Weak compactness of simplified nematic liquid flows in 2D

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Abstract

For any bounded, smooth domain Ω ⊂ R², we will establish the weak compactness property of solutions to the simplified Ericksen-Leslie system for both uniaxial and biaxial nematics, and the convergence of weak solutions of the Ginzburg-Landau type nematic liquid crystal flow to a weak solution of the simplified Ericksen-Leslie system as the parameter tends to zero. This is based on the compensated compactness property of the Ericksen stress tensors, which is obtained by the L⁰-estimate of the Hopf differential for the Ericksen-Leslie system and the Pohozaev type argument for the Ginzburg-Landau type nematic liquid crystal flow.

1 Introduction

Let Ω ⊂ R² be a bounded domain with smooth boundary, and N ⊂ R^L (for L ≥ 2) be a smooth compact Riemannian manifold without boundary, and 0 < T ≤ ∞. We formulate a generalized form of simplified Ericksen-Leslie system of nematic liquid crystals in which the director field takes values in N:

$$\begin{cases}
    u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla v \odot \nabla v), \\
    \nabla \cdot u = 0, \\
    v_t + u \cdot \nabla v = \Delta v + A(v)(\nabla v, \nabla v),
\end{cases}$$

in Ω × (0, T),

where (u(x, t), v(x, t), P(x, t)) : Ω × (0, T) → R² × N × R represents the fluid velocity field, the orientation director field of nematic material (into a general Riemannian manifold), and the pressure function respectively, (\nabla v \odot \nabla v)_{ij} = \nabla_x v \cdot \nabla_x v for i, j = 1, 2 represents the Ericksen-Leslie stress tensor, and A(y)(‘, ·) is the second fundamental form of N at the point y ∈ N.

The generalized system (1.1) covers the two important cases in nematic liquid crystals:

(1) For N = S², the system (1.1) becomes the simplified, uniaxial Ericksen-Leslie system first

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Consider the following Ginzburg-Landau energy functional for the director approximated system (cf. [13],[14]). More precisely, for a ny
\[ \text{dist}(\chi_{\text{map.}}) \delta \]

(2) For
\[ N = \{(y_1, y_2) \in S^2 \times S^2 \mid y_1 \cdot y_2 = 0\} \subset \mathbb{R}^6, \]
let \( v(x, t) = (n(x, t), m(x, t)) : \Omega \times (0, T) \rightarrow S^2 \times S^2 \) with \( n \cdot m = 0 \). Then the system (1.1) becomes the biaxial, Ericksen-Leslie system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= -\nabla \cdot (\nabla d \odot \nabla d), \\
\nabla \cdot u &= 0, \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d,
\end{align*}
\]
for \((u(x, t), d(x, t), P(x, t)) : \Omega \times (0, T) \rightarrow \mathbb{R}^2 \times S^2 \times \mathbb{R}\). In dimension two, the existence of a unique global weak solution has been proved in [12],[16], which satisfies the energy inequality and has at most finitely many singular times, see also [9]. Very recently, the authors in [10] have constructed example of singularity at finite time. In dimension three, a global weak solution has been proved in [12],[16], which satisfies the energy inequality.

This is a simplified version of the hydrodynamics of biaxial nematics model proposed by Grovers and Vertogen [3],[4],[5]. In dimensional two, the existence of a unique global weak solution has recently been shown in [19], which is smooth off at most finitely many singular times.

A strategy to construct a weak solution of (1.1) and (1.5) is to consider a Ginzburg-Landau approximated system (cf. [13],[14]). More precisely, for any \( \delta > 0 \) set the \( \delta \)-neighborhood of \( N \) by

\[ N_\delta = \{ y \in \mathbb{R}^L \mid \text{dist}(y, N) < \delta \}, \]
where \( \text{dist}(y, N) \) is the distance from \( y \) to \( N \). Let \( \Pi_N : N_\delta \to N \) be the nearest point projection map. There exists \( \delta_N = \delta(N) > 0 \) such that \( \text{dist}(y, N) \) and \( \Pi_N \) are smooth in \( N_{2\delta_N} \). Let \( \chi(s) \in C^\infty([0, \infty)) \) be a monotone increasing function such that

\[ \chi(s) = \begin{cases} 
  s, & \text{if } 0 \leq s \leq \delta_N^2, \\
  4\delta_N^2, & \text{if } s \geq 4\delta_N^2.
\end{cases} \]

Consider the following Ginzburg-Landau energy functional for the director \( v \)

\[ E_\varepsilon(u, v) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v, N)) \right). \]
Then the corresponding Ginzburg-Landau approximated system of (1.1) can be written as

\[
\begin{aligned}
&u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla v \odot \nabla v), \\
&\nabla \cdot u = 0, \\
v_t + u \cdot \nabla v = \Delta v - \frac{1}{\varepsilon^2} \chi'(\text{dist}^2(v, N)) \frac{d}{dv}(\text{dist}^2(v, N)).
\end{aligned}
\]

(1.4)

The main purpose of this paper is to study the weak compactness of solutions to the simplified Ericksen-Leslie system (1.1) and convergence of solutions of the Ginzburg-Landau approximation (1.4) to the simplified Ericksen-Leslie system (1.1). For this purpose, we will consider the following initial and boundary condition

\[ (u, v) |_{\partial Q_T} = (u_0, v_0) \]

(1.5)

where \(Q_T = \Omega \times (0, T)\) and \(\partial_p Q_T = (\Omega \times \{t = 0\}) \cup (\partial \Omega \times [0, T])\) is the parabolic boundary of \(Q_T\). We assume that

\[ u_0 |_{\partial \Omega} = 0, \quad v_0(x) \in N \text{ for a.e. } x \in \Omega, \]

(1.6)

and introduce the notations

\[
\begin{aligned}
H &= \text{closure of } C_0^\infty(\Omega, \mathbb{R}^2) \cap \{ f \mid \nabla \cdot f = 0 \} \text{ in } L^2(\Omega, \mathbb{R}^2), \\
J &= \text{closure of } C_0^\infty(\Omega, \mathbb{R}^2) \cap \{ f \mid \nabla \cdot f = 0 \} \text{ in } H_0^1(\Omega, \mathbb{R}^2), \\
H^1(\Omega, N) &= \{ f \in H^1(\Omega, \mathbb{R}^2) \mid f(x) \in N \text{ a.e. } x \in \Omega \}.
\end{aligned}
\]

We also assume that

\[ u_0 \in H, \quad v_0 \in H^1(\Omega, N). \]

(1.7)

Recall the definition of weak solutions of (1.1).

**Definition 1.1** A pair of maps \(u \in L^\infty([0, T], H) \cap L^2([0, T], J)\) and \(v \in L^2([0, T], H^1(\Omega, N))\) is called a weak solution to initial and boundary problem (1.1), (1.5) - (1.7), if

\[
- \int_{Q_T} \langle u, \xi \phi \rangle + \int_{Q_T} \langle u \cdot \nabla u, \xi \phi \rangle + \langle \nabla u, \xi \nabla \phi \rangle \\
= -\xi(0) \int_{\Omega} \langle u_0, \phi \rangle + \int_{Q_T} \langle \nabla v \odot \nabla v , \xi \nabla \phi \rangle,
\]

\[
- \int_{Q_T} \langle v, \xi \phi \rangle + \int_{Q_T} \langle u \nabla v , \xi \phi \rangle + \langle \nabla v, \xi \nabla \phi \rangle \\
= -\xi(0) \int_{\Omega} \langle v_0, \phi \rangle + \int_{Q_T} \langle A(v)(\nabla v, \nabla v), \xi \phi \rangle,
\]

(1.8)

for any \(\xi \in C^\infty([0, T])\) with \(\xi(T) = 0, \phi \in J\) and \(\phi \in H_0^1(\Omega, \mathbb{R}^3)\). Moreover, \((u, v) |_{\partial \Omega} = (u_0, v_0)\) in the sense of trace. The notion of a weak solution to the system (1.4) can be defined similarly.

Our first main theorem concerns the convergence of weak solutions of the system (1.4) to the system (1.1) as \(\varepsilon \to 0\). We remark that the existence of weak solutions to (1.4) has been established by [13] for \(N = S^2\) by the Galerkin method, which can be easily adapted to handle the case that \(N\) is a compact Riemannian manifold.
Theorem 1.2 For $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon)$ be a sequence of weak solutions to the Ginzburg-Landau approximated system \((1.4)\) with the initial and boundary condition \((1.5)-(1.7)\). Then there exists a weak solution \((u, v)\) of \((1.1)\) with the initial and boundary condition \((1.5)-(1.7)\) such that, after passing to subsequences,

$$u^\varepsilon \rightharpoonup u \text{ in } L^2([0,T], H^1(\Omega)), \quad v^\varepsilon \rightharpoonup v \text{ in } L^2([0,T], H^1(\Omega)).$$

In particular, the initial and boundary problem \((1.1)\) and \((1.5)-(1.7)\) admits at least one weak solution $u \in L^\infty([0,T], H) \cap L^2([0,T], J)$ and $v \in L^2([0,T], H^1(\Omega, N))$.

We would like to mention that when $\mathcal{N} = S^2$, the convergence of solutions of system \((1.4)\) to the system \((1.2)\) has recently been proved in two dimensional torus $T^2$ by Kortum in an interesting article [7]. In order to deal with convergence of the most difficult terms $\nabla d_\varepsilon \circ \nabla d_\varepsilon$ in the limit process, Kortum employed the concentration-cancellation method for the Euler equation developed by DiPerna and Majda [2] (see also [20]). Thanks to the rotational covariance of $\nabla d_\varepsilon \circ \nabla d_\varepsilon$, the test functions can be taken to a function of periodic one spatial variable ensuring the weak convergence of $\nabla d_\varepsilon \circ \nabla d_\varepsilon$ to $\nabla d \circ \nabla d$.

In this paper, we make some new observations on the Ericksen stress tensor $\nabla v \circ \nabla v$, which is flexible enough to handle any smooth domain $\Omega \subset \mathbb{R}^2$. Namely, by adding $-\frac{1}{2} |\nabla v^\varepsilon|^2 I_2$ to $\nabla v^\varepsilon \circ \nabla v^\varepsilon$, where $I_2$ is the $2 \times 2$ identity matrix, we have

$$\nabla v^\varepsilon \circ \nabla v^\varepsilon - \frac{1}{2} |\nabla v^\varepsilon|^2 I_2 = \frac{1}{2} \left( |\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2, 2\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon \rangle, |\partial_y v^\varepsilon|^2 - |\partial_x v^\varepsilon|^2 \right).$$

This is a matrix whose components constitute the Hopf differential of map $v^\varepsilon$, which are $|\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2$ and $\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon \rangle$. Since $v^\varepsilon$ is either an approximated harmonic map to $\mathcal{N}$ or a Ginzburg-Landau approximated harmonic map, we can develop its compensated compactness property by the Pohozaev type argument.

As a byproduct of the proof of Theorem 1.2 we obtain the following compactness for a sequence of weak solutions to the system \((1.1)\).

Theorem 1.3 Let $(u^k, v^k) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathcal{N}$ be a sequence of weak solutions to \((1.1)\), along with the initial and boundary condition \((u^k_0, v^k_0)\) satisfying \((1.6)\), such that

$$\sup_{k \geq 1} \left\{ \int_{Q_t} \left( |u^k|^2 + |\nabla v^k|^2 \right) + \int_{Q_t} \left( |\nabla u^k|^2 + |v^k + u^k \cdot \nabla v^k|^2 \right) \right\} < \infty,$$

\((1.9)\)

Furthermore, if we assume that

\((u^k_0, v^k_0) \rightharpoonup (u_0, v_0) \text{ in } L^2(\Omega) \times H^1(\Omega),$$

then there exists a weak solution \((u, v)\) of \((1.1)\) with the initial and boundary condition \((u_0, v_0)\) such that, after passing to subsequences,

$$u^k \rightharpoonup u \text{ in } L^2([0,T], H^1(\Omega)), \quad v^k \rightharpoonup v \text{ in } L^2([0,T], H^1(\Omega)).$$
2 Estimates on inhomogeneous Ginzburg-Landau equations

In this section, we will consider the inhomogeneous Ginzburg-Landau equation

$$\Delta v^\varepsilon - \frac{1}{\varepsilon^2} \chi' \left( \text{dist}^2(v^\varepsilon, \mathcal{N}) \right) \frac{d}{dv} \left( \text{dist}^2(v^\varepsilon, \mathcal{N}) \right) = \tau^\varepsilon \text{ in } \Omega. \quad (2.1)$$

Suppose

$$\sup_{0<\varepsilon \leq 1} E_\varepsilon(v^\varepsilon) = \int_{\Omega} \left( \frac{1}{2} |\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2} \chi (\text{dist}^2(v^\varepsilon, \mathcal{N})) \right) \leq \Lambda_1 < \infty, \quad (2.2)$$

and

$$\sup_{0<\varepsilon \leq 1} \|\tau^\varepsilon\|_{L^2(\Omega)} \leq \Lambda_2 < \infty. \quad (2.3)$$

Assume that there exist $v \in H^1(\Omega, \mathcal{N})$ and $\tau \in L^2(\Omega, \mathbb{R}^L)$ such that

$$\tau^\varepsilon \to \tau \text{ in } L^2(\Omega), \quad v^\varepsilon \rightharpoonup v \text{ in } H^1(\Omega).$$

Then we have

**Lemma 2.1** There exists $\delta_0 > 0$ such that if $v^\varepsilon \in H^1(\Omega, \mathbb{R}^L)$ is a family of solutions to (2.1) satisfying (2.2) and (2.3), and for $x_0 \in \Omega$ and $0 < r_0 < \text{dist}(x_0, \partial \Omega),$

$$\sup_{0<\varepsilon \leq 1} \int_{B_{r_0}(x_0)} \left( \frac{1}{2} |\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2} \chi (\text{dist}^2(v^\varepsilon, \mathcal{N})) \right) \leq \delta_0^2, \quad (2.4)$$

then there exists an approximated harmonic map $v \in H^1(B_{2r_0}(x_0), \mathcal{N})$ with tension filed $\tau$, i.e,

$$\Delta v + A(v)(\nabla v, \nabla v) = \tau, \quad (2.5)$$

such that as $\varepsilon \to 0,$

$$v^\varepsilon \to v \text{ in } H^1(B_{2r_0}(x_0)), \text{ and } \frac{1}{\varepsilon^2} \chi (\text{dist}^2(v^\varepsilon, \mathcal{N})) \to 0 \text{ in } L^1(B_{2r_0}(x_0)). \quad (2.6)$$

**Proof.** For any fixed $x_1 \in B_{2r_0}(x_0)$ and $0 < \varepsilon \leq \frac{r_0}{2},$ define $\tilde{v}^\varepsilon(x) = v^\varepsilon(x_1 + \varepsilon x) : B_1(0) \to \mathbb{R}^L.$

Then we have

$$\Delta \tilde{v}^\varepsilon = \chi'(\text{dist}(\tilde{v}^\varepsilon, \mathcal{N})) \frac{d}{dv}(\text{dist}(\tilde{v}^\varepsilon, \mathcal{N})) + \tilde{\tau}^\varepsilon \text{ in } B_1(0),$$

where $\tilde{\tau}^\varepsilon(x) = \varepsilon^2 \tau^\varepsilon(x_1 + \varepsilon x).$ Since

$$\|\Delta \tilde{v}^\varepsilon\|_{L^2(B_1(0))} \leq \left\| \chi'(\text{dist}(\tilde{v}^\varepsilon, \mathcal{N})) \frac{d}{dv}(\text{dist}(\tilde{v}^\varepsilon, \mathcal{N})) \right\|_{L^2(B_1(0))} + \|\tilde{\tau}^\varepsilon\|_{L^2(B_1(0))}$$

$$\leq C \left( \int_{\Omega \cap \{\text{dist}(v^\varepsilon, \mathcal{N}) \leq 2\delta_N\}} |\text{dist}(v^\varepsilon, \mathcal{N})|^2 \right)^{\frac{1}{2}} + \varepsilon \|\tau^\varepsilon\|_{L^2(\Omega)} \leq C + \Lambda_2.$$
Thus \( \tilde{v}^\varepsilon \in H^2(B_{r_2}^\varepsilon) \) and \( \| \tilde{v}^\varepsilon \|_{H^2(B_{r_2}^\varepsilon)} \leq C(1 + \Lambda_2) \). By Morrey’s inequality, we conclude that \( \tilde{v}^\varepsilon \in C^1(B_{r_2}^\varepsilon) \) and

\[
[\tilde{v}^\varepsilon]_{C^1(B_{r_2}^\varepsilon)} \leq C \| \tilde{v}^\varepsilon \|_{H^2(B_{r_2}^\varepsilon)} \leq C(1 + \Lambda_2).
\]

By rescaling, we get

\[
|\tilde{v}^\varepsilon(x) - \tilde{v}^\varepsilon(y)| \leq C(1 + \Lambda_2) \left( \frac{|x - y|}{\varepsilon} \right)^{\frac{1}{2}}, \quad \forall x, y \in B_{\varepsilon}(x_1).
\]

We claim that \( \text{dist}(\nu^\varepsilon, \mathcal{N}) \leq \delta_N \) on \( B_{r_2}^\varepsilon(x_0) \). Suppose it were false. Then there exists \( x_1 \in B_{r_2}^\varepsilon(x_0) \) such that \( \text{dist}(\nu^\varepsilon(x_1), \mathcal{N}) > \delta_N \). Then for any \( \theta_0 \in (0, 1) \) and \( x \in B_{\theta_0 \varepsilon}(x_1) \), it holds

\[
|\nu^\varepsilon(x) - \nu^\varepsilon(x_1)| \leq C \left( \frac{|x - x_1|}{\varepsilon} \right)^{\frac{1}{2}} \leq C \theta_0^{\frac{1}{2}} \leq \frac{1}{2} \delta_N,
\]

provided \( \theta_0 \leq \frac{\delta_N^2}{4 \varepsilon^2} \). It follows that

\[
\text{dist}(\nu^\varepsilon(x), \mathcal{N}) \geq \frac{1}{2} \delta_N, \quad \forall x \in B_{\theta_0 \varepsilon}(x_1),
\]

so that

\[
\int_{B_{\theta_0 \varepsilon}(x_1)} \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, \mathcal{N})) \geq \pi \delta_N^2 \theta_0^2,
\]

which contradicts to the assumption that

\[
\int_{B_{\theta_0 \varepsilon}(x_1)} \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, \mathcal{N})) \leq \int_{B_{r_2}^\varepsilon(0)} \left( \frac{1}{2} |\nabla v^\varepsilon|^2 + \frac{1}{2} \varepsilon \chi(\text{dist}^2(v^\varepsilon, \mathcal{N})) \right) \leq \delta_0^2
\]

for a sufficiently small \( \delta_0 > 0 \).

From \( \text{dist}(\nu^\varepsilon, \mathcal{N}) \leq \delta_N \) in \( B_{r_2}^\varepsilon(x_0) \), we may decompose \( \nu^\varepsilon \) into

\[
\nu^\varepsilon = \Pi_{\mathcal{N}}(\nu^\varepsilon) + \text{dist}(\nu^\varepsilon, \mathcal{N}) \nu(\Pi_{\mathcal{N}}(\nu^\varepsilon)) := \omega_\varepsilon + \zeta_\varepsilon \nu_\varepsilon,
\]

so that the equation of \( \nu_\varepsilon \) becomes

\[
\Delta \omega_\varepsilon + \Delta \zeta_\varepsilon \nu_\varepsilon + 2 \nabla \zeta_\varepsilon \nabla \nu_\varepsilon + \zeta_\varepsilon \Delta \nu_\varepsilon - \frac{1}{\varepsilon^2} \chi'(\nu^2_\varepsilon) \nabla \nu_\varepsilon \zeta_\varepsilon^2 = \tau_\varepsilon.
\]

(2.7)

Multiplying \( (2.7) \) by \( \nu_\varepsilon \), we get

\[
\Delta \zeta_\varepsilon = \langle \nabla \omega_\varepsilon, \nabla \nu_\varepsilon \rangle + \zeta_\varepsilon \| \nabla \nu_\varepsilon \|^2 + \frac{1}{\varepsilon^2} \chi'(\nu^2_\varepsilon) \langle \nabla \nu_\varepsilon \zeta_\varepsilon^2, \nu_\varepsilon \rangle + \tau_\varepsilon^1,
\]

(2.8)

where \( \tau_\varepsilon^1 = \langle \tau_\varepsilon, \nu_\varepsilon \rangle \). Plugging \( \Delta \zeta_\varepsilon \) into \( (2.7) \), we obtain

\[
\Delta \omega_\varepsilon + \langle \nabla \omega_\varepsilon, \nabla \nu_\varepsilon \rangle \nu_\varepsilon + \zeta_\varepsilon (\Delta \nu_\varepsilon + \| \nabla \nu_\varepsilon \|^2 \nu_\varepsilon) + 2 \langle \nabla \nu_\varepsilon, \nabla \zeta_\varepsilon \rangle = \tau_\varepsilon' \nu_\varepsilon + \tau_\varepsilon'' \nu_\varepsilon,
\]

(2.9)

where \( \tau_\varepsilon'' = \tau_\varepsilon - \tau_\varepsilon^1 \nu_\varepsilon \). Here we have used the fact

\[
\langle \nabla \zeta_\varepsilon^2, \nu_\varepsilon \rangle = \nabla \zeta_\varepsilon \zeta_\varepsilon^2.
\]
Applying the $W^{2,\frac{4}{3}}$-estimate for $(-\Delta + \frac{2}{\varepsilon^2})$ (see [8]), we obtain
\begin{align}
\|\nabla^2(\zeta_\frac{\eta^2}{\varepsilon^4})\|_{L^{\frac{4}{3}}} &\lesssim \|\zeta_\epsilon \Delta(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla \zeta_\epsilon \nabla(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla \omega_\epsilon \|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} \\
& \quad + \|\nabla \omega_\epsilon \|_{L^2} + \|\zeta_\epsilon\|_{L^\infty} \|\nabla \nu_\epsilon \|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\tau_\epsilon\|_{L^{\frac{4}{3}}}
\end{align}
(2.11)
where $A \lesssim B$ stands for $A \leq CB$ for some universal positive constant $C$.

For $\omega_\epsilon$, by a similar calculation we obtain
\begin{align}
\Delta(\omega_\epsilon \eta^2) &\lesssim -\langle \nabla \omega_\epsilon, \nabla(\nu_\epsilon \eta^2) \rangle \nu_\epsilon + \langle \nabla \omega_\epsilon, \nu_\epsilon \nabla(\eta^2) \rangle \nu_\epsilon \\
& \quad - \zeta_\epsilon \left[ \Delta(\nu_\epsilon \eta^2) - \nu_\epsilon \Delta(\eta^2) - 2\nabla \nu_\epsilon \nabla(\eta^2) \right] + \zeta_\epsilon \left[ |\nabla \nu_\epsilon \eta^2| - |\nu_\epsilon \nabla(\eta^2)| \right] \nu_\epsilon \\
& \quad - 2 \left[ \langle \nabla(\nu_\epsilon \eta^2), \nabla \zeta_\epsilon \rangle - \langle \nabla(\eta^2), \nabla \zeta_\epsilon \nu_\epsilon \rangle + \tau_\epsilon \nu_\epsilon^2 + \omega_\epsilon \Delta(\eta^2) + 2\nabla \omega_\epsilon \nabla(\eta^2) \right].
\end{align}
(2.12)
Applying the $W^{2,\frac{4}{3}}$-estimate, we obtain
\begin{align}
\|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} &\lesssim \|\nabla \omega_\epsilon\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla \omega_\epsilon\|_{L^{\frac{4}{3}}} + \|\zeta_\epsilon\|_{L^\infty} \|\Delta(\nu_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
& \quad + \|\zeta_\epsilon\|_{L^\infty} \left( 1 + \|\nu_\epsilon\|_{L^{\frac{4}{3}}} \right) + \|\zeta_\epsilon\|_{L^\infty} \|\nabla \nu_\epsilon \eta^2\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} \\
& \quad + \|\nabla \zeta_\epsilon\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla \zeta_\epsilon\|_{L^{\frac{4}{3}}} + \|\tau_\epsilon\|_{L^2}.
\end{align}
(2.13)
Therefore, we conclude that
\begin{align}
\|\nabla^2(\zeta_\frac{\eta^2}{\varepsilon^4})\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} &\lesssim \|\nabla v^\epsilon\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\zeta_\epsilon\|_{L^\infty} \|\nabla^2(\nu_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla v^\epsilon\|_{L^2} + \|\tau_\epsilon\|_{L^2}.
\end{align}
(2.14)
Since
\[v^\epsilon \eta^2 = \omega_\epsilon \eta^2 + \zeta_\epsilon \nu_\epsilon \eta^2\]
we have
\begin{align}
\|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} &\lesssim \|\nabla^2(\zeta_\epsilon \nu_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
\|\nabla^2(\nu_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} &\lesssim \|\zeta_\epsilon\|_{L^\infty} \|\nabla^2(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla \zeta_\epsilon\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla^2(\zeta_\epsilon \nu_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
\|\nabla^2(\zeta_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} &\lesssim \|\zeta_\epsilon\|_{L^\infty} \|\nabla^2(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla \zeta_\epsilon\|_{L^2} \|\nabla(\nu_\epsilon \eta^2)\|_{L^4} + \|\nabla^2(\zeta_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
\|\nabla \zeta_\epsilon\|_{L^2} &\lesssim \|\nabla \zeta_\epsilon\|_{L^2} + \|\nabla^2(\omega_\epsilon \eta^2)\|_{L^{\frac{4}{3}}} + 1.
\end{align}
(2.15)
Therefore, we have
\[
\left\| \nabla^2 (v^\varepsilon \eta^2) \right\|_{L^4} \lesssim \left\| \zeta \right\|_{L^\infty} \left\| \nabla^2 (v^\varepsilon \eta^2) \right\|_{L^4} + \left\| \nabla v^\varepsilon \right\|_{L^2} \left[ 1 + \left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} + \left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \right] + \left\| \tau \right\|_{L^2} + 1 \quad (2.16)
\]
Since \( \nu = \nu^\varepsilon \), we can directly calculate and show that
\[
\left\| \nabla^2 (\nu^\varepsilon \eta^2) \right\|_{L^4} \lesssim \left\| \nabla^{2} (v^\varepsilon \eta^2) \right\|_{L^4} + \left\| \nabla v^\varepsilon \right\|_{L^2} \left[ 1 + \left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \right] + 1. \quad (2.17)
\]
Therefore, we can conclude that
\[
(1 - C \left\| \zeta \right\|_{L^\infty}) \left\| \nabla^2 (v^\varepsilon \eta^2) \right\|_{L^4} \lesssim \left\| \nabla v^\varepsilon \right\|_{L^2} \left[ 1 + \left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \right] + \left\| \tau \right\|_{L^2} + 1. \quad (2.18)
\]
By Sobolev’s embedding, we have
\[
\left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \lesssim \left\| \nabla v^\varepsilon \right\|_{L^2} \left[ 1 + \left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \right] + \left\| \tau \right\|_{L^2} + 1. \quad (2.19)
\]
Taking \( \delta_0 \) small enough in the assumption (2.4), we conclude that
\[
\left\| \nabla (v^\varepsilon \eta^2) \right\|_{L^4} \lesssim \left\| \nabla v^\varepsilon \right\|_{L^2} + \left\| \tau \right\|_{L^2} + 1 \leq C (\delta_0, A_2). \quad (2.20)
\]
Substituting this into (2.19), we obtain which implies that
\[
\left\| \nabla^2 (v^\varepsilon \eta^2) \right\|_{L^4} \leq C (\delta_0, A_2). \quad (2.21)
\]
Hence \( v^\varepsilon \to v \) in \( H^1 (B_{\frac{3r_1}{4}} (x_0)) \).

By Fubini’s theorem, there exists \( r_1 \in \left[ \frac{r_0}{4}, \frac{2r_0}{3} \right] \)
\[
\int_{\partial B_{r_1} (x_0)} |\nabla \zeta|^2 \leq C \int_{B_{\frac{3r_0}{4}} (x_0)} |\nabla \zeta|^2 \leq C, \quad \int_{\partial B_{r_1} (x_0)} |\zeta|^2 \leq C \int_{B_{\frac{3r_0}{4}} (x_0)} |\zeta|^2 \leq C \varepsilon^2. \quad (2.22)
\]
Multiplying the equation of \( \zeta \) by \( \zeta \) and integrating by parts over \( B_{r_2} \), we obtain
\[
\int_{B_{r_1} (x_0)} \left( |\nabla \zeta|^2 + \frac{2}{\varepsilon^2} \chi (\zeta) \zeta^2 + |\nabla \eta|^2 \zeta^2 + \nabla \omega \nabla \nu \cdot \zeta \right) - \int_{\partial B_{r_1} (x_0)} \frac{\partial \zeta}{\partial v} \zeta = \int_{B_{r_1} (x_0)} \tau \zeta \quad (2.23)
\]
Then we have
\[
\int_{B_{r_1} (x_0)} \left( |\nabla \zeta|^2 + \frac{2}{\varepsilon^2} \zeta^2 \right) \leq C \left( \int_{\partial B_{r_1} (x_0)} |\nabla \zeta|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_{r_1} (x_0)} |\zeta|^2 \right)^{\frac{1}{2}} + C \left( \int_{B_{r_1} (x_0)} |\nabla \omega|^4 \right)^{\frac{1}{2}} \left( \int_{B_{r_1} (x_0)} |\zeta|^4 \right)^{\frac{1}{2}} \quad (2.24)
\]
\[
+ C \left( \int_{B_{r_1} (x_0)} |\tau|^2 \right)^{\frac{1}{2}} \left( \int_{B_{r_1} (x_0)} |\zeta|^2 \right)^{\frac{1}{2}} \leq C \varepsilon.
\]
Therefore we have that
\[
\frac{\zeta^2}{\varepsilon^2} \to 0 \quad \text{in} \quad L^1 (B_{r_1} (x_0)). \quad (2.25)
\]
Hence there exists $A$

By Fatou’s Lemma, we have

Combining this with (3.1), we obtain

Hence, by Aubin-Lions’ Lemma, there exists

This, combined with the equation (1.4), implies that there exists $t$

for (1.4) that for almost every $t$

The section is devoted to the proof of Theorem 1.2. First, recall from the global energy inequality for $\mathbf{1.4}$ that for almost every $t \in (0, T)$,

\[
\int_{\Omega \times \{t\}} (|u|^2 + |\nabla u|^2 + \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, N))) + 2\int_{Q_t} (|\nabla u|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) \leq E_0. \tag{3.1}
\]

This, combined with the equation (1.4), implies that there exists $p > 2$ such that

\[
\sup_{\varepsilon > 0} \left[ \|u_t^\varepsilon\|_{L_t^2 H_x^{-1} L^{1/3}_t W^{-2,p}} + \|v_t^\varepsilon\|_{L_t^{4/3} L_x^{4/3}} \right] < \infty. \tag{3.2}
\]

Hence, by Aubin-Lions’ Lemma, there exists $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0, T), \mathbb{R}^2)$ and $v \in L_t^\infty H_x^1 \cap L_t^\infty H_x^2(\Omega \times (0, T), N)$ such that after taking a subsequence,

\[
(u^\varepsilon, v^\varepsilon) \to (u, v) \text{ in } L^2(\Omega \times (0, T)), \quad (\nabla u^\varepsilon, \nabla v^\varepsilon) \rightharpoonup (\nabla u, \nabla v) \text{ in } L^2(\Omega \times (0, T)).
\]

Combining this with (3.1), we obtain

\[
v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon \rightharpoonup v_t + u \cdot \nabla v \text{ in } L^2(\Omega \times (0, T)).
\]

By the lower semi-continuity, we have

\[
\int_{Q_t} (|\nabla u|^2 + |v_t + u \cdot \nabla v|^2) \leq \liminf_{\varepsilon \to 0} \int_{Q_t} (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) < \infty. \tag{3.3}
\]

By Fatou’s Lemma, we have

\[
\int_0^t \liminf_{\varepsilon \to 0} \int_{\Omega} (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) \leq \liminf_{\varepsilon \to 0} \int_0^t \int_{\Omega} (|\nabla u^\varepsilon|^2 + |v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) \leq E_0. \tag{3.4}
\]

Hence there exists $A \subset [0, T]$ with full Lebesgue measure $T$ such that for any $t \in A$

\[
(u^\varepsilon(t), v^\varepsilon(t)) \to (u(t), v(t)) \text{ in } L^2 \times H^1. \tag{3.5}
\]
and
\[
\liminf_{\varepsilon \to 0^+} \int_\Omega (|\nabla u^\varepsilon|^2 + |u_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon|^2) (t) < \infty. \tag{3.6}
\]

Now we define the concentration set at \( t \) by
\[
\Sigma_t := \bigcap_{r>0} \left\{ \mathbf{x} \in \Omega : \liminf_{\varepsilon \to 0} \int_{B_r(\mathbf{x})} \frac{1}{2} |\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, \mathcal{N})) > \delta_0^2 \right\}, \tag{3.7}
\]
where \( \delta_0 \) is given by Lemma 2.1. By Lemma 2.2, it holds \(#(\Sigma_t) \leq C(E_0)\) and
\[v^\varepsilon(t) \to v(t) \text{ in } H^1_{\text{loc}}(\Omega \setminus \Sigma(t)).\]

We would first show that \( v \) is a weak solution of (1.1) by utilizing the geometric structure as in [1] (see also [15]). First notice that there exists a unit vector \( \nu^\varepsilon_N \perp T_{\nu^\varepsilon_N} \mathcal{N} \) such that
\[
\frac{d}{dv} \chi(\text{dist}^2(v^\varepsilon, \mathcal{N})) = 2\chi'(\text{dist}^2(v^\varepsilon, \mathcal{N})) \text{dist}(v^\varepsilon, \mathcal{N}) \nu^\varepsilon_N.
\]
Thus for any \( \phi \in C^0_0(\Omega, \mathbb{R}^L) \) and a.e. \( t \in (0, \infty) \) it holds
\[
\int_{\Omega \times \{t\}} \langle v_t^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon - \Delta v^\varepsilon, D\Pi_N(\Pi_N(v^\varepsilon)) \phi \rangle = 0.
\]
If we choose \( \phi \in C^\infty_0(\Omega \setminus \Sigma_t) \), then it follows from \( \nabla v^\varepsilon \to \nabla v \) in \( H^1_{\text{loc}}(\Omega \setminus \Sigma_t) \) that, after passing to the limit of the above equation,
\[
\int_{\Omega \times \{t\}} \langle v_t + u \cdot \nabla v, D\Pi_N(v) \phi \rangle = -\int_{\Omega \times \{t\}} \langle \nabla v, \nabla(D\Pi_N(v)) \phi \rangle.
\]
This implies that
\[v_t + u \cdot \nabla v - \Delta v = A_N(v)(\nabla v, \nabla v)\]
holds weakly in \( \Omega \setminus \Sigma_t \). Since \( \Sigma_t \) is a finite set, it also holds weakly in \( \Omega \) so that (1.1) holds.

Now, we proceed to verify \( u \) satisfies (1.1). First by the estimate (3.2), we have
\[u_t^\varepsilon \rightharpoonup u_t, \text{ in } L^2([0, T], H^{-1}) \cap L^2([0, T], W^{-2,p})\]
for some \( p > 2 \). For any \( \xi \in C^\infty([0, T]) \) with \( \xi(T) = 0, \varphi \in J \), since
\[
\int_{Q_T} u_t^\varepsilon \xi \varphi = -\int_{\Omega} u_0 \xi(0) \varphi - \int_{Q_T} u_t^\varepsilon \xi' \varphi,
\]
which, after taking \( \epsilon \to 0 \), implies that
\[
\int_{Q_T} u_t \xi \varphi = -\int_{\Omega} u_0 \xi(0) \varphi - \int_{Q_T} u_t \xi' \varphi.
\]

Claim: For any \( t \in A \), it holds
\[
0 = \int_{\Omega \times \{t\}} \langle \partial_t u^\varepsilon, \varphi \rangle + \int_{\Omega \times \{t\}} \langle u^\varepsilon \cdot \nabla u^\varepsilon, \varphi \rangle + \int_{\Omega \times \{t\}} \langle \nabla u^\varepsilon, \nabla \varphi \rangle + \int_{\Omega \times \{t\}} \langle \nabla \varphi, \nabla \varphi \rangle + \int_{\Omega \times \{t\}} (\nabla v^\varepsilon \circ \nabla v^\varepsilon) : \nabla \varphi
\]
\[
- \int_{\Omega \times \{t\}} \langle u_t, \varphi \rangle + \int_{\Omega \times \{t\}} \langle u \cdot \nabla u, \varphi \rangle + \int_{\Omega \times \{t\}} \langle \nabla u, \nabla \varphi \rangle + \int_{\Omega \times \{t\}} (\nabla v \circ \nabla v) : \nabla \varphi, \tag{3.8}
\]
for any \( \varphi \in J \).

For this claim, it suffices to show the convergence of Ericksen stress tensors, i.e.,

\[
\int_{\Omega \times \{t\}} (\nabla v^\varepsilon \otimes \nabla v^\varepsilon) : \nabla \varphi = \int_{\Omega \times \{t\}} (\nabla v \otimes \nabla v) : \nabla \varphi.
\]

For simplicity, we assume \( \Sigma_t = \{(0, 0)\} \subset \Omega \) consists of a single point at zero. Let \( \varphi \in C^\infty(\Omega, \mathbb{R}^2) \) be such that \( \text{div} \varphi = 0 \) and \((0, 0) \in \text{spt}(\varphi) \). Then we observe that by adding \(-\frac{1}{2} |\nabla v^\varepsilon|^2 \|_2 \), we have

\[
\int_{\Omega \times \{t\}} (\nabla v^\varepsilon \otimes \nabla v^\varepsilon) : \nabla \varphi = \int_{\Omega \times \{t\}} (\nabla v^\varepsilon \otimes \nabla v^\varepsilon - \frac{1}{2} |\nabla v^\varepsilon|^2 \|_2) : \nabla \varphi.
\]

While by direct computations, we have

\[
\nabla v^\varepsilon \otimes \nabla v^\varepsilon - \frac{1}{2} |\nabla v^\varepsilon|^2 \|_2 = \frac{1}{2} \begin{pmatrix} |\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2, & 2\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon \rangle \\ 2\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon \rangle, & |\partial_y v^\varepsilon|^2 - |\partial_x v^\varepsilon|^2 \end{pmatrix}.
\]

We can assume that there are two real numbers \( \alpha, \beta \) such that

\[
(|\partial_x v^\varepsilon|^2 - |\partial_y v^\varepsilon|^2)dxdy \to (|\partial_x v|^2 - |\partial_y v|^2)dxdy + \alpha \delta_{(0,0)},
\]

\[
\langle \partial_x v^\varepsilon, \partial_y v^\varepsilon \rangle dxdy \to \langle \partial_x v, \partial_y v \rangle dxdy + \beta \delta_{(0,0)},
\]

hold as convergence of Radon measures. Next we want to show

\[
\alpha = \beta = 0.
\]

Denote

\[
\Delta v^\varepsilon - \frac{1}{\varepsilon^2} \chi' \left( \text{dist}^2(v^\varepsilon, N) \right) \frac{d}{dv} \left( \text{dist}^2(v^\varepsilon, N) \right) = f^\varepsilon := \partial_t v^\varepsilon + u^\varepsilon \cdot \nabla v^\varepsilon
\]

and

\[
e^\varepsilon(v^\varepsilon) = \frac{1}{2} |\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2} \chi \left( \text{dist}^2(v^\varepsilon, N) \right).
\]

Now we derive the Pohozaev identity for \( v^\varepsilon \). For any \( X \in C^\infty_0(\Omega, \mathbb{R}^2) \), by multiplying the \( v^\varepsilon \) equation by \( X \cdot \nabla v^\varepsilon \) and integrating over \( B_r(0) \) we get

\[
\int_{\partial B_r(0)} (X^i v^\varepsilon_j) \cdot \left( v^\varepsilon_i \frac{x^i}{|x|} \right) - \int_{B_r(0)} X^i v^\varepsilon_j \cdot v^\varepsilon_i + \int_{B_r(0)} \text{div}X e^\varepsilon(v^\varepsilon) - \int_{\partial B_r(0)} e^\varepsilon(v^\varepsilon)(X \cdot \frac{x}{|x|}) = \int_{B_r(0)} (X \cdot \nabla v^\varepsilon) \cdot f^\varepsilon.
\]

If we choose \( X(x) = x \), then we have

\[
r \int_{\partial B_r(0)} \left| \frac{\partial v^\varepsilon}{\partial r} \right|^2 + \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, N)) - r \int_{\partial B_r(0)} e^\varepsilon(v^\varepsilon) = \int_{B_r(0)} |x| \frac{\partial v^\varepsilon}{\partial r} \cdot f^\varepsilon.
\]

Then

\[
\int_{\partial B_r(0)} e^\varepsilon(v^\varepsilon) = \int_{\partial B_r(0)} \left| \frac{\partial v^\varepsilon}{\partial r} \right|^2 + \frac{1}{r} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, N)) + O(\int_{B_r(0)} |\nabla v^\varepsilon||f^\varepsilon|).
\]
Integrating from \( r \) to \( R \), we have

\[
\int_{B_R(0) \setminus B_r(0)} e_\varepsilon(v^\varepsilon) - \int_{B_r(0)} e_\varepsilon(v^\varepsilon) = \int_{B_R(0) \setminus B_r(0)} \left| \frac{\partial v^\varepsilon}{\partial r} \right|^2 + \int_{B_r(0)} \frac{1}{r} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) d\tau \\
+ \int_r^R O\left( \int_{B_r(0)} |\nabla v^\varepsilon| |f^\varepsilon| \right) d\tau.
\]

(3.17)

Since \( \Sigma_t = \{(0,0)\} \), we can assume that

\[
e_\varepsilon(v^\varepsilon) dx \to \frac{1}{2} |\nabla v|^2 dx + \gamma \delta_{(0,0)}, \quad \text{in} \ B_\delta(0)
\]

as convergence of Radon measures, where \( \gamma \geq 0 \). Since \( t \in A \),

\[
\lim_{\varepsilon \to 0} \int_{B_r(0)} |f^\varepsilon| |\nabla v^\varepsilon| \leq \lim_{\varepsilon \to 0} \left( \int_{B_r(0)} |f^\varepsilon|^2 \right)^{\frac{1}{2}} \left( \int_{B_r(0)} |\nabla v^\varepsilon|^2 \right)^{\frac{1}{2}} \leq CE_0,
\]

Hence, by sending \( \varepsilon \to 0 \) we obtain from (3.17) that

\[
\int_{B_R(0) \setminus B_r(0)} \frac{1}{2} |\nabla v|^2 \geq \int_{B_R(0) \setminus B_r(0)} \left| \frac{\partial v}{\partial r} \right|^2 + \int_{B_r(0)} \frac{1}{r} \lim_{\varepsilon \to 0} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) d\tau + O(R).
\]

Sending \( r \to 0 \), we have

\[
\int_{B_R(0)} \frac{1}{2} |\nabla v|^2 \geq \int_{B_R(0)} \left| \frac{\partial v}{\partial r} \right|^2 + \int_{B_r(0)} \frac{1}{r} \lim_{\varepsilon \to 0} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) d\tau + O(R).
\]

From this, we claim that

\[
\frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) \to 0 \quad \text{in} \quad L^1(B_\delta).
\]

(3.19)

For, otherwise,

\[
\frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) dx \to \kappa \delta_{(0,0)}
\]

for some \( \kappa > 0 \), this implies

\[
\int_0^R \frac{1}{r} \lim_{\varepsilon \to 0} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) = \int_0^R \frac{\kappa}{r} d\tau = \infty,
\]

which is impossible.

Choosing \( X(x) = (x,0) \) in (3.14), we obtain that

\[
\frac{1}{2} \int_{B_r(0)} \left( |\partial_x v^\varepsilon|^2 - |\partial_x v^\varepsilon|^2 \right) + \int_{B_r(0)} \frac{1}{\varepsilon^2} \chi(\text{dist}^2(\varepsilon N)) \\
= \int_{B_r(0)} x \langle \partial_x v^\varepsilon, f^\varepsilon \rangle + \int_{\partial B_r(0)} \frac{x^2}{r} e_\varepsilon(v^\varepsilon) - \int_{\partial B_r(0)} x \langle \partial_x v^\varepsilon, \frac{\partial v^\varepsilon}{\partial r}\rangle.
\]

(3.20)

Observe that by Fubini’s theorem, for a.e. \( r > 0 \) it holds that

\[
\int_{\partial B_r(0)} x \langle \partial_x v^\varepsilon, \frac{\partial v^\varepsilon}{\partial r}\rangle \to \int_{\partial B_r(0)} x \langle \partial_x v, \frac{\partial v}{\partial r}\rangle,
\]

\[
\int_{\partial B_r(0)} \frac{x^2}{r} e_\varepsilon(v^\varepsilon) \to \frac{1}{2} \int_{\partial B_r} \frac{x^2}{r} |\nabla v|^2,
\]
and by (3.19),
\[ \int_{B_r(0)} \frac{1}{\varepsilon^2} \chi(\text{dist}^2(v^\varepsilon, N')) \to 0. \]

Furthermore,
\[ | \int_{B_r(0)} x(\partial_xv^\varepsilon, f^\varepsilon) | \leq C r \| f^\varepsilon \|_{L^2} \| \nabla v^\varepsilon \|_{L^2} = O(r). \]

Hence, by sending \( \varepsilon \to 0 \) in (3.20), we obtain
\[ \int_{B_r(0)} (|\partial_y v|^2 - |\partial_x v|^2) + \alpha = O(r), \]
this further implies \( \alpha = 0 \) after sending \( r \to 0 \).

Similarly, if we choose \( X(x) = (0, x) \) in (3.14) and pass the limit in the resulting equation, we can get that
\[ \int_{B_r(0)} \left\langle \partial_x v, \partial_y v \right\rangle + \beta = O(r). \]

Hence \( \beta = 0 \). This proves (3.12) and hence completes the proof of Claim.

Multiplying (3.8) by \( \xi \in C^\infty([0, T]) \) with \( \xi(T) = 0 \) and integrating over \([0, T]\), we conclude that \( u \) satisfies the (1.1) on \( Q_T \). The proof of Theorem 1.2 is complete.

4 Compactness of simplified Ericksen-Leslie system

This section is devoted to prove Theorem 1.3. First notice that since the sequence of weak solutions \((u^k, v^k)\) satisfies the assumption (1.9), and
\[ (u^k_0, v^k_0) \rightharpoonup (u_0, v_0) \quad \text{in} \quad L^2(\Omega) \times H^1(\Omega), \]
there exists \((u(x, t), v(x, t)) : \Omega \times (0, T) \to \mathbb{R}^2 \times N\) such that
\[ (u^k, v^k) \rightharpoonup (u, v) \quad \text{in} \quad L^2([0, T], H^1(\Omega)), \]
\[ v^k_t + u^k \cdot \nabla v^k \rightharpoonup v_t + u \cdot \nabla v \quad \text{in} \quad L^2([0, T], L^2(\Omega)). \]

Also it follows from (1.1) and (1.9) that there exists \( p > 2 \) that
\[ \sup_k \left[ \| u^k_t \|_{L^p_t H^{p-1}_x L^{2,p}_x} + \| v^k_t \|_{L^p_t H^{p-1}_x} \right] < \infty. \]

Hence, by Aubin-Lions’ Lemma we have that
\[ (u^k, v^k) \rightharpoonup (u, v) \quad \text{in} \quad L^2(Q_T) \times L^2(Q_T). \]

By the lower semi-continuity, we have
\[ \int_{Q_T} (|\nabla u|^2 + |v_t + u \cdot \nabla v|^2) \leq \liminf_{k \to \infty} \int_{Q_T} (|\nabla u^k|^2 + |v^k_t + u^k \cdot \nabla v^k|^2) \leq C_0. \]
By Fatou’s Lemma and (1.9), we have
\[ \int_0^t \liminf_{k \to \infty} \int_\Omega \left( |\nabla u_k|^2 + |v^k + u^k \cdot \nabla v^k|^2 \right) \leq \liminf_{k \to \infty} \int_{Q_{t_0}} \left( |\nabla u_k|^2 + |v^k + u^k \cdot \nabla v^k|^2 \right) \leq C_0. \]

Hence, there exists \( A \subset [0, T] \) with full Lebesgue measure \( T \), such that for all \( t \in A \)
\[ (u^k(t), v^k(t)) \to (u(t), v(t)), \quad \text{in } L^2(\Omega) \times H^1(\Omega) \] (4.4)
and
\[ \liminf_{k \to \infty} \int_\Omega \left( |\nabla u_k|^2 + |v^k + u^k \cdot \nabla v^k|^2 \right)(t) < \infty. \] (4.5)

Now we define the concentration set at time \( t \in (0, T) \) by
\[ \Sigma_t := \bigcap_{r > 0} \left\{ x \in \Omega : \liminf_{k \to \infty} \int_{B_r(x)} |\nabla v^k|^2 > \delta_0^2 \right\} \] (4.6)
where \( \delta_0 \) is small constant given by Theorem 1.2 in [21]. As in [21] (see also [18], [22]), we can show that for any \( t \in A \), it holds that \( \#(\Sigma_t) \leq C(E_0) \) and
\[ v^k(t) \to v \quad \text{in } H^1_{loc}(\Omega \setminus \Sigma_t). \] (4.7)

Similar to the proof of Theorem 1.2, we can show the weak limit \((u, v)\) satisfies the third equation of (1.1) in the weak sense. It remains to show that the first equation of (1.1) is also valid in the weak sense.

Similar to the proof of Theorem 1.2 to complete the proof of Theorem 1.3 it is suffices to show
\[ \lim_{k \to \infty} \int_{\Omega \times \{t\}} \left( \nabla v^k \cdot \nabla v^k \right) : \nabla \varphi = \int_{\Omega \times \{t\}} (\nabla v \cdot \nabla v) : \nabla \varphi, \quad \forall \varphi \in J. \] (4.8)

For simplicity, assume \( \Sigma_t = \{(0, 0)\} \subset \Omega \). Let \( \varphi \in C^\infty(\Omega, \mathbb{R}^2) \) be such that \( \text{div } \varphi = 0 \) and \((0, 0) \in \text{spt}(\varphi)\). By the same calculation as in (3.19), we have
\[ \nabla v^\varphi \cdot \nabla v^\varphi - \frac{1}{2} |\nabla v^\varphi|^2 = \frac{1}{2} \begin{pmatrix} |\partial_x v^\varphi|^2 - |\partial_y v^\varphi|^2, & 2\langle \partial_x v^\varphi, \partial_y v^\varphi \rangle \\ 2\langle \partial_x v^\varphi, \partial_y v^\varphi \rangle, & |\partial_y v^\varphi|^2 - |\partial_x v^\varphi|^2 \end{pmatrix}. \]

For any \( t \in A \), \( v^k(t) \) is an approximated harmonic maps from \( \Omega \) to \( \mathcal{N} \):
\[ \Delta v^k(t) + A(v^k)(\nabla v^k, \nabla v^k) = g^k(t) := v^k(t) + u^k \cdot \nabla v^k(t) \in L^2(\Omega). \] (4.9)

Recall the Hopf differential of \( v^k \) is defined by
\[ H^k = \left( \frac{\partial v^k}{\partial z} \right)^2 = |\partial_x v^k|^2 - |\partial_y v^k|^2 + 2i\langle \partial_x v^k, \partial_y v^k \rangle, \] (4.10)
where \( z = x + iy \in \mathbb{C} \). Then
\[ \frac{\partial H^k}{\partial z} = 2 \frac{\partial_x v^k}{\partial z} \frac{\partial^2 v^k}{\partial z \partial z} = 2 \Delta v^k \frac{\partial v^k}{\partial z} = 2 g^k(t) \cdot \frac{\partial v^k}{\partial z} := G^k. \] (4.11)
It is clear that
\[ \|G^k\|_{L^1(B_r)} \leq 2\|g^k(t)\|_{L^2(\Omega)} \left\| \frac{\partial v^k}{\partial z} \right\|_{L^2(\Omega)} \leq 2C_0. \] (4.12)

Therefore, for any \( z \in B_r(0) \)
\[ \mathcal{H}^k(z) = \int_{\partial B_{2r}(0)} \frac{\mathcal{H}^k(\omega)}{z-\omega} \, d\sigma + \int_{B_{2r}(0)} \frac{G^k(\omega)}{z-\omega} \, d\omega. \] (4.13)

By the Young inequality of convolutions, we obtain
\[ \|\mathcal{H}^k\|_{L^p(B_r)} \leq C(r,p)\|\mathcal{H}^k\|_{L^1(\partial B_{2r})} + \|\frac{1}{z}\|_{L^p} \|G^k\|_{L^1(B_{2r})} \leq C(r,p). \] (4.14)

for any \( 1 < p < 2 \). From this, we immediately conclude that
\[ |\partial_x v^k|^2 - |\partial_y v^k|^2 \rightarrow |\partial_x v|^2 - |\partial_y v|^2, \quad \langle \partial_x v^k, \partial_y v^k \rangle \rightarrow \langle \partial_x v, \partial_y v \rangle \text{ in } L^p(B_r(0)) \]
for any \( 1 < p < 2 \), which implies (4.8). This completes the proof of Theorem 1.3.

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