MODULI SPACES OF WITCH CURVES TOPOLOGICALLY REALIZE THE 2-ASSOCIAHAEDRA

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Abstract. For \( r \geq 1 \) and \( n \in \mathbb{Z}_{\geq 0} \setminus \{0\} \), we construct the compactified moduli space \( \overline{2M}_n \) of witch curves of type \( n \). We equip \( \overline{2M}_n \) with a stratification by the 2-associahedron \( W_n \), and prove that \( \overline{2M}_n \) is compact, second-countable, and metrizable. In addition, we show that the forgetful map \( \overline{2M}_n \to \mathcal{M}_r \) to the moduli space of stable disk trees is continuous and respects the stratifications.

1. Introduction

In [Bo1], the author constructed a collection of abstract polytopes (in particular, posets) called 2-associahedra. There is a 2-associahedron \( W_n \) for every \( r \geq 1 \) and \( n \in \mathbb{Z}_{\geq 0} \setminus \{0\} \), and they were introduced to model degenerations in the configuration space of witch curves, whose interior parametrizes configurations of \( r \) vertical lines in \( \mathbb{R}^2 \) with \( n \) marked points on the \( i \)-th line up to translations and positive dilations. By identifying \( \mathbb{R}^2 \cup \{\infty\} \simeq S^2 \), we can also view an element of the interior of \( 2M_n \) as a configuration of marked circles on \( S^2 \), where all the circles intersect at the south pole, up to Möbius transformations; both views are depicted in the following figure:

![Diagram of witch curves and circles on S^2](image)

The purpose of this paper is to construct the compactified configuration space \( \overline{2M}_n \), and to validate the construction of both \( W_n \) and \( \overline{2M}_n \) via the following main result:

**Theorem 1.1.** For any \( r \geq 1 \) and \( n \in \mathbb{Z}_{\geq 0} \setminus \{0\} \), \( \overline{2M}_n \) can be given the structure of a compact, second-countable, metrizable space stratified by \( W_n \). The forgetful map \( \overline{2M}_n \to \mathcal{M}_r \) to an associahedron can be upgraded to a continuous map \( \overline{2M}_n \to \overline{\mathcal{M}}_r \) to the moduli space of stable disk trees that respects the stratifications.

This result is an important step toward the author’s goal of defining a symplectic \( (A_\infty, 2) \)-category \( \text{Symp} \), in which the objects are certain symplectic manifolds and \( \text{hom}(M, N) := \text{Fuk}(M^\circ \times N) \). Indeed, \( (\overline{2M}_n) \) form the domain moduli spaces involved in the structure maps in \( \text{Symp} \). More progress toward the construction of \( \text{Symp} \) is described in [Bo2, BoWe, Bo1].

In §D of [McDSa], McDuff–Salamon equip the compactified moduli space \( \overline{\mathcal{M}}_r(\mathbb{C}) \) of marked genus-0 curves with a topology by including it into a product of \( \mathbb{C}P^1 \)'s via a collection of cross-ratio maps. This is the obvious approach to try here, too, but the author was unable to make this technique work in this context. Instead, we adapt the techniques from §5 of the same book, in which McDuff–Salamon equipped the compactified moduli space of stable maps into a symplectic manifold with a topology in which the convergent sequences are those which Gromov-converge.
While this necessitates a certain amount of topological overhead in our setting, an advantage is that it will be straightforward to adapt the current work to the setting of witch maps when such a result is needed.

1.1. **An example of Gromov convergence for witch curves.** As a coda to the introduction, we illustrate and motivate the definition of Gromov convergence in $\tilde{\mathcal{M}}_n$ by an example. For $\epsilon \in (0, \frac{1}{2})$, consider the following configuration in $\tilde{\mathcal{M}}_{10010}$:

In the limit as $\epsilon \to 0$, all lines but the right-most collide; the two marked points also collide. We resolve these collisions using the well-known technique of **soft rescaling**: whenever a marked point collides with a line (and in particular, with another marked point), we zoom in on the collision with just enough magnification that the colliding objects occupy a “window” of unit size. If, in this zoomed-in view, there are still colliding objects, we again rescale, and so on inductively.

A decision must be made about what to do when lines without marked points collide; here, we have decided to remember the fashion in which such lines collide, a choice which is motivated by considerations of pseudoholomorphic quilts. We implement this strategy by keeping track of the positions of the lines as points in $\mathbb{R}$ and performing soft rescaling on these configurations in parallel with our soft rescalings of the configurations of lines and points in $\mathbb{R}^2$.

Finally, we are ready to demonstrate soft rescaling for the family pictured above. This is shown in the following figure, where the left-most view is the original configuration, and the remaining configurations are the rescaled views. The arrows indicate that a configuration is produced by rescaling at the point that the arrow points to, with magnification labeling the arrow. In the bottom of the figure, we show the soft rescalings of the configurations of the line positions in $\mathbb{R}$.
We show the $\epsilon \to 0$ limit of this family in the following figure. On the right, we show an equivalent view: the planes with marked vertical lines are replaced with spheres with marked circles. In the tree of decorated spheres, the “nodal points” — where the south pole of one sphere is attached to one of the circles on another sphere — indicate that the upper sphere was a further rescaling of the lower sphere, centered at the attachment point.

It is important to note the role of the configurations of $x$-coordinates in $\mathbb{R}$ of the lines. (The limit of these configurations is depicted in the bottom of the most recent figure; on the right, it is shown as a tree of marked circles.) These limit of these configurations tracks the collisions of lines, and it is used to keep track of which lines in the original family collided to form which lines in the limiting tree of decorated spheres.

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2. Construction of $\overline{\mathcal{M}}_n$

In this section we prove Thm. 1.1. Specifically, in §2.1 we construct $\overline{\mathcal{M}}_n$; in §2.2 we show that every sequence in $\overline{\mathcal{M}}_n$ has a Gromov-convergent subsequence; in §2.3 we show that a Gromov-convergent sequence has a unique limit; and in §2.4 we define a topology on $\overline{\mathcal{M}}_n$ in which the convergent sequences are the Gromov-convergent ones. Before we begin, we make some preliminary remarks.

The compactified moduli space $\overline{\mathcal{M}}_r$ of disks with $r$ “input” and 1 “output” boundary marked points is well-known, though the author does not know of a detailed construction in the literature. Nearly all of the results we describe below for $\overline{\mathcal{M}}_n$ have analogues for $\overline{\mathcal{M}}_r$ — in particular, $\overline{\mathcal{M}}_r$ can be given a topology in which the convergent sequences are the Gromov-convergent ones, and with this topology it is compact, second-countable, metrizable, and stratified by $K_r$. Here is the definition of $\overline{\mathcal{M}}_r$:

Definition 2.1. A stable disk tree with $r \geq 2$ input marked points is a pair $(T, (x_\rho)_{\rho \in V_{\text{int}}(T)})$, where:

- $T$ is a stable rooted ribbon tree (RRT) with $r$ leaves.
- For $\rho \in V_{\text{int}}(T)$, $x_\rho \in \mathbb{R}^{\#_{\text{in}}(\rho)}$ is a tuple satisfying $x_{\rho,1} < \cdots < x_{\rho,\#_{\text{in}}(\rho)}$. 
We say that two stable disk trees \((T, (x_\rho))\), \((T', (x'_\rho))\) are isomorphic if there is an isomorphism of RRTs \(f: T \to T'\) and a function \(V_{\text{int}}(T) \to G_1: \rho \mapsto \phi_\rho\) such that:

\[
x'_{f(\rho), i} = \phi_\rho(x_\rho, i) \quad \forall \ \rho \in V_{\text{int}}(T).
\]

We denote the collection of stable disk trees with \(r\) input marked by \(SDT_r\), and we define the moduli space of stable disk trees with \(r\) input marked points \(\overline{\mathcal{M}_r}\) to be the set of isomorphism classes of stable disk trees of this type. For any stable RRT \(T\) with \(r\) leaves, define the corresponding strata \(SDT_{r,T} \subset SDT_r\), \(\overline{\mathcal{M}_{r,T}} \subset \overline{\mathcal{M}}_r\) to be the set of all stable disk trees (resp. isomorphism classes thereof) of the form \((T, (x_\rho))\). We say that a stable disk tree is smooth if its underlying RRT \(T\) has only one interior vertex; we denote a smooth stable witch curve by the tuple \(x \in \mathbb{R}'\) associated to the root.

Throughout this section we freely use the notation of [Bo1]. In that paper, tree-pairs were denoted \(T_b \overset{\pi}{\to} T_s\); here, we will use the notation \(T_b \overset{\pi}{\to} T_s\). Also, we make the conventions \(\overline{\mathcal{M}_1} := \text{pt}\), \(\overline{\mathcal{M}_{(1)}} := \text{pt}\).

2.1. Definition of \(\overline{\mathcal{M}_r}\) as a set, and Gromov convergence. In this subsection we define stable witch trees, isomorphism classes of which comprise \(\overline{\mathcal{M}_n}\). Throughout, we will denote by \(\mathbb{R}^2 \cup \{\infty\}\) the one-point compactification of \(\mathbb{R}^2\) (so \(\mathbb{R}^2 \cup \{\infty\} \cong S^2\)). We will make use of the reparametrization group \(G_2 := \mathbb{R}^2 \times \mathbb{R}_{>0}\) acting on \(\mathbb{R}^2\) by translations and positive dilations. This action of \(G_2\) on \(\mathbb{R}^2\) extends to an action on \(\mathbb{R}^2 \cup \{\infty\}\), by defining \(\phi(\infty) := \infty\) for every \(\phi \in G_2\). There is a projection \(p: G_2 \to G_1\), defined by sending \(((x, y) \mapsto (ax + b_1, ax + b_2)) \in G_2\) to \((x \mapsto ax + b_1) \in G_1\). We will overload notation and also denote by \(p\) the projection \(\mathbb{R}^2 \to \mathbb{R}^1\) onto the first factor.

Definition 2.2. A stable witch curve of type \(n \in \mathbb{Z}_{\geq 0} \setminus \{0\}\) is a triple \((2T = T_b \overset{\pi}{\to} T_s, (x_\rho)_{\rho \in V_{\text{int}}(T_s)}, (z_\alpha)_{\alpha \in V_{\text{comp}}(T_b)}\)

\[
\begin{align*}
&\text{where:} \\
&\quad \bullet \ T_b \text{ is a stable tree-pair of type } n. \\
&\quad \bullet \ For \ \rho \in V_{\text{int}}(T), \ x_\rho \in \mathbb{R}^{\#\text{in}^(n)} \text{ is a tuple satisfying } x_{\rho,1} < \cdots < x_{\rho,\#\text{in}^(n)}. \\
&\quad \bullet \ For \ \alpha \in V_{\text{comp}}(T_b), \ z_\alpha \in \mathbb{R}^2 \text{ is a collection}
\end{align*}
\]

\[
z_\alpha = \left\{ (z_{\alpha, i,j} = (x_{\alpha, i}, y_{\alpha, i})) \mid \begin{array}{l}
\text{in}(\alpha) = (\beta_1, \ldots, \beta_{\#\text{in}^(n)}), \\
1 \leq i \leq \#\text{in}^(\alpha), \ 1 \leq j \leq \#\text{in}^(\beta_i)
\end{array} \right\}
\]

satisfying \(x_{\alpha,1} < \cdots < x_{\alpha,\#\text{in}^(n)}\) and \(y_{\alpha,1} < \cdots < y_{\alpha,\#\text{in}^(\beta_i)}\) for every \(i\). Moreover, for \(\alpha \in V_{\text{comp}}(T_b)\) we require \((x_{\alpha,1}, \ldots, x_{\alpha,\#\text{in}^(\alpha)}) = (x_{\pi(\alpha),1}, \ldots, x_{\pi(\alpha),\#\text{in}^(\pi(\alpha))})\).

We say that two stable witch curves \((2T, (x_\rho)), (z_\alpha))\), \((2T', (x'_\rho)), (z'_\alpha))\) are isomorphic if there is an isomorphism of tree-pairs \(2f: 2T \to 2T'\) and functions \(V_{\text{int}}(T_s) \to G_1: \rho \mapsto \phi_\rho\) and \(V_{\text{comp}}(T_b) \to G_2: \alpha \mapsto \psi_\alpha\) such that:

\[
z'_{\alpha}(i,j) = \psi_\alpha(z_{\alpha, i,j}) \quad \forall \ \alpha \in V_{\text{comp}}(T_b), \quad x'_{\rho, i} = \phi_\rho(x_\rho, i) \quad \forall \ \rho \in V_{\text{int}}(T_s),
\]

\[
p(\psi_\alpha) = \phi_{\pi(\alpha)} \quad \forall \ \alpha \in V_{\text{comp}}(T_b).
\]

We denote the collection of stable witch curves of type \(n\) by \(SWC_n\), and we define the moduli space \(\overline{\mathcal{M}_n}\) of stable witch curves of type \(n\) to be the set of isomorphism classes of stable witch curves of this type. For any tree-pair \(2T\) of type \(n\), define the corresponding strata \(SWC_{n,2T} \subset SWC_n\), \(\overline{\mathcal{M}_{n,2T}} \subset \overline{\mathcal{M}_n}\) to be the set of all stable witch curves (resp. isomorphism classes thereof) of the form \((2T, (x_\rho)), (z_\alpha))\). We say that a stable witch curve is smooth if its underlying tree-pair \(2T\) has the property that \(V_{\text{int}}(T_s)\) and \(V_{\text{comp}}(T_b)\) each contain only one element; we denote a smooth stable witch curve by the pair \((x, z) \in \mathbb{R}^r \times \mathbb{R}^{\#in(n)}\) associated to the roots of \(T_s\) resp. \(T_b\). \(\triangle\)
If \((2T, (x_{\rho}), (z_{\alpha}))\) is a stable witch curve and \(\alpha \in V_{\text{comp}}(T_b)\), \(\beta \in V_{\text{comp}}(T_b) \cup \{\lambda_{ij}\}_{i,j}\) are distinct, then we define \(z_{\alpha\beta} \in \mathbb{R}^2 \cup \{\infty\}\) like so: Define \((\alpha = \gamma_1, \gamma_2, \ldots, \gamma_k = \beta)\) to be the elements of \(V_{\text{comp}}(T_b) \cup \{\lambda_{ij}\}_{i,j}\) through which the path from \(\alpha\) to \(\beta\) passes. If \(\gamma_2\) is closer to the root than \(\alpha\), then we define \(z_{\alpha\beta} := \infty\). If \(\gamma_2\) is the \(i\)-th incoming neighbor of \(\alpha\) and \(\gamma_3\) is the \(j\)-th incoming neighbor of \(\gamma_2\), then we define \(z_{\alpha\beta} := z_{\alpha,i,j}\). For distinct \(\rho \in V_{\text{int}}(T_s)\), \(\sigma \in V(T_s)\), we define \(x_{\rho\sigma}\) similarly. For \(\alpha, \beta\) as above, we define \(x_{\alpha\beta} := \rho(z_{\alpha\beta})\). For \(\alpha \in V_{\text{comp}}(T_b)\) and \(\rho \in V(T_s) \setminus \{\pi(\alpha)\}\), we define \(x_{\alpha\rho}\) like so: for \(\alpha \in V_{\text{comp}}^2(T_b)\), set \(x_{\alpha\rho} := x_{\pi(\alpha)\rho}\). For \(\alpha \in V_{\text{comp}}(T_b)\), define \(x_{\alpha\rho}\) like so:

\[
x_{\alpha\rho} := \begin{cases} 
  x_{\alpha,1}, & \lambda_i \in (T_s)_{\pi(\alpha)}, \\
  \infty, & \text{otherwise}.
\end{cases}
\]

Finally, for any \(\alpha \in V_{\text{comp}}(T_b)\), we denote \(z_{\alpha,\lambda_{\infty}} := \infty\), \(x_{\alpha,\lambda_{\infty}} := \infty\); here \(\lambda^T_{2b}\) and \(\lambda^T_{\infty}\) are formal quantities, rather than vertices in \(T_b\) resp. \(T_s\), which represent the fact that the root of the bubble tree and seam tree should be thought of as carrying a single “output” marked point.

We define the set of nodal points and set of special points of any interior vertex \(\alpha\) like so:

\[
Z^\text{node}_\alpha := \{z_{\alpha\beta} \mid \beta \in V_{\text{comp}}(T_b) \setminus \{\alpha\}\} \subset \mathbb{R}^2 \cup \{\infty\}, \\
Z^\text{spec}_\alpha := \{z_{\alpha\beta} \mid \beta \in V_{\text{comp}}(T_b) \cup \{\lambda_{ij}\}_{i,j} \setminus \{\alpha\}\} \subset \mathbb{R}^2 \cup \{\infty\}.
\]

Before we define Gromov convergence for tree-pairs, we need two preliminaries: a way to express the property that two vertices in \(V_{\text{comp}}(T_b)\) correspond to two spheres attached via a nodal point, and a notion of surjection for tree-pairs. The first notion is straightforward: for any tree-pair \(T_b \to T_s\), we say that \(\alpha, \beta \in V_{\text{comp}}(T_b) \cup \{\lambda_{ij}\}_{i,j}\) are contiguous if the path from \(\alpha\) to \(\beta\) consists of \(\alpha\), \(\beta\), and a third vertex (necessarily in \(V_{\text{seam}}(T_b)\)). The second notion is less obvious. If \(2T'\) is the result of making a single move on \(2T\) (in the sense of §3.1, [Bo1]), then there are evident maps \(T'_b \to T_b, T'_s \to T_s\). Composing these maps inductively, we see that for any tree-pairs with \(2T' < 2T\), there are induced maps \(T'_b \to T_b, T'_s \to T_s\). We call any map obtained in this fashion a tree-pair surjection. Note that for any tree-pair surjection \(2T' \to 2T\), the restriction \(T'_s \to T_s\) to seam trees is an RRT surjection.

**Definition 2.3.** A sequence \((2T', (x'_{\rho}), (z'_{\alpha})) \in \text{SWC}_n\) is said to **Gromov-converge** to \((2T, (x_{\rho}), (z_{\alpha}))\) if the following conditions hold:

- \((T'_b, (x'_{\rho}))\) Gromov-converges to \((T_b, (x_{\rho}))\) via some \(f' : T_s \to T'_s\) and \((\phi'_\rho) \subset G_1\).
- For \(\nu\) sufficiently large, there is a tree-pair surjection \(2f' : 2T \to 2T'\) covering \(f' : T_s \to T'_s\) and a collection of reparametrizations \((\psi'_{\alpha})_{\alpha \in V_{\text{comp}}(T_b)} \subset G_2\) such that the following hold:
  - **(Restriction)** For \(\alpha \in V_{\text{comp}}^2(T_b)\), \(p(\psi'_{\alpha}) = \phi'_{\pi(\alpha)}\).
  - **(Rescaling)** If \(\alpha, \beta \in V_{\text{comp}}(T_b)\) are contiguous, and if \(\nu_j\) is a subsequence such that \(f'_{\nu_j}(\alpha) = f'_{\nu_j}(\beta)\), then the sequence \(\psi'_{\alpha\beta} := (\psi'_{\alpha})^{-1} \circ \psi'_{\beta}\) converges to \(z_{\alpha\beta}\) u.c.s. away from \(z_{\beta\alpha}\).
  - **(Special Point)** If \(\alpha \in V_{\text{comp}}(T), \beta \in V_{\text{comp}}(T_b) \cup \{\lambda_{ij}\}_{i,j}\) are contiguous, and if \(\nu_j\) is a subsequence such that \(f'_{\nu_j}(\alpha) \neq f'_{\nu_j}(\beta)\), then:

\[
  z_{\alpha\beta} = \lim_{j \to \infty} (\psi'_{\alpha})^{-1}\left(z'_{\nu_j}\left(f'_{\nu_j}(\alpha), f'_{\nu_j}(\beta)\right)\right).
\]

If \((2T', (x'_{\rho}), (z'_{\alpha}))\) Gromov-converges to \((2T, (x_{\rho}), (z_{\alpha}))\) via \((\phi'_\rho)\) and \((\psi'_{\alpha})\), and \((\phi'_{\rho})_{\rho \in V_{\text{int}}(T_s)} \subset G_1\) and \((\psi'_{\alpha})_{\alpha \in V_{\text{comp}}(T'_b)} \subset G_2\) are any sequences of reparametrizations satisfying

\[
p(\psi'_{\alpha}) = \phi'_{\pi(\alpha)} \quad \forall \, \alpha \in V_{\text{comp}}^2(T_b),
\]


then \((2T^\nu, (\tilde{\phi}_\nu^\alpha(x^\nu)), (\tilde{\psi}_\nu^\alpha(z^\nu)))\) Gromov-converges to \((T, (x_\rho), (z_\alpha))\) via \((\tilde{\phi}^\nu_{f_\nu^\alpha(\rho)} \circ \tilde{\phi}^\nu_{f_\nu^\beta(\rho)})_{\rho \in V_{\text{int}}(T_\alpha)}\) and \((\tilde{\psi}^\nu_{f_\nu^\alpha(\alpha)} \circ \tilde{\psi}^\nu_{f_\nu^\beta(\alpha)})_{\alpha \in V_{\text{comp}}(T_\beta)}\).

The following lemma shows that Gromov convergence in \(\mathcal{M}_n\) actually implies a priori stronger versions of the (rescaling) and (special point) axioms; for simplicity, we state it in the case that the surjection \(2T \to 2T^\nu\) is fixed.

**Lemma 2.4.** Suppose that \((\tilde{2T}, (x_\rho^\nu), (z_\alpha^\nu)) \subset \text{SWC}_n\) Gromov-converges to \((2T, (x_\rho), (z_\alpha))\) via \(2f: 2T \to 2T, (\phi^\nu),\) and \((\psi^\nu)\). Then the following properties hold.

1. **(Rescaling’)** For any distinct \(\alpha, \beta \in V_{\text{comp}}(T_\beta)\) with \(f_\beta(\alpha) = f_\beta(\beta)\), the sequence \(\psi^\nu_{\alpha\beta}\) converges to \(z_{\alpha\beta}\) u.c.s. away from \(z_{\beta\alpha}\).

2. **(Special Point’)** For any \(\alpha \in V_{\text{comp}}(T_\beta)\), \(\beta \in V_{\text{comp}}(T_\beta) \cup \{\lambda_{ij}\}_{i,j}\) with \(f_\beta(\alpha) \neq f_\beta(\beta)\), the equality \(z_{\alpha\beta} = \lim_{\nu \to \infty} (\psi^\nu_{\alpha\beta})^{-1}(z_{f_\beta(\alpha)f_\beta(\beta)})\) holds.

**Proof.** (Rescaling’). Denote by \((\alpha = \gamma_1, \ldots, \gamma_k = \beta)\) the vertices in \(V_{\text{comp}}(T_\beta)\) through which the path from \(\alpha\) to \(\beta\) passes, and note that \(f_\beta(\alpha) = f_\beta(\beta)\) implies \(f_\beta(\alpha = \gamma_1) = f_\beta(\gamma_2) = \ldots = f_\beta(\gamma_k = \beta)\). (Indeed, it can be shown by induction that since \(2f\) is a tree-pair surjection, \(f_\beta\) is an RRT homomorphism.) We prove the claim by induction on \(k\). The \(k = 2\) case is exactly (rescaling). Suppose that we have proven the claim up to and including some particular \(k\); we now must prove the claim in the case that the path from \(\alpha\) to \(\beta\) has length \(k + 1\). By assumption, \(\psi^\nu_{\alpha\gamma_1}\) converges to \(z_{\alpha\gamma_1}\) u.c.s. away from \(z_{\gamma_1\alpha}\) and \(\psi^\nu_{\gamma_1\beta}\) converges to \(z_{\gamma_1\beta}\) u.c.s. away from \(z_{\beta\gamma_1}\). The fact that \((\gamma_1, \ldots, \gamma_{k+1})\) does not intersect itself implies \(z_{\gamma_k\beta} \neq z_{\gamma_k\alpha}\), \(z_{\alpha\gamma_k} = z_{\beta\gamma_k}\), and \(z_{\beta\gamma_k} = z_{\beta\alpha}\), so it follows that \(\psi^\nu_{\alpha\beta} = \psi^\nu_{\alpha\gamma_k} \circ \psi^\nu_{\gamma_k\beta}\) converges to \(z_{\alpha\beta}\) u.c.s. away from \(z_{\beta\alpha}\).

(Special Point’). Denote by \((\alpha = \gamma_1, \ldots, \gamma_k = \beta)\) the vertices in \(V_{\text{comp}}(T_\beta) \cup \{\lambda_{ij}\}_{i,j}\) through which the path from \(\alpha\) to \(\beta\) passes. We prove the claim by induction on \(k\). The \(k = 2\) case is exactly (special point). Suppose that we have proven the claim up to and including some particular \(k\); we now must prove the claim in the case that the path from \(\alpha\) to \(\beta\) includes \(k + 1\) elements of \(V_{\text{comp}}(T_\beta) \cup \{\lambda_{ij}\}_{i,j}\). If \(f_\beta(\gamma_{k-1}) = f_\beta(\beta)\), then the claim follows from the inductive hypothesis:

\[(8) \quad z_{\alpha\beta} = z_{\alpha\gamma_{k-1}} = \lim_{\nu \to \infty} (\psi^\nu_{\alpha\gamma_{k-1}})^{-1}(z_{f_\beta(\gamma_{k-1})}) \leq \lim_{\nu \to \infty} (\psi^\nu_{\alpha\gamma_{k-1}})^{-1}(z_{f_\beta(\alpha)f_\beta(\beta)}) = \lim_{\nu \to \infty} (\psi^\nu_{\alpha\beta})^{-1}(z_{f_\beta(\alpha)f_\beta(\beta)}).\]

Otherwise, we use the inductive hypothesis and the inequality \(z_{\gamma_{k-1}\beta} \neq z_{\gamma_{k-1}\alpha}\):

\[(9) \quad z_{\alpha\beta} = z_{\alpha\gamma_{k-1}} = \lim_{\nu \to \infty} (\psi^\nu_{\alpha\gamma_{k-1}})^{-1}((\psi^\nu_{\gamma_{k-1}\beta})^{-1}(z_{f_\beta(\gamma_{k-1})f_\beta(\beta)})) \leq \lim_{\nu \to \infty} (\psi^\nu_{\alpha\beta})^{-1}(z_{f_\beta(\alpha)f_\beta(\beta)}) = \lim_{\nu \to \infty} (\psi^\nu_{\alpha\beta})^{-1}(z_{f_\beta(\alpha)f_\beta(\beta)}).\]

Next, we prove a strengthening of (special point) in the case of a Gromov-convergent sequence of smooth stable witch curves.

**Lemma 2.5.** Suppose that \((x^\nu, z^\nu) \subset \text{SWC}_n\) Gromov-converges to \((2T, (x_\rho), (z_\alpha))\) via \((\phi^\nu)\) and \((\psi^\nu)\). For any \(\alpha \in V_{\text{comp}}(T_\beta)\) and \(\lambda_i \in V(T_\alpha)\), the equality \(x_{\alpha\lambda_i} = \lim_{\nu \to \infty} d((\psi^\nu_{\lambda_i})^{-1}(x^\nu))\) holds.

**Proof.** Step 1: If \(\alpha \in V_{\text{comp}}(T_\beta)\) is closer to the root than \(\beta \in V_{\text{comp}}(T_\beta)\), and we denote \(((\psi^\nu_{\lambda_i})^{-1} \circ \psi^\nu_{\alpha\lambda_i})(z) = a^\nu z + b^\nu\), then \(\lim_{\nu \to \infty} a^\nu = \infty\).

By (rescaling’), \((\psi^\nu_{\lambda_i})^{-1} \circ \psi^\nu_{\alpha\lambda_i}\) converges to \(\infty\) u.c.s. away from \(z_{\alpha\beta} \in \mathbb{R}^2\). The equality \(\lim_{\nu \to \infty} a^\nu = \infty\) follows.

Step 2: We prove the claim in the case that \(\alpha\) is further from the root from a vertex \(\beta \in V_{\text{comp}}(T_\beta)\) and closer to the root than a vertex \(\gamma \in V_{\text{comp}}(T_\beta)\).
Choose \( \beta, \gamma \) to be the closest vertices to \( \alpha \) having the properties just mentioned. First, fix \( i \) with the property that \( \lambda_i \) does not lie in \((T_s)_{\pi(\alpha)}\). Note that (SPECIAL POINT') and the stability of \( 2T \) implies that for some \( i', j \), the equality \( \lim_{\nu \to \infty} p((\psi_{\alpha}^\nu)^{-1}(z_{i'j})) = x_{\alpha,1} \in \mathbb{R} \) holds, hence
\[
\lim_{\nu \to \infty} p((\psi_{\alpha}^\nu)^{-1})(x_i) = x_{\alpha,1} \in \mathbb{R}.
\]

(Restriction) and (SPECIAL POINT') yield the inequality
\[
\lim_{\nu \to \infty} p((\psi_{\beta}^\nu)^{-1})(x_i) = x_{\pi(\beta)\lambda_i} < x_{\pi(\beta)\lambda_i'} = \lim_{\nu \to \infty} p((\psi_{\beta}^\nu)^{-1})(x_{i'}) = x_{\alpha,1}.
\]

Step 1, along with the last two displayed (in)equality, yields \( \lim_{\nu \to \infty} p((\psi_{\alpha}^\nu)^{-1})(x_i) = x_{\alpha,1} \).

**Step 3:** We prove the claim when \( \alpha \) does not satisfy the hypothesis of Step 2.

In this case, \( \pi(\alpha) \) must lie in \( \{\lambda_i\} \cup \{\alpha_{\text{root}}\} \). Suppose \( \pi(\alpha) = \alpha_{\text{root}} \). If \( r = 1 \), the claim clearly holds. Otherwise, choose \( \gamma \) to be the element of \( V_{\geq 2}^{\comp}(T_b) \) closest to \( \alpha \). For every \( i \), (RESTRICTION) and (SPECIAL POINT') yield the containment
\[
\lim_{\nu \to \infty} p((\psi_{\gamma}^\nu)^{-1})(x_i) = x_{\gamma, \lambda_i} \in \mathbb{R}.
\]

There exist \( i', j \) such that the equality \( \lim_{\nu \to \infty} p((\psi_{\alpha}^\nu)^{-1}(z_{i'j})) = x_{\alpha,1} \) holds, so Step 1 and the last displayed containment imply the claim. A similar argument can be made in the case that \( \pi(\alpha) \) is a leaf of \( T_s \).

\[
\Box
\]

2.2. Gromov compactness for \( \overline{2MC_n} \). This subsection is devoted to establishing the following result, which will later be used to show that the topology on \( \overline{2MC_n} \) is compact.

**Theorem 2.6.** Any sequence \( (2T^\nu, (x_\rho^\nu), (z_{\alpha}^\nu)) \subset SWC_n \) has a Gromov-convergent subsequence.

The central idea of the proof already occurs when the witch curves in the sequence are smooth. In this case, we prove this theorem inductively, on the total number of marked points. The idea is that when we add a new marked point to a Gromov-convergent sequence of smooth witch curves, there are four possibilities, illustrated in the following figure and made formal in Lemma 2.7.
Lemma 2.7. Suppose that a sequence $(x^r, z^r) \subset SDT_r$ of smooth stable witch curves Gromov-converges to $(2T, (x_\rho), (z_\alpha))$ via $\phi^r_\rho$ and $\psi^r_{\alpha}$, and that $(\zeta^r \in \mathbb{R}^2 \setminus z^r)$ is a sequence with the property that

$$
(13) \quad \zeta_\alpha := \lim_{r \to \infty} (\psi^r_\alpha)^{-1}(\zeta^r) \in \mathbb{R}^2 \cup \{\infty\}
$$

exists for every $\alpha \in V_{\text{comp}}(T_b)$. Then exactly one of the following conditions holds:

1. There exists a (unique) vertex $\alpha \in V_{\text{comp}}(T_b)$ such that $\zeta_\alpha \in \mathbb{R}^2 \setminus Z_{\alpha}^\text{spec}$.
2. There exists a (unique) contiguous pair $\alpha \in V_{\text{comp}}(T_b)$, $\lambda_{ij}$ such that $\zeta_\alpha = z_\alpha \lambda_{ij}$.
3. The root $\alpha_{\text{root}}$ has $\zeta_{\alpha_{\text{root}}} = \infty$.

Proof. We imitate the proof of Lemma 5.3.4, [McDSa].

Step 1: We prove the implication

$$
(14) \quad \alpha, \beta \in V_{\text{comp}}(T_b), \ z_\alpha \neq z_\beta \implies \zeta_\beta = z_\beta \beta.
$$

This follows from the (rescaling') part of Lemma 2.4 and the convergence of $(\psi^r_\alpha)^{-1}(\zeta^r)$ to $\zeta_\alpha \neq z_\alpha \beta$:

$$
(15) \quad \zeta_\beta = \lim_{r \to \infty} (\psi^r_\alpha)^{-1}(\zeta^r) = \lim_{r \to \infty} \psi^r_\beta \beta_\alpha((\psi^r_\alpha)^{-1}(\zeta^r)) = z_\beta \beta.
$$

Step 2: We prove the lemma.

We begin by proving that the four cases are mutually exclusive.

1. Suppose that $\alpha, \beta$ satisfy the condition in (3), and fix $\gamma \in V_{\text{comp}}(T_b) \setminus \{\alpha, \beta\}$. If $\gamma$ lies in $(T_b)_{\alpha \beta}$, then the inequality $\zeta_\gamma = z_\beta \beta \beta \neq z_\gamma \gamma$ and Step 1 imply $\zeta_\gamma = z_\gamma \gamma$, so none of (1), (2a), and (2b) hold. Otherwise, the inequality $\zeta_\alpha = z_\alpha \beta \beta \neq z_\alpha \gamma$ and Step 1 imply $\zeta_\gamma = z_\gamma \alpha$, so none of (1), (2a), and (2b) hold.
2. Suppose that (2b) holds. Step 1 implies that every $\alpha \in V_{\text{comp}}(T_b)$ has $\zeta_\alpha = \infty$, so neither (1) nor (2a) holds.
3. Suppose that $\alpha, \lambda_{ij}$ satisfy (2a). Step 1 implies that every $\beta \in V_{\text{comp}}(T_b) \setminus \{\alpha\}$ has $\zeta_\beta = z_\beta \alpha$, so (1) does not hold.

Next, we prove uniqueness in (1), (2a), and (3). In (1) and (2a), this is an immediate consequence of Step 1. To prove uniqueness in (3), suppose for a contradiction that $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are distinct pairs satisfying (3). Switching $\alpha$ and $\beta$ if necessary, we may assume that the paths from $\alpha'$ resp. $\beta'$ to $\alpha$ pass through $\beta$. Similarly, we may assume that the paths from $\alpha$ resp. $\beta$ to $\beta'$ pass through $\alpha'$. The inequality $\zeta_{\beta'} = z_{\beta'} \neq z_{\beta'} \beta'$ and Step 1 imply $\zeta_{\beta'} = z_{\beta' \beta'}$. This, together with the inequality $z_{\beta' \beta} \neq z_{\beta' \beta}$, imply $\zeta_{\beta'} = z_{\beta' \beta} \beta' \neq z_{\beta' \beta'}$, in contradiction with the assumption.

Finally, we show that at least one of these cases holds. Suppose that (1), (2a), and (2b) do not hold; we must show that (3) holds. The assumption implies that for every $\alpha \in V_{\text{comp}}(T_b)$, there exists a contiguous $\beta \in V_{\text{comp}}(T_b)$ with $\zeta_\alpha = z_\alpha \beta$. Define a path like so: First, choose any $\alpha_1 \in V_{\text{comp}}(T_b)$ and a contiguous vertex $\alpha_2 \in V_{\text{comp}}(T_b)$ with $\zeta_1 = z_1 \alpha_2$. Inductively continue this path by defining $\alpha_{k+1} \in V_{\text{comp}}(T_b)$ to be the vertex in $V_{\text{comp}}(T_b)$ contiguous to $\alpha_k$ satisfying $\zeta_k = z_{\alpha_k \alpha_{k+1}}$. The quotient of $T_b$ obtained by identifying each element of $V_{\text{comp}}(T_b)$ with its incoming neighbors is again a tree, and $(\alpha_1, \alpha_2, \ldots)$ is an infinite path in this quotient, so there must exist $k$ such that $\alpha_{k+2} = \alpha_k$. Then $\zeta_k = z_{\alpha_k \alpha_{k+1}}$, and $\zeta_{k+1} = z_{\alpha_k+1} \alpha_k$, so $\alpha_k, \alpha_{k+1}$ satisfy the condition in (3). □

Proof of Thm. 2.7. Step 1: For any $r \geq 2$ and $n \in \mathbb{Z}_{\geq 0}$ with $|n| = 1$, there is a bijection $\text{SDT}_r \to \text{SWC}_n$ which identifies Gromov-convergent sequences with Gromov-convergent sequences.
Fix \( n \) as above, where the only nonzero entry is \( n_{i_0} = 1 \). We begin by identifying stable RRTs with \( r \) leaves with stable tree-pairs of type \( n \).

- Given a stable RRT \( T \) with \( r \) leaves, we define a stable tree-pair \( 2T \) of type \( n \) like so: set \( T_s := T \). Define \( T_b \) by first setting \( T' \) to consist of all vertices in the path \([\alpha_{\text{root},T}, \lambda_{i_0}]\) and all incoming neighbors of these vertices; now, define \( T_b \) to be the result of inserting a dashed edge at \( \lambda_{i_0} \), and at every interior vertex of \( T' \) except the root. Here is an illustration of this process, in which an RRT with 5 leaves is sent to a tree-pair of type \((0,0,0,1,0)\):

- Given a stable tree-pair \( 2T \) of type \( n \), send it to its seam tree \( T_s \).

The fact that these maps are inverses follows from the stability condition on stable tree-pairs.

We now enhance this bijection to an identification of \( \mathcal{SDT}_r \) with \( \mathcal{SWC}_n \). Fix \( (T, (x_0)) \in \mathcal{SDT}_r \). Define \( 2T \) as above. Define \( (2T, (x_0), (z_0)) \in \mathcal{SWC}_n \) like so: for \( \alpha \in V_{\text{comp}}(T_b) \), choose \( i_0 \) with the property that the \( i_0 \)-th incoming neighbor \( \beta \) of \( \alpha \) has \( \text{in}(\beta) \neq 0 \), and set \( z_0 := \{(x_{\pi(\alpha)i_0}, 0)\} \).

It is straightforward to check that this indeed defines a bijection, and that it identifies Gromov-convergent sequences in \( \mathcal{SDT}_r \) with Gromov-convergent subsequences in \( \mathcal{SWC}_n \).

**Step 2:** If \((x^\nu, z^\nu) \subset \mathcal{SWC}_n\) is a sequence of smooth stable witch curves, then it has a Gromov-convergent subsequence.

We establish this claim by induction on \(|n|\). The base case \( \mathbf{n} = (2) \) follows from the fact that any two elements of \( \mathcal{SWC}_{(2)} \) are isomorphic, while the base case \( r \geq 2, |n| = 1 \) follows from Step 1.

Next, say that the claim has been proven up to some \(|\mathbf{n}| = a \geq 1\). Fix a sequence \((x^\nu, z^\nu) \in \mathcal{SWC}_n \) with \(|\mathbf{n}| = a\). Without loss of generality, we may choose \( i_0 \) such that the inequality \( z_{i_0,n_{i_0}} \geq z_{ij} \) holds for all \( i, j \). Define \( \tilde{\mathbf{n}} := (n_1, \ldots, n_{i_0-1}, n_{i_0}-1, n_{i_0+1}, \ldots, n_r) \) and \( \tilde{z}^\nu := \{z_{ij} \mid (i, j) \neq (i_0, n_{i_0})\} \).

By the inductive hypothesis, we may assume that \((x^\nu, z^\nu) \subset \mathcal{SWC}_n \) Gromov-converges to some \((2T', (x_0), (z_0))\). Set \( \tilde{z}^\nu := z_{i_0,n_{i_0}} \). Passing to a subsequence, we may assume that for every \( \alpha \in V_{\text{comp}}(T_b) \), the limit \( \zeta_\alpha := \lim_{\nu \to \infty} (\psi^\nu_\alpha)^{-1}(\tilde{z}^\nu) \) exists. We may now apply Lemma 2.7.

We divide the rest of the proof of this step into cases, depending on which case of Lemma 2.7 holds.

1. Fix \( \alpha \in V_{\text{comp}}(T_b) \) with the property \( \zeta_\alpha \in \mathbb{R}^2 \setminus Z_{\alpha_{\text{spec}}} \). We begin by defining a tree-pair \( 2T^\text{new} \) of type \( \mathbf{n} \). Set \( T^\text{new}_s := T_s \). Enlarge \( T_b \) to \( T^\text{new}_b \) like so: define \( T' \) to be the subtree of \( T_s \) consisting of the path \([\pi(\alpha), \lambda_{i_0}]\), together with all incoming neighbors of all vertices in this path. Insert a dashed edge at every interior vertex in \( T' \), including at its root. Add an incoming dashed edge to the vertex in \( T' \) corresponding to \( \lambda_{i_0} \). Finally, graft this tree into \( T_b \) by identifying its root with the vertex \( \beta \in \text{in}(\alpha) \) with the property that the path from \( \pi(\alpha) \) to \( \lambda_{i_0} \) passes through \( \pi(\beta) \); declare that the element we have just added to \( \text{in}(\beta) \) is maximal in \( \text{in}(\beta) \).

Next, we define a collection of reparametrizations \((\chi^\nu_\rho)\), where \( \gamma_\rho \) denotes the vertex we added to \( T^\text{new}_b \) corresponding to \( \rho \in \text{f}(\alpha), \lambda_{i_0} \). We characterize \( \chi^\nu_\beta \) by the following equations:

\[
p(\chi^\nu_\beta) = \phi^\nu_\rho, \quad (\chi^\nu_\beta)^{-1}(x_{i_0}, y_{i_0, n_{i_0}+1}) = ((\phi^\nu_\rho)^{-1}(x_{i_0}), 0).
\]

The sequence \((x^\nu, z^\nu)\) converges to \((2T^\text{new}, (x_0), (z_0))\) via \((\phi^\nu_\rho)\) and \((\psi^\nu_\alpha)\).
Suppose \( \zeta \). This case is similar to (2b), so we do not include all the details. As illustrated in Fig. 1, to obtain the new tree of quilted spheres we introduce a sphere with one seam between all the interior vertices of \( T \) at all the interior vertices of \( T' \) besides the root, and at the vertex of \( T' \) corresponding to \( \lambda_{i_0} \). Complete the construction of \( T'_{\text{new}} \) by introducing \( \alpha' := T'_{\text{root}} := \alpha' \), connect \( \alpha' \) by an incoming solid edge to a new vertex \( \alpha'' \), and connect both \( T_{\text{root}} \) and the root of \( T' \) to \( \alpha'' \) by dashed edges.

Next, we need to define reparametrizations \( \chi_{\alpha''}^{\nu} \), where \( \beta \nu \) is the vertex in \( T' \) corresponding to \( \rho \in [\alpha_{\text{root}}, T_i, T'_{\text{root}}] \). The definition of \( \chi_{\alpha''}^{\nu} \) is similar to the construction of the reparametrizations in (1). To define \( \chi_{\alpha''}^{\nu} \), choose \( i_1 \in [1, r] \) and \( j_1 \in [1, n_{i_1}] \) and characterize \( \chi_{\alpha''}^{\nu} \) by the equations:

\[
(\chi_{\alpha''}^{\nu})^{-1}(z_{i_0,n_{i_0}+1}) = (0,1), \quad (\chi_{\alpha''}^{\nu})^{-1}(z_{i_1,j_1}) = (0,0).
\]

This case is similar to (2b), so we do not include all the details. As illustrated in Fig. 1, to obtain the new tree of quilted spheres we introduce a sphere with one seam between the spheres corresponding to \( \alpha \) and \( \beta \), and the construction of the reparametrizations \( \chi^{\nu} \) corresponding to this new sphere is somewhat different than the constructions appearing in (2b); we therefore concentrate on this detail.

Suppose that \( \alpha, \beta \in V_{\text{comp}}(T_b) \) are contiguous and have the properties \( \zeta_\alpha = z_{\alpha \beta} \) and \( \zeta_\beta = z_{\beta \alpha} \). Assume w.l.o.g. that \( \beta \) is further from the root than \( \alpha \) is, which implies \( z_{\beta \alpha} = \infty \). We must construct a sequence \( (\chi^{\nu}_\alpha) \subset G_1 \) satisfying these conditions:

\begin{align*}
(\chi_1) & \quad (\chi^{\nu}_\alpha)^{-1}(z_{i_0,n_{i_0}+1}) = (0,1) \text{ for every } \nu. \\
(\chi_2) & \quad (\psi_\alpha^{\nu})^{-1} \circ \chi^{\nu} \text{ converges to 0 u.c.s. away from } \infty \text{ and } (\psi_\beta^{\nu})^{-1} \circ \chi^{\nu} \text{ converges to } \infty \text{ u.c.s. away from } z_{\alpha \beta}. \\
\end{align*}

To do so, we first note that we may assume \( z_{\alpha \beta} = 0 \); otherwise, set \( \xi \in G_2 \) to be translation by \( z_{\alpha \beta} \) and replace \( \psi_\alpha \), \( \psi_\beta \) and \( z_\alpha \) by \( \xi^{-1}(\psi_\alpha) \), \( \xi^{-1}(\psi_\beta) \) and \( \xi^{-1}(z_\alpha) \). The sequence \( \psi_\alpha^{\nu} = (\psi_\alpha^{\nu})^{-1} \circ \psi_\beta^{\nu} \) converges to 0 u.c.s. away from \( \infty \), so we may write \( \psi_\alpha^{\nu}(z) = \psi_\alpha^{\nu}(z) + b^{\nu} \) for \( a^{\nu} \in \mathbb{R} \), \( b^{\nu} \in \mathbb{R}^2 \) both converging to 0. Set \( w^{\nu} := (\psi_\alpha^{\nu})^{-1}(z_{i_0,n_{i_0}+1}) \); then we have

\[
\lim_{\nu \to \infty} w^{\nu} = \zeta_\beta = z_{\beta \alpha} = \infty, \quad \lim_{\nu \to \infty} a^{\nu} w^{\nu} = \lim_{\nu \to \infty} \psi_\alpha^{\nu}(w^{\nu}) = \zeta_\alpha = z_{\alpha \beta} = 0.
\]

(Restriction) and Lemma 2.2 imply that \( \text{Re}(w^{\nu}) \) is bounded; just as we assumed \( z_{\alpha \beta} = 0 \), we may therefore assume \( w^{\nu} = (0, e^{\nu}) \) for \( e^{\nu} \in \mathbb{R} \). Moreover, the inequality \( y_{i_0,n_{i_0}+1} \geq y_{i'j'} \) for all \( i', j' \) implies that \( e^{\nu} \) is eventually positive. The functions \( \chi^{\nu} \) defined by \( \chi^{\nu}(z) := (z - b^{\nu})/(a^{\nu} e^{\nu}) \) therefore lie in \( G_2 \) and satisfy these conditions:

\begin{align*}
(\xi_1) & \quad \chi^{\nu} \text{ converges to } \infty \text{ u.c.s. away from } 0. \\
(\xi_2) & \quad \xi^{\nu} \circ \psi_\alpha^{\nu} = (z \mapsto z/e^{\nu}) \text{ converges to } 0 \text{ u.c.s. away from } \infty.
\end{align*}
(3) The following identity holds for every $\nu$:

$$\left(\chi^{\nu'} \circ (\psi^{\nu}_\alpha)^{-1}\right)(z_{i_0,n_0+1}^\nu) = \left(\chi^{\nu'} \circ (\psi^{\nu}_\alpha)^{-1}\left((\psi^{-1}_\beta)^{-1}(z_{i_0,n_0+1}^\nu)\right)\right) = (\chi^{\nu'}) \circ (\psi^{\nu}_\alpha)(w^{\nu}) = (0, 1).$$

Now set

$$\chi^{\nu} := \psi^{\nu}_\alpha \circ (\chi^{\nu'}).$$

Then (3) implies $(\chi^{\nu'})^{-1}(z_{i_0,n_0+1}^\nu) = (0, 1)$, which establishes (1). (1) and (2) imply (2), so we have constructed a suitable rescaling sequence $(\chi^{\nu})$.

$$\square$$

2.3. Limits in $\overline{\mathcal{M}}_n$ are unique.

**Lemma 2.8.** Suppose that $(x^{\nu}, z^{\nu}) \subset \text{SWC}_n$ is a sequence of smooth stable witch curves that Gromov-converges to $(2T, (x^{\nu}_\nu), (z^{\nu}_\nu))$ via $(\phi^{\nu}_\nu)$ and $(\psi^{\nu}_\alpha)$, that $\alpha, \beta \in V_{\text{comp}}(T_b)$ are contiguous vertices, and that $(\chi^{\nu}) \subset G_2$ is a sequence with the property

$$\left(\psi^{\nu}_\alpha\right)^{-1} \circ \chi^{\nu} \to z_{\alpha\beta} \text{ u.c.s. away from } w_1, \quad (\psi^{\nu}_\beta)^{-1} \circ \chi^{\nu} \to z_{\beta\alpha} \text{ u.c.s. away from } w_2$$

for some $w_1, w_2 \in \mathbb{R}^2 \cup \{\infty\}$. If $\alpha$ is closer to the root than $\beta$, then $w_1 = \infty$; otherwise, $w_2 = \infty$. If $\lambda_{ij}$ lies in $(T_b)_{\alpha\beta}$, then $(\chi^{\nu})^{-1}(z_{ij}^{\nu})$ converges to $w_2$; otherwise, $(\chi^{\nu})^{-1}(z_{ij}^{\nu})$ converges to $w_1$. If $\nu_{\beta\lambda_{ij}} \neq \nu_{\beta\lambda_{ij}}$, then $p(\chi^{\nu})^{-1}(x_{ij}^{\nu})$ converges to $p(w_2)$; otherwise, $p(\chi^{\nu})^{-1}(x_{ij}^{\nu})$ converges to $p(w_1)$.

**Proof.** To prove the first claim, it suffices by symmetry to consider the case that $\alpha$ is closer to the root than $\beta$. In this case we have $z_{\alpha\beta} \in \mathbb{R}^2$, so the first equation in (23) and the equality $(\psi^{\nu}_\alpha^{-1} \circ \chi^{\nu}) = \infty$ imply $w_1 = \infty$.

To prove the second claim, it suffices by symmetry to consider the case that $\lambda_{ij}$ lies in $(T_b)_{\alpha\beta}$. Suppose that $(\chi^{\nu})^{-1}(z_{ij}^{\nu})$ does not converge to $w_2$. Passing to a subsequence, we may assume that there exists a compact set $K \ni w_2$ with $(\chi^{\nu})^{-1}(z_{ij}^{\nu}) \in K$ for all $\nu$. By hypothesis, we have $(\psi^{\nu}_\beta)^{-1}(z_{ij}^{\nu}) = (\psi^{\nu}_\beta)^{-1}(\chi^{\nu})(\chi^{\nu})^{-1}(z_{ij}^{\nu}) \to z_{\beta\alpha}$. On the other hand, (special point) implies that $(\psi^{\nu}_\beta)^{-1}(z_{ij}^{\nu})$ converges to $z_{\beta\lambda_{ij}}$; hence $z_{\beta\alpha} = z_{\beta\lambda_{ij}}$. This contradicts the assumption.

A similar argument proves the third claim. $$\square$$

**Lemma 2.9.** Suppose that $(x^{\nu}, z^{\nu}) \subset \text{SWC}_n$ and $(\psi^{\nu}_\alpha) \subset G_2$ are as in Lemma 2.8 and suppose that $(\chi^{\nu}) \subset G_2$ is a sequence of reparametrizations with the following properties:

(a) For every $i, j$ the limits $\xi_i := \lim_{\nu \to \infty} p(\chi^{\nu})^{-1}(z_{ij}^{\nu})$, $\xi_{ij} := \lim_{\nu \to \infty} (\chi^{\nu})^{-1}(z_{ij}^{\nu})$ exist.

(b) Define $Y_s := \{\xi_i\} \cup \{\infty\}$ and $Y_b := \{\xi_{ij}\} \cup \{\infty\}$. Either $\#Y_b \geq 3$, or $\#Y_b = 2$ and $\#Y_s \geq 3$.

Then there exists $\alpha \in V_{\text{comp}}(T_b)$ such that $(\psi^{\nu}_\alpha)^{-1} \circ \chi^{\nu}$ has a subsequence which converges uniformly to an element of $G_2$.

**Proof.** Step 1: If $(\tau^{\nu}) \subset G_2$ has no convergent subsequence, then it has a subsequence converging to $w$ u.c.s. away from $w'$ for some $w, w' \in \mathbb{R}^2 \cup \{\infty\}$.

Write $\tau^{\nu}(z) = a^{\nu}z + b^{\nu}$. After passing to a subsequence, we may assume that the limits

$$\lim_{\nu \to \infty} a^{\nu} = a^{\infty} \in \mathbb{R}_{\geq 0} \cup \{\infty\}, \quad \lim_{\nu \to \infty} b^{\nu} = b^{\infty} \in \mathbb{R}^2 \cup \{\infty\}$$

exist. It suffices to prove the claim for either $(\tau^{\nu})$ or $(\tau^{\nu})^{-1}$; replacing $\tau^{\nu}$ by $(\tau^{\nu})^{-1}$ if necessary, we may assume $a^{\infty} \in \mathbb{R}_{>0}$. By hypothesis, it cannot be that the containments $a^{\infty} \in \mathbb{R}_{>0}$, $b^{\infty} \in \mathbb{R}$ both hold. If $b^{\infty} = \infty$, then $\tau^{\nu}$ converges to $\infty$ u.c.s. away from $\infty$. If $a^{\infty} = 0$ and $b^{\infty} \in \mathbb{R}$, then $\tau^{\nu}$ converges to $b^{\infty}$ u.c.s. away from $\infty$.

**Step 2:** If $\tau^{\nu} := (\psi^{\nu}_\alpha)^{-1} \circ \chi^{\nu}$ has no uniformly-convergent subsequence, then after passing to a subsequence, $\tau^{\nu}$ converges to $w$ u.c.s. away from $w'$ for some $w \in Z^{\text{node}}_\alpha$ and $w' \in Y$. 

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By Step 1, we may pass to a subsequence such that $\tau^\nu$ converges to $w$ u.c.s. away from $w'$ for some $w, w' \in \mathbb{R}^2 \cup \{\infty\}$; it remains to show $w \in Z^{node}_{\alpha}$, $w' \in Y$. Suppose $w \notin Z^{node}_{\alpha}$. Then at most one $\lambda \in \{\lambda_{ij}\} \cup \{\lambda_\infty\}$ satisfies $z_{\alpha\lambda} = w$. In fact, the inequality $\# Y_b \geq 2$ implies that there is exactly one such $\lambda$. For simplicity, assume that there is either (i) no such $\lambda$, or (ii) the only such $\lambda = \lambda_\infty$; the case $w = z_{\alpha\lambda_{ij}}$ is similar. The reparametrizations $(\tau^\nu)^{-1}$ converge to $w'$ u.c.s. away from $\infty$ and $(\psi_{\alpha}^\nu)^{-1}(z_{ij}^\nu)$ converges to $z_{\alpha\lambda_{ij}} \neq \infty$ by (rescaling'), so for every $i, j$ we have

$$\xi_{ij} = \lim_{\nu \to \infty} (\chi^\nu)^{-1}(z_{ij}^\nu) = \lim_{\nu \to \infty} (\tau^\nu)^{-1}(\psi_{\alpha}^\nu)^{-1}(z_{ij}^\nu) = w'.$$

This and the inequality $\# Y_b \geq 2$ implies that $\# Y_b = 2$; also, (ii) must hold, i.e. $w = z_{\alpha\lambda_\infty} = \infty$.

Next, note that the facts $w \notin Z^{node}_{\alpha}$ and $w = \infty$ imply $\alpha = \alpha_{\text{root}}$, hence

$$\lim_{\nu \to \infty} (\psi_{\alpha}^\nu)^{-1}(z_{ij}^\nu) = z_{\alpha\text{root}\lambda_{ij}} \in \mathbb{R}^2 \ \forall i, j, \quad \lim_{\nu \to \infty} p(\psi_{\alpha}^\nu)^{-1}(x_{ij}^\nu) = x_{\alpha\text{root}\lambda_{i}} \in \mathbb{R} \ \forall i.$$  

The first of these equations, together with the convergence $\tau^\nu \to \infty$ u.c.s. away from $w'$, implies $p(\tau^\nu)^{-1} \to p(w')$ u.c.s. away from $\infty$. (Indeed, this is clear when $w' \in \mathbb{R}^2$. If $w' = \infty$, write $(\tau^\nu)^{-1}(z) = a^\nu z + b^\nu$; after passing to a subsequence, we may assume $a^\nu \to a^\infty \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $b^\nu \to b^\infty \in \mathbb{R}_\infty \cup \{\infty\}$. The convergence $(\tau^\nu)^{-1} \to \infty$ away from $\infty$ point implies that either $a^\infty = \infty, b^\infty = \infty$, or both. The first equation in (26) implies that we cannot have both $a^\infty \in \mathbb{R}_{\geq 0}$ and $b^\infty = \infty$: if so, every $\xi_{ij} = \lim_{\nu \to \infty} (\chi^\nu)^{-1}(z_{ij}^\nu) = \lim_{\nu \to \infty} (\tau^\nu)^{-1}(\psi_{\alpha}^\nu)^{-1}(z_{ij}^\nu)$ would equal $\infty$, contradicting the hypothesis $\# Y_b \geq 2$.) It follows that, for any $i$, we have

$$\zeta_{i} = \lim_{\nu \to \infty} p(\chi^\nu)^{-1}(x_{ij}^\nu) = \lim_{\nu \to \infty} p(\tau^\nu)^{-1}(p(\psi_{\alpha}^\nu)^{-1}(x_{ij}^\nu)) = p(w').$$

Therefore $\# Y_s \leq 2$. Together with the equality $\# Y_b = 2$, we have derived a contradiction.

A similar argument shows $w' \in Y_b$.

Step 3: If the conclusion of Lemma 2.9 does not hold, then there is a contradiction.

Suppose that no $\alpha \in V_{\text{comp}}(T_b)$ has the property that a subsequence of $(\psi_{\alpha_1}^\nu)^{-1} \circ \chi^\nu$ converges uniformly; we will construct a non-self-intersecting infinite sequence $(\alpha_1, \alpha_2, \ldots)$ in $V_{\text{comp}}(T_b)$ with every consecutive pair $\alpha_i, \alpha_{i+1}$ contiguous, a contradiction. We choose $\alpha_1$ to be any element of $V_{\text{comp}}(T_b)$. By Step 2, we may pass to a subsequence such that $(\psi_{\alpha_1}^\nu)^{-1} \circ \chi^\nu$ converges to $w_1 \in Z^{node}_{\alpha_1}$ u.c.s. away from $w'_1 \in Y$; define $\alpha_2 \in V_{\text{comp}}(T_b)$ to be the vertex contiguous to $\alpha_1$ with $w_2 = z_{\alpha_1\alpha_2}$. Inductively defining our sequence in this fashion, we obtain $(\alpha_1, \alpha_2, \ldots)$ with the property that $(\psi_{\alpha_i}^\nu)^{-1} \circ \chi^\nu$ converges to $z_{\alpha_i\alpha_{i+1}}$ u.c.s. away from $w'_i$. This path does not intersect itself: Indeed, assume that $\alpha_i = \alpha_{i+1}$ for some $i$. Then Lemma 2.8 with $\alpha := \alpha_1, \beta := \alpha_2$ implies $\# Y_b \leq 2$ and $\# Y_s \leq 2$, a contradiction. We have therefore constructed an infinite sequence in $V_{\text{comp}}(T_b)$ with each consecutive pair contiguous, a contradiction.

Theorem 2.10. Suppose that $(2T^\nu, (x_{ij}^\nu), (z_{ij}^\nu)) \subset SWC_n$ Gromov-converges to two stable witch curves $(2T, (x_\rho), (z_\rho))$ and $(2T, (x_\delta), (z_\delta))$. Then $(2T, (x_\rho), (z_\rho))$ and $(2T, (x_\delta), (z_\delta))$ are isomorphic.

Proof. Step 1: If $(x^\nu, z^\nu) \subset SWC_n$ is a sequence of smooth stable disk trees Gromov-converging to $(2T, (x_\rho), (z_\rho))$ and $(2T, (x_\delta), (z_\delta))$, then $(2T, (x_\rho), (z_\rho))$ and $(2T, (x_\delta), (z_\delta))$ are isomorphic.

Step 1A: After passing to a subsequence, the stable disk trees $(T_s, (x_\rho))$ and $(\bar{T}_s, (\bar{x}_\rho))$ are isomorphic.

This is a consequence of the Hausdorffness of $\mathcal{M}_r$. We may therefore assume $(T_s, (x_\rho)) = (\bar{T}_s, (\bar{x}_\rho))$. 


Step 1B: After passing to a subsequence, there is a unique bijection $g: V_{\text{comp}}(T_b) \to V_{\text{comp}}(\tilde{T}_b)$ such that the uniform limits exist.

\[(\psi_{\tilde{\alpha}})^{-1} \circ \psi_{\alpha} \quad \text{exists. Applying this reasoning with } T_b \]

Step 1b: \[(\psi_{\tilde{\alpha}})^{-1} \circ \psi_{\alpha} = ((\psi_{\tilde{\alpha}})^{-1} \circ \psi_{\alpha}) \circ ((\psi_{\alpha})^{-1} \circ \psi_{\alpha}) \]

converges to $z_{\beta \alpha}$ u.c.s. away from a single point. By applying this argument at every interior vertex of $\tilde{T}_b$, we obtain a uniquely-determined function $g: V_{\text{comp}}(\tilde{T}_b) \to V_{\text{comp}}(T_b)$ and a subsequence of our original data such that the uniform limit exists. Applying this reasoning with $T_b$ and $\tilde{T}_b$ interchanged shows that $g$ is invertible.

Step 1C: The reparametrizations $\chi_{\alpha}$ satisfy

\[(31) \quad \tilde{z}_{g(\alpha)\beta} = \chi_{\alpha}^{-1}(z_{\alpha\beta}), \quad \tilde{z}_{g(\alpha)\beta} = \chi_{\alpha}^{-1}(z_{\alpha \beta}), \quad \tilde{x}_{g(\alpha)\beta} = p(\chi_{\alpha})^{-1}(x_{\alpha \beta}). \]

The first equation follows from (32) and (special point∗):

\[(32) \quad \chi_{\alpha}^{-1}(z_{\alpha \beta}) = \chi_{\alpha}^{-1}\left(\lim_{\nu \to \infty} (\psi_{\alpha})^{-1}(z_{ij})\right) = \lim_{\nu \to \infty} (\psi_{\alpha})^{-1}(z_{ij}) = \tilde{z}_{g(\alpha)\beta} \]

A similar deduction proves the third equation. The second follows from (rescaling∗), the convergence of $\psi_{\alpha \beta}$ to $z_{\alpha \beta}$ u.c.s. away from $z_{\beta \alpha}$, and the convergence of $\tilde{z}_{g(\alpha)\beta}$ to $\tilde{z}_{g(\alpha)\beta}$ u.c.s. away from $\tilde{z}_{g(\alpha)\beta}$. Indeed, choosing $z \in \mathbb{R}^2 \setminus \{z_{\beta \alpha}, \chi_{\beta}(\tilde{z}_{g(\alpha)\beta})\}$, we have:

\[(33) \quad \chi_{\alpha}^{-1}(z_{\alpha \beta}) = \chi_{\alpha}^{-1}\left(\lim_{\nu \to \infty} (\psi_{\alpha})(z)\right) = \lim_{\nu \to \infty} (\psi_{\alpha})^{-1}(\tilde{z}_{g(\alpha)\beta})(z) = \lim_{\nu \to \infty} (\psi_{\alpha})^{-1}(\chi_{\beta}^{-1}(z)) = \tilde{z}_{g(\alpha)\beta} \]

Finally, the fourth equation follows from applying $p$ to the second equation.

Step 1D: We extend $g$ to a bijection $V(T_b) \to V(\tilde{T}_b)$.

We showed in Step 1b that $g: V_{\text{comp}}(T_b) \to V_{\text{comp}}(\tilde{T}_b)$ is a bijection. We now extend $g$ to a bijection between $V(T_b)$ and $V(\tilde{T}_b)$. First, set $g(\lambda_{ij}) := \tilde{\lambda}_{ij}$. Next, suppose that $\alpha$ is an element of $V_{\text{comp}}^2(T_b)$. By (restriction), $\in(\alpha)$ is in bijection with the limit set $\lim_{\nu \to \infty} p((\psi_{\alpha})^{-1}) (x_{ij}) \mid 1 \leq i \leq 1 \leq r \}$. It follows from Step 1b that there is an bijection

\[(34) \quad \{ \lim_{\nu \to \infty} p((\psi_{\alpha})^{-1}) (x_{ij}) \mid 1 \leq i \leq 1 \leq r \} = \{ \lim_{\nu \to \infty} p((\psi_{\alpha})^{-1}) (x_{ij}) \mid 1 \leq i \leq r \} \}

It follows from Lemma 2.5 that $g(\alpha)$ lies in $V_{\text{comp}}^2(\tilde{T}_b)$, so we can identify $\in(\alpha)$ and $\in(g(\alpha))$. A similar argument shows that if $\alpha$ lies in $V_{\text{comp}}^1(T_b)$, then $g(\alpha)$ lies in $V_{\text{comp}}^1(\tilde{T}_b)$, hence we can identify the incoming neighbor of $\alpha$ with that of $g(\alpha)$. We have now extended $g$ to a bijection $g: V(T_b) \to V(\tilde{T}_b)$.

Step 1E: Two vertices $\alpha, \beta$ in $V_{\text{comp}}(T_b)$ are contiguous if and only if (1) there is no $\gamma \in \{\lambda_{ij}\} \cup \{\lambda_{\alpha \beta}\}$ satisfying both $z_{\alpha \gamma} = z_{\alpha \beta}$ and $z_{\beta \gamma} = z_{\beta \alpha}$, and (2) there is no $\delta \in \{\lambda_{ij}\} \cup \{\lambda_{\alpha \beta}\}$ satisfying both
$x_{a\delta} = x_{a\beta}$ and $x_{\beta\delta} = x_{\beta\alpha}$. Vertices $\alpha \in V_{\text{comp}}(T_b)$, $\lambda_{ij}$ are contiguous if and only if there is no $\gamma \in V_{\text{comp}}(T_b)$ with $z_{\alpha \lambda_{ij}} \neq z_{\gamma \gamma}$.

Suppose that $\alpha, \beta \in V_{\text{comp}}(T_b)$ are contiguous, and fix $\gamma \in \{ \lambda_{ij} \} \cup \{ \lambda_{ij} \}^{\ast}$. Switching $\alpha$ and $\beta$ if necessary, we may assume that $\gamma$ lies in $(T_b)_{\alpha \beta}$. Then $z_{\beta \alpha} \neq z_{\beta \gamma}$. A similar argument produces $\delta \in \{ \lambda_i \} \cup \{ \lambda_{ij} \}^{\ast}$ with $x_{\beta \alpha} \neq x_{\beta \delta}$.

Next, we prove the contrapositive of the converse: Suppose that $\alpha, \beta \in V_{\text{comp}}(T)$ are not contiguous, and define $(\alpha = \gamma_1, \gamma_2, \ldots, \gamma_k = \beta)$ to be the vertices in $V_{\text{comp}}(T_b)$ through which the path from $\alpha$ to $\beta$ passes. Suppose that $\gamma_2$ lies in $V_{\text{comp}}^1$. Define $(\gamma_2 = \delta_1, \delta_2, \ldots, \delta_\ell)$ to be a path that starts at $\gamma_2$, passes first through $\delta_2 \notin \{ \alpha, \beta \}$, and terminates at a vertex in $\{ \lambda_{ij} \} \cup \{ \lambda_{ij} \}^{\ast}$ (possible due to the stability of $2T$). Then $z_{\delta_2 \gamma} = z_{\delta_2 \alpha}$ and $z_{\beta \delta} = z_{\beta \alpha}$. On the other hand, suppose that $\gamma_2$ lies in $V_{\text{comp}}(T_b)$. Define $(\gamma_2 = \rho_1, \rho_2, \ldots, \rho_\ell)$ to be a path in $T_b$ that starts at $\gamma_2$, passes first through $\rho_2 \notin \{ \pi(\alpha), \pi(\beta) \}$, and terminates at a vertex in $\{ \lambda_i \} \cup \{ \lambda_{ij} \}^{\ast}$. Then $x_{\rho_\ell \gamma} = x_{\rho_\ell \alpha}$, $x_{\beta \rho_\ell} = x_{\beta \alpha}$.

A similar, simpler argument proves the second assertion in Step 1e.

**Step 1F:** We show that $f$ extends to an isomorphism of RRTs, then complete Step 1.

It remains to prove the following facts:

- $g(\alpha_{\text{root}}) = \alpha_{\text{root}}$, where we denote $\alpha_{\text{root}} := \alpha_{\text{root}}^{T_b}$ and $\alpha_{\text{root}} := \alpha_{\text{root}}^{T_b}$.
- For $\alpha \in V_{\text{comp}}(T_b)$, $g$ induces a bijection from $\text{in}(\alpha)$ to $\text{in}(g(\alpha))$.
- For $\alpha, \beta \in V_{\text{comp}}(T_b)$ with $\beta$ an incoming neighbor of the $i$-th incoming neighbor of $\alpha$, $g(\beta)$ is an incoming neighbor of the $i$-th incoming neighbor of $g(\alpha)$.
- For $\alpha \in V_{\text{seam}}(T_b)$ and $\lambda_{ij} \in \text{in}(\alpha)$, $g(\lambda_{ij})$ lies in $\text{in}(g(\alpha))$.
- $g$ respects the ribbon tree structure of $T_b$ and $\bar{T}_b$.

First, we show $g(\alpha_{\text{root}}) = \bar{\alpha}_{\text{root}}$. Fix $\bar{\alpha} \in V_{\text{comp}}(\bar{T}_b) \setminus \{ g(\alpha_{\text{root}}) \}$, and write $\bar{\alpha} = g(\alpha)$ for some $\alpha \in V_{\text{comp}}(T_b)$. Step 1c implies $\bar{z}_{g(\alpha_{\text{root}})}^{\bar{\alpha}} = \bar{\alpha}^{-1} \bar{z}_{\alpha_{\text{root}}} \neq \infty$. Since $\bar{z}_{g(\alpha_{\text{root}})}^{\bar{\alpha}}$ is finite for every $\bar{\alpha} \in V_{\text{comp}}(\bar{T}_b) \setminus \{ g(\alpha_{\text{root}}) \}$, we must have $g(\alpha_{\text{root}}) = \bar{\alpha}_{\text{root}}$.

The second bullet is an immediate consequence of the construction of $g$ on $V_{\text{seam}}(T_b)$.

Next, fix $\alpha, \beta \in V_{\text{comp}}(T_b)$ with $\beta$ an incoming neighbor of the $i$-th incoming neighbor of $\alpha$. By Step 1d, there is no $\gamma \in \{ \lambda_{ij} \} \cup \{ \lambda_{ij} \}^{\ast}$ satisfying both $z_{\alpha \gamma} = z_{\beta \alpha}$ and $z_{\gamma \gamma} = z_{\gamma \alpha}$, nor is there $\delta \in \{ \lambda_i \} \cup \{ \lambda_{ij} \}^{\ast}$ satisfying both $x_{\alpha \delta} = x_{\alpha \beta}$ and $x_{\beta \delta} = x_{\beta \alpha}$. Together with Step 1c, it follows that there is no $\bar{\gamma} \in \{ \bar{\lambda}_{ij} \} \cup \{ \bar{\lambda}_{ij} \}^{\ast}$ with both $\bar{z}_{g(\alpha \gamma)}^{\bar{\alpha}} = \bar{z}_{g(\alpha \alpha) g(\gamma)}$ and $\bar{z}_{g(\beta \gamma)}^{\bar{\beta}} = \bar{z}_{g(\beta \beta) g(\gamma)}$, nor is there $\bar{\delta} \in \{ \bar{\lambda}_i \} \cup \{ \bar{\lambda}_{ij} \}^{\ast}$ with both $\bar{x}_{g(\alpha \delta)}^{\bar{\alpha}} = \bar{x}_{g(\alpha \alpha) g(\delta)}$ and $\bar{x}_{g(\beta \delta)}^{\bar{\beta}} = \bar{x}_{g(\beta \beta) g(\delta)}$. Step 1d now implies that $g(\alpha)$ and $g(\beta)$ are contiguous, and another application of Step 1d implies that $g(\bar{\alpha})$ is an incoming neighbor of the $i$-th incoming neighbor of $g(\alpha)$.

A similar argument to the previous paragraph shows that for $\alpha \in V_{\text{seam}}(T_b)$ and $\lambda_{ij} \in \text{in}(\alpha)$, $g(\lambda_{ij})$ lies in $g(\alpha)$.

It follows from Step 1c that for any $\alpha \in V(T_b)$, $g$ induces an order-preserving bijection from $\text{in}(\alpha)$ to $\text{in}(g(\alpha))$.

**Step 2:** The general case.

We begin by noting that for any $(2T, (x_\rho), (z_\alpha)) \in SWC_n$ and $\beta \in V_{\text{comp}}(T_b)$, we can associate a smooth stable witch curve. This association depends on whether $\beta$ lies in $V_{\text{comp}}^1(T_b)$ or $V_{\text{comp}}^2(T_b)$. If $\beta$ lies in $V_{\text{comp}}^1(T_b)$, we associate $(x_{\beta1}, z_\beta)$. Otherwise, we associate $(x_{\beta}, z_\beta)$.

**Step 2A:** If $(2T', (x'_\rho), (z'_\alpha))$ Gromov-converges to $(2T, (x_\rho), (z_\alpha))$ and $(x', z')$ is the sequence of smooth stable witch curves associated as in the previous paragraph to a vertex $\beta \in V_{\text{comp}}^1(T_b)$, then $(x', z')$ converges to a restriction of $(2T, (x_\rho), (z_\alpha))$. 


The only nontrivial part of this step is to spell out which restriction of $2T$ to use. Denote by $f: 2T \to 2T'$ the tree-pair surjection involved in the Gromov convergence of $(2T', (x'_\rho), (z'_\rho))$ to $(2T, (x_\rho), (z_\rho))$. First, suppose $\beta$ lies in $V^\geq_{\text{comp}}(T'_b)$. Define a tree-pair $2T|_\beta$ like so: $T_s|_\beta$ is the preimage under $f_s$ of $\pi(\beta)$ and its incoming neighbors. $T_b|_\beta$ is the preimage under $f_b$ of $\beta$, its incoming neighbors, and the incoming neighbors of its incoming neighbors. Then $2T|_\beta$ is a tree-pair, and it is straightforward to show that the smooth tree-pairs $(x^\nu, z^\nu)$ associated to $\beta$ Gromov-converge to the restriction of $(2T, (x_\rho), (z_\rho))$ to $2T|_\beta$. The same result can be proven in the case that $\beta$ lies in $V^1_{\text{comp}}(T_b)$: in this case, set $T_s|_\beta$ to be a single vertex.

**Step 2b:** We establish the general case.

We are now ready to prove Thm. 2.10 Since there are only finitely many isomorphism classes of tree-pairs of type $n$, we may pass to a subsequence and assume that $2T'' \equiv 2T'$ and that all the tree-pair surjections $2T \to 2T''$ and $2T \to 2T'$ coincide with maps $f: 2T \to 2T'$ and $\tilde{f}: \hat{2T} \to 2T'$. Since $(2T', (x'_\rho), (z'_\rho))$ Gromov-converges to $(2T, (x_\rho), (z_\rho))$, the smooth stable witch curves associated to $\beta$ Gromov-converge to the restriction of $(2T, (x_\rho), (z_\rho))$ to $2T|_\beta$, as in Step 2a. Similarly, $(x'^\nu(\beta), z'^\nu(\beta))$ Gromov-converges to the restriction of $(2T, (\tilde{x}_\rho), (\tilde{z}_\rho))$ to $2T'$. By Step 1, these two restrictions are isomorphic. Since this holds for every $\beta \in V_{\text{comp}}(T_b), (2T, (x_\rho), (z_\rho))$ and $(\tilde{T}, (\tilde{x}_\rho), (\tilde{z}_\rho))$ are isomorphic.

2.4. The definition and properties of the topology on $\overline{\mathcal{M}}_n$. Recall that if $X$ is a set and $C \subset X \times X^\mathbb{N}$ is an arbitrary collection of sequences and “limits”, we can define a topology $\mathcal{U}(C) \subset 2^X$ in which the open sets are those subsets $U \subset X$ having the property that for every $(x_0, (x_n)) \in C$ with $x_0 \in U$, $x_n$ is eventually in $U$. The following lemma gives sufficient conditions for the convergent sequences in $\mathcal{U}(C)$ to coincide with $C$.

**Lemma 2.11** (Lemma 5.6.5, [McDSa]). Let $X$ be a set and $C \subset X \times X^\mathbb{N}$ be a collection of sequences in $X$ that satisfies the property that if $(x_0, (x_n)) \in C$ and $(y_0, (x_n)) \in C$, then $x_0 = y_0$. Suppose that for every $x \in X$ there exists a constant $\epsilon_0(x) > 0$ and a collection of functions $X \to [0, \infty]: x' \mapsto \mu_\epsilon(x, x')$ for $0 < \epsilon < \epsilon_0(x)$ satisfying the following conditions.

(a) If $x \in X$ and $0 < \epsilon < \epsilon_0(x)$, then $\mu_\epsilon(x, x) = 0$.

(b) If $x \in X$, $0 < \epsilon < \epsilon_0(x)$, and $(x_n)_n \in X^\mathbb{N}$, then

$$
(35) \quad (x, (x_n)_n) \in C \iff \lim_{n \to \infty} \mu_\epsilon(x, x_n) = 0.
$$

(c) If $x \in X$, $0 < \epsilon < \epsilon_0(x)$, and $(x', (x_n)_n) \in C$, then

$$
(36) \quad \mu_\epsilon(x, x') < \epsilon \implies \limsup_{n \to \infty} \mu_\epsilon(x, x_n) \leq \mu_\epsilon(x, x').
$$

Then $C = \mathcal{C}(\mathcal{U}(C))$. Moreover, the topology $\mathcal{U}(C)$ is first countable and Hausdorff.

We will construct a topology on $\overline{\mathcal{M}}_n$ by using this lemma. To begin, we define the functions $\mu_\epsilon: \overline{\mathcal{M}}_n \to [0, \infty]$. 15
Definition 2.12. For any two stable witch curves \((2T, (x_\rho), (z_\alpha))\), \((\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))\) of type \(n\) and for any \(\epsilon > 0\), define a nonnegative real number \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha)))\) like so:

\[
\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))) := \min_{2f : 2T \to \tilde{2}T} \inf_{(\phi_\rho)_{\rho \in V_{Int}(T)} \in V_{Comp}(T_\mu)} \mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha)); 2f, (\phi_\rho), (\psi_\alpha)),
\]

where in the first line we take the minimum over all tree-pair surjections \(2f : 2T \to \tilde{2}T\) and the infimum over all tuples \((\phi_\rho) \subset G_1\) and \((\psi_\alpha) \subset G_2\) satisfying \(\psi_\alpha = \phi_\alpha(\alpha)\) for every \(\alpha \in V_{Comp}(T_\mu)\), and where in the second line we use the distance metrics on \(\mathbb{R} \cup \{\infty\}\) and \(\mathbb{R}^2 \cup \{\infty\}\) induced by identifying these spaces with round spheres. By convention, if there is no tree-pair surjection \(2T \to \tilde{2}T\), we set \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))) := \infty\). △

Remark 2.13. It is an immediate consequence of the definition that for any \((2T, (x_\rho), (z_\alpha))\), \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (2T, (x_\rho), (z_\alpha)))\) descends to \(\overline{\mathcal{M}_g}\).

Remark 2.14. The quantity \(\mu_\epsilon\) should be compared with a similar quantity, \(\rho_\epsilon\), which plays the analogous role in the definition of the Gromov topology on \(\overline{\mathcal{M}_n}\). (Compare also the analogous quantity used in §5, [McDSa] to define the topology on the space of stable maps.) For any two stable disk trees \((T, (x_\rho)), (\tilde{T}, (\tilde{x}_\rho))\) with \(r\) leaves and for \(\epsilon > 0\), \(\rho_\epsilon((T, (x_\rho)), (\tilde{T}, (\tilde{x}_\rho)))\) is defined like so:

\[
\rho_\epsilon((T, (x_\rho)), (\tilde{T}, (\tilde{x}_\rho))) := \min_{f : T \to \tilde{T}} \inf_{(\phi_\rho)_{\rho \in V_{Int}(T)}} \rho_\epsilon((T, (x_\rho)), (\tilde{T}, (\tilde{x}_\rho)); f, (\phi_\rho)),
\]

where in the first line we take the minimum over all RRT surjections \(f : T \to \tilde{T}\).

Lemma 2.15. Fix \((2T, (x_\rho), (z_\alpha)) \in SWC_n\). Then the following hold for every \(\epsilon > 0\):

- **(Convergence)** A sequence \((2T^\nu, (x_\rho^\nu), (z_\alpha^\nu)) \subset SWC_n\) Gromov-converges to \((2T, (x_\rho), (z_\alpha))\) if and only if \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (2T^\nu, (x_\rho^\nu), (z_\alpha^\nu)))\) converges to 0.
- **(Triangle)** If \((\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha)) \in SWC_n\) satisfies \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))) < \epsilon\) and the sequence \((2T^\nu, (x_\rho^\nu), (z_\alpha^\nu))\) Gromov-converges to \((\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))\), then

\[
\limsup_{\nu \to \infty} \mu_\epsilon((2T, (x_\rho), (z_\alpha)), (2T^\nu, (x_\rho^\nu), (z_\alpha^\nu))) \leq \mu_\epsilon((2T, (x_\rho), (z_\alpha)), (\tilde{2}T, (\tilde{x}_\rho), (\tilde{z}_\alpha))).
\]

Proof. **(Convergence)** If \((2T^\nu, (x_\rho^\nu), (z_\alpha^\nu))\) Gromov-converges to \((2T, (x_\rho), (z_\alpha))\) then it follows from (Rescaling‘) and (Special Point‘), and the analogous properties for Gromov convergence of stable disk trees, that \(\mu_\epsilon((2T, (x_\rho), (z_\alpha)), (2T^\nu, (x_\rho^\nu), (z_\alpha^\nu)))\) converges to 0.
Conversely, suppose μν((2T,(xρ),(zα))),(2Tν,(xρ),(zα′)) converges to 0. Then it is the case that for ν large enough there is an tree-pair surjection 2fν: 2T → 2Tν and tuples (φρ),(ψρ) with p(ψρ) = φα(α) for α ∈ Vcomp(Tb) such that the following inequality holds:

\[ \mu(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)) ; 2fν, (φρ),(ψρ)) \leq \mu(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)) + 2^{-ν}. \]

Since there are only finitely many tree-pair surjections with domain 2T, we may assume all the tree-pairs 2Tν are equal to a single 2T and all the maps 2fν: 2T → 2T are equal to a single 2f. First, we verify (rescaling). Fix contiguous α, β ∈ Vcomp(Tb) with fβ(α) = fβb(β); without loss of generality we may assume α is closer to the root than β, so zαβ = ∞. The convergence μν → 0 implies that (ψν)-1 ◦ ψν converges to zαβ uniformly on \( \mathbb{R}^2 \setminus B_1(∞) \), hence (ψν)-1 ◦ ψν converges to zαβ u.c.s. away from ∞. From this it follows that (ψν)-1 ◦ ψν converges to ∞ u.c.s. away from zαβ, so we have established (rescaling).

The (special point) requirement obviously holds. Finally, the inequality

\[ \rho\varepsilon_\nu((T, (xρ),(zα)), (T, (xρ),(zα))), (2T, (xρ),(zα)), (2Tν, (xρ),(zα′)), (2fν, (φρ),(ψρ))) \leq \mu(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)) + 2^{-ν}. \]

implies that the left-hand side converges to 0, so \( (T, (xρ),(zα)) \), \( (2T, (xρ),(zα)) \), \( (2Tν, (xρ),(zα′)), (2fν, (φρ),(ψρ))) \) Gromov-converges to \( (T, (xρ),(zα)) \) via \( fν \) and \( (φρ),(ψρ) \). We may conclude that \( (2Tν, (xρ),(zα′)) \) Gromov-converges to \( (2T, (xρ),(zα)) \).

(Triangle) The inequality \( \mu\varepsilon_\nu((2T,(xρ),(zα)),(2Tν,(xρ),(zα′)),2g,(χρ),(ξα)) < ϵ \) implies that there exists a tree-pair surjection \( 2g: 2T \to 2Tν \) and tuples \( (χρ),(ξα) \) such that \( (2Tν, (xρ),(zα′)) \) Gromov-converges to \( (2T, (xρ),(zα)) \).

\[ \mu(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)) ; 2fν, (φρ),(ψρ)) \leq \mu\varepsilon_\nu((2T,(xρ),(zα)),(2Tν,(xρ),(zα′)),2g,(χρ),(ξα)) < ϵ. \]

It follows that for every pair α, β ∈ Vcomp(Tb) with gα(α) ≠ gβ(β) we have

\[ d(zαβ, gα(β)) < ϵ; \]

similarly, for every ρ, σ ∈ Vint(Tb) with gs(α) ≠ gσ(α) we have

\[ d(xρσ, xρσ) < ϵ. \]

Now suppose that \( (2Tν, (xρ),(zα′)) \) Gromov-converges to \( (2T, (xρ),(zα)) \) via tree-pair surjections \( 2fν: 2T → 2Tν \) and reparametrizations \( (φρ),(ψρ) \). To prove (triangle), it suffices to prove the following equality:

\[ \mu(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)) ; 2fν, (φρ),(ψρ)) = \lim_{ν→∞} \mu\varepsilon(2T,(xρ),(zα)),(2Tν,(xρ),(zα′)),2fν, (φρ),(ψρ)) \]

Since there are only finitely many tree-pair surjections with domain 2T, we may assume \( 2Tν \equiv 2Tν' \) and \( 2fν \equiv 2f: 2T \to 2Tν' \). For any distinct α, β ∈ Vcomp(Tb) with gα(α) = gβ(β) we have \( ξβα = ξβα \equiv (ψνβ)(α)(ξβα) \equiv (ψνβ)(α)(ξβα) \), hence

\[ \sup_{(\mathbb{R}^2 \setminus B_1(∞)) \setminus B_1(zαβ)} d((ψνβ)(α)(ξβα), zβα) = \sup_{(\mathbb{R}^2 \setminus B_1(∞)) \setminus B_1(zαβ)} d((ψνβ)(α)(ξβα), zβα). \]

Similarly, for distinct ρ, τ ∈ Vint(Tb) with gρ(ρ) = gτ(τ), we have

\[ \sup_{(\mathbb{R}^2 \setminus B_1(∞)) \setminus B_1(xρσ)} d((ψνβ)(α)(ξβα), zβα) = \sup_{(\mathbb{R}^2 \setminus B_1(∞)) \setminus B_1(xρσ)} d((ψνβ)(α)(ξβα), zβα). \]

If α, β ∈ Vcomp(Tb) have gα(α) ≠ gβ(β) and \( fβb(α) = fβb(β) \), then (rescaling) implies that \( (ψνb)(α) \) converges to \( zβα \) u.c.s. away from \( zαβ \), hence by (41).
\( (\psi^\nu_{gb(\beta)} \circ \xi^\nu_{gb(\beta)})^{-1} \circ (\psi^\nu_{gb(\alpha)} \circ \xi^\nu_{gb(\alpha)}) \) converges to \( \xi^{-1}_\beta(z_{gb(\beta)} g_{gb(\alpha)}) \) uniformly on \( (\mathbb{R}^2 \cup \{ \infty \}) \setminus B_i(z_{\alpha \beta}) \).

We therefore have
\[
(46) \quad d\left( \xi^{-1}_\beta(z_{gb(\beta)} g_{gb(\alpha)}), z_{\beta \alpha} \right) = \lim_{\nu \to \infty} \sup_{(\mathbb{R}^2 \cup \{ \infty \}) \setminus B_i(z_{\alpha \beta})} d\left( (\xi^\nu_{gb(\beta)})^{-1} \circ \xi^\nu_{gb(\alpha)}, z_{\beta \alpha} \right).
\]

Similarly, it follows from (42) that if \( \rho, \sigma \in V_{\text{int}}(T_s) \) have \( g_s(\alpha) \neq g_s(\sigma) \) and \( f_s(g_s(\rho)) = f_s(g_s(\sigma)) \), we have
\[
(47) \quad d\left( \chi^{-1}_\sigma(z_{gs(\sigma)} g_s(\rho)), x_{\sigma \rho} \right) = \lim_{\nu \to \infty} \sup_{(\mathbb{R}^2 \cup \{ \infty \}) \setminus B_i(x_{\sigma \rho})} d\left( (\chi^\nu_\sigma)^{-1} \circ \chi^\nu_{\rho}, x_{\sigma \rho} \right).
\]

Finally, if \( \alpha, \beta \in V_{\text{comp}}(T_b) \) have \( f_b(g_b(\alpha)) \neq f_b(g_b(\beta)) \), then (SPECIAL POINT) implies the convergence of \( (\psi^\nu_{gb(\beta)})^{-1}(z_{gb(\beta)} g_{gb(\alpha)}) \) to \( z_{gb(\beta)} g_{gb(\alpha)} \), hence
\[
(48) \quad d\left( \xi^{-1}_\beta(z_{gb(\beta)} g_{gb(\alpha)}), z_{\beta \alpha} \right) = \lim_{\nu \to \infty} d\left( (\psi^\nu_{gb(\beta)} \circ \xi^\nu_{gb(\beta)})^{-1}(z_{gb(\beta)} g_{gb(\alpha)}) \right. \left. \circ \xi^\nu_{gb(\alpha)}, z_{\beta \alpha} \right).
\]

Similarly, if \( \rho, \sigma \in V_{\text{int}}(T_s) \) have \( f_s(g_s(\rho)) \neq f_s(g_s(\sigma)) \), then we have
\[
(49) \quad d\left( \chi^{-1}_\sigma(z_{gs(\sigma)} g_s(\rho)), x_{\sigma \rho} \right) = \lim_{\nu \to \infty} d\left( (\phi^\nu_{gs(\sigma)} \circ \chi^\nu_\sigma) \circ \chi^\nu_{\rho}, x_{gs(\sigma)} g_s(\rho) \right) \circ z_{\sigma \rho}.
\]

(44), (45), (46), (47), (48), and (49) together yield (43).

We now define the Gromov topology on \( \mathcal{M}_n \) to be \( \mathcal{U}(C) \), where \( C \) are the Gromov-convergent sequences. Moreover, we equip \( \mathcal{M}_n \) with the \( W_n \)-stratification defined by sending \( (2T, (x_\rho), (z_\alpha)) \) to \( 2T \). It is immediate from the definition of Gromov convergence that this map is continuous with respect to the Alexandroff topology on \( W_n \).

Proof of Thm. L17. It follows from Thm. 2.10 that Gromov-convergent sequences have unique limits. This, together with Rmk. 2.13 and Lemma 2.15, imply that Gromov-convergent sequences satisfy the hypotheses of Lemma 2.11. This proves that convergence in the Gromov topology on \( \mathcal{M}_n \) is equivalent to Gromov convergence, and that \( \mathcal{M}_n \) is first-countable and Hausdorff.

The rest of the proof of the topological properties of \( \mathcal{M}_n \) hinges on showing that \( \mathcal{M}_n \) is second-countable, just as the analogous result for \( \mathcal{M}_r \) depends similarly on showing that \( \mathcal{M}_r \) is second-countable. The proof of this result for \( \mathcal{M}_r \) in §5, [McDSa] contains a gap; McDuff–Salamon have communicated to the author a fix, which they intend to include in future editions of [McDSa]. This fix applies equally well to the current proof.

Finally, we observe that the forgetful map \( W_n \to K_r \) extends to a map \( \mathcal{M}_n \to \mathcal{M}_r \), sending \( (2T, (x_\rho), (z_\alpha)) \) to \( (T_s, (x_\rho)) \). This map sends Gromov-convergent sequences to Gromov-convergent sequences, and \( \mathcal{M}_n \) is first-countable, so this map is continuous.

\[
\square
\]

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