Commensurability of hyperbolic manifolds with geodesic boundary

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Abstract

Suppose $n \geq 3$, let $M_1, M_2$ be $n$-dimensional connected complete finite-volume hyperbolic manifolds with non-empty geodesic boundary, and suppose that $\pi_1(M_1)$ is quasi-isometric to $\pi_1(M_2)$ (with respect to the word metric). Also suppose that if $n = 3$, then $\partial M_1$ and $\partial M_2$ are compact. We show that $M_1$ is commensurable with $M_2$. Moreover, we show that there exist homotopically equivalent hyperbolic 3-manifolds with non-compact geodesic boundary which are not commensurable with each other.

We also prove that if $M$ is as $M_1$ above and $G$ is a finitely generated group which is quasi-isometric to $\pi_1(M)$, then there exists a hyperbolic manifold with geodesic boundary $M'$ with the following properties: $M'$ is commensurable with $M$, and $G$ is a finite extension of a group which contains $\pi_1(M')$ as a finite-index subgroup.

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A quasi-isometry between metric spaces is a (not necessarily continuous) map which looks biLipschitz from afar (see Section I for a precise definition). Even if they do not give information about the local structure of metric spaces, quasi-isometries usually capture the most important properties of their large-scale geometry.

Let $\Gamma$ be any group with a finite set $S$ of generators. Then a locally finite graph is defined which depends on $\Gamma$ and $S$ (such a graph is called a Cayley graph of $\Gamma$, see Section II). This graph is naturally endowed with a path metric, and metric graphs arising from different sets of generators are quasi-isometric to each other. Thus we can associate to any finitely generated group a well-defined quasi-isometry class of locally finite metric graphs. A lot of energies have been devoted in the last

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It is easily seen that (the Cayley graphs of) the fundamental groups of two compact hyperbolic $n$-manifolds without boundary are always quasi-isometric to each other. On the other hand, Schwartz proved in [Sch95] that in dimension greater than 2, the fundamental groups of two complete finite-volume non-compact hyperbolic manifolds without boundary are quasi-isometric to each other if and only if the manifolds themselves are commensurable with each other (see below for the definition of commensurability). In this paper we extend Schwartz’s result to the case of hyperbolic manifolds with non-empty geodesic boundary. Namely, let $n \geq 3$, let $M_1, M_2$ be $n$-dimensional connected complete finite-volume hyperbolic manifolds with non-empty geodesic boundary, and suppose that if $n = 3$, then $\partial M_1$ and $\partial M_2$ are compact. We prove that if $\pi_1(M_1)$ is quasi-isometric to $\pi_1(M_2)$, then $M_1$ is commensurable with $M_2$. Moreover, we show that there exist homotopically equivalent hyperbolic 3-manifolds with non-compact geodesic boundary which are not commensurable with each other.

We also prove the following quasi-isometric rigidity result for fundamental groups of hyperbolic manifolds with geodesic boundary. Let $n \geq 3$, let $M$ be an $n$-dimensional connected complete finite-volume hyperbolic manifold with non-empty geodesic boundary, and suppose that if $n = 3$, then $\partial M$ is compact. We show that if $G$ is a finitely generated group which is quasi-isometric to $\pi_1(M)$, then there exists a hyperbolic manifold with geodesic boundary $M'$ with the following properties: $M'$ is commensurable with $M$, and $G$ is a finite extension of a group which contains $\pi_1(M')$ as a finite-index subgroup. A similar quasi-isometric rigidity result for fundamental groups of complete finite-volume non-compact hyperbolic manifolds without boundary was proved by Schwartz in [Sch95].
The map $f$ is said to be a $k$-quasi-isometry if there exists a $k$-quasi-isometric embedding $g : X_2 \to X_1$ such that

$$d_1(x, g(f(x))) \leq k, \quad d_2(x', f(g(x'))) \leq k$$

for every $x \in X_1$, $x' \in X_2$.

In this case we say that $g$ is a $k$-pseudo-inverse of $f$, and that $(X_1, d_1)$ and $(X_2, d_2)$ are $k$-quasi-isometric to each other. A map is a quasi-isometry if it is a $k$-quasi-isometry for some $k \in \mathbb{R}$, and two spaces are quasi-isometric if they are $k$-quasi-isometric for some $k \in \mathbb{R}$.

Two quasi-isometries $f, g : X_1 \to X_2$ are equivalent if a constant $c > 0$ exists such that $d_2(f(x), g(x)) \leq c$ for every $x \in X_1$. Equivalence classes of quasi-isometries of a fixed metric space $X$ form the group $\text{QIsom}(X)$, which is called the quasi-isometry group of $X$.

### The Cayley graph of a group

Let $\Gamma$ be a group and $S \subset \Gamma$ be a finite set of generators for $\Gamma$. We also suppose that $S$ is symmetric, i.e. that $S = S^{-1}$, and that $1 \notin S$. For any $\gamma \in \Gamma$ let $|\gamma|_S$ be the minimal length of words representing $\gamma$ and having letters in $S$. If $\gamma_1, \gamma_2$ are elements of $\Gamma$ we set $d_S(\gamma_1, \gamma_2) = |\gamma_2^{-1}\gamma_1|_S$.

It is easily seen that $d_S$ is a distance which turns $\Gamma$ into a discrete metric space. Moreover, left translations by elements in $\Gamma$ act as isometries with respect to $d_S$.

We are now ready to define the Cayley graph $\mathcal{C}(\Gamma, S)$ of $\Gamma$ (with respect to $S$) as follows: the vertices of $\mathcal{C}(\Gamma, S)$ are the elements of $\Gamma$, two vertices $\gamma_1, \gamma_2 \in \Gamma$ are joined by an edge if and only if $d_S(\gamma_1, \gamma_2) = 1$ (i.e. if and only if $\gamma_1 = \gamma_2s$ for some $s \in S$), and $\mathcal{C}(\Gamma, S)$ does not contain multiple edges (i.e. two vertices are connected by at most one edge). It is easily seen that $d_S$ can be extended to a distance (still denoted by $d_S$) on $\mathcal{C}(\Gamma, S)$ which turns $\mathcal{C}(\Gamma, S)$ into a path metric space. Moreover, the action of $\Gamma$ onto itself by left translations can be extended to an isometric action of $\Gamma$ on $\mathcal{C}(\Gamma, S)$.

It is easily seen that if $S, S'$ are finite symmetric sets of generators for $\Gamma$, then $\mathcal{C}(\Gamma, S)$ is quasi-isometric to $\mathcal{C}(\Gamma, S')$. We say that a finitely generated group $\Gamma$ is quasi-isometric to a metric space $(X, d)$ if some (and then any) Cayley graph of $\Gamma$ is quasi-isometric to $(X, d)$. In particular, two finitely generated groups $\Gamma_1, \Gamma_2$ are quasi-isometric if some (and then any) Cayley graph of $\Gamma_1$ is quasi-isometric to some (and then any) Cayley graph of $\Gamma_2$. Every finite group is quasi-isometric to the trivial group. More in general, it is easily seen that every finite-index subgroup of any given group is quasi-isometric to the group itself. In a similar way, every finite extension of any given group is quasi-isometric to the group itself.

If $\Gamma'$ is a subgroup of $\Gamma$, we denote by $\Gamma/\Gamma'$ the set of left lateral classes of $\Gamma'$ in $\Gamma$, and by $[\Gamma : \Gamma'] = \#(\Gamma/\Gamma')$ the index of $\Gamma'$ in $\Gamma$. 

An important relationship with the universal covering

The (easy) result we are going to describe [Mil68, GdlH90] establishes perhaps the most important relationship between the geometry of a compact metric space and the metric properties of (a Cayley graph of) its fundamental group.

Let \((X, d)\) be a compact Riemannian manifold, possibly with boundary. Let \((\tilde{X}, \tilde{d})\) be the metric universal covering of \((X, d)\), let \(\tilde{x} \in \tilde{X}\) be a given basepoint and fix an identification of \(\pi_1(X)\) with the group of the covering automorphisms of \(\tilde{X}\). Then the map 

\[ \pi_1(X) \ni \gamma \mapsto \gamma(\tilde{x}) \in \tilde{X} \]

is a \(k\)-quasi-isometry between \(\pi_1(X)\) and \((\tilde{X}, \tilde{d})\), where \(k\) only depends on the diameter of \(X\) and the set of generators used to define the Cayley graph of \(\pi_1(X)\).

Since the universal covering of any compact hyperbolic \(n\)-manifold without boundary is isometric to \(\mathbb{H}^n\), this implies that the fundamental groups of any two compact hyperbolic \(n\)-manifolds without boundary are quasi-isometric to each other. On the contrary, Theorem 1.1 below implies that there exist pairs of compact hyperbolic manifolds with non-empty geodesic boundary having non-quasi-isometric fundamental groups.

Commensurability of hyperbolic manifolds

From now on we will always suppose \(n \geq 3\). Moreover, all manifolds will be connected. By hyperbolic \(n\)-manifold we will mean a complete finite-volume hyperbolic \(n\)-manifold with (possibly empty) geodesic boundary. Let \(N_1, N_2\) be hyperbolic \(n\)-manifolds. We say that \(N_1\) is commensurable with \(N_2\) if a hyperbolic \(n\)-manifold \(N_3\) exists which is the total space of a finite Riemannian covering both of \(N_1\) and of \(N_2\). Notice that in this case the fundamental group of \(N_3\) is a finite-index subgroup both of \(\pi_1(N_1)\) and of \(\pi_1(N_2)\), so \(\pi_1(N_1)\) is quasi-isometric to \(\pi_1(N_2)\). The main result of this paper shows that the converse of this statement is also true, under some additional conditions:

**Theorem 1.1.** Let \(N_1, N_2\) be hyperbolic \(n\)-manifolds with non-empty geodesic boundary and assume that if \(n = 3\) then \(\partial N_1\) and \(\partial N_2\) are compact. Suppose that \(\pi_1(N_1)\) is quasi-isometric to \(\pi_1(N_2)\). Then \(N_1\) is commensurable with \(N_2\).

The fundamental group as a group of isometries

We denote by \((\mathbb{H}^n, d^{\mathbb{H}})\) (or simply by \(\mathbb{H}^n\)) the hyperbolic \(n\)-space. Let \(N\) be a hyperbolic \(n\)-manifold with non-empty boundary and let \(\pi : \tilde{N} \to N\) be the universal covering of \(N\). By developing \(\tilde{N}\) in \(\mathbb{H}^n\) we can identify \(\tilde{N}\) with a convex polyhedron of \(\mathbb{H}^n\) bounded by a countable number of disjoint geodesic hyperplanes. The group of the automorphisms of the covering \(\pi : \tilde{N} \to N\) can be identified in a natural way with a discrete torsion-free subgroup \(\Gamma\) of \(\text{Isom}(\tilde{N}) \subset \text{Isom}(\mathbb{H}^n)\) such that \(N \cong \tilde{N}/\Gamma\). Also recall that there
exists an isomorphism $\pi_1(N) \cong \Gamma$, which is canonical up to conjugacy. With a slight abuse, from now on we will often refer to $\Gamma$ as to the fundamental group of $N$. Notice that $N$ uniquely determines $\Gamma$ up to conjugation by elements in $\text{Isom}(\mathbb{H}^n)$.

**The commensurator** Let $\Gamma_1, \Gamma_2$ be discrete subgroups of $\text{Isom}(\mathbb{H}^n)$. We say that an element $g \in \text{Isom}(\mathbb{H}^n)$ *commensurates* $\Gamma_1$ with $\Gamma_2$ if $g\Gamma_1g^{-1} \cap \Gamma_2$ has finite index both in $g\Gamma_1g^{-1}$ and in $\Gamma_2$. In this case we also say that $\Gamma_1$ and $\Gamma_2$ are commensurable with each other. We will see in Lemma 2.3 that two hyperbolic manifolds with non-empty geodesic boundary $N_1, N_2$ are commensurable if and only if their fundamental groups are.

If $\Gamma$ is a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$, the group of elements of $\text{Isom}(\mathbb{H}^n)$ which commensurate $\Gamma$ with itself is called the *commensurator* of $\Gamma$, and it is denoted by $\text{Comm}(\Gamma)$. Of course $\Gamma$ itself is a (not necessarily normal) subgroup of $\text{Comm}(\Gamma)$.

**Commensurator and quasi-isometry group** Let $G$ be a finitely generated group and $g$ be an element of $G$. We denote by $\ell_G(g) \in \text{QIsom}(G)$ the equivalence class of the quasi-isometry induced by the left translation by $g$. The map $\ell_G : G \to \text{QIsom}(G)$ is a homomorphism, whose image is not necessarily normal in $\text{QIsom}(G)$.

In what follows we will often denote just by $g$ the element $\ell_G(g) \in \text{QIsom}(G)$, and we will consider $G$ as a subgroup of $\text{QIsom}(G)$ via the homomorphism $\ell_G$. (In the cases we will be concerned with, the homomorphism $\ell_G$ is injective. Note however that this is not true in general: for example, if $G$ is Abelian then $\ell_G$ is the trivial homomorphism.) The following result will be proved in Section 4.

**Theorem 1.2.** Let $N$ be a hyperbolic $n$-manifold with non-empty boundary, let $\Gamma < \text{Isom}(\mathbb{H}^n)$ be the fundamental group of $N$ and if $n = 3$ also suppose that $\partial N$ is compact. Then the identity of $\Gamma$ extends to a canonical isomorphism between the commensurator of $\Gamma$ and the quasi-isometry group of $\Gamma$. In particular, we have $[\text{Comm}(\Gamma) : \Gamma] = [\text{QIsom}(\Gamma) : \Gamma]$.

If $G < \text{Isom}(\mathbb{H}^n)$ is the fundamental group of a non-compact complete finite-volume hyperbolic manifold without boundary, results of Margulis and Borel \cite{Bor81, Zim84} imply that $\text{Comm}(G)/G$ is finite if and only if $G$ is non-arithmetic. Moreover, a theorem of Schwartz \cite{Sch95} ensures that $[\text{Comm}(G) : G] = [\text{QIsom}(G) : G]$, so arithmeticity of $G$ turns out to depend only on the quasi-isometry type of $G$. The following result will be proved at the end of Section 2 and shows that things look very different in the case of non-empty geodesic boundary.

**Proposition 1.3.** Let $N$ be a hyperbolic $n$-manifold with non-empty boundary and let $\Gamma < \text{Isom}(\mathbb{H}^n)$ be the fundamental group of $N$. Then $\text{Comm}(\Gamma)/\Gamma$ is finite.

Thus, if $n \geq 4$ or $\partial N$ is compact, then $\text{QIsom}(\pi_1(N))/\pi_1(N)$ is a finite group.
Quasi-isometric rigidity  In Section 4 we will prove that quasi-isometries essentially preserve the property of being the fundamental group of a hyperbolic manifold with boundary:

**Theorem 1.4.** Let $N$ be a hyperbolic $n$-manifold with non-empty geodesic boundary, and if $n = 3$ also suppose that $\partial N$ is compact. Let $\Gamma < \text{Isom}(H^n)$ be the fundamental group of $N$ and $G$ be a finitely generated abstract group which is quasi-isometric to $\Gamma$. Then $G$ is a finite extension of a discrete subgroup $\Gamma' < \text{Isom}(H^n)$ which is commensurable with $\Gamma$.

By Selberg Lemma [Sel60], any finitely generated discrete subgroup of $\text{Isom}(H^n)$ contains a finite-index torsion-free subgroup, so Theorem 1.4 implies the following:

**Corollary 1.5.** Let $N$ and $G$ be as in the statement of Theorem 1.4. Then there exists a hyperbolic manifold $N'$ with the following properties: $N'$ is commensurable with $N$, and $G$ is a finite extension of a group which contains $\pi_1(N')$ as a finite-index subgroup.

**Counterexamples**  In Section 5 we will show that the hypotheses of Theorems 1.1, 1.2 cannot be weakened. Namely, we will prove the following:

**Theorem 1.6.** There exist non-commensurable hyperbolic 3-manifolds with non-compact geodesic boundary sharing the same fundamental group.

**Theorem 1.7.** A hyperbolic 3-manifold $M$ with non-compact geodesic boundary exists such that $\pi_1(M)$ has infinite index in $\text{QIsom}(\pi_1(M))$.

## 2 Hyperbolic manifolds with geodesic boundary

This section is devoted to a brief description of the most important topological and geometric properties of hyperbolic manifolds with geodesic boundary.

**Natural compactification**  Let $N$ be a hyperbolic $n$-manifold with non-empty geodesic boundary. Then $\partial N$, endowed with the Riemannian metric it inherits from $N$, is a hyperbolic $(n - 1)$-manifold without boundary (completeness of $\partial N$ is obvious, and the volume of $\partial N$ is proved to be finite in [Koj90]). It is well-known [Koj90, Koj94] that $N$ consists of a compact portion together with some cusps based on Euclidean $(n - 1)$-manifolds. More precisely, the $\varepsilon$-thin part of $N$ (see [Thu79]) consists of cusps of the form $T \times [0, \infty)$, where $T$ is a compact Euclidean $(n - 1)$-manifold with (possibly empty) geodesic boundary such that $(T \times [0, \infty)) \cap \partial N = \partial T \times [0, \infty)$. A cusp based on a closed Euclidean $(n - 1)$-manifold is therefore
disjoint from \( \partial N \) and is called \textit{internal}, while a cusp based on a Euclidean \((n - 1)\)-manifold with non-empty boundary intersects \( \partial N \) in one or two internal cusps of \( \partial N \), and is called a \textit{boundary cusp}. This description of the ends of \( N \) easily implies that \( N \) admits a natural compactification \( \overline{N} \) obtained by adding a closed Euclidean \((n - 1)\)-manifold for each internal cusp and a compact Euclidean \((n - 1)\)-manifold with non-empty geodesic boundary for each boundary cusp.

For later purposes, we observe that when \( n = 3 \), \( \overline{N} \) is obtained by adding to \( N \) some tori, Klein bottles, closed annuli and closed Möbius strips.

**Universal covering** Let \( \pi : \tilde{N} \to N \) be the universal covering of \( N \). We recall that \( \tilde{N} \) can be identified with a convex polyhedron of \( \mathbb{H}^n \) bounded by a countable number of disjoint geodesic hyperplanes \( S^i, i \in \mathbb{N} \). For any \( i \in \mathbb{N} \) let \( S^i_+ \) denote the closed half-space of \( \mathbb{H}^n \) bounded by \( S^i \) and containing \( \tilde{N} \), let \( S^i_- \) be the closed half-space of \( \mathbb{H}^n \) opposite to \( S^i_+ \) and let \( \Delta^i \) be the internal part of the closure at infinity of \( S^i_- \). Of course we have \( \tilde{N} = \bigcap_{i \in \mathbb{N}} S^i_+ \), so denoting by \( \tilde{N}_\infty \) the closure at infinity of \( \tilde{N} \) we obtain \( \tilde{N}_\infty = \partial \mathbb{H}^n \setminus \bigcup_{i \in \mathbb{N}} \Delta^i \). We also denote by \( \overline{\Delta}^i \) the closure at infinity of \( S^i \).

It is easily seen that any internal cusp of \( N \) lifts in \( \tilde{N} \) to the union of a countable number of disjoint horoballs, each of which is entirely contained in \( \tilde{N} \). Things become a bit more complicated when considering boundary cusps. First observe that if \( i \neq j \), then \( \overline{S}^i \cap \overline{S}^j \) is either empty or consists of one point in \( \partial \mathbb{H}^n \). It is shown in \cite{Koj90} that if \( q \in \overline{S}^i \cap \overline{S}^j \), then the intersection of \( \tilde{N} \) with a sufficiently small horoball centered at \( q \) projects onto a boundary cusp of \( N \). Conversely, any component of the preimage of a boundary cusp of \( N \) is the intersection of \( \tilde{N} \) with a horoball centered at a point which belongs to the boundary at infinity of two different components of \( \partial \tilde{N} \).

**Limit set and discreteness of \( \text{Isom}(\tilde{N}) \)** Let \( \Gamma < \text{Isom}(\tilde{N}) < \text{Isom}(\mathbb{H}^n) \) be the fundamental group of \( N \), let \( \Lambda(\Gamma) \) denote the limit set of \( \Gamma \) and set \( \Omega(\Gamma) = \partial \mathbb{H}^n \setminus \Lambda(\Gamma) \). Kojima has shown in \cite{Koj90} that \( \Lambda(\Gamma) = \tilde{N}_\infty \), so the round balls \( \Delta^i, i \in \mathbb{N} \) previously defined actually are the connected components of \( \Omega(\Gamma) \). Since \( \tilde{N}_\infty = \Lambda(\Gamma) \), we have that \( \tilde{N} \) is the intersection of \( \mathbb{H}^n \) with the convex hull of \( \Lambda(\Gamma) \), so \( N \) is the convex core (see \cite{Thu79}) of the hyperbolic manifold \( \mathbb{H}^n/\Gamma \). Thus \( \Gamma \) is geometrically finite and it uniquely determines \( N \). In Section \ref{sec:limit_set} we will need the following result, which is proved in \cite{Fri04} (see also \cite{KMS93}).

**Lemma 2.1.** Let \( j : S^{n-2} \to \Lambda(\Gamma) \) be a topological embedding. Then \( \Lambda(\Gamma) \setminus j(S^{n-2}) \) is path connected if and only if \( j(S^{n-2}) = \partial \Delta^l \) for some \( l \in \mathbb{N} \).

We also have the following:

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Lemma 2.2. The group Isom(\(\tilde{N}\)) of isometries of \(\tilde{N}\) is a discrete subgroup of Isom(\(\mathbb{H}^n\)).

Proof: Since \(\tilde{N}\) is connected and has non-empty interior, every element in Isom(\(\tilde{N}\)) uniquely determines an element in Isom(\(\mathbb{H}^n\)). Let now \(\{g_n\}_{n\in\mathbb{N}} \subset \text{Isom}(\tilde{N})\) be a sequence converging to the identity of \(\mathbb{H}^n\). If \(S^i, S^j\) are components of \(\partial N\) with \(\overline{S^i} \cap \overline{S^j} = \emptyset\), we denote by \(e_{ij} \subset \tilde{N}\) the unique shortest path joining \(S^i\) with \(S^j\) and we set

\[
E = \left( \bigcup_{\overline{S^i} \cap \overline{S^j} = \emptyset} e_{ij} \right).
\]

Now \(E\) is a \(\Gamma\)-invariant subset of \(\mathbb{H}^n\) and the limit set \(\Lambda(\Gamma)\) is not contained in the boundary at infinity of any hyperbolic hyperplane. Thus there exist components \(S^1, \ldots, S^{n+1}\) of \(\partial \tilde{N}\) such that \(\overline{S^i} \cap \overline{S^j} = \emptyset\) for all \(i, j = 1, \ldots, n + 1, i \neq j\), and the convex hull of the \(e_{ij}\)'s, \(i, j = 1, \ldots, n + 1, i \neq j\), has non-empty interior in \(\mathbb{H}^n\). Since \(g_k\) tends to the identity as \(k\) tends to \(\infty\), there exists \(M \gg 0\) such that \(g_k(S^i) = S^i\) for all \(i = 1, \ldots, n + 1, k \geq M\). Thus if \(k \geq M\) then \(g_k\) restricts to the identity on each \(e_{ij}\), \(i, j = 1, \ldots, n + 1, i \neq j\), whence on the convex hull of such \(e_{ij}\)'s, which has non-empty interior. This forces \(g_k = \text{Id}_{\mathbb{H}^n}\) for every \(k \geq M\), and proves that Isom(\(\tilde{N}\)) is discrete. \(\square\)

Three lemmas We are now ready to prove the following:

Lemma 2.3. Suppose \(N_1, N_2\) are hyperbolic n-manifolds with non-empty geodesic boundary, let \(\tilde{N}_i \subset \mathbb{H}^n\) be the universal covering of \(N_i\) and \(\Gamma_i < \text{Isom}(\tilde{N}_i) < \text{Isom}(\mathbb{H}^n)\) be the fundamental group of \(N_i\). Then \(N_1\) is commensurable with \(N_2\) if and only if \(\Gamma_1\) is commensurable with \(\Gamma_2\).

Proof: Suppose \(N_3\) is a finite Riemannian covering both of \(N_1\) and of \(N_2\) and let \(\Gamma_3 < \text{Isom}(\tilde{N}_3) < \text{Isom}(\mathbb{H}^n)\) be the fundamental group of \(N_3\). For \(i = 1, 2\), the covering projection \(p_i : N_3 \rightarrow N_i\) induces an isometry \(\tilde{p}_i : \tilde{N}_3 \rightarrow \tilde{N}_i\). Now conjugation by the isometry \(\tilde{p}_2 \circ \tilde{p}_1^{-1}\) takes \((p_1)_*(\Gamma_3)\) onto \((p_2)_*(\Gamma_3)\). Since \((p_i)_*(\Gamma_3)\) has finite index in \(\Gamma_i\), this implies that \(\Gamma_1\) is commensurable with \(\Gamma_2\).

On the other hand, let \(g \in \text{Isom}(\mathbb{H}^n)\) be an element such that \(\Gamma_3 = g\Gamma_1g^{-1} \cap \Gamma_2\) has finite index both in \(\Gamma_2\) and in \(g\Gamma_1g^{-1}\). Then \(\Lambda(\Gamma_3) = \Lambda(\Gamma_2) = g(\Lambda(\Gamma_1))\), whence \(\tilde{N}_2 = g(\tilde{N}_1)\) and \(N_2 = \tilde{N}_2/\Gamma_3\) is hyperbolic with non-empty boundary (and with universal covering \(\tilde{N}_3 = \tilde{N}_2\)). Of course the natural projection \(N_3 \cong N_2/\Gamma_3 \rightarrow \tilde{N}_2/\Gamma_2 = N_2\) is a finite Riemannian covering, and \(g^{-1} : \tilde{N}_3 \rightarrow \tilde{N}_1\) induces a finite Riemannian covering \(\tilde{N}_3/\Gamma_3 \cong N_3 \rightarrow N_1 \cong \tilde{N}_1/\Gamma_1\). Thus \(N_1\) is commensurable with \(N_2\). \(\square\)

The following result plays a crucial rôle in the proof of Theorem 1.11.
Lemma 2.4. Suppose $N_1, N_2$ are hyperbolic $n$-manifolds with non-empty geodesic boundary, let $\tilde{N}_i \subset \mathbb{H}^n$ be the universal covering of $N_i$ and $\Gamma_i < \text{Isom}(\tilde{N}_i) < \text{Isom}(\mathbb{H}^n)$ be the fundamental group of $N_i$. Suppose that $g \in \text{Isom}(\mathbb{H}^n)$ is such that $g(\tilde{N}_1) = \tilde{N}_2$. Then $g$ commensurates $\Gamma_1$ with $\Gamma_2$. In particular, $N_1$ is commensurable with $N_2$.

Proof: Let $\Gamma$ be the group of isometries of $\tilde{N}_1$. Of course we have $\Gamma_1 < \Gamma$ and $\Gamma_2 < \text{Isom}(\tilde{N}_2) = g\Gamma g^{-1}$. By Lemma 2.2, $\Gamma$ is discrete, so $\tilde{N}_1/\Gamma$ is isometric to an orbifold $N_{\text{orb}}$ of positive volume. Of course we also have $N_{\text{orb}} \cong \tilde{N}_2/(g\Gamma g^{-1})$, so $[g\Gamma g^{-1} : \Gamma_1] = \text{vol} N_1/\text{vol} N_{\text{orb}} < \infty$, $[g\Gamma g^{-1} : \Gamma_2] = \text{vol} N_2/\text{vol} N_{\text{orb}} < \infty$.

Thus $g\Gamma_1 g^{-1} \cap \Gamma_2$ has finite index in $g\Gamma g^{-1}$, whence a fortiori $g\Gamma_1 g^{-1} \cap \Gamma_2$ has finite index both in $g\Gamma_1 g^{-1}$ and in $\Gamma_2$. \hfill \Box

The following lemma readily implies Proposition 1.3.

Lemma 2.5. Let $N$ be a hyperbolic $n$-manifold with non-empty geodesic boundary, let $\tilde{N} \subset \mathbb{H}^n$ be the universal covering of $N$ and $\Gamma < \text{Isom}(\tilde{N}) < \text{Isom}(\mathbb{H}^n)$ be the fundamental group of $N$. Then $\text{Comm}(\Gamma) = \text{Isom}(\tilde{N})$ and $\text{Comm}(\Gamma)/\Gamma$ is finite.

Proof: By Lemma 2.2, $\text{Isom}(\tilde{N})$ is discrete, so $\tilde{N}/\text{Isom}(\tilde{N})$ is isometric to an orbifold $N_{\text{orb}}$ of positive volume. So $[\text{Isom}(\tilde{N}) : \Gamma] = \text{vol} N/\text{vol} N_{\text{orb}} < \infty$. \hfill (1)

Thus for every $g \in \text{Isom}(\tilde{N})$ we have $[\Gamma : \Gamma \cap g\Gamma g^{-1}] < \infty$, $[g\Gamma g^{-1} : \Gamma \cap g\Gamma g^{-1}] < \infty$.

This implies that $\text{Isom}(\tilde{N})$ is contained in $\text{Comm}(\Gamma)$.

On the other hand, if $g \in \text{Isom}(\mathbb{H}^n)$ belongs to $\text{Comm}(\Gamma)$ then $\Lambda(\Gamma) = \Lambda(g\Gamma g^{-1}) = g(\Lambda(\Gamma))$. Since $\tilde{N}$ is the hyperbolic convex hull of $\Lambda(\Gamma)$, it follows that $g(\tilde{N}) = \tilde{N}$, whence $g \in \text{Isom}(\tilde{N})$. Thus $\text{Comm}(\Gamma) = \text{Isom}(\tilde{N})$, and the conclusion follows from inequality (1). \hfill \Box

3 Quasi-isometries of hyperbolic polyhedra

Let $N$ be a hyperbolic $n$-manifold with non-empty geodesic boundary and denote by $\tilde{N} \subset \mathbb{H}^n$ the universal covering of $N$. 
Neutered hyperbolic polyhedra  Let $N^* \subset N$ be a compact core of $N$ whose preimage $\tilde{N}^*$ in $\tilde{N}$ is given by the complement in $\tilde{N}$ of a countable family of disjoint horoballs. We will refer to $\tilde{N}^*$ as to the neutered universal covering of $N$ (this terminology is taken from [Sch95]). The components of $\tilde{N} \setminus \tilde{N}^*$ will be called the removed ends of $\tilde{N}^*$, and they are of two kinds: those that project onto the internal cusps of $N$ are genuine horoballs completely contained in $\tilde{N}$, while a removed end that project onto a boundary cusp of $N$ is properly contained in a horoball centered at a point that belongs to the closure at infinity of two different components of $\partial \tilde{N}$.

For later purposes we insist that $N^* \subset N$ is chosen in such a way that the following conditions hold:  the distance between two distinct removed ends of $\tilde{N}^*$ is at least 1; a constant $c > 0$ exists such that the distance between any removed end of $\tilde{N}^*$ which projects onto an internal cusp of $N$ and the boundary of $\tilde{N}$ is equal to $c$; a constant $a > 0$ exists such that the boundary of any removed end of $\tilde{N}^*$ which projects onto a boundary cusp of $N$ is isometric to $\mathbb{R}^{n-2} \times [0,a]$. We point out that the last two conditions imply that every isometry of $\tilde{N}$ restricts to an isometry of $\tilde{N}^*$.

We observe that $\tilde{N}^*$ is no longer convex. Moreover, its boundary is partitioned into the following sets: the geodesic boundary $\partial g \tilde{N}^* = \partial \tilde{N}^* \cap \partial \tilde{N}$, which is given by the union of portions of geodesic hyperplanes, and the horospherical boundary $\partial h \tilde{N}^* = \partial \tilde{N}^* \setminus \partial \tilde{N}$, which is given by the union of portions of horospheres.

Recall that $d^H$ is the hyperbolic distance of $\mathbb{H}^n$ (which restricts course to a distance on $\tilde{N}^*$, still denoted by $d^H$) and let $d^*$ be the path distance induced on $\tilde{N}^*$. Of course we have $d^H \leq d^*$, but since $\tilde{N}^*$ is not convex, $d^H$ and $d^*$ are in fact quite different from each other. In general, $(\tilde{N}^*, d^*)$ is not quasi-isometric to $(\tilde{N}^*, d^H)$. However, it is easily seen that $d^H$ and $d^*$ are biLipschitz equivalent below any given scale, i.e. for any $c > 0$ there exists $b \geq 1$ such that

$$b^{-1} \cdot d^H(x,y) \leq d^*(x,y) \leq b \cdot d^H(x,y) \text{ for every } x,y \in \tilde{N}^* \text{ with } d^*(x,y) \leq c.$$  

Quasi-isometries preserve the horospherical boundary  Let $N_1, N_2$ be hyperbolic $n$-manifolds with non-empty geodesic boundary. From now on we also suppose that if $n = 3$, then $\partial N_1$ and $\partial N_2$ are both compact. Let $N_i^*$ be a compact core of $N_i$ as above, let $\tilde{N}_i \subset \mathbb{H}^n$ (resp. $\tilde{N}_i^* \subset \mathbb{H}^n$) be the universal covering (resp. the neutered universal covering) of $N_i$, and $\Gamma_i < \text{Isom}(\tilde{N}_i) < \text{Isom}(\mathbb{H}^n)$ be the fundamental group of $N_i$. We fix a finite set of generators for $\Gamma_i$ and suppose that $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is a $k_1$-quasi-isometry with $k_1$-pseudo-inverse $\varphi^{-1}$.

Since $N_i^* \subset N_i$ is compact, after fixing basepoints $\tilde{x}_i \in \tilde{N}_i$, we are provided with $k_2$-quasi-isometries $\varphi^* : \tilde{N}_1^* \rightarrow \tilde{N}_2^*$, $(\varphi^{-1})^* : \tilde{N}_2^* \rightarrow \tilde{N}_1^*$, one the $k_2$-pseudo-inverse of the other, where $\tilde{N}_i^*$ is endowed with the path metric $d_i^*$, and $k_2$ only depends on
Recall that our definition of neutered universal covering implies that a constant $a_i > 0$ exists such that any component of $\partial_h \bar{N}_i^s$ is isometric either to $\mathbb{R}^{n-1}$ or to $\mathbb{R}^{n-2} \times [0, a_i]$. Moreover, if $n = 3$ then the boundaries of $N_1$ and of $N_2$ are supposed to be compact. Thus if $H$ is a component of $\partial_h \bar{N}_i^s$, then $H$ is $r$-quasi-isometric to $\mathbb{R}^k$ for some $k \geq 2$, where $r$ only depends on $a_i$. Therefore an easy application of [Sch95] Lemma 3.1 implies the following:

**Lemma 3.1.** If $H_1$ is a component of $\partial_h \bar{N}_1^s$, then a unique component $H_2$ of $\partial_h \bar{N}_1^s$ exists such that $\varphi^*(H_1)$ is contained in the $r'$-neighbourhood of $H_2$, where $r'$ is a positive number depending solely on the geometry of $N_1^s, N_2^s$ and on $k_2$.

Recall that a map that stays at an uniformly bounded distance from a quasi-isometry is still a quasi-isometry. Moreover, up to swapping the indices, the statement of Lemma 3.1 also holds for $(\varphi^{-1})^*$. Thus, up to increasing $k_2$ by an amount which depends solely on $N_1, N_2$ and $k_2$ itself, we can suppose that $\varphi^*$ (resp. $(\varphi^{-1})^*$) takes $\partial_h \bar{N}_1^s$ (resp. $\partial_h \bar{N}_2^s$) to $\partial_h \bar{N}_1^s$ (resp. into $\partial_h \bar{N}_1^s$).

**Extending $\varphi^*$ to $\bar{N}_1$** We now describe how $\varphi^*$ can be extended to a quasi-isometry $\bar{\varphi}$ from $\bar{N}_1$ to $\bar{N}_2$. Even if the construction of $\bar{\varphi}$ is essentially the same as in [Sch95] Section 5], we outline it here, since some explicit properties of $\bar{\varphi}$ will be needed in the following paragraphs. We begin with the following:

**Lemma 3.2.** Let $(X, d_X), (Y, d_Y)$ be path metric spaces, let $k$ be a positive constant and suppose the maps $f : X \to Y$, $g : Y \to X$ have the following properties:

\[
d_Y(f(x), f(x')) \leq k \text{ for every } x, x' \in X \text{ with } d_X(x, x') \leq 1;
\]
\[
d_X(g(y), g(y')) \leq k \text{ for every } y, y' \in Y \text{ with } d_Y(y, y') \leq 1;
\]
\[
d_X(g(f(x)), x) \leq k \text{ for every } x \in X;
\]
\[
d_Y(f(g(y)), y) \leq k \text{ for every } y \in Y.
\]

Then $f$ and $g$ are $\max\{k, 3\}$-quasi-isometries.

**Proof:** Let $x, x'$ be points in $X$ and let $N \in \mathbb{N}$ be such that $N \leq d_X(x, x') < N + 1$. Since $X$ is a path metric space, there exist points $x_0 = x, x_1, \ldots, x_{N+1} = x'$ with $d_X(x_i, x_{i+1}) \leq 1$ for every $i = 0, \ldots, N$. Thus

\[
d_Y(f(x), f(x')) \leq \sum_{i=0}^{N} d_Y(f(x_i), f(x_{i+1})) \leq Nk + k \leq kd_X(x, x') + k.
\]

The same argument applied to $g$ shows that $d_X(g(y), g(y')) \leq kd_Y(y, y') + k$ for every $y, y' \in Y$. 

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Moreover, for every \( x, x' \in X \) we have
\[
d_X(x, x') \leq d_X(x, g(f(x))) + d_X(g(f(x)), g(f(x'))) + d_X(g(f(x')), x')
\leq k + kdy(f(x), f(x')) + k,
\]
whence \( k^{-1}d_X(x, x') - 3 \leq d_Y(f(x), f(x')). \) This shows that \( f \) is a \( \max\{k, 3\} \)-quasi-isometric embedding, and the conclusion follows since we can apply the same argument to \( g \).

Before going on, we fix some notation. For any \( q \in \partial \mathbb{H}^n, x \in \mathbb{H}^n \) we denote by \( H^q_x \) the horosphere of \( \mathbb{H}^n \) centered at \( q \) and containing \( x \), and by \( d_X^q \) the natural Euclidean path metric defined on \( H^q_x \). Moreover, we denote by \( \xi^q \) the projection of \( \mathbb{H}^n \) onto \( H^q_x \), \( i.e. \), the map which takes any point \( y \in \mathbb{H}^n \) to the intersection of \( H^q_x \) with the geodesic line containing \( y \) and having one endpoint at \( q \). It is easily seen that a universal constant \( c_1 > 0 \) exists such that
\[
d_X^q(x, \xi^q(x')) \leq c_1 \text{ for every } x, x' \in \mathbb{H}^n \text{ with } d^H(x, x') \leq 1.
\]

Suppose \( x \) is a point in \( \tilde{N}_1 \setminus \tilde{N}_1^* \). Then \( x \) belongs to exactly one removed end \( O \subset \mathbb{H}^n \). If \( q \in \partial \mathbb{H}^n \) is the center of \( O \), we denote by \( \eta_i(x) \) the intersection of \( \partial O \) with the geodesic line in \( \mathbb{H}^n \) containing \( x \) and having an endpoint at \( q \) (so \( \eta_i(x) = \xi^q(x) \), where \( y \) is any point belonging to \( \partial O \)).

We now define \( \bar{\varphi} : \tilde{N}_1 \to \tilde{N}_2 \) as follows: if \( x \) belongs to \( \tilde{N}_1^* \), then \( \bar{\varphi}(x) = \varphi^*(x) \); otherwise, we set \( \bar{\varphi}(x) = y \), where \( y \) is the unique point of \( \tilde{N}_2 \setminus \tilde{N}_2^* \) with \( \eta_2(y) = \varphi^*(\eta_1(x)) \) and \( d^H(y, \eta_2(y)) = d^H(x, \eta_1(x)) \). We say that \( \bar{\varphi} \) is the conical extension of \( \varphi^* \), and we denote by \( \bar{\varphi}^{-1} \) the conical extension of \( (\varphi^{-1})^* \).

**Proposition 3.3.** The map \( \bar{\varphi} : \tilde{N}_1 \to \tilde{N}_2 \) just constructed is a \( k_3 \)-quasi-isometry, where \( k_3 \) only depends on \( k_2 \).

**Proof:** Recall first that \( d_i^* \) and the restriction of \( d^H \) to \( \tilde{N}_1^* \) are biLipschitz equivalent below any given scale for \( i = 1, 2 \), so a constant \( c_2 > 0 \) exists such that
\[
d_i^*(y, y') \leq c_2 \text{ for every } y, y' \in \tilde{N}_1^* \text{ with } d^H(y, y') \leq 2. \tag{2}
\]

Let \( x, x' \) be points in \( \tilde{N}_1 \) with \( d^H(x, x') \leq 1 \). Since any two distinct removed ends of \( \tilde{N}_1 \) lie at distance at least 1 from each other, \( x \) and \( x' \) do not belong to different components of \( \tilde{N}_1 \setminus \tilde{N}_1^* \).

Suppose first that \( x, x' \in \tilde{N}_1^* \). Then inequality \( \tag{2} \) implies
\[
d^H(\bar{\varphi}(x), \bar{\varphi}(x')) \leq d_2^H(\bar{\varphi}(x), \bar{\varphi}(x')) = d_2^H(\varphi^*(x), \varphi^*(x')) \leq k_2c_2 + k_2. \tag{3}
\]

Suppose now that \( x \) and \( x' \) belong to the same removed end of \( \tilde{N}_1^* \) centered at \( q \) and set \( b = d^H(x, \eta_1(x)) \leq d^H(x', \eta_1(x')) \) (see Figure \( \square \)). Let also \( x'' = \xi^q(x') \) be
the projection of \( x' \) onto \( H^2 \). Of course we have \( d^\mathbb{H}(x', \eta_1(x')) - d^\mathbb{H}(x, \eta_1(x)) \leq 1 \) and \( d^\mathbb{H}(x, x'') \leq c_1 \). It is easily seen that \( d^\mathbb{H}_1(\eta_1(x), \eta_1(x')) = d^\mathbb{H}_2(\eta_1(x), \eta_1(x')) \leq c_1 \exp(b) \), whence \( d^\mathbb{H}_2(\varphi^*(\eta_1(x)), \varphi^*(\eta_1(x'))) \leq k_2c_1 \exp(b) + k_2 \). Observe now that \( \tilde{\varphi}(x), \tilde{\varphi}(x'), \tilde{\varphi}(x'') \) belong to the same removed end of \( \tilde{N}_2^* \). Suppose this end is centered at \( s \in \partial \mathbb{H}^3 \). Then by construction we have

\[
d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x')) \leq d^\mathbb{H}_2(\tilde{\varphi}(x), \tilde{\varphi}(x'')) = \exp(-b)d^\mathbb{H}_2(\varphi^*(\eta_1(x)), \varphi^*(\eta_1(x'))) \leq k_2c_1 + k_2 \exp(-b) \leq k_2(c_1 + 1),
\]

and

\[
d^\mathbb{H}_2(\tilde{\varphi}(x''), \tilde{\varphi}(x')) = d^\mathbb{H}_2(x'', x') \leq 1,
\]

whence

\[
d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x')) \leq d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x'')) + d^\mathbb{H}(\tilde{\varphi}(x''), \tilde{\varphi}(x')) \leq k_2(c_1 + 1) + 1.
\]

Finally suppose that \( x \notin \tilde{N}_1^*, x' \in \tilde{N}_1^* \) and set \( x'' = \eta_1(x) \). Of course we have \( d^\mathbb{H}(x, x'') \leq 1 \), whence \( d^\mathbb{H}_2(x'', x') \leq 2 \) by the triangular inequality. By construction we have \( d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x'')) = d^\mathbb{H}(x, x'') \leq 1 \). Moreover, the same argument yielding inequality (13) also shows that \( d^\mathbb{H}_2(\tilde{\varphi}(x''), \tilde{\varphi}(x')) \leq k_2c_2 + k_2 \). This readily implies

\[
d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x')) \leq d^\mathbb{H}(\tilde{\varphi}(x), \tilde{\varphi}(x'')) + d^\mathbb{H}(\tilde{\varphi}(x''), \tilde{\varphi}(x')) \leq k_2(c_2 + 1) + 1.
\]

Of course, analogous inequalities also hold for \( \tilde{\varphi}^{-1} \). Moreover, by the very construction of \( \tilde{\varphi}, \tilde{\varphi}^{-1} \) it follows that for all \( x \in \tilde{N}_1, y \in \tilde{N}_2 \) we have

\[
d^\mathbb{H}(x, \tilde{\varphi}^{-1}(\tilde{\varphi}(x))) \leq d^\mathbb{H}(\eta_1(x), \tilde{\varphi}^{-1}(\tilde{\varphi}(\eta_1(x)))) \leq d^\mathbb{H}_1(\eta_1(x), \tilde{\varphi}^{-1}((\tilde{\varphi}(\eta_1(x)))) \leq k_2,
\]

\[
d^\mathbb{H}(y, \tilde{\varphi}(\tilde{\varphi}^{-1}(y))) \leq d^\mathbb{H}(\eta_2(y), \tilde{\varphi}(\tilde{\varphi}^{-1}(\eta_2(y)))) \leq d^\mathbb{H}_2(\eta_2(y), \tilde{\varphi}(\tilde{\varphi}^{-1}(\eta_2(y)))) \leq k_2.
\]
Now the conclusion follows from Lemma 3.2.

Quasi-isometries preserve the geodesic boundary If \((X, d)\) is a metric space, a \(k\)-quasi-isometric embedding \(\alpha : \mathbb{R} \to X\) is called \(k\)-quasi-geodesic in \(X\). The following proposition establishes that a quasi-geodesic in hyperbolic space stays within a finite distance from a genuine geodesic (see e.g. [BP92, GdlH90] for a proof).

**Proposition 3.4.** Let \(\alpha : \mathbb{R} \to \mathbb{H}^n\) be a \(k\)-quasi-geodesic. Then there exist a positive number \(r\), depending solely on \(k\), and a geodesic \(\beta : \mathbb{R} \to \mathbb{H}^n\) such that the image of \(\alpha\) is contained in the \(r\)-neighbourhood of the image of \(\beta\).

A quasi-isometry between convex subsets of hyperbolic space naturally defines a continuous map between their closures at infinity [ET63, Tuk 85]:

**Proposition 3.5.** For \(i = 1, 2\) let \(W^i\) be a convex subset of \(\mathbb{H}^n\), let \(W^i_\infty \subset \partial \mathbb{H}^n\) be the boundary at infinity of \(W^i\) and suppose that \(f : W^1 \to W^2\) is a quasi-isometry. Then there exists a unique homeomorphism \(\partial f : W^1_\infty \to W^2_\infty\) having the following property: if \(\alpha\) is (the image of) a quasi-geodesic in \(W^1\) with endpoints \(x, x' \in W^1_\infty\), then \(f(\alpha)\) is (the image of) a quasi-geodesic in \(W^2_\infty\) with endpoints \(\partial f(x), \partial f(x') \in W^2_\infty\). Such a homeomorphism is called the extension of \(f\) to the conformal boundary of \(W^1\).

We are now ready to prove the following:

**Lemma 3.6.** If \(S_1\) is a component of \(\partial \tilde{N}_1\), then a component \(S_2\) of \(\tilde{N}_2\) exists such that \(\tilde{\varphi}(S_1)\) is contained in the \(r\)-neighbourhood of \(S_2\), where \(r\) is a positive number depending solely on \(k_3\).

**Proof:** Let \(\partial \tilde{\varphi} : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)\) be the extension of \(\tilde{\varphi}\) to the conformal boundary of \(\tilde{N}_1\). Since \(\partial \tilde{\varphi}\) is a homeomorphism, Lemma 2.1 implies that \(\partial \tilde{\varphi}\) takes the closure at infinity of \(S_1\) onto the closure at infinity of a connected component \(S_2\) of \(\partial \tilde{N}_2\).

Let \(\alpha\) be (the image of) a geodesic lying on \(S_1\). By Proposition 3.4, the set \(\tilde{\varphi}(\alpha)\) is contained in the \(r\)-neighbourhood of \(\alpha\) of \(\partial \tilde{\varphi}\) takes the endpoints of \(\alpha\) onto the endpoints of \(\beta\), so \(\beta\) lies on \(S_2\), and \(\tilde{\varphi}(\alpha)\) is contained in the \(r\)-neighbourhood of \(S_2\).

Thus, up to increasing \(k_3\) by an amount which depends solely on \(N_1, N_2\) and \(k_3\) itself, we can modify \(\tilde{\varphi}, \tilde{\varphi}^{-1}\) in such a way that the following conditions hold:

- \(\tilde{\varphi}, \tilde{\varphi}^{-1}\) are \(k_3\)-quasi-isometries between \((\tilde{N}_1, d^H)\) and \((\tilde{N}_2, d^H)\), which are one the \(k_3\)-pseudo-inverse of the other;
- \(\tilde{\varphi}\) (resp. \(\tilde{\varphi}^{-1}\)) takes \(\tilde{N}_1^*\) (resp. \(\tilde{N}_2^*\)) into \(\tilde{N}_2^*\) (resp. \(\tilde{N}_1^*\)).
• \(\bar{\varphi}\) (resp. \(\bar{\varphi}^{-1}\)) takes \(\partial_h \tilde{N}_1^s\) (resp. \(\partial_h \tilde{N}_2^s\)) into \(\partial_h \tilde{N}_1^s\) (resp. \(\partial_h \tilde{N}_1^s\));

• \(\bar{\varphi}, \bar{\varphi}^{-1}\) restrict to \(k_3\)-quasi-isometries between \((\tilde{N}_1^s, d_1^s)\) and \((\tilde{N}_2^s, d_2^s)\), which are one the \(k_3\)-pseudo-inverse of the other;

• \(\bar{\varphi}\) (resp. \(\bar{\varphi}^{-1}\)) is the conical extension of \(\varphi|_{N_1^s}\) (resp. of \(\varphi^{-1}|_{N_2^s}\));

• \(\bar{\varphi}\) (resp. \(\bar{\varphi}^{-1}\)) takes the geodesic boundary of \(\tilde{N}_1\) (resp. of \(\tilde{N}_2\)) into the geodesic boundary of \(\tilde{N}_1\) (resp. of \(\tilde{N}_1\)).

Also observe that if \(S, S'\) are distinct hyperplanes of \(\mathbb{H}^n\), then \(S\) is not contained in any \(\tau\)-neighbourhood of \(S'\), so \(\bar{\varphi}\) induces a bijection between the components of \(\partial \tilde{N}_1\) and the components of \(\partial \tilde{N}_2\).

**Mirroring along the boundary** Let now \(DN_i\) be the manifold obtained by mirroring \(N_i\) along its boundary. Since \(\partial N_i\) is totally geodesic, \(DN_i\) is a hyperbolic \(n\)-manifold without boundary. We can choose a universal covering \(p_i : \mathbb{H}^n \to DN_i\) in such a way that \(p_i\) restricts to the given universal covering \(\tilde{N}_i \to N_i\). The preimages under \(p_i\) of \(N_i\) and of its mirror copy define a tessellation \(\mathcal{T}_i\) of \(\mathbb{H}^n\) whose pieces are convex polyhedra isometric to \(\tilde{N}_i\). Since \(\bar{\varphi}\) induces a bijection between the components of \(\partial \tilde{N}_1\) and the components of \(\partial \tilde{N}_2\), the tessellations \(\mathcal{T}_1\) and \(\mathcal{T}_2\) are combinatorially equivalent to each other. Thus the map \(\bar{\varphi}\) can be extended to the whole of \(\mathbb{H}^n\) just by repeatedly mirroring along the hyperplanes which bound the pieces of \(\mathcal{T}_1\) and of \(\mathcal{T}_2\). More precisely, unique maps \(\phi, \phi^{-1} : \mathbb{H}^n \to \mathbb{H}^n\) exist with the following properties:

• \(\phi(\tilde{N}_1) = \tilde{N}_2, \phi^{-1}(\tilde{N}_2) = \tilde{N}_1\);

• \(\phi|_{\tilde{N}_1} = \phi, \phi^{-1}|_{\tilde{N}_2} = \phi^{-1}\);

• the image under \(\phi\) (resp. under \(\phi^{-1}\)) of a piece of \(\mathcal{T}_1\) (resp. of \(\mathcal{T}_2\)) is contained in a piece of \(\mathcal{T}_2\) (resp. of \(\mathcal{T}_1\));

• adjacent pieces of \(\mathcal{T}_1\) (resp. of \(\mathcal{T}_2\)) are taken by \(\phi\) (resp. by \(\phi^{-1}\)) into adjacent pieces of \(\mathcal{T}_2\) (resp. of \(\mathcal{T}_1\));

• let \(P_1, P_1'\) be adjacent pieces of \(\mathcal{T}_1\) with \(P_1 \cap P_1' = S_1\), and set \(\phi(P_1) = P_2, \phi(P_1') = P_2', S_2 = P_2 \cap P_2'\). If \(\sigma_1\) is the hyperbolic reflection along \(S_1\), then we have \((\phi \circ \sigma_1)|_{P_1} = (\sigma_2 \circ \phi)|_{P_1}, (\phi^{-1} \circ \sigma_2)|_{P_2} = (\sigma_1 \circ \phi^{-1})|_{P_2}\).

Let \(\Theta_i\) be the preimage under \(p_i\) of the double of \(N_i^s\). Then \(\Theta_i\) is obtained from \(\mathbb{H}^n\) by removing a countable set of horoballs at distance at least one from each other. Observe that for \(i, j = 1, 2\), if \(f : \Theta_i \to \Theta_j\) is a map taking the boundary of \(\Theta_i\) into
the boundary of \( \Theta_j \), then it does make sense to speak of the conical extension of \( f \), which is defined on the whole of \( \mathbb{H}^n \). We denote by \( T_i^* \) the tessellation of \( \Theta_i \) whose pieces are obtained by intersecting the pieces of \( \mathcal{T}_i \) with \( \Theta_i \). Every piece of \( T_i^* \) is the image of \( \tilde{N}_i^* \) under an element of \( \text{Isom}(\mathbb{H}^n) \). Let \( \delta_i^* \) be the path metric on \( \Theta_i \) induced by the hyperbolic Riemannian structure. If \( P_i^* \) is a piece of \( T_i^* \), then \((P_i^*, \delta_i^*|_{P_i^*})\) is isometric to \((\tilde{N}_i^*, d_i^*)\). Moreover, the restriction of \( \overline{\varphi} \) to any piece \( P_1^* \) of \( T_1^* \) defines a \( k_3 \)-quasi-isometry between \( P_1^* \) and the piece of \( T_2^* \) containing \( \overline{\varphi}(P_1^*) \), both endowed with their path distance.

**Lemma 3.7.** The maps \( \overline{\varphi}, \overline{\varphi}^{-1} \) are \( k_4 \)-quasi-isometries of \( \mathbb{H}^n \), where \( k_4 \) depends solely on \( k_3 \).

**Proof:** We first prove that \( \overline{\varphi}, \overline{\varphi}^{-1} \) restrict to quasi-isometries between \((\Theta_1, \delta_1^*)\) and \((\Theta_2, \delta_2^*)\) that are one the pseudo-inverse of the other. Since \( N_i^* \) is compact, it is easily seen that a positive constant \( c \) exists such that if \( x, x' \) belong to different connected components of the geodesic boundary of a piece \( P_i^* \) of \( T_i^* \), then \( \delta_i^*(x, x') \geq c \). Let us consider points \( x, x' \in \Theta_1 \) with \( cN \leq \delta_1^*(x, x') < c(N + 1) \).

Since \( \Theta_1 \) is a geodesic metric space, there exist an integer \( N' \leq N \) and points \( x_0 = x, x_1, \ldots, x_{N'+2} = x' \) with the following properties: \( \sum_{i=0}^{N'+1} \delta_1^*(x_i, x_{i+1}) = \delta_1^*(x, x') \), and \( x_i, x_{i+1} \) belong to the same closed piece of \( T_1^* \) for every \( i = 0, \ldots, N' + 1 \). Since \( \overline{\varphi} \) restricts to a \( k_3 \)-quasi-isometry on each piece of \( T_1^* \), we obtain

\[
\delta_2^*(\overline{\varphi}(x), \overline{\varphi}(x')) \leq \sum_{i=0}^{N'+1} \delta_2^*(\overline{\varphi}(x_i), \overline{\varphi}(x_{i+1})) \leq \sum_{i=0}^{N'+1} (k_3 \delta_1^*(x_i, x_{i+1}) + k_3) \leq k_3(1 + 1/c) \delta_1^*(x, x') + 2k_3.
\]

The same argument also shows that \( \delta_1^*(\overline{\varphi}^{-1}(y), \overline{\varphi}^{-1}(y')) \leq k_3(1 + 1/c) \delta_2^*(y, y') + 2k_3 \) for all \( y, y' \in \Theta_2 \). Suppose now that \( x \) belongs to a piece \( P_i^* \) of \( T_1^* \) and let \( P_2^* \) be the piece of \( T_2^* \) containing \( \overline{\varphi}(P_i^*) \). It is easily seen that \( \overline{\varphi}^{-1}(P_2^*) \) is contained in \( P_i^* \).

Moreover, by the very construction of \( \overline{\varphi} \) it follows that elements \( g_1, g_2 \) of \( \text{Isom}(\mathbb{H}^n) \) exist such that \( g_i(N_i^*) = P_i^*, \overline{\varphi}|_{P_i^*} = g_2 \overline{\varphi}^{-1} |_{P_i^*} \) and \( \overline{\varphi}^{-1}|_{P_i^*} = g_1 \overline{\varphi}^{-1}g_2^{-1}|_{P_i^*} \). This easily implies that \( \delta_1^*(x, \overline{\varphi}^{-1}(\overline{\varphi}(x))) = \delta_1^*(g_1^{-1}(x), \overline{\varphi}^{-1}(\overline{\varphi}(g_1^{-1}(x)))) \leq k_3 \). The same argument also shows that \( \delta_2^*(y, \overline{\varphi}(\overline{\varphi}^{-1}(y))) \leq k_3 \) for every \( y \in \Theta_2 \). By Lemma 3.2 this implies that \( \overline{\varphi}, \overline{\varphi}^{-1} \) restrict to quasi-isometries between \((\Theta_1, \delta_1^*)\) and \((\Theta_2, \delta_2^*)\) that are one the pseudo-inverse of the other.

Since \( \overline{\varphi}, \overline{\varphi}^{-1} \) are the conical extensions of their restrictions to \( \Theta_1, \Theta_2 \), the conclusion follows from the same argument yielding Proposition 3.3. \( \square \)

**Quasi-conformal homeomorphisms** Let an isometric identification of \( \mathbb{H}^n \) with the Poincaré disc model of hyperbolic \( n \)-space be fixed from now on, and put on \( \partial \mathbb{H}^n = S^{n-1} \) the Riemannian structure induced by the usual Euclidean metric on
We observe that such a structure is compatible with the canonical conformal structure of \( \partial \mathbb{H}^n \), and we denote by \( d_\partial \) the induced path metric on \( \partial \mathbb{H}^n \). For \( q \in \partial \mathbb{H}^n, \epsilon > 0 \) we set \( B(q, \epsilon) = \{ x \in \partial \mathbb{H}^n : d_\partial(q, x) \leq \epsilon \} \). Let \( f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n \) be a homeomorphism. For every \( q \in \partial \mathbb{H}^n \) we denote by \( K(f)_q \in [1, \infty] \) the number

\[
K(f)_q = \limsup_{\epsilon \to 0} \frac{\sup \{d_\partial(f(x), f(q)) : x \in B(q, \epsilon) \}}{\inf \{d_\partial(f(x), f(q)) : x \in B(q, \epsilon) \}}.
\]

The value \( K(f)_q \) gives a measure of how far \( f \) is from being conformal at \( q \), and does not depend on the choice of the identification of \( \mathbb{H}^n \) with the Poincaré model of hyperbolic \( n \)-space. We say that \( f \) is \( K \)-quasi-conformal at \( q \) if \( K(f)_q < K \), and that \( f \) is \( K \)-quasi-conformal if it is \( K \)-quasi-conformal at every point of \( \partial \mathbb{H}^n \). Moreover, \( f \) is said to be quasi-conformal if it is \( K \)-quasi-conformal for some \( K < \infty \).

Suppose \( f \) is differentiable at \( q \) and \( df_q \) is invertible, and recall that the Riemannian structure on \( \partial \mathbb{H}^n \) defines a metric on \( T(\partial \mathbb{H}^n)_q \) and on \( T(\partial \mathbb{H}^n)_f(q) \). If \( C \subset T(\partial \mathbb{H}^n)_q \) is a round sphere (not necessarily centered in \( 0 \)), then \( df_q(C) \subset T(\partial \mathbb{H}^n)_f(q) \) is an ellipsoid. We denote by \( K'(f)_q \) the ratio between the longest and the shortest axes of \( df_q(C) \), and observe that \( K'(f)_q \) is indeed well-defined, i.e., it does not depend on the choice of the sphere \( C \). The following lemma is straightforward:

**Lemma 3.8.** Let \( f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n \) be a homeomorphism, let \( q \in \partial \mathbb{H}^n \) and suppose that the differential of \( f \) at \( q \) exists and is invertible. Then \( K'(f)_q = K(f)_q \).

The following fundamental results are taken from [Mos68]:

**Proposition 3.9.** Let \( f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n \) be a quasi-conformal homeomorphism. Then the differential of \( f \) exists and is invertible almost everywhere (with respect to the Lebesgue measure).

**Proposition 3.10.** Suppose \( f : \partial \mathbb{H}^n \to \partial \mathbb{H}^n \) is a quasi-conformal homeomorphism that is 1-quasi-conformal almost everywhere (with respect to the Lebesgue measure). Then \( f \) is differentiable and 1-quasi-conformal everywhere, so it is the trace at infinity of a unique isometry of \( \mathbb{H}^n \).

**Extending \( \varphi \) to the conformal boundary** By Proposition 3.5, the quasi-isometries \( \varphi, \varphi^{-1} \) uniquely define extensions \( \partial \varphi, \partial \varphi^{-1} : \partial \mathbb{H}^n \to \partial \mathbb{H}^n \) that are one the inverse of the other (see also [Mos73, Thu79, BP92]). Moreover, \( \partial \varphi, \partial \varphi^{-1} \) are quasi-conformal homeomorphisms [Mos73].

Recall now that \( p_i : \mathbb{H}^n \to DN_i \) is the universal covering that extends the universal covering \( \tilde{N}_i \to N_i \). Let \( D\Gamma_i < \text{Isom}(\mathbb{H}^n) \) be the group of automorphisms of the covering \( p_i \). We have of course \( DN_i \cong \mathbb{H}^n/D\Gamma_i \) and \( \Gamma_i < D\Gamma_i \). A geodesic
hyperplane in \( \mathbb{H}^n \) is a face of \( \mathcal{T}_i \) if it is a boundary component of some piece of \( \mathcal{T}_i \). We set

\[
\mathcal{C}_i = \{ C \subset \partial \mathbb{H}^n : C \text{ is the boundary at infinity of a face of } \mathcal{T}_i \},
\]

\[
D \Lambda_i = D \Gamma_i \cdot \Lambda(\Gamma_i) \subset \partial \mathbb{H}^n ,
\]

and we observe that \( \bigcup_{C \in \mathcal{C}_i} C \subset D \Lambda_i \). Moreover, if \( C \in \mathcal{C}_1 \) then \( \partial \varphi(C) \in \mathcal{C}_2 \). It is well-known that the limit set of any geometrically finite subgroup of \( \text{Isom}(\mathbb{H}^n) \) has either null or full Lebesgue measure \([\text{Ahl}66]\) (recently, the same statement has been proved to hold for any discrete finitely generated subgroup of \( \text{Isom}(\mathbb{H}^n) \) \([\text{Ago}, \text{CG}]\)). Thus \( D \Lambda_i \) is a countable union of measure zero subsets of \( \partial \mathbb{H}^n \), and has itself zero Lebesgue measure.

**Deforming \( \varphi \) into an isometry** Recall that we have fixed on \( \partial \mathbb{H}^n \) a Riemannian metric induced by an identification of \( \mathbb{H}^n \) with the Poincaré disc model of hyperbolic \( n \)-space. We say that a point \( q \in \partial \mathbb{H}^n \) is secluded (with respect to \( \mathcal{T}_1 \)) if a sequence \( \{ B^i \}_{i \in \mathbb{N}} \) of round closed balls in \( \partial \mathbb{H}^n \) exists with the following properties: \( q \) belongs to the interior of \( B^i \) for every \( i \in \mathbb{N} \), the diameter of \( B^i \) tends to 0 as \( i \) tends to infinity, and the boundary of \( B^i \) is an element of \( \mathcal{C}_1 \) for every \( i \in \mathbb{N} \).

**Lemma 3.11.** Let \( q \) be a point in \( \partial \mathbb{H}^n \setminus D \Lambda_1 \). Then \( q \) is secluded with respect to \( \mathcal{T}_1 \).

**Proof:** Let \( \alpha : [0, \infty) \to \mathbb{H}^n \) be any geodesic ray with endpoint \( q \). If \( \alpha \) intersects a finite number of faces of \( \mathcal{T}_1 \), then \( q \) is contained in the closure at infinity of a piece of \( \mathcal{T}_1 \), whence in \( D \Lambda_1 \), a contradiction. Thus there exists an infinite set \( \{ F^i \}_{i \in \mathbb{N}} \) of faces of \( \mathcal{T}_1 \) such that \( F^i \) intersects the geodesic \( \alpha \) in a single point \( q^i = q \) in \( \mathbb{H}^n \). For \( i \in \mathbb{N} \) let \( C^i \) be the boundary at infinity of \( F^i \). Since \( q \notin D \Lambda_1 \), for every \( i \in \mathbb{N} \) we have \( q \notin C^i \), so a well-defined round ball \( B^i \subset \partial \mathbb{H}^n \) exists whose boundary is equal to \( C^i \) and whose interior contains \( q \). Also observe that (up to passing to a subsequence) we can assume that \( C^{i+1} \) is contained in \( B^i \) for every \( i \in \mathbb{N} \). In order to prove the lemma we only have to show that the diameter of \( B^i \) tends to 0 as \( i \) tends to infinity.

Suppose this is not true. In this case, since \( C^{i+1} \subset B^i \), it is easily seen that a limit round circle \( C^\infty \subset \partial \mathbb{H}^n \) exists such that \( \lim_{i \to \infty} C^i = C^\infty \) in the Hausdorff topology on closed subsets of \( \partial \mathbb{H}^n \). Let \( F^\infty \subset \partial \mathbb{H}^n \) be the hyperbolic hyperplane bounded by \( C^\infty \). Then \( \lim_{i \to \infty} F^i = F^\infty \) in the Hausdorff topology on closed subsets of \( \mathbb{H}^n \). Since the union of the faces of \( \mathcal{T}_1 \) is closed in \( \mathbb{H}^n \), this easily implies that \( F^\infty \) is a face of \( \mathcal{T}_1 \), so \( C^\infty \) belongs to \( \mathcal{C}_1 \) and is contained in \( D \Lambda_1 \). Since \( \lim_{i \to \infty} q^i = q \) we should have \( q \in F^\infty \cap \partial \mathbb{H}^n = C^\infty \), whence \( q \in D \Lambda_1 \), a contradiction. \( \square \)

**Lemma 3.12.** Let \( q \in \partial \mathbb{H}^n \) be secluded with respect to \( \mathcal{T}_1 \) and suppose that \( d(\partial \varphi)q \) exists and is invertible. Then \( \partial \varphi \) is \( 1 \)-quasi-conformal at \( q \).
Proof: Let us identify \( \partial \mathbb{H}^n \) with \( \mathbb{R}^{n-1} \cup \{\infty\} \) in such a way that \( q \) corresponds to \( 0 \in \mathbb{R}^{n-1} \). Without loss of generality, we can also assume that \( \partial \phi(q) = 0 \in \mathbb{R}^{n-1} \subset \partial \mathbb{H}^n \). Then there exists a family \( \{B^i\}_{i \in \mathbb{N}} \) of round balls in \( \mathbb{R}^{n-1} \subset \partial \mathbb{H}^n \) with the following properties: \( q \in \interior B^i \) for every \( i \in \mathbb{N} \); the Euclidean diameter of \( B^i \) tends to 0 as \( i \) tends to infinity; the boundary of \( B^i \) is an element \( C^i \) of \( C_1 \) for every \( i \in \mathbb{N} \).

Let \( \lambda_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \) be the linear dilation having ratio equal to the Euclidean diameter of \( B^i \). If we identify \( T(\partial \mathbb{H}^n) \) and \( T(\partial \mathbb{H}^n) \circ \phi(q) \) with \( T(\mathbb{R}^{n-1})_0 = \mathbb{R}^{n-1} \), then

\[
\lim_{i \to \infty} \lambda_i^{-1} \circ \partial \phi \circ \lambda_i = d(\partial \phi)_q \quad \text{uniformly on any compact subset of } \mathbb{R}^{n-1}.
\] (4)

Let \( B(0,1) \subset \mathbb{R}^{n-1} \) be the closed unitary ball centered in 0, and observe that the round sphere \( L^i = (\lambda_i)^{-1}(C^i) \) has Euclidean diameter equal to 1 and is contained in \( B(0,1) \). Since \( \partial \phi \) sends spheres in \( C_1 \) onto spheres in \( C_2 \), the sphere \( L^i \) is sent by \( \lambda_i^{-1} \circ \partial \phi \circ \lambda_i \) onto a round sphere. On the other hand, the linear map \( d(\partial \phi)_q \) sends \( L^i \) onto an ellipsoid \( E^i \). Let \( \nu_i \) be the ratio between the longest and the shortest axes of \( E^i \). Since \( d(\partial \phi)_q \) is invertible, there exists \( \delta > 0 \) such that the shortest axis of \( E^i \) is longer than \( \delta \) for every \( i \in \mathbb{N} \). Together with equality (4), this implies that \( \lim_{i \to \infty} \nu_i = 1 \). But \( \nu_i = K'(f)_q \) for every \( i \), whence \( K'(f)_q = 1 \). Now the conclusion follows from Lemma 3.8. \( \square \)

We recall that \( DA_i \) has zero Lebesgue measure, so by Proposition 3.9 Lemma 3.11 Lemma 3.12 and Proposition 3.10 the map \( \partial \phi \) is the trace at infinity of an element \( \widehat{\varphi} \) of \( \text{Isom}(\mathbb{H}^n) \).

Lemma 3.13. We have \( d^H(\varphi(x), \widehat{\varphi}(x)) \leq r \) for every \( x \in \mathbb{H}^n \), where \( r \) is a positive constant depending solely on \( k_4 \).

Proof: Let \( \alpha_1, \alpha_2 \subset \mathbb{H}^n \) be two geodesic lines intersecting perpendicularly at \( x \). By Proposition 3.4 the point \( \varphi(x) \) belongs both to the \( r' \)-neighbourhood of \( \widehat{\varphi}(\alpha_1) \) and to the \( r' \)-neighbourhood of \( \widehat{\varphi}(\alpha_2) \), where \( r' \) only depends on \( k_4 \). But \( \widehat{\varphi}(\alpha_1) \) intersects \( \widehat{\varphi}(\alpha_2) \) perpendicularly at \( \widehat{\varphi}(x) \), so \( d^H(\varphi(x), \widehat{\varphi}(x)) \leq r \), where \( r \) only depends on \( r' \), whence on \( k_4 \). \( \square \)

By construction, the trace at infinity of \( \widehat{\varphi} \) sends the boundary at infinity of any component of \( \partial \tilde{N}_1 \) into the boundary at infinity of a component of \( \partial \tilde{N}_2 \). Since \( \tilde{N}_i \) is the hyperbolic convex hull of its geodesic boundary for \( i = 1, 2 \), this implies that \( \widehat{\varphi} \) restricts to an isometry between \( \tilde{N}_1 \) and \( \tilde{N}_2 \).

4 Main results

For \( i = 1, 2 \) let \( N_i \) be a hyperbolic \( n \)-manifold with non-empty geodesic boundary, and suppose that if \( n = 3 \), then \( \partial N_i \) is compact. Let \( \tilde{N}_i \subset \mathbb{H}^n \) be the universal
covering of \( N_i \) and \( \Gamma_i \triangleleft \text{Isom}(\tilde{N}_i) < \text{Isom}(\mathbb{H}^n) \) be the fundamental group of \( N_i \). Fix a finite set \( S_i \) of generators of \( \Gamma_i \), fix a basepoint \( \tilde{x}_i \in \tilde{N}_i \), and consider \( \Gamma_i \) as a subset of \( \tilde{N}_i \) via the embedding \( \Gamma_i \ni \gamma \mapsto \gamma(\tilde{x}_i) \in \tilde{N}_i \). The following statement summarizes the results obtained in the preceding section.

**Theorem 4.1.** For any \( k \geq 1 \) there exists \( r = r(k) > 0 \) such that if \( \varphi : \mathcal{C}(\Gamma_1, S_1) \to \mathcal{C}(\Gamma_2, S_2) \) is a \( k \)-quasi-isometry, then an isometry \( \hat{\varphi} : \tilde{N}_1 \to \tilde{N}_2 \) exists such that \( d^{\hat{\varphi}}(\hat{\varphi}(\gamma), \varphi(\gamma)) \leq r \) for every \( \gamma \in \Gamma_1 \subset \tilde{N}_1 \).

**Quasi-isometry implies commensurability** We can now easily prove Theorem 1.1. Let \( N_1, N_2 \) be as above and suppose that \( \pi_1(N_1) \) is quasi-isometric to \( \pi_1(N_2) \). By Theorem 4.1, \( N_1 \) and \( N_2 \) have isometric universal coverings. By Lemma 2.4, this implies that \( N_1 \) is commensurable with \( N_2 \).

**Commensurator and quasi-isometry group** We now prove Theorem 1.2. Let \( N \) be a hyperbolic \( n \)-manifold with non-empty geodesic boundary, let \( \tilde{N} \subset \mathbb{H}^n \) (resp. \( \tilde{N}^* \subset \mathbb{H}^n \)) be the universal covering (resp. the neutered universal covering) of \( N \), and denote by \( d^* \) the path metric on \( \tilde{N}^* \). As before, we fix a point \( \tilde{x} \in \tilde{N}^* \subset \tilde{N} \), and consider the fundamental group \( \Gamma < \text{Isom}(\tilde{N}) < \text{Isom}(\mathbb{H}^n) \) of \( N \) as a subset of \( \tilde{N}^* \subset \tilde{N} \).

Let \( \varphi \) be a quasi-isometry of \( \Gamma \) into itself, and denote by \( [\varphi] \) the equivalence class of \( \varphi \) in \( \text{QIsom}(\Gamma) \). By Theorem 4.1, there exist \( r > 0 \) and an isometry \( \hat{\varphi} : \tilde{N} \to \tilde{N} \) such that \( d^{\hat{\varphi}}(\hat{\varphi}(\gamma), \varphi(\gamma)) \leq r \) for every \( \gamma \in \Gamma \). It is easily seen that this condition uniquely determines \( \hat{\varphi} \), and that \( \hat{\varphi} \) only depends on the equivalence class of \( \varphi \). Thus the map
\[
\rho : \text{QIsom}(\Gamma) \to \text{Isom}(\tilde{N}), \quad \rho([\varphi]) = \hat{\varphi}
\]
gives a well-defined group homomorphism.

Injectivity of \( \rho \) is obvious. Moreover, any isometry of \( \tilde{N} \) restricts to an isometry of \( (\tilde{N}^*, d^*) \), whence to a quasi-isometry of \( \Gamma \subset \tilde{N}^* \). This implies that \( \rho \) is surjective. Recall now that the left translation by an element \( \gamma \in \Gamma \) naturally defines an element of \( \text{QIsom}(\Gamma) \), still denoted by \( \gamma \). By construction we have \( d^{\hat{\varphi}}(\hat{\varphi}(\gamma), \rho(\gamma)(\tilde{x})) \leq r \) for every \( \gamma \in \Gamma \), and this easily implies that \( \rho(\gamma) = \gamma \) for every \( \gamma \in \Gamma \). This concludes the proof of Theorem 1.2.

**Quasi-isometric rigidity** We now prove Theorem 1.4. We keep the notation from the preceding paragraph. Let \( G \) be a finitely generated group and suppose \( \varphi : G \to \Gamma = \pi_1(N) \) is a \( k \)-quasi-isometry with \( k \)-pseudo-inverse \( \varphi^{-1} \). Without loss of generality, we can also suppose that \( \varphi(1_G) = 1_\Gamma \) and \( \varphi^{-1}(1_\Gamma) = 1_G \). Fix an element \( g \) in \( G \). The left translation by \( g \) defines an isometry of any Cayley graph.
of $G$, whence a $k'$-quasi-isometry $\mu(g)$ of $\Gamma$, where $k'$ depends on $k$ but not on $g$. By Theorem 4.1 there exist a constant $r > 0$ and an isometry $\nu(g) \in \text{Isom}(\tilde{N})$ such that $d^{\tilde{H}}(\mu(g)(\gamma)(\tilde{x}), \nu(g)(\gamma(\tilde{x}))) \leq r$ for every $\gamma \in \Gamma$. Taking $\gamma = 1_{\Gamma}$, we get in particular
\[
d^{\tilde{H}}(\varphi(g)(\tilde{x}), \nu(g)(\tilde{x})) \leq r, \tag{5}\]
where $r$ does not depend on $g$.

It is easily seen that the map $\nu : G \to \text{Isom}(\tilde{N})$ is a group homomorphism. We are going to show that $\nu$ has finite kernel and discrete image. Let $\{g_n\}_{n \in \mathbb{N}} \subset G$ be any sequence such that $\lim_{n \to \infty} \nu(g_n) = 1_{\Gamma}$. Up to discarding a finite number of terms, by equation (5) we can suppose that
\[
d^{\tilde{H}}(\varphi(g_n)(\tilde{x}), \tilde{x}) \leq r + 1 \quad \text{for every } n \in \mathbb{N}. \tag{6}\]

Since $\Gamma$ acts properly discontinuously on $\tilde{N}$, this implies that the set $\{\varphi(g_n)\}_{n \in \mathbb{N}} \subset \Gamma$ is finite, and since $\varphi : G \to \Gamma$ is a quasi-isometry, this gives in turn that $\{g_n\}_{n \in \mathbb{N}} \subset G$ is a finite set. Thus $\nu(G) \subset \text{Isom}(\tilde{N})$ is discrete and $\nu$ has finite kernel.

Let now $q$ be a point in $\Lambda(\Gamma) = \Lambda(\text{Isom}(\tilde{N}))$ and let $\{\gamma_n\}_{n \in \mathbb{N}} \subset \Gamma$ be a sequence with $\lim_{n \to \infty} \gamma_n(\tilde{x}) = q$. Since $\varphi$ is a $k$-quasi-isometry and $d^{\tilde{H}} \leq \delta^* \leq \delta^*$ on $\tilde{N}^*$, we have $d^{\tilde{H}}(\varphi(\varphi^{-1}(\gamma_n))(\tilde{x}), \gamma_n(\tilde{x})) \leq r'$, where $r'$ is a positive constant which does not depend on $n$. Thus, if $g_n = \varphi^{-1}(\gamma_n)$, by equation (5) we have
\[
d^{\tilde{H}}(\nu(g_n)(\tilde{x}), \gamma_n(\tilde{x})) \leq d^{\tilde{H}}(\nu(g_n)(\tilde{x}), \varphi(g_n)(\tilde{x})) + d^{\tilde{H}}(\varphi(g_n)(\tilde{x}), \gamma_n(\tilde{x})) \leq r''\]
where $r''$ is a positive constant which does not depend on $n$. Thus $\lim_{n \to \infty} \nu(g_n)(\tilde{x}) = \lim_{n \to \infty} \gamma_n(\tilde{x}) = q$, and $\Lambda(\nu(G)) = \Lambda(\Gamma)$. Recall now that if $\Delta$ is a geometrically finite subgroup of $\text{Isom}(\mathbb{H}^n)$ whose limit set is not equal to the whole of $\partial \mathbb{H}^n$, then any finitely generated subgroup of $\Delta$ is geometrically finite (see e.g. [Mor84, Proposition 7.1]). In our context, this implies that $\nu(G)$ is geometrically finite, so the convex core $N_G := \tilde{N} / \nu(G)$ is a finite-volume orbifold. If $N_{\text{orb}} = \tilde{N} / \text{Isom}(\tilde{N})$, then
\[
[\text{Isom}(\tilde{N}) : \nu(G)] = \text{vol } N_G / \text{vol } N_{\text{orb}} < \infty.
\]
This shows that $\nu(G)$ is commensurable with $\Gamma$.

5 Counterexamples

We now show that the conclusions of Theorems 1.1, 1.2 are no longer true if we consider hyperbolic 3-manifolds with non-compact geodesic boundary.
Non-commensurable manifolds sharing the same fundamental group
We begin by recalling that Thurston’s hyperbolization theorem for Haken manifolds \cite{Thu82} gives necessary and sufficient topological conditions on a 3-manifold to be hyperbolic with geodesic boundary:

**Theorem 5.1.** Let $M$ be a compact orientable 3-manifold with non-empty boundary, let $T$ be the set of boundary tori of $M$ and let $A$ be a family of disjoint closed annuli in $\partial M \setminus T$. Then $M = M \setminus (T \cup A)$ is hyperbolic if and only if the following conditions hold:

- the components of $\partial M$ have negative Euler characteristic;
- $M \setminus A$ is boundary-irreducible and geometrically atoroidal;
- the only proper essential annuli contained in $M$ are parallel in $M$ to the annuli in $A$.

The following proposition readily implies Theorem 1.6.

**Proposition 5.2.** Let $N$ be any orientable hyperbolic 3-manifold with compact non-empty geodesic boundary. Then there exists a hyperbolic 3-manifold with non-compact geodesic boundary which is homotopically equivalent to $N$ but not commensurable with $N$.

**Proof:** Let $\alpha$ be a simple essential loop in $\partial N$. We define $N'$ as $N \setminus \alpha$ and note that $N$ and $N'$ have a common compactification $\overline{N} = \overline{N}'$ such that $N = \overline{N} \setminus T$, $N' = \overline{N} \setminus (T \cup A)$, where $T$ is the family of the boundary tori of $\overline{N}$ and $A$ is a closed regular neighbourhood of $\alpha$ in $\partial N$. Moreover, it is easily seen that since $(\overline{N}, T, \emptyset)$ satisfies the assumptions of Theorem 5.1, so does $(\overline{N}, T, A)$, so $N'$ is hyperbolic. Of course $N'$ is homotopically equivalent to $N$.

Since $\partial N$ is compact, there do not exist different components of $\partial \tilde{N} \subset \mathbb{H}^3$ that meet in $\partial \mathbb{H}^n$. On the other hand, since $\partial N'$ is non-compact a pair $(S^1, S^2)$ of components of $\partial \tilde{N}' \subset \mathbb{H}^3$ exists such that $\overline{S^1} \cap \overline{S^2} = \{\text{pt.}\} \in \partial \mathbb{H}^3$. Thus $\tilde{N}$ is not isometric to $\tilde{N}'$, and $N$ is not commensurable with $N'$.

A hyperbolic 3-manifold with free non-Abelian fundamental group In this paragraph we prove Theorem 1.7. To this aim we construct a hyperbolic 3-manifold $M$ with non-compact geodesic boundary which is homotopically equivalent to a genus-2 handlebody. The fundamental group of $M$ is isomorphic to $\mathbb{Z} * \mathbb{Z}$, and it is well-known that (equivalence classes of) left translations define a subgroup of infinite index in the quasi-isometry group of $\mathbb{Z} * \mathbb{Z}$. On the other hand, by Lemma 2.5
Figure 2: The manifold $M$ is obtained by gluing in pairs the non-shadowed faces of the regular ideal octahedron along suitable isometries.

The group $\text{Comm}(\pi_1(M))/\pi_1(M)$ is finite, so $M$ provides the manifold required in the statement of Theorem 1.7.

The following explicit construction of $M$ is taken from Fri04. Let $O \subset \mathbb{H}^3$ be the regular ideal octahedron and let $v_1, \ldots, v_6$ be the vertices of $O$ as shown in Figure 2. We denote by $F_{ijk}$ the face of $O$ with vertices $v_i, v_j, v_k$. Let $g : F_{134} \to F_{156}$ be the unique isometry such that $g(v_1) = v_1, g(v_3) = v_6$ and $g(v_4) = v_5$, and $h : F_{236} \to F_{254}$ be the unique isometry such that $h(v_2) = v_2, h(v_3) = v_4, h(v_6) = v_5$. We define $M$ to be the manifold obtained by gluing $O$ along $g$ and $h$. Since all the dihedral angles of $O$ are right, it is easily seen that the metric on $O$ induces a complete finite-volume hyperbolic structure on $M$ such that the shadowed faces in Figure 2 are glued along their edges to give a non-compact totally geodesic boundary.

From a topological and combinatorial point of view, an ideal octahedron with four marked faces as in Figure 2 is equivalent to a truncated tetrahedron with the edges connecting truncation triangles removed, which is in turn equivalent to a “tetrapod” with six arcs connecting circular ends removed, as shown in Figure 3. Under this identification, the four shadowed ideal faces of $O$ correspond to the four regions into which the lateral surface of the tetrapod is cut by the 6 arcs, while the non-shadowed ideal faces of $O$ correspond to the four discs at the ends of the tetrapod. Therefore the manifold $M$ is obtained from the tetrapod by suitably gluing together in pairs the discs at its four ends. So this manifold is homeomorphic to a handlebody with boundary loops removed. Using this correspondence we can easily draw a picture.
of the natural compactification of $M$. This picture is shown in Figure 4. For a more detailed description of the natural compactification of hyperbolic 3-manifolds with non-compact geodesic boundary obtained by gluing regular ideal octahedra see [CFMP].

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