Theoretical description of the plaquette with exponential accuracy

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Abstract We review recent studies of the operator product expansion of the plaquette and of the associated determination of the gluon condensate. One first needs the perturbative expansion to orders high enough to reach the asymptotic regime where the renormalon behavior sets in. The divergent perturbative series is formally regulated using the principal value prescription for its Borel integral. Subtracting the perturbative series truncated at the minimal term, we obtain the leading non-perturbative correction of the operator product expansion, i.e., the gluon condensate, with superasymptotic accuracy. It is then explored how to increase such precision within the context of the hyperasymptotic expansion. The results fully confirm expectations from renormalons and the operator product expansion.

1 Introduction

The operator product expansion (OPE) \cite{1} is a fundamental tool for theoretical analyses in quantum field theories. Its validity is only proven rigorously within perturbation theory to arbitrary finite orders \cite{2}. The use of the OPE in a non-perturbative framework was initiated by the ITEP group \cite{3} (see also the discussion in Ref. \cite{4}), who postulated that the OPE of a correlator could be approximated by the following series:

\[
\text{correlator}(Q) \simeq \sum_{d} \frac{1}{Q^d} C_d(\alpha) \langle O_d \rangle ,
\]

where the expectation values of local operators $O_d$ are suppressed by inverse powers of a large external momentum $Q \gg \Lambda_{\text{QCD}}$, according to their dimensionality $d$. The Wilson coefficients $C_d(\alpha)$ encode the physics at momentum scales larger than $Q$. These are well approximated by perturbative expansions in the strong coupling parameter $\alpha$:

\[
C_d(\alpha) \simeq \sum_{n \geq 0} c_n \alpha^{n+1}.
\]

The large-distance physics is described by the matrix elements $\langle O_d \rangle$ that usually have to be determined non-perturbatively: $\langle O_d \rangle \sim A_{QCD}^d$.

It can hardly be overemphasized that (except for direct predictions of non-perturbative lattice simulations, e.g., on light hadron masses) all QCD predictions are based on factorizations that are generalizations of the above generic OPE.

In this short review, we summarize and discuss the recent results \cite{5,6,7,8,9,10} (see also the review \cite{11}), which validate the nonperturbative version of the OPE for the case of the plaquette in gluodynamics. This analysis utilizes lattice regularization. Then main advantage of this choice is that it enables us to use numerical stochastic perturbation theory (NSPT) \cite{12,13,14} to obtain perturbative expansion coefficients. This allows us to realize much higher orders than would have been possible with diagrammatic techniques. A disadvantage of the lattice scheme is that, at least in our discretization, lattice perturbative expansions converge slower than expansions in the $\overline{\text{MS}}$ coupling. This means that we have to go to comparatively higher orders to become sensitive to the asymptotic behavior. Many of the results obtained in a lattice scheme either directly apply to the $\overline{\text{MS}}$ scheme too or can easily (and in some cases exactly) be converted into this scheme. We also show how to obtain a theoretical controlled expression for the plaquette with exponential accuracy that is, in principle, systematically improbable using hyperasymptotic expansions, as developed in Refs. \cite{7,8,9,10} (see Refs. \cite{15,16} for original work in the context of ordinary differential equations).

The expectation value of the plaquette calculated in Monte Carlo (MC) simulations in lattice regularization

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with the standard Wilson gauge action [17] reads

$$\langle P \rangle_{MC} = \frac{1}{N^4} \sum_{x \in A_E} \langle P_x \rangle,$$

(3)

where $A_E$ is a Euclidean space-time lattice and

$$P_{x,\mu\nu} = 1 - \frac{1}{6} \text{Tr} \left( U_{x,\mu\nu} + U_{x,\mu\nu}^\dagger \right).$$

(4)

For details on the notation, see Ref. [5].

### 2 The plaquette: OPE in perturbation theory

$\langle P \rangle$ depends on the lattice extent $Na$, the spacing $\alpha$ and $\alpha = g^2/(4\pi) \equiv \alpha(a^{-1})$ (note that $\alpha$ is the bare lattice coupling and its natural scale is of order $\alpha^{-1}$). To compute this expectation value in strict perturbation theory, we Taylor expand in powers of $g$ before averaging over the gauge configurations (which we do using NSPT [12–14]). The outcome is a power series in $\alpha$:

$$\langle P \rangle_{\text{pert}}(N) = \frac{1}{Z} \left. \int [dU_{x,\mu}] \ e^{-S[U]} P[U] \right|_{\text{NSPT}} = \sum_{n \geq 0} p_n(N) \alpha^{n+1}.$$

The dimensionless coefficients $p_n(N)$ are functions of the linear lattice size $N$. We emphasize that they do not depend on the lattice spacing $a$, nor on the physical lattice extent $Na$, alone, but only on the ratio $N = (Na)/a$.

We are interested in the large-$N$ (i.e., infinite volume) limit. In this situation

$$\frac{1}{a} \gg \frac{1}{Na},$$

(5)

and it makes sense to factorize the contributions of the different scales within the OPE framework (in perturbation theory). The hard modes, of scale $\sim 1/a$, determine the Wilson coefficients, whereas the soft modes, of scale $\sim 1/(Na)$, can be described by expectation values of local gauge invariant operators. There are no such operators of dimension two. The renormalization group invariant definition of the gluon condensate

$$\langle G^2 \rangle = -\frac{2}{\beta_0} \left. \Omega \left[ \frac{\beta(\alpha)}{\alpha} G_{\mu\nu}^c G_{\mu\nu}^c \right] \Omega \right|_{\text{pert}} = \left. \Omega \left[ 1 + O(\alpha) \right] \frac{\alpha}{\pi} G_{\mu\nu}^c G_{\mu\nu}^c \right|_{\text{pert}} \Omega$$

(6)

is the only local gauge invariant expectation value of an operator of dimension $a^{-4}$ in pure gluodynamics. In the purely perturbative case discussed here, it only depends on the soft scale $1/(Na)$, i.e., on the lattice extent. On dimensional grounds, the perturbative gluon condensate $\langle G^2 \rangle_{\text{soft}}$ is proportional to $1/(Na)^4$, and the logarithmic $(Na)$ dependence is encoded in $\alpha[1/(Na)]$.

Therefore,

$$\frac{\pi^2}{36} a^4 \langle G^2 \rangle_{\text{soft}} = \frac{1}{N^4} \sum_{n \geq 0} f_n \alpha^{n+1} [1/(Na)],$$

(7)

and the perturbative expansion of the plaquette on a finite volume of $N^4$ sites can be written as

$$\langle P \rangle_{\text{pert}}(N) = P_{\text{pert}}(\alpha) + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{soft}}$$

$$+ O \left( \frac{1}{N^6} \right),$$

(8)

where

$$P_{\text{pert}}(\alpha) \equiv \sum_{n \geq 0} p_n \alpha^{n+1}$$

(9)

and $p_n$ are the infinite volume coefficients that we are interested in. The Wilson coefficient, $C_G$, which only depends on $\alpha$, is normalized to unity for $\alpha = 0$. It can be expanded in $\alpha$:

$$C_G(\alpha) = 1 + \sum_{k \geq 0} c_k \alpha^{k+1}.$$

(10)

Since the Wilson action is proportional to the plaquette $P$, $C_G$ is fixed by the conformal trace anomaly [18, 19]:

$$C_G^1(\alpha) = -\frac{2\pi}{\beta_0} \left( \frac{3}{\alpha^2} \right) = 1 + \frac{\beta_1}{\beta_0} \frac{\alpha}{4\pi} + \frac{\beta_2}{\beta_0} \left( \frac{\alpha}{4\pi} \right)^2$$

$$+ \frac{\beta_3}{\beta_0} \left( \frac{\alpha}{4\pi} \right)^3 + O(\alpha^4).$$

(11)

The $\beta$-function coefficients $\beta_j$ are known in the Wilson action lattice scheme for $j \leq 3$ (see Eq. (25) of Ref. [5]). $\beta_{\text{latt}}$ has been computed diagrammatically [20–22]. The value for $\beta_{\text{latt}}$ that we use [23] is an update of Ref. [24], and was obtained by calculating the normalization of the leading renormalon of the pole mass, and then assuming the corresponding $\overline{\text{MS}}$-scheme expansion to follow its asymptotic behavior from orders $\alpha^4$ onwards. Similar estimates, $\beta_{\text{latt}} \approx -1.37 \times 10^6$ up to $\beta_{3,\text{latt}} \approx -1.55 \times 10^6$, were found in Ref. [25] using a very different method. Note that $C_G(\alpha)$ is scheme dependent not only through $\alpha$, but also explicitly, due to its dependence on the higher $\beta$-function coefficients: $\beta_2$, etc. The $c_k$ depend on the $\beta_i$ with $i \leq k+1$ via (11). For $j \leq 3$ the coefficients $\beta_j$ are known in the Wilson action lattice scheme. For convenience, we also write

1 We define the $\beta$-function as $\beta(\alpha) = d\alpha/d\ln \mu = -\beta_0/(2\pi)\alpha^2 - \beta_1/(8\pi^2)\alpha^3 - \cdots$, i.e., $\beta_0 = 11$.  

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the expansion coefficients $c_k$ defined in (11) in terms of the constants that appear in
\[ A = \mu \exp \left\{ - \left[ \frac{2\pi}{\beta_0 \alpha(\mu)} + b \ln \left( \frac{1}{2} \frac{\beta_0 \alpha(\mu)}{2\pi} \right) \right] + \sum_{j \geq 1} s_j (-b)^j \left[ \frac{\beta_0 \alpha(\mu)}{2\pi} \right]^j \right\}, \] (12)
where $b = \beta_1/(2\beta_0^2)$, $s_1 = (\beta_0^2 - \beta_0/(4b\beta_0^2))$ and $s_2 = (\beta_0^3 - 2\beta_0\beta_1 + \beta_0^2\beta_3)/(16\beta_0^3)$ and
\[ c_0 = -\frac{b_0}{2\pi}, \quad c_1 = s_1 b \left( \frac{\beta_0}{2\pi} \right)^2, \]
\[ c_2 = -2s_2 b^2 \left( \frac{\beta_0}{2\pi} \right)^3. \] (13)
Combining Eqs. (7), (8) and (10) gives
\[ \langle P \rangle_{\text{pert}}(N) = \sum_{n \geq 0} \left[ p_n - \frac{f_n(N)}{N^4} \right] \alpha^{n+1} = \sum_{n \geq 0} p_n \alpha^{n+1} - \frac{1}{N^4} \left( 1 + \sum_{k \geq 0} c_k \alpha^{k+1}(a^{-1}) \right) \times \sum_{n \geq 0} f_n \alpha^{n+1}(Na)^{-1} + O \left( \frac{1}{N^6} \right), \] (14)
where $f_n(N)$ is a polynomial in powers of $\ln(N)$. Fitting this equation to the perturbative lattice results, the first 35 coefficients $p_n$ were determined in Ref. [5]. The results were confronted with the expectations from renormalons:
\[ p_n^\text{latt} \approx Z_p^\text{latt} \left( \frac{\beta_0}{2\pi a} \right)^n \Gamma(n + 1 + db) \Gamma(1 + db) \left[ 1 + \frac{20.08931}{n + db} + \cdots + \frac{5055 \pm 33}{(n + db)(n + db - 1)} + O \left( \frac{1}{n^7} \right) \right]. \] (15)
In Fig. 1, the infinite volume ratios $p_n/(np_{n-1})$ are compared to the expectation Eq. 16. The asymptotic behavior of the perturbative series due to renormalons is reached around orders $n \sim 27 - 30$, proving, for the first time, the existence of the renormalon in the plaquette. Note that incorporating finite volume effects is compulsory to see this behavior, since there are no infrared renormalons on a finite lattice. To parameterize finite size effects, the purely perturbative OPE Eq. (8) was used. The behavior seen in Fig. 1, although computed from perturbative expansion coefficients, goes beyond the purely perturbative OPE since it predicts the position of a non-perturbative object in the Borel plane.

3 The plaquette: OPE beyond perturbation theory

Since in NSPT one Taylor expands in powers of $g$ before averaging over the gauge variables, no mass gap is generated. In non-perturbative Monte-Carlo (MC) lattice simulations an additional scale, $A_{QCD} \sim 1/a e^{-2\pi/(\beta_0 \alpha)}$, is generated dynamically. However, we can always tune $N$ and $\alpha$ such that
\[ \frac{1}{a} \gg \frac{1}{Na} \gg A_{QCD}. \] (17)
In this small-volume situation, one encounters a double expansion in powers of $a/(Na)$ and $a A_{QCD}$ or, equivalently, $(Na) A_{QCD} \times a/(Na)$. The construction of the OPE is completely analogous to that of the previous section and one obtains\(^2\)
\[ \langle P \rangle_{\text{MC}} = \frac{1}{Z} \int [dU_{x,\mu}] e^{-S[U]} P[U] \bigg|_{\text{MC}} = P_{\text{pert}}(\alpha)(1) + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{MC}} + O(a^6). \] (18)
\(^2\) In the last equality, we approximate the Wilson coefficients by their perturbative expansions, neglecting the possibility of non-perturbative contributions associated to the hard scale $1/a$. These would be suppressed by factors $\sim \exp(-2\pi/\alpha)$ and, therefore, would be sub-leading relative to the gluon condensate.
In the last equality, we have factored out the hard scale $1/a$ from the scales $1/(Na)$ and $A_{\text{QCD}}$, which are encoded in $\langle G^2 \rangle_{\text{MC}}$. Exploiting the right-most inequality of Eq. (17), we can expand $\langle G^2 \rangle_{\text{MC}}$ as follows:

$$\langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{soft}} \left\{ 1 + \mathcal{O}(A_{\text{QCD}}^2(1/a)^2) \right\}.$$  
(19)

Hence, a non-perturbative small-volume simulation would yield the same expression as NSPT, up to non-perturbative corrections that can be made arbitrarily small by reducing $a$ and therefore $Na$, keeping $N$ fixed. In other words, $p_{n_\pi^{\text{NSPT}}}(N) = p_{n_\pi^{\text{MC}}}(N)$ up to non-perturbative corrections.

We can also consider the limit

$$\frac{1}{a} \gg A_{\text{QCD}} \gg \frac{1}{Na}.$$  
(20)

This is the standard situation realized in non-perturbative lattice simulations. Again the OPE can be constructed as in the previous section, Eq. (18) holds, and the $p_{n_\pi}$ and $c_{n_\omega}$-values are still the same. The difference is that now

$$\langle G^2 \rangle_{\text{MC}} = \langle G^2 \rangle_{\text{NP}} \left[ 1 + \mathcal{O}(A_{\text{QCD}}^2(1/a)^2) \right],$$  
(21)

where $\langle G^2 \rangle_{\text{NP}} \sim A_{\text{QCD}}^4$ is the so-called non-perturbative gluon condensate introduced in Ref. [3]. From now on we will call this quantity simply the “gluon condensate” $\langle G^2 \rangle$. Nevertheless, without further qualifications, this quantity is ill defined.

The perturbative sum and the leading nonperturbative correction in (18) are ill defined. The reason is that the perturbative series is divergent due to renormalons [26] (for a review see Ref. [27]) and other, subleading, instabilities. This makes any determination of $\langle G^2 \rangle$ ambiguous, unless we define how to truncate or how to approximate the perturbative series. Any reasonable definition consistent with $\langle G^2 \rangle \sim A_{\text{QCD}}^4$ can only be given if the asymptotic behavior of the perturbative series is under control. This has only been achieved recently in Ref. [5], where the perturbative expansion of the plaquette was computed up to $\mathcal{O}(\alpha^3)$. The observed asymptotic behavior was in full compliance with renormalon expectations.

Extracting the gluon condensate from the average plaquette was pioneered in Refs. [28–31], and many attempts followed during the next decades, see, e.g., Refs. [32–41]. Nevertheless, they suffered from insufficiently high perturbative orders and, in some cases, also finite volume effects. The failure to make a controlled contact to the asymptotic regime prevented a reliable lattice determination of $\langle G^2 \rangle$, where one could quantitatively assess the error associated with these determinations. This problem was first solved in Ref. [6]. In such paper, for the first time, the perturbative sum was computed with supersymmetric accuracy for the case of 4 dimensional SU(3) gluodynamics. This allowed to obtain a reliable determination of $\langle G^2 \rangle$ that scaled as $A_{\text{QCD}}^4$. One issue raised was to determine to which extent such a result was independent of the scheme used for the coupling constant. The answer to this question was given within the general framework of hyperasymptotic expansions. First in Ref. [7], where it was concluded that the error of using the superasymptotic approximation to the perturbative sum was of $\mathcal{O}(\sqrt{\alpha(1/a)Z} A_{\text{QCD}}^4)$, where $Z_P$ is the normalization of the leading renormalon. This error then sets the parametric precision of the determination of the gluon condensate using the superasymptotic approximation. Note that the scheme dependence of $Z_P$ and $A_{\text{QCD}}^4$ cancels each other. Therefore, the only remaining/leading scheme/scale dependence of the error is due to the $\sqrt{\alpha(1/a)}$ prefactor.

To improve over the asymptotic accuracy, the first step is to regularize the perturbative sum, which we do using the Principal Value (PV) prescription. Only after regularizing the perturbative sum, the definition of the gluon condensate is unambiguous and the operator product expansion of the plaquette reads

$$\langle P \rangle_{\text{MC}} = S_{\text{PV}} + \frac{\pi^2}{36} C_G(\alpha) a^4 \langle G^2 \rangle_{\text{PV}} + \mathcal{O}\left( (a\Lambda)^6 \right).$$  
(22)

This expression is, in practice, formal, as the exact expression of $S_{\text{PV}}$ is not known. This would require the exact knowledge of the Borel transform of the perturbative sum. Nevertheless, it is possible to obtain an approximate expression of it with a known parametric control of the error using its hyperasymptotic expansion. The accuracy of this expansion is limited from the information we get from perturbation theory. For the case at hand, we have

$$S_{\text{PV}} = \sum_{n=0}^{N_P} p_n \alpha^{n+1} + \Omega_{G^2}$$

$$+ \sum_{n=N_P+1}^{N'} [p_n - p_n^{(a_0)}] \alpha^{n+1} + \cdots,$$  
(23)

where $N'$ is the maximal order in perturbation theory that is included in the perturbative expansion. Within the hyperasymptotic counting, approximating $S_{\text{PV}}$ by $S_P \equiv \sum_{n=0}^{N_P} p_n \alpha^{n+1}$, the first term in (23), corresponds to the superasymptotic approximation, which we label as $(0,N_P)$. Adding $\Omega_{G^2}$ to the superasymptotic approximation corresponds to $(4,0)$ precision in the hyperasymptotic approximation and adding the last term corresponds to $(4,N')$ precision$^3$.

In (23), we take

$$N_P = 4 \frac{2\pi}{\beta_0 \alpha(1/a)} \left( 1 - \co(1/a) \right),$$  
(24)

as the order at which we truncate the perturbative expansion to reach the superasymptotic approximation.

$^3$ The labeling $(D,N)$ in general is defined in Refs. [8,9].
By default, we will take the smallest positive value of \( n \) that yields an integer value for \( N_P \), but we also explore the dependence of the result on \( n \). Note that the value of \( N_P \) used in Eq. (24) is slightly different from the value used in Ref. [6] to truncate the perturbative expansion with superasymptotic accuracy. In that reference, such number was named \( n_0 \) and was determined numerically. We will elaborate on this difference later.

\( \Omega_{G^2} \) is the terminator associated with the leading renomalon of the plaquette. It reads \([10]\)

\[
\Omega_{G^2} = \sqrt{\alpha(1/a)} K(P)e^{-\frac{6\eta}{\beta_0(1/a)}} \left( \frac{\beta_0(1/a)}{4\pi} \right)^{-2b} \left( 1 + \tilde{K}_1(P) \alpha/(1+a) \right) \left( 1 + \tilde{K}_2(P) \alpha^2/(1+a) + O(\alpha^3(1/a)) \right),
\]

where

\[
K^{(P)} = \frac{-Z_P}{\Gamma(1+4b)} \beta_0 \frac{1}{\pi} \beta_0^{1/2} \left( -\eta_c + \frac{1}{3} \right),
\]

\[
K_1^{(P)} = \frac{\beta_0/(4\pi)}{-\eta_c + \frac{1}{3}} \left[ -4bb_1 \left( \frac{\eta_c}{2} + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right],
\]

\[
K_2^{(P)} = \frac{\beta_0/(4\pi)^2}{-\eta_c + \frac{1}{3}} \left[ -4bb_1 \left( \frac{\eta_c}{2} + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right],
\]

\[
\tilde{K}_1^{(P)} = \tilde{K}_1^{(P)} - \frac{2b_\beta s_1}{\pi},
\]

\[
\tilde{K}_2^{(P)} = \frac{\beta_0/(4\pi)^2}{-\eta_c + \frac{1}{3}} \left[ -4bb_1 \left( \frac{\eta_c}{2} + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right] \left[ -4bb_1 \left( \frac{\eta_c}{2} + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right],
\]

where \( \eta_c \equiv -4b + \frac{8\pi}{3b} c - 1 \).

The value of \( Z_P \) was determined approximately (for \( n_f = 0 \)) in Ref. [5]:

\[
Z_P = (42 \pm 17) \times 10^4.
\]

Its error gives the major source of uncertainty in the determination of \( \Omega_{G^2} \), of the order of 40%. The other source of error is due to the fact that only approximate expressions are available for \( \Omega_{G^2} \), as we do not know the complete set of coefficients of the beta function in the lattice scheme. Nevertheless, we can study the convergence pattern of the weak-coupling expansion. Equation (25) yields a nicely convergent series with a controlled scheme dependence, as the weak coupling expansion is organized in terms of a single parameter: \( \alpha \). The error associated with truncating the expansion in (25) is estimated by observing the convergent pattern of the LO, NLO and NNLO results. From LO to NLO, in the worst cases, the differences are close but below 50%, and from NLO to NNLO, the differences are below 10%. One could then expect the NNLO contribution to be at the level of few percent, which can be neglected all together in comparison with the \( \sim 40\% \) error associated to \( Z_P \).

### 4 Determination of the gluon condensate

We now review the determination of the gluon condensate in Refs. [6, 10]. We determine the gluon condensate from the following equation:

\[
\langle G^2 \rangle_{PV} = \frac{36C_{G}^{-1}}{\pi^2 a^4} \left[ (P)_{MC} - S_{PV} \right]. \tag{32}
\]

If \( S_{PV} \) and \( \langle P \rangle_{MC} \) were known exactly, this equality is expected to hold up to corrections of \( O(a^2) \). Nevertheless, neither \( S_{PV} \) nor \( \langle P \rangle_{MC} \) are known exactly. On top of that, \( C_{G}^{-1} \) and the relation between \( a \) and \( \beta \) are also known in an approximated way. We now discuss how to determine them and their associated individual errors.

The MC data is taken from Ref. [42], restricting to the more precise \( N = 32 \) data and, to keep finite volume effects under control, to \( \beta \leq 6.65 \). We also limit ourselves to \( \beta \geq 5.8 \) to avoid large \( O(a^2) \) corrections. At very large \( \beta \)-values, obtaining meaningful results becomes challenging numerically: the individual errors both of \( \langle P \rangle_{MC}(\alpha) \) and of \( S_{PV}(\alpha) \) somewhat decrease with increasing \( \beta \). However, there is a very strong cancellation between these two terms, in particular at large \( \beta \)-values, since this difference decreases with \( a^{-1} \sim A_{\text{lat}}^{1/4} \exp(16\pi^2/\beta/33) \) on dimensional grounds, while \( \langle P \rangle_{MC} \) depends only logarithmically on \( a \). We illustrate this cancellation in Fig. 2.

Equation (12) is not accurate enough in the lattice scheme for the available \( \beta \)-values. Instead, the phenomenological parametrization of Ref. [43] (\( x = \beta - 6 \))

\[
a = r_0 \exp \left( -1.6804 - 1.7331x + 0.7849x^2 - 0.4428x^3 \right), \tag{33}
\]

obtained by interpolating non-perturbative lattice simulation results is used. Equation (33) was reported to be valid within an accuracy varying from 0.5% up to 1% in the range [43] \( 5.7 \leq \beta \leq 6.92 \), which includes the range \( \beta \in [5.8, 6.65] \) used here.

For the inverse Wilson coefficient

\[
C_{G}^{-1}(\alpha) = \frac{2\pi \beta(\alpha)}{\beta_0 \alpha^2} = 1 + \frac{\beta_1 \alpha}{\beta_0 4\pi}
\]
the corrections to $C_G = 1$ are small. However, the $O(\alpha^2)$ and $O(\alpha^3)$ terms are of similar sizes. We will account for this uncertainty in our error budget.

We now turn to $S_{PV}(\alpha)$. It is computed using the hyperasymptotic approximation. This introduces a parametric error according to the order we truncate this expansion. On top of that, the coefficients $p_n$, obtained in Ref. [5], are not known exactly. They carry statistical errors, and successive orders are correlated. Using the covariance matrix, also obtained in Ref. [5], the statistical error of $S_P(\alpha)$ can be calculated. In that reference, coefficients $p_n(N)$ were first computed on finite volumes of $N^4$ sites and subsequently extrapolated to their infinite volume limits $p_n$. This extrapolation is subject to parametric uncertainties that need to be estimated. In Ref. [5] and later in Ref. [10] the differences between determinations using $N \geq \nu$ points for $\nu = 9$ (the central values) and $\nu = 7$ were added as systematic errors to the statistical errors. We emphasize though, that the $\nu$-tral values) and illustrate this discussion in Fig. 3. All these error estimates scale with the parametric uncertainty predicted by theory $\sim O(e^{-4 \frac{m_0}{m_L (1+\gamma)}}) \sim O(\alpha^4 A^4)$ times $\sqrt{\alpha}$ (see the discussion in Refs. [8,9]).

If we increase the accuracy of the hyperasymptotic expansion by adding the term $\Omega_{G2}$ to the superasymptotic approximation, the parametric error decreases, and the accuracy reached is $(4,0)$ (note that the statistical error does not change). With this accuracy, the parametric error is $\sim O(e^{-4 \frac{m_0}{m_L (1+\gamma)}} (1+\log(3/2))) \sim O((aA)^{4(1+\log(3/2)))}$ (see the discussion in Refs. [8,9]). Note that $4\log(3/2) \simeq 1.6 < 2$. Therefore, these effects are parametrically more important than the next nonperturbative power corrections. Compared with the typical size of the term $\Omega_G$, these effects are suppressed by a factor of order $\sim O((aA)^{4(1+\log(3/2)))}$ in the energy range we do the fits, this yields suppression factors in the range $(aA_{\text{MC}})^{4\log(3/2)} \in (0.007, 0.05)$, where we have taken $A = A_{\text{MC}}$ to be more conservative. This discussion can be affected by powers of $\alpha$. It is expected that there is an extra suppression factor of $\alpha^{3/2}$ (as $\sqrt{\alpha}$ is already included in the terminating the real suppression factor would be of order $\alpha$). Depending on the scheme, the size of this extra factor is different. In any case, they go in the direction to make the estimate of the error smaller. We will not dwell further in this discussion of the parametric error of the $(4,0)$ hyperasymptotic accuracy, because we only approximately know $\Omega_{G2}$ and its error will hide the signal of these $O((aA)^{4(1+\log(3/2)))}$ effects. For $\Omega_G$ we use the analytic expression in (25) truncated at $O(\alpha^2)$. The error of this expression comes from $Z_P$, and from the
truncation of the weak coupling expansion of the terminant. The largest source of error comes from $Z_P$. Due to its size, this error overwhelms the parametric error associated to higher-order terms in the hyperasymptotic expansion.

Irrespective of the discussion of the error of the $(4,0)$ accuracy, it is nice to see that adding the terminant to the superasymptotic expression makes the jumps that we had with the superasymptotic approximation disappear. Adding the terminant also makes the resulting curve flatter. The dependence in $N_P$ (or in other words $c$) gets much milder too. We illustrate all this in Fig. 3.

In principle, we know perturbation theory to orders high enough to include the last term written in (23) and reach $(4,N')$ accuracy. Nevertheless, we find that the errors of $p_n$ for large $n$ hide the signal, and it is not possible to improve the prediction. The optimal value is given below in (35). For further details in the error analysis see Ref. [10].

5 Conclusions

For the first time ever, perturbative expansions at orders where the asymptotic regime is reached were obtained [5] and subtracted from non-perturbative Monte Carlo data with superasymptotic [6] and hyperasymptotic accuracy [10]. The most accurate value was obtained in this last reference:

$$\langle G^2\rangle_{PV}(n_f = 0) = 3.15(18) \, r_0^{-4}. \quad (35)$$

We emphasize that this result is independent of the scale and renormalization scheme used for the coupling constant. Even if the computation was made in the lattice scheme, the result is the same in the MS scheme within the accuracy of the computation. The limiting factor for improving the determination of the gluon condensate in pure gluodynamics is the error of perturbation theory. All systematic sources of error have their origin in the errors of perturbation theory (even what we call statistical errors of (32) are dominated by the statistical errors of the coefficients $p_n$). More precise values of these perturbative coefficients, and its knowledge to higher orders, would yield a more precise determination of the normalization of the renormalons, $Z_P$, and would allow working with hyperasymptotic accuracy $(4,N')$. Nowadays, if we try to reach this accuracy, we find that the error of the coefficients are too large to get accurate results. The situation with active light quarks is in an early stage but starts to be promising. The coefficients of the perturbative coefficients have been computed at finite volume in Ref. [44] for QCD with two massless fermions. More data at different volumes, and the infinite volume extrapolation of these coefficients, would then allow to give a determination of the gluon condensate in QCD with two massless fermions.

Overall, the OPE beyond perturbation theory has been validated. The scaling of the nonperturbative effects with the lattice spacing confirms the dimension $d = 4$. Dimension $d < 4$ slopes appear only when subtracting the perturbative series truncated at fixed preasymptotic orders. Therefore, these lower-dimensional “condensates” discussed in Ref. [45] or in Ref. [36] are nothing but approximate parametrizations of unaccounted perturbative effects, i.e., of the short-distance behavior. These will be observable dependent, unlike the non-perturbative gluon condensate. Such simplified parametrizations of the perturbative terms introduce unquantifiable errors and, therefore, are of limited phenomenological use. As illustrated in Fig. 4, even the effective dimension of such a “condensate” varies when truncating a perturbative series at different orders. In Refs. [46–48] various analyses, based on models such as string/gauge duality or Regge models, have been made claiming the existence of non-perturbative dimension two corrections. Our results strongly suggest that there may be flaws in these derivations.

The accurate value obtained for the gluon condensate, Eq. (35), is of a similar size as the intrinsic difference between (reasonable) subtraction prescriptions (see the discussion in Ref. [6]). This result contradicts the implicit assumption of sum rule analyses that the renormalon ambiguity is much smaller than leading non-perturbative corrections. The value of the gluon condensate obtained with sum rules can vary significantly due to this intrinsic ambiguity if determined using different prescriptions or truncating at different orders in perturbation theory. Clearly, the impact of this, e.g., on determinations of $\alpha_s$ from $\tau$-decays or from lattice simulations needs to be assessed carefully.

We finally mention that the nonzero value of $\langle G^2\rangle_{PV}$ shows that the PV regularization of the perturbative sum, even if computed exactly, would differ from the Monte Carlo simulation of the plaquette by a term of $\mathcal{O}(a^4 \Lambda^4)$. This may affect the conjecture that the resummation technique of the perturbative expansion proposed in Ref. [49] for the Adler function would not
need such nonperturbative corrections. This should be further investigated.

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