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The Born series for $S$-wave quartet $nd$ scattering at small cutoff values

Abstract Perturbative expansions, the Born series, of the scattering length and the amplitude of $S$-wave neutron-deuteron scattering for spin quartet channel below deuteron breakup threshold are studied in pionless effective field theory at small cutoff values. A three-body contact interaction is introduced when the integral equation is solved using the small cutoffs. After renormalizing the three-body interaction by using the scattering length, we expand the integral equation as the ordinary and inverted Born series. We find that the scattering length and the phase shift are considerably well reproduced with a few terms of the inverted Born series at relatively large cutoff values, $\Lambda \simeq 100$ MeV.

Keywords Inverted Born series · Small cutoffs · $S$-wave quartet $nd$ scattering

1 Introduction

After Weinberg suggested the application of chiral perturbation theory, a low energy effective field theory of QCD, to nuclear force [1], a lot of works have been done during last two decades. For reviews, see, e.g., Refs. [2; 3; 4; 5]. Meanwhile, new facilities for the rare isotope beams, in which exotic nuclei near the neutron (proton) drip line can be created, provide us a new opportunity for investigation [6; 7; 8]. Such a limit close to the drip line where the single neutron (proton) separation energy vanishes can be theoretically recognized as unitary limit (to be discussed below), and thus the unitary limit would be a good theoretical starting point to study the exotic nuclei near the nuclear drip line. In this work, we discuss a perturbative expansion of an integral equation, as the inverted Born series which can be related to the unitary limit, by sending the cutoff parameter to small values (along with the ordinary Born series). We study it in $S$-wave neutron-deuteron ($nd$) scattering for spin quartet channel in pionless effective field theory (EFT).

Pionless EFT [9; 10] is one of low energy EFTs for few nucleon systems, in which one chooses a typical momentum scale $Q$ smaller than the pion mass, $m_\pi$, and thus the pions are regarded as heavy degrees of freedom and integrated out of effective Lagrangian. Thus the large scale of the pionless EFT is $\Lambda \simeq m_\pi$, and the theory provides us a systematic expansion scheme in terms of $Q/\Lambda$. In applications of the pionless EFT, much more attention has been paid to $S$-wave $nd$ scattering for spin doublet ($S = 1/2$) channel because the one-nucleon-exchange interaction becomes “singular” [11; 12]. To control the singularity one promotes a three-body contact interaction to leading order (LO), and the coupling constant of the contact interaction exhibits so called limit-cycle when the scale parameter $\Lambda$ is sent to the asymptotic limit. This behavior is understood to be associated with “Efimov effect”. On the other hand, an application of pionless EFT to $S$-wave $nd$ scattering for spin quartet ($S = 3/2$) channel

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is regarded well known because the phase shift $\delta_0$ of the quartet $nd$ scattering is almost perfectly described by two effective range parameters in two-body deuteron channel, and no cutoff dependence is reported \cite{13, 13}.

In this work, we revisit the $S$-wave spin quartet $nd$ scattering below deuteron breakup threshold in pionless EFT and reduce the cutoff parameter $\Lambda$ smaller than $m_\pi$. The small scale limit might be interesting for studying, e.g., a relation between the pionless EFT and a Halo/Cluster EFT \cite{13} in which one may choose a typical scale smaller than the deuteron breakup momentum, and the deuteron is never broken into two nucleons and could be regarded as a cluster or an elementary field \cite{13}.

When the value of $\Lambda$ is sent to significantly smaller than $m_\pi$, the cutoff dependence emerges even in the quartet channel. Thus we introduce a three-body contact interaction when we solve the integral equation using the small cutoff values. After renormalizing the strength of the three-body interaction by using the scattering length $a_3$ of the $S$-wave spin quartet $nd$ scattering we expand the integral equation for the scattering length $a_3$ and the scattering amplitude in terms of the Born series up to next-to-next-to-leading order (NNLO) \cite{10}. We expand the Born series around so called trivial and non-trivial fixed point studied in the renormalization group analysis by Birse, McGovern, and Richardson \cite{17}. The trivial fixed point corresponds to weak coupling limit where all interactions vanish, whereas the non-trivial fixed point does to unitary limit where the scattering length becomes infinity or the binding energy vanishes. Around the unitary limit, the Born series is realized as the inverted Born series and expanded around the inverse of the amplitude \cite{18}. We find that, by reducing the cutoff value, both $a_3$ and $\delta_0$ are considerably well reproduced by a few terms of the inverted Born series at relatively large cutoff values, $\Lambda \simeq 100$ MeV.

This work is organized as the following. In Sec. 2, effective Lagrangian is displayed. In Sec. 3, integral equations for the scattering length and the amplitude are given, and the coupling of the three-body interaction is renormalized by the scattering length $a_3$. In Sec. 4, the scattering length and the amplitude are expanded in terms of the ordinary and inverted Born series, and the numerical results are obtained. Finally, in Sec. 5, the discussion and conclusions are presented.

2 Effective Lagrangian

The effective Lagrangian for the $S$-wave spin quartet $nd$ scattering in pionless EFT reads \cite{10, 12}

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_t + \mathcal{L}_3,$$

where $\mathcal{L}_N$ is the standard one-nucleon Lagrangian in the heavy-baryon formalism,

$$\mathcal{L}_N = N^\dagger \left\{ iv \cdot \mathbf{D} + \frac{1}{2m_N} \left[ (v \cdot \mathbf{D})^2 - D^2 \right] + \cdots \right\} N,$$

where $v^\mu$ is a velocity vector satisfying a condition $v^2 = 1$, $D_\mu$ is the covariant derivative, and $m_N$ is the nucleon mass. The dots denote higher order terms which are not relevant in the present work. $\mathcal{L}_t$ is dibaryon effective Lagrangian for $^3S_1$ state,

$$\mathcal{L}_t = \sigma t_i^\dagger \left\{ iv \cdot \mathbf{D} + \frac{1}{4m_N} \left[ (v \cdot \mathbf{D})^2 - D^2 \right] + \Delta_t \right\} t_i - y_t \left\{ t_i^\dagger \left[ N^T P_i^{(3S_1)} N \right] + h.c. \right\} + \cdots,$$

where $\sigma$ is a sign factor, $\sigma = \pm 1$, $t_i$ is a dibaryon field for spin triplet ($^3S_1$) state. $D_\mu$ is the covariant derivative for the dibaryon field. $\Delta_t$ is the mass difference between the dibaryon and two nucleons, $m_t = 2m_N + \Delta_t$, $y_t$ is a coupling constant for dibaryon-nucleon-nucleon ($dNN$) vertex. $P_i^{(3S_1)}$ is a projection operator for two nucleons in the $^3S_1$ state,

$$P_i^{(3S_1)} = \frac{1}{\sqrt{8}} \tau_2 \sigma_2 \sigma_i,$$

where $\tau_a$ and $\sigma_i$ are Pauli matrices for isospin and spin, respectively. Those two constants, $\Delta_t$ and $y_t$, and the sign factor $\sigma_1 (= -1)$ are determined by two effective range parameters, deuteron binding momentum $\gamma$ and effective range $\rho_d$, in $NN$ scattering for $^3S_1$ channel.

\footnote{We note that this is not an expansion scheme in EFT, but in terms of the Born series. We do not have a clear expansion parameter as $Q/\Lambda$.}
We obtain $L_3$, an effective Lagrangian of three-body contact interaction in terms of a nucleon and a dibaryon field for the $S$-wave spin quartet $nd$ state, as

$$L_3 = -\frac{m_N}{6} g(A) g^2 \Gamma^T \left[ (\sigma \cdot t)^T (\sigma \cdot t) + 3(\sigma \cdot t)(\sigma \cdot t)^T \right] N + \cdots.$$  \hspace{1cm} (5)

The expression of the interaction is the same as that of the spin projection operator of spin-1/2 and spin-1 field into spin-3/2 state. See, e.g., Eq. (A4.37) in Ref. 19. We note that the three-body interaction is a higher order term in the $S$-wave spin quartet $nd$ scattering in pionless EFT, and it is not necessary to be introduced when the value of the cutoff $A$ is about $m_\pi$ or larger, $A \geq m_\pi$, at leading order (LO). In addition, as we will see below, it becomes ineffective when we solve the integral equation using the ordinary cutoff value. We introduce it, however, so as to reproduce the scattering length $a_4$ in the quartet channel (and the strength of the coupling $g(A)$ is adjusted) when the $A$ value is reduced significantly smaller than $m_\pi$, $A < m_\pi$.

### 3 Integral equations

#### 3.1 Integral equation of $S$-wave $nd$ scattering amplitude for quartet channel

Diagrams for $S$-wave $nd$ scattering for the spin quartet ($S = 3/2$) channel are given in Fig. 1, whereas those for the two-body part in Fig. 1 the dressed dibaryon propagator, are given in Fig. 2. The integral equation of the $S$-wave spin quartet $nd$ scattering without the three-body contact interaction is known as

$$t(p,k) = -\frac{8\pi}{m_\rho d} K^{(a)}(p,k;E) \cdot \frac{1}{\pi} \int_0^A dq K^{(a)}(p,q;E) \cdot \frac{q^2 t(q,k)}{-\gamma + \frac{1}{2} \rho d (\gamma^2 - \frac{3}{4} q^2 + mE) + \sqrt{\frac{3}{4} q^2 - mE}},$$  \hspace{1cm} (6)

where $t(p,k)$ is the half-off-shell scattering amplitude and $p$ ($k$) is the magnitude of off-shell final state (on-shell initial state) relative three momentum in CM frame. $K^{(a)}(p,q;E)$ is the one-nucleon-exchange interaction,

$$K^{(a)}(p,q;E) = \frac{1}{2pq} \ln \left( \frac{p^2 + q^2 + pq - mE}{p^2 + q^2 - pq - mE} \right),$$  \hspace{1cm} (7)


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\[ P_{3/2}^{ij} = \frac{1}{6} (\sigma^i \sigma^j + 3 \delta^{ij} \sigma^j) = \frac{2}{3} \delta^{ij} - \frac{1}{3} \epsilon^{ijk} \sigma^k = \delta^{ij} - \frac{1}{3} \delta^i \sigma^j. \]
where \( E = -\frac{\gamma^2}{m_N^2} + \frac{3}{4m_N^2}k^2 \): \( \gamma \) and \( \rho_d \) are the deuteron binding momentum and the effective range, respectively. A sharp cutoff \( A \) is introduced in the loop integral.

We note that because a singularity at \( q \approx 197 \text{ MeV} \) which represents an unphysical deeply bound state appears in the propagator, the dressed dibaryon propagator is usually expanded in terms of the effective range, \( \rho_d \), to avoid an effect from the unphysical bound state. However, we are interested in reducing the cutoff value, less than \( A = 197 \text{ MeV} \), so we do not expand the dressed dibaryon propagator in the integral equation in Eq. (6).

Thus the on-shell T-matrix is given by
\[
T(k, k) = \sqrt{Z_d(k, k)} \sqrt{Z_d},
\]
where \( Z_d \) is the deuteron wavefunction normalization factor, \( Z_d = \gamma \rho_d/(1 - \gamma \rho_d) \). In terms of the half-off-shell scattering amplitude \( T(p, k) \), we have the integral equation as
\[
T(p, k) = -\frac{8\pi}{m_N^2 \rho_d} Z_d K^{(a)}(p, k; E) \left[ 1 - \frac{1}{\rho_d} \left( \frac{\gamma + \sqrt{\gamma^2 + \frac{3}{4}(q^2 - k^2)}}{\gamma + \sqrt{\gamma^2 + \frac{3}{4}(q^2 - k^2)}} \right) q^2 T(q, k) \right].
\]

### 3.2 Integral equation for scattering length

The scattering length, \( a_4 \), of the spin quartet \( nd \) scattering is obtained by the on-shell scattering amplitude with zero momentum as
\[
T(0, 0) = -\frac{3\pi}{m_N} a_4.
\]

Now we introduce a half off-shell amplitude (or a half off-shell scattering length) as
\[
T(p, 0) = -\frac{3\pi}{m_N} a(p, 0),
\]
where \( a(0, 0) = a_4 \). Thus we have an integral equation in terms of \( a(p, 0) \) as
\[
a(p, 0; A) = \frac{8Z_d}{3\rho_d} K^{(a)}(p, 0; -B_2) \left[ 1 - \frac{1}{\rho_d} \left( \frac{\gamma + \sqrt{\gamma^2 + \frac{3}{4}q^2}}{\gamma + \sqrt{\gamma^2 + \frac{3}{4}q^2}} \right) a(q, 0; A) \right],
\]
where we have included the \( A \) dependence in the scattering length \( a(p, 0; A) \) in the above expression though \( a(p, 0; A) \) is insensitive to \( A \) when \( A \approx m_\pi \) or larger. \( B_2 \) is the deuteron binding energy, \( B_2 = \gamma^2/m_N \).

Now we choose \( A = 197 \text{ MeV} \), and solve Eq. (11) numerically. We have two input parameters, \( \gamma \) and \( \rho_d \), whose values are known as \( \gamma = 45.7 \text{ MeV} \) and \( \rho_d = 1.76 \text{ fm} \), and thus have
\[
a^{\text{exp}}_4 \equiv a(0, 0; A = 197 \text{ MeV}) = 6.34 \text{ fm}.
\]
We reproduce results obtained by Bedaque, Hammer, and van Kolck [13, 14], \( a_4 = 6.33 \pm 0.10 \text{ fm} \), and Griesshammer [20], 6.354 \pm 0.020 \text{ fm}, up to NNLO in pionless EFT. It agrees well with those obtained by Bedaque and Griesshammer [21], 6.8 \pm 0.7 \text{ fm}, up to NLO in EFT with perturbative pions, H. Witala et al. [22], 6.321 ... 6.347 \text{ fm}, from various combinations of modern \( NN \) potentials and three-body forces, and the experimental datum [23],
\[
a^{\text{exp}}_4 = 6.35 \pm 0.02 \text{ fm}.
\]
In the left panel of Fig. 3, we numerically calculate and plot the half off-shell scattering length, \( a(p, 0; \Lambda) \) (fm) with \( \Lambda = 197 \) MeV as a function of off-shell momentum \( p \) (MeV). We find that the off-diagonal part of the scattering length quickly decreases and becomes almost negligible at \( p > m_\pi \). This fact may indicate the insensitivity of the amplitude for the quartet channel to the value of \( \Lambda \) when \( \Lambda \geq m_\pi \). In the right panel of Fig. 3, we plot the cutoff \( \Lambda \) dependence of the scattering length \( a(0, 0; \Lambda) \) (fm) without the three-body interaction. One can see that there is almost no \( \Lambda \) dependence at \( \Lambda > 100 \) MeV, whereas the significant cutoff dependence appears in the calculated scattering length \( a(0, 0; \Lambda) \) when the value of the cutoff is reduced less than \( \gamma (= 45.7 \) MeV).}

\[ a(p, 0; \Lambda) = 8 \frac{Z_d}{3 \rho_d} \left[ \frac{1}{\gamma^2 + p^2} + g(A) \right] + \frac{8}{3 \pi} \int_0^A dq \left[ K^{(a)}(p, q; -B_2) + g(A) \right] \frac{\gamma + \sqrt{\gamma^2 + \frac{4}{3} q^2}}{1 - \frac{1}{2} \rho_d \left( \gamma + \sqrt{\gamma^2 + \frac{4}{3} q^2} \right)} a(q, 0), \]

where we have removed the \( \Lambda \) dependence from the half off-shell scattering length \( a(p, 0) \) because the \( \Lambda \) dependence from the integral should be cancelled by \( g(A) \). In the limit that the cutoff \( \Lambda \) becomes large, at \( \Lambda = 197 \) MeV, the coupling constant \( g(A) \) should vanish, \( g(A) \rightarrow 0 \). On the other hand, in the limit that the cutoff \( \Lambda \) vanishes, one has

\[ a^{th}_1 = \frac{8}{3} \frac{Z_d}{\rho_d} \left( \frac{1}{\gamma^2} + g(0) \right), \]

where \( g(0) = -12.56 \cdots \text{fm}^2 \) to reproduce the value of \( a(0, 0; \Lambda = 197 \) MeV) in Eq. (13). Thus we consider a range of \( g(\Lambda) \), \(-12.56 \leq g(\Lambda) \leq 0 \) (fm\(^2\)), in this work.
In the left panel of Fig. 4 we numerically calculate $a(0,0)$ from Eq. (16) and plot curves of $a(0,0)$ with three fixed values of $\Lambda$, $\Lambda = 2\gamma$, $\gamma$, and $0.5\gamma$ as functions of $g(\Lambda)$. A dashed horizontal line for $a_{4}^{th}$ is also included in the figure. One can notice that the curve for $\Lambda = 2\gamma$ is insensitive to $g(\Lambda)$, except for a singular point which corresponds to a three-body bound state with zero binding energy, $B_{3} = 0$, even though the value of $g(\Lambda)$ is significantly changed. We note that the $S$-wave three-body force for the spin quartet channel may be suppressed due to the Pauli principle (by applying the antisymmetrization operator) in the conventional potential model calculation. We can reproduce the same effect (except for the appearance of the bound state) as that of the Pauli principle when we solve the integral equation using the normal cutoff value. The curves with $\Lambda = \gamma$ and $0.5\gamma$, on the other hand, show a sensitivity to $g(\Lambda)$ and vary widely and smoothly.

In the right panel of Fig. 4 we numerically calculate $g(\Lambda)$ from Eq. (16) and plot curves of $g(\Lambda)$ which reproduce $a_{4}^{th}$ (curve) and the three-body bound state with $B_{3} = 0$ (dashed curve) as functions of $\Lambda$. One can see that when the cutoff value is large, $\Lambda > 160$ MeV, the value of $g(\Lambda)$ almost vanishes. This may indicate the effect of the Pauli principle for the spin quartet channel. As the cutoff value is further reduced smaller than $\Lambda = 160$ MeV, we need nonzero value of $g(\Lambda)$ where the short range length scale of the theory becomes $r(=\Lambda^{-1}) > 1.24$ fm. This length scale might be regarded long enough to be out of the range of the Pauli principle. We note that the non-vanishing three-body contact interaction we obtained here may not be a genuine one, but correspond to the one induced by the exchanging nucleon propagator of the larger momentum than the value of $\Lambda$, which connects two two-body interactions and makes the effective three-body one at the small cutoff values. The similar observation that a three-body force is generated from two-body forces at small cutoffs in the SRG analysis is reported in Ref. [24]. One can also see that the three-body bound state with $B_{3} = 0$ appears when the strength of $g(\Lambda)$ becomes stronger than that of $g(\Lambda)$ which reproduces $a_{4}^{th}$ and $\Lambda$ is larger than about 60 MeV in the figure. We find that the curve of $g(\Lambda)$ for the three-body bound state with $B_{3} = 0$ varies smoothly, whereas that of $g(\Lambda)$ which reproduces $a_{4}^{th}$ has a plateau like shape at the middle of the range of $\Lambda$, $\Lambda \approx 60 \sim 100$ MeV. We use the curve of $g(\Lambda)$ which reproduces $a_{4}^{th}$ when studying the perturbation expansions, the ordinary and inverted Born series, in the following.

4 Perturbative expansions: the ordinary and inverted Born series

Now we expand the scattering length and the amplitude in terms of the ordinary and inverted Born series, as discussed in the introduction.
4.1 The ordinary and inverted Born series for the scattering length

Firstly, we study the expansion in terms of the ordinary and inverted Born series for the scattering length $a_4$. Thus the scattering length $a(0,0)$ in Eq. (16) is expanded as

$$a_4 = a(0,0) = \frac{8Z_d}{3\rho_d} [b_0 + b_1 + b_2 + \cdots],$$

(18)

where

$$b_0 = \frac{1}{\gamma^2} + g(\Lambda),$$

(19)

$$b_1 = \int_0^\Lambda dq \left[ \frac{1}{\gamma^2 + q^2} + g(\Lambda) \right]^2 F(q),$$

(20)

$$b_2 = \int_0^\Lambda dq \left[ \frac{1}{\gamma^2 + q^2} + g(\Lambda) \right] F(q)$$

$$\times \int_0^\Lambda dq' \left[ K^{(a)}(q, q', -B_2) + g(\Lambda) \right] F(q') \left[ \frac{1}{\gamma^2 + q'^2} + g(\Lambda) \right],$$

(21)

with

$$F(q) = -\frac{8}{3\pi} \frac{\gamma + \sqrt{\gamma^2 + \frac{4}{q^2}}}{1 - \frac{1}{\rho_d} \left( \frac{1}{\gamma + \sqrt{\gamma^2 + \frac{4}{q^2}}} \right)}.$$

(22)

This expansion corresponds to that around the weak coupling limit. On the other hand, we consider another expansion, the inverted Born series, of the scattering length as

$$\frac{1}{a_4} = \frac{3\rho_d}{8Z_d} \left[ \frac{1}{b_0} - \frac{b_1}{b_0^2} - \frac{b_2}{b_0^3} + \frac{b_3}{b_0^4} + \cdots \right].$$

(23)

In the left and right panel of Fig. 5, we numerically calculate and plot curves of the scattering length $a_4$ obtained from the ordinary and inverted Born series, respectively, as functions of $\Lambda$. Curves labeled by “LO” are results which include only the leading order term, $b_0$, those by “NLO” are results up to next-to-leading order (NLO) which include first two terms in the brackets in Eq. (18) and (23), and those by “NNLO” are results up to next-to-next-to leading order (NNLO) which include all terms in the brackets.
In the left panel of Fig. 5, one can see that a region of the cutoff $\Lambda$, where the curves of $a_4$ obtained from the terms up to NLO and NNLO in the ordinary Born expansion agree with $a_4^{th}$, is quite small, up to about 10 MeV. It would be a natural consequence of the perturbation around the weak coupling limit because the perturbation would converge when the scale $\Lambda$ becomes smaller than a typical scale of the process. In the present case, it may be $a_4^{th}$ ($1/a_4^{th} \approx 31.4$ MeV), and thus it converges when the cutoff $\Lambda$ becomes much smaller than $1/a_4^{th}$. In the right panel of Fig. 5 on the other hand, we find that the region where $a_4^{th}$ is reproduced is remarkably broaden, up to about 100 MeV due to the two or three terms (up to NLO and NNLO, respectively) of the inverted Born series.

4.2 The ordinary and inverted Born series for the scattering amplitude

Before expanding the integral equation for the scattering amplitude in terms of the ordinary and inverted Born series, we check how the three-body interaction $g(\Lambda)$ can reproduce the phase shift $\delta_0$ at the small cutoff values. In Fig. 6 we numerically calculate and plot the phase shift $\delta_0$ below deuteron breakup threshold as functions of $k$ (MeV). Curves are obtained by solving the integral equation at cutoff values, $\Lambda = 197, 140, 100, \text{ and } 60 \text{ MeV}$ with the three-body interaction $g(\Lambda)$ renormalized by $a_4^{th}$. Plus signs "+" denote results from a modern potential model (Av18) \[25\]. The deuteron breakup momentum is $k_{br} \approx 52.7$ MeV.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig6.png}
  \caption{Phase shift $\delta_0$ (deg.) of the $S$-wave nd scattering for spin quartet channel below deuteron breakup threshold as functions of $k$ (MeV). Curves are obtained by solving the integral equation at cutoff values, $\Lambda = 197, 140, 100, \text{ and } 60 \text{ MeV}$ with the three-body interaction $g(\Lambda)$ renormalized by $a_4^{th}$. Plus signs "+" denote results from a modern potential model (Av18) \[25\]. The deuteron breakup momentum is $k_{br} \approx 52.7$ MeV.}
  \label{fig:delta_k}
\end{figure}

We now expand the on-shell T-matrix in terms of the ordinary and inverted Born series up to NNLO. Thus the on-shell T-matrix is obtained as

$$T(k, k) = -\frac{8\pi Z_d}{m_N\rho_d} [B_0 + B_1 + B_2 + \cdots] \quad ,$$

with

$$B_0 = V(k, k) = K^{(0)}(k, k; E) + g(\Lambda),$$

$$B_1 = \int_0^A dq V(k, q)G(q, k)V(q, k),$$

$$\Lambda = 197\text{ MeV} \quad , \quad 140 \text{ MeV} \quad , \quad 100 \text{ MeV} \quad , \quad 60 \text{ MeV} \quad , \quad \text{Av}18 \quad + .$$
\[ B_2 = \int_0^A dqV(k, q)G(q, k) \int_0^A dq'V(q', k)V(q', k), \]  
(27)

where

\[ G(q, k) = \frac{8 \gamma + \sqrt{\gamma^2 + \frac{1}{q^2}}(q^2 - k^2)}{3\pi \left[ \gamma + \sqrt{\gamma^2 + \frac{1}{q^2}}(q^2 - k^2) \right]} q^2 - k^2 - i\epsilon. \]  
(28)

This corresponds to the expansion around the weak coupling limit. We also have an expansion around the inverse of the T-matrix as

\[ \frac{1}{T(k, k)} = \frac{m_N\rho_d}{8\pi Z_d} \left[ \frac{1}{B_0} - \frac{B_1}{B_0} - \frac{B_2}{B_0} + \frac{B_3}{B_0} + \cdots \right]. \]  
(29)

To calculate the phase shift \( \delta_0 \) from the ordinary Born series in Eq. (32) we employ the relation

\[ \delta_0 = \frac{1}{2\pi} \ln \left[ 1 + \frac{2km_N}{3\pi} T(k, k) \right], \]  
(30)

We note that when the amplitude expanded in the ordinary Born series is truncated, it breaks unitary condition and the phase shift becomes a complex number \( \delta_0 \). On the other hand, to calculate the phase shift from the inverted Born expansion in Eq. (29) we use the formula

\[ k \cot \delta_0 = \frac{3\pi}{m_N} \frac{1}{T(k, k)}, \]  
(31)

with

\[ \frac{1}{T_{\text{NLO}}(k, k)} = \frac{m_N\rho_d}{8\pi Z_d} \left[ \frac{1}{B_0} - \frac{ReB_1}{B_0} \right], \]  
(32)

\[ \frac{1}{T_{\text{NNLO}}(k, k)} = \frac{m_N\rho_d}{8\pi Z_d} \left[ \frac{1}{B_0} - \frac{ReB_1}{B_0} - \frac{ReB_2}{B_0} + \frac{(ReB_1)^2 - (ImB_1)^2}{B_0} + \cdot \cdot \cdot \right]. \]  
(33)

where \( B_0 \) has real part only. This expansion preserves the unitary condition, at least up to NNLO.

In Fig. 7 we numerically calculate and plot real and imaginary part of the phase shift \( \delta_0 \) of S-wave spin quartet nd scattering obtained from the truncated ordinary Born series in Eq. (24). Curves up to NLO are obtained by including first two terms, \( B_0 \) and \( B_1 \), and those up to NNLO by including all three terms, \( B_0, B_1, \) and \( B_2 \), in Eq. (24). We have fixed \( \Lambda = 13 \text{ MeV} \) because we found in the left panel of Fig. 5 that the scattering length \( a^0 \) are fairly well reproduced when the cutoff value is about up to 10 MeV. A line labeled by “Full” obtained from the calculation without the truncation at \( \Lambda = 197 \text{ MeV} \) is also included in the figure. One can see that the real parts of \( \delta_0 \) agree with the full result up to about 6 MeV for NLO and 9 MeV for NNLO. In addition, the imaginary parts sharply decrease around the edge of the cutoff value \( \Lambda = 13 \text{ MeV} \), and it indicates that the unitary condition is broken. The broken unitary condition up to NLO is also partly cured by including the higher order term, \( B_2 \), at NNLO.

In Figs. 8 and 9 we numerically calculate and plot curves of the phase shift \( \delta_0 \) obtained from the inverted Born series up to NLO and NNLO in Eqs. (22) and (33), respectively, where the cutoff values are chosen \( \Lambda = 140, 100, \) and 60 MeV. In addition, plus signs “+” labeled by “Full” in the figures denote the results from the calculation without the truncation at \( \Lambda = 197 \text{ MeV} \). We find in Fig. 8 that the curves considerably well converge to that of the full calculation as the cutoff value decreases, and the truncated inverted Born series up to NLO fairly well reproduces the result of the full calculation at \( \Lambda = 60 \text{ MeV} \). In Fig. 9 we can see that the convergence to the full result, as reducing the cutoff value, becomes faster due to the inclusion of the higher order terms.

\(^3\) This line is the same as that at \( \Lambda = 197 \text{ MeV} \) in Fig. 6.
5 Discussion and conclusions

In this work, we studied the perturbative expansions, as the ordinary and inverted Born series, of the integral equation for the S-wave nd scattering for the spin quartet channel below the deuteron breakup threshold in pionless EFT at the small cutoff values. The three-body contact interaction is introduced when the integral equation is solved by using the small cutoff values. After the strength of the three-body interaction is renormalized by using the scattering length $a_4$, we expand the integral equation for the scattering length and the amplitude, as the ordinary and inverted Born series, up to NNLO. We find that the scattering length (the phase shift) is considerably well reproduced by a few terms of the ordinary and inverted Born series as we reduce the cutoff values to about 10 MeV.
Fig. 9 Phase shift $\delta_0$(deg.) obtained from the inverted Born series up to NNLO with $\Lambda = 140$, 100, and 60 MeV. See the caption of Fig. 5 as well.

(10 MeV) and to about 100 MeV (60 MeV), respectively. Therefore, the inverted Born expansion in the present process can be a relevant approximation with a significantly larger valid momentum than the ordinary Born expansion.

In the present particular process, the $S$-wave spin quartet $nd$ scattering, one may regard that there is no advantage by sending $\Lambda$ to small values because the physical observables, the scattering length $a_4$ and the phase shift $\delta_0$, are well described (without fitting any unknown parameters) by the two effective range parameters in the deuteron channel at the usual large scale of the pionless theory, $\Lambda \approx m_\pi$. However, it could be a useful limit when one studies, e.g., a relation between pionless EFT and a Halo/Cluster EFT whose large scale smaller than $m_\pi$, such as a deuteron cluster theory for a reaction whose typical scale $Q$ is smaller than the deuteron breakup momentum \cite{10}.

In addition, it may be the interesting observation that the physical observables can be well described by a few terms of the inverted Born series, being closely related to the unitary limit, at the relatively large cutoff values. If this property were common in some class of the interactions and/or in the unitary limit, it could provide us a useful method to make a non-perturbative interaction perturbative and/or to be used in studies of the exotic nuclei near the drip line. Now we are studying the property of the inverted Born expansion by employing a renormalization group analysis, and it is to be reported separately.

Acknowledgements The author would like to thank K. Kubodera for reading the manuscript and Y.-H. Song for discussion. This work is supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0023061) and (2012R1A1A2009430).

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