Approximation of the Partition Number After Hardy and Ramanujan: An Application of Data Fitting Method in Combinatorics

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LI Wenwei
School of Mathematical Science, University of Science and Technology of China,
NO. 96, Road Jinzhai, Hefei, Anhui, P. R. China, 230026
E-Mail: liwenwei@ustc.edu

Abstract

Sometimes we need the approximate value of the partition number in a simple and efficient way. There are already several formulae to calculate the partition number \( p(n) \). But they are either inconvenient for most people (not majored in math) who do not want to write programs, or unsatisfying in accuracy. By bringing in two parameters in the Hardy-Ramanujan’s Asymptotic formula and fitting the data of the two parameters by least square method, iteration method and some other special designed methods, several revised elementary estimation formulae with high accuracy for \( p(n) \) are obtained. With these estimation formulae, the approximate value of \( p(n) \) can be calculated by a pocket calculator without programming function. The main difficulty is that the usual methods to fit the data of the two parameters by an elementary function is defective here. These method could be used in finding the fitting functions of some other complex data.

Key Words: Partition number, Estimation formula, Curve Fitting, Accuracy.
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Contents

1 Introduction 2
2 Main idea 4
3 Fit the Exponent 5
4 Fit the Denominator 8
5 Some Other Methods 10
  5.1 Modify the Denominator only 10
  5.2 Fit \( \text{R}_h(n) - p(n) \) 11
6 Approximate \( p(n) \) by Fitting \( \text{R}_h(n) - p(n) \) 12
  6.1 Result 1 12
  6.2 Result 2 13
  6.3 Result 3 14
7 Estimate \( p(n) \) When \( n \leq 100 \) 14
8 Conclusions 15
Acknowledgements 16
References 16
1 Introduction

The partition number \( p(n) \) is an interesting topic which attracts many attention. There are already a lot of literatures on many aspects of \( p(n) \). Many mathematicians, such as Euler, Hardy, Ramanujan, Rademacher, Newman, Erdős, Andrews, Berndt and Ono, have made important contribution to this topic. Some important literatures may be found in [1], or in the references of [21], [5], [4] and [17].

In recent years, a very important result dues to Ken Ono and his team who connected the partition function with the modular form and found the principles (refer [2], [9], [6] and [7]).

For a positive integer \( n \), an integer solution of the equation

\[
s_1 + s_2 + \cdots + s_q = n \quad (1 \leq s_1 \leq s_2 \leq \cdots \leq s_q, \quad q \geq 1),
\]

(1)

for all the possible integer \( q \) (where \( s_1, s_2, \ldots, s_q \) are unknowns) is called a partition of \( n \). The number of all the partitions of \( n \) is denoted by \( p(n) \), which is also called the partition number or the partition function.

In a lot of occasions, we need the value of \( p(n) \). There are already several formulae to calculate \( p(n) \).

In reference [10] (p.53, p.57) or [13], we may find the generation function of \( p(n) \) obtained by Euler:

\[
F(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n} = \prod_{i=1}^{\infty} (1-x^i)^{-1},
\]

(2)

and a formula

\[
p(n) = \frac{1}{2\pi i} \oint_C F(x) x^{n+1} \, dx,
\]

(3)

where \( C \) is a contour around the original point. Of course, we seldom use (3) to compute the value of \( p(n) \) in practical.

There is a recursion for \( p(n) \) ([10], p.55),

\[
p(n) = p(n-1) + p(n-2) - p(n-5) - \cdots + (-1)^{k-1} p \left( n - \frac{3k^2 + k}{2} \right) + \cdots.
\]

(4)

where

\[
k_1 = \left[ \frac{\sqrt{24n+1} - 1}{6} \right], \quad k_2 = \left[ \frac{\sqrt{24n+1} + 1}{6} \right],
\]

and assume that \( p(0) = 1 \). Here \([x]\) stands for the maximum integer that will not exceed the real number \( x \).

Equation (4) is much better for computing \( p(n) \). We can obtain the exact value of \( p(n) \) efficiently with a program based on it. But it is not convenient for many people who do not want to write programs. Further more, if we want to calculate \( p(n) \) by (4) by a small program written in C or some other general computer language, it is usually necessary to decide the size of the space in memory to store the results beforehand, which means we should know the approximate value of \( p(n) \) before the calculation started, (actually, here it is sufficient to know \([\log_2 p(n)+1] \) , where \([x]\) stands for the minimum integer that is greater than or equal to the real number \( x \) otherwise we have to do some extra work for overflow handling and consequently change the size of the space in memory to store the value of the variable that stands for \( p(n) \).

Obviously, the datatypes already defined in the C language itself are not suitable.

If we use the Dynamic Memory Allocation method, this problem is solved at the price of the program becoming a little more complicated. Actually, in a lot of cases, we can not decide the approximate size of the result, it is the best choice available.

If we can use maple, maximal, axiom or some other computer algebra systems, there is no need to consider this problem. But it is not always an option, especially when the function to do this job is part of a big program written in a compile language while mixing programming of an interpretative language and a compile language is nearly unavailable in most cases (with very few exceptions, such as mixing programming C and matlab).

The analysis of \( p(n) \) by contour integral with (3) (refer [10], p. 57) resulted a very good estimation of \( p(n) \),

\[
p(n) = \sum_{q=1}^{\infty} A_q(n) \cdot \phi_q(n) + O(n^{-1/2}),
\]

(6)

called the Hardy-Ramanujan formula (refer [11] and [10]), that 6 terms of this formula contain an error of 0.004 when \( n = 100 \), while 8 terms of this formula contain an error of 0.004 when \( n = 200 \). Here \( \alpha \) is an
arbitrary constant,

\[ \phi_q(n) = \frac{\sqrt{q}}{2\pi \sqrt{2}} \frac{d}{dn} \left( \frac{\exp\left(\frac{\pi}{q} \sqrt{\frac{2}{3}} (n - \frac{1}{24})\right)}{\sqrt{n - \frac{1}{24}}} \right) \]

(while \( p \) runs through the non-negative integers that are prime to \( q \) and less than \( q \)), \( \omega_{p,q} \) is a certain 24q-th root of unity, \( \left(\frac{a}{b}\right) \) is the Legendre symbol. \( b \) is an odd prime, and \( p' \) is any positive integer such that \( q \mid (1 + pp') \). When \( n \) is very large, \( p(n) \) is the integer nearest to \( \sum_{q=1}^{\lfloor \frac{\alpha \sqrt{n}}{\pi} \rfloor} A_q(n) \cdot \phi_q(n) \).

In [10] or [16], a convergent series for \( p(n) \) modified from [6] by Rademacher in 1937 is presented,

\[ p(n) = \sum_{q=1}^{\infty} A_q(n) \cdot \psi_q(n), \tag{7} \]

where \( A_q(n) \) is the same as mentioned above and

\[ \psi_q(n) = \frac{\sqrt{q}}{\pi \sqrt{2} \sqrt{n}} \frac{d}{dn} \left( \frac{\sinh\left(\frac{\pi}{q} \sqrt{\frac{2}{3}} (n - \frac{1}{24})\right)}{\sqrt{n - \frac{1}{24}}} \right). \]

Equations (6) or (7) are valuable in theory and can be used to calculate the value of \( p(n) \) with very high accuracy. But they are not convenient for practical usage especially when \( n \) is small, since it is very difficult for programmers, engineers or other ordinary people (not familiar with any computer algebra system softwares) since they are too complicated and they contain some special functions that most people (not majored in mathematics) are not familiar with. It is very difficult for them to use these two formulae to calculate \( p(n) \) on a pocket science calculator without programming function.

In references [21] or [3], we may find the famous asymptotic formula for \( p(n) \),

\[ p(n) \sim \frac{1}{4n \sqrt{3}} \exp\left(\sqrt{\frac{2}{3} \pi n^{3/2}}\right), \tag{8} \]

obtained by Godfrey Harold Hardy and Srinivasa Ramanujan in 1918 in the famous paper [11]. (Two different proofs can be found in [8] and [14]. The evaluation of the constants was shown in [13].) This formula will be called the Hardy-Ramanujan’s asymptotic formula in this paper. This asymptotic formula is with great importance in theory. Equation (5) is much more convenient than formulae (6) and (7) for ordinary people not majored in mathematics.

Let

\[ R_h(n) = \frac{1}{4n^{3/2}} \exp\left(\sqrt{\frac{2}{3} \pi \sqrt{n}}\right). \tag{9} \]

be the asymptotic function by Hardy and Ramanujan.

By the figure in reference [18], this asymptotic formula fits \( p(n) \) very well when \( n \) is huge. But when \( n \) is small, the relative error of \( R_h(n) \) to \( p(n) \) is not so satisfying as shown in Table 1 (when \( n \leq 1000 \)) on page 12. When \( n \leq 25 \), the relative error is greater than 9%; when 25 < \( n \leq 220 \), the relative error is greater than 3%; when \( n \leq 1000 \), the relative error is greater than 1.4%. From Figure 1.1 we will find out that the relative error is greater than 0.44% when 1000 ≤ \( n \leq 10000 \). Considering that \( p(n) \) is an integer and \( R_h(n) \) is definitely not, the round approximation of \( R_h(n) \) may be a little more accurate, but that does not help.

Although (9) is not so accurate when \( n \) is small, it provides some important clue for a more accurate formula for small \( n \).

By revising (6), some other estimation formulae with high accuracy is obtained here.

In Section 2 the main idea is introduced, two parameters \( C_1 \) and \( C_2 \) are brought in the Hardy-Ramanujan’s asymptotic formula, they will be fitted in sections 3 and 4 respectively. Sections 5 and 6 will show some other methods to obtain estimation formulae. Section 7 displays an estimation formula with more accuracy when \( n \leq 100 \).

The main difficulty is that it is too hard to obtain the appropriate functions to fit the data of \( C_1 \) (or \( C_2 \) or...
some others) generated here since we know very little about them and the usual methods to find fitting functions are invalid here. If we fit the data directly, the results are far from satisfactory, at least the accuracy is not as good as that of [8].

2 Main idea

There are many different ways to modify \( R_h(n) \), e.g. we could also construct a function \( p_1(n) \) to estimate \( R_h(n) - p(n) \), then \( R_h(n) - p_1(n) \) may reach a better accuracy when estimating \( p(n) \), or we can estimate the value of \( \frac{R_h(n)}{p(n)} \) by a function \( f_1(n) \) then estimate \( p(n) \) by \( \frac{R_h(n)}{f_1(n)} \), etc. The problem is that the accuracy of \( R_h(n) - p_1(n) \) is not so satisfying if we do not use the idea shown in (10) in Section 2, because the shape of the figure of \( \ln (R_h(n) - p(n)) \) is nearly the same as the shape of the figure of \( \ln (p(n)) \), at least we can not tell the difference by our eyes as shown on Figure 20 and Figure 21 (on page 12), though they are different in theory.

Since \( p(n) \sim R_h(n) \), we believe that an approximate formula with better accuracy may be in this form

\[
p(n) \approx \frac{1}{4\sqrt{3(n + C_2)}} \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n + C_1} \right). \tag{10}
\]

Where \( C_1 \) (or \( C_2 \)) may be a constant or a function of \( n \) that increases slowly than \( n \), so as to have

\[
\lim_{n \to \infty} \frac{1}{n^{\frac{1}{2}}} \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n + C_1} \right) = 1,
\]

\[
\lim_{n \to \infty} \frac{1}{n^{\frac{1}{2}}} \frac{R_h(n)}{p(n)} = 1.
\]

There are some other ways to modify \( R_h(n) \), we will discuss the details in section 5.

Table 1: The relative error of \( R_h(n) \) to \( p(n) \) when \( n \leq 1000 \).

| \( n \) | Rel-Err | \( n \) | Rel-Err | \( n \) | Rel-Err | \( n \) | Rel-Err | \( n \) | Rel-Err |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 87.67% | 16 | 11.60% | 40 | 7.34% | 220 | 3.05% | 520 | 1.97% |
| 2 | 35.76% | 17 | 12.03% | 50 | 6.54% | 240 | 2.92% | 540 | 1.93% |
| 3 | 36.35% | 18 | 10.91% | 60 | 5.95% | 250 | 2.86% | 560 | 1.90% |
| 4 | 22.00% | 19 | 11.25% | 70 | 5.56% | 240 | 2.79% | 580 | 1.86% |
| 5 | 27.74% | 20 | 10.43% | 80 | 5.13% | 250 | 2.69% | 600 | 1.83% |
| 6 | 17.11% | 21 | 10.32% | 90 | 4.83% | 250 | 2.52% | 640 | 1.77% |
| 7 | 21.78% | 22 | 9.90% | 100 | 4.57% | 250 | 2.44% | 680 | 1.72% |
| 8 | 16.08% | 23 | 10.05% | 110 | 4.35% | 250 | 2.37% | 720 | 1.67% |
| 9 | 17.50% | 24 | 9.49% | 120 | 4.16% | 250 | 2.29% | 760 | 1.63% |
| 10 | 14.53% | 25 | 9.56% | 130 | 3.99% | 250 | 2.21% | 800 | 1.58% |
| 11 | 16.02% | 26 | 9.16% | 140 | 3.84% | 250 | 2.14% | 840 | 1.55% |
| 12 | 12.91% | 27 | 9.15% | 150 | 3.71% | 250 | 2.06% | 880 | 1.51% |
| 13 | 14.22% | 28 | 8.82% | 160 | 3.69% | 250 | 2.10% | 920 | 1.48% |
| 14 | 12.56% | 29 | 8.11% | 170 | 3.49% | 250 | 2.05% | 960 | 1.44% |
| 15 | 12.86% | 30 | 8.50% | 200 | 3.20% | 250 | 2.01% | 1000 | 1.42% |

As we can not determine \( C_1 \) and \( C_2 \) at the same time because of technique problems, we may decide \( C_1 \) first then determine \( C_2 \), the main reason is that \( \frac{1}{(n + C_2)} \) and \( \frac{1}{n} \) differs very little when \( n \) is very huge, at least we believe that the difference is much less than the difference of \( \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n + C_1} \right) \) and \( \exp \left( \frac{\sqrt{3}}{3} \pi \sqrt{n} \right) \).

So, when \( n \gg 1 \), we believe

\[
p(n) \approx \frac{1}{4\sqrt{3n}} \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n + C_1} \right),
\]

\[\text{Figure 2: The graph of the data } \left( n, \frac{3}{2} \left( \frac{\ln (4n\sqrt{np(n)})}{\pi^2} \right)^2 \right).\]

\[\text{Figure 3: The graph of the data } \left( n, \frac{3}{2} \left( \frac{\ln (4n\sqrt{np(n)})}{\pi^2} \right)^2 \right).\]

\[\text{1 Usually, we will get the value of } C_1 \text{ and/or } C_2 \text{ from a number of pairs of } (n, p(n)) \text{ by the least square method, not from two pairs of } (n, p(n)) \text{ only. Many software can get efficiently}
\]

\[\text{the undetermined coefficients (by the least square method) by solving a system of (incompatible) linear equations, while it is very difficult to “solve” a system of tens or hundreds of transcendental equations that are incompatible.}\]

\[\text{2 It is not difficult to know that } \frac{1}{n(\pi^2)} \approx \frac{1}{n} \left( 1 - \frac{2}{n} \right), \text{ as } \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n + \delta} \right) \approx \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n} \right) \left( 1 + \frac{\delta}{n \sqrt{\frac{2}{3}} \pi} \right), \text{ when } \delta \ll n. \text{ Obviously, } \frac{\delta}{n} \ll \frac{1}{n \sqrt{\frac{2}{3}} \pi} \text{ when } \max(\delta, 1) \ll n.\]

\[\text{Table 0.1: The relative error of } R_h(n) \text{ to } p(n) \text{ when } n \leq 1000.}\]
3 Fit the Exponent

If we point the data \((n, C_1(n))\), i.e., 
\[
\left( n, \frac{3}{2} \cdot \frac{(\ln(4n\sqrt{3}p(n)))^2}{\pi^2} - n \right) \quad (n = 20k + 100, \; k = 1, 2, \cdots, 395)
\]
in the coordinate system, we will find that they lie in a straight line, as shown in the Figure 2 on page 4, which means that the Hardy-Ramanujan’s asymptotic formula is close to perfect. Here every tiny cycle stands for a data point.

Figure 4: The graph of the data \(\left( n, \frac{3}{2} \cdot \frac{(\ln(4n\sqrt{3}p(n)))^2}{\pi^2} \right) \) \((n \leq 80)\).

Figure 5: The graph of the data \((n, C_1(n))\) \((80 \leq n \leq 200)\).

hence \(4\sqrt{3}n \times p(n) \doteq \exp \left( \pi \sqrt{\frac{2}{3}(n + C_1)} \right)\), then

\[
C_1(n) \doteq \frac{3}{2} \cdot \frac{(\ln(4n\sqrt{3}p(n)))^2}{\pi^2} - n. \tag{11}
\]

If we point the data \(\left( n, \frac{3}{2} \cdot \frac{(\ln(4n\sqrt{3}p(n)))^2}{\pi^2} \right) \) \((n = 20k + 100, \; k = 1, 2, \cdots, 395)\) in the coordinate system, we will get the Figure 3 on page 4. Here the points when \(n \leq 120\) are not shown on Figure 3, partly because the deduction above is based on \(n \gg 1\), the main reason is that the points obviously do not lie in a curve when \(n \leq 120\), as shown on Figure 4 and Figure 5 (on page 5).

3 Fit the Exponent

Figure 6: The graph of a bad fitting curve of the data \((n, C_1(n))\)

Figure 3 looks like a logarithmic curve or a hyperbola. The author has tried many functions (by a small program written in MAPLE) like

\[
y = a \cdot (\ln(x^{e_1} + c_1))^{e_2} + b,
\]

where \(c_1, e_2\) and \(c_2\) are given constants while \(a\) and \(b\) are undermined coefficients to be decided. But none of them fits the data very well. A function

\[
y = a \cdot \left( \ln \left( \frac{7}{20} \cdot x - 16 \right) \right)^{29/32} + 2.5 + b,
\]

where \(a = 0.06656839293\) and \(b = -0.4166945066\), may fit the data better, but it is not as good as we expect, as shown on Figure 6 on page 5.

A hyperbola like \(y = \frac{a}{x} + b\) does not fit the data very well, either. Then we consider this type of functions

\[
y = \frac{a}{(x + c_2)^{e_2}} + b, \tag{12}
\]

where \(a, b, c_2\) and \(c_2\) are undetermined constants. This seems much better. For technique reason, we can not decide all the undetermined coefficients \(a, b, c_2, e_2\) at the same time.

These undetermined coefficients may be obtained in this way:

- A1. Give \(c_2\) and \(c_2\) initial values;
- A2. Fit the data \((n, C_1(n))\) by the least square method with Equation (12) and obtain the values of \(a\) and \(b\), then get the average error of the fitting function for the values of \(c_2, e_2, a, b\).

Because most computer algebra system (CAS) could not solve the system of many incompatible nonlinear equations by the least square method, or the time-consumption is unacceptable.

Here we use the square root of the mean square deviation

\[
s = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (y_i - f(x_i))^2}
\]
• A3. Reevaluate $c_2$ and $a$. Plot the points of the data \( \ln (n + c_2), \ln (b - C_1(n)) \) \((n = 20k + 100, k = 1, 2, \cdots, 395) \) in the coordinate system with the values of $b$ and $c_2$ just found, fit the data by the least square method with
\[
y = e_1 \cdot x + a_1
\]
and find the values of $a_1$ and $e_1$, then reevaluate $e_2$ and $a$ by
\[
e_2 = -e_1, a = -\exp(a_1);
\]

• A4. Reevaluate $c_2$. Plot the points of the data
\[
(n, \left(\frac{a}{C_1(n)^{1/e_2}}\right)^{1/e_2}) \quad (n = 20k + 100, k = 1, 2, \cdots, 395)
\]
in the coordinate system with the value of $b$ and the new values of $a$ and $e_2$, fit the data by the least square method with
\[
y = x + c_1
\]
and find the value of $c_1$, then reevaluate $e_2$ by $e_2 = c_2 - c_1$.

• A5. goto step 2 until a fitting function with the least average error is obtained.

For example, in step A1, the initial value could be set by $c_2 = 2.5, e_2 = 0.5$ (or some other values).

In step A2, if $c_2 = 2.5, e_2 = 0.5$, then $a = -0.0263583935, b = -0.3456348045$.

If we plot the figure of (12) with the value of $c_2, e_2, a, b$, and compare the figure with Figure 3 on page 4, we will get a graph nearly the same as Figure 3 (although there should be a little different, but we can not distinguish the difference by our eyes). The average error of the fitting function for the values of $c_2, e_2, a, b$ mentioned above is 1.074574171×10⁻⁵, which seems to be very tiny.

3 FIT THE EXPONENT

In Step A3, if $c_2 = 2.5, b = -0.3456365954, a_1 = -3.626380777, e_1 = -0.5012314726$.

After reevaluation, $e_2 = 0.5012314726, a = -0.02661232627$.

In Step A4, for the values of $b, c_2$ and $a$ mentioned before, after reevaluation $c_2 = 4.871833842$.

Figure 7: The graph of a good fitting curve of the data \((n, C_1(n))\)

Actually, only a few times of repeating the steps form A2 to A4, we will obtain a very good fitting function, as shown on Figure 7 on page 6.

For the initial value $c_2 = 2.5, e_2 = 0.5$, after repeating 41 times of the steps from A2 to A4, we will find a fitting function
\[
y = \left(\frac{a}{(x + 3.320623832)^{0.4963284361}}\right) - 0.3456286995,
\]
with a minimal average error 9.010349470×10⁻⁸.

After a few times more of iteration, a result with similar coefficients will be found but with a little more error.

There are some explanations about the steps above:

• (1). In step A4, we did not plot the points of the data \((n, \left(\frac{a}{C_1(n)^{1/e_2}}\right)^{1/e_2})\) because the
shape of the figure is not a horizontal line as shown on Figure 10 on page 7 (the points in the right hand side are not so smooth because only 10 significance digits are kept in the process, if more significance digits are calculated, it will be better). Actually, it is a little complicated. But it will not help us to obtain better values of the undetermined in (12) if we fit the data \( \left( n, \left( e^{\frac{a}{C_1(n)-n}} \right)^{1/\epsilon_2} - n \right) \) with a more accurate fitting function.

• (2). In step A3, if we do not reevaluate \( a \), the fitting parameters will not converge in general (even if we computing more significant figures in the process), or we can not continue the iterations steps at all since imaginary numbers appear.

• (3). If we started with a different initial value of \( c_2 \) and keep the initial value of \( c_2 \), such as \( c_2 = 15 \), after repeating 78 times of the steps from A2 to A4, we will find a fitting function

\[
y = \frac{-0.02503608938}{(x + 3.272445238)0.4962730054} - 0.3456286681\]

(14)

with a minimal average error \( 9.109686836 \times 10^{-8} \).

If we started with some different initial values for both \( c_2 \) and \( c_2 \), such as \( c_2 = 15 \) and \( c_2 = 0.7 \), (from Figure 2 on page 4, we will find that \( c_2 \) should be less that 1.0), we will get a similar result. After repeating 125 times of the steps from A2 to A4, we will find a fitting function

\[
y = \frac{-0.02503617719}{(x + 3.273513225)0.4962727258} - 0.3456286655\]

(15)

with a minimal average error \( 9.105941452 \times 10^{-8} \).

After that, \( c_2 \) and \( c_2 \) will decrease slowly and slowly, and the average error will increase little by little if we continue the steps from A2 to A4.

As concerned to the errors in computing, the valid value of the undetermined \( a \), \( b \), \( c_2 \) and \( c_2 \) should be \(-0.0259361, -0.34562866, 3.273, 0.49627\), the average absolute error of the fitting function of \( C_1(n) \) is about \( 9.1 \times 10^{-8} \).

Considering that (11) is an approximate formula, we may believe that the best value of \( c_2 \) is 0.5, as we prefer a simple exponent. Then it will be more convenient to obtain \( a \) and \( c_2 \).

Below \( c_2 \) is supposed to be 1/2, which means that the fitting function of \( C_1(n) \) is

\[
y = \frac{a}{\sqrt{x + c_2}} + b.\]

(16)

\[
\text{Figure 9: The graph of the data } \left( n, \left( e^{\frac{a}{C_1(n)-n}} \right)^{1/\epsilon_2} - n \right) \]

When \( c_2 \) is fixed to be 1/2, if we use the iteration method described above but keep the value of \( c_2 \) in step A3, i.e., substitute step A3 by

A3'. Reevaluate \( a \) by

\[
a = -\exp \left( \frac{1}{395} \sum_{k=1}^{395} \left( \ln (b - C_1(20k + 100)) - e_2 \cdot \ln (20k + 100 + c_2) \right) \right);\]

(that means we evaluate \( a \) twice in every loop) the sequence of fitting functions of \( C_1(n) \) will diverge. But we will obtain a converged sequence of the determinants if \( n \) ranges from 120 to 6000, (i.e., consider only the data \( \left( n, p(n) \right) \) when \( n = 20k + 100, k = 1, 2, \ldots, 295 \)). The fitting function of \( C_1(n) \) obtained in this way (when \( n \) ranges from 120 to 6000, step 20) is

\[
y = \frac{-0.02650620466}{\sqrt{x + 4.855479108}} - 0.3456326154, \]

\[\text{Figure 10: The graph of the data } \left( n, \left( e^{\frac{a}{C_1(n)-n}} \right)^{1/\epsilon_2} - n \right) \]

\[
\text{or equivalently, Plot the points of the data } \left( \ln (n + c_2), \ln (b - C_1(n)) \right) (n = 20k + 100, k = 1, 2, \ldots, 295) \text{ in the coordinate system with the values of } b, c_2 \text{ and } c_2 \text{ just found, fit the data by the least square method with } y = e_2 \cdot x + a_1 \text{ and find the values of } a_1, \text{ then reevaluate } a \text{ by } a = -\exp(a_1);\]
with the minimal average error $2.374935895 \times 10^{-7}$.\footnote{If we use the value of $c_1$ already found above, such as $c_2 = 3.273513225$ in (15), the fitting function is}

For the fixed value $1/2$ of $c_2$, if we continue use the iteration method described above but ignore step 3, which means we reevaluate $a$ only once in every loop, we will meet the same situation. The sequence of fitting functions of $C_1(n)$ will diverge if $n$ ranges from 120 to 8000 (or 6000) even if we calculate more significance digits (such as 18 significance digits) in the process, but it will converge if $n$ ranges from 120 to 4000. The fitting function of $C_1(n)$ obtained in this way (when $n$ ranges from 120 to 4000, step 20) is

$$
y = \frac{-0.02647712648}{\sqrt{x + 4.55083607}} - 0.345633305, \quad (18)
$$

with the minimal average error $1.993012726 \times 10^{-7}$ when the initial value of $c_2$ is 10 (iterated 4 times). But after more times of iteration, for several initial values of $c_2$ (such as 5, 10, 15, etc), the fitting functions converge to

$$
y = \frac{-0.0268 \cdots}{\sqrt{x + 4.888 \cdots}} - 0.345632760 \cdots, \quad (19)
$$

with the average error $2.68 \cdots \times 10^{-7}$.

Unlike the previous method, by the results mentioned above and some other results not mentioned here, the sequence of fitting functions of $C_1(n)$ usually converges to a function which is obviously different from the one with the minimal average error.

In order to get a fitting function with errors as tiny as possible, we can design another algorithm.

By the results described above, we know that $c_2$ is probably between 3 and 5, so we can find the fitting function of $C_1(n)$ and the corresponding average error for many values of $c_2$ in the possible range, then choose the one with minimal average error. To be cautious, we test the value of $c_2$ in the interval $[0.5, 15]$. The main steps are as below:

- (1) Initial $c_a, c_b, c_0, s_0, D_t, a_0, b_0$. Let $c_a := 0.5, c_b := 15, c_0 := 0, s_0 := 1, a_0 := 0, b_0 := 0, D_t := 8, s_t := 0.1$.

- (2) for $c_2$ from $c_a$ to $c_b$ by $s_t$ do
  
  Fit the data $(n, C_1(n))$ by the least square method with (10) and get the values of $a$ and $b$, then get the average error $s_1$ of the fitting function for the values of $c_2, a, b$; if $s_1 < s_0$, then let $c_0 := c_2, s_0 := s_1, a_0 := a, b_0 := b$; end if;

  - (3) If $D_t > 1$, then set $D_t := D_t - 1, c_a := c_0 - 5s_t, c_b := c_0 + 5s_t$;
    set $s_t := s_t / 10$; goto step (2);
  else, terminate the process.

Here the symbol “$x := y$” means that the variable $x$ is evaluated by the value of the variable $y$; in step (1), $D_t := 8$ means that we will get 8 significance digits of the value of $c_2$.

In the algorithm above, we have assumed implicitly that the average error is a smooth and continuous function of $a, b, c_2$ for the values of $x = 20k + 100, (k = 1, 2, \cdots, 395)$. For every $c_2$, we can get the value of $a$ and $b$, then obtain the the average error $s_1$, so $s_1$ could be believed as a convex and smooth function of $c_2$ (hence it will have only one minimum point) in the interval we are considering. This could be verified by plotting the figure of the curve $s_1 = s_1(c_2)$ in the given interval (although this work is not easy in practice).

If $n$ ranges from 120 to 8000 (step 20), we can get a fitting function of $C_1(n)$,

$$
y = \frac{-0.02651010067}{\sqrt{x + 4.8444724}} - 0.3456324524, \quad (20)
$$

with a minimal average error $2.446731760 \times 10^{-7}$.

If $n$ ranges from 120 to 6000 (step 20), the fitting function of $C_1(n)$ is,

$$
y = \frac{-0.02649625326}{\sqrt{x + 4.7152127}} - 0.3456327903, \quad (21)
$$

with a minimal average error $2.279396699 \times 10^{-7}$.

In the next section, (20) will be used to estimate $C_1(n)$, i.e.,

$$
C_1(n) \doteq -0.02651010067 \frac{1}{\sqrt{n + 4.8444724}} - 0.3456324524. \quad (22)
$$

4 Fit the Denominator

By (10) and (22), we have

$$
C_2(n) \doteq \frac{\exp \left( \frac{n}{3} \frac{C_1(n)}{\sqrt{n + C_1(n)}} \right)}{4 \sqrt[4]{3} \rho(n)} - n. \quad (23)
$$
If we point out the data \((n, C_2(n))\) \((1 \leq n \leq 80)\) on the coordinate system as shown on Figure 13 on page 9, we will immediately know that \(C_2(n)\) cannot be fit by a simple function. From the Figure 13 (or the value of \(C_2(n)\) calculated by a small program), it is clear that \(C_2(n)\) is very small when \(n > 40\), at least much less than \(n\), so there is no need to fit \(C_2(n)\) when \(n > 40\).

When \(n\) is odd, the points of \((n, C_2(n))\) in Figure 13 are above the horizontal-axis, it is not difficult to separate them into two parts and fit them by two cubic curves, as shown on Figure 11 and Figure 12. The two fitting functions are

\[
y = -1.548835311 \times 10^{-6} \times x^3 + 1.880663805 \times 10^{-4} \times x^2 - 0.008334098201 \times x + 0.1399798428,
\]
\[
y = -5.416501948 \times 10^{-6} \times x^3 + 5.728510889 \times 10^{-4} \times x^2 - 0.02125835759 \times x + 0.2882706948.
\]

For the points of \((n, C_2(n))\) under the horizontal-axis (when \(n\) is even) in Figure 13 we have to separate them into at least 4 parts so as to fit them smoothly, two or three parts are not convenient.

As a result, we have to fit \(C_2(n)\) by a hybrid function with at least 6 pieces, or fit \(p(n)\) by a piecewise-defined function with 7 pieces, which is very complicated. This seems to contradict with our purpose at the beginning of this paper.

From Figure 11 on page 9 we found that the value of \(C_2(n)\) are much less than \(n\) when \(n \geq 15\), so the error will be very tiny if we omit \(C_2(n)\). Hence we can calculate \(p(n)\) directly by

\[
R_{h1}(n) = \frac{1}{4\sqrt{3n}} \exp\left(\frac{2}{3} \pi \sqrt{n + \frac{a_1}{\sqrt{n + c_1}} + b_1}\right),
\]

(24)

where \(a_1 = -0.02651010067\), \(b_1 = -0.3456324524\) and \(c_1 = 4.8444724\).

The error of (24) to \(p(n)\) \((n \leq 1000)\) is shown on Table 2 on page 10. The accuracy is much better than (6). Although this fitting function is obtained when \(n \geq 120\), the relative error is less than \(6 \times 10^{-7}\) when \(n \geq 100\), less than 1\% when \(n \geq 26\), less than 1\% when \(n \geq 11\). When \(1000 \leq n \leq 3000\), the relative error is less than \(1 \times 10^{-8}\). When \(3000 \leq n \leq 10000\), the relative error is less than \(5.3 \times 10^{-9}\), as shown on Figure 14 on page 9. But the relative error is not so satisfying when \(n \leq 7\), especially when \(n = 1\).

Consider that \(p(n)\) is an integer, if we take the round approximation of (24),

\[
R'_{h1}(n) = \left[\frac{\exp\left(\sqrt{\frac{2}{3} \pi} \sqrt{n + \frac{a_1}{\sqrt{n + c_1}} + b_1}\right)}{4\sqrt{3n}} + \frac{1}{2}\right],
\]

(25)

(we may call it Hardy-Ramanujan’s revised estimation formula 1), it will solve perfectly the relative
error problem when \( n < 11 \), as shown on Table 3, although the relative error will increase very little for some \( n \), which is negligible. (The average relative error is less than \( 2 \times 10^{-8} \) when \( n \geq 200 \).

Take an example, when \( n = 100 \), \( R_{h2}(100) = 190569177 \), \( p(100) = 190569292 \), the difference is 115; when \( n = 200 \), \( R_{h2}(200) = 3972999059745 \), \( p(200) = 3972999029388 \), the difference is 30357. Although the errors are much greater than the error 0.004 of Hardy-Ramanujan formula with 6 terms \((n = 100)\) or 8 terms \((n = 200)\) (refer [11] or [16]), it contains only one term of elementary functions, and is convenient for a junior middle school student to calculate the value of \( p(n) \) with high accuracy.

### 5.1 Modify the Denominator only

If we assume that \( p(n) \approx \exp\left(\frac{\pi \sqrt{n}}{4\sqrt{3}n} \right) \), then

\[
C_2(n) \doteq \frac{1}{4\sqrt{3}p(n)} \exp\left(\frac{\pi \sqrt{2n}}{3} \right) - n,
\]

where

\[
y = 0.4432884566 \times \sqrt{x + 0.274078 + 0.1325096085} \]

to fit \( C_2(n) \) with an average error \( 3.65 \times 10^{-6} \).

Hence we can calculate \( p(n) \) by

\[
R_{h2}(n) = \left[ \frac{\exp\left(\sqrt{\frac{\pi}{2} \sqrt{n}}\right)}{4\sqrt{3}n} - \frac{1}{2} \right] ^2,
\]

when \( n \) is not so small.
Table 4: The relative error of $R_{h2}(n)$ to $p(n)$ when $n \leq 1000$.

| n     | Rel-Err |
|-------|---------|
| 1     | 14.93%  |
| 2     | -3.06%  |
| 3     | 3.96%   |
| 4     | -3.34%  |
| 5     | 3.84%   |
| 6     | -2.99%  |
| 7     | 2.36%   |
| 8     | -1.29%  |
| 9     | 0.88%   |
| 10    | -0.87%  |
| 11    | 1.12%   |
| 12    | -0.99%  |
| 13    | 0.68%   |
| 14    | -0.36%  |

Table 5: The relative error of $R_{h2}(n)$ to $p(n)$ when $n \leq 80$.

When $n \geq 40$, the relative error differs very little.

The error of $p(n)$ to $p(n)$ is shown on Table 3 on page 11 when $n < 1000$. The accuracy is much better than Table 2. Compared with Table 2, the accuracy are almost the same when $n \geq 1000$. When $1500 \leq n \leq 10000$, the relative error is obviously less than that of Table 2, as shown on Figure 10 on page 11 compared with Figure 15 on page 10. Which means that $R_{h2}(n)$ is more accurate than $R_{h1}(n)$. (If we change the range of $n$ of the data points, the accuracy of the fitting function obtained may not be so good.)

Consider that $p(n)$ is an integer, we can take the round approximation of $p(n)$.

$$R_{h2}(n) = \left\lfloor \frac{\exp\left(\sqrt{\frac{2}{3} \pi \sqrt{n}}\right)}{4 \sqrt{3} (n + a_2 \sqrt{n} + c_2 + b_2)} + \frac{1}{2} \right\rfloor,$$  \hspace{1cm} (27)

for small values of $n$. We may call it Hardy-Ramanujan’s revised estimation formula. The error of $R_{h2}(n)$ to $p(n)$ is shown on Table 5 (on page 11) when $n \leq 1000$.

5.2 Fit $R_{h2}(n)/p(n)$

At the beginning of section 2, some other methods to estimate $p(n)$ are mentioned, such as estimating the value of $\frac{R_{h2}(n)}{p(n)}$ by a function $f_1(n)$, then estimate $p(n)$ by $R_{h2}(n)/f_1(n)$.

The data $(n, \frac{R_{h2}(n)}{f_1(n)})$ $(n = 20k + 100, k = 1, 2, \cdots, 395)$ are shown on Figure 16 on page 12 (together with the figure of a fitting function). It is not difficult to find out that a function

$$y = 1 + \frac{1}{\sqrt{a_3 x + b_3}},$$

where $a_3 = 5.062307637$ and $b_3 = -75.65700620$, will fit the data very well, as shown on the figure, with an average error $1.41 \times 10^{-4}$. (because the data $(n, \left(\frac{R_{h2}(n)}{f_1(n)} - 1\right)^2$) lies exactly on a straight line $y = a_3 x + b_3$, as shown on Figure 19 on page 12.

Figure 16: The Relative Error of $R_{h2}(n)$ when $1000 \leq n \leq 10000$

Figure 17: The Relative Error of $R_{d3}(n)$ when $1000 \leq n \leq 1000$

So we have another fitting function for $p(n)$.

$$R_{d3}(n) = \frac{R_{h2}(n)}{1 + \sqrt{a_3 n + b_3}}.$$

However, this formula does not fit $p(n)$ very well when $n$ is small. When $n \leq 14$, the value of $R_{d3}(n)$ is an imaginary number. Unfortunately, when $n > 1000$, the error of $R_{d3}(n)$ to $p(n)$ is about 1000 times of the error of $R_{h2}(n)$, as shown on Figure 17 on page 11.

Actually, $R_{h2}(n)$ is in the form $\frac{R_{h2}(n)}{f_1(n)}$, since

$$\frac{\exp\left(\sqrt{\frac{2}{3} \pi \sqrt{n}}\right)}{4 \sqrt{3} (n + a_2 \sqrt{n} + c_2 + b_2)} = \frac{\exp\left(\sqrt{\frac{2}{3} \pi \sqrt{n}}\right)}{4 \sqrt{3} n^{3/2}}.$$

11
It is not difficult to verify that \( 1 + \frac{a_2 \sqrt{n+2} + b_2}{n} \) fits \( R_h(n) \) with very little error, \( 1 + \frac{1}{\sqrt{a_3 n + b_3}} \) will not reach that accuracy.

\[
\frac{R_h(n)}{1 + \frac{a_2 \sqrt{n+2} + b_2}{n}}.
\]

As \( 1 \) fits \( R_h(n) \), we wonder whether we can fit \( R_h(n) - p(n) \) by an expression similar like the right part of (28) such as \( \pi \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n} \right) \), where \( C_3(n) \) is a cubic function, or equivalently, fit \( \frac{\pi \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n} \right)}{12 \sqrt{2} n} \) by a cubic function \( C_3(n) \), from the data with the data \( (n, p(n)) \) \( (n = 20k + 60, k = 1, 2, \cdots, 397) \). The result is

\[
C_3(n) = a_1 n^3 + b_1 n^2 + c_1 n + d_1,
\]

where

\[
a_1 = 8.383485427,
\]

\[
b_1 = 130.0792015,
\]

\[
c_1 = -1.197477259 \times 10^5,
\]

\[
d_1 = 4.188653689 \times 10^7.
\]

Here \( c_1 \) and \( d_1 \) are very huge, which suggests that this result may not be so satisfying. As a sequence, if we estimate \( p(n) \) by

\[
F_3(n) = \frac{\pi \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n} \right)}{12 \sqrt{2} C_3(n)},
\]

the relative error differs very little with the relative error of \( R_h(n) \) to \( p(n) \) when \( n < 50 \), but the relative error is not satisfying when \( n < 280 \), as shown in Table 6 on page 12.

If we fit \( \frac{\pi \exp \left( \sqrt{\frac{2}{3}} \pi \sqrt{n} \right)}{12 \sqrt{2} n} \) by a function like

\[
C_3(n) = a_2 n^3 + b_2 n^2 + c_2 n^2 + d_2 n + e_2 n^2 + f_2 n^{0.5} + g_2,
\]

the result are even worse, since imaginary number appeared (as concerned to the data mentioned in this
Table 6: The relative error of $F_0(n)$ to $p(n)$ when $n \leq 1000$.

In the previous sub-section, we obtained the asymptotic order of $p(n) - p(n - 1)$, and revised it to fit $R_0(n) - p(n)$. Since $R_0(n)$ is always a little greater than $p(n)$, we may guess that there is a $t_0$ such that $R_0(n - t_0)$ is closer to $p(n)$ than $R_0(n)$. Then we can find the asymptotic order of $R_0(n) - R_0(n - t_0)$ and use the new asymptotic order to fit $R_0(n) - p(n)$.

6.2 Result 2

In the previous sub-section, we obtained the asymptotic order of $p(n) - p(n - 1)$, and revised it to fit $R_0(n) - p(n)$. Since $R_0(n)$ is always a little greater than $p(n)$, we may guess that there is a $t_0$ such that $R_0(n - t_0)$ is closer to $p(n)$ than $R_0(n)$. Then we can find the asymptotic order of $R_0(n) - R_0(n - t_0)$ and use the new asymptotic order to fit $R_0(n) - p(n)$.

By the same idea described in the algorithm mentioned on page 8, we can obtain the value $t_0 \approx 0.3594143172$.

When $n \gg 1$ and $t \gg 1$, we have

$$r(n) = R_0(n) - R_0(n - t) \approx \exp\left(\frac{\sqrt{\pi} \sqrt{n}}{4 \sqrt{3n}}\right) - \exp\left(\frac{\sqrt{\pi} \sqrt{n - t}}{4 \sqrt{3(n - t)}}\right)$$

$$\approx \exp\left(\frac{\sqrt{\pi} \sqrt{n}}{4 \sqrt{3n}}\right) - \exp\left(\frac{\sqrt{\pi} \sqrt{n - t}}{4 \sqrt{3(n - t)}}\right)$$

$$\approx \frac{\sqrt{\pi} \sqrt{n}}{4 \sqrt{3n}} - \frac{1}{n - t}$$

As

$$r(n) \sim \frac{\pi}{12 \sqrt{2} \sqrt{3}} \exp\left(\frac{\sqrt{\pi} \sqrt{n}}{\sqrt{3}}\right),$$

so we may consider fitting $R_0(n) - p(n)$ by

$$C_4(n) = a_2(n - t_0)^{1/5} + b_2(n - t_0) + c_2(n - t_0)^{0.5} + d_2.$$

Table 7: The relative error of $R_0(n)$ to $p(n)$ when $n \leq 1000$.

When $t_0 \approx 0.3594143172$, it is not difficult to find out that

$$a_2 = 1.039888529,$$

$$b_2 = -0.3305666395,$$

$$c_2 = 0.6134039843,$$

$$d_2 = -0.8582796393,$$

from the data $(n, p(n))$ ($n = 20k + 60$, $k = 1, 2, \ldots, 397$). Here none of the coefficients is very huge, which seems better than the previous result mentioned in this section. As a matter of fact, if we estimate $p(n)$ by

$$R_{13}(n) = R_0(n) - \sqrt{2\pi \ln t_0} \exp\left(\frac{\sqrt{\pi} \sqrt{n - t_0}}{24C_4(n)}\right),$$

the relative error of $R_{13}(n)$ to $p(n)$ when $n \leq 30$.

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the relative error of $R_{13}(n)$ to $p(n)$ when $n \leq 30$.
the relative error is very small even when \( n < 10 \) (except the cases when \( n = 1 \) or 2) as shown on Table 7 on page 13. This is the first time to have an estimation formula of \( p(n) \) which can reach a good accuracy without taking round approximation even when \( n < 10 \).

Further more, if we get the round value of \( R_{h_3}(n) \),

\[
R'_{h_3}(n) = \left| R_{h_3}(n) - \frac{\sqrt{2t_0\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n-t_0}}\right)}}{24C_4(n)} + \frac{1}{2} \right|
\]

(32)

the relative error to error is even less, especially when \( n = 15 \) or \( 1 < n < 12 \) (it reaches 0), as shown on Table 8 on page 13. The relative error is less than \( 3 \times 10^{-9} \) when \( 2500 < n < 10000 \), as shown on Figure 20 on page 12. This formula will be called Hardy-Ramanujan’s revised estimation formula 3.

### 6.3 Result 3

Now that we can fit \( R_{h_3}(n) \) by \( \sqrt{2t_0\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n-t_0}}\right)} \), where

\[
C_4(n) = a_2(n-t_0)^{1.5} + b_2(n-t_0) + c_2(n-t_0)^{0.5} + d_2,
\]

maybe we can also fit \( R_{h_3}(n) - p(n) \) by

\[
\frac{\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n}}\right)}{12\sqrt{2C_5(n)}}
\]

directly, where

\[
C_5(n) = a_3n^{1.5} + b_3n + c_3n^{0.5} + d_3,
\]

or equivalently, to fit \( \frac{\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n}}\right)}{12\sqrt{2C_5(n)}} - p(n) \) by a function \( C_5(n) \) in the form mentioned above.

We can easily obtain the value of the unknown coefficients in the equation above by the least square method,

\[
a_3 = 2.893270736,
b_3 = 0.4164546941,
c_3 = -0.08501098214,
d_3 = -0.4621004962.
\]

Again, none of the coefficients is very huge. As a result, the relative error of

\[
R'_{h_4}(n) = \left| R_{h_4}(n) - \frac{\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n}}\right)}{12\sqrt{2C_5(n)}} + \frac{1}{2} \right|
\]

(34)

to \( p(n) \) is also very small when \( n < 10 \) (even in the cases when \( n = 1 \) or 2) as shown on Table 9 on page 15. This is the first time to obtain an estimation formula of \( p(n) \) which can reach a very good accuracy even when \( n < 10 \).

Further more, if we get the round value of \( R_{h_4}(n) \),

\[
R''_{h_4}(n) = \left| R_{h_4}(n) - \frac{\pi\exp\left(\sqrt{\frac{2}{3}\pi\sqrt{n}}\right)}{12\sqrt{2C_5(n)}} + \frac{1}{2} \right|
\]

(35)

the relative error to error is even less, especially when \( n = 15 \) or \( 1 < n < 12 \) it reaches 0, as shown on Table 10 on page 15. The relative error is less than \( 1 \times 10^{-9} \) when \( 2500 < n < 10000 \), as shown on Figure 21 on page 12. That is much better than \( R_{h_3}(n) \) and \( R'_{h_3}(n) \), besides, it is more simple. This formula will be called Hardy-Ramanujan’s revised estimation formula 4.

### 7 Estimate \( p(n) \) When \( n \leq 100 \)

Until now, all the estimation function generated for \( p(n) \) are with very good accuracy when \( n \) is greater than 100, but they are not so accurate when \( n < 50 \). Although \( R'_{h_3}(n) \) and \( R'_{h_4}(n) \) are better than others, the relative error are still greater than 1%\(e\) for some values of \( n \).

On the other hand, in sections 3 and 4 when \( n < 100 \), it is nearly impossible to fit

\[
C_1(n) = \frac{3}{2} \cdot \left( \ln(4\sqrt{3}p(n)) \right)^2 - n
\]

In this section, we will give out a completely new estimation formula which is very accurate and very simple when \( n < 100 \) and is more simple and exact when \( n < 10 \).
Table 0.1: The relative error of

\[ n^{\sqrt{\frac{3p(n)}{\pi}}}-n \]

The relative error of \( R_{\text{h0}}(n) \) to \( p(n) \) when \( n \leq 1000 \).

| n \( \text{Rel-Err} \) | n \( \text{Rel-Err} \) | n \( \text{Rel-Err} \) | n \( \text{Rel-Err} \) | n \( \text{Rel-Err} \) |
|---|---|---|---|---|
| 1 | 5.4E-05 | 16 | -0.38% | 40 | -0.02% |
| 2 | -0.03% | 17 | -0.38% | 41 | -0.02% |
| 3 | -0.38% | 18 | -0.38% | 42 | -0.02% |
| 4 | -0.15% | 19 | -0.38% | 43 | -0.02% |
| 5 | -0.03% | 20 | -0.38% | 44 | -0.02% |
| 6 | -0.38% | 21 | -0.38% | 45 | -0.02% |
| 7 | -0.38% | 22 | -0.38% | 46 | -0.02% |
| 8 | -0.38% | 23 | -0.38% | 47 | -0.02% |
| 9 | -0.38% | 24 | -0.38% | 48 | -0.02% |
| 10 | -0.38% | 25 | -0.38% | 49 | -0.02% |
| 11 | -0.38% | 26 | -0.38% | 50 | -0.02% |
| 12 | -0.38% | 27 | -0.38% | 51 | -0.02% |
| 13 | -0.38% | 28 | -0.38% | 52 | -0.02% |
| 14 | -0.38% | 29 | -0.38% | 53 | -0.02% |
| 15 | -0.38% | 30 | -0.38% | 54 | -0.02% |

Table 0.2: The relative error of

\[ n^{\sqrt{\frac{3p(n)}{\pi}}}-n \]

The relative error of \( R_{\text{h2}}(n) \) to \( p(n) \) when \( n \leq 30 \).

When \( n \geq 22 \), the relative error differs very little.

Table 10: The relative error of \( R_{\text{h4}}(n) \) to \( p(n) \) when \( n \leq 30 \).

When \( n \geq 22 \), the relative error differs very little.

So we wander whether we can fit the data

\[ \left( n, \frac{\exp(\pi \sqrt{\frac{3p(n)}{\pi}})}{4\sqrt{3p(n)}} - n \right) (n = 3, 4, \ldots, 100) \]

by a piecewise function (with 2 pieces) so as to get a better estimation of \( p(n) \) when \( n \leq 100 \)?

The figure of the points of the data

\[ \left( n, \frac{\exp(\pi \sqrt{\frac{3p(n)}{\pi}})}{4\sqrt{3p(n)}} - n \right) (n = 3, 4, \ldots, 100) \]

are shown on Figure 24 (on page 15). It is not difficult to find that the even points (where \( n \) is even) roughly on a smooth curve, so are the odd points. If we try to fit them respectively, we will have the fitting function below:

\[
C_2'(n) = \begin{cases} 
0.4527092482 \times \sqrt{n} + 4.35278- & \text{if } n = 3, 5, 7, \ldots, 99; \\
0.4412187317 \times \sqrt{n} - 2.01699+ & \text{if } n = 4, 6, 8 \ldots, 100. 
\end{cases}
\] (36)

Hence we can calculate \( p(n) \) by

\[ R_{\text{h0}}(n) = \frac{\exp(\pi \sqrt{n})}{4\sqrt{3(n + C_2(n))}}, \text{ } 1 \leq n \leq 100. \] (37)

Consider that \( p(n) \) is an integer, we can take the round approximation of \( \left( \frac{\exp(\pi \sqrt{n})}{4\sqrt{3(n + C_2(n))}} \right) \), \( 1 \leq n \leq 100. \) (38)

The relative error of \( R_{\text{h0}}(n) \) (or \( R_{\text{h0}}'(n) \)) to \( p(n) \) are shown on Table 1 (or Table 12) on page 16. Compared with Table 5 on page 11 we will find that when \( n \geq 80 \), \( R_{\text{h2}}(n) \) is more accurate than \( R_{\text{h0}}'(n) \); when \( n \leq 50 \), \( R_{\text{h0}}'(n) \) is obviously better.

8 Conclusions

In this paper, we have presented several elementary estimation formulae with high accuracy to calculated \( p(n) \), that can be operated on a pocket science calculator without programming function.

When \( n \leq 80 \), we can use \( R_{\text{h2}}'(n) \) (Equation (38)) , with a relative error less than 0.004%; when \( n > 80 \), we can use \( R_{\text{h2}}(n) \) (Equation 27).
Table 0.1: The relative error of $F_4(n)$ to $p(n)$ when $n \leq 100$.

| $n$ | Rel-Err  | $n$ | Rel-Err  | $n$ | Rel-Err  | $n$ | Rel-Err  | $n$ | Rel-Err  |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 1   | -5.81%   | 21  | -6.05E-04| 41  | -1.04E-04| 61  | -1.94E-06| 81  | 2.44E-05 |
| 2   |          | 22  | 3.96E-04 | 42  | 1.54E-04 | 62  | 2.57E-05 | 82  | -3.79E-05|
| 3   | -1.97%   | 23  | -1.05E-05| 43  | -9.20E-05| 63  | 1.72E-06 | 83  | 2.51E-05 |
| 4   | 1.00%    | 24  | -9.13E-06| 44  | 1.55E-04 | 64  | 1.58E-05 | 84  | -4.18E-05|
| 5   | 0.90%    | 25  | 2.49E-04 | 45  | -9.48E-05| 65  | 8.04E-06 | 85  | 2.52E-05 |
| 6   | -0.91%   | 26  | 4.12E-04 | 46  | 1.36E-04 | 66  | 6.33E-06 | 86  | -4.85E-05|
| 7   | 0.64%    | 27  | -3.55E-04| 47  | -6.15E-05| 67  | 1.17E-05 | 87  | 2.50E-05 |
| 8   | -0.03%   | 28  | 1.92E-04 | 48  | 1.05E-04 | 68  | 2.90E-08 | 88  | -4.79E-05|
| 9   | 0.64%    | 29  | -1.64E-04| 49  | -5.30E-05| 69  | 1.46E-05 | 89  | 2.49E-05 |
| 10  | -0.35%   | 30  | 1.92E-04 | 50  | 1.02E-04 | 70  | 7.08E-06 | 90  | -5.08E-05|
| 11  | 0.08%    | 31  | -1.84E-04| 51  | -4.78E-05| 71  | 1.78E-05 | 91  | 2.44E-05 |
| 12  | -0.40%   | 32  | 2.86E-04 | 52  | 8.40E-05 | 72  | -1.43E-05| 92  | -5.31E-05|
| 13  | 0.90%    | 33  | -2.40E-04| 53  | -9.44E-05| 73  | 8.04E-06 | 93  | 2.31E-05 |
| 14  | 1.00%    | 34  | -9.13E-06| 54  | 1.55E-04 | 74  | 1.58E-05 | 94  | -5.54E-05|
| 15  | -1.97%   | 35  | 2.86E-04 | 55  | -9.20E-05| 75  | 2.11E-05 | 95  | -3.16E-05|
| 16  | 0.90%    | 36  | -2.40E-04| 56  | 8.04E-05 | 76  | -1.43E-05| 96  | -5.31E-05|
| 17  | -0.40%   | 37  | 2.86E-04 | 57  | 1.82E-05 | 77  | 2.30E-05 | 97  | 2.22E-05 |
| 18  | 0.64%    | 38  | 2.36E-04 | 58  | 6.26E-05 | 78  | -2.98E-05| 98  | -3.92E-05|
| 19  | -1.97%   | 39  | -1.52E-04| 59  | -7.03E-06| 79  | 2.49E-05 | 99  | 2.11E-05 |
| 20  | 1.82E-04 | 40  | 1.88E-04 | 60  | 3.21E-05 | 80  | -3.93E-05| 100| -6.06E-05|

Table 11: The relative error of $R_{10}(n)$ to $p(n)$ when $n \leq 100$.

When $n \geq 26$, the relative error differs very little.

Equations $[25]$, $[32]$ and $[35]$ are also very accurate although they are not as good as $[27]$.

By the construction of these estimation formulae, when $n \to \infty$, the relative error will approaches 0. (But the absolute error may approaches infinity).

If we can find the accurate expression of the coefficients $a_2 \pm 1.039888529$ in $[30]$, $l_0 \pm 0.3595143172$ in $[6,2]$ and $a_3 \pm 2.893270736$ in $[33]$, and can find the explanation in theory, we may gain better results.

The ideas described here could be used to acquire elementary estimation formulae in some other cases when approximate values are frequently wanted while the asymptotic formulae are less accurate than expectation and the methods to calculate the exact values are inconvenient, such as the computation of some kinds of restricted partition numbers if we have (or can deduce) the asymptotic formulae beforehand.

These methods to fitting $C_1(n)$ and $C_2(n)$ could also be used in searching for the fitting functions of some classes of data obtain in experiments if we want more accuracy.

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References

[1] Anonymous. “Bibliography on Partitions” in “Mathematical BBS”. Internet: http://elix.unife.it/Root/d-Mathematics/d-Number-theory/b-Partitions (accessed September 12, 2016), December 2007.

[2] Anonymous. Fractal Structure to Partition Function: Hidden structure in partition function . Internet: www.aimath.org/news/partition (accessed November 13, 2015), December 2013.

[3] T. M. Apostol. Functions of Number Theory, Additive Number Theory: Unrestricted Partitions. In F. W. J. Olver, D. W. Lozier, and R. F. Boisvert, editors, NIST Digital Library of Mathematical Functions (DLMF), August 2015. http://dlmf.nist.gov/27.14, Release 1.0.10 of 2015-08-07, (accessed November 12, 2015).

[4] C. Berg, C. Stump, L. Kariyawasam, G. Filipuk, and C. Reuter. Integer Partitions. http://www.findstat.org/IntegerPartitions (accessed September 8, 2016), Oct. 2015.

[5] J. H. Bruinier, A. Folsom, Z. A. Kent, and K. Ono. Recent Work on the Partition Function. In B. C. Berndt and D. Prasad, editors, Proceedings of the Legacy of Ramanujan Conference, New Delhi, volume 20 of Ramanujan Mathematical Society Lecture Notes, pages 139–151. Ramanujan Mathematical Society (RMS), 2013.

[6] F. Calegari. A Remark on a Theorem of Folsom, Kent, and Ono. Internet: http://www.math.uchicago.edu/~fcale/Files/FKO.pdf (accessed November 13, 2015), Jan. 2011.

[7] C. Clark. New math theories reveal the nature of numbers: Finite formula
REFERENCES

found for partition numbers. Internet: https://www.eurekalert.org/pub_releases/2011-01/eu-nmt011911.php (accessed September 8, 2016), Jan. 2011.

[8] P. Erdös. The Evaluation of the Constant in the Formula for the Number of Partitions of $n$. *Annals of Mathematics. Second Series*, 43(3):437–450, July 1942. MSC: 11P82 11P81, Zbl: 0061.07905.

[9] A. Folsom, Z. A. Kent, and K. Ono. $l$-adic properties of the partition function. *Advances in Mathematics*, 229(03):1586–1609, February 2012.

[10] M. Hall, Jr. A survey of combinatorial analysis. In I. Kaplansky and etc, editors, *Some aspects of analysis and probability*, volume IV of *Surveys in applied mathematics*, pages 35–104. John Wiley and Sons, Inc. [New York] and Chapman and Hall, Limited [London], 1958.

[11] G. H. Hardy and S. R. Ramanujan. Asymptotic Formulae in Combinatory Analysis. *Proceedings of the London Mathematical Society, s2*, XVII:75–115, January 1918.

[12] W.-W. Li. On the Number of Conjugate Classes of Derangements. *ArXiv e-prints*, Dec. 2016. arXiv:1612.08186 [math.CO].

[13] D. J. Newman. The evaluation of the constant in the formula for the number of partitions of $n$. *American Journal of Mathematics*, 73(3):599–601, July 1951.

[14] D. J. Newman. A simplified proof of the partition formula. *The Michigan Mathematical Journal*, 9(3):283–287, Jan. 1962. MR0142529, ZM 0105.26701.

[15] K. Ono. Arithmetic of the Partition Function. In J. Bustoz, M. E. H. Ismail, and S. K. Suslov, editors, *Special Functions 2000: Current Perspective and Future Directions*, volume 30 of *NATO Science Series, Series II: Mathematics, Physics and Chemistry*, pages 243–253. Kluwer Academic Publishers, 2001.

[16] H. Rademacher. A Convergent Series for the Partition Function $p(n)$. *Proceedings of the National Academy of Sciences of the United States of America*, 23(2):78–84, Feb. 1937. PMC1076871.

[17] N. J. A. Sloane. A000041, number of partitions of $n$ (the partition numbers). (Formerly M0663 N0244), in “The On-Line Encyclopedia of Integer Sequences (OEIS ®)”. Internet: http://oeis.org/A000041 (accessed September 20, 2015).

[18] N. J. A. Sloane. “A002865 as a graph” in “The On-Line Encyclopedia of Integer Sequences (OEIS ®)”. Internet: http://oeis.org/A002865/graph (accessed October 10, 2015).

[19] J. Sándor, D. S. Mitrinovic, and B. Crstici. Unrestricted partitions of an integer. In M. H. etc., editor, *Handbook of Number Theory, Volume 1*, volume 351 of *Mathematics and Its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995. § XIV.1.

[20] J. Sándor, D. S. Mitrinovic, and B. Crstici. Unrestricted partitions of an integer. In *Handbook of Number Theory I*, page 491. Springer Netherlands, 2006. Section XIV.1.

[21] E. W. Weisstein. “Partition Function P." From MathWorld – A Wolfram Web Resource. Internet: http://mathworld.wolfram.com/Partition-FunctionP.html (accessed September 20, 2015), 1999-2015.