Constantin Carathéodory axiomatic approach and Grigory Perelman thermodynamics for geometric flows and cosmological solitonic solutions

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Abstract We elaborate on statistical thermodynamics models of relativistic geometric flows as generalizations of G. Perelman and R. Hamilton theory centred around C. Carathéodory axiomatic approach to thermodynamics with Pfaffian differential equations. The anholonomic frame deformation method, AFDM, for constructing generic off-diagonal and locally anisotropic cosmological solitonic solutions in the theory of relativistic geometric flows and general relativity is developed. We conclude that such solutions cannot be described in terms of the Hawking–Bekenstein thermodynamics for hypersurface, holographic, (anti-) de Sitter and similar configurations. The geometric thermodynamic values are defined and computed for nonholonomic Ricci flows, (modified) Einstein equations, and new classes of locally anisotropic cosmological solutions encoding solitonic hierarchies.

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1 Introduction

Thermodynamics is a fundamental physical theory with various branches and applications in modern physics, engineering, biology, chemistry, information theory and mathematics, see reviews [1–5] and references therein. The equilibrium thermodynamics originated from the study of heat engines when the combination of mechanical and thermal concepts was done in an empirical way with further essential developments and contributions to statistical physics and ergodic theory, modern gravity, cosmology, etc.

Thermodynamic ideas and methods were developed and applied in the black hole, BH, physics [6–9], using the Bekenstein–Hawking entropy, and (using different concepts and constructions) in the proof of the Thurston–Poincaré conjecture due to Grigory Perelman [10], see original fundamental physical and mathematical works and reviews of geometric analysis and topological results in [11–16]. In a series of our and co-authors works, we studied possible implications of the approach elaborated by G. Perelman for geometric flow statistical thermodynamics in certain directions of modified and Einstein gravity and cosmology and astrophysics [17–20] and classical and quantum geometric information flow theory [21–23]. It was exploited the idea that the concept of Perelman W-entropy and associated statistical models present more general mathematical and physical possibilities compared to those elaborated for theories and solutions with Bekenstein–Hawking and another area-holographic type entropies.

Constantin Carathéory formulated the first systematic and axiomatic formulation of equilibrium thermodynamics [24,25]. In such an approach (see [5,26–28] on further extensions), the geometry of thermodynamics is symplectic and analogous to the structure of Hamilton mechanics and can be expressed through Pfaff forms and related systems of the first-order partial differential equations. The axiomatic treatment of thermodynamics caught the atten-
ition of a number of famous and well-known scientists [5,29–35] who recognized and in one case criticized [36] Carathéodory’s papers; for brief reviews, we cite [37–40].

In the present work, we will not get into the details of C. Carathéodory and G. Perelman achievements, see above-cited works and [37,38,41–44], on life and contributions in mathematics, physics, and education. We shall study only how the mathematic tools of axiomatic thermodynamics can be applied to relativistic generalizations of geometric flow theory and compute geometric thermodynamic values for locally anisotropic cosmological solutions. There will be considered also certain applications in modern cosmology. There are three main purposes of this article: (1) to show how the C. Carathéodory axiomatic approach to thermodynamics in the language of Pfaff forms can be extended in order to include in the scheme generalizations of the G. Perelman thermodynamics for relativistic geometric flows; (2) to consider possible applications of the anholonomic frame deformation method, AFDM, and study main properties of geometric evolution flows of locally anisotropic cosmological models (in particular with generic off-diagonal solitonic deformations of the Friedmann–Lemaître–Robertson–Walker, FLRW, metrics); (3) to provide explicit examples of how geometric flow thermodynamic values are computed for cosmological solitonic solutions which cannot be described by thermodynamic concepts elaborated in the framework of Bekenstein–Hawking entropy and generalizations.

This paper is organized as follows: In Sect. 2, we present an introduction into the theory of relativistic nonholonomic flows with modified F- and W-functionals and elaborate on respective statistical thermodynamic models. How the Carathéodory axiomatic approach can be extended in order to include some classes of generalized Ricci flows and solitons is considered. Then, in Sect. 3, we develop the AFDM and show that the important general decoupling and integration properties of geometric evolution flow and Ricci soliton equations are preserved for cosmological solitonic spaces. Possible locally anisotropic cosmological parameterizations are summarized in Table 1. We provide Table 2 summarizing the AFDM for generating such locally anisotropic cosmological solutions. In Sect. 4, we show how exact and parametric cosmological solitonic solutions can be constructed for relativistic geometric flow evolution equations. There are analysed explicit examples of computing respective W-entropy, thermodynamic values and Pfaffians. Finally, conclusions and perspectives are considered in Sect. 5. In Appendix A, we outline some results on Pfaffian differential equations. Appendix B contains necessary parameterizations for flows of cosmological solitonic metrics.

2 Generalized Carathéodory–Perelman thermodynamics and Ricci flows

In this section, we provide a brief introduction to the theory of relativistic geometric flows and analogous statistical thermodynamics for nonholonomic Einstein systems, NESs, see details in [17–20] and references therein. How such constructions can be formalized following the Carathéodory axiomatic approach is analysed.

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1 M. Planck and some other authors criticism “targeting quick results” was about the difficulty to provide a simple physical picture of the Carathéodory method and the concept of entropy together with sophisticated geometric methods unknown at that time to the bulk of physicists and mathematicians. At present, the functional analysis, measure theory and topology techniques are familiar to researchers publishing works in mathematical physics and geometry and physics.
2.1 Relativistic models of geometric flow thermodynamics

We consider a relativistic spacetime as in general relativity, GR. Geometrically, it is defined by a (pseudo-) Riemannian manifold \( V \) with a conventional splitting of dimension, \( \dim V = 4 = 2 + 2 \), and two-dimensional horizontal, \( h \), and two-dimensional vertical, \( v \), components (such decomposition will be useful for constructing exact solutions of systems of important physical equations). This induces dyadic decompositions of local bases and corresponding tangent bundles \( T V \) and, its dual, \( T^* V \). Being enabled with a metric \( g = (h g, v g) \) of a local pseudo-Euclidean signature \((+++−)\) and postulating local causality conditions as in special relativity theory, we model a curved spacetime as a Lorentzian manifold. We can always consider that such spacetimes are endowed with a double nonholonomic \( 2 + 2 \) and \( 3 + 1 \) splitting. (The first splitting will be used for elaborating new methods of constructing exact solutions, and the second splitting will be necessary for elaborating thermodynamical models.)

In this work, we say that a Lorentz manifold \( V \) is nonholonomic (in literature, there are also used equivalent terms like anholonomic, or non-integrable) if it is endowed with a \( h \)-and/or \( v \)-splitting defined by a Whitney sum defining a nonlinear connection, \( N \)-connection, structure \( N : \ T V = h V \oplus v V \), where \( T V \) is the tangent bundle on \( V \). Such a geometric structure is a fundamental one for elaborating various models of Finsler–Lagrange–Hamilton geometry which are determined in complete form if there are prescribed three fundamental geometric objects/structures (a nonlinear quadratic line element, a nonlinear connection and a distinguished connection which is adapted to a \( h \)-\( v \)-splitting). \( N \)-connections can be introduced also in (pseudo-) Riemannian geometry when (in local form) \( N = N^a_i (u) dx^i \otimes \partial_u \) is determined by for a corresponding set of coefficients \( \{ N^a_i \} \) which can be related to certain off-diagonal terms of metrics in certain local frames of coordinates. Corresponding subclasses of \( N \)-adapted (co) frames allow, for instance, nonholonomic dyadic decompositions of geometric and physical objects. Together with a so-called canonical nonholonomic deformations of linear connection structures (we shall use “hats” on geometric and physical objects adapted to such canonical nonholonomic frames), this allows to integrate (modified) Einstein and geometric flow equations in very general forms depending, in principle, on all spacetime coordinates and a geometric evolution parameter.

We shall use two important linear connections which can be constructed using the same metric structure:

\[
\begin{align*}
g \rightarrow \left\{ \begin{array}{ll}
\nabla : & \nabla g = 0; \quad \nabla T = 0, \quad \text{the Levi–Civita, LC, connection;} \\
\widehat{\nabla} : & \widehat{\nabla} g = 0; \quad h \widehat{T} = 0, \quad v \widehat{T} = 0. \quad \text{the canonical d-connection.} 
\end{array} \right.
\end{align*}
\]

In these formulas (see [17–20] for details on computing coefficients with respect to \( N \)-adapted and/or coordinate frames), the distinguished connection (d-connection) \( \widehat{\nabla} = (h \widehat{\nabla}, v \widehat{\nabla}) \) preserves under parallelism the decomposition \( N \) and \( \widehat{T}[g, N] \) is the corresponding torsion d-tensor. We use the terms d-tensor, d-connection, etc., for geometric objects adapted to an \( N \)-connection \( h \)-\( v \)-splitting. The LC-connection \( \nabla \) can be introduced without any \( N \)-connection structure, but the zero torsion condition in the case of generic off-diagonal metrics does not

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2 We parameterize the coordinates as \( u^\mu = (x^i, y^a) \), in brief, \( u = (x, y) \), where \( i, j, \ldots = 1, 2 \) and \( a, b = 3, 4 \), with small Greek indices \( \alpha, \beta, \ldots = 1, 2, 3, 4 \), when \( u^4 = y^4 = t \) is the time-like coordinate. We shall summarize on “up-low” repeating indices and use boldface symbols for spaces and geometric objects adapted to a \( N \)-connection splitting. For a double \( 2 + 2 \) and \( 3 + 1 \) splitting, the local coordinates are labelled \( u^a = (x^i, y^a) = (x^i, u^4 = t) \) for \( i, j, k = 1, 2, 3 \). The nonholonomic distributions can be \( N \)-adapted form for any open region \( U \subset V \) covered by a family of 3-d spacelike hypersurfaces \( \Sigma_t \) with a time-like parameter \( t \).
allow to prove a decoupling property and explicit integration of physically important systems of nonlinear partial differential equations, PDEs. Nevertheless, any geometric data \((g, V)\) can be distorted to some canonical ones \((\hat{g}, \hat{D})\) with decoupling of (modified) Einstein equations and encoding in general form various classes of physically important solutions

\[
\hat{D}[g, N] = V[g, N] + \hat{Z}[g, N].
\]

where \(\hat{Z}\) is the distortion d-tensor determined in standard algebraic form by the torsion tensor \(\hat{T}[g, N]\) of \(\hat{D}\). These values are completely defined by the metric d-tensor \(g = (h, v\hat{g})\) adapted to a prescribed \(N\). The values \(h \hat{T}\) and \(v\hat{T}\) denote respective torsion components which vanish on conventional h- and v-subspaces. There are also nontrivial components \(h v \hat{T}\) defined by certain anholonomy (equivalently, nonholonomic/non-integrable) relations. All geometric constructions on a Lorentz manifold \(V\) can be performed in a not adapted \(N\)-connection form with \(\nabla\) and/or in \(N\)-adapted form using \(\hat{D}\) from (1) or other type d-connections. The corresponding torsion d-tensor \(\hat{T}^\alpha_{\beta \gamma} ; \) Ricci d-tensor, \(\hat{R}_{\beta \gamma} ; \) scalar curvature \(s\hat{R} := g^\alpha\beta \hat{R}_{\alpha \beta \gamma} ; \) and Einstein d-tensor, \(\hat{E}_{\beta \gamma} := \hat{R}_{\beta \gamma} - \frac{1}{2} g_{\beta \gamma} s\hat{R}\) are defined and computed in standard forms as in metric-affine geometry and related via distortion formulas to respective values determined by \(\nabla\).

In the theory of Ricci flows of geometric objects on \(V\), it is considered an evolution positive parameter \(\tau, 0 \leq \tau \leq \tau_0\), which for thermodynamic models can identified with the temperature, or chosen to be proportional to a temperature parameter. For the geometric flow evolution of Riemannian metrics and respective statistical thermodynamic models, this was considered in G. Perelman’s famous preprint [10]. For evolution of (generalized) pseudo-Riemannian configurations, we can elaborate on two classes of (effective) geometric theories, when families of metrics 1] \(g(\tau) := g(\tau, u)\) are labelled by a conventional evolution (relativistic temperature parameter) or 2] \(g(\tau) := g(\tau, x^i)\) for an imaginary time-like coordinate \(u^a = (x^i, y^a) = (x^i, u^4 = i\tau)\), where \(i^2 = -1\) and, for simplicity, there used units when the fundamental speed of light is \(c = 1\). Hereafter, we shall write in brief only the dependence on evolution parameter, without spacetime or space coordinates if that will not result in ambiguities. In this work, we study only theories of class 1] when the evolution models are relativistic, encode solitonic waves for pseudo-Riemannian metric signatures, and can be characterized by relativistic thermodynamic models, see details in [17–19] and references therein. In such geometric and thermodynamic theories, we consider also flows of \(N\)-connections \(N(\tau) = N(\tau, u)\), canonical d-connections \(\hat{D}(\tau) = \hat{D}(\tau, u)\). On \(V\), we can introduce also families of Lagrange densities \(sL(\tau)\), for gravitational fields in a MGT or GR (when \(sL(\tau) = sR[\nabla(\tau)]\)), and \(\nabla L(\tau) = \nabla L[g(\tau), \hat{D}(\tau), \varphi(\tau)]\), as total Lagrangians for effective and matter fields which will be defined below for certain cosmological models with scalar fields \(\varphi(\tau) = \varphi(\tau, u)\).

For any region \(U \subset V\) with a 2+2 splitting \((N, g)\), we consider an additional structure of 3-d hypersurfaces \(\Sigma_t\) parameterized by time-like coordinate \(y^a = t\) for coordinates \(u^a = (x^i, y^a) = (x^i, t)\). The families of metrics can be represented as d-metrics with 3+1 splitting and \(N\)-adapted geometric evolution

\[
g(\tau) = g_{\alpha \beta}(\tau, u) d\epsilon^{\alpha\beta}(\tau) \otimes d\epsilon^{\alpha\beta}(\tau)
\]

\[
= q_i(\tau, x^k) dx^i \otimes dx^i + q_3(\tau, x^k, y^a) e^3(\tau) \otimes e^3(\tau)
\]

\[
- [q N(\tau, x^k, y^a)]^2 e^4(\tau) \otimes e^4(\tau),
\]

for \(e^{\mu}(\tau) = (e^i = dx^i, e^a(\tau) = dy^a + N^a_i(\tau) dx^i)\).
In (3), there are considered geometric flows of “shift” coefficients $q_i(\tau) = (q_i(\tau), q_3(\tau))$ related to flows of a 3-d metric $q_{ij}(\tau) = diag(q_i(\tau)) = (q_i(\tau), q_3(\tau))$ on a hypersurface $\Sigma$, if $q_3(\tau) = g_3(\tau)$ and $[q N(\tau)]^2 = g_4(\tau)$, where $q N(\tau)$ is a family of lapse functions. Here, it should be noted that we follow notations which are different from those in [45] for GR. In this work, it is used a left label $\tau$ in order to avoid ambiguities with the notations for the coefficients $N^i\mu$ of a N-connection. There are considered flows of N-adapted frames (4) determined by the flow evolution of N-connection coefficients.

In nonholonomic canonical variables, the relativistic versions of G. Perelman functionals (originally defined in [10] for flows of Riemannian metrics), in this work encoding also the geometric evolution of matter fields, are postulated [17–19] in the form

$$\mathcal{F}(\tau) = \int (4\pi)^{-2} e^{-\tilde{f}} \sqrt{|g|} d^4u (\hat{R} + \nabla L + |\hat{\delta}|^2)$$ (5) and

$$\mathcal{N}(\tau) = \int \hat{\mu} \sqrt{|g|} d^4u [\tau(\hat{R} + \nabla L + |\hat{\delta}|^2 + \hat{f} - 4].$$ (6)

In such formulas, a normalizing function $\hat{f}(\tau, u)$ can be a convenient one for elaborating certain topological/geometric/physical models or subjected to the conditions

$$\mathcal{V}(\tau) = \int \hat{\mu} \sqrt{|g|} d^4u = \int_{t_1}^{t_2} \int_{\Sigma_i} \hat{\mu} \sqrt{|g|} d^4u = 1,$$ (7)

for a classical integration measure $\hat{\mu} = (4\pi)^{-2} e^{-\tilde{f}}$ (a version of Carathéodory measure); and the Ricci scalar $\hat{\mu}$ is taken for the Ricci d-tensor $\hat{\delta}$ of a d-connection $\hat{D}$.

There is a series of arguments for writing the $\mathcal{F}$-functional (5) and $\mathcal{N}$-functional (6) in above forms:

1. Fixing variations of such functionals on a d-metric and respective matter field evolution scenarios, and considering self-similar configurations for a $\tau = \tau_0$, we obtain systems of nonlinear PDEs for relativistic Ricci solitons, which are equivalent to the gravitational field equations in nonholonomic variables,

$$\hat{R}_{ab} = \hat{\psi}_{ab}$$ (8)

for nonholonomic Einstein systems, NESs, if the normalizing function $\hat{f}$ is correspondingly chosen and the nonholonomic constraints for extracting Levi–Civita configurations are imposed to extract LC-configurations $\hat{D}_{|\tau=0} = \nabla$. The sources $\hat{\psi}_{ab}(\tau) = [\hat{\psi}_{ij}(\tau), \hat{\psi}_{ab}(\tau)]$ with coefficients defined with respect to N-adapted frames in (8) are of type $\hat{\psi}_{\mu\nu} = \hat{\psi}_{\mu\nu} + \nabla_{\mu\nu}^e$, where $\nabla^e_{\mu\nu}$ are effective sources determined by distortions of the linear connections and effective Lagrangians for gravitational fields. Such a source is not zero even in GR if there are nonzero distortions (2) from $\hat{D}$ to $\nabla$. A source for matter field, $\nabla_{\mu\nu}^e$, can be constructed using a N-adapted variational calculus for a Lagrange density $\mathcal{L}(g, \hat{D}, A, \psi)$, when

$$\nabla_{\mu\nu}^e = \chi (m \nabla_{\mu\nu} - \frac{1}{2} g_{\mu\nu} m \nabla) \rightarrow \chi (m T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} m T)$$

for [coefficients of $\hat{D}] \rightarrow$ [coefficients of $\nabla$]. In such formulas, we consider $m \nabla = g^{\mu\nu} m \nabla_{\mu\nu}$ for the N-adapted energy-momentum tensor

$$m \nabla_{\mu\nu} := - \frac{2}{\sqrt{|g_{\mu\nu}|}} \frac{\delta(\sqrt{|g_{\mu\nu}|} m \mathcal{L})}{\delta g^{\mu\nu}}.$$

(9)
For simplicity, we shall consider only Lagrange densities $^m\mathcal{L} = \phi \mathcal{L}(\mathbf{g}, \hat{\mathbf{D}}, \phi)$ determined by a scalar field $\phi(x, u)$ and/or geometric evolution of scalar fields $\phi(\tau) = \phi(\tau, x, u)$, when $^m\mathcal{T}_{\alpha\beta} = \phi \hat{T}_{\alpha\beta}$.

2. For three dimensional, 3-d, Riemannian metrics, there are obtained respective Lyapunov-type functionals as it was postulated in [10] and used for the proof of the Thurston–Poincaré conjecture.

3. The functional $\hat{W}(6)$ defines a nonholonomic canonical and relativistic generalization of the so-called W-entropy introduced in [10]. Various types of 4-d - 10-d $\hat{W}$-entropies and associated statistical and quantum thermodynamics values are used for elaborating models of classical and (commutative and noncommutative/supersymmetric) quantum geometric flows and geometric information flows, see [16–23] and references therein.

4. The functionals $\hat{F}$ and $\hat{V}$ result in generalized R. Hamilton equations [12] considered earlier in physics by D. Friedan [11] (respective proofs for and N-adapted variational calculus are presented in [17–20]):

$$\frac{\partial g_{ij}}{\partial \tau} = -2 \left( \hat{R}_{ij} - \hat{Y}_{ij} \right); \quad \frac{\partial g_{ab}}{\partial \tau} = -2 \left( \hat{R}_{ab} - \hat{Y}_{ab} \right);$$

$$\hat{R}_{ia} = \hat{R}_{ai} = 0; \quad \hat{R}_{ij} = \hat{R}_{ji}; \quad \hat{R}_{ab} = \hat{R}_{ba};$$

$$\frac{\partial}{\partial \tau} \hat{f} = -\hat{\nabla} \hat{f} + \hat{D} \hat{f}^2 - s \hat{R} + \hat{Y}_a,$$ (11)

where $\hat{\square}(\tau) = \hat{D}^\alpha(\tau) \hat{D}_\alpha(\tau)$ is used for the geometric flows of the d’Alembert operator. In nonholonomic canonical variables with $\hat{\mathbf{D}}$, such systems of nonlinear PDEs can be integrated in very general forms and restricted to describe the geometric evolution and Ricci soliton configurations of NESs, see details and proofs in references from the previous paragraph (point).

5. The measure $\mu \sqrt{|\mathbf{g}|} d^4u = (4\pi \tau)^{-2} e^{-\hat{f}} \sqrt{|\mathbf{g}|} d^4u$ consists an explicit example of a Carathéodory-type measure which allows to construct geometric and statistical thermodynamic models. Such a model was elaborated for the flow evolution of 3-d Riemannian metrics by G. Perelman and considered in the proof of the Thurston–Poincaré conjecture. Thermodynamic measures have more rich implications in various branches of topology and applications and provide a natural tool to understand the difficulties (ergodicity, approach to equilibrium, irreversibility, etc.) in the foundations of statistical physics and non-equilibrium thermodynamics, see discussions and references in [38–40,46–48]. Fixing a normalizing function $\hat{f}$, we prescribe an evolution scenarios with respective scales and phase space integral properties determined by geometric and physical data $(\mathbf{g}, \hat{\mathbf{D}}, \hat{\phi})$.

We can consider a geometric evolution model without $^m\mathcal{L}$ but with a re-defined functional measure $\mu \sqrt{|\mathbf{g}|} d^4u = (4\pi \tau)^{-2} e^{-\hat{f}(\nabla)} \sqrt{|\mathbf{g}|} d^4u$, where $f^\nabla$ is chosen to be a solution of this system of PDEs:

$$\text{tot} \mathcal{L} + [\hat{D} \hat{f}]^2 = [\hat{D}(f^\nabla)]^2 \text{ and } \tau (\text{tot} \mathcal{L} + | \hat{\mathbf{D}} \hat{f} | + \hat{\nu} [\hat{D}(f^\nabla)]^2 + \hat{f})$$

$$= \tau (| \hat{\mathbf{D}}(f^\nabla) | + | \hat{\nu} [\hat{D}(f^\nabla)]^2 + f^\nabla).$$ (12)

The solutions for such a $f^\nabla(\hat{f})$ and/or $\hat{f}(f^\nabla)$ can be found in an explicit form (usually, such a normalizing function can be approximated to a constant) for a large class of generic off-diagonal or diagonal solutions of systems (10) and (11). In such constructions, usually there are prescribed respective values of some generating functions and sources and effective cosmological constants subjected to certain nonlinear symmetry conditions, see details in [16–23] and Sect. 4. For various physical applications, it is enough to find a class of solutions of generalized R. Hamilton Eqs. (10) and to consider that the
geometric evolution is normalized by some functions $f^\varphi(\hat{f})$ and/or $\hat{f}(f^\varphi)$ subjected to conditions (11) and (12). In many cases, such normalizations can be performed with certain integration constants or for series expansions on a small parameter.

Using formulas (5), (6), (10), (11), we can elaborate on statistical thermodynamic models for geometric flows determined by data $(\mathbf{g}, \hat{\mathbf{D}}, A, f^\varphi)$ and applying concepts and formulas from statistical thermodynamics. It is considered a canonical ensemble at temperature $\beta^{-1} = T$, in this work $T$ is proportional to $\tau$, with partition function $Z = \int \exp(-\beta E) d\omega(E)$, where a measure $\omega(E)$ is defined as a density of states. In standard form, there are computed such important thermodynamical values: average flow energy, $\mathcal{E} = \langle E \rangle := -\partial \log Z / \partial \beta$; flow entropy, $S := \beta \langle E \rangle + \log Z$; flow fluctuation, $\eta := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z / \partial \beta^2$.

A geometric thermodynamic model can be constructed if we associate with (6) a respective thermodynamic generating functions (in this work, flows of $\varphi^\top \mathcal{L}$ are encoded into $f^\varphi(\hat{f}, \varphi^\top \mathcal{L})$ subjected to the conditions (12)),

$$\hat{\mathcal{Z}}[\mathbf{g}(\tau), f^\varphi] = \int (4\pi \tau)^{-2} e^{-\hat{f}(f^\varphi)} \sqrt{|\mathbf{g}|} d^4 u (-f^\varphi + 2), \text{ for } \mathbf{V}. \quad (13)$$

Hereafter, we shall not write functional dependencies on $\mathbf{g}(\tau)$ and $f^\varphi$ if it will not result in ambiguities.

Applying a similar variational calculus similar to that presented in details in [10,14–16] (in $N$-adapted form for frames (4) and d-connections $\hat{\mathbf{D}}$ to (13) and (6) and respective $3 + 1$ parameterizations of d-metrics (3), we define and compute analogous thermodynamic values for geometric evolution flows of NES

$$\hat{\mathcal{E}}(\tau) = -\tau^2 \int (4\pi \tau)^{-2} e^{-\hat{f}(f^\varphi)} \sqrt{|q_1 q_2 q_3 q N|} \delta^4 u \left[ -\delta R + |\hat{\mathbf{D}} f^\varphi|^2 - \frac{2}{\tau} \right],$$

$$\hat{\mathcal{S}}(\tau) = -\int (4\pi \tau)^{-2} e^{-\hat{f}(f^\varphi)} \sqrt{|q_1 q_2 q_3 q N|} \delta^4 u \left[ \tau \left( \delta R + |\hat{\mathbf{D}} f^\varphi|^2 \right) + f^\varphi - 4 \right],$$

$$\hat{\eta}(\tau) = -2\tau^4 \int (4\pi \tau)^{-2} e^{-\hat{f}(f^\varphi)} \sqrt{|q_1 q_2 q_3 q N|} \delta^4 u \left[ \delta R_{\alpha \beta} + \hat{\mathbf{D}}_\alpha \hat{\mathbf{D}}_\beta f^\varphi - \frac{1}{2\tau} g_{\alpha \beta} \right]. \quad (14)$$

In these formulas, $\delta^4 u$ contains $N$-elongated differentials and the data on matter fields and nonlinear symmetries are encoded in $f^\varphi(\hat{f}, \varphi^\top \mathcal{L})$. For fixed self-similar Riemannian configurations, the values $\hat{\mathcal{E}}$ and $\hat{\mathcal{S}}$ provide an equilibrium thermodynamic description of Ricci solitons. Such concepts can be considered along any causal curve on a Lorentz manifold when fixing certain normalization functions and nonlinear symmetries the conventional thermodynamic description holds true for evolution models of NESs. The fluctuation $\hat{\eta}(\tau)$ allows to include into consideration small perturbations of metrics and corresponding distortion values. Such a description defines a relativistic thermodynamic model which is irreversible and describes various types of nonlinear self-organizing, pattern forming, kinetic and/or stochastic processes, see examples and references in [16–23].

After Carathéodory had completed the proof of Poincaré recurrence theorem [49], he was the first who saw that measure theory is the natural language to discuss the problems of statistical physics and thermodynamics. The Poincaré hypothesis was formulated on topological properties of three-dimensional hypersurfaces endowed with Riemannian metrics. The proof of its generalized form as the Thurston–Poincaré conjecture was possible by introducing measures of type

$$\hat{M} = \hat{\mu} \sqrt{|\mathbf{g}|} d^4 u = (4\pi \tau)^{-2} e^{-\hat{f}} \sqrt{|\mathbf{g}|} d^4 u \quad (15)$$

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for Riemannian versions of functionals $\hat{F}(5)$ $\hat{W}(6)$, when with $\hat{D}_{\|T} = \nabla$, and a respective thermodynamic values (14). If $\sqrt{|g|}$ is considered for pseudo-Riemannian metrics (and various, for instance, Finsler–Lagrange–Hamilton generalizations [16,20]), we can speculate on respective generalized Poincaré recurrence theorem which can be reformulated in this form: The volume preserving dynamical transformations of an (effective) phase space with a measure $\hat{\mu}\sqrt{|g|}d^4u$ have the property that almost all points in any region of positive volume (excepting possible subsets of zero volume) will return back into their region after some finite time. For relativistic configurations, we consider 3+1 splitting and a Lorentz-type causality on respective spacetime and/or phase space (co) tangent Lorentz bundles. Of course, the return time for each point connected by a causal curve, is different but can be computed for a respective exact/parametric solution of the relativistic geometric flows and/or (modified) gravity theory (i.e. nonholonomic Ricci soliton configuration).

Following the Carathéodory measure theoretic idea [49] and Birkhoff’s approach to ergodicity [50,51], we can clarify the relation between ergodic and recurrent systems. In the case of geometric flows, we can define ergodicity by replacing Boltzmann’s sets with sets of nonzero volume measure which was generalized for nonholonomic manifolds and generalized Finsler spaces in [17,52–54]. Here, we note that geometric flows as ergodic systems are recurrent but not vice versa. Non-ergodic systems decompose into time-invariant ergodic sub-systems, and this property can be extended to relativistic flows defined along causal curves.

Mixing (it means that the statistical correlations decay and results in statistical regularity) of geometric flows implies ergodicity and is compatible with recurrence. Let us explain how it characterizes geometric evolution of NESs relating certain subsets $A, B, Y \subset \mathbf{V}$, when $S_t : Y \to Y$ is the geometric (gradient) flow evolution on the phase space $Y$, and there are satisfied nonzero measure conditions, $\hat{M}[A] \neq 0, \hat{M}[B] \neq 0$, for $\hat{M}[Y] = 1$ considered as a probability measure determined by (15), when $\lim_{t \to \pm \infty} \frac{\hat{M}[A \cap S_t B]}{\hat{M}[A]} = \frac{\hat{M}[B]}{\hat{M}[Y]}$. Such conditions for geometric evolution of NES determined by on families of 3-d hypersurfaces $\Xi_t$ mean that any set $B$ spreads over the phase space $Y$ so that for any (fixed set/hypersurface/window) the fraction of $B$ in $A$ approaches the fraction of $B$ over the whole phase space $Y$ (this is an uniform mixture of d-metrics).

Relativistic thermodynamic systems (14) with measure theoretic definitions of ergodicity and mixing for corresponding classes of solutions of (10) possess such properties on an open spacetime region $U \subset \mathbf{V}$,

1. as ergodic systems they have an unique equilibrium distribution defined by $g(\tau_0)$;
2. as mixing systems they approach an equilibrium state by $g(\tau_0)$;
3. the rates of approach to equilibrium are determined by the rates of decay of correlations which can be computed, for instance, for locally anisotropic cosmological solutions;
4. using exact solutions, we can study irreversibility as an unidirectional spontaneous evolution from present to future; this issue can addressed using operator theory and functional analysis (in this article, we do not consider such issues in Hilbert space and convex spaces, see references in [38]).

Nonholonomically deformed G. Perelman functionals (5) and (6) determine both the relativistic dynamical and thermodynamic (in general, with irreversible and non-equilibrium configurations) properties of NESs.

2.2 Carathéodory axiomatic thermodynamics and Ricci flows

Following the first seminal Carathéodory’s work [24], we state the main definitions of the concepts of states, equilibrium, energy and entropy, and thermodynamic coordinates. A geo-
metric flow state is given by any data \( [g_{\alpha \beta} = \{ q_{1,q_{2,q_{3}}(q,N)}], \mathbf{N}, f \}] \) defining a solution of the nonholonomic Ricci flow Eqs. (10) for a fixed normalizing function \( f \) and/or \( \tilde{f}(f) \) subjected to some conditions (11) and (12). The evolution parameter \( \tau \) can be identified with the temperature \( T \) in some conventional systems of references and chosen physical unities (in general, we can consider any convenient \( T(\tau) \); we use a “cal” symbol in order to avoid ambiguities when the capital letter \( T \) is used for torsion (1) or the energy-momentum tensor (9)). For such geometric data, we can always compute the statistical thermodynamic values \( \tilde{E}(\tau) \) and \( \tilde{S}(\tau) \), see (14). To elaborate on analogous thermodynamic models, we can consider a family of volumes \( \hat{V}(\tau) \neq 1 (7) \) when the condition \( \hat{V}(\tau) = 1 \) can be imposed by a corresponding \( f \) but \( d\hat{V}(\tau) \neq 0 \). This allows us to introduce a conventional pressure \( P \) and external work \( A \), and postulate that for any fixed \( \tau_0 \) the first law of thermodynamics for geometric flows

\[
d\tilde{E} = -Pd\hat{V}(\tau).
\]

Stating \( \tau_i \) and \( \tau_f \) for respective initial, and final states, we formulate

**Axiom 2.1** For any geometric flow thermodynamics of NESs, \( \tilde{E}(\tau_f) - \tilde{E}(\tau_i) + A = 0 \).

This axiom\(^3\) can be considered as the first postulate of the relativistic thermodynamics of Ricci flows. Reversibility for such systems can be introduced for self-similar configurations for a fixed \( \tau_0 \), i.e. for relativistic Ricci solitons defined equivalently by generalized Einstein equations (8). As dynamical equations, such (modified) gravitational and matter field equations possess reversible (at least in certain regions) solutions.\(^4\) Nevertheless, a general geometric flow evolution is described by irreversible Eqs. (10) and (11).

After that we can state the second axiom for relativistic geometric flows:\(^5\)

**Axiom 2.2** In neighbourhood of any self-similar configurations for a fixed \( \tau_0 \), i.e. of a nonholonomic Ricci soliton, there exists states/geometric data \( [g_{\alpha \beta} = \{ q_{1,q_{2,q_{3}}(q,N)}], \mathbf{N}, f \}] \) which are inaccessible as nonholonomic Ricci soliton systems for a fixed \( f \) but as accessible for some \( \tau \geq \tau_0 \) if there are nontrivial solutions of generalized Hamilton equations (10) and (11) relating \( \tau_0 \) as an initial state and a final state with \( \tau \).

\(^3\) A mathematical project usually starts as an axiomatic system starting with an ensemble of declarations/statements. This contains certain constructions, solutions of equations, and proofs of theorems. In the case of Euclidean geometry, the axioms are considered to be self-evident but various motivations and fundamental/experimental arguments are put forward for advanced theories related to physics and applications. As a typical axiomatic approach to modern thermodynamics can be considered [28,55], the axioms and certain definitions and “rules of interference” provide the basis for proving theorems. The word “postulate” is used in many cases instead of “axioms”. Here, we explain that in mathematics and logics the axioms are considered as general statements accepted without proofs. In their turns, postulates are used for some specific cases and can not be considered as “very general” statements. In many papers in non-mathematical journals oriented to mathematical physics and applications the axioms, definitions and rules of interference are not cited and related rules of interference are not sited but certain proofs and solutions are provided using corresponding mathematical tools. Such a geometric and PDE style will be used in this work.

\(^4\) For standard thermodynamic systems, i.e. not for the Ricci flows, this is just the internal energy and external work conservation law, i.e. the first postulate of thermodynamics.

\(^5\) Following Carathéodory (see also discussions and references in [37]), for standard thermodynamic systems the English version of such a famous second axiom is “In the neighbourhood of any equilibrium state of a system (of any number of thermodynamic coordinates), there exists states that are inaccessible by reversible adiabatic processes”. This axiom is better understood if it is used the Kelvin’s formulation of the second law of (standard, not geometric) thermodynamics “no cycle can exist whose net effect is a total conversion of heat into work”.

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In principle, one could be certain “un-physical” processes which may connect two geometric thermodynamic NES or general d-metric systems when $\hat{E}(\tau)$ and $\hat{S}(\tau)$ are computed for data $\{g_{\alpha\beta} = [q_1 q_2 q_3 (q_N)], N, f^\nabla\}$ which are not solutions of (10). Here, we note that in the axioms and definitions of the Carathéodory method there is no mention of heat, temperature or entropy because heat is regarded as derived value (and not a fundamental quantity) that appears as soon as the adiabatic restriction is removed. That was the weakness and strength of the approach to standard thermodynamics developed by the aid of the theory of Pfaffian equations. Born [29] was the first who centred the attention to the elegance of the new method but Planck [36] sharply criticized Carathéodory’s method considering that the Thomson–Clausius treatment was more reliable being much nearer to experimental evidence, i.e. to natural processes. For details, discussions and main references, we cite [37] and [38].

Theories of geometric Ricci flows are not elaborated similarly to typical thermodynamical systems characterizing engines and cycles with heat, or chemical reactions and cannot be studied in a phenomenological manner with engineering methods. The strength of axiomatic methods is that introducing in terms of geometric objects (measure defined by a metric, linear connection and corresponding curvature scalar and Ricci tensor) the concept of W-entropy, and related functionals and statistical thermodynamic constructions, G. Perelman was able to prove the Thurston–Poincaré conjecture. Such geometric methods are even more advanced and sophisticated than those used by C. Carathéodory and M. Planck’s criticism on mathematical “harshness” is not relevant for such geometric thermodynamic theories. Fundamental values similar to energy and entropy of thermodynamic systems can be defined and computed in rigorous mathematical forms (14), and one of the main goals of this paper is to show that the Carathéodory approach with Pfaff forms can be naturally extended (even in an almost phenomenological manner) to geometric flow models of NES and generalized gravity systems. We shall show that the Carathéodory method can be extended in such forms that to include in a relativistic form the original constructions with Riemannian metrics and statistical thermodynamic ideas.

2.2.1 Adiabatic approximation for nonholonomic Ricci solitons

The axiomatic thermodynamics of Carathéodory is based on the theory of Pfaffian differential equations (first studied by Pfaff, who proposed a general method of integrating PDEs of first order in 1814–1815), see [37], for a brief review and references, and Appendix A. We can elaborate on analogous “adiabatic” transformation of an “ideal” gas of evolution flows of relativistic systems $\{g_{\alpha\beta} = [11 q_2 q_3 (q_N)], N, f^\nabla\}$ defining a solution for respective nonholonomic deformations of Einstein equations (8).

In the approximation of ideal gas of NESs, for a nonholonomic Ricci soliton in a $\tau_0$, we state an equation of thermodynamic state

$$P\hat{V}(\tau_0) = \rho \tau_0 \text{ for } \hat{V}(\tau_0) = \text{const},$$

when the normalization (7) is not imposed, and (16), and write $d\hat{E} = -Pd\hat{V}(\tau_0) = C_v d\tau = dA$. Considering above formulas in local form, all values like $R$, $P$, $C_v$, etc., depend also on spacetime coordinates $u^\alpha$ or on space-like coordinates $x^i$ for cosmological configurations if we do not perform integration as in (14). Using the last two equations, we obtain

$$\frac{C_v}{\tau_0} d\tau + \frac{\rho}{\hat{V}(\tau_0)} d\hat{V} = 0.$$
For constant coefficients, such a Pfaff equation is exact (see Appendix A),
\[ \frac{\partial}{\partial \tau} \left( \frac{C_{v}}{\tau} \right) \bigg|_{\tau=\tau_0} = \frac{\partial}{\partial \tau} \left( \frac{C_{v}}{\tau} \right) \bigg|_{\tau=\tau_0} = 0, \]
i.e. the Schwarz equation (A.2) is satisfied. For such geometric flow and thermodynamic configurations, there is a solution of (18) as a function
\[ \phi(\tau_0, \hat{V}) = \int \frac{C_{v}}{\tau} d\tau \bigg|_{\tau=\tau_0} + \int \frac{\rho}{\hat{V}(\tau_0)} d\hat{V} = \text{const}. \]

Above formulas are similar to the well-known ones used for adiabatic transformations of an ideal gas when \( \tau_0^\gamma = 1 = \text{const} \), for \( \gamma_0 = C_p/C_v \) and \( \rho = C_p - C_v \). This is not surprising because all constructions are derived for corresponding approximations in the statistical thermodynamic energy of relativistic Ricci flows \( \hat{E} \) (14) computed for nonholonomic Ricci soliton configurations.

### 2.2.2 General transforms and integrating factors for ideal gases of nonholonomic Ricci solitons

For general geometric flows, the analogous systems became asymmetric which can be characterized by an equation of type
\[ \delta Q = d\hat{E} - \delta A, \]
where \( Q \) is a conventional “heat” related to nonholonomic Ricci flow evolution. (The term is introduced as for the usual thermodynamic systems which are not adiabatic.) In such cases, not each term of this equation can be a state function. If we consider the approximation of ideal gas (17) for NESs, we can write
\[ \delta Q = C_v d\tau + \frac{\rho \tau_0}{\hat{V}(\tau_0)} d\hat{V}. \]  
(19)

Because for this Pfaff form the Schwarz condition (A.2) is not satisfied, we conclude that \( Q \) is not a state thermodynamic function.

The value \( \delta A \) is not a state function because Eq. (A.2) is not satisfied for \( \delta A = \frac{\rho \tau_0}{\hat{V}(\tau_0)} d\hat{V} + 0 \cdot d\tau \). Nevertheless, \( \hat{E} \) is a state function because the Schwarz relation holds for
\[ d\hat{E} = C_v d\tau + 0 \cdot d\hat{V}. \]

The Schwartz condition (A.2) fails for \( \delta \phi = \frac{\rho \tau_0}{\hat{V}(\tau_0)} dP - \rho d\tau \). Using an integrating factor \( K = -1/P \), we satisfy the condition (A.3) which allows to find a solution (with possible total differential) of \( \hat{V} = \hat{V}(\tau_0, P) = \rho \tau_0/P \).

In a similar manner, we can find an integrating factor \( \tau_0 = \tau_0(T) \) for (19) when the Schwartz condition (A.3) is fulfilled. This allows us to define a new state function, the thermodynamic entropy \( S(T) \), when
\[ dS(T) := \frac{\delta Q}{\tau_0(T)} = \frac{d\hat{E} + Pd\hat{V}}{\tau_0(T)}. \]

In the ideal gas approximation for NESs, we obtain an exact differential \( dS(T) := \frac{C_v}{\tau_0(T)} d\tau + \frac{\rho}{\hat{V}(T)} d\hat{V} \).

It should be emphasized that the thermodynamic entropy \( S(T) \), in general, is different from the statistical geometric thermodynamic one \( \hat{S}(\tau) \) (14). For certain nonholonomic Ricci soliton configuration and flow evolution of NESs, we can chose such effective values \( \tau_0(T), C_v, \rho \) etc. in order to have \( S(T) = \hat{S}(\tau) \) for certain well-defined models with respective normalization functions and relativistic causal structures. The values \( \hat{E}(\tau) \) and
\( \hat{S} (\tau) \) are defined by a generalized W-entropy from an axiomatic approach to Ricci flows but extended to relativistic geometric flow. The method with Pfaff forms can be applied to such statistical/geometric thermodynamic models (which are different from standard thermodynamic ones) which allow to state the conditions when generalizations of G. Perelman thermodynamics can be described following the Carathéodory axiomatic approach.

Finally, we note that using homogeneous Pfaff forms as in [39] we can elaborate on relativistic models of Carathéodory–Gibbs–Perelman thermodynamics. In [40], a study of the Bekenstein–Hawking black hole thermodynamics [6–9] in Carathéodory approach was performed (such constructions can be extended to and black hole/cosmological/holographic models with conventional horizons). The geometric and statistical thermodynamic methods involving G. Perelman W-entropy are more general ones [17–20,23] because provide an unified approach to various classes of flow evolution and dynamical field theories when the thermodynamic ideas are not limited to horizon type configurations of some exact and parametric solutions.

3 Decoupling and integrability of cosmological solitonic flow equations

The goal of this section is to apply the anholonomic frame deformation method, AFDM, in order to show a general decoupling and integration property of the system of nonlinear PDEs (10) for locally anisotropic cosmological configurations encoding solitonic hierarchies. The geometric/physical objects for such effective statistical thermodynamical systems and corresponding Pfaff equations are determined by generating and integration functions and (effective) matter sources encoding geometric flow evolutions and solitonic configurations, see proofs and examples in [56–59]. Locally anisotropic cosmological solutions in gravity theories and corresponding inflation and dark matter and dark energy models were studied in [23,60,61].

3.1 How geometric cosmological flows can be encoded into solitonic hierarchies?

We model geometric evolution of a ‘prime’ cosmological metric, \( \hat{g} \), into a family ‘target’ d-metrics \( g(\tau) \) (3), when the nonholonomic deformations \( \hat{g} \rightarrow g(\tau) \) a modelled by \( \eta \)-polarization functions

\[
\begin{align*}
\mathbf{g}(\tau) &= \eta_{a}(\tau, x^{k}, t) \hat{g}_{a \circ}[\eta] \otimes \mathbf{e}^{a}[\eta] \\
&= \eta_{a}(\tau, x^{k})\hat{g}_{a}dx^{i} \otimes dx^{i} + \eta_{a}(\tau, x^{k}, t)\hat{h}_{a}e^{a}[\eta] \otimes \mathbf{e}^{a}[\eta],
\end{align*}
\]

\[
\mathbf{e}^{a}[\eta] = (dx^{i}, e^{a} = dy^{a} + \eta_{a}^{i} \hat{N}_{i}^{a}dx^{i}).
\]

The target N-connection coefficients are parameterized in the form \( \hat{N}_{i}^{a}(\tau, x^{k}, t) = \eta_{i}^{a}(\tau, x^{k}, t) \hat{N}_{i}^{a}(\tau, x^{k}, t) \). The values \( \eta_{i}(\tau) = \eta_{i}(\tau, x^{k}), \eta_{a}(\tau) = \eta_{a}(\tau, x^{k}, t) \) and \( \eta_{i}^{a}(\tau) = \eta_{i}^{a}(\tau, x^{k}, t) \) are gravitational polarization functions, or \( \eta \)-polarizations. Any target d-metric \( \mathbf{g}(\tau) \) defines a solution of the N-adapted Hamilton equations in canonical variables (10), or for relativistic nonholonomic Ricci soliton Eq. (8) with \( \tau = \tau_{0} \) which are equivalent to the canonical nonholonomic deformations of Einstein equations.

A cosmological prime metric \( \hat{g} = \hat{g}_{a\beta}(x^{i}, y^{\alpha})du^{a} \otimes du^{\beta} \) is parameterized in a general coordinate form with off-diagonal N-coefficients and/or represented equivalently in N-adapted form

\[
\hat{g} = \hat{g}_{a}(u)\mathbf{e}^{a} \otimes \mathbf{e}^{a} = \hat{g}_{i}(x)dx^{i} \otimes dx^{i} + \hat{g}_{a}(x, y)\mathbf{e}^{a} \otimes \mathbf{e}^{a},
\]
adapted matrices which are decomposed with respect to the flow direction: in the \(h\)-direction, 
\[
\hat{\gamma}(\tau, s)\text{ swept out }\hat{\gamma}.
\]

\[\hat{\gamma} = (\hat{e}_i, \hat{e}_1) = (e_i, e_1) = (\hat{y}^i, \hat{y}^1) = (y^i, y^1).
\]

\[N = e_1 = \hat{y}^1 = y^1.
\]

\[\hat{e}_a = e_a = dy^a + \hat{N}_i^a(u)dx^i, \quad \text{and} \quad \hat{e}_a = \hat{e}_a = \partial/\partial y^a.
\]

In general, such a d-metric \(\hat{g}(\tau) = \hat{g}_a(u)\) can be, or not, a cosmological solution of gravitational field equations in GR, but we impose the condition that under geometric evolution it transforms into a target metric (20) which must be an exact or parametric solution.

### 3.1.1 Generating cosmological solitonic hierarchies

Let us associate a non-stretching curve \(\gamma(\tau, l)\) on an Einstein manifold \(V\) to geometric evolution of a d-metric \(g(\tau)\), where \(\tau\) can be identified with the geometric flow parameter of temperature type and \(l\) is the arclength of the curve, see details in [53, 58, 59]. Such a curve is characterized by an evolution d-vector \(Y = \gamma_{l}\) and tangent d-vector \(X = \gamma_{\tau}\) for which \(g(X, X) = 1\). It also swept out \(\gamma(\tau, l)\) as a two-dimensional surface in \(T_{\gamma(\tau, l)}V \subset TV\).

On such nonholonomic configurations, we consider a coframe \(e \in T^\tau_{\tau}V_N \otimes (hp \oplus vp)\) constructed as a N-adapted \((SO(n) \oplus SO(m))\)-parallel basis along curve \(\gamma\). For 4-d nonholonomic Lorentz manifolds and their N-adapted flow evolution models, we consider that \(n = m = 4\) and model the evolution of 4-d Lorentzian d-metrics. The labels for respective dimension as \(n\) and \(m\) can be used in order to distinguish N-adapted decompositions into \(h\)-and \(v\)-, or \(cv\)-components.

Families of canonical d-connections \(\hat{D}(\tau)\) can be associated with respective families of linear d-connection 1-forms \(\Gamma(\tau) \in T^\tau_{\tau}V_N \otimes (so(n) \oplus so(m))\). Similar families of 1-forms can be introduced for other types of d-connections or for a LC-connection. We parameterize frame bases by linear d-connection 1-forms \(\gamma\)-components.

\[\hat{e}_hX = \gamma_hX \hat{v} e = \begin{bmatrix} 0 & (1, \vec{0}) \\ (-1, \vec{0})^T & h0 \end{bmatrix}, \quad \hat{e}_vX = \gamma_vX \hat{v} e = \begin{bmatrix} 0 & (1, \vec{0}) \\ (-1, \vec{0})^T & v0 \end{bmatrix}.
\]

For a \(n + m\) splitting, \(\hat{\Gamma}(\tau) = \begin{bmatrix} \hat{\Gamma}_hX(\tau) & \hat{\Gamma}_vX(\tau) \end{bmatrix}\), with

\[\hat{\Gamma}_hX(\tau) = \gamma_hX \hat{L} = \begin{bmatrix} 0 & (0, \vec{0}) \\ (-0, \vec{0})^T & \hat{L} \end{bmatrix} \in so(n + 1),
\]

where \(\hat{L} = \begin{bmatrix} 0 & \vec{v} \\ -\vec{v}^T & h0 \end{bmatrix} \in so(n), \quad \vec{v}^T \in \mathbb{R}^{n-1}, \quad h0 \in so(n - 1);
\]

and \(\hat{\Gamma}_vX(\tau) = \gamma_vX \hat{C} = \begin{bmatrix} 0 & (0, \vec{0}) \\ (-0, \vec{0})^T & \hat{C} \end{bmatrix} \in so(m + 1),
\]

where \(\hat{C} = \begin{bmatrix} 0 & \vec{v} \\ -\vec{v}^T & v0 \end{bmatrix} \in so(m), \quad \vec{v}^T \in \mathbb{R}^{m-1}, \quad v0 \in so(m - 1).
\]

Using a family of canonical d-connections \(\hat{D}(\tau)\), we can define certain families of N-adapted matrices which are decomposed with respect to the flow direction: in the \(h\)-direction,

\[\hat{e}_hY = \gamma_h \hat{h} e = \begin{bmatrix} 0 & (he\|, h\vec{e}\perp) \\ -(he\|, h\vec{e}\perp)^T & h0 \end{bmatrix},
\]

when \(\hat{e}_hY \in hp, \ (he\|, h\vec{e}\perp) \in \mathbb{R}^n, \ h\vec{e}\perp \in \mathbb{R}^{n-1},
\]

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for generating solitonic hierarchies with explicit dependence on geometric flow parameterizations related to geometric flows of 4-d Lorentzian metrics, we formulate such possibilities.

\[ E_{\text{R}} = \gamma_{vY} \cdot \hat{C} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \end{bmatrix} \in \mathfrak{so}(m + 1), \]

where \( h_{\sigma \tau} = \begin{bmatrix} 0 & \overrightarrow{\sigma} \\ - \overrightarrow{\sigma}^T \end{bmatrix} \in \mathfrak{so}(n), \overrightarrow{\sigma} \in \mathbb{R}^{n-1}, \ h_{\hat{\Theta}} \in \mathfrak{so}(n - 1). \)

Similar families of geometric objects and parameterizations can be constructed for the \( v \)-direction

\[ \mathbf{e}_{vY} = \gamma_{vY} \cdot \mathbf{e} = \begin{bmatrix} 0 & (v e_\|, v \overrightarrow{e}_\perp) \end{bmatrix}, \]

when \( \mathbf{e}_{vY} \in \gamma_{vY} \cdot \mathbf{e} \in \mathbb{R}^m, v \overrightarrow{e}_\perp \in \mathbb{R}^{m-1}, \)

and \( \hat{C}_{vY} = \gamma_{vY} \cdot \hat{C} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \end{bmatrix} \in \mathfrak{so}(m + 1), \)

where \( v_{\sigma \tau} = \begin{bmatrix} 0 & \overrightarrow{\sigma} \\ - \overrightarrow{\sigma}^T \end{bmatrix} \in \mathfrak{so}(m), \overrightarrow{\sigma} \in \mathbb{R}^{m-1}, \ v_{\hat{\Theta}} \in \mathfrak{so}(m - 1). \)

Adapting for general cosmological metrics the results proven in [53, 58, 59] for parameterizations related to geometric flows of 4-d Lorentzian metrics, we formulate such possibilities for generating solitonic hierarchies with explicit dependence on geometric flow parameter and on a time-like coordinate (and, for locally anisotropic cases, on space like coordinates):

- The 0 flows locally anisotropic cosmological spaces are convective (travelling wave) maps \( \gamma_{v} = \gamma_{v} \) distinguished as \( (h \gamma)_{v} = (h \gamma)_{vX} \) and \( (v \gamma)_{v} = (v \gamma)_{vX}. \) The classification of such maps depends on the type of cosmological d-metrics and d-connection structures.
- There are +1 flows defined as non-stretching mKdV maps describing geometric cosmological flows

\[-(h \gamma)_{v} = \hat{D}_{hX} (\tau, h \gamma)_{hX} + \frac{3}{2} \hat{D}_{hX} (\tau, h \gamma)_{hX} \cdot h_{\gamma} (h \gamma)_{hX}, \]

\[-(v \gamma)_{v} = \hat{D}_{vX} (\tau, v \gamma)_{vX} + \frac{3}{2} \hat{D}_{vX} (\tau, v \gamma)_{vX} \cdot v_{\gamma} (v \gamma)_{vX}, \]

and the +2, . . . flows as higher-order analogs.
- Finally, the -1 flows are defined by the kernels of families of canonical recursion h-operator

\[ h_{\hat{R}} (\tau) = \hat{D}_{hX} (\tau, h \gamma)_{hX} = 0 \text{ and } \hat{D}_{vY} (\tau, v \gamma)_{vX} = 0. \]

The families of canonical recursion d-operators \( \hat{R} (\tau) = (h \hat{R} (\tau), v \hat{R} (\tau)) \), are, respectively, related to bi-Hamiltonian structures for families of cosmological solitonic configurations. Such configurations are characterized also by respective Carathéodory and/or Perelman thermodynamic values with explicit dependence on a temperature parameter and a time-like coordinate.
3.1.2 Examples of solitonic cosmological distributions and nonlinear waves

The geometric flow evolution of any cosmological d-metric on a Lorentz manifold can be encoded into solitonic hierarchies. In this work, the geometric cosmological flow evolution is described by exact and parametric solutions of type \( g(\tau) = g(\tau, x^i, t) = [h g(\tau, x^i), v g(\tau, x^i, t)] \), with Killing symmetry on \( \partial_3 \) when in adapted coordinates the coefficients of such d-metrics do not depend on a space-like coordinate \( y^3 \). In principle, it is possible to construct more general classes of solutions with dependence on all spacetime coordinates and a temperature-like parameter, but general formulas and classification are technically cumbersome and we omit such considerations. For certain cosmological configurations, we shall consider d-metrics of type \( g(\tau) = g(\tau, t) = [h g(\tau), v g(\tau, t)] \), see Appendix B.

Cosmological solitonic waves:

We can consider nonlinear waves \( \tau = \tau(x^1, x^2, y^4) = \tau \) as solutions of solitonic 3-d equations

\[
\begin{align*}
\partial_1^2 \tau + \epsilon \partial_4(\partial_2 \tau + 6 \epsilon \partial_4 t + \partial_3^{444} t) &= 0, \quad \partial_1^2 \tau + \epsilon \partial_2(\partial_4 t + 6 \epsilon \partial_2 t + \partial_3^{222} t) = 0, \\
\partial_2^2 \tau + \epsilon \partial_4(\partial_1 \tau + 6 \epsilon \partial_4 t + \partial_3^{444} t) &= 0, \quad \partial_2^2 \tau + \epsilon \partial_1(\partial_4 t + 6 \epsilon \partial_1 t + \partial_3^{111} t) = 0, \\
\partial_3^2 \tau + \epsilon \partial_1(\partial_2 \tau + 6 \epsilon \partial_1 t + \partial_3^{111} t) &= 0, \quad \partial_3^2 \tau + \epsilon \partial_2(\partial_1 \tau + 6 \epsilon \partial_2 t + \partial_3^{222} t) = 0,
\end{align*}
\]

for \( \epsilon = \pm 1 \). These equations and their solutions can be redefined via frame/coordinate transforms for temperature–time cosmological generating functions.

Generating nonlinear temperature-time solitonic waves:

Geometric flows of cosmological metrics for a Lorentz manifold can be characterized by 3-d solitonic waves with explicit dependence flow parameter \( \tau \) defined by functions \( \tau(\tau, u) \) as solutions of such nonlinear PDEs:

\[
\tau = \begin{cases} 
\tau(x^1, x^2, t) \text{ as a solution of } \partial_3^2 \tau + \epsilon \partial_4[\partial_3 \tau + 6 \epsilon \partial_4 t + \partial_3^{444} t] = 0; \\
\tau(x^2, \tau, t) \text{ as a solution of } \partial_2^2 \tau + \epsilon \partial_3[\partial_3 \tau + 6 \epsilon \partial_3 t + \partial_3^{333} t] = 0; \\
\tau(t, x^1, t) \text{ as a solution of } \partial_1^2 \tau + \epsilon \partial_4[\partial_1 \tau + 6 \epsilon \partial_4 t + \partial_3^{111} t] = 0; \\
\tau(x^1, \tau, t) \text{ as a solution of } \partial_3^2 \tau + \epsilon \partial_4[\partial_3 \tau + 6 \epsilon \partial_4 t + \partial_3^{444} t] = 0; \\
\tau(t, x^2, t) \text{ as a solution of } \partial_2^2 \tau + \epsilon \partial_3[\partial_3 \tau + 6 \epsilon \partial_3 t + \partial_3^{333} t] = 0; \\
\tau(t, \tau, x^2) \text{ as a solution of } \partial_1^2 \tau + \epsilon \partial_3[\partial_3 \tau + 6 \epsilon \partial_3 t + \partial_3^{111} t] = 0.
\end{cases}
\]  

Applying general frame/coordinate transforms on respective solutions of Eq. (23), we construct cosmological solitonic waves parameterized by functions labelled in the form \( \tau = \tau(\tau, x^i) = \tau(\tau, x^i, t) \), or \( \tau(\tau, x^2, t) \).

In a similar form, we can consider other types of cosmoligic solitonic configurations determined, for instance, by sine-Gordon and various types of nonlinear wave configurations characterized by geometric curve flows, see details in [53,58,59] and references therein. Any solitonic hierarchy configuration with nonlinear waves of type \( \tau(\tau, u) \) or \( \tau = \tau(x^i, t) \) can be can be used as generating functions for certain classes of nonholonomic deformations of cosmological metrics. In this work, d-metrics of type \( g(\tau) \) (3) and/or (20) are d-tensor functionals of type

\[
g(\tau) = g[\tau(\tau, u)] = g[\tau] = (g_1[\tau], g_3[\tau])
\]

with polarization functions \( \eta_i(\tau) = \eta_i(\tau, x^k) = \eta_i[\tau], \eta_a(\tau) = \eta_a(\tau, x^k, y^b) = \eta_a[\tau] \) and \( \eta^d_b(\tau) = \eta^d_b(\tau, x^k, y^b) = \eta^d_b[\tau] \). A functional dependence \( [\tau] \) can be considered for multiple solitonic hierarchies with mixing (for instance, on some different solutions of equations of type (23) and/or (22)). This can be written conventionally in the form \( [\tau] = [t, i, \tau, \ldots] \) where
3.1.3 Table 1 with ansatz for cosmological geometric flows and solitonic hierarchies

In this work, we use brief notations of partial derivatives \( \partial_\alpha q = \partial q/\partial u^\alpha \) of a function \( q(x^k, y^\alpha) \), for instance, \( \partial_1 q = q^*_x = \partial q/\partial x^1, \partial_2 q = q^*_y = \partial q/\partial x^2, \partial_3 q = \partial q/\partial y^3 = \partial q/\partial \phi = q^\phi, \partial_4 q = \partial q/\partial t = \partial q = q^t \). Second-order derivatives are written in the form \( \partial_2^2 q = q^{\phi\phi}, \partial_2^2 q = \partial^2 q/\partial \phi^2 = \partial_2^2 q = q^{**} \). Partial derivatives on a flow parameter will be written in the form \( \partial_t = \partial/\partial \tau \). The \( \tau \)-evolution of any d-metric \( g(\tau) \) of type (3), (20) and (24) can be parameterized (using respective frame transforms) for respective local coordinates \((x^k, y^\alpha = t)\) and a common geometric flow evolution and/or curve flows temperature-like parameter \( \tau \),

\[
g(\tau) = e^{\psi(\tau, x^i)} = e^{\psi[1]}, \quad g_\alpha = \omega(\tau, x^i, y^\beta) h_\alpha(\tau, x^i, t) = \omega[1] h_\alpha[2], \quad N^3_i(\tau) = n_i(\tau, x^i, t) = n_i[4, 2], \quad N^3_i(\tau) = w_i(\tau, x^i, t) = w_i[2]. \tag{25}
\]

For simplicity, we can consider \( \omega = 1 \) for a large class of cosmological models with at least one Killing symmetry, for instance, on \( \partial_3 \).

We can introduce effective sources for geometric flows of NES (10) which by corresponding nonholonomic frame transforms and tetradic (vierbein) fields are parameterized in N-adapted form

\[
\text{eff } \hat{\Upsilon}_{\mu\nu}(\tau) = e^{\psi(\tau, x^i)} e^{\psi[1]}(\hat{\Upsilon}_{\mu\nu}(\tau) + \frac{1}{2} \partial_\tau g_{\mu\nu}(\tau)) = \left[ \text{eff } \hat{\Upsilon}(\tau, x^k) \delta^i_j, \hat{\Upsilon}(\tau, x^k, y^\alpha) \delta^\alpha_\beta \right].
\]

The values \( \text{eff } \hat{\Upsilon}(\tau) \) and \( \hat{\Upsilon}(\tau) \) can be taken as functions of certain solutions of nonlinear solitonic equations and then considered as generating data for (effective) matter sources and certain forms compatible with solitonic hierarchies for d-metrics (25). We write

\[
\hat{\mathcal{S}}[1] = \text{eff } \hat{\Upsilon}_\mu(\tau) = [ \hat{\mathcal{S}}[1] = \hat{\Upsilon}(\tau, x^i) \delta^i_j, \hat{\mathcal{S}}[2] = \hat{\Upsilon}(\tau, x^i, t) \delta^\alpha_\beta]. \tag{26}
\]

There are used “hat” symbols in order to emphasize that such values are used for systems of nonlinear PDEs involving a canonical d-connection.

We can work using canonical nonholonomic variables with functional dependence of d-metrics and prescribed effective sources on some cosmological solitonic hierarchies. In such cases, the system of nonholonomic entropic R. Hamilton equations (10) can be written in a formal nonholonomic Ricci soliton form (equivalently, as a nonholonomic deformation of Einstein equations when the geometric objects depend additionally on a temperature-like parameter \( \tau \) and for effective source (26)),

\[
\hat{R}_{\mu\nu}[1] = \hat{\mathcal{S}}_{\mu\nu}[1]. \tag{27}
\]

Table 1 (see below) summarizes the geometric data on nonholonomic 2+2 variables and corresponding ansatz which allow us to transform relativistic geometric flow equations and/or nonholonomic Ricci solitons into respective systems of nonlinear ordinary differential equations, ODEs, and partial differential equations, PDEs, determined by cosmological solitonic hierarchies.

In this paper, we study physically important cases when \( \hat{g} \) defines a cosmological metric in GR. For diagonalizable via coordinate transforms prime metrics, we can always find a coordinate system when \( N^b_i = 0 \). Non-singular nonholonomic deformations can be constructed.
| diagonal ansatz: PDEs $\rightarrow$ ODEs | AFDM: PDEs with decoupling: generating functions |
|---------------------------------------|--------------------------------------------------|
| radial coordinates $u^\alpha = (r, \theta, \varphi, t)$ | 2+2 splitting, $u^\alpha = (x^1, x^2, y^3, y^4 = t)$; flow parameter $\tau$ |
| LC-connection $\hat{\nabla}$ | $N : TV = hTV \oplus \nu TV$, locally $N = \{N_i^\mu(x, y)\}$ |
| diagonal ansatz $g_{\alpha\beta}(u)$ | canonical connection distortion $\hat{\mathbf{D}} = \nabla + \hat{Z}$ |
| $\hat{g}_{\alpha\beta} = \hat{g}_\alpha(t)$ for FLRW | $g_{\alpha\beta}(\tau) = \left[ g_{ij} + N_i^\alpha N_j^\beta h_{ab} N_i^b h_{cb} \right], \ 2 \times 2$ blocks |
| coord. transforms $e_\alpha = e_\alpha^{\prime} \partial_\alpha^{\prime}$, $e_\beta = e_\beta^{\prime} \partial_\beta^{\prime}$ | $g_{\alpha\beta}(\tau) = \left[ g_{ij} \right], h_{ab}(\tau)$, $\hat{g}_\alpha(\tau, x^k, y^\alpha) \hat{e}_a \hat{e}_b$ |
| $\hat{g}_\alpha(x^k, y^\alpha) \rightarrow \hat{g}_\alpha(t)$, $N_i^\alpha(x^k, y^\alpha) \rightarrow 0$. | $g_{\alpha\beta}(\tau) = g_{\alpha\beta}(\tau, x^i, y^4 = t)$ cosmological configurations |
| $\hat{\nabla}, Ric = \{ \hat{R}_{\beta\gamma} \}$ | Ricci tensors |
| $m \mathcal{L}[\phi] \rightarrow m \mathbf{T}_{\alpha\beta}[\phi]$ | $\hat{\mathbf{D}}, \hat{Ric} = \{ \hat{R}^{\prime}_{\beta\gamma} \}$ |
| trivial equations for $\hat{\nabla}$-torsion | sources |
| | $\hat{\mathcal{L}}[\hat{\phi}] = \hat{\mathcal{L}}[\phi]$ |
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for cosmological solutions with nontrivial functions \( \eta_\alpha = (\eta_1, \eta_\alpha, \eta_\alpha^d) \), and nonzero coefficients \( \hat{N}_h^d (u) \). For a d-metric (20), we can analyse the conditions of existence and properties of some target and/or prime cosmological solutions with solitonic waves and structure formation when, for instance, \( \eta_\alpha \to 1 \) and \( N_i^d \to \hat{N}_i^d \). The values \( \eta_\alpha = 1 \) and/or \( \hat{N}_i^d = 0 \) can be imposed as some special nonholonomic constraints on temperature–time cosmological flows.

3.2 Decoupling of geometric flow equations into cosmological solitonic hierarchies

In this subsection, we prove that the system of nonlinear PDEs (27) describing geometric flow evolution of cosmological NES encoding solitonic hierarchies can be decoupled in general form.

3.2.1 Canonical Ricci d-tensors for cosmological flows encoding solitonic hierarchies

We can chose certain systems of reference/coordinates when coefficients of a d-metric (25) and derived geometric objects like the canonical d-connection and corresponding curvature and torsion d-tensors do not depend on \( y^4 = t \) with respect to a class of N-adapted frames. Using a d-metric ansatz with \( \omega = 1 \) and a source \( \hat{\mathcal{S}}[\iota] = \{ \hat{h}\hat{\mathcal{S}}[\iota], \hat{v}\hat{\mathcal{S}}[\iota] \} \) (26), we compute the coefficients of the Ricci d-tensor and write the geometric flow modified Einstein equations (27) in the form

\[
\begin{align*}
\hat{R}_1[\iota] &= -2g_1g_2 h \hat{\mathcal{S}}; \\
\hat{R}_3[\iota] &= -2g_1g_2 h \hat{\mathcal{S}}; \\
\hat{R}_4 k(\tau) &= \frac{h_3}{2h_3} n_k^* + \left( \frac{3}{2} h_3^* + \frac{h_3}{h_4} h_4^* - \frac{n_k^*}{2h_4} \right) = 0; \\
\hat{R}_4 k(\tau) &= -w_k \left[ \left( \frac{h_3^*}{2h_3} \right)^2 + \frac{h_3^*}{2h_3} \frac{h_4^*}{2h_4} - \frac{h_3^*}{2h_3} \right] + \frac{h_3^*}{2h_3} \frac{\partial_k h_3}{\partial h_3} + \frac{\partial_k h_4}{\partial h_4} - \frac{\partial_k h_3^*}{2h_3} = 0.
\end{align*}
\]

Let us explain the decoupling property of this system of nonlinear PDEs: From equation (28), we can find \( g_1 \) (or, inversely, \( g_2 \)) for any prescribed functional of solitonic hierarchies encoded into a h-source \( \hat{h}\hat{\mathcal{S}}[\iota] \) and any given coefficient \( g_2(\tau, x^i) = g_2[\iota] \) (or, inversely, \( g_1(\tau, x^i) = g_1[\iota] \)) when the solitonic hierarchies for the coefficients of a h-metric in such coordinates depend on a temperature–like parameter but not on time like coordinates. We can integrate on time-like coordinate \( y^4 = t \) in (29) and define \( h_4(\tau, x^i, t) \) as a solution of first-order PDE for any prescribed v-source \( \hat{v}\hat{\mathcal{S}}[\iota] \) and given coefficient \( h_3(\tau, x^i, t) = h_3[\iota] \). Inversely, it is possible to define \( h_3(\tau, x^i, t) \) if \( h_4(\tau, x^i, t) = h_4[\iota] \) is given for such solutions we have to solve a second-order PDE. The coefficients of v-metrics involve, in general, different types of solitonic hierarchies even they are related via corresponding formulas to another classes of solitonic hierarchies prescribed for the effective v-source. We have to integrate two times on \( t \) in (30) in order to compute \( n_k(\tau, x^i, t) = n_k[\iota] \) for any defined \( h_3 \) and \( h_4 \). Introducing certain values of \( h_3 \) and \( h_4 \) in Eq. (31), we obtain a system of algebraic linear equations for \( w_k(\tau, x^i, t) = w_k[\iota] \). Here, it should be also emphasized that the cosmological
solitonic hierarchies encoded in the coefficients of a N-connection are different (in general) from those encoded in the coefficients of d-metric and nontrivial effective sources.

Using the decoupling property of nonlinear off-diagonal cosmological solitonic systems (28)–(31), we can integrate such PDEs step by step by prescribing, respectively, the effective sources, the h-coefficients, and v-coefficients, for geometric flows of d-metrics. The geometric evolution of such solutions involves a prescribed nonholonomic constraint on \( \partial_t g_{\mu\nu}(\tau) \) included in \( \tilde{\mathcal{F}}[t] \).

3.2.2 Nonlinear symmetries for cosmological solitonic generating functions and sources

Let us define the coefficients \( \alpha_i = h_4^* \partial_i \sigma, \beta = h_3^* \sigma^*, \gamma = (\ln |h_3|^{3/2} / |h_4|)^*, \) where

\[
\sigma = \ln |h_3^* / \sqrt{|h_3 h_4|}|
\]

(32)

for nonsingular values for \( h_4^* \neq 0 \) and \( \partial_t \sigma \neq 0 \) (we have to elaborate other methods if such conditions are not satisfied), we transform the system of solitonic nonlinear PDEs (28)–(31) into

\[
\psi^{**} + \psi'' = 2 h_3^* \partial_t \tilde{\mathcal{F}}[1t], \quad \sigma^* h_3^* = 2 h_3 h_4, \quad n_k^* \gamma n_k = 0, \quad \beta w_i - \alpha_i = 0.
\]

(33)

Such a system can be integrated in explicit and general forms (depending on the type of parameterizations, see details in [17,53] and some examples of cosmological solutions will be provided in next sections) if there are prescribed a generating function \( \psi(\tau) = \Phi(\tau, x^i, t) = \Psi[1t] := e^{\sigma} \) and generating sources \( h_3^* \) and \( v^* \). Here, we note that Levi-Civita, LC, conditions for extracting cosmological solitonic solution with zero torsion, can be transformed into a system of first-order PDEs

\[
(\partial_i - w_i \partial_t) \ln \sqrt{|h_3|} = 0, \quad \partial_t w_i = (\partial_i - w_i \partial_t) \ln \sqrt{|h_4|}, \quad \partial_t n_i = 0,
\]

\[
\partial_i n_k - \partial_k n_i, \quad \partial_k w_i = \partial_i w_k,
\]

(34)

which are considered as additional constraints on off-diagonal coefficients of metrics of type (25).

To generate exact and parametric solutions, we have to solve a system of two equations for \( \sigma \) in (32) and (33) involving four functions \( (h_3, h_4, v^* \tilde{\mathcal{F}}, \text{and } \Psi) \). We can check by corresponding computations that there is an important nonlinear symmetry which allows to redefine the generating function and the effective source and to introduce a family of effective cosmological constants \( \Lambda(\tau) \neq 0, \Lambda(\tau_0) = const \), not depending on spacetime coordinates \( u^\mu \). Such nonlinear transforms \( (\psi, v \tilde{\mathcal{F}}) \leftrightarrow (\Phi, \Lambda) \) are defined by formulas

\[
\Lambda[\psi^2[1t]]^* = [v \tilde{\mathcal{F}}[2t]][\Phi^2[1t]]^*, \quad \text{or } \Lambda[\psi^2[1t]] = \Phi^2[1t] v \tilde{\mathcal{F}}[2t] - \int dt \Phi^2[1t] v \tilde{\mathcal{F}}[2t]^*
\]

(35)

and allow us to introduce families of new generating functions \( \Phi(\tau, x^i, t) = \Phi[1t] \) and families of (effective) cosmological constants. The families of constants \( \Lambda(\tau) \) can be chosen for certain physical models and the geometric/physical data for \( \Phi \) encode nonlinear symmetries both for the generating functions and sources and respective cosmological solitonic hierarchies for \( v^* \tilde{\mathcal{F}} \), and \( \Psi \). In result of nonlinear symmetries, we can describe nonlinear systems of PDEs by two equivalent sets of generating data \( (\Psi, \Upsilon) \) or \( (\Phi, \Lambda) \). But such symmetries of
solitonic cosmological hierarchies are encoded into functionals with respective partial derivatives $\partial_i$ and/or integration on $dr$. To generate certain classes of solutions, we can work with effective cosmological constants but for other ones we have to consider generating sources. Finally, we note that modules in formulas (35) should be chosen in certain forms resulting in physically motivated nonlinear symmetries, relativistic causal models and thermodynamic values which are compatible with observational data in modern cosmology.

3.3 Integrability of geometric cosmological flow equations with solitonic hierarchies

We study properties of some classes of generic off-diagonal cosmological solutions of (27) determined by generated functions and sources with solitonic hierarchies.

3.3.1 Cosmological solutions for off-diagonal metrics and N-coefficients

By straightforward computations, we can verify that the system (28)–(31) represented in the form (33) (see similar details in [61,62]) can be solved by if the coefficients of a d-metric and N-connection are computed

$$g_i(\tau) = e^{\psi(\tau,x^i)}$$ as a solution of 2-d Poisson eqs. $\psi^{''} + \psi'' = 2h[1\iota]_t$;

$$g_3[3\iota] = h_3(\tau,x^i) = h_3^{[0]}(\tau,x^k) - \int dr \frac{(\Psi^2)^*}{4\sqrt{\gamma}} = h_3^{[0]}(\tau,x^k) - \Phi^2/4\Lambda(\tau);$$

$$4[4\iota] = h_4(\tau,x^i,t) = -\frac{(\Psi^2)[t]}{4(h_3^{[2\iota]}h_3[3\iota])^2} \frac{(\Psi^2)^*}{4(h_3^{[3\iota]}(\tau,x^k) - \int dr (\Psi^2)^*/4\sqrt{\gamma}]}$$

$$= -\frac{(\Phi^2)(\Phi^2)^*}{4h_3^{[3\iota]}(\tau,x^k)} \frac{[(\Phi^2)^*]^2}{4(\Lambda(\tau))(\tau,x^k)}; \quad (36)$$

$$N_k^3[3\iota] = n_k(\tau,x^i,t) = 1n_k(\tau,x^i) + 2n_k(\tau,x^i)$$

Finally, we note that modules in formulas (35) should be chosen in certain forms resulting in effective cosmological constants but for other ones we have to consider generating sources. We study properties of some classes of generic off-diagonal cosmological solutions of (27) determined by generated functions and sources with solitonic hierarchies.

In these formulas, there are considered different sets of solitonic hierarchies and respective integration functions $h_3^{[0]}(\tau,x^k)$, $1n_k(\tau,x^i)$, and $2n_k(\tau,x^i)$ encoding (non-) commutative parameters and integration constants but also nonlinear geometric flow scenarios on $\tau$ and cosmological evolution. These data and symmetries of solitonic hierarchies for generating geometric evolution data ($\Psi$, $\gamma$), or ($\Phi$, $\Lambda$), (all related by nonlinear differential/integral transforms (35)) can be prescribed in explicit form following certain topology/symmetry/asymptotic conditions and compatibility with observational data. The coefficients (36) define generic off-diagonal cosmological solitonic solutions with associated bi Hamilton structures if the corresponding anholonomy coefficients are not trivial. In general, such geometric flow cosmological solutions are with nontrivial nonholonomically induced d-torsions and solitonic hierarchies determined by evolution of N-adapted coefficients of
d-metric structures. We can impose additional nonholonomic constraints \(34\) in order to extract LC-configurations for cosmological metrics under geometric flow evolution.

### 3.3.2 Quadratic line elements for off-diagonal cosmological solitonic hierarchies

Instead of \(\Psi\) and/or \(\Phi\), we can consider as a generating function any coefficient \(h_3[3t] = h_3^{[0]} - \Phi^2/4\Lambda\), \(h_3^3(\tau) \neq 0\), and write formulas

\[
\Phi^2(\tau) = 4\Lambda \left( h_3[3t] - h_3^{[0]} \right), \quad (\Phi^2)^* = 4\Lambda (h_3)^* \quad \text{and} \quad (\Phi^*)^2 = \Lambda (h_3)^* \left( \frac{h_3^{[0]}}{h_3^3} \right) - 1.
\]

Using nonlinear symmetries \(35\), we find

\[
(\Psi^2)^* = 4 \left| \nu \nabla \left[ 2 t \right] \right| (h_3)^* \quad \text{and} \quad \Psi^2 = 4 \left| \nu \nabla \left[ 3 t \right] \right| h_3 - 4 \int dt \left| \nu \nabla \left[ 3 t \right] \right|^2 h_3.
\]

Such formulas determine corresponding functional \(\Psi\left[ \nu \nabla \left[ 3 t \right] , h_3^{[0]} \right] \) and \(\Phi[\Lambda, h_3, h_3^{[0]}]\). Introducing these values into respective formulas for \(h_3\), \(N_k^i\) and \(\nu \nabla \) in \(36\) and expressing the generating functions and the \(d\)-metric \(37\) with cosmological data \(25\) in terms of \(h_3\), for respective integration functions and effective sources for geometric evolution, we compute

\[
g_{3[3t]} = h_3(\tau, x^i, t) = h_3^{[0]}(\tau, x^k) - \int dt \frac{(\Psi^2)^*}{4 \nu \nabla} = h_3^{[0]}(\tau, x^k) - \Phi^2/4\Lambda(\tau);
\]
\[
g_{4[4t]} = h_4(\tau, x^i, t) = \frac{(\Phi^2)(\Phi^2)^*}{h_3^{[0]}(\Lambda(\tau) \int dt \nu \nabla (\Phi^2)^* )} = \frac{4(\Lambda(\tau))^*}{\int dt \nu \nabla (h_3)^*} ;
\]
\[
N_{k}^{3} \left[ 3 t \right] = n_k (\tau, x^i, t) = 1n_k(x^i, t) + 2n_k(x^i, t) \int dt \frac{(\Phi^2)^*}{4 \Lambda(\tau) \int dt \nu \nabla (\Phi^2)^* ||h_3||^2} = \frac{1}{\nu \nabla} \left( 1n_k(x^i, t) + 2n_k(x^i, t) \int dt \frac{(\Phi^2)^*}{4 \Lambda(\tau) \int dt \nu \nabla (\Phi^2)^* ||h_3||^2} \right).
\]

In result, the quadratic line element corresponding to this class of cosmological flow solutions in three equivalent forms:

\[
ds^2 = e^\psi(x^i) (\lambda dx^1)^2 + (dx^2)^2
\]

\[
\begin{cases}
-\frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1 \\
- \frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1, \quad \text{gener. funct.} h_3, \quad \text{source} \nu \nabla, \quad \text{or} \Lambda;
\end{cases}
\]

\[
\begin{cases}
-\frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1 \\
- \frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1, \quad \text{gener. funct.} \psi, \quad \text{source} \nu \nabla, \quad \text{or} \Lambda;
\end{cases}
\]

\[
\begin{cases}
-\frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1 \\
- \frac{1}{4\Lambda} \left( \left| \nu \nabla \left[ 3 t \right] \right|^2 \right) h_3^{[0]} \left[ \nu \nabla \left[ 3 t \right] \right]^2 dx^1, \quad \text{gener. funct.} \Phi, \quad \text{source} \nu \nabla, \quad \text{or} \Lambda;
\end{cases}
\]

The quadratic solitonic hierarchies determined by generating functions. A generating source \(\nu \nabla\) and effective cosmological constant \(\Lambda\) do not involve (in general) any solitonic behaviour.
Nonlinear symmetries \((35)\) mix different cosmological solitonic structures of generating functions and any cosmological functional for sources.

### 3.3.3 Off-diagonal Levi–Civita cosmological solitonic hierarchies

We can solve Eq. \((34)\) for zero torsion conditions considering special classes of generating functions and sources. For instance, we prescribe a \(\Psi(\tau) = \Psi(\tau, x^i, t)\) for which \((\partial_t \Psi)^* = \partial_t(\Phi^*)\) and fix a \(v^3(\tau, x^i, t) = v^3(\Psi) = v^3(\tau),\) or \(v^3 = \text{const}\). The nonlinear symmetries \((35)\) transforms into

\[
\Lambda \tilde{\Psi}^2 = \Phi^2|_{v^3|} - \int \dd t \Phi^2|_{v^3|^*} \text{ and } \Phi^2 = -4 \Lambda \tilde{h}^3(\tau, x^i, t), \tilde{\Psi}^2 = \int \dd t \; v^3 \tilde{h}^3_3.
\]

In the second case, the coefficient \(h_3(\tau) = \tilde{h}_3(\tau, x^i, t)\) can be considered also as generating function when \(h_4\) and the N-connection coefficients are computed using corresponding formulas an certain nonlinear symmetries and nonholonomic constraints. To generate zero torsion cosmologic solitonic hierarchies, we find some functions \(\tilde{A}(\tau) = A(\tau, x^i, t)\) and \(n(\tau) = n(\tau, x^i)\) when the coefficients of N-connection are

\[
n_k(\tau) = \tilde{n}_k(\tau) = \partial_k n(\tau, x^i) \text{ and } w_i(\tau) = \partial_i \tilde{A}(\tau) = \frac{\partial_i(\int \dd t \; v^3 \tilde{h}^3_3)}{v^3 \tilde{h}^3_3}.
\]

In result, the quadratic line elements for new classes of off-diagonal zero torsion locally anisotropic cosmologic solutions encoding solitonic hierarchies and defined as subclasses of solutions \((37)\),

\[
ds^2 = e^{\Psi(\tau, x^i)}(\dd x^1)^2 + (\dd x^2)^2 - \left\{ \begin{array}{ll}
\tilde{h}_3 \left[ \dd y^3 + (\partial_k n) \dd x^k \right] + \frac{(\tilde{n}_k)^2}{\int \dd t \; v^3 \tilde{h}^3_3} [\dd t + (\partial_k \tilde{A}) \dd x^k], & \text{gener. funct. } \tilde{h}_3, \\
\frac{(h_3^{[0]} - \int \dd t \; v^3 \tilde{h}^3_3 \tilde{n}_k^2)}{4 \; v^3} \left[ \dd y^3 + (\partial_k n) \dd x^k \right] + \frac{(\tilde{n}_k)^2}{\int \dd t \; v^3 \tilde{h}^3_3} [\dd t + (\partial_k \tilde{A}) \dd x^k], & \text{gener. funct. } \tilde{\Psi}, \\
\frac{(h_3^{[0]} - \int \dd t \; v^3 \tilde{h}^3_3 \tilde{n}_k^2)}{4 \; v^3} \left[ \dd y^3 + (\partial_k n) \dd x^k \right] + \frac{(\tilde{n}_k)^2}{\int \dd t \; v^3 \tilde{h}^3_3} [\dd t + (\partial_k \tilde{A}) \dd x^k], & \text{gener. funct. } \tilde{\Phi}.
\end{array} \right.
\]

\[(38)\]

For any value of flow parameter \(\tau\), such cosmological solitonic metrics are generic off-diagonal and define new classes of solutions which are different, for instance, from the FLRW metric. We may check if the anholonomy coefficients \(C_{\alpha \beta}^{[0]} = \{C_{i j}^{b} = \partial_b N_i^j, C_{j i}^{a} = \dd_j N_i^a - \dd_i N_j^a\}\) are not zero for solitonic values of \(N^3_i = \partial_i \tilde{A}\) and \(N^4_k = \partial_k n\) and conclude if certain metrics are or not generic off-diagonal. We can study certain nonholonomic cosmologic solitonic configurations determined, for instance, by data \((v^3, \tilde{\Psi}, h_3^{[0]}, \tilde{n}_k)\), with \(w_i = \partial_i \tilde{A} \to 0\) and \(\partial_k n \to 0\), see Appendix B.
3.4 Table 2: AFDM for constructing cosmological solitonic flows

We consider the coefficient \( h_3(\tau) = h_3(\tau, x^i, t) \) in (37) as a generating function. Such a value can be determined by a family of solitonic hierarchies, \( h_3(\tau) = h_3[3 t] \) with explicit dependence on a time-like coordinate \( t \). We can perform a deformation procedure for constructing a class of off-diagonal solutions with Killing symmetry on \( \partial_3 \) determined by cosmological solitonic hierarchies \( \hat{\mathcal{I}}[3] = [ h \hat{\mathcal{S}}[1 t], v \hat{\mathcal{S}}[2 t]] \) (26) and a parametric running cosmological constant \( \Lambda(\tau) \).

\[
\begin{align*}
\text{Such classes of cosmological solutions involve different types of solitonic hierarchies and,} \\
\text{in general, are with nontrivial nonholonomically induced torsion (it is possible to impose} \\
\text{nonholonomic constraints to LC-configurations (38)).}
\end{align*}
\]

4 Geometric cosmological flows and solitonic hierarchies

In this section, we consider applications of the anholonomic frame deformation method (AFDM, outlined in Tables 1 and 2) for constructing in explicit form exact and parametric off-diagonal cosmological solutions describing solitonic geometric flows.

4.1 Nonlinear PDEs for geometric flows with cosmological solitonic hierarchies

There will be studied solutions of modified Einstein equations (27) transformed into systems of nonlinear PDEs with decoupling (33).

4.1.1 Parametric cosmological solutions with additive solitonic sources

Let us introduce such conventions: we shall write that \( \int_0^\tau \hat{\mathcal{S}}(\tau) = \int_0^\tau \hat{\mathcal{S}}^a[1 t] \) (i.e. put a left label “0”) for a geometric flow source \( \hat{\mathcal{S}}(\tau) \) if it contains a term in \( \hat{\mathcal{Y}}^\mu_v(\tau) \) (26) defined as a source functional on a solitonic hierarchy \( [2 t] \). If it will be written \( \int_0^\tau \hat{\mathcal{S}}(\tau) \) without a left label “0”, we shall consider that such a term corresponds to a general (effective) \( \hat{\mathcal{Y}}^\mu_v(\tau) \) (not prescribing any solitonic configurations) encoding contributions from a distortion tensor \( \hat{Z} \). An effective source term \( \hat{\mathcal{S}}(\tau) \) determined by geometric flows (with left label “fl”) of the d-metric, \( \partial_\tau g'_{\alpha'\beta'}(\tau) \), in (26) is contained. It is of cosmological solitonic character if the d-metric coefficients are also cosmological solitons. We can consider cosmological solitonic hierarchies for Ricci soliton configurations, i.e. nonholonomic Einstein systems, with \( \hat{\mathcal{S}}(\tau) = 0 \).

For this class of solutions, we consider a source (26) with a left label \( a \) is used for “additive functionals”

\[
\begin{align*}
\mathcal{S}^a(\tau) = \mathcal{S}^a[1 t] = \mathcal{S}^a(\tau, x^i, t) = \hat{\mathcal{S}}^a[1 t] + \int_0^\tau \hat{\mathcal{S}}^a[2 t] + \int_0^\tau \hat{\mathcal{S}}^a[3 t].
\end{align*}
\]
Table 2  Off-diagonal cosmological flows with solitonic hierarchies Exact solutions of $\mathbf{R}_{\mu
u}(\tau) = \tilde{S}_{\mu
u}(\tau)$ (27) transformed into a system of nonlinear PDEs (28)–(31)

| d-metric ansatz with | $\text{Killing symmetry } \partial_1$ |
|----------------------|-----------------------------------|
| $\text{Effective matter sources}$ |
| $\text{Nonlinear PDEs (33)}$ |
| $\text{Generating functions: } h_3[3\ell]$, $\Psi(\tau, x^i, t) = e^{\sigma(\tau)} \Phi[\ell]$; $\text{integration functions: } h_3^{[0]}(\tau, x^k)$, $1 \eta_k(\tau, x^i)$, $2 \eta_k(\tau, x^i)$ |
| $\text{Off-diag. solutions, d--metric }$ |
| $\text{N-connec.}$ |
| $\text{LC-configurations (34)}$ |

| $ds^2 = g_i(\tau)(dx^i)^2 + g_o(\tau)(dy^i + N_i^o(\tau)dx^i)^2$, for $g_i = e^{\psi(\tau, x^i)}$, $g_o = h_o(\tau, x^i, t)$, $N_i^o = n_i(\tau, x^i, t)$, $N_i^0 = w_i(\tau, x^i, t), y^4 = t $; $\tilde{S}_{\mu\nu}(\tau) = [h_i \tilde{S}(\tau, x^i, t) \delta_{\mu\nu}^o, v^3(\tau, x^i, t) \delta_{\mu\nu}^o]$; $\partial_1 q^* = q$, $\partial_2 q = q^*, \partial_3 q = q^2$, $\partial_4 q = q^*$ |
| $\Sigma^\mu_{\nu}(\tau) = \frac{h^2}{\sqrt{\left|h_3^3(\tau)\right|}}$, $\psi^* + \psi'' = 2 h^3(\tau); \sigma = \ln \left|\frac{h_3^3}{\sqrt{\left|h_3^3(\tau)\right|}}\right|$, $\delta^o = h_o \tilde{S}(\tau, x^i, t) \delta^o_i$, $\Phi_1(\tau)$, $\sigma = \ln \left|\frac{h_3^3}{\sqrt{\left|h_3(\tau)\right|}}\right|$, $h_3(\tau) = h_3^{[0]} - \Phi^2/4 \Lambda(\tau), h_3^* \neq 0, \Lambda(\tau) \neq 0$ = const |
| $g_i(\tau) = e^{\psi(\tau, x^i)}$ as a solution of 2-d Poisson eqs. $\psi^* + \psi'' = 2 h^3(\tau);$ $\tilde{S}_{\mu\nu}(\tau) = [h_i \tilde{S}(\tau, x^i, t) \delta_{\mu\nu}^o, v^3(\tau, x^i, t) \delta_{\mu\nu}^o]$; $\tilde{S}_3(\tau) = h_3^{[0]} - \frac{1}{4} \int d\tau \left(\psi^*\right)^2 / \sqrt{\left|h_3^3(\tau)\right|}, h_4^{[0]} = -\frac{1}{4}\left(\psi^*\right)^2 / \sqrt{\left|h_3^3(\tau)\right|}$, see (36); $n_k(\tau) = 1 \eta_k + 2 \nu_k \int d\tau \left(\psi^*\right)^2 / \sqrt{\left|h_3^3(\tau)\right|}, w_i(\tau) = \partial_1 \psi / \sqrt{\left|h_3^3(\tau)\right|}, w_i^* = \left(\partial_1 - w_i \partial_1\right) \ln \sqrt{\left|h_3^3(\tau)\right|}$, $\partial_1 w_i = \eta_k n_k(\tau), n^3(\tau) = 0, \eta_k n_k(\tau) = \partial_k n_k(\tau) + \tilde{\Psi}_3(\tilde{\Psi})^* = \partial_k(\tilde{\Psi}_3)^* + \tilde{\Psi}_3(\tilde{\Psi})^*$ |
| $\text{and } v^3(\tau, x^i, t) = v^3(\tilde{\Psi}) = v^3, \text{ or } v^3 = \text{const.}$ |
Table 2 continued

| N-connections, zero torsion | \( \eta_k(\tau) = \mathring{\eta}_k(\tau) = \partial_k n(\tau, x^i) \) and \( w_i(\tau) = \partial_i \mathring{A}(\tau) = \begin{cases} \partial_i (\int \mathring{\hat{S}} \, \mathring{\hat{h}}^\alpha_\nu) / \mathring{\hat{h}}^\nu_\sigma; \\ \partial_i \mathring{\tilde{\psi}} / \mathring{\tilde{\psi}}^*; \\ \partial_i (\int \mathring{\hat{S}} (\Phi^2)^\sigma) / (\Phi)^* \mathring{\hat{S}}; \end{cases} \) |
| \( \mathring{\hat{g}} \rightarrow \mathring{\hat{g}} = [g_{i\alpha} = \eta_\alpha \mathring{g}_a, \eta_i^a \mathring{N}_i^a] \) |
| \( \mathring{\hat{g}} \rightarrow \mathring{\hat{g}} = [g_{i\alpha} = \eta_\alpha \mathring{g}_a, \eta_i^a \mathring{N}_i^a] \) |
| Prime metric for a cosm.sol. | \( ds^2 = \eta_1 [1] \mathring{g}_1 (x^i) [dx^1]^2 + \eta_2 [2] \mathring{g}_2 (x^i) [dx^2]^2 + \eta_3 [3] \mathring{g}_3 (x^i) [dx^3]^2 + \eta_4 [4] \mathring{g}_4 (x^i) [dx^4]^2 + \eta_1 [1] \mathring{N}_1^a (x^i) dx^i d\tau + \eta_i^a [i] \mathring{N}_i^a (x^k) dx^k] \) |
| Example of a prime metric | \( \mathring{g}_1 = \mathring{\hat{a}}^2 (\mathring{\eta}), \mathring{g}_2 = \mathring{\hat{a}}^2 (\mathring{\eta}), \mathring{g}_3 = \mathring{\hat{a}}^2 (\mathring{\eta}), \mathring{g}_4 = -1, \mathring{\eta} = \mathring{\eta}(t) \) |
| Solutions for polarization funct. | a FLRW or Bianchi-type solution; \( \eta_i(\tau) = e^{\psi(\tau, x^i)} / \mathring{\hat{g}}; \eta_4 \mathring{h}_4 = -4(\int \mathring{\hat{g}} [1/2] \mathring{\hat{g}}^2) \) |
| Polariz. funct. with zero torsion | \( \eta_3(\tau) = \eta_3(\tau, x^i, t) = \eta_3 [31] \) as a generating function; \( \mathring{\eta}_k^3 (\tau) \mathring{\hat{N}}^3_k = 16 \eta_k + \eta_k \mathring{\int} \mathring{d}f \, \mathring{g} (\int \mathring{3} \mathring{3} \mathring{h})^2 \eta_4^2 (\tau) \mathring{\hat{N}}_k^4 = \frac{\partial_i \mathring{\int} \mathring{d}f \, \mathring{g} (\int \mathring{3} \mathring{3} \mathring{h})^2 \eta_4^2 (\tau) \mathring{\hat{N}}_k^4}{\mathring{\hat{g}} (\int \mathring{3} \mathring{3} \mathring{h})^2} \) |
| \( \eta_4(\tau) = \frac{e^{\psi(\tau, x^i)} / \mathring{\hat{g}}; \eta_3 = \eta_3(\tau, x^i, t) \) as a generating function; \( \eta_4(\tau) = -4(\int \mathring{\hat{g}} [1/2] \mathring{\hat{g}}^2) \) |
| \( \eta_4(\tau) = - \frac{4(\int \mathring{\hat{g}} [1/2] \mathring{\hat{g}}^2) \eta_4^2 (\tau)}{\mathring{\hat{g}} (\int \mathring{3} \mathring{3} \mathring{h})^2}; \eta_4^2 (\tau) = \frac{\partial_4 \mathring{\hat{A}}}{\mathring{\hat{w}}}; \eta_4^2 (\tau) = \frac{\partial_4 \mathring{\hat{A}}}{\mathring{\hat{w}}}; \) |
In such source functionals, it is considered that we prescribe an effective cosmological solitonic hierarchy for matter fields even, in general, such gravitational and matter field interactions can be of non-solitonic type. The second equation (33) with cosmological source \( \psi (\tau, x^i, t) = v \hat{\Sigma}[t] \) can be integrated on time-like coordinate \( y^4 = t \). This allows us to construct off-diagonal cosmological metrics and generalized connections encoding solitonic hierarchies determined by a generating function \( h_3 (\tau, x^i, t) \) with Killing symmetry on \( \partial_3 \), by effective sources \( a \hat{\Sigma}[t] = (\hat{a} h_3[1,t], \hat{a} \hat{\Sigma}{2[t]} \) and an effective cosmological constant

\[
a \Lambda (\tau) = f^I \Lambda (\tau) + m \Lambda (\tau) + \dot{\Lambda}^I \Lambda (\tau).
\]  

(40)

This constant is related to \( a \hat{\Sigma}[t] \) (39) via nonlinear symmetry transforms (35).

Following the AFDM summarized in Table 2, we construct such a class of quadratic line elements for generic off-diagonal cosmological solutions determined by effective sources encoding solitonic hierarchies

\[
d s^2 = e^{\Psi_1 [t]} [(d x^1)^2 + (d x^2)^2 - \int d t \frac{[h_3^2 (\tau)]^2}{\int d t \hat{\Sigma}[t] h_3^2 (\tau)} h_3 (\tau)] d x^i + h_3 (\tau) [d y^3 + (1 n_k (\tau) + 4 n_k (\tau) \int d t \frac{[h_3^2 (\tau)]^2}{\int d t \hat{\Sigma}[t] h_3^2 (\tau)} h_3 (\tau)]^{5/2} d x^k] \].
\]

(41)

Such solutions can be nonholonomically constrained in order to extract LC-configurations. The formulas (41) can be re-defined equivalently in terms of generating functions \( \Psi (\tau, x^i, t) \) or \( \Phi (\tau, x^i, t) \) which can be of a non-solitonic character.

4.1.2 Einstein gravity with cosmological solitonic generating functions

Another class of cosmological NESs can be generated as (off-) diagonal cosmological solutions using generating functionals encoding cosmological solitonic hierarchies \( \Phi (\tau) = \Phi [i] \). Such functionals are subjected to nonlinear symmetries of type (35) and general effective sources \( \psi \hat{\Sigma} (\tau) \) which can be of non-solitonic character. The second equation into (33) transforms into

\[
\omega^*(\tau)[ \Phi [i], \Lambda (\tau)] h_3^2 (\tau)[\Phi [i], \Lambda (\tau)] = 2 h_4 (\tau)[\Phi [i], \Lambda (\tau)] h_3 (\tau)[\Phi [i], \Lambda (\tau)] \psi \hat{\Sigma} (\tau).
\]

This equation can be solved together with other Eqs. (28)–(31) following the AFDM, see Table 2.

The solutions for such cosmological configurations determined by general nonlinear functionals for generating functions can be written in all forms (37). We present here the quadratic line element corresponding only to solutions of third type parametrization

\[
d s^2 = e^{\Psi (\tau, x^k)} [(d x^1)^2 + (d x^2)^2 + (h_3^{[0]} (\tau, x^k) - \frac{(\Phi [i])^2}{4 \Lambda (\tau)}) [d y^3 (1 n_k (\tau, x^k) + 2 n_k (\tau, x^k) 
\int d t \frac{[(\Phi [i])^2]^*}{\Lambda (\tau) \int d t \psi \hat{\Sigma} \Lambda (\tau) [(\Phi [i])^2]^*} h_3^{[0]} (\tau, x^k) - \frac{(\Phi [i])^2}{4 \Lambda (\tau)}]^{5/2} d x^k]
\]

\[
- \frac{(\Phi [i])^2[(\Phi [i])^2]^*}{\Lambda (\tau) \int d t \psi \hat{\Sigma} \Lambda (\tau) [(\Phi [i])^2]^*} (h_3^{[0]} - \frac{(\Phi [i])^2}{4 \Lambda (\tau)})
\]

\[\text{Springer}\]
The zero torsion constraints (34) allow us to extract LC-configurations parameterized as a subclass of (42),

\[
\frac{1}{\Lambda(\tau)} \int dt \, v^3(\tau) \left( \frac{\dot{\Phi}^i(\tau)}{4\Lambda(\tau)} \right)^3
\]

where \( \dot{\Lambda}(\tau) \) and \( n(\tau) \) are also generating functions. Dualizing such solutions and their symmetries, we can generate stationary configurations.

4.1.3 Small N-adapted cosmological solitonic flow deformations

Let us analyse some classes of cosmological solitonic flow solutions with small parametric deformations for a well-known cosmological metric in GR (for instance, a FLRW or Bianchi one as in Appendix B) \( \tilde{g} = [\tilde{g}_i, \tilde{g}_a, \tilde{N}^i_b] \) when \( \delta_4 \tilde{g}_3 = \tilde{g}_3 = \neq 0 \). We formulate a geometric formalism for small generic off-diagonal parametric deformations of \( \tilde{g} \) into certain target cosmological solitonic metrics of type \( g \) (20) when

\[
ds^2 = \eta_i(\varepsilon, \tau) \tilde{g}_i(dx^i)^2 + \eta_a(\varepsilon, \tau) \tilde{g}_a(e^a)^2,
\]

\[
e^3 = dy^3 + n \eta_i(\varepsilon, \tau) \tilde{n}_i dx^i, \quad e^4 = dt + w \eta_i(\varepsilon, \tau) \tilde{w}_i dx^i.
\]

The coefficients \( \eta_a = \eta_a \tilde{g}_a, \, \omega \eta_i \tilde{w}_i, \, n \eta_i n_j \) in these formulas depend on a small parameter \( \varepsilon \), \( 0 \leq \varepsilon \ll 1 \), on evolution parameter \( \tau \) and on coordinates \( x^i \) and \( t \). We suppose that a family of (44) define a solution of cosmological solitonic flow equations described by a system of nonlinear PDEs with decoupling (33). The \( \varepsilon \)-deformations are parameterized in the form

\[
\eta_i(\varepsilon, \tau) = 1 + \varepsilon v_i(\varepsilon, \tau, x^k, t), \quad n \eta_i(\varepsilon, \tau, x^k, t) = 1 + \varepsilon n \eta_i(\varepsilon, \tau, x^k, t)
\]

where a generating function can be given by \( g_3(\tau) = \eta_3(\tau) \tilde{g}_3 = \eta_3(\tau, x^i, t) \tilde{g}_3(\tau, x^i, t) = [1 + \varepsilon v(\tau, x^i, t)] \tilde{g}_3 \), for \( v = v_3(\tau, x^i, t) \).

The deformations of \( h \)-components of a prime cosmological d-metric are \( \varepsilon \tilde{g}_i = \tilde{g}_i(1 + \varepsilon v_i) = e^{\psi(\tau, x^i)} \) for a solution of the 2-d Laplace equation in (33). For parameterizations

\[
\psi(\tau) = \psi(\tau, x^i) + \varepsilon \dot{\psi}(\tau, x^i) \quad \text{and} \quad h^3(\tau)(\tau) = h^3(\tau, x^i, t) + \varepsilon h^3(\tau, x^i, t),
\]

we compute the deformation polarization functions in the form \( v_i = e^{0 \psi} \tilde{g}_i, \, 0 \tilde{h}^3 \). The horizontal generating and source functions are solutions of \( 0 \psi + 0 \psi = 0 \tilde{h}^3 \) and \( 1 \psi' + 1 \psi'' = 1 \tilde{h}^3 \). Using \( \varepsilon \)-decompositions (45) and similar formulas for \( v \)-components, we compute \( \varepsilon \)-decomposition of the target cosmological solitonic d-metric and N-connection coefficients

\[
\dot{\tilde{g}}_i \eta_i(\tau) = e^{\psi(\tau, x^i)} \quad \text{as a solution of 2-d Poisson equations}
\]

\[
\varepsilon \tilde{g}_i(\tau) = [1 + \varepsilon e^{0 \psi} \tilde{g}_i, \, 0 \tilde{h}^3] \tilde{g}_i.
\]
also constructed as a solution of 2-d Poisson equations for $\psi$

$$\hat{g}_4 \eta_4(\tau) = -\frac{4[(\eta_3(\tau)\hat{g}_3)^{1/2}]^4}{\int \mathrm{d}t \, v^3(\tau)[\eta_3(\tau)\hat{g}_3]^3}$$

i.e. $\epsilon \hat{g}_4(\tau) = [1 + \epsilon \, v_4] \hat{g}_4$ for $v_4(\tau, x^i, t) = 2\frac{(v_3(\tau))}{\hat{g}_3} \left( \frac{\int \mathrm{d}t \, v^3(\tau)(v_3(\tau))(\eta_3(\tau)\hat{g}_3)]^3}{\int \mathrm{d}t \, v^3(\tau) \hat{g}_3^3} \right)$.

For such formulas, the system of coordinates is chosen in such a form that there are satisfied the condition $(\hat{g}_3^3)^2 = \hat{g}_4 \int \mathrm{d}t \, v^3(\tau)\hat{g}_3^3$, which allow to find $\hat{g}_4$ for any prescribed values $\hat{g}_3$ and $v^3(\tau)$.

The $\epsilon$-deformations of N-connection coefficients are computed

$$\eta_k^3(\tau) \hat{n}_k = 1n_k(\tau) + 16 \, 2n_k(\tau) \int \mathrm{d}t \, \left( \frac{(\eta_3(\tau)\hat{g}_3)^{-1/4})^2}{\int \mathrm{d}t \, v^3(\tau)(\eta_3(\tau)\hat{g}_3)^*} \right)$$

i.e. $\epsilon \eta_i(\tau) = [1 + \epsilon \, n_i(\tau)] \hat{n}_k = 0$ for $n_i(\tau, x^i, t) = 0$,

if the integration functions are chosen $1n_k(\tau) = 0$ and $2n_k(\tau) = 0$, and

$$\eta_4^4(\tau) \hat{w}_i = \frac{\partial_i \int \mathrm{d}t \, v^3(\tau)[\eta_3(\tau)\hat{g}_3]^e}{\int \mathrm{d}t \, v^3(\tau)[\eta_3(\tau)\hat{g}_3]^e}$$

i.e. $\epsilon \, w_i(\tau) = [1 + \epsilon \, w_i(\tau)] \hat{w}_i$ for $w_i(\tau, x^i, t) = \frac{\partial_i \int \mathrm{d}t \, v^3(\tau)(v_3(\tau))(\eta_3(\tau)\hat{g}_3)]^3}{\int \mathrm{d}t \, v^3(\tau) \hat{g}_3^3} - (v_3(\tau))^3 \hat{g}_3^3$, when a prime $\hat{w}_i = \partial_i \int \mathrm{d}t \, v^3(\tau)\hat{g}_3^3 / v^3(\tau)\hat{g}_3^3$ is well defined for some prescribed $v^3(\tau)$ and $\hat{g}_3^3$.

Finally, we conclude that $\epsilon$-deformed quadratic line elements for cosmological flow deformations can be written in a general form

$$ds^2 = \epsilon g_{\alpha\beta}(\tau, x^k, t) du^\alpha du^\beta = \epsilon g_{\alpha}(\tau, x^k)((dx^1)^2 + (dx^2)^2) + \epsilon g_{3}(\tau, x^k, t)(dy^3)^2$$

$$+ \epsilon h_{4}(\tau, x^k, t) [dt + \epsilon w_{i}(\tau, x^k, t)dx^i]^2.$$  

We can impose additional nonholonomic constraints (38) in order to extract LC-configurations for $\epsilon$-deformations with zero torsion.

4.2 FLRW metrics in (off-) diagonal cosmological media with solitonic hierarchies

Various classes of generic off-diagonal cosmological solitonic solutions can be constructed and parameterized in terms of $\eta$-polarization functions introduced in formulas (20) and applying the AFDM summarized in Tables 1 and 2. A primary cosmological d-metric can be parameterized in a necessary form as in Appendix (B) when $\hat{g} = [\hat{g}_i(x^i, t), \hat{g}_a = \hat{h}_a(x^i, t)]$, $\hat{N}_k^3 = \hat{n}_k(x^i, t), \hat{N}_k^4 = \hat{w}_k(x^i, t)$ (21) which for a FLRW configuration can be diagonalized by frame-coordinate transforms. A cosmological solitonic target metric $g$ can be generated by nonholonomic $\eta$-deformations, $\hat{g} \rightarrow g(\tau) = g_i(\tau, x^k) = \eta_i(\tau)\hat{g}_i, g_b(\tau, x^k, t) = \eta_b(\tau)\hat{g}_b, N^a_b(\tau, x^k, t) = \eta^a_b(\tau)\hat{N}^a_b$, and constrainted to the conditions to define exact and parametric solutions of the system of nonlinear PDEs with decoupling (33). A corresponding quadratic line element for $g$ can be parameterized in a form (20)

$$ds^2 = \eta_i(\tau, x^i, t)\hat{g}_i [dx^i]^2 + \eta_a(\tau, x^i, t)\hat{g}_a [dy^a + \eta_k^a(\tau, x^i, t)\hat{N}_k^a dx^k]^2,$$  

\[ Springer\]
with summation on repeating contracted low-up indices. The polarization values \( \eta_\alpha(\tau) \) and \( \eta_i^\alpha(\tau) \) are determined by geometric and cosmological solitonic flows and nonlinear interactions.

### 4.2.1 Cosmological solutions generated by solitonic sources

We prescribe that the effective \( v \)-source is determined by a solitonic hierarchy \( \hat{Y}(\tau, x^i, t) = \hat{v}\hat{S}[\ 2 t] \) (26) and compute the coefficients for a target d-metric (46) following formulas summarized in Table 2.

\[
\eta_i(\tau) = \frac{\psi(\tau, x^i)}{\hat{g}^{ij}}; \quad \eta_3(\tau) = \eta_3(\tau, x^i, t) \text{ as a generating function};
\]

\[
\eta_4(\tau) = -\frac{4[(\eta_3(\tau)\hat{h}_3(\tau))^2]^2}{\hat{h}_4[\ 2 t][\eta_3(\tau)\hat{h}_3(\tau)]};
\]

\[
\eta_k^3(\tau) = \frac{1}{\hat{w}_k} + 16 \frac{2n_k}{\hat{h}_k} \int dt \frac{[(\eta_3(\tau)\hat{h}_3(\tau)]^2}{\hat{v}\hat{S}[\ 2 t][\eta_3(\tau)\hat{h}_3(\tau)]};
\]

\[
\eta_i^4(\tau) = \frac{\partial_i}{\hat{w}_i} \hat{v}\hat{S}[\ 2 t][\eta_3(\tau)\hat{h}_3(\tau)]^*.
\]

(47)

with integration functions \( 1n_k(\tau, x^i) \) and \( 2n_k(\tau, x^i) \).

In (47), the gravitational polarization \( \eta_3(x^i, t) \) is taken as a (non-) singular generating function subjected to nonlinear symmetries of type (35) which can be written in the form

\[
\Phi^2 = -4 \Lambda(\tau)\hat{h}_3(\tau) = -4 \Lambda(\eta_3(\tau, x^i, t)\hat{h}_3(\tau, x^i, t) \text{ and}
\]

\[
(\Psi^2)^* = -\int dt \hat{v}\hat{S}[\ 2 t][\eta_3(\tau, x^i, t)\hat{h}_4(\tau, x^i, t)]^*.
\]

In this section, the values \( \Phi, \hat{h}_3 \) and \( \eta_3 \) may not encode solitonic hierarchies but \( \Psi \) and other coefficients of such target cosmological d-metric are solitonic ones if they are computed using \( \hat{v}\hat{S}[\ 2 t] \). We can constrain the coefficients (47) to a subclass of data generating target LC-configurations when the d-metrics satisfy the constraints (38) for zero torsion. The nonlinear functionals for the soliton \( v \)-source and (effective) cosmological constant can be changed into additive functionals \( \hat{v}\hat{S} \rightarrow \hat{v}\hat{S}[\ 1] \) and \( \Lambda \rightarrow \Lambda^*\hat{v}\hat{S}[\ 1] \) (39) and \( \Lambda^* \) (40).

### 4.2.2 FLRW metrics deformed by solitonic generating functions

Solutions with geometric flow and cosmological \( \eta \)-polarizations (44) can be constructed with coefficients of the d-metrics determined by nonlinear generating functionals \( \Phi[i] \), or prescribed additive functionals \( a\Phi[i] \) corresponding to (39). This includes terms with integration functions \( \hat{h}_3(\tau, x^i) \) for \( h_3[i] \) with explicit dependence on a time-like variable \( t \). Such configurations can be generated also by some prescribed cosmological data \( \hat{v}\hat{S}(\tau, x^i, t) \) and \( \Lambda(\tau) \), which are not obligatory of solitonic nature. We can compute corresponding nonlinear functionals \( \eta_3(\tau, x^i, t) \) (we omit here similar formulas for additive functionals \( a\eta_3(\tau, x^i, t) \)) using nonlinear symmetries (35) and related polarization functions

\[
\eta_3[i] = -\Phi_2^2[i]/4\Lambda(\tau)\hat{h}_3(\tau, x^i, t), \quad [\Psi^2(\tau)]^* = -\int dt \hat{v}\hat{S}(\tau, x^i, t)\hat{h}_3^*(\tau)
\]
\[
= - \int dt \, v \tilde{S}(\tau, x^i, t) [\eta_3[i][\hat{h}_3(x^i, t)]^*].
\]

We apply these formulas for the AFDM outlined in Table 2 and compute the coefficients of a cosmological solitonic d-metric of type (46),

\[
\eta_i(\tau) = \frac{e^{\psi(\tau, x^i)}}{g_i}; \eta_3(\tau) = \eta_3(\tau, x^i, t) = \eta_3[i] \text{ as a generating function;}
\]

\[
\eta_4 = -\frac{4(|\eta_3[i][\hat{h}_3]|^{1/2})^2}{\hat{h}_3|\int dt \, v \tilde{S}(\tau) \eta_3[i][\hat{h}_3]^*|};
\]

\[
\eta^3_k(\tau) = \frac{1n_k(\tau)}{n_k} + 16 \frac{2n_k(\tau)}{n_k} \int dt \, \frac{[\eta_3[i][\hat{h}_3]^{-1/4}]^*}{|\int dt \, v \tilde{S}(\tau)(\eta_3[i][\hat{h}_3]^*)|};
\]

\[
\eta^4_i(\tau) = \frac{\partial_i \int dt \, v \tilde{S}(\tau)(\eta_3[i][\hat{h}_3]^*)}{\hat{w}_i \int dt \, v \tilde{S}(\tau)(\eta_3[i][\hat{h}_3]^*)},
\]

(48)

for integrating functions \(_1n_k \tau, x^i\) and \(_2n_k \tau, x^i\).

Using (48), target cosmological solitonic off-diagonal metrics with zero torsion which solve (34) can be generated by polarization functions subjected to additional nonholonomic constraints

\[
\eta_i(\tau) = \frac{e^{\psi(\tau, x^i)}}{g_i}; \eta_3(\tau) = \tilde{\eta}_3(\tau, x^i, t) = \tilde{\eta}_3[i] \text{ as a generating function;}
\]

\[
\eta_4 = -\frac{4(|\tilde{\eta}_3[i][\hat{h}_3]^{1/2})^2}{\hat{h}_4|\int dt \, v \tilde{S}(\tau)(\tilde{\eta}_3[i][\hat{h}_3])^*|}; \eta^3_k(\tau) = \frac{\partial_k n(\tau)}{\tilde{\eta}_k}, \eta^4_i(\tau) = \frac{\partial_i \tilde{A}(\tau)}{\tilde{w}_k},
\]

for an integrating function \(n(\tau, x^i)\) and a generating function \(\tilde{A}(\tau, x^i, t)\).

The solutions constructed in this subsection describe certain nonholonomically deformed cosmological geometric flow configurations self-consistently imbedded into a solitonic gravitational evolution media which can model nonholonomic dark energy and dark energy configurations. A compatibility with observational data can be chosen for respective integration functions and constants and corresponding classes of solitonic hierarchies.

4.2.3 Cosmological deformations by solitonic sources and solitonic generating functions

General classes of cosmological solutions and nonholonomic deformations can be constructed with nonlinear solitonic functional both for generating functions and generating sources. Nonlinear superpositions of solutions of type (47) and (48) can be performed if the coefficients of d-metric are computed

\[
\eta_i(\tau) = \frac{e^{\psi(\tau, x^i)}}{g_i}; \eta_3(\tau) = \eta_3(\tau, x^i, t) = \eta_3[i 3i] \text{ as a generating function;}
\]

\[
\eta_4 = -\frac{4(|\eta_3[i][\hat{h}_3]^{1/2})^2}{\hat{h}_4|\int dt \, v \tilde{S}(\tau)(\eta_3[i][\hat{h}_3])^*|};
\]

\[
\eta^3_k(\tau) = \frac{1n_k(\tau)}{n_k} + 16 \frac{2n_k(\tau)}{n_k} \int dt \, \frac{[\eta_3[i][\hat{h}_3]^{-1/4}]^*}{|\int dt \, v \tilde{S}(\tau)(\eta_3[i][\hat{h}_3]^*)|};
\]
where \( \eta_1^k (\tau, x^k) \) and \( \eta_2^k (\tau, x^k) \) are integration functions. In formulas (49), we consider two different prescribed a nonlinear generating functional, \( \Phi[3\ell] \), and a nonlinear functional for source, \( \hat{\psi}[\ell] \), and running constant \( \Lambda (\tau) \) related via nonlinear symmetries of type (35). This allows us to compute a corresponding nonlinear functional \( \eta_3 (\tau, x^i, t) = \eta_3 [1, \ldots, \tau] \) and a polarization function

\[
\eta_3 (\tau) = - \Phi^2 [3\ell]/4 \Lambda (\tau) \hat{h}_3 (x^i, t), \quad (\Psi^2 (\tau))^* = - \int dt \hat{\psi}[\ell] h_3^* [3\ell] = - \int dt \hat{\psi}[\ell] [\eta_3 (\tau) \hat{h}_3 (x^i, t)]^*.
\] (50)

Imposing additional constraints (38) for a zero torsion, LC-cosmological solitonic metrics with geometric flows are generated.

In result, the quadratic line element corresponding to such classes of cosmological solutions (49) can be written for generating data (\( \Phi[3\ell], \Lambda (\tau) \)):

\[
d s^2 = e^{\psi (3\ell)} [(d\lambda^k)^2 + (d\zeta^k)^2 + \left( h_3^{(0)} (\tau, x^k) - \frac{(\Phi[3\ell])^2}{4\Lambda (\tau)} \right)]
\]

\[
\left[ \frac{1}{\Lambda (\tau) \int dt \hat{\psi}[\ell] [\Phi[3\ell]]^2] \right] \left( h_3^{(0)} (\tau, x^k) - \frac{(\Phi[3\ell])^2}{4\Lambda (\tau)} \right) - \frac{\left[ \frac{1}{\Lambda (\tau) \int dt \hat{\psi}[\ell] [\Phi[3\ell]]^2] \right] (\Phi[3\ell])^2}{4\Lambda (\tau)}
\]

\[
\int dt \frac{\partial}{\partial \tau} \left[ \frac{1}{\Lambda (\tau) \int dt \hat{\psi}[\ell] [\Phi[3\ell]]^2] \right] \eta_3 (\tau, t) \frac{1}{\hat{\psi}[\ell] [\Phi[3\ell]]^2} d\lambda^k
\] (51)

The data for a primary cosmological solution can be extracted using nonlinear symmetries (50), when \( \hat{g}_i = e^{\psi (x^i)} / \eta_i (\tau) \) and \( \hat{h}_3 (x^i, t) = - \Phi^2 [3\ell]/4 \Lambda (\tau) \eta_3 (\tau) \) are considered for certain values which for a \( \tau_0 \) are prescribed in some forms that the integration functions \( h_3^{(0)} (\tau, x^k), 1 n_k (\tau, x^k) \) and \( 2 n_k (\tau, x^k) \) encode a prime d-metric \( \hat{g} = [\hat{g}_i, \hat{g}_a, \hat{N}_b^j] \) (21). Such cosmological scenarios (51) describe geometric evolution for \( \tau > \tau_0 \) self-consistently imbedded into solitonic gravitational (dark energy) backgrounds and solitonic dark and/or standard matter.

4.3 Off-diagonal deformations of FLRW metrics by solitonic flow sources

We study how effective sources for geometric flows with cosmological solitonic hierarchies flows result in generic off-diagonal deformations and generalizations of a FLRW metric. Such nonholonomic deformations of geometric objects define new classes of exact solutions of systems of nonlinear PDEs (33). To apply the AFDM is necessary to define some nonholonomic variables which allow decoupling and integration of corresponding systems of equations describing N-adapted nonholonomic deformations of a prime d-metric \( \hat{g} = [\hat{g}_i, \hat{g}_a, \hat{N}_b^j] \) (21), see formulas for parameterizations of cosmological d-metrics in Appendix B. We cite [45] as a standard monograph on GR with necessary details on geometry of cosmological spaces and [20,23,60–62] for examples of...
nonholonomic deformations of BH and cosmological solutions in geometric flows and gravity theories.

4.3.1 Nonholonomic evolution of FLRW metrics with induced (or zero) torsion

We consider a primary cosmological d-metric parameterized as in Appendix (B) when \( \hat{g} = [\hat{g}_i(x^i, t), \hat{g}_{ij} = \hat{H}_a(x^i, t); \hat{N}_k^3 = \hat{n}_k(x^i, t), \hat{N}_k^4 = \hat{w}_k(x^i, t), ] \) (21) with \( \hat{g}_3^* \neq 0 \), which for a FLRW configuration can be diagonalized by frame/coordinate transforms. This allows us to construct nonholonomic cosmological deformations following the geometric formalism outlined in Sect. 4.1.3 and Table 2.

For general \( \eta \)-deformations (44) and constraints \( n_i = 0 \), the solitonic flow modifications of the FLRW metric are computed

\[
 ds^2 = e^{\psi(\tau, x^k)}[dx^1]^2 + [dx^2]^2 + \eta_3[3i]\hat{H}_3(e^3)^2 - \frac{4[(\eta_3[3i]\hat{H}_3)^{1/2})^2}{\int \tilde{\psi}([\eta_3[3i]\hat{H}_3]^*)}] \hat{H}_4(e^4)^2, \\
 e^3 = dy^3, \quad e^4 = dt + \frac{\partial_i}{\tilde{\psi}} \int \tilde{\psi}([\eta_3[3i]\hat{H}_3]^*) dx^i, \quad (52)
\]

where \( \eta_3(\tau) = \eta_3(\tau, x^k, t) = \eta_3[3i] \) is a generating function and \( \tilde{\psi}(\tau) = \tilde{\psi}([\eta_3[3i]]) \) is a flow generating source as in (49) and \( \psi(\tau, x^k) \) is a solution of a 2-d Poisson equation.

4.3.2 Small parametric modifications of FLRW metrics and effective flow sources

Let us elaborate on models of geometric cosmological flows for nonholonomic distributions describing \( \epsilon \)-deformations (45) for \( \hat{g}_3^* \neq 0 \) in target metrics of type (44). The corresponding quadratic line elements are

\[
 ds^2 = [1 + e^{0\psi(\tau, x^k)}] [dx^1]^2 + [1 + e^{\psi(\tau, x^k)}] \hat{g}_3(e^3)^2 \\
 + [1 + e^{\psi(\tau, x^k)}] \left( 2 \frac{[\tilde{\psi}_3(\tau)]^*}{\tilde{\psi}_3} - \frac{\int \tilde{\psi}([\tilde{\psi}_3]^*) dx^i}{\int dy^3 \tilde{\psi}_3} \right) \hat{g}_4(e^4)^2, \\
 e^3 = dx^3, \quad e^4 = dt + [1 + e^{\psi(\tau, x^k)}] \left( \frac{\partial_i}{\tilde{\psi}} \int \tilde{\psi}([\tilde{\psi}_3]^*) dx^i \right) \hat{g}_3(e^3)^2, \quad (53)
\]

In these formulas, \( 0\psi(\tau) = 0\psi(\tau, x^k) \) and \( 1\psi(\tau) = 1\psi(\tau, x^k) \) are solutions of 2-d Poisson equations with a generating \( h \)-source \( \hat{H}_3(\tau) = \hat{H}_3(\tau, x^k) = 0^h_3(\tau, x^k) + e^h_3(\tau, x^k) \) as described in Sect. 4.1.3; \( \tilde{\psi}_3(\tau) = \tilde{\psi}_3([\eta_3[3i]]) \) is a generating \( \psi \)-source with \( \epsilon \)-decomposition where \( \psi_3 = \psi(\tau, x^k, t) = \psi([\eta_3[3i]]) \) is a generating function. The cosmological flow solution (53) is for a N-adapted system of references and space coordinates \([x^i, y^3]\) for which the condition \( (\hat{g}_3^*)^2 = \hat{g}_4 \int \tilde{\psi}_3 \hat{g}_3^* \) allows us to compute well-defined coefficients \( \hat{g}_3 \) and \( \hat{w}_i = \partial_i [\tilde{\psi}_3 \hat{g}_3^*] / \tilde{\psi}_3 \hat{g}_3^* \) when there are prescribed certain values \( \tilde{\psi}_3 \) and \( \hat{g}_3^* \neq 0 \). We can fix the conditions \( \eta_3(\tau) = 0 \) and \( \eta_3(\tau) = 0 \) for which \( N_i^4 = n_i = 0 \) but even in such cases a nonzero coefficient \( N_i^4 \neq 0 \) results in nontrivial nonholonomic torsion and anholonomy coefficients. To extract LC-configurations, we can impose on \( \psi(\tau) \) and sources additional zero torsion constraints (38).

Considering for nonlinear symmetries of type (50) the formula \( \eta_3(\tau) = -\Phi^2[3i]/4 \Lambda(\tau) \hat{g}_3 \) for (53), we conclude that as a generating function (including solitonic hierarchies) can be...
used the value

$$\varepsilon v[3t] = -(1 + \Phi^2[3t]/4 \Lambda(\tau) \dot{g}_3)$$ or \(\Phi[3t] \simeq 2\sqrt{\Lambda(\tau) \dot{g}_3} \left(1 - \frac{\varepsilon}{2} v[3t]\right). \quad (54)$$

Other types of geometric flow and cosmologic solitonic hierarchies can be encoded into generated sources \(\dot{h}_0(\tau, x^k) = \dot{h}_0(\tau, x^k) + \varepsilon \dot{h}_0(\tau, x^k), \quad \dot{\Sigma}(\tau) = \dot{\Sigma}(t)\) and \((\Lambda(\tau), v[4t], \dot{h}_0(\tau, x^k) + \varepsilon \dot{h}_0(\tau, x^k))\).

4.4 W-entropy and thermodynamics of cosmological solitonic solutions

We provide explicit examples how G. Perelman’s W-entropy and related thermodynamic values can be computed for cosmological solitonic solutions under geometric flow evolution. To simplify formulas, we fix certain values for normalization and integration functions corresponding to a constant normalizing function, \(\tilde{f}(\tau) = \tilde{f}_0 = \text{const} = 0\), in (11).\(^6\) In result, the F- and W-functionals (5) are written

\[
\begin{align*}
\widehat{F} &= \frac{1}{8\pi^2} \int \tau^{-2} \sqrt{|g[\Phi(\tau, x^i, t)]|} \delta^4 u [h \Lambda(\tau) + \Lambda(\tau)], \\
\widehat{W} &= \frac{1}{4\pi^2} \int \tau^{-2} \sqrt{|g[\Phi(\tau, x^i, t)]|} \delta^4 u [h \Lambda(\tau) + \Lambda(\tau)]^2 - 1,
\end{align*}
\]

where \(\sqrt{|g[\Phi(\tau, x^i, t)]|} = \sqrt{|q_1q_2q_3(q, N)|} \]

\[
= 2e^{\psi(\tau, x^k)} |\Phi(\tau, x^i, t)| \sqrt{\frac{|[\Phi^2(\tau, x^i, t)]^*|}{|\Lambda(\tau) \int dt \tilde{\Sigma}(\tau, x^i, t)[\Phi^2(\tau, x^i, t)]^*|}} \quad (55)
\]

is computed for d-metrics parameterized in the form (3) with

\[
q_1(\tau) = q_2(\tau) = e^{\psi(\tau, x^k)}, \quad q_3(\tau) = \frac{-\Phi^2(\tau, x^i, t)}{4\Lambda(\tau)}, \quad |qN(\tau)|^2 = \frac{4|\Phi^2(\tau, x^i, t)^*|}{|\int dt \tilde{\Sigma}(\tau, x^i, t)[\Phi^2(\tau, x^i, t)]^*|}
\]

for \(h_4^{[0]} = 0\). The N-adapted differential \(\delta^4 u = dx^1 dx^2 e^4 = dx^1 dx^2 [dy^3 + n_l(\tau) dx^i] [dt + u_i(\tau) dx^i]\) is for N-connection coefficients with fixed integration functions \(n_k(\tau) = 0\) and \(2n_k(\tau) = 0\), when

\[
N_i = [n_l(\tau) = 0, \quad u_i(\tau) = \frac{\partial_i (\int dt \tilde{\Sigma}(\tau, x^i, t)[\Phi^2(\tau, x^i, t)]^*)}{\tilde{\Sigma}(\tau, x^i, t)[\Phi^2(\tau, x^i, t)]^*}].
\]

The statistical thermodynamic values can be computed using the thermodynamic generating function (13) corresponding to \(\tilde{W}(55)\) (for simplicity, with fixed normalization)

\[
\tilde{Z}[\Phi(\tau)] = \frac{1}{4\pi^2} \int \tau^{-2} d\Psi(\tau). \quad (56)
\]

\(^6\) Having defined such values in a convenient system of reference/coordinates, we can consider changing to any system of reference.
The effective integration volume functional \( dV(\tau) = dV(\psi(\tau, x^k), \Phi(\tau, x^i, t), v\hat{\mathcal{S}}(\tau, \Lambda(\tau))) \) is determined by data \((\psi(\tau), \Phi(\tau), v\hat{\mathcal{S}}(\tau), \Lambda(\tau))\) and computed

\[
dV(\tau) = e^{\psi(\tau)} \Phi(\tau) \sqrt{\left[\frac{|\Phi^2(\tau)|^n}{|\Lambda(\tau) \int \text{d}v\hat{\mathcal{S}}(\tau)[\Phi^2(\tau)]^n|}\right] \text{d}x \text{d}x^2 \text{d}y^3} \\
\left[\text{d}r + \frac{\partial_i \left(\int \text{d}y^3v\hat{\mathcal{S}}(\tau)[\Phi^2(\tau)]^n\right)}{v\hat{\mathcal{S}}(\tau)[\Phi^2(\tau)]^n} \text{d}x^i \right]. \tag{57}
\]

These formulas allow us to compute thermodynamic values for cosmological solitonic geometric flows

\[
\mathcal{\hat{E}}(\tau) = -\frac{1}{4\pi^2} \int \left[2h+2(\Lambda(\tau)+\Lambda(\tau)) - \frac{2}{\tau}\right] dV(\tau), \mathcal{\hat{S}}(\tau) = -\frac{1}{4\pi^2} \int (\Lambda(\tau)+\Lambda(\tau)) \tau dV(\tau). \tag{58}
\]

### 4.4.1 Thermodynamic values for cosmological solitonic generating functions and sources

We can consider Perelman and/or Carathéodory thermodynamic values are computed for a 3+1 splitting (3) determined by a cosmological solitonic d-metric (51) (for LC-configurations, we can consider (34)), when

\[
q_1 = q_2[\mathcal{h}^i] = e^{\psi[\mathcal{h}^i]}, q_3[\mathcal{3}^i] = -\frac{(\Phi[3^i])^2}{4\Lambda(\tau)}, \]

\[
[q N(\tau)]^2 = h_4[4^i] = -\frac{4\left[(\Phi[3^i])^2\right]^n}{\int \text{d}r \text{d}v\hat{\mathcal{S}}(\tau)(\Phi[3^i])^n}, \]

and \(N^i_l = \left[n_i(\tau) = 0, w_i(\tau) = \frac{\partial_i \left(\int \text{d}y^3v\hat{\mathcal{S}}(\tau)[\Phi^2(\tau)]^n\right)}{v\hat{\mathcal{S}}(\tau)[\Phi^2(\tau)]^n}\right]\).

For such coefficients and prescribed soliton hierarchies for the effective volume (57), we obtain

\[
dV[\mathcal{h}^i, 3^i] = e^{\psi[\mathcal{h}^i]} \Phi[3^i] \sqrt{\left[\frac{|\Phi^2[3^i]|^n}{|\Lambda(\tau) \int \text{d}r \text{d}v\hat{\mathcal{S}}(\tau)[\Phi^2[3^i]]^n|}\right] \text{d}x \text{d}x^2 \text{d}y^3} \\
\left[\text{d}r + \frac{\partial_i \left(\int \text{d}y^3v\hat{\mathcal{S}}(\tau)[\Phi^2[3^i]]^n\right)}{v\hat{\mathcal{S}}(\tau)[\Phi^2[3^i]]^n} \text{d}x^i \right].
\]

This volume element allows us to define the thermodynamic generating function (56) and compute respective thermodynamic values (58) for geometric flows of such solitonic hierarchies

\[
\mathcal{\hat{Z}}[\mathcal{h}^i, 3^i] = \frac{1}{4\pi^2} \int \text{dV}[\mathcal{h}^i, 3^i] \] and

\[
\mathcal{\hat{E}}[\mathcal{h}^i, 3^i] = -\frac{1}{4\pi^2} \int (\Lambda(\tau)+\Lambda(\tau)) \tau dV[\mathcal{h}^i, 3^i],
\]

\[
\mathcal{\hat{S}}[\mathcal{h}^i, 3^i] = -\frac{1}{4\pi^2} \int (\Lambda(\tau)+\Lambda(\tau)) - 2) \tau^{-2} dV[\mathcal{h}^i, 3^i].
\]
4.4.2 Small parametric cosmological solitonic and geometric flow thermodynamics

The d-metrics for such parametric solutions are described by quadratic elements (53) and generating functions $\Phi[3l, h_3] \simeq 2\sqrt{\Lambda(\tau)h_3}(1 - \frac{\epsilon}{2}v[3l])$ (54) and a primary cosmological metric. The $\epsilon$-decomposition for the respective effective volume form is

$$dV[h^i, \epsilon v[3l], i, h^3] = 2\epsilon|\psi[h^i]|[1 - \frac{\epsilon}{2}v[3l]] \left[ \frac{h_3}{\sqrt{\int dt v[3l]} * dx^1 dx^2 dy^3} \right]$$

which allows to compute corresponding thermodynamic generating function (56) and canonical energy and entropy (58) for geometric flow cosmological solitonic flow parametric $\epsilon$-deformations.

Finally, we emphasize that it is not possible to define and compute the Bekenstein–Hawking entropy for locally anisotropic cosmological solutions constructed in this section.

5 Conclusions, discussion, and perspectives

The axiomatic side of thermodynamics due to Constantin Carathéodory [24,25] and, further, the axiomatic treatment of physics were of constant and deep interest both to mathematicians and physicists (including D. Hilbert, W. Pauli, M. Born, and, in a critical sense, M. Planck) [63–66]. Relativistic generalization of Grigory Perelman geometric flow thermodynamics [10] allows us to include in the scheme and find geometric connections to mathematical physics, (modified) gravity theories, cosmology and (quantum) information theory [16–19,22,23].

In this article, we developed Carathéodory’s axiomatic approach to foundations of thermodynamics and statistical physics (considering Pfaff forms but also certain work on measure theory) and demonstrated also that his methods are useful for research and applications in modern gravity and cosmology theories. Although C. Carathéodory and G. Perelman geometric thermodynamic construction played a strategic role in finding solutions of most important and difficult problems in geometry and physics, their contributions have not yet properly appreciated by many physicists and mathematicians. This is probably due to the multi- and inter-disciplinary character of their works when further applications request both a “deep physical intuition” and “advanced mathematical education and very sophisticated geometric methods”. In our works, we try to establish a bridge between different communities of researchers.

Let us speculate on certain further perspectives on elaborating unified geometric methods to thermodynamics of geometric flows, gravity and cosmology, quantum information etc.:

1. Carathéodory’s works on measure theory [67,68], see further developments in [69] and (on symbolic shifts and ergodic theory) [51,70], seem to be useful in modelling information sources and processing [71,72] and (recent applications) quantum information theory [21–23].

2. The extended spectral decompositions are applicable for various conservative systems, complex systems and their macroscopic descriptions, with new possibilities for probabilistic prediction and control [47,48,52–54,73,74], in the theory of locally anisotropic kinetic
processes and diffusion [52–54], see also applications in noncommutative geometric flow theory and physics [16].

3. P. Finsler elaborated his geometry as a postgraduate of C. Carathéodory. Such Finsler–Lagrange–Hamilton theories and their modifications of Einstein gravity, geometric flows theories, cosmology and astrophysics have been recently axiomatized for nonholonomic Lorentz manifolds and (co) tangent bundles, see recent reviews in [75,76]. This paper should be considered as a cosmological partner of the works [20,77] on Finsler and other types modified black hole configurations and their generalized Perelman thermodynamics.

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A Pfaffian differential equations

Let us provide a brief introduction into the theory of Pfaff forms and thermodynamics, see details and references in [37–39]. A Pfaff differential form is

\[ \delta \phi = \sum_{I} X_I dz^I, \]

where \( I \) runs integer values (for simplicity, we consider \( I = 1, 2 \)) and \( \delta f \) is differential 1-form but may be not a differential of a real valued function \( \phi(z^I) \) of real variables \( z^I \), where \( \partial_I := \partial / \partial z^I \).

An equation

\[ \delta \phi = 0 \]  

(A.1)

is called a non-exact Pfaff equation. If \( \delta \phi = df = (\partial_I \phi)dz^I \) is an exact differential of a function \( \phi(z^I) \), i.e. we have an exact Pfaff equation, it is possible to integrate (A.1) along a path \( C \) connecting two points \( z_{[1]}^I \) and \( z_{[2]}^I \) (when \( \phi \) is path-independent) and express the solution in the form

\[ \phi = \int_C \delta \phi = \phi(z_{[2]}^I) - \phi(z_{[1]}^I) = \text{const}. \]

The H. A. Schwarz criterion is the necessary and sufficient condition to detect a total differential equation

\[ \partial_I X_J = \partial_J X_I, \text{ for } I \neq J, \text{ i.e. } \frac{\partial^2 \phi}{\partial x^1 \partial x^2} = \frac{\partial^2 \phi}{\partial x^2 \partial x^1}. \]  

(A.2)

In many cases, a non-exact Pfaffian with \( \partial_I X_J \neq \partial_J X_I \) can be transformed into an exact one by the aid of an integrating factor \( K(z^I) \), when the coefficients of \( \sum_I K_X_I dz^I \) satisfy the Schwarz condition

\[ \partial_I (K X_J) = \partial_J (K X_I), \text{ for } I \neq J. \]  

(A.3)

In such a case, the equation

\[ K \delta \phi = d(K \phi) = 0 \]  

(A.4)

can be integrated in an explicit form which allows us to find \( \phi \) for any prescribed \( K \) satisfying (A.3).
In a more general context, if we are not able to transform (A.1) into (A.4), we can additionally add to
\[ \delta(K\phi) = \sum_I KX_I dz^I \neq d(K\phi) \]
a differential of a new function \( B(z^I) \), \( dB = (\partial_I B)dz^I \) and search for such \( K \) and \( B \) when
\[ \partial_I (KX_I + B) = \partial_J (KX_J + B), \text{ for } I \neq J \text{ and } \delta(K\phi) + dB = d(K\phi + B). \]
In such a case, we can integrate
\[ d(K\phi + B) = 0 \] (A.5)
for any suitable \( K \) and \( B \) and find \( \phi \) in nonexplicit form from a so-called nonholonomic (non-integrable) function \( F(\phi, z^I) = const \). Usually, in thermodynamics we deal with equations of type (A.1) into (A.4), but on nonholonomic manifolds, equations of type (A.5) are involved.

B Parameterizations for families of cosmological d-metrics

We consider basic notations for quadratic line elements describing geometric flow evolutions and nonholonomic deformations of prime metrics into target cosmological ones.

B.1 Target d-metrics with geometric evolution of polarization functions

Families of target quadratic line elements can be represented in off-diagonal form, \( g_{\alpha\beta} = [g_i, h_a, n_i, w_i] \), and/or using \( \eta \)-polarization functions
\[ ds^2(\tau) = g_i(\tau, x^k)[dx^i]^2 + h_3(\tau, x^k, t)[dy^3 + n_i(\tau, x^k, t)dx^i]^2 \]
\[ + h_4(\tau, x^k, t)[dt + w_i(\tau, x^k, t)dx^i]^2 \]
\[ = \eta_i(\tau, x^k, t)\dot{g}_i(x^k, t)[dx^i]^2 + \eta_3(\tau, x^k, t)\dot{h}_3(x^k, t) \]
\[ [dy^3 + \eta_1^3(\tau, x^k, t)\dot{N}_i^3(x^k, t)dx^i]^2 \]
\[ + \eta_4(\tau, x^k, t)\dot{h}_4(x^k, t)[dt + \eta_4^4(\tau, x^k, t)\dot{N}_i^4(x^k, t)dx^i]^2 \]
\[ = \eta_i(\tau)\dot{g}_i[dx^i]^2 + \eta_3(\tau)\dot{h}_3[dy^3 + \eta_1^3(\tau)\dot{N}_i^3dx^i]^2 \]
\[ + \eta_4(\tau)\dot{h}_4[dt + \eta_4^4(\tau)\dot{N}_i^4dx^i]^2, \] (B.2)
where \( \tau \) is a temperature-like geometric evolution parameter and, for simplicity, we consider that prime metrics do not depend on such a parameter. There will be stated dependencies of type \( \eta_{\alpha\beta}(\tau) = \eta_{\alpha\beta}(\tau, x^k, t) \) if such not notations do not result in ambiguities. We consider a coordinate transform to a new time-like coordinate \( y^4 = t \rightarrow \zeta \) when \( t = t(x^i, \zeta) \).
\[ dt = \partial_tdx^i + (\partial t / \partial \zeta)d\zeta; \]
\[ d\zeta = (\partial t / \partial \zeta)^{-1}(dt - \partial_t dx^i), \text{ i.e. } (\partial t / \partial \zeta)d\zeta = (dt - \partial_t dx^i), \]
and rewrite the target d-metric using the new time variable \( \zeta \). For instance, the fourth term in (B.2) is computed
\[ \eta_4(\tau)\dot{h}_4[dt + \eta_4^4(\tau)\dot{N}_i^4dx^i]^2 = \eta_4(\tau)\dot{h}_4[\partial_t dx^k + (\partial t / \partial \zeta)d\zeta + \eta_4^4(\tau)\dot{N}_i^4dx^i]^2 \]
\[ = \eta_4(\tau)\dot{h}_4[(\partial_t dx^k + (\partial t / \partial \zeta)d\zeta + \eta_4^4(\tau)\dot{N}_i^4dx^i]^2 \]
\[ = \eta_4(\tau)\dot{h}_4[(\partial t / \partial \zeta)d\zeta + (\partial_t + \eta_4^4(\tau)\dot{N}_i^4)dx^i]^2 \]
In result, a new time coordinate geometric evolution and nonholonomic deformations of the FLRW metrics.

By definition, a quasi-FLRW configuration is stated by a diagonalized solution for a d-metric of type (B.3) below formulas relevant to (B.9)) by the polarization of the target cosmological factor, +

\[ \eta(\tau, \varsigma) = \eta_4(\tau, \varsigma) \]

In result, a new time coordinate \( \varsigma \) can be found from \( \partial t / \partial \varsigma = (\eta_4)^{-1} \) which results in

\[ d\varsigma = \eta_4(x^k, t) dt; \quad \varsigma = \int \eta_4(x^k, t) dt + \varsigma_0(x^k). \]

Such coordinates with flow parameter \( \tau \) and time-like \( \varsigma \) are useful for computations of geometric evolution and nonholonomic deformations of the FLRW metrics.

B.2 Off-diagonal and diagonal parameterizations of prime d-metrics

Let us consider a target line quadratic element for an off-diagonal cosmological solution written in the form \( (B.2) \). We can introduce an effective target locally anisotropic cosmological scaling factor \( \tilde{\alpha}^2(\tau, x^k, \varsigma) \neq \eta(\tau, x^k, \varsigma) \tilde{\alpha}^2(x^i, \varsigma) \) with gravitational polarization \( \eta(\tau, x^k, \varsigma) \) and prime cosmological scaling factor \( \tilde{\alpha}^2(\tau, x^k, \varsigma) \), which allows to consider limits \( \tilde{\alpha}(\tau, x^i, \varsigma) \to \tilde{\alpha}(\varsigma) \) with typical FLRW configurations. This can be performed following formulas

\[
\frac{ds^2}{d\tau^2} = \eta_3(\tau) \left( \eta_4(\tau) \frac{\eta_2(\tau)}{\eta_3(\tau)} \right) \theta_2[r^2]^2 + \tilde{h}_3[r^2y^2 + \eta_2^3(\tau) + \eta_4^2(\tau)] + \tilde{h}_4[r^2 + \eta_4^2(\tau)]
\]

such coordinates with flow parameter \( \tau \) and time-like \( \varsigma \) are useful for computations of geometric evolution and nonholonomic deformations of the FLRW metrics.

Considering a prime d-metric as a flat FLRW metric written in local coordinates \( \pi = [\pi^0(x^i, y^3, \varsigma)] = (x^i, y^3, \varsigma), \pi^2(x^i, y^3, \varsigma), \pi^3(x^i, y^3, \varsigma), \pi^4(x^i, y^3, \varsigma)] \), a d-metric \( (B.1) \) can be written in curved coordinate form \( \tilde{\alpha}^2(\pi) \), with local coordinated \( \pi^a \) using a prime cosmological scaling factor \( \tilde{\alpha}^2(\varsigma) \),

\[
ds^2 = \eta(x^k, \varsigma) \tilde{\alpha}^2(\tau, x^k, \varsigma) \frac{dr^2}{\tilde{\eta}_2(r^2, y^3, \varsigma)} + \tilde{h}_3(r^2, y^3, \varsigma) + \eta_2(r^2, y^3, \varsigma) + \tilde{h}_4(r^2, y^3, \varsigma) \rightarrow \tilde{\alpha}^2(\varsigma) \rightarrow \tilde{\alpha}^2(\varsigma).
\]

By definition, a quasi-FLRW configuration is stated by a diagonalized solution for a d-metric of type \( (B.3) \) when the integration functions and coordinates result in \( \tilde{\eta}_2(\tau, x^k, \varsigma) = 0 \).

Small nonholonomic deformations of such d-metrics can be parameterized \( \tilde{\eta}_2(\tau, x^k, \varsigma) \) (see below formulas relevant to \( (B.9) \)) by the polarization of the target cosmological factor, \( \eta(\tau, x^k, \varsigma) \) can be arbitrary one and not a value of \( 1 + \kappa x(\tau, x^k, \varsigma) \) with a small parameter \( \kappa \). We can consider a resulting scaling factor \( \tilde{\alpha}^2(\tau, x^k, \varsigma) \neq \eta(x, x^k, \varsigma) \tilde{\alpha}^2(x^i, \varsigma) \), with possible further re-parametrizations or limits to \( \tilde{\alpha}^2(\tau, \varsigma) \neq \eta(\tau, \varsigma) \tilde{\alpha}^2(\varsigma) \) encoding possible nonlinear off-diagonal and parametric interactions determined by systems of nonlinear PDEs.
B.3 Approximations for flows of target d-metrics

To study nonlinear properties of cosmological models is convenient to consider different types of parameterizations and approximations for nonholonomic deformations of a prime metric to a target d-metric (B.3) being under geometric flow evolution. For our purposes, there are important six classes of exact, or parametric, solutions which can be generated by a respective subclass of generating functions and/or generating sources and, for certain cases, making some diagonal approximations, or by introducing small ε-parameters. 

1. We can choose mutual re-parametrization of generating functions \((Ψ, Υ) \iff (Φ, Λ = const)\) and integrating functions when the coefficients of a family of target d-metric \(g_{αβ}(τ, ς)\) depend only a time-like coordinate \(ς\), when \(η(τ, x^k, ς) \rightarrow \tilde{η}(τ, ς)\), and \(a(τ, x^k, ς) \rightarrow \tilde{a}(τ, ς, ς)\) = \(\tilde{η}(τ, ς)\)\(\tilde{a}(σ)\). Respective families of linear quadratic elements (B.3) can be represented in the form

\[
ds^2(τ) = η(τ, ς)\tilde{a}(ς)[\tilde{η}(τ, ς)\tilde{g}_i(τ, ς)[dx^i]^2 + \hat{h}_3[dy^3 + \tilde{η}^3_k(τ, ς)\tilde{N}^3_k dx^k]^2] + \hat{h}_4[ds^2 + \tilde{η}^4_k(τ, ς)\tilde{N}^4_k dx^k]^2. (B.5)\]

With respect to coordinate bases, such families of cosmological solutions can be generic off-diagonal and could be chosen in some forms describing nonholonomic deformations of Bianchi cosmological models.

2. For FLRW prime configurations, we can consider families of generation functions and integration functions when the coefficients of a family of target d-metric and N-connection coefficients do not depend on space-like \(x^k\) being under geometric flow evolution and/or consider limits \(N^k_α → 0\). For such cases, we can transform families (B.5) into families of diagonal metrics

\[
ds^2(τ) = η(τ, ς)\tilde{a}(ς)[\tilde{η}(τ, ς)\tilde{g}_i(τ, ς)[dx^i]^2 + \hat{h}_3[dy^3]^2] + \hat{h}_4[dr^2]. (B.6)\]

3. Flow evolution with small parametric nonholonomic deformations of a prime metric into families of target off-diagonal cosmological solutions (B.3) can be approximated

\[
\tilde{η}(τ, x^k, ς) \simeq 1 + ε_i \tilde{χ}_1(τ, x^k, ς), η(τ, x^k, ς) \simeq 1 + ε_3 \chi(τ, x^k, ς), \tilde{η}^3_k(τ, x^k, ς) \simeq 1 + \tilde{ε}_3 \tilde{X}^3_k(τ, x^k, ς), \]

where small parameters \(ε_i, ε_3, \tilde{ε}_3\) satisfy conditions of type 0 \(\lesssim |ε_i|, |ε_3|, |\tilde{ε}_3|\) \(\ll 1\) and, for instance, \(\chi(τ, x^k, ς)\) is taken as a generating function. Such approximations restrict the class of generating functions subjected to nonlinear symmetries and may impose certain relations between such ε-constants and \(χ\)-functions. Corresponding quadratic line elements can be parameterized

\[
ds^2(τ) = [1 + ε_3 \chi(τ, x^k, ς)]\tilde{a}^2(ς)[1 + ε_i \tilde{χ}_1(τ, x^k, ς)]\tilde{g}_i[dx^i]^2 + \hat{h}_3[dy^3 + (1 + ε_3 \tilde{χ}^3_k(τ, x^k, ς))\tilde{N}^3_k dx^k]^2] + \hat{h}_4[ds^2 + (1 + ε^4_k \tilde{χ}^4_k(τ, x^k, ς))\tilde{N}^4_k dx^k]^2. (B.7)\]

Such τ-families of off-diagonal solutions define cosmological metrics with certain small independent fluctuations, for instance, a FLRW embedded self-consistently into a locally anisotropic background under geometric flow evolution.

4. We can consider also families of off-diagonal cosmological solutions with small parameters \(ε_i, ε_3, \tilde{ε}_3\) when the generating functions and d-metric and N-connection coefficients do not depend on space-like coordinates, which is typical for a number of cosmological models. For such approximations, the family of quadratic line element (B.7) transforms into

\[
ds^2(τ) = \frac{[1 + ε_3 \chi(τ, x^k, ς)]\tilde{a}^2(ς)[1 + ε_i \tilde{χ}_1(τ, x^k, ς)]\tilde{g}_i[dx^i]^2 + \hat{h}_3[dy^3 + (1 + ε_3 \tilde{χ}^3_k(τ, x^k, ς))\tilde{N}^3_k dx^k]^2] + \hat{h}_4[ds^2 + (1 + ε^4_k \tilde{χ}^4_k(τ, x^k, ς))\tilde{N}^4_k dx^k]^2. (B.8)\]

5. There are off-diagonal deformations, for instance, of a FLRW metric into a family of locally anisotropic cosmological solutions which can be constructed using only one small parameter \(κ = ε_i = ε_3 = \tilde{ε}_3\), and when the formulas (B.8) transform into

\[
ds^2(τ) = \frac{[1 + ε \chi(τ, x^k, ς)]\tilde{a}^2(ς)[1 + ε_i \tilde{χ}_1(τ, x^k, ς)]\tilde{g}_i[dx^i]^2 + \hat{h}_3[dy^3 + (1 + ε \tilde{χ}^3_k(τ, x^k, ς))\tilde{N}^3_k dx^k]^2] + \hat{h}_4[ds^2 + (1 + ε \tilde{χ}^4_k(τ, x^k, ς))\tilde{N}^4_k dx^k]^2. (B.9)\]

Such flows with ε-deformations can be generated by corresponding small ε-deformations of flows generating functions.
6. We can impose on families (B.9) the condition that the $\varepsilon$-deformations depend only on evolution temperature-like parameter and a time-like coordinate. This results in d-metrics

$$
\begin{align*}
\text{ds}^2(\tau) &= [1 + \varepsilon \chi(\tau, \varsigma)]\text{d} \omega^2(\varsigma)\left([1 + \varepsilon \tilde{\chi}_i(\tau, \varsigma)]\text{d}x^i\right)^2 \\
+ &h_3(\text{d}y^3 + (1 + \varepsilon \tilde{\chi}_k^3(\tau, \varsigma))N_k^4 \text{d}k^4)^2 + h_4(\text{d}x^4 + (1 + \varepsilon \tilde{\chi}_k^4(\tau, \varsigma))N_k^4 \text{d}k^4)^2
\end{align*}
$$

which can be considered as some ansatz used, for instance, for describing geometric evolution of quantum fluctuations of FLRW metrics.

In various classes of cosmological models with families of solutions with parametric $\varepsilon$-decompositions can be performed in a self-consistent form by omitting quadratic and higher order terms after a class of locally anisotropic solutions have been found for some general data ($\eta_{ia}, \eta_{ij}^a$). They are more general than approximate solutions found, for instance, for classical and quantum fluctuations of standard FLRW metrics and may involve flow evolution parameters of cosmological constants and generating functions and sources. For certain subclasses of generic off-diagonal solutions, we can consider that $\xi_i, \varepsilon a, \xi_i^a \sim \varepsilon$, when only one small parameter is considered for all coefficients of nonholonomic deformations.

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