Time evolution of nonadditive entropies: The logistic map

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(Dated: April 17, 2023)

Due to the second principle of thermodynamics, the time dependence of entropy for all kinds of systems under all kinds of physical circumstances always thrives interest. The logistic map $x_{t+1} = 1 - ax^2_t \in [-1,1]$ ($a \in [0,2]$) is neither large, nor closed, since it is dissipative. It exhibits, nevertheless, a peculiar time evolution of its natural entropy, which is the additive Boltzmann-Gibbs-‐Shannon one, $S_{BG} = -\sum_{i=1}^{W} p_i \ln p_i$, for all values of $a$ for which the Lyapunov exponent is positive, and the nonadditive one $S_q = \frac{1}{q-1} \sum_{i=1}^{W} p_i^q$ with $q = 0.2445 \ldots$ at the edge of chaos, where the Lyapunov exponent vanishes, $W$ being the number of windows of the phase space partition. We numerically show that, for increasing time, the phase-space-averaged entropy overshoots above its stationary-state value in all cases. However, when $W \to \infty$, the overshooting gradually disappears for the most chaotic case ($a=2$), whereas, in remarkable contrast, it appears to monotonically diverge at the Feigenbaum point ($a=1.4011 \ldots$). Consequently, the stationary-state entropy value is achieved from above, instead of from below, as it could have been a priori expected. These results raise the question whether the usual requirements – large, closed, and for generic initial conditions – for the second principle validity might be necessary but not sufficient.

1 - Introduction

Molecular physics may be seen as a more detailed description of solidly established laws of chemistry. Analogously, atomic physics may be seen as a more detailed description of solidly established laws of molecular physics. Nuclear physics, physics of elementary particles, evolve along the same lines. Epistemologically, it is tacitly required that, at each deeper and deeper description, the knowledge previously established on solid grounds is satisfactorily recovered at some adequate scale. Another paradigmatic example of the same path is general relativity which, in each deeper and deeper description, the knowledge previously established on solid grounds is satisfactorily recovered. Epistemologically, it is tacitly required that, at each deeper and deeper description, the knowledge previously established on solid grounds is satisfactorily recovered at some adequate scale. Another paradigmatic example of the same path is general relativity which, in each deeper and deeper description, the knowledge previously established on solid grounds is satisfactorily recovered.

In the realm of statistical mechanics, where the concept of coarse graining, hence of changes of scales, plays a fundamental role, a similar path is being followed since its formulation in the XIXth century, and also along the last three-‐four decades. The pioneering works of Boltzmann and Gibbs$^{[1,2]}$ established, upon undeniably solid bases, a magnificent theory which is structurally associated with the Boltzmann-‐Gibbs (BG) entropic functional

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \quad \text{with} \quad \sum_{i=1}^{W} p_i = 1,$$  \hspace{1cm} (1)

and consistent expressions for continuous or quantum variables, $k$ being a conventional positive constant adopted once forever (in physics, $k$ is chosen to be the Boltzmann constant $k_B$; in information theory and computational sciences, it is frequently adopted $k = 1$).

In the simple case of equal probabilities, this entropic functional is given by $S_{BG} = k \ln W$. Eq. (1) is generically additive$^{[4]}$. Indeed, if $A$ and $B$ are two probabilistically independent systems (i.e., $p_{ij}^{A+B} = p_i^A p_j^B$), we straightforwardly verify that $S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B)$. This celebrated entropic functional is consistent with thermodynamics.
for all systems whose $N$ elements are either independent or weakly interacting in the sense that only basically
correlations are involved. For example, if we have equal probabilities and the
number of accessible microscopic configurations is given by $W(N) \propto \mu^N$ ($\mu > 1; N \to \infty$),
then $S_{BG}(N)$ is extensive
as required by thermodynamics. Indeed $S_{BG}(N) = k \ln W(N) \sim k(\ln \mu)N$. But if the correlations are nonlocal
in space/time, $S_{BG}$ may become thermodynamically inadequate. Such is the case
of equal probabilities with say $W(N) \propto N^\nu$ ($\nu > 0; N \to \infty$): it immediately follows $S_{BG}(N) \propto \ln N$,
which violates thermodynamical extensivity. To satisfactorily approach cases such as this one, it was proposed in 1988
[5–8] to build a more general statistical mechanics based on the nonadditive entropic functional
\begin{equation}
S_q \equiv k \frac{1 - \sum p_i^q}{q - 1} = k \frac{1}{q - 1} \sum p_i \ln_q \frac{1}{p_i} = -k \sum p_i \ln_q p_i = -k \sum p_i \ln_{2-q} p_i \quad (q \in \mathbb{R}; S_1 = S_{BG}),
\end{equation}
with $\ln_q z \equiv \frac{z^{1-q}-1}{1-q}$ ($\ln_1 z = \ln z$) and its inverse $e_q^z \equiv [1 + (1 - q)z]^{1/(1-q)}$; ($e_1^z = e^z$; $[z]_+ = z$ if $z > 0$ and vanishes otherwise); for $q < 0$, it is necessary to exclude from the sum the terms with vanishing $p_i$. We easily verify that equal probabilities yield $S_q = k \ln_q W$, and that generically we have
\begin{equation}
S_q(A + B) = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q) \frac{S_q(A)}{k} \frac{S_q(B)}{k},
\end{equation}
and hence
\begin{equation}
S_q(A + B) = S_q(A) + S_q(B) + \frac{1 - q}{k} S_q(A) S_q(B). \tag{4}
\end{equation}
Consequently, in the $(1 - q)/k \to 0$ limit, we recover the $S_{BG}$ additivity.
For the anomalous class of systems mentioned above, namely $W(N) \propto N^\nu$, we obtain, \forall $\nu$, the extensive entropy $S_{1-1/\nu}(N) = k \ln_{1-1/\nu} W(N) \propto N$, as required by the
Legendre structure of thermodynamics (see [3] and references therein).

At this point let us remind that a general entropic functional $S_G(\{p_i\})$ is defined as
\begin{equation}
S_G = k \sum p_i f(p_i) \equiv k \sum p_i g(p_i). \tag{5}
\end{equation}
A $G$-generalized logarithm can be defined as $g(p_i) \equiv \ln_G(1/p_i)$. Consequently $S_G = k(\sigma_i)$, where $\sigma_i \equiv q \ln_G(1/p_i)$ is the surprise [11] or unexpectedness [12]; $\sigma(p)$ is assumed to be monotonically increase,
when $p_i$ decreases from 1 to 0, from 0 to its maximum value (which may be infinity). Only trace-form entropies can be written as the mean value of a surprise.

Moreover, an entropic functional $S_G(\{p_i\}; \{\eta\})$ is said composable if it satisfies, for two probabilistically independent
systems $A$ and $B$, the property $S_G = \frac{S(A) + S(B)}{k}$; $\{\eta\}$, where $\{\eta\}$ is a set of fixed indices characterizing the functional (e.g., for $S_q$, it is $\{q\}$); we use the notation $\{\eta\}$ to indicate absence of any such index. $F(x, (y; \{\eta\})$ is a smooth function of $(x, y)$ which depends on a (typically small) set of universal indices $\{\eta\}$ defined in such a way that $F(x, y; \{0\}) = x + y$ (additivity), which corresponds to the BG entropy. Additionally, $F(x, (x, \{\eta\})$ is assumed to satisfy $F(x, 0; \{\eta\}) = x$ (null-composability), $F(x, y; \{\eta\}) = F(y, x; \{\eta\})$ (symmetry), $F(x, (y, z; \{\eta\}) = F(F(x, y, z)$; $\{\eta\}) (associativity) (see details and thermodynamical motivation in [3] [10]).

We specially focus here on $S_q$ because the Enciso-Tempesta theorem [13] proves that this entropic functional is the
unique one which is simultaneously trace-form, composable, and contains $S_{BG}$ as a particular case.

In the present paper, we numerically exhibit the above mentioned epistemological path on the time evolution of
$S_q(t)$ associated with the paradigmatic one-dimensional dissipative logistic map. Through appropriate scaling of successively finer partitions of the phase-space into $W$ equal windows, we verify that, for both strong and weak chaos,
properties such as the entropy production per unit time and related phenomenology are satisfied.

The logistic map is defined as follows:
\begin{equation}
x_{t+1} = 1 - ax_t^2 \quad (x_t \in [-1, 1]; \, a \in [0, 2]; \, t = 0, 1, 2, \ldots).
\end{equation}

Depending on the value of the external parameter $a$, the corresponding Lyapunov exponent $\lambda$ can be positive, negative
or zero. When $\lambda > 0$ we say that the system is strongly chaotic: the simplest, and strongest, such case emerges
for $a = 2$, which implies $\lambda = \ln 2 > 0$. When $\lambda = 0$ and the corresponding value of $a$, noted $a_c$, is located
at the accumulation point of successive bifurcations, we say that the system is weakly chaotic. The most studied
such points occur at the edge of chaos, more precisely, at the so-called Feigenbaum-Coullet-Tresser point, with
$a_c = 1.40115518909205 \ldots$. In all cases, if we start from initial conditions such that the entropy nearly vanishes at
$t = 0$, we observe that, for all values of $q$, $S_q$ tends to increase (not necessarily in a monotonic manner) as time increases.
But it tends to increase linearly (thus providing a finite entropy production per unit time) only for an unique value of
the index $q$. For $a = 2$, the entropy which linearly increases with time, thus yielding a finite entropy production per unit time (satisfying the Pesin identity for the entropy production per unit time $K_{BG} \equiv \lim_{t \to \infty} S_{BG}(t)/t = \lambda$, is $S_{BG}$. In contrast, at the edge of chaos, the entropy which linearly increases with time is $S_q$ with $q = q_c \equiv 0.24448770134128 \ldots$ In fact, depending on the initial conditions, there are infinitely many such linearities (see 14-17 and references therein). This is why we present here the corresponding mean values over a natural set of initial conditions, similarly to what has been calculated in [19].

2 - Strong and weak chaos

We partition the interval $x \in [-1, 1]$ into $W$ equal little windows, and uniformly choose $M$ initial conditions, typically $M = 10^W$ (this number of initial conditions is sufficiently high for attaining proper estimates of entropies. In fact, the relative error $\epsilon = \frac{1}{t_{\max}} \sum_{t=1}^{t_{\max}} |S_{a,M}^i(M=10^W:t) - S_{a,M}^i(M=1000W:t)|/S_{a,M}^i(M=1000W:t)\}$ estimated with higher number of initial conditions, e.g. $M = 100W$ and $t_{\max} = 10000$ is $\epsilon < 2 \times 10^{-3}$), within one such interval (noted $j$). We denote $\{p_i\} (i = 1, 2, \ldots, W)$ the occupancy probabilities of all $W$ windows. At $t = 0$, we have $p_j = 1$ for the selected window, and $p_{i\neq j} = 0$ for all the other $(W-1)$ windows. Consequently $S_q(t)$ satisfies $S_q(0) = 0, \forall q$. For $a = 2$, the only value of $q$ for which we have a linear growth while approaching saturation is $q = 1$ (this procedure was initially proposed in [18]). We then repeat the operation for each one of the $W$ windows, and finally average the data for $S_q$: see Figs. 1 and 2.

FIG. 1: (Color online) Time dependence of the average $\langle S_q \rangle$ for $a = 2$. Left: Linear-linear representation. The slope of the linear part equals $\lambda = \ln 2$, consistently with the Pesin identity. For all values of $q \neq 1$, several values of $W$ yield curves that are superimposed, and consequently can not be clearly identified. Right: Log-linear representation. We verify that, for $q < 1$, a purely exponential behavior gradually emerges before saturation. Inset: The extrapolated slope for $q = 0$ yields 0.69.

FIG. 2: (Color online) Left: Data collapse of the results indicated in Fig. 1. The maxima observed here are consistent with Fig. 1 of [20]. Right: Extrapolation of the ratios.
We apply the same numerical procedure described above for $a = a_c$: see Figs. 3, 4 and 5.

Before concluding, let us emphasize that the robustness of the peaks in the time evolution of the entropy $S_q$ at $a = a_c$ has been numerically verified under four different circumstances, namely with regard to (i) the number of initial conditions within each one of the $W$ windows (Fig. 6); (ii) the precision used in the value of $a_c$ (bottom-right plot in Fig. 3); (iii) variations of the occupancy of phase space in the neighborhood of the multifractal attractor at $a = a_c$ (Fig. 7); (iv) variations of the map (Fig. 8).

4 - Final remarks

At this point, a few comments are certainly timely concerning the most distinguished non-trace-form entropic functional, namely the Renyi one \[21\]

$$S^R_q \equiv k \frac{\ln \sum_{i=1}^W p_i^q}{1 - q} = k \frac{\ln[1 + (1 - q)S_q/k]}{1 - q} \quad (q \in \mathbb{R}; S^R_1 = S_1 = S_{BG}),$$

hence

$$S^R_q/k = \ln e_{S_q/k}$$

and

$$S_q/k = \ln e_{S^R_q/k}$$

This additive (hence composable) entropic functional is, $\forall q$, a monotonic function of $S_q$, hence, under the same constraints, it is optimized by the same distribution which optimizes $S_q$. However, in contrast with $S_q$ which is
concave for all $q > 0$, $S_q^R$ is concave only for $0 < q \leq 1$. In addition to that, $S_q^R$ is Lesche-stable for all $q > 0$, whereas $S_q^R$ has this important experimental robustness only for its particular instance $q = 1$ \cite{23,24}. Moreover, if we consider the present nontrivial case $a = a_c$, we have that, in the $W \to \infty$ limit, $S_{q_c}(t) = \frac{k}{\rho} \ln \left[ 1 + (1 - q_c) S_{q_c}(t)/k \right] \propto \ln t$ for $t \gg 1$. Consequently, there is no constant finite Renyi-entropy production per unit time and, consistently, no Pesin-like identities can exist. Analogously, if we consider the stationary-state of a $N$-body system belonging to the previously mentioned anomalous class $W(N) \propto N^p$, we have that $S_{q_c}^R(N) \propto \ln N$ for any $q < 1$. In other words, $S_{q_c}^R(N)$ appears to be nonextensive and the Legendre structure of thermodynamics is therefore violated.

It is worthy to mention that composability is not necessary for a finite entropy production per unit time to exist (thus possibly qualifying for a Pesin-like identity to be satisfied). Such is the case, for instance, of the Kaniadakis entropy \cite{25} which is nonadditive, non-composable, trace-form, and yields nevertheless a finite entropy production per unit time.

The relevant influence of the precision used in the calculations has been exhibited as well: see Fig. 3. More specifically, the use of simple, double or quadruple precision sensibly enhances the overshooting of the time evolution of the entropy $S_{q_c}(t)$ at the Feigenbaum point. To reinforce these results, it would of course be necessary to use values of $a_c$ and of $q_c$ with consistently higher precision when $W$ increases. Still, the present indications are already clear enough.

Let us conclude by focusing on a relevant result of the present study. Our numerical simulations strongly indicate

\begin{align*}
\langle S_{q_c}/\ln W \rangle^\rho & \text{ Top left: Linear-linear representation aiming the collapse, for increasing } W, \text{ of the maxima; the slope at first inflection point, located at } t/\ln q_c \approx 0.02, \text{ is } 0.47 \pm 0.01; \langle S_{q_c}/\ln W \rangle^\rho \text{ attains its maximal value } 1.1 \pm 0.01 \text{ at } t/\ln q_c W^\rho = 5.3 \pm 0.3, \text{ with } \rho = 0.364 \pm 0.003. \\
\langle S_{q_c}/\ln_{q_c} W \rangle^\rho & \text{ Top right: The same as in Top-left but in linear-log representation. Bottom: Linear-log representation aiming the collapse, for increasing } W, \text{ of the stationary-state heights; this collapse occurs at height } = 1.35 \pm 0.04 \text{ with } \rho = 0.208 \pm 0.002.
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{(Color online) Data collapse of the results indicated in Fig. 3. Notice that, for $q < 1$, $\rho > 0$ and $W \gg 1$, $(\ln W)^\rho \sim [W^{1-q}/(1-q)]^\rho \propto W^{(1-q)/\rho}$, which reveals the (multi) fractal origin of $\rho$ (in the sense that $\rho$ is not an integer number). For $q \neq q_c$, there is no value of $\rho$ for which all the $a = a_c$ data can be collapsed through a single scaling with $(\ln W)^\rho$. Top left: Linear-linear representation aiming the collapse, for increasing $W$, of the maxima; the slope at first inflection point, located at $0.97 \pm 0.02$, is $0.47 \pm 0.01; \langle S_{q_c}/\ln q_c W \rangle^\rho$ attains its maximal value $1.1 \pm 0.01$ at $t/\ln q_c W^\rho = 5.3 \pm 0.3$, with $\rho = 0.364 \pm 0.003$. Top right: The same as in Top-left but in linear-log representation. Bottom: Linear-log representation aiming the collapse, for increasing $W$, of the stationary-state heights; this collapse occurs at height $= 1.35 \pm 0.04$ with $\rho = 0.208 \pm 0.002.$}
\end{figure}
FIG. 5: (Color online) $W \to \infty$ extrapolations of the maxima. Left: Abscissa scaled with $\ln W$. Right: Abscissa scaled with $\ln q_c W$. The data suggest that $\lim_{W \to \infty} \left[ \left( \langle S_q \rangle \right)_{\text{max}} / \langle S_q \rangle_{\text{stat}} \right]$ equals unity for $(a, q) = (2, 1)$ and diverges for $(a, q) = (a_c, q_c)$.

FIG. 6: (Color online) Influence of the number of initial conditions within each of the $W$ windows. Instance with $W = 2^{12}$ windows with two cases: $M = 10W$ initial conditions (black) and $M = 100W$ initial conditions (red). The similarity of the curves indicate that $M = 10W$ is a sufficiently large number of initial conditions for the purposes of the calculations.

FIG. 7: (Color online) Influence of the phase-space occupation in the neighborhood of the multifractal attractor at $a = a_c$ (see [26]). Uniform means that all the $W$ windows start with the same number of initial conditions; alternate means that the occupancy of each of the $W$ windows starts alternating the double of initial conditions and emptiness, the total number $WM$ of initial conditions in the interval $[-1, 1]$ remaining the same.
FIG. 8: (Color online) Peaks corresponding to the $z$-logistic map $x_{t+1} = 1 - a|x_t|^z$ for typical values of $z$. For $z = 1.50, 1.75, 2.00, 2.50, 3.00$ we have $(a_c, q_c) = (1.295509973160, -0.15), (1.355060756622, 0.11), (1.40115518909205, 0.2444877), (1.47054991523, 0.39), (1.52187878890, 0.47)$.

FIG. 9: (Color online) Influence of the precision of the calculations. We verify that, for small values of $W$, double precision (i.e., 16 digits) is more than enough for the present purposes, but, when $W$ sensibly increases, increasingly higher precision becomes necessary. Interestingly enough, low precision underestimates the maximal value of $\langle S_{q_c} \rangle$, whereas it overestimates its stationary-state value.

that, for both strong and weak chaos, the time behavior of $S_q$ ($q = 1$ for $a = 2$ and $q = q_c$ for $a = a_c$, respectively) at the $W \to \infty$ limit, consists in a diverging linear increase with finite slope. However, an important distinction arises on how $S_q(t)$ approaches its stationary-state value $S_q(\infty)$: it approaches from below for $a = 2$, which corresponds to
the naïve expectation, whereas it does so from above for $a = a_c$, which might be considered as unexpected. This – a priori surprising – behavior is due to the fact that, for relatively early times during the present evolution at fixed $W$, the entropy system tends to approach its maximal value $\ln W$. However, for later times, the system approaches its asymptotic stationary state, which is not uniform (it is instead $U$-shaped for $a = 2$ and multifractal for $a = a_c$).

The 2nd principle of thermodynamics is normally qualified for systems which are closed, very large, and for a generic initial situation. An intriguing question might arise: these conditions surely are necessary, but are they sufficient? Is there no need for also requiring a strongly chaotic internal dynamics (e.g., short-range interactions), which normally implies mixing and ergodicity for all or part of the system? At the light of the present results, this fundamental question appears to be an open one, surely deserving further study.

We acknowledge fruitful conversations with E.M.F. Curado and partial financial support by CNPq and Faperj (Brazilian agencies). We also acknowledge useful remarks from two anonymous referees which led to various numerical verifications of the robustness of the entropic peaks at the Feigenbaum point, thus enriching the manuscript.
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