Decoherence effects in interacting qubits under the influence of various environments

Sumanta Das and G S Agarwal

Department of Physics, Oklahoma State University, Stillwater, OK 74078, USA
E-mail: sumanta.das@okstate.edu and girish.agarwal@okstate.edu

Received 13 July 2009, in final form 10 August 2009
Published 6 October 2009
Online at stacks.iop.org/JPhysB/42/205502

Abstract
We study competition between the dissipative and coherent effects in the entanglement dynamics of two qubits. The coherent interactions are needed for designing logic gate operations with systems such as ion traps, semiconductor quantum dots and atoms. We show that the interactions lead to a phenomenon of periodic disentanglement and entanglement between the qubits. The disentanglement is primarily caused by environmental perturbations. The qubits are seen to remain disentangled for a finite time before getting entangled again. We find that the phenomenon is generic and occurs for a wide variety of models of the environment. We present analytical results for the time dependence of concurrence for all the models. The periodic disentanglement and entanglement behaviour is seen to be a precursor to the sudden death of entanglement (ESD) and can happen for environments which do not show ESD for non-interacting qubits. Further we also find that this phenomenon can even lead to delayed death of entanglement for correlated environments. (Some figures in this article are in colour only in the electronic version)

1. Introduction

It is now well understood that entanglement is the key resource for implementation of many quantum information protocols such as teleportation, cryptography, logic operations and quantum communications [1–6]. Bi-partite entanglement, i.e. entanglement among two quantum mechanical systems each envisaged as a quantum bit (a quantum mechanical two-level system analogous to a classical bit), has been found to be particularly important in this context. Numerous methods of producing qubit–qubit entanglement have been investigated during the past decade. A method that is of particular interest in the context of quantum logic gate operations with systems such as ion-traps and semiconductor nanostructures, relies on the coherent interactions among the qubits [7–16]. An earlier proposal by Barenco et al [7] has shown how one can implement a fundamental quantum gate such as the C-NOT gate using the dipole–dipole interaction among two quantum dots modelled as two qubits. This was followed by another proposal from DiVincenzo and Loss [8] in which they showed how the Heisenberg exchange interaction between two quantum dots can be used to implement universal one- and two-qubit quantum gates. In their model the qubit is realized as the spin of the excess electron on a single-electron quantum dot. They proposed the electrical gating of the tunnelling barrier between neighbouring quantum dots to create a Heisenberg coupling between the dots. Finally, they showed explicitly how by controlling the exchange coupling one can implement a quantum swap gate and XOR operation. Moreover, they also showed the implementation of single-qubit rotation using a pulsed magnetic field. Further in a later work Cirac and Zoller [9] discovered that by using the Coulombic interaction among two ions one can implement a two-qubit quantum logic gate operation. Clearly many proposals require interacting qubits for two-qubit quantum gates.

However, for a computation to progress efficiently one needs sustained entanglement among the qubits as they dynamically evolve in time. This can be achieved effectively if the quantum mechanical system under evolution is weakly interacting with its surrounding. In practice though, as the system evolves, the system–environment interaction becomes stronger thereby inhibiting loses in its initial coherence. This loss of quantum coherence is known as decoherence [17] and leads to degradation of entanglement. Thus, the study
of dynamical evolution of two entangled qubits coupled to environmental degrees of freedom is of fundamental importance in quantum information sciences. In recent years numerous studies have been done in this respect [18–24]. One study in particular predicted remarkable new behaviour in the entanglement dynamics of a bi-partite system. It reported that a mixed state of an initially entangled two-qubit system, under the influence of a pure dissipative environment, becomes completely disentangled in a finite time [23]. This was termed entanglement sudden death (ESD) [25] and was recently observed in two elegantly designed experiments with photonic qubits [26] and atomic ensemble [27]. Note that an earlier proposal has discussed a plausible experiment to observe ESD in cavity QED and trapped ion systems [28].

The phenomenon of ESD has motivated numerous theoretical investigations in other bi-partite systems involving pairs of atomic, photonic and spin qubits [29–32], multiple qubits [33] and spin chains [34–36]. Further ESD has also been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42]. Moreover, random matrix environments have been studied [43, 44]. These authors [44] also pointed out the differences in entanglement sudden death

Moreover, random matrix environments have been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42]. Furthermore, random matrix environments have been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42]. Additionally, random matrix environments have been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42].

The phenomenon of ESD has motivated numerous theoretical investigations in other bi-partite systems involving pairs of atomic, photonic and spin qubits [29–32], multiple qubits [33] and spin chains [34–36]. Further ESD has also been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42]. Moreover, random matrix environments have been studied [43, 44]. These authors [44] also pointed out the differences in entanglement sudden death

Moreover, random matrix environments have been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42]. Moreover, random matrix environments have been studied for different environments including collective vacuum noise [38], classical noise [39] and thermal noise [40–42].

In section 2 we discuss the model for two interacting qubits in contact with a simple dissipative environment and for the environment and show the existence of dark and bright periods for a wide variety of models for the environment. We show this explicitly by considering various models of the environment which induce correlated decays, pure and correlated dephasing of the qubits. All of these models exhibit the phenomenon of dark and bright periods even though some of them do not show ESD.

The organization of this paper is as follows. In section 2 we discuss the model for two interacting qubits in contact with a simple dissipative environment and formulate their dynamical evolution by solving the quantum-Liouville equation of motion. In section 3 we develop the theory to study the dynamics of entanglement of the two interacting qubits and calculate the time evolution of the concurrence under the influence of environmental perturbations. In section 4 we then study the entanglement dynamics of two interacting qubits under the influence of a pure dephasing environment. We find that the coherent qubit–qubit interaction not only leads to dark and bright periods in entanglement but also delays the onset of ESD. Further in section 5 we do a detailed study of the dynamics of the qubit–qubit entanglement for both non-interacting and interacting qubits for two different correlated models of the environment. In section 5.1 we focus on dissipative environments inducing correlated decay of the qubits. Here we find that for non-interacting qubits there is no ESD and even though entanglement vanishes for certain initial conditions at some instant, it gets partially regenerated quickly and then decays very slowly. When we include the interaction among the qubits, we find that the entanglement exhibits the phenomenon of dark and bright periods. We further study the behaviour of two-qubit entanglement for a pure correlated dephasing environment in section 5.2. We find that the correlated dephasing leads to delay of ESD in the absence of qubit–qubit interactions. We see that the degree of delay depends on the strength of the correlation. Here again when we include the qubit–qubit interaction, we observe dark and bright periods in entanglement with a much later onset of ESD. In each section we mention the earlier works. Finally in section 6 we summarize our findings and conclude with a future outlook.

2. Qubit–qubit interaction

The model that we consider for our study consists of two initially entangled interacting qubits, labelled A and B. Each qubit can be characterized by a two-level system with an excited state \( |e\rangle \) and a ground state \( |g\rangle \). Further we assume that the qubits interact independently with their respective environments. This leads to both local decoherence and loss of entanglement of the qubits. The decoherence, for instance, can arise due to spontaneous emission from the excited states. Figure 1 shows a schematic diagram of our model. The Hamiltonian for our model is then given by

\[
\mathcal{H} = \hbar \omega_0 (S_A^+ S_B^- + S_A^- S_B^+) + \hbar v (S_A^+ S_B^+ + S_A^- S_B^-),
\]

where \( v \) is the interaction between the two qubits and \( S_i^+ \), \( S_i^- \) \( (i = A, B) \) are the atomic energies, with raising and...
independently interact with their respective environments (baths) corresponding transition frequency. The qubits A and B an example say for qubit A this can be written as of the excited state and any initial coherences of the qubit. As induced by the vacuum fluctuation of the radiation field. For environment with the qubits. Note that in its simplest form the quantum-Liouville equation of motion which gives us information about the dynamics of the system can then be evaluated from the general framework of master equations. The time evolution the dynamics of two-qubit modelled as two-level atoms coupled to each other by an interaction parameter v. Here $|e\rangle$, $|g\rangle$ signify the excited and ground states and $\omega_0$ their corresponding transition frequency. The qubits A and B independently interact with their respective environments (baths) which leads to local decoherence as well as loss in entanglement.

lowering operators defined as $S_i^- = \frac{1}{\hbar}(|e\rangle\langle e| - |g\rangle\langle g|)$, $S_i^+ = |e\rangle\langle g| = (S_i^-)^\dagger$ respectively and obey angular momentum commutation algebra. We would use the two-qubit product basis given by

$$
|1\rangle = |e\rangle_A \otimes |e\rangle_B \quad |2\rangle = |e\rangle_A \otimes |g\rangle_B
$$

and

$$
|3\rangle = |g\rangle_A \otimes |e\rangle_B \quad |4\rangle = |g\rangle_A \otimes |g\rangle_B.
$$

Now as each qubit independently interacts with its respective environment, the dynamics of this interaction can be treated in the general framework of master equations. The time evolution of the density operator $\rho$ which gives us information about the dynamics of the system can then be calculated from the quantum-Liouville equation of motion

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + L\rho,$$  

where $L\rho$ includes the effect of the interaction of the environment with the qubits. Note that in its simplest form this can be considered to be a spontaneous emission process induced by the vacuum fluctuation of the radiation field. For the case of a simple dissipative environment with which the qubits are interacting independently, the effect will be decay of the excited state and any initial coherences of the qubit. As an example say for qubit A this can be written as

$$\dot{\rho}_{ee} = -2\gamma_A\rho_{ee},$$

$$\dot{\rho}_{eg} = -\gamma_A\rho_{eg}.\tag{4}$$

The above equation together with the normalization $\text{Tr}[\rho] = 1$ and the symmetry of the density matrix defines completely the dynamical system. The effect of environment as elucidated in equation (4) can be written in a compact form in terms of the atomic operators $S_i^+, S_i^-$ as

$$L\rho = -\sum_{j=A,B} \frac{\gamma_j}{2} (S_j^+ S_j^- \rho - 2 S_j^- \rho S_j^+ + \rho S_j^+ S_j^-), \tag{5}$$

where the terms $\gamma_A$ ($\gamma_B$) give the decay rate of qubit A (B) to the environment. We give the complete analytical solution of equation (3) in the basis defined by (2) for coupling to a dissipative environment (5) in appendix A.

3. Concurrence dynamics

To investigate the effect of interaction among the two qubits on decoherence, we need to study the dynamics of two-qubit entanglement. The entanglement for any bi-partite system is best identified by examining the concurrence \[60, 61\], an entanglement measure that relates to the density matrix of the system $\rho$. The concurrence for two qubits is defined as

$$C(t) = \max(0, \sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \sqrt{\lambda_4}}}). \tag{6}$$

where $\lambda$s are the eigenvalues of the non-Hermitian matrix $\rho(t)\tilde{\rho}(t)$ arranged in non-increasing order of magnitude. The matrix $\rho(t)$ is the density matrix for the two qubits and the matrix $\tilde{\rho}(t)$ is defined by

$$\tilde{\rho}(t) = (\sigma_i^{(1)} \otimes \sigma_i^{(2)})\rho^*(t)(\sigma_i^{(1)} \otimes \sigma_i^{(2)}), \tag{7}$$

where $\rho^*(t)$ is the complex conjugation of $\rho(t)$ and $\sigma_i$ is the well-known time-reversal operator for spin half systems in quantum mechanics. Note that the concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. Though in general the two-qubit density matrix $\rho$ will have all 16 elements, here we consider the initially entangled qubits to be in a mixed state \[23\] given by the density matrix

$$\rho \equiv \frac{1}{a|a\rangle\langle a| + d|4\rangle\langle 4| + (b + c)|\psi\rangle\langle \psi|},$$

$$a + b + c + d = 1, \tag{8}$$

where $a, b, c$ are independent parameters governing the nature of the initial state of the two entangled qubits. Note that the entanglement part of the state depends on the initial phase $\chi$. Following (8) one can see that the initial two-qubit density matrices have only six elements. In the matrix form $\rho$ is then given by

$$\rho(0) = \frac{1}{3} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & d & 0 \end{pmatrix}. \tag{9}$$

Here $z = e^{i\chi}\sqrt{bc}$ are the single photon coherences. Using the solution of the quantum-Liouville equation (A.1), it can be shown that the initial density matrix (9) preserves its form for all $t$. Finally we calculate the concurrence defined by (6) and (7) for the two qubits as

$$C(t) = \text{Max}(0, \tilde{C}(t)), \tag{10}$$

where $\tilde{C}(t)$ is given by

$$\tilde{C}(t) = 2[|\rho_{25}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}]. \tag{11}$$

Let us now consider a particular class of mixed states with a single parameter $a$ satisfying initially $a \geq 0$, $b = c = |z| = 1$ and $d = 1 - a$ \[23\]. Note that then (8) has the structure similar to a Werner state \[62\]. Using the dynamical evolution of the density matrix elements from appendix A and this set of initial conditions in (11), we obtain

$$\tilde{C}(t) = \frac{2}{3}e^{-\gamma t}[\cos^2\chi + \sin^2\chi \cos^2(2\nu t)]^{1/2}$$

$$- \sqrt{a(1 - a + 2w^2 + w^4d)}, \tag{12}$$

where
where \( w = \sqrt{1 - e^{-\nu t}} \). One can clearly see the dependence of \( \dot{C}(t) \) on the interaction \( \nu \) among the qubits and the initial phase \( \chi \). We see from (12) that in the absence of the interaction \( \nu \), concurrence becomes independent of the initial phase and yields the well-established result of Yu and Eberly [23].

Note that \( \dot{C}(t) \) can become negative if

\[
\dot{a}(1 - a + 2w^2 + w^4a) > (1 - \sin^2 \chi \sin^2(2vt)),
\]

in which case concurrence is zero and the qubits get disentangled. In figure 2 we show the time dependence of the entanglement by plotting equation (12) for \( \nu = 5\gamma, a = 0.4 \) and different values of the initial phase \( \chi \). The inset of figure 2 shows the long-time behaviour of entanglement for this case. We see from figure 2 that non-interacting qubits (\( \nu/\gamma = 0 \)) exhibit sudden death of entanglement (ESD) (visible more clearly in the inset) whereas when they interact (\( \nu/\gamma \neq 0 \)) the concurrence oscillates between zero and non-zero values. Thus, we see that the initially entangled qubits in the presence of the interaction \( \nu \) get repeatedly disentangled and entangled leading to dark and bright periods in the concurrence. The magnitude of bright periods diminishes with time and eventually at longer time this behaviour vanishes completely leading to death of entanglement (ESD). The length of a dark period is determined by condition (13).

Figure 2. Concurrence as a function of time for two initially entangled, interacting qubits with initial conditions \( b = c = |z| = 1.0 \) and two different initial phases \( \chi = \pi/4 \) (black curve) and \( \chi = \pi/2 \) (red/mid-grey curve).

In order to demonstrate the generic nature of our results, we consider other models of the environment. A model which has been successfully used in experiments [68] involves pure dephasing. The mathematical formulation for this kind of environmental model can be done via a master equation technique and is given by

\[
\mathcal{L}\rho = -\sum_{i=A,B} \Gamma_i (S_i^\dagger S_i^\dagger \rho - 2S_i^\dagger \rho S_i^\dagger z + \rho S_i S_i^\dagger) \quad (15)
\]

where \( 2\Gamma_A \) \((2\Gamma_B)\) is the dephasing rate of qubit A \( (B)\). Substituting (15) into (3) we get the equation for dynamical evolution of the qubits under the influence of this kind of environment. Note that in this model the populations do not decay as a result of the interaction with the environment whereas the coherences such as \( \rho_{23}(t) \) decay as \( \rho_{23}(0) e^{-i\Gamma_A \tau + \Gamma_B \tau} \). Let us now study the effect of the interaction \( \nu \) between the qubits on the dynamics of entanglement. We assume the same initial density matrix of equation (9) with the initial conditions \( d = 1 - a, b = c = |z| = 1 \) and are governed by the condition \( \sqrt{a(1-a) > |\cos(2vt)|} \). It is clearly visible from the plots that in the absence of any environment the amplitude of the bright periods does not diminish at all and thus the qubits get back their initial entanglement completely. This regeneration of entanglement is due to the inter-qubit interactions. Note that similar behaviour in concurrence dynamics (collapse and revival of entanglement) has been predicted in earlier studies of non-interacting qubits in atom cavity systems. For example, it was shown that for the double Jaynes–Cummings (JC) [63] model, with completely undamped non-interacting cavities, entanglement shows a periodic death and rebirth feature [64]. This was attributed to exchange of information between the finite number of cavity modes and the atoms—a new kind of temporary decoherence mechanism. In another work pairwise concurrence was calculated among four qubits, where the qubits were formed by the cavity modes and atoms [65]. Here again the JC-like interaction between the atom and the cavity gives rise to dark and bright periods in the entanglement dynamics of the qubits. It was shown that during the period when the concurrence between the cavities vanishes, the concurrence between the atoms reaches its peak and vice versa. This only happens because the cavities were assumed to be lossless with a finite number of modes and thus without environmental decoherence. Further it was shown that for qubits remotely located and in contact with their respective environment when driven independently by a single-mode quantized field, one gets dark and bright periods of entanglement instead of ESD, a feature similar to single atom behaviour in cavity quantum electrodynamics [66, 67]. These works [63–67] differ from ours as we focus on the effect of interaction among the qubits in the presence of a decohering environment. Note that in a more recent work it was shown how oscillators interacting with a correlated finite-temperature Markovian bath can lead to dark and bright periods in entanglement for certain initial conditions [49].

4. Pure dephasing of the qubits

In order to demonstrate the generic nature of our results, we consider other models of the environment. A model which has been successfully used in experiments [68] involves pure dephasing. The mathematical formulation for this kind of environmental model can be done via a master equation technique and is given by

\[
\mathcal{L}\rho = -\sum_{i=A,B} \Gamma_i (S_i^\dagger S_i^\dagger \rho - 2S_i^\dagger \rho S_i^\dagger z + \rho S_i S_i^\dagger) \quad (15)
\]
Appendix B that under pure dephasing, the corresponding dephasing rates are given by independently dephase to their respective environments (baths) which leads to decoherence and thus loss in entanglement. The corresponding dephasing rates are given by $\Gamma_A$ and $\Gamma_B$ respectively.

$a \geq 0$ to calculate the concurrence. One can clearly see from the solution of the quantum-Liouville equation given in appendix B that under pure dephasing, the form of the matrix in (9) is preserved for all time. Using the solutions of the master equation (3) derived in appendix B for the environment effects given by (15) and substituting into equations (10), (11) we get the time-dependent concurrence for this model to be

$$\tilde{C}_D(t) = \frac{2}{3} [e^{-\tau} [e^{-2\tau} \cos^2 \chi + \sin^2 \chi \cos(\Omega_1 \tau)] - \frac{1}{\Omega_1} \sin(\Omega_1 \tau)]^{1/2} - \sqrt{a(1-a)}],$$

(16)

where the suffix $D$ signifies that the concurrence is calculated for a dephasing environment and we assume $\Gamma_A = \Gamma_B = \Gamma$. Here $\tau = 4\Gamma t$ and $\Omega_1 = \sqrt{(2e/4\Gamma)^2 - (1/4)^2}$. For $v = 0$ we get $\tilde{C}_D(t) = \frac{2}{3} [e^{-2\tau} - \sqrt{a(1-a)}]$, which is independent of the initial phase $\chi$. We find death of entanglement for $\tau > \frac{1}{2} \ln[1/\sqrt{a(1-a)}]$. Note that Yu and Eberly [25] have considered this case earlier but for $a = 1$ only, in which case there is no ESD. In figure 5 we show the time dependence of entanglement for a purely dephasing model, for $a = 0.2$ and initial coherences governed by the phase $\chi$. From the figure we see that for $v \neq 0$, the two-qubit entanglement exhibits the phenomenon of dark and bright periods. Further we also see that for $v \neq 0$, dark and bright periods continue beyond the time when ESD occurs for non-interacting qubits. This kind of behaviour in the entanglement dynamics is found for other values of the parameter $a$ and interested readers are referred to [59] for further discussions on these.

5. Concurrence dynamics in correlated environmental models

5.1. Effect of correlated dissipative environment

We next consider an environment involving correlated decay and show how coupling to such an environment can lead to new effects in the entanglement dynamics for two qubit systems. We will consider the case of both non-interacting and interacting qubits for this model of the environment. To keep the analysis simple and get a better physical insight into the decoherence effect of this environment we will first study the case of non-interacting qubits. We assume as before that the qubits interact independently with their respective environments with decay rates $\gamma_A$ and $\gamma_B$. Further we assume that the qubits are close enough ($r \ll \lambda$, $r$ being the inter-qubit distance and $\lambda$ the wavelength of emitted radiation in the
process of a decay) such that they can undergo a correlated decay with decay rates \( \Gamma_{AB} \) (\( \Gamma_{BA} \)) for qubits A (B). Whether this would lead to further decoherence is a question we want to investigate. Note that the entanglement dynamics of two non-interacting two level atoms in the presence of dissipation caused by spontaneous emission was studied earlier in detail by Jakóbczyk and Jamróz [37]. They even considered a correlated model of a dissipative environment and showed the possible destruction of an initial entanglement and the possible creation of a transient entanglement between the atoms. Further they also discussed the question of non-locality and how it is influenced by the spontaneous emission by explicitly showing the violation of Bell-CSHS inequality. One of the chief differences between this work and ours is the initial density matrix \( \rho \) considered and the interaction introduced between the qubits. While we consider the possibility of both the qubits (atoms) being initially excited and show its important consequences on the decay dynamics, they have neglected this effect by putting \( \rho(0) = 0 \). We would show later in this paper (as can also be seen from their results) that the dissipative environment preserves the form of the initial \( \rho \). Hence, \( \rho(0) = 0 \) for all time in their case. Moreover, in a recent work the entanglement dynamics of two initially entangled qubits for a collective decoherence model was studied in context of ESD, by Ficek and Tanas [38]. They considered an initial density matrix of the form

\[
\rho = |\Psi_0\rangle\langle\Psi_0|; \quad |\Psi_0\rangle = \sqrt{p}|e_1, e_2\rangle + \sqrt{(1-p)}|g_1, g_2\rangle. \tag{17}
\]

It can be clearly seen that in this case the two qubits are initially prepared in an entangled state by the two-photon coherences. They further show that for this initial condition the single photon coherences are never generated. Moreover, the dipole–dipole interaction that they consider for the two-qubit system has no influence for this initial condition. Ficek and Tanas predicted dark periods and revival in the two-qubit entanglement in their work due to the correlated nature of the bath; we on the other hand consider the initial density matrix of the form (9) with single photon coherences and show that any coherent interaction among the qubits influences the entanglement dynamics at all later time.

We now include the effect of a dissipative environment with both independent and correlated decays of the qubits via a master equation technique given by

\[
\mathcal{L}\rho = - \sum_{j,k=A,B} \frac{\Gamma_{jk}}{2} (S_j^+ S_k^- \rho - 2 S_k^- S_j^- \rho + \rho S_j^+ S_k^-), \tag{18}
\]

\[
\Gamma_{jj} = \gamma_j.
\]

The time evolution of the density operator \( \rho \) which gives us information about the dynamics of the system can then be evaluated by solving the quantum-Liouville equation (3) with the environmental effect included by equation (18) and taking \( v = 0 \). Next as before we consider the qubits to be initially entangled with their initial state as a mixed state defined by the density matrix (9). We then solve the quantum-Liouville equation to study the dynamical evolution of the system. The reader is referred to appendix C for the explicit solution of the time-dependent density matrix elements. One can clearly see from appendix C that for this kind of model of the environment, as before, the initial density matrix preserves its form for all time \( t \). Now using appendix C in equations (10) and (11) and the initial conditions \( a \geq 0, d = 1 - a, b = c = |z| = 1 \), we obtain the concurrence dynamics of two initially entangled non-interacting qubits for this model of the environment as

\[
\tilde{C}(t) = \frac{2}{\sqrt{\pi}} e^{-y t} [(\cos x \cosh(\gamma t) - \sinh(\gamma t) + a\zeta(t))]^2 + \sin^2 x t \frac{1}{\sqrt{2}} - 3a(1 - \kappa(t)), \tag{19}
\]

where \( \zeta(t) \) and \( \kappa(t) \) are given by

\[
\zeta(t) = e^{-y t} \left[ \left( \frac{1 + \Gamma}{1 - \Gamma} \right) \left( e^{(1-\Gamma)\gamma t} - 1 \right) - \left( \frac{1 - \Gamma}{1 + \Gamma} \right) \left( e^{(1+\Gamma)\gamma t} - 1 \right) \right], \tag{20}
\]

\[
\kappa(t) = \frac{1}{2} a e^{-2\gamma t} \left[ 1 + \left( \frac{1 + \Gamma}{1 - \Gamma} \right) \left( e^{(1+\Gamma)\gamma t} - 1 \right) + \left( \frac{1 - \Gamma}{1 + \Gamma} \right) \left( e^{(1-\Gamma)\gamma t} - 1 \right) \right] + \frac{2}{3} e^{-\gamma t} [\cosh(\gamma t) - \cos x \sinh(\gamma t)]. \tag{21}
\]

For simplicity we have assumed equal decay rates of both the qubits, \( \gamma_A = \gamma_B = \gamma \) and \( \Gamma_{AB} = \Gamma_{BA} = \Gamma \). One can clearly see the dependence of \( \tilde{C}(t) \) on the correlated environmental effect given by \( \Gamma \) and the initial phase \( \chi \) in equation (19). We see from (19), (20) and (21) that for \( \Gamma = 0 \), concurrence becomes independent of the initial phase and yields the result of Yu and Eberly [23]. Note that \( \tilde{C}(t) \) can become negative if

\[
3a(1 - \kappa(t)) > [(\cos x \cosh(\gamma t) - \sinh(\gamma t) + a\zeta(t))]^2 + \sin^2 x t \left( \frac{1}{\sqrt{2}} - 3a(1 - \kappa(t)) \right), \tag{22}
\]

in which case concurrence is zero and the qubits get disentangled. To understand how the correlated decay of the qubits might affect their entanglement we study the analytical result of equation (19) for different values of the parameters \( a \) and \( x \). In figure 6 we show the time dependence of entanglement for \( a = 0.2 \) and two different values of initial phase \( \chi \) and correlated decay rate of \( \Gamma = 0.8\gamma \). Note that for \( \Gamma = 0 \), there is no ESD in this case [23] and concurrence monotonically goes to zero as \( t \to \infty \). For \( \Gamma \neq 0 \) we observe new behaviour in the entanglement of the qubits. Concurrence is seen to have a much slower decay in comparison to when \( \Gamma = 0 \). For a initial phase of \( \chi = \pi/4 \) we observe that the condition in equation (22) is satisfied and entanglement vanishes temporarily, i.e. the qubits get disentangled. The entanglement gets regenerated at some later time and finally goes to zero very slowly as \( t \to \infty \). Note that this disentanglement and re-entanglement phenomenon is non-periodic and is very sensitive to initial coherence among the qubits, for example it does not occur when the initial coherence is governed by the phase \( \chi = \pi/2 \). In figure 7 we plot concurrence for \( a = 0.4 \). For this value of a ESD is observed for \( \Gamma = 0 \) but not for \( \Gamma \neq 0 \). Instead we observe disentanglement and regeneration of entanglement among the
under the assumption that our initial two-qubit density matrix environment. Note that the solutions are essentially valid conditions no ESD occurs.

Thus, we find that the time interval between disentanglement of the qubits is given by

\[ \tau_d = \frac{2}{\gamma} \quad \text{when} \quad \chi = \pi/2, \quad \Gamma/\gamma = 0. \]

Figure 6. Time evolution of concurrence for \( a = 0.2, b = c = |z| = 1 \) and two different initial phases \( \chi \) for two non-interacting qubits in contact with a correlated dissipative environment. Here \( \Gamma/\gamma = 0 \) signifies the absence of any common bath for the qubits.

Figure 7. Time evolution of concurrence governed by the initial condition \( a = 0.4 \) for two non-interacting qubits in contact with a correlated dissipative environment. Here \( a, \Gamma/\gamma = 0 \) signifies the absence of any common bath in which case entanglement sudden death (ESD) is observed.

qubits for \( \chi = \pi/4 \). Here again we find no dark and bright periods nor any ESD for initial phase of \( \chi = \pi/2 \). Further, note that for the initial phase \( \chi = \pi/4 \) we have a longer time interval during which the qubits remain disentangled before getting entangled again, in comparison to the case for \( a = 0.2 \). Thus, we find that the time interval between disentanglement and regeneration of entanglement as well as the magnitude of regeneration strongly depends on the initial coherences of the initially entangled qubits. Hence we can conclude that for non-interacting qubits in contact with a dissipative correlated environment no ESD occurs.

Let us now consider the case of two initially entangled interacting qubits in contact with the correlated environment. The dynamical evolution of the system in the presence of the interaction \( v \) for the correlated model of an environment is evaluated in detail in appendix \( D \). We use the solutions of appendix \( D \) in (11) to calculate the concurrence for this environment. Note that the solutions are essentially valid under the assumption that our initial two-qubit density matrix \( \rho \) is given by equation (9). Further we consider as before that the two entangled qubit evolution is governed by the initial conditions

\[ a \geq 0, \quad d = 1 - a, \quad b = c = |z| = 1. \]

the time-dependent concurrence for two initially entangled interacting qubits becomes

\[ \ddot{C}(t) = \frac{\dot{a}}{2} e^{-\gamma t} \left[ t \left( \cos \chi \cosh(\Gamma t) - \sinh(\Gamma t) + a \zeta(t) \right)^2 + \cos^2(2vt) \sin^2 \chi \right]^{1/2} - \sqrt{3a[1 - \kappa(t)]}, \]

\[ C(t) = \max(0, \ddot{C}(t)), \]

where \( \zeta(t) \) and \( \kappa(t) \) are given by equations (20) and (21) respectively. The dependence of concurrence for \( \ddot{C}(t) > 0 \) on the interaction strength \( v \) between the qubits is clearly visible in equation (23). Further now we can see that the condition of complete disentanglement of the qubits is given by

\[ 3a[1 - \kappa(t)] > \left( \cos \chi \cosh(\Gamma t) - \sinh(\Gamma t) + a \zeta(t) \right)^2 + \cos^2(2vt) \sin^2 \chi. \]

When condition (25) is satisfied, \( \ddot{C}(t) < 0 \) and hence \( C(t) = 0 \). Next to study the effect of the qubit–qubit interaction on the entanglement dynamics we plot the time-dependent concurrence for different values of \( a \), initial phase \( \chi \) and correlated decay rates of \( \Gamma = 0.8\gamma \) in figures 8 and 9. We observe in the figures that for an initial phase of \( \chi = \pi/2 \) concurrence exhibits dark and bright periods at the initial time for both \( a = 0.2 \) and \( a = 0.4 \). For longer time the concurrence shows damped oscillatory behaviour. We attribute this effect to the competition between the fast inter-qubit interactions \( v \) and the environmental decays. For longer time the correlated decay becomes dominant and leads to a slow damped oscillatory decay of the entanglement. For \( \chi = \pi/4 \) the dark and bright periods are not very pronounced and are overshadowed very quickly by the correlated decay. Note that for this value of the initial phase we find that there exists a long period of time during which the qubits remain disentangled. At a much later time entanglement gets regenerated and increases initially and then starts decaying very slowly. This behaviour is quite different from the dark and bright periods seen for other models of the environment. Thus, we see that for interacting qubits there is no ESD for this model of the environment. Instead we find dark and bright periods with long period of disentanglement whose occurrence depends on the initial coherence.

5.2. Delay of ESD by correlated dephasing environment

Finally, we consider a purely correlated dephasing model of the environment and study the effect of such an environment on the entanglement dynamics of two qubits (see figure 10). Note that this kind of model is popular among solid state systems such as semiconductor quantum dots. We will study the behaviour of entanglement for both non-interacting and interacting qubits. As before to keep our analysis simple and to get a better physical insight into the question of decoherence for this kind of an environment we will first study the case of non-interacting qubits. We will then generalize our results by introducing the interaction among the qubits. For non-interacting qubits the Hamiltonian for our model is given by (1) with \( v = 0 \). The effect of the dephasing environment on
the qubits is included via a master equation technique and is given by

$$\mathcal{L}\rho = -\sum_{i=A,B} \Gamma_i \left( S_i^+ \rho S_i^- - S_i^- \rho S_i^+ + 2 \rho S_i^+ S_i^- \right)$$

$$- 2\Gamma_0 \left( S_A^+ S_B^- \rho - S_B^+ S_A^- \rho + \rho S_A^- S_B^+ - S_A^- \rho S_B^+ \right),$$

(26)

where $2\Gamma_A$ ($2\Gamma_B$) and $2\Gamma_0$ are respectively the independent and correlated dephasing rates of qubit A (B). The dynamical evolution of this system can then be studied by solving the quantum-Liouville equation (3) for $v = 0$ and including the effect of an environment by using (26). We now consider as earlier that the initial state of the two qubits is defined by the density matrix $\rho$ (9). Then the solution of the quantum-Liouville equation for this model of the environment is given by

$$\rho_{11}(t) = \frac{1}{2}a,$$

$$\rho_{22}(t) = \frac{1}{2}b,$$

$$\rho_{33}(t) = \frac{1}{2}c,$$

$$\rho_{12}(t) = \frac{1}{4} |z| e^{-i(\Gamma_A + \Gamma_B - 2\Gamma_0)t} e^{i\chi},$$

(27)

(28)

All other matrix elements of the two-qubit density matrix $\rho$ are zero. Now using the solutions of (29) it is straightforward to show that, for pure dephasing of the qubits, the form of matrix in (9) is preserved for all time. Note that in such a model the populations do not decay as a result of the interaction with the environment whereas the coherences such as $\rho_{23}(t)$ decay as $\sim \rho_{23}(0) e^{-\Gamma_A t + \Gamma_B t}$ or $\rho_{23}(t)$ for $\chi = 0$ or mod $\pi$. Let us now study the effect of correlated dephasing of the qubits on the dynamics of entanglement. For the initial conditions $d = 1 - a, b = c = |z| = 1$ and $a \geq 0$ using (27) in (11) we get the expression for time-dependent concurrence as

$$\hat{C}_D(t) = \frac{1}{2} \left[ e^{-2(\Gamma_A + \Gamma_B)t} - \sqrt{a(1-a)} \right],$$

(30)

where we have assumed $\Gamma_A = \Gamma_B = \Gamma$ for simplicity. From equation (30) it is clearly seen that in a purely dephasing environment entanglement among the qubits is independent of the initial coherence given by $\chi$ and depends only on $a$ and $\Gamma_0$. In figure 11 we plot the time dependence of concurrence for $a = 0.2$. We find that the effect of correlated dephasing is manifested in the delay of the onset of ESD. The time for the onset of ESD is given by $t \geq \frac{1}{4}(\Gamma - \Gamma_0) [1/\ln \sqrt{a(1-a)}]$. From the figure it is clearly visible that with an increase in the correlated decay $\Gamma_0$, the onset of ESD gets delayed further until $\Gamma_0 = \Gamma$, when concurrence becomes independent of the dephasing rates and is given by $C = \frac{1}{2} [1 - \sqrt{a(1-a)}]$. This situation represents a decoherence free subspace where concurrence becomes solely dependent on the value of $a$, i.e., the population of the excited state of the two qubits. Note that this kind of situation has already been tailored to study entanglement in the decoherence free subspace [68].

Let us now include the interaction among the qubits and study how this interaction might influence the entanglement dynamics for this model of the environment. The Hamiltonian of the two-qubit system and its coupling to the environment is then given by equations (1) and (26) respectively. To study the dynamics of entanglement we follow a similar process as described earlier. We use the solution of the quantum-Liouville equation derived explicitly in appendix E and substitute it into

**Figure 8.** Time evolution of concurrence for interacting qubits in contact with a correlated dissipative environment with a correlated decay rate of $\Gamma/\gamma = 0.8$. Here $b = \epsilon = |z| = 1$. A long period of disentanglement is observed for an initial phase $\chi = \pi/4$. Here the interaction strength among the qubits is taken to be $\nu/\gamma = 5.0$.

**Figure 9.** Time evolution of concurrence for interacting qubits in contact with a correlated dissipative environment for the same parameters as figure 8 and $a = 0.4$. The dark and bright periodic features are sustained for a longer time for an initial phase of $\chi = \pi/2$. A much longer period of disentanglement is now observed for $\chi = \pi/4$. 

**Figure 10.** Schematic diagram of two qubits modelled as two two-level atoms. Here $\omega_0$ is the transition frequency of the excited state $|e\rangle$ to the ground state $|g\rangle$. The qubits A and B independently dephase to their environments (baths) with dephasing rates of $\Gamma_A, \Gamma_B$ respectively. The qubits can also interact with the environment collectively when they are at proximity giving rise to correlated dephasing represented by the decay rate $\Gamma_0$.

$$\rho_{32}(t) = \rho_{23}(t), \quad \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t).$$

(29)

$$\rho_{32}(t) = \rho_{23}(t), \quad \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t).$$

(29)
equation (11) to calculate the time dependence of concurrence $C$. With the initial conditions $a = 1 - d$, $b = c = |z| = 1$, we get

\[
\hat{C}_D(t) = \frac{2}{3} \left[ e^{-(t-r)\Gamma_0} \left\{ e^{-2(t-r)\Gamma_0} \cos^2 \chi + \sin^2 \chi (\cos(\Omega't) \right)^2 \right]^{1/2} - \sqrt{a(1-a)} \right] \tag{31}
\]

\[
C(t) = \text{Max}(0, \hat{C}_D(t)) \tag{32}
\]

where $\Omega' = \sqrt{4v^2 - (\Gamma - \Gamma_0)}$ and we have assumed $\Gamma_A = \Gamma_B$. One can clearly see the dependence of concurrence on the interaction $v$ among the qubits for $\hat{C}_D > 0$. Note that due to the interaction among the qubits now concurrence becomes dependent on the initial phase $\chi$. To understand the behaviour of entanglement in the presence of the interaction ($v/\gamma = 5.0$) among the qubits we plot the time dependence of concurrence for different initial phase $\chi$ and correlated dephasing rates $\Gamma_0$ in figures 12–14. We consider the case $a = 0.2$ only to do a comparative study on the behaviour of concurrence in the presence and absence of inter-qubit interactions. Note that we have already discussed the effect of correlated dephasing on the two-qubit entanglement for this value of $a$. Let us now focus on any new feature that arises due to the qubit–qubit interactions. We can see clearly from figure 12 that for $v \neq 0$, $\chi = \pi/4$ the two qubit concurrence shows a damped oscillatory behaviour which leads to dark and bright periods at longer time before eventual death of entanglement. The generation of dark and bright periods is seen to delay the death of entanglement even further in comparison to that induced by correlated dephasing in the absence of qubit–qubit interactions. Moreover, in figure 13 we see that both the

**Figure 11.** Time evolution of concurrence for two non-interacting qubits in contact with a purely dephasing environment for the initial condition given by $a = 0.2$ and $b = c = |z| = 1$. The effect of correlated dephasing shows up as delay in the onset of ESD.

**Figure 12.** Time evolution of concurrence for two interacting qubits with the interaction strength $v/\gamma = 5.0$ in contact with a purely correlated dephasing environment and the initial conditions $a = 0.2, \chi = \pi/4, b = c = |z| = 1$. The (red grey online) curve corresponds to the concurrence of non-interacting qubits. Concurrence is seen to exhibit initial oscillations followed by dark and bright periods with eventual death of entanglement in the presence of interaction. The interaction also leads to delayed death of entanglement. Here $\Gamma_0$ is the correlated dephasing rate.

**Figure 13.** Time evolution of concurrence for two interacting qubits in contact with a correlated dephasing environment with the initial condition $a = 0.2, \chi = \pi/4, b = c = |z| = 1$. The effect of higher correlated dephasing manifests itself by increasing the periodicity of dark and bright features in concurrence. Here again we find that dark and bright periods are followed by death of entanglement.

**Figure 14.** Time evolution of concurrence for two interacting qubits in contact with a correlated dephasing environment with the initial condition $a = 0.2, \chi = \pi/4, b = c = |z| = 1$. Concurrence is seen to be sensitive to initial coherence among the two qubits. It does not exhibit initial oscillations for this value of $\chi$ but dark and bright periods with eventual death of entanglement in the presence of interaction. Here the interaction strength is taken to be $v/\gamma = 5.0$. 
oscillatory behaviour and dark and bright periods are enhanced with an increase in correlated dephasing rate. When we change the initial phase to $\pi/2$ for $\Gamma_0 = 0.2\Gamma$, we find (figure 14) no oscillatory behaviour in entanglement, rather a completely dark and bright periodic feature with eventual delayed death.

Thus, we see that the onset of dark and bright periods for this kind of environment model is profoundly influenced by the initial coherence of the two-qubit system.

The phenomenon of dark and bright periods in entanglement should have direct consequences for systems such as ion traps and quantum dots, the latter being currently the forerunner in implementation of quantum logic gates. The interaction between qubits considered in this paper is inherently present in these systems. In quantum dots for example, $\gamma^{-1} \sim$ few ns and one can get a very large range of the parameters $\Gamma^{-1}$ (1–100s of ps) [69]. Further the interaction strength $\nu$ can have a range between 1 $\mu$eV and 1 meV depending on gate biasing [13, 70–72]. An earlier study [73] reports $\gamma \sim$ 40–100 $\mu$eV and coupling strength $\sim$ 100–400 $\mu$eV, thereby making $\nu/\gamma \sim$ 1–10 for quantum dot molecules. Thus experimental parameters are in the range we used for our numerical calculation.

### 6. Summary

In summary we have done a detailed study of the decoherence effect for non-interacting and interacting initially entangled qubits in contact with different environments at zero temperature. We have shown how the interaction between qubits generates the phenomenon of dark and bright periods in the entanglement dynamics of an initially entangled two-qubit system in contact with different environments. We found this feature of dark and bright periods to be generic and occur for various models of the environment, as an example in a correlated dissipative environment we found the phenomenon of dark and bright periods in entanglement dynamics even though there was no sudden death of entanglement. Moreover, for purely dephasing models of the environment we found that the dark and bright period feature was sustained longer and delayed the sudden death. We found that there was no sudden death of entanglement for a correlated dissipative environment but rather depending on the initial coherences in the system entanglement could show a substantial slower decay and even the phenomenon of dark and bright periods. For a simple pure dephasing environment as well as for a correlated dephasing environment we have shown the existence of sudden death of entanglement. Due to correlated dephasing we found delayed death of entanglement. Further, in the correlated dephasing model we found that the onset of dark and bright periods was sensitive to the initial coherence in the system. The frequency of dark and bright periods was found to depend on the strength of interaction between the qubits as well as on the correlated decay and dephasing rates. As a future perspective it would be interesting to study the effect of the qubit–qubit interaction for environments having temperature fluctuations. Further it would also be interesting to extend our study to multi-qubit entanglement. An important class of states that can be treated for this purpose is the GHZ and W states. Moreover, cluster states [74] can also be considered as other probable candidates for the study of decoherence and loss of entanglement.

### Acknowledgments

This work was supported by NSF grant no CCF-0829860.

### Appendix A. Solution of the quantum-Liouville equation in the two-qubit product basis for two interacting qubits in contact with a dissipative environment

$$\rho_{11}(t) = \rho_{11}(0) e^{-2\gamma t},$$
$$\rho_{22}(t) = \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (1 + \cos(2\nu t))$$
$$+ \frac{1}{2} \rho_{33}(0) e^{-\gamma t} (1 - \cos(2\nu t)) + \rho_{11}(0) e^{-2\gamma t} (e^{\gamma t} - 1)$$
$$- \frac{1}{2} (\rho_{22}(0) - \rho_{33}(0)) e^{-\gamma t} \sin(2\nu t),$$
$$\rho_{33}(t) = \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (1 - \cos(2\nu t))$$
$$+ \frac{1}{2} \rho_{33}(0) e^{-\gamma t} (1 + \cos(2\nu t)) + \rho_{11}(0) e^{-2\gamma t} (e^{\gamma t} - 1)$$
$$+ \frac{1}{2} (\rho_{22}(0) - \rho_{33}(0)) e^{-\gamma t} \sin(2\nu t),$$
$$\rho_{12}(t) = \rho_{12}(0) e^{-\gamma t/2} \cos(\nu t) + i \rho_{13}(0) e^{-\gamma t/2} \sin(\nu t),$$
$$\rho_{13}(t) = \rho_{13}(0) e^{-\gamma t/2} \cos(\nu t) + i \rho_{12}(0) e^{-\gamma t/2} \sin(\nu t),$$
$$\rho_{14}(t) = \rho_{14}(0) e^{-\gamma t},$$
$$\rho_{23}(t) = \frac{1}{2} (\rho_{22}(0) - \rho_{33}(0)) \sin(2\nu t) e^{-\gamma t}$$
$$+ \frac{1}{2} \rho_{23}(0) e^{-\gamma t} (1 + \cos(2\nu t))$$
$$+ \frac{1}{2} \rho_{32}(0) e^{-\gamma t} (1 - \cos(2\nu t)),$$
$$\rho_{24}(t) = \rho_{24}(0) e^{-\gamma t/2} \cos(\nu t) - i \rho_{34}(0) e^{-\gamma t/2} \sin(\nu t)$$
$$- \rho_{12}(0) \left( \frac{1}{v^2 + (9/4)\gamma^2} \right) [2v e^{2\gamma t} + e^{-\gamma t/2} [2v \cos(\nu t)$$
$$- 3i\gamma \sin(\nu t))] - \rho_{13}(0) \left( \frac{1}{v^2 + (9/4)\gamma^2} \right) [3\gamma e^{-2\gamma t}$$
$$- e^{-\gamma t/2} [2v \sin(\nu t) + 3\gamma \cos(\nu t)]]],$$
$$\rho_{34}(t) = \rho_{34}(0) e^{-\gamma t/2} \cos(\nu t) - i \rho_{24}(0) e^{-\gamma t/2} \sin(\nu t)$$
$$- \rho_{12}(0) \left( \frac{1}{v^2 + (9/4)\gamma^2} \right) [3\gamma e^{-2\gamma t} - e^{-\gamma t/2} [2v \sin(\nu t)$$
$$+ 3\gamma \cos(\nu t)] - \rho_{13}(0) \left( \frac{1}{v^2 + (9/4)\gamma^2} \right) [2v e^{2\gamma t}$$
$$+ e^{-\gamma t/2} [2v \cos(\nu t) - 3i\gamma \sin(\nu t)]],$$

(A.1) and

$$\rho_{12}(t) = \rho_{23}^*(t), \rho_{23}(t) = \rho_{12}^*(t), \rho_{31}(t) = \rho_{13}^*(t),$$
$$\rho_{41}(t) = \rho_{14}^*(t), \rho_{42}(t) = \rho_{24}^*(t), \rho_{43}(t) = \rho_{34}^*(t), \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t).$$

Note that here we have considered $\gamma_A = \gamma_B = \gamma$. 
Appendix B. Solution of the quantum-Liouville equation in the two-qubit product basis for two interacting qubits in contact with a purely dephasing environment. The solutions correspond to an initial matrix of the form defined in equation (9)

\( \rho_{11}(t) = \rho_{11}(0), \) \hspace{1cm} (B.1)

\[ \begin{align*}
\rho_{22}(t) &= \frac{1}{2} \rho_{22}(0) \left[ 1 + e^{-(\Gamma_A + \Gamma_B)t}/2 \right] \left\{ \cos(2\Omega t) \\
&+ \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \\
&+ \frac{1}{2} \rho_{33}(0) \left[ 1 - e^{-(\Gamma_A + \Gamma_B)t}/2 \right] \left\{ \cos(2\Omega t) \\
&+ \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \\
&+ \frac{i[\rho_{23}(0) - \rho_{32}(0)]}{2\Omega} e^{-i(\Gamma_A + \Gamma_B)t}/2 \sin(2\Omega t), \hspace{1cm} (B.2) \\
\rho_{33}(t) &= \frac{1}{2} \rho_{22}(0) \left[ 1 - e^{-(\Gamma_A + \Gamma_B)t}/2 \right] \left\{ \cos(2\Omega t) \\
&+ \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \left\{ \cos(2\Omega t) + \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \\
&+ \frac{i[\rho_{23}(0) - \rho_{32}(0)]}{2\Omega} e^{-i(\Gamma_A + \Gamma_B)t}/2 \sin(2\Omega t), \hspace{1cm} (B.3) \\
\rho_{23}(t) &= \frac{1}{2} e^{-(\Gamma_A + \Gamma_B)t}/2 \rho_{23}(0) \left\{ e^{-(\Gamma_A + \Gamma_B)t}/2 + \cos(2\Omega t) \\
&+ \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \\
&+ \rho_{32}(0) \left\{ e^{-(\Gamma_A + \Gamma_B)t}/2 - \cos(2\Omega t) - \frac{(\Gamma_A + \Gamma_B)}{4\Omega} \sin(2\Omega t) \right\} \\
&- \frac{iv e^{-i(\Gamma_A + \Gamma_B)t}/2}{2\Omega} \sin(2\Omega t)[\rho_{22}(0) - \rho_{33}(0)], \hspace{1cm} (B.4) \\
\Omega &= \sqrt{\nu^2 - \left( \frac{\Gamma_A + \Gamma_B}{4} \right)^2}, \hspace{1cm} (B.5) \\
\rho_{32}(t) &= \rho_{23}^*(t), \hspace{1cm} \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t). \hspace{1cm} (B.6)
\end{align*} \]

All other elements of the density matrix \( \rho \) defined in the two-qubit product basis (2) remain zero for all time \( t \).

Appendix C. Solution of the quantum-Liouville equation in the two-qubit product basis for two non-interacting qubits in contact with a dissipative environment which results in a correlated decay. The solutions correspond to an initial matrix of the form defined in equation (9)

\[ \rho_{11}(t) = \rho_{11}(0) e^{-2\gamma t}, \hspace{1cm} (C.1) \]
\[ \rho_{22}(t) = \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (1 + \cosh(\Gamma_1 t)) \]
\[ - \frac{1}{2} \rho_{33}(0) e^{-\gamma t} (1 - \cosh(\Gamma_1 t)) \]
\[ - \rho_{11}(0) e^{-2\gamma t} \left( \frac{\gamma^2 + \Gamma_1^2}{\gamma^2 - \Gamma_1^2} \right) \]
\[ - \frac{1}{2} [\rho_{23}(0) + \rho_{32}(0)] e^{-\gamma t} \sinh(\Gamma_1 t) \]
\[ + \frac{1}{2} \rho_{11}(0) e^{-\gamma t} \left[ \frac{\gamma + \Gamma_1}{\gamma - \Gamma_1} \right] e^{\gamma t}, \hspace{1cm} (C.2) \]
\[ \rho_{33}(t) = \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (1 + \cosh(\Gamma_1 t)) \]
\[ - \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (1 - \cosh(\Gamma_1 t)) \]
\[ - \rho_{11}(0) e^{-2\gamma t} \left( \frac{\gamma^2 + \Gamma_1^2}{\gamma^2 - \Gamma_1^2} \right) \]
\[ + \frac{1}{2} \rho_{32}(0) e^{-\gamma t} \sinh(\Gamma_1 t) \]
\[ + \frac{1}{2} \rho_{11}(0) e^{-\gamma t} \left[ \frac{\gamma + \Gamma_1}{\gamma - \Gamma_1} \right] e^{\gamma t} + \left( \frac{\gamma - \Gamma_1}{\gamma + \Gamma_1} \right) e^{-\gamma t}, \hspace{1cm} (C.3) \]
\[ \rho_{23}(t) = \frac{1}{2} \rho_{23}(0) e^{-\gamma t} (1 + \cosh(\Gamma_1 t)) \]
\[ - \frac{1}{2} \rho_{23}(0) e^{-\gamma t} (1 - \cosh(\Gamma_1 t)) \]
\[ - \rho_{11}(0) e^{-2\gamma t} \left( \frac{2\gamma \Gamma_1}{\gamma^2 - \Gamma_1^2} \right) \]
\[ + \frac{1}{2} \rho_{11}(0) e^{-\gamma t} \left[ \frac{\gamma + \Gamma_1}{\gamma - \Gamma_1} \right] e^{\gamma t} + \left( \frac{\gamma - \Gamma_1}{\gamma + \Gamma_1} \right) e^{-\gamma t} \right], \hspace{1cm} (C.4) \]
\[ \rho_{23}(t) = \rho_{23}^*(t), \hspace{1cm} \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t). \hspace{1cm} (C.5) \]

All other elements of the density matrix \( \rho \) defined in the two-qubit product basis (2) remain zero for all time \( t \).

Appendix D. Solution of the quantum-Liouville equation in the two-qubit product basis for two interacting qubits in contact with a dissipative environment which results in a correlated decay. The solutions correspond to an initial matrix of the form defined in equation (9)

\[ \rho_{11}(t) = \rho_{11}(0) e^{-2\gamma t}, \hspace{1cm} (D.1) \]
\[ \rho_{22}(t) = \frac{1}{2} \rho_{22}(0) e^{-\gamma t} (\cos(2\nu t) + \cosh(\Gamma_1 t)) \]
\[ - \frac{1}{2} \rho_{33}(0) e^{-\gamma t} (\cos(2\nu t) - \cosh(\Gamma_1 t)) \]
\[-\rho_{11}(0) e^{-2\gamma t} \left( \frac{\gamma^2 + \Gamma_{12}^2}{\gamma^2 - \Gamma_{12}^2} \right) + \frac{1}{2} \rho_{11}(0) e^{-\gamma t} \left[ \left( \frac{\gamma + \Gamma_{12}}{\gamma - \Gamma_{12}} \right) e^{\Gamma_{12} t} + \left( \frac{\gamma - \Gamma_{12}}{\gamma + \Gamma_{12}} \right) e^{-\Gamma_{12} t} \right] \]

\[-\frac{1}{2} \left[ \rho_{23}(0) + \rho_{32}(0) \right] e^{-\gamma t} \sinh(\Gamma_{12} t) + \frac{1}{2} \left[ \rho_{23}(0) - \rho_{32}(0) \right] e^{-\gamma t} \sin(2\nu t). \quad (D.2)\]

\[\rho_{23}(t) = \frac{1}{2} \rho_{23}(0) e^{-\gamma t} \left( \cos(2\nu t) + \cosh(\Gamma_{12} t) \right) - \frac{1}{2} \rho_{23}(0) e^{-\gamma t} \left( \cos(2\nu t) - \cosh(\Gamma_{12} t) \right) - \frac{1}{2} \rho_{11}(0) e^{-\gamma t/2} \left( \frac{2\gamma^2}{\gamma^2 - \Gamma_{12}^2} \right) \]

\[+ \frac{1}{2} \rho_{11}(0) e^{-\gamma t} \left[ \left( \frac{\gamma + \Gamma_{12}}{\gamma - \Gamma_{12}} \right) e^{\Gamma_{12} t} + \left( \frac{\gamma - \Gamma_{12}}{\gamma + \Gamma_{12}} \right) e^{-\Gamma_{12} t} \right] \]

\[-\frac{1}{2} \left[ \rho_{23}(0) + \rho_{32}(0) \right] e^{-\gamma t} \sinh(\Gamma_{12} t) - \frac{1}{2} \left[ \rho_{23}(0) - \rho_{32}(0) \right] e^{-\gamma t} \sin(2\nu t). \quad (D.3)\]

\[\rho_{23}(t) = \frac{1}{2} \rho_{23}(0) e^{-\gamma t} \left( \cos(2\nu t) + \cosh(\Gamma_{12} t) \right) - \frac{1}{2} \rho_{23}(0) e^{-\gamma t} \left( \cos(2\nu t) - \cosh(\Gamma_{12} t) \right) - \rho_{11}(0) e^{-\gamma t/2} \left( \frac{2\gamma^2}{\gamma^2 - \Gamma_{12}^2} \right) \]

\[+ \rho_{11}(0) e^{-\gamma t} \left[ \left( \frac{\gamma + \Gamma_{12}}{\gamma - \Gamma_{12}} \right) e^{\Gamma_{12} t} + \left( \frac{\gamma - \Gamma_{12}}{\gamma + \Gamma_{12}} \right) e^{-\Gamma_{12} t} \right] \]

\[-\frac{1}{2} \left[ \rho_{23}(0) + \rho_{32}(0) \right] e^{-\gamma t} \sinh(\Gamma_{12} t) - \frac{1}{2} e^{-\gamma t/2} \left[ \rho_{23}(0) - \rho_{32}(0) \right] i \sin(2\nu t). \quad (D.4)\]

\[\rho_{32}(t) = \rho_{23}^*(t), \quad \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t). \quad (D.5)\]

All other elements of the density matrix $\rho$ defined in the two-qubit product basis (2) remain zero for all time $t$.

**Appendix E. Solution of the quantum-Liouville equation in the two-qubit product basis for two interacting qubits in contact with a purely dephasing environment. The solutions correspond to an initial matrix of the form defined in equation (9)**

\[\rho_{11}(t) = \rho_{11}(0), \quad (E.1)\]

\[\rho_{22}(t) = \frac{1}{2} \rho_{22}(0) \left[ 1 + e^{-\gamma_{\text{ff}} \Gamma_{12} t} \cos(2\Omega' t) + \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \right] \]

\[+ \frac{1}{2} \rho_{33}(0) \left[ 1 - e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \cos(2\Omega' t) \right] \]

\[+ \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \]

\[+ i [\rho_{23}(0) - \rho_{32}(0)] e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \sin(2\Omega' t), \quad (E.2)\]

\[\rho_{33}(t) = \frac{1}{2} \rho_{22}(0) \left[ 1 - e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \cos(2\Omega' t) \right] \]

\[+ \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \]

\[+ \frac{1}{2} \rho_{33}(0) \left[ 1 + e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \cos(2\Omega' t) \right] \]

\[- \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \]

\[-i [\rho_{23}(0) - \rho_{32}(0)] e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \sin(2\Omega' t), \quad (E.3)\]

\[\rho_{23}(t) = \frac{1}{2} e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \left[ \rho_{23}(0) \left[ e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} \right] \right. \]

\[\left. + \cos(2\Omega' t) + \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \right] \]

\[+ \rho_{32}(0) \left[ e^{-(\Gamma_A + \Gamma_B - 2\Gamma_0) t/2} - \cos(2\Omega' t) \right] \]

\[- \frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'} \sin(2\Omega' t) \]

\[+ i \sqrt{\frac{(\Gamma_A + \Gamma_B - 2\Gamma_0)}{4\Omega'}} \sin(2\Omega' t) [\rho_{23}(0) - \rho_{33}(0)]. \quad (E.4)\]

\[\Omega' = \sqrt{v^2 - \left( \frac{\Gamma_A + \Gamma_B - 2\Gamma_0}{4} \right)^2}, \quad (E.5)\]

\[\rho_{32}(t) = \rho_{23}^*(t), \quad \rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t). \quad (E.6)\]

All other elements of the density matrix $\rho$ defined in the two-qubit product basis (2) remain zero for all time $t$.

**References**

[1] Nielsen M and Chuang I 2004 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)

[2] Deutsch D 1985 Proc. R. Soc. Lond. A 400 97

[3] Shor P W 1994 Proc. 35th Ann. Symp. on the Foundations of Computer Science (Los Alamos, CA: IEEE Computer Society)

[4] Bennett C H and Brassard G 1984 Proc. IEEE Int.Conf. on Computers, Systems, and Signal Processing (Bangalore) p 175

Bennett C H et al 1993 Phys. Rev. Lett. 70 1895

Bennett C H and DiVincenzo D P 2000 Nature 404 247

[5] Bouwmeester D et al 1997 Nature 390 575

Riebe M et al 2004 Nature 429 734

Barrett M D et al 2004 Nature 429 737

Olmschenk S et al 2009 Science 323 486

[6] DiVincenzo D P 1995 Phys. Rev. A 50 1015

DiVincenzo D P 1995 Science 270 255
