Research Article

Laplace Transform Method for Pricing American CEV Strangles Option with Two Free Boundaries

Zhiqiang Zhou and Hongying Wu

School of Mathematics and Finance, Xiangnan University, Chenzhou 423000, China

Correspondence should be addressed to Zhiqiang Zhou; zqzhou@2014.swufe.edu.cn

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Laplace transform method (LTM) has a lot of applications in the evaluation of European-style options and exotic options without early exercise features. However, the Laplace transform methods for pricing American options have unsatisfactory accuracy and suffer from the instability. The aim of this paper is to develop a Laplace transform method for pricing American Strangles options with the underlying asset price following the constant elasticity volatility (CEV) models. By approximating the free boundaries, the Laplace transform is taken on a fixed space region to replace the moving boundaries space. After solving the linear system in Laplace space, Gaver-Stehfest formula (GSF) and hyperbola contour integral method (HCIM) are applied to compute the Laplace inversion. Numerical results show that the LTM-HCIM outperform the LTM-GSF in regard to the accuracy and stability for the option values.

1. Introduction

The Laplace transform methods for option pricing originate from the idea of randomizing the maturity in [1]. The Laplace transform methods are applied in the pricing options without early exercise features: Pelsser [2] for pricing double barrier options, Davydov and Linetsky [3] for pricing and hedging path dependent options under constant elasticity of variance (CEV) models, Sepp [4] for pricing double barrier options under double-exponential jump diffusion models, Cai and Kou [5] for pricing European options under mixed-exponential jump diffusion models, and Cai and Kou [6] for pricing Asian options under hyperexponential jump diffusion models.

This technique is known to work well for options without early exercise features, but, for American-style options, one difficulty has been perceived that the Black-Scholes-Merton PDE only holds where it is optimal to retain the option. Mallier and Aloabidi [7] develop a partial Laplace transform method for pricing American options in which the location of the free boundary for all values of the transform variable has to be determined by solving nonlinear integral equations. The approach is only conceptually appealing but practically ineffective in numerical implementations as for each transform variable one has to solve the nonlinear integral equations. In fact Mallier and Aloabidi [7] do not give the numerical computations.

A simple framework for Laplace transform methods is introduced by Zhu [8] for pricing American options under GBM models. This framework is later developed for evaluation of finite-lived Russian options by Kimura [9], pricing American options under CEV models by Wong and Zhao [10] and by Pun and Wong [11], pricing American options under hyperexponential jump-diffusion model by Leippold and Vasiljević [12], pricing stock loan (essentially an American option with time-dependent strike) by Lu and Putri [13], and pricing American options with regime switching by Ma et al. [14]. But this framework suffers from a drawback that numerical Laplace inversion such as Gaver-Stehfest (GS) and Gaver-Wynn-Rho (GWR) algorithm are not stable. Pun and Wong [11] explain the complex calculation of special functions as one of the possible reasons causing instability.

Zhou et al. [15] develop a new Laplace transform method for solving the free-boundary fractional diffusion equations arising in the American option pricing. By approximating the free boundary, the Laplace transform is taken on a fixed space region to replace the moving boundaries space. Then the
hyperbola contour integral method is exploited to restore the option values. The coefficient matrix has theoretically proven to be sectorial. Therefore, the highly accurate approximation and computational stability of the fast Laplace transform method are guaranteed.

In this paper, we develop the new Laplace transform method for solving American Strangles option pricing under CEV model. The Black-Scholes-Merton PDE of Strangles option is defined on the moving region \([S_p(\tau), S_\tau(\tau)]\), where \(S_p(\tau)\) is put side and \(S_\tau(\tau)\) is call side. The idea of the new method is modifying PDE into a new form on certain fixed region \([S_p, S_\tau]\). Then performing the Laplace transform leads to a ODE which involves the inverse functions \(\tau_p(S)\) and \(\tau_\tau(S)\) of \(S_p(\tau)\) and \(S_\tau(\tau)\), respectively. Using the finite difference method combined with the approximation of functions \(\tau_p(S)\) and \(\tau_\tau(S)\), where the optimal parameters of the approximation are obtained by minimizing the prescribed residual error, we obtain the numerical solution of the ODE in the Laplace space. In the final step, both the inversion Laplace Gaver-Stehfest formula (GSF) and the hyperbola contour integral method (HCIM) are used to recover the option value and the free boundary.

Numerical examples show the inverse Laplace GSF is unstable if the number of discrete integral nodes is greater than 20, so HCIM is an effective algorithm for computing Laplace inversion. However, the technics of spectral analysis in [15] cannot be applied over here, because the coefficients of PDEs arising from CEV Strangles option are not constant. This paper gives a convergence theorem by analysing the so-called Laplace transform iteration algorithm.

The remaining parts of this paper are arranged as follows: In Section 2, we develop the Laplace transform method for pricing American CEV Strangles option; in Section 3, we discuss the hyperbola contour integral method to compute inverse Laplace and give some stable conditions for HCIM. In Section 4, some examples are taken to confer the theoretical results of HCIM. Conclusions are given in the final section.

### 2. Evaluation of American Strangles under CEV Model

Assume that the underlying asset price is governed by CEV (see, e.g., [16]):

\[
dS_t = (r - q)S_t dt + \delta^{\beta + 1}\, dW_t, \tag{1}
\]

where \(r > 0\) is the risk-free interest rate, \(q \geq 0\) is the dividend yield, and \(W_t\) is standard Brownian motion. \(\delta\) represents the local volatility function, \(\beta\) and \(\delta\) can be interpreted as the elasticity of \(\sigma(S)\). \(\delta\) is the scale parameter fixing the initial instantaneous volatility at time \(t = 0\), i.e., \(\delta = \sigma(S_0)/S_0^{\beta}\). If \(\beta = 0\), the SDE (1) becomes the standard log-normal diffusion model or so-called geometric Brownian motion models (GBM). If \(\beta = -1/2\), the SDE (1) nests the Cox-Ingersoll-Ross (CIR) model. Let

\[
f(S, t) = \sup_{S_t \in \mathcal{S}_t} E\left[e^{-r(t-t)}\Pi(S_\tau) \mid S_t = S\right] \tag{2}
\]

be the price of an American Strangles position written on an underlying asset with price \(S\) at time \(t\) and the payoff

\[
\Pi(S_\tau) = \max(K_1 - S_\tau, 0) + \max(S_\tau - K_2, 0). \tag{3}
\]

This position is formed using a long put with strike \(K_1\) and a long call with strike \(K_2\). Let \(r = T - t\), \(n(S, r) = f(S, T - t)\), and the early exercise boundary on the put side be denoted by \(S_p(\tau)\), and the early exercise boundary on the call side be denoted by \(S_\tau(\tau)\). It is known that \(n(S, \tau)\) satisfies the Black-Scholes PDE:

\[
\frac{\partial n}{\partial \tau} = \frac{1}{2}\delta^2S^{2\beta+2} \frac{\partial^2 n}{\partial S^2} + (r - q)S \frac{\partial n}{\partial S} - rn, \tag{4}
\]

with initial condition

\[
n(S, 0) = \max(K_1 - S, 0) + \max(S - K_2, 0), \tag{5}
\]

and free boundary conditions

\[
\begin{align*}
\frac{\partial n}{\partial S} (S, \tau) &= K_1 - S, & n(S, \tau) &= -1, \quad S \geq S_p (\tau), \\
\frac{\partial n}{\partial S} (S, \tau) &= S - K_2, & n(S, \tau) &= 1, \quad S \leq S_\tau (\tau).
\end{align*} \tag{6}
\]

Chiarella and Ziogas [17] apply Fourier transform technique to derive a coupled integral equation system for the Strangles free boundaries, and then a numerical algorithm is provided to solve this system.

To establish the new Laplace transform method proposed by Zhou et al. [15], we first note some properties of early exercise boundary for American Strangle option under the CEV model (1). It is known from [18–20] that the free-boundaries, and then a numerical algorithm is provided to solve this system.
The discrete form of (10) is as follows: for $j = 1, 2, \ldots, N - 1$

\[
\frac{\Delta S}{2} S \frac{\partial^2 \nu^\infty}{\partial S^2} + \frac{(r - q) S \frac{\partial \nu^\infty}{\partial S}}{2} - r \nu^\infty = 0,
\]

with $S_p < S < S_c$, $\nu^\infty(S_p) = K_1 - S_p$, $\frac{d}{dS} \nu^\infty(S_p) = -1$, $\nu^\infty(S_c) = S_c - K_2$, $\frac{d}{dS} \nu^\infty(S_c) = 1$.

The values of $S_p$ and $S_c$ could be computed by secant method. Define uniform mesh $S_j = j \Delta S, j = 0, 1, \ldots, N$ and take larger enough number $S_{max} = S_N$ such that $S_{max} > S_c$. The discrete form of (10) is as follows:

\[
\left( \frac{\delta^2 S_j^{\beta+2}}{2 (\Delta S)^2} + \frac{(r - q) S_j}{2 \Delta S} \right) v_{j+1} + \left( \frac{\delta^2 S_j^{\beta+2}}{2 (\Delta S)^2} + \frac{(r - q) S_j}{2 \Delta S} \right) v_{j-1} = \left( \frac{\delta^2 S_j^{\beta+2}}{2 (\Delta S)^2} + \frac{(r - q) S_j}{2 \Delta S} \right) v_{j}.
\]

for $j = 1, 2, \ldots, N - 1$ with $v_j = \nu^\infty(S_j)$. Assuming $S_p = S_j$, and $S_c = S_j$, we have the pseudocode as Algorithm 1.

**Algorithm 1: Algorithm for solving $S_p$ and $S_c$.**

1. **Step 1.** Let $L_1 = 1, H_1 = j_1, L_2 = j_2, H_2 = N - 1$.
2. **Step 2.** Do the following loop:
   3. while $L_1 < H_1 - 1$ or $L_2 < H_2 - 1$
   4. Let $k_1 = \text{floor}(0.5 * (L_1 + H_1))$ and $k_2 = \text{floor}(0.5 * (L_2 + H_2))$.
   5. Solve linear system (13) with boundary $v_{k_1} = K_1 - S_{k_1}$ and $v_{k_2} = S_{k_2} - K_2$.
   6. Finite difference approximation (6).
   7. if $\frac{v_{k+1} - v_{k}}{\Delta S} < -1$ then
      8. $L_1 = k_1$;
   9. else
   10. $H_1 = k_1$;
   11. endif
   12. Finite difference approximation (7).
   13. if $\frac{v_{k} - v_{k-1}}{\Delta S} < 1$ then
      14. $L_2 = k_2$;
   15. else
   16. $H_2 = k_2$;
   17. endif
   18. endwhile
   19. **Step 3.** Output $S_p = S_{k_1}$ and $S_c = S_{k_2}$.

**Theorem 1.** With the same initial condition (5) and boundary conditions (6)-(7), the PDE (4) is equivalent to the following PDE:

\[
\frac{\partial \nu}{\partial \tau} = \frac{1}{2} \delta^2 S \frac{\partial^2 \nu}{\partial S^2} + \frac{(r - q) S \frac{\partial \nu}{\partial S}}{2} - r \nu
\]

\[
+ (r K_1 - q S) I_{(S \leq S_0)} (r K_2 - q S) I_{(S > S_0)}
\]

for $\tau \in (0, +\infty)$ and $S \in (S_p, S_c)$, where $I_{\{\cdot\}}$ is the indicator function.

**Proof.** (i) On the region $\tau \in (0, \infty), S \in (0, S_0(\tau))$, we know from the boundary condition (6) that $\nu(S, \tau) = K_1 - S$. Direct calculation shows that $\nu(S, \tau) = K_1 - S$ satisfies (14). (ii) On the region $\tau \in (0, \infty), S \in (S_0(\tau), S_c(\tau))$, both (4) and (14) have the same forms. (iii) On the region $\tau \in (0, \infty), S \in (S_c(\tau), \infty)$, we know from the boundary condition (7) that $\nu(S, \tau) = S - K_2$. Direct calculation shows that $\nu(S, \tau) = S - K_2$ satisfies (14). Combining (i), (ii), and (iii), we get this theorem.

Let $\tau_p(S)$ and $\tau_c(S)$ be the inverse functions of $S_p(\tau)$ and $S_c(\tau)$, respectively. Taking Laplace transform

\[
[\mathcal{L} \nu(\tau)](z) := \int_{0}^{\infty} e^{-z \tau} \nu(\tau) d\tau,
\]

\[
[\mathcal{L} I_{(S \leq S_p(\tau))}] = \int_{0}^{\infty} e^{-z \tau} I_{(S \leq S_p(\tau))} d\tau,
\]

\[
[\mathcal{L} I_{(S \geq S_c(\tau))}] = \int_{0}^{\infty} e^{-z \tau} I_{(S \geq S_c(\tau))} d\tau = \int_{0}^{\tau_p(S)} e^{-z \tau} d\tau
\]

\[
= \frac{1}{z} e^{-\tau_p(S)} = \frac{1}{z} (1 - e^{-\tau_p(S)}),
\]

\[
[\mathcal{L} I_{(S_p(\tau) \leq S \leq S_c(\tau))}] = \int_{0}^{\infty} e^{-z \tau} I_{(S_p(\tau) \leq S \leq S_c(\tau))} d\tau
\]

\[
= \int_{0}^{\tau_p(S)} e^{-z \tau} I_{(S \leq S_c(\tau))} d\tau = \int_{0}^{\tau_p(S)} e^{-z \tau} d\tau
\]

\[
= \frac{1}{z} e^{-\tau_p(S)} = \frac{1}{z} (1 - e^{-\tau_p(S)}),
\]
and

\[ \mathcal{L} \left[ I_{S \geq S_i(t)} \right] = \frac{1}{z} \left( 1 - e^{-z \tau_i(S)} \right), \quad (17) \]

to the both sides of (14), we get

\[
\begin{align*}
\mathcal{L}_z \hat{\nu} &= \frac{1}{2} \delta^2 \frac{\partial^2 \hat{\nu}}{\partial S^2} + (r - q) \frac{\partial \hat{\nu}}{\partial S} - r \hat{\nu} \\
&\quad + \max(K_1 - S, 0) + \max(S - K_2, 0) \\
&\quad + (r K_1 - q S) \frac{1}{z} \left( 1 - e^{-z \tau(S)} \right) \\
&\quad - (r K_2 - q S) \frac{1}{z} \left( 1 - e^{-z \tau_i(S)} \right) \\
&\quad + \max(K_1 - S, 0) + \max(S - K_2, 0) \\
&\quad + (r K_1 - q S) \frac{1}{z} \left( 1 - e^{-z \tau(S)} \right) \\
&\quad - (r K_2 - q S) \frac{1}{z} \left( 1 - e^{-z \tau_i(S)} \right)
\end{align*}
\]

(18)

on fixed space region \( S \in (S_p, S_c) \). The initial free boundary conditions (6) and (7) become the following forms:

\[
\begin{align*}
\hat{\nu}(S_p, z) &= \frac{1}{z} (K_1 - S_p), \\
\frac{\partial}{\partial S} \hat{\nu}(S_p, z) &= -\frac{1}{z}, \\
\hat{\nu}(S_c, z) &= \frac{1}{z} (S_c - K_2), \\
\frac{\partial}{\partial S} \hat{\nu}(S_c, z) &= -\frac{1}{z}.
\end{align*}
\]

(19)

(20)

Now, the free boundaries \( \tau_p(S) \) and \( \tau_c(S) \) can be approximated. To this end, we give the fit functions with unknown parameters \( p_i, \alpha_i \) for \( i = 1, 2 \):

\[
\begin{align*}
\tau_p(S) &= \begin{cases} 0, & S \geq S_p, \\
p_1 \left( - \log \frac{S - S_p}{S_p - S_c} \right)^{\alpha_1}, & S_p < S < S_c, \\
+\infty, & S = S_c,
\end{cases}
\end{align*}
\]

(21)

\[
\begin{align*}
\tau_c(S) &= \begin{cases} 0, & S \geq S_c, \\
p_2 \left( - \log \frac{S - S_c}{S_c - S_p} \right)^{\alpha_2}, & S_c < S < S_p, \\
+\infty, & S = S_p.
\end{cases}
\end{align*}
\]

(22)

Redefine uniform mesh \( S_j = S_p + j \Delta S, \quad j = 0, 1, \ldots, n + 1 \) with \( \Delta S = (S_c - S_p)/(n + 1) \). Equations (18) can be discretized as follows:

\[
(z I - A) \hat{\nu}(z) = g(z),
\]

(23)

with index \( n \) representing the number of space mesh partition. Matrix \( A \), solution vector \( \hat{\nu}(z) \), and RHS vector \( g(z) \) are defined as

\[
A(z) = \begin{bmatrix}
a_1 & b_1 & 0 & \cdots & 0 & 0 \\
& a_2 & b_2 & c_2 & \cdots & 0 \\
& & a_3 & b_3 & \cdots & 0 \\
& & & \ddots & \ddots & \ddots \\
& & & & 0 & 0 \\
0 & 0 & 0 & \cdots & b_{n-1} & c_{n-1} \\
0 & 0 & 0 & \cdots & b_n & a_n \\
\end{bmatrix}_{n \times n},
\]

(24)

\[
a_j = \frac{\delta^2 \frac{\partial^2 \hat{\nu}}{\partial S^2}}{(2 \Delta S)^2} - \frac{(r - q) S_j}{2 \Delta S},
\]

(25)

\[
b_j = -\frac{\delta^2 \frac{\partial^2 \hat{\nu}}{\partial S^2}}{(2 \Delta S)^2} - r,
\]

(26)

\[
g(z) = \begin{bmatrix} g_1(z), g_2(z), \ldots, g_n(z) \end{bmatrix}^T,
\]

(27)

where

\[
g_1 = a_1 (K_1 - S_p),
\]

(28)

\[
g_j = \max(K_1 - S_j, 0) + \max(S_j - K_2, 0) + \frac{1}{z} (r K_1 - q S_j) \left( 1 - e^{-z \tau_i(S_j)} \right) - \frac{1}{z} (r K_2 - q S_j) \left( 1 - e^{-z \tau(S_j)} \right),
\]

(29)

\[
g_n = c_n (S_c - K_2).
\]

(30)

Denote \( a_1(z) \) and \( a_n(z) \) to be the first row and the \( n \)th row of \( (z I - A)^{-1}(z) \), respectively. \( a_1(z) \) and \( a_n(z) \) can be obtained by solving linear equation:

\[
\begin{align*}
a_1(z) (z I - A) &= [1, 0, 0, \ldots, 0], \\
a_n(z) (z I - A) &= [0, 0, \ldots, 0, 1].
\end{align*}
\]

(31)

(32)

Then, we reach the expressions

\[
\begin{align*}
\hat{\nu}(S_1, z) &= a_1(z) g(z; p_1, p_2, \alpha_1, \alpha_2), \\
\hat{\nu}(S_n, z) &= a_n(z) g(z; p_1, p_2, \alpha_1, \alpha_2).
\end{align*}
\]

(33)

We find the parameters \( p_i, \alpha_i \) (\( i = 1, 2 \)) such that conditions (19) and (20) hold for all real values of \( z > 0 \). A practicable way is to consider a positive sequence \( \{z_1, z_2, \ldots, z_m\} \) which are used for numerical Laplace inversion and find the values
of unknown \( p_i \) and \( \alpha_i \) that optimizing the following objective function:

\[
\min_{p_i, \alpha_i, \alpha_i \geq 0} F(p_1, p_2, \alpha_1, \alpha_2) = \sum_{i=1}^{m} E_1 (i; p_1, p_2, \alpha_1, \alpha_2) + \sum_{i=1}^{n} E_2 (i; p_1, p_2, \alpha_1, \alpha_2)
\]

with

\[
E_1 (i; p_1, p_2, \alpha_1, \alpha_2) = \left( \frac{\tilde{v}(S_i, z_i) - \tilde{v}(S_0, z_i)}{\Delta S} \right)^2
- \frac{\partial}{\partial S} \tilde{v}(S_0, z_i)
+ \left( \frac{1}{z_i} \right)^2
\]

\[
E_2 (i; p_1, p_2, \alpha_1, \alpha_2) = \left( \frac{\tilde{v}(S_{i+1}, z_i) - \tilde{v}(S_m, z_i)}{\Delta S} \right)^2
- \frac{\partial}{\partial S} \tilde{v}(S_{i+1}, z_i)
+ \left( \frac{1}{z_i} \right)^2
\]

After determining \( r_p(S) \) and \( r_c(S) \) we can compute \( \tilde{v}_i(z) \) by solving linear system (23), and the value of American Strangles can be expressed in terms of the inverse Laplace transforms

\[
\tilde{v}^a(S, \tau) = \mathcal{L}^{-1} \left[ \tilde{v}^a(S_j, z) \right], \quad j = 1, 2, \ldots, n.
\]

Applying polynomial interpolations, we can obtain the option value \( v(S, \tau) \) for any values \( S \in (0, \infty) \). Moreover, the vector version of inverse Laplace can be formulated as

\[
v^a(\tau) = \mathcal{L}^{-1} \left[ \tilde{v}^a(z) \right].
\]

One of the classical numerical Laplace inversion is the Gaver-Stehfest formula (GSF) (see [21, 22]):

\[
v^a(\tau) \approx v^{a,L}(\tau) = \frac{\ln 2}{\tau} \sum_{k=1}^{L} C_k^{(L)} v^a \left( \frac{k \ln 2}{\tau} \right),
\]

where

\[
C_k^{(L)} = (-1)^{k+L/2} \frac{\ln 0}{\tau} \sum_{j=\lceil (k+1)/2 \rceil}^{\lceil L/2 \rceil} \frac{j^{L/2} (2j)!}{(L/2 - j)! (j-1)! (k-j)! (2j-k)!}.
\]

with \( L \) being taken as an even positive integer. Another effective numerical scheme for inverse Laplace transform is the so-called hyperbolic contour integral method (HCIM). We discuss HCIM in next section and give some examples, in Section 4, to illustrate the advantages of HCIM compared with GSF.

We finally summarize the Laplace transform method (NLTM) for pricing American Strangles as Algorithm 2.

### 3. Hyperbolic Contour Integral Method for Laplace Inversion

Many computational experiences show that the Gaver-Stehfest inverse Laplace formula (39) is unstable when \( L \geq 20 \) (see [21, 22]; also see Example 1 in this paper). So, it is very important to develop an appropriate numerical algorithm for restoring the option values. In this section, we discuss how to apply the Hyperbolic contour integral method to compute \( v^a(\tau) \) from \( v^a(\tau) \).

The inversion of Laplace transform is based on numerical integration of the Bromwich complex contour integral:

\[
v(S, \tau) = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+i\infty} e^{\tau z} \tilde{v}(S, z) \ dz, \quad \eta > \eta_0 ,
\]

where \( i = \sqrt{-1} \) and \( \eta_0 \) is the convergence abscissa. This means that all the singularities of \( \tilde{v}(S, z) \) (with respect to \( z \)) lie in the open half-plane \( \text{Re}(z) \leq \eta_0 \). The integral (41) is not well-suited for numerical integration. First, the exponential factor is highly oscillatory on the Bromwich line, \( z = \eta + iy, -\infty < y < +\infty \). Second, the transform \( \tilde{v}(S, z) \) typically decays slowly as \( |y| \to \infty \). One strategy for circumventing the slow decay is due to Talbot [23], who suggests that the Bromwich line be deformed into a contour \( \Gamma \) that begins and ends in the left half-plane, such that \( \text{Re}(z) \to -\infty \) at each end. On such a
contour, the exponential factor in (41) forces a rapid decay of the integrand as \( \text{Re}(z) \to -\infty \). This makes the integral well suited for approximation by the trapezoidal or midpoint rules. Owing to Cauchy’s theorem, such a deformation of contour is permissible as long as no singularities are traversed in the process, provided that \( \tilde{v}(S, z) \to 0 \) uniformly in \( \text{Re}(z) \leq \eta_0 \) as \( |z| \to \infty \).

Because the coefficient matrix \( A \) in (23) is not Toeplitz matrix, the technics of spectral analysis in [15] cannot be applied over here. To apply the idea of Talbot [23] and Zhou [15], we rewrite the linear system (23) as follows:

\[
(zI - \tilde{A}) \tilde{v}^n(z) = C\tilde{v}^n(z) + g(z),
\]

where \( I \) is the identity matrix.

\[
\tilde{A} = DB,
\]

\[
B = \frac{1}{(\Delta S)^2}
\begin{bmatrix}
-1 & 1/2 & 0 & \cdots & 0 & 0 \\
1/2 & -1 & 1/2 & \cdots & 0 & 0 \\
0 & 1/2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1/2 & -1 & 1/2 \\
0 & 0 & \cdots & 0 & 1/2 & -1
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
\delta S_1^{2\beta+2} & \delta S_2^{2\beta+2} & \cdots & \delta S_n^{2\beta+2} \\
\delta S_2^{2\beta+2} & \delta S_3^{2\beta+2} & \cdots & \delta S_{n-1}^{2\beta+2} \\
\vdots & \vdots & \ddots & \vdots \\
\delta S_n^{2\beta+2} & \delta S_1^{2\beta+2} & \cdots & \delta S_{n-2}^{2\beta+2} \\
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
-r & \frac{(r-q)S_1}{2\Delta S} & 0 & \cdots & 0 & 0 \\
\frac{(r-q)S_2}{2\Delta S} & -r & \frac{(r-q)S_2}{2\Delta S} & \cdots & 0 & 0 \\
\vdots & \vdots & -r & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{(r-q)S_{n-1}}{2\Delta S} & -r & \frac{(r-q)S_{n-1}}{2\Delta S} \\
0 & 0 & \cdots & 0 & \frac{(r-q)S_n}{2\Delta S} & -r
\end{bmatrix}
\]

and \( g(z) \) is defined by (27). Note that \( B \) is a Toeplitz matrix.

The linear system (42) can be solved by the following iteration algorithm:

\[
[\tilde{v}^n(z)]^{(\ell+1)} = (zI - \tilde{A})^{-1} \left( C[\tilde{v}^n(z)]^{(\ell)} + g(z) \right),
\]

\[
\ell = 0, 1, \ldots
\]

with \( [\tilde{v}^n(z)]^{(0)} = 0 \).

Let \( D^{-1} \)-inner product and the corresponding vector norm be defined by

\[
\langle v, v \rangle_{D^{-1}} = v^*D^{-1}v,
\]

\[
\|v\|_{D^{-1}} = \sqrt{\langle v, v \rangle_{D^{-1}}},
\]

\[
\forall v \in \mathbb{C},
\]

with \( D \) being defined by (44). Matrix \( D^{-1} \)-norm is induced by the vector \( D^{-1} \)-norm, i.e.,

\[
\|A\|_{D^{-1}} = \max_{v \neq 0} \frac{\|A\|_{D^{-1}}\|v\|_{D^{-1}}}{\|v\|_{D^{-1}}}, \ \forall A \in \mathbb{C}^{n \times n}.
\]
The theorem below shows that the iteration algorithm (46) is convergent if \( z \) falls outside of a sectorial region.

**Theorem 2.** (i) The spectrum \( \Lambda(\tilde{A}) \) of \( \tilde{A} \) defined by (43) lies in the sectorial region \( \Sigma_e \) for any values of \( e \in (0, \pi/2) \), i.e.,

\[
\Lambda(\tilde{A}) \subseteq \Sigma_e = \left\{ z \in C : |\arg(-z)| \leq e \right\}.
\]

(ii) For \( \tilde{A} \) defined by (43), we have

\[
\left\|(zI - \tilde{A})^{-1}\right\|_{D^{-1}} \leq \frac{1}{1 + \text{dist}(z, \Sigma_e)}, \quad z \notin \Sigma_e,
\]

with \( \Sigma_e \) being defined by (49) and \( \text{dist}(z, \Sigma_e) = \inf \{|z - \xi| : \xi \in \Sigma_e\} \).

(iii) Let \( \|C\|_{D^{-1}} \leq \gamma_0 \) with \( C \) being defined by (45) and assume \( \text{dist}(z, \Sigma_e) \geq \gamma \) such that \( 0 < \gamma_0 < 1 + \gamma \). Assuming \( \tilde{V}(z) \) solve (42), we have

\[
\left\|\tilde{V}(z)\right\|_D \leq \left( \frac{\gamma_0}{1 + \gamma} \right)^{[\ell]} \left\|\tilde{V}(z)\right\|_{D^{-1}}.
\]

Therefore, \([\tilde{V}(z)]^{[\ell+1]}\) is convergent to \( \tilde{V}(z) \) according to \( \|\cdot\|_{D^{-1}} \)-norm.

(iv) Under the assumptions of (iii), we have

\[
\left\|\tilde{V}(z)\right\|_{D^{-1}} \leq \frac{1}{1 + \gamma - \gamma_0} \left\|\tilde{g}(z)\right\|_{D^{-1}}.
\]

**Proof.** (i) Matrix \( B \) defined by (44) is a Toeplitz matrix and the generating function \( f(\eta) \) of \( B \) is

\[
f(\eta) = \frac{1}{(\Delta S)^2} \left( \frac{1}{2} e^{i\eta} - e^{-0i\eta} + \frac{1}{2} e^{-i\eta} \right) = \frac{1}{(\Delta S)^2} (\cos \eta - 1), \quad \eta \in (-\pi, \pi).
\]

From the above generating function \( f(\eta) \) and the discussion of Pang and Sun [24], we know the spectrum \( \Lambda(B) \) lies the sectorial region \( \Sigma_e \) for any values of \( e \in (0, \pi/2) \). Since \( D \) is positive definite, the spectrum \( \Lambda(\tilde{A}) = \Lambda(DB) \) also lies in the same sectorial region \( \Sigma_e \) (see [24, Lemma 3.5]).

(ii) The inequality (50) is proven in [24, Theorem 3.3].

(iii) From linear system (42) and iteration algorithm (46), we have

\[
\left\|\tilde{V}(z)\right\|_D^{[\ell+1]} \leq \left\|\tilde{V}(z)\right\|_{D^{-1}}^{[\ell]} \leq \left\|\tilde{V}(z)\right\|_D^{[\ell]}
\]

\[
\leq \frac{1}{1 + \text{dist}(z, \Sigma_e)} \left\|\tilde{g}(z)\right\|_{D^{-1}},
\]

for \( \ell = 0, 1, 2, \ldots \)

Therefore,

\[
\left\|\tilde{V}(z)\right\|_D^{[\ell+1]} \leq \frac{\gamma_0}{1 + \gamma} \left\|\tilde{V}(z)\right\|_{D^{-1}}^{[\ell]},
\]

\[
\leq \left( \frac{\gamma_0}{1 + \gamma} \right)^{[\ell+1]} \left\|\tilde{V}(z)\right\|_{D^{-1}}^{[\ell]},
\]

and

\[
\lim_{\ell \to \infty} \left\|\tilde{V}(z)\right\|_{D^{-1}}^{[\ell+1]} = 0,
\]

which means \([\tilde{V}(z)]^{[\ell+1]}\) is convergent to \( \tilde{V}(z) \) according to \( \|\cdot\|_{D^{-1}} \)-norm.

(iv) From linear equation (42) we have

\[
\left\|\tilde{V}(z)\right\|_{D^{-1}} \leq \left\|\tilde{g}(z)\right\|_{D^{-1}} + \|C\|_{D^{-1}} \left\|\tilde{V}(z)\right\|_{D^{-1}}
\]

\[
\leq \frac{\gamma_0}{1 + \gamma} \left\|\tilde{V}(z)\right\|_{D^{-1}} + \frac{1}{1 + \gamma} \left\|\tilde{g}(z)\right\|_{D^{-1}}.
\]

Consequently,

\[
\left\|\tilde{V}(z)\right\|_{D^{-1}} \leq \frac{1}{1 + \gamma - \gamma_0} \left\|\tilde{g}(z)\right\|_{D^{-1}},
\]

which ends the proof. \( \square \)

From Theorem 2, we see the iteration sequence \([\tilde{V}(z)]^{[\ell]}\), \( \ell = 0, 1, 2, \ldots \) and the limit \( \tilde{V}(z) \) are analytical if \( z \) lies far away from the sectorial region \( \Sigma_e \) defined by (49). Therefore, the convergence of iteration algorithm (46) requires that \( \|zI - \tilde{A}\|_{D^{-1}} \leq 1 \). This condition could be satisfied in next discussion.

The Laplace inversion of \( \tilde{V}(z) \) with respect to \( z \) can be expressed as

\[
\tilde{v}(\tau) = \frac{1}{2\pi i} \int_{\Gamma} e^{e^{\tau} \tilde{V}(z)} \, dz.
\]

This is the vector version of Laplace inversion (41). Weideman et al. [25] (also see Pang and Sun [24]) suggest the sectorial hyperbolic contour \( \Gamma \), which can be parameterized by

\[
\Gamma : z(\zeta) = \mu [1 + \sin (\zeta - \theta)], \quad -\infty < \zeta < \infty,
\]

where parameters \( \mu > 0 \) and \( \theta \) set the width and the asymptotic angle of the hyperbolic contour, respectively. Then substituting the contour (60) into (59) gives

\[
\tilde{v}(\tau) = \frac{1}{2\pi i} \int_{\Gamma} e^{e^{\tau} \tilde{V}(z)} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{e^{\tau} \zeta} \tilde{v}(\zeta) \, d\zeta.
\]
Discretization of this integral with uniform node spacing $h$ yields
\[ v^n(\tau) = \frac{h}{2\pi i} \sum_{k=-\infty}^{+\infty} e^{i(\xi_k)^T \tau} \varphi'(z(\xi_k)) + D_{E+}(h) \]
\[ + D_{E-}(h) \]
\[ = \frac{h}{2\pi i} \sum_{k=-L}^{L} e^{i(\xi_k)^T \tau} \varphi'(z(\xi_k)) + D_{E+}(h) \]
\[ + D_{E-}(h) + TE(hL) \]
\[ = v'^L(\tau) + D_{E+}(h) + D_{E-}(h) + TE(hL), \]
where $\xi_k = kh$ for the trapezoidal rule and $\xi_k = (k + 1/2)h$ for the midpoint rule, $L$ is the number of quadrature nodes, $D_{E+}(h)$ are the discretization errors, and $TE(hL)$ is the truncation error. Define
\[ g_{i}(\xi) = \frac{1}{2\pi i} e^{i(\xi)^T \tau} \varphi'(z(\xi)) \]
and assume the function $g_{i}(\xi + kd)$ to be analytic in the strip $d \in (d_+ , d_-)$ with $d_+ > 0$ and $d_- < 0$, then the discretization errors are bounded as
\[ \|D_{E+}(h)\| = \frac{M(d_+)}{e^{2\pi d_+ / h - 1}}, \]
\[ \|D_{E-}(h)\| = \frac{M(d_-)}{e^{-2\pi d_- / h - 1}}, \]
where
\[ M(d) = \int_{-\infty}^{+\infty} \|g_{i}(\xi + kd)\| d\xi, \ d = d_+ \text{ (or } d_-) \]
and $\|\|$ is some vector norm. The truncation error $TE(hL)$ can be approximated by the magnitude of the last term retained, i.e.,
\[ \|TE(hL)\| = O(\|g_{i}(hL)\|), \ L \to \infty. \]

We note that the estimates in (66) and (68) are extended in a straightforward manner, from the case $g_{i}(\xi)$ being scalar functions (see [24–26]) to the case when $g_{i}(\xi)$ takes value in a complex vector space.

To make all the singularities of the integrand in (61) to fall into the sectorial region $\Sigma$, defined by (49) for a given $\epsilon > 0$, the expressions of optimal parameters $h$, $\mu$, and $\theta$ must be chosen appropriately. By asymptotically matching $\|D_{E+}(h)\|$ and $\|TE(hL)\|$, $h$ and $\mu$ are derived as follows:
\[ h = \frac{\rho(\theta)}{L}, \]
\[ \mu = \frac{4\pi \theta - \pi^2 + 2\pi \epsilon L}{\rho(\theta)} \frac{L}{\tau}, \]
\[ \rho(\theta) = \cosh^{-1} \left( \frac{2\theta}{(4\theta - \pi + 2\epsilon) \sin \theta} \right), \]
and $\theta$ is a free parameter satisfying $\theta < \pi/2 - \epsilon$. With this choice the predicted convergence rate is given by
\[ \|v'^L(\tau) - v^n(\tau)\| \leq O(e^{-R(\theta)L}), \]
where
\[ R(\theta) = \frac{\pi^2 - 2\pi \theta - 2\pi \epsilon}{\rho(\theta)} \]

According to (70), we see that once the semiangle $\epsilon$ of the sectorial region (49) is given, then one can find the optimal $\theta$ by maximizing the function $R(\theta)$. Consequently, the optimal parameters $h$ and $\mu$ can be obtained from formulas (69).

Denoting $z_k = z(\xi_k)$ and $z' = z'(\xi_k)$, we have
\[ v'^L(\tau) = \frac{h}{2\pi i} \sum_{k=-L}^{L} e^{i(\xi_k)^T \tau} \varphi'(z_k) \approx v'^L(\tau) \]
\[ = \frac{h}{2\pi i} \sum_{k=-L}^{L} e^{i(\xi_k)^T \tau} \varphi'(z_k) [\varphi'(z_\xi)]^{(l)}. \]

Now, we give the convergence condition of the iteration algorithm (46). From [27, Theorem 3.3], the distance between $\Sigma_c$ and $\Gamma$ is given by
\[ \text{dist} (\Sigma_c, \Gamma) = \min_{P_{1,2} \in \Sigma_c, \Gamma} |P_1 - P_2| = \mu (1 - \sin \theta). \]

Inserting the above equation and the expression $\mu = ((4\pi \theta - \pi^2 + 2\pi \epsilon)/\rho(\theta))(L/\tau)$ into following inequality
\[ \left\| (2l - A) \right\|_{D^{-1}} \leq \frac{1}{1 + \text{dist}(\Sigma_c, \Gamma)} \]
we get
\[ \left\| (2l - A) \right\|_{D^{-1}} \leq \frac{1}{1 + ((4\pi \theta - \pi^2 + 2\pi \epsilon)(1 - \sin \theta)) \rho(\theta)(L/\tau)}. \]

Denote $\gamma = \mu(1 - \sin \theta) = \text{dist}(\Sigma_c, \Gamma)$. If we take $L$ such that
\[ \frac{L}{\tau} > \frac{\rho(\theta)(\gamma_0 - 1)}{(4\pi \theta - \pi^2 + 2\pi \epsilon)(1 - \sin \theta)}. \]
then $1 + \gamma > \gamma_0$ and the iteration algorithm (46) is convergent according to norm $\|\cdot\|_{D^{-1}}$. Inequality (76) requires that $L$ is large enough such that the hyperbolic contour $\Gamma$ is far away from the sectorial region $\Sigma_c$.

In following Theorem, we give an estimation for $\|C\|_{D^{-1}}$ and then give the minimum estimation for $L$.

**Theorem 3.** For matrix $C$ being defined by (45), we have the following estimation:
\[ \|C\|_{D^{-1}} \leq \gamma_0 = \frac{M \left( S_p, S_e \right)}{2\Delta S} \]
\[ \cdot \left[ \sqrt{8r^2 - 4r (r - q)} \right] (\Delta S)^2 + 3 (r - q)^2 S_p^2. \]
where $\bar{S}$ and $\underline{S}$ are defined by (8) in Section 2 and

$$ M(\bar{S}_p, \bar{S}_c) = \frac{\bar{S}_c \max_j S_j^\beta+1}{\bar{S}_p \min_j S_j^\beta+1} = \left( \frac{\bar{S}_c}{\bar{S}_p} \right)^\beta, \quad \beta \geq -1, $$
$$ M(\underline{S}_p, \underline{S}_c) = \frac{\underline{S}_c \max_j S_j^\beta+1}{\underline{S}_p \min_j S_j^\beta+1} = \left( \frac{\underline{S}_c}{\underline{S}_p} \right)^\beta, \quad \beta < -1. $$

(78)

If $\Delta S$ is small enough, then

$$ \|C\|_{D^{-1}} \leq \frac{\sqrt{3}(r-q)\bar{S}_p M(\bar{S}_p, \bar{S}_c)}{2\Delta S} = O \left( \frac{1}{\Delta S} \right), \quad \Delta S \to 0. $$

(79)

**Proof.** Firstly, we verified the inequality

$$ \|C\|_{D^{-1}} \leq \frac{\max_j S_j^{\beta+2}}{\min_j S_j^{\beta+1}} \|C\|, $$

where $\| \cdot \|$ represents the vector $\ell_2$-norm. Indeed,

$$ \|C\|_{D^{-1}} = \max_{v \neq 0} \frac{\|Cv\|_{D^{-1}}}{\|v\|_{D^{-1}}} = \max_{v \neq 0} \frac{\sqrt{(Cv)^{\top} D^{-1} Cv}}{\sqrt{v^{\top} D^{-1} v}} \\ \leq \max_{v \neq 0} \frac{\sqrt{1/\min_j \delta_j^{2\beta+2}}} {\sqrt{1/\max_j \delta_j^{2\beta+2}}} \frac{\|Cv\|}{\|v\|} = \frac{\max_j S_j^{\beta+2}}{\min_j S_j^{\beta+1}} \|C\|. $$

(81)

From definition (45), we have $C = (1/2\Delta S)D_1 C_1$ with

$$ C_1 = \begin{bmatrix}
-\frac{2r\Delta S}{S_1} (r-q) & 0 & \cdots & 0 & 0 \\
-\frac{2r\Delta S}{S_2} (r-q) & 0 & \cdots & 0 & 0 \\
0 & -\frac{2r\Delta S}{S_3} (r-q) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\frac{2r\Delta S}{S_{n-1}} (r-q) & 0 \\
0 & 0 & \cdots & 0 & -\frac{2r\Delta S}{S_n} (r-q)
\end{bmatrix} $$

and $D_1 = \text{Diag}[S_1, S_2, \ldots, S_n]$. By carefully calculating, we have

$$ C_1^\top C_1 = \left[ \epsilon_{jk} \right]_{n \times n} $$

(83)

where

$$ \epsilon_{jj} = \frac{4r^2 (\Delta S)^2}{S_j^2} + (r-q)^2, \quad j = 1, n, $$
$$ \epsilon_{jj} = \frac{8r^2 (\Delta S)^2}{S_j^2} + (r-q)^2, \quad j = 2, 3, \ldots, n-1, $$
$$ \epsilon_{j,j-1} = \frac{2r(r-q)(\Delta S)^2}{S_j S_{j+1}}, \quad j = 1, 2, \ldots, n, $$
$$ \epsilon_{j,j+1} = -\frac{(r-q)^2}{S_j S_{j-1}}, \quad j = 1, 2, \ldots, n, $$

and $c_{jk} = 0$ for $j = 1, 2, \ldots, n, k = j - 2$ and $k > j + 2$. By the Gershgorin circle theorem, each eigenvalue $\lambda(C_1^\top C_1)$ of $C_1^\top C_1$ falls into a certain Gershgorin disk $\mathcal{G}_j$, i.e.,

$$ \lambda(C_1^\top C_1) \in \mathcal{G}_j = \left\{ z : |z - \epsilon_{jj}| \leq \sum_{k=j-2, k \neq j}^{j+2} |c_{jk}| \right\} $$

(85)

for some index $j$. Incorporating (84) and (85), we get

$$ \lambda(C_1^\top C_1) \leq \max_j \left\{ \frac{8r^2 (\Delta S)^2}{S_j^2} + \frac{4r(r-q)(\Delta S)^2}{S_j S_{j+1}} + 3(r-q)^2 \min_j S_j^2 \right\} \right. $$

(86)

Therefore,

$$ \|C_1\| \leq \frac{\sqrt{\lambda(C_1^\top C_1)}}{\min_j S_j} \leq \frac{1}{\min_j S_j} \sqrt{(8r^2 - 4r(r-q))(\Delta S)^2 + 3(r-q)^2 \min_j S_j^2} $$

(87)

Incorporating (80) and (87), we have

$$ \|C\|_{D^{-1}} \leq \frac{\max_j S_j^{\beta+2}}{\min_j S_j^{\beta+1}} \|C\| \leq \frac{1}{2\Delta S} \min_j S_j^{\beta+1} \|D_1 C_1\| $$

(82)

$$ \leq \frac{1}{2\Delta S} \min_j S_j^{\beta+1} \|D_1\| \|C_1\| \leq \frac{1}{2\Delta S} \min_j S_j^{\beta+1} \max_j S_j^{\beta+1} $$

(88)

Let $\max_j S_j = \bar{S}_c, \min_j S_j = \underline{S}_p$ and $M(\bar{S}_p, \bar{S}_c) = (\bar{S}_c / \bar{S}_p) (\max_j S_j^{\beta+1} / \min_j S_j^{\beta+1})$; we finally obtain the estimate expression (77).

Finally, we give two remarks as follows. □
Remark 4. (i) For the fixed hyperbolic contour parameters $\theta$ and $\epsilon$, Theorem 3 and iteration convergence condition (76) show that we must take $L = O(1/\Delta S)$. (ii) For inverse Laplace formula $v_{n,L}^L(\tau)$ in (72), we solve $v_{1}^L(z)$ by directly computing linear equations $\hat{v}_{1}^L(z) = (1 - A)^{-1} g(z)$ (i.e., expression (23) in Section 2) if $L$ satisfies inequality (76) with $g_0$ being defined by (77).

Remark 5. From the upper estimate (52) for $\|v(z)\|_{D^2}$, we see the convergence of inverse Laplace dependents on the norm $\|g(z)\|_{D^2}$. Moreover, observing the expression of $g(z)$ defined by (27) and (29), the exponential terms $e^{-2\pi i [p(0),q(0)]}$ and $e^{-2\pi i [q(p),p(q)]}$ play important roles. Following the strategy of Zhou et al. [15], we split $g(z)$ into two parts $g_1(z) = g_1^j(z) + g_2^j(z)$ with

$$g_{1j} = \max \left( K_j - S_j , 0 \right) + \max \left( S_j - K_j , 0 \right)$$

$$+ \frac{1}{z} \left( r K_j - q S_j \right) - \frac{1}{z} \left( r K_j - q S_j \right)$$

$$- \frac{1}{z} \left( r K_j - q S_j \right) e^{-2\pi i (s_j)} 1_{\{r \geq s_j \}} (89)$$

$$g_{2j} = \frac{1}{z} \left( r K_j - q S_j \right) e^{-2\pi i (s_j)} 1_{\{r < s_j \}}$$

for $j = 1,2,\ldots,n$. Therefore, we have splitting expression $g(z) = g_1(z) + g_2(z)$. $e^{2\pi i} g_1(z)$ rapidly decays as $Re(z) \to -\infty$ if $z$ falls on the hyperbola contour $\Gamma$ defined by (60). To make the Laplace transform method convergent, we define an other hyperbola contour $\tilde{\Gamma}$:

$$\tilde{\Gamma} : z(\xi) = \mu \left[ 1 + \sin (K \theta + \theta) \right]$$

which begins and ends in the right half-plane, such that $Re(z) \to +\infty$ at each end. We select the same parameters $\epsilon, \mu,$ and $\theta$ as those of $\Gamma$. Obviously, $e^{2\pi i} g_2(z)$ rapidly decays as $Re(z) \to +\infty$ if $z$ falls on the hyperbola contour $\tilde{\Gamma}$ defined by (90). The inverse Laplace of $v_{1}^L(z)$ with respect to $z$ can be expressed as

$$v_{1}^L(\tau) = \frac{1}{2\pi i} \int_R e^{\frac{\tau}{1 - A}} g_1(z) + g_2(z) dz$$

$$= \frac{1}{2\pi i} \int_R e^{\frac{\tau}{1 - A}} \left( g_1(z) - g_2(z) \right) dz$$

$$+ \frac{1}{2\pi i} \int_R e^{\frac{\tau}{1 - A}} g_2(z) dz = v_{1}^L(\tau) + v_{2}^L(\tau),$$

where $v_{1}^L(\tau)$ and $v_{2}^L(\tau)$ have the predicted convergence rate $R(\theta)$ defined by formula (70).

4. Numerical Examples

In this subsection, we list some Strangles option values computed by the Laplace transform method and hyperbola contour integral method (labeled by “LTM-HCIM”). These results are compared with those obtained by LTM using inverse Laplace GSF (labeled by “LTM-GSF”) and FDM. The parameters in model (4)-(7) are taken as $T = 1$ (year), $r = 0.05$ (1/year), $q = 0.1$ (S), $K_1 = 1$ (S), $K_2 = 1.5$ (S), $S_0 = 1.5$, $\delta = 0.05$, $\delta_0 = 0.05$ and $\delta_{max} = 4K_2$. The values of parameters in HCIM are taken as $\theta = 0.8$ and $\epsilon = 0.4$. We take the number of space partition $N = 3000$ to compute $S_{\tilde{x}}$ and $S_{\tilde{y}}$ and take $n = 3000$ to optimise $r_\delta(S)$ and $r_\epsilon(S)$. The positive values $z_1 = 1 + 3i/m$ with $m = 6$ in expressions (35) and (36). The size of space mesh and the time mesh is set as $N = 2000$, $\Delta t = T/2000$ for FDM.

Example 1. Table 1 lists the American Strangles option prices with $g_0 = 0.4$, different values of $\beta$ and different values of $L$. From the table, we see LTM-GSF is unstable when $L > 20$, while the values of LTM-HCIM is very close to ones of FDM for $L = 10, 20, 26$. So, LTM-HCIM outperforms the LTM-GSF formula in regard to the accuracy and stability for the option values.

Example 2. This example illustrates the convergence rate of LTM-HCIM for the number $n$ of space mesh and the number $L$ for HCIM. Table 2 lists the space convergence rate with $L = 20$ and fixed $N = 3000$. The column labeled “Value” shows the option values at stock price $S_0 = 1.5$; the column labeled “Err” is defined by
Table 2: Values of American Strangles at time $t = 0$ (i.e., $\tau = T$) with $K_1 = 1$, $K_2 = 1.5$, $r = 0.05$, $q = 0.1$, $S_0 = 1.5$, $L = 20$, different values of $\sigma_0$, different values of $\beta$, and different space partition $n$.

| $n$  | Value  | $\sigma_0 = 0.2$, $\beta = 0.5$ | Rate  | Value  | $\sigma_0 = 0.4$, $\beta = -0.5$ | Rate  |
|-----|--------|---------------------------------|-------|--------|---------------------------------|-------|
| 24  | 0.150660 | 3.3277e-003                    | 0.6061| 0.262272| 2.7302e-003                    | 1.5515|
| 49  | 0.149394 | 2.1862e-003                    | 0.5367| 0.260271| 9.314e-004                     | 0.9603|
| 99  | 0.149262 | 1.5070e-003                    | 0.5154| 0.260335| 4.7871e-004                    | 1.0611|
| 199 | 0.149211 | 1.0543e-003                    | 0.5076| 0.260319| 2.2943e-004                    | 0.9601|
| 399 | 0.149187 | 7.4158e-004                    | 0.5037| 0.260274| 1.1793e-004                    | 0.9812|
| 799 | 0.149191 | 5.2302e-004                    | 0.5018| 0.260276| 5.9737e-005                    | 0.9920|
| 1599| 0.149189 | 3.6937e-004                    | —     | 0.260275| 3.0034e-005                    | —     |
| 3199| 0.149189 | —                               | —     | 0.260275| —                               | —     |

\[
\text{Err}(n) = \sqrt{\Delta S(n)} \left\| v^{n,L}(\tau) - \mathcal{F} \left[ v^{2n,L}(\tau) \right] \right\|, \tag{92}
\]

where $\mathcal{F}[v^{2n,L}(\tau)]$ is the linear interpolation of $v^{2n,L}(\tau)$ from $2n$-mesh to $n$-mesh, $\| \cdot \|$ is the $\ell_2$ vector norm and $\Delta S(n) = (S_n - S_{n+1})/(n + 1)$, and the convergence rate is defined by

\[
\text{Rate}(n) = \log_2 \frac{\text{Err}(n)}{\text{Err}(2n)}. \tag{93}
\]

From Table 2, we see the convergence rate of LTM-HCIM is 0.5 and 1.0 for $\beta = 0.5$ and $\beta = -0.5$, respectively. It is very interesting to investigate the relationship between the convergence rates and the values of $\beta$ and we leave this topic to the future.

We regard the computational solution $v^{n,L}(\tau)$ with $n = 1599$, $L = 30$ as the reference option prices, i.e., $v^{1} (\tau) \approx v^{1599,30} (\tau)$, and the LTM-HCIM time-errors are defined by

\[
\text{Err}(L) = \sqrt{\Delta S(1600)} \left\| v^{1599,L}(\tau) - v^{1599,30}(\tau) \right\|, \tag{94}
\]

for different values of $L$. Figure 1 plots the LTM-HCIM errors in log-scale v.s. $L$. From this Figure we see the numerical convergence rates are consistent with the predicted rates

\[
\left\| v^{n,L}(\tau) - v^1(\tau) \right\| \leq \mathcal{O}\left(e^{-R(\beta)L}\right), \tag{95}
\]

for moderate values of $L$.

5. Conclusions

An efficient Laplace transform method was developed for pricing American Strangles option under CEV diffusion. Numerical examples show LTM is a fast algorithm to solve PDEs with free boundaries and the hyperbola contour integral method has higher numerical accuracy and good stability for numerical Laplace inversion. The method developed here can be extended to the pricing of other American options, such as American options with regime switching and with jump–diffusion process.

Data Availability

All Matlab codes supporting the results of this study are available from the corresponding author on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Hongying Wu carried out the experiments in Section 4 and Zhiqiang Zhou made the main contributions to the
other sections. All authors read and approved the final manuscript.

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