A novel variational form of the Schatten-$p$ quasi-norm

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Abstract

The Schatten-$p$ quasi-norm with $p \in (0, 1)$ has recently gained considerable attention in various low-rank matrix estimation problems offering significant benefits over relevant convex heuristics such as the nuclear norm. However, due to the nonconvexity of the Schatten-$p$ quasi-norm, minimization suffers from two major drawbacks: 1) the lack of theoretical guarantees and 2) the high computational cost which is demanded for the minimization task even for trivial tasks such as finding stationary points. In an attempt to reduce the high computational cost induced by Schatten-$p$ quasi-norm minimization, variational forms, which are defined over smaller-size matrix factors whose product equals the original matrix, have been proposed. Here, we propose and analyze a novel variational form of Schatten-$p$ quasi-norm which, for the first time in the literature, is defined for any continuous value of $p \in (0, 1]$ and decouples along the columns of the factorized matrices. The proposed form can be considered as the natural generalization of the well-known variational form of the nuclear norm to the nonconvex case i.e., for $p \in (0, 1)$. Notably, low-rankness is now imposed via a group-sparsity promoting regularizer. The resulting formulation gives way to SVD-free algorithms thus offering lower computational complexity than the one that is induced by the original definition of the Schatten-$p$ quasi-norm. A local optimality analysis is provided which shows that we can arrive at a local minimum of the original Schatten-$p$ quasi-norm problem by reaching a local minimum of the matrix factorization based surrogate problem. In addition, for the case of the squared Frobenius loss with linear operators obeying the restricted isometry property (RIP), a rank-one update scheme is proposed, which offers a way to escape poor local minima. Finally, the efficiency of our approach is empirically shown on a matrix completion problem.

1 Introduction

Recently, nonconvex heuristics such as $\ell_p$ and $\mathcal{S}_p$ (Schatten-$p$) quasi-norms\footnote{The $\mathcal{S}_p$ quasi-norm (raised to the power $p$) is defined as $\|X\|_{\mathcal{S}_p}^p = \sum_{i=1}^{\text{rank}(X)} \sigma_i^p(X)$, where $\sigma_i(X)$ is the $i$th singular value of $X$.} with $0 < p < 1$ have been shown to significantly outperform their convex counterparts, i.e., $\ell_1$ and nuclear norms, in sparse/low-rank vector/matrix recovery problems, for a wide range of applications\cite{BD2019}. Indeed, for applications such as recovering low-rank matrices from limited measurements the $\mathcal{S}_p$ quasi-norm for $p \in (0, 1)$ is known theoretically to perform at least as well, if not better than the nuclear norm.
(the closest convex approximation to the rank function), provided that one can find the minimum $S_p$ solution \[^5\]. This result is somewhat intuitive in the sense that it is known that the $S_p$ quasi-norm provides an increasingly tight approximation to the rank of a matrix as $p \to 0$; however, the price to be paid is the fact that the $S_p$ quasi-norm is no longer convex for $p \in (0, 1)$, which can significantly complicate efficiently solving models that employ the $S_p$ quasi-norm.

Over the past few years, several works have been presented in the literature which aim to provide practical methods to approximately solve $S_p$ quasi-norm minimization problems. \([2, 3, 6–8]\). By and large, these works typically consider problems of the form

$$
\min_{X \in \mathbb{R}^{m \times n}} \ell(Y, X) + \lambda \|X\|_{S_p}^p
$$

where $Y$ is a data matrix and $\ell(Y, X)$ is a loss function which serves as a measure of how well the adopted model fits the data. Clearly, due to nonconvexity, the task of finding a minimizer of (21) can be quite challenging and potentially limits what can be guaranteed theoretically about solving problems with form (21). Moreover, traditional iterative $S_p$ quasi-norm minimization algorithms typically rely on singular value decomposition steps (SVDs) at each iteration, whose computational cost ($O(\min(mn^2, m^2n))$) may be cumbersome in the high dimensional and large-scale data regime.

With the aim to address the high computational cost of $S_p$ quasi-norm minimization, variational forms of the $S_p$ quasi-norm have been proposed in the literature. Such approaches first: 1) represent a given matrix $X \in \mathbb{R}^{m \times n}$ as the product of matrices $U \in \mathbb{R}^{m \times d}$ and $V \in \mathbb{R}^{n \times d}$, i.e., $X = UV^T$ and then 2) define a regularization term over $U$ and $V$ whose minimum equals the value of the $S_p$ quasi-norm. For example, when $p = 1$, the $S_p$ quasi-norm reduces to the nuclear norm $\|X\|_* = \sum_{i=1}^{\min(m, n)} \sigma_i(X)$, for which several variational forms have been proposed in the literature \([9]\):

$$
\|X\|_* = \min_{d \in \mathbb{N}_+} \min_{U, V} \sum_{i=1}^d \|u_i\|_2 \|v_i\|_2 = \min_{d \in \mathbb{N}_+} \min_{U, V} \sum_{i=1}^d \frac{1}{2} (\|u_i\|_2^2 + \|v_i\|_2^2). \tag{2}
$$

Then, instead of working with a problem with form (21) one can instead consider an equivalent\[^2\] problem in the $(U, V)$ space:

$$
\min_{d \in \mathbb{N}_+} \min_{U, V} \ell(Y, UV^T) + \lambda \sum_{i=1}^d \|u_i\|_2 \|v_i\|_2 = \min_{d \in \mathbb{N}_+} \min_{U, V} \ell(Y, UV^T) + \lambda \sum_{i=1}^d \frac{1}{2} (\|u_i\|_2^2 + \|v_i\|_2^2) \tag{3}
$$

Notably, although variational forms allow one to define problems in terms of $(U, V)$ instead of $X$, which potentially results in significant computational advantages for large-scale data, the resulting problems are not general nonconvex w.r.t. $U$ and $V$ by construction due to the presence of the matrix product $UV^T$. However, despite this challenge, for the case of the convex nuclear norm ($p = 1$) global optimality guarantees can still be obtained for solving problems in the $(U, V)$ domain, such as (5), by exploiting properties of the variational form \([10]\).

Capitalizing on the merits of the variational form of the nuclear norm, different alternative definitions of the $S_p$ quasi-norm have been proposed. In \([3]\), a variational form of $S_p$ is defined for $p = \{\frac{1}{2}, \frac{3}{2}\}$ via norms defined over the matrix factors $U$ and $V$. Along the same lines, a generalized framework which is valid for any value of $p$ was proposed in \([11]\); however, these forms typically require one to compute SVDs of the $(U, V)$ factors to evaluate the (quasi)-norm on $X$, which can be quite limiting computationally in the large-data setting. Recently, a new variational form of the $S_p$ quasi-norm was introduced in \([4]\), which, contrary to previous approaches of \([11, 12]\), is defined over the columns of the matrix factors via group-sparsity imposing norms and are thus SVD-free. Nevertheless, a major shortcoming of the variational form proposed in \([4]\) is that it is only defined for discrete values of $p$ rather than for an arbitrary $p \in (0, 1)$. As a consequence, it is not a full generalization of the variational form of the nuclear norm. Finally, all existing approaches for defining variational forms of the $S_p$ quasi-norm lack any optimality analysis or study with regard to optimizing the problem in the resulting $(U, V)$ space, and a significant question is whether such forms introduce additional ‘spurious’ local minima into the optimization landscape (i.e., a local minimum might exist for $(U, V)$, but there is no local minimum if one is optimizing w.r.t. $X = UV^T$).

**Paper contributions.** In this paper, we propose a novel variational form of the $S_p$ quasi-norm which, similarly to \([4]\), is defined over the columns of the matrix factors $U$ and $V$, but unlike \([4]\) our

\[^2\]Here equivalence is meant to imply that a solution for $(U, V)$ also gives a solution for $X = UV^T$. 

\[\text{2}\]
While the above variational form is general for an arbitrary value of \( p \in (0, 1) \), further, the proposed variational form of the \( S_p \) quasi-norm is a direct generalization of the celebrated variational form of the nuclear norm (2) to the nonconvex case i.e., for \( p \in (0, 1) \), with a clear and intuitive link showing the generalization. In addition, by minimizing the proposed form, the resulting algorithms induce column sparsity on the factors after only a few iterations, eventually converging to matrix factors whose number of nonzero columns equals to their rank. In that sense, a rank revealing decomposition of the matrix \( X \) is obtained directly from the optimization w.r.t. \((U, V)\).

Our second contribution is to present a theoretical analysis which shows that local minimizers of the factorized problem give rise to local minimizers of the original problem defined w.r.t. to \( X \). Our optimality analysis makes use of nonconvex optimization theory [13] and is general in the sense that it can potentially be applied to other variational forms of the \( S_p \) quasi-norm as well as to general concave singular value penalty functions on the condition that such functions are subdifferentially regular (see supplement). As a result of our analysis, one has the computational benefits imparted by the variational form with assurances that no additional poor local minima are introduced into the problem.

Our third contribution is to provide a strategy for escaping from “bad” stationary points by making use of rank-one updates for the case of the squared loss functions composed with linear operators that satisfy the restricted isometry property (RIP). In particular, when optimizing w.r.t. \((U, V)\) one must choose an initialization for the rank (i.e., the number of columns in \( U \) and \( V \)), and with a poor initial choice (e.g., initializing with too small of an initial rank) can result in convergence to poor local minima, which can be escaped by our rank-one update strategy by growing the rank of the solution.

In summary, the current paper goes beyond state-of-the-art by a) generalizing the ubiquitous variational form of the nuclear norm (\( S_p \) for \( p = 1 \)) to the nonconvex case \( p \in (0, 1) \) and b) providing a fruitful insight on the implications arising by transforming the original nonconvex problem into the factorized space via vis-a-vis the landscape properties of the newly formulated nonconvex objective function.

### 2 Prior Art

The variational definition of the nuclear norm (\( S_p \) for \( p = 1 \)) given in [2], has been widely utilized in a variety of problems. For example, formulations similar that in [3] have been employed for problems such as collaborative filtering [9], robust PCA [14], etc. In a similar vein, a few efforts for providing alternative formulations of the \( S_p \) quasi-norms for \( p \in (0, 1) \) have recently appeared in the literature. In [12], the authors proposed variational definitions for the \( S_{1/2} \) quasi-norm i.e.,

\[
\|X\|_{S_{1/2}} = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{d \times d}, V \in \mathbb{R}^{d \times d}} \frac{1}{2} \|U\|_2^2 \|V\|_2^2 = \|U\|_F^2 + \|V\|_F^2.
\]

In [11], generalized variational forms of the \( S_p \) quasi-norm for any \( p \in (0, 1) \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) were derived:

\[
\|X\|_{S_p}^p = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{d \times d}, V \in \mathbb{R}^{d \times d}} \|U\|_{S_{p_1}}^p \|V\|_{S_{p_2}}^p = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{d \times d}, V \in \mathbb{R}^{d \times d}} \frac{p_1}{p_1 + p_2} \|U\|_{S_{p_1}}^p + \frac{p_2}{p_1 + p_2} \|V\|_{S_{p_2}}^p.
\]

While the above variational form is general for an arbitrary value of \( p \in (0, 1) \), note that in the general case one still needs to compute SVDs of \( U \) and \( V \) to evaluate the variational form due to the \( S_p \) quasi-norms on the factors, which can be a hindrance in large-scale problems. To alleviate this problem, the authors of [4] recently proposed two different variational definitions of the \( S_p \) quasi-norm defined for discrete values of \( p, p' = \frac{q}{q+1} \) and \( p' = \frac{2q}{q+2} \) with \( q = \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \} \), as

\[
\|X\|_{S_p}^p = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{d \times d}, V \in \mathbb{R}^{d \times d}} \frac{1}{(1 + \frac{1}{q})^2} \sum_{i=1}^{d} \frac{1}{q} \|u_i\|_2^q + \alpha \|v_i\|_2
\]

and

\[
\|X\|_{S_p}^{p'} = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{d \times d}, V \in \mathbb{R}^{d \times d}} \frac{1}{(\frac{1}{2} + \frac{1}{q})^2} \sum_{i=1}^{d} \frac{1}{q} \|u_i\|_2^q + \frac{\alpha}{2} \|v_i\|_2.
\]
It should be noted that both of the above $S_p$ quasi-norms are defined over the columns of the factors $U$ and $V$. Thus they provide potentially much simpler forms to evaluate in the $(U, V)$ domain due to the fact that one does not need to compute any SVDs. Further, the forms in (6) and (7) can also be interpreted as providing a group-sparsity regularization on the columns of $(U, V)$, which sets entire columns of $(U, V)$ to 0. Despite these advantages, we note that existing forms are not natural generalizations of the classical result for the nuclear norm (2) and they are not valid for all continuous values $p \in (0, 1]$. Moreover, they have been proposed without a theoretical analysis of local or global optimality. Next, we propose an alternative variational form that addresses all of these shortcomings.

3 The variational form of the Schatten-$p$ quasi-norm

In this section, we introduce a variational form of the $S_p$ (Schatten-$p$) quasi-norm which is defined for any continuous value of $p \in (0, 1]$. For its derivation, we invoke simple arguments of classical linear algebra for the properties of concave sums of singular values that date back to the 1970s [15].

**Theorem 1.** (Theorem 3, [15]) Let $A, B, C$ be $m \times n$ matrices with $m \geq n$ such that $C = A + B$. For any permutation invariant, non-decreasing and concave function $F : \mathbb{R}^{2n} \cup 0 \to \mathbb{R}$, it holds

$$F(c_1, c_2, \ldots, c_n, 0, 0, \ldots, 0) \leq F(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

where $a_i, b_i, c_i$ for $i = 1, 2, \ldots, n$ denote the singular values of matrices $A, B, C$.

The proposed variational form can be immediately derived from Theorem 1 as follows.

**Theorem 2.** Let $0 < p \leq 1$, $X \in \mathbb{R}^{m \times n}$ and $\text{rank}(X) = r$ where $r \leq d \leq \min(m, n)$. Then:

$$\|X\|_S^p = \min_{d \in \mathbb{N}_+} \min_{U \in \mathbb{R}^{m \times d}, V \in \mathbb{R}^{n \times d}} \sum_{i=1}^d \|u_i\|_2^p \|v_i\|_2^p \quad \text{s.t.} \quad UV^T = X$$

$$= \min_{d \in \mathbb{N}_+} \sum_{i=1}^d \left(\|u_i\|_2^2 + \|v_i\|_2^2\right)^{p/2} \quad \text{s.t.} \quad UV^T = X$$

**Proof.** Let us define $F : \mathbb{R}^{2n} \to \mathbb{R}$ as $F(x_1, x_2, \ldots, x_{2n}) = \sum_{i=1}^{2n} f(x_i)$ where $f(x_i) = |x_i|^p$ with $p \leq 1$. Clearly, $F$ satisfies the conditions of Theorem 1, i.e., $F$ is permutation invariant, non-decreasing and concave. Now, since $\text{rank}(X) \leq d$, we can express $X$ as the product of matrices $U \in \mathbb{R}^{m \times d}$ and $V \in \mathbb{R}^{n \times d}$, i.e., $X = UV^T = \sum_{i=1}^d u_i v_i^T$, where $d \geq r$. Let us define $A = u_1 v_1^T$ and $B = \sum_{i=2}^d u_i v_i^T$. Matrices $u_1 v_1^T$ and $B$ consist of no more than 1 and $d - 1$ non-zero singular values, respectively. From Theorem 1, we get $F(\sigma_1(X), \sigma_2(X), \ldots, \sigma_r(X), 0, \ldots, 0_{2n-r}) \leq F(\sigma_1(u_1 v_1^T), 0, \ldots, 0_{n-1}, \sigma_1(B), \sigma_2(B), \ldots, \sigma_{d-1}(B), 0, \ldots, 0_{n-(d-1)})$, which can be written as

$$\sum_{i=1}^r \sigma_i^p(X) \leq \sigma_1^p(u_1 v_1^T) + \sum_{i=1}^{d-1} \sigma_i^p(B).$$

Applying the same decomposition step followed for $X$ to $B$ and repeating the process for $d-2$ steps leads to the following inequality:

$$\sum_{i=1}^r \sigma_i^p(X) \leq \sum_{i=1}^d \sigma_i^p(u_i v_i^T).$$

Then since $\sigma_1(u_i v_i^T) = \|u_i v_i^T\|_2^p = \|u_i\|_2 \|v_i\|_2$ and $\|u_i\|_2 \|v_i\|_2 \leq \frac{1}{2} (\|u_i\|_2^2 + \|v_i\|_2^2)$, we obtain the two forms appearing in the RHS of (9). To show that the inequality is actually achieved, and thus the minimum of the RHS of (9) is equal to the LHS, observe that the minimum is always achieved for $U = U_X \Sigma_1^{1/2}$ and $V = V_X \Sigma_2^{1/2}$, where $X = U_X \Sigma X V_X^T$ is the singular value decomposition of $X$. \qed

It should be noted that a general result of the same type as ours was developed recently appeared in [16] for concave singular value functions. The proof of [16] is also simple, however it was based on a more complicated result (Theorem 4.4. of [17]), instead of the simpler Theorem 1 of [15] applied in our case. Surprisingly, classical linear algebra results from [15,17] appear to have been overlooked.
when deriving variational forms of the $S_p$ quasi-norm in the machine learning literature, leading to more complicated derivations of alternative variational forms of the $S_p$ quasi-norm, such as (6) and (7) in [4], which are defined only for discrete values of $p$. In contrast, our proposed variational form can be defined for any continuous value of $p \in (0, 1)$. In addition, our proposed variational form is a natural generalization of the variational form of the nuclear norm for $p = 1$ given in (2). Moreover, unlike the forms in (4) and (5), our form depends only on the columns of $U$ and $V$, which will allow more efficient optimization methods (which do not require SVD calculations at each optimization iteration) for matrix factorization problems with $S_p$ quasi-norm regularization, such as [21], by restating them w.r.t the matrix factors $U, V$ as follows:\footnote{Note that as in [4] all variational forms allow the number of columns in $(U, V)$ to be arbitrary, but we will omit this for notational simplicity.}

$$
\min_{X} l(Y, X) + \lambda \|X\|_{S_p}^p = \min_{U, V} l(Y, UV^T) + \lambda \sum_{i=1}^{d} \|u_i\|_2^p \|v_i\|_2^p \\
= \min_{U, V} l(Y, UV^T) + \frac{\lambda}{2p} \sum_{i=1}^{d} (\|u_i\|_2^2 + \|v_i\|_2^2)^p.
$$

## 4 Analysis of the Landscape of the Non-Convex Objective

In this section, we present an analysis of the problem defined in (12) elaborating on the conditions that ensure that a local minimum of the variationally defined $S_p$ quasi-norm minimization problem, i.e., a pair $(\hat{U}, \hat{V})$, gives rise to a matrix $\hat{X} = \hat{U} \hat{V}^T$ which is a local minimum of the original $S_p$ quasi-norm regularized objective function defined in (21). This is motivated by the fact that when one changes the domain from $X$ to $(U, V)$ then it is possible to introduce additional ‘spurious’ local minima into the problem, i.e., a point which is local minima w.r.t. $(U, V)$ but such that $X = UV^T$ is not a local minima w.r.t. $X$. It has been shown that in certain settings, such as semidefinite programming problems in standard form [13], solving problems in the factorized domain will not introduce additional local minima, but as problems involving the $S_p$ quasi-norm with $p \in (0, 1)$ are inherently non-convex even in the $X$ domain, it is unclear whether such results can be generalized to this setting. Here we provide a positive result and show that under mild conditions local minima w.r.t. $(U, V)$ will correspond to local minima w.r.t. $X$. In addition, we also consider a special case of the general problem in [21] where the loss is chosen to be the squared Frobenius composed with RIP linear operators. In this setting we propose a rank-one update strategy to escape poor stationary points and local minima by growing the rank of the factorization.

### 4.1 Properties of Local Minima in Factorized Schatten-$p$ Norm Regularized Problems

The general idea that motivates this section is similar to what has been used to study similar variational forms defined for other general norms on $X$ [10,19]. Concretely, the key approach employed in this prior work is to relate the local minima of nonconvex objective functions that are defined w.r.t. matrices $U$ and $V$ to global minima of convex objective functions over $X$ such that $X = UV^T$. However, it is worth emphasizing that since in our case we focus on $S_p$ quasi-norms for values of $p \in (0, 1)$, both the problems w.r.t. to $X$ and $U, V$ are nonconvex. Evidently, this presents considerable challenges in deriving conditions for global optimality w.r.t. either $X$ or $(U, V)$. Moreover, finding even a local minimum of a nonconvex problem can be NP-hard in general. Interestingly, as we show in Theorem 3 when certain conditions are satisfied we ensure that no decreasing directions of the objective function w.r.t. $X$ exist, thus showing that we have arrived at a local minimum of (12) and (21).

The first step in establishing optimality properties of the $S_p$ quasi-norm regularized problem given in (21) is to derive the subgradients of the $S_p$ quasi-norm. Note that conventional definitions of subgradients are not valid in the case of the $S_p$ quasi-norm due to the lack of convexity. That said, generalized notions of subgradients (referred to as regular subgradients, [13]) can be utilized to account for local variations of the objective functions. In the following Lemma, we derive the regular subdifferential of the $S_p$ quasi-norm.
Lemma 1. Let $X = UV^T$ where $U \in \mathbb{R}^{m \times r}$, $\Sigma \in \mathbb{R}^{r \times r}$ and $V \in \mathbb{R}^{n \times r}$, denote the singular value decomposition of $X \in \mathbb{R}^{m \times n}$. The regular subdifferential of $\|X\|_p^p$ is given by

$$\hat{\partial}\|X\|_p^p = \{U\text{diag}(\hat{\sigma}(\|\sigma_+(X)\|_p^p)^T + W : U^TW = 0, WV = 0\} \tag{13}$$

where $\sigma_+(X)$ is the vector of the positive singular values of $X$.

Note that the form of the expressions derived for regular subgradients of Schatten-$p$ quasi-norm resemble the ones of subgradients of the nuclear norm, however the former a) are locally defined and b) require no constraint on the spectral norm of $W$.

Next we relate local minima of the factorized problem defined over $U, V$ to local minima of the original $S_p$ quasi-norm problem defined w.r.t. $X$ in [21].

Theorem 3. Let $(\hat{U}, \hat{V})$ be a local minimizer of the factorized $S_p$ quasi-norm regularized objective functions defined in [12], where $\hat{U} = U\Sigma^{\frac{1}{2}}$ and $\hat{V} = V\Sigma^{\frac{1}{2}}$, matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times m}$, consisting of $r < m \leq n$ nonzero orthonormal columns, and $\Sigma$ is a $m \times m$ real-valued diagonal matrix. Assume matrix $X = \hat{U}\hat{V}^T = U\Sigma V^T$ with $\text{rank}(X) \leq r$. $\hat{X}$ is a local minimizer of the $S_p$ regularized objective function defined over $X$ in [21].

Theorem 3 in fact says that we can always obtain a local minimizer of the original objective function defined over $X$ once we reach a pair $(\hat{U}, \hat{V})$, which is a local minimizer of the factorized $S_p$ regularized objective function, on the condition that it gives rise to a low-rank matrix $\hat{X}$ (which is akin to requiring that $(U, V)$ is sufficiently parameterized). While the above result also requires the nonzero columns of $(U, V)$ to be orthogonal, we note that this is not a significant limitation in practice as one can always perform a single SVD of $X = \hat{U}\hat{V}^T$ to find a factorization with the required form which has objective value in [12] less than or equal to that of $(\hat{U}, \hat{V})$.

### 4.2 Rank One Updates for Escaping Poor Local Minima

Next we consider the special case where the loss function is the squared loss composed with a linear operator (here we show one of the two variational forms that we propose, but everything in this section applies equivalently to the other form), which gives the following form:

$$\min_{U, V} \frac{1}{2}\|Y - A(\hat{U}\hat{V}^T)\|_F^2 + \frac{\lambda}{2p}\sum_{i=1}^{d}(\|u_i\|_2^2 + \|v_i\|_2^2)^p, \tag{14}$$

where $A$ is a general linear operator. We are then interested in whether a rank-one descent step can be found after converging to a stationary point or local minima. That is, can we augment our $(\hat{U}, \hat{V})$ factors by adding an additional column which will improve the objective function and allow us to escape from poor local minima or stationary points. In particular, we want to solve the following:

$$\min_{\hat{u}, \hat{v}} \frac{1}{2}\|Y - A(\hat{U}\hat{V}^T + \hat{u}\hat{v}^T)\|_F^2 + \lambda\sum_{i=1}^{d}\left(\|u_i\|_2^2 + \|v_i\|_2^2\right)^p + \lambda\left(\|\hat{u}\|_2^2 + \|\hat{v}\|_2^2\right)^p. \tag{15}$$

Note that in the $p = 1$ case this is the variational form of a nuclear norm problem, and if the optimal $(\hat{u}, \hat{v})$ is the all zero vectors, then this is sufficient to guarantee the global optimality of $(\hat{U}, \hat{V})$ for the problem in [14] with $p = 1$ [10]. Further, for the $p = 1$ case it can be shown that the solution to (15) can be solved via a singular vector problem [10]. Namely, the optimal $(\hat{u}, \hat{v})$ will be a scaled multiple of the largest singular vector pair of the matrix $A^*(Y - A(\hat{U}\hat{V}^T))$, where $A^*$ denotes the adjoint operator of $A$.

For the $p < 1$ case solving (15) may not be sufficient to guarantee global optimality, but this can still provide a practical solution for improving a given stationary point solution. One difficulty, however, is that there is always a local minima around the origin for problem (15) w.r.t. $(\hat{u}, \hat{v})$ [20] (i.e., for any $(\hat{u}, \hat{v})$ with sufficiently small magnitude the objective in (15) will be greater than for $(\hat{u}, \hat{v}) = (0, 0)$). As a result, we need to test if for a given direction $(\hat{u}, \hat{v})$ there exists a (arbitrarily large) scaling of $(\hat{u}, \hat{v})$ which reduces the objective. Specifically, note that (15) is equivalent to solving the following:

$$\arg\min_{\tau \geq 0, u, v} \frac{1}{2}\|Y - A(\hat{U}\hat{V}^T + \tau^2 uv^T)\|_F^2 + \lambda\tau^{2p} \text{ s.t. } \|u\|_2 = 1, \|v\|_2 = 1$$

$$= \arg\min_{\tau \geq 0, u, v} -\tau^2\langle A^*(R), uv^T \rangle + \frac{1}{2}\tau^4\|A(uv^T)\|_F^2 + \lambda\tau^{2p} \text{ s.t. } \|u\|_2 = 1, \|v\|_2 = 1 \tag{16}$$
where \( R = Y - A(UV^T) \). If the optimal solution for \( \tau \) is at 0, then there is no rank-1 update that can decrease the objective, while if the optimal \( \tau > 0 \) then a rank-1 update can reduce the objective. Here we will consider the special case when the linear operator, \( A \), satisfies the RIP condition for rank-1 matrices with constant \( \delta \), which is defined as follows:

**Definition 1.** Let \( A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^q \) be a linear operator and assume without loss of generality that \( m \leq n \). We define the restricted isometry constant \( \delta \geq 0 \) as the smallest number such that

\[
(1 - \delta) \|X\|_F \leq \|A(X)\| \leq (1 + \delta) \|X\|_F
\]

holds for any rank-one matrix \( X \), where \( \|A(X)\| \) denotes the spectral norm of \( A(X) \).

From this, we then have the following rank-one update step:

**Proposition 1.** For a linear operator \( A \) which is RIP on rank-1 matrices with constant \( \delta = 0 \) the solution to \( (16) \) \((u_{\text{opt}}, v_{\text{opt}})\) is given by the largest singular vector pair of \(-A^*(R)\). Further, let \( \mu = \frac{2 - 2p}{2 - p} \sigma(A^*(R)) \), where \( \sigma(\cdot) \) denotes the largest singular value. Then the optimal value of \( \tau \) is given as:

\[
\tau_{\text{opt}} = \begin{cases} 
\sqrt{\mu} - \frac{\lambda}{\mu^{1-p}} \frac{\sigma(A^*(R))}{1 + \mu^{2-p}} \leq 0 \\
0 \quad \text{otherwise}
\end{cases}
\]  

Additionally, for a linear operator \( A \) which is RIP on rank-1 matrices with constant \( \delta > 0 \), then the above solution will be optimal to within \( \mathcal{O}\left(\frac{1}{\mu^2} \delta \right) \) in objective value. Further, the above rank-1 update is guaranteed to reduce the objective provided \( \lambda - \frac{\mu}{2(1-p)} \sigma(A^*(R)) + \frac{1}{2\mu^{2-p}} < \frac{1}{2} \mu^2 \delta \).

## 5 Application to Schatten-\( p \) Norm Regularized Matrix Completion

In this section we apply our framework to the matrix completion problem, where the goal is to estimate the missing entries of a data matrix by leveraging inherent low-rank structures in the data. We assume that we partially observe a noisy version of the true matrix \( X_{\text{true}} \), i.e.,

\[
P_Z(Y) = P_Z(X_{\text{true}} + E),
\]

where \( Z \) contains the indexes of the known elements of \( Y \), \( P_Z \) is a projection operator on this set and \( E \) contains zero-mean i.i.d. Gaussian elements with variance \( \sigma^2 \).

Assuming a low-rank representation of \( X_{\text{true}} \), we wish to find \( X = UV^T \) using our proposed variational form of the \( S_p \) quasi-norm by solving the following problem:

\[
\min_{U, V} \frac{1}{2} \|P_Z(Y - UV^T)\|_F^2 + \frac{\lambda}{2p} \sum_{i=1}^d (\|u_i\|_2^2 + \|v_i\|_2^2)^p.
\]  

The minimization task w.r.t. to matrix factors \( U, V \) does not lead to closed form updates for \( U, V \) due to both the form of the loss function and the non-separability of the regularization term for \( p < 1 \). That said, a block successive upper-bound minimization strategy is employed, which makes use of local tight upper-bounds of the objective function for updating each of the matrix factors. The resulting algorithm does not entail singular value decomposition (SVD) steps and its computational complexity is in the order of \( \mathcal{O}(|Z|d) \). Analytical details of the algorithm are provided in the supplement.

Recall also from Theorem that the proposed objective in is equivalent to a \( S_p \) quasi-norm regularized matrix completion formulation \( \frac{1}{2} \|P_Z(Y - X)\|_F^2 + \lambda \|X\|_{S_p}^2 \). Hence, generalization error bounds that have been provided for the \( S_p \) quasi-norm (Theorem 3 of ), which show that the matrix completion error can potentially be decreased as \( p \to 0 \), also apply here. The latter, is also empirically verified in the following subsection (see Figure).

### 5.1 Experimental Results

In this section we provide simulated and real data experimental results that advocate the merits of the proposed variational form of the \( S_p \) quasi-norm in the case of the matrix completion problem. For comparison purposes, the iterative reweighted nuclear norm (IRNN) algorithm of , which makes use of a reweighted strategy for minimizing various concave penalty functions over the vector of singular values of the original matrix \( X \), is utilized. In our case, IRNN is set up for minimizing the
In addition, the softImpute-ALS algorithm of [23], which minimizes the variational form of the nuclear norm (see eq. (2)) and the factored group sparse (FGSR) algorithm for \(S_p\) quasi-norm minimization of [4], are also used as baseline methods. Note that the FGSR algorithm is based on the minimization of the variational \(S_p\) quasi-norm form given in (7) and hence is an algorithm which is closest to our approach. The regularization parameters for all algorithms are carefully selected so that they all attain their best performance. All experiments are conducted on a MacBook Pro with 2.6 GHz 6-Core Intel Core i7 CPU and 16GB RAM using MATLAB R2019b.

Simulated data. We randomly generate a low-rank matrix of size 500 × 500 and rank r. The low-rank matrix is contaminated by additive i.i.d. Gaussian noise with \(\sigma\) selected so that we get different SNR values for the input data matrix given to the algorithms. A subset of the elements of the noisy matrix is then selected uniformly at random. For each experiment, we report the average results of 10 independent runs. The relative recovery error is used as a performance metric defined as \(\text{RE} = \frac{\|P\bar{Z}(\hat{X} - X_{true})\|_F}{\|P\bar{Z}(X_{true})\|_F}\) (where \(\bar{Z}\) contains the indices of the missing entries and \(\hat{X}\) denotes the estimated \(X\)). In Figure 1 we see that the proposed algorithms (named Variational Schatten-\(p\) with \(p = \{0.3, 0.5\}\) in Figures 1(a) and 1(b)) perform equal or better than competing methods in virtually all situations. In all examined cases it can be seen that the softImpute-ALS, which uses the variational form of the nuclear norm shows poor performance as compared to the other algorithms that minimize either the original \(S_p\) quasi-norm or variational definitions thereof, highlighting the benefit of using the \(S_p\) quasi-norm with \(p < 1\). In Table 1 we report the relative recovery error (average of 10 independent runs) as well as the (median) estimated final rank of the solution obtained when the algorithms are initialized with different rank initializations. As can be observed, the proposed algorithm is significantly more robust to errors in the rank of the initialization. Notably, the rank-one updating scheme detailed in Section 4.2, which allows the proposed algorithms to escape poor local minima, further enhances robustness to rank initialization by promoting its convergence to the true rank even in the more challenging scenario whereby the true rank is underestimated at initialization.

Real data. We next test the algorithms on the MovieLens-100K dataset, [24] which contains 100,000 ratings (integer values from 1 to 5) for 943 movies by 1682 users. We examine two cases corresponding to sampling rates of 50% and 25% of the known entries. For each case we initialize all matrix factorization based algorithms with ranks ranging from 10 to 50 with step-size 5. From Figure 3 it can be seen that the two versions of proposed algorithm corresponding to \(p = 0.5\) and \(p = 0.3\) performs comparably to FGSR-2/3 and FGSR-0.5 while outperforming IRNN and softImpute-ALS in terms of the normalized mean average error (NMAE). The latter, is shown to be vulnerable to
Table 1: Relative recovery error (RE) and (median) estimated rank $\hat{r}$ vs rank initialization. True rank $r = 20$, missing rate=0.4 and SNR=10dB.

| initial rank (r) | Variational Schatten-0.5 | | Variational Schatten-0.3 | | FGSR-2/3 | | FGSR-0.5 |
|------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
|                  | RE $\hat{r}$ | no | RE $\hat{r}$ | yes | RE $\hat{r}$ | no | RE $\hat{r}$ | yes | RE $\hat{r}$ | no | RE $\hat{r}$ | yes | RE $\hat{r}$ | no |
| 0.5r             | 0.2241 | 20 | 0.5260 | 10 | 0.2425 | 18 | 0.5263 | 10 | 0.5259 | 10 | 0.5259 | 10 |
| 0.75r            | 0.1450 | 20 | 0.3253 | 10 | 0.1699 | 19 | 0.3254 | 10 | 0.3267 | 10 | 0.3253 | 10 |
| r                | 0.1228 | 20 | 0.1228 | 20 | 0.1225 | 20 | 0.1225 | 20 | 0.1320 | 20 | 0.1230 | 20 |
| 1.25r            | 0.1231 | 20 | 0.1231 | 20 | 0.1219 | 20 | 0.1219 | 20 | 0.1320 | 20 | 0.1227 | 20 |
| 1.5r             | 0.1231 | 20 | 0.1230 | 20 | 0.1223 | 20 | 0.1223 | 20 | 0.1325 | 20 | 0.1231 | 20 |

rank initialization unlike the remaining algorithms, which minimize different versions of the $S_p$ quasi-norm.

Figure 3: NMAE vs rank initialization for fraction of observed entries of 0.5 (left) and 0.25 (right).

6 Conclusions

In this work, a novel variational form of the Schatten-$p$ quasi-norm, which generalizes the popular variational form of the nuclear norm to the nonconvex case for $p \in (0, 1)$ is introduced. A local optimality analysis is provided, which shows how local minima of the variational problem correspond to local minima of the original one. A rank-one update scheme is given for the case of the Frobenius loss functions with RIP linear operators, which allows one to escape poor local minima. The merits of the proposed algorithm w.r.t. relevant state-of-the-art approaches are demonstrated on the matrix completion problem.

Broader Impact

Low-rank modeling and estimation is a fundamental tool in machine learning and has numerous applications such as matrix completion and recommendation systems. As a result, understanding models for low-rank modeling and estimation is critical to understanding any potential biases and failure risks of such models. The proposed work offers new insights when it comes to the optimality properties of these problems, which might be of further interest to scientists studying relevant nonconvex theory. Moreover, our results provide theoretical insights and guidance which might be of interest to practitioners in guaranteeing and understanding the performance of their models.

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The $S_p$ regularized objective function defined over $X$ and the factorized objective functions defined over matrices $U$ and $V$ are given next,

$$L(X) = l(Y, X) + \lambda \|X\|_{S_p}^p$$  \hspace{1cm} (21)

$$L_1(U, V) = l(Y, UV^T) + \lambda \sum_{i=1}^d \|u_i\|_p^p \|v_i\|_2^p$$  \hspace{1cm} (22)

$$L_2(U, V) = l(Y, UV^T) + \lambda \frac{1}{2p} \sum_{i=1}^d (\|u_i\|_2^2 + \|v_i\|_2^2)^p.$$  \hspace{1cm} (23)

A Technical background and proofs of Lemma 1 and Theorem 3

First we provide definitions and the technical lemmas which are necessary for deriving: a) the regular subgradients of the Schatten-$p$ raised to $p$ quasi-norm for $p \in (0, 1)$ and b) the dual relationship between subderivatives and regular subgradients. Both are key ingredients of the proof of Theorem 3.

**Definition 2** ([25]). A function is called as a singular value function if it is extended real-valued, defined on $\mathbb{R}^{m \times n}$ of the form $f \circ \sigma : \mathbb{R}^{m \times n} \to \mathbb{R}$, where $f : \mathbb{R}^q \to [-\infty, +\infty]$, $q \leq \min(m, n)$, is absolutely symmetric i.e., it is invariant to permutations and changes of the signs of its arguments.

Based on Definition 1, we can say that Schatten-$p$ quasi-norm of matrix $X$ is a singular value function arising from the $\ell_p$ quasi-norm, which is absolutely symmetric and is applied on the vector of the singular values of $X$.

Next we provide the following notions of general, regular and horizon subgradients, which generalize traditional subgradients of convex functions to the case of nonconvex ones.

**Definition 3** ([13]). Let $f : \mathbb{R}^n \to \mathbb{R}$ and a point $\bar{x}$ with $f(\bar{x})$ finite. A vector $u \in \mathbb{R}^n$ is:

- a regular subgradient of $f$ at $\bar{x}$ i.e., $u \in \hat{\partial} f(\bar{x})$, if
  $$f(x) \geq f(\bar{x}) + \langle u, x - \bar{x} \rangle + o(\|x - \bar{x}\|)$$  \hspace{1cm} (24)

- a general subgradient of $f$ at $\bar{x}$ i.e., $u \in \partial f(\bar{x})$, if there exist sequences $x^i \to \hat{x}$ (i.e., $x^i \to \hat{x}$ with $f(x^i) \to f(\hat{x})$) and $u^i \in \hat{\partial} f(x^i)$, with $u^i \to u$.  
The sets $\partial f(\bar{x}), \partial f(\bar{x}), \partial^\infty f(\bar{x})$ are called regular, general and horizon subdifferential of $f$ at $\bar{x}$, respectively.

The above definitions implicitly assume that subgradients define hyperplanes that locally bound the function from below. Hence, they are also called as lower subgradients.

The following lemma (Theorem 7.1 of [25]) relates the set of general subgradients i.e., the general subdifferential, of an absolutely symmetric function $f$ with that of the corresponding singular value function.

**Lemma 2** (Theorem 7.1 of [25]). Let $X = U\Sigma^TV$ where $U \in \mathbb{R}^{m \times r}, \Sigma \in \mathbb{R}^{r \times r}$ and $V \in \mathbb{R}^{n \times r}$, denote the singular value decomposition of $X \in \mathbb{R}^{m \times n}$. The general subdifferential of a singular value function $f \circ \sigma$ at $X$ is given by the formula

$$\partial(f \circ \sigma)(X) = U \text{diag}(\partial f(\sigma(X)))V^T$$

where $\partial f(\sigma(X))$ is the general subdifferential of $f(\sigma(X))$. The regular and horizon subdifferential of $(f \circ \sigma)(X)$ i.e., $\hat{\partial}(f \circ \sigma)(X)$ and $\partial^\infty(f \circ \sigma)(X)$ can be similarly derived.

Subderivatives generalize the notion of one-sided directional derivatives and are defined as follows

**Definition 4.** Let $f : \mathbb{R}^q \to \mathbb{R}$ where $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ and a point $\bar{x}$ where $f(\bar{x})$ is finite. The subderivative of $f$ at $\bar{x}$ is defined as

$$df_{\bar{x}}(w) = \lim_{\tau \downarrow 0} \inf_{w \to w} \frac{f(\bar{x} + \tau w) - f(\bar{x})}{\tau}$$

A critical property that ensures the dual relationship between regular subgradients and subderivatives is the so-called subdifferential regularity. The following lemma can be utilized for examining whether a function is subdifferentially regular or not.

**Lemma 3.** ([13]) Let a function $f : \mathbb{R}^q \to \mathbb{R}$ and a point $\bar{x}$ with $f(\bar{x})$ finite and $\partial f(\bar{x}) \neq \emptyset$. $f$ is subdifferentially regular at $\bar{x}$ if and only if $f$ is locally lower semi-continuous at $\bar{x}$ with

$$\partial f(\bar{x}) = \hat{\partial} f(\bar{x}) = \partial^\infty f(\bar{x})$$

where $\hat{\partial} f(\bar{x})$ denotes the horizon set of the set of the general subgradients of $f$.

By Definition 3 we have that a vector $u \in \mathbb{R}^q$ is a regular subgradient of $f(x)$ at $\bar{x}$ i.e., $u \in \hat{\partial} f(\bar{x})$ if

$$\lim_{x \to \bar{x}} \inf_{u \in \hat{\partial} f(\bar{x})} \frac{f(x) - f(\bar{x}) - \langle u, x - \bar{x} \rangle}{|x - \bar{x}|} \geq 0$$

Clearly the regular subgradient of $f_i(x_i) = |x_i|^p$ for $x_i \in (-\infty, 0) \cup (0, +\infty)$ boils down to the gradients thereof hence the corresponding regular subdifferential sets are singletons and coincide with the general ones. At $\bar{x} = 0$, the regular subdifferential of $f_i$ is the interval $(-\infty, +\infty)$. That being said, we have

$$\hat{\partial} f_i = \begin{cases} -p & x \in (-\infty, 0) \\ \frac{1}{(-x_i)^{1-p}} & x_i = 0 \\ p & x \in (0, +\infty) \end{cases}$$

The regular subdifferential of $\|x\|_p^p$ can thus be obtained as $\partial \|x\|_p^p = [\hat{\partial} f_1(x_1), \hat{\partial} f_2(x_2), \ldots, \hat{\partial} f_n(x_n)]$.

**Lemma 4.** The $\ell_p$ quasi-norm raised to the power of $p$ defined as $\|x\|_p^p = \sum_{i=1}^n |x_i|^p$ where $x \in \mathbb{R}^q$, is subdifferentially regular.
**Proof.** Let us define the function \( f_i(x_i) = |x_i|^p \). The \( \ell_p \) quasi-norm can be written as \( \|x\|_p^p = \sum_{i=1}^{q} |x_i|^p \). Since the \( \ell_p \) quasi-norm is separable, regular, general and horizon subgradient can be taken by cartesian products of the subgradients of \( f_i \), for \( i = 1, 2, \ldots, q \). Moreover, \( \|x\|_p^p \) is regular at \( x \) when \( f_i \) are regular at \( x_i \) for all \( i \). Hence, we focus on \( f_i(x_i) = |x_i|^p \). \( f_i(x_i) \) is smooth and thus regular in \( \mathbb{R} - \{0\} \). At \( 0, f_i(0) \) is nondifferentiable and \( \partial f_i(0) = (-\infty, +\infty) \). Evidently, the set of regular subgradients is closed and coincides with \( \partial f_i(0) \) (see Definition 3). Moreover, for the set of horizon subgradients at \( 0 \) we have \( \partial^\infty f_i(0) = \{ -\infty, +\infty \} \equiv \partial f_i(0)^\infty \). Hence, due to Lemma 3 we can conclude the proof.

Having shown that the \( \ell_p \) quasi-norm for \( p \in (0, 1) \) is subdifferentially regular, we can now show that the singular value function arising by the \( \ell_p \) quasi-norm, i.e., the Schatten-\( p \) quasi-norm is also subdifferentially regular.

**Lemma 5.** The Schatten-\( p \) quasi-norm with \( p \in (0, 1) \) is a subdifferentially regular function.

**Proof.** See Corollary 7.5 of [25].

Lemma 3 is critical in the proof of the Theorem 3 since it allows us to use the dual relationship between subderivatives and the regular subgradients of the Schatten-\( p \) quasi-norm. The following theorem provides a dual correspondence between the general subgradients of a subdifferentially regular function and its subderivatives.

**Theorem 4 (Theorem 8.30 of [13]).** If a function \( f \) is subdifferentially regular at \( \tilde{X} \) then one has \( \partial f(\tilde{X}) \neq \emptyset \iff df(\tilde{X}) \neq -\infty \) and

\[
df_{\tilde{X}}(W) = \sup\{<Q, W>|Q \in \partial f(\tilde{X})\}
\]

with \( \partial f(\tilde{X}) \) closed and convex.

Next, the proof of Lemma 1, which gives the expressions for the regular subgradients of \( \mathcal{S}_p \) quasi-norm raised to \( p \) is provided.

**Proof of Lemma 1**

**Proof.** From Lemma 2, we have that a matrix \( Y \) lies in \( \hat{\partial}(f \circ \sigma)(X) \) if and only if \( \sigma(Y) \in \hat{\partial} f(\sigma(X)) \) and there exists a simultaneous singular value decomposition of the form \( X = U \operatorname{diag}(\sigma(X)) V^\top \). Note that we herein assume the full singular value decomposition i.e., matrices \( U, V \) are orthogonal of size \( m \times m \) and \( n \times n \), respectively. By representing \( \tilde{U} = [U \ U_\perp] \) and \( \tilde{V} = [V \ V_\perp] \), where \( U_\perp \in \mathbb{R}^{m \times m-r}, V_\perp \in \mathbb{R}^{n \times n-r}, \Sigma_+ = \operatorname{diag}(\sigma_+(X)) \) and \( \operatorname{colsp}(U_\perp), \operatorname{colsp}(V) \perp \operatorname{colsp}(V_\perp) \), we rewrite the singular value decomposition of \( X \) in the form

\[
X = [U \ U_\perp] \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} [V \ V_\perp]
\]

(31)

Note that since \( U_\perp \) and \( V_\perp \) correspond to zero singular values of \( X \), they are not uniquely defined. Next, going back to the form of a regular subgradient \( Y \) of \( \|X\|_{\mathcal{S}_p}^p \) and based on the above-defined expression of the full singular value decomposition of \( X \) we have,

\[
Y = [U \ U_\perp] \begin{bmatrix} \hat{\partial} \|\Sigma_+(X)\|_p^p & 0 \\ 0 & 0 \end{bmatrix} [V \ V_\perp]
\]

(32)

where \( D \) contain elements in \( (-\infty, +\infty) \) on its main diagonal and zeros elsewhere. eq. (32) can be written in a more compact form as

\[
Y = U \hat{\partial} \|\Sigma_+(X)\|_p^p V^\top + U_\perp D V_\perp^\top
\]

and by setting \( W = U_\perp D V_\perp^\top \), we get the expression for the regular subdifferential of \( \|X\|_{\mathcal{S}_p}^p \).

**Lemma 6.** Let \( (\tilde{U}, \tilde{V}) \) with \( \tilde{U} = U \Sigma^{\frac{1}{2}} \) and \( \tilde{V} = V \Sigma^{\frac{1}{2}} \) be a stationary point of (2.3), where non all-zero columns of matrices \( U \in \mathbb{R}^{m \times d} \) and \( V \in \mathbb{R}^{n \times d} \) are orthonormal, and \( \Sigma \) is a \( d \times d \)
real-valued diagonal matrix. The pair \((\hat{U}, \hat{V})\) satisfies the following equations

\[
\nabla l(Y, \hat{U}\hat{V}^\top)\hat{V} + \hat{U}\partial \|\Sigma\|_{S_p}^p \geq 0
\]

\[
\nabla l(Y, \hat{U}^\top \hat{V})^\top \hat{U} + \hat{V}\partial \|\Sigma\|_{S_p}^p \geq 0
\]

**Proof.** Since \((\hat{U}, \hat{V})\) is a local minimum of the objective defined in (23), it is also a stationary point of it. Moreover, by defining \(f_1(x) = x_2^p\) we can get

\[
\partial u_i \sum_{i=1}^d f_i \left( \frac{1}{2}(\|u_i\|_2^2 + \|v_i\|_2^2) \right) = u_i \partial f_i \left( \frac{1}{2}(\|u_i\|_2^2 + \|v_i\|_2^2) \right)
\]

Hence, by using the chain product rule w.r.t. to \(U\) we can get

\[
\partial U \sum_{i=1}^d f_i \left( \frac{1}{2}(\|u_i\|_2^2 + \|v_i\|_2^2) \right) = U \text{ diag} \left[ \partial f_i \left( \frac{1}{2}(\|u_i\|_2^2 + \|v_i\|_2^2) \right) \right]_{i=1,2,...,d}
\]

(36) estimated at \((\hat{U}, \hat{V})\) becomes

\[
\partial U \sum_{i=1}^d f_i \left( \frac{1}{2}(\|u_i\|_2^2 + \|v_i\|_2^2) \right) \bigg|_{(\hat{U}, \hat{V})} = U \text{ diag}[\partial f_i(\sigma_i)]_{i=1,2,...,d} = \hat{U}\partial \|\Sigma\|_{S_p}^p
\]

In addition to eq. (36), we have

\[
\nabla_U l(Y, UV^\top) = \nabla l(Y, UV^\top)\hat{V}
\]

Eq. (37) and (38) give rise to the following condition for the stationary point \((\hat{U}, \hat{V})\) of (23)

\[
\nabla l(Y, \hat{U}\hat{V}^\top)\hat{V} + \hat{U}\partial \|\Sigma\|_{S_p}^p \geq 0
\]

Following a similar process for partial gradients w.r.t \(V\) we get the second stationary point condition for \(L_2\)

\[
\nabla l(Y, \hat{U}^\top \hat{V})^\top \hat{U} + \hat{V}\partial \|\Sigma\|_{S_p}^p \geq 0
\]

Next we move on the proof of Theorem 3. The proof builds upon and significantly extends the results presented in Theorem 2 of [16]. In particular, contrary to the latter which applies to a specific regularizer characterized by some convenient properties (e.g. weak convexity), in our case by using the dual correspondence between subderivatives and regular subgradients, we can get similar results for any concave singular value penalty function as long as it is subdifferentially regular. Note that subdifferential regularity is a quite general condition and ensures that the set regular subgradients coincides with the set of the general ones. Moreover, Theorem 3 can be considered as an enhanced version of Theorem 2 of [16], since it that local minima of the variational \(S_p\) regularized problem correspond to local minima of the original problem defined over \(X\). That said, it goes one step further from Theorem 2 of [16] which shows that directional derivatives are nonnegative w.r.t. low-rank perturbations.

**Proof of Theorem 3**

**Proof.** Since \((\hat{U}, \hat{V})\) is a local minimizer of \(L_1(U, V)\) and \(L_2(U, V)\) defined in (22) and (23), respectively, we can define a small perturbation of \((\hat{U}, \hat{V})\) i.e., \((U_t, V_t)\)

\[
U_t, V_t^\top = \hat{U}\hat{V}^\top + t\hat{U}\hat{V}^\top
\]
with $\hat{U} \in \mathbb{R}^{m \times m}$, $\hat{V} \in \mathbb{R}^{n \times m}$ assuming that $m \leq n$, such that
\[ L_i(\hat{U}, V_t) - L_i(\bar{U}, \bar{V}) \geq 0, \tag{42} \]
for $i = 1, 2$ and $t \searrow 0$.

Moreover, since $\hat{U}$ and $\hat{V}$ are defined as $\hat{U} = U\Sigma_\perp^{1/2}$ and $\hat{V}\Sigma_\perp^{1/2}$ where $U$, $V$ are of size $m \times m$ and $n \times m$, respectively, contain orthonormal columns, and $\Sigma$ is a diagonal $m \times m$ matrix, we have
\[ L_1(\hat{U}, \hat{V}) = L_2(\hat{U}, \hat{V}) = L(\hat{X}) \tag{43} \]
Let us define
\[ \hat{X} + t\hat{X} = U_t V_t^T \tag{44} \]
where the product $\hat{U}\hat{V}^T$ of (41) has been replaced by $\hat{X}$.

Our goal is to show that $\hat{X}$ is local minimum of $L(X)$ i.e., $L(\hat{X} + t\hat{X}) - L(\hat{X}) \geq 0$ for $t \searrow 0$. Due to (42) and (43) the result immediately arises if we show that $L(\hat{X} + t\hat{X}) = L_i(U_t, V_t)$ for $i = 1, 2$.

$L(X)$ consists of the sum of a differentiable loss function denoted as $l(Y, X)$ and the nonconvex matrix function $\|X\|_{sp}$ and its subderivatives at $\hat{X}$ w.r.t $X$ are defined as
\[ dL_{\hat{X}}(\hat{X}) = \lim_{t \searrow 0} \inf_{\hat{Z} \to \hat{X}} \frac{L(\hat{X} + t\hat{Z}) - L(\hat{X})}{t} \tag{45} \]
From the above definition and due to the continuity of $L(X)$ it becomes evident that nonnegative subderivatives imply $L(\hat{X} + t\hat{X}) - L(\hat{X}) \geq 0$ for $t \searrow 0$.

Next, without loss of generality we assume, that the columns of $\hat{U}$ and $\hat{V}$ belong to the subspace formed by the direct sum of the columnspace of $\hat{U}$ (resp. $\hat{V}$) and the columnspace of a matrix $U_\perp \in \mathbb{R}^{m \times (m-r)}$ (resp. $V_\perp \in \mathbb{R}^{n \times (n-r)}$) with $\text{rank}(U_\perp) \leq (m-r)$ (resp. $\text{rank}(V_\perp) \leq (n-r)$) such that $\text{colsp}(\hat{U}) \perp \text{colsp}(U_\perp)$ (resp. $\text{colsp}(\hat{V}) \perp \text{colsp}(V_\perp)$). That said we get
\[ \hat{U} = UA_\perp + U_\perp B \tag{46} \]
\[ \hat{V} = V\tilde{C} + \hat{V}_\perp D \tag{47} \]
Since matrices $A, C$ of size $m \times m$ and $B, D$ of sizes $(m-r) \times m$ and $(n-r) \times m$ respectively, are arbitrary, there is no loss of generality by expressing $\hat{U}, \hat{V}$ as in (46) and (47). By (46) and (47) we hence have
\[ \hat{U}\hat{V}^T = UAC_\perp^T + U_\perp K_1 A_\perp^T + U_\perp K_2 B_\perp^T + U_\perp K_3 C_\perp^T + U_\perp K_4 D_\perp^T \tag{48} \]
\[ = U\Sigma_\perp K_1^T V_\perp^T + U_\perp K_2 V_\perp^T + U_\perp K_3 V_\perp^T + U_\perp K_4 V_\perp^T \tag{49} \]

By using Theorem 3 we can employ the dual relationship between the subderivatives of $L(X)$ defined in (45) and the regular subgradients thereof,
\[ dL_{\hat{X}}(\hat{X}) = \sup\{\langle Q, \hat{X} \rangle \mid Q \in \partial \|\hat{X}\|_{sp} \} + \langle \nabla l(Y, \hat{X}), \hat{X} \rangle \tag{50} \]
In the following, we will show that the values of the subderivatives in (50), remain unaffected if we assume that $K_1$ is a diagonal matrix and matrices $K_2, K_3$ vanish. In doing so, both matrices $\hat{U}$ and $\hat{V}$ are conveniently expressed in a desirable and simplified form.

Focusing on the first term of (50) and by using the form for the subgradients of $\|X\|_{sp}$ given in Lemma 1 and $\hat{X}$ where $\hat{X} = U_X\Sigma_+ V_X$, with $U_X \in \mathbb{R}^{m \times r}$, $V_X \in \mathbb{R}^{n \times r}$ and $\Sigma_+ \in \mathbb{R}^{p \times r}$ denotes the singular value decomposition of $X$ we have,
\[ \langle U_X \partial \Sigma_+ \| X \|_p^r V_X^T + W, \hat{U}\hat{V}^T \rangle = \langle U_X \partial \Sigma_+ \| X \|_p^r V_X^T, \hat{U}\hat{V}^T \rangle + \langle W, \hat{U}\hat{V}^T \rangle \tag{51} \]
By using (49), the first term of (51) is rewritten as
\[ \langle U_X \partial \Sigma_+ \| X \|_p^r V_X^T, \hat{U}\hat{V}^T \rangle = \langle U_X \partial \Sigma_+ \| X \|_p^r V_X^T, UK_1 V_\perp^T + U_\perp K_2 V_\perp^T + U_\perp K_3 V_\perp^T + U_\perp K_4 V_\perp^T \rangle \]
\[ = \langle U_X \partial \Sigma_+ \| X \|_p^r V_X^T, UK_1 V_\perp^T \rangle \tag{52} \]
We have $U_X \hat{U} = [\Sigma^\frac{1}{2} \ 0_{m-r}]$ and $V_X \hat{V} = [\Sigma^\frac{1}{2} \ 0_{m-r}]$ where $0_{m-r}$ denote $m - r$ zero columns, and next we define with a slight abuse of notation $\hat{\Sigma}^\frac{1}{2} = [\Sigma^\frac{1}{2} \ 0_{m-r}]$. From (52) we have
\[
\langle U_X \partial | \Sigma_+ \parallel p V_X, \hat{U} \hat{V}^T \rangle = \langle \partial | \Sigma_+ \parallel p, \hat{\Sigma}^\frac{1}{2} K_1 \hat{\Sigma}^\frac{1}{2} \parallel T \rangle
\] (53)
Based on the above analysis, it becomes evident that (53) is invariant to the values of non-diagonal elements of $K_1$. Finally, for the second term of (51), we have
\[
\langle W, \hat{U} \hat{V}^T \rangle = \langle W, UK_4 V^T + \hat{UK}_2 \hat{V}^T + \hat{UK}_3 V^T + \hat{UK}_4 \hat{V}^T \rangle = \langle W, \hat{UK}_4 \hat{V}^T \rangle
= \langle D, K_4 \rangle
\] (54)
where the last equality by definition of $W$ according to Lemma 1. Recall that $D \in \mathbb{R}^{(m-r) \times (n-r)}$ contains elements in the interval $(-\infty, +\infty)$ on its main diagonal and zeros elsewhere and hence the term $D$, $K_4$, is also invariant to non-diagonal elements of $K_1$.

Let us now focus on the second term of the subderivative defined in (50) i.e., $\langle \nabla l(Y, \hat{X}), \hat{X} \rangle$. We have
\[
\langle \nabla l(Y, \hat{UV}^T), \hat{UK}_1 \hat{V}^T \rangle = \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_1 \hat{V}^T \rangle = \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_1 \hat{V}^T \rangle
+ \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_2 \hat{V}^T \rangle + \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_3 \hat{V}^T \rangle + \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_4 \hat{V}^T \rangle
\] (55)
\[
\langle \nabla l(Y, \hat{UV}^T), \hat{UK}_1 \hat{V}^T \rangle = \langle \nabla l(Y, \hat{V}^T), \hat{UK}_1 \hat{V}^T \rangle
\] (56)
Focusing on the first term of (56) we get
\[
\langle \nabla l(Y, \hat{UV}^T), \hat{UK}_1 \hat{V}^T \rangle = \langle \nabla l(Y, \hat{V}^T), \hat{UK}_1 \hat{V}^T \rangle
\] (57)
due to Lemma 6 takes the form
\[
\langle \nabla l(Y, \hat{V}^T), \hat{UK}_1 \hat{V}^T \rangle = -\langle \hat{U} \partial | \Sigma_+ \parallel p, \hat{UK}_1 \rangle = \langle \hat{V} \partial | \Sigma_+ \parallel p, \hat{SK}_1 \rangle
\] (58)
Following a similar analysis as above we can easily see that (58) is likewise invariant to non-diagonal elements of $K_1$. For the second term of (57) and again due to Lemma 6 we have
\[
\langle \nabla l(Y, \hat{UV}^T), \hat{UK}_2 \hat{V}^T \rangle = \langle \nabla l(Y, \hat{UV}^T), \hat{UK}_2 \hat{V}^T \rangle = 0
\] (59)
It can be easily noticed that the same holds for the third term of (56) involving $K_3$. Hence we have shown that the values of the subderivatives defined in (45) for the given $X = UV^T$ where $UV^T$ takes the form given in (59) are not affected by $K_2, K_3$ and non-diagonal elements of $K_1$.

That being said we can now simplify the expression of $\hat{X} = \hat{UV}^T$ without loosing generality as
\[
\hat{UV}^T = U_X TV_X^T + U_{X,\perp} PV_{X,\perp}^T
\] (60)
where $U_{X,\perp} PV_{X,\perp}^T$ arises by the singular value decomposition of $\hat{U}_{\perp} K_4 \hat{V}_{\perp}^T$. Note that $T, P$ are $r \times r$ and $m - r \times m - r$ diagonal matrices containing the $r$ nonzero diagonal elements of $K_1 \Sigma$ and the singular values of $\hat{U}_{\perp} K_4 \hat{V}_{\perp}^T$, respectively. With the simplified form of (60) we now go back to the expression of $U_t V_t^T$ in (44) assuming $t \to 0$,
\[
U_t V_t^T = \hat{UV}^T + t \hat{UV}^T
\] (61)
\[
= U_X \Sigma_+ V_X^T + t(U_X TV_X^T + U_{X,\perp} PV_{X,\perp}^T)
\] (62)
\[
= [U_X \ U_{X,\perp}] \begin{bmatrix} \Sigma_+ + tT & 0 \\ 0 & (tP)^\frac{1}{2} \end{bmatrix} \begin{bmatrix} V_X \\ V_{X,\perp} \end{bmatrix}
\] (63)
(63) can be viewed as the singular value decomposition of matrix $X + t\hat{X}$.

By setting $U_t = [U_X \ U_{X,\perp}] \begin{bmatrix} \Sigma_+ + tT & 0 \\ 0 & (tP)^\frac{1}{2} \end{bmatrix}$ and $V_t = [V_X \ V_{X,\perp}] \begin{bmatrix} \Sigma_+ + tT & 0 \\ 0 & (tP)^\frac{1}{2} \end{bmatrix}$ it becomes evident that $\mathcal{L}(X + t\hat{X}) = \mathcal{L}_i(U_t, V_t)$ for $i = 1, 2$, which concludes the proof. \hfill \qed
B  Proof of Proposition 1

Proof. If we first consider the case with $\delta = 0$, then since $\|u\|_2 = 1$ and $\|v\|_2 = 1$ we need to solve the following

$$\arg\min_{\tau \geq 0, u, v} \{ f(\tau, u, v) = -\tau^2 (A^*(R), uv^T) + \frac{1}{2} \tau^4 + \lambda \tau^{2p}\} \ \text{s.t.} \ \|u\|_2 = 1, \ \|v\|_2 = 1$$  \hspace{1cm} (64)

Note that for any non-negative $\tau$ the solution to the above w.r.t. $(u, v)$ is given by the largest singular vector pair of $A^*(R)$. Substituting this into the above equation gives the following problem:

$$\arg\min_{\tau \geq 0} -\tau^2 \sigma(A^*(R)) + \frac{1}{2} \tau^4 + \lambda \tau^{2p} \hspace{1cm} (65)$$

Clearly for $\tau = 0$ the above equation is equal to 0, so we are left to test whether $\exists \tau > 0$ such that the above is strictly less than 0. For $p \in (0, 1), \tau > 0$, this results in the following:

$$-\tau^2 \sigma(A^*(R)) + \frac{1}{2} \tau^4 + \lambda \tau^{2p} < 0 \iff \lambda - \tau^2 - 2p \sigma(A^*(R)) + \frac{1}{2} \tau^{2p} < 0$$  \hspace{1cm} (66)

Finding the critical points of the above w.r.t. $\tau$ we get:

$$-(2 - 2p) \tau^{1-2p} \sigma(A^*(R)) + \frac{1}{2} - 2p \tau^{3-2p} = 0 \iff \tau = \pm \sqrt[2-p]{\frac{2 - 2p}{2-p} \sigma(A^*(R))}$$  \hspace{1cm} (67)

Since only the positive root is feasible we can substitute it into (66) to test whether the minimum is strictly negative.

To see the result with $\delta > 0$, note that result is given follow the same approach, with the exception that the objective in (64) is no longer the exact objective we need minimize, but rather we need to minimize:

$$\hat{f}(\tau, u, v) = -\tau^2 (A^*(R), uv^T) + \frac{1}{2} \tau^4 + \lambda \|A(\hat{u}v^T)\|_F^2 + \lambda \tau^{2p}$$  \hspace{1cm} (70)

The result is then completed by noting that for $\|u\|_2 = \|v\|_2 = 1$ and any $\tau \geq 0$ the following bound is provided by restricted isometry:

$$|\hat{f}(\tau, u, v) - f(\tau, u, v)| = |\frac{1}{2} \tau^4 \|A(\hat{u}v^T)\|_F^2 - \frac{1}{2} \tau^4| \leq \frac{1}{2} \delta \tau^4$$  \hspace{1cm} (71)

\[\square\]

C  The proposed Variational Schatten-$p$ matrix completion algorithm

In this section we present the matrix completion minimization algorithm for variational $S_p$ regularized objective function.

Recall that the factorized $S_p$ regularized objective function is given as,

$$\min_{U, V} \mathcal{L}(U, V), \ \text{where}$$  \hspace{1cm} (72)

$$\mathcal{L}(U, V) = \frac{1}{2} \|X - U(V^T)\|_F^2 + \lambda \sum_{i=1}^{d} (\|u_i\|_2^2 + \|v_i\|_2^2)^p.$$  \hspace{1cm} (73)

The proposed Variational Schatten-$p$ algorithm is based on the ideas stemming from the block successive minimization framework (BSUM), [21][25]. That is, it assumes local tight upper-bounds of the objective (73), for updating the matrix factors $U$ and $V$. More specifically, following the relevant low-rank matrix factorization based matrix completion algorithm of [27], matrices $U$ and $V$ are updated by minimizing approximate second order Taylor expansions of (73), i.e.,

$$U^{t+1} = \min_{U} g(U|U^t, V^t)$$  \hspace{1cm} (74)

and

$$V^{t+1} = \min_{V} q(V|U^{t+1}, V^t)$$  \hspace{1cm} (75)
where the superscript \( t \) denotes the iteration number and \( g(U|U^t, V^t) \) and \( q(V|U^{t+1}, V^t) \) have the following form

\[
g(U|U^t, V^t) = \mathcal{L}(U^t, V^t) + \text{trace}\{(U - U^t)^\top \nabla_U \mathcal{L}(U^t, V^t)\}
+ \frac{1}{2} \text{vec}(U - U^t)^\top \hat{H}_{U^t} \text{vec}(U - U^t)
\tag{76}
\]

\[
g(V|U^{t+1}, V^t) = \mathcal{L}(U^{t+1}, V^t) + \text{trace}\{(V - V^t)^\top \nabla_V \mathcal{L}(U^{t+1}, V^t)\}
+ \frac{1}{2} \text{vec}(V - V^t)^\top \hat{H}_{V^t} \text{vec}(V - V^t)
\tag{77}
\]

The approximate Hessian matrices \( \hat{H}_{U^t} \) and \( \hat{H}_{V^t} \) are of block diagonal form of sizes \( md \times md \) and \( nd \times nd \), respectively, and are defined as follows

\[
\hat{H}_{U^t} = \mathbf{I}_m \otimes \hat{H}_{U^t}
\tag{78}
\]

\[
\hat{H}_{V^t} = \mathbf{I}_n \otimes \hat{H}_{V^t}
\tag{79}
\]

where \( \otimes \) denotes the Kronecker product,

\[
\hat{H}_{U^t} = V^t, U^t + \lambda W_{(U^t, V^t)}
\tag{80}
\]

\[
\hat{H}_{V^t} = U^{t+1}, V^t + \lambda W_{(U^{t+1}, V^t)}
\tag{81}
\]

and \( W_{(U^t, V^t)} = \text{diag}(p(||u_i||^2 + ||v_i||^2)^{p-1}) \). Since the objective function is non-smooth at the origin for \( p \leq 1 \), a pruning process is followed i.e., columns of matrix factors \( U \) and \( V \) whose energy is below a threshold are deleted. In doing so, division by zero is always avoided in the computation of \( W_{(U^t, V^t)} \).

Moreover, it can be easily shown that both \( g(U|U^t, V^t) \) and \( q(V|U^{t+1}, V^t) \) are local tight upper-bounds of the original objective function since the difference of the approximate Hessian matrices from the true ones result to a positive semi-definite matrix (see \[27\] for details). Finally, convergence to stationary points of the original objective function can be established since all criteria required according to the BSUM framework, are satisfied, \[21\]. The outline of the algorithm is given in Algorithm 1. Note that the rank-one update strategy described in Section 4.2 (see Proposition 1) of the main paper is also adopted after the convergence of the algorithm to a stationary point for escaping potential poor local minima. The algorithm is assumed to converge when the relative error of successive reconstructions of matrix \( X \) defined as \( \frac{||U^{t+1}V^{t+1} - U^tV^t||_F}{||U^0V^0||_F} \) becomes less than a threshold (denoted as \( \text{tol} \) in Algorithm 1) or the maximum iteration number is reached.

**Algorithm 1** The proposed Variational Schatten-\( p \)-matrix completion algorithm

**Inputs:** \( P_2(Y), \lambda, \text{thres} = 10^{-5}, \text{MaxIter} = 1000, \text{tol} = 10^{-4} \)

**Initialize** \( U^0 \in \mathbb{R}^{m \times d}, V^0 \in \mathbb{R}^{n \times d} \)

\( t = 0 \)

**for** \( t = 1 : \text{MaxIter} \)

\( U^{t+1} = U^k - \nabla_U \mathcal{L}(U^t, V^k) \hat{H}_{U^t}^{-1} \)

\( V^{t+1} = V^k - \nabla_V \mathcal{L}(U^{t+1}, V^k) \hat{H}_{V^t}^{-1} \)

**Column Pruning:**

**if** \( ||u_i||_2 \leq \text{thres} \) or \( ||v_i||_2 \leq \text{thres} \) **then**

**Delete** \( u_i \) and \( v_i \), \((i = 1, 2, \ldots, d)\).

**end if**

\( t = t + 1 \)

**if** \( \text{converged} \) and rank-one update condition is satisfied **then**

**Add** rank-one matrix for escaping poor local minima (see Proposition 1).

**end if**

**end for**

**Output:** \( U^{t+1}, V^{t+1} \)