Asymptotic formulas for the number of the partitions into summands of the form $\lfloor \alpha m \rfloor$

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Abstract

Let $\alpha > 1$ be an irrational number. We establish asymptotic formulas for the number of partitions of $n$ into summands and distinct summands, chosen from the sequence $\{\lfloor \alpha m \rfloor \}_{m \in \mathbb{N}}$. This improves some results of Erdős and Richmond established in 1977.

1 Introduction and statement of results

A partition of an integer $n$ is a sequence of non-increasing positive integers whose sum equals $n$. The study of the asymptotic behavior of various types of integer partition has a long history, see Hardy and Ramanujan [HR18], Ingham [Ing41], Roth and Szekeres [RS54], Meinardus [Mei54] and Richmond [Ric75] for example. One of the most celebrated result is the asymptotic formula for $p(n)$, the number of unrestricted partitions of $n$. Hardy and Ramanujan [HR18] proved

$$p(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi \sqrt{2n/3}},$$ (1.1)

as integer $n \to +\infty$. They also established an asymptotic formula for $q(n)$, the number of partitions of $n$ with unequal parts, that is

$$q(n) \sim \frac{1}{4 \cdot 3^{1/4} \cdot n^{3/4}} e^{\pi \sqrt{n/3}},$$

as integer $n \to +\infty$.

Let $\alpha > 1$ be an irrational number and $\lfloor x \rfloor$ denotes the largest integer $\leq x$. As a natural extension which has not previously appeared in the literature, Erdős and Richmond [ER78] investigate the asymptotic behaviour of the numbers $p_\alpha(n)$ and $q_\alpha(n)$, the number of partitions of $n$ into summands and distinct summands, respectively, chosen from the sequence $\{\lfloor \alpha m \rfloor \}_{m \in \mathbb{N}}$. They gave asymptotic formulas with an error term of $p_\alpha(n)$ and $q_\alpha(n)$ for almost all $\alpha$ (in the Lebesgue sense).
Partition into summands of the form $|\alpha m|$

To introduce the main results of Erdös and Richmond [ER78] conveniently, we introduce the definition of *irrationality measure*. Let $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$ and $\alpha \in \mathbb{R}$. Recall that $\mu \in \mathbb{R}$ is an irrationality exponent of $\alpha$ if

$$0 < q^{-1}\|q\alpha\| < q^{-\mu},$$

has (at most) finitely many solutions $q \in \mathbb{N}$. We denote by $\mu(\alpha)$ the infimum of such irrationality exponents $\mu$ and call it the irrationality measure of $\alpha$. If $\mu(\alpha) = \infty$ then we call $\alpha$ a *Liouville number*. We note that the irrationality measure of an irrational number is always $\geq 2$.

One of the critical issues in [ER78] is the convergence of the following Dirichlet series

$$J_\alpha(s) := \sum_{\ell \geq 1} \tilde{B}_1(\alpha \ell) \ell^s, \quad (1.2)$$

where $\tilde{B}_1(x) = \{x\} - 1/2$, and $\{x\} := x - \lfloor x \rfloor$ is the fractional part of $x$. Hardy and Littlewood [HL22, p. 248, (a)] proved that if $\mu(\alpha) \in [2, \infty)$ then $J_\alpha(s)$ is convergent for $s > 1 - 1/(\mu(\alpha) - 1)$\footnote{The condition stated in [HL22, pp. 213, Equation (1.331)] is

$$n^h |\sin(\alpha n \pi)| \geq A > 0$$

for all $n \in \mathbb{N}$. This is equivalent to our definition for irrationality exponent when letting $\mu = 1 + h$.}. Thanks to this result, Erdös and Richmond [ER78] established asymptotic formulas with error term of $p_\alpha(n)$ and $q_\alpha(n)$ when irrational number $\alpha > 1$ has a finite irrationality measure $\mu(\alpha)$, see [ER78, Theorem 2]\footnote{The condition stated in [ER78] is there exist $\lambda \in \mathbb{R}$ such that

$$|\ell^{1+\lambda+\varepsilon} \sin(\alpha \ell \pi)| \to \infty,$$

holds for any $\varepsilon > 0$, as integer $\ell \to \infty$. This is equivalent to our definition for irrationality exponent when letting $\mu = 2 + \lambda$.}. However, for $\alpha > 1$ being a Liouville number, that is $\mu(\alpha) = \infty$, they only can prove

$$\log q_\alpha(n) = \pi \sqrt{\frac{n}{3\alpha}} + O(n^\varepsilon)$$

and

$$\log p_\alpha(n) = \pi \sqrt{\frac{2n}{3\alpha}} + O(n^\varepsilon),$$

for any $\varepsilon > 0$.

In this paper, we are interesting the asymptotic formulas of $p_\alpha(n)$ and $q_\alpha(n)$, when $\alpha > 1$ being a Liouville number. The main result of this paper is the following Theorem 1.1.

\footnote{Note that in the statement of [ER78, Theorem 2], as well as its proof, there exist serval typos. For the corrected leading asymptotic formulas of $p_\alpha(n)$ and $q_\alpha(n)$ with $\mu(\alpha) < \infty$, see Theorem 1.1 of this paper.}

\section*{1. Theorem 1.1.}
Theorem 1.1. Let \( \alpha > 1 \) be irrational number. As \( n \to \infty \)
\[
q_\alpha(n) \sim \frac{\exp \left( \pi \sqrt{\frac{2\alpha}{3\alpha}} \right)}{2^{2-1/2\alpha}(3\alpha)^{1/4}n^{3/4}},
\]
and
\[
p_\alpha(n) = \frac{\exp \left( \frac{2\pi \sqrt{\frac{2\alpha}{3\alpha}}}{n^{1-1/4\alpha}} \right)}{n^{1-1/4\alpha+\alpha(1)}}.
\]
(1.3)

Furthermore, if the series (1.2) of \( J_\alpha(1) \) is convergent, then
\[
p_\alpha(n) \sim \frac{\exp \left( \frac{2\pi \sqrt{\frac{2\alpha}{6\alpha}}}{\Lambda_\alpha n^{1-1/4\alpha}} \right)}{\Lambda_\alpha n^{1-1/4\alpha}},
\]
(1.4)

where
\[
\Lambda_\alpha = 4\sqrt{3}(\pi e^{-\gamma})^{1/20} \left( \frac{\alpha}{6} \right)^{1/4\alpha} \prod_{\ell \geq 1} \left( 1 - \frac{\{\alpha\ell\}}{\alpha\ell} \right) e^{\frac{1}{12\alpha}}.
\]
(1.5)

Remark 1.1. Theorem 1.1 is new for \( \alpha > 1 \) being a Liouville number. Surprisingly, for all irrational numbers \( \alpha > 1 \), we can find an asymptotic formula of type (1.1) for \( q_\alpha(n) \). However, for \( p_\alpha(n) \) if there is no additional assumption, such as the convergence of the series (1.2) of \( J_\alpha(1) \), we can only get (1.3) at present.

By using a result of Ostrowski [Ost22] on Diophantine approximation, we give an effective condition for the validity of the asymptotic formula (1.4).

Corollary 1.2. If there exists a function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) of which is decreasing for all sufficiently large \( x \) such that
\[
1 + \log x + 1 \right)^{-1-\delta}, (\delta > 0), \text{ then } \psi(x) \text{ satisfies the conditions of Corollary 1.2. If } \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ has a finite irrationality measure, that is there exist a } \lambda \geq 2 \text{ such that}
\]
\[
q^{-1}||q\alpha|| > e^{-\psi(q)}.
\]
(1.7)

Then the series (1.2) of \( J_\alpha(1) \) is convergent. Further, (1.4) in Theorem 1.1 holds.

Remark 1.2. It might be an interesting problem that whether the series (1.2) of \( J_\alpha(1) \) is convergent for all irrational numbers \( \alpha \).

Remark 1.3. Taking \( \psi(x) = x(\log x + 1)^{-1-\delta}, (\delta > 0), \text{ then } \psi(x) \text{ satisfies the conditions of Corollary 1.2. If } \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ has a finite irrationality measure, that is there exist a } \lambda \geq 2 \text{ such that}
\]
\[
q^{-1}||q\alpha|| \gg q^{-\lambda},
\]
for all integer \( q \geq 1 \). Then, since \( q^{-\lambda} \gg e^{-\psi(q)} \), we have \( \alpha > 1 \) satisfies the conditions of Corollary 1.2. Therefore, we can see from Wolfram Web Resources [Wei20] that \( \alpha \) can take all irrational algebraic numbers greater than 1, 1 + log 2, log 3, e, \pi, \pi^2, \zeta(3) \) and so on.
We shall give some numerical data to support Theorem 1.1 and Corollary 1.2. Let us consider the cases of $\alpha = \sqrt{2}$. One can prove in Appendix A that

$$\Lambda_{\sqrt{2}} \in (5.7331, 5.7339).$$

Denoting by

$$\hat{q}_{\sqrt{2}}(n) = n^{-3/4}e^{\pi\sqrt{2}}$$

and

$$\hat{p}_{\sqrt{2}}(n) = n^{-1 + 1/4\sqrt{2}}e^{\pi\sqrt{2}}.$$

Then, with the help of Mathematica, the use of Theorem 1.1 gives

$$\hat{q}_{\sqrt{2}}(n) \sim (4.493 \cdots)q_{\sqrt{2}}(n), \text{ and } \hat{p}_{\sqrt{2}}(n) \sim (5.773 \cdots)p_{\sqrt{2}}(n). \quad (1.8)$$

We illustrate some of our results in the following (All computations are done in Mathematica).

| $n$  | $q_{\sqrt{2}}(n)$ | $\hat{q}_{\sqrt{2}}(n)$ | $\hat{q}_{\sqrt{2}}(n)/q_{\sqrt{2}}(n)$ |
|------|------------------|--------------------------|---------------------------------------|
| 50   | 552              | 2568.04                  | $\sim 4.65225$                        |
| 100  | 28870            | 13302.60                 | $\sim 4.60774$                       |
| 200  | 9582142          | 4.38416 \cdot 10^7       | $\sim 4.57534$                       |
| 400  | 43472367216      | 1.97845 \cdot 10^{11}    | $\sim 4.55105$                       |
| 800  | 7972258288121676 | 3.61410 \cdot 10^{16}    | $\sim 4.53334$                       |
| 1600 | 273790497114888860128182 | 1.23780 \cdot 10^{24}     | $\sim 4.52097$                       |

| $n$  | $p_{\sqrt{2}}(n)$ | $\hat{p}_{\sqrt{2}}(n)$ | $\hat{p}_{\sqrt{2}}(n)/p_{\sqrt{2}}(n)$ |
|------|------------------|--------------------------|---------------------------------------|
| 25   | 560              | 3412.03                  | $\sim 6.09291$                       |
| 50   | 28086            | 167998.0                 | $\sim 5.98154$                       |
| 100  | 8892735          | 5.26274 \cdot 10^7       | $\sim 5.91802$                       |
| 200  | 38427241214      | 2.25740 \cdot 10^{11}    | $\sim 5.87448$                       |
| 400  | 6706078262183805 | 3.91959 \cdot 10^{16}    | $\sim 5.84843$                       |
| 800  | 219091729965354807601257 | 1.27599 \cdot 10^{24}     | $\sim 5.82401$                       |

Comparing the above two tables with (1.8), we see that the numerical data supports our main result Theorem 1.1 and Corollary 1.2.

2 The proof of the main results

In the section, we prove Theorem 1.1 and Corollary 1.2. Let $\alpha > 1$ be an irrational number and $t > 0$. Denoting by

$$L_{\alpha}(t) = - \sum_{\ell \geq 1} \log \left( 1 - e^{-t[\alpha\ell]} \right).$$
The following Proposition 2.1 follows from Erdős and Richmond [ER78, Theorem 1].

**Proposition 2.1.** As $n \to +\infty$

$$q_\alpha(n) \sim \frac{\exp(L_\alpha(x) - L_\alpha(2x) + nx)}{\sqrt{2\pi(L_\alpha''(n) - 4L_\alpha''(2x))}},$$

where $x \in \mathbb{R}_+$ solve the equation: $L'_\alpha(x) - 2L'_\alpha(2x) + n = 0$. Furthermore, as $n \to +\infty$

$$p_\alpha(n) \sim \frac{\exp(L_\alpha(y) + ny)}{\sqrt{2\pi L_\alpha''(n)}},$$

where $y \in \mathbb{R}_+$ solve the equation: $L'_\alpha(y) + n = 0$.

We see from Erdős and Richmond [ER78] that Proposition 2.1 follows from the work of Roth and Szekeres [RS54], as well as the equidistribution properties of the sequence $\{\lfloor\alpha m\rfloor\}_{m \in \mathbb{N}}$. Under above Proposition 2.1, Theorem 1.1 will follows from the following proposition.

**Proposition 2.2.** Let $t \to 0^+$. For any irrational $\alpha > 1$ we have

$$L_\alpha(t) = \frac{\pi^2}{6\alpha t} + \frac{1 - \alpha^{-1}}{2} \log t + o(|\log t|),$$

$$L_\alpha(t) - L_\alpha(2t) = \frac{\pi^2}{12\alpha t} - \frac{1 - \alpha^{-1}}{2} \log 2 + o(1),$$

$$L'_\alpha(t) = -\frac{\pi^2}{6\alpha t^2} + \frac{1 - \alpha^{-1}}{2t} + o\left(\frac{1}{t}\right),$$

and

$$L''_\alpha(t) = \frac{\pi^2}{3\alpha t^3} + O\left(\frac{1}{t^2}\right).$$

Furthermore, if the series (1.2) of $J_\alpha(1)$ is convergent then

$$L_\alpha(t) = \frac{\pi^2}{6\alpha t} + \frac{1 - \alpha^{-1}}{2} \log t + c_\alpha + o(1),$$

where

$$c_\alpha = \frac{\gamma}{2\alpha} - \frac{1}{2} \log (2\pi) + \frac{1 - \alpha^{-1}}{2} \log \alpha + \sum_{\ell \geq 1} \left(\frac{1}{2\alpha \ell} + \log \left(1 - \frac{\{\alpha \ell\}}{\alpha \ell}\right)\right).$$

This proposition is a direct consequence of Proposition 3.1, Lemma 3.2, Lemma 3.2, and Lemma 3.4 of Section 3.
We now prove Theorem 1.1. We just give the proof for $p_\alpha(n)$, the proof for $q_\alpha(n)$ is similar. Using Proposition 2.1 and Proposition 2.2, we find that

$$n + \frac{\pi^2}{6\alpha y^2} - \frac{1 - \alpha^{-1}}{2y} (1 + o(1)) = 0,$$

as $n \to \infty$. This immediately implies

$$y = \frac{\pi}{\sqrt{6\alpha n}} - \frac{1 - \alpha^{-1}}{4n} (1 + o(1)), \quad (2.1)$$

as $n \to \infty$. Substituting (2.1) into Proposition 2.1 and Proposition 2.2, by simplification we obtain the proof of Theorem 1.1 for $p_\alpha(n)$.

We now give the sketch of the proof Corollary 1.2. Clearly, we just need to prove under the conditions of Corollary 1.2, the series (1.2) of $J_\alpha(1)$ is convergent. Using integration by parts for Riemann-Stieltjes integrals, the convergence of the series (1.2) easily follows the following.

**Proposition 2.3.** Let $\alpha$ be satisfies the conditions of Corollary 1.2. Then, there exist a constant $c_0 > 0$ such that

$$\sum_{1 \leq \ell \leq x} \tilde{B}_1(\alpha \ell) \ll \frac{x\psi(c_0 \log x)}{\log x},$$

for all sufficiently large $x$.

**Proof.** Since for any integer $q \geq 1$,

$$q^{-1}\|qa\| > e^{-q\psi(q)}$$

means that for any $p \in \mathbb{Z}$ with $\gcd(p, q) = 1$, there exist a constant $C > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > Ce^{-q\psi(q)}.$$

Then, we can use the same idea for the cases of $\alpha$ such that

$$\left| \alpha - \frac{p}{q} \right| > e^{-\lambda q}, \quad (\lambda > 0 \text{ is a constant}),$$

in Ostrowski [Ost22, p.83] to prove

$$S(x) = \sum_{1 \leq \ell \leq x} \tilde{B}_1(\alpha \ell) \ll \frac{x\psi(c_0 \log x)}{\log x}, \quad (2.2)$$

for sufficiently large $x$, where $c_0 > 0$ is a constant. This completes the proof of the proposition. \qed
3 The proof of Proposition 2.2

The following proposition gives a very well decomposition of the logarithm of the generating function $L_\alpha(t)$. From which we can find the main contribution of the asymptotics of the generating function.

Proposition 3.1. Let $t > 0$. We have

$$L_\alpha(t) = L_1(\alpha t) + 2^{-1}t D(\alpha t) + R_\alpha(t) + E_\alpha(t),$$

where

$$D(t) = \sum_{n \geq 1} \frac{1}{e^{nt} - 1}, \quad R_\alpha(t) = \sum_{\ell \geq 1} \frac{t \bar{B}_1(\alpha \ell)}{e^{\alpha \ell} - 1},$$

and

$$E_\alpha(t) = \sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} \int_0^u \frac{du}{e^{\alpha \ell} - 1} \int_0^{(\alpha \ell - v)t/2} K \left( \frac{(\alpha \ell - v)t}{2} \right) \frac{dv}{(\alpha \ell - v)^2}$$

with $K(u) = u^2 / \sinh^2(u)$.

Proof. First of all, by a direct calculation, we find that

$$L_\alpha(t) = -\sum_{\ell \geq 1} \log \left( 1 - e^{-\alpha \ell t} \right) + \sum_{\ell \geq 1} \int_{\ell \{\alpha \ell\}}^{\alpha \ell t} d \log \left( 1 - e^{-u} \right)$$

$$= L_1(\alpha t) + \sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} \int_0^{u} \frac{t du}{e^{\alpha \ell t} - 1}. $$

For the second sum above, we split that

$$\sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} \int_0^{u} \frac{t du}{e^{\alpha \ell t} - 1} = \sum_{\ell \geq 1} \frac{t \{\alpha \ell\}}{e^{\alpha \ell t} - 1} + \sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} \int_0^{u} \frac{t^2 e^{(\alpha \ell - u)t}}{(e^{\alpha \ell t} - 1)^2} dv$$

$$= \frac{t}{2} \sum_{\ell \geq 1} \frac{1}{e^{\alpha \ell t} - 1} + \sum_{\ell \geq 1} \frac{t \bar{B}_1(\alpha \ell)}{e^{\alpha \ell t} - 1} + E_\alpha(t),$$

where $\bar{B}_1(x) = \{x\} - 1/2$, and

$$E_\alpha(t) = \sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} \int_0^{u} \left[ \frac{(\alpha \ell - v)t/2}{\sinh((\alpha \ell - v)t/2)} \right]^2 \frac{dv}{(\alpha \ell - v)^2}. $$

This completes the proof of the proposition. \qed

In the following content, we estimate each component of the above decomposition of the logarithm of the generating function $L_\alpha(t)$. 

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Lemma 3.2. Let \( t \to 0^+ \). Denoting by
\[
\hat{L}_\alpha(t) = L_1(\alpha t) + 2^{-1} t D(\alpha t),
\]
then we have
\[
\hat{L}_\alpha(t) = \frac{\pi^2}{6\alpha t} + \frac{1-\alpha^{-1}}{2} \log t + \frac{1-\alpha^{-1}}{2} \log \alpha + \frac{\gamma}{2\alpha} - \frac{1}{2} \log (2\pi) + O(t^{1/2}),
\]
\[
\hat{L}_\alpha'(t) = -\frac{\pi^2}{6\alpha t^2} + \frac{1-\alpha^{-1}}{2t} + O(t^{-1/2}),
\]
and
\[
\hat{L}_\alpha''(t) = \frac{\pi^2}{3\alpha t^2} + O \left( \frac{1}{t^2} \right).
\]

Proof. Let \( t \to 0^+ \). Notice that
\[
D(t) = \sum_{n \geq 1} \frac{e^{-nt}}{1-e^{-nt}} = \sum_{n \geq 1} \tau(n)e^{-nt},
\]
where \( \tau(n) = \sum_{d|n} 1 \) is the divisor function. Using the well-known fact for divisor function \( \tau(n) \) that
\[
\sum_{n \leq x} \tau(n) = (\log x + 2\gamma - 1)x + O(x^{1/2}),
\]
we have
\[
t \sum_{n \geq 1} n^j \tau(n)e^{-nt} = t \int_{1}^{\infty} x^j e^{-xt} \left( \sum_{n \leq x} \tau(n) \right) dx
\]
\[
= t \int_{1}^{\infty} x^j e^{-xt} d \left( (\log x + 2\gamma - 1)x + O(t^{1/2-j}) \right)
\]
\[
= t^{-j} \int_{t}^{\infty} x^j (\log x + 2\gamma - \log t)e^{-x} dx + O(t^{1/2})
\]
\[
= j!t^{-j}(\gamma - \log t) + t^{-j} \int_{0}^{\infty} x^j (\log x + \gamma)e^{-x} dx + O(t^{1/2-j}).
\]

Further, by note that
\[
\int_{0}^{\infty} (\log x + \gamma)e^{-x} dx = 0 \quad \text{and} \quad \int_{0}^{\infty} x(\log x + \gamma)e^{-x} dx = 1,
\]
which immediately implies
\[
tD(t) = \gamma - \log t + O(t^{1/2}), \quad (tD(t))' = -t^{-1} + O(t^{-1/2}), \quad \text{and} \quad (tD(t))'' \ll t^{-2},
\]

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by a direct calculation. Finally, together with the well-known transform relation that

\[ L_1(t) = -\frac{t}{24} - \frac{1}{2} \log(2\pi) + \frac{1}{2} \log t + \frac{\pi^2}{6t} + L_1 \left( \frac{4\pi^2}{t} \right), \]

see for example [HR18, Equation (1.42)], and above estimates for \( D(t) \) we immediately obtain the proof of the lemma.

**Lemma 3.3.** Let \( t \to 0^+ \). We have

\[ E_\alpha(t) = E_\alpha(0) + o(1) \]

with

\[ E_\alpha(0) = -\sum_{\ell \geq 1} \left( \frac{\{\alpha \ell\}}{\alpha \ell} + \log \left( 1 - \frac{\{\alpha \ell\}}{\alpha \ell} \right) \right), \]

and for each integer \( k \geq 1 \),

\[ E^{(k)}_\alpha(t) \ll_k t^{1-k}. \]

**Proof.** The proof for the value of \( E_\alpha(0) \) is direct by use of Lebesgue’s Dominated Convergence Theorem. We now prove the estimates for the derivative of \( E_\alpha(t) \). Using the definition of \( E_\alpha(t) \), we have

\[ E^{(k)}_\alpha(t) = \sum_{\ell \geq 1} \int_0^{\{\alpha \ell\}} du \int_0^u K^{(k)} \left( \frac{(\alpha \ell - v)t}{2} \right) \frac{dv}{2^k(\alpha \ell - v)^{2-k}}. \]

Note that

\[ K^{(k)}(u) \ll_k \min(1, e^{-u}), \]

we have

\[ E^{(k)}_\alpha(t) \ll_k \sum_{1 \leq \ell \leq 1/t} t^{k-2} + \sum_{\ell > 1/t} e^{-\alpha t/2} \ell^{k-2} \ll_k t^{1-k}, \]

as \( t \to 0^+ \). This completes the proof of lemma.

We finally prove the following lemma which plays an important role in this paper.

**Lemma 3.4.** As \( t \to 0^+ \)

\[ R'_\alpha(t) = o(t^{-1}), \text{ and } R''_\alpha(t) \ll t^{-2}. \]

Furthermore,

\[ R_\alpha(t) = o(|\log t|) \text{ and } R_\alpha(t) - R_\alpha(2t) = o(1). \]

Moreover, if the series (1.2) of \( J_\alpha(1) \) is convergent then

\[ R_\alpha(t) = \alpha^{-1} J_\alpha(1) + o(1). \]
Partition into summands of the form \(|\alpha m|

**Proof.** Let \( t \to 0^+ \). The proof of the estimate for \( R''_\alpha(t) \) is a direct calculation. In fact, by the definition of \( R'_\alpha(t) \),

\[
R''_\alpha(t) = \frac{d^2}{dt^2} \sum_{\ell \geq 1} t \tilde{B}_1(\alpha \ell) e^{\alpha \ell t} - 1
\]

\[
= \sum_{\ell \geq 1} \tilde{B}_1(\alpha \ell) \frac{d^2}{du^2} \bigg|_{u=\alpha t} u e^u - 1
\]

\[
\ll \sum_{1 \leq \ell \leq 1/t} \ell + \sum_{\ell > 1/t} \ell(\ell t)e^{-\alpha t} \ll t^{-2}.
\]

We now give the proof of estimate for \( R'_\alpha(t) \). Since the sequence \( \{\tilde{B}_1(\alpha \ell)\}_{n \in \mathbb{Z}} \) is equidistributed in \([-1/2, 1/2)\) for all \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), we have

\[
S_\alpha(x) := \sum_{1 \leq \ell \leq x} \tilde{B}_1(\alpha \ell) = o(x), \quad (3.1)
\]

as \( x \to +\infty \). Using the definition of \( R'_\alpha(t) \), we have

\[
R'_\alpha(t) = \sum_{\ell \geq 1} e^{\alpha \ell t} - 1 - \alpha \ell t e^{\alpha \ell t} \tilde{B}_1(\alpha \ell)
\]

\[
= \int_1^\infty e^{\alpha xt} - 1 - \alpha xt e^{\alpha xt} \frac{dS_\alpha(x)}{x} dS_\alpha(x)
\]

\[
= \int_{\alpha t}^\infty S_\alpha(u/\alpha t) \left( \frac{e^u - 1 - ue^u}{(e^u - 1)^2} \right)' \, du.
\]

Thus the using of (3.1) implies that

\[
R'_\alpha(t) \ll \int_{\alpha t}^{(\alpha t)^{1/2}} \left| \frac{u}{\alpha t} \right| \, du + o \left( \frac{1}{\alpha t} \int_{(\alpha t)^{1/2}}^{\infty} u \left( \frac{e^u - 1 - ue^u}{(e^u - 1)^2} \right)' \, du \right)
\]

\[
\ll 1 + o(t^{-1}) = o(t^{-1}).
\]

Therefore,

\[
R_\alpha(t) = R_\alpha(1) + \int_1^t R'_\alpha(u) \, du
\]

\[
\ll 1 + \int_{1/\log(1/t)}^1 \frac{du}{u} + o \left( \int_t^{1/\log(1/t)} \frac{du}{u} \right) = o(\log t),
\]

and

\[
R_\alpha(t) - R_\alpha(2t) = t \int_1^t R'_\alpha(tu) \, du \ll t \times o \left( \int_1^2 \frac{1}{tu} \, du \right) = o(1).
\]
Moreover, if the series (1.2) of \( J_\alpha(1) \) is convergent then the use of integration by parts for Riemann-Stieltjes integrals implies

\[
R_\alpha(t) - \alpha^{-1} J_\alpha(1) = \sum_{\ell \geq 1} \left( \frac{\alpha \ell t}{e^{\alpha \ell t} - 1} - 1 \right) \frac{\tilde{B}_1(\alpha \ell)}{\alpha \ell}
\]

\[
\ll \sum_{1 \leq \ell \leq t^{-1/2}} |\alpha \ell t| \frac{1}{\alpha \ell} + \left| \int_{t^{-1/2}}^{\infty} \left( \frac{\alpha xt}{e^{\alpha xt} - 1} - 1 \right) \frac{d}{dx} \left( \sum_{1 \leq \ell \leq x} \tilde{B}_1(\alpha \ell) \right) dx \right|
\]

\[
= \sum_{1 \leq \ell \leq t^{-1/2}} t + \left| \int_{t^{-1/2}}^{\infty} \left( \frac{\alpha xt}{e^{\alpha xt} - 1} - 1 \right) d \left( \alpha^{-1} J_\alpha(1) + o(1) \right) \right|
\]

\[
\ll t^{1/2} + o(1) \left( t^{1/2} + \int_{t^{-1/2}}^{\infty} \left| \frac{d}{dx} \frac{\alpha xt}{e^{\alpha xt} - 1} \right| dx \right) = o(1),
\]

which completes the proof of the lemma.

\[\square\]

### A Numerical approximation for \( \Lambda_\alpha \)

In this appendix we investigate the numerical approximation for \( \Lambda_\alpha \). Denoting by

\[
\Pi_\alpha = \prod_{\ell \geq 1} \left( 1 - \frac{\alpha \ell}{\alpha \ell} \right) e^{-\frac{1}{2\alpha \ell}},
\]

then using (1.5) we see that \( \Lambda_\alpha = 4\sqrt{3}(\pi e^{-\gamma})^{1/2}(\alpha/6)^{1/4} \Pi_\alpha \). Taking a logarithm of (A.1) we obtain

\[
\log \Pi_\alpha = \sum_{\ell \geq 1} \left( \frac{1}{2\alpha \ell} - \log \left( 1 - \frac{\alpha \ell}{\alpha \ell} \right) \right)
\]

\[
= \left( \sum_{1 \leq \ell \leq N} + \sum_{\ell > N} \right) \left( \frac{1}{2\alpha \ell} - \log \left( 1 - \frac{\alpha \ell}{\alpha \ell} \right) \right)
\]

\[
=: \Sigma_m(N) + \Sigma_e(N),
\]

(A.2)

where \( N > 10 \) is an integer will be chosen for give a good numerical approximation for \( \Lambda_\alpha \).

We now bound the error term \( \Sigma_e(N) \). We rewritten the sum of \( \Sigma_e(N) \) as

\[
\Sigma_e(N) = - \sum_{\ell > N} \frac{\tilde{B}_1(\alpha \ell)}{\alpha \ell} - \sum_{\ell > N} \left( \log \left( 1 - \frac{\alpha \ell}{\alpha \ell} \right) + \frac{\alpha \ell}{\alpha \ell} \right)
\]

\[
= - \Sigma_{1e}(N) + \Sigma_{2e}(N).
\]

(A.3)
Partition into summands of the form $|\alpha m|$

It is not difficult to give a bound for the second sum above that

$$0 < \sum_{\ell > N} \left( \frac{\log \left( 1 - \frac{1}{\alpha \ell} \right)}{\alpha \ell} + \frac{1}{\alpha \ell} \right) < -\int_N^\infty \left( \frac{\log \left( 1 - \frac{1}{\alpha x} \right)}{\alpha x} + \frac{1}{\alpha x} \right) \, dx < \frac{3\alpha N - 2}{6\alpha N(\alpha N - 1)}. \quad \text{(A.4)}$$

Using part integration to $\sum_{\ell > N}$ we have

$$\sum_{\ell > N} = -\int_N^\infty \frac{1}{\alpha x} \, dS_\alpha(x) = \frac{S_\alpha(N)}{\alpha N} - \frac{1}{\alpha} \int_N^\infty \frac{S_\alpha(x)}{x^2} \, dx. \quad \text{(A.5)}$$

We now focus on the approximation of $\Lambda_\alpha$ to a class of irrational numbers $\alpha$ in which the partial quotients of the continued fraction expansion of $\alpha$ are bounded. In other world, $\alpha$ has the following continued fraction expansion

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

with all $a_j \leq A$ for some $A > 0$. In this cases Ostrowski [Ost22, pp. 80–81] proved that

$$|S_\alpha(x)| \leq \frac{3}{2} A \log x, \quad \text{(A.6)}$$

for all $x > 10$. Substituting (A.6) in (A.5) we find that

$$|\sum_{\ell > N}| \leq \frac{3A \log N}{\alpha N} + \frac{3A}{2\alpha N}.$$ 

Combining (A.2)–(A.4), and above we obtain

$$|\log \Pi_\alpha - \sum_{m}(N)| < \frac{3A \log N}{\alpha N} \left( \log N + \frac{1}{2} \right) + \frac{3\alpha N - 2}{6\alpha N(\alpha N - 1)}. \quad \text{(A.7)}$$

We now give the numerical approximation for $\Lambda_{\sqrt{2}}$. Note that $\sqrt{2} = [1; 2, 2, 2, \ldots]$, that is the partial quotients of the continued fraction expansion are bounded by 2. Hence from (A.7) we have

$$|\log \Pi_{\sqrt{2}} - \sum_{m}(N)| < \frac{3\sqrt{2}}{N} \left( \log N + \frac{1}{2} \right) + \frac{3N - \sqrt{2}}{6N(2N - 1)}.$$ 

Taking $N = 10^6$, then using Mathematica we find that

$$\log \Pi_{\sqrt{2}} = -0.127496 + 6.11 \times 10^{-5} \theta,$$

for some $\theta \in (-1, 1)$. Hence

$$\Lambda_{\sqrt{2}} = 4\sqrt{3}(\pi e^{-\gamma})^{1/2\sqrt{2}}(\sqrt{2}/6)^{1/4\sqrt{2}}\Pi_{\sqrt{2}} \in (5.7731, 5.7739).$$
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