Sufficient conditions on planar graphs to have a relaxed DP-3-colorability

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Abstract

It is known that DP-coloring is a generalization of a list coloring in simple graphs and many results in list coloring can be generalized in those of DP-coloring. In this work, we introduce a relaxed DP-coloring which is a generalization if a relaxed list coloring. We also shows that every planar graph $G$ without 4-cycles or 6-cycles is DP-$(k,d)^*$-colorable. It follows immediately that $G$ is $(k,d)^*$-choosable.

1 Introduction

Every graph in this paper is finite, simple, and undirected. Embedding a graph $G$ in the plane, we let $V(G)$, $E(G)$, and $F(G)$ denote the vertex set, edge set, and face set of $G$. For $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of $G$ induced by $U$. For $X, Y \subseteq V(G)$ where $X$ and $Y$ are disjoint, we let $E_G(X,Y)$ be the set of all edges in $G$ with one endpoint in $X$ and the other in $Y$.

The concept of choosability was introduced by Vizing in 1976 [15] and by Erdős, Rubin, and Taylor in 1979 [9], independently. A $k$-list assignment $L$ of a graph $G$ assigns a list

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L(v) (a set of colors) with |L(v)| = k to each vertex v. A graph G is L-colorable if there is a proper coloring f where f(v) ∈ L(v). If G is L-colorable for every k-assignment L, then we say G is k-choosable.

Dvořák and Postle [7] introduced a generalization of list coloring in which they called a correspondence coloring. But following Bernshteyn, Kostochka, and Pron [3], we call it a DP-coloring.

**Definition 1.** Let L be an assignment of a graph G. We call H a cover of G if it satisfies all the followings:
(i) The vertex set of H is \[\bigcup_{u \in V(G)} \{\{u\} \times L(u)\} = \{(u, c) : u \in V(G), c \in L(u)\}\];
(ii) H[\{u \times L(u)\}] is a complete graph for every u \in V(G);
(iii) For each uv ∈ E(G), the set E_H(\{u\} \times L(u), \{v\} \times L(v)) is a matching (maybe empty).
(iv) If uv /∈ E(G), then no edges of H connect \{u\} \times L(u) and \{v\} \times L(v).

**Definition 2.** An (H, L)-coloring of G is an independent set in a cover H of G with size |V(G)|. We say that a graph is DP-k-colorable if G has an (H, L)-coloring for every k-assignment L and every cover H of (G. The DP-chromatic number of G, denoted by \(\chi_{DP}(G)\), is the minimum number k such that G is DP-k-colorable.

If we define edges on H to match exactly the same colors in L(u) and L(v) for each uv ∈ E(H), then G has an (H, L)-coloring if and only if G is L-colorable. Thus DP-coloring is a generalization of list coloring. This also implies that \(\chi_{DP}(G) \geq \chi_l(G)\). In fact, the difference of these two chromatic numbers can be arbitrarily large. For graphs with average degree d, Bernshteyn [2] showed that \(\chi_{DP}(G) = \Omega(d/\log d)\), while Alon [1] showed that \(\chi_l(G) = \Omega(\log d)\).

Dvořák and Postle [7] showed that \(\chi_{DP}(G) \leq 5\) for every planar graph G. This extends a seminal result by Thomassen [13] on list colorings. On the other hand, Voigt [16] gave an example of a planar graph which is not 4-choosable (thus not DP-4-colorable). It is of interest to obtain sufficient conditions for planar graphs to be DP-4-colorable. Kim and Ozeki [10] showed that planar graphs without k-cycles are DP-4-colorable for each k = 3, 4, 5, 6. Kim and Yu [11] extended the result on 3- and 4-cycles by showing that planar graphs without triangles adjacent to 4-cycles are DP-4-colorable.

The concept of improper choosability was independently introduced by Škrekovski [12], and Eaton and Hull [8]. A graph G is (L, d)^∗-colorable if there is a coloring f where f(v) ∈ L(v) such that every subgraph induced by vertices with the same color has maximum degree...
at most $d$. If $G$ is $(L,d)^*$-colorable for every $k$-assignment $L$, then we say $G$ is $(k,d)^*$-choosable.

In 1986, Cowen, Cowen, and Woodall [6] constructed a planar graph that is not $(3,1)^*$-choosable. Many sufficient conditions for planar graphs to be $(3,1)^*$-choosable are studied. Zhang [17] showed that every planar graph without 5-cycles or 6-cycles is $(3,1)^*$-choosable. Chen and Raspaud [4] proved that every planar graph without 4-cycles adjacent to 3- or 4-cycles is $(3,1)^*$-choosable. Chen, Raspaud, and Wang [5] proved that every planar graph without adjacent triangles or 6-cycles is $(3,1)^*$-choosable.

Inspired by DP-coloring, we define a generalization of a relaxed list coloring as follows.

**Definition 3.** Let $H$ be a cover of a graph $G$ with respect to a list assignment $L$. A $d$-representative set of $G$ is a set of vertices $S$ in $H$ such that

1. $|S| = |V(G)|$,
2. $u \neq v$ for any two different members $(u,c)$ and $(v,c')$ in $S$, and
3. $H[S]$ has maximum degree at most $d$.

An $(H,L,d)$-coloring of $G$ is a $d$-representative set. We say that a graph is $DP-(k,d)^*$-colorable if $G$ has an $(H,L,d)$-coloring for every $k$-assignment $L$ and every cover $H$ of $G$. Since only $d = 1$ is considered in this paper, we write a representative set instead of a 1-representative set.

If we define edges on $H$ to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(H)$, then $G$ has an $(H,L,d)$-coloring if and only if $G$ is $(L,d)^*$-colorable. This follows immediately that $G$ is $DP-(k,d)^*$-colorable implies $G$ is $(k,d)^*$-choosable.

In this work, we have the following result.

**Theorem 1.** Every planar graph without 4-cycles or 6-cycles is $DP-(3,1)^*$-colorable.

## 2 Structure Obtained from Condition on Cycles

First, we introduce some notations and definitions. A $k$-vertex ($k^+$-vertex, $k^-$-vertex, respectively) is a vertex of degree $k$ (at least $k$, at most $k$, respectively). The same notations are applied to faces. A $(d_1,d_2,\ldots,d_k)$-face $f$ is a face of degree $k$ where vertices on $f$ have degree $d_1,d_2,\ldots,d_k$ in a cyclic order. A $(d_1,d_2,\ldots,d_k)$-vertex $v$ is a vertex of degree $k$ where faces incident to $v$ have degree $d_1,d_2,\ldots,d_k$ in a cyclic order. A $(3,4^+,4^+)$-face $f$ is called a pendant 3-face of $v$ if $v$ is not a vertex of $f$ but adjacent to a 3-vertex of $f$.

Let $G$ be a graph without 4-cycles or 6-cycles. The following property is straightforward.
Proposition 4. A 3-face in $G$ does not share exactly one edge with 6$^-$-faces.

Proposition 4 yields the following two Propositions.

Proposition 5. If $f$ is a pendant 3-face of $v$ where $u$ is a 3-vertex in $f$, then two faces that are incident to both $v$ and $u$ are 7$^+$-faces.

Proposition 6. Every vertex $v$ is incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces.

3 Structure of Minimal Non DP-(3,1)-colorable Graphs

Definition 7. Let $H$ be a cover of $G$ with a list assignment $L$. Let $G' = G - F$ where $F$ is an induced subgraph of $G$. A list assignment $L'$ is a restriction of $L$ on $G'$ if $L'(u) = L(u)$ for each vertex in $G'$. A graph $H'$ is a restriction of $H$ on $G'$ if $H' = H[\{v \times L(v) : v \in V(G')\}]$. Assume $G'$ has an $(H', L', 1)$-coloring with a representative set $R'$ in $H'$ such that $|R'| = |V(G)| - |V(F)|$.

A residual list assignment $L^*$ of $F$ is defined by

$$L^*(x) = L(x) - \bigcup_{ux \in E(G)} \{c' \in L(x) : (u, c)(x, c') \in E(H) \text{ and } (u, c) \in I'\}$$

for each $x \in V(F)$.

A residual cover $H^*$ is defined by $H^* = H[\{x \times L^*(x) : x \in V(F)\}]$.

From above definitions, we have the following fact.

Lemma 8. Assume $G$ has an induced subgraph $G'$ and a cover $H$ with a list assignment $L$. Let $H'$ be a restriction of $H$ on $G'$. If $G'$ has an $(H', L', 1)$-coloring with a representative set $R'$ in $H'$ such that $|R'| = |V(G)| - |V(F)|$, then a residual cover $H^*$ is a cover of $F$ with an assignment $L^*$. Furthermore, if $F$ is $(H^*, L^*, 1)$-colorable, then $G$ is $(H, L, 1)$-colorable.

Proof. One can check from the definitions of a cover and a residual cover that $H^*$ is a cover of $F$ with an assignment $L^*$.

Suppose that $F$ is $(H^*, L^*, 1)$-colorable. Then $H^*$ has a representative set $R^*$ with $|R^*| = |F|$. It follows from Definition 7 that no edges connect $H^*$ and $R'$. Additionally, $R'$ and $R^*$ are disjoint. Altogether, we have that $R = R' \cup R^*$ is a representative set in $H$ with $|R| = (|V(G)| - |V(F)|) + |V(F)| = |V(G)|$. Thus $G$ is $(H, L, 1)$-colorable.

From now on, let $G$ be a minimal non DP-4-colorable graph.
Lemma 9. Each vertex in $G$ is a $3^+$-vertex.

Proof. Suppose to the contrary that $G$ has a vertex $x$ degree at most 2. Let $L$ be a 3-assignment and let $H$ be a cover of $G$ such that $G$ has no $(H, L, 1)$-coloring. By the minimality of $G$, the subgraph $G' = G - x$ admits $(H', L', 1)$-coloring where $L'$ (and $H'$) is a restriction of $L$ (and $H$, respectively) in $G'$. Thus there is a representative set $R'$ with $|R'| = |G'|$ in $H'$. Consider a residual list assignment $L^*$ on $x$. Since $|L(x)| = 3$ and $d(x) \leq 2$, we obtain $|L^*(x)| \geq 1$. Clearly, $\{(x, c)\}$ where $c \in L^*(x)$ is a representative set in $G[\{x\}]$. Thus $G[\{x\}]$ is $(H^*, L^*, 1)$-colorable. It follows from Lemma 8 that the graph $G$ is $(H, L, 1)$-colorable, a contradiction. \qed

Lemma 10. Each neighbor of 3-vertex in $G$ is a $4^+$-vertex.

Proof. Suppose to the contrary that there are adjacent 3-vertices $u$ and $v$. Let $L$ be a 3-assignment and let $H$ be a cover of $G$ such that $G$ has no $(H, L, 1)$-coloring. By the minimality of $G$, the subgraph $G' = G - \{u, v\}$ admits an $(H', L', 1)$-coloring where $L'$ (and $H'$, respectively) is a restriction of $L$ (and $H$, respectively) in $G'$. Thus there is a representative set $R'$ with $|R'| = |G'|$ in $H'$. Consider a residual list assignment $L^*$ on $G[\{u, v\}]$. We have $|L^*(u)|$ and $|L^*(v)| \geq 1$. Clearly, $\{(v, c), (u, c')\}$ where $c \in L^*(v)$ and $c' \in L^*(u)$ is a representative set in $G[\{u, v\}]$. We obtain a representative set $R^*$ with $|R^*| = 2$. Thus $G[\{u, v\}]$ is $(H^*, L^*, 1)$-colorable. It follows from Lemma 8 that $G$ is $(H, L, 1)$-colorable, a contradiction. \qed

Lemma 11. Each 4-vertex in $G$ is adjacent to at most two 3-vertices.

Proof. Suppose to the contrary that a 3-vertex $v$ is adjacent to three 3-vertices, $u_1, u_2,$ and $u_3$. It follows from Lemma 10 that $u_i$ is not adjacent to $u_j$ for $i, j \in \{1, 2, 3\}$. Let $L$ be a 3-assignment and let $H$ be a cover of $G$ such that $G$ has no $(H, L, 1)$-coloring. By the minimality of $G$, the subgraph $G' = G - \{v, u_1, u_2, u_3\}$ admits an $(H', L', 1)$-coloring where $L'$ (and $H'$, respectively) is a restriction of $L$ (and $H$, respectively) in $G'$. Thus there is a representative set $R'$ with $|R'| = |G'|$ in $H'$. Consider a residual list assignment $L^*$ on $G[\{v, u_1, u_2, u_3\}]$. Since $|L(v)| = 3$ for every $v \in V(G)$, we have $|L^*(v)| \geq 2$ and $|L^*(u_i)| \geq 1$ for $i \in \{1, 2, 3\}$. Let $H^*$ be an residual cover of $G[\{v, u_1, u_2, u_3\}]$. First of all, we choose a color $c_i$ from $L^*(u_i)$. So there is a color $c$ in $L^*(v)$ such that $(v, c)$ is adjacent to at most one of $\{(u_i, c_i)\}$ for $i \in \{1, 2, 3\}$. Clearly, $\{(v, c), (u_1, c_1), (u_2, c_2), (u_3, c_3)\}$ where $c \in L^*(v)$ and $c_i \in L^*(u_i)$ for $i \in \{1, 2, 3\}$ is a representative set in $G[\{v, u_1, u_2, u_3\}]$. Thus we
obtain a representative set \( R^* \) with \( |R^*| = 4 = |G[\{v, u_1, u_2, u_3\}]| \). Thus \( G[\{v, u_1, u_2, u_3\}] \) is \((H^*, L^*, 1)\)-colorable. It follows from Lemma 8 that \( G \) is \((H, L, 1)\)-colorable, a contradiction. \( \square \)

From Lemma 10 we obtain the upper bound of the number of incident 3-vertices of a face in \( G \).

**Corollary 12.** Each face in \( G \) is incident to at least \( \frac{d(f)}{2} \) 3-vertices.

### 4 Main Result

**Theorem 2.** Every planar graph without 4-cycles or 6-cycles is DP-(3,1)*-colorable.

**Proof.** Suppose that \( G \) is a minimal counterexample. Then each vertex in \( G \) is a \( 3^+ \)-vertex by Lemma 9. The discharging process is as follows. Let the initial charge of a vertex \( u \) in \( G \) be \( \mu(u) = 2d(u) - 6 \) and the initial charge of a face \( f \) in \( G \) be \( \mu(f) = d(f) - 6 \). Then by Euler’s formula \( |V(G)| - |E(G)| + |F(G)| = 2 \) and by the Handshaking lemma, we have

\[
\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12.
\]

Now, we establish a new charge \( \mu^*(x) \) for all \( x \in V(G) \cup F(G) \) by transferring charge from one element to another and the summation of new charge \( \mu^*(x) \) remains \(-12\). If the final charge \( \mu^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), then we get a contradiction and the proof is completed.

The discharging rules are

(R1) Every \( 4^+ \)-vertex sends charge 1 to each incident 3-face.

(R2) Every \( 4^+ \)-vertex sends charge \( \frac{1}{3} \) to each incident 5-face.

(R3) Every \( 4^+ \)-vertex sends charge \( \frac{1}{3} \) to each pendent 3-face.

(R4) Every \( 7^+ \)-face sends charge \( \frac{1}{3} \) to each incident 3-vertex.

(R5) Every 3-vertex sends charge \( \frac{2}{3} \) to each incident 3-face.

Next, we show that the final charge \( \mu^*(u) \) is nonnegative.

**CASE 1:** A 3-vertex \( v \).

If \( v \) is not incident to any 3-face, then \( \mu^*(v) \geq 0 \). If \( v \) is incident to a 3-face, then \( v \) is a \((3,7^+,7^+)\)-vertex by Proposition 4. Thus \( \mu^*(v) \geq \mu(v) - \frac{2}{3} + 2 \times \frac{1}{3} = 0 \) by (R4) and (R5).

**CASE 2:** A 4-vertex \( v \).

It follows from by Proposition 6 that \( v \) is incident to at most two 3-faces. If \( v \) is incident
to two 3-faces, then \( v \) is a \((3, 7^+, 3, 7^+)\)-vertex by Proposition 4 and \( v \) has no any pendent 3-faces by Lemma 5. Thus \( \mu^*(v) \geq \mu(v) - 2 \times 1 = 0 \) by (R1). If \( v \) is incident to exactly one 3-face, then \( v \) is a \((3, 7^+, 5^+, 7^+)\)-vertex by Proposition 4. Moreover, \( v \) has at most two pendent 3-faces by Lemma 11. Thus \( \mu^*(v) \geq \mu(v) - 1 - 3 \times \frac{1}{3} = 0 \) by (R1), (R2), and (R3).

If \( v \) is not incident to any 3-face, then \( v \) has at most two pendent 3-faces by Lemma 11. Thus \( \mu^*(v) \geq \mu(v) - 6 \times \frac{1}{3} = 0 \) by (R2) and (R3).

\[ \text{CASE 3: A } 5^+-\text{vertex } v. \]

To facilitate the calculation, we redefine the discharging rule for \( v \) and its incident faces \( f_1, f_2, \ldots, f_{d(v)} \). Let \( v \) send charge \( \frac{2}{3} \) to each incident face. We have \( \mu^*(v) \geq (2d(v) - 6) - d(v) \times \frac{2}{3} \geq 0 \). Now, let each non 3-face \( f_i \) send charge \( \frac{1}{6} \) to each adjacent 3-face of \( v \) and each adjacent pendent 3-face of \( v \). That means the remaining charge is received by \( v \) as follows,

(1) Each 3-face of \( v \) receive charge at least \( \frac{2}{3} + 2 \times \frac{1}{6} = 1 \) by Proposition 4.

(2) Each 5+-face of \( v \) receive charge at least \( \frac{2}{3} - 2 \times \frac{1}{6} = \frac{1}{3} \) by Proposition 5.

(3) Each pendant 3-face of \( v \) receive charge at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) by Proposition 5.

One can see that charge of each \( f_i \) is at least that obtains from (R1), (R2), and (R3). Thus \( \mu^*(v) \geq 0 \).

\[ \text{CASE 4: A 3-face } f. \]

It follows from Lemma 10 that \( f \) is a \((3^+, 4^+, 4^+)\)-face. If \( f \) is a \((3, 4^+, 4^+)\)-face, then \( f \) is a pendant 3-face of some 4+-vertex by Lemma 10. Thus \( \mu^*(f) \geq \mu(f) + 2 \times 1 + \frac{2}{3} + \frac{1}{3} = 0 \) by (R1), (R3), and (R5). If \( f \) is a \((4^+, 4^+, 4^+)\)-face, then \( \mu^*(f) \geq \mu(f) + 3 \times 1 = 0 \) by (R1).

\[ \text{CASE 5: A } 5^+-\text{face } f. \]

If \( f \) is a 5-face, then \( f \) is incident to at least three 4+-vertices by Corollary 12. Thus \( \mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0 \) by (R2). If \( f \) is a 7-face, then \( f \) is incident to at most three 3-vertices by Corollary 12. Thus \( \mu^*(f) \geq \mu(f) - 3 \times \frac{1}{3} = 0 \) by (R4). If \( f \) is a 8+-face, then \( f \) is incident to at most \( \frac{d(f)}{2} \) 3-vertices by Corollary 12. Thus \( \mu^*(f) \geq \mu(f) - \frac{d(f)}{2} \times \frac{1}{3} > 0 \) from (R4).

This completes the proof.

\[ \square \]

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