Effects of Nanolayer and Second Order Slip on Unsteady Nanofluid Flow Past a Wedge

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Abstract: This paper presents the study of unsteady nanofluids flow and heat transfer past a wedge with second order velocity slip and temperature jump. The model is modified by considering the existence of a nanolayer together with the effects of thermophoresis and Brownian motion. The fundamental equations were transformed into ordinary differential equations by a new set of similarity transformations and solved by using the homotopy analysis method (HAM). We determined that the error reached $10^{-6}$ and the effectiveness of HAM was attained. The influence of second-order slip on the fluid skin-friction coefficient was analyzed and we determined that the Nusselt number decreases and skin friction coefficient rises with an increase in the thickness of the nanolayer.

Keywords: nanolayer; velocity slip; temperature jump; HAM; wedge

1. Introduction

Nanofluids have been used in electronic information induction, scientific experiments recently. Nanofluids, which consist of nanometer-sized solid particle or tubes suspended in base fluids, are solid–liquid composite materials [1]. Nanofluids can flow smoothly with such ultrafine nanoparticles without clogging. Researchers [2] have proven that nanofluids can enhance the thermal transport property of the original fluid. On the other hand, some people have also noticed the issue that layered solid-like structures will be formed between the liquid and solid molecules. Nanolayers form a thermal bridge between a bulk fluid and a solid nanoparticle [3]. Some experiments [4,5] have proven that the nanolayer has a major effect on the thermal conductivity of nanofluid, while the particle diameter is no more than 10 nm. The thickness of the nanolayer can improve the heat transfer capacity of the nanofluid. However, we know little about the relationship between the thermal properties of the fluid and the nanolayer. It is very necessary to take the nanolayer into account and it indicates the direction for the development of new coolant [5].

In many practical situations, the velocity of free flow varies with time, which leads to instability of flow. For many engineering problems, such as periodic fluid motion, the unsteady boundary layer plays a vital role [7]. Unsteady magnetohydrodynamic (MHD) Falkner–Skan flow of Casson nanofluid was studied by Imran Ullah et al. [8]. Hydro-magnetic unsteady channel flow of nanofluid with heat transfer was studied by Awan et al. [9]. Unsteady flow and heat transfer past a shrink sheet in nanofluids was studied by Jahan et al. [10]. Some scholars also studied non-Newtonian nanofluid [11–13].

With further research on nanofluids, scholars have shown greater attention to the slip boundaries than the nonslip boundary conditions. Babu et al. [14] studied three-dimensional MHD slip flow of nanofluids over a stretching sheet with thermophoresis and Brownian motion effects; Usman et al. [15] studied thermal and velocity slip effects on Casson nanofluid flow; Abbas et al. [16] studied slip effects on stagnation point flow of a micro polar nanofluid; Muhammad Ramzan et al. [17] studied the partial
slip effect of MHD micropolar nanofluid flow on rotating disks; Seth et al. [18] studied the transient flow of MHD nanofluid, considering Navier’s slip boundary condition. However, the literature only discusses the first order slip model. The degree of slip at the boundary depends on a number of interfacial parameters [19]. The Navier’s slip condition breaks down at higher shear rates. The values calculated by using second order slip boundary conditions are closer to the experimental value [20].

The unsteady fluid slip-flow issue has gained extensive interest from researchers, and some mathematical models have been solved by numerical methods. The main numerical methods used are the Runge–Kutta method, the shooting method, and the finite difference method [21]. Sobamowo et al. [22] studied the flow of nanofluid under the effect of slip conditions by numerical methods. Job et al. [23] studied unsteady MHD free convection nanofluid flows by the mixed finite element method. Because most of the fundamental governing equations have a strongly nonlinear and non-conventional nature, obtaining good approximate solutions through traditional methods is difficult. In order to solve this problem, the homotopy analysis method [24] was found to be a worthy and applicable method.

In the paper, we implement the homotopy analysis method to investigate unsteady nanofluids flow and heat transfer based on high order velocity slip and temperature jump. The equations are transformed by a new different similarity transformation and analyzed by HAM. The effects of nanolayer, velocity slip parameter, temperature jump on the Nusselt number and skin-friction coefficient are discussed. The effects of Biot number, solid volume fraction, thickness of nanolayer, and wedge angle on velocity and temperature are found as well.

2. Mathematical Formulation of the Problem

When adding nanoparticles to the base liquid, a layer of thin film will be covered around the nanoparticles. The nanolayer can improve the thermal conductivity of particles in the liquid suspension. Assuming that each particle layer can be combined with nanoparticles, the result is that the nanolayer can increase the volume concentration $\phi_e$, which can be calculated as [25]:

$$\phi_e = (1 + \frac{\delta}{r})^3 \phi_r$$  \hspace{1cm} (1)

where $r$ is the radius of the nanoparticles and $\delta$ is the thickness of the nanolayer. Nanofluids are not a simple mixture of liquid and solid particles. The physical properties characterizing the nanofluids are strongly affected by the characteristics of the suspended particle and base fluids, such as particle morphology, chemical structure of particles and base fluids. Furthermore, the effective density of nanofluid $\rho_{nf}$, effective heat capacity $(\rho C_p)_{nf}$, thermal conductivity $\kappa_{nf}$, thermal diffusivity $\alpha_{nf}$, effective dynamic viscosity $\mu_{nf}$, electrical conductivity $\sigma_{nf}$, respectively, depend on the nanoparticle volume fraction $\phi_e, \delta, r$ as follows [26]:

$$\rho_{nf} = (1 - \phi_e)\rho_{bf} + \phi_e \rho_{sp}, \quad \frac{\kappa_{bf}}{\kappa_{nf}} = \frac{\kappa_{sp} + 2\kappa_{bf} + \phi_e(\kappa_{bf} - \kappa_{sp})}{\kappa_{sp} + 2\kappa_{bf} - 2\phi_e(\kappa_{bf} - \kappa_{sp})},$$  \hspace{1cm} (2)

$$\frac{(\rho C_p)_{nf}}{(\rho C_p)_{bf}} = (1 - \phi_e)(\rho C_p)_{bf} + \phi_e(\rho C_p)_{sp}, \quad \alpha_{nf} = \frac{\kappa_{nf}}{(\rho C_p)_{nf}}, \quad \mu_{nf} = \frac{\mu_{bf}}{(1 - \phi_e)^{2.5}},$$  \hspace{1cm} (3)

$$\sigma_{nf} = \sigma_{bf}[1 + \frac{3\phi_e}{(\sigma_{sp}/\sigma_{bf}) + 2} - \phi_e(\sigma_{sp}/\sigma_{bf} - 1)],$$  \hspace{1cm} (4)

where subscripts $sp, bf, nf$ correspond to solid particle, base fluid, and nanofluid, respectively.
2.1. Mathematical Modeling Analysis

Assuming that the flow is two-dimensional, laminar flow, and the research of nanofluid is a volatile and incompressible fluid, we also hypothesize that the influence of fluid flow in the process of evaporation and surface tension can be ignored. Considering the effect of the nanolayer and Brownian motion and high order slip, the corresponding mathematical model around the wedge is constructed as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{U}{\rho_f} \frac{\partial^2 u}{\partial y^2} + \frac{\sigma_{nf} B_0^2}{\rho_f} [u - U(x)] + \left[ \phi_c (\rho \beta^*_{sp})_{ir} + (1 - \phi_c) (\rho \beta^*)_{bf} \right] \frac{\partial}{\partial y} (T - T_\infty) \cos \left( \frac{\beta \pi}{2} \right), \tag{6}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\alpha_{nf}}{\rho_f c_p} \frac{\partial^2 T}{\partial y^2} + \frac{\tau}{\rho_f c_p} \left( \frac{\partial T}{\partial x} \right)^2 + \frac{q_0}{\rho_f c_p} (T - T_\infty) - \frac{1}{(\rho c_p)_{nf}} \frac{\partial}{\partial y} \frac{\rho}{\sigma^2}, \tag{7}
\]

where \( u \) and \( v \) are the velocity components along the \( x \) and \( y \) directions, respectively, \( U(x) = \frac{dx}{dt} \) is the uniform velocity, \( a \) denotes the initial stretching rate and is a positive constant, \( \lambda \) is the stretching parameter, the angle of the wedge is denoted by \( \beta \pi \), and \( T \) indicates the temperature inside the boundary layer, \( B_0 \) is the magnetic field along the \( y \)-axis, \( T_\infty \) denotes the temperature far from the surface and is a constant, \( Q_0 \) is the heat generation coefficient, \( \tau = \frac{(\rho c_p)_{sp}}{(\rho c_p)_{bf}} \) is the ratio of effective heat capacity. \( D_f \) is the thermophoresis diffusion coefficient, \( \beta^* \) is the thermal expansion coefficients, \( g \) is the gravitational force due to acceleration [27].

In this paper, Rosseland’s diffusion approximation was adopted for the radiative heat flux \( q_r \), and the expression was written as [28]:

\[
q_r = \frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y}, \tag{8}
\]

where \( \sigma^* \) is the Stefan–Boltzmann constant, \( k^* \) denotes the Rosseland mean spectral absorption coefficient. Then, we expanded \( T^4 \) in a Taylor series about \( T_\infty \) and neglected higher terms to gain:

\[
T^4 = 4T^3_\infty T - 3T^4_\infty. \tag{9}
\]

Thus, we obtained:

\[
q_r = -\frac{16\sigma^* T^3_\infty}{3k^*} \frac{\partial T}{\partial y}. \tag{10}
\]

Based on the first order and second order velocity slip and temperature jump, the boundary conditions based on Kandasamy et al. [29] were modified as follows.

When \( t = 0 \),

\[
u = 0, \quad T = T_0, \tag{11}
\]

when \( t > 0 \),

\[
u = L_1 \frac{\partial u}{\partial y} + L_2 \frac{\partial^2 u}{\partial y^2}, \quad \nu = 0, \quad h[T - T_0] = K_1 \frac{\partial T}{\partial y} + K_2 \frac{\partial^2 T}{\partial y^2}, \quad \text{at } y = 0, \tag{12}
\]

\[
u = U(x), \quad T = T_\infty \quad \text{as } \quad y \to \infty, \tag{13}
\]

where \( T_0 \) is a hot fluid at temperature, \( L_1 \) and \( L_2 \) denote the slip parameters of velocity, \( K_1 \) and \( K_2 \) are the jump parameters of temperature, \( h \) is a heat transfer coefficient.
For the sake of deriving a simplified model by converting partial differential equations into ordinary differential equations, a stream function \( \psi \) was introduced in this paper such that \( u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x} \).

First, we scaled a set of points \((x, y, \psi, u, v)\):

\[
\Gamma : x' = xe^{\alpha_1}, y' = ye^{\alpha_2}, \psi' = \psi e^{\alpha_3}, u' = ue^{\alpha_4}, v' = ve^{\alpha_5}.
\] (14)

Then, we introduced Lie-group transformations to obtain the relationships between the parameters \( \alpha_i, i = 1, \ldots, 5 \). Based on the above Lie-group transformations, the stream function and similar parameter can be prescribed as follows:

\[
\eta = \sqrt{\frac{a}{v_b\lambda_f y_f}} \psi, \quad \psi = \sqrt{\frac{a v_b f}{\lambda_f}} x f(\eta), \quad \theta = \frac{T - T_\infty}{T_0 - T_\infty}, U = \frac{a x}{\lambda_f}.
\] (15)

For the convenience of calculation, we defined the following parameters:

\[
\gamma = \frac{Gr_b f}{Re_b f^2}, v_b f = \frac{\mu_b f}{\rho_b}, Gr_b f = \frac{g \beta g (T_0 - T_\infty) x^3}{v_b^2}, Re_b f = \frac{U x}{v_b f},
\] (16)

\[
Ha = \frac{\sigma_v f B_0^2 x}{\rho_b U}, \quad Pr = \frac{v_b f}{\alpha_b f}, \quad Q = \frac{Q_0 x}{(\rho C_p)_b U}, \quad N = \frac{16 \sigma^* T_\infty^3}{3 \alpha^*},
\] (17)

\[
Nt = \frac{D_T}{v_T}(T_0 - T_\infty), \quad Bi = \frac{h x}{k_b f}, Re_b f^{-1}, K = Kn_x Re_b f^2, \quad Kn_x = \frac{L}{x}.
\] (18)

The governing fundamental equations were transformed into the following third order ordinary differential equations after transformation:

\[
f'''' + A (D f' + f f'' + 1 - f'^2) + AB \gamma \cos \left(\frac{\beta n}{2}\right) - Ha C (f' - 1) = 0,
\] (19)

\[
(1 + NF) \theta'' + \frac{1}{2} d Pr E \theta' \eta + Pr E \theta' f + \frac{N t}{Pr} E \theta'^2 + \frac{Q Pr E \theta}{2} = 0.
\] (20)

Accordingly, boundary conditions were transferred into the following form:

\[
f'(0) = K f''''(0) + \frac{K^2}{L} f''''(0), \quad f(0) = 0, \quad f'\left(\infty\right) = 1,
\] (21)

\[
\theta(0) = \frac{1}{Bi G} \theta'(0) + \frac{K}{L B i G} \theta''''(0) + 1, \quad \theta\left(\infty\right) = 0,
\] (22)

where:

\[
A = (1 - \phi_2)^{2.5}[1 - \phi_e + \phi_e \left(\frac{\partial \phi_e}{\partial y_f}\right)], \quad B = \left[\frac{1 - \phi_e}{1 - \phi_e + \phi_e \left(\frac{\partial \phi_e}{\partial y_f}\right)} + \frac{\beta_g}{f} \left(\frac{\partial \phi_e}{\partial y_f}\right) \left(1 - \phi_e\right) \left(\frac{\partial \phi_e}{\partial y_f}\right) \phi_e\right],
\] (23)

\[
C = (1 - \phi_e)^{2.5}, \quad D = \frac{1}{a}, \quad E = (\rho C_p)_b \frac{v_b f}{v_b f}, \quad F = \frac{1}{v_b f}, \quad G = \frac{v_b f}{v_b f}, \quad d = -\frac{d_m^2}{a}.
\]

When the fluid flows over a wedge, it will create skin friction drag \( C_f \). Nu is defined as the dimensionless temperature gradient of the fluid near the wall. It determines the strength of the convective heat transfer fluid. The local Nusselt number and skin friction coefficient are important physical parameters in the study of fluid flow and heat transfer, and they are defined as follows:

\[
C_f = \frac{2 \mu_{nf} \partial u}{\rho_b f U \partial y} \bigg|_{y = 0}, \quad C_f R e_b f^{1/2} = \frac{2}{(1 - \phi_e)^{2.5}} f''''(0),
\] (24)
\[
N_{u_x} = \frac{xq_{\omega}}{\kappa_{ff}(T_0 - T_{\infty})}, \quad q_\omega = -\kappa_{nf} \frac{\partial T}{\partial y} \bigg|_{y=0}, \quad N_{u_x} \Re y^{-1} = -\frac{1}{C} \theta'(0).
\]

2.2. Application of HAM

Due to the strongly nonlinear and uncommon feature of the third order ordinary differential equations, we used the homotopy analysis method to get an approximate analytical solution.

Firstly, we defined that: \( \lambda_1 = K, \lambda_2 = \kappa^2, \omega_1 = \frac{1}{6\lambda_1}, \omega_2 = \kappa \), assuming the initial approximations were:

\[
f_0(\eta) = a_0 + a_1 \eta + a_2 \eta e^{-\eta}, \quad \theta_0(\eta) = b_0 + b_1 e^{-\eta}.
\]

Substituting the initial guess into the boundary conditions, we could get:

\[
f_0(\eta) = x + \frac{1}{3\lambda_2 - 2\lambda_1 - 1} x e^x, \quad \theta_0(\eta) = \frac{1}{a_1/a_2 + 1} e^{-\eta}.
\]

Then, selecting the linear operator:

\[
L_f = \frac{d^3f}{d\eta^3} + \frac{d^2f}{d\eta^2} + L_\theta = \frac{d^2\theta}{d\eta^2} + \frac{d\theta}{d\eta},
\]

and they both satisfied the following conditions:

\[
L_f[C_1 + C_2 \eta + C_3 \eta e^{-\eta}] = 0, \quad L_\theta[C_4 + C_5 \eta e^{-\eta}] = 0.
\]

After that, the zero-order deformation equations could be constructed as follows:

\[
(1 - p)L_f[F(\eta; p) - f_0(\eta)] = p h_f N_f[F(\eta; p)],
\]

\[
(1 - p)\theta_f[\Theta(\eta; p) - \theta_0(\eta)] = p h_\theta N_\theta[\Theta(\eta; p), F(\eta; p)],
\]

where:

\[
N_f[F(\eta; p), \Theta(\eta; p)] = \frac{\partial^3 F}{\partial \eta^3} + A(D \frac{\partial F}{\partial \eta} + \frac{\partial F}{\partial \eta} \frac{\partial^2 F}{\partial \eta^2} + 1) + \left( \frac{\partial F}{\partial \eta} \right)^2 + HAC \left( 1 - \frac{\partial F}{\partial \eta} \right) + AB \Theta \cos \left( \frac{\beta \pi}{2} \right)
\]

\[
N_\theta[F(\eta; p), \Theta(\eta; p)] = (1 + NF_1) \frac{\partial \Theta}{\partial \eta^2} + \text{PrEF} \frac{\partial \Theta}{\partial \eta} + Q \text{PrE} \Theta + \text{PrEN} \left( \frac{\partial \Theta}{\partial \eta} \right)^2 + \frac{1}{2} \text{Pr} \text{Pe} \frac{\partial \Theta}{\partial \eta^2}.
\]

Differentiating the zero-order deformation Equations (30)–(31) m-times with respect to \( p \) at \( p = 0 \) and then dividing the resulting expression by \( m! \), we could attain high-order equations:

\[
L_f[F_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R'_m(\eta),
\]

\[
L_\theta[\Theta_m(\eta) - \chi_m \Theta_{m-1}(\eta)] = h_\theta R''_m(\eta).
\]

High order equations satisfied the boundary conditions:

\[
f_m(0) = f'_m(+) = 0, \quad f'_m(0) = \lambda_1 f''_m(0) + \lambda_2 f'''_m(0),
\]

\[
\theta_m(+) = 0, \quad \theta_m(0) = \omega_1 \theta_m'(0) + \omega_2 \theta_m''(0),
\]

where:

\[
f_m(\eta) = \left. \frac{\partial^m F(\eta)}{\partial \eta^m} \right|_{\eta=0}, \quad \theta_m(\eta) = \left. \frac{\partial^m \Theta(\eta)}{\partial \eta^m} \right|_{\eta=0}, \quad F = f_0(\eta) + \sum_{m=1}^{+\infty} f_m(\eta) p^m, \quad \Theta = \theta_0(\eta) + \sum_{m=1}^{+\infty} \theta_m(\eta) p^m.
\]
3. Results and Discussion

First of all, we determined the value of auxiliary parameter $h_m$ through the BVPh2.0 (Shanghai Jiao Tong University, shanghai, China; http://numericaltank.sjtu.edu.cn/BVPh2_0.htm) procedure package. Table 1 shows the error values of same auxiliary parameters and different auxiliary parameters.

| Order | Same $h_m$ | Different $h_m$ |
|-------|------------|-----------------|
| 2     | 0.09804597363984649 | 0.10971050115341624 |
| 4     | 0.01695228455734058  | 0.01987321691368215 |
| 6     | 0.00149718658281736  | 0.00111284953763120 |
| 8     | 0.00092006200460696  | 0.0010328583202702 |
| 10    | 0.00016652968112583  | 0.00023106953667965 |
| 12    | 0.00028937817232142  | 0.0001866850247371 |
| 14    | 0.00004625312077364  | 0.00003407739648355 |
| 16    | 0.00004411186515886  | 0.00003247720532823 |
| 18    | 0.00004138348805906  | 0.00003498110747688 |
| 20    | 0.00000913314203849  | 0.00000687359503864 |

Through the above form, we can obtain that the error reached $10^{-3}$ in the sixth order solution, which reached the standard of the engineering calculation. We defined the error as the following:

$$\int_{0}^{\infty} \left[ f''' + Af'f'' + 1 - f'^2 \right] + AB\eta \cos\left(\frac{\beta\pi}{2}\right) - H\alpha C\left(f' - 1\right) d\eta.$$  

With the increase of the order, we can determine that the error reached $10^{-6}$ and the rate of convergence of the two methods was similar, so we chose the same parameters in this paper.

The first step of HAM is to acquire the value of auxiliary parameter $h_m$. Figures 1 and 2 are plotted by the case that when other parameters are constant ($\beta = 0.15, \gamma = 0.1, H = 2.0, Pr = 6.5, Nt = 0.5, Q = 0.3$), the range of $h_m$ was $[-0.3, 0.3]$. This paper selected $h_m = -0.1$ for the next calculation.

![Figure 1. $h_m$ vs $f''(0)$](image-url)
which reduced heat transfer and the viscosity together with the resistance of the nanofluids.

The increase in fluid resistance caused the increase in surface skin friction coefficient in the meantime. In addition, the effect of various solid particles and solid volume fraction on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$, and this trend is also accordance with their definitions. The increase in thickness of nanolayer led to an increase in the solid volume fraction of nano-particles, which added to heat transfer and the viscosity together with the resistance of the nanofluids. Obviously, this gave rise to an increase in $N_{uxRe}^{-1/2}bf$ and $C_{fRe}^{1/2}bf$.

After determining the value of $h_{m}$, this paper analyzed the effects of various physical parameters on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$.

Table 2 reflects the effect of radius on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$, and this trend is consistent with its definition. The increase in radius led to the decrease in the solid volume fraction of nano-particles, which reduced heat transfer and the viscosity together with the resistance of the nanofluids.

Table 2. The effect of radius on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$ with $\beta=0.25$, $\lambda_{1}=0.6$, $\lambda_{2}=0.5$.

| Radius | $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$ | $C_{f}\text{Re}^{1/2}_{bf}$ |
|--------|-----------------------------------|-----------------------------|
| 5      | -0.891502                         | 1.26755                     |
| 6      | -0.854912                         | 1.24286                     |
| 7      | -0.830858                         | 1.22730                     |
| 8      | -0.813950                         | 1.21633                     |
| 9      | -0.801466                         | 1.20772                     |
| 10     | -0.791884                         | 1.20219                     |
| 11     | -0.784313                         | 1.19735                     |
| 12     | -0.778192                         | 1.19350                     |

In other words, the increase in radius caused the reduction of $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$ and $C_{f}\text{Re}^{1/2}_{bf}$.

Table 3 points to the effect of the thickness of the nanolayer on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$, and this trend is also accordance with their definitions. The increase in the thickness of nanolayer led to an increase in the solid volume fraction of nano-particles, which added to heat transfer and the viscosity with the resistance of the nanofluids. Obviously, this gave rise to an increase in $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$ and $C_{f}\text{Re}^{1/2}_{bf}$.

Figures 3 and 4 illustrate the effects of various solid particles and solid volume fraction on $C_{f}\text{Re}^{1/2}_{bf}$ and $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$, and this change conforms to its mathematical calculation formulas. Nanoparticles cause a change in the thermal conductivity of the mixture, which affects the flow of the fluid. The specific analysis is shown below. The increase in solid volume fraction causes a rise in the heat transfer and the viscosity as well as the resistance of the nanofluids. From the physical meaning of the Nusselt number we could determine that the increase in thermal conductivity led to the decrease in $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$. The increase in fluid resistance caused the increase in surface skin friction coefficient in the meantime. In addition, the effect of various solid particles on $\text{Nu}_{x}\text{Re}^{-1/2}_{bf}$ was greater than that of $C_{f}\text{Re}^{1/2}_{bf}$. 

![Figure 2. $h_{m} \theta'(0)$](image-url)
Table 3. The effect of the thickness of nanolayer on $C_fRe^{1/2}_{bf}$ and $Nu_xRe^{-1/2}_{bf}$.

| Thickness | $Nu_xRe^{-1/2}_{bf}$ | $C_fRe^{1/2}_{bf}$ |
|-----------|----------------------|---------------------|
| 0         | -0.721204            | 1.15796             |
| 1         | -0.747387            | 1.17419             |
| 2         | -0.778192            | 1.1935              |
| 3         | -0.81395             | 1.21633             |
| 4         | -0.854912            | 1.24286             |
| 5         | -0.901176            | 1.27453             |
| 6         | -0.952644            | 1.31082             |
| 7         | -1.00893             | 1.35419             |
| 8         | -1.06935             | 1.40348             |
| 9         | -1.13284             | 1.46137             |
| 10        | -1.19796             | 1.52895             |

Figure 3. The effect of $\phi$ on $Nu$.

Figure 4. The effect of $\phi$ on $C_f$.

Figures 5 and 6 show the effects of thermophoresis parameter $Nt$ on $C_fRe^{1/2}_{bf}$ and $Nu_xRe^{-1/2}_{bf}$. Thermophoresis parameter $Nt$ of nano-fluid in the fluid was used to measure the temperature distribution. When the thermophoresis parameter increased, the thermophoretic force helped to migrate the nanoparticles from the region of high temperature to low temperature area, which caused the temperature and the thermal efficiency to rise, resulting in an increase in $Nu_xRe^{-1/2}_{bf}$. $Nt$ principally affected $Nu_xRe^{-1/2}_{bf}$, and the influence of $Nt$ on $C_fRe^{1/2}_{bf}$ only presented a little increase trend. Researchers have shown that velocity slip and temperature jump have a crucial influence on the performance of the fluid at micro or nano scales.
conductivity increased, leading to the rise of the wall temperature of the nanofluid. The volume fraction \( \omega \) both decreased. When the temperature of the second order jump is zero, it is a dividing line. When the first order temperature jump, the wall temperature increased. Therefore \( C_{bf} \) in the first order velocity slip, the wall velocity decreased, which caused the decrease in \( C_f \). This affected the velocity by way of the change of fluid flow. With the increase in the first order velocity slip, the wall velocity decreased, which caused the decrease in \( C_{bf} \) and \( \nu_x \operatorname{Re}^{-1/2} \). With the increase in the second order velocity slip, the result was the opposite. The temperature jump parameters mainly affected the dimensionless temperature. With the increase in the first order temperature jump, the wall temperature increased. Therefore \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \) both decreased. When the temperature of the second order jump is zero, it is a dividing line. When \( \omega_2 \leq 0 \), \( C_f \operatorname{Re}^{1/2} \) decreased and \( \nu_x \operatorname{Re}^{-1/2} \) increased with the rise of the second order temperature jump; but while \( \omega_2 > 0 \), \( C_f \operatorname{Re}^{1/2} \) rose and \( \nu_x \operatorname{Re}^{-1/2} \) fell.

Table 4 demonstrates the effect of the first order and the second order velocity slip and temperature jump on \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \). Velocity slip parameters were used to measure the sliding loss in fluid flow, and the sliding loss affected the velocity by way of the change of fluid flow. With the increase in the first order velocity slip, the wall velocity decreased, which caused the decrease in \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \). With the increase in the second order velocity slip, the result was the opposite. The temperature jump parameters mainly affected the dimensionless temperature. With the increase in the first order temperature jump, the wall temperature increased. Therefore \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \) both decreased. When the temperature of the second order jump is zero, it is a dividing line. When \( \omega_2 \leq 0 \), \( C_f \operatorname{Re}^{1/2} \) decreased and \( \nu_x \operatorname{Re}^{-1/2} \) increased with the rise of the second order temperature jump; but while \( \omega_2 > 0 \), \( C_f \operatorname{Re}^{1/2} \) rose and \( \nu_x \operatorname{Re}^{-1/2} \) fell.

**Figure 5.** Effect of \( Nt \) on \( C_f \).

**Figure 6.** The effect of \( Nt \) on \( \nu_x \).

Table 4 demonstrates the effect of the first order and the second order velocity slip and temperature jump on \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \). Velocity slip parameters were used to measure the sliding loss in fluid flow, and the sliding loss affected the velocity by way of the change of fluid flow. With the increase in the first order velocity slip, the wall velocity decreased, which caused the decrease in \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \). With the increase in the second order velocity slip, the result was the opposite. The temperature jump parameters mainly affected the dimensionless temperature. With the increase in the first order temperature jump, the wall temperature increased. Therefore \( C_f \operatorname{Re}^{1/2} \) and \( \nu_x \operatorname{Re}^{-1/2} \) both decreased. When the temperature of the second order jump is zero, it is a dividing line. When \( \omega_2 \leq 0 \), \( C_f \operatorname{Re}^{1/2} \) decreased and \( \nu_x \operatorname{Re}^{-1/2} \) increased with the rise of the second order temperature jump; but while \( \omega_2 > 0 \), \( C_f \operatorname{Re}^{1/2} \) rose and \( \nu_x \operatorname{Re}^{-1/2} \) fell.

Figure 7 demonstrates the effect of solid volume fraction on dimensionless temperature. When the volume fraction of nanoparticles rose, the thickness of the thermal boundary layer and the thermal conductivity increased, leading to the rise of the wall temperature of the nanofluid. The volume fraction of fluid mainly affected the heat transfer resistance, so the change in speed curve was not very obvious. Figure 8 demonstrates the effect of the thickness of the nanolayer on dimensionless temperature. The thickness of nanolayer primarily affected the solid volume fraction. The solid volume fraction rose with the increase in the thickness of the nanolayer, which gave rise to the decline in wall temperature.
Table 4. The effect of particle and slip parameters on $C_f \text{Re}^{1/2}_{bf}$ and $\text{Nu}_x \text{Re}^{-1/2}_{bf}$.

| Parameter | Value | $\text{Nu}_x \text{Re}^{-1/2}_{bf}$ | $C_f \text{Re}^{1/2}_{bf}$ |
|-----------|-------|------------------------------------|----------------------------|
| $\lambda_1$ | 0     | -0.739726                          | 1.279140                   |
|           | 0.5   | -0.737890                          | 0.987631                   |
|           | 1     | -0.736706                          | 0.810298                   |
|           | 1.5   | -0.735953                          | 0.678731                   |
|           | 2     | -0.735382                          | 0.586453                   |
| $\lambda_2$ | -1    | -0.736851                          | 0.829446                   |
|           | -0.5  | -0.739202                          | 1.184720                   |
|           | 0     | -0.744404                          | 1.996120                   |
|           | 0.3   | -0.752506                          | 3.308360                   |
|           | 0.5   | -0.767527                          | 5.848600                   |
| $\omega_1$ | 0     | -1.060260                          | 0.944702                   |
|           | 1     | -0.471135                          | 0.944367                   |
|           | 2     | -0.407875                          | 0.944307                   |
|           | 3     | -0.302604                          | 0.944202                   |
| $\omega_2$ | -2    | -0.210557                          | 0.918785                   |
|           | -1    | -0.372054                          | 0.918769                   |
|           | -0.5  | -0.603914                          | 0.918768                   |
|           | 0     | -1.839810                          | 0.916633                   |
|           | 0.5   | 2.405830                           | 0.914590                   |
|           | 1     | 0.681384                           | 0.918776                   |
|           | 2     | 0.294131                           | 0.918839                   |

Figure 7. The effect of $\phi$ on temperature.
Figure 8. The effect of $\delta$ on temperature.

Figure 9 shows the effect of the angle $\beta$ of the wedge on dimensionless velocity. With the increase in the wedge angle, the thermal diffusion reduced, which caused the drop in the temperature of the fluid. This change increased the impact of buoyancy, resulting in the decrease in the driving force of the fluid, which caused the decrease in the fluid velocity. Figure 10 reflects $Bi$ on the dimensionless temperature. The Biot number reflects the distribution of temperature field in the object under unsteady heat conduction. The smaller $Bi$ is, the smaller the internal thermal resistance and the bigger the external thermal resistance. With an increase in $Bi$, the ratio of the heat resistance increased, which gave rise to the reduction in wall temperature.
4. Conclusions

Considering the effects of the thickness of the nanolayer, temperature jump, and velocity slip, the model of unsteady nanofluid flow and heat transfer was modified in this paper. Meanwhile, both thermophoresis and Brownian motion were considered, and the differential equation was solved by HAM. Some of the main finds of this paper can be summarized as follows:

An increase in the thickness of the nanolayer causes an increase in the local Nusselt number $Nu_x Re^{-1/2}$ and the skin friction coefficients $C_f Re^{1/2}$, together with the decrease in wall temperature. An enhancement of the thermophoresis parameter gives rise to an increase in $Nu_x Re^{-1/2}$ and $C_f Re^{1/2}$.

With an increase in $\lambda_1$ and $\omega_1$, $C_f Re^{1/2}$ and $Nu_x Re^{-1/2}$ reduce. With an increase in the second order velocity slip parameter $\lambda_2$, the result is the opposite. When the second order temperature jump parameter $\omega_2 \leq 0$, $C_f Re^{1/2}$ decreases and $Nu_x Re^{-1/2}$ increases, with an enhancement of $\omega_2$; but while $\omega_2 > 0$, $C_f Re^{1/2}$ rises and $Nu_x Re^{-1/2}$ reduces.

An increase in solid volume fraction $\phi$ causes a decrease in the wall temperature. With an increase in the Biot number the dimensionless wall temperature drops. An increase in the angle of the wedge $\beta$ leads to a decrease in wall velocity.

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