Directed cycles have the edge-Erdős-Pósa property

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Abstract

In this short note we prove that for every $k \in \mathbb{N}$ there is a $t_k \in \mathbb{N}$ such that for every digraph $G$ there are either $k$ edge-disjoint directed cycles in $G$ or a set $X$ of at most $t_k$ edges such that $G - X$ contains no directed cycle.

1 Introduction

If $\mathcal{F}$ is a family of (directed or undirected) graphs, we say that $\mathcal{F}$ has the vertex/edge-Erdős-Pósa property if there is a sequence $(t_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$ and every (di-)graph $G$, either $G$ contains $k$ vertex/edge-disjoint subgraphs each isomorphic to a member of $\mathcal{F}$, or there is a set $X$ of at most $t_k$ vertices/edges such that $G - X$ contains no subgraph isomorphic to a member of $\mathcal{F}$.

Many classes of undirected graphs are known to have the vertex-Erdős-Pósa property but only few of them are also investigated regarding the edge property. A summary of some results can be found in [4] and in a table in [1].

When we address the Erdős-Pósa property in directed graphs only a few results are known — and nearly all of them regarding the vertex version. In 1996, Reed et al. in [5] proved that directed cycles in digraphs have the vertex-Erdős-Pósa property. The bound of $t_k$ in terms of $k$ is extremely large.

Theorem 1 (Reed et al. [5]). For every $k \in \mathbb{N}$ there is a $t_k \in \mathbb{N}$ such that for every digraph $G$ there are either $k$ vertex-disjoint directed cycles in $G$ or a set $X \subseteq V(G)$ of size at most $t_k$ such that $G - X$ contains no directed cycle.

To the best of our knowledge the edge-version for directed cycles has been an open problem so far. In this paper we will prove that it is indeed true.

2 Proof of the theorem

Even et al. [2] showed a correspondence between edge-hitting sets in $G$ and vertex-hitting sets in $L(G)$ and for that they also defined the directed line graph. We will show an analogous statement for edge-disjoint and vertex-disjoint directed cycles and can then prove our main theorem.

Let $G = (V(G), E(G))$ be a digraph. The directed line graph $L(G)$ of $G$ has $E(G)$ as its vertex set and two directed edges $(a, b), (c, d)$ of $G$ are joined by a directed edge from $(a, b)$ to $(c, d)$ in $L(G)$ if and only if $b = c$. If $T$ is a subgraph of $L(G)$, we denote by $G[T]$ the minimal subgraph of $G$ that has edge set $V(T)$. For a vertex $u \in V(G)$ we define $E(u)$ as the set of all edges that have $u$ as their first or second vertex. It is easy to see that the subdigraph $L_u = L(G)[E(u)]$ contains no directed cycle.

Lemma 2. For every directed cycle $C$ in $G$, $C' = L(G)[E(C)]$ is a directed cycle in $L(G)$. For every directed cycle $C'$ in $L(G)$, $C = G[C']$ contains a directed cycle in $G$.

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Proof. Let \( u_1, \ldots, u_\ell \) be the vertices of \( C \). Therefore, the vertices of \( C' \) are \( v_1 = (u_1, u_2), \ldots, v_\ell = (u_\ell, u_1) \) which are all distinct as \( u_1, \ldots, u_\ell \) are distinct. The vertices \( v_i \) and \( v_{i+1} \pmod{\ell} \) in \( L(G) \) are joined by a directed edge in \( L(G) \) because the endvertex of \( v_i \) is the startvertex of \( v_{i+1} \pmod{\ell} \). Hence, \( C' \) is a directed cycle.

Let \( C' \) be a directed cycle in \( L(G) \) with vertex set \( \{v_i = (u_i^1, u_i^2) : i = 1, \ldots, \ell\} \). As \( v_i \) and \( v_{i+1} \pmod{\ell} \) are joined by a directed edge, we have \( u_i^2 = u_{i+1}^1 \pmod{\ell} \) for every \( i = 1, \ldots, \ell \).

Now, \( G[C'] \) has vertex set \( \{u_i^1, u_i^2\} \) where not all vertices are necessarily distinct and the edge set consists of all edges \( v_i = (u_i^1, u_{i+1}^1) \pmod{\ell} \), \( i = 1, \ldots, \ell \). Let \( i < j \) be integers such that \( u_i^1 = u_j^1 \) and such that \( j - i \) is minimal. It then follows that the graph \( G[\{v_i, \ldots, v_j\}] \) is a directed cycle contained in \( G[C'] \). \( \square \)

Now we can prove our main theorem.

**Theorem 3.** For every \( k \in \mathbb{N} \) there is a \( t_k \in \mathbb{N} \) such that for every digraph \( G \) there are either \( k \) edge-disjoint directed cycles in \( G \) or a set \( X \subseteq E(G) \) of size at most \( t_k \) such that \( G - X \) contains no directed cycle.

**Proof.** Apply Theorem 1 to the directed line graph \( L(G) \) of \( G \). If the theorem returns \( k \) vertex-disjoint directed cycles \( C_1, \ldots, C_k \) in \( L(G) \), their preimages \( G[C_i] \) in \( G \) are edge-disjoint and each contain a directed cycle in \( G \), by Lemma 2. Hence, \( G \) contains \( k \) edge-disjoint directed cycles.

If the theorem returns a vertex hitting set \( X \) of size \( t_k \) in \( L(G) \), the same set \( X \) (as a set of edges) is an edge hitting set for directed cycles in \( G \). Namely, if \( C \) was a directed cycle in \( G - X \), then, by Lemma 2, the digraph \( L(G)[C] \) would be a directed cycle in \( L(G) \) that avoids \( X \) contradicting the choice of \( X \) as a hitting set in \( L(G) \). \( \square \)

3 Discussion

Very lately, Kawarabayashi and Kreutzer mentioned in their paper [3] on the directed grid theorem that for any \( \ell \in \mathbb{N} \), the class of directed cycles of length at least \( \ell \) have the vertex-Erdős-Pósa property. So is it possible to use the method above also for long directed cycle? Unfortunately not. Although the image of a long cycle in \( G \) is a long cycle in \( L(G) \) again, the preimage of a long cycle in \( L(G) \) may consist of many short cycles in \( G \).

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References

[1] H. Bruhn, M. Heinlein, and F. Joos, Frames, A-paths and the Erdős-Pósa-property, to appear in SIAM Journal on Discrete Mathematics.

[2] G. Even, J. (Seffi) Naor, B. Schieber, and M. Sudan, Approximating minimum feedback sets and multicut in directed graphs, Algorithmica 20 (1998), no. 2, 151–174.

[3] K. Kawarabayashi and S. Kreutzer, The directed grid theorem, arXiv:1411.5681 (2014).

[4] J.-F. Raymond and D. Thilikos, Recent techniques and results on the Erdős-Pósa property, to appear in Disc. App. Math.

[5] Bruce Reed, Neil Robertson, Paul Seymour, and Robin Thomas, Packing directed circuits, Combinatorica 16 (1996), no. 4, 535–554.