Escape Times in Fluctuating Metastable Potential and Acceleration of Diffusion in Periodic Fluctuating Potentials

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Abstract

The problems of escape from metastable state in randomly flipping potential and of diffusion in fast fluctuating periodic potentials are considered. For the overdamped Brownian particle moving in a piecewise linear dichotomously fluctuating metastable potential we obtain the mean first-passage time (MFPT) as a function of the potential parameters, the noise intensity and the mean rate of switchings of the dichotomous noise. We find noise enhanced stability (NES) phenomenon in the system investigated and the parameter region of the fluctuating potential where the effect can be observed. For the diffusion of the overdamped Brownian particle in a fast fluctuating symmetric periodic potential we obtain that the effective diffusion coefficient depends on the mean first-passage time, as discovered for fixed periodic potential. The effective diffusion coefficients for sawtooth, sinusoidal and piecewise parabolic potentials are calculated in closed analytical form.

Key words: Brownian motion, Noise enhanced stability, Metastable state, Diffusion coefficient
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1 Mean first-passage times in dichotomously fluctuating metastable potential

The problem of escape from metastable state in fluctuating potential is of great importance to many natural systems, ranging from physical and chemical systems to biological complex systems. The characteristic feature of all these complex nonequilibrium systems is that they are open systems, which are in contact with a fluctuating environment. A typical problem is the enhancement of stability of metastable states in fluctuating potentials due to the external noise [1,2,3,4,5]. The noise enhanced stability (NES) phenomenon was observed experimentally and numerically in various physical systems. This effect implies that a system remains in the metastable state for a longer time than in deterministic case and mean first-passage time (MFPT) has a maximum at some noise intensity. Specifically NES phenomenon was found in the transient dynamics of a periodically driven overdamped particle in a noisy cubic potential [4], in a tunnel diode by investigating the escape time in a periodically driven metastable system in a strong forcing regime [1]. More recently it was found that several properties of the noise-induced stability in one-dimensional maps and in fluctuating bistable potentials are related to NES effect [6]. The noise induced slowing down exhibited by the mobility of an overdamped particle moving in a periodic potential and in an inhomogeneous medium [7], the noise induced order in one-dimensional map of the Belousov-Zhabotinsky reaction [8], and the one-peak structure in the MFPT of a bistable kinetic model driven by correlated noises [9] are akin to the NES phenomenon. Previous theoretical papers analyzed NES phenomenon in systems with metastable and unstable states of fixed potential or periodically driving potential [1,2,3,4,6]. However the model of randomly switching metastable state is more realistic when we describe the generation process of the carrier traps in semiconductors. Let us consider one-dimensional overdamped Brownian motion in randomly switching potential profile

\[
\frac{dx}{dt} = -\frac{dU(x)}{dx} + a\eta(t) + \xi(t). \tag{1}
\]

where \(x(t)\) is the displacement in time \(t\), \(\xi(t)\) is the white Gaussian noise with zero mean and \(\langle \xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)\), \(\eta(t)\) is Markovian dichotomous process which takes the values \(\pm 1\) with the mean rate of switchings \(\nu\). We assume that potential \(U(x)\) has reflecting boundary at the point \(x = 0\) and absorbing boundary at the point \(x = b\) \((b > 0)\). Exact results for MFPTs for Brownian diffusion in switching potentials were first derived in [10]. Here we use equations from [10] but different conditions at the reflecting boundary [11]. Thus following ref.[10] we obtain the coupled differential equations for the MFPTs of our system (1)
Fig. 1. The piecewise linear potential $U(x)$ with metastable and unstable states.

$$DT^\prime\prime + [a - U^\prime (x)] T^\prime + \nu (T_- - T_+) = -1,$$
$$DT^\prime\prime - [a + U^\prime (x)] T^\prime + \nu (T_+ - T_-) = -1.$$ (2)

Here $T_+(x)$ and $T_-(x)$ are respectively the MFPTs for positive $\eta(0) = +1$ and negative $\eta(0) = -1$ initial value of dichotomous noise. In accordance with ref. [11] the conditions at reflecting boundary $x = 0$ and absorbing boundary $x = b$ are $T'_\pm(0) = 0$ and $T_{\pm}(b) = 0$. Let us introduce for convenience two new auxiliary functions $T = (T_+ + T_-)/2$ and $\theta = (T_+ - T_-)/2$. So that from Eqs. (2) we obtain

$$DT^\prime\prime - U^\prime (x) T^\prime + a\theta' = -1,$$
$$D\theta^\prime\prime - U^\prime (x) \theta' + aT' - 2\nu\theta = 0.$$ (3)

with the following boundary conditions: $T'(0) = \theta'(0) = 0$ and $T(b) = \theta(b) = 0$. By eliminating $T(x)$ from Eqs. (3) we arrive at third-order linear differential equation for $\theta(x)$

$$\theta^\prime\prime\prime - \frac{2U'(x)}{D}\theta'' + \left[\frac{U'^2(x)}{D^2} - \frac{U''(x)}{D} - \gamma^2\right]\theta' + \frac{2\nu U'(x)}{D^2}\theta = \frac{a}{D^2},$$ (4)

where $\gamma = \sqrt{a^2/D^2 + 2\nu/D}$. Let us assume that in Eq. (1) $U(x) = k(L - x) \cdot 1(x - L)$, where $1(x)$ is step function and $0 < L < b$. As we can see from Fig. 1 we have a metastable state for $\eta(t) = -1$ and an unstable state for $\eta(t) = +1$. We will consider the MFPT $T_+(0)$, i.e. MFPT for initial unstable state with starting position of Brownian particle at the origin. In the absence of switchings and thermal diffusion the dynamical MFPT $T_+(0)$ equals

$$T_+(0) = \frac{L}{a} + \frac{b - L}{k + a}.$$ (5)
We can solve Eqs. (3) and (4) separately for regions $0 < x < L$ and $L < x < b$. After some algebra we obtain finally the MFPT $T_+(0)$ as

$$T_+(0) = T(0) + \theta(0) = c_1 \frac{\gamma^2 D^2}{a^2} + c_2 - \frac{a^2}{\gamma^4 D^3}. \quad (6)$$

The unknown constants $c_1$ and $c_2$ can be determined from the boundary conditions and the continuity conditions of the functions $\theta(x), \theta'(x), T(x), T'(x)$ at the point $x = L$.

2 Analysis of noise enhanced stability phenomenon conditions

By expanding the Eq. (6) in power series on $D$, for very low intensity we obtain

$$T_+(0) \simeq 2L \frac{\nu^2}{a^2} + \frac{b - L}{k} - \frac{\beta (1 - \beta)}{2\nu} \left(1 - e^{-s}\right) + \frac{D}{a^2} F(s, \beta, \omega) \quad (7)$$

where

$$F(s, \beta, \omega) = \frac{\beta^3 [2 + s (1 + \beta^2)]}{(1 - \beta) (1 - \beta^2)} \cdot e^{-s} + \frac{\beta (1 - \beta^2 - 2\beta^3)}{2 (1 - \beta^2)} \left(1 - e^{-s}\right) - \frac{5 + \beta}{2 (1 + \beta)} + 2\omega \left(\frac{1}{1 - \beta^2} - \frac{3}{\beta}\right) - \frac{2\omega^2}{\beta^2} \quad (8)$$

and $\beta, s$ and $\omega$ are dimensionless parameters

$$\beta = \frac{a}{k}, \quad \omega = \frac{\nu L}{k}, \quad s = \frac{2\omega}{1 - \beta^2} \left(\frac{b}{L} - 1\right).$$

The sign of the function $F(s, \beta, \omega)$ in Eq. (7) determines the condition to observe the NES phenomenon in considered system. We can write this condition by the following inequality

$$F(s, \beta, \omega) > 0. \quad (9)$$

Let us analyze the structure of NES phenomenon region on the plane $(\beta, \omega)$. In the case of very slow switching $\nu \to 0$ ($\omega \to 0$, $s \to 0$) Eq. (9) takes the form
Fig. 2. The shaded areas are the region of the plane \((\omega, \beta)\) where the NES effect takes place.

\[
\beta > 0.802; \quad \omega < \frac{2\beta^2 (1 - \beta) - 5\beta (1 - \beta^2)^2 / 2}{6 (1 - \beta^2)^2 - 2\beta (1 - \beta^2) + \beta^2 (3\beta^2 - 1) (b/L - 1)}. \tag{10}
\]

In the case of \(\beta \approx 1\) we obtain from Eqs. (8), (9) and (10)

\[
\omega < \frac{1 - \beta}{b/L - 1}; \quad \frac{1}{2} + \frac{5}{2} (1 - \beta) < \omega < \frac{1}{2(1 - \beta)}.
\]

In Fig. 2 are shown two NES regions (shaded areas) on the plane \((\beta, \omega)\). As we can see from Fig. 2, the NES effect occurs only at the values of \(\beta\) near 1, i.e. at very small steepness \(k - a = k(1 - \beta)\) of the reverse potential barrier for the metastable state (see Fig. 1). This NES region is different from that obtained for the case of periodical dichotomous force [2]. On our opinion, this narrowing of NES phenomenon area in comparison with the case of periodically driven metastable states is due to the uncertainty of metastable state starting time. The NES effect disappears when we choose the absorbing boundary at the point \(y = L\).

### 3 Acceleration of diffusion in fluctuating periodic potentials

The problem of diffusion in a periodic potential is related to a rich variety of physical situations such as diffusion of atoms in crystals, synchronization of oscillations, fluctuations of a Josephson supercurrent and so on. An exact expression for diffusion coefficient was independently obtained in the overdamped limit by various methods for an arbitrary periodic potential and for sinusoidal potential [12]. Recently the mean velocity and effective diffusion coefficient of Brownian particle moving in a tilted periodic potential have been found in [13]. The case of diffusion in fast fluctuating periodic field was first investigated in ref. [14], where the exact result for diffusion coefficient in the
sawtooth potential was derived. In this section we generalize this result to
the case of arbitrary potential profiles (see also [15]). We will find that the
problem of calculating an effective diffusion coefficient in fluctuating periodic
potential reduces to mean first-passage problem similar to the case of fixed
potential profile [16]. We consider an overdamped Brownian particle in fluctu-
tating periodic potential \( U(x) \) whose dynamics is governed by the Langevin
equation

\[
\frac{dx}{dt} = -\frac{dU(x)}{dx} \cdot \zeta(t) + \xi(t),
\]  

(11)

where \( x(t) \) is the displacement in time \( t \), \( \xi(t) \) and \( \zeta(t) \) are statistically inde-
dependent Gaussian white noises with zero means and intensities \( 2D \) and \( 2D_\zeta \)
respectively. Further we assume that potential \( U(x) \) is even function with pe-
riod \( L \) and place the origin in one of the potential minima. Following ref. [12]
we determine the effective diffusion coefficient as the limit

\[
D_{\text{eff}} = \lim_{t \to \infty} \frac{\langle x^2(t) \rangle}{2t}
\]  

(12)

because we have \( \langle x(t) \rangle = 0 \). From Eq. (11) the Fokker-Planck equation for
probability density \( W(x, t) \)\(^1\) is

\[
\frac{\partial W}{\partial t} = D_x \frac{\partial}{\partial x} U'(x) \frac{\partial}{\partial x} U'(x) W + D \frac{\partial^2 W}{\partial x^2}.
\]  

(13)

We can choose arbitrary initial condition for Eq. (13), since we look at the
asymptotic behavior of mean square coordinate of Brownian particles. We
place therefore all Brownian particles in the origin at \( t = 0 \): \( W(x, 0) = \delta(x) \).
Because of periodicity of the potential, the diffusion process can be coarsely
conceived as consecutive transitions of Brownian particle from points of poten-
tial minima \( x_m = mL \) to nearest neighboring points \( x_{m \pm 1} \). The transition time
represents the escape time over left or right absorbing boundaries \( x = x_{m \pm 1} \)
for particle starting from the point \( x = x_m \), i.e. the random first-passage time.
Thus, we can consider as in [13], the discrete process

\[
\tilde{x}(t) = \sum_{k=0}^{n(0,t)} q_i,
\]  

(14)

where \( q_i \) are random increments of jumps with values \( \pm L \) and \( n(0,t) \) de-
notes the total number of jumps in the time interval \( (0, t) \). In the asymptotic

\(^1\) We interpret the stochastic differential equation (11) in Stratonovich’s sense.
limit \( t \to \infty \) the ”fine structure” of a diffusion is unimportant, and the random processes \( x(t) \) and \( \tilde{x}(t) \) become statistically equivalent. In particular, \( \langle x^2(t) \rangle \simeq \langle \tilde{x}^2(t) \rangle \). Further, the random increments \( q_i \) and the random times \( \tau_j \) between jumps are statistically independent of one another because of the markovianity of \( x(t) \) and have the same probability distributions \( W(q) \) and \( w(\tau) \) respectively. Because of the symmetry of potential \( U(x) \) the probability density \( W(q) \) reads

\[
W(q) = \frac{1}{2} \left[ \delta(q - L) + \delta(q + L) \right].
\]

(15)

Let us calculate the second moment \( \langle \tilde{x}^2(t) \rangle \) from Eqs. (14), (15)

\[
\langle \tilde{x}^2(t) \rangle = \sum_{n=0}^{\infty} P_n(t) \sum_{k=0}^{n} \sum_{l=0}^{n} \langle q_k q_l \rangle = \sum_{n=0}^{\infty} P_n(t) \sum_{k=0}^{n} \langle q_k^2 \rangle = L^2 \langle n(0,t) \rangle,
\]

(16)

where \( P_n(t) \) is the probability of \( n \) jumps in the time interval \((0,t)\). On the other hand in the limit \( t \to \infty \) we have that

\[
\langle n(0,t) \rangle \simeq \frac{t}{\tau},
\]

(17)

where \( \tau = \langle \tau_j \rangle \) is the MFPT for Brownian particle with initial position \( x = 0 \) and absorbing boundaries at \( x = \pm L \). After substitution of Eqs. (16), (17) in Eq. (12) we arrive finally at the exact result

\[
D_{\text{eff}} = \frac{L^2}{2\tau},
\]

(18)

which was first derived in ref. [16] for fixed periodic potential. In fluctuating periodic potentials therefore the calculation of diffusion coefficient \( D_{\text{eff}} \) reduces to mean first-passage time problem. According to Eq. (13) we must solve the following equation with conjugated kinetic operator

\[
D \tau''(x) + D \zeta U'(x) \frac{d}{dx} [U'(x) \tau'(x)] = -1
\]

(19)

and boundary conditions \( \tau(-L) = 0, \tau(L) = 0 \). Then we must put \( x = 0 \), i.e. find \( \tau = \tau(0) \). Solving Eq. (19) with these boundary conditions, we finally obtain the following exact formula for effective diffusion coefficient of Brownian particle in fast fluctuating periodic potential \( U(x) \)
As evident from Eq. (20), $D_{\text{eff}} > D$ for an arbitrary potential profile $U(x)$, therefore diffusion of Brownian particles accelerates in comparison with the case $U(x) = 0$. This result fully confirms the assumption previously proposed in [14]. We emphasize that the value of diffusion constant does not depend on the height of potential barriers, as for fixed potential [12], but it depends on its gradient $U''(x)$. We can explain the phenomenon of diffusion acceleration directly from Eq. (18). The potential barriers change places through a random modulation and Brownian particles move from point $x = 0$ to point $x = L$ more rapidly in comparison with free diffusion case, i.e. in the average particles move downhill for the most part of the distance. Let us consider particular shapes of potential $U(x)$.

(a) For the sawtooth profile $U(x) = 2E|x|/L$ at $|x| \leq L/2$ we easily obtain the Malakhov’s exact result [14]

$$D_{\text{eff}} = D + D_{\zeta} \frac{4E^2}{L^2}. \quad (21)$$

(b) For sinusoidal potential $U(x) = E \sin^2(\pi x/L)$ from Eq. (20) we have

$$D_{\text{eff}} = \frac{\pi^2 D (1 + \gamma^2)}{4K^2 \left( \gamma / \sqrt{1 + \gamma^2} \right)}, \quad \gamma = \frac{\pi E}{L} \sqrt{\frac{D_{\zeta}}{D}}, \quad (22)$$

where $K(k)$ is the complete elliptic integral of the first kind $(0 < k < 1)$. We derive from Eq. (22) at small strength $D_{\zeta}$ of modulating white noise ($\gamma \ll 1$) the following approximating expression

$$D_{\text{eff}} \simeq D + D_{\zeta} \frac{\pi^2 E^2}{2L^2}.$$  

This formula coincides with approximate result [14] obtained on the coarse assumption of Gaussian probability density $W(x,t)$ although the real probability density is multi-modal. In opposite case $\gamma \gg 1$ using the asymptotic formula for elliptic integral

$$K(k) \simeq \ln \frac{4}{\sqrt{1 - k^2}} \quad (k \to 1),$$

we find from Eq. (22)
According to Eq. (23) the effective diffusion coefficient increases with intensity $D_\zeta$ of modulating noise but more slowly than linear law (21).

(c) Finally for piecewise parabolic periodic potential profile

$$U(x) = \begin{cases} 8E(x/L)^2, & |x| \leq L/4 \\ E \left[ 1 - 8(x/L - 1/2)^2 \right], & L/4 \leq x \leq 3L/4 \end{cases}$$

we get from the exact formula (20) the following result

$$D_{\text{eff}} = \frac{Dm^2}{\ln^2 \left( m + \sqrt{1 + m^2} \right)}, \quad m = \frac{4E}{L} \sqrt{\frac{D_\zeta}{D}}.$$  \hfill (24)

At comparatively small intensity $D_\zeta$ ($m \ll 1$) we obtain

$$D_{\text{eff}} \simeq D + D_\zeta \frac{16E^2}{3L^2}$$  \hfill (25)

that is quite similar to the formula obtained for sinusoidal potential. Moreover, the dependence of effective diffusion coefficient on large $D_\zeta$ is similar to the law (23) for sinusoidal potential profile.

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