Multiple commutation relations in the models with $\mathfrak{gl}(2|1)$ symmetry

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Abstract

We consider quantum integrable models with $\mathfrak{gl}(2|1)$ symmetry. We derive a set of multiple commutation relations between the monodromy matrix entries. These multiple commutation relations allow us to obtain different representations for Bethe vectors.

Key words: Bethe ansatz, monodromy matrix, commutation relations.

1 Introduction

The algebraic Bethe ansatz is an efficient method for finding the spectra of quantum Hamiltonians [1–3]. It is also known that this method is well suited for calculating correlation functions of quantum models [4–7]. The latest problem, within the framework of the algebraic Bethe ansatz reduces to the calculating scalar products of Bethe vectors. In their turn, the scalar products mentioned above appear to be a particular case of multiple commutation relations (MCR) of the monodromy matrix entries. The subject of this paper are MCR in the $\mathfrak{gl}(2|1)$-based models.

In the models solvable by the algebraic Bethe ansatz, the determining is an $RRT$-relation

$$R(u,v) \cdot (T(u) \otimes I) \cdot (I \otimes T(v)) = (I \otimes T(v)) \cdot (T(u) \otimes I) \cdot R(u,v).$$

(1.1)

Here $R(u,v)$ is an $R$-matrix acting in the tensor product $V \otimes V$ of an auxiliary vector spaces $V$. The monodromy matrix $T(u)$ acts in $V \otimes \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of a quantum model. Equation (1.1) holds in the tensor product $V \otimes V \otimes \mathcal{H}$. It sets the commutation relations between the matrix elements $T_{ij}(u)$ of the monodromy matrix $T(u)$. At the same time, to calculate scalar products and form factors one should know the MCR of the type

$$[T_{ij}(u_1) \ldots T_{ij}(u_n), T_{kl}(v_1) \ldots T_{kl}(v_m)].$$

(1.2)
Formally, knowing \([T_{ij}(u), T_{kl}(v)]\) one can calculate the commutator in (1.2). However, a straightforward use of the RTT-relation in this case, leads to extremely complex expressions that hardly can be used for further analysis.

The MCR in the \(\mathfrak{gl}(2)\)-invariant models and their \(q\)-deformations are well known.\textsuperscript{[5, 8, 9]} Some generalizations to the models with \(\mathfrak{gl}(3)\)-invariant \(R\)-matrix were considered in\textsuperscript{10}. In the present paper we deal with the models based on the superalgebra \(\mathfrak{gl}(2|1)\). Among the models described by this algebra, the most famous is the supersymmetric t-J model, which plays an important role in the condensed matter physics \textsuperscript{[11, 12]}. The algebraic structure of this model as well as the application of the algebraic Bethe ansatz were considered in\textsuperscript{[13–18]}.

As explained above, the MCR are needed for calculating the scalar products of Bethe vectors. However, in models with high rank symmetry, to construct the Bethe vectors is already a non-trivial task. This problem was considered by different approaches\textsuperscript{[10, 19–23]}. A generalization of a trace formula for the Bethe vectors\textsuperscript{23} for the case of superalgebras \(\mathfrak{gl}(m|n)\) was obtained in\textsuperscript{24}. Recently, explicit expressions for the Bethe vectors in the models with \(\mathfrak{gl}(2|1)\) and \(\mathfrak{gl}(1|2)\) symmetries were found in\textsuperscript{25}. In this paper we use a method of MCR and prove the equivalence of different representations for the \(\mathfrak{gl}(2|1)\) Bethe vectors. We also obtain new formulas for them.

The article is organized as follows. In section 2 we introduce the model under consideration and describe necessary notation. Section 3 is devoted to the most simple MCR. In section 4 we formulate more complex MCR, which allow us to prove different representations for the Bethe vectors in section 5. In sections 6, 7 we prove the MCR formulated in section 4. In appendices A and B we gather some identities necessary for the proof.

2 Basic notions and notation

For \(\mathfrak{gl}(2|1)\)-based models an auxiliary vector space \(V\) is a \(\mathbb{Z}_2\)-graded space \(C^{2|1}\) with a basis \(\{e_1, e_2, e_3\}\). We call the vectors \(\{e_1, e_2\}\) even, while \(e_3\) is odd. Respectively, we introduce a party function on the set of indices as \([1] = [2] = 0\) and \([3] = 1\).

The \(R\)-matrix in (1.1) has the form

\[
R(u, v) = I \otimes I + g(u, v)P \quad g(u, v) = \frac{c}{u - v},
\]

(2.1)

where \(c\) is a constant and \(P\) is a graded permutation matrix \(\textsuperscript{26}\). The tensor product in (1.1) also is graded leading to the set of commutation relations between the monodromy matrix entries \(T_{ij}\):

\[
[T_{ij}(u), T_{kl}(v)] = (-1)^{i(j|k) + (j|l) + [k][l]} g(u, v) \left( T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v) \right),
\]

(2.2)

where we have introduced a graded commutator as

\[
[T_{ij}(u), T_{kl}(v)] = T_{ij}(u)T_{kl}(v) - (-1)^{[i] + [j]}(k + [l])T_{kl}(v)T_{ij}(u).
\]

(2.3)

Relabeling in (2.2) the subscripts \(i \leftrightarrow k, j \leftrightarrow l\), and replacing \(u \leftrightarrow v\) we obtain one more commutation relation

\[
[T_{ij}(u), T_{kl}(v)] = (-1)^{[i](j|l) + [j][l]} g(u, v) \left( T_{il}(u)T_{kj}(v) - T_{il}(v)T_{kj}(u) \right).
\]

(2.4)
Relations (2.2), (2.4) are the starting point of our study.

Let us describe now the notation used below. We mostly use the same notation as we did in our papers concerning the models with $gl(3)$ symmetry (see e.g. [10]). Apart from the function $g(x, y)$ we also introduce two functions

$$f(x, y) = 1 + g(x, y) = \frac{x - y + c}{x - y}, \quad h(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{x - y + c}{c}. \quad (2.5)$$

The functions introduced above have the following obvious properties:

$$g(x, y) = -g(y, x), \quad h(x, y) = \frac{1}{g(x, y - c)}, \quad f(x - c, y) = \frac{1}{f(y, x)}. \quad (2.6)$$

Extending the $\mathbb{Z}_2$-grading to the operators $T_{ij}$ by

$$[T_{ij}(u)] = [i] + [j], \quad \text{mod } 2, \quad (2.7)$$

we distinguish between even operators (i.e. $[T_{ij}(u)] = 0$) and odd operators (i.e. $[T_{ij}(u)] = 1$). It follows from the commutation relations (2.2), (2.4) that any even operator $T_{ij}(u)$ commutes with itself for arbitrary values of the argument $u$. On the other hand, one has for the odd operators $T_{ij}(u)$

$$h(v_1, v_2)T_{j3}(v_1)T_{j3}(v_2) = h(v_2, v_1)T_{j3}(v_1)T_{j3}(v_2), \quad h(v_2, v_1)T_{3j}(v_1)T_{3j}(v_2) = h(v_1, v_2)T_{3j}(v_2)T_{3j}(v_1), \quad j = 1, 2. \quad (2.8)$$

Therefore it is convenient to introduce symmetric products of the odd operators.

**Definition 2.1.** Let $\bar{v} = \{v_1, \ldots, v_n\}$. Define

$$T_{j3}(\bar{v}) = \frac{T_{j3}(v_1) \cdots T_{j3}(v_n)}{\prod_{n \geq \ell > m \geq 1} h(v_\ell, v_m)}, \quad j = 1, 2, \quad (2.9)$$

and

$$T_{3k}(\bar{v}) = \frac{T_{3k}(v_1) \cdots T_{3k}(v_n)}{\prod_{n \geq \ell > m \geq 1} h(v_m, v_\ell)}, \quad k = 1, 2. \quad (2.10)$$

Obviously, the operator products introduced above are symmetric over the parameters $\bar{v} = \{v_1, \ldots, v_n\}$.

Let us formulate a convention on the notation. We denote sets of variables by bar: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc. For the sake of generality, we will sometimes use such the notation, even in cases where the set consists of only one element. Individual elements of the sets are denoted by Latin subscripts: $w_j$, $u_k$ etc. The subsets complementary to the individual elements are denoted by $\bar{u}_k$: $\bar{u}_k = \bar{u} \setminus u_k$. As a rule, the number of elements in the sets is not shown explicitly in the equations, however we give these cardinalities in special comments to the formulas. The notation $\bar{u} \pm c$ means that all the elements of the set $\bar{u}$ are shifted by $\pm c$: $\bar{u} \pm c = \{u_1 \pm c, \ldots, u_n \pm c\}$.

Subsets of variables (including subsets consisting of one element) are labeled by Greek subscripts: $\bar{u}_\alpha$, $\bar{v}_\sigma$, $\bar{w}_\beta$ etc. The union of the sets is denoted by braces: $\bar{w} = \{\bar{u}, \bar{v}\}$. A subset complementary to the set $\bar{u}_\alpha$ is denoted by $\bar{u}_\bar{\alpha}$: $\bar{u}_\bar{\alpha} = \bar{u} \setminus \bar{u}_\alpha$. The notation $\bar{u} \Rightarrow \{\bar{u}_\alpha, \bar{u}_\bar{\alpha}\}$ means
that the set \( \bar{u} \) is divided into two subsets \( \bar{u}_a \) and \( \bar{u}_\bar{a} \). Similarly, the notation \( \bar{v} \Rightarrow \{ \bar{v}_\alpha, \bar{v}_\beta, \bar{v}_\gamma \} \) means that the set \( \bar{v} \) is divided into three disjoint subsets. Hereby, \( \{ \bar{v}_\alpha, \bar{v}_\beta \} = \bar{v}_\gamma \), \( \{ \bar{v}_\alpha, \bar{v}_\gamma \} = \bar{v}_\beta \), and \( \{ \bar{v}_\gamma, \bar{v}_\beta \} = \bar{v}_\alpha \). We assume that the elements in every subset of variables are ordered in such a way that the sequence of their subscripts is strictly increasing. We call this ordering natural order.

In order to avoid too cumbersome formulas we use shorthand notation for products of commuting operators. Similarly to (2.9), (2.10) we introduce

\[
T_{ij}(\bar{u}) = \prod_{u_k \in \bar{u}} T_{ij}(u_k), \quad \text{for } [i] + [j] = 0, \mod 2. \tag{2.11}
\]

One should follow the same prescription for the products of the functions \( g, f, h \). Namely, if such the function depends on a set of variables, this means that one should take the product over the corresponding set. For example,

\[
h(v, \bar{u}) = \prod_{u_j \in \bar{u}} h(v, u_j); \quad f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k); \quad g(\bar{v}_\alpha, \bar{v}_\bar{a}) = \prod_{v_j \in \bar{v}_\alpha} \prod_{v_k \in \bar{v}_\bar{a}} g(v_j, v_k). \tag{2.12}
\]

To conclude this section we introduce a partition function of the six-vertex model with domain wall boundary conditions (DWPF) \( K(\bar{u}|\bar{v}) \). This function plays an important role in the MCR. It depends on two sets of variables \( \bar{u} \) and \( \bar{v} \), such that \( \# \bar{u} = \# \bar{v} \). By definition \( K(\emptyset|\emptyset) = 1 \). Otherwise, if \( \# \bar{u} = \# \bar{v} = n, n > 0 \), then the function \( K(\bar{u}|\bar{v}) \) has the following determinant representation:

\[
K(\bar{u}|\bar{v}) = \Delta'(\bar{u})\Delta(\bar{v})h(\bar{u}, \bar{v})\det_n \left( \frac{g(u_j, v_k)}{h(u_j, v_k)} \right), \tag{3.13}
\]

where \( \Delta'(\bar{u}) \) and \( \Delta(\bar{v}) \) are

\[
\Delta'(\bar{u}) = \prod_{1 \leq j < k \leq n} g(u_j, u_k), \quad \Delta(\bar{v}) = \prod_{n \geq k \geq 1} g(v_j, v_k). \tag{3.14}
\]

It is easy to see that \( K(\bar{u}|\bar{v}) \) is symmetric over \( \bar{u} \) and symmetric over \( \bar{v} \), however \( K(\bar{u}|\bar{v}) \neq K(\bar{v}|\bar{u}) \). Some useful properties of the DWPF are gathered in appendix A.

## 3 MCR of operators belonging to the same row or column

In this section we consider relatively simple MCR. They occur if the operators \( T_{ij} \) and \( T_{kl} \) belong to the same row or column of the monodromy matrix, i.e. either \( i = k \) or \( j = l \).

**Proposition 3.1.** Let \( \# \bar{u} = n, \# \bar{v} = m \), and \( \{ \bar{u}, \bar{v} \} = \bar{w} \). Then for \( i, j, k < 3 \)

\[
T_{ij}(\bar{u})T_{ik}(\bar{v}) = (-1)^n \sum K(\bar{w}_\alpha|\bar{u} + c)f(\bar{w}_\alpha, \bar{w}_\alpha)T_{ik}(\bar{w}_\alpha)T_{ij}(\bar{w}_\alpha). \tag{3.1}
\]

For \( i, j < 3 \)

\[
T_{i3}(\bar{u})T_{j3}(\bar{v}) = (-1)^n h(\bar{v}, \bar{u}) \sum K(\bar{u}|\bar{w}_\alpha + c)g(\bar{w}_\alpha, \bar{w}_\alpha)T_{j3}(\bar{w}_\alpha)T_{i3}(\bar{w}_\alpha), \tag{3.2}
\]


\( T_{ij}(\bar{\mu})T_{i3}(\bar{v}) = \sum h(\bar{\mu}_{\bar{\alpha}}, \bar{u})g(\bar{\alpha}_{\bar{\alpha}}, \bar{\mu}_{\bar{\alpha}})T_{i3}(\bar{\mu}_{\bar{\alpha}})T_{ij}(\bar{\alpha}_{\bar{\alpha}}), \) \hfill (3.3)

\[ \sum h(\bar{\mu}, \bar{\alpha}_{\bar{\alpha}})g(\bar{\alpha}_{\bar{\alpha}}, \bar{\mu})T_{ij}(\bar{\alpha}_{\bar{\alpha}})T_{3i}(\bar{\mu}). \] \hfill (3.4)

For \( i < 3 \)

\[ T_{33}(\bar{\mu})T_{3i}(\bar{v}) = \sum h(\bar{\mu}, \bar{\alpha}_{\bar{\alpha}})g(\bar{\alpha}_{\bar{\alpha}}, \bar{\mu}_{\bar{\alpha}})T_{3i}(\bar{\alpha}_{\bar{\alpha}})T_{33}(\bar{\alpha}_{\bar{\alpha}}). \] \hfill (3.5)

\[ T_{3i}(\bar{\mu})T_{33}(\bar{v}) = \sum h(\bar{\alpha}_{\bar{\alpha}}, \bar{v})g(\bar{\alpha}_{\bar{\alpha}}, \bar{\mu}_{\bar{\alpha}})T_{33}(\bar{\alpha}_{\bar{\alpha}})T_{3i}(\bar{\alpha}_{\bar{\alpha}}), \] \hfill (3.6)

All the sums are taken over partitions \( \bar{\alpha} \Rightarrow \{ \bar{\alpha}_{\bar{\alpha}} \} \) with \( \#\bar{\alpha} = n \) and \( \#\bar{\alpha} = m \).

Proposition 5.3 gives us the MCR for the operators belonging to the same row of the monodromy matrix. In order to obtain the commutation relations for the operators from the same column it is enough to apply antimorphism of the algebra \( 11 \) [25].

\[ \psi(T_{ij}(u)) = (-1)^{|i||j|+|i|}T_{ij}(u), \quad \psi(T_{ij}(u)T_{kl}(v)) = (-1)^{|T_{ij}|T_{kl}}\psi(T_{kl}(v))\psi(T_{ij}(u)). \] \hfill (3.7)

Clearly, that after the action of \( \psi \) onto equations (3.1)–(3.6) we obtain MCR for the operators belonging to the same column of the monodromy matrix.

The proofs of (3.1)–(3.5) are similar one to each other. They use induction over the numbers of the operators and summation lemmas B.1, B.2. The proof of (3.1) (based on lemma B.2) was given in [10]. In order to show how lemma B.1 works we consider the proof of (3.3).

It follows from the commutation relations (2.4) that

\[ T_{ij}(u)T_{i3}(v) = f(v, u)T_{i3}(v)T_{ij}(u) + g(u, v)T_{i3}(u)T_{ij}(v). \] \hfill (3.8)

It is easy to see that this equation coincides with (3.3) for \( n = m = 1 \). Then one can use double induction over \( n \) and \( m \). However, for \( n = 1 \) or \( m = 1 \) it is better to apply the standard scheme of the algebraic Bethe ansatz. Let for definiteness \( n = 1 \) and \( m > 1 \). Then moving the operator \( T_{ij}(u) \) through the product \( T_{i3}(\bar{v}) \) we obtain the following result

\[ T_{ij}(u)T_{i3}(\bar{v}) = \Lambda T_{i3}(\bar{v})T_{ij}(u) + \sum_{k=1}^{m} \Lambda_k T_{i3}(\{ \bar{v}_k, u \})T_{ij}(v_k). \] \hfill (3.9)

Here \( \Lambda \) and \( \Lambda_k \) are some rational coefficients to be determined, and we recall that \( \bar{v}_k = \bar{v} \setminus v_k \).

Obviously, in order to obtain the term proportional to \( \Lambda \), the operator \( T_{ij}(u) \) should go to the right preserving its original argument. This immediately gives

\[ \Lambda = f(\bar{v}, u). \] \hfill (3.10)

Let us find now \( \Lambda_1 \). The product \( T_{i3}(\{ \bar{v}_1, u \})T_{ij}(v_1) \) arises if and only if the operators \( T_{ij}(u) \) and \( T_{i3}(v_1) \) exchange their arguments:

\[ T_{ij}(u)T_{i3}(\bar{v}) = T_{ij}(u)T_{i3}(v_1)\frac{T_{i3}(\bar{v}_1)}{h(\bar{v}_1, v_1)} = g(u, v_1)T_{i3}(u)T_{ij}(v_1)\frac{T_{i3}(\bar{v}_1)}{h(\bar{v}_1, v_1)} + UWT, \] \hfill (3.11)

where \( UWT \) means unwanted terms. Generically, we call a term unwanted if it does not contribute to the desirable coefficient. In the case under consideration, the term proportional to
Comparing equations (3.15) and (3.5) at $T_g$ cannot obtain $u$ in the denominator all the terms of the sum vanish as soon as $u$ in $\bar{v}_1$. It remains to transform the product $T_{i3}(u)T_{i3}(\bar{v}_1)$ into $T_{i3}(\{\bar{v}_1, u\})$, and we finally obtain

$$
\Lambda_1 = g(u, v_1)g(\bar{v}_1, v_1)h(\bar{v}_1, u).
$$

Due to the symmetry of $T_{i3}(\bar{v})$ over $\bar{v}$ we conclude that

$$
\Lambda_k = g(u, v_k)g(\bar{v}_k, v_k)h(\bar{v}_k, u).
$$

Thus,

$$
T_{ij}(u)T_{i3}(\bar{v}) = f(\bar{v}, u)T_{i3}(\bar{v})T_{ij}(u) + \sum_{k=1}^{m} g(u, v_k)g(\bar{v}_k, v_k)h(\bar{v}_k, u)T_{i3}(\{\bar{v}_k, u\})T_{ij}(v_k).
$$

Comparing equations (3.15) and (3.5) at $n = 1$ we see that the first term in (3.15) corresponds to the partition $\check{w}_\alpha = u$ and $\check{w}_\sigma = \bar{v}$. The other terms in (3.15) appear if we set $\check{w}_\alpha = v_k$ and $\check{w}_\sigma = \{\bar{v}_k, u\}$. Thus, (3.5) is proved for $n = 1$ and $m$ arbitrary.

Generalization for $n > 1$ can be done via induction over $n$. Suppose that (3.5) holds for some $n - 1$. Then acting successively with $T_{ij}(\bar{u}_n)$ and $T_{ij}(u_n)$ on the product $T_{i3}(\bar{v})$ we obtain

$$
T_{ij}(u_n)T_{ij}(\bar{u}_n)T_{i3}(\bar{v}) = \sum h(\bar{w}_\alpha, \bar{u}_n)g(\bar{w}_\alpha, \check{w}_\alpha)h(\bar{w}_\sigma, u_n)g(\bar{w}_\sigma, \check{w}_\sigma)T_{i3}(\bar{w}_\alpha)\bar{T}_{ij}(\check{w}_\alpha)\bar{T}_{ij}(\bar{w}_\alpha).
$$

Here we first have partitions of the set $\{\bar{u}_n, \bar{v}\}$ into subsets $\check{w}_\alpha$ and $\check{w}_\sigma$ with $\#\check{w}_\alpha = n - 1$ and $\#\check{w}_\alpha = m$. Then we combine $u_n$ and $\check{w}_\alpha$ and divide this set into subsets $\check{w}_\rho$ and $\check{w}_\sigma$ with $\#\check{w}_\rho = 1$ and $\#\check{w}_\sigma = m$. As a result we have divided the set $\check{w} = \{\bar{u}, \bar{v}\}$ into three subsets $\{\check{w}_\alpha, \check{w}_\rho, \check{w}_\sigma\}$ in such a way that $u_n \notin \check{w}_\alpha$. Substituting $\check{w}_\alpha = \{\check{w}_\rho, \check{w}_\sigma\} \setminus u_n$ into (3.16) we arrive at

$$
T_{ij}(\bar{u})T_{i3}(\bar{v}) = \sum h(\bar{w}_\rho, \bar{u}_n)h(\bar{w}_\sigma, \bar{u}_n)g(\bar{w}_\rho, \check{w}_\alpha)g(\bar{w}_\sigma, \check{w}_\alpha)
\frac{h(\check{u}_n, \check{w}_\alpha)g(\check{u}_n, \check{w}_\alpha)}{h(u_n, \check{w}_\alpha)g(u_n, \check{w}_\alpha)}
\times h(\bar{w}_\sigma, u_n)g(\bar{w}_\sigma, \check{w}_\sigma)T_{i3}(\check{w}_\sigma)\bar{T}_{ij}(\check{w}_\sigma)\bar{T}_{ij}(\check{w}_\sigma).
$$

Here the sum is taken over partitions of the set $\{\bar{u}, \bar{v}\}$ into subsets $\{\check{w}_\alpha, \check{w}_\rho, \check{w}_\sigma\}$. Observe that we have got rid of the restriction $u_n \notin \check{w}_\alpha$. Indeed, due to the factor $g(u_n, \check{w}_\alpha)$ in the denominator all the terms of the sum vanish as soon as $u_n \in \check{w}_\alpha$.

Let us set $\check{w}_\sigma = \bar{w}_\mu$, and (3.17) turns into

$$
T_{ij}(\bar{u})T_{i3}(\bar{v}) = \sum h(\bar{w}_\mu, \check{u}_n)h(\bar{w}_\mu, \bar{u}_n)g(\bar{w}_\mu, \bar{w}_\mu)g(\bar{w}_\mu, \bar{w}_\mu)
\frac{h(\check{u}_n, \check{w}_\mu)g(\check{u}_n, \check{w}_\mu)}{h(u_n, \check{w}_\mu)g(u_n, \check{w}_\mu)}
\times g(\bar{w}_\mu, \check{w}_\mu)\frac{g(u_n, \check{w}_\mu)}{h(\check{w}_\mu, u_n)}.
$$
One can say that here we first divide the set \( \{ \bar{u}, \bar{v} \} \) into subsets \( \{ \bar{w}_\mu, \bar{w}_\rho \} \), and then divide the subset \( \bar{w}_\mu \) into \( \bar{w}_a \) and \( \bar{w}_\rho \). The sum over partitions \( \bar{w}_\mu \to \{ \bar{w}_a, \bar{w}_\rho \} \) (see the terms in the parenthesis in (3.18)) can be computed explicitly via lemma B.1

\[
\sum g(\bar{w}_\rho, \bar{w}_a) g(u_n, \bar{w}_\rho) \frac{g(u_n, \bar{w}_\rho)}{h(\bar{w}_\rho, \bar{u}_n)} = -\sum g(\bar{w}_\rho, \bar{w}_a) g(u_n, \bar{w}_\rho) g(\bar{w}_a, \bar{u}_n - c) = \frac{h(u_n, \bar{u}_n) g(u_n, \bar{w}_\mu)}{h(\bar{w}_\mu, \bar{u}_n)}. \tag{3.19}
\]

Substituting this into (3.18) we finally arrive at

\[
T_{ij}(\bar{u})T_{i3}(\bar{v}) = \sum h(\bar{w}_\rho, \bar{u}) g(\bar{w}_\rho, \bar{w}_\rho) T_{i3}(\bar{w}_\rho) T_{ij}(\bar{w}_\mu), \tag{3.20}
\]

what coincides with (3.18) at \#\bar{u} = n up to the labels of the subsets.

Other equations of proposition 3.1 can be proved in the similar manner.

4 MCR of operators belonging to different rows and columns

MCR of the operators belonging to the different rows and columns of the monodromy matrix are much more sophisticated than the ones considered above. We will consider only one specific example of these MCR. This example is important, because it allows one to obtain different representations for the Bethe vectors.

**Definition 4.1.** For \#\bar{u} = a and \#\bar{v} = b define two operators

\[
X_{a,b}(\bar{u}, \bar{v}) = \sum g(\bar{v}_\alpha, \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) h(\bar{u}_\alpha, \bar{u}_\alpha) T_{13}(\bar{u}_\alpha) T_{12}(\bar{u}_\alpha) T_{23}(\bar{v}_\alpha) T_{22}(\bar{v}_\alpha), \tag{4.1}
\]

and

\[
Y_{a,b}(\bar{u}, \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) h(\bar{u}_\alpha, \bar{u}_\alpha) T_{13}(\bar{v}_\alpha) T_{12}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{22}(\bar{v}_\alpha). \tag{4.2}
\]

In both equations the sum is taken over partitions \( \bar{v} \to \{ \bar{v}_\alpha, \bar{v}_\alpha \} \) and \( \bar{u} \to \{ \bar{u}_\alpha, \bar{u}_\alpha \} \) with the restriction \#\bar{u}_\alpha = \#\bar{v}_\alpha = n, \text{ where } n = 0, 1, \ldots, \min(a, b) \).

The operator (4.1) was considered in [25]. There it was proved that it possesses the following recursion:

\[
X_{a,b}(\bar{u}, \bar{v}) = T_{12}(u_a) X_{a-1,b}(\bar{u}_a, \bar{v}) + \sum g(\bar{v}_\rho, u_a) f(\bar{v}_\rho, \bar{u}_a) g(\bar{v}_\rho, \bar{v}_\rho) T_{13}(u_a) X_{a-1,b-1}(\bar{u}_a, \bar{v}_\rho) T_{22}(\bar{v}_\rho). \tag{4.3}
\]

Here the sum is taken over partitions \( \bar{v} = \{ \bar{v}_\rho, \bar{v}_\rho \} \), where the subset \( \bar{v}_\rho \) consists of one element. Recall that \( \bar{u}_a = \bar{u} \setminus u_a \).

**Proposition 4.1.** The operator (4.2) satisfies recursion (4.3)

\[
Y_{a,b}(\bar{u}, \bar{v}) = T_{12}(u_a) Y_{a-1,b}(\bar{u}_a, \bar{v}) + \sum g(\bar{v}_\rho, u_a) f(\bar{v}_\rho, \bar{u}_a) g(\bar{v}_\rho, \bar{v}_\rho) T_{13}(u_a) Y_{a-1,b-1}(\bar{u}_a, \bar{v}_\rho) T_{22}(\bar{v}_\rho). \tag{4.4}
\]

The notation is the same as in (4.3).
The proof of this proposition will be given in the following sections. Here we would like to mention only that proposition (4.1) yields

\[ Y_{a,b}(\bar{u}, \bar{v}) = X_{a,b}(\bar{u}, \bar{v}), \quad \forall \ a, b. \]  

(4.5)

Indeed, it is clear that

\[ Y_{0,b}(\emptyset, \bar{v}) = X_{0,b}(\emptyset, \bar{v}) = T_{23}(\bar{v}), \quad \forall \ b. \]  

(4.6)

Then using recursions (4.3) and (4.4) we arrive at (4.5).

Equation (4.5) can be considered as MCR of the operators \( T_{12}(\bar{u}) \) and \( T_{23}(\bar{v}) \). Indeed, we have

\[ X_{a,b}(\bar{u}, \bar{v}) = T_{12}(\bar{u}) T_{23}(\bar{v}) + \ldots, \]
\[ Y_{a,b}(\bar{u}, \bar{v}) = T_{23}(\bar{v}) T_{12}(\bar{u}) + \ldots, \]  

(4.7)

where dots mean the terms proportional to the operators \( T_{13} \) and \( T_{22} \). Thus, equation (4.5) provides us with a multiple commutator \([T_{12}(\bar{u}), T_{23}(\bar{v})]\) in terms of the operators \( T_{13}, T_{22} \), and the products of \( T_{12} \) and \( T_{23} \) with less number of arguments.

## 5 Different representations for Bethe vectors

MCR (4.5) allow us to find different representations for the Bethe vectors in the models with \( gl(2|1) \) symmetry. Recall that Bethe vectors are special polynomials in operators \( T_{ij}(u) \) with \( i \leq j \) applied to the pseudovacuum vector \( \Omega \). This vector possesses the following properties:

\[ T_{ii}(u)\Omega = \lambda_i(u)\Omega, \]
\[ T_{ij}(u)\Omega = 0, \quad i > j. \]  

(5.1)

Here \( \lambda_i(u) \) are some scalar functions depending on a specific model.

The Bethe vectors depend on two sets of variables (Bethe parameters) \( \bar{u} \) and \( \bar{v} \). We denote the Bethe vectors \( \Phi_{a,b}(\bar{u}; \bar{v}) \). It was proved in [23] that they have the following explicit form:

\[ \Phi_{a,b}(\bar{u}; \bar{v}) = X_{a,b}(\bar{u}; \bar{v}) \Omega, \]  

(4.4)

where \( X_{a,b} \) is given by (4.1). As we have proved that \( X_{a,b}(\bar{u}; \bar{v}) = Y_{a,b}(\bar{u}; \bar{v}) \), we immediately obtain an alternative representation \( \Phi_{a,b}(\bar{u}; \bar{v}) = Y_{a,b}(\bar{u}; \bar{v}) \Omega \), where \( Y_{a,b} \) is given by (4.2). Thus, we have the following explicit formulas for the Bethe vectors:

\[ \Phi_{a,b}(\bar{u}; \bar{v}) = \sum g(\bar{u}_a, \bar{u}_a) f(\bar{v}_\alpha, \bar{v}_\alpha) h(\bar{u}_a, \bar{u}_a) T_{13}(\bar{u}_a) T_{12}(\bar{u}_a) T_{23}(\bar{v}_\alpha) \lambda_2(\bar{v}_\alpha) \Omega, \]
\[ \Phi_{a,b}(\bar{u}; \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\alpha, \bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_a) \lambda_2(\bar{u}_a) \Omega. \]  

(5.2)

Here we have used that \( T_{22}(u)\Omega = \lambda_2(u)\Omega \) and extended the convention on the shorthand notation (2.12) to the products of the functions \( \lambda_2(u) \).

Due to proposition (4.1) we can easily obtain another representations

\[ \Phi_{a,b}(\bar{u}; \bar{v}) = \sum g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{u}_a, \bar{u}_a) h(\bar{v}_\alpha, \bar{v}_\alpha) \times T_{12}(\bar{u}_a) T_{13}(\bar{u}_a) T_{23}(\bar{v}_\alpha) \lambda_2(\bar{v}_\alpha) \Omega, \]
\[ \Phi_{a,b}(\bar{u}; \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{v}_\alpha, \bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{12}(\bar{u}_a) \lambda_2(\bar{u}_a) \Omega. \]  

(5.3)
Let us prove, for instance, the second equation (5.3). Consider the sum over partitions \( \bar{\nu} \Rightarrow \{ \bar{\nu}, \bar{\alpha} \} \) in the second equation (5.2) for \( n = \# \bar{\nu} \) fixed. Using (A.2) we obtain

\[
\sum K(\bar{\nu}|\bar{\alpha})g(\bar{\nu}, \bar{\alpha}) T_{13}(\bar{\nu}) T_{23}(\bar{\nu}) = (-1)^n \sum K(\bar{\nu}|\bar{\alpha}) f(\bar{\nu}, \bar{\alpha}) K(\bar{\nu}|\bar{\beta} + c) g(\bar{\nu}, \bar{\beta}) T_{23}(\bar{\nu}) T_{13}(\bar{\nu}), \tag{5.4}
\]

where we have an additional sum over partitions \( \bar{\nu} \Rightarrow \{ \bar{\nu}, \bar{\beta} \} \). The partitions \( \bar{\nu} \Rightarrow \{ \bar{\nu}, \bar{\alpha} \} \) and \( \bar{\nu} \Rightarrow \{ \bar{\nu}, \bar{\beta} \} \) are independent except that \( \# \bar{\nu} = \# \bar{\beta} = n \). Thus, we can take the sum over partitions \( \bar{\nu} \Rightarrow \{ \bar{\nu}, \bar{\alpha} \} \) in the r.h.s. of (5.4). For this we first transform the DWPF \( K(\bar{\nu}|\bar{\beta} + c) \) using (A.2)

\[
K(\bar{\nu}|\bar{\beta} + c) = (-1)^{b-n} K(\{ \bar{\nu}, \bar{\alpha} \}|\{ \bar{\beta} + c, \bar{\alpha} + c \})
\]

Substituting this into (5.4) we obtain

\[
(-1)^n \sum K(\bar{\nu}|\bar{\alpha}) f(\bar{\nu}, \bar{\alpha}) K(\bar{\nu}|\bar{\beta} + c) = (-1)^{b-n} \sum K(\bar{\nu}|\bar{\alpha}) f(\bar{\nu}, \bar{\alpha}) K(\bar{\beta} - c|\bar{\nu})
\]

where we have used (3.9) and (A.2). Substituting this into (5.4) we arrive at

\[
\sum K(\bar{\nu}|\bar{\alpha}) g(\bar{\nu}, \bar{\alpha}) T_{13}(\bar{\nu}) T_{23}(\bar{\nu}) = \sum K(\bar{\nu}|\bar{\alpha}) f(\bar{\nu}, \bar{\alpha}) g(\bar{\nu}, \bar{\alpha}) T_{23}(\bar{\nu}) T_{13}(\bar{\nu}). \tag{5.5}
\]

It remains to replace here \( \bar{\nu} \rightarrow \bar{\nu} \) and \( \bar{\nu} \rightarrow \bar{\alpha} \), and we obtain the second representation (5.3).

Similarly, starting with the first representation (5.2) and using summation formula (5.4) we find the first representation (5.3).

To conclude this section we give also explicit representations for the dual Bethe vectors. They belong to the dual space and can be obtained from (5.2), (5.3) via antimorphism (3.7). This antimorphism sends the pseudovacuum vector to its dual: \( \psi(\Omega) = \Omega^\dagger \). The vector \( \Omega^\dagger \) belongs to the dual space and possesses the properties

\[
\Omega^\dagger T_{ii}(u) = \lambda_i(u) \Omega^\dagger, \quad \Omega^\dagger T_{ij}(u) = 0, \quad i < j. \tag{5.6}
\]

Here the functions \( \lambda_i(u) \) are the same as in (5.1).

Acting with \( \psi \) onto (5.2), (5.3) we find

\[
\Phi_{a,b}^\dagger(\bar{u}, \bar{v}) = (-1)^{b(b-1)/2} \sum g(\bar{u}, \bar{u}) f(\bar{u}, \bar{u}) h(\bar{u}, \bar{u}) \times \lambda_2(\bar{u}) \Omega^\dagger T_{32}(\bar{u}) T_{21}(\bar{u}) T_{31}(\bar{u}), \tag{5.7}
\]

\[
\Phi_{a,b}^\dagger(\bar{u}, \bar{v}) = (-1)^{b(b-1)/2} \sum K(\bar{u}|\bar{u}) f(\bar{u}, \bar{u}) g(\bar{u}, \bar{u}) \times \lambda_2(\bar{u}) \Omega^\dagger T_{21}(\bar{u}) T_{21}(\bar{u}) T_{31}(\bar{u}), \tag{5.8}
\]

9
and

\[ \Phi_{a,b}(\bar{u}, \bar{v}) = (-1)^{b(b-1)/2} \sum g(\bar{u}_\alpha, \bar{u}_\beta) f(\bar{v}_\alpha, \bar{u}_\alpha) g(\bar{v}_\beta, \bar{v}_\alpha) h(\bar{u}_\alpha, \bar{u}_\alpha) \times \lambda_2(\bar{v}_\alpha) \Omega T_{32}(\bar{v}_\alpha) T_{31}(\bar{u}_\alpha) T_{21}(\bar{u}_\alpha), \]  

\[ \Phi_{a,b}(\bar{u}, \bar{v}) = (-1)^{b(b-1)/2} K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{v}_\alpha, \bar{v}_\alpha) \times \lambda_2(\bar{v}_\alpha) \Omega T_{21}(\bar{u}_\alpha) T_{31}(\bar{v}_\alpha) T_{32}(\bar{v}_\alpha). \]

(5.10)

6 MCR (4.5) for \( a = 1 \)

For \( a = 1 \) equation (4.5) takes the form

\[ [T_{12}(u), T_{23}(\bar{v})] = \sum g(u, \bar{v}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) \left( T_{13}(u) T_{23}(\bar{v}_\alpha) T_{22}(\bar{v}_\alpha) - T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{22}(u) \right). \]  

(6.1)

Here the sum is taken over partitions of the set \( \bar{v} \) into subsets \( \bar{v}_\alpha \) and \( \bar{v}_\beta \) with a restriction \( \# \bar{v}_\alpha = 1 \). This MCR can be proved via the standard method of the algebraic Bethe ansatz. Indeed, due to (2.4) we have

\[ [T_{12}(u), T_{23}(\bar{v})] = \prod_{1 \leq j < l \leq b} \frac{1}{h(v_j, v_l)} \sum_{k=1}^{b} g(u, v_k) \times T_{23}(v_1) \ldots T_{23}(v_{k-1}) \left( T_{13}(u) T_{22}(v_k) - T_{13}(v_k) T_{22}(u) \right) T_{23}(v_{k+1}) \ldots T_{23}(v_b). \]  

(6.2)

Then moving the operator \( T_{13} \) to the left and the operator \( T_{22} \) to the right we eventually obtain the following result

\[ [T_{12}(u), T_{23}(\bar{v})] = \sum_{k=1}^{b} \left( \Lambda_k T_{13}(u) T_{23}(\bar{v}_k) T_{22}(v_k) + \tilde{\Lambda}_k T_{13}(v_k) T_{23}(\bar{v}_k) T_{22}(u) \right) \]  

\[ + \sum_{j,k=1 \atop j \neq k}^{b} M_{jk} T_{13}(v_j) T_{23}(\{\bar{v}_{j,k}, u\}) T_{22}(v_k). \]  

(6.3)

Here \( \Lambda_k, \tilde{\Lambda}_k, \) and \( M_{jk} \) are some rational coefficients. The set \( \bar{v}_{j,k} \) is \( \bar{v} \setminus \{v_j, v_k\} \). Due to the symmetry of \( T_{23}(\bar{v}) \) over \( \bar{v} \) it is enough to find \( \Lambda_1, \tilde{\Lambda}_b, \) and \( M_{21} \). For this we need particular cases of equations (5.1), (6.2)

\[ T_{22}(v) T_{12}(\bar{u}) = f(v, \bar{u}) T_{12}(\bar{u}) T_{22}(v) + \sum g(\bar{u}_\rho, v) f(\bar{u}_\rho, \bar{u}_\rho) T_{12}(v) T_{12}(\bar{u}_\rho) T_{22}(\bar{u}_\rho), \]  

(6.4)

\[ T_{23}(\bar{u}) T_{13}(v) = (-1)^{b} f(v, \bar{u}) T_{13}(v) T_{23}(\bar{u}) + \sum g(v, \bar{u}_\rho) g(\bar{u}_\rho, \bar{u}_\rho) h(v, \bar{u}_\rho) T_{13}(\bar{u}_\rho) T_{23}(\{v, \bar{u}_\rho\}). \]  

(6.5)

Here the sums are taken over partitions \( \bar{u} \Rightarrow \{\bar{u}_\rho, \bar{u}_\rho\} \) with \( \# \bar{u}_\rho = 1 \).
Let us find $\Lambda_1$. Then it is clear that only the term $k = 1$ in \[6.2\] contributes into this coefficient. Indeed, all other terms cannot produce $T_{22}(v_1)$ in the extreme right position. Thus, presenting $T_{23}(\bar{v})$ as
\[
T_{23}(\bar{v}) = T_{23}(v_1) \frac{T_{23}(\bar{v}_1)}{h(v_1, v_1)},
\]
we obtain
\[
[T_{12}(u), T_{23}(\bar{v})] = g(u, v_1)(T_{13}(u)T_{22}(v_1) - T_{13}(v_1)T_{22}(u)) \frac{T_{23}(\bar{v}_1)}{h(\bar{v}_1, v_1)} + UWT. \tag{6.7}
\]
Clearly, the product $T_{13}(v_1)T_{22}(u)$ does not contribute to the desirable coefficient. In the remaining product $T_{13}(v_1)T_{22}(u)$ the operator $T_{13}(u)$ should be kept in its extreme left position, while $T_{22}(v_1)$ should be moved to the extreme right position. Hereby, the argument $v_1$ of $T_{22}$ should be preserved. Then we arrive at
\[
\Lambda_1 = g(u, v_1)g(\bar{v}_1, v_1). \tag{6.8}
\]
Similarly, calculating $\tilde{\Lambda}_b$ we should consider only the term with $k = b$ in \[6.2\]
\[
[T_{12}(u), T_{23}(\bar{v})] = g(u, v_b)\frac{T_{23}(\bar{v}_b)}{h(v_b, \bar{v}_b)} (T_{13}(u)T_{22}(v_b) - T_{13}(v_b)T_{22}(u)) + UWT. \tag{6.9}
\]
Now $T_{22}(u)$ should be kept in its extreme right position, while $T_{13}(v_b)$ should be moved to the left preserving its argument. We find
\[
\tilde{\Lambda}_b = -g(u, v_b)g(\bar{v}_b, v_b). \tag{6.10}
\]
Finally, it is easy to see that $M_{21} = 0$. Indeed, in order to obtain $M_{21}$ we should take only the term with $k = 1$ in \[6.2\]. In other words we arrive at \[6.7\]. Otherwise we cannot obtain $T_{22}(v_1)$ in the extreme right position. But the operator $T_{13}$ is already in the extreme left position, hence, it cannot depend on $v_2$. Thus, we conclude that $M_{21} = 0$.

Using now the symmetry properties we find
\[
\Lambda_k = g(u, v_k)g(\bar{v}_k, v_k), \quad \tilde{\Lambda}_k = -g(u, v_k)g(\bar{v}_k, v_k), \quad M_{jk} = 0, \tag{6.11}
\]
and hence,
\[
[T_{12}(u), T_{23}(\bar{v})] = \sum_{k=1}^{b} g(u, v_k)g(\bar{v}_k, v_k) \left( T_{13}(u)T_{23}(\bar{v}_k)T_{22}(v_k) - T_{13}(v_k)T_{23}(\bar{v}_k)T_{22}(u) \right), \tag{6.12}
\]
what is equivalent to \[6.1\].

7 MCR (4.5) for $a$ and $b$ arbitrary

In order to show that the operator $Y_{a,b}(\bar{u}, \bar{v})$ satisfies recursion \[4.1\] we act with $T_{12}(u_a)$ on the operator $Y_{a-1,b}(\bar{u}_a, \bar{v})$
\[
T_{12}(u_a) Y_{a-1,b}(\bar{u}_a, \bar{v}) = \sum K(\bar{v}_a|u_a)f(\bar{v}_a, u_\alpha)g(\bar{v}_\alpha, \bar{v}_a) T_{12}(u_a)T_{13}(\bar{v}_a)T_{23}(\bar{v}_a)T_{12}(\bar{u}_\alpha)T_{22}(\bar{u}_a). \tag{7.1}
\]
Here the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_\alpha, \bar{v}_\beta\}$ and $\bar{u}_\alpha \Rightarrow \{\bar{u}_\alpha, \bar{u}_\beta\}$.

The goal is to move $T_{12}(u_a)$ to the right in such a way that eventually to have all the operators in the following order:

$$T_{13} \ T_{23} \ T_{12} \ T_{22}. \quad (7.2)$$

The analysis of the commutation relations shows that moving $T_{12}(u_a)$ to the right we might obtain several types of the operator products

$$T_{13}(\bar{v}_\beta) \ T_{23}(\bar{v}_\gamma) \ T_{12}(\bar{u}_\gamma) \ T_{12}(u_a) \ T_{22}(\bar{u}_\beta), \quad (7.3)$$

$$T_{13}(\bar{v}_\beta) \ T_{23}(\bar{v}_\gamma) \ T_{12}(\bar{u}_\gamma) \ T_{22}(\bar{u}_\beta) \ T_{22}(u_a), \quad (7.4)$$

$$T_{13}(u_a) \ T_{13}(\bar{v}_\beta) \ T_{23}(\bar{v}_\gamma) \ T_{12}(\bar{u}_\gamma) \ T_{22}(\bar{u}_\beta) \ T_{22}(\bar{v}_\rho), \quad (7.5)$$

$$T_{13}(u_a) \ T_{13}(\bar{v}_\beta) \ T_{23}(\bar{v}_\gamma) \ T_{12}(\bar{u}_\gamma) \ T_{12}(\bar{v}_\rho) \ T_{22}(\bar{u}_\beta). \quad (7.6)$$

The first two types (7.3) and (7.4) coincide with the operator products in the definition of $Y_{a,b}(\bar{v}, \bar{v})$. We call such term wanted terms (WT). The remaining terms are unwanted (UWT). We will see that the terms of the type (7.5) cancel the sum over partitions in (4.4), while the terms of the type (7.6) cancel them selves.

### 7.1 Wanted terms

In this section we pay attention to the wanted terms only. The wanted terms are characterized by the property that the operators $T_{13}$ and $T_{23}$ depend only on the variables from the set $\bar{v}$, while the operators $T_{12}$ and $T_{22}$ depend only on the variables from the set $\bar{u}$.

Moving $T_{12}(u_a)$ through the product $T_{13}(\bar{v}_\alpha)$ we use (3.3). Let us write it in the form

$$T_{12}(u) T_{13}(\bar{v}_\alpha) = f(\bar{v}_\alpha, u) T_{13}(\bar{v}_\alpha) T_{12}(u) + \sum g(u, \bar{v}_\rho) g(\bar{v}_\sigma, \bar{v}_\rho) T_{13}(u) T_{13}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho), \quad (7.7)$$

where the sum is taken over partitions $\bar{v}_\alpha \Rightarrow \{\bar{v}_\rho, \bar{v}_\sigma\}$ with $\# \bar{v}_\rho = 1$. Substituting this into the r.h.s. of (7.1) we obtain

$$T_{12}(u_a) Y_{a-1,b}(\bar{u}_a, \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{v}_\alpha, \bar{u}_\alpha) T_{13}(\bar{v}_\alpha) T_{12}(u_a)$$

$$+ g(u_a, \bar{v}_\rho) g(\bar{v}_\sigma, \bar{v}_\rho) T_{13}(u_a) T_{13}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.8)$$

The second term in the square brackets is unwanted, as it contains $T_{13}(u_a)$. Therefore we postpone its consideration and arrive at

$$T_{12}(u_a) Y_{a-1,b}(\bar{u}_a, \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{v}_\alpha, \bar{u}_\alpha) T_{13}(\bar{v}_\alpha) T_{12}(u_a) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha) + UWT. \quad (7.9)$$

Now we use (7.1):

$$T_{12}(u_a) Y_{a-1,b}(\bar{u}_a, \bar{v}) = \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) f(\bar{v}_\alpha, \bar{u}_\alpha) T_{13}(\bar{v}_\alpha)$$

$$\times \left\{ g(u_a, \bar{v}_\rho) g(\bar{v}_\sigma, \bar{v}_\rho) \left( T_{13}(u_a) T_{23}(\bar{v}_\sigma) T_{22}(\bar{v}_\rho) - T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\sigma) T_{22}(u_a) \right) + T_{23}(\bar{v}_\alpha) T_{12}(u_a) \right\} T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha) + UWT \quad (7.10)$$
Here we have an additional partition \( \bar{v}_a \Rightarrow \{ \bar{v}_p, \bar{v}_a \} \) with \( \# \bar{v}_p = 1 \).

The contribution proportional to \( T_{13}(u_a) T_{23}(\bar{v}_\sigma) T_{22}(\bar{v}_p) \) is again unwanted. The term proportional to \( T_{23}(\bar{v}_a) T_{12}(u_a) \) already contains all the operators in the necessary order. Thus, we obtain the first contribution to the wanted terms

\[
W_1 = \sum K(\bar{v}_a|\bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_a, \bar{v}_a) f(\bar{v}_a, u_a) T_{13}(\bar{v}_a) T_{23}(\bar{v}_a) T_{12}(u_a) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.11}
\]

It remains to consider the last contribution in (7.10), which is proportional to the product \( T_{13}(\bar{v}_a) T_{23}(\bar{v}_\sigma) T_{22}(u_a) \). Substituting \( \bar{v}_a = \{ \bar{v}_\sigma, \bar{v}_p \} \) in this term we obtain

\[
T_{12}(u_a) Y_{\bar{a}-1,6}(\bar{u}_a, \bar{v}) - W_1 = \sum K(\bar{v}_p|\bar{u}_a) f(\bar{v}_p, \bar{v}_a) g(\bar{v}_p, \bar{v}_a) f(\bar{v}_a, u_a) \times g(\bar{v}_\sigma, \bar{v}_p) g(\bar{v}_p, u_a) T_{13}(\bar{v}_a) T_{13}(\bar{v}_\sigma) T_{23}(\bar{v}_a) T_{22}(u_a) T_{22}(\bar{u}_a) + UWT. \tag{7.12}
\]

In order to complete the calculation we should move \( T_{22}(u_a) \) through the product \( T_{12}(\bar{u}_a) \). However, before doing this we can take the sum over partitions \( \bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \} \). Indeed, we have

\[
T_{13}(\bar{v}_a) T_{13}(\bar{v}_p) = h(\bar{v}_p, \bar{v}_a) T_{13}(\{ \bar{v}_a, \bar{v}_p \}) = h(\bar{v}_a, \bar{v}_p) T_{13}(\bar{v}_\sigma). \tag{7.13}
\]

Hence,

\[
T_{12}(u_a) Y_{\bar{a}-1,6}(\bar{u}_a, \bar{v}) - W_1 = \sum K(\bar{v}_a|\bar{u}_a) f(\bar{v}_a, \bar{v}_a) f(\bar{v}_a, u_a) g(\bar{v}_a, \bar{v}_a) f(\bar{v}_a, u_a) \times T_{13}(\bar{v}_\sigma) T_{23}(\bar{v}_a) T_{22}(u_a) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a) + UWT. \tag{7.14}
\]

Thus, the sum over partitions \( \bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \} \) involves only the rational functions in the square brackets of (7.14). We have

\[
\sum_{\bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \}} K(\bar{v}_a|\bar{u}_a) f(\bar{v}_a, \bar{u}_a) f(\bar{v}_a, u_a) g(\bar{v}_a, \bar{u}_a) = f(\bar{v}_\sigma, u_a) \sum_{\bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \}} K(\bar{v}_a|\bar{u}_a) f(\bar{v}_a, \bar{v}_a) \frac{f(\bar{v}_p, \bar{v}_a)}{h(\bar{v}_p, u_a)}
\]

\[
= -f(\bar{v}_\sigma, u_a) \sum_{\bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \}} K(\bar{v}_a|\bar{u}_a) (u_a - c|\bar{v}_p) f(\bar{v}_p, \bar{v}_a) \tag{7.15}
\]

where we used \( K(u_a - c|\bar{v}_p) = -1/h(\bar{v}_p, u_a) \). We see that the sum over partitions is reduced to lemma [3.2]. Hence, we arrive at

\[
\sum_{\bar{v}_\sigma \Rightarrow \{ \bar{v}_a, \bar{v}_p \}} K(\bar{v}_a|\bar{u}_a) f(\bar{v}_a, \bar{u}_a) f(\bar{v}_a, u_a) g(\bar{v}_a, \bar{u}_a) = K(\bar{v}_\sigma|\{ \bar{u}_a, u_a \}), \tag{7.16}
\]

what gives us

\[
T_{12}(u_a) Y_{\bar{a}-1,6}(\bar{u}_a, \bar{v}) - W_1 = \sum K(\bar{v}_a|\{ \bar{u}_a, u_a \}) f(\bar{v}_a, \bar{u}_a) g(\bar{v}_a, \bar{v}_a) \times T_{13}(\bar{v}_a) T_{23}(\bar{v}_a) T_{22}(u_a) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a) + UWT. \tag{7.17}
\]

Here we have relabeled \( \bar{v}_\sigma \) by \( \bar{v}_a \) and \( \bar{v}_\sigma \) by \( \bar{v}_a \).
It remains to move the operator $T_{22}(u_a)$ through the product $T_{12}(\bar{u}_\alpha)$. It can be done via (3.1). In our case this equation takes the form

$$T_{22}(u_a)T_{12}(\bar{u}_\alpha) = f(u_a, \bar{u}_\alpha)T_{12}(\bar{u}_\alpha)T_{22}(u_a) + \sum g(\bar{u}_\rho, u_a)f(\bar{u}_\rho, \bar{u}_\sigma)T_{12}(u_a)T_{12}(\bar{u}_\sigma)T_{22}(\bar{u}_\rho),$$

where the sum is taken over partitions $\bar{u}_\alpha \Rightarrow \{\bar{u}_\rho, \bar{u}_\sigma\}$ with $\#\bar{u}_\rho = 1$. Substituting this equation to (7.17) we find

$$T_{12}(u_a) Y_{a-1, b}(\bar{u}_\alpha, \bar{v}) - W_1 = W_2 + W_3 + UWT,$$  \hspace{1cm} (7.19)

where

$$W_2 = \sum K(\bar{v}_\alpha|\{\bar{u}_\alpha, u_a\})f(\bar{u}_\alpha, \bar{u}_\alpha)f(u_a, \bar{u}_\alpha)g(\bar{v}_\alpha, \bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha) T_{22}(u_a)T_{22}(\bar{u}_\alpha),$$ \hspace{1cm} (7.20)

and

$$W_3 = \sum K(\bar{v}_\alpha|\{\bar{u}_\alpha, u_a\})f(\bar{u}_\alpha, \bar{u}_\alpha)g(\bar{v}_\alpha, \bar{v}_\alpha) g(\bar{u}_\rho, u_a)f(\bar{u}_\rho, \bar{u}_\sigma)$$

$$\times T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(u_a)T_{12}(\bar{u}_\sigma)T_{22}(\bar{u}_\rho)T_{22}(\bar{u}_\alpha).$$ \hspace{1cm} (7.21)

Observe that the contribution $W_2$ can be written in the form

$$W_2 = \sum_{u_a \in \bar{u}_\alpha} K(\bar{v}_\alpha|\{\bar{u}_\alpha, u_a\})f(\bar{u}_\alpha, \bar{u}_\alpha)g(\bar{v}_\alpha, \bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha).$$ \hspace{1cm} (7.22)

Here, in distinction of the original formula (7.1) we have the sum over partitions of the complete set $\bar{u}$ (i.e. including $u_a$). However, we have the restriction $u_a \in \bar{u}_\alpha$.

In (7.21) we can take the sum over partitions $\bar{u}_\beta \Rightarrow \{\bar{u}_\alpha, \bar{u}_\rho\}$. Setting there $\{\bar{u}_\alpha, \bar{u}_\rho\} = \bar{u}_\beta$ we obtain

$$W_3 = \sum \left[ K(\bar{v}_\alpha|\{\bar{u}_\alpha, u_a\})g(\bar{u}_\rho, u_a)f(\bar{u}_\sigma, \bar{u}_\rho)g(\bar{v}_\alpha, \bar{v}_\alpha)\right] f(\bar{u}_\sigma, \bar{u}_\sigma)g(\bar{v}_\alpha, \bar{v}_\alpha)$$

$$\times T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(u_a)T_{12}(\bar{u}_\sigma)T_{22}(\bar{u}_\rho)T_{22}(\bar{u}_\alpha).$$ \hspace{1cm} (7.23)

The sum of the terms in the square brackets can be computed via (3.10)

$$\sum_{\bar{u}_\beta \Rightarrow \{\bar{u}_\alpha, \bar{u}_\rho\}} K(\bar{v}_\alpha|\{\bar{u}_\alpha, u_a\})g(\bar{u}_\rho, u_a)f(\bar{u}_\sigma, \bar{u}_\rho) = \left(f(\bar{u}_\sigma, u_a) - f(\bar{v}_\alpha, u_a)\right)K(\bar{v}_\alpha|\bar{u}_\beta).$$ \hspace{1cm} (7.24)

Thus, we arrive at

$$W_3 = \sum K(\bar{v}_\alpha|\bar{u}_\alpha)\left(f(\bar{u}_\alpha, u_a) - f(\bar{v}_\alpha, u_a)\right)g(\bar{v}_\alpha, \bar{v}_\alpha)$$

$$\times T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(u_a)T_{12}(\bar{u}_\alpha)T_{22}(\bar{u}_\alpha),$$ \hspace{1cm} (7.25)

where we relabeled $\bar{u}_\beta$ by $\bar{u}_\alpha$ and $\bar{u}_\sigma$ by $\bar{u}_\alpha$.

We see that

$$W_1 + W_3 = \sum K(\bar{v}_\alpha|\bar{u}_\alpha)\left(f(\bar{u}_\alpha, u_a) - f(\bar{v}_\alpha, u_a)\right)g(\bar{v}_\alpha, \bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(u_a)T_{12}(\bar{u}_\alpha)T_{22}(\bar{u}_\alpha)$$

$$= \sum K(\bar{v}_\alpha|\bar{u}_\alpha)\left(f(\bar{u}_\alpha, u_a) - f(\bar{v}_\alpha, u_a)\right)g(\bar{v}_\alpha, \bar{v}_\alpha) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha)T_{22}(\bar{u}_\alpha).$$ \hspace{1cm} (7.26)
Thus, we have again the sum over partitions of the complete set \( \bar{u} \), however, now the restriction is \( u_a \in \bar{u}_a \). Hence all together we obtain

\[
\sum_{k=1}^{3} W_k = \sum K(\bar{u}_a | u_a) f(\bar{u}_a, \bar{u}_\bar{a}) g(\bar{v}_a, \bar{v}_\bar{a}) T_{13}(\bar{v}_a) T_{12}(\bar{u}_\bar{a}) T_{22}(\bar{u}_a) = Y_{a,b}(\bar{u}, \bar{v}). \quad (7.27)
\]

Thus, we have shown that the contribution of the wanted terms to the action of the operator \( T_{12}(u_a) \) onto \( Y_{a-1,b}(\bar{u}_a, \bar{v}) \) produces the operator \( Y_{a,b}(\bar{u}, \bar{v}) \). It remains to prove that the unwanted terms do not contribute to the final result.

### 7.2 Cancellation of unwanted terms

We focus now at the unwanted terms. We denote their contributions by symbols \( U_k \). Later we shell split these contributions into two groups: the terms of the type (7.5) will be denoted by \( Z_k \); the terms of the type (7.6) will be denoted by \( \bar{Z}_k \).

#### 7.2.1 First contribution

We start with term proportional to \( T_{13}(u_a) T_{23}(\bar{v}_\sigma) T_{22}(\bar{v}_\rho) \) in (7.10). Substituting there \( \bar{v}_\bar{a} = \{ \bar{v}_\sigma, \bar{v}_\rho \} \) we obtain

\[
U_1 = \sum K(\bar{v}_a | \bar{u}_a) f(\bar{v}_a, \bar{u}_\bar{a}) g(\bar{v}_\sigma, \bar{v}_\bar{a}) f(\bar{v}_a, \bar{u}_a) g(\bar{v}_\sigma, \bar{v}_a) g(\bar{v}_\rho, \bar{v}_a) g(\bar{v}_\rho, \bar{u}_a)
\times T_{13}(\bar{v}_a) T_{13}(u_a) T_{13}(\bar{v}_\sigma) T_{22}(\bar{v}_\rho) T_{12}(\bar{u}_\bar{a}) T_{22}(\bar{u}_a).
\tag{7.28}
\]

Moving \( T_{13}(u_a) \) to the left via

\[
T_{13}(\bar{v}_a) T_{13}(u_a) = T_{13}(u_a) T_{13}(\bar{v}_a) \frac{h(u_a, \bar{v}_a)}{h(\bar{v}_a, u_a)},
\tag{7.29}
\]

we find

\[
U_1 = T_{13}(u_a) \sum K(\bar{v}_a | \bar{u}_a) f(\bar{v}_a, \bar{u}_\bar{a}) g(\bar{v}_\sigma, \bar{v}_\bar{a}) f(\bar{v}_a, \bar{u}_a) g(\bar{v}_\sigma, \bar{v}_a) g(\bar{v}_\rho, \bar{v}_a) g(\bar{v}_\rho, \bar{u}_a)
\times T_{13}(\bar{v}_a) T_{23}(\bar{v}_\sigma) T_{22}(\bar{v}_\rho) T_{12}(\bar{u}_\bar{a}) T_{22}(\bar{u}_a).
\tag{7.30}
\]

Now we move the operator \( T_{22}(\bar{v}_\rho) \) through the product \( T_{12}(\bar{u}_\bar{a}) \) via (6.4)

\[
U_1 = T_{13}(u_a) \sum K(\bar{v}_a | \bar{u}_a) f(\bar{v}_a, \bar{u}_\bar{a}) g(\bar{v}_\sigma, \bar{v}_\bar{a}) f(\bar{v}_a, \bar{u}_a) g(\bar{v}_\sigma, \bar{v}_a) g(\bar{v}_\rho, \bar{v}_a) g(\bar{v}_\rho, \bar{u}_a)
\times \left[ f(\bar{v}_\rho, \bar{u}_\bar{a}) T_{12}(\bar{u}_\bar{a}) T_{22}(\bar{v}_\rho) + g(\bar{v}_\rho, \bar{v}_a) f(\bar{u}_\rho, \bar{v}_a) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\rho) T_{22}(\bar{u}_\rho) \right] T_{22}(\bar{u}_a).
\tag{7.31}
\]

Here we have an additional sum over partitions \( \bar{u}_\bar{a} = \{ \bar{u}_\mu, \bar{u}_\tau \} \) with \( \# \bar{u}_\tau = 1 \). The first term in the square brackets contributes to the type (7.25):

\[
Z_1 = T_{13}(u_a) \sum K(\bar{v}_a | \bar{u}_a) f(\bar{v}_a, \bar{u}_\bar{a}) g(\bar{v}_\sigma, \bar{v}_\bar{a}) f(\bar{v}_a, \bar{u}_a) g(\bar{v}_\sigma, \bar{v}_a) g(\bar{v}_\rho, \bar{v}_a) g(\bar{v}_\rho, \bar{u}_a)
\times T_{13}(\bar{v}_a) T_{23}(\bar{v}_\sigma) T_{12}(\bar{u}_\bar{a}) T_{22}(\bar{u}_a) T_{22}(\bar{v}_\rho).
\tag{7.32}
\]
The second term in the square brackets contributes to the type (7.6). Substituting there $\bar{u}_\alpha = \{\bar{u}_\mu, \bar{u}_\tau\}$ we obtain

$$
\tilde{Z}_1 = T_{13}(u_a) \sum \left[ K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\tau) g(\bar{u}_\tau, \bar{v}_\rho) \right] f(\bar{u}_\alpha, \bar{u}_\mu) f(\bar{v}_\rho, \bar{v}_\rho) g(\bar{v}_\sigma, \bar{v}_\alpha) \\
\times f(u_a, \bar{v}_\alpha) g(u_a, \bar{v}_\rho) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho) T_{22}(\bar{u}_\tau) T_{22}(\bar{u}_\alpha). \quad (7.33)
$$

Here the sum over partitions $\bar{u}_\mu \Rightarrow \{\bar{u}_\alpha, \bar{u}_\tau\}$ can be taken. The product $f(\bar{u}_\alpha, \bar{u}_\mu) f(\bar{u}_\tau, \bar{u}_\tau)$ combines into $f(\bar{u}_\mu, \bar{u}_\mu)$, while the sum of the terms in the square brackets is a particular case of lemma [B.32]

$$
\sum_{\bar{u}_\mu \Rightarrow \{\bar{u}_\alpha, \bar{u}_\tau\}} K(\bar{v}_\alpha | \bar{u}_\alpha) g(\bar{u}_\alpha, \bar{v}_\rho) f(\bar{u}_\alpha, \bar{u}_\tau) = -f(\bar{u}_\mu, \bar{v}_\rho) K(\{\bar{v}_\rho - c, \bar{v}_\alpha\} | \bar{u}_\mu). \quad (7.34)
$$

Substituting this into (7.33) and relabeling $\bar{u}_\mu$ by $\bar{u}_\alpha$ and $\bar{u}_\mu$ by $\bar{u}_\alpha$ we find

$$
\tilde{Z}_1 = -T_{13}(u_a) \sum f(\bar{u}_\alpha, \bar{v}_\rho) K(\{\bar{v}_\rho - c, \bar{v}_\alpha\} | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\rho, \bar{v}_\rho) g(\bar{v}_\sigma, \bar{v}_\alpha) \\
\times f(u_a, \bar{v}_\alpha) g(u_a, \bar{v}_\rho) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.35)
$$

### 7.2.2 Second contribution

We now turn back to the term with $T_{13}(u_a)$ in (7.34). Substituting there $\bar{v}_\alpha = \{\bar{v}_\sigma, \bar{v}_\rho\}$ we obtain

$$
U_2 = T_{13}(u_a) \sum K(\{\bar{v}_\sigma, \bar{v}_\rho\} | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\sigma, \bar{v}_\sigma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho) \\
\times T_{13}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho) T_{23}(\bar{v}_\alpha) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.36)
$$

Now we move $T_{12}(\bar{v}_\rho)$ through the product $T_{23}(\bar{v}_\alpha)$ via (6.1)

$$
U_2 = T_{13}(u_a) \sum K(\{\bar{v}_\sigma, \bar{v}_\rho\} | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\sigma, \bar{v}_\sigma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho) \\
\times T_{13}(\bar{v}_\sigma) \left\{ g(\bar{v}_\mu, \bar{v}_\tau) g(\bar{v}_\mu, \bar{v}_\tau) \left[ T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{22}(\bar{v}_\tau) - T_{13}(\bar{v}_\tau) T_{23}(\bar{v}_\mu) T_{22}(\bar{v}_\rho) \right] \\
+ T_{23}(\bar{v}_\alpha) T_{12}(\bar{v}_\rho) \right\} T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.37)
$$

Here we have an additional sum over partitions $\bar{v}_\alpha \Rightarrow \{\bar{v}_\mu, \bar{v}_\tau\}$, where $\bar{v}_\tau$ consists of one element.

The term with the product $T_{23}(\bar{v}_\alpha) T_{12}(\bar{v}_\rho)$ gives direct contribution to the type (7.6)

$$
\tilde{Z}_2 = T_{13}(u_a) \sum K(\{\bar{v}_\sigma, \bar{v}_\rho\} | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\sigma, \bar{v}_\sigma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho) \\
\times T_{13}(\bar{v}_\sigma) T_{23}(\bar{v}_\alpha) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.38)
$$

In the remaining terms of (7.37) we can partly take the sums over partitions. Consider the first term in the square brackets in (7.37) setting there $\bar{v}_\alpha = \{\bar{v}_\mu, \bar{v}_\tau\}$

$$
U_2^{(1)} = T_{13}(u_a) \sum K(\{\bar{v}_\sigma, \bar{v}_\rho\} | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\mu, \bar{v}_\sigma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho) \\
\times g(\bar{v}_\rho, \bar{v}_\tau) g(\bar{v}_\mu, \bar{v}_\tau) T_{13}(\bar{v}_\alpha) T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{22}(\bar{v}_\tau) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.39)
$$
In this formula, the set \( \bar{v} \) is divided into four subsets \( \{ \bar{v}_r, \bar{v}_\rho, \bar{v}_\sigma, \bar{v}_\mu \} \) with \( \#\bar{v}_r = \#\bar{v}_\rho = 1 \). Combining \( \{ \bar{v}_\sigma, \bar{v}_\rho \} \) into \( \bar{v}_\gamma \), we obtain

\[
U_2^{(1)} = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_r, \bar{v}_\gamma) [f(\bar{v}_\rho, \bar{v}_\sigma) g(\bar{v}_r, \bar{v}_\tau) g(u_a, \bar{v}_\rho)]
\times T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.40}
\]

The sum over partitions \( \bar{v}_\gamma \Rightarrow \{ \bar{v}_\sigma, \bar{v}_\rho \} \) (see the terms in the square brackets) can be easily computed via contour integration:

\[
\sum_{\bar{v}_\gamma \Rightarrow \{ \bar{v}_\sigma, \bar{v}_\rho \}} f(\bar{v}_\rho, \bar{v}_\sigma) g(\bar{v}_r, \bar{v}_\rho) g(u_a, \bar{v}_\rho) = g(u_a, \bar{v}_r)(f(\bar{v}_r, \bar{v}_\gamma) - f(u_a, \bar{v}_\gamma)). \tag{7.42}
\]

Substituting this into (7.40) we obtain

\[
U_2^{(1)} = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_r, \bar{v}_\gamma) (f(\bar{v}_r, \bar{v}_\gamma) - f(u_a, \bar{v}_\gamma))
\times T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.43}
\]

The second term in the square brackets of (7.37) can be treated similarly. Again setting \( \bar{v}_\alpha = \{ \bar{v}_\mu, \bar{v}_r \} \) we obtain

\[
U_2^{(2)} = T_{13}(u_a) \sum K(\{ \bar{v}_\sigma, \bar{v}_\tau \} | \bar{u}_a) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\mu, \bar{v}_\sigma) g(\bar{v}_r, \bar{v}_\sigma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho)
\times g(\bar{v}_r, \bar{v}_\rho) g(\bar{v}_\rho, \bar{v}_\gamma) T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.44}
\]

This time we combine \( \{ \bar{v}_\sigma, \bar{v}_\tau \} \) into \( \bar{v}_\gamma \) and find

\[
U_2^{(2)} = T_{13}(u_a) \sum K(\{ \bar{v}_\sigma, \bar{v}_\tau \} | \bar{u}_a) f(\bar{v}_\tau, \bar{v}_\sigma) g(\bar{v}_r, \bar{v}_\sigma) [f(\bar{v}_\rho, \bar{u}_a) g(\bar{v}_\rho, \bar{v}_\gamma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho)]
\times T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.45}
\]

The sum over partitions \( \bar{v}_\gamma \Rightarrow \{ \bar{v}_\sigma, \bar{v}_\tau \} \) (see the terms in the square brackets) can be computed via (7.37)

\[
\sum_{\bar{v}_\gamma \Rightarrow \{ \bar{v}_\sigma, \bar{v}_\tau \}} K(\{ \bar{v}_\sigma, \bar{v}_\tau \} | \bar{u}_a) f(\bar{v}_\mu, \bar{v}_\sigma) g(\bar{v}_r, \bar{v}_\mu) = K(\bar{v}_\gamma | \bar{u}_a) (f(\bar{v}_\rho, \bar{u}_a) - f(\bar{v}_\rho, \bar{v}_\gamma)). \tag{7.46}
\]

Substituting this into (7.45) we arrive at

\[
U_2^{(2)} = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_a) (f(\bar{v}_\rho, \bar{u}_a) - f(\bar{v}_\rho, \bar{v}_\gamma)) f(\bar{u}_a, \bar{u}_a) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_\rho, \bar{v}_\rho) g(u_a, \bar{v}_\rho)
\times T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_a) T_{22}(\bar{u}_a). \tag{7.47}
\]
It remains to combine the contributions (7.47) and (7.48) (relabeling $\bar{v}_\tau$ by $\bar{v}_\rho$ in (7.48)), what gives us

$$U_2^{(1)} + U_2^{(2)} = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times (f(\bar{v}_\rho, \bar{u}_\alpha) - f(u_a, \bar{v}_\gamma)) T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{22}(\bar{v}_\rho) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha). \quad (7.48)$$

### 7.2.3 Final cancellations

To achieve our goal we should move $T_{22}(\bar{v}_\rho)$ in (7.48) through the product $T_{12}(\bar{u}_\alpha)$ via (7.4).

$$U_2^{(1)} + U_2^{(2)} = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times (f(\bar{v}_\rho, \bar{u}_\alpha) - f(u_a, \bar{v}_\gamma)) T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{22}(\bar{v}_\rho)$$

$$+ g(u_\rho, \bar{v}_\rho) f(\bar{u}_\rho, \bar{u}_\sigma) T_{12}(\bar{u}_\sigma) T_{22}(\bar{u}_\rho) T_{22}(\bar{u}_\alpha). \quad (7.49)$$

Here we obtain additional partitions $\bar{u}_\alpha \Rightarrow \{ \bar{u}_\alpha, \bar{v}_\rho \}$, where $\bar{v}_\rho$ consists of one element. The first term in the square brackets contributes to the terms of the type (7.48).

$$Z_2 = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) f(\bar{v}_\rho, \bar{u}_\alpha) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times (f(\bar{v}_\rho, \bar{u}_\alpha) - f(u_a, \bar{v}_\gamma)) T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{22}(\bar{u}_\alpha) T_{22}(\bar{v}_\rho). \quad (7.50)$$

Replacing here $\bar{v}_\gamma \to \bar{v}_\alpha$ and $\bar{v}_\mu \to \bar{v}_\sigma$ we obtain

$$Z_2 = T_{13}(u_a) \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) f(\bar{v}_\rho, \bar{u}_\alpha) g(\bar{v}_\sigma, \bar{v}_\alpha) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times (f(\bar{v}_\rho, \bar{u}_\alpha) - f(u_a, \bar{v}_\alpha)) T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\sigma) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha) T_{22}(\bar{v}_\rho). \quad (7.51)$$

Combining this contribution with (7.49) we find

$$Z_1 + Z_2 = T_{13}(u_a) \sum K(\bar{v}_\alpha | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\sigma, \bar{v}_\alpha) f(\bar{v}_\rho, \bar{u}_\alpha) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\sigma) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha) T_{22}(\bar{v}_\rho), \quad (7.52)$$

and hence,

$$Z_1 + Z_2 = \sum g(u_a, \bar{v}_\rho) f(\bar{v}_\rho, \bar{u}_\alpha) g(\bar{v}_\beta, \bar{v}_\rho) T_{13}(u_a) Y_{a-1,b-1}(\bar{u}_\alpha, \bar{v}_\rho) T_{22}(\bar{v}_\rho). \quad (7.53)$$

We see that this contribution cancels the sum over partitions in the recursion (7.4).

The second term in the square brackets of (7.49) contributes to the terms of the type (7.48), therefore we denote it by $Z_3$. We have

$$Z_3 = T_{13}(u_a) \sum K(\bar{v}_\gamma | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) f(\bar{v}_\rho, \bar{u}_\alpha) g(\bar{v}_\mu, \bar{v}_\gamma) g(\bar{v}_\beta, \bar{v}_\rho) g(u_a, \bar{v}_\rho)$$

$$\times (f(\bar{v}_\rho, \bar{u}_\alpha) - f(u_a, \bar{v}_\gamma)) T_{13}(\bar{v}_\gamma) T_{23}(\bar{v}_\mu) T_{12}(\bar{u}_\sigma) T_{22}(\bar{u}_\rho) T_{22}(\bar{u}_\alpha), \quad (7.54)$$

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We see that we can take the sum over partitions \( \bar{u}_\alpha \) in the square brackets and the product where we have used lemma B.2. The second contribution comes from the sum where we substituted \( \bar{u}_\beta \) where again we have used lemma B.2 and 1.

Thus, we have two contributions. The first one comes from the sum

\[
\sum_{\bar{u}_\alpha \Rightarrow \{ \bar{u}_\alpha, \bar{u}_\beta \}} K(\bar{u}_\beta | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\beta)| \bar{u}_\beta) = -f(\bar{u}_\mu, \bar{v}_\rho) K(\{ \bar{v}_\rho - c, \bar{v}_\gamma \}| \bar{u}_\mu),
\]

where we have used lemma B.2. The second contribution comes from the sum

\[
\sum_{\bar{u}_\mu \Rightarrow \{ \bar{u}_\alpha, \bar{u}_\rho \}} K(\bar{v}_\rho | \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\rho)| f(\bar{u}_\rho, \bar{u}_\mu) = -f(\bar{v}_\rho, \bar{u}_\mu) \sum_{\bar{v}_\alpha \Rightarrow \{ \bar{u}_\alpha, \bar{u}_\rho \}} K(\bar{v}_\alpha | \bar{u}_\beta) f(\bar{u}_\beta, \bar{u}_\gamma) \frac{f(\bar{u}_\rho, \bar{u}_\mu)}{h(\bar{v}_\rho, \bar{u}_\rho)}
\]

where again we have used lemma B.2 and \( 1/h(\bar{v}_\rho, \bar{u}_\rho) = -K(\bar{u}_\rho | \bar{v}_\rho + c) \).

The sum (7.56) produces the contribution

\[
\bar{Z}_3^{(1)} = T_{13}(u_a) \sum K(\{ \bar{v}_\rho - c, \bar{v}_\gamma \}| \bar{u}_\sigma) f(\bar{u}_\sigma, \bar{v}_\rho) f(\bar{u}_\alpha, \bar{v}_\gamma) f(\bar{u}_\beta, \bar{v}_\rho) g(\bar{u}_\rho, \bar{v}_\rho) g(\bar{u}_\alpha, \bar{v}_\rho) \times T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\sigma) T_{22}(\bar{u}_\sigma),
\]

where we used \( \bar{u}_\mu = \bar{u}_\sigma \). After relabeling the subsets \( \bar{u}_\gamma \rightarrow \bar{v}_\alpha, \bar{v}_\mu \rightarrow \bar{v}_\sigma, \bar{u}_\sigma \rightarrow \bar{u}_\alpha \), and \( \bar{u}_\rho \rightarrow \bar{u}_\beta \), we arrive at

\[
\bar{Z}_3^{(1)} = T_{13}(u_a) \sum K(\{ \bar{v}_\rho - c, \bar{v}_\alpha \}| \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{v}_\rho) f(\bar{u}_\alpha, \bar{v}_\alpha) f(\bar{u}_\alpha, \bar{u}_\beta) g(\bar{v}_\gamma, \bar{v}_\alpha) g(\bar{v}_\gamma, \bar{v}_\beta) g(\bar{u}_\rho, \bar{v}_\rho) g(\bar{u}_\alpha, \bar{v}_\rho) \times T_{13}(\bar{v}_\alpha) T_{23}(\bar{v}_\sigma) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha).
\]

Comparing this equation with (7.56) we see that \( \bar{Z}_3^{(1)} = -\bar{Z}_4 \).

Similarly, the sum (7.57) produces the term

\[
\bar{Z}_3^{(2)} = -T_{13}(u_a) \sum K(\{ \bar{v}_\rho, \bar{v}_\gamma \}| \bar{u}_\sigma) f(\bar{u}_\sigma, \bar{v}_\gamma) g(\bar{v}_\rho, \bar{v}_\gamma) g(\bar{u}_\rho, \bar{v}_\rho) g(\bar{u}_\alpha, \bar{v}_\rho) \times T_{13}(\bar{v}_\rho) T_{23}(\bar{v}_\mu) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\sigma) T_{22}(\bar{u}_\sigma).
\]

Here we again relabel the subsets \( \bar{v}_\gamma \rightarrow \bar{v}_\sigma, \bar{v}_\rho \rightarrow \bar{v}_\alpha, \bar{u}_\rho \rightarrow \bar{u}_\beta, \) and \( \bar{u}_\beta \rightarrow \bar{u}_\alpha \), what gives us

\[
\bar{Z}_3^{(2)} = -T_{13}(u_a) \sum K(\{ \bar{v}_\rho, \bar{v}_\sigma \}| \bar{u}_\alpha) f(\bar{u}_\alpha, \bar{u}_\alpha) g(\bar{v}_\alpha, \bar{v}_\alpha) g(\bar{v}_\rho, \bar{v}_\rho) g(\bar{u}_\rho, \bar{v}_\rho) \times T_{13}(\bar{v}_\sigma) T_{23}(\bar{v}_\alpha) T_{12}(\bar{v}_\rho) T_{12}(\bar{u}_\alpha) T_{22}(\bar{u}_\alpha).
\]

Comparing this expression with (7.58) we see that \( \bar{Z}_3^{(2)} = -\bar{Z}_2 \). Thus, the terms of the type (7.6) do cancel them selves, and we see that unwanted terms do not contribute to the final result.

Thus, we have proved, that the operator \( Y_{a,b}(\bar{u}, \bar{v}) \) satisfies recursion (4.1).
A Properties of DWPF

Let $\bar{u} = \{u_1, \ldots, u_n\}$ and $\bar{v} = \{v_1, \ldots, v_n\}$. The DWPF is symmetric function of $\bar{u}$ and symmetric function of $\bar{v}$. It behaves as $1/u_n$ (resp. $1/v_n$) as $u_n \rightarrow \infty$ (resp. $v_n \rightarrow \infty$) at other variables fixed. It has simple poles at $u_j = v_k$. The behavior of $K$ near these poles can be expressed in terms of DWPF with less number of arguments:

$$K(\bar{u}|\bar{v}) \bigg|_{u_n \rightarrow v_n} = g(u_n, v_n) f(v_n, \bar{v}) f(\bar{u}, u_n) K(\bar{u}_n|\bar{v}_n) + \text{reg},$$

(A.1)

where $\text{reg}$ means the regular part at $u_n \rightarrow v_n$ and we recall that $\bar{u}_n = \bar{u} \setminus u_n$ and $\bar{v}_n = \bar{v} \setminus v_n$.

One can also easily check that the DWPF possesses the properties:

$$K(\bar{u}, z - c|\bar{v}, z) = K(\{\bar{u}, z\}|\{\bar{v}, z + c\}) = -K(\bar{u}|\bar{v}),$$

$$K(\bar{u} - c|\bar{v}) = K(\bar{u}|\bar{v} + c) = (-1)^n \frac{K(\bar{v}|\bar{u})}{f(\bar{v}, \bar{u})}$$

(A.2)

$$K(\bar{u}|\bar{v}) \bigg|_{c \rightarrow -c} = K(\bar{v}|\bar{u}).$$

B Summation formulas

B.1 Summation of Cauchy determinants

Lemma B.1. Let $\bar{w}$, $\bar{u}$, and $\bar{v}$ be sets of complex variables, such that $\# \bar{u} = m_1$, $\# \bar{v} = m_2$, and $\# \bar{w} = m_1 + m_2$. Then

$$\sum g(\bar{w}_\alpha, \bar{u}) g(\bar{w}_\alpha, \bar{v}) g(\bar{w}_\alpha, \bar{w}_\alpha) = \frac{g(\bar{w}, \bar{u}) g(\bar{w}, \bar{v})}{g(\bar{u}, \bar{v})}.$$  

(B.1)

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_\alpha$ and $\bar{w}_\beta$, such that $\# \bar{w}_\alpha = m_1$ and $\# \bar{w}_\beta = m_2$.

Proof. The proof is based on a well known explicit representation for the Cauchy determinant. Let $\# \bar{u} = \# \bar{v} = n$. Then

$$g(\bar{u}, \bar{v}) = \Delta_n(\bar{u}) \Delta'_n(\bar{v}) \det_{n} \left( g(u_j, v_k) \right).$$  

(B.2)

Substituting (B.2) into (B.1) we obtain

$$\sum g(\bar{w}_\alpha, \bar{u}) g(\bar{w}_\alpha, \bar{v}) g(\bar{w}_\alpha, \bar{w}_\alpha) = \sum \Delta_{m_1}(\bar{w}_\alpha) \Delta'_{m_1}(\bar{u}) \det_{m_1} \left( g(w_{\alpha_j}, u_k) \right) \Delta_{m_2}(\bar{w}_\alpha) \Delta'_{m_2}(\bar{v}) \det_{m_2} \left( g(w_{\alpha_j}, v_k) \right) g(\bar{w}_\alpha, \bar{w}_\alpha).$$  

(B.3)
Obviously,
\[
\Delta_{m_1}(\bar{w}_\alpha)\Delta_{m_2}(\bar{w}_\alpha)g(\bar{w}_\alpha, \bar{w}_\alpha) = (-1)^{|P_{\alpha, \bar{\alpha}}|}\Delta_{m_1+m_2}(\bar{w}), \tag{B.4}
\]
where \(P_{\alpha, \bar{\alpha}}\) is the permutation mapping the union \(\{\bar{w}_\alpha, \bar{w}_\bar{\alpha}\}\) of the naturally ordered subsets \(\bar{w}_\alpha\) and \(\bar{w}_\bar{\alpha}\) into the naturally ordered set \(\{w_1, \ldots, w_{m_1+m_2}\}\). Then we arrive at
\[
\sum g(\bar{w}_\alpha, \bar{u})g(\bar{w}_\bar{\alpha}, \bar{v})g(\bar{w}_\alpha, \bar{w}_\alpha) = \Delta_{m_1+m_2}(\bar{w})\Delta'_{m_1}(\bar{u})\Delta'_{m_2}(\bar{v}) \sum (-1)^{|P_{\alpha, \bar{\alpha}}|} \det(g(w_{\alpha_j}, u_k)) \det(g(w_{\bar{\alpha}_j}, v_k)). \tag{B.5}
\]
The obtained sum is nothing but a development of a determinant with respect to \(m_1\) columns:
\[
\sum (-1)^{|P_{\alpha, \bar{\alpha}}|} \det(g(w_{\alpha_j}, u_k)) \det(g(w_{\bar{\alpha}_j}, v_k)) = \det_{m_1+m_2}(g(w_j, \gamma_k)), \tag{B.6}
\]
where \(\gamma = \{\bar{u}, \bar{v}\} = \{u_1, \ldots, u_{m_1}, v_1, \ldots, v_{m_2}\}\). Thus, we obtain
\[
\sum g(\bar{w}_\alpha, \bar{u})g(\bar{w}_\bar{\alpha}, \bar{v})g(\bar{w}_\alpha, \bar{w}_\alpha) = \Delta_{m_1+m_2}(\bar{w})\Delta'_{m_1}(\bar{u})\Delta'_{m_2}(\bar{v}) \det_{m_1+m_2}(g(w_j, \gamma_k)). \tag{B.7}
\]
It remains to use \((\text{B.2})\) for the determinant of \(g(w_j, \gamma_k)\):
\[
\det_{m_1+m_2}(g(w_j, \gamma_k)) = \frac{g(\bar{w}, \bar{u})g(\bar{w}, \bar{v})}{\Delta_{m_1+m_2}(\bar{w})\Delta'_{m_1+m_2}(\bar{u}, \bar{v})}. \tag{B.8}
\]
Substituting this into \((\text{B.7})\) we immediately arrive at \((\text{B.1})\).

**Lemma B.2.** Let \(\bar{w}, \bar{u},\) and \(\bar{v}\) be sets of complex variables, such that \(\#\bar{u} = m_1, \#\bar{v} = m_2,\) and \(\#\bar{w} = m_1 + m_2\). Then
\[
\sum K(\bar{w}_\alpha|\bar{u})K(\bar{v}|\bar{w}_\alpha)f(\bar{w}_\alpha, \bar{w}_\alpha) = (-1)^{m_1}f(\bar{w}, \bar{u})K_{m_1+m_2}(\{\bar{u} - c, \bar{v}\})|\bar{w}). \tag{B.9}
\]
The sum is taken with respect to all partitions of the set \(\bar{w}\) into subsets \(\bar{w}_\alpha\) and \(\bar{w}_\bar{\alpha}\), such that \(#\bar{w}_\alpha = m_1\) and \(#\bar{w}_\bar{\alpha} = m_2\).

The proof of this Lemma is given in \([29]\).

**Lemma B.3.** Let \(\#\bar{u} = \#\bar{v} = n\). Then for arbitrary complex \(\xi\)
\[
\sum K(\bar{v}|\{\bar{u}_\alpha, \xi\})g(\bar{u}_\alpha, \xi)f(\bar{u}_\alpha, \bar{u}_\alpha) = (f(\bar{u}, \xi) - f(\bar{v}, \xi))K(\bar{v}|\bar{u}). \tag{B.10}
\]
Here the sum is taken over partitions \(\bar{u} \Rightarrow \{\bar{u}_\alpha, \bar{u}_\bar{\alpha}\}\) with \(#\bar{u}_\alpha = 1\).

**Proof.** Since both parts of \((\text{B.10})\) decrease at \(\xi \to \infty\), it is enough to consider residues in the poles at \(\xi = u_j\) and \(\xi = v_j\). In the r.h.s. they are quite obvious. If \(\xi = u_j\) in the l.h.s., then the pole occurs if and only if \(u_j \in \bar{u}_\alpha\). Since \(#\bar{u}_\alpha = 1\), we conclude that \(\bar{u}_\alpha = u_j\), and there is only one term in the sum over partitions. Thus, we arrive at
\[
\sum K(\bar{v}|\{\bar{u}_\alpha, \xi\})g(\bar{u}_\alpha, \xi)f(\bar{u}_\alpha, \bar{u}_\alpha)|_{\xi = u_j} = g(u_j, \xi)f(u_j, u_j)K(\bar{v}|\bar{u}). \tag{B.11}
\]

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what coincides with the residue in the r.h.s. 

If $\xi = v_j$, then $K(\bar{v}|\{\bar{u}_{\alpha}, \xi\})$ has a pole, and we use recursion \([A.1]:\)

$$
\sum K(\bar{v}|\{\bar{u}_{\alpha}, \xi\})g(\bar{u}_{\alpha}, \xi)f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) \bigg|_{\xi \rightarrow v_j} = g(v_j, \xi)f(\bar{v}_j, v_j) \sum f(v_j, \bar{u}_{\alpha})K(\bar{v}_j|\bar{u}_{\alpha})g(\bar{u}_{\alpha}, v_j)f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) + \text{reg.} \quad (B.12)
$$

Now the sum over partitions can be calculated via lemma \([B.2]:\)

$$
\sum f(v_j, \bar{u}_{\alpha})K(\bar{v}_j|\bar{u}_{\alpha})g(\bar{u}_{\alpha}, v_j)f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) = -f(v_j, \bar{u}) \sum K(\bar{v}_j|\bar{u}_{\alpha})f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) \bigg|_{h(v_j, \bar{u})} = f(v_j, \bar{u}) \sum K(\bar{v}_j|\bar{u}_{\alpha})K(\bar{u}_{\alpha}|v_j + c)f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) = -K(\bar{v} | \bar{u}), \quad (B.13)
$$

where we used $1/h(v_j, \bar{u}) = -K(\bar{u}_{\alpha}|v_j + c)$. Hence,

$$
\sum K(\bar{v}|\{\bar{u}_{\alpha}, \xi\})g(\bar{u}_{\alpha}, \xi)f(\bar{u}_{\alpha}, \bar{u}_{\alpha}) \bigg|_{\xi \rightarrow v_j} = -g(v_j, \xi)f(\bar{v}_j, v_j)K(\bar{v} | \bar{u}) + \text{reg}, \quad (B.14)
$$

what coincides with the residue in the r.h.s. of \((B.10)\).

**Corollary B.1.** Under the conditions of lemma \([B.3]:\)

$$
\sum K(\{\bar{v}_{\alpha}, \xi\}|\bar{u})g(\bar{v}_{\alpha}, \xi)f(\bar{v}_{\alpha}, \bar{v}_{\alpha}) = (f(\xi, \bar{u}) - f(\xi, \bar{v}))K(\bar{v} | \bar{u}). \quad (B.15)
$$

*Here the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\alpha}, \bar{v}_{\alpha}\}$ with $\#\bar{v}_{\alpha} = 1$.**

This identity follows from \((B.10)\) after the replacements $c$ by $-c$ and $\bar{u} \leftrightarrow \bar{v}$.

**References**

[1] L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan, *Quantum Inverse Problem. I*, Theor. Math. Phys. 40 (1979) 688–706.

[2] L. D. Faddeev and L. A. Takhtajan, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Usp. Math. Nauk 34 (1979) 13; Russian Math. Surveys 34 (1979) 11 (Engl. transl.).

[3] L. D. Faddeev, in: Les Houches Lectures Quantum Symmetries, eds A. Connes et al, North Holland, (1998) 149.

[4] V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge: Cambridge Univ. Press, 1993.

[5] N. Kitanine, J. M. Maillet, V. Terras, *Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field*, Nucl. Phys. B 567 (2000) 554–582, arxiv:math-ph/9907019.
[6] N. Kitanine, K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, Form factor approach to dynamical correlation functions in critical models, J. Stat. Mech. 1209 (2012) P09001, arXiv:1206.2630.

[7] F. Göhmann, A. Klümper, A. Seel, Integral representations for correlation functions of the XXZ chain at finite temperature, J. Phys. A 37 (2004) 7625–7652, arXiv:hep-th/0405089.

[8] A. G. Izergin and V. E. Korepin, The Quantum Inverse Scattering Method Approach to Correlation Functions, Commun. Math. Phys. 94 (1984) 67–92.

[9] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Spin-spin correlation functions of the XXZ-1/2 Heisenberg chain in a magnetic field, Nucl. Phys. B 641 (2002) 487–518, arXiv:hep-th/0201045.

[10] S. Belliard, S. Pakuliak, E. Ragoucy, N. A. Slavnov, Bethe vectors of GL(3)-invariant integrable models, J. Stat. Mech. (2013) P02020, arXiv:1210.0768.

[11] F.C. Zhang, T.M. Rice, Effective Hamiltonian for the superconducting Cu oxides, Phys. Rev. B 37 (1988) 3759–3761.

[12] P. Schlottmann, Integrable narrow-band model with possible relevance to heavy Fermion systems, Phys. Rev. B 36 (1987) 5177–5185.

[13] P.A. Bares and G. Blatter, Supersymmetric t-J model in one dimension: Separation of spin and charge, Phys. Rev. Lett. 64 (1990) 2567–2570.

[14] S. Sarkar, Bethe-ansatz solution of the t-J model, J. Phys. A 23 (1990) L409–L414.

[15] P.A. Bares, G. Blatter, M. Ogata, Exact solution of the t-J model in one dimension at 2t = ±J: Ground state and excitation spectrum, Phys. Rev. B 44 (1991) 130–154.

[16] S. Sarkar, The supersymmetric t-J model in one dimension, J. Phys. A 24 (1991) 1137–1152.

[17] F.H.L. Essler and V. E. Korepin, Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric t-J model, Phys. Rev. B 46 (1992) 9147–9162.

[18] A. Foerster and M. Karowski, Algebraic properties of the Bethe ansatz for an spl(2,1)-supersymmetric t-J model, Nucl. Phys. B 396 (1993) 611–638.

[19] S. Khoroshkin, S. Pakuliak, Weight function for $U_q(\widehat{sl}_3)$ Theor. Math. Phys. 145:1 (2005) 1373–1399, arXiv:math/0610433.

[20] S. Khoroshkin, S. Pakuliak, V. Tarasov, Off-shell Bethe vectors and Drinfeld currents, J. Geom. Phys. 57:8 (2007) 1713–1732, arXiv:math/0610517.

[21] L. Frappat, S. Khoroshkin, S. Pakuliak, E. Ragoucy, Bethe Ansatz for the Universal Weight Function, Ann. H. Poincaré 10 (2009) 513–548, arXiv:0810.3135.
[22] S. Belliard, S. Pakuliak, E. Ragoucy, *Bethe Ansatz and Bethe Vectors Scalar Products*, SIGMA 6 (2010) 094, arXiv:1012.1455.

[23] V. Tarasov and A. Varchenko, *Combinatorial formulae for nested Bethe vector*, SIGMA 9 (2013) 048 and arXiv:math.QA/0702277.

[24] S. Belliard and É. Ragoucy, *The nested Bethe ansatz for 'all' closed spin chains*, J. Phys. A 41 (2008) 295202, arXiv:math-ph/0804.2822.

[25] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Bethe vectors for models based on the super-Yangian \( Y(gl(m|n)) \)*, arXiv:1604.02311.

[26] P.P. Kulish and E.K. Sklyanin, *On the solution of the Yang-Baxter equation*, Zap. Nauchn. Semin. LOMI 95 (1980) 129–160; J. Sov. Math. 19 (1982) 1596 (Engl. transl.).

[27] V. E. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. 86 (1982) 391–418.

[28] A. G. Izergin, *Partition function of the six-vertex model in a finite volume*, Dokl. Akad. Nauk SSSR 297 (1987) 331–333; Sov. Phys. Dokl. 32 (1987) 878–879 (Engl. transl.).

[29] S. Belliard, S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Highest coefficient of scalar products in \( SU(3) \)-invariant integrable models*, J. Stat. Mech. Theory Exp. (2012) P09003, arXiv:1206.4931.