AN APPROXIMATION THEOREM IN CLASSICAL MECHANICS

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Abstract. A theorem by K. Meyer and D. Schmidt says that the reduced three-body problem in two or three dimensions with one small mass is approximately the product of the restricted problem and a harmonic oscillator [7]. This theorem was used to prove dynamical continuation results from the classical restricted circular three-body problem to the three-body problem with one small mass.

We state and prove a similar theorem applicable to a larger class of mechanical systems. We present applications to spatial (N+1)-body systems with one small mass and gravitationally coupled systems formed by a rigid body and a small point mass.

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1. Introduction. Consider the Newtonian three-body problem with one of the bodies having small mass. If the latter is negligible so that it does not affect the motion of the other two bodies, then the dynamics decouples and it consists of two sub-systems. One describes the dynamics of the large bodies, called the primaries;
the other describes the motion of the infinitesimal body in the gravitational field of the primaries. The restricted three body problem consists in finding the dynamics of the infinitesimal body when the dynamics of the primaries is assumed known. The restricted circular three-body problem refers to the situation when the primaries are in a relative equilibrium with respect to the translational and rotational symmetry, that is, when they move on circular paths with constant angular velocity about their (fixed) centre of mass.

The meaning of the Meyer and Schmidt theorem [7] stated in the Abstract is that, in the three-body problem with the small mass taken as an expansion parameter, after the reduction of the translational and rotational symmetries, the dynamics at first order is given by the sum of two Hamiltonians: one is the Hamiltonian of the restricted circular three body problem; the other is the Hamiltonian of the linearised dynamics at the relative equilibrium of the primaries, where the latter is understood as an equilibrium in the symplectic reduced space of the relative two body problem. (Note that the latter is a one degree of freedom harmonic oscillator.) Provided certain non-degeneracy conditions hold, this theorem was successfully used in proving various dynamical continuation results from the restricted circular three body problem to the three-body problem (see [5]).

The proof of Meyer and Schmidt’s Theorem (MSTh) relies on applying appropriate scalings (blow-ups) and approximation theory. In the spatial case it is accomplished by taking advantage of the low dimensionality of the relative two-body problem (i.e., the motion of the primaries). A generalization to the \((N + 1)\)-body problem is feasible (see problem 9 pp.103 in [6]), with the observation that one needs to be aware of the normal form of the reduced dynamics on the symplectic reduced space of the \(N\)-body problem.

In this paper we take a closer look at the MSTh and state and prove its analogue for a wider class of mechanical systems. We apply our findings to the spatial \((N + 1)\)-body problem and gravitationally coupled bodies.

We start by clarifying the meaning of restricted problems in mechanics. Essentially, one deals with a coupled system formed by what we call the driver and the driven systems. When the coupling is negligible, the dynamics of the driver is determined independently. Once this is distinguished, the dynamics of the driven system writes as a time-dependent restricted problem, with the dynamics of the driver appearing explicitly in the equations. For instance, in the case of the three-body problem with one small mass, the driver system in formed by the two large bodies, and the driven system the small third body.

In the particular case when the driver is taken at an equilibrium, the restricted driven problem is autonomous, with the equilibrium point of the driven system appearing as a parameter. When continuous symmetries are present and when the driver is a relative equilibrium, the dynamics of the restricted driven system can be rendered as autonomous, provided it is set in a non-inertial moving frame aligned with the group velocity of the driver. For example, in the restricted circular three-body problem, the dynamics of the infinitesimal mass is considered in the rotating frame of the primaries.

We prove that if in the restricted problem the driver is in an equilibrium, a small non-zero coupling with the driven system causes the driver to perform small oscillations about its equilibrium position. A similar phenomenon occurs in the symmetric case when the driver is in a relative equilibrium, with the modification
that the small oscillations take place in the driver system’s symplectic reduced space about its relative equilibrium understood as an equilibrium.

The proofs are developed on the Hamiltonian side and, as in [7], rely on finding appropriate scalings and approximations. When the system is non-symmetric and the driver is at an equilibrium, the methodology is straightforward. For Lie symmetric systems, with the driver at a relative equilibrium, the main theorem’s proof involves general symplectic reduction theory and the description of the linearization operator at a relative equilibrium developed by Patrick in [8].

The approximation results here may be used in proving the continuation of dynamical features from restricted to un-restricted problems, all essentially being applications of the implicit function theorem (provided generic non-degeneracy conditions hold). Since many such results are specific to the system in discussion, we mention only two of these, namely the continuation of (relative) equilibria and (relative) periodic orbits. We flag some interesting examples (other than the three-body problem) which could be investigated in a “continuation” context, and leave for future work their detailed study.

The paper is organized as follows: in Section 2 we introduce “restricted” problems as they appear in classical mechanics. In Section 3 we present the continuation method for non-symmetric systems with the driver at an equilibrium. We start with the example of the planar double pendulum and proceed to prove a general statement. Section 4 considers Lie symmetric systems in the case when the driver is at a relative equilibrium. The main result is Theorem 4.1. Section 5 presents two applications: the \((N + 1)\)-body problem with one small mass, and gravitationally coupled systems formed by a rigid body and a small point mass. We conclude with some final remarks.

2. Restricted problems in mechanics. A classical example of a restricted problem is the restricted three body problem of celestial mechanics; it consists in the dynamics of an small mass point under the gravitational attraction of two large bodies (idealized as mass points) under the assumption that the small (infinitesimal) mass does not affect the large bodies’ dynamics. Another example is that of a double pendulum with a small mass at the free end. This system consists of two point masses \(m_1\) and \(m_2\) subject to gravity, with \(m_1\) connected to a fixed point by a rigid rod, and \(m_2\) hanging from \(m_1\) by another rigid rod; if \(m_2\) is infinitesimally small, then it will not affect the dynamics of \(m_1\) but it will be dragged by it. In this case, the dynamics of \(m_2\) may be called then the restricted double pendulum problem.

The examples above are included in more general of systems class defined by Lagrangians of the form \(L : T(Q_1 \times Q_2) \to \mathbb{R}\),

\[
L(q_1, q_2, \dot{q}_1, \dot{q}_2) = L^{(1)}(q_1, \dot{q}_1) + \varepsilon^2 L^{(2)}(q_1, q_2, \dot{q}_1, \dot{q}_2),
\]

where the configurations space \(Q_1 \times Q_2\) is the product of two smooth finite-dimensional manifolds. We call the system given by \(L^{(1)}\) the driver system, and the system defined by \(L^{(2)}\) the driven system. The choice of using \(\varepsilon^2\) instead of \(\varepsilon\) is a convenience which is motivated by some later calculations. The Euler-Lagrange equations are

\[
\frac{d}{dt} \left( \frac{\partial L^{(1)}}{\partial \dot{q}_1} + \varepsilon^2 \frac{\partial L^{(2)}}{\partial q_1} \right) - \frac{\partial L^{(1)}}{\partial q_1} + \varepsilon^2 \frac{\partial L^{(2)}}{\partial q_1} = 0.
\]

(2)
\[ \varepsilon^2 \frac{d}{dt} \left( \frac{\partial L^{(2)}}{\partial \dot{q}_2} \right) = \varepsilon^2 \frac{\partial L^{(2)}}{\partial q_2} \]  

(3)

Simplifying \( \varepsilon^2 \) in the second equation, and then taking \( \varepsilon^2 \) infinitesimally small, the system becomes

\[ \frac{d}{dt} \left( \frac{\partial L^{(1)}}{\partial \dot{q}_1} \right) = \frac{\partial L^{(1)}}{\partial q_1} \]  

(4)

\[ \frac{d}{dt} \left( \frac{\partial L^{(2)}}{\partial \dot{q}_2} \right) = \frac{\partial L^{(2)}}{\partial q_2} \]  

(5)

Consider a solution \( q_1(t) \) of the driver Lagrangian \( L^{(1)} \) and substitute it into \( L^{(2)} \). We obtain then a time-dependent Lagrangian

\[ L_R(q_2, \dot{q}_2; q_1(t), \dot{q}_1(t)) \]  

(6)

which we define as the Lagrangian of the restricted problem derived from (1).

In general, \( L_R \) is time-dependent. In the particular case when \( q_1(t) = q_1^e \) is an equilibrium of \( L^{(1)} \), the Lagrangian \( L_R \) becomes time-independent, with \( q_1^e \) as an external parameter, i.e. \( L_R = L_R(q_1, \dot{q}_1; q_1^e) \).

If \( L^{(1)} \) is invariant under the action of a Lie group \( G \), then its dynamics may accept relative equilibria. Recall that relative equilibrium is a solution of the form \( q_1(t) = \exp(\xi t)q_1^e \) for some \( \xi \in g \) and \( q_1 \in Q_1 \), where \( g \) denotes the Lie algebra of the \( G \). One can then pass the dynamics of \( L^{(2)} \) in the moving frame of the \( \exp(\xi t)q_1^e \) so that it will become autonomous; in this case, the group velocity \( \xi \) and the point \( q_1^e \) are fixed parameters.

For example, in the classical three body problem with one small mass, the driver Lagrangian \( L^{(1)} \) describes the relative motion of the two large masses (the primaries). Well-known restricted problem are:

- the **elliptical restricted three-body problem**, when the relative motion \( q_1(t) \) of the primaries is a solution of the Kepler problem;
- the **circular restricted three-body problem**, when the primaries move on circular trajectories with constant angular velocity around their common centre of mass.

The Hamiltonian of the restricted problem is defined by taking the Legendre transform of \( L_R \)

\[ p_2 = \frac{\partial L_R}{\partial \dot{q}_2}(q_2, \dot{q}_2; q_1(t), \dot{q}_1(t)). \]  

(7)

and calculating \( H_R(q_2, p_2; q_1(t), \dot{q}_1(t)) \). Note that if we apply the Legendre transform to the coupled Lagrangian (1), we obtain

\[ H(q_1, q_2, p_1, p_2) = H^{(1)}(q_1, p_1) + \varepsilon^{-2} H^{(2)}(q_1, q_2, p_1, p_2; \varepsilon). \]  

(8)

The dependency of \( H^{(2)} \) on \( \varepsilon \) obstructs the retrieval of the restricted Hamiltonian \( H_R \), as it is not clear what scalings should be applied in order to have a well-defined \( \varepsilon \)-expansion of \( H \).

In what follows we consider the simpler cases when \( q_1(t) \) is an equilibrium or a relative equilibrium.
3. Continuation near an equilibrium for non-symmetric systems.

3.1. A case study: The planar double pendulum. Consider the double pendulum, that is two point masses $m_1$ and $m_2$ subject to gravity, with $m_1$ connected to a fixed point by a rigid rod of length $l_1$, and $m_2$ hanging from $m_1$ by a rigid rod of length $l_2$. If we denote $\theta_1$ and $\theta_2$ the angles of $l_1$ and $l_2$ with the vertical direction, respectively, the Lagrangian is

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} \left( l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 \right) + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) + m_1 g l_1 \cos \theta_1 + m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2).$$

If $m_2 = \varepsilon^2$ is small, we have

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = L^{(1)}(\theta_1, \dot{\theta}_1) + \varepsilon^2 L^{(2)}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2),$$

where

$$L^{(1)}(\theta_1, \dot{\theta}_1) = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + m_1 g l_1 \cos \theta_1$$

(which is the Lagrangian of a single pendulum of mass $m_1$), and

$$L^{(2)}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2} \left( l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 \right) + g (l_1 \cos \theta_1 + l_2 \cos \theta_2).$$

The single pendulum $m_1$ has two equilibria, $\theta_1(t) = 0$ and $\theta_1(t) = \pi$, with the pendulum pointing down and up, respectively. Consider the equilibrium $\theta_1(t) = 0$, and substitute this solution into $L_2$. We obtain the restricted double pendulum Lagrangian:

$$L_R(\theta_2, \dot{\theta}_2) = \frac{1}{2} l_2^2 \dot{\theta}_2^2 + gl_2 \cos \theta_2 + gl_1$$

which, after dropping the constant term $gl_1$, is the Lagrangian of a single pendulum of unit mass. Applying the Legendre transform, the restricted Hamiltonian is

$$H_R(\theta_2, p_2) = \frac{1}{2l_2^2} p_2^2 - gl_2 \cos \theta_2.$$

Now let us consider the Hamiltonian of the double pendulum

$$H(\theta_1, \theta_2, p_1, p_2) = \frac{p_1^2}{2l_1^2 \left[ m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]} - \frac{p_1 p_2 \cos(\theta_1 - \theta_2)}{l_1 l_2 \left[ m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]} + \left( \frac{m_1 + m_2}{m_2} \right) \frac{p_2^2}{2l_2^2 \left[ m_1 + m_2 \sin^2(\theta_1 - \theta_2) \right]} - (m_1 + m_2) g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2.$$

If $m_2 = \varepsilon^2$, expanding $H$ in terms of $\varepsilon$ leads to

$$H(\theta_1, \theta_2, p_1, p_2) = H^{(1)}(\theta_1, p_1) + \varepsilon^2 H^{(2)}(\theta_1, \theta_2, p_1, p_2; \varepsilon)$$

where

$$H^{(1)}(\theta_1, p_1) = \frac{p_1^2}{2l_1^2 m_1} - m_1 g l_1 \cos \theta_1$$

and

$$H_2(\theta_1, \theta_2, p_1, p_2; \varepsilon) = \frac{1}{\varepsilon^2} \frac{p_2^2}{2l_2^2 m_1} - \frac{p_2^2}{2l_2^2 m_1} \sin^2(\theta_1 - \theta_2) - \varepsilon^2 \frac{p_2^2}{2l_2^2 m_1} \cos^2(\theta_1 - \theta_2) \sin^2(\theta_1 - \theta_2) - \varepsilon^2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2) + O(\varepsilon^4).$$
Let \((\theta_1, p_1) = (0, 0)\) be the stable equilibrium of the \(H_1\). Applying
\[
\theta_1 = \varepsilon \Theta_1, \quad p_1 = \varepsilon P_1
\]
and
\[
\theta_2 = \Theta_2, \quad p_2 = \varepsilon^2 P_2,
\]
and re-scaling \(H \to \varepsilon^{-2}H\) we obtain
\[
H(\Theta_1, \Theta_2, P_1, P_2) = \left( \frac{P_1^2}{2l_1^2 m_1} + \frac{m_1 g l_1}{2} \Theta_1^2 \right) + \left( \frac{P_2^2}{2l_2^2} - gl_2 \cos \Theta_2 \right) + O(\varepsilon)
\]
where we ignore the constant term. Thus, the double pendulum with a hanging small mass \(m_2\) is given at the first order by the product of a harmonic oscillator (the normal mode of the \(m_1\) pendulum near its stable equilibrium) and a simple pendulum of unit mass. By applying arguments based on the implicit function theorem, we can deduce then various continuation results from the restricted to the unrestricted problem with \(m_2\) small (e.g. continuation of equilibria, periodic solutions and bifurcations).

The example above shows that the right scalings (the blow up near the equilibrium of \(H^{(1)}\) and the re-scaling of \(H^{(2)}\)) are key. A straightforward generalization of this case study is the subject of the next subsection.

### 3.2. Statement for non-symmetric systems.

Let \(P\) be a finite-dimensional symplectic vector space. By definition, a symplectic scaling on \(P\) of multiplier \(\alpha > 0\), is a smooth invertible change of variables \(z \to z(\alpha)\) and such that, in canonical coordinates, we have
\[
\mathcal{J} = \alpha^{-1} \left[ \frac{\partial z}{\partial Z} \right] \left[ \frac{\partial z}{\partial Z} \right]^T
\]
where \(\mathcal{J}\) is the canonical structure matrix (see [5]). If \(H : P \to \mathbb{R}\) is a Hamiltonian, \(H = H(z)\), then the equations of motion in the new variables \(Z\) are
\[
\dot{Z} = \frac{\partial Z}{\partial \dot{Z}} \mathcal{J} \left( \frac{\partial Z}{\partial \dot{Z}} \right)^T dH(Z) \bigg|_{z(Z)}
\]
that is
\[
\dot{Z} = \alpha^{-1} \mathcal{J} dH \bigg|_{z(Z)}
\]
so practically, in coordinates \(Z\), one has to multiply the Hamiltonian by \(\alpha^{-1}\) and substitute \(z\) by \(z(Z)\). It is usual then to applying the time-parametrization \(\alpha^{-1} \frac{d}{d\tau} = \frac{d}{dt}\) we obtain the equations of motion of the scaled Hamiltonian
\[
\dot{Z} = \mathcal{J} dH \bigg|_{z(Z)}. \tag{22}
\]

**Remark 1.** The blow-up transformation at a point \(z_0\) given by \(z = z(Z) := z_0 + \varepsilon Z\) is a symplectic scaling of multiplier \(\varepsilon^{-2}\) and
\[
\dot{Z} = \varepsilon^{-2} \mathcal{J} dH \bigg|_{z = z_0 + \varepsilon Z}. \tag{23}
\]

Let \(P\) and \(V\) be a symplectic manifold and a symplectic vector space, respectively, together with a family of Hamiltonians \(\{H(z_1, z_2; \varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}\), \(\varepsilon_0 > 0\) fixed, of the form
\[
H : P_1 \times V \to \mathbb{R}
\]
\[
H(z_1, z_2; \varepsilon) = H^{(1)}(z_1) + H^{(2)}(z_1, z_2; \varepsilon). \tag{24}
\]
Let \( z^e \) be an equilibrium of \( H^{(1)} \) and consider a system of coordinates near \( z^e \) so that \( P \) is identified locally with a symplectic vector space. Assume that there is a symplectic scaling \( z \rightarrow z_2(Z_2; \varepsilon) \) of multiplier \( \varepsilon^{-2} \), so that

\[
H^{(2)}(z_1, z_2; \varepsilon) = H^{(2)}(z_1, z_2(Z_2; \varepsilon); \varepsilon) = \varepsilon^2 \tilde{H}^{(2)}(z_1, Z_2) + O(\varepsilon^3), \quad \forall z_1 \in P_1.
\]  

(25)

Under the transformation \( z_1 \rightarrow z_1(Z_1) := z^e_1 + \varepsilon Z_1 \), in coordinates \((Z_1, Z_2)\), the Hamiltonian becomes

\[
\tilde{H}(Z_1, Z_2; \varepsilon) = H^{(1)}(z^e_1 + \varepsilon Z_1) + H^{(2)}(z^e_1 + \varepsilon Z_1, Z_2)
\]

\[
= H^{(1)}(z^e_1 + \varepsilon Z_1) + \varepsilon^2 \tilde{H}^{(2)}(z^e_1 + \varepsilon Z_1, Z_2) + O(\varepsilon^3)
\]  

(26)

Expanding at \( H^{(1)} \) at \( z^e_1 \) and \( H^{(2)} \) in the first argument at \((z^e_1, Z_2)\), and ignoring the constant term, we have

\[
\tilde{H}(Z_1, Z_2; \varepsilon) = \varepsilon^2 \frac{1}{2} Z_1^T \left[ d^2H^{(1)}(z^e_1) \right] Z_1 + \varepsilon^2 \tilde{H}^{(2)}(Z_2; z^e_1) + O(\varepsilon^3)
\]  

(27)

Using the time reparametrization \( \tilde{t} \rightarrow \varepsilon^{-2} \tilde{t} \), the dynamics is given by

\[
\tilde{H}(Z_1, Z_2; \varepsilon) = \frac{1}{2} Z_1^T \left[ d^2H^{(1)}(z^e_1) \right] Z_1 + \tilde{H}^{(2)}(Z_2; z^e_1) + O(\varepsilon). 
\]  

(28)

Note that \( [d^2H^{(1)}(z^e_1)] \) is the Hessian matrix of \( H^{(1)} \) at \( z^e_1 \); equivalently, the map \( Z_1 \rightarrow (1/2)d^2H^{(1)}(z^e_1)(Z_1, Z_1) \) gives the linearization of the Hamiltonian vectorfield \( X_{H^{(1)}} \) at \( z^e_1 \). Thus we have proven:

Theorem 3.1. In the above context, the product of the linearized dynamics of \( H^{(1)} \) at \( z^e_1 \) and the dynamics of \( H^{(2)}(Z_2; z^e_1) \) is an \( O(\varepsilon) \) approximation of the dynamics of the Hamiltonian (24).

Recall that for a differential equations system an equilibrium is called elementary (or non-degenerate) if its linearization matrix is non-singular (see [1], [5]). Also, a periodic solution of a Hamiltonian system is elementary (or non-degenerate) if all, but two of its multipliers, are different of 1. The implicit function theorem applied to the equilibria and the periodic solutions of Hamiltonian (28) (see also [5]) leads to the following:

Corollary 1. Consider a Hamiltonian of the form (28) and assume that \( z^e_1 \) is elementary. Then any elementary equilibrium \( z^e_2 \) of \( H^{(2)}(Z_2; z^e_1) \) can be continued as an equilibrium of the Hamiltonian (28).

Remark 2. The continued equilibria are equilibria of the Hamiltonian (24).

Corollary 2. Consider a Hamiltonian of the form (28) and let \( \varphi(t) \) be a non-degenerate periodic solution of \( H^{(2)}(Z_2; z^e_1) \) with period \( T_0 \) and multipliers \( 1, 1, \lambda_i, \tilde{\lambda}_i^{-1} \). If \( \lambda_i \neq 1 \) for all \( i \) and \( T_0 \neq 0 \mod 2\pi \), then the Hamiltonian (28) has a periodic solution of the form \( Z_1(t; \varepsilon) = O(\varepsilon), Z_2(t; \varepsilon) = \varphi(t) + O(\varepsilon) \) of period \( T = T_0 + O(\varepsilon) \).

Remark 3. The continued periodic solutions of (28) are periodic solutions of the Hamiltonian (24).

4. Systems with Lie symmetries. In this Section we extend the re-scaling method and continuation results presented above to systems with Lie symmetries.
4.1. General theory. Let \((P, \Omega)\) be a symplectic vector space and consider the free and proper action of a Lie group \(G\) on \(P\). We denote by \(g\) and \(g^*\) the Lie algebra and the Lie coalgebra of \(G\), respectively. Let \(H : P \to \mathbb{R}\) be a Hamiltonian invariant under \(G\) and assume that the symplectic form \(\Omega\) is also \(G\)-invariant. Then the dynamics benefits from additional first integrals which are given by the momentum map. Recall that by definition, the momentum map \(J : P \to g^*\) is a map such that for every \(\xi \in g\), we have
\[
\langle J(z), \xi \rangle = J_\xi(z)
\]
where \(J_\xi : P \to \mathbb{R}\) is the Hamiltonian function for the infinitesimal generator vector field
\[
\xi_P : P \to TP
\]
\[
z \to \xi_P(z) := \frac{d}{dt} \big|_{t=0} (\exp(t\xi) \cdot z).
\]
Specifically, the Hamiltonian vector field \(X_{J_\xi}\) associated to \(J_\xi\) is such that \(X_{J_\xi}(z) = \xi_P(z)\) for all \(z \in P\). Since \(P\) is a vector space, \(J_\xi(z)\) momentum map takes the form where
\[
J_\xi(z) = \frac{1}{2} \Omega(z, \xi_P(z)).
\]

Denote by \(G_\mu := \{g \in G \mid g \cdot \mu = \mu\}\) the isotropy group of a fixed momentum \(\mu \in g^*\), where \(G\) acts on \(g^*\) by the co-adjoint action \((4)\). If the action of \(G\) on \(P\) (or at least on the level set \(J^{-1}(\mu)\)) is free and proper, the quotient submanifold \(P_\mu := J^{-1}(\mu)/G_\mu\) is smooth. Let \(\pi : J^{-1}(\mu) \to P_\mu\) be the canonical projection and consider \(h : P_\mu \to \mathbb{R}\) defined by \(h \circ \pi = H\). By the Symplectic Reduction Theorem ([Meyer; Marsden-Weinstein]; see [3]), there is unique symplectic structure \(\Omega_\mu\) on \(P_\mu\) such that dynamical solutions of the system \((P, \Omega, H)\) project into dynamical solutions of the reduced Hamiltonian \((P_\mu, \Omega_\mu, h)\).

Many symmetric systems accept relative equilibria solutions, that is dynamical trajectories that are also one-parameters orbits of the symmetry group \((3)\). Formally, a point \(z^e \in P\) is a relative equilibrium if there is a \(\xi \in g\) such that the curve
\[
z(t) = \exp(t\xi) \cdot z^e \quad \text{for some} \quad z^e \in P
\]
is a solution of the Hamiltonian vector field \(X_H\). The value \(\xi\) is called the drift velocity of the relative equilibrium. The point \(z^e\) is determined as a critical point of the Hamiltonian \(H\) restricted to a momentum level set with the drift velocity \(\xi\) playing the role of a Lagrange multiplier:
\[
H_\xi(z) := H(z) - J_\xi(z) = H(z) - \langle J(z), \xi \rangle.
\]
The function \(H_\xi\) is usually called the augmented Hamiltonian or the Hamiltonian in a moving frame with drift velocity \(\xi\). Let \(\mu_e\) be the momentum of the relative equilibrium \(\exp(t\xi) \cdot z^e\). If \(\mu_e\) is a regular value for \(J\) and then the group action is free and proper, then the relative equilibrium projects into a unique equilibrium in \(P_\mu\).

For any \(z \in P\), denote by \([z] \in P_\mu\) the class of equivalence of \(z \in P_\mu\). A relative equilibrium \(z^e\) corresponds to an equilibrium \([z^e]\) in \(P_\mu\). The reduced linearization at \([z^e]\) is the linearization of the reduced Hamiltonian \(h_\mu\) at the equilibrium \([z^e]\). If the spectrum of the reduced linearization does not contain zero, then the relative equilibrium is called reductively elementary (or non-degenerate) (see [5], [8]). Also, a relative periodic solution, that is a periodic solution of the reduced Hamiltonian \(h_{\mu_e}\),
is called reductively elementary (or non-degenerate) if all, but two of its multipliers are different from 1.

4.2. Continuation near a relative equilibrium.

**Theorem 4.1.** Let \( (P, \Omega_1) \) be a symplectic manifold and \( (V, \Omega_2) \) a symplectic vector space, together with the diagonal action of a Lie group \( G \). Assume that the \( G \) actions on \( P \) and \( V \) are free, proper and with \( G \)-equivariant momentum maps which we denote \( J^{(1)} \) and \( J^{(2)} \), respectively.

For \( \varepsilon \in (0, \varepsilon_0) \), \( \varepsilon_0 > 0 \) small, consider the family of \( G \)-invariant Hamiltonians \( \{ H(z_1, z_2; \varepsilon) \}_{\varepsilon \in (0, \varepsilon_0)} \)

\[
H(z_1, z_2; \varepsilon) : P \times V \to \mathbb{R}
\]

\[
H(z_1, z_2; \varepsilon) = H^{(1)}(z_1) + H^{(2)}(z_1, z_2; \varepsilon)
\]

where \( H^{(1)} \) is \( G \)-invariant as well. Assume that there is a symplectic scaling \( z_2 \to z_2(Z_2) \) of multiplier \( \varepsilon^{-2} \), so that

\[
H^{(2)}(z_1, z_2(Z_2); \varepsilon) = \varepsilon^2 \tilde{H}^{(2)}(z_1, Z_2) + O(\varepsilon^3), \quad \forall z_1 \in P_1.
\]

Let \( z_1^* \) be a relative equilibrium of \( H^{(1)} : P \to \mathbb{R} \) with drift velocity \( \xi \) and momentum \( \mu \). For every \( \varepsilon \) fixed, denote by \( h_{\mu, \varepsilon} \) the reduced Hamiltonian of \( H(z_1, z_2; \varepsilon) \).

Then the product of the dynamics of the linearization at the equilibrium \( [z_1^*] \) of reduced Hamiltonian \( h_{\mu, \varepsilon}^{(1)} : (J^{(1)})^{-1}/G_{\mu, \varepsilon} \to \mathbb{R} \) of \( H^{(1)} \) and the dynamics of \( \tilde{H}^{(2)}(Z_2; z_1^*) \) in the moving frame of \( z_1^* \) is an \( O(\varepsilon) \) approximation of the dynamics of the reduced Hamiltonian \( h_{\mu, \varepsilon} \).

**Proof.** Since the \( G \) action is diagonal on \( P \times V \), its momentum map is

\[
J(z_1, z_2) = J^{(1)}(z_1) + J^{(2)}(z_2).
\]

Consider the dynamics near \( z_1^* \in P \) in a local system of coordinates, so that locally \( P \) is identified with a symplectic vector space. We now consider the scalings

\[
z_1 \to z_1(Z_1) := z_1^* + \varepsilon Z_1, \quad z_2 \to z_2(Z_2)
\]

on \( P \) and \( V \), respectively, both symplectic and of multiplier \( \varepsilon^{-2} \). Under this scalings, the Hamiltonian function \( J_\xi, \xi \in \mathfrak{g} \), becomes

\[
J_\xi(z_1, z_2) = J^{(1)}_\xi(z_1) + J^{(2)}_\xi(z_2) = \frac{1}{2} \Omega_1(z_1, \xi_{\Omega_1}(z_1)) + \frac{1}{2} \Omega_2(z_2, \xi_{\Omega_2}(z_2))
\]

\[
= \frac{1}{2} \Omega_1(z_1^* + \varepsilon Z_1, \xi_{\Omega_1}(z_1^* + \varepsilon Z_1)) + \varepsilon^4 \frac{1}{2} \Omega_2(z_2(Z_2), \xi_{\Omega_2}(z_2(Z_2)))
\]

\[
= \frac{1}{2} \Omega_1(z_1^*, \xi_{\Omega_1}(z_1^*)) + \varepsilon (\Omega_1(Z_1, \xi_{\Omega_1}(z_1^*)) + \Omega_1(z_1^*, \xi_{\Omega_1}(Z_1))) + O(\varepsilon^3)
\]

\[
= J^{(1)}_\xi(z_1^*) + \varepsilon \left( dJ^{(1)}_\xi(z_1^*) \right)(Z_1) + O(\varepsilon^2)
\]

So in a series expansion in \( \varepsilon \), the momentum map takes the form

\[
J(z_1, z_2) = J_\varepsilon(z_1, z_2) = J^{(1)}_\varepsilon(z_1^*) + \varepsilon (dJ^{(1)}_\varepsilon(z_1^*))(Z_1) + O(\varepsilon^2)
\]

Thus

\[
J_\varepsilon^{-1}(\mu) = \{(Z_1, Z_2) \in P \times V | J^{(1)}_\varepsilon(z_1^*) + \varepsilon (dJ^{(1)}_\varepsilon(z_1^*))(Z_1) + O(\varepsilon^2) = \mu \}_\varepsilon.
\]
Since \( J^{(1)}(z_1^\varepsilon) = \mu_\varepsilon \) we have
\[
J^{-1}_\varepsilon(\mu_\varepsilon) = \{(Z_1, Z_2) \in P \times V \mid \varepsilon \left(dJ^{(1)}_\varepsilon(z_1^\varepsilon)\right)(Z_1) + O(\varepsilon^2) = 0\}.
\]

(38)

or, for \( \varepsilon > 0 \),
\[
J^{-1}_\varepsilon(\mu_\varepsilon) = \{(Z_1, Z_2) \in P \times V \mid \left(dJ^{(1)}_\varepsilon(z_1^\varepsilon)\right)(Z_1) + O(\varepsilon) = 0\}.
\]

(39)

In particular, the map \( \varepsilon \rightarrow J^{-1}_\varepsilon \) is smooth and it extends at \( \varepsilon = 0 \) to
\[
0 \rightarrow J^{-1}_0(\mu_\varepsilon) = \{(Z_1, Z_2) \in P \times V \mid \left(dJ^{(1)}_\varepsilon(z_1^\varepsilon)\right)(Z_1) = 0\}
= \ker dJ^{(1)}_\varepsilon(z_1^\varepsilon) \times V.
\]

(40)

Also, since the \( G \)-actions on \( P \) and \( V \) are free, the reduced spaces \( \tilde{P}_\varepsilon = J^{-1}_\varepsilon(\mu)/G_{\mu_\varepsilon} \) are smooth manifolds for all \( \varepsilon \in [0, \varepsilon_0] \).

We return now to the Hamiltonian (31). First we write it in a frame with moving velocity \( \xi_\varepsilon \):
\[
H_{\xi_\varepsilon}(z_1, z_2; \varepsilon) = H^{(1)}(z_1) - J^{(1)}_\varepsilon(z_1) + H^{(2)}(z_1, z_2; \varepsilon) - J^{(2)}_\varepsilon(z_2).
\]

(41)

Under the transformations (34), it becomes
\[
H_{\xi_\varepsilon}(z_1, z_2; \varepsilon) = H^{(1)}(z_1^\varepsilon + \varepsilon Z_1) - J^{(1)}_\varepsilon(z_1^\varepsilon + \varepsilon Z_1)
+ H^{(2)}(z_1^\varepsilon + \varepsilon Z_1, z_2(Z_2); \varepsilon) - J^{(2)}_\varepsilon(z_2(Z_2)).
\]

(42)

Expanding \( H_{\xi_\varepsilon} \) in terms of \( \varepsilon \), taking into account that \( z_1^\varepsilon \) is a relative equilibrium (and so \( z_1^\varepsilon \) is a critical point of \( H^{(1)}_{\xi_\varepsilon} \)), and ignoring the constant term \( H^{(1)}(z_1^\varepsilon) \), we have
\[
H_{\xi_\varepsilon}(z_1, z_2; \varepsilon) \equiv H_{\xi_\varepsilon, \varepsilon}(Z_1, Z_2) = \varepsilon^2 \frac{1}{2} Z_1^T \left(d^2 H^{(1)}_{\xi_\varepsilon}\right) Z_1
+ \varepsilon^2 \tilde{H}^{(2)}_{\xi_\varepsilon}(z_1^\varepsilon, Z_2) + O(\varepsilon^3).
\]

(43)

Since \( z_1^\varepsilon \) becomes a fixed parameter in the \( \tilde{H}^{(2)}_{\xi_\varepsilon} \) expression, from now on we denote \( H^{(2)}(Z_2; z_1^\varepsilon) := H^{(2)}(z_1^\varepsilon, Z_2) \). The scaling \( H_{\xi_\varepsilon, \varepsilon} \rightarrow \varepsilon^{-2} H_{\xi_\varepsilon, \varepsilon} \) brings the dynamics to its standard Hamiltonian form, with the Hamiltonian given by
\[
H_{\xi_\varepsilon, \varepsilon} : J^{-1}_\varepsilon(\mu_\varepsilon) \rightarrow \mathbb{R}
\]

(44)

\[
H_{\xi_\varepsilon, \varepsilon}(Z_1, Z_2) = \frac{1}{2} Z_1^T \left(d^2 H^{(1)}_{\xi_\varepsilon}\right) Z_1 + \tilde{H}^{(2)}_{\xi_\varepsilon}(z_1^\varepsilon, Z_2) + O(\varepsilon).
\]

(45)

For any \( \varepsilon \in [0, \varepsilon_0) \), since the reduced spaces are well-defined, by the general symplectic reduction theory, the dynamics of \( H_{\xi_\varepsilon, \varepsilon} \) projects into the dynamics of the reduced Hamiltonian
\[
h_{\mu_\varepsilon, \varepsilon} : J^{-1}_\varepsilon(\mu_\varepsilon)/G_{\mu_\varepsilon} \rightarrow \mathbb{R}.
\]

(46)

While we do not have an explicit expression for \( h_{\xi_\varepsilon, \varepsilon} \) for \( \varepsilon > 0 \), we can find an expression for \( h_{\mu_\varepsilon, 0} \). At \( \varepsilon = 0 \), we have
\[
H_{\xi_\varepsilon, 0}(Z_1, Z_2) = \frac{1}{2} Z_1^T d^2 H^{(1)}_{\xi_\varepsilon}(z_1^\varepsilon) Z_1 + \tilde{H}^{(2)}_{\xi_\varepsilon}(Z_2; z_1^\varepsilon)
\]

(47)
The Hamiltonian $H_{\xi,0}(Z_1, Z_2)$ is given by the sum of two uncoupled terms, with the first term $G_\mu$-symmetric. We observe that in fact

$$Z_1 \to \frac{1}{2} Z_1^T d^2 H_{\xi}(z_1^e) Z_1, \quad Z_1 \in \text{Ker} dJ(\xi)(z_1^e)$$  \hspace{1cm} (48)

is the Hamiltonian of the linear vector field $dX_{H_{\xi}(z_1^e)}(z_1^e)$ restricted to the invariant subspace $\text{Ker} dJ(\xi)(z_1^e)$ (see [8]). We have $(J(\xi))^{-1}(\mu_e) \simeq G_{\mu}dJ(\xi)(z_1^e)$ and, since $G$-action on $P$ is free

$$\left( J(\xi)(\mu_e) \right)^{-1}/G_{\mu} \simeq \text{Ker} dJ(\xi)(z_1^e)/T_{z_1^e}(G_{\mu} \cdot z_1^e).$$  \hspace{1cm} (49)

In coordinates, the reduced space is expressed by taking a complement to $g_{\mu_e} \cdot z_1^e$ in $\text{Ker} dJ(\xi)(z_1^e)$. We denote such a complement by $W$, i.e.

$$W \oplus (g_{\mu_e} \cdot z_1^e) = \text{Ker} dJ(\xi)(z_1^e).$$  \hspace{1cm} (50)

Recall that the reduced dynamics of $H(\xi)$ is given by $h_{\mu_e}^{(1)} : (J(\xi))^{-1} / G_{\mu_e} \to \mathbb{R}$ and that in the reduced space the point $[z_1^e] \in (J(\xi))^{-1} / G_{\mu_e} \simeq W$ is an equilibrium. Then the linearization of $h_{\mu_e}^{(1)}$ at $[z_1^e]$ is given by the Hamiltonian

$$h_{\mu_e, \text{lin}}^{(1)} : W \to \mathbb{R}$$

$$h_{\mu_e, \text{lin}}^{(1)}([Z_1]) := \frac{1}{2} ([Z_1])^T \left[ d^2 H_{\xi}(z_1^e) \right] \bigg|_W [Z_1].$$  \hspace{1cm} (51)

where $\left[ d^2 H_{\xi}(z_1^e) \right] \bigg|_W$ denotes the restriction of the operator $d^2 H_{\xi}(z_1^e)$ to $W$. In conclusion, the dynamics of $H_{\xi, \varepsilon}$ at $\varepsilon = 0$ drops to

$$h_{\mu_e,0} : W \times V \to \mathbb{R}$$

$$h_{\mu_e,0}([Z_1], Z_2) = h_{\mu_e, \text{lin}}^{(1)}([Z_1]) + \tilde{h}_\xi^{(2)}(Z_2; z_1^e)$$  \hspace{1cm} (52)

$$= \frac{1}{2} ([Z_1])^T \left[ d^2 H_{\xi}(z_1^e) \right] \bigg|_W [Z_1] + \tilde{h}_\xi^{(2)}(Z_2; z_1^e)$$  \hspace{1cm} (53)

and it provides a $O(\varepsilon)$ approximation for the reduced dynamics given by $h_{\mu_e, \varepsilon}$. \hspace{1cm} \Box

The analogues of Corollaries 1 and 2 are:

**Corollary 3.** In the above context, assume that $[z_1^e]$ is reductively elementary equilibrium of the reduced Hamiltonian $h_{\mu}^{(1)}$ and let $z_2^e$ be an elementary equilibrium of $\tilde{h}_\xi^{(2)} (Z_2; z_1^e)$. Then the equilibrium $(0, Z_2^e)$ of $h_{0, \mu_e}$ persists for small $\varepsilon > 0$ as an equilibrium of $[Z_1^e(\varepsilon), Z_2^e(\varepsilon)]$ of the reduced Hamiltonian $h_{\mu_e, \varepsilon}$. The $[Z_1^e(\varepsilon), Z_2^e(\varepsilon)]$ equilibria are in fact relative equilibria (after a reparamerization) for $H_{\varepsilon}(Z_1, Z_2; \varepsilon)$.

**Corollary 4.** In the above context, consider an elementary periodic solution $\varphi(t)$ of $\tilde{h}_\xi^{(2)} (Z_2; z_1^e)$ with period $T_0$ and multipliers $1, 1, \lambda_i, \lambda_i^{-1}, \tilde{\lambda}_i, \tilde{\lambda}_i^{-1}$. If $\lambda_i \neq 1$ for all $i$ and $T_0 \neq 0 \mod 2\pi$, then the periodic solution $(0, \varphi(t))$ of $h_{\mu_e,0}$ persists for small $\varepsilon$ as a periodic solution $[Z_1^e(t; \varepsilon), Z_2^e(t; \varepsilon)]$ of $h_{\mu_e, \varepsilon}$ and its period is $T(\varepsilon) = T_0 + O(\varepsilon)$. The periodic solutions $[Z_1^e(t; \varepsilon), Z_2^e(t; \varepsilon)]$ in fact relative periodic orbits (after a reparamerization) for $H_{\varepsilon}(Z_1, Z_2; \varepsilon)$. 


5. Applications.

5.1. The classical \((N+1)\)-body problem with one small mass. Consider the spatial classical \((N+1)\)-body problem. Without loosing generality, we set the origin of the reference frame in the centre of mass. Further, we describe the configuration of the system in Jacobi coordinates \((q_1, q_2, \ldots, q_N)\) where \(q_1\) is the relative vector between the points masses \(m_1\) and \(m_2\) and, \(q_k, k = 2, 3, \ldots, N\) is the position vector of the \(k + 1\) mass point \(m_{k+1}\) with respect to the centre of mass of the sub-system formed by \(m_1, m_2, \ldots, m_k\). Let \(m_{N+1}\) have its mass much smaller than all other masses (the primaries), so that \(m_{N+1} = \varepsilon^2\). Denote the momenta by \(p_1, p_2, \ldots, p_N\) and, to simplify notation, let \(q := (q_1, q_2, \ldots, q_N)\) and \(p := (p_1, p_2, \ldots, p_{N-1})\). Then the Hamiltonian of the system is

\[
H : (\mathbb{R}^{3N} \setminus \{\text{collisions}\}) \times \mathbb{R}^{3N} \rightarrow \mathbb{R}
\]

\[
H = H^{(1)}(q, p) + H^{(2)}(q_N, p, p_N),
\]

with

\[
H^{(1)}(q, p) = \frac{1}{2} p^T M p + \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{d_{ij}}
\]

\[
H^{(2)}(q, p) = \frac{p_N^2}{2M_N} + \sum_{1 \leq i \leq N} \frac{m_i m_{N+1}}{d_{i(N+1)}}
\]

where \(M := \text{diag}(M_k^{-1})\) is the \((N-1) \times (N-1)\) matrix of reduces masses defined by

\[
M_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad M_k = \frac{m_{k+1}(m_1 + m_2 + \ldots + m_k)}{m_1 + m_2 + \ldots + m_{k+1}}, \quad k = 2, 3, \ldots, N.
\]

and \(d_{ij}\) is the distance between masses \(m_i\) and \(m_j\).

This system is invariant under the co-tangent lifted diagonal action of \(SO(3)\), which is free for non-zero momenta and for all non-collinear configurations. Assume that \(m_{N+1} = \varepsilon^2\). We verify now that the conditions of Theorem 4.1 are fulfilled.

Under the assumption that \(m_{N+1} = \varepsilon^2\) we identify \(z_1 = (q, p), z_2 = (q_N, p_N)\) and

\[
P := \left(\mathbb{R}^{3(N-1)} \setminus \{\text{collisions and collinear configurations}\}\right) \times \mathbb{R}^{3(N-1)}
\]

\[
V = \mathbb{R}^3 \setminus \{\text{collisions with primaries}\} \times \mathbb{R}^3
\]

\[
H = H^{(1)}(q, p) + H^{(2)}(q, p_N, p_N; \varepsilon)
\]

where \(H_1\) is given by (55) and

\[
H^{(2)}(q, p, q_N, p_N; \varepsilon) = \frac{(m_1 + m_2 + \ldots + m_N) + \varepsilon^2 P_N}{\varepsilon^2 (m_1 + m_2 + \ldots + m_N)} \cdot \frac{p_N^2}{2} + \varepsilon^2 \sum_{1 \leq i \leq N} \frac{m_i}{d_{i(N+1)}}.
\]

Under the scaling

\[
(q_N, p_N) \rightarrow (Q_N, \varepsilon^2 P_N)
\]

the symplectic form \(\Omega_2 := dp_N \wedge dq_N\) on \(V\) and the Hamiltonian fulfill the conditions 1. and 2. of Theorem 4.1. Thus

**Proposition 1.** The dynamics of the \((N + 1)\)-body problem with one small mass is approximately the product of the linearisation of the \(N\)-body \(m_1, m_2, \ldots, m_N\) system near a relative equilibrium and the restricted \((N + 1)\)-body problem in the rotating frame of relative equilibrium of the primaries \(m_1, m_2, \ldots, m_N\).
In the light of Corollaries 3 and 4 we state:

**Proposition 2.** Consider the \((N + 1)\) body problem with one small mass. Let the non-small masses (primaries) be in a reductively elementary relative equilibrium. Then any elementary equilibrium of the restricted \((N + 1)\)-body problem continues into a relative equilibrium of the \((N + 1)\) problem with one small mass.

**Proposition 3.** Consider the \((N + 1)\) body problem with one small mass. Let the non-small masses (primaries) be in a reductively elementary relative equilibrium. Any periodic elementary orbit of the restricted \((N + 1)\) body problem continues into a periodic orbit of the reduced \((N + 1)\) problem with one small mass.

### 5.2. Gravitationally coupled systems.

Consider a system formed by a rigid body and a mass point (or a homogeneous spherical rigid body). Choose a spatial coordinate system with origin at the centre of mass of the system, which we assume remains fixed. If \(q_i\) is the vector from the centre of mass of the system to the centre of mass of each body, let \(q = q_2 - q_1\) be the relative vector between the two centres of mass. The attitude of the rigid body is specified by a rotation matrix \(R\) from its reference configuration \(B\), around its own centre of mass. Thus the configuration space is \(Q := SO(3) \times \mathbb{R}^3 \setminus \{\text{collisions}\}\).

Let the coefficient of inertia matrix of the body, with respect to the origin, be

\[
\mathcal{J} := \int_B XX^T \, dm(X).
\]

The kinetic term is given

\[
K_{\text{rot}} := \frac{1}{2} \text{tr} \left( \dot{R} \mathcal{J} R^T \right)
\]

where \(X\) is the position (label) of a given particle in the body the reference configuration, \(dm(X)\) is the mass element. As known, this is a metric on \(TSO(3)\) invariant with respect to the left action of \(SO(3)\) on itself. Thus

\[
K_{\text{rot}} := \frac{1}{2} \ll \dot{R}, R \gg := \frac{1}{2} \text{tr} \left( \dot{R} \mathcal{J} R^T \right) = \frac{1}{2} \text{tr} \left( \hat{\xi} \hat{\xi}^T \right) = \frac{1}{2} \ll \hat{\xi}, \hat{\xi} \gg,
\]

where \(\hat{\xi} := R^{-1} \dot{R} \in so(3)\) is the rigid body’s angular velocity. Using the hat map isomorphism \(\mathbb{R}^3 \simeq so(3)\)

\[
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \rightarrow \hat{\xi} = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},
\]

we retrieve a well-known equation for the kinetic energy

\[
K_{\text{rot}} = \frac{1}{2} \langle \xi, \dot{\xi} \rangle = \frac{1}{2} \xi^T \mathbb{I} \xi,
\]

where \(\mathbb{I} := \int_B (\|X\|^2 I_3 - XX^T) \, dm(X)\) is the moment of inertia tensor (matrix).

Let \(m_1\) and \(m_2\) be the mass of the rigid body and the mass point, respectively, We have that the relative displacement kinetic energy of the system is

\[
K_{\text{displ}} := \frac{1}{2} m \|\dot{q}\|^2
\]
where the \( m := \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass. Thus the total kinetic energy of the systems is

\[
K = K_{rot} + K_{disp} := \frac{1}{2} \langle \dot{R}, \dot{R} \rangle + \frac{1}{2} m \| \dot{q} \|^2
\]

Finally, the Lagrangian \( L : T(SO(3) \times \mathbb{R}^3) = TSO(3) \times T\mathbb{R}^3 \rightarrow \mathbb{R} \) is given by

\[
L(R, q, \dot{R}, \dot{q}) = \frac{1}{2} \langle \dot{R}, \dot{R} \rangle + \frac{1}{2} m \| \dot{q} \|^2 - V(R, q)
\]

where

\[
V(R, q) := Gm_2 \int_{B} \frac{1}{\| q + RX \|} dm(X)
\]

is the gravitational potential. A restricted problem derived from the model above is the motion of infinitesimal mass point in the gravitational field of a rigid body. Using the methodology in Section 2, one may write the corresponding Lagrangian and the Hamiltonian.

We call the restricted point mass - rigid body problem the motion of an infinitesimal mass in the gravitational field a rigid body in the case when the rigid body is at a relative equilibrium. As known, physically, relative equilibria of rigid bodies are steady rotations about one of the principal axis on inertia.

Using the Legendre transform \( \Pi = \partial L / \partial \dot{R} \), we pass the dynamics on the Hamiltonian side and obtain:

\[
H : T^*SO(3) \times T^*\mathbb{R}^3 \rightarrow \mathbb{R}
\]

\[
H(R, q, \Pi, p) = \frac{1}{2} \text{tr} \left( \Pi J^{-1} \Pi \right) + \frac{p^2}{2m} + V(R, q).
\]

or

\[
H(R, q, \Pi, p) = H^{(1)}(R, \Pi) + H^{(2)}(R, q, \Pi, p)
\]

with

\[
H^{(1)}(R, \Pi) := \frac{1}{2} \text{tr} \left( \Pi J^{-1} \Pi \right) \quad \text{and} \quad H^{(2)}(R, \Pi, q, p) := \frac{1}{2m} p^2 + V(R, q).
\]

The spatial action of \( SO(3) \) on \( Q \) is the diagonal left multiplication,

\[
A \cdot (R, q) = (AR, Aq).
\]

This action is linear and proper. We assume that the rigid body is non-symmetric so that the \( SO(3) \) action is free for all momenta \( \mu \neq 0 \).

**Remark 4.** The Hamiltonian of the rigid body-mass point system is invariant with respect to the \( SO(3) \)-action and therefore it may be reduced. The standard method is to apply Poisson reduction, which essentially consists in describing the motion in the body coordinates of the rigid body. The (Poisson) reduced system is further endowed with a Casimir invariant and the reduced space is obtained by restricting the dynamics to a Casimir level set.

In the context of the present paper, we employ symplectic reduction, but in order to apply Theorem 4.1 we do not need to describe neither the symplectic reduced space, nor the reduced Hamiltonian.
To apply Theorem 4.1 we take $P = T^*SO(3)$ and $V = T^*\mathbb{R}^3$. Let the total angular momentum be $\mu \neq 0$ and assume that $m = \varepsilon^2$ is small. The Hamiltonian becomes

$$H(R, q, \Pi, p; \varepsilon) = H^{(1)}(R, \Pi) + H^{(2)}(R, q, \Pi, p; \varepsilon).$$

(66)

with

$$H^{(1)}(R, \Pi) := \frac{1}{2} \text{tr} \left( \Pi J^{-1} \Pi \right)$$

(67)

and

$$H^{(2)}(R, \Pi, q, p; \varepsilon) := \frac{p^2}{2\varepsilon^2} + G\varepsilon^2 \int_B \frac{1}{\|q + RX\|} \, dm(X).$$

(68)

It can be verified that under the scaling $(q, p) \rightarrow (Q, \varepsilon^2 P)$ condition (32) is fulfilled. Note that $H^{(1)}$ is the (un-reduced) Hamiltonian of the rigid body with a fixed point. Recall that the symplectic reduced space of the rigid body at momentum $\mu \neq 0$ is the co-adjoint sphere $O_\mu$ of radius $|\mu|$. For more on the rigid body dynamics, see [2] or [4].

Let $(R^e, \Pi^e)$ a relative equilibrium of $H^{(1)}$ and denote $(s^e, \sigma^e)$ be the corresponding equilibrium of the symplectic reduced Hamiltonian $h^{(1)}_{\mu e} : O_{\mu e} \rightarrow \mathbb{R}$. Applying Theorem 4.1 we obtain that for $\varepsilon$ small the dynamics of the symplectic reduced Hamiltonian $h^{(1)}_{\mu e},\varepsilon$ of $H$ is given within $O(\varepsilon)$ by the product of the dynamics of the linearization at the equilibrium $(s^e, \sigma^e)$ of $h^{(1)}$ and the dynamics of the Hamiltonian of the restricted point mass - rigid body problem. Differently stated, the reduced dynamics is given by

$$h^{(1)}_{\mu e,\varepsilon} : T_{\mu e} \mathcal{O} \times T^*\mathbb{R}^3 \rightarrow \mathbb{R}$$

$$h^{(1)}_{\mu e,\varepsilon}(\eta, q, p) = h^{(1),lin}_{\mu e}(\eta) + H_R(q, p; R^e, \Pi^e) + O(\varepsilon)$$

(69)

where $h^{(1),lin}_{\mu e}$ is the Hamiltonian of the linearization at a steady state of the rigid body and $H_R$ is the Hamiltonian of the restricted point mass - rigid body problem. For instance, if the rigid body is at a stable steady state (i.e., it rotates about its long or short axis), then $h^{(1),lin}_{\mu e}$ is the Hamiltonian of a one-degree of freedom harmonic oscillator. We write Corollaries 3 and 4 in the present context:

**Corollary 5.** Consider the restricted point mass - rigid body problem and assume that the rigid body is at a stable steady state. Then any non-degenerate equilibrium of the restricted problem continue as a relative equilibrium in the reduced unrestricted problem with a small mass point.

**Corollary 6.** Consider the restricted point mass - rigid body problem and assume that the rigid body is at a stable steady state. Then any non-degenerate periodic orbit of the restricted problem continue as a periodic orbit in the reduced unrestricted problem with a small mass point.

6. **Final remarks.** A notable example left out from our paper is that of a double spherical pendulum with one small mass at the non-fixed end. In this case, Theorem 4.1 cannot be applied directly, since the phase space of the driven system is the manifold $T^*S^2$ (where $S^2$ is the two dimensional sphere) rather than a vector space. Nonetheless, one may avoid this issue by embedding $S^2$ in $\mathbb{R}^3$ and working with constraints. Another example is that of a symmetric (oblate or prolate) spheroidal
rigid body gravitationally coupled with a small mass point. Here, when in the uncoupled problem the spheroid rotates about its symmetry axis, the action of the spatial group of rotations is not free and thus Theorem 4.1 is not applicable. These investigations are left for future work.

Provided certain non-degeneracy conditions hold, Theorem 4.1 could be used to show that various dynamical features (e.g. bifurcations or invariant tori) continue from the restricted to non-restricted problems. These kind of results are developed by Meyer and Schmidt [7] in the framework of the three body problem. While a purely theoretical approach in this direction is possible, such studies usually require extra information on the non-restricted system; for this reason, we believe that they would be more interesting in the context of a specific problem.

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