RIBBON CONCORDANCE AND THE MINIMALITY OF TIGHT FIBERED KNOTS

TETSUYA ABE AND KEIJI TAGAMI

Abstract. Agol proved that ribbon concordance forms a partial ordering on the set of knots in the 3-sphere. In this paper, we prove that all tight fibered knots are minimal in this partially ordered set. We also give the table of prime minimal knots up to 8-crossings except for $8_{15}$.

1. Introduction

In 1981, Gordon [6] introduced the notion of ribbon concordance of knots in the 3-sphere $S^3$, and conjectured that ribbon concordance forms a partial ordering $\leq$ on the set of knots in $S^3$ ([6, Conjecture 1.1]). Recently, Agol [3] solved this conjecture affirmatively.

Throughout this paper, we only consider knots in $S^3$. Let $\mathcal{K}$ be the set of knots. We are interested in global properties of the partially ordered set $(\mathcal{K}, \leq)$. The first property is that the connected components of $(\mathcal{K}, \leq)$ are exactly the concordance classes of knots, which follows from the following lemma.

Lemma 1.1 (e.g. [6, Section 6], [18, Lemma 2.5]). Let $K_0$ and $K_1$ be knots in $S^3$. Then $K_0$ is concordant to $K_1$ if and only if there exists a common stabilization of $K_0$ and $K_1$, that is, a knot $K_2$ such that $K_0 \leq K_2$ and $K_1 \leq K_2$.

In knot concordance theory, one of the most important questions is the following.

Question 1 ([6, Question 6.3]). Does every concordance class contain a unique minimal representative?

In other words, Question 1 asks whether, for each knot $K$, there exists a unique minimal knot $K_{\text{min}}$ with $K_{\text{min}} \leq K$ or not. This question is divided into Questions 2 and 3 as follows.

Question 2. For each knot $K$, does there exist a minimal knot $K_{\text{min}}$ with $K_{\text{min}} \leq K$?

Question 3 ([6, Question 6.1]). Let $K_{\text{min}}$ be a minimal knot. If $K_{\text{min}}$ is concordant to a knot $K$, then is $K_{\text{min}} \leq K$? In particular, in a given concordance class, is a minimal representative unique?

An affirmative answer to the next question will imply an affirmative answer to Question 2.

Question 4 ([6, Question 6.2]). If $K_1 \geq K_2 \geq \ldots$, does there exist some $m$ such that $K_n = K_m$ for all $n \geq m$?

Note that Question 4 is true if one of $K_i$ ($i = 1, 2, \ldots$) is fibered (Corollary [3.4]). Therefore Question 2 is true for fibered knots $K$. Question 3 is closely related to the slice-ribbon conjecture, which is a major open problem in low-dimensional topology. Here, the unknot is minimal in $(\mathcal{K}, \leq)$, for example, see [6]. In Section 2, we give
Theorem 1.2. All tight fibered knots are minimal in \((K, \preceq)\).

Recall that a fibered knot in \(S^3\) is called tight if the corresponding open book decomposition to the fibered knot supports the tight contact structure in \(S^3\). Theorem 1.2 is a slight generalization of Baker’s result in [4]. It is well known that a torus knot is tight. Therefore Theorem 1.2 recovers Gordon’s result as follows.

Corollary 1.3. All torus knots are minimal in \((K, \preceq)\).

We also give the table of prime minimal knots up to 8-crossings except for 8\(_{15}\).

Theorem 1.4. The following knots are minimal in \((K, \preceq)\).

\[
3_1, 4_1, 5_1, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_4, 7_5, 7_6, 7_7,
8_1, 8_2, 8_3, 8_4, 8_5, 8_6, 8_7, 8_{12}, 8_{13}, 8_{14}, 8_{16}, 8_{17}, 8_{18}, 8_{19}, 8_{21}.
\]

We do not know whether the positive alternating knot 8\(_{15}\) is minimal in \((K, \preceq)\) or not. For the details, see Table 1. We conclude this section with the following question.

Question 5. Are strongly quasi-positive knots minimal in \((K, \preceq)\)? How about positive knots?

This question is motivated by the fact that tight fibered knots are strongly quasi-positive due to Hedden [7].

2. Terminologies

In this section, we recall some basic definitions. Throughout this paper, all manifolds and knots are oriented, and we will work in the smooth category.

Let \(K_0\) and \(K_1\) be knots. An embedded annulus \(C \subset S^3 \times [0,1]\) is called a concordance from \(K_0\) to \(K_1\) if \(\partial(S^3 \times [0,1], C) = (-S^3 \times \{0\}, -K_0 \times \{0\}) \cup (S^3 \times \{1\}, K_1 \times \{1\})\).

We say that \(K_0\) is concordant to \(K_1\) if there is a concordance from \(K_0\) to \(K_1\). If the projection \(S^3 \times [0,1] \to [0,1]\) is a Morse function when restricted to \(C\) with no local maxima, then \(C\) is called a ribbon concordance from \(K_0\) to \(K_1\). We say that \(K_0\) is ribbon concordant to \(K_1\), denoted by \(K_0 \preceq K_1\) or \(K_1 \succeq K_0\), if there is a ribbon concordance from \(K_0\) to \(K_1\). This convention is the same as that in [21], and different from Gordon’s one found in [11, 3, 6]. However, we use the symbol \(\preceq\) to describe the same meaning. Note that \(K_0 \preceq K_1\) if and only if \(K_1\) is obtained from the disjoint union of \(K_1\) and the \(n\)-component unlink by attaching \(n\) bands. For example, the trefoil \(3_1\) is ribbon concordant to the knots 8\(_{10}\) and 8\(_{11}\), see Figure 1. For a more fascinating presentation of the two-bridge knot 8\(_{11}\), see [2].

3. Proof of Theorem 1.2

In this section, we first prove Lemma 3.1. As a simple application of Lemma 3.1, we prove that the unknot is minimal in \((K, \preceq)\) (Corollary 3.2). After that, we prove Theorem 1.2

Ozsváth and Szabó [17] proved that the knot Floer homology detects the Seifert genus of a given knot \(K\). More precisely, they proved that

\[g_3(K) = \max\{a \in \mathbb{Z} \mid \text{HFK}(K, a) \neq 0\},\]

where \(\text{HFK}\) is the knot Floer homology. By \(g_3(K)\), we denote the 4-ball genus of \(K\). When we study the minimality of knots, the following lemma is useful.
Lemma 3.1. Let $K$ be a knot with $g_3(K) = g_4(K)$. If there exists a knot $K'$ with $K' \leq K$, then

$$g_4(K') = g_3(K).$$

This lemma is an immediate corollary of the result in [21]. For the sake of the readers, we give a full proof.

Proof. Note that we have $\widehat{HFK}(K', g_3(K')) \neq 0$. Zemke [21] proved that $K' \leq K$ induces an injective homomorphism $\widehat{HFK}(K') \to \widehat{HFK}(K)$. Therefore $\widehat{HFK}(K, g_3(K')) \neq 0$. This implies that

$$g_3(K') \leq \max\{a \in \mathbb{Z} | \widehat{HFK}(K, a) \neq 0\} = g_3(K).$$

We will prove that $g_3(K) \leq g_3(K')$. By the assumption, we have $g_3(K) = g_4(K)$. Furthermore, we have $g_4(K) = g_4(K')$ since the 4-ball genus is a concordance invariant. Therefore we have

$$g_3(K) = g_4(K) = g_4(K') \leq g_3(K').$$

This implies that $g_3(K') = g_3(K)$. \hfill \Box

As an application of Lemma 3.1, we prove the following.

Corollary 3.2. The unknot $U$ is minimal in $(K, \leq)$.

Proof. Let $K$ be a knot with $K \leq U$. By Lemma 3.1, $g_3(K) = g_3(U) = 0$. Hence $K = U$. \hfill \Box

The following is an alternative proof of Corollary 3.2 using Khovanov homology.

An alternative proof of Corollary 3.2. Suppose that $K \leq U$. Levine and Zemke [11] proved that $K \leq U$ induces an injective homomorphism $Kh(K) \to Kh(U) = \mathbb{Z} \oplus \mathbb{Z}$, where $Kh$ is the Khovanov homology. Here $Kh(K)$ contains $\mathbb{Z} \oplus \mathbb{Z}$ since there is a spectral sequence from the Khovanov homology to the Lee homology. This implies that $Kh(K) = \mathbb{Z} \oplus \mathbb{Z} = Kh(U)$. By the celebrated theorem of Kronheimer and Mrowka [10], the Khovanov homology detects the unknot. Therefore, we have $K = U$. \hfill \Box

Now we are ready to prove the main result.

Proof of Theorem 1.2. Let $K$ be a tight fibered knot. Hedden [7] proved that a fibered knot is tight if and only if the knot is strongly quasi-positive. Therefore $g_3(K) = g_4(K)$ (e.g. [12, 19]). Suppose that $K' \leq K$. By Lemma 3.1, $g_3(K') = g_3(K)$. Ni [16] proved that the knot Floer homology detects the fiberedness of a given knot $K$. More precisely,

$$\text{rank } \widehat{HFK}(K, g_3(K)) = 1 \text{ if and only if } K \text{ is fibered.}$$

Using $g_3(K') = g_3(K)$, we have

$$0 \neq \text{rank } \widehat{HFK}(K', g_3(K')) = \text{rank } \widehat{HFK}(K', g_3(K)) \leq \text{rank } \widehat{HFK}(K, g_3(K)) = 1,$$
where the inequality follows from the injectivity of $\widehat{HF}(K') \to \widehat{HF}(K)$ and the last equality follows from the fiberedness of $K$. This means that rank $\widehat{HF}(K', g_3(K')) = 1$. Therefore $K'$ is also fibered. Baker [3] proved that, based on Gordon’s result [6, Lemma 3.4], $K$ is minimal among all fibered knots in $S^3$. Therefore $K' = K$. 

**Remark 3.3.** Subsequently, we noticed the following argument. For more details, see Miyazaki’s paper [15]. Silver [20] observed that Rapaport’s conjecture on the knot-like groups implies the following.

If $K_0 \leq K_1$ and $K_1$ is fibered, then $K_0$ is also fibered.

This statement is true since Kochloukova [9] solved Rapaport’s conjecture in 2006. This fact dramatically simplifies the proof of Theorem 1.2.

**Corollary 3.4.** Let $K_1 \geq K_2 \geq \cdots \geq K_i \geq K_{i+1} \geq \cdots$, be sequence of knots. Suppose that one of $K_i$ ($i = 1, 2, \ldots$) is fibered, then there exists some $m$ such that $K_n = K_m$ for all $n \geq m$.

**Proof.** Without loss of generality, we can suppose that $K_1$ is fibered. By Remark 3.3, $K_i$ is fibered for all positive integers $i$. By [6, Lemma 3.4], $K_i \neq K_{i+1}$ implies

$$g(K_i) = d(K_i)/2 > d(K_{i+1})/2 = g(K_{i+1}),$$

where $d(L)$ is the degree of the Alexander polynomial of a knot $L$. Therefore, there exists some $m$ such that $K_n = K_m$ for all $n \geq m$. □

4. **Proof of Theorem 1.4**

In this section, we prove Theorem 1.4 and give the table of prime minimal knots up to 8-crossings except for 815, see Table 1.

**Proof of Theorem 1.4.** Gordon [6, Theorem 1.3] proved that a Q-anisotropic and transfinitely nilpotent knot is minimal in $(K, \leq)$). Here we have the following.

- Two-bridge knots and fibered knots are transfinitely nilpotent ([6]).
- A knot with an irreducible Alexander polynomial is Q-anisotropic ([6, Corollary 4.2]).

By using these results, we can prove that the following knots are minimal in $(K, \leq)$.

31, 41, 51, 62, 63, 71, 72, 73, 74, 75, 76, 77, 81, 82, 83, 84, 86, 87, 812, 813, 814, 816, 817, 820.

We will prove that the knots 85, 818, 819, and 821 are minimal in $(K, \leq)$.

**The minimality of 85 and 821:** the knot 85 is fibered and $g_3(85) = 3$. If $K \leq 85$, then $K$ is fibered by Remark 3.3 and $g_3(K) \leq 3$ by [6, Lemma 3.4]. Since $85\# - K$ is a slice knot, we have

$$\Delta_{85}(t)\Delta_{-K}(t) = \Delta_{85\#-K} = f(t)f(t^{-1})$$

for some $f(t) \in \mathbb{Z}[t]$, where $-K$ denotes the mirror image of $K$ with reversed orientation, $\Delta_L(t)$ is the Alexander polynomial of a knot $L$, and $\hat{\pm}$ means "is equal to, up to multiplication by a unit". Here we have

$$\Delta_{85}(t) \equiv (t^2 - t + 1)(t^4 - 2t^3 + t^2 - 2t + 1).$$

Therefore $\Delta_{-K}(t) = \Delta_{85}(h(t))$ for some $h(t) \in \mathbb{Z}[t]$. The inequality $g_3(K) \leq 3$ implies that $h(t) \equiv 1$. Hence $g_3(K) = g_3(85)$. This implies that $K = 85$ by [6, Lemma 3.4]. Therefore 85 is minimal in $(K, \leq)$.

By the same argument, we can prove that the knot 821 is also minimal in $(K, \leq)$.

**The minimality of 818:** the knot 818 is fibered and $g_3(818) = 3$. If $K \leq 818$, then $K$ is fibered and $g_3(K) \leq 3$ by the same argument. Livingston [13] proved that the concordance genus of 818 is three, in particular, $g_3(K) \geq 3$. Hence $g_3(K) = 3$ and this implies that $K = 818$ by [6, Lemma 3.4]. Therefore 818 is minimal in $(K, \leq)$.
The minimality of $8_{19}$: the knot $8_{19}$ is minimal in $(\mathcal{K}, \leq)$ since it is a torus knot, see Corollary 1.3.

As a summary, we obtain Table 1. Here

- The term “two-bridge” asks whether a given knot is two-bridge or not.
- The term “fibered” asks whether a given knot is fibered or not.
- The Alexander polynomial is presented in the form such that each factor is irreducible.
- The term “SQ” asks whether a given knot is strongly quasipositive or not.
- The term “minimality” asks whether a given knot is minimal in $(\mathcal{K}, \leq)$ or not.

In Table 1, we referred to KnotInfo [13].

| Name | two-bridge | fibered | $\Delta_K(t)$ | SQ | minimality |
|------|------------|---------|---------------|----|------------|
| $3_1$ | Yes | Yes | $1 - t + t^2$ | Yes | Yes |
| $4_1$ | Yes | Yes | $1 - 3t + t^2$ | No | Yes |
| $5_1$ | Yes | Yes | $1 - t + t^2 - t^3 + t^4$ | Yes | Yes |
| $5_2$ | Yes | No | $2 - 3t + 2t^2$ | Yes | Yes |
| $6_1$ | Yes | No | $(2t - 1)(t - 2)$ | No | $U \leq 6_1$ |
| $6_2$ | Yes | Yes | $1 - 3t + 3t^2 - 3t^3 + t^4$ | No | Yes |
| $6_3$ | Yes | Yes | $1 - 3t + 5t^2 - 3t^3 + t^4$ | No | Yes |
| $\tau_1$ | Yes | Yes | $1 - t + t^2 - t^3 + t^4 - t^5 + t^6$ | Yes | Yes |
| $\tau_2$ | Yes | No | $3 - 5t + 3t^2$ | Yes | Yes |
| $\tau_3$ | Yes | No | $2 - 3t + 3t^2 - 3t^3 + 2t^4$ | Yes | Yes |
| $\tau_4$ | Yes | No | $4 - 7t + 4t^2$ | Yes | Yes |
| $\tau_5$ | Yes | No | $2 - 4t + 5t^2 - 4t^3 + 2t^4$ | Yes | Yes |
| $\tau_6$ | Yes | Yes | $1 - 5t + 7t^2 - 5t^3 + t^4$ | No | Yes |
| $\tau_7$ | Yes | Yes | $1 - 5t + 9t^2 - 5t^3 + t^4$ | No | Yes |
| $\tau_8$ | Yes | No | $3 - 7t + 3t^2$ | No | Yes |
| $\tau_9$ | Yes | Yes | $1 - 3t + 3t^2 - 3t^3 + 3t^4 + t^5$ | No | Yes |
| $\tau_{10}$ | Yes | No | $4 - 9t + 4t^2$ | No | Yes |
| $\tau_{11}$ | Yes | No | $2 - 5t + 5t^2 - 5t^3 + 2t^4$ | No | Yes |
| $\tau_{12}$ | Yes | Yes | $(t^2 - t + 1)(t^2 - 2t + t^3 - 2t + 1)$ | No | Yes |
| $\tau_{13}$ | Yes | No | $2 - 6t + 7t^2 - 6t^3 + 2t^4$ | No | Yes |
| $\tau_{14}$ | Yes | Yes | $1 - 3t + 5t^2 - 5t^3 + 5t^4 - 3t^5 + t^6$ | No | Yes |
| $\tau_{15}$ | No | No | $(2 - 2t + t^2)(1 - 2t + 2t^2)$ | No | $U \leq 8_8$ |
| $\tau_{16}$ | Yes | Yes | $(-1 + t - 2t^2 + t^3)(-1 + 2t - t^2 + t^3)$ | No | $U \leq 8_9$ |
| $\tau_{17}$ | No | Yes | $(1 - t + t^2)^2$ | No | $3_1 \leq 8_{19}$ |
| $\tau_{18}$ | Yes | No | $(2t - 1)(t - 2)(t^2 - t + 1)$ | No | $3_1 \leq 8_{19}$ |
| $\tau_{19}$ | Yes | Yes | $1 - 7t + 13t^2 - 7t^3 + t^4$ | No | Yes |
| $\tau_{20}$ | Yes | No | $2 - 7t + 11t^2 - 7t^3 + 2t^4$ | No | Yes |
| $\tau_{21}$ | Yes | No | $2 - 8t + 11t^2 - 8t^3 + 2t^4$ | No | Yes |
| $\tau_{22}$ | No | No | $(1 - t + t^2)(3 - 5t + 3t^2)$ | Yes | Unknown |
| $\tau_{23}$ | No | Yes | $1 - 4t + 8t^2 - 9t^3 + 8t^4 - 4t^5 + t^6$ | No | Yes |
| $\tau_{24}$ | No | Yes | $1 - 4t + 8t^2 - 11t^3 + 8t^4 - 4t^5 + t^6$ | No | Yes |
| $\tau_{25}$ | No | Yes | $(1 - 3t + t^2)(1 - t + t^2)^2$ | No | Yes |
| $\tau_{26}$ | Yes | Yes | $(1 - t + t^2)(1 - t + t^2)$ | Yes | Yes |
| $\tau_{27}$ | Yes | Yes | $(1 - t + t^2)(1 - t + t^2)$ | No | $U \leq 8_{20}$ |
Acknowledgements. The first author thanks Sang Youl Lee for his interest to the table of minimal knots in the conference “The 13th KOOK-TAPU Joint Seminar on Knots and Related Topics” on 26–28 July 2022, which leads us to Theorem 1.4. The first author was supported by the Research Promotion Program for Acquiring Grants in Aid for Scientific Research (KAKENHI) in Ritsumeikan University. The second author was supported by JSPS KAKENHI Grant numbers JP18K13416 and JP22K13923.

REFERENCES

[1] T. Abe and K. Tagami, Fibered knots with the same 0-surgery and the slice-ribbon conjecture, Math. Res. Lett. 23 (2016), no. 2, 303–323.
[2] T. Abe and K. Tagami, A generalization of the slice-ribbon conjecture for two-bridge knots and $t_n$-move, preprint (2022).
[3] I. Agol, Ribbon concordance of knots is a partial order, arXiv:2201.03626 (2022).
[4] K. L. Baker, A note on the concordance of fibered knots, J. Topol., 9 (2016), no. 1, 1–4.
[5] A. Daemi and C. Scaduto, Chern-Simons functional, singular instantons, and the four-dimensional clasp number, arXiv:2007.13160 (2020).
[6] C. McA. Gordon, Ribbon Concordance of Knots in the 3-sphere, Math. Ann. 257 (1981), no. 2, 157–170.
[7] M. Hedden, Notions of positivity and the Ozsváth-Szabó concordance invariant, J. Knot Theory Ramifications. 19 (2010), no. 5, 617–629.
[8] H. Imori, Instanton knot invariants with rational holonomy parameters and an application for torus knot groups, arXiv:2108.13998 (2021).
[9] H. Kochloukova, On a conjecture of E. Rapaport Strasser about knot-like groups and its pro-p version, Journal of Pure and Applied Algebra, 204 (2006), Issue 3, 536–554.
[10] P. B. Kronheimer and T. S. Mrowka, Khovanov homology is an unknot-detector, Publ. Math. Inst. Hautes Études Sci. 113 (2011), 97–208.
[11] A. S. Levine and I. Zemke, Khovanov homology and ribbon concordances, Bull. Lond. Math. Soc. 51 (2019), no. 6, 1099–1103.
[12] C. Livingston, Computations of the Ozsváth-Szabó knot concordance invariant, Geom. Topol. 8 (2004), 735–742.
[13] C. Livingston, The concordance genus of a knot, II, Algebr. Geom. Topol. 9 (2009), 167–185.
[14] C. Livingston and A. H. Moore, KnotInfo: Table of Knot Invariants, knotinfo.math.indiana.edu, August 12, 2022.
[15] K. Miyazaki, A note on genera of band-connected sums that are fibered, J. Knot Theory Ramifications. 27 (2018), no. 12, 1871002.
[16] Y. Ni, Knot Floer homology detects fibred knots, Invent. Math. 170 (2007), no. 3, 577–608.
[17] P. Ozsváth and Z. Szabó, Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334.
[18] M. Powell and C. Julia, Slice Knots: Knot Theory in the 4th Dimension, based on lectures of Peter Teichner (2010).
[19] A. N. Shumakovitch, Rasmussen invariant, slice-Bennequin inequality, and sliceness of knots, J. Knot Theory Ramifications 16 (2007), no. 10, 1403–1412.
[20] D. S. Silver, On knot-like groups and ribbon concordance, Journal of Pure and Applied Algebra, 82 (1992), Issue 1, 99–105.
[21] I. Zemke, Knot Floer homology obstructs ribbon concordance, Ann. of Math. 190 (2019) no. 3, 931–947.

Faculty of Science and Engineering, Ritsumeikan University, Kusatsu-Shiga, Japan
Email address: tabe@fc.ritsumei.ac.jp

Department of The Faculty of Economics Sciences, Hiroshima Shudo University, Hiroshima, 731-3195 JAPAN
Email address: ktagami@shudo-u.ac.jp