The absorber hypothesis of electrodynamics

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We test the absorber hypothesis of the action-at-a-distance electrodynamics for globally-bounded solutions of a finite-particle universe. We find that the absorber hypothesis forbids globally-bounded motions for a universe containing only two charged particles, otherwise the condition alone does not forbid globally-bounded motions. We discuss the implication of our results for the various forms of electrodynamics of point charges.

INTRODUCTION

Action-at-a-distance electrodynamics was formulated in 1945 [1], which is relatively late in electromagnetic history. The original motivation was the regularity of the point-charge limit, but the theory has so many desirable physical properties that it is surprising it was not considered earlier for the other reasons. In the original articles, Wheeler and Feynman [1] further promoted the absorber hypothesis, whereby action-at-a-distance electrodynamics reduces to Dirac’s electrodynamics of point charges with retarded-only fields [2]. The absorber hypothesis [1] states that the universe absorbs all future and past radiation, i.e., that the universal far-fields vanish at all times. If true, the absorber hypothesis would be useful to approximate the electrodynamics of the other charges of a large bounded universe, thereby avoiding a many-body problem, but the approximation has never been tested. Here we show that the absorber hypothesis fails for a universe consisting of two charges with globally-bounded interparticle distances. Otherwise, for universes with three- or more- point charges the absorber hypothesis alone is no obstacle for globally-bounded motions. For these possible universal motions it remains to be understood if the absorber hypothesis is an additional property that holds for a special class of globally-bounded orbits, like neutrally-stable orbits for example, or just true for any globally bounded orbit of a universe containing a large enough number of charges.

The absorber hypothesis, henceforth denoted A.H., [1] is most easily expressed in terms of the advanced and retarded electromagnetic field tensor of each particle, respectively \( F^{(i)ad} \) and \( F^{(i)ret} \), as

\[
\sum_i [F^{(i)ret} + F^{(i)adv}] = o\left(\frac{1}{r}\right),
\]

when \( r \to \infty \), as discussed in [2]. The fall faster than \((1/r)\) defined by Eq. [1] implies the vanishing of the semi-sum of the radiation fields, which is called perfect absorption. As argued by Wheeler and Feynman [1], Eq. [1] includes a combination of incoming and outgoing waves, and the only way to achieve a cancellation at all times is if both the retarded sum and the advanced sum vanish separately, i.e.,

\[
\sum_i F^{(i)ret} = o\left(\frac{1}{r}\right), \quad (2)
\]

\[
\sum_i F^{(i)adv} = o\left(\frac{1}{r}\right), \quad (3)
\]

when \( r \to \infty \). In the sequel of their work of 1945, Wheeler and Feynman used Eqs. (2) and (3) to establish the important condition

\[
\sum_i [F^{(i)ret} - F^{(i)adv}] = o\left(\frac{1}{r}\right), \quad (4)
\]

for the source-free retarded-minus-advanced field, which implies its vanishing everywhere by the Cauchy problem. The various electrodynamics of point charges originated from Maxwell’s electrodynamics, by a procedure inaugurated by Lorentz [2, 4, 5]. Maxwell’s equations are time-reversible and therefore the general solution is given by a linear combination of the retarded Green function and the advanced Green function [6], each combination generating a different electrodynamics of point charges [5]. In the general case the far-electric field of a point charge involves an arbitrary parameter \( \chi \) [6],

\[
E = \frac{1}{2}(1-\chi)E^{adv} + \frac{1}{2}(1+\chi)E^{ret}, \quad (5)
\]

while the far-magnetic field is given by

\[
B = \frac{1}{2}(1-\chi)n^+ \times E^{adv} - \frac{1}{2}(1+\chi)n^- \times E^{ret}, \quad (6)
\]

where unit vectors \( n^\pm \) point away from the advanced/retarded position of the charge, respectively [6] and in our unit system the speed of light is \( c = 1 \). For
a spatially bounded universe we can take a sphere of radius much larger than the universal radius, such that \( n^+ = n^- = n \), and the Poynting vector \( \mathbf{P} = \mathbf{E} \times \mathbf{B} \) evaluated with Eqs. (5) and (6) is

\[
\mathbf{P} = \frac{1}{4} ((1 - \chi)^2 \mathbf{E}_{adv}^2 - (1 + \chi)^2 \mathbf{E}_{ret}^2) n
\]

(7)

where single bars denote Euclidean modulus, and we have used the transversality of the far-fields, \( n \cdot \mathbf{E}_{ret} = n \cdot \mathbf{E}_{adv} = 0 \).

Notice that the A.H. is a stronger condition than the vanishing of the Poynting flux of the semi-sum field obtained from Eq. (7) with \( \chi = 0 \). For instance, the A.H. implies the vanishing of the flux Eq. (7) for any \( \chi \) because \( \mathbf{E}_{ret}^2 = \mathbf{E}_{adv}^2 = 0 \) by Eqs. (2) and (3), but not vice-versa. For example the circular two-body orbit of the action-at-a-distance theory has a vanishing angular average of \( \mathbf{P} \) but it does not satisfy the A.H. (neither Eq. (2) nor Eq. (3) hold). Therefore, studying orbits with a vanishing Poynting flux of retarded fields is relevant for two different electromagnetic theories of point charges: (i) In the action-at-a-distance electrodynamics it is a necessary condition for globally bounded two-body orbits satisfying the A.H., because the A.H. requires \( \mathbf{E}_{ret} = 0 \), but it is not sufficient to apply the condition to the retarded-far-electric field \( \mathbf{E}_{ret}^1(t, \mathbf{n}) \) only, since the far-magnetic field is proportional to \( \mathbf{B}_{ret}^1(t, \mathbf{n}) \) by

\[
\mathbf{B}_{ret}^1(t, \mathbf{n}) = \mathbf{n} \times \mathbf{E}_{ret}^1.
\]

(8)

The retarded far-electric field of a charge \( q_1 \) at a space-time point \( (t, Rn) \) is given by the Lienard-Wiechert formula

\[
\mathbf{E}_{ret}^1(t, \mathbf{n}) = \frac{q}{4 \pi} \frac{\mathbf{n} \times [(\mathbf{n} - \mathbf{v}_1(t_1) \times \mathbf{a}_1(t_1))] [(1 - \mathbf{n} \cdot \mathbf{v}(t_1))^2 R]}{(1 - \mathbf{n} \cdot \mathbf{v}(t_1))^2 R}. \]

(9)

In Eq. (9), unit vector \( \mathbf{n} \) points from the charge 's retarded position \( (t_1, x_1) \) to the space-time point \( (t, Rn) \) while \( \mathbf{v}_1(t_1) \) and \( \mathbf{a}_1(t_1) \) are respectively the Cartesian velocity and Cartesian acceleration of the point charge in the past light-cone of the observation point \( (t, Rn) \). The time of particle 1 in light-cone with \( (t, Rn) \) is given by

\[
t_1 = t - |x_1(t_1) - Rn|,
\]

(10)

where single bars stand for Cartesian distance. Equation (10) is approximated at large values of \( R \) by

\[
t_1 = t - R + \mathbf{n} \cdot \mathbf{x}_1(t_1).
\]

(11)

Equation (11) defines \( t_1 \) as an implicit function of time \( t \) with derivative

\[
\frac{dt_1}{dt} = \frac{1}{(1 - \mathbf{n} \cdot \mathbf{v}_1(t_1))}.
\]

(12)

Using Eq. (12), the far-field (9) can be expressed simply as

\[
\mathbf{E}_{ret}^1(t, \mathbf{n}) = \frac{q_1}{R} \frac{\mathbf{n} \times [\mathbf{x}_1(t_1)]}{dt^2} |(1 - \mathbf{n} \cdot \mathbf{v}(t_1))^2 R|,
\]

(13)

where \( \mathbf{x}_1(t_1) \) is the position of particle 1 at time \( t_1 \). The retarded time \( t_1 \) is a function of time \( t \) by Eq. (11), and condition (2) reduces to the vanishing of Eq. (13). We henceforth adopt a unit system where the speed of light is \( c = 1 \). To apply conditions (2) and (3) to a spatially bounded one-body orbit takes an inertial frame and a sphere of large radius \( R \) centered at the origin. The space-time points \( (t, Rn) \) on the sphere are specified by the time \( t \) and the unit vector \( \mathbf{n} \). As far as necessary condition is concerned, it suffices to consider the condition for the far-retarded fields, Eq. (2), the advanced condition Eq. (3) representing exactly the same obstruction. It is also sufficient to apply the condition to the retarded-far-electric field \( \mathbf{E}_{ret}^1(t, \mathbf{n}) \) only, since the far-magnetic field is proportional to \( \mathbf{B}_{ret}^1(t, \mathbf{n}) \) by

\[
\mathbf{B}_{ret}^1(t, \mathbf{n}) = \mathbf{n} \times \mathbf{E}_{ret}^1.
\]

(8)

UNIVERSE WITH A SINGLE CHARGE

We henceforth adopt a unit system where the speed of light is \( c = 1 \). To apply conditions (2) and (3) to a spatially bounded one-body orbit takes an inertial frame and a sphere of large radius \( R \) centered at the origin. The space-time points \( (t, Rn) \) on the sphere are specified by the time \( t \) and the unit vector \( \mathbf{n} \). As far as a necessary condition is concerned, it suffices to consider the condition for the far-retarded fields, Eq. (2), the advanced condition
\[ v_1(t_1) = (1 - n \cdot v_1(t_1)) \frac{\partial f_1}{\partial t} n. \] (16)

According to Eq. (11) one can vary \( n \) in a cone with axis along \( x_1(t_1) \neq 0 \) while leaving \( t_1 \) and \( t \) fixed so that the left-hand side of Eq. (14) is fixed. This is seen to be impossible because the right-hand side of Eq. (16) varies unless

\[ \frac{\partial f_1}{\partial t} = 0, \] (17)

which must be the case. Therefore \( v_1(t_1) = 0 \) and the particle is resting at the origin of some inertial frame for the only consistent solution. No surprises arise in this one-charge case, the A.H. requires only that the charge is resting in some inertial frame. Since an isolated charge suffers no force in the action-at-a-distance theory, it moves at a constant velocity and we can always find an inertial frame where it is resting, so that the A.H. is always fulfilled!

UNIVERSE CONSISTING OF TWO CHARGES

Unlike the case of a single charge, where a trivial resting orbit is acceptable, a two-body bounded orbit with both charges resting a finite distance apart is unacceptable because the Coulombian attraction would cause an acceleration incompatible with constant rest. Assuming a globally bounded motion exists, it should be inside a sphere of radius \( \rho << R \). The Poynting vector of the retarded fields on the surface of this sphere is

\[ P_{\text{ret}} = -|E_1^{\text{ret}} + E_2^{\text{ret}}|^2 n. \] (18)

Here we consider only the case of opposite charges. Our next result is true for the case of two arbitrary charges as well, but the mathematical details are surprisingly more elaborate and since the protonic and the electronic charges are known to be equal to an incredible precision\(^9\), we consider here only the equal-charge problem. Henceforth charge 1 is supposed positive and equal to \( q \) while charge 2 is negative and equal to \( -q \). Again the absorber condition (2) is equivalent to the vanishing of the flux of the retarded fields, i.e.,

\[ E_1^{\text{ret}} + E_2^{\text{ret}} = \frac{q n}{R} \times \frac{d^2}{dt^2} (x_1(t_1) - x_2(t_2)) = 0. \] (19)

The minus sign in Eq. (19) is because the charges are opposite. Analogously to the time of particle 1, in Eq. (19) the time of particle 2 is given by

\[ t_2 = t - R + n \cdot x_2(t_2). \] (20)

Notice also that Eqs. (11) and (20) yield an implicit relation between \( t_1 \) and \( t_2 \),

\[ t_1 - t_2 = n \cdot (x_1(t_1) - x_2(t_2)). \] (21)

Equation (19) has a general bounded solution of type

\[ x_1(t_1) - x_2(t_2) = D(n) + n f(t, n) + t S(n), \] (22)

where again \( D(n) \) is an arbitrary bounded function of \( n \) satisfying \( n \cdot D = 0 \), \( f(t, n) \) is a \( C^2 \) bounded function of time and \( S(n) \) is a bounded vector function of \( n \). It follows from Eqs. (21) and (22) that \( f(t, n) = (t_1 - t + R) - (t_2 - t + R) \) and the condition of a globally-bounded orbit implies \( S(n) = 0 \). We therefore rewrite Eq. (22) as

\[ x_1(t_1) - x_2(t_2) = D(n) + [(t_1 - t) - (t_2 - t)]n. \] (23)

The derivative of Eq. (23) respect to time yields

\[ \frac{v_1(t_1)}{(1 - n \cdot v_1(t_1))} - \frac{v_2(t_2)}{(1 - n \cdot v_2(t_2))} = K_{12} n, \] (24)

where

\[ K_{12} = \frac{1}{(1 - n \cdot v_1(t_1))} - \frac{1}{(1 - n \cdot v_2(t_2))}. \] (25)

For arbitrary \( t_1 \) and \( t_2 \) it is possible to move \( n \) in a cone with axis along \( x_1(t_1) - x_2(t_2) \neq 0 \) in a way that fixes \( t_1 \) and \( t_2 \). It is important to observe that the time \( t \) of the observation point also changes along this variation, as needed by Eqs. (11) and (20). Along this variation, the unitary vector \( n \) describes a cone that can be at the best tangent to the plane defined by the fixed vectors \( v_1(t_1) \) and \( v_2(t_2) \). Therefore the only possibility is that \( K_{12} = 0 \), such that the velocity vectors on the left-hand side of Eq. (24) must be collinear. It is further possible to show with Eq. (25) using \( K_{12} = 0 \) and Eq. (24) that

\[ v_1(t_1) = v_2(t_2), \] (26)

which can be used to prove that each velocity is piecewise constant, as follows. Since Eq. (26) is valid for arbitrary \( t_1 \) and \( t_2 \) satisfying the light-cone condition (21), one can fix \( t_1 \) while moving \( t_2 \) to a maximal interval by playing with \( t \) and \( n \), such that the velocity is constant in the maximal interval determined by Eq. (21). It is interesting to notice that the velocity must remain constant for the maximal time equal to the interparticle separation predicted by Eq. (21), i.e.,

\[ a = |x_1(t_1) - x_2(t_2)|. \] (27)

The distance \( a \) is itself constant while the particles have the same velocity, as of Eq. (26). If the particles never collide, the particle separation \( |x_1(t_1) - x_2(t_2)| \) must be bounded from below, and the only physical motion having piecewise-constant velocity and piecewise-constant
separation on bounded intervals as derived above would be motion of both particles with the same constant velocity. This motion would be impossible because the Coulombian force from the other particle at a finite distance would necessarily produce acceleration. The remaining options left would be spiky orbits with a discontinuous velocity at constant particle separation, where the two particles jump together, and it would be unphysical. Our conclusion is then that for two isolated charges there is no globally-bounded \( C^2 \) orbit satisfying the A.H. Our analysis naturally came down to collinear collision orbits, which have zero angular momentum like the quantum ground-state of the hydrogen atom. In the action-at-a-distance theory these collision orbits are possible and have been calculated in Ref. [15] and could be the candidates to satisfy the A.H. at least marginally (for example, with minimal radiative losses over a very large time). Unfortunately such collisions terminates after a finite time[13] and form a composite particle which does not radiate because it moves as one charge, so that again one does not have two charges in non-trivial bounded motion. For the other forms of electrodynamics the very existence of collision orbits is problematic;—There is yet no result available for two-body motion with arbitrary masses in the Lorentz-Dirac theory, and surprisingly the only existing result used an infinitely-massive second particle and concluded that no collision solution exists[17].

Last, for three- or more particle systems no incompatibility is found as far as the A.H. is concerned. In the Lorentz-Dirac theory it would be necessary that a globally-bounded solution did not radiate energy so that our result suggests that such solution does not exist for two-charge universes. It remains to be studied if either a third distant charge can already support a globally-bounded physical orbit in the Dirac theory or if a large number of charges is needed. The above result does not necessarily put an end to the interest in the Lorentz-Dirac theory, but it suggests the supporting universe should contain more-than-two charges in globally-bounded motion, which seems to be the case of our universe.

DISCUSSIONS AND CONCLUSION

We have seen that the A.H. forbids globally bounded orbits for a universe with only two charges. For universes with three- or more charges the A.H. condition alone does not exclude globally-bounded electromagnetic orbits. Moreover, since Eq. (24) does not involve interparticle distances, the globally-bounded non-radiating orbit could even be such that the third charge is significantly separated from the other two. However, since the A.H. is only a necessary condition, we can not conclude that globally bounded orbits satisfying the A.H. do exist for more-than-two-charges universes, and for that it would be necessary to deal with the neutral-delay equations of motion, or to go by inspection of known solutions[7, 8]. The investigation of the possible solutions of these neutral-delay equations is still an open problem. Non-radiating motions of spatially-extended charge distributions with retarded-only fields have already been studied[10], but so far none with only three point charges was reported in the literature.

The consequences of our results depend if one is using the Dirac theory with retarded fields or the action-at-a-distance electrodynamics, as follows: (i) For the Dirac theory our results strongly suggest that no globally-bounded solution exists at all for the Lorentz-Dirac equation with two charges, because such orbit would be leaking energy to infinity. This physical limitation of the Dirac theory makes it impossible to consider the other charges of a large universe as a perturbation for a globally-bounded two-body orbit, but rather these other charges are an essential ingredient for the bounded motion to exist. In Ref.[11] the vanishing of the flux of retarded fields, Eq. (2), was used as a condition for long-lived orbits:—If these orbits conjectured in Ref.[11] are only long-lived and eventually ionize or decay, there is no contradiction with our present results. Otherwise if these orbits are to be globally-bounded, our present result requires at least a third charge somewhere, in the best scenario for the Dirac theory. (ii) On the other hand, for the action-at-a-distance electrodynamics bounded two-body orbits do exist[7, 8], and it would be physically sensible to consider the other charges of a large universe as a perturbation for a globally-bounded two-body orbit. Notice that the Schoenberg-Schild orbits[7, 8] do not satisfy the A.H. (2), in agreement with our result that holds for any globally-bounded orbit. These circular orbits do no leak energy to infinity because the total flux of the semi-sum field vanishes, even though the far-fields themselves do no vanish and neither the flux of the retarded-fields vanishes. The natural physical interpretation of condition (2) as used in Ref.[11] is as follows:—Let us promote the A.H. to a condition that is respected by the dynamics of a stable universe with a large number of charges. Along such dynamics, if a two-body circular orbit is formed, the other universal charges must provide reaction far-fields to add linearly to these and enforce the A.H., since no bounded two-body orbit can possibly satisfy the A.H. alone. In this perspective, the globally-bounded two-body orbit is disturbing the universe by disturbing the A.H. boundary condition at infinity, and it would be physically desirable to minimize the strength of the offending retarded far-fields of the two-body orbit, so that lesser universal reaction is required to enforce the A.H. For that the mechanism suggested in Ref.[11], i.e., interference with a beat of the fast solenoidal motion of a deformed two-body orbit, would be a physically desirable perturbation. It remains to be researched if such solenoidal two-body orbits do exist as novel non-
trivial solutions of unperturbed two-body motion in the action-at-a-distance theory. In a large bounded universe the far-fields of the other particles must be included in the flux calculation, which is implemented in the random electrodynamics in the form of a non-A.H. boundary condition for the far-fields. The inclusion of the universal far-fields in some estimates for the physical magnitudes of solenoidal two-body orbits was implemented using a model called dissipative Fokker electrodynamics in Ref. [12].

In has been common use in the derivations of point-charge electrodynamics that a particular choice of Green function is enough to describe the most general physical dynamics, and it is even desirable that it should be so for simplicity. Of course one can not rule out that a particular choice leads to equations with no solution of physical interest, or even worse, that the Green function yielding the correct physics could be different for different problems. Assuming a particular choice to be enough for all physical situations, the next question is what should that choice be. It is important to stress that the advanced/retarded Green functions involve respectively a future or a past position measured at another spatial point by another clock synchronized a la Einstein, which is not contradictory to causality like it would be using the Newtonian future in a Newtonian mechanics, for example. In the theory of relativity one is not allowed to compare times that are not measured at the same point. To falsify the ”known” future of the other spatial point, one would have to travel to this spatial point. By the definition of the light-cone, traveling at the speed of light should arrive precisely when the predicted future happens, so that no falsification of the ”known future” is possible. The remaining rationale for the most popular choice (retarded-only Green function) are based on the analysis of a single charge acted upon by non-electromagnetic forces that start suddenly. For such globally-bounded motions the charges have accelerated all the way back in the infinite past and the electromagnetic far-fields exist for all times regardless of the choice of the Green function. For example, along a circular motion with a small velocity the future field and the past field approximately coincide for an increasing series of distances approximately proportional to multiples of the rotation period, so that the retarded Green function and the advanced Green function predict very similar field-patterns anyway. Moreover the equations of motion obtained using the advanced Green function involve the advanced acceleration of the other particle, which can be solved for this most-advanced acceleration yielding delay-only equations of motion, Eq. (22) of Ref. [14], a construction that needs no future information to define solutions. We are of the opinion that the study of many-body electromagnetic-only problems should guide the choice of the Green’s function, and not vice-versa. For that the knowledge of the possible bounded motions of a finite number of charges is essential [16]. Here we have shown that bounded two-body orbits are impossible in the Lorentz-Dirac theory, so that interesting dynamics necessitates three bodies at the best in the Dirac theory. The inexistence of unperturbed globally-bounded two-body motions is a shortcoming of the Dirac theory if one is looking for a sensible dynamical system to describe atomic physics classically. On the other hand, there are several selling points for the action-at-a-distance theory: (i) a known one-parameter family of bounded circular-orbits for two-body motion [8], (ii) the regularity of the point-charge limit and the absence of self-interaction and runaways [8], (iii) The ill-posedness of the backward equations of motion in any electrodynamics of point charges but the action-at-a-distance theory, i.e., the backward equations of motion define the acceleration as a function of its derivative and second derivative in the past, Eq. (25) of Ref. [14]. As discussed in Ref. [14], along a backward integration one constructs an acceleration that is only continuous until the first breaking point, but later on its first and second derivatives in the past are needed. Since these extra derivatives exist only if the third and fourth derivatives of the initial history existed, recursively one should have started with a $C^\infty$ initial segment. This recursion makes it impossible to start with generic data, so that if a solution ever existed one should start a backward integration from very special $C^\infty$ data, which would be spoiled by the numerical calculations. Moreover, it turns out that backward integration would be absolutely necessary because the forward equations have explosive runaway solutions [14], so that something is not well for the Lorentz-Dirac theory. On the contrary, action-at-a-distance electrodynamics yields neutral-delay backward equations for two-body motion, rather than ill-posed, and last (v) Inside the action-at-a-distance theory, bounded universes are not required to satisfy the A.H., even though the A.H. can provide an improved stability to counter energy losses, as follows:—If only the energy flux of the semi-sum field vanishes, an offending perturbation of size $\varepsilon$ in the retarded far-field (or in the advanced far-field) perturbs the energy flux Eq. (8) at $O(\varepsilon)$, while if the A.H. holds the perturbed flux is $O(\varepsilon^2)$ by Eq. (7). Therefore the A.H. can be thought as a stronger boundary condition for stable universal dynamics in the action-at-a-distance theory, while in the Dirac theory the A.H. is necessary for a vanishing energy flux.
[1] J. A. Wheeler and R. P. Feynman Rev. Mod. Phys. 17, 157 (1945); J. A. Wheeler and R. P. Feynman 21, 425 (1949).
[2] P. A. M. Dirac, Proceedings of the Royal Society of London, ser. A 167, 148 (1938).
[3] F. Hoyle and J. V. Narlikar, Lectures on Cosmology and Action at a Distance Electrodynamics, World Scientific, London (1996).
[4] J. S. Nodvik, Ann. Phys. (N.Y.) 28 225 (1964).
[5] C. Jayaratnam Eliezer, Reviews of Modern Physics 19 (1947).
[6] J. D. Jackson, Classical Electrodynamics Second Edition, John Wiley and Sons, New York (1975) (Eq. 6.61 of page 224).
[7] M. Schoenberg, Physical Review 69, 211 (1946).
[8] A. Schild, Physical Review 131, 2762 (1963).
[9] A. Staruszkiewicz, Acta Physica Polonica B 33, 2041 (2002).
[10] G. H. Goedcke, Physical Review 135 B 281 (1964) and J. B. Arnett and G. H. Goedcke, Physical Review 168, 1424 (1968).
[11] J. De Luca, Physical Review E 73, 026221 (2006).
[12] T. Boyer, Physical Review D 11, 790 (1975).
[13] J. De Luca, Physical Review E 71, 056210 (2005).
[14] J. De Luca, J. Math. Phys. 48, 012702 (2007).
[15] E. B. Hollander and J. De Luca, Chaos 14, 1093 (2004).
[16] D. J. Louis-Martinez, Phys. Lett. A 320, 103 (2003).
[17] C. J. Eliezer, Proc. Cambridge Philos. Soc. 39, 173 (1943).