Infinite horizon Stochastic Optimal Control for Volterra equations with completely monotone kernels

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Abstract

The aim of the paper is to study an optimal control problem on infinite horizon for an infinite dimensional integro-differential equation with completely monotone kernels, where we assume that the noise enters the system when we introduce a control. We start by reformulating the state equation into a semilinear evolution equation which can be treated by semigroup methods. The application to optimal control provide other interesting result and require a precise description of the properties of the generated semigroup. The main tools consist in studying the differentiability of the forward-backward system with infinite horizon corresponding with the reformulated problem and the proof of existence and uniqueness of of mild solutions to the corresponding HJB equation.

Keywords: Abstract integro-differential equation, Analytic semigroup, Backward Stochastic differential equations Elliptic PDEs, Hilbert spaces, Mild Solutions

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1. Introduction

In this paper we study a stochastic optimal control problem with infinite horizon for an infinite dimensional integral equation of Volterra type on a separable Hilbert space. Our starting point is a controlled stochastic Volterra equation of the form

\[
\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{t} a(t-s)u(s)ds &= Au(t) + f(u(t)) \\
&\quad + g[r(u(t), \gamma(t)) + \dot{W}(t)], \quad t \in [0, T] \\
u(t) &= u_0(t), \quad t \leq 0.
\end{aligned}
\]  

(1)

for a process \( u \) in a Hilbert space \( H \), where \( W(t), \ t \geq 0 \) is a cylindrical Wiener process defined on a suitable probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with values into a (possibly different) Hilbert space \( \Xi \); the kernel \( a \) is completely monotonic, locally integrable and singular at 0; \( A \) is a linear operator which generates an analytical semigroup; \( f \) is a Lipschitz continuous function from \( H \) into itself; \( g \)
is bounded linear mapping from $H$ into $L_2(\Xi, H)$ (the space of Hilbert-Schmidt operators from $\Xi$ to $H$). The function $r$ is a bounded Borel measurable mapping from $H \times U$ into $\Xi$. Also, the control is modelled by the predictable process $\gamma$ with values in some specified subset $U$ (the set of control actions) of a third Hilbert space $U$. We notice that the control enters the system together with the noise. This special structure is imposed by our techniques; however the presence of the operator $r$ allows more generality.

The optimal control problem that we wish to treat in this paper consists in minimizing the functional cost

$$J(u_0, \gamma) = \mathbb{E} \int_0^\infty e^{-\lambda t} \ell(u(t), \gamma(t))dt,$$

where $\ell: H \times U \to \mathbb{R}$ is a given real bounded function and $\lambda$ is any positive number.

Our first task is to provide a semigroup setting for the problem, by the state space setting first introduced by Miller in [3] and Desch and Miller [1] and recently revised, for the stochastic case, in Homan [29], Bonaccorsi and Desch [3], Bonaccorsi and Mastrogiacomo [4]. Within this approach, equation (1) is reformulated into an abstract evolution equation without memory on a different Hilbert space $X$. Namely, we rewrite equation (1) as

$$\begin{cases}
  dx(t) = Bx(t)dt + (I - B)Pf(Jx(t))dt \\
   + (I - B)Pg(r(Jx(t), \gamma(t)))dt + dW(t) \\
  x(0) = x,
\end{cases}$$

(3)

where the initial datum $x$ is obtained from $u_0$ by a suitable transformation. Moreover, $B$ is the infinitesimal generator of an analytic semigroup $e^{tB}$ on $X$. $P: H \to X$ is a linear mapping which acts as a sort of projection into the space $X$. $J: D(J) \subset X \to H$ is an unbounded linear functional on $X$, which gives a way going from $x$ to the solution to problem (1). In fact, it turns out that $u$ has the representation

$$u(t) = \begin{cases}
  Jx(t), & t > 0, \\
  u_0(t), & t \leq 0.
\end{cases}$$

For more details, we refer to the original papers [3, 29]. Further, the optimal control problem, reformulated into the state setting $X$, consists in minimizing the cost functional

$$\tilde{J}(x, \gamma) = \mathbb{E} \int_0^\infty e^{-\lambda t} \ell(Jx(t), \gamma(t))dt$$

(where the initial condition $u_0$ is substituted by $x$ and the process $u$ is substituted by $Jx$). It follows that $\gamma$ is an optimal control for the original Volterra equation if and only if it is an optimal control for that state equation (3).

To our knowledge, this paper is the first attempt to study optimal control problems with infinite horizon for stochastic Volterra equations. In order to
handle the control problem we introduce the weak reformulation; in this setting we wish to perform the standard program of synthesis of the optimal control, by proving the so-called fundamental relation for the value function and then solving in the weak sense the closed loop equation. The main step is to study the stationary Hamilton-Jacobi-Bellman equation corresponding with our problem. In our case, this is given by

\[ L v(x) = \lambda v(x) - \psi(x, \nabla v(x))(I - B)Pg(Jx)), \quad x \in X, \]  

where \( L \) is the infinitesimal generator of the Markov semigroup corresponding to the process \( x \):

\[ L[h](x) = \frac{1}{2} \text{Tr}[(I - B)Pg \nabla^2 h(x)g^* P^*(I - B)^*] + \langle Bx + (I - B)Pf(Jx), \nabla h(x) \rangle. \]

and \( \psi \) is the Hamiltonian of the problem, defined in terms of \( \ell \), i.e.

\[ \psi(x, z) := \inf_{\gamma \in U} \{ \ell(Jx, \gamma) + z \cdot r(x, \gamma) \}. \]  

Here \( \nabla h(x) \in H^* \) denotes the Gâteaux derivative at point \( x \in H \) and \( \nabla^2 h \) is the second Gâteaux derivative, identified with an element of \( L(H) \) (the vector space of linear and continuous functionals on \( H \)). Equations of type (HJB) have been studied by probabilistic approach in several finite dimensional situations [9, 15, 41, 42] and extended in infinite dimension by Fuhrman and Tessitore [25]. In particular, an equation like (HJB) is studied without any nondegeneracy assumption on \( g \) and a unique Gâteaux differentiable mild solution to (HJB) is found by an approach based on backward stochastic differential equations (BSDEs).

The main difficulty, in our case, is due to the presence of unbounded terms such as \((I - B)Pg\) which forces us to prove extra regularity for the solution to (HJB).

In this paper we develop the BSDE techniques along the lines initiated by Fuhrman and Tessitore in [25] and Hu and Tessitore [30]. In particular, the BSDE corresponding with our problem is given by

\[ dY(\tau) = \lambda Y(\tau)dt - \psi(x(\tau; x), Z(\tau))dt + Z(\tau)dW(\tau), \tau \geq 0 \]  

together with a suitable growth condition (which substitutes the final condition of the finite horizon case). In the above formula, \( x(\cdot; x) \) is the solution of the equation (3) starting from \( x \in X \) at time \( t = 0 \).

Our purpose is to prove that the solution of (BSDE) exists and it is unique for any positive \( \lambda \) and that the mild solution of the (HJB) equation is given in terms of the solution of the (BSDE). In particular, if we set \( v(x) := Y(0) \) it follows that \( v \) is the mild solution of (HJB). As notice before, in our case it is not enough to prove that \( v \) is once (Gâteaux) differentiable to give sense to equation (HJB). Indeed the occurrence of the term \( \nabla v(\cdot)(I - B)Pg \), together
with the fact that $P$ maps $H$ into an interpolation space of $D(B)$, forces us to prove that the map $h \mapsto \nabla v(x)(I - B)^\theta P g[h]$ for suitable $\theta \in (0, 1)$ extends to a continuous map on $H$. To do that we start proving that this extra regularity holds, in a suitable sense, for the state equation \[^{3}\] and then that it is conserved if we differentiate in Gâteaux sense the backward equation with respect to the process $x$. On the other side, we can prove that if the map $h \mapsto \nabla v(x)(I - B)^\theta[h]$ extends to a continuous function on $H$ then the processes $t \mapsto v(x(t; x))$ and $W$ admit joint quadratic variation in any interval $[t, T]$ and this is given by

$$\int_t^T \nabla v(x(\tau; x))(I - B)P g \, d\tau.$$ 

This is result is standard and is done by an application of the Malliavin calculus (on a finite time horizon).

We can then come back to the control problem and using the probabilistic representation of the unique mild solution to equation \[^{HJB}\] we easily show existence of an optimal feedback law. Indeed we are able to prove that the so called fundamental relation which states that $J(x, \gamma) \geq v(x)$ and equality holds if and only if the feedback law

$$\gamma(\tau) \in \Gamma(x^\gamma(\tau; x), \nabla v(x^\gamma(\tau; x))(I - B)P g(x(\tau; x))))$$

is verified, with $\Gamma(x, v)$ the set of minimizers in \[^{1}\] and $x^\gamma(\cdot; x)$ the solution of equation \[^{3}\] corresponding with the control $\gamma$. We refer to Section 3 for precise statements and additional results.

The paper is organized as follows: in the next section we give some useful notation and we introduce the main assumptions on the coefficients of the problem. In Section 2 we reformulate the problem into a semilinear abstract evolution equation and we study the properties of leading operator. In Section 4 we prove the first main result of the paper: we determine existence and uniqueness of the solution of the reformulated equation \[^{3}\]. In Section 5 we turn back to the original Volterra equation \[^{1}\] in order to establish even in this case existence and uniqueness of the solutions. In section 6 we give some properties of the forward equation corresponding with the reformulated uncon trolled problem. In Section 7 we study existence, uniqueness and properties of the backward stochastic equation \[^{B}SDE\]. In Section 8 we proceed with the study of the Hamilton Jacobi Bellman equation and, finally, in section 9 we employ the results proved in the previous sections in order to perform the standard synthesis of the optimal control.

2. Notations and main assumptions

The norm of an element $x$ of a Banach space $E$ will be denoted by $|x|_E$ or simply $|x|$ if no confusion is possible. If $F$ is another Banach space, $L(E, F)$ denotes the space of bounded linear operators from $E$ to $F$, endowed with the usual operator norm.
The letters $\Xi$, $H$, $U$ will always denote Hilbert spaces. Scalar product is denoted $\langle \cdot, \cdot \rangle$, with a subscript to specify the space, if necessary. All Hilbert space are assumed to be real and separable.

By a cylindrical Wiener process with values in a Hilbert space $\Xi$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we mean a family $(W(t))_{t \geq 0}$ of linear mappings from $\Xi$ to $L^2(\Omega)$, denoted $\xi \mapsto \langle \xi, W(t) \rangle$ such that

1. for every $\xi \in \Xi, (\langle \xi, W(t) \rangle)_{t \geq 0}$ is a real (continuous) Wiener process;
2. for every $\xi_1, \xi_2 \in \Xi$ and $t \geq 0$, $E(\langle \xi_1, W(t) \rangle \langle \xi_2, W(t) \rangle) = \langle \xi_1, \xi_2 \rangle$.

$(\mathcal{F}_t)_{t \geq 0}$ will denote the natural filtration of $W$, augmented with the family of $\mathbb{P}$-null sets. The filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. All the concepts of measurability for stochastic processes refer to this filtration. By $\mathcal{B}(\Gamma)$ we mean the Borel $\sigma$-algebra of any topological space $\Gamma$.

In the sequel we will refer to the following class of stochastic processes with values in an Hilbert space $K$:

1. $L^p(\Omega; L^2(0, T; K))$ defines, for $T > 0$ and $p \geq 1$, the space of equivalence classes of progressively measurable processes $y : \Omega \times [0, T) \to K$, such that

$$
|y|_{L^p(\Omega; L^2(0, T; K))}^p := E \left[ \int_0^T |y(s)|_K^2 \, ds \right]^{p/2} < \infty.
$$

Elements of $L^p(\Omega; L^2(0, T; K))$ are identified up to modification.

2. $L^p(\Omega; C([0, T]; K))$ defines, for $T > 0$ and $p \geq 1$, the space of equivalence classes of progressively measurable processes $y : \Omega \times [0, T) \to K$, with continuous paths in $K$, such that the norm

$$
|y|_{L^p(\Omega; C([0, T]; K))}^p := E \left[ \sup_{t \in [0, T]} |y(t)|_K^p \right]
$$

is finite. Elements of $L^p(\Omega; C([0, T]; K))$ are identified up to indistinguishability.

We also recall notation and basic facts on a class of differentiable maps acting among Banach spaces, particularly suitable for our purposes (we refer the reader to Fuhrman and Tessitore [24] or Ladas and Lakshmikantham [33, Section 1.6] (1970) for details and ). Let now $X, Y, V$ denote Banach spaces. We say that a mapping $F : X \to V$ belongs to the class $G^1(X, V)$ if it is continuous, Gâteaux differentiable on $X$, and its Gâteaux derivative $\nabla F : X \to L(X, V)$ is strongly continuous.

The last requirement is equivalent to the fact that for every $h \in X$ the map $\nabla F(\cdot)h : X \to V$ is continuous. Note that $\nabla F : X \to L(X, V)$ is not continuous in general if $L(X, V)$ is endowed with the norm operator topology; clearly, if it happens then $F$ is Fréchet differentiable on $X$. It can be proved that if $F \in G^1(X, V)$ then $(x, h) \mapsto \nabla F(x)h$ is continuous from $X \times X$ to $V$; if, in addition, $G$ is in $G^1(V, Z)$ then $G(F)$ is in $G^1(X, Z)$ and the chain rule holds:
\( \nabla(G(F))(x) = \nabla G(F(x)) \nabla F(x) \). When \( F \) depends on additional arguments, the previous definitions and properties have obvious generalizations.

Moreover, we assume the following.

**Hypothesis 2.1.**

1. The kernel \( a : (0, \infty) \to \mathbb{R} \) is completely monotonic, locally integrable, with \( a(0+) = +\infty \). The singularity in 0 shall satisfy some technical conditions that we make precise in Section 3.
2. \( A : D(A) \subset H \to H \) is a sectorial operator in \( H \). Thus \( A \) generates an analytic semigroup \( e^{tA} \).
3. The function \( f : H \to H \) is measurable and continuously Gâteaux differentiable; moreover there exist constants \( L > 0 \) and \( C > 0 \) such that
   \[
   |f(u) - f(v)| \leq L|u - v|, \quad u, v \in H; \\
   |f(0)| + \|\nabla u f(u)\|_{L(H)} \leq C, \quad u \in H.
   \]
4. The mapping \( g \) belongs to \( L_2(\Xi, H) \) (the space of Hilbert-Schmidt operators from \( \Xi \) to \( H \));
5. The function \( r : H \times U \to \Xi \) is Borel measurable and there exists a positive constant \( C > 0 \) such that
   \[
   |r(u_1, \gamma) - r(u_2, \gamma)| \leq C|u_1 - u_2|, \quad u_1, u_2 \in H, \gamma \in U; \\
   |r(u, \gamma)|_{\Xi} \leq C, \quad u \in H, \gamma \in U.
   \]
6. The process \( (W(t))_{t \geq 0} \) is a cylindrical Wiener process defined on a complete probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with values in the Hilbert space \( \Xi \).

The initial condition satisfies a global exponential bound as well as a linear growth bound as \( t \to 0 \):

**Hypothesis 2.2.**

1. There exist \( M_1 > 0 \) and \( \omega > 0 \) such that \( |u_0(t)| \leq M_1 e^{\omega t} \) for all \( t \leq 0 \);
2. There exist \( M_2 > 0 \) and \( \tau > 0 \) such that \( |u_0(t) - u_0(0)| \leq M_2 |t| \) for all \( t \in [-\tau, 0] \);
3. \( u_0(0) \in H_\varepsilon \) for some \( \varepsilon \in (0, 1/2) \).

Concerning the cost functional \( \ell \) we make the following general assumptions:

**Hypothesis 2.3.** The function \( \ell : H \times U \to \mathbb{R} \) is continuous and bounded.

We consider the following notion of weak solution for the Volterra equation (1).

**Definition 2.4.** We say that a process \( u = (u(t))_{t \geq 0} \) is a weak solution to equation (1) if \( u \) is an adapted, \( p \)-mean integrable (for any \( p \geq 1 \), continuous
\[ H \text{-valued predictable process and the identity} \]

\[
\int_{-\infty}^{t} a(t-s)(u(s),\zeta)_{H}ds = \langle \bar{u}, \zeta \rangle_{H} + \int_{0}^{t} \langle u(s), A^{*} \zeta \rangle_{H}ds \\
+ \int_{0}^{t} \langle f(u(s)), \zeta \rangle ds + \int_{0}^{t} \langle g r(u(s), \gamma(s)), \zeta \rangle_{H}ds + \langle gW(t), \zeta \rangle
\]

holds \( \mathbb{P} \text{-a.s.} \) for arbitrary \( t \in [0,T] \) and \( \zeta \in D(A^{*}) \), with \( A^{*} \) being the adjoint of the operator \( A \) and

\[
\bar{u} = \int_{-\infty}^{0} a(-s)u_{0}(s)ds.
\]

3. The analytical setting

A completely monotone kernel \( a : (0, \infty) \rightarrow \mathbb{R} \) is a continuous, monotone decreasing function, infinitely often derivable, such that

\[
(-1)^{n} \frac{d^{n}}{dt^{n}} a(t) \geq 0, \quad t \in (0, \infty), \quad n = 0, 1, 2, \ldots
\]

By Bernstein’s theorem, \( a \) is completely monotone if and only if there exists a positive measure \( \nu \) on \( [0, \infty) \) such that

\[
a(t) = \int_{(0, \infty)} e^{-\kappa t} \nu(d\kappa), \quad t > 0.
\]

Under the assumption \( a \in L^{1}(0,1) \), it holds that the Laplace transform \( \hat{a} \) is well defined and it is given in terms of \( \nu \) by

\[
\hat{a}(s) = \int_{(0, \infty)} \frac{1}{s + \kappa} \nu(d\kappa).
\]

We introduce the quantity

\[
\alpha(a) = \sup \left\{ \rho \in \mathbb{R} : \int_{c}^{\infty} s^{\rho-2} \frac{1}{\hat{a}(s)} ds < \infty \right\}
\]

and we make the following assumption:

**Hypothesis 3.1.** \( \alpha(a) > 1/2 \).

**Remark 3.2.** It is known from the theory of deterministic Volterra equations that the singularity of \( a \) helps smoothing the solution. We notice that \( \alpha(a) \) is independent on the choice of \( c > 0 \) and this quantity describes the behavior of the kernel near 0; by this way we ensure that smoothing is sufficient to keep the stochastic term tractable.
It is known that we can associate to any completely monotone kernel \( a \), by means of Bernstein’s Theorem \([43, \text{ pag. 90}]\), a measure \( \nu \) on \([0, +\infty)\) such that
\[
a(t) = \int_{[0, +\infty)} e^{-\kappa t} \nu(d\kappa).
\]
From the required singularity of \( a \) at \( 0^+ \) we obtain that \( \nu([0, +\infty)) = a(0^+) = +\infty \) while for \( s > 0 \) the Laplace transform \( \hat{a} \) of \( a \) verifies
\[
\hat{a}(s) = \int_{[0, +\infty)} \frac{1}{s + \kappa} \nu(d\kappa) < +\infty.
\]
Under the assumption of complete monotonicity of the kernel, a semigroup approach to a type of abstract integro-differential equations encountered in linear viscoelasticity was introduced in \([17]\) and extended to the case of Hilbert space valued equations in \([3]\).

We will see that this approach allow to treat the case of semilinear, stochastic integral equations; we start for simplicity with the equation
\[
\frac{d}{dt} \int_{-\infty}^{t} a(t-s)u(s) \, ds = Au(t) + f(t), \quad t \in [0, T]
\]
\[
u(t) = u_0(t), \quad t \leq 0,
\]
where \( f \) belongs to \( L^1(0, T; X) \). Moreover, we introduce the following identity, which comes from Bernstein’s theorem
\[
\int_{-\infty}^{t} a(t-s)u(s) \, ds = \int_{-\infty}^{t} \int_{[0, +\infty)} e^{-\kappa(t-s)} \nu(d\kappa) u(s) \, ds = \int_{[0, +\infty)} x(t, \kappa) \nu(d\kappa);
\]
here \( x(t, \kappa) \) is function given by
\[
x(t, \kappa) = \int_{-\infty}^{t} e^{-\kappa(t-s)} u(s) \, ds.
\]
Formal differentiation yields
\[
\frac{\partial}{\partial t} x(t, \kappa) = -\kappa x(t, \kappa) + u(t),
\]
while the integral equation \([8]\) can be rewritten
\[
\int_{[0, +\infty)} (-\kappa x(t, \kappa) + u(t)) \nu(d\kappa) = Au(t) + f(t).
\]
Now, the idea is to use equation \([8]\) as the state equation, with \( Bx = -\kappa x(\kappa) + u \), while \([9]\) enters in the definition of the domain of \( B \).

In our setting, the function \( x(t, \cdot) \) will be considered the state of the system, contained in the state space \( X \) that consists of all Borel measurable functions \( y : [0, +\infty) \to H \) such that the seminorm
\[
\|y\|_X^2 := \int_{[0, +\infty)} (\kappa + 1)|y(\kappa)|_H^2 \nu(d\kappa)
\]
is finite. We shall identify the classes $y$ with respect to equality almost everywhere in $\nu$.

Let us consider the initial condition. We introduce the space

$$\tilde{X}_0 := \{ u : \mathbb{R}^- \to H : \exists M > 0 \text{ and } \omega > 0 \text{ s.t. } |u(t)| < Me^{\omega t}, t \leq 0 \}$$

and we endow it with a positive inner product

$$\langle u, v \rangle_{\tilde{X}} = \int \int [a(t+s) - a'(t+s)][u(-s), v(-t)]_H \, ds \, dt;$$

then, setting $\tilde{N}_0 = \{ u \in \tilde{X}_0 : \langle u, u \rangle_{\tilde{X}} = 0 \}$, $\langle \cdot, \cdot \rangle_{\tilde{X}}$ is a scalar product on $\tilde{X}_0/\tilde{N}_0$; we define $\tilde{X}$ the completion of this space with respect to $\langle \cdot, \cdot \rangle_{\tilde{X}}$. We let the operator $Q : \tilde{X} \to X$ be given by

$$x(0, \kappa) = Qu_0(\kappa) = \int_{-\infty}^{0} e^{\kappa s} u_0(s) \, ds. \quad (10)$$

It has been proved in [3, Proposition 2.5] that the operator $Q$ is an isometric isomorphism between $\tilde{X}$ and $X$. This operator maps the initial value of the stochastic Volterra equation in the initial value of the abstract state equation. Different initial conditions of the Volterra equation generate different initial conditions of the state equation.

Hypothesis 2.2 is necessary in order to have a greater regularity on the initial value of the state equation. In fact in this case [3, Proposition 2.20] shows that $Qu_0$ belongs to $X_\eta$ for $\eta \in (0, \frac{1}{2})$.

**Remark 3.3.** We stress that under our assumptions we are able to treat, for instance, initial conditions for the Volterra equation of the following form

$$u_0(t) = \begin{cases} 0, & (\infty, \infty); \\ \bar{u}, & [\infty, 0] \end{cases}$$

provided $\bar{u}$ has a suitable regularity.

We quote from [3] the main result concerning the state space setting for stochastic Volterra equations in infinite dimensions.

**Theorem 3.4** (State space setting). Let $A, a, \alpha(a), W$ be given above; choose numbers $\eta \in (0, 1), \theta \in (0, 1)$ such that

$$\eta > \frac{1}{2} (1 - \alpha(a)), \quad \theta < \frac{1}{2} (1 + \alpha(a)), \quad \theta - \eta > \frac{1}{2}. \quad (11)$$

Then there exist

1) a separable Hilbert space $X$ and an isometric isomorphism $Q : \tilde{X} \to X$, 
2) a densely defined sectorial operator $B : D(B) \subset X \to X$ generating an analytic semigroup $e^{tB}$ with growth bound $\omega_0$, 

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3) its real interpolation spaces $X_\rho = (X, D(B))_{(\rho,2)}$ with their norms $\| \cdot \|_\rho$,

4) linear operators $P : H \to X_\theta$, $J : X_\eta \to H$

such that the following holds:

For each $x_0 \in X$, the problem (6) is equivalent to the evolution equation

$$\begin{align*}
x'(t) &= Bx(t) + (I - B)Pf(t) \\
x(0) &= x_0
\end{align*}$$

(12)
in the sense that if $u_0 \in \tilde{X}_0$ and $x(t; x_0)$ is the weak solution to Problem (12) with $x_0 = Q u_0$, then $u(t; u_0) = Jx(t; x_0)$ is the unique weak solution to Problem (6).

It is remarkable that $B$ generates an analytic semigroup, since in this case we have at our disposal a powerful theory of optimal regularity results. In particular, besides the interpolation spaces $X_\theta$ introduced in Theorem 3.4, we may construct the extrapolation space $X_{-1}$, i.e., a Sobolev space of negative order associated to $e^{tB}$.

Under some additional condition it is possible to prove exponential stability of the semigroup $e^{tB}$. As we will see, this is especially needed in the study of the optimal control problem with infinite horizon. In particular we impose the following

**Hypothesis 3.5.** There exists $\sigma > 0$ such that the function $e^{\sigma t} a(t)$ is completely monotonic.

Consequently, we obtain that $B$ is of negative type.

**Proposition 3.6.** The real parts of the spectrum of $B$ are bounded by some $\omega_0 < 0$. Consequently the semigroup $e^{tB}$ decays exponentially.

**Proof.** We proceed as in Bonaccorsi and Desch. All we need to show is that $0$ is in the resolvent set of $B$. Once this is proved, the spectral bound is negative since the spectrum is confined to a sector. From this exponential decay follows, since analytic semigroups have spectrum determined growth. Now we notice that $\hat{a}(s)$ exists in the set $\mathbb{C} \setminus (-\infty, -\sigma]$ and $0$ is in the resolvent set of $A$, by assumption. Moreover, we can give the explicit expression of $R(s, B)$, which is

$$[R(s, B)x](\kappa) = \frac{1}{s + \kappa} \left[ x(\kappa) + R(s\hat{a}(s)) \int_{[0,\infty)} \frac{\kappa}{s + \kappa} x(\kappa) \nu(d\kappa) \right]$$

(the calculation is straightforward and can be found in [8, Lemma 3.5]). Then it is easily seen that $R(s, B)$ can be extended to a neighborhood of $0$. $\square$

The semigroup $e^{tB}$ extends to $X_{-1}$ and the generator of this extension, that we denote $B_{-1}$, is the unique continuous extension of $B$ to an isometry between $X$ and $X_{-1}$. See for instance [18, Definition 5.4] for further details.
Remark 3.7. In the sequel, we shall always denote the operator with the letter $B$, even in case where formally $B_{-1}$ should be used instead. This should cause no confusion, due to the similarity of the operators.

Remark 3.8. If we apply to problem (1) the machinery introduced above, we obtain, on the space $X$, the stochastic Cauchy problem

\[
\begin{aligned}
\frac{dx(t)}{dt} &= Bx(t)dt + (I - B)Pf(Jx(t))dt \\
&
+ gr(Jx(t), \gamma(t))dt + (I - B)Pg dW(t)
\end{aligned}
\]

for $0 \leq t \leq T$ and initial condition $x \in X_\eta$. The above expression is only formal since the coefficients do not belong to the state space; however, we can give a meaning to the mild form of the equation and we can also prove existence and uniqueness of the solution. Moreover, given the solution $x$ of (13), we obtain the solution of the original Volterra equation by setting $v(t) = Jx(t)$. This facts will be the object of the next sections.

4. The state equation: existence and uniqueness

In this section, motivated by the construction in Section 3, we shall establish existence and uniqueness result for the following stochastic uncontrolled Cauchy problem on the space $X$ defined in Section 3:

\[
\begin{aligned}
\frac{dx(t)}{dt} &= Bx(t)dt + (I - B)Pf(Jx(t))dt \\
&
+ (I - B)Pg r(Jx(t), \gamma(t))dt + (I - B)Pg dW(t)
\end{aligned}
\]

for $0 \leq t \leq T$ and initial condition $x \in X_\eta$. The above expression is only formal since the coefficients do not belong to the state space; however, we can give a meaning to the mild form of the equation:

Definition 4.1. We say that a continuous, $X$-valued, predictable process $x = (x(t))_{t \geq 0}$ is a (mild) solution of the state equation (13) if $P$-a.s.,

\[
x(t) = e^{(t-s)B}x + \int_s^t e^{(t-\sigma)B}(I - B)Pf(Jx(\sigma))d\sigma
\]

\[
+ \int_s^t e^{(t-\sigma)B}(I - B)Pg r(Jx(\sigma), \gamma(\sigma))d\sigma + \int_s^t e^{(t-\sigma)B}(I - B)Pg dW(\sigma).
\]

Let us state the main existence result for the solution of equation (13).

Theorem 4.2. Under Hypotheses 2.1, 2.2, for an arbitrary predictable process $\gamma$ with values in $\mathcal{U}$, for every $0 \leq t \leq T$ and $x \in X_\eta$, there exists a unique adapted process $x \in L^p_x(\Omega, \mathcal{C}([0, T]; X_\eta))$ solution of (13). Moreover, the estimate

\[
\mathbb{E} \sup_{t \in [0, T]} ||x(t)||_{X_\eta}^p \leq C(1 + ||x||_{X_\eta}^p)
\]
holds for some positive constant $C$ depending on $T$ and the parameters of the problem.

Proof. The proof of the above theorem proceeds, basically, on the same lines as the proof of Theorem 3.2 in Bonaccorsi and Mastrogiacomo [4] (2009). First, we define a mapping $K$ from $L^p(\Omega; C([0, T]; X_{\eta}))$ to itself by the formula

$$K(x)(t) := e^{tB}x + \Lambda(x)(t) + \Delta(x)(t) + \Gamma(t), \quad (16)$$

where the second, third and last term in the right side of (16) are given by

$$\Lambda(x)(t) = \int_0^t e^{(t-\tau)B}(I - B)P f(Jx(\tau))d\tau \quad (17)$$
$$\Delta(x)(t) = \int_0^t e^{(t-\tau)B}(I - B)P g\gamma(Jx(\tau), \gamma(\tau))d\tau \quad (18)$$
$$\Gamma(t) = \int_0^t e^{(t-\tau)B}(I - B)P g dW(\tau) \quad (19)$$

Then, we will prove that the mapping $K$ is a contraction on $L^p(\Omega; C([0, T]; X_{\eta}))$ with respect to the equivalent norm

$$| | |x| | |_p := E \sup_{t \in [0, T]} e^{-\beta pt}||x(t)||_{_{L^p}}$$

where $\beta > 0$ will be chosen later. For simplicity we write $\Lambda(t)$ instead of $\Lambda(x)(t)$.

Our first step is to prove that $\Gamma, \Delta$ and $\Lambda$ are well-defined mappings on the space $L^p(\Omega; C([0, T]; X_{\eta}))$ and to give estimates on their norm. We choose $\delta$ small enough such that $1 + \eta - \theta + 1/p < \delta << 1/2$ and define

$$y_\eta(\sigma) := \int_0^\sigma (t - \sigma)^{-\delta}e^{(t-\sigma)B}(I - B)^\theta P g dW(\sigma).$$

Since the semigroup $e^{tB}$ is analytic, $P$ maps $H$ into $X_\theta$ and $g \in L_2(\Xi, H)$, an application of Lemma 7.2 in [33], yields: $E \int_0^T |y_\eta(\sigma)|^p < \infty$. In particular $y \in L^p([0, T]; X)$, $\mathbb{P}$-a.s. Moreover, if we set

$$(R_\delta \phi)(t) = \int_0^t (t - \sigma)^{\delta - 1}e^{(t-\sigma)B}(I - B)^{1+\eta - \theta} \phi(\sigma)d\sigma,$$

then in [14, Proposition A.1.1.] it is proved that $R_\delta$ is a bounded linear operator from $L^p([0, T]; X)$ into $C([0, T]; X)$. Finally, by stochastic Fubini-Tonelli Theorem we can rewrite:

$$(R_\delta y_\eta)(t) = \int_0^t \int_0^\tau (t - \tau)^{\delta - 1}(\tau - \sigma)^\delta (I - B)^{\eta + 1}e^{(t-\sigma)B}P g dW(\sigma) \ d\tau$$
$$= \left(\int_0^1 (1 - \tau)^{\delta - 1}\tau^{-\delta}d\tau\right)(I - B)^\eta \Gamma(t)$$
and conclude that $\Gamma(t) \in L^p(\Omega, C([0, T]; X_\eta))$.

In a similar (and easier) way it is possible to show that $\Lambda(\cdot, t)$ and $\Delta(\cdot, t)$ belong to $L^p(\Omega, C([0, T]; X_\eta))$, hence, we conclude that $\mathcal{K}$ maps $L^p(\Omega; C([0, T]; X_\eta))$ into itself.

Now we claim that $\mathcal{K}$ is a contraction in $L^p(\Omega, C([0, T]; X_\eta))$. In fact, by straightforward estimates we can write

$$
\|\Lambda(x)(t) - \Lambda(y)(t)\|\leq C_{L,T} \beta^{1/2 + \delta} |x - y|_p^{1/2} + \delta - (\theta - \eta) |x - y|_p.
$$

Therefore $\mathcal{K}$ is Lipschitz continuous from $L^p(\Omega, C([0, T]; X_\eta))$ into itself; further, we can find $\beta$ large enough such that $C_{L,T}(2\beta)^{1/2 + \delta + \eta - \theta} < 1$. Hence $\mathcal{K}$ becomes a contraction on the time interval $[0, T]$ and by a classical fixed point argument we get that there exists a unique solution of the equation (13) on $[0, T]$.

**Remark 4.3.** In the following it will be also useful to consider the uncontrolled version of equation (13), namely:

$$
\begin{align*}
\{ & d x(t) = B x(t) dt + (I - B) P f(J x(t)) dt + (I - B) P g dW(t) \\
     & x(0) = x.
\end{align*}
$$

We will refer to (20) as the forward equation. We then notice that existence and uniqueness for the above equation can be treated in an identical way as in the proof of Theorem 4.2.

5. The controlled stochastic Volterra equation

In this section we prove existence and uniqueness of solution for the original Volterra equation (1). As a preliminary step for the sequel, we state two results of existence and uniqueness for (a special case of) the original Volterra equation. The proofs can be found in [5, Section 2].

The first result concerns with the solution to the linear deterministic Volterra equation.

**Proposition 5.1.** The linear equation

$$
\frac{d}{dt} \int_{-\infty}^{t} a(t-s) u(s) \, ds = Au(t), \quad t \in [0, T]
$$

$$
u(t) = 0, \quad t \leq 0.
$$

has a unique weak solution $u \equiv 0$.

The second result deals with existence and uniqueness of the Stochastic Volterra equation with non-homogeneous terms. The result comes from a generalization of [3, Theorem 3.7], where the case $f(t) \equiv 0$ is treated.
Proposition 5.2. In our assumptions, let \( x_0 \in X_\eta \) for some \( \frac{1-\alpha(a)}{2} < \eta < \frac{1}{2} \alpha(a) \). Given the process

\[
x(t) = e^{tB} x_0 + \int_0^t e^{(t-s)B} (I - B) Pf(s) \, ds + \int_0^t e^{(t-s)B} (I - B) Pg \, dW(s)
\]

(22)

we define the process

\[
u(t) = \begin{cases} Jx(t), & t \geq 0, \\ u_0(t), & t \leq 0. \end{cases}
\]

(23)

Then \( u(t) \) is a weak solution to problem

\[
\frac{d}{dt} \int_{-\infty}^t a(t-s)u(s) \, ds = Au(t) + f(t) + gW(t), \quad t \in [0, T]
\]

\[
u(t) = u_0(t), \quad t \leq 0.
\]

(24)

After the preparatory results stated above, here we prove that main result of existence and uniqueness of solutions of the original controlled Volterra equation (1).

Theorem 5.3. Assume Hypotheses 2.1 and 2.2. Let \( \gamma \) be an admissible control and \( x \) be the solution to problem (3) (associated with \( \gamma \)) whose existence is proved in Theorem 4.2. Then the process

\[
u(t) = \begin{cases} u_0(t), & t \leq 0 \\ Jx(t), & t \in [0, T] \end{cases}
\]

(25)

is the unique solution of the stochastic Volterra equation

\[
\frac{d}{dt} \int_{-\infty}^t a(t-s)u(s) \, ds = Au(t) + f(\tilde{u}(t)) + g \left[ r(u(t), \gamma(t)) + \tilde{W}(t) \right], \quad t \in [0, T]
\]

\[
u(t) = u_0(t), \quad t \leq 0.
\]

(26)

Proof. We propose to fulfill the following steps: first, we prove that the affine equation

\[
\frac{d}{dt} \int_{-\infty}^t a(t-s)u(s) \, ds = Au(t) + f(\tilde{u}(t)) + g \left[ r(\gamma(t), \tilde{u}(t)) + \tilde{W}(t) \right], \quad t \in [0, T]
\]

\[
u(t) = u_0(t), \quad t \leq 0.
\]

(27)

defines a contraction mapping \( Q : \tilde{u} \mapsto \tilde{u} \) on the space \( L_2^2(\Omega; C([0, T]; H)) \). Therefore, equation (27) admits a unique solution.

Then we show that the process \( u \) defined in (25) satisfies equation (27). Accordingly, by the uniqueness of the solution, the thesis of the theorem follows.
**First step.** We proceed to define the mapping

\[ Q : L^p(\Omega; C([0, T]; H)) \to L^p(\Omega; C([0, T]; H)) \]

where \( Q(\tilde{u}) = u \) is the solution of the problem

\[
\frac{d}{dt} \int_{-\infty}^t a(t-s)u(s) \, ds = Au(t) + f(\tilde{u}(t)) + g \left[ r(\tilde{u}(t), \gamma(t)) + \tilde{W}(t) \right], \quad t \in [0, T]
\]

\[
u(t) = u_0(t), \quad t \leq 0.
\]  

(28)

Let \( \tilde{u}_1 \) and \( \tilde{u}_2 \) be two processes belonging to \( L^p(\Omega; C([0, T]; H)) \) and take \( u_1 = Q(\tilde{u}_1) \) and \( u_2 = Q(\tilde{u}_2) \). It follows from the uniqueness of the solution, proved in Proposition (21), that the solution \( u_i(t) \) \( (i = 1, 2) \) has the representation

\[
u_i(t) = \begin{cases} J v_i(t), & t \in [0, T] \\ u_0(t), & t \leq 0 \end{cases}
\]

where

\[
v_i(t) = e^{tB}v_0 + \int_0^t e^{(t-s)B} (I - B) Pg r(\tilde{u}_i(s), \gamma(s)) \, ds \\
+ \int_0^t e^{(t-s)B} (I - B) Pf(\tilde{u}_i(s)) \, ds + \int_0^t e^{(t-s)B} (I - B) Pg dW(s).
\]

In particular,

\[
U(t) = u_1(t) - u_2(t) = \begin{cases} J (v_1(t) - v_2(t)), & t \in [0, T] \\ 0, & t \leq 0 \end{cases}
\]

then

\[
E \sup_{t \in [0, T]} e^{-\beta pt} |U(t)|^p \leq \|J\|_{L(X, H)}^p \quad \quad \text{and} \quad \quad E \sup_{t \in [0, T]} e^{-\beta pt} \|v_1(t) - v_2(t)\|_H^p.
\]

Now we notice that the quantity on the right hand side can be treated as in Theorem 4.2 and the claim follows.

**Second step**

It follows from the previous step that there exists at most a unique solution \( u \) of problem (27); hence it only remains to prove the representation formula (25) for \( u \).

Let \( \hat{f}(t) = f(Jx(t)) + gr(Jx(t), \gamma(t)) \); it is a consequence of Proposition 5.2 that \( u \), defined in (25), is a weak solution of the problem

\[
\frac{d}{dt} \int_{-\infty}^t a(t-s)u(s) \, ds = Au(t) + \hat{f}(t) + gW(t), \quad t \in [0, T]
\]

\[
u(t) = u_0(t), \quad t \leq 0,
\]  

(29)
and the definition of $\tilde{f}$ implies that $u$ is a weak solution of

$$\frac{d}{dt} \int_{-\infty}^{t} a(t-s)u(s)\, ds = Au(t) + f(Jx(t)) + g\left[r(Jx(t), \gamma(t)) + \dot{W}(t)\right], \quad t \in [0, T]$$

$$u(t) = u_0(t), \quad t \leq 0,$$

that is problem (27).

6. The forward equation

In this section we state some properties of the forward equation (13) corresponding with our problem.

In the following, $x$ will denote the solution of the uncontrolled equation (13). Under the assumption introduced in the previous sections we can give the regular dependence of $x$ on the initial condition $x$. This result will be used later in order to characterize the solution of the stationary HJB in terms of the solution of a suitable stochastic backward differential equation and, consequently, to characterize the optimal control for our problem.

**Proposition 6.1.** Under Hypotheses 2.1 and 3.5, for any $p \geq 1$ the following holds.

1. The map $x \mapsto x(t; x)$ defined on $X_\eta$ and with values in $L^p(\Omega, C([0, T]; X_\eta))$ is continuous.

2. The map $x \mapsto x(t; x)$ has, at every point $x \in X_\eta$, a Gâteaux derivative $\nabla_x x(t; x)$ and, for every $h \in X_\eta$, the following equation holds $\mathbb{P}$-a.s.:

$$\nabla_x x(t; x)[h] = e^{tB}h + \int_{0}^{t} e^{(t-s)(I-B)}P \nabla_u f(Jx(\tau; x))J\nabla_x x(\tau; x)[h]d\tau.$$

Finally, $\mathbb{P}$-a.s., we have

$$|\nabla_x x(t; x)[h]| \leq C|h|,$$

for all $t > 0$ and some $C > 0$.

**Proof.** Points 1 and 2 are proved, for instance, in [11, Proposition 6.2]. To prove the 3 we simply notice that

$$\left\| \int_{0}^{t} e^{tB}(I-B)P \nabla_u f(Jx(\tau; x))J\nabla_x x(\tau; x)[h]d\tau \right\|_{\eta}$$

$$\leq |P|_{L(H,X)} \|\nabla_u f\|_{L(H,H)} |J|_{L(X_\eta,H)} \int_{0}^{t} e^{-\omega s-1-\gamma t} |\nabla_x x(s; x)[h]|ds.$$

By application of Gronwall’s lemma we thus obtain

$$|\nabla_x x(t; x)[h]| \leq C|h|, \quad t \geq 0, \quad \mathbb{P} - a.s.$$

where $C$ is a positive constant independent of $t$ and $x$. \qed
As we will see later, an important point in order to study the HJB equation corresponding with our problem is that of extending the map \( h \mapsto \nabla_x x(t; x) (I - B)^{1-\theta} P g[h] \) - a priori defined on \( X_{1-\theta} \) - to the whole space \( X \). This result is stated below.

**Proposition 6.2.** Under assumptions \([2, 1]\) and \([3.5]\) the map \( h \mapsto \nabla_x x(t; x) (I - B)^{1-\theta} P g[h] \) - a priori defined on \( X_{1-\theta} \) - can be extended to the whole space \( X \) and it is continuous from \( [0, T] \times X \times X \) to \( L^\infty(\Omega; C([0, T]; X_\eta)) \) for any \( T > 0 \). Finally, there exists a constant \( C > 0 \) such that

\[
|\nabla_x x(t; x) (I - B)^{1-\theta} P g[h]| \leq C|h|, \tag{33}
\]

for all \( t \geq 0, x \in X_\eta, h \in X \), with \( C \) independent of \( t \) and \( x \).

**Proof.** We proceed by proving that the norm of \( \nabla_x x(t; x) (I - B)^{1-\theta} P g \) in the space of linear operators on \( X \) is finite. In fact, taking into account Proposition \([0, 1]\) we see that, for any \( h \in X_{1-\theta} \), the process \( \nabla_x x(t; x) (I - B)^{1-\theta} P g[h] \) satisfies the following equation:

\[
\nabla_x x(t; x) (I - B)^{1-\theta} [h] = e^{tB} (I - B)^{1-\theta} h
+ \int_0^t e^{(t-\tau)B} (I - B) P \nabla_x f(Jx(\tau; x)) J \nabla_x x(\tau; x) (I - B)^{1-\theta} [h] d\tau.
\]

Hence, recalling that \( \nabla_x f \) is bounded and Proposition \([3.6]\) we can estimate \( |\nabla_x x(t; x) (I - B)^{1-\theta} P g[h]| \) as follows:

\[
|\nabla_x x(t; x) (I - B)^{1-\theta} P g[h]| \leq e^{-\omega t} \theta^{-1} \\
+ C_f |J| \int_0^t |\nabla_x x(t; x) (I - B)^{1-\theta} P g[h]| e^{-(t-\tau)\omega} (\tau - t)^{\theta-1} |h| d\tau
\]

Now the bound \((33)\) follows by an easy application of Gronwall’s lemma, while for continuity, we can refer to \([11, Proposition\ 6.2]\). \(\square\)

7. The backward equation on an infinite horizon

In this section we consider the backward stochastic differential equation in the unknown \( (Y, Z) \):

\[
Y(\tau) = Y(T) + \int_\tau^T (\lambda Y(\sigma)) - \psi(x(\sigma; x), Z(\sigma)) d\sigma + \int_\tau^T Z(\sigma) dW(\sigma), \tag{34}
\]

where \( 0 \leq \tau \leq T < \infty, \lambda > 0, x(\cdot; x) \) is the solution of the uncontrolled equation \(\([14]\)\) and \(\psi \) is the Hamiltonian function relative to the control problem described in Section \([1]\). More precisely, for \( x \in X_\eta, z \in \Xi^* \) we have

\[
\psi(x, z) := \inf \{ \ell(Jx, \gamma) + z \cdot r(J x, \gamma) \mid \gamma \in \mathcal{U} \}.
\]

We require the following assumption on \( \psi \):

\[
\psi(x, z) := \inf \{ \ell(Jx, \gamma) + z \cdot r(J x, \gamma) \mid \gamma \in \mathcal{U} \}.
\]

We require the following assumption on \( \psi \):
Hypothesis 7.1. 1. $\psi$ is uniformly Lipschitz continuous in $z$, with Lipschitz constant $K$, that is:

$$|\psi(x,z_1) - \psi(x,z_2)| \leq K\|z_1 - z_2\| \Xi^\ast.$$ 

2. $\sup_{x \in X} |\psi(x,0)| := M < \infty$.

3. The map $\psi$ is Gâteaux differentiable on $X_\eta \times \Xi^\ast$ and the maps $(x,h,z) \mapsto \nabla_x \psi(x,z)[h]$ and $(x,z,\zeta) \mapsto \nabla_z \psi(x,z)[\zeta]$ are continuous on $X_\eta \times X \times \Xi^\ast$ and $X_\eta \times \Xi^\ast \times \Xi^\ast$ respectively.

The existence and uniqueness of solution to (34) under Hypothesis 7.1 was first studied (even though for more general coefficients $\psi$) by Briand and Hu in [8] and successively by Royer in [44]. Their result in valid when $W$ is a finite dimensional Wiener process but the extension to the case in which $W$ is a Hilbert-valued Wiener process is immediate.

In our context the result reads as follows.

Proposition 7.2. Assume Hypothesis 2.3 and 7.1. Then we have:

1. For any $x \in X_\eta$, there exists a solution $(Y,Z)$ to BSDE (34) such that $Y$ is a continuous process bounded by $M\lambda$, and $Z \in L^2(\Omega; L^2(0,\infty; \Xi))$ with $\mathbb{E} \int_0^\infty e^{-2\lambda s} |Z(s)|^2 ds < \infty$. Moreover, the solution is unique in the class of processes $(Y,Z)$ such that $Y$ is continuous and uniformly bounded, and $Z$ belongs to $L^2_{loc}(\Omega, L^2(0,\infty; \Xi^\ast))$. In the following we will denote such a solution by $Y(\cdot; x)$ and $Z(\cdot; x)$.

2. Denoting by $(Y(\cdot; x,n), Z(\cdot; x,n))$ the unique solution of the following BSDE (with finite horizon):

$$Y(\tau; x,n) = \int_\tau^n (\psi(x(\sigma); x, Z(\sigma))) - \lambda Y(\sigma))d\sigma + \int_\tau^n Z(\sigma; x,n)dW(\sigma),$$

(36)

then $|Y(\tau; x,n)| \leq \frac{M}{\lambda}$ and the following convergence rate holds:

$$|Y(\tau; x,n) - Y(\tau; x)| \leq \frac{M}{\lambda} \exp \{-\lambda(n - \tau)\}.$$ 

Moreover,

$$\mathbb{E} \int_0^\infty e^{-2\lambda \sigma} \|Z(\sigma; x,n) - Z(\sigma; x)\|^2 d\sigma \to 0.$$ 

3. For all $T > 0$ and $p > 1$, the map $x \mapsto (Y(\cdot; x)|_{[0,T]}, Z(\cdot; x)|_{[0,T]})$ is continuous from $X_\eta$ to the space $L^p(\Omega, C([0,T]; \mathbb{R})) \times L^p(\Omega, L^2([0,T]; \Xi^\ast))$.

We need to study the regularity of $Y(\cdot; x)$. More precisely we would like to show that $Y(0; x)$ is Gâteaux differentiable with respect to the initial condition $x$ and that both $Y(0; x)$ and $\nabla_x Y(0; x)$ turn out to be bounded.

The following result is one of the crucial point of the paper.

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Theorem 7.3. Under Hypotheses 2.1, 3.5, 2.3 and 7.1, the map \( x \mapsto Y(0; x) \) is Gâteaux differentiable. Moreover, \( |Y(0; x)| + |\nabla_x Y(0; x)| \leq c. \)

Proof. The argument follows essentially the proof of 31, Theorem 3.1. An important point is the boundedness of \( \nabla_x x(t, x)[h] \) (\( \mathbb{P} \)-a.s. and for any \( t \geq 0 \)), which was proved in Proposition 6.1. We first recall that, under our assumptions (see 24, Proposition 5.2), the map \( x \mapsto (Y(t; x, n), Z(t; x, n)) \) considered in Proposition 7.2 is Gâteaux differentiable from \( X_t \rightarrow \mathcal{L} \) to \( LP(\Omega; C(0, T; \mathbb{R}) \times LP(\Omega; L^2(0, T; \Xi^*)) \) for all \( p \geq 2 \). Denoting by \( \nabla_x Y(t; x, n)[h], \nabla_x Z(t; x, n)[h] \) the partial Gâteaux derivative with respect to \( x \) in the direction \( h \in X \), the processes \( \nabla_x Y(t; x, n)[h], \nabla_x Z(t; x, n)[h] \) solves the equation, \( \mathbb{P} \)-a.s.

\[
\nabla_x Y(\tau; x, n)[h] = \int_0^\tau (\psi(x(\sigma; x), Z(\sigma; x, n))\nabla_x x(\sigma; x) d\sigma - \int_0^\tau \lambda \nabla_x Y(\sigma; x, n)[h] d\sigma + \int_0^\tau Z(\sigma; x, n) dW(\sigma),
\]

We notice that by the assumption made on \( \nabla_x \psi, \nabla_x \psi \) and Proposition 5.2, we have

\[
|\nabla_x x(t; x)[h]| \leq C|h| \quad \text{and} \quad |\nabla_x \psi(x(t; x), Z(t; x))| \leq C.
\]

Therefore by the same arguments based on Girsanov transform as in 7, Lemma 3.1, we obtain

\[
\sup_{t \in [0, T]} |\nabla_x Y(t; x, n)| \leq C|h|, \quad \mathbb{P} \text{-a.s.}
\]

and, again as in the proof of 7, Lemma 3.1, applying Itô formula to \( e^{-2M} |\nabla_x Z(t; x, n)|^2 \), we get:

\[
\mathbb{E} \int_0^\infty e^{-2M} (|\nabla_x Y(t; x, n)|^2 + |\nabla_x Z(t; x, n)|^2) dt < C|h|^2.
\]

Let now \( \mathcal{M}^{2,-2\lambda} \) be the Hilbert space of all couples of \( \{y, z\} \) for \( t \geq 0 \)-adapted processes \((y, z)\), where \( y \) has values in \( \mathbb{R} \) and \( z \in \Xi^* \), such that

\[
\mathbb{E} \int_0^\infty e^{-2M} (|y(t)|^2 + |z(t)|^2) dt < C|h|^2.
\]

Fix now \( x \in X_t \) and \( h \in X \). Then there exists a subsequence of

\[
\{(\nabla_x Y(t; x, n)h, \nabla_x Z(t; x, n)h, \nabla_x Y(0; x, n)h)\}
\]

such that \( \{(\nabla_x Y(t; x, n)h, \nabla_x Z(t; x, n)h)\} \) converges weakly to \( (U^1(x, h), V^1(x, h)) \) in \( \mathcal{M}^{2,-2\lambda} \) and \( \nabla_x Y(0; x, n)h \) converges to \( \xi(x, h) \in \mathbb{R} \). Proceeding as in 31, Theorem 3.1, we see that the convergence of \( \nabla_x Y(t; x, n)h \) is, in reality, in \( L^2(0; T; \mathbb{R}) \) for all \( T > 0 \) and, moreover, that \( \lim_{n \to \infty} \nabla_x Y(0; x, n)h \) exists and
Finally, for any \( T > 0 \) with \( \Pi(\cdot) \) introduced in Section 1. The main point is to prove that the mapping \( s \mapsto \nabla \Phi(t, x) \) is well defined as operators from \( L^1(U) \) to \( \mathbb{R} \) and the claim is proved.

Moreover it hold that \( U^1(t; x) = U(t; x)h \) for any fixed \( h \in X \). Summarizing up, we have that \( U(0; x)h = \lim_{n \to \infty} \nabla_x Y(0; x, n)h \) exists, it is linear and verifies \( U(0; x)h \leq C|\theta| \) for every \( h \) fixed. Finally, it is continuous in \( x \) for every \( h \) fixed.

Finally, for \( t > 0 \) we have

\[
\lim_{t \to 0} \frac{Y(0; x + th) - Y(0; x)}{t} = \lim_{t \to 0} \lim_{n \to \infty} \frac{Y(0; x + th, n) - Y(0; x, n)}{t} = \lim_{t \to 0} \int_{0}^{1} \nabla_x Y(0; x + th\theta)h d\theta = \lim_{t \to 0} \int_{0}^{1} Y(0; x + th\theta)d\theta = U(0; x)h,
\]

and the claim is proved.

Starting from the Gâteaux derivatives of \( Y \) and \( Z \), we introduce suitable auxiliary processes which allow ourselves to express \( Z \) in terms of \( \nabla Y \) and \( (I - B)^{1 - \theta} \) and then get the fundamental relation for the optimal control problem introduced in Section 1. The main point is to prove that the mappings \((h, x) \mapsto \nabla_x Y(t; x)(I - B)^{1 - \theta} P_g[h] \) and \((h, x) \mapsto \nabla_x Z(t; x)(I - B)^{1 - \theta} P_g \) are well defined as operators from \( X \times X \) respectively in \( L^\infty(\Omega; C([0, T]; \mathbb{R})) \) and \( L^\infty(\Omega; C([0, T]; \mathbb{R})) \).

**Proposition 7.4.** For every \( p \geq 2 \), \( \beta < 0 \), \( x \in X_\eta \), \( h \in X \) there exists two processes

\[
\{ \Pi(t; x)[h] : t \geq 0 \} \quad \text{and} \quad \{ Q(t; x)[h] : t \geq 0 \}
\]

with \( \Pi(t; x)[h] \in L^p(\Omega; C([0, T]; X_\eta)) \) and \( Q(t; x)[h] \in L^p(\Omega; C([0, T]; X^*)) \) for any \( T > 0 \) and such that if \( x \in X_\eta \) then \( \mathbb{P} \)-a.s. the following identifications hold:

\[
\Pi(t; x)[h] = \begin{cases} 
\nabla_x Y(t; x)(I - B)^{1 - \theta} [h], & t > 0; \\
\nabla_x Y(0; x)(I - B)^{1 - \theta} [h], & t = 0;
\end{cases} \quad \text{(37)}
\]

\[
Q(t; x)[h] = \begin{cases} 
\nabla_x Z(t; x)(I - B)^{1 - \theta} [h], & t > 0; \\
0, & t = 0;
\end{cases} \quad \text{(38)}
\]
Moreover, the map \( (x, h) \mapsto \Pi(\cdot; x)[h] \) is continuous from \( X_\infty \times X \) to \( L^p(\Omega; C_\beta([0, T]; \mathbb{R})) \) and the map \( (x, h) \mapsto Q(\cdot; x)[h] \) is continuous from \( X_\infty \times X \) into \( L^p(\Omega; C_\beta([0, T]; \Xi^*)) \) and both maps are linear with respect to \( h \). Finally, there exists a positive constant \( C \) such that

\[
\mathbb{E}\|\Pi(0; x)\| \leq C \tag{39}
\]

**Proof.** For any \( n \in \mathbb{N} \), we introduce a suitable stochastic differential equation on \([0, n]\) which should give a sequence \( (\Pi^{(n)}(\cdot; x)[h], Q^{(n)}(\cdot; x)[h]) \) of approximating processes for \((\Pi(\cdot; x), Q(\cdot; x))\); more precisely, for fixed \( p \geq 2 \), \( x \in X_\infty \) and \( h \in X \) we consider the equation

\[
\Pi^{(n)}(t; x)[h] = \int_t^n \nu^{(n)}(t; x)h dt + \int_t^n \nabla_x \psi(x(t; x), Z(t; x))Q^{(n)}(t; x)[h] dt - \lambda \int_t^n \Pi^{(n)}(t; x)[h] dt + Q^{(n)}(t; x)[h] dW(t) \quad t \in [0, n] \tag{40}
\]

where

\[
\nu^{(n)}(t; x)h = 1_{[0, n]}(t) \nabla_x \psi(x(t; x), Z(t; x)) \nabla_x x(t; x)(I - B)^{1-\theta} Pgh.
\]

The solution to (40) exists (see [11, Proposition 7.5]) and the maps \( x \mapsto \Pi^{(n)}(\cdot; x)[h] \) are continuous from \( X_\infty \times X \) to \( L^p(\Omega; C([0, n]; X_\infty)) \) and \( Q^{(n)}(\cdot; x)[h] \in L^p(\Omega; C([0, n]; \Xi^*)) \). Further, if \((Y(\cdot; n, x), Z(\cdot; n, x))\) are the processes introduced in Proposition 7.2 then the following identifications hold

\[
\Pi(t; x)^{(n)}[h] = \begin{cases} 
\nabla_x Y(t; x, n)(I - B)^{1-\theta}[h], & t \in [0, n]; \\
\nabla_x Y(0; x, n)(I - B)^{1-\theta}[h], & t = 0;
\end{cases} \tag{41}
\]

\[
Q(t; x)^{(n)}[h] = \begin{cases} 
\nabla_x Z(t; x, n)(I - B)^{1-\theta}[h], & t \in [0, n]; \\
0, & t = 0;
\end{cases} \tag{42}
\]

for any \( x \in X_\infty \) and \( h \in X \). Hence for any \( n \in \mathbb{N} \), \((\Pi^{(n)}(t; x), Q^{(n)}(t; x))_{t \in [0, n]}\) extend the mappings

\[
h \mapsto \nabla_x Y(t; x, n)(I - B)^{1-\theta} Pgh \quad h \mapsto \nabla_x Z(t; x, n)(I - B)^{1-\theta} Pgh, \quad t \in [0, n].
\]

Moreover, we have the estimates

\[
\mathbb{E} \sup_{t \in [0, n]} \|\Pi^{(n)}(t; x)\|_{\mathbb{R}^p}^p e^{-p\beta t} + \mathbb{E} \left( \int_0^n e^{-2\beta r} \|\Pi^{(n)}(r; x)[h]\|^2 \, dr \right)^{p/2} \\
+ \mathbb{E} \left( \int_0^n e^{-2\beta r} \|Q^{(n)}(r; x)[h]\|^2 \, dr \right)^{p/2} < \infty, \tag{43}
\]

for suitable \( \beta > 0 \). It remains to prove that the processes \((\Pi^{(n)}(\cdot; x), Q^{(n)}(\cdot; x))\) converge to some pair \((\Pi(\cdot; x), Q(\cdot; x))\) and that the identifications (37) and (38) hold. For the convergence of \((\Pi^{(n)}(\cdot; x), Q^{(n)}(\cdot; x))\), we notice that \((\Pi^{(n)}(\cdot; x), Q^{(n)}(\cdot; x))\)
solves a BSDE wht bounded coefficients. In fact, by the assumption made on \(\nabla_x\psi, \nabla_z\psi\) and Proposition 6.2, we have
\[
|v^{(n)}(t; x)h| \leq C|h| \quad \text{and} \quad |\nabla_z\psi(x(t); Z(t; x))| \leq C.
\]
Hence, following [30, Theorem 3.1] or Theorem 7.3 above, we conclude that \(\Pi(0; x)h = \lim_{n \to \infty} \Pi^{(n)}(0; x)h\) exists, it is linear, verifies \(|\Pi(0; x)h| \leq C|h|\) for every \(h\) fixed and it is continuous in \(x\) for every \(h\) fixed. Finally, since on \(X_{\theta - 1}\) the processes \((\Pi^{(n)}(\cdot; x), Q^{(n)}(\cdot; x))\) extends the processes
\[
(\nabla_x Y(t; x, n)(I - B)^{1 - \theta} \bar{h}, \nabla_z Z(t; x, n)(I - B)^{1 - \theta} \bar{h}),
\]
on the same space we have
\[
\Pi(0; x)h = \lim_{n \to \infty} \nabla_x Y(0; x, n)(I - B)^{1 - \theta} Pg[h] = \nabla_x Y(0; x)(I - B)^{1 - \theta} [h].
\]

\[\Box\]

**Corollary 7.5.** Setting \(v(x) = Y(0; x)\), we have that \(v\) is Gâteaux differentiable with respect to \(x\) on \(X_\eta\) and the map \((x, h) \mapsto \nabla v(x)[h]\) is continuous.

Moreover, for \(x \in X_\eta\), the linear operator \(h \mapsto \nabla v(x)(I - B)^{1 - \theta} [h]\) - a priori defined for \(h \in \mathcal{D}\) - has an extension to a bounded linear operator from \(X\) into \(\mathbb{R}\), that we denote by \([\nabla v(I - B)^{1 - \theta}](x)\).

Finally, the map \((x, h) \mapsto [\nabla v(I - B)^{1 - \theta} Pg](x)\) is continuous as a mapping from \(X_\eta \times X\) into \(\mathbb{R}\) and there exists \(C > 0\) such that
\[
||\nabla v(I - B)^{1 - \theta}(x)[h]|_X| \leq C|h|_X,
\]
for \(x \in X_\eta, h \in X\).

**Proof.** First of all, we notice that \(Y(0; x)\) is deterministic, since \((\mathcal{F}_t)_{t \geq 0}\) is generated by the Wiener process \(W\) and \(Y(\tau)\) is \(\mathcal{F}_\tau\)-adapted.

Moreover, in Theorem 7.3 we proved that the map \(x \mapsto Y(\cdot; x)\) is continuous and Gâteaux differentiable with values in \(L^p(\Omega; C([0, T]; \mathbb{R}))\); consequently, it follows that \(x \mapsto v(x) = Y(0; x)\) is Gâteaux differentiable with values in \(\mathbb{R}\).

Next, we notice that \(\Pi(0; x) = \nabla_x Y(0; x)(I - B)^{1 - \theta} [h]\). The existence of the required extension and its continuity are direct consequence of Proposition 7.3 and estimate (51) follows from (49).

\[\Box\]

We are now in the position to give a meaning to the expression \(\nabla_x Y(t; x)(I - B)^{1 - \theta}\) and, successively, to identify it with the process \(Z(t; x)\). To this end we quote from [11] a preliminary result where we investigate the existence of the joint quadratic variation of \(W\) with a process of the form \(\{w(t, x(t)) : t \in [0, T]\}\) for a given function \(w : [0, T] \times X \to \mathbb{R}\), on an interval \([0, s] \subset [0, T]\). In order to simplify the exposition we omit the proof. We only notice that the proof requires the study of the Malliavin derivative of \(x\) and this can be done in the same way as in [11, Section 6.1].
Proposition 7.6. Suppose that $w \in C([0,T] \times X_\eta; \mathbb{R})$ is Gâteaux differentiable with respect to $x$, and that for every $s < T$ there exist constant $K$ (possibly depending on $s$) such that

$$|w(t,x)| \leq K, \quad |\nabla w(t,x)| \leq K, \quad t \in [0,s], x \in X. \quad (45)$$

Let $\eta$ and $\theta$ satisfy condition (11) in Theorem 3.4. Assume that for every $t \in [0,T)$, $x \in X_\eta$ the linear operator $k \mapsto \nabla w(t,x)(I - B)^{-\theta}k$ (a priori defined for $k \in D$) has an extension to a bounded linear operator $X \rightarrow \mathbb{R}$, that we denote by $[\nabla w(I - B)^{-\theta}](t,x)$. Moreover, assume that the map $(t,x,k) \mapsto [\nabla w(I - B)^{-\theta}](t,x)k$ is continuous from $[0,T] \times X_\eta \times X$ into $\mathbb{R}$. For $t \in [0,T), x \in X_\eta$, let $\{x(t,s,x), t \in [s,T]\}$ be the solution of equation (20). Then the process $\{w(t,x(t,s,x)), t \in [s,T]\}$ admits a joint quadratic variation process with $\mathcal{W}$, for every $j \in \mathbb{N}$, on every interval $[s,t] \subset [s,T)$, given by

$$\int_s^t [\nabla w(I - B)^{-\theta}](r,x(r,s,x))(I - B)^{\theta}P_{gej}dr. \quad (46)$$

the above result allows to identify the process $Z(\cdot;x)$ with $[\nabla v(I - B)^{-\theta}](x(\cdot;x))(I - B)^{\theta}P_g$, as we can see in the result below.

Corollary 7.7. For every $x \in X_\eta$ we have, $\mathbb{P}$-a.s.

$$Y(t,x) = v(x(t,x)), \quad \text{for all } t \geq 0; \quad \text{for almost all } t \geq 0. \quad (47)$$

Proof. We need to consider the equation (which is slightly more general than (43))

$$\begin{cases}
\mathbf{d}x(t) = Bx(t)dt + (I - B)P\mathbf{f}(Jx(t))dt \\
\mathbf{x}(s) = x,
\end{cases} \quad (48)$$

for $t$ varying $[s, \infty)$. We set $x(t) = x$ for $t \in [0,s)$ and we denote by $(x(t,s,x))_{t \geq 0}$ the solution, to indicate dependence on $x$ and $s$. By an obvious extension of the results concerning the BSDE (44) stated above, we can solve the backward equation (44) with $x$ given by (48); we denote the corresponding solution by $(Y(t,s,x), Z(t,s,x))$ for $t \geq 0$. Thus $(Y(t,x), Z(t;x))$ coincides with $(Y(t,0,x), Z(t,0,x))$ that occurs in the statement of Proposition 7.2.

Now let us prove (46). We start from the well-known equality: for $0 \leq r \leq T$, $\mathbb{P}$-a.s.

$$x(t,s,x) = x(t,r,x(r,s,x)), \quad \text{for all } t \in [s,T].$$

It follows easily from the uniqueness of the backward equation (44) that $\mathbb{P}$-a.s.

$$Y(t,s,x) = Y(t,r,x(r,s,x)), \quad \text{for all } t \in [s,T].$$
In particular, for every $0 \leq r \leq s \leq t$,
\begin{align*}
Y(t; s, x(s; r, x)) &= Y(t; r, x), \quad t \in [s, \infty), \\
Z(t; s, x(s; r, x)) &= Z(t; r, x), \quad t \in [s, \infty).
\end{align*}

Since the coefficients of (48) do not depend on time, we have
\begin{align*}
x(\cdot; 0, x) \overset{(d)}{=} x(\cdot + t, t, x), \quad t \geq 0,
\end{align*}
where $\overset{(d)}{=}$ denotes equality in distribution. As a consequence we obtain
\begin{align*}
(Y(\cdot; 0, x), Z(\cdot; 0, x)) \overset{(d)}{=} (Y(\cdot + t, t, x), Z(\cdot + t, t, x)), \quad t \geq 0,
\end{align*}
where both sides of the equality are viewed as random elements with values in the space $C(\mathbb{R}_+; \mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{X}^*)$. In particular
\begin{align*}
Y(0; 0, x) \overset{(d)}{=} Y(t; t, x),
\end{align*}
and since they are both deterministic, we have
\begin{align*}
Y(0; 0, x) = Y(t; t, x), \quad x \in X_\eta, t \geq 0
\end{align*}
so that we arrive at (46).

To prove (47) we consider the joint quadratic variation of $(Y(t; x))_{t \in [0, T]}$ and $W$ on an arbitrary interval $[0, t] \subset [0, T)$; from the backward equation (34) we deduce that it is equal to $\int_0^t Z(r; x)dr$. On the other side, the same result can be obtained by considering the joint quadratic variation of $(v(x(t; x)))_{t \in [0, T]}$ and $W$. Now by an application of Proposition 7.6 (whose assumptions hold true by Corollary 7.5) leads to the identity
\begin{align*}
\int_0^t Z(r; x)dr = \int_0^t [\nabla v(I - B)^{-1}](r; x(r; x))(I - B)^{\theta}P_g(x(r; x))dr,
\end{align*}
and (47) is proved. \hfill \square

8. The stationary Hamilton Jacobi Bellman equation

Now we proceed as in [25]. Let us consider again the solution $x(t; x)$ of equation (13) and denote by $(P_t)_{t \geq 0}$ its transition semigroup:
\begin{align*}
P_t[h](x) = \mathbb{E}h(x(t; x)), \quad x \in X_\eta, 0 \leq t,
\end{align*}
for any bounded measurable $h : X \to \mathbb{R}$. We notice that by the bound (47) this formula is true for every $h$ with polynomial growth. In the following $P_t$ will be considered as an operator acting on this class of functions.
Let us denote by $L$ the generator of $P_t$:

$$L[h](x) = \frac{1}{2} \text{Tr}[(I - B) Pg \nabla^2 h(x) g^* (I - B)^*] + \langle Bx + (I - B) Pf(Jx), \nabla h(x) \rangle,$$

where $\nabla h$ and $\nabla^2 h$ are first and second Gâteaux derivatives of $h$ at the point $x \in X$ (here we are identified with elements of $X$ and $L(X)$ respectively). This definition is formal, since it involves the terms $(I - B) Pg$ and $(I - B) Pf$ which - a priori - are not defined as elements of $L(X)$ and the domain of $L$ is not specified.

In this section we address solvability of the nonlinear stationary Kolmogorov equation:

$$L[v(\cdot)](x) = \lambda v(x) - \psi(x, \nabla v(x)(I - B) Pg), \quad x \in X. \quad (HJB)$$

This is a nonlinear elliptic equation for the unknown function $v : X_\eta \to \mathbb{R}$. We define the notion of solution of the (HJB) by means of the variation of constant formula:

**Definition 8.1.** We say that a function $v : X_\eta \to \mathbb{R}$ is a mild solution of the Hamilton - Jacobi - Bellman equation (HJB) if the following conditions hold:

1. $v \in C(X; \mathbb{R})$ and there exist $C \geq 0$ such that $|v(x)| \leq C, \ x \in X$;
2. $v$ is Gâteaux differentiable with respect to $x$ on $X_\eta$ and the map $(x, h) \mapsto \nabla v(x)(h)$ is continuous $X_\eta \times X \to \mathbb{R}$;
3. For all $x \in X_\eta$ the linear operator $k \mapsto \nabla v(x)(I - B)^{1-\theta} k$ (a priori defined for $k \in D$) has an extension to a bounded linear operator on $X \to \mathbb{R}$, that we denote by $[\nabla v(I - B)^{1-\theta}](x)$. Moreover the map $(x, k) \mapsto [\nabla v(I - B)^{1-\theta}](x)k$ is continuous $X_\eta \times X \to \mathbb{R}$ and there exist constants $C \geq 0$ such that

   $$\|\nabla v(I - B)^{1-\theta}(x)\|_{L(X)} \leq C, \quad x \in X_\eta,$$

4. The following equality holds for every $x \in X_\eta$:

   $$v(x) = e^{-\lambda T} P_T[v](x) - \int_0^T e^{-\lambda s} P_s \left[\psi(\nabla v(I - B)^{1-\theta})(I - B)^{\theta} Pg)(x)\right] ds. \quad (49)$$

**Remark 8.2.** We notice that, by assumption, $|\psi(x, z)| \leq C$; moreover, we saw in the preceding sections how to give a meaning to the term

$$\nabla v(x)(I - B)Pg.$$

Hence if $v$ is a function satisfying the bound required in Definition 8.1, we have

$$|\psi(x, [\nabla v(I - B)^{1-\theta}](x)(I - B)^{\theta} Pg)| \leq C$$

and formula (49) is meaningful.
Now we are ready to prove that the solution of the equation (HJB) can be defined by means of the solution of the BSDE associated with the control problem (13).

**Theorem 8.3.** Assume Hypotheses 2.1, 2.2, 2.3 and 7.1; then there exists a unique mild solution of the HJB equation. The solution is given by the formula

$$v(x) = Y(0; 0, x) = Y(t; t, x), \quad t \geq 0,$$

where \((x, Y, Z)\) is the solution of the forward-backward system (13) and (34).

**Proof.** We start by proving existence. In particular, we prove that \(v\), given by (50), is a solution of HJB. Hence, we set:

$$v(x) := Y(0; 0, x).$$

By Corollary 7.7 the function \(v\) has the regularity properties stated in Definition 8.1. In order to verify that equality (49) holds we first fix \(x \in X_\eta\). We notice that

$$\psi(\cdot, [\nabla v(I - B)^{1-\theta}] (\cdot)(I - B)^\theta P g)(x) = \psi(\cdot, [\nabla Y(I - B)^{1-\theta}] (\cdot)(I - B)^\theta P g)(x)$$

and we recall that

$$[\nabla v(I - B)^{1-\theta}](x(t; 0, x))(I - B)^\theta P g = Z(t; 0, x), \quad 0 \leq t.$$

Hence

$$P_t \left[ \psi(\cdot, [\nabla v(I - B)^{1-\theta}] (\cdot)(I - B)^\theta P g) (x(t; 0, x)) \right] = \mathbb{E}[\psi(x(t; 0, x), Z(t; 0, x))].$$

On the other hand, applying the Itô formula to the backward equation gives

$$e^{-\lambda t}Y(t; t, x) - e^{-\lambda T}Y(T; t, x) + \int_t^T e^{-\lambda r}Z(r; t, x)dW(r)$$

$$= - \int_t^T e^{-\lambda r} \psi(x(r; t, x), Z(r; t, x))dr,$$

for any \(0 \leq t \leq T < \infty\). Taking the expectation and recalling again Corollary 7.7 we obtain

$$e^{-\lambda t}v(x) = e^{-\lambda T} \mathbb{E}v(x(T; t, x)) - \mathbb{E} \int_t^T e^{-\lambda r} \psi(x(r; t, x), Z(r; t, x))dr$$

and substituting in the integral the expression obtained in (51) we get the required equality (49). This completes the proof of the existence part.
Now we consider uniqueness of the solution. Let \( v \) denote a mild solution. We look for a convenient expression for the process \( v(x(t; t, x)) \), \( 0 \leq t \leq r \leq T < \infty \). By (49) and the definition of \( P_t \), for any \( y \in X \) we have

\[
v(y) = e^{-\lambda(T-t)}E[v(x(T - t; 0, y))] \\
- \int_{0}^{T-t} e^{-\lambda r} E \left[ \psi(x(r; 0, y), \nabla v(I - B)^{1-\theta}(r, x(r; 0, y))(I - B)^{\theta} P_g) \right] dr.
\]

Set \( y = x(t; 0, x) \). Then equality (52) rewrites as

\[
v(x(t; 0, x)) = e^{-\lambda(T-t)}E[v(x(T - t; 0, x(t; 0, x)))] \\
- \int_{0}^{T-t} e^{-\lambda r} E \left[ \psi(x(r; 0, x(t; 0, x)), \nabla v(I - B)^{1-\theta}(x(r; 0, x(t; 0, x)))(I - B)^{\theta} P_g) \right] dr.
\]

Moreover, recalling that for any \( r \in [t, T] \) the equality

\[x(r; t, x(t; 0, x)) = x(r; 0, x)\]

hold \( \mathbb{P} \)-a.s. we obtain

\[
v(x(t; 0, x)) = e^{-\lambda(T-t)}E[v(x(T; t, x))] \\
- \int_{0}^{T-t} e^{-\lambda r} E \left[ \psi(x(r + t; 0, x), \nabla v(I - B)^{1-\theta}(x(r + t; 0, x))(I - B)^{\theta} P_g) \right] dr.
\]

Since \( x(t; 0, x) \) is \( \mathcal{F}_t \)-measurable, we can replace the expectation by the conditional expectation given \( \mathcal{F}_t \):

\[
e^{-\lambda T}v(x(t; 0, x)) = e^{-\lambda T} E_{\mathcal{F}_t} \left[ v(x(T; t, x)) \right] \\
- E_{\mathcal{F}_t} \left[ \int_{t}^{T} e^{-\lambda \tau} \psi(x(r + t; 0, x), \nabla v(I - B)^{1-\theta}(x(r + t; 0, x))(I - B)^{\theta} P_g) d\tau \right].
\]

and, by change of variable, we get:

\[
e^{-\lambda T}v(x(t; 0, x)) = e^{-\lambda T} E_{\mathcal{F}_t} \left[ v(x(T; t, x)) \right] \\
- E_{\mathcal{F}_t} \left[ \int_{t}^{T} \psi(x(r; 0, x), \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P_g) d\tau \right]
= E_{\mathcal{F}_t} \left[ \xi \right] \\
+ E_{\mathcal{F}_t} \left[ \int_{0}^{T} \psi(x(r; 0, x), \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P_g) d\tau \right],
\]

where we have defined

\[
\xi := e^{-\lambda T}v(x(T; t, x)) \\
- \int_{0}^{T} \psi(x(r; 0, x), \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P_g) d\tau.
\]
9. Synthesis of the optimal control

Since we assume polynomial growth for \( v \) and \( \nabla v \), therefore \( \xi \) is square integrable. Since \( (\mathcal{F}_t)_{t \geq 0} \) is generated by the Wiener process \( W \), it follows that there exists \( \bar{Z} \in L^2_T(\Omega \times [0, T]; L_2(\mathbb{R})) \) such that

\[
\mathbb{E}[\xi] = \mathbb{E}[\xi] + \int_0^t \bar{Z}(r) dW(r), \quad t \in [0, T].
\]

An application of the Itô formula gives

\[
v(x(t; 0, x)) = \mathbb{E}[\xi] + \int_0^t e^{\lambda r} \bar{Z}(r) dW(r) + \lambda \int_0^t \mathbb{E}[\xi] + \int_0^t \psi(r, x(r; t, x), \nabla v(I - B)^{1-\theta})(x(r; t, x))(I - B)^{\theta} P g dW(r), \quad (53)
\]

We conclude that the process \( v(x(t; s, x)) \), \( t \in [s, T] \) is a real continuous semimartingale.

For \( \xi \in \mathbb{R} \), let us define \( W^\xi \) by \( W^\xi(t) = (\xi, W(t)) \) and let us consider the joint quadratic variation process of \( W^\xi \) with both sides of (53). Applying Proposition 7.3, we obtain, \( \mathbb{P} \)-a.s.,

\[
\int_0^t \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P g dW(r) = \int_0^t \bar{Z}(r) dr.
\]

Therefore, for a.a. \( t \in [0, T] \), we have \( \mathbb{P} \)-a.s. \( \mathbb{E}[\xi] = \int_0^t \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P g dW(r) = \bar{Z}(r) \), so substituting into (53) we obtain, for \( t \in [0, T] \),

\[
v(x(t; 0, x)) = v(x) + \int_0^t \nabla v(I - B)^{1-\theta}(x(r; t, x))(I - B)^{\theta} P g dW(r)
\]

and

\[
\int_0^t \psi(r, x(r; t, x), \nabla v(I - B)^{1-\theta})(x(r; 0, x))(I - B)^{\theta} P g dW(r).
\]

Comparing with the backward equation (54) we notice that the pairs

\( (Y(t; 0, x), Z(t; 0, x)) \)

and

\( (v(x(t; 0, x)), \nabla v(I - B)^{1-\theta}(x(r; 0, x))(I - B)^{\theta} P g = \bar{Z}(r)) \)

solve the same equation. By uniqueness, we have \( Y(t; 0, x) = v(x(t; 0, x)), t \in [0, T] \), and setting \( t = 0 \) we obtain \( Y(t; t, x) = v(x) \).

9. Synthesis of the optimal control

In this section we proceed with the study of the optimal control problem associated with the stochastic Volterra equation

\[
\begin{align*}
\frac{d}{dt} \int_{-\infty}^{t} a(t - s) u(s) ds &= Au(t) + f(u(t)) \\
&\quad + g(u(t)) [r(u(t), \gamma(t)) + W(t)], \quad t \in [0, T] \\
u(t) &= u_0(t), \quad t \leq 0.
\end{align*}
\]

(54)
for a process $u$ with values in the Hilbert space $H$. Here $f$ and $r$ are the nonlinear functions introduced in Hypothesis 2.1 and $\gamma = \gamma(\omega, t)$ is the control variable, which is assumed to be a predictable real-valued process $\mathcal{F}_t$-adapted. The optimal control that we wish to treat consists in minimizing over all admissible controls a cost functional of the form
\[ J(u_0, \gamma) = \mathbb{E} \int_0^\infty e^{-\lambda t} \ell(u(t), \gamma(t)) dt, \] (55)
where $\ell : H \times U \to \mathbb{R}$ is a given real-valued function.

We will work under assumptions 2.1 and 7.1.

To handle the control problem, we first restate equation (54) in an evolution setting and we provide the synthesis of the optimal control by using the forward-backward system approach.

As it has been proved in Section 3, given a control process $\gamma$ and any $u_0 \in H$, we can rewrite the problem (54) in the following abstract form
\[
\begin{align*}
\{ & \quad dx(t) = Bx(t)dt + (I - B)f(Jx(t))dt \\
& \quad + (I - B)Pg(r(Jx(t), \gamma(t))dt + dW(t)) \\
& \quad x(0) = x.
\end{align*}
\]
Here $X$ is a suitable separable Hilbert space, $B$ is a densely defined sectorial operator on a domain $D(B) \subset X$, $P$ is a linear operator from $H$ with values into a real interpolation space $X_{\theta}$ ($\theta \in (0, 1)$) between $D(B)$ and $X$ and $J$ is a linear operator from $X_{\eta}$ ($\eta \in (0, 1)$) into $H$; finally $x \in X_{\eta}$ (see Theorem 3.4 for more details).

In this setting the cost functional will depend on $x$ and $\gamma$ and is given by
\[ J(x, \gamma) = \mathbb{E} \int_0^\infty e^{-\lambda s} \ell(Jx(s), \gamma(s)) ds; \] (56)
(with an abuse of notation we still denote the rewritten cost functional as $J$).

We notice that for all $\lambda > 0$ the cost functional is well defined and $J(x, \gamma) < \infty$ for all $x \in X_{\eta}$ and all a.c.s. There are different ways to give precise meaning to the above problem; one of them is the so called weak formulation and will be specified below.

In the weak formulation the class of admissible control system (a.c.s.) is given by the set $U := (\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, W, \tilde{\gamma})$, where $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a complete probability space; the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ verifies the usual conditions, the process $\tilde{W}$ is a Wiener process with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ and the control $\tilde{\gamma}$ is an $\mathcal{F}_t$-predictable process taking values in some subset $U$ of $X$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. With an abuse of notation, for given $x \in X_{\eta}$, we associate to every a.c.s. a cost functional $J(x, U)$ given by the right side of (56). Although formally the same, it is important to note that now the cost is a functional of the a.c.s. and not a functional of $\tilde{\gamma}$ alone. Any a.c.s. which minimizes $J(x, \cdot)$, if it exists, is called optimal for the control problem starting from $x$ at time $t$. 

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the weak formulation. The minimal value of the cost is then called the optimal cost. Finally we introduce the value function \( V : X_\eta \to \mathbb{R} \) of the problem as:

\[
V(x) = \inf_{\gamma \in U} J(x, \gamma), \quad x \in X_\eta,
\]

where the infimum is taken over all a.c.s. \( U \).

Optimal control problems on infinite horizon of the type considered in this paper (with coefficients having the properties listed in Hypotheses 2.1, 2.3, 7.1) have been exhaustively studied by Fuhrman and Tessitore in [24, 25] and Hu and Tessitore in [30], compare Theorem 5.1. Within their approach the existence of an optimal control is related to the existence of the solution of a suitable forward backward system (FBSDE) that is a system in which the coefficients of the backward equation depend on the solution of the forward equation. Moreover, the optimal control can be selected using a feedback law given in terms of the solution to the corresponding FBSDE.

We recall that the Hamiltonian corresponding to our control problem is given by

\[
\psi(x, z) = \inf_{\gamma \in U} \{ \ell(Jx, \gamma) + z \cdot r(Jx, \gamma) \}, \quad x \in X_\eta, z \in \Xi^*,
\]

and we define the following set

\[
\Gamma(x, z) = \{ \gamma \in U : \ell(Jx, \gamma) + z \cdot r(Jx, \gamma) = \psi(x, z) \},
\]

\[
t \in [0, T], \quad x \in X_\eta, \quad z \in \Xi^*.
\]

(57)

For further use we require an additional property of the function \( \psi \):

**Hypothesis 9.1.** For all \( x \in X_\eta, \ z \in \Xi^* \) there exists a unique \( \Gamma(x, z) \) that realizes the minimum in (57). Namely:

\[
\psi(x, z) = \ell(Jx, \Gamma(x, z)) + z \cdot r(Jx, \Gamma(x, z)),
\]

with \( \Gamma \in C(X_\eta \times \Xi^*; U) \).

Now, let us consider an arbitrary set-up \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W})\) and

\[
\tilde{x}(t) = e^{tB}x + \int_0^t e^{(t-\sigma)B}(I - B)Pf(\tilde{x}(\sigma))d\sigma + \\
\int_0^t e^{(t-\sigma)B}(I - B)Pg d\tilde{W}(\sigma), \quad 0 \leq t \leq T < \infty.
\]

(58)

By Theorem 4.2 stated in Section 4 equation (58) is well-posed and the solution \((\tilde{x}(t))_{t \geq 0}\) is a continuous process in \( X_\eta \), adapted to the filtration \((\tilde{\mathcal{F}}_t)_{t \geq 0}\). Moreover, the law of \((\tilde{W}, \tilde{x})\) is uniquely determined by \( x, B, f \) and \( g \). We define the process

\[
\tilde{W}^U(t) = \tilde{W}(t) - \int_0^t r(\tilde{x}(s), \tilde{\gamma}(s))ds, \quad 0 \leq t \leq T,
\]

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and we note that, since \( r \) is bounded, by the Girsanov theorem there exists a probability measure \( P \) on \( (\Omega, \mathcal{F}) \) such that \( W^U \) is a Wiener process under \( P \). Rewriting equation (58) in terms of \( W^U \) we get that \( \tilde{x} \) solves the controlled state equation (in weak sense)

\[
\tilde{x}(t) = x + \int_0^t e^{(t-\sigma)B(I-B)P(\tilde{x}(\sigma))}d\sigma + \int_0^t e^{(t-\sigma)B(I-B)Pgd\tilde{W}^U(\sigma)} + \int_0^t e^{(t-\sigma)A(I-B)Pr(\tilde{x}(s),\tilde{\gamma}(s))}d\sigma.
\]

We notice that for all \( \lambda > 0 \) the cost functional is well defined and \( J(x,\gamma) < \infty \) for all \( x \in X_\eta \) and all a.c.s. \( U \).

By Theorem 8.3, for all \( \lambda > 0 \) the stationary Hamilton-Jacobi-Bellman equation relative to the above stated problem, namely:

\[
Lv(x) = \lambda v(x) - \psi(x,\nabla v(x)(I-B)P), \quad x \in X_\eta \quad (HJB)
\]

admits a unique mild solution, in the sense of Definition 8.1. Here \( L \) is the infinitesimal generator of the Markov semigroup corresponding to the process \( x \):

\[
L\phi(x) = \frac{1}{2}\text{Tr}((I-B)Pg^*P^*g(I-B)^*\nabla^2\phi(x)) + \langle Bx + (I-B)Pf(x), \nabla\phi(x) \rangle.
\]

Let \( v \) be the unique mild solution of equation (HJB). Consider the following finite horizon backward equation (with respect to probability \( \tilde{P} \) and to the filtration generated by \( \{\tilde{W}_t : t \in [0,T]\} \)):

\[
\tilde{Y}(t) - \nu(\tilde{x}(T)) + \int_t^T \tilde{Z}d\tilde{W}(\sigma) = \int_t^T \psi(\tilde{x}(\sigma),\tilde{Z}(\sigma))d\sigma - \lambda \int_t^T \tilde{Y}(\sigma)d\sigma, \quad (60)
\]

where \( \psi \) is the Hamiltonian function. It was proved in Section 7 that there exists a solution \( (\tilde{x},\tilde{Y},\tilde{Z}) \) of the forward-backward system (58)-(60) on the interval \( [0,T] \), where \( \tilde{Y} \) is unique up to indistinguishability and \( Z \) is unique up to modification. Moreover from the proof of Theorem 4.8 it follows that the law of \( (\tilde{Y},\tilde{Z}) \) is uniquely determined by the law of \( (\tilde{W},\tilde{x}) \) and \( \psi \). To stress the dependence on the initial datum \( x \), we will denote the solution of (58) and (60) by \( \{(\tilde{x}^x(t),\tilde{Y}^x(t),\tilde{Z}^x(t)), t \in [0,T]\} \). Theorem 8.3 and uniqueness of the solution of system (58)-(60), yields that

\[
\tilde{Y}^x(t) = v(\tilde{x}^x(t)), \quad \tilde{Z}^x(t) = [\nabla v(I-B)^{1-\theta}|(x^x(t))(I-B)^\theta]P.
\]

Hence the solution of the (HJB) equation is the value function of the control problem; this allows to construct the optimal feedback law. The relevance of the Hamilton-Jacobi-Bellman equation to our control problem is explained in the following main result of this section:
Theorem 9.2. Assume Hypotheses 7.1 and 9.1 and suppose that \( \lambda > 0 \). Then the following holds:

1. For all a.c.s. we have \( J(x, \gamma) \geq v(x) \).
2. The equality holds if and only if the following feedback law is verified by \( \gamma \) and \( \tilde{x}^\gamma \):

\[
\gamma(t) = \Gamma(\tilde{x}(t), \nabla v(\tilde{x}(t))(I-B)Pg), \quad \mathbb{P} \text{ - a.s. for } a.e. t \geq 0.
\] (61)

Finally, there exists at least an a.c.s. \( \mathcal{U} \) verifying (61). In such a system, the closed loop equation admits a solution

\[
\begin{aligned}
d\tilde{x}(t) &= B\tilde{x}(t)dt + (I-B)Pg\tilde{x}(t))dt + (I-B)Pg \left( r(\tilde{x}(t), \Gamma(\tilde{x}(t), [\nabla v(\tilde{x}(t))(I-B)^{1-\theta}(I-B)^{\theta}Pg])dt + d\tilde{W}^\mathcal{U}(t) \right), \quad t \geq 0 \\
\tilde{x}(0) &= x \in X_\eta,
\end{aligned}
\] (62)

and if \( \tilde{\gamma}(t) = \Gamma(\tilde{x}(t), [\nabla v(\tilde{x}(t))(I-B)^{1-\theta}(I-B)^{\theta}Pg) \) then the couple \( (\tilde{\gamma}, \tilde{x}) \) is optimal for the control problem.

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