Delocalized Entanglement of Atoms in optical Lattices

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We show how to detect and quantify entanglement of atoms in optical lattices in terms of correlations functions of the momentum distribution. These distributions can be measured directly in the experiments. We introduce two kinds of entanglement measures related to the position and the spin of the atoms.

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Experiments on atoms in optical lattices have recently become very attractive playgrounds to investigate basic issues in the context of quantum information theory [1 2 3]. The high degree of control reached in those experiments should allow us to prepare a large variety of entangled states, to analyze their physical properties, and to verify and quantify the presence of entanglement. However, in order to carry out these investigations, a clear definition of entanglement with a clear physical meaning should be given, and ways to detect it should be explored [4]. For atoms in optical lattices those questions are far from being trivial, since one has to consider different degrees of freedom (such as atom numbers and internal levels) and take into account the presence of super-selection rules, as well as the fact that in most of those experiments we only have access to certain collective properties.

In this letter we will define and explore certain kinds of entanglement which are relevant in current experiments dealing with bosonic and fermionic atoms in optical lattices. We will concentrate on the entanglement properties between different sites of the optical lattice (i.e. in second quantization), since in this case a clear meaning as a resource for quantum information tasks can be assigned to those definitions. Furthermore, we will restrict ourselves here to the simplest case of bipartite entanglement. Since it is very hard in practice to address atoms at different lattice sites, we define the delocalized bipartite reduced density operators for a state \( \rho \) on an optical lattice by

\[
\rho_{AB} = \sum_{m} \rho_{(m,m+x)},
\]

where \( \rho_{(m,m+x)} \) denotes the restriction of \( \rho \) to the sites \( m \) and \( m + x \) of the lattice. Since we do not want to rely on any form of addressability, we assume the lattice to be (approximately) infinite. To simplify matters, we consider here only a 1D lattice, but everything holds as well when the state is defined on 2D or 3D lattices. We will study the entanglement of \( \rho_{AB} \) by means of experimental feasible collective measurements, that can be translated directly into expectation values of \( \rho_{AB} \). Note that, for this one to one correspondence between measurement results and expectation values of \( \rho_{AB} \), we define this state to be unnormalized. This definition of \( \rho_{AB} \) is useful for entanglement detection, however for a quantitative investigation, we need to include the normalization in the definition. Due to the infinite number of lattice sites the trace of \( \rho_{AB} \) is infinite, such that a straightforward normalization fails to give any usable quantitative information. Therefore we have to take into account that, for a finite number of atoms, most of the lattice sites are empty and do not contribute to any measurement. The obvious solution is to restrict the summation to a finite part of the lattice, such that \( \rho_{AB} \) can be normalized. But this makes only sense if we can ensure the atoms to be located in a relative small region of the lattice, what requires either some form of addressability or an extra assumption about the localization of the state \( \rho \). A more general solution is to define the state \( \rho'_{AB} \) as the (normalized) projection of \( \rho_{AB} \) to the subspace where at least one atom is present, i.e., project out the zero atom subspace of \( \rho_{AB} \). Even if the atoms are evenly spread over the whole lattice, as long as the number of atoms is small, the projection of \( \rho_{AB} \) has a finite trace. Unfortunately, we can not give the exact trace in terms of experimental feasible measurements, but we can bound the trace which still allows us to derive lower bounds for the entanglement. Through the definition of \( \rho_{AB} \) and \( \rho'_{AB} \) we are mixing different contributions of different sites, which implies that the entanglement we define will be somehow delocalized between different pairs of sites which are separated by a distance \( x \). Note, that \( \rho_{AB} \) depends on the distance \( x \), which can be freely chosen. We will give lower bounds to the entanglement of formation for \( \rho'_{AB} \). Note, that due to the translational symmetry of \( \rho_{AB} \), we can never reach a maximally entangled state for \( \rho'_{AB} \). The maximally possible entanglement of formation that can be reached is 0.285. [5]

We will consider two situations: firstly, that in which the atomic internal levels are not involved, and therefore we will deal with different occupation numbers; secondly, the one in which entanglement occurs between different internal states of the atoms in each site. As we will show, the first case is very simple to characterize and one can already claim that this kind of entanglement has been created in several experiments carried out so far. The second kind of entanglement is much subtler, and require...
more sophisticated measurements in order to prove the existence of entanglement.

The trapped atoms are described in second quantization by the annihilation and creation operators. We will assume two internal levels \(a\) and \(b\) for each atom, in which \(a_m, a_m^\dagger\) resp. \(b_m, b_m^\dagger\) denote the annihilation and creation operators of atoms in level \(a\) resp. \(b\) at site \(m\).

**Measurements:** One kind of measurement that is feasible in experiments is to turn off the lattice potential and look at the density \(n_a(x,t) = \langle \psi_a(x,t) \rangle_{\psi_a(x,t)}\) (resp. \(n_b(x,t)\)) of the expanding atom cloud, where \(\psi_a(x,t)\) is the bosonic or fermionic field operator for the internal level \(a\). Furthermore, one can measure the density-density correlations \(c_{ab}(x,x',t) = \langle \psi_a(x,t)^\dagger \psi_a(x',t) \psi_b(x',t) \psi_b(x,t) \rangle\) and in a similar way \(c_{aa}(x,x',t), c_{bb}(x,x',t)\) and \(c_{ab}(x,x',t)\). In a long time of flight approximation the new positions of the particles can be calculated from the momentum distribution \(k \approx mx/(\hbar t)\), due to the relation we can measure the momentum distribution \(n_a(k) = \lim_{t \to \infty} n_a(\frac{mk}{\hbar}, t)\) and their correlations \(c_{xy}(k,k') = \lim_{t \to \infty} c_{xy}(\frac{mk}{\hbar},\frac{mk'}{\hbar}, t)\). This momentum distribution in second quantization is given by

\[
n_a(k) \approx \sum_{n,m,} \hat{w}_n(k) \hat{w}_m(k)^\ast \langle a_n^\dagger a_m \rangle \tag{2}
\]

and the density-density correlations by

\[
c_{ab}(k,k') \approx \sum_{n,m,n',m'} \hat{w}_n(k) \hat{w}_m(k)^\ast \hat{w}_{n'}(k') \hat{w}_{m'}(k')^\ast \langle a_n^\dagger a_m b_{n'}^\dagger b_{m'} \rangle , \tag{3}
\]

where \(\hat{w}_n(k)\) denotes the Fourier transformed Wannier function at time zero at site \(n\). And in an analogous manner we get \(n_b(k), c_{aa}(k,k'), c_{bb}(k,k')\) and \(c_{ab}(k,k')\).

**Occupation number entanglement:** We assume now the simple case, where we only have one type of atom, say that in level \(a\). Due to the atom number conservation the possible product states in second quantization are restricted to be of the form

\[
| \ldots, n_0, n_1, n_2, \ldots \rangle, \tag{4}
\]

because it is not allowed to have superpositions between states with different number of atoms. Here \(n_i\) denotes the occupation number of site \(i\). We define a state to be entangled, if it can not be decomposed into states of the form \(| \ldots, n_0, n_1, n_2, \ldots \rangle\). Testing separability simplifies in this case to check whether a given state is diagonal in \(| \ldots, n_0, n_1, n_2, \ldots \rangle\). Note, that entanglement defined with respect to such a super-selection rule is in general a less powerful resource for quantum information tasks, because the entanglement properties can only be seen/used when having access to several copies of the state \(| \ldots, n_0, n_1, n_2, \ldots \rangle\). To detect this kind of entanglement we look at the momentum distribution \(| \ldots, n_0, n_1, n_2, \ldots \rangle\). A simple observation is that for all product states we get a \(\delta(n = m)\) in the sum, because \(a_n^\dagger a_m\) maps any state of form \(| \ldots, n_0, n_1, n_2, \ldots \rangle\) to a orthogonal one if \(n \neq m\). For this reason we get for any separable state that \(n_{ab}(k) = N\) is just proportional to the total number operator and is independent from \(k\), i.e., the momentum distribution is flat (up to the envelope Wannier-functions). Any non flat momentum distribution indicates an entangled multipartite state, something that has been already observed in several experiments 3. For a more quantitative statement about the entanglement of \(| \ldots, n_0, n_1, n_2, \ldots \rangle\) we look at the Fourier transformation of the momentum distribution of \(\rho\)

\[
\langle Q_x \rangle_{\rho} := \int dk e^{-i k x d} \langle n_a(k) \rangle_{\rho} = \sum_m \langle a_m^\dagger a_{m+x} \rangle_{\rho}. \tag{5}
\]

Here we have used, that multiplication by a phase \(e^{i k x d}\) in momentum representation results in a shift in position, i.e., \(\hat{w}_n(k) e^{-i k x d} = \hat{w}_{n+x}(k)\) and that two at different places located Wannier functions are orthogonal, i.e., \(\int dk \hat{w}_{n+x}(k) \hat{w}_m^\ast(k) = \delta(n + x, m)\). \(x\) is taken to be an arbitrarily integer and \(d\) denotes the lattice constant. For \(\rho\) we can give an interpretation in terms of an expectation value of the bipartite state \(\rho_{AB}\):

\[
\langle Q_x \rangle_{\rho} := \langle a_A^\dagger a_B \rangle_{\rho_{AB}}. \tag{6}
\]

Assume now the idealized case where the occupation number of every site is restricted to be either one or zero, defining this way exactly one qubit per site. Then \(\rho_{AB}\) is a two qubit density matrix and \(\langle a_A^\dagger a_B \rangle = \langle 01 | \rho_{AB} | 10 \rangle\) is exactly one off-diagonal element, where \(|0\rangle\) and \(|1\rangle\) denote empty or occupied sites. Due to the super-selection rules it is the only allowed off-diagonal element and defines the entanglement properties of the state. The state \(\rho_{AB}\) can be decomposed into two parts. A separable part spanned by the vectors \(|11\rangle, |00\rangle\) and the part spanned by the vectors \(|10\rangle, |01\rangle\) that contains entanglement if \(|01\rangle | \rho_{AB} | 10 \rangle \neq 0\). For this part we now want to estimate a lower bound for entanglement of formation 3. Note, that we use here a definition for entanglement of formation respecting the super-selection rules. Assuming now a normalized state \(\rho'_{AB}\) with given off-diagonal element \(\lambda\), it can easily be shown, that a pure state completely supported on the \(|10\rangle, |01\rangle\) subspace having the same off-diagonal element \(\lambda\) gives us a lower bound to the entanglement of formation. Exploiting this leads to a lower bound for the entanglement given by 

\[
E_{af}(\rho'_{AB}) \geq S \left( \frac{1}{2} \left[ 1 - \sqrt{1 - 4 |\lambda|^2} \right] \right),
\]

where \(S(x) = -x \log(x) - (1 - x) \log(1 - x)\) denotes the von Neumann entropy. To estimate \(|\lambda|\) for our \(\rho'_{AB}\) we first have to find a bound for the trace of the unnormalized \(\rho'_{AB}\). This can be given by \(2 \langle N \rangle\), since the the reduced densities states in \(| \ldots, n_0, n_1, n_2, \ldots \rangle\) cover two times the whole lattice. Therefore we can conclude that \(|\lambda| \geq \frac{\langle Q_x \rangle_{\rho}}{2 \langle N \rangle}\). While the assumed restriction of maximal one atom per site matches perfectly for fermions, in the bosonic
case we can still give a bound if the following constraint can be guaranteed, e.g., verified by further measurements \( \mathbf{R} \). The expected number of sites with more than one atom has to be smaller than \( \epsilon \langle N \rangle \) and the maximally occupation number of one site has to be smaller than \( r \). Under these conditions, the measurement result is still close to the off-diagonal element \( \langle 01 \vert \rho_{AB} \vert 10 \rangle \). The error coming from overpopulated sites can bounded by \( (2\epsilon r + 4\sqrt{\epsilon}) \langle N \rangle \) (see Appendix) leading to

\[
E_{of}(\rho_{AB}) \geq S \left( \frac{1}{2} - 1 \sqrt{1 - \left( \frac{\langle Q_x \rangle}{\langle N \rangle} \right)^2} \right). 
\]

**Internal level entanglement:** We now consider the case, where we have two level atoms in the lattice. In the ideal situation we would have exactly one atom per site such that the internal levels realize one qubit. In this case \( \rho_{AB} \) is again a two qubit state without any restriction due to the conservation laws. To detect this stronger kind of entanglement it is not enough to look independently at the momentum distribution of level \( a \) and \( b \), but we have to look at the correlation \( \mathbf{R} \) between momentum distributions \( c_{ab}(k,k') \). By properly chosen Fourier transformations in \( k \) and \( k' \) we define

\[
\langle Q_x^{ab} \rangle := \int dk \, dk' \, e^{ikx} \langle c_{ab}(k,k') \rangle \quad (8)
\]

and in an analog way \( \langle Q_x^{aa} \rangle \) and \( \langle Q_x^{bb} \rangle \). Here we again use the fact that the integrals over Wannier functions on different sites leads to delta functions. We furthermore assume a situation where we can restrict these sums to the case \( m = m' \).

\[
\langle Q_x^{ab} \rangle = \sum_m \langle a_m^+ a_{m+x} b_{m+x} b_m \rangle, \quad (10)
\]

We will discuss later in the section dephasing how this condition can be realized by adding extra magnetic fields such that the \( m \neq m' \) terms vanish.

**The one atom per site case:** To illustrate the basic idea, we assume now the idealized situation, where we have a state \( \rho \) for which we can ensure, that in every site is exactly one atom. Note, that in this case we can use \( \mathbf{10} \) without assuming any extra magnetic fields, because the terms with \( m \neq m' \) vanish already because of the assumption. Equation \( \mathbf{10} \) can now interpreted as the expectation value

\[
\langle Q_x^{ab} \rangle_{\rho} := \langle a_A^+ a_B b_B^+ b_A \rangle_{\rho_{AB}}, \quad (11)
\]

of a bipartite density matrix \( \rho_{AB} \) as defined in \( \mathbf{11} \). We want now to calculate the overlap of the state \( \rho_{AB} \) with a maximally entangled state, i.e., the fidelity \( f_\phi(\rho_{AB}') = \langle \phi \vert \rho_{AB}' \vert \phi \rangle \), where \( \phi \) will be one of the Bell-state defined by

\[
\phi_{\pm} = \frac{1}{\sqrt{2}} (\vert 10 \rangle_A \vert 01 \rangle_B \pm \vert 01 \rangle_A \vert 10 \rangle_B)
\]

Here \( \vert 10 \rangle_A \) denotes the atom of Alice being in the \( a \) and \( \vert 01 \rangle_A \) being in the \( b \) level and in analog way for Bob. We claim that now, that

\[
\langle Q_x^{ab} + Q_x^{ba} \rangle_{\rho} = - \langle \phi^- \vert \rho_{AB} \vert \phi^- \rangle + \langle \phi^+ \vert \rho_{AB} \vert \phi^+ \rangle + \langle N \rangle \quad (12)
\]

where the \( (\pm) \) in \( \mathbf{12} \) distinguishes between the bosonic and the fermionic case. This can easily checked, by calculating the expectation values for an arbitrary pure state

\[
\vert \Phi \rangle = \lambda_{00} \vert 10 \rangle_A \vert 01 \rangle_B + \lambda_{01} \vert 01 \rangle_A \vert 10 \rangle_B \quad (13)
\]

\[
+ \lambda_{10} \vert 10 \rangle_A \vert 01 \rangle_B + \lambda_{11} \vert 01 \rangle_A \vert 10 \rangle_B.
\]

We get that \( \langle Q_x^{ab} \rangle_{\phi} = \lambda_{00} \vert \phi^- \rangle + \langle \phi^+ \rangle_{\phi} \) such that the sum is equal to \( -\langle \phi^- \rangle + \langle \phi^+ \rangle \). Furthermore we obtain that

\[
\langle Q_x^{aa} \rangle_{\rho} = \langle a_A^+ a_B a_B^+ a_A \rangle_{\rho_{AB}}, \quad (14)
\]

and in an analog way \( \langle Q_x^{bb} \rangle \). It is easily verified, that \( \langle a_A^+ a_B b_B^+ b_A \rangle \) is the bosonic state and in analogous manner for \( \langle Q_x^{ba} \rangle \). It is is easy verified, that \( \langle a_A^+ a_B b_B^+ b_A \rangle = \langle b_A^+ a_B a_B^+ b_A \rangle = 1 \) for bosonic and fermionic case. Note, that we can get fidelities with further maximally entangled states by applying a global unitary \( U \otimes \cdots \otimes U \) to \( \rho \). This translates to apply \( U \otimes U \) to \( \rho_{AB} \). While the overlap with \( \vert \phi^- \rangle \) stays unchanged, we get that

\[
\langle \phi^+ \vert U \otimes U \rho_{AB} (U \otimes U)^\dagger \vert \phi^+ \rangle = \langle \phi^+ \vert \rho_{AB} \vert \phi^+ \rangle,
\]

where \( \langle \phi_U \rangle = 1 \otimes (UU^T) \).
$U$. The fidelities $f_{\phi_+}$ and $f_{\phi_-}$ of a state directly gives us lower bounds to several entanglement measurements by comparing it to Bell-diagonal or isotropic states $|\mathbb{I}\rangle$, e.g.,

$$E_{of}(\rho_{AB}) \geq S \left( \frac{1}{2} \left[ 1 - \sqrt{1 - (1 - 2f_{\phi_\pm})^2} \right] \right). \quad (15)$$

The general case: We consider now the general case where the number of atoms per site is arbitrarily. In this case it is difficult to give a lower bound to $\langle \phi_{\pm} | \rho_{AB} | \phi_{\pm} \rangle$, because the four Bell-states do not build a basis in the larger Hilbert-space and it is therefore not possible to derive the fidelity from $[10]$. But it still makes sense to define $\Lambda := \pm (\langle \phi_+ \rangle - \langle \phi_- \rangle)$, because the four Bell-states do not build a basis in the larger Hilbert-space. Given a value of $\Lambda$ it can be shown that the bound $[15]$ still holds with $f_{\phi_\pm}$ now replaced by $f_{\phi_\pm} = \frac{1}{2} \Lambda$, even though $f_{\phi_\pm}$ has now no direct interpretation as fidelity. Note, that $\Lambda$ is a direct bound for the projection number of $|\mathbb{I}\rangle$ and can be used itself to quantify the entanglement. Our goal is to give a lower bound for $f_{\phi_\pm}$ for a state $\rho_{AB}$ that we define in this case as the projection $\rho_{AB}$ to the subspace with 2 or more atoms. We assume, that number of defective sites $D$ should be bounded by $D \leq \epsilon \langle N \rangle$. As defect counts every site having two or more atoms. Furthermore we assume the maximal occupation number of $a$ and $b$ to be less than $r$. A straightforward calculation (see Appendix) leads to

$$f_{\phi_+}^B \geq \frac{\langle \pm (Q^{ab} + Q^{ba}) + (2 - 4\epsilon^2)N - Q^{\prime ab} - Q^{\prime ba} \rangle}{2 \langle N \rangle}$$

$$f_{\phi_-}^F \geq \frac{\langle \pm (Q^{ab} + Q^{ba}) + Q^{aa} + Q^{bb} - 4\epsilon N \rangle}{2 \langle N \rangle}$$

for the bosonic and fermionic cases, which can be used to estimate a lower bound for $\rho_{AB}^L$.

Dephasing with magnetic field: We discuss now the possibilities of eliminating the terms of $[9]$ where $m \neq m'$ by dephasing, i.e., by destroying any delocalization of atoms over several sites. If we write $\hat{F} = \sum_k \lambda_K |K\rangle$ in a product basis $|K\rangle = |a_{\infty}^{\prime\prime}, \ldots, a_{\infty}^{\prime}, b_{\infty}^{\prime\prime}, \ldots, b_{\infty}^{\prime}\rangle$, where $k_{i}^{(a/b)}$ denotes the number of atoms in site $i$ in level $a/b$, then the unwanted terms that contribute to the sum are of the form

$$\lambda_K^* \lambda_{K'} \langle K | a_i^a a_{m+x}^b b_i^b b_{m+x}^a | K' \rangle,$$

where $m \neq m'$. To give a nonzero value, $|K\rangle$ has to be equal to $|K'\rangle$ although $|K\rangle$ has an extra atom in $m$ and $m'+x$ and a missing atom in $m+x$ and $m'$, which implies defects in this pair of sites. We will use this displaced atoms and an additional inhomogeneous magnetic field to introduce some random phase to $[10]$ and thereby guarantee that these terms vanish. Let us assume a magnetic field, that is proportional to $k^2$, where $k$ is the number of the lattice site. Then the state $|K'\rangle$ gets, up to a global phase, a phase of $e^{i(m+x)^2 + (m'+x)^2}$, whereas $|K\rangle$ picks up $e^{i(m^2t + (m'+x)^2)}$. So $|K\rangle$ and $|K'\rangle$ get a relative phase of $e^{i((m^2 - m')x)}$ [10]. By a properly chosen set of times, we can randomize the phases such that for given $x$ all terms with $m \neq m'$ vanish in average, while the terms with $m = m'$ stay unchanged. Note that even without magnetic field it seems to be quite unlikely, that all terms of the form $[10]$ sum up to a nonzero contribution. To give some non vanishing amount, the defects have to be correlated in a very unlikely way. In detail, one defect located at $m$ or $m+x$ has to be correlated to a defect located at $m'$ or $m'+x$ and in addition all these cases has to be correlated with each other. If we assume, that defects occur only randomly and therefore are uncorrelated, $[10]$ already vanish.

In conclusion, we have defined two figures of merit to quantify entanglement for atoms trapped in optical lattices, by only measuring density correlation functions of the expanded atomic cloud, without requiring any addressability of the original lattice setup. One set of measurement data can be used to study entanglement at arbitrarily distances $x$. We discuss bounds in the cases where the defects in the lattice can be bounded. We acknowledge support from EU projects SCALA and DFG-Forschungsgruppe 635.
Appendix

Calculating errors for the off-diagonal element: \( \rho_{AB} \) can be written as a direct sum \( \sum_n \rho_n \), where \( \rho_n \) is the \( n \)-atom subspace. The off-diagonal element \( (01|\rho_{AB}|10) \) is exactly given by \( \langle Q_x \rangle_{\rho_n} \). So we have to bound all the absolute values of \( \langle Q_x \rangle_{\rho_n} \) for \( n > 1 \). Note, that all \( \rho_n \) for \( n > 2 \) has at least one defect. Therefore we can bound the trace by two times the number of defects \( (2\langle N \rangle) \) and therefore the total contribution by \( 2\varepsilon \langle N \rangle \). We need to take twice the number of defects, because the reduced states in the cover two times the whole lattice. More complicated is to find a bound \( \rho_2 \), because 2 atoms do not imply automatically a defect. We therefore look at a pure state supported on \( \rho_2 \)

\[ |\psi_2 \rangle = \lambda_0 |11 \rangle + \lambda_1 |20 \rangle + \lambda_2 |02 \rangle. \tag{17} \]

\( \langle Q_x \rangle_{\psi_2} \) easily calculates to \( (\lambda_0 \lambda_1^* + \lambda_0^* \lambda_2)\sqrt{2} \), with \( \sum_i |\lambda_i|^2 = 1 \). We now try to find an upper bound for its absolute value. First lets assume, that all \( \lambda_i \) are real and positive leading to \( \lambda_0 (\lambda_1 + \lambda_2) \sqrt{2} \). For given \( \lambda_0 \) this is maximized for \( \lambda_1 = \lambda_2 \). Defining \( \lambda' = \frac{1}{\sqrt{2}} \lambda_1 + \lambda_2 \), we get \( \langle Q_x \rangle_{\psi_2} \leq 2 \lambda' \sqrt{2} \leq 2 \lambda' \sqrt{2} \leq 2 \sqrt{\lambda_1^2 + \lambda_2^2} \) as upper bound. Now, we use that \( \sqrt{2} \leq \frac{1}{2\sqrt{R}} x + \frac{\sqrt{2}}{2} \) for any positive parameter \( R \), which gives us \( \langle Q_x \rangle_{\psi_2} \leq \frac{1}{\sqrt{R}} (\lambda_1^2 + \lambda_2^2) + \sqrt{R} \). Here \( \lambda_1^2 + \lambda_2^2 \) is now exactly the probability for a defect. For the unnormalized \( \rho_2 = \sum_i q_i |\psi_2 \rangle \langle \psi_2 | \) we get \( \langle Q_x \rangle_{\rho_2} \leq \frac{1}{\sqrt{R}} \sum_i q_i (\lambda_1^2 + \lambda_2^2) + \sqrt{R} \). The sum \( \sum_i q_i (\lambda_1^2 + \lambda_2^2) \) is now the probability to found a defect in \( \rho_2 \), that can be bounded by \( 2\varepsilon \langle N \rangle \). Furthermore, the trace of \( \rho_2 \) can be bounded by \( \text{tr}(\rho_2) \leq \langle N \rangle \), leading to \( \langle Q_x \rangle_{\rho_2} \leq \frac{1}{\sqrt{R}} + \sqrt{R} \). We need a free to choose \( R = \sqrt{\varepsilon} \) and get finally \( \langle Q_x \rangle_{\rho_2} \leq 4\sqrt{\varepsilon} \). So we get \( |\langle 01|\rho_{AB}|10 \rangle| \geq |\langle Q_x \rangle_{\rho} | - (4\sqrt{\varepsilon} + 2\varepsilon) \langle N \rangle \)

Calculating errors for fidelity: We want to bound \( \Lambda = 2f_{\phi\pm} - 1 = (\phi_+ + \phi_-) + (\psi_+ + \psi_-) \). Given the state \( \rho_{AB} = \sum_n \rho_n \) written as direct sum. We are now interested in \( \Lambda_{\rho_{AB}} \) which is equal to \( \Lambda_{\rho_2} \), because \( \rho \) is supported on the 2-atom subspace. Since \( (12) \) holds also on the bigger 2-atom subspace \( 2f_{\phi\pm}(\rho_2) - 1 \) equals \( \Lambda_{\rho_2} \). \( \Lambda_{\rho_1} \) gives no contribution and can therefore neglected. For this reason, we only have to take defects with two or more atoms per site into account. We therefore have only to bound \( 2f_{\phi\pm}(\rho^+) - 1 \), where \( \rho^+ = \sum_{n>2} \rho_n \). Note, that this sum runs by assumption only up to \( r^4 \). The trace of \( \rho^+ \) is smaller than \( 2D < 2\varepsilon \langle N \rangle \), since for \( n > 2 \) there exists always a defect and \( 2f_{\phi\pm}^B - 1 \) can be bounded by \( 4r^2 \) for bosons and by 4 for fermions.