A note on Euler number of locally conformally Kähler manifolds

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Abstract

Let $M^{2n}$ be a compact Riemannian manifold of non-positive (resp. negative) sectional curvature. We call $(M, J, \theta)$ a (bounded) locally conformally Kähler manifold if the lifted Lee form $\tilde{\theta}$ on the universal covering space of $M$ is $d$ (bounded). We show that if $M^{2n}$ is homeomorphic to a $d$ (bounded) LCK manifold, then its Euler number satisfies the inequality $(-1)^n \chi(M^{2n}) \geq$ (resp. $> 0$).

Keywords. LCK manifold, Euler number

1 Introduction

In the early 19th century, Hopf proposed the following conjecture:

“Let $(M^{2n}, g)$ be a closed $2n$-dimensional Riemannian manifold. If the sectional curvature of $M^{2n}$ is negative, then the Euler number $\chi(M^{2n})$ of $M^{2n}$ satisfies the inequality $(-1)^n \chi(M^{2n}) > 0$.”

This conjecture is true in dimensions 2 and 4 [3]. In dimension 2, by the Gauss-Bonnet formula, one can see that a closed Riemannian surface with negative section curvature has negative Euler number. In dimension 4, Chern [3] proved that negative sectional curvature implies that Gauss-Bonnet integrand is pointwise positive.

Dodziuk [7] has proposed to settle the Hopf conjecture using the Atiyah index theorem for coverings (see [1]). In this approach, one is required to prove a vanishing theorem for $L^2$ harmonic $k$-forms, $k \neq n$, on the universal covering of $M^{2n}$. The vanishing of these $L^2$ Betti numbers implies, by Atiyah’s result, that $(-1)^n \chi(M^{2n}) \geq 0$. The strict inequality $(-1)^n \chi(M^{2n}) > 0$ follows provided one can establish the existence of nontrivial $L^2$ harmonic $n$-forms on the universal cover. The program outlined above was carried out by Gromov [9] when the manifold in question is Kähler and is homotopy equivalent to a compact manifold with strictly negative sectional curvatures. The center idea is that Gromov introduced the notion of Kähler hyperbolicity. Gromov [9] points out that a bounded closed $k$-form, $k \geq 2$, on a complete simply-connected manifold whose sectional curvatures are bounded above by a negative constant is
automatically $d$(bounded) then he proved the above conjecture in the Kähler case. In order to attack Hopf Conjecture in the Kählerian case when $K \leq 0$ by extending Gromov’s idea, Cao-Xavier \[14\] and Jost-Zuo \[10\] independently introduced the concept of Kähler non-ellipticity, which includes nonpositively curved compact Kähler manifolds, and showed that their Euler characteristics satisfies

\[-1^n \chi(M^{2n}) \geq 0.\]

Inspired by Kähler geometry, Tan-Wang-Zhou gave the definition of symplectic parabolic manifold \[15\]. By a well known result that a closed symplectic manifold satisfies the hard Lefschetz property if and only if de Rham cohomology consists with the new symplectic cohomology, they proved that if $(M, \omega)$ is a $2n$-dimensional closed symplectic parabolic manifold which satisfies the hard Lefschetz property, then the Euler number satisfies

\[-1^n \chi(M^{2n}) \geq 0.\]

Let $(M, J, g)$ be a Hermitian manifold of complex dimension $n$, where $J$ denotes its complex structure, and $g$ its Hermitian metric. There is an interesting structure in $M$. We call $g$ a locally conformal Kähler metric if $g$ is conformal to some local Kählerian metric in the neighborhood of each point of $M^{2n}$. In many situations, the LCK structure becomes useful for the study of topology in non-kähler geometry \[11, 12\]. The aim of this paper is to establish its validity for all $n$ in the $d$(bounded) locally conformally Kähler manifold, see Definition 2.8.

**Theorem 1.1.** Let $M^{2n}$ be a compact $2n$-dimensional Riemannian manifold of non-positive (resp. negative) curvature. If $M^{2n}$ is homeomorphic to a $d$(bounded) locally conformally Kähler manifold, then the Euler number of $M^{2n}$ satisfies the inequality $(-1)^n \chi(M^{2n}) \geq (\text{resp.} >) 0$.

## 2 Preliminaries

### 2.1 $L^2$-Hodge number

Let $(M, g)$ be a complete Riemannian manifold. Let $\Omega^k(M)$ and $\Omega^k_0(M)$ denote the smooth $k$-forms on $M$ and the smooth $k$-forms with compact support on $M$. Let $\langle \cdot, \cdot \rangle$ denote the pointwise inner product on $\Omega^*(M)$ given by $g$ and duality. The global inner product is defined by

\[\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle d\text{Vol}_g.\]

We also write $|\alpha|^2 = \langle \alpha, \alpha \rangle, \|\alpha\|_{L^2(M, g)} = \int_M |\alpha|^2 d\text{Vol}_g$ and let

\[\Omega^k_2(M, g) = \{ \alpha \in \Omega^k(M) : \|\alpha\|_{L^2(M, g)} < \infty \}.

We denote by $\mathcal{H}^k_2(M, g)$ its space of $L^2$-harmonic $k$-forms, that is to say the space of $L^2$ $k$-forms which are closed and co-closed:

\[\mathcal{H}^k_2(M, g) = \{ \alpha \in \Omega^k_2(M, g) : d\alpha = d^*\alpha = 0 \},\]

where

\[d : \Omega^k_0(M) \rightarrow \Omega^{k+1}_0(M)\]
is the exterior differential operator and
\[ d^* : \Omega^{k+1}_0(M) \rightarrow \Omega^k_0(M) \]
its formal adjoint. The operator \( d \) does not depend on \( g \) but \( d^* \) does.

We assume throughout this subsection that \( (M, g, J) \) is a compact complex \( n \)-dimensional manifold with a Hermitian metric \( g \), and \( \pi : (\tilde{M}, \tilde{g}, \tilde{J}) \rightarrow (M, g, J) \) its universal covering with \( \Gamma \) as an isometric group of deck transformations. Let \( \mathcal{H}^k_{(2)}(M) \) be the spaces of \( L^2 \)-harmonic \( k \)-forms on \( \Omega^{p,q}_{(2)}(\tilde{M}) \), the squared integrable \( k \)-forms on \( (\tilde{M}, \tilde{g}) \), and denote by \( \dim_{\Gamma} \mathcal{H}^k_{(2)}(\tilde{M}) \) the Von Neumann dimension of \( \mathcal{H}^k_{(2)}(\tilde{M}) \) with respect to \( \Gamma \) [1] [13]. Its precise definition is not important in our article but only the following two basic facts are needed, see [9] [13].

1. \( \dim_{\Gamma} \mathcal{H}^k_{(2)}(M) = 0 \Leftrightarrow \mathcal{H}^k_{(2)}(M) = \{0\} \),
2. \( \dim_{\Gamma} \mathcal{H} \) is additive. Given
   \[ 0 \rightarrow \mathcal{H}_1 \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_3 \rightarrow 0, \]
   one have
   \[ \dim_{\Gamma} \mathcal{H}_2 = \dim_{\Gamma} \mathcal{H}_1 + \dim_{\Gamma} \mathcal{H}_3. \]

We denote by \( h^k_{(2)}(M) \) the \( L^2 \)-Hodge numbers of \( M \), which are defined to be
\[ h^k_{(2)}(M) := \dim_{\Gamma} \mathcal{H}^k_{(2)}(\tilde{M}), \quad (0 \leq p, q \leq n). \]
It turns out that \( h^k_{(2)}(M) \) are independent of the Hermitian metric \( g \) and depend only on \( (M, J) \). By the \( L^2 \)-index theorem of Atiyah [1], we have the following crucial identities between \( \chi(M) \) and the \( L^2 \)-Hodge numbers \( h^k_{(2)}(M) \):
\[ \chi(M) = \sum_{k=0}^{n} (-1)^k h^k_{(2)}(M). \]

**Remark 2.1.** \( L^2 \)-Betti number are not homotopy invariants for complete non-compact manifolds. But Dodziuk [6] proves that the class of the representation of \( \Gamma \) on the space of \( L^2 \)-harmonic forms is a homotopy invariant of \( M \). In particular the \( \Gamma \)-dimension (in the sense of Von Neumann) of the space of \( L^2 \)-harmonic forms does not depend on the chosen \( \Gamma \)-invariant metric.

### 2.2 Kähler parabolic

Let \( (M, g) \) be a Riemannian manifold. A differential form \( \alpha \) is called \( d \)(bounded) if there exists a form \( \beta \) on \( M \) such that \( \alpha = d\beta \) and
\[ \|\beta\|_{L^\infty(M,g)} = \sup_{x \in M} |\beta(x)|_g < \infty. \]
It is obvious that if \( M \) is compact, then every exact form is \( d \)(bounded). However, when \( M \) is not compact, there exist smooth differential forms which are exact but not \( d \)(bounded). For instance, on \( \mathbb{R}^n \), \( \alpha = dx^1 \wedge \cdots \wedge dx^n \) is exact, but it is not \( d \)(bounded).
Let’s recall some concepts introduced in \cite{4,10}. A differential form $\alpha$ on a complete non-compact Riemannian manifold $(M, g)$ is called $d$-(sublinear) if there exist a differential form $\beta$ and a number $c > 0$ such that $\alpha = d\beta$ and

\[
|\alpha(x)|_g \leq c,
\]

\[
|\beta(x)|_g \leq c(1 + \rho_g(x, x_0)),
\]

where $\rho_g(x, x_0)$ stands for the Riemannian distance between $x$ and a base point $x_0$ with respect to $g$.

The concept of $d$-(sublinear) is both natural and flexible. We recall the following classical fact pointed out by Gromov \cite{2,1} (negative case) and Cao-Xavier \cite{4} (non-positive case).

**Theorem 2.2.** Let $M$ be a complete simply-connected manifold of non-positive (resp. negative) sectional curvature and $\alpha$ a bounded closed $k$-form on $M$. Then $\alpha$ is $d$-(sublinear) for $k \geq 1$ (resp. $d$-(bounded) for $k \geq 2$).

**Proposition 2.3.** If $(M, g)$ is a closed smooth Riemannian manifold of non-positive (resp. negative) sectional curvature and $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ its universal covering. If $\alpha$ is a closed $k$-form on $M$, then the lifted $k$-form $\tilde{\alpha} := \pi^*\alpha$ is $d$-(sublinear) for $k \geq 1$ (resp. $d$-(bounded) for $k \geq 2$).

The following results are the main theorems in \cite{4,9,10}.

**Theorem 2.4.** Let $(M, g)$ be a complete $2n$-dimensional Kähler manifold with a $d$-(sublinear) Kähler form $\omega$. Then $H^k_{(2)}(M, g) \neq \{0\}$, when $k \neq n$, i.e.,

\[
h^k_{(2)}(M) = 0, \ k \neq n.
\]

Furthermore, if the Kähler form $\omega$ is $d$-(bounded), then $H^n_{(2)}(M, g) \neq \{0\}$, i.e,

\[
h^n_{(2)}(M) \geq 1.
\]

**Remark 2.5.** The proof for the vanishing type results in all cases is a direct application of the $L^2$ versions Lefschetz theorem. The real hard part is the nonvanishing results in negative case, where a careful analysis on the lower bound of the eigenvalues of the Laplacian on $L^2$-harmonic forms was carried out in \cite{9}.

### 2.3 Locally conformally Kähler manifold

In this section we state several equivalent definitions of the notion of a locally conformal Kähler (LCK) manifold. Let $(M, J, g)$ be a complex manifold of $\dim_C = n > 1$, where $J$ denotes its complex structure and $g$ its Hermitian metric. Locally conformally Kähler (LCK) manifolds are, by definition, complex manifolds admitting a Kähler covering with deck transformation acting by Kähler homotheties. An equivalent definition is that there is an open cover $\{U_i\}$ of $M$ and a family $\{f_i\}$ of $C^\infty$ functions $f_i : U_i \to \mathbb{R}$ so that each local metric $g_i = \exp(-f_i)g|_{U_i}$ on $U_i$ is Kählerian. Then the metrics $e^{-f_i}g_i$ glue to a global metric whose associated 2-form $\omega$ satisfies the integrability condition $d\omega = \theta \wedge \omega$, thus being locally conformal with the Kähler metrics $g_i$. Here $\theta|_{U_i} = df_i$. The closed 1-form $\theta$ is called the **Lee form**. This gives another definition of an LCK structure, which will be used in this paper \cite{8}.
Definition 2.6. Let \((M, J, g)\) be a complex Hermitian manifold, \(\dim \mathbb{C} M > 1\), with
\[
d\omega = \theta \wedge \omega,
\]
where \(\theta\) is a closed 1-form. Then \(M\) is called a \textbf{locally conformally Kähler (LCK)} manifold.

A compact LCK manifold never admits a Kähler structure, unless the cohomology class \(\theta \in H^1(M)\) vanishes [16].

If \((M, g)\) is a compact Riemannian manifold and \(\pi: (\tilde{M}, \tilde{g}) \to (M, g)\) its universal covering, there is a very known result as follows.

Proposition 2.7. ([10, Proposition 1]) If \(\alpha\) is a closed 1-form on a compact Riemannian manifold \(M\), then the lifted 1-form \(\tilde{\alpha} := \pi^*\alpha\) on the universal covering space \(\tilde{M}\) is \(d\) (sublinear).

Following Proposition 2.7, it implies that the lifted Lee form \(\pi^*\theta\) is \(d\) (sublinear) with respect to metric \(\pi^*g\). In this article, we now introduce a class of LCK manifold as follows.

Definition 2.8. Let \((M, J, g)\) be a compact \(2n\)-dimensional LCK manifold with the Lee form \(\theta\). We denote by \((\tilde{M}, \tilde{J}, \tilde{g})\) the universal covering space of \((M, J, g)\) and \(\tilde{\theta} := df\) the lifted Lee form. We call \((M, J, g)\) a \(d\) (bounded) LCK manifold, if \(f\) is bounded.

Example 2.9. A manifold \((M, J, g)\) called \textbf{globally conformal Kähler (GCK)} if there is a \(C^\infty\) function \(f: M \to \mathbb{R}\) such that the metric \(e^{-f}g\) is Kählerian, i.e. the Lee form \(\theta = df\). Thus a closed, GCK manifold \((M, J, g)\) is a \(d\) (bounded) manifold because the lifted function \(\pi^*f\) on \(\tilde{M}\) is also bounded. One can see that if the first Betti number \(b^1(M) = 0\) (for example, \(M\) is simply-connected) then \(M\) is GCK. In particular, the universal covering space of a LCK manifold is GCK.

Remark 2.10. A compact LCK (but not GCK) manifold cannot have strictly positive sectional curvature because (by a classical result of J. Synge [14]) it would be simply connected and thus GCK.

3 Euler number of LCK manifold

If \(g_1 = e^f g_2\) are two conformally equivalent Riemannian metrics on a smooth \(2n\)-dimensional manifold \(M\), then we have the equality, see [5, Proposition 5.2]:
\[
\mathcal{H}^n_{(2)}(M, g_1) = \mathcal{H}^n_{(2)}(M, g_2).
\]

The spaces of \(L^2\) harmonic forms depend only on the \(L^2\) structures, hence if \(g_1\) and \(g_2\) are two Riemannian metrics on a manifold \(M\) \((g_1, g_2\) need not to be complete) which are quasi-isometric that is for a certain constant
\[
C^{-1} g_1 \leq g_2 \leq C g_1,
\]
then clearly the Hilbert spaces \( \Omega^k_{(2)}(M, g_1) \) and \( \Omega^k_{(2)}(M, g_2) \) are the same with equivalent norms. Hence the quotient spaces defining reduced \( L^2 \) cohomology are the same, that is
\[
\mathcal{H}^k_{(2)}(M, g_1) = \mathcal{H}^k_{(2)}(M, g_2).
\]
In fact, the spaces \( \mathcal{H}^k_{(2)}(M, g) \) are biLipschitz-homotopy invariants of \( (M, g) \), see [5, Proposition 3.1].

**Proposition 3.1.** Let \((M, J, g)\) be a compact \( 2n \)-dimensional \( d \)(bounded) LCK manifold with the Lee form \( \theta \). If the sectional curvature of \( M \) is non-positive (resp. negative), then there exist a 1-form \( \eta \) and a function \( f \) on the universal covering space \((\tilde{M}, \tilde{J}, \tilde{g})\) such that
\[
(1) \quad \tilde{\theta} = df,
\]
where \( \tilde{\theta} \) is the lifted 1-form on \( \tilde{M} \),
\[
(2) \quad e^{-f}\tilde{\omega} = d\eta,
\]
where \( \tilde{\omega} \) is the lifted Kähler form of \((\tilde{M}, \tilde{g})\),
\[
(3) \quad |\eta(x)|_{e^{-f}\tilde{g}} \leq c(e^{-f}\tilde{g}(x, x_0) + 1)
\]
(resp. \( |\eta(x)|_{e^{-f}\tilde{g}} \leq c \)), where \( c \) is uniform positive constant.

In particular, in non-positive sectional curvature case, for any \( k \neq n \),
\[
\mathcal{H}^k_{(2)}(\tilde{M}, e^{-f}\tilde{g}) = \{0\}
\]
and in negative sectional curvature case,
\[
\mathcal{H}^n_{(2)}(\tilde{M}, e^{-f}\tilde{g}) \neq \{0\}.
\]

**Proof.** Let \( \pi : (\tilde{M}, \tilde{g}) \rightarrow (M, g) \) be the universal covering map, \( \omega \) the Kähler form and \( \theta \) the Lee form on \( M \). By the definition of \( d \)(bounded) LCK manifold, it implies that there exists a bounded function \( f \) on \( \tilde{M} \) such that the lifted Lee form
\[
\tilde{\theta} := \pi^*\theta = df.
\]
Noting that
\[
d(e^{-f}\tilde{\omega}) = -df \wedge e^{-f}\tilde{\omega} + e^{-f}d\tilde{\omega} = e^{-f}\tilde{\omega} \wedge (\tilde{\theta} - df) = 0
\]
and
\[
|e^{-f}\tilde{\omega}|_{\tilde{g}} \leq e^{-f}|\tilde{\omega}|_{\tilde{g}} < \infty.
\]
Hence the 2-form \( e^{-f}\tilde{\omega} \) is a bounded closed form on \((\tilde{M}, \tilde{g})\). If the sectional curvature of \( M \) is non-positive (resp. negative), it follows from Theorem 2.2 that
\[
e^{-f}\tilde{\omega} = d\eta
A note on Euler number of locally conformally Kähler manifolds

and

$$|\eta(x)|_{\bar{g}} \leq c(1 + \rho_{\bar{g}}(x, x_0))(\text{resp. } |\eta(x)|_{\bar{g}} \leq c),$$

where $\eta$ is a one-form on $\bar{M}$ and $c$ is a uniform positive constant. The metric on $\bar{M}$ induced by the Kähler form $e^{-f}\bar{\omega}$ is $e^{-f}\bar{g}$. The metrics $\bar{g}$ and $e^{-f}\bar{g}$ are quasi-isometric, since there exists a uniform positive constant $c'$ such that

$$\frac{1}{c'}\rho_{\bar{g}}(x, x_0) \leq \rho_{e^{-f}\bar{g}}(x, x_0) \leq c'\rho_{\bar{g}}(x, x_0).$$

We then have (1) if the sectional curvature of $M$ is non-positive, then

$$|\eta(x)|_{e^{-f}\bar{g}} = e^{-\frac{f}{2}}|\eta(x)|_{\bar{g}} \leq e''(\rho_{\bar{g}}(x, x_0) + 1) \leq e''(\rho_{e^{-f}\bar{g}}(x, x_0) + 1);$$

(2) if the sectional curvature of $X$ is negative, then

$$|\eta(x)|_{e^{-f}\bar{g}} = e^{-\frac{f}{2}}|\eta(x)|_{\bar{g}} \leq e''.$$

Here $c''$ is a uniform positive constant only depends on $f$ and the metric $g$. Following from Theorem 2.4, then for any $k \neq n$, $\mathcal{H}^k_{(2)}(\bar{M}, e^{-f}\bar{g}) = \{0\}$ (non-positive case) and $\mathcal{H}^n_{(2)}(\bar{M}, e^{-f}\bar{g}) \neq \{0\}$ (negative case).

We then have

**Theorem 3.2.** Let $M$ be a compact $2n$-dimensional $d$(bounded) LCK manifold. If the sectional curvature of $M$ is non-positive (resp. negative), then the Euler characteristic of $M$ satisfies the inequality $(-1)^n\chi(M) \geq (\text{resp. } >) 0$.

**Proof.** Nothing that the metrics $\bar{g}$, $e^{-f}\bar{g}$ are quasi-isometric. Then following Proposition 3.1 in non-positive sectional curvature case, we obtain that the spaces of $L^2$-harmonic $k$-form on $(\bar{M}, \bar{g})$ satisfy

$$\mathcal{H}^k_{(2)}(\bar{M}, \bar{g}) = \{0\}, \forall k \neq n$$

and in negative sectional curvature case

$$\mathcal{H}^n_{(2)}(\bar{M}, \bar{g}) \neq \{0\}.$$

The Atiyah index theorem for covers [1] then gives $(-1)^n\chi(M) \geq (\text{resp. } >) 0$.

**Corollary 3.3.** Let $M^{2n}$ be a compact $2n$-dimensional Riemannian manifold of non-positive (resp. negative) curvature. If $M^{2n}$ is homeomorphic to a GCK manifold, then the Euler characteristic of $M^{2n}$ satisfies the inequality $(-1)^n\chi(M^{2n}) \geq (\text{resp. } >) 0$.

**Proof.** The conclusion follows form Theorem 3.2 and the GCK manifold is $d$(bounded).

In [4], the authors shown that the property of $d$(sublinearity) has homotopy invariance.
Lemma 3.4. (Lemma 3) Let $F : M_1 \to M_2$ be a smooth homotopy equivalence between two compact Riemannian manifolds, $\pi : \tilde{M}_i \to M_i$ the universal covering maps for $i = 1, 2$. Then, for any closed differential form $\alpha$ on $M_2$, $\pi^*(\alpha)$ is $d$(sublinear) (resp. $d$(bounded)) on $\tilde{M}_2$ if the form $(F \circ \pi)^*(\alpha)$ is $d$(sublinear) (resp. $d$(bounded)) on $\tilde{M}_1$.

Proof of Theorem 1.1 Since $F$ is a homotopy equivalence, there exists a smooth map $G : M_2 \to M_1$ such that both $F \circ G$ and $G \circ F$ are homotopic to the identity maps. Clearly, the maps $F$ and $G$ can be lifted to the universal covering spaces. Let then $\tilde{F} : \tilde{M}_1 \to \tilde{M}_2$ and $\tilde{G} : \tilde{M}_2 \to \tilde{M}_1$ be the lifted maps, so that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{M}_1 & \xrightarrow{\tilde{F}} & \tilde{M}_2 \\
\pi & & \pi \\
M_1 & \xrightarrow{F} & M_2 \\
\pi & & \pi \\
M_1 & \xrightarrow{G} & M_1
\end{array}
\]

Since $M_1$ is compact LCK manifold, the Lee form $\theta$ is bounded and $\pi$ is a local isometry, $\pi^*\theta$ is a bounded form on $\tilde{M}_1$. By the assumption, there is a bounded function $f$ on $\tilde{M}_1$ such that $\pi^*\theta = df$.

Following the idea in Lemma 3.4, the assumption and the commutativity of the diagram imply that the form

$$(G \circ \pi)^*\theta = (\pi \circ \tilde{G})^*\theta = \tilde{G}^*(\pi^*\theta) = \tilde{G}^*df = d(\tilde{G}^*f)$$

is $d$(bounded). Thus the form $e^{-\tilde{G}^*f}(G \circ \pi)^*\omega$ on $\tilde{M}_2$ is closed because

$$d e^{-\tilde{G}^*f}(G \circ \pi)^*\omega = e^{-\tilde{G}^*f}(-d(\tilde{G}^*f) \wedge (G \circ \pi)^*\omega + d(G \circ \pi)^*\omega)$$

$$= e^{-\tilde{G}^*f}(-(G \circ \pi)^*\theta \wedge (G \circ \pi)^*\omega + d(G \circ \pi)^*\omega)$$

$$= e^{-\tilde{G}^*f}(G \circ \pi)^*(-\theta \wedge \omega + d\omega) = 0.$$ 

Following Theorem 2.2, it implies that $e^{-\tilde{G}^*f}(G \circ \pi)^*\omega$ on $\tilde{M}_2$ is $d$(sublinear) (resp. $d$(bounded)). Thus the lifted Kähler form $\tilde{F}^*(e^{-\tilde{G}^*f}(G \circ \pi)^*\omega)$ on $\tilde{M}_1$ is $d$(sublinear) (resp. $d$(bounded)) as well. On the other hand, by the commutativity of the diagram

$$(G \circ F \circ \pi)^*\omega = (G \circ \pi \circ \tilde{F})^*\omega.$$ 

We then have

$$\tilde{F}^*(e^{-\tilde{G}^*f}(G \circ \pi)^*\omega) = e^{-(\tilde{G} \circ \tilde{F})^*f}(G \circ \pi \circ \tilde{F})^*\omega = e^{-(\tilde{G} \circ \tilde{F})^*f}(G \circ F \circ \pi)^*\omega.$$ 

Following Theorem 2.4, it implies that the spaces of $L^2$-harmonic $k$-forms with respect to metric $e^{-(\tilde{G} \circ \tilde{F})^*f}(G \circ F \circ \pi)^*g_1$ satisfies

$$\mathcal{H}^k_{(2)}(\tilde{M}_1, e^{-(\tilde{G} \circ \tilde{F})^*f}(G \circ F \circ \pi)^*g_1) = \{0\}, \forall k \neq n,$$

in non-positive sectional curvature case, and

$$\mathcal{H}^k_{(2)}(\tilde{M}_1, e^{-(\tilde{G} \circ \tilde{F})^*f}(G \circ F \circ \pi)^*g_1) \neq \{0\},$$

and
in negative sectional curvature case.

The metric on $M_1$ induced by $(G\circ F)^*\omega$ is $(G\circ F)^*g_1$. The universal covering space of $(M_1, (G\circ F)^*\omega)$ is $(\tilde{M}_1, (G \circ F \circ \pi)^*\omega)$. We denote by $\mathcal{H}^k_{(2)}(\tilde{M}_1, (G \circ F \circ \pi)^*g_1)$ the spaces of $L^2$-harmonic $k$-forms on $\Omega^k_{(2)}(M_1)$ with respect to metric $(G \circ F \circ \pi)^*g_1$. The Euler characteristic $\chi(M_1)$ is a topology invariant. The Atiyah index theorem for covers \[1\] then gives

$$\chi(M_1) = \sum_{k=0}^{-1} (-1)^k h_{(2)}^k(M_1),$$

where

$$h_{(2)}^k(M_1) := \dim \mathcal{H}^k_{(2)}(\tilde{M}_1, (G \circ F \circ \pi)^*g_1), \forall 0 \leq k \leq n.$$

Since the metrics $e^{-(G\circ F)^*f}(G\circ F\circ \pi)^*g_1, (G\circ F\circ \pi)^*g_1$ are quasi-isometric, we obtain that $h_{(2)}^k(M_1) = 0$, when $k \neq n$ in non-positive sectional curvature case and $h_{(2)}^n(M_1) = 0$ in negative sectional curvature case. Thus the Euler number of $M_1$ satisfies $(-1)^n\chi(M_1) \geq$ (resp. $>0$).

\[\square\]

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