Curvature estimates for spacelike graphic hypersurfaces in Lorentz–Minkowski space $\mathbb{R}^{n+1}_1$

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Abstract  
In this paper, we can obtain curvature estimates for spacelike admissible graphic hypersurfaces in the $(n+1)$-dimensional Lorentz–Minkowski space $\mathbb{R}^{n+1}_1$, and through which the existence of spacelike admissible graphic hypersurfaces, with prescribed 2-nd Weingarten curvature and Dirichlet boundary data, defined over a strictly convex domain in the hyperbolic plane $\mathbb{H}^n(1) \subset \mathbb{R}^{n+1}_1$ of center at origin and radius 1, can be proven.

KEYWORDS  
curvature estimates, Dirichlet boundary condition, Lorentz–Minkowski space, spacelike hypersurfaces

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1 | INTRODUCTION

Throughout this paper, let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional $(n \geq 2)$ Lorentz–Minkowski space with the following Lorentzian metric:

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$ 

In fact, $\mathbb{R}^{n+1}_1$ is an $(n+1)$-dimensional Lorentz manifold with index 1. Denote by

$$\mathbb{H}^n(1) = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_1 | x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\},$$

which is exactly the hyperbolic plane of center $(0,0,\ldots,0)$ (i.e., the origin of $\mathbb{R}^{n+1}_1$) and radius 1 in $\mathbb{R}^{n+1}_1$. Clearly, from the Euclidean viewpoint, $\mathbb{H}^2(1)$ is one component of a hyperboloid of two sheets.

Assume that

$$\mathcal{C} := \{(x, u(x)) | x \in M^n \subset \mathbb{H}^n(1)\}$$ (1)

is a spacelike graphic hypersurface defined over some bounded piece $M^n \subset \mathbb{H}^n(1)$, with the boundary $\partial M^n$, of the hyperbolic plane $\mathbb{H}^n(1)$, where $\sup_{M^n} \frac{|Du|}{u} \leq \rho < 1$. Let $x$ be a point on $\mathbb{H}^n(1)$, which is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. By the abuse of notations, let $\partial_i$ be the corresponding coordinate vector fields on $\mathbb{H}^n(1)$ and $\sigma_{ij} = g_{\mathbb{H}^n(1)}(\partial_i, \partial_j)$ be the induced Riemannian metric on $\mathbb{H}^n(1)$. Of course, $\{\sigma_{ij}\}_{i,j=1,2,\ldots,n}$ is also the met-
Gric on $M^n \subset \mathcal{H}^n(1)$. Denote by $u_i := D_i u$, $u_{ij} := D_j D_i u$, and $u_{ijk} := D_k D_j D_i u$ the covariant derivatives of $u$ w.r.t. the metric $g_{\mathcal{H}^n(1)}$, where $D$ is the covariant connection on $\mathcal{H}^n(1)$. Let $\nabla$ be the Levi–Civita connection of $G$ w.r.t. the metric $g := u^2 g_{\mathcal{H}^n(1)} - dr^2$ induced from the Lorentzian metric $\langle \cdot, \cdot \rangle_L$ of $\mathbb{R}^{n+1}_1$. Clearly, the tangent vectors of $G$ are given by

$$X_i = (1, Du) = \partial_i + u_i \partial_r, \quad i = 1, 2, \ldots, n.$$  

The induced metric $g$ on $G$ has the form

$$g_{ij} = \langle X_i, X_j \rangle_L = u^2 \sigma_{ij} - u_i u_j,$$

its inverse is given by

$$g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} + \frac{u^i u^j}{u^2 u^2} \right),$$

and the future-directed timelike unit normal of $G$ is given by

$$\nu = \frac{1}{\nu} \left( \partial_r + \frac{1}{u^2} u^i \partial_i \right),$$

where $u^i := \sigma^{ij} u_j$ and $\nu := \sqrt{1 - u^{-2} |Du|^2}$ with $Du$ the gradient of $u$. Of course, in this paper, we use the Einstein summation convention—repeated superscripts and subscripts should be made summation from 1 to $n$. The second fundamental form of $G$ is

$$h_{ij} = -\langle \nabla_{X_j} X_i, \nu \rangle_L = \frac{1}{\nu} \left( u_{ij} + u \sigma_{ij} - \frac{2}{u} u_i u_j \right),$$

(2)

with $\nabla$ the covariant connection in $\mathbb{R}^{n+1}_1$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the principal curvatures of $G$, which are actually the eigenvalues of the matrix $(h_{ij})_{n \times n}$ w.r.t. the metric $g$. The so-called $k$-th Weingarten curvature at $X = (x, u(x)) \in G$ is defined as

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$  

(3)

Remark 1.1.

(1) Clearly, $\sigma_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is actually the mean curvature $H$ of $G$ at $X$, while $\sigma_n = \lambda_1 \lambda_2 \cdots \lambda_n$ denotes the Gauss–Kronecker curvature of $G$ at $X$. Since $G$ is a spacelike hypersurface in $\mathbb{R}^{n+1}_1$, when $n = 2$ the intrinsic Gauss curvature of $G$ at $X$ should be $-\sigma_n$.

(2) As explained and shown by López [24], (in suitable orientation) the mean curvature $H$ of a surface in $\mathbb{R}^3_1$ satisfies $H = \text{tr}(A)$, where $\epsilon = -1$ if the surface is spacelike while $\epsilon = 1$ if the surface is timelike, and $\text{tr}(A)$ stands for the trace of the second fundamental form $A$. However, in his setting, each component $h_{ij}$ of $A$ has exactly the opposite sign with the one we have used here (i.e., $h_{ij} = \langle \nabla_{X_j} X_i, \nu \rangle_L$ in [24]). But, if we use López’s setting here, for the spacelike graphic hypersurface $G$, the mean curvature $H$ is the same with our treatment here since $\epsilon = -1$ and $H = -\text{tr}(A)$. Hence, there is no essential difference between our setting here and López’s. One might find that for curves and surfaces in $\mathbb{R}^3_1$, López’s setting is more convenient than the one we have used here. Both settings have been used by us in previous works—see, for example, [12, 14] for the setting here and [10, 13] for López’s.

(3) In [9], Gao and Mao first considered the evolution of spacelike graphic hypersurface, defined over a convex piece of $\mathcal{H}^n(1)$ and contained in a time cone in $\mathbb{R}^{n+1}_1$ ($n \geq 2$), along the inverse mean curvature flow (IMCF for short) with zero Neumann boundary condition (NBC for short), and showed that this flow exists for all the time, the spacelike graphic property of the evolving hypersurfaces is preserved along flow, and after suitable rescaling, the rescaled hypersurfaces converge to a piece of the spacelike graph of a constant function defined over $\mathcal{H}^n(1)$ as time tends to infinity. Recently, the anisotropic versions of this conclusion (both in $\mathbb{R}^{n+1}_1$ and more general Lorentz manifold $M^n \times \mathbb{R}$) have been solved (see [10, 11]). Besides, the lower dimensional case has also been discussed (see [13]). If the IMCF in [9] was
replaced by the inverse Gauss curvature flow (IGCF for short), we can obtain the long-time existence and the asymptotical behavior of the new flow (see [12]). There is one more thing we would like to mention here—as revealed in (3) of [9, Remark 1.1], although a new setting for the mean curvature \( k \) (different from López’s mentioned in (2) above) has been used therein, but for the flow problem considered in [9] there would not have been an essential difference between two settings if opposite orientations were used for the timelike unit normal vector in the IMCF equation. This kind of phenomena happen in the research of differential geometry. For instance, one might find that there at least exist two definitions for the (1,3)-type curvature tensor on Riemannian manifolds, which have opposite sign, but essentially same fundamental equations (such as the Gauss equation, the Codazzi equation, the Ricci identity) can be derived provided necessary settings have been made.

(4) One can easily find that boring trouble on sign would happen if one uses López’s setting in [24] (for the second fundamental form, the mean curvature, etc.) to deal with the PCPs in \( \mathbb{R}^{n+1} \). Based on this reason, we prefer to go back to our treatment in [14] whose definitions for \( h_{ij} \) and \( H \) are the same with ones here. Through this philosophy, we use the setting \( \sigma_\alpha = \lambda_1 \lambda_2 \cdots \lambda_n \) for the Gauss–Kronecker curvature in our study of IGCF with zero NBC in \( \mathbb{R}^{n+1} \). Of course, in this situation, the orientation for the timelike unit normal vector in the flow equation should be past-directed.

We also need the following conception:

**Definition 1.2.** For \( 1 \leq k \leq n \), let \( \Gamma_k \) be a cone in \( \mathbb{R}^n \) determined by
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_\lambda(\lambda) > 0, \ l = 1, 2, \ldots, k \}.
\]
A smooth spacelike graphic hypersurface \( \mathcal{G} \subset \mathbb{R}^{n+1} \) is called \( k \)-admissible if at every point \( X \in \mathcal{G}, (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k \).

In this paper, we investigate the curvature estimates and then the existence of solutions for a class of nonlinear partial differential equations (PDEs for short) given as follows:
\[
\begin{align*}
\sigma_k &= \psi(x, u, \vartheta), & x \in M^n \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}, & k = 1, 2, \ldots, n, \\
u &= \varphi, & x \in \partial M^n,
\end{align*}
\]  
(4)

where \( \psi \), depending on \( X, \vartheta := \langle X, \nu \rangle_L \), and \( \varphi \) are functions defined on \( M^n \). The regularity requirements on functions \( \psi \) and \( \varphi \) would be mentioned in curvature estimates below. Obviously, by (2), we know that \( \sigma_k \) in (4) should be determined by the graphic function \( u \) and its derivatives. Based on this fact, if necessary, sometimes we also write \( \sigma_k \) as \( \sigma_k[u] \) to emphasize this connection. This simplification will be used similarly in the sequel.

**Remark 1.3.**

(1) Clearly, (4) is a prescribed curvature problem (PCP for short) with Dirichlet boundary condition (DBC for short). It is reasonable and feasible to consider the PCP
\[
\sigma_k = \psi(x, u, \vartheta)
\]  
(5)
over \( \mathcal{H}^n(1) \) or a piece of it. In fact, (i) if \( k = 1 \) and \( \psi = a \) for some positive constant \( a > 0 \) in (5), then \( \mathcal{G} \) should be \( \mathcal{H}^n(\frac{\lambda}{a}) \) or a piece of it; (ii) if \( k = n \) and \( \psi = a > 0 \) in (5), then \( \mathcal{G} \) should be \( \mathcal{H}^n(\frac{\sqrt{\lambda}}{a}) \) or a piece of it. Obviously, in these two cases, the graphic function \( u(x) \) should be constant. Naturally, one might try to know more except these relatively simple examples.

(2) Assume that \( \Omega \subset \mathbb{R}^n \) is smooth bounded and strictly convex, and that \( \psi \) is a smooth positive function. For spacelike graphic hypersurfaces \( \mathcal{G} := \{(x, u(x)) \in \mathbb{R}^{n+1} | x \in \Omega \} \) defined over \( \Omega \subset \mathbb{R}^n \), Huang [21] considered the following PCP:
\[
\begin{align*}
\sigma_k &= \psi(x, u, w), & x \in \Omega, \\
u &= \varphi, & x \in \partial \Omega,
\end{align*}
\]  
(6)
where \( w = 1 / \sqrt{1 - |Du|^2} \), and showed the existence of solutions to (6) provided \( \varphi \) is spacelike, affine, and \( \psi_1^1(x, u, w) \) has extra growth assumption and convexity in \( w \). It is easy to know that the future-directed timelike unit normal vector \( \tilde{\nu} \) of spacelike graphic hypersurfaces \( \tilde{G} \) therein should be

\[
\tilde{\nu} = \frac{\partial_x + u \delta_{ij} \partial_j}{\sqrt{1 - |Du|^2}} = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}}.
\]

and \( w = -\langle \varepsilon_{n+1}, \tilde{\nu} \rangle_L \), with \( \varepsilon_{n+1} = (0, \ldots, 0, 1) \) the unit basis of the \( x_{n+1} \)-axis of \( \mathbb{R}^{n+1} \). This interesting fact leads to an observation:

• Although a spacelike graphic hypersurface defined over \( M^n \subset \mathcal{H}^n(1) \) is also spacelike graphic over \( \Omega \subset \mathbb{R}^n \) and vice versa, since there exists at least a diffeomorphism between \( \Omega \) and \( M^n \). However, \( w \) cannot be equal to \( \hat{\vartheta} \) identically by this diffeomorphism. Therefore, essentially the PCP (4) should be different from Huang’s (6).

(3) The PCPs (with or without boundary condition) in Euclidean space or even more general Riemannian manifolds were extensively studied—see, for example, [5, 6, 19, 25] and the references therein for details. Affected by the study of geometry of submanifolds, it is natural to consider PCPs in the pseudo-Riemannian context. In fact, except Huang’s interesting result mentioned above, many other important results on PCPs in pseudo-Riemannian manifolds have been obtained. For instance, in the Lorentz–Minkowski space or general Lorentz manifolds, Bartnik [2], Bartnik-Simon [3], and Gerhardt [15, 16] solved the Dirichlet problem for the prescribed mean curvature equation, Delanoë [7] and Guan [18] considered the prescribed Gauss–Kronecker curvature equation with DBC, while Bayard [4], Gerhardt [17], and Urbas [26] worked for the prescribed scalar curvature equation.

For the PCP (4), first, we can get the following curvature estimate:

**Theorem 1.4.** Suppose that \( u \in C^4(M^n) \cap C^2(\overline{M^n}) \) is a spacelike, \( k \)-admissible solution of the PCP (4), \( 0 < \varphi \in C^\infty(\overline{M^n}) \), and that \( \psi_1^1(X, \vartheta) \) is convex in \( \vartheta \) and satisfies

\[
\frac{\partial \psi_1^1(X, \vartheta)}{\partial \vartheta} \cdot \vartheta \geq \psi_1^1(X, \vartheta) \quad \text{for fixed } X \in \mathcal{G}.
\] (7)

Then, the second fundamental form \( A \) of \( \mathcal{G} \) satisfies

\[
\sup_{M^n} ||A|| \leq C \left( 1 + \sup_{\partial M^n} ||A|| \right).
\]

(8)

where \( C \) depends only on \( n, ||\varphi||_{C^1(\overline{M^n})}, ||\psi||_{C^2(\overline{M^n} \times [\inf_{\partial M^n} u, \sup_{\partial M^n} u] \times \mathbb{R})} \).

**Remark 1.5.** It is not hard to find some \( \varphi \) satisfying assumptions in Theorem 1.4. For instance, (i) \( \psi(x, u, \vartheta) = (-\vartheta)^p h(x, u) \) for \( p \geq k \); (ii) \( \psi(x, u, \vartheta) = e^{-p \vartheta} h(x, u) \) for \( p \geq k \).

An interior curvature estimate can be obtained in the case that \( \varphi \) is affine and satisfies the strict version of (7).

**Theorem 1.6.** Suppose that \( u \in C^4(M^n) \cap C^2(\overline{M^n}) \) is a spacelike, \( k \)-admissible solution of the PCP (4), \( 0 < \varphi \in C^\infty(\overline{M^n}) \), and that \( \psi_1^1(X, \vartheta) \) is convex in \( \vartheta \) and satisfies

\[
\frac{\partial \psi_1^1(X, \vartheta)}{\partial \vartheta} \cdot \vartheta > \psi_1^1(X, \vartheta) \quad \text{for fixed } X \in \mathcal{G}.
\]

(9)

Furthermore, suppose that \( M^n \subset \mathcal{H}^n(1) \) is \( C^2 \) and uniformly convex, and that \( \varphi \) is spacelike and affine. If \( u \in C^4(M^n) \) is a spacelike, \( k \)-admissible solution of the PCP (5), then
\[
\sup_{\mathcal{M}^n} |A| \leq C(\mathcal{M}^n)
\]
for any \(\mathcal{M}^n \subset M^n\), where \(C(\mathcal{M}^n)\) depends only on \(n, \zeta, M^n, \text{dist}(\mathcal{M}^n, \partial M^n)\), \(\|\varphi\|_{C^{1,1}(\mathcal{M}^n)}\) and \(\|\psi\|_{C^{1,1}(\mathcal{M}^n \times \inf_{\partial M^n} u, \sup_{\partial M^n} u) \times \mathbb{R})}\).

**Remark 1.7.**

(1) The positive constant \(\zeta\) here will be determined clearly in the proof of Theorem 1.6 in Section 4.2.
(2) Here, \(\text{dist}(\mathcal{M}^n, \partial M^n)\) characterizes the Riemannian distance between \(\mathcal{M}^n\) and \(\partial M^n\), and of course, depends on the induced metric \(\{\sigma_{i,j}\}_{i,j=1,2,\ldots,n}\) on \(\mathcal{H}^n(1)\).

Combining the above curvature estimates and the boundary \(C^2\)-estimates proven in the sequel, together with the method of continuity, we can get the existence and uniqueness of solutions to the PCP (4) with \(k = 2\) as follows:

**Theorem 1.8.** Suppose that \(M^n\) is a smooth bounded domain of \(\mathcal{H}^n(1)\) and is strictly convex, while \(\psi\) is a smooth positive function, \(\psi_u \leq 0\), and \(\psi^\frac{1}{2}\) is convex in \(\vartheta\) satisfying

\[
\frac{\partial \psi^\frac{1}{2}(x, u, \vartheta)}{\partial \vartheta} \cdot \vartheta \geq \psi^\frac{1}{2}(x, u, \vartheta) \quad \text{for fixed } (x, u) \in M^n \times \mathbb{R}.
\]

Then, for any spacelike, affine function \(\varphi\), there exists a uniquely smooth spacelike, \(2\)-admissible graphic hypersurface \(\mathcal{G}\) (defined over \(M^n\)) with the prescribed curvature \(\psi\) and Dirichlet boundary data \(\varphi\).

**Remark 1.9.** In Section 3.1, we use the comparison principle, so we add the condition \(\psi_u \leq 0\) in order to obtain the existence of solution for the PCP (4) with \(k = 2\). It is not hard to find some \(\psi\) satisfying assumptions in Theorem 1.8. For instance, (i) \(\psi(x, u, \vartheta) = (-\vartheta)^p h(x, u), h_u \leq 0\) for \(p \geq k\); (ii) \(\psi(x, u, \vartheta) = e^{-p\vartheta^2} h(x, u), h_u \leq 0\) for \(p \geq k\).

**Remark 1.10.**

(1) In the PCP (4), if \(\sigma_k = \sigma_k(\lambda(A))\) was replaced by \(\sigma_k(\lambda(A))\)

\[
\sigma_k(\lambda(A)) \quad \text{with } 2 \leq k \leq n, 0 \leq l \leq k - 2,
\]

then the a priori estimates for solutions to the corresponding Dirichlet problem of a class of Hessian quotient equations can be obtained under suitable assumptions, which leads to the existence and uniqueness of solutions for some \(k\)—see [8] for details.
(2) Clearly, if \(l = 0\), then the \((k, l)\)-Hessian quotient \(\frac{\sigma_k(\lambda(A))}{\sigma_l(\lambda(A))}\) becomes \(\sigma_k(\lambda(A))\), which implies that the PCP considered in [8] covers (4) as a special case. This leads to the fact that the a priori estimates obtained therein, which of course is much complicated than the one shown in this paper, can be used directly in the usage of Schauder theory in the proof of existence of solutions to the PCP (4) shown in Section 6.
(3) We have already shown that it is reasonable and feasible to consider PCPs (with DBC) on bounded domains in \(\mathcal{H}^n(1) \subset \mathbb{R}^{n+1}\) through Theorem 1.8 here and [8]. Based on this fact, one can try to extend the existing results on the PCPs to this setting. We prefer to leave this attempt to readers who are interested in this topic and we believe that our work here and [8] would give some guidance.

The paper is organized as follows. Some useful formulas for spacelike graphic hypersurfaces defined over \(M^n \subset \mathcal{H}^n(1)\) will be introduced in Section 2. Parts of these formulas were shown by us first in [9] and were also mentioned in some works later (see, e.g., [10]–[8]). In Section 3, we will give the \(C^1\) estimate for the PCP (4). Curvature estimates in Theorems 1.4 and 1.6 will be proven in Section 4, and boundary \(C^2\)-estimates will be proven in Section 5. The proof of Theorem 1.8 will be shown in the last section.
2 | SOME ELEMENTARY FORMULAS

As shown in [14, Section 2], we have the following fact:

**FACT.** Given an \((n + 1)\)-dimensional Lorentz manifold \((\overline{N}^{n+1}, \overline{g})\), with the metric \(\overline{g}\), and its spacelike hypersurface \(N^n\). For any \(p \in N^n\), one can choose a local Lorentzian orthonormal frame field \(\{e_0, e_1, e_2, \ldots, e_n\}\) around \(p\) such that, restricted to \(N^n\), \(e_1, e_2, \ldots, e_n\) form orthonormal frames tangent to \(N^n\). Taking the dual coframe fields \(\{\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n\}\) such that the Lorentzian metric \(\overline{g}\) can be written as \(\overline{g} = -\zeta_0^2 + \sum_{i=1}^{n} \zeta_i^2\). Making the convention on the range of indices

\[
0 \leq I, J, K, \ldots \leq n; \quad 1 \leq i, j, k \ldots \leq n,
\]

and doing differentials to forms \(\zeta_i\), one can easily get the following structure equations:

\[
\begin{align*}
(\text{Gauss equation}) \quad & R_{ijkl} = \overline{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}), \\
(\text{Codazzi equation}) \quad & h_{ij,k} - h_{ik,j} = \overline{R}_{0ijk}, \\
(\text{Ricci identity}) \quad & h_{ij,kl} - h_{ij,kl} = \sum_{m=1}^{n} h_{mj}R_{mijkl} + \sum_{m=1}^{n} h_{im}R_{mlijkl},
\end{align*}
\]

where \(R\) and \(\overline{R}\) are the curvature tensors of \(N^n\) and \(\overline{N}^{n+1}\), respectively. Clearly, in our setting here, all formulas mentioned above can be used directly with \(\overline{N}^{n+1} = \mathbb{R}^{n+1}_1\) and \(\overline{g} = \langle \cdot, \cdot \rangle_L\).

For the spacelike graphic hypersurface \(G \subset \mathbb{R}^{n+1}_1\) given by (1) and \(X = (x, u(x)) \in G\), set \(X_{ij} := \partial_i \partial_j X - \Gamma^k_{ij}X_k\) with \(\Gamma^k_{ij}\) the Christoffel symbols of the metric on \(G\). Then it is easy to know

\[
h_{ij} = -\left\langle X_{ij}, v \right\rangle_L,
\]

and have the following identities:

\[
\begin{align*}
(\text{Gauss formula}) \quad & X_{ij} = h_{ij} v, \\
(\text{Weingarten formula}) \quad & v_{,i} = h_{ij} X^j.
\end{align*}
\]

Using (10), (11), and (12) with the fact \(\overline{R} = 0\) in our setting, we have

\[
R_{ijkl} = h_{ij}h_{kl} - h_{ik}h_{jl},
\]

\[
\nabla_k h_{ij} = \nabla_j h_{ik}, \quad \text{(i.e., } h_{ij,k} = h_{ik,j}),
\]

and

\[
\Delta h_{ij} = (\sigma_1)_{ij} - \sigma_1 h_{ik}h_{jk}^\prime + h_{ij}|A|^2,
\]

where as usual \(\nabla, \Delta\) denote the gradient and the Laplace operators on \(G\), respectively. Here, the comma “,” in subscript of a given tensor means doing covariant derivatives. Besides, we make an agreement that, for simplicity, in the sequel, the comma “,” in subscripts will be omitted unless necessary.

**Remark 2.1.** Similar to the Riemannian case, the derivation of the formula (17) depends on Equations (15) and (16).

We also need the following fact:

**Lemma 2.2.** Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n\) and \(k = 0, 1, 2, \ldots, n\). Denote by \(\sigma_k(\lambda)\) defined as (3) the \(k\)th elementary symmetric function of \(\lambda_1, \lambda_2, \ldots, \lambda_n\). Also set \(\sigma_0 = 1\). Denote by \(\sigma_k(\lambda|i)\) the symmetric function with \(\lambda_i = 0\). Then for any \(1 \leq i \leq n\), one has
\[
\sigma_{k+1}(\lambda) = \sigma_{k+1}(\lambda|i) + \lambda_i \sigma_k(\lambda|i),
\]
\[
\sum_{i=1}^{n} \lambda_i \sigma_k(\lambda|i) = (k+1)\sigma_{k+1},
\]
\[
\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda),
\]
\[
\frac{\partial \sigma_{k+1}(\lambda)}{\partial \lambda_i} = \sigma_k(\lambda|i),
\]

and
\[
\sum_{i=1}^{n} \lambda_i^2 \sigma_k(\lambda|i) = \sigma_1(\lambda)\sigma_{k+1}(\lambda) - (2k+2)\sigma_{k+2}(\lambda).
\]

**Proof.** The above properties of \(\sigma_k\) can be obtained by direct calculations, which we prefer to omit here. \(\square\)

For any equation
\[
F(A) = f(\lambda_1, \lambda_2, \ldots, \lambda_n),
\] (18)
where \(A\) is the second fundamental form of the spacelike graphic hypersurface \(G \subset \mathbb{R}^{n+1}_1\) with \(\lambda_1, \lambda_2, \ldots, \lambda_n\) its principal curvatures. We can prove the following two conclusions:

**Lemma 2.3.** For the function \(F\) defined by (18) and the quantity \(\vartheta\) given in the PCP (4), one has
\[
F^{ij} \nabla_i \nabla_j \nu = \nu F^{ij} h^m_j h_{im} + F^{ij} \nabla_i h^m_j X_m, \\
\Delta \vartheta = \sigma_1 + \nabla^i \sigma_1 \langle X, X_i \rangle_L + |A|^2 \vartheta.
\]

**Proof.** By the Weingarten formula (14), it follows that
\[
\nabla_i \nabla_j \nu = \nabla_i \left( h^m_j X_m \right) = \nabla_i h^m_j X_m + h^m_j h_{im} \nu.
\]
The second assertion in Lemma 2.3 can be obtained as follows:
\[
\Delta \vartheta = g^{mn} \nabla_m \nabla_n \langle X, \nu \rangle_L \\
= g^{mn} \nabla_m (h^l_i \langle X, X_i \rangle_L) \\
= \nabla^i \sigma_1 \langle X, X_i \rangle_L + \sigma_1 + |A|^2 \vartheta,
\]
by using the Gauss formula (13) and also (14). \(\square\)

**Lemma 2.4.** For the function \(F\) defined by (18), we have
\[
F^{ij} \nabla_i \nabla_j \sigma_1 = -F^{ij, pq} \nabla^k h_{ij} \nabla_k h_{pq} + F^{ij} h^m_j h_{im} \sigma_1 - F^{ij} h_{ij} |A|^2 + \Delta f
\]
and
\[
F^{ij} \nabla_i \nabla_j h_{mn} = -F^{ij, pq} \nabla_n h_{ij} \nabla_m h_{pq} + F^{ij} h^l_j h_{im} h_{ln} - F^{ij} h^l_m h_{ij} h_{ln} + \nabla_m \nabla_n f.
\]

**Proof.** Using (17), it follows that
\[
F^{ij} \nabla_i \nabla_j \sigma_1 = F^{ij} h^m_j h_{im} \sigma_1 - F^{ij} h_{ij} |A|^2 + F^{ij} \Delta h_{ij}.
\]
On the other hand, by direct calculation, one has

\[
\Delta F = \Delta f = g^{kl} \nabla_k \nabla_l F = g^{kl} \nabla_k (F^{ij} \nabla_j h_{ij}) = F^{ij,pq} \nabla^k h_{ij} \nabla_k h_{pq} + F^{ij} \Delta h_{ij}.
\]

The first assertion can be obtained by combining the above two identities. The second assertion of Lemma 2.4 can be proven similarly.

Remark 2.5. Clearly, in the proofs of Lemmas 2.3 and 2.4, we know that \( F^{ij} = \partial F / \partial h_{ij} \), \( F^{ij,pq} = \partial^2 F / \partial h_{ij} \partial h_{pq} \).

3 \ C^1 \ ESTIMATE

3.1 \ Boundary estimate

Let \( s^+ \) be the solution of the following Dirichlet problem:

\[
\begin{aligned}
\sigma_1[s] &= n \left( \frac{\psi(x, u, \vartheta)}{C^k_n} \right)^{1/k}, & x \in M^n, \\
\sigma_n[s] &= \left( \frac{\psi(x, u, \vartheta)}{C^k_n} \right)^{n/k}, & x \in M^n, \\
s &= \varphi, & x \in \partial M^n.
\end{aligned}
\]

From the Mac–Laurin development, we have

\[
\sigma_1[u] \geq \sigma_1[s^+].
\]

The comparison principle for the mean curvature operator gives \( u \leq s^+ \) in \( M^n \), and thus \( \frac{\partial u}{\partial \nu} \geq \frac{\partial s^+}{\partial \nu} \). In order to get a lower barrier, let \( s^- \) be the solution of the following Dirichlet problem:

\[
\begin{aligned}
\sigma_1[s] &= k \left( \frac{\psi(x, u, \vartheta)}{C^k_n} \right)^{1/k}, & x \in M^n, \\
\sigma_n[s] &= \left( \frac{\psi(x, u, \vartheta)}{C^k_n} \right)^{n/k}, & x \in M^n, \\
s &= \varphi, & x \in \partial M^n.
\end{aligned}
\]

Also from the Mac–Laurin development, we have

\[
\sigma_n[u] \leq \sigma_n[s^-].
\]

So \( u \geq s^- \) in \( M^n \), and thus \( \frac{\partial u}{\partial \nu} \leq \frac{\partial s^-}{\partial \nu} \).

3.2 \ Maximum principle

The upper bound on \( Du \) amounts to an upper bound on \( W := \frac{1}{u} = 1 / \sqrt{1 - |Du|^2} \), where \( \pi := \ln u \). Therefore, it would follow from the boundary estimate once one can prove that \( W e^{\pi} \) cannot attain an interior maximum for \( S \) sufficiently large under control.
Proposition 3.1. Let \( u \) be the admissible solution of the PCP (4). Then,

\[
\sup_{\mathbb{M}^n} \mathcal{W} \leq \left( \sup_{\mathbb{M}^n} \mathcal{W} \right) e^{S_2 \left( 2 \sup_{\mathbb{M}^n} |\phi| + \text{diam}(M^n) \right)},
\]

where as usual \( \text{diam}(M^n) \) stands for the diameter of the bounded domain \( M^n \subset \mathbb{H}^n(1) \).

Proof. By contradiction, suppose that \( \sup_{\mathbb{M}^n} \mathcal{W} e^{S_\pi} \) is achieved at an interior point \( x_0 \in M^n \). At \( x_0 \), we choose a nice basis for the convenience of computations, that is, let \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal basis of \( T_{x_0} M^n \) (i.e., the tangent space at \( x_0 \) diffeomorphic to \( \mathbb{R}^n \)) such that \( D\pi(x_0) = |D\pi(x_0)| e_1 \), and moreover, the matrix \( \left( (D^2\pi(x_0))_{ij} \right)_{(n-1)\times(n-1)} \) \( 2 \leq i, j \leq n \), is orthogonal under the basis \( \{e_2, \ldots, e_n\} \). Since \( |\pi_1| \leq |D\pi| \) on \( \mathbb{M}^n \) and \( \pi_1(x_0) = |D\pi(x_0)| \). The function

\[
\ln \left( \frac{1}{1 - \pi_1^2} \right) + S\pi = -\frac{1}{2} \ln \left( 1 - \pi_1^2 \right) + S\pi
\]

has a maximum at \( x_0 \) as well. Hence, at \( x_0 \), for any \( i \in \{1, \ldots, n\} \), one has

\[
\frac{\pi_{ii}\pi_1}{1 - \pi_1^2} + S\pi_i = 0.
\]

So, the matrix of the curvature operator is diagonal, with diagonal entries \( \frac{1}{u v} \left( 1 + \frac{\pi_{ii}}{u v^2} \right) \). Moreover, still at \( x_0 \), one has \( \pi_{111} \pi_1 \leq -\pi_{11}^2 - \frac{2(\pi_{12}^2 + \pi_{13}^2)}{1 - \pi_1^2} S\pi_{11} (1 - \pi_1^2) \), and for \( i > 1 \), \( \pi_{ii1} \pi_1 \leq -(1 - \pi_1^2) S\pi_{ii} \). Then, we have

\[
\sum_{i=1}^n \frac{\partial \sigma_k}{\partial \lambda_i} \cdot \lambda_{i,1} = \sum_{i=1}^n \frac{\partial \sigma_k}{\partial \lambda_i} \cdot h_{i,1} = \psi_1.
\]

Since \( h_{i,1} = \frac{1}{u v} \left( 1 + \frac{\pi_{ik}}{u v^2} \right) \), we have

\[
h_{i,1} = \frac{3\pi_1 \pi_{1i}^2}{u v^5} + \frac{\pi_{i11}}{u v^3} - \frac{\pi_1}{u v} = \frac{\pi_1(3S^2 - 1)}{u v} + \frac{\pi_{111}}{u v^3},
\]

\[
h_{i,1} = \frac{\pi_{i11} \pi_{ii}}{u v^3} + \frac{\pi_{i11}}{u v} + \frac{\pi_{i1}}{u v^3} - \frac{\pi_1}{u v} - \frac{\pi_{1i}}{u v}
\]

\[
= \frac{\pi_{i1}}{u v} - \frac{\pi_{i1} \pi_{ii}(S + 1)}{u v} - \frac{\pi_i(S + 1)}{u v} \quad \text{for } i > 1.
\]

The differentiated equation, multiplied by \( \pi_1 \), becomes:

\[
\frac{\partial \sigma_k}{\partial \lambda_i} \left( \frac{\pi_1^2(3S^2 - 1)}{u v} + \frac{\pi_{111} \pi_1}{u v^3} \right)
\]

\[
+ \sum_{i \geq 2} \frac{\partial \sigma_k}{\partial \lambda_i} \left( \frac{\pi_{i11} \pi_1}{u v} - \frac{\pi_{i1} \pi_{ii}(S + 1)}{u v} - \frac{\pi_i(S + 1)}{u v} \right) = \pi_1 \psi_1.
\]

From the maximum conditions, we have

\[
\frac{\pi_1^2(3S^2 - 1)}{u v} + \frac{\pi_{111} \pi_1}{u v^3} \leq \frac{\pi_1^2(3S^2 - 1)}{u v}.
\]
and, since $\pi_{1ii} = \pi_{i11} - \pi_1$, we have
\[
\frac{\pi_{1i1}\pi_1}{uv} - \frac{\pi_{1}^2 + (S + 1)}{uv} - \frac{\pi_1(S + 1)}{uv} \\
\leq -\frac{1}{u} uS\pi_{1ii} - \frac{\pi_{1}^2 + (S + 1)}{uv}.
\]

Then, we can infer
\[
\frac{\partial \sigma_k}{\partial \lambda_1} \cdot \frac{\pi_{1}^2(S^2 - 1)}{uv} - \sum_{i \geq 2} \frac{\partial \sigma_k}{\partial \lambda_i} \left( \frac{1}{u} uS\pi_{ii} + \frac{\pi_{1}^2 + S + 1}{uv} + \frac{\pi_{1}^2 + S + 1}{uv} \right) \geq \pi_1\psi_1,
\]
and finally can obtain
\[
-k\sigma_k(\pi_{1}^2 + S) + (n - k + 1)\sigma_{k-1} \cdot \frac{2S + 1 + \frac{\partial \sigma_k}{\partial \lambda_1} \left( \frac{uS(1 - S)}{u} - \frac{2S + 1}{uv} \right)}{uv} \geq \pi_1\psi_1.
\]

We hope
\[
(n - k + 1)\sigma_{k-1} \cdot \frac{2S + 1 + \frac{\partial \sigma_k}{\partial \lambda_1} \left( \frac{uS(1 - S)}{u} - \frac{2S + 1}{uv} \right)}{uv} \leq 0,
\]
which is equivalent to
\[
\sigma_{k-1}(\lambda) \left( u^2S(1 - S) + (n + 1)(2S + 1) \right) \leq 0.
\]

Since $\pi_{1}^2 \leq \rho^2 < 1$, choosing $S = S_1$ large enough such that $\frac{S(S_1 - 1)}{2S_1 + 1} \geq \frac{n+1}{1-\rho^2}$, so we have
\[
k\sigma_k S \leq \sup_{M^n} |D\psi|.
\]

Then, choosing $S_2 > \max \left\{ \frac{\sup |D\psi|}{k \inf \psi} \cdot S_1 \right\}$, we reach a contradiction. \hfill \Box

## 4 CURVATURE ESTIMATES

### 4.1 The first curvature estimate

We write (4) in the form
\[
F(A) = \frac{1}{\pi_1^k}(A) = \psi_1^k(X, \vartheta) = f(X, \vartheta) \quad \text{for any } X \in \mathcal{G}.	ag{21}
\]

Proof of Theorem 1.4. Consider the function
\[
W(A) = \sigma_1(A),
\]
which attains its maximum value at some $X_0 = (x_0, u(x_0)) \in \mathcal{G}$. If $x_0 \in \partial M^n$, then our claim (8) follows directly. Now, we try to prove this claim in the case that $x_0 \notin \partial M^n$. Choose the frame fields $e_1, e_2, ..., e_n, \nu$ at $X_0$ such that $e_1, e_2, ..., e_n \in T_{X_0} \mathcal{G}$ at $X_0$ and $(h_{ij})_{i,j=0}^n$ is diagonal at $X_0$ with eigenvalues $h_{11} \geq h_{22} \geq \cdots \geq h_{nn}$. Here, as usual, $T_{X_0} \mathcal{G}$ denotes the tangent space of the graph hypersurface $\mathcal{G}$ at $X_0$. For each $i = 1, ..., n$, we have
\[
\nabla_i \sigma_1 = 0 \quad \text{at } X_0.
\]
Therefore, at $X_0$, it follows that
\begin{equation}
0 \geq F^{ij} \nabla_i \nabla_j \sigma_1
= -F^{ij,pq} \nabla_i h_{ij} \nabla_p h_{pq} + F^{ij} h_{im} h_{mj} \sigma_1 - F^{ij} h_{ij} |A|^2 + \Delta f.
\end{equation}

Since $f$ is convex in $\vartheta$, together with Lemma 2.3, we have
\begin{equation}
\Delta f = \frac{\partial^2 f}{\partial X^\alpha \partial X^\beta} \nabla_i X^\alpha \nabla_i X^\beta + 2 \frac{\partial^2 f}{\partial X^\alpha \partial \vartheta} \nabla_i X^\alpha \nabla_i \vartheta
+ \frac{\partial^2 f}{\partial \vartheta^2} |\nabla \vartheta|^2 - c_1 \sigma_1 - c_2
\geq \frac{\partial f}{\partial \vartheta} \Delta \vartheta + \frac{\partial^2 f}{\partial \vartheta^2} |\nabla \vartheta|^2 - c_1 \sigma_1 - c_2,
\end{equation}
where positive constants $c_1$, $c_2$ depend on $||\varphi||_{C^1(M^n)}$, $||\psi||_{C^2(M^n, \inf u, \sup u)}$, and $X^\alpha := \langle X, \partial_x \rangle L$, $\alpha = 1, 2, ..., n + 1$.

Obviously, $\partial_1, \partial_2, ..., \partial_n$ are the corresponding coordinate vector fields on $\mathcal{H}^n(1)$, $\partial_{n+1} := \partial_r$. Putting (23) into (22) yields
\begin{equation}
0 \geq F^{ij} \nabla_i \nabla_j \sigma_1
\geq -F^{ij,pq} \nabla_i h_{ij} \nabla_p h_{pq} + F^{ij} h_{im} h_{mj} \sigma_1
+ (\frac{\partial f}{\partial \vartheta} - f) |A|^2 - c_1 \sigma_1 - c_2
\end{equation}
where we have used (7) and the concavity of $F$. On the other hand, by Lemma 2.2, one has
\begin{equation}
F^{ij} h_{im} h_{mj} = \frac{1}{k} \sigma_{k-1}^{k-1} \left[ \sigma_k \sigma_1 - (k+1) \sigma_{k+1} \right]
\geq \frac{1}{n} \sigma_k^{k} \sigma_1,
\end{equation}
where the last inequality can be derived from the Newton inequalities for $\sigma_{k+1} > 0$,
\begin{equation}
\frac{\sigma_{k+1} \sigma_{k-1}}{C_{k+1} C_{k-1}} \leq \left( \frac{\sigma_k}{C_k} \right)^2.
\end{equation}
Taking (25) into (24), it is easy to know that $\sigma_1$ is bounded. Then the conclusion of Theorem 1.4, that is, (8), follows naturally.

\[ \square \]

4.2 The second curvature estimate

Let
\[ \mathcal{P}(\lambda) := F(A) = \sigma_k^+ (A) = f(X, \vartheta) \quad \text{for any } X \in \mathcal{G}. \]

Set
\begin{equation}
\sigma_k^+ (\lambda_1, ..., \lambda_n) = \mathcal{P}(\lambda_1, ..., \lambda_n),
\end{equation}

\[ \square \]
\[ \text{tr} P^{ij} = \sum_{i=1}^{n} P^{ii}, \quad P_i = \frac{\partial P}{\partial \lambda_i}. \]  

(27)

First, we list a useful lemma, which can be found in, for example, [1, 25, 26].

**Lemma 4.1.** For any symmetric matrix \( \eta = (\eta_{ij}) \), we have

\[
F^{ij, pq} \eta_{ij} \eta_{pq} = \sum_{i,j} \frac{\partial^2 P}{\partial \lambda_i \partial \lambda_j} \eta_{ij} \eta_{jj} + \sum_{i \neq j} \frac{P_i - P_j}{\lambda_i - \lambda_j} \eta_{ij}^2.
\]  

(28)

The second term on RHS of (28) is nonpositive if \( \lambda \) is concave, and it is interpreted as the limit if \( \lambda_i = \lambda_j \).

**Proof of Theorem 1.6.** Let \( \eta = \varphi - u \) and, as before, for any point \( x_0 \in M^n \), \( X_0 = (x_0, u(x_0)) \). Denote by \( \omega \) the constant function, whose graph is the hyperbolic plane of center at origin and radius \( R \) (i.e., \( \mathcal{H}^n(R) \)), lying above the graph of \( \varphi \) such that \( \omega(x_0) = \varphi(x_0) \) and \( D\omega(x_0) = D\varphi(x_0) \).

Then, for large enough \( R \) and small enough \( \epsilon > 0 \), we have \( F[(\omega - \epsilon)(A)] < F[u(A)] \) in \( M^n_{\epsilon} := \{x \in M^n | \omega(x) - \epsilon < \varphi(x)\} \subset M^n \) and \( \omega(x) - \epsilon = \varphi(x) \geq u \) on \( \partial M^n_{\epsilon} \). By the comparison principle, we then have \( u \leq \omega(x) - \epsilon \) in \( M^n_{\epsilon} \).

Consequently \( (\varphi - u)(x_0) \geq \epsilon \), so we have \( \eta > 0 \) in \( M^n_{\epsilon} \).

We now consider the function

\[ G = \eta^2 e^{\Psi(\varphi)} h_{ij} X_i X_j, \]

achieving its maximum value at some \( X_0 \in C \), where \( \alpha \geq 1 \), \( \Psi \) is a function determined later, and satisfies \( \Psi' := \frac{\partial \Psi}{\partial \varphi} \geq 0 \).

Without loss of generality, one may choose the frame fields \( e_1 = \chi, e_2, \ldots, e_n, \nu \) such that \( e_1, e_2, \ldots, e_n \in T_{X_0} C, \nabla e_i e_j = 0 \) at \( X_0 \) for all \( i, j = 1, \ldots, n \), and \((h_{ij})_{n \times n} \) is diagonal at \( X_0 \) with eigenvalues \( h_{11} \geq h_{22} \geq \cdots \geq h_{nn} \).

At \( X_0 \), for each \( i = 1, \ldots, n \), one has

\[
\alpha \nabla_i \eta + \Psi' \nabla_i \varphi + \nabla i \eta \geq 0,
\]

\[
\alpha \left( \frac{\nabla_i \nabla J \eta}{\eta} - \frac{\nabla_i \eta \nabla J \eta}{\eta^2} \right) + \Psi' \nabla_i \varphi + \Psi' \nabla J \eta + \frac{\nabla i \eta h_{11}}{h_{11}^2} - \frac{\nabla i \eta h_{11} \nabla J \eta}{h_{11}^2} \leq 0.
\]

Therefore, by Lemma 2.4, we have

\[
0 \geq \alpha F_i^j \left( \frac{\nabla_i \nabla J \eta}{\eta} - \frac{\nabla_i \eta \nabla J \eta}{\eta^2} \right) + \Psi' F_i^j \nabla_i \varphi + \Psi' F_i^j \nabla J \eta + \frac{\nabla i \eta h_{11}}{h_{11}^2} - \frac{\nabla i \eta h_{11} \nabla J \eta}{h_{11}^2} + \frac{\nabla i \eta f}{h_{11}} - \frac{1}{h_{11}} \frac{\nabla i \eta f h_{11} h_{11}}{h_{11}^2}.
\]

We also find that

\[
F_i^j \nabla_i \nabla J \varphi = \partial F_i^j \eta_{im} h_{jm} + f + \nabla J f(X, X_i)L.
\]

Consequently,

\[
0 \geq \alpha F_i^j \left( \frac{\nabla_i \nabla J \eta}{\eta} - \frac{\nabla_i \eta \nabla J \eta}{\eta^2} \right) + \Psi' F_i^j \nabla_i \varphi + \Psi' F_i^j \nabla J f(X, X_i)L + f h_{11}
\]

\[
+ \left( \Psi' \varphi + 1 \right) F_i^j \eta_{im} h_{jm} + \frac{\nabla i \eta f}{h_{11}} - \frac{1}{h_{11}} \frac{F_i^j \eta_{im} \nabla J f_{11} h_{11}}{h_{11}^2}.
\]

Since \( f \) is convex in \( \varphi \), we have
\[ \nabla_1 f = \frac{\partial f}{\partial X^\alpha} \nabla_1 X^\alpha + \frac{\partial f}{\partial \vartheta} \nabla_1 \vartheta, \]
\[ \nabla_1 \nabla_1 f = \frac{\partial^2 f}{\partial X^\alpha \partial X^\beta} \nabla_1 X^\alpha \nabla_1 X^\beta + \frac{\partial^2 f}{\partial X^\alpha \partial \vartheta} \nabla_1 X^\alpha \vartheta + \frac{\partial^2 f}{\partial X^\alpha \partial \vartheta} |\nabla_1 \vartheta|^2 + \frac{\partial f}{\partial X^\alpha} \nabla_1 V_1 X^\alpha + \frac{\partial f}{\partial \vartheta} \nabla_1 V_1 \vartheta \]
\[ \geq \frac{\partial f}{\partial \vartheta} \nabla_1 \vartheta - c_3 h_{11} - c_4 \]
\[ = \frac{\partial f}{\partial \vartheta} (\vartheta h_{11}^2 + \nabla_1 h_{11}(X, X_1)_L) - c_3 h_{11} - c_4, \]
where \( c_3, c_4 \) are positive constants depending on \( ||\varphi||_{C^1(M^n)} \) and \( ||\psi||_{C^2(M^n, \inf \{u, \sup u\} \times \mathbb{R})} \). Inserting this into (30) yields

\[ 0 \geq \alpha F_{ij} \left( \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \right) + \psi'' F_{ij} \nabla_i \vartheta \nabla_j \vartheta + \psi' \nabla_i f(X, X_1)_L + \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} \]
\[ + \left( \psi' \vartheta + 1 \right) F_{ij} h_{im} h_{jm} + \frac{\partial f}{\partial \vartheta} \frac{\nabla_1 h_{11}(X, X_1)_L}{h_{11}} - \frac{1}{h_{11}} F_{ij, pq} \nabla_1 h_{ij} \nabla_1 h_{pq} \]
\[ - \frac{F_{ij} \nabla_1 h_{11} \nabla_1 h_{11}}{h_{11}^2} - c_3, \]
where we have assumed that \( h_{11} \) is sufficiently large. Otherwise, the assertion of Theorem 1.6 holds.

Next, we assume that \( \varphi \) has been extended to be constant in the \( \vartheta \) direction. Therefore,

\[ \nabla_i \nabla_j \eta = \sum_{\alpha, \beta = 1}^n \frac{\partial^2 \varphi}{\partial X^\alpha \partial X^\beta} \nabla_1 X^\alpha \nabla_1 X^\beta + \sum_{\alpha = 1}^n \frac{\partial \varphi}{\partial X^\alpha} \nabla_1 \nabla_1 X^\alpha - u_{ij} \]
\[ \geq \sum_{\alpha = 1}^n \frac{\partial \varphi}{\partial X^\alpha} \nu^\alpha h_{ij} - c_5 h_{ij} \nu, \]
where \( c_5 > 0 \) depends on \( ||\varphi||_{C^1(M^n)} \) and we have again used Gaussian formula and the assumption that \( \varphi \) is affine. Consequently,

\[ F_{ij} \nabla_i \nabla_j \eta \geq \left( \sum_{\alpha = 1}^n \frac{\partial \varphi}{\partial X^\alpha} \nu^\alpha \right) F_{ij} h_{ij} \geq -c_6, \]
where positive constant \( c_6 \) depends on \( c_5, ||\psi||_{C^0(M^n, \inf \{u, \sup u\} \times \mathbb{R})} \) and \( ||\varphi||_{C^1(M^n)} \). Combining (31) and (32), at \( X_0 \), we have

\[ 0 \geq \frac{c_6 \alpha}{\eta} - \alpha F_{ij} \frac{\nabla_i \nabla_j \eta}{\eta^2} + \psi'' F_{ij} \nabla_i \vartheta \nabla_j \vartheta + \psi' \nabla_i f(X, X_1)_L + \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} \]
\[ + \left( \psi' \vartheta + 1 \right) F_{ij} h_{im} h_{jm} + \frac{\partial f}{\partial \vartheta} \frac{\nabla_1 h_{11}(X, X_1)_L}{h_{11}} - \frac{1}{h_{11}} F_{ij, pq} \nabla_1 h_{ij} \nabla_1 h_{pq} \]
\[ - \frac{F_{ij} \nabla_1 h_{11} \nabla_1 h_{11}}{h_{11}^2} - c_3. \]

We now estimate the remaining terms in (33), and divide the argument into two cases.

**Case 1.** Assume that there exists a positive constant \( \xi \) to be determined such that

\[ h_{nn} \leq -\xi h_{11}. \]
Using the critical point condition (29), we have

\[
F^{ij} \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2} = F^{ij} \left( \frac{\nabla_i \eta}{\eta} + \Psi' \nabla_i \vartheta \right) \left( \frac{\nabla_j \eta}{\eta} + \Psi' \nabla_j \vartheta \right) 
\leq (1 + \varepsilon^{-1}) \alpha^2 F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + (1 + \varepsilon) (\Psi')^2 F^{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

for any \( \varepsilon > 0 \). Since \( |\nabla \eta| \leq c_7(\tilde{M}^n) \), so

\[
F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} \leq c_8 \text{tr} F^{ij} \eta^2,
\]

where \( c_8 > 0 \) depend on \( c_7 \). Therefore, at \( X_0 \), we have

\[
0 \geq -\frac{c_6 \alpha}{\eta} - c_9 \left[ \alpha + (1 + \varepsilon^{-1}) \alpha^2 \right] \frac{\text{tr} F^{ij} \eta^2}{\eta^2} + \left[ \Psi'' - (1 + \varepsilon) (\Psi')^2 \right] F^{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

\[
+ \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} + \left( \Psi' \vartheta + 1 \right) F^{ij} h_{im} h_{jm} - c_3
\]

(35)

where \( c_9 := \max\{1, c_8\} \) and the concavity of \( F(A) \) has been used. On the other hand, from (29), the last two terms of the RHS of (35) are bounded from below:

\[
\frac{\partial f}{\partial \vartheta} \nabla_i h_{11} \langle X, X_i \rangle_L + \Psi' \nabla_i f \langle X, X_i \rangle_L
\]

\[
= \left( \Psi' \nabla_i f - \alpha \frac{\partial f}{\partial \vartheta} \frac{\nabla_i \eta}{\eta} - \frac{\partial f}{\partial \vartheta} \Psi' \nabla_i \vartheta \right) \langle X, X_i \rangle_L
\]

\[
= \left( \Psi' \frac{\partial f}{\partial X^\beta} \nabla_i X^\beta - \alpha \frac{\partial f}{\partial \vartheta} \frac{\nabla_i \eta}{\eta} - \frac{\partial f}{\partial \vartheta} \Psi' \nabla_i \vartheta \right) \langle X, X_i \rangle_L
\]

\[
\geq -\frac{c_{10} \alpha}{\eta} - c_{11},
\]

where \( c_{10} \) is a positive constant depending on \( c_7 \), \( \|\varphi\|_{C^1(\tilde{M}^n)}, \|\psi\|_{C^1(\tilde{M}^n \times [\inf_{\tilde{M}^n} u, \sup_{\tilde{M}^n} u] \times \mathbb{R})} \), and \( c_{11} > 0 \) depends on \( \|\varphi\|_{C^1(\tilde{M}^n)}, \|\Psi\|_{C^1(\tilde{M}^n \times [\inf_{\tilde{M}^n} u, \sup_{\tilde{M}^n} u] \times \mathbb{R})} \). Therefore,

\[
0 \geq -\frac{c_{12} \alpha}{\eta} - c_9 \left[ \alpha + (1 + \varepsilon^{-1}) \alpha^2 \right] \frac{\text{tr} F^{ij} \eta^2}{\eta^2} + \left[ \Psi'' - (1 + \varepsilon) (\Psi')^2 \right] F^{ij} \nabla_i \vartheta \nabla_j \vartheta
\]

\[
+ \left( \frac{\partial f}{\partial \vartheta} \cdot \vartheta - f \right) h_{11} + \left( \Psi' \vartheta + 1 \right) F^{ij} h_{im} h_{jm} - c_{13},
\]

(36)

where constant \( c_{12} > 0 \) depends on \( c_9, c_{10} \), and constant \( c_{13} > 0 \) depends on \( c_3 \) and \( c_{11} \). By the Weingarten formula (14), it follows that

\[
F^{ij} \nabla_i \vartheta \nabla_j \vartheta = F^{ij} h_{ij} \langle X, X_i \rangle_L \langle X, X_i \rangle_L \leq c_{14} F^{ij} h_{im} h_{jm},
\]

where \( c_{14} \) is a positive constant depending on \( \|\varphi\|_{C^1(\tilde{M}^n)} \), and then we can take a function \( \Psi \) satisfying

\[
\Psi'' - (1 + \varepsilon) (\Psi')^2 \leq 0.
\]

(37)
Since $M^n$ is bounded and $C^2$, there exists a positive constant $a = a(\rho) > \sup_{M^n} u$ such that

$$\frac{-2}{3}a \leq \vartheta < - \sup_{M^n} u.$$ 

Let us take

$$\Psi(\vartheta) = -\log(2a - \vartheta),$$

so we have (37) and

$$\Psi' \vartheta + 1 + c_{14}(\Psi'' - (1 + \varepsilon)(\Psi')^2) \geq \frac{1}{2} \quad \text{for } \varepsilon \leq \frac{a^2}{c_{14}}.$$ 

From (36), together with

$$F^{ij}h_{im}h_{jm} = F^{ii}h_{ii}^2 \geq \frac{\zeta^2}{n}h_{11}^2 \text{tr}F^{ij},$$

which follows from the assumption (34) and the fact $F^{nn} \geq \frac{1}{n} \text{tr}F^{ij}$, at $X_0$, we have that

$$0 \geq -\frac{c_{12}\alpha}{\eta} - c_9 \left[\alpha + (1 + \varepsilon^{-1})\alpha^2\right] \frac{\text{tr}F^{ij}}{\eta^2} + \left(\frac{\partial f}{\partial \vartheta} \cdot \vartheta - f\right)h_{11} + \frac{\zeta^2}{2n}h_{11}^2 \text{tr}F^{ij} - c_{13},$$

which implies an upper bound

$$\eta h_{11} \leq \frac{c_{15}}{5} \quad \text{at } X_0,$$

since

$$\text{tr}F^{ij} = \frac{(n-k+1)\sigma_{k-1}}{kf^{k-1}} > 0,$$

where $c_{15}$ is a positive constant depending on $c_9, c_{12}, c_{13}, \alpha, M^n, ||\varphi||_{C^0(\Omega)},$ |

**Case 2.** We now assume that

$$h_{nn} \geq -\zeta h_{11}. \quad (38)$$

Since $h_{11} \geq h_{22} \geq \cdots \geq h_{nn}$, we have

$$h_{ii} \geq -\zeta h_{11} \quad \text{for all } i = 1, \ldots, n.$$ 

For a positive constant $\tau$, assume to be 4, we divide $\{1, \ldots, n\}$ into two parts as follows:

$$I = \{i : p^{ii} \leq 4p^{11}\}, \quad J = \{j : p^{jj} > 4p^{11}\},$$

where $p^{ii} := \frac{\partial p}{\partial h_{ii}} = P_i$ is evaluated at $\lambda(X_0)$. Then, for each $i \in I$, by (29), we have

$$P_i \frac{|\nabla_i h_{11}|^2}{h_{11}^2} = P_i \left(\frac{\nabla_i \eta}{\eta} + \Psi' \nabla_i \vartheta\right)^2 \leq (1 + \varepsilon^{-1})\alpha^2 P_i \frac{|\nabla_i \eta|^2}{\eta^2} + (1 + \varepsilon)(\Psi')^2 P_i |\nabla_i \vartheta|^2$$
for any $\varepsilon > 0$. For each $j \in J$, we have

$$\alpha P_j \frac{|\nabla_j \eta|^2}{\eta^2} = \alpha^{-1} P_j \left( \frac{\nabla_j h_{11}}{h_{11}} + \Psi' \nabla_j \hat{\vartheta} \right)^2$$

$$\leq \frac{1 + \varepsilon}{\alpha} (\Psi')^2 P_j |\nabla_j \hat{\vartheta}|^2 + \frac{1 + \varepsilon^{-1}}{\alpha} P_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}$$

for any $\varepsilon > 0$. Consequently,

$$\alpha \sum_{i=1}^n P_i \frac{|\nabla_i \eta|^2}{\eta^2} + \sum_{i=1}^n P_i \frac{|\nabla_i h_{11}|^2}{h_{11}^2}$$

$$\leq \left[ \alpha + (1 + \varepsilon^{-1})\alpha^2 \right] \sum_{i \in I} P_i \left( \frac{|\nabla_i \eta|^2}{\eta^2} + (1 + \varepsilon)(\Psi')^2 \sum_{i \in I} P_i |\nabla_i \hat{\vartheta}|^2 \right)$$

$$+ \frac{1 + \varepsilon}{\alpha} (\Psi')^2 \sum_{j \in J} P_j |\nabla_j \hat{\vartheta}|^2 + \left[ 1 + (1 + \varepsilon^{-1})\alpha^{-1} \right] \sum_{j \in J} P_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}$$

$$\leq 4n \left[ \alpha + (1 + \varepsilon^{-1})\alpha^2 \right] P_1 \frac{|\nabla_1 \eta|^2}{\eta^2} + (1 + \varepsilon)(1 + \alpha^{-1})(\Psi')^2 \sum_{i=1}^n P_i |\nabla_i \hat{\vartheta}|^2$$

$$+ \left[ 1 + (1 + \varepsilon^{-1})\alpha^{-1} \right] \sum_{j \in J} P_j \frac{|\nabla_j h_{11}|^2}{h_{11}^2}.$$
We now prove the claim. Using the concavity of $\mathcal{P}$, Lemma 4.1, and the Codazzi equation (16), we can obtain

$$\frac{1}{h_{11}} P_{ij, pq} \nabla_i h_{ij} \nabla_k h_{pq} \geq - \frac{2}{h_{11}} \sum_{j \in J} \frac{P_i - P_j}{\lambda_i - \lambda_j} |\nabla_j h_{11}|^2.$$ 

We then need to show that

$$\frac{2(\lambda_i - \lambda_j)}{h_{11}} \geq (1 + c_1 \alpha^{-1}) \frac{P_j}{h_{11}}$$

for each $j \in J$ provided that $\alpha$ is sufficiently large.

Set $\delta = c_1 \alpha^{-1}$, and then we need to show

$$(1 - \delta) \lambda_i \lambda_1 \geq 2 \lambda_1 \lambda_1 - (1 + \delta) \lambda_j \lambda_j$$

for $j \in J$ provided that $\delta > 0$ is sufficiently small. We show this if either $\lambda_j \geq 0$ or $\lambda_j \leq 0$ and $|\lambda_j| \leq \zeta \lambda_1$ for a sufficiently small positive constant $\zeta$.

Since $j \in J$, so we have $\lambda_j > 4 \lambda_1$. Therefore, if $\lambda_j \geq 0$, then (41) is satisfied if $\delta = 1/4$. On the other hand, if $\lambda_j \leq 0$, then $|\lambda_j| \leq \zeta \lambda_1$ by (38), and therefore (41) is again satisfied if $\delta = 1/4$ and $\zeta = 1/5$.

The proof of Theorem 1.6 is finished.

\[\square\]

5 C² BOUNDARY ESTIMATES

Throughout this section, we just assume that $M^n$ is strictly convex. By N. M. Ivochkina, M. Lin, N. S. Trudinger [22, 23] and Pierre Bayard [4], we have the following inequality: \[\exists B_0 = B_0(n, k) \text{ such that in } \Gamma_k, \forall i \in \{1, \ldots, k\},\]

$$\frac{\partial \sigma_i}{\partial \lambda_i} \cdot \lambda_i^2 \leq \lambda_i \sigma_k + B_0 \sum_{l \neq i} \frac{\partial \sigma_i}{\partial \lambda_l} \cdot \lambda_l^2.$$ \hspace{1cm} (42)

Let $x_0$ be a boundary point, and \{e₁, ..., eₙ\} be an adapted basis at $x_0$, and we know that $\sup_{M^n} \frac{|Du|}{u} = \sup_{M^n} |D\pi| \leq \rho < 1$.

**Lemma 5.1.** Let $g : \overline{M^n} \cap \overline{B}_r(x_0) \times B(0, 1) \rightarrow \mathbb{R}, (x, p) \mapsto g(x, p)$ be a function of class $C^2$, concave with respect to $p$, where $B_r(x_0) := \{x \in \mathbb{R}^{n+1} \mid |x - x_0| \leq r\}$, $B(0, 1) := \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}$, and $\mathcal{W} = g(\cdot, Du) - \frac{B}{2} \sum_{s=1}^{n-1} (u_s - u_s(x_0))^2$. If $D_{r, \rho}$ denotes the compact $\overline{M^n} \cap \overline{B}_r(x_0) \times \overline{B}(0, \rho)$, for $B = B \left( n, k, \rho, B_\delta, \|g\|_{1, D_{r, \rho}} \right)$ sufficiently large, $\mathcal{W}$ satisfies on $M^n \cap B_r(x_0)$ the inequality:

$$\sum_{i, j} \frac{\partial \sigma_i}{\partial q_{ij}} (u) \mathcal{W}_{ij} \leq B_1 \left( 1 + |D\mathcal{W}| + \sum_{i, j} \frac{\partial \sigma_i}{\partial q_{ij}} (u) \mathcal{W}_{ij} + \sigma_{k-1}(u) \right),$$

where $B_1 = B_1 \left( n, k, M^n, \psi, \rho, B_\delta, \|g\|_{2, D_{r, \rho}} \right)$.

**Proof.** Let us denote by $\xi_\alpha, \alpha = 1, ..., n$, vectors of $\mathcal{H}^n(1)$ induced by the map $x \mapsto X := (x, u(x))$, an orthonormal basis of principle vectors of $T_X G$, and $(\eta_\alpha^2)$ such that, $\forall \alpha \in \{1, ..., n\}, e_\alpha = \sum_{\alpha=1}^{n} \eta_\alpha^2 e_\alpha$. Define $(\tilde{\xi}_\alpha)$ such that, $\forall \alpha \in \{1, ..., n\}, \tilde{\xi}_\alpha = \sum_{\alpha=1}^{n} \tilde{\xi}_\alpha^2 e_\alpha$. We thus have from the definition $(\tilde{\eta}_\alpha^2) = (\tilde{\xi}_\alpha^2)^{-1}$.

We will use the Greek letters and the Latin letters for derivatives in the basis $\{\xi_\alpha, \alpha = 1, ..., n\}$ and $\{e_s, s = 1, ..., n\}$, respectively. For instance, $u_{\alpha\beta}$ and $u_{s\alpha}$ will denote, respectively, $D^2 u(\xi_\alpha, \xi_\beta)$ and $D^2 u(e_s, \xi_\alpha)$. In view of the choice of the $\xi_\alpha$, the quantities $\frac{1}{10} (u_{\alpha\beta} + u_{\alpha\alpha}^2 / u) \alpha = 1, ..., n$, are the principal curvatures of the graph $G$ of $u$. The inequality in Lemma 5.1 may then be written as

$$\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \mathcal{W}_{\alpha\alpha} \leq B_1 \left( 1 + |D\mathcal{W}| + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \mathcal{W}_{\alpha\alpha}^2 + \sigma_{k-1} \right).$$
For the first and second derivatives of $\mathcal{W}$, we have $\forall \alpha \in \{1, \ldots, n\}$,

\[ \mathcal{W}_\alpha = g_\alpha + \sum_{i=1}^{n} g_{p_i} u_{i\alpha} - B \sum_{s=1}^{n-1} u_{s\alpha}(u_s - u_s(x_0)), \]

and

\[ \mathcal{W}_{\alpha\alpha} = g_{\alpha\alpha} + 2 \sum_{i=1}^{n} g_{p_i} u_{i\alpha\alpha} + \sum_{s,t=1}^{n} g_{p_s p_t} u_{s\alpha\alpha} + \sum_{s=1}^{n} g_{p_s} u_{s\alpha\alpha\alpha} - B(u_{s\alpha\alpha}(u_s - u_s(x_0)) + u_{s\alpha\alpha}^2). \]

The following formula can represent the third derivatives of $u$:

\[ (\sigma_k(u))_j = \frac{\partial \sigma_k(u)}{\partial \lambda_{\alpha j}} \cdot \lambda_{\alpha j} \]

\[ = \frac{\partial \sigma_k(u)}{\partial \lambda_{\alpha}} \cdot \left[ u_{\alpha} \left( \frac{u_{\alpha\alpha}}{u} - \frac{u_{\alpha} u}{u^2} \right) \right] \cdot \left( u_{\alpha\alpha\alpha} + u - \frac{2 u_{\alpha} u}{u^2} \right) + \left( u_{\alpha\alpha\alpha} + u + \frac{4 u_{\alpha} u_{\alpha\alpha}}{u} + \frac{2 u_{\alpha}^2 u}{u^2} \right). \]

(43)

Since $u_{im} = u_{nmi} - u_i$, by the use of $u_{k\alpha} = \tilde{\eta}_{\alpha s} u_{\alpha\alpha\alpha}$, we get:

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \mathcal{W}_{\alpha\alpha\alpha} = \sum_{l=1}^{n} g_{p_l} \psi_l - B \sum_{s=1}^{n-1} \psi_s(u_s - u_s(x_0)) \]

\[ + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \left\{ g_{\alpha\alpha\alpha} + 2 \sum_{i=1}^{n} \tilde{\eta}_{\alpha s} g_{p_s} u_{\alpha s} + \sum_{s,t=1}^{n} g_{p_s p_t} u_{s\alpha\alpha\alpha} \right\} \]

\[ - Bu^2 \sum_{s=1}^{n-1} |\tilde{\eta}_{\alpha s}|^2 + \left( 4 \pi_{\alpha\alpha\alpha} - \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right) \mathcal{W}_{\alpha\alpha} + \left( \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} - 4 \pi_{\alpha\alpha\alpha} \right) g_{\alpha} \]

\[ - \sum_{i=1}^{n} g_{p_i} \left[ 2u_i(1 + \pi_{\alpha\alpha\alpha}^2) - \pi_{\alpha\alpha\alpha}^2 \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right] + B \sum_{s=1}^{n-1} \left[ 2u_s(1 + \pi_{\alpha\alpha\alpha}^2) - \pi_{\alpha\alpha\alpha}^2 \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right] (u_s - u_s(x_0)), \]

where $\pi = \log u$. It is easy to estimate

\[ \sum_{l=1}^{n} g_{p_l} \psi_l - B \sum_{s=1}^{n-1} \psi_s(u_s - u_s(x_0)) \leq b_1(1 + |DW|), \]

where positive constant $b_1$ depends on $\psi$. Moreover, the term

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( g_{\alpha\alpha\alpha} + 2 \sum_{i=1}^{n} \tilde{\eta}_{\alpha s} g_{p_s} u_{\alpha s} + \left( \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} - 4 \pi_{\alpha\alpha\alpha} \right) g_{\alpha} - \sum_{i=1}^{n} g_{p_i} \left[ 2u_i(1 + \pi_{\alpha\alpha\alpha}^2) - \pi_{\alpha\alpha\alpha}^2 \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right] \right) \]

is easily estimated by $b_2 \left( (\sigma_{k-1} + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} |\lambda_{\alpha}\|), \right)$, positive constant $b_2$ depends on $\|g\|_{2,p_\alpha}, \rho, M^n, k$. Recalling that $g$ is concave w.r.t. $p$, for all $\alpha \in \{1, \ldots, n\}$, $\sum_{l=1}^{n} g_{p_l} \tilde{\eta}_{\alpha s} \tilde{\eta}_{\alpha t} \leq 0$, and then we thus finally get the estimate of $\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \mathcal{W}_{\alpha\alpha\alpha}$:

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \mathcal{W}_{\alpha\alpha\alpha} \leq b_3 \left( 1 + |DW| + \sigma_{k-1} + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} |\lambda_{\alpha}\| \right) + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( -Bu^2 \sum_{s=1}^{n-1} |\tilde{\eta}_{\alpha s}|^2 \right) \]

\[ + \left( 4 \pi_{\alpha\alpha\alpha} - \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right) \mathcal{W}_{\alpha\alpha} + B \sum_{s=1}^{n-1} \left[ 2u_s(1 + \pi_{\alpha\alpha\alpha}^2) - \pi_{\alpha\alpha\alpha}^2 \pi_{\alpha\alpha\alpha} \lambda_{\alpha\alpha} \right] (u_s - u_s(x_0)) \]
\[ \begin{align*}
&\leq b_3 \left(1 + |DW| + \sigma_k - 1 + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot |\lambda_{\alpha}| \right) \\
&\quad + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( -b_4 B \lambda_{\alpha}^2 \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2 + b_5 |W_{\alpha}| + b_6 |\lambda_{\alpha}||W_{\alpha}| \right),
\end{align*} \]

(44)

where the positive constant \( b_3 \) depends on \( b_1, b_2 \), positive constants \( b_4, b_5, b_6 \) depend on \( \rho, M^n \). By [4, Lemma 4.3], we denote \( \delta_{\varepsilon} := \delta_{\varepsilon}(\varepsilon, \rho, n) \), where \( \varepsilon \in (0, 1) \). Let us consider two types of points: either \( \forall \alpha \in \{1, \ldots, n\} \), \( \sum_{s=1}^{n-1} |\tilde{\eta}_{\alpha,s}|^2 \geq \delta_{\varepsilon} \), or \( \exists \alpha \in \{1, \ldots, n\} \) (e.g., \( \alpha = 1 \)) such that \( \sum_{s=1}^{n-1} |\tilde{\eta}_{\alpha,s}|^2 < \delta_{\varepsilon} \).

For the points of the first type, one has

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \left( -b_4 B \lambda_{\alpha}^2 \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2 + b_5 |W_{\alpha}| + b_6 |\lambda_{\alpha}||W_{\alpha}| \right) \]

where the positive constant \( b_5 \) depends on \( b_6, b_7, b_8 \), and the positive constant \( b_6 \) depends on \( b_6, b_7, b_8 \). Since the term \( \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot |\lambda_{\alpha}| \) in (44) can be estimated by

\[ b_3 \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot |\lambda_{\alpha}| \leq \frac{b_{10}}{\delta_{\varepsilon}} \sigma_{k-1} + \frac{\delta_{\varepsilon}}{2} \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2, \]

where the positive constant \( b_{10} \) depends on \( b_3 \), taking \( B \) large enough, we can get a bound on \( \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot W_{\alpha,\alpha} \) of the expected form.

For the points of the second type, let us consider two cases:

**First case.** \( \lambda_1 \leq 0 \). The inequality (42) then becomes

\[ \frac{\partial \sigma_k}{\partial \lambda_1} \cdot \lambda_1^2 \leq b_0 \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2. \]

(45)

From [4, Lemma 4.3], for \( \alpha \geq 2, \exists \delta_{\varepsilon'} := \delta_{\varepsilon'}(n, \rho, \varepsilon) \), s.t. \( \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2 \geq \delta_{\varepsilon'} \). Hence,

\[ \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2 \leq \frac{1}{\delta_{\varepsilon'}} (\sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2 \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2), \]

and by using inequality (45), it follows that

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2 \leq \frac{b_0 + 1}{\delta_{\varepsilon'}} (\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha,\alpha}^2 \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2). \]

Thus, we can obtain

\[ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( -b_4 B \lambda_{\alpha}^2 \sum_{s=1}^{n-1} |\eta_{s,\alpha}|^2 + b_5 |W_{\alpha}| + b_6 |\lambda_{\alpha}||W_{\alpha}| \right) \]

\[ \leq \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( b_5 |W_{\alpha}| + b_6 |\lambda_{\alpha}||W_{\alpha}| - \frac{B}{b_{11}} \lambda_{\alpha,\alpha}^2 \right) \]

\[ \leq \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_{\alpha}} \left( b_5 (1 + \frac{b_6}{b_{11}}) W_{\alpha}^2 + b_6 \delta_{\varepsilon} \lambda_{\alpha,\alpha}^2 - \frac{B}{b_{11}} \lambda_{\alpha,\alpha}^2 \right), \]
with \( b_{11} = \frac{b_{5} + 1}{b_{k} \delta \epsilon} \), and
\[
b_{3} \sum_{\alpha=1}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} |\lambda_{\alpha}| \leq b_{12} \sigma_{k-1} + \frac{1}{2b_{13}} \sum_{\alpha=1}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha}^{2},
\]
where positive constants \( b_{12}, b_{13} \) depend on \( b_{3} \). Now one can give the expected bound on \( \sum_{\alpha=1}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} \cdot \mathbb{W}_{\alpha} \) with \( B \) large enough.

**Second case.** \( \lambda_1 > 0 \). Then, one has
\[
\sum_{\alpha=1}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} (b_{5} |\mathbb{W}_{\alpha}| + b_{6} |\lambda_{\alpha}||\mathbb{W}_{\alpha}|) = \frac{\partial \sigma_{k}}{\partial \lambda_{1}} (b_{5} |\mathbb{W}_{1}| + b_{6} |\lambda_{1}||\mathbb{W}_{1}|) + \sum_{\alpha=2}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} (b_{5} |\mathbb{W}_{\alpha}| + b_{6} |\lambda_{\alpha}||\mathbb{W}_{\alpha}|)
\]
\[
\leq b_{14} \frac{\partial \sigma_{k}}{\partial \lambda_{1}} \cdot \lambda_{1} \mathbb{W}_{1} + b_{15} \sum_{\alpha=1}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} \cdot \mathbb{W}_{\alpha}^{2} + b_{16} \sum_{\alpha=2}^{n} \frac{\partial \sigma_{k}}{\partial \lambda_{\alpha}} \cdot \lambda_{\alpha}^{2},
\]
where the positive constants \( b_{14}, b_{15} \) depend on \( b_{5}, b_{6} \). Let us bound \( Q = \frac{\partial \sigma_{k}}{\partial \lambda_{1}} \cdot \lambda_{1} \mathbb{W}_{1} \), \( Q = \mathbb{W}_{1} \left( \psi - \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} \right) \) with \( \psi = \frac{\partial \sigma_{k}}{\partial \lambda_{1}} \cdot \lambda_{1} + \psi_{1} \), \( \psi_{1} \) is positive. We must again consider two cases:

If \( \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} \geq 0 \),
\[
|Q| \leq |\mathbb{W}_{1}| \left( \psi - \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} \right) \leq |\mathbb{W}_{1}| \psi \leq b_{16} |D\mathbb{W}|,
\]
where the positive constant \( b_{16} \) depends on \( \psi \).

If \( \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} < 0 \),
\[
|Q| = -|\mathbb{W}_{1}| \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} + |\mathbb{W}_{1}| \psi \leq -|\mathbb{W}_{1}| \frac{\partial \sigma_{k+1}}{\partial \lambda_{1}} + b_{16} |D\mathbb{W}|.
\]
(46)

Since
\[
\mathbb{W}_{1} = g_{1} + \sum_{t=1}^{n} g_{p_{t}} u_{t} \tilde{\eta}_{t}^{1} - B \sum_{s=1}^{n-1} \tilde{\eta}_{s}^{1} (u_{s} - u_{s}(x_{0})).
\]
If \( |\sum_{t=1}^{n} g_{p_{t}} \tilde{\eta}_{t}^{1} - B \sum_{s=1}^{n-1} \tilde{\eta}_{s}^{1} (u_{s} - u_{s}(x_{0}))| \) is less than \( \tilde{\beta} \) and \( |g_{1}| \) is less than \( b_{17} \), then
\[
|\mathbb{W}_{1}| \leq \tilde{\beta} \lambda_{1} + b_{17}.
\]
Besides, \( \forall s \in \{1, \ldots, n-1\}, |\tilde{\eta}_{s}^{1}| \leq (\delta_{\epsilon})^{1/2} \). Choosing \( \epsilon = \epsilon (n, \rho, B) \) sufficiently small such that
\[
B (\delta_{\epsilon})^{1/2} \leq 1,
\]
and then, with such a choice of \( \epsilon \) we get: \( \forall s \in \{1, \ldots, n-1\}, |B \tilde{\eta}_{s}^{1}| \leq 1. \) Let us then take \( \tilde{\beta} \) as
\[
\tilde{\beta} = \sup_{D_{\gamma, \rho}} \left( \sum_{t=1}^{n} |g_{p_{t}}| + 2(n-1) \right).
\]
(48)
Hence,

\[-|\mathbb{W}_1| \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \leq - \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \tilde{\beta} \lambda_1 - b_{17} \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \]

\[\leq - \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \tilde{\beta} \lambda_1 + b_{17} \left( \frac{\partial \sigma_k}{\partial \lambda_1} \cdot \lambda_1 - \psi \right).\]

(49)

Let us find a bound on \(\frac{- \partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1\):

\[
\sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2 = \sum_{\alpha=2}^n \left( \psi - \frac{\partial \sigma_{k+1}}{\partial \lambda_\alpha} \right) \lambda_\alpha
\]

\[= \psi \sigma_1(\lambda|1) - \sum_{\alpha=2}^n \frac{\partial \sigma_{k+1}}{\partial \lambda_\alpha} \cdot \lambda_\alpha
\]

\[= \psi \sigma_1(\lambda|1) - \sum_{\alpha=1}^n \frac{\partial \sigma_{k+1}}{\partial \lambda_\alpha} \cdot \lambda_\alpha + \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1
\]

\[= \psi \sigma_1(\lambda|1) - (k+1) \sigma_{k+1} + \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1.\]

Let us note that \(\sigma_{k+1} = \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1 + \frac{\partial \sigma_{k+2}}{\partial \lambda_1}\). Hence,

\[- \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1 = \frac{1}{k} \left( \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2 + (k+1) \frac{\partial \sigma_{k+2}}{\partial \lambda_1} - \psi \cdot \sigma_1(\lambda|1) \right).\]

Recalling \(\psi \sigma_1(\lambda|1) \geq 0\) and (42), we have

\[- \frac{\partial \sigma_{k+1}}{\partial \lambda_1} \cdot \lambda_1 \leq \frac{1}{k} (1 + (k+1) R_0) \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2.\]

(50)

Last, we estimate \(\frac{\partial \sigma_k}{\partial \lambda_1} \cdot \lambda_1\) as follows:

\[
\frac{\partial \sigma_k}{\partial \lambda_1} \cdot \lambda_1 = k \sigma_k - \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha
\]

\[\leq k \sigma_k + \frac{1}{4 \gamma} \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha + \gamma \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2
\]

\[\leq b_{18} (1 + \sigma_{k-1}) + \gamma \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2,\]

(51)

where the positive constant \(b_{18}\) depends on \(\psi, k, \gamma,\) and \(\gamma\) is to be specified. With \(\gamma\) sufficiently small, inequalities (46), (49), (50), and (51) then give

\[|Q| \leq b_{19} (1 + |D\mathbb{W}| + \sigma_{k-1}) + b_{20} \sum_{\alpha=2}^n \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2,\]
where the positive constant $b_{19}$ depends on $\tilde{\beta}, k, \psi, b_{16}, b_{17}, b_{18}$, the positive constant $b_{20}$ depends on $\tilde{\beta}, b_{17}, B_0$. Recalling the inequality

$$\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \mathbb{W}_{\alpha \alpha} \leq b_3 \left( 1 + |D\mathbb{W}| + \sigma_{k-1} + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot |\lambda_\alpha| \right) - b_4 B \delta_i' \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2$$

$$+ \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} (b_3 |\mathbb{W}_{\alpha}| + b_6 |\lambda_\alpha||\mathbb{W}_{\alpha}|),$$

with estimates (51),

$$b_3 \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot |\lambda_\alpha| \leq b_{21} (1 + \sigma_{k-1}) + b_{22} \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2$$

and

$$\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} (b_3 |\mathbb{W}_{\alpha}| + b_6 |\lambda_\alpha||\mathbb{W}_{\alpha}|) \leq b_{23} (1 + |D\mathbb{W}| + \sigma_{k-1}) + b_{24} \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2 + b_{18} \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \mathbb{W}_{\alpha}^2,$$

where the positive constant $b_{21}$ depends on $b_3, b_{18}$, the positive constant $b_{22}$ depends on $b_3, \gamma$, the positive constant $b_{23}$ depends on $b_{14}, b_{19}$, and the positive constant $b_{24}$ depends on $b_{20}$. Let $B = B \left( n, k, \rho, \|\mathbb{g}\|_{1, D_r^\alpha}, B_0 \right)$ be large enough, independent of $\varepsilon$, and compatible with (47). Then, we have

$$\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \mathbb{W}_{\alpha \alpha} \leq b_{25} \left( 1 + |D\mathbb{W}| + \sigma_{k-1} + \sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \mathbb{W}_{\alpha}^2 \right)$$

$$+ b_{26} \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2 - b_4 B \delta_i' \sum_{\alpha=2}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \lambda_\alpha^2,$$

where the positive constant $b_{25}$ depends on $b_3, b_{15}, b_{21}, b_{23}$, and the positive constant $b_{26}$ depends on $b_{22}, b_{24}$. Now, we can give the expected bound on $\sum_{\alpha=1}^{n} \frac{\partial \sigma_k}{\partial \lambda_\alpha} \cdot \mathbb{W}_{\alpha \alpha}$ with $B$ large enough. This completes the proof of Lemma 5.1.

Setting $\mathbb{W} = \exp \left( -\tilde{\mathbb{W}}_1 \mathbb{g}(x_0, D_u(x_0)) \right) - \exp \left( -\tilde{\mathbb{W}}_1 \mathbb{W} \right) - b |x - x_0|^2$, from Lemma 5.1 and an appropriate $b$ as in [22], we can get the following crucial inequality.

**Lemma 5.2.** For $b = b \left( n, B_1, \|\mathbb{g}\|_{0, D_r^\alpha}, B \right)$ sufficiently large, $\mathbb{W}$ satisfies

$$\sum_{i,j} \frac{\partial \sigma_k}{\partial q_{ij}} (u) \mathbb{W}_{ij} \leq B_2 (1 + |D\mathbb{W}|)$$

on $M^n \cap B_r(x_0)$, where $B_2 = B_2 \left( B_1, b, r, \|\mathbb{g}\|_{0, D_r^\alpha}, B \right)$.

**Lemma 5.3.** Let $\tilde{\psi} \in C^2 (\partial M^n \cap B_r(x_0))$ and $a_0 \in \mathbb{R}$. The function $\tilde{v} = -a_0 |x - x_0|^2 - \tilde{h}(d) + \psi(x')$, with $\tilde{h}(d) = B_3 \left( 1 - e^{-|D\mathbb{g}|} \right)$, satisfies: for any positive function $p, Dp \in \mathbb{R}^n$, s.t. $\frac{|Dp|}{p} \leq \rho < 1$, $\forall i \in \{1, \ldots, k\}$, $F_i(Dp, D^2\mathbb{u}) > 0$, and

$$\sum_{i,j} \frac{\partial F_k}{\partial q_{ij}} (u) \mathbb{W}_{ij} \geq B_2 (1 + |D\mathbb{W}|)$$

on $M^n \cap B_r(x_0)$

(52)

for suitable parameters $B_3, B_4$, depending only on $n, k, M^n, \psi, \rho, a_0, \|\tilde{\psi}\|_{2, \partial M^n \cap B_r(x_0)}$, and $B_2$. 


Here, $d$ denotes the distance function to the boundary of $M^n$, $x = (x', x_n)$ in a given adapted basis at the boundary-point $x_0$, and $F_k(Dp, q) = F_k(g^{ij}(Dp)q)$, $g^{ij}(Dp) = \frac{1}{p^2} \left( \sigma^{ij} + \frac{p^l p^l}{p^2 - |Dp|^2} \right)$, $F_k(q)$ denotes the $k$th symmetric function of the eigenvalues of $q$. Particularly, $F_k(u) = \sigma_k(u)$.

To prove Lemma 5.3, we will need the following fact.

**Lemma 5.4.** If $q$ is a symmetric nonnegative matrix, then for any positive function $p$, $Dp \in \mathbb{R}^n$, s.t. $\frac{|Dp|}{p} \leq \rho < 1$, $\forall k \in \{1, \ldots, n\}$,

$$F_k(Dp, q) \geq \left( \frac{1}{p^2} \right)^k F_k(q).$$

**Proof.** Set $\forall k \in \{1, \ldots, n\}, \forall (Dp, q) \in B(0, 1) \times S_n(\mathbb{R})$, where $S_n(\mathbb{R})$ denotes $n \times n$ order real symmetric matrix,

$$F_k(Dp, q) = F_k(g^{ij}(Dp)q).$$

The expression (53), independent of the orthonormal basis of $\mathbb{R}^n$, is chosen to express $p$ and $q$. Take an orthonormal basis $\{e_1, \ldots, e_n\}$ with $e_1$ directed along $Dp$. Then,

$$g^{ij}(Dp)q = \begin{pmatrix}
1/p^2 & \cdots & 1/p^2 \\
1/p^2 & \cdots & 1/p^2 \\
\vdots & \ddots & \vdots \\
1/p^2 & \cdots & 1/p^2
\end{pmatrix} q = \begin{pmatrix}
\frac{1}{p^2 - |Dp|^2} q_{11} & \frac{1}{p^2 - |Dp|^2} q_{12} & \cdots & \frac{1}{p^2 - |Dp|^2} q_{1n} \\
\frac{1}{p^2 - |Dp|^2} q_{21} & \frac{1}{p^2 - |Dp|^2} q_{22} & \cdots & \frac{1}{p^2 - |Dp|^2} q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{p^2 - |Dp|^2} q_{n1} & \frac{1}{p^2 - |Dp|^2} q_{n2} & \cdots & \frac{1}{p^2 - |Dp|^2} q_{nn}
\end{pmatrix}.
$$

Thus,

$$F_k(g^{ij}(Dp)q) = \frac{1}{p^2 - |Dp|^2} \left( \frac{1}{p^2} \right)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \begin{vmatrix}
q_{i_1 i_2} & \cdots & q_{i_1 i_k} \\
q_{i_2 i_1} & \cdots & q_{i_2 i_k} \\
\vdots & \ddots & \vdots \\
q_{i_k i_1} & \cdots & q_{i_k i_k}
\end{vmatrix} + \left( \frac{1}{p^2} \right)^k \sum_{I=(i_1, \ldots, i_k)} q_{i_1 i_2} \cdots q_{i_k i_k} \geq \left( \frac{1}{p^2} \right)^k F_k(q).$$

Using $\frac{|Dp|}{p} \leq \rho < 1$, since the determinants in the first sum are nonnegative when $q$ is nonnegative, Lemma 5.4 follows. \qed

**Proof of Lemma 5.3.** We first show that for any positive function $p$, $Dp \in \mathbb{R}^n$, s.t. $\frac{|Dp|}{p} \leq \rho < 1$, $\forall l \in \{1, \ldots, k\}$,

$$F_l(Dp, q) \geq B_l(1 + |D\vartheta|)^l,$$  \hspace{1cm} (54)

where $B_l$ is as large as desired, and if $B_3$, $B_4$ are suitable parameters. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{H}^n(1)$, $q \in S_n(\mathbb{R})$. Denote by $Dp'$ the component of $Dp$ on $\{e_1, \ldots, e_{n-1}\}$, by $q' \in S_{n-1}(\mathbb{R})$ the restriction of $q$ on $\{e_1, \ldots, e_{n-1}\}$, and by $q_{(n,n)}$ the $n \times n$ matrix deduced from $q$ by setting its $(n, n)$ coefficient to 0. It is easy to show that

$$F_l(Dp, q) = \frac{1}{p^2 - |Dp'|^2} q_{nn} F_{l-1} \left( Dp', q' \right) + O(|q_{(n,n)}|),$$  \hspace{1cm} (55)

where $O(|q_{(n,n)}|)$ denotes quantities estimated by $C|q_{(n,n)}|$ with $C$ depending on $Dp$, $p$. Let $x \in M^n \cap B_r(x_0)$, and $\{e_1, \ldots, e_n\}$ be an adapted basis at the boundary-point $y$ minimizing the distance between $x$ and $\partial M^n$. We get

$$F_l(Dp, D^2\vartheta) = \frac{1}{p^2 - |Dp'|^2} \sigma_{nn} F_{l-1} \left( Dp', D^2\vartheta' \right) + O(|D^2\vartheta_{(n,n)}|),$$  \hspace{1cm} (56)
where

\[
D^2 \tilde{v} = \tilde{h} \begin{pmatrix}
-\frac{2a_0}{\tilde{h}'} + \frac{\kappa_1}{1 - \kappa_1 d'} - \frac{2a_0}{\tilde{h}'} + \frac{\kappa_{n-1}}{1 - \kappa_{n-1} d'} - \frac{2a_0}{\tilde{h}'} - \tilde{h}'' \\
\end{pmatrix}
\begin{pmatrix}
\frac{\partial^2 \tilde{\phi}}{\partial \psi \partial \chi_j} \\
0 & \cdots & 0
\end{pmatrix}_{1 \leq i, j \leq n-1}.
\]

Let us bound \( F_{l-1}(Dp', D^2 \tilde{v}') \) from below: From the definition of \( F_{l-1} \), one knows that

\[
\left( \frac{1}{\tilde{h}'} \right)^{l-1} F_{l-1}(Dp', D^2 \tilde{v}') \text{ tends to } F_{l-1}(g^l(Dp') \text{diag}(\kappa_1, \ldots, \kappa_{n-1})),
\]

where \( d \) tends to 0, \( \tilde{h}' \) tends to +\( \infty \), \( \kappa_1, \ldots, \kappa_{n-1} \) denote the principal curvatures of \( \partial M^n \) associated with \( \{e_1, \ldots, e_{n-1}\} \). By Lemma 5.4, (57) can be estimated from below. Taking \( r \) small and \( \tilde{h}' \) large, we thus obtain the estimate

\[
F_{l-1}(Dp', D^2 \tilde{v}') \geq \delta \left( \frac{1}{\tilde{h}'} \right)^l,
\]

where \( \delta \) is a positive constant under control. This allows the estimate of \( F_l(Dp, D^2 \tilde{v}) \): from (56), we have

\[
F_l(Dp, D^2 \tilde{v}) \geq \frac{1}{\tilde{h}'} \frac{1}{\tilde{h}'} F_k(Du, D^2 \tilde{v}),
\]

where \( F_k^{-1}(u) \) is itself estimated from below. In view of (54), \( F_k^{-1}(Du, D^2 \tilde{v}) \) is larger than \( B_3 (1 + |D\tilde{v}|) \) if \( B_3 \) and \( B_4 \) are suitable parameters, where \( B_5 \) is as large as desired, that is, Lemma 5.3 holds for sufficiently large \( B_3, B_4 \) under control. \( \square \)

**Estimate of mixed second derivatives.** Let \( \{e_1, \ldots, e_n\} \) still denote an adapted basis at the boundary-point \( x_0 \) and let \( t \in \{1, \ldots, n-1\} \). Our purpose is to estimate \( u_{t\gamma}(x_0) \). For any \( x \in M^n \cap B_r(x_0) \), let \( \tilde{\xi} = e_t + \tilde{\rho}(x') e_n \), where \( \tilde{\rho} \) denotes the function locally defined on \( T_{x_0} \partial M^n \) whose graph is \( \tilde{\partial} M^n \). Set

\[
g(x, p) = \langle p, \tilde{\xi}(x) \rangle = p_t + \tilde{\rho}(x') e_n.
\]

Using Lemma 5.1 and Lemma 5.2, there exists the \( B = B \left( n, k, \rho, B_3, \|g\|_{1, D_r, \rho} \right), B_1 = B_1 \left( n, k, M^n, \psi, \rho, B_0, \|g\|_{2, D_r, \rho} \right), \) and \( b = b \left( n, B_1, \|g\|_{0, D_r, \rho}, B \right) \) sufficiently large such that the function

\[
\tilde{W} = \exp \left( -B_1 u_{\tilde{\xi}}(x_0) \right)
- \exp \left( -B_1 u_{\tilde{\xi}} + \frac{BB_1}{2} \sum_{s=1}^{n-1} (u_s - u_s(x_0))^2 \right) - b|x - x_0|^2
\]
satisfies
\[ \sum_{i,j} \frac{\partial^2 F_k}{\partial q_{i,j}}(u) \overline{\nabla}_{ij} \leq B_2 (1 + |D\overline{\nabla}|), \]
where \( B_2 = B_2 \left( B_1, b, r, \|g\|_{0,D_1,p}, B \right) \). Define
\[ \tilde{\varphi}(x') = \exp \left( -B_1 \varphi_t(x_0) \right) \exp \left( -B_1 \varphi_t(x') \right), \]
\[ \exp \left( BB_1 \sum_{s=1}^{n-1} \left( \varphi_s(x') - \varphi_s(x_0) \right)^2 + 2B_1 B \left( |\varphi_n(x')|^2 + 1 \right) |D\tilde{\varphi}(x')|^2 \right) \]
and \( \tilde{v} = -a_0 |x - x_0|^2 - \tilde{h}(d) + \tilde{\varphi}(x') \), where \( \tilde{h}(d) = B_3 \left( 1 - e^{-B_{4d}} \right) \). For suitable constants \( a_0, B_3, B_4 \), from Lemma 5.3, we have \( \tilde{v} \leq \overline{\nabla} \) on \( \partial (M^n \cap B_r(x_0)) \) and \( \sum_{i,j} \frac{\partial^2 F_k}{\partial q_{i,j}}(u) \tilde{v}_{ij} \geq B_2 (1 + |D\tilde{v}|) \) on \( \overline{M^n} \cap B_r(x_0) \). By the comparison principle in [4, Lemma 4.5], we have \( \tilde{v}(x_0) = \overline{\nabla}(x_0) \), and then since \( \tilde{v}(x_0) \leq \overline{\nabla}_n(x_0) \), that is, \( \tilde{v}_n(x_0) \leq B_1 u_n(x_0) \exp \left( -B_1 \varphi_t(x_0) \right) \). In other words,
\[ u_n(x_0) \geq B_6, \]
where \( B_6 = B_6 \left( n, k, M^n, \psi, \rho, \|\varphi\|_{3,\overline{M^n}} \right) \). To estimate \( u_n(x_0) \) from above, we do the similar argument with \( g(x, p) = -p_{t} - \tilde{\rho}(x') p_{n} \).

**Estimate of normal second derivatives.** Now, we want to estimate an upper bound on \( h_{nn} \). A lower bound on \( h_{nn} \) easily follows from \( \sigma_1(u) > 0 \) and the estimates of tangential and mixed second derivatives. An upper bound on \( h_{nn} \) on \( \partial M^n \) amounts to a lower bound on
\[ A_k = \frac{\partial F_k}{\partial h_{nn}}(u) \]
on \( \partial M^n \) by a positive quantity under control, since
\[ F_k(u) = A_k \cdot h_{nn} + B_k = \psi, \]
where \( B_k \) depends only on \( u, Du \), the tangential and mixed second derivatives of \( u \), which are already estimated.

**Lemma 5.5.** \( A_k \) is given by
\[ A_k = \frac{1}{u^2} \frac{u^2 - |\partial \tilde{\varphi}|^2}{u^2 v^2} \cdot F_{k-1}(\partial \varphi, \partial^2 \varphi + u_{\rho} \partial \tilde{\varphi}), \]
where \( \partial \) denotes the tangential gradient and \( \tilde{\varphi} \) the future-directed unit normal to \( \partial M^n \).

We continue to proceed as follows: Estimate \( h_{nn} \) via the previous method at a point \( y \) of \( \partial M^n \), where \( F_{k-1}(\partial \varphi, \partial^2 \varphi + u_{\rho} \partial \tilde{\varphi}) \) is minimum. It implies a lower bound on \( F_{k-1}(\partial \varphi, \partial^2 \varphi + u_{\rho} \partial \tilde{\varphi}) \) because
\[ F_{k-1}(\partial \varphi, \partial^2 \varphi + u_{\rho} \partial \tilde{\varphi})(y) \geq B_6 F_k(u)(y). \]

For the last inequality, see [22], and the positive constant \( B_6 \) depends on \( n, k, \rho, M^n \). From the definition of \( y \), the function \( F_{k-1}(\partial \varphi, \partial^2 \varphi + u_{\rho} \partial \tilde{\varphi}) \) admits itself a lower bound on \( \partial M^n \), and so does \( A_k \), which yields an estimate on the second normal derivatives at every boundary point.

Let us take \( g(x, p) = F_{k-1} \left( \partial \varphi(x'), \partial^2 \varphi(x') + \langle p, \tilde{\varphi}(x') \rangle \partial \tilde{\varphi}(x') \right) \), where \( x = (x', x_n) \) in the basis \( \{e_1, ..., e_n\} \). A priori \( g \) is concave with respect to \( p \) only for \( k = 2 \), then \( g(x, p) = F_1 \left( \partial \varphi(x'), \partial^2 \varphi(x') + \langle p, \tilde{\varphi}(x') \rangle \partial \tilde{\varphi}(x') \right) \). The rest is almost the same as the argument in [4, pp. 27–28], so we omit it.
Remark 5.6. Nearly the whole part of our proof in this section is valid for any \( k = 1, 2, \ldots, n \). However, the auxiliary function \( g(x, p) \) in Lemma 5.1 is concave with respect to \( p \), and then we need to use the constraint \( k = 2 \) to estimate the double normal second derivatives on the boundary.

6 | EXISTENCE AND UNIQUENESS

At the end, we can show the existence and uniqueness of solutions to the PCP (4) as follows:

Proof of Theorem 1.8. Clearly, the PCP (4) is equivalent with the following Dirichlet problem:

\[
\begin{aligned}
\sigma_k(u, Du, D^2u) &= \psi(x, u, \vartheta(u, Du)), & x \in M^n \subset \mathbb{R}^{n+1}, \\
u &= \varphi, & x \in \partial M^n,
\end{aligned}
\]

and the method of continuity can be used to get the existence of its solutions. We divide the argument into three steps as follows:

**Step 1.** For each \( t \in [0, 1] \), consider the following problem \(^8\)

\[
\begin{aligned}
t \sigma_k(u, Du, D^2u) + (1-t)\Delta u &= \psi(x, u, \vartheta(u, Du)), & x \in M^n, \\
u &= \varphi, & x \in \partial M^n.
\end{aligned}
\]

Clearly, for \( t = 0 \), (58) corresponds to the Dirichlet problem of the Laplace operator. Let \( \omega = u - \varphi \), and then (58) is equivalent to

\[
\begin{aligned}
t \sigma_k(\omega + \varphi, D(\omega + \varphi), D^2(\omega + \varphi)) + (1-t)\Delta(\omega + \varphi)
&= \psi(x, \omega + \varphi, \vartheta((\omega + \varphi), D(\omega + \varphi))), & x \in M^n, \\
\omega &= 0, & x \in \partial M^n.
\end{aligned}
\]

Now, we set

\[
\mathcal{X} := \left\{ \omega \in C^{2,\alpha}(\overline{M^n}) | \omega = 0 \text{ on } \partial M^n \right\}
\]

and

\[
\mathcal{P}(\omega, t) := t \sigma_k(\omega + \varphi, D(\omega + \varphi), D^2(\omega + \varphi)) + (1-t)\Delta(\omega + \varphi) - \psi(x, \omega + \varphi, \vartheta((\omega + \varphi), D(\omega + \varphi))).
\]

Then, the solvability of (59) is equivalent to find a function \( \omega \in \mathcal{X} \) such that \( \mathcal{P}(\omega, t) = 0 \) in \( M^n \).

Set

\[
I = \{ t \in [0, 1] | \text{ there exists a } \omega \in \mathcal{X} \text{ such that } \mathcal{P}(\omega, t) = 0 \}.
\]

By the standard Schauder theory for the Laplace operator (see, e.g., [20, Chapter 5]), we know that 0 \( \in I \). The rest is to show 1 \( \in I \). To do this, we need to prove that I is both open and closed in [0,1].

**Step 2.** We first show that I is open. Note that \( \mathcal{P} : \mathcal{X} \times [0, 1] \to C^2(\overline{M^n}) \) is of class \( C^1 \) and using its Fréchet derivative, we have a uniformly elliptic operator with \( C^\alpha \)-coefficients. The Fréchet derivative here is given by

\[
\begin{aligned}
\mathcal{F}_\omega(\omega, t)(\vartheta) := & \lim_{\varepsilon \to 0} \frac{\mathcal{P}(\omega + \varepsilon \vartheta, t) - \mathcal{P}(\omega, t)}{\varepsilon}.
\end{aligned}
\]

By the linear Schauder theory, \( \mathcal{F}_\omega(\omega, t) \) is an invertible operator from \( \mathcal{X} \) to \( C^2(\overline{M^n}) \). Suppose \( t_0 \in I \), that is, \( \mathcal{P}(\omega_{t_0}, t_0) = 0 \) for some \( \omega_{t_0} \in \mathcal{X} \). By the implicit function theorem, for any \( t \) close to \( t_0 \), there is a unique \( \omega_t \in \mathcal{X} \), close to \( \omega_{t_0} \) in the \( C^{2,\alpha} \)-norm, satisfying \( \mathcal{F}(\omega_t, t) = 0 \). Hence, \( t \in I \) for all such \( t \), and so I is open.
**Step 3.** For the closedness, by the lower order estimates in Section 3, the curvature estimates in Section 4 (i.e., Theorems 1.4, 1.6) and boundary $C^2$ estimates in Section 5, we know that any $\omega$ in $\mathcal{X}$ of $F(\omega, t) = 0$ in $\mathcal{M}^n$ satisfies a uniform $C^{2,\alpha}$-estimate, independent of $t$, that is,

$$|\omega|^1_{C^{2,\alpha}(\mathcal{M}^n)} \leq C,$$  

independent of $t$.

Using Arzelà–Ascoli theorem, the closedness of $I$ follows directly.

Therefore, by the above argument, we know that $I$ is the whole unit interval. Then, the function $\omega^1$ is our desired solution of (59) corresponding to $t = 1$. The uniqueness of solutions to the PCP (4) can be obtained by directly using the comparison principle to the $\sigma_k$ operator. This completes the proof.

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**Conflict of Interest Statement**

The authors declare that there are no conflicts of interests regarding the publication of this paper.

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**Endnotes**

1 The reason why we call $\mathcal{H}^n(1)$ a hyperbolic plane is that it is a simply connected Riemannian $n$-manifold with constant negative curvature and is geodesically complete.

2 Clearly, for accuracy, here $D^i u$ should be $D_i u$. In the sequel, without confusion and if needed, we prefer to simplify covariant derivatives like this. In this setting, $u_{ij} := D_i D_j u$, $u_{ik} := D_i D_k D_j u$ mean $u_{ij} = D_i D_j u$ and $u_{ik} = D_i D_j D_k u$, respectively.

3 Provided the dimension constant is neglected.

4 Also different from the one here.

5 Clearly, in (1) of Remark 1.10 here, $\sigma_k(\lambda(\cdot))$ denotes the $k$th elementary symmetric function of eigenvalues of a given tensor—the second fundamental form $A$.

6 Using similar arguments to [3, 7], one can easily get the existence of solutions to the Dirichlet problems (19) and (20), respectively.

7 This can be assured, since $\varphi$ is defined on $\mathcal{M}^n$ and of course one can require its extension to the normal bundle of $\mathcal{M}^n$ to be constant.

8 Clearly, the operator $\Delta$ in the Dirichlet problem (58) should be the Laplacian on $\mathcal{M}^n \subset \mathcal{H}^n(1)$. In fact, this happens to all symbols $\Delta$ in Section 6. For convenience and if without confusion, we abuse the notation $\Delta$, which in this paper was used to stand for the Laplacian on different geometric objects (i.e., on the convex piece $\mathcal{M}^n$ or the spacelike graphic hypersurface $G$).

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