SOLVABILITY OF THE INITIAL-BOUNDARY VALUE PROBLEM OF THE NAVIER-STOKES EQUATIONS WITH ROUGH DATA

TONGKEUN CHANG AND BUM JA JIN

Abstract. In this paper, we study the initial and boundary value problem of the Navier-Stokes equations in the half space. We prove the unique existence of weak solution \( u \in L^q(R^n_+ \times (0,T)) \) with \( \nabla u \in L^q_{loc}(R^n_+ \times (0,T)) \) for a short time interval when the initial data \( h \in B^{-\frac{n+2}{q}}_{q \cdot q}(R^n_+) \) and the boundary data \( g \in L^q(0,T;B^{-\frac{n+2}{q}}_{q \cdot q}(R^n_+)) + L^q(R^n \times (0,T)) \) with normal component \( g_n \in L^q(0,T;\dot{B}^{-\frac{n+2}{q}}_{\frac{q}{2}}(0,T)) \), \( n + 2 < q < \infty \) are given.

2000 Mathematics Subject Classification: primary 35K61, secondary 76D07.

Keywords and phrases: Stokes equations, Navier-Stokes equations, nonhomogeneous initial data, nonhomogeneous boundary data, weak solutions, Half space.

1. Introduction

Let \( R^n_+ = \{ x \in R^n | x_n > 0 \} \), \( n \geq 2 \) and \( 0 < T < \infty \). Let us consider the nonstationary Navier-Stokes equations

\[
\begin{align*}
  u_t - \Delta u + \nabla p &= f - \text{div}(u \otimes u), & \text{div } u &= 0, & \text{in } R^n_+ \times (0,T), \\
  u|_{t=0} &= h, & u|_{x_n=0} &= g.
\end{align*}
\] (1.1)

There are abundant literatures for the study of the Navier-Stokes equations with homogeneous boundary data. See [3, 7, 28, 34] and references therein for the half space problem. See also [3, 9, 14, 15, 17, 21, 22, 23, 24, 29, 31] and the references therein for the problems in other domains such as whole space, a bounded domain, or exterior domain.

Over the past decade, the Navier-Stokes equations with the nonhomogeneous boundary data have been studied actively. See [1, 4, 5, 32, 40] and references therein for the half space problem. See also [4, 5, 10, 11, 12, 13, 18, 19, 20, 29] and the references therein for the problems in other domains such as whole space, a bounded domain, or exterior domain.

In [18, 19, 20, 33], the solvabilities of bounded or exterior domain problem have been studied for a boundary data in anisotropic space \( B^{\alpha-\frac{n}{q}-\frac{2}{r}}_{q_0}(\partial \Omega \times (0,T)) \), \( \alpha > \frac{1}{q} \) (with \( q > \frac{n+2}{\alpha+1} \)), where \( g \in B^{\frac{\alpha}{q_0}}_{q_0}(S \times (0,T)) \) means the zero extension of \( g \) to \( S \times (-\infty,T) \) is in \( B^{\frac{\alpha}{q_0}}_{q_0}(S \times (-\infty,T)) \). On the other hand, in [11, 12, 13, 18, 19, 20, 29] a rough boundary data have been considered.

H. Amann [4] showed unique maximal solution \( u \in L^r_{loc}(0,T^*, H^\frac{1}{2}_q(\Omega)) \), \( 3 < q < r < \infty \), \( \frac{1}{r} + \frac{3}{q} \leq 1 \) for some maximal time \( T^* \) in any domain in \( R^3 \) with nonempty compact smooth boundary when a
nonzero initial data in $B_q^{-\frac{1}{q}}(\Omega) \cap L^p_q(\Omega)$ and nonzero boundary data in $L^{r}_loc(\mathbb{R}^n_+; W_q^{-\frac{1}{q}} + \frac{1}{q}(\partial \Omega))$ are given. J.E.Lewis[32] showed a global in time existence of solution in $L^p(\mathbb{R}^n_+; L^q(\mathbb{R}^n_+))$ for small data $\eta \in L^q(R^n_+ \cap L^r(R^n_+)$ and $g \in L^q(\mathbb{R}^n_+; L^r(R^n_+))$ with $r_1, r_2, p, q, r, d < \infty, r_1 \leq n < r_2, \frac{n-1}{r} + \frac{2}{q} = 1,$ and $\frac{2}{r} + \frac{2}{q} = 1$. K.A.Voss[40] showed the existence of a global in time self-similar solution for small data $h \in \dot{B}_{6, \infty}^{-\frac{1}{2}}(\mathbb{R}^3_+) \cap \dot{B}_{4, \infty}^{-\frac{1}{2}}(\mathbb{R}^3_+)$ and $t^{\frac{5}{6}}g(t) \in L^\infty(\mathbb{R}^n_+; L^3(\mathbb{R}^2))$ with $g_n = 0$. M.Fernandes de Almeida and L.C.F. Ferreira[1] showed the existence of global in time solution in the framework of Morrey space for a small data $h \in M_{p, n-p}(\mathbb{R}^n_+), t^{\frac{5}{6}}\nabla g \in BC(\mathbb{R}^n_+, M_{r, n-p}(\mathbb{R}^n_+))$ and $t^{\frac{5}{6}}\nabla g_n \in BC(\mathbb{R}^n_+, M_{r, n-p}(\mathbb{n}^n_+)), 2 < p, q < \infty, 1 < r < \infty.$

In particular, R. Farwig, H. Kozono and H. Sohn[12] showed the local in time existence of a very weak solution $u \in L^q(0, T; L^q(\Omega))$ in an exterior domain when nonzero initial in $B_q^{-\frac{1}{q}}$ and nonzero boundary data in $L^q(0, T; W_q^{-\frac{1}{q}}(\partial \Omega))$ for $\frac{2}{s} + \frac{2}{q} = 1, 2 < s < \infty, 3 < q < \infty$ are given (Precisely speaking, in [12] a nonzero divergence is considered).

In this paper, we show the unique existence of $u \in L^q(R^n_+ \times (0, T))$ with $\nabla u \in L^\infty(R^n_+ \times (0, T))$ for the Navier-Stokes equations (1.1) for a small time interval $(0, T)$ with the initial $h \in B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)$ and the boundary data $g \in L^q(0, T; B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)) + L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T))$ with $g_n \in L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T)), q > n + 2.$ Our result could be compared with the one in [12]. The case $q = r = 5$ in [12] coincides with the case $q = 5$ in our result, except the fact that our result cover larger class for $g'$ (the tangential component of the boundary data) since $L^q(0, T; B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)) + L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T)) \supseteq L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T)).$

The following is the main result of this paper.

**Theorem 1.1.** Let $\infty > q > n+2.$ Assume that $h \in B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)$ with $div h, 0, g \in L^q(0, T; B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)) + L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(\mathbb{R}^n_+))$ with $g_n \in L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T)).$ Then there is $T^*(0 < T^* < \infty)$ so that the Navier-Stokes equations (1.1) has a unique weak solution $u \in L^q(\mathbb{R}^n_+ \times (0, T^*))$ with $\nabla u \in L^\infty_{loc}(\mathbb{R}^n_+ \times (0, T^*)).$

The space $L^q(0, T; B_q^{-\frac{1}{q}}(\mathbb{R}^n_+)) + L^q(\mathbb{R}^n_+; B_q^{-\frac{1}{r}}(0, T))$ coincides with anisotropic Besov space $B_q^{-\frac{1}{q}, -\frac{1}{r}}(\mathbb{R}^n_+ \times \mathbb{R}^n_+)$ (see section 2). Our result is optimal in the sense that the spaces for the initial and the boundary data cannot be enlarged for our solution class. Our arguments in this paper are based on the elementary estimates of the heat operator and the Laplace operator. The solution representation in section 5.1 could be useful to study asymptotic behavior of the solution.

Before proving Theorem 1.1 we have studied the initial and boundary value problem of the Stokes equations in $\mathbb{R}^n_+ \times (0, T)$ as follows:

$$u_t - \Delta u + \nabla p = f, \quad \text{div} u = 0, \text{in} \mathbb{R}^n_+ \times (0, T),$$

$$u|_{t=0} = h, \quad u|_{x_n=0} = g. \quad (1.2)$$
There are various literatures for the solvability of the Stokes equations \([12]\) with homogeneous or nonhomogeneous boundary data. See \([7, 16, 17, 24, 25, 27, 28, 34, 36]\), and references therein for the Stokes problem with homogeneous boundary data. See \([18, 19, 20, 26, 27, 33, 34, 35]\), and references therein for the Stokes problem with nonhomogeneous boundary data.

In \([18, 19, 20, 26, 27, 33, 34, 35]\), a boundary data in anisotropic space \(B^{\frac{\alpha}{q}}_{q,0}(\partial \Omega \times \mathbb{R}_+)\), \(\alpha > \frac{1}{q}\) has been considered. J.P. Raymond\([33]\) showed the unique existence of weak solution \(u \in B^{\frac{\alpha}{q}}_{q,0}(\Omega \times (0, T))\), \(0 \leq s \leq 2\) in a bounded domain when a nonzero initial data in \(L^2(\Omega)\) and nonzero boundary data in \(H^1(0, T; H^{-1}_x(S))\) are given. In \([12]\), R. Farwig, H. Kozono and H. Sohr also showed the existence of a very weak solution \(u \in L^s(0, T; L^q(\Omega))\) (of Stokes equations) in an exterior domain when nonzero initial in \(B^{\frac{\alpha}{q}}_{q,s}\) and nonzero boundary data in \(L^s(0, T; W^{-\frac{\alpha}{q}}(\partial \Omega))\) for \(1 < s < \infty, 3 < q < \infty\) are given.

The following states our result on the unique solvability of the Stokes equations \([12]\).

**Theorem 1.2.** Let \(1 < q < \infty\) Assume that \(h \in B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)\) with \(\text{div} h = 0\), and \(g \in B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)\). In addition, if \(1 < q \leq 3\), then we assume that \(g - \Gamma \ast_x \hat{h} \in B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)\) for some \(\hat{h} \in B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)\) which is a solenoidal extension of \(h\) to \(\mathbb{R}^n\). Let \(f = \text{div} F, F \in L^p(\mathbb{R}^n \times \mathbb{R}_+)\) for some \(p\) with \(\alpha_1 = 1 - \frac{2}{p} + \frac{n-1}{q} > 0\). Then there is a unique weak solution \(u \in L^q(\mathbb{R}^n \times (0, T))\) with \(\nabla u \in L^p_{\text{loc}}(\mathbb{R}^n \times (0, T))\) satisfying the following inequality

\[
\|u\|_{L^q(\mathbb{R}^n \times (0, T))} \leq c \max\{1, T^{\frac{1}{q}}\} \|h\|_{B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)} + \max\{1, T^{\frac{1}{q}}\} \|g - \Gamma \ast_x \hat{h}\|_{B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)} + cT^{\frac{\alpha}{q}} \|F\|_{L^p(\mathbb{R}^n \times (0, T))}, \quad q > 3,
\]

and

\[
\|u\|_{L^q(\mathbb{R}^n \times (0, T))} \leq c \max\{1, T^{\frac{1}{q}}\} \|h\|_{B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)} + \max\{1, T^{\frac{1}{q}}\} \|g - \Gamma \ast_x \hat{h}\|_{B^{\frac{-\alpha}{q}}_{q}(\mathbb{R}^n)} + cT^{\frac{\alpha}{q}} \|F\|_{L^p(\mathbb{R}^n \times (0, T))}, \quad 1 < q \leq 3.
\]

We organize this paper as follows. In section 2 we introduce the notations and the function spaces. In section 3 the preliminary estimates in anisotropic spaces for the heat operator, Riesz operator, and Poisson operator are given. In section 4 we consider Stokes equations \([12]\) with the zero force and the zero initial velocity, and give the proof of Theorem 1.1. In section 5 we complete the proof of Theorem 1.2 with the help of Theorem 1.1 and the preliminary estimates in section 3. In section 6 we give the proof of Theorem 1.1 applying the estimate of Theorem 1.2 to the approximate solutions.
2. Notations and Definitions

We denote by \( x' \) and \( x = (x', x_n) \) the points of spaces \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^n \), respectively. The multiple derivatives are denoted by \( D_x^k D_{x'}^l = \frac{\partial^{k+l}}{\partial x^k \partial x'_l} \) for multi index \( k \) and nonnegative integer \( m \). For vector field \( f = (f_1, \ldots, f_n) \) on \( \mathbb{R}^n \), we write \( f' = (f_1, \ldots, f_{n-1}) \) and \( f = (f', f_n) \). Throughout this paper we denote by \( c \) various generic constants. Let \( \mathbb{R}^n_+ = \{ x = (x', x_n) : x_n > 0 \} \), \( \mathbb{R}^m_+ = \{ x = (x', x_n) : x_n \geq 0 \} \), \( \mathbb{R}_+ = (0, \infty) \).

For the Banach space \( X \) and interval \( I \), we denote by \( X' \) the dual space of \( X \), and by \( L^p(I; X), 1 \leq p \leq \infty \) the usual Bochner space. For \( 0 < \theta < 1 \) and \( 1 < p < \infty \), denote by \( (X, Y)_{\theta, p} \) the real interpolation of the Banach space \( X \) and \( Y \). For \( 1 \leq p \leq \infty \), we write \( p' = \frac{p}{p-1} \).

Let \( \Omega \) be a \( m \)-dimensional Lipschitz domain, \( m \geq 1 \). Denote by \( C_0^\infty (\Omega) \) stands for the collection of all complex-valued infinitely differentiable functions in \( \mathbb{R}^m \) compactly supported in \( \Omega \). Let \( 1 \leq p \leq \infty \) and \( k \) be a nonnegative integer. The norms of usual Lebesgue space \( L^p(\Omega) \), the usual Sobolev space \( W^k_p(\Omega) \) (Slobodetski space \( W^s_p(\Omega) \) for noninteger \( s > 0 \)) and the usual homogeneous Sobolev spaces \( \dot{W}^k_p(\Omega) \) are written by \( \| \cdot \|_{L^p(\Omega)}, \| \cdot \|_{W^k_p(\Omega)}, \| \cdot \|_{\dot{W}^k_p(\Omega)} \), respectively. Note that \( W^0_p(\Omega) = \dot{W}^0_p(\Omega) = L^p(\Omega) \). For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), denote by \( B^s_{p,q}(\Omega) \) and \( \dot{B}^s_{p,q}(\Omega) \), \( 1 \leq p, q \leq \infty \) the usual Besov spaces and the homogeneous Besov spaces, respectively. For the simplicity, set \( B^s_p(\Omega) = B^s_{p,p}(\Omega) \) and \( \dot{B}^s_p(\Omega) = \dot{B}^s_{p,p}(\Omega) \).

Denote by \( \dot{B}^s_p(\Omega) \) the set of distributions \( f \in B^s_p(\mathbb{R}^m) \) which is supported in \( \Omega \) with norm \( \| f \|_{\dot{B}^s_p(\Omega)} = \| f \|_{B^s_p(\mathbb{R}^m)} < \infty \). It is known that \( B^s_p(\Omega) = \dot{B}^s_p(\Omega) = \dot{B}^s_p(\Omega) \) if \( 0 \leq s < \frac{1}{p} \), \( \dot{B}^s_p(\Omega) = (B^{-s}_p(\Omega))^\prime \) if \( s < 0 \) (when \( \Omega \) is Lipschitz domain).

It is also known that \( B^s_p(\Omega) = L^p(\Omega) \cap \dot{B}^s_p(\Omega) \) for \( s > 0 \); \( B^s_p(\Omega) = L^p(\Omega) + \dot{B}^s_p(\Omega) \) for \( s < 0 \); \( B^s_p(\Omega) = (L^p(\Omega), W^s_2(\Omega))_{\theta, p} \) for \( 0 < s < s_1 \); \( B^s_p(\mathbb{R}^m) = (B^s_2(\mathbb{R}^m), B^s_2(\mathbb{R}^m))_{\theta, p} \) for \( s = (1 - \theta)s_1 + \theta s_2 \) for \( 0 < \theta < 1 \). See [6, 17, 24, 35, 39] for more properties of the Besov spaces.

Let \( I \) be an interval of \( \mathbb{R} \). For \( k \in \mathbb{N} \cup \{ 0 \} \), denote by \( W^{2k, k}_{q}(\mathbb{R} \times I) \) and \( \dot{W}^{2k, k}_{q}(\mathbb{R} \times I) \) the usual anisotropic Sobolev space (Slobodetski space \( W^s_q(\mathbb{R} \times I) \) for noninteger \( s > 0 \)) and the usual homogeneous anisotropic Sobolev space, respectively. Note that \( W^{0,0}_p(\mathbb{R} \times I) = \dot{W}^{0,0}_p(\mathbb{R} \times I) = L^p(\mathbb{R} \times I) \).

Now, we introduce anisotropic Besov space and its properties (see chapter 4 of [38], chapter 5 of [39], and chapter 3 of [24] for the definition of anisotropic spaces and their properties although different notations were used in each books).

Define anisotropic Besov space \( B^s_{p,q} (\mathbb{R}^m \times \mathbb{R}) \) by

\[
B^s_{p,q} (\mathbb{R}^m \times \mathbb{R}) = \begin{cases} 
L^p(\mathbb{R}; B^s_p(\mathbb{R}^m)) \cap L^p(\mathbb{R}^m; B^s_p(\mathbb{R})) & \text{if } s > 0, \\
L^p(\mathbb{R}; B^s_p(\mathbb{R}^m)) + L^p(\mathbb{R}^m; B^s_p(\mathbb{R})) & \text{if } s < 0, \\
(B^{-1, -\frac{1}{q}}_q(\mathbb{R}^m \times \mathbb{R}), B^{1, \frac{1}{q}}_q(\mathbb{R}^m \times \mathbb{R}))_{1,q} & \text{if } s = 0.
\end{cases}
\]
The homogeneous anisotropic Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^m \times \mathbb{R}) \) is defined analogously. The above definition is equivalent to the definitions in \[2, 38\].

Let \( \Omega \) be a Lipshitz domain of \( \mathbb{R}^n \) and \( I \) be an interval in \( \mathbb{R} \). Let \( \mathcal{D}' \) be the distributions on \( \Omega \times I \). For \( s \in \mathbb{R} \), an anisotropic Besov space \( B^s_{q}(\Omega \times I) \) is defined by

\[
B^s_{q}(\Omega \times I) := \{ f \in \mathcal{D}' \mid f = F|_{\Omega \times I} \text{ for some } F \in B^s_{q}(\mathbb{R}^n \times \mathbb{R}) \}
\]

with norm \( \| f \|_{B^s_{q}(\Omega \times I)} := \inf \{ \| F \|_{B^s_{q}(\mathbb{R}^n \times \mathbb{R})} : F \in B^s_{q}(\mathbb{R}^n \times \mathbb{R}) \text{ with } F|_{\Omega \times I} = f \} \). The homogeneous anisotropic spaces \( \dot{B}^s_{p,q}(\Omega \times I) \) is defined analogously.

Denote by \( B^s_{p,q}(\Omega \times [0,T]) \) the set of distributions \( f \in B^s_{p,q}(\Omega \times (-\infty,T)) \) which is supported in \( \Omega \times (0,T) \) with \( \| f \|_{B^s_{p,q}(\mathbb{R}^n \times (-\infty,T))} \). It is known that \( B^s_{p,q}(\Omega \times [0,T]) = B^s_{p,q}(\Omega \times [0,T]) = B^s_{\tilde{p},\tilde{q}}(\Omega \times (0,T)) \) if \( 0 \leq s < \frac{\tilde{p}}{p} \) and \( B^s_{\tilde{p},\tilde{q}}(\Omega \times (0,T)) \) if \( s < 0 \).

The properties of the anisotropic Besov spaces are comparable with the properties of Besov spaces: \( \dot{B}^s_{p,q}(\Omega \times I) = L^p(\mathbb{R}^n; B^s_{p,q}(\Omega \times I)) \) and \( B^s_{p,q}(\Omega \times I) = L^p(\mathbb{R}^n; B^s_{p,q}(\Omega \times I)) \) for \( s > 0 \); \( \dot{B}^s_{\tilde{p},\tilde{q}}(\Omega \times I) = L^\tilde{p}(\mathbb{R}^n; \dot{B}^s_{\tilde{p},\tilde{q}}(\Omega \times I)) \) and \( B^s_{\tilde{p},\tilde{q}}(\Omega \times I) = L^\tilde{p}(\mathbb{R}^n; B^s_{\tilde{p},\tilde{q}}(\Omega \times I)) \) for \( s < 0 \); \( B^s_{\tilde{p},\tilde{q}}(\Omega \times I) = (L^\tilde{p}(\mathbb{R}^n; B^s_{\tilde{p},\tilde{q}}(\Omega \times I)))_{\tilde{q}^{-1},\tilde{p}} \) for \( 0 < s < 2k \); \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^m \times \mathbb{R}) = (B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^m \times \mathbb{R}))_{\tilde{q}^{-1},\tilde{p}} \) for \( 0 < \theta < 1 \), \( \alpha = (1 - \theta)\alpha_1 + \theta\alpha_2 \) for any real number \( \alpha_1 < \alpha_2 \). See \[2, 38, 39\] for more properties of the anisotropic Besov spaces.

**Definition 2.1** (Weak solution to the Stokes equations). Let \( 1 < q < \infty \). Let \( h, g, f = \text{div} \mathcal{F} \) satisfy the same hypothesis as in Theorem 1.2. Then a vector field \( u \in L^p(\mathbb{R}^n_+ \times (0,T)) \) with \( \nabla u \in L^q_{\text{loc}}(\mathbb{R}^n_+ \times (0,T)) \) is called a weak solution of the Stokes system 1.2 if the following conditions are satisfied:

- (In case \( \infty > q > 3 \))
  \[
  - \int_0^T \int_{\mathbb{R}^n_+} u \cdot \Delta \Phi dxdt = \int_0^T \int_{\mathbb{R}^n_+} u \cdot \Phi_t - F : \nabla \Phi dxdt + < h, \Phi(\cdot, 0)>_{\mathbb{R}^n_+} - < g, \frac{\partial \Phi}{\partial x_n}>_{\mathbb{R}^n_+} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ 
  \]
  for each \( \Phi \in C_0^\infty(\mathbb{R}^n_+ \times (0,T)) \) with \( \text{div} \Phi = 0, \Phi|_{x_n=0} = 0 \), where \( < \cdot, \cdot>_{\mathbb{R}^n_+} \) denotes the duality paring between \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \) and \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \) and \( < \cdot, \cdot>_{\mathbb{R}^n_+} \) denotes the duality paring between \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \) and \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \).

- (In case \( 1 < q \leq 3 \))
  \[
  - \int_0^T \int_{\mathbb{R}^n_+} (u - v) \cdot \Delta \Phi dxdt = \int_0^T \int_{\mathbb{R}^n_+} u \cdot \Phi_t - F : \nabla \Phi dxdt - < g - v|_{x_n=0}, \frac{\partial \Phi}{\partial x_n}>_{\mathbb{R}^n_+} \times \mathbb{R}^n_+ 
  \]
  for each \( \Phi \in C_0^\infty(\mathbb{R}^n_+ \times (0,T)) \) with \( \text{div} \Phi = 0, \Phi|_{x_n=0} = 0 \), where \( v = \Gamma_t * \tilde{h} \) and \( < \cdot, \cdot>_{\mathbb{R}^n_+} \) denotes the duality paring between \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \) and \( B^s_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+) \).
Definition 2.2 (Weak solution to the Navier-Stokes equations). Let $\infty > q > n + 2$. Let $h, g$ satisfy the same hypothesis as in Theorem 17. Then a vector field $u \in L^q(\mathbb{R}_+^n \times (0,T))$ with $\nabla u \in L^p_{loc}(\mathbb{R}_+^n \times (0,T))$ for some $1 < p \leq q$ is called a weak solution of the Navier-Stokes system if the following conditions are satisfied:

$$
- \int_0^T \int_{\mathbb{R}^n} u \cdot \Delta \Phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} u \cdot (\Phi_t + (u \otimes u) : \nabla \Phi) \, dx \, dt + h, \Phi(\cdot,0) >_{\mathbb{R}^n} - g, \frac{\partial \Phi}{\partial x_n} >_{\mathbb{R}^n-1 \times \mathbb{R}^+}
$$

for each $\Phi \in C_0^\infty(\mathbb{R}_+^n \times [0,T))$ with $\text{div}_x \Phi = 0$, $\Phi|_{x_n=0} = 0$.

3. Preliminaries.

3.1. Basic Theories. According to the usual trace theorem, if $u \in B^s_p(\mathbb{R}_+^n)$ ($u \in W^s_p(\mathbb{R}_+^n)$), then $u|_{x_n=0} \in B^{s-\frac{1}{p}}_{p,\frac{n}{p}}(\mathbb{R}^{n-1})$ for $s > \frac{1}{p}$ (see [3]), and if $u \in B^{s,\frac{2}{p}}_p(\mathbb{R}_+^n \times (0,T))$ ($u \in W^{s,\frac{2}{p}}_p(\mathbb{R}_+^n \times (0,T))$), then $u|_{x_n=0} \in B^{s,\frac{2}{p} - \frac{2}{p}}_{p,\frac{n}{p}}(\mathbb{R}^{n-1} \times (0,T))$ for $s > \frac{2}{p}$ and $u|_{t=0} \in B^{s,\frac{2}{p}}_{p,\frac{n}{p}}(\mathbb{R}_+^n)$ for $s > \frac{2}{p}$ (see [2, 38, 39]).

On the other hand, for a solenoidal vector field $u \in L^p(\mathbb{R}_+^n)$, $1 < p < \infty$ it holds that $u_n \in B^{\frac{1}{p}}_p(\mathbb{R}^{n-1})$ with

$$
\|u_n\|_{B^{\frac{1}{p}}_p(\mathbb{R}^{n-1})} \leq c\|u\|_{L^p(\mathbb{R}^n)}.
$$ (3.1)

Let $R = (R_1, \cdots, R_n)$ be the Riesz operator on $\mathbb{R}^n$. It is the well known fact that $R_i$ is bounded operator from $B^{s}_p(\mathbb{R}^n)$ to $B^{s}_p(\mathbb{R}^n)$ (from $W^{k}_p(\mathbb{R}^n)$ to $W^{k}_p(\mathbb{R}^n)$, $k = 0, \pm 1, \cdots$) for $s \in \mathbb{R}$ and $1 < p < \infty$ (see [37] for the reference). Using the fact that $B^{s,-\frac{2}{p}}_p(\Omega \times (0,T)) = L^p(0,T; B^s(\Omega) \cap L^p(\Omega; B^{\frac{2}{p}}_p(0,T)))$ for $s > 0$ and $R_i$ is self-adjoint operator, the following boundedness property holds for anisotropic Besov spaces as follows:

$$
\|Rf\|_{B^{s,-\frac{2}{p}}_p(\mathbb{R}^n \times (0,T))} \leq c\|f\|_{B^{s,-\frac{2}{p}}_p(\mathbb{R}^n \times (0,T))}, s \in \mathbb{R}, 1 < q < \infty,
$$ (3.2)

$$
(\|Rf\|_{W^{k}_q(\mathbb{R}^n \times (0,T))}) \leq c\|f\|_{W^{k}_q(\mathbb{R}^n \times (0,T))}, k = 0, 1, 2, \cdots).
$$ (3.3)

3.2. Estimate of the heat operator. Define three types of heat operator $T_1, T_2, T_1^* , T_2^*$ by

$$
T_1 f = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \Gamma(x - y, t - s)f(y, s)dy \, ds,
$$

$$
T_2 g = \int_{-\infty}^{t} \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', x_n, t - s)g(y', s)dy' \, ds,
$$

$$
T_1^* f(y, s) = \int_{s}^{\infty} \int_{\mathbb{R}^n} \Gamma(x - y, t - s)f(x, t)dx \, dt,
$$

$$
T_2^* g(y, t) = \int_{s}^{\infty} \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', y_n, t - \tau)g(x', \tau)dx' \, dt.
$$

Observing that $T_1^*$ is the adjoint operator of $T_1$, we can derive the following estimate for $T_1$ and $T_1^*$. 


Lemma 3.1. Let $1 < p < \infty$ and $0 \leq \alpha \leq 2$.

\[
\|T_1f\|_{W^{2k}_p(\mathbb{R}^n \times \mathbb{R})} + \|T_1^*f\|_{W^{2k}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{B^{2k-2,k-1}_p(\mathbb{R}^n \times \mathbb{R})}, \quad k = 0, 1,
\]

\[
\|T_1f\|_{B^{\alpha}_p(\mathbb{R}^n \times \mathbb{R})} + \|T_1^*f\|_{B^{\alpha}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{B^{\alpha-2,\alpha-1}_p(\mathbb{R}^n \times \mathbb{R})}, \quad 0 < \alpha < 2.
\]

Using the result of Lemma 3.1, the following estimate for $T_2$ and $T_2^*$ can be derived.

Lemma 3.2. Let $1 < p < \infty$ and $0 \leq \alpha \leq 2$. Then

\[
\|T_2g\|_{W^{2k}_p(\mathbb{R}^n \times \mathbb{R})} + \|T_2^*g\|_{W^{2k}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|g\|_{B^{2k-1-\frac{1}{q},k-\frac{1}{4}}(\mathbb{R}^n \times \mathbb{R})}, \quad k = 0, 1,
\]

\[
\|T_2g\|_{B^{\alpha}_p(\mathbb{R}^n \times \mathbb{R})} + \|T_2^*g\|_{B^{\alpha}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|g\|_{B^{\alpha-1-\frac{1}{q},\alpha-\frac{1}{4}}(\mathbb{R}^n \times \mathbb{R})}, \quad 0 < \alpha < 2.
\]

The above lemma will be useful for the proof of Theorem 4.1 and also for the estimate of $\Gamma_i * h|_{x_n=0}$.

Lemma 3.3. Let $1 < q < \infty$. Define the heat operator $T_0$ by $T_0h = \int_{\mathbb{R}^n} \Gamma(x-y,t)h(y)dy$. Then,

\[
\|T_0h\|_{L^q(\mathbb{R}^n \times (0,T))} \leq c \max\{1, T^{\frac{2}{q}}\}\|h\|_{B^{\alpha}_q(\mathbb{R}^n)},
\]

(3.4)

\[
\|T_0h|_{x_n=0}\|_{B^{\alpha}_q(\mathbb{R}^n \times (0,T))} \leq c \max\{1, T^{\frac{2}{q}}\}\|h\|_{\bar{B}^{\alpha}_q(\mathbb{R}^n)},
\]

(3.5)

Lemma 3.4. Let $1 < p,q < \infty$ with $1 - (n+2)(\frac{1}{p} - \frac{1}{q}) > 0$. For $f \in L^p(\mathbb{R}^n \times (0,T))$, define $u$ by $u(x,t) = \int_0^t \int_{\mathbb{R}^n} D_s \Gamma(x-y,t-s)f(y,s)dyds$. Let $\alpha_1 = 1 - (n+2)(\frac{1}{p} - \frac{1}{q})$. Then $u \in W^{1\frac{k}{2}}_{q\theta}(\mathbb{R}^n \times (0,T))$ with

\[
\|u\|_{W^{1\frac{n}{2}}_{q\theta}(\mathbb{R}^n \times (0,T))} \leq c \max\{1, T^{\alpha_1}\}\|f\|_{L^p(\mathbb{R}^n \times (0,T))},
\]

\[
\|u\|_{L^q(\mathbb{R}^n \times (0,T))} \leq c T^{\alpha_1}\|f\|_{L^p(\mathbb{R}^n \times (0,T))}.
\]

Moreover, $u|_{x_n=0} \in B^{\frac{n+1}{2} - \frac{1}{q}}(\mathbb{R}^n \times (0,T))$ with

\[
\|u|_{x_n=0}\|_{B^{\frac{n+1}{2} - \frac{1}{q}}(\mathbb{R}^n \times (0,T))} \leq c T^{\alpha_1}\|f\|_{L^p(\mathbb{R}^n \times (0,T))}.
\]

4. Stokes equations (1.2) with $f = 0$ and $h = 0$

Let $(w,r)$ be the solution of the equations

\[
w_t - \Delta w + \nabla r = 0, \quad div w = 0, \quad w|_{t=0} = 0, \quad w|_{x_n=0} = G.
\]

(4.1)

Let

\[
K_{ij}(x,t) = -2\delta_{ij} D_{x_i} \Gamma(x,t) + 4D_{x_i} \int_0^t \int_{\mathbb{R}^n} D_{x_i} \Gamma(z,t) D_{x_i} N(x-z)dz,
\]

where $\Gamma$ and $N$ are the fundamental solutions of the heat equation and the Laplace equation in $\mathbb{R}^n$, respectively, that is,

\[
\Gamma(x,t) = \begin{cases} 
\frac{c}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\
0 & \text{if } t \leq 0,
\end{cases}
\]

and

\[
N(x) = \begin{cases} 
\frac{1}{\omega_n (2-n)|x|^{n-2}} & \text{if } n \geq 3, \\
\frac{1}{2\pi} \log |x| & \text{if } n = 2.
\end{cases}
\]
In [34], an explicit formula for $w$ of the Stokes equations (4.1) with boundary data $G = (G', 0)$ is obtained by

$$w_i(x, t) = \sum_{j=1}^{n-1} < K_{ij}(x' - \cdot, x_n, t - \cdot), G_j >_{\mathbb{R}^{n-1} \times \mathbb{R}^+}. \quad (4.2)$$

Here $< \cdot, \cdot >$ is a duality pairing between $B_{q}^{-\alpha+\frac{\beta}{q} - \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (0, T))$ and $B_{q0}^{\alpha-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R}^+)$ (if $G$ is a function, then $< K_{ij}(x' - \cdot, x_n, t - \cdot), G_j >_{\mathbb{R}^{n-1} \times \mathbb{R}^+} = \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s)G_j(y', s)dy'ds$).

**Theorem 4.1.** Let $1 < q < \infty$ and $0 < T < \infty$. Let $G \in B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R}^+)$ with $G_n = 0$. Let $w$ be the vector field defined by (4.2). Then $w \in L^q(\mathbb{R}^n_+ \times (0, T))$ with

$$\|w\|_{L^q(\mathbb{R}^n_+ \times (0, T))} \leq c \max\{1, T^{\frac{1}{q'}}\} \|G\|_{B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (0, T))}.$$

**Proof.** By the definition of the space $B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (0, T))$, the zero extension of $G$ is in $B_{q}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (-\infty, T))$. Again, by the definition of the space $B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (-\infty, T))$, there is $\tilde{G} \in B_{q}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R})$ with $\tilde{G}|_{\mathbb{R}^{n-1} \times (0, T)} = G$, supp$\tilde{G} \subset \mathbb{R}^{n-1} \times (0, \infty)$ and $\|\tilde{G}\|_{B_{q}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c \|G\|_{B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (0, T))}.$

Hence, without loss of generality, we assume that $G \in B_{q}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R})$, supp $G \subset \mathbb{R}^{n-1} \times \mathbb{R}^+$ with $\|G\|_{B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c \|G\|_{B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times (0, T))}$ (By density argument, we may assume that $G \in C_0^\infty(\mathbb{R}^{n-1} \times \mathbb{R})$).

According to [3], $w$ can be rewritten by the following form

$$w_i(x, t) = -T G_i(x, t) - 4\delta_{in} I \left( \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} TG_j \right)(x, t) + 4 \frac{\partial}{\partial x_i} S \left( \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} TG_j \right)(x, t), \quad i = 1, \cdots, n, \quad (4.3)$$

where $T, S$ and $I$ are defined by

$$TG_i(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - y', x_n, t - \tau) G_i(y', \tau)dy'd\tau, \quad (4.4)$$

$$If(x, t) = \int_{\mathbb{R}^{n-1}} N(x' - y', 0) f(y', x_n, t)dy', \quad (4.5)$$

$$Sf(x, t) = \int_{0}^{x_n} \int_{\mathbb{R}^{n-1}} N(x - y)f(y, t)dy. \quad (4.6)$$

Observe that $T = D_{x_n}T_2$, where $T_2$ is the heat operator defined in section 3. From Lemma 3.2 we have

$$\|TG\|_{L^q(\mathbb{R}^n_+ \times \mathbb{R})} \leq c \|G\|_{B_{q0}^{-\frac{\beta}{q} + \frac{n-1}{q}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \quad (4.7)$$

Direct computation shows that for $1 \leq j \leq n - 1$

$$I \left( \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} TG_j \right) = \sum_{j=1}^{n-1} R'_j TG_j, \quad (4.8)$$
where \( R' = (R'_1, \ldots, R'_{n-1}) \) is \( n - 1 \) dimensional Riesz operator. By the well known property of Riesz operator we have

\[
\| \mathcal{I} \left( \frac{\partial}{\partial x_j} TG_j \right) \|_{L^s(\mathbb{R}^2 \times \mathbb{R})} dt \leq c \sum_{j=1}^{n-1} \| TG_j \|_{L^s(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c \| G \|_{\dot{B}^s_{q'} - \dot{B}^s_q(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{4.9}
\]

Let \( f(x, t) = \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} TG_j(x, t) \). Direct computation also shows that \( Sf \) solves

\[
\Delta Sf(x, t) = \text{div} F \quad \text{in} \quad \mathbb{R}^n_+ \quad \text{for each} \quad t > 0, \quad Sf|_{x_n=0} = 0, \tag{4.10}
\]

where

\[
F_j := -\frac{1}{2} TG_j, \quad j = 1 \cdots, n-1, \quad F_n := \mathcal{I} \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} TG_j(x, t). \tag{4.11}
\]

By the solution representation of Laplace equation (4.10), \( Sf \) can be rewritten by the formula

\[
Sf(x, t) = -\int_{\mathbb{R}^n_+} (N(x - y) - E(x - y^*)) \text{div} F(y, t) dy.
\]

Using Calderon Zygmund inequality, we have

\[
\| D_x Sf \|_{L^s(\mathbb{R}^n_+ \times \mathbb{R})} \leq c \| F \|_{L^s(\mathbb{R}^n_+ \times \mathbb{R})} \leq c \| G \|_{\dot{B}^s_{q'} - \dot{B}^s_q(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{4.12}
\]

Combining (4.10), (4.11) and (4.12), we have

\[
\| u \|_{L^s(\mathbb{R}^n_+ \times \mathbb{R})} \leq c \| G \|_{\dot{B}^s_{q'} - \dot{B}^s_q(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{4.13}
\]

On the other hand, by Young’s theorem and Minkovski’s theorem, we have

\[
\| u \|_{L^s(\mathbb{R}^n_+ \times (0, T))} \leq c T^{\frac{1}{2s}} \| G \|_{L^s(\mathbb{R}^{n-1} \times (0, T))}. \tag{4.14}
\]

Recall that \( B^s_{q_0}(\Omega \times (0, T)) = \dot{B}^s_{q_0}(\Omega \times (0, T)) + L^q(\Omega \times (0, T)) \) for \( s < 0 \). Combining from (4.13) and (4.14), we have

\[
\| u \|_{L^s(\mathbb{R}^n_+ \times (0, T))} \leq c \max\{1, T^{\frac{1}{2s}}\} \| G \|_{\dot{B}^s_{q_0} - \dot{B}^s_q(\mathbb{R}^{n-1} \times (0, T))}. \tag{4.15}
\]

\[\square\]

**Remark 4.2.** Let \( G^*_i(x, y, t) = D_{x_j} \int_0^t \int_{\mathbb{R}^{n-1}} \Gamma(z - y^*, t)D_x N(x - z) dz, \quad y^* = (y', -y_n). \) It is known that

\[
|D^*_i D^*_j \tilde{K}_{ij}(x', y', x_n, t)| \leq c \frac{t^{s + \frac{n-1}{2}} (|x' - y'|^2 + t)^{-\frac{n+1}{2}} (x_n^2 + t)^{\frac{n-1}{2}}}{(x_n^2 + t)^{\frac{n+1}{2}}}, \tag{4.16}
\]

where \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \) (see Proposition 2.5 of [30]). Using the properties of heat kernel \( \Gamma_t \) and the estimates of \( G^*_i \), we have

\[
|D^*_i D^*_j D_x \tilde{K}_{ij}(x' - y', x_n, t)| \leq c \frac{t^{s + \frac{n-1}{2}} (|x' - y'|^2 + x_n^2 + t)^{-\frac{n+1}{2}} (x_n^2 + t)^{\frac{n-1}{2}}}{(x_n^2 + t)^{\frac{n+1}{2}}}. \]
Using this estimate of $K_{ij}$, direct computation shows that
\[ \|D_xw(\cdot, x_n, t)\|_{L^q(\mathbb{R}^n \times (0, T))} \leq \frac{ct^2}{q^2}x_n^{-2} \|G\|_{L^q(\mathbb{R}^n \times (0, T))}. \]
and
\[ \|D_xw(\cdot, x_n, t)\|_{L^q(\mathbb{R}^n \times (0, T))} \leq \frac{ct^2}{q^2}x_n^{-2} \|G\|_{L^q(\mathbb{R}^n \times (0, T))}. \]

5. Proof of Theorem 1.2

Let us consider the Stokes equations (1.2) with general nonhomogeneous data $h, f, g$ with $f = \text{div}F$. Below, we give a solution formula of the Stokes equations (1.2) decomposed by four vector field, $v, V, \nabla \phi$ and $w$ which will be defined in section 5.1.

5.1. Solution formula. Let $\tilde{F}$ be an extension of $F$ to $\mathbb{R}^n \times \mathbb{R}^+$, and let $\tilde{f} = \text{div}\tilde{F}$. Define the projection operator $\mathbb{P}$ by
\[ [\mathbb{P}\tilde{f}]_j(x, t) = \delta_{ij}\tilde{f}_i + D_xD_x\int_{\mathbb{R}^n} N(x - y)\tilde{f}_i(y, t)dy = \delta_{ij}\tilde{f}_i + R_iR_j\tilde{f}_i, \]
and define $Q$ by
\[ Q\tilde{f} = -D_x\int_{\mathbb{R}^n} N(x - y)\tilde{f}_i(y, t)dy. \]
Then
\[ \text{div } \mathbb{P}\tilde{f} = 0 \text{ in } \mathbb{R}^n \times (0, T) \text{ and } \tilde{f} = \mathbb{P}\tilde{f} + \nabla Q\tilde{f}. \]

Define $V$ by
\[ V(x, t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s)\mathbb{P}\tilde{f}(y, s)dyds. \] (5.1)
Observe that $V$ satisfies the equations
\[ V_t - \Delta V = \mathbb{P}\tilde{f}, \text{ div } V = 0 \text{ in } \mathbb{R}^n \times (0, T), \quad V|_{t=0} = 0 \text{ on } \mathbb{R}^n. \] (5.2)
Observe that $[\mathbb{P}\tilde{f}]_j = D_x\left(\delta_{ij}\tilde{F}_{ki} + R_iR_j\tilde{F}_{ki}\right)$ for $\tilde{f} = \text{div}\tilde{F}$. Hence $V$ can be rewritten by
\[ V_j(x, t) = -\int_0^t \int_{\mathbb{R}^n} D_y\Gamma(x - y, t - s)\left(\delta_{ij}\tilde{F}_{ki} + R_iR_j\tilde{F}_{ki}\right)(y, s)dyds. \] (5.3)

Let $\tilde{h}$ be an extension of $h$ satisfying that
\[ \text{div } \tilde{h} = 0 \text{ in } \mathbb{R}^n. \]
Define $v$ by
\[ v(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)\tilde{h}(y)dy. \] (5.4)
Observe that $v$ satisfies the equations
\[ v_t - \Delta v = 0, \text{ div } v = 0 \text{ in } \mathbb{R}^n \times (0, T), \quad v|_{t=0} = \tilde{h} \text{ on } \mathbb{R}^n. \] (5.5)
Define \( \phi \) by
\[
\phi(x, t) = 2 \int_{\mathbb{R}^n-1} N(x' - y', x_n) \left( g_n(y', t) - v_n(x', 0, t) - V_n(x', 0, t) \right) dy'.
\] (5.6)

Observe that
\[
\Delta \phi = 0, \nabla \phi|_{x_n=0} = \left( R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}), g_n - v_n|_{x_n=0} - V_n|_{x_n=0} \right).
\] (5.7)

Note that \( \nabla \phi|_{t=0} = 0 \) if \( g_n|_{t=0} = h_n|_{x_n=0} \).

Let \( G = (G', 0) \), where
\[
G' = (G_1, \cdots, G_{n-1}) = g' - v'|_{x_n=0} - V'|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}).
\] (5.8)

Note that \( G'|_{t=0} = 0 \) if \( g|_{t=0} = h|_{x_n=0} \). Let \( w \) be the vector field defined by the formula \( R \) with boundary data \( G = (G', 0) \) for \( G' \) as defined in \([5.8]\). Then,
\[
u = w + \nabla \phi + v + V \text{ and } p = r - \phi_t + \mathbb{Q} \mathbf{f}
\] (5.9)
satisfies formally the nonstationary following Stokes equations \([1.2]\).

5.2. Estimates of \( u = v + V + \nabla \phi + w \). Choose \( \tilde{h} \in B_q^{-\frac{2}{n}}(\mathbb{R}^n) \) so that \( \tilde{h}|_{\mathbb{R}^n_+} = h \) and \( \text{div} \tilde{h} = 0 \).

Choose \( \mathcal{F} \in L^p(\mathbb{R}^n \times \mathbb{R}^n_+) \) so that \( \mathcal{F}|_{\mathbb{R}^n_+} = \mathcal{F} \). Let \( \tilde{f} = \text{div} \mathcal{F} \).

Let \( V, v \) and \( \phi \) be the corresponding vector fields defined by \([5.3], [5.4], \) and \([5.6] \), respectively, and let \( w \) be defined by \([4.2] \) with \( G \) as defined by \([5.5]\).

1) From Lemma \([3.3]\) we have with
\[
\|w\|_{L^q(\mathbb{R}^n_+, (0, T))} \leq c \max\{1, T^{\frac{2}{q}}\} \|h\|_{B_q^{-\frac{2}{n}}(\mathbb{R}^n_+)}.
\]

2) By the \( L^p \) boundedness of the Riesz operator (see \([3.2]\)) we have \( \|R_i R_j \tilde{F}_{kl}\|_{L^p(\mathbb{R}^n_+)} \leq c \|\mathcal{F}\|_{L^p(\mathbb{R}^n_+)} \). Hence, from Lemma \([3.4]\) we have
\[
\|V\|_{L^q(\mathbb{R}^n_+, (0, T))} \leq c T^{\frac{2}{q'}} \|\mathcal{F}\|_{L^p(\mathbb{R}^n_+, (0, T))}.
\]

3) Since \( \text{div} V = 0 \), \( \text{div} v = 0 \) in \( \mathbb{R}^n_+ \times (0, T), v_n \) and \( V_n \) have trace (see \([3.1]\)) with
\[
\|V_n(t)|_{x_n=0}\|_{B_{\frac{p}{2}}(\mathbb{R}^{n-1})} \leq c \|V(t)\|_{L^q(\mathbb{R}^n_+)} \quad \text{and} \quad \|v_n(t)|_{x_n=0}\|_{B_{\frac{p}{2}}(\mathbb{R}^{n-1})} \leq c \|v(t)\|_{L^q(\mathbb{R}^n_+)}.
\] (5.10)

This leads to the estimate
\[
\|v_n|_{x_n=0}\|_{L^q(0, T; B_{\frac{p}{2}}(\mathbb{R}^{n-1}))} \leq \|v\|_{L^q(0, T; L^s(\mathbb{R}^n_+))} = \|v\|_{L^q(\mathbb{R}^n_+, (0, T))},
\] (5.11)
\[
\|V_n|_{x_n=0}\|_{L^q(0, T; B_{\frac{p}{q}}(\mathbb{R}^{n-1})))} \leq \|V\|_{L^q(0, T; L^s(\mathbb{R}^n_+))} = \|V\|_{L^q(\mathbb{R}^n_+, (0, T))}.
\] (5.12)

4) Let \( P_{x_n} \) be the Poisson operator defined by
\[
P_{x_n} f(x) = c_n \int_{\mathbb{R}^n-1} \frac{x_n}{\|x' - y'\|^2 + x_n^2} f(y') dy',
\]
which satisfies the Laplace equation
\[-\Delta P_{x_n} f = 0 \text{ in } \mathbb{R}^n_+, P_{x_n} f|_{x_n=0} = f.\]
Observe that
\[ D_{x_n} \phi(x, t) = 2 \int_{\mathbb{R}^{n-1}} D_{x_n} N(x' - y', x_n) (g_n(y', t) - v_n(y', 0, t) - \dot{V}_n(y', 0, t)) dy' \]
\[ = P_{x_n}(g_n - v_n|_{x_n=0} - \dot{V}_n|_{y_n=0}), \]
\[ D_{x'} \phi(x, t) = 2 \int_{\mathbb{R}^{n-1}} D_{x_n} N(x' - y', x_n) R'(g_n - v_n|_{y_n=0} - \dot{V}_n|_{y_n=0})(y', t) dy' \]
\[ = P_{x_n} R'(g_n - v_n|_{y_n=0} - \dot{V}_n|_{y_n=0}). \]

Since Poisson operator $P_{x_n}$ is bounded from $B_q^{-\frac{1}{2}}$ to $L^q$ (see [37] for the reference) and Riesz operator $R'_i$ is $L^q$ bounded, we have that
\[ \|\nabla \phi\|_{L^q(\mathbb{R}^n \times (0, T))} \leq c \|g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0}\|_{L^q(0, T; B_q^{-\frac{1}{2}}(\mathbb{R}^{n-1})))} + c \|R'(g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0})\|_{L^q(0, T; B_q^{-\frac{1}{2}}(\mathbb{R}^{n-1})))} \]
\[ \leq c \|g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0}\|_{L^q(0, T; B_q^{-\frac{1}{2}}(\mathbb{R}^{n-1})))} + \|v\|_{L^q(\mathbb{R}^n \times (0, T))} + \|\dot{V}\|_{L^q(\mathbb{R}^n \times (0, T))}. \]

5) Applying the last estimates of Lemma 5.5, $v|_{x_n=0} \in B_q^{-\frac{1}{2} - \frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))$ with
\[ \|v\|_{x_n=0}|_{B_q^{-\frac{1}{2} - \frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))} \leq c \max\{1, T^\frac{1}{2p}\} \|h\|_{B_q^{-\frac{1}{2}}(\mathbb{R}^n)}. \]

Applying the last estimates of Lemma 5.4, $V|_{x_n=0} \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$ with
\[ \|V|_{x_n=0}|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} \leq T^\frac{1}{2q_0} \|\phi\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)}. \]

Since $R_i$ is $L^q$ bounded (see [3.2]), $R'(g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0}) \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$ with
\[ \|R'(g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0})\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} \leq \|(g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0})\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))}. \]

In the end, we conclude that $G' = g' - v'|_{x_n=0} - V'|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - \dot{V}_n|_{x_n=0}) \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$ with
\[ \|G'\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} \leq c (\|g\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} + \max\{1, T^\frac{1}{2q_0}\} \|h\|_{B_{q_0}^{-\frac{1}{2}}(\mathbb{R}^n)} + T^\frac{1}{2q_0} \|\phi\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)}). \]

Recall that if $q > 3$, then $B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T)) = B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$. Hence we conclude that $G \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$.

If $1 < q \leq 3$, then from the fact that $V|_{x_n=0} \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$ and from the hypothesis $g - v|_{x_n=0} = g - \Gamma_{i *} \tilde{h}|_{x_n=0} \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$, we still conclude that $G \in B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))$ with some modification that
\[ \|G'\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} \leq c (\|g - \Gamma_{i *} \tilde{h}|_{x_n=0}\|_{B_{q_0}^{-\frac{1}{2} - \frac{1}{2q_0}}(\mathbb{R}^{n-1} \times (0, T))} + T^\frac{1}{2q_0} \|\phi\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)}. \]
6) Finally, applying Theorem 4.1 to the fact that \( G = (G', 0) \in B_{q_0}^{-\frac{1}{4}, -\frac{1}{2}t}(\mathbb{R}^{n-1} \times (0, T)) \), we conclude that \( w \in L^q(\mathbb{R}_+^n \times (0, T)) \) with

\[
\|w\|_{L^q(\mathbb{R}^n_+ \times (0,T))} \leq c \max\{1, T^\frac{1}{q_0}\} \|G\|_{B_{q_0}^{-\frac{1}{4}, -\frac{1}{2}t}(\mathbb{R}^{n-1} \times (0,T))}.
\]

This completes the proof of the estimate of the solution in Theorem 1.2.

5.3. Regularity. Using the estimate of the heat kernel \( \Gamma_t \), direct computation of \( v = \Gamma_t \ast \tilde{h} \) leads to the estimate that

\[
\|\nabla v\|_{L^q(\mathbb{R}^n_+)} \leq c t^{-\frac{1}{2}} \|h\|_{L^q(\mathbb{R}^n)}
\]

and

\[
\|\nabla v\|_{L^q(\mathbb{R}^n_+)} \leq c t^{-\frac{1}{2} - \frac{1}{q}} \|h\|_{B_{q_0}^{-\frac{1}{4}, -\frac{1}{2}t}(\mathbb{R}^n_+)}.
\]

According to Lemma 3.4, \( V_j(x, t) = -\int_0^t \int_{\mathbb{R}^n} Dv_k \Gamma(x - y, t - s) \left( \partial_{ij} \tilde{F} + R_i R_j \tilde{F}_{ki} \right)(y, s) dy ds \in W_p^{1, 1}(\mathbb{R}^n \times (0, T)) \). Using the estimate of the Poisson kernel \( Dv_k \mathcal{N}(x' - y', x_n) \), direct computation of \( \phi = \int_{\mathbb{R}^n} \mathcal{N}(x' - y', x_n)(g_n(y', t) - v_n(y', 0, t) - V_n(y', 0, t)) dy' \) leads to the estimate that

\[
\|\nabla^2 \phi\|_{L^q(\mathbb{R}^{n-1} \times (0, T))} \leq c x_n^{-1} \|g_n - v_n|_{x_n=0} - V_n|_{x_n=0}\|_{L^q(\mathbb{R}^{n-1} \times (0, T))}
\]

and

\[
\|\nabla^2 \phi\|_{L^q(\mathbb{R}^{n-1} \times (0, T))} \leq c x_n^{-1 - \frac{1}{q}} \|g_n - v_n|_{x_n=0} - V_n|_{x_n=0}\|_{L^q(\mathbb{R}^{n-1} \times (0, T))}.
\]

Finally, recall Remark 4.2 that

\[
\|D_x w(\cdot, x_n, t)\|_{L^q(\mathbb{R}^{n-1})} \leq c t^\frac{1}{2} x_n^{-\frac{1}{2}} \|G\|_{L^q(\mathbb{R}^{n-1} \times (0, T))}.
\]

and

\[
\|D_x w(\cdot, x_n, t)\|_{L^q(\mathbb{R}^{n-1})} \leq c t^\frac{1}{2} x_n^{-\frac{1}{2} - \frac{1}{q}} \|G\|_{B_{q_0}^{-\frac{1}{4}, -\frac{1}{2}t}(\mathbb{R}^{n-1} \times (0, T))},
\]

where \( G = (G', 0), G' = g' - v'|_{x_n=0} = V'|_{x_n=0} = R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}) \). Therefore, we conclude that \( \nabla u = \nabla (v + V + \nabla \phi + w) \in L^p_{t,x}(\mathbb{R}_+^n \times (0, T)) \).

5.4. Uniqueness. Suppose that \((u_1, p_1)\) and \((u_2, p_2)\) are very weak solutions of the Stokes equations 1.2 in the class \( L^q(\mathbb{R}_+^n \times (0, T)) \), then \( u_1 - u_2 \) satisfies the variational formulation

\[
\int_0^T \int_{\mathbb{R}_+^n} (u_1 - u_2) \cdot (-\phi_t - \Delta \phi + \nabla \pi) dx dt = 0
\]

for any \( \phi \in C^\infty_0(\mathbb{R}_+^n \times [0, T]) \) and \( \{ \phi \in C^\infty_0(\mathbb{R}_+^n \times (0, T)) \mid \text{div} \phi(\cdot, t) = 0 \text{ for all } t \in (0, T) \} \}. \) Since \( \{-\phi_t - \Delta \phi + \nabla \pi : \phi \in C^\infty_0(\mathbb{R}_+^n \times [0, T])\} \) is dense in \( L^\infty(\mathbb{R}_+^n \times [0, T]) \), we conclude that \( u_1 - u_2 = 0 \) a.e. in \( \mathbb{R}_+^n \times (0, T) \). Therefore, the uniqueness of the solution of the Stokes system 1.2 holds in the class \( L^q(\mathbb{R}_+^n \times (0, T)) \).
6. Nonlinear problem

In this section we would like to give a proof of Theorem 1.1. For the purpose of it, we construct approximate solutions and then derive uniform convergence in \( L^q(\mathbb{R}_+^n \times (0, T)) \). For the uniform estimates, bilinear estimates should be preceded.

Choose \( p = \frac{q}{2} \). Then \( \alpha_1 = 1 - (n + 2)(\frac{1}{p} - \frac{1}{q}) > 0 \) for any \( q > n + 2 \) and

\[
\|(u \otimes v)\|_{L^p(\mathbb{R}^n \times (0, T))} \leq c\|u\|_{L^q(\mathbb{R}^n \times (0, T))}\|v\|_{L^q(\mathbb{R}^n \times (0, T))}
\]

for any \( u, v \in L^q(\mathbb{R}^n \times (0, T)) \).

6.1. Proof of Theorem 1.1. In this section we would like to construct a solution of the Navier-Stokes equations 1.1.

6.1.1. Approximating solution. Let \((u^1, p^1)\) be the solution of the equations

\[
\begin{align*}
    &u^1_t - \Delta u^1 + \nabla p^1 = 0, \quad \text{div} u^1 = 0, \quad \text{in} \ \mathbb{R}_+^n \times (0, T), \\
    &u^1|_{t=0} = h, \quad u^1|_{x_n=0} = g.
\end{align*}
\]

Let \( m \geq 1 \). After obtaining \((u^1, p^1), \ldots, (u^m, p^m)\) construct \((u^{m+1}, p^{m+1})\) which satisfies the equations

\[
\begin{align*}
    &u^{m+1}_t - \Delta u^{m+1} + \nabla p^{m+1} = f^m, \quad \text{div} u^{m+1} = 0, \quad \text{in} \ \mathbb{R}_+^n \times (0, T), \\
    &u^{m+1}|_{t=0} = h, \quad u^{m+1}|_{x_n=0} = g,
\end{align*}
\]

where \( f^m = -\text{div}(u^m \otimes u^m) \).

6.1.2. Uniform boundedness. Let \( q > n + 2 \). By the result of Theorem 1.2 we have

\[
\|u^1\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq c_1 \left( \max\{1, T^{\frac{1}{q'}}\} \|h\|_{B_q^\frac{-1}{q'}(\mathbb{R}_+^n)} + \max\{1, T^{\frac{1}{2q'}}\} \|g\|_{B_q^\frac{-1}{2q'}(\mathbb{R}_+^n \times (0, T))} \right)
\]

\[
+ \|g_n\|_{L^q(0, T; B_{q'}^\frac{-1}{q'}(\mathbb{R}^n))},
\]

According to bilinear estimate 6.1, choosing \( p = \frac{q}{2} \), we have

\[
\|(u^m \otimes u^m)\|_{L^p(\mathbb{R}^n \times (0, T))} \leq c\|u^m\|_{L^q(\mathbb{R}_+^n \times (0, T))}^2.
\]

Hence, we have

\[
\|u^{m+1}\|_{L^q(\mathbb{R}^n_+ \times (0, T))} \leq c_1 \left( \max\{1, T^{\frac{1}{q'}}\} \|h\|_{B_q^\frac{-1}{q'}(\mathbb{R}_+^n)} + \max\{1, T^{\frac{1}{2q'}}\} \|g\|_{B_q^\frac{-1}{2q'}(\mathbb{R}_+^n \times (0, T))} \right)
\]

\[
+ \|g_n\|_{L^q(0, T; B_{q'}^\frac{-1}{q'}(\mathbb{R}^n))} + T^{\frac{1}{2} - \frac{n+2}{2q'}} \|u^m\|_{L^q(\mathbb{R}_+^n \times (0, T))}^2.
\]

Set

\[
M_0 = \|h\|_{B_q^\frac{-1}{q'}(\mathbb{R}_+^n)} + \|g\|_{B_q^\frac{-1}{2q'}(\mathbb{R}_+^n \times (0, T))} + \|g_n\|_{L^q(0, T; B_{q'}^\frac{-1}{q'}(\mathbb{R}^n))}.
\]

Choose \( M > 2c_1M_0 \). Then (6.4) leads to the estimate

\[
\|u^1\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq c_1 M_0 < M \text{ for } T \leq 1.
\]
Under the condition that \( \|u^m\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq M \), (6.5) leads to the estimate

\[
\|u^{m+1}\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq c_1 M_0 + c_1 T^{\frac{2}{q}} M^2 \text{ for } T \leq 1.
\]

Choose \( 0 < T \leq \frac{1}{(2c_1 M)^2} \), together with the condition \( T \leq 1 \). Then by the mathematical induction argument we can conclude that

\[
\|u^m\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq M \text{ for all } m = 1, 2 \ldots
\]

6.1.3. **Uniform convergence.** Let \( U^m = u^{m+1} - u^m \) and \( P^m = p^{m+1} - p^m \). Then \( U^m \) satisfies the equations

\[
U^m_t - \Delta U^m + \nabla P^m = -\text{div}(u^m \otimes U^{m-1} + U^{m-1} \otimes u^{m-1}), \quad \text{div} U^m = 0, \text{ in } \mathbb{R}_+^n \times (0, T),
\]

\[
U^m|_{t=0} = 0, \quad U^m|_{x_n=0} = 0,
\]

By the result of Theorem 1.2, we have

\[
\|U^m\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq c_2 T^{\frac{2}{q} - \frac{n+2}{q}} \|u^m \otimes U^{m-1} + U^{m-1} \otimes u^{m-1}\|_{L^p(\mathbb{R}_+^n \times (0, T))}
\]

\[
\leq c_2 T^{\frac{2}{q} - \frac{n+2}{q}} \|u^m\|_{L^q(\mathbb{R}_+^n \times (0, T))} \|U^{m-1}\|_{L^q(\mathbb{R}_+^n \times (0, T))} + \|u^{m-1}\|_{L^q(\mathbb{R}_+^n \times (0, T))} \|U^{m-1}\|_{L^q(\mathbb{R}_+^n \times (0, T))}.
\]

Choose \( 0 < T \leq \frac{1}{(2c_1 M)^2} \), together with the condition \( T \leq \frac{1}{(2c_1 M)^2} \) and \( T \leq 1 \). Then, the above estimate leads to the

\[
\|U^m\|_{L^q(\mathbb{R}_+^n \times (0, T))} \leq \frac{1}{2} \|U^{m-1}\|_{L^q(\mathbb{R}_+^n \times (0, T))}.
\]

(6.6) implies that the infinite series \( \sum_{k=1}^{\infty} U^k \) converges in \( L^q(\mathbb{R}_+^n \times (0, T)) \). Again it means that \( u^m = u^1 + \sum_{k=1}^{m} U^k \) converges to \( u^1 + \sum_{k=1}^{\infty} U^k \) in \( L^q(\mathbb{R}_+^n \times (0, T)) \). Set \( u := u^1 + \sum_{k=1}^{\infty} U^k \).

6.2. **Existence and regularity.** Let \( u \) be the same one constructed by the previous section. In this section, we will show that \( u \) satisfies weak formulation of Navier-Stokes equations, that is, \( u \) is a weak solution of Navier-Stokes equations with appropriate distribution \( p \).

Let \( \Phi \in C^\infty_0(\mathbb{R}_+^n \times (0, T)) \) with \( \text{div} \Phi = 0 \) and \( \Phi|_{x_n=0} = 0 \). Observe that

\[
- \int_0^T \int_{\mathbb{R}_+^n} u^{m+1} \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}_+^n} u^{m+1} \cdot \Phi_t + (u^{m} \otimes u^m) \cdot \nabla \Phi dx dt + \langle h, \Phi(\cdot, 0) \rangle_{\mathbb{R}_+^n} - \langle g, \frac{\partial \Phi}{\partial x_n} \rangle_{\mathbb{R}_+^n \times \mathbb{R}_+^n}.
\]

Now send \( m \) to the infinity, then, since \( u^m \to u \) in \( L^q(\mathbb{R}_+^n \times (0, T)) \), we have

\[
- \int_0^T \int_{\mathbb{R}_+^n} u \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}_+^n} u \cdot \Phi_t + (u \otimes u) \cdot \nabla \Phi dx dt + \langle h, \Phi(\cdot, 0) \rangle_{\mathbb{R}_+^n} - \langle g, \frac{\partial \Phi}{\partial x_n} \rangle_{\mathbb{R}_+^n \times \mathbb{R}_+^n}.
\]

(6.7)
Since \( u \) satisfies the Stokes equations \([12]\) with \( f = -\text{div} \mathcal{F} = -\text{div}(u \otimes u) \), \( u \) can be decomposed by \( u = v + V + \nabla \phi + w \), where

\[
v(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t) \tilde{h}(y) dy,
\]

\[
V_j(x, t) = \int_0^t \int_{\mathbb{R}^n} D_{x_j} \Gamma(x - y, t - s) [\delta_{ij} \tilde{u}_k \tilde{u}_i + R_i R_j \tilde{u}_k \tilde{u}_i](y, s) dy ds,
\]

\[
\phi(x, t) = 2 \int_{\mathbb{R}^n} N(x' - y', x_n) \left( g_n(y', t) - v_n(y', 0, t) - V_n(y', 0, t) \right) dy',
\]

\[
w_i = \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^n} K_{ij}(x' - y', x_n, t - s) G_j(y', s) dy ds,
\]

for \( G = (g' - V'|_{x_n=0} - v'|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}, 0) \). Observe that \( \delta_{ij} \tilde{u}_k \tilde{u}_i + R_i R_j \tilde{u}_k \tilde{u}_i \in L^4(\mathbb{R}^n_+ \times (0, T)) \). According to Lemma 5.3, \( V_j(x, t) \in W^{1, \frac{4}{2}}(\mathbb{R}^n \times (0, T)) \). On the other hand, by the same argument in section 5.3, we have

\[
\| \nabla v \|_{L^q(\mathbb{R}^n_+ \times (0, T))} \leq c t^{-\frac{1}{2}} \| h \|_{L^q(\mathbb{R}^n_+)},
\]

\[
\| \nabla v \|_{L^q(\mathbb{R}^n_+ \times (0, T))} \leq c t^{-\frac{1}{2}} \| h \|_{B^q_{\infty, \infty}(\mathbb{R}^n_+)}.
\]

Therefore, we conclude that \( \nabla u = \nabla(v + V + \nabla \phi + w) \in L^{\frac{4}{n}}(\mathbb{R}^n_+ \times (0, T)) \). This leads to the conclusion that \( u \) is a weak solution of the Navier-Stokes equations \([11]\).

6.3. Uniqueness. Let \( v \in L^q(\mathbb{R}^n_+ \times (0, T)) \) be another solution of Naiver-Stokes equations \([11]\) with pressure \( q \). Then \( u - v \) satisfies the equations

\[
(u-v)_t - \Delta(u-v) + \nabla(p - q) = -\text{div}(u \otimes (u-v) + (u-v) \otimes v) \text{ in } \mathbb{R}^n_+ \times (0, T),
\]

\[
\text{div}(u-v) = 0, \text{ in } \mathbb{R}^n_+ \times (0, T),
\]

\[
(u-v)|_{t=0} = 0, \quad (u-v)|_{x_n=0} = 0.
\]

Applying Theorem 1.2 to the above Stokes equations for \( u-v \), then we have

\[
\|u-v\|_{L^q(\mathbb{R}^n_+ \times (0,T_1))} \leq c T_1^{\frac{1}{2} - \frac{n+2}{2q}} \|u \otimes (u-v) + (u-v) \otimes v\|_{L^p(\mathbb{R}^n_+ \times (0,T_1))}
\]

\[
\leq c T_1^{\frac{1}{2} - \frac{n+2}{2q}} (\|u\|_{L^q(\mathbb{R}^n_+ \times (0,T_1))} + \|v\|_{L^q(\mathbb{R}^n_+ \times (0,T_1))}) \|u-v\|_{L^q(\mathbb{R}^n_+ \times (0,T_1))}, \quad T_1 \leq T.
\]
If we take $T_1 \leq \frac{1}{\mathfrak{m}_2^2}(\|u\|_{L^q(\mathbb{R}_+^n \times (0,T_1))} + \|v\|_{L^q(\mathbb{R}_+^n \times (0,T_1))} + 1)^2$ together with $T_1 \leq 1$, then the above inequality leads to the conclusion that

$$\|u - v\|_{L^q(\mathbb{R}_+^n \times (0,T_1))} = 0$$

that is, $u \equiv v$ in $\mathbb{R}_+^n \times (0,T_1)$.

By the same argument, we can show that

$$\|u - v\|_{L^q(\mathbb{R}_+^n \times (T_1,2T_1))} = 0$$

that is, $u \equiv v$ in $\mathbb{R}_+^n \times (T_1,2T_1)$.

After iterating this procedure finitely many times, we obtain the conclusion that $u = v$ in $\mathbb{R}_+^n \times (0,T)$.

**Appendix A. Proof of Lemma 3.1**

The following is well known estimates (see section 4.3 in [30] for the reference):

$$\|T_1 f\|_{\dot{W}^{2-1,q}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^q(\mathbb{R}^n \times \mathbb{R})}. \quad (A.1)$$

Observe that $T_1^*$ is adjoint operator of $T_1$, since

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} T_1 \psi(x,t) dx dt = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} T_1^* \phi(y,s) \psi(y,s) dy ds.$$

Observe that $D_y^2 T_1^* f$, $D_y T_1^* f$ have $L^p$ Fourier multipliers since the Fourier transform of $T_1^* f$ is

$$\hat{T}_1^* f = \frac{1}{|\xi - \eta|^p} f(\xi, \eta).$$

By the well known theory for the multiplier (see [37]) we have

$$\|T_1^* f\|_{\dot{W}^{2-1,q}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}, \quad 1 < p < \infty. \quad (A.2)$$

Since $T^*$ is the adjoint operator of $T$, (A.2) implies that

$$\|T_1 f\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{W}^{2-1,q}(\mathbb{R}^n \times \mathbb{R})}. \quad (A.3)$$

and (A.1) implies that

$$\|T_1^* f\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{W}^{2-1,q}(\mathbb{R}^n \times \mathbb{R})}. \quad (A.4)$$

Applying real interpolation theory to (A.3) and (A.1), we complete the proof of the estimate $T_1 f$ in Lemma 3.1 for $2 > \alpha > 0$. Also, applying real interpolation theory to (A.4) and (A.2), we complete the proof of the estimate $T_1^* f$ in Lemma 3.1 for $2 > \alpha > 0$.

**Appendix B. Proof of Lemma 3.2**

The following is well known estimates (see section 4.3 in [30] for the reference):

$$\|T_2 g\|_{\dot{W}^{2-1,q}(\mathbb{R}^n \times \mathbb{R})} \leq c\|g\|_{B^{1-1/q}_{q,\frac{q}{q-1}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \quad (B.1)$$

Firstly, let us derive the estimate of $T_2 g$. Observe the identity

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} T_2 g(x,t) \phi(x,t) dx dt = \langle g, T_1^* \phi \rangle_{g_n=0} > \quad (B.2)$$
holds for \( \phi \in C_0^\infty(\mathbb{R}_n^+ \times \mathbb{R}) \), where \( T_1^\ast \phi(y, s) = \int_s^\infty \int_{\mathbb{R}^n} \Gamma(x - y, t - s) \phi(x, t) dx dt \) and \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \dot{B}_q^{1 - \frac{1}{2} + \frac{1}{2q} + \frac{1}{2q} + \frac{1}{2q}}(\mathbb{R}^{n-1} \times \mathbb{R}) \) and \( \dot{B}_{q'}^{1 + \frac{1}{2} - \frac{1}{2q'} - \frac{1}{2q'} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R}) \). From the result of Lemma 3.1 we have

\[
\| T_1^\ast \phi \|_{\dot{W}^{2,1}_q(\mathbb{R}^n \times \mathbb{R})} \leq c \| \phi \|_{L^q(\mathbb{R}^n \times \mathbb{R})} = \| \phi \|_{L^q(\mathbb{R}^n \times \mathbb{R})}. \tag{B.3}
\]

By trace theorem, we have

\[
\| T_1^\ast \phi \|_{\dot{W}^{2,1}_q(\mathbb{R}^n \times \mathbb{R})} \leq c \| T_1^\ast \phi \|_{\dot{W}^{2,1}_q(\mathbb{R}^n \times \mathbb{R})} \leq c \| \phi \|_{L^q(\mathbb{R}^n \times \mathbb{R})}. \tag{B.4}
\]

Apply the estimate (B.4) to (B.2), we have

\[
\| T_2g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} \leq c \| g \|_{\dot{B}^{-1 + \frac{1}{q} - \frac{1}{2q} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R})} \tag{B.5}
\]

Applying real interpolation theory to (B.5) and (B.1), we complete the proof of the estimate of \( T_2g \) in Lemma 3.2.

Analogously, we can derive the estimate of \( T_2^\ast g \), observing the identity

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} T_2^\ast g(y, s) dy ds = < g, T_1 \phi > \tag{B.6}
\]

holds for \( \phi \in C_0^\infty(\mathbb{R}_n^+ \times \mathbb{R}) \), where \( T_1 \phi(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s) \phi(y, s) dy ds \) and \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \dot{B}_q^{1 - \frac{1}{2} + \frac{1}{2q} + \frac{1}{2q} + \frac{1}{2q} + \frac{1}{2q}}(\mathbb{R}^{n-1} \times \mathbb{R}) \) and \( \dot{B}_{q'}^{1 + \frac{1}{2} - \frac{1}{2q'} - \frac{1}{2q'} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R}) \). By the same procedure as for the estimate of \( T_2f \) (We omit the details), we can obtain the estimate of \( T_2^\ast g \) that

\[
\| T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} \leq c \| g \|_{\dot{B}^{-1 + \frac{1}{q} - \frac{1}{2q} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{B.7}
\]

Since \( D_{x_k} D_{x_l} T_2^\ast g = T_2^\ast (D_{x_k} D_{x_l} g) \) for \( k, l \neq n \), and \( D_x T_2^\ast g = T_2^\ast (D_x g) \) we have

\[
\sum_{k \neq n} \| D_{x_k}^2 T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} + \| D_x T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} \leq c \| g \|_{\dot{B}^{-1 + \frac{1}{q} - \frac{1}{2q} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{B.8}
\]

Since \( D_{x_k}^2 T_2^\ast g = -D_x T_2^\ast g - \sum_{n \neq m} D_{x_k}^2 T_2^\ast g \), we again have

\[
\| \Delta_y T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} + \| D_x T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} \leq c \| g \|_{\dot{B}^{-1 + \frac{1}{q} - \frac{1}{2q} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{B.9}
\]

By the well known elliptic theory \( T_2^\ast g|_{y_n = 0} = 0 \) implies that

\[
\| D_y^2 T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})} \leq c \| \Delta_y T_2^\ast g \|_{L^q(\mathbb{R}^n \times \mathbb{R})}. \tag{B.10}
\]

Combining all the above estimates we conclude that

\[
\| T_2^\ast g \|_{W^{2,1}_q(\mathbb{R}^n \times \mathbb{R})} \leq c \| g \|_{\dot{B}^{-1 + \frac{1}{q} - \frac{1}{2q} - \frac{1}{2q'}}(\mathbb{R}^{n-1} \times \mathbb{R})}. \tag{B.10}
\]

Applying real interpolation theory to (B.7) and (B.10), we complete the proof of the estimate of \( T_2^\ast g \) in 3.2 for \( 0 < \alpha < 2 \).
Appendix C. Proof of Lemma 3.3

• First we would like to derive the estimate of $T_0 h = \Gamma_t * h$.

Let us consider the case $h \in \dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)$. Observe the identity $\int_0^\infty \int_{\mathbb{R}^n} T_0 h(x, t) \phi(x, t) dx dt = < h, T^*_1 \phi |_{s=0} >$ holds for $\phi \in C_0^\infty (\mathbb{R}^n \times \mathbb{R})$, where $T^*_1 \phi(y, s) = \int_s^\infty \int_{\mathbb{R}^n} \Gamma(x - y, t - s) \phi(x, t) dx dt$, and $< \cdot, \cdot >$ is the duality pairing between $\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)$ and $\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^n)$. From the result of Lemma 3.1 we have

$$\| T^*_1 \phi \|_{W^{2,1}_q (\mathbb{R}^n \times \mathbb{R})} \leq c \| \phi \|_{L^q (\mathbb{R}^n \times \mathbb{R})}.$$  

By trace theorem this implies that

$$\| T^*_1 \phi |_{s=0} \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)} \leq c \| T^*_1 \phi \|_{W^{2,1}_q (\mathbb{R}^n \times \mathbb{R})} \leq c \| \phi \|_{L^q (\mathbb{R}^n \times \mathbb{R})}.$$  

Hence, we have

$$< h, T^*_1 \phi |_{s=0} > \leq c \| h \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)} \| T^*_1 \phi \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)} \leq c \| h \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)} \| \phi \|_{L^q (\mathbb{R}^n \times \mathbb{R})}.$$  

Again this leads to the conclusion that

$$\| T_0 h \|_{L^q (\mathbb{R}^n \times \mathbb{R})} \leq c \| h \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)}.$$  

(C.1)

On the other hand, by Young’s theorem we have

$$\| T_0 h \|_{L^r (\mathbb{R}^n \times (0, T))} \leq c T^\frac{1}{r} \| h \|_{L^r (\mathbb{R}^n)}.$$  

(C.2)

From the fact that $B^s_q (\mathbb{R}^n) = \dot{B}^s_q (\mathbb{R}^n) + L^s (\mathbb{R}^n)$ for $s < 0$, (C.1) and (C.2) imply that

$$\| T_0 h \|_{L^r (\mathbb{R}^n \times (0, T))} \leq c \max \{ 1, T^\frac{1}{r} \} \| h \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)}.$$  

(C.3)

• Now, we will derive the estimate of $T_0 h |_{x_n=0} = \Gamma_t * f |_{x_n=0}$.

Let $h \in \dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)$. Observe the identity

$$< T_0 h, \phi >= < h, T^*_2 \phi |_{s=0} >,$$  

(C.4)

holds for any $\phi \in C_0^\infty (\mathbb{R}^{n-1} \times \mathbb{R})$, where $T^*_2 \phi(y, s) = \int_s^\infty \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', y_n, t - s) \phi(x', t) dx'dt$ and $< \cdot, \cdot >$ is the duality pairing between $\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)$ and $\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^n)$ . From the result of Lemma 3.2 $T^*_2 \phi \in \dot{W}^{2,1}_q (\mathbb{R}^n_+ \times \mathbb{R})$ with

$$\| T^*_2 \phi \|_{\dot{W}^{2,1}_q (\mathbb{R}^n_+ \times \mathbb{R})} \leq c \| \phi \|_{\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^{n-1} \times \mathbb{R})}.$$  

By Trace theorem, this implies that $T^*_2 \phi |_{s=0} \in \dot{B}^{\frac{2}{q}}_q (\mathbb{R}^n_+)$ with

$$\| T^*_2 \phi |_{s=0} \|_{\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^n_+)} \leq c \| \phi \|_{\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^{n-1} \times \mathbb{R})}.$$  

Hence

$$< h, T^*_2 \phi |_{s=0} > \leq c \| h \|_{\dot{B}^{-\frac{2}{q}}_q (\mathbb{R}^n)} \| \phi \|_{\dot{B}^{\frac{2}{q}}_q (\mathbb{R}^{n-1} \times \mathbb{R})}.$$
Applying the above estimate to (C.4), we conclude that $T_0h|_{x_n=0} \in B^{\frac{1}{q'}-\frac{1}{q}}_q(\mathbb{R}^{n-1} \times \mathbb{R})$ with

$$
\|T_0h|_{x_n=0}\|_{B^{\frac{1}{q'}-\frac{1}{q}}_q(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|h\|_{B^{\frac{1}{q'}}_q(\mathbb{R}^n)}. \tag{C.5}
$$

On the other hand, by Young’s inequality we have

$$
\|T_0h|_{x_n=0}\|_{L^s((\mathbb{R}^{n-1} \times (0, T))} \leq cT^{\frac{1}{q'}-\frac{1}{q}}\|h\|_{L^s(\mathbb{R}^n)}. \tag{C.6}
$$

Recall the fact that $B^{\frac{1}{q}+\frac{1}{q}}_q(\Omega \times (0, T)) = B^{\frac{1}{q}+\frac{1}{p}}_q(\Omega \times (0, T)) + L^q(\Omega \times (0, T))$ for $s < 0$. Combining (C.5) and (C.6), we have

$$
\|T_0h|_{x_n=0}\|_{B^{\frac{1}{q'}-\frac{1}{q}}_q(\mathbb{R}^{n-1} \times (0, T))} \leq c\max\{1, T^{\frac{1}{q'}-\frac{1}{q}}\}\|h\|_{B^{\frac{1}{q'}}_q(\mathbb{R}^n)}, \tag{C.7}
$$

**APPENDIX D. PROOF OF LEMMA 3.4**

Note that $u = D_xT_1\tilde{f}$. By Lemma 3.3, the following estimate holds

$$
\|u\|_{W^{1, q}_p(\mathbb{R}^{n} \times \mathbb{R})} \leq \|T_1\tilde{f}\|_{W^{1, q}_p(\mathbb{R}^{n} \times \mathbb{R})} \leq c\|\tilde{f}\|_{L^p(\mathbb{R}^{n} \times \mathbb{R})}. \tag{D.1}
$$

On the other hand, by Young’s inequality we have

$$
\|u\|_{L^q(\mathbb{R}^{n} \times (0, T))} \leq cT^{\frac{1}{q'}-\frac{1}{q}}\|\tilde{f}\|_{L^p(\mathbb{R}^{n} \times (0, T))}, \tag{D.2}
$$

where $\alpha_1 = 1 - (n + 2)(\frac{1}{p} - \frac{1}{q}) > 0$.

Now we will derive the estimate of $u|_{x_n=0}$ in $B^{\frac{1}{q}+\frac{1}{q}}_q(\mathbb{R}^{n-1} \times (0, T))$. Choose $q_1 = \frac{(n+1)q}{n+2}$ so that $q_1 < q$ and $(\beta_1 := 1 - (n + 2)(\frac{1}{p} - \frac{1}{q_1}) > \frac{1}{q_1}$. By Young’s inequality

$$
\|u|_{x_n=0}\|_{L^{q_1}(\mathbb{R}^{n-1} \times (0, T))} \leq cT^{\frac{1}{q_1}-\frac{1}{q_1}}\|\tilde{f}\|_{L^{p}(\mathbb{R}^{n} \times (0, T))} \tag{D.2}
$$

Observe that $L^{q_1}(\mathbb{R}^{n-1} \times (0, T)) \subset B^{\frac{1}{q}+\frac{1}{q}}_q(\mathbb{R}^{n-1} \times (0, T))$ for $-\frac{n+1}{q_1} = -\frac{1}{q} - \frac{n+1}{q}$. Note that

$$
\beta_1 - \frac{1}{2q_1} = \frac{1}{2}(1 - (n + 2)(\frac{1}{p} - \frac{1}{q_1})) - \frac{1}{2q_1} = \frac{1}{2}(1 - (n + 2)(\frac{1}{p} - \frac{1}{q_1})) = \frac{\alpha_1}{2}.
$$

This completes the proof of Lemma 3.4.

**REFERENCES**

[1] M.F. de Almeida and L.C.F. Ferreira, *On the Navier-Stokes equations in the half-space with initial and boundary rough data in Morrey spaces*, J. Differential Equations 254, no. 3, 1548-1570 (2013).

[2] H. Amann, *Anisotropic function spaces and maximal regularity for parabolic problems. Part 1. Function spaces*, Jindřich Nečas Center for Mathematical Modeling Lecture Notes, 6, Matfyzpress, Prague, vi+141 (2009).

[3] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. 2, no. 1, 16-98 (2000).

[4] H. Amann, *Navier-Stokes equations with nonhomogeneous Dirichlet data*, J. Nonlinear Math. Phys. 10, suppl. 1, 1-11 (2003).

[5] H. Amann, *Nonhomogeneous Navier-Stokes Equations with Integrable Low-Regularity Data*, J. Nonlinear Math. Phys. 10, suppl. 1, 1-11 (2003).

[6] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[7] M. Cannone, F. Planchon and M. Schonbek, *Strong solutions to the incompressible Navier-Stokes equations in the half-space*, Comm. Partial Differential Equations 25, no. 5-6, 903-924 (2000).

[8] T. Chang and H.J. Choe, *Maximin modulus estimate for the solution of the Stokes equations*, J. Differential Equations 254, no. 7, 2682-2704 (2013).

[9] E.B. Fabes, B.F. Jones and N.M. Rivière, *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Ration. Mech. Anal. 45, 222-240 (1972).
[10] R. Farwig and H. Kozono, *Weak solutions of the Navier-Stokes equations with non-zero boundary values in an exterior domain satisfying the strong energy inequality*, J. Differential Equations 256, no. 7, 2633-2658 (2014).
[11] R. Farwig, G.P. Galdi and H. Sohr, *Very Weak Solutions of Stationary and Instationary Navier-Stokes Equations with Nonhomogeneous Data*, Nonlinear elliptic and parabolic problems, 113-136, Progr. Nonlinear Differential Equations Appl., 64, Birkhäuser, Basel, 2005.
[12] R. Farwig, H. Kozono, and H. Sohr, *Very weak solutions of the Navier-Stokes equations in exterior domains with nonhomogeneous data*, J. Math. Soc. Japan 59, no. 1, 127-150 (2007).
[13] R. Farwig, H. Kozono, and H. Sohr, *Global weak solutions of the Navier-Stokes equations with nonhomogeneous boundary data and divergence*, Rend. Semin. Mat. Univ. Padova 125, 51-70 (2011).
[14] Y. Giga, *Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations 62, no. 2, 186-212 (1986).
[15] Y. Giga and T. Miyakawa, *Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial velocity and Morrey spaces*, Comm. Partial Differential Equations 14, no. 5, 577-618 (1989).
[16] M. Giga, Y. Giga and H. Sohr, *$L^p$ estimates for the Stokes system, Functional analysis and related topics*, 1991 (Kyoto), 5567, Lecture Notes in Math., 1540, Springer, Berlin (1993).
[17] Y. Giga and H. Sohr, *Abstract $L^p$ estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal. 102, no. 1, 72-94 (1991).

[18] G. Grubb, *Nonhomogeneous time-dependent Navier-Stokes problems in $L_p$ Sobolev spaces*, Differential Integral Equations 8, no. 5, 1013-1046 (1995).
[19] G. Grubb, *Nonhomogeneous Dirichlet Navier-Stokes problems in low regularity $L_p$ Sobolev spaces*, J. Math. Fluid Mech. 3, no. 1, 57-81 (2001).
[20] G. Grubb and V.A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand. 69, no. 2, 217-290 (1992).
[21] D. Hitimie, *The resolution of the Navier-Stokes equations in anisotropic spaces*, Revista Matemática Iberoamericana Vol. 15, No. 1, 1-35 (1999).
[22] T. Kato, *Strong solutions of the Navier-Stokes equation in Morrey spaces*, Bol. Soc. Brasil. Mat. (N.S.) 22, no. 2, 127-155 (1992).
[23] T. Kato, *Strong $L^p$ solutions of the Navier-Stokes equation in $\mathbb{R}^n$, with applications to weak solutions*, Math. Z. 187, no. 4, 471-480 (1984).
[24] T. Kato and G. Ponce, *Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p(\mathbb{R}^2)$*, Rev. Mat. Iberoamericana 2, no. 1-2, 73-88 (1986).
[25] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. 157, no. 1, 22-35 (2001).
[26] H. Koch and V. A. Solonnikov, *$L_q$-Estimates for a solution to the nonstationary Stokes equations*, Journal of Mathematical Sciences, Vol. 106, No. 3, 3042-3072 (2001).
[27] H. Koch and V. A. Solonnikov, *$L_q$-estimates of the first-order derivatives of solutions to the nonstationary Stokes problem*, Nonlinear problems in mathematical physics and related topics, I, Int. Math. Ser. (N. Y.), 1, Kluwer/Plenum, New York, 203-218 (2002).
[28] H. Kozono, *Global $L^n$-solution and its decay property for the Navier-Stokes equations in half-space $R^n_+$*, J. Differential Equations 79, no. 1, 79-88 (1989).
[29] H. Kozono, and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations 19, no. 5-6, 959-1014 (1994).
[30] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968.
[31] P.G. Lemarié-Rieusset, *The Navier-Stokes equations in the critical Morrey-Campanato space*, Rev. Mat. Iberoam. 23, no. 3, 897-930 (2007).
[32] J.E. Lewis, *The initial-boundary value problem for the Navier-Stokes equations with data in $L^p$*, Indiana Univ. Math. J. 22, 739-761 (1972/73).
[33] J.-P. Raymond, *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 24, no. 6, 921-951 (2007).
[34] V.A. Solonnikov, *Estimates of the solutions of the nonstationary Navier-Stokes system*, Boundary value problems of mathematical physics and related questions in the theory of functions, 7, Zap. Naum. Sem. LOMI. 38, 153-231 (1973).
[35] V.A. Solonnikov, *$L^p$-estimates for solutions to the initial boundary-value problem for the generalized Stokes system in a bounded domain*, Function theory and partial differential equations. J. Math. Sci. (New York) 105 (2001), no. 5, 2448-2484.
[36] V.A. Solonnikov, *Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator*, (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 123-156; translation in Russian Math. Surveys 58 (2003), no. 2, 331-365.
[37] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.
[38] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, 78, Birkhäuser Verlag, Basel, 1983.
[39] H. Triebel, *Theory of Function Spaces. III*, Monographs in Mathematics, 100, Birkhäuser Verlag, Basel, 2006.
[40] K.A. Voss, *Self-similar solutions of the Navier-Stokes equation*, Thesis (Ph.D.) Yale University, 1996.
[41] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Tokyo 33, 131-174 (1986).
[42] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators, Second edition*, Johann Ambrosius Barth, Heidelberg, 1995.

Department of Mathematics, Yonsei University, Seoul, 136-701, South Korea
*E-mail address:* chang7357@yonsei.ac.kr

Department of Mathematics, Mokpo National University, Muan-gun 534-729, South Korea
*E-mail address:* bumjajin@hanmail.net