We study the Lagrangian structure of relativistic Vlasov systems, such as the relativistic Vlasov-Poisson and the relativistic quasi-electrostatic limit of Vlasov-Maxwell equations. We show that renormalized solutions of these systems are Lagrangian and that these notions of solution, in fact, coincide. As a consequence, finite-energy solutions are shown to be transported by a global flow. Moreover, we extend the notion of generalized solution for “effective” densities, and we prove the existence of such solutions. Finally, under a higher integrability assumption of the initial condition, we show that solutions have every energy bounded, even in the gravitational case. These results extend to our setting those recently obtained for the Vlasov-Poisson system in a series of papers by Ambrosio, Colombo, and Figalli; here, we analyze relativistic systems and also consider the contribution of the magnetic force into the evolution equation.

**KEYWORDS**
Lagrangian flows, relativistic Vlasov equation, renormalized solutions, transport equations

**MSC CLASSIFICATION**
35F25, 35Q83, 34A12, 37C10

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**1 | INTRODUCTION**

**1.1 | Overview**

In this paper, we are interested in the Lagrangian structure of relativistic Vlasov systems. These systems describe the evolution of a nonnegative distribution function \( f : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty) \) under the action of a self-consistent acceleration:

\[
\begin{align*}
\partial_t f_t + \hat{v} \cdot \nabla_x f_t + (E_t + \hat{v} \times B_t) \cdot \nabla_v f_t &= 0 & \text{in } (0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3; \\
\rho_t(x) &= \int_{\mathbb{R}^3} f_t(x,v) \, dv & \text{in } (0,\infty) \times \mathbb{R}^3; \\
E_t(x) &= \sigma_E \int_{\mathbb{R}^3} J_t(y) K(x-y) \, dy & \text{in } (0,\infty) \times \mathbb{R}^3; \\
B_t(x) &= \sigma_B \int_{\mathbb{R}^3} J_t(y) \times K(x-y) \, dy & \text{in } (0,\infty) \times \mathbb{R}^3.
\end{align*}
\]  

(1.1)

Here, \( f_t(x,v) \) denotes the distribution of particles with position \( x \) and velocity \( v \) at time \( t \), \( \hat{v} \) := \((1 + |v|^2)^{-1/2}v \) is the velocity of particles (we assume the speed of light is \( c = 1 \)), \( \sigma_E \in \{0,\pm 1\} \), \( \sigma_B \in \{0,1\} \), and \( K : \mathbb{R}^3 \to \mathbb{R}^3 \) is given by \( K(x) = (4\pi)^{-1}x/|x|^3 \).
Such systems are very important in mathematical physics and appear in a variety of physical models. Typically, $\rho_t$ and $J_t$ represent the density of particles and the relativistic particle current density and $E_t$ and $B_t$ the electric and magnetic fields, respectively. We postpone a derivation of (1.1) and a more complete description of its physical significance to Appendix A, but summarize what (1.1) models depending on the values of $\sigma_E$, $\sigma_B$, and critical charge $q_c$:

**Relativistic Vlasov-Poisson equations**: charged particles under a self-consistent electric field or particles under self-consistent electric and gravitational fields with particle charge $q > q_c$ if $\sigma_E = 1$, $\sigma_B = 0$; motion of galaxy clusters under a gravitational field or particles under self-consistent electric and gravitational fields with particle charge $q < q_c$ if $\sigma_E = -1$, $\sigma_B = 0$ (see, for instance, Rein \(^1\), Section 1 and references therein);

**Relativistic Vlasov-Biot-Savart equations**: charged particles under a self-consistent magnetic field; particles under self-consistent quasi-electrostatic (QES) electromagnetic and gravitational fields with particle charge $q = q_c$ if $\sigma_E = 0$ and $\sigma_B = 1$;

**QES relativistic Vlasov-Maxwell equations**: charged particles under a self-consistent QES electromagnetic field; particles under self-consistent QES electromagnetic and gravitational fields with particle charge $q > q_c$ if $\sigma_E = \sigma_B = 1$;

**Relativistic gravitational Vlasov-Biot-Savart equations**: charged particles under self-consistent magnetic and gravitational fields; particles under self-consistent quasi-magnetostatic (QMS) electromagnetic and gravitational fields with particle charge $q < q_c$ if $\sigma_E = -1$ and $\sigma_B = 1$.

Note we allow $\sigma_B = \sigma_E = 0$, that is, (1.1) to be the linear transport equation, but its theory is classical and we shall not consider it. Moreover, the fact that the critical charge evolution system coincides with the Vlasov-Biot-Savart system suggests that the displacement current $\partial_t E_t$ behaves like a lower order term; see (A2) in Appendix A. This is well-known in Electrodynamics\(^2\); Maxwell predicted theoretically as a correction of Ampère’s law. Nonetheless, we show that it behaves like a lower order term in the magnetic potential energy; see Lemma 4.5 and Remark 4.2.

Concerning the existence of classical solutions of (1.1), we refer to literature\(^3–5\) where the existence of local solutions for the relativistic Vlasov-Poisson system is established. As mentioned in Rein \(^1\), Section 1.5 very little is known regarding the existence of global solutions for general initial data. Existence results can be found, for instance, for spherically and axially symmetric initial data; see Glassey and Schaeffer\(^6,7\). In the aforementioned results, higher integrability assumptions and moment conditions on the initial data are required. However, in order to be more physically relevant, it is best to avoid such hypotheses even though classical solutions may fail to exist. We thus consider renormalized and generalized solutions, which allow us to establish a Lagrangian structure for the system, global existence results, and (under suitable energy bounds) a global in time maximal regular flow, as we explain in the next section.

### 1.2 Main results

For our purposes, a crucial observation is that (1.1) can be written as

$$\partial_t f_t + b_t \cdot \nabla_{x,v} f_t = 0,$$

where, for each fixed $t > 0$, the vector field $b_t : \mathbb{R}^6 \to \mathbb{R}^6$ is given by $b_t(x,v) = (\hat{v}, E_t + \hat{v} \times B_t)$. Moreover, this vector field is divergence-free, since

$$\nabla_{x,v} \cdot b_t = \nabla_v \cdot (\hat{v} \times B_t) = (\nabla_v \times \hat{v}) \cdot B_t - \hat{v} \cdot (\nabla_v \times B_t) = 0.$$

By the transport nature of (1.2), it is expected that solutions have a Lagrangian structure, meaning that the initial condition $f_0$ is transported to $f_t$ by an associated flow. In the weak regularity regime, however, the existence of such flow is not guaranteed by the classical Cauchy-Lipschitz theory. Indeed, since $K$ is locally integrable, we have $E_t, B_t \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ whenever $f_t \in L^1(\mathbb{R}^6)$, so that $b_t$ is only in the space $L^1_{\text{loc}}(\mathbb{R}^6; \mathbb{R}^6)$.

Since $b_t$ is divergence-free, (1.2) can be rewritten as

$$\partial_t f_t + \nabla_{x,v} \cdot (b_t f_t) = 0.$$

\(^1\)This terminology, albeit not standard, is in analogy to the Vlasov-Poisson system, since the magnetic field obeys the Biot-Savart law.
The latter can be interpreted in the distributional sense provided \( b_t f_t \) is locally integrable which, however, does not follow only from the assumption \( f_t \in L^1(\mathbb{R}^6) \). To treat this problem, we introduce a function \( \beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) such that

\[
\partial_t \beta(f_t) + \nabla_{x,v} \cdot (b_t \beta(f_t)) = 0 \tag{1.3}
\]

whenever \( f_t \) is a smooth solution of (1.2); of course, such equality holds since \( b_t \) is divergence-free and due to the chain rule. Hence, \( b_t \beta(f_t) \in L^1_{\text{loc}} \), which leads to the concept of a renormalized solution, as in the celebrated results by DiPerna-Lions.  

**Definition 1.1** (Renormalized solution). For a Borel vector field \( b \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6; \mathbb{R}^6) \), we say that a Borel function \( f \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6) \) is a renormalized solution of (1.2) starting from \( f_0 \) if (1.3) holds in the sense of distributions, that is,

\[
\int_{\mathbb{R}^6} \phi_0(x,v) \beta(f_0(x,v)) \, dx \, dv + \int_0^T \int_{\mathbb{R}^6} \left[ \partial_t \phi_t(x,v) + \nabla_{x,v} \phi_t(x,v) \cdot b_t(x,v) \right] \beta(f_t(x,v)) \, dx \, dv \, dt = 0 \tag{1.4}
\]

for all \( \phi \in C^1_c([0,T] \times \mathbb{R}^6) \) and \( \beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

Moreover, \( f \in L^\infty((0,T); L^1(\mathbb{R}^6)) \) is called a renormalized solution of (1.1) starting from \( f_0 \) if, by setting

\[
\rho_t(x) := \int_{\mathbb{R}^6} f_t(x,v) \, dv, \quad E_t(x) := \sigma_E \int_{\mathbb{R}^6} \rho_t(y) K(x-y) \, dy, \\
J_t(x) := \int_{\mathbb{R}^6} \hat{\nu} f_t(x,v) \, dv, \quad B_t(x) := \sigma_B \int_{\mathbb{R}^6} J_t(y) \times K(x-y) \, dy, \quad \text{and} \quad b_t(x,v) := (\hat{\nu},E_t(x) + \hat{\nu} \times B_t(x)), \tag{1.5}
\]

we have that \( f_t \) satisfies (1.4), for every \( \phi \in C^1_c([0,T] \times \mathbb{R}^6) \), with \( b_t \) as in (1.5).

Observe that the integrability assumption \( f_t \in L^1(\mathbb{R}^6) \) is used so that \( \rho_t, J_t, E_t, \) and \( B_t \) are well defined. From now on, we refer to \( E_t \) and \( B_t \) as the electric and the magnetic fields, respectively, even though \( E_t \) may represent a gravitational field as well; see Appendix A.

Our first main result shows that distributional or renormalized solutions of (1.1) are in fact Lagrangian solutions. This gives a characterization of solutions of (1.1), since Lagrangian solutions are generally stronger than renormalized or distributional solutions.

**Theorem 1.1.** Let \( T > 0 \) and \( f \) be a nonnegative function. Assume \( f \in L^\infty((0,T); L^1(\mathbb{R}^6)) \) is weakly continuous in the sense that

\[
t \mapsto \int_{\mathbb{R}^6} f_t(\varphi) \, dx \, dv \quad \text{is continuous for any} \quad \varphi \in C_c(\mathbb{R}^6).
\]

Assume further that:

(i) \( f \in L^\infty((0,T); L^\infty(\mathbb{R}^6)) \) and \( f_t \) is a distributional solution of (1.1) starting from \( f_0 \); or

(ii) \( f_t \) is a renormalized solution of (1.1) starting from \( f_0 \).

Then, \( f_t \) is a Lagrangian solution transported by the Maximal Regular Flow \( X(t,x) \) associated to \( b_t(x,v) = (\hat{\nu},E_t(x) + \hat{\nu} \times B_t(x)) \) (see Definition 2.2 and Definition 2.4), starting from \( 0 \). In particular, \( f_t \) is renormalized.

Next, in Definition 3.1, we introduce the concept of generalized solutions, which allows the electromagnetic field to be generated by effective densities \( \rho^{\text{eff}} \) and \( J^{\text{eff}} \). This may be interpreted as particles vanishing from the phase space but still contributing in the electromagnetic field in the physical space. In fact, generalized solutions are renormalized if the number of particles is conserved in time, as follows from Lemma 3.1. This indicates that, should renormalized solutions fail to exist, there must be a loss of mass/charge as \( \hat{\nu} \) approaches the speed of light.

Our second main theorem provides, under minimal assumptions on the initial datum, the global existence of generalized solutions.
Theorem 1.2 (Existence of generalized solutions). Let $f_0 \in L^1(\mathbb{R}^6)$ be a nonnegative function. Then there exists a generalized solution $(\rho_t^g, J_t^g, E_t^g)$ of (1.1) starting from $f_0$ (see Definition 2.1). Moreover, the map

$$t \in [0, \infty) \mapsto f_t \in L^1_{\text{loc}}(\mathbb{R}^6)$$

is continuous and the solution $f_t$ is transported by the Maximal Regular Flow associated to field $b_t^{\text{eff}}(x, v) = (\hat{v}, E_t^{\text{eff}} + \hat{v} \times B_t^{\text{eff}})$.

In view of Theorem 1.2, if we assume higher integrability on the initial datum and bounded initial energy, we can prove the existence of a global Lagrangian solution. Moreover, we show strong continuity of densities and fields and that each energy remains bounded in later times. Furthermore, we emphasize that our result holds even in the gravitational case $\sigma_E = -1$.

Theorem 1.3 (Existence of global Lagrangian solution). Let $f_0$ be a nonnegative function with every energy bounded (see Definition 4.1). Then there exists a global Lagrangian (hence renormalized) solution $f_t \in C([0, \infty); L^1(\mathbb{R}^6))$ of (1.1) with initial datum $f_0$, and the flow is globally defined on $[0, \infty)$ for $f_0$-almost every $(x, v) \in \mathbb{R}^6$, with $f_t$ being the image of $f_0$ through the incompressible flow.

Moreover, the following properties hold:

(i) the densities $\rho_t, I_t$ and the fields $E_t, B_t$ are strongly continuous in $L^1_{\text{loc}}(\mathbb{R}^6)$;

(ii) for every $t \geq 0$, $f_t$ has every energy bounded independently of time.

1.3 Structure of the paper

The paper is organized as follows. As previously mentioned, a discussion of the physical interpretation of (1.1) is presented in Appendix A. In Section 2, we prove Theorem 1.1. More explicitly, we rely on the machinery for nonsmooth vector fields developed in\(^9\) to prove the equivalence of renormalized and Lagrangian solutions. Moreover, in Corollary 2.1, we show that if the electromagnetic and relativistic energies are integrable in $[0, T]$, then its associated flow is globally defined in time. In Section 3, we extend the notion of generalized solutions from Ambrosio et al.\(^10,\) Definition 2.6 to our setting (see Definition 3.1) in order to allow for an “effective” density current of particles (along with the corresponding “effective” density of particles), and we prove the existence of a Lagrangian solution with the “effective” acceleration (Theorem 1.2). Finally, in Section 4, we prove Theorem 1.3 under the condition of each bounded energy (see Definition 4.1), obtaining a globally defined flow and a solution of (1.1) for all range of $\sigma_E$ and $\sigma_B$.

2 LAGRANGIAN SOLUTION AND ASSOCIATED FLOW

In this section, we prove Theorem 1.1, which says that Lagrangian and renormalized solutions of (1.1) are equivalent. For this, we use the machinery developed in Ambrosio et al.\(^10,\) Sections 4 and 5 combined with a version of Ambrosio et al.\(^10,\) Theorem 4.4 that we show holds for our vector field $b$ as well. From now on, we denote by $\mathcal{M} := \mathcal{M}(\mathbb{R}^6)$ the space of measures in $\mathbb{R}^6$ with finite total mass, by $\mathcal{M}_+ \mathcal{M}(\mathbb{R}^6)$ the space of nonnegative measures with finite total mass, by $\mathcal{AC}(I; \mathbb{R}^6)$ the space of absolutely continuous curves on the interval $I$ with values in $\mathbb{R}^6$, and by $\mathcal{L}^6$ the Lebesgue measure in $\mathbb{R}^6$. We begin with the preliminary definitions of renormalized solutions, of regular and of regular and maximal regular flows:

Definition 2.1 (Regular flow). Fix $r_1 < r_2$ and $B \subseteq \mathbb{R}^6$ a Borel set. For a Borel vector field $b : (r_1, r_2) \times \mathbb{R}^6 \rightarrow \mathbb{R}^6$, we say that $X : [r_1, r_2] \times B \rightarrow \mathbb{R}^6$ is a regular flow with field $b$ when

(i) for a.e. $x \in B$, we have that $X(\cdot, x) \in \mathcal{AC}([r_1, r_2]; \mathbb{R}^6)$ and that it solves the equation $\dot{x}(t) = b(x(t))$ a.e. in $(r_1, r_2)$ with initial condition $X(r_1, x) = x$ (of course, the absolutely continuity hypothesis of $X(\cdot, x)$ is to ensure that it is a.e. differentiable);

(ii) there exists $C > 0$ such that $X(t, \cdot)_B(\mathcal{L}^6 \cap B) \leq C \mathcal{L}^6$ for all $t \in [r_1, r_2]$. Note that $C$ can depend on the particular flow $X$.

Here, we denote by $X_\# \mu$ the pushforward of a measure $\mu$ by $X$ and by $\nu \ll B$ the restriction of the measure $\nu$ to the set $B$.

Definition 2.2 (Maximal regular flow). For every $s \in (0, T)$, a Borel map $X(\cdot, s, \cdot)$ is said to be a maximal regular flow (starting at $s$) if there exist two Borel maps $T^+_s X : \mathbb{R}^6 \rightarrow (s, T], T^-_s X : \mathbb{R}^6 \rightarrow [0, s)$ such that $X(\cdot, s, x)$ is defined in $(T^+_s X(\cdot), T^-_s X(\cdot))$ and
(i) for a.e. \( x \in \mathbb{R}^6 \), we have that \( X(\cdot, s, x) \in \text{AC}(T_{s,X}^-, T_{s,X}^+; \mathbb{R}^6) \) and that it solves the equation \( \dot{x}(t) = b_1(x(t)) \) a.e. in \((T_{s,X}^-, T_{s,X}^+)) \) with \( X(s, x, x) = x \);

(ii) there exists a constant \( C > 0 \) such that \( X(t, s, \cdot, \cdot) \in \mathcal{L}(\mathcal{Z}_s; \mathbb{R}^6) \) for \( 0 < t < T_{s,X}^+ \), \( s \leq T_{s,X} \), and \( \lim_{t \to T_{s,X}^-} |X(t, s, x)| = \infty \) for all \( t \in [0, T] \). As before, this constant \( C \) can depend on \( \mathbf{X} \) and \( s \);

(iii) for a.e. \( x \in \mathbb{R}^6 \), either \( T_{s,X}^+ = T \) and \( X(\cdot, s, x) \in \mathcal{L}(\mathcal{Z}_s; \mathbb{R}^6) \), or \( \lim_{t \to T_{s,X}^-} |X(t, s, x)| = \infty \). Analogously, either \( T_{s,X} = 0 \) and \( X(\cdot, s, x) \in \mathcal{L}(\mathcal{Z}_s; \mathbb{R}^6) \), or \( \lim_{t \to T_{s,X}^-} |X(t, s, x)| = \infty \).

The following lemma (compare with Ambrosio et al.\(^{10}\), Theorem 4.4), combined with the facts that \( b_1 \) is divergence-free in the sense of distribution a.e. in time and that \( b \in L^\infty((0, T); L^1_{moc}(\mathbb{R}^6; \mathbb{R}^6)) \), provides a sufficient condition to the existence and the uniqueness of a maximal regular flow for the continuity equation.

**Lemma 2.1.** Let \( b : (0, T) \times \mathbb{R}^6 \to \mathbb{R}^6 \) be given by \( b_1(x, v) = (b_{11}(v), b_{22}(x, v)) \), where

\[
b_{11}(x, v) = K \ast \rho_t(x) + b_{11}(v) \times \int_{\mathbb{R}^1} K(y - x) \times dJ_t(y) = : K \ast \rho_t(x) + b_{11}(v) \times \tilde{b}_{11}(x),
\]

with \( \rho \in L^\infty((0, T); \mathcal{M}_+(\mathbb{R}^3)) \) and \( |J| \in L^\infty((0, T); \mathcal{M}_+(\mathbb{R}^3)) \). Then, \( b \) satisfies the following: for any nonnegative \( \tilde{\rho} \in L^\infty(\mathbb{R}^3) \) with compact support and any closed interval \([a, b] \subset (0, T)\), both continuity equations

\[
\frac{d}{dt} \rho_t \pm \nabla_{x,v} \cdot (b_1 \rho_t) = 0 \text{ in } (a, b) \times \mathbb{R}^6
\]

have at most one solution in the class of all nonnegative weakly* continuous functions \([a, b] \ni t \mapsto \tilde{\rho}_t \) with \( \rho_a = \tilde{\rho} \) and \( \cup_{t \in [a, b]} \text{Supp} \rho_t \subset \mathbb{R}^6 \).

**Proof.** Let \( \mathcal{P}(X) \) be the set of probability measures on \( X \) and

\[
e_t : C([0, T]; \mathbb{R}^6) \longrightarrow \mathbb{R}^6
\]

the evaluation map at time \( t \), which means \( e_t(\eta) := \eta(t) \). By the same argument as in Ambrosio et al.\(^{10}\), Theorem 4.4, it is enough to show that given \( \eta \in \mathcal{P}(C([0, T]; B_{R \times B_{R}})) \) for some \( R > 0 \) concentrated on integral curves of \( b \) such that \((e_t)_\# \eta \leq C_{\eta} \mathcal{Z}_s^6\) for all \( t \in [0, T] \), the disintegration \( \eta_t \) of \( \eta \) with respect to \( e_0 \) is a Dirac delta for \( \tilde{\rho} \)-a.e. \( x \). Recall that the disintegration of \( \eta \) with respect to \( e_0 \) is a family of measures \( \eta_t \) such that, for all \( E \in C([0, T]; B_{R \times B_{R}}) \),

\[
\eta(E) = \int_{\mathbb{R}^6} \eta_t(E \cap e_0^{-1}(x)) dx.
\]

For this purpose, Ambrosio et al.\(^{10}\) consider the function

\[
\Phi_{\delta, \zeta}(t) := \int_0^T \int_0^{\infty} \log \left( 1 + \frac{|y^1(t) - \eta^1(t)|}{\zeta \delta} + \frac{|y^2(t) - \eta^2(t)|}{\delta} \right) d\mu(x, \eta, \gamma),
\]

where \( \delta, \zeta \in (0, 1) \) are small constants to be chosen later, \( t \in [0, T] \), \( \tilde{\rho} : = (e_0)_\# \eta, d\mu(x, \eta, \gamma) : = d\eta_t(\gamma) d\eta_t(\eta) d\tilde{\rho}(x)^\dagger \), with notation

\[
\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{R}^3 \times \mathbb{R}^3,
\]

and assume by contradiction that \( \eta_t \) is not a Dirac delta for \( \tilde{\rho} \)-a.e. \( x \), which means that there exists a constant \( a > 0 \) such that

\[
\int_0^T \left( \int_0^T \min(|\gamma(t) - \eta(t)|, 1) dt \right) d\mu(x, \eta, \gamma) \geq a.
\]

Note that \( \mu \in \mathcal{P}(\mathbb{R}^3 \times C([0, T]; \mathbb{R}^3)^2) \) and \( \Phi_{\delta, \zeta}(0) = 0. \)
Indeed, if $\eta_0$ is a Dirac delta for $\tilde{\rho}$-a.e. $x$, the integrand above would vanish.

Moreover, they show that, without loss of generality, by assuming $a \leq 2T$, it holds
\[
\Phi_{\delta, \xi}(t_0) \geq \frac{a}{2T} \log \left(1 + \frac{a}{2\delta T}\right).
\]

Now, computing the time derivative of $\Phi_{\delta, \xi}$, we have that
\[
\frac{d\Phi_{\delta, \xi}}{dt}(t) \leq \iint \left(\frac{|b_{11}(y^2(t)) - b_{11}(\eta^2(t))|}{\zeta \delta + |y^2(t) - \eta^2(t)|} + \frac{\zeta |b_{11}(y^2(t)) \times (b_{22}(y^1(t)) - b_{22}(\eta^1(t)))|}{\zeta \delta + |y^1(t) - \eta^1(t)|}\right)
\]
\[
+ \frac{\zeta |(b_{11}(y^2(t)) - b_{11}(\eta^2(t))) \times (b_{22}(y^1(t)) - b_{22}(\eta^1(t)))|}{\zeta \delta + |y^1(t) - \eta^1(t)|} + \frac{\eta |K \ast \rho_1(y^1(t)) - K \ast \rho_1(\eta^1(t))|}{\zeta \delta + |y^1(t) - \eta^1(t)|}\bigg) \, d\mu(x, \eta, \gamma).
\]

By our assumption on $b_{11}$, the first summand is easily estimated using the Lipschitz regularity of $b_{11}$ in $B_R$:
\[
\iint \frac{|b_{11}(y^2(t)) - b_{11}(\eta^2(t))|}{\zeta \delta + |y^2(t) - \eta^2(t)|} \, d\mu(x, \eta, \gamma) \leq \frac{||Vb_1||_{L^\infty(0,T;B_R)}}{\zeta}.
\]

Analogously, the third summand is estimated using that $b_{22}$ is locally integrable and the Lipschitz regularity of $b_{11}$ in $B_R$:
\[
\iint \frac{\zeta |(b_{11}(y^2(t)) - b_{11}(\eta^2(t)) \times (b_{22}(y^1(t)) - b_{22}(\eta^1(t)))|}{\zeta \delta + |y^2(t) - \eta^2(t)|} \, d\mu(x, \eta, \gamma)
\]
\[
\leq \zeta \||Vb_1||_{L^\infty(0,T;B_R)}||b_{22}||_{L^\infty(0,T;L^1(B_R))}.
\]

For the second term, we have
\[
\iint \frac{\zeta |(b_{11}(y^2(t)) \times (b_{22}(y^1(t)) - b_{22}(\eta^1(t)))|}{\zeta \delta + |y^1(t) - \eta^1(t)|} \, d\mu(x, \eta, \gamma)
\]
\[
\leq C ||b_1||_{L^\infty(0,T;B_R)} \iint \frac{\zeta |K \ast \tilde{\rho}(y^1(t)) - K \ast \tilde{\rho}(\eta^1(t))|}{\zeta \delta + |y^1(t) - \eta^1(t)|} \, d\mu(x, \eta, \gamma),
\]
where $\tilde{\rho}(y) := \sup_{t \in [0, T]} |J_t|(y)$. Since $J_t \in L^\infty((0, T); \mathcal{M}(\mathbb{R}^d))$, its total variation is well-defined and has finite measure, thus
\[
\tilde{\rho} \in L^\infty((0, T); \mathcal{M}_+(\mathbb{R}^d)).
\]

By Ambrosio et al.,\textsuperscript{10} Theorem 4.4 estimate (4.13) we have that\textsuperscript{2}
\[
\iint \frac{\zeta |K \ast \tilde{\rho}(y^1(t)) - K \ast \tilde{\rho}(\eta^1(t))|}{\zeta \delta + |y^1(t) - \eta^1(t)|} \, d\mu(x, \eta, \gamma) \leq C \zeta \left(1 + \log \left(\frac{C}{\zeta \delta}\right)\right),
\]
where $\tilde{\rho} \in L^\infty((0, \infty); \mathcal{M}_+(\mathbb{R}^d))$ and $C$ depends only on $\sup_{t \in (0, T)} |\tilde{\rho}(\mathbb{R}^d)|$ and $R$. Hence, the second and fourth terms can be estimated by (2.5).

Then, using (2.3), (2.4), and (2.5), one can integrate (2.2) with respect to time in $[0, t_0]$ to obtain
\[
\frac{d\Phi_{\delta, \xi}}{dt}(t_0) \leq C t_0 \left(\frac{1}{\zeta} + \zeta \log \left(\frac{C}{\zeta \delta}\right) + \frac{1}{\delta}\right) + \zeta \log \left(\frac{1}{\zeta}\right),
\]
where $C$ is a constant depending only on $R$, $b_{11}$, $\sup_{t \in (0, T)} |\rho_1(\mathbb{R}^d)|$, and $\sup_{t \in (0, T)} |\tilde{\rho}(\mathbb{R}^d)|$. Choosing first $\zeta > 0$ small enough in order to have $C t_0 \zeta < a/(2T)$ and then letting $\delta \to 0$, we find a contradiction with (2.1), concluding the proof. \qed
The existence, uniqueness, and a semigroup property for the maximal regular flow follow at once from Ambrosio et al. \(^9\), Theorems 5.7, 6.1, 7.1; see also Ambrosio et al. \(^10\), Theorem 4.3 for a concise statement.

Next, we define generalized flows (analogous to Definition 2.1 and Lagrangian solutions. For this, we define \(\overline{\mathbb{R}^6} = \mathbb{R}^6 \cup \{\infty\}\), and given a open set \(A \subseteq [0, \infty), AC_{\text{loc}}(A; \overline{\mathbb{R}^6})\) the set of continuous curves \(\gamma : A \rightarrow \mathbb{R}\) that are absolutely continuous when restricted to any closed interval in \(A\).

**Definition 2.3** (Generalized flow). For a Borel vector field \(b : (0, T) \times \mathbb{R}^6 \rightarrow \mathbb{R}^6\), the measure \(\eta \in \mathcal{M}_+(C([0, T]; \overline{\mathbb{R}^6}))\) is said to be a generalized flow of \(b\) if \(\eta\) is concentrated on the (Borel) set

\[
\Gamma := \{\eta \in C([0, T]; \overline{\mathbb{R}^6}) : \eta \in AC_{\text{loc}}(\{\eta \neq \infty\}; \mathbb{R}^6)\text{ and } \eta(t) = b_\iota(\eta(t)) \text{ for a.e. } t \in \{\eta(t) \neq \infty\}\}.
\]

The generalized flow is regular if there exists \(C \geq 0\) such that

\[
(e_t)_\iota \eta|_{\mathbb{R}^6} \leq C\mathcal{L}_{\mathbb{R}^6} \forall t \in [0, T].
\]

**Definition 2.4** (Transported measures and Lagrangian solutions). Let \(b : (0, T) \times \mathbb{R}^6 \rightarrow \mathbb{R}^6\) be a Borel vector field having a maximal regular flow \(X\), and \(\eta \in \mathcal{M}_+(C([0, T]; \overline{\mathbb{R}^6}))\) such that \((e_t)_\iota \eta\) is absolutely continuous with respect to \(\mathcal{L}_{\mathbb{R}^6}\), for all \(t \in [0, T]\). We say that \(\eta\) is transported by \(X\) if, for all \(s \in [0, T]\), \(\eta\) is concentrated on

\[
\{\eta \in C([0, T]; \overline{\mathbb{R}^6}) : \eta(s) = \infty \text{ or } (\eta(\cdot) = X(\cdot, s, \eta(s))) \text{ in } (T_{s}^{-}_X(\eta(s)), T_{s}^{+}_X(\eta(s)))\}.
\]

Moreover, let \(\rho \in L^\infty((0, T); L^1_{\text{loc}}(\mathbb{R}^6))\) be a nonnegative a distributional solution of the continuity equation, weakly continuous on \([0, T]\) in the sense that for all \(\varphi \in C_c(\mathbb{R}^6)\), the map \(t \mapsto \int \varphi \rho_t\) is weakly continuous. We say that \(\rho_t\) is a Lagrangian solution if there exists \(\eta \in \mathcal{M}_+(C([0, T]; \overline{\mathbb{R}^6}))\) transported by \(X\) with \((e_t)_\iota \eta = \rho_t\mathcal{L}_{\mathbb{R}^6}\) for every \(t \in [0, T]\).

By Ambrosio et al., \(^10\), Theorem 4.7 we have that for \(b\) as in Lemma 2.1, regular generalized flows are transported by its maximal regular flow \(X\). We are now ready to prove Theorem 1.1.

**Proof of Theorem.** Notice that the vector field \(b\) satisfies \(b \in L^\infty((0, T); L^1_{\text{loc}}(\mathbb{R}^6; \mathbb{R}^6))\), is divergence-free, and satisfies the uniqueness of bounded compactly supported nonnegative distributional solutions of the continuity equation (see Lemma 2.1). Therefore, by Ambrosio et al., \(^10\), Theorem 5.1 we deduce that: if (i) holds, then \(f\) is a Lagrangian solution; if (ii) holds and it is not bounded, then \(\beta(f)\) is a Lagrangian solution, where we choose \(\beta(s) = \arctan(s)\). In particular, by Ambrosio et al., \(^10\), Theorem 4.10 we have that \(f_t\) is a renormalized solution.

We have a direct corollary that provides conditions to obtain a globally defined flow, that is, to avoid a finite-time blow up.

**Corollary 2.1.** Fix \(T > 0\) and let \(f \in L^\infty((0, T); L^1(\mathbb{R}^6))\) be a nonnegative renormalized solution of (1.1) (as in Definition 1.1). Assume that

\[
\int_0^T \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x, v) dx dv dt + \int_0^T \int_{\mathbb{R}^6} \frac{1}{2} |E_t|^2 + \frac{1}{2} |B_t|^2 dx dt < \infty,
\]

that is, that the relativistic and the electromagnetic energies are integrable in time. Then,

(i) The maximal regular flow \(X(t, \cdot)\) associated to \(b_t = (\dot{v}, E_t + \dot{v} \times B_t)\) and starting from 0 is globally defined on \([0, T]\) for \(f_0\)-a.e. \((x, v)\);

(ii) \(f_t\) is the image of \(f_0\) through this flow, that is, \(f_t = X(t, \cdot) f_0 = f_0 \circ X^{-1}(t, \cdot)\) for all \(t \in [0, T]\):

\[
\int_{\mathbb{R}^6} \phi(x, v) f_t(x, v) dx dv = \int_{\mathbb{R}^6} \phi(X(t, x, v)) f_0(x, v) dx dv
\]

for all \(\phi \geq 0, t \in [0, T]\);
(iii) the map

$$[0, T] \ni t \mapsto \int_{\mathbb{R}^3} \psi(f_1(x, v)) \, dx \, dv$$

is constant in time for all Borel $\psi : [0, \infty) \to [0, \infty)$.

Proof. By Theorem 1.1, the solution is transported by the maximal regular flow associated to $b_t(x, v) = (\dot{v}, E_t(x) + \dot{v} \times B_t(x))$. Moreover, since $f_t$ is renormalized, $g_t := \frac{2}{\pi} \arctan f_t : (0, T) \times \mathbb{R}^3 \to [0, 1]$ is a solution of the continuity equation with vector field $b_t$. Since $g_t^2 \leq g_t \leq f_t$ and $|\dot{v}| < 1$, we have

$$I := \int_0^T \int_{\mathbb{R}^3} \frac{|b_t(x, v)| g_t(x, v)}{(1 + |x|^2 + |v|^2)^{1/2}) \log(2 + |x|^2 + |v|^2)^{1/2}} \, dx \, dv \, dt$$

$$\leq C \int_0^T \int_{\mathbb{R}^3} f_t \, dx \, dv \, dt + \int_0^T \int_{\mathbb{R}^3} |E_t| + |B_t| g_t \, dx \, dv \, dt$$

$$\leq \left( \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^2 \log(2 + |v|)} \, dv \right) \left( \int_{\mathbb{R}^3} |E_t|^2 + |B_t|^2 \, dx \, dt \right) + C \int_0^T \int_{\mathbb{R}^3} (1 + |v|) f_t \, dx \, dv \, dt.$$

By (2.6) and $(1 + |v|)^2 \leq \sqrt{2(1 + |v|^2)}$, we conclude $I$ is bounded.

Now, by the no blow-up criterion in Ambrosio et al.,10, Proposition 4.11 we obtain that the maximal regular flow $X$ of $b$ is globally defined on $[0, T]$, whence (i) follows. Moreover, the trajectories $X(t, x, v)$ belong to $AC([0, T]; \mathbb{R}^6)$ for $g_0$-a.e. $(x, v) \in \mathbb{R}^6$, and $g_t = X(t, \cdot) \circ g_0 = g_0 X^{-1}(t, \cdot)$. Since $f_t = \tan \left( \frac{\pi}{2} g_t \right)$ and the map $[0, 1) \ni s \mapsto \tan \left( \frac{\pi}{2} s \right) \in [0, \infty)$ is a diffeomorphism, we obtain that $f_t = X(t, \cdot) \circ f_0 = f_0 X^{-1}(t, \cdot)$, whence (ii) follows. In particular, for all Borel functions $\psi : [0, \infty) \to [0, \infty)$ we have

$$\int_{\mathbb{R}^3} \psi(f_1) \, dx \, dv = \int_{\mathbb{R}^3} \psi(f_0) \circ X^{-1}(t, \cdot) \, dx \, dv = \int_{\mathbb{R}^3} \psi(f_0) \, dx \, dv,$$

where the second equality follows by the incompressibility of the flow, which gives (iii). \qed

Remark 2.1. As in Ambrosio et al.,10, Remark 2.1 given $0 \leq s \leq t \leq T$, with our previous results, it is possible to reconstruct $f_t$ from $f_s$ by using the flow, that is, $f_t = X(t, s, \cdot) u_0(f_s)$.

3 | EXISTENCE OF GENERALIZED SOLUTION

We now introduce the concept of a generalized solution, which allows the electromagnetic field to be generated by effective densities $\rho_{\text{eff}}$ and $J_{\text{eff}}$. We may interpret it as particles vanishing from the phase space but still contributing in the electromagnetic field in the physical space. Thus, it is natural to assume that $\rho_t^{\text{eff}}$ may be larger than $\rho_t$, but it is bounded by the initial particle density $\rho_0$. Moreover, we assume that the particle current density $J_{\text{eff}}$ is relativistic and compatible with $\rho_t^{\text{eff}}$, that is, $|J_t^{\text{eff}}| < \rho_t^{\text{eff}}$ and satisfies the continuity equation (see (3.2a), (3.2b), and (3.2c) below).

**Definition 3.1** (Generalized solution). Given $\bar{f} \in L^1(\mathbb{R}^6)$, let $f \in L^\infty((0, \infty); L^1(\mathbb{R}^6))$ be a nonnegative function, $\rho_t^{\text{eff}} \in L^\infty((0, \infty); \mathcal{M}_+(\mathbb{R}^3))$, and $(J_t^{\text{eff}})_i \in L^\infty((0, \infty); \mathcal{M}_+(\mathbb{R}^3))$ for each component $i \in \{1, 2, 3\}$. We say that the triplet $(f_t, \rho_t^{\text{eff}}, J_t^{\text{eff}})$ is a (global in time) generalized solution of (1.1) starting from $\bar{f}$ if, setting

$$\rho_t(x) := \int_{\mathbb{R}^3} f_t(x, v) \, dv, \quad E_t^{\text{eff}}(x) := \sigma \int_{\mathbb{R}^3} K(x - y) \, d\rho_t^{\text{eff}}(y),$$

$$J_t(x) := \int_{\mathbb{R}^3} \dot{v} f_t(x, v) \, dv, \quad B_t^{\text{eff}}(x) := \sigma \int_{\mathbb{R}^3} K(y - x) \times dJ_t^{\text{eff}}(y),$$

$$b_t^{\text{eff}}(x, v) := (\dot{v} E_t^{\text{eff}}(x) + \dot{v} \times B_t^{\text{eff}}(x)),$$




the following hold: \( f \) is a renormalized solution of the continuity equation with vector field \( b_i^\text{eff} \) starting from \( f \),

\[
\rho_t \leq \rho_i^\text{eff}, \quad |J_t^\text{eff}| < \rho_i^\text{eff} \quad \text{as measures for a.e. } t \in (0, \infty), \tag{3.2a}
\]

\[
\rho_i^\text{eff}(\mathbb{R}^3) \leq \|f_0\|_{L^1(\mathbb{R}^3)} \quad \text{for a.e. } t \in (0, \infty), \quad \text{and}
\]

\[
\partial_t \rho_i^\text{eff} + \nabla \cdot J_t^\text{eff} = 0 \quad \text{with initial condition } \tilde{\rho} = \int_{\mathbb{R}^3} f \, dv, \text{ that is,}
\]

\[
\int_{\mathbb{R}^3} \phi_0 \, d\tilde{\rho} + \int_0^{\infty} \int_{\mathbb{R}^3} (\partial_t \phi_i^\text{eff} + \nabla \phi_i \cdot \nabla \rho_i^\text{eff}) \, dt = 0 \quad \forall \phi \in C_c^1([0, \infty) \times \mathbb{R}^3). \tag{3.2c}
\]

Notice that, by the Radon-Nikodym's Theorem, combined with (3.2a), there exists a vector field \( V^\text{eff} \in L^\infty((0, \infty); L^1(\rho^\text{eff}; \mathbb{R}^3)) \) such that \( d_t^\text{eff} = V^\text{eff} \, d\rho_t^\text{eff} \) and \( |V_t^\text{eff}(x)| < 1 \) for a.e. \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \). This is analogous to the continuity equation associated to (1.1) with initial condition \( \rho_0 \), which is obtained by integrating (1.1) with respect to \( \nu \) over the whole domain \( \mathbb{R}^3 \):

\[
\int_{\mathbb{R}^3} \phi_0 \, d\rho_0 + \int_0^{\infty} \int_{\mathbb{R}^3} (\partial_t \phi_t + \nabla \phi_t \cdot \nabla \rho_i^\text{eff}) \, dt = 0 \quad \forall \phi \in C_c^1([0, \infty) \times \mathbb{R}^3), \tag{3.3}
\]

where \( V := J/\rho \in L^\infty((0, \infty); L^1(\rho^\text{eff}; \mathbb{R}^3)) \) satisfies \( d_t^i = V_i \, d\rho_t \) and \( |V_i(x)| < 1 \) for a.e. \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \). Notice that the second condition in (3.2a) and (3.2c) are not present in the definition of generalized solutions of the non-relativistic Vlasov-Poisson system\(^{10}\), Definition 2.6, and are imposed in order to allow an effective magnetic field, which reduces to the self-consistent one if the number of particles is conserved.

To see that Definition 3.1 is in fact a generalization of Definition 1.1, we remark that (3.2a) and (3.2b) that, if the number of particles is conserved a.e. in time, that is, if \( \|f_t\|_{L^1(\mathbb{R}^3)} = \|f_0\|_{L^1(\mathbb{R}^3)} \) for a.e. \( t \), then \( \rho_t^\text{eff} = \rho_t \). Indeed, if the conservation holds, we have

\[
\rho_t(\mathbb{R}^3) \leq \rho_t^\text{eff}(\mathbb{R}^3) \leq \|f_0\|_{L^1(\mathbb{R}^3)} = \rho_0(\mathbb{R}^3) = \rho_t(\mathbb{R}^3),
\]

thus \( \rho_t(\mathbb{R}^3) = \rho_t^\text{eff}(\mathbb{R}^3) \). Now, if there exists \( E \subset \mathbb{R}^3 \) and \( t > 0 \) such that \( \rho_t(E) < \rho_t^\text{eff}(E) \), then

\[
(\rho_t^\text{eff} - \rho_t)(\mathbb{R}^3) \geq (\rho_t^\text{eff} - \rho_t)(E) > 0,
\]

a contradiction.

Moreover, by (3.2c) and (3.3), we have that \( \rho_t \) satisfy the continuity equation with both velocities \( V_t \) and \( V_t^\text{eff} \) with initial condition \( \rho_0 \). The following lemma gives that \( V = V^\text{eff} \), whence \( J = J^\text{eff} \).

**Lemma 3.1.** Assume \( \rho_t \) satisfies the continuity equation with the same initial condition and both vector fields \( V, V^\text{eff} \). Assume further that \( V, V^\text{eff} \) satisfy

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|V_t(x)| + |V^\text{eff}_t(x)|}{1 + |x|} \, d\rho_t(x) \, dt < \infty.
\]

Then, \( V = V^\text{eff} \).

**Proof.** Consider a (convex) class \( \mathcal{D}_b \) of measured-value solutions \( \mu_t \in \mathcal{M}_+(\mathbb{R}^3) \) of continuity equation with vector field \( b_t \), satisfying

\[
0 \leq \mu_t' \leq \mu_t \quad \Rightarrow \quad \mu_t' \in \mathcal{D}_b
\]

whenever \( \mu_t' \) still solves the continuity equation with vector field \( b_t \), and the integrability condition

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|b_t(x)|}{1 + |x|} \, d\mu_t(x) \, dt < \infty.
\]
Notice that $\rho_t \in \mathcal{L}_V \cap \mathcal{L}_{V^{\mathrm{eff}}}$ for all $T > 0$, so that by the results of DiPerna and Lions,\textsuperscript{11} we have

$$\rho_t = X(t, \cdot)_\# \rho_0 = X^{\mathrm{eff}}(t, \cdot)_\# \rho_0 \quad \forall t \in [0, T].$$

(3.4)

where $X$ and $X^{\mathrm{eff}}$ are $\mathcal{L}_V$ and $\mathcal{L}_{V^{\mathrm{eff}}}$ Lagrangian flows, respectively, that is, $X(t, \cdot)$ and $X^{\mathrm{eff}}(t, \cdot)$ are (unique) absolutely continuous functions in $[0, T]$ starting from $\rho_0$ (at time 0) such that

$$\dot{X}(t, \cdot) = V_t(X(t, \cdot)), \quad \dot{X}^{\mathrm{eff}}(t, \cdot) = V^{\mathrm{eff}}_t(X^{\mathrm{eff}}(t, \cdot)),$$

$$X(0, \cdot) = X^{\mathrm{eff}}(0, \cdot) = \mathrm{Id}$$

for $\rho_0$-almost everywhere. By (3.4) and the uniqueness of $X$ and $X^{\mathrm{eff}}$, we conclude that $V_t = V^{\mathrm{eff}}_t$. $\square$

It follows that, if the number of particles is conserved in time, then generalized solutions are renormalized ones. This observation indicates that a generalized solution which is not renormalized must lose mass/charge as the velocity approaches the speed of light.

Next, our goal is to prove the global existence of generalized solutions $f_t$ for any nonnegative $f_0 \in L^1(\mathbb{R}^6)$ (Theorem 1.2). In order to do so, we need to establish the existence of a (unique) distributional solution with smooth kernel and initial data. More precisely, we show that by smoothing the kernel $K$ and with nonnegative initial condition in $C^\infty(\mathbb{R}^6)$, we obtain a classical solution of (1.1). To avoid any confusion with the notation of Theorem 1.2 and Theorem 1.3, we denote by $K := \eta * K$ and by $g$ the smoothed kernel by $\eta \in C^\infty_c(\mathbb{R}^3)$ and the initial condition, respectively.

**Proposition 3.1.** Let $g \in C^\infty_c(\mathbb{R}^6)$ be a nonnegative function. Then, there exists a unique nonnegative Lagrangian solution $f \in C^\infty([0, \infty) \times \mathbb{R}^6)$ of the smoothed system (1.1):

$$\begin{align*}
\partial_t f_1 + \partial_x \cdot \nabla_x f_1 + (E_t + \partial_x \times B_t) \cdot \nabla_x f_1 &= 0 & \text{in} & (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3; \\
p_1(x) &= \int_{\mathbb{R}^3} f_1(x,v) \, dv, \quad J_1(x) = \int_{\mathbb{R}^3} \partial_v f_1(x,v) \, dv & \text{in} & (0, \infty) \times \mathbb{R}^3; \\
E_t(x) &= \sigma \int_{\mathbb{R}^3} \rho(y) K(x - y) \, dy & \text{in} & (0, \infty) \times \mathbb{R}^3; \\
B_t(x) &= \sigma \int_{\mathbb{R}^3} J_1(y) \times K(x - y) \, dy & \text{in} & (0, \infty) \times \mathbb{R}^3; \\
f_0(x,v) &= g(x,v) & \text{in} & \mathbb{R}^3 \times \mathbb{R}^3. 
\end{align*}
$$

(3.5)

**Proof.** In this proof, we adapt ideas and techniques from Rein.\textsuperscript{1, Section 1} By induction, we construct a sequence of smooth functions $f^n$ with initial condition $g$ which converges to a solution of (3.5). For $n = 1$, let $f^1$ be a solution of the linear transport equation

$$\begin{align*}
\partial_t f^1(x,v) + \nabla_x \cdot (\partial_v f^1)(x,v) &= 0, & \text{in} & (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3; \\
f^1_0(x,v) &= g(x,v) 
\end{align*}$$

which gives that

$$f^1_0(x,v) = g(x - tv, v) \in C^\infty([0, \infty) \times \mathbb{R}^6).$$

Moreover, we have that $f^1$ is a Lagrangian solution, since there exists a unique solution $Z^0(t, \cdot) := (X^0, V^0)(t, \cdot)$ of

$$\begin{align*}
\dot{Z}(t, \cdot) &= b^0(Z(t, \cdot)); \\
Z(0, \cdot) &= \mathrm{Id},
\end{align*}$$

where $b^0(x,v) := (\partial_v, 0)$. Hence,

$$f^1_\ast = g \circ Z^0(t), \quad \|f^1_\ast\|_{L^1(\mathbb{R}^6)} = \|g\|_{L^1(\mathbb{R}^6)}, \quad \text{and} \quad \|f^1_\ast\|_{L^\infty(\mathbb{R}^6)} = \|g\|_{L^\infty(\mathbb{R}^6)}.$$

Now, for $n \geq 2$, assume that there exists a smooth Lagrangian function

$$f^n \in L^\infty([0, \infty) \times \mathbb{R}^6) \cap L^\infty([0, \infty); L^1(\mathbb{R}^6))$$

which satisfies
\[
\begin{align*}
\partial_t f^n_t(x, v) + \nabla_x \cdot (b^n_t f^n_t)(x, v) &= 0, \\
[f^n_0(x, v)] &= g(x, v),
\end{align*}
\]  
(3.6)

where
\[
\begin{align*}
E^n_t(x) &= \sigma_t \int \rho^n_t(y) \mathcal{K}(x - y) \, dy, \\
B^n_t(x) &= \sigma_t \int J^n_t(y) \times \mathcal{K}(x - y) \, dy, \\
b^n_t(x, v) &= (\dot{v}, E^n_t + \dot{v} \times B^n_t)(x, v),
\end{align*}
\]
and define $f^{n+1}$ as a solution of (3.6) with vector field $b^n_t$. Notice that $b^n_t$ is divergence-free, and since $f^n$ and $\mathcal{K}$ are smooth, we obtain that $b^n_t$ is also smooth. Moreover, we have $b^n_t \in L^\infty(0, \infty; W^{k, \infty}(\mathbb{R}^6; \mathbb{R}^6))$ for all $k \in \mathbb{N}$, since by Young’s inequality (recall that $|J^n| < \rho^n_a.e.$)
\[
\|D_{x,v} b^n_t\|_{L^\infty(0,\infty; L^\infty(\mathbb{R}^6; \mathbb{R}^6))} \leq C \left( 1 + \|K\|_{L^1(\mathbb{R}_t; \mathbb{R}^3)} \|D^k \eta\|_{L^\infty(\mathbb{R}^3)} \|\rho^n\|_{L^\infty(0,\infty; L^1(\mathbb{R}^3))} + \|K\|_{L^\infty(\mathbb{R}^3; \mathbb{R}^3)} \|D^k \eta\|_{L^1(\mathbb{R}^3)} \|\rho^n\|_{L^\infty(0,\infty; L^1(\mathbb{R}^3))} \right).
\]  
(3.7)

Thus, for all $t \geq 0$, we have a smooth incompressible flow $Z^n(t) = (X^n_t, V^n_t)(t)$, which satisfies
\[
\begin{align*}
\dot{Z}(t, \cdot) &= b^n_t(Z(t, \cdot)); \\
Z(0, \cdot) &= \text{Id},
\end{align*}
\]  
(3.8)

and the following properties hold:
\[
f^n_{t+1} = g \circ Z^n(t), \quad \|f^n_{t+1}\|_{L^1(\mathbb{R}^6)} = \|g\|_{L^1(\mathbb{R}^3)}, \quad \text{and} \quad \|f^n_{t+1}\|_{L^\infty(\mathbb{R}^3)} = \|g\|_{L^\infty(\mathbb{R}^3)},
\]  
(3.9)

Now, we want to exploit the fact that (recall that $g \in C^\infty$)
\[
|f^n_{t+1} - f^n_t| \leq C|Z^n(t) - Z^{n-1}(t)|
\]  
(3.10)

to show that $f^n$ is a Cauchy sequence in $C([0, T] \times \mathbb{R}^6)$. For this purpose, notice that (we omit the $t$ and $(x, v)$ arguments for a cleaner presentation)
\[
|X^n(s) - X^{n-1}(s)| \leq \int_s^t |V^n(\tau) - V^{n-1}(\tau)| + |V^n(\tau)| \left| \frac{1}{\sqrt{1 + |V^n(\tau)|^2}} - \frac{1}{\sqrt{1 + |V^{n-1}(\tau)|^2}} \right| \, d\tau
\]
\[
\leq \int_s^t |V^n(\tau) - V^{n-1}(\tau)| + \sqrt{1 + |V^n(\tau)|^2} - \sqrt{1 + |V^{n-1}(\tau)|^2} \, d\tau.
\]
Thus, by the Mean Value Theorem, we conclude that
\[
|X^n(s) - X^{n-1}(s)| \leq 2 \int_s^t |V^n(\tau) - V^{n-1}(\tau)| \, d\tau.
\]
Moreover, define $E^n$ and $B^n$ as in (3.5) with densities $\rho^n$ and $J^n$, respectively. Now, by the same procedure as before combined with the uniform boundedness of $B^n$ (by (3.7) and (3.9)), we have
\[
|V^n(s) - V^{n-1}(s)| \leq C \int_s^t |E^n_t(X^n(\tau)) - E^{n-1}_t(X^{n-1}(\tau))| + |B^n_t(X^n(\tau)) - B^{n-1}_t(X^{n-1}(\tau))| + |V^n(\tau) - V^{n-1}(\tau)| \, d\tau.
\]
By (3.7) and (3.9), $E^n$ and $B^n$ are uniformly bounded with respect to $n$ and $t$, thus

$$|E^n_0(X^n(t)) - E^n_0(X^{n-1}(t))| \leq \|(E^n_0 - E^{n-1}_0)(X^n(t))| + |E^n_0 - E^{n-1}_0| + C\|X^n(t) - X^{n-1}(t)|,$$

and, analogously,

$$|B^n_0(X^n(t)) - B^n_0(X^{n-1}(t))| \leq \|B^n_0 - B^{n-1}_0\|L^\infty(\mathbb{R}^d) + C\|X^n(t) - X^{n-1}(t)|.$$

Hence, we obtain that

$$|Z^n(s) - Z^{n-1}(s)| \leq C\int_0^s \|E^n_0 - E^{n-1}_0\|L^\infty(\mathbb{R}^d) + \|B^n_0 - B^{n-1}_0\|L^\infty(\mathbb{R}^d) + \|Z^n(t) - Z^{n-1}(t)\|\,dt.$$

Thus, by Gronwall’s inequality, we conclude that

$$|Z^n(t) - Z^{n-1}(t)| \leq C\int_0^t \|E^n_0 - E^{n-1}_0\|L^\infty(\mathbb{R}^d) + \|B^n_0 - B^{n-1}_0\|L^\infty(\mathbb{R}^d)\,dt.$$

Now, by (3.9), we have that $f^n \in C^\infty_c$, which combined with (3.10) and Young’s inequality gives

$$\|f^{n+1}_t - f^n_t\|L^\infty(\mathbb{R}^d) \leq C\int_0^t \|\rho^n - \rho^{n-1}\|L^\infty(\mathbb{R}^d) + \|J^n_t - J^{n-1}_t\|L^\infty(\mathbb{R}^d,\mathbb{R}^d)\,dt \leq C\int_0^t \|f^n - f^{n-1}\|L^\infty(\mathbb{R}^d)\,dt. \tag{3.11}$$

Therefore, by induction, we have, for all $T > 0$,

$$\|f^{n+1}_t - f^n_t\|L^\infty(\mathbb{R}^d) \leq C\frac{T^n}{n!}, \quad t \in [0, T],$$

and we conclude that $f^n$ converges uniformly to a function $f \in C([0, \infty) \times \mathbb{R}^6)$ on any compact subset of $[0, \infty)$. Moreover, by (3.9), we have $f_t = g \circ Z(t)$, and $f \in L^\infty([0, \infty); L^1(\mathbb{R}^6)) \cap L^\infty([0, \infty) \times \mathbb{R}^6)$, where

$$Z(t, \cdot) := \lim_{n \to \infty} Z^n(t, \cdot).$$

Notice that $f_t$ has compact support (since $g \in C^\infty_c$); thus, $\rho^n$ and $J^n$ converge to $\rho$ and $J$ in $C([0, \infty) \times \mathbb{R}^6)$, respectively. Therefore, $E^n$ and $B^n$ converge to $E$ and $B$; thus, $\mathbf{b}^n$ converges to $\mathbf{b}$ in $C([0, \infty) \times \mathbb{R}^6)$. By the same computation as (3.7), we have in fact that $\mathbf{b} \in C([0, \infty); W^{k,\infty}(\mathbb{R}^d))$ for all $k \in \mathbb{N}$, and we conclude, by passing the limit in (3.8), that $Z \in C^1([0, \infty); C^\infty_c(\mathbb{R}^d))$, and we have $f \in C^1([0, \infty); C^\infty(\mathbb{R}^6))$. By iteration, $f$ is a smooth nonnegative Lagrangian solution of (3.5), where $Z \in C^\infty([0, \infty) \times C^\infty(\mathbb{R}^6))$ solves

$$\begin{cases}
\dot{Z}(t, \cdot) = \mathbf{b}(Z(t, \cdot)); \\
Z(0, \cdot) = 1d.
\end{cases} \tag{3.12}$$

In particular, we have that $f \in C^\infty_c([0, \infty) \times \mathbb{R}^6)$.

To prove the uniqueness part, assume there exist Lagrangian solutions $f$ and $\tilde{f}$ of (3.5). Thus,

$$f_t := g \circ Z(t) \text{ and } \tilde{f}_t := g \circ \tilde{Z}(t),$$

where both $Z, \tilde{Z}$ solve (3.12). Thus, we may repeat the proof of (3.11) for $f_t - \tilde{f}_t$ to obtain

$$\|f_t - \tilde{f}_t\|L^\infty(\mathbb{R}^d) \leq C\int_0^t \|f_t - \tilde{f}_t\|L^\infty(\mathbb{R}^d)\,dt,$$

and we conclude by Gronwall’s inequality that $f \equiv \tilde{f}$.

We are now able to prove our second main result.
Proof of Theorem. Our proof follows the same general structure of the proof of Ambrosio et al.\textsuperscript{10, Theorem 2.7}; we begin by approximating $f$ as an $L^1$ limit of $f^n$ (Steps 1 and 2), which was already shown in Ambrosio et al.\textsuperscript{10}; then, we approximate $(\rho^n f^n, J^n f^n)$ and show that the electromagnetic field of the approximation converges to the effective field $(E_{\text{eff}}, B_{\text{eff}})$ (Steps 3 and 4); finally, in Step 5, we combine stability results for the continuity equation obtained in Ambrosio et al.\textsuperscript{10, Section 5} to take limits in the approximated system and conclude that the limiting solution is transported by the limit of the incompressible flow, as wanted.

**Step 1: Approximating solutions.**

Consider $K^n := K * \eta^n$, where $\eta^n(x) := n^3 \eta(nx)$, and $\eta$ is a standard convolution kernel in $\mathbb{R}^3$. Let $f_0^n \in C^\infty_c(\mathbb{R}^6)$ be a sequence such that

\begin{equation}
    f_0^n \rightharpoonup f_0 \quad \text{in} \quad L^1(\mathbb{R}^6).
\end{equation}

Moreover, denote $f^n$ the smooth solution of (1.1) with initial condition $f^n_0$ and kernel $K^n$ (see Proposition 3.1), and its respective charge density, electric field, density current, and magnetic field defined by

\begin{align*}
    \rho^n_t(x) &:= \int_{\mathbb{R}^3} \rho^n_t(x,v) \, dv, \quad E^n_t(x) := \sigma_E \int_{\mathbb{R}^3} \rho^n_t(x,y) K^n(x-y) \, dy, \\
    J^n_t(x) &:= \int_{\mathbb{R}^3} \hat{\nu} f^n_t(x,v) \, dv, \quad \text{and} \quad B^n_t(x) := \sigma_B \int_{\mathbb{R}^3} J^n_t(x,y) \times K^n(x-y) \, dy. 
\end{align*}

Since $K^n$ is smooth and vanishes at infinity, we have $E^n, B^n \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^3))$, nonetheless without a uniform bound with respect to $n$ (see the proof of (3.7)). Thus, $B^n_t := (\hat{\nu} E^n_t + \hat{\nu} \times B^n_t)$ is a Lipschitz divergence-free vector field, and its flow $X^n(t): \mathbb{R}^6 \rightarrow \mathbb{R}^6$ is well defined and incompressible; thus, by the theory for the transport equation, for all $t \in [0, \infty)$ and each component $i \in \{1, 2, 3\}$,

\begin{equation}
    f^n_t = f^n_0 \circ X^n(t)^{-1} \quad \text{and} \quad \|J^n_t\|_{L^1(\mathbb{R}^3)} \leq \|\hat{\nu} f^n_t\|_{L^1(\mathbb{R}^3)} < \|\rho^n_t\|_{L^1(\mathbb{R}^3)} = \|f^n_t\|_{L^1(\mathbb{R}^3)} = \|f^n_0\|_{L^1(\mathbb{R}^3)}. \tag{3.14}
\end{equation}

Assume, without loss of generality, that $\mathcal{L}^6(\{f_0 = k\}) = 0$ for every $k \in \mathbb{N}$; otherwise, consider $\mathcal{L}^6(\{f_0 = k + \tau\}) = 0$ for some $\tau \in (0, 1)$. Then, for all $k$, we deduce

\begin{equation}
    f^n_k := I_{\{k \leq f^n \leq k+\tau\}} f^n_0 \rightharpoonup f^k := I_{\{k \leq f \leq k+\tau\}} f_0 \quad \text{in} \quad L^1(\mathbb{R}^6). \tag{3.15}
\end{equation}

Thus, by defining $f^{n,k}_t := I_{\{k \leq f^n \leq k+\tau\}} f^n_t$, we have that $f^{n,k}_t$ is a distributional solution of the continuity equation (with vector field $b^n_t$) and $f^n_0$ initial datum. Moreover, we have

\begin{equation}
    f^{n,k}_t = I_{\{k \leq f^n \leq k+\tau\}} f^n_0 \circ X^n(t)^{-1}, \quad \|f^{n,k}_t\|_{L^1(\mathbb{R}^6)} = \|f^n_0\|_{L^1(\mathbb{R}^6)} \quad \forall t \in [0, \infty). \tag{3.16}
\end{equation}

**Step 2: Limit in phase space.**

By construction, $(f^{n,k})_{n \in \mathbb{N}}$ is a nonnegative uniformly bounded sequence. Hence, there exists $f^k \in L^\infty((0, \infty) \times \mathbb{R}^6)$ such that

\begin{equation}
    f^{n,k} \rightharpoonup f^k \quad \text{weakly* in} \quad L^\infty((0, \infty) \times \mathbb{R}^6) \quad \text{as} \quad n \rightarrow \infty \quad \forall k \in \mathbb{N}. \tag{3.17}
\end{equation}

Moreover, for any $K \subset \mathbb{R}^6$, and any bounded function $\phi : (0, \infty) \rightarrow (0, \infty)$ with compact support, we use the test function $\phi(t) I_{K(x,v)} \text{sign}(f^n_0(x,v))$ for the previous two weak convergence combined with Fatou’s lemma, the convergence of $(f^{n,k})_{n \in \mathbb{N}}$, and (3.16) to obtain

\begin{equation}
    \int_0^\infty \phi(t) \|f^k_t\|_{L^1(K)} dt \leq \left( \int_0^\infty \phi(t) dt \right) \liminf_{n \rightarrow \infty} \|f^{n,k}_0\|_{L^1(\mathbb{R}^6)} = \left( \int_0^\infty \phi(t) dt \right) \|f^n_0\|_{L^1(\mathbb{R}^6)},
\end{equation}

Since $\phi$ was arbitrary the supremum among all compact subset $K \subset \mathbb{R}^6$, we obtain

\begin{equation}
    \|f^k_t\|_{L^1(K)} \leq \|f^n_0\|_{L^1(\mathbb{R}^6)} \quad \text{for a.e.} \quad t \in (0, \infty), \tag{3.18}
\end{equation}
so, in particular, \( f^k \in L^\infty((0, \infty); L^1(R^6)) \). Moreover, by defining \( f = \sum_{k=0}^{\infty} f^k \), we have

\[
\| f \|_{L^1(R^6)} \leq \| f_0 \|_{L^1(R^6)} \quad \text{for a.e. } t \in [0, \infty).
\] (3.19)

Noticing that \( f^n = \sum_{k=0}^{\infty} f^{n,k} \), by fixing \( \varphi \in L^\infty((0, T) \times R^6) \), (3.16), and (3.18), we have for all \( k_0 \geq 1 \),

\[
\left| \int_0^T \int_{R^6} \varphi(f^n - f) \, dx \, dv \, dt \right| \leq \sum_{k=0}^{k-1} \left| \int_0^T \int_{R^6} \varphi(f^{n,k} - f^k) \, dx \, dv \, dt \right| + T \| \varphi \|_{L^\infty((0,T) \times R^6)} \sum_{k=k_0}^{\infty} \left( |f^{n,k}_0| + |f^k_0| \right) \, dx \, dv.
\]

Now, by the convergence (3.17) the first term vanishes as \( n \to \infty \). Thus, by convergences (3.13) and (3.15), we have

\[
\limsup_{n \to \infty} \left| \int_0^T \int_{R^6} \varphi(f^n - f) \, dx \, dv \, dt \right| \leq 2T \| \varphi \|_{L^\infty((0,T) \times R^6)} \| f_0 \|_{L^1(R^6)}.
\]

Letting \( k_0 \to \infty \) and since \( \varphi \in L^\infty \) was arbitrary, we conclude

\[
f^n \rightharpoonup f \quad \text{weakly in } L^1((0, T) \times R^6).
\] (3.20)

**Step 3: Limit in physical densities.**

Since \( \{ \rho^n \}_{n \in \mathbb{N}} \) and \( \{ J^n_i \}_{n \in \mathbb{N}} \) are bounded sequences in \( L^\infty((0, \infty); \mathcal{M}_+(R^3)) \) and \( L^\infty((0, \infty); \mathcal{M}(R^3)) \), respectively, for each component \( i \in \{1, 2, 3\} \) (see (3.14)), and \( L^\infty((0, \infty); \mathcal{M}_+(R^3)) = [L^1((0, \infty); C_0(R^3))]^* \), there exist \( \rho^{\text{eff}} \in L^\infty((0, \infty); \mathcal{M}_+(R^3)) \) and \( J^{\text{eff}}_i \in L^\infty((0, \infty); \mathcal{M}(R^3)) \) such that

\[
\rho^n \rightharpoonup \rho^{\text{eff}} \quad \text{weakly* in } L^\infty((0, \infty); \mathcal{M}_+(R^3));
\]

\[
J^n_i \rightharpoonup J^{\text{eff}}_i \quad \text{weakly* in } L^\infty((0, \infty); \mathcal{M}(R^3)).
\] (3.21)

for each component \( i \in \{1, 2, 3\} \). Hence, by the lower semicontinuity of the norm under weak* convergence, we have

\[
\esssup_{t \in (0, \infty)} |\rho^{\text{eff}}_i|(R^3) \leq \lim_{n \to \infty} \sup_{t \in (0, \infty)} \| \rho^n_i \|_{L^1(R^3)} = \lim_{n \to \infty} \| \rho^n_0 \|_{L^1(R^3)} = \| f_0 \|_{L^1(R^3)}.
\] (3.22)

Now, fixing a nonnegative function \( \varphi \in C_c((0, \infty) \times R^3) \), by (3.20) and (3.21), we obtain that

\[
\int_0^\infty \int_{R^3} \varphi_t(x) \, d\rho^{\text{eff}}_i(x) \, dt \geq \lim_{R \to \infty} \liminf_{n \to \infty} \int_0^\infty \int_{R^3 \times \Omega_R} f_i^n(x,v) \varphi_t(x) \, dv \, dx \, dt
\]

\[
= \int_0^\infty \int_{R^3} f_i(x,v) \varphi_t(x) \, dv \, dx \, dt = \int_0^\infty \int_{R^3} \varphi_t(x) \, d\rho_t(x) \, dt.
\]

Moreover, by recalling that \( |\tilde{v}| < 1 \), we have

\[
\int_0^\infty \int_{R^3} \varphi_t(x) \, d\rho^{\text{eff}}_i(x) \, dt = \lim_{n \to \infty} \int_0^\infty \int_{R^3} f_i^n(x,v) \varphi_t(x) \, dv \, dx \, dt
\]

\[
> \lim_{n \to \infty} \int_0^\infty \int_{R^3} |\tilde{v}| \, f_i^n(x,v) \varphi_t(x) \, dv \, dx \, dt
\]

\[
\geq \int_0^\infty \int_{R^3} \varphi_t(x) \, d|J^{\text{eff}}_i|(x) \, dt.
\]
Thus,
\[
\rho_t \leq \rho_{t}^{\text{eff}}, \quad |J_{t}^{\text{eff}}| < \rho_{t}^{\text{eff}} \quad \text{as measures for a.e. } t \in (0, \infty).
\]  
(3.23)

Finally, by the same argument to show (3.3), we notice that
\[
\int_{\mathbb{R}^3} \phi_0 d\rho_{0}^t + \int_0^\infty \int_{\mathbb{R}^3} \left( \partial_t \phi_t d\rho_t^n + \nabla \phi_t \cdot dJ_{t}^{n} \right) dt = 0 \quad \forall \phi \in C_0^1([0, \infty) \times \mathbb{R}^3).
\]

Hence, by (3.13) and (3.21), we conclude by taking the limit \( n \to \infty \) that
\[
\int_{\mathbb{R}^3} \phi_0 d\rho_{0}^t + \int_0^\infty \int_{\mathbb{R}^3} \left( \partial_t \phi_t d\rho_t^{\text{eff}} + \nabla \phi_t \cdot dJ_{t}^{\text{eff}} \right) dt = 0 \quad \forall \phi \in C_0^1([0, \infty) \times \mathbb{R}^3).
\]
i.e.,
\[
\partial_t \rho_t^{\text{eff}} + \nabla \cdot J_{t}^{\text{eff}} = 0 \quad \text{as measures with initial condition } \rho_0.
\]  
(3.24)

**Step 4: Limit of vector fields.**

Using the definitions in (3.1), we claim that
\[
\mathbf{b}^n \rightharpoonup \mathbf{b}^{\text{eff}} \quad \text{weakly in } L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^6; \mathbb{R}^6)
\]
(3.25)

and that, for every ball \( B_R \subset \mathbb{R}^3 \),
\[
[E^n + \dot{\nu} \times B^n](x + h) \rightharpoonup [E^n + \dot{\nu} \times B^n](x) \quad \text{as } |h| \to 0 \quad \text{in } L^1_{\text{loc}}((0, \infty); L^1(B_R)), \quad \text{uniformly in } n.
\]
(3.26)

For this purpose, we first prove that the sequence \((\mathbf{b}^n)_{n \in \mathbb{N}}\) is bounded in \( L^p_{\text{loc}}((0, \infty) \times \mathbb{R}^6; \mathbb{R}^6) \) for every \( p \in [1, 3/2] \). Indeed, by using Young’s inequality, for every \( t \geq 0, n \in \mathbb{N}, \text{and } r > 0 \),
\[
\|R_t^n\|_{L^p(B, \mathbb{R}^3)} + \|E_t^n\|_{L^p(B, \mathbb{R}^3)} \leq \|(J_t^n) \ast \eta^n\|_{L^p(\mathbb{R}^3)} + \|(\rho_t^n \ast \eta^n) \ast K\|_{L^p(\mathbb{R}^3)}.
\]

The first term can be controlled by
\[
\|(J_t^n) \ast \eta^n\|_{L^p(\mathbb{R}^3)} \leq \|(J_t^n)\|_{L^1(\mathbb{R}^3)} \|\eta^n\|_{L^\infty(\mathbb{R}^3)} \|\mathbf{b}^{\text{eff}}\|_{L^p(B, \mathbb{R}^3)} + \mathcal{L}^3(B_r)^{1/p} \|\eta^n\|_{L^1(\mathbb{R}^3)} \|\mathbf{b}^{\text{eff}}\|_{L^p(B, \mathbb{R}^3)}.
\]

Likewise, the second term can be bounded by
\[
\|(\rho_t^n \ast \eta^n) \ast K\|_{L^p(\mathbb{R}^3)} \leq \|(\rho_t^n \ast \eta^n)\|_{L^1(\mathbb{R}^3)} \|K\|_{L^p(B, \mathbb{R}^3)} + \mathcal{L}^3(B_r)^{1/p} \|(\rho_t^n \ast \eta^n)\|_{L^1(\mathbb{R}^3)} \|K\|_{L^\infty(\mathbb{R}^3 \setminus B_r)}.
\]

Thus, up to subsequences, the sequence \((\mathbf{b}_n)_{n \in \mathbb{N}}\) converges weakly in \( L^p_{\text{loc}} \). We now claim that for every \( \varphi \in C_c((0, \infty) \times \mathbb{R}^3) \),
\[
\lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^3} (E_t^n + \dot{\nu} \times B_t^n) \varphi_t dx dt = \int_0^\infty \int_{\mathbb{R}^3} (E_t^{\text{eff}} + \dot{\nu} \times B_t^{\text{eff}}) \varphi_t dx dt.
\]
Indeed, denoting $T_\varphi$ the upper time support of $\varphi$, we have

\[
\left| \int_0^\infty \int_{\mathbb{R}^3} (E^n_t + \dot{\varphi} \times B^n_t) \varphi_t \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^3} (B^n_t \sim \dot{\varphi} \times B^n_t) \varphi_t \, dx \, dt \right| \\
\leq \int_0^\infty \int_{\mathbb{R}^3} (\rho^n_t - \rho^n_{t\text{eff}}) \varphi_t \ast K \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^3} (\rho^n_t \ast K - \rho^n_t \ast \eta^n_t) \, dx \, dt \\
+ \int_0^\infty \int_{\mathbb{R}^3} (J^n_t - J^n_{t\text{eff}}) \times \varphi_t \ast K \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^3} J^n_t \ast (\varphi_t \ast K - \rho^n_t \ast \eta^n_t) \, dx \, dt \\
\leq \int_0^\infty \int_{\mathbb{R}^3} (\rho^n_t - \rho^n_{t\text{eff}}) \varphi_t \ast K \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^3} (J^n_t - J^n_{t\text{eff}}) \times \varphi_t \ast K \, dx \, dt \\
+ T_\varphi(\|\rho^n\|_{L^\infty((0,\infty);L^1(\mathbb{R}^3))} + \|J^n\|_{L^1((0,\infty);L^3(\mathbb{R}^3))}) \|\varphi \ast K - \rho^n_t \ast \eta^n_t\|_{L^1(\mathbb{R}^3)}.
\]

By the weak convergence (3.21) and the fact that $\varphi \ast K$ is a bounded continuous function, the first and second terms vanish as $n \to \infty$. Moreover, the last term also vanishes, since the first factor is bounded by $C\|f_0\|_{L^1(\mathbb{R}^3)}$, where $C > 0$ is a universal constant and $\varphi \ast K \ast \eta^n$ converges uniformly to $\varphi \ast K$ in $(0,\infty) \times \mathbb{R}^3$. Thus, we have proven (3.25).

We now prove (3.26). For this purpose, we combine the fact that $K \in W^{a,p}(\mathbb{R}^3;\mathbb{R}^3)$ for every $a < 1$ and $p < 3/(2 + a)$, and Young’s inequality to obtain

\[
\|E^n_t + \dot{\varphi} \times B^n_t\|_{W^{a,p}(B_R;\mathbb{R}^3)} \leq C(R)(\|\rho^n_t\|_{L^3} + \|J^n_t\|_{L^3}) \ast \eta^n_t\|_{L^1(\mathbb{R}^3)}.
\]

Combining $\|\eta^n\|_{L^1(\mathbb{R}^3)} = 1$ with (3.14), we can bound the right term independently of $n$ and $t$, which combined with the embedding of fractional Sobolev spaces and Nikolsky spaces to gives

\[
\|b^n(\cdot + h) - b^n(\cdot)\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)} \leq C(p,a,R,\|b^n\|_{W^{a,p}(B_R;\mathbb{R}^3)}) |h|^a \forall |h| \leq R,
\]

and (3.26) follows.

**Step 5.** By (3.25) and (3.26), we can apply the stability result from DiPerna and Lions\textsuperscript{11, Theorem II.7} to deduce that $f^k$ is a weakly continuous distributional solution of the continuity equation with vector field $b^n_{\text{eff}}$ and starting from $f^k_0$ for every $k \in \mathbb{N}$. We now exploit the linearity of the continuity equation to show that $F^m := \sum_{k=1}^m f^k$ is also a bounded distributional solution for every $m \in \mathbb{N}$. Using the same arguments as in the proof of Theorem 1.1, we obtain that $F^m$ is a renormalized solution for every $m \in \mathbb{N}$. Since $F^m \to f$ strongly in $L^1_{\text{loc}}((0,\infty) \times \mathbb{R}^6)$ as $m \to \infty$, we obtain that $f$ is a renormalized solution of the continuity equation with vector field $b^n_{\text{eff}}$ and starting from $f_0$, which combined with (3.19) (3.22), (3.23), and (3.24) proves that the trio $(f_t, f^n_t, J^n_t)$ is a generalized solution starting from $f_0$ according to Definition 3.1.

To show that $f$ is transported by the maximal regular flow associated to $b^n_{\text{eff}}$, we simply use that each $f^k$ is transported (once again with the same argument as in Theorem 1.1) combined with the definition of $f$ and (3.19). Finally, by Ambrosio et al.,\textsuperscript{10, Theorem 4.10} we conclude that the map

\[
[0,\infty) \ni t \mapsto f_t \in L^1_{\text{loc}}(\mathbb{R}^6)
\]

is continuous.

\[\square\]

## 4 FINITE ENERGY SOLUTIONS

Up to now, we have established the existence of a generalized solution (see Theorem 1.2) and that renormalized and generalized solutions coincide in case the mass/charge is conserved in time. In this section, we investigate whether the existence of renormalized solutions can be shown under the more natural condition that the initial total energy is bounded, that is,

\[
\mathcal{E}_0 := \int_{\mathbb{R}^3} \sqrt{1 + |V|^2} f_0(x,\nu) \, dx \, dv + \frac{\sigma_R}{2} \int_{\mathbb{R}^3} (H \ast \rho_0) \rho_0 \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (H \ast J_0) \cdot J_0 \, dx < \infty,
\] (4.1)
where the first term is the relativistic (initial) total energy, the second and third are the (initial) electric and magnetic potential energies, respectively, and \( H(x) := (4\pi |x|)^{-1} \). For this purpose, we recall that, by integrating the first equation of (1.1) with respect to \((x,v)\) on the whole domain \(\mathbb{R}^6\), gives that the relativistic energy (formally) satisfies

\[
\frac{d}{dt} \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x,v) \, dx \, dv = \int_{\mathbb{R}^6} \dot{\varrho} \cdot (E_t + \varrho \times B_t) f_t(x,v) \, dx \, dv = \int_{\mathbb{R}^3} E_t \cdot J_t \, dx.
\]

Now, Poynting’s Theorem gives that the relativistic Vlasov-Maxwell equation has its electromagnetic total energy (formally) conserved, that is,

\[
\int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x,v) \, dx \, dv + \frac{1}{2} \int_{\mathbb{R}^3} |E_t|^2 + |B_t|^2 \, dx = \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_0(x,v) \, dx \, dv + \frac{1}{2} \int_{\mathbb{R}^3} |E_0|^2 + |B_0|^2 \, dx,
\]

while for the system (1.1) we obtain a similar expression (see (4.4) below):

\[
\int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x,v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx = \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_0(x,v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_0) \rho_0 \, dx. \tag{4.2}
\]

Although (4.2) is formal, we shall exploit a semicontinuity argument to show it in the form of an inequality (see the proof of Theorem 1.3):

\[
\int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x,v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx \leq \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_0(x,v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_0) \rho_0 \, dx. \tag{4.3}
\]

Notice that the magnetic potential energy does not appear in the conservation above. On the other hand, one can (formally) integrate by parts the electric and magnetic energy to obtain the relations

\[
\int_{\mathbb{R}^3} |E_t|^2 \, dx = \int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx;
\]

\[
\int_{\mathbb{R}^3} |B_t|^2 \, dx = \int_{\mathbb{R}^3} (H \ast J_t) \cdot J_t \, dx - \int_{\mathbb{R}^3} (\nabla \cdot (H \ast J_t))^2 \, dx. \tag{4.4}
\]

We can interpret \( H \ast \rho_t \) and \( H \ast J_t \) as the electric potential and magnetic vector potential, respectively (see \(^2\)). Notice that, on one hand, electric potential energy is fully converted into electric energy. On the other hand, magnetic potential energy is converted into magnetic energy and displacement current \( \partial_t E_t \), since

\[
- \int_{\mathbb{R}^3} (\nabla \cdot (H \ast J_t))^2 \, dx = \int_{\mathbb{R}^3} \nabla \cdot (H \ast J_t) \partial_t (H \ast \rho_t) \, dx = \int_{\mathbb{R}^3} (H \ast J_t) \cdot \partial_t E_t \, dx. \tag{4.5}
\]

Moreover, we obtain (formally) that the magnetic potential energy is nonnegative for a.e. \( t \in [0, \infty) \). We observe that (4.2) and (4.4) do not have any magnetic energy terms, so that it is unclear whether the classical energy estimate \( \mathcal{E}_t < \mathcal{E}_0 \) holds, where the total energy of the system at time \( t \) is given by

\[
\mathcal{E}_t := \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t(x,v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (H \ast J_t) \cdot J_t \, dx.
\]

**Remark 4.1.** Although the formal argument that leads to (4.4) suggests the magnetic potential energy is nonnegative, we rigorously justify it in the proof of Lemma 4.5. Hence, (4.1) implies that the right-hand side of (4.3) is bounded.
Remark 4.2. By (4.2) and (4.5), we (formally) have

\[ \int_{\mathbb{R}^3} |\tilde{B}_t|^2 \, dx = \int_{\mathbb{R}^3} A_t \cdot (J_t + \partial_t E_t) \, dx, \tag{4.6} \]

where \( A_t := H \ast J_t \) is the magnetic vector potential. Since we can interpret \( \partial_t E_t \) as a density current, one might define the magnetic vector potential as \( H \ast (J_t + \partial_t E_t) \), and therefore (4.5) does not provide a relation between magnetic energy and magnetic potential energy. We claim that (4.6) still holds if \( A_t = H \ast (J_t + \partial_t E_t) \); thus, we may interpret \( \partial_t E_t \) as a lower order term. Indeed, define a magnetic field with density current \( J_t^* := J_t + \partial_t E_t \), that is, \( \tilde{B} = \nabla \times (H \ast J_t^*) \), and a calculation analogous to (4.2) gives that

\[ \int_{\mathbb{R}^3} |\tilde{B}_t|^2 \, dx = \int_{\mathbb{R}^3} (H \ast J_t^*) \cdot J_t^* \, dx - \int_{\mathbb{R}^3} (\nabla \cdot (H \ast J_t^*))^2 \, dx. \tag{4.7} \]

Notice that \( \nabla \cdot (H \ast J_t^*) = H \ast (\nabla \cdot J + \partial_t \rho_t) = 0 \), hence the last term vanishes. Moreover, since \( E_t \) is irrotational, \( B_t = \tilde{B}_t \); thus, combining (4.6) and (4.7), we conclude that

\[ \int_{\mathbb{R}^3} (H \ast \partial_t E_t) \cdot (J_t + \partial_t E_t) \, dx = 0. \]

Therefore, had we defined the magnetic vector potential as \( H \ast J_t \), (4.6) would be unaltered.

Notice that if \( \sigma_E = 1 \), a bound as (4.3) gives that each energy term of \( \mathcal{E}_t \) is bounded, since \( |J| < \rho \) a.e. in space-time. However, it does not provide, in general, control of relativistic energy and electric and magnetic potential energies if \( \sigma_E = -1 \) or \( \sigma_E = 0 \). If we also assume a higher integrability of \( f_0 \) and a suitable smallness condition on its norm, the next lemma can be used to bound each energy.

**Lemma 4.1.** Let \( f \in L^1(\mathbb{R}^6) \cap L^q(\mathbb{R}^6) \) be a nonnegative function for some \( q \geq 1 \) and \( \sqrt{1 + |\cdot|^2} \, f \in L^1(\mathbb{R}^6) \). Set \( p := \frac{4q-3}{3q-2} \). Then \( \rho = \int_{\mathbb{R}^3} f(\cdot, v) \, dv \in L^p(\mathbb{R}^3) \) and there exists a constant \( C > 0 \) depending only on \( q \) such that

\[ \|\rho\|_{L^p(\mathbb{R}^3)} \leq C \|\sqrt{1 + |\cdot|^2} \, f\|_{L^q(\mathbb{R}^6)}^{\theta} \|f\|_{L^1(\mathbb{R}^6)}^{1-\theta}, \]

where \( \theta := \frac{3q-1}{4q-3} \).

**Proof.** Choose \( R > 0 \) and split the integral of \( \rho \) on the sets \( \{|v| < R\} \) and \( \{|v| \geq R\} \). For each \( x \in \mathbb{R}^3 \), we can write

\[ \rho(x) \leq R^{3q-1/q} \|f(x, \cdot)\|_{L^q(\mathbb{R}^6)}+ R^{-1} \|\sqrt{1 + |\cdot|^2} \, f(x, \cdot)\|_{L^q(\mathbb{R}^6)}. \]

By minimizing the right-hand side with respect to \( R \), we have

\[ \rho(x) \leq C \|\sqrt{1 + |\cdot|^2} \, f(x, \cdot)\|_{L^q(\mathbb{R}^6)}^{3q/(4q-3)} \|f(x, \cdot)\|_{L^1(\mathbb{R}^6)}^{q/(4q-3)}. \]

At last, the result follows by taking the \( L^p \)-norm on \( \rho \) and by using Hölder’s inequality.

As anticipated, if \( f_0 \) satisfies

\[ f_0 \in \begin{cases} L^1(\mathbb{R}^6) & \text{if } \sigma_E = 1; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) & \text{if } \sigma_E = 0; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) \text{ and } \|f_0\|_{L^{3/2}(\mathbb{R}^6)} \leq \epsilon & \text{if } \sigma_E = -1 \end{cases} \tag{4.8} \]

for some suitable \( \epsilon > 0 \), the previous lemma allows us to bound each relativistic energy, electric and magnetic potential. Indeed, by Calderón-Zygmund estimates and the Sobolev embedding, we have that

\[ \|H \ast \rho_t\|_{L^2(\mathbb{R}^3)} \leq C \|D^2(H \ast \rho_t)\|_{L^{5/2}(\mathbb{R}^3)} \leq C \|\rho_t\|_{L^{5/3}(\mathbb{R}^3)} \tag{4.9} \]
for some universal constant $C > 0$. Combining (4.9) with Hölder’s inequality and Lemma 4.1 with $p = 6/5$ and $q = 3/2$ gives

$$
\int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx \leq \|H \ast \rho_t\|_{L_p(\mathbb{R}^3)} \|\rho_t\|_{L_q(\mathbb{R}^3)} \leq C \|\rho_t\|_{L_q(\mathbb{R}^3)}^2
$$

$$
\leq C \sqrt{1 + |v|^2} \|f_0\|_{L_p(\mathbb{R}^3)} \|f_0\|_{L_q(\mathbb{R}^3)} \tag{4.10}
$$

Notice that $\|f\|_{L_p([0,\infty) ; L_q(\mathbb{R}^3) \cap B_0)} \leq \|f_0\|_{L_q(\mathbb{R}^3)}$ when the solution is built by approximation (see (3.19)). Hence, if (4.3) holds, we already have a bound of the relativistic energy in the pure magnetic case $\sigma_E = 0$, and by the previous bound, we obtain the following boundedness of the magnetic and electric potential energies (recall that $|J| < \rho$ a.e. in space-time):

$$
\int_{\mathbb{R}^3} (H \ast J_t) \cdot J_t \, dx \leq \int_{\mathbb{R}^3} (H \ast \rho_t) \rho_t \, dx \leq C \|f_0\|_{L_q(\mathbb{R}^3)} \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0(x, v) \, dx \, dv.
$$

Now, in the repulsive case $\sigma_E = -1$, we obtain by (4.3) and (4.10) that

$$
(1 - C \|f\|_{L_p([0,\infty) ; L_q(\mathbb{R}^3) \cap B_0)}) \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0(x, v) \, dx \, dv \leq \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0(x, v) \, dx \, dv - \int_{\mathbb{R}^3} (H \ast \rho_0) \rho_0 \, dx.
$$

Assuming that $f$ is built by approximation as before and that $\|f_0\|_{L_q(\mathbb{R}^3)} < 1/C = : \epsilon$, we have a bound of the relativistic energy; therefore, by (4.10), the electric and magnetic potential energies are bounded as well. This motivates the following:

**Definition 4.1.** We say that $f_0$ has every energy bounded if (4.1) and (4.8) hold. Moreover, if $f_1$ also satisfies (4.3) for almost every $t \in [0, \infty)$, then we say that $f_1$ has every energy bounded.

**Remark 4.3.** Notice that we need stronger assumptions on the initial data compared to the nonrelativistic Vlasov-Poisson case for $\sigma_E = -1$, where it is only needed that $f_0 \in L^{9/7}(\mathbb{R}^3)$, with no smallness assumption (see\(^\text{13}\)). This is due to the fact that classical kinetic energy grows as $|v|^2$, whereas the relativistic energy as $|v|$.

We now prove that if $f_0$ has every energy bounded, then we have a smooth sequence $(f_n^0)_{n \in \mathbb{N}}$ and a sequence of mollified kernels $(H \ast \eta^k)_{n \in \mathbb{N}}$ with uniform bounded energy. We denote by $L^\infty_c$ the space of bounded measurable functions with compact support.

**Lemma 4.2.** Let $\eta^k(x) := k^3 \eta(kx)$, where $\eta$ is a standard convolution kernel in $\mathbb{R}^3$. Let $f_0$ be a nonnegative function with every bounded energy. Then there exists a sequence $(f_n^0)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^6)$ converging pointwise to $f_0$ and a sequence $(k_n)_{n \in \mathbb{N}}$ such that $k_n \rightarrow \infty$ and, by setting $\rho_0^n = f_{R_n} f_0^*(\cdot, v) \, dv$ and $J_0^n = f_{R_n} \tilde{f}_0^*(\cdot, v) \, dv$,

$$
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0^n(x, v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \eta^k \ast \rho_0^n) \rho_0^n \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (H \ast \eta^k \ast J_0^n) \cdot J_0^n \, dx
$$

$$
= \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0(x, v) \, dx \, dv + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H \ast \rho_0) \rho_0 \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (H \ast J_0) \cdot J_0 \, dx.
$$

**Proof.** We split the proof in three steps: in Step 1, we assume that $f_0 \in L^\infty_c(\mathbb{R}^6)$ and approximate it by a sequence of smooth functions with compact support; in Step 2, we obtain the desired limit without the mollification of $H$; in Step 3, we introduce the mollification of the kernel $\eta^k \ast H$, and conclude that the limit holds if we extract a subsequence of $k$ which depends on $n$.

**Step 1:** $f_0 \in L^\infty_c(\mathbb{R}^6)$. Consider a sequence of smooth functions $f_n^0$ which converge pointwise such that $\|f_n^0\|_{L_p(\mathbb{R}^3)} \leq \|f_0\|_{L_p(\mathbb{R}^3)}$ and $\text{supp} f_n^0 \subset B_R$, for all $n$, for some $R > 0$. Thus, we have $J_0^n \subset \text{supp} f_0^n \subset B_R$. Moreover, there holds $|H \ast J_0^n| < H \ast \rho_0^n < \infty$, $H \ast \rho_0^n \rightarrow H \ast \rho_0$, and $H \ast J_0^n \rightarrow H \ast J_0$ in $L^p_{\text{loc}}$, for every $p$, and we conclude by the dominated convergence theorem that
\[
\begin{align*}
\lim_{n \to \infty} & \left( \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_n^0(x, v) \, dv \, dx + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H * \rho_0^n) \rho_0^n \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (\rho_0^n + J_0^n) \cdot J_0^n \, dx \right) \\
& = \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} f_0(x, v) \, dv \, dx + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} (H * \rho_0) \rho_0 \, dx + \frac{\sigma_B}{2} \int_{\mathbb{R}^3} (\rho_0 + J_0) \cdot J_0 \, dx.
\end{align*}
\]

(4.11)

**Step 2:** \( f_0 \in L^1(\mathbb{R}^6) \) without mollification of \( H \). By Step 1, it is enough to approximate \( f_0 \) by \( \{ f_n^0 \}_{n \in \mathbb{N}} \subset L^\infty_c(\mathbb{R}^6) \) with converging energies to obtain (4.11). For this purpose, define

\[
f_n^0(x, v) := \min \{ n, 1, \mathbb{I}_{B_n}(x, v) f_0(x, v) \}, \ (x, v) \in \mathbb{R}^6.
\]

Since \( H \geq 0 \), the first two integrands on the left-hand side of (4.11) converges monotonically, and we conclude by monotone convergence. Since \( |(H * J_0^n) \cdot J_0^n| < (H * \rho_0) \rho_0 \) a.e., and \((H * \rho_0) \rho_0 \) is integrable (since \( f_0 \) has every energy bounded), we conclude that the last integral on the left-hand side converges by the dominated convergence.

**Step 3:** **Approximation of the kernel.** Given \( \{ f_n^0 \}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^6) \) provided by the previous two steps, we have

\[
\lim_{k \to \infty} \left( \int_{\mathbb{R}^3} (H * \eta_n^k \ast \rho_0^n) \rho_0^n \, dx + \int_{\mathbb{R}^3} (H * \eta_n^k + J_0^n) \cdot J_0^n \, dx \right) = \int_{\mathbb{R}^3} (H * \rho_0^n) \rho_0^n \, dx + \int_{\mathbb{R}^3} (H * J_0^n) \cdot J_0^n \, dx
\]

for every fixed \( n \). Hence, there exists \( k_n \) sufficiently large such that

\[
\left| \int_{\mathbb{R}^3} (H * \eta_n^k \ast \rho_0^n) \rho_0^n \, dx + \int_{\mathbb{R}^3} (H * \eta_n^k + J_0^n) \cdot J_0^n \, dx - \int_{\mathbb{R}^3} (H * \rho_0^n) \rho_0^n \, dx - \int_{\mathbb{R}^3} (H * J_0^n) \cdot J_0^n \, dx \right| \leq \frac{1}{n},
\]

and the lemma is proved.

In what follows, we need the following lemma\(^{10, \text{Lemma 3.3}} \) that we state for convenience of the reader.

**Lemma 4.3.** Let \( T > 0 \) and \( \phi \in C_c((0, T)) \) be a nonnegative function. Then, for every sequence \( \{ \rho_n \}_{n \in \mathbb{N}} \subset C([0, T]; \mathcal{M}_+(\mathbb{R}^3)) \) such that

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \rho_n(t) < \infty
\]

and

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^3} \phi(t) (\rho_n(t) - \rho_t) \right| = 0 \text{ for every } \rho \in C_c^\infty(\mathbb{R}^3).
\]

we have

\[
\int_0^T \phi(t) \int_{\mathbb{R}^3} H * \rho_t(x) \, dx \, dt \leq \liminf_{n \to \infty} \int_0^T \phi(t) \int_{\mathbb{R}^3} H * \eta_n^* \ast \rho_n(x) \, dx \, dt.
\]

(4.12)

Although the previous lemma is enough for \( \sigma_E \in \{ 0, 1 \} \), we need a slightly higher integrability assumption in the gravitational case \( \sigma_E = -1 \). This is due to the fact that we obtain (4.3) by a lower semicontinuity argument, and (4.13) is not sufficient if the electric potential energy is nonpositive. Nonetheless, if \( \rho \in L^{6/5} \), we obtain (4.13) with a limit and an equality; this is the content of the next lemma.

**Lemma 4.4.** Let \( \rho^n, \rho \in L^\infty([0, T]; L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)) \) as in Lemma 4.3. Moreover, assume that

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \| \rho^n(t) \|_{L^{6/5}(\mathbb{R}^3)} < \infty.
\]

(4.14)

Then

\[
\lim_{n \to \infty} \int_0^T \phi(t) \int_{\mathbb{R}^3} H * \eta_n^k \ast \rho_n^k(x) \, dx \, dt = \int_0^T \phi(t) \int_{\mathbb{R}^3} H * \rho_t(x) \, dx \, dt.
\]

(4.15)
Proof. Notice that
\[
\int_{\mathbb{R}^3} H \ast \eta^n \ast \rho_t^n(x) - H \ast \rho_t(x) \rho_t(x) \, dx = \int_{\mathbb{R}^3} H \ast \eta^n \ast (\rho_t^n(x) - \rho_t(x)) \rho_t^n(x) \, dx \\
+ \int_{\mathbb{R}^3} H \ast (\eta^n \ast \rho_t(x) - \rho_t(x)) \rho_t^n(x) \, dx \\
+ \int_{\mathbb{R}^3} H \ast \rho_t(x)(\rho_t^n(x) - \rho_t(x)) \, dx \\
= : I_1 + I_2 + I_3.
\]

Now, by (4.9) and Hölder inequality, we obtain
\[
|I_2| \leq C \| \eta^n \ast \rho_t - \rho_t \|_{L^{5/4}(\mathbb{R}^3)} \sup_{n \in \mathbb{N}} \| \rho_t^n \|_{L^{5/4}(\mathbb{R}^3)}.
\]

Letting \( n \to \infty \), we obtain that \( I_2 \) vanishes. We now define \( \zeta_k \in C_c^\infty(\mathbb{R}^3) \) as a cutoff function in the annular set \( B_k \setminus B_{1/k} \), namely,
\[
\begin{cases}
\zeta_k = 1 & \text{in } B_k \setminus B_{1/k}; \\
\zeta_k = 0 & \text{in } B_{k+1} \cup B_{1/(k+1)}; \\
0 \leq \zeta_k \leq 1 & \text{in } \mathbb{R}^3.
\end{cases}
\]

We write \( I_3 \) as
\[
|I_3| \leq \left| \int_{\mathbb{R}^3} H \ast \rho_t(x)(\rho_t^n(x) - \rho_t(x)) \zeta_k(x) \, dx \right| + \left| \int_{\mathbb{R}^3} H \ast \rho_t(x)(\rho_t^n(x) - \rho_t(x))(1 - \zeta_k(x)) \, dx \right|
\]

We want to take first the limit \( n \to \infty \) and after \( k \to \infty \) to be able to use (4.12). Now, by (4.9), we obtain
\[
\left| \int_{\mathbb{R}^3} H \ast \rho_t(x)(\rho_t^n(x) - \rho_t(x))(1 - \zeta_k(x)) \, dx \right| \leq \left| \int_{B_k \setminus B_{k+1} \cup B_k} H \ast \rho_t(x)(\rho_t^n(x) - \rho_t(x)) \, dx \right|
\]
\[
\leq C \| \rho_t \|_{L^{5/4}(\mathbb{R}^3)} \sup_{n \in \mathbb{N}} \| \rho_t^n \|_{L^{5/4}(B_k \setminus B_{k+1} \cup B_k)}.
\]

Defining measures \( \mu^n := (\rho_t^n - \rho_t)^{6/5} \, dx \) and \( \mu := \sup_{n \in \mathbb{N}} \mu^n \), (4.14) and the continuity from below for measures gives that
\[
\lim_{k \to \infty} \sup_{n \in \mathbb{N}} \int_{B_k \setminus B_{k+1}} (\rho_t^n - \rho_t)^{6/5} \, dx = \lim_{k \to \infty} \mu_t(B_k \setminus B_{k+1}) = \mu_t \left( \bigcap_{k=1}^{\infty} B_{1/k} \right) + \mu_t \left( \bigcap_{k=1}^{\infty} B_{k} \right) = 0,
\]
and we conclude that the second term vanishes as \( k \to \infty \). Now, we bound the first term by
\[
\| H \ast \rho_t \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})} \left| \int_{\mathbb{R}^3} \zeta_k(\rho_t^n(x) - \rho_t(x)) \, dx \right|.
\]

By Young's inequality, we have
\[
\| H \ast \rho_t \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})} \leq \| H \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})} \| \rho_t \|_{L^{1}(\mathbb{R}^3)} < \infty.
\]

Hence, by (4.12), \( I_3 \) vanishes as \( n \to \infty \) and \( k \to \infty \). Analogously, we have
\[
|I_1| = \left| \int_{\mathbb{R}^3} H \ast \eta^n \ast \rho_t^n(x)(\rho_t^n(x) - \rho_t(x)) \, dx \right|
\]
\[
\leq \| H \ast \rho_t^n \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})} \left| \int_{\mathbb{R}^3} \zeta_k(\rho_t^n(x) - \rho_t(x)) \, dx \right|
\]
\[
+ C \sup_{n \in \mathbb{N}} \| \rho_t^n \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})} \sup_{n \in \mathbb{N}} \| \rho_t^n - \rho_t \|_{L^{6/5}(B_{k+1} \setminus B_{k+2})},
\]
and by the same argument as before, \( I_1 \) vanishes as \( n \to \infty \) and \( k \to \infty \), and the lemma follows. \( \blacksquare \)
Next, we analyze (4.4) rigorously for $|J| < \rho \in L^1(\mathbb{R}^3)$. The following lemma gives (4.4) with an inequality so that, in particular, the magnetic potential energy is nonnegative. Actually, our argument yields the same result for $|J| < \rho \in \mathcal{M}_+(\mathbb{R}^3)$.

**Lemma 4.5.** For $|J| < \rho \in L^1(\mathbb{R}^3)$, we have

\[
\int_{\mathbb{R}^3} |\nabla (H * \rho)|^2 \, dx \leq \int_{\mathbb{R}^3} (H * \rho) \, dx; \\
\int_{\mathbb{R}^3} |\nabla \times (H * J)|^2 \, dx \leq \int_{\mathbb{R}^3} (H * J) \cdot J \, dx - \int_{\mathbb{R}^3} (\nabla \cdot (H * J))^2 \, dx.
\]

(4.16)

In particular, we obtain that the magnetic potential energy is nonnegative.

**Proof.** We split the proof as in the proof Lemma 4.2:

**Step 1:** $J, \rho \in L^\infty_c(\mathbb{R}^3)$. Consider first $\rho, J$ compactly supported smooth functions, and perform an integration by parts to obtain

\[
\int_{B_k} |\nabla (H * \rho)|^2 \, dx = \int_{B_k} (H * \rho) \, dx + \int_{\partial B_k} H * \rho \nabla (H * \rho) \cdot v_B \, d\mathcal{H}^2; \\
\int_{B_k} |\nabla \times (H * J)|^2 \, dx = \int_{B_k} (H * J) \cdot J \, dx - \int_{\mathbb{R}^3} (\nabla \cdot (H * J))^2 \, dx \\
- \int_{\partial B_k} [(H * J) \times (\nabla \times (H * J))] \cdot v_B \, d\mathcal{H}^2 \\
+ \int_{\partial B_k} \nabla \cdot (H * J) H * J \cdot v_B \, d\mathcal{H}^2.
\]

The same identity holds for $J_i, \rho \in L^\infty_c(\mathbb{R}^3)$ by approximation for each component $i \in \{1, 2, 3\}$. Since $H * \mu$ and $\nabla (H * \mu)$ decay as $R^{-1}$ and $R^{-2}$ when evaluated at $\partial B_k$ for all $\mu \in L^\infty_c(\mathbb{R}^3)$, the boundary terms vanish as $R \to \infty$, and we obtain that (4.16) holds with an equality.

**Step 2:** $J, \rho \in L^1(\mathbb{R}^3)$. We consider the truncations

\[
\rho^n := \min\{n, I_{B_k}(x, v) \rho\}, J^n_i := \min\{n, I_{B_k}(x, v) J_i\}.
\]

Since $H \geq 0$, by monotone convergence and Step 1 we obtain that

\[
\int_{\mathbb{R}^3} (H * \rho) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} (H * \rho^n) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla (H * \rho^n)|^2 \, dx.
\]

Moreover, since $|J| < \rho$, by the dominated convergence theorem and Step 1, we obtain that

\[
\int_{\mathbb{R}^3} (H * J) \cdot J \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} (H * J^n) \cdot J^n \, dx \\
= \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla \times (H * J^n)|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^3} (\nabla \cdot (H * J^n))^2 \, dx.
\]

Assuming without loss of generality that $(H * \rho) \rho \in L^1(\mathbb{R}^3)$, we get bounded sequences $(\nabla (H * \rho^n))_{n \in \mathbb{N}}, (\nabla \cdot (H * J^n))_{n \in \mathbb{N}},$ and $(\nabla \times (H * J^n))_{n \in \mathbb{N}}$ in $L^2$. Since each sequence converges in the sense of distributions to $\nabla (H * \rho), \nabla \cdot (H * J),$ and $\nabla \times (H * J)$, respectively, by the lower semicontinuity of the $L^2$-norm with respect to weak convergence we conclude (4.16).

Finally, we prove our third main result.

□
Proof of Theorem. The proof of existence of renormalized solutions begins similarly to the proof of Theorem 1.2: let \( \{ f^n \}_{n \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^6) \) and \( \{ k_n \}_{n \in \mathbb{N}} \) given by Lemma 4.2. By Steps 1-3 in the proof of Theorem 1.2 we get a sequence of smooth functions \( f^n \) satisfying (1.1) with initial condition \( f^n_0 \) and kernel \( k^n \) (see Proposition 3.1) such that

\[
\begin{align*}
\lim_{n \to \infty} f^n &\to f \text{ weakly in } L^1([0, T) \times \mathbb{R}^6) \text{ for any } T > 0; \\
\rho^n &\to \rho_{\text{eff}} \text{ weakly* in } L^\infty([0, \infty) ; \mathcal{M}_+(\mathbb{R}^3)); \\
|J_{\text{eff}}| &< \rho_{\text{eff}} \text{ as measures;}
\end{align*}
\]

(4.17)

Moreover, since (4.2) holds for classical solutions and \( f_0 \) has every energy bounded, we obtain

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, \infty)} \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f^n_t \, dx \, dv \leq C,
\]

(4.19)

and by the lower semicontinuity of the relativistic energy we deduce that, for every \( T > 0 \),

\[
\int_0^T \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_t \, dx \, dv \, dt \leq \liminf_{n \to \infty} \int_0^T \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f^n_t \, dx \, dv \, dt \leq CT.
\]

(4.20)

We now claim that \( \rho_{\text{eff}} = \rho \) and, consequently, \( J = J_{\text{eff}} \), where \( |J| < \rho \in L^\infty((0, T); L^1(\mathbb{R}^6)) \) as in (1.1). For this, consider \( \zeta_k : \mathbb{R}^6 \to [0, 1] \) a nonnegative function which equals 1 inside \( B_k \) and 0 in \( B_{k+1}^c \) and compute

\[
\int_0^\infty \int_{\mathbb{R}^6} (\rho^n_t - \rho_t) \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^6} (f^n_t(x, v) - f_t(x, v)) \varphi_t(x) \zeta_k(v) \, dv \, dt \\
+ \int_0^\infty \int_{\mathbb{R}^6} f^n_t(x, v) \varphi_t(x)(1 - \zeta_k(v)) \, dv \, dt \\
+ \int_0^\infty \int_{\mathbb{R}^6} f_t(x, v) \varphi_t(x)(\zeta_k(v) - 1) \, dv \, dt.
\]

By the weak convergence in \( L^1 \) in (4.17), the first term vanishes as \( n \to \infty \). The second and third terms can be estimated using (4.19) and (4.20):

\[
\int_0^\infty \int_{\mathbb{R}^6} f^n_t(x, v) \varphi_t(x)(1 - \zeta_k(v)) \, dv \, dt + \int_0^\infty \int_{\mathbb{R}^6} f_t(x, v) \varphi_t(x)(\zeta_k(v) - 1) \, dv \, dt
\]

\[
\leq \frac{CT_{\varphi} \| \varphi \|_{L^\infty((0, \infty) \times \mathbb{R}^3)}}{k},
\]

where \( T_{\varphi} \) is support of \( \varphi \) with respect to time. Letting \( k \to \infty \), we conclude that \( \rho^n \) converges to \( \rho \) weakly* in \( L^\infty((0, \infty) ; \mathcal{M}_+(\mathbb{R}^3)) \), which combined with (4.17) gives that \( \rho = \rho_{\text{eff}} \). Hence, by (4.17) and Lemma 3.1, we conclude that \( J = J_{\text{eff}} \), and in Steps 4 and 5 in the proof of Theorem 1.2, we obtain a global Lagrangian (hence renormalized) solution \( f_t \in C((0, \infty); L^1_{loc}(\mathbb{R}^6)) \) of (1.1) with initial datum \( f_0 \).

Next, we prove properties by a lower semicontinuous argument on the energy of \( f^n \). 

Step 1: Bound on the total energy for \( \mathcal{L}^1 \)-almost every time. We use the weak convergence of \( f^n \) (see (4.17)) with test function \( \psi(t) \sqrt{1 + |v|^2} \chi_t(x, v) \), where \( \psi \in C^\infty_c((0, \infty)) \) and \( \chi_t \in C^\infty_c(\mathbb{R}^6) \) are nonnegative functions, with \( \chi_t \) being a cutoff between \( B_t \) and \( B_{t+1} \), we obtain
\[
\int_0^\infty \int_{\mathbb{R}^6} f_i(x,v) \sqrt{1 + |v|^2} \phi(t) \chi_r(x,v) \, dv \, dx \, dt \leq \liminf_{n \to \infty} \int_0^\infty \phi(t) \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i^n(x,v) \, dv \, dx \, dt.
\]

Taking the supremum with respect to \( r \), we deduce that
\[
\int_0^\infty \phi(t) \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i(x,v) \, dv \, dx \, dt \leq \liminf_{n \to \infty} \int_0^\infty \phi(t) \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i^n(x,v) \, dv \, dx \, dt.
\]

(4.21)

Since \( \phi \) is arbitrary, we have that \( \sqrt{1 + |v|^2} f_i \in L^1_{\text{loc}}(\mathbb{R}^6) \) for almost every \( t \). Moreover, since we can decompose the density current as \( J = V \rho \) (see remark after Definition 3.1), where \( |V| < 1 \) a.e. in spacetime, we have that
\[
\sup_{t \in [0,\infty)} \int_{\mathbb{R}^3} |V_i(x)| \, d\rho_i(x) < \infty;
\]

hence, by Ambrosio\textsuperscript{14}, Theorem 8.1.2, we have that \( \rho_i \) has a weakly* continuous representative. Furthermore, since \( \rho^n \) satisfies a similar continuity equation, by the proof of Ambrosio\textsuperscript{14}, Theorem 8.1.2 we have that
\[
\left| \int_{\mathbb{R}^3} (\rho^n_i - \rho^\star_i) \phi \, dx \right| \leq \|\phi\|_{C^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |V^n_i(x)\rho^n_i \, dx | \, dr \leq C|t - s|
\]

for all \( \phi \in C^\infty_c(\mathbb{R}^3) \), which gives that the map \( t \mapsto \int_{\mathbb{R}^3} \phi \, d\rho^n_i \) is equicontinuous. By the weak* convergence of \( \rho^n \) to \( \rho \) in \( L^\infty((0,\infty) \times L^1(\mathbb{R}^3)) \), we have a uniform boundedness; thus, Arzelà-Ascoli theorem implies that
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^3} \phi \, d(\rho^n_i - \rho_i) \right| = 0 \text{ for every } \phi \in C^\infty_c(\mathbb{R}^3).
\]

(4.22)

Combining the above with the fact that \( \rho^n_i \) is uniformly bounded with respect to \( n \) and \( t \), by Lemma 4.3, we obtain
\[
\int_0^\infty \phi(t) \int_{\mathbb{R}^3} H \ast \rho_i(x) \, d\rho_i(x) \, dt \leq \liminf_{n \to \infty} \int_0^\infty \phi(t) \int_{\mathbb{R}^3} H \ast \eta^n \ast \rho^n_i(x) \, d\rho^n_i(x) \, dt.
\]

(4.23)

Combining (4.21), (4.23), and (4.2), we conclude that for \( \sigma_E \in \{0, 1\} \),
\[
\int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i(x,v) \, dv \, dx + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H \ast \rho_i(x) \, dx \right) \, dt
\]
\[
\leq \liminf_{n \to \infty} \int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i^n(x,v) \, dv \, dx + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H \ast \eta^n \ast \rho^n_i(x) \, dx \right) \, dt
\]
\[
= \left( \int_0^\infty \phi(t) \, dt \right) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_0(x,v) \, dv \, dx + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H \ast \rho_0(x) \, dx \right)
\]

The case \( \sigma_E = -1 \) is subtler: By (4.21) and (4.2), we have that
\[
\int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i(x,v) \, dv \, dx - \frac{1}{2} \int_{\mathbb{R}^3} H \ast \rho_i(x) \, dx \right) \, dt
\]
\[
\leq \liminf_{n \to \infty} \int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_i^n(x,v) \, dv \, dx - \frac{1}{2} \int_{\mathbb{R}^3} H \ast \rho_i(x) \, dx \right) \, dt
\]
\[
\leq \left( \int_0^\infty \phi(t) \, dt \right) \left( \int_{\mathbb{R}^6} \sqrt{1 + |v|^2} f_0(x,v) \, dv \, dx - \frac{1}{2} \int_{\mathbb{R}^3} H \ast \rho_0(x) \, dx \right)
\]
\[
+ \frac{1}{2} \limsup_{n \to \infty} \int_0^\infty \phi(t) \left( \int_{\mathbb{R}^3} H \ast \eta^n \ast \rho^n_i(x) \, dx - H \ast \rho_i(x) \, dx \right) \, dt
\]
Notice that by (4.20), (4.18), and (4.9), we have for every $T > 0$,
\[
\sup_{t \in [0,T]} \|\rho_t\|_{L^{\infty}(\mathbb{R}^3)} + \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \|\rho_t^n\|_{L^{\infty}(\mathbb{R}^3)} < \infty.
\]
Thus, by Lemma 4.4, we obtain that the last term equals 0. Since $\phi$ was arbitrary and since $f_0$ has every energy bounded, we conclude that $f_t$ has every energy bounded for $L^1$-almost every $t \in (0, \infty)$.

**Step 2: Bound on the total energy for every time.** Notice that the relativistic and electric potential energy is lower semicontinuous with respect to the strong $L^1_{\text{loc}}$ and weak* $\mathcal{M}_+$ convergences, respectively. Hence, by the continuity of $t \mapsto f_t \in L^1((\mathbb{R}^3))$ and $t \mapsto \rho_t \in \mathcal{M}_+(\mathbb{R}^3)$ for the $L^1_{\text{loc}}$ and weak* $\mathcal{M}_+$ convergences, respectively, combined with Step 1, we have that for $t_n \to \tilde{t}$ in $[0, \infty)$ such that (4.3) holds for all $t_n$, we may pass the limit and obtain (4.3) for $t = \tilde{t}$.

**Step 3: Strong $L^1_{\text{loc}}$ continuity of the $\rho, J$.** Given $t \in [0, \infty)$, let $t_n \to t$. Fix $r > 0$, and for any $R > 0$
\[
\int_{B_r} \int_{\mathbb{R}^3} |f_{t_n} - f_t| \ dv \ dx \leq \int_{B_r} \int_{\mathbb{R}^3} |f_{t_n} - f_t| \ dv \ dx + R^{-1} \int_{B_r} \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} (f_{t_n} + f_t) \ dv \ dx.
\]
By the uniform boundedness of the relativistic energy with respect to time and the $L^1_{\text{loc}}$ continuity of $f_t$, by taking the limit in $n$ and then in $R$, we conclude that $\rho_{t_n} \to \rho_t$ in $L^1_{\text{loc}}$. Moreover, since $|v| < 1$, we have
\[
\int_{B_r} |J_{t_n} - J_t| \ dx \to \int_{B_r} |J_{t_n} - J_t| \ dv \ dx \to 0,
\]
thus $J_{t_n} \to J_t$ in $L^1_{\text{loc}}$. Finally, since $K \in L^1_{\text{loc}}$ and $|J|(\mathbb{R}^3) < \rho(\mathbb{R}^3) < \infty$, we conclude that $E_t, B_t$ are also strongly continuous in $L^1_{\text{loc}}(\mathbb{R}^3)$.

**Step 4: Globally defined flow.** We can combine the fact that $f_t$ has every energy bounded and Lemma 4.5 to obtain that $E_t, B_t \in L^\infty(\mathbb{R}^3)$, thus by Corollary 2.1 we conclude that the trajectories of the maximal regular flow starting at any given $t$ do not blow up for $f_t$-almost every $(x, v) \in \mathbb{R}^6$.

**Step 5: Strong $L^1$-continuity of $f$.** By Theorem 1.1 and $L^1_{\text{loc}}$-continuity of $f_t$, we deduce that finite energy solutions conserve mass, i.e., $\rho_t(\mathbb{R}^3) = \rho_0(\mathbb{R}^3)$ for every $t \in [0, \infty)$. In particular, solutions are strongly continuous in $L^1(\mathbb{R}^6)$ and not only $L^1_{\text{loc}}(\mathbb{R}^6)$ (see Ambrosio et al. 10, Theorem 4.10).

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**CONFLICT OF INTEREST**
This work does not have any conflicts of interest.

**ORCID**

Diego Marcon [https://orcid.org/0000-0001-6352-7705](https://orcid.org/0000-0001-6352-7705)

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APPENDIX A: DERIVATION

The relativistic Vlasov equation describes the evolution of a function $f : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ under the action of a self-consistent acceleration $A : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$:

$$\partial_t f_t(x, v) + \dot{v} \cdot \nabla_x f_t(x, v) + A_t(x, v) \cdot \nabla_v f_t(x, v) = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (A1)$$

In this paper, we consider the acceleration given by

$$A_t(x, v) = g_t(x) + \frac{q}{m} (E_t(x) + \dot{v} \times B_t(x)),$$

where $g_t$, $E_t$, and $B_t$ are the Newtonian gravitational, electric, and magnetic fields, respectively, and $q$ and $m$ are the particle charge and mass. Newtonian gravity implies that $g_t = Gm \nabla(-\Delta)^{-1} \rho_t$, where $G$ is the gravitational constant and $\rho_t$ the density of particles. We study the case in which the electromagnetic field satisfies one of the quasi-static limits of Maxwell's equations (see, for instance, Manfredi\textsuperscript{15} and references therein):

$$\nabla \cdot E_t = \frac{q}{\varepsilon_0} \rho_t, \quad \nabla \cdot B = 0, \quad \nabla \times E_t = 0, \quad \nabla \times B_t = \frac{q}{\varepsilon_0} J_t + \partial_t E_t, \quad (A2)$$

or

$$\nabla \cdot E_t = \frac{q}{\varepsilon_0} \rho_t, \quad \nabla \cdot B = 0, \quad \nabla \times E_t = -\partial_t B_t, \quad \nabla \times B_t = \frac{q}{\varepsilon_0} J_t, \quad (A3)$$

where $J_t$ is the relativistic particle current density. Equations (A2) and (A3) are known as the quasi-electrostatic (QES) and quasi-magnetostatic (QMS) limits, respectively. The solution of (A2) can be written as

$$E_t = -\frac{q}{\varepsilon_0} \nabla(-\Delta)^{-1} \rho_t \quad \text{and} \quad B_t = \frac{q}{\varepsilon_0} \nabla \times (-\Delta)^{-1} J_t,$$

while the solution of (A3) is

$$E_t = -\frac{q}{\varepsilon_0} \nabla(-\Delta)^{-1} \rho_t - \frac{q}{\varepsilon_0} \partial_t (-\Delta)^{-1} J_t \quad \text{and} \quad B_t = \frac{q}{\varepsilon_0} \nabla \times (-\Delta)^{-1} J_t.$$
Notice that the leading term in the QES limit is the electric field whereas in the QMS it is the magnetic field. Hence, in the QES case, we can write $A_t$ in terms of $\rho_t$ and $J_t$ only:

$$A_t(x, v) = \left(\frac{q^2}{4\pi \varepsilon_0 m} - Gm\right) \int_{\mathbb{R}^3} \rho_t(y) \frac{x - y}{|x - y|^3} \, dy + \frac{q^2}{4\pi \varepsilon_0 m} \, \dot{v} \times \int_{\mathbb{R}^3} J_t(y) \times \frac{x - y}{|x - y|^3} \, dy,$$

where $\varepsilon_0$ is the electric permittivity. Next, define the critical charge $q_c$ as

$$q_c := \pm \sqrt{4\pi \varepsilon_0 Gm}.$$

If $q > q_c$, we have that the electric field is stronger and, up to redefining of $\rho_t$ and $J_t$, we may write the acceleration as

$$A_t(x, v) = \int_{\mathbb{R}^3} \rho_t(y)K(x - y) \, dy + \dot{v} \times \int_{\mathbb{R}^3} J_t(y) \times K(x - y) \, dy.$$

Analogously, if $q < q_c$, we can write

$$A_t(x, v) = -\int_{\mathbb{R}^3} \rho_t(y)K(x - y) \, dy + \dot{v} \times \int_{\mathbb{R}^3} J_t(y) \times K(x - y) \, dy.$$

In both cases, if we drop the magnetic field (since it is a lower order term), we have the relativistic Vlasov-Poisson system. Moreover, notice that in the critical case $q = q_c$, we only have the magnetic force acting in the evolution Equation (A1), which is exactly the same as if we only consider the leading term in the QMS limit, that is, the relativistic Vlasov-Biot-Savart system.