Spontaneous Breaking of Flavor Symmetry and Parity in the Nambu-Jona-Lasinio Model with Wilson Fermions

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Abstract

We study the lattice Nambu-Jona-Lasinio model with two flavors of Wilson fermions in the large $N$ limit, where $N$ is the number of ‘colors’. For large values of the four-fermion coupling we find a phase in which both, flavor symmetry and parity, are spontaneously broken. In accordance with general expectations there are three massless pions on the phase boundary, but only two of them remain massless inside the broken phase. This is analogous to earlier results obtained in lattice QCD, indicating that this behavior is a very general feature of the Wilson term.
1. INTRODUCTION

Recently, Bitar and Vranas\[1\] presented an extensive study of the lattice Nambu-Jona-Lasinio model. This model, although interesting in its own right as an effective low-energy theory of the strong interactions, can serve as an ideal testing ground for the properties of the Wilson term. This is because the model can be easily studied in the large $N$ limit, where in this case $N$ refers to the number of colors. In this letter we are concerned primarily with the nature of the symmetry breaking induced by the Wilson term. We will show that in the large $N$ limit there exists a phase in which both parity and flavor symmetry are spontaneously broken. It is the existence of this phase transition that is responsible for the occurrence of massless pions in the model despite the fact that chiral symmetry is explicitly broken by the Wilson term. The same phenomenon is also responsible for the masslessness of the pion in lattice QCD\[3\]. The existence of this phase was missed by Bitar and Vranas. Nevertheless, most of our results in the symmetric phase are in agreement with theirs.

The model under consideration is defined through the action

\[
S = \sum_{x,y} \sum_{i=1}^{N} \left\{ \bar{\psi}^i(x) \left[ M(x,y) + M^t(x,y) \right] \psi^i(y) + \frac{2\beta}{N} [\sigma^2(x) + \pi^2(x)] \delta(x,y) \right\},
\]

where $\psi^i(x)$ has also two flavor components in addition to the indices shown. The combination $\beta/N$ will be referred to as $\tilde{\beta}$ below. The matrix $M(x,y)$ is defined by

\[
M(x,y) = \frac{1}{2} \sum_{\mu} \left[ (\gamma_\mu - r) \delta(x + \mu, y) - (\gamma_\mu + r) \delta(x - \mu, y) \right]
+ \delta(x, y) \left[ 4r + m + \sigma(x) + i\gamma_5 \vec{\pi}(x) \cdot \vec{r} \right].
\]

In the above action auxiliary fields $\pi$ and $\sigma$ have been introduced to decouple the four fermion interaction\[4\]. The terms proportional to the parameter $r$ come from the Wilson term. In the following we will set $r = 1$. The Wilson term is of order $O(a)$ ($a$ is the lattice spacing) in the naive continuum limit but nevertheless it has a pronounced effect on the theory. It gives the fermionic doublers masses on the order of the momentum space cutoff, $O(q/a)$, and in QCD it also serves to produce the correct anomaly of the flavor singlet axial current in the continuum limit\[2\]. Note that the Wilson term explicitly breaks chiral symmetry.

As it is well known, the Nambu-Jona-Lasinio model in the absence of the Wilson term breaks chiral symmetry spontaneously at a critical value $\beta_c$. Above $\beta_c$, the theory is chirally symmetric for $m = 0$, and $<\bar{\psi}^i\psi^i>$ serves as an order parameter for the transition. With the Wilson term in place however, $<\bar{\psi}^i\psi^i>$ is always non-zero and chiral symmetry is explicitly broken. Massless pions, if they exist in the model, therefore cannot considered to be the Goldstone bosons of broken chiral symmetry. We will show below that massless pions do exist and that they must considered to be Goldstone bosons of spontaneously broken flavor symmetry. In addition to the Goldstone pions there is one remaining pion which is massless only on the phase boundary and is associated with the spontaneous breaking of a discrete space-time symmetry, parity. In general, when there are $n_f$ flavors, there are $n_f^2 - 2$
Goldstone bosons and one mode which is massless only on the phase boundary. The flavor singlet pseudo-scalar meson is always massive.

In Sec. 2 we derive the formulas for the condensate and the masses in the large $N$ limit. In Sec. 3 we present the results of our calculation, and finally Sec. 4 contains a short discussion of our results.

2. LARGE $N$ APPROXIMATION

We start by integrating out the fermion fields in Eq. (1) obtaining the following effective action for the auxiliary fields:

$$ S = -\frac{N}{2} \left[ Tr(\log M) + Tr(\log M^\dagger) \right] + 2N\tilde{\beta} \sum_x \left[ \sigma^2(x) + \pi^2(x) \right], $$

where the trace extends over space, color, flavor and spin degrees of freedom. Since the action is proportional to $N$ we can proceed in the large $N$ limit by evaluating the functional integral around the stationary point of the action. We write

$$ \sigma(x) \sim \sigma_s + \frac{\delta \sigma(x)}{\sqrt{N}}, \quad \bar{\pi}(x) \sim \pi_s \hat{e}_3 + \frac{\delta \bar{\pi}(x)}{\sqrt{N}}, \quad N \to \infty, $$

where we have accounted for fluctuations around the translationally invariant saddle point. Note, that we have allowed for the possibility that the isovector $\bar{\pi}$ might develop a non-zero condensate which we have arbitrarily chosen to point in the 3-direction in isospin space. Such a condensate of course breaks both, parity and flavor symmetry.

In momentum space, the inverse propagator at the saddle point is given by

$$ \hat{M}_s(p) = i \sum_\mu \gamma_\mu \sin p_\mu + (4 - \sum_\mu \cos p_\mu) + m + \sigma_s + i\gamma_5 \tau_3 \pi_s. $$

The effective potential evaluated at the saddle point is

$$ V_{eff} = -2Nn_f \int \frac{d^4 p}{(2\pi)^4} \log[g(p)] + Nn_f \tilde{\beta}(\sigma^2_s + \pi^2_s) $$

where the function $g(p)$ is defined as

$$ g(p) = \sum_\mu \sin^2 p_\mu + \pi^2_s + [w(p) + m_q]^2. $$

Following Bitar and Vranas[1] we define the constituent quark mass $m_q = m + \sigma_s$ and the function $w(p) = 4 - \sum_\mu \cos p_\mu$. Demanding that the linear term in the fluctuations around the saddle point vanish, leads to the two gap equations

$$ 0 = \frac{\sigma_s}{2} - \int \frac{d^4 p}{(2\pi)^4} \frac{\sigma_s + m + w(p)}{g(p)}, $$

$$ 0 = \frac{\pi_s}{2} - \int \frac{d^4 p}{(2\pi)^4} \frac{\pi_s}{g(p)}. $$

2
Obviously, $\pi_s = 0$ is always a solution to the last equation, but as it turns out it is not always the one that minimizes the effective potential.

The inverse propagators are obtained from the quadratic fluctuations and are given by

$$G^{-1}_\sigma(k) = \frac{\tilde{\beta}}{2} - \int \frac{d^4p}{(2\pi)^4} \sum_\mu \sin(p_\mu + \frac{k_\mu}{2}) \sin(p_\mu - \frac{k_\mu}{2}) + \pi_s^2 - [w(p + \frac{k}{2}) + m_q][w(p - \frac{k}{2}) + m_q] \frac{g(p + \frac{k}{2})g(p - \frac{k}{2})}{g(p + \frac{k}{2})g(p - \frac{k}{2})},$$

and

$$G^{-1}_\pi^a(k) = 2\pi_s^2 \delta^{a,3} \int \frac{d^4p}{(2\pi)^4} \frac{1}{g(p + \frac{k}{2})g(p - \frac{k}{2})} + \frac{\tilde{\beta}}{2} - \int \frac{d^4p}{(2\pi)^4} \sum_\mu \sin(p_\mu + \frac{k_\mu}{2}) \sin(p_\mu - \frac{k_\mu}{2}) + \pi_s^2 + [w(p + \frac{k}{2}) + m_q][w(p - \frac{k}{2}) + m_q] \frac{g(p + \frac{k}{2})g(p - \frac{k}{2})}{g(p + \frac{k}{2})g(p - \frac{k}{2})}.$$  

In the last equation we have used that the pion propagator is diagonal in flavor space. Note, that $G^{-1}_\pi^a(0) = 0$ for $a = 1, 2$ due to Eq. (9) when $\pi_s \neq 0$.

It is very difficult to calculate the mass, $m$, for the pseudo-scalar meson from the complex zeros of the inverse propagator in Eq. (10). To obtain the mass of the pions, which is expected to be small close to the phase boundary, we can define the pion wave function renormalization constant $Z_{\pi^a}$ and pion mass $m_{\pi^a}$ such that

$$\lim_{k \to 0} G^{-1}_{\pi^a}(k) = Z_{\pi^a}^{-1} \left( k^2 + m_{\pi^a}^2 \right).$$

Unless $Z_{\pi^a}$ becomes infinite, the pion masses can be computed from

$$m_{\pi^a}^2 = \lim_{k \to 0} \frac{\frac{\partial}{\partial k^2} G^{-1}_{\pi^a}(k)}{G^{-1}_{\pi^a}(k)}.$$  

Note that since parity is broken in the phase in which $\pi_s \neq 0$, the mixed $\pi - \sigma$ propagator is nonvanishing. Physical states in this phase are not eigenstates of parity and are obtained by diagonalizing the mass matrix.

### 3. RESULTS

To make our results comparable with those presented in [1], we choose $N = 2$ so that $\beta = 2\tilde{\beta}$ in the following. We solved the gap equations (8-9) numerically on a $10^4$-lattice using a simple Newton procedure with the bare parameters ranging form $0 < \beta \leq 2.5$ and $-2 \leq m \leq -6$, each in steps of 0.01. We compared the results of the calculation where $\pi_s$ is allowed to take a non-zero solution to those results that are obtained for $\pi_s$ set to zero. We found throughout that $\pi \neq 0$ usually minimizes the effective potential in Eq. (6), if such a solution exists. Some care must be taken in this calculation due to threefold solutions of the gap equation for $\sigma$ when $\pi_s = 0$ and $\beta < 0.75$. In all cases we determined the solutions
for \( \pi_s \) and \( \sigma_s \) that minimize the effective potential. The results for \( \sigma_s \) and \( \pi_s \) from this calculations are plotted in Figs. 1 and 2.

In Fig. 3, the continuous line represents the phase boundary between the regions where \( \pi_s = 0 \) and the region where parity-flavor symmetry is broken and \( \pi_s \neq 0 \). Note that the region with \( \pi_s \neq 0 \) disappears for \( \beta > 1.41 \). We have extended the calculation for \( m = -4 \) far into the weak-coupling regime where \( \beta \) is large but there do not appear to be any more non-zero solutions for \( \pi_s \). Along the dashed line in Fig. 3, the mass \( m_q \) of the constituent quark vanishes. The intersection of the line where \( m_q = 0 \) with the phase boundary at \( \beta \approx 0.33 \) and \( m \approx -2.7 \) is the continuum chiral limit and corresponds reasonably well with the prediction of [1].

Close to the phase boundary the pion masses are expected to be small. Thus, they can be calculated using Eq. (13). We find that the masses for all three pions are equal on the side of the phase boundary where \( \pi_s = 0 \) [as can be seen directly from Eq. (11)]. As one approaches the phase boundary we find that \( m_\pi^2 \sim m - m_c \). Although this is of course the behavior expected of a theory which breaks chiral symmetry spontaneously, the reader should understand that here it is simply a consequence of the fact that the critical exponent of the flavor-parity breaking transition has its mean field value at large \( N \). In the broken phase, the masses of the \( \pi^{1,2} \) remain zero, they are the Goldstone modes corresponding to the two unbroken generators of flavor symmetry. The mass of \( \pi^3 \) on the other hand is zero only at the critical point. It is to be considered as the inverse correlation length of an Ising like parity breaking transition. This behavior of the pion masses is summarized in Fig. 4. As we mentioned before, we were not able to obtain any useful results for the sigma mass. Its expected behavior is such that it is nonvanishing throughout the phase diagram [3].

Finally, we would like to mention that we were unable to find massless pions for values of \( \beta > 1.41 \). This seems to contradict what was found in [1] (see in particular Fig. 8 of this reference). On the basis of our interpretation of the phase structure of the model one does not expect a massless particle in that region since there are no phase transitions there.

4. CONCLUSION

We have analyzed the phase structure of the Nambu-Jona-Lasinio model on the lattice with Wilson fermions in the large \( N \) limit. We found that in analogy to lattice QCD with Wilson fermions there exists a phase in which flavor and parity symmetry are spontaneously broken. As opposed to what is expected to happen in lattice QCD, the region of broken symmetry is a simply connected, compact region, symmetric around \( m = -4 \). There appears to be no interesting phase structure at weak coupling. Since a complicated phase structure has been found at weak coupling of the lattice Gross-Neveu model (an asymptotically free four-fermi model in 2 dimensions) with Wilson fermions [4], the absence of such an interesting phase structure at weak coupling may be related to the fact that the Nambu-Jona-Lasinio model is non-asymptotic free. We feel that the most interesting aspect of our results is that one is able to obtain both parity and flavor violation in a vector-like theory in a well defined
approximation scheme. Due to the presence of the Wilson term, well known theorems\cite{5} about the absence of such breaking do not apply. We hope that the results presented here will help to make it easier to accept the parity-flavor breaking scenario for lattice QCD as well. Also, in some of the recent proposals for putting chiral fermions on the lattice, one is forced to work in a region where $m < m_c$ (the broken region in our language)\cite{6}. A good understanding of the nature of this phase is therefore very important.

ACKNOWLEDGEMENTS

This manuscript was authored under Contract No. DE-AC02-76-CH00016 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.
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FIGURE CAPTIONS

FIGURE 1: $\pi_s$ as function of the bare parameters $m$ and $\beta$, obtained from the gap equations in (8-9). The value of $\pi_s$ is non-zero only in a near-circular region (see Fig. 3) between $0 < \beta < 1.41$ and $-5.5 < m < -2.5$ that is symmetric with respect to the axis where $m = -4$. Note that for small $\beta$, $\pi_s$ appears to develop a singularity similar to $\exp\{-(4 + m)^2 / \beta\} / \sqrt{\beta}$.

FIGURE 2: $\sigma_s$ as function of the bare parameters $m$ and $\beta$, obtained from the gap equations in (8-9). The value of $\sigma_s$ is antisymmetric with respect to the axis where $m = -4$, and passes continuously through zero on this axis for all values of $\beta > 0$.

FIGURE 3: Phase diagram for the Nambu-Jona-Lasinio model in Eq. (1) with spontaneously broken parity-flavor symmetry. $\pi_s$ is non-zero only in the near-circular region that is symmetric with respect to the dotted line where $m = -4$. Everywhere outside of that region $\pi_s = 0$, and the theory maintains the symmetry. Along the dashed line the quark mass $m_q$ vanishes. The intersection of the line where $m_q = 0$ with the phase boundary where $m_\pi = 0$ is the continuum chiral limit of the lattice theory.

FIGURE 4: Pion masses near to the phase boundary for some generic $\beta = 0.58$. For $m > m_c = -2.804$, all pion masses are equal and non-zero. The pion masses vanish for $m = m_c$, and, while $m_{\pi^2}$ becomes finite, the masses for $\pi^{(1,2)}$ remain zero in the phase where parity-flavor symmetry is broken and $m < m_c$. 
Figure 1
