Errett Bishop theorems on Complex Analytic Sets: 
Chow’s Theorem Revisited and Foliations of 
Compact Leaves on Kähler Manifolds

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Abstract
In this paper, we present a series of seemingly unrelated results of Complex Analysis which are, in fact, connected via a different approach to their proofs using the results of Errett Bishop of volumes, extensions, and limits of analytic varieties. We start with a brief introduction to the tools developed by Bishop and show their usefulness by proving Chow’s theorem via a technique suggested a long time ago in a beautiful book by Gabriel Stolzenberg, then we show some of the relationships between the theory of analytic subsets and classical results of complex-analytic functions. We finish with the original contributions of this paper which consists of applications of these tools to the theory of holomorphic foliations with alternative and, we believe, simpler proofs to Edwards, Millet, and Sullivan’s impactful result for foliations with compact leaves in the case of complex foliations in Kähler manifolds and J. V. Pereira’s global stability result for holomorphic foliations on compact Kähler manifolds.

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1 Introduction

The purpose of this paper is first to survey Errett Bishop’s results on the limit and the continuation of analytic sets with bounded volume. We list these statements here as theorems 5.2, 5.1, and 6.2. These theorems were originally published in 1964 in an article titled Conditions for the Analyticity of certain sets (see [1]). The presentation of the results we show here are expository and inspired by the 1966 Gabriel Stolzenberg’s book Volumes, Limits, and Extensions of Analytic Varieties (see [10]) which also was the catalyst for the exploration of Chow’s theorem and stability of holomorphic foliations on Kähler manifolds via Bishop’s results.

In 1949, Wei-Liang Chow published the paper “On Compact Complex Analytic Varieties”, which contains the proof of a long-standing conjectured result that gives a deep link between analytic geometry and projective algebraic geometry. The result, now known as Chow’s theorem, states that every closed analytic subvariety of the complex projective space $\mathbb{CP}^n$ is algebraic. The history regarding this result is related to the development of important results of analytic sets of the 20th century. The original proof of Chow was done via analytic simplexes (an analytic simplex is a topological simplex in $\mathbb{CP}^n$ that is biholomorphic with the standard simplex in some complex affine space). Later in 1953 Reinhold Remmert and Karl Stein proved that the closure of a purely $k$-dimensional analytic set is also an analytic set under conditions which, in particular, imply that the complex cone of a projective subvariety is also an analytic set. Jean-Pierre Serre in his famous 1956 paper Géométrie Algébrique et Géométrie Analytique colloquially known as GAGA proves the same statement with a totally different set of tools.

The proof we present here does not reference directly the algebraic properties of the ring of germs of analytic functions nor do we use properties of quasi-coherent sheaves; instead, we are concerned with local properties of varieties and the Hausdorff measure of said variety respectively. We think this approach is very attractive and easier to understand for people who are not very much familiarized with the modern categorical language of algebraic geometry. Furthermore, the theory presented here is linked to areas of mathematics that are not usually
associated with Chow’s result. In addition, Bishop’s results imply both Chow’s and Remmert-
Stein’s theorems directly, meaning that this view is just as efficient and profound as Remmert-
Stein’s proof.

We present more results that link the theory of complex algebraic sets and some well-
known theorems in complex analysis (1). Furthermore, a careful study of Bishop’s results
pointed us to apply them to foliations with compact leaves on Kähler manifolds in a manner
similar to Edwards, Millet, and Sullivan in [3] and J. V. Pereira’s result in [8].

2 Preliminaries

We begin with some basic definitions and terminology which can be found, for instance, in
[2].

Definition 1. Let $M$ be a complex manifold. A subset $A \subset M$ is called a complex analytic
subset of $M$, if for each $p \in M$ there exists an open neighborhood $U \subset M$ of $p$ and finitely
many holomorphic functions $f_1, \ldots, f_k : U \to \mathbb{C}$ such that

$$A \cap U = \{ z \in U | f_1(z) = \cdots = f_k(z) = 0 \}.$$ 

Definition 2. A subset $A \subset M$ is called complex analytic set, if for each $p \in A$ there exists
an open neighborhood $U$ of $p$ and finitely many holomorphic functions $f_1, \ldots, f_k : U \to \mathbb{C}$
such that $A \cap U = \{ z \in U : f_1(z) = \cdots = f_k(z) = 0 \}$

Remark 1. The subtle difference in the definitions ($p \in M$ vs $p \in A$) is that a complex
analytic subset of a complex manifold is a closed subset of the manifold. A complex analytic
set is locally closed but not necessarily closed. For example, a non-empty open subset of
$\mathbb{C}^n$ is an analytic manifold but it is not closed of $\mathbb{C}^n$.

We denote the sheaf of holomorphic functions on $\mathbb{C}^n$ by $\mathcal{H}_n$ and for $U \subset \mathbb{C}^n$ an open
subset, the ring of holomorphic functions on $U$ will be denoted by $\mathcal{H}_n(U)$ or simply $\mathcal{H}(U)$,
thus $\mathcal{H}_n := \mathcal{H}(\mathbb{C}^n)$. Given an open subset $U$ of $\mathbb{C}^n$ and a finite subset $\{f_1, \ldots, f_k\} \subset \mathcal{H}(U)$
we will denote the locus consisting of points where all of these functions vanish by

$$V_U(f_1, \ldots, f_k) := \{ z \in U | f_1(z) = \cdots = f_k(z) = 0 \}$$ vanishing locus of $f_i$, $i = 1, \ldots, n$

In the case of $\mathbb{C}^n$, an **analytic subset** of $\mathbb{C}^n$ is a locally ringed subset $X \subset \mathbb{C}^n$, meaning
that for every $x \in X$ there is a neighborhood $U$ of $x$ and a finite subset $\{f_1, \ldots, f_k\} \subset \mathcal{H}(U)$
with $X \cap U = V_U(f_1, \ldots, f_k)$, together with the local ring

$$\mathcal{H}_X := \mathcal{H}_n/I(X),$$
where $I(X) := \{ f \in \mathcal{H}_n | f|_{\partial U} = 0 \}$.

Definition 3. An **analytic space** $(X, \mathcal{H}_X)$ is a topological Hausdorff space $X$ together with
a local ring structure $\mathcal{H}_X$ that is locally isomorphic to an analytic subset of $\mathbb{C}^n$. We call a
neighborhood with its local isomorphism a **chart**.

An **analytic subspace** of an analytic space $(X, \mathcal{H}_X)$ consists of a subset $Y \subset X$ such that
for every $y \in Y$ there is a chart $(U, \varphi)$ around $y$ with

$$\varphi(Y \cap U) = V_U(\varphi(\eta_1), \ldots, \varphi(\eta_k)).$$
and \( \{\eta_1, \ldots, \eta_k\} \subset H_X \), naturally we have a local ring structure \((Y, H_Y)\) given by \( H_Y := H_n/I(Y) \), as before.

**Definition 4.** A subset \( N \subset M \) of the complex \( n \)-dimensional manifold \( M \) is called a complex submanifold of \( M \) of dimension \( d \) \((0 \leq d \leq n)\), if it’s closed and for each \( p \in N \) there is an open neighborhood \( U \subset M \) of \( p \) and a biholomorphic map \( \varphi : U \to V \subset \mathbb{C}^n \), where \( V \) is an open subset (so that \( \varphi \) is a holomorphic chart), so that \( \varphi(N \cap U) = \varphi(U) \cap \mathbb{C}^d \).

**Remark 2.** It follows from the holomorphic version of the Implicit Function Theorem that a complex submanifold of a complex manifold \( M \) is an analytic subset of \( M \).

**Definition 5.** Let \( M \) be a complex manifold. An immersed complex submanifold of \( M \) is a subset \( N \) endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold and a complex structure with respect to which the inclusion map \( N \hookrightarrow M \) is a holomorphic immersion. In other words: \( N \) is the image of a holomorphic injective immersion of a complex manifold. Since embeddings with closed images are the same thing as proper injective immersions, the image of a complex manifold under a proper injective immersion is a complex submanifold.

We denote complex projective space of dimension \( n \) by \( \mathbb{CP}^n \); this is a complex manifold, and therefore an analytic space with its usual affine charts. Algebraic subsets on \( \mathbb{C}^{n+1} \) consist of the zeroes of a finite number of polynomials in \( \mathbb{C}[z_0, \ldots, z_n] \). Note that using projective coordinates, the set of vanishing points of homogeneous polynomials are well defined on \( \mathbb{CP}^n \), and therefore algebraic subsets of \( \mathbb{CP}^n \) are well defined. Analytic and algebraic subsets of \( \mathbb{CP}^n \) are going to be called projective analytic subsets and projective algebraic subsets respectively. The following theorem states the equality of the notions of projective analytic subsets and projective algebraic subsets:

**Theorem 2.1 (Chow).** Every closed projective analytic subset is a projective algebraic subset.

In order to prove this statement, we proceed with some properties of volumes of purely \( k \)-dimensional analytic subvarieties and their Hausdorff measures, then we proceed to enunciate one of the key theorems for this proof, a result that we call Bishop’s proper mapping theorem.

### 3 Volumes of analytic sets and Wirtinger’s Inequality

Volumes and metrics have a natural relationship that we are going to exploit throughout this article. In the context of complex analytic spaces and manifolds, Kähler geometry is special for its interplay between the metric and the associated volumes, both for the whole manifold and for its submanifolds. Related to this is the result due to Wirtinger that, among other things, implies that Kähler submanifolds of a Kähler manifolds minimize the volumes of their respective homological class, this will be of crucial importance when dealing with the volume function of the leaves on a compact Kähler manifold in theorem 9.1.

**Definition 6.** Let \( M \) be a complex manifold with an integrable almost-complex structure \( J : TM \to TM \). A Hermitian metric on \( M \) is a smooth family real bilinear forms \( \{h_p\}_p \) for all \( p \in M \) where \( h_p : T_pM \times T_pM \to \mathbb{C} \), such that

- \( h_p(JX, JY) = h_p(X, Y) \) \( \forall \{X, Y\} \subset T_pM \) \( \forall p \in M \).
- \( h_p(X, JX) > 0 \) \( \forall X \in T_pM \setminus \{0\} \) \( \forall p \in M \).
- \( h_p(X, Y) = \overline{h_p(Y, X)} \) \( \forall \{X, Y\} \subset T_pM \) \( \forall p \in M \).
We will also denote $h(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. We call a manifold with a Hermitian metric an Hermitian manifold.

We note here that a Hermitian metric $h$ has an associated Riemannian metric and an associated 2-form given by the real and imaginary part of $h$ and vice versa. A Riemannian manifold of even dimension $(M, g)$ with a complex structure $J$ has a natural Hermitian metric given by

$$h(X, Y) := g(X, Y) - i\omega(X, Y).$$

Where $\omega$ is its associated 2-form given by $\omega(u, v) = g(Ju, v)$ which is clearly antisymmetric since $J^2 = -\text{Id}_M$.

**Definition 7.** Let $(M, h)$ be a Hermitian manifold with Hermitian metric $h$ and associated 2-form $\omega$, $h$ is a Kähler metric if $d\omega = 0$. We call a manifold with a Kähler metric a Kähler manifold. The associated 2-form will be called a Kähler form or a Kähler symplectic form.

Being a Kähler manifold has some strong topological restrictions. For example, powers of the Kähler form $\omega^k$ are nontrivial representatives of cohomology classes in $H^{2k}(M; \mathbb{R})$, meaning that these groups are never trivial. More important for our purposes is the fact that, by our definition of submanifold, it follows that every complex submanifold (or immersed complex submanifold) of a Kähler manifold is also Kähler since the closedness of the 2-form follows from the compatibility of $J$ and the derivative under the pullback $f^*$. Other consequences are the following:

**Theorem 3.1** (Wirtinger’s inequality). Let $V$ be a real vector space of even dimension $2k$ endowed with a positive-definite inner product $g$, symplectic form $\omega$, and an almost-complex structure $J$ linked, as before by $\omega(u, v) = g(Ju, v), \forall u, v \in V$. Then for any orthonormal vectors $u_1, \ldots, u_{2k}$ the following inequality holds:

$$\left(\omega \wedge \cdots \wedge \omega\right)^k(u_1, \ldots, u_{2k}) \leq k!.\$$

There is equality if and only if the span of $u_1, \ldots, u_{2k}$ is closed under the action of $J$.

**Corollary 1** (Complex submanifolds of Kähler manifolds minimize volume). Let $M$ be a Kähler manifold with Kähler form $\omega_M$ and let $f : N \to M$ be a closed and oriented immersion of an oriented real manifold of real dimension $2k$. Let $\omega = f^* \omega_M$, then

$$\int_N \frac{\omega^k}{k!} \leq \int_N d\text{Vol}_N \quad \text{where } d\text{Vol}_N \text{ is the volume form of } N,$$

and the equality holds if and only if $N$ is a complex submanifold of $M$.

**Theorem 3.2.** Any complex submanifold of a Kähler manifold is a minimal submanifold.

**Corollary 2.** Let $N$ be a complex compact submanifold with boundary of a Kähler manifold $M$, then $N$ is a volume-minimizing submanifold in its homology class $H_{2k}(M, \partial N, \mathbb{Z})$, meaning that for any real submanifold $X$ of real dimension $2k$ and common boundary $\partial N$ that is homologous to $N$, has the following inequality

$$\text{Vol}_{2k}(N) \leq \text{Vol}_{2k}(X).$$

Let $\omega = -\text{Im}<\cdot, \cdot>$ be the standard 2-form of the standard Euclidean Kähler metric in $\mathbb{C}^n$ and let $M$ be a complex submanifold of $\mathbb{C}^n$ by Wirtinger’s inequality, if $\text{Vol}_{2k}(M)$ is the
volume of $M$ given by the Riemannian structure $g = Re \langle \cdot, \cdot \rangle|_M$, then
\[
\text{Vol}_{2k}(M) = \frac{1}{k!} \int_M \omega^k.
\] (2)

If $X$ is a purely $k$-dimensional analytic subset of $\mathbb{C}^n$ and $\Sigma(X)$ is its singular locus, then $M = X \setminus \Sigma(X)$ is a complex manifold and since $\Sigma(X)$ is an analytic subset of $X$ of lesser dimension its volume is negligible, i.e.
\[
\text{Vol}_{2k}(X) = \frac{1}{k!} \int_X \omega^k = \frac{1}{k!} \int_{X \setminus \Sigma(X)} \omega^k = \text{Vol}_{2k}(X \setminus \Sigma(X)).
\]

### 4 Calibrated manifolds

The Wirtinger inequality relates the symplectic and volume forms of a hermitian inner product. It implies that the normalized exterior powers of the Kähler form of a Kähler manifold are calibrations.

**Definition 8 (Calibrated manifolds).** A calibrated $n$-manifold is a Riemannian manifold $(M, g)$ of dimension $n$ equipped with a differential $p$-form $\varphi$ ($0 \leq p \leq n$) with the following properties

1. $\varphi$ is closed, meaning $d\varphi = 0$, where $d$ is the exterior derivative.
2. For any $x \in M$ and any oriented $p$-dimensional subspace $\xi$ of $T_x M$, $\varphi|_\xi = \lambda \text{Vol}_\xi$ with $\lambda \leq 1$.

When referring to Riemannian manifolds, $\text{Vol}_\xi$ denotes the volume form of $\xi$ with respect to the Riemannian metric $g$. The $p$-from its called a $p$-calibration.

For $x \in M$, set $G_x(\varphi) = \{ \xi \subset T_x M \mid \varphi|_\xi = \text{Vol}_\xi \}$. In order for the theory to be nontrivial, we need $G_x(\varphi)$ to be nonempty. Let $G(\varphi)$ denote the union of all $G_x(\varphi)$ with $x \in M$.

**Definition 9 (Calibrated submanifold).** A $p$-dimensional calibrated submanifold of a manifold $M$ with calibration $\varphi$ is an oriented submanifold $\Sigma$ such that the calibration restricted to the tangent bundle of $\Sigma$ equals the induced volume form of the submanifold $\varphi|_{T\Sigma} = \text{Vol}_\Sigma$. Equivalently $T\Sigma \subset G(\varphi)$.

For calibrated manifolds and submanifolds the volume-minimizing property of submanifolds in the same homological class is proven by the following one-line argument
\[
\int_\Sigma \text{Vol}_\Sigma = \int_\Sigma \varphi = \int_{\Sigma'} \varphi \leq \int_{\Sigma'} \text{Vol}_{\Sigma'}
\] (3)

where the first equality holds because $\Sigma$ is calibrated, the second equality is Stoke’s theorem (as $\varphi$ is closed), and the last inequality holds because $\varphi$ is a calibration. Here $\Sigma$ is a calibrated submanifold of $M$ and $\Sigma'$ is any submanifold in the same homology class of $\Sigma$. Note that a Kähler manifold $(M, \omega)$ is a calibrated manifold with its calibrations given by the Kähler form and its powers $\{\omega^k \mid 1 \leq k \leq \dim(M)\}$. Meaning that $(M, \omega^k)$ is a calibrated manifold for every $k \in \{1, \ldots, \dim(M)\}$, and the previous one-line argument is a version of Wirtinger’s inequality.
5 Hausdorff measure and Bishop’s theorems

Besides the volumes of analytic sets, we will study the Hausdorff measure of said sets in order to determine their dimension. We denote the $\delta$-Hausdorff measure of a subset $S \subset \mathbb{C}^n$ by $H^d_\delta(S)$, see [10][ch. 3]. Here is a brief introduction to it and some of its properties.

Let $(X, \rho)$ be a metric space and let $S \subset X$, then let the diameter of $S$ be defined by $\text{diam}(S) = \sup \{ \rho(x, y) \mid x, y \in S \}$. That is to say, the diameter of a set is the distance between the farthest two points in the set.

**Definition 10.** Let $S$ be any subset of $X$, and $\delta > 0$ a real number. We define the Hausdorff outer Measure of dimension $d \in \mathbb{R}^+$ bounded by $\delta$ (written $H^d_\delta(S)$) by:

$$H^d_\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(U_i))^d \mid S \subset \bigcup_{i=0}^{\infty} U_i, \ \text{diam}(U_i) < \delta \right\}.$$  \hspace{1cm} (4)

Were the infimum is taken over all countable covers of $S$ by sets $U_i \subset X$ satisfying $\text{diam}(U_i) < \delta$.

If we allow $\delta$ to approach zero, the infimum is taken over a decreasing collection of sets, and therefore $H^d_\delta$ increases. We can conclude that

$$H_d(S) := \lim_{\delta \to 0} H^d_\delta(S) = \sup_{\delta > 0} H^d_\delta(S)$$

exists, but may be infinite. We call this limit Hausdorff Outer Measure of dimension $d$. We will explore Hausdorff Outer Measure more rigorously soon, but first, we will provide a few comments.

1. If $H_d(S) < \infty$, and $d < \kappa$, then $H_\kappa(S) = 0$.
2. If $f : X \to Y$ is a Lipschitz continuous function with Lipschitz constant $\lambda$, then for any $\delta \in \mathbb{R}^+$ and $S \subset X$, the following inequality holds

$$H_d(f(S)) \leq \lambda^d H_d(S).$$

3. If $X = \mathbb{R}^n$ and $S = M$ is a smooth submanifold of dimension $k \in \mathbb{Z}^+$, then the volume of $M$ as a submanifold is related to its Hausdorff measure by the formula

$$\text{Vol}_k(M) = \alpha_k H_k(S), \quad \alpha_k = \frac{\pi^{k/2}}{\Gamma(k/2 + 1)},$$

where $\Gamma(z)$ is the Euler’s gamma function. The constant $\alpha_k$ is the volume of the unit ball in $\mathbb{R}^k$.

Related to the Hausdorff measure is the Hausdorff metric defined for $K_1$ and $K_2$ compact subsets of a metric space $(X, d)$ as

$$d_h(K_1, K_2) = \max \left\{ \sup_{x \in K_1} d(x, K_2), \ \sup_{y \in K_2} d(K_1, y) \right\}.$$
The Hausdorff metric allows us to define a convenient notion of convergence of closed subsets; let \( \{ S_n \} \) be any sequence of closed subsets of \( X \), then we say that \( S_n \xrightarrow{h} S \), if for every \( K \subset X \) compact we have

\[ d_h(K \cap S_n, K \cap S) \to 0. \]

The first of Bishop’s results that we present here is very useful for understanding some of the analytical properties of limits (as defined before) of purely \( k \)-dimensional analytical subvarieties.

**Theorem 5.1.** [Sequence theorem] Let \( \{ V_n \} \) be a sequence of purely \( k \)-dimensional subvarieties of a domain \( \Omega \subset \mathbb{C}^n \) such that \( \forall \ n \in \mathbb{N} \), for every \( \rho \in \mathbb{R} \) a positive constant, then for the Hausdorff measure we have \( H_{2k+1}(S) = 0 \), moreover, \( V \) is a purely \( k \)-dimensional analytic subvariety of \( \Omega \).

As a direct application of this result, one can show the following very useful proposition (see [10][ch. 4]).

**Theorem 5.2.** [Bishop’s proper mapping theorem] Let \( \Omega \subset \mathbb{C}^n \) be a domain that contains 0 and let \( S \subset \Omega \) be a closed subset. If \( H_{2k+1}(S) = 0 \), then there is a suitable coordinate change of \( \mathbb{C}^n \), say, \( (z_1, \ldots, z_n) \) and a neighborhoods, \( \Omega_k \subset \mathbb{C}^k \) and \( \Omega_{n-k} \subset \mathbb{C}^{n-k} \), such that \( 0 \in \Omega_k \times \Omega_{n-k} \subset \Omega \) and the projection \( \pi_k : \Omega_k \times \Omega_{n-k} \to \Omega_k \), \( \pi_k(z,w) := z \), is a proper map.

**Remark 3.** When \( S \) is a purely \( k \)-dimensional analytic subvariety and not just a closed subset, then the following theorem implies that the projection \( \pi \) is a ramified analytic covering.

**Theorem 5.3.** Let \( \Omega = \Omega_k \times \Omega_{n-k} \) be a open subset of \( \mathbb{C}^n \) and \( \pi : \Omega \to \Omega_k \) the projection \( (z_k, z_{n-k}) \to z_k \). Let \( A \subset \Omega \) be a analytic subset such that \( \pi : A \to \Omega_k \) is proper. Then \( A' = \pi(A) \) is an analytic subset in \( \Omega_k \) and \( |\pi^{-1}[z_k] \cap A| \) is locally finite in \( \Omega \).

See [2][pp. 47].

With this theorem, the regular points of an analytic set can be characterized.

**Corollary 3.** Let \( 0 \in A \subset \mathbb{C}^n \) be an analytic set. The point \( 0 \in A \) is a regular point if and only if there is an open set \( 0 \in U \subset \mathbb{C}^n \) and a coordinate plane at \( \mathbb{C}_k \) such that the projection \( \pi_l : A \cap U \to \mathbb{C}_k \cap U \) is one to one.

### 6 Consequences of the proper mapping theorem

As mentioned, this result by Bishop can be used to prove many other important results (see [10]) one of the most significant is the proof of Remmert-Stein’s theorem, this was generalized and proved by Bishop in [1].

**Theorem 6.1.** [Remmert-Stein] Let \( \Omega \subset \mathbb{C}^n \) be an open subset and \( Y \) an analytic subset of \( \Omega \) and let \( X \) be a analytic subset of \( \Omega \setminus Y \). If \( Y \) is of dimension at most \( k-1 \) and \( X \) is of pure dimension \( k \), then the closure of \( X \) in \( \Omega \), \( \overline{X} \cap \Omega \), is an analytic subset of \( \Omega \).
This is an essential step toward the proof of Chow’s theorem if one is trying to avoid using categorical methods and quasi-coherent sheaves. This is because Remmert and Stein’s result implies that the Cone($X$) of a projective analytic subset of dimension $k$, $X \subset \mathbb{C}P^n$ is an analytic subset of dimension $k + 1$ in $\mathbb{C}^{n+1}$, where the cone is defined by

$$\text{Cone}(X) := \pi^{-1}[X] \cup \{0\}, \quad \pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n.$$  

Here $\pi$ is the usual projection of $\mathbb{C}^{n+1} \setminus \{0\}$ onto the projective space, so clearly Cone($X$) = $\pi^{-1}[X]$, and since $\pi$ is an analytic projection, $\pi^{-1}[X]$ is an analytic subset. Then from this point on, the classical proof is to use the fact that the cone is homothetic-invariant to show that the ideal of locally defined holomorphic functions that vanish at the cone has a countable basis. Then with Hilbert’s basis theorem, it is easy to prove the fact that the ring of germs of holomorphic functions is Noetherian. This shows that Cone($X$) is in fact algebraic; see [2], but the same result can be proved without algebraic methods with an equally simple proof using only the geometric and analytical tools we have presented thus far. We start the proof of Chow’s theorem by citing another consequence of the proper mapping theorem and we give a sketch of the proof.

**Theorem 6.2.** [Bishop] Let $V$ be a purely $k$ dimensional subvariety of $\mathbb{C}^n$ and let $B(R, 0)$ be the standard ball in $\mathbb{C}^n$ of radius $R$. If there is a constant $C \in \mathbb{R}^+$ such that

$$\text{Vol}_{2k}(V \cap B(R, 0)) \leq CR^{2k} \quad \forall R \in \mathbb{R}^+,$$

then $V$ is algebraic.

**Sketch of the proof:** Let $\{R_n\} \subset \mathbb{R}^+$ be an unbounded sequence, $R_n \to \infty$ and define $V_n$ as the image of $V \cap B(0, R_n)$ by the homothety $z \mapsto z/R_n$, then $\{V_n\}$ is a sequence of analytic sets of the unit ball with $\text{Vol}_{2k}(V_n) < C$, and such that $0 \in \lim_{n \to \infty} V_n$. Then by the proper mapping theorem there is a neighborhood of $0$, $\Delta = \Delta_k \times \Delta_{n-k}$ such that the projection on the first factor $\pi_k$ is a $\sigma$-sheeted branched covering for each $V_n \cap \Delta$, since $R_n \to \infty$ the balls $B(0, R_n)$ cover the whole of $\mathbb{C}^{n+1}$ and we deduce that the projection $\pi_k$ restricted to $V$ is a $\sigma$-sheeted branched covering. From this let us construct a set of canonically defining functions for $V$; for each $z \in \mathbb{C}^n$ we denote

$$\pi_k^{-1}[\pi_k(z)] \cap V = \{\alpha_1(z), \ldots, \alpha_\sigma(z)\},$$

and let $P_\alpha : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be

$$P_\alpha(z, w) := \langle z - \alpha_1(z), w \rangle \ldots \langle z - \alpha_\sigma(z), w \rangle,$$

we observe that if $z \in V$, $P_\alpha(z, w) = 0$, so for $z$ outside of $V$ we can chose a $w$ such that $\langle z - \alpha_j(z), w \rangle \neq 0$ for all $j$. Now by expanding $P_\alpha$ in powers of $w$

$$P_\alpha(z, w) = \sum_{|\mu| \leq \sigma} \eta_\mu(z)w^\mu$$

then because $\pi_k$ is a branched covering the functions $\eta_\mu$ are analytic. By applying a homothety for a suitable $R_n$, one can prove using Cauchy’s estimates that the functions $\eta_\mu$ are in fact polynomials of degree at most $\sigma - |\mu|$ thereby proving the theorem.
7 Proof of Chow’s theorem via Bishop’s theorems

Chow via Bishop. By Theorem 6.1, \( \text{Cone}(X) := \pi^{-1}[X] \) is an analytic set of \( \mathbb{C}^{n+1} \), and if \( X \subset \mathbb{C}P^n \) is of dimension \( k \) (real dimension \( 2k \)), then \( \text{Cone}(X) \) has dimension \( k + 1 \). Now the mapping \( \pi : S^{2k+1} \to \mathbb{C}P^n \) known as Hopf’s Fibration, is Riemannian fibration from the sphere with its canonical metric and the Fubini-Study metric on projective space, it fibers \( \mathbb{C}P^n \) into circles (of length \( 2\pi \)), so by Fubini’s theorem:

\[
\text{Vol}_{2k+1}(\text{Cone}(X) \cap S^{2n+1}) = 2\pi \text{Vol}_{2k}^\mathbb{CP}(X) := A,
\]

where \( \text{Vol}^\mathbb{CP} \) is the “projective volume”, meaning the volume form of the complex projective space given by the Fubini-Study metric. Clearly \( A \) is finite since \( X \) is compact, this means that the volume of the intersection \( \text{Cone}(X) \cap B(R, 0) \) grows at most polynomially as \( R \to \infty \), since \( \text{Cone}(X) \) is homothetic-invariant and using polar coordinates we see that

\[
\text{Vol}_{2k+2}(\text{Cone}(X) \cap B(R, 0)) = \int_{\text{Cone}(X) \cap B(R, 0)} dr \wedge \sigma_r,
\]

where \( \sigma_r \) is the \( 2k + 1 \) volume form of \( S(0, r) \cap X \) with \( S(0, r) := \{\|z\| = r\} \) with \( 0 \leq r \leq R \). We can find a bound for the volume of \( \text{Cone}(X) \cap B(R, 0) \) as follows:

\[
\int_{(\text{Cone}(X) \cap B(R, 0))} dr \wedge \sigma_r = \int_0^R r^{2k+1}dr \int_{\text{Cone}(X) \cap S^{2n+1}} \sigma_1 = \frac{A}{2k+2} R^{2k+2},
\]

so setting \( C = \frac{A}{2k+2} \) by Theorem 6.2 this means \( \text{Cone}(X) \) is algebraic and therefore \( X \) is also algebraic.

\( \square \)
8 Proof of Montel’s Theorem via Bishop

We now proceed to prove a generalization of Montel’s result of compact subsets of holomorphic functions on the unit disk. We note that there is already another way to prove a version of Montel’s result that was proved by Lelong in [7], although it is formulated using the language of integration currents.

Definition. A family $F$ of analytic functions on a domain $\Omega \subset \mathbb{C}^n$ is normal in $\Omega$ if every sequence of functions $\{f_n\}_{n \in \mathbb{N}} \subseteq F$ contains either a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}} \subset F$ which converges to a limit function $f \neq \infty$, uniformly on each compact subset of $\Omega$, or a subsequence which converges uniformly to $\infty$ on each compact subset.

The open unit ball on $\mathbb{C}^n$ will be denoted $B$ with its closure on $\mathbb{C}^n$ denoted by $\overline{B}$. Let $A(B) := C(\overline{B}) \cap H(B)$, be the Banach algebra of holomorphic functions in $B$ that are continuous on $\overline{B}$ and therefore also on $S := \partial B$. We will prove the following result

Theorem 8.1 (Montel’s Theorem). Let $F \subset A(B)$, a set of locally bounded functions, i.e. for every $z \in B$ there exists an open neighborhood $z \in \Omega \subset B$ such that

$$\|f|_{\Omega}\|_{\infty} = \sup\{|f| | z \in \Omega\} \leq C_\Omega,$$

for some $C \in \mathbb{R}^+$, then $F$ is a normal set.

We are going to prove this by means of Bishop’s results. First, we note that the graph of a holomorphic function $f : B \to \mathbb{C}$ is an analytic subvariety of pure dimension $n$ on $B \times \mathbb{C}$ since $\Gamma_f = V_B(w-f(z)) = \{(z,w) : w - f(z) = 0 \}, z \in B, w \in \mathbb{C}$ (2), where $\Gamma_f$ denotes the graph of $f$, and $z = (z_1, \ldots, z_n)$. Also if $\{f_{i\Omega}\}$ are bounded for all $f \in F$, then $\text{Vol}_{2n}(\Gamma_{f_{i\Omega}})$ have uniformly bounded volume as we shall see.
Proof. Let $\epsilon \in (0, 1)$ and we take $B_\epsilon = B(1 - \epsilon, 0) \subset B$, since $B_\epsilon$ is a compact subset of $B$, we have that $F$ is uniformly bounded in $B_\epsilon$. Let $C \in \mathbb{R}^+$ be a bound for $|f(z)|$ for all $z \in B_\epsilon$ and $f \in F$, then by Cauchy’s integral formula for the open ball (see [9][Ch. 3, 3.2.4.]), if $S_z := \partial B_z$, then

$$f(z) = \int_{S_z} \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma(\zeta) \quad \forall z \in B_\epsilon \setminus S_z,$$

where $\sigma$ is the usual Lebesgue measure in $S_z$. Taking the derivative with respect to $z_j$ we obtain

$$\frac{\partial f}{\partial z_j}(z) = \int_{S_z} \frac{f(\zeta)\zeta_j}{(1 - \langle z, \zeta \rangle)^{n+1}} d\sigma(\zeta) \quad \forall z \in B_\epsilon \setminus S_z,$$

therefore, we have the following bound for the partial derivatives of all $f \in F$ and $z \in B_\epsilon \setminus S_z$,

$$|\frac{\partial f}{\partial z_j}(z)| \leq C |\text{Vol}_{2n-1}(S_z)(1 - \epsilon)| \frac{\text{Vol}_{2n-1}(S_z)(1 - \epsilon)}{1 + (1 - \epsilon)^2}.$$

Now, to show that the hypotheses of theorem 5.1 are satisfied, we use this constant bound of the derivatives to show that for a sequence of graphs of functions $\{f_n\}_{n \in \mathbb{N}}$, the $2n$ dimensional volumes of their graphs are uniformly bounded in $B_\epsilon$ by a constant $M \in \mathbb{R}^+$. This is because the volume of $\Gamma_f$ is given by

$$\text{Vol}_{2n}(\Gamma_f) = \int_{B_\epsilon} |\lambda(z)| dx_1 \ldots dx_n dy_1 \ldots dy_n,$$

where $z_j = x_j + iy_j$ and $\lambda : B_\epsilon \to \mathbb{R}$ is given by the pullback of the parametrization function $\varphi(z) = (z, f(z))$

$$\varphi^\ast \omega = \lambda(z)(dx_1 \wedge dy_1) \wedge \ldots \wedge (dx_1 \wedge dy_n),$$

with $\omega$ being the volume form of $\Gamma_f \subset B \times \mathbb{C}$. A straightforward calculation shows:

$$\lambda(z) = \sqrt{1 + \|\nabla f\|^2},$$

with

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

Therefore, the volumes of all the graphs are uniformly bounded by (11) and (12). Now the only thing left to show is the convergence with respect to the Hausdorff metric of a subsequence of $\{\Gamma_{f_n}\}$ in $B_\epsilon$. This follows from the fact that $|f(z)|$ is uniformly bounded in $B_\epsilon$ for all $f \in F$. Since the image of all $f_n$ are inside a compact set, it therefore follows that for $z \in S_t$, there exists a convergent subsequence $\{f_{n_k}(z)\}$ and by Cauchy’s integral formula $\{f_{n_k}(z)\}$ converges uniformly on $B_\epsilon \setminus S_z$. This means that $\Gamma_{f_{n_k}} \to \Gamma_f$ in the Hausdorff metric with $\Gamma_f \subset B_\epsilon \times \mathbb{C}$, and by theorem 5.1 we know $\Gamma_f|_{B_\epsilon}$ is an analytic subvariety of pure complex dimension $n$. Moreover, it is clearly the graph of a holomorphic function $f : B_\epsilon \setminus S_z \to \mathbb{C}$ since at each fiber $\{z_0\} \times \mathbb{C}$, the intersection of each graph $\Gamma_{f_{n_k}}$ gives a unique point $(z_0, f_{n_k}(z_0))$. If there were two distinct points $(z_0, w_1)$ and $(z_0, w_2)$ at the intersection $\{\{z_0\} \times \mathbb{C}\} \cap \Gamma_f$, then taking a compact set $K$ containing $\{f_{n_k}(z_0)\} \cup \{w_1, w_2\}$, by
Hausdorff convergence we would have that the limit of \( \{ f_n(z_0) \} \) is not unique. Thus \( \Gamma_f \) has to be the graph of a holomorphic function in \( B_{\epsilon} \) for all \( \epsilon \in (0, 1) \). By analytic continuation, we thus define \( f \) in all of \( B \).

**Remark 4.** As previously noted there is a version of Montel’s theorem proved by Lelong in terms of the integration currents defined by analytic subvarieties of pure dimension \( k \). An integration current defined by an analytic subvariety of dimension \( k \), \( A \) is the linear operator

\[
t(A)[\phi] = \int_A \phi, \quad \text{where } \phi \text{ is a differential } (k, k) \text{ form with compact support.}
\]

In this case the boundedness refers to the norm of the aforementioned operators, where we say that a family \( F \) of purely \( k \) dimensional analytic subsets are locally bounded if \( \{ \| t(A) \| \mid A \in F \} \) are bounded for every compact set. See [7]

The following table shows the similarities between the results shown here and others in the classical theory of holomorphic functions of one variable.

| Complex Analysis | Theory of analytic sets |
|------------------|-------------------------|
| **Liouville’s theorem** | **Bishop’s theorem (Theorem 6.2)** |
| If \( |f(z)| \leq CR^k \) for \( |z| \leq R \) for all \( R \in \mathbb{R}^+ \) with \( f \) entire, \( k \) a positive integer and \( C \) a positive constant, then \( f \) is a polynomial. | If \( \text{Vol}_{2k}(X \cap B(R, 0)) \leq CR^{2k} \) for all \( R \in \mathbb{R}^+ \) with \( C \) a positive constant and \( X \) an analytic subvariety, then \( X \) is algebraic. |
| **Riemann’s extension theorem.** | **Bishop’s generalization of Remmert-Stein’s theorem.** |
| If \( f : (\Omega \setminus E) \subset \mathbb{C} \to \mathbb{C} \) is a holomorphic function and \( E \) is a compact subset of capacity 0, then \( f \) is extendible to a holomorphic function on the whole region \( \Omega \). | Let \( U \subset \mathbb{C}^n \) be a bounded open subset and let \( B \subset U \) be closed with \( X \subset U \setminus B \) a purely \( k \) dimensional analytic subset such that \( B \subset \overline{X} \). If \( B \) has capacity 0 relative to the algebra of analytic functions on \( X \) that are continuously extendible to \( \overline{X} \), and if it exists \( f : U \to \mathbb{C}^k \) proper on \( B \) with \( f(B) \) not an open connected subset of \( \mathbb{C}^k \), then \( X \cap U \) is an analytic subset of \( U \) (see [1][Theorem 4]). |
| **Montel’s compactness theorem.** | **Lelong’s normal currents theorem (a consequence of Bishop’s sequence theorem) see [7]** |
| Let \( F \) be a family of locally bounded holomorphic functions \( f_i : \Delta \to \mathbb{C} \). Then \( F \) is a normal family if and only if \( F \) is locally uniformly bounded. | Let \( F \) be a family of analytic subvarieties and let \( t(F) \) be the set of integration currents defined by \( F \), suppose \( t(F) \) is locally bounded (in terms of the operator norm). Then \( t(F) \) is a normal family if and only if its locally uniformly bounded. |
9 Foliations on Kähler manifolds with all leaves compact

In this final section, we apply Bishop’s sequence theorem (theorem 5.1) to prove a version of [3][Theorem 1], for the particular case of compact complex foliations on Kähler manifolds with all leaves compact, also we prove that a holomorphic foliation in a compact Kähler manifold with at least one compact leaf with finite holonomy, then all of its leaves are compact.

Theorem 9.1. Let $M$ be a compact connected Kähler manifold of complex dimension $n$ and $\mathcal{F}$ a holomorphic foliation with leaves of complex dimension $d < n$ and with all leaves compact, then:

1. The $2d$-dimensional volume (with respect to the Kähler metric) of the leaves is uniformly bounded.

2. The quotient space $M/\mathcal{F}$ is a complex orbifold, with singularities corresponding to leaves with non-trivial holonomy (which by the first proposition is a finite group).

Proof. First, let us define the volume function $\nu : M \to \mathbb{R}^+$ given by $z \mapsto \text{Vol}_{2d}(\mathcal{L}_z)$. Now for the open and dense set of generic leaves

$$H_0 = \{ x \in M \mid \mathcal{L}_x \text{ has zero holonomy} \},$$

where $\mathcal{L}_x$ denotes the leaf through $x$, $\nu$ is continuous. Let $\{z_n\} \subset H_0$ be a sequence, without loss of generality, suppose that each $z_i$ is on a different leaf $\mathcal{L}_{z_i}$ of $\mathcal{F}$ and suppose that $z_n \to z \in H_0$. Since all the leaves are compact it is clear that $\mathcal{L}_{z_n} \to \mathcal{L}$ for the Hausdorff metric, where $\mathcal{L} \subset M$ is a non-empty closed set. Now, let $\mathcal{L}_z$ be the leaf containing $z$, since $\mathcal{L}_z$ has zero holonomy, by the generalization of Reeb’s stability theorem (see [11]), there exists a tubular neighborhood of $\mathcal{L}_z$, say $U$, which is biholomorphic to $\mathcal{L}_z \times D$, where $D \subset \mathbb{C}^{n-d}$ is an open disk (ball) and such that $U$ is a saturated open subset of $M$ with every leaf of $U$ mapped biholomorphically to the sets $\mathcal{L}_z \times \{x\}$. Therefore, every leaf in $U$ is homologous to $\mathcal{L}_z$ and by Corollary 2, all leaves in $U$ have the same volume. Since $z \in U$ and $z_n \to z$ there is a large enough positive integer $N$ such that all leaves $\mathcal{L}_{z_n}$ have the same volume for $k > N$. Therefore, by theorem 5.1, $\mathcal{L}$ is an analytic subvariety of $U$ of complex dimension $d$ with its volume equal to $\lim_{n \to \infty} \text{Vol}_{2d}(\mathcal{L}_{z_n})$. Since tangency to $\mathcal{F}$ is defined locally by the null space of $d$ holomorphic 1-forms, by Hausdorff convergence this tangency is preserved on the limit, so $\mathcal{L}$ is tangent to $\mathcal{F}$ and therefore $\mathcal{L} = \mathcal{L}_z$. Now, $\nu$ is not continuous in general but rather lower semicontinuous. Semicontinuity can be proved by showing that the leaf space $M/\mathcal{F}$ is Hausdorff (see [3][p. 20]), which we will prove, but more than that, the volume function $\nu$ is in fact discretely lower-semicontinuous, meaning that for any $n \in \mathbb{Z}^+$, $z \in M$ and $\epsilon \in \mathbb{R}^+$, there is a small enough neighborhood of $z$ such that either

$$\nu(y) > n \nu(z) \quad \text{or} \quad |\nu(y) - k \nu(z)| < \epsilon \quad \text{for some} \quad k \in \{1, \ldots, n\}.$$

We prove this fact locally. Given a tubular neighborhood of a leaf $\mathcal{L}_z$, say $W$ there is a bundle retraction $\rho : W \to \mathcal{L}_z$ with $\rho^{-1}(x)$ homeomorphic to a disk. For every leaf $\mathcal{L}_y$, the restriction $\rho|_{W \cap \mathcal{L}_y} : (W \cap \mathcal{L}_y) \to \mathcal{L}_z$ is a codimension zero submersion, if $y$ is sufficiently close to $z$, then $\mathcal{L}_y \subset W$ and also the image under $\rho$ of the leaf $\mathcal{L}_y$ covers all of $\mathcal{L}_z$. Therefore, by compactness and analyticity of the leaves, $\mathcal{L}_y$ is a finitely sheeted covering space of $\mathcal{L}_z$ with
covering transformation \( \rho|_{\mathcal{L}_i} \) which proves the discrete lower-semicontinuity of \( \nu \). Clearly the previous argument is valid for open saturated subsets \( U \subset M \). With this, we proceed to show that the set where \( \nu \) is not locally bounded, also known as the “bad set”:

\[
B := \{ x \in M \mid \nu \text{ is not bounded in a neighborhood of } x \},
\]

is a saturated compact set of codimension greater or equal to 2 (see [4]). By definition \( B \) is the union of leaves for which \( \nu \) is not bounded around them therefore its a saturated set, that is closed since for any convergent sequence of points in \( B, x_n \to x \) there is another sequence of points \( \{x_{n_k}\} \) arbitrarily close to \( x_n \) such that \( \{\nu(x_{n_k})\} \) is unbounded. Therefore since \( \{x_n\} \) is arbitrarily close to \( x \), by a diagonal argument there is a sequence \( \{x_{n_l}\} \) arbitrarily close to \( x \) such that \( \nu(x_{n_l}) \) is unbounded, therefore \( B \) is a saturated closed (compact) subset.

Now let \( \{x_n\} \) be a sequence in \( B \), since \( B \) is a closed compact saturated set and all leaves are compact, there is a finite covering of \( B \) by open saturated sets \( U_j \) such that for any convergent subsequence \( \{x_{n_k}\} \) and their corresponding leaves \( \mathcal{L}_{n_k} \) converge in \( U_j \cap B \) for some \( j \) since \( U_j \cap B \) is a saturated open set in \( B \) such that eventually every leaf \( \mathcal{L}_{n_k} \) stays in \( U_j \), therefore by the same argument for lower semi-continuity applied to \( U_j \), possibly by shrinking it a little, the volume of \( \{U \cap \mathcal{L}_{n_k}\} \) is uniformly bounded and by theorem 5.1, we have that \( \{\mathcal{L}_{n_k}\} \) converges to a closed analytic subset of \( B \subset M \). Since \( \{x_n\} \) was arbitrarily and we showed that every convergent subsequence has bounded volumes in a neighborhood of \( B \) then the volume is therefore bounded in any neighborhood of \( B \), therefore \( B = \emptyset \).

The second assertion follows easily from Thurston’s generalization of Reeb’s theorem [11], since for every leaf with null holonomy \( \mathcal{L} \) we have an open laminated set \( U \) biholomorphic to \( \mathcal{L} \times D \) so locally \( M/\mathfrak{F} \) is homeomorphic to \( D \). Furthermore, \( M/\mathfrak{F} \) is Hausdorff since every leaf is compact, so if \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are two distinct leaves, then there are \( \{\epsilon_1, \epsilon_2\} \subset \mathbb{R}^+ \) such that the sets

\[
D_i := \{ z \in M \mid d_{\mathcal{L}_i}(z, L) < \epsilon_i \}, \quad i \in \{1, 2\},
\]

are disjoint, so intersecting with a laminated tubular neighborhood of \( \mathcal{L}_i \) we have that \( M/\mathfrak{F} \) is Hausdorff. Finally, if \( \mathcal{L} \) has non-trivial holonomy, then by the boundedness of the volume function, the holonomy group \( H(\mathcal{L}) \) is finite (see [3] [p. 20]) and \( M/\mathfrak{F} \) is locally homeomorphic to \( D/H(\mathcal{L}) \), where \( \mathcal{L} \) has a tubular neighborhood homeomorphic to \( \mathcal{L} \times D \).

We finish this paper with a proof of a result of Vito Antonio Pereira which guarantees the compactness of all leaves in a compact Kähler manifold provided that it is known that at least ONE of the leaves is compact and has finite holonomy (see [8][Theorem 1]).

**Theorem 9.2.** Let \( M \) be a compact connected Kähler manifold of dimension \( n \) and \( \mathfrak{F} \) be a holomorphic foliation of codimension \( q < n \) for which there is at least one compact leaf with finite holonomy, then all leaves are compact with finite holonomy or equivalently their volumes are uniformly bounded and the leaf space is Hausdorff.

**Proof.** Let us denote by \( \mathcal{L}_0 \) to the compact leaf with finite holonomy of \( \mathfrak{F} \). The generalized Reeb stability theorem asserts that there exists a saturated neighborhood, say \( \Omega \) of \( \mathcal{L} \) such that \( \Omega \) is biholomorphic to a product \( \mathcal{L} \times D \) with \( D \subset \mathbb{C}^q \) an open polydisk and such that every leaf in \( \Omega \) is a finitely sheeted covering of \( \mathcal{L} \) given by a retraction of \( \Omega \) to \( \mathcal{L} \), therefore every
Leaf in $\Omega$ is compact with finite holonomy. Let us define the following set

$$U := \{ x \in M \mid \text{the leaf through } x \text{ is compact with finite holonomy} \}. $$

Clearly $L \subset \Omega \subset U$ and therefore $U$ is a non-empty saturated set. Also by the same argument as before every leaf in $U$ has a tubular saturated neighborhood comprised of compact leaves with finite holonomy, therefore for every point $z \in U$ there is an open saturated neighborhood $W$ of $z$ such that $z \in W \subset U$ i.e. $U$ is an open saturated set. Note that $U$ is the union of all saturated open sets with all its leaves compact and with finite holonomy, another way to describe it is as the maximal open set of saturated sets with all its leaves compact and with finite holonomy. We show that this set is also closed to finish the proof. Let $\{ z_n \} \subset U$ be a convergent sequence with limit $z_n \to z_0$, since $\{ z_n \}$ is convergent, in particular is a Cauchy sequence. Moreover, let $L_n$ be the leaf through $z_n$ and $L_0$ the leaf through $z_0$, since $z_k \in U$, every $L_k$ is compact with finite holonomy, to show $z_0 \in U$ we need to show $L_0$ is compact and has finite holonomy, first since $\{ z_k \}$ is a Cauchy sequence, the leaves $\{ L_k \}$ are eventually arbitrarily close between them. Since all of them are compact submanifolds of a Kähler manifold, they minimize the volume of their homology classes respectively, therefore there is a $N \in \mathbb{N}$ such that for all $\{ L_k \mid k \geq N \}$ are in the same homological class, therefore eventually the volume of $\{ L_k \}$ is constant. Now for any foliated chart $V$ around $z_0$, take the sequence $L_k \cap V$ in the intersection $V \cap U$, since $z_0$ is a limit point of $U$, $U \cap V \neq \emptyset$, moreover by convergence there is a $N \in \mathbb{N}$ such that for $k \geq N$, $z_k \in V$, since $V$ is a foliated chart, $V$ is biholomorphic to a product $W_q \times W_{n-q}$ and every leaf $\{ L_k \cap V \}_{k \geq N}$ is the set $\varphi^{-1}(W_q \times \{ z_k \})$, here we abused the notation a little bit by identifying the points of the convergent sequence with their corresponding transversal coordinates. Its clear that $V \cap L_k \rightarrow V \cap L_0$ as closed subsets of $U \cap V$, since $z_k \rightarrow z_0$ and the plaques $W_q \times \{ z_k \}$ clearly converge to $W_q \times \{ 0 \}$ by continuity of $\varphi$. Therefore since $L_k$ are compact connected and with finite linear holonomy and bounded volume, by Bishop’s sequence theorem (theorem 5.1), $\{ L_k \}$ converges to a closed (compact) analytic subset of $L_0$ that is tangent to $\tilde{\mathfrak{F}}$ at $z_0$ therefore $L_k \rightarrow L_0$ since leaves are connected by definition. This means that $L_0$ is compact analytic set with finite holonomy, therefore $U = M$ since its both open and closed and $M$ is connected.

\[ \Box \]

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**DECLARATIONS**

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