M-theory FDA, Twisted Tori and Chevalley Cohomology†

Pietro Fré∗

Dipartimento di Fisica Teorica, Universitá di Torino, & INFN - Sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy

Abstract

The FDA algebras emerging from twisted tori compactifications of M-theory with fluxes are discussed within the general classification scheme provided by Sullivan’s theorems and by Chevalley cohomology. It is shown that the generalized Maurer Cartan equations which have appeared in the literature, in spite of their complicated appearance, once suitably decoded within cohomology, lead to trivial FDA.s, all new p–form generators being contractible when the 4–form flux is cohomologically trivial. Non trivial $D = 4$ FDA.s can emerge from non trivial fluxes but only if the cohomology class of the flux satisfies an additional algebraic condition which appears not to be satisfied in general and has to be studied for each algebra separately. As an illustration an exhaustive study of Chevalley cohomology for the simplest class of SS algebras is presented but a general formalism is developed, based on the structure of a double elliptic complex, which, besides providing the presented results, makes possible the quick analysis of compactification on any other twisted torus.

E-mail:
*) fre@to.infn.it

† This work is supported in part by the European Union RTN contract MRTN-CT-2004-005104 and by the Italian Ministry of University (MIUR) under contract PRIN 2003023852
1 Introduction

Recently considerable attention has been devoted to flux compactifications of String Theory and M–theory, since this provides mechanisms to stabilize the moduli fields \[1, 2, 3, 6, 7, 8, 9\]. Within this context a particularly interesting class of flux compactifications is constituted by those on twisted tori. This is the contemporary understanding of the Scherk-Schwarz \[4\] mechanism of mass generation from extra dimensions. As it was explained by Hull and Reid-Edwards \[5\], twisted tori are just Lie group manifolds \(G\) modded by the action of some discrete subgroup \(\Delta \subset G\) which makes them compact.

A general pattern only recently elucidated is the relation between fluxes and gauge algebras. After dimensional reduction in presence of flux compactifications one ends up with some lower dimensional gauged supergravity and a particularly relevant question is that about the structure of the gauge Lie algebra in relation with the choice of the flux. Generalizing a concept originally introduced in \[10\], the authors of \[13\] have provided a very elegant and intrinsic classification of supergravity gaugings in terms of the so called embedding matrix and of its transformation properties under the duality group \(U\). Later, in various papers, the relation between the entries of the embedding matrix and the geometrical fluxes has been elucidated for various cases of compactifications \[14, 12, 15\]. Hence an obvious question is that relative to the gauge algebra emerging from flux compactifications on twisted tori. This question has been addressed in a recent series of papers \[22, 23, 24, 25\] and it has been advocated that the gauge algebraic structures emerging in \(D = 4\) supergravity from M–theory reduction on twisted tori with fluxes do not fall in the class of Lie algebras \(G\) rather have to be understood in the more general context of Free Differential Algebras. A priori this is not too surprising since M–theory itself, as all other higher dimensional supergravities, is based on the gauging of an FDA. The algebraic structure that goes under this name was independently discovered at the beginning of the eighties in Mathematics by Sullivan \[20\] and in Physics by the present author in collaboration with R. D’Auria \[18\]. Free Differential Algebras (FDA) are a categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (\(p\)–forms) of rank greater than one. The very existence of FDA.s is a consequence of Chevalley cohomology of ordinary Lie algebras and Sullivan has provided us with a very elegant classification scheme of these algebras based on two structural theorems rooted in the set up of such an elliptic complex.

As I already noted in a paper of about two decades ago \[17\], FDA.s have the additional fascinating property that, differently from ordinary Lie algebras they already encompass their own gauging. Indeed the first of Sullivan’s structural theorems, which is in some sense analogous to Levi’s theorem for Lie algebras, states that the most general FDA is a semidirect sum of a so called minimal algebra \(\mathbb{M}\) with a contractible one \(\mathbb{C}\). The generators of the minimal algebra are physically interpreted as the connections or potentials, while the contractible generators are physically interpreted as the curvatures. The real hard–core of the FDA is the minimal algebra and it is obtained by setting the contractible generators (the curvatures) to zero. The structure of the minimal algebra \(\mathbb{M}\), on its turn, is beautifully determined by Chevalley cohomology of \(\mathbb{G}\). This happens to be the content of Sullivan’s second structural theorem.

In the present paper my goal is that of recasting the findings and the equations presented
in [22, 23, 24, 25] into the general scheme of FDA.s. In particular I want to unveil the cohomological interpretation of those equations and answer the basic question, namely what is the structure of the minimal algebra \( \mathbb{M} \). My result is that notwithstanding their apparent complicated appearance, when appropriately decoded within the language of cohomology, the zero curvature Maurer Cartan equations presented in [24, 25] actually define a trivial minimal algebra \( \mathbb{M} = \mathbb{G} \), all higher degree generators being contractible. This is necessarily the case if the 4–form flux is cohomologically trivial. Non trivial 4–form fluxes might instead lead to non trivial \( D = 4 \) minimal algebras. However this happens if and only if a certain algebraic relation is satisfied by the cohomology class to which the flux is assigned. I will spell out explicitly such an algebraic condition showing that it is not generically satisfied by simple non degenerate SS algebras. It can be, instead, satisfied for degenerate algebras.

In this paper my main goal is to establish a compact, index free, formalism to deal with twisted tori compactifications which unveils the underlying mathematical set up, that of a double elliptic complex. Indeed the basic objects we deal with are space–time \( p \)-forms valued in Chevalley \( q \)-cochains and we have two boundary operators, one acting on the base space and one on the Chevalley fibre. Once reformulated in this way the spectrum of generators of the \( D = 4 \) FDA is immediately read off from the Chevalley cohomology groups of the fibre Lie algebra \( \mathbb{G} \). The generalized structure constants of the FDA are also elegantly rewritten in terms of the Poincaré pairing form on the Chevalley complex and the question whether the minimal algebra is trivial or not can also be answered within this framework.

The paper is organized as follows.

In section 2 I summarize the set up of Chevalley cohomology. In subsections 2.1, 2.2 and 2.3 I calculate the cohomology groups of SS algebras and construct an explicit basis for their cochain spaces.

In section 3 I recall the general set up of FDA.s and I summarize Sullivan structural theorems. In subsection 3.1 I discuss the non trivial FDA.s associated with SS algebras.

In section 4 I illustrate the working of Sullivan’s theorem with the example of the super FDA of M–theory.

In section 5 I elaborate the cohomological analysis of twisted tori compactifications and I set up the double elliptic complex formalism. Then in subsection 5.3 I single out the cohomological interpretation of the zero curvature equations showing that they always lead to a trivial minimal algebra, all \( p \)-form generators being contractible. In the last section 5.4 I introduce the curvature and I discuss the bearing of cohomologically non trivial fluxes.

Finally section 6 contains my conclusions.

## 2 Chevalley Cohomology

As a necessary preparatory step for my discussion let me shortly recall the setup of the Chevalley elliptic complex leading to Lie algebra cohomology. This will also fix my notations and conventions.

Let us consider a (super) Lie algebra \( \mathbb{G} \) identified through its structure constants \( \tau^{IJK} \).
which are alternatively introduced through the commutation relation of the generators \(^1\)

\[ [T_I, T_K] = \tau_{JK}^I T_I \]  

(2.1)
or the Cartan Maurer equations:

\[ \partial e^I = \frac{1}{2} \tau_{JK}^I e^J \wedge e^K \]  

(2.2)

where \( e^I \) is an abstract set of left–invariant 1–forms. The isomorphism between the two descriptions (2.1) and (2.2) of the Lie algebra is provided by the duality relations:

\[ e^I (T_J) = \delta^I_J \]  

(2.3)

A \( p \)-cochain \( \Omega^{[p]} \) of the Chevalley complex is just an exterior \( p \)-form on the Lie algebra with constant coefficients, namely:

\[ \Omega^{[p]} = \Omega_{I_1 \ldots I_p} e^{I_1} \wedge \ldots \wedge e^{I_p} \]  

(2.4)

where the antisymmetric tensor \( \Omega_{I_1 \ldots I_p} \in \wedge^p \text{adj} G \) which belongs to the \( p \)-th antisymmetric power of the adjoint representation of \( G \) has constant components. Using the Maurer cartan equations (2.2) the coboundary operator \( \partial \) has a pure algebraic action on the Chevalley cochains:

\[ \partial \Omega^{[p]} = \partial \Omega_{I_1 \ldots I_{p+1}} e^{I_1} \wedge \ldots \wedge e^{I_{p+1}} \]

\[ \partial \Omega_{I_1 \ldots I_{p+1}} = (-)^{p-1} \frac{p}{2} \tau_{[I_1 I_2} \Omega_{I_{p+1}]} R \]  

(2.5)

and Jacobi identities guarantee the nilpotency of this operation \( \partial^2 = 0 \). The cohomology groups \( H^{[p]} (G) \) are constructed in standard way. The \( p \)-cocycles \( \Omega^{[p]} \) are the closed forms \( \partial \Omega^{[p]} = 0 \) while the exact \( p \)-forms or \( p \)-coboundaries are those \( \Lambda^{[p]} \) such that they can be written as \( \Lambda^{[p]} = \partial \Phi^{[p-1]} \) for some suitable \( p - 1 \)-forms \( \Phi^{[p-1]} \). The \( p \)-th cohomology groups is spanned by the \( p \)-cocycles modulo the \( p \)-coboundaries. Calling \( C^p (G) \) the linear space of \( p \)-chains the operator \( \partial \) defined in eq.(2.5) induces a sequence of linear maps \( \partial_p \):

\[ C^0 (G) \overset{\partial_0}{\longrightarrow} C^1 (G) \overset{\partial_1}{\longrightarrow} C^2 (G) \overset{\partial_2}{\longrightarrow} C^3 (G) \overset{\partial_3}{\longrightarrow} C^4 (G) \overset{\partial_4}{\longrightarrow} \ldots \]  

(2.6)

and we can summarize the definition of the Chevalley cohomology groups in the standard form used for all elliptic complexes:

\[ H^{(p)} (G) \equiv \frac{\ker \partial_p}{\text{Im} \partial_{p-1}} \]  

(2.7)

On the Chevalley complex it is also convenient to introduce the operation of contraction with a tangent vector and then of Lie derivative.

\(^1\)In view of my main goal which is the analysis of FDA.s emerging in bosonic M-theory compactifications I adopt a pure Lie algebra notation. Yet every definition presented here has a straightforward extension to superalgebras and indeed when I will recall the discussion of how the FDA of M-theory emerges from the application of Sullivan structural theorems is within the scope of super Lie algebra cohomology.
The contraction operator $i_X$ associates to every tangent vector, namely to every element of the Lie algebra $X \in G$ a linear map from the space $C^p(G)$ of the $p$-cochains to the space $C^{p-1}(G)$ of the $p-1$-cochains:

$$\forall X \in G ; \quad i_X : C^p(G) \mapsto C^{p-1}(G) \quad (2.8)$$

Explicitly we set:

$$\forall X = X^MT_M \in G ; \quad i_X \Omega[p] = p X^M \Omega_{MI_1...I_{p-1}} e^{I_1} \wedge \ldots \wedge e^{I_{p-1}} \quad (2.9)$$

Next I introduce the Lie derivative $\ell$ which to every element of the Lie algebra $X \in G$ associates a map from the space of $p$–cochains into itself:

$$\forall X = X^MT_M \in G ; \quad \ell_X : C^p(G) \mapsto C^p(G) \quad (2.10)$$

The map $\ell_X$ is defined as follows:

$$\ell_X \equiv i_X \circ \partial + \partial \circ i_X \quad (2.11)$$

and satisfies the necessary property in order to be a representation of the Lie algebra:

$$[\ell_X, \ell_Y] = \ell_{[X,Y]} \quad (2.12)$$

By explicit calculation we find in components:

$$\ell_X \Omega[p] = (-)^{p-1} p X^M \tau^R_{M[I_1} \Omega_{I_2I_3...I_{p}]} R e^{I_1} \wedge \ldots \wedge e^{I_p} \quad (2.13)$$

Furthermore if $X$ and $Y$ are any two $G$ Lie algebra–valued space–time forms, respectively of degree $x$ and $y$, by direct use of the above definitions, you can easily verify the following identity which holds true on any $p$–cochain $C[p]$:

$$(i_X \circ \ell_Y + (-)^{xy+1} \ell_Y) C[p] = -i_{[X,Y]} C[p] \quad (2.14)$$

and which will of great help in my following analysis.

### 2.1 The pairing form and Poincaré duality on the Chevalley complex

Chevalley cochains are antisymmetric tensors on a finite dimensional vector space, namely the Lie algebra $G$. For this reason the space $C^p$ is a finite dimensional and can in fact be decomposed into subspaces of predictable dimension. Let us call

$$d = \dim G \quad (2.15)$$

Then the vector space of all cochains $C$ has dimension $2^d$ and we can write:

$$C = C^0 \oplus C^1 \oplus C^2 \oplus \ldots \oplus C^d$$

$$2^d = 1 + \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{d} \quad (2.16)$$
Each finite dimensional $p$-cochain space can now be decomposed into the following triplet of subspaces:

$$C^n = \Gamma^n \oplus \partial \Xi^{n-1} \oplus \Xi^n$$  \hspace{1cm} (2.17)

where, by definition we have posed:

$$\partial \Xi^{n-1} \equiv \text{Im} \partial_{n-1}$$  
$$\Gamma^p \equiv \text{orthogonal complement of Im} \partial_{n-1} \text{ in ker } \partial_n$$  
$$\Xi^n \equiv \text{orthogonal complement of ker } \partial_n \text{ in } C^n$$  \hspace{1cm} (2.18)

We give a name to the dimensions of these subspaces:

$$\dim \Gamma^n \equiv h_n; \quad \dim \partial \Xi^{n-1} \equiv \wp_n; \quad \dim \Xi^n \equiv r_n$$  \hspace{1cm} (2.19)

and we have the obvious sum rules:

$$h_n + \wp_n + r_n = \binom{d}{n}$$  \hspace{1cm} (2.20)

**Volume preserving algebras** From now on I restrict my attention to Lie algebras $\mathcal{G}$ that are characterized by the additional property:

$$\tau^M_{\ MI} = 0$$  \hspace{1cm} (2.21)

imposed on the structure constants (see \[23\, 24\, 25\]). For such algebras we immediately prove the following lemma. \(\forall X \in \mathcal{G}\):

$$\ell_X \text{ Vol} = 0$$  \hspace{1cm} (2.22)

where

$$\text{Vol} = \frac{1}{d!} \epsilon_{I_1I_2...I_d} e^{I_1} \wedge ... \wedge e^{I_d}$$  \hspace{1cm} (2.23)

is the the volume $d$-form. Then the orthogonal decomposition can be better formulated by introducing a pairing form on the space cochains. This will lead to Poincaré duality. Let me consider two forms of complementary degree $\Omega[p]$ and $\Psi[d-p]$. I define their scalar product by setting:

$$\Omega[p] \wedge \Psi[d-p] \equiv \langle \Omega[p], \Psi[d-p]\rangle \text{ Vol}$$  \hspace{1cm} (2.24)

Next I can prove a very important property of the pairing form I have just introduced which holds true for volume preserving algebras, namely under the additional condition \(2.21\). To this effect note that for such subalgebras for which eq. \(2.21\) is true, any arbitrary $d-1$ form is closed:

$$\forall \Omega[d-1] \in C^{d-1} : \partial \Omega[d-1] = 0$$  \hspace{1cm} (2.25)

The proof is elementary. The components of $\partial \Omega[d-1]$ are $\tau^M_{[I_1I_2} \Omega_{I_3I_4...I_{d-1}]M}$ from which it follows that the index $M$ being different from $I_3I_4...I_{d-1}$ is necessarily equal to either $I_1$ or $I_2$ and
this makes the contribution zero in view of eq. (2.21). Secondly I note that:

\[
0 = \partial (\Omega^{p-1} \wedge \Psi^{[d-p]}) \\
= \partial \Omega^{p-1} \wedge \Psi^{[d-p]} + (-)^{p-1} \Omega^{[p-1]} \wedge \partial \Psi^{[d-p]} \\
= \left( \langle \partial \Omega^{[p-1]} , \Psi^{[d-p]} \rangle + (-)^{p-1} \langle \Omega^{[p-1]} , \partial \Psi^{[d-p]} \rangle \right) \ast \text{Vol}
\]

(2.26)

Hence I conclude that we have the very important formula realizing Poincaré duality:

\[
\langle \partial \Omega^{[p-1]} , \Psi^{[p]} \rangle = (-)^{p-1} \langle \Omega^{[p-1]} , \partial \Psi^{[p]} \rangle
\]

(2.27)

Equation (2.27) can be used to derive recursion relation that determine the values of the numbers \( \wp_n \) and \( r_n \) in terms of the formal Hodge numbers \( h_n \). The first observation is that, thanks to (2.27) the subspace \( \partial \Xi^{[d-n-1]} \), namely \( \text{Im} \partial_{d-n-1} \) is orthogonal to the space of \( n \)-cycles, i.e. to \( \Gamma^{[n]} \oplus \partial \Xi^{[n-1]} \):

\[
\langle \Gamma^{[n]} \oplus \partial \Xi^{[n-1]} , \partial \Xi^{[d-n-1]} \rangle = \langle \partial \left( \Gamma^{[n]} \oplus \partial \Xi^{[n-1]} \right) , \Xi^{[d-n-1]} \rangle = 0
\]

(2.28)

The above equation implies that:

\[
\varphi_{d-n} = r_n = \left( \begin{array}{c} d \\ n \end{array} \right) - h_n - \varphi_n
\]

(2.29)

Applying now the same relation in the opposite direction, namely exchanging \( (d-n) \leftrightarrow n \), we obtain:

\[
\varphi_n = \left( \begin{array}{c} d \\ d-n \end{array} \right) - h_{d-n} - \varphi_{d-n}
\]

(2.30)

and summing together eq. (2.29) with eq. (2.30) we obtain the standard Poincaré duality on cohomology groups:

\[
h_{d-n} = h_n
\]

(2.31)

On the other hand, from their very definition we can obtain a recursion relation on the numbers \( \varphi_n \):

\[
\varphi_n = \left( \begin{array}{c} d \\ n-1 \end{array} \right) - h_{n-1} - \varphi_{n-1}
\]

(2.32)

which can be solved, once combined with eq.s (2.31) and (2.24). In table 1 we exhibit the complete table of the relevant numbers \( h_n, \varphi_n, r_n \) for a 7–dimensional Lie algebra, which will be the case of interest for our subsequent study of M–theory compactification on the so called twisted tori. From table 1 we realize that the following identity also follows, from the above relations, in full generality:

\[
\varphi_n = r_{n-1}
\]

(2.33)

Two other important formal properties of the pairing form are the following ones. Let \( W \in \mathcal{G} \) be a space–time 1-form valued element of the Lie algebra. Considering M–theory compactifications on twisted tori we will see the role of such an object. At the level of the present
Table 1: Decomposition of Chevalley cochain spaces for a 7-dimensional Lie algebra $\mathbb{G}$. As one sees the entire table is parametrized by three independent numbers $h_{3,2,1}$, the dimensions of the cohomology groups $H^{1,2,3}(\mathbb{G})$, respectively.

| $n$ | $h_n$ | $\varphi_n$ | $r_n$ | $h_n + \varphi_n + r_n = \binom{7}{n}$ |
|-----|-------|-------------|-------|--------------------------------------|
| 7   | $h_7 = h_0 = 1$ | 0           | 0     | 1                                    |
| 6   | $h_6 = h_1$    | $7 - h_1$  | 0     | 7                                    |
| 5   | $h_5 = h_2$    | $14 - h_2 + h_1$ | $7 - h_1$ | 21                                  |
| 4   | $h_4 = h_3$    | $21 - h_3 + h_2 - h_1$ | $14 - h_2 + h_1$ | 35                                  |
| 3   | $h_3$          | $14 - h_2 + h_1$ | $21 - h_3 + h_2 - h_1$ | 35                                  |
| 2   | $h_2$          | $7 - h_1$   | $14 - h_2 + h_1$ | 21                                  |
| 1   | $h_1$          | 0           | $7 - h_1$ | 7                                    |
| 0   | $h_0 = 1$      | 0           | 0     | 1                                    |

The mathematical discussion $W$ is just a Lie algebra element that also anticommutes with the Chevalley 1–forms $e^I$. This being the case we have:

$$\langle \Psi^{[d-p+1]} , i_W \Omega^{[p]} \rangle = \langle i_W \Psi^{[d-p+1]} , \Omega^{[p]} \rangle$$  \hspace{1cm} (2.34)

Using the pairing form and relying on its properties, I can now construct a complete orthonormal basis for the space of Chevalley chains. I introduce the following index conventions. In degree $p \leq \frac{d}{2}$ we introduce two type of indices. The unbarred ones

$$\alpha(p) = 1, \ldots, h_p = h_{d-p}$$  \hspace{1cm} (2.35)

which enumerate the cohomology classes and will be attributed to an orthonormal basis of forms for the spaces $\Gamma^{[p]}$:

$$\langle \Gamma_{\alpha(p)}^{[p]} , \Gamma_{\beta(p)}^{[d-p]} \rangle = \delta_{\alpha(p)\beta(p)}$$  \hspace{1cm} (2.36)

and the barred ones:

$$\bar{\alpha}(p) = 1, \ldots, r_p = r_{d-p-1}$$  \hspace{1cm} (2.37)

which enumerate the basis elements for the space $\Xi^{[p]}$, namely the space of those $p$–cochains whose derivative is strictly non zero $\partial \Xi^{[p]} \neq 0$:

$$\langle \Xi_{\bar{\alpha}(p)}^{[p]} , \partial \Xi_{\bar{\beta}(p)}^{[d-p-1]} \rangle = \delta_{\bar{\alpha}(p)\bar{\beta}(p)}$$  \hspace{1cm} (2.38)

Furthermore we always have the following orthogonality relations:

$$\langle \Gamma^{[p]} , \partial \Xi^{[d-p-1]} \rangle = 0$$
For the same reason we also have:

Indeed we have $\Xi = 0$. For the very definition of Lie derivative (2.11), it follows that:

Some additional useful general relations

Table 2: Construction of a well adapted basis of $n$–cochains for a 7–dimensional Lie algebra.

| $\Gamma[1]_a$ ↔ $\Gamma[6]_a$ | $\langle \Gamma[1]_a, \Gamma[6]_b \rangle = \delta_{a,b}$ | $\alpha, \beta = 1, \ldots, h_1$ |
|-----------------------------|---------------------------------|----------------------------------|
| $\Xi[1]_a$ ↔ $\Xi[5]_a$    | $\langle \Xi[1]_a, \partial \Xi[5]_a \rangle = \delta_{a,b}$ | $\bar{\alpha}, \bar{\beta} = 1, \ldots, 7 - h_1$ |
| $\Gamma[2]_a$ ↔ $\Gamma[5]_a$ | $\langle \Gamma[2]_a, \Gamma[5]_b \rangle = \delta_{a,b}$ | $a, b = 1, \ldots, h_2$ |
| $\Xi[2]_a$ ↔ $\Xi[4]_a$    | $\langle \Xi[2]_a, \partial \Xi[4]_a \rangle = \delta_{a,b}$ | $\bar{a}, \bar{b} = 1, \ldots, 14 - h_2 + h_1$ |
| $\Gamma[3]_x$ ↔ $\Gamma[4]_x$ | $\langle \Gamma[3]_x, \Gamma[4]_y \rangle = \delta_{x,y}$ | $x, y = 1, \ldots, h_3$ |
| $\Xi[3]_x$ ↔ $\Xi[3]_x$    | $\langle \Xi[3]_x, \partial \Xi[3]_y \rangle = \delta_{x,y}$ | $\bar{x}, \bar{y} = 1, \ldots, 21 - h_3 + h_2 - h_1$ |

which are true by definition of the spaces $\Xi[p]$ and $\Gamma[p]$. In the case of a 7–dimensional Lie algebra $G$, which is that relevant for twisted tori compactifications, in order to avoid a too clumsy notation I distinguish among the various $\alpha(p)$ and $\bar{\alpha}(p)$ indices by changing alphabet type: since the relevant values of $p$ are just three this is possible. Hence by definition I set:

$$
\alpha(1), \beta(1) \ldots \equiv \alpha, \beta = 1, \ldots, h_1 \\
\bar{\alpha}(1), \bar{\beta}(1) \ldots \equiv \bar{\alpha}, \bar{\beta} = 1, \ldots, 7 - h_1 \\
\alpha(2), \beta(2) \ldots \equiv a, b \ldots = 1, \ldots, h_2 \\
\bar{\alpha}(2), \bar{\beta}(2) \ldots \equiv \bar{a}, \bar{b} \ldots = 1, \ldots, 14 - h_2 + h_1 \\
\alpha(3), \beta(3) \ldots \equiv x, y \ldots = 1, \ldots, h_3 \\
\bar{\alpha}(3), \bar{\beta}(3) \ldots \equiv \bar{x}, \bar{y} \ldots = 1, \ldots, 21 - h_3 + h_2 - h_1
$$

Correspondingly I can construct a complete orthonormal basis of cochains as it is displayed in table 2.

**Some additional useful general relations** There are still some general relations that I can prove and that will be of great help in my subsequent analysis of Free Differential Algebras of M-theory. From the definition of pairing (2.21) we easily conclude that for volume preserving algebras (2.21) and $\Psi[p], \Omega^{[d-p]}$ any two cochains of complementary degree we have:

$$
\langle \ell_W \Psi[p], \Omega^{[d-p]} \rangle = (-)^p \langle \Psi[p], \ell_W \Omega^{[d-p]} \rangle
$$

From the very definition of Lie derivative (2.11), it follows that:

$$
\langle \ell_W \Gamma(p), \Gamma^{(d-p)} \rangle = 0
$$

Indeed we have $\ell_W \Gamma(p) = \partial i_W \Gamma(p)$ since $\partial \Gamma(p) = 0$ and $\langle \partial i_W \Gamma(p), \Gamma^{(d-p)} \rangle = \langle i_W \Gamma(p), \partial \Gamma^{(d-p)} \rangle = 0$. For the same reason we also have:

$$
\langle \ell_W \Gamma(p), \partial \Xi^{(d-p-1)} \rangle = 0
$$

(2.43)
This orthogonality relations implies that:

\[ \ell_W \Gamma^{(p)} \subset \partial \Xi^{(p-1)} \]  \hspace{2cm} (2.44)

Similarly we have:

\[ \ell_W \Xi^{(p)} \subset \Gamma^{(p)} \oplus \Xi^{(p)} \oplus \partial \Xi^{(p-1)} \]
\[ \ell_W \partial \Xi^{(p-1)} \subset \partial \Xi^{(p-1)} \] \hspace{2cm} (2.45)

All together eq.s (2.44) and (2.45) imply that the representation of the algebra \( G \) on the space of \( p \)-cochains is always upper triangular:

\[
\ell_W \left( \begin{array}{ccc}
\Xi^{(p)} \\
\Gamma^{(p)} \\
\partial \Xi^{(p-1)}
\end{array} \right) = \left( \begin{array}{ccc}
\ast & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & \ast
\end{array} \right) \left( \begin{array}{c}
\Xi^{(p)} \\
\Gamma^{(p)} \\
\partial \Xi^{(p-1)}
\end{array} \right)
\]  \hspace{2cm} (2.46)

**Decomposition of \( p \)-cochains along the three subspaces** Once we have have constructed an orthogonal basis, each Chevalley \( p \)-cochain can be decomposed along it and we can define its projections onto the three subspaces \( \Gamma^{(p)} \), \( \partial \Xi^{(p-1)} \) and \( \Xi^{(p-1)} \).

Let \( \Omega^{(p)} \) be a generic \( p \)-cochain. I can set:

\[
\Omega^{(p)} = P_\perp^{(p)} [\Omega] \oplus \partial Q^{(p-1)} [\Omega] \oplus P_{\parallel}^{(p)} [\Omega]
\]  \hspace{2cm} (2.47)

where

\[
P_\perp^{(p)} [\Omega^{(p)}] \in \Gamma^{(p)} \; ; \; \partial Q^{(p-1)} [\Omega^{(p)}] \in \partial \Xi^{(p-1)} \; ; \; P_{\parallel}^{(p)} [\Omega^{(p)}] \in \Xi^{(p)}
\]  \hspace{2cm} (2.48)

The projection onto the three orthogonal subspaces can be performed using the pairing form and the orthogonal basis. Explicitly I can write:

\[
P_\perp^{(p)} [\Omega^{(p)}] = \sum_{\alpha^{(p)}=1}^{h_p} \langle \Omega^{(p)} , \Gamma^{(d-p)} \rangle_{\alpha^{(p)}} \Gamma^{(p)}_{\alpha^{(p)}}
\]

\[
P_{\parallel}^{(p)} [\Omega^{(p)}] = \sum_{\alpha^{(p)}=1}^{r_p} \langle \Omega^{(p)} , \partial \Xi^{(d-p-1)} \rangle_{\alpha^{(p)}} \Xi^{(p)}_{\alpha^{(p)}}
\]

\[
Q_{\parallel}^{(p-1)} [\Omega^{(p)}] = \sum_{\alpha^{(p)-1}=1}^{r_{p-1}} \langle \Omega^{(p)} , \Xi^{(d-p-1)} \rangle_{\alpha^{(p)-1}} \Xi^{(p-1)}_{\alpha^{(p)-1}}
\]  \hspace{2cm} (2.49)

The above formulae will be very useful in my analysis of the FDA emerging from M–theory compactifications on twisted tori
2.2 Calculation of the relevant cohomology groups for SS algebras

As a concrete illustration of the general scheme I consider the example of Scherk Schwarz algebras introduced in [24] and already used as a toy model in [23, 25]. The algebra \( G \) underlying such an example is identified by the following choice of the structure constants:

\[
I = 0, i; \quad i = 1, \ldots, 6
\]

\[
\tau^I_{JK} = \begin{cases} 
\tau^i_{0k} = T^i_j = \text{antisymmetric matrix} \\
0 & \text{otherwise}
\end{cases} \tag{2.50}
\]

This means that the Maurer Cartan equations (2.2) take the following specific form:

\[
\begin{align*}
\partial e^0 &= 0 \\
\partial e^i &= e^0 \wedge T^i_j e^j 
\end{align*} \tag{2.51}
\]

The \( 6 \times 6 \) antisymmetric matrix \( T^i_j \) can always be skewed diagonalized by means of automorphisms of the algebra:

\[
T^i_j = \begin{pmatrix}
m_1 \epsilon & 0_{2\times 2} & 0_{2\times 2} \\
0_{2\times 2} & m_2 \epsilon & 0_{2\times 2} \\
0_{2\times 2} & 0_{2\times 2} & m_3 \epsilon
\end{pmatrix}
\]

\[
\epsilon = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\tag{2.52}
\]

and the Maurer Cartan equations (2.51) turn out to define seven different non isomorphic algebras, depending on the eigenvalue pattern structure. From our previous discussion we learned that there are three relevant Chevalley Cohomology groups \( H^{4,2,1}(G) \). In this section I calculate them for the algebras (2.51).

**Computation of \( H^2(G) \)** I begin with the second cohomology group. I want to compute its dimension and the structure of the tensors which correspond to its elements. The first task is to find the 2–cocycles

\[
\partial \Sigma^{[2]} = 0 \iff \tau^{R}_{[IJ]} \Sigma_{K|R} = 0 \tag{2.53}
\]

Using the specific form (2.50) of the structure constants we reduce condition (2.53) to the form

\[
\tau^{r}_{[IJ]} \Sigma_{K|r} = 0
\]

and we have two possibilities for the free indices, either \( IJK = ijk \) or \( IJK = 0jk \). In the first case, due to (2.50), the equation is automatically satisfied and there is no condition on the tensor \( \Sigma \). For the second choice of free indices I get instead the following condition:

\[
0 = \tau^{r}_{0j} \Sigma_{k|r} = \frac{1}{3} \left( \tau^{r}_{0j} \Sigma_{kr} + \tau^{r}_{jk} \Sigma_{0r} + \tau^{r}_{k0} \Sigma_{jr} \right) = \frac{1}{3} \left[ (T \cdot \sigma)_{jk} + (T \cdot \sigma)_{kj} \right] \tag{2.54}
\]

\[
(2.55)
\]
where I have named $\sigma$ the antisymmetric matrix $\Sigma_{rs}$. Due to antisymmetry of both matrices the last line of equation (2.55) can be read as

$$[T, \sigma] = 0 \quad (2.56)$$

Hence the space of 2–cocycles is spanned by the 6 tensors with arbitrary components $\Sigma_{0r}$ (no-condition on them) plus the solutions of eq. (2.56). Let us study its meaning.

Being antisymmetric both $\sigma$ and $T$ are elements of the SO(6) Lie algebra. Once $T$ is skew diagonalized it belongs to the Cartan subalgebra of SO(6) and I can identify the skew eigenvalues $m_i$ with its components along a basis of Cartan generators:

$$T = \mathbb{i} (m_1 \mathcal{H}^1 + m_2 \mathcal{H}^2 + m_3 \mathcal{H}^3) \in \text{CSA} \subset \text{SO}(6) \quad (2.57)$$

Let me now parametrize the most general form that $\sigma$ can have:

$$\sigma = \mathbb{i} \left( f_i \mathcal{H}^i \right) + x_\alpha X^\alpha + y_\alpha Y^\alpha$$

$$X^\alpha = \left( E^\alpha - E^{-\alpha} \right)$$

$$Y^\alpha = \mathbb{i} \left( E^\alpha + E^{-\alpha} \right) \quad (2.58)$$

where $\alpha \in \Delta_+(D_3)$ are the positive roots of the $D_3$ simple algebra, $E^{\pm\alpha}$ the step-up (step-down) operators associated to each root and $\mathcal{H}_i$ its Cartan generators. All Cartan generators commute with a Cartan element by definition, hence the 3–parameters $f_i$ are free. On the other hand we have:

$$[T, X^\alpha] = \alpha(T) Y^\alpha$$

$$[T, Y^\alpha] = -\alpha(T) X^\alpha \quad (2.59)$$

Hence for each positive root $\alpha$ orthogonal to $T$ there is a pair $(x_\alpha, y_\alpha)$ of additional parameters in $\sigma$, besides the three $f_i$. Let me recall the form of positive roots for the $D_3$ root systems. They are six and precisely $\epsilon_i \pm \epsilon_j$, having denoted by $\epsilon_i$ an orthonormal basis in $\mathbb{R}^3$. Hence I can write:

$$\alpha_1(T) = m_1 - m_2 \quad ; \quad \alpha_2(T) = m_1 + m_2$$

$$\alpha_3(T) = m_1 - m_3 \quad ; \quad \alpha_4(T) = m_1 + m_3$$

$$\alpha_5(T) = m_2 - m_3 \quad ; \quad \alpha_6(T) = m_2 + m_3 \quad (2.60)$$

I can now discuss the number of roots orthogonal to $T$, in relation with the eigenvalue structure. If $m_i$ are all non vanishing and different from each other then no $\alpha_i(T)$ vanishes. Correspondingly the number of parameters in $\sigma$ is just three and the dimension of the 2-cocycle space is $\text{dim}[\ker \partial_2] = 6 + 3 = 9$, where 6 are the tensors $\Sigma_{0r}$, and 3 the skew diagonal $\sigma.s$. On the other hand if the $m_i$ are all non zero but two are equal among themselves, say $m_1 = m_2 \neq m_3$, it is evident that there is just one root orthogonal to $T$. In this case the dimension of the 2-cocycle space grows. We have $\text{dim}[\ker \partial_2] = 6 + 3 + 2 \times 1 = 11$. If all eigenvalues $m_i$ are equal and non zero, $m_1 = m_2 = m_3$, there are three roots orthogonal to $T$ and we have $\text{dim}[\ker \partial_2] = 6 + 3 + 2 \times 3 = 15$. It should be noted that the same conclusion holds true if the eigenvalues are equal in absolute value but opposite in sign. This discussion explains the first
Table 3: Chevalley Cohomology groups of the Lie algebra defined by the Maurer Cartan equations in eq.(2.51). As one sees the cohomology crucially depends on the eigenvalue structure of the $6 \times 6$ matrix $T$.

three entries in the second column of table 3. Before addressing the remaining cases where one or more of the skew eigenvalues $m_i$ vanish, let me discuss the image of the $\partial_1$ map in the cohomology sequence (2.6). This space is spanned by the trivial 2–cocycles of the form:

$$\Sigma^{[2]} \in \text{Im} \partial_1 \Rightarrow \Sigma^{[2]} = \partial U^{[1]}$$

In components this means tensors of the form

$$\tau^{Ji}_{JK} U_I \Rightarrow \tau^{i}_{0k} U_i = T^i_j U_j$$

where $U_i$ is an arbitrary 6-vector. If the matrix $T$ is non degenerate all tensors $\Sigma_{0r}$ can be put in such a form: it suffices to pose $U_i = (T^{-1})^i_j \Sigma_{0j}$. Hence when no eigenvalue $m_i$ vanishes, we have $\text{dim}[\text{Im} \partial_1] = 6$ and this explains also the first three entries in the 4-th column of table 3. Indeed we always have:

$$\text{dim}[H^p] = \text{dim}[\ker \partial_p] - \text{dim}[\text{Im} \partial_{p-1}]$$

On the contrary if some skew eigenvalue vanish the dimension of $\text{dim}[\text{Im} \partial_1]$ decreases since $T$ projects onto a smaller space. In particular if one eigenvalue vanishes, say $m_3 = 0$ we have:

$$\text{dim}[H^p] = \text{dim}[\ker \partial_p] - \text{dim}[\text{Im} \partial_{p-1}]$$

On the contrary if some skew eigenvalue vanish the dimension of $\text{dim}[\text{Im} \partial_1]$ decreases since $T$ projects onto a smaller space. In particular if one eigenvalue vanishes, say $m_3 = 0$ we have:

| $G$ | $\dim \ker \partial_2$ | $\dim \text{Im} \partial_1$ | $\mathbf{h}_2$ | $\dim \ker \partial_1$ | $\mathbf{h}_1$ | $\dim \ker \partial_4$ | $\dim \text{Im} \partial_3$ | $\mathbf{h}_4$ |
|-----|------------------------|--------------------------|-------------|-----------------|-------------|-----------------|----------------|-------------|
| $m_1 \neq m_2$ | 9 = 6 + 3 | 6 | 3 | 1 | 1 | 23 = 20 + 3 | 20 | 3 |
| $\neq m_3$ | | | | | | | | |
| $m_1 = m_2$ | 11 = 6 + 3 | 6 | 5 | 1 | 1 | 25 = 20 + 3 | 20 | 5 |
| $\neq m_3$ | +2 x 1 | | | | | +2 x 1 | | |
| $m_1 = m_2$ | 15 = 6 + 3 | 6 | 9 | 1 | 1 | 29 = 20 + 3 | 20 | 9 |
| = $m_3$ | +2 x 3 | | | | | +2 x 3 | | |
| $m_1 \neq m_2$, | 9 = 6 + 3 | 4 | 5 | 1 + 2 | 3 | 23 = 20 + 3 | 16 | 7 |
| $m_3 = 0$ | | | | | | | | |
| $m_1 = m_2$ | 11 = 6 + 3 | 4 | 7 | 1 + 2 | 3 | 25 = 20 + 3 | 12 | 13 |
| $m_3 = 0$ | +2 x 1 | | | | | +2 x 1 | | |
| $m_1 \neq 0$, | 13 = 6 + 3 | 2 | 11 | 1 + 4 | 5 | 27 = 20 + 3 | 12 | 15 |
| $m_2 = m_3 = 0$ | +2 x 2 | | | | | +2 x 2 | | |
| $m_1 = m_2$ | 21 = 6 + 3 | 0 | 21 | 1 + 6 | 7 | 35 = 20 + 3 | 0 | 35 |
| $m_3 = 0$ | +2 x 6 | | | | | +2 x 6 | | |
\[ \dim[\text{Im}\partial_1] = 4, \] if two vanish \( m_2 = m_3 = 0 \) we get \( \dim[\text{Im}\partial_1] = 2 \) and when all vanish we have \( \dim[\text{Im}\partial_1] = 0. \)

Keeping this information in mind let me return to discuss the space of cocycles in the degenerate case when some eigenvalues vanish. The first case to consider is \( m_1 \neq m_2 \neq 0, m_3 = 0 \). For these eigenvalues no root is orthogonal to \( T \) hence the space of 2–cocycles has dimension \( \dim[\ker\partial_2] = 6 + 3 = 9 \). The decreased dimension of the 2–boundary space explains the increase in the number of cohomology classes, \( h_2 = 5 \) (see table 3). If \( m_3 = 0 \) and \( m_1 = m_2 \) are equal, there is just one root orthogonal to \( T \) and we obtain \( \dim[\ker\partial_2] = 6 + 3 + 2 = 11 \) which leads to \( h_2 = 7 \). When two eigenvalues vanish \( m_2 = m_3 \) there are two roots orthogonal to \( T \) and we get \( \dim[\ker\partial_2] = 6 + 3 + 2 \times 2 = 13 \), leading to \( h_2 = 11 \). Finally when all eigenvalues vanish all six roots are orthogonal to \( T \) and we obtain \( \dim[\ker\partial_2] = 6 + 3 + 2 \times 6 = 21 \) leading to \( h_2 = 21 \). This completes the explanation of the second, third and fourth columns of table 3, namely the computation of \( h_2 \) for all the seven different algebras encoded in eq.(2.51).

**Computation of \( H^1(G) \)** This is fairly simple. A 1-form \( \Sigma^{[1]} \) is just a vector in the adjoint representation \( \Sigma_I \) and the cocycle condition is:

\[
\tau^F_{IJ} \Sigma_F = 0 \quad (2.64)
\]

As we did above we have to distinguish two cases in the choice of the free indices \( IJ \). Either \( IJ = ij \) or \( IJ = 0j \). In the first case we obtain no condition on the vector \( \Sigma_I \). In the second case we obtain:

\[
0 = \tau^f_{ij} \Sigma_i = T^j_i \Sigma_i \quad (2.65)
\]

As long as the matrix \( T \) is non degenerate it does not admit null eigenvectors and hence the space of 1–cocycles is 1–dimensional, it just spanned by the vectors of the form: \( \Sigma_0 \neq 0, \Sigma_i = 0. \) This explains the first three entries in the fifth column of table 3 Then one easily realizes that for each vanishing skew eigenvalue \( m_i = 0 \) we have a pair of null eigenvectors and this explains the remaining entries in the fifth column. The fact that the sixth column coincides with the fifth, namely that each 1-cocycle is also a cohomology class follows from the fact that there are no 1–coboundaries, since in Chevalley cohomology we cannot produce a 1–cochain starting from a 0–chain, that is just from a constant number. This concludes the calculation of \( H^1(G) \)

**Computation of \( H^4(G) \)** We are interested both in \( H^3(G) \) and in \( H^4(G) \), but by Poincaré duality these cohomology groups are isomorphic and it suffices to calculate one of the two. It turns out that \( H^4(G) \) is easier. A 4-cochain \( \Sigma^{[4]} \) is represented by a 4–index tensor \( \Sigma_{IJKL} \) and the cocycle condition is:

\[
0 = \tau^R_{[IJ} \Sigma_{KLM]R} \quad (2.66)
\]

Once again we have to distinguish two cases in the choice of the free indices \( IJKLM \), either \( ijk\ell m \) or \( 0jk\ell m \). In the first case the cocycle condition is identically satisfied and we get no restriction on the components of \( \Sigma^{[4]} \). In the second case the equation splits again in two sectors:

\[
\tau^r_{[0i} \Sigma_{k\ell m]}r = 0 \quad \Rightarrow \quad \Sigma_{0k\ell m} = \text{free}
\]

\[
\tau^r_{0[i} \Sigma_{k\ell m]}r = 0 \quad \Rightarrow \quad T^r_{[i} \Sigma_{k\ell m]}r = 0
\]

(2.67)
Hence we have to analyze the equation in the right-bottom corner of eq. (2.67). $\Sigma_{k\ell mr}$ is a rank 4 antisymmetric tensor in $d = 6$. Hence we can write it as:

$$
\Sigma_{k\ell mr} = \epsilon_{k\ell mruv} \sigma^{uv}
$$

(2.68)

where $\sigma^{uv}$ is a rank 2 antisymmetric tensor, namely a matrix. In this way we can translate eq. (2.67) as follows:

$$
0 = \epsilon^{abijklm} T_{i}^{r} \Sigma_{k\ell mr} = \epsilon^{abijklm} \epsilon_{k\ell mruv} T_{i}^{r} \sigma^{uv}
$$

$$
\Sigma_{k\ell mr} = \delta^{abi} T_{i}^{r} \sigma^{uv}
$$

(2.69)

Since $T_{i}^{r}$ is antisymmetric, there is no contribution from $i = r$, what remains is

$$
0 = T_{v}^{a} \sigma^{hv} - T_{v}^{b} \sigma^{av}
$$

(2.70)

This is the same equation (see (2.56)) we have already solved in the calculation of $H^{2}(G)$. Hence in the various cases the dimensions of $\ker \partial_{4}$ are

$$
\dim \ker \partial_{4} = 20 + \# \text{ of solutions of } [T, \sigma] = 0
$$

(2.71)

This explains all the entries in the seventh column of table [3]. It remains to calculate the dimension of $\text{Im} \partial_{3}$. To this effect we argue in the following way. The space $\partial \Xi^{[3]}$ is spanned by tensors of the form:

$$
\Sigma_{0ijk} = T_{[i}^{r} U_{jk]r}
$$

(2.72)

where $U_{jkr}$ is an arbitrary rank 3 antisymmetric tensor in $d = 6$. Hence the operation described by equation (2.12) defines a linear map from the space of rank 3 antisymmetric tensors into itself:

$$
\mathcal{T} : \bigwedge^{3} \mathbb{R} \rightarrow \bigwedge^{3} \mathbb{R}
$$

(2.73)

It suffices to derive the $20 \times 20$ matrix $\mathcal{T}$ and calculate its rank, namely the number of non-vanishing eigenvalues, depending on the eigenvalue structure of $T$. If we choose a lexicographic order for the independent components of a rank 3 antisymmetric tensor, i.e. $U_{123}, U_{124}, \ldots, U_{456}$, the explicit form of the matrix $\mathcal{T}$ is given below:

$$
\mathcal{T} = 
\begin{pmatrix}
0 & m_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-m_{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -m_{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -m_{3} & 0 & 0 & 0 & 0 & 0 & 0 & m_{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(2.74)
and calculating its determinant we find:

$$\text{Det } \mathcal{T} = m_1^4 m_2^4 m_3^4 \left( m_1^4 + (m_2^2 - m_3^2)^2 - 2 m_1^2 (m_2^2 + m_3^2) \right)^2 \quad (2.75)$$

Furthermore the 10 independent skew eigenvalues of $\mathcal{T}$ (which is antisymmetric) are:

$$\begin{align*}
\lambda_1 &= m_1 \\
\lambda_2 &= -m_1 \\
\lambda_3 &= m_2 \\
\lambda_4 &= -m_2 \\
\lambda_5 &= m_3 \\
\lambda_6 &= -m_3 \\
\lambda_7 &= m_1 + m_2 + m_3 \\
\lambda_8 &= -m_1 + m_2 + m_3 \\
\lambda_9 &= m_1 - m_2 + m_3 \\
\lambda_{10} &= m_1 + m_2 - m_3
\end{align*} \quad (2.76)$$

Inspection of eq. (2.76) suffices to explain the entries in the eighth column of table 3. Indeed we can write:

$$\dim \text{Im } \partial_3 = 20 - 2 \times (\# \text{ of null skew eigenvalues of } \mathcal{T}) \quad (2.77)$$

This concludes my discussion of the cohomology groups.

### 2.3 The orthogonal basis of $p$ cochains in one example

As a further illustration of the general set up I choose the case of the SS algebra with generic non vanishing eigenvalues $m_1 \neq m_2 \neq m_3 \neq 0$ and for such an algebra I perform the explicit construction of the orthogonal basis introduced in table 2. In this case table 3 takes the following explicit form:

| $n$ | $h_n$ | $\varphi_n$ | $r_n$ |
|-----|-------|------------|------|
|  7  |  1    |  0        |  0   |
|  6  |  1    |  6        |  0   |
|  5  |  3    | 12        |  6   |
|  4  |  3    | 20        | 12   |
|  3  |  3    | 12        | 20   |
|  2  |  3    |  6        | 12   |
|  1  |  1    |  0        |  6   |
|  0  |  1    |  0        |  0   |

Let me construct one by one all the elements of the basis. I start from $\Gamma^7$. This is just the volume form:

$$\Gamma^7 = \text{Vol} = \frac{1}{7!} \varepsilon_{I_1 I_2 \ldots I_7} e^{I_1} \wedge e^{I_2} \wedge \ldots \wedge e^{I_7} \quad (2.79)$$
Its dual is the number 1:

\[ \Gamma^{[0]} = 1 \]  

which indeed fulfills the relation:

\[ \Gamma^{[7]} \wedge \Gamma^{[0]} = \text{Vol} \]  

Next I turn to the pair \( \Gamma^{[6]} \) and \( \Gamma^{[1]} \). Here we have \( h_1 = h_6 = 1 \) and it suffices to pose:

\[ \Gamma^{[1]} = e^0 ; \quad \Gamma^{[6]} = \frac{1}{6!} \epsilon_{i_1 \ldots i_6} e^{i_1} \wedge \ldots \wedge e^{i_6} \Rightarrow \Gamma^{[1]} \wedge \Gamma^{[6]} = \text{Vol} \]  

To complete the analysis in dimensions 1 and 6, I have to construct the pairs \( \Xi^{[1]}_{\alpha} \) and \( \Xi^{[5]}_{\beta} \), where \( \alpha, \beta = 1, \ldots, 6 \). To this effect let me consider the symmetric, non degenerate \( 6 \times 6 \) matrix:

\[ \mathcal{M} = 5! T^2 \]  

and let \( \overrightarrow{w}_\alpha \) be an orthonormal system of 6 vectors with respect to the scalar product defined by \( \mathcal{M} \):

\[ \overrightarrow{w}_\alpha \cdot \overrightarrow{w}_\beta \equiv w_{\alpha p} \mathcal{M}^p_r w^r_\beta = \delta_{\alpha \beta} \]  

In terms of these vectors I set:

\[ \Xi^{[1]}_{\alpha} = w_{\alpha i} T^r e^r \]
\[ \Xi^{[5]}_{\beta} = w^i_\beta \epsilon_{i j_1 \ldots j_5} e^{j_1} \wedge \ldots \wedge e^{j_5} \]  

and I immediately obtain:

\[ \partial \Xi^{[5]}_{\beta} = 5 e^0 \wedge w^i_\beta \epsilon_{i j_1 \ldots j_5} e^{j_1} \wedge \ldots \wedge e^{j_5} \]  

Then by means of straightforward algebraic manipulations, you can verify that:

\[ \Xi^{[1]}_{\alpha} \wedge \partial \Xi^{[5]}_{\beta} = \overrightarrow{w}_\alpha \cdot \overrightarrow{w}_\beta \text{Vol} \]  

as requested.

The next step is the construction of the pair \( \Gamma^{[2]}_a \), \( \Gamma^{[5]}_b \). To this effect let \( \sigma_a \) be a basis of three \( 6 \times 6 \) matrices in the CSA of \( \text{SO}(6) \), commuting, by definition, with \( T \) and normalized in the following way:

\[ \text{Tr} (\sigma_a \sigma_b) = - \frac{1}{4! 2!} \delta_{ab} \]  

Then I pose:

\[ \Gamma^{[2]}_a = \sigma_{a j_1 j_2} e^{j_1} \wedge e^{j_2} \]
\[ \Gamma^{[5]}_b = e^0 \wedge \epsilon_{i_1 \ldots i_4 a b} \sigma^u v e^{i_1} \wedge \ldots \wedge e^{i_4} \]  

and we have both \( \partial \Gamma^{[2]}_a = 0 \) as we already know, and \( \partial \Gamma^{[5]}_b = 0 \) for the simple reason that there is already one \( e^0 \) and the boundary operator introduces a second. Furthermore one easily verifies that:

\[ \Gamma^{[2]}_a \wedge \Gamma^{[5]}_b = - 4! 2! \times \text{Tr} (\sigma_a \sigma_b) \text{Vol} \]
Let me now turn my attention to the construction of the pair \( \Xi^{[2]}_{\bar{a}} \) and \( \Xi^{[4]}_{\bar{b}} \). For this purpose let me recall that the remaining 12 generators of SO(6), which are not in the CSA can be organized into a set of matrices \( E_{\bar{a}} \) fulfilling the relations:

\[
[T, E_{\bar{a}}] = K_{\bar{a}\bar{b}} E_{\bar{b}}
\]

\[
\text{Tr} (E_{\bar{a}} E_{\bar{b}}) = \delta_{\bar{a}\bar{b}}
\]

where \( K_{\bar{a}\bar{b}} \) is an antisymmetric 12 \( \times \) 12 matrix. Indeed \( E_{\bar{a}} \) are constructed with the step operators associated with roots of the SO(6) Lie algebra and \( T \) is an element of the CSA. Then I can just pose:

\[
\Xi^{[2]}_{\bar{a}} = E_{\bar{a}} |_{ij} e^{i} \wedge e^{j}
\]

\[
\Xi^{[4]}_{\bar{b}} = \frac{1}{4!2!} (K^{-1})_{b\bar{c}} e_{j1...j4uv} E^{uv}_{\bar{c}} e^{j1} \wedge ... \wedge e^{j4}
\]

(2.92)

Applying the boundary operator I find:

\[
\partial \Xi^{[4]}_{\bar{b}} = \frac{1}{3!2!} e^{0} \wedge T^{j1} e_{j1...j4uv} E^{uv}_{\bar{c}} e^{j2} \wedge ... \wedge e^{j4}
\]

(2.93)

and by direct computation I obtain:

\[
\Xi^{[2]}_{\bar{a}} \wedge \partial \Xi^{[4]}_{\bar{b}} = (K^{-1})_{b\bar{c}} \text{Tr} ([E_{\bar{a}}, T] E_{\bar{c}}) \times \text{Vol} = \delta_{\bar{a}\bar{b}} \times \text{Vol}
\]

(2.94)

as requested.

It remains, to be studied, the pair of sectors in grade 3 and 4. The basis for the space \( \Xi^{[4]} \) has already been constructed in eq. (2.92). As for the basis of the dual spaces \( \Gamma^{[3]} \) and \( \Gamma^{[4]} \) it suffices to pose:

\[
\Gamma^{[4]}_{x} = \epsilon_{i1...i4uv} \sigma_{x}^{uv} e^{i1} \wedge ... \wedge e^{i4}
\]

\[
\Gamma^{[3]}_{y} = e^{0} \wedge \sigma_{y[uv} e^{u} \wedge e^{v}
\]

(2.95)

where \( \sigma_{x} \) are the same 6 \( \times \) 6 matrices in the CSA of SO(6) introduced in equation (2.88). You can immediately verify that in this way the forms are already correctly normalized:

\[
\Gamma^{[3]}_{y} \wedge \Gamma^{[4]}_{x} = \delta_{xy} \times \text{Vol}
\]

(2.96)

Finally I have to define the basis for the space \( \Xi^{[8]} \) its conjugate being in this case \( \partial \Xi^{[3]} \). To this effect let me recall the 20 \( \times \) 20 matrix \( T \) introduced in eq. (2.74). Formally it was defined in the following way. Let \( U_{i1i2i3|\bar{x}} \) be the lexicographic basis of rank three antisymmetric tensors with \( \bar{x} = 1, \ldots, 20 \) normalized as follows:

\[
U_{i1i2i3|\bar{x}} U_{j1j2j3|\bar{y}} \epsilon_{i1i2i3j1j2j3} = \Omega_{\bar{x}\bar{y}}
\]

(2.97)

where \( \Omega_{\bar{x}\bar{y}} \) is some 20 \( \times \) 20 non degenerate antisymmetric matrix. Then I have defined:

\[
T_{[i}^{r} U_{jk]r|\bar{x}} = \mathcal{T}_{\bar{x}\bar{y}} U_{ijk|\bar{y}}
\]

(2.98)
and I have obtained the explicit result (2.74). Let me now set:

\[ \Xi^{[3]} \bar{x} = U_{i_1 i_2 i_3 \bar{x}} e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \]  

(2.99)

A straightforward calculation immediately shows that:

\[ \partial \Xi^{[3]} \bar{x} \wedge \Xi^{[3]} \bar{y} = \{ \Omega, \mathcal{T} \} \bar{x} \bar{y} \times \text{Vol} \]  

(2.100)

The matrix \( \{ \Omega, \mathcal{T} \} \) is symmetric, being the anticommutator of two antisymmetric matrices and non degenerate. By a suitable change of basis we can always reduce it to a be a delta.

This concludes my discussion and illustration of Chevalley cohomology with a particular attention to the aspects relevant for M-theory compactification on twisted tori.

I next turn to recall the very concept and the structural theorems relative to free differential algebras.

### 3 General Structure of FDA.s and Sullivan’s theorems

As I have already recalled in the introduction Free Differential Algebras (FDA) are a natural categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (p–forms) of rank greater than one.

FDA.s were independently discovered in Mathematics by Sullivan [20] and in Physics by the present author in collaboration with R. D’Auria [18]. The original name given to this algebraic structure by D’Auria and me was that of Cartan Integrable Systems. Later, recognizing the conceptual identity of our supersymmetric construction with the pure bosonic constructions considered by Sullivan, we also turned to its naming FDA which has by now become generally accepted. In this section my purpose is that of recalling the definition of FDA.s and two structural theorems by Sullivan which show how all possible FDA.s are, in a sense to be described, cohomological extensions of normal Lie algebras or superalgebras.

Another question which is of utmost relevance in all physical applications is that of gauging of FDA.s. Just in the same way as physics gauges standard Lie algebras by means of Yang Mills theory through the notion of gauge connections and curvatures one expects to gauge FDA.s by introducing their curvatures. A surprising feature of the FDA setup which was noticed and explained by me in a paper of 1985 [17] is that differently from Lie algebras the algebraic structure of FDA already encompasses both the notion of connection and the notion of curvature and there is a well defined mathematical way of separating the two which relies on the two structural theorems by Sullivan. In this section I ultra shorty summarize this essential mathematical lore as a preparation for my analysis of twisted tori compactifications. It is indeed my goal to show how the recent findings in this realm fit and are properly interpreted within the cohomological set-up of Chevalley cohomology and the cohomological classification scheme of FDA.s Adopting this language and its appropriate technical tools will also allow me to correct some imprecise statements appeared in the literature and provide a general scheme for a systematic analysis of all cases of interest.
Definition of FDA  The starting point for FDA.s is the generalization of Maurer Cartan equations. As we already emphasized in section 2 a standard Lie algebra is defined by its structure constants which can be alternatively introduced either through the commutators of the generators as in eq. (2.1) or through the Maurer Cartan equations obeyed by the dual 1–forms as in eq. (2.2). The relation between the two descriptions is provided by the duality relation in eq. (2.3). Adopting the Maurer Cartan viewpoint FDA.s can now be defined as follows. Consider a formal set of exterior forms \( \{ \theta^A(p) \} \) labelled by the index \( A \) and by the degree \( p \) which may be different for different values of \( A \). Given this set we can write a set of generalized Maurer Cartan equations of the following type:

\[
d\theta^A(p) + \sum_{n=1}^{N} C^A(p)_{B_1(p_1)...B_n(p_n)} \theta^{B_1(p_1)} \wedge \ldots \wedge \theta^{B_n(p_n)} = 0
\]  

(3.1)

where \( C^A(p)_{B_1(p_1)...B_n(p_n)} \) are generalized structure constants with the same symmetry as induced by permuting the \( \theta \) s in the wedge product. They can be non–zero only if:

\[
p + 1 = \sum_{i=1}^{n} p_i
\]

(3.2)

Equations (3.1) are self-consistent and define an FDA if and only if \( dd\theta^A(p) = 0 \) upon substitution of (3.1) into its own derivative. This procedure yields the generalized Jacobi identities of FDA.s.2

Classification of FDA and the analogue of Levi theorem: minimal versus contractible algebras  A basic theorem of Lie algebra theory states that the most general Lie algebra \( \mathcal{A} \) is the semidirect product of a semisimple Lie algebra \( \mathcal{L} \) called the Levi subalgebra with \( \text{Rad}(\mathcal{A}) \), namely with the radical of \( \mathcal{A} \). By definition this latter is the maximal solvable ideal of \( \mathcal{A} \). Sullivan [20] has provided an analogous structural theorem for FDA.s. To this effect one needs the notions of minimal FDA and contractible FDA. A minimal FDA is one for which:

\[
C^A(p)_{B(p+1)} = 0
\]

(3.3)

This excludes the case where a \((p+1)\)–form appears in the generalized Maurer Cartan equations as a contribution to the derivative of a \( p \)–form. In a minimal algebra all non differential terms are products of at least two elements of the algebra, so that all forms appearing in the expansion of \( d\theta^A(p) \) have at most degree \( p \), the degree \( p + 1 \) being ruled out.

On the other hand a contractible FDA is one where the only form appearing in the expansion of \( d\theta^A(p) \) has degree \( p + 1 \), namely:

\[
d\theta^A(p) = \theta^{A(p+1)} \Rightarrow d\theta^{A(p+1)} = 0
\]

(3.4)

A contractible algebra has a trivial structure. The basis \( \{ \theta^A(p) \} \) can be subdivided in two subsets \( \{ \Lambda^A(p) \} \) and \( \{ \Omega^B(p+1) \} \) where \( A \) spans a subset of the values taken by \( B \), so that:

\[
d\Omega^B(p+1) = 0
\]

(3.5)
for all values of $B$ and

$$d\Lambda^A(p) = \Omega^A(p+1) \quad (3.6)$$

Denoting by $\mathcal{M}^k$ the vector space generated by all forms of degree $p \leq k$ and $C^k$ the vector space of forms of degree $k$, a minimal algebra is shortly defined by the property:

$$d\mathcal{M}^k \subset \mathcal{M}^k \wedge \mathcal{M}^k \quad (3.7)$$

while a contractible algebra is defined by the property

$$dC^k \subset C^{k+1} \quad (3.8)$$

In analogy to Levi’s theorem, the first theorem by Sullivan states that: The most general FDA is the semidirect sum of a contractible algebra with a minimal algebra.

**Sullivan’s first theorem and the gauging of FDA.s** Twenty years ago in [17] I observed that the above mathematical theorem has a deep physical meaning relative to the gauging of algebras. Indeed I proposed the following identifications:

1. The contractible generators $\Omega^A(p+1) + \ldots$ of any given FDA $A$ are to be physically identified with the curvatures.

2. The Maurer Cartan equations that begin with $d\Omega^A(p+1)$ are the Bianchi identities.

3. The algebra which is gauged is the minimal subalgebra $\mathcal{M} \subset A$.

4. The Maurer Cartan equations of the minimal subalgebra $\mathcal{M}$ are consistently obtained by those of $A$ by setting all contractible generators to zero.

**Sullivan’s second structural theorem and Chevalley cohomology** The second structural theorem proved by Sullivan deals with the structure of minimal algebras and it is constructive. Indeed it states that the most general minimal FDA $\mathcal{M}$ necessarily contains an ordinary Lie subalgebra $G \subset \mathcal{M}$ whose associated 1-form generators we can call $e^I$, as in equation (2.2). Additional $p$-form generators $A[p]$ of $\mathcal{M}$ are necessarily, according to Sullivan’s theorem, in one-to-one correspondence with Chevalley $p + 1$ cohomology classes $\Gamma^{p+1}(e)$ of $G \subset \mathcal{M}$. Indeed, given such a class, which is a polynomial in the $e^I$ generators, we can consistently write the new higher degree Maurer Cartan equation:

$$\partial A[p] + \Gamma^{p+1}(e) = 0 \quad (3.9)$$

where $A[p]$ is a new object that cannot be written as a polynomial in the old objects $e^I$.

Considering now the FDA generated by the inclusion of the available $A[p]$, one can inspect its Chevalley cohomology: the cochains are the polynomials in the extended set of forms $\{ A, e^I \}$ and the boundary operator is defined by the enlarged set of Maurer Cartan equations. If there are new cohomology classes $\Gamma^{p+1}(e, A)$, then one can further extend the FDA by including new $p$-generators $B[p]$ obeying the Maurer Cartan equation:

$$\partial B[p] + \Gamma^{p+1}(e, A) = 0 \quad (3.10)$$

$^3$For detailed explanations on this see again, apart from the original article [20] the book [21].
The iterative procedure can now be continued by inspecting the cohomology classes of type \( \Gamma^{p+1} \) \((e, A, B)\) which lead to new generators \( C^p \) and so on. Sullivan’s theorem states that those constructed in this way are, up to isomorphisms, the most general minimal FDA.s.

To be precise, this is not the whole story. There is actually one generalization that should be taken into account. Instead of absolute Chevalley cohomology one can rather consider relative Chevalley cohomology. This means that rather then being \( G \)-singlets, the Chevalley \( p \)-cochains can be assigned to some linear representation of the Lie algebra \( G \). In this case eq. (2.4) is replaced by:

\[
\Omega^\alpha[p] = \Omega_{I_1...I_p}^\alpha e^{I_1} \wedge \ldots \wedge e^{I_p}
\]

where the index \( \alpha \) runs in some representation \( D \):

\[
D : \; T_I \rightarrow [D (T_I)]^\alpha_\beta
\]

and the boundary operator is now the covariant \( \nabla \):

\[
\nabla \Omega^\alpha[p] = \partial \Omega^\alpha[p] + e^I \wedge [D (T_I)]^\alpha_\beta \Omega^\beta[p]
\]

Since \( \nabla^2 = 0 \), we can repeat all previously explained steps and compute cohomology groups. Each non trivial cohomology class \( \Gamma^{p+1}(e) \) leads to new \( p \)-form generators \( A^\alpha[p] \) which are assigned to the same \( G \)-representation as \( \Gamma^{p+1}(e) \). All successive steps go through in the same way as before and Sullivan’s theorem actually states that all minimal FDA.s are obtained in this way for suitable choices of the representation \( D \), in particular the singlet.

### 3.1 Non trivial FDA extensions of the SS algebras and anticipations on twisted tori compactifications of M–theory

In sections 2.2 and 2.3 I presented a detailed study of the cohomology groups for the Scherk–Schwarz algebras (2.51), which are of interest in M–theory compactifications on twisted tori. In order to illustrate the bearing of Sullivan’s structural theorems, let me describe the minimal FDA.s that could be constructed starting from such algebras \( G \). For instance, given the cohomology groups, one could extend the original algebra with \( h_2(G) \) new 1–form generators \( A^a_1 \), by writing:

\[
dA^a_1 + \Gamma^a_2 = 0 \; \; ; \; \; a = 1, \ldots, h_2(G)
\]

Similarly one could extend the FDA with \( h_3(G) \) two-forms \( A^x_2 \) by writing:

\[
dA^x_2 + \Gamma^x_3 = 0 \; \; ; \; \; x = 1, \ldots, h_3(G)
\]

In components the above non–trivial minimal algebra would lead to the following curvature definitions, upon extension with the contractible generators according to first Sullivan’s theorem:

\[
G^I \equiv dW^I + \frac{1}{2} \tau^I_{JK} W^J \wedge e^K
\]

\[
F^a_2 \equiv dA^a_1 + \Gamma^a_2 W^I \wedge W^J
\]

\[
F^x_3 \equiv dA^x_2 + \Gamma^x_3 W^I \wedge W^J \wedge W^K
\]
Note that in the above algebra the index carried by the new $p$–form generators is lower just as in the cohomology class that generates it. I stress this because it turns out that in the FDA's algebras generated by twisted tori compactifications this will not be the case and those FDA's will in general be different from the above. Sullivan's structural theorem creates an association between cohomology classes and new forms $p$ generators of the FDA which goes as follows:

$$\forall \Gamma^{(p+1)}_x \in H^{(p+1)}(G) \Rightarrow \exists \text{ new generator } A^{[p]}_x \text{ of degree } p$$  \hspace{1cm} (3.17)

Also in M–theory compactifications there is an association between cohomology classes and new generators of the FDA but, as we will see, it rather as follows:

$$\forall \Gamma^{(p)}_x \in H^{(p)}(G) \Rightarrow \exists \text{ new generator } \Sigma^{[3-p]}_x \text{ of degree } 3 - p$$  \hspace{1cm} (3.18)

Eq. (3.18) takes origin by the fact that $\Sigma^{[3-p]}_x \wedge \Gamma^{(p)}_x$ will a contribution to the development of the three–form potential of M–theory. For this reason the index carried by the new generator is not in the same lower position as in the cohomology class rather it is in the upper position, i.e. transforms contravariantly with respect to any symmetry that acts on Chevalley cocycles.

On the other hand Sullivan’s structural theorem holds true for all FDA's, also those emerging from twisted tori compactifications. Hence the new generators $\Sigma^{[3-p]}_x$ associated with $p$–cohomology classes can be non trivial, namely not contractible ones, if and only if they are suitable paired with $(4 - p)$ Chevalley cocycles $\Gamma^{(4-p)}_x$ which, from M–theory reduction, should automatically appear in the appropriate position as second member of the equation. In other words, from such reductions we obtain a non trivial FDA, if and only if:

$$\Gamma^{(p)}_x \in H^{(p)}(G) \Rightarrow \Sigma^{[3-p]}_x \wedge \Gamma^{(4-p)}_x \in H^{(p)}(G)$$  \hspace{1cm} (3.19)

It will turn out that, generically, the situation (3.19) is not realized. There are new generators associated with cohomology classes but they are in general contractible and the minimal FDA is nothing else but the original algebra $G$. I postpone an enlarged discussion of this point to the end of section 5.4 after deriving the details of M–theory reduction.

4 The super FDA of M theory and its cohomological structure

Sullivan's theorems have been introduced and proved for Lie algebras and their corresponding FDA extensions but they hold true, with obvious modifications, also for superalgebras $G_s$ and for their FDA extensions. Actually, in view of superstring and supergravity, it is precisely in the supersymmetric context that FDA's have found their most relevant applications. In this section, as an illustration of the general set up and also as an introduction to my specific interest, which is M–theory compactifications, I present the structure of the M–theory FDA, by recalling the results of [18] and [17]. Within this context I will also be able to illustrate the bearing of a quite relevant question: is an FDA $\mathbb{M}$ always equivalent to a normal Lie algebra $\hat{G} \supset G$ larger than the Lie algebra of which $\mathbb{M}$ is a cohomological extension? How to
mathematically formulate and answer such a question I will show below by recalling results of [18] and also more recent literature [26, 27, 28].

Let me begin by writing the complete set of curvatures, plus their Bianchi identities. This will define the complete FDA:

\[ A = M \sqcup C \]  

(4.1)

The curvatures being the contractible generators \( C \). By setting them to zero we retrieve, according to Sullivan’s first theorem, the minimal algebra \( M \). This latter, according instead to Sullivan’s second theorem, has to be explained in terms of cohomology of the normal subalgebra \( G \subset M \), spanned by the 1–forms. In this case \( G \) is just the \( D = 11 \) superalgebra spanned by the following 1–forms:

1. the vielbein \( V^a \)
2. the spin connection \( \omega^{ab} \)
3. the gravitino \( \psi \)

The higher degree generators of the minimal FDA \( M \) are:

1. the bosonic 3–form \( A^{[3]} \)
2. the bosonic 6-form \( A^{[6]} \).

The complete set of curvatures is given below ([18] [17]):

\[
\begin{align*}
T^a &= \mathcal{D}V^a - \frac{i\psi}{2} \wedge \Gamma^a \psi \\
R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \\
\rho &= \mathcal{D}\psi \equiv d\psi - \frac{i}{4} \omega^{ab} \wedge \Gamma_{ab} \psi \\
F^{[4]} &= dA^{[3]} - \frac{i}{2} \psi \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b \\
F^{[7]} &= dA^{[6]} - \frac{15}{8} F^{[4]} \wedge A^{[3]} - \frac{15}{2} \psi V^a \wedge \bar{\psi} V^b \wedge \Gamma_{ab} \psi \wedge A^{[3]} \\
&\quad - i \frac{1}{2} \psi \wedge \Gamma_{a_1 \ldots a_5} \psi \wedge V^{a_1} \wedge \ldots \wedge V^{a_5} \\
R^{ab} &= R^{ab}_{\phantom{ab}cd} V^c \wedge V^d + i \rho_{mn} \left( \frac{1}{2} \Gamma^{abmn} \psi \wedge V^a + \frac{i}{2} \Gamma^{a_1 \ldots a_4 m} \psi \wedge V^m \right) F^{a_1 \ldots a_4} \\
&\quad + \psi \wedge \Gamma^{mn} \psi F^{mnab} + \frac{1}{24} \psi \wedge \Gamma^{abc_1 \ldots c_4} \psi F^{c_1 \ldots c_4} \\
\end{align*}
\]  

(4.2)

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities, which for brevity I do not explicitly write (see [17]). The dynamical theory is defined, according to a general constructive scheme of supersymmetric theories, by the principle of rheonomy (see [21] ) implemented into Bianchi identities. Indeed there is a unique rheonomic parametrization of the curvatures \( (4.2) \) which solves the Bianchi identities and it is the following one:

\[
\begin{align*}
T^a &= 0 \\
F^{[4]} &= F^{a_1 \ldots a_4} V^{a_1} \wedge \ldots \wedge V^{a_4} \\
F^{[7]} &= \frac{1}{84} F^{a_1 \ldots a_4} V^{b_1} \wedge \ldots \wedge V^{b_7} \epsilon_{a_1 \ldots a_4 b_1 \ldots b_7} \\
\rho &= \rho_{a_1 a_2} V^{a_1} \wedge V^{a_2} - i \frac{1}{2} \left( \Gamma^{a_1 a_2 a_3} \psi \wedge V^{a_4} + \frac{i}{2} \Gamma^{a_1 \ldots a_4 m} \psi \wedge V^m \right) F^{a_1 \ldots a_4} \\
R^{ab} &= R^{ab}_{\phantom{ab}cd} V^c \wedge V^d + i \rho_{mn} \left( \frac{1}{2} \Gamma^{abmn} \psi \wedge V^a + \frac{i}{2} \Gamma^{a_1 \ldots a_4 m} \psi \wedge V^m \right) F^{a_1 \ldots a_4} \\
&\quad + \psi \wedge \Gamma^{mn} \psi F^{mnab} + \frac{1}{24} \psi \wedge \Gamma^{abc_1 \ldots c_4} \psi F^{c_1 \ldots c_4} \\
\end{align*}
\]  

(4.3)
The expressions \([1.3]\) satisfy the Bianchi's provided the space–time components of the curvatures satisfy the following constraints

\[
0 = D_m F^{mc_1c_2c_3} + \frac{1}{96} \epsilon^{c_1c_2c_3a_1a_2} F_{a_1...a_4} F_{a_5...a_8}
\]

\[
0 = \Gamma^{abc} \rho_{bc}
\]

\[
R_{a m}^{c n} = 6 F^{ac_1c_2c_3} F^{bc_1c_2c_3} - \frac{1}{2} \delta^a_b F^{c_1...c_4} F^{c_1...c_4}
\]  

which are the space–time field equations.

### 4.1 The minimal FDA and cohomology

Setting \(T^a = R^{ab} = \rho = F_4 = F_7 = 0\) in eqs. \([4.2]\) we obtain the Maurer Cartan equations of the minimal algebra \(\mathcal{M}\). In particular we have:

\[
dA^3 = \Gamma^4 (V, \psi) \equiv \frac{1}{2} \bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b
\]

\[
dA^6 = \Gamma^7 (V, \psi, A^3) \equiv \frac{15}{2} V^a \wedge V^b \wedge \bar{\psi} \wedge \Gamma_{ab} \psi \wedge A^3 + \frac{1}{2} \bar{\psi} \wedge \Gamma_{a_1...a_5} \psi \wedge V^{a_1} \wedge ... \wedge V^{a_5}
\]

The reason why the three–form generator \(A^3\) does exist and also why the six–form generator \(A^6\) can be included is, in this set up, a direct consequence of the cohomology of the super Poincaré algebra in \(D = 11\), via Sullivan’s second theorem. Indeed the 4–form \(\Gamma^4 (V, \psi)\) defined in the first line of eq. \([4.5]\) is a cohomology class of the super Poincaré Lie algebra whose Maurer Cartan equations are the first three of eqs. \([4.2]\) upon setting \(T^a = R^{ab} = \rho = 0\). We have:

\[
d\Gamma^4 (V, \psi) = 0
\]

and there is no \(\Phi^3 (V, \psi)\) such that \(\Gamma^4 (V, \psi) = d\Phi^3 (V, \psi)\). In this way we see that \(M2\)–branes and \(M5\)–branes, namely the dynamical objects that make up M-theory and which respectively couple to the forms \(A^3\) and \(A^6\) are just an yield of Chevalley cohomology of the super Poincaré algebra.

### 4.2 FDA equivalence with larger (super) Lie algebras

I can now address the question I posed at the beginning of this section. Are FDA.s eventually equivalent to normal (super) Lie algebras? For minimal algebras the question can be nicely rephrased in the following way: can a non trivial cohomology class of a Lie algebra \(\mathbb{G}\) be trivialized by immersing \(\mathbb{G}\) into a larger algebra \(\hat{\mathbb{G}}\)? Indeed by adding new 1–form generators \(\phi^p\) which, together with the generators \(e^I\) of \(\mathbb{G}\) satisfy the Maurer Cartan equations of the larger algebra \(\hat{\mathbb{G}} \supset \mathbb{G}\), it may happen that we are able to construct a polynomial \(\Phi^{p-1} (e, \phi)\) such that:

\[
d\Phi^{p-1} (e, \phi) = \Gamma^{p} (e)
\]

In this case the generator \(A^{p-1}\) of the FDA associated with the cohomology class \(\Gamma^{p} (e)\) can be simply deleted by the list of independent generators and simply identified with the polynomial \(\Phi^{p-1} (e, \phi)\).
In these terms the question was already posed twenty three years ago by D’Auria and me in [18] obtaining a positive answer [18] which has been recently revisited in [26, 27].

The enlarged algebra \( \hat{G} \) contains, besides the generators of \( G \) a bosonic 1–form \( B^{a_1 a_2} \) which is in the rank two antisymmetric representation of the Lorentz group, a bosonic 1–form \( B^{a_1 a_2 \ldots a_5} \) which is in the rank five antisymmetric representation and finally a fermionic 1–form \( \eta \) which is in the spinor representation just as the generator \( \psi \).

The Maurer Cartan equations of \( \hat{G} \) are:

\[
\begin{align*}
0 &= R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega_{cb} \\
0 &= T^a \equiv D\psi^a - \frac{1}{2} \psi^{a} \wedge \Gamma^a \psi \\
0 &= T^{a_1 a_2} \equiv DB^{a_1 a_2} - \frac{1}{2} \psi^{a_1 a_2} \psi \\
0 &= T^{a_1 \ldots a_5} \equiv DB^{a_1 \ldots a_5} - \frac{i}{12} \psi^{a_1 \ldots a_5} \psi \\
0 &= \rho \equiv D\psi \equiv d\psi - \frac{1}{4} \omega^{ab} \wedge \Gamma^{ab} \psi \\
0 &= \sigma \equiv D\eta - i \delta \Gamma^a \psi \wedge V^a - \gamma_1 \Gamma^a \psi \wedge B^{ab} - \gamma_2 \Gamma^{a_1 \ldots a_5} \psi \wedge B^{a_1 \ldots a_5}
\end{align*}
\]  

(4.11)

These Maurer Cartan equations are consistent, namely closed provided the following equation is satisfied by the coefficients:

\[
\delta + 10 \gamma_1 - 720 \gamma_2 = 0
\]  

(4.14)

Using all the generators of \( \hat{G} \) one can construct a cubic polynomial

\[
\Phi^3 [(V, \psi, B^{(2)}, B^{(5)}) = \lambda B^{a_1 a_2} \wedge V^{a_1} \wedge V^{a_2} + \alpha_1 B^{a_1 a_2} \wedge B^{a_2 a_3} \wedge B^{a_3 a_4} + \alpha_2 B^{b_1 a_1 \ldots a_4} \wedge B^{b_1 b_2} \wedge B^{b_2 a_1 \ldots a_4} \\
+ \alpha_3 \epsilon^{a_1 \ldots a_5 b_1 \ldots b_5} B^{a_1 \ldots a_5} \wedge B^{b_1 \ldots b_5} \wedge V^{m} \\
+ \alpha_4 \epsilon^{m_1 \ldots m_6 n_1 \ldots n_5} B^{m_1 m_2 m_3 p_1 p_2} \wedge B^{m_4 m_5 m_6 p_1 p_2} \wedge B^{n_1 \ldots n_5} \\
i \beta_1 \psi \wedge \Gamma^a \eta \wedge V^a + \beta_2 \psi \wedge \Gamma^{a_1 a_2} \eta \wedge B^{a_1 a_2} + i \beta_3 \psi \wedge \Gamma^{a_1 \ldots a_5} \eta \wedge B^{a_1 \ldots a_5}
\]

(4.15)

such that:

\[
d\Phi^3 [(V, \psi, B^{(2)}, B^{(5)}) = \Gamma^{[4]} (V, \psi) \equiv \frac{1}{2} \psi \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b
\]  

(4.16)

The coefficients appearing in (4.15) are completely fixed by eq. (4.14) for any of the 1–parameter family of algebras described by (4.8–4.13). Indeed the closure condition (4.14) is one equation on three parameters which are therefore reduced to two. One of them, say \( \gamma_1 \) can be reabsorbed into the normalization of the extra fermionic generator \( \eta \), but the other remains essential and its value selects one algebra within a family of non isomorphic ones. Following [26] it is convenient to set:

\[
\delta = 2 \gamma_1 (s + 1) ; \quad \gamma_2 = 2 \gamma_1 \left( \frac{s}{6!} + \frac{1}{5!} \right)
\]  

(4.17)

and \( s \) is the parameter which parametrizes the inequivalent algebras \( \hat{G}_s \). For each of them we have a solution of eq. (4.16) realized by

\[
\begin{align*}
\alpha_1 &= \frac{2 (3 + s)}{15 s^2} ; \quad \alpha_4 = \frac{-(6 + s)^2}{250 s^2} \\
\alpha_2 &= \frac{-(6 + s)^2}{720 s^2} ; \quad \alpha_3 = \frac{140 (6 + s)^2}{432000 s^2} \\
\lambda &= \frac{6 + 2 s + s^2}{5 s^2} ; \quad \beta_1 = \frac{3 - 2 s}{10 s^2 \gamma_1} \\
\beta_2 &= \frac{3 + s}{20 s^2 \gamma_1} ; \quad \beta_3 = \frac{6 + s}{2400 s^2 \gamma_1}
\end{align*}
\]  

(4.18)
In the original paper by D’Auria and myself \[18\] only two of this infinite class of solutions were found, namely those corresponding to the values:

\[ s = -1 \quad ; \quad s = \frac{3}{2} \] (4.19)

which are the roots of the equation \( \lambda = 1 \). Indeed we had imposed this additional condition which is unnecessary as it has been shown in \[26, 27\] where the more general solution (4.17,4.18) has been found.

It remains to be seen whether the equivalence between the minimal FDA \( \mathcal{M} \) and the Lie algebra \( \hat{\mathcal{G}} \) can be promoted to a dynamical equivalence between their gaugings. In other words can we consistently parametrize the curvatures of \( \hat{\mathcal{G}} \) in such a way that identifying the three form \( \mathbf{A}^{[3]} \) with the polynomial \( \Phi^{[3]} \), the rheonomic parametrizations (4.3) are automatically reproduced? This is a rather formidable algebraic problem and to the present time we have not been able to answer it in the positive or negative way\(^4\). My goal in the present paper is different. I rather want to concentrate on the bosonic FDA.s which emerge in twisted tori compactifications and clarify some issues that were left open in the current literature on this topic. As I already stressed, my presentation of the M-theory super FDA had a double purpose. On one hand I wanted to introduce the structure whose bosonic sector I will utilize in the analysis of compactifications, on the other hand it was my purpose to illustrate the working of Sullivan theorems and Chevalley cohomology in well known examples. The same concepts will be at work in the proposed object of study.

5 Compactification of M-theory on Twisted Tori and FDA

If I delete the fermionic generator \( \psi \), the FDA of M–theory, introduced in eq.s (4.2) reduces to the following system:

\[
\begin{align*}
R^{\hat{a}\hat{b}} &= d\omega^{\hat{a}\hat{b}} - \omega^{\hat{a}\hat{c}} \wedge \omega^{\hat{c}\hat{b}} \\
T^{\hat{a}} &= D V^{\hat{a}} \\
F^{[4]} &= dA^{[3]} \\
F^{[7]} &= dA^{[6]} - 15 F^{[4]} \wedge A^{[3]}
\end{align*}
\] (5.1)

where I have put a hat on the \( D = 11 \) Lorentz indices \( \hat{a}, \hat{b}, \ldots \). This is just a preparatory step for the next one, namely compactification on a so called twisted torus. This is just a fancy name utilized in current literature for a 7–dimensional group manifold \( \mathcal{G} \) (possibly modded by the action of some discrete subgroup \( \Delta \subset \mathcal{G} \) which makes it compact) whose Lie algebra

\(^4\text{This problem is currently considered in a collaboration by L. Castellani, P. Fré, F. Gargiulo and K. Rulik but results are difficult to be obtained mostly because of computer time limits in the massive algebraic simplifications which turn out to be needed. It must also be noted that the algebras defined by equations (4.8-4.13) and by some authors named D’Auria-Fré algebras have been discussed as a possible basis for a Chern-Simons formulation of fundamental M–theory [28, 29]. They have also been retrieved as part of a wider set of gauge algebras by Castellani [30], using his method of extended Lie derivatives.}\)
assume to have structure constants $\tau^I_{JK}$ and whose left–invariant 1–forms
\[
e^I = e^I_M(y) \, dy^M
\] (5.2)
have to be identified with the abstract forms $e^I$ utilized in Chevalley cohomology (see eq. (2.2)). The idea is that of performing the dimensional reduction around the vacuum:
\[
\mathcal{M}_{11} = \mathcal{M}_4 \times \mathcal{G}/\Delta
\] (5.3)
in presence of a flux for the $F^{[4]}$ field strength and to analyze the structure of the $D = 4$ FDA which emerges from the $D = 11$ FDA by this token. To this effect and following the conventions of the recent papers [23–25], I adopt the following notations and ansätze. The $D = 11$ vielbein is split as follows:
\[
V^\hat{a} = \begin{cases} 
V^a = E^a & ; \ a = 0, 1, 2, 3 \\
V^I = e^I + W^I & ; \ I = 4, 5, 6, 7, 8, 9, 10
\end{cases}
\] (5.4)
Then I consider the spin connection of the $D = 11$ theory and I split it as follows:
\[
\hat{\omega}^{\hat{a}\hat{b}} = \begin{cases} 
\hat{\omega}^{ab} = 0 \\
\hat{\omega}^{aI} = 0 \\
\hat{\omega}^{IJ} = \omega^{IJ}_{(W)} + \Delta \omega^{IJ}
\end{cases}
\] (5.5)
The connection $\omega^{IJ}_{(W)}$ is defined in such a way that with respect to it there is torsion and the torsion tensor is precisely related to the structure constants, namely, by definition we set:
\[
\mathcal{D}_{(W)} V^I = G^I + \frac{1}{2} \tau^I_{JK} V^J \wedge V^K
\] (5.6)
where
\[
G^I \equiv dW^I + \frac{1}{2} \tau^I_{JK} W^J \wedge W^K
\] (5.7)
is the curvature 2–form of the space–time gauge field $W^I$ gauging the algebra $\mathcal{G}$.
The connection $\omega^{IJ}_{(W)}$ is completely fixed by the position (5.6). Indeed on one hand we must have:
\[
0 = \mathcal{D}_{(W)} V^I - \Delta \omega^{JK} \wedge V^K \eta_{JK}
= \mathcal{D}_{(W)} V^I + \Delta \omega^{IJ} \wedge V^J
\] (5.8)
while on the other hand eq. (5.6), must be true. The solution is uniquely given by setting:
\[
\omega^{IJ}_{(W)} = -\tau^I_{JM} W^M
\] (5.9)
5.1 Expansion of the 3-form potential, types of differential forms and their notations

Next one parametrizes the three form $A^{[3]}$ of M-theory FDA according to the chosen compactification, namely decomposes it along the basis of vielbein, as follows:

$$A^{[3]} = C_{IJK}^0 V^I \wedge V^J \wedge V^K + A^{[1]}_{IJ} \wedge V^I \wedge V^J + B^{[2]}_I \wedge V^I + A^{[3]}$$

(5.10)

The convention followed in equation (5.10) and which I adopt throughout is that $p$-forms in $D = 11$ space are denoted with boldfaced characters, $A^{[p]}, B^{[q]}, C^{[r]}, \ldots$ and $[p]$ specifies their degree in such a space. On the other hand differential forms in $D = 4$ are denoted by capital normal letters $A^{[p]}, B^{[q]}, C^{[r]}, \ldots$, $[p]$ mentioning their degree in $D = 4$.

Since the algebraic structure underlying all these considerations is a double elliptic complex I will introduce a third type of differential forms denoted by capital calligraphic letters $A^{[q,p]}, B^{[q,p]}, C^{[q,p]}, \ldots$ and characterized not by one, rather by two degrees, one with respect to $D = 4$ space-time and one with respect to the internal group manifold $G$. Explicitly we have:

$$A^{[q,p]} = A^{[q]}_{I_1 \ldots I_p} \wedge e^{I_1} \wedge \ldots \wedge e^{I_p}$$

(5.11)

According to these notations the exterior derivative operator $d$ in $D = 11$ can be rewritten as the sum of two anticommuting boundary operators as it happens in all double elliptic complexes:

$$d = d + \partial \quad ; \quad d^2 = 0$$
$$0 = \partial d + d \partial \quad ; \quad d^2 = 0 \quad ; \quad \partial^2 = 0$$

(5.12)

To complete my set of conventions I also introduce a standard notation for $p$-forms on the group manifold $G$. They will be denoted by capital Greek letters $\Omega^{[p]}, \Sigma^{[q]}, \Theta^{[r]}, \ldots$, so that:

$$\Omega^{[p]} = \Omega_{I_1 \ldots I_p} e^{I_1} \wedge \ldots \wedge e^{I_p}$$

(5.13)

When the coefficients $\Omega_{I_1 \ldots I_p}$ are constant tensors, then $\Omega^{[p]}$ becomes a cochain in the Chevalley complex for Lie algebra cohomology as summarized in section 2.

I can now reinterpret the meaning of the connection $\omega(W)$ introduced in eq.(5.9) from the point of view of Chevalley cohomology. Consider a space-time $q$-form valued $p$-cochain of the Chevalley elliptic complex. This means an object of the type introduced in eq.(5.11). Following the so far introduced notations I can calculate the $D(W)$ derivative of the space-time form $A^{[q]}_{I_1 \ldots I_p}$ which plays the role of components for a Chevalley $p$-cochain. Utilizing the definition of eq.(5.9) I find

$$D(W)A^{[q]}_{I_1 \ldots I_p} = dA^{[q]}_{I_1 \ldots I_p} - (-)^{p-1} pW^M \tau^R_{M[I_1} A^{[q]}_{I_2I_3 \ldots I_p]}R$$

(5.14)

Next I consider the new $(q + 1)$-form valued $p$-cochain whose components are given by $D(W)A^{[q]}_{I_1 \ldots I_p}$, namely:

$$\mathcal{E}^{[q+1,p]} \equiv D(W)A^{[q]}_{I_1 \ldots I_p} e^{I_1} \wedge \ldots \wedge e^{I_p}$$

(5.15)
Recalling eq. (2.13) I obtain the following identification:

\[ E[q+1,p] = d A[q,p] - \ell W A[q,p] \quad (5.16) \]

Eq. (5.16) provides a tool of great help in order to translate all formulae into an intrinsic notation which exposes their meaning in Chevalley cohomology. Indeed I can identify the covariant derivative \( D(W) \) introduced in ref.s \[23, 25\] with the following index-free operator:

\[ D(W) \equiv d - \ell W \quad (5.17) \]

which maps \( A[q,p] \)-forms into \( A[q+1,p] \)-forms. Its fundamental property is encoded into the following identity:

\[ D^2(W) = -\ell G \quad (5.18) \]

### 5.2 Expansion of the 4-form field strength with flux

According to the bosonic reduction of M-theory FDA spelled out in eq.s (5.1) we can calculate the expansion of the 4-form field strength:

\[ F[4] = dA[3] \quad (5.19) \]

Following the approach of \[23, 25\] I assume that there is an internal flux namely I assume that in the vacuum background configuration \( F[4] \) is non zero, rather it is equal to an internal constant 4-form:

\[ F[4] = \Pi[0,4] \]

\[ \Pi[0,4] = g_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \quad (5.20) \]

The Bianchi identity \( dF[4] = 0 \) implies that the form \( \Pi[0,4] \) is closed in the Chevalley complex, namely that it is a Chevalley 4-cocycle of the Lie algebra \( G \):

\[ \partial \Pi[0,4] = 0 \quad (5.21) \]

In component form eq. (5.21) is equivalent to:

\[ \tau^L_{[PQ} g_{IJK]L} = 0 \quad (5.22) \]

Calculating explicitly \( dA[3] \) from the expansion ansatz (5.10) I obtain the following result:

\[ \hat{F}[4] \equiv F[4] - \Pi[0,4] \]

\[ = F^{[4]} + I_{[I} \wedge V^{I} + F_{IJ}^{[2]} \wedge V^{I} \wedge V^{J} + F_{IJK}^{[1]} \wedge V^{I} \wedge V^{J} \wedge V^{K} + F_{IJKL}^{[0]} \wedge V^{I} \wedge V^{J} \wedge V^{K} \wedge V^{L} \quad (5.23) \]
where the curvatures have the following form identical to that calculated in [25] 5:

\[
\begin{align*}
F_{[0]}^{ijkl} &= -g_{ijkl} + \frac{3}{2} \tau^M_{[ij} C_{kl]M}^{[0]} \\
F_{[1]}^{ij} &= D(W) C_{ij} + \tau^L_{[ij} A_{kl]L}^{[1]} - 4g_{ijkl} W^L \\
F_{[2]}^{ij} &= D(W) A_{ij}^{[1]} + \frac{1}{2} \tau^L_{ij} B_{kl}^{[2]} - 6g_{ijkl} W^L \wedge W^M + 3 C_{ijkl}^{[0]} G^L \\
F_{[3]}^{i} &= D(W) B_{i}^{[2]} - 2 G^j \wedge A_{ij}^{[1]} - 4g_{ijkl} W^j \wedge W^k \wedge W^L \\
F_{[4]} &= dA_{[3]} - g_{ijkl} W^i \wedge W^j \wedge W^k \wedge W^L + B_{i}^{[2]} \wedge G^i
\end{align*}
\]

(5.24)

It is now my goal to rewrite formulae (5.24) in a more intrinsic notation that better exposes their cohomological meaning within the framework of Chevalley cohomology. I introduce the definitions:

| Potentials | Curvatures |
|------------|------------|
| Flux\[^{0,4}\] = \Pi^{[0,4]} | \mathcal{F}^{[0,4]} = F_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^L |
| \mathcal{A}^{[0,3]} = C_{ijk} e^i \wedge e^j \wedge e^k | \mathcal{F}^{[1,3]} = F_{ijkl} e^i \wedge e^j \wedge e^k |
| \mathcal{A}^{[1,2]} = A_{ij} e^i \wedge e^j | \mathcal{F}^{[2,2]} = F_{ij} e^i \wedge e^j |
| \mathcal{A}^{[2,1]} = B_{i}^{[2]} e^i | \mathcal{F}^{[3,1]} = F_{i}^{[3]} e^i |
| \mathcal{A}^{[3,0]} = A_{i}^{[3]} | \mathcal{F}^{[4,0]} = F^{[4]} |

(5.25)

and by means of them I can rewrite eqs (5.24) as follows:

\[
\begin{align*}
\mathcal{F}^{[0,4]} &= -\Pi^{[0,4]} + \partial \mathcal{A}^{[0,3]} \\
\mathcal{F}^{[1,3]} &= (d - \ell_W) \mathcal{A}^{[0,3]} - \partial \mathcal{A}^{[1,2]} + i_W \Pi^{[0,4]} \\
&\equiv D(W) \mathcal{A}^{[0,3]} - \partial \mathcal{A}^{[1,2]} + i_W \Pi^{[0,4]} \\
\mathcal{F}^{[2,2]} &= (d - \ell_W) \mathcal{A}^{[1,2]} + \partial \mathcal{A}^{[2,1]} + \frac{1}{2} i_W \circ i_W \Pi^{[0,4]} + i_G \mathcal{A}^{[0,3]} \\
&\equiv D(W) \mathcal{A}^{[1,2]} + \partial \mathcal{A}^{[2,1]} + \frac{1}{2} i_W \circ i_W \Pi^{[0,4]} + i_G \mathcal{A}^{[0,3]} \\
\mathcal{F}^{[3,1]} &= (d - \ell_W) \mathcal{A}^{[2,1]} - i_G \mathcal{A}^{[1,2]} - \frac{1}{6} i_W \circ i_W \Pi^{[0,4]} \\
&\equiv D(W) \mathcal{A}^{[2,1]} - i_G \mathcal{A}^{[1,2]} - \frac{1}{6} i_W \circ i_W \Pi^{[0,4]} \\
\mathcal{F}^{[4,0]} &= dA_{[3]} - \frac{1}{24} i_W \circ i_W \circ i_W \circ i_W \Pi^{[0,4]} + i_G \mathcal{A}^{[2,1]}
\end{align*}
\]

(5.26)

5The notations used here are identical to those of [25] with just an exception. The Kaluza Klein gauge field coming from the vielbein was named $A^L$ in [25], using the same letter $A$ as used for the gauge fields coming from the 3–form. Here we changed notation to $W^L$, while the gauge fields coming from the 3–form remained noted by $A_{...}$. This change in notation is not just capricious. The Chevalley cohomological nature of $W$ is just quite different from that of its relatives $A$. The Kaluza Klein vector is associated not with forms on the group manifold rather with tangent vectors. It is a section of the tangent rather than of the cotangent bundle of $G$ and its role in the Chevalley complex is quite different since we are supposed to perform Lie derivatives of Chevalley forms along $W$. For this reason the change in notation was unavoidable. According to this, the covariant derivative $D^{(\tau)}$ of [25] has been renamed $D^{(W)}$ mentioning the tangent vector $W$ out of which it is constructed. The need of such renaming is clear when one considers the already discussed transcription of $D^{(W)}$. 

30
5.3 Cohomological interpretation of the zero curvature equations and the Minimal FDA.s

According to the general structural theory of FDA.s briefly recalled in the introduction and in section 3, the minimal part of any FDA is defined by setting all of its curvatures to zero, the switching on of curvatures corresponding to the introduction of contractible generators \[17\]. Furthermore any minimal FDA, in agreement with Sullivan’s theorems should have an interpretation as an extension of a Lie algebra by means of its cohomology classes. I am therefore primarily interested in singling out the structure of the minimal algebra emerging from M-theory compactifications on twisted tori. To this effect I just have to consider the generalized Maurer Cartan equations which emerge by setting to zero the \[F_{p,q}\] curvatures defined in eq.s(5.26) and analyze their implications and cohomological meaning.

Let us begin with the first equation \(F_{0,4} = 0\). This implies that the flux \(\Pi_{0,4}\) is actually an exact form:

\[
\Pi_{0,4} = \partial A_{0,3} \tag{5.27}
\]

Furthermore since \(\Pi_{0,4}\) is constant it also implies that the antisymmetric tensor \(C_{IJK}\) is constant as well:

\[
dA_{0,3} = 0 \tag{5.28}
\]

Consider next the second equation \(F_{1,3} = 0\). Substituting into it the previous result (5.27) we obtain:

\[
0 = (d - \ell_W) A_{0,3} - \partial A_{1,2} + i_W \partial A_{0,3}
\]

The general solution of eq.(5.29) is given by:

\[
A_{1,2} = \Sigma_{1,2} - i_W A_{0,3} \tag{5.30}
\]

where \(\Sigma_{1,2}\) is a 1–form valued cocycle of Chevalley cohomology. Therefore, using the standard basis introduced in table 2 we can write the expansion:

\[
\begin{align*}
\Sigma_{1,2} &= \Sigma_{1,2}^\perp \oplus \Sigma_{1,2}^\parallel \\
\Sigma_{1,2}^\perp &= \sum_{a=1}^{h_2} Z^a_{[1]} \Gamma_a^{[2]} \\
\Sigma_{1,2}^\parallel &= \partial \Upsilon^{[1,1]} \\
\Upsilon^{[1,1]} &= \sum_{\alpha=1}^{7-h_1} \Upsilon^\alpha_{[1]} \Xi^\alpha
\end{align*}
\tag{5.31}
\]

where \(Z^a_{[1]}\) are \(h_2\) new gauge 1–forms in \(D = 4\) and similarly \(\Upsilon^\alpha_{[1]}\) are other \(7 - h_1\) such gauge fields.

Let us now analyze the third equation \(F_{2,2} = 0\). Inserting into it the previous results we obtain:

\[
0 = (d - \ell_W) \Sigma_{1,2} - (d - \ell_W) i_W A_{0,3} + \frac{1}{2} i_W \circ i_W \partial A_{0,3} + i_G A_{0,3} + \partial A_{2,1} \tag{5.32}
\]
By formal manipulations, using the definition of the Lie derivative (2.11) and the definition of the contraction operation (2.8), eq. (5.32) can be rewritten as follows:

\[ 0 = d\Sigma^{[1,2]} - \partial (i_W \Sigma^{[1,2]} - \frac{1}{2} i_W \circ i_W A^{[0,3]} - A^{[2,1]}) + Q^{[2,2]} \]  

where the remaining form \(Q^{[2,2]}\) defined below is actually identically zero:

\[ Q^{[2,2]} \equiv i_G A^{[0,3]} - i_d W A^{[0,3]} + \frac{1}{2} (\ell_W \circ i_W + i_W \circ \ell_W) A^{[0,3]} = 0 \]  

Eq. (5.34) follows immediately from the definition of the curvature \(G\) in eq. (5.7) and from the identity:

\[ - i_{[W,W]} C^{[p]} \circ i_W (\ell_W + \ell_W \circ i_W) C^{[p]} \]  

holding true for any \(p\)-cochain and following from eq. (2.14). In order to interpret equation (5.33), we just have to recall the orthogonal decomposition (5.31) to separate the form \(\Sigma^{[1,2]}\)

Equation (5.36) simply states that the gauge–fields \(Z^r_{[1]}\) associated with the \(h_2\) harmonic 2–form of \(G\) are abelian and just contribute a contractible sector to the \(D = 4\) FDA:

\[ dZ^r_{[1]} = 0 \]  

Eq. (5.37) needs instead a little more elaboration. I have to recall that, for 1–cycles \(C^{[1,1]}\), the Lie derivative (2.11) is just given by the first term only:

\[ \ell_W C^{[1,1]} = i_W \circ \partial C^{[1,1]} \]  

The reason is that on 0-chains the Chevalley boundary operator \(\partial\) is identically zero and hence \(\partial i_W C^{[1,1]} \equiv 0\). Taking this observation into account equation (5.37) is solved by setting:

\[ \Sigma^{[2,1]} = \Sigma^{[2,1]}_{\perp} + dY^{[1,1]} + i_W \Sigma^{[1,2]} - \frac{1}{2} i_W \circ i_W A^{[0,3]} - A^{[2,1]} \]  

where \(\Sigma^{[2,1]}_{\perp}\) is a 2–form valued Chevalley 1-cocycle:

\[ \partial \Sigma^{[2,1]} = 0 \]  

Recalling our basis conventions in table 2 we can write:

\[ \Sigma^{[2,1]}_{\perp} = \sum_{\alpha=1}^{h_1} B_{[2]}^\alpha \Gamma^{[1]}_\alpha \]  

Next I analyze the implications of the equation \(F^{[3,1]} = 0\). Inserting eq. (5.30) and the first of eqs (5.40) into the definition of \(F^{[3,1]}\) (see eqs (5.26)) I obtain four type of contributions:

\[ F^{[3,1]} = T(\Sigma^{[2,1]}_{\perp}) + T(\Sigma^{[1,2]}_{\perp}) + T(Y^{[1,1]}) + T(A^{[0,3]}) \]  

32
those in $\Sigma^{[2,1]}_\bot$, those in $\Sigma^{[1,2]}_\bot$, those in $\Upsilon^{[1,1]}$ and those in $\mathcal{A}^{[0,3]}$, respectively. Let me elaborate them separately. I begin with $\mathcal{T}(\Sigma^{[2,1]}_\bot)$ and I find:

$$\mathcal{T}(\Sigma^{[2,1]}_\bot) = (d - \ell W) \Sigma^{[2,1]}_\bot = (d - \partial i_W) \Sigma^{[2,1]}_\bot = d\Sigma^{[2,1]}_\bot$$

(5.44)

The first step in (5.44) follows since $\partial \Sigma^{[2,1]}_\bot = 0$, the third because $i_W \Sigma^{[2,1]}_\bot$ is a 0–cochain in Chevalley cohomology. Next I consider the $\mathcal{T}(\Sigma^{[1,2]}_\bot)$ contribution. Here I utilize the identity:

$$-\ell W \circ i_W \Sigma^{[1,2]}_\bot = \frac{1}{2} i_{[W,W]} \Sigma^{[1,2]}_\bot$$

(5.45)

which follows since $\Sigma^{[1,2]}_\bot$ is a cocycle $\partial \Sigma^{[1,2]}_\bot = 0$. By means of this identity we can evaluate:

$$\mathcal{T}(\Sigma^{[1,2]}_\bot) = (d - \ell W) i_W \Sigma^{[1,2]}_\bot - i_G \Sigma^{[1,2]}_\bot = -i_W d\Sigma^{[1,2]}_\bot$$

(5.46)

Let us now consider the terms in $\Upsilon^{[1,1]}$. Here we have:

$$\mathcal{T}(\Upsilon^{[1,1]}) = D^2(W) \Upsilon^{[1,1]} - i_G \Upsilon^{[1,1]} = 0$$

(5.47)

Finally let us consider the terms $\mathcal{T}(\mathcal{A}^{[0,3]})$. Here we have:

$$\mathcal{T}(\mathcal{A}^{[0,3]}) = D(W) \left(-\frac{1}{2} i_W \circ i_W \mathcal{A}^{[0,3]} \right) + i_G \circ i_W \mathcal{A}^{[0,3]} - \frac{1}{6} i_W \circ i_W \circ i_W \partial \mathcal{A}^{[0,3]}$$

$$= \frac{1}{2} i_W \circ \ell W \circ i_W \mathcal{A}^{[0,3]} - \frac{1}{6} i_W \circ i_W \circ i_W \partial \mathcal{A}^{[0,3]}$$

$$= 0$$

(5.48)

The last line of eq.(5.48) follows from an explicit evaluation of the two terms in the second line.

**The Minimal FDA in $D = 4$**  We can now summarize our results. The 0–curvature equations

$$\mathcal{F}^{[p,q]} = 0$$

(5.49)

lead to the following $D = 4$ bosonic free differential algebra. Besides the seven $W^I$ one–forms gauging the original algebra $\mathbb{G}$, there are $h_3$ zero-forms $\Phi_0^x$, $h_2$ additional gauge one–forms $Z^a_a$ and $h_1$ gauge two–forms $B^\alpha_{[2]}$ and $h_0 = 1$ three-forms

$$\overline{A}^\ell_{[3]} \equiv A^{[3]}_3 - C_{IJK} W^I \wedge W^J \wedge W^K$$

(5.50)

closing the algebra:

$$dW^I + \frac{1}{2} \tau'_{JK} W^J \wedge W^K = 0 \ ; \ I = 1, \ldots, \dim \mathbb{G}$$

$$d\Phi_0^x = 0 \ ; \ x = 1, \ldots, h_3(\mathbb{G})$$

$$dZ^a_a = 0 \ ; \ a = 1, \ldots, h_2(\mathbb{G})$$

(5.51)

$$dB^\alpha_{[2]} = 0 \ ; \ \alpha = 1, \ldots, h_1(\mathbb{G})$$

$$d\overline{A}^\ell_{[3]} = 0 \ ; \ \ell = 1, \ldots, h_0(\mathbb{G}) = 1$$
where $h_{0,1,2,3}$ are the dimensions of the cohomology groups of $G$. In addition there are also $\varphi_2$ gauge fields $\Upsilon^{\bar{a}}_{[1]}$ on which no condition is imposed, where $\varphi_2 = 7 - h_1$ is the dimension of $\text{Im} \partial_1$. These gauge fields are non-physical, they are the gauge degrees of freedom of the 2-forms. It is evident from their structure that the minimal part of the FDA is simply given by the original algebra $G$ which is a standard Lie algebra, the remaining three equations, defining instead contractible generators.

With reference to the D=4 FDA summarized in eq. (5.51) we can now summarize the complete parametrization of the original potentials $A^{[p,q]}$ displayed in table (5.25). The independent objects are the following five:

\[
\begin{align*}
\Sigma_{\perp}^{[0,3]} &= \sum_{\bar{x}=1}^{h_3} \Phi_{[0]}^x \Gamma^{[3]}_x \\
\Sigma_{\perp}^{[1,2]} &= \sum_{\bar{a}=1}^{h_2} Z^{\bar{a}}_{[1]} \Gamma^{[2]}_a \\
\Sigma_{\perp}^{[2,1]} &= \sum_{\bar{\alpha}=1}^{h_1} B^{\bar{\alpha}}_{[2]} \Gamma^{[1]}_{\alpha} \\
\Sigma_{\perp}^{[3,0]} &= \sum_{\ell=1}^{h_0=1} \bar{A}^{[3]}_{[\ell]} \Gamma^{[0]}_{\ell} \\
\Upsilon^{[0,3]} &= \sum_{\bar{x}=1}^{21-h_3+h_2-h_1} \phi^{\bar{x}} \Xi^{[3]}_{\bar{x}} \\
\Upsilon^{[0,2]} &= \sum_{\bar{a}=1}^{14-h_2+h_1} \phi^{\bar{a}} \Xi^{[2]}_{\bar{a}} \\
\Upsilon^{[1,1]} &= \sum_{\bar{\alpha}=1}^{7-h_1} \Upsilon^{\bar{\alpha}}_{[1]} \Xi^{[1]}_{\bar{\alpha}} 
\end{align*}
\]

(5.52)

where $\phi^{\bar{a}}, \phi^{\bar{x}}$ are $35 - h_3$ scalar fields coming from the M–theory three–form subdivided into the two sectors of 3–cochains orthogonal to the harmonic cycles. Their vev.s are arbitrary numbers at the present level and they can be seen as moduli of the algebra. Similarly, as I have already stressed, the $7 - h_1$ one–forms $\Upsilon^{\bar{\alpha}}_{[1]}$ are just gauges.

In terms of the objects defined in equations (5.52) the potentials $A^{[p,q]}$ are explicitly parametrized as follows and this is the most general solution of the zero curvature equations:

\[
\begin{align*}
\Pi^{[0,4]} &= \partial \Upsilon^{[0,3]} \\
A^{[0,3]} &= \Sigma_{\perp}^{[0,3]} + \Upsilon^{[0,3]} + \partial \Upsilon^{[0,2]} \\
A^{[1,2]} &= \Sigma_{\perp}^{[1,2]} + \partial \Upsilon^{[1,1]} - i_W \Sigma_{\perp}^{[0,3]} - i_W \Upsilon^{[0,3]} \\
A^{[2,1]} &= \Sigma_{\perp}^{[2,1]} + D^W \Upsilon^{[1,1]} + i_W \Sigma_{\perp}^{[1,2]} - \frac{1}{2} i_W \circ i_W \Sigma_{\perp}^{[0,3]} - \frac{1}{2} i_W \circ i_W \Upsilon^{[0,3]} \\
A^{[3,0]} &= \Sigma_{\perp}^{[3,0]} + \frac{1}{6} i_W \circ i_W \circ i_W \Sigma_{\perp}^{[0,3]} + \frac{1}{6} i_W \circ i_W \circ i_W \Upsilon^{[0,3]} 
\end{align*}
\]

(5.53)
Conclusive Remarks on the Minimal FDA  Equations (5.51), (5.52) and (5.53) are written in a rather pedantic way, but this is done in order to stress the conceptual implications of the zero curvature equations. As we have already remarked, according to the general theory of FDA.s (see section 3) the new generators of the algebra should be in one–to–one correspondence with the cohomology classes of the original algebra and indeed they are. Yet the correspondence is that anticipated in eq.(3.18) and not that involved in Sullivan’s second theorem, namely that of eq.(3.17). Indeed the number $h_0$ which is just trivially equal to one predicts the number of $D=4$ three–forms, the number $h_1$ predicts the number of two–forms, the number $h_2$ predicts the number of additional one–forms. Finally we expect that $h_3$ should predict the number of essential 0–forms, or scalars. At this level we do not see it clearly, since all the 35 scalars are somehow constants, both those associated with non trivial $h_3$–classes and those associated with boundaries. The vev.s of all the scalars are to be regarded as moduli. In any case the $D=4$ FDA defined by the zero curvature equations is a trivial FDA since all new generators are contractible, as we have already remarked.

5.4 Introducing the FDA curvatures and non trivial fluxes

It is now fairly easy to deform the minimal FDA by introducing curvatures and, as I shall emphasize, it is only in presence of these latter that the flux $\Pi[0,4]$ can be cohomologically non trivial. So let us introduce the curvatures of degree $[0,4]$ by setting:

$$
\mathcal{F}^{[0,4]} = \mathcal{R}^{[0,4]}_\perp + \mathcal{R}^{[0,4]}_\partial
$$

$$
\mathcal{R}^{[0,4]}_\partial = \partial Q^{(3)} (\Pi^{[0,4]}) - \partial A^{[0,3]}
$$

$$
\mathcal{R}^{[0,4]}_\perp = P^{(4)}_\perp (\Pi^{[0,4]}) \equiv \Delta^{[0,4]} = \sum_{x=1}^{h_4} \mu^x \Gamma^{[4]}_x
$$

where $\mu^x$ are $h_4 = h_3$ constant parameters. The crucial observation is that the zero curvature equations I have previously solved are modified by the addition to each curvature $\mathcal{F}^{[p,q]}$ of terms of the following form:

$$
\mathcal{F}^{[1,3]} \rightarrow \mathcal{F}^{[1,3]} + i_W \Delta^{[0,4]}
$$

$$
\mathcal{F}^{[2,2]} \rightarrow \mathcal{F}^{[2,2]} + \frac{1}{2} i_W \circ i_W \Delta^{[0,4]}
$$

$$
\mathcal{F}^{[3,1]} \rightarrow \mathcal{F}^{[3,1]} - \frac{1}{6} i_W \circ i_W \circ i_W \Delta^{[0,4]}
$$

$$
\mathcal{F}^{[4,0]} \rightarrow \mathcal{F}^{[4,0]} - \frac{1}{24} i_W \circ i_W \circ i_W \Delta^{[0,4]}
$$

Let us see the implications of these additions. Modifying eq.(5.29) I obtain:

$$
\mathcal{F}^{[1,3]} = (d - i_W) A^{[0,3]} - \partial A^{[1,2]} + i_W \partial A^{[0,3]} + i_W \Delta^{[0,4]}
$$

$$
\mathcal{F}^{[2,2]} = d A^{[0,3]} + \partial (A^{[1,2]} + i_W A^{[0,3]}) + i_W \Delta^{[0,4]}
$$

The 3–cochain $i_W \Delta^{[0,4]}$ has generically a non trivial projection along all three subspaces $\Gamma^{[3]}$, $\partial \Xi^{[2]}$ and $\Xi^{[3]}$ (see eqs (2.47) and (2.49)). Defining:

$$
\mathcal{R}^{[1,3]}_\perp = d \Sigma^{[0,3]} + P^{(3)}_\perp (i_W \Delta^{[0,4]})
$$
\[ R_{||}^{[1,3]} = d\gamma^{[0,3]} + P_{||}^{(3)} (i_W \Delta^{[0,4]}) \]
\[ R_{\phi}^{[1,3]} = \partial \left[ -d\gamma^{[0,2]} - i_W \Sigma^{[0,3]} - i_W \gamma^{[0,3]} + Q_{||}^{(2)} (i_W \Delta^{[0,4]}) \right] \]

(5.57)

the curvature \( F^{[1,3]} \) is decomposed into its three independent projections.

\[ F^{[1,3]} = R_{||}^{[1,3]} + R_{\phi}^{[1,3]} + R_{\Sigma}^{[1,3]} \]

(5.58)

In a similar way I can decompose all the other curvatures. For instance let me focus on the curvature \( F^{[3,1]} \) and consider the transverse component \( R_{\Sigma}^{[1,3]} \). Here I find:

\[ R_{\Sigma}^{[3,1]} = d\Sigma^{[2,1]} + P_{\Sigma}^{(1)} (-i_W d\Sigma^{[1,2]} - \frac{1}{2} i_W \circ i_W dA^{[0,3]}) + P_{\Sigma}^{(1)} (-\frac{1}{6} i_W \circ i_W \circ i_W \Delta^{[0,4]}) \]

const term

(5.59)

In the above equation as in the first of eqs. (5.37) you can see the conditions under which a non trivial FDA could emerge from flux compactifications on twisted tori. All terms in (5.59) are derivative terms, except the last one which is defined by the non trivial flux \( \Delta^{[0,4]} \). Recalling eq. (2.49), from (5.59) we extract:

\[ F[B]^\alpha = dB_{[2]}^\alpha - \frac{1}{6} \langle i_W \circ i_W \circ i_W \Delta^{[0,4]} , \Gamma^{[6]}_\alpha \rangle + \text{derivative terms of lower degree forms} \]

(5.60)

\[ \Lambda^{[3]}_\alpha (W) = \langle i_W \circ i_W \circ i_W \Delta^{[0,4]} , \Gamma^{[6]}_\alpha \rangle \]

(5.61)

The object

(5.61)

is a cubic polynomial in the 1–forms \( W^I \). According to Sullivan’s second theorem it is necessarily a cohomology class of \( G \), or zero.

The condition for non trivial deformations is therefore that \( \Lambda^{[3]}_\alpha (W) \) should be non zero or that

\[ \Lambda^{[2]}_\alpha (W) = \langle i_W \circ i_W \Delta^{[0,4]} , \Gamma^{[5]}_\alpha \rangle \]

(5.62)

should be non zero. When \( \Lambda^{[3]}_\alpha (W) \neq 0 \) we have non trivial 2–form generators. On the other hand when \( \Lambda^{[2]}_\alpha (W) \neq 0 \) we have non trivial 1–form generators, namely the Lie algebra \( G \) is extended to some bigger algebra.

Let us now immediately verify that \( \Lambda^{[3]}_\alpha (W) = 0 \) in the case of SS algebras with all different eigenvalues \( 0 \neq m_1 \neq m_2 \neq m_3 \). In this case the flux \( \Delta^{[0,4]} \) is a linear combination of the forms \( \Gamma^{[4]}_x \) presented in eq. (2.35). Correspondingly we have:

\[ i_W \circ i_W \circ i_W \Delta^{[0,4]} \propto W^{x_1} \wedge W^{x_2} \wedge W^{x_3} \wedge \epsilon^m \mu^x \epsilon_{i_1 i_2 i_3 m n} \sigma_{x}^{uv} \]

(5.63)

The form \( \Gamma^{[6]} \) is instead given in eq. (2.82) and it is immediately evident that the wedge product \( i_W \circ i_W \circ i_W \Delta^{[0,4]} \wedge \Gamma^{[6]} \) is zero since \( \Gamma^{[6]} \) contains already all the six \( \epsilon^i \), so that necessarily there are in the wedge product two identical \( \epsilon^i \). Consider instead \( \Lambda^{[2]}_\alpha (W) \) for the same case of SS algebra. Here things are different. On one hand we have:

\[ i_W \circ i_W \Delta^{[0,4]} \propto W^{x_1} \wedge W^{x_2} \wedge \epsilon^m \wedge \epsilon^n \mu^x \epsilon_{i_1 i_2 m n} \sigma_{x}^{uv} \]

(5.64)
on the other hand, $\Gamma_b^{[5]}$, as given in eq.(2.89), contains $e^0$ and four $e^i$. It follows that $\Lambda_a^{[2]}(W)$ is non-zero rather it is of the form:

$$
\Lambda_a^{[2]}(W) \propto \mu^x e_{i_1i_2mnuv} \sigma_{ma}^{mn} \sigma_x^{uv} W^{i_1} W^{i_2}
$$

(5.65)

It means that we have an enlargement of the $\mathcal{G}$ Lie algebra induced by the non trivial flux. It is interesting to check how Sullivan theorem is anyhow verified. According to Sullivan’s second theorem the 2–form $\Lambda_a^{[2]}(W)$ should be a harmonic 2–form of $\mathcal{G}$, namely should be a linear combination of $\Gamma_a^{[2]}$ as displayed in eq.(2.89) with $e^I$, replaced by $W^I$. Indeed it is! To see it it suffices to recall that, by definition, the matrices $\sigma_{1,2,3}$ are SO(6) Cartan generators, namely they are skew diagonal:

$$
\mu^x \sigma_x = \begin{pmatrix}
0 & \mu_1 & 0 & 0 & 0 & 0 \\
-\mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_2 & 0 & 0 \\
0 & 0 & -\mu_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & 0 \\
0 & 0 & 0 & 0 & -\mu_3 & 0 
\end{pmatrix}
$$

(5.66)

As a consequence of this $\Lambda_a^{[2]}(W)$ is of the form:

$$
\begin{align*}
\Lambda_1^{[2]}(W) &\propto \mu_2 \sigma_3^{uv} W^u \wedge W^v + \mu_3 \sigma_2^{uv} W^u \wedge W^v \\
\Lambda_2^{[2]}(W) &\propto \mu_1 \sigma_3^{uv} W^u \wedge W^v + \mu_3 \sigma_1^{uv} W^u \wedge W^v \\
\Lambda_3^{[2]}(W) &\propto \mu_1 \sigma_2^{uv} W^u \wedge W^v + \mu_2 \sigma_1^{uv} W^u \wedge W^v
\end{align*}
$$

(5.67)

which is precisely what it is required by Sullivan’s theorem!

Let me finally discuss how also the Maurer Cartan equations of the 2–forms can become non trivial in presence of a non trivial 4–form flux, if the SS algebra is degenerate. To this effect let me consider the case $m_1 \neq 0$, $m_2 = m_3 = 0$. Here the fourth cohomology group is larger and we also have harmonic 4–forms of the type:

$$
\Gamma^{(4)} \propto e^0 \wedge e^{i_1} \wedge e^{i_2} \wedge e^{i_3} U_{i_1i_2i_3}
$$

(5.68)

provided the tensor $U_{ijk}$ vanishes when one of its indices takes the values 1 or 2. If we use such a harmonic form in the construction of $\Delta^{[0,4]}$, namely of the flux, then $\Lambda_x^{[3]}(W)$ as defined by equation (5.61) can receive non vanishing contributions and we can have a genuine deformation also of the 2–form Maurer Cartan equations.

6 Conclusions and Perspectives

In this paper I have recast the analysis of Free Differential Algebras emerging from M–theory compactifications on so called twisted tori, into the language of Chevalley cohomology and within the framework of the structural theorems on FDA.s due to Sullivan. The main results that I have obtained are:
1. Establishing a general and compact double elliptic complex formalism which enables the analysis of all explicit cases within the cohomology framework of Lie algebras. In particular I have established general formulae for the calculation of the spectrum of $p$-forms in terms of Hodge numbers of the Lie algebra $G$ and of the generalized structure constants in terms of Poincaré pairing on the Chevalley complex.

2. Explicit calculation of the cohomology for the simplest SS algebras and construction of an explicit orthogonal basis for the $p$-cochain spaces.

3. A general cohomological analysis of the zero curvature equations which reveals how the minimal FDA is always trivial if the 4-form flux is cohomologically trivial.

4. An explicit algebraic criterion to be satisfied by the 4-form flux in order to generate non-trivial FDA.s

The applications and future development lines of the presented results are several. In particular it is worth mentioning:

1. Analysis of a wider class of algebras $G$ within the presented framework and a systematic search on the structure of FDA obtained thereof, in relation with the available 4-form fluxes.

2. Revisited analysis of the dual gauge algebras in relation with the presented cohomological set up.

3. Extension of the present analysis from M-theory to Type IIB compactifications.

4. Use of twisted tori backgrounds and of the formalism presented here as a case study in relation with the open problem of gauging M-theory algebras $^{48,4,13}$ $^{30}$

Work on these issues is on agenda.
References

[1] For an exhaustive review on flux compactifications see A.R. Frey Warped Strings: Self dual flux and contemporary compactifications hep-th/0308156.

[2] S. Gukov, Solitons, superpotentials and calibrations Nucl. Phys. B574 (2000) 169 hep-th/9911011 S. Gukov, C. Vafa and E. Witten Nucl. Phys. B584 (2000) 69 hep-th/9906070

[3] G.L. Cardoso, G. Dall’Agata, D. Luest, P. Manousselis and G. Zoupanos Non Kaehler string backgrounds and their five torsion classes Nucl. Phys. B652 (2003) 5 hep-th/0211118. G.L. Cardoso, G. Curio, G. Dall’Agata and D. Luest, BPS action and superpotential for heterotic string compactifications with fluxes JHEP 10 (2003) 004 hep-th/0306088 G.L. Cardoso, G. Curio, G. Dall’Agata and D. Luest, Heterotic string theory on non–kaeler manifolds with H-flux and gaugino condensate Fortsch. Phys. 52 (2004) 483 hep-th/03100021.

[4] J. Sherk and J.H. Schwarz Phys. Lett. B82 (1979) 60, J. Sherk and J.H. Schwarz How to get masses form extra dimensions Nucl. Phys. B153 (1979) 61

[5] C.M. Hull, R.A. Reid-Edwards Flux Compactifications of String Theory on Twisted Tori hep-th/0503114

[6] N. Kaloper and R.C. Myers The O(dd) story of massive supergravity JHEP 05 (1999) 010 hep-th/9901045

[7] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi New supersymmetric string compactifications JHEP 03 (2003) 061 hep-th/0211182.

[8] M.B. Schulz Superstring orientifolds with torsion: O5 orientifolds of torus fibrations and their massless spectra Fortsch. Phys. 52 (2004) 963 hep-th/0406001.

[9] J.P. Derendinger, C. Kounnas, P.M. Petropoulos, F. Zwirner, Superpotentials in IIA compactifications with fluxes hep-th/0411276

[10] F. Cordaro, P. Frè, L. Gualtieri, P. Termonia, M. Trigiante, N=8 gaugings revisited: an exhaustive classification, Nucl.Phys. B532 (1998) 245-279, hep-th/9804056.

[11] L. Andrianopoli, F. Cordaro, P. Frè, L. Gualtieri, Non-Semisimple Gaugings of D=5 N=8 Supergravity and FDA,s, Class.Quant.Grav. 18 (2001) 395-414, hep-th/0009048 L. Andrianopoli, F. Cordaro, P. Frè, L. Gualtieri, Non-Semisimple Gaugings of D=5 N=8 Supergravity, Fortsch.Phys. 49 (2001) 511-518, hep-th/0012203

[12] C. Angelantonj, S. Ferrara and M. Trigiante, JHEP 0310, 015, (2003), Phys. Lett. B582 (2004) 263

[13] B. de Wit, H. Samtleben and M. Trigiante Phys. Lett. B583 (2004) 338 hep-th/0311224
[14] C. Angelantonj, R. D’Auria, S. Ferrara and M. Trigiante Phys. Rev. Letters B583 (2004) 331

[15] L. Andrianopoli, M.A. Lledó, M. Trigiante The Scherk Schwarz mechanism as a flux compactification with internal torsion hep-th/0502083.

[16] E. Cremmer and B. Julia Supergravity Theory in eleven dimensions, Phys Lett. B76 (1978) 409, The SO(8) supergravity Nucl. Phys. B159 (1979) 141.

[17] P. Fré, Comments on the 6–index photon in D=11 supergravity and the gauging of free differential algebras, Class. Quant. Grav. 1 (1984) L81.

[18] R. D’Auria and P. Fré, Geometric supergravity in D=11 and its hidden supergroup Nucl. Phys. B201 (1982) 101.

[19] L. Castellani, P. Fré, F. Giani, K. Pilch and P. van Nieuwenhuizen Gauging of D=11 supergravity Ann. Phys. 146 (1983) 35.

[20] D. Sullivan Infinitesimal computations in topology Bull. de l’Institut des Hautes Etudes Scientifiques, Publ. Math. 47 (1977)

[21] L. Castellani, R. D’Auria, P. Fré Supergravity and superstrings: a geometric perspective, World Scientific, Singapore 1991.

[22] G. Dall’Agata and S. Ferrara Gauged supergravity algebras from twisted tori compactifications with fluxes arXiv:hep-th/0502066

[23] R. D’Auria, S. Ferrara, M. Trigiante, E7(7) symmetry and dual gauge algebra of M-theory on a twisted seven torus arXiv:hep-th/0504108

[24] G. Dall’Agata, R. D’Auria and S. Ferrara Compactifications on twisted tori with fluxes and free differential algebras arXiv:hep-th/0503122

[25] R. D’Auria, S. Ferrara, M. Trigiante, Curvatures and potential of M-theory in D=4 with fluxes and twist arXiv:hep-th/0507225

[26] I.A. Bandos, J.A. de Azcarraga, J.M. Izquierdo, M. Picon, O. Varela, On the underlying gauge group structure of D=11 supergravity Phys. Lett. B596 (2004) 145 [hep-th/0406020]

[27] I.A. Bandos, J.A. de Azcarraga, M. Picon, O. Varela, On the formulation of D=11 Supergravity and the composite nature of its three form field Ann. of Physics 317 (2005) 238 [hep/th0409100]

[28] H. Nastase Towards a Chern-Simons M-theory of Osp(1|32) × Osp(1|32) hep-th/0306269

[29] P. Horava, M-theory as a holographic field theory Phys. Rev. D59 (1999) 04004 hep-th/9712130

[30] L. Castellani Lie derivative along antisymmetric tensors and the M-theory superalgebra hep-th/0508213