Cooperative Control of Linear Multi-Agent Systems via Distributed Output Regulation and Transient Synchronization

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Abstract—A wide range of multi-agent coordination problems including reference tracking and disturbance rejection requirements can be formulated as a cooperative output regulation problem. The general framework captures typical problems such as output synchronization, leader-follower synchronization, and many more. In the present paper, we propose a novel distributed regulator for groups of identical and non-identical linear agents. We consider global external signals affecting all agents and local external signals affecting only individual agents in the group. Both signal types may contain references and disturbances. Our main contribution is a novel coupling among the agents based on their transient state components or estimates thereof in the output feedback case. This coupling achieves transient synchronization in order to improve the cooperative behavior of the group in transient phases and guarantee a desired decay rate of the synchronization error. This leads to a cooperative reaction of the group on local disturbances acting on individual agents. The effectiveness of the proposed distributed regulator is illustrated by a vehicle platooning example and a coordination example for a group of four non-identical 3-DoF helicopter models.

Index Terms—Cooperative Control; Distributed control; Multi-agent systems; Regulator theory; Linear output feedback; Synchronization.

I. INTRODUCTION

In a variety of modern man-made systems, it is desirable to synthesize a cooperative behavior among individual dynamical agents, similarly to bird flocks and fish schools observed in nature. Examples include multi-vehicle coordination and formation flight problems, robot cooperation in production lines, power balancing in micro-grids, and many more. Of particular interest are distributed control laws which require only local information exchange between neighboring subsystems and no centralized data collection or processing entity. The main advantages of distributed control algorithms are scalability for large networks of dynamical systems, flexibility with respect to addition and removal of subsystems, and robustness with respect to failure of individual subsystems. The following overview of selected publications on distributed control methods for multi-agent systems serves as introduction and motivation for the methods developed in the present paper.

A. Related Work

A fundamental cooperative control problem is the consensus or synchronization problem for groups of linear dynamical agents. The consensus problem has been studied extensively over the past decade, starting from single-integrator agents [1], [2], to double-integrator agents [3], to identical general linear agents [4]–[8], and to non-identical general linear agents [9]–[12]. All these studies focus on consensus and synchronization of the states or outputs of the autonomous agents in the group. Under the distributed control law, the closed-loop system as a whole is an autonomous system, i.e., it has no external inputs signals.

From a practical point of view, it is desirable to influence the behavior of the group via external reference signals. A solution to this problem is the leader-follower setup [7], [13]–[15]. The idea is to select a particular agent as leader for the group or introduce a virtual leader and design the distributed control law such that all agents synchronize to this leader. The motion of the group is then controlled through the motion of the active leader.

Moreover, it is important to consider external disturbances acting on the multi-agent system, to analyze the performance of the closed-loop system, and to incorporate disturbance rejection or attenuation requirements in the design procedure. Rejection of constant disturbances is addressed in [16]–[18]. Disturbance attenuation and $\mathcal{H}_\infty$ performance criteria are addressed in [19]–[21].

Reference tracking and disturbance rejection problems can be formulated in the general framework of output regulation which was developed in the 1970s, [22], [23]. The basic setup consists of a so-called exosystem, an autonomous system that generates all external signals (references as well as disturbances) acting on the plant, and a description of the plant. The signal generated by the exosystem is referred to as the generalized disturbance. The tracking and regulation requirements are formulated in terms of a regulation error depending on the plant state and the external signals. The objective is to find a control law, also called regulator, which ensures internal stability of the plant and asymptotic convergence to zero of the regulation error for all initial conditions. For the details on the output regulation theory, the reader is referred to the books [24], [25], and [26].

It has been proposed in [27], [28] to formulate multi-agent coordination problems with external reference and disturbance signals as a synchronized output regulation problem. Since then, a cooperative output regulation theory for a large class of practically relevant cooperative control problems is under development. The problem setup in [27], [28] consists of an autonomous exosystem and a group of identical linear agents, which are affected by the signal generated by the exosystem, and a tracking error for each agent which shall converge to zero. The problem becomes a cooperative control problem since, by assumption, not all agents have access to the external process.
signal. In particular, the group is divided into a group of informed agents which are able to reconstruct the external signal, and uninformed agents which are dependent on information exchange with informed agents in order to solve their regulation task. The proposed distributed regulator consists of local feedback laws designed according to the classical output regulation theory and a distributed estimator for the external signal. It has been shown in [29] that the cooperative output regulation problem generalizes existing solutions for the leader-follower problem. In [30], agents with non-identical and uncertain dynamics are considered. The main limitation of the proposed solution is that the underlying communication graph is assumed to have no loops. This assumption is relaxed in [31]. Instead, the follower agents are assumed to have identical nominal dynamics. The papers [32] and [33] extend the results of [27], [28] and present a solution to the cooperative output regulation problem with non-identical agents based on state feedback and output feedback, respectively. Each agent is described by a generalized plant in which all matrices may be different for different agents. Cooperation among the agents is again required since the uninformed agents are not able to reconstruct the external signal locally. The solution proposed in [33] consists of three components: local feedback laws which are constructed based on the classical output regulation theory for each agent; local observers for the state of each agent; and a distributed observer for the generalized disturbance. Further developments in this area focus on robust cooperative output regulation for uncertain agent dynamics [34], [35], [36]. All studies mentioned above consider a single autonomous exosystem generating all reference and disturbance signals acting on the group. In [37], each agent has an additional local exosystem that generates local reference signals. However, only reference signals and no disturbance signals are considered. Multiple exosystems are also considered in [38].

First, we introduce one global exosystem generating signals affecting all agents and a local exosystem for each agent generating local references and disturbances. This exosystem structure leads to distributed estimators of lower dimension compared to previous works, where only one single exosystem (of possibly high dimension) is considered which generates all external signals acting on the group.

Second, we formulate the overall cooperative control problem as a single centralized overall output regulation problem. The solvability conditions for the overall output regulation problem and its particular structure allow us to derive necessary and sufficient solvability conditions for the distributed regulation problem.

Third, we present a distributed regulator based on output feedback which solves the cooperative output regulation problem. Our main contribution is to introduce a novel coupling in the distributed regulator based on the transient state components of the agents. We present a design method for these couplings which allows to impose performance specifications such as a minimum decay rate of the synchronization error of the group. The novel distributed regulator ensures a cooperative reaction of the group on external disturbances.

Moreover, a detailed numerical example of four 3-DoF helicopters with non-identical model parameters is presented in order to illustrate the design procedure for groups of non-identical agents and demonstrate the performance of the novel distributed regulator with transient synchronization.

As an auxiliary result, we present a design method based on LMIs for the coupling gain in networks of identical linear agents such that the poles corresponding to the synchronization error are placed within a specified region. This procedure is the key to incorporate performance specifications such as a minimum decay rate in the cooperative control design.

C. Outline

Section II contains mathematical preliminaries. Section III presents the problem setup and some auxiliary results. The distributed regulator for general non-identical linear agents is presented in Section IV. In Section V, an extension of the distributed regulator is derived which guarantees exponential stability of the synchronization error with a desired decay rate. The derivation is based on the assumption that the agents have identical dynamics. A vehicle platooning example illustrates the results. In Section VI, the assumption of identical agent dynamics is relaxed. It is shown how the coupling can be designed in case of non-identical agents based on robust control methods and the procedure is demonstrated in an example with four non-identical 3-DoF helicopters. Section VII concludes the paper.

II. Preliminaries

The sets of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. The open left half plane, imaginary axis, and open right half plane of \( \mathbb{C} \) are denoted by \( \mathbb{C}_- \), \( \mathbb{C}_0 \), and \( \mathbb{C}_+ \), respectively. For \( z \in \mathbb{C} \), \( \bar{z} \) is the complex conjugate, \( \text{Re}(z) \) is the real part and \( \text{Im}(z) \) is the imaginary part. The spectrum of a matrix \( A \in \mathbb{C}^{n \times n} \) is denoted by \( \sigma(A) \subset \mathbb{C} \) and \( A \) is called stable.
if $\sigma(A) \subset \mathbb{C}^-$, i.e., all its eigenvalues have negative real part. If $A$ is real and $\sigma(A) \subset \mathbb{C}^-$, it is also called Hurwitz. $\text{diag}(M_k)$ and $\text{stack}(M_k)$ denote a block diagonal matrix and a vertical stack of matrices with blocks $M_k$, $k = 1, \ldots, N$, respectively. For a set of vectors $v_k \in \mathbb{R}^n$, $k = 1, \ldots, N$, $v \in \mathbb{R}^{Nn}$ denotes the stack vector $v = [v_1^T \cdots v_N^T]^T$. For a vector $v \in \mathbb{R}^n$, $\text{diag}(v)$ is a diagonal matrix with the entries of $v$ on the diagonal. The identity matrix of dimension $N$ is $I_N$ and the vector of ones is $1$.

For a transfer function matrix $G$, $\|G\|_\infty$ denotes its $\mathcal{H}_\infty$ norm. The symbol $\otimes$ denotes the Kronecker product.

III. Problem Setup

A. Agent Models

The dynamics of the non-identical agents are described by linear state-space models. The index set of the agents is defined as $\mathcal{N} = \{1, \ldots, N\}$, where $N$ is the number of agents in the group. The dynamics of the undisturbed agents are described by

$$\dot{x}_k = A_k x_k + B_k u_k$$

where $x_k (t) \in \mathbb{R}^d_k$ is the state and $u_k (t) \in \mathbb{R}^d_k$ is the control input of agent $k \in \mathcal{N}$. The coordination problem is formulated in terms of the generalized plant

$$\dot{x}_k = A_k x_k + B_k u_k + B_k^d d^x + B_k^d d^x_k$$

$$y_k = C_k x_k + D_k u_k + D_k^d d^x + D_k^d d^x_k$$

$$e_k = C_k^+ x_k + D_k^+ u_k + D_k^+ d^x + D_k^+ d^x_k$$

where $y_k (t) \in \mathbb{R}^{d_k}$ is the measurement output of agent $k$ and $d^x(k) \in \mathbb{R}^{d^x_k}$, $d^x_j(k) \in \mathbb{R}^{d^x_j_k}$ are external signals specified next. The regulation error $e_k (t) \in \mathbb{R}^{d_k}$ is defined such that asymptotic tracking and disturbance rejection is equivalent to $e_k (t) \to 0$ as $t \to \infty$ for all initial conditions.

B. External Signals

We consider two types of external input signals which affect the group: a global signal that affects all agents and local signals that affect individual agents in the group. Each of these signals represents a generalized disturbance which may consist of reference signals and disturbances. While the problem formulation captures this general case, we may think of the global signal as a pure reference signal and include all disturbances into the local signals. We assume that all signals belong to a known family of signals. The global signal $d^x(t) \in \mathbb{R}^{d^x}$ is a solution of the autonomous linear system

$$d^x = S^x d^x$$

where $\sigma(S^x) \subset \mathbb{C}^0 \cup \mathbb{C}^+$. The local signal $d^x_j(k) \in \mathbb{R}^{d^x_j_k}$ acting on agent $k$ is a solution of the autonomous linear system

$$d^x_j(k) = S^x_j d^x_j(k)$$

where $\sigma(S^x_j) \subset \mathbb{C}^0 \cup \mathbb{C}^+$ for all $k \in \mathcal{N}$.

C. Information Structure

All agents have communication capabilities. The communication topology is described by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, with node set $\mathcal{N} = \{1, \ldots, N\}$ corresponding to the agent index set and edge set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$. A directed edge $(j, k) \in \mathcal{E}$ corresponds to possible information flow from agent $j \in \mathcal{N}$ to agent $k \in \mathcal{N}$. We think of these edges as communication channels. Hence, each agent can transmit local measurements or controller states. There is no forwarding, in the sense that each agent exchanges information only with its direct neighbors and not with its two-hop or multi-hop neighbors. The neighbor set of agent $k$ is the subset $\mathcal{N}_k \subset \mathcal{N}$ from which $k$ can receive information, i.e., $\mathcal{N}_k = \{j \in \mathcal{N} : (j, k) \in \mathcal{E}\}$. We call a directed graph $\mathcal{G}$ connected if it contains a directed spanning tree (which is sometimes called quasi strongly connected in the literature), and strongly connected if there is a directed path from every node to every other node, i.e., every node is the root of a spanning tree. The adjacency matrix $A_{\mathcal{G}}$ of $\mathcal{G}$ is defined element-wise by $a_{ij} = 1 \Leftrightarrow (j, k) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. In case $\mathcal{G}$ is connected, its Laplacian matrix $L_{\mathcal{G}} = \text{diag}(A_{\mathcal{G}}) - A_{\mathcal{G}}$ has exactly one zero eigenvalue $\lambda_1 (L_{\mathcal{G}}) = 0$ and all other eigenvalues $\lambda_k (L_{\mathcal{G}}), k \in \mathcal{N} \setminus \{1\}$, have positive real parts $\Re (\lambda_k (L_{\mathcal{G}})) > 0$, cf. [2]. In the following, the eigenvalues of $L_{\mathcal{G}}$ are shortly denoted by $\lambda_k, k \in \mathcal{N}$, and the numbering is always chosen such that $\lambda_1 = 0$. For further details on graph theory, see [2], [8], [9] and the references therein.

D. The Distributed Output Regulation Problem

The group objective under consideration consists of two parts: asymptotic tracking of reference signals and asymptotic disturbance rejection. Typically, we are interested in synchronization problems which can be formulated as tracking problem of a common, global reference signal. Moreover, our focus will be on the cooperative behavior of the group in transient phases. The cooperative control problem is formally stated as follows.

Problem 1. For each $k \in \mathcal{N}$, find a distributed regulator

$$\dot{z}_k = A_{kk}^+ z_k + B_{kk}^+ y_k + \sum_{j \in \mathcal{N}_k} \left( A_{kj}^+ z_j + B_{kj}^+ y_j \right)$$

$$u_k = C_{kk}^+ z_k + D_{kk}^+ y_k + \sum_{j \in \mathcal{N}_k} \left( C_{kj}^+ z_j + D_{kj}^+ y_j \right)$$

such that the following two conditions are satisfied:

P1) If $d^x(0) = 0$ and $d^y_j(0) = 0$, $k \in \mathcal{N}$, then, for all initial conditions $x_k(0) = x_{k,0}$ and $z_k(0) = z_{k,0}$,

$$\lim_{t \to \infty} x_k(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} z_k(t) = 0.$$

P2) For all initial conditions $d^x(0) = d^x_0$, $d^y_j(0) = d^y_{j,0}$, $x_k(0) = x_{k,0}$, and $z_k(0) = z_{k,0}$, $k \in \mathcal{N}$,

$$\lim_{t \to \infty} e_k(t) = 0.$$

Problem 1 for the generalized plants (2) is a very general problem formulation that captures many particular practically relevant problems such as reference tracking and disturbance...
rejection, cf., [24], [26]. Due to the distributed structure of the controller (5), this problem formulation captures distributed cooperative control problems.

\textbf{E. The Overall Output Regulation Problem}

Problem 1 is a distributed output regulation problem due to the structure imposed on the regulator (5). In case of a complete graph, i.e., all-to-all communication, (5) becomes a centralized dynamic output feedback controller of the form

\begin{align}
\dot{z} &= A'z + B' y \\
u &= C'z + D'y,
\end{align}

(6a) (6b)

where \( \dot{z}, y, u \) are the stack vectors of \( z_k, y_k, u_k, k \in \mathbb{N}, \) respectively. Let \( x, e \) be the stack vectors of \( x_k, e_k, k \in \mathbb{N}, \) and \( d = [d_k^T \ d_k^T \cdots d_k^T]^T. \) Then, the overall cooperative control problem can be formulated as a single classical output regulation problem by combining all agents (2) into one large generalized plant and one large exosystem as follows:

\begin{align}
\dot{x} &= Ax + Bu + B_d d \\
y &= Cx + Du + D_d d \\
e &= C_e x + D_e u + D_{ed} d
\end{align}

(7a) (7b) (7c)

and

\[ \dot{d} = S d, \]

(8)

with the matrices given by \( A = \text{diag}(A_k), B = \text{diag}(B_k), B_d = [\text{stack}(B_{d_k^g}) \ \text{diag}(B_{d_k^g})], C = \text{diag}(C_k), D = \text{diag}(D_k), D_d = [\text{stack}(D_{d_k^g}) \ \text{diag}(D_{d_k^g})], C_e = \text{diag}(C_{e_k}), D_e = \text{diag}(D_{e_k}), D_{ed} = [\text{stack}(D_{d_{ed}}^g) \ \text{diag}(D_{d_{ed}}^g)] \) and

\[ S = \begin{bmatrix} S^g & 0 \\ 0 & \text{diag}(S_{e_k}^g) \end{bmatrix}. \]

The distributed output regulation problem can only be solved if the overall output regulation problem has a centralized solution of the form (6). Hence, we study the necessary conditions for the solvability of the overall output regulation problem and exploit the structure in order to derive necessary conditions for the local output regulation problems.

\textbf{Theorem 1} ([26, Theorem 1.14]). \textbf{Let the pair} \((A, B)\) \textbf{be stabilizable and the pair}

\[ \begin{bmatrix} A & B_d \\ 0 & S \end{bmatrix} \begin{bmatrix} \Pi & \Gamma \end{bmatrix} \]

be detectable and suppose that \( \sigma(S) \subset \mathbb{C}^0 \cup \mathbb{C}^+. \) Then, Problem 1 has a centralized solution (6), if and only if the regulator equation

\[ \begin{bmatrix} A & B_d \\ C_e & D_e \end{bmatrix} \begin{bmatrix} \Pi & 0 \\ 0 & 0 \end{bmatrix} S + \begin{bmatrix} D_d & 0 \end{bmatrix} = 0 \]

(10)

is solvable with a solution \( \Pi, \Gamma. \)

In case all conditions in Theorem 1 are fulfilled, a centralized regulator can be constructed as follows. Choose \( F \) and \( \Lambda \) such that \( A - BF \) and

\[ \begin{bmatrix} A & B_d \\ 0 & S \end{bmatrix} - \Lambda \begin{bmatrix} C & D_d \end{bmatrix} \]

are stable. Define \( G = \Gamma + F \Pi \) where \( \Pi, \Gamma \) solve (10). Then, the controller is given by

\[ \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} u + L (y - \hat{y}) \]

\[ u = [-F \ G] \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}, \]

where \( \hat{y} = \xi \hat{x} + D_d \hat{d} + D u. \) The control law \( u = -Fx + Gd \) is referred to as the full information control law. Since \( x \) and \( d \) are not directly accessible, the observer is constructed to obtain estimates \( \hat{x} \) and \( \hat{d}. \) This controller is of the form (6) with \( z = [\xi^T \ d^T]^T. \)

\textbf{Lemma 1.} The regulator equation (10) for the overall output regulation problem is solvable, if and only if the local regulator equations

\[ \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} \Pi_k & \Gamma_k \end{bmatrix} - \begin{bmatrix} \Pi_k^g & 0 \\ 0 & 0 \end{bmatrix} S^g + \begin{bmatrix} B_{d_k^g} \end{bmatrix} D_{d_k^g} = 0 \]

(11)

and

\[ \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} \Pi_k & \Gamma_k \end{bmatrix} - \begin{bmatrix} \Pi_k^e & 0 \\ 0 & 0 \end{bmatrix} S^e + \begin{bmatrix} B_{d_k^e} \end{bmatrix} D_{d_k^e} = 0 \]

(12)

are solvable with a solution \( \Pi_k^g, \Gamma_k^g, \Pi_k^e, \) and \( \Gamma_k^e \) for all \( k \in \mathbb{N}. \)

\textbf{Proof:} We prove the statement by showing that every solution \( \Pi, \Gamma \) of (10) yields a solution \( \Pi_k^g, \Gamma_k^g, \Pi_k^e, \Gamma_k^e, \) \( k \in \mathbb{N} \) of (11), (12), and vice versa. Consider (10) and partition \( \Pi, \Gamma, \) according to

\[ \Pi = \begin{bmatrix} \Pi_1 & \cdots & \Pi_N \\ \vdots & \ddots & \vdots \\ \Pi_N & \cdots & \Pi_{1N} \end{bmatrix}, \]

(13)

\[ \Gamma = \begin{bmatrix} \Gamma_1 & \cdots & \Gamma_N \\ \vdots & \ddots & \vdots \\ \Gamma_N & \cdots & \Gamma_{1N} \end{bmatrix}. \]

(14)

The first equation \( A \Pi + B \Gamma - \Pi S + B_d \) of (10) yields

\[ 0 = \begin{bmatrix} A_1 & \Pi_{11} & \cdots & \Pi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ A_N & \Pi_{N1} & \cdots & \Pi_{NN} \end{bmatrix} + \begin{bmatrix} B_1 \Gamma_1 & B_1 \Gamma_{11} & \cdots & B_1 \Gamma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ B_N \Gamma_N & B_N \Gamma_{N1} & \cdots & B_N \Gamma_{NN} \end{bmatrix} - \begin{bmatrix} \Pi_1 S_1 & \cdots & \Pi_1 S_N \\ \vdots & \ddots & \vdots \\ \Pi_N S_1 & \cdots & \Pi_N S_N \end{bmatrix} + \begin{bmatrix} B_{d_1}^g & \cdots & B_{d_1}^e \\ \vdots & \ddots & \vdots \\ B_{d_N}^g & \cdots & B_{d_N}^e \end{bmatrix} \]

(15)

Eq. (15) can be decomposed into the set of equations

\[ A_k \Pi_k^g + B_k \Gamma_k^g - \Pi_k^g S + B_{d_k}^g = 0 \]

(16)
and 
\[ A_k \Pi^f_k + B_k \Gamma^f_k - \Pi^f_k S^f_k + B^f_k = 0 \] (17)

and for \( j \neq k \), 
\[ A_k \Pi^f_{kj} + B_k \Gamma^f_{kj} - \Pi^f_k S^f_{kj} = 0. \] (18)

The second equation \( \mathcal{C}_e \Pi + \mathcal{D}_e \Gamma + \mathcal{D}_edd = 0 \) of (10) yields 
\[
0 = \begin{bmatrix}
C^\ell \Pi^g & C^\ell \Pi^g_1 & \cdots & C^\ell \Pi^g_N \\
C^\ell \Pi^g_N & C^\ell \Pi^g_1 & \cdots & C^\ell \Pi^g_N \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
D^\ell \Gamma^g_1 & D^\ell \Gamma^g_1 & \cdots & D^\ell \Gamma^g_1_N \\
\vdots & \vdots & \ddots & \vdots \\
D^\ell \Gamma^g_N & D^\ell \Gamma^g_N & \cdots & D^\ell \Gamma^g_N_N \\
D^\ell \Gamma^g_1 & D^\ell \Gamma^g_1 & \cdots & D^\ell \Gamma^g_1_N \\
\vdots & \vdots & \ddots & \vdots \\
D^\ell \Gamma^g_N & D^\ell \Gamma^g_N & \cdots & D^\ell \Gamma^g_N_N \\
\end{bmatrix}
\] (19)

Eq. (19) can be decomposed into the set of equations
\[ C^\ell \Pi^g_k + D^\ell \Gamma^g_k + D^\ell d^g = 0 \] (20)

and 
\[ C^\ell \Pi^g_{kj} + D^\ell \Gamma^g_{kj} + D^\ell d^g = 0 \] (21)

and for \( j \neq k \), 
\[ C^\ell \Pi^g_{kj} + D^\ell \Gamma^g_{kj} = 0. \] (22)

("Only if") Suppose that (13), (14) solve (10). Then, according to (16), (20) and (17), (21), the regulator equations (11) and (12) are solved by \( \Pi^g_k, \Gamma^g_k \) and \( \Pi^g_{kj}, \Gamma^g_{kj} \). ("If") Suppose that \( \Pi^g_k, \Gamma^g_k \) and \( \Pi^g_{kj}, \Gamma^g_{kj} \) solve (11) and (12). It is easy to see that (18) and (22) can be satisfied by the choice \( \Pi^g_{kj} = 0 \) and \( \Gamma^g_{kj} = 0 \) for \( k, j \in N \) with \( k \neq j \). Consequently, (13), (14) with \( \Pi^g_{kj} = 0 \) and \( \Gamma^g_{kj} = 0 \) for \( k, j \in N \), \( k \neq j \), solve (10).

\[ \text{F. List of Assumptions} \]

**Assumption 1.** The pair \((A_k, B_k)\) is stabilizable for all \( k \in N \).

**Assumption 2.** The pair
\[
\begin{bmatrix}
A_k \\
0
\end{bmatrix}
\begin{bmatrix}
B^f_k \\
S^f_k
\end{bmatrix}
= 
\begin{bmatrix}
C_k \\
D^f_k
\end{bmatrix}
\]
is detectable for all \( k \in N \).

**Assumption 3.** Agent 1 has direct access to the signal \( d^g \).

**Assumption 4.** The regulator equations
\[ A_k \Pi^g_k + B_k \Gamma^g_k - \Pi^g_k S^g + B^g_k = 0 \] (23a)

\[ C^\ell_k \Pi^g_k + D^\ell_k \Gamma^g_k + D^\ell d^g = 0 \] (23b)

and
\[ A_k \Pi^g_{kj} + B_k \Gamma^g_{kj} - \Pi^g_k S^g_{kj} + B^g_{kj} = 0 \] (24a)

\[ C^\ell_k \Pi^g_{kj} + D^\ell_k \Gamma^g_{kj} + D^\ell d^g = 0 \] (24b)

have a solution \( \Pi^g_k, \Gamma^g_k, \Pi^g_{kj}, \Gamma^g_{kj} \) for all \( k \in N \).

**Assumption 5.** The communication topology is described by a directed connected graph \( \mathcal{G} \) and node 1 is the root of a spanning tree.

In the following, we discuss each assumption separately in order to point out the importance.

**Assumption 1** is equivalent to stabilizability of the pair \((A, B)\) due to the structure of (7). Moreover, it is required for **P1** since the plant (1) may be unstable.

**Assumption 2** is necessarily satisfied if the pair (9) is detectable due to the structure of (7), (8). The assumption that (9) is detectable causes no loss of generality, as discussed in [24]. Note that detectability of (9) does not imply detectability of all pairs
\[
\begin{bmatrix}
A_k & B^f_k & B^g_k \\
S^f_k & 0 & 0 \\
S^g_k & 0 & 0
\end{bmatrix}
\begin{bmatrix}
D_k \\
D^f_k \\
D^g_k
\end{bmatrix}
\]

The pair (9) is detectable if (25) is detectable for at least one agent \( k \in N \). This agent must be the root of a spanning tree in order to solve the distributed regulation problem. The agents have to cooperate in order to obtain an estimate of \( d^g \).

**Assumption 3** along with **Assumption 2** guarantees detectability of (9). Following [33], we call agent 1 the informed agent. Note that **Assumption 3** can easily be relaxed to detectability of the pair (25) for agent \( k = 1 \). We work with the stricter assumption for ease of presentation.

**Assumption 4** is necessarily satisfied if there exists a centralized regulator solving **Problem 1**. It is well known that the solvability of the regulator equation (10) is a necessary and sufficient condition for the solvability of the output regulation problem, as stated in **Theorem 1**. Due to the structure of the overall generalized plant (7) and exosystem (8), the existence of a solution to (10) is equivalent to the existence of solutions to the local regulator equations (23), (24), as shown in **Lemma 1**. Hence, there is no loss of generality in this assumption.

**Assumption 5** is required for the construction of a distributed regulator, cf., [33].

\[ \text{G. Synchronization with Pole Placement Constraints} \]

Before we present our solution to Problem 1 in the following Section IV, we briefly discuss the state synchronization problem for a group of identical linear agents via static diffusive couplings since we will encounter two such state synchronization problems in the remainder of this paper. We are interested in a guaranteed performance of such networks in terms of a desired decay rate of the synchronization error. For this purpose, we propose a novel design procedure for suitable coupling gains by pole placement constraints.

**Lemma 2** ([8]). Consider a group of \( N \) identical linear agents \( x_k = A x_k + B u_k \), where \( x_k \in \mathbb{R}^n \). \( A \) is not Hurwitz, and \((A,B)\) is stabilizable. There exists a coupling gain matrix \( K \) such that the interconnected closed-loop system with static diffusive couplings \( u_k = K \sum_{j=1}^{N} a_{kj} (x_j - x_k) \), where \( a_{kj} \) are the entries of \( A \), satisfies \( x_k(t) - x_k(t) \to 0 \) as \( t \to \infty \) for all \( k, j \in N \) and all initial conditions, if and only if the directed graph \( \mathcal{G} \) describing the communication topology is connected.
Design procedures for a suitable coupling gain $K$ can be found in [8], [7]. These procedures always lead to a suitable $K$ if all assumptions of Lemma 2 are satisfied. Here, we present a novel design procedure based on LMIs which takes into account performance specifications in terms of pole placement constraints. For this purpose, we make use of the LMI region introduced in [39], which is defined as

$$\mathcal{R} = \{ z \in \mathbb{C} : L + zM + zM^T < 0 \},$$

for some fixed real matrices $L = L^T$ and $M$. The region $\mathcal{R}$ is a convex subset of $\mathbb{C}$.

**Theorem 2.** Suppose that all assumptions of Lemma 2 are satisfied and let $\mathcal{R} \subset \mathbb{C}^-$ be an LMI region defined by $L = L^T$ and $M$. If there exist real matrices $Y > 0$ and $Z$ such that for all $k \in \mathbb{N}\backslash\{1\}$, the LMIs

$$L \otimes Y + M \otimes (AY - \lambda_k BZ) + M^T \otimes (AY - \lambda_k BZ)^T < 0 \quad (26)$$

are satisfied, then the coupling gain $K = ZY^{-1}$ ensures state synchronization. Moreover, the poles of all modes corresponding to the synchronization error $\hat{x}''$ (defined in the proof) are contained in $\mathcal{R}$.

**Proof:** The closed-loop system consisting of $\dot{x}_k = Ax_k + Bu_k$ and $u_k = K \sum_{j=1}^{N} a_{kj} (x_j - x_k)$ is given by

$$\dot{x} = (I_N \otimes A - L \otimes BK)x,$$

We apply the state transformation $\tilde{x} = (T^{-1} \otimes I_n)x$ as introduced by [4], where the matrix $T$ is chosen such that

i) $A = T^{-1}L \otimes T$ is upper-triangular,

ii) the first column of $T$ is the vector of ones $1$,

iii) the first row of $T^{-1}$ is $p^T$, with $p \otimes L \otimes 0^T$ and $p^T 1 = 1$.

Note that $1$ is the right eigenvector and $p^T$ is the normalized left eigenvector of $L \otimes$ corresponding to the zero eigenvalue. This state transformation yields

$$\dot{\tilde{x}} = (I_N \otimes A - L \otimes BK)\tilde{x}.$$

The matrix $(I_N \otimes A - L \otimes BK)$ is upper block triangular with blocks $A - \lambda_k BK$ on the diagonal and $\lambda_1 = 0$. Let $\tilde{x}$ be partitioned into $\tilde{x}'$ and $\tilde{x}''$. Then, it follows that

$$\dot{\tilde{x}}' = A\tilde{x}' - \lambda_2 BK \tilde{x}'' + \lambda_1 BK \tilde{x}'', \quad \dot{\tilde{x}}'' = \bar{\Lambda} \tilde{x}''.$$

Let $T$ be partitioned as $T = [1 \ T'']$. Then, it is easy to see that

$$x = (T \otimes I_n)\tilde{x} = (1 \otimes I_n)\tilde{x}' + (T'' \otimes I_n)\tilde{x}''.$$ Since $1$ is linearly independent of the columns of $T''$, state synchronization, i.e., $x_j(t) - x_j(t) \to 0$ as $t \to \infty$ for all $k, j \in \mathbb{N}$, is equivalent to $\tilde{x}''(t) \to 0$ as $t \to \infty$. Consequently, the state component $\tilde{x}''$ can be seen as the synchronization error of the network. Due to the block-triangular structure of the system matrix, the problem under consideration reduces to a simultaneous stabilization problem for the $N - 1$ systems on the diagonal

$$\dot{x}_k = (A - \lambda_k BK)\tilde{x}_k$$

for $k = 2, \ldots, N$ via $K$ and with the pole placement constraint expressed by the LMI region $\mathcal{R}$. Note that the eigenvalues $\lambda_k$ of $L \otimes$ are in general complex since $\mathcal{S}$ is directed. If there exists a real symmetric matrix $Y > 0$ such that

$$L \otimes Y + M \otimes (A - \lambda_k BK)Y + M^T \otimes (A - \lambda_k BK)^T < 0, \quad (27)$$

then all poles of $A - \lambda_k BK$ are contained in $\mathcal{R}$ [39]. The change of variable $Z = KY$ leads to the LMIs (26), which proves the statement.

**Remark 1.** For real eigenvalues $\lambda_k$, the existence of a real matrix $Y > 0$ such that (27) is satisfied, is necessary and sufficient for all poles of $A - \lambda_k BK$ to be contained in $\mathcal{R}$ [39]. The requirement that $Y$ be real instead of complex Hermitian [40] may cause conservatism in the design procedure in Theorem 2. Moreover, conservatism is caused by the fact that we search for a common Lyapunov function, i.e., a common $Y$, for all $k \in \mathbb{N}\backslash\{1\}$.

Fig. 1. Two exemplary LMI regions.

**Theorem 3.** Consider a group of agents (2) with exosystems (3), (4). Suppose that Assumptions 1–5 are satisfied. Then, a distributed regulator that solves Problem 1 can be constructed as follows:

- For all $k \in \mathbb{N}$, choose $F_k$ such that $A_k - B_k F_k$ is Hurwitz.

IV. THE DISTRIBUTED REGULATOR

**Theorem 3.** Consider a group of agents (2) with exosystems (3), (4). Suppose that Assumptions 1–5 are satisfied. Then, a distributed regulator that solves Problem 1 can be constructed as follows:

- For all $k \in \mathbb{N}$, choose $F_k$ such that $A_k - B_k F_k$ is Hurwitz.
• For all $k \in \mathbb{N}$, find a solution for (23) and (24) and set
  \[ G_k^d = \Gamma_k^d + F_k \Pi_k^d \]  
  \[ G_k^g = \Gamma_k^g + F_k \Pi_k^g \]  
  (28a)  
  (28b)

• Set $\hat{d}_k^d = d_k^d$, and for all $k \in \mathbb{N} \setminus \{1\}$,
  \[ \hat{d}_k^g = S^k \hat{d}_k^g + K \sum_{j \in N_k} (d_j^g - \hat{d}_k^g), \]  
  (29)

where $K$ is chosen such that $S^k \Lambda_k K$ is stable for the non-zero eigenvalues $\lambda_k$, $k = 2, \ldots, N$, of the Laplacian $L_0$.

• For all $k \in \mathbb{N}$, choose $L_k$ such that
  \[ \begin{bmatrix} C_k & D_k^p \end{bmatrix} - L_k \begin{bmatrix} C_k & D_k^p \end{bmatrix} \]

is Hurwitz and construct the observers

\[ \begin{bmatrix} \dot{x}_k^d \\ \dot{x}_k^g \\ \dot{d}_k^d \\ \dot{d}_k^g \\ \end{bmatrix} = \begin{bmatrix} A_k & B_k^d & 0 & 0 \\ B_k^d & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k^d \\ x_k^g \\ d_k^d \\ d_k^g \end{bmatrix} + \begin{bmatrix} B_k & B_k^g \end{bmatrix} \begin{bmatrix} u_k \\ \dot{d}_k^g \end{bmatrix} + L_k (y_k - \hat{y}_k) \]

(30)

where $\hat{y}_k = C_k \hat{x}_k^d + D_k^p \hat{d}_k^g + D_k^d \hat{d}_k^d + D_k u_k$.

Finally, the control law for each agent $k \in \mathbb{N}$ is given by

\[ u_k = -F_k \hat{x}_k^d + G_k \hat{x}_k^g + G_k d_k^d. \]

(32)

**Proof:** Define the observer errors $e_k^d = x_k - \hat{x}_k$, $e_k^g = d_k - \hat{d}_k$, $e_k^{d^g} = d_k^g - \hat{d}_k^g$. The observer errors $e_k^d$ and $e_k^{d^g}$ satisfy

\[ \begin{bmatrix} \dot{e}_k^d \\ \dot{e}_k^g \\ \dot{e}_k^{d^g} \\ \end{bmatrix} = \begin{bmatrix} A_k & B_k^d & 0 & 0 \\ B_k^d & 0 & 0 & 0 \\ \end{bmatrix} \begin{bmatrix} e_k^d \\ e_k^g \\ e_k^{d^g} \end{bmatrix} + \begin{bmatrix} B_k & B_k^g \end{bmatrix} \begin{bmatrix} u_k \\ \dot{d}_k^g \end{bmatrix} + L_k (y_k - \hat{y}_k) \]

(31)

\[ \begin{bmatrix} e_k^d \\ e_k^g \\ e_k^{d^g} \end{bmatrix} = \begin{bmatrix} A_k & B_k^d & 0 & 0 \\ B_k^d & 0 & 0 & 0 \\ \end{bmatrix} \begin{bmatrix} e_k^d \\ e_k^g \\ e_k^{d^g} \end{bmatrix} + \begin{bmatrix} B_k & B_k^g \end{bmatrix} \begin{bmatrix} u_k \\ \dot{d}_k^g \end{bmatrix} + L_k \begin{bmatrix} C_k & D_k^p \end{bmatrix} \begin{bmatrix} e_k^d \\ e_k^g \\ e_k^{d^g} \end{bmatrix} \]

(33)

The observer error $e_k^{d^g}$ satisfies $e_k^{d^g} = 0$ and for $k \in \mathbb{N} \setminus \{1\}$,

\[ e_k^{d^g} = d_k^g - \hat{d}_k^g = S^k e_k^{d^g} - K \sum_{j \in N_k} (d_j^g - \hat{d}_k^g) \]

\[ = S^k e_k^{d^g} + K \sum_{j \in N_k} (e_j^{d^g} - e_k^{d^g}). \]

By Assumption 5, $\mathbb{S}$ contains a spanning tree rooted at 1. Hence, a suitable gain matrix $K$ in (29) exists by Lemma 2. A design procedure is given in Theorem 2. Consequently, for all $k \in \mathbb{N}$, $\varepsilon_k^{d^g}(t) = d_k^g(t) - \hat{d}_k^g(t) \to 0$ as $t \to \infty$. Hence, it follows that for all $k \in \mathbb{N}$, $\varepsilon_k^d(t) \to 0$ and $\varepsilon_k^{d^g}(t) \to 0$ as $t \to \infty$ since (30) is Hurwitz by construction.

Next, we define the error variable

\[ \varepsilon_k = x_k - \Pi_k^d d_k^d - \Pi_k^g d_k^g. \]

(34)

Note that $\varepsilon_k$ is the transient state component of agent $k$. In particular, it is the state component in the complement of the subspace $V_k^p = \{(x_k, d_k^d, d_k^g) : x_k = \Pi_k^d d_k^d + \Pi_k^g d_k^g\}$.

As discussed in [24], the equations (23a), (24a) express the fact that the subspace $V_k^p$ is a controlled invariant subspace of the system

\[ \begin{bmatrix} \dot{x}_k \\ \dot{d}_k^d \\ \dot{d}_k^g \\ \end{bmatrix} = \begin{bmatrix} A_k & 0 & 0 \\ 0 & S^k & 0 \\ 0 & 0 & S_k^g \end{bmatrix} \begin{bmatrix} x_k \\ d_k^d \\ d_k^g \end{bmatrix} + \begin{bmatrix} B_k \\ 0 \\ 0 \end{bmatrix} u_k \]

and $V_k^p$ is rendered invariant by $u_k = \Gamma_k^d d_k^d + \Gamma_k^g d_k^g$. Moreover, this subspace is annihilated by the regulation error map (2c) due to (23b), (24b). As we will show next, $V_k^p$ is rendered asymptotically stable by the proposed distributed regulator, and $\varepsilon_k(t)$ indeed converges to zero for all initial conditions.

Using (34), (2a), (3), (4), (32), (33), (23a), (24a), and (28), we can compute

\[ \dot{\varepsilon}_k = \dot{x}_k - \dot{\Pi}_k^d d_k^d - \dot{\Pi}_k^g d_k^g \]

\[ = A_k x_k + B_k u_k + B_k^d d_k^d + B_k^g d_k^g - \Pi_k^d S^k d_k^d - \Pi_k^g S_k^g d_k^g \]

\[ + B_k^d d_k^d + B_k^g d_k^g - \Pi_k^d S^k d_k^d - \Pi_k^g S_k^g d_k^g \]

\[ = A_k x_k + B_k \left( -F_k \hat{x}_k^d + G_k \hat{x}_k^g + G_k d_k^d \right) \]

\[ + B_k^d d_k^d + B_k^g d_k^g - \Pi_k^d S^k d_k^d - \Pi_k^g S_k^g d_k^g \]

\[ = (A_k - B_k F_k) (\varepsilon_k + \Pi_k^d d_k^d + \Pi_k^g d_k^g) + B_k F_k \varepsilon_k \]

\[ + B_k G_k^d d_k^d - B_k G_k^g d_k^g + B_k G_k^d d_k^g - B_k G_k^g d_k^d \]

\[ + B_k G_k^d d_k^d + B_k G_k^g d_k^g - \Pi_k^d S^k d_k^d - \Pi_k^g S_k^g d_k^g \]

(35)

Since the observer errors $\varepsilon_k^d, \varepsilon_k^{d^g}$ vanish asymptotically and $A_k - B_k F_k$ is Hurwitz by construction, we can conclude that $\varepsilon_k(t) \to 0$ as $t \to \infty$ for all $k \in \mathbb{N}$. If $d_k^0(0) = 0$ and $d_k^d(0) = 0$ for all $k \in \mathbb{N}$, then $x_k(t) = \varepsilon_k(t) \to 0$ as $t \to \infty$, and since the observer errors vanish asymptotically, also $\hat{x}_k(t) \to 0$, $\hat{d}_k^g(t) \to 0$, and $\hat{d}_k^d(t) \to 0$ as $t \to \infty$. Consequently, $\textbf{P1}$ is fulfilled.
It remains to show that the regulation errors $e_k$ converge to zero for all initial conditions. Using (2c), (32), (33), (23b), (24b), and (28), we can compute

$$
e_k = C_k x_k + D_k^w u_k + D_k^{rd} d^s + D_k^{rd} d^f_k = C_k (x_k + \Pi_k^d d^s + \Pi_k^d d^f_k) + \Pi_k^d u_k + D_k^{rd} d^s + D_k^{rd} d^f_k$$

$$= C_k \varepsilon_k + (C_k \Pi_k^d + D_k^{rd} d^s + (C_k \Pi_k^d + D_k^{rd} d^f_k) + D_k^{rd} d^s + D_k^{rd} d^f_k$$

$$= C_k \varepsilon_k + (C_k \Pi_k^d + D_k^{rd} d^s + (C_k \Pi_k^d + D_k^{rd} d^f_k)$$

$$= C_k \varepsilon_k + (C_k \Pi_k^d + D_k^{rd} d^s + (C_k \Pi_k^d + D_k^{rd} d^f_k)$$

$$= C_k \varepsilon_k + (C_k \Pi_k^d + D_k^{rd} d^s + (C_k \Pi_k^d + D_k^{rd} d^f_k)$$

$$= C_k \varepsilon_k + (C_k \Pi_k^d + D_k^{rd} d^s + (C_k \Pi_k^d + D_k^{rd} d^f_k)$$

Both the state component $x_k$ and the observer errors $\varepsilon_k, \varepsilon_k^{rd}, \varepsilon_k^{rpd}$ vanish asymptotically. Hence, $e_k(t) \to 0$ as $t \to \infty$ for all $k \in \mathbb{N}$, i.e., Problem 1 is solved.

**Remark 2.** Problem 1 can be solved by the distributed regulator proposed in [33], when (8) is regarded as a single large exosystem. However, according to the formulation in [33], each agent reconstructs the full vector $d$ of all reference and disturbance signals acting on the group via a distributed estimation protocol, which leads to dynamic controllers of high order and is unnecessary. Our distributed regulator takes the structure of (8) into account: each agent estimates only the global generalized disturbance as well as the local generalized disturbance acting on itself.

The distributed regulator constructed in Theorem 3 solves Problem 1. The following example shows that this result allows to solve a platooning problem and illustrates the distributed regulator. Moreover, this examples points out a limitation of the control scheme and motivates the extension of the distributed regulator, which will be presented in the next section.

**Example 1.** Consider a group of $N = 5$ vehicles in a platoon, each modeled as a double-integrator system of the form

$$\dot{x}_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k,$$

for all $k \in \mathbb{N}$. Moreover, there is a virtual leader generating a reference signal for the group, given by $\dot{d}^s = S^s d^s$ with $S^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. The state $x_k \in \mathbb{R}^2$ consists of the vehicle position $s_k$ and velocity $v_k$, i.e., $x_k = [s_k, v_k]^T$. The input $w_k$ is a local disturbance acting on vehicle $k$, e.g., due to a mass change, gear shift, or other external influence. We assume a sinusoidal disturbance $w_k$ for agent 2 and constant disturbances for all other agents.

![Fig. 2. Communication graph and external signals of the platoon.](image)

The task is to find a control law $u_k$ for each vehicle such that the following requirements are met for all initial conditions:

1. The velocities $v_k$ of all vehicles converge to the velocity commanded by the virtual leader:

$$\forall k \in \mathbb{N} : \lim_{t \to \infty} v_k(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$$

2. The relative distance of each vehicle with respect to the virtual leader converges to a desired constant value $r_k$:

$$\forall k \in \mathbb{N} : \lim_{t \to \infty} s_k(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = r_k.$$

The external local disturbance $w_k$ and local reference $r_k$ are combined into the local generalized disturbance signals $d_k$. For agent 2, we have

$$d_2^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} d_2^s, \quad \begin{bmatrix} 2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} d_2^s.$$

For all other agents, we have $d_k^s = [r_k, w_k]^T$ and

$$d_k^f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} d_k^f.$$

This leads to $B_2^d = B_{w2} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $B_2^d = B_{w2} \begin{bmatrix} 0 & 1 \end{bmatrix}$ for all other agents. We assume that each vehicle can communicate with its follower and predecessor in the platoon as illustrated by the graph in Fig. 2. The platooning problem can be formulated as a distributed output regulation problem with regulation errors defined as

$$e_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} d_2^s + \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} d_k^f.$$

Note that for $k = 2$, $D_k^{rd}$ has a third column of zeros. We assume that only agent 1 has access to the global reference signal $d^s$ and all agents have access to their local reference signal $r_k$ and to their state $x_k$. Local disturbance estimators as in (31) are used in order to reconstruct $w_k$.

Simulation results for two different choices of the control gain $F_k$ and with local exogenous input signals according to Fig. 3 are shown in Fig. 4. The distributed regulation problem is solved in both cases, i.e., the reference signals are tracked and the disturbance signals are rejected asymptotically. This examples illustrates an inherent limitation of the control scheme: The group does not react cooperatively on local
disturbances acting on individual agents. The steps in the constant disturbances \( w_4 \) and \( w_5 \) at \( t = 15 \) and \( t = 45 \), respectively, are rejected by the local controllers of these agents. As can be seen in Fig. 4, a more aggressive choice of the control gains \( F_k \) leads to a faster disturbance rejection. But the other agents in the group have no information about the disturbance and cannot adjust their actions to this situation. A more desirable and cooperative reaction on the local disturbances would be that the other vehicles in the platoon slow down or accelerate in order to maintain the desired relative distances during the transient phase. Maintaining the desired inter-vehicle distances is of higher importance than maintaining the desired velocity for a vehicle platoon. Such performance requirements cannot be taken into account explicitly with the distributed regulator described in Theorem 3.

Up to now, the proposed control strategy is a hierarchical control strategy. Cooperation happens only on the network level in order to spread the reference signal \( d^s \) all over the network. Such a hierarchical approach has many benefits but also has its drawbacks, cf., [18]. In particular, there is no cooperative reaction on the individual external disturbances.

V. IMPROVING THE TRANSIENT BEHAVIOR: IDENTICAL AGENTS

The distributed regulator presented in Section IV solves the distributed output regulation problem and realizes a cooperative behavior of the group. In this section, we show that the cooperative behavior can be improved significantly by a suitable extension of the distributed regulator. As motivated in Example 1, it is desirable to establish a cooperative reaction on local disturbances, meaning that the group disagreement or synchronization error is kept small in transient phases. For this purpose, we present a novel distributed regulator with additional couplings among the agents which stabilize the synchronization error and guarantee a desired performance. This is our main contribution and improvement of the distributed regulator compared to previous works [33].

In a first step, we consider groups of agents with identical dynamics. In particular, we impose the following assumption.

**Assumption 6.** Assume that for all \( k \in \mathbb{N} \),

\[
A_k = A, \quad B_k = B, \quad C_k = C^e, \quad D_k = D^e, \quad F_k = F.
\]

Note that not all matrices in (2) are required to be identical for the individual agents. In particular, the measurement output maps (2b) are allowed to be non-identical, and the local exosystems (4) as well as the generalized disturbance input matrices in (2a), (2c), can be non-identical. This leaves great flexibility in the problem formulation despite Assumption 6. Moreover, Assumption 6 will be relaxed in Section VI.

Our goal is to improve the cooperative behavior of the group in transient phases. The disagreement of the group can be quantified based on the transient state components \( \epsilon_k \) defined in (34). For this purpose, we consider the transient synchronization errors defined by

\[
\epsilon_k^t = \epsilon_k - \frac{1}{N} \sum_{j=1}^{N} \epsilon_j.
\]  

From (36), we know that \( \epsilon_k = (C^e - D^e F) \epsilon_k \) under Assumption 6 and without observer errors. Since \( C^e - D^e F \) is identical for all \( k \in \mathbb{N} \), agreement of \( \epsilon_k \) corresponds to agreement of the regulation errors \( \epsilon_k \). In the following, we propose a distributed regulator which solves Problem 1 and, at the same time, exponentially stabilizes the synchronization errors \( \epsilon_k^t \) with a certain desired decay rate \( \gamma > 0 \). In order to guarantee that this decay rate can be achieved, we refine Assumption 1 as follows.

**Assumption 1’.** The pair \( (A_k + \gamma I_{n_k^e}, B_k) \), where \( \gamma > 0 \), is stabilizable for all \( k \in \mathbb{N} \).

We start with the full information case. This allows to present the main idea in a clear way and will be instrumental in the proof of the output feedback case.

**Lemma 3** (Full information regulator). Consider a group of agents (2) with exosystems (3), (4). Suppose that Assumptions 1’, 4, 5 and 6 are satisfied. Then, a distributed full-information regulator that solves Problem 1 and additionally achieves exponential stability of synchronization errors \( \epsilon_k^t \) with decay rate \( \gamma > 0 \) can be constructed as follows:
Choose $F$ such that $A - BF$ is Hurwitz.

For all $k \in \mathbb{N}$, find a solution for (23) and (24) and set
\[
G_k^i = T_{ik}^T + FT_{ik}^T \\
G_k = T_{ik}^T + FT_{ik},
\]

Choose $H$ such that for $k = 2, \ldots, N$,
\[
\max\{\Re(\mu), \mu \in \sigma(A - BF - \lambda_k BH)\} < -\gamma, \quad (38)
\]
i.e., the maximal real part of all eigenvalues of the matrices $A - BF - \lambda_k BH$ is smaller than $-\gamma$.

Finally, the control law for each agent $k \in \mathbb{N}$ is given by
\[
 u_k = -F \dot{x}_k + G_k^e \dot{e}_k + d_k^e + H \sum_{j \in N_k} (e_j - e_k). \quad (39)
\]

Proof: Analogously to (35), the dynamics of $\dot{e}_k$ can be computed as
\[
 \dot{e}_k = (A - BF) \dot{e}_k + BH \sum_{j \in N_k} (e_j - e_k). \quad (40)
\]
The novel term in the control law (39) couples the transient state components of the agents. Eq. (40) is a diffusively coupled network of $N$ identical stable linear systems. The same transformation as in the proof of Theorem 2, i.e.,
\[
 \hat{e} = (T^{-1} \otimes I_n) \epsilon,
\]
leads to
\[
 \dot{\hat{e}} = (I_N \otimes (A - BF) - \Lambda \otimes BH) \hat{e}.
\]
Let $\hat{e}$ be partitioned into $\hat{e}'$ and $\hat{e}''$. Then,
\[
 \dot{\hat{e}}' = (A - BF) \hat{e}' + \Lambda \hat{e}'
\]
and
\[
 \dot{\hat{e}}'' = \begin{bmatrix} 0 & \cdots & \cdots & \cdots \\ & 0 & \cdots & \cdots \\ & & \cdots & \cdots \end{bmatrix} \hat{e}''.
\]
By (38) the maximal real part of the latter matrix is smaller than $-\gamma$. Note that such a gain $H$ exists due to Assumptions 1’ and Lemma 2. Hence, there exists a constant $\hat{c} > 0$ such that
\[
 \|\hat{e}'(t)\| \leq \hat{c} \|\hat{e}'(0)\| e^{-\gamma t},
\]
for all $t \geq 0$, cf., [41]. We define the projection matrix $P = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T$ and partition $T = [T_1 \cdots T_N]$. Then, the stack vector of synchronization errors (37) is given by $\epsilon' = (P \otimes I_n) \epsilon = (PT \otimes I_n) \hat{e} = (PT'' \otimes I_n) \epsilon''$ since $P1 = 0$. Hence, it follows that there exists a constant $c > 0$ such that
\[
 \|\epsilon''(t)\| \leq c \|(P \otimes I_n) \epsilon(0)\| e^{-\gamma t}.
\]
Consequently, the synchronization error is exponentially stable with a decay rate of at least $\gamma$.

The matrix $A - BF$ is Hurwitz by the choice of $F$, which guarantees that $\hat{e}'(t) \to 0$ as $t \to \infty$. Therefore, it holds that $\hat{e}'(t) \to 0$ as $t \to \infty$. Since $e_k = (C^e - D^F) \epsilon_k$, this shows that Problem 1 is solved.

The novel control law (39) solves Problem 1 and additionally enforces synchronization of the regulation errors with a desired decay rate and therefore has the desired effect, that is, a cooperative reaction of the group on disturbances acting on individual agents. The gain matrices $F$ and $H$ allow to tune separately the local disturbance rejection of each agent and the synchronization of the group. Theorem 2 serves as a design method for $H$.

The full information control law (39) is impractical since the agents do not have direct access to their state and generalized disturbance signals. The following theorem shows that (39) can be implemented based on observers, analogously to (32) in Theorem 3. In order to guarantee that the desired decay rate can be achieved, we refine Assumption 2.

**Assumption 2’**. The pair
\[
 \left( \begin{bmatrix} A_k & B_k^d \\ 0 & 0 \end{bmatrix} + \eta_l \left( I_{s_g} + n_d^e \right) \right), \left( \begin{bmatrix} C_k & D_k^d \end{bmatrix} \right),
\]
where $\eta > 0$, is detectable for all $k \in \mathbb{N}$.

**Theorem 4** (Distributed output feedback regulator). Consider a group of agents (2) with exosystems (3), (4). Suppose that Assumptions 1’, 2’, 3, 4, 5 and 6 are satisfied. Construct a distributed regulator as in Theorem 3 and choose $L_k$ and $K$ such that the corresponding systems are exponentially stable with decay rate $\eta > \gamma > 0$. Choose $H$ as in (38). Then, the control law
\[
 u_k = -F \dot{x}_k + G_k^e \dot{e}_k + G_k^d + H \sum_{j \in N_k} (\dot{e}_j - \dot{e}_k), \quad (41)
\]
solves Problem 1 and additionally achieves exponential stability of the synchronization errors $\epsilon_k^e$ with decay rate $\gamma > 0$.

Proof: The novel term in (41) has no influence on the dynamics of the observer errors $\epsilon_k^e, \epsilon_k^{d^e}, \epsilon_k^{d^d}$. They obey the same dynamics as in the proof of Theorem 3. Moreover, by assumption, there exist constants $c_k > 0$ such that for all $t \geq 0$,
\[
 \begin{bmatrix} \epsilon_k^e(t) \\ \epsilon_k^{d^e}(t) \\ \epsilon_k^{d^d}(t) \end{bmatrix} \leq c_k \begin{bmatrix} \epsilon_k^e(0) \\ \epsilon_k^{d^e}(0) \\ \epsilon_k^{d^d}(0) \end{bmatrix} e^{-\gamma t}. \quad (42)
\]
Note that
\[
 \dot{\epsilon}_k = \dot{x}_k - \Pi_k \epsilon_k - \Pi_k^d \epsilon_k^{d^d} - \Pi_k^d \epsilon_k^{d^d} = \epsilon_k - \epsilon_k^e + \Pi_k \epsilon_k^{d^e} + \Pi_k^d \epsilon_k^{d^d}.
\]
Analogously to the proof of Theorem 3, the dynamics of $\epsilon_k^e$ can be computed as
\[
 \dot{\epsilon}_k = (A - BF) \epsilon_k + BH \sum_{j \in N_k} (e_j - e_k) + \delta_k(t),
\]
where $\delta_k(t)$ captures the influence of the observer errors
\[
 \delta_k(t) = BF \epsilon_k^e - BG_k^e \epsilon_k^{d^e} - BG_k^d \epsilon_k^{d^d} - BH \sum_{j \in N_k} \left( \epsilon_j^e - \Pi_j \epsilon_j^e - \Pi_j \epsilon_j^{d^e} + \Pi_j \epsilon_j^{d^d} + \Pi_k \epsilon_k^{d^e} + \Pi_k^d \epsilon_k^{d^d} \right).
\]
We define the stack vector $\delta = [\delta_1^T \cdots \delta_N^T]^T$. From (42) it follows that there exists a constant $c_\delta$ such that for all $t \geq 0$,
\[
 \|\delta(t)\| \leq c_\delta \|\delta(0)\| e^{-\gamma t}. \quad (43)
\]
Analogously to the proof of Lemma 3, we obtain
\[
 \dot{\hat{e}} = (I_N \otimes (A - BF) - \Lambda \otimes BH) \hat{e} + (T^{-1} \otimes I_n) \delta.
\]
With (43) and since $\eta > \gamma$, it follows that $\epsilon''(t) \to 0$ exponentially as $t \to \infty$ with decay rate $\gamma$. 

Moreover, we also have $\mathbf{e}(t) \to 0$ as $t \to \infty$. Since $\mathbf{e}(t) \to 0$ as $t \to \infty$ for all $k \in \mathbb{N}$, the coupling term in (41) vanishes asymptotically. Consequently, analogously to the proof of Theorem 3, it holds that $e_k(t) \to 0$ as $t \to \infty$ and Problem 1 is solved.

**Example 2.** We consider the same setup as in Example 1. Now, we use the novel distributed regulator as described in Theorem 4. Simulation results with local exogenous input signals according to Fig. 3 are shown in Fig. 5. The effect of the disturbances on the inter-vehicle distances is indeed rejected much more efficiently, compared to Example 1. The platoon reacts cooperatively on the local disturbances and maintains small synchronization errors. The two different simulations in Fig. 5 with different choices of the control gains $F$ and $H$ illustrate the flexibility of the control design. Depending on the requirements, it is possible to put more emphasis on the disturbance rejection or on the synchronization.

**VI. IMPROVING THE TRANSIENT BEHAVIOR: NON-IDENTICAL AGENTS**

In this section, we aim at relaxing Assumption 6. In case of non-identical agents, an analogue coupling term as in (39) leads to a diffusively coupled network of non-identical stable linear systems in (40). In this case, it is hard to find a suitable coupling gain $H$. In case of non-identical state dimensions $n_k$ such couplings cannot be realized at all. We exclude this case and treat the non-identical agents as perturbed versions of a nominal system. We present a design method for $H$ based on this nominal system and use robustness arguments in order to prove exponential stability of the heterogeneous network. For ease of presentation, we discuss only the full information case in the following. The output feedback implementation can be carried out analogously to Theorem 4.

Suppose that we implement a control law like (41) in a group of non-identical agents, i.e.,

$$u_k = -F_k x_k + G_k \dot{d}_k + G_k \dot{d}_k + H \sum_{j \in \mathbb{N}_k} (\varepsilon_j - \varepsilon_k)$$

(44)

where $F_k$, $G_k$, $G_k'$ are designed as in Theorem 3 and $H$ is a coupling gain to be specified in the following. Then, analogously to (40), the error variables $\varepsilon_k$ obey the dynamics

$$\dot{\varepsilon}_k = (A_k - B_k F_k) \varepsilon_k + B_k H \sum_{j \in \mathbb{N}_k} (\varepsilon_j - \varepsilon_k),$$

where $A_k - B_k F_k$ is Hurwitz. The key assumption in the following is that the systems

$$\dot{\varepsilon}_k = (A_k - B_k F_k) \varepsilon_k + B_k \nu_k$$

(45)

with artificial inputs $\nu_k$ have similar dynamics. In particular, we express each system as perturbed version of the same nominal system $P$ with uncertainty $\Delta_k$, according to

$$P : \begin{cases} \dot{\varepsilon}_k = \tilde{A} \varepsilon_k + \tilde{B} \nu_k + \tilde{B}^m \omega_k \\ \zeta_k = C \zeta_k + D \nu_k \\ \Delta_k = \Delta_k \zeta_k, \end{cases}$$

as illustrated in Fig. 6. Note that the regulation error $\epsilon_k = (C^* - FD^*) \zeta_k$ can be interpreted as output of the system $P$. The fact that the systems (45) are similar is formally expressed in the following assumption.

**Assumption 7.** The matrix $\tilde{A}$ is Hurwitz. The uncertainties $\Delta_k$ are proper real-rational stable transfer matrices and satisfy $\|\Delta_k\|_\infty < \eta^k$, where $\eta^k > 0$, for all $k \in \mathbb{N}$.

The following procedure yields a description of the systems (45) as in (46). We define the nominal system $G$ based on the average matrices

$$\bar{A} = \frac{1}{N} \sum_{k=1}^{N} (A_k - B_k F_k), \quad \bar{B} = \frac{1}{N} \sum_{k=1}^{N} B_k,$$

i.e., the transfer function matrix from $\nu_k$ to $\varepsilon_k$ is given by

$$G(s) = (sI - \bar{A})^{-1} \bar{B}.$$

Note that we choose $\varepsilon_k$ as output since we want to construct a state feedback controller for the nominal system later on. At this point, we have to check whether $\bar{A}$ is Hurwitz. If it is not Hurwitz, the description (46) must be constructed in a different way such that $\bar{A}$ is Hurwitz. If it is Hurwitz, we proceed as follows. We define the individual transfer function matrices

$$G_k(s) = (sI - (A_k - B_k F_k))^{-1} B_k.$$

![Fig. 6. Nominal agent dynamics $P$ with individual perturbation $\Delta_k$.](image)
Then, we express \( G_k(s) \) as perturbed version of the nominal system \( G(s) \) with dynamical additive uncertainty \( \Delta_k \), i.e.,
\[
G_k(s) = G(s) + \Delta_k(s),
\]
where \( \Delta_k \) is simply obtained from \( \Delta_k(s) = G_k(s) - G(s) \). Since \( G(s) \) and \( G_k(s) \) are proper real-rational and stable, \( \Delta_k \) is proper real-rational and stable as well. The bound \( \eta^\Delta \) in Assumption 7 is computed as
\[
\eta^\Delta = \max_{k \in N} \| \Delta_k \|_{\infty}.
\]
With the matrices \( B^\omega = \tilde{B}, C^\xi = L^\eta, \) and \( D^\xi = 0 \), and the uncertainties \( \Delta_k \) obtained from the procedure above, the systems (45) are represented as in (46) and Assumption 7 is satisfied.

With couplings
\[
v_k = H \sum_{j \in N_k} (e_j - e_k),
\]
between the systems and the representation (46) for each agent, we obtain
\[
T^{\omega\xi}_k: \begin{align*}
\dot{e}_k = (I_N \otimes \tilde{A} - L^\eta \otimes \tilde{B}H) e + (I_N \otimes B^\omega) \omega \quad (47a) \\
\dot{\xi}_k = (I_N \otimes C^\xi - L^\eta \otimes D^\xi H) e \\
\Delta : \quad \omega = \text{diag}(\Delta_k) \dot{\xi}.
\end{align*}
\]
This is a diffusively coupled network of uncertain stable linear systems, which is illustrated in Fig. 7. The next step is to design \( H \) such that the \( \mathcal{H}_{\infty} \) norm of the network with input \( \omega \) and output \( \xi \) is minimized. For this purpose, we follow the ideas of [19] and extend the synthesis procedure such that a desired decay rate \( \gamma > 0 \) is achieved for the synchronization error. Note that a lower bound on the achievable \( \mathcal{H}_{\infty} \) performance is given by the autonomous system since we consider only diffusive couplings and \( \lambda_1 = 0 \).

**Lemma 4.** Let the graph \( \mathcal{G} \) be undirected and connected. Then, the network \( T^{\omega\xi} \) is exponentially stable and satisfies
\[
\| T^{\omega\xi} \|_{\infty} < \eta
\]
for a given \( \eta > 0 \) and achieves state synchronization exponentially with a decay rate \( \gamma > 0 \), if and only if the \( N \) systems
\[
T^{\omega\xi}_k: \begin{align*}
\dot{e}_k = (\tilde{A} - \lambda_k \tilde{B}H) \tilde{e}_k + B^\omega \omega_k \\
\dot{\xi}_k = (C^\xi - \lambda_k D^\xi H) \tilde{e}_k
\end{align*}
\]
are exponentially stable and satisfy \( \| T^{\omega\xi}_k \|_{\infty} < \eta \) for all \( k \in N \), and additionally for \( k = 2, \ldots, N \), have a decay rate \( \gamma \).

**Proof:** Since \( \mathcal{G} \) is undirected and connected there exists an orthogonal matrix \( U \) such that \( U^T L^\omega U = \Lambda = \text{diag}(\lambda_k) \) and \( \lambda_1 = 0 \) is the top left diagonal element of \( \Lambda \). With the coordinate transformation \( \tilde{e} = (U^T \otimes I_N) e \), \( \omega = (U^T \otimes I_N) \omega \),
\[
\tilde{e} = (I_N \otimes \tilde{A} - \Lambda \otimes \tilde{B}H) \tilde{e} + (I_N \otimes B^\omega) \omega
\]
\[
\tilde{\xi} = (I_N \otimes C^\xi - \Lambda \otimes D^\xi H) \tilde{e}.
\]
The latter system is decomposed into the decoupled systems
\[
\begin{align*}
\dot{e}_k &= (\tilde{A} - \lambda_k \tilde{B}H) \tilde{e}_k + B^\omega \omega_k \\
\dot{\xi}_k &= (C^\xi - \lambda_k D^\xi H) \tilde{e}_k
\end{align*}
\]
Hence, exponential stability of \( T^{\omega\xi} \) corresponds to exponential stability of these systems for all \( k \in N \). Analogously to the proof of Lemma 3, the state components \( k = 2, \ldots, N \) correspond to the synchronization error of the network. Exponential stability with decay rate \( \gamma \) of these systems is equivalent to state synchronization with decay rate \( \gamma \) as stated in the proof of Lemma 3. Moreover, since \( \| U \|_{\infty} = \| U^T \|_{\infty} = 1 \) and the maximal singular value of a block diagonal matrix equals the maximum of the maximal singular values of each block, it follows that \( \| T^{\omega\xi} \|_{\infty} = \max_k \| T^{\omega\xi}_k \|_{\infty} < \eta \).

**Theorem 5.** Let the graph \( \mathcal{G} \) be undirected and connected. Suppose that Assumption 7 is satisfied and let \( \eta < 1/\eta^\Delta \) and \( \gamma > 0 \). Suppose that there exist \( X > 0, Y > 0, Z \) such that the LMIs
\[
\begin{bmatrix}
\tilde{A}^T X + X \tilde{A} - X B^\omega (C^\xi)^T \\
(B^\omega)^T X - \eta I \\
C^\xi & 0 & -\eta I
\end{bmatrix} < 0
\]
and, for \( k = 2, \ldots, N \),
\[
\begin{bmatrix}
\tilde{A}^T Y + Y \tilde{A} - Y (C^\xi - \lambda_k D^\xi Z)^T \\
(C^\xi Y - \lambda_k D^\xi Z)^T - \eta I \\
0 & -\eta I
\end{bmatrix} < 0
\]
are satisfied, where \( \Xi = (\tilde{A} Y - \lambda_k \tilde{B}Z) + (\tilde{A} Y - \lambda_k \tilde{B}Z) + 2\gamma Y \).
Then, the network (47) with \( H = Z^T Y^{-1} \) is exponentially stable.

**Proof:** The \( N \) subsystems \( T^{\omega\xi}_k \) are composed of the transformed systems
\[
P: \begin{align*}
\dot{e}_k = A \tilde{e}_k - \lambda_k B \tilde{V}_k + B^\omega \omega_k \\
\dot{\xi}_k = C^\xi \tilde{e}_k - \lambda_k D^\xi \tilde{V}_k
\end{align*}
\]
and state feedback law \( \tilde{V}_k = H \tilde{e}_k \). The key idea is to design one single controller \( \tilde{V}_k = H \tilde{e}_k \) guaranteeing the desired \( \mathcal{H}_{\infty} \) norm \( \eta \) for all \( N \) subsystems \( P \) as well as the desired decay rate \( \gamma \) for \( k = 2, \ldots, N \). According to the Bounded Real Lemma [42], the closed loop of \( P \) with control law \( \tilde{V}_k = H \tilde{e}_k \) is asymptotically stable and has \( \mathcal{H}_{\infty} \) norm less than or equal \( \eta \), if and only if there exists \( X > 0 \) such that
\[
\begin{bmatrix}
\Theta \\
(B^\omega)^T X - \eta I \\
C^\xi - \lambda_k D^\xi H
\end{bmatrix} < 0,
\]
where \( \Theta = (\tilde{A} - \lambda_k \tilde{B}H)^T X + X (\tilde{A} - \lambda_k \tilde{B}H) \). Since \( \lambda_1 = 0 \), the best achievable \( \mathcal{H}_{\infty} \) norm is limited by the uncontrollable system.
For $k = 1$, the matrix inequality above simplifies to the analysis LMI (48), which allows to find a lower bound on the achievable $H_{\infty}$ norm $\eta$. For $k = 2, \ldots, N$, synthesis LMIs for $H$ are derived as follows. In order to guarantee the desired decay rate $\gamma$, we add $2\gamma X$ to the upper left block $\Theta$. This guarantees that the real parts of all corresponding eigenvalues are no larger than $-\gamma$. Then, we multiply the matrix inequality from both sides with $\text{diag}(X^{-1}, I, I)$ and perform the change of variables $Y = X^{-1}$ and $Z = KY$. This yields the LMIs (49) for $k = 2, \ldots, N$. The corresponding control gain is obtained from $H = ZY^{-1}$.

By Lemma 4, it follows that $T^{\alpha_\zeta}$ is exponentially stable and $\|T^{\alpha_\zeta}\|_\infty < \eta$. Since $\eta < 1/\eta^k$ by assumption, exponential stability of (47) is a direct consequence of the Small Gain Theorem, cf., [25].

The design procedure for a suitable coupling gain $H$ for a group of non-identical agents is summarized as follows. First, express the non-identical systems (45) as perturbed versions of a common nominal plant $P$ with uncertainties $\Delta_k$ according to (46) such that Assumption 7 is satisfied with $\eta^k$ as small as possible. Second, design a gain matrix $H$ according to Theorem 5 for some $\eta < 1/\eta^k$ and a desired $\gamma > 0$. Finally, examine the design and evaluate the behavior of the closed-loop system.

Remark 3. The nominal network has $n^g$ poles which correspond to the synchronous motion of the variables $e_k$ and $(N - 1)n^g$ poles which correspond to the synchronization error and which, by design of the coupling gain $H$, have a decay rate of $\gamma$. In presence of the perturbations, the network (47) is still guaranteed to be exponentially stable by Theorem 5. However, the perturbations introduce a coupling between the $n^g$ modes of the synchronous motion and the $(N - 1)n^g$ modes of the synchronization error. The procedure in Theorem 5 does not guarantee robust pole placement of the $(N - 1)n^g$ poles such that their real parts are smaller than $-\gamma$ in presence of the perturbations. It rather guarantees robust stability and nominal performance of the network. The influence from the $n^g$ modes with lower decay rate to the synchronization error may shift poles and result in slower convergence of the synchronization error. Nevertheless, the additional coupling term can be expected to improve the convergence speed of the synchronization error significantly. For details on the robust pole placement problem, the reader is referred to [43].

Remark 4. Note that the design procedure in Theorem 5 can be extended to pole placement constraints in terms of general LMI regions analogously to Theorem 2, based on the results of [43]. In this case, the LMIs (49) have to be replaced by

\[
\begin{bmatrix}
\Xi_R & M_1^T \otimes B^{0g} & M_1^T \otimes (C^g Y - \lambda_k D^g Z)^T \\
M_2 \otimes (C^g Y - \lambda_k D^g Z) & -\eta I & 0 \\
-M_2 \otimes (B^{0g} Y - \lambda_k B Z)^T & 0 & -\eta I 
\end{bmatrix} < 0,
\]

where $\Xi_R = L \otimes I + M \otimes (\bar{A} Y - \lambda_k \bar{B} Z) + M^T \otimes (\bar{A} Y - \lambda_k \bar{B} Z)^T$ and $M_1^T M_2 = M$ is a decomposition such that $M_1$ and $M_2$ have full column rank. Such a decomposition can be obtained from the singular value decomposition of $M$.

Example 3. Here, our main results are applied to a coordination problem of four Quanser 3-DoF Lab Helicopters\(^1\).

Suppose we have a group of helicopters modeled by

\[
x_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_k^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p_k^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p_k^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d_k
\]

\[
y_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \hat{d}_k
\]

for $k \in \mathbb{N}$. The state $x_k \in \mathbb{R}^6$ consists of the travel angle $\alpha_k$, pitch angle $\beta_k$, and elevation angle $\gamma_k$ and the respective angular velocities, i.e., $\dot{x}_k = [\alpha_k \dot{\alpha}_k \beta_k \dot{\beta}_k \gamma_k \dot{\gamma}_k]^T$. The measurement output $y_k$ consists of the three absolute angles. The parameters of the four helicopters are given in vector form by $p_1 = [0.7 1.17 6.0 0.58]^T$, $p_2 = [0.5 1.05 5.8 0.50]^T$, $p_3 = [0.6 1.10 6.1 0.63]^T$, $p_4 = [0.7 1.25 5.6 0.48]^T$. Note that the helicopters are non-identical due to the different parameter values assumed for each system.

The objective is that all helicopters track a reference signal for travel and elevation angles while asymptotically rejecting the external disturbances. The reference is generated by

\[
d^g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d^g,
\]

where $r_a = [1 \ 0 \ 0] d^g$ is a ramp signal and reference for the travel angles and where $r_\gamma = [0 \ 0 \ 1] d^g$ is a constant signal and reference for the elevation angles. Consequently, the regulation errors are defined as

\[
e_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} d^g.
\]

Each helicopter is affected by constant disturbances $d_k$, as forces on the rotors, which is described by the local exosystems with $S_k = \text{diag}(0, 0)$ for all $k \in \mathbb{N}$. The communication graph $\mathcal{G}$ is an undirected cycle.

In a first step, we construct a distributed output regulator according to Theorem 3. The control gains $F_k$ are designed as LQR with weight matrices $Q = \text{diag}(50, 1, 1, 1, 100, 50)$ and $R = \text{diag}(2, 2)$. The observer poles are placed between $-10$ and $-12$ on the real axis. We simulate the closed-loop system and apply a random piecewise constant disturbance signal with magnitude in the range of $-2.5N$ to $2.5N$ each helicopter. Fig. 8 shows the corresponding simulation result. Next, we extend the distributed regulator (32) by couplings as in (44) in order to improve the cooperative behavior of the group and synchronize the transient state components. The control gains $F_k$ are now designed with a higher weight $R = \text{diag}(4, 4)$ on the control signals. Following the procedure described in Section VI, we compute a nominal model $G$.\(^1\) http://www.quanser.com/products/3dof_helicopter
We obtain \( \eta \) the singular value plots of the transfer matrices \( G_k \) for \( \Delta_k = \eta \) (Example 3)

\[
\Delta_k = \eta
\]

coupling gain \( H \) based on the LMIs (Example 3).

Fig. 9. Singular value plots of the additive uncertainties \( \Delta_k \) (Example 3).

\[
\Delta_k \leq \max_k \| \Delta_k \|_{\infty} = 0.3129. \text{ We compute a suitable coupling gain } H \text{ based on the LMIs (50) where we choose } \eta = 0.95(\alpha^2)^{-1} \text{ and the pole placement region } \mathcal{S}(3,30,\pi/3) \text{ ensuring a decay rate of } \gamma = 3. This leads to a feasible design of } H. \text{ Simulation results with the same setup as before and the extended distributed regulator are shown in Fig. 10. Obviously, the cooperative behavior is improved considerably compared to Fig. 8. The synchronization error of the elevation angles remains very small in the transient phases. The group of non-identical helicopters reacts cooperatively on the local external disturbances, as desired.}

Fig. 10. Simulation result with distributed regulator according to Theorem 3 and additional coupling term as in (44), with coupling gain \( H \) designed according to Theorem 5 and (50) (Example 3).

\[
\begin{align*}
\alpha & [\text{rad}] \\
\gamma & [\text{rad}] \\
\text{Singular Values [dB]} & \\
\text{Frequency [rad/s]} & \\
\end{align*}
\]

VII. CONCLUSION

The cooperative output regulation problem captures a wide range of practical multi-agent coordination problems. In this paper, we have presented a novel distributed regulator which solves the coordination problem and additionally allows to improve and tune the synchronization error dynamics of the group. A novel coupling term based on the transient state components of each agent allows to impose a desired exponential decay rate on the synchronization error among agents, which leads to a significant improvement of the cooperative behavior of the group in transient phases. Under the novel control law, the group is able to react cooperatively on external disturbances acting on individual agents. We have discussed a vehicle platooning example and a coordination example for a group of four 3-DoF helicopters in order to emphasize the importance of a cooperative reaction on disturbances and in order to illustrate the design procedure and effectiveness of the novel distributed regulator with transient synchronization.

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