RUAN’S CONJECTURE ON SINGULAR SYMPLECTIC FLOPS

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Abstract. We prove that the orbifold quantum ring is preserved under singular symplectic flops. Hence we verify Ruan’s conjecture for this case.

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1. Introduction

One of deep discovery in Gromov-Witten theory is its intimate relation with the birational geometry. A famous conjecture of Ruan asserts that any two $K$-equivalent manifolds have isomorphic quantum cohomology rings ([Ruan]) (see also [Wang]). Ruan’s conjecture was proved by Li-Ruan for smooth algebraic 3-folds ([LR]) almost ten years ago. Only recently, it was generalized to simple flops and Mukai flops in arbitrary dimensions by Lee-Lin-Wang ([LLW]). In a slightly different context, there has been a lot of activities regarding Ruan’s conjecture in the case of McKay correspondence.

On the other hand, it is well known that the appropriate category to study the birational geometry is not smooth manifolds. Instead, one should consider the singular manifolds with terminal singularities. In the complex dimension three, the terminal singularities are the finite quotients of hypersurface singularities and hence the deformation of them are orbifolds. It therefore raises the important questions if Ruan’s conjecture still holds for the orbifolds where there are several very interesting classes of flops. This is the main topic of the current article.

Li-Ruan’s proof of the case of smooth 3-folds consists of two steps. The first step is to interpret flops in the symplectic category, then, they use almost complex deformation to reduce the problem to the simple flop; the second step is to calculate the change of quantum cohomology under the simple flop. The description of a smooth simple flop is closely related to the conifold singularity

$$W_1 = \{(x, y, z, t) | xy - z^2 + t^2 = 0\}.$$  

In [CLZZ], we initiate a program to understand the flop associated with the singularities

$$W_r = \{(x, y, z, t) | xy - z^{2r} + t^2 = 0\} / \mu_r(a, -a, 1, 0).$$

$W_r$ appears in the list of terminal singularities in [K]. The singularities without quotient are also studied in [La] and [BKL].

The program is along the same framework of that in [LR]. The first step is to describe the flops with respect to $W_r$ symplectically. This is done in the previous paper ([CLZZ]). Our main theorem in this paper is

**Theorem 1.1.** Suppose that $Y^s$ is a symplectic 3-fold with orbifold singularities of type $W_{r_1}, \ldots, W_{r_n}$ and $Y^{sf}$ is its singular flop, then

$$QH_{CR}(Y^s) = QH_{CR}(Y^{sf}).$$

Theorem 1.1 verifies Ruan’s conjecture in this particular case. We should mention that Ruan also proposed a simplified version of the above conjecture in terms of Ruan cohomology $RH_{CR}$ which has been established in [CLZZ] as well. Furthermore, our previous results enters the proof of this general conjecture in a crucial way.
The technique of the proof is a combination of the degeneration formula of orbifold Gromov-Witten invariants, the localization techniques and dimension counting arguments. The theory of relative orbifold Gromov-Witten invariants and its degeneration formula involves heavy duty analysis on moduli spaces and will appear elsewhere [CLS].

The paper is organized as following. We first describe the relative orbifold GW-invariants and state the degeneration formula (without proof) (§2). Then, we summary the result of [CLZZ] on the singular symplectic flops and Ruan cohomology (§3). The heart of the proof is a detail analysis of relative orbifold GW-theory on local models (§4 and §5). The main theorem is proved in §6.

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2. Relative orbifold Gromov-Witten theory and the degeneration formula

2.1. The Chen-Ruan Orbifold Cohomologies. Let $X$ be an orbifold. For $x \in X$, if its small neighborhood $U_x$ is given by a uniformization system $(\tilde{U}, G, \pi)$, we say $G$ is the isotropy group of $x$ and denoted by $G_x$. Let

$$T = \left( \bigcup_{x \in X} G_x \right) / \sim.$$

Here $\sim$ is certain equivalence relation. For each $(g) \in T$, it defines a twisted sector $X_{(g)}$. At the mean while, the twisted sector is associated with a degree-shifting number $\iota(g)$. The Chen-Ruan orbifold cohomology is defined to be

$$H^*_\text{CR}(X) = H^*(X) \oplus \bigoplus_{(g) \in T} H^{*-2\iota(g)}(X_{(g)}).$$

For details, readers are referred to [CR1].

2.2. Orbifold Gromov-Witten invariants. Let $\overline{M}_{g,n,A}(X)$ be the moduli space of representable orbifold morphism of genus $g$, $n$-marked points and $A \in H_2(X, \mathbb{Z})$ (cf. [CR2], [CR3]). By specifying the monodromy

$$h = ((h_1), \ldots, (h_n))$$

at each marked points, we can decompose

$$\overline{M}_{g,n,A}(X) = \bigsqcup_{h} \overline{M}_{g,n,A}(X, h).$$

Let

$$\text{ev}_i : \overline{M}_{g,n,A}(X, h) \to X_{(h_i)}, 1 \leq i \leq n$$
be the evaluation maps. The primary orbifold Gromov-Witten invariants are defined as
$$
\langle \alpha_1, \ldots, \alpha_n \rangle_{g,A}^X = \int_{\overline{M}_{g,n,A}(X,h)} m \prod_{i=1}^n ev_i^*(\alpha_i),
$$
where $\alpha_i \in H^*(X_{(h_i)})$.

In particular, set
$$
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,0},
\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR} + \sum_{A \neq 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,A}.
$$

2.3. Ring structures on $H^*_{CR}(X)$. Let $V$ be a vector space over $R$. Let
$$
h : V \otimes V \to R
$$
be a non-degenerate pairing and
$$
A : V \otimes V \otimes V \to R
$$
be a triple form. Then it is well known that one can define a product $*$ on $V$ by
$$
h(u * v, w) = A(u, v, w).
$$
Different $A$’s give different products.

**Remark 2.1.** Suppose we have $(V, h, A)$ and $(V', h', A')$. A map $\phi : V \to V'$ induces an isomorphism (with respect to the product) if $\phi$ is a group isomorphism and
$$
\phi^* h' = h, \quad \text{and} \quad \phi^* A' = A.
$$

Now let $V = H^*_{CR}(X)$ and $h$ be the Poincare pairing on $V$. If
$$
A(\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR},
$$
it defines the Chen-Ruan product. If
$$
A(\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle,
$$
it defines the Chen-Ruan quantum product. We denote the ring to be $QH^*_{CR}(X)$.

2.4. Moduli spaces of relative stable maps for orbifold pairs. For the relative stable maps for the smooth case, there are two equivariant versions. One is on the symplectic manifolds with respect to cylinder ends, each of which admits a Hamiltonian $S^1$ action ([LR]), the other is on the closed symplectic manifolds with respect to divisors([LR, Li]). This is also true for orbifolds. We adapt the second version here.

Let $X$ be a symplectic orbifold with disjoint divisors
$$
\{Z_1, \ldots, Z_k\}.
$$
For simplicity, we assume $k = 1$ and $Z = Z_1$. 
By a relative stable map in \((X, Z)\), we mean a stable map \(f \in \mathcal{M}_{g,n,A}(X)\) with additional data that record how it intersects with \(Z\). Be precisely, suppose
\[ f : (\Sigma, z) \rightarrow X. \]
The additional data are collected in order:

- Set \( \mathbf{x} = f^{-1}(Z) = \{x_1, \ldots, x_k\} \).
  We call \(x_i\) the relative marked points. The rest of marked points are denoted by
  \( \mathbf{p} = \{p_1, \ldots, p_m\} \), i.e., \( \mathbf{z} = \mathbf{x} \cup \mathbf{p} \);
- Let \( \mathbf{g} = ((g_1),\ldots,(g_k)) \) denote the monodromy of \(f\) (with respect to \(Z\)) at each point in \(\mathbf{x}\). The rest are denoted by
  \( \mathbf{h} = ((h_1),\ldots,(h_m)) \) which are the monodromy of \(f\) (with respect to \(X\)) at each point in \(\mathbf{p}\).
- the multiplicity of the tangency \(\ell_j\) of \(f\) with \(Z\) at \(z_j = f(x_j)\) is defined by the following: Locally, the neighborhood of \(z_j\) is given by
  \[ (V \times \mathbb{C} \rightarrow \tilde{V})/G_{z_j}, \]
  where \(\tilde{V}/G_{z_j} \subset Z\). Suppose the lift of \(f\) is
  \[ \tilde{f} : \tilde{\mathcal{D}} \rightarrow \tilde{V} \times \mathbb{C} \]
  \[ \tilde{f}(t) = (v(t), u(t)) \]
  Suppose the multiplicity of \(u\) is \(\alpha\) and \(g_j \in G_{z_j}\) acts on the fiber over \(z_j\) with multiplicity \(c\). Then the multiplicity is set to be
  \[ \ell_j = \frac{\alpha \cdot c}{|g_j|}. \]
  We say \(f\) maps \(x_j\) to \(Z_{g_j}\) at \(\ell_j z_j\). Set
  \[ \ell = (\ell_1, \ldots, \ell_k). \]
  We may write
  \[ f^{-1}(Z) = \ell \cdot \mathbf{x} = \sum_{j=1}^{k} \ell_j x_j. \]
  As a relative object, we say \(f\) is in the moduli space of relative map
  \[ \mathcal{M}_{g,n,A}(X, Z, \mathbf{h}, \mathbf{g}, \ell). \]
  We denote the map by
  \[ f : (\Sigma, \mathbf{p}, \ell \cdot \mathbf{x}) \rightarrow (X, Z). \]
We now describe the compactification of this moduli space. The construction is similar to the smooth case ([LR]).

The target space of a stable relative map is no longer \( X \). Instead, it is extended in the following sense: let \( L \rightarrow Z \) be the normal bundle of \( Z \) in \( X \) and \( PZ = \mathbb{P}(L \oplus \mathbb{C}) \) be its projectification, then given an integer \( b \geq 0 \), we have an extended target space
\[
X^b := X \cup \bigcup_{1 \leq \alpha \leq b} PZ^\alpha.
\]
Here \( PZ^\alpha \) denotes the \( \alpha \)-th copy of \( PZ \). Let \( Z^0_\alpha \) be the 0-section and \( Z^\infty_\alpha \) be the \( \infty \)-section of \( PZ^\alpha \). \( X \) is called the root component of \( X^b \). \( Z^0_\alpha \) is called the divisor of \( X^b \) and is (again) denoted by \( Z \).

**Definition 2.1.** A relative map in \( X^b \) consists of following data: on each component, there is a relative map: on the root component, the map is denoted by \( f_0 : (\Sigma_0, p_0, l_0 \cdot x_0) \rightarrow (X, Z) \); and on each component \( PZ^\alpha \), the map is denoted by \( f^\alpha : (\Sigma^\alpha, p^\alpha, l^\alpha \cdot x^\alpha \cup l^\alpha \cdot \hat{x}^\alpha) \rightarrow (PZ^\alpha, Z^0_\alpha \cup Z^\infty_\alpha) \).

Here \( l^\alpha \cdot x^\alpha = f^{-1}(Z^0_\alpha) \) and \( \hat{l}^\alpha \cdot \hat{x}^\alpha = f^{-1}(Z^\infty_\alpha) \). Moreover, we require \( f^\alpha \) at \( Z^0_\alpha \) matches \( f^{\alpha+1} \) at \( Z^\infty_{\alpha+1} \). (see Remark 2.2.)

We denote such a map by \( f = (f_0, f^1, \ldots, f^b) \).

Set \( x = x^b \) and \( g_j = g_{x_j}^b, \quad g = (g_1, \ldots, g_{|x|}) \)
\[
l_j = l^b_j, \quad l = (l_1, \ldots, l_{|x|}).
\]

We say that \( f \) maps \( x_j \) to the divisor \( Z \) of \( X^b \) at \( l_j z_j \in Z(g_j) \). Similarly, \( h \) records the twisted sector for \( p = p^0 \cup \bigcup_{\alpha} p^\alpha \).

The homology class \( A \) in \( X \) represented by \( f \) can be defined properly. Collect the data
\[
\Gamma = (g, A, h, g, l), \quad \mathcal{T} = (g, l).
\]

We say that \( f \) is a relative orbifold map in \( X^b \) of type \((\Gamma, T)\). Denote the moduli space by \( \tilde{M}_{\Gamma, T}(X^b, Z) \).

**Remark 2.2.** Let \( f^\alpha \) and \( f^{\alpha+1} \) be as in the definition. Suppose that
- \( f^\alpha \) maps \( x_j^\alpha \in x^\alpha \) to \( (Z^0_\alpha)(g_j^\alpha) \) at \( l^\alpha k_j z_j^0 \);
- \( f^{\alpha+1} \) maps \( \hat{x}_i^{\alpha+1} \in \hat{x}^{\alpha+1} \) to \( (Z^\infty_{\alpha+1})(g_i^{\alpha+1}) \) at \( \hat{l}_i^{\alpha+1} \hat{z}_i^{\alpha+1} \),
then by saying that \( f^\alpha \) at \( Z^\alpha_0 \) matches \( f^\alpha+1 \) at \( Z^\alpha_\infty \) we mean that
\[
\ell^\alpha_i = \bar{\ell}^\alpha+1_i, \quad z^\alpha_i = \bar{z}^\alpha+1_i, \quad \text{and} \quad g^\alpha_i = \bar{g}^\alpha+1_i.
\]

Note that there is a \( \mathbb{C}^* \) action on \( PZ^\alpha_\alpha \). Let \( T \) be the product of these \( b \) copies of \( \mathbb{C}^* \). Then \( T \) acts on \( \tilde{\mathcal{M}}_{\Gamma,T}(X^\sharp_b, Z) \). Define
\[
\mathcal{M}_{\Gamma,T}(X^\sharp_b, Z) = \tilde{\mathcal{M}}_{\Gamma,T}(X^\sharp_b, Z)/T.
\]

It is standard to show that

**Proposition 2.3.** There exists a large integer \( B \) which depends on topological data \( (\Gamma, T) \) such that \( \mathcal{M}_{\Gamma,T}(X^\sharp_b, Z) \) is empty when \( b \geq B \).

Hence,

**Definition 2.2.** The compactified moduli space is
\[
\overline{\mathcal{M}}_{\Gamma,T}(X, Z) = \bigcup_{b \in \mathbb{Z} \geq 0} \mathcal{M}_{\Gamma,T}(X^\sharp_b, Z).
\]

The following technique theorem is proved in [CLS]

**Theorem 2.4.** \( \overline{\mathcal{M}}_{\Gamma,T}(X, Z) \) is a smooth compact virtual orbifold without boundary with virtual dimension
\[
2c_1(A) + 2(\dim_{\mathbb{C}} X - 3)(1 - g) + 2 \sum_{i=1}^{m} (1 - \iota(h_i)) + 2 \sum_{j=1}^{n} (1 - \iota(g_j) - [\ell_j]),
\]

where \( [\ell_j] \) is the largest integer that is less or equal to \( \ell_j \).

2.5. **Relative orbifold Gromov-Witten invariants.** Let \( \overline{\mathcal{M}}_{\Gamma,T}(X, Z) \) be the moduli space given above. There are evaluation maps
\[
ev^X_i : \overline{\mathcal{M}}_{\Gamma,T}(X, Z) \to X_{(h_i)}, \quad \text{ev}^X_i(f) = f(p_i), 1 \leq i \leq m; \quad \text{and}
\]
\[
ev^Z_j : \overline{\mathcal{M}}_{\Gamma,T}(X, Z) \to Z_{(g_j)}, \quad \text{ev}^Z_j(f) = f(x_j), 1 \leq j \leq k.
\]

Then for
\[
\alpha_i \in H^*(X_{(h_i)}), 1 \leq i \leq m, \quad \beta_j \in H^*(Z_{(g_j)}), 1 \leq j \leq k
\]
the relative invariant is defined as
\[
\langle \alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_k, T \rangle_{\Gamma}^{(X, Z)}
\]
\[
= \frac{1}{|\text{Aut}(T)|} \int_{\overline{\mathcal{M}}_{\Gamma,T}(X, Z)} \prod_{i=1}^{m} (\text{ev}^X_i)^* \alpha_i \prod_{j=1}^{k} (\text{ev}^Z_j)^* \beta_j.
\]

In this paper, we usually set
\[
a = (\alpha_1, \ldots, \alpha_m), \quad b = (\beta_1, \ldots, \beta_k),
\]
then the invariant is denoted by \( \langle a | b, T \rangle_{\Gamma}^{(X, Z)} \).
Moreover, if $\Gamma = \coprod \gamma$, the relative invariants (with disconnected domain curves) is defined to be the product of each connected component

$$\langle a|b, T\rangle_{\Gamma}^{(X,Z)} = \prod_{\gamma} \langle a|b, T\rangle_{\Gamma_{\gamma}}^{(X,Z)}.$$

2.6. The degeneration formula. The symplectic cutting also holds for orbifolds. Let $X$ be a symplectic orbifold. Suppose that there is a local $S^1$ Hamiltonian action on $U \subset X$. We assume that

$$U \cong Y \times (-1,1)$$

and the projection onto the second factor

$$\pi_2: U \to (-1,1)$$

gives the Hamiltonian function. $Y \times \{0\}$ splits $X$ into two orbifolds with boundary $Y$, denoted by $X^{\pm}$. Then the routine symplectic cutting gives the degeneration

$$\pi: X \to \bar{X}^{+} \cup_{Z} \bar{X}^{-}.$$

Topologically, $\bar{X}^{\pm}$ is obtained by collapsing the $S^1$-orbits of the boundaries of $X^{\pm}$.

There are maps

$$\pi_*: H_2(X) \to H_2(X^{+} \cup_{Z} X^{-}), \quad \pi^*: H^*(X^{+} \cup_{Z} X^{-}) \to H^*(X).$$

For $A \in H_2(M)$ we set $[A] \subset H_2(X)$ to be $\pi_*^{-1}(\pi_*(A))$ and denote $\pi_*(A)$ by $(A^+, A^-)$. On the other hand, for $\alpha^{\pm} \in H^*(X^{\pm})$ with $\alpha^+_Z = \alpha^-_Z$, it defines a class on $H^*(X^{+} \cup_{Z} X^{-})$ which is denoted by $\langle \alpha^+, \alpha^- \rangle$. Let $\alpha = \pi^*(\alpha^+, \alpha^-)$.

**Theorem 2.5.** Suppose $\pi: X \to X^{+} \cup_{Z} X^{-}$ is the degeneration. Then

$$(2.1) \quad \langle a \rangle_{\Gamma}^{X} = \sum_{\eta} \sum_{T} C_{\eta} \langle a^+|b^I, T\rangle_{\Gamma^+}^{(X^+, Z)} \langle a^-|b^I, T\rangle_{\Gamma^-}^{(X^-, Z)}.$$

Notations in the formula are explained in order. $\Gamma$ is a data for Gromov-Witten invariants, it includes $(g, [A])$; $(\Gamma^+, \Gamma^-, I_\rho)$ is an admissible triple which consists of (possible disconnected) topological types $\Gamma^\pm$ with the same relative data $T$ under the identification $I_\rho$ and they glue back to $\Gamma$. (For instance, one may refer to [HLR] who interpret $\Gamma$’s as graphs and then the gluing has an obvious geometric meaning); the relative classes $\beta^i \in b^I$ runs over a basis of $Z_{(g_i)}$ and at the mean while $\beta_i$ runs over the dual basis; finally

$$C_{\eta} = \frac{1}{\text{Aut}(T)} \prod_{i=1}^{k} \ell_i,$$

for $T = (g, I)$.  

3. Singular symplectic flops

3.1. Local models and local flops. Locally, we are concern those resolutions of
\[ \tilde{W}_r = \{(x, y, z, t) | xy - z^{2r} + t^2 = 0\} \]
and their quotients. \( \tilde{W}_r \setminus \{0\} \) inherits a symplectic form \( \tilde{\omega}_r \) from \( \mathbb{C}^4 \).

By blow-ups, we have two small resolutions of \( \tilde{W}_r \):
\[ \tilde{W}_r^s = \{(x, y, z, t, [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 | xy - z^{2r} + t^2 = 0, \frac{p}{q} = \frac{x}{z^r - t} = \frac{y}{z^r + t}\} \]
\[ \tilde{W}_r^{sf} = \{(x, y, z, t, [p, q]) \in \mathbb{C}^4 \times \mathbb{P}^1 | xy - z^{2r} + t^2 = 0, \frac{p}{q} = \frac{x}{z^r + t} = \frac{y}{z^r - t}\} \]

Let \( \tilde{\pi}_r^s : \tilde{W}_r^s \to \tilde{W}_r \) and \( \tilde{\pi}_r^{sf} : \tilde{W}_r^{sf} \to \tilde{W}_r \) be the projections. The exceptional curves \( (\tilde{\pi}_r^s)^{-1}(0) \) and \( (\tilde{\pi}_r^{sf})^{-1}(0) \) are denoted by \( \tilde{\Gamma}_r^s \) and \( \tilde{\Gamma}_r^{sf} \) respectively. Both of them are isomorphic to \( \mathbb{P}^1 \).

Let \( \mu_r = \langle \xi \rangle, \xi = e^{\frac{2\pi i}{r}} \) be the cyclic group of \( r \)-th roots of 1. We denote its action on \( \mathbb{C}^4 \) by \( \mu_r(a, b, c, d) \) if the action is given by
\[ \xi \cdot (x, y, z, t) = (\xi^a x, \xi^b y, \xi^c z, \xi^d t). \]
Then \( \mu_r(a, -a, 1, 0) \) acts on \( \tilde{W}_r \), and naturally extending to its small resolutions. Set
\[ W_r = \tilde{W}_r / \mu_r, \quad W_r^s = \tilde{W}_r^s / \mu_r, \quad W_r^{sf} = \tilde{W}_r^{sf} / \mu_r. \]

Similarly,
\[ \Gamma_r^s = \tilde{\Gamma}_r^s / \mu_r, \quad \Gamma_r^{sf} = \tilde{\Gamma}_r^{sf} / \mu_r. \]

We call that \( W_r^s \) and \( W_r^{sf} \) are the small resolutions of \( W_r \). We say that \( W_r^{sf} \) is the flop of \( W_r^s \) and vice versa. They are both orbifolds with singular points on \( \Gamma_s \) and \( \Gamma_r^{sf} \). Note that the symplectic form \( \tilde{\omega}_r \) reduces to a symplectic form \( \omega_r^\circ \) on \( W_r \).

It is known that

**Proposition 3.1.** For \( r \geq 2 \), the normal bundle of \( \tilde{\Gamma}_r^s \) (\( \tilde{\Gamma}_r^{sf} \)) in \( \tilde{W}_r^s \) (\( \tilde{W}_r^{sf} \)) is \( \mathcal{O} \oplus \mathcal{O}(-2) \).

**Proof.** We take \( \tilde{W}_r^s \) as an example. For the set \( \Lambda_p = \{q \neq 0\} \), set \( u = p/q \). Then \((u, z, y)\) gives a coordinate chart for \( \Lambda_p \). Similarly, for the set \( \Lambda_q = \{p \neq 0\} \),
set \( v = q/p \). Then \((v, z, x)\) gives a coordinate chart for \( \Lambda_q \). The transition map is given by

\[
\begin{align*}
v &= u^{-1}; \\
z &= z; \\
x &= -u^2 y + 2uz^r.
\end{align*}
\]

By linearize this equation, it is easy to get the conclusion. q.e.d.

**Corollary 3.2.** For \( r \geq 2 \), the normal bundle of \( \Gamma_r^s (\Gamma_r^{sf}) \) in \( W_r^s (W_r^{sf}) \) is \((\mathcal{O} \oplus \mathcal{O}(-2))/\mu_r\).

On \( \tilde{\Gamma}_r^s (\tilde{\Gamma}_r^{sf}) \), there are two special points. In term of \([p, q]\) coordinates, they are

\[ 0 = [0, 1]; \quad \infty = [1, 0]. \]

We denote them by \( p^s \) and \( q^s \) (\( p^{sf} \) and \( q^{sf} \)) respectively. After taking quotients, they become singular points. By the proof of Proposition 3.1, the uniformization system of \( p^s \) is

\[ \{(p, x, y, z, t) | x = t = 0\} \]

with \( \mu_r \) action given by

\[ \xi(p, y, z) = (\xi p, \xi y, \xi z). \]

At \( p^s \), for each given \( \xi^k = \exp(2\pi ik/r) \), \( 1 \leq k \leq r \), there is a corresponding twisted sector \([\text{CR1}]\). As a set, it is same as \( p^s \). We denote this twisted sector by \([p^s]_k\). For each twisted sector, a degree shifting number is assigned. We conclude that

**Lemma 3.3.** For \( \xi^k = \exp(2\pi ik/r) \), \( 1 \leq k \leq r \), the degree shifting

\[ \iota([p^s]_k) = 1 + \frac{k}{r}. \]

**Proof.** This follows directly from the definition of degree shifting. q.e.d.

Similar results hold for the singular point \( q^s \). Hence we also have twisted sector \([q^s]_k\) and

\[ \iota([q^s]_k) = 1 + \frac{k}{r}. \]

Similarly, on \( W^{sf} \), there are twisted sectors \([p^{sf}]_k, [q^{sf}]_k\) and

\[ \iota([p^{sf}]_k) = \iota([q^{sf}]_k) = 1 + \frac{k}{r}. \]

### 3.2. Torus action.

We introduce a \( T^2 \)-action on \( \tilde{W}_r \):

\[(t_1, t_2)(x, y, z, t) = (t_1 t_2^x, t_1^{-1} t_2^y, t_2 z, t_2^r t). \]

For an action \( t_1 t_2 \), we write the weight of action by \( a\lambda + bu \). This action naturally extends to the actions on all models generated from \( \tilde{W}_r \), such as \( W_r, W_r^s \) and \( W_r^{sf} \).
It then induces an action on $\tilde{\Gamma}_s^r (\tilde{\Gamma}_s^r)^f$:

$$(t_1, t_2)[p, q] = [t_1 p, q].$$

Recall that the normal bundle of $\tilde{\Gamma}_s^r$ in $\tilde{W}_s^r$ is $O \oplus O(-2)$.

**Lemma 3.4.** The action weights at $O_p$ and $O_q$ are $u$. The action weights at $O_p(-2)$ and $O_q(-2)$ are $-\lambda + ru$ and $\lambda + ru$.

**Proof.** This follows directly from the model given by §3.1. q.e.d.

it is easy to verify that $p^s, q^s (p^sf, q^sf)$ are fixed points of the action.

On the other hand, there are four special lines connecting to these points that are invariant with respect to the action. Let us look at $\tilde{W}_s^r$. For the point $p^s$, two lines are in $\Lambda_p$ and are given by

$$\tilde{L}_{p,y}^s = \{x = z = t = 0, u = 0\},$$
$$\tilde{L}_{p,z}^s = \{x = y = 0, z^r + t = 0, u = 0\}.$$  

To the point $q^s$, two lines are in $\Lambda_q$ and are given by

$$\tilde{L}_{q,x}^s = \{y = z = t = 0, v = 0\},$$
$$\tilde{L}_{q,z}^s = \{x = y = 0, z^r - t = 0, v = 0\}.$$  

Similarly, for $\tilde{W}_s^r$ we have

$$\tilde{L}_{p,y}^sf = \{x = z = t = 0, u = 0\},$$
$$\tilde{L}_{p,z}^sf = \{x = y = 0, z^r - t = 0, u = 0\},$$
$$\tilde{L}_{q,x}^sf = \{y = z = t = 0, v = 0\},$$
$$\tilde{L}_{q,z}^sf = \{x = y = 0, z^r + t = 0, v = 0\}.$$  

Correspondingly, these lines in $W_s^r$ and $W_s^{sf}$ are denoted by the same notations without tildes.

**Remark 3.5.** Note that the defining equations for the pairs $L_{q,x}^s$ and $L_{q,x}^sf$, $L_{p,y}^s$ and $L_{p,y}^{sf}$ are same.

3.3. **Symplectic orbi-conifolds and singular symplectic flops.** An orbi-conifold ([CLZZ]) is a topological space $Z$ with a set of (singular) points

$$P = \{p_1, \ldots, p_k\}$$

such that $Z - P$ is an orbifold and for each $p_i \in P$ there exists a neighborhood $U_i$ that is isomorphic to $W_{r_i}$ for some integer $r_i \geq 1$. By a symplectic structure on $Z$ we mean a symplectic form $\omega$ on $Z - P$ and it is $\omega_{p_i}^o$ in $U_i$. We call $Z$ a symplectic orbi-conifold. There exists $2^k$ resolutions of $Z$. Let $Y^s$ be such a resolution, its flop is defined to be the one that is obtained by flops each local model of $Y^s$. We denote it by $Y^{sf}$. In [CLZZ] we prove that

**Theorem 3.6.** $Y^s$ is a symplectic orbifold if and only if $Y^{sf}$ is.
So $Y_{sf}$ is called the (singular) symplectic flop of $Y_s$ and vice versa.

Now for simplicity, we assume that $Z$ contains only one singular point $p$ and is smooth away from $p$. Suppose $Y_s$ and $Y_{sf}$ are two resolutions that are flops of each other and, locally, $Y_s$ contains $W_s^r$ and $Y_{sf}$ contains $W_{sf}^r$. Then

$$H^r_{CR}(Y_s) = H^r_s(Y_s) \oplus \bigoplus_{k=1}^r \mathbb{C}[p^s]_k \oplus \bigoplus_{k=1}^r \mathbb{C}[q^s]_k;$$

$$H^r_{CR}(Y_{sf}) = H^r_s(Y_{sf}) \oplus \bigoplus_{k=1}^r \mathbb{C}[p^{sf}]_k \oplus \bigoplus_{k=1}^r \mathbb{C}[q^{sf}]_k.$$

**Lemma 3.7.** There are natural isomorphisms

$$\psi_k : H^k(Y_s) \to H^k(Y_{sf}).$$

**Proof.** We know that

$$Y_s - \Gamma_s = Y_{sf} - \Gamma_{sf}.$$  

We also have the exact sequence

$$\cdots \to H^k(Y, Y \setminus \Gamma) \to H^k(Y) \to H^k(Y, Y \setminus \Gamma) \to H^{k+1}(Y, Y \setminus \Gamma) \to \cdots$$

and

$$H^k(Y, Y \setminus \Gamma) \cong H^k_c(\mathcal{O} \oplus \mathcal{O}(-2)) \cong H^{k-4}(\mathbb{P}^1).$$

$Y$ is either $Y_s$ or $Y_{sf}$ and $\Gamma$ is the exceptional curve in $Y$.

Suppose we have $\omega^s \in H^k(Y_s)$. Suppose $X_{\omega^s}$ is its Poincare dual. If $k > 2$, we may require that $X_{\omega^s} \cap \Gamma_s = \emptyset$. Hence $X_{\omega^s}$ is in

$$Y_s \setminus \Gamma_s = Y_{sf} \setminus \Gamma_{sf}.$$  

Using this, we get a class $\omega_{sf} \in H^k(Y_{sf})$. Set $\psi_k(\omega^s) = \omega_{sf}$.

If $k \leq 2$, since

$$H^m_{comp}(\mathcal{O} \oplus \mathcal{O}(-2)) = 0, m \leq 3,$$

we have

$$H^k(Y_s) \cong H^k(Y_s \setminus \Gamma_s) \cong H^k(Y_{sf} \setminus \Gamma_{sf}) \cong H^k(Y_{sf}).$$

The isomorphism gives $\psi_k$. q.e.d.

On the other hand, we set

$$\psi_o([p^s]_k) = [p^{sf}]_k, \quad \psi_o([q^s]_k) = [q^{sf}]_k,$$

Totally, we combine $\psi_k$ and $\psi_o$ to get a map

(3.2) \quad $\Psi^* : H^r_{CR}(Y_s) \to H^r_{CR}(Y_{sf}).$

It can be shown that

**Proposition 3.8.** $\Psi^*$ preserves the Poincare pairing.
Without considering the extra classes from twisted sectors, the proof is standard. When the cohomology classes from twisted sectors are involved, it is proved in [CLZZ].

On the other hand, there is a natural isomorphism

$$\Psi : H_2(Y^s) \rightarrow H_2(Y^{sf})$$

with $$\Psi([\Gamma^s_r]) = -[\Gamma^{sf}_r]$$.

Now suppose that we do the symplectic cutting on $$Y^s$$ and $$Y^{sf}$$ at $$W^s_r$$ and $$W^{sf}_r$$ respectively. Then

$$\pi^s_s : \overset{\text{degenerate}}{Y^s} \rightarrow Y^s \cup Z_{M^s_r};$$
$$\pi^{sf}_s : \overset{\text{degenerate}}{Y^{sf}} \rightarrow Y^{sf} \cup Z_{M^{sf}_r}.$$ 

It is clear that $$M^s_r$$ and $$M^{sf}_r$$ are flops of each other. Then similarly, we have a map

$$\Psi^*_r : H^*_{orb}(M^s_r) \rightarrow H^*_{orb}(M^{sf}_r).$$

It is easy to see that the diagram

$$H^*_{CR}(Y^s) \xrightarrow{\pi^*_s \Sigma^*} H^*_{CR}(Y^{sf})$$

commutes.

3.4. Ruan cohomology rings. As explained in [2,3] the cohomology ring structure is defined via a triple form $$\mathcal{A}$$. In the current situation, we can define a ring structure on $$Y^s$$ (and $$Y^{sf}$$) that plays a role between Chen-Ruan (classical) ring structure and Chen-Ruan quantum ring structure. The triple forms on $$Y^s$$ and $$Y^{sf}$$ are given by

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_R = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR} + \sum_{A = d[\Gamma^s_r], d > 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,A} q^d_s, \tag{3.4}$$

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{R} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{CR} + \sum_{A = d[\Gamma^{sf}_r], d > 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,A} q^d_{sf}, \tag{3.5}$$

respectively. Here $$q_s$$ and $$q_{sf}$$ are formal variables that represent classes $$[\Gamma^s_r]$$ and $$[\Gamma^{sf}_r]$$. They define Ruan rings $$RH(Y^s)$$ and $$RH(Y^{sf})$$. In [CLZZ], we already proved that

**Theorem 3.9.** $$\Psi^*$$ gives the isomorphism $$RH(Y^s) \cong RH(Y^{sf})$$. 
4. Relative Gromov-Witten theory on $M^s_r$ and $M^{sf}_r$

4.1. Local models $M^s_r$ and $M^{sf}_r$. $M^s_r$ and $M^{sf}_r$ are obtained from $W^s_r$ and $W^{sf}_r$ by cutting at infinity. We explain this precisely.

We introduce an $S^1$-action on $C^4_{\gamma}(x,y,z,t) = (\gamma^r x, \gamma^r y, \gamma z, \gamma^r t)$.

Using this action, we collapse $\tilde{W}_r$ at $\infty$. The infinity divisor is identified as $\tilde{Z} = \tilde{W}_r^s \cap S^7_{S^1}$.

By this way, we get an orbifold with singularity at $0$, denoted by $\tilde{M}_r$. By blowing-up $\tilde{M}_r$ at $0$, we have $\tilde{M}^s_r$ and $\tilde{M}^{sf}_r$. The $T^2$-action can then naturally extend to $\tilde{M}_r$, $\tilde{M}^s_r$ and $\tilde{M}^{sf}_r$. By taking quotients, we have $M^s_r$ and $M^{sf}_r$.

$M^s_r$ and $M^{sf}_r$ are the collapsing of $W^s_r$ and $W^{sf}_r$ at infinity. Note that the $T^2$-action given in §3.2 also acts on these spaces.

Let $\tilde{P} := P(r, r, 1, r, 1)$ be the weighted projective space. Then $\tilde{M}_r$ can be embedded in $\tilde{P}$ and is given by the equation $xy - z^{2r} + t^2 = 0$.

The original $\tilde{W}_r$ is embedded in $\{w \neq 0\}$ and $\tilde{Z}$ is in $\{w = 0\}$. $\mu_r$-action extends to $\tilde{P}$ and is given by the equation $\xi(x,y,z,t,w) = (\xi^a x, \xi^{-a} y, \xi z, t, w)$.

Then $M^s_r$ is embedded in $\tilde{P}/\mu_r$. Set $Z = \tilde{Z}/\mu_r$. To understand the local behavior of $M^s_r$ and $M^{sf}_r$ at $Z$, it is sufficient to use this model at $\{w = 0\}$.

We now study the singular points at $Z$. Combining the $S^1$ and $\mu_r$ actions, we have

$$(\gamma, \xi)(x, y, z, t, w) = (\gamma^r \xi^a x, \gamma^r \xi^{-a} y, \gamma \xi z, \gamma^r t, \gamma w), (\gamma, \xi) \in S^1 \times Z_r.$$ 

**Lemma 4.1.** There are four singular points

- $\zeta = [1, 0, 0, 0, 0];$
- $\eta = [0, 1, 0, 0, 0];$
- $\zeta^+ = [0, 0, 1, 1, 0];$
- $\zeta^- = [0, 0, 1, -1, 0]$ 

and a singular set

$S = \{xy + t^2 = 0, z = 0\}$

on $Z$. Their stabilizers are $Z_{r^2}, Z_{r^2}, Z_r, Z_r$ and $Z_r$ respectively.

**Proof.** Take a point $(x, y, z, t, w)$. We find those points with nontrivial stabilizers.

Case 1, assume that $z \neq 0$. Then $\gamma \xi = 1$. Therefore,

$$(\gamma, \xi)(x, y, z, t, w) = (\gamma^r \xi^a x, \gamma^r \xi^{-a} y, \gamma \xi z, \gamma^r t, 0) = (\xi^a x, \xi^{-a} y, z, t).$$
In order to have nontrivial stabilizers, we must have \( x = y = 0 \). Therefore, only \( 3^+ \) and \( 3^- \) survive. Their stabilizer are both \( \mathbb{Z}_r \).

Case 2, assume that \( z = 0 \). If \( t \neq 0 \), \( \gamma \) should be a \( r \)-root. Then
\[
(\gamma, \xi)(x, y, z, t, w) = (\xi^a x, \xi^{-a} y, 0, t, 0) = (\xi^a x, \xi^{-a} y, z, t).
\]
Hence, when \( xy \) are not 0, the set \( \{xy + t^2 = 0\} \) has the stabilizer \( \mathbb{Z}_r \).

Case 3, assume that \( z = t = 0 \). Then \( xy = 0 \) by the equation. Hence we can only have \( x \) and \( y \). Clearly, their stabilizers are both \( \mathbb{Z}_r^2 \). q.e.d.

We now look at the local models at \( 3^+ \) and \( 3^- \). The coordinate chart at \( 3^+ \) is given by \((x, y, w)\) and the action is
\[
(4.1) \quad \xi(x, y, w) = (\xi^{-a} x, \xi^a y, \xi w), \xi \in \mathbb{Z}_r.
\]
The model at \( 3^- \) is same.

Now look at the local models at \( r \) and \( \eta \). At \( r \), the local coordinate chart is given by \((z, t, w)\). The action is given by
\[
(4.2) \quad \xi(z, t, w) = (\xi\eta z, \xi^r t, \xi w), \text{ where } \eta^{-a} \xi^r = 1, \xi \in \mathbb{Z}_r^2.
\]
Suppose
\[
(4.3) \quad \eta = \exp 2\pi i \mu, 0 \leq \mu < 1.
\]
Similarly, at \( \eta \), the local coordinate chart is given by \((z, t, w)\). The action is given by
\[
(4.4) \quad \xi(z, t, w) = (\xi\eta z, \xi^r t, \xi w), \text{ where } \eta^{-a} \xi^r = 1, \xi \in \mathbb{Z}_r^2.
\]

For points on \( S \), the action on the normal direction is given by
\[
(4.5) \quad \xi(z, w) = (\xi z, \xi w), \xi \in \mathbb{Z}_r.
\]

Recall that we have four lines described in §3.2. They are now being four lines in \( M_{sr} (M_{sf}^r) \). Take \( M_r^s \) as an example. We have
\[
L_{p,y}^s : p^s \leftrightarrow r; \quad L_{q,x}^s : q^s \leftrightarrow r;
\]
\[
L_{p,z}^s : p^s \leftrightarrow 3^-; \quad L_{q,z}^s : q^s \leftrightarrow 3^+.
\]
In the table, for each line we give the name of the curve and the ends it connects.

For \( M_r^{sf} \), we have
\[
L_{p,y}^{sf} : p^{sf} \leftrightarrow \eta; \quad L_{q,x}^{sf} : q^{sf} \leftrightarrow r;
\]
\[
L_{p,z}^{sf} : p^{sf} \leftrightarrow 3^+; \quad L_{q,z}^{sf} : q^{sf} \leftrightarrow 3^-.
\]

Regarding the \( T^2 \)-action, we have following two lemmas. The proof is straightforward, we leave it to readers.

**Lemma 4.2.** The fixed points on \( M_r^s (M_r^{sf}) \) are \( p^s, q^s, (p^{sf}, q^{sf}) \) on \( \Gamma_r^s (\Gamma_r^{sf}) \) and \( r, \eta, 3^+, 3^- \) on \( Z \).
Lemma 4.3. In $M^s_r$, the invariant curves with respect to the torus action are $L^s_{p,y}, L^s_{q,x}, L^s_{p,z}$ and $L^s_{q,z}$.

4.2. Relative Moduli spaces for the pair $(M^s_r, Z)$. We explain the relative moduli spaces for the pair $(M^s_r, Z)$. Similar explanations can be applied to $(M^{s,f}_r, Z)$.

Let $\Gamma = (g, A, h, g, l)$, $T = (g, l)$ be as before. Let $\overline{\mathcal{M}}_{\Gamma, T}(M^s_r, Z)$ be the moduli space. Recall that the virtual dimension of the moduli space is

$$\dim = c_1(A) + \sum_{i=1}^{m} (1 - \iota(h_i)) + \sum_{j=1}^{k} (1 - [\ell_j] - \iota(g_j)).$$

Here we use the complex dimension.

First, we note that $[c_1(M^s_r) \cdot Z] = (r + 2)[c_1(L_Z) \cdot Z] = \sum_{x \in X} (r + 2)\ell_x$.

Therefore

$$\dim = \sum_{i=1}^{m} (1 - \iota(h_i)) + \sum_{j=1}^{k} ((r + 2)\ell_j + 1 - [\ell_j] - \iota(g_j))$$

Set

$$u_i = 1 - \iota(h_i), 1 \leq i \leq m,$$

$$v_j = (r + 2)\ell_j + 1 - [\ell_j] - \iota(g_j), 1 \leq j \leq k.$$ 

Then

$$\dim = \sum_{i=1}^{m} u_i + \sum_{j=1}^{k} v_j.$$ 

For $u_i$ and $v_j$ we have following facts:

1. If $x_j$ is mapped to $x$ with $g_j = \exp(2\pi i \frac{\alpha}{r^2})$. Then
   $$\ell_j = [\ell_j] + \frac{\alpha}{r^2}.$$ 
   So
   $$v_j = (r + 2)(\frac{\alpha}{r^2} + [\ell_j]) + 1 - ([\ell_j] + \frac{\alpha}{r^2} + \{\frac{\alpha}{r^2} + \mu\} + \{\frac{\alpha}{r}\}).$$ 
   Here $\{z\} := z - [z]$. Note that this can be
   $$(r + 1)[\ell_j] + n - \mu, n \geq 1.$$ 
   Here $\mu$ is defined in (4.3). The degree shifting numbers are given by (4.2).

2. If $x_j$ is mapped to $y$ with $g_j = \exp(2\pi i \frac{\alpha}{r^2})$, it is same as the previous case.
(3) If $x_j$ is mapped to $S$ with $g_j = \exp(2\pi i \frac{\omega}{r})$, then

$$v_j = (r+1)[\ell_j] + \alpha + 1.$$ 

The degree shifting numbers are given by (4.5).

(4) If $x_j$ is mapped to $z^+$ with $g_j = \exp(2\pi i \frac{\alpha}{r})$. Then

$$\ell_j = [\ell_j] + \frac{\alpha}{r}.$$ 

So

$$v_j = (r+1)[\ell_j] + \alpha + \frac{\alpha}{r}.$$ 

Here the degree shifting numbers are given by (4.1).

(5) If $x_j$ is mapped to $z^-$ with $g_j = \exp(2\pi i \frac{\alpha}{r})$, then it is same as the previous case.

(6) whenever $g_j = 1$

$$v_j = (r+1)\ell_j + 1.$$ 

(7) when $h_i = \exp(2\pi i \frac{\alpha}{r})$, $u_i = -\frac{\alpha}{r}$.

(8) when $h_i = 1$, $u_i = 1$.

4.3. **Admissible data** $(\Gamma, T, a)$. Suppose we are computing the relative Gromov-Witten invariant

$$(4.6) \quad \langle a | b, T \rangle^{(M^*, Z)}_{\Gamma}.$$ 

Let $|\alpha|$ denote the degree of a form $\alpha$. Set

$$N = \dim - \sum_{i=1}^{m} |\alpha_i|.$$ 

On the other hand, set

$$N' = \sum_{j=1}^{k} \dim(Z_{g_j}).$$ 

**Definition 4.1.** The data $(\Gamma, T, a)$ is called admissible if $N \leq N'$.

By the definition, we have

**Lemma 4.4.** If $(\Gamma, T, a)$ is not admissible, the invariant (4.6) is 0.

In this paper, we may assume that:

**Assumption 4.5.** (i) $|\alpha_i| = 0$ for all $p_i$, (ii) $|a| \leq 3$.

Since

$$N - N' = \sum_{i=1}^{m} u_i + \sum_{j=1}^{k} (v_j - \dim(Z_{g_j})), $$

we make the following definition.

**Definition 4.2.** we say that $u_i$ is the contribution of marked point $p_i$ to $N - N'$ and $v_j - \dim(Z_{g_j})$ is that of $x_j$. 
Proposition 4.6. Suppose that Assumption 4.5 holds. If \((\Gamma, T, a)\) is admissible, then one of the following cases holds:

1. \(x\) consists of only one smooth point, then \(p\) consists of three singular points \((p_1, p_2, p_3)\) such that 
   \[\iota(h_1) + \iota(h_2) + \iota(h_3) = 5.\]
   For this case, \(N = N'\).
2. if \(x\) contains a point \(x\) maps to \(z^-\) or \(z^+\), then one of the following should hold:
   \(|a| = 2, \iota(h_1) + \iota(h_2) = 3 + \frac{1}{r};\)
   \(|a| = 3, \iota(h_1) + \iota(h_2) + \iota(h_3) = 4 + \frac{1}{r}.\)
3. \(x\) consists of only singular points, the multiplicities at \(x \in x\) are all less than 1. Furthermore, \(x\) can not be mapped to either \(3^+\) or \(3^-\).

Moreover, \(\ell_x \leq 1.\)

Proof. First, we suppose that \(x\) is smooth point. It contributes \((r + 1)l_x - 1\) to \(N\) and contributes 2 to \(N'\). Hence its contribution to \(N = N'\) is 
\[(r + 1)l_x - 1 \geq r.\]

If \(x\) a singular point, its contribution to \(N = N'\) is given by the following list
- \(x \to r\) but not in \(S\), the contribution is \((r + 1)[l_x] + n - \mu, n \geq 1;\)
- \(x \to S,\) the contribution is \((r + 1)[l_x] + \alpha + 1,\)
- \(x \to 3^+\), the contribution is \((r + 1)[l_x] + \alpha + \alpha/r.\)

Note that they are all positive. Hence we conclude that, if \(x\) contains a smooth point \(x,\) then only the following situation survives: \(|x| = 1, r = 2, |a| = 3,\) and
\[\iota(h_1) + \iota(h_2) + \iota(h_3) = 5.\]
Furthermore, \(N = N'.\)

Now suppose that \(x\) contains a point \(x\) mapping to \(3^-\) (or \(3^+\)), only the following situation survives: \(\alpha = 1\) and one of the following holds:
\[|a| = 2, \iota(h_1) + \iota(h_2) = 3 + \frac{1}{r};\]
\[|a| = 3, \iota(h_1) + \iota(h_2) + \iota(h_3) = 4 + \frac{1}{r}.\]

The rest admissible data belong to the third case. q.e.d.

5. Vanishing results on relative invariants

5.1. Localization via the torus action. The torus action \(T^2\) on \(M^*_r\) induces an action on the moduli space. We now study the fix loci of the moduli space with respect to the action. We use the notations in \([22]\) for \(X = M^*_r.\)
A relative stable map
\[ f = (f_0^1, f_1^1, \ldots, f_k^1, \ldots, f_b^k) \]
is invariant if and only if each \( f \) is invariant. Since we only consider the invariants for admissible data, the invariant maps in such moduli spaces only have \( f^0 \). In fact, the fact \( \ell_x \leq 1 \) in Proposition 4.6 implies this.

\( f^0 \) is a stable map in \( X \) whose components are invariant maps (maybe constant map) and nodal points are mapped to fix points, which are \( p^s, q^s \) and \( r, q, s^+, s^- \). The constant map should also map to these points, while the nontrivial invariant curves should cover one of those four lines or \( \Gamma_s \).

Set \( FT = \{ p^s, q^s, r, q, s^+, s^- \} \), \( IC = \{ \Gamma_s, L_p, y, L_q, x, L_p, z, L_q, z \} \).

As in the Gromov-Witten theory, we introduce graphs to describe the components of fix loci. We now describe the graph \( T \) for \( f^0 \). Let \( V_T \) and \( E_T \) be the set of vertices and edges of \( T \).
- each vertex is assigned to a connected component of the pre-image of \( FT \); on each vertex, the image point is recorded;
- each edge is assigned to the component that is non-constant map; the image with multiplicity is recorded;
- on each flag, a twisted sector (or the group element of the sector) is recorded.

Let \( F_T \) be the fix loci that correspond to the graph \( T \). Since \( T \) only describes \( f^0 \), \( F_T \) may contain several components. Let \( \mathfrak{F} \) be the collection of graphs and \( \mathcal{F} \) be the collection of \( F_T \).

We recall the virtual localization formula. Suppose that \( \Omega \) is a form on \( \overline{M}_{\Delta}(X, Z) \) and \( \Omega_T \) is its equivariant extension if exists. Then
\[
I(\Omega) = \int_{\overline{M}_{\Delta}(X, Z)}^{vir} \Omega = \sum_{T \in \mathfrak{T}} \int_{F_T} \frac{\Omega_T}{e_{T^2}(N_{F_T}^{vir})},
\]
Here, \( N^{vir}_{F_T} \) is the virtual normal bundle of \( F_T \) in the virtual moduli space, \( e_{T^2} \) is the \( T^2 \)-equivariant Euler class of the bundle. The right hand side is a function in \( (\lambda, u) \), which is rational in \( \lambda \) and polynomial in \( u \). We denote each term in the summation as \( \iota_{F_T}^{\Omega} (\lambda, u) \) and the sum by \( I^{\Omega}(\lambda, u) \).

**Lemma 5.1.** Let \( (\Gamma, \mathcal{T}, \mathbf{a}) \) be an admissible data in Proposition 4.6. Then the nontrivial relative invariants
\[ \langle \mathbf{a} | \mathbf{b} \rangle_{\Gamma, \mathcal{T}}^{(M^*, Z)} \]
can be computed via localization.

**Proof.** It is sufficient to show that the forms in \( \mathbf{a} \) and \( \mathbf{b} \) have equivariant extensions. By our assumption, we always take \( \alpha \in \mathbf{a} \) to be 1. It is already equivariant.

Now suppose that \( \beta_j \) is assigned to \( x_j \). If \( Z_{(g_j)} \) is a single point, \( \beta_j \) are taken to be 1, which is equivariant. If \( Z_{g_j} = S \), \( \beta_j \) is either a 0 or 2-form,
both have equivariant representatives. The last case is that $x$ is smooth. For this case, since $N = N'$ (cf. case (1) in Proposition 4.6), we must have \( \deg(\beta) = 4 \) to get nontrivial invariants. Since $\beta$ is of top degree, it has an equivariant representative as well. q.e.d.

5.2. Vanishing results on $I_{F_T}^\Omega(\lambda, u)$, (I). By localization, we have

$$I(\Omega) = I^\Omega(\lambda, u) = \sum_{T \in \mathcal{T}} I^\Omega_{F_T}(\lambda, u).$$

Since the left hand side is independent of $u$, we have

$$I(\Omega) = \lim_{u \to 0} I^\Omega(\lambda, u) = \sum_{T \in \mathcal{T}} \lim_{u \to 0} I^\Omega_{F_T}(\lambda, u).$$

**Theorem 5.2.** Suppose that $|x| > 0$. If $T$ contains an edge $e_0$ that records a map cover $\Gamma^s_r$, then

$$\lim_{u \to 0} I_{F_T}(\lambda, u) = 0.$$  

Edge $e_0$ records a map:

$$f_0 : S^2 \to \Gamma^s_r.$$  

$f_0$ can be either a smooth or an orbifold map. Hence, we restate the theorem as,

**Proposition 5.3.** If the map $f_0$ for $e_0$ is smooth, then

$$\lim_{u \to 0} I_{F_T}(\lambda, u) = 0.$$  

and

**Proposition 5.4.** If the map $f_0$ for $e_0$ is singular, then

$$\lim_{u \to 0} I_{F_T}(\lambda, u) = 0.$$  

Clearly, Proposition 5.3 and 5.4 imply Theorem 5.2.

Suppose that $T$ is a graph. Let $C$ be a curve in $F_T$. First assume that all components in $C$ are smooth. Then copied from [GP], we have

$$0 \to \mathcal{O}_{C} \to \bigoplus_{\text{vertices}} \mathcal{O}_{C_v} \oplus \bigoplus_{\text{edges}} \mathcal{O}_{C_e} \to \bigoplus_{\text{flags}} \mathcal{O}_F \to 0,$$

then (write $E = f^*TM^s_r$)

$$0 \to H^0(C, E) \to \bigoplus_{\text{vertices}} H^0(C_v, E) \oplus \bigoplus_{\text{edges}} H^0(C_e, E) \to \bigoplus_{\text{flags}} E_p(F) \to H^1(C, E)$$

$$\to \bigoplus_{\text{edges}} H^1(C_v, E) \oplus \bigoplus_{\text{edges}} H^1(C_e, E) \to 0.$$
Please refer to [GP] for flags. Here, by \( p(F) \) we mean the fixed point assigned to the flag. Hence the contribution of \( H^1/H^0 \) is

\[
\frac{H^1(C, E)}{H^0(C, E)} = \bigoplus_{\text{vertices}} E^\text{val}(v)-1 \oplus \bigoplus_{\text{vertices}} H^1(C_e, E) \oplus \bigoplus_{\text{edges}} H^0(C_e, E)
\]

We translate each term to the equivariant Euler class, i.e., a polynomial in \( \lambda \) and \( u \).

If \( C \) is not smooth, each space in the long complex should be replaced by the invariant subspace with respect to the proper finite group actions. Hence, each term in the right hand side of (5.1) should be replaced accordingly.

Recall that

\[
I^{Ω}_{\Omega T} (λ, u) = \int_{F_T} \frac{Ω_{T2}}{e_{T2}(N^{\text{vir}}_{F_T})}.
\]

It is known that the equivariant form of \( H^1/H^0 \) gives

\[
\frac{1}{e_{T2}(N^{\text{vir}}_{F_T})}.
\]

It is easy to show that for each \( e \), whose map \( f_e \) does not \( \Gamma_r^s \),

\[
e_{T2} \left( \frac{H^1(C_e, f^*_e E)}{H^0(C_e, f^*_e E)} \right)
\]

contains no either \( u \) or \( u^{-1} \). We now focus other terms in \( H^1/H^0 \).

**Claim 1:** if \( f_0 \) is smooth, the equivariant Euler form of the above term \( H^1/H^0 \) contains a positive power of \( u \).

We count the possible contributions for \( u \). Case 1, there is \( v = p^s \) or \( q^s \) with \( \text{val}(v) > 1 \), we then have \( u^{\text{val}(v)-1} \); Case 2, there is a component \( C_e \) that is a multiple of \( \Gamma_r^s \), we then actually have \( u \) in the denominator for \( H^0(C_e, f^*_0 O) \) and \( ru \) in the numerator for \( H^1(C_e, f^*_0 O(-2)) \). So they cancel out. Hence we have the claim. \( \square \)

**Claim 2:** if \( f_0 \) is an orbifold map, the equivariant Euler of \( H^1/H^0 \) contains a factor of positive power of \( u \).

Suppose \( f_0 \) is given by

\[
f_0 : [S^2] \to Γ^s_r \subset W^s_r.
\]

Such a map can be realized by a map

\[
\tilde{f}_0 : S^2 \to \tilde{Γ}^s_r \subset \tilde{W}^s_r
\]

with a quotient by \( \mathbb{Z}_r \). Here \([S^2] = S^2/\mathbb{Z}_r \).

Unlike the smooth case, \( H^0([S^2], f^*_0 E) \) contains no \( u \). By the computations given below, in Corollary 5.7 we conclude that \( H^1([S^2], f^*_0 E) \) contains a factor \( u \). \( \square \)
We compute $H^1([S^2], f_0^*E)$ and its weight.

Suppose that $f_0$ is a $d$-cover. Then on the sphere $\tilde{S}^2$, the torus action weight at 0 is $\lambda/d$ and at $\infty$ is $-\lambda/d$, and suppose that $\mathbb{Z}_r$ action is $\mu$ at 0 and $\mu^{-1}$ at $\infty$; for the pull-back bundle $\mathcal{O}(-2d)$ the torus action weight at fiber over 0 is $-\lambda + ru$ and at fiber over $\infty$ is $\lambda + ru$, the $\mathbb{Z}_r$ action are $\mu^{-d}$ and $\mu^d$ for the fibers over at 0 and $\infty$. These data are ready for us to compute the action on $H^1([S^2], f_0^*E)$.

By Serre-duality, we have

$$H^1(S^2, \mathcal{O}(-2d)) = \left( H^0(S^2, \mathcal{O}(2d-2)) \right)^*.$$ 

The induced torus action weights on $\mathcal{O}(2d-2)$ at the fibers over 0 and $\infty$ are

$$\frac{d-1}{d}\lambda - ru, \quad \frac{1-d}{d}\lambda - ru,$$

respectively. The induced $\mathbb{Z}_r$ action are $\mu^{d-1}$ and $\mu^{-d+1}$ for fibers on $\mathcal{O}(2d-2)$ over 0 and $\infty$.

**Lemma 5.5.** The sections of $H^0(S^2, \mathcal{O}(2d-2))$ are given by

$$\{x^a y^b | a + b = 2d - 2, a, b \geq 0\}.$$ 

The torus action weight for section $x^a y^b$ is $\frac{d-1}{d}\lambda - ru$. The action of $\mu$ is $\mu^{d-1-a}$.

Hence

**Lemma 5.6.** The section $x^a y^b$ that is $\mathbb{Z}_r$-invariant if and only if $r|d-1-a$, and the torus action weight is $\frac{d-1-a}{d}\lambda - ru$.

**Corollary 5.7.** $H^1([S^2], f_0^*\mathcal{O}(-2))$ contains a factor $ru$.

**Proof.** By the above lemma, we know that $x^{d-1} y^{d-1}$ is $\mathbb{Z}_r$-invariant and it is action weight is $-ru$. By taking the dual, the corresponding factor is $ru$. q.e.d.

**Proof of Proposition 5.3** Claim 1 implies the proposition.

**Proof of Proposition 5.4** Claim 2 implies the proposition.

5.3. Vanishing results on $I^{\Omega}_{F^+}(\lambda, u)$, (II). Now suppose that $(\Gamma, T, a)$ is an admissible data given in Proposition 4.6.

**Theorem 5.8.** If $(\Gamma, T, a)$ is of case (1) and (2) in Proposition 4.6

$$\langle a|b \rangle^{(\mathbb{M}^*, Z)} = 0.$$ 

**Proof.** Suppose that

$$\langle a|b \rangle^{(\mathbb{M}^*, Z)} = I(\Omega)$$

for some $\Omega$ which has equivariant extension $\Omega_{\mathbb{F}^2}$. (cf. Lemma 5.1).
Let $F_T$ be a fixed component of the torus action. It contributes 0 to the invariant unless the fixed curves $C \in F_T$ contains no component covering $\Gamma^s$ (cf. Theorem 5.2). It is easy to conclude that $C$ must contain a ghost map $f: (S^2, q_1, q_2, \ldots, q_l) \to M^s$ such that the image of $f$ is either $p^s$ or $q^s$ and sum of the degree shifting numbers for all twisted sectors defined by $q_i$ is $2 + l$. Suppose $p^s = f(S^2)$. Again, we claim that $I_{\Omega}^s \cdot (\lambda, u)$ contains factor $u$. In fact, $e_{\Omega}^s (H^1(S^2, f^* O_p^s)) = u$. This is proved in [CH]. Hence, it is easy to see the claim follows. q.e.d.

**Definition 5.1.** If
\[
\lim_{u \to 0} I_{F_T}^s (\lambda, u) = 0,
\]
we say the component $F_T$ contributes trivial to the invariant $I(\Omega)$.

**Corollary 5.9.** Let $(\Gamma, T, a)$ be admissible, and
\[
I(\Omega) = \langle a | b \rangle^{(M^s, Z)}_{T, a}.
\]
If $F_T$ contributes nontrivial to the invariant, then for any curve $C \in F_T$, all its connected components in the root component $M^s$ must cover $L^s_{p,y}$ or $L^s_{q,x}$.

**Proof.** By the previous theorem, the admissible data must be of the 3rd case in Proposition 4.6. Hence, points in $C \cap Z$ must be either $\tau$ or $\eta$. Therefore, the invariant curves that $C$ lives on must be $L^s_{p,y}$ and $L^s_{q,x}$. q.e.d.

### 6. Proof of the Main Theorem

Combining §2.3 and Theorem 3.9, we reduce Theorem 1.1 to

**Theorem 6.1.**
\[
\sum_{A \in \mathbb{Z} \Gamma^s_+} \langle \alpha^s_1, \alpha^s_2, \alpha^s_3 \rangle_A = \sum_{A \in \mathbb{Z} \Gamma^s_+} \langle \Psi^* \alpha^s_1, \Psi^* \alpha^s_2, \Psi^* \alpha^s_3 \rangle_A.
\]

The rest of the section is devoted to the proof of this theorem.

**Remark 6.2.** We should point out that ”$=$” is rather strong from the point of view of Ruan’s conjecture. Usually, it is conjectured that
\[
\langle \alpha^s_1, \alpha^s_2, \alpha^s_3 \rangle \equiv \langle \Psi^* \alpha^s_1, \Psi^* \alpha^s_2, \Psi^* \alpha^s_3 \rangle.
\]
By ”$\equiv$”, we mean that both sides equal up to analytic continuations. This is necessary when classes $[\Gamma^s]$ and $[\Gamma^s]$ involved. For example, in [CLZZ] we proved
\[
\langle \alpha^s_1, \alpha^s_2, \alpha^s_3 \rangle \equiv \langle \Psi^* \alpha^s_1, \Psi^* \alpha^s_2, \Psi^* \alpha^s_3 \rangle
\]
for Theorem 5.4.

But for the invariants that correspond to $A \neq d[\Gamma^s]$, it turns out that we do not need the analytic continuation argument, the reason is because of Theorem 3.3 as long as $[\Gamma^s]$ (so is $[\Gamma^s]$) appears, the invariant vanishes.
6.1. Reducing the comparison to local models. We now apply the degeneration formula to reduce the comparing three point functions only on local models. We explain this: consider a three point function

\[ \langle \alpha_1^s, \alpha_2^s, \alpha_3^s \rangle_{0, A_s} \]

First, we observe that \( [A_s] = A_s \) since \( \pi_s \) has no kernel (cf [LR]). Denote the topology data by

\[ \Gamma^s = (0, A_s) \]

and forms by \( \alpha^s = (\alpha_1^s, \alpha_2^s, \alpha_3^s) \). Correspondingly, on \( Y^sf \) we introduce

\[ \alpha^{sf} = (\Psi^*)^{-1} \alpha^s, \]

\[ A^{sf} = \Psi \alpha A_s, \]

\[ \Gamma^{sf} = (0, A^{sf}). \]

We write \( \Gamma^{sf} = \Psi(\Gamma^s) \).

Consider the degenerations

\[ \pi_s : Y^s \xrightarrow{\text{degenerate}} M^s_r \cup_Z Y^-, \]

\[ \pi^{sf} : Y^sf \xrightarrow{\text{degenerate}} M^{sf}_r \cup_Z Y^-. \]

Let \( \eta^s = (\Gamma^+, \Gamma^-, I_\rho) \) be a possible splitting of \( \Gamma^s \). Correspondingly,

\[ \Psi(\eta^s) := (\Psi_r(\Gamma^+, \Gamma^-, I_\rho) \]

gives a splitting of \( \Psi(\Gamma^s) \). On the other hand, suppose that

\[ \alpha^s = \pi_s^* (\alpha^+, \alpha^-). \]

Then by the diagram (3.3),

\[ \alpha^{sf} := \Psi^*(\alpha^s) = \pi^{sf}_* (\Psi_r(\alpha^+, \alpha^-)). \]

Proposition 6.3. Suppose that \( \alpha^s \) and \( \beta \) are given on \( M^s_r \), then

\[ \langle \alpha^+, \beta, T \rangle_{\Gamma^+, \rho} = \langle \Psi^*_r(\alpha^+), \beta, T \rangle_{\Psi_r(\Gamma^+, \rho)} \]

Unlike \( \langle \rangle^{(\Psi^*_r, Z)} \), here \( \langle \rangle^{(\Psi^*_r, Z)} \) only sums over all admissible data.

Proposition 6.4. Proposition 6.3 \( \Rightarrow \) Theorem 6.1.

Proof. Applying the degeneration formula to \( \langle \alpha^s \rangle_{\Gamma^s} \), we have

\[ \langle \alpha^s \rangle_{\Gamma^s} = \sum_I \sum_{\eta = (\Gamma^+, \Gamma^-, I_\rho)} C_{\eta} \langle \alpha^+, \beta, T \rangle_{\Gamma^+, \rho} \langle \beta, T \rangle_{\Gamma^-, \rho} \]

Similarly,

\[ \langle \alpha^{sf} \rangle_{\Gamma^{sf}} = \sum_I \sum_{\eta^{sf} = (\Gamma^+, \Gamma^-, I_\rho)} C_{\eta} \langle \alpha^+, \beta, T \rangle_{\Gamma^+, \rho} \langle \beta, T \rangle_{\Gamma^-, \rho} \]

Here \( \ast^{sf} \) is always the correspondence of \( \ast^s \) via \( \Psi \) or \( \Psi_r \).
Since only admissible data contributes on the right hand sides of two equations, (6.1) implies
\[(a^s)^Y_s = (a^{sf})^{Y_{sf}}\]
which is exactly what Theorem [6.1] asserts.

6.2. **Proof of Proposition 6.3.** We now proceed to proof Proposition 6.3. Since the moduli spaces in local models admit torus actions. By localizations, we know the contributions only come from those fix loci. So it is sufficient to compare fix loci and the invariants they contribute.

For \((M^s, Z)\), let \(T\) be a graph, and \(F^s_T\) be the component of fix loci. Then by Corollary [5.9] \(F^s_T\) makes a nontrivial contribution only when each curves in \(F_T\) consists of only components on \(L^s_{p,y}\) and \(L^s_{q,x}\).

Similarly, for \((M^{sf}, Z)\), \(F^{sf}_T\) makes a nontrivial contribution only when each curves in \(F^{sf}_T\) consists of only components on \(L^{sf}_{p,y}\) and \(L^{sf}_{q,x}\).

Since, the flop identifies \(L^s_{p,y} \leftrightarrow L^{sf}_{p,y}, L^s_{q,x} \leftrightarrow L^{sf}_{q,x}\) and their normal bundles, therefore the flop identifies \(F^s_T\) and \(F^{sf}_T\) and their virtual normal bundles in their moduli spaces. Hence,
\[I_{F^s_T}(\lambda, u) = I_{F^{sf}_T}(\lambda, u)\]
Proposition 6.3 then follows.

**References**

[BKL] J. Bryant, S. Katz, N. Leung, Multiple covers and the integrality conjecture for rational curves in CY threefolds, J. Algebraic Geometry 10(2001), no.3.,549-568.

[CH] B. Chen, S. Hu, A de Rham model of Chen-Ruan cohomology ring of abelian orbifolds, Math. Ann. 2006 (336) 1, 51-71.

[CL] B. Chen, A-M. Li, Symplectic Virtual Localization of Gromov-Witten invariants, arXiv:math.DG/0610370.

[CLS] B. Chen, A-M. Li, S. Sun, Relative Gromov-Witten invariants and glue formula, in preparation.

[CLZZ] B. Chen, A-M. Li, Q. Zhang, G. Zhao, Singular symplectic flops and Ruan cohomology, accepted by Topology.

[CT] B. Chen, G. Tian, Virtual orbifolds and Localization, arXiv:math.DG/0610369.

[CR1] W. Chen, Y. Ruan, A new cohomology theory for orbifold, AG/0004129, Commun. Math. Phys., 248(2004), 1-31.

[CR2] W. Chen, Y. Ruan, orbifold Gromov-Witten theory, AG/0011149. Cont. Math., 310, 25-86.

[CR3] W. Chen, Y. Ruan, orbifold quantum cohomology, Preprint AG/0005198.

[FP] Faber, R. Pandharipande, Hodge integrals and Gromov-Witten theory, Invent. Math., 139 (2000), 173C199, math.AG/9810173

[F] R. Friedman, Simultaneous resolutions of threefold double points, Math. Ann. 274(1986) 671-689.

[GP] T. Graber, R. Pandharipande,Localization of virtual classes. Invent. Math. 135 (1999), no. 2, 487-518.

[Gr] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math., 82 (1985), 307-347.
[HLR] J. Hu, T.-J. Li, Y. Ruan, Birational cobordism invariance of uniruled symplectic manifolds, to appear on Invent. Math.

[HZ] J. Hu, W. Zhang, Mukai flop and Ruan cohomology, Math. Ann. 330, No.3, 577-599 (2004).

[K] J. Kollár, Flips, Flops, Minimal Models, Etc., Surveys in Differential Geometry, 1(1991),113-199.

[La] Henry B. Laufer, On $\mathbb{CP}^1$ as an exceptional set, In recent developments in several complex variables, 261-275, Ann. of Math. Studies 100, Princeton, 1981.

[LLW] Y.-P. Lee, H.-W. Lin C.-L. Wang, Flops, Motives and Invariance of Quantum Rings, To appear in Ann. of Math.

[L] E. Lerman, Symplectic cuts, Math Research Let 2(1995) 247-258

[LR] A-M. Li, Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145, 151-218(2001)

[LZZ] A-M. Li, G. Zhao, Q. Zheng, The number of ramified covering of a Riemann surface by Riemann surface, Commu. Math. Phys. 213(2000), 3, 685-696.

[Li] J. Li, Stable morphisms to singular schemes and relative stable morphisms, JDG 57 (2001), 509-578.

[Reid] M. Reid, Young Person's Guide to Canonical Singularities, Proceedings of Symposia in Pure Mathematics, V.46 (1987).

[R1] Y. Ruan, Surgery, quantum cohomology and birational geometry, math.AG/9810039

[R2] Y. Ruan, Virtual neighborhoods and pseudo-holomorphic curves, alg-geom/9611021

[S] I. Satake, The Gauss-Bonnet theorem for V-manifolds, J. Math. Soc. Japan 9(1957), 464-492.

[STY] I. Smith, R.P. Thomas, S.-T. Yau, Symplectic conifold transitions, SG/0209319. J. Diff. Geom., 62(2002), 209-232.

[Wang] C.-L. Wang, K-equivalence in birational geometry, in "Proceeding of Second International Congress of Chinese Mathematicians (Grand Hotel, Taipei 2001)", International Press 2003.