Robust optimal estimation of the mean function from discretely sampled functional data

Ioannis Kalogridis and Stefan Van Aelst

Department of Mathematics, KU Leuven, Belgium

August 4, 2020

Abstract

Estimating the mean of functional data is a central problem in functional data analysis, yet most current estimation procedures either unrealistically assume completely observed trajectories or lack robustness with respect to the many kinds of anomalies one can encounter in the functional setting. To remedy these deficiencies we introduce the first optimal class of robust estimators for the estimation of the mean from discretely sampled functional data. The proposed method is based on M-type smoothing spline estimation with repeated measurements and is suitable for densely observed trajectories as well as for sparsely observed trajectories that are subject to measurement error. Our analysis clearly delineates the role of the sampling frequency in the determination of the asymptotic properties of the M-type smoothing spline estimators: for commonly observed trajectories, the sampling frequency dominates the error when it is small but ceases to be important when it is large. On the other hand, for independently observed trajectories the sampling frequency plays a more limited role as the asymptotic error is jointly determined by the sampling frequency and the sample size. We illustrate the excellent performance of the proposed family of estimators relative to existing methods in a Monte-Carlo study and a real-data example that contains outlying observations.

Keywords: Functional data, sparse functional data, M-estimators, smoothing splines, reproducing kernel Hilbert spaces.

1 Introduction

In recent years, technological innovations and improved storage capabilities have led practitioners to observe and record increasingly complex high-dimensional data that are characterized by an underlying functional structure. Such data are nowadays commonly referred to as functional data and relevant research has been enjoying considerable popularity, following works such as Ramsay (1982), Ramsay and Dalzell (1991); Rice and Silverman (1991) and Ramsay and Silverman (2005). Despite the fact that nowadays the field of
functional data analysis has become very broad with many specialized subpaths [see, e.g., Ferraty and Vieu (2006); Horváth and Kokoszka (2012); Kokoszka and Reimherr (2017)], special attention is still reserved for inferences regarding the mean function $\mu$ of a random function $X$, which is viewed as an element of $(\Omega, \mathcal{A}, \mathbb{P})$ with sample paths in some nicely-behaved function space.

In the early days of functional data analysis, it was commonly assumed that a sample of fully observed curves $X_1, \ldots, X_n$ was readily available to practitioners with the implication that mean estimation could be optimally performed through the use of the sample mean of the curves, see, for example, (Horváth and Kokoszka, 2012, Chapter 2) and Hsing and Eubank (2015, Chapter 8.1). More recently, emphasis has been placed on the more realistic setting of discretely observed curves, possibly distorted by additive random noise. One of the first contributions in that direction was made by Yao et al. (2005), who proposed local linear estimation with repeated measurements and obtained uniform rates of convergence for the case of sparsely observed trajectories. Degras (2008) derived consistency conditions for a broad family of weighted linear estimators and anticipated later developments by remarking that the rate of convergence with respect to the $L_2$ norm can at most be of order $1/n$ for densely observed data. The delicate interplay between sample size and sampling frequency was further illustrated by Li and Hsing (2010), who showed that the uniform rate of convergence is in between the parametric and non-parametric rates, the exact rate depending on the sampling frequency. A major breakthrough was finally obtained by Cai and Yuan (2011), who established minimax rates of convergence and showed that these optimal rates can be achieved by a least-squares smoothing spline estimator with repeated measurements.

A common drawback for the all above estimation procedures is their lack of resistance towards outlying observations, which comes as a consequence of their reliance on the minimization of $L_2$ distances. Resistant estimation of central tendency in functional data is a problem that has received significantly less attention in the literature and what is more, to the best of our knowledge, all available estimators require completely observed trajectories. Cuesta-Albertos and Fraiman (2006) proposed a functional equivalent of the trimmed mean estimator and showed its consistency in the ideal setting of fully observed trajectories. The robust spatial median has received the lion’s share of attention in the literature. Gervini (2008) proved its root-n convergence rate for fully observed finite-dimensional trajectories. These convergence results were greatly strengthened by Cardot et al. (2013) to cover separable Hilbert spaces of functions. An even deeper treatment of the spatial median was offered by Chakraborty and Chaudhuri (2014), who established general Glivenko-Cantelli and Donsker-type results for the empirical spatial distribution process in general Banach spaces. More recently, Sinova et al. (2018) proposed a broad family of M-estimators for functional location and derived its consistency, again under the finite-dimensional assumption of the complete trajectories.

Practitioners who seek robust estimates often overcome the discreteness of the data and the measurement error either by a pre-smoothing step or by altogether ignoring these problematic aspects of the data and then applying any of the aforementioned robust estimators. Although due to the lack of a better alternative these are popular strategies, it should be
stressed that their theoretical side-effects are not well-understood and consequently it is impossible to decide on best practices. The natural question that then arises is whether it is possible to construct theoretically-optimal robust estimators for the functional mean that can operate in the ubiquitous functional data setting of discretely observed curves and measurement errors.

We show in this chapter that the answer to this query is in the affirmative by considerably extending the theoretical results of Cai and Yuan (2011) to cover a large class of estimators, which includes, for example, the resistant quantile and Huber-type smoothing splines. Furthermore, our theoretical development does not require identical distributions for the measurement errors or any moments, thereby allowing for a considerable relaxation of the conditions of Cai and Yuan (2011). In practice, such estimators may be efficiently computed with the convenient B-spline representation and well-established iterative algorithms, so that the associated computational burden is minimal, even if the trajectories are densely observed.

2 The family of M-type smoothing spline estimators

Throughout this section we shall assume that $X(\cdot)$ is a second-order process on $[0, 1]$ with mean function $\mu$. Let $X_1, \ldots, X_n$ denote $n$ independent and identically distributed copies of $X$. Our goal is to recover the mean function $\mu(\cdot) := \mathbb{E}\{X(\cdot)\}$ from noisy observations of the discretized curves:

$$Y_{ij} = X_i(T_{ij}) + \zeta_{ij}, \quad j = 1, \ldots, m_i \quad \text{and} \quad i = 1, \ldots, n,$$

where $T_{ij}$ are sampling points and $\zeta_{ij}$ are random noise variables that were assumed by Yao et al. (2005); Cai and Yuan (2011) to be independent of the $X_i$ and iid with zero mean and finite variance. In what follows it will be convenient to view the errors $\zeta_{ij}$ as part of $n$ independent copies of the process $\zeta : (\Omega, \mathcal{A}, P) \times [0, 1] \to \mathbb{R}$, that is, $\zeta_{ij} = \zeta_i(T_{ij})$. Expressing model (1) in terms of mean-deviations, we may equivalently write

$$Y_{ij} = \mu(T_{ij}) + \epsilon_i(T_{ij}), \quad j = 1, \ldots, m_i \quad \text{and} \quad i = 1, \ldots, n,$$

where $\epsilon_i(T_{ij}) = X_i(T_{ij}) - \mu(T_{ij}) + \zeta_i(T_{ij})$ denotes the error process associated with the $i$th response. The problem thus becomes one of regression with repeated measurements and correlated errors.

There is a plethora of parametric and non-parametric estimators that may be potentially up for the problem at hand, but smoothing spline estimation seems to arise naturally after one imposes the smoothness of the sample paths $t \mapsto X(t, \omega)$. Let $W^{r,2}([0, 1])$ denote the Hilbert-Sobolev space of order $r$, that is,

$$W^{r,2}([0, 1]) = \{ f : [0, 1] \to \mathbb{R}, f \text{ has } r-1 \text{ absolutely continuous derivatives} \}
$$

$$f^{(1)}, \ldots, f^{(r-1)} \quad \text{and} \quad \int_0^1 |f^{(r)}(x)|^2 dx < \infty\}.$$

3
This separable Hilbert space be understood as the completion of the space of \( r \)-times continuously differentiable functions under a suitable norm, just as the \( L^p([0, 1]) \) spaces are the completion of \( C([0, 1]) \) spaces under their norm. See Adams and Fournier (2003) for more details. Let us assume that the sample paths of \( X \) belong to \( W^{r, 2}([0, 1]) \) almost surely for some \( r \geq 1 \), viz,

\[
\mathbb{E}\left\{ \int_0^1 |X^{(r)}(t)|^2 dt \right\} < \infty.
\]

In general, \( r \) is determined a priori by the practitioner and reflects how smooth she anticipates the sample paths to be. Conditions ensuring this regularity of the sample paths are given by Scheuerer (2010) and are of particularly simple form for weakly stationary processes. Let

\[
K(s, t) := \mathbb{E}\{ (X(s) - \mu(s))(X(t) - \mu(t)) \},
\]

denote the covariance function of the process. It is not difficult to see that for weakly stationary processes \( K(s, t) = K(t - s, 0) := \Phi(t - s) \), since the covariance function is symmetric and only depends on \( |t - s| \). Corollary 1 of Scheuerer (2010) shows that \( t \mapsto X(t, \omega) \in W^{r, 2}([0, 1]) \) almost surely, provided that \( \Phi^{(2r)}(0) \) exists. That is, the covariance function is \( 2r \) times differentiable at the origin. For non-stationary processes, the conditions are more involved, although they similarly require smoothness of the covariance function. The interested reader is referred to Scheuerer (2010) for more general results.

Assuming now that the sample paths take values in the Hilbert-Sobolev space of order \( r \), the convexity of the semi-norm \( f \mapsto \| f^{(r)} \|_2 \) and Jensen’s inequality imply that

\[
\int_0^1 |\mu^{(r)}(t)|^2 dt \leq \mathbb{E}\left\{ \int_0^1 |X^{(r)}(t)|^2 dt \right\} < \infty,
\]

so that the mean function also belongs to the Hilbert-Sobolev space of order \( r \). An intuitively appealing robust estimator for \( \mu \) may therefore be obtained by solving

\[
\min_{f \in W^{r, 2}([0, 1])} \left\{ \frac{1}{n} \sum_{i=1}^n \frac{1}{m_i} \sum_{j=1}^{m_i} \rho(Y_{ij} - f(T_{ij})) + \lambda \int_0^1 |f^{(r)}(t)|^2 dt \right\},
\]

for some convex nonnegative loss function \( \rho \) satisfying \( \rho(0) = 0 \) and a penalty parameter \( \lambda > 0 \), whose positivity is needed to make the problem well-defined. The two extreme cases \( \lambda \downarrow 0 \) and \( \lambda \to \infty \) illustrate the compromise smoothing spline estimators seek to achieve. In the former case, the solution will become arbitrarily close to an natural interpolating spline to the \( Y_{ij} \) of order \( 2r \). On the other hand, for \( \lambda \to \infty \), the dominance of the penalty term in (3) will force \( \int_0^1 |\hat{\mu}_n^{(r)}(t)|^2 dt = 0 \), where \( \hat{\mu}_n \) denotes a solution of (3). A Taylor expansion with integral remainder of \( \hat{\mu}_n \) about 0 shows that

\[
\hat{\mu}_n(t) = P_r(t) + \int_0^1 \frac{\hat{\mu}_n^{(r)}(x)}{(r - 1)!} (t - x)^{r-1} dx,
\]
where $P_r(t)$ is the Taylor polynomial of order $r$. But by the Schwarz inequality,
\[
\left| \int_0^1 \hat{\mu}_n^{(r)}(x) (t-x)^{r-1} dx \right| \leq \frac{|t|^{r-1/2}}{(r-1)!(2r-1)^{1/2}} \left\{ \int_0^1 |\hat{\mu}_n^{(r)}(x)|^2 dx \right\}^{1/2},
\]
so that the integral term will be equal to zero. This implies that for large $\lambda$, $\hat{\mu}_n$ will be roughly equal to its Taylor polynomial of degree at most $(r-1)$. For in between values of $\lambda$, as Eubank (1999) notes, the penalty functional may be viewed as providing a bound on how far $\mu$ is allowed to depart from a polynomial.

While for general convex $\rho$-functions it cannot be assumed that the solution is unique (Eggermont and LaRiccia, 2009, Chapter 17), for $\lambda > 0$ and $\sum_{i=1}^n m_i \geq r$ it can be shown that a solution to the above variational problem may be found in the space of $2r$th order natural splines with knots at the unique $T_{ij}$ ($i = 1, \ldots, n$, $j = 1, \ldots, m_i$), see Theorem 1 of Kalogridis (2020). Since this is a finite-dimensional space, with dimension equal to the number of unique knots, a solution to (3) may be expediently found by solving a ridge-type regression problem; Section 4 provides the details.

The proposed estimator is a generalization of the robust smoothing spline introduced by Huber (1979) for the case of independent and identically distributed errors. It is clear that the square loss $\rho(x) = x^2$ fulfils the requirements on $\rho$ and one obtains the least-squares smoothing spline with repeated measurements proposed by Rice and Silverman (1991) and theoretically investigated by Cai and Yuan (2011). However, the benefit of the above formulation is that it includes loss functions that increase less rapidly as their argument becomes larger in absolute value, so that some resistance towards outlying observations is achieved. A far from exhaustive list of examples includes $L^q$ estimates with $\rho_q(x) = |x|^q, 1 \leq q < 2$, quantile estimates with $\rho_\alpha(x) = |x| + (2\alpha - 1)x, 0 < \alpha < 1$ and Huber’s function given by
\[
\rho_k(x) = \begin{cases} x^2, & |x| \leq k \\ 2k(|x| - k/2), & |x| > k, \end{cases}
\]
for some $k > 0$ that controls the blending of square and absolute losses. The resistant least-absolute deviations estimator (LAD) may be obtained with $q = 1$ or $\alpha = 1/2$ or may be smoothly approximated with the Huber estimator as $k \downarrow 0$.

If $\psi = \rho'$ exists everywhere, a Gateaux differential argument shows that the minimizer $\hat{\mu}_n$ must satisfy
\[
- \frac{1}{n} \sum_{i=1}^n \frac{1}{m_i} \sum_{j=1}^{m_i} \psi(Y_{ij} - \hat{\mu}_n(T_{ij})) g(T_{ij}) + 2\lambda \int_0^1 \hat{\mu}_n^{(r)}(t) g^{(r)}(t) dt = 0,
\]
for all $g \in \mathcal{W}^{r,2}([0,1])$. If we restrict attention to the space of $2r$th order natural splines with knots at the unique $T_{ij}$, then the above only needs to hold for $g$ in that space. In general, however, for $\rho'$ that have jumps, such as sign$(x)$, the right-hand side of (4) will not be exactly equal to zero so that a different approach is required in order to identify the minimizer, see, for example, (Eggermont and LaRiccia, 2009, Chapter 19).
3 Asymptotic properties

Following Cai and Yuan (2011) we shall examine the asymptotic properties of M-type smoothing spline estimators in two very different scenarios: the common design case and the independent design case. In the case of common design, the curves are recorded at the same locations, that is,

\[ T_{1j} = T_{2j} = \ldots = T_{nj}, \quad j = 1, \ldots, m, \]

while in the case of independent design we shall allow \( T_{ij} \) to depend on \( i \), provided that these locations cover the interval \([0, 1]\), in a sense that will be made more precise shortly.

In both cases, letting \( \|f\|_2 = (\int_{[0, 1]} |f(x)|^2 dx)^{1/2} \) for \( f \in L^2([0, 1]) \) denote the standard \( L^2([0, 1]) \) norm, our aim is to establish convergence rates with respect to the norm \( \|f\|_{r,\lambda} = (\|f\|_2^2 + \lambda \|f^{(r)}\|_2^2)^{1/2} \), for \( f \in W^r,2([0, 1]) \subset L^2([0, 1]) \). It is not difficult to see that \( \| \cdot \|_{r,\lambda} \) is equivalent to the frequently used Sobolev norm \( \| \cdot \|_{r,1} \), albeit not uniformly in \( \lambda \). The benefit of considering convergence with respect to \( \| \cdot \|_{r,\lambda} \) is that we will be able to describe the convergence rates of the derivatives of \( \hat{\mu}_n \), thereby extending the results of Cai and Yuan (2011) also in this direction. Comparing our assumptions with those of Cai and Yuan (2011) reveals that the price that we pay for these additional results comes in the form of slightly less flexibility in the rate of decay of the penalty parameter in the common design and more stringent conditions on the design points \( T_{ij} \) in the independent design.

Starting with the common design, notice that the M-type estimator \( \hat{\mu}_n \) may now be defined as the solution of

\[
\min_{f \in W^r,2([0, 1])} \left\{ \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(Y_{ij} - f(T_j)) + \lambda \int_{0}^{1} |f^{(r)}(t)|^2 dt \right\}. \tag{5}
\]

The assumptions that will be required for our theoretical development are as follows.

(A1) \( \rho \) is a convex function on \( \mathbb{R} \) with right and left derivatives \( \psi_+ \) and \( \psi_- \) and \( \psi \) is any choice of the subgradient of \( \rho \).

(A2) The error processes \( \epsilon_i(\omega, t) : (\Omega, \mathcal{A}, P) \times [0, 1] \to \mathbb{R}, i = 1, \ldots, n \) are independent and identically distributed.

(A3) There exist constants \( \kappa \) and \( M_1 \) such that for all \( x \in \mathbb{R} \) and \( |y| < \kappa \),

\[ |\psi(x + y) - \psi(x)| \leq M_1 \]

(A4) There exists a constant \( M_2 \) such that

\[ \sup_j \mathbb{E} \{ \psi(\epsilon_{1j} + u) - \psi(\epsilon_{1j}) \}^2 \leq M_2 |u|, \]

as \( u \to 0 \).
\((A5)\) \(\mathbb{E}\{\psi(\epsilon_{1j})\} = 0, \sup_j \mathbb{E}\{|\psi(\epsilon_{1j})|^2\} \leq c_0\) for some \(c_0 > 0\) and there exist positive constants \(\delta_j, j = 1, \ldots, m\), such that \(0 < \inf_j \delta_j \leq \sup_j \delta_j < \infty\) and

\[
\mathbb{E}\{\psi(\epsilon_{1j} + u)\} = \delta_j u + o(u), \quad \text{as} \quad u \to 0.
\]

\((A6)\) The family of discretization points \(T_j\) satisfies \(T_0 := 0 < T_1 < T_2 < \ldots, < T_m < T_{m+1} := 1\) and

\[
\max_{0 \leq j \leq m} |T_{j+1} - T_j| \leq cm^{-1}
\]

for some constant \(c > 0\).

Assumption \((A1)\) is standard in regression M-estimation, see, e.g., Wu (2007); Li et al. (2011). It ensures that (3) has a solution in \(W^{1,2}([0, 1])\) and that all solutions will asymptotically be contained in the same ball of \(W^{1,2}([0, 1])\), so that one need not worry about ”bad” solutions in the limit. Assumption \((A2)\) specifies the permissible error structure. It is a substantially weaker than the assumption of \((Cai and Yuan, 2011)\), for we do not require the independence of the measurements errors associated with the \(i\)the response: \(\zeta_{ij}, j = 1, \ldots, m\).

Assumptions \((A3)\)– \((A5)\) permit the use of discontinuous score functions by imposing instead some regularity on the error process and its finite-dimensional distributions. It is clear that they are satisfied for smooth \(\psi\)-functions. However, as \((Bai and Wu, 1994)\) show that \((A4)\) may hold with the tighter \(|u|^2\) on the right hand side even if \(\psi\) has jumps. The first part of \((A5)\) ensures the Fisher-consistency of the estimates. For symmetric \(\rho\) functions, it is automatically satisfied if each \(\epsilon_{1j}, j = 1, \ldots, m\) possesses a symmetric distribution. The second moment condition is satisfied for bounded score functions without any additional assumptions on the error, thereby showing that one can do away with the second moment assumption on the error employed by Cai and Yuan (2011). The final part of \((A5)\) implies that the functions \(m_j(t) := \mathbb{E}\{\psi(\epsilon_{1j} + u)\}\) have strictly positive derivatives at \(u = 0\). This is necessary for the minimum to be well-separated in the limit. Similarly, it is not a stringent assumption and can be shown to hold for several interesting \(\rho\)-functions. We now list a few interesting examples to illustrate this point.

Example 1 (Least squares). Consider the linear score function \(\psi(x) = 2x\) corresponding to the least-squares smoothing spline estimator. Then \((A6)\) holds with \(\delta_j = 2, j = 1, \ldots, m\) provided that \(\mathbb{E}\{\epsilon_{1j}\} = 0, \mathbb{E}\{|\epsilon_{1j}|^2\} = \sigma^2, j = 1, \ldots, m\), as in Cai and Yuan (2011).

Example 2 (LAD and quantile regression). Consider M-estimation with \(\rho(x) = |x|\). Then provided that each \(\epsilon_{1j}, j = 1, \ldots, m\) has a distribution function \(F_j\) symmetric about zero and a positive density \(f_j\) on an interval about zero,

\[
\mathbb{E}\{\text{sign}(\epsilon_{1j} + t)\} = 2f_j(0)t + o(t), \quad \text{as} \quad t \to 0,
\]

so that \((A6)\) holds with \(\delta_j = 2f_j(0)\) provided that the median of each \(\epsilon_{1j}, j = 1, \ldots, m\) is zero and all marginal densities are bounded away from zero on an interval about zero. This easily generalizes to M-estimation with \(\rho_\alpha(x) = |x| + (2\alpha - 1)x\), provided that in this case one views the regression function as the \(\alpha\)-quantile function, that is, \(\Pr(Y_{ij} \leq \mu(T_j)) = \alpha\).
Example 3 (Huber). Here, for \( k > 0 \), \( \psi_c(x) = 2xI(|x| \leq c) + 2c\text{sign}(x)I(|x| > c) \) and we may assume that each \( F_j \) is absolutely continuous and symmetric about zero so that

\[
\mathbb{E}\{\psi_k(\epsilon_{1j} + t)\} = (2F_j(k) - 1) t + o(t), \quad \text{as} \quad t \to 0.
\]

It is clear that (A6) holds in this case with \( \delta_j = 2F_j(k) - 1 \) provided that \( \inf_j F_j(k) > 1/2 \).

Example 4 (Smooth \( \psi \)-functions). All monotone everywhere differentiable \( \psi \) functions with bounded second derivative \( \psi'' \), such as \( \rho(x) = \log(\cosh(x)) \) or \( \rho(x) = x^2 \log(1 + e^{-x}) - 2 \log(2) \), satisfy (A6) if \( 0 < \inf_j \mathbb{E}\{\psi'(\epsilon_{1j})\} \leq \sup_j \mathbb{E}\{\psi'(\epsilon_{1j})\} < \infty \). This is a straightforward generalization of the classical Fisher-consistency condition for smooth \( \psi \) functions, see (Cox, 1983; Cunningham et al., 1991).

Finally, condition (A6) also exists in (Cai and Yuan, 2011) and ensures that the sampling points are unique and are observed at a sufficiently regular grid. It is similarly a weak assumption that can be shown to hold, for example, if \( T_j = 2j/(2m + 2) \), \( j = 1, \ldots, m \) and many other designs.

With these assumptions we may now state our first main result. To ensure consistency of the estimators we shall assume that the discretization points become more numerous as \( n \to \infty \). In other words, we will require that \( m \to \infty \) as a function of \( n \). This is a natural requirement in order to examine consistency in norms that involve integrals over finite domains.

**Theorem 1.** Assume (A1)-(A6) and further that as \( n \to \infty \), \( m(n) \) and \( \lambda(n) \) vary in such a way that \( m(n) \to \infty \), \( \lambda(n) \to 0 \) and

\[
\lim_{n \to \infty} n^{(r-1)\lambda^{3/2}} = \lim_{n \to \infty} nm\lambda^{1/r} = \infty \quad \text{and} \quad \liminf_{n \to \infty} m\lambda^{1/2r} \geq 2c_r,
\]

where \( c_r \) is the constant of Lemma 2. Then all sequences of \( M \)-type smoothing splines \( \widehat{\mu}_n \) satisfy

\[
||\widehat{\mu}_n - \mu||_{r, \lambda}^2 = O_P\left(n^{-1} + m^{-2r} + \lambda\right).
\]

Furthermore, if \( \psi(x) \) is Lipschitz-continuous the condition \( \lim_n n^{r-1}\lambda^{3/2} = \infty \) may be replaced by \( \lim_n n^{(2r-1)/2} \lambda = \infty \).

The result should be contrasted with the very different asymptotic error decomposition of \( M \)-type smoothing splines in classical nonparametric regression (Cox, 1983; Kalogridis, 2020). Effectively, the only similarity is the role of \( \lambda \), which needs to tend to zero in order for the estimation bias to become negligible. The manner, however, in which \( \lambda \) needs to tend to zero varies depending on the smoothness of the score function \( \psi(x) \). If \( \psi(x) \) is discontinuous, as in the case of quantile estimation, taking \( \lambda \sim (2c_r)^{2r}(m^{-2r} + n^{-1}) \) ensures that the required limit conditions are satisfied for \( r > 2 \). On the other hand, with the same choice of \( \lambda \) the limit conditions for smoother score functions are satisfied even for \( r > 1 \). Thus, smoother score functions tend to allow for greater flexibility in the choice of \( r \) and the smoothing parameter \( \lambda \).

Assuming now that \( \lambda \sim (2c_r)^{2r}(m^{-2r} + n^{-1}) \) and the limit conditions are satisfied, we are led to

\[
||\widehat{\mu}_n - \mu||^2 \leq ||\widehat{\mu}_n - \mu||^2_{r, \lambda} = O_P\left(n^{-1} + m^{-2r}\right),
\]

8
and these are the optimal rates of convergence under the common design (Cai and Yuan, 2011). It is very interesting to observe that a phase transition between the two sources of error occurs at $m \asymp n^{1/2r}$. Indeed, if $m \gg n^{1/2r}$ then the asymptotic error behaves like $n^{-1}$, which is the rate of convergence for many functional location statistics that assume the curves are observed in their entirety, see (Horváth and Kokoszka, 2012) and (Gervini, 2008) for the functional mean and median respectively. On the other hand, if $m = O(n^{1/2r})$ then the asymptotic error behaves like $m^{-2r}$, which is the error associated with piecewise polynomial interpolation of a $W^{r,2}([0,1])$-function (DeVore and Lorentz, 1993). The results confirm the intuitive notion that as long as the grid is dense, the error will decay with the parametric rate $n^{-1}$. However, Theorem 1 also shows that for sparsely observed data the discretization error cannot be ignored in the way many currently available robust estimation procedures require.

As a consequence of Theorem 1 we may now obtain rates of convergence for the derivatives $\hat{\mu}_n^{(s)}$, $s = 1, \ldots, r - 1$ and tightness of $\hat{\mu}_n^{(r)}$ in the $L^2([0,1])$ norm. These are summarized in Corollary 1.

**Corollary 1.** Under the assumptions of Theorem 1

$$||\hat{\mu}_n^{(j)} - \mu^{(j)}||_2^2 = \lambda^{-j/r} O_p \left( n^{-1} + m^{-2r} + \lambda \right),$$

for all $j \leq r$.

Since, by assumption, $\lambda \to 0$ as $n \to \infty$, the corollary implies that derivatives of higher order are more difficult to estimate, ceteris paribus. This is not a surprising conclusion, as differentiability is a local property and one is expected to need a larger sample size and higher grid resolution in order to examine local properties with the same degree of precision.

We now turn to the problem of mean estimation when trajectories are possibly observed at different points, in other words the case of independent design. We aim to establish the optimality of general M-type estimators in this setting and the assumptions that we will be needing are for the most part straightforward generalizations of the assumptions governing the case of common design.

(B1) (A1).

(B2) The error processes $\epsilon_i(\omega, t) : (\Omega, \mathcal{A}, P) \times [0,1] \to \mathbb{R}, i = 1, \ldots, n$ are independent.

(B3) (A3).

(B4) There exists a constant $M_2$ such that

$$\sup_{i,j} \mathbb{E} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \}^2 \leq M_2 |u|,$$

as $u \to 0$. 

9
(B5) $\mathbb{E}\{\psi(\epsilon_{ij})\} = 0$, $\sup_{i,j} \mathbb{E}\{|\psi(\epsilon_{ij})|^2\} \leq c_0$ and there exist positive constants $\delta_{ij}, (i = 1, \ldots, n) (j = 1, \ldots, m_i)$ such that $0 < \inf_{i,j} \delta_{ij} \leq \sup_{i,j} \delta_{ij} < \infty$ and $\mathbb{E}\{\psi(\epsilon_{ij} + u)\} = \delta_{ij}u + o(u), \text{ as } u \to 0$.

(B6) Let $[0,1] = \bigcup_{i=1}^{n} I_i$ denote a partition of $[0,1]$ into disjoint intervals $I_i$. For the family of discretization points $\{T_{ij}\}$ we require $\{T_{ij}\}_{j=1}^{m_i} \in I_i$ and that $\{T_{ij}\}_{j=1}^{m_i}$ are quasi-uniform in $I_i$ in the sense of Eggermont and LaRiccia (2009). That is, there exists a constant $c_r > 0$ such that, for all $n \geq 2$ and all $f \in \mathcal{W}^{1,1}([0,1])$,

$$\left| \frac{1}{m_i} \sum_{j=1}^{m_i} f(T_{ij}) - \int_{I_i} f(t) dt \right| \leq \frac{c_r}{m_i} \int_{I_i} |f'(t)|dt,$$

for all $i = 1, \ldots, n$.

Clearly, assumption (B2) is weaker than assumption (A2), as it implies nothing about the finite-dimensional distributions. Assumptions (B3)–(B5) accommodate for the need of distinct discretization points $T_{ij}$, while assumption (B6) requires that the sampling points are somewhat dense in $[0,1]$. Thus, while we do not require each trajectory to be densely sampled throughout $[0,1]$, we do require that each trajectory covers its (possibly small) subinterval in a nice enough way so that the Riemann approximation to the integral of a $\mathcal{W}^{1,1}([0,1])$-function results in a small remainder, according to the EulerMaclaurin formula. This assumption is geared towards sparse longitudinal data observed on different subdomains.

A quantity that turns out to play a central role in the asymptotic properties of the estimator in the independent design is the harmonic mean of $m_1, \ldots, m_n$ given by

$$m := \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \right)^{-1}.$$

See Li and Hsing (2010) for a similar observation in the context of local linear estimation. Notice that even if $m_1 = \ldots = m_n$ we are still in a different setting than the common design. The reason is that the sampling points need not coincide and $m$ cannot always be assumed to tend to infinity, as the trajectories may be sparsely sampled.

With the above assumptions we obtain the following result

**Theorem 2.** Assume (B1)–(B6) and further that as $n \to \infty$, $m(n)$ varies with $n$ and $\lambda(n) \to 0$ in such a way that

$$\lim_{n \to \infty} n^{2r} \lambda = \lim_{n \to \infty} n^{(r-1)} \lambda^{3/2} = \lim_{n \to \infty} (mn)^r \lambda = \infty.$$

Then, every sequence of M-type smoothing spline estimators $\hat{\mu}_n$ satisfies

$$||\hat{\mu}_n - \mu||^2_{p,\lambda} = O_P \left((mn\lambda^{1/2r})^{-1} + n^{-1} + \lambda \right).$$

Furthermore, if $\psi(x)$ is Lipschitz-continuous the condition $\lim_n n^{r-1} \lambda^{3/2} = \infty$ may be replaced by $\lim_n n^{(2r-1)/2} \lambda = \infty$.
Taking $\lambda \asymp (mn)^{-2r/2r+1}$, the limit assumptions are satisfied provided, for example, that $m \asymp n^v$ with $v < (r - 1)(2r + 1)/3r - 1$. With this choice of $\lambda$ we are lead to
\[
||\hat{\mu}_n - \mu||^2_2 = O_P \left( (mn)^{-2r/(2r+1)} + n^{-1} \right),
\]
which are the optimal rates of convergence for the independent design, Cai and Yuan (2011). Similarly to the common design case, the phase transition occurs at $m \asymp n^{1/2r}$ although in the present case the conclusions are more subtle. For $m >> n^{1/2r}$ it may be yet again seen that the asymptotic error behaves like $n^{-1}$. However, a rather interesting phenomenon occurs when $m = O(n^{1/2r})$. In this case, the asymptotic error behaves $(mn)^{-2r/(2r+1)}$, which would be the optimal nonparametric rate of convergence had we possessed $mn$ independent observations at $mn$ different sites (Cox, 1983; Kalogridis, 2020).

It is quite remarkable that in the case of independent design a high rate of convergence can be obtained even if $m$ does not tend to infinity. In this scenario the asymptotic error would behave like $n^{-2r/(2r+1)}$ and thus it would be as if we had no repeated measurements for each curve. The intuition for this seemingly bizarre conclusion is that eventually we will have $n >> m$, so that the impact of repeated measurements would become negligible in the limit.

Corollary 2 completes our theoretical results by establishing convergence rates for the derivatives in the independent design.

**Corollary 2.** Under the assumptions of Theorem 2
\[
||\hat{\mu}_{n}^{(j)} - \mu^{(j)}||^2_2 = \lambda^{-j/r} O_P \left( (mn\lambda^{1/2r})^{-1} + n^{-1} + \lambda \right),
\]
for all $j \leq r$.

While this type of design has been considered previously (see, e.g., Yao et al. (2005); Li and Hsing (2010); Cai and Yuan (2011)), to the best of our knowledge rates of convergence for the derivatives of $\hat{\mu}_n$ have not been established before. Note that derivative estimation can be useful in situations where the qualitative features of $\mu$ need to be examined in more detail, for example when modelling growth data.

### 4 Computation and smoothing parameter selection

As discussed in Section 2, there exists at least one solution of (3) in the space of natural splines of order $2r$ with knots at the unique $T_{ij}$. Thus we may restrict attention to the linear subspace of natural splines for the computation of the estimator. Assume for simplicity that all $T_{ij}$ are distinct and let $a = \min_{ij} T_{ij} > 0$ and $b = \max_{ij} T_{ij} < 1$. Then the natural spline has $v := \sum_{i=1}^n m_i - 2$ interior knots and we may write
\[
\mu(t) = \sum_{k=1}^{v+2r} \mu_k B_k(t),
\]
where $\hat{\mu}_j$ are coefficients and the $B_j$ are the B-spline basis functions of order $2r$ with knots at the interior $T_{ij}$. For $\rho$-functions whose derivative $\psi$ exists everywhere, a fast computational algorithm may be developed along the lines of the well-known iteratively reweighted least squares (IRLS) procedure (Maronna et al., 2006; Huber and Ronchetti, 2009), which we outline below for this case.

First, note that since $\hat{\mu}_n$ lies in a linear subspace, it suffices to find the vector $\mu = (\mu_1, \ldots, \mu_{v+2r})^\top$ such that

$$-\frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \psi (Y_{ij} - B_{ij}^\top \hat{\mu}) B_{ij} + 2\lambda \Omega \hat{\mu} = 0, \quad (6)$$

for $B_{ij} = (B_1(T_{ij}), \ldots, B_{v+2r}(T_{ij}))^\top$ and $\Omega_{kl} = \langle B_k^{(r)}, B_l^{(r)} \rangle_2$ ($k = 1, \ldots, v+2r$) ($j = 1, \ldots, v+2r$). Initially, it may seem that this formulation ignores the boundary constraint that govern natural splines but, as Hastie et al. (2009, pp. 161-162) note, the penalty term automatically imposes them. The reasoning is as follows: if that were not the case, it would always be possible to find a 2$r$th order natural interpolating spline that leaves the first term in (3) unchanged, but due to it being a polynomial of order $r$ outside of $[a, b]$ the penalty seminorm would be strictly smaller.

Let $r_{ij} := Y_{ij} - B_{ij}^\top \hat{\mu}$ denote the $ij$th residual and $W(x) := \psi(x)/x$ the weight function, then equation (6) can be rewritten as

$$-\frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} W(r_{ij})(Y_{ij} - B_{ij}^\top \hat{\mu}) B_{ij} + 2\lambda \Omega \hat{\mu} = 0.$$

Based on the residuals and weights corresponding to a current estimate $\hat{\mu}_c$ the above expression can be used to update the estimate by

$$\hat{\mu}_{c+1} = \left( -\frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} W(r_{ij}) B_{ij} B_{ij}^\top + 2\lambda \Omega \right)^{-1} \left( -\frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} W(r_{ij}) B_{ij} Y_{ij} \right).$$

The IRLS algorithm consists of iterating the above updating step until convergence, which is guaranteed irrespective of the starting values and yields a stationary point of (6), under mild conditions on $\rho$ (Huber and Ronchetti, 2009) that include the boundedness of $W(x)$ near zero.

For the non-smooth quantile loss function $\rho_\alpha(x)$ the above algorithm fails on account of its kink at the origin. However, the easily implementable recipe of (Nychka et al., 1995) may be used in order to produce an approximate solution of (3). In particular, the algorithm may be used with the smooth approximation

$$\tilde{\rho}_\alpha(x) = \begin{cases} \rho_\alpha(x) & |x| \geq \epsilon \\ \alpha x^2 / \epsilon & 0 \leq x < \epsilon \\ (1 - \alpha) x^2 / \epsilon & -\epsilon < x \leq 0, \end{cases}$$

12
for some small $\epsilon > 0$. With $\tilde{\rho}_\alpha$ as the objective function an approximate quantile smoothing spline estimate may be computed with the IRLS algorithm and thus without the need of a computationally burdensome quadratic program.

To determine the penalty parameter $\lambda$ in a data-driven way we propose to select $\lambda$ that minimizes the generalized cross-validation criterion

$$GCV(\lambda) := \frac{\sum_{i=1}^{n} m_i^{-1} \sum_{j=1}^{m_i} W(r_{ij})|r_{ij}|^2}{n(1 - \text{Tr}\, H(\lambda))^2},$$

where the residuals $r_{ij}$ and the pseudo-influence matrix $H(\lambda)$ are those obtained upon convergence of the IRLS algorithm. The GCV criterion employed herein is a generalization of the criterion proposed by Cunningham et al. (1991). A similar criterion is also proposed in (Cantoni and Ronchetti, 2001).

## 5 Finite-sample performance

In this section we compare the numerical performance of the proposed robust M-type smoothing spline estimator with a number of existing proposals. In particular, we include the least-squares smoothing spline estimator of Rice and Silverman (1991); Cai and Yuan (2011), the least-squares local linear estimator of Yao et al. (2005); Degras (2011) and the robust functional M-estimator of Sinova et al. (2018) in the comparison. For simplicity and for ease of comparison with the functional M-estimator of Sinova et al. (2018), which at the very least requires densely sampled trajectories, we confine ourselves to the case of common design. First, we briefly review the construction of the local linear estimator and the functional M-estimator.

Let $K : \mathbb{R} \to [0, 1]$ denote a symmetric nonnegative kernel (weight) function on $\mathbb{R}$ that is Lipschitz continuous. Then, for any given point $t$ the local linear estimator of Yao et al. (2005); Degras (2011) estimates $\mu(t)$ by $\hat{\mu}_{LSLP}(t) = \hat{\beta}_0$, where $(\hat{\beta}_0, \hat{\beta}_1)$ solves the local linear problem

$$\min_{(\beta_0, \beta_1) \in \mathbb{R}^2} \sum_{j=1}^{m} \{ \bar{Y}_j - \beta_0 - \beta_1 (T_j - t) \}^2 K \left( \frac{T_j - t}{h} \right).$$

Here, $\bar{Y}_j$ denotes the response average at $T_j$ and $h$ denotes the smoothing parameter (i.e., the bandwidth). In our implementation we use the Gaussian kernel for $K$. The bandwidth $h$ is chosen by the direct plug-in methodology, which essentially works by replacing the unknown functionals in the asymptotic expression of the optimal bandwidth with Gaussian kernel estimates. See, e.g., (Fan and Gijbels, 1996) for further details on local polynomial regression.

Let $\rho$ denote a real-valued loss function, the Huber loss in our implementation. Then for fully observed iid trajectories $X_1, \ldots, X_n$, Sinova et al. (2018) propose to estimate $\mu(t)$ by $\hat{\mu}_{HF}$ such that

$$\hat{\mu}_{HF} = \arg\min_{f \in \mathcal{C}^2([0,1])} \frac{1}{n} \sum_{i=1}^{n} \rho(||X_i - f||_2).$$

(7)
The well-known spatial median of (Gervini, 2008) can be recovered from (7) by setting \( \rho(x) := x \). The tuning parameter was set equal to 0.70 corresponding to 85% efficiency in the Gaussian location model and the estimator was computed through the IRLS variant provided by Sinova et al. (2018). Wherever necessary, \( \mathcal{L}^2([0,1]) \)-norms and inner products were computed with trapezoidal Riemann approximations.

In our simulation experiments we are particularly interested in the effect of the sampling frequency, the level of noise and the effect of atypical observations on the estimates. To reflect these considerations we have generated independent and identically distributed curves according to the Karhunen-Love expansion

\[
X(t) = \mu(t) + 2^{1/2} \sum_{k=1}^{50} W_k \frac{\sin((k-1/2)\pi t)}{(k-1/2)\pi},
\]

for \( t \in (0,1) \), for a mean function \( \mu(t) \), which we take to be either \( \sin(6\pi t)(t+1) \) or \( 3\exp((0.25-t)/0.3^2) \), and independent and identically distributed random variables \( W_k \) following the \( t_5 \) distribution. The use of the \( t_5 \) distribution instead of the commonly used standard Gaussian distribution implies that some of the curves exhibit some outlying behaviour, so that our samples will contain observations with possibly large influence.

Once the curves have been generated we discretize them in \( m \in \{20, 50\} \) equispaced points \( 0 < T_1 < \ldots < T_m < 1 \) and generated the noisy observations according to

\[
Y_{ij} = X_i(T_j) + \sigma \zeta_{ij},
\]

with \( \sigma > 0 \) a constant that controls the level of noise in the data. We consider the values \( \sigma \in \{0.2, 0.5, 1\} \), reflecting low, medium and high levels of noise, respectively. The errors \( \zeta_{ij} \) are iid random variables generated according to each of the following distributions

- standard Gaussian,
- \( t_3 \) distribution,
- a convex combination of independent Gaussians with means equal to 0 and 12, unit variances and weights equal to 0.90 and 0.10, respectively,
- Tukey’s Slash distribution, i.e., the quotient of independent standard Gaussian and \((0,1)\)-uniform random variables.

Figure 1 presents examples of typical set of observations under standard Gaussian errors and small noise with the first and second mean-function, respectively.
Figure 1: Two representative samples with mean functions equal to \(\sin(6\pi t)(t + 1)\) (left) and \(3e^{-(t-0.25)/0.3^2}\) (right), \(\sigma = 0.2\) and standard Gaussian errors \(\zeta_{ij}\). The mean functions are depicted as solid black lines.

For this comparison we consider the M-type smoothing spline estimator based on the Huber function with tuning constant equal to 0.70, similarly as for the functional M-estimator, and henceforth denote this estimator by \(\hat{\mu}_{HSP}\). The least-squares smoothing spline estimator described by Cai and Yuan (2011) is denoted by \(\hat{\mu}_{LSSP}\). These authors do not recommend a method to select the penalty parameter in finite samples. Hence, similarly as for the M-type smoothing spline estimator we use the aforementioned GCV criterion, setting all weights equal to 1 in the least squares setting. Both spline estimators were fitted with \(r = 2\), which means that we consider the frequently used cubic splines. All estimators were implemented in the freeware \(R\) (R Core Team, 2018). For the local linear estimator we used the package \texttt{KernSmooth} (Wand and Ripley, 2016) while the functional M-estimator was implemented through a custom-made function according to the algorithm provided by (Sinova et al., 2018).

The criterion that we employ to evaluate the performance of the estimators is the mean-square error, given by

\[
\text{MSE} = \frac{1}{m} \sum_{j=1}^{m} \{\hat{\mu}(T_j) - \mu(T_j)\}^2,
\]

which for large grids is an approximation to the \(L^2([0,1])\) distance. Tables 1 and 2 below report the mean-squared errors and their standard errors based on 1000 simulated datasets of size \(n = 60\). In our experience such a moderately small sample size occurs very often in practice, see e.g., the well-known Canadian Weather dataset (Ramsay and Silverman, 2005).
| $\sigma$ | $m$  | Dist. | $\hat{\mu}_{\text{LSSP}}$ | Mean | SE  | $\hat{\mu}_{\text{HSP}}$ | Mean | SE  | $\hat{\mu}_{\text{LSLP}}$ | Mean | SE  | $\hat{\mu}_{\text{HF}}$ | Mean | SE  |
|---------|------|-------|-----------------|------|-----|-----------------|------|-----|-----------------|------|-----|-----------------|------|-----|
|        |      |       | G.              | 0.028| 0.001| 0.026           | 0.001| 0.001| 0.430           | 0.001| 0.002| 0.222           | 0.001| 0.001|
| 20     |      |       | $t_3$           | 0.031| 0.001| 0.029           | 0.001| 0.001| 0.432           | 0.001| 0.002| 0.266           | 0.001| 0.001|
|        |      |       | M. G.           | 0.089| 0.002| 0.061           | 0.001| 0.001| 0.440           | 0.002| 0.008| 0.086           | 0.002| 0.001|
| 0.2    |      |       | Sl.              | 2e+05| 2e+05| 0.049           | 2e+04| 2e+04| 0.068           | 0.001| 0.008| 0.088           | 0.002| 0.002|
|        |      |       | G.              | 0.029| 0.001| 0.025           | 0.001| 0.001| 0.300           | 0.001| 0.023| 0.001           | 0.001| 0.001|
|        |      |       | $t_3$           | 0.027| 0.001| 0.025           | 0.001| 0.001| 0.299           | 0.001| 0.024| 0.001           | 0.001| 0.001|
| 50     |      |       | M. G.           | 0.099| 0.002| 0.056           | 0.001| 0.002| 0.092           | 0.002| 0.088| 0.002           | 0.026| 0.001|
|        |      |       | Sl.              | 6310 | 52000| 0.034           | 345.58| 242.064| 0.105           | 0.001| 0.105| 0.001           | 0.001| 0.001|
|        |      |       | G.              | 0.031| 0.001| 0.030           | 0.001| 0.001| 0.428           | 0.002| 0.266| 0.001           | 0.001| 0.001|
| 20     |      |       | $t_3$           | 0.035| 0.002| 0.028           | 0.001| 0.001| 0.702           | 0.006| 0.349| 0.005           | 0.001| 0.001|
|        |      |       | M. G.           | 0.428| 0.006| 0.082           | 0.002| 0.002| 2743            | 1.471.92| 0.208| 0.003           | 0.001| 0.001|
| 0.5    |      |       | Sl.              | 9582 | 5963 | 0.104           | 2743 | 1.471.92| 0.208           | 0.003| 0.001| 0.001           | 0.001| 0.001|
|        |      |       | G.              | 0.028| 0.001| 0.025           | 0.001| 0.001| 0.303           | 0.001| 0.266| 0.001           | 0.001| 0.001|
|        |      |       | $t_3$           | 0.035| 0.002| 0.028           | 0.001| 0.001| 0.371           | 0.001| 0.335| 0.001           | 0.001| 0.001|
| 50     |      |       | M. G.           | 0.437| 0.005| 0.075           | 0.002| 0.002| 0.414           | 0.005| 0.406| 0.004           | 0.001| 0.001|
|        |      |       | Sl.              | 3271 | 846  | 0.050           | 551.4 | 269.80 | 0.418           | 0.012| 0.003| 0.001           | 0.001| 0.001|
|        |      |       | G.              | 0.042| 0.001| 0.043           | 0.001| 0.001| 0.448           | 0.002| 0.041| 0.001           | 0.001| 0.001|
| 20     |      |       | $t_3$           | 0.071| 0.001| 0.049           | 0.001| 0.001| 0.470           | 0.003| 0.061| 0.001           | 0.001| 0.001|
|        |      |       | M. G.           | 1.71 | 0.014| 0.123           | 1.792 | 1.179 | 1.792           | 0.013| 0.012| 0.001           | 0.001| 0.001|
| 1      |      |       | Sl.              | 7e+07| 7e+07| 0.264           | 1.792 | 1.792 | 0.663           | 0.006| 0.008| 0.008           | 0.001| 0.001|
|        |      |       | G.              | 0.037| 0.001| 0.032           | 0.001| 0.001| 0.398           | 0.001| 0.400| 0.001           | 0.001| 0.001|
|        |      |       | $t_3$           | 0.070| 0.002| 0.035           | 0.001| 0.001| 0.051           | 0.001| 0.065| 0.001           | 0.001| 0.001|
| 50     |      |       | M. G.           | 1.68 | 0.012| 0.099           | 1.529 | 1.490 | 1.490           | 0.012| 0.011| 0.011           | 0.013| 0.013|
|        |      |       | Sl.              | 8e+04| 6e+04| 0.088           | 6205  | 3337  | 1.467           | 0.013| 0.013| 0.013           | 0.013| 0.013|

Table 1: Mean and standard error of the MSE for the competing estimators over 1000 datasets of size $n = 60$ with mean function $\mu(t) = \sin(6\pi t)(t + 1)$. Best performances are in bold.
The results indicate that the least-squares smoothing spline estimator, \( \hat{\mu}_{LSSP} \), behaves well under all noise and discretization settings if the errors follow a Gaussian distribution. However, its performance quickly deteriorates as soon as the errors deviate from the Gaussian ideal. Similar remarks apply for the local linear estimator \( \hat{\mu}_{LSLP} \), except that its behavior is not good for the sinusoidal mean function when \( m \) is small. The reason for this lesser performance is that \( \hat{\mu}_{LSLP} \) regularly oversmooths in this case and thus misses the peaks and troughs of the sinusoidal mean function. We believe that its performance may be greatly improved if more effort is invested in the selection of its bandwidth, which naturally would come at the cost of a slower implementation.

Comparing the robust estimators \( \hat{\mu}_{HSP} \) and \( \hat{\mu}_{HF} \) in detail reveals that \( \hat{\mu}_{HF} \) is highly sensitive to the level of noise, when that is accompanied by heavy-tailed measurement errors. In particular, for the Gaussian mixture and the Slash distribution \( \hat{\mu}_{HF} \) tends to perform as badly as the the least-squares estimator \( \hat{\mu}_{LSSP} \) with respect to both mean functions. By contrast, \( \hat{\mu}_{HSP} \) exhibits a high degree of resilience towards high levels of noise and small

Table 2: Mean and standard error of the MSE for the competing estimators over 1000 datasets of size \( n = 60 \) with mean function \( \mu(t) = 3 \exp((0.25 - t)/0.3^2) \). Best performances are in bold.

| \( \sigma \) | \( m \) | Dist. | \( \hat{\mu}_{LSSP} \) | \( \hat{\mu}_{HSP} \) | \( \hat{\mu}_{LSLP} \) | \( \hat{\mu}_{HF} \) |
|---|---|---|---|---|---|---|
| 0.2 | 20 | G. | 0.025 | 0.001 | 0.022 | 0.001 | 0.025 | 0.002 | **0.021** | 0.001 |
| t₃ | 0.029 | 0.001 | **0.025** | 0.001 | 0.030 | 0.001 | 0.026 | 0.001 |
| M. G. | 0.089 | 0.002 | **0.055** | 0.002 | 0.087 | 0.002 | 0.086 | 0.002 |
| Sl. | 1788 | 944 | **0.030** | 0.001 | 660.7 | 428 | 0.068 | 0.001 |
| 0.5 | 50 | G. | 0.028 | 0.001 | 0.023 | 0.001 | 0.028 | 0.001 | 0.024 | 0.001 |
| t₃ | 0.026 | 0.001 | **0.021** | 0.001 | 0.027 | 0.001 | 0.023 | 0.001 |
| M. G. | 0.090 | 0.003 | **0.055** | 0.002 | 0.087 | 0.003 | 0.092 | 0.002 |
| Sl. | 1e+04 | 9e+03 | **0.023** | 0.001 | 720.1 | 462.8 | 0.105 | 0.001 |
| 1 | 20 | Gaus. | 0.030 | 0.001 | 0.026 | 0.001 | 0.032 | 0.001 | 0.027 | 0.001 |
| t₃ | 0.034 | 0.001 | **0.027** | 0.001 | 0.035 | 0.002 | 0.034 | 0.001 |
| M. G. | 0.438 | 0.006 | **0.075** | 0.002 | 0.402 | 0.006 | 0.364 | 0.005 |
| Sl. | 1744 | 582.9 | **0.039** | 0.003 | 437.4 | 146.9 | 0.205 | 0.002 |
| 50 | 0.5 | G. | 0.027 | 0.001 | 0.022 | 0.001 | 0.027 | 0.001 | 0.026 | 0.001 |
| t₃ | 0.034 | 0.001 | **0.024** | 0.001 | 0.030 | 0.001 | 0.034 | 0.001 |
| M. G. | 0.433 | 0.006 | **0.068** | 0.002 | 0.382 | 0.005 | 0.405 | 0.005 |
| Sl. | 6e+05 | 6e+05 | **0.030** | 0.001 | 3e+04 | 2e+04 | 0.419 | 0.004 |
| 1 | 50 | G. | 0.038 | 0.001 | 0.031 | 0.001 | 0.038 | 0.001 | 0.031 | 0.001 |
| t₃ | 0.072 | 0.002 | **0.036** | 0.001 | 0.057 | 0.002 | 0.063 | 0.001 |
| M. G. | 1.70 | 0.014 | **0.101** | 0.002 | 1.541 | 0.014 | 1.707 | 0.014 |
| Sl. | 2e+05 | 2e+05 | **0.058** | 0.001 | 3e+05 | 2e+05 | 0.673 | 0.008 |
| 1 | 1 | G. | 0.037 | 0.001 | 0.026 | 0.001 | 0.031 | 0.001 | 0.039 | 0.001 |
| t₃ | 0.070 | 0.001 | **0.028** | 0.001 | 0.035 | 0.001 | 0.063 | 0.001 |
| M. G. | 1.70 | 0.012 | **0.089** | 0.002 | 1.475 | 0.012 | 1.508 | 0.010 |
| Sl. | 6e+04 | 4e+04 | **0.42** | 0.001 | 4200 | 2642 | 1.48 | 0.013 |
grid resolutions. In particular, the estimator maintains a consistent performance towards all kinds of symmetric contamination considered. Its performance deteriorates with the asymmetric mixture of Gaussians but clearly not to the same extent as its competitors. Overall, the present simulation experiments suggest that robust smoothing spline estimates may be used to good effect in a variety of settings that greatly diminish the performance of their competitors.

6 Real-data example: global population growth

To illustrate the practical usefulness of robust estimates for central tendency, we analyse a dataset consisting of yearly population growth data from 210 countries between 1960 and 2018. The data originate from surveys of the United Nations and the whole dataset may be found in the website of the World Bank https://data.worldbank.org/indicator/SP.POP.GROW?name_desc=false. A plot of the time series is given in the left panel of figure 2 below. Our goal in this analysis is to obtain a smooth curve that best describes the population trend for most countries of our sample.

![Population growth/decline rates of 217 countries between 1960-2018 with three extreme outliers. Right: Huber-type and least-squares smoothing spline estimates.](image)

Figure 2: Left: population growth/decline rates of 217 countries between 1960-2018 with three extreme outliers. Right: Huber-type and least-squares smoothing spline estimates.

However, an obstacle in this analysis is that the sample is quite inhomogeneous, as the diversity across counties results in very disparate population growth. The practical implication of these disparities is that the dataset contains a non-negligible number of isolated and persistent outliers, according to the taxonomy of Hubert et al. (2015). These observations are countries that either consistently exhibited substantially lower or higher growth rates or countries that exhibited one-off spikes or drop-offs in their population growths. For exam-
ple, the immigration of several hundreds of thousands of Jews to the otherwise small state of Israel immediately after the collapse of the Soviet Union greatly boosted the host country’s growth rates, albeit temporarily.

To help explain the global population dynamics throughout this 60-year period, we have computed both the least-squares estimator of Rice and Silverman (1991); Cai and Yuan (2011) and the Huber-type estimator proposed herein. These are shown in the right panel of Figure 2, where to better appreciate their differences we have focused our attention to growth rates between 1 and 2.5 per cent, which both estimators suggest that are typical. Interestingly, both estimates point towards an overall decline in population growth from 1960 onwards. However, the least-squares estimates consistently indicate higher growth rates in comparison to the robust estimates. This difference becomes especially pronounced between 2000 and 2010, where the least-squares estimates even suggest a temporary increase in the rate of growth.

In view of the left panel of Figure 2, these differences may be attributable to the few countries that exhibited rapid population growth in the 1970s, 1980s and 2000s. It turns out that the largest upward spikes are caused by population developments in three counties, namely the United Arab Emirates, Qatar and the small island of Saint Martin which are highlighted in the left panel of Figure 2. In order to fully assess the effect of these outlying trajectories on the estimates, one can remove them from the sample and compare the estimates based on the reduced dataset with the estimates on the original dataset. The right panel of figure 3 shows the estimates for the reduced dataset.

Comparing these estimates with those in Figure 2 reveals that while the Huber-type estimate has barely changed, the least-squares estimate has undergone significant change due to the removal of these three counties. In particular, the least-squares estimate of the mean function has now moved much closer to the robust estimate of the mean function and the suspicious bump between 2000 and 2010 has vanished. The differences between the least squares and robust estimates for the original data can thus be attributed almost completely to population changes in only three counties, which constitute a mere 1.4 per cent of the sample size. This example illustrates the extreme sensitivity of least-squares smoothing spline estimates to atypical observations and the remarkably stable performance of robust M-type smoothing spline estimates in a variety of situations. We may think of numerous examples in other fields of science, such as economics, medicine and astronomy where resistant, highly efficient methods of inference can be very useful.
Figure 3: Left: population growth/decline rates of the sample without Qatar, the United Arab Emirates and Saint Martin. Right: Huber-type(—) and least-squares(--·--) smoothing spline estimates from the reduced sample.

7 Concluding remarks

The results of the present chapter establish the optimality of a broad family of smoothing spline estimators under both the common and the independent/spase designs. From here, there are several possible research directions worth pursuing. Of particular importance are robust inferences with respect to the covariance structure of second-order processes. Despite its importance, this problem has not received enough attention and, to the best of our knowledge, current works, such as (Panaretos and Kraus, 2012), require completely observed trajectories. Robust dispersion estimates from discretely sampled functional data may be constructed by a straightforward extension of the framework proposed in our work, replacing the one-dimensional smoothing spline with a two-dimensional thin plate spline (Wahba, 1990; Wood, 2017). Robust estimation of the covariance structure would then allow for robust functional principal component analysis from discretely sampled data in the manner outlined in Yao et al. (2005); Li and Hsing (2010).

Another important area for the application of robust smoothing spline estimators is functional regression and its variants, which have attracted great interest in recent years. To date, most estimation proposals assume that the functional predictor is fully observed and consequently the discreteness of the data is essentially ignored. A smoothing spline estimator for discretely observed curves based on the MM principle was proposed by Maronna and Yohai (2013) as a robustification of the least squares proposal of Crambes et al. (2009), but without any theory. We are confident that under appropriate conditions this MM-type smoothing spline estimator can achieve the optimal rates of convergence, as defined in (Crambes et al., 2009), whilst also providing a much safer estimation method in practice.
We aim to study these topics in detail as part of our future work.

8 Appendix: Proofs of the theoretical results

We begin by recalling some useful definitions and auxiliary results. The Hilbert-Sobolev space $W^{r,2}([0,1])$ is a reproducing kernel Hilbert space under the inner product

$$\langle f, g \rangle_{r,\lambda} = \langle f, g \rangle_2 + \lambda \langle f^{(r)}, g^{(r)} \rangle_2,$$

as Eggermont and LaRiccia (2009) show that there exists a constant $c_r$ such that, for all $0 < \lambda < 1$, $x \in [0,1]$ and all $f \in W^{r,2}([0,1])$,

$$|f(x)| \leq \lambda^{-1/4r} c_r \|f\|_{r,\lambda},$$

with $\|f\|_{r,\lambda} = |\langle f, f \rangle_{r,\lambda}|^{1/2}$. The constant $c_r$ does not depend on $x$, therefore this defines an embedding of $W^{r,2}([0,1]$ in $C([0,1])$, according to the definition given in (Adams and Fournier, 2003, page 9).

As a consequence, for all small $\lambda$ there exists a symmetric function $R_{r,\lambda}(x,y)$, the reproducing kernel, such that $R_{r,\lambda}(x,\cdot) \in W^{r,2}([0,1])$ for every $x \in [0,1]$ and for every $f \in W^{r,2}([0,1])$,

$$f(x) = \langle f, R_{r,\lambda}(x,\cdot) \rangle_{r,\lambda}.$$

By the symmetry of $R_{r,\lambda}(x,y)$, we obtain

$$\|R_{r,\lambda}(x,\cdot)\|_{r,\lambda}^2 = \langle R_{r,\lambda}(x,\cdot), R_{r,\lambda}(\cdot, x) \rangle_{r,\lambda} = R_{r,\lambda}(x,x) \leq c_r \lambda^{-1/4r} \|R_{r,\lambda}(x,\cdot)\|_{r,\lambda}$$

for $\lambda \in (0,1)$. It follows that

$$\sup_{x \in [0,1]} \|R_{r,\lambda}(x,\cdot)\|_{r,\lambda} \leq c_r \lambda^{-1/4r}. \quad (9)$$

The interested reader is referred to (Adams and Fournier, 2003; Eggermont and LaRiccia, 2009) for more details on Sobolev spaces and reproducing kernels.

We first state a lemma concerning minimization of convex semi-continuous functionals in the Hilbert space. The lemma was proven in (Kalogridis, 2020).

**Lemma 1.** Let $\mathcal{H}$ denote a real Hilbert space of functions endowed with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and associated norm $\|\cdot\|_{\mathcal{H}}$ and let $L : \mathcal{H} \to \mathbb{R}_+$ denote a convex lower semicontinuous functional. Then, if

$$L(0) < \inf_{\|f\|_{\mathcal{H}} = 1} L(f),$$

there exists a minimizer of $L$ in the unit ball $\{ f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq 1 \}$.

We next state a useful lemma concerning the approximation of an integral of a smooth $W^{r,2}([0,1])$-function with a sum.
Lemma 2. Under \((A5)\) there exists a constant \(c_r > 0\) such that, for all \(f \in W^{r,2}([0,1])\) and \(m \geq 2\),
\[
\left| \frac{1}{m} \sum_{j=1}^{m} |f(T_j)|^2 - \int_0^1 |f(t)|^2 dt \right| \leq \frac{c_r}{m^{1/2r}} \left\{ \int_0^1 |f(t)|^2 dt + \lambda \int_0^1 |f^{(r)}(t)|^2 dt \right\}.
\]

Proof. The lemma may be proven by slightly adjusting and combining Lemmas 2.24 and 2.27 of Eggermont and LaRiccia (2009), along with the inequality \((m - 1)^{-1} \leq 2m^{-1}\), which is valid for all \(m \geq 2\).

Proof of Theorem 1. Let \(L_n(f)\) denote the objective function, that is,
\[
L_n(f) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(Y_{ij} - f(T_j)) + \lambda \int_0^1 |f^{(r)}(t)|^2 dt.
\]
Further, let \(Q\) denote the operator associated with \(r\)th order spline interpolation. In other words, for every \(f \in W^{r,2}([0,1])\), \(Q(f)\) is the solution to
\[
\min_{g \in W^{r,2}([0,1])} \int_0^1 |g^{(r)}(t)|^2 dt \text{ subject to } g(T_j) = f(T_j), \quad j = 1, \ldots, m.
\]
Let \(R(\mu) := \mu - Q(\mu)\) denote the spline interpolation error for \(\mu\) and for convenience write \(R_j := R(\mu)(T_j)\). Furthermore, let \(g := Q(\mu) - f\). It is easy to see that minimizing \(L_n(f)\) is equivalent to minimizing
\[
L_n(g) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(\epsilon_{ij} + R_j + g(T_j)) + \lambda \|g^{(r)}\|_2^2 + \lambda \|Q(\mu)^{(r)}\|_2^2 - 2\lambda (Q(\mu)^{(r)}, g^{(r)})_2.
\]
Writing \(C_n := n^{-1} + m^{-2r} + \lambda\), we aim to show that for every \(\epsilon > 0\) there exists \(D_\epsilon \geq \) such that
\[
\lim_{n \to \infty} \Pr \left( \inf_{\|g\|_{r,\lambda} = D} L_n(C_n^{1/2}g) > L_n(0) \right) \geq 1 - \epsilon.
\]
By Lemma 1 this would entail the existence of a minimizer \(\hat{g}_n\) such that \(\|\hat{g}_n\|_{r,\lambda}^2 = O_P(C_n)\), provided that we can establish the convexity and lower semicontinuity of \(L_n\) on \(W^{r,2}([0,1])\). If that is indeed the case, using the one-to-one relation between \(g\) and \(f\) we would obtain
\[
\|\hat{\mu}_n - Q(\mu)\|_{r,\lambda}^2 = O_P(C_n).
\]
To check the convexity and lower semicontinuity of \(L_n\) we observe that the former follows easily from \((A1)\) and the convexity of the map \(f \to ||f^{(r)}||_2^2\), as a composition of a convex
function and an increasing convex function on $[0, \infty)$. To establish the lower semicontinuity of $L_n$, it suffices to observe that by (A1) $\rho$ is continuous on $\mathbb{R}$ and by the Sobolev embedding theorem convergence of $\{f_k\}_k \in W^{r,2}([0,1])$ implies pointwise convergence at each $x \in [0,1]$. Furthermore, the semi-norm $||f^{(r)}||^2_{r,\lambda}$ is continuous with respect to convergence in $W^{r,2}([0,1])$, under, for example, the norm $||\cdot||_{r,\lambda}$. Since the sum of two continuous functions is continuous, we have proved the continuity of the objective function, hence its lower semi-continuity as well.

Writing now

$$||\hat{\mu}_n - \mu||_{r,\lambda}^2 \leq 2||\hat{\mu}_n - Q(\mu)||_{r,\lambda}^2 + 2||Q(\mu) - \mu||_{r,\lambda}^2,$$

(12)

reveals that if (10) were to hold, the claim would follow from the properties of spline interpolation. In particular, it is well-known [see, e.g., (DeVore and Lorentz, 1993, Theorem 7.3)] that

$$||\mu - Q(\mu)||_{r,\lambda}^2 \leq \text{const.} \max_{1 \leq j \leq m} |T_{j+1} - T_j|^{2r} \int_0^1 |\mu^{(r)}(t)|^2 dt = O(m^{-2r}),$$

(13)

where the last bound follows from (A5) and the fact that $\mu \in W^{r,2}([0,1])$. On the other hand, Theorem 3.4 of Schultz (1970) and (A4) imply that the spline interpolant can be chosen so that

$$||\mu - Q(\mu)||_{r,\lambda}^2 \leq \text{const.} ||\mu^{(r)}||_{2}^2.$$

(14)

From (13) and (14) we have

$$||Q(\mu) - \mu||_{r,\lambda}^2 = ||Q(\mu) - \mu||_2^2 + \lambda ||(Q(\mu) - \mu)^{(r)}||_{2}^2 = O(m^{-2r}) + \lambda O(1) = O(C_n),$$

(15)

as, by definition of $C_n$, $m^{-2r} + \lambda < C_n$. Thus, under (10), from (11) we would obtain

$$||\hat{\mu}_n - \mu||_{r,\lambda}^2 = O_P(C_n) + O(C_n) = O_P(C_n),$$

which is the desired result. It thus remains to establish (10). To that end, decompose
\[ L_n(C_n^{1/2}g) - L_n(0) \]

as follows

\[
L_n(C_n^{1/2}g) - L_n(0) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho (\epsilon_{ij} + R_T(\mu)(T_j) + C_n^{1/2}g(T_j)) \\
- \frac{1}{nm} \sum_{j=1}^{m} \rho (\epsilon_{ij} + R_T(\mu)(T_j)) + \lambda C_n ||g^{(r)}||_2^2 \\
- 2\lambda C_n^{1/2} \langle Q_T(\mu)^{(r)}, g^{(r)} \rangle_2
\]

\[
= \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_T(\mu) + C_n^{1/2}g(T_j)} \mathbb{E}\{\psi(\epsilon_{ij} + u)\} \, du \\
+ \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_T(\mu) + C_n^{1/2}g(T_j)} [\psi(\epsilon_{ij} + u) - \mathbb{E}\{\psi(\epsilon_{ij} + u)\}] \, du \\
+ \lambda C_n ||g^{(r)}||_2^2 - 2\lambda C_n^{1/2} \langle Q_T(\mu)^{(r)}, g^{(r)} \rangle_2,
\]

which, after adding and subtracting \( \psi(\epsilon_{ij}) \) and \( \mathbb{E}\{\psi(\epsilon_{ij})\} = 0 \) from the second integrand, may be equivalently rewritten as

\[
L_n(C_n^{1/2}g) - L_n(0) := I_1(g) + I_2(g) + I_3(g) + I_4(g),
\]

with

\[
I_1(g) := \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_j + C_n^{1/2}g(T_j)} \mathbb{E}\{\psi(\epsilon_{ij} + u)\} \, du + \lambda C_n ||g^{(r)}||_2^2
\]

\[
I_2(g) := \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_j + C_n^{1/2}g(T_j)} [\psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij})] \\
- \mathbb{E}\{\psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij})\} \, du
\]

\[
I_3(g) := -C_n^{1/2} \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(\epsilon_{ij}) g(T_j)
\]

\[
I_4(g) := -2C_n^{1/2} \lambda \langle Q_T(\mu)^{(r)}, g^{(r)} \rangle_2.
\]

By the superadditivity of the infimum we have the lower bound

\[
\inf_{||g||_{r,\lambda} = D} \{L_n(C_n^{1/2}g) - L_n(0)\} \geq \inf_{||g||_{r,\lambda} = D} I_1(g) + \inf_{||g||_{r,\lambda} = D} I_2(g) \\
+ \inf_{||g||_{r,\lambda} = D} I_3(g) + \inf_{||g||_{r,\lambda} = D} I_4(g).
\]

We will show that for sufficiently large \( D \), \( \inf_{||g||_{r,\lambda} = D} I_1(g) \) is positive and dominates all other terms in the decomposition. For this, we need to determine the order of each one of the four terms.
Starting with \( \inf_{\|g\|_{r,\lambda} = D} I_4(g) \), first observe that Theorem 3.4 of (Schultz, 1970) implies

\[
\| (Q(\mu))^{(r)} \|_2 \leq \text{const.} \| \mu^{(r)} \|_2
\]

for some positive constant. The Schwarz inequality now yields

\[
\inf_{\|g\|_{r,\lambda} = D} I_4(g) \leq \sup_{\|g\|_{r,\lambda} = D} I_4(g) \leq 2C_n^{1/2} \lambda^{1/2} \| Q(\mu)^{(r)} \|_2 \sup_{\|g\|_{r,\lambda} = D} \lambda^{1/2} \| g^{(r)} \|_2
\]

\[
\leq \text{const.} D \lambda^{1/2} C_n^{1/2} \| \mu^{(r)} \|_2
\]

\[
= O(1) DC_n, \quad (16)
\]

where we have used \( \mu \in \mathcal{W}^{r,2}([0,1]) \), \( \lambda^{1/2} < C_n^{1/2} \) and \( \lambda^{1/2} \| g^{(r)} \|_2 \leq \| g \|_{r,\lambda} \), by definition of the norm \( \| \cdot \|_{r,\lambda} \).

Turning to \( I_3(g) \), observe that by assumption (A2) the errors \( \epsilon_{ij} = \epsilon_i(T_j) \) are independent in \( i \), by the measurability of the projections, and are also identically distributed for each \( j \). With the reproducing property we thus obtain,

\[
|I_3(g)| = \frac{C_n^{1/2}}{mn} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(\epsilon_{ij}) \langle \mathcal{R}_{r,\lambda}(T_j, \cdot), g \rangle_{r,\lambda} \right|
\]

\[
= \frac{C_n^{1/2}}{nm} \left| \langle \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(\epsilon_{ij}) \mathcal{R}_{r,\lambda}(T_j, \cdot), g \rangle_{r,\lambda} \right|
\]

\[
\leq \frac{C_n^{1/2}}{nm} \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(\epsilon_{ij}) \mathcal{R}_{r,\lambda}(T_j, \cdot) \right\|_{r,\lambda} \| g \|_{r,\lambda},
\]

and

\[
E \left\{ \frac{1}{nm} \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(\epsilon_{ij}) \mathcal{R}_{r,\lambda}(T_j, \cdot) \right\|_{r,\lambda}^2 \right\}
\]

\[
= \frac{1}{nm^2} \sum_{j=1}^{m} \sum_{l=1}^{m} E \{ \psi(\epsilon_{ij}) \psi(\epsilon_{il}) \} \mathcal{R}_{r,\lambda}(T_j, T_l)
\]

\[
\leq \frac{c_0}{nm^2} \sum_{j=1}^{m} \sum_{l=1}^{m} \mathcal{R}_{r,\lambda}(T_j, T_l),
\]

by the Schwarz inequality and assumption (A4). Recall that for each \( x \in [0,1] \), \( \mathcal{R}_{r,\lambda}(x, \cdot) \in \mathcal{W}^{r,2}([0,1]) \). Since, by assumption, \( m(n) \to \infty \) as \( n \to \infty \) and \( |T_{j+1} - T_j| \leq \text{const.} m^{-1} \) a quadrature argument now shows that

\[
\frac{1}{m^2} \sum_{j=1}^{m} \sum_{l=1}^{m} \mathcal{R}_{r,\lambda}(T_j, T_l) = \int_0^1 \int_0^1 \mathcal{R}_{r,\lambda}(x, y) dx dy + O(1)
\]

\[
= O(1).
\]
From this it follows that
\[ \sup_{\|g\|_{r,\lambda} = D} |I_3(g)| = O_P(1)DC_n^{1/2}n^{-1/2} = O_P(1)DC_n, \] (17)
by definition of \(C_n\).

Before examining the terms \(I_1(g)\) and \(I_2(g)\) let us first note that by the reproducing property
\[ C_n^{1/2} \max_{j \leq m} |g(T_j)| = C_n^{1/2} \max_{j \leq m} \|g, R_{r,\lambda}(\cdot, T_j)\|_{r,\lambda} \leq C_n^{1/2} \|g\|_{r,\lambda} \max_{j \leq m} \|R_{r,\lambda}(\cdot, T_j)\|_{r,\lambda} \leq c_r DC_n^{1/2} \lambda^{-1/4r}. \]
Similarly,
\[ \max_{j \leq m} |R_j| \leq \text{const.} \lambda^{-1/4r} |R(\mu)|_{r,\lambda} \leq C_n^{1/2} \lambda^{-1/4r}, \]
where the last inequality follows from (13). Our limit conditions now ensure that
\[ \lim_{n \to \infty} C_n^{1/2} \max_{j \leq m} |g(T_j)| = \lim_{n \to \infty} \max_{j \leq m} |R_j| = 0. \]
With this in mind, applying Fubini’s theorem and (A3) now yields
\[
\int_{R_j} \int_{R_j + C_n^{1/2}g(T_j)} \mathbb{E}\{\psi(\epsilon_{ij} + u)\}du = \int_{R_j} \int_{R_j + C_n^{1/2}g(T_j)} \{\delta_j u + o(u)\}du \\
= \frac{C_n}{2} \delta_j |g(T_j)|^2 \{1 + o(1)\} \\
+ \delta_j C_n^{1/2} R_j g(T_j) \{1 + o(1)\},
\]
for all large \(n\). Consequently,
\[
\frac{1}{m} \sum_{j=1}^{m} \int_{R_j} \int_{R_j + C_n^{1/2}g(T_j)} \mathbb{E}\{\psi(\epsilon_{ij} + u)\}du = \\
\frac{C_n}{2m} \sum_{j=1}^{m} \delta_j |g(T_j)|^2 \{1 + o(1)\} \\
+ \frac{C_n^{1/2}}{m} \sum_{j=1}^{m} \delta_j R_j g(T_j) \{1 + o(1)\} \\
:= I_{11}(g) + I_{12}(g),
\]
say. We now establish a lower bound for \(I_{11}(g)\) and an upper bound for \(I_{12}(g)\). From (A3) and the quadrature lemma (Lemma 2) we obtain
\[
I_{11}(g) \geq \inf_j \delta_j \frac{C_n}{2m} \sum_{j=1}^{m} |g(T_j)|^2 \{1 + o(1)\} \\
\geq \text{const.} C_n \left( \|g\|^2_2 - \frac{c_r}{m \lambda^{1/2} r} \|g\|^2_{r,\lambda} \right) \{1 + o(1)\}, \quad (18)
\]
as well as

\[ |I_{12}(g)| \leq C_n^{1/2} \sup_j \frac{1}{m} \sum_{j=1}^m |g(T_j)||R_j|\{1 + o(1)\} \]

\[ \leq \text{const. } C_n^{1/2} \left\{ \frac{1}{m} \sum_{j=1}^m |g(T_j)|^2 \right\}^{1/2} \left\{ \frac{1}{m} \sum_{j=1}^m |R_j|^2 \right\}^{1/2} \{1 + o(1)\} \]

By (A5) and Lemma 2 we may approximate the sums with integrals to obtain

\[ \frac{1}{m} \sum_{j=1}^m |g(T_j)|^2 \leq ||g||_2^2 + \frac{c_r}{m \lambda^{1/2r}} ||g||_{r,\lambda}^2 \]

\[ \leq \left(1 + \frac{c_r}{m \lambda^{1/2r}}\right) ||g||_{r,\lambda}^2 \]

\[ \leq \text{const. } ||g||_{r,\lambda}^2, \]

where we have used ||g||_2 \leq ||g||_{r,\lambda} and the limit condition \( \lim \inf_{n \to \infty} m \lambda^{1/2r} \geq 2c_r \). Likewise

\[ \frac{1}{m} \sum_{j=1}^m |R_j|^2 \leq ||R(\mu)||_2^2 + \frac{c_r}{m \lambda^{1/2r}} (||R(\mu)||_2^2 + \lambda||R(\mu)^{(r)}||_2^2) \]

\[ = O(m^{-2r}) + O(1) \left( O(m^{-2r}) + \lambda O(1) \right) \]

\[ = O(m^{-2r}) + O(\lambda), \]

by (11). From this and the fact that \( m^{-r} + \lambda^{1/2} < C_n^{1/2} \) we may conclude

\[ \sup_{||g||_{r,\lambda} = D} |I_{12}(g)| \leq \text{const. } DC_n \{1 + o(1)\}, \quad (19) \]

for all large \( n \).

Combining now (18) and (19) shows

\[ I_1(g) \geq \text{const. } C_n \left( ||g||_2^2 - \frac{c_r}{m \lambda^{1/2r}} ||g||_{r,\lambda}^2 \right) \{1 + o(1)\} + \text{const. } C_n \lambda ||g^{(r)}||_2^2 + O(1) DC_n \]

\[ = \text{const. } C_n \|g\|_{r,\lambda}^2 \left( 1 - \frac{c_r}{m \lambda^{1/2r}} \right) \{1 + o(1)\} + O(1) DC_n, \]

\[ \geq 1/2 \text{ for large } n \]

since, by assumption, \( \lim \inf_{n \to \infty} m \lambda^{1/2r} \geq 2c_r \). Consequently,

\[ \inf_{||g||_{r,\lambda} = D} I_1(g) \geq D^2 C_n \{1 + O(D^{-1}) + o(1)\}. \quad (20) \]

To conclude the proof we will now use a covering argument and a concentration inequality to show that

\[ \sup_{||g||_{r,\lambda} \leq D} |I_2(g)| = o_P(1) DC_n, \quad (21) \]
which in combination with (16), (17) and (20) would yield the result of the theorem. To that end note that we may write

\[ I_2(g) = n^{-1} \sum_{i=1}^{n} I_{in}(g), \]

with

\[ I_{in}(g) := m^{-1} \sum_{j=1}^{m} \int_{0}^{-C_n^{1/2} g(T_j)} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du. \]

By (A2) \( I_{in}(i = 1, \ldots, n) \) are independent random variables. They are also uniformly bounded on \( B_D := \{ g \in \mathcal{W}^{r,2}([0,1]) : \|g\|_{r,\lambda} \leq D \} \), since \( \max_{j \leq m} C_n^{1/2} |g(T_j)| \to 0 \) as \( n \to \infty \) and therefore by (8)

\[ \max_{i \leq n} |I_{in}(g)| \leq 2M_1 C_n^{1/2} \max_{j \leq m} |g(T_j)| \leq 2M_1 C_n^{1/2} \lambda^{-1/4} D. \]  \( (22) \)

The total variance may bounded on \( B_D \) as follows

\[
\sum_{i=1}^{n} \text{Var} \{ I_{in}(g) \} \leq m^{-2} \sum_{i=1}^{n} \mathbb{E} \left\{ \sum_{j=1}^{m} \int_{R_j}^{R_j+C_n^{1/2} g(T_j)} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du \right\}^2 \\
\leq \sum_{i=1}^{n} m^{-1} \sum_{j=1}^{m} \mathbb{E} \left\{ \int_{R_j}^{R_j+C_n^{1/2} g(T_j)} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du \right\}^2 \\
\leq M_2 n m^{-1} \max_{j \leq m} C_n^{1/2} |g(T_j)| \left| \int_{R_j}^{R_j+C_n^{1/2} g(T_j)} |u| du \right| \\
\leq nC_n^{1/2} \max_{j \leq m} |g(T_j)| m^{-1} \sum_{j=1}^{m} \{ 3|R_j|^2 + 2|g(T_j)|^2 \} \\
\leq n M_2^* C_n^{3/2} \lambda^{-1/4} D^3, \]  \( (23) \)

for some \( M_2^* > 0 \).

For any \( (g_1, g_2) \in B_D \times B_D \) we also have

\[
|I_{in}(g_1) - I_{in}(g_2)| = \left| m^{-1} \sum_{j=1}^{m} \int_{0}^{-C_n^{1/2} g_1(T_j)} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du - \mathbb{E} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} | du \right| \\
\leq 2M_1 C_n^{1/2} \|g_1 - g_2\|_{\infty}. \]  \( (24) \)
By our assumptions \( \lambda \in (0, 1) \) for all large \( n \) and therefore for all \( f \in \mathcal{W}^{r, 2}([0, 1]) \)
\[
||f||_{r, \lambda} \geq \lambda^{1/2}||f||_{r, 1}.
\]
Hence
\[
B_D \subset \{ f \in \mathcal{W}^{r, 2}([0, 1]) : ||f||_{r, 1} \leq \lambda^{-1/2}D \} := \mathcal{Z}_D.
\]
Let \( N(\varepsilon, \mathcal{F}, || \cdot ||_\infty) \) denote the \( \varepsilon \)-covering number of a set of functions \( \mathcal{F} \) in the sup-norm \( || \cdot ||_\infty \). By Proposition 6 of (Cucker and Smale, 2001) we deduce the existence of a constant \( K > 0 \) such that for all \( \varepsilon > 0 \) we have
\[
\log N(\varepsilon, \mathcal{Z}_D, || \cdot ||_\infty) \leq K \left( \frac{D}{\lambda^{1/2} \varepsilon} \right)^{1/r},
\]
from which we obtain \( \log N(\varepsilon, B_D, || \cdot ||_\infty) \leq KD^{1/r}(\lambda^{1/2} \varepsilon)^{-1/r} \). Fix \( \varepsilon > 0 \). From the above we have
\[
N((8M_1)^{-1}C_n^{1/2}D\varepsilon, B_D, || \cdot ||_\infty) \leq \exp \left( K \left\{ \frac{8M_1}{\varepsilon C_n^{1/2} \lambda^{1/2}} \right\}^{1/r} \right).
\]
For short, let \( N \) denote this covering number and let \( B_j \) \((j = 1, \ldots, N)\) denote the balls that cover \( B_D \). Select \( g_j \in B_j \) \((j = 1, \ldots, N)\) such that each \( g_j \) is also in \( B_D \). This is possible by the definition of the covering number \( N \), for if some \( B_j \) do not contain any points of \( B_D \) then one can construct an even smaller cover by omitting these \( B_j \).

By (24), the construction of the cover now implies that
\[
\sup_{g \in B_D} |I_2(g)| \leq \max_{j \leq N} |I_2(g_j)| + \max_{j \leq N} \sup_{g \in B_j} |I_2(g) - I_2(g_j)|
\leq \max_{j \leq N} |I_2(g_j)| + \varepsilon/2.
\]
Boole’s and Bernstein’s inequalities [see, e.g., (van der Vaart, 1998)] along with (22) and (23) now reveal
\[
\Pr \left( \sup_{g \in B_D} |I_2(g)| \geq \varepsilon DC_n \right) \leq \sum_{j=1}^{N} \Pr \left( |I_2(g_j)| \geq \varepsilon DC_n/2 \right)
\leq 2N \exp \left( -\frac{1}{8} \frac{n \lambda^{1/4} \varepsilon^2 D^2 C_n^2}{M_1^* D^3 C_n^{3/2} + \varepsilon/2 M_1^* D^2 C_n^{3/2}} \right)
\sim 2 \exp \left( -K_1 n C_n^{1/2} \lambda^{1/4} + K_2 C_n^{-1/2} \lambda^{-1/2} \right),
\]
for some constants \( K_1, K_2 > 0 \), depending only on \( D \) and \( \varepsilon \). It is easy to see that by our limit assumptions \( n C_n^{1/2} \lambda^{1/4} \to \infty \) and
\[
(n C_n^{1/2} \lambda^{1/4} \lambda^{1/2} C_n^{1/2} \lambda^{1/2} \lambda^{1/2} C_n) \geq n^{(r-1)/r} \lambda^{3/2r} \to \infty.
\]
Therefore,

\[ \lim_{n \to \infty} \Pr \left( \sup_{g \in \mathcal{B}_D} |I_2(g)| \geq \epsilon DC_n \right) = 0, \]

and (21) is thus established. The first claim of the theorem follows from comparing (16), (17), (20) and (21): \( \inf_{\|g\|_{r, \lambda} = D} I_1(g) \) is positive and dominates all other terms for sufficiently large \( D \) and \( n \).

To see how the Lipschitz condition on \( \psi \) may be used to obtain simpler limit conditions note that for \( \psi \) that is Lipschitz (22) and (23) may be bounded to the order of \( C_n \lambda^{1/2r} D^2 \) and \( C_n^2 \lambda^{-1/2r} D^4 \) respectively. The claim then follows from an application of Bernstein’s inequality under the revised limit conditions.

\[ \square \]

**Proof of Corollary 1.** The proof can be deduced from Eggermont and LaRiccia (2009, Chapter 13, Lemma 2.17) that establishes the inequality

\[ \{ \| f \|_2^2 + \lambda^{j/r} \| f^{(j)} \|_2^2 \}^{1/2} \leq \text{const.}_{j,r} \| f \|_{r, \lambda}. \]

for all \( j \leq r \) and \( f \in \mathcal{W}^{r, 2}([0, 1]) \) with the constant depending only on \( j \) and \( r \). Since \( \mathcal{W}^{r, 2}([0, 1]) \) is a vector space, Theorem 1 now implies that for any \( 1 \leq j \leq r \)

\[ \lambda^{j/r} \| \hat{\mu}_n^{(j)} - \mu^{(j)} \|_2^2 \leq \text{const.}_{j,r} \| \hat{\mu}_n - \mu \|_{r, \lambda}^2 = O_P \left( n^{-1} + m^{-2r} + \lambda \right). \]

The result follows by the positivity of \( \lambda \).

\[ \square \]

**Lemma 3.** Under (B5) there exists a constant \( c_r > 0 \) such that for all \( f \in \mathcal{W}^{r, 2}([0, 1]) \) and all \( n \geq 2 \),

\[ \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{1}{m_i} |f(T_{ij})|^2 - \int_0^1 |f(t)|^2 dt \right| \leq \frac{c_r}{\lambda^{1/2r} nm} \| f \|_{r, \lambda}^2. \]

**Proof.** The proof follows verbatim from the proof of Lemma 2.27 of (Eggermont and LaRiccia, 2009, Chapter 13).

\[ \square \]

**Proof of Theorem 2.** Let \( L_n(f) \) denote the objective function, that is,

\[ L_n(f) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \rho (Y_{ij} - f(T_{ij})) + \lambda \int_0^1 |f^{(r)}(t)|^2 dt, \]

and write \( C_n = (nm \lambda^{1/2r})^{-1} + n^{-1} + \lambda \). It will be shown that for every \( \epsilon > 0 \) there exists \( D_\epsilon \geq 1 \) such that

\[ \lim_{n \to \infty} \Pr \left( \inf_{\|g\|_{r, \lambda} = D} L_n(\mu + C_n^{1/2} g) > L_n(\mu) \right) \geq 1 - \epsilon. \]

(25)
As previously, this would imply the existence of a minimizer in the ball \( \{ f \in \mathcal{W}^{r,2}([0,1]) : \| f - \mu \|^2_{r,\lambda} \leq DC_n \} \) with high probability. Decomposing \( L_n(\mu + C_n^{1/2} g) - L_n(\mu) \) we may write

\[
L_n(\mu + C_n^{1/2} g) - L_n(\mu) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \int_0^{C_n^{1/2} g(T_{ij})} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du
\]

\[
- \frac{C_n^{1/2}}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} g(T_{ij}) \psi(\epsilon_{ij}) + \lambda C_n \| g^{(r)} \|^2_2
\]

\[
+ 2\lambda C_n^{1/2} (\mu^{(r)}, g^{(r)})_2
\]

\[
:= I_1(g) + I_2(g) + I_3(g),
\]

with

\[
I_1(g) := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \int_0^{C_n^{1/2} g(T_{ij})} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du + C_n \lambda \| g^{(r)} \|^2_2
\]

\[
I_2(g) := \frac{C_n^{1/2}}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} g(T_{ij}) \psi(\epsilon_{ij})
\]

\[
I_3(g) := 2\lambda C_n^{1/2} (\mu^{(r)}, g^{(r)})_2.
\]

We proceed to determine the order of \( \inf_{\| g \|_{r,\lambda} = b} \mathbb{E}\{ I_1(g) \} \), \( \sup_{\| g \|_{r,\lambda} \leq D} | I_1(g) - \mathbb{E}\{ I_1(g) \} | \), \( \sup_{\| g \|_{r,\lambda} \leq D} | I_2(g) | \) and \( \sup_{\| g \|_{r,\lambda} \leq D} | I_3(g) | \). Starting with the latter, with the Schwarz inequality we immediately obtain

\[
| I_3(g) | \leq 2C_n^{1/2} \lambda \| \mu^{(r)} \|_2 \| g^{(r)} \|_2 \leq \text{const.} C_n^{1/2} \lambda^{1/2} \| g^{(r)} \|_2 \leq \text{const.} \| g \|_{r,\lambda} C_n,
\]

since \( \mu \in \mathcal{W}^{r,2}([0,1]) \), \( \lambda^{1/2} < C_n \) and \( \lambda^{1/2} \| g^{(r)} \|_2 \leq \| g \|_{r,\lambda} \). It follows that

\[
\sup_{\| g \|_{r,\lambda} \leq D} | I_3(g) | \leq \text{const.} DC_n. \quad (26)
\]

On the other hand, with the reproducing property and the Schwarz inequality we obtain

\[
| I_2(g) | \leq C_n^{1/2} \left\| \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \psi(\epsilon_{ij}) \mathcal{R}_{r,\lambda}(T_{ij}, \cdot) \right\|_{r,\lambda} \| g \|_{r,\lambda},
\]

in the same manner as in the proof of Theorem 1. Now, by the independence of the error processes,

\[
\mathbb{E}\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \psi(\epsilon_{ij}) \mathcal{R}_{r,\lambda}(T_{ij}, \cdot) \right\|_{r,\lambda}^2 \right\}
\]

\[
\leq \frac{c_0}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{l=1}^{m_i} \frac{1}{m_i^2} \mathcal{R}_{r,\lambda}(T_{ij}, T_{il})
\]

31
and since for every \( y \in [0, 1], \ R_{r,\lambda}(\cdot, y) \in W^{r,2}([0, 1]) \), we obtain by assumption (B5) and the reproducing property
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{l=1}^{m_i} \frac{1}{m_i^2} R_{r,\lambda}(T_{ij}, T_{il}) = O \left( \frac{1}{n} \right) + O \left( \frac{1}{n^2} \sum_{i=1}^{n} \frac{1}{m_i} \lambda^{1/2r} \right) = O(1)(n^{-1} + (nm\lambda^{1/2r})^{-1}).
\]

Given that \( n^{-1} + (nm\lambda^{1/2r})^{-1} \leq C_n \), conclude that
\[
\sup_{\|g\|_{r,\lambda} \leq D} |I_2(g)| = O_P(1)DC_n.
\] (27)

Turning to \( I_1 \), a similar derivation with the reproducing property as in the proof of Theorem 1 shows
\[
\lim_{n \to \infty} C_n^{1/2} \max_{1 \leq i \leq n} \max_{1 \leq j \leq m_i} |g(T_{ij})| = 0,
\]
provided that \( \lim_{n \to \infty} n\lambda^{1/2r} = \lim_{n \to \infty} nm\lambda^{1/r} = \infty \), both of which hold by hypothesis. Applying Fubini’s theorem and assumption (B3) now shows
\[
\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \int_{0}^{C_n^{1/2}g(T_{ij})} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du \right\} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \int_{0}^{C_n^{1/2}g(T_{ij})} \{ \delta_{ij} u + o(u) \} du \\
= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m_i} \sum_{j=1}^{m_i} \delta_{ij} C_n |g(T_{ij})|^2 \{ 1 + o(1) \} \\
\geq \inf_{ij} \delta_{ij} \sum_{i=1}^{n} \frac{C_n}{nm_i} \sum_{j=1}^{m_i} |g(T_{ij})|^2 \{ 1 + o(1) \},
\]
whence with the help of Lemma 3 we obtain
\[
\inf_{\|g\|_{r,\lambda} = D} \mathbb{E}\{I_1(g)\} \geq \text{const.} D^2C_n \{ 1 + o(1) \},
\] (28)
where the constant is strictly positive.

Letting \( R_n(g) := I_1(g) - \mathbb{E}\{I_1(g)\} \), to conclude the proof, we will now show
\[
\sup_{\|g\|_{r,\lambda} \leq D} |R_n(g)| = o_P(1)DC_n.
\] (29)

Clearly, we have
\[
R_n(g) = n^{-1} \sum_{i=1}^{n} I_{in}(g),
\]
with \( I_{in}(g) \) \((i = 1, \ldots, n)\) independent random variables given by

\[
I_{in}(g) = m_i^{-1} \sum_{j=1}^{m_i} \int_0^{-C_n^{1/2} g(T_{ij})} \left\{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \right\} du - E \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du.
\]

Since by assumption \((B3)\) has uniformly bounded local increments, \((8)\) reveals that \( I_{in}(g) \) \((i = 1, \ldots, n)\) are uniformly bounded on \( B_D = \{ f \in W^{r,2}([0,1] : \| f \|_{r,\lambda} \leq D \}. \) In particular,

\[
\max_{i \leq n} |I_{in}(g)| \leq 2M_1 c_r D C_n^{1/2} \lambda^{-1/4r}, \quad \tag{30}
\]

for all \( g \in B_D \). A variance computation under condition \((B4)\) and Lemma 2 also shows

\[
\sum_{i=1}^{n} \text{Var} \{ I_{in}(g) \} \leq \sum_{i=1}^{n} m_i^{-2} \mathbb{E} \left\{ \int_0^{-C_n^{1/2} g(T_{ij})} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du \right\}^2
\]

\[
\leq \sum_{i=1}^{n} m_i^{-1} \sum_{j=1}^{m_i} \mathbb{E} \left\{ \int_0^{-C_n^{1/2} g(T_{ij})} \{ \psi(\epsilon_{ij} + u) - \psi(\epsilon_{ij}) \} du \right\}^2
\]

\[
\leq M_2 \sum_{i=1}^{n} m_i^{-1} \sum_{j=1}^{m_i} C_n^{1/2} g(T_{ij}) \left| \int_0^{-C_n^{1/2} g(T_{ij})} u du \right|
\]

\[
\leq M_2 2^{-1} c_r \lambda^{-1/4r} C_n^{3/2} \sum_{i=1}^{n} m_i^{-1} \sum_{j=1}^{m_i} g(T_{ij})^2
\]

\[
\leq n M_2^* D^3 \lambda^{-1/4r} C_n^{3/2}, \quad \tag{31}
\]

for some \( M_2^* > 0 \) and all \( g \in B_D \). Furthermore, it is easy to see that for \((g_1, g_2) \in B_D \times B_D\) one has

\[
|I_{in}(g_1) - I_{in}(g_2)| \leq 2M_1 C_n^{1/2} \| g_1 - g_2 \|_{\infty}. \quad \tag{32}
\]

Fix \( \epsilon > 0 \) and cover the ball \( B_D \) with a collection of balls \( B_j \) \((j = 1, \ldots, N)\) such that the radius of each ball in the sup-norm does not exceed \((8M_1)^{-1} DC_n^{1/2} \epsilon. \) The argument in the proof of Theorem 1 shows that we can have

\[
N \leq \exp \left( K \left\{ \frac{8M_1}{C_n^{1/2} \lambda^{1/2} \epsilon} \right\}^{1/r} \right),
\]

for some universal constant \( K > 0 \). Selecting \( g_j \in B_D \cap B_j \) \((j = 1, \ldots, N)\) from \((29)-(32)\)
and Bernstein's inequality it may now be seen that

\[
    \Pr \left( \sup_{g \in B_D} |R_n(g)| \geq \epsilon DC_n \right) \leq \sum_{j=1}^{N} \Pr \left( |R_n(g_j)| \geq \epsilon DC_n/2 \right)
\]

\[
\leq 2N \exp \left( -\frac{1}{8} \frac{n\lambda^{1/4} \epsilon^2 D^2 C_n^2}{M_2^* D^3 C_n^{5/2} + \epsilon/2M_1^* D^2 C_n^{5/2}} \right)
\]

\[
\sim 2 \exp \left( -K_1 nC_1^{1/2} \lambda^{1/4r} + K_2 C_n^{-1/2r} \lambda^{-1/2r} \right),
\]

for some constants \( K_1, K_2 > 0 \), depending only on \( D \) and \( \epsilon \). By our limit assumptions,

\[
(nC_n^{1/2} \lambda^{1/4r})^2 \geq n\lambda^{1/2r} \to \infty,
\]

and

\[
(nC_n^{1/2} C_n^{1/2r} \lambda^{3/4r})^2 \geq n^{(r-1)/r} \lambda^{3/2r} \to \infty.
\]

Therefore,

\[
\lim_{n \to \infty} \Pr \left( \sup_{g \in B_D} |R_n(g)| \geq \epsilon DC_n \right) = 0,
\]

and since \( \epsilon > 0 \) was arbitrary, (29) is established. The claim of the theorem follows through comparison of (26)–(29). In the case of a Lipschitz \( \psi \) the claim follows exactly the same way as in the proof of Theorem 1 with tighter bounds on (30) and (31).

\( \square \)

Proof of Corollary 2. This follows immediately from the embedding used in Corollary 1. \( \square \)

References

Adams, R.A., and Fournier, J.J.F. (2003) Sobolev Spaces, 2nd ed., Elsevier/Academic Press, Amsterdam.

Bai, Z.D., and Wu, Y. (1994) Limiting Behavior of M-estimators of Regression Coefficients in High Dimensional Linear models I. Scale-Dependent Case, J. Multivariate Anal. 51, 211–239.

Cai, T.T., and Yuan, M. (2011) Optimal estimation of the mean function based on discretely sampled functional data: Phase transition, Ann. Stat. 39, 2330–2355.

Cantoni, E. and, Ronchetti, E.M. (2001) Resistant selection of the smoothing parameter for smoothing splines, Statistics and Computing 11, 141–146.

Cardot, H., Cénac, P., and Zitt, P.A. (2013) Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm, Bernoulli 19, 18–43.
Chakraborty, A., and Chaudhuri, P. (2014) The spatial distribution in infinite dimensional spaces and related quantiles and depths, Ann. Stat. 42 1203–1231.

Cox, D.D. (1983) Asymptotics for M-type smoothing splines, Ann. Stat. 11 530–551.

Crambes, C., Kneip, A., and Sarda, P. (2009) Smoothing splines estimators for functional linear regression, Ann. Statist. 37 35–72.

Cucker, F. and Smale, S. (2001) On the Mathematical Foundations of Learning, Bul. Amer. Math. Soc. 39 1–49.

Cuesta-Albertos, J.A., and Fraiman, R. (2006) Impartial trimmed means for functional data, in: R.Y. Liu, R. Serfling, D. Souvaine (Eds.), Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications, American Mathematical Society, pp. 121–146.

Cunningham, J., Eubank, R.L., and Hsing, T. (1991) M-type smoothing splines with auxiliary scale estimation, Comput. Stat. Data Anal. 11 43–51.

Degras, D.A. (2008) Asymptotics for the nonparametric estimation of the mean function of a random process, Stat. Prob. Lett. 78 2976–2980.

Degras, D.A. (2011) Simultaneous confidence bands for nonparametric regression with functional data, Stat. Sinica 21 1735–1765.

DeVore, R.A., and Lorentz, G.G. (1993) Constructive Approximation, Springer, New York.

Eggermont, P.P.B., and LaRiccia, V.N. (2009) Maximum Penalized Likelihood Estimation, Volume II: Regression, Springer, New York.

Eubank, R.L. (1999) Nonparametric Regression and Spline Smoothing, 2nd ed., CRC Press, New York.

Fan, J., and Gijbels, I. (1996) Local polynomial modelling and its applications, CRC Press, Suffolk, UK.

Ferraty, F., and Vieu, P. (2006) Nonparametric Functional Data Analysis: Theory and Practice, Springer, New York.

Gervini, D. (2008) Robust functional estimation using the median and spherical principal components, Biometrika 95 587–600.

Hastie, T.J., Tibshirani, R.J., and Friedman, J. (2009) The elements of statistical learning: data mining, inference, and prediction, 2nd ed., Springer, New York.

Horváth, L., and Kokoszka, P. (2012) Inference for Functional Data With Applications, Springer, New York.
Hsing, T., and Eubank, R. (2015) Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators, Wiley, New York.

Huber, P.J. (1979) Robust Smoothing, in: R.L. Launer, G.N. Wilkinson (Eds.), Robustness in Statistics, Academic Press, New York, pp. 33–47.

Huber, P.J., and Ronchetti, E.M., Robust Statistics, 2nd ed., Wiley, Hoboken, NJ, 2009.

Hubert, M., Rousseeuw, P.J., and Segaert, P. (2015) Multivariate functional outlier detection, Stat. Methods Appl. 24 177–202.

Kalogridis, I. (2020) Asymptotics for M-type smoothing splines with non-smooth objective functions, arXiv paper, https://arxiv.org/abs/2002.04898.

Kokoszka, P., and Reimherr, M. (2017) Introduction to Functional Data Analysis, CRC Press, Boca Raton, FL.

Li, Y., and Hsing, T. (2010) Uniform convergence rates for nonparametric regression and principal component analysis in functional/longitudinal data, Ann. Stat. 38 3321–3351.

Li, G., Peng, H., and Zhu, L. (2011) Nonconcave penalized M-estimation with a diverging number of parameters, Stat. Sinica 21 391–419.

Maronna, R.A., Martin, D., and Yohai, V.J. (2006) Robust Statistics, Wiley, Chichester.

Maronna, R.A., and Yohai, V.J. (2013) Robust functional linear regression based on splines, Comput. Statist. Data Anal. 65 46–55.

Nychka, D., Martin, D., Haaland, P., and O’Connell, M. (1995) A nonparametric regression approach to syringe grading for quality improvement, J. Amer. Stat. Assoc. 90 1171-1178.

Panaretos, V.M., and Kraus, D. (2012) Dispersion operators and resistant second-order functional data analysis, Biometrika 99 813–832.

R Core Team (2018) R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria.

Ramsay, J.O. (1982) When the data are functions, Psychometrika 47 379–396.

Ramsay, J.O., and Dalzell, C.J. (1991) Some Tools for Functional Data Analysis, J. R. Stat. Soc. Ser. B 53 539–561.

Ramsay, J.O., and Silverman, B.W. (2005) Functional Data Analysis, Wiley, New York.

Rice, J.A., and Silverman, B.W. (1991) Estimating the mean and covariance structure nonparametrically when the data are curves, J. R. Stat. Soc. Ser. B 53 233–243.

Scheuerer, M. (2010) Regularity of the sample paths of a general second order random field, Stoch. Proc. Appl. 120 1879–1897.
Schultz, M.H. (1970) Error Bounds for Polynomial Spline Interpolation, Math. Comp. 24 507–515.

Sinova, B., González-Rodríguez, G., and Van Aelst, S. (2018) M-estimators of location for functional data, Bernoulli 24 2328–2357.

Wahba, G. (1990) Spline models for observational data, Siam, Philadelphia, Pen.

Wand, M.P., and Ripley, B. (2006) kernSmooth: Functions for kernel smoothing for Wand & Jones (1995), R package version 0.1-16.

Wood, S.N. (2017) Generalized Additive Models: An Introduction with R, 2nd ed., CRC Press, Boca Raton, FL.

Wu, W.B. (2007) M-estimation of linear models with dependent errors, Ann. Stat. 35 495–521.

van der Vaart, A.W. (1998) Asymptotic Statistics, Cambridge University Press, New York, NY.

Yao, F., Müller, H.-G., and Wang, J.-L. (2005) Functional data analysis for sparse longitudinal data, J. Amer. Stat. Assoc. 100 577–590.

Yohai, V.J., and Maronna, R.A. (1979) Asymptotic Behavior of M-Estimators for the Linear Model, Ann. Stat. 7 258–268.