Quantum Hall effect in a $p$-type heterojunction with a lateral surface quantum dot superlattice

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The quantization of Hall conductance in a $p$-type heterojunction with lateral surface quantum dot superlattice is investigated. The topological properties of the four-component hole wavefunction are studied both in $r$- and $k$-spaces. New method of calculation of the Hall conductance in a 2D hole gas described by the Luttinger Hamiltonian and affected by lateral periodic potential is proposed, based on the investigation of four-component wavefunction singularities in $k$-space. The deviations from the quantization rules for Hofstadter "butterfly" for electrons are found, and the explanation of this effect is proposed. For the case of strong periodic potential the mixing of magnetic subbands is taken into account, and the exchange of the Chern numbers between magnetic subbands is discussed.

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I. INTRODUCTION

Quantum states and transport of 2D Bloch electrons in a magnetic field are the phenomena which show extremely rich variety of physical and topological properties. The fascinating physical problems occurring here are caused by the mutual effects of the lattice periodic potential and the non-periodic vector potential of a uniform magnetic field. It is known that the former leads to the energy band structure while the latter tends to form discrete energy levels. The parameter which plays an important role in the problem is the magnetic flux $\Phi$ penetrating the lattice elementary cell. If the flux is equal to a rational number $p/q$ of flux quanta $\Phi_0 = \frac{2\pi\hbar c}{|e|}$ where $p$ and $q$ are mutually prime integers, it is possible to define a new set of translations on the lattice, called magnetic translations $1, 2$ for which the quasimomentum is a good quantum number. For example, if the vector potential of uniform magnetic field $B$ be chosen in Landau gauge $A = (0, Bx, 0)$, and $\Phi/\Phi_0 = p/q$, the simplest form of magnetic translations on a square lattice with the period $a$ is $x \rightarrow x + qa, y \rightarrow y + ma$ where $n$ and $m$ are integers. So, the magnetic elementary cell is $q$ times larger in $x$ direction, and the corresponding magnetic Brillouin zone (MBZ) is defined as following:

$$-\pi/qa \leq k_x \leq \pi/qa, \quad -\pi/a \leq k_y \leq \pi/a.$$  

(1)

When the quasimomentum runs over the MBZ (1), the energy varies in a band which is called a magnetic subband. When the amplitude of periodic potential $V_0$ is smaller than the cyclotron energy $\hbar\omega_c$ one can neglect the influence of neighboring Landau levels and may obtain the set of $p$ magnetic subbands originating from a single level $3$. If several electron Landau levels are taken into account, the periodic potential leads to the mixing between magnetic subbands originating from different levels $4$–$6$. However, the in-plane properties of the electron wavefunction remain the same both for coupled and uncoupled Landau levels. For example, one can see that, regardless to the particular form and the number of Landau levels taken into account, the electron wavefunction gains an additional phase under the magnetic translations. The relation between the translated and the initial wavefunctions in magnetic field is known as the generalized Bloch conditions (or Peierls conditions) $7$,$2$

$$\psi_{k_x,k_y}(x,y,z) = \psi_{k_x,k_y}(x + qa,y + a,z) \exp(-ik_xqa) \times \exp(-ik_ya) \exp(-2\piipy/a).$$  

(2)

It follows from (2) that the wavefunction gains the phase $2\pi p$ after the circulation along the boundary of the magnetic unit cell. As a result, the magnetic field forces the periodic part $u_k(r) = \exp(-ikr)\psi_k(r)$

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of the wavefunction to have $-p$ vorticity in the magnetic unit cell which indicates that there are at least $p$ zeros of the wavefunction per each magnetic cell\footnote{8}. This result has a topological nature because of its independance of the shape and the amplitude of periodic potential.

During last years the researchers have investigated several significant theoretical and experimental features of the systems where a 2D electron gas with additional periodic potential is in the regime of quantum Hall effect (QHE). If a single Landau level is splitted by a 2D periodic potential which has the area of elementary cell corresponding to $p/q$ of flux quanta penetrating the cell, the spectrum transforms to the system of $p$ magnetic subbands grouped near the unperturbed level. One might expect that each of magnetic subbands gives a Hall conductance $\sigma_H$ equal to $e^2/\hbar$, but according to Laughlin each subband must carry an integer multiple of the Hall current carried by the entire Landau level which is equal to $e^2/h$.

For the first time the confirmation of this rule which describes the quantization of Hall conductance in periodically modulated 2D systems has been obtained by Thouless, Kohmoto, Nightingale, and den Nijs in their pioneer paper\footnote{3}. They have studied in details a simple quasi-1D model of a strongly anisotropic lattice for which an explicit expression for $\sigma_H$ has been derived. If the Fermi energy falls into the $n$th gap of the $N$th splitted Landau level, the Hall conductance can be written as following:

$$\sigma = \frac{e^2}{h}(t_r + N - 1),$$

where $t_r$ is an integer obtained from the Diophantine equation

$$t_r p + s_r q = r.$$\footnote{4}

Equation (4) has integer solutions for some integer values of $s_r, |s_r| \leq p/2$. It was found further that the quantization of $\sigma_H$ in periodically modulated systems has a topological nature. Namely, the value of $\sigma_H$ for a fully occupied magnetic subband is related to the number and the type of the wavefunction singularities in $k$-space. Kohmoto has shown that these singularities determine the first Chern number for a particular magnetic subband which is, in units of $-e^2/h$, exactly the Hall conductance of this subband\footnote{9}.

An original method for calculation of the Hall conductance of 2D electron gas affected by weak periodical modulation has been proposed by Usov\footnote{9}. He has shown that the value of $\sigma_H$ is related to the winding numbers $S_m, m = 1, 2, \ldots$ of the wavefunction singularities in the extended MBZ (19). These singularities are direct consequence of a non-trivial topology of the MBZ or EMBZ, and a winding number $S_m$ is determined as the phase mismatch at the beginning and at the end of a circulation around the singular point $k_m$. As a result, the Hall conductance of a fully occupied subband $\alpha$ is given by:\footnote{9}

$$\sigma^\alpha_H = \frac{e^2}{h} \left[ \frac{1}{p} + \frac{q}{p} \sum_m S_m \right].$$\footnote{5}

The topological features of the problem have been discussed for the first time by Novikov\footnote{10}. Namely, the formation of $p$ magnetic subbands near the Landau level was treated as a fiber bundle of magnetic Bloch functions on a $T^2$-torus which is the MBZ (1). This problem has also been discussed by Avron, Seiler and Simon using homotopy theory\footnote{11}. The generalization of the proof of existance of the topological invariant to the situation where many-body interaction and substrate disorder are present has been obtained by Niu, Thouless, and Wu\footnote{12}. It should be mentioned that Simon\footnote{13} made a connection between the topological invariant and Berry’s geometrical phase factor\footnote{14} in the quantum adiabatic theorem. The Berry’s phase links the Hall conductance with a 2D integral over the MBZ (1) of the so called Berry curvature which is a $k$-dependent function, and it can be written in the spirit of the Kubo formula for the conductance\footnote{15\textsuperscript{-}17}.

Another approach to the calculation of Hall conductance is based on the St\v{s}eda formula\footnote{18}. If the Fermi level is located in the energy gap, the Hall conductance is given by

$$\sigma_H = e c \frac{\partial N(E)}{\partial B},$$\footnote{6}

where $N(E)$ is the number of states per unit area having energy lower than the gap energy. The formula (6) has been widely used for calculations of the Hall conductance of 2D electron gas with periodical modulation, even in the presence of Landau level coupling\footnote{5}. It was also applied to 3D systems\footnote{19\textsuperscript{-}21} were the generalization of (6) is known as the Kohmoto-Halperin-Wu formula\footnote{21}. However, the application of St\v{s}eda formula (6) to the systems with multi-component wavefunction is not justified and thus we shall focus on the analytical approach described above.

During last decade the number of experimental studies have been performed in order to investigate a 2D electron gas laterally modulated by a surface superlattice of quantum dots (antidots). Such a
system is convenient for investigation of both classical effects (commensurability of the lattice periods and cyclotron radius, transition to chaos, etc.) and of the energy spectrum consisting of magnetic subbands. For example, the oscillations of longitudinal magnetoresistance have been detected under the conditions where classical cyclotron radius $2R_c$ envelopes an integer number of antidots or numerous reflections from one antidot occur\cite{22,23}. It should be mentioned that the effects of randomly distributed impurities on collision broadening and transport scattering rate in 2D electron gas with periodical modulation, which consideration is of great importance for experimentators, have also been studied theoretically\cite{24}. The first experimental evidences of electron Landau levels splitted into the set of magnetic subbands have been obtained by the measuremets of the longitudinal magnetoresistance\cite{25}. Then, the Hall resistance in a laterally modulated 2D electron gas have been studied experimentally and the confirmations of subband energy spectrum have been found\cite{26}.

The experiments in $p$-type heterojunctions without periodic potential have also become possible due to the progress in technology which substantially improved the quality of $p$ channels in GaAs/AlGaAs heterojunctions\cite{33}. Thus, almost all intriguing phenomena found for 2D electron systems were also observed in 2D hole channels. It should be noted that their theoretical investigation has been carried out much earlier, for example, the studies of fractional quantum Hall effect in a 2D hole gas have been performed\cite{28}. In several recent publications the magnetotransport in 2D hole gas with lateral periodic modulation was studied\cite{29,30}. The corresponding quantum-mechanical model describing the hole subband spectra and the four-component magnetic Bloch wavefunctions has been derived by us recently, and the magnetooptical properties of laterally modulated 2D holes have also been studied\cite{31}. As soon as the transport experiments in laterally modulated 2D hole gas have started\cite{29}, it is now needful to derive a quantum-mechanical description of transport phenomena in such systems, and, in particular, of the quantum Hall effect.

In the current paper we present a new method of calculation of the Hall conductance in a 2D hole gas affected by lateral periodic potential. In Sec. II we briefly describe the magnetic hole Bloch states in a $p$-type heterojunction subjected to a magnetic field and affected by lateral periodic potential, which have been studied by us previously\cite{31}. In Sec. III we generalize the method derived by Usov\cite{9} for calculation of the Hall conductance in a system studied in Sec. II where the charged particle is described by a four-component eigenfunction of the Luttinger Hamiltonian. We find an unusual behavior of the Hall conductance as a function of the Fermi energy in comparence to the well-known dependance obtained for Hofstadter "butterfly"\cite{32} for the electrons. We propose an explanation of this effect by evaluating the role of the off-diagonal terms of the Luttinger Hamiltonian which provide a highly non-equidistant character of hole Landau levels. The quantization of the Hall conductance has been investigated both at weak and at strong periodic potential. In the latter case we’ve taken into account the magnetic subband mixing which leads to the exchange of the Chern numbers between magnetic subbands, changing their impact to the Hall conductance. We believe that the differences between the quantization of Hall conductance in $n$- and $p$-type heterojunctions which have been predicted by us can be observed experimentally. In Sec. IV we give the summary of our results.

II. MAGNETIC HOLE STATES IN LATERALLY MODULATED HETEROJUNCTION

The holes are studied near the upper edge of GaAs $p$-like valence band located at $k = 0$. We assume that the external magnetic field is oriented along (001) crystal direction which is perpendicular to the heterojunction plane ($xy$). The 2D holes are described in the $|J; m_J\rangle$ basis by the $4 \times 4$ Luttinger Hamiltonian where both magnetic field and confinement potential $V_h(z)$ of a single heterojunction are taken into account\cite{32,33}. In addition, the periodic potential $V(x, y)$ of a quantum dot superlattice is introduced which simplest form is\cite{6}

$$V(x, y) = V_0 \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a}.$$  \hspace{1cm} (7)

Here $a$ is the superlattice period and the case $V_0 < 0$ ($>0$) corresponds to the periodic electric potential generated by quantum dot (antidot) superlattice. The Hamiltonian for magnetic Bloch hole quantum states written in atomic units $\hbar = m_0 = 1$ takes the following form in the no-warping approximation\cite{31}:

$$H_L = \begin{pmatrix}
H_{11} & \gamma_3 \sqrt{\frac{\hbar eB}{c}} k_y a & 0 \\
H_{22} & 0 & -\gamma_3 \sqrt{\frac{\hbar eB}{c}} k_y a \\
H_{33} & 0 & \gamma_3 \sqrt{\frac{\hbar eB}{c}} k_y a \\
H_{44} & \gamma_3 \sqrt{\frac{\hbar eB}{c}} k_y a & 0
\end{pmatrix},$$  \hspace{1cm} (8)
where

\[ H_{11} = -(\gamma_1/2 - \gamma_2)k_x^2 - (eB/c) \left[ (\gamma_1 + \gamma_2) \left( a^+ a + \frac{1}{2} \right) + \frac{3}{2} \right] + V_h(z) + V(x, y), \]

\[ H_{22} = -(\gamma_1/2 + \gamma_2)k_x^2 - (eB/c) \left[ (\gamma_1 - \gamma_2) \left( a^+ a + \frac{1}{2} \right) - \frac{1}{2}\kappa \right] + V_h(z) + V(x, y), \]

\[ H_{33} = -(\gamma_1/2 + \gamma_2)k_x^2 - (eB/c) \left[ (\gamma_1 - \gamma_2) \left( a^+ a + \frac{1}{2} \right) + \frac{1}{2}\kappa \right] + V_h(z) + V(x, y), \]

\[ H_{44} = -(\gamma_1/2 - \gamma_2)k_x^2 - (eB/c) \left[ (\gamma_1 + \gamma_2) \left( a^+ a + \frac{1}{2} \right) - \frac{3}{2}\kappa \right] + V_h(z) + V(x, y). \]

The lower half of matrix (8) is obtained by Hermitian conjugation. In Eq.(8) \( a \) is an annihilation operator, \( e \) is a modulus of elementary charge, \( \gamma_1, \gamma_2, \gamma_3 \) and \( \kappa \) are the material bulk parameters which are well-known for GaAs. The hole energy is counted as negative from the upper edge of the valence band throughout the paper. In the effective mass approximation the \( k_z \) component of quasimomentum in (8) is replaced by its operator form \( k_z = -i\partial/\partial z \). This substitution is performed at \( B = 0 \) and \( V(x, y) = 0 \) which yields an infinite set of doubly degenerate heavy and light hole subbands energies and eigenfunctions \( \psi_j^z(z), \nu = 1, 2, \ldots \). The \( z \)-dependent envelope functions \( C_j^z(z) \) at finite \( B \) can be constructed as superpositions of zero-field functions \( \psi_j^z(z) \). Now let the periodic potential (7) be applied, corresponding to the rational magnetic flux through the elementary cell with the area \( S = a^2 \): \n
\[ \frac{BS}{\Phi_0} = \frac{BS}{2\pi\hbar c/e} = \frac{p}{q}. \]  

If the condition (9) is satisfied, any of four components \( \psi_j^3 \) of the vector of hole envelope functions becomes a magnetic Bloch function classified by \( k_x \) and \( k_y \) quantum numbers varying in the MBZ (1), and the total hole quantum state can be written as following:

\[ \psi_{k_x, k_y} (r) = \psi_{k_x, k_y}^1 (r) \left[ \frac{3}{2}, \frac{3}{2} \right] + \psi_{k_x, k_y}^2 (r) \left[ \frac{3}{2}, -\frac{1}{2} \right] + \psi_{k_x, k_y}^3 (r) \left[ \frac{3}{2}, 1 \right] + \psi_{k_x, k_y}^4 (r) \left[ \frac{3}{2}, -\frac{3}{2} \right]. \]  

The translational properties of each component of the envelope function (10) in \((xy)\) plane are the same as for the single-component electron wavefunction\(^{32}\). In particular, every component of (10) satisfies to the Peierls condition (2). Hence, one can write the components \( \psi_{k_x, k_y}^j (r) \) of (10) as a superposition of the Landau quantum states\(^{3-6,31}\), namely

\[ \psi_{k_x, k_y}^j (r) = \frac{1}{L a \sqrt{q}} \sum_\nu C_j^\nu (z) \sum_{N=1}^{p} d_{j\nu N} (k_x, k_y) \sum_{l=-L/2}^{L/2} u_N (x - x_0 - lqa - nqa/p) \times \]

\[ \times \exp \left( i k_x \left[ \frac{lqa + nqa}{p} \right] \right) \exp \frac{2\pi i y l p + n a}{a} \exp (i k y y). \]  

It should be mentioned that the set of basis functions for the hole states in magnetic subbands splitted from interacting hole Landau levels has more complicated structure than those for electrons (the latter is discussed, for example, in Refs.5,6). Namely, at the absence of periodic potential the four-component eigenvector of Luttinger Hamiltonian in a single subband of size quantization \( \nu \) has the form\(^{32}\):

\[ F_N^{\nu} = e^{ik_y} (C_1^\nu (z) u_{N-2}, C_2^\nu (z) u_N, C_3^\nu (z) u_{N-1}, C_4^\nu (z) u_{N+1}). \]  

In Eq.(12) \( u_N(x) \) is a harmonic oscillator wavefunction which vanish for negative values of its index. Below we shall discuss in details the structure of expression (11).

First, we should restrict ourselves to some limited number of size quantization subbands to be taken into account. In heterojunctions with typical hole concentration \( n_h = 5 \times 10^{11} \text{cm}^{-2} \) and depletion-layer density \( N_{\text{dep}} = 10^{15} \text{cm}^{-3} \) only the lowest hole subband of size quantization is occupied\(^{32,33}\). Hence, in the expression (11) for the hole state it seems to be relevant to consider only several subbands of size quantization neighbouring to the lowest one. Besides, for each subband of size quantization in Eq.(11)
we take into account only several Landau levels \( N \). During the investigation of hole states (11) in this Sec. we consider the first three subbands of size quantization which corresponds to two heavy- and one light-hole levels. The basis for the hole state (11) at \( V(x,y) = 0 \) consists of the following four-component vectors:

\[
e^{ik_y} (0, 0, 0, C_1(\nu)u_0), \quad e^{ik_y} (0, C_1(\nu)u_0, 0, C_1(\nu)u_1),
\]

\[
e^{ik_y} (C_1(\nu)u_0, C_2(\nu)u_2, C_3(\nu)u_1, C_4(\nu)u_3), \quad e^{ik_y} (0, 0, 0, C_4(\nu)u_0),
\]

\[
e^{ik_y} (0, 0, C_4(\nu)u_0, 0, C_4(\nu)u_1),
\]

\[(13)\]

where the upper index \( \nu = 1,2 \) labels the first or the second subband of size quantization, and the light-hole components for the second subband have been removed. It is easy to see that each term in (13) has the form of (12) with particular values of \( \nu \) and \( N \). It should be noted that the group of neighbouring hole levels may not be classified by a monotonous sequence \( N = -1,0,1,\ldots \) in Eq.(12) which is the fundamental difference between hole and electron Landau level (the latter are labeled by increasing index). In the presence of periodic potential \( V(x,y) \) for which the condition (9) is satisfied, each of Landau levels is splitted into \( p \) subbands. The summation over \( n = 1,\ldots,p \) in Eq.(11) corresponds to this splitting which is a general feature both for the electron and hole magnetic Bloch states.\(^3\)-\(^6\),\(^31\). To define the limits for indices \((\nu, N)\) in (11), one should fix the \(|J;m_j\rangle\) projection \( j = 1,2,3,4 \) and than take the sum of \( j\)-th components of all vectors in (13) with coefficients \( d_{j\nu Nn}(k_x,k_y) \). The total number of non-zero components in the set of basis vectors (13) is equal to 11 which is smaller than the total number of available components \( 4 \times 6 = 24 \) due to the vanish of those components of (12) which have negative indices. Hence, after substituting the total hole wavefunction (10) - (11) into the Schrödinger equation with the Hamiltonian (8) one obtains the \( 11p \times 11p \) eigenvalue problem for the \( 11p \) coefficients \( d_{j\nu Nn}(k_x,k_y) \) in every of 11p hole magnetic subbands.

In the Introduction we’ve mentioned that the wavefunction of a Bloch electron has at least \( p \) zeros per magnetic cell if the magnetic flux is equal to \( p/q \) of flux quanta, which is a consequence of the Peierls condition (2). It is interesting to generalize this result for a multi-component wavefunction. Namely, if \( \theta^j_k \) denotes the phase of the \( j\)-th periodic part \( u^j_k(r) = \exp(-ikr)\psi^j_k(r) \) of the hole component \( \psi^j_k(r) \) defined by (11), one can introduce the vorticity \( \Gamma_j \) for each component as following:

\[
\Gamma_j = \frac{1}{2\pi} \oint \frac{\partial \theta^j_k(x,y)}{\partial l} dl
\]

where the integration path is taken along the boundary of magnetic unit cell in the counterclockwise direction. It was mentioned above that the condition (2) is valid for every component of the vector of hole envelope functions. So, it is not surprising that the vorticity (14) is equal for all hole components:

\[
\Gamma_j = -p, \quad j = 1,2,3,4.
\]

However, the position inside magnetic cell and the total number of zeros can be different for each of the \(|J;m_j\rangle\) components due to their particular form of basic functions and coefficients in (11). It should be mentioned also that the total number of zeros per magnetic cell can be greater than \( p \) because of the opposite signs of some vorticities. All of these results are shown on Fig.1 where the probability distributions of all four hole envelope functions are plotted at \( k_x = k_y = 0 \) in a non-overlapped magnetic subband at \( p/q = 5 \). The zeros are shown as black circles of different size corresponding to their order (see the inset). One can see that different hole components have the different number and (in general) the different order of zeros. Some of the zeros are located on the sides and in the corners of magnetic cell which is reflected by semi- and quarter-circle areas. It should be mentioned that despite of the different positions and the number of zeros for each of \(|J;m_j\rangle\) components, the total vorticity per one magnetic cell (15) is equal for all components at any \((k_x,k_y)\) in all magnetic subbands. This result reflects the topological nature of the wavefunction vorticity (14).
FIG. 1. Probability distributions for the $|J; m_J\rangle$ hole components $J = 1 - 4$ (a - d) in a magnetic subband are shown at $p/q = 5$ and $k_x = k_y = 0$. Darker areas correspond to the greater values of the wavefunction modulus. The positions of wavefunction zeros are marked as black circles with diameter proportional to their order. The zeros located on the sides and in the corners of magnetic cell which are plotted by semi- and quarter-circle areas, respectively.
III. QUANTIZATION OF HALL CONDUCTANCE

The Hall conductance $\sigma_H$ is quantized in units of $e^2/h$ as soon as the Fermi energy lays in the energy gap. The value of $\sigma_H$ is determined by the sum of partial conductances of filled magnetic subbands. Thus, we shall study at first the Hall conductance of one fully occupied magnetic subband $\alpha$. In the absence of disorder and at zero temperature, its contribution to Hall conductance is given by $^{3,24,8,9}$

$$\sigma_H^\alpha = \frac{e^2}{\pi^2h} \int \text{Im} \left( \frac{\partial u_k}{\partial k_y} \frac{\partial u_k}{\partial k_x} \right) d^2k$$

(16)

where $u_k = \Psi_{k_x,k_y}(r)e^{-ikr}$ is the periodic part of the Bloch function in the current subband $\alpha$. For the four-component hole state (10) one obtains from (16) the following expression for $\sigma_H$:

$$\sigma_H^\alpha = \frac{e^2}{\pi^2h} \sum_{j=1}^{4} \int \text{Im} \left( \frac{\partial u_k^{(j)}}{\partial k_y} \frac{\partial u_k^{(j)}}{\partial k_x} \right) d^2k$$

(17)

where $u_k^{(j)} = \Psi_{k_x,k_y}^{(j)}(r)e^{-ikr}$ and $\Psi_{k_x,k_y}^{(j)}(r)$ is defined by (11). In this Section we shall focus on the magnetic subbands originating from the lowest subband of size quantization and thus will ignore the second heavy-hole subband, omitting the last three vectors in (13). This will reduce the number of non-zero components from 11 to 7. After substituting $u_k^{(j)}$ into (16) and taking into account the orthogonality and normalization of the basis functions in (11), one may express the Hall conductance (17) through the partial derivations of the components $d_{\nu \nu' \nu''}(k_x,k_y)$ describing the quantum state. For brevity, in the following we shall replace the set of indices $(\nu \nu' \nu'')$ by a single index $n = 1, \ldots, 7p$ which runs sequentially all the required values. We get

$$\sigma_H = \frac{e^2}{\pi^2h} \int \text{Im} \left[ \frac{i\hbar^2}{2}\sigma_H^\alpha + \sum_{n=1}^{7p} \frac{\partial d_n}{\partial k_y} \frac{\partial d_n}{\partial k_x} \right] d^2k$$

(18)

The expression (18) has been initially derived $^9$ for the Hall conductance of a magnetic subband splitted from a single electron Landau level. We claim that (18) is valid for the case of several interacting electron or hole levels as long as the spectrum is non-degenerate. The differences with the single-level problem is only in the size of the matrix equation which is now equal to $N \cdot p$ instead of $p$. The orthogonality and normalization of the basis functions used for construction of the hole envelope function (11) is of the same kind as the basis set for a single-level problem. This feature allows us to expand directly the approach used by Usov for the case of several interacting levels. So, we use the expression (18) for calculations of the Hall conductance for the hole Landau levels which are coupled even at zero superlattice potential by the off-diagonal elements of the Luttinger Hamiltonian.

It is evident from (18) that for calculation of the Hall conductance one should study first the analytical properties of coefficients $d_n$, as the functions of quasimomentum. First, one can transform the 2D integral (18) into a 1D contour integral. In order to simplify the integration and to reduce it to the summation of the winding numbers over the singularities (see the right side of (5)), one has to introduce the extended magnetic Brillouin zone (EMBZ) which is derived from the previously determined magnetic Brillouin zone (1) by extending it $p/q$ times in the $k_y$ direction:

$$-\pi/qa \leq k_x \leq \pi/qa, \quad -p\pi/qa \leq k_y \leq p\pi/qa$$

(19)

It was shown by Usov $^9$ that the integration along the "boundaries" of the EMBZ (19) gives no impact to the value of $\sigma_H$ which is explicitly determined only by the contour integrals around the singularities (winding numbers). We shall briefly repeat the outline of the derivation of this result. One can choose a representaion for which one of the components of the vector $d = (d_1, \ldots, d_{7p})$, say, $d_1$, is real. The singular points $k_m$, $m = 1, 2, \ldots$ are thus determined as points where $d_1(k_m) = 0$. The other components will be written as $d_j = |d_j| \exp(i\theta_j)$, $j = 2, 3, \ldots$. The winding numbers appear to be equal for any of these components. To be specific, let us consider the calculation of the winding numbers for $d_2$. At a particular singular point $k_m$ its winding number $S_m$ can be calculated as an algebraic sum of rotations (modulo $2\pi$) of the vector with the components $(Re d_2, Im d_2)$. A typical behavior of $d_1$ and both $Re d_2$ and $Im d_2$ in two magnetic subbands is illustrated on Figures 2-3 for the magnetic flux $p/q = 3/2$, $a = 80$ nm and the amplitude of periodic potential $V_0 = 0.7$ meV which corresponds to the case of non-overlapped magnetic subbands.
FIG. 2. Typical behavior of two components of the eigenvector $\mathbf{d}$ describing the hole quantum state in the representation with real $d_1$. The eigenvector is taken for a magnetic subband with the Hall conductivity equal to $+1$ in units $e^2/h$. The magnetic flux $p/q = 3/2$, $a = 80$ nm and the amplitude of periodic potential $V_0 = 0.7$ meV which corresponds to the case of non-overlapped magnetic subbands. Darker areas correspond to the greater values of the $d_1,2$ modulus, and the negative parts are shaded with lines. The contours $L$ and the direction of circulation around the singularity are shown schematically.

The contours $L_{1,2}$ of circulations around the singularities are shown schematically. It is evident from Figures 2–3 that while approaching the singular point where $d_1 = 0$ which is marked by black dot, both real and imaginary parts of $d_2$ have different limits depending on the direction in $(k_x, k_y)$ plane and thus do not have a true limit in this point. The impact of the component $d_j$ at a singular point $k_m$ to the Hall conductance is proportional to $|d_j|^2 S'_{j,m}$ where $S'_{j,m}$ is the winding number for the component $d_j$. It was shown⁸ that for all components $j = 2, 3, \ldots, N_p$ the winding numbers are equal, $S'_{j,m} = S_m$. So, the summation over all components gives the impact to the Hall conductance provided by a singular point $k_m$:

$$\sum_j |d_j|^2 S'_{j,m} = S_m \sum_j |d_j|^2 = S_m,$$  \hspace{1cm} (20)

where we’ve used the normalization of the vector $\mathbf{d} = (d_1, \ldots, d_{N_p})$. As soon as the winding numbers are calculated, the Hall conductance of a particular magnetic subband is given by Eq.(5). By examining the expression (5), one can mention that the first term in the square brackets is just the contribution of one out of $p$ subbands to the free-electron Hall conductivity. The remaining term in (5) is stipulated by the presence of periodic potential and by a non-trivial topology of the wavefunction in the EMBZ (19). At least one singular point can be found for every magnetic subband when $k$ runs over EMBZ. As a result, the expression in brackets in (5) is always an integer, which is the first Chern class of magnetic subband⁸.
FIG. 3. Same as Fig.3 but for a subband with Hall conductivity equal to $-1$.

The quantization of $\sigma_H$ as a function of the number of filled magnetic subbands (or, equivalently, of the position of the Fermi level) is shown on Figures 4 - 5 both for non-overlapped and overlapped magnetic subbands. When the amplitude $V_0$ of the periodic potential (7) is smaller than the distance $\Delta E_{12}$ between neighbouring Landau levels, none of the subbands are overlapped (see Fig.4) and possible deviations in the quantization of $\sigma_H$ from the sequences obtained for the Hofstadter "butterfly" are caused by the non-equidistant character of hole Landau levels which leads to hole subband spectra with another structure than the "butterfly" for electrons. The quantization of $\sigma_H$ in the gaps between non-overlapped subbands as a function of Fermi energy is plotted on Fig.4. It should be noted that when the Fermi level is swept through a subband centered at $E_n$ (or through a region of overlapped subbands, see below), the Hall conductivity interpolates smoothly between the adjacent quantized values. These values are shown by solid lines on Figures 4 - 5 and the interpolation curves are the dashed lines.
If the amplitude $V_0$ is increased, the neighbouring magnetic subbands from different hole Landau levels can touch each other at some point in the MBZ. This touch means that the degeneracy of the spectrum has occurred, and the application of the expression (5) is invalid. However, the further increase of $V_0$ leads to the repulsion of the touched subbands and to the breaking of the degeneracy. So, one can use (5) at higher $V_0$ when the condition $V_0 > \Delta E$ is satisfied and some of magnetic subbands are overlapped but the spectrum remains to be non-degenerate. The spectrum for $V_0 = 3$ meV is shown on the inset of Fig.5. One can see that the number and the maximum width of gaps on Fig.5 has decreased with respect to the system of non-overlapped subbands on Fig.4 which will reduce the number and maximum width of Hall plateaus. For convenience, on Fig.5 we’ve labeled the remaining gaps and the corresponding Hall plateaus by numbers. Again the dashed line serves as a guide to the eye and it qualitatively shows the impact to the Hall conductance provided by those subbands which are fully occupied at the current position of $E_F$ when it is swept through the region of overlapped subbands. The unusually high and low, even negative values of $\sigma_H$ are marked by arrows. It should be stressed that the differences between the behavior of $\sigma_H$ on Figures 4 and 5 are provided by only two changes in the $\sigma^\alpha_H$ for subbands $\alpha = 4$ and $\alpha = 8$ (see Fig.4). We found that these subbands have been degenerated at some intermediate values of $V_0$ which are greater than on Fig.4 but lower than on Fig.5. According to the topological point of view\textsuperscript{8,10}, the subband touches have caused the exchange of the Chern classes $\Delta c = \pm q$ between these subbands where $q = 2$ in our examples. It can be easily seen that such exchange ($−2$ for subband 4 and $+2$ for subband 8) exactly transforms the quantization shown on Fig.4 to the dependence on Fig.5. We suppose that the novell features of the quantization of Hall conductance in laterally modulated hole gas which have been discussed in this paper can be detected in transport experiments.
FIG. 5. (Top inset) Overlapped hole magnetic subbands originating from the same hole Landau levels as on Fig. 4 but splitted by stronger periodic potential with $V_0 = 3$ meV. The numbers label energy gaps. (Bottom) Quantization of $\sigma_{xy}$ in the energy gaps labeled on the inset as a function of the Fermi level position. The deviations from the quantization sequence for Hofstadter "butterfly" are marked by arrows.

IV. SUMMARY

We presented a new method of calculation of the Hall conductance in a 2D hole gas affected by lateral periodic potential which is a generalization of the method derived by Usov\textsuperscript{9} for a system where the charged particle is described by a four-component eigenfunction of the Luttinger Hamiltonian. An unusual behavior of the Hall conductance in comparance to the well-known dependance for Hofstadter "butterfly" as a function of the Fermi energy is found for holes. The explanation of this effect is proposed by evaluating the role of the off-diagonal terms of the Luttinger Hamiltonian which provide a highly non-equidistant character of hole Landau levels. The quantization of the Hall conductance is investigated both at weak and at strong periodic potential. In the latter case the magnetic subband mixing is taken into account which leads to the exchange of the Chern numbers between magnetic subbands, changing their impact to the Hall conductance. The experimental observation of the differences between the quantization of Hall conductance in n- and p-type heterojunctions is discussed.

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