Boundary state of superstring in open string channel

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Abstract: We derive boundary state of superstring in the open string channel. It describes the superconformal field theory of open string emission and absorption by D-brane. We define the boundary state by conformal mappings from upper half plane with operators inserted at two points corresponding to the corners of semi-infinite strip. We obtain explicit oscillator forms analytically for the fermion and superconformal ghost sectors. For the fermion sector we compare this with numerical result obtained by using naive boundary condition.

Keywords: D-branes, Boundary Quantum Field Theory, Intersecting branes models.
1. Introduction

In two dimensional conformal field theory, D-branes are described by boundary states. They belong to the closed string sector and realize the boundary conditions associated with the D-branes on the worldsheet. By taking inner product with various closed string states, one can describe the emission or absorption of closed string by the D-brane (for review articles, for example [1]).

In the previous papers [2, 3], we proposed an analog of the boundary state in the open string channel (see also [4] where similar state is discussed in a different context). As the original one, it represents the emission and absorption of the open string by the D-brane (say Σ). Since the open string itself should be attached to (other) D-brane(s) (say Ξ_l, Ξ_r for left and right ends of the open string), such state is relevant when these D-branes intersect,

\[ \Sigma \cap \Xi_i \neq \text{null} \quad (i = l, r). \]

In the following we call such state as the open boundary state or OBS in short.
Figure 1: Three coordinates used in the text. $w$ is natural to define the boundary conditions of OBS. $\zeta = \cos w$ is used to define the correlation function of the boundary conformal field theory. $z = e^{-iw}$ is used in the operator formalism. The last two are used in the next section.

As D-brane (or the boundary state) itself describes the black hole (brane) with unit charge, OBS describes a solitonic excitation on the world volume of D-brane $\Sigma_{l,r}$. In the previous notation, suppose we take $\Xi_l = \Xi_r \equiv \Xi$ as a $D(p+4)$-brane and $\Sigma$ a $Dp$-brane embedded in $\Xi$. Then the associated OBS describes point-like instanton configuration on $\Xi$ with unit charge.

In the previous paper, we constructed OBS for bosonic string and studied its properties in detail. We derived its oscillator representation for the bosonic field $X$ and $bc$ ghost. Inner product between two OBS represents an amplitude whose world sheet has a rectangle shape with its edges surrounded by various D-branes. We found that BRST invariance of OBS gives nontrivial constraint on the D-branes which are attached to the edges at each corner.

The purpose of this paper is to give a similar construction of OBS for superstring case. This is a nontrivial step since there are a few technical problems which do not show up in the bosonic case. In §2, we derive the boundary condition which should be imposed on OBS. It implies that OBS can be expressed in the form $\exp\left(\frac{1}{2} \sum_{rs} \psi_r K_{rs} \psi_s\right)|0\rangle$ where $K_{rs}$ is an infinite size matrix. This approach, however, has an ambiguity that some operator insertions at the corners do not change the boundary condition. In this sense, the boundary condition alone does not fix the matrix $K_{rs}$ uniquely. In §3, we solve this problem by other method developed in string field theory §5 where the correlation function is used to define the vertex. Since the correlation function is unique once the operator insertions at the corners are given, one can uniquely fix the matrix $K$ in §4. It also simplifies the derivation of the constraints from BRST invariance of OBS through CFT in §5. Finally in §6, we provide some applications of OBS and point out some unsolved issues.

2. Boundary conditions

The worldsheet diagram of an open string emitted from the D-brane $\Sigma$ is given by a half infinite strip (see the first figure in Fig.2). We use variable $w$ ($w = \sigma + i\tau$) to parametrize this region as $0 \leq \sigma \leq \pi, \quad \tau \geq 0$. Two endpoints of the open string correspond to $\sigma = 0, \pi$ and they are attached to the D-branes $\Xi_{l,r}$. The D-brane $\Sigma$ from which the open string is emitted corresponds to the edge $0 \leq \sigma \leq \pi, \tau = 0$. Let us call these boundaries as the “left (or right) boundary” and “bottom boundary”.
The boundary conditions for the left and right edges are
\[
\bar{\partial}X(\sigma = 0, \tau) = -\epsilon_l \partial X(\sigma = 0, \tau), \quad \bar{\partial}X(\sigma = \pi, \tau) = -\epsilon_r \partial X(\sigma = \pi, \tau), \quad (2.1)
\]
\[
\tilde{\psi}(\sigma = 0, \tau) = i\eta_l \psi(\sigma = 0, \tau), \quad \tilde{\psi}(\sigma = \pi, \tau) = -i\eta_r \psi(\sigma = \pi, \tau), \quad (2.2)
\]
and the boundary condition for the bottom edge is
\[
\bar{\partial}X(\sigma, \tau = 0) = \epsilon_b \partial X(\sigma, \tau = 0), \quad (2.3)
\]
\[
\tilde{\psi}(\sigma, \tau = 0) = -\eta_b \psi(\sigma, \tau = 0). \quad (2.4)
\]
The parameters \(\epsilon_{l,r,b}\) and \(\eta_{l,r,b}\) take values 1 or -1 and each sign describes Dirichlet or Neumann boundary conditions. We note that the coefficient of the right hand side of (2.2) has an extra factor of the imaginary unit \(i\) if we compare it with (2.4). It comes from the conformal transformation \(w \to \pm iw\) for the weight 1/2 field \(\psi\). Those factors disappear in \(\zeta\) frame which will be discussed in the next section (see (3.2)) where boundary conditions are set only on the real axis.

We can replace anti-holomorphic fields by holomorphic ones by using the doubling trick. By the boundary conditions (2.1) and (2.2), we set
\[
\partial X(\sigma, \tau) \equiv -\epsilon_l \bar{\partial}X(-\sigma, \tau) \quad \text{for} \quad -\pi < \sigma < 0, \quad (2.5)
\]
\[
\psi(\sigma, \tau) \equiv -i\eta_r \tilde{\psi}(-\sigma, \tau) \quad \text{for} \quad -\pi < \sigma < 0. \quad (2.6)
\]
We define the chiral fields on \(-\pi \leq \sigma \leq \pi\). \(\partial X\) and \(\psi\) need to satisfy the following periodicity conditions
\[
\partial X(w + 2\pi) = \epsilon_l \epsilon_r \partial X(w), \quad \psi(w + 2\pi) = -\eta_l \eta_r \psi(w). \quad (2.7)
\]
We extend them to the whole upper half plane by these conditions. Combining the boundary conditions (2.3) and (2.4) with the doubling tricks (2.5) and (2.6), we obtain the boundary conditions which define OBS \(|B^\alpha\rangle\),
\[
[\partial X(\sigma, 0) + \epsilon_l \epsilon_b \partial X(-\sigma, 0)]|B^\alpha\rangle = 0 \quad (2.8)
\]
\[
[\tilde{\psi}(\sigma, 0) + i\eta_l \eta_b \epsilon(\sigma) \psi(-\sigma, 0)]|B^\alpha\rangle = 0, \quad (2.9)
\]
where \(\epsilon(\sigma) \equiv \text{sign} (\sin \sigma)\). The step function in (2.9) is the origin of the complication of OBS for fermions. It is indispensable to make the boundary condition for \(0 < \sigma < \pi\) consistent with that for \(-\pi < \sigma < 0\) with the pure imaginary factor. The boundary conditions for \(\beta\gamma\) ghost or supercurrent \(T_F\) take a similar form since the factor of \(i\) comes from the half-integer conformal weight of the fermion field.

**OBS for \(X\) and \((b,c)\)-ghost** The boundary condition for \(X\) (2.8) was solved in [2, 3] and we obtained the explicit form of OBS in terms of the oscillators. Let us briefly recall the result. Our convention of the oscillator expansions are given in appendix A. The boundary conditions are rewritten compactly in the following way
\[
(\alpha_n + \epsilon_l \epsilon_b \alpha_{-n}) |B^\alpha_{\epsilon_l \epsilon_b}\rangle = 0, \quad (2.10)
\]
where index \( n \) runs over positive integers when \( \epsilon_l \epsilon_r = 1 \), and over positive half odd integers when \( \epsilon_l \epsilon_r = -1 \). The solution of this condition is
\[
\langle B^o \rangle_{\epsilon_l, \epsilon_r}^{\epsilon_b} \propto \exp \left( -\epsilon_l \epsilon_b \sum_{n>0} \frac{1}{2n} \sigma_{-n}^2 \right) |\text{zero mode}\rangle_{\epsilon_l, \epsilon_r}^{\epsilon_b},
\]
where \( |\text{zero mode}\rangle_{\epsilon_l, \epsilon_r}^{\epsilon_b} \) represents the Fock vacuum times the zero-mode wave function.

When \( (\epsilon_l, \epsilon_r) \neq (+1,+1) \), we do not have non-trivial zero-mode and it is simply the Fock vacuum. For \( (\epsilon_l, \epsilon_r) = (+1,+1) \), the zero mode wave function becomes nontrivial. An appropriate choice is \( \delta(p) \) for \( \epsilon_b = 1 \) and \( \delta(x - x_0) \) with \( x_0 \in \mathbb{R} \) for \( \epsilon_b = -1 \).

**OBS for the \((b,c)\)-ghost system** is defined by the following boundary conditions
\[
[c(\sigma) + c(-\sigma)]|B^o\rangle = [b(\sigma) - b(-\sigma)]|B^o\rangle = 0.
\]
It is solved in terms of oscillators as \[2\]
\[
|B^o\rangle^{\text{gh}} = \exp \left( \sum_{n>0} c_{-n} b_{-n} \right) c_0 c_1 |\Omega\rangle,
\]
where \( |\Omega\rangle \) is SL(2,\( \mathbb{R} \))-invariant vacuum for the fermionic ghost.

**OBS for \( \psi \)** One may in principle apply the same strategy to obtain OBS for the fermion. The boundary condition in terms of oscillators is obtained by Fourier transformation of the boundary condition \(2.9\),
\[
(c(\sigma) + c(-\sigma))|B^o\rangle = [b(\sigma) - b(-\sigma)]|B^o\rangle = 0,
\]
where the index \( r \) is \( -\infty < r < \infty \), \( \eta = \eta_b \eta_l \), and the infinite dimensional matrix \( N \) is the Fourier transform of the step function. In NS sector, it takes the following form,
\[
N_{rs} = \begin{cases} 0 & (r + s = 0) \\ \frac{1}{\pi (r + s)} & (r + s \neq 0) \end{cases}
\]
where \( r, s \) run over half odd integers. The matrix \( N \) satisfies \( \sum_s N_{rs} N_{st} = \delta_{r,t} \). We decompose it into \( 2 \times 2 \) blocks,
\[
N = \begin{pmatrix} N_{r,-s} & N_{r,s} \\ N_{r,-s} & N_{r,s} \end{pmatrix} = \begin{pmatrix} n_{rs} & \tilde{n}_{rs} \\ -\tilde{n}_{rs} & -n_{rs} \end{pmatrix},
\]
where indices \( r, s \) run over positive half-integers. \( N^2 = 1 \) implies that \( n, \tilde{n} \) satisfy
\[
n^2 - \tilde{n}^2 = 1, \quad n\tilde{n} = \tilde{n}n = n^T, \quad \tilde{n} = \tilde{n}^T.
\]
We decompose this relation in terms of the creation and annihilation part,
\[
(\psi_r - \sum_{s>0} K_{rs}(\eta)\psi_{-s})|B^o\rangle = 0,
\]
where $r > 0$. The matrix $K$ is written in terms of $n, \tilde{n}$ as,

$$K(\eta) \equiv \eta(1 - \eta n)^{-1}\tilde{n} = -\eta\tilde{n}^{-1}(1 + \eta n) = -K(\eta)^T. \quad (2.19)$$

The condition (2.18) is easily solved as

$$|B^o\rangle = \exp\left(\frac{1}{2} \sum_{r,s>0} K(\eta)_{rs} \psi_{-r}\psi_{-s}\right) |\text{vac}\rangle \quad (2.20)$$

where $|\text{vac}\rangle$ is the Fock vacuum of NS sector.

Naively the construction of OBS for Ramond sector is similar. However, the treatment of zero mode becomes tricky. Related, and even more serious problem is that it is possible to insert some operators at the corners which do not affect the boundary condition. It implies that the boundary condition alone does not fix OBS uniquely. Numerical explanation of this problem is given in appendix [4].

For this reason, we will not pursue this line of argument in the following. Instead, we will use the technique of string field theory which solves these problems automatically.

### 3. Definition of OBS through correlation functions

From this section, we will use the conformal field theory technique to derive OBS instead of using the boundary condition directly.

The relation between the two is similar to that for the two alternative definitions of the interaction vertex in string field theory. The first one is to express the gluing condition of the strings by the delta functionals and derive the oscillator form of the vertex by solving the constraint. The treatment in the previous section is analogous to this one. The second definition is to use the correlation functions [5] of the worldsheet glued by the vertex. One can use a conformal transformation of this worldsheet into the disk or the upper half plane and calculate the correlation function by evaluating the disk amplitude. The vertex (or more precisely Neumann coefficient) is expressed in terms of the moments of this correlation function.

For the definition of OBS, one can map the worldsheet of the semi-infinite strip in the $w$-plane into the upper half plane with insertions of local fields by (see Fig.2)

$$\zeta = \cos w. \quad (3.1)$$

Three edges of the semi-infinite strip ($\tau \geq 0, 0 \leq \sigma \leq \pi$) are mapped into three regions of the real axis in the $\zeta$-coordinate, $\zeta > 1, -1 < \zeta < 1, \zeta < -1$.

When $\phi$ is $\partial X$ ($h = 1$) and $\psi$ ($h = 1/2$), the boundary conditions (2.1), (2.2), (2.3), (2.4) are replaced by

$$\bar{\partial}X(\bar{\zeta}) = \epsilon_i \partial X(\zeta), \quad \bar{\psi}(\bar{\zeta}) = \eta_i \psi(\zeta), \quad (3.2)$$

where index $i$ represents $l, r, b$ for $\zeta > 1, \zeta < 1, -1 < \zeta < 1$, respectively. We use one of these conditions to replace the anti-chiral field by the chiral field in the lower half plane as
the doubling trick. Suppose we take it to the region $\zeta > 1$. The field $\partial X$ (or $\psi$) will have a branch cut at $-1 < \zeta < 1$ if the parameters satisfy $\epsilon_1\epsilon_b = -1$ (or $\eta_1\eta_b = -1$). It implies that we need an appropriate operator which changes the boundary condition inserted at $\zeta = 1$. Similarly an operator insertion at $\zeta = -1$ is needed when $\epsilon_1\epsilon_b = -1$ (or $\eta_1\eta_b = -1$). Let $O_{\pm 1}$ be such operators which are needed at $\zeta = \pm 1$. For the bosonic field $X^\mu$, the operator which changes the boundary condition is the twist field $\sigma$ of conformal weight 1/16, which appears in the $\mathbb{Z}_2$ orbifold CFT [6]. For the fermionic field $\psi_\mu$, the corresponding operator is the spin field. $O_{\pm 1}$ is an appropriate product of them which depend on the parameters $\epsilon_i, \eta_i$.

We define OBS for $\phi$ as a state which reproduces the correlation function

$$\langle \Omega | \phi_1^{(z)}(z_1) \cdots \phi_n^{(z)}(z_n) | B^o \rangle = \langle \phi_1^{(z)}(\zeta_1) \cdots \phi_n^{(z)}(\zeta_n) O_1 \rangle \left( \frac{d\zeta_1}{dz_1} \right)^{b_1} \cdots \left( \frac{d\zeta_n}{dz_n} \right)^{b_n}. \quad (3.3)$$

The left hand side is the expression in the operator formalism whereas the right hand side is the correlation function on the $\zeta$-plane with operator insertions. The left hand side can be computed with Wick’s theorem with the propagator defined as the two point function in the form (3.1) (see appendix C), and the behaviors of the left and right hand sides at singularities are identical. Since correlation functions are determined uniquely by the behavior at singularities, this expression should be true for any number of insertions. The coordinate $z$ is defined by $z = e^{-iw}$ (Fig.2) and it is suitable to describe CFT in the operator formalism. The boundary associated with OBS corresponds to a unit circle $|z| = 1$, so the open string propagates from the unit circle to $z = \infty$. When $\phi$ is a free field such as $\partial X$ or $\psi$, one can obtain the explicit form of OBS from the knowledge of two point correlation functions as will be explained in the next section.

There are some advantages to use the correlation functions to define OBS instead of the boundary condition (2.14). As we noted, the boundary condition at $0 < \sigma < \pi$ does not fix OBS uniquely since one may have many types of insertions at the corners ($\sigma = 0, \pi$) which do not affect the boundary condition. On the other hand, we have no ambiguity in the definition of OBS (3.3) since the correlation function is unique once we choose the operator insertions at the corners. We can also avoid technical difficulty in solving equations including infinite dimensional matrices appearing in (2.14) and (2.18). As we will see below, we can easily obtain explicit oscillator form of OBS by starting (3.3).

To describe OBS for superstring, we have to take the product of OBS for each field as

$$|B^o\rangle = \prod_{\mu = 0}^9 |B^o_{X^\mu}\rangle \otimes \prod_{\mu = 0}^9 |B^o_{\psi^\mu}\rangle \otimes |B^o_{gh}\rangle |B^o_{gh}\rangle,$$  \quad (3.4)

where $|B^o_{X^\mu}\rangle$, $|B^o_{\psi^\mu}\rangle$, $|B^o_{gh}\rangle$, and $|B^o_{gh}\rangle$ are OBS in the boson, fermion, $(b,c)$ ghost, and the $(\beta, \gamma)$ superconformal ghost sectors, respectively. OBS in each sector is defined by (3.3).

In superstring, the boundary conditions in bosonic sector and fermionic sector must be correlated to define the supercurrent $T_F$ consistently. We introduce $s_a = \pm 1$ ($a = l, b, r$) to represent the boundary conditions for $T_F$,

$$\bar{T}_F(\bar{\zeta}) = s_a T_F(\zeta), \quad a = l, b, r,$$  \quad (3.5)
along the real axis of $\zeta$ as eq.(3.3). Since the supercurrent is given by $T_F = \psi^\mu \partial X_\mu$, the relation $s_a = \epsilon^\mu_0 \eta^\mu_0$ must hold for each pair of $\psi^\mu$ and $X^\mu$. ($\epsilon^\mu_0$ and $\eta^\mu_0$ are defined for each direction $\mu = 0, \ldots, 9$.) We also need to choose appropriate superconformal ghost sector of the inserted vertices depending on the boundary conditions $s_a$. If $s_a$ changes at $\zeta = \pm 1$, we need to insert the vertex operator of the form $(\prod \sigma \prod S) e^{-\phi/2}$ which represents R vacuum, and otherwise we insert that of NS vacuum of the form $(\prod 1 \prod \sigma S) e^{-\phi}$. We should carefully distinguish the sectors of vertex operators at the corners from the sector of OBS itself. The latter is defined by the combination of the left and right boundary conditions while the sectors of vertices are determined by the boundary conditions at $\zeta = \pm 1$ (namely the corners in $w$ coordinate) If two vertices are (NS,NS) or (R,R), OBS is in the NS-sector, and if the vertices are (NS,R) or (R,NS), OBS is in the R-sector.

4. Explicit forms of OBS

4.1 Fermion sector

In the following construction of OBS in the fermion sector, we need to use boundary changing operators for fermion fields. For this reason it is convenient to define complex fermions $\bar{\psi}_\pm = (\bar{\psi}_1 \pm i\bar{\psi}_2)/\sqrt{2}$ and bosonize them as $\psi_\pm = e^{\pm iH}$.

Let $e^{i x H(-1)}$ and $e^{i y H(1)}$ be the inserted operators at the two points $\zeta = \pm 1$. The two charges $x$ and $y$ should be chosen appropriately according to the boundary conditions. If the boundary condition changes at $\zeta = -1$, $x$ must be half odd integer, while if the boundary condition does not change $x$ must be integer. The charge $y$ also should be chosen in the same way depending on whether the boundary condition changes at $\zeta = +1$.

We can determine OBS from the relation

$$\langle \Omega | e^{-i(x+y)H(z)} \psi_{\pm}(z_1) \psi_{\mp}(z_2) | B^{\alpha}_{\psi_{\pm}} \rangle_{xy}$$

$$= \langle e^{-i(x+y)H(z)} \psi_{\pm}(z_1) \psi_{\mp}(z_2) e^{i x H(-1)} e^{i y H(1)} \rangle \left( \frac{\partial \zeta_1}{\partial z_1} \right)^{1/2} \left( \frac{\partial \zeta_2}{\partial z_2} \right)^{1/2}. \tag{4.1}$$

The insertion of $e^{-i(x+y)H}$ at the infinity is necessary to match the fermion number. OBS $| B^{\alpha}_{\psi_{\pm}} \rangle_{xy}$ satisfying this relation has $U(1)$ charge $x + y$, and we take the ansatz

$$| B^{\alpha}_{\psi_{\pm}} \rangle_{xy} = e^{i(x+y)H(0)} | \Omega \rangle,$$ \tag{4.2}

where $\cdot \cdot \cdot$ is the normal ordering defined on the ‘vacuum’ state $e^{i(x+y)H}|\Omega\rangle$. Namely, we define creation and annihilation operators as follows:

creation: $\psi^+_{-x-y-r}, \psi^-_{x+y-r}$, annihilation: $\psi^+_{-x-y+r}, \psi^-_{x+y+r}$, \quad ($r = 1, 3, 5, \ldots$). \tag{4.3}

By substituting the ansatz (4.2) into the left hand side of (4.1), we obtain

$$\text{l.h.s of (4.1)} = D_{x+y}(z_1, z_2) + \oint \frac{dz}{2\pi i} \oint \frac{dz'}{2\pi i} D_{x+y}(z_1, z) K(z, z') D_{x+y}(z', z_2). \tag{4.4}$$
where $D_{x+y}(z, z')$ is the propagator defined on the state $e^{i(x+y)H}\Omega$, and is given by

$$D_{x+y}(z, z') = \langle \Omega | e^{-i(x+y)H(\infty)} \psi^+_x(z) \psi^-_{z'}(z') e^{i(x+y)H(0)} | \Omega \rangle = \left( \frac{z}{z'} \right)^{x+y} \frac{1}{z - z'} . \quad (4.5)$$

Let us assume that the function $K(z, z')$ is analytic in the region $|z|, |z'| > 1$ and damps sufficiently fast at infinity. This will be confirmed after we obtain explicit form of the function $K$. With this assumption, we can show that the contour integrals in (4.4) pick up only the contribution from the poles of propagators at $z = z_1$ and $z' = z_2$, and we obtain

$$\text{l.h.s of (4.1)} = D_{x+y}(z_1, z_2) - K^{xy}(z_1, z_2) \quad (4.6)$$

On the other hand, the right hand side of (4.1) is easily computed as

$$\text{r.h.s of (4.1)} = D_{x+y}(z_1, z_2) \sqrt{\frac{1 - \frac{1}{z_1^2}}{1 - \frac{1}{z_2^2}}} \left( \frac{1 + \frac{1}{z_1}}{1 + \frac{1}{z_2}} \right)^{2x} \left( \frac{1 - \frac{1}{z_1}}{1 - \frac{1}{z_2}} \right)^{2y} . \quad (4.7)$$

Comparing (4.6) and (4.7) we obtain the function $K(z_1, z_2)$ as

$$K^{xy}(z_1, z_2) = \frac{1}{z_1 z_2} \left( \frac{z_1}{z_2} \right)^{x+y} K^{xy} \left( \frac{1}{z_1}, \frac{1}{z_2} \right) , \quad (4.8)$$

where the function $K^{xy}$ is defined by

$$K^{xy}(u, v) = \frac{1}{u - v} \left( \frac{\sqrt{(1 - u^2)(1 - v^2)}}{1 - uv} \right) \left( \frac{1 + u}{1 + v} \right)^{2x} \left( \frac{1 - u}{1 - v} \right)^{2y} - 1 . \quad (4.9)$$

The function $K^{xy}(u, v)$ is analytic in the region $|u|, |v| < 1$. The potential singularity at $u = v$ is canceled by zero of the factor in the parenthesis. With this fact we confirm the assumption which we used to perform the contour integrals in (4.4). This behavior of the function $K^{xy}$ also guarantees that $K^{xy}$ can be expanded with respect to $u$ and $v$ in the region $|u|, |v| < 1$ as

$$K^{xy}(u, v) = \sum_{m,n=0}^{\infty} K^{xy}_{mn} u^m v^n . \quad (4.10)$$

A way to compute the explicit forms of the coefficients is given in appendix D. With the coefficients $K^{xy}_{mn}$ we can explicitly give OBS in the oscillator form.

$$|B\rangle =: \exp \left( \sum_{n,m=0}^{\infty} \psi^-_{m-1/2+x+y} K^{xy}_{m,n} \psi^+_{n-1/2-x-y} \right) : e^{i(x+y)H(0)}|\Omega\rangle . \quad (4.11)$$

Note that the indices of fermion oscillators run over all creation operators defined in (4.3).

The matrix $K$ thus defined should agree with $K$ in (2.13) expressed in terms of infinite dim. matrices. We confirmed it in appendix E. We note that there exists discrepancies when $(x, y) \neq (0, 0)$. They are due to the nontrivial operator insertion at the corners.
4.2 Superconformal ghost sector

Let us construct OBS in the superconformal ghost sector. We denote OBS defined with the inserted operators $e^{p\phi(-1)}$ and $e^{q\phi(1)}$ by $|B_{scg}^0\rangle_{pq}$ where $p$ and $q$ are the picture of the inserted vertex operators. The picture of OBS itself is $p+q$. OBS in the superconformal ghost sector can be determined by using the following relation:

$$
\langle \Omega | e^{-(p+q+2)\phi(\infty)} \gamma(z_1) \beta(z_2) | B_{scg}^0 \rangle_{pq} \\
= \langle e^{-(p+q+2)\phi(\infty)} \gamma(\zeta_1) \beta(\zeta_2) e^{p\phi(-1)} e^{q\phi(1)} \rangle \left( \frac{d\zeta_1}{dz_1} \right)^{-1/2} \left( \frac{d\zeta_2}{dz_2} \right)^{3/2}.
$$

(4.12)

We take the ansatz

$$
|B_{scg}^0\rangle_{pq} = \exp \left( \oint dz \oint dz' \beta(z) \tilde{K}^{pq}(z, z') \gamma(z') \right) e^{(p+q)\phi(0)} |\Omega\rangle.
$$

(4.13)

By substituting this ansatz into the left hand side of (4.12), we obtain

$$
l.h.s \text{ of (4.12)} = D_{p+q}(z_1, z_2) - \tilde{K}^{pq}(z_1, z_2),
$$

(4.14)

where $D_{p+q}$ is the propagator defined by

$$
D_{p+q}(z, z') = \langle \Omega | e^{-(p+q+2)\phi(\infty)} \gamma(z) \beta(z') e^{(p+q)\phi(0)} |\Omega\rangle = \left( \frac{z}{z'} \right)^{p+q} \frac{1}{z - z'}.
$$

(4.15)

The right hand side of (4.12) is easily computed as

$$
r.h.s \text{ of (4.12)} = D_{p+q}(z_1, z_2) \sqrt{\left( 1 - \frac{1}{z_1} \right) \left( 1 - \frac{1}{z_2} \right) \left( \frac{1}{1 + \frac{1}{z_1}} \right) \left( \frac{1}{1 + \frac{1}{z_2}} \right)^{2p-1} \left( \frac{1 - \frac{1}{z_1}}{1 - \frac{1}{z_2}} \right)^{2q-1}}.
$$

(4.16)

By comparing (4.14) and (4.16) we obtain

$$
\tilde{K}^{pq}(z_1, z_2) = \frac{1}{z_1 z_2} \left( \frac{z_1}{z_2} \right)^{p+q} K_{p-1/2,q-1/2} \left( \frac{1}{z_1}, \frac{1}{z_2} \right),
$$

(4.17)

where $K$ is the function defined in (4.9). With the expansion coefficients in (4.10) we can give the oscillator form of OBS:

$$
|B_{scg}^0\rangle_{pq} = \exp \left( \sum_{m,n=0}^{\infty} \beta_{m+p+q-3/2} K_{mn}^{p-1/2,q-1/2} \gamma_{n-p-q+1/2} \right) e^{(p+q)\phi(0)} |\Omega\rangle.
$$

(4.18)

5. BRST invariance of OBS

In this section, we derive the constraint from the BRST invariance of the OBS. Let $Q_B$ be the BRST charge and $j_B$ be corresponding BRST current. In the bosonic case [2], we have seen that the BRST invariance,

$$
Q_B |B^0\rangle = 0
$$

(5.1)
implies that the number of twist fields (namely the number of ND sector) at each corner must be 16. The computation in the operator formalism performed in \cite{2} was complicated but can be understood more directly. By the correspondence between the operator formalism and the correlation function (3.3), the insertion of $Q_B = \int dz j_B(z)$ in front of $|B^o\rangle$ is equivalent to insertion of $\int d\zeta j_B(\zeta)$ where the contour surrounds the points $\zeta = \pm 1$ associated with the corners (see Fig. 5). As shown in the figure, this contour can be deformed to two semi-circles around $\zeta = \pm 1$. The BRST invariance (5.1) is thus reduced to the BRST invariance of operators inserted there. The operator takes the form $c(\pm 1) \prod_{\mu \in \text{DN}} \sigma^\mu(\pm 1)$ where $\mu$ runs over the Dirichlet-Neumann directions. BRST invariance requires the dimension of the insertion is zero. Since the conformal dimension of $\sigma$ is $1/16$ and that of $c$ is $-1$, the number of twist fields should be 16. We note that the Dirichlet-Neumann sector here is that for the open string which interpolates between the bottom and the left (or right) boundaries. For example, suppose the D-brane ($\Sigma_{l,r}$) where the open string is attached is D25-brane, the D-brane ($\Sigma$) described by OBS should be D9-brane.

Although the proof of BRST invariance in the operator formalism seems to be hopelessly difficult for superstring due to the complicated form of OBS, we may use this argument to find the constraint from the BRST invariance. The problem is again reduced to the operator insertion at $\zeta = \pm 1$.

We have to be careful in that the open string interpolates the bottom and left (or right) boundaries and we have to specify NS or R sector for such open strings. We will call them as NS$^c$ or R$^c$ sector where the superfix $c$ implies the corner.

For NS$^c$ sector, the natural ghost insertion is $ce^{-\phi}$ which has dimension $-1/2$. On the other hand, in the DN direction we have to insert $\sigma S$ which has dimension $1/8$. In DD and NN sector, we do not have operator insertion of matter sector. Therefore in order to have cancellation of conformal dimensions, the number of DN directions must be four.

For R$^c$ sector, the ghost insertion is $ce^{-\phi/2}$ which has conformal dimension $-5/8$. In the matter sector, in DN direction we have insertion of $\sigma$ and in DD and NN directions we have insertion of $S$. In both cases the operator from matter sector has dimension $1/16$. Therefore, the conformal dimension always cancels between matter and ghost sector. Namely there is no constraint coming from the R sector.

To summarize, if we require the BRST invariance of OBS in both NS and R sectors, the number DN directions should be four. It coincides with the usual consistency condition in the intersecting D-branes.
6. Conclusion and Discussion

In this paper, we carried out the explicit construction of OBS for superstring. We have encountered a few technical challenges compared with the bosonic case \cite{2,3} which include the ambiguity of operator insertion at the corners. Nevertheless we arrived a unique expression for both fermion and superconformal ghost. The concrete form of OBS is more complicated than the bosonic case. The computation of the inner product between them, which was easy in the bosonic case, becomes technically more difficult and we could not carry it out in this paper.

There are a few applications of OBS which may be interesting in the future. One direction is to explore the relation with string field theory (SFT). So far, the boundary state is mainly used in SFT as the source term \cite{8}. Since the usual boundary state belongs to the closed string sector, we need closed SFT in order to introduce such coupling. However, so far our understanding of closed SFT has not been complete. On the other hand, if we use OBS, we can use it as the source term to open SFT where we have a standard formulation. Namely, Witten’s action is redefined in the presence of D-brane as,

\[
S = \frac{1}{2} \int \Psi \star Q \Psi + \frac{g}{3} \int \Psi \star \Psi \star \Psi + \int \Psi \star \langle B^o \rangle .
\] (6.1)

This gives a natural introduction of D-brane in open SFT. It will be interesting to explore the consequences of such coupling. For example, the idempotency relation,

\[
\langle B^o \rangle \star \langle B^o \rangle \propto \langle B^o \rangle
\] (6.2)

which was shown in \cite{3} (as a generalization for closed string relation \cite{9}) appears as a consistency condition of such coupling. We will come back to this issue in the forthcoming paper \cite{10}.

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A. Notations

We give here the notations. In the bosonic string sector, the oscillator expansions of $\partial X$
with various boundary conditions are given by

\[ X^{(NN)}(w, \bar{w}) = \hat{x} - \alpha' \hat{p}(w - \bar{w}) + i \left( \frac{\alpha'}{2} \right)^{\frac{3}{2}} \sum_{m \neq 0} \frac{1}{m} \alpha_m (e^{imw} + e^{-im\bar{w}}), \tag{A.1} \]

\[ X^{(DD)}(w, \bar{w}) = x + \frac{y - x}{2\pi} (w + \bar{w}) + i \left( \frac{\alpha'}{2} \right)^{\frac{3}{2}} \sum_{m \neq 0} \frac{1}{m} \alpha_m (e^{imw} - e^{-im\bar{w}}), \tag{A.2} \]

\[ X^{(DN)}(w, \bar{w}) = x + i \left( \frac{\alpha'}{2} \right)^{\frac{3}{2}} \sum_{r \in \mathbb{Z} + 1/2} \frac{1}{r} \alpha_r (e^{irw} - e^{-ir\bar{w}}), \tag{A.3} \]

\[ X^{(ND)}(w, \bar{w}) = x + i \left( \frac{\alpha'}{2} \right)^{\frac{3}{2}} \sum_{r \in \mathbb{Z} + 1/2} \frac{1}{r} \alpha_r (e^{irw} + e^{-ir\bar{w}}). \tag{A.4} \]

The commutation relation for mode variables is,

\[ [\alpha_n, \alpha_m] = n \delta_{n+m,0}, \quad [\hat{x}, \hat{p}] = i. \tag{A.5} \]

Let us review the notations of the fermionic string sector. The fermionic string is represented by a worldsheet field \( \psi_{\mu} \), where \( \mu \) is the space time index. (We often abbreviate this space time index.) The oscillator expansions and the operator product expansion are

\[ \psi(z) = \sum_r \frac{\psi_r}{z^{r+1/2}}, \quad \psi(w) = (-i)^{1/2} \sum_r \psi_r e^{irw}, \tag{A.6} \]

\[ \psi(z_1) \psi(z_2) \sim \frac{1}{z_1 - z_2}, \quad \{ \psi_r, \psi_s \} = \delta_{r+s,0}, \tag{A.7} \]

where the index \( r \) is the integer or half-integer, and is determined by the periodicity condition of \( \psi \). When considering the spin operator, one spin field involves two space time directions, namely the Dirac fermions on the worldsheet are necessary,

\[ \psi_\pm \equiv \frac{1}{\sqrt{2}} (\psi^1 \pm i \psi^2), \quad \psi_+(z_1) \psi_-(z_2) \sim \frac{1}{z_1 - z_2}, \quad \{ \psi_r^+, \psi_s^- \} = \delta_{r+s,0}. \tag{A.8} \]

where indices \( \pm \) represent the U(1) charge \( \pm 1 \) of the fields, defined by the current \( j_{U(1)} = \psi_+ \psi_- \).

Their bosonized forms are defined in the following way,

\[ \psi_\pm \cong e^{\pm iH}, \quad H(z_1)H(z_2) \sim - \ln(z_1 - z_2). \tag{A.9} \]

The spin fields are defined in the following way,

\[ S_\pm \cong \exp \left( \pm \frac{i}{2} H \right) \tag{A.10} \]

The background charge is equal to 0. Thus, net U(1) charge of all fields between \( \langle \Omega \rangle \) and \( |\Omega\rangle \) in the correlation function must be equal to 0. Let \( |x\rangle \) be \( e^{ixH(0)}|\Omega\rangle \), which is used in the construction of OBS. Then the dual state \( \langle \tilde{x} | \) is \( \langle \Omega | e^{-ixH(\infty)} \).
Our notations of the \((\beta, \gamma)\)-superconformal ghost system is the standard one \cite{7}. The conformal weight of \(\gamma (\beta)\) is \(-1/2\) \((3/2)\), respectively. The background charge of this system is \(Q = 2\). Their oscillator expansions and the operator product expansions are

\[
\gamma(z) = \sum_r \frac{\gamma_r}{z^{r-1/2}}, \quad \beta(z) = \sum_r \frac{\beta_r}{z^{r+3/2}}, \quad (A.11)
\]

\[
\gamma(w) = (\gamma_r e^{iw}, \beta_r e^{iw}), \quad (A.12)
\]

\[
\gamma(z_1)\beta(z_2) \sim \frac{1}{z_1 - z_2}, \quad \beta(z_1)\gamma(z_2) \sim -\frac{1}{z_1 - z_2}, \quad [\gamma_r, \beta_s] = \delta_{r+s,0}. \quad (A.13)
\]

The bosonizations are

\[
\gamma \cong e^{\phi}\eta, \quad \beta \cong (\partial \xi) e^{-\phi}, \quad \phi(z_1)\phi(z_2) \sim -\ln(z_1 - z_2), \quad \xi(z_1)\eta(z_2) \sim \frac{1}{z_1 - z_2}, \quad (A.14)
\]

In our calculation, \((\eta, \xi)\) does not appear. The U(1) charge is defined by the U(1) current \(j = -\beta \gamma = -\partial \phi\). Thus, \(\gamma (\beta)\) has U(1) charge \(1\) \((-1\)). In the bosonized form, we can define the following vacua

\[
|q\rangle \equiv e^{q\phi(0)}|\Omega\rangle. \quad (A.15)
\]

By definition, \(|q\rangle\) has U(1) charge \(q\). The conformal weight of \(e^{q\phi}\) is \(-q(q + 2)/2\). The background charge of this system is \(Q = 2\). Thus net U(1) charge of all fields between \(|\Omega\rangle\) and \(|\Omega\rangle\) in the correlation function must be equal to \(-2\). Then, the dual bra vacuum \(|\bar{q}\rangle\) satisfying \(<\bar{q}|q\rangle = 1\) is defined by \(<\bar{q}| \equiv <\Omega|e^{(-2-q)\phi(\infty)}\).

**B. OBS in the bosonic string sector**

The simplest way to obtain OBS in the bosonic sector is to directly use the boundary condition \((2.10)\), and in \cite{4} OBS is obtained in this way. It is of course possible to use the method of conformal mapping to construct OBS as we have done in the text for the fermion and superconformal ghost sectors. We demonstrate it in the following.

To make the derivation as parallel as possible to the other cases, we here use a complex chiral boson \(Z\) and its conjugate \(\bar{Z}\) with the OPE

\[
Z(z)\bar{Z}(z') \sim -\log(z - z'). \quad (B.1)
\]

The inserted operators used in the definition of OBS in this case are \(\sigma^{2p}(-1)\) and \(\sigma^{2q}(1)\), where \(\sigma(z)\) is the twist operator for the boson fields \(Z\) and \(\bar{Z}\). The numbers \(p, q = 0, 1/2\) are chosen according to the boundary conditions. We use the notation

\[
\sigma^1 \equiv \sigma, \quad \sigma^0 \equiv \text{identity operator.} \quad (B.2)
\]

It is also convenient to define

\[
\sigma^2(z) \equiv \delta(Z(z)) \sim \lim_{\epsilon \to 0} \sigma(z + \epsilon)\sigma(z). \quad (B.3)
\]
This operator changes the SL(2, R) vacuum into the position eigenstate

$$\sigma^2(0)|\Omega\rangle = |Z = 0\rangle. \quad (B.4)$$

With these notation, the defining equation of OBS is given as

$$\langle \Omega|\sigma^{2-2p-2q}(\infty)Z(z_1)\partial\bar{Z}(z_2)|B^\sigma_{Z}\rangle_{pq}$$

$$= (\sigma^{2-2p-2q}(\infty)Z(\zeta_1)\partial\bar{Z}(\zeta_2)\sigma^{2p(-1)\sigma^{2q}(1)}) \frac{\partial \bar{\zeta}_2}{\partial \bar{z}_2}. \quad (B.5)$$

We insert $\sigma^2$ at infinity when $p = q = 1/2$. Although this is not necessary to obtain non-vanishing amplitude, it is convenient because it removes the divergence associated with the infinite volume of the $Z$ space, and it makes the expression (B.5) similar to the corresponding equations in the fermion and superconformal cases. We take the following ansatz:

$$|B^\sigma_{Z}\rangle_{pq} =: \exp \left( \oint \frac{dz}{2\pi i} \oint \frac{dz'}{2\pi i} \partial \bar{Z}(z)K(z, z')Z(z') \right) : \sigma^{2p+2q}(0)|\Omega\rangle. \quad (B.6)$$

By substituting this ansatz into the left hand side of (B.5) we obtain

$$\text{l.h.s. of (B.5)} = D_{p+q}(z_1, z_2) - K(z_1, z_2), \quad (B.7)$$

where the propagator $D_{p+q}$ is given by

$$D_{p+q}(z, z') \equiv \langle \Omega|\sigma^{2-2p-2q}(\infty)Z(z)\partial\bar{Z}(z')\sigma^{2p+2q}(0)|\Omega\rangle = \left( \frac{z}{z'} \right)^{p+q} \frac{1}{z - z'}. \quad (B.8)$$

The right hand side of (B.7) is

$$\text{r.h.s. of (B.5)} = D_{p+q}(z_1, z_2) \sqrt{(1 - \frac{1}{z_1^2})(1 - \frac{1}{z_2^2})} \left( \frac{1}{1 + z_1} \right)^{2p-\frac{1}{2}} \left( \frac{1}{1 + z_2} \right)^{2p-\frac{1}{2}} \quad (B.9)$$

This is quite similar to (4.7) and (4.16), and differs from them only by the powers of the last two factors, which are due to the difference of the conformal dimensions of the fields. By comparing (B.7) and (B.9), we obtain

$$K(z_1, z_2) = \frac{1}{z_1 z_2} \left( \frac{z_1}{z_2} \right)^{p+q} \mathcal{K}^{p-1/4,q-1/4} \left( \frac{1}{z_1}, \frac{1}{z_2} \right). \quad (B.10)$$

In this case, the function $\mathcal{K}^{p-1/4,q-1/4}$ does not include square roots and is simplified as

$$\mathcal{K}^{p-1/4,q-1/4}(u, v) = \frac{(-)^{2q}u^{(1-2p)(1-2q)}v^{4pq}}{1 - uv}. \quad (B.11)$$

We can easily compute the expansion coefficients of this function and obtain the explicit form of OBS in the bosonic sector

$$|B^\sigma_{Z}\rangle_{pq} = \exp \left( -(-)^{2q} \sum_{r>0} \frac{\alpha_-}{r} \right) \sigma^{2p+2q}(0)|\Omega\rangle, \quad (B.12)$$

where the index $r$ runs over positive integer (positive half odd integer) when $2(p + q)$ is even (odd).
C. \(n\)-point functions

When we define OBS, we used only the 2-point function

\[ D(z_1, z_2) = \langle 0|\phi(z_1)\phi(z_2)|B^o\rangle, \quad (C.1) \]

where \(\phi\) are various types of fields, \(|B\rangle\) is OBS in the corresponding sector, and \(\langle 0\rangle\) is an appropriately chosen vacuum state. If we want to compute general \(n\)-point correlation functions of the form

\[ \langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n)|B^o\rangle, \quad (C.2) \]

we can use Wick’s theorem with the propagator \((C.1)\). Namely, the amplitude is given as the sum of contributions of all pairings of the operators \(\phi_k\) \((k = 1, \ldots, n)\), and the contribution of each pairing is obtained by replacing each pair by the propagator \((C.1)\). We prove this fact in what follows. The proof applies to any free field if we replace \(\phi\), \(|B\rangle\), \(\langle 0\rangle\), etc. by appropriate fields and states. We will not distinguish them here.

In order to prove the statement above, the following identity is useful:

\[ \phi(z)|B^o\rangle = \oint_{|z'|>|z|} \frac{dz'}{2\pi i} \phi^{ct}(z')|B^o\rangle D(z', z), \quad (C.3) \]

where the integration contour is a circle of radius \(|z'| > |z|\) and \(\phi^{ct}\) is the creation part of \(\phi\) on the vacuum \(|0\rangle\), which is the vacuum state used in the ansatz \(|B\rangle = \exp(\phi K \phi) : |0\rangle\). Let us first prove this equation. We decompose the operator \(\phi(z)\) on the right hand side to annihilation part \(\phi^{an}(z)\) and creation part \(\phi^{ct}(z)\). For annihilation part, by using \(|B\rangle = \exp(\phi K \phi) : |0\rangle\), we obtain

\[ \phi^{an}(z)|B^o\rangle = \oint_{|z'|>|z|} \frac{dz'}{2\pi i} \oint_{|z''|>|z'|} \frac{dz''}{2\pi i} \phi^{ct}(z')|B^o\rangle K(z', z'')D(z'', z) \]

\[ = -\oint_{|z'|>|z|} \frac{dz'}{2\pi i} \phi^{ct}(z')|B^o\rangle K(z', z), \quad (C.4) \]

where \(D(z, z')\) is the propagator defined by \(D(z, z') = \langle 0|\phi(z)\phi(z')|0\rangle\). We performed \(z''\) integral in the same way as the integral in \((4.4)\). We deformed the contour outward, and used the fact that the integral around \(z'' = \infty\) vanishes. Only the pole of the propagator \(D(z'', z)\) contributes to this integral. For the \(z'\) integral, we deformed the contour from a circle inside \(z\) to a circle outside \(z\) by using the regularity of the function \(K(z', z)\) at \(z' = z\).

The creation operator part is rewritten as

\[ \phi^{ct}(z)|B^o\rangle = \oint_{|z'|>|z|} \frac{dz'}{2\pi i} \phi^{ct}(z')|B^o\rangle D(z', z). \quad (C.5) \]

We used the operator identity \(\oint_{|z'|>|z|} \phi^{ct}(z') D(z', z) = \phi^{ct}(z)\). If we use the relation like \((4.6)\) we can see that the sum of \((C.4)\) and \((C.5)\) is nothing but the right hand side of \((C.3)\), and we have proven the relation \((C.3)\).

We apply the formula \((C.3)\) to the rightmost operator in the correlation function \((C.2)\), we obtain

\[ \oint_{|z'_{n-1}|>|z|>|z_n|} \frac{dz}{2\pi i} \langle 0|\phi_1(z_1)\phi_2(z_2)\cdots\phi_{n-1}(z_{n-1})\phi^{ct}_n(z)|B\rangle D(z, z_n) \quad (C.6) \]
By the Wick contraction of the operator $\phi^T_n(z)$ and other operators $\phi_k(z_k)$ ($k = 1, \ldots, n-1$), we obtain
\[
\oint \frac{dz}{|z_n-1|>|z|>|z_n|} \sum_{k=1}^{n-1} \pm \langle 0 | \phi_1(z_1) \cdots \phi_k(z_k) \cdots \phi_{n-1}(z_{n-1}) | B \rangle D(z_k, z) D(z, z_n) \tag{C.7}
\]

The sign in the summand should be chosen appropriately according to the statistics of the operators. By deforming the integration contour, this can be rewritten as the sum of pole contributions,
\[
\sum_{k=1}^{n-1} \pm \langle 0 | \phi_1(z_1) \cdots \phi_k(z_k) \cdots \phi_{n-1}(z_{n-1}) | B \rangle D(z_k, z_n) \tag{C.8}
\]

If we iterate this procedure we obtain the correlation function as a combination of propagator (C.1).

**D. Explicit form of $\mathcal{K}$**

In this appendix we briefly explain how we expand the function
\[
\mathcal{K}^{xy}(u, v) = \frac{1}{u - v} \left( \frac{\sqrt{(1 - u^2)(1 - v^2)}}{1 - uw} \right) \left( \frac{1 + u}{1 + v} \right)^{2x} \left( \frac{1 - u}{1 - v} \right)^{2y} - 1 \right) \tag{D.1}
\]

We first divide $\mathcal{K}^{xy}$ into two parts $\mathcal{K}^1$ and $\mathcal{K}^2$ defined by
\[
\mathcal{K}^{xy} = \mathcal{K}^1 + \mathcal{K}^2, \quad \mathcal{K}^1 \equiv \frac{P(u)Q(v) - 1}{u - v}, \quad \mathcal{K}^2 \equiv -\frac{P(u)Q(v)}{1 - uv} \tag{D.2}
\]

$P(u)$ and $Q(v)$ are functions of $u$ and $v$, respectively:
\[
P(u) = (1 + u)^{2x+1/2}(1 - u)^{2y-1/2}, \quad Q(v) = (1 + v)^{-2x-1/2}(1 - v)^{-2y+1/2}. \tag{D.3}
\]

If the functions $\mathcal{K}^1$ and $\mathcal{K}^2$ did not include the factors $1/(v - u)$ and $1/(1 - uv)$, which were not factorized into functions of $u$ and $v$, we would easily obtain the expansion of the function $\mathcal{K}^1$ and $\mathcal{K}^2$. The unwanted factors can be removed by acting appropriate differential operators on these functions (we follow a similar computation in [11]).

\[
(u \partial_u + v \partial_v + 1)\mathcal{K}^1 = \left[ \frac{1/2 + 2x}{(1 + u)(1 + v)} + \frac{1/2 - 2y}{(1 - u)(1 - v)} \right] P(u)Q(v) \tag{D.4}
\]

\[
(u \partial_u - v \partial_v - 2x - 2y)\mathcal{K}^2 = \left[ \frac{1/2 + 2x}{(1 + u)(1 + v)} - \frac{1/2 - 2y}{(1 - u)(1 - v)} \right] P(u)Q(v) \tag{D.5}
\]

The right hand side of these equations consists of only factorized terms, and from these equations we obtain the expansion coefficients of function $\mathcal{K}^1$ and $\mathcal{K}^2$ as
\[
\mathcal{K}^{mn}_{mn} = \frac{1}{m + n + 1} \left[ \left( \frac{1}{2} + 2x \right) P^+_m Q^+_n + \left( \frac{1}{2} - 2y \right) P^-_m Q^-_n \right], \tag{D.6}
\]

\[
\mathcal{K}^{nm}_{mn} = \frac{1}{m - n - 2x - 2y} \left[ \left( \frac{1}{2} + 2x \right) P^+_m Q^+_n - \left( \frac{1}{2} - 2y \right) P^-_m Q^-_n \right]. \tag{D.7}
\]
The coefficients $K_{mn}$ in (1.10) is the sum of these two coefficients. We defined $P_n^\pm$ and $Q_n^\pm$ as the coefficients of the expansions
\[
P(u) = \sum_{n=0}^{\infty} P_n^\pm u^n, \quad Q(v) = \sum_{n=0}^{\infty} Q_n^\pm v^n.
\]

(E.8)

E. Numerical comparison of $K^{xy}$ for NS sector

In this appendix, we compare the matrix $K$ computed in §2 (eq. (2.19)) with that in §4 (eq. (4.13)). Since we cannot perform the analytic computation of the products of infinite matrices in (2.19), we have to perform a numerical analysis. We truncate matrices $n, \tilde{n}$ by $500 \times 500$ and numerically carry out the matrix product. When $\eta = 1$ (or $\eta_l = \eta_r = \eta_b$), the matrix $K$ in the second expression ($-\tilde{n}^{-1}(1 + n)$) gives (we keep only first six entries in the following),
\[
\begin{pmatrix}
0. & 0.499996 & 0. & 0.124994 & 0. & 0.062493 \\
-0.500003 & 0. & 0.624992 & 0. & 0.187489 & 0. \\
0. & -0.625007 & 0. & 0.624989 & 0. & 0.195299 \\
-0.125004 & 0. & -0.625010 & 0. & 0.632798 & 0. \\
0. & -0.187509 & 0. & -0.632826 & 0. & 0.632796 \\
-0.0625048 & 0. & -0.195324 & 0. & -0.632829 & 0.
\end{pmatrix}.
\]

(E.1)

This is in good agreement with $K^{00}$,
\[
\begin{pmatrix}
0. & 0.5 & 0. & 0.125 & 0. & 0.0625 \\
-0.5 & 0. & 0.625 & 0. & 0.1875 & 0. \\
0. & -0.625 & 0. & 0.625 & 0. & 0.195313 \\
-0.125 & 0. & -0.625 & 0. & 0.632813 & 0. \\
0. & -0.1875 & 0. & -0.632813 & 0. & 0.632813 \\
-0.0625 & 0. & -0.195313 & 0. & -0.632813 & 0.
\end{pmatrix}.
\]

(E.2)

When $\eta = \eta_l\eta_r = -1$, (2.19) gives
\[
\begin{pmatrix}
0. & -1.5 & 0. & -0.875016 & 0. & -0.687531 \\
1.502 & 0. & -0.373998 & 0. & -0.0617483 & 0. \\
0. & 0.374998 & 0. & -0.875008 & 0. & -0.429703 \\
0.878016 & 0. & 0.876508 & 0. & -0.491057 & 0. \\
0. & 0.0624983 & 0. & 0.492182 & 0. & -0.773449 \\
0.691281 & 0. & 0.431578 & 0. & 0.774855 & 0.
\end{pmatrix}.
\]

(E.3)

It should agree with $K^{-1,1}$, but
\[
\begin{pmatrix}
-2. & -1.5 & -1. & -0.875 & -0.75 & -0.6875 \\
1.5 & 0. & -0.375 & 0. & -0.0625 & 0. \\
-1. & 0.375 & -0.5 & -0.875 & -0.375 & -0.429688 \\
0.875 & 0. & 0.875 & 0. & -0.492188 & 0. \\
-0.75 & 0.0625 & -0.375 & 0.492188 & -0.28125 & -0.773438 \\
0.6875 & 0. & 0.429688 & 0. & 0.773438 & 0.
\end{pmatrix}.
\]

(E.4)
The agreement is limited to the components $K_{nm}$ with $n + m =$ odd! The non-vanishing components $n + m =$ even come from the oscillator insertion at the corner. Such extra term cancels if we combine $(K^{-1,1} + K^{1,-1})/2$ and then it coincides with (2.13).

In this way, we have seen that the formula (2.19) gives a correct formula only when there are no operator insertions at the corners. If such insertion is necessary, one should use (4.10) derived from the correlation function.

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