Duality and the Fractional Quantum Hall Effect

A.P.Balachandran¹, L. Chandar¹, B. Sathiapalan²

¹ Department of Physics, Syracuse University, Syracuse, NY 13244-1130, U.S.A.
² Department of Physics, Pennsylvania State University, 120, Ridge View Drive, Dunmore, PA-18512, U.S.A.

Abstract

The edge states of a sample displaying the quantum Hall effect (QHE) can be described by a 1+1 dimensional (conformal) field theory of $d$ massless scalar fields taking values on a $d$-dimensional torus. It is known from the work of Naculich, Frohlich et al. and others that the requirement of chirality of currents in this scalar field theory implies the Schwinger anomaly in the presence of an electric field, the anomaly coefficient being related in a specific way to Hall conductivity. The latter can take only certain restricted values with odd denominators if the theory admits fermionic states. We show that the duality symmetry under the $O(d,d;\mathbb{Z})$ group of the free theory transforms the Hall conductivity in a well-defined way and relates integer and fractional QHE’s. This means, in particular, that the edge spectra for dually related Hall conductivities are identical, a prediction which may be experimentally testable. We also show that Haldane’s hierarchy as well as certain of Jain’s fractions can be reproduced from the Laughlin fractions using the duality transformations. We thus find a framework for a unified description of the QHE’s occurring at different fractions. We also give a simple derivation of the wave functions for fractions in Haldane’s hierarchy.
1. Introduction

Fractional Quantum Hall Effect (FQHE) is the phenomenon of quantized conductance at values that are fractions of what simple considerations suggest for a system of non-interacting electrons. There have been many attempts to understand this phenomenon. Following the original work of Laughlin describing the ground state of the FQHE for a filling fraction $\nu = \frac{1}{m}$ (where $m$ is odd), there have been extensions describing the FQHE for a general rational $\nu$. The theory of Jain has the extra appeal of establishing a connection between the integer and fractional effects. This is important phenomenologically (apart from being aesthetically appealing), since experimentally there does not seem to be much difference between the physics at Hall plateaus corresponding to integer and fractional filling factors.

Many of these models can be described by Chern-Simons theories in the interior of the disc. The advantage of using Chern-Simons theories is that they have observables only at the boundary of the manifold (here a disc). In particular, these theories assert that quantum Hall effect can be described by observables living at the edge of the sample. A simple way to motivate these edge currents is from classical considerations. An edge for the Hall sample arises because of the existence of a potential barrier that prevents the electrons from escaping the finite region. We thus expect a radially outward electric field at the boundary. This will now create a Hall current tangential to the boundary because of the presence of a magnetic field perpendicular to the sample. This Hall current is the edge current mentioned above. This argument can be made precise by considering the quantum mechanical wave functions of the electrons confined by a potential barrier as has been done by Halperin. In this work he has also shown the intimate connection between this current and the integer quantization of Hall conductivity. There have subsequently been many papers emphasizing the importance of
edge states in understanding aspects of both Integer Quantum Hall Effect (IQHE) and FQHE [14, 15, 16, 17, 18].

In this work, we utilize the fact that the Chern-Simons theories mentioned above are equivalent to theories of chiral bosons living at the edge of the sample [11, 12]. We thus use a 1+1 dimensional theory consisting of $d$ massless scalar fields with the target manifold being a $d$-torus. We can then write down an action similar to that used in various dimensional compactification schemes for string theories [19, 20]. These theories have certain duality symmetries [21, 22, 23] which in the context of Hall effect, have interesting consequences. When we couple the system to electromagnetism, it is seen that the requirement of chirality of the fields and currents relates the Hall conductivity to the so-called “metric” on the target torus [8, 9, 14]. If we assume that the scalar fields have definite electromagnetic couplings, then, for different choices of the metric, we get different Hall conductivities. Now, with the help of the duality symmetry, we can relate integer and fractional values, and by choosing specific elements of the generalised duality transformations, we can get the hierarchical values predicted by Haldane as well as Jain’s fractions. In this way, we can relate the integer and fractional conductivities in these schemes. The important prediction that arises as a consequence is that the spectra of edge excitations are identical for the two conductivities so related. This prediction may be experimentally testable.

Using conformal field theory techniques, we also give a simple derivation of the hierarchical wave functions for electrons and quasiparticles that have appeared in the literature [3, 4, 25, 26, 27, 28]. We will see later in Sections 3 and 4 that this prescription works only so long as we are in the lowest Landau level.

An important result we have used in the course of this work is due to Naculich [29] (and also to Frohlich et al. [9, Wilczek [15] and others). It is that the imposition of chirality on the above scalar field theory leads to the Schwinger anomaly [31] (the anomaly coefficient
being related in a known way \cite{30, 29} to Hall conductivity).

In Section 2, we specialize to the case of one scalar field with its radius $R$ of compactification held arbitrary to begin with. By imposing chirality and requiring the existence of fermionic states in the theory, we get restrictions on this radius. We then gauge this theory preserving chirality, and show how this radius of compactification is related to Hall conductivity. By using the standard $R \rightarrow \frac{1}{R}$ duality that exists for this theory, we then relate Hall conductivities at two fractions which are inverses of each other. By allowing many scalar fields at the boundary, we then show that the fractions compatible with the restrictions obtained on $R$ are always those with odd denominators. These fields incidentally can be interpreted in terms of fields for the electron and the flux tubes attached to it.

In Section 3, we study the response of Hall conductivity to the more general $O(d, d; \mathbb{Z})$ “symmetry” that exists for the free field theory with many scalar fields. We will see that this symmetry makes the problem much richer but also much more complicated.

In Section 4, we motivate a choice of the internal “metric” for the scalar field theory. Then we show how we can obtain both Haldane’s continued fractions as well as certain of Jain’s fractions from subsets of the $O(d, d; \mathbb{Z})$ transformations of Section 3.

In Section 5, we present some conclusions and describe a few open questions.

2. Duality and FQHE for a Chiral Boson

2.1. The Chiral Constraint on the Left-Right Symmetric Boson

We will imagine for the rest of the paper that the Hall system is on a disc $D$ with a circular boundary $\partial D$. In this situation, the excitations at the edge are described by massless fields in 1+1 dimensions. They give the edge currents mentioned in Section 1. Since a fermionic theory in 1+1 dimensions can always be bosonised, we may as well work
with scalar fields. To begin with, we assume that the theory is a free theory with just one
scalar field. We assume that it is valued in a circle. The action for this theory is

$$A_0 = \frac{R^2}{8\pi} \int dt \int_0^{2\pi} dx (\partial_\mu \phi)(\partial^\mu \phi)$$  \hspace{1cm} (2.1)$$

where our spacetime metric has diagonal elements $(1, -1)$ [and zero elsewhere] and $\phi(x)$ is identified with $\phi(x) + 2\pi$.

In the above, we have set the radius of the disc (not to be confused with the radius of compactification in the target space of the scalar field) equal to 1. Restoring the actual radius will rescale the spectrum of the Hamiltonian of the above action by a factor $1/(\text{radius})$.

Note that by assumption, $e^{i\phi}$ is valued in a circle, $e^{i\phi(x)} \in S^1$. The latter condition enables the existence of solitons, similar to sine-Gordon solitons, in this theory. As usual some of these solitons will later on be interpreted as fermions.

The coefficient $R^2$ outside the integral in (2.1) is, as of now, arbitrary.

The above action by itself defines a left-right symmetric theory whereas the real Hall system describes a situation which is chiral. We thus need to impose chirality as a constraint on the states of the theory described by (2.1). If we choose to impose the constraint that the field is left-moving, then

$$\partial_- \phi \equiv \frac{1}{2}(\dot{\phi} - \phi') = 0.$$  \hspace{1cm} (2.2)$$

[Here and below, we define $x_\pm = t \pm x$. So $dx^\mu \partial_\mu = dx^+ \partial_+ + dx^- \partial_-$ where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$, and $A_\mu dx^\mu = A_+ dx^+ + A_- dx^-$ where $A_\pm = \frac{1}{2}(A_0 \pm A_1)$. Also $\partial_\pm$ refer to differentiations with $x_\pm$ as independent variables while $\partial_0, \partial_1$ refer to differentiations with $t$ and $x$ as independent variables.] We should then check if such a constraint is compatible with the equation of motion that arises from the action (2.1). This equation of motion is

$$\partial_\mu \partial^\mu \phi \equiv \frac{1}{4} \partial_+ \partial_- \phi = 0,$$  \hspace{1cm} (2.3)$$
which is clearly compatible with (2.2).

But this will no longer be true when we gauge the action (2.1) with an external electromagnetic field described by the potential $A_\mu$ to obtain

$$S_0[\phi, A] = \frac{R^2}{8\pi} \int d^2 x (D_\mu \phi)^2 - \frac{1}{4k^2} \int d^2 x F_{\mu\nu} F^{\mu\nu},$$

$$D_\mu \phi \equiv \partial_\mu \phi - eA_\mu,$$  \hspace{1cm} (2.4)

$k$ being a constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. [The field $A_\mu$ is of course an addition to the vector potential describing the magnetic field ever present on the Hall sample and responsible for instance for the Landau levels.] This gauging has been done in accordance with the gauge transformation law

$$e^{i\phi} \rightarrow e^{i(\phi + \epsilon)},$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \epsilon$$  \hspace{1cm} (2.5)

so that $e^{i\phi}$ transforms like the phase of a charged Higgs field.

The equations of motion of the above action are

$$\frac{1}{k^2} \partial^\nu F_{\nu\mu} = \frac{eR^2}{4\pi} (D_\mu \phi),$$

$$\partial^\mu (D_\mu \phi) = 0.$$  \hspace{1cm} (2.6)

The constraint (2.2) for the ungauged theory should now be replaced by the gauge invariant constraint

$$D_- \phi \equiv (\partial_- \phi - eA_-) = 0.$$  \hspace{1cm} (2.7)

However this constraint turns out to be incompatible with equation (2.6) above. This can be seen as follows:

$$0 = \partial_+ D_- \phi = \frac{1}{4} (\partial_0 + \partial_1)(D_0 \phi - D_1 \phi)$$

$$\Rightarrow \partial_\mu \left( \frac{eR^2}{4\pi} D^\mu \phi \right) = -\frac{e^2 R^2}{4\pi} E,$$

$$E := \partial_0 A_1 - \partial_1 A_0.$$  \hspace{1cm} (2.8)
Thus the gauge invariant chirality constraint leads to an equation inconsistent with the last equation in (2.4). Note that equation (2.8) is identical to the Schwinger anomaly equation for the $U(1)$ current of a chiral fermion [31]. But we have obtained it here without appealing to the existence of fermions. The only input that we need is chirality. We thus arrive at the interesting result that chirality in a bosonic theory can lead to anomalies. Moreover, the anomaly coefficient is determined completely by the constant $R^2$. [In our context, these results are originally due to Naculich [29]. See also Frohlich [9], Wilczek [15] and Alvarez-Gaume and Witten [32].]

Let us return to (2.8). To obtain this equation, the action (2.4) has to be augmented to the new action

$$S(\phi, A) = S_0(\phi, A) + \frac{eR^2}{4\pi} \int d\phi A \equiv S_0(\phi, A) + \frac{eR^2}{4\pi} \int d^2x \epsilon^{\mu
u} \partial_\mu \phi A_\nu,$$

so that its equations of motion are now compatible with the chirality constraint. The current $j^\mu$ for this theory is

$$j^\mu \equiv -\delta \frac{\delta}{\delta A_\mu} [S(\phi, A) + \frac{1}{4k^2} \int d^2x F_{\mu\nu} F^{\mu\nu}] = \frac{eR^2}{4\pi} (D^\mu \phi + \epsilon^{\mu\nu} \partial_\nu \phi)$$

and its equation of motion gives the anomaly equation

$$\partial_\mu j^\mu = -\frac{e^2 R^2}{4\pi} E.$$ (2.11)

The current $j^\mu$ is not gauge invariant. But we want to identify the current at the edge with the electromagnetic current there in a Hall sample. It should therefore be gauge invariant unlike $j^\mu$. It should also be chiral whereas $j^- = j_0 - j_1$ is non-zero. Thus $j^\mu$ cannot be the electromagnetic current at the edge.

We can however overcome both these drawbacks of $j^\mu$ by replacing $\partial_\nu$ in (2.10) by the covariant derivative $D_\nu$. We then get the current

$$J^\mu = \frac{eR^2}{4\pi} (D^\mu \phi + \epsilon^{\mu\nu} D_\nu \phi).$$ (2.12)
This is the current with the “covariant”, and not the “consistent” anomaly [33, 29].

There is a very nice interpretation for such a modification of the current from \( j^\mu \) to \( J^\mu \) which also shows why \( J^\mu \) is the physical edge current. (Similar arguments have appeared in [9].) This is as follows. The modified action \( S(\phi, A) \) which contains also the anomaly term is clearly not invariant under the gauge transformations (2.5). The physical reason for this is the fact that the boundary of the Hall sample is not isolated. Thus there can be a flow of charge from (to) the edge to (from) the interior. We therefore do not expect the charge at the boundary to be conserved by itself. But since we do know that the total charge (which includes the charge in the interior of the disc as well) is conserved, the action describing dynamics in the bulk should change under the above gauge transformations (2.5) in such a way that the total action is gauge invariant [30, 29, 9, 15]. The simplest such action in the bulk is the Chern-Simons term

\[
-\frac{e^2 R^2}{4\pi} \int_{D \times \mathbb{R}^1} A dA = -\frac{e^2 R^2}{4\pi} \int_{D \times \mathbb{R}^1} d^3 x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \\
A := A_\mu dx^\mu, \quad \epsilon^{\mu\nu\lambda} = \text{Levi-Civita symbol with } \epsilon^{0r\theta} = 1, \quad (2.13)
\]

(apart from other gauge invariant terms like the Maxwell term). [Here \( \mathbb{R}^1 \) accounts for time. Also wedge symbols between differential forms will be omitted in this paper.] Since

\[
\delta \int_{D \times \mathbb{R}^1} A dA = 2 \int_{D \times \mathbb{R}^1} dA \delta A - \int_{\partial D \times \mathbb{R}^1} A \delta A, \quad (2.14)
\]

we see that the current that we obtain from this action has a bulk piece as well as an edge piece. Also the latter contribution is precisely the one that changes \( j^\mu \) to \( J^\mu \) [29]!

It is interesting to note that the full action (that is, the action in the interior plus the action at the boundary) can be written in the manifestly gauge invariant form [15]

\[
S_{total} = S_0 - \frac{R^2}{4\pi} \int_{D \times \mathbb{R}^1} D\phi d(D\phi), \\
D\phi d(D\phi) \equiv d^3 x \epsilon^{\mu\nu\lambda} (D_\mu \phi) \partial_\nu (D_\lambda \phi). \quad (2.15)
\]
Now, using the equations of motion of the action (2.9), we get the anomaly equation

$$\partial_{\mu} j^{\mu} = -\frac{e^2 R^2}{2\pi} E$$ \hspace{1cm} (2.16)

Note that the anomaly term in (2.16) is twice that in (2.11).

On the other hand, using arguments as in [30, 34] (see also [24]), we know that the anomaly coefficient is the same as the Hall conductivity $\sigma_H$.

Let us recapitulate this argument briefly [30, 24]. The anomaly equation (2.16) tells us that the charge is not conserved. In fact the rate of change of charge at the boundary is $-\frac{e^2 R^2}{2\pi} \int dE$. Now, a Hall system obeys the Hall equation in the bulk which in particular gives rise to the equation $J^r = -\sigma_H E_\theta$ in the bulk (where $r, \theta$ refer to polar coordinates on the disk). From this equation we get the rate of change of charge in the bulk to be equal to $\sigma_H \int_{\partial D} dE$. Since charge that is escaping from bulk has to go to the boundary, (2.16) gives the relation

$$\sigma_H = \frac{e^2 R^2}{2\pi}.$$ \hspace{1cm} (2.17)

It is satisfying to note that this is the same as the Hall conductivity that one would have obtained from the Chern-Simons action (2.13) (along with the Maxwell term) in the interior. Thus, for the action (2.13), the current in the bulk is

$$J^\mu = \frac{e^2 R^2}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda,$$ \hspace{1cm} (2.18)

from which we get

$$J^i = -\frac{e^2 R^2}{2\pi} \epsilon^{ij} E_j,$$

$$\epsilon^r = -\epsilon^{\theta r} = 1,$$ \hspace{1cm} (2.19)

which is exactly the statement that the Hall conductivity is $\frac{e^2 R^2}{2\pi}$.

Let us now return to the gauge invariant constraint (2.7). We will now see that elementary quantum theory imposes non-trivial restrictions on the allowed $R^2$ because of
this constraint. Firstly, we have as a consequence of (2.7),

\[
\int_0^{2\pi} dx (D_0 \phi - D_1 \phi) = 0. \tag{2.20}
\]

Let

\[
\Pi = \frac{R^2}{4\pi} (D_0 \phi + eA_1) \tag{2.21}
\]

denote the momentum density conjugate to \(\phi\). We then have the equal time commutation relations (CR’s)

\[
\begin{align*}
[\phi(x), \phi(x')] &= 0, \\
[\Pi(x), \Pi(x')] &= 0, \\
[\phi(x), \Pi(x')] &= i\delta(x - x'). \tag{2.22}
\end{align*}
\]

[Often we will not indicate the time-dependence of the fields and their modes as they will all be at equal times.] In terms of \(\Pi\), (2.20) reads

\[
\int_0^{2\pi} dx \left( \frac{4\pi \Pi}{R^2} - \phi' \right) = 0, \\
\phi' := \partial_1 \phi. \tag{2.23}
\]

We next briefly analyse the properties of \(\Pi\) and \(\phi'\).

From the CR of \(\phi\) and \(\Pi\), we get

\[
[\phi(x), \int dx' \Pi(x')] = i \tag{2.24}
\]

at equal times. Thus \(\int dx' \Pi(x')\) is canonically conjugate to \(\phi\), or rather to the spatially constant mode \(\phi_0\) of \(\phi\).

Now since \(\phi(x)\) is identified with \(\phi(x) + 2\pi\), the dependence of wave functions on \(\phi_0\) should satisfy

\[
\begin{align*}
\psi(\phi_0 + 2\pi) &= e^{i2\pi \alpha} \psi(\phi_0), \\
\psi'(\phi_0 + 2\pi) &= e^{i2\pi \alpha} \psi'(\phi_0) \tag{2.25}
\end{align*}
\]
where $\alpha$ is a real number taking values in $[0,1]$. (Only the dependence of $\psi$ on $\phi_0$ has been displayed here.) As is well known, $\alpha$ can be interpreted in terms of a flux passing through the circle in the target space (the circle on which the field $\phi$ takes values).

It follows from (2.25) that

$$\psi_m(\phi_0) = e^{i(m+\alpha)\phi_0}, \quad m = 0, \pm 1, \pm 2, \ldots \quad (2.26)$$

are the eigenfunctions of

$$p := \int dx \Pi(x) \equiv -i \frac{\partial}{\partial \phi_0} \quad (2.27)$$

while the corresponding eigenvalues are $m + \alpha$. Therefore

$$\text{Spec } p \equiv \text{Spectrum of } p = \{m + \alpha : m \in \mathbb{Z}\}. \quad (2.28)$$

Also, the identification of $\phi(x)$ with $\phi(x) + 2\pi$ gives rise to the following condition on $\int dx \phi'(x)$:

$$\int dx \phi'(x) = 2\pi N, \quad N \in \mathbb{Z}. \quad (2.29)$$

In our current approach, there is no operator in the theory which can change the winding number. We can therefore regard all states as having a fixed winding number $N$, different choices of $N$ giving different quantisations of (2.1).

With $N$ fixed, it follows from (2.23) that $p$ too has a fixed value $(m + \alpha)$ in a given quantisation where

$$\frac{4\pi(m + \alpha)}{R^2} - 2\pi N = 0.$$

It gives, for $N \neq 0$,

$$R^2 = \frac{2(m + \alpha)}{N}. \quad (2.30)$$

Since $R^2 > 0$, this implies in particular that

$$\frac{m + \alpha}{N} > 0 \text{ if } N \neq 0, \quad (2.31)$$
while if $N = 0$, then so is $p$. Thus just requiring chirality imposes the above non-trivial restriction on the possible choices of $R^2$.

The other non-trivial condition we obtain is from requiring the existence of spinorial states in the theory. This requirement is physically important because we know that the Hall system must (unlike superconductivity) be described microscopically using particles with fermionic statistics. For example, properties of the Hall fluid like incompressibility follow only if the fundamental excitations are fermionic. Such a requirement translates into requiring that the operator which rotates the whole system by $2\pi$ has $-1$ as one of its eigenvalues. The corresponding eigenstate would then be spinorial and would describe a fermionic excitation.

Now, the operator that generates spatial translations along the boundary of the disc is

$$\hat{P} =: \int_0^{2\pi} \Pi \phi' := \hat{P}_0 + \hat{P}_{\text{oscillators}},$$

where the double dots refer to normal ordering with respect to the oscillator modes while the subscripts 0 and oscillators indicate the splitting of the operator appearing in the integrand into contributions from the zero and oscillatory modes respectively. The zero modes here refer to all modes which are not oscillator modes and hence include the spatially constant as well as the winding (soliton) modes.

The operator that translates by a distance $2\pi$ (along the edge) is the same as the operator which physically rotates the system by $2\pi$. This operator is

$$e^{i2\pi \hat{P}} = e^{i2\pi \hat{P}_0} e^{i2\pi \hat{P}_{\text{oscillators}}} = e^{i2\pi \hat{P}_0}$$

The last equality above is because the second factor acts as identity on all states. The easiest way to see this is by noticing that any eigenstate of the number operator (in the Fock space of the oscillator modes) is an eigenstate of $\hat{P}_{\text{oscillator}}$ with an integer as the
eigenvalue. Thus \( e^{i2\pi \hat{p}_{\text{oscillator}}} \) has 1 as eigenvalue on all such states. Since these states form a basis, we have the above result that this factor acts as the identity operator.

Therefore,

\[
e^{i2\pi \hat{p}} = e^{i2\pi \hat{P}_0} = e^{i\phi \int dx \phi'(x)}
\]

which has the eigenvalues

\[
e^{i2\pi (m+\alpha)N}.
\]

Since we require the above operator to have \(-1\) as one of its eigenvalues, it follows that there must exist some \(m, \alpha \) and \(N\) such that

\[
2\pi (m+\alpha)N = \pi \times \text{odd integer}.
\]

If we limit the possibilities of \(\alpha\) to just 0 or \(\frac{1}{2}\) (corresponding to the wave function in (2.25) being periodic or antiperiodic), then the above condition can be rewritten as

\[
MN = \text{odd integer},
\]

\[
M := 2(m + \alpha),
\]

\[
M, N \in \mathbb{Z}.
\]

It follows from (2.36) that \(M\) and \(N\) are both odd integers (and hence nonzero).

The condition (2.35) can also be obtained by requiring the existence of anticommuting vertex operators (VO’s) that create chiral fermions from the vacuum as we shall see in Section 2.2.

Thus for \(\alpha = 0\) or \(\frac{1}{2}\), what we have in summary, for the case of one chiral boson, is

\[
\text{Spectrum of } p \equiv \text{Spec } (p) = \left\{ \frac{M}{2} \right\},
\]

\[
\int dx \phi'(x) = 2\pi N,
\]

\[
R^2 = \frac{M}{N} > 0,
\]

\[
M, N \in 2\mathbb{Z} + 1.
\]
This restriction on the possible choices of $R^2$ along with (2.17) leads to the result:

$$
\sigma_H = \frac{e^2 M}{2\pi N}.
$$

(2.38)

Thus the Hall conductivity here is quantized in fractions with not just odd denominators, but also odd numerators.

It is important to emphasize here that had we not modified the current (2.10) to (2.12) to take into account the contribution from the interior of the disc, we would have obtained a value for the Hall conductivity which would have been physically wrong because it would have predicted the existence of only even denominators (contrary to what is experimentally observed).

If we have many independent scalar fields at the boundary, $\sigma_H$ will be the total of such fractions. Thus $\sigma_H$ can take any fractional value with odd denominators (there being no condition that the numerator is also odd now). To arrive at this result, we of course also need the assumption that the charges $q_i$ of the various scalar fields are all integer multiples of the electronic charge $e$.

One further point about (2.38) deserves mention. This has to do with the perplexing fact that we obtain fractions with $M \neq 1$ (in fact equal to any odd number) even with a single chiral boson. Normally, one obtains only fractions with one in the numerator with a single chiral boson. The reason for this is that the states permitted in our theory contain all states of the form $|\pm M_0, \pm N_0\rangle$, $|\pm 2M_0, \pm 2N_0\rangle$, ... where $R^2 = \frac{M_0}{N_0}$ with $M_0, N_0$ being co-prime (and odd). [The arguments in these states indicate the eigenvalues of $2p$ and the winding number. They are vacua for the oscillator modes.]

From the definition (2.12) of the current $\mathcal{J}^\mu$, we see that the charge defined as $\int dx \mathcal{J}^0$ has the spectrum of values

$$
\frac{e}{2}(M + R^2 N) - \frac{e^2 M}{2\pi N} \int dx A_1 = eM - \frac{e^2 M}{2\pi N} \int dx A_1.
$$

(2.39)

This equation shows that the absolute value for the charge of the vacuum itself (which is
the state $|0, 0\rangle$ with no oscillator excitations) could be fractional since $\int dx A_1$ could be an arbitrary real number. In this case, the entire spectrum for the charge is also fractional, though integer spaced (in units of $e$). Also the fundamental fermions of our theory are seen to have charges $= \pm e M_0$ with respect to the vacuum.

If we require the electron to exist as one of the excitations of the theory, then we are forced to have $M_0 = 1$. In that case the Hall conductivity is of the form $\frac{e^2}{2\pi} \frac{1}{N_0}$.

**Duality and the QHE**

We now come to the next result of this paper. This has to do with the duality “symmetry” ("T-duality") that exists for the action (2.1) under the transformation $R \rightarrow \frac{1}{R}$ [21 22]. That this is a “symmetry” for the Hamiltonian (in the sense of leaving its spectrum invariant) can be seen most simply by looking at its explicit expression. Since only the “zero” modes (the spatially constant and solitonic modes) of the theory contain information about $R$ (it being possible to scale the non-zero modes freely), we need focus attention also only on these zero modes. The Hamiltonian restricted to these modes is

$$H_0 = \frac{R^2}{8\pi} \left[ \frac{8\pi p^2}{R^4} + \left( \frac{\int dx \phi'(x)}{2\pi} \right)^2 2\pi \right]$$

For the states $|M, N\rangle$ introduced above, we have

$$p|M, N\rangle = \frac{M}{2}|M, N\rangle,$$

$$\frac{1}{2\pi} \int dx \phi'(x)|M, N\rangle = N|M, N\rangle,$$

and so

$$H_0|M, N\rangle = \frac{1}{4} (\frac{M^2}{R^2} + N^2 R^2)|M, N\rangle.$$  \hspace{1cm} (2.42)

Hence the spectrum of $H_0$ remains invariant under $R \rightarrow \frac{1}{R}$ [the eigenvalue of $H_0$ remaining unaltered if $|M, N\rangle$ is replaced by $|N, M\rangle$ after this transformation].

By itself, the above result is common knowledge. The interest in it for us is for the following reasons. $R^2$ is allowed to have only certain rational values. Up to factors,
it is also the Hall conductivity for the theory on the disc whose edge excitations are
described by the action \( \mathcal{L} \). As a consequence, \( R \rightarrow \frac{1}{R} \) takes us from a Hall conductivity
\( \sigma_H = \frac{e^2 M}{2 \pi N} \) to a new Hall conductivity \( \sigma'_H = \frac{e^2 N}{2 \pi M} \). This gives a novel way of relating Hall
conductivities at two fractions which are inverses of each other. In particular, it relates
the integer Hall effect to the fractional Hall effect at one of the Laughlin fractions.

Physically, what this means is the following: the two theories described by \( R \) and \( \frac{1}{R} \)
are identical so long as they are free. [In fact they are unitarily related in the sense that
there exists a unitary transformation on the operators which transforms the Hamiltonian
\( H_0 \) into another of the same form with \( R \) replaced by \( \frac{1}{R} \).] However, these theories are
no longer identical after gauging since electromagnetism breaks the symmetry. The Hall
conductivities they give correspond to reciprocal filling fractions. This is the interesting
result that we explore further in the subsequent sections. Note that this result, in
particular, means that the *spectra of edge excitations at these two filling fractions are
identical.*

### 2.2. Mode expansion of the Chiral Boson

Until now, we have been working with a Lagrangian description of a scalar field theory
having both left and right moving modes and then imposing chirality as a constraint on
the system. We could instead have followed the procedure (standard in conformal field
theory) of writing the scalar field itself as the sum of a left moving piece \( \phi_l \) and a right
moving piece \( \phi_r \), each of which is separately compactified on a circle:

\[
\phi(x, t) = \phi_l(x, t) + \phi_r(x, t).
\] (2.43)

The \( \phi_l \) and \( \phi_r \) here depend respectively only on \( x_+ (= t + x) \) and \( x_- (= t - x) \) so that the
t dependence of these fields will sometimes not be specified explicitly. In this approach,
we impose chirality by the constraints

\[
\frac{\partial}{\partial x_-} \phi_r^{(0)}(x, t)|_t = \frac{\partial}{\partial x_-} \phi_r^{(+)}(x, t)|_t = 0,
\] (2.44)
on the physical states, where the superscripts 0 and + refer to the zero and positive frequency components of $\frac{\partial}{\partial x} \phi_r(x,t)$.

The mode expansions in the original theory with the action (2.1) are

$$\phi(x,t) = q + N x + \frac{2}{R^2} p t + \frac{1}{R} \sum_{n>0} \left[ \frac{a_n}{\sqrt{n}} e^{-inx} + \frac{a_n^*}{\sqrt{n}} e^{inx} + \frac{b_n}{\sqrt{n}} e^{-inx} + \frac{b_n^*}{\sqrt{n}} e^{inx} \right],$$

$$\Pi(x,t) = \frac{p}{2\pi} - \frac{iR}{4\pi} \sum \left[ \sqrt{n} a_n e^{-inx} - \sqrt{n} a_n^* e^{inx} + \sqrt{n} b_n e^{-inx} - \sqrt{n} b_n^* e^{inx} \right].$$ \hspace{1cm} (2.45)

The commutators of the operators in these expansions follow from those of $\phi$ and $\Pi$ \hspace{1cm} (2.22), the non-vanishing commutators being

$$[q,p] = i,$$

$$[a_n, a_n^*] = \delta_{nn} = [b_n, b_n^*].$$ \hspace{1cm} (2.46)

We write the mode expansions of $\phi_l$ and $\phi_r$ as

$$\phi_l(x,t) = q_l + \frac{p_l}{R^2} x + \frac{1}{R} \sum_{n>0} \left[ \frac{a_n}{\sqrt{n}} e^{-inx} + \frac{a_n^*}{\sqrt{n}} e^{inx} \right],$$

$$\phi_r(x,t) = q_r + \frac{p_r}{R^2} x - \frac{1}{R} \sum_{n>0} \left[ \frac{b_n}{\sqrt{n}} e^{-inx} + \frac{b_n^*}{\sqrt{n}} e^{inx} \right].$$ \hspace{1cm} (2.47)

Here, in view of (2.43), we have

$$q = q_l + q_r,$$

$$p = \frac{p_l + p_r}{2},$$

$$N = \frac{p_l - p_r}{R^2}.$$ \hspace{1cm} (2.48)

We now impose the following commutation relation on $q_{l,r}$ and $p_{l,r}$ consistently with (2.46):

$$[q_t, p_t] = i = [q_r, p_r],$$

All other commutators of these operators = 0. \hspace{1cm} (2.49)

Note that $q_{l,r}$ and $p_{l,r}$ are not completely determined by $q$, $p$ and $N$. Rather we have introduced the additional operator

$$\theta = \frac{R^2}{2} (q_l - q_r).$$ \hspace{1cm} (2.50)
conjugate to \( N \),

\[
[\theta, N] = i
\]  

in writing the expansions (2.47).

We now look at the consequence of the chirality condition (2.44) on \( \vert \rangle \). Because of (2.44) and (2.47) we have the result

\[
p_r \vert \rangle = b_n \vert \rangle = 0.
\]  

(2.52)

Since \( p_r = p - \frac{R^2}{2} N \), we find (2.30) again for \( N \neq 0 \):

\[
R^2 = \frac{2p}{N} = \frac{p_l}{N} \text{ because } p_r = 0.
\]  

(2.53)

In this equation \( p, p_l, p_r \) and \( N \) are to be interpreted as the eigenvalues of the corresponding operators on any physical state where they are all diagonal. We shall use such a convention whenever convenient to economise on symbols. It will be clear from the context if we are referring to operators or their eigenvalues.

The following consequence of the chirality constraint is to be noted. Since \( 2 \times [ \text{any eigenvalue of } p ] \) is odd on our spinorial states by (2.37), (2.48) and (2.52) show that

\[
\text{Spec } p_l \equiv \{ \text{the eigenvalues of } p_l \} = 2\mathbb{Z} + 1
\]  

(2.54)

on the physical subspace of spinorial states.

Let \( \vert 0 \rangle \) denote the vacuum state. It is annihilated by \( p_r, p_l \) and all annihilation operators. We next construct the vertex operators (VO’s) that create solitons from \( \vert 0 \rangle \). Since we are in the left-handed chiral subspace, we consider only the VO’s constructed out of \( \phi_l \). We thus consider the Fubini-Veneziano VO [23]

\[
V(x_+) := e^{iM\phi(x_+)},
\]  

(2.55)

where the normal ordering symbol indicates that the \( q_l \)’s and \( a_n^\dagger \)’s stand to the left of the \( p_l \)’s and \( a_n \)’s. [The \( t \)-dependence of the operators here are given by their \( x_+ \)-dependences.]

17
Now since $[p_t, V(x_+)] = MV(x_+)$, $M$ has to be an odd integer if $V(x_+)|0\rangle$ is to be a spinorial (physical) state. Similarly since

$$[N, V(x_+)] = \left[\frac{p_t}{R^2}, V(x_+)\right] = \frac{M}{R^2}V(x_+),$$

(2.56)

$\frac{M}{R^2}$ is the soliton number of $V(x_+)|0\rangle$ and so has to be an odd integer too. By the way we see that $p_t$ and $N$ for the state $V(x_+)|0\rangle$ automatically satisfy the chirality constraint (2.52).

The charge of the state $V(x'_+)|0\rangle$ (with respect to the vacuum; see (2.39)) is calculated by considering the following commutation relation at equal times: [All quantities below are at equal times and hence their t-dependences are not shown.]

$$[\mathcal{J}_0(x), V(x')] = e\frac{R^2}{4\pi}[D_0\phi(x) + D_1\phi(x), V(x')] = e\frac{R^2}{4\pi}[\dot{\phi}(x) + \phi'(x), V(x')]$$

$$= e[\Pi(x) + \frac{R^2}{4\pi}\phi'(x), V(x')]$$

$$= Me\delta(x - x')V(x').$$

(2.57)

Thus the charge of $V(x'_+)|0\rangle$ is $Me$ and it is located at the point $x'$.

A calculation using methods standard in string and conformal field theories shows that the requirement

$$[V(x), V(x')]_+ = 0 \text{ for } x \neq x'$$

(2.58)

at equal times forces $p_t N$ (and hence $p_t$ and $N$) to be odd, a condition also required if $V(x)|0\rangle$ is to be spinorial. Thus the spinorial excitation described by $V(x)|0\rangle$ obeys Fermi statistics.

The crucial advantage of the approach followed in this sub-section is that we can now calculate the ‘$n$-particle’ wave function of the minimum energy state for the theory that lives at the boundary. The minimum energy state of $n$ particles, each with winding number $\frac{M}{R^2}$, does not have any oscillatory excitations and is hence

$$|\langle nMe \rangle| = e^{inMq_0}|0\rangle.$$

(2.59)
Thus the $n$-particle wave function is (see also [26])

$$\Psi(x_1, x_2, \ldots, x_n) = \langle (nMe) | V(x_1)V(x_2) \ldots V(x_n) | 0 \rangle^*.$$  \hfill (2.60)

This expression gives, on using the mode expansions (2.47) and the CR’s (2.46, 2.49),

$$\Psi(x_1, x_2, \ldots, x_n) = \prod_{0<i<j}^n \left( e^{-ix_i} - e^{-ix_j} \right)^{M^2 \over 2\pi} = \prod_{0<i<j}^n \left( e^{-ix_i} - e^{-ix_j} \right)^{MN}.$$  \hfill (2.61)

Now there is an intimate relationship between (2.61) and Laughlin’s wave function for filling fraction $R^2 = 1/|N_0|$ when $V(x)|0\rangle$ has charge $-e$ (so that $M_0 = -1$). This was first pointed out by Fubini, and Fubini and Lutken [26] and studied further by many authors [8, 14, 16, 17, 18, 27, 28]. The significance of this relationship is particularly clear in our approach. Thus let $z = \xi + i\eta$ be the complex coordinate on the disk $D$ of unit radius so that $|z| = 1$ at its boundary $\partial D$. The Laughlin wave function for $n$ particles of charge $-e$ for filling fraction $1/|N_0|$ in the “symmetric” gauge is

$$\chi(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_n, \eta_n) \equiv \chi(\{\xi_i, \eta_i\}) = e^{-\frac{1}{4\pi} \sum_{i=1}^n |z_i|^2} \prod_{i<j} (z_i - z_j)^{|N_0|},$$  \hfill (2.62)

where $l$ is the cyclotron radius of the electron in the given magnetic field and $z_i = \xi_i + i\eta_i$ is the coordinate of the $i$th particle on the disk. Equation (2.62) shows that $e^{-\frac{1}{4\pi} \sum_{i=1}^N |z_i|^2} \chi(\{\xi_i, \eta_i\})$ is an anti-holomorphic function in all $z_i$ on $D$. [Here we have chosen the measure defining the scalar product of wave functions to be the standard Lebesgue measure $\prod_i d\xi_i d\eta_i$.]

Now suppose we are given this information, namely that $e^{-\frac{1}{4\pi} \sum_{i=1}^N |z_i|^2} \chi(\{\xi_i, \eta_i\})$ is an anti-holomorphic function in all $z_i$ on $D$. Then its values when all $z_i$ are restricted to $\partial D$ completely determines it everywhere on $D$ by analytic continuation.

For the ground state wave function (2.61) at the boundary, this analytic continuation gives precisely the Laughlin wave function (2.62). We can also of course construct excited state wave functions at the edge and the corresponding analytically continued wave
functions. It is in this manner that the Laughlin wave functions get related to correlation functions of the conformal field theory at the edge.

It is important to note that for this $M = -1$ case, we can only get the filling fractions $1, 1/3, 1/5, \ldots$.

For pedagogical reasons, let us also record the wave function of the state $a_k^\dagger |nMe\rangle$ which has an oscillator mode as well excited. The wave function of this state at $\partial D$ is

$$\tilde{\Psi}(x_1, x_2, \ldots, x_n) = \langle (nMe) | a_k V(x_1) V(x_2) \ldots V(x_n) | 0 \rangle^*$$  \hspace{1cm} (2.63)

Since

$$V(x)^{-1} a_k V(x) = a_k + i \frac{M}{\sqrt{kR}} e^{ikx},$$  \hspace{1cm} (2.64)

we readily find that

$$\tilde{\Psi}(x_1, x_2, \ldots, x_n) = -i \frac{M}{\sqrt{kR}} \left( \sum_{i=1}^{n} e^{-ikx_i} \right) \Psi(x_1, x_2, \ldots, x_n)$$  \hspace{1cm} (2.65)

which continues antianalytically to the wave function

$$\tilde{\Psi}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) = -i \frac{M}{\sqrt{kR}} \left( \sum_{i} \bar{z}_i^k \right) \Psi(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$$  \hspace{1cm} (2.66)

on $D$.

In these calculations, if from the beginning we had retained only right-moving chiral modes at the edge, and imposed the hypothesis that wave functions on $D$ are holomorphic functions on $D$ up to the factor $e^{-\frac{1}{4\pi} \sum |z_i|^2}$, the result would be holomorphic versions of (2.61, 2.66). If both left- and right- moving modes are excited at $\partial D$, the wave function at $\partial D$ factorizes into a product of wave functions for left- and right- moving pieces. On requiring the wave function on $D$ to be the product of the anti-holomorphic continuation of the former and holomorphic continuation of the latter and the overall factor $\exp(-\frac{1}{4\pi} \sum |z_i|^2)$, we find as usual that it is uniquely determined. The coefficient of $\exp(-\frac{1}{4\pi} \sum |z_i|^2)$ in this wave function of course contains both $z$’s and $\bar{z}$’s.
This approach to wave functions on $D$ from those on $\partial D$, being quite general, will also work when the particle considered is not an electron but some other quasiparticle (so that the charge $ke$ of its state: $e^{ik\phi(x_+)} : |0\rangle$ is not $-e$). For example, we can constrain any allowed excitation ["quasiparticle"] by requiring that the wave function of the two-particle system consisting of the electron and the quasiparticle is single-valued on $D$ (in addition to fulfilling the condition of anti-holomorphicity outlined above). Assuming that its restriction to $\partial D$ is the ground state wave function of edge dynamics, this edge wave function is seen to be

$$\langle (k - 1)e| : e^{-i\phi(x_1)} :: e^{ik\phi_l(x_2)} : |0\rangle^* = (e^{-ix_1} - e^{-ix_2})^{-k|N_0|}. \quad (2.67)$$

The wave function on $D$ is thus

$$e^{-\frac{1}{4\pi}(|z_1|^2 + |z_2|^2)(\bar{z}_1 - \bar{z}_2)^{-k|N_0|}} \quad (2.68)$$

It is devoid of poles and branch cuts (and so finite and single-valued) only if

$$k|N_0| \in \mathbb{Z}^- \text{ or } k \in \frac{\mathbb{Z}^-}{|N_0|} \quad (2.69)$$

where $R^2 = 1/|N_0|$. This reproduces a well-known result in Laughlin’s theory [2].

The quasihole excitation of Laughlin is obtained by the choice $k = -1/|N_0|$ and has previously been discussed by Fubini and Lutken [26]. A treatment of quasiparticle excitations is also developed in their work.

3. $O(d, d; \mathbb{Z})$ and FQHE for $d$ Chiral Bosons

3.1. Chiral Constraint on Left-Right Symmetric Bosons

In this section, we generalise the action (2.1) to one with many scalar fields. The motivation for doing this is that FQHE at fractions involving numerators not equal to 1 is believed to be properly described by excitations around many filled Landau levels.
Since every filled Landau level should give rise to one chiral field at the edge, one may expect that many chiral fields are required to describe these excitations. We shall see later however that there are difficulties in obtaining the wave functions of more than one filled Landau level in this approach. One other reason for several scalar fields could be that the different fields correspond to the edge fields of the different excitations that are permitted in the interior (namely, the statistical gauge fields and the successive vorticial excitations that appear in the usual hierarchical approaches). In this approach, the first of the many scalar fields alone describes the electronic edge current while the remaining fields correspond to the edge currents of the statistical and vorticial fields. It is this approach which also gives wave functions that can be interpreted as the ground state wave functions of the appropriate filling fractions. The same is not possible with the approach using many filled Landau levels as will be discussed below.

We will discuss the above-mentioned relation of bulk and edge excitations in some detail in a work under preparation [10].

With many scalar fields at the edge, the action is

\[
A'_0 = \frac{1}{8\pi} G_{ij} \int dt \int_0^{2\pi} dx \partial_\mu \phi_i^\mu \partial_\mu \phi^j
\]  

(3.1)

where \( \phi^j(x) \) is identified with \( \phi^j(x) + 2\pi \). \( G \) here is an invertible positive definite symmetric matrix with constant elements.

As in string theories [19, 20], we can also add a topological term to the above which leaves the equations of motion unaffected to obtain the action

\[
A_0 = \frac{1}{8\pi} G_{ij} \int d^2x \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{8\pi} B_{ij} \int d^2x \partial_\mu \phi^i \partial_\nu \phi^j \epsilon^{\mu\nu}
\]  

(3.2)

Here \( B \) is a constant antisymmetric matrix:

\[
B_{ij} = -B_{ji}.
\]  

(3.3)
Chirality for this theory can be enforced by imposing the condition
\[ \partial_- \phi^i \equiv \frac{1}{2}(\dot{\phi}^i - \phi'^i) = 0 \] (3.4)
which eliminates the right-moving modes. As before, we should check that such a constraint is compatible with the equations of motion of the action (3.2). Since the latter are
\[ \partial_+ \partial_- \phi^i = 0, \] (3.5)
the two are mutually compatible.

However, as previously, this is no longer true when we gauge (3.2): the gauge invariant chirality constraint and the equations of motion from the gauged version of (3.2) are not consistent as we will see below.

The gauged action is
\[ S_0(\phi^i, A) = \frac{1}{8\pi} G_{ij} \int d^2x \, D_\mu \phi^i D^\mu \phi^i + \frac{1}{8\pi} B_{ij} \int d^2x \, \partial_\mu \phi^i \partial_\nu \phi^j \epsilon^{\mu\nu} \],
\[ D_\mu \phi^i \equiv \partial_\mu \phi^i - Q^i A_\mu, \] (3.6)
where \( Q^i \) is the charge associated with \( \dot{\phi}^i \). The last term in (3.2) is not gauged here because it is a topological term and is gauge invariant. We do not therefore alter it in the process of gauging.

Imposition of the gauge invariant constraint
\[ D_- \phi^i = (\partial_- \phi^i - Q^i A_-) = 0 \] (3.7)
once again leads to an inconsistency with the equations of motion
\[ \partial_\mu D^\mu \phi^i = 0 \] (3.8)
of (3.6) because
\[ 0 = \partial_+ (D_- \phi^i) = \frac{1}{4} (\partial_\mu (D^\mu \phi^i) + Q^i E) \]
\[ \Rightarrow \partial_\mu (D^\mu \phi^i) = -Q^i E. \] (3.9)
Thus imposition of chirality necessitates augmentation of the action \((3.10)\) by an anomaly term \(\frac{Q^i G_{ij}}{4\pi} \int d\phi^j A\) for compatibility with the equations of motion:

\[
S[\phi, A] = S_0[\phi, A] + \frac{Q^i G_{ij}}{4\pi} \int d^2x \epsilon^{\mu\nu} \partial_\mu \phi^j A_\nu. \tag{3.10}
\]

Just as before we first define a current \(j^\mu = -\frac{\delta S}{\delta A^\mu} = \frac{Q^i G_{ij}}{4\pi} (D^\mu \phi^j + \epsilon^{\mu\nu} \partial_\nu \phi^j)\) and then “covariantise” it to get a current \(J^\mu\):

\[
J^\mu = \frac{Q^i G_{ij}}{4\pi} (D^\mu \phi^j + \epsilon^{\mu\nu} D_\nu \phi^j). \tag{3.11}
\]

This current is not only gauge invariant, it also satisfies \(J^- = 0\) as an identity. Thus this is to be interpreted as the physical Hall current as opposed to \(j^\mu\) which is neither gauge invariant nor chiral. As in the case with a single chiral boson, here too we can interpret this modification of \(j^\mu\) as arising out of a Chern-Simons term in the bulk required for gauge invariance of the total action.

Using \((3.9)\) and \((3.11)\), we get

\[
\partial_\mu J^\mu = -\frac{Q^i G_{ij} Q^j}{2\pi} E. \tag{3.12}
\]

Thus

\[
\sigma_H = \frac{Q^i G_{ij} Q^j}{2\pi}. \tag{3.13}
\]

Having obtained \(\sigma_H\) in terms of \(G_{ij}\), we can now see the conditions that a semiclassical theory (where electromagnetism is treated classically) will impose on \(G_{ij}\) and \(B_{ij}\).

Letting \(\Pi_i\) denote the momentum field canonically conjugate to \(\phi^i\), we get, from \((3.11)\),

\[
\Pi_i = \frac{G_{ij}}{4\pi} D_0 \phi^j + \frac{B_{ij}}{4\pi} \phi^j + \frac{G_{ij} Q^j}{4\pi} A_1. \tag{3.14}
\]

We also have as a consequence of \((3.7)\)

\[
\frac{G_{ij}}{4\pi} \int_0^{2\pi} dx (D_0 \phi^j - D_1 \phi^j) = 0. \tag{3.15}
\]
Therefore, in view of (3.14),

\[ \int dx(\Pi_i - \frac{1}{4\pi}(G_{ij} + B_{ij})\phi^j) = 0. \]  

(3.16)

Now as in (2.28),

\[ \text{Spec } p_i = \{ m_i + \alpha_i : m_i \in \mathbb{Z}, 0 \leq \alpha_i < 1 \}, \]

\[ p_i := \int dx\Pi_i, \]  

(3.17)

\[ \alpha_i \text{'s being fixed in a given theory. Also} \]

\[ \int_0^{2\pi} dx\phi^i = 2\pi N^i, \quad N^i \in \mathbb{Z}, \]  

(3.18)

\[ N^i \text{ being winding numbers.} \]

Equation (3.16) leads to the condition

\[ (G_{ij} + B_{ij})N^j = 2(m_i + \alpha_i) \]  

(3.19)

on \( m_i + \alpha_i \) and \( N^i \), which is analogous to (2.30).

As in Section 2.1, we get further constraints by looking at the generator of translations

\[ \hat{P} =: \int_0^{2\pi} \Pi_i \phi^i := \hat{P}_0 + \hat{P}_{\text{oscillators}} \]  

(3.20)

(the subscripts here having the same meaning as they did in (2.32)) from which we arrive at the operator \( \hat{R}(2\pi) \) that rotates the system by \( 2\pi \):

\[ \hat{R}(2\pi) = e^{i2\pi\hat{P}} = e^{i2\pi\hat{P}_0} = e^{i2\pi(m_i + \alpha_i)N^i}. \]  

(3.21)

Requiring that there exists a spinorial state \( |\rangle \) in the theory [so that \( \hat{R}(2\pi)|\rangle = -|\rangle \)] gives

\[ 2(m_i + \alpha_i)N^i = \text{odd integer} \]  

(3.22)

on that state. Exactly, the same condition can be obtained by requiring appropriate vertex operators to anticommute as we shall see later.
We adopt the choices 0 and $\frac{1}{2}$ for $\alpha_i$. We are thus allowing the wave functions to be only either periodic or antiperiodic in the spatially constant modes $\phi^i_0$ of $\phi^i$. With this assumption,

$$M_i = 2(m_i + \alpha_i) \in \mathbb{Z} \quad (3.23)$$

and (3.22) takes the form

$$M_i N^i = \text{odd integer on spinorial states.} \quad (3.24)$$

Equations (3.17), (3.18), (3.19) and (3.23) can now be rewritten as

$$(G_{ij} + B_{ij}) N^j = M_i, \quad (3.25)$$

$$\text{Spec } p_i = \frac{1}{2} M_i, \quad (3.26)$$

$$\int dx \phi'^i = 2\pi N^i, \quad (3.27)$$

$$M_i, N^i \in \mathbb{Z}. \quad (3.28)$$

$O(d, d; \mathbb{Z})$ and QHE

For the action (3.2) also, we have a generalisation [23] of the duality transformation of the action (2.1). Again the easiest way to see this is by looking at the part $H_0$ of the Hamiltonian containing only the zero modes since, as is well known, this is the only part that has information about $G_{ij}$ and $B_{ij}$. [As in Section 2, zero modes here refer to all modes which are not oscillator modes.] The operator $H_0$ is defined by

$$H_0 |M, N\rangle = \frac{1}{4} \begin{pmatrix} M_i & N^i \end{pmatrix} \begin{pmatrix} (G^{-1})^{ij} & -(G^{-1})^{ik} B_{kj} \\ B_{ik}(G^{-1})^{kj} & G_{ij} - B_{ik}(G^{-1})^{kl} B_{lj} \end{pmatrix} \begin{pmatrix} M_j \\ N^j \end{pmatrix} |M, N\rangle \quad (3.29)$$

where $M_i$ and $N^i$ satisfy the conditions (3.24) and (3.23) and $|M, N\rangle$ is the vacuum state for the oscillator modes.

We now show that the group $O(d, d; \mathbb{Z})$ acts on the matrix

$$\mathcal{M} = \begin{pmatrix} G^{-1} & -G^{-1} B \\ B G^{-1} & G - B G^{-1} B \end{pmatrix} \quad (3.30)$$
and transforms the Hamiltonian to a new one with the same spectrum as the original Hamiltonian. Much of these results are known \[23\].

Let
\[
\eta = \begin{pmatrix}
0_{d \times d} & 1_{d \times d} \\
1_{d \times d} & 0_{d \times d}
\end{pmatrix}
\] (3.31)
where the subscripts refer to the dimensionality of the matrices. It defines a metric with signature \((+,+,\ldots;-, -, \ldots)\) with an equal number of +’s and −’s. The real \(2d \times 2d\) matrices \(g\) fulfilling
\[
g^T \eta g = \eta
\] (3.32)
thus constitute the group \(O(d, d; R) \equiv O(d, d)\).

Now one can check that \(M^T \eta M = \eta\) so that
\[
M \in O(d, d).
\] (3.33)

Furthermore \(g^T M g = M'\) can also be written in the form (3.30) with \(G\) and \(B\) replaced by \(G'\) and \(B'\) respectively. The sketch of the proof is as follows:

Firstly, we note that \(M'\), like \(M\), is symmetric and is an element of \(O(d, d)\). Furthermore, it can be checked that the top left \(d \times d\) sub-matrix of \(M'\) is positive definite (and hence non-singular) provided the same is true of the corresponding sub-matrix of \(M\) (as is the case because of our assumption that \(G\) is a positive definite symmetric matrix). Using these properties, it can be shown that \(M'\) can be written in the form (3.30) for some \(G'\) and \(B'\).

Thus \(O(d, d)\) acts on matrices of the form \(M\), or equally well, on the set of pairs \((G, B)\). It thus transforms the action \(A_0\) to a new one with a changed \(G\) and \(B\).

But \(O(d, d)\) does not act on \((M, N)\), as it does not preserve the condition \(M_i, N^i \in \mathbb{Z}\). It is only \(O(d, d; \mathbb{Z})\) that does so.

Now \(O(d, d; \mathbb{Z})\) also preserves the remaining conditions (3.24) and (3.25) on \((M, N)\). This is so because they are respectively equivalent to the manifestly \(O(d, d; \mathbb{Z})\) compatible
conditions

\[
\frac{1}{2} \begin{pmatrix} M & N \end{pmatrix} \eta \begin{pmatrix} M \\ N \end{pmatrix} \in 2\mathbb{Z} + 1, \tag{3.34}
\]

\[
\mathcal{M} \begin{pmatrix} M \\ N \end{pmatrix} = \eta \begin{pmatrix} M \\ N \end{pmatrix}, \tag{3.35}
\]

the action of \( O(d, d; \mathbb{Z}) \) being

\[
\mathcal{M} \rightarrow \mathcal{M}' = g^T \mathcal{M} g, \quad \begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \begin{pmatrix} M' \\ N' \end{pmatrix} = g^{-1} \begin{pmatrix} M \\ N \end{pmatrix}, g \in O(d, d; \mathbb{Z}). \tag{3.36}
\]

It now follows readily that the spectrum of the Hamiltonian defined by \((G, B)\) is precisely the same as that of the one defined by \((G', B')\). \( O(d, d; \mathbb{Z}) \) hence generalises the duality transformation of Section 2. Note however that the set of pairs \((M, N)\) fulfilling our conditions are not necessarily the same in the two theories.

The summary of what we have seen in this section is the following. By considering a theory described by \((3.1)\) on the boundary of a disc and by imposing chirality on the fields and currents, we see that the theory on the disc has a Hall conductivity given by \(\sigma_H = \frac{1}{2\pi} Q^i G_{ij} Q^j\). Moreover, the \(G_{ij}\) and \(B_{ij}\) have to satisfy certain relations \((3.24,3.25)\) in order that they describe a chiral theory that permits the existence of fermionic excitations.

Furthermore, there exists an \(O(d, d; \mathbb{Z})\) group of transformations that leaves the spectrum of the Hamiltonian of \((3.2)\) invariant. This happens because these transformations on the matrix \(\mathcal{M}\) and the corresponding transformations on \((M, N)\) are compatible with the conditions \((3.24,3.25)\).

Let \(G'\) and \(B'\) be the transform of \(G\) and \(B\) under an element of \(O(d, d; \mathbb{Z})\). If \(Q^i\)'s are held fixed under this transformation, the corresponding Hall conductivities are \(\sigma_H = \frac{1}{2\pi} Q^i G_{ij} Q^j\) and \(\sigma'_H = \frac{1}{2\pi} Q^i G'_{ij} Q^j\). We thus see that we have a way of relating \(\sigma_H\) with \(\sigma'_H\) using the group \(O(d, d; \mathbb{Z})\).

In the next section, we will use this result to arrive at the hierarchy due to Haldane \[\text{[3]}\] as well as the Jain fractions \[\text{[5]}\].
3.2. Mode expansions of Chiral Bosons

We begin with the mode expansions of $\phi_i$ and $\Pi_i$ for the theory described by the action (3.2). [See also the definitions (3.17, 3.18)]:

$$\phi_i(x,t) = q_i + N^i x + (G^{-1})^{ij} (2p_j - B_{jk} N^k) t + \sum_{n>0} \frac{1}{\sqrt{n}} [a_n e^{-in(t+x)} + a_n^* e^{in(t+x)}] + \sum_{n>0} \frac{1}{\sqrt{n}} [b_n e^{-in(t-x)} + b_n^* e^{in(t-x)}],$$

$$\Pi_i(x,t) = \frac{p_i}{2\pi} - \frac{i}{4\pi} \sum_{n>0} \sqrt{n} [(G + B)_{ij} (a_n^* e^{-in(t+x)} - a_n e^{in(t+x)})] - \frac{i}{4\pi} \sum_{n>0} \sqrt{n} [(G - B)_{ij} (b_n e^{-in(t-x)} - b_n^* e^{in(t-x)})].$$

The non-zero commutators between the operators occurring here are given by

$$[q^i, p_j] = i\delta^i_j,$$

$$[a_n^i, a_m^{j*}] = \delta_{nm} (G^{-1})^{ij},$$

$$[b_n^i, b_m^{j*}] = \delta_{nm} (G^{-1})^{ij}.$$ (3.38)

The expansion of $\phi^i(x,t)$ can also be rewritten in the slightly more convenient form

$$\phi^i(x,t) = q^i + (G^{-1})^{ij} p_{lj}(t + x) + (G^{-1})^{ij} p_{rj}(t - x) + \sum_{n>0} \frac{1}{\sqrt{n}} [a_n^i e^{-in(t+x)} + a_n^{i*} e^{in(t+x)}] + \sum_{n>0} \frac{1}{\sqrt{n}} [b_n^i e^{-in(t-x)} + b_n^{i*} e^{in(t-x)}],$$

$$p_{li} := p_i + \frac{1}{2} (G + B)_{ij} N^j,$$

$$p_{ri} := p_i - \frac{1}{2} (G - B)_{ij} N^j.$$ (3.39)

From the definition of $p_l$ and $p_r$ in (3.39), we have the relations

$$p = \frac{1}{2} [(G + B) G^{-1} p_l + (G - B) G^{-1} p_r],$$

$$N = G^{-1} (p_l - p_r).$$ (3.40)

If we now introduce the operators $q^l_i$ and $q^r_i$ conjugate to $p_{li}$ and $p_{ri}$ respectively and commuting with all other operators, then we see that $q^l_i + q^r_i$ is conjugate to $p_i$ and

(continued...)

[Note: The rest of the paragraph text continues here, but is not fully visible in the image provided.]
moreover commutes with \( N^i \) and the oscillator modes. Thus \( q_\ell^i + q_r^i \) is the same as \( q^i \). We can now decompose \( \phi^i \) into left- and right-moving fields \( \phi_r^i \) and \( \phi_l^i \) with the help of these operators thus:

\[
\begin{align*}
\phi_l^i(x_+) &= q^i + (G^{-1})^{ij} p_{ij} x_+ + \sum_{n>0} \frac{1}{\sqrt{n}} [a^i_n e^{-i n x_+} + \text{h. c.}], \\
\phi_r^i(x_-) &= q^i + (G^{-1})^{ij} p_{rj} x_- + \sum_{n>0} \frac{1}{\sqrt{n}} [b^i_n e^{-i n x_-} + \text{h. c.}], \\
x_+ := t + x , & \quad x_- := t - x.
\end{align*}
\]

(3.42)

Note that \( \phi^i(x, t) = \phi_l^i(x_+) + \phi_r^i(x_-) \) and that \( \phi_l \) and \( \phi_r \) commute with each other.

It should of course be verified that the above mode decomposition is preserved in time. For this to be true, we need

\[
\begin{align*}
[q^i_l, H] &= i(G^{-1})^{ij} p_{ij}, \\
[q^i_r, H] &= i(G^{-1})^{ij} p_{rj}.
\end{align*}
\]

(3.43)

These are indeed satisfied as can be checked by rewriting (3.29) in terms of the variables \( p_l \) and \( p_r \):

\[
H_0 |p_l, p_r \rangle = \frac{1}{2} \begin{pmatrix} p_{li} & p_{ri} \end{pmatrix} \begin{pmatrix} (G^{-1})^{ij} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix} \begin{pmatrix} p_{lj} \\ p_{rj} \end{pmatrix} |p_l, p_r \rangle.
\]

(3.44)

The chirality constraint can be imposed as previously by requiring

\[
\frac{\partial \phi_r^{(0)}}{\partial x_-}| \rangle = \frac{\partial \phi_r^{(+)}}{\partial x_-}| \rangle = 0,
\]

(3.45)

or

\[
p_r | \rangle = b_n | \rangle = 0
\]

(3.46)

on any allowed physical state \( | \rangle \). The vanishing of \( p_r \) on a physical state \( | \rangle \) with winding numbers \( N^i \) and eigenvalues \( M_i \) for \( 2p_i \) gives back the result (3.25):

\[
(G + B)_{ij} N^j = M_i.
\]
The $M_i$’s here are constrained to be integers if we require the $2p_i$’s to have an integral spectrum. If we denote the eigenvalues of $p_{li}$ by $M_{li}$, then we see from (3.40) and (3.46) that

$$2p = (G + B)G^{-1}p_l, \quad \text{or} \quad M = (G + B)G^{-1}M_l$$

(3.47)

Therefore, using also (3.25), we get

$$G_{ij}N^j = M_{li}.$$ 

(3.48)

It should be noted here that there is no requirement that the $M_{li}$’s are integers.

The vertex operator which when acting on the vacuum creates a state with winding numbers $N^i$ and $p_{li} = M_{li}$ (such that (3.48) is fulfilled) is

$$V(x_+) = e^{iM_{li}\phi_l(x_+)}.$$ 

(3.49)

As in the calculation in (2.57), we can calculate the charge of the state $V(x_+)|0\rangle$ by commuting the charge density operator $\mathcal{J}^0(x')$ (obtained from (3.11)) and $V(x)$ at equal times. [Here by the charge of a state, we mean the deviation of its charge from that of the vacuum.] We find that this charge is

$$M_{li}Q^i$$ 

(3.50)

and that it is localised at the point $x$.

The requirement

$$[V(x), V(x')]_+ = 0$$ 

(3.51)

at equal times leads to the condition

$$M_{li}(G^{-1})_{ij}M_{lj} \in 2\mathbb{Z} + 1.$$ 

(3.52)

Using (3.48), (3.25), and the facts that $G$ and $B$ are symmetric and antisymmetric respectively, we now have the following set of identities:

$$M_{li}(G^{-1})_{ij}M_{lj} = N^iM_{li}$$
\[ N^i G_{ij} N^j \]
\[ = N^i (G + B)_{ij} N^j \]
\[ = N^i M_i. \] 

(3.53)

Thus the condition under which (3.51) is satisfied is same as the condition written down in (3.24) which was needed for the existence of spinorial states.

The formalism developed in this sub-section can be used to find the \( n \)-particle wave function of the minimum energy \( n \)-particle state at the boundary. Calculations analogous to those leading to (2.61) give

\[ \Psi(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) = \prod_{i,j=1; i<j}^n (\bar{z}_i - \bar{z}_j)^{M_{ik}N_k} \]

(3.54)

where \( z_i = e^{ix_i}, x_i \) being coordinates on the circle. (The exponent here can also be written as \( M_{ik}N_k \) since these two expressions are equal owing to (3.53).)

We can of course think of obvious generalisations of the above wave function when the particles are not all identical. For example, we can consider the state having \( n_1 \) particles with charge \( M_{i1}(Q^1) \), \( n_2 \) particles with charge \( M_{i2}(Q^2) \) etc. up to \( n_K \) particles with charge \( M_{iK}(Q^K) \). For the particular case where \( M_{i1}(r) = \delta_{ri} \), (chosen for the case of specificity) the wave function of the minimum energy state at the boundary is

\[ \prod_{i,j=1; i<j}^{n_1} (\bar{z}_i^{(1)} - \bar{z}_j^{(1)})(G^{-1})^{11} \prod_{k,l; k<l}^{n_2} (\bar{z}_k^{(2)} - \bar{z}_l^{(2)})(G^{-1})^{22} \prod_{i=1}^{n_1} \prod_{k=1}^{n_2} (\bar{z}_i^{(1)} - \bar{z}_k^{(2)})(G^{-1})^{12} \ldots, \] 

(3.55)

where \( z_i^{(r)} \) is the coordinate of the \( i \)th particle of type \( r \) (that is, created by the operator \( e^{i\phi_i^r(x_i)} \)).

As in sub-section 2.2, we can analytically continue these wave functions into the interior of the disc and they should then give the anti-holomorphic part of the ground state wave functions for the theory in the disc. However, the wave functions so obtained are not the multi-electron wave functions that one would have got for filling fractions involving many filled Landau levels. This is because all these wave functions are anti-holomorphic by
construction whereas we expect wave functions of electrons occupying higher Landau levels to contain holomorphic pieces too. We thus see that the present approach is inadequate for describing a situation with many filled Landau levels.

However, the other interpretation of the scalar fields as representing the edge currents of the different particles (electrons and vorticial charges/fluxons of the successive hierarchies) makes sense because then we are still only in the lowest Landau level. It is therefore this approach that we shall focus on while computing the Hall conductivities. Thus in particular, we shall associate the first scalar field alone with an electronic charge and ascribe zero charges for the remaining scalar fields when we later use (3.13) to calculate conductivities in Section 4.

Now, note that if these wave functions are to be well-defined (singularity free) on the disk, then the entries of the matrix $G^{-1}$ have to be integers:

$$ (G^{-1}) \in \mathbb{Z}. $$

(3.56)

Hence the wave function (3.55) is either symmetric or antisymmetric under interchange of the order of the vertex operators creating two different kinds of particles signifying that they are bosons or fermions. In other words, we are allowing the wave functions to be only either single-valued or double-valued under the transport of one particle around another.

The integrality condition (3.56) is analogous to the integrality condition $(R^2)^{-1} \in \mathbb{Z}$ coming from (2.61) for $M = 1$.

Now we shall see in the next section that an appropriate choice of the matrix $G + B$ reproduces the Haldane hierarchy under a sequence of $O(d, d; \mathbb{Z})$ transformations.

The matrix $(G + B)^{-1}$ appropriate for Haldane’s hierarchy is given in (4.30). We also note here that the corresponding wavefunctions given by (3.55) match the known hierarchical wave functions [3, 4, 27, 28].
4. $O(d, d; \mathbb{Z})$ and the Hierarchy

We now have a theory of $d$ chiral massless scalars describing the edge states of a quantum Hall system.

Physical motivations for certain choices of $G$ come from the “hierarchy” scheme of Haldane [3] or from the approach of Jain [5]. Perhaps the simplest way to get restrictions on $G$ from the hierarchy schemes is to first construct the Chern-Simons (CS) mean field theories for these schemes on the disk. The CS Lagrangian has previously been used by many authors [6, 7, 8, 9] for the QHE. In a paper under preparation [10], we will also discuss its use to get the Haldane hierarchy. Now these CS models contain many connection one-forms and a matrix $\tilde{G}$ coupling them. There are also scalar fields at the edge of the disk with an anomaly term containing $G$ as in (3.10). Just as in the previous Sections, neither the edge nor the bulk actions are separately gauge invariant, but one can show that their sum is, provided $\tilde{G} = G$. In this way the hierarchy schemes can restrict $G$.

We will elaborate on these considerations involving the hierarchy schemes in [10]. Here we will content ourselves with discussing a method to obtain the continued fractions of the Haldane hierarchy for the Hall conductivity $\frac{1}{2\pi} Q^i G_{ij} Q^j$ starting from the following choice for $G^{-1}$:

$G^{-1} = \begin{bmatrix}
m & 0 & 0 & \ldots \\
0 & 2p_1 & 0 & 0 & 0 \\
0 & 0 & 2p_2 & 0 & 0 \\
\vdots & 0 & 0 & 2p_3 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$ (4.1)

Such a choice with

$m \in 2\mathbb{Z} + 1, \quad p \in \mathbb{Z}
$ (4.2)

is the simplest choice for $G^{-1}$ compatible with physical requirements. It generalises the $R^{-2}$ for the single scalar field in (2.1) to $d$ uncoupled scalar fields. At the same time, it
ensures that the first field alone permits fermionic vertex operators while the rest permit only bosonic vertex operators. Thus it permits only one electron field, namely the unique fermionic vertex operator, which is as we want (see the first paragraph of Section 3.1 in this connection).

Let us first summarize our point of view. The $O(d,d;\mathbb{Z})$ transformations are “symmetries” of the ungauged theory, in the sense that there is a well-defined mapping from the states of a theory with given $G$ and $B$ to the states of a theory with transformed $G$ and $B$ such that the spectrum of the Hamiltonian remains invariant. However once we gauge the theory, this “symmetry” is broken explicitly. We first select the set of charges $Q^i$ as follows:

$$Q^1 = e, \quad Q^2 = Q^3 = Q^4 = \ldots = 0.$$  \hspace{1cm} (4.3)

They are so chosen that the first field alone has the charge of an electron while the rest have zero charge. If we now transform $G$ and $B$ keeping the $Q^i$ fixed, we get a new value for the Hall conductivity. One can advance the hypothesis that the states thus related are similar in their dynamical properties. We would then be saying that the “strong” interactions between the electrons respect this $O(d,d;\mathbb{Z})$ “symmetry”, but the (weak) external electromagnetic interactions break the “symmetry”. The symmetry being broken in a well-defined way by a term in the action that transforms in a definite way, we get definite predictions relating physical properties of one system with that of the $O(d,d;\mathbb{Z})$ transformed system. This is reminiscent of the isospin symmetry of strong interactions broken by electromagnetism. Thus, in our approach, the underlying dynamics of the Hall system obeys the duality “symmetry” of the ungauged scalar theory describing the edge states, but this is broken by the coupling with the (weak) external electromagnetic field.

A choice of $G$ and $B$ determines the matrix $\mathcal{M}$ in (3.30), which is an element of $O(d,d)$. When we act on $\mathcal{M}$ with group elements of $O(d,d;\mathbb{Z})$, we move along an orbit. Different points on an orbit correspond to systems with identical spectrum, but with
different response to external electromagnetic fields. We will first illustrate the action of $O(d, d; \mathbb{Z})$ on the $G$ given by (4.1). Later we will discuss another choice that also generates a similar hierarchy of filling fractions.

Let us begin by recalling the discussion of Section 2, where there is just one scalar field. The Hall conductivity in that model is given by (2.38):

$$
\sigma_H = \frac{e^2}{2\pi} \frac{M}{N}; \ M, N \in 2\mathbb{Z} + 1.
$$

Now, the charge of the particle associated with $(M, N)$ is $Me$. On requiring that there is an electron in the theory, we see that our choice of $R^2$ must admit a pair $(-1, -|N_0|)$ and hence also $(1, |N_0|)$ with $R^2 = 1/|N_0|$. Thus we get filling fractions $1, 1/3, 1/5, \ldots$, excitations characterized by $(M, N) = m(1, |N_0|) \ m \in \mathbb{Z}$, and charges and winding numbers $me$ and $m|N_0|$ respectively. The elementary excitations have charges $\pm e$ and corresponding winding numbers $\pm |N_0|$.

Since $R^2 \to 1/R^2$ under duality, it maps $(M, N)$ to $(N, M)$ or $m(1, |N_0|)$ to $m(|N_0|, 1)$. We can now get integral filling factors while the elementary excitations have charges $\pm |N_0|e$ and winding numbers $\pm 1$.

Let us next turn to the two-scalar case. The analogue of the constraint $R^2 = M/N$ is the chirality condition

$$(G + B)_{ij}N^j = M_i, \ i = 1, 2. \tag{4.4}$$

Let us consider

$$
B = 0, \ G^{-1} = (G + B)^{-1} = \begin{pmatrix} m & 0 \\ 0 & 2p \end{pmatrix}, \tag{4.5}
$$

with $Q^1 = e, \ Q^2 = 0$ (see (1.13)). On using (3.13), we get $\sigma_H = \frac{e^2}{2\pi m}$. The filling fraction $\nu$ is therefore equal to $1/m$. In this formula, $m$ has to be chosen to be odd since otherwise there will be no solution for the two conditions (B.24) and (B.25). Thus, the filling fraction is forced to obey the odd denominator rule for the above choice of $G$ and $B$. 
Let us define the transformations $E^{(v)}$ and $D$ by

$$E^{(v)} : G \to G, \quad E^{(v)} : B \to B + \epsilon^{(v)},$$

$$D : G + B \to (G + B)^{-1}. \quad (4.6)$$

Here $\epsilon^{(v)}$ is an antisymmetric matrix (having the same dimensionality as $G$ or $B$) which satisfies

$$\epsilon_{ij}^{(v)} = -\epsilon_{ji}^{(v)} = \delta_{vi}\delta_{v+1,j} - (i \leftrightarrow j). \quad (4.7)$$

It can be checked that these two transformations are both elements of $O(d, d; \mathbb{Z})$ and that the corresponding group elements (that act on $M$ as in (3.36)) are:

$$E^{(v)} \equiv \begin{pmatrix} 1 & -\epsilon^{(v)} \\ 0 & 1 \end{pmatrix},$$

$$D \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.8)$$

Here $1$ and $\epsilon^{(v)}$ denote unit and antisymmetric matrices of the appropriate dimension. These definitions will be used for general $d \neq 2$ as well.

Let us now specialize to the case $d = 2$. Then $v$ has only one value and hence will be omitted. For this $d$, consider the sequence of transformations

$$(G+B) \xrightarrow{D} (G+B)^{-1} \equiv (G'+B') \xrightarrow{E} (G'+B'+\epsilon) \equiv (G''+B'') \xrightarrow{D} (G''+B'')^{-1} \equiv (G'''+B'''). \quad (4.9)$$

After the first transformation, we get

$$(G' + B') = \begin{pmatrix} m & 0 \\ 0 & 2p \end{pmatrix}. \quad (4.10)$$

The value of the filling fraction $\nu$ is now $m$. At the end of the second transformation, we get

$$(G'' + B'') = \begin{pmatrix} m & 1 \\ -1 & 2p \end{pmatrix}. \quad (4.11)$$
so that the filling fraction \( \nu \) is still equal to \( m \). At the end of the third and final transformation, we get

\[
(G''' + B''') = \frac{1}{2pm + 1} \begin{pmatrix} 2p & 1 \\ -1 & m \end{pmatrix},
\]

which gives the filling fraction

\[
\nu = \frac{2p}{2pm + 1} = \frac{1}{m + \frac{1}{2p}}.
\]  

(4.13)

Thus we have for the action of the triplet \( D * E * D \) on \( \nu \),

\[
\frac{1}{m} \xrightarrow{D} m \xrightarrow{E} m \xrightarrow{D} \frac{1}{m + \frac{1}{2p}}.
\]  

(4.14)

Calling the action \( D * E^{(v)} * D \) as \( \tilde{E}^{(v)} \), we see that we can write a corresponding element of \( O(d,d; \mathbb{Z}) \) for it using (4.8):

\[
\tilde{E}^{(v)} \equiv \begin{pmatrix} 1 & 0 \\ -\epsilon^{(v)} & 1 \end{pmatrix}.
\]

(4.15)

Written in this form, the interpretation of \( \tilde{E}^{(v)} \) is quite clear. Namely, it is that transformation on \( M \) with the property that the corresponding transformation on \( (M,N) \) (see (3.36)) leaves \( M \) invariant and changes \( N \) alone. The condition that \( M \) and not \( M_l \) is left unchanged may be some cause for concern at first sight. Thus for example the invariance of charge \( M_l Q^i \) requires the invariance of \( M_l \). However, a little thought shows that it is precisely this feature of the above transformations that also permits fractional charges for the transformed theory even though the original theory had only integral charges (since \( B = 0 \) and \( M = M_l \) for the original theory).

One can do such a transformation for the more general case with \( d \) scalar fields. If we start with

\[
B = 0, \quad G^{-1} = (G + B)^{-1} = \text{diag}(m, 2p_1, 2p_2, \ldots)
\]

(4.16)

We can perform the sequence of transformations \( \tilde{E}^{(d-1)} * \tilde{E}^{(d-2)} * \ldots \tilde{E}^{(1)} \), where \( \tilde{E}^{(v)} \) is
defined in (4.15), to get a transformed \((G + B)^{-1}\) which looks as follows:

\[
(G + B)^{-1} = \begin{bmatrix}
m & 1 & 0 & \cdots \\
-1 & 2p_1 & 1 & 0 & \cdots \\
0 & -1 & 2p_2 & 1 & 0 & \cdots \\
& & 0 & -1 & 2p_3 & 1 & \cdots \\
& & & & & & \cdots \\
& & & & & & \cdots \\
\end{bmatrix}.
\] (4.17)

Once again it is easy to see that we relate the state with \(\nu = \frac{1}{m}\) to the state with

\[
\nu = \frac{1}{m + \frac{1}{2p_1 + 2p_2 + \cdots}}.
\] (4.18)

[See [8] for a similar calculation performed with a matrix like the one in (4.17) except for the fact that the matrix in [8] is symmetric.] Thus we can generate the entire hierarchy by means of these generalized duality transformations on the Laughlin fractions [2]!

While the orbit considered above does generate the required filling fractions, the theories at its distinct points have, in general, no vertex operators that create excitations with the charge of an electron. We shall see this in more detail in the following.

In the one scalar case, we have already seen that electron operators exist when \(R^2 = \frac{1}{|N_0|}\) (but not when \(R^2 = |N_0|\), with \(|N_0|\) being odd.

Let us turn to the two scalar case. The conditions that have to be satisfied are:

a) \((G + B)_{ij}N^j = M_i\)

with \(M, N\) being integers,

b) \(G_{ij}N^j = M_{ii},\; M_{11} = -1,\)

and

c) \(M, N \in \mathbb{Z};\; M_iN^i = 2\mathbb{Z} + 1.\)

The condition on \(M_{11}\) in (4.20) comes from the fact that the corresponding vertex operator has charge equal to the electronic charge [see (3.50)] if we also use our assumption (4.3) that \(Q^i = e\delta_{ii}.\)
It is easy to check that whereas the couplet \((G, B)\) of (4.13) can satisfy (4.19) to (4.21), the couplet \((G'', B'')\) of (4.12) cannot satisfy these equations (equation (4.20) being the one that is incompatible with this couplet). Thus there are no vertex operators with the charge of an electron here. If we insist on the existence of excitations with electronic charge (which is necessary if we want to identify the edge wave functions with the bulk wave functions as done in Sections 2 and 3), then we are forced to consider alternative sequences of \((G + B)\)’s.

With this in mind, let us consider the sequence of transformations given in (4.13) but with a different starting point:

\[
(G + B)^{-1} = \begin{pmatrix} m & 0 \\ 2p & 2 \end{pmatrix},
\]

Then we have the following:

\[
(G + B) = \frac{1}{2pm} \begin{pmatrix} 2p & 0 \\ -2 & m \end{pmatrix},
\]

\[
(G' + B') = \begin{pmatrix} m & 0 \\ 2p & 2 \end{pmatrix},
\]

\[
(G'' + B'') = \begin{pmatrix} m & 1 \\ 1 & 2p \end{pmatrix},
\]

\[
(G''' + B''') = \frac{1}{2pm - 1} \begin{pmatrix} 2p & -1 \\ -1 & m \end{pmatrix}.
\]

Thus we have a sequence similar to (4.14):

\[
\frac{1}{m} \rightarrow m \rightarrow \frac{E}{m} \rightarrow m \rightarrow \frac{1}{m - \frac{1}{2p}}.
\]

It can be checked now that (4.19) - (4.21) can be satisfied for the starting matrix (4.23) and the final matrix (4.26). For (4.26), it is trivial to check these relations because it is a symmetric matrix (so that \(G = G + B\)). We present a solution for the case where \((G + B)\) is given by (4.23). One finds:

\[
N^2 = 2p(N^1 + m),
\]
\[ M_{l_1} = -1, \]
\[ M_{l_2} = \frac{(2pm - 1)N_1}{2pm} + m, \]
\[ M_1 = \frac{N_1}{m}, \]
\[ M_2 = N_1 + m - \frac{N_1}{pm}. \quad (4.28) \]

Everything is expressed in terms of \( N_1 \). We see clearly that \( pm \), and hence also \( m \), should be factors of \( N_1 \) if \( M_1 \) and \( M_2 \) are to be integers. Furthermore, one can check that \( N.M \) is odd, as required, if \( \frac{(N_1)^2}{m} = N_1(\frac{N_1}{m}) \) is odd. As \( m \) divides \( N_1 \), we thus have that \( N_1 \) and \( m \) must be odd. As \( p \) divides the odd \( N_1 \), \( p \) too must be odd. Thus both \( m \) and \( p \) have to be odd. It is easy to satisfy all these conditions. Once they are all satisfied, \( e^{iM_i\phi_i} \) is the electron operator.

Finally, in the case of \( d \) scalars, by analogy with (4.22), one can start with
\[
(G + B)^{-1} = \begin{pmatrix} m & 0 & 0 & 0 & \ldots \\ 2 & 2p_1 & 0 & 0 & \ldots \\ 0 & 2 & 2p_2 & 0 & \ldots \\ 0 & 0 & 2 & 2p_3 & \ldots \end{pmatrix} \quad (4.29)
\]
instead of (4.1) and get

The transformed \((G + B)^{-1}\) will then be
\[
\begin{pmatrix} m & 1 & 0 & 0 & \ldots \\ 1 & 2p_1 & 1 & 0 & \ldots \\ 0 & 1 & 2p_2 & 1 & \ldots \\ 0 & 0 & 1 & 2p_3 & \ldots \end{pmatrix} \quad (4.30)
\]

after the sequence of transformations \( \tilde{E}^{(d-1)} \ast \tilde{E}^{(d-2)} \ast \ldots \tilde{E}^{(1)} \). The transformed filling fraction will then be
\[
\nu = \frac{1}{m - \frac{1}{2p_1 - \frac{1}{2p_2 - \ldots}}} \quad (4.31)
\]

We also briefly note here that certain of Jain’s fractions \([5]\) can be obtained with two scalar fields so that \( d = 2 \). They are given by formulae very similar to what we have as the final filling fractions in (4.14) or (4.27) except that \( m \) and \( 2p \) are interchanged. Thus, to obtain these fractions, there remains one more “duality” transformation to be
implemented after applying $\tilde{E} (= D \ast E \ast D)$ on the couplet $(G, B)$ defined from (4.5) or (4.22). The corresponding $O(2, 2; \mathbb{Z})$ matrix of this last “duality” transformation is

$$S = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}$$

(4.32)

Jain also has an additional hierarchy of fractions to arrive at which two other operations are also needed. They are

$$\nu \rightarrow 1 - \nu,$$

$$\nu \rightarrow 1 + \nu.$$

(4.33)

We, however, do not obtain these transformations of the filling fractions using the duality transformations.

Nevertheless, we find it very intriguing that most of the interesting filling fractions can be obtained using duality transformations. This implies in particular that the *spectra of edge excitations at these different fractions are all identical*! We do not yet have a good physical argument explaining why this happens. Clearly it is worth studying the effects of all the $O(d, d; \mathbb{Z})$ generators in a more systematic way.

## 5. Conclusions

Let us summarize what has been done in this paper. We have described the edge states of a QHE sample by a multi-component massless scalar field. The scalar fields are compactified on torii. The zero modes thus live on a periodic lattice with some “metric” $G$. There is also an antisymmetric matrix $B$ that describes a mixing of momenta and winding modes. This theory is then coupled to electromagnetism. Since the edge currents of the Hall system are chiral, we impose chirality on the currents of the scalar fields. This makes the theory anomalous and one has to add an extra term to the action to describe
this anomaly. The coefficient of this term is proportional to the Hall conductivity $\sigma_H$. Actually, for reasons of gauge invariance, the full action also has a Chern-Simons term with a uniquely determined overall coefficient proportional to $\sigma_H$. Now, when we further require that there exist vertex operators describing electrons, we find that $\sigma_H$ is forced to be a rational multiple of $\frac{e^2}{2\pi}$. We thus derive the quantization of $\sigma_H$ in a simple way. By analytically continuing the wave functions of the minimum energy states at the boundary, we also precisely get the Laughlin wave functions describing the ground states of the multi-particle quantum theory in the interior of the disc.

We emphasize that while the calculations that fix the Hall conductivity are being performed for the 1+1 dimensional field theory at the edge of the disc, the considerations of Section 2 show that there is a definite connection between the interior physics that can be described by a CS theory and the conformal field theory on the boundary [11, 12, 15]. In fact it is this connection which guarantees the chirality and gauge invariance of the edge current. Indeed, a person ‘living’ at the edge of the disk, and unaware of anything in the interior, could have deduced many of the basic features of the quantum Hall effect just by requiring gauge invariance and chirality of the physical currents.

Finally, and perhaps most interestingly, we find that $\sigma_H$ can be transformed in well-defined ways by applying generalized duality transformations, which are the familiar $O(d,d;\mathbb{Z})$ transformations of compactified string theories [19, 21, 22, 23]. In particular we can relate the integer QHE’s to the fractional QHE’s including those that occur in most of the hierarchical schemes. An important prediction we find is that the spectra of edge excitations at the fractions related by these transformations are all identical. We find it rather remarkable that such transformations should exist. A physical interpretation of their meaning would be of interest for the study of the dynamics of the strongly correlated electron system in the interior of the disc.

ACKNOWLEDGEMENTS
We would also like to thank T.R.Govindarajan, A.Momen, John Varghese and especially D.Karabali for many discussions. A.P.B. also thanks C.Callan for bringing reference [29] to his attention.

The work of APB and LC was supported by a grant from the Department of Energy under contract number DE-FG02-85ER40231.

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