The Hyperbolic Clifford Algebra of Multivecfors

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Abstract

In this paper we give a thoughtful exposition of the Clifford algebra $\mathcal{Cl}(H_V)$ of multivecfors which is naturally associated with a hyperbolic space $H_V$, whose elements are called vecfors. Geometrical interpretation of vecfors and multivecfors are given. Poincaré automorphism (Hodge dual operator) is introduced and several useful formulas derived. The role of a particular ideal in $\mathcal{Cl}(H_V)$ whose elements are representatives of spinors and resume the algebraic properties of Witten superfields is discussed.
1 Introduction

In this paper we give a thoughtful presentation of the hyperbolic Clifford algebra of a real \( n \)-dimensional space \( V \). We start recalling in Section 2.1 the construction of a hyperbolic space \( H_V = (V \oplus V^*, \langle \cdot, \cdot \rangle) \) endowed with a "duality" non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) and then show how to relate \( H_V \) with the hyperbolic space associated with the exterior direct sums of pairs \((V, b)\) and \((V, -b)\) where \( b \) is an arbitrary symmetric bilinear form on \( V \) of arbitrary signature. In Section 2.2 we introduce an orientation for \( H_V \) and in Section 2.3 we give a geometrical interpretation for the elements of \( H_V \) that we call vecfors. In Section 3 we study the endomorphisms of \( H_V \) and introduce the concept of isotropic extensions. In Section 4.1 we introduce \( (\bigwedge H_V, \langle \cdot, \cdot \rangle) \) the exterior algebra of multivecfors endowed with a canonical bilinear form \( \langle \cdot, \cdot \rangle \) induced by \( \langle \cdot, \cdot \rangle \). Left and right contractions of multivecfors are introduced and their properties are detailed. In Section 4.2 we introduce the Poincaré automorphism (Hodge dual) on \( (\bigwedge H_V, \langle \cdot, \cdot \rangle) \) and study its properties. Section 4.3 discusses a differential algebra structure on \( \bigwedge H_V \). In Section 5 we introduce \( \mathcal{C}l(H_V) \) the Clifford algebra of multivecfors (or mother algebra), and recall that it is the algebra of endomorphisms of \( \bigwedge H_V \), i.e., \( \mathcal{C}l(H_V) \simeq \text{End}(\bigwedge H_V) \). A grandmother algebra \( \mathcal{C}l(H_V^2) \simeq \text{End}(\mathcal{C}l(H_V)) \) where \( H_V^2 \) is a second order hyperbolic structure is also discussed. In Section 6 we study the structure of a particular minimal ideal in \( \mathcal{C}l(H_V) \) whose elements (representatives of spinors) represent the algebraic properties of superfields. In Section 7 we present our conclusions.

2 Hyperbolic Spaces

2.1 Definition; Basic Properties

Let \( V, V^* \) be a pair of dual \( n \)-dimensional vector spaces over the real field \( \mathbb{R} \). We call hyperbolic structure over \( V \) to the pair

\[ H_V = (V \oplus V^*, \langle \cdot, \cdot \rangle) \]

Footnote: \( V \oplus V^* \) means the exterior direct sum of the vector spaces \( V \) and \( V^* \).
where \( \langle \cdot, \cdot \rangle \) is the non-degenerate symmetric bilinear form of index \( n \) defined by
\[
\langle x, y \rangle = x^*(y) + y^*(x)
\]
for all \( x = x_* \oplus x^* \in H_V \).

The elements of \( H_V \) will be referred as vecfors. A vecfor \( x = x_* \oplus x^* \in H_V \) will be said to be
(i) positive, if \( x^*(x_*) > 0 \),
(ii) null, if \( x^*(x_*) = 0 \),
(iii) negative, if \( x^*(x_*) < 0 \).

If \( x^*(x_*) = \pm 1 \), then we say that \( x = x_* \oplus x^* \) is a unit vecfor.

Let \( H^*_V \) denote the dual space of \( H_V \), i.e., \( H^*_V = ((V \oplus V^*)^*, \langle \cdot, \cdot \rangle, >^{-1}) \), where \( \langle \cdot, \cdot \rangle, >^{-1} \) is the reciprocal of the bilinear form \( \langle \cdot, \cdot \rangle \). We have the following natural isomorphisms:
\[
H^*_V \simeq H_{V*} \simeq H_V
\]
where \( H_{V*} \) is the hyperbolic space over \( V^* \). Therefore, \( H_V \) is an “auto-dual” space, i.e., it may be canonically identified with its dual space.

The spaces \( V \) and \( V^* \) are naturally identified with their images in \( H_V \) under the inclusions \( i_* : V \to V \oplus V^* \), \( x_* \mapsto i_*x_* = x_* \oplus 0 \equiv x_* \), and \( i^* : V^* \to V \oplus V^* \), \( x^* \mapsto i^*x^* = 0 \oplus x^* \equiv x^* \). Then,
\[
\langle x_*, y_* \rangle = 0, \quad \langle x^*, y^* \rangle = 0, \quad \text{and} \quad \langle x^*, y_* \rangle = x^*(y_*),
\]
for all \( x_*, y_* \in V \subset V \oplus V^* \) and \( x^*, y^* \in V^* \subset V \oplus V^* \). This means that \( V \) and \( V^* \) are maximal totally isotropic subspaces of \( H_V \) and any pair of dual basis \( \{e_1, \ldots, e_n\} \) in \( V \) and \( \{\theta^1, \ldots, \theta^n\} \) in \( V^* \) determines a Witt basis in \( H_V \), i.e., a basis satisfying
\[
\langle e_i, e_j \rangle = 0, \quad \langle \theta^i, \theta^j \rangle = 0, \quad \text{and} \quad \langle \theta^i, e_j \rangle = \delta^i_j.
\]

**Remark.** More generally, to each subspace \( S \subset V \) (or, analogously, \( S^* \subset V^* \)) we can associate a maximal totally isotropic subspace \( I(S) \subset H_V \) as follows: Define the null subspace \( S' \subset V^* \) of \( S \subset V \) by
\[
S' = \{x^* \in V^*, \ x^*(y_*) = 0 \ \forall y_* \in S\}.
\]
We mention without proof that the null subspaces satisfy the properties:
(i) \( S'' = S \),
(ii) \( S_1 \subset S_2 \implies S'_2 \subset S'_1 \),
(iii) \( (S_1 + S_2)' = S'_1 \cap S'_2 \),
(iv) \( (S_1 \cap S_2)' = S'_1 + S'_2 \),
(v) \( \dim S + \dim S' = \dim V \),
(vi) \( S^* = \frac{V^*}{S'} \).

Now construct the space

\[ I(S) = S \oplus S' \]

which is a \( n \)-dimensional vector subspace of \( H_V \), according to property (v).

From the definition of \( S' \), we get \( [x, y] > 0 \) for all \( x, y \in I(S) \), proving thus that \( I(S) \) is a maximal totally isotropic subspace of \( H_V \). The null space of \( I(S) \) is the space \( I'(S) = S' \oplus S \simeq I(S) \), whereas the dual space \( I^*(S) \) of \( I(S) \) is given by the quotient

\[ I^*(S) = \frac{H_V^*}{I(S)} \simeq \frac{H_V}{I(S)} \]

Of course, \( I^*(S) \) is itself a maximal totally isotropic subspace of \( H_V \). The above relation also means that

\[ H_V \cong I(S) \oplus I^*(S) \]

for any \( S \subset V \).

To each Witt basis \( \{ e_1, \ldots, e_n, \theta^1, \ldots, \theta^n \} \) of \( H_V \), we can associate an orthonormal basis \( \{ \sigma_1, \ldots, \sigma_{2n} \} \) of \( H_V \) by letting

\[ \sigma_k = \frac{1}{\sqrt{2}} (e_k \oplus \theta^k) \quad \text{and} \quad \sigma_{n+k} = \frac{1}{\sqrt{2}} (\bar{e}_k \oplus \theta^k) \]

where \( \bar{e}_k = -e_k \), \( k = 1, \ldots, n \). We have,

\[ j \sigma_k, \sigma_l > = \delta_{kl}, \quad j \sigma_{n+k}, \sigma_{n+l} > = -\delta_{kl}, \quad \text{and} \quad j \sigma_k, \sigma_{n+l} > = 0, \]

\( k, l = 1, \ldots, n \). If \( x = x_\star \oplus x^* \), with \( x_\star = x_\star^k e_k \) and \( x^* = x_\star^k \theta^k \), then the components \((x^k, x^{n+k})\) of \( x \) with respect to \( \sigma_k \) are

\[ x^k = \frac{1}{\sqrt{2}} (x_\star^k + x^k_\star) \quad \text{and} \quad x^{n+k} = \frac{1}{\sqrt{2}} (x_\star^k - x^k_\star). \quad (1) \]
The base vectors $\sigma_{n+k}$ are obtained from the $\sigma_k$ through the involution $x = x_* \oplus x^* \mapsto \bar{x}$, where

$$\bar{x} = (-x_*) \oplus x^*$$

is the (hyperbolic) conjugate of the vector $x$. Note that $\bar{x}$ is orthogonal to $x$ and that its square has opposite sign to that of $x$, e.g.,

$$|\bar{x}, x| = 0 \quad \text{and} \quad |\bar{x}, \bar{x}| = -|x, x|.$$

With the components of $x$ given by Eq. 1, the components $(\bar{x}^k, \bar{x}^{n+k})$ of $\bar{x}$ in the basis $\{\sigma_k\}$ would be given by

$$\bar{x}^k = \frac{1}{\sqrt{2}}(x^*_k - x^*_*) \quad \text{and} \quad \bar{x}^{n+k} = \frac{1}{\sqrt{2}}(x^*_k + x^*_*).$$

**Remark.** The hyperbolic conjugation $: H_V \to H_V$ together with the canonical bilinear form $\langle , \rangle$, enable us to define a bilinear form $[ , ]$ by

$$[x, y] = \langle \bar{x}, y \rangle$$

for all $x, y \in H_V$. For $x = x_* + x^*$ and $y = y_* + y^*$, it holds

$$[x, y] = x^*(y_*) + y^*(x_*)$$

and it follows that

$$[x, y] = -[y, x]$$

meaning that $[ , ]$ is an antisymmetric bilinear form on $H_V$.

Due to the auto-duality of $H_V$ stated by the isomorphisms $H^*_V \cong H_{V^*} \cong H_V$, we can use the notation $\{\sigma^1, \ldots, \sigma^{2n}\}$ to indicate the dual basis $\{\sigma_1, \ldots, \sigma_{2n}\}$ as well the reciprocal basis of this same basis. As elements of $H^*_V$ the expressions of the $\sigma^k$’s are

$$\sigma^k = \frac{1}{\sqrt{2}}(\theta^k + e_k) \quad \text{and} \quad \sigma_k = \frac{1}{\sqrt{2}}(\bar{\theta}^k + e_k).$$

Another remarkable result on the theory of hyperbolic spaces is the following

**Proposition 1.** Given an arbitrary non-degenerate symmetric bilinear form $b$ on $V$, there is an isomorphism

$$H_V \cong (V, b) \oplus (V, -b) = H_{b^*}$$
Proof. See, e.g., [2]. The isomorphism is given by the mapping \( \rho_b : H_V \to H_{bV} \), by \( x^* + x^* \mapsto x^+ + x^- \), where

\[
x_\pm = \frac{1}{\sqrt{2}} (b^* x^* \pm x^*)
\]

with \( b^* x^* := b^* (x^* , ) \) for all \( x^* \in V^* \), where \( b^* \), the reciprocal of \( b \) is a bilinear form in \( V^* \) such that \( b_{ik} b_{kj} = \delta_{ij}, b_{ij} = b(e_i, e_j), b_{ik} = b^*(\theta^i, \theta^k) \).

Observe that the image of the basis \( \{ \sigma_1, \ldots, \sigma_{2n} \} \) of \( H_V \) under the above isomorphism is the basis \( \{ e_1, \ldots, e_{2n} \} \) of \( H_{bV} \) given by

\[
e_k = \frac{1}{\sqrt{2}} \left[ (e_k + e^k) \oplus (e_k - e^k) \right],
\]

and

\[
e_{n+k} = \frac{1}{\sqrt{2}} \left[ (e_k - e^k) \oplus (e_k + e^k) \right],
\]

where \( e^k = b^{kl} e_l, k = 1, 2, \ldots, n \). The basis \( \{ e_k \} \) need not to be a \( b \)-orthonormal basis of \( V \); but in case it is, the above expressions become:

\[
e_k = \begin{cases} e_k \oplus 0, & k = 1, \ldots, p \\ 0 \oplus e_k & k = p + 1, \ldots, n \end{cases},
\]

and

\[
e_{n+k} = \begin{cases} 0 \oplus e_k, & k = 1, \ldots, p \\ e_k \oplus 0 & k = p + 1, \ldots, n \end{cases},
\]

where \( p \) is the index of \( b \).

Remark. The hyperbolic structure \( H_{V \oplus V'} \) of the direct sum of two vector spaces \( V \) and \( V' \) satisfies

\[
H_{V \oplus V'} \simeq H_V \oplus H_{V'},
\]

where \( \langle , \rangle_{V \oplus V'} = \langle , \rangle_V \oplus \langle , \rangle_{V'} \). Taking \( V' = V^* \) and writing \( H_V^2 = H_{H_V} \) we conclude that

\[
H_V^2 \simeq H_V \oplus H_{V'}.
\]

Moreover, it follows from Proposition 1, reminding that \( H_V \) is a metric space, that

\[
H_V^2 \simeq H_V \oplus H_{-V},
\]

6
where $H_{-V} = (V \oplus V^* , -\langle , \rangle)$. $H_V^2$ is called second order hyperbolic space.

To a pair of dual basis $\{e_1, \ldots, e_n\}$ in $V$ and $\{\theta^1, \ldots, \theta^n\}$ in $V^*$, we associate an orthonormal basis $\{\Sigma_1, \ldots, \Sigma_{4n}\}$ of $H_V^2$ by the relations

\[
\Sigma_k = \frac{1}{\sqrt{2}} \sigma_k \oplus \sigma^k
\]
\[
\Sigma_{n+k} = \frac{1}{\sqrt{2}} \sigma_{n+k} \oplus \sigma^{n+k}
\]
\[
\Sigma_{2n+k} = \frac{1}{\sqrt{2}} (-\sigma_k) \oplus \sigma^k
\]
\[
\Sigma_{3n+k} = \frac{1}{\sqrt{2}} (-\sigma_{n+k}) \oplus \sigma^{n+k}
\]

$k = 1, \ldots, n$, where $\{\sigma_k\}$ and $\{\sigma^k\}$ are like before.

### 2.2 Orientation

Besides a canonical metric structure, a hyperbolic space is also provided with a canonical sense of orientation, induced by the $2n$-vector $\sigma \in \bigwedge^{2n} H_V$ given by

\[
\sigma = \sigma_1 \wedge \ldots \wedge \sigma_{2n}
\]

where $\{\sigma_1, \ldots, \sigma_{2n}\}$ is the orthonormal basis of $H_V$ naturally associated with the dual basis $\{e_1, \ldots, e_n\}$ of $V$ and $\{\theta^1, \ldots, \theta^n\}$ of $V^*$. Note that $\sigma$ satisfies

\[
\langle i \sigma, \sigma \rangle = (-1)^n.
\]

Here, $\langle \cdot , \cdot \rangle$ denotes the natural extension [8] of the metric of $H_V$ to the space $\bigwedge^{2n} H_V$.

The reason for saying that $\sigma$ defines a canonical sense of orientation is that it is independent on the choice of the basis for $V$. To see this it is enough to verify that

\[
\sigma = e_* \wedge \theta^*
\]

where

\[
e_* = e_1 \wedge \ldots \wedge e_n \quad \text{and} \quad \theta^* = \theta^1 \wedge \ldots \wedge \theta^n
\]

Thus, under a change of basis in $V$, $e_*$ transforms as $\lambda e_*$, $\lambda \neq 0$, whereas $\theta^*$ transforms as $\lambda^{-1}\theta^*$, so that $\sigma$ remains unchanged.
2.3 Geometrical Interpretation of Vectors

We start recalling that, with respect to its underlying vector space, every vector is represented by an arrow (oriented line segment) pointing outward the origin (zero vector), with the operations of multiplication by a scalar and the operation of sum of vectors by the parallelogram rule having their usual interpretation.

However, it is often useful or necessary to interpret the elements of a given vector space with respect to another one, related to the first by some prescription. In particular, our aim here is to show how the elements of the hyperbolic space $H_V$ are represented taking the space $V$ as the “basis” for the representation. That is, we attribute to elements of $V$ their usual interpretation as arrows and we want to describe elements of $H_V$ as collections of such arrows.

Since vectors are pairs constituted by a vector and a linear form, the first question is how a linear form $\alpha \in V^*$ is represented within $V$. The answer is that it is described as an ordered pair of $(n-1)$-dimensional parallel hyperplanes $(S_0(\alpha), S_1(\alpha))$, given by

$$S_0(\alpha) = \{x \in V, \alpha(x) = 0\} \quad \text{and} \quad S_1(\alpha) = \{x \in V, \alpha(x) = 1\}$$

It should be noted that only $S_0(\alpha)$, which indeed is the null subspace spanned by $\alpha$, is a vector subspace of $V$. The notation $S_a(\alpha), a \in \mathbb{R}$, will be used here to indicate the set of vectors $x \in V$ satisfying the equation $\alpha(x) = a$. The sequence $\{S_a(\alpha), a \in \mathbb{R}\}$ can also be used as an alternative representation of a linear form.

If $\alpha' = a\alpha$, $a \neq 0$, we have $S_0(\alpha') = S_0(\alpha)$, while

$$S_1(\alpha') = \{x \in V, \alpha'(x) = 1\}$$
$$= \{x \in V, \alpha(x) = 1/a\}$$
$$= \{x \in V, x = y/a, y \in S_1(\alpha)\}$$

which denotes the affine $(n-1)$-dimensional hyperplane obtained from $S_1(\alpha)$ by “dilating” each vector of $S_1(\alpha)$ by a factor $1/a$. If $a > 1$, such a dilation is indeed a contraction: multiplying a linear form by a factor greater than one yields the linear form represented by affine hyperplanes parallel to those representing the original linear form, but which are closer from each other, with respect to the original hyperplanes, by the factor
1/a. Analogously, for 0 < a < 1, the hyperplanes representing $\alpha'$ will be separated from each other, with respect to the hyperplanes representing $\alpha$, by a factor $1/a > 1$, thus representing a “true” dilation in $V$.

For $a = -1$, the hyperplane $S_1(\alpha')$ will be constituted by the negatives of the vectors in $S_1(\alpha)$, so that both hyperplanes will be separated from $S_0(\alpha)$ by the same amount, but will be given by the (“anti-”) mirror image of each other.

Given now $\alpha, \beta \in V^*$, their sum $\alpha + \beta \in V^*$ is represented in $V$ as follows: The hyperplane $S_0(\alpha + \beta)$ is given by the unique $(n-1)$-dimensional affine hyperplane containing the intersections $S_1(\alpha) \cap S_1(-\beta)$ and $S_1(-\alpha) \cap S_1(\beta)$ (note that these intersections are $(n-2)$-dimensional affine hyperplanes, unless $\alpha = \beta$ when we have $S_0(\alpha + \beta) = S_0(\alpha)$). In turn, $S_1(\alpha + \beta)$ is given by the unique hyperplane containing the intersections $S_0(\alpha) \cap S_1(\beta)$ and $S_0(\beta) \cap S_1(\alpha)$. Observe also that we can take advantage from the fact that the hyperplanes representing a linear form are parallel and to obtain $S_0(\alpha + \beta)$ as the unique hyperplane containing the origin which is parallel to $S_1(\alpha + \beta)$.

With the above constructions, the geometrical meaning of the vectors become apparent: an element $x = x \oplus \alpha \in H_V$ is represented by an arrow $\vec{x}$ together with a pair of hyperplanes $S_0(\alpha)$ and $S_1(\alpha)$. The sum of two vectors is obtained geometrically by summing up independently the representations of their vector parts (according to the parallelogram rule) and the representations of their linear form parts (according to the sum rule stated above). The multiplication of a vector $x$ by a scalar $a$, in turn, has the interesting property that its vector part is represented by an arrow “dilated” by a factor $a$ with respect to the original one, while the hyperplanes representing its linear form part are “contracted” by the same factor with respect to the original one.

We had classified a vector $x = x \oplus \alpha$ by the value of the contraction $\alpha(x)$. Null vectors were defined by the condition $\alpha(x) = 0$, which is the defining condition of the hyperplane $S_0(\alpha)$. Therefore, null vectors stand geometrically for vectors such that the vector part is “parallel” to the linear form part.

For a non-null vector, the contraction $\alpha(x)$ gives the amount by which the vector $x$ should be contracted, and possibly reflected, in order to get a “unitary” vector with respect to $\alpha$, i.e., a vector with its tip lying on the hyperplane $S_1(\alpha)$. Positive vectors, for which $\alpha(x) > 0$, are such that their
vector part have the same “orientation” of their linear form part, so that only a “contraction” need to be done in order to normalize them. In turn, the vector parts of negative vectors, have opposite “orientation” of those of their linear form parts, so that they need to be reflected through the origin, in addition to being contracted, in order to be normalized.

3 Endomorphisms of \( H_V \)

3.1 Isotropic Extensions

Given a linear mapping \( \phi : V \to V \) (i.e., \( \phi \in \text{End} \, V \)), we define \( \phi^* : V^* \leftarrow V^* \), the dual mapping of \( \phi \), by

\[
(\phi^* \alpha) x = \alpha(\phi x)
\]

for all \( x \in V \) and \( \alpha \in V^* \). The operation \( \phi \mapsto \phi^* \), \( \phi \in \text{End} \, V \) is linear and satisfy

(i) \( \phi^{**} = \phi \),
(ii) \( (\phi \psi)^* = \psi^* \phi^* \),
(iii) \( \ker \phi^* = (\im \phi)' \), \( \im \phi^* = (\ker \phi)' \),
(iv) \( \det \phi^* = \det \phi \), \( \text{tr} \phi^* = \text{tr} \phi \),
(v) \( \phi(S) \subset S \implies \phi^*(S') \subset S' \), \( S \subset V \).

Every linear mapping \( \phi \in \text{End} \, V \) (or, equivalently, \( \phi^* \in \text{End} \, V^* \)) induces a linear mapping \( I(\phi) \in \text{End} \, H_V \), defined by

\[
I(\phi) = \phi \oplus \phi^*.
\]

We call \( I(\phi) \) isotropic extension of \( \phi \) \( (\phi^*) \) to \( H_V \). The reason for this nomenclature is that, due to the fifth property above, if \( S \subset V \) is stable
under $\phi$, then $I(S)$ is stable under $I(\phi)$, i.e.,

$$I(\phi)(S \oplus S') \subset S \oplus S'.$$

**Example.** Vecfors and Endomorphisms. To each vecfor $x = x_* \oplus x^* \in H_V$ corresponds a linear mapping $x : V \rightarrow V$, given by

$$x(y_*) = x^*(y_*)x_*,$$

for all $y_* \in V$. The dual $x^*$ of $x$ is given, obviously, by

$$x^*(y^*) = y^*(x_*)x^*,$$

for all $y^* \in V^*$. 

**Example.** Hyperbolic Projections. We can associate to a non-null vecfor $x = x_* \oplus x^* \in H_V$ a projection operators $P_x : V \rightarrow V$ and $P^x : V^* \leftarrow V^*$ by the relations

$$P_x y_* = \frac{x^* y_*}{x^* x_*} x_* \quad \text{and} \quad P^x y^* = \frac{y^* x_*}{x^* x_*} x^*.$$

The operator $P^x$ is the dual of the operator $P_x$. Indeed, $(P^x y^*)(y_*) = y^*(P_x y_*) = (x^* y_*/x^* x_*) y^*(x_*) = (y^* x_*/x^* x_*) x^* y_* = (P^x y^*)(y_*)$ for all $y_* \in V$ and $y^* \in V^*$. 

The vector $P_x y_*$ is given geometrically by the intersection of the line containing the vector $x_*$ with the $S_1$-hyperplane associated with the linear form $x^*/x^*(y_*)$, while $P^x y^*$ is the linear form parallel to $x^*$, whose $S_1$-hyperplane contains the intersection of $S_1(y^*)$ with the vector $x_*$. 

The isotropic extension of $P_x$ to $H_V$, $I(P_x) = P_x$, is given by

$$P_x = P_x \oplus P^x.$$

The operator $P_x$ is itself a projection, since $P_x^2 = P_x^2 \oplus (P^x)^2 = P_x \oplus P^x = P_x$. Moreover $P_x$ is self-dual with respect to the bilinear form $\langle \cdot, \cdot \rangle$, in the sense that

$$\langle P_x y, z \rangle = \langle y, P_x z \rangle.$$

In other words, if $P^x$ denotes the dual of the transformation $P_x$, then

$$P^x = P_x.$$
Figure 2: Geometrical representation of the projections induced by hyper-vectors: (a) $P_x y_*$ is the vector obtained from the intersection of $x_*$ with the $S_1$-hyperplane of the linear form $x^*/x^*(y_*)$; (b) $P^x y_*$ is the linear form parallel to $x^*$ whose $S_1$-hyperplane crosses the intersection of $x_*$ with the $S_1$-hyperplane of $y^*$.

For the orthonormal basis $\{\sigma_1, \ldots, \sigma_{2n}\}$ related to a pair of dual basis $\{e_1, \ldots, e_n\}$ in $V$ and $\{\theta^1, \ldots, \theta^n\}$ in $V^*$, the matrix representations of the projection operators $P_{\sigma_k}$ and $P_{\sigma_{n+k}}$ are

$$(P_{\sigma_k}) = (P_{\sigma_{n+k}}) = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with the factors ‘1’ appearing in the positions $k$ and $n + k$ of the diagonal.

**Example.** Hyperbolic Reflections. Reflections form another important class of transformations in the theory of linear spaces. And it turns out that every non-null vector $x \in H_V$ can also be associated with a reflection $R_x : V \to V$ and with a reflection $R^x : V^* \leftarrow V^*$ given respectively by

$$R_x y_* = y_* - \frac{2}{x^*(y_*)} x_* x^*$$

and

$$R^x y_* = y^* - \frac{2}{x^*(x_*)} x^*$$

for all $y_* \in V$ and $y^* \in V^*$, where $x = x_* \oplus x^* \in H_V$. Once again, the mappings $R_x$ and $R^x$ are duals of each other.

The isotropic extension of $R_x$ to the space $H_V$, $I(R_x) = R_x$ is given by

$$R_x = R_x \oplus R^x$$

and it should be noted that $R_x$ is an orthogonal mapping with respect to the bilinear form $\langle \cdot, \cdot \rangle$ of $H_V$, i.e.,

$$\langle R_x y, R_x z \rangle = \langle y, z \rangle$$
for all \( y, z \in H_V \). This relation can also be written as

\[
R^x R_x = 1
\]

with \( R^x \) denoting the dual of \( R_x \) and 1 being the identity mapping of \( H_V \).

With respect to our basis \( \{ \sigma_1, \ldots, \sigma_{2n} \} \), the matrix representations of the mappings \( R_{\sigma_k} \) and \( R_{\sigma_{n+k}} \) are

\[
(R_{\sigma_k}) = (R_{\sigma_{n+k}}) = \text{diag}(1, \ldots, 1, -1, \ldots, 1, -1, \ldots, 1)
\]

where the factors ‘\(-1\)’ appears in positions \( k \) and \( n + k \) of the diagonal.

## 4 Exterior Algebra of a Hyperbolic Space

The exterior algebra \( \bigwedge H_V \) of the hyperbolic structure \( H_V \) is the pair

\[
\bigwedge H_V = (\bigwedge (V \oplus V^*), \hat{\langle}, \rangle)
\]

where

\[
\bigwedge (V \oplus V^*) = \sum_{r=0}^{2n} \bigwedge^r (V \oplus V^*)
\]

is the exterior algebra of \( V \oplus V^* \) and \( \hat{\langle}, \rangle \) is the canonical bilinear form on \( \bigwedge (V \oplus V^*) \) induced by the bilinear form \( \hat{\langle}, \rangle \) of \( H_V \), e.g., for simple elements \( x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots \wedge y_r \in \bigwedge^r H_V, \hat{\langle}, \rangle \) is given by

\[
\hat{\langle} x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots \wedge y_r \hat{\rangle} = \text{Det} \begin{pmatrix} |x_1, y_1| & \cdots & |x_1, y_r| \\ \vdots & \ddots & \vdots \\ |x_r, y_1| & \cdots & |x_r, y_r| \end{pmatrix},
\]
and it is extended by linearity and orthogonality to all of the algebra $\bigwedge H_V$. Of course, due to the isomorphisms $H_V^* \simeq H_V^* \simeq H_V$, we have

$$\bigwedge H_V^* \simeq \bigwedge H_V^* \simeq \bigwedge H_V,$$

and it follows that

$$\bigwedge H_V \simeq \left( \bigwedge H_V \right)^*,$$

i.e., $\bigwedge H_V$ is itself an auto-dual space. The elements of $\bigwedge H_V$ will be called multivectors.

Grade involution, reversion and conjugation in the algebra $\bigwedge H_V$ are defined as usual[2]: For a homogeneous multivecfor $u \in \bigwedge^r H_V$,

$$\hat{u} = (-1)^r u, \quad \tilde{u} = (-1)^{\frac{1}{2}(r-1)} u, \quad \bar{u} = (-1)^{\frac{1}{2}(r+1)} u$$

and we recall that every element $u \in \bigwedge H_V$ is uniquely decomposed into a sum

$$u = u_+ + u_-$$

with $u_+$ and $u_-$, the even and the odd part of $u$, given respectively by

$$u_+ = \frac{1}{2}(u + \hat{u}) \quad \text{and} \quad u_- = \frac{1}{2}(u - \hat{u})$$

The spaces $\bigwedge V$ and $\bigwedge V^*$ are identified with their images in $\bigwedge H_V$ under the homomorphisms $i^w : \bigwedge V \to \bigwedge H_V$ and $i_w : \bigwedge V^* \to \bigwedge H_V$ by

$$i^w(x_1 \wedge \ldots \wedge x_r) = (x_1 \oplus 0) \wedge \ldots \wedge (x_r \oplus 0) \equiv x_1 \wedge \ldots \wedge x_r$$

and

$$i_w(x_1^* \wedge \ldots \wedge x_r^*) = (x_1^* \oplus 0) \wedge \ldots \wedge (x_r^* \oplus 0) \equiv x_1^* \wedge \ldots \wedge x_r^*.$$

Then, for $u_*, v_* \in \bigwedge V \subset \bigwedge H_V$ and $u^*, v^* \in \bigwedge V^* \subset \bigwedge H_V$, we have

$$\hat{i}u_*, v_* > 0,$$

$$\hat{i}u^*, v^* > 0,$$

$$\hat{i}u^*, v_* > u^*(v_*). \quad (2)$$

---

[2] See, e.g., [8].
Thus, $\bigwedge V$ and $\bigwedge V^*$ are totally isotropic subspaces of $\bigwedge H_V$. But they are no longer maximal: $\langle \cdot, \cdot \rangle$ being neutral, the dimension of a maximal totally isotropic subspace is $2^{2n-1}$, whereas $\dim \bigwedge V = \dim \bigwedge V^* = 2^n$. For elements $u = u_\ast \wedge u^\ast, v = v_\ast \wedge v^\ast \in \bigwedge H_V$ with $u_\ast \in \bigwedge^r V$, $u^\ast \in \bigwedge^s V^*$, $v_\ast \in \bigwedge^s V$, and $v^\ast \in \bigwedge^r V^*$ it holds
\[ \langle u, v \rangle = (-1)^{rs} u^\ast(v_\ast) v^\ast(u_\ast). \]

**Proposition.** There is the following natural isomorphism:
\[ \bigwedge H_V \simeq \bigwedge V \hat{\otimes} \bigwedge V^* \]
where $\hat{\otimes}$ denotes the graded tensor product. Moreover, being $b$ a non-degenerate bilinear form on $V$, it holds also
\[ \bigwedge H_V \simeq \bigwedge H_{bV} \hat{\otimes} \bigwedge H_{-bV}. \]

For a proof see, e.g., Greub [1]. The first of the above isomorphisms is given by the mapping $\bigwedge V \hat{\otimes} \bigwedge V^* \to \bigwedge H_V$ by
\[ u_\ast \hat{\otimes} u^\ast \mapsto i^w u_\ast \wedge i_w u^\ast, \]
for all $u_\ast \in \bigwedge V$ and $u^\ast \in \bigwedge V^*$. Under this mapping, we can make the identification
\[ \bigwedge^r H_V = \sum_{p+q=r} \bigwedge^p V \hat{\otimes} \bigwedge^q V^*. \]

### 4.1 Contractions

A left contraction $\downarrow : \bigwedge H_V \times \bigwedge H_V \to \bigwedge H_V$ and a right contraction $\uparrow : \bigwedge H_V \times \bigwedge H_V \to \bigwedge H_V$ are introduced in the algebra $\bigwedge H_V$ in the usual way (see, e.g., [3]), i.e., by
\[ \langle u \downarrow v, w \rangle = \langle v, \hat{u} \wedge w \rangle, \]
and
\[ \langle v \uparrow u, w \rangle = \langle v, w \wedge \hat{u} \rangle, \]
for all \( u, v, w \in \bigwedge H_V \). These operations have the general properties

\[
\begin{align*}
x \downarrow y = x \downarrow \downarrow y = [x, y], \\
1 \downarrow u = u, \quad x \downarrow = 1 \downarrow x = 0, \\
(u \downarrow v)^\ast = \hat{u} \downarrow \hat{v}, \quad (u \downarrow v)^\ast = \hat{u} \downarrow \hat{v}, \\
(u \downarrow v)^\ast = \hat{u} \downarrow \hat{v}, \quad (u \downarrow v)^\ast = \hat{u} \downarrow \hat{v}, \\
u \downarrow (v \downarrow w) = (u \wedge v) \downarrow w, \\
(u \downarrow v) \downarrow w = u \downarrow (v \wedge w), \\
(u \downarrow v) \downarrow w = u \downarrow (v \wedge w), \\
x \downarrow (u \wedge v) = (x \downarrow u) \wedge v + \hat{u} \wedge (x \downarrow v), \\
(u \wedge v) \downarrow x = u \wedge (v \wedge x) + (u \wedge x) \wedge \hat{v}, \\
x \wedge (u \downarrow v) = \hat{u} \downarrow (x \wedge v) - (\hat{u} \downarrow x) \downarrow v \\
(u \downarrow v) \wedge x = (u \wedge x) \downarrow \hat{v} - u \downarrow (x \downarrow \hat{v}) \\
u_+ \downarrow v = v \downarrow u_+, \quad u_- \downarrow v = \hat{v} \downarrow \hat{u}_-
\end{align*}
\]

Moreover, from Eqs. \( 2 \) we get (there are analogous results for right contraction)

\[
\begin{align*}
u_\ast \downarrow v_\ast & = 0 \quad \text{and} \quad u_\ast \downarrow v_\ast = 0 \quad (3)
\end{align*}
\]

for all \( u_\ast, v_\ast \in \bigwedge V \) and \( u_\ast, v_\ast \in \bigwedge V^\ast \), so that for elements of the form \( u = u_\ast \wedge u^\ast \) and \( x = x_\ast \oplus x^\ast \), it holds

\[
x \downarrow u = (x^\ast \downarrow u_\ast) \wedge u^\ast + \hat{u}_\ast \wedge (x_\ast \downarrow u^\ast)
\]

For more details about the properties of left and right contractions, see, e.g., \( [3, 8] \).

### 4.2 Poincaré Automorphism (Hodge Dual)

Define now the Poincaré automorphism (or Hodge dual) \( \ast : \bigwedge H_V \to \bigwedge H_V \) by

\[
\ast u = \hat{u} \downarrow \sigma,
\]

16
for all \( u \in \bigwedge H_V \). The inverse \( \star^{-1} \) of this operation is given by
\[
\star^{-1} u = \tilde{\sigma} \downarrow \tilde{u}.
\]
The following general properties of the Hodge duality holds true
\[
\star \sigma = (-1)^n, \quad \star^{-1} \sigma = 1
\]
\[
\int \star u, \star v = (-1)^n \langle u, v \rangle
\]
\[
\star(u \wedge v) = \tilde{v} \downarrow \star u
\]
\[
\star^{-1}(u \wedge v) = (\star^{-1} v) \downarrow \tilde{u}
\]
\[
\star(u \downarrow v) = (\star^{-1} v) \wedge \tilde{u}
\]
\[
\star^{-1}(u \downarrow v) = (\star^{-1} v) \wedge \tilde{u}
\]
For \( x = x_\ast \oplus x^\ast \in H_V \subset \bigwedge H_V \) we have, since \( x_\ast \downarrow e_\ast = 0 \) and \( x^\ast \downarrow \theta^\ast = 0 \) that
\[
\star x = (x^\ast \downarrow e_\ast) \wedge \theta^\ast - e_\ast \wedge (\theta^\ast \downarrow x_\ast),
\]
and it follows that, for \( u_\ast \in \bigwedge V \subset \bigwedge H_V \) and \( u^\ast \in \bigwedge V^\ast \subset \bigwedge H_V \),
\[
\star u^\ast = (\tilde{u}^\ast \downarrow e_\ast) \wedge \theta^\ast = D\# u^\ast \wedge \theta^\ast
\]
and
\[
\star u_\ast = e_\ast \wedge (\theta^\ast \downarrow \tilde{u}_\ast) = e_\ast \wedge D\# u_\ast
\]
where we introduced the Poincaré isomorphisms \( D\# : \bigwedge V^\ast \to \bigwedge V \) and \( D\# : \bigwedge V \to \bigwedge V^\ast \) by (see, e.g. \([1]\)),
\[
D\# u^\ast = \tilde{u}^\ast \downarrow e_\ast \quad \text{and} \quad D\# u_\ast = \theta^\ast \downarrow \tilde{u}_\ast
\]
For an element of the form \( u = u_\ast \wedge u^\ast \) with \( u_\ast \in \bigwedge V \) and \( u^\ast \in \bigwedge V^\ast \),
\[
\star u = D\# u^\ast \wedge D\# u_\ast
\]

4.3 Differential Algebra Structure on \( \bigwedge H_V \)

Provide the spaces \( \bigwedge V \) and \( \bigwedge V^\ast \) with structure of differential algebras, picking up linear operators \( \partial_\ast : \bigwedge V \to \bigwedge V \) and \( \partial^\ast : \bigwedge V^\ast \to \bigwedge V^\ast \), both
satisfying
\[ \mathfrak{d}^2 = 0 \]
\[ \mathfrak{d} \wedge + \mathfrak{d} = 0 \]
\[ \mathfrak{d}(u \wedge v) = (\mathfrak{d}u) \wedge v + \hat{u} \wedge (\mathfrak{d}v) \]

with \( \mathfrak{d} \equiv \mathfrak{d}_* \) or \( \mathfrak{d} \equiv \mathfrak{d}^* \) and \( u, v \in \bigwedge V \) or \( u, v \in \bigwedge V^* \), accordingly.

The pair \((\mathfrak{d}_*, \mathfrak{d}^*)\) induces a differential operator \( \mathfrak{d} \) on \( \bigwedge H_V \), defined as the unique differential operator (with respect to the grade involution of \( \bigwedge H_V \)) satisfying
\[ \mathfrak{d} \ i^w = i^w_\mathfrak{d} \quad \text{and} \quad \mathfrak{d} \ i_w = i_w^* \mathfrak{d}, \]
where \( i^w \) and \( i_w \) are the canonical homomorphisms from \( \bigwedge V \) and \( \bigwedge V^* \) to \( \bigwedge H_V \). That is to say, for an element of \( \bigwedge H_V \) of the form \( u = u_* \wedge u^* \) with \( u_* \in \bigwedge V \) and \( u^* \in \bigwedge V^* \), \( \mathfrak{d} \) acts as
\[ \mathfrak{d}u = (\mathfrak{d}_*u_*) \wedge u^* + \hat{u}_* \wedge (\mathfrak{d}^*u^*) \]

The simpler example of this construction is given by the left [or analogously the right] contraction. Indeed,
\[ \mathfrak{d}_* = x^* \downarrow \quad \text{and} \quad \mathfrak{d}^* = x_* \downarrow \]
for some \( x^* \in V \) and \( x_* \in V^* \), are easily seen to be differential operators on \( \bigwedge V \) and \( \bigwedge V^* \), respectively. Of course, they induce on \( \bigwedge H_V \) the operator
\[ \mathfrak{d} = x \downarrow, \]
with \( x = x_* \oplus x^* \).

5 Clifford Algebra of a Hyperbolic Space: The Mother Algebra

Introduce in \( \bigwedge H_V \) the Clifford product of a vector \( x \in H_V \) by an element \( u \in \bigwedge H_V \) by
\[ xu = x \downarrow u + x \wedge u \]
and extend this product by linearity and associativity to all of the space $\bigwedge H_V$. The resulting algebra is isomorphic to the Clifford algebra $\mathcal{C}\ell(H_V)$ of the hyperbolic structure $H_V$ and will thereby be identified with it.

We call $\mathcal{C}\ell(H_V)$ the mother algebra (or the hyperbolic Clifford algebra) of the vector space $V$. The even and odd subspaces of $\mathcal{C}\ell(H_V)$ will be denoted respectively by $\mathcal{C}\ell^{(0)}(H_V)$ and $\mathcal{C}\ell^{(1)}(H_V)$, so that

$$\mathcal{C}\ell(H_V) = \mathcal{C}\ell^{(0)}(H_V) \oplus \mathcal{C}\ell^{(1)}(H_V)$$

and the same notation of the exterior algebra is used for grade involution, reversion, and conjugation in $\mathcal{C}\ell(H_V)$, which obviously satisfy

$$(uv)^\wedge = \hat{u}\hat{v}, \quad (uv)^\sim = \check{v}\check{u}, \quad (uv)^\bar{\wedge} = \bar{\check{v}}\bar{\check{u}}$$

For vectors $x, y \in H_V$, we have the relation

$$xy + yx = 2\langle x, y \rangle$$

so that for the basis elements $\{\sigma_k\}$ it holds

$$\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl},$$
$$\sigma_{n+k}\sigma_{n+l} + \sigma_{n+l}\sigma_{n+k} = -2\delta_{kl},$$
$$\sigma_k\sigma_{n+l} = -\sigma_{n+l}\sigma_k.$$

In turn, for the Witt basis $\{e_k, \theta^k\}$, we have instead

$$e_ke_l + e_le_k = 0,$$
$$\theta^k\theta^l + \theta^l\theta^k = 0,$$
$$\theta^ke_l + e_l\theta^k = 2\delta^k_l.$$

The Clifford product have the following general properties

$$u \wedge \sigma = u\sigma, \quad \sigma \wedge u = \sigma u$$
$$j_u, v w >= j\hat{v}u, w >= ju\hat{w}, v >$$
$$x \wedge u = \frac{1}{2}(xu + \hat{u}x), \quad u \wedge x = \frac{1}{2}(ux + x\hat{u})$$
$$x \wedge u = \frac{1}{2}(xu - \hat{u}x), \quad u \wedge x = \frac{1}{2}(ux - x\hat{u})$$
Moreover, for an element of the form $u = u_\ast \wedge u^\ast$, with $u_\ast \in \bigwedge V$ and $u^\ast \in \bigwedge V^\ast$,

$$xu = (x^\ast u_\ast) \wedge u^\ast + \hat{u}_\ast \wedge (x_\ast u^\ast)$$

and it is also important to note that the square of the volume 2n-vector $\sigma$ satisfy

$$\sigma^2 = 1$$

** Proposition 2.** There is the following natural isomorphism

$$\mathcal{C} \ell(H_V) \simeq \text{End}(\bigwedge V).$$

In addition, being $b$ a non-degenerate symmetric bilinear form on $V$, it holds also

$$\mathcal{C} \ell(H_V) \simeq \mathcal{C} \ell(H_{bV}) \simeq \mathcal{C} \ell(V, b) \otimes \mathcal{C} \ell(V, -b).$$

** Proof.** The first isomorphism is given by the extension to $\mathcal{C} \ell(H_V)$ of the Clifford map $\varphi : H_V \to \text{End}(\bigwedge V)$ by $x \mapsto \varphi_x$, with

$$\varphi_x(u_\ast) = \frac{1}{\sqrt{2}} (x^\ast \wedge u_\ast + x_\ast \wedge u_\ast),$$

for all $u_\ast \in \bigwedge V$. The second isomorphism, in turn, is induced from the Clifford map

$$x_\ast \oplus x^\ast \mapsto x_+ \hat{\otimes} 1 + 1 \hat{\otimes} x_-, $$

with

$$x_\pm = \frac{1}{\sqrt{2}} (b^\ast x^\ast \pm x_\ast).$$

**Corollary.** The even and odd subspaces of the hyperbolic Clifford algebra are

$$\mathcal{C} \ell^0(H_V) \simeq \text{End}(\bigwedge^{(0)} V) \oplus \text{End}(\bigwedge^{(1)} V)$$
and
\[ Cℓ^{(1)}(HV) \simeq L(\bigwedge^{(0)}V, \bigwedge^{(1)}V) \oplus L(\bigwedge^{(1)}V, \bigwedge^{(0)}V) \]
where \(L(V,W)\) denotes the space of the linear mappings from \(V\) to \(W\) and \(\bigwedge^{(0)}V\) and \(\bigwedge^{(1)}V\) denote respectively the spaces of the even and of odd elements of \(\bigwedge V\).

**Remark.** Recalling the definition of the second order hyperbolic structure, \(H^{2}_V\), it follows from Proposition 2 that
\[ Cℓ(H^{2}_V) \simeq \text{End}(Cℓ(H_V)) \]
The algebra \(Cℓ(H^{2}_V)\) may be called *grandmother* algebra of the vector space \(V\).

6 **Ideals in \(Cℓ(H_V)\). Superfields**

It is a well known result that minimal ideals in Clifford algebras are the spaces representing spinors (for more details see, e.g., [6, 8]). Given a pair of dual basis \(\{e_1, \ldots, e_n\}\) in \(V\) and \(\{θ^1, \ldots, θ^n\}\) in \(V^*\) the \(n\)-form volume element \(θ^* = θ^1 ∧ ... ∧ θ^n\). Taking into account Eq.(3) (and the analogous ones for the right contraction) we immediately realize that \(\bigwedge V^*\) (the space of multiforms) is a minimal left ideal of the hyperbolic Clifford algebra \(Cℓ(H_V)\), i.e., we have [5]
\[ \bigwedge V^* = Cℓ(H_V)θ^*. \]
A typical element of \(ψ ∈ \bigwedge V^*\) which is a *spinor* in \(Cℓ(H_V)\) is then written as:
\[ ψ = s + v_μθ^μ + \frac{1}{2}f_{μν}θ^μθ^ν + ... + pθ^1 ∧ ... ∧ θ^n, \]
where \(s, v_μ, f_{μν}, ... , p \in \mathbb{R}\).

The algebraic structure described above can be transferred for spin manifolds through the construction of the Hyperbolic Clifford bundle \(Cℓ(H(M)) := Cℓ(TM ⊕ T^∗M, 1, >)\) and the elements of \(Cℓ(H(M))θ^*\) are Witten superfields[9].

\[^3\text{Of course, now } θ^* ∈ \text{sec} \bigwedge T^∗M \hookrightarrow \text{sec} Cℓ(H(M)) \text{ refers to a volume element in } M \text{ defined by a section of the coframe bundle, i.e., } \{θ^i\} ∈ \text{sec } F(M).\]
It is opportune to recall here that in [6, 7] we showed that Dirac spinors can be rigorously represented as equivalence classes of (even) sections of the Clifford bundle of differential forms $\mathcal{Cl}(T^*M, g)$ of a Lorentzian spacetime, and in this sense the electron field is a kind of superfield. More details on this issue can be found in [8].

7 Conclusions

In this paper we introduced and studied with details the Clifford algebra $\mathcal{Cl}(H_V)$ of multivecfors of a hyperbolic space $H_V$, and give the geometrical meaning of those objects. We recall that $\mathcal{Cl}(H_V)$ is the algebra of endomorphisms of $\bigwedge V$, something that make mutivecfors really important objects, although they are not yet general enough to represent the operators we use in differential geometry.

Besides that we showed that $\bigwedge V^*$ is a particular minimum ideal in $\mathcal{Cl}(H_V)$ whose elements are representatives of spinors and which describe the algebraic properties of Witten superfields. We end with the observation that Clifford and Grassmann algebras can be generalized in several different ways and in particular the approach in [4] that deals with Peano spaces seems particularly interesting to be further studied concerning construction of hyperbolic structures. We will return to this issue in a forthcoming paper.

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4The general operator need in the study of differential geometry are the extensors (and extensor fields) of the hyperbolic space that will be presented in a series of forthcoming papers.
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