Properties of Spin and Orbital Angular Momenta of Light

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This paper analyses the algebraic and physical properties of the spin and orbital angular momenta of light in the quantum mechanical framework. The consequences of the fact that these are not angular momenta in the quantum mechanical sense are worked out in mathematical detail. It turns out that the spin part of the angular momentum has continuous eigenvalues. Particular attention is given to the paraxial limit, and to the definition of Laguerre Gaussian modes for photons as well as classical light fields taking full account of the polarization degree of freedom.

I. INTRODUCTION

There has been great interest for some time now in the angular momentum properties of the Maxwell field [1], in particular its proposed separation into what have been called spin and orbital angular momentum of light [2]. In addition to many theoretical investigations [3–21] extensive experimental work [22–28] has also been devoted to understand these concepts.

In an earlier work [29] a unified framework for studying these novel properties of light, in both classical and quantum domains, has been presented. In particular, the fact that the spin and orbital parts of the total angular momentum are not truly quantum mechanical angular momenta at all has been emphasized.

The aim of the present paper is to carry this study further and in particular to analyse in full detail the quantum mechanical properties of the spin angular momentum of light at the one photon level. It is seen that the eigenvalues and eigenvectors of the spin angular momentum are very different from those of any true angular momentum as understood in quantum mechanics. The essential roles of polarization and transversality of light are brought out, and new vectorial Laguerre–Gauss fields including polarization in the paraxial regime are constructed.

The contents of this paper are organised as follows. Section II reviews the formulation of the free Maxwell equations in a particularly economical form using the complex transverse analytic signal vector potential. The seven basic constants of motion following from Poincaré invariance are expressed in terms of analytic signal vector potential and electric field. The spin and orbital angular momenta, SAM and OAM, which are also real constants of motion, are identified. The description of a general solution of the Maxwell equations using a complex transverse vector function on wave vector space, and a natural Lorentz invariant Hilbert space made up of such functions, is outlined. Canonical quantisation is recalled, and the operator forms of the seven hermitian constants of motion, as well as of the SAM and OAM, are listed.

The rest of this paper deals essentially with one photon states. In Section III some of the properties of the SAM and OAM operators are worked out. The connections to the helicity operator, the component of the total angular momentum in the momentum direction, are obtained and its properties are described. Helicity is a well-defined concept in terms of the generators of the Poincaré group. The fact that the SAM components are commutative, and that along with the total angular momentum they generate a Euclidean group, is brought out. The helicity operator is seen to be invariant under this Euclidean group. For later comparison, the discrete set of complete orthonormal eigenfunctions of total angular momentum are recorded. Section IV solves completely the problem of eigenvalues and eigenfunctions for the SAM along with helicity. It is emphasised that these are ideal non normalisable eigenfunctions, as the eigenvalues of the SAM components are continuous. The contrast with the total angular momentum eigenfunctions is explicitly seen. To emphasize this aspect, the properties of SAM in a normalised simultaneous eigenfunction of the helicity and the third component of total angular momentum are worked out. It is shown that such an eigenfunction can never be an eigenfunction of the third component of SAM as well; the SAM components have a nontrivial variance matrix in such a state. Section V is devoted to
an analysis of the paraxial regime. It is recalled that it is appropriate to perform canonical quantisation before considering the paraxial limit. The approximate nature of this limit, and the correspondingly approximate consequence of transversality, are both clearly brought out. These considerations, combined with the paraxial limit of the general simultaneous eigen functions of helicity and third component of total angular momentum, lead to the development of Laguerre Gaussian mode functions for the vector Maxwell field. The helicity eigenvalue of plus or minus $\hbar$ appears as a third label added to the two that enumerate the modes in the scalar optical case. Section VI is devoted to Concluding Remarks.

II. CONSTANTS OF MOTION AND QUANTIZATION OF THE FREE MAXWELL FIELD

We begin with the classical free Maxwell equations written in terms of the complex positive frequency analytic signal vector potential $\mathbf{A}^{(+)}(x)$ where $x \equiv (x, t)$. The basic (first order) equation of motion (EOM) is

$$i \frac{\partial}{\partial t} \mathbf{A}^{(+)}(x, t) = (\hat{\omega} \mathbf{A}^{(+)})(x, t),$$

$$\hat{\omega} = c(-\nabla^2)^{1/2}. \quad (2.1)$$

This is consistent with the transversality constraint

$$\nabla \cdot \mathbf{A}^{(+)}(x, t) = 0. \quad (2.2)$$

The initial data is specified by $\mathbf{A}^{(+)}(x, 0)$. The analytic signal electric and magnetic fields can be regarded as derived quantities at each instant of time:

$$\mathbf{E}^{(+)}(x) = \frac{i}{c}(\hat{\omega} \mathbf{A}^{(+)})(x), \quad \mathbf{B}^{(+)}(x) = \nabla \times \mathbf{A}^{(+)}(x). \quad (2.3)$$

They are also transverse and obey first order EOM similar to $\mathbf{A}^{(+)}$ in (2.1). For convenience we will use both $\mathbf{A}^{(+)}$ and $\mathbf{E}^{(+)}$ in various important expressions.

From the relativistic invariance of the Maxwell equations we obtain seven constants of motion (COM) which have no explicit time dependence – momentum $P$, energy $P^0$, and total angular momentum $\mathbf{J}$ (all real):

$$P_j = \frac{1}{2\pi c} \int d^3x \mathbf{E}^{(+)}(x)^* \cdot \partial_j \mathbf{A}^{(+)}(x),$$

$$P^0 = \frac{1}{2\pi} \int d^3x \mathbf{E}^{(+)}(x)^* \cdot \partial^0 \mathbf{A}^{(+)}(x),$$

$$J_j = \frac{1}{2\pi c} \int d^3x \mathbf{E}^{(+)}(x)^* (\delta_{mn}(x \wedge \nabla)_j + \epsilon_{jmn}) A_n^{(+)}(x). \quad (2.4)$$

Here $(\partial^0 \equiv \partial / \partial x_0, x_0 = -x^0 = -ct)$. The two terms in the total angular momentum $\mathbf{J}$ are identified as the orbital angular momentum (OAM) and spin angular momentum (SAM) respectively of the free field, and both are real COM’s:

$$L_j = \frac{1}{2\pi c} \int d^3x \mathbf{E}^{(+)}(x)^* (x \wedge \nabla)_j A_m^{(+)}(x),$$

$$S_j = \frac{1}{2\pi c} \int d^3x \epsilon_{jmn} E_m^{(+)}(x)^* A_n^{(+)}(x). \quad (2.5)$$

These will be studied in detail in the sequel.

The general solution of (2.1) and (2.2) can be written in terms of a complex transverse function $\mathbf{v}(\mathbf{k})$ of the real wave vector $\mathbf{k} \in \mathbb{R}^3$:

$$\mathbf{A}^{(+)}(x, t) = \frac{c}{2\pi} \int \frac{d^3k}{\sqrt{\omega}} e^{ik \cdot x} \mathbf{v}(\mathbf{k}),$$

$$\mathbf{E}^{(+)}(x, t) = \frac{i}{2\pi} \int \frac{d^3k}{\sqrt{\omega}} e^{ik \cdot x} \mathbf{v}(\mathbf{k}),$$

$$\mathbf{k} \cdot \mathbf{v}(\mathbf{k}) = 0, \quad \omega = ck = c|\mathbf{k}|, \quad \mathbf{k} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{x} - \omega t. \quad (2.6)$$

Thus the most general free Maxwell field is given equally well by $\mathbf{A}^{(+)}(x)$ or $\mathbf{v}(\mathbf{k})$. The seven COM’s (2.4) can be expressed in terms of $\mathbf{v}(\mathbf{k})$:

$$P_j = \int d^3k \ k_j \mathbf{v}(\mathbf{k})^* \cdot \mathbf{v}(\mathbf{k}), \quad P^0 = \int d^3k \ \omega \mathbf{v}(\mathbf{k})^* \cdot \mathbf{v}(\mathbf{k})$$

$$J_j = \int d^3k \ \mathbf{v}_m(\mathbf{k})^* (-i \delta_{mn}(\mathbf{k} \wedge \nabla)_j - i\epsilon_{jmn}) \mathbf{v}_n(\mathbf{k}),$$

$$\tilde{\delta}_j = \frac{\partial}{\partial k_j}. \quad (2.7)$$

The OAM and SAM are

$$L_j = -i \int d^3k \ \mathbf{v}_m(\mathbf{k})^* (\mathbf{k} \wedge \nabla)_j \mathbf{v}_n(\mathbf{k}),$$

$$S_j = -i \int d^3k \ \mathbf{v}_m(\mathbf{k})^* \epsilon_{jmn} \mathbf{v}_n(\mathbf{k}). \quad (2.8)$$

At the classical level we define a Hilbert space $\mathcal{H}$ by using a metric in the space of amplitudes $\mathbf{v}(\mathbf{k})$:

$$\mathcal{M} = \{ \mathbf{v}(\mathbf{k}) \mid \mathbf{k} \cdot \mathbf{v}(\mathbf{k}) = 0, \quad ||\mathbf{v}||^2 = \int d^3k \ \mathbf{v}(\mathbf{k})^* \cdot \mathbf{v}(\mathbf{k}) < \infty \}. \quad (2.9)$$

The norm $||\mathbf{v}||$ is Lorentz invariant. The space $\mathcal{M}$ will play an important role after quantization to which we now turn.

The process of canonical quantization involves replacing the classical amplitudes $\mathbf{v}(\mathbf{k}), \mathbf{v}(\mathbf{k})^*$ by vectorial operators $\sqrt{\omega} \mathbf{a}(\mathbf{k}), \sqrt{\omega} \mathbf{a}(\mathbf{k})^*$ obeying the canonical commutation relations (CCR) on a suitable Hilbert space $\mathcal{H}$:

$$[\mathbf{a}_j(\mathbf{k}), \mathbf{a}_l(\mathbf{k}')^\dagger] = \left( \delta_{jl} - \frac{k_jk_l}{|\mathbf{k}|^2} \right) \delta^{(3)}(\mathbf{k} - \mathbf{k}'),$$

$$[\mathbf{a}, \mathbf{a}^\dagger] = [\mathbf{a}^\dagger, \mathbf{a}] = 0,$n

$$\mathbf{k} \cdot \mathbf{a}(\mathbf{k}) = \mathbf{k} \cdot \mathbf{a}(\mathbf{k})^\dagger = 0. \quad (2.10)$$

The field operators are

$$\hat{\mathbf{A}}^{(+)}(x) = \frac{c}{2\pi} \sqrt{\hbar} \int \frac{d^3k}{\sqrt{\omega}} e^{ik \cdot x} \hat{\mathbf{a}}(\mathbf{k}),$$

$$\hat{\mathbf{E}}^{(+)}(x) = \frac{i}{2\pi} \sqrt{\hbar} \int \frac{d^3k}{\sqrt{\omega}} e^{ik \cdot x} \hat{\mathbf{a}}(\mathbf{k}). \quad (2.11)$$
The operator forms of the classical COM’s are the hermitian operators

\[ \hat{P}^\mu = \int d^3k \ h \omega \ \hat{a}(k) \cdot \hat{a}(k); \]
\[ \hat{P}_j = \int d^3k \ h k_j \ \hat{a}(k) \cdot \hat{a}(k); \]
\[ \hat{J}_j = -i \hbar \int d^3k \ \delta_m(k) (\delta_m(k \wedge \nabla)_j + \epsilon_{jmn}) \ \hat{a}_n(k); \] (a)
\[ \hat{L}_j = -i \hbar \int d^3k \ \delta_m(k) (k \wedge \nabla)_j \hat{a}_m(k), \]
\[ \hat{S}_j = -i \hbar \int d^3k \ \delta_m(k) \epsilon_{jmn} \hat{a}_n(k). \]

(2.12)

The commutation relations among the former are determined by the Poincaré group structure:

\[ [\hat{P}^\mu, \hat{P}^\nu] = 0; \]
\[ [\hat{J}, \hat{P}^\mu] = 0; \]
\[ [\hat{J}, \hat{P}_j] = i \hbar \epsilon_{jlm} \hat{P}^l; \]
\[ [\hat{J}, \hat{J}_j] = i \hbar \epsilon_{jlm} \hat{J}_l. \] (2.13)

We will examine the important operator properties of the OAM and SAM, \( \hat{L}_j \) and \( \hat{S}_j \), in the next Section.

The Hilbert space \( \mathcal{H} \) on which the CCR’s (2.10) are realized irreducibly is the direct sum of subspaces \( \mathcal{H}_n, n = 0, 1, 2, \ldots \), made up of states with definite total photon number \( n \). Thus \( \mathcal{H}_0 \) is the one dimensional subspace of no photon states (multiples of the vacuum state \( |0 \rangle \)); \( \mathcal{H}_1 \) is the subspace of single photon states; and so on. The importance of the classical Hilbert state \( \mathcal{M} \), Eq. (2.9), is that there is a one to one correspondence \( \mathcal{M} \leftrightarrow \mathcal{H}_1 \), given by the following structure:

\[ \mathbf{v}(k) \in \mathcal{M}, \quad |v\rangle = \hat{a}(v)\dagger |0\rangle \in \mathcal{H}_1, \]
\[ \hat{a}(v) = \frac{1}{\sqrt{\hbar}} \int d^3k \ \mathbf{v}(k) \cdot \hat{a}(k), \]
\[ \hat{a}(v)^\dagger = \frac{1}{\sqrt{\hbar}} \int d^3k \ \mathbf{v}(k) \cdot \hat{a}(k)^\dagger; \]
\[ [\hat{v}(v), \hat{v}'(v')] = \frac{(v, v')}{\hbar} \epsilon; \]
\[ \hat{a}_j(k)|v\rangle = \frac{1}{\sqrt{\hbar}} v_j(k) |0\rangle. \] (2.14)

The inner products among one photon states in \( \mathcal{H}_1 \) are essentially the classical inner products in \( \mathcal{M} \):

\[ \langle v' | v \rangle = (v', v) / \hbar \]

(2.15)

III. OPERATOR PROPERTIES OF TOTAL, ORBITAL AND SPIN ANGULAR MOMENTUM OF PHOTONS

We now take up a detailed analysis of the operators \( \hat{L}, \hat{S} \) representing the OAM and SAM of the quantized Maxwell field respectively. For our purposes it suffices to restrict these (and other) operators to one-photon states in \( \mathcal{H}_1 \). Their actions on a one-photon wavefunction \( v(k) \) can be expressed in a succinct manner. For \( \hat{P}^\mu \) and \( \hat{J} \) we have:

\[ (\hat{P}^\mu)(v)_j(k) = h \omega v_j(k), \quad (\hat{P}_j)(v)(k) = \hbar k_j v_j(k), \]
\[ (\hat{J}_j)(v)(k) = -i \hbar \left( (k \wedge \nabla)_j v_j(k) + \epsilon_{jmn} v_n(k) \right). \] (3.1)

For \( \hat{L} \) and \( \hat{S} \) we find:

\[ (\hat{L}_j)(v)(k) = -i \hbar \left( (k \wedge \nabla)_j v_j(k) + \frac{k_j}{|k|^2} (k \wedge v(k))_l \right), \]
\[ (\hat{S}_j)(v)(k) = i \hbar \frac{k_j}{|k|^2} (k \wedge v(k)). \] (3.2)

Two operator relations follow easily:

\[ \hat{P} \cdot \hat{L} = 0, \quad \hat{P} \wedge \hat{S} = 0. \] (3.3)

The helicity operator \( \hat{W} \) is defined in terms of Poincaré group generators as

\[ \hat{W} = \frac{\hat{P} \cdot \hat{J}}{\sqrt{\hat{P} \cdot \hat{P}}}. \] (3.4)

With (3.3) this simplifies to

\[ \hat{W} = \frac{\hat{P} \cdot \hat{S}}{\sqrt{\hat{P} \cdot \hat{P}}}. \] (3.5)

We next easily find some operator product relations:

\[ \hat{J} \cdot \hat{S} = \hat{S} \cdot \hat{S} = \hat{W}^2 = \hbar^2. \] (3.6)

Therefore we also have

\[ \hat{L} \cdot \hat{S} = 0. \] (3.7)

Turning to commutators, while Eqs. (2.13) are part of the Poincaré Lie algebra, we now find these additional ones:

\[ [\hat{J}_1, \hat{W}] = 0; \]
\[ [\hat{J}_1, \hat{L}_m \text{ or } \hat{S}_m] = i \hbar \epsilon_{imn} (\hat{L}_n \text{ or } \hat{S}_n); \]
\[ [\hat{S}_1, \hat{P}_m \text{ or } \hat{S}_m \text{ or } \hat{W}] = 0, \quad [\hat{L}_1, \hat{W}] = 0. \] (3.8)

As expected, \( \hat{W} \) is a rotational scalar while \( \hat{L} \) and \( \hat{S} \) are vectors. The six hermitian operators \( \hat{J} \) and \( \hat{S} \), all having the dimensions of action, realise the Lie algebra of a Euclidean group \( E(3) \). This is distinct from the Euclidean subgroup \( E(3) \) of the Poincaré group, generated by \( \hat{J} \) and \( \hat{P} \).

The result for \( \hat{W}^2 \) in Eq. (3.6) seems counterintuitive, since all \( \hat{P} \) and \( \hat{S} \) commute pairwise and all have continuous eigenvalues. The reason of course is the result \( \hat{P} \wedge \hat{S} = 0 \).

The operators \( \hat{J} \) constitute a quantum mechanical angular momentum. Thus the eigenvalues of \( \hat{J} \cdot \hat{J} \) and \( \hat{J}_3 \) are \( l(l + 1)\hbar^2 \) and \( \hbar m \) respectively, for \( l = 1, 2, \ldots \), and \( m = l, l-1, \ldots, -l \) for photons. As is well known, their
simultaneous eigenfunctions form a complete orthonormal basis for transverse vector functions of the unit wave vector $\hat{k} \in S^2$ [30, 31]:

$$(\hat{J}_z, \hat{S}_z) Y_{lm}^{(a)}(\hat{k}) = \{\hbar^2 l(l+1), m\hbar\} Y_{lm}^{(a)}(\hat{k}), \quad a = 1, 2;$$

$$Y_{lm}^{(1)}(\hat{k}) = \frac{1}{\sqrt{l(l+1)}} (-i \hat{k} \wedge \nabla) Y_{lm}(\hat{k}),$$

$$Y_{lm}^{(2)}(\hat{k}) = \hat{k} \wedge Y_{lm}^{(1)}(\hat{k});$$

$$\int_{S^2} d\Omega(\hat{k}) Y_{lm,m'}^{(a)}(\hat{k})^* \cdot Y_{lm}^{(a)}(\hat{k}) = \delta_{a',a} \delta_{l',l} \delta_{m',m};$$

$$\sum_{a=1}^{2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} Y_{lm,j}^{(a)}(\hat{k}) Y_{lm,j'}^{(a)}(\hat{k}')^* = \delta^{(2)}(\hat{k}, \hat{k}') (\delta_{jj'} - \frac{k_j k_{j'}}{|k|^2}).$$

(3.9)

Here $Y_{lm}(\hat{k})$ are the usual spherical harmonics and $\delta^{(2)}(\hat{k}, \hat{k}')$ is the two dimensional surface Dirac delta function over $S^2$.

As we will see, since the $\hat{S}$ are not an angular momentum, their eigenvalues and eigenvectors have very different characters.

IV. SPIN AND HELICITY EIGENFUNCTIONS, VARIANCE MATRIX FOR SPIN

Now we consider the eigenvalues and eigenvectors of the SAM $\hat{S}$. Since the four operators $\hat{S}_y, \hat{W}$ commute pairwise, they can all be simultaneously diagonalized. As the $\hat{S}_y$ transform as a three dimensional vector under spatial rotations, we see from Eqs. (3.6) that the possible eigenvalues for $\hat{S}$ and $\hat{W}$ have the forms

$$\hat{S} \rightarrow \hbar s, \quad \hat{W} \rightarrow \hbar w, \quad s \in S^2, w = \pm 1.$$  

(4.1)

It follows that while $\hat{W}$ possesses normalizable eigenvectors, for eigenvectors of $\hat{S}$ we must use delta function normalization on $S^2$ (cf Eq. (3.9)).

Based on the actions given in Eqs. (3.1),(3.2), we can easily construct the corresponding (ideal) eigenvectors in $H_1$. To handle $\hat{W}$, we need to choose, for each $\hat{k} \in S^2$, a pair of transverse mutually orthogonal circular polarization vectors $\epsilon^{(\pm)}(\hat{k})$. In terms of the spherical polar angles $\theta, \varphi$ of $\hat{k} \in S^2$, their definitions and important properties are as follows (with $C$ for cos and $S$ for sin):

$$\epsilon^{(\pm)}(\hat{k}) = \frac{e^{i\varphi}}{\sqrt{2}} (C\theta C\varphi - i S\varphi, C\theta S\varphi + i C\varphi, -S\theta),$$

$$\epsilon^{(-)}(\hat{k}) = \epsilon^{(\pm)}(\hat{k})^* = \frac{e^{-i\varphi}}{\sqrt{2}} (C\theta C\varphi + i S\varphi, C\theta S\varphi - i C\varphi, -S\theta);$$

$$\hat{k} \cdot \epsilon^{(a)}(\hat{k}) = 0, \quad a = \pm; \quad \epsilon^{(a)}(\hat{k}) \cdot \epsilon^{(b)}(\hat{k}) = \delta_{a,b};$$

$$\hat{k} \wedge \epsilon^{(a)}(\hat{k}) = -i a \epsilon^{(a)}(\hat{k}); \quad \epsilon^{(\pm)}(\hat{k}) \wedge \epsilon^{(-)}(\hat{k}) = \hat{k}.$$  

(4.2)

As is well known, transverse circular polarization vectors defined smoothly all over $S^2$ do not exist [32-34]. The above choices are well defined at $\theta = 0$ but multivalued at $\theta = \pi$. Their behaviours under parity are useful, and read:

$$\epsilon^{(a)}(\hat{k}) = i a e^{i a \varphi} \epsilon^{(-a)}(\hat{k}), \quad a = \pm,$$  

(4.3)

so

$$\hat{k} \wedge \epsilon^{(a)}(\hat{k}) = i a \epsilon^{(a)}(-\hat{k}).$$  

(4.4)

After some straightforward analysis, the (ideal) simultaneous eigenvectors of $\hat{S}_y, \hat{W}$ can be found up to arbitrary ‘radial’ functions:

$$\langle s, w | s, w \rangle = \int_{S^2} d\Omega(\hat{k}) \delta^{(2)}(\hat{k}, \hat{k}') \delta^{(2)}(\hat{k}, \hat{k}) \epsilon^{(\pm)}(\hat{k}), \quad \text{any } a(k, s, w).$$  

(4.5)

The inner products have the form expected from orthonormality:

$$\langle s', w' | s, w \rangle = \int d^3 k (|s', w') \rangle (|s, w \rangle)^*$$

$$= \delta_{w, w'} \delta^{(2)}(s', s) \int_0^{\infty} k^2 dk a'(k, s, w) a(k, s, w).$$  

(4.6)

As for the completeness property, we omit the factor $a(k, s, w)$ in Eq. (4.5) and find for the angular part:

$$\sum_{w=\pm 1} \int_{S^2} d\Omega(\hat{s}) \left( \delta^{(2)}(\hat{k}, \hat{k}) \epsilon^{(\pm)}(\hat{s}) \right)^* \left( \delta^{(2)}(\hat{k}', \hat{k}) \epsilon^{(\pm)}(\hat{s}) \right)$$

$$= \delta^{(2)}(\hat{k}, \hat{k}') \left( \delta_{j, j'} - \frac{k_j k_{j'}}{|k|^2} \right).$$  

(4.7)

This is to be compared to the last line in (3.9) : while the right hand sides are the same, the left hand sides have very different structures, due to the differences between $\hat{J}$ and $\hat{S}$.

The fact that $\hat{S}$ has continuous eigenvalues (not at all like a quantum mechanical angular momentum), hence no normalisable eigenvectors, has important consequences. We illustrate this by examining the properties of $\hat{S}$ in a normalized simultaneous eigenvector of $\hat{J}_3$ and $\hat{W}$. This has the general form :

$$\hat{J}_3 \rightarrow \hbar m, \quad \hat{W} \rightarrow \hbar w;$$

$$\langle v_{m,w} | \epsilon^{(m-w)\varphi}(\hat{k}) \rangle = \langle v_{m,w} | \epsilon^{(m)\varphi}(\hat{k}) \rangle e^{i(m-w)\varphi};$$

$$\langle v_{m,w} | v_{m,w} \rangle = 2 \pi \int_0^{\infty} k^2 dk \int_0^{\pi} \sin \theta d\theta |a(k, m, w, \theta)|^2 = 1.$$  

(4.8)

Here $a(k, m, w, \theta)$ is arbitrary. Let us now define an associated probability distribution $p(x)$ over $[-1, 1]$ in the
polar angle $\theta$ with $x = \cos \theta$, as follows:

$$p(x) = 2\pi \int_0^\infty k^2 dk |a(k, m, w, \theta)|^2 \geq 0,$$

$$\int_{-1}^1 dx \, p(x) = 1.$$  \hspace{1cm} (4.9)

The normalisation condition (4.8) implies that $p(x)$ is not of delta function type, so it describes a non trivial spread and variance in $x$. Then using Eqs. (3.2) and (4.8) we find the expectation values of the SAM:

$$\langle \mathbf{v}_{m,w} | \hat{S}_I | \mathbf{v}_{m,w} \rangle = \hbar \int_0^\infty k^2 dk \int_{S^2} d\Omega(k) |a(k, m, w, \theta)|^2 \hat{k}_I$$

$$= \hbar \langle x \rangle_0 \delta_{0,3},$$

$$\langle f(x) \rangle_0 = \int_{-1}^1 dx f(x) p(x).$$  \hspace{1cm} (4.10)

Going a step further, we can obtain the expectation values of quadratics in the ‘spin’ components as a $3 \times 3$ matrix:

$$\langle \mathbf{v}_{m,w} | \hat{S}_I \hat{S}_I | \mathbf{v}_{m,w} \rangle = \hbar^2 \left( \int_0^\infty k^2 dk \int_{S^2} d\Omega(k) |a(k, m, w, \theta)|^2 \hat{k}_I \hat{k}_I \right)$$

$$= \hbar^2 \text{diag} \left( \frac{1}{2} \langle (1-x^2) \rangle_0, \frac{1}{2} \langle (1-x^2) \rangle_0, \langle x \rangle_0 \right).$$  \hspace{1cm} (4.11)

Therefore the SAM variance matrix in the normalized state $|\mathbf{v}_{m,w}\rangle$ is, using (4.10),

$$V = \hbar^2 \text{diag} \left( \frac{1}{2} \langle (1-x^2) \rangle_0, \frac{1}{2} \langle (1-x^2) \rangle_0, \langle (\Delta x)^2 \rangle_0 \right),$$

$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle_0 - \langle x \rangle_0^2.$$  \hspace{1cm} (4.12)

From the statements made above regarding the nature of the probability distribution $p(x)$, it is clear that the spread $\langle (\Delta x)^2 \rangle$ in $\hat{S}_I$ is strictly positive, $\langle (\Delta x)^2 \rangle > 0$. So in any normalized state $|\mathbf{v}_{m,w}\rangle$ with well defined $\hat{J}_3$ and $\hat{W}$, there is always a spread in the values of the components of $\hat{S}$. In particular even though both $\hat{J}_3$ and $\hat{W}$ commute with $\hat{S}_I$, the normalised eigenvector $|\mathbf{v}_{m,w}\rangle$ of $\hat{J}_3$ and $\hat{W}$ cannot be a simultaneous eigenvector of $\hat{S}_I$ as well, whatever be the choice of $a(k, m, w, \theta)$. By the same token, the state $|\mathbf{v}_{m,w}\rangle$ can never be an eigenvector of the third component $\hat{L}_3$ of OAM, for any choice of $a(k, m, w, \theta)$.

V. PARAXIAL REGIME AND VECTOR LAGUERRE-GAUSS MODES

In the previous Sections we have discussed on the one hand the exact simultaneous eigenfunctions of the total squared angular momentum $\hat{J}^2$ and its component $\hat{J}_3$, and on the other hand those of the three components of the SAM $\hat{S}$ and the helicity $\hat{W}$. These are collected together in Eqs. (3.9) and Eqs. (4.5), (4.6) respectively. In both cases, only angular and polarization dependences are involved. In the general $\hat{S}, \hat{W}$ eigenfunction in Eq. (4.5) for example, an arbitrary, ‘radial’ function $a(k, s, w)$ appears. Similarly in the general simultaneous eigenvector of $\hat{J}_3, \hat{W}$ in Eq. (4.8) an arbitrary function $a(k, m, w, \theta)$ is present.

Now we turn to the physically very important paraxial regime. As argued in earlier work [29], it is reasonable to consider the paraxial limit after canonical quantization has been completed and the photon picture of light has been obtained. Thus once Eqs. (2.10) and their consequences and interpretation are in hand, in the subsequent analysis based on Eqs. (2.14) we limit the choices of $\mathbf{v}(k) \in \mathcal{M}$ to those having the paraxial property. That is, the paraxial approximation is made on the choice of $\mathbf{v}(k)$ within $\hat{a}(\mathbf{x})$ and $\hat{a}(\mathbf{v})$, not in the canonical quantization rule $\mathbf{v}(k) \to \sqrt{\hbar} \hat{a}(k), \mathbf{v}(k)^* \to \sqrt{\hbar} \hat{a}(k)^*$ in any sense. ‘Paraxial photons’ are to be understood in this way.

The paraxial region in wave vector space is defined (in an approximate way) as consisting of those $k$ vectors whose transverse components $k_\perp$ are much smaller than their (positive) longitudinal components:

$$|k_\perp| << k, \quad k_3 \approx k - k_\perp^2/2k.$$  \hspace{1cm} (5.1)

A photon wave function $\mathbf{v}(k)$ is paraxial if it is negligible outside the paraxial region:

$$\mathbf{v}(k) \approx 0$$ unless $k$ paraxial.  \hspace{1cm} (5.2)

In that case, transversality determines $v_3(k)$ in terms of $v_\perp(k)$:

$$v_3(k_\perp,k) \approx - \left( 1 + \frac{k_\perp^2}{2k^2} \right) \frac{k_\perp \cdot v_\perp(k_\perp,k)}{k}. $$  \hspace{1cm} (5.3)

The longitudinal component is one order of magnitude smaller than the transverse components.

One way in which the paraxial property for $\mathbf{v}(k)$ can be achieved is if each component $v_j(k)$ is a common transverse Gaussian factor times a polynomial in $k_\perp$. This requires that there be a transverse width $w_0$ and some minimum wave vector magnitude $k_{\text{min}} > 0$, and

$$\mathbf{v}(k_\perp,k) = \left( \frac{a_\perp(k_\perp,k)}{c(k_\perp,k)} \right) e^{-w_0^2 k_\perp^4/4},$$

$$w_0 >> \lambda_{\text{max}} = 2\pi/k_{\text{min}},$$

$$c(k_\perp,k) \approx - \left( 1 + \frac{k_\perp^2}{2k^2} \right) \frac{k_\perp \cdot a_\perp(k_\perp,k)}{k},$$  \hspace{1cm} (5.4)

with $a_\perp$ and $c$ polynomial in $k_\perp$.

We can now connect with the exact $\hat{J}_3, \hat{W}$ eigenfunctions in Eq. (4.8), and their paraxial limits, in this way. For given eigenvalues, $hm, hw$ of $\hat{J}_3, \hat{W}$ the eigenfunction in Eq. (4.8) contains the arbitrary function $a(k, m, w, \theta)$ as a factor. To make this eigenfunction paraxial means
to impose suitable conditions on this free function. The paraxial (small $\theta$) limits of $\epsilon^{(\pm)}(k)$ are:

$$
\epsilon^{(+)}(k) \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{i}{\sin \theta e^{i\phi}} \\ -\frac{i}{\cos \theta e^{-i\phi}} & 1 \end{pmatrix}; \quad \epsilon^{(-)}(k) \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i\theta e^{-i\phi} & -1 \end{pmatrix}.
$$

(5.5)

In scalar paraxial optics the important family of Laguerre–Gaussian (LG) mode functions have the general structure of (5.4)–polynomials times a Gaussian factor in transverse variables. These are defined using cylindrical coordinates, so we have the connection:

$$
k = k(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) = (\rho \cos \varphi, \rho \sin \varphi, k_3); \quad \rho = k \sin \theta, \quad k_3 = k \cos \theta, \quad k_3^2 = \rho^2 + k_3^2.
$$

(5.6)

For small $\theta$, we have $\rho \simeq k \theta$, $k_3 \simeq k - \rho^2/2k$. The LG mode functions are labelled by two integers: $p = 0, 1, 2, \cdots, m = 0, \pm 1, \pm 2, \cdots$; and they are

$$
\phi_{m,p}(k_\perp) = \frac{w_0}{\sqrt{2\pi}} \frac{p!}{(p + |m|)!} e^{im\varphi} \left(\frac{iw_0 \rho}{\sqrt{2}}\right)^{|m|} L_p^{|m|}\left(\frac{w_0^2 \rho^2}{2}\right) e^{-w_0^2 \rho^2/4}.
$$

(5.7)

Comparing Eq. (5.4) with Eqs. (4.8),(5.5),(5.7) we are led for each given $m$ to two choices:

$$
w = 1 \quad a(k, m, +1, \theta) \rightarrow \phi_{m-1,p}(k_\perp); \quad \nu_{m,+1,p}(k_\perp, k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{i}{\theta e^{i\phi}} \\ -\frac{i}{\cos \theta e^{-i\phi}} & 1 \end{pmatrix} \phi_{m-1,p}(k_\perp); \quad (a)
$$

$$
w = -1 \quad a(k, m, -1, \theta) \rightarrow \phi_{m+1,p}(k_\perp);
\nu_{m,-1,p}(k_\perp, k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{i}{\theta e^{i\phi}} \\ -\frac{i}{\cos \theta e^{-i\phi}} & 1 \end{pmatrix} \phi_{m+1,p}(k_\perp). \quad (b)
$$

(5.8)

To leading paraxial order, these are the complete–transverse vector LG mode fields. We stress that these are eigenfunctions of the total angular momentum component $\hat{J}_3$ and helicity $\hat{W}$ with respective eigenvalues $hm, \pm \hbar$. In addition to the labels $m, p$ in Eq. (5.7) in the scalar case, now the third helicity label $w = \pm$ also appears.

VI. CONCLUDING REMARKS

We have presented a careful analysis of the properties of the so-called spin and orbital angular momenta of light, in the quantum domain, as they apply to single photon states. It has been known for some time that these operators, which are hermitian constants of motion, do not have the spectral properties expected of an angular momentum in the sense of quantum mechanics. Thus the photon spin is not such an angular momentum. Its components do not have discrete quantised eigenvalues. It is a result of transversality of the Maxwell field that there is no position operator for the photon, therefore no way of separating the total angular momentum into well defined and independent spin and orbital parts. The terms ‘spin’ and ‘orbital’ angular momenta of light are thus misnomers which however cannot now be corrected.

We show by explicit construction that there exist ideal (non normalisable) eigenvectors for all three spin components simultaneously. One can, of course, construct normalisable wave packets out of these eigenvectors, involving small patches over the sphere $S^2$. At the classical level it is an interesting challenge to produce wave fields corresponding to such solutions of the Maxwell equations. The helicity and the three spin components do possess simultaneous ideal eigenvectors, with their eigenvalues being chosen independently. However a normalised eigenvector of a component of the total angular momentum and helicity can never be an eigenvector of that component of the spin as well.

We recall that a noteworthy feature of the formalism developed in [29] and briefly recapitulated here is the one to one correspondence between classical radiation field configurations and the quantum description thereof at the single photon level. This leads one to expect that some of the results arising from the peculiar features of the ‘spin’ and ‘orbital’ angular momentum operators at the one photon level, as discussed here ought to have measurable signatures at the classical level as well. 

Finally we draw attention to the paraxial vectorial Laguerre-Gauss fields which are a physically relevant and nontrivial generalisation of the enormously useful scalar paraxial mode fields of the same name. It is an experimental challenge to create such fields, and to bring out their characteristic signatures.

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