On Some Ramsey Numbers for Quadrilaterals Versus Wheels

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Abstract For given graphs $G_1$ and $G_2$, the Ramsey number $R(G_1, G_2)$ is the least integer $n$ such that every 2-coloring of the edges of $K_n$ contains a subgraph isomorphic to $G_1$ in the first color or a subgraph isomorphic to $G_2$ in the second color. Surahmat et al. proved that the Ramsey number $R(C_4, W_n) \leq n + \lceil (n - 1)/3 \rceil$. By using asymptotic methods one can obtain the following property: $R(C_4, W_n) \leq n + \sqrt{n} + o(1)$. In this paper we show that in fact $R(C_4, W_n) \leq n + \sqrt{n} - 2 + 1$ for $n \geq 11$. Moreover, by modification of the Erdős-Rényi graph we obtain an exact value $R(C_4, W_{q^2+1}) = q^2 + q + 1$ with $q \geq 4$ being a prime power. In addition, we provide exact values for Ramsey numbers $R(C_4, W_n)$ for $14 \leq n \leq 17$.

Keywords Ramsey numbers · Quadrilateral · Wheels

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1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let $G$ be such a graph. The vertex set of $G$ is denoted by $V(G)$, the edge set of $G$ by $E(G)$, and the number of edges in $G$ by $e(G)$. Let $d(v)$ be the
degree of vertex \( v \), and let \( d_1(v) \) and \( d_2(v) \) denote the number of the edges incident to \( v \) colored with the first and the second color, respectively. By \( \delta_i(G) \) we denote the minimum degree of \( G \) in color \( i \). The open neighborhood in color \( i \) of vertex \( v \) in graph \( G \) is \( N_i(v) = \{ u \in V(G) \mid \{ u, v \} \in E(G) \text{ and } \{ u, v \} \text{ is colored with color } i \} \). Define \( G[S] \) to be the subgraph of \( G \) induced by the set of vertices \( S \subset V(G) \). Let \( P_n \) (resp. \( C_n \)) be the path (resp. cycle) on \( n \) vertices. A wheel \( W_n \) is a graph on \( n \) vertices obtained from a \( C_{n-1} \) by adding one vertex \( w \) and making \( w \) adjacent to all vertices of the \( C_{n-1} \).

For given graphs \( G_1, G_2 \), the Ramsey number \( R(G_1, G_2) \) is the smallest integer \( n \) such that if we arbitrarily color the edges of the complete graph of order \( n \) with 2 colors, then it always contains a monochromatic copy of \( G_1 \) colored with the first color or a monochromatic copy of \( G_2 \) colored with the second color. A coloring of the edges of \( n \)-vertex complete graph with 2 colors is called a \( (G_1, G_2; n) \)-coloring if it does not contain a subgraph isomorphic to \( G_1 \) colored with the first color nor a subgraph isomorphic to \( G_2 \) colored with the second color.

The Turán number \( t(n, G) \) is the maximum number of edges in any \( n \)-vertex graph which does not contain a subgraph isomorphic to \( G \). A graph on \( n \) vertices is said to be extremal with respect to \( G \) if it does not contain a subgraph isomorphic to \( G \) and has exactly \( t(n, G) \) edges.

Some well known theorems will be used to prove the main result of this paper.

**Theorem 1** (Ore [3]) Let \( G \) be a graph on \( n \) (\( n \geq 3 \)) vertices. If \( d(v) + d(w) \geq n \) for every pair of non-adjacent vertices \( v \) and \( w \) of \( G \), then \( G \) is Hamiltonian.

**Theorem 2** (Rosta [7], Faudree and Schelp [2]) For all integers \( n \geq 5 \)

\[
R(C_4, C_n) = \text{max}(n + 1, 7).
\]

**Theorem 3** (Reiman [6]) For all integers \( n \geq 4 \)

\[
t(n, C_4) < \frac{1}{4}n(1 + \sqrt{4n - 3}).
\]

Several results have been obtained for wheels and quadrilaterals. Surahmat et al. [8] showed that \( R(C_4, W_m) = 9, 10 \) and \( 9 \) for \( m = 4, 5 \) and \( 6 \) respectively. Independently, Kung-Kuen Tse [10] showed that \( R(C_4, W_m) = 10, 9, 10, 9, 11, 12, 13, 14, 16 \) and \( 17 \) for \( m = 4, 5, 6, 7, 8, 9, 10, 11, 12 \) and \( 13 \), respectively. In 2005, Surahmat et al. [9] obtained property that \( R(C_4, W_n) \leq n + \lceil (n - 1)/3 \rceil \). Suppose that we have an admissible coloring of \( K_m \) without \( C_4 \) in color 1 and without \( W_n \) in color 2. Asymptotically we have a well-known property that \( t(n, C_4) \approx \frac{1}{2}n^2 \). Since \( R(C_4, C_{n-1}) = n \) for \( n \geq 7 \), we obtain \( \frac{1}{4}m(m-n) \approx \frac{1}{2}m^2 \) which implies that \( m - n \approx \sqrt{m} \) and \( R(C_4, W_n) = n + \sqrt{n} + o(1) \). The main result of this work is the following.

**Theorem 4** For all integers \( n \geq 11 \)

\[
R(C_4, W_n) \leq n + \left\lfloor \sqrt{n} - \frac{3}{2} \right\rfloor + 1.
\]
2 Main Theorem

Proof (Theorem 4) For simplicity of notation, we set \( k = [\sqrt{n - 2}] \). Let us consider a graph \( G = K_{n+k+1} \) and its decomposition \( G = G_1 \cup G_2 \), where \( V(G) = V(G_1) = V(G_2) \) and \( E(G_i) \) consists of all edges of \( G \) in \( i \)th color. Suppose that for graph \( G \) there is a \( (C_4, W_n; n + k + 1) \)-coloring and let us consider such coloring.

First let us assume that there is a vertex \( v \in V(G) \) such that \( d_1(v) \leq k \). Then \( d_2(v) \geq n \) and by \( R(C_4, C_{n-1}) = n \) we immediately obtain a \( W_n \) in the second color.

Now, suppose that \( \delta_1(G) \geq k + 2 \). Let us consider integer \( p \) such that \( n \in \{(p - 1)^2 + 2, \cdots, p^2 + 1\} \). Then \( k = p - 1 \). Let \( s = n - (p - 1)^2 \), one can see that \( 2 \leq s \leq 2p \). In this case the minimum possible number of edges in color 1 in \( G \) is

\[
\left[ \frac{1}{2} (n + k + 1) \delta_1(G) \right] \geq \frac{1}{4} (n + k + 1)(2p + 2) \geq
\]

\[
\geq \frac{1}{4} (n + k + 1) \left( 1 + \sqrt{4(p^2 + p + 1) - 3} \right) \geq
\]

\[
\geq \frac{1}{4} \left( n + k + 1 \right)(1 + \sqrt{4(p^2 - p + 1 + s) - 3}) \geq
\]

\[
\geq \frac{1}{4} \left( n + k + 1 \right)(1 + \sqrt{4(n + k + 1) - 3}) > t(n + k + 1, C_4),
\]

a contradiction.

The last case to consider is \( \delta_1(G) = k + 1 \). In this case \( G_1 \) has at most \( t(n + k + 1, C_4) = \left\lfloor \frac{(n+k+1)\delta_1(G)}{2} \right\rfloor + A \) edges. Similarly to the previous case let us consider integer \( p \) such that \( n \in \{(p - 1)^2 + 2, \cdots, p^2 + 1\} \). Then \( k + 1 = p \). Let us take vertex \( v \in V(G) \) such that \( d_1(v) = k + 1 \), subgraph \( G' = G_2[N_2(v)] \) and two vertices \( v_1, v_2 \in V(G') \), where the edge \( \{v_1, v_2\} \in E(G_1) \). Then \( |V(G')| = n - 1 \) and in subgraph \( G' \) we have \( d_2(v_1) + d_2(v_2) = 2(n - 2) - (d_1(v_1) + d_1(v_2)) \). We have the following

Claim \( d_1(v_1) + d_1(v_2) \leq 2\delta_1(G) + A \) or \( d_1(v_1) + d_1(v_2) \leq 2\delta_1(G) + A + 1 \) depending on the parity of \( \delta_1(G) \) and \( (n + k + 1) \).

Proof If \( \delta_1(G) \) and \( |V(G)| = (n + k + 1) \) are odd, then it is impossible that for all vertices \( w \in V(G) \) we have \( d_1(w) = \delta_1(G) \). In the worst situation, when all \( A \) edges are adjacent to \( v_1 \) or \( v_2 \), we have that \( d_1(v_1) + d_1(v_2) \leq 2\delta_1(G) + A + 1 \).

We will prove that \( d_2(v_1) + d_2(v_2) \geq n - 1 \) for all vertices \( v_1, v_2 \in V(G') \) such that \( \{v_1, v_2\} \in E(G_1) \). In this case we obtain a contradiction because by Ore’s Theorem subgraph \( G' \) contains a \( C_{n-1} \) and \( G \) contains a \( W_n \) in the second color.

The remaining part of the proof is divided into three parts.
Table 1  Values needed to prove that $d_2(v_1) + d_2(v_2) \geq n - 1$ for $11 \leq n \leq 17$

| $n$ | 11   | 12   | 13   | 14   | 15   | 16   | 17   |
|-----|------|------|------|------|------|------|------|
| $|V(G)| = n + k + 1$ | 15   | 16   | 17   | 18   | 19   | 20   | 21   |
| $n - 1$ | 10   | 11   | 12   | 13   | 14   | 15   | 16   |
| $t(|V(G)|, C_4)$ | 30   | 33   | 36   | 39   | 42   | 46   | 50   |
| $A$ | 0    | 1    | 2    | 3    | 4    | 6    | 8    |
| $d_2(v_1) + d_2(v_2) \geq$ | 10   | 11   | 12   | 13   | 14   | 14   | 14   |

Table 2  Values needed to prove that $d_2(v_1) + d_2(v_2) \geq n - 1$ for $18 \leq n \leq 26$

| $n$ | 18   | 19   | 20   | 21   | 22   | 23   | 24   | 25   | 26   |
|-----|------|------|------|------|------|------|------|------|------|
| $|V(G)| = n + k + 1$ | 23   | 24   | 25   | 26   | 27   | 28   | 29   | 30   | 31   |
| $n - 1$ | 17   | 18   | 19   | 20   | 21   | 22   | 23   | 24   | 25   |
| $t(|V(G)|, C_4)$ | 56   | 59   | 63   | 67   | 71   | 76   | 80   | 85   | 90   |
| $A$ | –    | –    | 0    | 2    | 3    | 6    | 7    | 10   | 12   |
| $d_2(v_1) + d_2(v_2) \geq$ | 21   | 24   | 25   | 26   | 26   | 26   | 26   | 26   | 25   |

1. $11 \leq n \leq 17$

In this case $\delta_1(G) = p = 4$. The exact values of $t(n, C_4)$ are known for all $n \leq 21$, see [1]. In addition, this paper covers all extremal graphs. Table 1 contains all values needed to prove the inequality $d_2(v_1) + d_2(v_2) \geq n - 1$.

One can see that for all $11 \leq n \leq 15$ the proof is complete. For case $n = 16$ let us consider the graph $G_1$. If it is the only extremal graph for $t(20, C_4)$ [1] then its maximum degree is 5, so by Ore’s Theorem $G'$ contains a $C_{15}$ and $G$ contains a $W_{16}$ in the second color. If $|E(G_1)| \leq 45$, then $A \leq 5$ and $d_2(v_1) + d_2(v_2) \geq 15$. By similar considerations in case $n = 17$, if $G_1$ is the only extremal graph for $t(21, C_4)$ [1] then $G'$ contains a $C_{16}$ and $G$ contains a $W_{17}$. If $|E(G_1)| = 49$ and there exists a vertex $w \in V(G)$ such that $d_1(w) = 8$, then we obtain a $C_4$ in color 1 in $G$ (consider $\delta_1(G) = 4$ and all possible edges in color 1 from $N_1(w)$ to the remaining vertices of $G$). If $d_1(w) \leq 7$ for all vertices $w \in V(G)$, then by Ore’s Theorem $G'$ contains a $C_{16}$ and $G$ contains a $W_{17}$. Then $A \leq 6$ and $d_2(v_1) + d_2(v_2) \geq 16$ and we are done.

2. $18 \leq n \leq 26$ In this case $\delta_1(G) = p = 5$. The exact values and extremal graphs for $t(n, C_4)$ are known for all $22 \leq n \leq 31$, see [11]. Table 2 presents all values needed to finish the checking the inequality $d_2(v_1) + d_2(v_2) \geq n - 1$ for $18 \leq n \leq 26$. We will mark with ‘−’ the case when $A < 0$.

3. $n \geq 27$

In this case $p \geq 6$. We have that in $G'd_2(v_1) + d_2(v_2) \leq 2\delta_1(G) + 1 + A$, then in $G'd_2(v_1) + d_2(v_2) \geq 2(n - 2) - (2\delta_1(G) + 1 + A) = 2n - 2p - 5 - A$. In order to finish the proof we have to show that $2n - 2p - 5 - A \geq n - 1$, i.e. $A \leq n - 2p - 4$. Observe that $w(n, p) = t(n + p, C_4) - \left\lfloor \frac{(n+p)p}{2} \right\rfloor \leq \frac{1}{4}(n + p)(1 + \sqrt{(n + p)^2 - 3}) - \left\lfloor \frac{(n+p)p}{2} \right\rfloor$ is an increasing function of $n$, i.e. $w(n_1, p) > w(n_2, p)$ if $n_1 > n_2$. Then, the maximal possible value of $A$ holds for $n = p^2 + 1$. For even $p$ we have that $t(n + p, C_4) \leq \frac{(p^2 + p + 1)(p + 1)}{2} - \frac{1}{2}$ and
\[ \left\lfloor \frac{n+p}{2} \right\rfloor = \left( \frac{p^2+p+1}{2} \right)p. \]

For odd \( p \) we have that \( t(n+p, C_4) \leq \frac{(p^2+p)(p+1)}{2} \) and \( \left\lceil \frac{n+p}{2} \right\rceil = \left( \frac{p^2+p}{2} + \frac{1}{2} \right)p \). In both situations we obtain that \( A \leq \frac{p^2+p}{2} \) and for all \( p \geq 6, A \leq p^2-2p-3 \).

\( \square \)

Taking \( n = q^2 + 1 \) in Theorem 4, we have

**Corollary 5** For all integers \( q \), \( q \geq 4 \)

\[ R(C_4, W_{q^2+1}) \leq q^2 + q + 1. \]

### 3 Erdős-Rényi Graph

Let \( q \) be a prime power. The famous Erdős-Rényi graph \( ER(q) \), first constructed by Erdős and Rényi in 1962, was studied in detail by Parsons in [4]. We know the following properties of \( ER(q) \):

- \( ER(q) \) has \( q^2 + q + 1 \) vertices, \( q + 1 \) vertices with degree \( q \) and \( q^2 \) vertices with degree \( q + 1 \)
- \( ER(q) \) does not contain a subgraph \( C_4 \)
- in \( ER(q) \) there are no two adjacent vertices of degree \( q \)
- in \( ER(q) \) no vertex of degree \( q \) belongs to a subgraph \( K_3 \)

Let \( H(q) \) denote the subgraph of \( ER(q) \) obtained by deleting one vertex of degree \( q \). By the third property of \( ER(q) \), the subgraph \( H(q) \) contains \( 2q \) vertices with degree \( q \) and \( q^2 - q \) vertices with degree \( q + 1 \). One can observe that for all vertices \( w \), the degree \( d(w) \) in the complement of \( H(q) \) is at most \( q^2 - 1 \). By this fact, the complement of \( H(q) \) does not contain a \( W_{q^2+1} \), so there exists a \( (C_4, W_{q^2+1}; q^2 + q) \)-coloring. By this fact and by Corollary 5 we have the following

**Theorem 6** For \( q \geq 4 \) being a prime power

\[ R(C_4, W_{q^2+1}) = q^2 + q + 1. \]

### 4 Exact Values for Small Wheels

Up to date values for \( R(C_4, W_n) \) are known only for \( n \leq 13 \). We determined the next four values as follows:

**Theorem 7**

1. \( R(C_4, W_{14}) = 18 \),
2. \( R(C_4, W_{15}) = 19 \),
3. \( R(C_4, W_{16}) = 20 \),
4. \( R(C_4, W_{17}) = 21 \).

**Proof** By Theorem 6 we immediately obtain \( R(C_4, W_{17}) = 21 \). In order to determine an upper bound for all remaining cases we use Theorem 4. For a lower bound we present appropriate matrix of critical coloring (see Fig. 1). These matrices were obtained by using simulated annealing to find \( C_4 \)-free graphs with a minimum degree 4. \( \square \)
Fig. 1  Lower bound for $R(C_4, W_n)$, $14 \leq n \leq 16$

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