Quantum metrology for a general Hamiltonian parameter

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Quantum metrology enhances the sensitivity of parameter estimation using the distinctive resources of quantum mechanics such as entanglement. It has been shown that the precision of estimating an overall multiplicative factor of a Hamiltonian can be increased to exceed the classical limit, yet little is known about estimating a general Hamiltonian parameter. In this paper, we study this problem in detail. We find that the scaling of the estimation precision with the number of systems can always be optimized to the Heisenberg limit, while the time scaling can be quite different from that of estimating an overall multiplicative factor. We derive the generator of local parameter translation on the unitary evolution operator of the Hamiltonian, and use it to evaluate the estimation precision of the parameter and establish a general upper bound on the quantum Fisher information. The results indicate that the quantum Fisher information generally can be divided into two parts: one is quadratic in time, while the other oscillates with time. When the eigenvalues of the Hamiltonian do not depend on the parameter, the quadratic term vanishes, and the quantum Fisher information will be bounded in this case. To illustrate the results, we give an example of estimating a parameter of a magnetic field by measuring a spin-½ particle, and compare the results for estimating the amplitude and the direction of the magnetic field.

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I. INTRODUCTION

Quantum metrology [1, 2] is a scheme that uses entanglement to increase the precision of parameter estimation by quantum measurements beyond the limit of its classical counterpart. In classical parameter estimation, the estimation precision scales as $\nu^{-\frac{1}{2}}$, where $\nu$ is the number of rounds of measurement. The scaling can be rewritten as $(N\nu)^{-\frac{1}{2}}$, where $N$ is the number of qubits used in each round, for parameter estimation by quantum measurements if the $N$ qubits are not entangled. This scaling is often termed as the standard quantum limit (SQL) [3], which characterizes the precision limit of quantum measurements in the presence of the shot noise. A more fundamental imprecision of quantum measurement originates from the Heisenberg uncertainty principle, which is one of the most fundamental properties of quantum mechanics, due to the probabilistic nature of quantum measurements. Research in quantum metrology has shown that with the assistance of $n$-qubit entanglement, the optimal scaling of the estimation precision can be raised to $N^{-1}\nu^{-\frac{1}{2}}$, the Heisenberg limit, implying an improvement of $N^{\frac{1}{2}}$ over the SQL.

Quantum metrology is rooted in the theory of quantum estimation, which was pioneered by Helstrom [4] and Holevo [5], who proposed the parameter-based uncertainty relation. Braunstein, Caves and Milburn [6, 7] developed that theory from the view of the Cramér-Rao bound [8], which characterizes how well a parameter can be estimated from a probability distribution, and obtained the optimal Fisher information over different quantum measurement schemes for a given parameter-dependent quantum state. This is often called quantum Fisher information.

Given the importance of precision measurement in different fields of physics and engineering, the quantum Fisher information has attracted great interest from researchers. Giovannetti, Lloyd and Maccone [1] found that the scaling of the quantum Fisher information has an $N^{\frac{1}{2}}$ improvement than its classical counterpart if an $N$-qubit maximally entangled state is used. This stimulated the emergence of quantum metrology, which has been applied to different quantum systems to raise the precision of measurements.

The optimality of quantum metrology in terms of the scaling of the measurement precision was proved in [9] for different initial states and measurement schemes, and also by [10] from the view of the query complexity of a quantum network. Moreover, when there is interaction among the $N$ entangled qubits or the Hamiltonian is nonlinear, the measurement precision can be further increased to beyond the Heisenberg limit [11, 10].

Many applications of quantum metrology have been found, including to quantum frequency standards [17, 18], optical phase estimation [19, 20], atomic clocks [26, 31], atomic interferometers [31], quantum imaging [32, 33], and quantum-enhanced positioning and clock synchronization [34]. The quantum Fisher information has also been studied in open systems [35–41], along with growing research on protocols assisted by error correction [42–44]. Moreover, quantum metrology with nonlinear Hamiltonians has received considerable attention [12, 13, 15–52]. Reviews of the field of quantum metrology should be refereed to [1, 2].

Studies of quantum metrology have mainly focused on the precision of measuring an overall multiplicative factor of a Hamiltonian, e.g., the parameter $g$ in a Hamiltonian...
the optimal scaling of the measurement precision with the number $N$ of systems, is still $N^{-1}$, but the time scaling will be different. In detail, it will be shown that the quantum Fisher information can generally be divided into two parts: one is linear in the time $t$, corresponding the variation of the eigenvalues, and the other is oscillatory, corresponding to the variation of the eigenvectors of the Hamiltonian. The oscillating part is bounded no matter how long the time $t$ is. We will obtain an upper bound on the Fisher information for the general case.

The study of this problem will extend the current knowledge of quantum metrology to a more general case, and more kinds of precision measurements could benefit from this extension, especially those that go beyond phase or frequency measurement. For instance, as we will show as an example in the paper, it can enhance the precision of measuring the direction of a magnetic field by a spin-$\frac{1}{2}$ system, which is useful for calibrating the field. Therefore, the results of this paper will be useful to both theory and experiments in quantum metrology.

### II. BACKGROUND

Let us first review some concepts of the estimation theory, and their quantum counterparts. The task of parameter estimation is to determine a parameter from a set of data which depend on the parameter. A general procedure for estimating a parameter is: first acquire a set of data $x_1, \cdots, x_\nu$ which obey a probability distribution dependent on the parameter $f_g(x)$, where $g$ is the parameter to estimate; then estimate $g$ from $x_1, \cdots, x_\nu$ by a certain estimator, and obtain the estimated value $\hat{g}$.

While there are many different estimation strategies, such as the method of moments and maximum likelihood estimation, the performances of those strategies differ. One of the most important benchmarks of a strategy is the estimation precision, which is usually characterized by the estimation error $\delta g$:

$$\delta g \equiv \frac{\hat{g} - g}{|d\hat{g}/dg|} g,$$  \hspace{1cm} (1)

where the factor $|d\hat{g}/dg|$ is to eliminate the local difference in the units between the estimator and the real parameter for different $g$. If the estimation procedure is repeated many times, the estimated value $\hat{g}$ may have fluctuations. So an appropriate measure to quantify the performance of an estimator is the root mean square error (RMSE) of the estimation results $\sqrt{\frac{1}{\nu} \sum_{i=1}^{\nu} (\delta g_i)^2}$. A cornerstone of the classical theory of parameter estimation is the Cramér-Rao bound, which bounds the precision limit of an estimator by the following relation:

$$\langle (\delta g)^2 \rangle \geq \frac{1}{\nu F(g)} + \langle (\delta g)^2 \rangle,$$  \hspace{1cm} (2)

where $F_g$ is the Fisher information, defined as

$$F_g = \int (\partial_x \ln f_g(x))^2 f_g(x)dx.$$  \hspace{1cm} (3)

The second term on the right side of (2), $\langle (\delta g)^2 \rangle$, characterizes the bias of the estimator. If the estimator is unbiased, i.e., $\langle \hat{g} \rangle = g$, then $\langle (\delta g)^2 \rangle = 0$.

The achievability (or the tightness) of the Cramér-Rao bound is addressed by the Fisher theorem. Fisher proved that for asymptotically large $\nu$, the Cramér-Rao bound can always be achieved by maximum likelihood estimation (MLE), and the estimation result is unbiased. Because of this property, MLE has been widely adopted in parameter estimation protocols.

In quantum metrology, one measures a parameter dependent state, say $\rho_g$, to estimate $g$. The process of a quantum metrology protocol splits into two stages. First, measure the state in some basis [more generally, perform a positive operator-valued measure (POVM) on it], and record the measurement result. When such a measurement is repeated $\nu$ times for the same $\rho_g$, we will acquire $\nu$ measurement results. These results depend on $g$, so they can be used as sample data to estimate $g$.

The second stage is estimating the parameter $g$ based on the measurement results by some appropriate estimation strategy. The precision of the estimation is bounded by (2) as usual. The complexity of quantum metrology comes from the many different choices of the measurements (or POVMs). Different choices lead to different precisions of the estimation results. The aim of quantum metrology is to increase the estimation precision by optimizing the measurement basis (or POVM).

Braunstein and Caves obtained the optimal Fisher information for all POVMs for a given $\rho_g$, which is the so-called quantum Fisher information, through the logarithmic derivative $L_g$:

$$F_g^{(Q)} = \text{Tr}(L_g \rho_g L_g).$$  \hspace{1cm} (4)

The logarithmic derivative $L_g$ has several different but equivalent definitions. The most common is the symmetric logarithmic derivative (SLD), defined as $\partial_g \rho_g = (L_g \rho_g + \rho_g L_g)/2$. $L_g$ in this definition is Hermitian, and the quantum Fisher information $F_g^{(Q)}$ can be simplified to $\text{Tr}(\rho_g^2 L_g^2)$. In the eigenbasis of $\rho_g$, an explicit form
of $L_g$ can be found:

$$L_g = 2 \sum_{i,j} \frac{\langle \eta_i | \partial_i \rho_g | \eta_j \rangle}{\eta_i + \eta_j} |\eta_i \rangle \langle \eta_j|,$$

(5)

where the $\eta_i$’s are the eigenvalues of $\rho_g$ and the $|\eta_i \rangle$’s are the corresponding eigenstates.

In the current literature of quantum metrology, most research interest has been focused on estimating an overall multiplicative factor of a Hamiltonian; for example, estimating $g$ in a Hamiltonian $gH$. Usually an initial pure state $|\Psi\rangle$ is used to undergo evolution by the Hamiltonian so that

$$\rho_g = \exp(-igtH)|\Psi\rangle \langle \Psi| \exp(igtH).$$

(6)

In such a case, the quantum Fisher information $F_g^{(Q)}$ can be simplified to

$$F_g^{(Q)} = 4t^2 \langle \Psi | \Delta H^2 | \Psi \rangle.$$

(7)

It can be proved \[9\] that $\langle \Psi | \Delta H^2 | \Psi \rangle_{\max} = \frac{1}{4}(E_{\max} - E_{\min})^2$ where $E_{\max}$ and $E_{\min}$ are the maximal and minimal eigenvalues of $H$ respectively. Since $E_{\max}$ and $E_{\min}$ grow linearly with the number of systems $N$, $F_g^{(Q)} \propto N^2$, which is the origin of the $\sqrt{N}$ improvement of the precision scaling in quantum metrology compared with the SQL.

III. $N$ SCALING OF QUANTUM FISHER INFORMATION

Now we turn to the major problem of this paper. We are interested in quantum metrology for a general parameter in a Hamiltonian. Both the eigenvalues and the eigenstates of the Hamiltonian may depend on the parameter. We mainly consider the scaling of the quantum Fisher information with the number of the systems $N$ in this section, and leave the more general results for the next section.

We first introduce the general framework about how to derive the quantum Fisher information of estimating a Hamiltonian parameter. Suppose the Hamiltonian of a single system is $H_g$, the initial state of the system is $\rho_0$, and the parameter we want to estimate is $g$. After the evolution under the Hamiltonian, the state of the system becomes $\rho_g = U_g \rho_0 U_g^\dagger$, where $U_g = \exp(-itH_g)$. The sensitivity of $\rho_g$ to the parameter $g$ can be characterized by the generator of the local parameter translation from $\rho_0$ to $\rho_{g+dg}$, where $dg$ is an infinitesimal change of $g$.

In detail, when $g$ is changed to $g + dg$, $\rho_g$ is updated to $\rho_{g+dg} = U_{g+dg} \rho_0 U_{g+dg}^\dagger$. Since $U_{g+dg} \approx U_g + \partial_g U_g dg$, the translation from $\rho_g$ to $\rho_{g+dg}$ can be written as

$$\rho_{g+dg} \approx (U_g + dg \partial_g U_g) \rho_0 (U_g^\dagger + dg \partial_g U_g^\dagger) = (I + dg (\partial_g U_g) U_g^\dagger) U_g \rho_0 U_g^\dagger (I + dg U_g \partial_g U_g^\dagger) \approx \exp(-ih_g dg) \rho_g \exp(ih_g dg),$$

(8)

where

$$h_g = i(\partial_g U_g) U_g^\dagger.$$

(9)

Here, $h_g$ is the generator of parameter translation with respect to $g$, and the subscript $g$ is to indicate that this generator is local in $g$. It can be shown \[3, 7\] that when the initial state of the system is a pure state $|\Psi\rangle$, the quantum Fisher information of the evolved state $U_g |\Psi\rangle$ is

$$F_g^{(Q)} = 4 \langle \Psi | \Delta h_g^2 | \Psi \rangle,$$

(10)

And the variance of $h_g$ is maximized when $|\Psi\rangle = \frac{1}{\sqrt{\lambda_{\max}}} (\lambda_{\max} - \lambda_{\min}) (e^{i\varphi} |\lambda_{\min}\rangle) (e^{i\varphi} \text{is an arbitrary phase})$, so the maximal quantum information is

$$F_g^{(Q)} = (\lambda_{\max}(h_g) - \lambda_{\min}(h_g))^2,$$

(11)

where $\lambda_{\max}(h_g)$ and $\lambda_{\min}(h_g)$ are the maximal and minimal eigenvalues of $h$ respectively.

When there are $N$ systems, the total Hamiltonian is $H_{g,\text{total}} = H_{g,1} + \cdots + H_{g,N}$, where $H_{g,i}$ is the Hamiltonian for the $i$-th system alone, i.e., $H_{g,i} = I^{\otimes i-1} \otimes H_g \otimes I^{\otimes N-i}$. Since $[H_i, H_j] = 0, \forall i, j = 1, \cdots, N$, we have

$$h_{g,\text{total}} = i \frac{\partial h_{g,\text{total}}}{\partial g} = h_g \cdot \sum_{i=1}^{N} \otimes h_{g,i}.$$

(12)

As $H_{g,1}, \cdots, H_{g,N}$ are the same Hamiltonian on different systems, it is obvious that

$$\lambda_{\max}(h_{g,\text{total}}) = N \lambda_{\max}(h_g),$$

$$\lambda_{\min}(h_{g,\text{total}}) = N \lambda_{\min}(h_g).$$

(13)

So according to \[11\],

$$\max F_g^{(Q)}_{\text{total}} = N^2 F_g^{(Q)}_{\text{max}},$$

(14)

where $F_g^{(Q)}_{\text{total}}$ is the total quantum Fisher information of the $N$ systems.

Eq. (14) is interesting, since it implies that the optimal scaling of the total Fisher information using $N$ systems can always reach $N^2$, which beats the classical scaling limit and is universal for estimating an arbitrary parameter in the Hamiltonian. Of course, if there are interactions among the $N$ systems, the optimal scaling of the Fisher information may be even higher, which has been found for estimating an overall multiplicative factor of a Hamiltonian \[11, 13\]. In that case, the total Hamiltonian becomes $H_{g,\text{total}} = \sum_{i=1}^{N} h_{g,i}$ if there are $k$-body interactions among the $N$ systems. Obviously, the total Hamiltonian can grow nonlinearly with $N$ in general, so the quantum Fisher information may increase faster than $N^2$. Such a case is beyond the scope of this paper, and we do not consider it in detail here.
IV. QUANTUM FISHER INFORMATION FOR GENERAL HAMILTONIAN PARAMETERS

In this section, we study quantum metrology for general Hamiltonian parameters in detail. We consider only single systems here, and focus on the time scaling of the quantum Fisher information, since the scaling with the number of systems was treated in the previous section. It can be seen from Eq. (11) that the key to the quantum Fisher information $F^{(Q)}_g$, the generator $h_g$ of the local parameter translation from $U_g$ to $U_{g+\lambda}$, so our main effort is to derive $h_g$ in the following.

A. Result for $t \ll 1$

First, we study the derivative of $\exp(-itH(g))$ with respect to $g$ which is needed in (9). This derivative is nontrivial, since $H_g$ does not commute with $\partial_y H_g$ in general. To obtain this derivative, we start from an integral formula for the derivative of an operator exponential [54]:

$$\frac{\partial \exp(-i\beta H(\lambda))}{\partial \lambda} = -i \int_0^\beta \exp(-i\mu H(\lambda)) \frac{\partial H(\lambda)}{\partial \lambda} \exp(i\mu - i\beta) H(\lambda) \mu \text{d}\mu,$$

where $\mu, \beta \in \mathbb{R}$. By this formula and according to the definition of $h_g$ (9), we get

$$h_g = \int_0^t \exp(-i\mu H_g) \partial_y H_g \exp(i\mu H_g) \mu \text{d}\mu. \quad (16)$$

When $t \ll 1$, the first-order approximation of (16) is

$$h_g \approx t \partial_y H_g.$$

As shown in Appendix II, we can get an upper bound for the quantum Fisher information in this case:

$$F^{(Q)}_{g,\max} \leq \frac{t^2}{2} \text{Tr}(\partial_y H_g)^2. \quad (17)$$

B. Result for general $t$

For larger $t$, direct calculation of the integral in (16) is not easy. Of course, one can use the Baker–Campbell–Hausdorff formula to expand the integrand, but that will yield an infinite series that is difficult to treat. So we resort to a different approach to work out $h_g$, which was first proposed in [54].

Denote the integrand of (16) as $Y(\mu)$:

$$Y(\mu) = \exp(-i\mu H_g) \frac{\partial H_g}{\partial y} \exp(i\mu H_g). \quad (18)$$

The derivative of $Y(\mu)$ with respect to $\mu$ satisfies

$$\frac{\partial Y}{\partial \mu} = -i[H_g, Y], \quad (19)$$

and the initial condition is $Y(0) = \partial_y H_g$.

To solve the differential equation (19), consider the following eigenvalue equation:

$$[H_g, \Gamma] = \lambda \Gamma. \quad (20)$$

In this equation, $H_g$ can be treated as a superoperator acting on $\Gamma$. To distinguish $H_g$ as a superoperator from that as an operator, we denote the superoperator of $H_g$ as $H_g$, and (20) can be rewritten as

$$H_g \Gamma = \lambda \Gamma. \quad (21)$$

It is easy to verify that $H_g$ is an Hermitian superoperator. (See Appendix I.) Therefore, $H_g$ has $d^2$ real eigenvalues, some of which may be degenerate. Suppose the eigenvalues of $H_g$ are $\lambda_1, \cdots, \lambda_{d^2}$, and that $\lambda_k = 0$ for $k = 1, \cdots, r$, $\lambda_k \neq 0$ for $k = r + 1, \cdots, d^2$, and denote the corresponding orthonormal eigenvectors as $\Gamma_1, \cdots, \Gamma_{d^2}$, satisfying $\text{Tr}(\Gamma_k^\dagger \Gamma_j) = \delta_{ij}$. Then $\partial_y H_g$ can be decomposed as

$$\partial_y H_g = \sum_{k=1}^{d^2} c_k \Gamma_k, \quad (22)$$

where $c_k = \text{Tr}(\Gamma_k^\dagger \partial_y H_g)$. Since $Y(\mu)$ can also be decomposed in terms of $\Gamma_1, \cdots, \Gamma_{d^2}$, and $Y(0) = \partial_y H_g$, the solution of Eq. (19) is

$$Y(\mu) = \sum_{k=1}^{d^2} \text{Tr}(\Gamma_k^\dagger \partial_y H_g) e^{-i\lambda_k \mu} \Gamma_k. \quad (23)$$

Now, we can insert the above solution for $Y(\mu)$ into (16), and since the first $r$ eigenvalues of $H_g$ are zero,

$$h_g = t \sum_{k=1}^r \text{Tr}(\Gamma_k^\dagger \partial_y H_g) \Gamma_k$$

$$- i \sum_{k=r+1}^{d^2} \frac{1 - e^{-i\lambda_k t}}{\lambda_k} \text{Tr}(\Gamma_k^\dagger \partial_y H_g) \Gamma_k. \quad (24)$$

Eq. (24) is the general solution for $h_g$. When one obtains the eigenvalues and eigenvectors of $H_g$ from (20) and plugs them into (24), $h_g$ can then be derived.

If we know the eigenvalues and eigenstates of $H_g$ (as an ordinary operator), the solution for $h_g$ (24) can be greatly simplified. Suppose $H_g$ has $n_g$ different eigenvalues $E_1, \cdots, E_{n_g}$, the degeneracy of $E_k$ is $d_k$, and the eigenstates corresponding to $E_k$ are $|E_k^{(1)}\rangle, \cdots, |E_k^{(d_k)}\rangle$. The eigenvectors and eigenvalues of $H_g$ are

$$\Gamma_k^{(ij)} = |E_k^{(i)}\rangle \langle E_k^{(j)}|, \quad h_k^{(ij)} = E_k - E_i. \quad (25)$$

It is obvious that the degeneracy of the zero eigenvalue is $d_1^2 + \cdots + d_{n_g}^2$, and the corresponding eigenvectors are
\[ \Gamma^{(ij)}_{kk}, i, j = 1, \ldots, d_k, k = 1, \ldots, n_g. \] The coefficients of these eigenvectors in \( h_g \) are
\[
\text{Tr}(\Gamma^{(ij)}_{kl} \partial \mathcal{H}_g) = \left\langle E_k^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle = \partial \mathcal{H}_g E_k \delta_{ij}. \tag{26}
\]

The eigenvectors with nonzero eigenvalues of \( \mathcal{H}_g \) are \( \Gamma^{(ij)}_{kl}, k \neq l \), and their coefficients in \( h_g \) are
\[
\text{Tr}(\Gamma^{(ij)}_{kl} \partial \mathcal{H}_g) = \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle = E_k \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle + E_l \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle \tag{27}
\]
where we have used \( \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle + \left\langle \partial \mathcal{H}_g E_l^{(ij)} \right| E_k^{(ij)} \right\rangle = \partial \mathcal{H}_g E_k^{(ij)} \).

Plugging (25), (26) and (27) into (24), we finally have
\[
h_g = t \sum_{k=1}^{n_g} \frac{\partial E_k}{\partial g} P_k + 2 \sum_{k \neq l \neq i \neq j = 1}^d d_k d_l \sum_{l=1}^d e^{-i(E_k - E_l) t} \sin \frac{(E_k - E_l)}{2} \left( \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle \left\langle E_k^{(ij)} \right| \left\langle E_l^{(ij)} \right\rangle, \tag{28}
\]
where \( P_k \) is the projection onto the eigensubspace corresponding to \( E_k \): \( P_k = \sum_{l=1}^d \left| E_k^{(ij)} \right\rangle \langle E_k^{(ij)} \right\rangle \). We have used \( 1 - e^{-i(E_k - E_l) t} = 2i \text{exp} \frac{(E_k - E_l) t}{2} \sin \frac{(E_k - E_l) t}{2} \).

The form of \( h_g \) in (28) implies that the quantum Fisher information \( F_g^{(Q)} \) can be divided into two parts: one is due to the dependence of the eigenvalues \( E_k \) on \( g \), and this part is linear in the time \( t \); the other is due to the dependence of the eigenstates \( \left| E_k^{(ij)} \right\rangle \) on \( g \), and that part oscillates with time.

When the dimension of the system is low, one may find the eigenvalues and the eigenstates of the Hamiltonian explicitly, so Eq. (28) is a more direct and compact result for \( h_g \). However, if the dimension of the system is very high, e.g., a condensed matter system, then the eigenvalues and the eigenstates will be extremely difficult to obtain, and the general result (24) will be more helpful. In this case, the eigenvalues and eigenstates of \( \mathcal{H}_g \) are still unavailable, but one can get some knowledge of the quantum Fisher information \( F_g^{(Q)} \) from the symmetry of the Hamiltonian.

For example, if \( H \) is invariant under a unitary operation \( U = \text{exp}(-i \Omega) \), then \( [\mathcal{H}_g, \Omega] = 0 \), which implies that \( \Omega \) is an eigenvector of \( \mathcal{H}_g \) with eigenvalue zero. Thus one can calculate the coefficient \( \text{Tr}(\Omega \partial \mathcal{H}_g) \) and check whether \( \partial \mathcal{H}_g \) belongs to the support of \( \partial \mathcal{H}_g \). If it does, then the quantum Fisher information \( F_g^{(Q)} \) will scale as \( t^2 \) when \( t \gg 1 \). So we can see that even lacking details about the eigenvalues and eigenvectors of \( \mathcal{H}_g \), (24) can give some information about the scaling of \( F_g^{(Q)} \) through the symmetry of \( \mathcal{H}_g \).

C. Upper bound on the quantum Fisher information \( F_g^{(Q)} \)

From (24) or (28), we can obtain an upper bound on the quantum Fisher information \( F_g^{(Q)} \).

First, we note that
\[
\left\langle \Delta h_g^{2} \right\rangle_{\text{max}} \leq \frac{1}{2} \text{Tr}(h_g^{2}), \tag{29}
\]
(see Appendix II for a proof), so from (10) and (24), we can derive that
\[
F_g^{(Q)} \leq 2t^2 \sum_{k=1}^{r} \left| \text{Tr}(\Gamma^{(ij)}_{kl} \partial \mathcal{H}_g) \right|^2 + 8 \sum_{k=r+1}^{d} \frac{\left| \text{Tr}(\Gamma^{(ij)}_{kl} \partial \mathcal{H}_g) \right|^2}{\lambda_k^2} \sin \frac{\lambda_k t}{2}. \tag{30}
\]

And when we know the eigenvalues and eigenstates of the Hamiltonian \( \mathcal{H}_g \), the upper bound can be simplified to
\[
F_g^{(Q)} \leq 2t^2 \sum_{k=1}^{n_g} d_k \left| \partial \mathcal{H}_g E_k \right|^2 + 8 \sum_{k \neq l \neq i \neq j = 1}^d \left| \text{sin} \right(\frac{E_k - E_l}{2}) \left| \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle \left\langle E_k^{(ij)} \right| \left\langle E_l^{(ij)} \right\rangle. \tag{31}
\]

In particular, if the eigenvalues of \( \mathcal{H}_g \) are independent of \( g \), the upper bound of \( F_g^{(Q)} \) will not grow as \( t^2 \) when \( t \) is large, and the bound becomes
\[
F_g^{(Q)} \leq 8 \sum_{k \neq l \neq i \neq j = 1}^d \left| \text{sin} \right(\frac{E_k - E_l}{2}) \left| \left\langle E_l^{(ij)} \right| \partial \mathcal{H}_g \right| E_k^{(ij)} \right\rangle \left\langle E_k^{(ij)} \right| \left\langle E_l^{(ij)} \right\rangle. \tag{32}
\]

In this case the quantum Fisher information \( F_g^{(Q)} \) is always finite, no matter how long the time \( t \) is, in sharp contrast to the time scaling of the Fisher information for estimating an overall multiplicative factor of a Hamiltonian.

V. Example: A Spin-\( \frac{1}{2} \) Particle in a Magnetic Field

In this section, we consider an example to illustrate the results in the previous sections. We study the quantum Fisher information in estimating a parameter of a magnetic field by measuring a spin-\( \frac{1}{2} \) particle in the field.

Suppose the magnetic field is \( B \hat{n}_g \), where \( B \) is the amplitude of the magnetic field, and \( \hat{n}_g = (\cos \theta, 0, \sin \theta) \), gives its direction. The parameter \( \theta \) denotes the angle between the direction of the magnetic field and the \( z \)
axis. Now we place a spin-$\frac{1}{2}$ particle, e.g., an electron, in this magnetic field, and our task is to estimate the angle $\theta$ by measuring this particle.

The interaction Hamiltonian between the particle and the magnetic field is

$$H_\theta = B(\cos \theta \sigma_x + \sin \theta \sigma_z),$$

where $\sigma_x$ and $\sigma_z$ are Pauli operators. We have assumed $e = m = c = 1$ in the above Hamiltonian for simplicity.

The eigenvalues of $H_\theta$ are $\pm B$, and the corresponding eigenstates are

$$|+\rangle = \left(\frac{\cos(\frac{\pi}{4} - \frac{\theta}{2})}{\sin(\frac{\pi}{4} - \frac{\theta}{2})}, |\theta\rangle = \left(\frac{\sin(\frac{\pi}{4} - \frac{\theta}{2})}{\cos(\frac{\pi}{4} - \frac{\theta}{2})}\right) \right).$$

According to (28), the generator $h$ of the local translation with respect to the parameter $\theta$ for an evolution of time $t$ is

$$h = B \begin{pmatrix} e^{iBt} & 0 \\ 0 & e^{-iBt} \sin Bt \end{pmatrix}. \quad (35)$$

The eigenvalues of $h$ are $\pm B \sin Bt$, so the maximum quantum Fisher information is

$$F^{(Q)}_{\max} = 4B^2 \sin^2 Bt. \quad (36)$$

We can also extend this result to a more general case. Suppose the direction of the magnetic field $\vec{n}_\theta$ has an arbitrary form with $||\vec{n}_\theta|| = 1$, then the Hamiltonian of the interaction between the particle and the magnetic field is

$$H_\theta = B\vec{n}_\theta \cdot \vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli operators.

We can obtain $h$ from (28):

$$h = B \sin Bt (\cos Bt \partial_\theta \vec{n}_\theta + i \sin Bt \partial_\theta \vec{n}_\theta \times \vec{n}_\theta) \cdot \vec{\sigma}. \quad (38)$$

Since $\partial_\theta \vec{n}_\theta \times \vec{n}_\theta$ is orthogonal to $\partial_\theta \vec{n}_\theta$, and $||\partial_\theta \vec{n}_\theta \times \vec{n}_\theta|| = ||\partial_\theta \vec{n}_\theta||$, the eigenvalues of $h$ are

$$\pm B ||\partial_\theta \vec{n}_\theta|| \sin Bt. \quad (39)$$

Therefore, the maximum quantum Fisher information of estimating $\theta$ is

$$F^{(Q)}_{\max} = 4B^2 ||\partial_\theta \vec{n}_\theta||^2 \sin^2 Bt. \quad (40)$$

From (36) and (40), we can see that the maximum quantum Fisher information oscillates with the time $t$, and the period of the oscillation is $\frac{2\pi}{B}$. This implies that the maximum quantum Fisher information is always bounded in this case, and the upper bound is $4B^2 ||\partial_\theta \vec{n}_\theta||^2$. This is in sharp contrast to the case where the parameter to estimate is an overall multiplicative factor of the Hamiltonian. (Compare to the amplitude case below.) In that case, the maximum quantum Fisher information grows as $t^2$, and is unbounded as $t \to \infty$.

By way of comparison, if instead we want to estimate a parameter in the amplitude $B_\theta$ of the magnetic field, where $g$ is the parameter to estimate, and the direction of the magnetic field $\vec{n}_\theta$ is fixed, then

$$h = \partial_\theta B_\theta \vec{n}_\theta \cdot \vec{\sigma}. \quad (41)$$

In this case, the maximum quantum Fisher information is

$$F^{(Q)}_{\max} = 4(\partial_\theta B_\theta)^2 t^2, \quad (42)$$

which recovers the time scaling $t^2$, which is known in quantum metrology for phase estimation.

The maximum quantum Fisher information $F^{(Q)}_{\max}$ for estimating $\theta$ has an intuitive physical picture. The derivative $\partial_\theta \vec{n}_\theta$ characterizes how fast the direction $\vec{n}_\theta$ changes with the parameter $\theta$. If $\vec{n}_\theta$ changes fast with the parameter $\theta$, it will be more sensitive to distinguish different $\theta$, so the precision of estimating $\theta$ will be higher.

VI. CONCLUSION

In summary, in this paper we studied quantum metrology for estimating a general parameter of a Hamiltonian. We obtained the generator $h_\theta$ of the infinitesimal parameter translation with respect to $g$, of which the variance is the quantum Fisher information, and also a general upper bound on the quantum Fisher information. The results show that the optimal scaling of the quantum Fisher information with the number of systems can always reach the Heisenberg limit, but the time scaling can be different from that of estimating an overall multiplicative factor. We considered estimating a parameter of a magnetic field by measuring a spin-$\frac{1}{2}$ particle as an example to illustrate the results, and compared estimating a parameter of the magnetic field amplitude to estimating a parameter of the magnetic field direction. When estimating a parameter of the magnetic field amplitude, the time scaling of the quantum Fisher information is $t^2$, but when estimating the parameter of the magnetic field direction, the quantum Fisher information oscillates as a sine function of $t$. This example clearly shows the difference between estimating an overall multiplicative factor and estimating a general parameter, and gives a physical picture illustrating the general results.

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APPENDIX A: PROOF OF THE HERMICITY OF $\mathcal{H}_g$

Suppose $\{\sigma_1, \cdots, \sigma_d\}$ is an orthonormal basis in the operator space, then the $(i,j)$-th element of the superoperator $\mathcal{H}_g$ is

$$ (\mathcal{H}_g)_{ij} = \text{Tr}(\sigma_i^\dagger [H_g, \sigma_j]), \quad (43) $$

and

$$ (\mathcal{H}_g)_{ij}^\dagger = \text{Tr}(\sigma_i^\dagger [H_g, \sigma_j]). \quad (44) $$

If $\mathcal{H}_g$ is Hermitian, it must satisfy that $(\mathcal{H}_g)_{ij} = (\mathcal{H}_g)_{ij}^\dagger$. We can check whether this is true directly from $43$ and $44$. Note that

$$ (\mathcal{H}_g)_{ij} - (\mathcal{H}_g)_{ij}^\dagger = \text{Tr}(\sigma_i^\dagger [H_g, \sigma_j]) + \text{Tr}(\sigma_i^\dagger [H_g, \sigma_j]) $$

$$ = \text{Tr}(\{H_g, \sigma_i^\dagger \sigma_j\}) $$

$$ = 0, \quad (45) $$

so this proves the hermicity of $\mathcal{H}_g$.

APPENDIX B: PROOF OF EQ. (29)

First, we note that $[9]$.

$$ \langle \Delta h_g^2 \rangle_{\text{max}} = \frac{1}{4}(\lambda_{\text{max}} - \lambda_{\text{min}})^2, \quad (46) $$

where $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ are the maximum and minimum eigenvalues of $h$, respectively.

On one hand, $|\lambda_{\text{max}} - \lambda_{\text{min}}| \leq |\lambda_{\text{max}}| + |\lambda_{\text{min}}|$, so

$$ \langle \Delta h_g^2 \rangle_{\text{max}} \leq \frac{(|\lambda_{\text{max}}| + |\lambda_{\text{min}}|)^2}{2} \leq \frac{|\lambda_{\text{max}}|^2 + |\lambda_{\text{min}}|^2}{2}, \quad (47) $$

where the second inequality follows from the well-known power mean inequality: for any real positive numbers $x_1, \cdots, x_n$ and nonzero $p, q$,

$$ \left( \frac{x_1^p + \cdots + x_n^p}{n} \right)^{\frac{1}{p}} \leq \left( \frac{x_1^q + \cdots + x_n^q}{n} \right)^{\frac{1}{q}}, \quad \text{if } p \geq q. \quad (48) $$

If we take $q = 1$ and $p = 2$, it will produces $47$.

On the other hand,

$$ \text{Tr}(h_g^2 \mathcal{H}) = \sum_k |\lambda_k|^2 \geq |\lambda_{\text{max}}|^2 + |\lambda_{\text{min}}|^2, $$

where $\lambda_k$ runs over all eigenvalues of $h_g$, so we have

$$ \langle \Delta h_g^2 \rangle_{\text{max}} \leq \frac{1}{2} \text{Tr}(h_g^2 \mathcal{H}), \quad (49) $$

which proves Eq. (29).
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