Dual Actions for Born–Infeld and $Dp$-Brane Theories*

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Abstract Dual actions with respect to $U(1)$ gauge fields for the Born–Infeld and $Dp$-brane theories are reexamined. Taking into account an additional condition, i.e. a corollary to the field equation of the auxiliary metric, one obtains an alternative dual action that does not involve the infinite series in the auxiliary metric given by [M. Abou Zeid and C.M. Hull, Phys. Lett. B 428 (1998) 277], but just picks out the first term from the series formally. New effective interactions of the theories are revealed. That is, the new dual action gives rise to an effective interaction in terms of one interaction term rather than infinitely many terms of different (higher) orders of interactions physically. However, the price paid for eliminating the infinite series is that the new action is not quadratic but highly nonlinear in the Hodge dual of a $(p−1)$-form field strength. This non-linearity is inevitable under the requirement that the two dual actions are equivalent.

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1 Introduction and Summary

It is a common point of view that the remarkable progress of string theory[1] is the discovery of Dirichlet $p$-branes.[2] Geometrically, the $Dp$-branes are $(p + 1)$-dimensional hypersurfaces that are embedded in a higher dimensional spacetime. Dynamically, they are solitonic solutions to string equations that are “branes” on which open strings attach with Dirichlet boundary conditions. The dynamics of $Dp$-branes is induced by the open strings and governed in general by the action of the Born–Infeld type.[3] Recently, many different actions for generalizations of the Born–Infeld have been proposed to describe the effective worldvolume theories of $Dp$-branes. For instance, see Refs. [4–18] where quite interesting are the action with quadratic abelian field strengths[14] and its conformal invariant development.[15] The motivation for introducing an auxiliary metric[14] and restoring a conformal symmetry[15] lies particularly in simplifying quantization, which originates from the fact that the string action[19] with an auxiliary worldsheet metric and a conformal invariance greatly simplifies the analysis of string theory and allows a covariant quantization.[20] One more recent development,[18] nevertheless, depends on the introduction of two independent auxiliary metrics and gives various new actions some of which possess a so-called doubly conformal invariance.

In this paper we reexamine dual actions with respect to $U(1)$ gauge fields for Born–Infeld and $Dp$-brane theories. Our main point is that one can eliminate the auxiliary metric, which has some advantages we shall show later and is contrary to the treatment in Ref. [14] where one eliminates the field strength instead and then gets the dual version. In Ref. [14], the dual action is quadratic in a new two-form field strength that can be solved in terms of the Hodge dual of a $(p−1)$-form field strength, and involves an infinite series in the auxiliary metric. This infinite series includes in fact infinitely many different (higher) orders of interactions of the auxiliary metric and other fields. The quadratic form originates from its mother action that is quadratic in the abelian field strength. Here we give an alternative dual action in which this infinite series somehow does not appear but just its first term remains when we take into account an additional condition that was unnoticed before, i.e. one of corollaries to the field equation of the auxiliary metric. Besides its formalism that contains finite terms, physically this type of actions gives an effective interaction in which infinitely many terms of different (higher) orders of interactions are replaced by only the lowest order of the interaction of the auxiliary metric and other fields. However, the price paid is that the new dual action is highly nonlinear in the new two-form field strength which is, as already mentioned, the Hodge dual of a $(p−1)$-form field strength. It is the non-linearity that the two dual actions are equivalent classically. The reason lies in the difference of the two actions that is an infinite series of higher orders of interactions of the auxiliary metric and other fields. If there were no non-linearity, the two actions would not be equivalent because the infinitely many interacting terms could never be equal to a surface term. In a sense, our result uncovers a new phenomenon existed in classically equivalent theories of the BIons and $Dp$-branes, that is, the interaction expressed by an infinite

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series with an independent auxiliary metric is effective to the one that picks out only the first term from the series but with a constrained auxiliary metric.

2 Dual Actions for Born–Infeld

For the sake of convenience to compare our result with that of Ref. [14], we use the same notation as that adopted by that reference. Let us start with the mother action for the Born–Infeld theory in \( p + 1 \) spacetime dimensions proposed in Ref. [14],

\[
S = -\frac{T_p}{4} \int d^{p+1}x (-g)^{1/4} (-\gamma)^{1/4} \times [\gamma^{\mu\nu}(g_{\mu\nu} - g^{\rho\sigma} F_{\mu\rho} F_{\sigma\nu}) - (p - 3)],
\]

which is quadratic in the gauge field strength \( F_{\mu\nu} \) as mentioned above. Various symbols stand for as follows:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}
\]

is the field strength of an abelian gauge field \( A_\mu \), some Greek lowercase letters, for example, \( \mu, \nu, \rho, \sigma \), running over 0, 1, \ldots, \( p \), are used as spacetime indices and \( g_{\mu\nu} \) is the spacetime metric. \( \gamma_{\mu\nu} \) is the auxiliary metric that is introduced in order to rewrite the Born–Infeld action as the quadratic form in \( F_{\mu\nu} \). Different from the case occurred earlier,[13] here the auxiliary metric is symmetric. Moreover, \( g^{\mu\nu} \) and \( \gamma^{\mu\nu} \) mean the inverse of \( g_{\mu\nu} \) and \( \gamma_{\mu\nu} \), respectively, and \( g \equiv \det(g_{\mu\nu}), \gamma \equiv \det(\gamma_{\mu\nu}) \).

In order to derive the dual form of Eq. (1) with respect to the gauge field \( A_\mu \), we impose a Lagrange multiplier term upon the mother action and thus construct such an action

\[
S' = -\frac{T_p}{4} \int d^{p+1}x \left\{ (-g)^{1/4} (-\gamma)^{1/4} \left[ \gamma^{\mu\nu}(g_{\mu\nu} - g^{\rho\sigma} F_{\mu\rho} F_{\sigma\nu}) - (p - 3) \right] + 2 \hat{H}^{\mu\nu}(F_{\mu\nu} - \partial_\mu A_\nu) \right\},
\]

where \( \hat{H}^{\mu\nu} \) is introduced as an auxiliary tensor field and \( F_{\mu\nu} \) is regarded at present as an independent tensor field. Now varying Eq. (3) with respect to \( \hat{H}^{\mu\nu} \) simply gives the definition of the abelian field strength Eq. (2), together with which Eq. (3) turns back to the mother action Eq. (1). This does not provide anything new but just shows the classical equivalence between the two actions. However, varying Eq. (3) with respect to \( A_\mu \) leads to the equation that \( \hat{H}^{\mu\nu} \) satisfies,

\[
\partial_\mu \hat{H}^{\mu\nu} = 0, \tag{4}
\]

which can be solved in terms of the Hodge dual of a \((p-1)\)-form field strength \( \partial_\mu \tilde{A}_{\sigma_1 \cdots \sigma_{p-1}} \),

\[
\hat{H}^{\mu\nu} = \frac{1}{(p - 1)!} \epsilon^{\mu\nu\rho_1 \cdots \rho_{p-2}} \partial_\rho \tilde{A}_{\sigma_1 \cdots \sigma_{p-2}}, \tag{5}
\]

where \( \epsilon^{\mu\nu\rho_1 \cdots \rho_{p-2}} \) is the alternating tensor density in \( p+1 \) spacetime dimensions and \( \tilde{A}_{\sigma_1 \cdots \sigma_{p-2}} \) is a \((p-2)\)-form potential that is introduced for solving \( \hat{H}^{\mu\nu} \). Next, dealing with \( F_{\mu\nu} \) as an independent variable, we obtain its field equation by making variation of Eq. (3) with respect to this tensor field,

\[
-(-g)^{1/4} (-\gamma)^{1/4} (g^{\rho\sigma} F_{\rho\sigma} \gamma^{\mu\nu} + \gamma^{\rho\sigma} F_{\rho\sigma} g^{\mu\nu}) = 2 \hat{H}^{\mu\nu}, \tag{6}
\]

where \( \hat{H}^{\mu\nu} \) is given at present by the solution Eq. (5).

In the present stage, the usual way of deriving the dual of Eq. (1), as adopted in Ref. [14], is to solve from Eq. (6) the tensor field \( F_{\mu\nu} \) in terms of \( \gamma_{\mu\nu} \), \( g_{\mu\nu} \), and \( \hat{H}^{\mu\nu} \) and then to substitute the solution into Eq. (3). Because of the complexity of Eq. (6), such a solution contains an infinite series in the auxiliary metric \( \gamma_{\mu\nu} \). Nevertheless, the merit of the corresponding dual action is obvious, that is, this dual action is quadratic in the new two-form field strength \( \hat{H}^{\mu\nu} \).

\[\footnote{For simplicity but without losing generality, let the parameter \( \Lambda \) be unit in this note.}\]

Instead of solving Eq. (6) directly, we provide an alternative way to deal with this equation. Considering the variation of Eq. (3) with respect to \( \gamma_{\mu\nu} \), we have the field equation of the auxiliary metric,

\[
\gamma^{\mu\nu} = g^{\mu\nu} - F_{\rho\sigma} g^{\mu\rho} F_{\nu\sigma}, \tag{7}
\]

which takes the same form as that derived from Eq. (1), i.e., the addition of the Lagrange multiplier to Eq. (1) does not change the formulation of the field equation of the auxiliary metric. Note that \( F_{\mu\nu} \) in Eq. (3) should be treated as an implicit functional of \( \gamma_{\mu\nu} \) that is now constrained by Eq. (6) when we derive Eq. (7), while \( F_{\mu\nu} \) of Eq. (1) is independent of the auxiliary metric. It is easier to derive Eq. (7) from Eq. (1) as the two actions, Eq. (1) and Eq. (3), are equivalent classically. Alternatively, one derives from the action Eq. (3) the equations of motion firstly for \( \gamma_{\mu\nu} \) and then for \( F_{\mu\nu} \), which gives rise to the same formulations as Eq. (7) and Eq. (6), respectively. The reason is obvious, that is, the equations of motion for \( F_{\mu\nu} \) and \( \gamma_{\mu\nu} \) are independent of the order they are deduced from the same action.

Several corollaries of the field equation of the auxiliary metric Eq. (7) can be obtained, among which the useful one for our purpose takes the form

\[
g^{\mu\rho} F_{\rho\sigma} \gamma^{\sigma\nu} = \gamma^{\mu\rho} F_{\rho\sigma} g^{\sigma\nu}, \tag{8}
\]

see Appendix for its proof. With this relation, \( F_{\mu\nu} \) can be solved easily from Eq. (6),

\[
F_{\mu\nu} = -(-g)^{-1/4} (-\gamma)^{-1/4} \frac{1}{2} \gamma_{\mu\nu} \hat{H}^{\rho\sigma} g_{\rho\sigma}, \tag{9}
\]

which, different from that given by Ref. [14], is no longer an infinite series. Now substituting Eq. (9) into Eq. (3), we thus derive an alternative dual action
In principle, an equivalent dual action which is obtained when Eq. (9) is substituted into Eq. (7). In fact this expression reveals in physics an effective interaction that is realized by the price of the non-linearity in the new two-form field strength $\tilde{H}^{\mu\nu}$ as emphasized above (see also Eq. (11) below). The new dual action seems to be quadratic in $\tilde{H}^{\mu\nu}$, but in fact highly nonlinear. The reason is that the auxiliary metric $\tilde{\gamma}_{\mu\nu}$ in Eq. (10) is no longer a free variable but an implicit functional of $g_{\mu\nu}$ and $\tilde{H}^{\mu\nu}$ as follows:

$$\gamma_{\mu\nu} = g_{\mu\nu} - (-g)^{-1/2}(\gamma)_{\mu\nu} + (-g)^{-1/2} H^{\rho\sigma} \gamma_{\rho\sigma} H^{-\lambda\nu} g_{\lambda\nu},$$

(11)

which is obtained when Eq. (9) is substituted into Eq. (7). In principle, an equivalent dual action $S_D[g_{\mu\nu}, \tilde{H}^{\mu\nu}]$ can then be derived by solving $\gamma_{\mu\nu}$ from Eq. (11) and substituting the solution into Eq. (10), but in practice this procedure is quite difficult to carry out because of the non-linearity in $\gamma_{\mu\nu}$ in Eq. (11). Although the procedure is not performed both to our result and to that of Ref. [14], the explicit relation, i.e. Eq. (11) involved by the three variables ($\gamma_{\mu\nu}$, $g_{\mu\nu}$, and $\tilde{H}^{\mu\nu}$) is provided here while to obtain such a relation would be hard and tedious from the action with the infinite series in $\gamma_{\mu\nu}$ by means of a perturbative way suggested in Ref. [14].

Anyway, we give an alternative dual action with respect to U(1) gauge fields for the Born–Infeld theory and note that it is characterized by a new interaction formula. As to the motivation, we try to reveal as many dual-hips as possible existed in both the BIons and bosons and bosonic $p$-branes in the previous work.

We can write several different formulations of the dual action Eq. (10) which might be of interest. The first is related to the following definition of $H^{\mu\nu}$,

$$H^{\mu\nu} \equiv (-g)^{-1/4}(\gamma)^{-1/4} \tilde{H}^{\mu\nu},$$

(12)

with which the dual action takes a seemingly elegant form

$$S_D = -\frac{T_p}{4} \int d^{p+1}x (-g)^{-1/4} (\gamma)^{-1/4} \left\{ [\gamma_{\mu\nu} (g_{\mu\nu} - (p - 3))] + (-g)^{-1/2}(\gamma)^{-1/2} H^{\mu\sigma} \gamma_{\mu\rho} g_{\rho\sigma} \tilde{H}^{\nu\rho} \right\}. $$

(10)

Note that the new dual action does not involve any infinite series and in particular the term related to $\tilde{H}^{\mu\nu}$ is, just formally, the first term of that infinite series.\[14\] In fact this expression reveals in physics an effective interaction that is realized by the price of the non-linearity in the new two-form field strength $\tilde{H}^{\mu\nu}$ as emphasized above (see also Eq. (11) below). The new dual action seems to be quadratic in $\tilde{H}^{\mu\nu}$, but in fact highly nonlinear. The reason is that the auxiliary metric $\tilde{\gamma}_{\mu\nu}$ in Eq. (10) is no longer a free variable but an implicit functional of $g_{\mu\nu}$ and $\tilde{H}^{\mu\nu}$ as follows:

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(10)

3 Dual Actions for Dp-branes

As the Dp-brane kinetic term takes the form of the Born–Infeld type,[3] the dualization is therefore quite similar. The starting point is the action[14]

$$S = -\frac{T_p}{4} \int d^{p+1}x (-g)^{-1/4}(\gamma)^{-1/4} \left[ \gamma^{\mu\nu} (g_{\mu\nu} - g^{\rho\sigma} F_{\mu\rho} F_{\sigma\nu}) - (p - 3) \right],$$

(18)

where

$$F_{\mu\nu} \equiv F_{\mu\nu} - B_{\mu\nu},$$

(19)

$\phi$, $g_{\mu\nu}$, and $B_{\mu\nu}$ are pullbacks to the worldvolume of the background dilaton, spacetime metric and NS antisymmetric two-form fields, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $A_\mu(\xi)$ the U(1) worldvolume gauge field. $\gamma_{\mu\nu}$ is the auxiliary worldvolume metric that is introduced for the same purpose as that of the Born–Infeld case. Now Greek lowercase letters, $\mu, \nu, \rho, \ldots,$
running over 0, 1, . . . , p are utilized as indices in the worldvolume that is spanned by p + 1 arbitrary parameters \( \xi^\mu \). The next step is to add a Lagrange multiplier term to the above action and therefore one obtains its classically equivalent form

\[
S' = -\frac{T_p}{4} \int d^{p+1} \xi \{ e^{-\phi} (-g)^{1/4} (-\gamma)^{1/4} [\gamma_{\mu\nu} (g_{\mu\nu} - g^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu}) - (p - 3)] + 2 \bar{H}^{\mu\nu} (F_{\mu\nu} - \partial_{[\mu} A_{\nu]} \} .
\]

Making variation of Eq. (20) with respect to \( A_\mu \) leads to the same expression as Eq. (5) in which \( \bar{H}^{\mu\nu} \) has been solved in terms of a \((p - 2)\)-form potential \( A_{1, \ldots, p-2} \) in the worldvolume. Moreover, doing for the two-form \( F_{\mu\nu} \) gives its field equation

\[
-e^{-\phi} (-g)^{1/4} (-\gamma)^{1/4} (g^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} + \gamma^{\mu\nu} F_{\rho\sigma} g^{\rho\sigma}) = 2 \bar{H}^{\mu\nu} ,
\]

which looks like Eq. (6) formally just with the replacement of \( F_{\mu\nu} \) by \( e^{-\phi} F_{\mu\nu} \). The following step is to derive from either Eq. (18) or Eq. (20) the field equation of the auxiliary metric,

\[
\gamma_{\mu\nu} = g_{\mu\nu} - F_{\mu\rho} g^{\rho\sigma} F_{\sigma\nu} ,
\]

which, through a similar proof to that of the Born–Infeld theory, also leads to a useful corollary

\[
g^{\mu\rho} F_{\rho\sigma} \gamma_{\sigma\nu} = \gamma^{\mu\rho} F_{\rho\sigma} g^{\sigma\nu} .
\]

Using this relation, one can solve \( F_{\mu\nu} \) easily from Eq. (21),

\[
F_{\mu\nu} = -e^{\phi} (-g)^{-1/4} (-\gamma)^{-1/4} \gamma_{\mu\nu} \bar{H}^{\rho\sigma} g_{\rho\sigma} .
\]

The final step is thus to substitute Eq. (24) into Eq. (20) and at last one deduces an alternative dual action for the \( Dp \)-branes,

\[
S_D = -\frac{T_p}{4} \int d^{p+1} \xi e^{-\phi} (-g)^{1/4} (-\gamma)^{1/4} \left[ [\gamma_{\mu\nu} g_{\mu\nu} - (p - 3)] + 2 e^{\phi} (-g)^{-1/4} (-\gamma)^{-1/4} \bar{H}^{\mu\nu} B_{\mu\nu} + e^{2\phi} (-g)^{-1/2} (-\gamma)^{-1/2} \bar{H}^{\mu\sigma} \gamma_{\mu\rho} g_{\rho\sigma} \bar{H}^{\rho\nu} \right] .
\]

One can see that this kind of dual actions has nothing to do with any infinite series in \( \gamma_{\mu\nu} \), and that the term related to the square of \( \bar{H}^{\mu\nu} \) is just the first term of that infinite series appeared in Ref. [14]. However, \( \gamma_{\mu\nu} \) in Eq. (25) is not free but constrained by

\[
\gamma_{\mu\nu} = g_{\mu\nu} - e^{\phi} (-g)^{-1/2} (-\gamma)^{-1/2} \gamma_{\mu\rho} \bar{H}^{\rho\sigma} \gamma_{\sigma\nu} \bar{H}^{\lambda\nu} .
\]

Equation (25), together with Eq. (26), shows a kind of physically effective interactions that contains finite terms. It is the non-linearity in \( \bar{H}^{\mu\nu} \) that this action is classically equivalent to that of Ref. [14]. Incidentally, when \( \bar{H}^{\mu\nu} \) is replaced by \( H^{\mu\nu} \) defined by

\[
H^{\mu\nu} \equiv e^{\phi} (-g)^{-1/4} (-\gamma)^{-1/4} \bar{H}^{\mu\nu} ,
\]

the dual action Eq. (25) becomes much simpler in formalism,

\[
S_D = -\frac{T_p}{4} \int d^{p+1} \xi e^{-\phi} (-g)^{1/4} (-\gamma)^{1/4} \left[ [\gamma_{\mu\nu} g_{\mu\nu} (\gamma_{\mu\nu} + \gamma_{\rho\mu} H^{\mu\rho} H^{\nu\sigma})] + 2 H^{\mu\nu} B_{\mu\nu} - (p - 3) \right] .
\]

The further discussions on the dual actions of \( Dp \)-branes are similar to that of the Born–Infeld and omitted here.

**Appendix: Proof of the Formula: \( G^{-1} F G^{-1} \)**

Here \( G \), \( \Gamma \), and \( F \) stand for the matrices of \( g_{\mu\nu} \), \( \gamma_{\mu\nu} \), and \( F_{\mu\nu} \), respectively. As a result, the above formula is the matrix form of the component one \( g^{\mu\nu} F_{\rho\sigma} \gamma^{\rho\sigma} = \gamma^{\mu\nu} F_{\rho\sigma} g^{\rho\sigma} \). The starting point of this proof is the field equation of the auxiliary metric Eq. (7) whose matrix form can be written as \( \Gamma = G - FG^{-1} \).

**Lemma** If two matrices \( A \) and \( B \) are commutative, i.e. \( AB = BA \), and one of them is invertible, say \( A \), then \( A^{-1} \) and \( B \) must be commutative, i.e. \( A^{-1} B = BA^{-1} \).

The proof of this lemma is obvious, that is, multiplying by \( A^{-1} \) to the left and right successively on both sides of \( AB = BA \) leads directly to \( A^{-1} B = BA^{-1} \).

Let us turn to the proof of the formula. Rewrite the field equation of the auxiliary metric as

\[
\Gamma = G (1 - G^{-1} F G^{-1} F) ,
\]

whose inverse thus takes the form

\[
\Gamma^{-1} = (1 - G^{-1} F G^{-1} F)^{-1} G^{-1} .
\]

Multiplying by \( F \) to the right on both sides of the above equation gives

\[
\Gamma^{-1} F = (1 - G^{-1} F G^{-1} F)^{-1} G^{-1} F .
\]

If let \( A \equiv 1 - G^{-1} F G^{-1} F \) and \( B \equiv G^{-1} F \), with the above lemma Eq. (31) becomes

\[
\Gamma^{-1} F = G^{-1} F (1 - G^{-1} F G^{-1} F)^{-1} = G^{-1} F G^{-1} ,
\]

where \((1 - G^{-1} F G^{-1} F)^{-1} = \Gamma^{-1} G \) has been used to the last equality. Equation (32) is just the formula after \( G^{-1} \) is multiplied to the right on both sides of the equation.
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