A SHARP LOWER BOUND FOR MIXED-MEMBERSHIP ESTIMATION

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Consider an undirected network with \( n \) nodes and \( K \) perceivable communities, where some nodes may have mixed memberships. We assume that for each node \( 1 \leq i \leq n \), there is a probability mass function \( \pi_i \) defined over \( \{1, 2, \ldots, K\} \) such that

\[
\pi_i(k) = \text{the weight of node } i \text{ on community } k, \quad 1 \leq k \leq K.
\]

The goal is to estimate \( \{\pi_i, 1 \leq i \leq n\} \) (i.e., membership estimation).

We model the network with the degree-corrected mixed membership (DCMM) model [8]. Since for many natural networks, the degrees have an approximate power-law tail, we allow severe degree heterogeneity in our model.

For any membership estimation \( \{\hat{\pi}_i, 1 \leq i \leq n\} \), since each \( \pi_i \) is a probability mass function, it is natural to measure the errors by the average \( \ell^1 \)-norm

\[
\frac{1}{n} \sum_{i=1}^{n} \| \hat{\pi}_i - \pi_i \|_1.
\]

We also consider a variant of the \( \ell^1 \)-loss, where each \( \| \hat{\pi}_i - \pi_i \|_1 \) is re-weighted by the degree parameter \( \theta_i \) in DCMM (to be introduced).

We present a sharp lower bound. We also show that such a lower bound is achievable under a broad situation. More discussion in this vein is continued in our forthcoming manuscript [7].

The results are very different from those on community detection. For community detection, the focus is on the special case where all \( \pi_i \) are degenerate; the goal is clustering, so Hamming distance is the natural choice of loss function, and the rate can be exponentially fast.

The setting here is broader and more difficult: it is more natural to use the \( \ell^1 \)-loss, and the rate is only polynomially fast.

1. Introduction. Consider an undirected network \( \mathcal{N} = (V, E) \), where \( V = \{1, 2, \ldots, n\} \) is the set of nodes and \( E \) is the set of (undirected) edges. Let \( A \in \mathbb{R}^{n,n} \) be the adjacency matrix where

\[
A(i, j) = \begin{cases} 
1, & \text{if nodes } i \text{ and } j \text{ have an edge,} \\
0, & \text{otherwise.}
\end{cases}
\]

1 \leq i, j \leq n,

The diagonals of \( A \) are zero since we do not allow for self-edges. Suppose the network has \( K \) perceivable communities (i.e., clusters)

\[
\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_K.
\]
and a node may belong to more than one cluster (i.e., mixed memberships). For each node $1 \leq i \leq n$, suppose there exists a Probability Mass Function (PMF) $\pi_i = (\pi_i(1), \pi_i(2), \ldots, \pi_i(K))' \in \mathbb{R}^K$ such that

$$\pi_i(k) \text{ is the "weight" of node } i \text{ on } C_k, \quad 1 \leq k \leq K.$$ 

We call node $i$ “pure” if $\pi_i$ is degenerate (i.e., one entry is 1 and the other entries are 0) and “mixed” otherwise. The primary interest is to estimate $\pi_i, 1 \leq i \leq n$.

Estimating mixed memberships is a problem of great interest in social network analysis [1, 2, 8, 16]. Take the Polbook network [13] for example. Each node is a book on US politics for sale in Amazon.com, and there is an edge between two nodes if they are frequently co-purchased. Jin et al. (2017) [8] modeled this network with a two-community (“Conservative” and “Liberal”) mixed membership model, where the estimated mixed membership of a node describes how much weight this book puts on “Conservative” and “Liberal”.

We are interested in the optimal rate of convergence associated with membership estimation. Below, we introduce a model and present a sharp lower bound. We show that the lower bound is achievable in a broad class of situations where we allow severe degree heterogeneity.

1.1. Model. Consider the degree-corrected mixed membership (DCMM) model [8]. Recall that $A$ is the adjacency matrix. DCMM assumes that

$$\{A(i, j) : 1 \leq i < j \leq n\} \text{ are independent Bernoulli variables,}$$

where the Bernoulli parameters are different. For a symmetric non-negative matrix $P \in \mathbb{R}^{K,K}$ and a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)'$, where $\theta_i > 0$ is the degree heterogeneity parameter of node $i$, DCMM models

$$\mathbb{P}(A(i, j) = 1) = \theta_i \theta_j \cdot \pi_i' P \pi_j, \quad 1 \leq i < j \leq n.$$ 

To ensure model identifiability, we assume

$$P \text{ is non-singular and have unit diagonals.}$$

We now calibrate DCMM with a matrix form. Introduce the two matrices $\Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^{n,n}$ and $\Pi = [\pi_1, \pi_2, \ldots, \pi_n]' \in \mathbb{R}^{n,K}$. Then,

$$A = \left[ \Omega - \text{diag}(\Omega) \right] + W, \quad \Omega = \Theta \Pi \Pi' \Theta, \quad W = A - \mathbb{E}[A].$$
Here $\Omega$ is a low-rank matrix ($\text{rank}(\Omega) = K$) containing Bernoulli parameters and $W$ is a generalized Wigner matrix.

DCMM can be viewed as extending the mixed membership stochastic block model (MMSB) [1] to accommodate degree heterogeneity, and can also be viewed as extending the degree-corrected block model (DCBM) [11] to accommodate mixed memberships. DCMM is similar to the overlapping continuous community assignment model (OCCAM) [16], where the difference is that DCMM regards each membership vector $\pi_i$ as a PMF with a unit $\ell^1$-norm while OCCAM models that each $\pi_i$ has a unit $\ell^2$-norm (which seems hard to interpret).

**Remark.** The identifiability condition of DCMM is different from that of DCBM. In DCBM, even when $P$ is singular, the model can still be identifiable. However, in DCMM, since there are many more free parameters, the full rank assumption of $P$ is required for identifiability.

**An example.** Let’s look at an example with $K = 2$ and

$$P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$ 

If nodes $i$ and $j$ are both pure nodes, then there are three cases:

$$\mathbb{P}(A(i,j) = 1) = \theta_i \theta_j \begin{cases} a, & i, j \in C_1, \\ c, & i, j \in C_2, \\ b, & i \in C_1, j \in C_2 \text{ or } i \in C_2, j \in C_1. \end{cases}$$

As a result, in the special case with all nodes being pure, the “signal” matrix $\Omega$ has the form

$$\Omega = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \\ a & b \\ b & c \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix},$$

where the matrix $\Pi P \Pi'$ can be shuffled to a block-wise constant matrix by some unknown permutation:

$$\Pi P \Pi' = \begin{bmatrix} a & b & a & b \\ b & c & b & c \\ a & b & a & b \\ b & c & b & c \end{bmatrix} \xrightarrow{\text{permute}} \begin{bmatrix} a & a & b & b \\ a & a & b & b \\ a & a & b & b \\ b & b & c & c \end{bmatrix}.$$
In general cases where the nodes have mixed memberships, $\Omega$ has a similar form, except that $\Pi$ is no longer a matrix of 0’s and 1’s and $\Pi \Pi'$ can no longer be shuffled to a block-wise constant matrix.

$$
\Omega = 
\begin{bmatrix}
\theta_1 & \cdots & \theta_n \\
\vdots & \ddots & \vdots \\
0.7 & 0.3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

$\Pi \Pi'$

1.2. Loss functions. Given estimators $\hat{\Pi} = [\hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_n]'$, since each $\pi_i$ is a PMF, it is natural to measure the (unweighted) $\ell^1$-estimation error:

$$
\mathcal{H}(\hat{\Pi}, \Pi) = n^{-1} \sum_{i=1}^{n} \|\hat{\pi}_i - \pi_i\|_1.
$$

(1.4)

We also consider a variant of the $\ell^1$-error where $\|\hat{\pi}_i - \pi_i\|_1$ is reweighed by the degree parameter $\theta_i$. Write $\bar{\theta} = n^{-1} \sum_{i=1}^{n} \theta_i$, $\theta_{\max} = \max_{1 \leq i \leq n} \theta_i$, and $\theta_{\min} = \min_{1 \leq i \leq n} \theta_i$. Define the degree-weighted $\ell^1$-estimation error as

$$
\mathcal{L}(\hat{\Pi}, \Pi) = n^{-1} \sum_{i=1}^{n} \left(\frac{\theta_i}{\bar{\theta}}\right)^{1/2} \|\hat{\pi}_i - \pi_i\|_1.
$$

(1.5)

When $\theta_{\max}/\theta_{\min}$ is bounded, the above loss functions are equivalent in the sense that for a constant $C > 1$,

$$
C^{-1}\mathcal{H}(\hat{\Pi}, \Pi) \leq \mathcal{L}(\hat{\Pi}, \Pi) \leq C\mathcal{H}(\hat{\Pi}, \Pi).
$$

However, when there is severe degree heterogeneity (i.e., $\theta_{\max}/\theta_{\min} \gg 1$), the weighted $\ell^1$-loss is more convenient to use: The minimax rate for $\mathcal{H}(\hat{\Pi}, \Pi)$ depends on all $\theta_i$ in a complicated form, but the minimax rate of $\mathcal{L}(\hat{\Pi}, \Pi)$ is a simple function of $\bar{\theta}$.

**Remark.** The weights in (1.5) are motivated by the study of an oracle case where all true parameters of DCMM are known except for $\pi_i$ of one node $i$. In this case, there exists an oracle estimator $\hat{\pi}_{i0}$ such that

$$
\|\hat{\pi}_{i0} - \pi_i\|_1 = (\theta_i/\bar{\theta})^{-1/2} \cdot O((n\bar{\theta}^2)^{-1/2})
$$

with high probability. It motivates us to re-weight $\|\hat{\pi}_i - \pi_i\|$ by $(\theta_i/\bar{\theta})^{1/2}$. 

1.3. Lower bound. In the asymptotic analysis, we fix \( K \) and \( P \in \mathbb{R}^{K,K} \) and let \((\Theta, \Pi)\) change with \( n \). Our results take the form: For each \( \theta \) in a broad class \( Q_n^*(K,c) \) (to be introduced), we provide a minimax lower bound associated with a class of \( \Pi \).

Given \( \theta \in \mathbb{R}_+^n \), let \( \theta_{(1)} \leq \theta_{(2)} \leq \ldots \leq \theta_{(n)} \) be the sorted values of \( \theta_i \)'s. For a constant \( c \in (0,1/K) \), introduce

\[
Q_n^*(K,c) = \{ \theta \in \mathbb{R}_+^n : \bar{\theta} \geq n^{-1/2} \log(n), \theta_{(cK,n)} \geq n^{-1/2} \log(n) \}
\]

Denote by \( G_n(K) \) the set of all matrices \( \Pi \in \mathbb{R}^{n,K} \) such that each row \( \pi_i \) is a PMF. Given \( \Pi \in G_n(K) \), let

\[
\mathcal{N}_k = \{ 1 \leq i \leq n : \pi_i = e_k \}, \quad \mathcal{M} = \{ 1, 2, \ldots, n \} \setminus (\mathcal{N}_1 \cup \ldots \cup \mathcal{N}_K),
\]

where \( e_1, e_2, \ldots, e_K \) are the standard bases of \( \mathbb{R}^K \). It is seen that \( \mathcal{N}_k \) is the set of pure nodes of community \( k \) and \( \mathcal{M} \) is the set of mixed nodes. Fix \((K,c)\) and an integer \( L_0 \geq 1 \). Introduce

\[
\tilde{G}_n(K,c,L_0;\theta) = \left\{ \Pi \in G_n(K) : \right. \\
|\mathcal{N}_k| \geq cn, \text{ for } 1 \leq k \leq K; \\
\sum_{i \in \mathcal{N}_k} \theta_i^2 \geq c||\theta||^2, \text{ for } 1 \leq k \leq K; \\
\left. \right| \text{there is } L \leq L_0, \text{ a partition } \mathcal{M} = \bigcup_{\ell=1}^L \mathcal{M}_\ell, \text{ PMF's } \gamma_1, \ldots, \gamma_L, \\
\text{where } \min_{j \neq \ell} \| \gamma_j - \gamma_\ell \| \geq c, \min_{1 \leq \ell \leq L, 1 \leq k \leq K} \| \gamma_\ell - e_k \| \geq c, \\
\text{such that } |\mathcal{M}_\ell| \geq c|\mathcal{M}| \geq \frac{\log^2(n)}{\theta^2}, \max_{i \in \mathcal{M}_\ell} \| \pi_i - \gamma_\ell \| \leq \frac{1}{\log(n)} \}. 
\]

**Theorem 1.1** (Lower bound of the weighted \( \ell^1 \)-error). Fix \( K \geq 2, c \in (0,1/K) \), and a nonnegative symmetric matrix \( P \in \mathbb{R}^{K,K} \) that satisfies (1.3). Suppose the DCMM model (1.1)-(1.2) holds. As \( n \to \infty \), there are constants \( C_0 > 0 \) and \( \delta_0 \in (0,1) \) such that, for any \( \theta \in Q_n^*(K,c) \),

\[
\inf \sup_{\Pi \in \tilde{G}_n(K,c,L_0;\theta)} \mathbb{P} \left( \mathcal{L}(\hat{\Pi}, \Pi) \geq \frac{C_0}{\sqrt{n\theta^2}} \right) \geq \delta_0.
\]

When \( \theta_{\max} \leq C\theta_{\min} \), the unweighted and weighted \( \ell^1 \)-errors are equivalent, and we also have a lower bound for the unweighted \( \ell^1 \)-error:

**Corollary 1.1** (Lower bound of the unweighted \( \ell^1 \)-error). Suppose the conditions of Theorem 1.1 hold. As \( n \to \infty \), there are constants \( C_1 > 0 \) and \( \delta_0 \in (0,1) \) such that, for any \( \theta \in Q_n^*(K,c) \) satisfying \( \theta_{\max} \leq C\theta_{\min} \),

\[
\inf \sup_{\Pi \in \tilde{G}_n(K,c,L_0;\theta)} \mathbb{P} \left( \mathcal{H}(\hat{\Pi}, \Pi) \geq \frac{C_1}{\sqrt{n\theta^2}} \right) \geq \delta_0.
\]
Remark. Our results allow for severe degree heterogeneity; for \( \theta \in \mathcal{Q}_n^*(K, c) \), it is possible that \( \theta_{\max}/\theta_{\min} \gg 1 \). In addition, we allow for sparse networks because \( \theta \in \mathcal{Q}_n^*(K, c) \) only requires that the average node degree grows with \( n \) in a logarithmic rate.

Remark. The lower bounds here are different from those on community detection [15, 3]. For community detection, the focus is on the special case where all \( \pi_i \) are degenerate; the goal is clustering, so Hamming distance is the natural choice of loss function, and the rate can be exponentially fast. The setting here is broader and more difficult: it is more natural to use the \( \ell^1 \)-loss, and the rate is only polynomially fast.

1.4. Achievability. Jin et al. [8] proposed a method Mixed-SCORE for estimating \( \pi_i \)'s. The Mixed-SCORE is a fast and easy-to-use spectral approach, and can be viewed as an extension of Jin’s SCORE [5, 4, 9]. However, SCORE is originally designed for community detection, and to extend it to membership estimation, we need several innovations; see [5, 8] for details. It turns out that Mixed-SCORE is also rate-optimal.

The following theorem follows directly form Theorem 1.2 of [8]:

**Theorem 1.2 (Upper bound).** Fix \( K \geq 2 \), \( c \in (0, 1/K) \), and a nonnegative symmetric irreducible matrix \( P \in \mathbb{R}^{K \times K} \) that satisfies (1.3). Suppose the DCMM model (1.1)-(1.2) holds. Let \( \hat{\Pi} \) be the Mixed-SCORE estimator. As \( n \to \infty \), for any \( \theta \in \mathcal{Q}_n^*(K, c) \) with \( \theta_{\max} \leq C\theta_{\min} \) and any \( \Pi \in \tilde{G}_n(K, c, L_0) \), with probability \( 1 - o(n^{-3}) \),

\[
\mathcal{L}(\hat{\Pi}, \Pi) \leq C\mathcal{H}(\hat{\Pi}, \Pi) \leq \frac{C \log(n)}{\sqrt{n\theta^2}}.
\]

In the case that \( \theta_{\max} \leq C\theta_{\min} \), the upper bound and lower bound have matched, and the minimax rate of convergence for both weighted and unweighted \( \ell^1 \)-errors is

\[
(n\theta^2)^{-1/2},
\]
up to a multiple-log\( (n) \) factor.

For more general settings where \( \theta_{\max}/\theta_{\min} \) is unbounded, in a forthcoming manuscript Jin and Ke [7], we demonstrate that

- The minimax rate of convergence for the weighted \( \ell^1 \)-loss \( \mathcal{L}(\hat{\Pi}, \Pi) \) is still \( (n\theta^2)^{-1/2} \), up to a multiple-log\( (n) \) factor.
- The minimax rate of convergence for the unweighted \( \ell^1 \)-loss \( \mathcal{H}(\hat{\Pi}, \Pi) \) depends on individual \( \theta_i \)'s in a more complicated form.
- Mixed-SCORE achieves the minimax rate for a broad range of settings.
At the heart of the upper bound arguments is some new node-wise large deviation bounds we derived; see our forthcoming manuscript [7]. On a high level, the technique is connected to the post-PCA entry-wise bounds in Jin et al. [6] and Ke and Wang [12], but is for very different settings. The main interest of [6] is on gene microarray analysis, where we discuss three interconnected problems: subject clustering, signal recovery, and global testing; see also Jin and Wang [10] on IF-PCA. The main interest of [12] is on topic estimation in text mining.

As far as we know, Jin et al. [6] is the first paper that has carefully studied post-PCA entry-wise bounds. The bounds are crucial for obtaining sharp bounds on the clustering errors by PCA approaches.

2. Proof of Theorem 1.1. We introduce a subset of $\tilde{G}_n(K, c, L_0; \theta)$:

$$G_n^*(K, c; \theta) = \{ \Pi \in G_n(K) : |N_k| \geq cn, \text{ for } 1 \leq k \leq K; \sum_{i \in N_k} \theta_i^2 \geq c\|\theta\|^2, \text{ for } 1 \leq k \leq K; \|\pi_i - (1/K)1_K\| \leq 1/\log(n), \text{ for } i \in \mathcal{M} \}.$$

Since $G_n^*(K, c; \theta) \subset \tilde{G}_n(K, c, L_0; \theta)$, for any estimator $\hat{\Pi}$,

$$\sup_{\Pi \in \tilde{G}_n(K, c, L_0; \theta)} \mathbb{P}(L(\hat{\Pi}, \Pi) \geq C_0 s_n) \geq \sup_{\Pi \in G_n^*(K, c; \theta)} \mathbb{P}(L(\hat{\Pi}, \Pi) \geq C_0 s_n).$$

Hence, it suffices to prove the lower bound for $\Pi \in G_n^*(K, c; \theta)$.

We need the following lemma, which is adapted from Theorem 2.5 of [14]. We recall that $G_n(K)$ is the set of all matrices $\Pi \in \mathbb{R}^{n,K}$ each row of which is a PMF in $\mathbb{R}^K$.

**Lemma 2.1.** For any subset $G_n^* \subset G_n(K)$, if there exist $\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(J)} \in G_n^*$ such that:

(i) $\mathcal{L}(\Pi^{(j)}, \Pi^{(k)}) \geq 2C_0 s_n$ for all $0 \leq j \neq k \leq J$,

(ii) $\frac{1}{J+1} \sum_{j=0}^J KL(P_j, P_0) \leq \beta \log(J),$

where $C_0 > 0$, $\beta \in (0, 1/8)$, $P_j$ denotes the probability measure associated with $\Pi^{(j)}$, and $KL(\cdot, \cdot)$ is the Kullback-Leibler divergence, then

$$\inf_{\Pi} \sup_{\Pi \in G_n^*} \mathbb{P}(\mathcal{L}(\hat{\Pi}, \Pi) \geq C_0 s_n) \geq \frac{\sqrt{J}}{1+\sqrt{J}} \left(1 - 2\beta - \sqrt{\frac{2\beta}{\log(J)}}\right).$$

As long as $J \to \infty$ as $n \to \infty$, the right hand side is lower bounded by a constant.

By Lemma 2.1, it suffices to find $\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(J)} \in G_n^*(K, c)$ that satisfy the requirement of Lemma 2.1. Below, we first consider the case $K = 2$ and then generalize the proofs to $K \geq 3$. 
2.1. The case $K = 2$. We re-parametrize the model by defining $a \in (0, 1]$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in [-1, 1]^n$ through

\[(2.8) \quad P = \begin{bmatrix} 1 & 1 - a \\ 1 - a & 1 \end{bmatrix}, \quad \pi_i = \left( \frac{1 + \gamma_i}{2}, \frac{1 - \gamma_i}{2} \right), \quad 1 \leq i \leq n.\]

Since there is a one-to-one mapping between $\Pi$ and $\gamma$, we instead construct $\gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(n)}$. Without loss of generality, we assume $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$. Let $n_1 = \lfloor cn \rfloor$ and $n_0 = n - 2n_1$. Introduce

\[\gamma^* = \left( 0, 0, \ldots, 0, 1, 1, \ldots, 1, -1, -1, \ldots, -1 \right)^t.\]

Note that $\gamma_i^* \in \{ \pm 1 \}$ implies that node $i$ is a pure node and $\gamma_i^* = 0$ indicates that $\pi_i^* = (1/2, 1/2)$. From the Varshamov-Gilbert bound for packing numbers [14, Lemma 2.9], there exist $J_0 \geq 2^{n_0/8}$ and $\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(J_0)} \in \{0, 1\}^{n_0}$ such that $\omega^{(j)} = (0, 0, \ldots, 0)'$ and $\|\omega^{(j)} - \omega^{(\ell)}\|_1 \geq n_0/8$, for all $0 \leq j \neq \ell \leq J_0$. Let $J = 2J_0$, $\omega^{(0)} = \omega^{(v)}$, and $\omega^{(2\ell \pm 1)} = \pm \omega^{(\ell)}$ for $1 \leq \ell \leq J_0$. Then, the resulting $\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(J)}$ satisfy that:

(a) min$_{0 \leq j \neq \ell \leq J} \|\omega^{(j)} - \omega^{(\ell)}\|_1 \geq n_0/8.$

(b) For any real sequence $\{h_i\}_{i=1}^{n_1}$, $\sum_{\ell=0}^J \sum_{i=1}^{n_0} h^{(\ell)} = 0$.

For a properly small constant $c_0 > 0$ to be determined, letting $\delta_n = c_0 (n\tilde{\theta})^{-1/2}$, we construct $\gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(J)}$ by

\[(2.9) \quad \gamma^{(\ell)} = \gamma^* + \delta_n \left( v \circ \omega^{(\ell)}, 0, 0, \ldots, 0 \right), \quad \text{with} \quad v = \left( \frac{1}{\sqrt{\theta_1}}, \frac{1}{\sqrt{\theta_2}}, \ldots, \frac{1}{\sqrt{\theta_{n_0}}} \right).\]

We then use the one-to-one mapping (2.8) to obtain $\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(J)}$. To check that each $\Pi^{(\ell)}$ belongs to $\mathcal{G}_n^*(K, c; \theta)$, we notice that $\|\pi^{(\ell)} - (1/2, 1/2)\| = O(\theta^{1/2}) = O(\theta^{1/2})$ for $1 \leq i \leq n_0$; $\theta^{1/2}$ is the $(2cn)$-smallest value of $\theta_1, \ldots, \theta_n$ and it satisfies that $\theta^{1/2} \geq n^{1/2} \log(n)$; additionally, $\theta \geq n^{1/2} \log(n)$; it follows that $\|\pi^{(\ell)} - (1/2, 1/2)\| = O(c_0 / \log(n))$; hence, $\Pi^{(\ell)} \in \mathcal{G}_n^*(K, c; \theta)$ as long as $c_0$ is appropriately small.

What remains is to show that the requirements (i)-(ii) in Lemma 2.1 are satisfied for $s_n = (n\tilde{\theta})^{-1/2}$. Consider (i). Note that for any $0 \leq j \neq \ell \leq J$,

\[\mathcal{L}(\Pi^{(j)}, \Pi^{(\ell)}) = \min \left\{ \frac{1}{n\sqrt{\theta}} \sum_{i=1}^{n} \sqrt{\theta_i} |\gamma^{(j)}_i \pm \gamma^{(\ell)}_i| \right\}.\]
For “−”, the term in the brackets is at most $\delta_n \sum_{i=1}^{n_0} \theta_i^{-1/2} \leq n_0 \delta_n \theta_{n_0}^{-1/2} = o(n)$; for “+”, this term is at least $4n_1 \geq 4en$. Therefore, the minimum is achieved at “−”. Furthermore, we have

$$L(\Pi^{(j)}, \Pi^{(\ell)}) = \frac{1}{n\sqrt{\theta}} \sum_{i=1}^{n} \sqrt{\theta_i} |\gamma_i^{(j)} - \gamma_i^{(\ell)}| = \frac{\delta_n}{n\sqrt{\theta}} \|\omega^{(j)} - \omega^{(\ell)}\|_1 \geq \frac{n_0 \delta_n}{8n\sqrt{\theta}},$$

where the last inequality is due to Property (a) of $\omega^{(0)}, \ldots, \omega^{(J)}$. Since $\delta_n = c_0(n\theta)^{-1/2}$ and $n_0 \geq (1 - cK)n$, the right hand side is lower bounded by $(c_0 c_0/8) \cdot (n \theta^2)^{-1/2}$. This proves (i).

We now prove (ii). Note that $KL(P_{\ell}, P_0) = \sum_{1 \leq i < j \leq n} \Omega_{ij}^{(\ell)} \log(\Omega_{ij}^{(\ell)}/\Omega_{ij}^{(0)})$. Additionally, the parametrization (2.8) yields that

$$\Omega_{ij} = \theta_i \theta_j \left[(1 - a/2) + (a/2) \gamma_i \gamma_j\right], \quad 1 \leq i \neq j \leq n.$$ 

Since $\gamma_i^{(0)} = \gamma_i^{(\ell)}$ for all $i > n_0$, if both $i, j > n_0$, then $\Omega_{ij}^{(\ell)} = \Omega_{ij}^{(0)}$ and the pair $(i, j)$ has no contribution to $KL(P_{\ell}, P)$. Therefore, we can write

$$\frac{1}{J + 1} \sum_{\ell=0}^{J+1} KL(P_{\ell}, P_0) = \frac{1}{J + 1} \sum_{\ell=0}^{J+1} \left( \sum_{1 \leq i < j \leq n_0} + \sum_{1 \leq i \leq n_0, n_0 < j \leq n} \right) \Omega_{ij}^{(\ell)} \log(\Omega_{ij}^{(\ell)}/\Omega_{ij}^{(0)})$$

$$\equiv (I) + (II).$$

First, consider (I). From (2.9) and (2.10), for all $1 \leq i < j \leq n_0$, we have $\Omega_{ij}^{(0)} = \theta_i \theta_j (1 - a/2)$ and

$$\Omega_{ij}^{(\ell)} = \Omega_{ij}^{(0)} (1 + \Delta_{ij}^{(\ell)}), \quad \text{where} \quad \Delta_{ij}^{(\ell)} = \frac{a}{2 - a} \frac{\delta_n^2}{\sqrt{\theta_i \theta_j} \omega_i^{(\ell)} \omega_j^{(\ell)}}.$$

Write $\Delta_{\text{max}} = \max_{1 \leq i < j \leq n_0, 1 \leq \ell \leq J} |\Delta_{ij}^{(\ell)}|$. Since $\theta_1 \geq \ldots \geq \theta_{n_0} \gg n^{-1/2}$, we have $\Delta_{\text{max}} = O(n^{1/2} \delta_n^2) = O((n \theta^2)^{-1/2}) = o(1)$. By Taylor expansion, 

$$(1 + t) \ln(1 + t) = t + O(t^2) \leq 2|t| \quad \text{for} \ t \text{ sufficiently small.}$$

Combining the above gives

$$\Omega_{ij}^{(\ell)} \log(\Omega_{ij}^{(\ell)}/\Omega_{ij}^{(0)}) = \Omega_{ij}^{(0)} (1 + \Delta_{ij}^{(\ell)}) \ln(1 + \Delta_{ij}^{(\ell)})$$

$$\leq 2 \Omega_{ij}^{(0)} |\Delta_{ij}^{(\ell)}|$$

$$\leq a \delta_n^2 \sqrt{\theta_i \theta_j} \cdot |\omega_i^{(\ell)} \omega_j^{(\ell)}|$$
\[ \leq a\delta_n^2 \sqrt{\theta_i \theta_j}, \]

where the third line is due to (2.12) and the expression \( \Omega_{ij}^{(0)} \). It follows that

(2.13) \[ (I) \leq a\delta_n^2 \sum_{1 \leq i < j \leq n_0} \sqrt{\theta_i \theta_j} \leq a\delta_n^2 \left( \sum_{1 \leq i \leq n_0} \sqrt{\theta_i} \right)^2. \]

Next, consider (II). For \( i \leq n_0 \) and \( j > n_0 \), \( \Omega_{ij}^{(0)} = \theta_i \theta_j (1 - a/2) \) and

(2.14) \[ \Omega_{ij}^{(\ell)} = \Omega_{ij}^{(0)} (1 + \tilde{\Delta}_{ij}^{(\ell)}), \quad \text{with} \quad \tilde{\Delta}_{ij}^{(\ell)} = \gamma_j^{(\ell)} \cdot \frac{a}{2 - a \sqrt{\theta_i}} \cdot \omega_i^{(\ell)} \quad \text{and} \quad \gamma_j^{(\ell)} \in \{ \pm 1 \}. \]

Write \( \tilde{\Delta}_{\text{max}} = \max_{1 \leq i \leq n_0, n_0 < j \leq n, 1 \leq \ell \leq J} |\tilde{\Delta}_{ij}^{(\ell)}| \). Similar to the bound for \( \Delta_{\text{max}} \), we have \( \tilde{\Delta}_{\text{max}} = O\left(n^{1/4} \delta_n\right) = O\left((n \theta)^{-1/4}\right) = o(1) \). Also, by Taylor expansion, \((1 + t) \ln(1 + t) = t + t^2 / 2 + O(|t|^3) \leq t + t^2 \) for \( t \) sufficiently small.

Combining the above gives

(2.15) \[ \Omega_{ij}^{(\ell)} \log(\Omega_{ij}^{(\ell)} / \Omega_{ij}^{(0)}) = \Omega_{ij}^{(0)} (1 + \Delta_{ij}^{(\ell)}) \ln(1 + \Delta_{ij}^{(\ell)}) \]

\[ \leq \Omega_{ij}^{(0)} \tilde{\Delta}_{ij}^{(\ell)} + \Omega_{ij}^{(0)} (\tilde{\Delta}_{ij}^{(\ell)})^2. \]

Motivated by (2.15), we first bound

(II1) \[ \equiv \frac{1}{J + 1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} \Omega_{ij}^{(0)} \tilde{\Delta}_{ij}^{(\ell)} \]

\[ = \frac{1}{J + 1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} a\delta_n \cdot \theta_j \gamma_j^{(\ell)} \cdot \sqrt{\theta_i \omega_i^{(\ell)}} \]

\[ = \frac{a\delta_n}{2(J + 1)} \left( \sum_{j=n_0+1}^{n} \theta_j \gamma_j^{(\ell)} \right) \cdot \sum_{\ell=0}^{J} \sum_{i=1}^{n} \sqrt{\theta_i \omega_i^{(\ell)}} \]

\[ = 0, \]

where we have used Property (b) of \( \omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(J)} \). We then bound

(II2) \[ \equiv \frac{1}{J + 1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} \Omega_{ij}^{(0)} (\tilde{\Delta}_{ij}^{(\ell)})^2 \]

\[ = \frac{1}{J + 1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} \frac{a^2 \delta_n^2}{4 - 2a} \cdot \theta_j \cdot |\omega_i^{(\ell)}| \]
Let \( \eta \) be a nonzero vector such that \( \theta = 1 \) for each \( i \). Introduce \( \pi^*_i = \frac{1}{n_0} \), \( \pi^*_1, \ldots, \pi^*_n \) as the standard basis vectors of \( \mathbb{R}^K \). Introduce

\[
\Pi^* = \left( \frac{1}{n_0} \mathbf{1}_K, \ldots, \frac{1}{n_0} \mathbf{1}_K, e_1, \ldots, e_1, \ldots, e_K, \ldots, e_K \right)'.
\]

Let \( \omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(J)} \in \{0, 1\}^{n_0} \) be the same as above. Let \( \delta_n = c_0(n\bar{\theta})^{-1/2} \) be a constant to be determined. Write \( \Pi^{(\ell)} = [\pi^{(\ell)}_1, \ldots, \pi^{(\ell)}_n]' \). For each \( 0 \leq \ell \leq J \), we construct \( \pi^{(\ell)}_i = [\pi^{(\ell)}_i, \ldots, \pi^{(\ell)}_n]' \) by

\[
\pi^{(\ell)}_i = \pi^*_i + \begin{cases} 
\omega^{(\ell)} \cdot (\delta_n/\sqrt{\theta_i}) \cdot \eta, & 1 \leq i \leq n_0 \\
0, & n_0 + 1 \leq i \leq n.
\end{cases}
\]
Same as before, we show the requirements (i)-(ii) in Lemma 2.1 are satisfied. Note that

\[ \mathcal{L}(\Pi^{(j)}, \Pi^{(\ell)}) = \frac{\delta_n \|\eta\|_1}{n\sqrt{\theta}} \|\omega^{(j)} - \omega^{(\ell)}\|_1 \geq \frac{n_0 \delta_n \|\eta\|_1}{8n\sqrt{\theta}} = (c_0 \|\eta\|_1 / 8) \cdot (n\theta^2)^{-1/2}. \]

Hence, (i) holds for \( s_n = (n\theta^2)^{-1/2} \).

It remains is to prove (ii). Let (I) and (II) be defined in the same way as in (2.11). We aim to find expressions similar to those in (2.12) and (2.14) and then generalize the bounds of (I) and (II) for \( K = 2 \) to a general \( K \). For preparation, we first derive an expression for \( \Omega_{ij} \). Introduce \( \tilde{\pi}_i = \pi_i - \frac{1}{K} \mathbf{1}_K \in \mathbb{R}^K \). Note that \( \pi_i^0 \mathbf{1}_K = 1 \). By direct calculations,

\[ \Omega_{ij} = \theta_i \theta_j (1 - \tilde{\pi}_i^0 \tilde{\pi}_j) = \theta_i \theta_j (1 - \tilde{\pi}_i \tilde{\pi}_j) = \theta_i \theta_j (\pi_i - \frac{1}{K} \mathbf{1}_K) - \theta_i \theta_j (\frac{1}{K} \mathbf{1}_K) \]

(2.18)  

Consider (I). Since \( \tilde{\pi}_i^{(0)} \) is a zero vector for all \( 1 \leq i \leq n_0 \), we have

\[ \Omega_{ij}^{(0)} = \theta_i \theta_j (1 - \tilde{\bar{a}}), \quad \text{with} \quad \tilde{\bar{a}} = \frac{1}{K} \mathbf{1}_K \tilde{\pi}_1 \],  

if \( 1 \leq i \neq j \leq n_0 \).

Furthermore, for \( 1 \leq i \leq n_0 \), \( \tilde{\pi}_i^{(\ell)} \propto \eta \), where it follows from (2.17) that \( \eta \tilde{\pi}_1 \mathbf{1}_K = 0 \). Hence, the middle two terms in (2.18) are zero. As a result,

\[ \Omega_{ij}^{(\ell)} = \theta_i \theta_j (1 - \bar{a}) - \frac{\delta_n^2}{\sqrt{\theta_i \theta_j}} \cdot \eta \tilde{\pi}_i \omega_i^{(\ell)} \omega_j^{(\ell)}, \quad 1 \leq i \neq j \leq n_0. \]

Combining the above, we find that for all \( 1 \leq i \neq j \leq n_0 \),

(2.19)  

\[ \Omega_{ij}^{(\ell)} = \Omega_{ij}^{(0)} (1 + \Delta_{ij}^{(\ell)}), \quad \text{where} \quad \Delta_{ij}^{(\ell)} = \frac{-(\eta \tilde{\pi}_i \mathbf{1}_K) \bar{a}}{1 - \bar{a}}. \]

This provides a counterpart of (2.12) for a general \( K \). Same as before, we have the bound: \( \Delta_{\max} \equiv \max_{1 \leq i \neq j \leq n_0} \max_{0 \leq \ell \leq J} |\Delta_{ij}^{(\ell)}| = O(n^{1/2} \delta_n) = o(1) \). Following the proof of (2.13), we find that

(2.20)  

\[ (I) \leq C_1 \delta_n^2 \left( \sum_{i=1}^{n_0} \sqrt{\theta_i} \right)^2 \leq C_1 n_0 \delta_n^2 \left( \sum_{i=1}^{n_0} \theta_i \right), \quad \text{where} \quad C_1 = |\eta \tilde{\pi}_1 \mathbf{1}_K| \cdot |\bar{a}|. \]

Consider (II). In this case, we need to calculate \( \Omega_{ij} \), \( 1 \leq i \leq n_0, n_0 < j \leq n. \) Recall that \( \tilde{\pi}_i^{(0)} \) is a zero vector. Write \( \{n_0 + 1, n_0 + 2, \ldots, n\} = \cup_{k=1}^K N_k \),
where \( \mathcal{N}_k = \{1 \leq j \leq n : \pi_j(0) = e_k\} \). For \( j \in \mathcal{N}_k \), it holds that \( \hat{\pi}_j(0) + \frac{1}{K} \mathbf{1}_K = e_k \). Combining the above with (2.18), we find that for \( 1 \leq k \leq K \),

\[
\Omega^{(0)}_{ij} = \theta_i \theta_j (1 - \tilde{b}_k), \quad \text{with} \quad \tilde{b}_k \equiv \frac{1}{K} e'_k \hat{P} \mathbf{1}_K, \quad \text{if } 1 \leq i \leq n_0, j \in \mathcal{N}_k.
\]

Additionally, we have \( (\hat{\pi}^{(\ell)}_i)' \hat{P} \mathbf{1}_K \propto \eta' \hat{P} \mathbf{1}_K = 0 \) and \( \hat{\pi}_{ij}(0) + \frac{1}{K} \mathbf{1}_K = e_k \). It follows from (2.18) that

\[
\Omega^{(\ell)}_{ij} = \theta_i \theta_j (1 - \tilde{b}_k) - \theta_i \theta_j (\hat{\pi}^{(\ell)}_i)' \hat{P} \hat{\pi}^{(\ell)}_j
\]

\[
= \theta_i \theta_j (1 - b_k) - \theta_i \theta_j \cdot \frac{\omega_i^{(\ell)} \delta_n}{\sqrt{\theta_i}} \cdot \eta' \hat{P} (e_k - \frac{1}{K} \mathbf{1}_K)
\]

\[
= \theta_i \theta_j (1 - b_k) - \theta_i \theta_j \cdot \frac{\omega_i^{(\ell)} \delta_n}{\sqrt{\theta_i}} \cdot (\eta' \hat{P} e_k),
\]

where the last equality is because of \( \eta' \hat{P} \mathbf{1}_K = 0 \). As a result, for \( 1 \leq i \leq n_0 \) and \( j \in \mathcal{N}_k \),

\[
(2.21) \quad \Omega^{(\ell)}_{ij} = \Omega^{(0)}_{ij} (1 + \tilde{\Delta}^{(\ell)}_{ij}), \quad \text{where} \quad \tilde{\Delta}^{(\ell)}_{ij} = -\left(\eta' \hat{P} e_k\right) \cdot \frac{1}{1 - b_k \sqrt{\theta_i}} \omega_i^{(\ell)}.
\]

This provides a counterpart of (2.14) for a general \( K \). Same as before, let \( \tilde{\Delta}_{\max} \equiv \max_{1 \leq i \neq j \leq n_0} \max_{0 \leq \ell \leq J} |\tilde{\Delta}^{(\ell)}_{ij}| \), and it is seen that \( \tilde{\Delta}_{\max} = O(n^{1/4} \delta_n) = o(1) \). Again, by Taylor expansion, we have (2.15). It follows that

\[
(II) \leq (II_1) + (II_2),
\]

where \((II_1)\) and \((II_2)\) are defined the same as before. Using (2.21), we have

\[
(II_1) = \frac{1}{J+1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j \in \mathcal{N}_k} \Omega^{(0)}_{ij} \tilde{\Delta}^{(\ell)}_{ij}
\]

\[
= \frac{\delta_n}{J+1} \left( \sum_{k=1}^{K} \sum_{j \in \mathcal{N}_k} -\left(\eta' \hat{P} e_k\right) \theta_j \right) \cdot \sum_{\ell=0}^{J} \sum_{i=1}^{n} \sqrt{\theta_i} \omega_i^{(\ell)}
\]

\[
= 0,
\]

where the last equality is due to Property (b) for \( \omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(J)} \). Using (2.21) again, we have

\[
(II_2) = \frac{1}{J+1} \sum_{\ell=0}^{J} \sum_{i=1}^{n_0} \sum_{j \in \mathcal{N}_k} \Omega^{(0)}_{ij} \tilde{\Delta}^{(\ell)}_{ij}^2
\]
\[
\frac{\delta_n^2}{1 - \hat{b}} \cdot \left( \sum_{1 \leq k \leq K} (\eta_i' \hat{P}_e)_k^2 \sum_{j \in N_k} \theta_j \right) \cdot \max_{0 \leq \ell \leq J} \| \omega^{(\ell)} \|_1 \\
\leq \frac{\delta_n^2}{1 - \hat{b}} \cdot \max_{1 \leq k \leq K} (\eta_i' \hat{P}_e)_k^2 \cdot \left( \sum_{j = n_0 + 1}^n \theta_j \right) \cdot n_0.
\]

Combining the above gives

\[(2.22) \quad (II) \leq C_2 n_0 \delta_n^2 \left( \sum_{i = n_0 + 1}^n \theta_i \right), \quad \text{where} \quad C_2 = \frac{1}{1 - \hat{b}} \max_{1 \leq k \leq K} (\eta_i' \hat{P}_e)_k^2.
\]

We note that (2.20) and (2.22) server as the counterpart of (2.14) and (2.16), respectively. Similarly as in the case of \( K = 2 \), we obtain (ii) immediately. The proof for a general \( K \) is now complete. \( \square \)

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