Lightcone reference for total gravitational energy

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Abstract

We give an explicit expression for gravitational energy, written solely in terms of physical spacetime geometry, which in suitable limits agrees with the total Arnowitt-Deser-Misner and Trautman-Bondi-Sachs energies for asymptotically flat spacetimes and with the Abbot-Deser energy for asymptotically anti-de Sitter spacetimes. Our expression is a boundary value of the standard gravitational Hamiltonian. Moreover, although it stands alone as such, we derive the expression by picking the zero-point of energy via a “lightcone reference.”

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Introduction

It is satisfying to learn that each of the accepted notions of total gravitational energy can be expressed as a limit of (the on-shell value of) the standard gravitational Hamiltonian $H$. Indeed, i. the Arnowitt-Deser-Misner (ADM) energy \footnote{Null infinity coincides with spatial infinity for asymptotically anti-de Sitter solutions.} associated with spatial infinity for asymptotically flat spacetimes, ii. the Trautman-Bondi-Sachs (TBS) energy \footnote{As written by Brown and York.} associated with null infinity for asymptotically flat spacetimes, and iii. the Abbott-Deser (AB) energy \footnote{The ADM correspondence was noted first in Refs. \cite{5}, while the TBS correspondence was shown in Ref. \cite{6}. For asymptotically anti-de Sitter spacetimes, that the surface integral (1) defines the correct “energy” (actually “conserved mass” is a better term in this context) at infinity was shown first in Ref. \cite{8}. Ref. \cite{8} spells out the relationship between the integral (1) and the original definition of Abbott and Deser.} associated with infinity for asymptotically anti-de Sitter spacetimes can all be written as (a limit of) the following Hamiltonian boundary value: \footnote{As written by Brown and York.}

$$H|_B = (8\pi)^{-1} \int_B d^2x \sqrt{\sigma} N (k - k|_{\text{ref}})$$  \hspace{1cm} (1)

Here $B$ is a large nearly spherical two-surface with coordinate radius $r$, tending to an infinite-area round sphere as $r \to \infty$. $N$ is a smearing (lapse) function, which is unity for the asymptotically flat scenarios and grows like $r$ in the asymptotically anti-de Sitter case, and $d^2x \sqrt{\sigma}$ is the proper $B$ area element. $B$ is embedded in a three-slice $\Sigma$ of the ambient physical spacetime $M$, and $k$ is the mean curvature of $B$ as embedded in $\Sigma$. Usually, $k|_{\text{ref}} = k|_{\text{ref}}(\sigma_{ab})$ denotes an arbitrary function of the intrinsic $B$ metric $\sigma_{ab}$. However, to ensure that (1) gives rise to the correct asymptotic energies as $r \to \infty$, we define it in this paragraph via the following prescription.

- Embed $B$ isometrically in a static-time slice $\Sigma$ of a reference spacetime $M$, which will either be Minkowski spacetime or an anti-de Sitter spacetime. For example, when defining the ADM energy, embed $B$ isometrically in Euclidean three-space $E^3$.

- Compute $k|_{\text{ref}}$ as the mean curvature of $B$ as embedded in $\Sigma$.

Physically, the proper surface integral of $(8\pi)^{-1} Nk|_{\text{ref}}$ is set as the energy zero-point, and (1) expresses the energy of $\Sigma$ relative to the energy of $\Sigma$. This choice for the zero-point determines the correct asymptotic energies. However, by following this prescription, we only define the surface integral (1) implicitly. Indeed, obtaining a closed-form expression for $k|_{\text{ref}}$ is tantamount to actually solving a stubborn embedding problem. For all but simple cases this is a hopeless proposition.

In this brief report, we show that the explicit alternative

$$H|_B = (8\pi)^{-1} \int_B d^2x \sqrt{\sigma} N (k + \sqrt{2R + 4/\ell^2})$$  \hspace{1cm} (2)

\footnote{Null infinity coincides with spatial infinity for asymptotically anti-de Sitter solutions.}
to \( \frac{1}{1} \) also determines the correct asymptotic energies. Here \( \mathcal{R} \) is the \( B \) Ricci scalar, and \( \ell \) is the radius of curvature for an asymptotically anti-de Sitter solution, related to the (negative) cosmological constant \( \Lambda \) by \( \ell = (-3/\Lambda)^{1/2} \). One must take \( \ell \to \infty \) for asymptotically flat spacetimes, in which case the second term under the radical vanishes. Now setting

\[
k|_{\text{ref}} = -\sqrt{2\mathcal{R} + 4/\ell^2},
\]

we pick a different zero-point, namely the proper surface integral of the new \((8\pi)^{-1}Nk|_{\text{ref}}\), which agrees with the previous one in the \( r \to \infty \) limit. As both \( k|_{\text{ref}} \)'s are determined solely by the \( B \) metric \( \sigma_{ab} \), both (1) and (2) are Hamiltonian boundary values as shown by Brown and York; and we may view both expressions as Brown-York quasilocal energies ("quasilocal" as \( B \) is finite before the \( r \to \infty \) limit is taken). [5]

We demonstrate the asymptotic equivalence of (1) and (2) by comparing the leading-order terms in the separate asymptotic expansions for \( k|_{\text{ref}} \). Now, although the analysis below is certainly valid for case ii., in which \( B \) tends to a round-sphere cut of future null infinity, this limit differs conceptually from the other two. Hence, we postpone commenting on this case until the final section. We assume that the \( \Sigma \) Cauchy data obeys the required asymptotic conditions associated with either an asymptotically flat solution [8] or an asymptotically anti-de Sitter solution [9]. In both cases, the physical \( \Sigma \) three-metric \( h_{ij} \) approaches a fixed background metric \( f_{ij} \). We need not consider the details of this approach, but we do note that in each case the fall-off conditions on \( h_{ij} \) imply that the Ricci scalar of \( B \) has the following asymptotic expansion:

\[
\mathcal{R} \sim 2r^{-2} + \mathcal{R}^3 r^{-3}.
\]

The \( \mathcal{R} \) remainder need only be \( o(r^{-3}) \), i.e. it falls off faster than \( r^{-3} \). Notice that the leading term is the scalar curvature for a round sphere of radius \( r \). (It is often the case that the coefficient \( \mathcal{R}^3 \) actually vanishes identically, so after the leading term the next non-zero term comes at a higher power of inverse radius. Moreover, even if \( \mathcal{R}^3 \) does not vanish, a simple argument based on the Gauß-Bonnett Theorem shows that it is a pure divergence on the unit sphere. [4] These points do not affect the analysis here.) The expansion (4) completely determines the asymptotic behavior of the energy zero-points we consider. Indeed, adopting (3) in the asymptotically flat case, we compute

\[
k|_{\text{ref}} \sim -2r^{-1} - \frac{1}{2}\mathcal{R}^3 r^{-2},
\]

The leading term in (5) is exact for a round sphere of radius \( r \) embedded in \( E^3 \). Likewise, again defining \( k|_{\text{ref}} \) with (3), in the asymptotically anti-de Sitter case we get

\[
k|_{\text{ref}} \sim -2/\ell - \ell r^{-2} - \frac{1}{2} \mathcal{R}^3 r^{-3}.
\]

In the next section, we show that with \( k|_{\text{ref}} \) constructed via the prescription listed in the first paragraph, we obtain the exact same expansions as (5) and (6). As this prescription defines a Hamiltonian boundary value (1) which determines
the correct asymptotic energies, we are assured that (2) does as well. We remark that upon multiplication by \((8\pi)^{-1}N\) and subsequent integration over \(B\), the leading term in (3) and the first two leading terms in (4) diverge in the \(r \to \infty\) limit. These terms cancel corresponding divergent terms which arise in the proper \(B\) integral of the physical \((8\pi)^{-1}Nk\).

As given in (3), \(k^{\text{ref}}\) is built solely with the intrinsic \(B\) metric \(\sigma^{ab}\); and, therefore, the surface integral (2) stands alone as a bona-fide Hamiltonian boundary value which determines the correct asymptotic energies. Nevertheless, via an auxiliary isometric embedding of \(B\) into a lightcone of the ambient reference spacetime, we can physically motivate the choice (3). In the second section we discuss this motivation, the “lightcone reference,” in some detail.

1 Static-time reference

To follow the prescription outlined in the first paragraph of the introduction, start with the line-element for the reference spacetime \(M\),

\[
ds^2 = -F(R)dt^2 + F^{-1}(R)dR^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \(F(R) = 1\) for Minkowski spacetime and \(F(R) = (1 + R^2/\ell^2)\) for an anti-de Sitter spacetime. In both cases \(R\) is the round-sphere areal radius. Now consider the surface \(B\) embedded in a constant-\(t\) slice \(\Sigma\) associated with the above line element. Take \(m^a\) and \(\bar{m}^a\) as a complex null dyad tangent to \(B\), and for notational ease let \(k_{ab}\) stand for \((k^{\text{ref}})^{ab}\), the extrinsic curvature of \(B\) as embedded in \(\Sigma\). Notice that

\[
2k^a_{\bar{m}b} = 2k^{ab}_{m\bar{m}}\quad \text{and} \quad k^a_{mm} = k_{ab}m^a m^b
\]

respectively capture the trace and trace-free pieces of \(k_{ab}\). The embedding constraints satisfied by the intrinsic and extrinsic geometry of \(B \subset \Sigma\) are the following:

\[
R_{m\bar{m}mm} = [k^a_{\bar{m}m} - k_{mm}k_{\bar{m}m}] - \frac{1}{2}R
\]

\[
0 = \bar{\delta}k^a_{\bar{m}m} - \bar{\delta}k_{mm}.
\]

Here \(\bar{\delta}\) is the full \(B\) “eth” operator and \(R_{m\bar{m}mm} := m^i \bar{m}^j m^k \bar{m}^l R_{ijkl}\), where \(R_{ijkl}\) is the \(\Sigma\) Riemann tensor. \(R_{ijkl}\) vanishes for a static-time slice of Minkowski spacetime, but when \(\Sigma\) is a static-time slice of an anti-de Sitter spacetime \(R_{m\bar{m}mm}\) is non-zero. Indeed, were \(B\) a round sphere of radius \(R = r\), we would have \(R_{m\bar{m}mm} = 1/\ell^2\).

Let us first consider that case that \(\Sigma\) is Euclidean three-space \(E^3\). In this case, were \(B\) a round sphere of radius \(R = r\), we would have \(k_{\bar{m}m} = -1/r\) and \(k_{mm} = 0\). Therefore, as \(B\) tends to an infinite-area round sphere, we posit the expansions

\[
k_{\bar{m}m} \sim -r^{-1} + k_{m\bar{m}}^2 r^{-2}
\]

\[
k_{mm} = O(r^{-2}).
\]

\[\text{We use \((i,j,\cdots)\) as } \Sigma \text{ indices, whereas we have used \((a,b,\cdots)\) as } B \text{ indices.}\]
Plugging these expansions along with \( (4) \) into \( (8) \), we can consistently match up leading powers of \( r \) to find the exact same expansion as given in \( (5) \).

Now consider the other case when \( \Sigma \) is a static slice of an anti-de Sitter spacetime. Then, were \( B \) again a round sphere of radius \( R = r \), we would have \( k_{mm} = 0 \) and

\[
k_{\bar{m}m} = -r^{-1} \sqrt{1 + r^2/\ell^2}.
\]

This suggests the Ansätze

\[
k_{\bar{m}m} \sim -1/\ell - \frac{1}{2} \ell r^{-2} + k_{m\bar{m}} r^{-3} \quad (13)
k_{mm} = O(r^{-2}) \quad (14)
\]

Moreover, we show below that with \( m^k \) tangent to our slightly distorted sphere \( B \),

\[
R_{m\bar{m}m\bar{m}} = 1/\ell^2 + O(r^{-4}) \quad (15)
\]

With this result, \( (4) \), and \( (14) \) plugged into \( (8) \), we consistently obtain the same expansion as given in \( (6) \). Notice that in both cases above we have not used the embedding constraint \( (9) \), although we would need to were we also seeing the leading asymptotics of \( k_{mm} \).

Let us verify equation \( (13) \). Consider the natural foliation of \( \Sigma \) and its associated triad \( \{ p, q, \bar{q} \} \). Here \( q^k \) is a complex dyad tangent to the round-sphere leaves of \( \Sigma \), and \( p^k \) is the outward-pointing normal of a round sphere in \( \Sigma \). As \( B \) is not perfectly round, at each point on \( B \) we have an expansion of the form

\[
m^k = \alpha p^k + \beta q^k + \gamma \bar{q}^k
\]

with complex expansion coefficients \( \alpha \), \( \beta \), and \( \gamma \). The coefficients must satisfy the orthogonality constraints \( \alpha^2 + 2|\beta|\gamma = 0 \) and \( \alpha^2 + |\beta|^2 + |\gamma|^2 = 1 \). Now as \( B \) approaches a round sphere for large \( r \), these constraints enforce the fall-off conditions \( \alpha = O(r^{-1}), \beta = 1 + O(r^{-2}), \) and \( \gamma = O(r^{-2}) \). Consider the explicit expressions for the components of the \( \Sigma \) Riemann tensor with respect to \( \{ p, q, \bar{q} \} \) (all are either zero or \( \pm 1/\ell^2 \), and in particular \( R_{q\bar{q}q\bar{q}} = 1/\ell^2 \)), as well as the above orthogonality relationships between \( \alpha \), \( \beta \), and \( \gamma \). Using these, we find that

\[
R_{m\bar{m}m\bar{m}} = (|\beta|^4 + 2|\alpha|^2 + |\gamma|^4 - 2|\beta|^2|\gamma|^2)/\ell^2. \quad (17)
\]

Finally, with the above fall-off for \( \alpha \) and \( \beta \) the second orthogonality constraint implies that \( |\beta|^4 + 2|\alpha|^2 = 1 + O(r^{-4}) \), establishing \( (13) \).

### 2 Lightcone reference

In this section we derive the choice \( (3) \). We begin by rewriting the \( M \) line-element in terms of an outgoing null coordinate \( w := t - \ell \arctan(R/\ell) \), thereby obtaining

\[
ds^2 = -F(R)dw^2 - 2dw dR + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (18)
\]
for which constant-$w$ surfaces are outgoing null cones. Pick one such cone $N$ and embed $B$ in it. One can imagine a spherical cross section of $N$ which is “pushed up and down” the null generators of $N$ in an angle-dependent fashion. Under such a transformation the metric of the cross section undergoes a conformal transformation to become isometric with $B$. We then work with the chain of inclusions $B \subset N \subset M$, and further consider a three-slice $\Sigma$ which spans $B$. As $B$ is not perfectly round, $\Sigma$ is not a static slice of $M$. Further, in general $\Sigma$ will not be a moment of time symmetry, and hence will have a non-vanishing extrinsic curvature tensor $K_{ij}$. Projection into $B$ of the free indices of $K_{ij}$ defines an extrinsic curvature tensor $l_{ab}$ on $B$. Now, since $B$ is realized as a 2-surface in $M$, the embedding $B \subset M$ must satisfy the following constraint:

\[
(k)^2 - k_{ab}k_{ab} - (l)^2 + l_{ab}l_{ab} - R = 2R_{m\bar{m}m\bar{m}}, \tag{19}
\]

where $R_{m\bar{m}m\bar{m}} := R_{\mu\nu\lambda\sigma}m^\mu \bar{m}^\nu m^\lambda \bar{m}^\sigma$ denotes projection of the M Riemann tensor $R_{\mu\nu\lambda\sigma}$ into $B$ (again $m^\mu$ is the complex null dyad tangent to $B$). There are of course other constraint equations associated with the embedding $B \subset M$, but we need not consider them here.

To construct the lightcone reference, first note that the geodetic congruence $N$ is sheer-free, which means that the complex shear $\varsigma := k_{mm} + l_{mm}$ of $N$ vanishes. The vanishing of $\varsigma$ thus implies that the trace-free piece of $k_{ab}$ equals minus the trace-free piece of $l_{ab}$; whence Eq. (19) becomes

\[
(k)^2 - (l)^2 - 2R = 4R_{m\bar{m}m\bar{m}}. \tag{20}
\]

Now, by performing a simple coordinate transformation on the above line-element, one can quickly argue that

\[
R_{m\bar{m}m\bar{m}} = 1/\ell^2 \tag{21}
\]

even for our distorted surface $B$ (this is also the result were $B$ perfectly round). Of course, the projection (21) vanishes straightaway if $M$ is Minkowski spacetime. Next, we choose $\Sigma$ by the condition $l = 0$. This sets to zero the component of gravitational momentum normal to $B$, defining $\Sigma$ as the “rest frame” associated with $B$. With $l = 0$, we can immediately solve (20) for $k$ to get (3). Notice that the $\Sigma$ of this section would also be a static-time slice, were $B$ a round sphere (in $M$ static-time slices intersect lightcones in round spheres). Therefore, as $B$ tends to a round sphere in the $r \to \infty$ limit, it is not surprising that the static-time and lightcone references determine the same asymptotic energies.

### 3 Null infinity limit

In closing, let us return to case ii., the Trautman-Bondi-Sachs energy associated with null infinity for asymptotically flat spacetimes. In this case, as well as being a surface drawn in $\Sigma$, $B$ is also a cut of an outgoing null congruence $N \subset M$, and

\[^5\text{With } (\mu, \nu, \cdots) \text{ as } M \text{ indices.}\]
whether $r$ is an areal or affine radius we have an expansion for the $B$ curvature scalar which matches \((4)\). Then for either choice of zero-point we determine an expansion for $k|_{\text{ref}}$ of the form \((5)\). Therefore, \((6)\) has the correct form to give the Trautman-Bondi-Sachs energy in the limit that $B$ becomes a round two-sphere cut of null infinity.

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\[\text{This was also noted in} \ [12]\]