathcal{H}_2\). These can be either pure states or density matrices with equal determinants. Suppose Bob wants to learn the parity bit (exclusive-OR) of an \(n\)-bit string of these bits and not the specific value of each bit. The parity bit is described by one of two density matrices \(\rho_0^{(n)}\) and \(\rho_1^{(n)}\) which lie in a \(2^n\)-dimensional Hilbert space \(\mathcal{H}_{2^n}\). We solve the case of the distinguishability of these parity density matrices when the two possible signals (of each single bit) are sent with equal probabilities. Let \(x\) be any classical string of \(n\) such bits, and \(\rho_x = \rho(1^\text{st\ bit}) \cdots \rho(n^\text{th\ bit})\) be the density matrix made up of the tensor product of the signaling states \(\rho(i)\) corresponding to the \(i^{\text{th}}\) bit of \(x\). Formally, we distinguish between the two density matrices:

\[
\rho_0^{(n)} = \frac{1}{2^{n-1}} \sum_{x \mid p(x) = 0} \rho_x \quad \text{and} \quad \rho_1^{(n)} = \frac{1}{2^{n-1}} \sum_{x \mid p(x) = 1} \rho_x ,
\]

where the sum is over all possible strings with the same parity (each sent with equal probability \(1/2^n\)) and \(p(x)\) is the parity function of \(x\). We show a simple way to write the parity density matrices and we calculate the optimal mutual information which can be obtained.

Parity bits are often used in quantum cryptography \([7,10]\), where they play a crucial role in error-correction and privacy amplification \([11,13]\). The question of security of quantum cryptography is yet open, and our results may have several implications for attacking this issue. In particular, the special case where the two signaling states have large overlap is important in the analysis of the security of quantum key distribution against powerful multi-particle eavesdropping attacks.
In Section II we find a simple way to write the density matrices of the parity bit for any \( n \); We show that they can be put in a block diagonal form and we explain the importance of that fact. In Section III we investigate the distinguishability of the parity matrices; The optimal measurement which distinguishes them is found to be a standard (von Neumann) measurement in an entangled basis (which is a generalization of the Bell basis of two particles); We calculate exactly the optimal mutual information which is derived on the parity bit by performing that optimal measurement. In Section IV we show that for two almost fully overlapping states, the optimal mutual information \( I_M \) decreases exponentially with the length of the string; This is the main result of our paper due to its possible importance to quantum cryptography. While exponentially small, this optimal information is nevertheless considerably greater than the information that would have been obtained by measuring each bit separately and classically combining the results of these measurements, thus, we prove the advantage of such “joint” measurements. Going back to the parity matrices obtained in Section II we are also able to calculate the maximal deterministic (conclusive) information; This is done in Section V where we also confirm a result previously obtained by Huttner and Peres [14] for two bits. In Section VI we repeat the calculation of the optimal mutual information for the more general case where the bits are represented by non-pure states (in \( \mathcal{H}_2 \) and with equal determinants). In Section VII we briefly discuss the implications of our results to the security of quantum cryptography.

II. DENSITY MATRICES FOR PARITY BITS

Let Alice send \( n \) bits. The possible values of a single bit (0 or 1) are represented by

\[
\psi_0 = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \quad \text{and} \quad \psi_1 = \begin{pmatrix} \cos \alpha \\ -\sin \alpha \end{pmatrix}
\]

respectively. In terms of density matrices these are:

\[
\rho_0^{(1)} = \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix} \quad \text{and} \quad \rho_1^{(1)} = \begin{pmatrix} c^2 & -sc \\ -sc & s^2 \end{pmatrix},
\]

where we use a shorter notation \( s \equiv \sin \alpha; \ c \equiv \cos \alpha \), for convenience, and the superscript \((1)\) is explained in the following paragraph.

The parity bit of an \( n \)-bit string is the exclusive-OR of all the bits in the string. In other words, the parity is 1 if there are an odd number of 1’s and 0 if there are an even number. The parity density matrices of \( n \) bits will be denoted as \( \rho_0^{(n)} \) and \( \rho_1^{(n)} \) in case the parity is ‘0’ and ‘1’ respectively. Using these density matrices we define also the total density matrix \( \rho^{(n)} \equiv \frac{1}{2}(\rho_0^{(n)} + \rho_1^{(n)}) \) and the difference density matrix \( \Delta^{(n)} \equiv \frac{1}{2}(\rho_0^{(n)} - \rho_1^{(n)}) \), so that

\[
\rho_0^{(n)} = \rho^{(n)} + \Delta^{(n)} \quad \text{and} \quad \rho_1^{(n)} = \rho^{(n)} - \Delta^{(n)}. \tag{4}
\]

The one-particle density matrices (equation 3) also describe the parities of one particle, and therefore we can calculate
\[ \rho^{(1)} = \frac{1}{2}(\rho_0^{(1)} + \rho_1^{(1)}) = \begin{pmatrix} c^2 & 0 \\ 0 & s^2 \end{pmatrix}, \]  
\[ \Delta^{(1)} = \frac{1}{2}(\rho_0^{(1)} - \rho_1^{(1)}) = \begin{pmatrix} 0 & sc \\ sc & 0 \end{pmatrix}. \]

The density matrices of the parity bit of two particles are:

\[ \rho_0^{(2)} = \frac{1}{2}(\rho_0^{(1)} \rho_0^{(1)} + \rho_1^{(1)} \rho_1^{(1)}) \]
\[ \rho_1^{(2)} = \frac{1}{2}(\rho_0^{(1)} \rho_1^{(1)} + \rho_1^{(1)} \rho_0^{(1)}) \]

where the multiplication is a tensor product. The total density matrix is

\[ \rho^{(2)} = \frac{1}{2}(\rho_0^{(2)} + \rho_1^{(2)}) \]
\[ = \frac{1}{4}[\rho_0^{(1)} (\rho_0^{(1)} + \rho_1^{(1)}) + \rho_1^{(1)} (\rho_1^{(1)} + \rho_0^{(1)})] \]
\[ = \rho^{(1)} \rho^{(1)}, \]

which, by using the basis

\[ |b_0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \]
\[ |b_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \]
\[ |b_2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]
\[ \text{and} \quad |b_3\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

in \( \mathcal{H}_4 \), can be written as

\[ \rho^{(2)} = \rho^{(1)} \rho^{(1)} = \begin{pmatrix} c^4 & 0 & 0 & 0 \\ 0 & c^2 s^2 & 0 & 0 \\ 0 & 0 & c^2 s^2 & 0 \\ 0 & 0 & 0 & s^4 \end{pmatrix}. \]

The difference density matrix is

\[ \Delta^{(2)} = \frac{1}{2}(\rho_0^{(2)} - \rho_1^{(2)}) \]
\[ = \frac{1}{4}[\rho_0^{(1)} (\rho_0^{(1)} - \rho_1^{(1)}) + \rho_1^{(1)} (\rho_1^{(1)} - \rho_0^{(1)})] \]
\[ = \Delta^{(1)} \Delta^{(1)} = \begin{pmatrix} 0 & 0 & 0 & c^2 s^2 \\ 0 & 0 & c^2 s^2 & 0 \\ 0 & c^2 s^2 & 0 & 0 \\ c^2 s^2 & 0 & 0 & 0 \end{pmatrix}. \]
The density matrices of the parity bit of \( n \) particles can be written recursively:

\[
\rho^{(n)}_0 = \frac{1}{2} (\rho^{(1)}_0 \rho^{(n-1)}_0 + \rho^{(1)}_1 \rho^{(n-1)}_1),
\]

\[
\rho^{(n)}_1 = \frac{1}{2} (\rho^{(1)}_0 \rho^{(n-1)}_1 + \rho^{(1)}_1 \rho^{(n-1)}_0),
\]

leading to

\[
\rho^{(n)} = \frac{1}{2} (\rho^{(0)}_0 + \rho^{(0)}_1) = \rho^{(1)} \rho^{(n-1)},
\]

and

\[
\Delta^{(n)} = \frac{1}{2} (\rho^{(0)}_0 - \rho^{(0)}_1) = \Delta^{(1)} \Delta^{(n-1)}.
\]

Using these expressions recursively we get

\[
\rho^{(n)} = (\rho^{(1)})^n
\]

which is diagonal, and

\[
\Delta^{(n)} = (\Delta^{(1)})^n
\]

which has non-zero terms only in the secondary diagonal. The density matrices \( \rho^{(n)}_0 \) and \( \rho^{(n)}_1 \) are now immediately derived for any \( n \) using equation (14):

\[
\rho^{(n)}_0 = (\rho^{(1)})^n + (\Delta^{(1)})^n
\]

and

\[
\rho^{(n)}_1 = (\rho^{(1)})^n - (\Delta^{(1)})^n.
\]

As an illustrative example we write \( \rho_0 \) and \( \rho_1 \) for two particles:

\[
\rho^{(2)}_0 = \begin{pmatrix}
    c^4 & 0 & 0 & c^2 s^2 \\
    0 & c^2 s^2 & c^2 s^2 & 0 \\
    0 & c^2 s^2 & c^2 s^2 & 0 \\
    c^2 s^2 & 0 & 0 & s^4
\end{pmatrix};
\]

\[
\rho^{(2)}_1 = \begin{pmatrix}
    c^4 & 0 & 0 & -c^2 s^2 \\
    0 & c^2 s^2 & -c^2 s^2 & 0 \\
    0 & -c^2 s^2 & c^2 s^2 & 0 \\
    -c^2 s^2 & 0 & 0 & s^4
\end{pmatrix}.
\]

The only non-zero terms in the parity density matrices are the terms in the diagonals for any \( n \), thus the parity density matrices have an X-shape in that basis.

The basis vectors can be reorganized to yield block-diagonal matrices built of \( 2 \times 2 \) blocks. The original basis vectors (see, for example, equation 8), \( |b_i\rangle \), are simply \( 2^n \)-vectors where the \( i \)-th element of the \( i \)-th basis vector is 1 and all other elements are 0 (\( i \) ranges from 0 to \( 2^n - 1 \)). The new basis vectors are related to the old as follows:

\[
|b'_i\rangle = |b_{i/2}\rangle \quad \text{for even} \ i \quad \text{and} \quad |b'_i\rangle = |b_{2^n-(i+1)/2}\rangle \quad \text{for odd} \ i.
\]
The parity density matrices are now, in the new basis (we omit the ' from now on as we will never write the matrices in the original basis):

\[
\rho_p^{(n)} = \begin{pmatrix}
B_p^{[j=1]} & 0 & \ldots & 0 \\
0 & B_p^{[j=2]} & \ldots & 0 \\
0 & 0 & \ldots & B_p^{[j=2^{(n-1)}]}
\end{pmatrix}
\]  

(19)

where the subscript \( p \) stands for the parity (0 or 1). Each of the 2x2 matrices has the form

\[
B_p^{[j]} = \begin{pmatrix}
c^{2(n-k)} s^{2k} & \pm c^n s^n \\
\pm c^n s^n & c^{2k} s^{2(n-k)}
\end{pmatrix}
\]

(20)

with the plus sign for \( p = 0 \) and the minus sign for \( p = 1 \), and 0 \( \leq k \leq n \), and all these density matrices satisfy \( \det B_p^{[j]} = 0 \). The first block \((j = 1)\) has \( k = 0 \); there are \( \binom{n}{1} \) blocks which have \( k = 1 \) or \( k = n - 1 \); there are \( \binom{n}{2} \) \( j \)’s which have \( k = 2 \) or \( k = n - 2 \), etc. This continues until \( k = (n - 1)/2 \) for odd \( n \). For even \( n \) the process continues up to \( k = n/2 \) with the minor adjustment that there are only \( \frac{1}{2} \binom{n}{n/2} \) \( j \)’s of \( k = n/2 \). This enumeration groups blocks which are identical or identical after interchange of \( k \) and \( n - k \) and accounts for all \( 2^n / 2 \) blocks. We will see later that blocks identical under interchange of \( k \) and \( n - k \) will contribute the same mutual information about the parity bit, thus we have grouped them together.

With the density matrices written in such a block-diagonal form of 2x2 blocks the problem of finding the optimal mutual information can be analytically solved. It separates into two parts:

- Determining in which of \( 2^n / 2 \) orthogonal 2d subspaces (each corresponding to one of the 2x2 blocks) the system lies.

- Performing the optimal measurement within that subspace.

The subspaces may be thought of as \( 2^n / 2 \) parallel channels, one of which is probabilistically chosen and used to encode the parity by means of a choice between two equiprobable pure states within that subspace (these two states are pure because the \( B_0 \) and \( B_1 \) matrices each have zero determinant). We shall present in the next section the optimal measurement that yields the optimal mutual information transmissible through such a two-pure-state quantum channel. The channel then corresponds to a classical binary symmetric channel (BSC), i.e. a classical one-bit-in one-bit-out channel whose output differs from its input with some error probability \( p_j \) independent of whether the input was 0 or 1. The optimal mutual information in each subchannel is the optimal mutual information of a BSC with error probability \( p_j \) and is \( I_2(p_j) = 1 - H(p_j) \), with \( H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \), the Shannon entropy function. The optimal mutual information \( I_M \) for distinguishing \( \rho_1^{(n)} \) from \( \rho_1^{(n)} \) can thus be expressed as an average over the optimal mutual information of the subchannels:

\[
I_M = \sum_{j=1}^{2^n/2} q_j I_2(p_j),
\]  

(21)
where \( q_j = \text{Tr} \hat{B}_j^p = \text{Tr} B_1^p \) is the probability of choosing the \( j \)’th subchannel. The BSC error probability \( p_j \) for the \( j \)’th subchannel depends on the subchannel’s 2x2 renormalized density matrices \( \hat{B}_j^p = B_j^p / q_j \), and is easily calculated once the optimal measurement is found. For each subchannel the \( q_j \) and renormalized 2x2 matrices look like

\[
q_j = c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)}
\]

and

\[
\hat{B}_j^p = \begin{pmatrix}
\pm c^n s^n & c^{2(n-k)} s^{2k} \\
\pm c^n s^n & c^{2k} s^{2(n-k)}
\end{pmatrix}.
\]

In our previous example of \( n = 2 \) the matrices are put in a block diagonal form:

\[
\rho_0^{(2)} = \begin{pmatrix}
c^4 & c^2 s^2 & 0 & 0 \\
c^2 s^2 & s^4 & 0 & 0 \\
0 & 0 & c^2 s^2 & c^2 s^2 \\
0 & 0 & c^2 s^2 & c^2 s^2
\end{pmatrix}; \quad \rho_1^{(2)} = \begin{pmatrix}
c^4 & -c^2 s^2 & 0 & 0 \\
-c^2 s^2 & s^4 & 0 & 0 \\
0 & 0 & c^2 s^2 & -c^2 s^2 \\
0 & 0 & -c^2 s^2 & c^2 s^2
\end{pmatrix},
\]

so that \( q_{j=1} = c^4 + s^4; q_{j=2} = 2c^2 s^2; \) and

\[
\hat{B}_p^{[j=1]} = \begin{pmatrix}
c^4 & \pm c^2 s^2 \\
\pm c^2 s^2 & s^4
\end{pmatrix}; \quad \hat{B}_p^{[j=2]} = \begin{pmatrix}
1/2 & \pm 1/2 \\
\pm 1/2 & 1/2
\end{pmatrix}.
\]

**III. OPTIMAL INFORMATION IN A PARITY BIT**

Two pure states or two density matrices in \( \mathcal{H}_2 \) with equal determinants can always be written (in an appropriate basis) in the simple form

\[
\rho_0 = \begin{pmatrix}
a_1 & a_2 \\
a_2 & a_3
\end{pmatrix}; \quad \rho_1 = \begin{pmatrix}
a_1 & -a_2 \\
-a_2 & a_3
\end{pmatrix}
\]

with \( a_i \) real positive numbers such that \( \text{Tr} \rho_p = a_1 + a_3 = 1 \). For the two pure states of equation \( 4 \), say, for the polarization states of a photon, it is easy to see (and can be formally proven \( 5,6 \)) that a standard measurement in an orthogonal basis symmetric to the two states optimizes the mutual information (and also minimizes the average error probability). The angle between one basis vector and the polarization state is \( \frac{\pi}{4} \pm \alpha \). The measurement results in an error with probability

\[
P_e = \sin^2\left(\frac{\pi}{4} - \alpha\right) = \frac{1 - \cos\left(\frac{\pi}{2} - 2\alpha\right)}{2} = \frac{1 - \sin(2\alpha)}{2},
\]

and with the same error probability for both inputs, thus, leading to a binary symmetric channel (BSC). The optimal information of such a channel is well known and is

\[
I_{BSC} = I_2(P_e).
\]
Note that the overlap of the two-states is cos(2\(\alpha\)), thus, for two pure states in any dimension, the optimal information \(I_2(\frac{1-\sin(2\alpha)}{2})\) is a simple function of the overlap. The density matrices of such pure states (equation 3) can be written as \(\rho_i = (\mathbb{1} + \sigma \cdot r_i)/2\) with the \(\sigma\) being the Pauli matrices and \(r = (\pm \sin 2\alpha, 0, \cos 2\alpha)\) being a three dimensional vector which describes a spin direction. Using this notation any density matrix is described by a point in a three dimensional unit ball, called the Bloch sphere. The pure states are points on the surface of that sphere (also called the Poincare sphere). With the density matrix notation the optimal basis for distinguishing the states is the \(x\) basis (note that the angle between the basis vector and the state is doubled in this notation). The measurement of the two projectors

\[
A_\rightarrow = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_\leftarrow = 1/2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]

yields

\[
P_e = \text{Tr} \rho_1 A_\rightarrow = \frac{1}{2} - a_2,
\]

which recovers the result of equation (27) in case of pure states of equation (3). However, the treatment of density matrices is more general and this is the optimal measurement also in the case of non-pure states with equal determinants [5,6], when \(\rho_i\) of equation (3) are replaced by \(\rho_i^{dm}\) of equation (69) and (70) of section VI, and this case is also described by a BSC. The only difference between the matrices is that \(\text{Det} \rho_p = 0\) for pure states and \(0 \leq \text{Det} \rho_p \leq \frac{1}{4}\) for density matrices.

Instead of measuring the density matrices in the \(x\) direction we perform the following unitary transformation on the density matrices

\[
U = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

(31)

to obtain \(\rho' = U \rho U^\dagger\) which is then measured in the \(z\) basis. Note that the transformation transform the original \(z\)-basis to \(x\)-basis (the motivation for this approach will be understood when we discuss the 2x2 blocks of the parity matrices). The new density matrices are

\[
\rho_0' = \left( \begin{array}{cc} \frac{1}{2} + a_2 & \frac{a_1 - a_3}{2} \\ \frac{a_1 - a_3}{2} & 1/2 - a_2 \end{array} \right), \quad \rho_1' = \left( \begin{array}{cc} \frac{1}{2} - a_2 & \frac{a_1 - a_3}{2} \\ \frac{a_1 - a_3}{2} & 1/2 + a_2 \end{array} \right)
\]

(32)

and their measurement yields the probability \(\frac{1}{2} \pm a_2\) to derive the correct (plus) and the wrong (minus) answers (as we obtained before), leading to optimal mutual information of

\[
I_2(\frac{1}{2} - a_2),
\]

(33)

which depends only on \(a_2\). Note that the same information is obtained in case \(a_1\) and \(a_3\) are interchanged.
The naive way to derive information on a parity bit is to derive the optimal information on each particle separately and calculate the information on the parity bit. We call this individual or single-particle measurement. It is the best Bob can do in case he has no quantum memory in which to keep the particles (which, usually arrive one at a time) or he has no ability to perform more advanced joint measurements. The optimal error-probability for each particle is \( r \equiv P_e^{(1)} = \frac{1 - \sin 2\alpha}{2} \). The probability of deriving the wrong parity bit is equal to the probability of having an odd number of errors on the individual particles \( P_{e}^{(n)} = \sum_{j=odd}^{n} \binom{n}{j} r^j (1 - r)^{n-j} \).

To perform the sum over only odd \( j \) we use the formulas

\[
(p + q)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-j} q^j \quad \text{and} \quad (p - q)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-j} (-q)^j ,
\]

to derive

\[
\sum_{j=odd}^{n} \binom{n}{j} p^{n-j} q^j = \frac{(p + q)^n - (p - q)^n}{2}.
\]

Assigning \( q = r \) and \( p = 1 - r \) we get

\[
P_{e}^{(n)} = \sum_{j=odd}^{n} \binom{n}{j} r^j (1 - r)^{n-j} = \frac{1^n - (1 - 2r)^n}{2} = \frac{1}{2} - \left( \frac{1 - 2r}{2} \right)^n.
\]

The mutual information \( I_S \) in this single-particle measurement is

\[
I_S = I_2(P_{e}^{(n)}) = I_2 \left( \frac{1}{2} - \frac{(\sin 2\alpha)^n}{2} \right),
\]

using \( r = \frac{1 - \sin(2\alpha)}{2} \).

A lot of useless side-information is also obtained (e.g., on the individual bits). This fact indicates that Bob might be able to do much better by concentrating on deriving only useful information. The optimal measurement for finding mutual information on the parity bit is not a single-particle measurement, but is instead a measurement on the full \( 2^n \)-dimensional Hilbert space of the system. In general, optimizing over all possible measurement is a very difficult task unless the two density matrices in \( \mathcal{H}_{2^n} \) are pure states. However, in the preceding section we have shown how to reduce the problem to that of distinguishing the \( 2 \times 2 \) blocks of our block-diagonal parity matrices. We now have only to apply the optimal single-particle measurement to the \( 2 \times 2 \) \( \hat{B}^{[j]} \)'s of equation 23 and use the result in equation 21.

The error probability (equation 30) for distinguishing the \( \hat{B}^{[j]} \)'s is seen to be:

\[
p_j = \frac{1}{2} - \frac{c^n s^n}{c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)}},
\]

\[
\]

9
from which the information $I_2(p_j)$ in each channel is obtained.

Plugging the error probability $p_j$ (equation [37]) and the probability of choosing the $j$'th subchannel $q_j$ (equation [22]) into [21], the optimal information on the parity bit is now:

$$I_M = \frac{2^n}{2} \sum_{j=1}^{2^n/2} \left( c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)} \right) I_2 \left( \frac{1}{2} - \frac{c^2 s^n}{c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)}} \right). \quad (38)$$

In the simple case of orthogonal states ($\alpha = \frac{\pi}{4}$) all these density matrices are the same and we get $q_j = \left(\frac{1}{2}\right)^{n-1}$, $p_j = 0$ and $I_M = 1$ as expected.

- A brief remark is in order at this stage. The transformation to the $x$ basis for each 2 by 2 matrix, $\hat{B}_p^{(n,k)}$ is actually a transformation from a product basis to a fully entangled basis of the $n$ particles. That basis is a generalization of the Bell basis of [14].

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n \pm \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_n \quad (39)$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n \pm \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_n \quad (40)$$

etc. The Bell basis for two particles is frequently used and its basis contains the EPR singlet state and other three orthogonal fully entangled states.

For large $n$, the number of blocks is exponentially large and performing the summation required in equation [38] is impractical, since all the $2^{n-1}$ matrices must be taken into account. However, that problem can be simplified by realizing that all blocks with a given $k$, as well as all blocks with $k$ and $n-k$ interchanged, contribute the same information to the total. This is easily seen in equation [38] where both the weight and the argument of $I_2$ are symmetric in $k$ and $n-k$. The optimal mutual information for even $n$ is then

$$I_M^{\text{even}} = \sum_{k=0}^{n/2-1} \binom{n}{k} q_k I_2(p_k) + \frac{1}{2} \binom{n}{n/2} q_{n/2} I_2(p_{n/2}) \quad (41)$$

and for odd $n$

$$I_M^{\text{odd}} = \sum_{k=0}^{n-1} \binom{n}{k} q_k I_2(p_k) \quad (42)$$

As an example we calculate $I_M$ for $n = 2$ (of course, the counting argument is not needed in that case). This particular result complements the result in [14] where the deterministic information of such a system is considered (see also Section [7]). In the new basis ([22]) the density matrices of (equation [25]) become

$$\hat{B}_0^{(n=2,k=0)} = \begin{pmatrix} 1/2 + \frac{c^2}{c^4 + s^4} & 1/2 - \frac{c^2}{c^4 + s^4} \\ \frac{1}{2} c^4 + \frac{s^4}{s^4} & \frac{1}{2} c^4 + \frac{s^4}{s^4} \end{pmatrix}; \quad \hat{B}_1^{(n=2,k=0)} = \begin{pmatrix} 1/2 - \frac{c^2}{c^4 + s^4} & 1/2 - \frac{c^2}{c^4 + s^4} \\ \frac{1}{2} c^4 + \frac{s^4}{s^4} & \frac{1}{2} c^4 + \frac{s^4}{s^4} \end{pmatrix}$$
and
\[
\hat{B}'_{0}^{(n=2,k=1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{B}'_{1}^{(n=2,k=1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\] (43)

We use the notation \( S = 2sc = \sin 2\alpha; \ C = c^2 - s^2 = \cos 2\alpha \) (hence, \( c^4 - s^4 = C \) and \( c^4 + s^4 = \frac{1+C^2}{2} \)) to obtain \( q_1 = 2c^2s^2 = \frac{S}{2} \), \( p_1 = 0 \), \( q_0 = \frac{1}{2}(1+C^2) \) and \( p_0 = \frac{C^2}{1+C^2} \) (the \( q_j \)s were obtained in the previous section). The mutual information of the parity of two bits is obtained using equation (21)
\[
I_M = q_0I_2(p_0) + q_1I_2(p_1)
= \frac{1}{2}(1+C^2)I_2\left(\frac{C^2}{1+C^2}\right) + \frac{S^2}{2}.
\] (44)

IV. INFORMATION ON THE PARITY BIT OF ALMOST FULLY OVERLAPPING STATES

The case of almost fully overlapping states is extremely important to the analysis of eavesdropping attacks on any quantum key distribution scheme as will be discussed in Section VII. In this case the angle \( \alpha \) is small so \( s \equiv \sin \alpha \simeq \alpha \) and \( c \equiv \cos \simeq 1 - \frac{\alpha^2}{2} \). To observe the advantage of the joint measurement, let us first calculate the optimal information obtained by individual measurements. In that case, equations (35 and (27) yield
\[
P^e_{\alpha} = \frac{1}{2} - \frac{(2\alpha)^n}{2}.
\] (45)

For small \( \eta \) the logarithmic function is approximated by
\[
\log(\frac{1}{2} \pm \eta) = \frac{\ln(\frac{1}{2} \pm \eta)}{\ln 2} \approx -1 \pm \frac{2}{\ln 2} \eta - \frac{2}{\ln 2} \eta^2,
\] (46)
from which the mutual information
\[
I_2(\frac{1}{2} - \eta) = 1 - H(\frac{1}{2} - \eta) = 1 + (\frac{1}{2} + \eta) \log(\frac{1}{2} + \eta) + (\frac{1}{2} - \eta) \log(\frac{1}{2} - \eta)
\approx \frac{2}{\ln 2} \eta^2
\] (47)
is obtained. Using this result and assigning \( \eta = (2\alpha)^n/2 \), the information (to first order) obtained by the optimal single-particle measurement is
\[
I_S = \frac{2}{\ln 2} \frac{(2\alpha)^{2n}}{4} = \frac{(2\alpha)^{2n}}{2 \ln 2}.
\] (48)

We use the same approximations and equations (37) and (22) to calculate the leading terms in the optimal mutual information. For \( k = \frac{n}{2} \) (\( n \) even) we get \( p_k = 0 \) (regardless of the small angle) and
\[ I_2(p_\frac{n}{2}) = 1. \] (49)

For \( k < \frac{n}{2} \) we get \( p_k \approx \frac{1}{2} - \frac{n^2}{4\pi} \approx \frac{1}{2} - \alpha^{-2k} \) which yields (using equation (47) with \( \eta = \alpha^{-2k} \))

\[ I_2(p_k) \approx \frac{2}{\ln 2} \alpha^{2n-4k}. \] (50)

The coefficient \( q_k = \alpha^{2k} \) for \( k < \frac{n}{2} \) and \( q_k = 2\alpha^{2k} \) for \( k = \frac{n}{2} \), so that

\[ q_k I_2(p_k) \approx \frac{2}{\ln 2} \alpha^{2(n-k)} \] (51)

for \( k < \frac{n}{2} \), and

\[ q_k I_2(p_k) \approx 2\alpha^n \] (52)

for \( k = \frac{n}{2} \). The dominant terms are those with the largest \( k \), that is, \( k \) closest to \( \frac{n}{2} \). The next terms are smaller by two orders in \( \alpha \). The number of density matrices with these \( k \)'s are also the largest (up to a factor of 2 in case of even \( n \)). Therefore, the terms \( k = \frac{n}{2} \) for even \( n \) and \( k = \frac{n-1}{2} \) for odd \( n \) are the dominant terms in the final expression. Thus, for almost fully overlapping states, the mutual information is

\[ I_{M \text{ even}} \approx \frac{1}{2} \binom{n}{\frac{n}{2}} 2\alpha^n = \alpha^n \left( \frac{n}{\alpha} \right) \] for even \( n \), and

\[ I_{M \text{ odd}} \approx \binom{n}{\frac{n-1}{2}} \frac{2}{\ln 2} \alpha^{n+1} \] (53)

for odd \( n \).

These expressions can be further simplified. The number of density matrices of any type is bounded (for large \( n \)) using Stirling formula (see [13] in the chapter on Reed-Solomon codes)

\[ \binom{n}{k} < \frac{2^n H(k/n)}{\sqrt{2\pi(k/n)(1-k/n)n}}. \] (54)

For \( k \) near \( \frac{n}{2} \), \( \eta \equiv \frac{1}{2} - \frac{k}{n} \) is small, and the standard approximation \([14]\): \( H \approx 1 - O(\eta^2) = 1 - O \left( (\frac{1}{4} - k/n)^2 \right) < 1 \) is used to derive \( \binom{n}{k} < \frac{2^n}{\sqrt{2\pi(k/n)(1-k/n)n}} \). Using also \( k/n(1 - k/n) \approx \frac{1}{4} - \eta^2 \), we derive

\[ \binom{n}{k} < \frac{2^n}{\sqrt{\pi n}} (1 + O(\eta^2)). \] (55)

Thus the leading term in \( I_M \) is
\[ I_M < \frac{2^n}{\sqrt{\frac{\pi}{2n}}} \alpha^n = (2\alpha)^n/\sqrt{\frac{\pi}{2n}} \]  

(56)

for even \( n \) and

\[ I_M < \frac{2^n}{\sqrt{\frac{\pi}{2n}}} \frac{2}{\ln^2 2} \alpha^{n+1} = \frac{2}{\ln 2} (2\alpha)^n/\sqrt{\frac{\pi}{2n}} < (2\alpha)^n/\sqrt{\frac{\pi}{2n}} \]  

(57)

for odd \( n \) (using \( \alpha < \ln 2/2 \)). We see that we could keep a better bound for odd \( n \) but for simplicity we consider the same bound for both even and odd \( n \)’s.

We can now compare the optimal information \( I_M \) from a joint measurement on all \( n \) particles to the optimal information \( I_S \) from separate measurements (cf. eq. 48):

\[
I_M = O(1) \times (2\alpha)^n/\sqrt{n} \\
I_S = O(1) \times (2\alpha)^{2n}.
\]

(58)

Since \( \alpha \) is a small number (corresponding to highly overlapping signal states), the joint measurement is superior to the individual measurement by a factor of \( O((2\alpha)^n) \). However, it is only superior by a polynomial factor, since

\[ I_M \approx (I_S)^2. \]  

(59)

V. DETERMINISTIC INFORMATION ON THE PARITY BIT

For a single particle Bob can perform a different kind of individual measurement which is not optimal in terms of average mutual information but is sometimes very useful [1,4]. It yields either a conclusive result about the value of that bit or an inconclusive one, and Bob will know which of the types of information he has succeeded in obtaining. Such a measurement corresponds to a binary erasure channel [1,4,14]. With probability \( p_2 \) of an inconclusive result, the mutual information is \( I_{p_2} = 1 - p_2 \). The minimal probability for an inconclusive result is \( \cos 2\alpha \) leading to \( I_{p_2} = 1 - \cos 2\alpha \) [1]. This result is obtained by performing a generalized measurement (Positive Operator Value Measure [1,14,19]) on the system or a standard measurement performed on a larger system which contains the system and an auxiliary particle [1,14]. Note that this results in less mutual information than the optimal measurement for one-particle mutual information. If Bob uses this type of measurement on each particle separately his deterministic single-particle information about the parity bit is \((1 - \cos 2\alpha)^n\).

We now use the block-diagonal density matrices derived in section [1] to derive also the optimal deterministic information on the parity bit. We note that each of the 2x2 blocks in the block-diagonal density matrices is the density matrix of a pure state, so we may replace the optimal measurement in each subchannel with the optimal deterministic measurement and proceed as before. The total optimal deterministic information is easily calculated by replacing \( I(\pi) \) in [11] and [12] by \( I(p_{\pi}) = 1 - p_{\pi} \). To find the minimal \( P_{\pi} \) we write each of the normalized density matrices \( \hat{B}^{(n,k)} \) as pure states with some angle \( \gamma \):
\[
\begin{pmatrix}
\cos \gamma \\
\sin \gamma
\end{pmatrix}
\quad \text{and} \quad 
\begin{pmatrix}
\cos \gamma \\
- \sin \gamma
\end{pmatrix}
\tag{60}
\]

Comparing with equation (23)

\[
p_\gamma = \cos(2 \gamma) = \cos^2 \gamma - \sin^2 \gamma = \frac{c^{2(n-k)} s^{2k} - c^{2k} s^{2(n-k)}}{c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)}},
\tag{61}
\]

hence

\[
I(p_{\gamma_k}) = 1 - \frac{c^{2(n-k)} s^{2k} - c^{2k} s^{2(n-k)}}{c^{2(n-k)} s^{2k} + c^{2k} s^{2(n-k)}}.
\tag{62}
\]

The total information is

\[
I_{D}^{\text{even}} = \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} q_k I(p_{\gamma_k}) + \frac{1}{2} \binom{n}{\frac{n}{2}} q_{\frac{n}{2}} I(p_{\gamma_{n/2}}),
\tag{63}
\]

for even \( n \), and

\[
I_{D}^{\text{odd}} = \sum_{k=0}^{n-1} \binom{n}{k} q_k I(p_{\gamma_k})
\tag{64}
\]

for odd \( n \).

For \( n = 2 \) we recover a result previously obtained by Huttner and Peres [14] by performing the optimal POVM on the first pair of density matrices of equation (25), and a measurement in the entangled basis (as before) on the second. The probability of an inconclusive result is

\[
\cos^2 \gamma - \sin^2 \gamma = \frac{c^{4} - s^4}{c^{4} + s^4} = \frac{2C}{1+C^2},
\]

hence the optimal deterministic information is

\[
1 - \frac{2C}{1+C^2},
\]

leading to the total deterministic information

\[
I_{D} = q_1 I_{D}(p_1) + q_2 I_{D}(p_2) = \frac{1}{2} (1 + C^2)(1 - 2 \frac{C}{1+C^2}) + \frac{S^2}{2} = 1 - C,
\tag{65}
\]

which is exactly the result obtained by Huttner and Peres (note, however, that they used an angle which is \( \frac{\pi}{4} - \alpha \) hence derived \( 1 - S \) for the deterministic information).

For almost overlapping states (small \( \alpha \)) the dominant terms are still the same as in the case of optimal information. The term \( q_k \) is as before and the information in each port is

\[
I(p_{\gamma_k}) = 1
\tag{66}
\]

for \( k = \frac{n}{2} \) and

\[
I(p_{\gamma_k}) = 1 - \frac{c^{2n-4k} - s^{2n-4k}}{c^{2n-4k} s^{2n-4k}} = 1 - (1 - \alpha^{2n-4k})^2 = 2\alpha^{2n-4k}
\tag{67}
\]

for \( k < \frac{n}{2} \). Taking into consideration only the dominant term we get

\[
I_{D}^{\text{even}} \approx \binom{n}{\frac{n}{2}} \alpha^n
\]

for even \( n \) which is the same as the optimal information, and

\[
I_{D}^{\text{odd}} \approx \binom{n}{\frac{n-1}{2}} 2\alpha^{n+1}
\tag{68}
\]

for odd \( n \) which is smaller than the optimal mutual information by a factor of \( \frac{1}{\ln 2} \).
VI. PARITY BIT FOR DENSITY MATRICES

The previous discussion assumed that $\rho_p^{(1)}$ are pure states. The generalization to the case of density matrices with equal determinants is straightforward. Let the bit ‘0’ and the bit ‘1’ be represented by

$$\rho_{0}^{dm} = \begin{pmatrix} c^2 & sc - r \\ sc - r & s^2 \end{pmatrix},$$

and

$$\rho_{1}^{dm} = \begin{pmatrix} c^2 & -(sc - r) \\ -(sc - r) & s^2 \end{pmatrix},$$

with $s = \sin \alpha$ etc., and $r < sc$ which contains the most general density matrices of the desired type. On the Poincare sphere these density matrices have the same $z$ components as the previously written pure states but smaller $x$ components (hence smaller angle $\alpha'$). We could choose other ways of representing these density matrices, e.g., with similar $x$ components and smaller $z$ components. Such representations were more appropriate for comparison with pure states (since they yield the same mutual information for a single particle) but less convenient for showing that the previous result is easily generalized.

Clearly

$$\rho^{(1)}_{dm} = \frac{1}{2}(\rho_{0}^{(1)} + \rho_{1}^{(1)}) = \begin{pmatrix} c^2 & 0 \\ 0 & s^2 \end{pmatrix},$$

and

$$\Delta^{(1)}_{dm} = \frac{1}{2}(\rho_{0}^{(1)} - \rho_{1}^{(1)}) = \begin{pmatrix} 0 & sc - r \\ sc - r & 0 \end{pmatrix},$$

The total density matrix doesn’t change and the difference density matrix has terms $(sc - r)^n$ instead of $(sc)^n$. Reorganizing the basis vectors we again get the block diagonal matrices where each of the 2 by 2 matrices has the form

$$B_p^{(n,k)} = \begin{pmatrix} c^{2(n-k)}s^{2k} & \pm(cs - r)^n \\ \pm(cs - r)^n & c^{2k}s^{2(n-k)} \end{pmatrix}.$$

When normalized, these density matrices have the form of equation 26 and are optimally distinguished by measuring them in the $x$ direction. Transforming to the $x$ basis as before we get the same

$$q_k = c^{2(n-k)}s^{2k} + c^{2k}s^{2(n-k)},$$

as before, and

$$p_k = \frac{1}{2} - \frac{(cs - r)^n}{c^{2(n-k)}s^{2k} + c^{2k}s^{2(n-k)}}.$$
The total information can now be calculated as before by assigning these $p_k$ and $q_k$ into equations (42 and 41). Thus, the case of mixed states is also analytically solved for any number of bits, and the influence of mixing on the optimal mutual information is through the $p_k$s.

The case of $\alpha = \frac{\pi}{4}$ (when the matrices commute) is of special interest due to its importance to quantum bit commitment [10]. Equation (73) yields

$$B_{pk}^{(n,k)} = \left( \begin{array}{cc} \left(\frac{1}{2}\right)^n & \pm\left(\frac{1}{2} - r\right)^n \\ \pm\left(\frac{1}{2} - r\right)^n & \left(\frac{1}{2}\right)^n \end{array} \right)$$

independent of $k$. Normalizing and transforming to the entangled basis we get $p = \frac{1}{2} - \frac{1}{2}(1 - 2r)^n$ and $q = (\frac{1}{2})^{n-1}$. There are $2^{n-1}$ matrices like that so

$$I_M = 2^{n-1}(\frac{1}{2})^{n-1}I_2\left(\frac{1}{2} - \frac{1}{2}(1 - 2r)^n\right) = I_2\left(\frac{1}{2} - \frac{1}{2}(1 - 2r)^n\right).$$

However, in this case each particle actually carries a classical information hence the collective measurement cannot improve the derived information. Indeed, the one-particle density matrices yields probability $r$ of deriving an error, leading to total error probability of equation (35), $P_e(n) = \frac{1}{2} - \frac{1}{2}(1 - 2r)^n$, and total information $I_S = I_2\left(\frac{1}{2} - \frac{1}{2}(1 - 2r)^n\right)$ as expected. Note that the mixing by itself induces an exponential decay of the amount of information, a fact used in [10].

Calculating the optimal information for small $\alpha$ and any $r$ is possible but complicated. Another alternative which is much simpler is to find a bound on the optimal information using pure states with the same angle, $\alpha'$, on the Poincare sphere, using

$$\tan 2\alpha' = \frac{\sin 2\alpha - 2r}{\cos 2\alpha},$$

or using an alternative form for the mixed states [69] and [70].

**VII. IMPLICATIONS**

Protocols in quantum cryptography use parity bits. Quantum bit commitment [10], quantum oblivious transfer [21] and quantum key distribution [11] protocols use parities of publicly announced subsets of the transmitted bits for both error-correction and privacy amplification (PA). When used for error-correction, subset parities are publicly announced in order to identify errors and correct them, and this is a crucial step in real channels since it could leak information to Eve. When used for PA [12,13] (say, to derive one final bit) a subset parity is agreed to be the final secret bit, and this technique is used to limit the adversary’s information to an exponentially small fraction of a bit. PA is effective when particles are not measured together (see discussions in [13] and in [22]), and presumably

\[1\] This case, including the X-shape of the parity matrices, was solved independently by D. Mayers [20].
also if all measurements are completed before the specification of the subsets used in PA
is publicly announced. But it is still an open question whether it is effective also when the
adversary can use this specification to *choose* her attack. Our result provides the optimal
measurement which can be done to find a parity bit and therefore is crucial for such
analysis. In particular cases, when almost fully overlapping states are used, we proved
two complementary results regarding that optimal measurement:

- The optimal information is much larger than the one obtained be measuring each
  bit separately.
- The optimal measurement still yields exponentially small information. Thus we
  proved an *effectiveness result*: classical PA techniques are effective against *any*
quatum measurement.

The discussion so far treats an imaginary scenario which is very general but is not good
as a cryptographic protocol.

Realistic protocols are very complicated (e.g., due to the use of error correction), hence,
are more difficult to analyze. However, to emphasize the importance of the “effectiveness
result” just mentioned let us consider a different scenario which is common in quantum
key distribution schemes: Alice and Bob are the legitimate users who try to establish
a secret key. They use any binary scheme and Alice sends *n* particles through a noisy
channel to Bob. An adversary, Eve, is trying to learn information on their key. She gets
the particles one at a time, interacts with each one of them weakly, and send it forward
to Bob. She must interact weakly with all particles if she wants to induce only small
error rate (or else she could attack strongly only few of the particles but PA is already
proven effective against that type of attacks). The classical PA is also effective if Eve
cannot use its specification to attack all bits together, but in reality she can do it if she
has a quantum memory. Although the specification is announced after the transmission
is over, Eve can keep information in the quantum state of a system which has interacted
with all the transmitted particles, and use it after all the specification is announced. Such
attack has never been analyzed in the case of real (noisy) channels and devices.\(^2\) The
“effectiveness result” might allow one to prove security against a “collective” attack (first
described by E. Biham and T. Mor) in which Eve learns the PA specification and the
error-correction data and uses them to choose the optimal measurement. In that attack
Eve attaches a separate probe to each particle via transluscent attack (defined in \([17]\)),
using a quantum memory to keep the probes for a later time, and measuring them together
after receiving all relevant data. Eve must attack weakly since she doesn’t want to induce
large error rate, so she obtains (for each transmitted particle) a probe with two almost
overlapping pure states (or density matrices). Hence, the “effectiveness result” can be
used to prove that her information on the final string is exponentially small in the length
of the initial string.

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\(^2\) The case of error-free channels is completely solved, a consequence of \([23,24]\). The case of
error-free devices (but real channels) can be solved due to the possibility of purifying singlets
\([25,26]\).
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