EXPLICITLY SOLVABLE SYSTEMS OF FIRST-ORDER DIFFERENCE EQUATIONS WITH HOMOGENEOUS POLYNOMIAL RIGHT-HAND SIDES

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In this short paper we identify special systems of (arbitrary number) \( N \) of first-order Difference Equations with nonlinear homogeneous polynomials of arbitrary degree \( M \) in their right-hand sides, which feature very simple explicit solutions. A novelty of these findings is to consider special systems characterized by constraints involving both their parameters and their initial data.

The general system of an arbitrary number \( N \) of first-order Difference Equations (DEs) with homogeneous polynomials of arbitrary degree \( M \) on their right-hand sides reads as follows:

\[ \tilde{z}_n(s) = \sum_{m_\ell}^{(M)} \{ c_{nm_1m_2...m_N} [z_1(s)]^{m_1} [z_2(s)]^{m_2} \cdots [z_N(s)]^{m_N} \}, \]

\[ n = 1, 2, ..., N. \]  

(1)

\textbf{Notaion.} Above and hereafter \( s \) is the discrete independent variable taking all nonnegative integer values, \( s = 0, 1, 2, ... \); the \( N \) dependent variables \( z_n(s) \) with \( n = 1, 2, ..., N \) are (possibly complex) numbers, and ascertaining their \( s \)-evolution from the set of \( N \) initial data \( z_n(0) \) is our main task; the symbol \( \tilde{z}_n(s) \) denotes the forward-shifted dependent variable,

\[ \tilde{z}_n(s) \equiv z_n(s + 1), \quad n = 1, 2, ..., N; \]  

(2)

the symbol \( \sum_{m_\ell}^{(M)} \) denotes the sum running over all nonnegative values of the \( N \) nonnegative integer parameters (indices and exponents) \( m_\ell \) subject to the restrictions

\[ m_\ell \geq 0, \quad \sum_{\ell=1}^{N} (m_\ell) = M. \]  

(3)

implying that the polynomials in \( N \) variables \( z_n(s) \) in the right-hand sides of the \( N \) DEs \( \text{I} \) are all homogeneous of degree \( M \), being characterized by the \( s \)-independent coefficients \( c_{nm_1m_2...m_N} \).

The findings reported in this paper are the extension to discrete time of the somewhat analogous results for systems of first-order Ordinary Differential Equations (ODEs) reported in \( \text{I} \); indeed, its presentation occasionally follows verbatim the text of \( \text{I} \). Simple as they are, they are to the best of our knowledge new, being based on a somewhat unconventional approach: to identify explicitly solvable cases of the system \( \text{I} \) by introducing constraints involving, in addition to the coefficients \( c_{nm_1m_2...m_N} \), also the initial data \( z_n(0) \) (which, in applicable contexts, may play the role of control elements, allowing to manipulate the time evolution of the system).

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Remark 1. In this paper we focus on systems with homogeneous polynomial right-hand sides, see (1); but clearly—as pointed out by the extension [2] of the results of [1]—these findings can be extended to more general homogeneous functions than polynomials (see [3] [4]).

Our main result is the following

Proposition. The system (1) features the special solution

\[ z_n(s) = z_n(0) \left[ z_N(0) \right]^{M-1} Z^{(M-1)/(M-1)}, \quad n = 1, 2, ..., N, \tag{4a} \]

provided there hold the following \( N \) explicit algebraic constraints on the \textit{a priori arbitrary} parameter \( Z \), the coefficients \( c_{nm_1m_2...m_M} \) and the \( N \) initial data \( z_n(0) \):

\[ Z = (r_n)^{-1} \sum_{m_\ell} (M) \left\{ c_{nm_1m_2...m_N} \prod_{\ell=1}^{N-1} [(r_{\ell})^{m_\ell}] \right\}, \quad n = 1, 2, ..., N, \tag{4b} \]

where (here and hereafter)

\[ r_n \equiv z_n(0)/z_N(0). \tag{4c} \]

Remark 2. The proof that (6) satisfies the system of ODEs (1) is elementary: just insert (4a) in (1) and verify that, thanks to (3) and (4b), the \( N \) DEs (1) are satisfied.

Remark 3. Note that only the ratios of the \( N \) initial data play a role in the constraints (4b).

Remark 4. The system of \( N \) algebraic equations (4b) generally determines—for any given assignment of the \textit{a priori arbitrary} coefficients \( c_{nm_1m_2...m_N} \)—\( N \) out of the \( N+1 \) quantities \( Z \) and \( z_n(0) \) (of the latter, only their ratio); but it is also possible to select \textit{ad libitum} \( N \) elements out of the \textit{complete} set of data \( Z \), \( c_{nm_1m_2...m_M} \) and \( z_n(0) \) (of the latter, only their ratio), and to then consider these selected elements as those to be determined—by the \( N \) conditions (4b)—in terms of the remaining \textit{arbitrarily assigned} elements in the \textit{complete} set of these data. If one chooses to satisfy these \( N \) conditions by solving the \( N \) equations (4b) for \( N \) of the coefficients \( c_{nm_1m_2...m_M} \)—or for the parameter \( Z \) and \( N-1 \) of the coefficients \( c_{nm_1m_2...m_M} \)—then this task can be generally performed explicitly, since the relevant \textit{algebraic} equations to be solved are then \textit{linear} in the unknown quantities; otherwise these determinations require the solution of \textit{nonlinear} equations, a task which can be performed explicitly only rarely in an algebraic setting; but which can generally be performed, with \textit{arbitrary} approximation, in a \textit{numerical} context.

Example. Assume for instance \( N = 2 \) and \( M = 4 \), so that the system (1) reads as follows (note below the notational simplification):

\[ \ddot{z}_n(s) = \sum_{m=0}^{4} c_{nm} \left[ z_1(s) \right]^{4-m} \left[ z_2(s) \right]^m, \quad n = 1, 2, \tag{5} \]

featuring 2 dependent variables \( z_n(s) \) and 10 \textit{a priori arbitrary} coefficients \( c_{nm} \) \( (n = 1, 2; m = 0, 1, 2, 3, 4) \). Then the solution (4b) reads as follows:

\[ z_n(s) = z_n(0) \left[ z_2(0) \right]^{4-1} Z^{(4-1)/3}, \quad n = 1, 2, ..., N, \tag{6a} \]

and the 2 conditions (4b) read as follows:

\[ Z = r^{-1} \sum_{m=0}^{2} (c_{1m} r^m) = \sum_{m=0}^{2} (c_{2m} r^m), \tag{6b} \]

with \( r \equiv z_1(0)/z_2(0) \), namely

\[ \sum_{m=0}^{2} [(c_{1m} - c_{2m} r) r^m] = 0. \tag{6c} \]

These 2 algebraic constraints can of course be \textit{explicitly} solved for any 2 of the 10 coefficients \( c_{nm} \) in terms of the other 8 coefficients \( c_{nm} \) and of the 2 \textit{arbitrary} data \( Z \) and the ratio \( r \equiv z_1(0)/z_2(0) \); or alternatively for \( Z \) and only 1 of the 10 coefficients \( c_{nm} \) in terms of the other 9 coefficients \( c_{nm} \) and of the ratio \( r \equiv z_1(0)/z_2(0) \); with many other possibilities left to the imagination of the interested reader.
Final Remark. As already noted above, the mathematics behind the results reported above is rather elementary. Yet these findings do not seem to have been advertised so far, while their applicable potential is clearly vast; so—especially among applied mathematicians and practitioners of the various scientific disciplines where systems such as those discussed above play a role—a wider knowledge of them seems desirable; for instance via their inclusion in standard compilations of solvable equations such as those collected in the website EqWorld.

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