Algebraic Connectivity Invariance of Network-of-Networks Derived from a Graph Product

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Abstract: This paper presents algebraic connectivity invariance of a network-of-networks derived from a graph product. The network is modeled as an interconnected system of several kinds of similar subsystems via a certain structure, where several vertices of each subsystem are attached to the structure. The algebraic connectivity of this network can be characterized by using an appropriate network having homogeneous subsystems under a certain condition, which is investigated in this paper. In particular, a necessary and sufficient condition for adding links (edges) to subsystems which preserve the algebraic connectivity is clarified.

Key Words: algebraic approaches, complex systems, control theory, graph theoretic models, network topologies.

1. Introduction

Multi-agent systems can be modeled with a graph, and the spectrum of its graph Laplacian characterizes the dynamics of the system [1]. For example, algebraic connectivity—that is, the second smallest eigenvalue of the Laplacian—can be a measure of the connectedness of the graph [2]–[4]. It also gives us the convergence rate of the multi-agent system in the worst case. Since adding and/or deleting edges change the graph Laplacian, optimizing the algebraic connectivity has been investigated too [5]–[7].

On the other hand, by considering large systems as a network-of-networks, i.e., a hierarchical network constructed by interconnecting subsystems, analysis and/or synthesis methods for the systems have been explored in various ways. See, for example, [8]–[15]. In particular, networks of clusters have been considered in [8],[9],[11],[14], where several estimates of the rate of convergence/consensus have been derived. In this context, the authors of the present paper have proposed a computation method of the exact value of the algebraic connectivity of a time-invariant undirected network-of-networks having a graph product structure [16], where the algebraic connectivity is known to give the exact rate of consensus. The system considered is called the graph product, which consists of several homogeneous subsystems connected with each other according to an interconnected graph. Then the algebraic connectivity is readily obtained from characteristic values of the small graphs, i.e., the subsystem and the interconnection.

In this paper, we present algebraic connectivity invariance of a class of networks-of-networks derived from a graph product, where the network has a heterogeneous substructure. In the system, each subsystem has the same number of agents (that is, the same number of vertices) and has a structure which can be different, where several vertices of subsystems form interconnections with other subsystems, respectively. Then we derive a necessary and sufficient condition for adding edges to subsystems of the original graph with preserving its algebraic connectivity, which provides substantial characteristics for keeping the rate of convergence. To this end, we first define positive definite matrices with graph Laplacian of the subsystem and eigenvalues of the interconnection. We also calculate the eigenvectors of certain positive matrices defined above and eigenvectors of the graph Laplacian corresponding to the interconnection. The condition is expressed with the eigenvectors and graph Laplacian corresponding to edge addition. That is, an eigenstructure of a particular matrix plays an important role in establishing this rigorous condition, which provides a new insight into systems theory of networks-of-networks.

The outline of this paper is as follows. In Section 2, we address notation, mathematical basics, and graph Laplacian. In Section 3, we introduce a hierarchical network to be considered and state our problem setting. In Section 4, we present the main result of this paper. We clarify the necessary and sufficient conditions for adding edges to subsystems which preserves the algebraic connectivity. In Section 5, we give some numerical examples which illustrate the main result. Then in Section 6, we make some concluding remarks.

Note that an earlier preliminary version of this paper was presented at a conference [17], where only a sufficient condition of the main result of this paper was established. The present paper contains a more comprehensive result including the necessary and sufficient condition and several numerical examples.

2. Preliminaries

In this section, we address preliminaries such as notation, mathematical basics, and graph Laplacian.

The set of real numbers is denoted by \( \mathbb{R} \). The cardinality of a set \( S \) is denoted by \( |S| \). Let \( A^T \) denote the transpose of \( A \). For a symmetric matrix \( A = A^T \in \mathbb{R}^{n \times n} \), \( \lambda_i(A) \) \( (i = 1, 2, \ldots, n) \) denote its eigenvalues in non-decreasing order, i.e.,

\[
\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)
\]

If \( B - A \) is a positive semidefinite matrix for symmetric matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \), that is, when \( x^T(B - A)x \geq 0 \) holds with

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any real vector \( x \in \mathbb{R}^n \),
\[
\lambda_i(A) \leq \lambda_i(B), \quad i = 1, 2, \ldots, n \tag{1}
\]
hold. The identity matrix of order \( n \) is denoted by \( I_n \in \mathbb{R}^{n \times n} \), and its \( k \)-th column is denoted by \( e(n, k) \). The vector whose all \( n \) entries are 1 is denoted by \( 1_n \). Let \( A \otimes B \) denote the Kronecker product of matrices \( A \) and \( B \). We recall properties of the Kronecker product: (i) When both \( AB \) and \( CD \) are defined, \( (A \otimes C)(B \otimes D) = (AB) \otimes (CD) \). (ii) \( A \otimes B = 0 \) if and only if \( A = 0 \) or \( B = 0 \).

Consider an undirected graph \( G = (V, E) \), where \( V = \{1, \ldots, n\} \) and \( E \subseteq V \times V \) denote the vertex set and the edge set of \( G \), respectively. Its graph Laplacian \( L(G) \in \mathbb{R}^{n \times n} \) is defined by \( L(G) = D(G) - A(G) \), where \( D(G) \) and \( A(G) \) are the degree matrix and the adjacency matrix of \( G \), respectively, that is, \( D(G) = \text{diag}(d_1, d_2, \ldots, d_n) \), where \( d_i \) denotes the degree of vertex \( i \), while for \( A(G) = [a_{ij}] \) with \( i = 1, 2, \ldots, n \), \( a_{ij} = 1 \) when \( (i, j) \in E \) and \( a_{ij} = 0 \) otherwise. From the construction, since we can represent \( L(G) \) as
\[
L(G) = \sum_{(i, j) \in E} (e(n, i) - e(n, j))(e(n, i) - e(n, j))^T,
\]
any graph Laplacian is symmetric and positive semidefinite, i.e., \( L(G) = L(G)^T \) and \( L(G) \geq 0 \). Furthermore, \( L(G)1_n = 0 \), which means that the smallest eigenvalue of any graph Laplacian is equal to zero, i.e., \( \lambda_1(L(G)) = 0 \). It is known that \( \lambda_2(L(G)) > 0 \) if and only if \( G \) is a connected graph. See [3]. Thus we call \( \lambda_2(L(G)) \) the algebraic connectivity of a graph \( G \).

### 3. Problem Formulation

We consider a hierarchical multi-agent system with \( n_C \) subsystems having undirected graph structures \( S_i, i = 1, 2, \ldots, n_C \), whose dynamics can be expressed by
\[
\dot{x}_i(t) = -L(S_i)x_i(t) + \sum_{k \in K} e(n_S, k)w_{ik}(t),
\]
\[
z_{ik}(t) = e^1(n_S, x_i(t)), \quad i = 1, 2, \ldots, n_C \quad \forall k \in K \tag{2}
\]
where \( x_i(t) \in \mathbb{R}^{n_S} \) denotes the state of the \( i \)-th subsystem and \( K \subseteq \{1, 2, \ldots, n_S\} \) denotes the set of vertices which interact with other subsystems. One should notice that in our problem setting, all subsystems have the same \( K \).

The subsystems are connected by an undirected graph structure \( C \) and their interaction is given by
\[
w_k(t) = -L(C)z_k(t) \quad \forall k \in K, \tag{3}
\]
where
\[
z_k(t) = \begin{bmatrix} z_{k,1}(t) & z_{k,2}(t) & \cdots & z_{k,n_C}(t) \end{bmatrix}^T,
\]
\[
w_k(t) = \begin{bmatrix} w_{k,1}(t) & w_{k,2}(t) & \cdots & w_{k,n_C}(t) \end{bmatrix}^T,
\]
and \( w_{k,i}(t) \in \mathbb{R} \) denotes the connective input to the vertex \( k \) of the \( i \)-th subsystem and \( z_{k,i}(t) \in \mathbb{R} \) denotes the connective output from the vertex \( k \) of the \( i \)-th subsystem.

We rewrite the subsystems (2) and the interconnection (3) as
\[
\dot{x}(t) = -L_Sx(t) + \sum_{k \in K} E_D(k)w_k(t),
\]
\[
z_k(t) = E_D^T(k)x(t),
\]
\[
w_k(t) = -L_Cz_k(t) \quad \forall k \in K,
\]
where \( x(t) \in \mathbb{R}^n \) \((n = n_Cn_S)\) which is defined by
\[
x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) & \cdots & x_n^T(t) \end{bmatrix}^T
\]
denotes a state of the entire system and
\[
E_D(k) = \text{blkdiag}(e(n_S, k), e(n_S, k), \ldots, e(n_S, k)) \in \mathbb{R}^{n \times n},
\]
\[
L_S = \text{blkdiag}(L(S_1), L(S_2), \ldots, L(S_{n_S})),
\]
\[
L_C = L(C)
\]
are coefficient matrices for all \( k \in K \).

From the above, a state equation of the entire system
\[
\dot{x}(t) = \left( L_S + \sum_{k \in K} E_D(k)L_CE_D^T(k) \right)x(t)
\]
is derived. That is, a graph Laplacian of a graph \( G \) can be expressed by
\[
L(G) = L_S + \sum_{k \in K} E_D(k)L_CE_D^T(k).
\]
The graph \( G \) which corresponds to the structure of the entire system is constructed by attaching the vertex \( k \) of each undirected graph \( S_i \) \((i = 1, 2, \ldots, n_C)\) to the vertex \( i \) of a graph \( C \) respectively for all \( k \in K \), where \( K \subseteq \{1, 2, \ldots, n_S\} \) (see also Figs. 1 and 2 in Section 5). It is known that the worst case speed of convergence of the entire system can be estimated by the algebraic connectivity of the graph Laplacian (5). See [18].

In this paper, we propose a characterization of a network-of-networks having this heterogeneous substructure based on its algebraic connectivity \( \lambda_2(L(G)) \). Especially, we investigate a necessary and sufficient condition such that the algebraic connectivity of the entire system with heterogeneous subsystems is equal to that of a specific large system with homogeneous subsystems. This enables us to compute the algebraic connectivity of large systems with heterogeneous subsystems by using the previous result [16] for the systems with homogeneous subsystems.

### 4. Main Result

Before introducing the main result of this paper, we define
\[
Z_i = L(S_i) + \lambda_i(L(C))M(n_S, K)
\]
for \( i = 1, 2, \ldots, n_C \), given undirected graphs \( C \) and \( S^* \) with a set \( K \subseteq \{1, 2, \ldots, n_S\} \), where \( L(C) \in \mathbb{R}^{n_S \times n_S}, L(S^*) \in \mathbb{R}^{n_S \times n_S} \), and
\[
M(n_S, K) = \sum_{k \in K} e(n_S, k)e(n_S, k)^T.
\]

In [16], the authors considered the so-called graph product case where \( S_1, S_2, \ldots, S_{n_S} \) in (4) are identical to \( S^* \). In this case, \( L(G) \) in (5) can be written as
\[
L(G^*) = I_{n_S} \otimes L(S^*) + \sum_{k \in K} E_D(k)L_CE_D^T(k).
\]

More precisely, \( L(G^*) \) is the graph Laplacian of the graph \( G^* \) which is constructed by attaching the vertex \( k \) of an undirected graph \( S^* \) to the vertex \( i \) of a graph \( C \) respectively for all \( k \in K \), where \( K \subseteq \{1, 2, \ldots, n_S\} \). For this \( L(G^*) \), the following lemma was established in [16].
Lemma 1 The eigenvalues of $L(G^*)$ are given by $\lambda_i(Z_i)$, $i = 1, 2, \ldots, n_S$, $j = 1, 2, \ldots, n_C$. Furthermore, its algebraic connectivity $\lambda_2(L(G^*))$ is given by

$$\lambda_2(L(G^*)) = \min\{\lambda_1(Z_2), \lambda_2(Z_1)\}.$$  

The outline of the proof of this lemma is as follows. The spectrum of $L(G^*)$ is equal to the union of the spectrum of $Z_i$ for $i = 1, \ldots, n_C$ because $L(G^*)$ can be transformed into a block diagonal matrix $blkdiag(Z_1, \ldots, Z_{n_C})$ with an orthogonal matrix $U_C \otimes I_{n_S}$. 

$$(U_C^T \otimes I_{n_S})L(G^*)(U_C \otimes I_{n_S}) = blkdiag(Z_1, Z_2, \ldots, Z_{n_C}),$$

which means that the eigenvalues of $L(G^*)$ are given by $\lambda_i(Z_i)$. Since $Z_{l+1} - Z_l \geq 0$ for $l = 1, \ldots, n_C - 1$, we see that $\lambda_2(L(G^*))$ is $\lambda_1(Z_2)$ or $\lambda_2(Z_1)$ from $\lambda_2(L(G^*)) = \lambda_1(Z_1) = 0$ and the inequality (1). Thus we obtain $\lambda_2(L(G^*)) = \min\{\lambda_1(Z_2), \lambda_2(Z_1)\}$. A detailed proof can be found in [16].

The following theorem is the main result of this paper, which clarifies a necessary and sufficient condition for graphs with heterogeneous subsystems to have the same algebraic connectivity with a certain graph with homogeneous subsystems.

Theorem 1 Define $L(G)$ by (5), i.e., the graph $G$ is constructed by attaching the vertex $k$ of each undirected graph $S_j$ ($i = 1, 2, \ldots, n_C$) to the vertex $i$ of an undirected graph $C$ respectively for all $k \in K$. Select an undirected graph $S^*$ satisfying

$$L(S^*) - L(S_i) \geq 0, \quad i = 1, 2, \ldots, n_C.$$  

(7)

Define $Z_i$ by (6) with this $S^*$. Assume that $\lambda_1(Z_i)$ and $\lambda_2(Z_i)$ are distinct from each other and the other $\lambda_i(Z_i)$, $i = 1, 2, \ldots, n_S$, $j = 1, 2, \ldots, n_C$. Then the following statements hold.

Case 1) When $\lambda_1(Z_2) < \lambda_2(Z_1)$,

$$\lambda_2(L(G)) = \lambda_1(Z_2)$$

holds if and only if either

$$[L(S_i) - L(S^*)] \xi_2 = 0 \quad \text{or} \quad (\text{equal\: to} \: \lambda_2(L(C)))$$

(8)

holds for each $i$, where $i = 1, 2, \ldots, n_C$. Here $\xi_2$ and $u_2$ are the eigenvectors of $\lambda_1(Z_2)$ and $\lambda_2(L(C))$, respectively.

Case 2) When $\lambda_1(Z_2) > \lambda_2(Z_1)$,

$$\lambda_2(L(G)) = \lambda_2(Z_1)$$

holds if and only if

$$[L(S_i) - L(S^*)] \xi_1 = 0$$

(9)

holds for each $i$, where $i = 1, 2, \ldots, n_C$. Here $\xi_1$ is the eigenvector of $\lambda_2(Z_1)$.

The meaning of this theorem can be evident by referring to Lemma 1. The lemma gives an explicit expression of the algebraic connectivity of $G^*$ whose subsystems are identical to each other, while Theorem 1 says that the same algebraic connectivity is often obtained even for $G$ which has heterogeneous subsystems. We, therefore, see that Theorem 1 characterizes graphs whose algebraic connectivity can be obtained via small graph analysis given in Lemma 1, which gives us a computational advantage. Here it should also be noted that any graph $S_I$ which is obtained by adding edges to the graph $S^*$ satisfies the assumption (7). In this sense, Theorem 1 gives an interesting characterization of hierarchical graphs whose algebraic connectivity is identical to that of a graph having homogeneous subsystems.

We then state the following lemma, which is used for the proof of Theorem 1.

Lemma 2 Under the assumption (7) of Theorem 1,

$$\lambda_2(L(G)) = \lambda_2(L(G^*))$$

if an eigenvalue of $L(G)$ is $\lambda_2(L(G^*))$.

Proof of Lemma 2. One should notice that

$$L(G^*) = (I_{n_C} \otimes L(S^*)) + \sum_{k=1}^{n_S}(I_{n_C} \otimes e(n_S, k))L(C)(I_{n_C} \otimes e(n_S, k))^T$$

$$= (I_{n_C} \otimes L(S^*)) + \sum_{k=1}^{n_S}(L(C) \otimes M(n_S, K)),$$

$$L(G) \equiv L(G^*) + \sum_{i=1}^{n_C} (e(n_C, i) e(n_C, i)^T \otimes (L(S_i) - L(S^*))$$

hold by (4) and (5). Thus under the assumption (7) of Theorem 1, we have

$$L(G) - L(G^*) \geq 0,$$

which means $\lambda_2(L(G)) \geq \lambda_2(L(G^*))$ by the inequality (1), while we always have $\lambda_1(L(G)) = \lambda_1(L(G^*)) = 0$. We, therefore, see that $\lambda_2(L(G)) = \lambda_2(L(G^*))$ if an eigenvalue of $L(G)$ is $\lambda_2(L(G^*))$. 

\textbf{Proof of Theorem 1.} Notice that

$$[Z_1 - \lambda_2(Z_1)I_{n_C}] \xi_1 = 0, \quad \xi_1 \neq 0, \quad (10)$$

$$[Z_2 - \lambda_1(Z_2)I_{n_C}] \xi_2 = 0, \quad \xi_2 \neq 0, \quad (11)$$

$$[L(C) - \lambda_2(L(C))I_{n_C}] u_2 = 0, \quad u_2 \neq 0$$

(12)

hold by the definition of the eigenvector. Since $L(C)$ is a symmetric and positive semidefinite matrix, it can be diagonalized as

$$\Lambda_C = U_C^T L(C) U_C$$

$$= \text{diag}(\lambda_1(L(C)), \lambda_2(L(C)), \ldots, \lambda_{n_C}(L(C)))$$

by an orthogonal matrix $U_C$, where

$$U_C = \begin{bmatrix} u_1 & u_2 & \cdots & u_{n_C} \end{bmatrix}, \quad U_C U_C^T = I_{n_C}.$$  

It should be noticed that $u_2$ of this $U_C$ is in fact $u_2$ of (12). Then we see that

$$U_C^T \otimes I_{n_S})L(G^*)(U_C \otimes I_{n_S})$$

$$= I_{n_C} \otimes L(S^*) + \sum_{k=1}^{n_S}(\Lambda_C \otimes M(n_S, K))$$

$$= blkdiag(Z_1, Z_2, \ldots, Z_{n_C})$$

by the property of the Kronecker product, where these $Z_i$, $i = 1, 2, \ldots, n_C$, have been defined in (6). Thus there exist the eigenvectors $\xi_1$ and $\xi_2$ for the eigenvalues $\lambda_2(Z_1)$ and $\lambda_1(Z_2)$ which have also been defined in (10) and (11).
Now we consider two cases. Case 1) \( \lambda_1(Z_2) < \lambda_2(Z_2) \). In this case, we see that

\[
\lambda_2(L(G^*)) = \lambda_1(Z_2) \tag{13}
\]

from Lemma 1. Furthermore, the eigenvector \( \xi_{G_2} \) for the eigenvalue \( \lambda_2(L(G^*)) \) of \( L(G^*) \) is represented as

\[
\xi_{G_2} = (U_C \otimes I_u)(e(n_C, 2) \otimes \xi_2)
\]

\[
= u_2 \otimes \xi_2. \tag{14}
\]

In fact, (11) and (12) imply that \( \xi_{G_2} \neq 0 \), and the equation

\[
[L(G^*) - \lambda_1(Z_2)I_u]\xi_{G_2} = 0
\]

holds from the identity

\[(\text{blkdiag}(Z_1, Z_2, \ldots, Z_m) - \lambda_1(Z_2)I_u)(e(n_C, 2) \otimes \xi_2) = 0,\]

where (11) is employed.

(Sufficiency) When the condition (8) holds, we see

\[
\xi_1 = (U_C \otimes I_u)(e(n_C, 1) \otimes \xi_1) = u_1 \otimes \xi_1.
\]

Thus we have

\[
\lambda_1(L(G^*)) = \lambda_2(L(G^*)^T).
\]

Thus we obtain

\[
\lambda_2(L(G^*))v_2^T v_2 = v_2^T L(G^*)v_2
\]

\[
= v_2^T \underbrace{[L(G) - L(G^*)]}_{\lambda_2(L(G^*))}v_2 + v_2^T L(G^*)v_2
\]

\[
= \lambda_2(L(G^*))v_2^T v_2, \tag{16}
\]

where \( v_2 \) is the eigenvector corresponding to \( \lambda_2(L(G^*)) \), i.e.,

\[
[L(G) - \lambda_2(L(G^*))I_u]v_2 = 0, \quad v_2 \neq 0.
\]

Since \( \mathbf{I}_u \) is the eigenvector corresponding to \( \lambda_1(L(G)) \) of \( L(G) \), we see that \( \mathbf{I}_u^2 v_2 = 0 \). This \( \mathbf{I}_u \) is also the eigenvector corresponding to \( \lambda_1(L(G^*)) \) of \( L(G^*) \). Thus we have

\[
\lambda_2(L(G^*))v_2^T v_2 \leq v_2^T L(G^*)v_2, \tag{17}
\]

because the algebraic connectivity can be characterized by

\[
\lambda_2(L(G^*)) = \min_{\sum y \neq 0} \frac{\sum \lambda_i L(G^*)^T y}{\sum y^T y},
\]

which is a consequence of the well-known Courant-Fischer principle [4]. These relations (16) and (17) together with \( L(G) - L(G^*) \geq 0 \) from (7) imply that

\[
v_2^T [L(G) - L(G^*)]v_2 = 0,
\]

that is,

\[
\lambda_2(L(G^*))v_2^T v_2 = v_2^T L(G^*)v_2,
\]

which means that \( v_2 \) is an eigenvector corresponding to \( \lambda_2(L(G^*)) \) of \( L(G^*) \). Since \( \lambda_2(L(G^*)) \) is distinct from the assumption of the theorem, such an eigenvector can be represented as \( \xi_{G_2} \) of (14). Thus we see that (15) holds as well.

Since the condition (15) holds, it turns out that (8) should hold. The necessity for Case 1 is then proved.

Case 2) \( \lambda_1(Z_2) > \lambda_2(Z_2) \). In this case, we see that

\[
\lambda_2(L(G^*)) = \lambda_2(Z_1)
\]

from Lemma 1. Furthermore, the eigenvector \( \xi_{G_1} \) for the eigenvalue \( \lambda_2(L(G^*)) \) of \( L(G^*) \) is represented as

\[
\xi_{G_1} = (U_C \otimes I_u)(e(n_C, 1) \otimes \xi_1)
\]

\[
= u_1 \otimes \xi_1.
\]

In fact, the condition \( \xi_1 \neq 0 \) in (10) together with the fact that all of elements of \( u_1 \) are identical and nonzero implies that \( \xi_{G_1} \neq 0 \), and the equation

\[
[L(G^*) - \lambda_2(Z_1)I_u]\xi_{G_1} = 0
\]

holds from the identity

\[(\text{blkdiag}(Z_1, Z_2, \ldots, Z_m) - \lambda_2(Z_1)I_u)(e(n_C, 1) \otimes \xi_1) = 0,\]

where (10) is employed.

The rest of the proof of Case 2 is parallel to that of Case 1. In fact, replacing \( \lambda_1(Z_2) \) and \( \xi_{G_2} \) with \( \lambda_2(Z_1) \) and \( \xi_{G_1} \), we have the proof, where the condition (8) is simplified to the condition (9) because \( e(n_C, 1)^T u_1 \) is always not zero.

\[\square\]

5. Numerical Examples

We consider the graphs in Fig. 1. The algebraic connectivity of each graph is given by

\[
\lambda_2(L(S^{(0)})) = 0.4384, \quad \lambda_2(L(S^{(1)})) = 0.4384,
\]

\[
\lambda_2(L(S^{(2)})) = 1.4384, \quad \lambda_2(L(S^{(3)})) = 0.6571,
\]

\[
\lambda_2(L(S^{(4)})) = 1.2679, \quad \lambda_2(L(C)) = 1.0000.
\]

The unit eigenvectors \( u_1, u_2, u_3, \) and \( u_4 \) corresponding to \( \lambda_1(L(C)), \lambda_2(L(C)), \lambda_3(L(C)), \) and \( \lambda_4(L(C)) \) are given by

\[
U_C = [u_1 \quad u_2 \quad u_3 \quad u_4]
\]

\[
= \begin{bmatrix}
0.5000 & 0.4082 & 0.7071 & -0.2887 \\
0.5000 & 0.4082 & -0.7071 & -0.2887 \\
0.5000 & 0.0000 & 0.0000 & 0.8660 \\
0.5000 & -0.8165 & 0.0000 & -0.2887
\end{bmatrix}. \tag{18}
\]

Since the graphs \( S^{(1)}, S^{(2)}, S^{(3)}, \) and \( S^{(4)} \) include the graph \( S^{(0)} \),

\[
(L(S^{(1)}) - L(S^{(0)})) \geq 0, \quad (L(S^{(2)}) - L(S^{(0)})) \geq 0,
\]

\[
(L(S^{(3)}) - L(S^{(0)})) \geq 0, \quad (L(S^{(4)}) - L(S^{(0)})) \geq 0
\]
hold. In Fig. 2, we show how we simplify the graph with a hierarchical structure. In Fig. 2 (a), vertices 2 and 8 of each subsystem form an interconnection with other subsystems, respectively. We simplify Fig. 2 (a) to Fig. 2 (b) by simplifying each subsystem and writing the set $K$ of vertices which forms an interconnection in the figure. In this way, a graph in Fig. 2 (a) with $K = \{2, 8\}$ is the same with the graph in Fig. 2 (b).

Now we consider the following two cases of examples.

5.1 The Case of $\lambda_1(Z_2) < \lambda_2(Z_1)$

When $K$ is equal to $K_1 = \{2, 3, 5, 7\}$, we have

$$\lambda_2(L(G^4)) = 0.4298,$$

where we also have

$$(L(S^{(0)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(1)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(2)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(3)}) - L(S^{(0)}))e_0 \neq 0, \\ (L(S^{(4)}) - L(S^{(0)}))e_0 \neq 0.$$}

Then Theorem 1 says that

$$\lambda_2(L(G)) = \lambda_1(Z_2) = 0.4384$$

for a graph $G$ which is constructed by attaching the vertex $k$ of undirected graphs $S_i$ ($i = 1, 2, \ldots, n_C$) to the vertex $i$ of a graph $C$ respectively for all $k \in K_1$, where $S_i$ can be selected from $S^{(0)}$, $S^{(1)}$, or $S^{(2)}$. The graphs $S^{(3)}$ or $S^{(4)}$ can also be selected as subsystems since $e(n_C, 3)^T u_2 = 0$ in $U_C$ of (18).

When we computed the algebraic connectivity of the graphs in Figs. 3 and 4 which satisfy the condition (8) of Case 1 in Theorem 1, we have

$$\lambda_2(L(G^4)) = 0.4298, \quad \lambda_2(L(G^{(1)})) = 0.4298, \quad \lambda_2(L(G^{(2)})) = 0.4298, \quad \lambda_2(L(G^{(3)})) = 0.4298.$$

That is,

$$(L(S^{(0)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(1)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(2)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(3)}) - L(S^{(0)}))e_0 \neq 0, \\ (L(S^{(4)}) - L(S^{(0)}))e_0 \neq 0.$$}

We remark that the graph $G^{(4)}$ is a graph obtained by exchanging $S^{(3)}$ and $S^{(2)}$ in the graph $G^{(3)}$. Since we have $e(n_C, 3)^T u_2 = 0, G^{(2)}$ satisfies the condition in Theorem 1, while $G^{(4)}$ does not.

5.2 The Case of $\lambda_1(Z_2) > \lambda_2(Z_1)$

When $K$ is equal to $K_2 = \{1, 2, 3, 5, 7, 8\}$, we have

$$\lambda_2(L(G)) = 0.4384$$

where we also have

$$(L(S^{(0)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(1)}) - L(S^{(0)}))e_0 = 0, \\ (L(S^{(2)}) - L(S^{(0)}))e_0 \neq 0, \\ (L(S^{(3)}) - L(S^{(0)}))e_0 \neq 0, \\ (L(S^{(4)}) - L(S^{(0)}))e_0 \neq 0.$$}

We remark that graphs which satisfy the condition (9) of Case 2 are not the same with Case 1, which results from changing $K$ from $K_1$ to $K_2$. Theorem 1 says that

$$\lambda_2(L(G)) = \lambda_2(Z_1) = 0.4384.$$
for a graph $G$ which is constructed by attaching the vertex $k$ of undirected graphs $S_i$ ($i = 1, 2, \ldots, n_C$) to the vertex $i$ of a graph $C$ respectively for all $k \in K_2$, where $S_i$ can be selected from $S(0)$ or $S(1)$.

When we computed the algebraic connectivity of the graphs in Fig. 3 which satisfy the condition (9) of Case 2 in Theorem 1, we have

$$\lambda_2(L(G')) = 0.4384, \quad \lambda_2(L(G^{(1)})) = 0.4384.$$  

That is,

$$\lambda_2(L(G')) = \lambda_2(L(G^{(1)}))$$

actually holds in this case.

On the other hand, when systems do not satisfy the condition (9) of Case 2 in Theorem 1 like those in Fig. 4 and Fig. 5, we obtain

$$\lambda_2(L(G^{(2)})) = 0.6277, \quad \lambda_2(L(G^{(3)})) = 0.6277,$$

$$\lambda_2(L(G^{(4)})) = 0.6630, \quad \lambda_2(L(G^{(5)})) = 0.6762.$$  

That is, the equation $\lambda_2(L(G)) = \lambda_2(Z_4)$ cannot hold.

6. Concluding Remarks

In this paper, we proposed the algebraic connectivity invariance of a class of networks-of-networks derived from a graph product, where the network has a heterogeneous substructure. Each subsystem has the same number of agents, while their structure can differ. In each subsystem, agents connected to the entire system, that is, agents interact with other subsystems. Theorem 1, that is, the main result of this paper, clarified the necessary and sufficient condition for such systems to have the same algebraic connectivity with a certain graph with homogeneous subsystems. This condition is given by the eigenstructure of $Z_4$, which provides a new insight into systems theory of networks-of-networks. Then the main result was demonstrated through numerical examples.

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