ORTHOGONALLY ADDITIVE MAPPINGS ON
HILBERT MODULES

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Abstract. In this paper, we study the representation of orthogonally additive mappings acting on Hilbert C*-modules and Hilbert H*-modules. One of our main results shows that every continuous orthogonally additive mapping $f$ from a Hilbert module $W$ over $\mathcal{K}(\mathcal{H})$ or $\mathcal{HS}(\mathcal{H})$ to a complex normed space is of the form $f(x) = T(x) + \Phi(\langle x, x \rangle)$ for all $x \in W$, where $T$ is a continuous additive mapping, and $\Phi$ is a continuous linear mapping.

Let $\mathcal{A}$ be a C*-algebra or an H*-algebra, $(W, \langle ., . \rangle)$ be a Hilbert module over $\mathcal{A}$, and $G$ be a complex normed space. A continuous mapping $W \to G$ is said to be orthogonally additive if for all $x, y \in W$,

$$\langle x, y \rangle = 0 \implies f(x + y) = f(x) + f(y).$$

In this paper, we study the representation of orthogonally additive mappings. If $T : W \to G$ is a continuous additive mapping, and $\Phi : \mathcal{A} \to G$ is a continuous mapping, then clearly the mapping $f : W \to G$ defined by

$$f(x) = T(x) + \Phi(\langle x, x \rangle) \text{ for all } x \in W$$

(1)
is a continuous orthogonally additive mapping. One of our main goals is to show that the converse also holds true if $\mathcal{A}$ is a C*-algebra of compact operators or an H*-algebra. In particular, this answers [23, Problem 27] affirmatively, not only for Hilbert H*-modules, but for Hilbert C*-modules over a C*-algebra of compact operators as well. Other related problems in [23] have been also solved in [11, 12, 13].

Orthogonally additive mappings have been extensively studied from many aspects. See the survey [17] and the references therein for the representation of orthogonally additive mappings on orthogonality spaces. Refer to [20, 21, 22] for the connection between the existence

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of even orthogonally additive mappings and inner product spaces. Recently, several mathematicians have obtained some interesting results on orthogonally additive polynomials. See, e.g., [7, 9, 10, 16], among others.

The rest of this paper is organized as follows. In Section 1, we give some necessary background and set up some notation. Section 2 deals with \( \perp \)-additive mappings on abelian groups. The context there is very general, so the results there may be also useful in future. Applying the results of Section 2, we obtain the representation of orthogonally additive mappings in general Hilbert modules in Section 3. In Section 4, we strengthen the results of Section 3 in the case when \( \mathcal{A} \) is \( \mathcal{K}(\mathcal{H}) \) or \( \mathcal{H}S(\mathcal{H}) \). The main result is then generalized to any Hilbert module over a \( C^* \)-algebra of compact operators or \( H^* \)-algebra in Section 5. In the last section, we obtain the representation of orthogonally additive mappings on \( \mathcal{B}^2(\mathcal{H}_1, \mathcal{H}_2) \).

1. Preliminaries

In this section, we give some necessary background and set up our notation.

1.1. Hilbert \( C^* \)-modules and Hilbert \( H^* \)-modules. Hilbert modules arise as generalizations of a complex Hilbert space when the complex field is replaced by a \( C^* \)-algebra or an \( H^* \)-algebra. The idea of replacing the complex numbers by the elements of a \( C^* \)-algebra first appeared in the work of Kaplansky [14] and by the elements of a proper \( H^* \)-algebra in the work of Saworotnow [18].

A \( C^* \)-algebra is a complex Banach \( * \)-algebra \( (\mathcal{A}, \| \cdot \|) \) such that \( \| a^*a \| = \| a \|^2 \) for all \( a \in \mathcal{A} \). An \( H^* \)-algebra is a complex Banach \( * \)-algebra \( (\mathcal{A}, \| \cdot \|) \), whose underlying Banach space is a Hilbert space with respect to the inner product \( \langle \cdot , \cdot \rangle \) satisfying \( \langle ab, c \rangle = \langle b, a^*c \rangle \) and \( \langle ba, c \rangle = \langle b, ca^* \rangle \) for all \( a, b, c \in \mathcal{A} \). The trace-class associated with an \( H^* \)-algebra \( \mathcal{A} \) is defined as the set \( \tau(\mathcal{A}) = \{ ab : a, b \in \mathcal{A} \} \); it is a self-adjoint two-sided ideal of \( \mathcal{A} \) which is dense in \( \mathcal{A} \).

Some examples of \( C^* \) algebras are \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{K}(\mathcal{H}) \), the algebras of all bounded operators, resp. all compact operators, on some complex Hilbert space \( \mathcal{H} \). An example of an \( H^* \)-algebra is \( \mathcal{H}S(\mathcal{H}) \), the algebra of all Hilbert-Schmidt operators on \( \mathcal{H} \).

An element \( a \) in a \( C^* \)-algebra \( \mathcal{A} \) is called positive \( (a \geq 0) \) if it is self-adjoint and has nonnegative spectrum. An element \( a \) in an \( H^* \)-algebra \( \mathcal{A} \) is called positive \( (a \geq 0) \) if \( \langle ax, x \rangle \geq 0 \) for all \( x \in \mathcal{A} \). If \( \mathcal{A} \) is a \( C^* \)-algebra (resp. an \( H^* \)-algebra) then every positive \( a \in \mathcal{A} \) (resp. \( a \in \tau(\mathcal{A}) \)) can be written as \( a = b^*b \) for some \( b \in \mathcal{A} \).
Let \( \mathcal{A} \) be a \( C^* \)-algebra or an \( H^* \)-algebra. Let \( W \) be an algebraic right \( \mathcal{A} \)-module which is a complex linear space with a compatible scalar multiplication, i.e. \( \lambda(xa) = (\lambda x)a = x(\lambda a) \) for all \( x \in W, a \in \mathcal{A}, \lambda \in \mathbb{C} \). The space \( W \) is called a (right) inner product \( \mathcal{A} \)-module if there exists a generalized inner product, that is, a mapping \( \langle \cdot, \cdot \rangle \) from \( W \times W \) to \( \mathcal{A} \) if \( \mathcal{A} \) is a \( C^* \)-algebra, and to \( \tau(\mathcal{A}) \) if \( \mathcal{A} \) is an \( H^* \)-algebra, having the following properties:

(i) \( \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \) for all \( x, y, z \in W \),
(ii) \( \langle x, y a \rangle = \langle x, y \rangle a \) for all \( x, y \in W \) and \( a \in \mathcal{A} \),
(iii) \( \langle x, y \rangle^* = \langle y, x \rangle \) for all \( x, y \in W \),
(iv) \( \langle x, x \rangle \geq 0 \) for all \( x \in W \), and \( \langle x, x \rangle = 0 \Leftrightarrow x = 0 \).

If \( W \) is an inner product module over \( (\mathcal{A}, \| \cdot \|) \), then for \( x \in W \) we write \( \|x\|_W = \|\langle x, x \rangle\|^{1/2} \). If \( W \) is complete with respect to this norm, then it is called a Hilbert \( \mathcal{A} \)-module, or a Hilbert \( C^* \)-module (resp. \( H^* \)-module) over the \( C^* \)-algebra (resp. \( H^* \)-algebra) \( \mathcal{A} \).

We shall use the symbol \( \langle W, W \rangle \) for the linear span of all inner products \( \langle x, y \rangle, x, y \in W \). A Hilbert \( \mathcal{A} \)-module \( W \) is said to be full if \( \langle W, W \rangle \) is dense in \( \mathcal{A} \). Notice that \( \mathcal{A} \) is a (full) Hilbert \( \mathcal{A} \)-module via \( \langle x, y \rangle = x^*y \) for all \( x, y \in \mathcal{A} \).

Let \( w \in W \). If \( \langle w, w \rangle = e \) is a projection in \( \mathcal{A} \), then \( w = we \). Indeed,

\[
\langle w - we, w - we \rangle = \langle w, w \rangle - \langle w, w \rangle e - e \langle w, w \rangle + e \langle w, w \rangle e = 0
\]

(see the paragraph before [6] Lemma 1]). This property will be used frequently later.

The main difference between Hilbert \( C^* \)-modules and Hilbert \( H^* \)-modules is the fact that Hilbert \( H^* \)-modules can be equipped with the structure of a complex Hilbert space. Although both structures obey
the same axioms as ordinary Hilbert spaces (except that the inner product takes values in a more general structure than in the field of complex numbers), there are some properties that differ Hilbert \( C^* \)-modules from Hilbert spaces. For example, a closed submodule \( V \) of a Hilbert \( C^* \)-module \( W \) need not be (orthogonally) complemented, that is, \( V \oplus V^\perp \neq W \) in general, where \( V^\perp \) denotes \( \{x \in W : \langle x, y \rangle = 0 \text{ for all } y \in V\} \). However, Hilbert \( C^* \)-modules over compact operators share many nice properties with Hilbert spaces; in particular all closed submodules of such modules are complemented.
We shall deal with Hilbert $C^*$-modules over $C^*$-algebras of compact operators, and Hilbert $H^*$-modules. These structures possess orthonormal bases. More precisely, if $W$ is a Hilbert $A$-module, where $A$ is a $C^*$-algebra of compact operators or an $H^*$-algebra, then there exists a net $\{w_i : i \in I\}$ generating a dense submodule of $W$, such that $\langle w_i, w_i \rangle$ is a minimal projection in $A$, and $\langle w_i, w_j \rangle = 0$ for $i \neq j$. All these orthonormal bases have the same cardinal number which is called the orthogonal dimension of $W$ and denoted by $\dim_A W$. More details on orthonormal bases for Hilbert $C^*$-modules over $C^*$-algebras of compact operators can be found in [6], and for Hilbert $H^*$-modules in [8].

1.2. Notation and conventions. Let $W$ be a Hilbert $C^*$-module (resp. $H^*$-module) over a $C^*$-algebra (resp. an $H^*$-algebra) $A$. We simply use Hilbert $A$-module or Hilbert module over $A$ to denote either of them.

If $(\mathcal{H}, (\cdot, \cdot))$ is a Hilbert space and $\xi, \eta \in \mathcal{H}$, then by $\xi \otimes \eta$ we denote the rank one operator defined by $(\xi \otimes \eta)(\nu) = (\nu, \eta)\xi$ for all $\nu \in \mathcal{H}$.

All spaces are assumed to be over complex numbers.

If $G$ is an abelian group, we always use “+” as its group operation.

“Orthogonally additive mapping(s)” are abbreviated as “o. a. m.”.

2. $\perp$-additive mappings on Abelian Groups

Let $W$ and $G$ be abelian groups. Suppose that $\perp$ is a binary relation on $W$. We shall say that a mapping $f : W \to G$ is $\perp$-additive if for all $x, y \in W$

$$x \perp y \implies f(x + y) = f(x) + f(y),$$

and that a mapping $F : W \times W \to G$ is $\perp$-preserving if for all $x, y \in W$

$$x \perp y \implies F(x, y) = 0.$$

Let us recall that a mapping $T : W \to G$ is called additive if $T(x + y) = T(x) + T(y)$ for all $x, y \in W$, a mapping $B : W \times W \to G$ is called biadditive if it is additive in both variables, a mapping $Q : W \to G$ is called quadratic if $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ for all $x, y \in W$, and if $W$ is a complex vector space then a mapping $S : W \times W \to G$ is called sesquilinear if it is linear in the first and conjugate linear in the second variable.

Lemma 2.1. Let $W$ be an abelian group with a binary relation $\perp$, and $V, G$ be uniquely 2-divisible abelian groups. Suppose that there exist additive mappings $\varphi, \psi : V \to W$ with the following properties:

$$\varphi(V) \perp \psi(V) \quad \text{and} \quad (\varphi + \psi)(V) \perp (\varphi - \psi)(V).$$  

(2)
Let \( W_0 := \varphi(V) + \psi(V) \leq W \). If \( f : W \to G \) is a \( \perp \)-additive mapping, then the following holds:

(i) If \( f \) is odd (resp. even), then \( f \) is additive (resp. quadratic) on \( W_0 \).

(ii) If \( x \perp y \) implies \((-x) \perp (-y)\), then there exist mappings \( T : W \to G \) and \( B : W \times W \to G \) such that \( T \) is additive on \( W_0 \), \( B \) is \( \perp \)-preserving symmetric biadditive on \( W_0 \times W_0 \), and

\[
f(x) = T(x) + B(x, x) \quad \text{for all } x \in W_0.
\]

**Proof.** (i) Since \( \varphi \) and \( \psi \) are additive, clearly they are odd. Now using the \( \perp \)-additivity of \( f \), additivity of \( \varphi \) and \( \psi \), as well as the properties given in (2), we obtain

\[
\begin{align*}
f(\varphi(x) + \varphi(y)) + f(\psi(x) - \psi(y)) &= f(\varphi(x + y)) + f(\psi(x) - y)) \\
&= f(\varphi(x + y) + \psi(x) - y)) \\
&= f(\varphi(x) + \varphi(y) + \psi(x) - \psi(y)) \\
&= f((\varphi + \psi)(x) + (\varphi - \psi)(y)) \\
&= f((\varphi + \psi)(x)) + f((\varphi - \psi)(y)) \\
&= f(\varphi(x) + \psi(x)) + f(\psi(y) - \psi(y)) \\
&= f(\varphi(x)) + f(\psi(x)) + f(\varphi(y)) + f(-\psi(y))
\end{align*}
\]

(3)

for all \( x, y \in V \).

First assume that \( f \) is odd. Switching \( x \) and \( y \) in (3) gives

\[
\begin{align*}
f(\varphi(x) + \varphi(y)) + f(\psi(y) - \psi(x)) &= f(\varphi(y)) + f(\psi(y)) + f(\varphi(x)) + f(-\psi(x)).
\end{align*}
\]

(4)

Add (3) and (4) and use that \( f \) is odd to get

\[
2f(\varphi(x) + \varphi(y)) = 2f(\varphi(x)) + 2f(\varphi(y)),
\]
equivalently,

\[
f(\varphi(x) + \varphi(y)) = f(\varphi(x)) + f(\varphi(y))
\]
as \( G \) is uniquely 2-divisible. Thus \( f \) is additive on \( \varphi(V) \). Then (3) reduces to

\[
f(\psi(x) - \psi(y)) = f(\psi(x)) + f(-\psi(y)),
\]

that is,

\[
f(\psi(x) + \psi(y)) = f(\psi(x)) + f(\psi(y))
\]
as \( \psi \) is odd. Hence, \( f \) is additive on \( \psi(V) \) as well. It is now easy to verify that \( f \) is additive on \( W_0 \).
Now assume that \( f \) is even. Put \( y = x \) in (3) to get
\[
f(2\varphi(x)) + f(0) = 2f(\varphi(x)) + 2f(\psi(x)),
\]
then put \( y = -x \) in (3) to get
\[
f(0) + f(2\psi(x)) = 2f(\varphi(x)) + 2f(\psi(x)).
\]
Comparing (3) and (6) yields \( f(2\varphi(x)) = f(2\psi(x)) \), that is, \( f(\varphi(2x)) = f(\psi(2x)) \) for all \( x \in V \). Since \( V \) is uniquely 2–divisible, we have
\[
f(\varphi(x)) = f(\psi(x)) \quad \text{for all} \quad x \in V.
\]
Then \( f(\psi(x) - \psi(y)) = f(\varphi(x) - \varphi(y)) \) for all \( x, y \in V \). This together with (3) implies
\[
f(\varphi(x) + \varphi(y)) + f(\varphi(x) - \varphi(y)) = 2f(\varphi(x)) + 2f(\varphi(y)),
\]
so \( f \) is quadratic on \( \varphi(V) \). Then it follows from (7) that \( f \) is also quadratic on \( \psi(V) \). Therefore \( f \) is quadratic on \( W_0 \).

(ii) Set
\[
T(x) = \frac{1}{2}(f(x) - f(-x)), \quad F(x) = \frac{1}{2}(f(x) + f(-x)) \quad \text{for all} \quad x \in W.
\]
Then \( T \) is odd and \( \perp \)-additive. By (i), \( T \) is additive on \( W_0 \). Furthermore, \( F \) is even and \( \perp \)-additive. Again by (i), \( F \) is quadratic on \( W_0 \). Then \( F(0) = 0 \), so
\[
F(x + x) + F(x - x) = 2F(x) + 2F(x)
\]
yields \( F(2x) = 4F(x) \) for all \( x \in W_0 \).

Define
\[
B(x, y) = \frac{1}{4}(F(x + y) - F(x - y)) \quad \text{for all} \quad x, y \in W.
\]
Since \( F \) is even, \( B \) is symmetric. It is well-known that \( B \) is biadditive (on \( W_0 \), but in the sequel we prove this fact for reader’s convenience. Obviously, \( B(x, 0) = B(0, x) = 0 \) and \( B(x, x) = \frac{1}{4}(F(2x) - F(0)) = F(x) \) for all \( x \in W_0 \). Since \( F \) is quadratic on \( W_0 \), for all \( x, y, u \in W_0 \) we have
\[
4B(x + y, 2u) = F(x + y + 2u) - F(x + y - 2u)
= F((x + u) + (y + u)) + F((x + u) - (y + u))
- F((x - u) - (y - u)) - F((x - u) + (y - u))
= 2(F(x + u) + F(y + u)) - 2(F(x - u) + F(y - u))
= 8B(x, u) + 8B(y, u).
\]
Since $G$ is uniquely 2–divisible, this implies
\[ B(x + y, 2u) = 2B(x, u) + 2B(y, u). \] (8)
Inserting $y = 0$ and $x = z$ yields
\[ B(z, 2u) = 2B(z, u). \]
If we put $x + y$ instead of $z$, using (8) we get
\[ B(x + y, u) = B(x, u) + B(y, u). \]
Hence, $B$ is biadditive on $W_0 \times W_0$. Finally,
\[ f(x) = T(x) + F(x) = T(x) + B(x, x) \]
for all $x \in W_0$. Notice that, for all $x, y \in W_0$, $x \perp y$ implies
\[
2B(x, y) = B(x, y) + B(y, x) \\
= B(x + y, x + y) - B(x, x) - B(y, y) \\
= (f(x + y) - T(x + y)) - (f(x) - T(x)) - (f(y) - T(y)) \\
= (f(x + y) - f(x) - f(y)) - (T(x + y) - T(x) - T(y)) = 0.
\]
Hence $B(x, y) = 0$, namely, $B$ is $\perp$-perserving on $W_0 \times W_0$.

**Remark 2.2.** By the definitions of $T$ and $B$, it is easy to see that they are uniquely determined by $f$. Actually,
\[
T(x) = \frac{1}{2}(f(x) - f(-x)), \\
B(x, y) = \frac{1}{8}(f(x + y) + f(-x - y) - f(x - y) - f(-x + y))
\]
for all $x, y \in W$. The reason why we only have
\[ f(x) = T(x) + B(x, x) \]
for all $x \in W_0$, instead of $W$, is because it is only known that $F(x) = B(x, x)$ for all $x \in W_0$.

**Lemma 2.3.** Let $W, V, G$ be normed spaces, and $\perp$ be a binary relation on $W$ such that $x \perp y$ implies $(-x) \perp (-y)$. Suppose that there are continuous linear mappings $\varphi, \psi : V \to W$ with the following properties:
\[ \varphi(V) \perp \psi(V) \] and \[ (\varphi + \lambda\psi)(V) \perp (\varphi - \lambda\psi)(V) \] for $\lambda \in \{1, i\}$. (9)
Let $W_0 := \varphi(V) + \psi(V) \leq W$.
If $f : W \to G$ is a continuous $\perp$-additive mapping, then there exist continuous mappings $T : W \to G$ and $S : W \times W \to G$ such that $T$
is additive on \( W_0 \), \( S \) is sesquilinear on \( W_0 \times W_0 \) with the property that for all \( x, y \in W_0 \), \( x \perp y \) implies \( S(x, y) + S(y, x) = 0 \), and
\[
f(x) = T(x) + S(x, x) \quad \text{for all} \quad x \in W_0.
\]
Furthermore, if we also assume that \( x \perp y \) implies \( x \perp iy \), then \( S \) is \( \perp \)-preserving on \( W_0 \times W_0 \).

**Proof.** By Lemma 2.1 (ii), there exist mappings \( T : W \rightarrow G \) and \( B : W \times W \rightarrow G \) such that \( T \) is additive on \( W_0 \), \( B \) is \( \perp \)-preserving symmetric biadditive on \( W_0 \times W_0 \), and
\[
f(x) = T(x) + B(x, x) \quad \text{for all} \quad x \in W_0.
\]
Since \( f \) is continuous, clearly so are \( T \) and \( B \) (see Remark 2.2).

Since \( \psi \) is linear and \( B \) is \( \perp \)-preserving on \( W_0 \times W_0 \), it follows from (2) that for all \( x, y \in V \) and \( \lambda \in \{1, i\} \) we have
\[
0 = B((\varphi + \lambda \psi)(x), (\varphi - \lambda \psi)(y))
= B(\varphi(x) + \lambda \psi(x), \varphi(y) - \lambda \psi(y))
= B(\varphi(x), \varphi(y)) + B(\psi(\lambda x), \varphi(y))
- B(\varphi(x), \psi(\lambda y)) - B(\lambda \psi(x), \lambda \psi(y))
= B(\varphi(x), \varphi(y)) - B(\lambda \psi(x), \lambda \psi(y)).
\]
This implies
\[
B(i\varphi(x), i\psi(y)) = B(\varphi(x), \varphi(y)) = B(\psi(x), \psi(y))
\]
and
\[
B(i\varphi(x), i\varphi(y)) = B(\varphi(ix), \varphi(iy)) = B(\psi(ix), \psi(iy))
= B(i\psi(x), i\psi(y)) = B(\varphi(x), \varphi(y)).
\]
Hence,
\[
B(ix, iy) = B(x, y)
\]
for all \( x, y \in \varphi(V) + \psi(V) = W_0 \).

Since \( B \) is biadditive and continuous on \( W_0 \times W_0 \), it is also \( \mathbb{R} \)-bilinear on \( W_0 \times W_0 \). Define \( S : W \times W \rightarrow G \) by
\[
S(x, y) = B(x, y) + iB(x, iy).
\]
Then, for all \( x, y \in W_0 \),
\[
S(ix, y) = B(ix, y) + iB(ix, iy) = B(ix, y) + iB(x, y)
= i(B(x, y) - iB(ix, y)) = i(B(x, y) + iB(x, iy))
= iS(x, y),
\]
and analogously
\[
S(x, iy) = -iS(x, y).
\]
Since the mapping $B$ is continuous $\mathbb{R}$-bilinear on $W_0 \times W_0$, the mapping $S$ is continuous $\mathbb{R}$-bilinear on $W_0 \times W_0$ as well. However, from the above we conclude that $S$ is continuous sesquilinear on $W_0 \times W_0$.

Also, notice that

\[
S(x,y) + S(y,x) = B(x,y) + iB(x, iy) + B(y, x) + iB(y, ix)
\]

\[
= 2B(x,y) + iB(x, iy) + iB(ix, y) = 2B(x,y)
\]

for all $x, y \in W_0$. In particular, we get

\[
S(x,x) = B(x,x)
\]

for all $x \in W_0$.

Then $S$ is a continuous sesquilinear mapping on $W_0 \times W_0$ with the properties that for all $x, y \in W_0$,

\[
x \perp y \implies S(x,y) + S(y,x) = 0
\]

and

\[
f(x) = T(x) + S(x,x)
\]

for all $x \in W_0$.

Furthermore, if $x \perp y$ implies $x \perp iy$, then for all $x, y \in W_0$ satisfying $x \perp y$ we have

\[
S(x,y) + S(y,x) = 0 \text{ and } -iS(x,y) + iS(y,x) = 0.
\]

Hence $S(x,y) = 0$, that is, $S$ is $\perp$-preserving on $W_0 \times W_0$.

By the definition of $S$ and Remark 2.2 we conclude that $S$ is also uniquely determined by $f$ and

\[
S(x,y) = \frac{1}{8} \left( f(x+y) + if(x+iy) - f(x-y) - if(x-iy) + f(-x-y) + if(-x-iy) - f(-x+y) - if(-x+iy) \right)
\]

for all $x, y \in W$.

It should be also mentioned that the mapping $T$ is $\mathbb{R}$-linear on $W_0$ since it is continuous and additive on $W_0$, but it is not $\mathbb{C}$-linear in general.

3. O. A. M. on Hilbert modules

Let $(W, \langle \cdot, \cdot \rangle)$ be a Hilbert $\mathcal{A}$-module and let $G$ be an abelian group. We shall study $\perp$-additive mappings and $\perp$-preserving mappings for the binary relation $\perp$ on $W$ given by

\[
x \perp y \iff \langle x,y \rangle = 0.
\]

A mapping $f : W \to G$ is said to be orthogonally additive if

\[
\langle x,y \rangle = 0 \implies f(x+y) = f(x) + f(y).
\]
A mapping $B : W \times W \to G$ is said to be orthogonality preserving if
\[ \langle x, y \rangle = 0 \implies B(x, y) = 0. \]

A morphism between Hilbert $A$-modules $V$ and $W$ is a mapping $\varphi : V \to W$ satisfying $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$. It is clear that morphisms are continuous mappings, and it is not difficult to verify that they are also $A$-linear mappings, that is, linear mappings satisfying $\varphi(ax) = \varphi(x)a$ for all $x \in V$ and $a \in A$.

**Theorem 3.1.** Let $W$ be a Hilbert $A$-module, $V$ be a submodule of $W$, and $\varphi : V \to W$ be a morphism such that $\varphi(V) \subseteq V^\perp$. Let $W_0 := V \oplus \varphi(V) \leq W$. Suppose that $G$ is a uniquely 2–divisible abelian group and that $f : W \to G$ is an o. a. m. Then the following holds:

(i) There exist mappings $T : W \to G$ and $B : W \times W \to G$ such that $T$ is additive on $W_0$, $B$ is symmetric biadditive orthogonality preserving on $W_0 \times W_0$, and

\[ f(x) = T(x) + B(x, x) \quad \text{for all } x \in W_0. \]

(ii) If $G$ is a normed space and $f$ is continuous, then there exist continuous mappings $T : W \to G$ and $S : W \times W \to G$ such that $T$ is additive on $W_0$, $S$ is sesquilinear orthogonality preserving on $W_0 \times W_0$, and

\[ f(x) = T(x) + S(x, x) \quad \text{for all } x \in W_0. \]

**Proof.** Set $x \perp y$ if and only if $\langle x, y \rangle = 0$. Let $id : V \to V$ be the identity mapping. Since $\varphi(V) \subseteq V^\perp$, we have $\langle \varphi(V), id(V) \rangle = 0$. Furthermore, for all $x, y \in V$, and $\lambda \in \{1, i\}$,

\[ \langle (\varphi + \lambda \cdot id)(x), (\varphi - \lambda \cdot id)(y) \rangle = \langle \varphi(x), \varphi(y) \rangle - \langle x, y \rangle = 0. \]

Then (i) and (ii) follow from Lemma 2.1 (ii) and Lemma 2.3 respectively.

We write $U \sim V$ if $U$ and $V$ are unitarily equivalent Hilbert $C^*$-modules over a $C^*$-algebra $A$, that is, if there exists a mapping $u : U \to V$ such that there is a mapping $u^* : V \to U$ satisfying $\langle ux, y \rangle = \langle x, u^*y \rangle$ for all $x \in U$, $y \in V$, and

\[ u^*u = id_U, \quad uu^* = id_V. \]

It is clear that $u$ is surjective and $\langle u(x), u(y) \rangle = \langle x, y \rangle$ for all $x, y \in U$.

A closed submodule $V$ of a Hilbert $C^*$-module $W$ is said to be complemented if $W = V \oplus V^\perp$, and fully complemented if $V$ is complemented and $V^\perp \sim W$. 
Corollary 3.2. Let $V$ be a fully complemented submodule of a Hilbert $C^*$-module $W$, $G$ be a uniquely 2–divisible abelian group, and $f : W \to G$ be an o. a. m. Then there exist mappings $T : W \to G$ and $B : W \times W \to G$ such that $T$ is additive on $V$, $B$ is symmetric biadditive orthogonality preserving on $V \times V$, and

$$f(x) = T(x) + B(x,x) \quad \text{for all} \quad x \in V.$$ 

Furthermore, if $G$ is a normed space and $f$ is continuous then there exist a continuous mapping $T : W \to G$ which is additive on $V$, and a continuous mapping $S : W \times W \to G$, which is sesquilinear and orthogonality preserving on $V \times V$, such that

$$f(x) = T(x) + S(x,x) \quad \text{for all} \quad x \in V.$$ 

Proof. Since $V$ is a fully complemented submodule of $W$ there exists a linear operator $u : W \to V^\perp$ such that $\langle u(x), u(y) \rangle = \langle x, y \rangle$ for all $x, y \in W$. Set $\varphi = u|_V$. Then $\varphi : V \to V^\perp$ satisfies $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$. So it remains to apply Theorem 3.1. \hfill \Box

4. O. A. M. on Hilbert $K(H)$-modules and Hilbert $HS(H)$-modules

In this section, $\mathcal{A}$ always denotes $K(H)$ or $HS(H)$. Let $W$ be a Hilbert $\mathcal{A}$-module and $e \in \mathcal{A}$ be a rank one projection in $\mathcal{A}$. Then there exists an orthonormal basis $\{w_i : i \in I\}$ for $W$ such that $\langle w_i, w_i \rangle = e$ for all $i \in I$ (see [6] Remark 4 (d)] for Hilbert $K(H)$-modules, and [5] Proposition 1.5] for Hilbert $HS(H)$-modules). The following lemma will allow us to deal with yet another suitable orthonormal basis for $W$.

Lemma 4.1. Let $W$ be a Hilbert $\mathcal{A}$-module with $\dim \mathcal{A} W \leq \dim \mathcal{H}$, and $\{\xi_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}$. Then there exists an orthonormal basis $\{w_i : i \in J \subseteq I\}$ for $W$ such that $\langle w_i, w_i \rangle = \xi_i \otimes \xi_i$ for all $i \in J$.

Proof. Let us fix an arbitrary $j_0 \in I$. Then there exists an orthonormal basis $\{g_i : i \in J\}$ for $W$ such that $\langle g_i, g_i \rangle = \xi_{j_0} \otimes \xi_{j_0}$ for all $i \in J$. Since $\dim \mathcal{A} W \leq \dim \mathcal{H}$, we assume $J \subseteq I$. Define, for all $i \in J$,

$$w_i = g_i(\xi_{j_0} \otimes \xi_i).$$

Then $\langle w_i, w_j \rangle = 0$ if $i \neq j$ and

$$\langle w_i, w_i \rangle = \langle g_i(\xi_{j_0} \otimes \xi_i), g_i(\xi_{j_0} \otimes \xi_i) \rangle$$

$$= (\xi_i \otimes \xi_{j_0})(g_i, g_i)(\xi_{j_0} \otimes \xi_i)$$

$$= (\xi_i \otimes \xi_{j_0})(\xi_{j_0} \otimes \xi_{j_0})(\xi_{j_0} \otimes \xi_i)$$

$$= \xi_i \otimes \xi_i$$
for all \( i \in J \). Furthermore, for all \( x \in W \),
\[
x = \sum_{i \in J} g_i \langle g_i, x \rangle \\
= \sum_{i \in J} g_i \langle g_i(\xi_{j_0} \otimes \xi_{j_0}), x \rangle \\
= \sum_{i \in J} g_i (\xi_{j_0} \otimes \xi_{j_0}) \langle g_i, x \rangle \\
= \sum_{i \in J} g_i (\xi_{j_0} \otimes \xi_i) (\xi_i \otimes \xi_{j_0}) \langle g_i, x \rangle \\
= \sum_{i \in J} g_i (\xi_{j_0} \otimes \xi_i) \langle g_i(\xi_{j_0} \otimes \xi_i), x \rangle \\
= \sum_{i \in J} w_i \langle w_i, x \rangle.
\]

By [6, Theorem 1], \( \{ w_i : i \in J \} \) is an orthonormal basis for \( W \).

**Remark 4.2.** Let \( \mathcal{H} \) be a Hilbert space with \( \dim \mathcal{H} = \aleph_0 \), \( W \) be a Hilbert \( A \)-module such that \( \dim_A W = \aleph_0 \), and \( \{ \xi_i : i \in \mathbb{N} \} \) be an orthonormal basis for \( \mathcal{H} \). By Lemma 4.1 there exists an orthonormal basis \( \{ w_i : i \in \mathbb{N} \} \) for \( W \) such that \( \langle w_i, w_i \rangle = \xi_i \otimes \xi_i \) for all \( i \in \mathbb{N} \). Let \( a \in A \). Since
\[
\| \sum_{i=m}^n w_i a \|_W^2 = \| \sum_{i=m}^n \sum_{i=m}^n w_i a \| \|
= \| \sum_{i=m}^n a^* \langle w_i, w_i \rangle a \| = \| \sum_{i=m}^n a^* (\xi_i \otimes \xi_i) a \|,
\]
and \( \sum_{i=1}^\infty a^* (\xi_i \otimes \xi_i) a = a^* a \), the sequence \( (\sum_{i=1}^n w_i a)_{n=1}^\infty \) is a Cauchy sequence in \( W \), so it converges. Hence, for all \( a \in A \), \( \sum_{i=1}^\infty w_i a \in W \) and \( \langle \sum_{i=1}^\infty w_i a, \sum_{i=1}^\infty w_i a \rangle = a^* a \).

Before giving the main result of this section, we provide a representation result of sesquilinear orthogonality preserving mappings \( S : W \times W \to G \). This result is of independent interest.

**Proposition 4.3.** Let \( W \) be a Hilbert \( A \)-module and \( G \) be a normed space. If \( S : W \times W \to G \) is a continuous sesquilinear orthogonality preserving mapping, then there is a unique linear mapping \( \Phi : \langle W, W \rangle \to G \) such that
\[
S(x, y) = \Phi(\langle y, x \rangle) \quad \text{for all} \quad x, y \in W.
\]

Furthermore, if \( \mathcal{H} \) is finite dimensional or \( \dim \mathcal{H} = \dim_A W = \aleph_0 \), then \( \Phi \) can be extended to a continuous linear mapping on \( A \).
Proof. Let \( \{\xi_i : i \in I\} \) be an orthonormal basis for \( \mathcal{H} \) and let \( e_i = \xi_i \otimes \xi_i \) for all \( i \in I \). Fix an arbitrary \( i_0 \in I \). Let \( \{w_j : j \in J\} \) be an orthonormal basis for \( W \) such that \( \langle w_j, w_j \rangle = e_{i_0} \) for all \( j \in J \). Then for all \( j, k \in J \) and all \( a, b \in \mathcal{A} \) we have

\[
\langle w_j a - w_k a, w_j b + w_k b \rangle = a^* \langle w_j, w_j \rangle b - a^* \langle w_k, w_k \rangle b
\]

which is not true in general if \( i \neq k \), and hence

\[
0 = S(w_j a - w_k a, w_j b + w_k b) = S(w_j a, w_j b) - S(w_k a, w_k b).
\]

In particular, for \( b = e_{i_0} \), one obtains

\[
S(w_j a, w_j) = S(w_j a, w_j e_{i_0}) = S(w_k a, w_k e_{i_0}) = S(w_k a, w_k)
\]

for all \( j, k \in J \). Hence the mapping \( \Phi_{i_0} : \mathcal{A} \to G \), defined by

\[
\Phi_{i_0}(a) = S(w_k a, w_k),
\]

does not depend on \( k \in J \). It is clear that \( \Phi_{i_0} \) is linear. Notice that

\[
\|\Phi_{i_0}(a)\| = \|S(w_k a, w_k)\| \leq \|S\| \cdot \|w_k\|_W \cdot \|a\|
\]

and let

\[
\Phi_{i_0}(\langle ye_{i_0}, x \rangle) = S(w_k \langle ye_{i_0}, x \rangle, w_k) = \sum_{j \in J} S(w_k \langle ye_{i_0}, w_j \rangle \langle w_j, x \rangle, w_k)
\]

Since \( i_0 \in I \) is arbitrary, it follows that

\[
S(x, y) = S(x, \sum_{i \in I} y e_i) = \sum_{i \in I} S(x, y e_i) = \sum_{i \in I} \Phi_i(\langle ye_i, x \rangle).
\]
Let us define $\Phi : \langle W, W \rangle \to G$ by

$$\Phi(a) = \sum_{i \in I} \Phi_i(e_i a).$$

By (11), the mapping $\Phi$ is well-defined and $S(x, y) = \Phi(\langle y, x \rangle)$ for all $x, y \in W$. Since all $\Phi_i$ are linear, $\Phi$ is linear as well. Uniqueness of such $\Phi$ is obvious.

If $H$ is finite dimensional then $\sum_{i \in I} \Phi_i(e_i a)$ converges for all $a \in A$. If $\dim H = \dim A W = \aleph_0$ then by Lemma 4.1 there exists an orthonormal basis $\{v_i : i \in I\}$ for $W$ such that $\langle v_i, v_i \rangle = \xi_i \otimes \xi_i$ for all $i \in I$. By Remark 4.2, $\sum_{j \in I} v_j a \in W$ for all $a \in A$, so for all $a, b \in A$ we have

$$S(\sum_{j \in I} v_j b, \sum_{j \in I} v_j a^*) = \sum_{i \in I} \Phi_i(\langle \sum_{j \in I} v_j a^* e_i, \sum_{j \in I} v_j b \rangle) = \sum_{i \in I} \Phi_i(\sum_{j \in I} e_i(a v_j, v_j) b) = \sum_{i \in I} \Phi_i(e_i a b).$$

Hence $\sum_{i \in I} \Phi_i(e_i a)$ converges for all $a \in A^2 = A$. It means that in the cases when $\dim H$ is finite or $\dim H = \dim A W = \aleph_0$ we can extend $\Phi$ from $\langle W, W \rangle$ to $A$ if we define

$$\Phi(a) = \sum_{i \in I} \Phi_i(e_i a).$$

Finally, let us prove that $\Phi : A \to G$ is bounded. If $H$ is finite dimensional, this immediately follows from (10). Now assume that $\dim H = \dim A W = \aleph_0$. Then for every $a \in A$ we have, by Remark 4.2

$$\|\Phi(a^* a)\| = \|\Phi((\sum_{i \in I} v_i a, \sum_{i \in I} v_i a))\| = \|S(\sum_{i \in I} v_i a, \sum_{i \in I} v_i a)\| \leq \|S\| \|\sum_{i \in I} v_i a\|^2_W = \|S\| \|a^* a\|. $$

Thus $\Phi$ is bounded on positive elements on $A$. Therefore it is bounded on $A$.

We are now ready to prove our main result of this section.
Theorem 4.4. Let \( W \) be a Hilbert \( \mathcal{A} \)-module such that \( \dim_\mathcal{A} W \geq 2 \). Let \( G \) be a uniquely 2–divisible abelian group, and \( f : W \to G \) be an o. a. m. Then the following holds:

(i) There exist a unique additive mapping \( T : W \to G \) and a unique symmetric biadditive orthogonality preserving mapping \( B : W \times W \to G \) such that

\[
f(x) = T(x) + B(x, x) \text{ for all } x \in W.
\]

(ii) If \( G \) is a normed space and \( f \) is continuous, then there are a unique continuous additive mapping \( T : W \to G \) and a unique linear mapping \( \Phi : \langle W, W \rangle \to G \) such that

\[
f(x) = T(x) + \Phi(\langle x, x \rangle) \text{ for all } x \in W.
\]

Furthermore, if \( \mathcal{H} \) is finite dimensional or \( \dim \mathcal{H} = \dim_\mathcal{A} W = \aleph_0 \), then \( \Phi \) can be extended to a continuous linear mapping on \( \mathcal{A} \).

Proof. (i) First assume that \( W \) is either finite dimensional with \( \dim_\mathcal{A} W = 2n \), or \( \dim_\mathcal{A} W \geq \aleph_0 \). If \( \dim_\mathcal{A} W = 2n \) then let \( V \) be a closed submodule of \( W \) such that \( \dim_\mathcal{A} W = n \); if \( \dim_\mathcal{A} W \geq \aleph_0 \) then let \( V \) be a closed submodule of \( W \) such that \( \dim_\mathcal{A} V = \dim_\mathcal{A} V^\perp = \dim_\mathcal{A} W \). Let \( \{ w_i : i \in I \} \) be an orthonormal basis for \( W \) such that \( \langle w_i, w_i \rangle = e \) for all \( i \in I \), where \( e \) is a fixed rank one projection in \( \mathcal{A} \). Let \( \{ w_i : i \in I_1 \subseteq I \} \) be an orthonormal basis for \( V \) and \( \{ w_i : i \in I_2 \subseteq I \} \) be an orthonormal basis for \( V^\perp \). Let \( \varphi : V \to V^\perp \) be an isomorphism between the bases of \( V \) and \( V^\perp \). It remains to apply Theorem 3.1 (i). Notice that \( V \oplus \varphi(V) = V \oplus V^\perp = W \).

Now assume that \( W \) is finite dimensional with \( \dim_\mathcal{A} W = 2n + 1 \). From the above we conclude that the desired conclusion is true for \( f \) restricted to any \( 2n \)-dimensional closed submodule of \( W \). Let \( X \) be a closed submodule of \( W \) such that \( \dim_\mathcal{A} X = 1 \). Then \( \dim_\mathcal{A} X^\perp = 2n \). Let \( Z \) be a closed submodule of \( W \) such that \( \dim_\mathcal{A} Z = 2 \), and \( X \subset Z \). The statement is true both on \( Z \) and \( X^\perp \), hence on \( W \).

(ii) Combining the proofs of (i) above and Theorem 3.1 (ii), we can find a unique continuous additive mapping \( T : W \to G \) and a unique continuous sesquilinear orthogonality preserving mapping \( S : W \times W \to G \) such that

\[
f(x) = T(x) + S(x, x) \text{ for all } x \in W.
\]

Then apply Proposition 4.3 to end the proof.

We should mention that the condition \( \dim_\mathcal{A} W \geq 2 \) is essential in Theorem 4.4, as shown in the following example.
Example 4.5. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product given by $\langle \xi, \eta \rangle = \eta \otimes \xi$. It is known that $\dim_{\mathcal{A}} \mathcal{H} = 1$ (\cite{6} Example 1] and \cite{5} Example 2.3]). Notice that $\langle \xi, \eta \rangle = 0$ if and only if $\xi = 0$ or $\eta = 0$. Then every odd mapping on $\mathcal{H}$ (taking values in a uniquely 2–divisible abelian group) is orthogonally additive, but not additive in general. For example, fix $0 \neq \eta_0 \in \mathcal{H}$ and define $f(\xi) = (\xi, \eta_0)\xi \otimes \xi$ for all $\xi \in \mathcal{H}$.

The following example shows that in the case when $\mathcal{H}$ is infinite dimensional and $\dim_{\mathcal{A}} W$ is finite, the mapping $\Phi$ from Theorem \cite{4} cannot be extended to a continuous linear mapping on $\mathcal{A}$.

Example 4.6. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Let $W = \mathcal{H} \oplus \mathcal{H}$ be a Hilbert $\mathcal{A}$-module with the coordinate operations and the $\mathcal{A}$-valued inner product given by $\langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2$.

Then $\dim_{\mathcal{A}} W = 2$ (see \cite{6} Theorem 3] and \cite{8} Section 2]). To distinguish the above notation, we use $(\,,\,)_{\mathcal{H}}$ for the inner product on $\mathcal{H}$. Define $f : W \to \mathbb{C}$ by $f((\xi_1, \xi_2)) = (\xi_1, \xi_1)_{\mathcal{H}} + (\xi_2, \xi_2)_{\mathcal{H}}$ for all $\xi_1, \xi_2 \in \mathcal{H}$.

We claim that $f$ is an orthogonally additive mapping. Indeed, let $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}$ be such that $0 = \langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2$.

Then one can easily check that $(\xi_1, \eta_1)_{\mathcal{H}} + (\eta_1, \xi_1)_{\mathcal{H}} + (\xi_2, \eta_2)_{\mathcal{H}} + (\eta_2, \xi_2)_{\mathcal{H}} = 0$.

Thus

\[
\begin{align*}
 f((\xi_1, \xi_2) + (\eta_1, \eta_2)) &= f((\xi_1 + \eta_1, \xi_2 + \eta_2)) \\
 &= (\xi_1 + \eta_1, \xi_1 + \eta_1)_{\mathcal{H}} + (\xi_2 + \eta_2, \xi_2 + \eta_2)_{\mathcal{H}} \\
 &= (\xi_1, \xi_1)_{\mathcal{H}} + (\eta_1, \eta_1)_{\mathcal{H}} + (\xi_1, \xi_2)_{\mathcal{H}} + (\eta_1, \xi_2)_{\mathcal{H}} + (\xi_2, \eta_1)_{\mathcal{H}} + (\eta_2, \eta_1)_{\mathcal{H}} \\
 & \quad + (\xi_2, \xi_2)_{\mathcal{H}} + (\eta_2, \eta_2)_{\mathcal{H}} + (\eta_2, \xi_2)_{\mathcal{H}} + (\eta_2, \eta_2)_{\mathcal{H}} \\
 &= (\xi_1, \xi_1)_{\mathcal{H}} + (\xi_2, \eta_2)_{\mathcal{H}} + (\eta_2, \xi_2)_{\mathcal{H}} + (\eta_2, \eta_2)_{\mathcal{H}} \\
 &= f((\xi_1, \xi_2)) + f((\eta_1, \eta_2)).
\end{align*}
\]

This shows that $f$ is an orthogonally additive mapping. It is clear that $f$ is even.

In what follows, we prove that there is no continuous linear mapping $\Phi : \mathcal{A} \to \mathbb{C}$ such that $f(w) = \Phi(\langle w, w \rangle)$ for all $w \in W$. To the contrary, assume that there is such a mapping $\Phi$. Let $\{\xi_n : n \in \mathbb{N}\}$ be an
orthonormal basis for $\mathcal{H}$ and let $E_n = \xi_n \otimes \xi_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we set $T_n = \sum_{k=1}^{n} \frac{1}{k} E_k \in \langle W, W \rangle$. Since the sequence $(T_n)$ converges to $T = \sum_{k=1}^{\infty} \frac{1}{k} E_k \in A$ and $\Phi$ is continuous, we conclude that the sequence $(\Phi(T_n))$ converges as well. However,

$$\Phi(T_n) = \sum_{k=1}^{n} \frac{1}{k} \Phi(E_k) = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \Phi(2E_k)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \Phi(\xi_k \otimes \xi_k + \xi_k \otimes \xi_k)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \Phi((\langle \xi_k, \xi_k \rangle, (\xi_k, \xi_k)))$$

$$= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} f((\xi_k, \xi_k))$$

$$= \sum_{k=1}^{n} \frac{1}{k}$$

does not converge; a contradiction.

The following result is an immediate consequence of Theorem 4.4 (see [6, Example 2]).

**Corollary 4.7.** Let $\mathcal{H}$ be a Hilbert space with $2 \leq \dim \mathcal{H} \leq \aleph_0$, and the orthogonality on $A$ be defined by

$$x \perp y \iff x^* y = 0.$$ 

Assume that $G$ is a uniquely 2-divisible abelian group, and that $f : A \to G$ is an o. a. m. Then the following holds:

(i) There exist a unique additive mapping $T : A \to G$ and a unique symmetric biadditive orthogonality preserving mapping $B : A \times A \to G$ such that

$$f(x) = T(x) + B(x, x) \text{ for all } x \in A.$$ 

(ii) If $G$ is a normed space and $f$ is continuous, then $T$ is continuous and there exists a unique continuous linear mapping $\Phi : A \to G$ such that

$$f(x) = T(x) + \Phi(x^* x) \text{ for all } x \in A.$$ 

The following example demonstrates that the underlying algebra is $\mathcal{K}(\mathcal{H})$ (instead of just being a $C^*$-algebra of compact operators) is essential in Corollary 4.7 (and Theorem 4.4). The same example shows that we cannot take an arbitrary $H^*$-algebra instead of $\mathcal{HS}(\mathcal{H})$. 


Example 4.8. Let $\mathcal{H}$ be a separable Hilbert space. Fix an orthonormal basis $\{\xi_i\}$ for $\mathcal{H}$. As usual, we represent operators on $\mathcal{H}$ as matrices with respect to $\{\xi_i\}$. Let $\mathcal{D}$ be the norm closed subalgebra of $\mathcal{A}$ consisting of diagonal operators. Then $\mathcal{D}$ is a Hilbert module over itself; let us notice that $\mathcal{D}$ is commutative. Then $(x,y) = 0$ if and only if $x^*y = y^*x = yx^* = xy^* = 0$. Define $f : \mathcal{D} \to \mathcal{D}$ by $f(x) = x(x^*)^2$. Then $f$ is an odd orthogonally additive mapping, but it is clearly not additive.

5. O. A. M. ON HILBERT $C^*$-MODULES OVER A $C^*$-ALGEBRA OF COMPACT OPERATORS AND HILBERT $H^*$-MODULES

Let $\mathcal{A}$ be an arbitrary $C^*$-algebra of compact operators. By [2], Theorem 1.4.5

$$\mathcal{A} = \bigoplus_{j \in J} \mathcal{K}(\mathcal{H}_j) = \{(a_j) \in \prod_{j \in J} \mathcal{K}(\mathcal{H}_j) : \lim_{j \in J} \|a_j\| = 0\}.$$ 

Let $W$ be a Hilbert $\mathcal{A}$-module. We may assume that $W$ is full. If $W_j$ denotes the closed linear span of $W\mathcal{K}(\mathcal{H}_j)$, then each $W_j$ is a (full) Hilbert $\mathcal{K}(\mathcal{H}_j)$-module and $W$ is the outer direct sum of $W_j$’s:

$$W = \bigoplus_{j \in J} W_j = \{(w_j) \in \prod_{j \in J} W_j : \lim_{j \in J} \|w_j\| = 0\}.$$ 

(see [6], Introduction or [19]).

Now let $\mathcal{A}$ be an arbitrary $H^*$-algebra. By [1], Theorems 4.2 and 4.3, $\mathcal{A}$ is the orthogonal sum $\bigoplus_{j \in J} \mathcal{A}_j$ where each $\mathcal{A}_j$ is a simple $H^*$-algebra which is a minimal closed ideal of $\mathcal{A}$ and $\mathcal{A}_j = \mathcal{HS}(\mathcal{H}_j)$ for some Hilbert space $\mathcal{H}_j$. Then every $a \in \mathcal{A}$ can be written as $a = \sum_{j \in J} a_j$ with $a_j \in \mathcal{HS}(\mathcal{H}_j)$ and $\|a\|^2 = \sum_{j \in J} \|a_j\|^2$. Let $W$ be a Hilbert $\mathcal{A}$-module. We may assume that $W$ is faithful (i.e. it has zero annihilator in $\mathcal{A}$).

According to [8], Theorem 2.3 there exists a family $\{W_j : j \in J\}$ such that each $W_j$ is a (faithful) Hilbert $\mathcal{HS}(\mathcal{H}_j)$-module and $W$ is the mixed product of $W_j$’s:

$$W = \times_{j \in J} W_j = \{(w_j) \in \prod_{j \in J} W_j : \sum_{j \in J} \|w_j\|^2 < \infty\}.$$ 

Theorem 5.1. Let $\mathcal{A} = \bigoplus_{j \in J} \mathcal{A}_j$ be a $C^*$-algebra of compact operators, resp. an $H^*$-algebra, with $\mathcal{A}_j = \mathcal{K}(\mathcal{H}_j)$, resp. $\mathcal{A}_j = \mathcal{HS}(\mathcal{H}_j)$. Let $W = \bigoplus_{j \in J} W_j$ be a Hilbert $\mathcal{A}$-module with $W_j$ a Hilbert $\mathcal{A}_j$-module such that $\dim\mathcal{A}_j W_j = \dim\mathcal{H}_j = \aleph_0$ for each $j \in J$. Let $G$ be a normed space and let $f : W \to G$ be a continuous o. a. m. Then there exist a continuous additive mapping $T : W \to G$ and a continuous linear mapping $\Phi : \mathcal{A} \to G$ such that

$$f(x) = T(x) + \Phi(\langle x, x \rangle) \quad \text{for all} \quad x \in W.$$
Proof. Define $f_j = f|_{W_j}$ for each $j \in J$. Then $f_j : W_j \to G$ is a continuous o. a. m. By Theorem 4.4 there exist a continuous additive mapping $T_j : W_j \to G$ and a continuous linear mapping $\Phi_j : A_j \to G$ such that

$$f_j(x_j) = T_j(x_j) + \Phi_j(\langle x_j, x_j \rangle) \quad \text{for all} \quad x_j \in W_j.$$  

Define $T : W \to G$ by

$$T(x) = \frac{1}{2}(f(x) - f(-x)).$$

If we write $x = \sum_{j \in J} x_j$ with $x_j \in W_j$, then

$$T(x) = \frac{1}{2} \sum_{j \in J} (f_j(x_j) - f_j(-x_j)) = \sum_{j \in J} T_j(x_j).$$

This implies that $T$ is an additive mapping; it is continuous since $f$ is continuous.

Let $\{w_i : i \in I\}$ be an orthonormal basis for $W$ and let $\{w_i^j : i \in I_j\} \subseteq \{w_i : i \in I\}$ be an orthonormal basis for $W_j$. By Lemma 4.1 without loss of generality we can assume $\langle w_i^j, w_i^j \rangle = \xi_i^j \otimes \xi_i^j$ where $\{\xi_i^j : i \in I_j\}$ is an orthonormal basis for $H_j$. Let $a_j \in A_j$. Then

$$\Phi_j(a_j^*a_j) = \Phi_j(\sum_{i \in I_j} a_j^* \langle \xi_i^j \otimes \xi_i^j \rangle a_j) = \sum_{i \in I_j} \Phi_j(a_j^* \langle w_i^j, w_i^j \rangle a_j)$$

$$= \sum_{i \in I_j} \Phi_j(\langle w_i^j a_j, w_i^j a_j \rangle) = \Phi_j(\sum_{i \in I_j} w_i^j a_j, \sum_{i \in I_j} w_i^j a_j).$$

Notice that $\sum_{i \in I} w_ia$ converges for all $a \in A$. In fact, if $a = \sum_{j \in J} a_j$ with $a_j \in A_j$, then Remark 4.2 implies that $\sum_{i \in I_j} w_i^j a_j \in W_j$ and

$$\langle \sum_{i \in I_j} w_i^j a_j, \sum_{i \in I_j} w_i^j a_j \rangle = a_j^*a_j.$$  

Then

$$\sum_{i \in I_j} w_i^j a_j \|_{W_j} = \|\sum_{i \in I_j} w_i^j a_j, \sum_{i \in I_j} w_i^j a_j\|^{\frac{1}{2}}$$

$$= \|a_j^*a_j\|^{\frac{1}{2}} = \|a_j\|.$$  

Hence $\sum_{j \in J} \sum_{i \in I_j} w_i^j a_j \in W$, that is, $\sum_{i \in I} w_ia \in W$. For $a \in A$ define $\Phi(a^*a) = \frac{1}{2}(f(\sum_{i \in I} w_ia) + f(- \sum_{i \in I} w_ia))$. If we write $a = \sum_{j \in J} a_j$...
with \( a_j \in A_j \), then

\[
\Phi(a^*a) = \frac{1}{2} \sum_{j \in J} \left( f_j \left( \sum_{i \in I_j} w_i^j a_j \right) + f_j \left( - \sum_{i \in I_j} w_i^j a_j \right) \right) = \sum_{j \in J} \Phi_j \left( \sum_{i \in I_j} w_i^j a_j, \sum_{i \in I_j} w_i^j a_j \right) = \sum_{j \in J} \Phi_j(a_j^*a_j).
\]

Using the fact that every \( a \in A \) can be written as a linear combination of four positive elements (i.e., those of the form \( x^*x \) for some \( x \in A \)), we define \( \Phi(a) = \sum_{j \in J} \Phi_j(a_j) \) for all \( a = \sum_{j \in J} a_j \in A \). Since each \( \Phi_j \) is linear, \( \Phi \) is linear as well. Since

\[
\|\Phi(a^*a)\| \leq \|f\| \cdot \left\| \sum_{i \in I} w_i^j a_j \right\| = \|f\| \cdot \|a\|
\]

for every \( a \in A \), \( \Phi \) is continuous. Finally,

\[
f(x) = T(x) + \Phi(\langle x, x \rangle) \quad \text{for all} \quad x \in W.
\]

Corollary 5.2. Let \( \mathcal{A} = \bigoplus_{j \in J} A_j \) be a \( C^* \)-algebra of compact operators, resp. an \( H^* \)-algebra, with \( A_j = \mathcal{K}(\mathcal{H}_j) \), resp. \( A_j = \mathcal{HS}(\mathcal{H}_j) \), such that \( 2 \leq \dim \mathcal{H}_j \leq \aleph_0 \) for each \( j \in J \). Let \( G \) be a normed space and let \( f : \mathcal{A} \to G \) be a continuous o. a. m., with respect to the orthogonality defined by

\[
x \perp y \iff x^*y = 0.
\]

Then there exist a unique continuous additive mapping \( T : \mathcal{A} \to G \) and a unique continuous linear mapping \( \Phi : \mathcal{A} \to G \) such that

\[
f(x) = T(x) + \Phi(\langle x, x \rangle) \quad \text{for all} \quad x \in \mathcal{A}.
\]

Let us mention that Example 4.8 also provides a counterexample for Corollary 5.2 in the case when \( \mathcal{A} = W = \bigoplus_{j \in J} A_j \) with \( A_j = W_j = \mathcal{K}(\mathcal{H}_j) \) or \( \mathcal{HS}(\mathcal{H}_j) \) and \( \dim A_j W_j = \dim \mathcal{H}_j = 1 \) for all \( j \in J \).

6. O. A. M. on \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \)

The aim of this section is to prove an analogue of Corollary 4.7 for \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) instead of \( \mathcal{K}(\mathcal{H}) \) and \( \mathcal{HS}(\mathcal{H}) \).

Proposition 6.1. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces with \( \dim \mathcal{H}_1, \dim \mathcal{H}_2 \geq 2 \). Let \( G \) be a uniquely \( 2 \)-divisible abelian group and let \( f : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G \) be an o. a. m., with respect to the orthogonality defined by

\[
x \perp y \iff x^*y = 0.
\]
Then there exist a unique additive mapping $T : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G$ and a symmetric biadditive orthogonality preserving mapping $B : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \times \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G$ such that

$$f(x) = T(x) + B(x, x) \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2).$$

If $\mathcal{H}$ is a Hilbert space such that $\dim \mathcal{H} \geq 2$, $G$ is a Banach space and $f$ is continuous, then $T$ is continuous and there exists a unique continuous linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to G$ such that

$$f(x) = T(x) + \Phi(x^* x) \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}).$$

**Proof.** Let us emphasize that $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert $\mathcal{B}(\mathcal{H}_1)$-module with respect to the inner product $\langle x, y \rangle = x^* y$. If $\mathcal{H}_2$ is infinite dimensional, let $K$ be a closed subspace of $\mathcal{H}_2$ such that both $K$ and $K^\perp$ are infinite dimensional. If $\dim \mathcal{H}_2 = 2n$, let $K$ be an $n$-dimensional subspace of $\mathcal{H}_2$. Let $U : K \to K^\perp$ be unitary. Then $\varphi : \mathcal{B}(\mathcal{H}_1, K) \to \mathcal{B}(\mathcal{H}_1, K^\perp)$, $\varphi(A) = UA$ is an isomorphism. Notice that $\mathcal{B}(\mathcal{H}_1, K) \oplus \varphi(\mathcal{B}(\mathcal{H}_1, K)) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. It remains to apply Theorem 3.1 (i).

If $\dim \mathcal{H}_2 = 2n + 1$, let $K$ be a 1-dimensional subspace of $\mathcal{H}_2$. Then $\dim K^\perp = 2n$ and according to the above, the statement holds on $\mathcal{B}(\mathcal{H}_1, K^\perp)$. Let $M$ be a 2-dimensional subspace of $\mathcal{H}_2$ containing $K$. Again, according to the above, the statement also holds true on $\mathcal{B}(\mathcal{H}_1, M)$, hence finally on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

The second statement can be proved in a similar way, but using Theorem 3.1 (ii) instead of Theorem 3.1 (i), and then applying the results from [3] (see also [4, Theorem 1.1]) to represent $S$ via $\Phi$. 

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