On Dual Formulation for Higher Spin Gauge Fields in \((A)dS_d\)

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Abstract

We obtain dual actions for spin \(s \geq 2\) massless fields in \((A)dS_d\) by solving different algebraic constraints in the same first-order theory. Flat space dual higher spin actions obtained by Boulanger, Cnockaert and Henneaux [1] by solving differential constraints are shown to result from our formulation in a sort of quasi-classical approximation for the flat limit. The case of \(s = 2\) is considered in detail.

1. Introduction

The study of dual formulations of classically equivalent field theories has long history (see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 1] and references therein) and is important for a deeper understanding of different faces of superstring theory. In the paper [1] it was shown that the dual formulations [6, 7] of totally symmetric higher spin (HS) massless fields in flat space can be conveniently obtained from the first-order formalism developed in [10, 11]. The idea was to interpret the equations for the frame-like HS fields \(e\) (just the frame \(e^a_\mu\) for the spin-2 case) as differential constraints on the Lorentz connection-type fields \(\omega\). Solving these differential constraints and plugging the resulting expressions back into the original HS action one obtains equivalent description of the HS dynamics in terms of \(\omega\). The main comment of these notes is that the flat limit
description of [11] results from our description within the stationary phase expansion with respect to small $\lambda^2$. The deformation to $(A)dS_d$ is therefore useful for the analysis of dualities avoiding procedures of resolving differential restrictions inevitable in flat space. Our analysis is based on the results of [11] where the gauge form of the description of 4$d$ HS dynamics of [10] was extended to the case of AdS$_d$.

The content of the rest of the paper is as follows. In Section 2, by using the MacDowell-Mansouri formulation of gravity with cosmological term [12], we obtain a dual action for a spin-2 field propagating on the AdS$_d$ background and then take the flat limit within the stationary phase approximation in $\lambda^2$. In Section 3 dual actions for arbitrary spins $s \geq 2$ in $(A)dS_d$ are obtained and their flat limit is analysed. Conclusions are given in Section 4. Appendix contains formulae for $\tau$-operators of [11] used in this paper.

2. Spin-2 case

2.1 MacDowell-Mansouri formulation

As shown by MacDowell and Mansouri [12], 4$d$ gravity with the negative cosmological constant can be formulated in terms of gauge fields of the AdS$_4$ algebra $o(3,2)$. It was shown in [12] that, up to a topological term, Einstein-Hilbert action with the cosmological constant admits an equivalent form

$$S^M = -\frac{\lambda^{-2}}{4\kappa^2} \int \varepsilon_{abcd} \mathcal{R}^{ab} \wedge \mathcal{R}^{cd},$$ (2.1)

with

$$\mathcal{R}^{ab} = d\Omega^{ab} + \Omega^{a}{}_{c} \wedge \Omega^{cb} - \lambda^2 E^{a} \wedge E^{b},$$ (2.2)

where $E^a$ is the frame 1-form and $\Omega$ is the Lorentz connection 1-form.

The straightforward $d$-dimensional generalization of the $4d$ MacDowell-Mansouri pure gravity action is [13]

$$S = -\frac{\lambda^{-2}}{4(d-3)!} \int \varepsilon_{c_{1} \ldots c_{d-4}abfg} E^{c_{1}} \wedge \ldots \wedge E^{c_{d-4}} \wedge \mathcal{R}^{ab} \wedge \mathcal{R}^{fg}.$$ (2.3)

An AdS$_d$ space of radius $\lambda^{-1}$ is described by the frame $E = h$ and Lorentz connection $\Omega = \omega_0$ satisfying the equations

$$d\omega_0^{ab} + \omega_0^a{}_{c} \wedge \omega_0^{cb} - \lambda^2 h^a \wedge h^b = 0,$$ (2.4a)

$$dh^a + \omega_0^a{}_{c} \wedge h^c = 0,$$ (2.4b)

where $a, b = 0 \ldots d - 1$. Clearly, any solution of (2.4) is a solution of the field equations of the action (2.3). In fact, it is the most symmetric vacuum solution of
the theory \cite{23}. The de Sitter case is considered analogously with negative $\lambda^2$. For definiteness we will refer below to the AdS case.

The perturbative expansion near the AdS background is

$$\Omega = \omega_0 + \omega, \quad E = h + e,$$

where $e$ and $\omega$ are dynamical (fluctuational) parts of the gauge fields. Then the linear part of $R^{ab}$ is

$$R^{ab} = D\omega^{ab} - \lambda^2 (h^a \wedge e^b - h^b \wedge e^a), \quad (2.5)$$

with

$$D\omega^{ab} = d\omega^{ab} + \omega_0^c a \wedge \omega^{cb} + \omega_0^c b \wedge \omega^{ac},$$

i.e. $D$ is the background Lorentz covariant differential.

Hence, $d$-dimensional free action of a massless spin-2 field is

$$S^{(2)} = -\frac{\lambda^{-2}}{4(d-3)!} \int \varepsilon_{c_1...c_{d-4}abfg} h^{c_1} \wedge \ldots \wedge h^{c_{d-4}} \wedge R^{ab} \wedge R^{fg}. \quad (2.6)$$

It is invariant under the linearized gauge transformations of the form

$$\delta e^a = D\varepsilon^a - h_b \varepsilon^{ab}, \quad (2.7a)$$
$$\delta \omega^{ab} = D\varepsilon^{ab} - \lambda^2 (h^a \varepsilon^b - h^b \varepsilon^a), \quad (2.7b)$$

where $\varepsilon^a$ and $\varepsilon^{ab} = -\varepsilon^{ba}$ are arbitrary gauge parameters.

### 2.2 Dual action

After conversion of world indices into Lorentz ones by the background frame field the spin-2 action reads\footnote{Here $d\mu$ is the invariant volume-element $\text{det}(h^\mu_c)d^dx$.}

$$S^{(2)} = -\frac{\lambda^{-2}}{16(d-3)!} \int d\mu \, \delta^{mnpq}_{abcd} R_{mn|}^{ab} R_{pq|}^{cd}, \quad (2.8)$$

where

$$R_{cd|}^{ab} \equiv h^\mu_d \mu R_{\mu\nu|}^{ab} = D_\nu \omega_{d|}^{ab} - \lambda^2 (\delta_a^c \delta^b_d - \delta_a^d \delta^b_c) - (c \leftrightarrow d) \quad (2.9)$$

and we use notation

$$\delta^{abcd}_{mnpq} = \frac{1}{(d-4)!} \varepsilon_{e_1...e_{d-4}mnpq} \varepsilon^{e_1...e_{d-4}abcd}. \footnote{Here $d\mu$ is the invariant volume-element $\text{det}(h^\mu_c)d^dx.$}$$
The vertical dash in $R_{cd|}^{ab}$ and $\omega_{d|}^{ab}$ separates converted world indices from the Lorentz ones.

From (2.8) and (2.9) one obtains

$$S^{(2)}[\omega, e] = \frac{-\lambda^{-2}}{4(d-3)} \int d\mu [\delta_{abfg}^{|cd} D_{c\omega_{d|}^{ab}} D_{h\omega_{j|}^{fg}|}]$$

$$- 8\lambda^{2}(d-3)(\omega_{c}^{ab} + \delta_{e}^{b}\omega_{d|}^{bd} - \delta_{d}^{b}\omega_{e|}^{ad}) D_{a}e_{b}^{c}$$

$$+ 4\lambda^{4}(d-3)(d-2)(e_{a}^{b}e_{b}^{a} - e_{a}^{b}e_{b}^{a})]. \quad (2.10)$$

Let us introduce the field $Y_{ab|c}$

$$Y_{ab|c} = \omega_{c|ab} + \eta_{ac}\omega_{d|b}^{d} - \eta_{bc}\omega_{d|a}^{d}.$$

Up to a total derivative, the action (2.10) can be rewritten as

$$S^{(2)}[Y, e] = \int d\mu [Y_{ab|c} Y_{ac|b} - \frac{1}{d-2} Y_{ab|b} Y_{ac|c} - 2D_{a}Y_{ab|c}e_{b}^{c}$$

$$- \lambda^{2}(d-2)(e_{a}^{b}e_{b}^{a} - e_{a}^{b}e_{b}^{a})]. \quad (2.11)$$

Variation of the action (2.11) with respect to $e_{a}^{c}$ gives

$$\lambda^{2}e_{a}^{b} = - \frac{1}{d-2} \left[ D_{a}Y_{bc|}^{c} + \frac{1}{d-1} \delta_{b}^{c}D_{c}Y_{cd|}^{d}\right], \quad (2.12a)$$

$$\lambda^{2}e_{a}^{a} = - \frac{1}{(d-2)(d-1)} D_{a}Y_{ab|}. \quad (2.12b)$$

Using (2.12) one can get rid of the frame field in (2.11) to obtain the second-order action

$$S^{(2)}[Y] = \int d\mu \left[ Y_{ab|c} Y_{ac|b} - \frac{1}{d-2} Y_{ab|b} Y_{ac|c} - \lambda^{-2}\frac{1}{d-2} D_{a}Y_{ab|c}D_{c}Y_{cc|b}$$

$$+ \frac{\lambda^{-2}}{(d-2)(d-1)} D_{a}Y_{ab|c}D_{c}Y_{cd|b}\right]. \quad (2.13)$$

### 2.3 Flat theory as a classical limit

The generating functional $Z$ for the action (2.13) is

$$Z = N \int DY f[Y] e^{i\tilde{S}[Y]}, \quad (2.14)$$

where $N$ is a normalization factor,

$$f[Y] = e^{i \int d\mu (Y_{ab|c} Y_{ac|b} - \frac{1}{d-2} Y_{ab|b} Y_{ac|c})}$$
and
\[
\tilde{S}[Y] = \frac{\lambda^{-2}}{d-2} \int d\mu \left( -D_a Y^{ab|c} D_c Y^{cd|b} + \frac{1}{d-1}D_a Y^{ab|b} D_c Y^{cd|d} \right).
\]

To evaluate this integral in the flat limit \( \lambda \to 0 \) one can apply the stationary phase method. Stationary points of \( \tilde{S}[Y] \) satisfy the equation
\[
D_b D_a Y^{ad|c} - \delta^c_d D_b D_a Y^{ae|c} - (b \leftrightarrow c) = 0.
\]

It is readily seen that, up to the terms proportional to \( \lambda^2 \) which vanish in the flat limit, (2.15) admits a solution of the form
\[
Y_{ab|d}^{\ d} = D_e Y_{abc|d}^{\ e},
\]

where \( Y_{ab|d}^{\ d} = Y_{[ab|d]} \) and \([...]\) mean antisymmetrization.

The key point is that the solutions (2.16) are dominating because \( D_a Y_{abc|d}^{\ d} = \lambda^2 Y_{eab|c} \) and therefore \( \tilde{S}[Y] \sim \lambda^2 \). Contribution from other stationary solutions is suppressed when \( \lambda \to 0 \) since, because of the boundary terms, \( \tilde{S}[Y] \sim \lambda^{-2} \).

Thus, in the flat limit the generating functional has the form
\[
Z \propto \int \mathcal{D}X e^{i \int d\mu (Y_{abc|d}^{\ d} - \frac{1}{d-2}Y_{abc|d} Y^{eac|c})} \bigg|_{Y_{ab|d}^{\ d} = \partial_i Y_{abc|d}^{\ e}}
\]

(2.17)

that is the flat space action is equivalent to
\[
S_{fl}[Y] = \int d^d x \left( \partial_a Y_{abc|d} \partial_b Y^{eac|d} - \frac{1}{d-2} \partial_a Y_{abc|d} \partial_b Y^{eac|d} \right),
\]

(2.18)

which is just the action obtained in [6, 1].

We see that in our approach the dual flat space action results from some sort of quasi-classical approximation with \( \lambda \) being a counterpart of Planck constant in quantum mechanics. From this interpretation it follows in particular that naive deformation of the dual flat space formulation to the AdS case with \( \lambda \neq 0 \) may be difficult unless all-order corrections of the quasi-classical expansion are taken into account, that restores the formulation in terms of the Lorentz connection as a dynamical variable. Hopefully, such interpretation may help to answer some of the questions on the structure of interacting dual theories.

3. Higher spins

3.1 Frame-like formulation

As shown in [11], a totally symmetric massless field of spin \( s \) in any dimension can be described by a collection of 1-forms \( dx^\mu \omega_{a(t)}^{a(s-1),b(t)} (s-1 \geq t \geq 0) \) which are\(^2\)For simplicity throughout this paper we use convention of [10] that symmetrized groups of indices are designated by one and the same letter and a number of indices is given in parentheses,
symmetric in the Lorentz indices $a$ and $b$ separately and satisfy
\begin{align}
\omega_k|_{ab}^{a(s-1),b(t-1)} &= 0, \quad (3.1a) \\
\omega_k|_{c}^{a(s-3)c,b(t)} &= 0, \quad (3.1b)
\end{align}
where symmetrization over $s$ Lorentz indices $a$ is assumed in the first relation $3.1(a)$. From $(3.1)$ it follows that all traces of fiber Lorentz indices are also zero.

Note that the 1-forms $\omega^{a(s-1),b(t)}$ satisfying $(3.1)$ are described by the traceless Young tableaux

\[ \begin{array}{ccc}
1 & 2 & \\
3 & 4 & \\
\end{array} \]

The linearized curvatures have the structure
\[ R^{a(s-1),b(t)} = D\omega^{a(s-1),b(t)} + \tau_-(\omega)^{a(s-1),b(t)} - \lambda^2 \tau_+(\omega)^{a(s-1),b(t)}, \quad (3.2) \]
with
\[ \begin{align*}
\tau_-(\omega)^{a(s-1),b(t)} &= \alpha h_c \wedge \omega^{a(s-1),b(t)c}, \\
\tau_+(\omega)^{a(s-1),b(t)} &= \beta \Pi(h^d \wedge \omega^{a(s-1),b(t-1)}),
\end{align*} \]
where $\Pi$ is the projection operator to the irreducible representation described by the traceless Young tableaux of the Lorentz algebra $o(d-1,1)$ with $s-1$ and $t$ cells in the first and the second rows, respectively. $\alpha$ and $\beta$ are some coefficients which depend on $s, t$ and $d$ and are fixed so that
\[ (\tau_-)^2 = 0, \quad (\tau_+)^2 = 0, \quad D^2 - \lambda^2 \{\tau_-, \tau_+\} = 0. \]
The curvatures $(3.2)$ are invariant under the linearized gauge transformations:
\[ \delta \omega^{a(s-1),b(t)} = D\varepsilon^{a(s-1),b(t)} + \tau_-(\varepsilon)^{a(s-1),b(t)} - \lambda^2 \tau_+(\varepsilon)^{a(s-1),b(t)}, \quad (3.3) \]
where $\varepsilon^{a(s-1),b(t)}$ are arbitrary gauge parameters possessing the symmetry and tracelessness properties analogous to those of $\omega^{a(s-1),b(t)}$ in $(3.1)$. Explicit expressions for $\tau_+$ and $\tau_-$ are given in Appendix.

The quadratic action functional for the massless spin-$s$ field has the form
\[ S^{(s)} = \lambda^{-2} \int \sum_{p=0}^{s-2} \chi^{-2p} \frac{[(p + 1)!]^2}{(d + p - 3)!} \varepsilon_{c_1...c_d} h^{e_5} \wedge ... \wedge h^{e_d} \wedge R^{e_1a(s-2),e_2b(p)} \wedge R^{e_3a(s-2),e_4b(p)}. \quad (3.4) \]
It is fixed up to an overall factor by the condition that its variation with respect to the extra fields $\omega^{a(s-1),b(t)}$ with $t \geq 2$ is identically zero
\[ \frac{\delta S^{(s)}}{\delta \omega^{a(s-1),b(t)}} \equiv 0 \quad \text{for } t \geq 2. \quad (3.5) \]
e.g. $a(s-1)$ means $\{a_1...a_{s-1}\}$.
3.2 Dual action for higher spins

Since the extra fields $\omega^{a(b(t))}$ with $t \geq 2$ contribute to the action only through surface terms, it is enough to take into account the terms with the curvatures $R^{a(b(1))}$ and $R^{a(b(2))}$ in (3.5). Using explicit expressions for the operators $\tau_\pm$ from (11) (see Appendix) one obtains

$$R_{a(s-1),b} = D\omega_{a(s-1),b} + \lambda^2 [h_b \wedge e_{a(s-1)} - h_a \wedge e_{a(s-2)b} + \frac{s-2}{d+s-4} (h^c \wedge e_{cba(s-3)} \eta_{aa} - h^c \wedge e_{ca(s-2)} \eta_{ab})]$$

(3.6a)

and

$$R_{a(s-1),b(2)} = \frac{\lambda^2}{\sqrt{d(s-1)(d+s-3)}} \left[ \frac{(s-2)(d+s-3)}{d-2} h^c \wedge \omega_{a(s-1),c} \eta_{bb} - (d+s-3)(s-2) h^b \wedge \omega_{a(s-1),b} + (d+s-3)(s-1) h_a \wedge \omega_{a(s-2),b} \right. \right.$$  

$$- (s-2)(s-1) h^c \wedge \omega_{ca(s-3),b} \eta_{aa} + (s-3)(s-1) h^c \wedge \omega_{ca(s-2),b} \right.$$  

$$+ (s-2)(s-1) \frac{h_c \wedge \omega_{b(2)a(s-3),c} \eta_{aa}}{d-2} - \frac{(d+2s-6)(s-1)}{d-2} h_c \wedge \omega_{a(s-2),b,c} \delta_b^b \right]$$

(3.6b)

where $\eta_{ab}$ is the flat metric.

From (3.4) one obtains

$$S^{(s)} = \frac{1}{2(d-3)(d-2)} \int d\mu \{ \mathcal{L}_2[\omega] + \lambda^2 \mathcal{L}_1[D\omega, e] \},$$

(3.7)

with

$$\mathcal{L}_1[e, D\omega] \cong \frac{d-2}{4} \delta_{c_1c_2c_3c_4}^{d_1d_2d_3d_4} R_{d_1d_2} c_1 a(s-2), c_2 R_{d_3d_4} c_3 a(s-2), c_4 ,$$  

(3.8a)

$$\mathcal{L}_2[\omega] \cong \lambda^{-4} \delta_{c_1c_2c_3c_4}^{d_1d_2d_3d_4} R_{d_1d_2} c_1 a(s-2), c_2 R_{d_3d_4} c_3 a(s-2), c_4 ,$$

(3.8b)

where $\cong$ indicates that extra fields have been neglected. Substitution of (3.6) into (3.8) gives

$$\frac{\mathcal{L}_1}{d-2} \cong \frac{\lambda^4 s^2(d-3)(d-2)}{(s-1)^2} (e_{c|a(s-2)} c_{d a(s-2)d} - e_{c|a(s-2)} d_{a(s-2)c}$$

$$- 2\lambda^2 s(d-3) \frac{(A_{bc|} d a(s-2), c_{d| a(s-2)d} + A_{bc} c a(s-2), d_{a(s-2)b}$$

$$+ A_{bc} d a(s-2), c_{d| a(s-2)b} + \frac{1}{4} \delta_{c_1c_2c_3c_4} A_{b_1b_2} c_1 a(s-2), c_2 A_{b_3b_4} c_3 a(s-2), c_4$$

(3.9a)
and

\[ L_2 \simeq -\frac{(d + s - 3)(d - 3) + s^2 - s - 1}{s - 1} \omega_{cd}^{a(s-1),c} \omega_{d[a(s-1),d} \]
\[ + (1 + (d + s - 3)(d - 3)(s - 1)) \omega_{c(a(s-2),b}^a \omega_{d]}^{a(s-2),b} - (c \leftrightarrow d), \]

(3.9b)

where \( A_{cd}^{a(s-1),b} = D_c \omega_{d[a(s-1),b} - D_d \omega_{c]}^{a(s-1),b} \). As expected, \( L_2[\omega] \) vanishes for the spin-2 case.

The gauge transformations \((3.3)\) for \( \omega_{c[a(s-1),b} \) have the form

\[ \delta \omega_{c[a(s-1),b} = D_c \varepsilon_{a(s-1),b} + \varepsilon_{a(s-1),bc} + \eta_{cb} \varepsilon_{a(s-1)b} - \eta_{a\varepsilon_{a(s-2)b}} 
+ \frac{s - 2}{d + s - 4} \eta_{a\varepsilon_{a(s-3)c}} \frac{s}{d + s - 4} \eta_{ba \varepsilon_{a(s-2)c}}, \]

(3.10)

where \( \varepsilon_{a(s-1)}, \varepsilon_{a(s-1),b}, \varepsilon_{a(s-1),b(2)} \) are arbitrary traceless parameters possessing the same symmetry properties as the respective connection 1-forms \( \omega_{a(s-1),b(t)} \).

The field \( \omega_{c[a(s-1),b} \) contains the following irreducible Lorentz traceless components

As follows from \((3.10)\), the second tableau with two cells in the second row is pure gauge.

On the other hand the field \( B_{bc[a(s-1)] = \omega_{b[a(s-1),c} - \omega_{c[a(s-1),b} \) has the following components

Comparing the Lorentz tensor patterns of the fields \( B_{bc[a(s-1)] \) and \( \omega_{b[a(s-1),c} \) and taking into account the gauge invariance, one concludes that the action \((3.7)\) can be expressed in terms of \( B_{bc[a(s-1)] \). The final result is

\[ S^{(s)}[e, B] = \frac{s}{2(d - 2)} \int d\mu \left( \frac{\lambda^2 s(d - 2)}{(s - 1)^2} (e_{c[a(s-2)]} e_{d]}^{a(s-2)d} - e_{a[a(s-2)d} e_{d]}^{a(s-1)}) \]
\[ + \frac{4(d - 2)}{s - 1} D_{[c e [d]}^{a(s-1)} (B_{a]^{c[a(s-2)] d]} - \frac{1}{2(s - 1)} B_{cd}^{a(s-1)}) \]
\[ + 2 \delta_c^a B_d^{d|e} a(s-2) + (s - 2) \delta_a^c B_{e a]}^{d|e} a(s-3)^d \]
\[ + (d + s - 4)(B_{cb}^{c[a(s-2)] B_d^{d[da(s-2]} + (s - 1) B_{ca]}^{c[a(s-2)] B_d^{d[da(s-2)}} \]
\[ + B_{bc[a(s-1)] B^{ab}[ca(s-2)]} = \frac{1}{2(s - 1)} B_{bc[a(s-1)] B^{bc[a(s-1)]}) \}. \]

(3.11)

Let us introduce the field \( Y_{cd}^{a(s-1)} \)

\[ Y_{cd}^{a(s-1)} = B_{cd}^{a(s-2)} - \frac{1}{2(s - 1)} B_{cd}^{a(s-1)} - 2 \delta_c^a B_{db}^{a(s-2)b} \]
\[ + (s - 2) \delta_c^a B_d^{a(s-3)b} - (c \leftrightarrow d), \]

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which has the properties

\[ Y_{cd}^{a(s-1)} = Y_{cd}^{a(s-1)} \text{ and } Y_{cd}^{a(s-3)b} = 0. \quad (3.12) \]

Now the action \((3.11)\) can be rewritten in the form

\[
S^{(s)}[e, Y] = \int d\mu \left( \frac{\lambda^2}{2(s-1)^2} \left( e_c[a(s-2)] e_d^{a(s-2)d} - e_a[a(s-2)] e_d^{a(s-1)} \right) + \frac{s}{s-1} \frac{1}{d+s-4} \left( (s-3)Y_{bc}[a(s-2)] Y^{c[a(s-3)d]} - (s-2)Y_{bc}[a(s-2)] Y^{ad[c(a(s-3))]} \right) + \frac{s-2}{2(s-1)} Y_{bc}[a(s-1)] Y^{bc[a(s-1)]} \right). \quad (3.13)
\]

Variation of the action \((3.13)\) with respect to \(e_c[a(s-1)]\) yields

\[
\lambda^2 e_{a[a(s-2)]} = -\frac{s-1}{s(d-2)} \left( D_c Y^{cd|a(s-1)} - \frac{s-1}{d+s-3} \delta^d_a D_c Y^{cb|a(s-2)b} \right), \quad (3.14a)
\]

\[
\lambda^2 e_{d[a(s-1)]} = \frac{(s-1)^2}{s(d-2)(d+s-3)} D_c Y^{cd|a(s-2)d}. \quad (3.14b)
\]

Taking into account that \(e_{b[a(s-1)]} = s e_{b[a(a(s-1))]} - (s-1) e_{a[a(s-2)b]}\) one obtains

\[
\lambda^2 e_{d[a(s-1)]} = -\frac{s-1}{s(d-2)} \left[ \frac{(s-1)(s-3)}{d+s-3} \delta^d_a D_c Y^{cb|a(s-2)b} + (s-1) D_c Y^{c[a(s-2)d]} - (s-2) D_c Y^{c[a(s-1)]} \right] - \frac{(s-1)(s-2)}{d+s-3} \eta_{aa} D_c Y^{cb|a(s-3)d}. \quad (3.15)
\]

Plugging this expression back into \((3.13)\) one gets the dual action in terms of the field \(Y\)

\[
S^{(s)}[Y] = \frac{1}{2(d-2)} \int d\mu \left( \lambda^2 \left[ \frac{(s-1)(s-3)}{d+s-3} D_c Y^{cb|a(s-2)b} D_f Y^{fd[a(s-2)]} - (s-2) D_c Y^{c[a(s-1)]} D_f Y^{f[a(s-1)]} + (s-1) D_c Y^{c_e[a(s-1)]} D_f Y^{f[a(s-2)]} \right] + (s-1)(d-2) \left( Y_{bc}[a(s-1)] Y^{ba[c(a(s-2))]} + \frac{s-2}{2(s-1)} Y_{bc}[a(s-1)] Y^{bc[a(s-1)]} \right) \right). \quad (3.16)
\]

For \(s = 2\) one recovers the action \((2.13)\) of Section 2.2.
3.3 Flat limit

The flat limit of the action (3.16) is obtained analogously to the spin-2 case. Firstly, one writes the equation for stationary points of the part of the action singular in $\lambda$:

$$0 = (s - 1)D_f D_c Y^{e}_{a|a(s-2)d} - (s - 2)D_f D_c Y^{e}_{d|a(s-1)} + \frac{s - 1}{d + s - 3}[ (s - 3)D_f D_c Y^{eb}_{ba(s-2)}\eta_{ad} - (s - 2)D_f D_c Y^{eb}_{bda(s-3)}\eta_{ad}] - (d \leftrightarrow f) + O(\lambda^2),$$  

(3.17)

where the properties (3.12) were taken into account.

One observes that, up to the terms proportional to $\lambda^2$, the equation (3.17) has a solution

$$Y_{bc|a(s-1)} = D_d Y^{d}_{bc|a(s-1)},$$  

(3.18)

where

$$Y_{bcd|a(s-1)} = Y_{[bcd]|a(s-1)}, \quad Y_{bcd|a(s-3)e} = 0.$$

The solutions (3.18) are dominating because $D_b Y^{b}_{c|a(s-1)} \sim \lambda^2$, i.e. no boundary terms contribute for this case. Hence, the part of the action (3.16) singular in $\lambda$ becomes proportional to $\lambda^2$ being restricted to (3.18). Contribution from other stationary solutions is suppressed due to non-zero boundary terms.

Thus, up to an overall factor, the action (3.16) is

$$S_{fl}[Y] = \int d^d x \left\{ -\partial_e Y^{e}_{bc|a(s-1)}\partial_f Y^{fba|ca(s-2)} + \frac{s - 2}{2(s - 1)}\partial_e Y^{e}_{bc|a(s-1)}\partial_f Y^{f|a(s-1)} \
\quad + \frac{1}{d + s - 4}[ (s - 3)\partial_e Y^{e}_{bc|a(s-2)} b \partial_f Y^{f|d|a(s-2) d} \
\quad - (s - 2)\partial_e Y^{e}_{bc|a(s-2)} b \partial_f Y^{f|d|a(s-3) d}] \right\}.$$  

(3.19)

This is precisely the action of [1].

4. Conclusion

The main conclusion of this paper is that flat space dual actions for massless fields of spins $s \geq 2$ result from a kind of quasi-classical approximation in which inverse (A)dS$_d$ radius $\lambda$ is interpreted as a small parameter analogous to the Planck constant in quantum mechanics. For $\lambda \neq 0$, the action we start with is manifestly equivalent to the original HS action because it results from resolution of algebraic constraints which express frame-like fields in terms of (derivatives of) Lorentz connection-like fields. The resulting dual action contains inverse powers of $\lambda$, so that its flat limit can be conveniently analysed within stationary phase approximation. Since the free dual HS actions are shown to result from the quasi-classical expansion in $\lambda^2$, our approach
can also be useful for the analysis of dual formulations of interacting theories in the flat and $(A)dS_d$ cases.

It should be mentioned that, unfortunately, the important case of the spin-1 field cannot be handled in our approach since there is no auxiliary Lorentz-like connection for spin-1 (a related property is that the spin-1 field equations do not contain curvature-dependent mass-like terms in the $(A)dS_d$ space-time).

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**Appendix. Operators $\tau_{\pm}$**

Here we reproduce some relevant formulae of [11].

The fields $dx^b\omega_{\mu|a(s),b(t)}$ can be conveniently described in terms of “generating functions”, which are “state vectors” in the Fock space generated by the operators $a^a_n$, $a^{+a}_{m}$ ($n, m = 1, 2$) obeying Heisenberg commutation relations,

$$[a^a_n, a^{+a}_m] = \delta_{nm}\eta^{ab}, \quad [a^a_n, a^b_m] = [a^{+a}_n, a^{+b}_m] = 0. \quad (1)$$

Given $p$-form $A_{a(s),b(t)}$, we introduce a “Fock vector”

$$|A(p, s, t)\rangle = a_1^{+a_1} \ldots a_s^{+a_s} a_2^{+b_2} \ldots a_t^{+b_t} A_{a(s),b(t)}|0\rangle. \quad (2)$$

Consider the subspace extracted by the conditions

$$a_2^{b_2} a_{1b_1}^+] A(p, s, t)|0\rangle = 0, \quad (3a)$$

$$a^n_a a_{mb}|A(p, s, t)\rangle = 0, \quad (3b)$$

$$N_1|A(p, s, t)\rangle = (s - 1)|A(p, s, t)\rangle, \quad (3c)$$

where $N_\alpha = a^{+a}_\alpha a_{mb}$.

The linearized HS curvatures $|R(2, s, t)\rangle$ are

$$|R(2, s, t)\rangle = D|\omega(1, s, t)\rangle + \tau_-|\omega(1, s, t + 1)\rangle - \lambda^2 \tau_+|\omega(1, s, t - 1)\rangle. \quad (4)$$
Here $D$ is the Lorentz covariant differential with respect to background Lorentz connection and $\tau_{\pm}$ have the following form

$$\tau_- = \left[\frac{(d + N_1 + N_2 - 3)(N_1 - N_2 + 1)}{d + 2N_2 - 2}\right]^{1/2} h^b a_{2b}, \quad (5a)$$

$$\tau_+ =\left[(d + N_1 + N_2 - 4)(N_1 - N_2 + 2)(d + 2N_2 - 4)\right]^{-1/2} \times \{ (d + N_1 + N_2 - 4)[(N_1 - N_2 + 1)h b a_{2b}^+ - h b a_{1}^{+} a_{1c}] + (a_{1}^{+} a_{1b}^+)(a_{2}^{+} a_{1c})h^a a_{1a} - \frac{1}{d + 2N_2 - 6}(a_{1}^{+} a_{1b}^+)(a_{2}^{+} a_{1c})^2 h^a a_{2a} \}

- (N_1 - N_2)(a_{1}^{+} a_{2b}^+)^2 h^a a_{1a} + \frac{d + 2N_1 - 4}{d + 2N_2 - 6}(a_{1}^{+} a_{2b}^+)(a_{2}^{+} a_{1c})h^a a_{2a} - \frac{(N_1 - N_2 + 1)(d + N_1 + N_2 - 4)}{d + 2N_2 - 6}(a_{2}^{+} a_{2b}^+)^2 h^a a_{2a}\} . \quad (5b)$$

All non-polynomial functions of $N_{1,2}$ are understood as usual by

$$f(N_1, N_2)|A(p, s, t)\rangle = f(s, t)|A(p, s, t)\rangle .$$

The curvatures (4) are invariant under the linearized gauge transformations (3.3)

$$\delta|\omega(1, s, t)\rangle = D|\varepsilon(0, s, t)\rangle + \tau_-|\varepsilon(0, s, t + 1)\rangle - \lambda^2 \tau_+|\varepsilon(0, s, t - 1)\rangle , \quad (6)$$

where $|\varepsilon(0, s, t)\rangle$ is an arbitrary gauge parameter.

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