Dynamic Renormalization Group Study of the $\phi^4$ Model with Colored Noise

J. García-Ojalvo$^{a,b}$, J.M. Sancho$^b$ and H. Guo$^c$

(a) Departament de Física i Enginyeria Nuclear,
Escola Tècnica Superior d’Enginyers Industrials de Terrassa
Universitat Politècnica de Catalunya, Colom 11, E-08222 Terrassa, Spain.

(b) Departament d’Estructura i Constituents de la Matèria, Facultat de Física
Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain.

(c) Department of Physics, McGill University
Rutherford Building, 3600 University Street
Montréal, Québec, Canada H3A 2T8.

Abstract

The non-conserved $\phi^4$ model defined by a Langevin equation with external non-white noise is studied by means of the Dynamic Renormalization Group. The correlation time of the noise changes the critical point location but does not affect the critical exponents up to order $\varepsilon^2$. The same effect is obtained when the correlation length of the noise is considered. These results are shown to be in agreement with previous numerical simulations.
I. INTRODUCTION

Langevin equations in spatially-extended systems have been extensively used in the studies of equilibrium and nonequilibrium phenomena where fluctuations are important. These equations are stochastic partial differential equations in which the fluctuations are introduced through a noise term, whose stochastic properties are chosen according to the physical situation. Fluctuations of internal (thermal) origin represent microscopic degrees of freedom, and since they evolve in spatial and temporal scales much shorter than those of the gross variables of the system, they are assumed to be uncorrelated (delta correlated) in space and time. In this case they are modelled by a gaussian-and-white noise process in space and time. On the other hand, if the origin of the fluctuations is external to the system, the possibility of a noise with some spatial or temporal structure has to be considered.

Langevin equations have been used in many different scenarios, one of which has been for many years the study of Dynamical Critical Phenomena [1,2]. The methodology used there is a suitable extension of the Renormalization Group techniques developed for the study of static critical phenomena [3,4]. The technique has also been generalized to study nonequilibrium systems presenting some kind of scale invariance or criticality [5,6].

In this paper we will follow this methodology as it was implemented in Refs. [5,6] to study the nonequilibrium critical properties of a Langevin-equation model with a non-white (colored) noise.

Among the different models that can be considered we have chosen the simplest one, in which the order parameter is not conserved. This model is useful for describing order-disorder transitions. In this non-conserved case the equation of motion of the field variable is

\[
\frac{\partial \psi(x, t)}{\partial t} = -\frac{\delta F[\psi]}{\delta \psi(x, t)} + \xi(x, t),
\]

(1.1)

where \( x \) is the position in a \( d \)-dimensional space, \( F[\psi] \) is the Ginzburg-Landau free-energy functional.
\[ F[\psi] = \int d^d x \left\{ \frac{1}{2} r \psi^2 + \frac{1}{2} \mu \left| \nabla \psi \right|^2 + \frac{1}{4} u \psi^4 \right\} . \] \quad (1.2)

and the correlation of the noise is

\[ \langle \xi(x, t) \xi(x', t') \rangle = 2D \delta(x - x') \delta(t - t'), \] \quad (1.3)

where \( D = k_B T \) to ensure the correct equilibrium steady state. This is known as model A in the notation of Ref. [2]. This model comes from a coarse-graining procedure applied to a spin model [7]. Its behavior can be easily understood by means of a mean-field analysis of the potential given by the Ginzburg-Landau free energy. When \( r \) is positive this potential has a minimum at zero field, which corresponds to a coarse-grained field equal to 0 (i.e. the spins are randomly up or down). A negative value of \( r \) leads to a potential with two nonzero symmetrical minima (the spins have a preferred direction, either up or down). The role of noise in this scheme is a disordering one: when \( r \) is negative and the intensity of the noise increases, the potential barrier between the two minima is more easily surpassed by the spins. Thus, although the potential shape does not change, the coarse-grained field will eventually become zero (for a high enough value of the intensity of the noise): the spins will be disordered because their thermal energy (i.e. the noise intensity) is so high that they do not see the potential barrier.

Our aim in this paper is to study the model (1.1-1.2) in the special situation characterized by the fact that the noise term \( \xi(x, t) \) is non-delta correlated. In this case there is no fluctuation-dissipation relation and the steady state is not the equilibrium one. Hence we are facing with a pure non-equilibrium problem. In a first approach to this problem we will assume that the correlation is of the Ornstein-Uhlenbeck type:

\[ \langle \xi(x, t) \xi(x', t') \rangle = \frac{D}{\tau} e^{-\frac{|t-t'|}{\tau}} \delta^d(x - x') \] \quad (1.4)

This is a suitable way of modelling the noise in the case when it is external. \( \tau \) is its correlation time. It can be seen that in the limit \( \tau \to 0 \) the noise becomes white and the equilibrium situation is recovered provided \( D = k_B T \). Naïvely, one might say that an increasing value
of $\tau$ makes the noise ”softer”, taking it farther from the white character (given by $\tau \to 0$) and reducing its effective intensity. This effect has indeed been found numerically [8] and approximate techniques borrowed from stochastic theory have been developed in order to understand this phenomenon [9]. In this paper we perform dynamic Renormalization-Group (DRG) calculations on this model in order to study the effects of the correlation time of the noise in the non-equilibrium critical behavior of the system.

This paper is organized as follows: in next Section we present a preliminary scaling analysis to obtain some of the critical characteristics of our model. In Sect. II the Dynamic Renormalization Group results are presented. In Sect. IV we summarize the results corresponding to a noise colored in both space and time. The comparison between our theoretical results and those obtained from simulations is given in Sect. V. Technical details of the DRG calculations are presented in the Appendix.

II. PRELIMINARY ANALYSIS

A glimpse of the relevance of the different terms in (1.2) at the critical point can be obtained by studying the effect of a scale transformation on the dynamical equation (1.1). Let this scale transformation be defined by

$$x = b \tilde{x}; \quad t = b^z \tilde{t}; \quad \psi = b^{-a} \tilde{\psi},$$

(2.1)

with $b > 1$, which means that one is focusing on long distances (characteristic of the critical point). $a$ is related to the equilibrium exponent $\eta$ via $a = \frac{1}{2}(d - 2 + \eta)$ [2]. Introducing these changes into Eq. (1.1) and making use of the expression of the Ginzburg-Landau functional (1.2) one can see that the dynamical equation becomes

$$\frac{\partial \tilde{\psi}}{\partial \tilde{t}} = -rb^z \tilde{\psi} + \mu b^{z-2} \nabla^2 \tilde{\psi} - u b^{z-2a} \tilde{\psi}^3 + b^{a+z} \xi.$$  

(2.2)

Now we define a rescaled noise $\tilde{\xi} = b^{a+z} \xi$, which has a new intensity

$$\tilde{D} = D b^{2a+z-d}$$

(2.3)
and a correlation time $\tilde{\tau} = \tau / b^z$. Moreover, in order to keep the shape of the dynamical equation after the scale transformation, we also define the following rescaled parameters:

$$
\tilde{r} = r b^z, \quad \tilde{\mu} = \mu b^{z-2}, \quad \tilde{u} = u b^{z-2a}
$$

(2.4)

Now we place ourselves in the critical region of the system (which in a first approximation is supposed to be located near $r = 0$) where, in the absence of the nonlinear term ($u = 0$), the equation remains invariant under the scale transformation if the exponents $a$ and $z$ are chosen to be

$$
a = a_0 = \frac{d - 2}{2} \quad (\eta_0 = 0)
$$

$$
z = z_0 = 2
$$

(2.5)

These are the trivial exponents of model (1.1-1.2) with colored noise, which coincide with the ones corresponding to the white-noise case. They are valid when the nonlinearity of the model does not play a role in the critical behavior of the system, and this occurs beyond a critical dimension which we can now determine. As we have seen in (2.4), the nonlinear coefficient $u$ transforms upon rescaling with an exponent $z-2a$. By making use of the trivial values of $z$ and $a$, one finds that this exponent happens to be $z_0 - 2a_0 = 4 - d \equiv \varepsilon$. Thus when $d > d_c = 4$, the nonlinearity decreases under rescaling ($b > 1$), so that $u$ is irrelevant. On the other hand, for $d < d_c$, $u$ grows under rescaling, and hence it is now relevant. This analysis is independent of the character of the noise, i.e. it does not depend on whether the noise is white or colored. In fact, since $z > 0$ and $\tilde{\tau} = \tau b^{-z}$, the correlation time $\tau$ is always an irrelevant parameter, so that one should not expect any effect of it in the universal properties of the model. However it has, as we will see, influence on the system.

III. DRG RESULTS

The DRG is going to be implemented in Fourier space, so the first thing we must do is to transform our Langevin equation (1.1) conveniently. Let us define the Fourier transform of a field variable $\psi$ as:
\[
\psi(x, t) = \frac{1}{(2\pi)^{d+1}} \int \psi(k, \omega) e^{i(k \cdot x - \omega t)} \, dk \, d\omega
\] (3.1)

where \( k \) is a vector in a d-dimensional (Fourier) space. According to these definitions, the Fourier transform of Eq. (1.1) can be written as

\[
\psi(k, \omega) = \psi^0(k, \omega) - u G_0(k, \omega) \int_{k_1} \int_{k_2} \psi(k_1, \omega_1) \psi(k_2, \omega_2) \psi(k - k_1 - k_2, \omega - \omega_1 - \omega_2)
\] (3.2)

where \( \int_{k, \omega} \) stands for \( \frac{1}{(2\pi)^{d+1}} \int_{d} d\omega \). \( \psi^0 \) is the zeroth-order approximation (in \( u \)) to the field in momentum space:

\[
\psi^0(k, \omega) = G_0(k, \omega) \xi(k, \omega)
\] (3.4)

and \( G_0 \) is the propagator:

\[
G_0(k, \omega) = \frac{1}{r + \mu k^2 - i\omega}
\] (3.5)

On the other hand, by using definition (3.1) in Eq. (1.4) we can calculate the correlation of the noise in Fourier space:

\[
< \xi(k, \omega) \xi(k', \omega') > = (2\pi)^{d+1} \frac{2D}{1 + \omega^2 \tau^2} \delta^d(k + k') \delta(\omega + \omega')
\] (3.6)

Diagrammatic version of Eq. (3.2)

FIG. 1

DRG will now be applied to Eq. (3.2). In order to simplify this procedure, a diagrammatic notation is very convenient. The diagram version of Eq. (3.2) is shown in Fig. 1. Thick lines stand for the field, whereas a thin line represents the zeroth order approximation of the field and a vertex stands for the two integrals (over \( k_1 \) and \( k_2 \)) and \( u \). The propagator
is represented by a thin line with an arrow. As can be seen in this figure, momentum through a vertex must be conserved.

In this work we have decided to carry out a DRG analysis using the standard momentum shell integration scheme. While details of this method are well documented in Ref. [1], here we briefly review the procedure, which consists of the two following steps:

a. **Momentum Shell Integration.** The first step of the DRG is to eliminate the short wavelength modes (i.e. integrate out modes in the momentum shell $\Lambda e^{-l} < |k| < \Lambda$, where $l > 1$ and $\Lambda$ is the momentum upper cutoff), since we are interested in the long wavelength behavior of the system.

b. **Space Rescaling.** After the momentum shell integration step, we rescale space in such a way that the full momentum space is recovered. This leads to the differential equations satisfied by the running coupling constants. A fixed point analysis of these equations will then reveal the scaling properties of the long wavelength correlations which we are seeking.

Explicit details of the DRG calculations for our model are summarised in the Appendix. The DRG results are obtained from the analysis of the *differential flow equations* [1,4], defined as the infinitesimal variation of the parameters when the renormalization step $l$ is very small. From (A6), they are simply:

$$\frac{d\bar{r}}{dl} = z\bar{r} + 3 \bar{u} K_4 (1 - \mu \tau) \quad (3.7a)$$
$$\frac{du}{dl} = (z - 2a)u - 9u\bar{u} K_4 \quad (3.7b)$$
$$\frac{d\mu}{dl} = (z - 2) \mu \quad (3.7c)$$
$$\frac{dD}{dl} = (2a + z - d) D \quad (3.7d)$$

where the following parameters have been defined:

$$\bar{r} \equiv \frac{r}{\mu}, \quad \bar{u} \equiv \frac{uD}{\mu^2} \quad (3.8)$$

The fixed point of this transformation is the one at which these derivatives are zero. Physically it represents the critical point of the system. Therefore, by studying its position as a function of $\tau$ we will be able to study the influence of the time correlation of the noise
in the disordering transition induced by the noise intensity. By imposing this invariance on Eqs. (3.7c) and (3.7d) one finds that the values of the exponents $a$ and $z$ up to $O(\varepsilon)$ are the trivial ones already obtained in Sec. II. Invariance of the static parameters (Eqs. (3.7a) and (3.7b)) shows that the fixed point is given by:

$$\bar{r}^*=\frac{\varepsilon}{6}(1-\mu^*\tau), \quad \bar{u}^*=\frac{\varepsilon}{9K_4}$$ (3.9)

For $\tau=0$ (white noise) the critical value of $r$ is negative, which is reasonable: in the presence of an additive (disordering) noise, the system will be disordered even for a small negative value of $r$. When $\tau \neq 0$ the critical value of $r$ is nearer 0, which means that the disordering effect of the noise will somehow become diminished because of its correlation in time. Thus the effect of the time correlation of the noise is a ”softening” one, as expected and explained above. It is worth noting that in the white-noise limit our results coincide with the known values appearing in the literature [1,2,4].

Concerning the evaluation of the critical exponents of the system, this can be done in a simple way once the recursion relations are known. By making use of standard techniques [4] involving the evaluation of the eigenvalues of the recursionrelation matrix it is easily seen that there is no contribution of $\tau$ (up to first order) to any of the exponents. Hence one can conclude from this analysis that a correlation in time of the fluctuations affecting a $\phi^4$ non-conserved model influences the position of the critical point of the system (as observed numerically [5]), but not its universal properties (through its critical exponents).

IV. COLORED NOISE IN SPACE AND TIME

A more realistic assumption in relation to external noise is the existence of a non-delta correlation also in space. Intuition tells us that the influence of a non-zero correlation length of the noise will be qualitatively the same as the effect of the correlation time, i.e. an ordering one. This was checked numerically and also explained theoretically in Ref. [6]. A DRG argument concerning this behavior can be made in a straightforward way following
the procedure previously described. The noise is chosen to be a generalization of Eq. (1.4) with no delta correlation in space. This can be done in a simple way by assuming that it obeys the Langevin equation

\[ \dot{\xi}(\vec{r}, t) = -\frac{1}{\tau} (1 - \lambda^2 \nabla^2) \xi + \frac{1}{\tau} \eta(\vec{r}, t), \quad (4.1) \]

where \(\eta(\vec{r}, t)\) is a gaussian white noise with correlation (1.3). This is an extension of the Ornstein-Uhlenbeck process, where the laplacian term takes into account the coupling of the field at different points [10], so that \(\lambda\) is the correlation length. The correlation of this noise in Fourier space can be seen to be

\[ <\xi(k, \omega)\xi(k', \omega')> = \frac{2D}{(1 + \lambda^2 k^2)^2 + \omega^2 \tau^2} \delta^d(k + k') \delta(\omega + \omega'), \quad (4.2) \]

which can be compared to Eq. (3.6).

The scaling analysis of Sec. II applied to this new situation shows that \(\lambda\) changes as \(\tilde{\lambda} = \lambda/b\), so that it is an irrelevant parameter (as \(\tau\)). Nevertheless we aim to analyze, as in the case of the correlation time, its nonuniversal effects on the system.

The contribution of \(\lambda\) to the diagrams which renormalize \(r\) and \(u\) can be easily obtained. The new results for the corresponding differential flow equations are

\[ \frac{d\bar{r}}{dl} = z \bar{r} + 3 \bar{u} K_4 \frac{1}{(1 + \lambda^2)^2} \left( 1 - \frac{\mu \tau}{1 + \lambda^2} \right) \quad (4.3a) \]
\[ \frac{du}{dl} = (z - 2a)u - 9u \bar{u} K_4 \frac{1}{(1 + \lambda^2)^2} \quad (4.3b) \]

which should be compared to Eqs. (3.7a-3.7b). These new contributions of \(\lambda\) modify the value of the fixed point in such a way that \(\bar{r}^*\) and \(\bar{u}^*\) become:

\[ \bar{r}^* = -\frac{\varepsilon}{6} \left( 1 - \frac{\mu^* \tau}{1 + \lambda^2} \right), \quad \bar{u}^* = \frac{\varepsilon}{9 K_4} \left( 1 + \lambda^2 \right)^2 \quad (4.4) \]

This modification does not lead to variations of the eigenvalues of the matrix associated to the transformation, so that critical exponents are not changed from the white-noise case, as expected.
Hence we can conclude that the noises used here, either (3.6) or (4.2), with a finite correlation time and length, change the position of the critical point but not the critical exponents, when compared to the white-noise case (1.3). This is due to the way this kind of noises behave (see (2.3), for instance). Should the correlation of the noise decay as a power law, then, according to the scaling analysis of Ref. [6], critical exponents would be changed.

V. COMPARISON WITH NUMERICAL RESULTS

In Refs. [8,9] model A (1.1) with noises (1.4) and (4.2) was studied by means of a numerical simulation in 2-d. Two main results were obtained there. Firstly, it was established that either $\tau$ or $\lambda$ stabilize the system, in the sense that the critical value of the noise intensity is enlarged. Secondly, critical exponents were also evaluated by means of a finite-size scaling analysis, and their values were found to be similar (within error bars) to the accepted values for the white-noise case. Since the numerical evidence of the critical-noise-intensity shift as a function of $\tau$ and $\lambda$ is much clearer, that is what we want to explain in the light of the above DRG results.

The simulation model, defined by fixed values of $r$, $\mu$ and $u$, was

$$\frac{\partial \psi(\vec{x},t)}{\partial t} = \frac{1}{2} \left( \psi - \psi^3 + \nabla^2 \psi \right) + \xi(\vec{x},t)$$

(5.1)

where the only independent parameters were the noise parameters $D$, $\tau$ and $\lambda$.

Let us now consider Eq. (1.1) at the critical point with the parameters $r^*$, $u^*$, $D^*$, $\mu^* = 1$, $\tau$ and $\lambda$. By means of a change of variables, one can recover the simulation model (5.1) with a critical noise intensity given by

$$D_c = \frac{1}{2} \frac{\bar{u}^*}{\bar{r}^* \epsilon/2} = D_c(\tau = 0, \lambda = 0) \left( 1 + \lambda^2 \right) + \mathcal{O}(\tau^\epsilon)$$

(5.2)

where $D_c(\tau = 0, \lambda = 0) = \frac{1}{3K_4} = \frac{8\pi^2}{3}$.

Certainly one cannot expect a good agreement between a calculation in 4-d extended to 2-d and a numerical simulation of a 2-d model in a finite discrete lattice. In this sense,
the predicted value for $D_c(\tau = 0, \lambda = 0)$ is very different from the numerical result (0.38). Nevertheless, the ratio $D_c(\tau, \lambda)/D_c(\tau = 0, \lambda = 0)$ gives better results, as one can see in Fig. 2. One can thus conclude that RG calculations have given a reasonable explanation of the critical results for model A when a colored noise is considered.

**Fig. 2**

*Ratio of the critical noise intensity in the colored case to its value in the white case versus $\tau$ (from Ref. [9]). The lines correspond to the DRG result ([5,2]).*

**ACKNOWLEDGMENTS**

This research was supported in part by the Dirección General de Investigación Científica y Técnica (Spain) under Project No. PB90-0030. H.G. is supported by the Natural Sciences and Engineering Research Council of Canada, and le Fonds pour la Formation des Chercheurs
et l’Aide à la Recherche de la Province du Québec.

**APPENDIX: CALCULATIONS OF THE DRG AT ONE-LOOP ORDER**

In order to eliminate the outer (high-momentum) modes from Eq. (3.2) we break the integral term (the third diagram in Fig. 1) into modes located in the inner hypersphere (i.e. those with $0 < |k| < \Lambda e^{-l}$), which will be called *external modes*; and modes located in the outer shell (i.e. those with $\Lambda e^{-l} < |k| < \Lambda$), which will be referred to as *internal modes*. This leads to a rewriting of Eq. (3.2), whose diagrammatic representation is shown in Fig. 3 for both the internal and external modes. Fields and generators depending on internal modes are represented by means of a slashed line, and those depending on external modes are left unchanged.

![Diagram](attachment:diagram.png)

*Evolution equations for the external (a) and internal (b) modes ready for the perturbation procedure*

**Fig. 3**

The evolution equation for the internal field (Fig. 3b) can be solved iteratively up to $O(u^2)$, and the result can be introduced in the internal contributions to the integral term of the corresponding dynamical equation for the external field (Fig. 3a). This leads to a renormalized equation where internal modes appear only in the noise term. After averaging
out this internal noise, the remaining renormalized equation is the one represented in Fig. 4.

\[ -u G_0^< (k, \omega) \int_{k_1}^{>} \int_{\omega_1}^{>} \psi^< (k_2, \omega_2) \left< \psi^{0>} (k_1, \omega_1) \psi^{0>} (k - k_1 - k_2, \omega - \omega_1 - \omega_2) \right> = \\
- u G_0^< (k, \omega) \psi^< (k, \omega) \int_{k_1}^{>} \int_{\omega_1}^{>} \frac{2D}{1 + \omega^2} G_0^> (k_1, \omega_1) G_0^> (-k_1, -\omega_1) \]  

(A1)

where definition of \( \psi^0 \) (3.4) and correlation of the noise in Fourier space (3.6) have been used. We will denote this result as shown in the second member of Fig. 4. The superindex \( < (>) \) on \( G_0 \) and \( \psi \) denotes an external (internal) mode.

\[ \left< \begin{array}{c} k \\ k_1 \\ k_2 \end{array} \right> = \begin{array}{c} k \\ -k_1 \end{array} \]

Interpretation of the diagram which renormalizes \( r \)

FIG. 5

Now we will be able to write down the transformation (recursion) relations for the pa-
rameters $r$, $\mu$ and $u$. We are in the zone where $u$ is relevant $(d < d_c = 4)$ but are also supposing it to be small (which will presumably occur for $d \sim 4$). Therefore if these recursion relations are to be an expansion around $u = 0$ they should also be an expansion in $\varepsilon = 4 - d$. Moreover, $r$ will also be supposed to be small in the following calculations (we are interested in the critical point).

It should be noted that the diagram in Fig. 5 contains only one thick line, and will therefore renormalize the diagram at the left member of the figure, whereas the second bubble contains three thick lines, so that it will add to the one-vertex diagram at the right member of the figure. That means, as we will see, that the first bubble renormalizes $r$ and the second one $u$. On the other hand, it is easy to see that $\mu$ is not renormalized at $O(u)$, since the one-thick-line bubble does not contain any term proportional to $k^2$.

Concerning the computation of the diagrams, it should be said that the frequency integrals can be evaluated by means of contour integrations in the complex plane, and the integrals in momentum space can also be calculated by assuming $r \sim 0$ and a very small $\tau$. These integrals should be expanded in $\varepsilon = 4 - d$, but since they are all multiplied by $u$, which is also small, we only need their zero order in $\varepsilon$. Thus we evaluate them in $d = 4$. We also let $\Lambda = 1$. Moreover, since the parameter $l$ is the step of the renormalization procedure, and we want this procedure to be continuous, we will assume $l \ll 1$. After all these considerations, the final result can be shown to be

$$r_I = r + 3 \frac{uD}{\mu} K_4 (1 - \mu \tau) \ l \quad (A2a)$$

$$u_I = u - 9 \frac{u^2 D}{\mu^2} K_4 \ l \quad (A2b)$$

$$\mu_I = \mu, \quad (A2c)$$

where $K_d$ is $(2\pi)^{-d}$ times the surface of the unit $d$-dimensional hypersphere:

$$K_d \equiv 2^{1-d} \pi^{-d/2} \Gamma(d/2) \quad (A3)$$

We also want to know how the noise intensity $D$ changes under the effects of the DRG. This is not given by the analysis explained above, because the parameter $D$ does not appear
explicitly in Eq. (3.2). To make it come out explicitly one can ”autocorrelate” this equation, i.e. multiply the equation for $\psi(k, \omega)$ by the same equation for $\psi(k', \omega')$. This can be done diagrammatically in a simple way. It can be seen that the first diagram renormalizing $D$ is a two-vertex diagram, which means that it is of $O(u^2)$. Therefore the recursion relation for $D$ up to the order we are considering is simply

$$D_I = D$$  \hspace{1cm} (A4)

It is worth noting that there is no first-order correction in $\tau$ neither for $\mu$, $u$ nor $D$.

Finally, we want the DRG procedure to lead to a renormalized equation as similar as possible to the original one. At this moment both equations are identical in form, the only difference being the range of the variable in momentum space to which they are applied. Indeed, the renormalized equation only governs the evolution of the external modes. In order to eliminate this difference, we shall perform the following rescalation:

$$\tilde{k} = e^l k, \quad \tilde{\omega} = e^z \omega, \quad \tilde{\psi}_r = e^{al} \psi_r$$  \hspace{1cm} (A5)

This scale transformation is identical to the one described in Sec. II with $b = e^l$. There we could see that the rescalation led to effective values of the parameters of the model ((2.3) and (2.4)). Hence, by combining the momentum-shell integration process and the space rescalation, one finds the following final discrete recursion relations for the model parameters up to first order in $l$:

$$\tilde{r} = r_I e^{zl} \simeq r + l \left[ z r + 3 K_4 \frac{uD}{\mu} (1 - \mu \tau) \right]$$  \hspace{1cm} (A6a)

$$\tilde{u} = u_I e^{l(z-2a)} \simeq u + l \left[ (z-2a)u - 9 K_4 \frac{u^2 D}{\mu^2} \right]$$  \hspace{1cm} (A6b)

$$\tilde{\mu} = \mu_I e^{l(z-2)} \simeq \mu + l(z-2) \mu$$  \hspace{1cm} (A6c)

$$\tilde{D} = D_I e^{l(2a+z-d)} \simeq D + l(2a + z - d) D$$  \hspace{1cm} (A6d)
REFERENCES

[1] S.K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, 1976).

[2] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[3] K.G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).

[4] M.E. Fisher, *Scaling, Universality and Renormalization Group Theory*, Lectures Notes in Physics, Vol. 186 (Springer, Berlin, 1983).

[5] D. Forster, D.R. Nelson and M.J. Stephen, Phys. Rev. A **16**, 732 (1977).

[6] E. Medina, T. Hwa, M. Kardar and Y.C. Zhang, Phys. Rev. A **39**, 3053 (1989).

[7] J.S. Langer, Ann. Phys. **65**, 53 (1971).

[8] J. García-Ojalvo, J.M. Sancho and L. Ramírez-Piscina, Phys. Lett. A **168**, 35 (1992).

[9] J. García-Ojalvo and J.M. Sancho, ”Colored noise in spatially-extended systems”, to appear in Phys. Rev. E (1994).

[10] J. García-Ojalvo, J.M. Sancho and L. Ramírez-Piscina, Phys. Rev. A **46**, 4670 (1992).