Deficit of reactor antineutrinos
at distances smaller than 100 m and inverse $\beta$–decay

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We analyse a change in a deficit of reactor antineutrinos at distances smaller than 100 m by changing the lifetime of the neutron from $\tau_n = 885.7$ s to $\tau_n = 879.6$ s, calculated for the axial coupling constants $\lambda = -1.2694$ and $\lambda = -1.2750$, respectively, in order to get a result corresponding to the new world average value $\tau_n = 880.1(1.1)$ s. We calculate the angular distribution and cross section for the inverse $\beta$–decay, taking into account the contributions of the “weak magnetism” and the neutron recoil to next–to–leading order in the large baryon mass expansion and the radiative corrections of order $\alpha/\pi \sim 10^{-3}$, calculated to leading order in the large baryon mass expansion. We obtain an increase of a deficit of reactor antineutrinos of about 0.73%. We discuss a universality of radiative corrections to order $\alpha$ to the neutrino (antineutrino) reactions induced by weak charged currents, pointed out by Kurylov, Ramsey-Musolf and Vogel (Phys. Rev. D 67, 035502 (2003)), and calculate the antineutrino–energy spectrum of the neutron $\beta^–$–decay to order $\alpha/\pi$ and taking into account the contributions of the “weak magnetism” and the proton recoil.

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I. INTRODUCTION

In this paper we analyse a deficit of the reactor antineutrinos at distances smaller than 100 m from reactors. As has been pointed out in Ref.[1] the ratio of the observed event rate of antineutrinos, emitted by reactor and detected at distances smaller than 100 m, to the predicted rate is of about 0.943(23). This implies an existence of a deficit of antineutrinos of about 5.7%. Such a deficit of antineutrinos may be, for example, explained by the electron–sterile antineutrino oscillations, where a mass of sterile antineutrinos is of about $m_\nu \sim 1\text{ eV}$ [1] (see also Ref.[2]). The experiment on the observation of the sterile antineutrinos from reactors at distances $6 – 13\text{ m}$ has been recently proposed by in Ref.[3].

The yield of reactor antineutrinos is being detected by the inverse $\beta$–decay $\bar{\nu}_e + p \rightarrow n + e^+$ in terms of the yield of the positrons, produced by antineutrinos in the energy region $2\text{ MeV} \leq E_\nu \leq 8\text{ MeV}$ [1]. The calculation of the angular distribution and cross section of the inverse $\beta$–decay $\bar{\nu}_e + p \rightarrow n + e^+$, induced by reactor antineutrinos, was calculated in Ref.[4] in the non–relativistic approximation and without radiative corrections. Then, in Refs.[5, 6] the obtained results were applied to the experimental analysis of the limits on the parameters of the electron antineutrino oscillations. The account for the contributions of the “weak magnetism” and the neutron recoil, calculated to next–to–leading order in the large baryon mass expansion, and the radiative corrections, calculated to leading order in the large baryon mass expansion, has been carried out in Refs.[7]–[12]. A comparison of the radiative and recoil corrections in the neutron $\beta^–$–decay $n \rightarrow p + e^- + \bar{\nu}_e$ to the inverse $\beta$–decay $\bar{\nu}_e + p \rightarrow n + e^+$ has been performed in Ref.[13] within the heavy–baryon chiral perturbation theory (HB$\chi$PT). The same approach has been also used in [10]–[13]. The authors of the papers [7]–[12] discussed the results on the inverse $\beta$–decay in connection with measurements of the $\theta_{13}$ mixing angle of the antineutrino mass eigenstates and electron–sterile antineutrino oscillations [2].

A reactor antineutrino deficit of about 5.7% has been observed in [1] at the level of 98.6% C.L. by using the theoretical cross section for the inverse $\beta^–$–decay, calculated for the lifetime of the neutron $\tau_n = 885.7$ s or the axial coupling constant $\lambda = -1.2694$. It is important to emphasise that the lifetime of the neutron $\tau_n = 885.7$ s disagrees with recent world average value $\tau_n = 880.1(1.1)$ s [2]. Thus one may expect [3] that a reduction of the lifetime of the neutron from $\tau_n = 885.7$ s to $\tau_n = 880.1(1.1)$ s might lead to an increase of a deficit of reactor antineutrinos of about

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0.7%. Since the theoretical value $\tau_n = 879.6\text{ s}$ of the lifetime of the neutron, calculated in Ref.[14], agrees well with the world average one $\tau_n = 880.1(1.1)\text{ s}$ [2], below we follow Ref.[14] and calculate the inverse $\beta$–decay.

Thus, in connection with an analysis of a deficit of reactor antineutrinos we calculate the angular distribution and cross section for the inverse $\beta$–decay by taking into account the contributions of the “weak magnetism” and the neutron recoil to next–to–leading order in the large baryon mass expansion and the radiative corrections of order $\alpha/\pi$, calculated to leading order in the large baryon mass expansion, where $\alpha = 1/137.036$ is the fine–structure constant [2].

The paper is organised as follows. In section II we give a numerical analysis of the cross section for the inverse $\beta$–decay, calculated in Appendices A, B and C. We discuss the yield of positrons, induced by reactor antineutrinos, in connection with a deficit of reactor antineutrinos, observed in [1]. In section III we analyse the asymmetry of the inverse $\beta$–decay as an alternative method for measurements of the axial coupling constant $\lambda$ and the determination of the correlation coefficient $a_0$, describing in the neutron $\beta^–$–decay correlations between the 3-momenta of the electron and antineutrino (see [15, 16] and [14]). We calculate also the average value $\langle \cos \theta_{e\bar{\nu}} \rangle$ as a function of the antineutrino energy. In section II we confirm a universality of the radiative corrections, calculated to order $\alpha/\pi \sim 10^{-3}$, to the neutrino (antineutrino) reactions with the electron (positron) in the final state, pointed out by Kurylov, Ramsey-Musolf and Vogel [17, 18]. We show that the radiative corrections to the cross sections for the reactions of the neutrino (antineutrino) disintegration of the deuterium, calculated in [17, 18], coincide with the radiative corrections to the cross section for the inverse $\beta$–decay.

In addition we have calculated the antineutrino–energy spectrum of the neutron $\beta^–$–decay by taking into account the radiative corrections to order $\alpha/\pi$ and the contributions of the “weak magnetism” and the proton recoil to next–to–leading order in the proton mass expansion. In section V we summarise the obtained results and discuss their connection to possible existence of sterile neutrinos. In Appendices A, B and C we give detailed calculations i) of the correlation coefficients $A(E_\nu), B(E_\nu)$ and $C(E_\nu)$ (see Eq.(1)) to next–to–leading order in the large baryon mass expansion, caused by the “weak magnetism” and the neutron recoil, ii) of the radiative corrections to the correlation coefficient $A(E_\nu)$ (see Eq.(1)) or to the cross section for the inverse $\beta$–decay (see Eq.(4)) and iii) of the radiative corrections to the correlation coefficient $B(E_\nu)$ (see Eq.(1)) or to the asymmetry $B_{\exp}(E_\nu)$ (see Eq.(7)) and $\langle \cos \theta_{e\bar{\nu}} \rangle$ (see Eq.(8)). In Appendix D we calculate the cross section for the radiative inverse $\beta$–decay by taking into account the contributions of the proton–photon interaction to leading order in the large proton mass expansion. Such contributions are important for a gauge invariance of the amplitude and the final expression for the cross section for the radiative inverse $\beta$–decay.

II. DEFICIT OF REACTOR ANITNEUTRINOS

The angular distribution or the differential cross section for the inverse $\beta$–decay can be written in the following general form (see Appendix A and Eq.(A-70))

$$\frac{d\sigma(E_\nu, \cos \theta_{e\bar{\nu}})}{d \cos \theta_{e\bar{\nu}}} = \frac{1}{2} \sigma_0 \left( A(E_\nu) \left( 1 + \frac{\alpha}{\pi} f_A(\bar{E}) \right) + B(E_\nu) \left( 1 + \frac{\alpha}{\pi} f_B(\bar{E}) \right) \beta \cos \theta_{e\bar{\nu}} + C(E_\nu) \beta^2 \cos^2 \theta_{e\bar{\nu}} \right) \bar{k} E, \quad (1)$$

where $\theta_{e\bar{\nu}}$ is an antineutrino–positron correlation angle, $\bar{E} = E_\nu - \Delta$, $\bar{k} = \sqrt{(E_\nu - \Delta)^2 - m_e^2}$ and $\bar{\beta} = \bar{k}/\bar{E}$ are the energy, momentum and velocity of the positron with $\Delta = m_n - m_p = 1.2934 \text{ MeV}$, calculated at $m_n = 939.5654 \text{ MeV}$ and $m_p = 938.2720 \text{ MeV}$ [2]. The correlation coefficients $A(E_\nu), B(E_\nu)$ and $C(E_\nu)$ are calculated to next–to–leading order in the large baryon mass or large $M$ expansion, caused by the “weak magnetism” and the neutron recoil (see Appendix A), where $M = (m_n + m_p)/2$ is the average nucleon mass [14]. To leading order in the large $M$ expansion the correlation coefficients are equal to $A(E_\nu) = 1, B(E_\nu) = a_0$ and $C(E_\nu) = 0$, where $a_0$ is the correlation coefficient of correlations between the 3–momenta of the electron and antineutrino in the neutron $\beta^–$–decay [15]. The constant $\sigma_0$ is equal to

$$\sigma_0 = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{\pi} \left( 1 + \Delta_R \right) = \frac{2\pi^2}{\tau_n f(E_\nu, Z = 1)} \frac{1}{(1 + \delta_R)}, \quad (2)$$

where $\lambda$ is the axial coupling constant [15], $G_F = 1.1664 \times 10^{-11} \text{ MeV}^{-2}$ and $V_{ud} = 0.97427(15)$ are the Fermi weak coupling constant and the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix element [2], $\tau_n$ is the lifetime of the neutron, $f(E_\nu, Z = 1)$ and $E_\nu = (m_n^2 - m_p^2 + m_e^2)/2m_n = 1.2927 \text{ MeV}$ are the Fermi integral and the end–point energy of the electron–energy spectrum of the neutron $\beta^–$–decay [14], respectively. The Fermi integral, calculated in [14] by taking into account the contributions of the “weak magnetism” and the proton recoil to next–to–leading order in the large $M$ expansion, is related to the phase–space factor $f$, defined by $f(E_\nu, Z = 1) = m_n^3 f$. The phase–space factor $f$ depends slightly on the axial coupling constant $\lambda$ and for $\lambda = -1.2694$ [1] and for $\lambda = -1.2750$ [15] it is equal to $f = 1.6894$ [14]. In turn, the lifetime of the neutron $\tau_n$, being inversely proportional to the factor $(1 + 3\lambda^2)$, for
the basis of our revision of a deficit of reactor antineutrinos, observed in \[1\].

\[\tau = \theta \]

where defined by \[5–7\] the large \(A_\lambda\) by changing the axial coupling constant \(\lambda\) defined by \[1, 3\]. The later is important for the planning experiments on the detection of sterile antineutrinos at distances \((6 - 13)\) m from the reactor by Serebrov et al. \[3\]. Then, \((1 + \delta_R)(1 + \Delta_R) = 1 + R_C\) are the radiative corrections \[19–24\] integrated over the electron–energy spectrum of the neutron \(\beta^-\)–decay (see also \[14\]), where \(\delta_R = 0.01505\) is defined by one–photon exchanges only \[23, 24\] and \(\Delta_R = 0.02381\) is caused by electroweak boson exchanges and QCD corrections \[23, 24\]. The numerical values of \(\delta_R\) and \(\Delta_R\), obtained in \[23, 24\] and discussed in \[15\], have been also confirmed in \[14\] (see discussion below Eq.(68) of Ref.\[14\]). The functions \((\alpha/\pi) f_\Lambda(\bar{E})\) and \((\alpha/\pi) f_B(\bar{E})\) define the radiative corrections to the correlation coefficients \(A(\bar{E})\) and \(B(\bar{E})\), respectively, calculated to leading order in the large \(M\) expansion (see Appendices A, B and C). Since the radiative corrections \((\alpha/\pi) f_\Lambda(\bar{E})\) and \((\alpha/\pi) f_B(\bar{E})\) do not depend on the axial coupling constant \(\lambda\), they do not influence on the change of a deficit of reactor antineutrinos by changing the axial coupling constant \(\lambda\) from \(\lambda = 1.2694\) to \(\lambda = 1.2750\).

A deficit of reactor antineutrinos may be observed by measuring the yield of positrons in the inverse \(\beta^-\)–decay, defined by \[5–7\]

\[Y_{e^+} = \int_{(E_\bar{\nu})_{\text{min}}}^{(E_\bar{\nu})_{\text{max}}} dE_\bar{\nu} \sigma(\bar{E}_\nu) n(E_\bar{\nu}),\]

where \(n(E_\bar{\nu})\) is the reactor antineutrino flux \[5–7\] (see also \[1\]) for the antineutrino energy region \((E_\bar{\nu})_{\text{min}} = 2\) MeV \(\leq E_\bar{\nu} \leq (E_\bar{\nu})_{\text{max}} = 8\) MeV and \(\sigma(E_\bar{\nu})\) is the cross section for the inverse \(\beta^-\)–decay. Integrating the angular distribution Eq.(1) over \(\cos \theta_\bar{\nu}\) in the limits \(-1 \leq \cos \theta_\bar{\nu} \leq +1\) (see Appendix A) we obtain the cross section for the inverse \(\beta^-\)–decay

\[\sigma(E_\bar{\nu}) = \sigma_0 \left( A(E_\bar{\nu}) + \frac{1}{2} C(E_\bar{\nu}) \beta^2 \right) \left( 1 + \frac{\alpha}{\pi} f_\Lambda(\bar{E}) \right) \bar{k}\bar{E} .\]

For the derivation of Eq.(4) we have neglected the contributions of the terms, which are of order \((\alpha/\pi)(E_\bar{\nu}/M) \sim 10^{-5}\) in the antineutrino energy region \(2\) MeV \(\leq E_\bar{\nu} \leq 8\) MeV.

For the numerical analysis of the cross section for the inverse \(\beta^-\)–decay Eq.(4) we use analytical expressions for the correlation coefficients \(A(\bar{E}_\nu)\) and \(C(\bar{E}_\nu)\), given in Appendix A (see Eq.(A-24) and Eq.(A-75)), and the function \((\alpha/\pi) f_\Lambda(\bar{E})\) and \((\alpha/\pi) f_B(\bar{E})\) are in analytical agreement with the results, obtained in \[7–11\]. For the antineutrino energy region \(2\) MeV \(\leq E_\bar{\nu} \leq 8\) MeV the function \((\alpha/\pi) f_\Lambda(\bar{E})\) is plotted in Fig.1.

A numerical analysis of a deficit of reactor antineutrinos has been carried out in \[1\] for the lifetime of the neutron \(\tau_n = 885.7\) s, corresponding the axial coupling constant \(\lambda = -1.2694\). This lifetime of the neutron does not agree with the recent world average value \(\tau_n = 880.1(1.1)\) s \[2\], whereas the lifetime of the neutron \(\tau_n = 879.6(1.1)\) s, calculated for the axial coupling constant \(\lambda = -1.2750(9)\) \[14\], agrees well with the world average value \(\tau_n = 880.1(1.1)\) s. This is the basis of our revision of a deficit of reactor antineutrinos, observed in \[1\].

The cross sections for the inverse \(\beta^-\)–decay, calculated for \(\lambda = -1.2750\) and \(\lambda = -1.2694\), defined in the antineutrino energy region \(2\) MeV \(\leq E_\bar{\nu} \leq 8\) MeV, are shown in Fig. 2 (left). Since the cross sections for the inverse \(\beta^-\)–decay, calculated for \(\lambda = -1.2750\) and \(\lambda = -1.2694\), are practically indistinguishable, in Fig. 2 (right) we plot the ratio

\[\frac{\sigma(\bar{E}_\nu)}{\sigma_0} = \frac{A(\bar{E}_\nu)}{A_0} \left( 1 + \frac{\alpha}{\pi} f_\Lambda(\bar{E}) \right) \left( 1 + \frac{\alpha}{\pi} f_B(\bar{E}) \right) \bar{k}\bar{E} .\]
FIG. 2: (Color online) The cross sections for the inverse $\beta$-decay (a), calculated for $\lambda = -1.2750$ and $\lambda = -1.2694$, and the ratio $R(E_\nu) = \Delta \sigma(E_\nu)/\sigma(E_\nu)$ (b) in the antineutrino energy region $2 \text{MeV} \leq E_\nu \leq 8 \text{MeV}$.

\[ R(E_\nu) = \frac{\Delta \sigma(E_\nu)}{\sigma(E_\nu)} \text{,} \]

\[
R(E_\nu) = \frac{6\lambda \Delta \lambda}{1 + 3\lambda^2} \left[ \frac{1}{M} \frac{2}{3} \left( \frac{1}{2} + \frac{1}{\lambda} \right) E_\nu \right] \times \left[ 1 - \frac{1}{M} \frac{1}{3\lambda^2} \right] \left( \lambda^2 - (\kappa + 1) \lambda \right) \Delta + 4(\kappa + 1) \lambda E_\nu - (\lambda^2 + 2(\kappa + 1) \lambda + 1) \frac{m_e^2}{E} \right].
\]

The ratio $R(E_\nu)$ defines a relative deviation of the cross section for the inverse $\beta$-decay, calculated at $\lambda = -1.2694$, from the cross section, calculated at $\lambda = -1.2750$. Since in the energy region $2 \text{MeV} \leq E_\nu \leq 8 \text{MeV}$ the ratio $R(E_\nu)$ depends slightly on the antineutrino energy and a maximum of the reactor antineutrino–energy spectrum is smeared over the region $2 \text{MeV} \leq E_\nu \leq 4 \text{MeV}$ (see Fig. 12 of Rev.[1]), one may set that on average a variation of the axial coupling constant $\lambda$ from $\lambda = -1.2694$ to $\lambda = -1.2750$ changes the value of the cross section for the inverse $\beta$-decay in of about 0.734 %. Such a change of the cross section agrees well with a change of the lifetime of the neutron in 0.69 % from $\tau_n = 855.7 \text{s}$ [1] to $\tau_n = 879.6 \text{s}$ [14].

Since in the antineutrino energy region $2 \text{MeV} \leq E_\nu \leq 8 \text{MeV}$ the ratio $R(E_\nu)$ depends slightly on the antineutrino energy, the proposed analysis of the cross section for the inverse $\beta$-decay implies that a deficit $\Delta Y_\nu = 5.7 \%$ of the reactor antineutrinos, observed in [1] for the lifetime of the neutron $\tau_n = 855.7 \text{s}$ and the axial coupling constant $\lambda = -1.2694$, should be increased up to $\Delta Y_\nu \approx 6.434 \%$ including 0.734 %, as has been pointed out in [3]. We would like to emphasise that the obtained increase of a deficit of antineutrinos does not depend on the radiative corrections to the inverse $\beta$-decay, which are cancelled in the ratio $R(E_\nu)$ due to their independence of the axial coupling constant.

III. ASYMMETRY OF INVERSE $\beta$-DECAY AS TOOL FOR MEASUREMENT OF CORRELATION COEFFICIENT $a_0$

In the neutron $\beta^-$-decay the correlation coefficient $a_0$ is a quantitative characteristic of correlations between 3-momenta of the electron and antineutrino, calculated to leading order in the large proton mass expansion [15] (see also [14]). Since the antineutrino in the final state of the neutron $\beta^-$-decay is hard to detect, for the experimental determination of $a_0$ one should measure correlations of the 3-momenta of the decay proton and electron [14]. The inverse $\beta$-decay may be a good laboratory for a measurement of the correlation coefficient $a_0$. In terms of the angular distribution of the cross section for the inverse $\beta$-decay we may define the asymmetry $B_{\exp}(E_\nu)$ as

\[
B_{\exp}(E_\nu) = \frac{\frac{d\sigma(E_\nu, \cos \theta_{\ell\nu})}{d\cos \theta_{\ell\nu}}|_{\theta_{\ell\nu}=0}}{\frac{d\sigma(E_\nu, \cos \theta_{\ell\nu})}{d\cos \theta_{\ell\nu}}|_{\theta_{\ell\nu}=\pi}} \cdot \frac{\frac{d\sigma(E_\nu, \cos \theta_{\ell\nu})}{d\cos \theta_{\ell\nu}}|_{\theta_{\ell\nu}=\pi}}{\frac{d\sigma(E_\nu, \cos \theta_{\ell\nu})}{d\cos \theta_{\ell\nu}}|_{\theta_{\ell\nu}=0}}.
\]

Substituting Eq.(1) into Eq.(6) we obtain

\[
B_{\exp}(E_\nu) = \frac{B(E_\nu)\beta}{A(E_\nu) + C(E_\nu)\beta^2} \left( 1 + \frac{\alpha}{\pi} (f_B(E) - f_A(E)) \right).
\]

Since the experimental data on the asymmetry $B_{\exp}(E_\nu)$ may be fitted by only one parameter $\lambda$ and at the neglect of the $1/M$ corrections the asymmetry $B_{\exp}(E_\nu)$ is equal to $B_{\exp}(E_\nu) = a_0 \beta$, the asymmetry $B_{\exp}(E_\nu)$ may be a good
The radiative corrections \((\alpha/\pi) f_A(\bar{E})\) and \((\alpha/\pi) f_B(\bar{E})\), defined by Eq.(A-71) and Eq.(A-72) with \((\alpha/\pi) f_A^{ fragmentation}(\bar{E})\) and \((\alpha/\pi) f_B^{ fragmentation}(\bar{E})\), given by the integrals over the positron energy in Eq.(A-64) and Eq.(A-65), respectively. The green line is calculated for the radiative corrections, given by the analytical expressions in Eq.(B-20) and Eq.(C-16), respectively.

In Fig. 5 we plot \(\langle \cos \theta_{e\bar{e}} \rangle\) in the region of the antineutrino energy 2 MeV \(\leq E_{\bar{\nu}} \leq 8\) MeV. In comparison with [9], the

\[
\langle \cos \theta_{e\bar{e}} \rangle = \frac{\int_{-1}^{+1} \cos \theta_{e\bar{e}} \frac{d\sigma(E_{\bar{\nu}}, \cos \theta_{e\bar{e}})}{d\cos \theta_{e\bar{e}}} \cos \theta_{e\bar{e}}}{\int_{-1}^{+1} \frac{d\sigma(E_{\bar{\nu}}, \cos \theta_{e\bar{e}})}{d\cos \theta_{e\bar{e}}} \cos \theta_{e\bar{e}}} = \frac{B(E_{\bar{\nu}}) \beta}{3A(E_{\bar{\nu}}) + C(E_{\bar{\nu}}) \beta^2} \left(1 + \frac{\alpha}{\pi} (f_B(\bar{E}) - f_A(\bar{E}))\right). \tag{8}
\]

We would like to note that the asymmetry \(B_{\text{exp}}(E_{\bar{\nu}})\) acquires a maximal absolute value in the region of the antineutrino energies 2 MeV \(\leq E_{\bar{\nu}} \leq 4\) MeV in the vicinity of the maximum of the antineutrino–energy spectrum. Such a property of the asymmetry \(B_{\text{exp}}(E_{\bar{\nu}})\) makes meaningful a measurement of the axial coupling constant \(\lambda\) and the determination of the correlation coefficient \(a_0\) from the inverse \(\beta\)-decay as a method of the determination of the correlation coefficient \(a_0\) alternative to the electron–proton energy distribution and the proton–energy spectrum, discussed in [14].

Then we follow [9] and calculate the average value of \(\langle \cos \theta_{e\bar{e}} \rangle\). We obtain

\[
\langle \cos \theta_{e\bar{e}} \rangle = \frac{\int_{-1}^{+1} \cos \theta_{e\bar{e}} \frac{d\sigma(E_{\bar{\nu}}, \cos \theta_{e\bar{e}})}{d\cos \theta_{e\bar{e}}} \cos \theta_{e\bar{e}}}{\int_{-1}^{+1} \frac{d\sigma(E_{\bar{\nu}}, \cos \theta_{e\bar{e}})}{d\cos \theta_{e\bar{e}}} \cos \theta_{e\bar{e}}} = \frac{B(E_{\bar{\nu}}) \beta}{3A(E_{\bar{\nu}}) + C(E_{\bar{\nu}}) \beta^2} \left(1 + \frac{\alpha}{\pi} (f_B(\bar{E}) - f_A(\bar{E}))\right). \tag{8}
\]
average value \( \langle \cos \theta_{e\nu} \rangle \), given by Eq.(8), is improved by the contributions of the radiative corrections.

IV. UNIVERSALITY OF RADIATIVE CORRECTIONS TO ORDER \( \alpha/\pi \) TO NEUTRINO (ANTINEUTRINO) REACTIONS, INDUCED BY WEAK CHARGED CURRENTS, AND ANTINEUTRINO ENERGY SPECTRUM OF NEUTRON \( \beta^- \)-DECAY

As has been pointed out by Kurylov, Ramsey-Musolf and Vogel [17, 18], the radiative corrections to order \( \alpha/\pi \sim 10^{-3} \), calculated to the cross sections for a class of nuclear reactions, induced by weak charged currents and involving neutrino (antineutrino) in the initial state with electron (positron) in the final state, are universal. Such a universality has been proved in [17, 18] for the reactions of the neutrino (antineutrino) disintegration of the deuteron, caused by charged weak currents. As has been found in [17, 18], the cross sections for the neutrino (antineutrino) reactions, induced by charged weak currents, can be written in the following form

\[
\sigma_{CC}(E) = \sigma_{CC}^{(\text{tree})}(E) \left( 1 + \frac{\alpha}{\pi} g(E) \right),
\]

where \( \sigma_{CC}^{(\text{tree})}(E) \) and \( \sigma_{CC}(E) \) are the cross sections for the reaction under consideration, calculated to leading and to next-to-leading order in \( \alpha/\pi \), respectively, \( E \) is the energy observed in the detector. The function \( g(E) \) is given by a sum of one–virtual photon exchanges, bremsstrahlung, electroweak boson exchanges and QCD corrections. Using Eq.(B-7) the function \( g(E) \), calculated in [17], can be given by

\[
g(E) = \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) + \frac{3}{4} - \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) - \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{2} \left( \frac{2\beta}{1 + \beta} \right) + \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) - \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) + \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) - \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) + C_R,
\]

where the constant \( C_R \) is defined by the contributions of electroweak boson exchanges and QCD corrections. The calculation of the integrals one can perform analytically by using the procedure, expounded in Appendices B and C. As a result we arrive at the analytical expressions, adduced for these integrals in the Appendix to Ref.[17]. Using these expressions we may transcribe the function \( g(E) \) into the form

\[
g(E) = \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) + \frac{23}{8} + 2 \ln \left( \frac{2\beta}{1 + \beta} \right) - 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) + 4 \left( \frac{2\beta}{1 + \beta} \right) + \frac{3}{8} \left( \beta^2 + \frac{7}{3} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) + \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) + \frac{1}{1 + \beta} \left( \frac{2\beta}{1 + \beta} \right) + C_{WZ} = f_A(E) + C_{WZ}.
\]

where we have redefined the contributions of the electroweak boson exchanges and QCD corrections. The function \( f_A(E) \), given by Eq.(A-76) (see also Eq.(B-20)), defines the radiative corrections to the cross section for the inverse \( \beta^-\)-decay Eq.(4), induced by the charged weak currents. Thus we may rewrite Eq.(9) as follows

\[
\sigma_{CC}(E) = \sigma_{CC}^{(\text{tree})}(E) \left( 1 + \Delta_R \right) \left( 1 + \frac{\alpha}{\pi} f_A(E) \right),
\]
where $\Delta_R = (\alpha/\pi) C_W Z$. According to [17], the constant $\Delta_R$ may contain also the contributions, depending on a nuclear structure of a target.

As has been pointed by Sirlin [25] (see also [17, 18]), the radiative corrections to the antineutrino (neutrino) energy spectra of the $\beta-$decays are described by the function $f_A(E)$. Indeed, one may show that the antineutrino energy spectrum of the neutron $\beta^-$decay takes the form

$$
\frac{d\lambda_n(E_\bar{\nu})}{dE_\bar{\nu}} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{2\pi^3} (1 + \Delta_R) \bar{k}_e E_{\bar{\nu}} F(\bar{E}_e, Z = 1) E_{\bar{\nu}}^2 \left( 1 + \frac{\alpha}{\pi} f_A(\bar{E}_e) \right) \zeta(\bar{E}_e),
$$

where $E_{\bar{\nu}} = E_0 - E_\nu$ and $\bar{k}_e = \sqrt{E_{\bar{\nu}}^2 - m^2_n}$ are the energy and momentum of the electron, $E_0 = (m^2_n - m^2_p + m^2_\nu)/2m_n = 1.2927$ MeV is the end-point energy of the electron energy spectrum of the neutron $\beta^-$decay and $F(\bar{E}_e, Z = 1)$ is the Fermi function, describing the electron–proton Coulomb final-state interaction [14]. To leading order in the large $M$ expansion the function $\zeta(\bar{E}_e)$ is equal to unity, $\zeta(\bar{E}_e) = 1$. The deviations from unity are defined to order $1/M$ and caused by the contributions of the “weak magnetism” and the proton recoil.

Thus, the analysis, carried out above, confirms a universality of the radiative corrections to the cross sections for a class of neutrino (antineutrino) reactions, induced by charged currents and calculated to order $\alpha/\pi \sim 10^{-3}$. A nice review of the radiative corrections in precision electroweak physics has been recently written by Sirlin and Ferroglia [26].

We would like to emphasise that the function $\zeta(\bar{E}_e)$ cannot be obtained from the function $\zeta(E_e)$, calculated in [14], by the replacement $E_e \to \bar{E}_e$ that is valid for the transition probability of the neutron $\beta^-$decay [25], calculated to leading order in the large $M$ expansion. A deviation of $\zeta(\bar{E}_e)$ from $\zeta(E_e)$ is attributable to different phase–space factors $\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu)$ [14], where $\bar{k}_\nu$ is a 3–momentum of the antineutrino. For the antineutrino energy spectrum of the neutron $\beta^-$decay the phase–space factor $\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu)$ is defined by

$$
\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu) = \int_{m_n}^\infty \frac{d\lambda_n(E_\bar{\nu})}{dE_\bar{\nu}} \left| \frac{f_{\bar{\nu}}(E_\bar{\nu})}{E_\bar{\nu}} \right| k_e E_{\bar{\nu}} dE_e = \frac{1}{1 - \frac{E_\bar{\nu}}{m_n} \left( 1 - \frac{1}{\beta} \cos \theta_{\bar{\nu}} \right) k_e E_{\bar{\nu}} |f(E_e) = 0|, \tag{14}
$$

where $f(E_e) = m_n - E_p - E_\bar{\nu} - E_e$, $E_p = \sqrt{(k_e + \bar{k}_\nu)^2 + m_p^2}$, $k_e = \sqrt{E_e - m_e^2}$ and the electron energy $E_e$ is equal to

$$
E_e = \frac{E_0 - E_\bar{\nu}}{1 - \frac{E_\bar{\nu}}{m_n} \left( 1 - \frac{1}{\beta} \cos \theta_{\bar{\nu}} \right)}. \tag{15}
$$

The exact expression for the phase–space factor $\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu)$ is

$$
\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu) = \sqrt{1 + 2 \left( \frac{1 - \beta^2}{\bar{\beta}^2} \right) \frac{E_\bar{\nu}}{m_n} \left( 1 - \beta \cos \theta_{\bar{\nu}} \right) \left( 1 - \frac{1}{2} \frac{E_\bar{\nu}}{m_n} \left( 1 - \beta \cos \theta_{\bar{\nu}} \right) \right)^2 \left( 1 - \frac{E_\bar{\nu}}{m_n} \left( 1 - \frac{1}{\beta} \cos \theta_{\bar{\nu}} \right) \right)^2 \left( 1 - \frac{E_\bar{\nu}}{m_n} \left( 1 - \frac{1}{\beta} \cos \theta_{\bar{\nu}} \right) \right)^2}. \tag{16}
$$

A detailed analysis of the antineutrino–energy and angular $(E_\bar{\nu}, \cos \theta_{\bar{\nu}})$ distribution and the antineutrino–energy spectrum we are planning to perform in our forthcoming publication. Here we give the antineutrino–energy spectrum in the antineutrino–energy region $1 \gg E_\bar{\nu}/\beta^2 M$ or $E_\bar{\nu} \ll (E_0 - m_e)(1 - m_e/2M)$. In this energy region the phase–space factor $\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu)$ takes the form

$$
\Phi_{\beta^-}(\bar{k}_e, \bar{k}_\nu) = 1 + \frac{E_\bar{\nu}}{M} \left( 1 + \frac{2\bar{\beta}^2}{\beta^2} - \frac{2 + \bar{\beta}^2}{\beta} \cos \theta_{\bar{\nu}} \right) = 1 + \frac{E_\bar{\nu}}{M} \left( 1 + \frac{2\bar{\beta}^2}{\beta^2} - \frac{2 + \bar{\beta}^2}{\beta^2} \frac{k_e \cdot \bar{k}_\nu}{E_e E_\bar{\nu}} \right). \tag{17}
$$

Using the results, obtained in [14] (see Eq.(A-17) of Ref.[14]) and the expansions

$$
E_e = \bar{E}_e \left( 1 + \frac{E_\bar{\nu}}{M} \left( 1 - \bar{\beta} \cos \theta_{\bar{\nu}} \right) \right),
$$

$$
k_e = \bar{k}_e \left( 1 + \frac{1}{\beta^2} \frac{E_\bar{\nu}}{M} \left( 1 - \beta \cos \theta_{\bar{\nu}} \right) \right),
$$

$$
\beta = \bar{\beta} \left( 1 + \frac{1 - \bar{\beta}^2}{\beta^2} \frac{E_\bar{\nu}}{M} \left( 1 - \bar{\beta} \cos \theta_{\bar{\nu}} \right) \right), \tag{18}
$$
for the function $\zeta(E_e)$ we obtain the following expression

$$
\zeta(E_e) = 1 + \frac{1}{1 + 3\lambda^2} \frac{1}{M} \left[ \frac{(1 + 3\lambda^2) E_{\bar{\nu}}}{\beta^2} + \left( 7\lambda^2 + 4(\kappa + 1) + 1 \right) E_{\bar{\nu}} - 2\lambda \left( \lambda + \kappa + 1 \right) E_0 \right] - \left( \lambda^2 - 2(\kappa + 1) + 1 \right) \frac{m_e^2}{E_e}.
$$

(19)

Analysing the lifetime of the neutron by integrating the antineutrino–energy spectrum one can show that in the vicinity of the end–point energy $(E_0 - m_e)$ of the antineutrino–energy spectrum the contribution of the term, proportional to $1/\beta^2$, behaves as $\alpha ((E_0 - m_e)/M) t^\beta$.

Taking into account in the function $\zeta(E_e)$ the terms, non–singular in the limit $\beta \to 0$, we get the lifetime of the neutron equal to $\tau_n = 880.6(1.1)$ s. The lifetime of the neutron, calculated from the antineutrino–energy spectrum with the function $\zeta(E_e)$, given by Eq.(19), by the integration over the antineutrino–energy spectrum in the limits $0 \leq E_{\bar{\nu}} \leq (E_0 - m_e)(1 - m_e/2M)$ is equal to $\tau_n = 879.9(1.1)$ s. Thus, the contribution of the term, proportional to $1/\beta^2$, is of order 0.7 s or 0.08%, which is smaller compared with the theoretical uncertainty of the lifetime of the neutron $\Delta \tau_n = \pm 1.1$ s or $\pm 0.13%$. Both values of the lifetime of the neutron $\tau_n = 880.6(1.1)$ s and $\tau_n = 879.9(1.1)$ s, calculated from the antineutrino–energy spectrum, agree well with the lifetime of the neutron $\tau_n = 879.6(1.1)$ s, calculated in [14] by integrating over the electron–energy spectrum, and the world average value $\tau_n = 880.1(1.1)$ s [2].

V. CONCLUSION AND DISCUSSION

We have analysed a deficit of reactor antineutrinos at distances smaller than 100 m. We have carried out this analysis having investigated the yield of antineutrinos in terms of the yield of positrons from the inverse $\beta$–decay $\bar{\nu}_e + p \to n + e^+$. The positrons are produced by reactor antineutrinos in the antineutrino energy region 2 MeV $\leq E_{\bar{\nu}} \leq 8$ MeV. The cross section for the inverse $\beta$–decay we have calculated by taking into account i) the contributions of the “weak magnetism” and neutron recoil to next–to–leading order in the large baryon mass expansion and ii) the radiative corrections to order $\alpha/\pi$, caused by one–virtual photon exchanges and the radiative inverse $\beta$–decay, calculated to leading order in the large baryon mass expansion. The radiative corrections defined by the electroweak boson exchanges and the QCD corrections are taken in terms of the parameter $\Delta R = 0.02381$ [15, 23] (see also [14]). We have shown that the calculation of the cross section for the inverse $\beta$–decay for the axial coupling constant $\lambda = -1.2750$ and the lifetime of the neutron $\tau_n = 879.6(1.1)$ s [14] increases the yield of positrons of about 0.734% with respect to the yield of positrons, calculated for the axial coupling constant $\lambda = -1.2694$ and lifetime of the neutron $\tau_n = 885.7$ s [1]. As a result, a deficit of reactor antineutrinos at distances smaller than 100 m is of about $\Delta Y_{\bar{\nu}} = 6,434\%$. Such an increase of a deficit of reactor antineutrinos, which for the first time has been pointed out in [3], makes meaningful a search for sterile antineutrinos at distances $(6 - 13)$ m from reactors, proposed by Serebrov et al. [3].

The dependence of the angular distribution for the inverse $\beta$–decay on $\cos \theta_{\bar{\nu}e}$ allows to define the asymmetry $B_{\exp}(E_{\bar{\nu}})$ of the position yields in the $\theta_{\bar{\nu}e} = 0$ and $\theta_{\bar{\nu}e} = \pi$ directions. In the non–relativistic approximation or to leading order in the large baryon mass expansion the asymmetry $B_{\exp}(E_{\bar{\nu}})$ is proportional to the correlation coefficient $a_0$, describing in the neutron $\beta^–$decay and in the non–relativistic approximation correlations between the 3–momenta of the electron and antineutrino [15] (see also [14]). The experimental analysis of the asymmetry $B_{\exp}(E_{\bar{\nu}})$ may be treated as an alternative method for a measurement of the axial coupling constant $\lambda$ and a determination of the correlation coefficient $a_0$ (see [16] and [14]). We have shown that the absolute values of the asymmetry $B_{\exp}(E_{\bar{\nu}})$ are maximal in the antineutrino energy region $2$ MeV $\leq E_{\bar{\nu}} \leq 4$ MeV, where the antineutrino–energy spectrum is maximal. Since at experiment positrons are emitted forward and backward into a solid angle $\Delta \Omega_{12} = 2\pi(\cos \theta_1 - \cos \theta_2)$, we may define the asymmetry $B_{\exp}(E_{\bar{\nu}}, \theta_2, \theta_1)$ as follows

$$
B_{\exp}(E_{\bar{\nu}}, \theta_2, \theta_1) = \frac{1}{2} \frac{B(E_{\bar{\nu}}) B(\cos \theta_1 + \cos \theta_2)}{A(E_{\bar{\nu}}) + \frac{1}{3} C(E_{\bar{\nu}}) \beta^2 (\cos^2 \theta_1 + \cos \theta_1 \cos \theta_2 + \cos^2 \theta_2)} \left( 1 + \frac{\alpha}{\pi} (f_B(\bar{E}) - f_A(\bar{E})) \right).
$$

(20)

For $\theta_2 = \theta_1 = 0$ that corresponds an emission of positrons into a zero solid angle forward and backward we arrive at Eq.(7). A definition of the asymmetry $B_{\exp}(E_{\bar{\nu}}, \theta_2, \theta_1)$ is similar to a definition of the asymmetry $A_{\exp}(E_e)$ in the neutron $\beta^–$decay [14], describing correlations between the neutron spin and the electron 3–momentum.

We have calculated the average value of $\langle \cos \theta_{\bar{\nu}e} \rangle$ as a function of the antineutrino energy $E_{\bar{\nu}}$. In comparison with the results, obtained by Vogel and Beacom [9], we have improved the average value $\langle \cos \theta_{\bar{\nu}e} \rangle$ by taking into account the contributions of the radiative corrections.

In our approach the radiative corrections are defined by the functions $(\alpha/\pi) f_A(\bar{E})$ and $(\alpha/\pi) f_B(\bar{E})$. They are calculated to leading order in the large baryon mass expansion. The function $(\alpha/\pi) f_A(\bar{E})$ and $(\alpha/\pi) f_B(\bar{E})$ define
the radiative corrections to the correlation coefficient \(A(E_\nu)\) or to the cross section for the inverse \(\beta\)–decay and the correlation coefficient \(B(E_\nu)\) or to the asymmetry \(R_{\text{exp}}(E_\nu)\), respectively. They are in analytical agreement with the results, obtained by Vogel [7], Fayans [9], Fukugita and Kubota [10] and Raha, Myhrer and Kubodera [11]. We would like to note that the authors of Ref.[11] calculated the phase–space factor Eq.(A-14) by using a non–relativistic approximation for a total energy of the neutron. This leads to an increase of the cross section for the inverse \(\beta\)–decay by a factor \((1 + \Delta/M)\) in comparison to our result and to the result, obtained by Vogel and Beacom [9].

In Appendix D we have calculated the angular and photon–energy distribution of the radiative inverse \(\beta\)–decay by taking into account the contribution of the proton–photon interaction to leading order in the large proton mass expansion. Such a contribution is responsible for a gauge invariance of the amplitude of the radiative inverse \(\beta\)–decay and a gauge invariant calculation of the angular and photon–energy distribution of the radiative inverse \(\beta\)–decay. The performed analysis confirms our results, obtained in Appendix A by using the Coulomb gauge for a description of physical degrees of freedom of an emitted real photon.

Finally we have confirmed a universality of the radiative corrections to order \(\alpha/\pi \sim 10^{-3}\), described by the function \(f_A(E)\) for the cross section for the inverse \(\beta\)–decay. Such a universality of the radiative corrections to order \(\alpha/\pi\) for the neutrino (antineutrino) reactions, induced by weak charged currents, has been proved by Kurylov, Ramsey-Musolf and Vogel by example of the neutrino (antineutrino) disintegration of the deuteron with the electron (positron) in the final state [17, 18]. Following [17, 18] and using the results, obtained by Sirin [25], one may extend such a universality of the radiative corrections, given by the function \(f_A(E)\), on the neutrino (antineutrino) energy spectra. We have calculated the antineutrino–energy spectrum of the neutron \(\beta\)–decay by taking into account the radiative corrections to order \(\alpha/\pi \sim 10^{-3}\) and the contributions of the “weak magnetism” and the proton recoil to next–to–leading order in the large \(M\) expansion. We have found that the large \(M\) expansion is not well defined for the antineutrino–energy spectrum of the neutron \(\beta\)–decay. Restricting the antineutrino energies from above by the inequality \(1 \gg E_\nu/\beta^2 M\), we have calculated the contributions of the “weak magnetism” and the proton recoil. We have shown that the non–singular terms of the antineutrino–energy spectrum give the lifetime of the neutron equal to \(\tau_n = 880.6(1.1)\) s. The contributions of the non–singular terms together with the contribution of the term proportional to \(1/\beta^2\), integrated over the antineutrino–energies in the limits \(0 \leq E_\nu \leq (E_0 - m_e)(1 - m_e/2M)\), change the lifetime of the neutron as \(\tau_n = 879.9(1.1)\) s. Both values of the lifetime of the neutron \(\tau_n = 880.6(1.1)\) s and \(\tau_n = 879.9(1.1)\) s agree well with the lifetime of the neutron \(\tau_n = 879.6(1.1)\) s, calculated in [14] by integrating the electron–energy spectrum of the neutron \(\beta\)–decay and the world average value \(\tau_n = 880.1(1.1)\) s [2]. A more detailed analysis of the antineutrino–energy and angular distribution \((E_\nu, \cos \theta_{e\nu})\) of the neutron \(\beta\)–decay and the lifetime of the neutron, obtained from the antineutrino–energy spectrum, we are planning to perform in our forthcoming publication.

An alternative analysis of a total cross section for the inverse \(\beta\) decay has been proposed by Strumia and Vissani [29]. The authors have calculated a cross section for the inverse \(\beta\) decay by using i) the radiative corrections, calculated by Kurylov, Ramsey-Musolf and Vogel [18], ii) vector, axial-vector and weak magnetism form-factors, taken in the dipole approximation, and iii) the contribution of the pseudoscalar form-factor, calculated within current algebra technique with PCAC (Partial Conservation of Axial Current) hypothesis in the one–pion–exchange approximation [30]. Because of these form-factors the obtained expression of the cross section for the inverse \(\beta\) decay can be applied to an analysis of antineutrino–proton inelastic scattering in the energy region, going beyond the energy region of reactor antineutrinos. In the non–relativistic approximation for baryons and to order \(1/M\) a total cross section for the inverse \(\beta\) decay, calculated in [29], should be reduced to ours and as well as to the expressions, obtained in [7]–[11].

VI. ACKNOWLEDGEMENTS

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VII. APPENDIX A: AMPLITUDE AND CROSS SECTION OF INVERSE $\beta$–DECAY

1. Next–to–leading order $1/M$ corrections, caused by weak magnetism and neutron recoil

For the calculation of the cross section for the inverse $\beta$–decay we use the following Hamiltonian of weak lepton–nucleon interactions \[14\]

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\bar{\psi}_n(x)\gamma_\mu(1 + \gamma^5)p_\mu(x)] + \frac{\kappa}{2M} \partial^\nu [\bar{\psi}_n(x)\sigma_{\mu\nu}p_\nu(x)] \right\} [\bar{\psi}_\nu(x)\gamma^\mu(1 - \gamma^5)p_\mu(x)] \quad \text{(A-1)}$$

invariant under time reversal, where $\psi_p(x)$, $\psi_n(x)$, $\psi_\nu(x)$ and $\psi_\nu(x)$ are the field operators of the proton, neutron, electron (positron) and neutrino (antineutrino), respectively, $\gamma^\mu$, $\gamma^5$ and $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ are the Dirac matrices \([27]\) and $\kappa = \kappa_p - \kappa_n = 3.7058$ is the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the proton $\kappa_p = 1.7928$ and the neutron $\kappa_n = -1.9130$ and measured in nuclear magneton \([2]\). For numerical calculations we use $\lambda = -1.2750(9)$ \([15]\), $G_F = 1.664 \times 10^{-11}$ MeV$^{-2}$ and $|V_{ud}| = 0.97427(15)$ \([2]\). The value of the CKM matrix element $|V_{ud}| = 0.97427(15)$ agrees well with $|V_{ud}| = 0.97425(22)$, measured from the superallowed $0^+ \rightarrow 0^+$ nuclear $\beta^–$-decays \([28]\). The inverse $\beta$–decay possesses a threshold. In the laboratory frame or in the rest frame of the proton the inverse neutrino threshold energy is equal to $(E_\nu)_{\text{thr}} = (m_n + m_e)^2 - m_p^2)/2m_p = 1.806067(30)$ MeV, calculated for $m_n = 939.565379(21)$ MeV, $m_p = 938.272046(21)$ MeV and $m_e = 0.510999$ MeV \([2]\).

Following \([14]\) we define the amplitude of the inverse $\beta$–decay $\bar{\nu}_e + p \rightarrow n + e^+$ as

$$M(\bar{\nu}_e p \rightarrow n e^+) = 2m_p \frac{G_F}{\sqrt{2}} V_{ud} \mathcal{M}_\beta, \quad \text{(A-2)}$$

where the amplitude $\mathcal{M}_\beta$ is calculated in the rest frame of the proton and the non–relativistic approximation for the neutron, taking into account the contributions of the weak magnetism and the neutron recoil. It reads \([31]\)

$$\mathcal{M}_\beta = \left(1 + \frac{\Delta}{2M} \right) \left\{ [\phi^+_n \phi_p][\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] - \lambda [\phi^+_n \bar{\sigma} \phi_p] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \frac{\lambda}{2M} [\phi^+_n (\bar{\sigma} \cdot \bar{k}_n) \phi_p] \right\}$$

$$\times [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n - i \frac{\kappa + 1}{2M} [\phi^+_n (\bar{\sigma} \times \bar{k}_n) \phi_p] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] - \frac{\bar{k}_n}{2M} [\phi^+_n \phi_p][\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n]],$$

where $\phi_p$ and $\phi_n$ are Pauli spinor functions of the proton and neutron, $v$ and $v_n$ are Dirac bispinor functions of the positron and antineutrino, respectively, $M = (m_n + m_e)/2$ is the average nucleon mass \([14]\) and $\Delta = m_n - m_p$. The factor $(1 + \Delta/2M)$ comes from the normalisation factor $\sqrt{2m_p(E_n + m_n)}$ of the Dirac bispinor wave functions of the proton and neutron, divided by $2m_p$. To order $1/M$ this gives

$$\sqrt{2m_p(E_n + m_n)} = 1 + \frac{\Delta}{2M}.$$  \quad \text{(A-4)}$$

Using the Dirac equations for free antineutrino and positron $\bar{v}_\nu(\bar{k}_p \cdot \bar{\gamma}) = E_\nu \bar{v}_\nu \gamma^0$ and $(\bar{k} \cdot \bar{\gamma}) = (E\gamma^0 + m_e)v$, respectively, we transcribe the last term in Eq.\(\text{(A-3)}\) into the form

$$-\frac{\bar{k}_n}{2M} [\phi^+_n \phi_p] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] = -\frac{\Delta}{2M} [\phi^+_n \phi_p][\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \frac{m_e}{2M} [\phi^+_n \phi_p][\bar{v}_\nu (1 + \gamma^5)v_n]. \quad \text{(A-5)}$$

Substituting Eq.\(\text{(A-5)}\) into Eq.\(\text{(A-3)}\) we obtain

$$\mathcal{M}_\beta = [\phi^+_n \phi_p][\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] - \lambda [\phi^+_n \bar{\sigma} \phi_p] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \frac{\lambda}{2M} [\phi^+_n (\bar{\sigma} \cdot \bar{k}_n) \phi_p] \right\}$$

$$\times [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n - i \frac{\kappa + 1}{2M} [\phi^+_n (\bar{\sigma} \times \bar{k}_n) \phi_p] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \frac{m_e}{2M} [\phi^+_n \phi_p][\bar{v}_\nu (1 + \gamma^5)v_n]], \quad \text{(A-6)}$$

where we have denoted $\bar{\lambda} = \lambda(1 + \Delta/2M)$. The hermitian conjugate amplitude takes the form

$$\mathcal{M}^\dagger_\beta = [\phi_p \phi_n][\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] - \bar{\lambda} [\phi_p \bar{\sigma} \phi_n] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \bar{\lambda} [\phi_p (\bar{\sigma} \cdot \bar{k}_n) \phi_n] \right\}$$

$$\times [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n + i \frac{\kappa + 1}{2M} [\phi_p (\bar{\sigma} \times \bar{k}_n) \phi_n] \cdot [\bar{v}_\nu \gamma^0(1 - \gamma^5)v_n] + \frac{m_e}{2M} [\phi_p \phi_n][\bar{v}_\nu(1 - \gamma^5)v_n]]. \quad \text{(A-7)}$$
In the rest frame of the proton the cross section for the inverse $\beta$–decay is defined by

$$\sigma(E_\nu) = G_F^2 |V_{ud}|^2 \frac{m_p}{2E_\nu} \int \frac{1}{2} \sum_{\text{pol}} |M_\beta|^2 (2\pi)^4 \delta^4(k + k_n - k_p - k_\nu) \frac{d^3k}{(2\pi)^3} \frac{d^3k_n}{2E_n},$$  \hspace{1cm} (A-8)

where we have summed over the polarisations of the neutron and positron and averaged over the polarisations of the proton. The 4–momenta of the neutron, positron, proton and antineutrino are defined by $k_n = (E_n, \vec{k}_n)$, $k = (E, \vec{k})$, $k_p = (m_p, \vec{0})$ and $k_\nu = (E_\nu, \vec{k}_\nu)$, respectively. Having integrated over the 3–momentum of the neutron and the positron energy we arrive at the expression

$$\sigma(E_\nu) = G_F^2 |V_{ud}|^2 \int_{(\cos \theta_{e\bar{\nu}})_{\text{min}}}^{(\cos \theta_{e\bar{\nu}})_{\text{max}}} \frac{1}{2} \sum_{\text{pol}} |M(\bar{\nu}_e p \to ne^+)|^2 \Phi(E_\nu, k, \cos \theta_{e\bar{\nu}}) d\cos \theta_{e\bar{\nu}},$$  \hspace{1cm} (A-9)

where $\cos \theta_{e\bar{\nu}} = \vec{k}_\nu \cdot \vec{E}/E_\nu k$ and $\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}})$ is the phase–space factor equal to

$$\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}}) = \frac{1}{16\pi} \frac{m_p}{E_\nu} \int \delta(E + \sqrt{m_n^2 + (\vec{k}_n - \vec{k})^2 - m_p - E_\nu}) \frac{k}{E_n} dE = \frac{1}{16\pi} \frac{m_p}{E_\nu} \frac{k}{E_n + E_\nu - E_\nu \frac{E}{k} \cos \theta_{e\bar{\nu}}} =$$

$$= \frac{1}{16\pi} \frac{m_p}{E_\nu} \frac{k}{m_p} \frac{1}{E_\nu} \frac{k}{1 + \frac{E_\nu}{m_p} \left(1 - \frac{1}{\beta} \cos \theta_{e\bar{\nu}}\right)} = \frac{1}{16\pi} \frac{k}{E_\nu} \frac{1}{1 + \frac{E_\nu}{m_p} \left(1 - \frac{1}{\beta} \cos \theta_{e\bar{\nu}}\right)},$$  \hspace{1cm} (A-10)

where $\beta = k/E$ is the positron velocity. For the derivation of the phase–factor $\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}})$ we have used energy $E_n + E = m_p + E_\nu$ and 3–momentum $\vec{k}_\nu = \vec{k_n} + \vec{k}$ conservation. The positron energy is given by

$$E = \frac{E_\nu - \frac{m_n^2 - m_p^2 - m_e^2}{2m_p}}{1 + \frac{E_\nu}{m_p} (1 - \cos \theta_{e\bar{\nu}})}.$$  \hspace{1cm} (A-11)

A behaviour of the phase–space factor $\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}})$ as a function of the antineutrino energy $E_\nu$ may affect the upper limit $(\cos \theta_{e\bar{\nu}})_{\text{max}}$ and may lead to a deviation from $(\cos \theta_{e\bar{\nu}})_{\text{max}} = +1$. Indeed, the phase–factor becomes negative for

$$\cos \theta_{e\bar{\nu}} > \beta \left(1 + \frac{m_p}{E_\nu}\right).$$  \hspace{1cm} (A-12)

Thus, if $\beta \geq 1/(1 + m_p/E_\nu)$ the integration over $\cos \theta_{e\bar{\nu}}$ may be carried out in the limits $-1 \leq \cos \theta_{e\bar{\nu}} \leq +1$. In turn, for $\beta < 1/(1 + m_p/E_\nu)$ the region of the integration over $\cos \theta_{e\bar{\nu}}$ is restricted from above as $-1 \leq \cos \theta_{e\bar{\nu}} \leq (\cos \theta_{e\bar{\nu}})_{\text{max}}$. However, one may show that the antineutrino energy region, obeying the constraint $\beta < 1/(1 + m_p/E_\nu)$, is located very close to threshold and does not affect on the value of the cross section. Below we integrate over $\cos \theta_{e\bar{\nu}}$ in the limits $-1 \leq \cos \theta_{e\bar{\nu}} \leq +1$. As a result, the cross section Eq.(A-9) is defined by

$$\sigma(E_\nu) = G_F^2 |V_{ud}|^2 \int_{-1}^{+1} \frac{1}{2} \sum_{\text{pol}} |M(\bar{\nu}_e p \to ne^+)|^2 \Phi(E_\nu, k, \cos \theta_{e\bar{\nu}}) d\cos \theta_{e\bar{\nu}}.$$  \hspace{1cm} (A-13)

Now we may proceed to the analysis of the cross section for the inverse $\beta$–decay to next–to–leading order in the large $M$ expansion keeping the terms of order $1/M$.

To order $1/M$ of the large $M$ expansion the phase–space factor $\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}})$ is

$$\Phi(E_\nu, k, \cos \theta_{e\bar{\nu}}) = \frac{1}{16\pi} \frac{k}{E_\nu} \left[1 - \frac{E_\nu}{M} \left(1 - \frac{1}{\beta} \cos \theta_{e\bar{\nu}}\right)\right].$$  \hspace{1cm} (A-14)

The phase–space factor, taken in the form of the expansion Eq.(A-14), is always positive for all antineutrino energies $E_\nu \geq (E_\nu)_{\text{thr}}$. The positron energy $E$, calculated to next–to–leading order in the large $M$ expansion, is equal to

$$E = \bar{E} \left[1 - \frac{E_\nu}{M} \left(1 - \frac{1}{\beta} \cos \theta_{e\bar{\nu}} + \frac{\Delta^2 - m_e^2}{2EE_\nu}\right)\right],$$  \hspace{1cm} (A-15)
where we have denoted \( \tilde{E} = E_\nu - \Delta, \bar{\beta} = \tilde{k}/\tilde{E} \) and \( \bar{k} = \sqrt{E^2 - m_e^2} \). For numerical calculations we use \( \Delta = m_n - m_p = 1.293332 \text{ MeV} \) [2]. For the absolute value of the momentum \( k \) and velocity \( \beta \) of the positron we obtain the following expressions

\[
k = \bar{k}\left[1 - \frac{E_\nu}{M} \frac{1}{\beta^2} \left(1 - \bar{\beta} \cos \theta_{e\bar{\nu}} + \frac{\Delta^2 - m_e^2}{2 E E_\nu}\right)\right].
\]  
(A-16)

and

\[
\beta = \bar{\beta}\left[1 - \frac{E_\nu}{M} \frac{1 - \bar{\beta}^2}{\beta^2} \left(1 - \bar{\beta} \cos \theta_{e\bar{\nu}} + \frac{\Delta^2 - m_e^2}{2 E E_\nu}\right)\right].
\]  
(A-17)

The squared absolute value of the amplitude \( \mathcal{M}_\beta \), summed over the polarisations of interacting particles, is equal to

\[
\sum_{\text{pol}} |\mathcal{M}_\beta|^2 = 16 \left\{ (1 + 3\bar{\lambda}^2) E E_\nu + (1 - \bar{\lambda}^2) \bar{k} \cdot \bar{k}_\nu + \frac{1}{M}\left[-m_e^2 E_\nu - \lambda^2(E E_\nu^2 - E_\nu k^2) - \lambda^2(E_\nu - E) \bar{k} \cdot \bar{k}_\nu + 2(\kappa + 1) \lambda (E E_\nu^2 + E_\nu k^2) - 2(\kappa + 1) \lambda (E_\nu - E) \bar{k} \cdot \bar{k}_\nu \right]\right\}.
\]  
(A-18)

For the calculation of the cross section we transcribe the r.h.s. of Eq.(A-15) into the form

\[
\sum_{\text{pol}} |\mathcal{M}_\beta|^2 = 16 (1 + 3\lambda^2) E E_\nu \left(\tilde{A}(E_\nu) + \bar{B}(E_\nu) \frac{\bar{k} \cdot \bar{k}_\nu}{E E_\nu}\right) = 16 (1 + 3\lambda^2) E E_\nu \left(\tilde{A}(E_\nu) + \bar{B}(E_\nu) \beta \cos \theta_{e\bar{\nu}},
\right)
\]  
(A-19)

where we have denoted

\[
\tilde{A}(E_\nu) = 1 + \frac{1}{M} + \frac{1}{1 + 3\lambda^2} \left(3\lambda^2 \Delta - (\lambda^2 - 2(\kappa + 1) \lambda) E_\nu + (\lambda^2 + 2(\kappa + 1) \lambda) E - (\lambda^2 + 2(\kappa + 1) \lambda + 1) \frac{m_e^2}{E}\right),
\]

\[
\bar{B}(E_\nu) = \frac{1 - \lambda^2}{1 + 3\lambda^2} + \frac{1}{M} + \frac{1}{1 + 3\lambda^2} \left(-\lambda^2 \Delta - (\lambda^2 + 2(\kappa + 1) \lambda) E_\nu + (\lambda^2 - 2(\kappa + 1) \lambda) E\right) = a_0 + \frac{1}{M} + \frac{1}{1 + 3\lambda^2} \left(-\lambda^2 \Delta - (\lambda^2 + 2(\kappa + 1) \lambda) E_\nu + (\lambda^2 - 2(\kappa + 1) \lambda) E\right).
\]  
(A-20)

In terms of the antineutrino energy \( E_\nu \) the correlation coefficients \( \tilde{A}(E_\nu) \) and \( \bar{B}(E_\nu) \) read

\[
\tilde{A}(E_\nu) = 1 + \frac{1}{M} + \frac{1}{1 + 3\lambda^2} \left(2(\lambda^2 - (\kappa + 1) \lambda) \Delta + 4(\kappa + 1) \lambda E_\nu - (\lambda^2 + 2(\kappa + 1) \lambda + 1) \frac{m_e^2}{E}\right),
\]

\[
\bar{B}(E_\nu) = a_0 + \frac{1}{M} + \frac{1}{1 + 3\lambda^2} \left(-2(\lambda^2 - 2(\kappa + 1) \lambda) \Delta - 4(\kappa + 1) \lambda E_\nu\right),
\]  
(A-21)

where \( \bar{E} = E_\nu - \Delta \). The angular distribution for the inverse \( \beta \)-decay takes the form

\[
\frac{d\sigma(E_\nu, \cos \theta_{e\bar{\nu}})}{d\cos \theta_{e\bar{\nu}}} = (1 + 3\lambda^2) \frac{G^2_F |V_{ud}|^2}{2\pi} \left(\tilde{A}(E_\nu) + \bar{B}(E_\nu) \beta \cos \theta_{e\bar{\nu}}\right) k \bar{E} \left[1 - \frac{E_\nu}{M} \left(1 - \frac{1}{\beta} \cos \theta_{e\bar{\nu}}\right)\right].
\]  
(A-22)

Using the expansions Eq.(A-15) - Eq.(A-17) we may transcribe the angular distribution Eq.(A-24) into the form

\[
\frac{d\sigma(E_\nu, \cos \theta_{e\bar{\nu}})}{d\cos \theta_{e\bar{\nu}}} = (1 + 3\lambda^2) \frac{G^2_F |V_{ud}|^2}{2\pi} \left(\tilde{A}(E_\nu) + \bar{B}(E_\nu) \beta \cos \theta_{e\bar{\nu}} + C(E_\nu) \bar{\beta}^2 \cos^2 \theta_{e\bar{\nu}}\right) k \bar{E},
\]  
(A-23)

where we have denoted

\[
A(E_\nu) = \tilde{A}(E_\nu) - \frac{1 + 2\beta^2}{\beta^2} \frac{E_\nu}{M} \left(1 + \frac{1 + \beta^2}{1 + 2\beta^2} \frac{\Delta^2 - m_e^2}{2 E E_\nu}\right),
\]

\[
B(E_\nu) = \bar{B}(E_\nu) + \frac{2 + \bar{\beta}^2}{\bar{\beta}^2} \frac{E_\nu}{M} \left[1 - a_0 \left(1 + \frac{2}{2 + \bar{\beta}^2} \frac{\Delta^2 - m_e^2}{2 E E_\nu}\right)\right],
\]

\[
C(E_\nu) = a_0 \frac{3}{\beta^2} \frac{E_\nu}{M}.
\]  
(A-24)
For the derivation of these expansions we have used the expansions of the following products

\[ kE = \frac{k}{E} \left[ 1 - \frac{1 + \frac{\beta^2}{2} E^2}{M} \left( 1 + \frac{\Delta^2 - m^2}{2E} \right) + \frac{1}{\beta} \frac{\Delta^2}{M} \cos \theta_{e\nu} \right], \]

\[ kE\beta = \bar{k}E\beta \left[ 1 - \frac{2 + \frac{\beta^2}{2} E^2}{M} \left( 1 + \frac{\Delta^2 - m^2}{2E} \right) + \frac{2}{\beta} \frac{\Delta^2}{M} \cos \theta_{e\nu} \right], \]

\[ kE \left[ 1 - \frac{E}{M} \left( 1 - \frac{1}{\beta \cos \theta_{e\nu}} \right) \right] = \bar{k}E \left[ 1 - \frac{1 + \frac{\beta^2}{2} E^2}{M} \left( 1 + \frac{\Delta^2 - m^2}{2E} \right) + \frac{1}{\beta} \frac{\Delta^2}{M} \cos \theta_{e\nu} \right], \]

\[ kE \beta \left[ 1 - \frac{E}{M} \left( 1 - \frac{1}{\beta \cos \theta_{e\nu}} \right) \right] = \bar{k}E\beta \left[ 1 - \frac{2 + \frac{\beta^2}{2} E^2}{M} \left( 1 + \frac{\Delta^2 - m^2}{2E} \right) + \frac{3}{\beta} \frac{\Delta^2}{M} \cos \theta_{e\nu} \right]. \]  

(A-25)

The correlation coefficients \( A(E_{e\nu}), B(E_{e\nu}) \) and \( C(E_{e\nu}) \) are calculated in agreement with the results, obtained by Vogel and Beacom [9]. Now let us take into account the contributions of the radiative corrections, which we calculate to leading order in the large \( M \) expansion.

2. Radiative corrections, caused by one–virtual photon exchanges

For this aim of the calculation of the contributions of one–virtual photon exchanges we use the results, obtained in [14] for the neutron \( \beta^- \)–decay. The amplitude of the inverse \( \beta^- \)–decay with one–virtual photon exchanges we write as follows

\[ M^{(\gamma)}(\bar{v}_{e\nu} \rightarrow ne^+) = 2m_p \frac{G_F}{\sqrt{2}} V_{ud} \left( M_{pp}^{(\gamma)} + M_{e^+e^+}^{(\gamma)} + M_{pe^+}^{(\gamma)} \right), \]  

(A-26)

where the amplitudes \( M_{pp}^{(\gamma)} \) and \( M_{e^+e^+}^{(\gamma)} \) are related to the contributions of the proton and positron self–energy corrections, and the amplitude \( M_{pe^+}^{(\gamma)} \) is induced by the proton–positron vertex correction. They are defined by

\[ 2m_p M_{pp}^{(\gamma)} = e^2 \int \frac{d^4q}{(2\pi)^4} iD_{\alpha\beta}(q) \left( \bar{u}_n \gamma^\mu (1 + \gamma^5) \frac{1}{m_p - k_p - i0} \gamma^\alpha \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\beta \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\alpha \bar{u}_p \right) \]  

\[ \left[ \bar{v}_p \gamma_\mu (1 - \gamma^5) \right] \]  

\[ + \left[ \bar{u}_n \gamma^\mu (1 + \gamma^5) \right] \frac{1}{m_p - k_p - i0} \left( -\delta m_p + \frac{Z_2^{(p)}}{2} \right) \left[ \bar{v}_p \gamma_\mu (1 - \gamma^5) \right], \]

\[ 2m_p M_{e^+e^+}^{(\gamma)} = e^2 \left[ \bar{u}_n \gamma^\mu (1 + \gamma^5) \frac{1}{m_e + k + i0} \gamma^\alpha \frac{1}{m_e + \bar{k} + \bar{q} - i0} \gamma^\beta \frac{1}{m_e + \bar{k} + \bar{q} - i0} \gamma^\alpha \bar{u}_p \right] \]  

\[ \left[ \bar{v}_p \gamma_\mu (1 - \gamma^5) \right] \]  

\[ + \left[ \bar{u}_n \gamma^\mu (1 + \gamma^5) \right] \frac{1}{m_e + \bar{k} + \bar{q} - i0} \left( -\delta m_e + \frac{Z_2^{(e)}}{2} \right) \left[ \bar{v}_p \gamma_\mu (1 - \gamma^5) \right], \]

\[ 2m_p M_{pe^+}^{(\gamma)} = -e^2 \int \frac{d^4q}{(2\pi)^4} iD_{\alpha\beta}(q) \left( \bar{u}_n \gamma^\mu (1 + \gamma^5) \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\alpha \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\beta \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\beta \frac{1}{m_p - k_p - \bar{q} - i0} \gamma^\alpha \bar{u}_p \right) \]  

(A-27)

where \( \delta m_p, Z_2^{(p)} \) and \( \delta m_e, Z_2^{(e)} \) are renormalisation constants of masses and wave functions of the proton and positron, respectively, and \( D_{\alpha\beta}(q) \) is the photon propagator

\[ D_{\alpha\beta}(q) = \frac{1}{q^2 + i0} \left( g_{\alpha\beta} - \xi g_{\alpha\beta} \frac{\delta q^2}{q^2} \right), \]  

(A-28)

and \( \xi \) is a gauge parameter. Following [14] we replace the amplitudes \( M_{pp}^{(\gamma)}, M_{e^+e^+}^{(\gamma)} \) and \( M_{pe^+}^{(\gamma)} \) by the expressions

\[ 2m_p M_{pp}^{(\gamma)} \rightarrow 2m_p M_{pp}^{(SC)} = - \frac{\alpha}{8\pi} \left[ \bar{u}_n W^\mu u_p \right] \left[ \bar{v}_p O_{\mu\nu} \right] \int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q) \frac{2k_p + q^\alpha (2k_p + q)^\beta}{(q^2 + 2k_p \cdot q + i0)^2}, \]  

\[ 2m_p M_{e^+e^+}^{(\gamma)} \rightarrow 2m_p M_{e^+e^+}^{(SC)} = - \frac{\alpha}{8\pi m_e} \left[ \bar{u}_n W^\mu u_p \right] \int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q) \frac{2k_p^\alpha + q^\alpha (2k_p^\beta + q^\beta)}{(q^2 + 2k_p \cdot q + i0)^2}, \]  

\[ 2m_p M_{pe^+}^{(\gamma)} \rightarrow 2m_p M_{pe^+}^{(SC)} = \frac{\alpha}{4\pi} \int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q) \frac{2k_p^\alpha + q^\alpha}{q^2 + 2k_p \cdot q + i0} \frac{2k_p^\beta + q^\beta}{q^2 + 2k_p \cdot q + i0}, \]  

(A-29)
where the abbreviation (SC) means “Sirlin’s Correction” with $W^\mu = \gamma^\mu(1 + \lambda \gamma^5)$ and $O_\mu = \gamma_\mu(1 - \gamma^5)$ [14]. The sum of the amplitudes
\[
\mathcal{M}_{RC}^{(SC)} = \mathcal{M}_{pp}^{(SC)} + \mathcal{M}_{e^+ e^+}^{(SC)} + \mathcal{M}_{pe^+}^{(SC)}
\]
\[
\text{(A-30)}
\]
is invariant under a gauge transformation $D_{\alpha\beta}(q) \to D_{\alpha\beta}(q) + e(q^2)q_\alpha q_\beta$, where $e(q^2)$ is an arbitrary function. As has been shown in [14] the deviations of the amplitudes $\mathcal{M}_{pp}^{(SC)}$, $\mathcal{M}_{e^+ e^+}^{(SC)}$ and $\mathcal{M}_{pe^+}^{(SC)}$ from the amplitudes $\mathcal{M}_{pp}^{(SC)}$, $\mathcal{M}_{e^+ e^+}^{(SC)}$ and $\mathcal{M}_{pe^+}^{(SC)}$ do not depend on the positron energy and can be absorbed by the renormalisation constants of the Fermi coupling constant $G_F$ and the axial coupling constant $\lambda$ [14]. Following [14] and using for the regularisation of the ultra–violet and infrared divergent contributions the Pauli-Villars regularisation and the finite–photon mass (FPM) regularisation, respectively, we obtain
\[
2m_p \mathcal{M}_{pp}^{(SC)} = \left[ \bar{u}_n W^\mu u_p \right] \left[ \frac{\alpha}{2\pi} \right] \left[ -\frac{1}{2} \ln \left( \frac{\Lambda}{m_p} \right) - \ln \left( \frac{\mu}{m_p} \right) - \frac{3}{4} \right]
\]
\[
+ \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right] + \left[ \bar{v}_n O_\mu \gamma^\nu v \right] \left[ -\frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right],
\]
\[
\text{(A-31)}
\]
where $\Lambda$ and $\mu$ are the ultra–violet cut–off and the finite–photon mass, respectively, and $L(z)$ is the Spence function, defined by [32]–[35]
\[
L(z) = \int_0^z \frac{dt}{t} \ln |1 - t|.
\]
\[
\text{(A-32)}
\]
Due to gauge invariance of the sum of the amplitudes $\mathcal{M}_{pp}^{(SC)}$, $\mathcal{M}_{e^+ e^+}^{(SC)}$ and $\mathcal{M}_{pe^+}^{(SC)}$ we have used the Feynman gauge $\xi = 0$ [14].

Summing up the amplitudes $\mathcal{M}_{pp}^{(SC)}$, $\mathcal{M}_{e^+ e^+}^{(SC)}$ and $\mathcal{M}_{pe^+}^{(SC)}$ we obtain the radiative corrections to the amplitude of the inverse $\beta$–decay, caused by one–virtual photon exchanges. We get
\[
2m_p \mathcal{M}_{RC}^{(SC)} = \frac{\alpha}{2\pi} \left[ \bar{u}_n W^\mu u_p \right] \left[ \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) - \frac{11}{8} + \ln \left( \frac{m_e}{m_p} \right) \right]
\]
\[
+ \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \left[ \bar{v}_n O_\mu \gamma^\nu v \right] \left[ -\frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right],
\]
\[
\text{(A-33)}
\]
Since the amplitude $2m_p \mathcal{M}_{pe^+}^{(SC)}$ defines the energy dependence of the radiative corrections we give the calculation of the amplitude $2m_p \mathcal{M}_{pe^+}^{(SC)}$ in more detail. Using the Feynman gauge $\xi = 0$ we transcribe $2m_p \mathcal{M}_{pe^+}^{(SC)}$ into the form
\[
2m_p \mathcal{M}_{pe^+}^{(SC)} = \frac{\alpha}{4\pi} \left[ \bar{u}_n W^\mu u_p \right] \left[ \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) - \frac{11}{8} + \ln \left( \frac{m_e}{m_p} \right) \right]
\]
\[
- \frac{1}{4} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \left[ \bar{v}_n O_\mu \gamma^\nu v \right] \left[ -\frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right],
\]
\[
\text{(A-34)}
\]
The contribution of the first three integrals is equal to (see Appendix D and Eq.(D-12) of Ref. [14])
\[
\int \frac{d^4 q}{\pi^2} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_\cdot q + i0} + \int \frac{d^4 q}{\pi^2} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_\cdot q + i0}
\]
\[
- \int \frac{d^4 q}{\pi^2} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_\cdot q + i0} = 2 \ln \left( \frac{\Lambda}{m_e} \right) + 1,
\]
\[
\text{(A-35)}
\]
where $\Lambda$ is an ultra–violet cut–off. After the merging of the denominators (see Eq.(C-3) - Eq.(C-5) of Ref.[14]), the shift of the virtual momentum and the Wick rotation we reduce the fourth and fifth integrals in Eq.(A-34) to the form [14]

$$4(k \cdot k_p) \int \frac{d^4q}{\pi^2} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_p \cdot q + i0} \frac{1}{q^2 + 2k \cdot q + i0} = -2E_{m_p} \int_0^1 \frac{dx}{p^2(x)} \ell n\left[\frac{p^2(x)}{\mu^2}\right]$$

(A-36)

and

$$-2i \int \frac{d^4q}{\pi^2} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_p \cdot q + i0} \frac{1}{q^2 + 2k \cdot q + i0} \bar{v}_\mu O_\mu \sigma_\alpha \beta k^\alpha k^\beta p_{\mu} \nu = -2i[\bar{v}_\mu O_\mu \sigma_\alpha \beta k^\alpha k^\beta p_{\mu} \nu] \int_0^1 \frac{dx}{p^2(x)}$$

(A-37)

respectively, where $p(x) = kx + k_p(1 - x)$, $p^2(x) = m_e^2 x^2 + m_p^2 (1 - x)^2 + 2m_e m_p \gamma x(1 - x)$ and $\gamma = 1/\sqrt{1 - \beta^2}$. For the calculation of the integral over $x$ in Eq.(A-36) we transform it as follows

$$\int_0^1 \frac{dx}{p^2(x)} \ell n\left[\frac{p^2(x)}{\mu^2}\right] = \frac{1}{m_e m_p c} \int_0^1 \frac{dx}{(a - x)^2} = \ell n\left[\frac{m_e m_p c}{\mu^2}(a - x)^2 - b^2\right]$$

(A-38)

where we have denoted

$$a = \rho - \gamma, b = \sqrt{\gamma^2 - 1}, c = \frac{1}{\rho + 2\gamma}$$

(A-39)

with $\rho = m_p/m_e$. Making a change of variables $a - x = \beta \coth \phi$ [14] we obtain

$$\int_0^1 \frac{dx}{p^2(x)} \ell n\left[\frac{p^2(x)}{\mu^2}\right] = \frac{1}{m_e m_p c} \int_{\phi_1}^{\phi_2} d\phi \ell n\left[\frac{m_e m_p c}{\mu^2} \frac{b^2}{\sinh^2 \phi}\right] =$$

$$= \frac{1}{m_e m_p c} \ell n\left[\frac{4m_e m_p c}{\mu^2} \frac{b^2}{\phi_2 - \phi_1 - (\phi_2 - \phi_1)} + L(e^{-2\phi_2}) - L(e^{-2\phi_1})\right]$$

(A-40)

where the last two terms are the Spence functions [32]–[35]. The limits of the integration $\phi_2$ and $\phi_1$ are equal to

$$\phi_2 = \frac{1}{2} \ell n\left(\frac{1 - a - b}{1 - a + b}\right) = \frac{1}{2} \ell n\left(\frac{\rho - \gamma - \sqrt{\gamma^2 - 1}}{\rho - \gamma + \sqrt{\gamma^2 - 1}}\right)$$

$$\phi_1 = \frac{1}{2} \ell n\left(\frac{a + b}{a - b}\right) = \frac{1}{2} \ell n\left(\frac{\rho - \gamma + \sqrt{\gamma^2 - 1}}{\rho - \gamma - \sqrt{\gamma^2 - 1}}\right)$$

(A-41)

Keeping the leading order contributions in the large $\rho$ expansion we arrive at the expression

$$\int_0^1 \frac{dx}{p^2(x)} \ell n\left[\frac{p^2(x)}{\mu^2}\right] = \frac{1}{m_e m_p \sqrt{\gamma^2 - 1}} \left\{ \ell n\left[\frac{4m_e^2}{\mu^2} (\gamma^2 - 1)\right] \frac{1}{2} \ell n\left(\frac{\gamma + \sqrt{\gamma^2 - 1}}{\gamma - \sqrt{\gamma^2 - 1}}\right) - \frac{1}{4} \ell n^2\left(\frac{\gamma + \sqrt{\gamma^2 - 1}}{\gamma - \sqrt{\gamma^2 - 1}}\right) + L\left(\frac{\gamma - \sqrt{\gamma^2 - 1}}{\gamma + \sqrt{\gamma^2 - 1}}\right) - L(1)\right\}$$

(A-42)

In terms of the positron velocity $\beta$ it reads

$$\int_0^1 \frac{dx}{p^2(x)} \ell n\left[\frac{p^2(x)}{\mu^2}\right] = \frac{1}{Em_p \beta} \left\{ \ell n\left[\frac{4m_e^2}{\mu^2} \beta^2 \frac{1}{1 - \beta^2}\right] \frac{1}{2} \ell n\left(\frac{1 + \beta}{1 - \beta}\right) - \frac{1}{4} \ell n^2\left(\frac{1 + \beta}{1 - \beta}\right) + L\left(\frac{1 - \beta}{1 + \beta}\right) - L(1)\right\}$$

(A-43)

where $E = m_e/\sqrt{1 - \beta^2}$. Using the relation for the Spence functions [32]–[35]

$$L\left(\frac{1 - \beta}{1 + \beta}\right) - L(1) = -L\left(\frac{2\beta}{1 + \beta}\right) - \ell n\left(\frac{2\beta}{1 + \beta}\right) \ell n\left(\frac{1 + \beta}{1 - \beta}\right)$$

(A-44)
the r.h.s. of Eq. (A-43) may be written in the following form

\[ \int_0^1 \frac{dx}{p^2(x)} \ln \left( \frac{p^2(x)}{\mu^2} \right) = \frac{1}{E m_p \beta} \left\{ \ln \left[ \frac{4m_e^2}{\mu^2} \beta^2 \right] \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) - L \left( \frac{2 \beta}{1 + \beta} \right) \right\}. \] (A-45)

This gives

\[ 4(k \cdot k_p) \int \frac{d^4q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_p \cdot q + i0} \frac{1}{q^2 + 2k \cdot q + i0} = 2 \ln \left( \frac{\mu}{m_e} \right) \beta \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{2\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{2}{\beta} L \left( \frac{2\beta}{1 + \beta} \right). \] (A-46)

The integral over \( x \) in Eq. (A-37) is equal to

\[ \int_0^1 \frac{dx}{p^2(x)} = \frac{1}{2m_e m_p \beta c} \left[ (b + a) \ln \left( \frac{b + a - 1}{b + a} \right) - (b - a) \ln \left( \frac{b - a + 1}{b - a} \right) \right]. \] (A-47)

Keeping the leading order contributions in the large \( \rho \) expansion we obtain

\[ \int_0^1 \frac{dx}{p^2(x)} = \frac{1}{E m_p \beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right). \] (A-48)

As a result, the integral in Eq. (A-37) is

\[ -2i \int \frac{d^4q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_p \cdot q + i0} \frac{1}{q^2 + 2k \cdot q + i0} [\bar{\nu}_\rho \gamma^\mu \sigma_{\alpha \beta} q^\alpha k_p^\beta v] = -2i [\bar{\nu}_\rho \gamma^\mu \sigma_{\alpha \beta} k_p^\beta v] \times \frac{1}{E m_p} \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) = [\bar{\nu}_\rho \gamma^\mu \sigma_{\alpha \beta} k_p^\beta v] \frac{1}{E m_p} \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right), \] (A-49)

where we have used the Dirac equation \((\slashed{k} \cdot \gamma) v = (E^0 + m_e) v\) for the positron wave function. Summing up the contributions, defined by Eq. (A-36), Eq. (A-46) and Eq. (A-49), we arrive at the expression for the amplitude \(2m_p \mathcal{M}_{\rho e^+}^{(SC)}\), given in Eq. (A-31).

The angular distribution for the inverse \( \beta \)-decay, including the contributions of one–virtual photon exchanges, takes the form

\[ \frac{d\sigma(E_{\bar{\nu}}, \cos \theta_{\bar{\nu}})}{d\cos \theta_{\bar{\nu}}} = \left( 1 + 3\lambda^2 \right) \frac{G_F^2 |V_{ud}|^2}{2\pi} \left( A(E_{\bar{\nu}}, \mu) + B(E_{\bar{\nu}}, \mu) \beta \cos \theta_{\bar{\nu}} + C(E_{\bar{\nu}}) \beta^2 \cos^2 \theta_{\bar{\nu}} \right) \bar{k} E, \] (A-50)

where we have denoted

\[ A(E_{\bar{\nu}}, \mu) = \left( 1 + \frac{\alpha}{\pi} f(E, \mu) \right) + \frac{1}{M} \left[ 1 + 3\lambda^2 \left( 2(\lambda^2 - (\kappa + 1) \lambda) \Delta + 4(\kappa + 1) \lambda E_{\bar{\nu}} \right) - (\lambda^2 + 2(\kappa + 1) \lambda + 1) \frac{m_e^2}{E} \right] - \frac{1 + 2\beta^2}{\beta^2} E_{\bar{\nu}} \frac{1}{M} \left[ 1 + \frac{1 + \beta^2}{1 + 2\beta^2} \Delta^2 - m_e^2 \right], \]

\[ B(E_{\bar{\nu}}, \mu) = a_0 \left( 1 + \frac{\alpha}{\pi} g(E_{\bar{\nu}}, \mu) \right) + \frac{1}{M} \left[ 1 + 3\lambda^2 \left( - 2(\lambda^2 - 2(\kappa + 1) \lambda) \Delta - 4(\kappa + 1) \lambda E_{\bar{\nu}} \right) + \frac{2 + \beta^2}{\beta^2} \frac{E_{\bar{\nu}}}{M} \left[ 1 - a_0 \left( 1 + \frac{2 + \beta^2}{2E_{\bar{\nu}}} \Delta^2 - m_e^2 \right) \right] \right] \] (A-51)

The functions \( f(E, \mu) \) and \( g(E_{\bar{\nu}}, \mu) \) are given by

\[ f(E, \mu) = \frac{3}{2} \ln \left( \frac{m_e}{\mu} \right) - \frac{11}{8} + \frac{1}{\beta} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right) - \frac{1}{4\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{\beta}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right), \]

\[ g(E_{\bar{\nu}}, \mu) = \frac{3}{2} \ln \left( \frac{m_e}{\mu} \right) - \frac{11}{8} + \frac{1}{\beta} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right) - \frac{1}{4\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{\beta}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right). \] (A-52)
The functions $f(\tilde{E}, \mu)$ and $g(\tilde{E}, \mu)$ are different, since the term, proportional to $[\tilde{\nu}_\nu O_\mu \gamma^0 v]$, gives a contribution only to the correlation coefficient $A(\tilde{E}, \mu)$. A dependence of the functions $f(\tilde{E}, \mu)$ and $g(\tilde{E}, \mu)$ on the infrared regularisation scale $\mu$ should be removed by taking into account the contribution of the radiative inverse $\beta$–decay, i.e. the contribution of the reaction $\tilde{\nu}_e + p \rightarrow n + e^+ + \gamma$, where $\gamma$ is a photon on mass–shell.

3. Radiative corrections, caused by radiative inverse $\beta$–decay

The amplitude of the radiative inverse $\beta$–decay is

$$ M(\tilde{\nu}_e p \rightarrow n e^+ \gamma) = \varepsilon^* \alpha M_{\alpha}(\tilde{\nu}_e p \rightarrow n e^+ \gamma) =$$

$$= e \frac{G_F}{\sqrt{2}} V_{ud} \left\{ \sum_{n} \frac{1}{m_e + k + \hat{q} - i0} \varepsilon^* v \right\} - \left\{ \sum_{n} \frac{1}{m_p - k_p + \hat{q} - i0} \varepsilon^* u_p \right\} [\tilde{\nu}_\nu O_\mu v], \quad (A-53) $$

where $q = (\omega, \hat{q} = \omega \hat{n})$ and $\varepsilon^*$ are the 4–momentum and 4–polarisation vector of a photon, obeying the constraint $q \cdot \varepsilon^* = 0$. The amplitude $M_{\alpha}(\tilde{\nu}_e p \rightarrow n e^+ \gamma)$ is gauge invariant $q^a M_{\alpha}(\tilde{\nu}_e p \rightarrow n e^+ \gamma) = 0$. For the calculation of the cross section for the reaction $\tilde{\nu}_e + p \rightarrow n + e^+ + \gamma$ we use the Coulomb gauge $\varepsilon = (0, \varepsilon)$ and keep the leading order contributions in the large baryon mass expansion [14]. In such an approximation we define the amplitude of the reaction $\tilde{\nu}_e + p \rightarrow n + e^+ + \gamma$ as follows [14]

$$ M(\tilde{\nu}_e p \rightarrow n e^+ \gamma) = - e G_F \frac{V_{ud}}{\sqrt{2}} \frac{m_p}{\omega} \frac{M_{\beta\gamma}}{E - k \cdot \tilde{n}}. \quad (A-54) $$

The amplitude $M_{\beta\gamma}$ and its hermitian conjugate $M^\dagger_{\beta\gamma}$ are equal to

$$ M_{\beta\gamma} = [\varphi_\alpha_\mu \varphi_\nu] [\tilde{\nu}_\nu \gamma^0 (1 - \gamma^5) Q \nu] - \lambda [\varphi_\alpha_\mu \tilde{\sigma} \varphi_\nu] \cdot [\tilde{\nu}_\nu \gamma^0 (1 - \gamma^5) Q \nu], \quad (A-55) $$

and

$$ M^\dagger_{\beta\gamma} = [\varphi_\alpha_\mu \varphi_\nu] [\tilde{\nu}_\nu \gamma^0 (1 - \gamma^5) \gamma \nu] - \lambda [\varphi_\alpha_\mu \tilde{\sigma} \varphi_\nu] \cdot [\tilde{\nu}_\nu \gamma^0 (1 - \gamma^5) \gamma \nu], \quad (A-56) $$

where $Q = 2 k \cdot \varepsilon^* + \hat{q} \varepsilon^*$ and $\tilde{Q} = \gamma^0 Q^\dagger \gamma^0 = 2 k \cdot \varepsilon + \hat{q} \varepsilon [14]$. The absolute squared value of the amplitude $M_{\beta\gamma}$, summed over the polarisations of the neutron and positron and averaged over polarisations of the proton, is given by

$$ \frac{1}{2} \sum_{\text{pol}} |M_{\beta\gamma}|^2 = 2 \text{tr} \{ \tilde{k} \tilde{Q} \gamma^0 \gamma^0 Q (1 - \gamma^5) \} + 2 \lambda^2 \text{tr} \{ \tilde{k} \tilde{Q} \gamma^0 \gamma^0 Q (1 - \gamma^5) \} =$$

$$= 2 (1 + 3 \lambda^2) E_\nu \left\{ \text{tr} \{ \tilde{k} \tilde{Q} \gamma^0 Q (1 - \gamma^5) \} + a_0 \frac{\tilde{k}}{E_\nu} \cdot \text{tr} \{ \tilde{k} \tilde{Q} \gamma^0 Q (1 - \gamma^5) \} \right\}, \quad (A-57) $$

The calculation of the traces we carry out by using the following formula

$$ \frac{1}{4} \text{tr} \{ \tilde{k} \tilde{Q} \gamma^0 \tilde{Q} (1 - \gamma^5) \} = 4 (k \cdot \varepsilon^*) (k \cdot \varepsilon) (k + q)^\mu + 2 (k \cdot q) \varepsilon^* \varepsilon, \quad (A-58) $$

where the ellipsis denotes the terms, which do not contribute to the cross section for the radiative inverse $\beta$–decay with unpolarised photons. One may obtain these terms using Eq. (B-9) in Appendix B of Ref.[14]. The angular and photon–energy distribution of the radiative inverse $\beta$–decay is

$$ \frac{d^2 \sigma^{(\gamma)}(E_\nu, \cos \theta_{e\bar{\nu}})}{d\omega d\cos \theta_{e\bar{\nu}}} = (1 + 3 \lambda^2) \frac{\alpha}{2 \pi} \frac{G_F}{\pi} \frac{|V_{ud}|^2}{2 \pi} \frac{1}{k E} \int \frac{dQ_{\hat{n}}}{4 \pi} \left[ \frac{(\beta^2 - (\beta \cdot \hat{n})^2)^2}{(1 - \beta \cdot \hat{n})^2} \left( 1 + \frac{\omega}{E} \right) + \frac{1}{1 - \beta \cdot \hat{n}} \frac{\omega^2}{E^2} \right] \times$$

$$+ a_0 \frac{\tilde{k}}{E_\nu} \cdot \left[ \frac{(\beta^2 - (\beta \cdot \hat{n})^2)^2}{(1 - \beta \cdot \hat{n})^2} \left( \frac{\beta \cdot \hat{n}}{E} + \frac{\beta - \hat{n}}{1 - \beta \cdot \hat{n}} \frac{\omega}{E} + \frac{\hat{n}}{1 - \beta \cdot \hat{n}} \frac{\omega^2}{E^2} \right) \right], \quad (A-59) $$

where $E = \tilde{E} - \omega$ and $\beta = k/E = \sqrt{1 - m_e^2/E^2}$. Up to a common factor the expression in curl brackets coincides...
with Eq.(29) of Ref.[10] and Eq.(9) of Ref.[11]. The integrals over directions of a photon momentum are equal to

\[ \int \frac{d\Omega}{4\pi} \frac{\beta^2 - (\vec{\beta} \cdot \vec{n})^2}{(1 - \vec{\beta} \cdot \vec{n})^2} = 2 \left[ \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right], \]

\[ \int \frac{d\Omega}{4\pi} \frac{1}{1 - \vec{\beta} \cdot \vec{n}} = 1 + \left[ \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right], \]

\[ \int \frac{d\Omega}{4\pi} \frac{\vec{n} \cdot (\beta^2 - (\vec{\beta} \cdot \vec{n})^2)}{(1 - \vec{\beta} \cdot \vec{n})^2} = \beta \left\{ -1 - \left( 1 - \frac{3}{\beta^2} \right) \left[ \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right] \right\}, \]

\[ \int \frac{d\Omega}{4\pi} \frac{\vec{\beta} - \vec{n} (\beta^2 - (\vec{\beta} \cdot \vec{n})^2)}{1 - \vec{\beta} \cdot \vec{n}} = \beta \left\{ 1 + \left( 1 - \frac{1}{\beta^2} \right) \left[ \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right] \right\}, \]

\[ \int \frac{d\Omega}{4\pi} \frac{\vec{n}}{1 - \vec{\beta} \cdot \vec{n}} = \frac{\vec{\beta}}{\beta^2} \left[ \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 1 \right]. \]  

(A-60)

Substituting Eq.(A-60) into Eq.(A-59) the angular and photon-energy distribution of the radiative inverse \( \beta \)-decay is

\[ \frac{d\sigma}{d\omega d\theta_{\psi \varphi}} (E_\psi, \cos \theta_{\psi \varphi}) = (1 + 3\lambda^2) \frac{\alpha G_F^2 |V_{ud}|^2}{2\pi} k E \left\{ \left[ \left( 1 + \frac{\omega}{E} + \frac{\omega^2}{E^2} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\omega^2}{E^2} \right] + a_0 \left( 1 + \frac{\omega}{\beta^2 E} + \frac{1}{2\beta^2 E^2} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \beta \cos \theta_{\psi \varphi} \right\}. \]  

(A-61)

Apart from kinematic factors, the r.h.s. of Eq.(A-61) reproduces the photon-electron energy spectrum of the radiative \( \beta^- \)-decay of the neutron (see Eq.(B-11) of Ref.[14]). Having integrated over the photon energy we obtain the following angular distribution

\[ \frac{d\sigma}{d\theta_{\psi \varphi}} (E_\psi, \cos \theta_{\psi \varphi}) = (1 + 3\lambda^2) \frac{\alpha G_F^2 |V_{ud}|^2}{2\pi} k E \left\{ \int_0^{E-m_e} \frac{d\omega}{\omega} k E \left[ \left[ \left( 1 + \frac{\omega}{E} + \frac{\omega^2}{E^2} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\omega^2}{E^2} \right] + a_0 \left( 1 + \frac{\omega}{\beta^2 E} + \frac{1}{2\beta^2 E^2} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \beta \cos \theta_{\psi \varphi} \right\}. \]  

(A-62)

The integral over the photon energy is infrared divergent. For the regularisation of the infrared divergences we propose to rewrite the r.h.s. of Eq.(A-62) as follows

\[ \frac{d\sigma}{d\theta_{\psi \varphi}} (E_\psi, \cos \theta_{\psi \varphi}) = (1 + 3\lambda^2) \frac{\alpha G_F^2 |V_{ud}|^2}{2\pi} k E \left\{ \int_0^{E-m_e} \frac{d\omega}{\omega} k E \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \beta \cos \theta_{\psi \varphi} \right\} + f_A^\gamma (\bar{E}) + f_B^\gamma (\bar{E}). \]  

(A-63)

The functions \( f_A^\gamma (\bar{E}) \) and \( f_B^\gamma (\bar{E}) \) are defined by

\[ f_A^\gamma (\bar{E}) = \frac{1}{kE} \int_{m_e}^\bar{E} \frac{dE}{E - \bar{E}} \left\{ kE \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \bar{E} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\}, \]  

(A-64)

\[ f_B^\gamma (\bar{E}) = \frac{1}{kE\beta} \int_{m_e}^\bar{E} \frac{dE}{E - \bar{E}} \left\{ kE \beta \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \bar{E} \beta \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\}, \]  

(A-65)

where \( k = \sqrt{E^2 - m_e^2}, \beta = k/E, \bar{k} = \sqrt{\bar{E}^2 - m_e^2}, \bar{\beta} = \bar{k}/\bar{E} \) and \( \bar{E} = E_\psi - \Delta \). In Appendices B and C we give analytical expressions for functions \( f_A^\gamma (\bar{E}) \) and \( f_B^\gamma (\bar{E}) \), respectively.
The regularisation of the infrared divergent contribution runs as follows (see Eq.(B-18) - Eq.(B-26) in Appendix B of Ref.[14])

\[
\int_{0}^{\tilde{E}} \frac{d\omega}{\omega} \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \rightarrow \int_{0}^{\tilde{E}} \frac{d\omega q^2}{\omega^3} \int \frac{d\Omega_{\tilde{E}}}{4\pi} \left[ \frac{\beta^2 - (\bar{v} \cdot \bar{\omega})^2}{(1 - \beta \cdot \bar{v})^2} \right] = \int_{0}^{\epsilon_{\text{max}}} \frac{dv\nu^2 \beta^2}{1 - v^2} 2 \int_{-1}^{+1} \frac{dv(1 - v^2 x^2)}{(1 - \beta \nu x)^2} =
\]

\[
= \epsilon \ln \left( \frac{2(\tilde{E} - m_e)}{\mu} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + 1 + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right),
\]

(A-66)

where \( q = |\vec{q}| \), \( \omega = \sqrt{q^2 + \mu^2} \) and \( \bar{v} = \vec{q}/\omega \). As a result, we define the angular distribution for the radiative inverse \( \beta \)-decay as follows [14]

\[
\frac{d\sigma^{(\gamma)}(E_{\nu}, \cos \theta_{e\nu})}{d\cos \theta_{e\nu}} = (1 + 3\lambda^2) \frac{A(E_{\nu})}{\pi} G_{\nu}^2 |V_{ud}|^2 \frac{1}{2\pi} k \tilde{E} \left( g_{\beta \gamma}^{(1)}(\tilde{E}, \mu) + a_{0} g_{\beta \gamma}^{(2)}(\tilde{E}, \mu) \bar{\beta} \cos \theta_{e\nu} \right),
\]

(A-67)

where the functions \( g_{\beta \gamma}^{(1)}(\tilde{E}, \mu) \) and \( g_{\beta \gamma}^{(2)}(\tilde{E}, \mu) \) are equal to

\[
g_{\beta \gamma}^{(1)}(\tilde{E}, \mu) = \epsilon \ln \left( \frac{2(\tilde{E} - m_e)}{\mu} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + 1 + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right) + f_{A}(\tilde{E})
\]

(A-68)

and

\[
g_{\beta \gamma}^{(2)}(\tilde{E}, \mu) = \epsilon \ln \left( \frac{2(\tilde{E} - m_e)}{\mu} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + 1 + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{4\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{\beta} L \left( \frac{2\beta}{1 + \beta} \right) + f_{B}(\tilde{E}).
\]

(A-69)

Summing up the contributions of one–virtual photon exchanges and the radiative inverse \( \beta \)-decay we obtain the following angular distribution of the inverse \( \beta \)-decay

\[
\frac{d\sigma(\vec{E}_{\nu}, \cos \theta_{e\nu})}{d\cos \theta_{e\nu}} = (1 + 3\lambda^2) \frac{G_{\nu}^2 |V_{ud}|^2}{2\pi} \left( 1 + \Delta_{R} \right) \left[ A(\vec{E}_{\nu}) \left( 1 + \frac{\alpha}{\pi} f_{A}(\tilde{E}) \right) + B(\vec{E}_{\nu}) \left( 1 + \frac{\alpha}{\pi} f_{B}(\tilde{E}) \right) \bar{\beta} \cos \theta_{e\nu} \right]
\]

(A-70)

where the functions \( f_{A}(\tilde{E}) \) and \( f_{B}(\tilde{E}) \), describing the radiative corrections, are equal to

\[
f_{A}(\tilde{E}) = \frac{3}{2} \frac{m_{p}}{m_{e}} - \frac{3}{8} + \epsilon \ln \left( \frac{2(\tilde{E} - m_e)}{\mu} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{2\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{2}{\beta} L \left( \frac{2\beta}{1 + \beta} \right)
\]

(A-71)

and

\[
f_{B}(\tilde{E}) = \frac{3}{2} \frac{m_{p}}{m_{e}} - \frac{3}{8} + \epsilon \ln \left( \frac{2(\tilde{E} - m_e)}{\mu} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \frac{1}{2\beta} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{2}{\beta} L \left( \frac{2\beta}{1 + \beta} \right)
\]

(A-72)

The contribution \( \Delta_{R} = (\alpha/\pi) C_{WZ} = 0.0238 \) with \( C_{WZ} = 10.249 \) is defined by electroweak boson exchanges and QCD corrections. Such a contribution has been calculated in [22–24]. The numerical value \( \Delta_{R} = 0.0238 \), calculated in Appendix D of Ref.[14], agrees also well with that \( \Delta_{R} = 0.024 \), used in [1].

The cross section for the inverse \( \beta \)-decay is defined by the integral

\[
\sigma(\vec{E}_{\nu}) = \int_{-1}^{+1} \frac{d\sigma(\vec{E}_{\nu}, \cos \theta_{e\nu})}{d\cos \theta_{e\nu}} d\cos \theta_{e\nu}.
\]

(A-73)

Having integrated over \( \cos \theta_{e\nu} \) we obtain

\[
\sigma(\vec{E}_{\nu}) = (1 + 3\lambda^2) \frac{G_{\nu}^2 |V_{ud}|^2}{\pi} \left( 1 + \Delta_{R} \right) \tilde{E} \left( A(\vec{E}_{\nu}) + \frac{1}{3} C(\vec{E}_{\nu}) \bar{\beta}^2 \right) \left( 1 + \frac{\alpha}{\pi} f_{A}(\tilde{E}) \right).
\]

(A-74)
Using Eq.(A-21) and Eq.(A-24) the cross section for the inverse β-decay we transcribe into the form

\[ \sigma(E_\beta) = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{\pi} (1 + \Delta_R) \bar{k} E \left[ 1 + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left( 2(\lambda^2 - (\kappa + 1)\lambda) \Delta + 4(\kappa + 1)\lambda E_\beta \right) - \left( \lambda^2 + 2(\kappa + 1)\lambda + 1 \right) \left( \frac{m^2}{\bar{E}} \right) - \frac{1 + 2\beta^2}{\beta^2} \frac{E_\beta}{M} \left( 1 + \frac{1 + \beta^2}{1 + 2\beta^2} \Delta^2 - \frac{m^2}{2E E_\beta} \right) + a_0 \frac{E_\beta}{M} \right] \left( 1 + \frac{\alpha}{\pi} f_A(\bar{E}) \right), \]  

(A-75)

where \( \bar{E} = E_\beta - \Delta \) and \( \bar{k} = \sqrt{(E_\beta - \Delta)^2 - m^2} \).

The cross section Eq.(A-75) is in analytical agreement with the cross sections, calculated by Vogel and Beacom [9] (see Eq.(12) and Eq.(13) of Ref.[9]) and by Raha, Myhrer and Kubodera [11] (see Eq.(20) of Ref.[11]) and Erratum to Ref.[11]), with the replacement \( f_A(\bar{E}) \rightarrow f_V(\bar{E}) \) and \( f_A(\bar{E}) \rightarrow f_R(\bar{E}) \), respectively. The functions \( f_V(\bar{E}) \) and \( f_R(\bar{E}) = 1_2 \delta_{\text{out}}(\bar{E}) \) describe the radiative corrections, calculated by Vogel [7] and by Raha, Myhrer and Kubodera [11]. Using the results, obtained in [7] and [8], one may show that the radiative corrections to the correlation coefficient \( A(E_\beta) \), calculated by Vogel [7] (see Eq.(19) of Ref.[7]), Fayans [8] (see Eq.(25) and Eq.(II.14) of Ref.[8]) and Raha, Myhrer and Kubodera [11] (see Eq.(12) of Ref.[11]), are defined by the function \( f(\bar{E}) = f_V(\bar{E}) = f_R(\bar{E}) \) equal to

\[ f(\bar{E}) = \frac{3}{2} \ln \left( \frac{m_\beta}{m_e} \right) + \frac{23}{8} + 2 \ln \left( \frac{2\beta}{1 + \beta} \right) \left[ \frac{1 + \beta}{1 - \beta} \right] - 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) \]  

(A-76)

In Appendix B (see Eq.(B-20)) we show that the function \( f_A(\bar{E}) \) agrees fully with the function \( f(\bar{E}) \).

**VIII. APPENDIX B: ANALYTICAL EXPRESSIONS FOR FUNCTIONS \( f_A^{(\gamma)}(\bar{E}) \) AND \( f_A(\bar{E}) \)**

In this Appendix we propose a detailed calculation of the integrals, defining the function \( f_A^{(\gamma)}(\bar{E}) \), and give the analytical expression for the function \( f_A(\bar{E}) \). Making a change of variables \( E = m_e \cosh \varphi \) and \( \bar{E} = m_e \cosh \bar{\varphi} \), we get

\[ \frac{1}{kE} \int_{m_e}^{kE} \frac{dE}{E - \bar{E}} \left\{ kE \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \bar{kE} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\} = \]

\[ = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \left\{ \sinh \varphi \cosh \varphi \left( \frac{\varphi}{\tanh \varphi} - 1 \right) - \sinh \varphi \cosh \varphi \left( \frac{\varphi}{\tanh \varphi} - 1 \right) \right\} \]

\[ = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi - \cosh \varphi \sinh \varphi \sinh \varphi \cosh \varphi \cosh \varphi \cosh \varphi = \]

\[ = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi \sinh \varphi \cosh \varphi \cosh \varphi = \]

\[ = \int_1 + \int_2 + \int_3. \]  

(B-1)

The calculation of the integral \( I_1 \) runs as follows

\[ I_1 = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi (\cosh^2 \varphi - \cosh^2 \varphi) = -\frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi + \cosh \varphi = \]

\[ = -3 \frac{\varphi}{\tanh \varphi} + \frac{\varphi}{\sinh \varphi \cosh \varphi} + \frac{5}{2} = - \frac{5 + \beta^2}{4\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{5}{2}, \]  

(B-2)

The integral \( I_1 \) is equal to

\[ I_1 = - \frac{5 + \beta^2}{4\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{5}{2}. \]  

(B-3)

For the integral \( I_2 \) we propose the following calculation

\[ I_2 = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi (\varphi - \varphi) \cosh \varphi = \frac{2}{\tanh \varphi} \int_0^{\varphi} d\varphi \sinh \varphi \cosh \varphi - \cosh \varphi (\varphi - \varphi) = \]

\[ = \frac{2}{\tanh \varphi} \left[ - \varphi \ln \left( \cosh \varphi - 1 \right) + \int_0^{\varphi} d\varphi \ln \left( \cosh \varphi - \cosh \varphi \right) \right]. \]  

(B-4)
The remained integral in Eq.(B-4) we calculate by changing of variable \( u = e^\varphi \) and \( \bar{u} = e^{\bar{\varphi}} \). This gives

\[
\int_0^{\bar{\varphi}} d\varphi \, \ln(\cosh \varphi - \cosh \bar{\varphi}) = -\ln n \bar{\varphi}u + \int_1^{\bar{u}} \frac{du}{u} \ln(\bar{u} - u) + \int_1^{\bar{u}} d\bar{u} \ln \left( 1 - \frac{1}{\bar{u}u} \right) =
\]

\[
= -\ln n \bar{\varphi}u + \ln^2 \bar{u} + \int_1^{1/\bar{u}^2} \frac{dx}{x} \ln(1-x) = -\ln n \bar{\varphi}u + \ln^2 \bar{u} + L(1) - L(1/\bar{u}^2) = -\ln n \bar{\varphi} + \bar{\varphi}^2 + L(1) - L(e^{-2\bar{\varphi}}) =
\]

\[
= -\ln n \frac{1}{2} \left( 1 + \frac{\beta}{1 - \beta} \right) + \frac{1}{4} \ln^2 \left( 1 + \frac{\beta}{1 - \beta} \right) + L(1) - L \left( \frac{1 - \beta}{1 + \beta} \right) = -\ln n \frac{1}{2} \left( 1 + \frac{\beta}{1 - \beta} \right) + \frac{1}{4} \ln^2 \left( 1 + \frac{\beta}{1 - \beta} \right) + L \left( \frac{2\beta}{1 + \beta} \right)
\]

\[
+ \ln \left( \frac{2\beta}{1 + \beta} \right) \ln \left( 1 + \frac{\beta}{1 - \beta} \right), \tag{B-5}
\]

where we have used Eq.(A-44). Thus for the integral \( I_2 \) we obtain the expression

\[
I_2 = \frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} \frac{d\varphi \sinh \varphi}{\cosh \varphi - \cosh \bar{\varphi}} \frac{1}{(\varphi - \bar{\varphi}) \cosh^2 \varphi} = \frac{2}{\tanh \varphi} \int_0^{\varphi} \frac{d\varphi \sinh \varphi}{\cosh \varphi - \cosh \bar{\varphi}} (\varphi - \bar{\varphi}) =
\]

\[
= \frac{2}{\tanh \varphi} \left[ -\varphi \ln(\cosh \varphi - 1) - \ln \frac{\bar{\varphi} + \varphi^2}{\bar{\varphi} + L(1)} - L(e^{-2\varphi}) \right] = \frac{2}{\beta} \left[ -\frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 - \sqrt{1 - \beta^2}}{\sqrt{1 - \beta^2}} \right) \right]
\]

\[
- \frac{\ln n}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{4} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right) + L \left( \frac{2\beta}{1 + \beta} \right) + \ln \left( \frac{2\beta}{1 + \beta} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) \tag{B-6}
\]

where we have used the relation

\[
\ln \left( \frac{1 - \sqrt{1 - \beta^2}}{\sqrt{1 - \beta^2}} \right) = -\frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) + \ln \left( \frac{\beta}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right). \tag{B-7}
\]

The integral \( I_2 \) is equal to

\[
I_2 = \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right). \tag{B-8}
\]

We calculate the integral \( I_3 \) as follows

\[
I_3 = -\frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} \frac{d\varphi \sinh \varphi}{\cosh \varphi - \cosh \bar{\varphi}} \frac{\sinh \varphi \cosh \varphi - \sinh \bar{\varphi} \cosh \bar{\varphi}}{\cosh \varphi - \cosh \bar{\varphi}} = -\frac{2}{\sinh \varphi \cosh \varphi} \int_0^{\varphi} \frac{d\varphi \sinh \varphi}{\cosh \varphi - \cosh \bar{\varphi}}
\]

\[
\times \left[ \sinh \varphi \left( \cosh \varphi - \cosh \bar{\varphi} \right) + \cosh \bar{\varphi} \left( \sinh \varphi - \sinh \bar{\varphi} \right) \right] = -\frac{2}{\sinh \varphi \cosh \varphi} \left[ -\frac{1}{2} \int_0^{\varphi} d\varphi \sinh^2 \varphi + \cosh \bar{\varphi} \right]
\]

\[
\times \int_0^{\varphi} \frac{d\varphi \sinh \varphi}{\cosh \varphi - \cosh \bar{\varphi}} \left[ \sinh \varphi \cosh \varphi \ln(\cosh \varphi - \cosh \bar{\varphi}) \right] = -\frac{2}{\sinh \varphi \cosh \varphi} \left[ \frac{1}{2} \left( \varphi - \sinh \varphi \cosh \bar{\varphi} \right) - \sinh \varphi \cosh \bar{\varphi} \ln(\cosh \varphi - 1) \right]
\]

\[
+ \cosh \bar{\varphi} \int_0^{\varphi} d\varphi \cosh \varphi \ln(\cosh \varphi - \cosh \bar{\varphi}) \right]. \tag{B-9}
\]

For the calculation of the last integral in Eq.(B-9) we make a change of variables \( u = e^\varphi \) and \( \bar{u} = e^{\bar{\varphi}} \). This gives

\[
\int_0^{\varphi} d\varphi \cosh \varphi \ln(\cosh \varphi - \cosh \bar{\varphi}) = \frac{1}{2} \ln n \int_1^{\bar{u}} du \left( 1 + \frac{1}{u^2} \right) + \frac{1}{2} \int_1^{\bar{u}} du \left( 1 + \frac{1}{u^2} \right) \ln \left( (\bar{u} - u) \left( 1 - \frac{1}{\bar{u}u} \right) \right) =
\]

\[
= -\ln n \sinh \bar{\varphi} + \frac{1}{2} \int_1^{\bar{u}} du \ln(\bar{u} - u) + \frac{1}{2} \int_1^{\bar{u}} du \ln \left( 1 - \frac{1}{\bar{u}u} \right) + \frac{1}{2} \int_1^{\bar{u}} du \ln \left( 1 - \frac{1}{\bar{u}u} \right) \left. \ln \left( 1 - \frac{1}{\bar{u}u} \right) \right. \tag{B-10}
\]

The integrals over \( u \) are equal to

\[
\frac{1}{2} \int_1^{\bar{u}} du \ln(\bar{u} - u) = \frac{1}{2} \left( \bar{u} - 1 \right) \ln \bar{n}u + \frac{1}{2} \left( \bar{u} - 1 \right) \ln \left( 1 - \frac{1}{\bar{u}u} \right) - \frac{1}{2} \left( \bar{u} - 1 \right),
\]

\[
\frac{1}{2} \int_1^{\bar{u}} du \ln(\bar{u} - u) = \frac{1}{2} \left( \bar{u} - 1 \right) \ln \bar{n}u + \frac{1}{2} \left( 1 - \frac{1}{\bar{u}} \right) \ln \left( 1 - \frac{1}{\bar{u}} \right) - \frac{1}{2} \ln \bar{n}u,
\]
\[
\frac{1}{2} \int_1^u du \ln (1 - \frac{1}{u}) = \frac{1}{2} \left( \ln \left( \frac{1}{u} \right) - \frac{1}{2} \left( 1 - \frac{1}{u} \right) \ln \left( 1 - \frac{1}{u} \right) - \frac{1}{2} \ln u, \right.
\]
\[
\frac{1}{2} \int_1^u du \ln (1 - \frac{1}{u}) = \frac{1}{2} \left( \ln \left( \frac{1}{u} \right) - \frac{1}{2} \left( 1 - \frac{1}{u} \right) \ln \left( 1 - \frac{1}{u} \right) - \frac{1}{2} \ln u - \frac{1}{2} \left( \frac{1}{u} - \frac{1}{u} \right) \ln u = \left( \frac{1}{u} - \frac{1}{u} \right) - \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u} \right) \ln u - \frac{1}{2} \left( \frac{1}{u} - \frac{1}{u} \right) \ln u - \frac{1}{2} \left( \frac{1}{u} - \frac{1}{u} \right). \right. \tag{B-11}
\]

The sum of the integrals over \( u \) is
\[
\frac{1}{2} \int_1^u du \ln (\bar{u} - u) + \frac{1}{2} \int_1^u du \ln (\bar{u} - u) + \frac{1}{2} \int_1^u du \ln (1 - \frac{1}{u}) = \left( \frac{1}{u} - \frac{1}{u} \right) \ln \left( \frac{1}{u} \right) - \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u} \right) \ln u - \frac{1}{2} \left( \frac{1}{u} - \frac{1}{u} \right) = (2 \ln 2 - 1) \sinh \varphi + 2 \sinh \varphi \ln \sinh \varphi - \varphi \cosh \varphi. \tag{B-12}
\]

The integral of Eq. (B-10) is
\[
\int_0^\varphi d\varphi \cosh \varphi \ln (\cosh \varphi - \cosh \varphi) = (\ln 2 - 1) \sinh \varphi + 2 \sinh \varphi \ln \sinh \varphi - \varphi \cosh \varphi. \tag{B-13}
\]

In terms of \( \varphi \) and \( \bar{\beta} \) the integral \( I_3 \) reads
\[
I_3 = 3 - 2 \ln 2 + \bar{\varphi} \left( \frac{1}{\tanh \varphi} + \tanh \bar{\varphi} \right) + 2 \ln (\cosh \varphi - 1) - 4 \ln (\sinh \varphi) = \]
\[
= 3 - 2 \ln 2 + \frac{1}{2} \left( \frac{1}{\beta} + \beta \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) + 2 \ln \left( \frac{1 - \sqrt{1 - \beta^2}}{\sqrt{1 - \beta^2}} \right) - 4 \ln \left( \frac{\beta}{\sqrt{1 - \beta^2}} \right) = \]
\[
= 3 + \left( -1 + \frac{1}{2} \beta + \frac{\beta}{2} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \ln \left( \frac{2 \beta}{1 + \beta} \right) - \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right). \tag{B-14}
\]

For the integral \( I_3 \) we obtain the expression
\[
I_3 = 3 + \left( -1 + \frac{1}{2} \beta + \frac{\beta}{2} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \ln \left( \frac{2 \beta}{1 + \beta} \right) - \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right). \tag{B-15}
\]

Substituting Eq. (B-3), Eq. (B-8) and Eq. (B-15) into Eq. (B-1) we obtain
\[
\frac{1}{kE} \int_{m_c}^E dE \left\{ KE \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - \bar{K}E \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\} = \frac{11}{2} + \left( -1 - \frac{3}{4} \beta + \frac{\beta}{4} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) \]
\[
+ \frac{2}{\beta} \left[ 2 \beta \ln \left( \frac{2 \beta}{1 + \beta} \right) + \ln \left( \frac{2 \beta}{1 + \beta} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{1}{2} \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) \right] \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right]. \tag{B-16}
\]

Now we calculate the integral
\[
\frac{1}{kE} \int_{m_c}^E dE k \left( \frac{\bar{E} + E}{2E} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\bar{E} - E}{E} = \frac{1}{\sinh \varphi \cosh \varphi} \int_{\varphi}^{\varphi} d\varphi \sinh^2 \varphi \left[ \cosh \varphi + \cosh \varphi \left( \frac{\varphi}{\tanh \varphi} \right) \right] \tag{B-17}
\]
\[
+ \cosh \varphi \sinh \varphi \left( \frac{1}{\sinh \varphi} \right) \sinh \varphi \left( \frac{1}{\cosh \varphi} \right) \int_{\varphi}^{\varphi} d\varphi \left[ \varphi \sinh \varphi \left( \cosh \varphi + \cosh \varphi \right) - 2 \sinh^2 \varphi \right] = \frac{1}{\sinh \varphi \cosh \varphi} \int_{\varphi}^{\varphi} d\varphi \left[ \sinh \varphi \right] \tag{B-18}
\]
\[
\times \varphi \sinh \varphi + \frac{1}{2} \varphi \sinh \varphi \left( \frac{3}{4} \cosh^2 \varphi \right) - \frac{9}{4} \sinh \varphi \cosh \varphi \left( \frac{3}{4} \cosh^2 \varphi \right) = \frac{\varphi}{\sinh \varphi \cosh \varphi} \left[ \sinh \varphi \left( \frac{3}{4} \cosh^2 \varphi \right) - \frac{9}{4} \sinh \varphi \cosh \varphi \right] = \frac{\varphi}{\sinh \varphi \cosh \varphi} \left[ \frac{3}{4} \cosh^2 \varphi - \frac{9}{4} \sinh \varphi \cosh \varphi \right] = \frac{\varphi}{\sinh \varphi \cosh \varphi} \left[ \frac{3}{4} \cosh^2 \varphi - \frac{9}{4} \sinh \varphi \cosh \varphi \right]. \tag{B-19}
\]

Thus, the integral under consideration is equal to
\[
\frac{1}{kE} \int_{m_c}^E dE k \left( \frac{\bar{E} + E}{2E} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\bar{E} - E}{E} = \frac{1}{2} \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{9}{4} \right]. \tag{B-18}
\]

As a result, the function \( f_A^{(1)}(\bar{E}) \) is given by
\[
f_A^{(1)}(\bar{E}) = \frac{13}{4} + \left( -1 + \frac{3}{8} \bar{\beta} - \frac{3}{8} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{2}{\beta} \left[ 2 \beta \ln \left( \frac{2 \beta}{1 + \beta} \right) + \ln \left( \frac{2 \beta}{1 + \beta} \right) \right] \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right]. \tag{B-19}
\]
Summing up Eq.(A-71) and Eq.(B-19) for the function $f_A(E)$ we obtain the following analytical expression

$$f_A(E) = \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) + \frac{23}{8} + 2 \ln \left( \frac{2\beta}{1+\beta} \right) \left[ \frac{1}{\beta} \ln \left( \frac{1+\beta}{1-\beta} \right) - 2 \right] + \frac{4}{\beta} L_2 \left( \frac{2\beta}{1+\beta} \right) + \frac{3}{8} \left( \frac{7}{3} \right) \frac{1}{\beta} \ln \left( \frac{1+\beta}{1-\beta} \right)$$

Thus, we have shown that the function $f_A(E)$ coincides with the function $f(E)$, calculated by Vogel [7], Fayans [8] and Raha, Myhrer and Kudohara [11] (see Eq.(A-76)).

**IX. APPENDIX C: ANALYTICAL EXPRESSION FOR FUNCTIONS $f_B^\gamma(E)$ AND $f_B(E)$**

In this Appendix we give a detailed calculation of the integrals, defining the function $f_B^\gamma(E)$. Making a change of variables $E = m_e \cosh \varphi$ and $E = m_e \cosh \tilde{\varphi}$ for the first integral in Eq.(A-65) we obtain

$$\frac{1}{kE} \int_{m_e}^{E} \frac{dE}{E-E} \left\{ kE \beta \left[ \frac{1}{\beta} \ln \left( \frac{1+\beta}{1-\beta} \right) - 2 \right] - kE \beta \left[ \frac{1}{\beta} \ln \left( \frac{1+\beta}{1-\beta} \right) - 2 \right] \right\} = \frac{2}{\sinh^2 \varphi} \int_0^\varphi \frac{d\varphi}{\sinh \varphi + \cosh \varphi} \left[ \sinh^2 \varphi \times \left( \frac{\varphi}{\tanh \varphi} - 1 \right) \right] - \frac{2}{\sinh^2 \varphi} \frac{\varphi}{\cosh \varphi + \sinh \varphi} \left[ - \varphi \sinh \varphi (\cosh \varphi - \cosh \varphi) + \cosh \varphi \left( \varphi - \varphi \right) \sinh \varphi + \varphi \cosh \varphi (\sinh \varphi - \varphi \varphi) \right] = I_1 + I_2 + I_3 + I_4.$$  

The calculation of the integral $I_1$:

$$I_1 = - \frac{2}{\sinh^2 \varphi} \int_0^\varphi d\varphi \sinh^2 \varphi \varphi = - \frac{2}{\sinh^2 \varphi} \left( \varphi^2 \sinh^2 \varphi - \frac{1}{8} \cosh 2 \varphi \varphi + \frac{1}{8} \frac{1}{4} \varphi^2 \right) =$$

$$= \frac{1}{2} \frac{\varphi}{\tanh \varphi} + \frac{1}{2} \frac{\varphi^2}{\sinh^2 \varphi} = \frac{1}{2} - \frac{1}{2} \frac{1}{2 \beta \ln \left( \frac{1+\beta}{1-\beta} \right)} + \frac{1}{8 \beta^2} \frac{1}{\ln \left( \frac{1+\beta}{1-\beta} \right)}.$$  

The integral $I_1$ is equal to

$$I_1 = \frac{1}{2} - \frac{1}{2 \beta} \ln \left( \frac{1+\beta}{1-\beta} \right) + \frac{1}{8 \beta^2} \frac{1}{\ln \left( \frac{1+\beta}{1-\beta} \right)}.$$  

The calculation of the integral $I_2$:

$$I_2 = \frac{2}{\sinh \varphi \tanh \varphi} \int_0^\varphi \frac{d\varphi}{\cosh \varphi - \cosh \varphi} \sinh^2 \varphi \varphi = \frac{2}{\sinh \varphi \tanh \varphi} \left( \int_0^\varphi \frac{d\varphi}{\cosh \varphi - \cosh \varphi} \left( \cosh^2 \varphi - \cosh^2 \varphi \right) + \sinh^2 \varphi \times \int_0^\varphi \frac{d\varphi}{\cosh \varphi - \cosh \varphi} \right) =$$

$$= \frac{2}{\sinh \varphi \tanh \varphi} \left[ - \cosh \varphi + \frac{1}{2} \varphi^2 \cosh \cosh \varphi + \sinh^2 \varphi \right] = \frac{2}{\sinh \varphi \tanh \varphi}[ -1 + \cosh \varphi \varphi + \frac{1}{2} \varphi^2 \cosh \cosh \varphi + \sinh^2 \varphi].$$  

The calculation of the integral in Eq.(C-4) we carry out by changing variables $u = e^x$ and $\tilde{u} = e^{\tilde{x}}$. We get

$$\int_0^\varphi \frac{d\varphi}{\cosh \varphi - \cosh \varphi} = 2 \int_1^\varphi \frac{du}{(u-u)(u)(u)} = \frac{1}{\sinh \varphi} \int_1^u \frac{du}{u} - \frac{1}{\sinh \varphi} \int_1^u \frac{du}{u} =$$

$$= \frac{1}{\sinh \varphi} \int_1^{e^{-\varphi}} \frac{dt}{1-\tilde{u}} + \frac{2\tilde{\varphi}}{\sinh \varphi} \int_1^{e^{-\varphi}} \frac{dt}{1-\tilde{u}} + \frac{1}{\sinh \varphi} \int_1^{e^{-\varphi}} \frac{dt}{1-\tilde{u}} = \frac{1}{\sinh \varphi} \int_1^{e^{-\varphi}} \frac{dt}{1-\tilde{u}} = \frac{1}{\sinh \varphi} L(1-e^{-\varphi}) - \frac{2\tilde{\varphi}^2}{\sinh \varphi} =$$

$$= \mathcal{I}(1) = \frac{1}{\sinh \varphi} L(1-e^{-\varphi})$$

$$+ \frac{2}{\sinh \varphi} L(1-e^{-\varphi}) - \frac{1}{\sinh \varphi} L(1-e^{-2\varphi}) =$$

$$= \frac{1}{\sinh \varphi} L(1-e^{-\varphi}).$$  

(C-5)
In terms $\bar{\varphi}$ the integral $I_2$ is

$$I_2 = \frac{2}{\sinh \varphi \tanh \varphi} \left[ -1 + \cosh \bar{\varphi} - 2 \sinh \bar{\varphi} \ln(1 + e^{-\varphi}) + \frac{1}{2} \varphi^2 (\cosh \bar{\varphi} - \sinh \bar{\varphi}) + 2 \sinh \bar{\varphi} L(1 - e^{-\varphi}) \right. \\
\left. - \sinh \bar{\varphi} L(1 - e^{-2\varphi}) \right] = \frac{2}{\sinh \varphi \tanh \varphi} + \frac{2}{\tanh^2 \varphi} - \frac{4\bar{\varphi}}{\tanh \varphi} \left[ \ln(1 + e^{-\varphi}) + \varphi^2 \left( \frac{1}{\tanh^2 \varphi} - \frac{1}{\tanh \varphi} \right) \right] \\
+ \frac{4}{\tanh \varphi} L(1 - e^{-\varphi}) - \frac{2}{\tanh \varphi} L(1 - e^{-2\varphi}).$$

(C-6)

The integral $I_2$ is equal to

$$I_2 = 2 \left( \frac{\sqrt{1 - \beta^2}}{\beta} - \frac{\ln \left( \frac{1 + \beta}{1 - \beta} \right)}{\beta} \right) \left( 1 + \sqrt{1 - \beta^2} \right) + \frac{1 - \beta}{4\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{4}{\beta} L \left( 1 - \sqrt{1 - \beta^2} \right)$$

$$- \frac{2}{\beta} L \left( \frac{2\beta}{1 + \beta} \right).$$

(C-7)

The calculation of the integral $I_3$:

$$I_3 = \frac{2\bar{\varphi}}{\sinh \varphi \tanh \varphi} \int_0^{\bar{\varphi}} \frac{d\varphi}{\cosh \varphi - \sinh \varphi} \left( \sinh \varphi - \sinh \bar{\varphi} \right) = \frac{2\bar{\varphi}}{\sinh \varphi \tanh \varphi} \left[ - \sinh \bar{\varphi} \ln(\cosh \bar{\varphi} - 1) \right. \\
\left. + \int_0^{\bar{\varphi}} d\varphi \cosh \varphi \ln(\cosh \varphi - \sinh \varphi) \right] = \frac{2\bar{\varphi}}{\sinh \varphi \tanh \varphi} \left[ - \sinh \bar{\varphi} \ln(\cosh \bar{\varphi} - 1) + (\ln \bar{\varphi} - 1) \sinh \bar{\varphi} \right. \\
\left. + 2 \sinh \bar{\varphi} \ln \sinh \bar{\varphi} \sinh \bar{\varphi} \right],$$

where we have used Eq.(B-13). In terms of $\bar{\varphi}$ the integral $I_3$ is equal to

$$I_3 = - \frac{2\bar{\varphi}}{\tanh \varphi} \ln(\cosh \bar{\varphi} - 1) + 2(\ln \bar{\varphi} - 1) \frac{\varphi}{\tanh \varphi} + \frac{4\bar{\varphi}}{\tanh \varphi} \ln(\sinh \bar{\varphi}) - \frac{2\varphi^2}{\tanh^2 \varphi}.$$

(C-8)

For the integral $I_3$ we obtain the following expression

$$I_3 = - \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 - \sqrt{1 - \beta^2}}{\beta} \right) + (\ln \bar{\varphi} - 1) \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right)$$

$$\ln \left( \frac{\beta}{\sqrt{1 - \beta^2}} \right)$$

$$- \frac{1}{2\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) = \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) + \frac{1}{\beta} \ln \left( \frac{2\beta}{1 + \beta} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{1}{2\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right)$$

$$- \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right).$$

(C-10)

The calculation of the integral $I_4$ is as follows:

$$I_4 = \frac{2}{\sinh^2 \varphi} \int_0^{\bar{\varphi}} d\varphi \sinh \varphi (\cosh \varphi + \sinh \varphi) = \frac{2}{\sinh^2 \varphi} \left[ 1 - \cosh \varphi + \frac{3}{2} \sinh^2 \varphi \right] = 3 - 2 \frac{\sqrt{1 - \beta^2}}{\beta^2} \left( 1 - \sqrt{1 - \beta^2} \right).$$

(C-11)

The integral $I_4$ is equal to

$$I_4 = 3 - 2 \frac{\sqrt{1 - \beta^2}}{\beta^2} \left( 1 - \sqrt{1 - \beta^2} \right).$$

(C-12)

Summing up the contributions we obtain

$$\frac{1}{kE\beta} \int_{m_c}^{E} \frac{dE}{E - E} \left\{ kE\beta \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] - kE\beta \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\} = \frac{7}{2} + \frac{2}{\beta^2} (1 - \sqrt{1 - \beta^2})^2$$

$$- \frac{2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( 1 + \sqrt{1 - \beta^2} \right) + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) + \frac{1}{\beta} \ln \left( \frac{2\beta}{1 + \beta} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right)$$

$$- \frac{3}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( \frac{1 - \beta^2}{8\beta^2} \right) \ln \left( \frac{1 + \beta}{1 - \beta} \right) + \frac{4}{\beta} L \left( 1 - \sqrt{1 - \beta^2} \right) - \frac{2}{\beta} L \left( \frac{2\beta}{1 + \beta} \right).$$

(C-13)
The calculation of the second integral Eq.(A-65) is as follows:

\[
\frac{1}{kE\beta} \int_{m_e}^E dE \frac{1}{2} (E + E) \left[ \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] = \frac{1}{\sinh^2 \varphi} \int_0^\varphi d\varphi \left[ \cosh \varphi \cosh \varphi + \varphi \cosh^2 \varphi - \cosh \varphi \sinh \varphi - \sinh \varphi \cosh \varphi \right] = -\frac{11}{4} + 2 \frac{\sqrt{1 - \beta^2}}{\beta^2}
\]

Summing up the contributions we obtain the function \( f_B^f(\beta) \)

\[
f_B^f(\beta) = \frac{3}{4} + \frac{2}{\beta^2} (1 - \sqrt{1 - \beta^2}) - \frac{2}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( 1 + \sqrt{1 - \beta^2} \right) + \frac{1}{2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \ln \left( 1 - \sqrt{1 - \beta^2} \right) - \frac{2}{\beta} \ln \left( \frac{2\beta}{1 + \beta} \right)
\]

As a result the function \( f_B(\beta) \) is given by

\[
f_B(\beta) = \frac{3}{2} \ln \left( \frac{m_p}{m_e} \right) + \frac{3}{8} + \frac{2}{\beta^2} (1 - \sqrt{1 - \beta^2}) + \frac{4}{\beta} \ln \left( 1 - \sqrt{1 - \beta^2} \right) - \frac{1 - 4\beta + 3\beta^2}{16\beta^2} \ln^2 \left( \frac{1 + \beta}{1 - \beta} \right)
\]

where we have used the relations

\[
\ln \left( \frac{1 + \sqrt{1 - \beta^2}}{1 - \sqrt{1 - \beta^2}} \right) = 2 \ln \left( \frac{\sqrt{1 + \beta} + \sqrt{1 - \beta}}{\sqrt{1 + \beta} - \sqrt{1 - \beta}} \right),
\]

\[
\ln \left( 1 + \sqrt{1 - \beta^2} \right) = \frac{1}{2} \ln \left( \frac{\sqrt{1 + \beta} + \sqrt{1 - \beta}}{\sqrt{1 + \beta} - \sqrt{1 - \beta}} \right) + \frac{1}{2} \ln \left( \frac{2\beta}{1 + \beta} \right).
\]

The radiative corrections, defined by the function \((\alpha/\pi) f_B(\beta)\), agrees well with the results, obtained by Fukugita and Kubota [10] and Raha, Myhrer and Kubodera [11].

X. APPENDIX D: AMPLITUDE AND CROSS SECTION OF RADIATIVE INVERSE \(\beta\)-DECAY WITH ACCOUNT FOR CONTRIBUTION OF PROTON–PHOTON INTERACTION

In this Appendix we calculate the differential cross section for the radiative inverse \(\beta\)-decay by taking into account the contribution of the proton–photon interaction. Such a contribution is important for a gauge invariant calculation of the amplitude and the cross section for the radiative inverse \(\beta\)-decay.

The amplitude of the radiative inverse \(\beta\)-decay, taking into account the contribution of the proton–photon interactions, is defined by (see Eq.(A-53))

\[
M(\bar{\nu}_e p \rightarrow n e^+ \gamma)_{\lambda'} = \varepsilon'_{\lambda' \nu} M_\alpha(\bar{\nu}_e p \rightarrow n e^+ \gamma) = e \frac{G_F}{\sqrt{2}} V_{ud}
\]

\[
\times \left\{ [\bar{u}_n W^\mu u_p] \frac{\varepsilon'_{\lambda' \nu}}{m_e + k + \hat{q} - i0} \varepsilon'_{\lambda' \nu} - [\bar{u}_n W^\mu u_p] \frac{1}{m_p - k_p + \hat{q} - i0} \varepsilon'_{\lambda' \nu} \right\} v_p O_{\mu} v, \quad (D-1)
\]

where \( q = (\omega, \hat{q} = \omega \hat{n}) \) and \( \varepsilon'_{\lambda' \nu} \) are the 4–momentum and 4–polarisation vector of a photon with \( \lambda' = 1, 2 \). The polarisation vector \( \varepsilon'_{\lambda' \nu} \) obeys the constraint \( q \cdot \varepsilon'_{\lambda' \nu} = 0 \). The amplitude \( M_\alpha(\bar{\nu}_e p \rightarrow n e^+ \gamma) \) is gauge invariant. Indeed,
replacing \( \varepsilon_\lambda^a \rightarrow q^a \) and using the Dirac equations \((\hat{k} + m_c)v = 0\) and \((\hat{k}_p - m_p)u_p = 0\) for the positron and the proton, we obtain \( q^a M_{\lambda a}(\tilde{\nu}_e p \rightarrow n e^+ \gamma) = 0 \).

Since the proton and neutron in the radiative inverse \( \beta \)-decay are non–relativistic, the amplitude of the radiative inverse \( \beta \)-decay can be given by the expression

\[
M(\tilde{\nu}_e p \rightarrow n e^+ \gamma)_{\lambda'} = - e \frac{G_F}{\sqrt{2}} V_{ud} \frac{m_p}{\omega} \left( \frac{M^{(e)}_{\lambda'}(k^\prime \cdot n)}{E - \hat{k} \cdot n} - \frac{M^{(p)}_{\lambda'}(k_p \cdot n)}{k_p \cdot n} \right),
\]

(D-2)

where \( n = q/\omega = (1, \vec{n}) \) is a 4–vector, normalised by \( n^2 = 0 \). The amplitudes \( M^{(e)}_{\lambda'} \) and \( M^{(p)}_{\lambda'} \) and their hermitian conjugate \( M^{(e)\dagger}_{\lambda'} \) and \( M^{(p)\dagger}_{\lambda'} \) are equal to

\[
M^{(e)}_{\lambda'} = [\varphi_{p\lambda}^\dagger | \hat{\nu}_e \gamma^0(1 - \gamma^5) Q_{\lambda'} v] - \lambda \left[ \varphi_{n\lambda}^\dagger \bar{\sigma} \varphi_{n\lambda} \right] [\bar{\nu}_e \gamma^0(1 - \gamma^5) Q v],
\]

\[
M^{(p)}_{\lambda'} = 2k_p \cdot \varepsilon^\lambda_{\lambda'} \left[ [\varphi_{p\lambda}^\dagger | \hat{\nu}_e \gamma^0(1 - \gamma^5) v] - \lambda \left[ \varphi_{n\lambda}^\dagger \bar{\sigma} \varphi_{n\lambda} \right] [\bar{\nu}_e \gamma^0(1 - \gamma^5) v] \right],
\]

(D-3)

and

\[
M^{(e)\dagger}_{\lambda'} = [\varphi_{p\lambda}^\dagger | \hat{\nu}_e \gamma^0(1 - \gamma^5) v] - \lambda \left[ \varphi_{n\lambda}^\dagger \bar{\sigma} \varphi_{n\lambda} \right] [\bar{\nu}_e \gamma^0(1 - \gamma^5) v],
\]

\[
M^{(p)\dagger}_{\lambda'} = 2k_p \cdot \varepsilon^\lambda_{\lambda'} \left[ [\varphi_{p\lambda}^\dagger | \bar{\nu}_e \gamma^0(1 - \gamma^5) v] - \lambda \left[ \varphi_{n\lambda}^\dagger \bar{\sigma} \varphi_{n\lambda} \right] [\bar{\nu}_e \gamma^0(1 - \gamma^5) v] \right],
\]

(D-4)

where \( Q_{\lambda'} = 2k \cdot \varepsilon^\lambda_{\lambda'} + \hat{q} \varepsilon^\lambda_{\lambda'} \) and \( \bar{Q}_{\lambda'} = \gamma^0 Q_{\lambda'}^\dagger \gamma^0 = 2k \cdot \varepsilon^\lambda_{\lambda'} + \hat{\varepsilon}_{\lambda'} \hat{q} [14] \). For the calculation of the contributions of the proton–photon interaction we have kept only the leading terms in the large proton mass expansion.

The squared absolute value of the amplitude Eq.(D-2), averaged over polarisations of the proton and summed over polarisations of the neutron and positron is defined by

\[
\frac{1}{2} \sum_{\text{pol.}} |M(\tilde{\nu}_e p \rightarrow n e^+ \gamma)_{\lambda'}|^2 = \pi \alpha G_F^2 |V_{ud}|^2 \frac{m_p}{\omega^2} \left\{ \frac{1}{(E - \hat{k} \cdot \vec{n})^2} \sum_{\text{pol.}} |M^{(e)}_{\lambda'}|^2 + \frac{1}{(k_p \cdot n)^2} \sum_{\text{pol.}} |M^{(p)}_{\lambda'}|^2 \right\} - \frac{1}{(E - \hat{k} \cdot \vec{n})(k_p \cdot n)} \sum_{\text{pol.}} \left( M^{(e)\dagger}_{\lambda'} M^{(p)\dagger}_{\lambda'} M^{(e)}_{\lambda'} + M^{(p)}_{\lambda'} M^{(e)\dagger}_{\lambda'} M^{(e)}_{\lambda'} \right) \]

(D-5)

Since the first term has been calculated in Appendix A (see Eq.(A-57) and Eq.(A-58)), we calculate below the second and the third terms only. We get

\[
\frac{1}{2} \sum_{\text{pol.}} |M^{(e)}_{\lambda'}|^2 = 32E_{\nu}(1 + 3\lambda^2) \left\{ \left[ (k \cdot \varepsilon^\lambda_{\lambda'})(k \cdot \varepsilon^\lambda_{\lambda'}) \left( 1 + \frac{\omega}{E} \right) - (E - \hat{k} \cdot \vec{n}) \right] \frac{\omega}{E} \frac{1}{2} \left( (k \cdot \varepsilon^\lambda_{\lambda'}) \varepsilon^0_{\lambda'} + (k \cdot \varepsilon^\lambda_{\lambda'}) \varepsilon^0_{\lambda'} \right) \right.
\]

\[
- \left. (E - \hat{k} \cdot \vec{n}) \frac{\omega^2}{E} \left( \varepsilon^\lambda_{\lambda'} \cdot \varepsilon_{\lambda'} \right) \right\} + a_0 \frac{k_p}{E_{\nu}} \left[ (k \cdot \varepsilon^\lambda_{\lambda'})(k \cdot \varepsilon^\lambda_{\lambda'}) \left( \beta + \hat{n} \frac{\omega}{E} \right) - (E - \hat{k} \cdot \vec{n}) \frac{\omega}{E} \frac{1}{2} \left( (k \cdot \varepsilon^\lambda_{\lambda'}) \varepsilon^\lambda_{\lambda'} + (k \cdot \varepsilon^\lambda_{\lambda'}) \varepsilon^\lambda_{\lambda'} \right) \right.
\]

\[
- \left. (E - \hat{k} \cdot \vec{n}) \frac{\omega^2}{E} \left( \varepsilon^\lambda_{\lambda'} \cdot \varepsilon_{\lambda'} \right) \right\} \right. \right\}
\]

(D-6)

\[
\frac{1}{2} \sum_{\text{pol.}} |M^{(p)}_{\lambda'}|^2 = 32E_{\nu}(1 + 3\lambda^2) \left( k_p \cdot \varepsilon^\lambda_{\lambda'}(k_p \cdot \varepsilon^\lambda_{\lambda'}) \left( 1 + a_0 \frac{k \cdot \hat{k}_p}{EE_{\nu}} \right) \right)
\]

(D-7)

and

\[
\frac{1}{2} \sum_{\text{pol.}} \left( M^{(e)\dagger}_{\lambda'} M^{(p)\dagger}_{\lambda'} M^{(e)}_{\lambda'} + M^{(p)}_{\lambda'} M^{(e)\dagger}_{\lambda'} M^{(e)}_{\lambda'} \right) = 32E_{\nu}(1 + 3\lambda^2) \left( (k_p \cdot \varepsilon^\lambda_{\lambda'})(k_p \cdot \varepsilon^\lambda_{\lambda'}) \left( 1 + a_0 \frac{k \cdot \hat{k}_p}{EE_{\nu}} \right) \right) \]

(D-8)

The contribution of one–virtual photon exchanges is invariant under gauge transformation of the photon propagator \( D_{\mu\nu}(q) \rightarrow D_{\mu\nu}(q) + c(q^2) q_\mu q_\nu \), where \( c(q^2) \) is an arbitrary function and \( q^2 \neq 0 \). Due to such an invariance one can calculate the contribution of one–virtual photon exchanges in any gauge [20]. Following [20] we have calculated the contribution of one–virtual photon exchanges in the Feynman gauge (see also see also Appendix B of [14]).

One may show that the squared absolute value of the amplitude Eq.(D-2), averaged over the polarisations of the proton and summed over the polarisations of the neutron and positron and given by Eqs.(D-5) - (D-8), is invariant
under the gauge transformation $\varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{\nu} \to \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{\nu} + c \eta^{\mu} \eta^{\nu}$, where $c$ is an arbitrary constant. Indeed, making such a gauge transformation in Eq.(D-5) one obtains an additional contribution, proportional to the constant $c$

$$\delta \frac{1}{2} \sum_{\text{pol.}} |M(\bar{\nu}_e p \rightarrow n e^+ \gamma)_{\lambda}|^2 = c \frac{32 E E_{\bar{\nu}}}{2 \pi} (1 + 3 \lambda^2) \pi \alpha G_F^2 |V_{ud}|^2 \frac{m_e^2}{\omega^2} \left\{ \frac{1}{(E - \vec{k} \cdot \vec{n})^2} \left( \frac{k \cdot q}{E} \right)^2 \left( 1 + \frac{\omega}{E} \right) \right\}$$

$$+ (E - \vec{k} \cdot \vec{n})(k \cdot q) \frac{\omega^2}{E} + a_0 \frac{\vec{k}_\nu}{E_{\bar{\nu}}} \left\{ \left( \frac{\omega}{E} \right)^2 \left( 1 + \frac{\omega}{E} \right) \right\} + \frac{k_p \cdot q}{(k_p \cdot n)^2} \left( 1 + a_0 \frac{\vec{k}_\nu}{E_{\bar{\nu}}} \right) \beta \right\} = 0,$$  \hspace{1cm} (D-9)

where we have used the relations $q = \omega n = \omega(1, \vec{n}), \ E - \vec{k} \cdot \vec{n} = (k \cdot q)/\omega$ and $(k_p \cdot n) = (k_p \cdot q)/\omega$. Thus, due to gauge invariance of Eq.(D-5) for the summation over polarisation one may use any gauge. However, it is obvious that one has to sum over the physical degrees of freedom of the real photons, which are defined by the polarisation vector $\varepsilon_{\lambda'} = (0, \vec{e}_{\lambda'})$ [36]. The polarisation vector $\vec{e}_{\lambda'}$ has the following properties [36]

$$\vec{q} \cdot \varepsilon_{\lambda'} = \vec{q} \cdot \varepsilon_{\lambda'} = 0,$$

$$\varepsilon_{\lambda'} \cdot \varepsilon_{\lambda''} = \delta_{\lambda \lambda'},$$

$$\sum_{\lambda'=1,2} \varepsilon_{\lambda'} \varepsilon_{\lambda'} = \delta^{ij} - \frac{q_i q_j}{\omega^2} = \delta^{ij} - n^i n^j.$$  \hspace{1cm} (D-10)

Summing over physical degrees of freedom of a real photon we arrive at the expression

$$\frac{1}{2} \sum_{\lambda'=1,2} \sum_{\text{pol.}} |M(\bar{\nu}_e p \rightarrow n e^+ \gamma)_{\lambda}|^2 = 32 E E_{\bar{\nu}}(1 + 3 \lambda^2) \pi \alpha G_F^2 |V_{ud}|^2 \frac{m_e^2}{\omega^2} \left\{ \frac{\beta^2 - (\vec{\beta} \cdot \vec{n})^2}{(1 - \vec{\beta} \cdot \vec{n})^2} \left( \frac{\omega}{E} \right)^2 \left( 1 + \frac{1}{1 - \vec{\beta} \cdot \vec{n}} \right) E^2 \right\}$$

$$+ a_0 \frac{\vec{k}_\nu}{E_{\bar{\nu}}} \left\{ \frac{\beta^2 - (\vec{\beta} \cdot \vec{n})^2}{(1 - \vec{\beta} \cdot \vec{n})^2} \left( \frac{\omega}{E} \right)^2 \right\} \left( \frac{\beta - \vec{n}}{1 - \vec{\beta} \cdot \vec{n}} - \frac{\vec{n} \cdot \omega}{E} + \frac{\vec{n} \cdot \omega^2}{E^2} \right) \right\}.$$  \hspace{1cm} (D-11)

This gives the angular and photon–energy distribution of the radiative inverse $\beta$–decay, defined by (see Eq.(A-59))

$$\frac{d^2 \sigma(\gamma)(E_{\bar{\nu}}, \cos \theta_{\bar{\nu}})}{d \omega d \cos \theta_{\bar{\nu}}} = (1 + 3 \lambda^2) \frac{\alpha}{2 \pi} \frac{G_F^2 |V_{ud}|^2}{\omega} E \int d\Omega_{\bar{\nu}} \frac{1}{4\pi} \left\{ \beta \frac{(1 + \vec{\beta} \cdot \vec{n})^2}{(1 - \vec{\beta} \cdot \vec{n})^2} \left( \frac{\omega}{E} \right)^2 \left( \frac{1}{1 - \vec{\beta} \cdot \vec{n}} \right) E^2 \right\}$$

$$+ a_0 \frac{\vec{k}_\nu}{E_{\bar{\nu}}} \left\{ \frac{\beta^2 - (\vec{\beta} \cdot \vec{n})^2}{(1 - \vec{\beta} \cdot \vec{n})^2} \left( \frac{\omega}{E} \right)^2 \right\} \left( \frac{\beta - \vec{n}}{1 - \vec{\beta} \cdot \vec{n}} - \frac{\vec{n} \cdot \omega}{E} + \frac{\vec{n} \cdot \omega^2}{E^2} \right) \right\},$$  \hspace{1cm} (D-12)

where $E = \bar{E} - \omega$ and $\beta = k/E = \sqrt{1 - m_e^2/E^2}$. Up to a common factor the expression in curl brackets coincides with Eq.(29), obtained by Fukugita and Kubota in Ref.[10], and Eq.(9), obtained by Raha, Myhrer and Kudohera in Ref.[11].

We would like to notice that the contributions of the radiative inverse $\beta$–decay to the correlation coefficients $A(E)$ and $B(E)$, described by the functions $g(61)(\beta)$ and $g(62)(\beta)$ in the paper by Fukugita and Kubota [10] (see Eq.(32) in Ref.[10]), in our notation are equal to

$$\frac{1}{2} g(61)(\beta) = g(61)(E, \mu) = \ell n \left( \frac{m_e}{\mu} \right) \left[ \frac{1}{\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{17}{4} \ell n \left( \frac{2 \bar{\beta}}{1 + \beta} \right) \left( \frac{1}{\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right) + \frac{3}{\beta} \ell n \left( \frac{2 \bar{\beta}}{1 + \beta} \right),$$

$$+ \frac{1}{4\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - 4 \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{7}{8} \ell n \left( \frac{1 + \beta}{1 - \beta} \right)$$  \hspace{1cm} (D-13)

and

$$\frac{1}{2} g(62)(\beta) = g(62)(E, \mu) = \ell n \left( \frac{m_e}{\mu} \right) \left[ \frac{1}{\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{7}{4} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{2}{\beta} \ell n \left( \frac{2 \bar{\beta}}{1 + \beta} \right) + \frac{4}{\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) + \ell n \left( \frac{2 \bar{\beta}}{1 + \beta} \right) + \ell n \left( \frac{1 + \beta}{1 - \beta} \right)$$

$$\times \left[ \frac{1}{\beta} \ell n \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right].$$  \hspace{1cm} (D-14)
where in the function $g^{(1)}(\beta)$, given by Eq.(31) of Ref.[10], we have summed up the first two terms. It is easy to check that our results are in agreement with the results, obtained by Fukugita and Kubota [10]. For the derivation of Eqs.(D-13) and (D-14) we have used the definitions of the functions $f_{\gamma}(\bar{E}, \mu)$ and $g_{\beta\gamma}(\bar{E}, \mu)$, given by Eqs.(A-68) and (A-69), respectively, and the functions $f_{\gamma}^{(\gamma)}(\bar{E})$ and $g_{B}^{(\gamma)}(\bar{E})$, given by Eqs.(B-19) and (C-15), respectively, and the relations Eqs.(B-8) and (C-17).

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