Geometric measures of redundancy and irrelevance tradeoff exponent to choose suitable delay times for continuous systems

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Abstract

Using the concept of the geometric measures of redundance and irrelevance tradeoff exponent (RITE), we present a new method to determine suitable delay times for continuous systems. After applying the RITE algorithm to both simulation and experimental observations, we find the results obtained are close to those obtained from the criterion of the average mutual information (AMI), while the RITE algorithm has the following advantages: simple implementation, reasonable computational cost and robust performance against observational noise.

Key words: Delay time, RITE

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1 INTRODUCTION

Since the embedding theorem of Takens [1] appeared, a number of papers have been published on criteria for estimating a suitable delay time for a nonlinear time series. One criterion, based on the second order autocorrelation (SOAC), chooses the time as the delay when the SOAC first becomes zero or drops to a certain fraction of its initial value [2]. This method is simple to implement, but it lacks a universal fraction for different systems to obtain suitable delay times 2. As a generalisation of the above idea, Albano et al.

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2 For example, the first zero criterion is successful for the Rössler system but it fails for the Lorenz system.

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[3] proposed a heuristic idea. They take the time at the consistent extrema of different higher-order autocorrelation functions as candidates for a suitable embedding window, therefore if we choose an embedding dimension, we also choose a delay time. As a further step, having noticed that the SOAC is actually a linear measure of dependence, Fraser & Swinney [5] introduced an important statistic, mutual information, based on information theory. Mutual information is a nonlinear measure of dependence between two data sets, for a scalar time series, we can use the average mutual information (AMI) to select a proper delay time. The criterion is to take the time at the first local minimum of the AMI as the desired delay time. Mutual information is a valuable concept, but it is rather complex to implement. In addition, it was found its performance was not very robust for small data sets [7].

From other viewpoints, some criteria were proposed based on the utilization of the geometric information of the reconstructed attractor in embedding space. Buzug and Pfister [8] devised the fill-factor algorithm to determine a suitable delay time by examining the attractor’s expansion in embedding space. It will be selected as the suitable delay time when the fill-factor is maximized. But this algorithm also takes into account the situation of "overfolding", and more seriously, it will fail to yield significant delays if the attractor has more than one unstable focus [8]. As a solution, Buzug and Pfister designed another algorithm, integral local deformation (ILD). This algorithm will choose a suitable delay time when the attractor’s local minimum deformation is achieved. Comparatively, this algorithm needs substantially more computational time than the fill-factor algorithm, and as we will indicate in the later section, it might be more suitable to use this algorithm to choose embedding window rather than delay time.

There have been many other criteria proposed. For example, Rosenstein [4] developed an approach named reconstruction signal strength resting on the concepts of redundancy error and irrelevance error, their approach is computationally efficient and can obtain a satisfactory performance, but the criterion for choosing suitable delay times is somewhat empirical. In this communication we do not intend to provide a detailed review, readers are invited to refer to the literature [9] and [10] and references therein for more details.

In the remaining sections, firstly we will propose a new algorithm to choose suitable delay times based on the concept of the geometric measures of redundancy and irrelevance tradeoff exponent (RITE). Then we will examine the performance of this algorithm by applying it to data sets from both simulation and experimental observations. Finally we have a summary.
In a very recent paper Cellucci and coworkers [11] state their viewpoint on embedding methods as: A circular logic has resulted in which embedding criteria are assessed by an adjudicating criterion which is itself an embedding criterion. Following this viewpoint, we learn that the best embedding criteria might differ under different adjudicating criteria. Hence we would like to elucidate that we do not seek the best embedding criteria for different adjudicating criteria, instead we intend to let our adjudicating criterion fit as many cases as possible.

As we have known, sufficiently high embedding dimension is a necessary but not sufficient condition to form an embedding reconstruction according to the embedding theorem of Takens. To be an embedding by itself will impose two constraints on the reconstruction mapping $\Psi : \mathbb{R} \to \mathbb{R}^m$, where $m$ is embedding dimension. One is that $\Psi$ shall be a one-to-one mapping, the other is that the derivative mapping $D \cdot \Psi$ shall also be one-to-one [6], where $D$ denotes the differentiating operator on $\Psi$.

In practice, although some delay times no longer lead to an embedding reconstruction (unlike the ideal situations), which will be discussed in the following content, it is hoped there are at least some others remaining. We note that, these remaining delay times equivalently lead to an embedding in the sense of characterizing the reconstructed attractor, although some particular values might indeed facilitate the analysis of a time series. Hence our adjudicating criterion is to guarantee the reconstruction mapping to be an embedding, and even if we obtain different delay times from different algorithms, we still consider them as suitable candidates for an embedding reconstruction.

For a delay time embedding reconstruction, a scalar time series $\{x_i : i = 1, 2, \ldots, N\}$ is used to construct vectors $\vec{X}_i = (x_i, x_{i+\tau}, \ldots, x_{i+(m-1)\tau})$ in $\mathbb{R}^m$, where $m$ is embedding dimension and $\tau$ is delay time. Now let us consider the effects of different delay times on the reconstructed attractor. Without losing generality, we confine our discussions to the two-dimensional embedding space $x_{i+\tau}$ vs. $x_i$. Fig. 1 demonstrates the reconstructed attractors of the Lorenz system [6] for three different delay times. When $\tau$ is too small, then $x_{i+\tau}$ will be very close to $x_i$ due to the continuity of the manifold. Therefore the pair points $(x_i, x_{i+\tau})$ will distribute around the unity line $x_{i+\tau} = x_i$ as indicated in Fig. 1 (a). But in practice, the presence of noise will let an embedding vector $\vec{X}_i = (x_i, x_{i+\tau}, \ldots, x_{i+(m-1)\tau})$ distributed as a "ball" rather than a point in metric space $\mathbb{R}^m$. The balls of adjacent vectors might intersect with each other, hence the reconstruction mapping $\Psi : \mathbb{R} \to \mathbb{R}^m$ is not one-to-one and no longer an embedding. When delay time $\tau$ is too large, say $\tau = 32$ as adopted in Fig. 1 (c), the reconstructed attractor is overfolded and does not
Fig. 1. Effects of different delay times on the reconstructed attractor of the Lorenz system in the two-dimensional embedding space. (a) delay time=2; (b) delay time=8; (c) delay time=32.

preserve the geometric structure of the original attractor comparing to those in panel (a) and (b), which also means the reconstruction is not an embedding.

From the viewpoint of information theory, delay time $\tau$ is too small means that $x_{i+\tau}$ contain mainly redundant information of $x_i$. This is called *redundance*. If delay time is too large, then for chaotic systems, $x_{i+\tau}$ will be irrelevant to $x_i$, hence $x_{i+\tau}$ contains no information of $x_i$. This is known as *irrelevance*. As Liebert and Schuster [12] have argued that, we shall consider not only the effect of redundance but also that of irrelevance in estimate of suitable delay times. Therefore a tradeoff shall be achieved between redundance and irrelevance so as to guarantee the reconstruction mapping to be an embedding. We define following statistic, namely *redundance and irrelevance tradeoff exponent (RITE)*, to measure the tradeoff,

\[
RITE = \frac{\rho(x_i, x_{i+\tau}) \langle x_i^2 \rangle + (1 - \rho(x_i, x_{i+\tau})) \langle x_i \rangle^2 }{ \langle x_i^2 \rangle + \langle x_i \rangle^2 } \tag{1}
\]

where $\langle \cdot \rangle$ denotes the expectation taken over time $i$ and

\[
\rho(x_i, x_{i+\tau}) = \frac{\text{cov}(x_i, x_{i+\tau})}{\text{var}(x_i)} = \frac{\langle x_i x_{i+\tau} \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2} \tag{2}
\]

where $\rho(x_i, x_{i+\tau})$ is the SOAC, $\text{cov}(x_i, x_{i+\tau})$ and $\text{var}(x_i)$ are the covariance with delay time $\tau$ and the variance of the time series $\{x_i\}$ respectively. After simplifications, we have

\[
RITE = \frac{\langle x_i x_{i+\tau} \rangle}{\langle x_i^2 \rangle + \langle x_i \rangle^2} \tag{3}
\]

As we shall see in the following content, Eqn. (3) is only a constant affine
transformation of the SOAC if directly applied to the original time series \( \{x_i\} \). Before that let us first interpret the meaning of Eqn. (1). We take \( \langle x_i^2 \rangle \) as the case of complete redundancy for the measure \( \langle x_i x_{i+\tau} \rangle \), when delay time \( \tau \) tends to zero and referring to \( x_{i+\tau} \) brings no more information of \( x_i \). Conversely, \( \langle x_i \rangle^2 \) is the case of complete irrelevance for the measure \( \langle x_i x_{i+\tau} \rangle \), when \( x_{i+\tau} \) is irrelevant and thus uncorrelated to \( x_i \), hence \( \langle x_i x_{i+\tau} \rangle \) is reduced to \( \langle x_i \rangle^2 \). The SOAC \( \rho(x_i, x_{i+\tau}) \) plays the role to measure the redundancy between \( x_{i+\tau} \) and \( x_i \) with a weight of \( \langle x_i^2 \rangle / (\langle x_i \rangle^2 + \langle x_i \rangle^2) \), while \( 1 - \rho(x_i, x_{i+\tau}) \) denotes the measure of irrelevance with the assigned weight of \( \langle x_i \rangle^2 / (\langle x_i \rangle^2 + \langle x_i \rangle^2) \). Starting from \( \tau = 0 \), as delay time \( \tau \) increases, the redundancy measure \( \rho(x_i, x_{i+\tau}) \) shall usually decrease while the irrelevance measure \( 1 - \rho(x_i, x_{i+\tau}) \) shall increase, hence a natural criterion is to choose the suitable delay time at the first local minimum of RITE, which guarantees the reconstruction to be an embedding in an optimal way according to Eqn. (1).

If directly applying the measure of RITE to measure the original scalar time series \( \{x_i\} \), we can find from Eqn. (1) it is a trivial measure with the same performance as that of the SOAC since \( \langle x_i^2 \rangle \) and \( \langle x_i \rangle^2 \) are both independent of delay time \( \tau \). A remedy is that, we can equivalently characterize the reconstructed attractor in the two-dimensional embedding space \( x_{i+\tau} \) vs. \( x_i \) instead of in the time domain.

Let \( (x_i, x_{i+\tau}) \) denote the vector from the origin to point \( (x_i, x_{i+\tau}) \) in the two dimensional embedding space, as shown in Fig. 2, we have the distance \( d_i \) of the pair points \( (x_i, x_{i+\tau}) \) to the identity line \( x_{i+\tau} = x_i \) expressed by:
Fig. 3. Figure in the left panel indicates the average integral local deformation vs. delay time for the time series from the Lorenz system with 9000 data points. The number of reference points is 500, radius for neighbour searching is set to 5. Embedding dimension \( m \) varies from 2 to 6 (from upper to lower) and delay time increases from 1 to 50. Figure in the right panel adopts the same parameters as the left for calculations except that the time series is shorter, consisting of only 1200 data points.

\[
d_i = \frac{1}{\sqrt{2}} |x_{i+\tau} - x_i| \tag{4}
\]

where \(|\cdot|\) denotes the distance in Euclidean space. The projection length \( p_i \) of vector \((x_i, x_{i+\tau})\) onto the identity line is:

\[
p_i = \frac{1}{\sqrt{2}} |x_{i+\tau} + x_i| \tag{5}
\]

Therefore the angle between vector \((x_i, x_{i+\tau})\) and the identity line is:

\[
\theta_i = \tan^{-1} \left( \frac{|x_{i+\tau} - x_i|}{x_{i+\tau} + x_i} \right) \tag{6}
\]

From Eqn. (4), (5) and (6), we obtain three new time series \( \{d_i\}, \{p_i\} \) and \( \{\theta_i\} \) derived from the original one which consist of geometric description variables of the reconstructed attractor in the two-dimensional embedding space. These geometric variables shall also be continuous in the time domain since all of the three above transforms are continuous. We apply the measure of \( RITE \) to these geometric variables with the same criterion to choose suitable delay times as having stated above, i.e., a suitable delay time will be chosen at the first local minimum of the geometric measures of \( RITE \).
3 NUMERICAL RESULTS

We note that if the origin in the embedding space of a time series is marginal to or even outside of the reconstructed attractor, the sensitivity of the geometric measures of RITE to different delay times will be significantly reduced. We therefore conduct the following smooth affine transform on the original time series \( \{x_i\} \).

\[
y_i = \frac{x_i - \langle x_i \rangle}{\sqrt{\text{var}(x_i)}}
\]

The new time series \( \{y_i\} \) shall have the same dynamical properties in the time domain as the original time series has, while it takes the origin of embedding space as the "center" of its reconstructed attractor in a statistical sense. With this consideration, we prefer to studying the time series \( \{y_i\} \) rather than \( \{x_i\} \). In addition, we will discard the scale factor \( 1/\sqrt{2} \) of both Eqn. (4) and (5) in all of our calculations without affecting the results.

We will study the simulation data sets from the Lorenz and Rössler systems [6]. For the Lorenz system, the equations are:

\[
\begin{align*}
\dot{x}(t) &= \sigma(y(t) - x(t)) \\
\dot{y}(t) &= rx(t) - y(t) - x(t)z(t) \\
\dot{z}(t) &= x(t)y(t) - bz(t)
\end{align*}
\]

with parameters \( \sigma = 10, r = 28, c = 8/3 \) and the sampling time \( \Delta t_s = 0.02s \).

For the Rössler system, the equations are:

\[
\begin{align*}
\dot{x}(t) &= -y(t) - z(t) \\
\dot{y}(t) &= x(t) + ay(t) \\
\dot{z}(t) &= b + z(t)(x(t) - c)
\end{align*}
\]

with parameters \( a = 0.15, b = 0.20, c = 10.00 \) and the sampling time \( \Delta t_s = 0.1s \).

We will also apply the geometric measures of RITE to the sunspot record from year 1700 to year 1987 and infant respiratory data during stage 4 sleep (S4) [13]. In addition, we will calculate delay times chosen by the ILD and AMI algorithms for the comparison purpose. Our results are listed in Table 1.

Although the ILD algorithm was originally designed to determine suitable delay times \( \tau \), it might be more appropriate to utilize it in establishing embedding
Table 1  
Delay times chosen by the algorithms of ILD, AMI and the geometric measures of RITE.

| Data set  | ILD  | AMI  | Geometric measures of RITE |
|-----------|------|------|---------------------------|
|           | τ/m  |      | distance | projection | angle |
| Lorenz    | 9/4  | 8    | 9        | 12         | 10    |
| Rössler   | 10/3 | 16   | 16       | 14         | 13    |
| Sunspot   | 2    | 3    | 2        | 3          | 2     |
| S4        | 5/5  | 8    | 7        | 8          | 5     |

window \( m \cdot \tau \). As indicated in the left panel of Fig. 3, when using Eqn. (24) in Ref. [8] for calculations, for the Lorenz system the products of embedding dimensions \( m \) (\( m > \)correlation dimension \( d_c \)) and their corresponding delay times \( \tau \) at the first local minimum of the average ILD are nearly a constant of 36. This conclusion also holds for data sets of the Rössler system and S4. In contrast, the sunspot record has consistent local minima and the products of \( m \cdot \tau \) do not keep constant. This still does not contradict our conclusion as the sunspot record is an extremely short time series. As shown in the right panel of Fig. 3, the constant embedding window will vanish when the time series from the Lorenz system is shorter, instead a consistent local minimum appears at \( \tau = 8 \).

Since embedding window \( m \cdot \tau \) remains constant, different delay times will be obtained from the ILD algorithm for different embedding dimensions, nevertheless, we think the ILD algorithm can still indicate how to obtain a proper embedding reconstruction with sufficiently high embedding dimension. Firstly we need to choose an optimal embedding dimension for each data set (except for the sunspot record) under the criterion of global False Nearest Neighbours (GFNN) [15] [11], then we can obtain the corresponding delay time according to embedding window \(^3\). For the sunspot record we choose delay time at the first consistent local minimum of the average ILD. The results are indicated in Table 1.

From Table 1 we find that, loosely speaking, the results of different algorithms are close to each other. As we have stated in the previous section, although delay times chosen by different geometric measures of RITE and the other two algorithms are usually different, we still take all of them as the suitable candidates for an embedding reconstruction.

\(^3\) It has to admit it is somewhat "circular" in this situation, since the choice of the optimal embedding dimension by GFNN algorithm in turn needs to take the suitable delay time as a parameter. In our calculation, we use the suitable delay time obtained by the AMI algorithm in Table 1 as the parameter to determine the optimal embedding dimension for each data set (except for the sunspot record).
Fig. 4. Nonlinear prediction error of local constant model v.s. delay time. Embedding dimensions used in the model are 4, 3, 3 and 5 for the Lorenz system, the Rössler system, the sunspot record and data set of S4 respectively. The ranges of delay time are all from 1 to 60. The NLPEs corresponding to delay times in Table 1 chosen by different algorithms for each data set are marked with diamonds. We use the program zeroth in TISEAN package [14] for our calculations.

We use the nonlinear prediction error (NLPE) to verify the reconstruction quality of our choice. As we have known, local constant model [16] utilises nearest neighbours for nonlinear prediction, when sufficiently high embedding dimension is reached, most of the effect of false nearest neighbours will be excluded. When embedding dimension and the radius of neighbour searching are fixed, the NLPE will only depend on delay time. Hence the NLPE can qualitatively determine whether our choice for an embedding reconstruction is acceptable, as the prediction error of a suitable delay time shall achieve a tradeoff between being too small and being too large if the time series is not completely predictable or completely unpredictable. In Fig. 4, the NLPEs corresponding to delay times listed in Table 1 chosen by different algorithms for each data set are marked with diamonds. As we can find, certain tradeoff for each geometric measure of RITE is indeed achieved.

Now let us examine the computational cost of each algorithm listed in Table 1. Let \( N \) denotes the data set size of time series \( \{x_i\} \), then the ILD algorithm approximately requires \( O(N_{ref} \times (N \ln N)) \) unit operations on searching nearest neighbours for each embedding dimension and each delay time, where \( N_{ref} \) is the number of reference points. The AMI algorithm needs about \( O(N^2) \) unit operations to calculate joint probability distribution for each delay time, while the RITE algorithm will be faster than both of them, undergoing about
Table 2
Delay times chosen by the geometric measures of RITE for the time series from the Lorenz and Rössler systems contaminated with observational Gaussian white noises.

| Noise level (%) | Lorenz system | Rössler system |
|----------------|--------------|---------------|
|                | distance     | projection    | angle | distance | projection | angle |
| 0              | 9            | 12            | 10    | 16       | 14         | 13    |
| 3              | 9            | 12            | 10    | 16       | 14         | 13    |
| 6              | 9            | 12            | 9     | 16       | 14         | 13    |
| 9              | 9            | 12            | 10    | 16       | 14         | 13    |
| 12             | 9            | 12            | 8     | 16       | 14         | 2     |

$O(N)$ unit operations on both the transforms over the original data set and the calculations of expectation for each delay time.

We will also test the robustness of the geometric measures of RITE against observational Gaussian white noise $N(0, \delta^2)$. Noise level is defined as the ratio of $\delta$ to $\delta_s$, where $\delta_s$ is the standard deviation of the original scalar time series $\{x_i\}$ before the transform of Eqn. (7). As indicated in Table 2, using delay times chosen at noise level zero as the references, we find both the distance and the projection measures of RITE are rather robust against observational noise, noise level up to 12% still can not affect the choices of delay time. As expected, the angle measure of RITE is more sensitive to noise. For the Lorenz system, small fluctuations of the choice appear when noise level is higher than 6%. For the Rössler system, the performance seems better. The odd choice $\tau = 2$ at noise level 12% follows our criterion suggested above, which is due to a small spike on the curve of the angle measure of RITE vs. delay time, while the next local minimum is exactly at delay time $\tau = 13$. Although the robustness against observational noise of the geometric measures of RITE might vary from system to system, we believe in general it is satisfactory.

4 CONCLUSION

It has been a difficult problem to set up a universal criterion for the choice of delay time. Average mutual information is the most preferred statistic used for choosing delay time since it has a valuable physical meaning, but it requires a complicated implementation algorithm. To achieve higher accuracy, more complex implementation and more running time are needed. Also it does not deal very well with short time series. Comparatively, the RITE algorithm intends to provide an optimal choice of delay time with the objective to guarantee the reconstruction to be an embedding. Our calculates indicate that the
RITE algorithm performs well on a variety of time series of various lengths and even in the presence of substantial noise. We therefore feel that such a simple algorithm should be preferred to the more complex implementation suggested previously.

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