Quantum Group Based Theory for Antiferromagnetism and Superconductivity: Proof and Further Evidence

Sher Alam\textsuperscript{1}, S. M. Mamun\textsuperscript{3}, T. Yanagisawa\textsuperscript{2}, M. O. Rahman\textsuperscript{1} and J.A.S. Termizi

\textsuperscript{1}Photonics, Nat. Inst. of AIST, Tsukuba, Ibaraki 305-8568, Japan
\textsuperscript{2}Nanoelectronics, Nat. Inst. of AIST, Tsukuba, Ibaraki 305-8568, Japan
\textsuperscript{3}GUAS & Photon Factory, KEK, Tsukuba, Ibaraki 305, Japan \textsuperscript{4}Department of Physics, Islamia College, Peshawar, NWFP, Pakistan.

Abstract

Previously one of us presented a conjecture [APF-4 Proceedings] to model antiferromagnetism and high temperature superconductivity and their 'unification' by quantum group symmetry rather than the corresponding classical symmetry in view of the critique by Baskaran and Anderson of Zhang's classical SO(5) model. This conjecture was further sharpened, experimental evidence and the important role of 1-d systems [stripes] was emphasized and moreover the relationship between quantum groups and strings via WZWN models were given in [Phys. Lett A272, (2000)]. In this brief note we give and discuss mathematical proof of this conjecture, which completes an important part of this idea, since previously an explicit simple mathematical proof was lacking. Moreover an independent calculation [IC/99/2] which constructs the generators forming SO(5) algebra not only supports our previous conjecture but provides a check on our calculations. It is important to note that in terms of physics that the arbitariness [freedom] of the d-wave factor $g^2(k)$ is tied to quantum group symmetry whereas in order to recover classical SO(5) one must set it to unity in an adhoc manner. We intuitively expect that this freedom may be related to psuedogap behaviour in cuprates.
I. INTRODUCTION

The parent compounds of HTSC cuprates superconductors are Antiferromagnetic [AF] Mott insulators. These materials turn superconducting [SC] upon doping. The existence of pseudo-gap [PG] in these materials is yet another interesting feature. Spin glass behaviour has been reported in some of the cuprates [such as LSCO]. It is also of interest to test the notion of quantum phase transitions and the effects of quantum critical points at higher temperatures in these and related materials. Thus it is natural to understand the different states or phases of these materials from a consistent theoretical construction.

With some of these points in mind, previously one of us presented a conjecture [1] to model antiferromagnetism and high temperature superconductivity and their ‘unification’ by quantum group symmetry rather than the corresponding classical symmetry in view of the critique by Baskaran and Anderson of Zhang’s classical SO(5) model. This conjecture was further sharpened, experimental evidence and the important role of 1-d systems [stripes] was emphasized and moreover the relationship between quantum groups and strings via WZWN models were given in [2].

There are several ways to implement the quantum group symmetry, such as starting from Hamiltonian which has this symmetry. For example it is known that the Hubbard model at half-filling has exact quantum group symmetry in 1-d. A generalization of the Hubbard model with full quantum group symmetry away from half-filling and 1-d has been given by Schupp [3]. Another method is to consider the 1-d Hamiltonians such as Hubbard with quantum group symmetry and use these to study the 2-d and 3-d HTSC problem via the string formalism [4]. In addition one can formulate schemes of quantum group symmetry breaking. Yet another approach is to start with a classical groups that are relevant to AF and SC embed them in a larger classical group and then consider the quantum deformation of this group. Several choices exist [2]. The main purpose of this note is to consider the choice SO(5) [4].
II. QUANTUM SO(5) GROUP

It is clear that in order to construct the quantum SO(5) algebra we must consider the deformation of ordinary commutation relations. For the bosonic harmonic oscillator SU(2) $q$ was considered by Biedenharn \cite{7} and Macfarlane \cite{8}. In the same manner we can deform the anticommutation relationships of the fermion operators $c$ and $c^\dagger$

$$c_{k,i}c_{l,j}^\dagger + (g(k)g(l))^{-1}c_{l,j}^\dagger c_{k,i} = \left(g^{-2}(k)\right)^Q \delta_{kl}\delta_{ij},$$

$$g(l)c_{k,i}c_{l,j} + g(k)c_{l,j}c_{k,i} = 0,$$

(1)

where $g$ is the deformation function, and $Q$ is defined such that the following relations hold

$$[Q, c_{k,i}] = -c_{k,i},$$

$$[Q, c_{k,i}^\dagger] = c_{k,i}^\dagger,$$

$$Q = N - M,$$

(2)

where $N$ and $M$ are respectively the number of electrons and number of sites.

To determine $Q$ we set $i = j$ and $k = l$ which reduces Eq. (1) to

$$c_{k,i}c_{k,i}^\dagger + (g(k)g(l))^{-2}c_{k,i}^\dagger c_{k,i} = \left(g^{-2}(k)\right)^Q,$$

$$c_{k,i}c_{k,i} + c_{k,i}c_{k,i} = 0,$$

(3)

and on using Eqs. (2) and (3) after a short calculation we obtain

$$Q = \frac{1}{2} \sum_k \left(g^2(k)\right)^Q \left(g^{-2}(p)c_{k,i}^\dagger c_{k,i} - c_{k,i}c_{k,i}^\dagger\right).$$

(4)

Having obtained the U(1) charge generator $Q$ let us incorporate it into a bigger group [here taken to be SO(5)] by taking it in conjunction with SO(3) spin rotation representing AF \cite{2}. The deformed version of rotation generators of SO(3) can be written in our case as

*We note that U(1) is equivalent to rotation group in 2d i.e. SO(2) and SO(3) has the same Lie Algebra as SU(2)
\begin{align*}
S_1 &= \frac{1}{2} \sum_k (g(k)^2)^{Q-1} c_{k,i}^\dagger (\sigma^x)_{ij} c_{k,j}, \\
S_2 &= \frac{1}{2} \sum_k (g(k)^2)^{Q-1} c_{k,i}^\dagger (\sigma^y)_{ij} c_{k,j}, \\
S_3 &= \frac{1}{2} \sum_k (g(k)^2)^{Q-1} c_{k,i}^\dagger (\sigma^z)_{ij} c_{k,j}. \\
\end{align*}

From standard group theory it is known that the generators $L_{ab}$, $a, b = 1, \ldots, N$ of the SO(N) “rotation” groups are $N(N-1)/2$ in number, and satisfy the following fundamental relation,

\[ [L_{ab}, L_{cd}] = i(\delta_{bd}L_{ac} - \delta_{ac}L_{bd} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc}). \tag{6} \]

We note that the action of $L_{ab}$ on any N-dimensional vector $s_a$ with $a = 1, \ldots, N$ is given by

\[ [L_{ab}, s_c] = i(\delta_{ac}s_b - \delta_{bc}s_a). \tag{7} \]

For example the familiar [rotation group in 3-d space] SO(3) has three generators. For SO(5) there are ten generators. These generators [matrices] will act on the 5-d vector $s_a$ with $a = 1, \ldots, 5$, where $s_1$ and $s_5$ are the SC part of the 5-d SO(5) vector and can be readily identified with d-wave SC-order parameters, viz,

\begin{align*}
    s_1 &= \Delta + \Delta^\dagger, \\
    s_5 &= i(\Delta - \Delta^\dagger), \\
    \Delta &= -i\frac{1}{4} \sum_k |g(k)|^2 (g(k)^2)^{Q} c_{k,i}(\sigma^y)_{ij} c_{-k,j}, \\
    g(k) &= \cos(k_x) - \cos(k_y),
\end{align*}

and $s_2$, $s_3$ and $s_4$ which represent the AF part are given by

\begin{align*}
    s_2 &= -\frac{1}{2} (-1)^Q \sum_k (g(k)^2)^{Q} c_{k+A,i}^\dagger (\sigma^x)_{ij} c_{k,j}, \\
    s_3 &= -\frac{1}{2} (-1)^Q \sum_k (g(k)^2)^{Q} c_{k+A,i}^\dagger (\sigma^y)_{ij} c_{k,j}, \\
    s_4 &= -\frac{1}{2} (-1)^Q \sum_k (g(k)^2)^{Q} c_{k+A,i}^\dagger (\sigma^z)_{ij} c_{k,j}, \\
    A &= (\pi, \pi, \pi)
\end{align*}

\[ (9) \]
where $\mathbf{A}$ is the AF ordering vector. So far we have identified four of the ten symmetry generators of SO(5) group, namely $Q, S_1, S_2, S_3$. The remaining six can be notated by $\Pi_a$ [$a = 1, 2, 3]$ and $\Pi_a^\dagger$ and which for the quantum deformed SO(5) read as

$$\Pi_1 = -\frac{1}{2}(-1)^Q \sum_k |g(k)| (g(k)^2)^{Q} c_{k+A,i} (\sigma^y \sigma^x)_{ij} c_{-k,j},$$

$$\Pi_2 = -\frac{1}{2}(-1)^Q \sum_k |g(k)| (g(k)^2)^{Q} c_{k+A,i} (\sigma^y \sigma^y)_{ij} c_{-k,j},$$

$$\Pi_3 = -\frac{1}{2}(-1)^Q \sum_k |g(k)| (g(k)^2)^{Q} c_{k+A,i} (\sigma^y \sigma^z)_{ij} c_{-k,j}.$$

(10)

Now the ten generators $L_{ab}$ can be easily constructed from $Q, S_1, S_2, S_3, \Pi_1, \Pi_2, \Pi_3, \Pi_1^\dagger, \Pi_2^\dagger, \Pi_3^\dagger$ and satisfy the relation Eq. [4] as they should. An independent calculation which constructs the deformed SO(5) algebra by Duc and Thang [10] provides a check on our results and also confirms our earlier conjecture [1,2].

A very important feature of quantum deformed SO(5) is that $g(k)$ not only appears in the SC order parameter but also in the AF-order parameters, see Eqs.8-9, unlike classical SO(5) where it only appears in SC order parameters. In classical SO(5) of Zhang although a unification of AF and SC is posited, it does not imply that the pairing mechanism is AF fluctuations. Moreover classical SO(5) lacks a mechanism for pseudogap besides other problem. In contrast SO(5) quantum deformed algebra relates the AF and SC in non-trivial ways, for example, $g(k)$ appears in both AF and SC parameters. Plus one can go further and boldly suggest that pseudogap region is a consequence of the competition between the AF and SC orders. The pseudogap may be related to short-range AF correlations. Since the deformation parameter enters into the fundamental anticommutation relations Eq. [1] one can think of quantum symmetry groups as a classification scheme for the various physical phenomena in strongly correlated systems, such as fractionalization of electron, Luttinger liquid behaviour, pseudogap and the deviations [anomalies] of Fermi surface away from its normal form due to Luttinger Liquid behaviour. In a more specific scenario we can think of the deformation of anticommutation relations Eq. [1] as representative of the correlation-renormalization dressing effects of fermionic operators. Thus we can think in a general sense that the quantum symmetry could provide a classification of strongly correlated behaviour of
Mott Insulators. Yet another interesting feature of quantum groups is its relation to various topological orders, this has not been exploited.

We note that in order to close the SO(5) algebra i.e. to satisfy Eq. 4, Zhang \[4\] imposed the condition $g^2(k) = 1$. Henley \[5\] made a very useful observation that by taking $g(k) = \text{sgn} (\cos(k_x) - \cos(k_y))$ one can close the SO(5) algebra without invoking the adhoc restriction of Zhang \[4\]. Yet another very significant observation is that d-wave SC does not follow from the classical SO(5) of Zhang \[4\] since one other inversion-symmetric choice $g(k) = \text{sgn} (\cos(k_x) + \cos(k_y))$ is also enough to obtain the closure of the SO(5) algebra.\[†\]

III. MICROSCOPIC HAMILTONIANS: COMMENTS

What about the role of phonons in quantum group scenario for HTSC and related materials? In any case, whatever point of view one adopts, one must answer the question: What is the precise role of phonons in cuprates? i.e. do they play any role?, if not, how can we prove that this is so? To do so, one way is that we must take a look at the various microscopic Hamiltonians related to the basic Hubbard model which share with it the common feature of having quantum group symmetries but are not restricted to half-filling and to 1-d.\[‡\] An extended Hubbard Hamiltonian (EHH) [i.e. with phonons] with generalized quantum group symmetries away from half-filling was proposed by Montorsi and Rasetti \[9\]. Using mathematical arguments [i.e. Hopf algebra] Schupp claimed that the quantum symmetry of EHH in \[9\] exists only on 1-d lattices and in appropriate approximation is still restricted to half-filling. To this end Schupp \[8\] suggested an EHH with full quantum symmetry, having

\[†\]This was also realized by Henley \[5\], “Interestingly , one other simple, inversion-symmetric choice would also satisfy the condition (4): $\eta_k = \text{sgn} (\cos(k_x) + \cos(k_y))$. That variant of SO(5), which entails “extended s-wave” pairing, appears free from internal contradictions (contrary to a suggestion in Ref.[1]).” We note that “condition (4)” refers to closure condition in Henley \[5\] notation, and Ref.[1] in Henley \[5\] is reference \[4\] here.

\[‡\]In another scenario we can stay with 1-d, but go to the string formulation \[11,12\]. Indeed this may be a more fruitful approach as we have reasoned in \[2,11,12\] since low-dimensions [i.e.1-d, and 2-d] seem to play an important role in physics of cuprates in both normal and SC states from both experimental and theoretical points of views.
the symmetry group $SU(2) \times SU(2)/Z_2$, not restricted to half-filling and 1-d lattices. Indeed this Hamiltonian can be regarded as a realization of our suggestion [2] to use the quantum symmetry group SO(4) [since $SU(2) \times SU(2)/Z_2$ is equivalent to SO(4)] for a theory of AF and SC of cuprates. The EHH in [3] [see Eq. 23 in [3]] has six-free parameters, its first three terms make up the Hubbard Model but not restricted to half-filling. This Hamiltonian also satisfies one of previous aims, that is to find a non-trivial relation [unification] of the t-J and Hubbard models. Indeed the last term of EHH in [3] is like the hopping term in the t-J model with the hopping strength depending on the occupation of sites. Moreover and importantly after deformation this t-J like term is the origin of the non-trivial quantum symmetries of the EHH in [3]. In summary the EHH in [3] is one way of realizing the SO(4) quantum group scenario for a theory of cuprates suggested in [2], plus it also provides a non-trivial way to relate Hubbard and t-J models. However much work has to done, namely a definitive answer on the role of phonons in cuprates, and physical reasoning behind this.

IV. EXPERIMENTAL SUPPORT FOR OUR SCENARIO

The 1-d behavior is intimately tied to quantum group symmetry. We thus expect that the physics of low-dimensional [1-d and 2-d] systems in particular the ones which involve correlated electrons can be understood in a consistent and elegant manner by using quantum groups. It is important to note that the 1-d behavior of magnetic fluctuation was predicted by our theory [2] before the experiment. The fluctuations associated with charge were regarded as 1-d, whereas magnetic fluctuations were regarded as 2-d. However it was predicted [2] and experimentally shown [13] that magnetic fluctuations are also 1-d. Thus in our scenario all relevant degrees of freedom are 1-d instead of quasiparticle in the sense of Landau. In one scenario of our formulation superconductivity arises as a dressed stripe phase. We think it may be possible to write down transformations equivalent [such as one writes in BCS, i.e. Bogoliubov] which can map the 1-d states onto the 2-d and 3-d superconducting phase. Another consequence of our formulation is the carrier inhomogeneity. We have previously emphasized the carrier [electron] inhomogeneity in our modelling of HTSC materials [2]. In contrast many models of HTSC assume that charge carrier introduced by
doping distribute uniformly, leading to an electronically homogeneous system, as in normal metals. However recent experimental work [14] confirms our intuition, which is encouraging. This inhomogeneity is expected to be manifested in both the local density of states spectrum and superconducting gap. This inhomogeneity we suspect appears in the temperature dependent XANES spectra of Zn-doped LSCO samples [15]. In order to confirm the spin-charge separation and 1-d behaviour we have also suggested SET based experiments for the HTSC and related materials [16]. It is interesting to note that a recent paper on pseudogap and quantum-transition phenomenology in HTSC cuprates by Tallon et al., [17] supports our suggestion that pseudogap may be related to short-range AF correlations.

V. CONCLUSIONS

In conclusion it has been shown explicitly that one can indeed construct a closed quantum group SO(5) algebra which relates elegantly AF with SC without any adhoc restrictions on the d-wave factor $g^2(k)$, as previously suggested in [1,2]. It is important to note that in terms of physics that the arbitariness [freedom] of the d-wave factor $g^2(k)$ is tied to quantum group symmetry whereas in order to recover classical SO(5) one must set it to unity in an adhoc manner. Moreover the d-wave factor appears not only in the SC order parameter but also in the AF order parameters, thus relating them in a non-trivial manner. We have also briefly commented on the realization of our previous conjecture [2] in context of SO(4) quantum group, where phonons can be incorporated at least for 1-d case [3] and beyond 1-d and half-filling by using EHH [3] in a scenario which relates t-J, Hubbard and related Hamiltonians via twisting ala Hopf algebras [3].

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