Regimes of the Vishniac–Ryu Decelerating Shock Instability

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Abstract

Here we revisit the derivation of the instability of dense shocked layers, originally developed by Vishniac and Ryu. Our motivation is that density profiles found in actual astrophysical and laboratory systems often do not match the assumptions in that paper. In order to identify the anticipated theoretical growth rates for various circumstances, one must first revisit the derivation and allow for the possibility that the density scale length differs, in magnitude and/or in sign, from the isothermal scale height. This analysis leads us to find regimes of purely convective instability and also of Vishniac stabilization of this instability, in addition to some new regimes of Vishniac behavior. We also identify a typographical error in the original paper that matters for quantitative evaluation of growth rates.

Key words: hydrodynamics – instabilities – supernovae: general – turbulence

1. Introduction

Thin layers of matter are common in the universe. They are found wherever radiative losses of energy become large, for example, in old supernova remnants (Blondin et al. 1998) and in shocks as they emerge from supernovae (Enßn & Burrows 1992). They may perhaps also be found elsewhere, for example, in collisions of porous asteroids with dense objects. They are also found in laboratory systems that produce strongly radiating, shocked layers (Reighard et al. 2006) or blast waves in certain gases (Grun et al. 1991; Ditmire & Edens 2008). Since these layers are typically moving, they tend to accumulate mass and to decelerate. Vishniac (1983) recognized that there is an important imbalance between the internal pressure driving a thin layer and the ram pressure associated with accumulating matter. The first always acts along the normal to the surface of the thin layer, while the second acts in the direction of the incoming flow. In the presence of a modulation in the layer, this imbalance causes mass to accumulate in the lagging material. In response, the lagging material decelerates less rapidly than the leading material at lower density, causing the modulation to invert, after which the process repeats. Under various geometrical assumptions, this fundamental process can produce either power-law or exponential growth. Bertschinger (1986) explored the problem of a thin dense layer accumulating mass behind a radiative shock, solving for the internal profile of the layer. He confirmed the results of Vishniac (1983) and also showed that the layer must in fact be pressure-driven in order to have instability.

A few years later, Vishniac & Ryu (1989) examined the behavior of a thin layer with internal structure. In the following, we refer to Vishniac and Ryu as VR and to their seminal paper on instabilities in shocked, decelerating layers as VR89. In VR89, VR also examined the stability of self-similar, spherical shock waves. Our focus here will be on the evolution of thin, planar layers, which might represent a phase in the evolution of a small segment of a spherical system or the evolution of a laboratory system.

A number of simulation studies have explored the behavior of such layers. Strickland & Blondin (1995) found fluctuations growing in a cool dense layer produced by a radiative shock. Blondin et al. (1998) examined the transition from a Sedov–Taylor blast wave to a radiatively collapsed shock. Michaut et al. (2012) and Cavet et al. (2011) examined the long-term behavior of a similar system, using a distinct value of \(\gamma\) for the dense shell, to approximate the behavior of a cooling layer. Their work concluded, as did Mac Low & Norman (1993) for the blast-wave case, that the instability develops as anticipated by Vishniac, but then dies out in the long run as the shell thickens and the Mach number of the shock decreases, leaving behind only some internal structuring.

Some other related works are also worth mentioning. Badjin et al. (2016) have noted that care must be taken in simulations if one is to avoid the excitation of numerical instabilities that can mimic the physical instabilities of interest here. Vishniac (1994) considered shock-bounded slabs, later simulated by Blondin & Marks (1996). They found a similar instability, which has been called the Nonlinear Thin Shell Instability, but it is not of direct interest here. Robinson & Pasley (2018) have recently emphasized that ionization and other real-gas (i.e., non-polytropic) effects can enhance the onset of VR-type instabilities in systems that might otherwise be thought to be stable. This is consistent with our view that most of the relevant large-scale mechanics depend only on length scales which, for real gases, may not be well described by simple polytropic closures.

Of note here is that VR assumed the shocked layer to be isothermal. In the presence of deceleration, this leads to an exponential density profile that decreases behind the shock front. However, there are good reasons to consider other possibilities. There are many other possible profiles. Depending on the heat transport dynamics, the profile might be adiabatic. In general, the density in the profile may be determined by physics other than simple hydrodynamics and might not be described well by any value of polytropic index, \(\gamma\). If the maximum density after radiative collapse is limited by the...
accumulation of magnetic pressure, as discussed in Blondin et al. (1998), then the density profile might either increase or decrease behind the shock, depending on the magnetic structure within the unshocked medium. In the early phases of laboratory radiative shocks, and likely also in shocks emerging from supernovae, the radiation transport physics causes the layer density to increase away from the shock, producing a slope having the opposite sign of that produced by the isothermal case. Other possibilities arise if the shock is accelerating, as can happen in a steep density gradient (Zeldovich & Razier 1966) or if the pressure driving the shock is caused to gradually increase. The qualitative argument above would suggest that modulations might grow unstably without oscillations in such a case. In addition, one can recognize this case as a thin-shell variant on the Rayleigh–Taylor instability. We have considered both accelerating and decelerating shocks in the presence of density profiles of arbitrary sign. We find unstable behavior that differs from that discussed in VR89, for cases whose assumptions differ from theirs, including some regimes of very rapid instability. We also find a typographical error in VR89 of interest to anyone needing to do a precise evaluation. We present our calculation and discuss these results here.

In a related but separate branch of inquiry, Ryu & Vishniac (1987) examined the stability of blast waves that begin with a self-similar, Sedov–Taylor structure. By assuming that the large-scale structure of the material is governed by polytropic gas laws, they expressed the parameters of both the VRI and the Sedov–Taylor blast wave on common terms. They found that the overstability can outrun the expansion of the blast wave and appear at large scales if γ is less than about 1.2, which corresponds to an 11-fold density jump. This type of closure for the large-scale structure is not of direct interest to us here, but did lead to a variety of subsequent works (Ryu & Vishniac 1991; Nishi 1992; Mac Low & Norman 1993; Nishi & Kamaya 2000; Kushnir et al. 2005; Sanz et al. 2016) elucidating the detailed behavior of such systems. In particular, Ryu & Vishniac (1991) described a case, for a blast wave whose large-scale structure is determined by a polytropic index γ, in which a convectively unstable, post-blast-wave layer could be stabilized by oblique shock relations. We discuss this work further in Section 4.3. Here we proceed by retaining mechanical length scales, the relationships between which are not closed by assuming a polytropic γ. We recover some parallel results to behaviors found in a space defined by such an assumption, but can now elucidate both their origin and behavior over a larger space of possible parameters.

In the following, we first present the definition of the problem and discuss the quasi-steady, hydrostatic structure of the layer. We then summarize the derivation of the dispersion relation in Section 3. In Section 4, we explore the unstable behavior for a wide range of conditions—decelerating or accelerating layers having density gradients that point either outward or inward. In Section 5, we examine the behavior of the modulations of the various relevant variables. Section 6 then concludes the paper. In the appendices, we rework the derivation of VR89 for the more general case. We first develop the equations that describe the dynamic behavior in Appendix B, which is followed with a discussion of the boundary conditions in Appendix C. Here in particular is where one must closely attend to the difference between the interior density scale length, L, and the isothermal scale height, h, which involves the magnitude of the deceleration. We combine the results to date in Appendix D to obtain a dispersion relation that equals that found in VR89 in the correct limit.

2. Problem Definition and Hydrostatic Structure

Figure 1 shows a schematic of the system being analyzed. A shocked layer, sustained by an internal pressure \( p_i \), slowly decelerates as it accumulates mass from the swept-up external material at initial density \( \rho_E \). The entire layer decelerates at a rate \( \dot{V}_s \), taken to be given. The deceleration is slow enough that the layer sustains a quasi-steady hydrostatic structure. Here when we refer to the shock front, we refer to the initial density jump (technically the shock itself) combined with any rapid, subsequent density increase, generally due to radiative cooling. We take the postshock pressure, in this sense, to be \( \rho_E V_s^2 \), for shock speed \( V_s \). In detail, the pressure should be multiplied by \( (1 - \rho_i / \rho_E) \), which is very close to 1 for the large density jumps of interest here.

Under the influence of the deceleration, the pressure decreases through the layer, of thickness \( H \), so that
\[
\rho(z) = \rho_E V_s^2 - \dot{V}_s \int_0^z \rho(z') dz', \quad \text{so that}
\]
\[
p_i = \rho_E V_s^2 - \sigma V_s,
\]
in which the areal density of the layer is \( \sigma \). It is worth highlighting the fact that the entire original argument of Vishniac rests on having \( p_i \) be constant along the layer, even as the layer evolves. This requires that any ripples grow and propagate at subsonic speeds. This remains true for most but not all of the circumstances explored below.

The density structure of the layer depends upon its history and its properties. If the layer is isothermal, then the density profile is
\[
\rho(z) = \rho_E e^{-z/h},
\]
in which the isothermal scale height \( h \) is
\[
h = c_s^2 / V_s,
\]
where the isothermal sound speed is \( c_i = \frac{\sqrt{\gamma} \rho}{\rho \gamma} \). Other profiles are possible for other assumptions regarding the properties of the dense layer, as discussed above. In the present paper, we will characterize the actual density profile by a scale length, \( L = -1/(\partial \ln \rho/\partial z) \) (with the sign anticipating a density decrease deeper into the shocked layer). We will characterize the adiabatic sound speed \( \sqrt{\gamma} c_a \) as the value corresponding to adiabatic fluctuations of small amplitude, so that \( \partial p/\partial \rho = \gamma c_a^2 \), with the partial derivative taken at constant entropy, with \( \gamma \) being the usual adiabatic or polytropic index. The isothermal limit of VR89 then corresponds to \( \gamma = 1 \) and to \( L = h \).

We refer to unit vectors in the direction of the deceleration and in a transverse direction (see Figure 1) as \( \hat{x} \) and \( \hat{z} \), respectively. The hydrostatic velocity profile is as required to keep the mass flux through the shocked layer constant, so that \( u_r = -\hat{x} V_r (1 - \rho E/\rho(z)) \) in the laboratory frame while \( u = \hat{z} u(z) = \hat{z} V_r \rho E/\rho(z) \) in the shock frame. Note that \( u(z) \) is very small on the scale of \( V_s \).

3. Structure and Results of the Derivation

The derivation, detailed in Appendices A–D, begins with the Euler equations for the mass density and the momentum density. It assumes small perturbations about the hydrostatic profile and that the fluctuations of pressure are adiabatic and thus barotropic. This produces a set of differential equations that are linear in the magnitude of the perturbations, assumed to have amplitudes \( \propto e^{i(\omega t - kx)} \). The general solution to these equations involves three coefficients, each associated with some spatial structure. Two of these follow from the assumption that for these, one can ignore terms involving \( \partial \), \( \partial _z \), which are the first-order perturbation in the \( z \)-component of velocity. The third applies at any location, such as the shock front, where terms involving \( \partial _z \), \( \partial _v \), become very large. In this way, one finds equations for the components of the spatial structure of the system. All of these turn out to involve the density scale length \( L \); none of them involve \( h \).

The boundary conditions then enable one to relate the coefficients and find a dispersion relation. The first boundary condition imposes continuity of velocity across the shock. The other two boundary conditions impose the correct behavior of the pressure, which must have a known value at the shock front and which must be continuous at the internal interface. The two pressure boundary conditions involve the isothermal scale height \( h \), but not \( L \).

The dispersion relation is most compactly written by normalizing \( \omega \) to the acoustic frequency \( kc_i \) so that \( \omega_n = \omega/(kc_i) \). The resulting dispersion relation is

\[
\omega_n^2 - \Omega \omega_n^2 - \Pi = 0,
\]

where \( \omega_n = \omega/(kc_i) \), so that \( \omega_n = \omega/(kc_i) \). The resulting dispersion relation is

\[
\omega_n^4 - \Omega \omega_n^2 - \Pi = 0,
\]

with

\[
\Omega = \frac{H}{\gamma H} \left[ \frac{(\kappa H)^2}{\gamma H} + \frac{2}{\phi} - \frac{H}{\gamma H} \right]
\]

and

\[
\Pi = -\frac{H}{\kappa H} \left[ \frac{(Q/\phi)}{\phi} + \frac{1}{\phi} - \frac{H}{\gamma H} \right]
\]

The variable \( Q \), which arises during the spatial solution and is

\[
Q = \sqrt{1 + 4(kL)^2} \left(1 - \frac{\omega_n^2}{\gamma} \right)
\]

In VR89, the factor of 4 in the middle term in Equation (8) is missing. This typographical error does not affect the structure of the solutions but does affect their numerical evaluation. We now proceed to work with Equations (5) through (8) to analyze and evaluate the unstable behavior under various assumptions.

4. The Growth Behavior Based on the Equations Found Here

4.1. Analysis of the Dispersion Relation

It is useful to look at the roots of Equation (5). These can at times be simply interpreted, as \( \Pi \) is real whenever \( \omega_n \) is purely real or imaginary. The four roots are

\[
\omega_n = \pm \sqrt{\Omega \pm \sqrt{(\Omega^2 + 4\Pi)}}/2.
\]

Figure 2 shows these, for \( \Omega = 1 \), which is the case in VR89.

One sees that there is a purely stable zone for \( -0.25 \omega_n^2 < \Omega < 0 \), so that this zone shrinks or expands with \( \Omega \). In the limit of large \( \kappa = kL \), small \( kH \), and large \( kH \) such that \( \Omega \to 1 \), one finds

\[
\Pi \to -\frac{1}{kHh} = \frac{-\lambda^2}{4\pi^2 h H},
\]

from which

\[
-\frac{\lambda^2}{4\pi^2 h H} < 1 \quad \text{so} \quad \lambda < \pi \sqrt{h H} \text{ for stability.}
\]

The minimum unstable wavelength also can be written as \( \lambda_{\text{min}} = \pi \sqrt{h H} \). For \( \Pi > 0 \), the unstable roots for \( \omega_n \) are purely imaginary or purely real, independent of \( \Omega \), and \( \Pi \) is self-consistently real. The root corresponding to exponential growth in this case is

\[
\Im[\omega_n] = -\sqrt{(\Omega - \sqrt{(\Omega^2 + 4\Pi)})}/2.
\]

If \( \Omega < 0 \) and of sufficient magnitude, for example, when \( kH \ll 1 \), then the growth rate becomes \( \sqrt{2\Omega} \rightarrow \sqrt{2}/(KH) \), independent of \( \Pi \).

For moderate values of \( \Omega \) (and positive \( h \)), it is \( \Pi \) that determines the behavior, so long as \( \omega_n \) is purely real or purely
imaginary. We can analyze the sign-determining parts of the structure term, written as

$$S = -\left[\frac{[Q/\phi]}{\tanh[Q/\phi]} + \frac{1}{\phi} - \frac{H}{h}\right]$$  \hspace{1cm} (13)

where $Q$ is positive definite, and the first term within the square brackets is positive and $>1$ and symmetric in $Q/\phi$. One has

$$\left|\frac{Q}{\phi}\right| = \sqrt{\frac{H^2}{4L^2} + (kH)^2\left(1 - \frac{\omega_n^2}{\gamma}\right)}.$$  \hspace{1cm} (14)

This term is not readily $>1$, since $L$ tends to be not much smaller than $H$, and $kH$ tends to be not much greater than $1$, while $\omega_n^2$ is often a significant fraction of $1$. Correspondingly, the first term in the square brackets in Equation (13) is only slightly greater than $1$. Setting this term equal to $1$ in Equation (7), one can solve for $\Pi = -1/4$ to find thresholds for

$$\frac{h}{H} = \frac{2}{(kH)^2} \left(1 + \frac{H}{2L} \pm \sqrt{1 + \frac{H}{L} + \frac{H^2}{4L^2} - \frac{1}{(kH)^2}}\right).$$  \hspace{1cm} (15)

These thresholds establish a range of $h/H$ that is unstable for positive $L$, with stable behavior at smaller and larger values. Alternatively, solving the same equation for $kH$ and converting to $\lambda/H$, one finds

$$\frac{\lambda}{H} = \pi \frac{h}{H} \left(\frac{h}{H} + \frac{H}{2L} - 1\right)^{-1},$$  \hspace{1cm} (16)

which establishes a lower limit on $\lambda/H$ for instability. In addition, the above discussion lets us see that, when $\phi$ becomes small while $H/h$ and $kH$ do not, the $1/\phi$ term dominates both the values of $\Omega$ and $\Pi$. In this case, there is a region where

$$0 < \frac{\sqrt{\Omega^2 + 4\Pi}}{\Omega} < 1,$$  \hspace{1cm} (17)

in which $\omega$ is real, and there is no instability.

Returning to Equation (13), one can see that positive $\Pi$ occurs when $\phi \lesssim -1 + H/h$, corresponding to $L < 0$ and $|L| \gtrsim H(1-H/h)$, requiring $h > H$ to have instability. This case is then unstable for all $Q$. As $k$ or $h$ go to infinity, one can see in Equation (7) that $\Pi$ goes to zero, and the remaining solution is $\omega = \pm \omega_{ac} = \pm c_{ac}$, corresponding to sound waves being all that is left in the system.

Thus we see in Figure 2 that, when $\Pi = 0$, there are only the acoustic solutions at $\omega_n = 1$. When $\Pi > 0$, one sees a purely real branch above $\omega_n = 1$, which corresponds to an “augmented acoustic mode,” and a purely imaginary branch, which corresponds to $L < 0$. The negative $\Pi$ cases correspond to “classical” Vishniac modes, with both the minimum unstable wavelength feature and the compound real and imaginary parts when it is unstable.

It is straightforward, using a computational mathematics program, to find and display the largest unstable root of Equation (5), working with whichever variables may be most convenient. (In most regimes there is only one unstable root, but at times there are two.) As usual, care must be taken to find the correct root across the domain of interest. We proceed to consider several cases. Because the values and ratio of $L$ and $h$ have substantial effects on the resulting solutions, we consider variations in these parameters across some orders of magnitude. The index $\gamma$ only varies from 1 to $5/3$. The only case we identified in which its effects are significant is that in VR89, in which one constrains $L$ to equal $h$. We discuss why below. In most cases below, we show frequencies and growth rates only for $\gamma = 5/3$. The reader can readily obtain solutions for other values of $\gamma$, if needed.

4.2. A Decelerating Layer with Positive L

We first consider the case VR89, in which $L = h$ and $\gamma = 1$. Using Equation (7), we found the unstable root and displayed it as follows. We show $L/H = \phi/2$ on the abscissa and $\lambda/H = 2\pi/(kH)$ on the ordinate. To give the growth rate a common scale across the plot, we plot the ratio of $-\Im(\omega)$ to the acoustic frequency for $\lambda = H$, $2\pi c_{ac}/H$. This produces Figures 3(a) and 4, showing growth rate and real frequency, respectively. One sees that, across the unstable region, the absolute growth rate peaks for small $\lambda/H$ and for $L \gtrsim H$, but remains significant over a large region. Figure 3(b) shows the growth rate for $L = h$ and $\gamma = 5/3$, although this is not self-consistent. Here the threshold on the left, at lower values of $L/H$, is seen to have increased in comparison to the $\gamma = 1$ case.
We show why this happens below. In addition, one sees the effect discussed in the introduction, that for fixed $\lambda$, an increase of $H$ over time will eventually stabilize the modulations. One sees in Figure 4 that the real frequency is near $k c_s$ of $H$ for $\lambda/H$; normalized (positive) scale length, $L/H$; and $L = h$. The dashed line shows the minimum unstable wavelength of Equation (11).

Turning to the effect of having different $L$ and $h$, but still with $L > 0$, we show the growth rate for various parameters in Figure 5. Panels (a) and (b) show the growth rate in terms of different variables. In panel (a), the region of instability has a lower boundary and a boundary to the upper left. Both correspond to $\Pi \sim -1/4$. The lower dashed curve shows the result of Equation (16). The continuous dashed curve, which approximately follows both stability boundaries, corresponds to two roots in the evaluation of $[\sqrt{\Omega^2 + 4]\Pi]/\Omega = 0$, using the approximation that $Q/\phi \sim \tanh[Q/\phi]$. A precise evaluation of $[\sqrt{\Omega^2 + 4]\Pi]/\Omega = 0$ traces out the stability boundary in detail; it also shows that this quantity remains below 1 throughout the rest of the plot. As $h/H$ varies, the unstable region displayed in this plane moves with the dashed curve, primarily by moderate up-and-down motion of the lower boundary. Here again the normalization by $k c_s$ obscures the fact that the largest absolute growth rates are in the middle of the plot, for $\lambda$ not far above the threshold value.

Figure 5(b) shows the effect of variations in $h$. One sees that, for fixed $\lambda/H$, the growth rate depends much more strongly on the rate of deceleration (through $h/H$) than it does on the density scale length (through $L/H$). The upper boundary of the unstable region is as expected based on Equation (15). By contrast, as one approaches the lower boundary of Equation (15), the unstable roots move onto the imaginary axis so that $\omega_n$ becomes purely imaginary and quite large. Physically, the smallness of $h/H$, corresponding to large decelerations, implies that the dense part of the layer becomes quite thin, so increased instability is not a surprise. Here the modulations grow without propagating and have the potential to strongly affect the structure of the layer. However, once the modulation amplitude becomes large enough, its growth becomes supersonic and so would drive shocks into the interior. We did not allow for this case in our analysis above; it seems likely that the actual behavior would weaken the growth from that point forward.

We can also best explain the sensitivity to $\gamma$ seen in Figure 3 by referring to Figure 5(b). One can see that, at small $L/H$, the...
growth rate contours for low growth rates become nearly parallel to the thin dashed curve showing \( L = h \). Small changes to the overall contours cause the intersection of the threshold with the \( L = h \) curve to move a comparatively long distance. What happens mathematically is that the increase in \( \gamma \) causes the value of \( \Omega \) to increase above 1, which turns out to imply that \( \Pi \) must become more negative to produce instability. In no other case discussed in the present paper does the variation in \( \gamma \) cause significant changes to the contour plots. Results for any specific value of \( \gamma \) are straightforward to obtain.

While these results do not directly apply to spherical expansions, they should not be too far off for the behavior of comparatively short wavelengths within a small-enough segment of a spherical shell. RV87 considered power-law expansions having radius \( R \propto t^n \), in both planar and spherical geometry. For such expansions, one can show that both \( H \) and \( h \propto t^n \), so the highest unstable mode number does not change. If one approximates \( h \sim 3H \). In addition, one can see by examining typical cases of such adiabatic blast waves that \( L \lesssim H \). It would appear from Figure 5 that such systems will access a comparatively small region in parameter space, near these values. The growth is relatively weak there, and can be stabilized entirely if \( L/H \) is small enough, at the actual value of \( h/H \). Such adiabatic systems seem unlikely to be able to access the strongly unstable zone seen in Figure 5(b) for \( h < H \).

In addition, when \( \omega_n \) becomes purely imaginary and quite large, the assumption \( Q/\phi \sim \tanh[Q/\phi] \) becomes invalid and \( h/H \) must decrease further to reach \( \Pi = 1/4 \), as can be seen in Figure 5(c). In this regime, \( \Omega \ll 0 \) and one reaches the condition identified above where \( \omega_n \to -i\sqrt{2}/(kh) \). Aside from the extreme behavior we find here for unusually small \( h \), the primary effect of separating \( L \) and \( h \) in the analysis is to open up the possibility for instability for a much larger range of \( L \), and higher \( L/\lambda \), than one finds in the VR case shown in Figure 3.

4.3. A Decelerating Layer with Negative \( L \)

Under some circumstances, a decelerating layer may contain a density profile in which the density increases from the shock location toward the boundary with the interior matter. One example is that of the early evolution of a radiating, shocked layer that can become optically thick, occurring in shock breakout from supernovae and in laboratory radiative-shock experiments. Another example is that of a collapsing shock in an old supernova remnant that is traversing a region of increasing average magnetic field.

Because the shell is denser than the matter interior to it, the inner surface of the shell remains Rayleigh–Taylor stable in this case. Within the shell, in contrast, the pressure gradient is parallel to the entropy gradient and opposed to the density gradient, so that internal gravity waves are expected to be convectively unstable. The characteristic frequency of these oscillations, within thick layers such as Earth’s atmosphere, is the Brunt–Väisälä frequency (Väisälä 1925; Brunt 1927), which, for acceleration \( g \), is

\[
\omega_{BV} = \sqrt{g\nabla \ln(\rho)} \text{, which here is} \tag{18}
\]

\[
\omega_{BV} = \sqrt{\frac{V_s}{L}} = \frac{\sqrt{c_s^2}}{(hL)} \text{, so that} \tag{19}
\]

implying that the modes are purely unstable when \( h > 0 \) and \( L < 0 \). In the case of interest in the present section, the mechanisms of convective instability and those of the Vishniac instability are both at play. By studying the roots of Equation (5), we explored this regime.

Figure 6 shows the growth rate in (a) and real frequency in (b) for this case with \( h = 10H \). Near the upper boundary on the top plot and toward the left, one has \( \Omega \ll 0 \) and finds large growth, while the real frequency has become zero. The growth rate increases toward the upper-left corner and is \( 3[\omega_n] \sim 1/\sqrt{(kh)(kL)} = \omega_{BV}/(k\varepsilon) \), and we recover the standard convective growth rate, but note that what is growing here are ripples of the entire layer combined with mass clumping. The small value of \( L \) localizes most of the matter near the inner boundary, creating a layer that is effectively thinner than \( H \). Here, the modulations grow without propagating and have the potential to strongly affect the structure of the layer. However, as \( kL \) becomes \( \ll 1 \), the solutions may change because the terms we dropped involving \( u/L \) start to become significant and
may alter the behavior. (This caution also applies below.) In addition, once the modulation amplitude becomes large enough, its growth becomes supersonic and so would drive shocks into the interior. We did not allow for this case in our analysis above; it seems likely that the actual behavior would weaken the growth from that point forward.

However, as $|L|$ increases, which might be a consequence of convective instability, and for $\lambda/H \sim 10$ to 100, the frequency of the modulations soon develops a real component, and the waves begin to propagate. The dashed curve in Figure 6(b) shows where, at some value of $|L|$, the parameter $\Omega$ becomes zero so that the dispersion is now $\omega^2 = \Pi$, which always has a negative imaginary (i.e., unstable) root. This is a new regime of the Vishniac instability, apparently unidentified previously, in which one always finds instability. This regime with $h > 0$ and $L < 0$ features propagating modes having a real frequency typically near $kc_s$ and having a growth rate near $0.1kc_s$, occurring at a wavelength near $\lambda = 2\pi h/\sqrt{1 + h/|L|}$.

Figure 6(a) shows that the modes are stabilized as the wavelength decreases. The dashed curve again shows where $\Pi = -1/4$ from Equation (16). As $h/H$ varies, the unstable region displayed in the plane of Figure 6(a) moves with the dashed curve, primarily by moderate up-and-down motion of the lower boundary. Here again, within the region where $\Pi < -1/4$ of Equation (16), the growth rate remains insensitive to $|L|/H$ at large values and the waves have a real frequency so they propagate. The growth rate decreases as $\lambda/H$ does, reflecting a decrease in $|\Pi|$ to small values. Along its lower boundary, $\Omega$ is near 1 and the modulations become stable near the dashed curve. We identify this as Vishniac stabilization of the convective instability, in which the thin-shell modulations prevent the development of unstable convection. The Vishniac dynamics appear to be smoothing out short-wavelength density maxima faster than they can grow.

Ryu & Vishniac (1991) described a related case, where a convectively unstable post-blast-wave layer could be stabilized by the oblique shock relations. They defined the large-scale structure of the adiabatic blast wave using a polytropic $\gamma$ and described their results in terms of conditions on $\gamma$. Our results provide a much clearer interpretation of the stabilization they find. The appearance of the stabilized region of parameter space here reveals that this stabilization is completely disconnected from the usual Vishniac stabilization for blast waves of $\gamma \geq 1.2$, as the latter depends on the large-scale, Sedov–Taylor dynamics which are not present here. Furthermore, by assuming neither blast-wave dynamics nor polytropic gas closures, we open this stabilization mechanism to more complex possibilities, including non-adiabatic mechanisms, which could fix the length scales relative to one another. Radiative transport in the presence of opacity or material composition gradients is just one possibility that could align $L$ and $h$ such that either the layer is Vishniac-stabilized or the region of propagating modes near $\Omega = 1$ is realized.

In this case we have found instability for all values of $L$ and $\lambda$ above a threshold. There is an extremely unstable case when $L/H$ becomes small, which we identify as standard (Brun–Väisälä) convection. There is a regime with $\Omega \sim 0$ in which one always finds propagating and growing waves, and there is what we describe as Vishniac stabilization of the convective mode at short wavelengths. This range of effects may apply to the early phases of radiatively collapsing shocks, depending upon the details and changing as they evolve.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Contours of growth rate normalized to $kc_s$, for isothermal scale height $h/H = -10$ for variations in $\lambda = 10H$ and normalized scale length, $|L|/H$. (a) $L > 0$. (b) $L < 0$.}
\end{figure}

\subsection{4.4. Accelerating Layers}

As discussed in the introduction, there also are cases in which a thin, shocked layer might accelerate, notably including shock breakout from supernovae. This reverses the hydrostatic pressure gradient, so that the inner surface of the dense layer becomes Rayleigh–Taylor unstable. Here our separation of the density scale length and the isothermal scale height enables us to examine the combination of Rayleigh–Taylor dynamics with the compressible dynamics of the thin shell. To our knowledge this has not been looked at previously. It is not surprising, though, to find purely growing modulations in this case. These have the potential to disrupt the shell, and likely more so than pure Rayleigh–Taylor modulations without mass clumping.

In terms of the mathematics, this case corresponds to $h < 0$. Examining Equation (7), one can see that $\Pi > 0$ in this case, unless $\phi = 2L/H$ is very small and negative. Here the instability will weaken to unimportance wherever $\Pi$ becomes small. As one would expect from the basic physics and also from Figure 2, the unstable roots of Equation (9) lie on the imaginary axis throughout the entire regime of interest, having no real part. Figure 7 shows the growth rate for both signs of $L$. Here there are no stability thresholds as such, but $\Pi$ becomes quite small, and $\Omega$ does not become large and negative, near the lower axis in both images and near the left axis in Figure 7(b).
Near the upper-left corner of Figure 7(b), \( \Omega \) does become large and negative. Along the upper axis, we again recover the Brunt–Väisälä growth rate \( -\gamma [\omega_x] \sim 1/(k h)(k L) \). In addition, since the absolute growth rates found here are of order \( k c_s \), one finds that the ratio of the thin-shell growth rate to the Rayleigh–Taylor growth rate at an embedded interface, \( \sqrt{k V_s} \), is \( \sim \sqrt{k h} \), becoming large for weak decelerations when \( h \gg \lambda \).

Overall, for the case of accelerating layers, we find instability for a very large range of conditions, with a growth rate within a factor of a few \( k c_s \).

5. Structure within the Layer

It may be helpful to display some aspects of the structure corresponding to these conditions. Figure 8 shows the relevant amplitudes for \( \lambda/H = 2 \pi/(k H) = 10, L = 10 H, \) and \( h = 10 H \). Shown in the figure is \( \delta/B_-, v_z/(c_s B_-), v_z/(c_B B_-), \) and \( \Delta z/(H B_-) \), as indicated. Since \( B_- \) is orders of magnitude smaller than 1, the physical amplitudes shown in the plot are all small. Panel (a) shows the modulations at the shock front, and panel (b) shows them at the interior interface.

Attending first to \( \Delta z \) (the gray curve) and \( v_z \) (the short-dashed curve), one can see that these variables are out of phase, as they should be. One can also see, by comparing the two parts of the figure, that the layer is thickening and thinning across the wave cycle. Comparing \( \Delta z \) and the density increase \( \delta \), one sees that these are close to but not exactly in phase at the interior boundary and nearly \( \pi \) radians out of phase at the shock. The regions of largest density do occur where the layer is thickest. Examining the long-dashed curve showing \( v_x \), one can see that the mass flow converges and then diverges near alternate zeros of \( \Delta z \) and \( \delta \). Considering the sense of these variations, it becomes clear that the modulations are propagating to the left, as is consistent with our imposed structure.

6. Conclusion

Herein, we have revisited the classic derivation of the instability of dense, shocked layers supported from within, which was first provided by Vishniac & Ryu (1989). Our aim has been to separate the role of the density scale length from that of the isothermal scale height, for the case of thin planar layers. We find that the density gradient enters into the spatial structure determined by the basic differential equations, while the isothermal scale height appears in the boundary conditions. This opens up the possibility of having distinct behavior depending on the sign of the density gradient and on whether the layer is accelerating or decelerating. All four kinds of structure may occur, both in nature and in laboratory astrophysics experiments. We have explored these behaviors for relevant ranges of density scale length, isothermal scale height, and ratio of unstable wavelength to layer thickness. We find that the unstable growth rate is often insensitive to the density scale length over large ranges, but is almost always sensitive to the isothermal scale height (and thus to the rate of deceleration of the shocked layer). The growth rate itself is typically within an order of magnitude of the acoustic frequency.

The unstable modulations most often propagate along the layer and involve both concentrations of matter and thickening of the layer. But there are some parameter regimes in which there is no propagation and the modulations grow in place. The combination of pure growth and mass clumping has the potential to disrupt the shell. For decelerating shocks, these purely growing modes occur under two circumstances. They appear when the isothermal scale height becomes smaller than the layer thickness and the density profile decreases away from the shock, or when the density scale length becomes smaller than the layer thickness. Purely growing modes also appear for all circumstances when the shocked layer is accelerating, in which case one has a compressible, thin-shell variant of the Rayleigh–Taylor instability.

The results found here may be useful based on the observation of long-wavelength structures. Any thin, decelerating shells that do not develop clumps may lie in the stable regime seen in Figure 6 above. The observation of large, stationary modulations or of bullet formation from decelerating thin shells would be indicative of the convective regime discussed in Section 4.3, while if the shell is accelerating, one would expect the thin-shell Rayleigh–Taylor mode, in which mass clumping by the Vishniac mechanism could accentuate the breakup of the shell. By contrast, propagating structures that move laterally along the shock surface might be described as Vishniac clumps.

The results and formulae herein should also be useful in assessing the potential importance and actual growth rate of the Vishniac–Ryu instability in a variety of experimental and computational applications.

We close with a speculation about one potential implication of these results. As the shock moves outward through a core-collapse supernova, each density transition produces a dense shell between a reverse shock and a composition interface (Guzman & Plewa 2009). This shell, or its outer surface, may develop structure from the Rayleigh–Taylor instability as well. During this phase the star remains optically very thick and the...
pressure in the shocked matter is almost entirely due to radiation. Later, as the density of a given shell becomes small enough, the radiation it contains will escape to the surface. In response, one would expect the shell to become much thinner because the pressure driving it will be sustained from within the star. Calculating the detailed behavior during this phase is complex; it appears not to have received much attention in simulations. Depending on the specifics, such dense shells could become violently unstable via the mechanisms described here. This would have the potential to be the missing link that could become violently unstable via the mechanisms described in the introduction, there are many mechanisms that may affect the overall, gradually established density profile, independent of its equation of state as such. In contrast, the small-amplitude fluctuations, which are roughly on the acoustic timescale, will generally be well described by a polytropic model, for some specific value of $\gamma$. In the usual case that active heat transport mechanisms have small effects on the acoustic timescale, the appropriate value of $\gamma$ will be that corresponding to isentropic compression of the medium of interest (not necessarily 5/3). In the unlikely limit of rapid-enough heat transport, by some mechanism, they go (Mihalas & Weibel-Mihalas 1984; Drake 2018) to isothermal behavior well described by taking $\gamma = 1$. Beyond that, $\gamma$ turns out to play a very minor role in the mathematics. In practice, the only place where it made a noticeable difference in the results is when one applies the constraint of VR89 that $L = h$. While taking $L = h$ was a perfectly fine assumption for a first theoretical paper, it is in fact unlikely in practice. Layers that accumulate mass via shocks tend to develop internal density and temperature gradients.

Recognizing that $\nabla \cdot [\rho(z) \, u(z)] = 0$, we find from these equations

$$\rho \delta_p + \rho \nabla \cdot \delta \rho = 0,$$

$$\rho \delta_t v_t + \rho u \delta_t \rho = 0,$$

$$\rho \delta_t v_z + \rho V_s \delta_t \rho = 0.$$

The background, hydrostatic profiles were described above. We take $\delta p = \gamma c_s^2 \delta \rho_p$, which employs our adiabatic, barotropic assumption about small fluctuations of the pressure. Throughout the following, the VR89 results are recovered by taking $\gamma = 1$. (We were asked by the referee to expand on differences in equation of state between that relevant to the density profile and that relevant to the small-amplitude fluctuations. As we discussed in the introduction, there are many mechanisms that may affect the overall, gradually established density profile, independent of its equation of state as such. In contrast, the small-amplitude fluctuations, which are roughly on the acoustic timescale, will generally be well described by a polytropic model, for some specific value of $\gamma$. In the usual case that active heat transport mechanisms have small effects on the acoustic timescale, the appropriate value of $\gamma$ will be that corresponding to isentropic compression of the medium of interest (not necessarily 5/3). In the unlikely limit of rapid-enough heat transport, by some mechanism, they go (Mihalas & Weibel-Mihalas 1984; Drake 2018) to isothermal behavior well described by taking $\gamma = 1$. Beyond that, $\gamma$ turns out to play a very minor role in the mathematics. In practice, the only place where it made a noticeable difference in the results is when one applies the constraint of VR89 that $L = h$. While taking $L = h$ was a perfectly fine assumption for a first theoretical paper, it is in fact unlikely in practice. Layers that accumulate mass via shocks tend to develop internal density and temperature gradients.)

Recognizing that $\nabla \cdot [\rho(z) \, u(z)] = 0$, we find from these equations

$$\partial_t \delta_p + \rho \nabla \cdot \delta \rho = 0,$$

$$\rho \delta_t v_t + \rho u \delta_t \rho = -\gamma c_s^2 \delta_p,$$

$$\rho \delta_t v_z + \rho V_s \delta_t \rho = -\gamma c_s^2 \delta_p.$$
derivatives in $z$ are large. The equations become

$$i\omega \delta + ikv_c + \partial_z v_z - \frac{v_z}{L} = 0,$$ (33)

$$i\omega v_z + ik\gamma_c^2 \delta = 0,$$ and

$$i\omega v_z + \gamma_c^2 \partial_z \delta = 0,$$ from which

$$\left(\frac{\omega^2 - k^2\gamma_c^2}{\gamma_c^2}\right)\delta - \frac{\gamma_c^2}{L}\partial_z \delta + \gamma_c^2 \partial_z^2 \delta = 0,$$ or

$$\left(\frac{\omega^2}{\gamma_c^2} - k^2\right)\delta - \frac{1}{L}\partial_z \delta + \partial_z^2 \delta = 0.$$ (37)

These have a solution

$$\delta = B_+ \exp\left[\frac{z}{2L}(1 + Q)\right] + B_- \exp\left[\frac{z}{2L}(1 - Q)\right],$$ where

$$Q = \sqrt{1 + 4k^2L^2 - 4\frac{\omega^2L^2}{\gamma_c^2}}.$$ (39)

Here we have found a significant typo, as VR have $Q$ without the factors of 4. Note that the sign of $Q$ has no impact, as the solution is invariant for a change in its sign. The sign of $L$ affects the direction in which $\delta$ increases, but the solution remains valid for either sign of $L$. We also follow VR and define $\lambda_+ = \gamma (1 + Q)/(2L)$ and $\lambda_- = \gamma (1 - Q)/(2L)$ for convenience. Equations (34) and (35) then give us

$$v_z = -\frac{k\gamma_c^2}{\omega} \delta$$ and

$$v_z = i\frac{\omega}{\omega} (B_+\lambda_+ \exp[\lambda_+z/\gamma] + B_-\lambda_- \exp[\lambda_-z/\gamma]).$$ (41)

We note that $v_z$ and $\delta$ are in phase, and that both are out of phase with $v_x$ (and thus in phase with the displacement of the matter, $\Delta z$). (Also we note that $v_z$ is mislabeled as $v_z$ in VR89, Equation (4d)).

To find the third solution of Equations (30)–(32) (after dropping the terms involving $u/L$), VR sought a solution in which the previously dropped $ud\Omega$ terms are dominant. Such a solution would only be significant near a very sharp change in magnitude, as at the shock. If one takes $\partial_z \gg 1/L$ and sets $\delta = 0$ in the equations, then Equation (30) implies $\partial_z v_z = -ikv_z$. Then we have from Equation (31),

$$v_z = A \exp\left(-i \int \frac{\omega}{u} dz'\right),$$ which gives

$$v_z = A \frac{\mu}{\omega} \exp\left(-i \int \frac{\omega}{u} dz'\right).$$ (43)

It will prove useful to note that $\omega$ does not depend on $z$ and to express $u(z)$ using the relations in Section 2. Then this part of the overall solution for $v_z$ can be written

$$v_z = A \frac{p_k}{\rho(z) V_z \omega} \exp\left(-i \frac{\omega V_z}{p_k} \int_0^z \rho(z') dz'\right)$$ so

$$v_z(0) = A \frac{c^2 k}{\omega}.$$ (45)

Also note $v_z(0) = A$. Now the general solution involves the sum of Equations (40) and (42) or Equations (41) and (43), and

$$v_z(0) = \frac{kr}{i V_z \omega}. $$ (46)

This confirms Equation 6(a) in VR89.

The other two boundary conditions arise from the necessary behavior of the pressure at the boundaries. Suppose that, near a boundary $b$ at a location $w = 0$, there is an initial pressure profile $p_{o, b}(w)$, and that the instantaneous pressure profile is $p_{o, A}(w)$ and the deviation of the boundary is $\zeta$. Note that the pressure is also a function of $\rho$. Taylor-expanding the pressure near the location $w = 0$, we have

$$p_{o}(\zeta) = p_{o, b}(0) + \left[\frac{\partial p_{o, b}}{\partial \omega}\right] \zeta + \left[\frac{\partial p_{o, b}}{\partial \rho}\right] \delta, \text{in which}$$ (47)

$$\left[\frac{\partial p_{o, b}}{\partial \omega}\right] = -\rho \dot{V}_z \text{ and } \left[\frac{\partial p_{o, b}}{\partial \rho}\right] = \chi_c^2 \text{, so}$$ (48)
\[ p_v(\zeta) = p_o(0) - \rho V_0 \zeta + \delta_p \gamma c_s^2. \]  
(49)

At a shock front, which we designate as location \( \zeta = \Delta z(0) \), we have
\[ p_v(\zeta) = p_o(0) + \left[ \frac{\partial p_v}{\partial V_z} \right]_0 V_z. \]
(50)

Using this result in Equation (49), recognizing that positive \( V_z \) slows the shock here, and writing the initial postshock pressure as \( \rho c_s^2 \),
\[- \frac{2 \rho c_s^2 V_z}{V_t} = -\rho \dot{V}_z + \delta_p \gamma c_s^2, \]
which gives
\[ \delta(0) = \frac{V_t \Delta z(0)}{\gamma c_s^2} - \frac{2 V_z}{V_t} = \frac{\Delta z(0)}{\gamma h} - \frac{2}{V_z} \].
(51)

The term involving \( V_z(0) \) is very small, as \( V_z(0) \ll u \) and we showed above that \( u \ll V_z \), so we, like VR, ignore this term. We then relate \( \Delta z \) to \( V_z \) to obtain
\[ \delta(0) = \frac{\Delta z(0)}{\gamma h} = \frac{V_z(0)}{i \gamma c_s h}. \]
(53)

This confirms Equation 6(b) in VR89.

Next consider the internal boundary with \( z = H \) corresponding to \( w = 0 \) in Equation (49), and with \( \zeta = \Delta z(H) \) now. Because lateral sonic flow sustains constant pressure along isopotentials throughout the medium, we have \( p_v(\Delta z(H)) = p_o(H) = p_i \) and Equation (49) becomes, with \( \delta_p \) evaluated at \( z = H \),
\[ \delta_p \gamma c_s^2 = \rho \dot{V}_z \Delta z(H), \]
from which
\[ \delta(H) = \frac{\Delta z(H)}{\gamma h} = \frac{V_z(H)}{i \gamma c_s h}, \]
(54)

confirming Equation (7) in VR89. It is worth mentioning that the derivation of the boundary conditions follows as well for an accelerating shock wave, which corresponds in Equations (53) and (54) to negative \( h \).

In the present section, we have found boundary conditions that depend not only on various structural variables, but also on the isothermal scale height \( h \). Notably, the boundary conditions do not depend at all upon the density scale length \( L \).

**Appendix D**

**Development of the Dispersion Relation**

The first step here is to apply the general solution to the boundary conditions (BCs). Equation (46) gives us
\[ A - \frac{k^2 \gamma c_s^2}{\omega} (B_+ + B_-) = \frac{V_t k}{\omega} \left( B_+ \frac{i c_s^2 \lambda_+}{\omega} + B_- \frac{i c_s^2 \lambda_-}{\omega} \right) \]
\[ + A \frac{c_s^2 k V_t}{\omega \omega}, \]
which is
\[ A \left[ 1 - \frac{c_s^2 k^2}{\omega^2} \right] = \frac{k c_s^2}{\omega} \left( B_+ \frac{\gamma + i V_t \lambda_+}{\omega} \right) \]
\[ + B_+ \frac{\gamma + i V_t \lambda_-}{\omega}. \]
(55)

This is written as it is to make it easy to see that it is identical to Equation 9(a) in VR. If one considers \( [V_z, \lambda_\pm]/\omega \), recognizing that \( \omega \) is of order \( kc \), or smaller, one can note that \( \lambda_\pm/k \) is never large while \( V_z/c_k \) is always large. One concludes that \( [V_z, \lambda_\pm]/\omega \gg \gamma \). This leads us to recast the previous equation as
\[ A \left[ 1 - \frac{c_s^2 k^2}{\omega^2} \right] = \frac{i V_t k c_s^2}{\omega^2} [B_+ \lambda_+ + B_- \lambda_-]. \]
(57)

Turning to Equation (53), it gives
\[ B_+ + B_- = \frac{c_s^2 k^2}{\omega^2 \gamma h} (B_+ \lambda_+ + B_- \lambda_-) - \frac{i c_s^2 k}{V_t \omega^2 \gamma h}, \]
(58)

which we can rewrite as
\[ B_+ \left( 1 - \frac{c_s^2 \lambda_+}{\omega^2 \gamma h} \right) + B_- \left( 1 - \frac{c_s^2 \lambda_-}{\omega^2 \gamma h} \right) = \frac{c_s^2 k^2}{V_t \omega^2 \gamma h} \]
\[ = \frac{-c_s^2 k^2 - 1}{\omega^2 i V_t \gamma h}. \]
(59)

Note that the first line here is identical to Equation 9(b) in VR89. The difference in the factors of \( A \) between Equations (57) and (52) is that Equation (46) has \( V_z(0) \) on the left-hand side while Equation (53) has \( \delta(0) \), and the general solution for \( \delta \) does not involve \( A \). The right-hand side of Equation (46) is \( V_z(0) k V_t/(\omega) \). One can carry through to find a dispersion relation by progressively eliminating the coefficients in these equations.

We first eliminate \( A \) from the above two boundary conditions. Doing this, we find
\[ 0 = B_+ \left( 1 - \frac{c_s^2 k^2}{\omega^2} - \frac{c_s^2 \lambda_+}{\omega^2 \gamma h} \right) + B_- \left( 1 - \frac{c_s^2 k^2}{\omega^2} - \frac{c_s^2 \lambda_-}{\omega^2 \gamma h} \right). \]
(60)

matching Equation (12) in VR. We can see what that implies, using the third BC.

Turning to Equation (54), we first note that, at the internal interface, the background velocity changes slowly so we take \( A = 0 \). Equation (54) then gives
\[ B_+ e^{\lambda_+ \gamma h / \gamma} + B_- e^{\lambda_- \gamma h / \gamma} = \frac{c_s^2}{\omega^2 \gamma h} (B_+ \lambda_+ e^{\lambda_+ \gamma h / \gamma} + B_- \lambda_- e^{\lambda_- \gamma h / \gamma}). \]
(61)

Rearranging and noting that \( \lambda_+ - \lambda_- = \gamma Q/L \), one has
\[ B_+ e^{\gamma Q h / L} \left( 1 - \frac{c_s^2}{\omega^2 \gamma h} \lambda_+ \right) + B_- \left( 1 - \frac{c_s^2}{\omega^2 \gamma h} \lambda_- \right) = 0. \]
(62)

This matches VR Equation 8(a), if one takes \( h = L \) and identifies \( e^{\gamma Q h / L} = \beta^{1/\beta} \), which follows from their relation that \( H = L \ln(1/\beta) \), and allows the factors of \( \gamma \) in our definition of \( \lambda_\pm \) to cancel those in the denominators. Note that in this exponential term, \( L \) represents the density gradient and has no necessary relation to any other scale height, such as \( h \). But their definition of \( \beta \) as \( p_i/\rho V_s^2 \) introduces such connections. We steer clear of them here, for now, by sticking with the...
exponential expression involving $L$. Note also that, at this point, the factors involving $\gamma$ entirely cancel out and do not enter the remainder of the solution except through $Q$. To proceed, we introduce a dimensionless variable $\eta = L/h$, which will equal 1 for the case of VR89.

It aids the derivation to introduce some dimensionless variables. Taking $\omega_{\text{at}} = c_0k$, we define $\omega_n = \omega_{\text{at}} / \omega_{\text{at}}$ along with $\kappa = kL$. We note that then $e^{iH\Omega / \kappa} = e^{iHQ/\kappa}$, and that $Q^2 = 1 + 4\kappa^2(1 - \omega_n^2)$. With these definitions, Equations (60) and (62) can be reworked to read

\[
B_+ e^{iHQ/\kappa} (2\kappa^2 \omega_n^2 - \eta (Q + 1)) + B_- (2\kappa^2 \omega_n^2 + \eta (Q - 1)) = 0, \quad \text{and} \quad B_+ (2\kappa^2 \omega_n^2 - 2\kappa^2 - \eta (Q + 1)) + B_- (2\kappa^2 \omega_n^2 - 2\kappa^2 + \eta (Q - 1)) = 0.
\]

We note again that the solution of these equations is independent of the sign attached to $Q$. Changing the sign of $Q$ only switches the roles of $B_+$ and $B_-$. These two equations have the solution

\[
0 = e^{iHQ/\kappa} (2\kappa^2 \omega_n^2 - \eta (Q + 1)) \times (2\kappa^2 \omega_n^2 - 2\kappa^2 + \eta (Q - 1)) - (2\kappa^2 \omega_n^2 + \eta (Q - 1))(2\kappa^2 \omega_n^2 - 2\kappa^2 - \eta (Q + 1)).
\]

With some algebra, this unpacks to give

\[
0 = \frac{\eta(1 - \eta/\gamma)}{\kappa^2} = \left(1 + \frac{1}{\kappa(h)} - \frac{1}{\gamma(\kappa h)^2}\right)
\]

and

\[
\Omega = \left(1 + \frac{\eta(1 - \eta/\gamma)}{\kappa^2}\right) = \left(1 + \frac{1}{\kappa(h)} - \frac{1}{\gamma(\kappa h)^2}\right) \quad \text{and}
\]

\[
\Pi = \frac{-\eta}{2\kappa^2} \left(\frac{Q}{\tanh(kHQ/(2\kappa))} - (2\eta - 1)\right), \quad \text{or}
\]

\[
\Pi = \frac{-\eta}{2\kappa^2} \left(\frac{kHQ/(2\kappa)}{\tanh(kHQ/(2\kappa))} - \frac{kH}{2\kappa} (2\eta - 1)\right).
\]

We recast these as

\[
\Omega = \frac{H/h}{(kh)^2} \left(\frac{(kh)^2}{H/h} + \frac{\phi}{\gamma} - \frac{\phi}{h}\right), \quad \text{and}
\]

\[
\Pi = \frac{-H/h}{(kh)^2} \left(\frac{Q/\phi}{\tanh(Q/\phi)} + \frac{\phi}{\gamma} - \frac{\phi}{h}\right).
\]

in which we have introduced the variable $\phi = 2L/h$ in Equation (71) for future convenience, and note that $\kappa = kH\phi/2$ and that $2\eta/\phi = H/h$. In this way, the variable that changes sign with $L$ is $\phi$. We are also set up to use $H/h = H/h/\phi = 2$ as a parameter, and note that it too can change sign. Equation (68) is equivalent to Equation 13(a) in VR89, with $e^{iHQ/\kappa} = \beta^{-\alpha}$, again consistent with $H = L \ln(1/\beta)$ as a definition of $\beta$. The simple case of VR89 has $L = h$ and thus $\phi = 2h/H$. If one takes $L = h$, then $\Omega = 1$, and one can rework our Equation (71) to obtain from (66) Equation (13) in VR89.

If one considers the region beyond the interface (the “interior”), Equations (33)–(39) apply, but with a change in $L$ to a value for that region, $L_0$, and $z$ replaced by $(z - H)$ in Equation (38). Since the disturbance must decay at large $z$, $B_+ = 0$. The boundary condition that will set $B_-$ is that $v_\text{r}$ must be continuous across the interface. Since the equations and boundary conditions describing the interior of the thin shell are sufficient to define the dispersion relation and the solutions there, the dynamics of the interior respond to those of the thin layer. Thus, we do not need to further discuss the response of the interior.

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**References**

Badjin, D. A., Glazyrin, S. I., Manukovskiy, K. V., & Blinnikov, S. I. 2016, MNRAS, 459, 2188

Bertschinger, E. 1986, ApJ, 304, 154

Blondin, J. M., & Marks, B. 1996, NewA, 1, 235

Blondin, J. M., Wright, E. B., Borkowski, K. J., & Reynolds, S. P. 1998, ApJ, 500, 342

Brent, D. 1927, QJRMS, 53, 30

Cavet, C., Michaut, C., Roy, F., et al. 2011, Ap&SS, 336, 183

Ditmire, T., & Edens, A. D. 2008, LPRv, 2, 400

Druk, R. P. 2018, High Energy Density Physics: Fundamentals, Inertial Fusion and Experimental Astrophysics (2nd ed.; Berlin: Springer)

Ennsman, L., & Burrows, A. 1992, ApJ, 393, 742

Fesen, R. A. 2001, ApJS, 133, 161

Grun, J., Stricker, J., Manka, C., et al. 1991, PhRvL, 66, 2738

Hughes, J. P., Rakowski, C. E., Burrows, D. N., & Slane, P. O. 2000, ApJL, 528, L109

Kushnir, D., Waxman, E., & Shvarts, D. 2005, ApJ, 634, 407

Mac Low, M. M., & Norman, M. L. 1993, ApJ, 407, 207

Michaut, C., Cavet, C., Bouquet, S. E., Roy, F., & Nguyen, H. C. 2012, ApJ, 759, 78

Mihalas, D., & Weibel-Mihalas, B. 1984, Foundations of Radiation Hydrodynamics, Vol. 1 (Oxford: Oxford Univ. Press)

Nishi, R. 1992, PThPh, 87, 347

Nishi, R., & Kamaya, H. 2000, ApJ, 532, 1172

Pettigrew, A. B., Drake, R. P., Dannenberg, K. K., et al. 2006, PhPl, 13, 082901

Robinson, A. P. L., & Pasley, J. 2018, PhPl, 25, 052701

Ryu, D., & Vishniac, E. T. 1987, ApJ, 313, 820

Ryu, D., & Vishniac, E. T. 1991, ApJ, 368, 411

Sanz, J., Bouquet, S. E., Michaut, C., & Miniere, J. 2016, PhPl, 23, 062114

Strickland, R., & Blondin, J. M. 1995, ApJ, 449, 727

Väisälä, V. 1925, CPMCM, 2, 46

Vishniac, E., & Ryu, D. 1989, ApJ, 337, 917

Vishniac, E. T. 1983, ApJ, 274, 154

Vishniac, E. T. 1994, ApJ, 428, 186

Zeldovich, Y. B., & Raiser, Y. P. 1966, Physics of Shock Waves and High-temperature Hydrodynamic Phenomena, Vol. 1 (2002nd ed.; New York: Dover)