Conformal linear gravity in de Sitter space

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Abstract

It has been shown that the theory of linear conformal quantum gravity must include a tensor field of rank-3 and mixed symmetry [1]. In this paper, we obtain the corresponding field equation in de Sitter space. Then, in order to relate this field with the symmetric tensor field of rank-2, $\mathcal{K}_{\alpha\beta}$ related to graviton, we will define homomorphisms between them. Our main result is that if one insists $\mathcal{K}_{\alpha\beta}$ to be a unitary irreducible representation of de Sitter and conformal groups it must satisfy a filed equation of order 6, which is obtained.

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1 Introduction

Gravitational fields are long range and seems to travel with the speed of light, in the first approximation, at least, their equations are expected to be conformally invariant (CI). Einstein’s theory of gravitation, in the background field method ($g_{\mu\nu} = g^{BG}_{\mu\nu} + h_{\mu\nu}$) and linear approximation, can be considered as a theory of massless symmetric tensor field of rank-2, $h_{\mu\nu}$ on a fixed background $g^{BG}_{\mu\nu}$, such as de Sitter space. It is well known that the massless fields propagate on the light-cone and are invariant under the conformal group $SO(2,4)$. For spin $s \geq 1$ they are invariant under the gauge transformation as well.

On the other hand, Einstein’s theory of gravity seems perfect as a classical theory. Experimental data have confirmed it and have ruled out several possible alternatives. However, as a quantum theory it is less satisfactory since, as soon as one couples to matter, the first order quantum corrections lead to a divergent $S$ matrix. These divergences are nonrenormalizable.

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Einstein’s classical theory of gravitation, as well as equation of \( h_{\mu\nu} \) is not CI thus could not be considered as a comprehensive universal theory of gravitational fields.

In de Sitter (dS) space, mass is not an invariant parameter for the set of observable transformations under the dS group \( SO(1,4) \). Concept of light-cone propagation, however, does exist and leads to the conformal invariance. “Massless” is used in reference to propagation on the dS light-cone (conformal invariance). The term “massive” is refereed to fields that in their Minkowskian limit (zero curvature) reduce to massive Minkowskian fields [2]. The conformal invariance, and the light-cone propagation, constitutes the basis for constructing “massless” field in dS space.

In previous papers, we used Dirac’s six cone formalism to obtain CI equations for the scalar, vector [3], and rank-2 symmetric tensor [4] fields which transformed according to the unitary irreducible representation (UIR) of dS group. The conformal space and six-cone formalism was first used by Dirac to obtain the CI equations [5]. This formalism developed by Mack and Salam [6] and many others [7]. This approach to conformal symmetry leads to the best path to exploit the physical symmetry in contrast to approaches based on group theoretical treatment of state vector spaces. This is essentially because in the latter approach it would be much more difficult to see how to break the symmetry down to Poincaré invariance [6].

Barut and Böhm [2] have shown that for the physical representation of the conformal group (UIR), the value of the conformal Casimir operator is 9. But according to calculation of Binegar et al [1] for the tensor field of rank-2 and conformal degree 0, this value becomes 8. Therefore tensor field of rank-2 does not correspond to any UIR of the conformal group. In other words, the tensor field that carries physical representations of the conformal group must be a tensor field of higher rank.

In this paper we propose and study a mixed symmetry tensor field of rank-3, \( \Psi_{abc} \), with conformal degree zero, which transforms according to the UIR of the conformal group [1, 8, 9]. By mixed symmetry we mean

\[
\Psi_{abc} = -\Psi_{bac}, \quad \sum_{\text{cycl}} \Psi_{abc} = 0,
\]

while a field of conformal degree zero satisfies \( u^d \partial_d \Psi_{abc} = 0 \), \( a, b, c, d \equiv 0, 1, ..., 5 \), where \( u^d \) are the coordinates in \( \mathbb{R}^6 \). We then project this field to dS space and define homomorphisms between the projected field, \( F_{\alpha\beta\gamma} \), and rank-2 symmetric tensor field \( K_{\alpha\beta} \) on dS space (\( \alpha, \beta \equiv 0, 1, ..., 4 \)). It has been shown that if one insists \( K_{\alpha\beta} \) to transform according to the UIRs of dS and conformal groups it must satisfy a field equation of order 6.

The paper is organized as follows. Section 2 is devoted to a brief review of the notations. In this section we recall Dirac’s manifestly covariant formalism of mixed symmetry tensor fields on the six-cone and their projection to de Sitter space. Section 3 introduces CI wave equation with the subsidiary conditions i.e., transversality and divergencelessness. Section 4 is devoted to define homomorphisms between \( F_{\alpha\beta\gamma} \) and \( K_{\alpha\beta} \) on de Sitter space. Finally a brief conclusion and an outlook for further investigation has been presented.

2 Notation

The dS metric is a solution of the cosmological Einstein’s equation with positive constant \( \Lambda \). Recent astrophysical data indicate that our universe might currently be in a dS phase [10].
The importance of dS space has been primarily ignited by the study of the inflationary model of the universe and quantum gravity \[11\]. The de Sitter space is identical to four dimensional one-sheeted hyperboloid (intrinsic) embedded in five dimensional flat space (ambient)

\[
X_H = \{ x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} \}, \quad \alpha, \beta = 0, 1, 2, 3, 4,
\]

where \( \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1) \) and \( H \) is the Hubble parameter.

The concept of conformal space was used by Dirac \[5\] to demonstrate the field equations for spinor and vector fields in 1 + 3 dimensional space-time in manifestly CI form. The conformal group \( SO(2, 4) \) acts nonlinearly on Minkowski coordinates. Dirac proposed a manifestly conformally covariant formulation in which the Minkowski coordinates are replaced by coordinates on which \( SO(2, 4) \) acts linearly. The resulting theory is then formulated on a 5 dimensional hypercone (named Dirac’s six-cone) in a 6 dimensional space. Dirac’s six-cone, or Dirac’s projection cone, is defined by

\[
u^2 \equiv (\nu^0)^2 - \vec{\nu}^2 + (\nu^5)^2 = \eta_{ab} u^a u^b = 0, \quad \eta_{ab} = \text{diag}(1, -1, -1, -1, -1, 1),
\]

where \( u^a \in \mathbb{R}^6 \), and \( \vec{\nu} \equiv (u^1, u^2, u^3, u^4) \). Reduction to four dimensional (physical space-time) is achieved by projection, that is by fixing the degrees of homogeneity of all fields. Wave equations, subsidiary conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. These are well-defined operators that map tensor fields to tensor fields with the same rank on cone \( v^2 = 0 \) \[3, 12\].

We consider tensors of specific symmetry type that are transverse, divergenceless and traceless,

a) transversality, \( u_a \Psi^{ab\cdots} = 0 \),

b) tracelessness, \( \Psi_a^{ab\cdots} = 0 \),

c) divergencelessness, \( \text{Grad}_a \Psi^{ab\cdots} = 0 \), where the operator \( \text{Grad}_a \) unlike \( \partial_a \) is intrinsic on the cone, and is defined by \[12\]:

\[
\text{Grad}_a \equiv u_a \partial_b \partial^b - (2\hat{N}_5 + 4) \partial_a.
\]

The action of second order Casimir operator of conformal group on \( \Psi \) is \[1, 2, 12\]:

\[
Q_2 \Psi^{cd\cdots} = \frac{1}{2} L_{ab} L^{ab} \Psi^{cd\cdots} = \left(-u^2 \partial^2 + \hat{N}_5 (\hat{N}_5 + 4) - 2N + n_1 (n_1 + 4) + n_2 (n_2 + 2) + n_3^2 \right) \Psi^{cd\cdots},
\]

where \( n_1 \geq n_2 \geq n_3 \geq 0 \) are integers that label the symmetry type according to the lengths of the rows of the Yang diagrams and \( L_{ab} \) are the generators of the conformal Lie algebra. \( \Psi \) is a tensor field of a definite rank and a definite symmetry. \( N \) is the rank of the tensor field \( \Psi^{abc\cdots} \) and \( \hat{N}_5 \) is the conformal-degree operator defined by:

\[
\hat{N}_5 \equiv u^a \partial_a.
\]
On the cone \((u^2 = 0)\), the second order Casimir operator of the conformal group, \(Q_2\), reduces to a constant. Therefore it is not a suitable operator to define CI wave equations. For example, for rank-2 symmetric tensor field \(\Psi^{cd}\), we have
\[
Q_2 \Psi^{cd} = \left( \hat{N}_5 (\hat{N}_5 + 4) + 8 \right) \Psi^{cd},
\]
and for a mixed symmetry rank-3 tensor field \(\Psi^{abc}\) we have
\[
Q_2 \Psi^{abc} = \left( \hat{N}_5 (\hat{N}_5 + 4) + 9 \right) \Psi^{abc}.
\]
It is clear that this operator cannot lead to wave equations on the cone since it is a constant. So intrinsic wave operators are used to obtain wave equations on the cone. These operators exist only in exceptional cases. For tensor fields of degree \(-1, 0, 1, \ldots\), the intrinsic wave operators are \(\partial^2\), \((\partial^2)^2\), \((\partial^2)^3\), \ldots, respectively [12]. Thus, the following CI system of equations has been utilized on the cone [3]:
\[
\begin{cases}
(\partial_\alpha \partial^\alpha)^n \Psi = 0, \\
\hat{N}_5 \Psi = (n - 2) \Psi.
\end{cases}
\tag{2.5}
\]
Other CI conditions can be added to the above system in order to restrict the space of the solutions. In order to project the coordinates on the cone \(u^2 = 0\) to the dS space, we choose the following relation:
\[
\begin{cases}
x_\alpha = (Hu^5)^{-1} u_\alpha, \\
x_5 = Hu^5.
\end{cases}
\tag{2.6}
\]
Note that \(x^5\) becomes superfluous when we deal with the projective cone. It is easy to show that various intrinsic operators introduced previously now read as:

1. the conformal-degree operator \((\hat{N}_5)\)
   \[
   \hat{N}_5 = x_5 \frac{\partial}{\partial x_5},
   \tag{2.7}
   \]

2. the conformal gradient \((Grad_\alpha)\)
   \[
   Grad_\alpha = -x_5^{-1} \{ H^2 x_\alpha [Q_0 - \hat{N}_5 (\hat{N}_5 - 1)] + 2 \bar{\partial}_\alpha (\hat{N}_5 + 1) \},
   \tag{2.8}
   \]
   where \(\bar{\partial}_\alpha\) is tangential (or transverse) derivative in de Sitter space
   \[
   \bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad x \cdot \partial = 0.
   \]
   \(\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta\) is the transverse projector. \(Q_0 = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -H^{-2} (\bar{\partial})^2\) is the scalar Casimir operator.

3. and the powers of d’Alembertian \((\partial_\alpha \partial^\alpha)^n\), which act intrinsically on field of conformal degree \((n - 2)\),
   \[
   (\partial_\alpha \partial^\alpha)^n = -H^{2n} x_5^{-2n} \prod_{j=1}^{n} [Q_0 + (j + 1)(j - 2)].
   \tag{2.9}
   \]

4
We have shown [3] that for scalar and vector fields, the simplest CI system of equations is obtained through setting \( n = 1 \) in (2.5), i.e. the field with conformal degree \(-1\). Resulting field equations are transformed according to the UIRs of \( SO(1,4) \). In the flat limit \((H \to 0)\), the CI equation for the vector field reduces exactly to the Maxwell equation [13]. For a symmetric tensor field of rank-2, the CI system (2.5) with \( n = 1 \) leads to [4] (for simplicity from now on we take \( H = 1 \)):

\[
(Q_0 - 2)K_{\alpha\beta} + \frac{2}{3}S(\bar{\partial}_\beta + 2x_\beta)\bar{\partial} \cdot K_{\alpha} - \frac{1}{3}\theta_{\alpha\beta}\bar{\partial} \cdot \bar{\partial} \cdot K = 0 .
\]  

(2.10)

By imposing the traceless and divergenceless conditions on the tensor field \( K_{\alpha\beta} \), which are necessary for UIRs of dS group, the CI equation (2.10) reduces to \(^1\)

\[
(Q_0 - 2)K_{\alpha\beta} = 0, \text{ or } (Q_2 + 4)K_{\alpha\beta} = 0 .
\]

The solution of this CI field equation corresponds to a representation of discrete series, namely \( \Pi_{2,1}^\pm \) [14, 15]. However, this equation does not coincide with any UIR of the Poincaré group. In the flat limit. Note that in the flat limit the CI equation (2.10) reduces to the CI massless spin-2 wave equation of order-2 in four dimensional Minkowski space which was found by Barut and Xu [4]; they have found this equation by varying the coefficients of various terms in the standard equation [16].

If we take \( n = 2 \) in (2.5) we will obtain the following CI system [4]

\[
(Q_2 + 4)[(Q_2 + 6)K_{\alpha\beta} + D_{2\alpha}\partial_2.K_{\beta}] + \frac{1}{3}D_{2\alpha}D_{1\beta}\bar{\partial} \cdot \bar{\partial} \cdot K + \frac{1}{3}\theta_{\alpha\beta}(Q_0 + 6)\bar{\partial} \cdot \bar{\partial} \cdot K = 0 ,
\]

\[
Q_1\bar{\partial} \cdot K_{\alpha} + \frac{2}{3}D_{1\alpha}\bar{\partial} \cdot \bar{\partial} \cdot K + \frac{1}{6}Q_1D_{1\alpha}\bar{\partial} \cdot \bar{\partial} \cdot K = 0 ,
\]

\[
K' = 0 .
\]  

(2.11)

By imposing the traceless and divergenceless conditions on the tensor field \( K_{\alpha\beta} \), the CI system (2.11) becomes

\[
(Q_0 - 2)Q_0K_{\alpha\beta} = 0, \text{ or } (Q_2 + 4)(Q_2 + 6)K_{\alpha\beta} = 0 .
\]  

(2.12)

The solution of this CI field equation corresponds to the two representations of discrete series, namely \( \Pi_{2,1}^\pm \) and \( \Pi_{2,2}^\pm \) [14, 15]. However as mentioned, symmetric tensor field of rank-2 does not correspond to any UIR of the conformal group. In the next section we will study a mixed symmetry tensor.

3 Conformally invariant field equation

Considering the conformal invariance in the dS space, we consider a mixed symmetry rank-3 tensor field \( \Psi_{abc} \) in Dirac’s null-cone. We classify the degrees of freedom of this tensor field in de Sitter space by \(^2\)

\[
F_{\alpha\beta\gamma} = \Psi_{\alpha\beta\gamma} + x_\alpha x^\cdot \Psi_{\beta\gamma} + x_\beta x^\cdot \Psi_{\alpha\gamma} + x_\gamma x^\cdot \Psi_{\alpha\beta} + x_\alpha x_\beta x^\cdot x^\cdot \Psi_{\gamma} + x_\alpha x_\gamma x^\cdot x^\cdot \Psi_{\beta} .
\]  

(3.1)

\(^1\) \( Q_2 \) (\( Q_1 \)) is the Casimir operator of the dS group for the spin-2 (spin-1) field, mathematical details can be found in [13, 14].

\(^2\) we have used the notation \( x^\alpha x^\gamma \Psi_{\alpha\beta\gamma} \equiv x^\cdot \Psi_{\beta} \).
\[ \mathcal{T}_{\alpha\beta} = x \cdot \Psi_{\alpha\beta} + x \cdot \Psi_{\alpha\beta} + x \cdot \Psi_{\alpha\beta} + x_{\alpha} x \cdot x \cdot \Psi_{\beta\gamma} + x_{\beta} x \cdot x \cdot \Psi_{\alpha\gamma}, \quad (3.2) \]

\[ K_{\alpha} = x \cdot x \cdot \Psi_{\alpha\beta}, \quad (3.3) \]

\[ \phi = x \cdot x \cdot x \cdot \Psi = 0, \]

where \( F_{\alpha\beta\gamma} \) is a mixed symmetry rank-3 tensor field and \( \mathcal{T}_{\alpha\beta} \) and \( K_{\alpha} \) are tensors of rank two and one on dS space respectively \( (x^\alpha \mathcal{T}_{\alpha\beta} = x^\beta \mathcal{T}_{\alpha\beta} = x^\alpha K_{\alpha}) \). The fields \( \Psi_{5ab} \) and their contraction and multiplication with \( x \), are auxiliary fields and do not need to be transformed to dS space.

We find CI equations in Dirac’s null-cone by setting \( n = 2 \) in (2.5). Then followed by projection we obtain CI equations in dS space

\[
\begin{aligned}
(Q_0 - 2)Q_0 \Psi_{\alpha\beta\gamma} &= 0, \\
\bar{N}_a \Psi_{\alpha\beta\gamma} &= 0.
\end{aligned} \quad (3.4)
\]

The following conditions can be added to the above system to restrict the space of solutions:

a) transversality \( u_{\alpha} \Psi_{\alpha\beta\gamma} = 0 \), that results in

\[ x^5(\Psi_{5\beta\gamma} + x \cdot \Psi_{5\beta\gamma}) = 0, \quad (3.5) \]

b) divergencelessness \( \text{Grad}_a \Psi_{\alpha\beta\gamma} = 0 \), that results in

\[
\partial \cdot \Psi_{\beta\gamma} = -x \cdot \partial x \cdot \Psi_{\beta\gamma}, \quad \text{or} \quad \bar{\partial} \cdot \Psi_{\beta\gamma} = -x \cdot \Psi_{\beta\gamma}. \quad (3.6)
\]

We combine (3.2), (3.3) and (3.1) to get :

\[
Q_0(Q_0 - 2)x \cdot \Psi_{\beta\gamma} = 0, \quad Q_0(Q_0 - 2)x \cdot x \cdot \Psi_{\alpha\gamma} = 0, \quad \bar{\partial}_\alpha x \cdot \Psi_{\beta\gamma} = \Psi_{\gamma\beta\gamma}. \quad (3.7)
\]

We can write \( F_{\alpha\beta\gamma} \) as

\[ F_{\alpha\beta\gamma} = \frac{1}{4} x_{\gamma} A_{\alpha\beta} + \Psi_{\alpha\beta\gamma} + x_{\alpha} x \cdot \Psi_{\beta\gamma} + x_{\beta} x \cdot \Psi_{\alpha\gamma}, \quad (3.8) \]

where we used the following identities

\[
A_{\alpha\beta} \equiv \bar{\partial}^\gamma F_{\alpha\beta\gamma} - x_{\alpha} F_{\gamma\beta} + x_{\beta} F_{\gamma\alpha} = \\
4(x \cdot \Psi_{\alpha\beta} + x_{\alpha} x \cdot x \cdot \Psi_{\beta\gamma} + x_{\beta} x \cdot x \cdot \Psi_{\alpha\gamma}), \quad (3.9)
\]

\[
\frac{1}{2} \bar{\partial} \cdot F_{\alpha\gamma} = x \cdot \Psi_{\alpha\gamma}, \quad F_{\alpha\beta} - \frac{1}{2} x_{\beta} \bar{\partial} \cdot F_{\alpha\gamma} = \Psi_{\alpha\beta} + x \cdot x \cdot \Psi_{\beta\gamma}.
\]

So the operation of \( Q_0(Q_0 - 2) \) on \( F_{\alpha\beta\gamma} \) leads to

\[
Q_0(Q_0 - 2)(F_{\alpha\beta\gamma} - \frac{1}{4} x_{\gamma} A_{\alpha\beta}) = \\
-4(\bar{\partial}_\alpha + 3x_{\alpha})(Q_0 - 2)x \cdot \Psi_{\beta\gamma} - 4(\bar{\partial}_\beta + 3x_{\beta})(Q_0 - 2)x \cdot \Psi_{\alpha\gamma}. \quad (3.10)
\]
Multiplying above equation by \( x_\beta \) results in

\[
(Q_0 - 2)x \cdot \Psi_{\alpha \gamma} = \frac{1}{8}(Q_0 - 2)(4\bar{\partial} \cdot F_{\alpha \gamma} - A_{\alpha \gamma} - x\gamma \bar{\partial} \cdot A_{\alpha}),
\]

(3.11)
similarly we have

\[
(Q_0 - 2)x \cdot \Psi_{\beta \gamma} = \frac{1}{8}(Q_0 - 2)(4\bar{\partial} \cdot F_{\beta \gamma} - A_{\gamma \beta} - x\gamma \bar{\partial} \cdot A_{\beta}).
\]

(3.12)
Finally, from Eq.s (3.7), (3.8) and (3.9), the following CI field equation is obtained for the mixed symmetry tensor field \( F_{\alpha \beta \gamma} \) in de Sitter space

\[
2Q_0(Q_0 - 2)(F_{\alpha \beta \gamma} - \frac{1}{4}x\gamma A_{\alpha \beta}) + (\bar{\partial}_\alpha + 3x\alpha)(Q_0 - 2)(4\bar{\partial} \cdot F_{\beta \gamma} - A_{\gamma \beta} - x\gamma \bar{\partial} \cdot A_{\beta})
\]

\[
+ (\bar{\partial}_\beta + 3x\beta)(Q_0 - 2)(4\bar{\partial} \cdot F_{\alpha \gamma} - A_{\alpha \gamma} - x\gamma \bar{\partial} \cdot A_{\alpha}) = 0.
\]

(3.13)

It is important to note that the solution of this field equation is a physical state of the conformal group which transforms according to the UIR of this group. In the next section we will consider its transformation according to UIRs of the de Sitter group \( SO(1,4) \).

### 4 Group theoretical content

In order to obtain the relation between the rank-3 mixed symmetry tensor field \( F_{\alpha \beta \gamma} \) and a massless spin-2 field \( K_{\alpha \beta} \) (UIR of dS group), we define following homomorphisms between them.

There are different definitions. Here we consider two cases.

#### 4.1 Simplest case

The simplest homomorphism can be defined as:

\[
F_{\alpha \beta \gamma} = \bar{z}_\alpha K_{\beta \gamma} - \bar{z}_\beta K_{\alpha \gamma},
\]

(4.1)
where \( \bar{z}_\alpha = \theta_{\alpha \beta} z^\beta \) and \( z^\beta \) is a constant vector field. Replacing the equation (4.1) in the field equation (3.10) and after some calculation, we obtain

\[
Q_0(Q_0 - 2)\{12(x \cdot z)K_{\beta \gamma} + 4(z \cdot \bar{\partial})K_{\beta \gamma} - 5\bar{z}_\beta \bar{\partial} \cdot K_{\gamma} - 4x_\beta z. \cdot K_{\gamma} - \bar{z}_\gamma \bar{\partial} \cdot K_{\beta}
\]

\[
-3x_\gamma (x \cdot z)\bar{\partial} \cdot K_{\beta} - x_\gamma (z \cdot \bar{\partial})\bar{\partial} \cdot K_{\beta} + x_\beta x_\gamma z. \bar{\partial} \cdot K + x_\gamma \bar{z}_\beta \bar{\partial} \cdot \bar{\partial} \cdot K \} = 0.
\]

(4.2)
By imposing the traceless and divergenceless conditions which are necessary for associating \( K_{\alpha \beta} \) with the UIR of the dS group, we get

\[
7[Q_0]^2 K_{\beta \gamma} - 46Q_0 K_{\beta \gamma} - 64K_{\beta \gamma} = 0, \text{ or } (Q_0 - 2)(7Q_0 - 32)K_{\beta \gamma} = 0.
\]

(4.3)
Clearly this equation is not compatible with equation (2.12) of the de Sitter linear gravity. In other words this homomorphism cannot lead to any UIR of dS group. Now we consider another possibility.
4.2 Second case

Now we try the following definition, which is deduced from the field strength tensor of electromagnetic potential:

\[ F_{\alpha\beta\gamma} \equiv \left( \partial_{\alpha} + x_{\alpha} \right) K_{\beta\gamma} - \left( \partial_{\beta} + x_{\beta} \right) K_{\alpha\gamma}. \]  

(4.4)

By substituting (4.4) into the (3.10), we find

\[ Q_0(Q_0 - 2) \left[ 4(Q_0 - 2) K_{\beta\gamma} - Q_0 x_{\beta} \partial \cdot K_{\beta} + 3 \partial_{\beta} \partial \cdot K_{\gamma} + 7 x_{\beta} \partial \cdot K_{\gamma} - x_{\gamma} \partial_{\beta} \partial \cdot K_{\beta} 
- \partial_{\gamma} \partial \cdot K_{\beta} - 2 x_{\beta} x_{\gamma} \partial \cdot K - x_{\gamma} \partial_{\beta} \partial \cdot K \right] = 0. \]  

(4.5)

Similarly by imposing the traceless and divergenceless conditions \((K' = 0 = \partial \cdot K)\), we obtain

\[ (Q_0 - 2)^2 Q_0 K_{\beta\gamma} = 0, \quad \text{or equivalently,} \quad (Q_2 + 4)^2(Q_2 + 6) K_{\alpha\beta} = 0. \]  

(4.6)

It is clear that this CI field corresponds to the two representations of discrete series, namely \(\Pi_{\pm 2,1}^\pm\) (twofold) and \(\Pi_{\pm 2,2}^\pm\). The representations \(\Pi_{\pm 2,2}^\pm\) of the discrete series have a Minkowskian interpretation and have a unique extension to a direct sum of two UIRs \(C(3; 2, 0)\) and \(C(-3; 2, 0)\) of the conformal group with positive and negative energies, respectively [2, 17]. \(\Pi_{\pm 2,2}^\pm\) restricts to the massless UIR \(P^>(0, 2)\) \((\mathcal{P}<(0, 2))\) of the Poincaré group with positive (negative) energy. Similar statements hold for \(\Pi_{\pm 2,2}^-\) with negative helicity (namely \(\mathcal{P}^<(0, -2))\). Moreover, equation (4.6) can be written in the intrinsic coordinates as [4, 15]:

\[ \left( \Box^3 + 8 \Box^2 + 24 \Box + 48 \right) h_{\mu\nu} = 0, \]  

(4.7)

and in the metric signature \((- + + +)\), we have:

\[ \left( \Box^3 - 8 \Box^2 + 24 \Box - 48 \right) h_{\mu\nu} = 0. \]  

(4.8)

Therefore if one insists \(K_{\alpha\beta}\) or equivalently \(h_{\mu\nu}\) to transform according to the UIR of dS and conformal groups it must satisfy a field equation of order 6.

4.3 Fierz representation

The spin-2 field can be described in two ways, which are called the Einstein frame and the Fierz frame representations. The most common one, the Einstein frame, uses a symmetric tensor of rank-2, \(K\) to represent the field. In the Fierz frame this role is played by a mixed symmetry tensor of rank-3, \(F\). Such an object has 20 independent components. It has been shown that it must obey a further condition in order to represent only one single spin-2 field, otherwise it represents two spin-2 fields [8, 9].

Now we try the Fierz representation. Transformation of the Fierz representation from intrinsic coordinate to the ambient space results in [4, 15]:

\[ F_{\alpha\beta\gamma} \equiv \left( \partial_{\alpha} + x_{\alpha} \right) K_{\beta\gamma} - \left( \partial_{\beta} + x_{\beta} \right) K_{\alpha\gamma} 
+ \theta_{\beta\gamma} \left( \partial_{\alpha} K - \partial \cdot K_{\alpha} - x_{\alpha} K \right) - \theta_{\alpha\gamma} \left( \partial_{\beta} K - \partial \cdot K_{\beta} - x_{\beta} K \right). \]  

(4.9)
It is interesting to note that by imposing the traceless and divergenceless conditions, which are necessary for associating with UIRs, Fierz representation reduces to the previous case, and one can associate the same UIR of the dS group. The difference between these two cases is that the latter is an isomorphism between the mixed symmetry rank-3 tensor field $F$ and symmetric rank-2 tensor field $K$.

5 Conclusion

It was pointed out that Einstein’s theory of gravitation, in the background field method, $g_{\mu\nu} = g_{\mu\nu}^{BG} + h_{\mu\nu}$, can be considered as a theory of massless symmetric tensor field of rank-2 on a fixed background, such as dS space. Massless fields propagate on the light cone and then their equations must be CI. Contrary to Maxwell equation, Einstein’s equation of gravitation, as well as equation of $h_{\mu\nu}$, is not conformally invariant.

In our previous paper [4] we used a symmetric rank-2 tensor field $\Psi_{ab}$ and Dirac’s six-cone formalism to obtain CI field equation for $\mathcal{K}_{\alpha\beta}$ in dS space. Although the equation was CI, it did not transform according to the UIRs of the conformal group i.e. it was not physical state of this group. Binegar et al [1] have shown that mixed symmetry tensor field of rank-3 transforms according to UIRs of the conformal group. In this paper, by definition homomorphisms between this tensor field and $\mathcal{K}_{\alpha\beta}$, we obtained a CI equation which can be interpreted as the UIR of the conformal and dS groups. It has been shown that if we want $\mathcal{K}_{\alpha\beta}$ to be a physical state of the dS and conformal groups simultaneously, it must satisfy a field equation of order 6. So, conformal gravity seems to be like a $R^3$ gravity theory. As a future work, it may be possible to find a CI gravitational field which in its linear approximation gives this linear physical equation.

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