On Relativistic Corrections to Microlensing Effects: Applications to the Galactic Black Hole

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ABSTRACT
The standard treatment of gravitational lensing by a point mass lens \( M \) is based on a weak-field deflection angle \( \hat{\alpha} = 2/\sqrt{x_0} \), where \( x_0 = r_0 c^2 / 2GM \) with \( r_0 \) the distance of closest approach to the mass of a lensed light ray. It was shown that for a point mass lens, the total magnification and image centroid shift of a point source remain unchanged by relativistic corrections of second order in \( 1/x_0 \). This paper considers these issues analytically taking into account the relativistic images, under three assumptions \( \text{A1–A3} \), for a Schwarzschild black hole lens with background point and extended sources having arbitrary surface brightness profiles. The assumptions are:

- \( \text{A1:} \) The source is close to the line of sight and lies in the asymptotically flat region outside the black hole lens;
- \( \text{A2:} \) The observer-lens and lens-source distances are significantly greater than the impact parameters of the lensed light rays; and
- \( \text{A3:} \) The distance of closest approach of any light ray that does not wind around the black hole on its travel from the source to the observer, lies in the weak-field regime outside the black hole. We apply our results to the Galactic black hole for lensing scenarios where \( \text{A1–A3} \) hold. We show that a single factor characterizes the full relativistic correction to the weak-field image centroid and magnification. As the lens-source distance increases, the relativistic correction factor strictly decreases. In particular, we find that for point and extended sources about 10 pc behind the black hole, which is a distance significantly outside the tidal disruption radius of a sun-like source, the relativistic correction factor is minuscule, of order \( 10^{-14} \). Therefore, for standard lensing configurations, any detectable relativistic corrections to microlensing by the Galactic black hole will most likely have to come from sources significantly closer to the black hole.

Key words: black holes; strong field regime – gravitational lensing: photometry and astrometry.

1 INTRODUCTION
Microlensing describes gravitational lensing of a source whose multiple images are not resolved. Two fundamental microlensing observables are the total magnification (photometry) and image centroid shift (astrometry) of images of a lensed source. These observables have important astrophysical applications such as determining the mass and distance to the lens, angular radius of the source, etc. (see, e.g., Paczynski 1996, Paczyński 1998, Boden, Shao & van Buren 1998, Jeong, Han & Park 1999, Gaudi & Petters 2002, and references therein).

A natural issue to explore is how are the photometry and astrometry of a source being lensed by a point mass lens. This could have important implications for the testability of general relativity’s predictions about how the gravitational field of a black hole affects light rays. Indeed, the standard theoretical framework for point mass microlensing is based on relativistic calculations to first-order in \( 1/x_0 \) about a Schwarzschild black hole, where

\[
x_0 = \frac{r_0}{2 \xi}, \quad r_\xi = \frac{GM_\bullet}{c^2},
\]

where \( M_\bullet \) is the black hole’s mass and \( r_\xi \) the gravitational radius. Ebina et al. (2000) found that to second-order in \( 1/x_0 \), the relativistic corrections appear in the position and magnification of images due to a point mass lens, while no such correction appears in the total magnification. Lewis & Wang (2001) also showed that no relativistic correction...
to second-order in $1/x_0$ occurs for the associated image centroid shift. In this paper, we extend the work of the previous authors by determining an analytical expression under assumptions A1–A3 for the full Schwarzschild black hole relativistic correction of the image centroid shift, which includes the total magnification, for point and extended sources with arbitrary surface brightness.

The full relativistic corrections will then be applied to the case of the massive black hole at the center of our Galaxy. Microlensing by the Galactic black hole has been studied by several authors in the weak-field limit of the black hole (e.g., Wardle & Yusef-Zadeh 1993, Alexander & Sternberg 1999, Alexander & Loeb 2000, Alexander 2001). In addition, the precession of star orbits in the strong-field regime of the black hole was considered by, e.g., Jaroszynski (1998a, 1999), Fragile & Mathews (2004), Virbhadra & Ellis (2004). Given a numerical treatment of the magnifications of several relativistic images for a point source being lensed by a Schwarzschild black hole lens at the Galactic center. They considered sources within our Galaxy that are far away from the black hole (about 8.5 kpc), while we shall consider sources as close as 10 pc to the black hole. Analytical work on magnification due to a Schwarzschild lens was also done by several authors for point and/or extended sources with uniform brightness profiles (e.g., Ohaman 1987, Frittelli, King & Newman 2000, Bozza et al. 2001, Eiroa, Romero & Torres 2002). As noted above, our microlensing treatment will apply not only to the magnification, but the image centroid of extended sources with arbitrary surface brightness.

We shall also show that a single factor approximates the relativistic corrections to the weak-field total magnification and image centroid due to a Schwarzschild black hole lens at the Galactic center. The same factor applies to either a point or extended source. Estimates of this factor will be given for lens-source distances ranging from 10 pc to 100 pc. In principle, the magnification and image centroid of sources closer to the Galactic black hole should provide stronger relativistic microlensing signatures. We shall show that even for sources about 10 pc behind the black hole, the full relativistic correction is still negligible.

Section 2 reviews some basic results about lensing by a Schwarzschild black hole. In Section 3, we compute explicitly the image centroid due to a Schwarzschild black hole acting on point and extended sources with arbitrary brightness profiles. Our image centroid formula expresses the relativistic image centroid in terms of the weak-field image centroid due to a point mass lens. Section 4 estimates the relativistic corrections to microlensing by the Galactic black hole.

### 2 IMAGE POSITION AND MAGNIFICATION

We review some basic analytical results about the image positions and magnification due to a Schwarzschild black hole lens.

#### 2.1 Lens Equation and Bending Angle

A Schwarzschild black hole of mass $M_\bullet$ is the unique static, spherically symmetric, vacuum (i.e., Ricci flat) spacetime that is Minkowski at infinity (e.g., [Wald 1984, p. 119]):

$$ds^2 = -(1 - \frac{r_s}{r}) c^2 dt^2 + (1 - \frac{r_s}{r})^{-1} dr^2 + r^2 d\Omega^2$$

where $r_s$ is the Schwarzschild radius

$$r_s = 2r_g = 3 \frac{M_\bullet}{M_\odot} \text{ km}$$

and $d\Omega^2$ the standard metric on a 2-sphere of radius 1.

Relative to the optical axis through the observer and center of the black hole, let $\beta$ denote the angular position of a point source behind the black hole. Note that a source between the observer and the black hole is also lensed; in principle, even the observer is lensed. Such situations will not be treated in this paper. We shall always assume the standard lensing configuration of a source behind the black hole.

The capture cross section $\sigma$ of a light ray in a Schwarzschild spacetime is $\sigma = \pi b_{\text{crit}}^2$, where $b_{\text{crit}} = 3\sqrt{3} r_g$ is the critical apparent impact parameter (e.g., Wald 1984, p.144). Consequently, any light ray that makes it from the source to the observer will have an apparent impact parameter $b$ greater than $b_{\text{crit}}$. The classification of null geodesics in the Schwarzschild geometry divides such source-to-observer light rays into those that loop around the black hole at least once before arriving at the observer and those that travel directly to the observer without looping (cf. Misner, Thorne & Wheeler 1973, p. 674).

Let $D_{LS}$, $D_L$, and $D_S$ be the distances from lens to source, observer to lens, and observer to source, resp. Denote the angular position of a lensed image relative to the line of sight by $\vartheta$. A negative angle $-\vartheta$ will correspond to an angular position on the side of the black hole opposite to the angular position $\vartheta$. A trigonometric argument yields the lens equation, which relates $\beta$ and $\vartheta$ as follows (Virbhadra & Ellis 2000):

$$\tan \beta = \tan \vartheta - \frac{D_{LS}}{D_S} (\tan \vartheta + \tan(\hat{\alpha} - \vartheta)), \quad \text{(1)}$$

where $\hat{\alpha}$ is the bending angle of the lensed ray. The lensed images of a light source at $\beta$ are the solutions $\vartheta$ of the lens equation. The magnification $\mu$ of a lensed image $\vartheta$ is the ratio of the solid angle subtended at the distance to the source by the lensed image to the solid angle of the unlensed source. Explicitly,

$$\mu(\vartheta) = \left| \frac{\sin \beta(\vartheta)}{\sin \vartheta} \cdot \frac{d\vartheta}{d\beta(\vartheta)} \right|^{-1} \cdot \text{.} \quad \text{(2)}$$

The lens equation cannot be solved analytically in closed form, so approximations appropriate to the lensing scenario of interest to us will be employed.

Our working assumptions are:

**A1**: The source is assumed to be close to the line of sight, i.e., $|\beta| << 1$, and lie in the asymptotically flat region outside the black hole, i.e., $D_{LS} >> r_s$.

**A2**: In anticipation of later applications to the Galactic black hole, we suppose that $D_L >> b$ and $D_{LS} >> b$, where $b$ is the apparent impact parameter of any light ray from the source to the observer.

**A3**: For any light ray from the source to the observer that do not wind around the black hole, we shall assume that its
The integral in (5) is an elliptic integral and can be expressed

\[ \int_0^\infty \frac{dx}{\sqrt{1-x^2}} = \pi, \]

where

\[ P(x) = x^2[(x/x_0)^2(1 - 1/x_0) - (1 - 1/x)], \quad x_0 = r_0/r_* \]

The integral in (3) is an elliptic integral and can be expressed in terms of an elliptic integral of the first kind. In fact, set

\[ P(x) = \frac{1 - 1/x_0}{x_0^2} X(x) \]

with

\[ X(x) = x \left( x^3 - \frac{x_0^2}{1 - 1/x_0} (x - 1) \right). \]

The zeros of the quartic \( X(x) \) are

\[ r_1 = x_0, \quad r_2 = \frac{x_0}{2(1 - 1/x_0)} \left[ 1 - x_0 + \sqrt{3 + 2x_0 + x_0^2} \right], \]

\[ r_3 = \frac{x_0}{2(1 - 1/x_0)} \left[ 1 - x_0 - \sqrt{3 + 2x_0 + x_0^2} \right], \quad r_4 = 0. \]

Note that \( r_1 > r_2 > r_3 > r_4 \). It can be shown (e.g., Hancock 1958, p. 47) that

\[ \int_0^\infty \frac{dx}{\sqrt{X(x)}} = \frac{2}{(r_1 - r_3)(r_2 - r_4)} F(\phi, k), \]

where

\[ F(\phi, k) = \int_0^\phi (1 - k \sin^2 \phi)^{-1/2} d\phi \]

is an elliptic integral of the first kind with arguments

\[ \phi = \sin^{-1} \sqrt{\frac{r_2 - r_4}{r_1 - r_4}}, \quad k = \frac{(r_1 - r_3)(r_2 - r_4)}{(r_1 - r_3)(r_2 - r_4)}. \]

Using (3), we plot the bending angle \( \hat{\alpha} \) in Figure 1. In addition, Taylor expanding the integrand of in (6) yields the following expansion for the bending angle (Virbhadra & Ellis 2000, Eiroa, Romero & Torres 2002).

\[ \hat{\alpha}(x_0) = \frac{2}{x_0} + \left( \frac{15}{16} \pi - 1 \right) \frac{1}{x_0^3} + O \left( \frac{1}{x_0} \right). \]

For light rays that travel directly from the source to the observer without looping around the black hole, assumption A3 yields that \( 1/x_0 << 1 \). By (6), the bending angle takes the standard weak-field form:

\[ \hat{\alpha}(x_0) \approx \frac{2}{x_0}. \]

Because \( |\hat{\alpha}| << 1 \) in this case and \( |\hat{\vartheta}| << 1 \) by (4), we obtain \( |\hat{\alpha} - \hat{\vartheta}| << 1 \) and, hence,

\[ \tan(\hat{\alpha} - \hat{\vartheta}) \approx \hat{\alpha} - \hat{\vartheta}. \]
For the source-to-observer rays that do not loop, we have $1/x_0 \approx r_*/(\vartheta D_L)$. Combined with equations (3) and (4), we see that the lens equation reduces to the usual weak-field lens equation:

$$\beta = \vartheta - \frac{D_{LS}}{D_S} \hat{\alpha} = \vartheta - \frac{\theta_E^2}{\vartheta},$$  \hspace{1cm} (9)

where

$$\theta_E = \sqrt{2D_{LS}r_* / D_L D_S}.$$  \hspace{1cm} (10)

For rays that loop around the black hole at least once before arriving at the observer, the bending angle $\hat{\alpha}$ is near a multiple of $2\pi$ (cf. Figure 6). Such light rays are close to the unstable photon orbit at radius $3r_g$ (e.g., Chandrasekhar 1992, p. 132). In this case, the bending angle has the form $\hat{\alpha} = 2\pi + \Delta \hat{\alpha}_n$, where $n \geq 1$ is the number of times the light ray loops around the black hole and $|\Delta \hat{\alpha}_n| < < 1$ (representing a small angular deviation from a multiple of $2\pi$). Consequently,

$$\tan(\alpha - \vartheta) \approx \Delta \alpha_n - \vartheta.$$  \hspace{1cm} (11)

Equation (11), along with A1 and A2, yields the following strong-field lens equation for rays circling near $3r_g$ (Bozza et al. 1999): \hspace{1cm} (see text)

$$\beta = \vartheta - \frac{D_{LS}}{D_S} \Delta \alpha_n.$$  \hspace{1cm} (12)

We now turn to the solutions of the weak- and strong-field lens equations.

2.4 Relativistic Images

If $\beta \neq 0$, then we know from Section 2.3 that on opposite sides of the black hole there are a primary image $\vartheta^+_n$ and secondary image $\vartheta^-_n$ of the source:

$$\vartheta^+_n = \frac{\theta_E}{2} \left[ (\beta/\theta_E) \pm \sqrt{(\beta/\theta_E)^2 + 4} \right] \approx \pm \theta_E + \frac{\beta}{2}.$$  \hspace{1cm} (13)

The primary and secondary images have magnification

$$\mu^+_n = \frac{(\beta/\theta_E)^2 + 2}{2(\beta/\theta_E)^2 + 4} \pm \frac{1}{2} \approx \frac{\theta_E}{2|\beta|}. \hspace{1cm} (14)$$

Note that assumption A1 was used for the approximations in (13) and (14). The total magnification is

$$\mu_{tot} = \mu^+_n + \mu^-_n = \frac{\theta_E}{|\beta|}.$$  \hspace{1cm}

To simplify expressions like (13) and (14) that involve $\theta_E$, we shall scale all angles as follows:

$$u = \frac{\beta}{\theta_E}, \hspace{1cm} \theta = \frac{\vartheta}{\theta_E}.$$  \hspace{1cm}

The weak-field image positions and magnifications now become

$$\vartheta^+_n = \frac{u}{2} \pm 1, \hspace{1cm} \mu^+_n = \frac{1}{2|u|}, \hspace{1cm} \mu_{tot} = \frac{1}{|u|}.$$  \hspace{1cm} (15)

Note that the angular image positions and magnifications in (15) for the weak field regime coincide with that obtained from lensing by a point mass.

2.3 Relativistic Einstein rings

If $\beta = 0$, then the source is lensed into the formally infinitely magnified Einstein ring mentioned in Section 2.2, along with an infinite sequence of relativistic Einstein rings near $3r_g$.

The angular radii $\vartheta^E_n$ of the relativistic Einstein rings are given by the solutions of the strong-field lens equation (12) for $\beta = 0$ (Bozza et al. 2001):

$$\vartheta^E_n = \vartheta \left( A_0 e^{-(2n+1)\pi} + B_0 \right), \hspace{1cm} (16)$$

where

$$\vartheta = \frac{r_*}{D_L}, \hspace{1cm} A_0 = \sqrt{3} \left( \frac{18}{2 + \sqrt{3}} \right)^2, \hspace{1cm} B_0 = 3\sqrt{3} / 2.$$  \hspace{1cm}

Note that $\vartheta^E_n$ is approximately the angle spanned by the Schwarzschild radius. For simplicity, set

$$d^+_n = A_0 \vartheta \left( e^{-(2n+1)\pi} + B_0 \right), \hspace{1cm} (17)$$

and rewrite $\vartheta^E_n$ as

$$\vartheta^E_n = d^+_n + \vartheta^E B_0.$$  \hspace{1cm}

Since $\vartheta^E_n << 1$, we have $d^+_1 < 1$ (because $A_0 e^{-3\pi} \approx 3.25 \times 10^{-3}$). Consequently, each term in the strictly decreasing sequence $\{d^+_n\}_{n=1}^\infty$ obeys

$$d^+_n \leq d^+_1 < 1, \hspace{1cm} n = 1, 2, \ldots .$$

2.2 Einstein ring, Primary and Secondary Images

The solutions of the weak-field lens equation (1) are well known (e.g., see Petters, Levine & Wambsganss 2001; pp. 187-191, for a detailed treatment).

If $\beta = 0$ (source is on the optical axis), then the source is lensed into a formally infinitely magnified circle, called an \textit{Einstein ring}, which has an angular radius $\theta_E$.

If $\beta \neq 0$ (source is off the optical axis), then on opposite sides of the black hole there are a primary image $\vartheta^+_n$ and secondary image $\vartheta^-_n$ of the source:

$$\vartheta^+_n = \frac{\theta_E}{2} \left[ (\beta/\theta_E) \pm \sqrt{(\beta/\theta_E)^2 + 4} \right] \approx \pm \theta_E + \frac{\beta}{2}.$$  \hspace{1cm} (13)

The primary and secondary images have magnification

$$\mu^+_n = \frac{(\beta/\theta_E)^2 + 2}{2(\beta/\theta_E)^2 + 4} \pm \frac{1}{2} \approx \frac{\theta_E}{2|\beta|}. \hspace{1cm} (14)$$

Note that assumption A1 was used for the approximations in (13) and (14). The total magnification is

$$\mu_{tot} = \mu^+_n + \mu^-_n = \frac{\theta_E}{|\beta|}.$$  \hspace{1cm}

To simplify expressions like (13) and (14) that involve $\theta_E$, we shall scale all angles as follows:

$$u = \frac{\beta}{\theta_E}, \hspace{1cm} \theta = \frac{\vartheta}{\theta_E}.$$  \hspace{1cm}

The weak-field image positions and magnifications now become

$$\vartheta^+_n = \frac{u}{2} \pm 1, \hspace{1cm} \mu^+_n = \frac{1}{2|u|}, \hspace{1cm} \mu_{tot} = \frac{1}{|u|}.$$  \hspace{1cm} (15)

Note that the angular image positions and magnifications in (15) for the weak field regime coincide with that obtained from lensing by a point mass.
where $\mu$ is $\Theta$ it suffices to study $\Theta$.

In this section, we derive a formula using (23)–(25) to the case of a Reissner-Nordström black hole.

The center-of-light or centroid of all the images of a source are then given as follows (Bozza et al. 2001, Eiroa, Romero & Torres 2002).

$$\Theta_{\text{cent}} = \Theta_{\text{cent}}^R + \Theta_{\text{cent}}^E$$

Equations (26), (27), and (30) simplify to

$$\Theta_{\text{cent}}^R = \frac{2}{\mu_R} \left( \frac{D_S}{D_L} \right)^2 \frac{u}{|u|} \sum_{n=1}^{\infty} \theta_n^E (d_n^*)^2.$$
where
\[ \mu_{\mathrm{ext}} = \frac{\int_D du \, S(u) \mu_{\mathrm{ext}}(u)}{\int_D du \, S(u)} \]
is the weak-field extended source total magnification. Readers are referred to Witt & Mao (1994) for a discussion of \( \mu_{\mathrm{ext}} \) for a point mass lens.

The image centroid of the extended source is
\[ \Theta_{\mathrm{ext}}^{\mathrm{cent}} = \frac{\int_D du \, S(u) \Theta_{\mathrm{cent}} \mu_{\mathrm{ext}}(u)}{\int_D du \, S(u) \, \mu_{\mathrm{ext}}(u)}. \]

Equation (38) gives
\[ \Theta_{\mathrm{cent}}^{\mathrm{ext, wf}} = C_{\mathrm{cent}} \Theta_{\mathrm{ext, wf}}^{\mathrm{cent}}, \]
where
\[ \Theta_{\mathrm{ext, wf}}^{\mathrm{cent}} = \frac{\int_D du \, S(u) \Theta_{\mathrm{cent}}^{\mathrm{ext, wf}}(u) \, \mu_{\mathrm{ext}}(u)}{\int_D du \, S(u) \, \mu_{\mathrm{ext}}(u)} \]
is the weak-field extended image centroid — see Mao & Witt (1994) for a treatment of \( \Theta_{\mathrm{ext, wf}}^{\mathrm{cent}} \) with a point mass lens. Thus, as in the point source case, the quantities \( C_R \) and \( D_R \) determine the relativistic corrections to the extended source image centroid, as well as the total magnification, for a point mass lens.

4 APPLICATIONS TO THE GALACTIC BLACK HOLE LENS

In this section, we estimate the relativistic corrections to the weak-field total magnification and image centroid for the case of the Galactic black hole under A1–A3.

We begin with estimates of some needed quantities. The constants \( C_0(1) \) and \( C_0(2) \) are given approximately as follows:
\[ C_0(1) \approx 8.47 \times 10^{-3}, \quad C_0(2) \approx 2.75 \times 10^{-5}. \]

For the massive black hole at the Galactic center, we have (Ghez et al., 1993):
\[ M_\bullet \approx 2.6 \times 10^6 M_\odot, \quad D_L \approx 8.5 \text{ kpc}. \]
The linear and angular Schwarzschild radii of the black hole are
\[ r_\bullet \approx 5.23 \times 10^{-2} \text{ AU}, \quad \vartheta_\bullet = \frac{r_\bullet}{D_L} \approx 6.46 \text{ mas}. \]

Since the relativistic images are near \( 3r_\bullet/2 \), those on opposite sides of the black hole have angular spacings of order \( 3\vartheta_\bullet \approx 19.38 \text{ mas} \), which is outside the resolving capabilities of near-future instruments. The expected resolution of the Space Interferometry Mission to be launched in 2009 is about 10 mas. Hence, the source is microlensed by the strong-field region near the black hole.

The (weak-field) angular Einstein radius as a function of the lens-source distance \( x = D_{LS} \) is given by
\[ \theta_E(x) = \left( \frac{2\vartheta_\bullet x}{D_L + x} \right)^{1/2} \approx 1.65 \sqrt{\frac{x}{D_L + x}} \text{ as}. \]

The function \( \theta_E(x) \) has a positive derivative and so is strictly increasing — see Figure 2. The angular diameter of the Einstein ring is approximately the angular spacing between the primary and secondary images. For \( 10 \text{ pc} \leq x \leq 100 \text{ pc} \), the range of the angular spacing is \( 0.1 \text{ as} \lesssim 2\theta_E(x) \lesssim 0.4 \text{ as} \), which is at the boundary of the resolving capabilities of present day instruments.

We also have
\[ \theta_\bullet(x) = \frac{\vartheta_\bullet}{\theta_E(x)} \approx 3.922 \times 10^{-6} \sqrt{\frac{D_L + x}{x}} \]
and by (34) we get
\[ D_R(x) \approx 2 \times 10^{-13} \left( \frac{D_L + x}{x} \right). \]

Let us now verify that assumptions A1–A3 hold. We are supposing that \( |\beta| \ll 1 \) and \( D_{LS} \geq 10 \text{ pc} \), which by (40) yields \( D_{LS}/r_\bullet \approx 3.82 \times 10^7 \gg 1 \), so assumption A1 is satisfied. Now, the largest value \( b_{\max} \) of the impact parameter of light rays from the source to the observed occurs for the rays that travel to the observer without looping around the black hole. These rays passing through the nearly flat region outside the black hole, which means that the associated impact parameter and distance of closest approach are approximately equal. Equation (33) shows that the angular positions of the closest approach of the light rays are given approximately by \( \pm \theta_E \) (since \( |\beta| \ll 1 \)). It follows that \( b_{\max} \) can be approximated by the linear Einstein ring radius, \( r_E(x) = D_L \theta_E(x) \) (Figure 2). The function \( x/r_E(x) \) is
strictly increasing with a minimum value of approximately \(x/\tau_E(x) \gtrsim 4.38 \times 10^3 \gg 1\) for \(x \gtrsim 10\) pc — see Figure 3. In addition, since \(r_E(x)\) is also strictly increasing with maximum \(r_E^{\text{max}} \approx 8 \times 10^{-6} D_L\) for \(x \gg D_L\) (see (1)), it follows that \(D_L/r_E \gtrsim D_L/r_E^{\text{max}} \approx 1.3 \times 10^5 \gg 1\). Consequently, assumption A2 holds. Now, since \(r_*/\tau_E(x)\) is strictly decreasing with a maximum value of order \(10^{-4}\) for \(x \gtrsim 10\) pc, the primary and secondary images lie in the nearly flat region outside the black hole. In particular, assumption A3 is also satisfied.

We add that since \(r_*/x\) is at most of order \(10^{-6}\) for \(x \gtrsim 10\) pc, a source at distance \(x\) lies in a region of space that is “flatter” than the region near the linear Einstein radius \(r_E(x)\) (since \(r_*/r_E(x) \ll 10^{-4}\)). Furthermore, for a source star of mass \(M_\star\) and radius \(R_\star\), the closest it can get to the Galactic black hole without being thorn apart is given roughly by the tidal disruption radius of the source (e.g., Magorrian & Tremaine 1999):

\[
r_{\text{tidal}} \approx 137.5 \left( \frac{\eta_\text{E}^2 M_\star}{2.6 \times 10^6 M_\odot} \right)^{1/3} \left( \frac{M_\odot}{M_\star} \right)^{1/3} R_\star,
\]

where \(\eta_\text{E}\) is a parameter of order unity that depends on the model of the source. For \(M_\star \approx M_\odot\) and \(R_\star \approx R_\odot\), the tidal radius is then of order

\[
r_{\text{tidal}} \approx 137.5 R_\odot \approx 0.64 \text{ AU}.
\]

Hence, a source at \(x \gtrsim 10\) pc is significantly outside its tidal disruption radius.

For the Galactic black hole, equation (27) shows that the relativistic correction to the weak-field total magnification is

\[
C_R(x) \approx 2.1 \times 10^{-18} \left( \frac{D_L + x}{x} \right)^{3/2}.
\]

Since the derivative of \(C_R(x)\) is always negative, the correction \(C_R(x)\) is a strictly decreasing function of the lens-source distance \(x\) — see Figure 3. The figure shows that the relativistic correction \(C_R\) to the magnification is at most of order \(5 \times 10^{-14}\) for sources with \(D_L \gtrsim 10\) pc.

Turning to the relativistic image centroid, equations (3) and (12) yield that the derivative of \(D_R(x)/C_R(x)\) is strictly decreasing with increasing lens-source distance — Figure 3. Furthermore, the figure yields that \(C_R(x)\) is at most of order \(10^{-14}\) and \(D_R(x)/C_R(x)\) is at most of order \(10^{-24}\) for \(x \gtrsim 10\) pc, so the relativistic image centroid factor \(C(x)\) can be approximated as follows:

\[
C_{\text{cent}}(x) \approx 1 - C_R(x) \quad \text{for } x \gtrsim 10\text{ pc}.
\]

It follows that the relativistic correction \(C_R\) to the weak-field total magnification also serves as a relativistic correction to the weak-field image centroid. The value \(C_R(x) = 1\) corresponds to the weak-field case, which clearly holds approximately for \(x \gtrsim 10\) pc. In light of Figure 3, measurable relativistic corrections to the total magnification and image centroid due to standard gravitational lensing by the Galactic black hole would most likely require sources very close to the black hole.

Figure 3. The top graph shows that \(D_{LS} \gg \tau_E\) for \(D_{LS} \gtrsim 10\) pc, while the bottom illustrates that \(r_*/\tau_E \ll 1\). The latter implies that the primary and secondary images lie in the nearly flat region outside the Galactic black hole.

5 CONCLUSION

Previous work on gravitational lensing by the black hole at the Galactic center investigated the relativistic corrections to the weak-field total magnification and image centroid to second order in \(1/x_0 = 2GM/(r_0 c^2)\), where \(r_0\) is the distance of closest approach of the light ray to the black hole. It was shown recently that for a point mass lens the total magnification and image centroid shift of a point source remain unchanged by relativistic corrections of second order in \(1/x_0\). We computed the relativistic corrections for a Schwarzschild black hole lens under assumptions A1–A3. These corrections were applied to the case of the massive black hole at the Galactic center. We found that the weak-field magnification and image centroid have approximately the same relativistic correction. This correction is a strictly decreasing function of the lens-source distance \(D_{LS}\). For \(D_{LS} \gtrsim 10\) pc, the relativistic correction is of order at most \(10^{-14}\), a minuscule correction. Hence, for standard lensing configurations, a nontrivial relativistic correction to microlensing by the Galactic black hole would likely have to come from sources deep inside the black hole’s potential well.
Figure 4. The quantities $C_R$ and $D_R C_R$ as a function of the distance $D_{LS}$ of the source from the Galactic black hole, where $10 \text{ pc} \leq D_{LS} \leq 100 \text{ pc}$. Note that these distances are significantly outside the tidal disruption radius, $r_{\text{tidal}} \approx 0.64 \text{ AU}$, for a sun-like source. The bottom figure guarantees that $C_R$ describes the relativistic correction to both the weak-field image centroid and total magnification (see discussion in text). The correction is clearly negligible for sources $D_{LS} \gtrsim 10 \text{ pc}$.

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