AN INTEGRAL EXPRESSION OF THE FIRST NON-TRIVIAL ONE-COCYCLE OF THE SPACE OF LONG KNOTS IN $\mathbb{R}^3$

KEICHI SAKAI

Abstract. Our main object of study is a certain degree-one cohomology class of the space $K_3$ of long knots in $\mathbb{R}^3$. We describe this class in terms of graphs and configuration space integrals, showing the vanishing of some anomalous obstructions. To show that this class is not zero, we integrate it over a cycle studied by Gramain. As a corollary, we establish a relation between this class and ($\mathbb{R}$-valued) Casson’s knot invariant. These are $\mathbb{R}$-versions of the results which were previously proved by Teiblyum, Turchin and Vassiliev over $\mathbb{Z}/2$ in a different way from ours.

1. Introduction

A long knot in $\mathbb{R}^n$ is an embedding $f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ that agrees with the standard inclusion $\iota(t) = (t, 0, \ldots, 0)$ outside $[-1, 1]$. We denote by $K_n$ the space of long knots in $\mathbb{R}^n$ equipped with $C^\infty$-topology.

In [7] a cochain map $I : D^* \rightarrow \bigOmega^*_{DR}(K_n)$ from certain graph complex $D^*$ was constructed for $n > 3$. The cocycles of $K_n$ corresponding to trivalent graph cocycles via $I$ generalize an integral expression of finite type invariants for (long) knots in $\mathbb{R}^3$ (see [1, 2, 11, 17]). In [13] the author found a nontrivalent graph cocycle $\Gamma \in D^*$ and proved that, when $n > 3$ is odd, it gives a non-zero cohomology class $[I(\Gamma)] \in H^3_{DR}(K_n)$. On the other hand, when $n = 3$, some obstructions to $I$ being a cochain map (called anomalous obstructions; see for example [17, §4.6]) may survive, so even the closedness of $I(\Gamma)$ was not clear. However, the obstructions for trivalent graph cocycles $X$ (of “even orders”) in fact vanish [1], hence the map $I$ still yields closed zero-forms $I(X)$ of $K_3$ (they are finite type invariants). This raises our hope that all the obstructions for any graphs may vanish and hence the map $I$ would be a cochain map even when $n = 3$.

In this paper we will show (in Theorem 2.4) that the obstructions for the nontrivalent graph cocycle $\Gamma$ mentioned above also vanish, hence the map $I$ yields the first example of a closed one-form $I(\Gamma)$ of $K_3$. To show that $[I(\Gamma)] \in H^3_{DR}(K_3)$ is not zero, we will study in part how $I(\Gamma)$ fits into a description of the homotopy type of $K_3$ given in [3, 4, 5]. It is known that on each component $K_3(f)$ that contains $f \in K_3$, there exists a one-cycle $G_f$ called the Gramain cycle [9, 3, 14, 16]. The Kronecker pairing gives an isotopy invariant $V : f \mapsto \langle I(\Gamma), G_f \rangle$. We show in

2000 Mathematics Subject Classification. 58D10; 55P48, 57M25, 57M27, 81Q30.

Key words and phrases. The space of long knots; configuration space integrals; non-trivalent graphs; an action of little cubes; Gramain cycles; Casson’s knot invariant.

The author is partially supported by Grant-in-Aid for Young Scientists (B) 21740038, The Sumitomo Foundation, The Iwanami Fujukai Foundation, and JSPS Research Fellowships for Young Scientists 228006.
Theorem 3.1 that \( V \) coincides with Casson's knot invariant \( v_2 \), which is characterized as the coefficient of \( z^2 \) in the Alexander-Conway polynomial. This result will be generalized in Theorem 5.6 for one-cycles obtained by using an action of little two-cubes operad on the space \( \tilde{K}_3 \) of framed long knots \( 4 \).

Closely related results have appeared in \([14, 16]\), where the \( \mathbb{Z}/2 \)-reduction of a cocycle \( v^3 \) of \( K_n \) (\( n \geq 3 \)), appearing in the \( E_1 \)-term of Vassiliev's spectral sequence \([15]\), was studied. A natural quasi-isomorphism \( D^* \to E_0 \otimes \mathbb{R} \) maps our cocycle \( \Gamma^* \) to \( v^3 \). In this sense, our results can be seen as “lifts” of those in \([14, 16]\) to \( \mathbb{R} \).

The invariant \( v_2 \) can also be interpreted as the linking number of collinearity manifolds \([3]\). Notice that in each formulation (including the one in this paper) the value of \( v_2 \) is computed by counting some collinearity pairs on the knot.

## 2. Construction of a close differential form

### 2.1. Configuration space integral

We briefly review how we can construct (closed) forms of \( K_n \) from graphs. For full details see \([7, 14]\).

Let \( X \) be a graph in a sense of \([7, 17]\) (see Figure 2.1 for examples). Let \( v_i \) and \( v_f \) be the numbers of the interval vertices (or \( i \)-vertices for short; those on the specified oriented line) and the free vertices (or \( f \)-vertices; those which are not interval vertices) of \( X \), respectively. With \( X \) we associate a configuration space

\[
C_X := \left\{ (f: x_1, \ldots, x_{v_i}; x_{v_i+1}, \ldots, x_{v_f}) \in \mathcal{K}_n \times \text{Conf}(\mathbb{R}^3, v_i) \times \text{Conf}(\mathbb{R}^n, v_f) \mid f(x_i) \neq x_j \text{ for any } 1 \leq i \leq v_i \leq j \leq v_i + v_f \right\}
\]

where \( \text{Conf}(M, k) := M^{\times k} \setminus \bigcup_{1 \leq i < j \leq k} \{ x_i = x_j \} \) for a space \( M \).

Let \( e \) be the number of the edges of \( X \). Define \( \omega_X \in \Omega^{(n-1)e}_\text{DR}(C_X) \) as the wedge of closed \((n-1)\)-forms \( \varphi^*_X \text{vol}_{S^{n-1}} \), where \( \varphi_X : C_X \to S^{n-1} \) is the Gauss map, which assigns a unit vector determined by two points in \( \mathbb{R}^n \) corresponding to the vertices adjacent to an edge \( \alpha \) of \( X \) (for an \( i \)-vertex corresponding to \( x_i \in \mathbb{R}^3 \), we consider the point \( f(x_i) \in \mathbb{R}^n \)). Here we assume that \( \text{vol}_{S^{n-1}} \) is \("\text{(anti)symmetric}\"\), namely \( \iota^* \text{vol}_{S^{n-1}} = (-1)^n \text{vol}_{S^{n-1}} \) for the antipodal map \( \iota : S^{n-1} \to S^{n-1} \). Then \( I(X) \in \Omega^{(n-1)e-\nu_i-n\nu_f}(K_n) \) is defined by

\[
I(X) := (\pi_X)_\ast \omega_X,
\]

the integration along the fiber of the natural fibration \( \pi_X : C_X \to K_n \). This fiber is a subspace of \( \text{Conf}(\mathbb{R}^3, v_i) \times \text{Conf}(\mathbb{R}^n, v_f) \). Such integrals converge, since the fiber can be compactified in such a way that the forms \( \varphi^*_X \text{vol}_{S^{n-1}} \) are still well-defined on the compactification (see \([2, \text{Proposition 1.1}]\)). We extend \( I \) linearly onto \( \mathcal{D}^* \), a cochain complex spanned by graphs. The differential \( \delta \) of \( \mathcal{D}^* \) is defined as a signed sum of graphs obtained by “contracting” the edges one at a time.

One of the results of \([7]\) states that \( I : \mathcal{D}^* \to \Omega^\ast_{\text{DR}}(K_n) \) is a cochain map if \( n > 3 \). The proof is outlined as follows. By the generalized Stokes theorem, \( dI(X) = \pm (\pi_X)^\ast \omega_X \), where \( \pi_X \) is the restriction of \( \pi_X \) to the codimension one strata of the boundary of the (compactified) fiber of \( \pi_X \). Each codimension one stratum corresponds to a collision of subconfigurations in \( C_X \), or equivalently to \( A \subset V(X) \cup \{ \infty \} \) (here \( V(X) \) is the set of vertices of \( X \)) with a consecutiveness property: if two \( i \)-vertices \( p, q \) are in \( A \), then all the other \( i \)-vertices between \( p \) and \( q \) are in \( A \). Here \("\in A\" \) means that the points \( x_l \) (\( l \in A \)) escape to infinity. When \( \infty \notin A \), the interior \( \text{Int} \Sigma_A \) of the corresponding stratum \( \Sigma_A \) to \( A \) is described by
Here

\begin{itemize}
  \item $X_A$ is the maximal subgraph of $X$ with $V(X_A) = A$, and $X/X_A$ is a graph obtained by collapsing the subgraph $X_A$ to a single vertex $v_A$;
  \item $B_A = S^{n-1}$ if $A$ contains at least one $i$-vertex, and $B_A = \{\ast\}$ otherwise;
  \item If $A$ consists of $i$-vertices $i_1, \ldots, i_s$ ($s > 0$) and $f$-vertices $i_{s+1}, \ldots, i_{s+t}$, then
\end{itemize}

\begin{equation}
\hat{B}_A := \left\{ \left( v; (x_{i_1}, \ldots, x_i; x_{i_{s+1}}, \ldots, x_{i_{s+t}}) \right) \in S^{n-1} \times \text{Conf} (\mathbb{R}^1, s) \times \text{Conf} (\mathbb{R}^n, t) \mid x_{i_p} \neq x_i \text{ for any } 1 \leq p \leq s < q \leq s + t \right\} / \sim
\end{equation}

where $\sim$ is defined as

\begin{equation}
(v; (x_{i_1}, \ldots, x_i; x_{i_{s+1}}, \ldots, x_{i_{s+t}})) \sim (v; (a(x_{i_1} + r), \ldots, a(x_i + r); a(x_{i_{s+1}} + rv), \ldots, a(x_{i_{s+t}} + rv)))
\end{equation}

for any $a \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}$ (if $A$ consists only of $t$ $f$-vertices, then

\begin{equation}
\hat{B}_A := \text{Conf} (\mathbb{R}^n, t) / (\mathbb{R}_{>0} \times \mathbb{R}^n),
\end{equation}

where $\mathbb{R}_{>0} \times \mathbb{R}^n$ acts on $\text{Conf} (\mathbb{R}^n, t)$ by scaling and translation; 

\begin{itemize}
  \item $\rho_A$ is the natural projection;
  \item when $A$ contains at least one $i$-vertices, $D_A : C_X/X_A \to S^{n-1}$ maps $(f; (x_i))$ to $f'(v_{x_A}) / f'(v_{x_A})$.
\end{itemize}

We omit the case $\infty \in A$; see \cite{JJ} Appendix.

By properties of fiber integrations and pullbacks, the integration of $\omega_X$ along $\text{Int} \Sigma_A$ can be written as $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D_A(\rho_A)_*\hat{\omega}_{X_A})$, where $\hat{\omega}_{X_A} \in \Omega^*_{DR}(\hat{B}_A)$ is defined similarly to $\omega_X \in \Omega^*_{DR}(C_X)$.

The stratum $\Sigma_A$ is called \textit{principal} if $|A| = 2$ \textit{hidden} if $|A| \geq 3$, and \textit{infinity} if $\infty \in A$. Since two-point collisions correspond to contractions of edges, we have $dI(X) = I(\delta X)$ modulo the integrations along hidden and infinity faces. When $n > 3$, the hidden/infinity contributions turn out to be zero; in fact $(\rho_A)_*\hat{\omega}_{X_A} = 0$ if $n > 3$ and if $A$ is not principal (see \cite{JJ} Appendix) or the next Example 2.1. This proves that the map $I$ is a cochain map if $n > 3$.

\textbf{Example 2.1.} Here we show one example of vanishing of an integration along a hidden face $\Sigma_A$. Let $X$ be the seventh graph in Figure 2.1 and $A := \{1, 4, 5\}$. Then in (2.1), $B_A = S^{n-1}$ since $A$ contains an $i$-vertex 1, and

\begin{equation}
\hat{B}_A = \{(v; x_1, x_4, x_5) \in S^{n-1} \times \mathbb{R}^1 \times \text{Conf} (\mathbb{R}^n, 2) \mid x_1 v \neq x_4, x_5 \}/ \sim,
\end{equation}

where $(v; x_1, x_4, x_5) \sim (v; a(x_1 + r); a(x_4 + rv), a(x_5 + rv))$ for any $a > 0$ and $r \in \mathbb{R}^1$.

The subgraph $X_A$ consists of three vertices $1, 4, 5$ and three edges $14, 15$ and $45$. The open face $\text{Int} \Sigma_A$, where three points $f(x_1), x_4$ and $x_5$ collide with each other, is a hidden face and is described by the square (2.1). Then the integration of $\omega_X$
along $\text{Int } \Sigma_A$ is $(\pi_{X/X_A})_*(\omega_{X/X_A} \wedge D^*_A(\rho_A), \hat{\omega}_{X_A})$, where

$$\hat{\omega}_{X_A} = \varphi^*_{14} \text{vol}_{S^{n-1}} \wedge \varphi^*_{15} \text{vol}_{S^{n-1}} \wedge \varphi^*_{45} \text{vol}_{S^{n-1}} \in \Omega^{3(n-1)}_{DR}(\hat{B}_A);$$

$$\varphi_{1j} := \frac{x_j - x_1 v}{|x_j - x_1 v|} \quad (j = 4, 5), \quad \varphi_{45} := \frac{x_5 - x_4}{|x_5 - x_4|}.$$ 

In this case we can prove that $(\rho_A)_* \hat{\omega}_{X_A} = 0$, hence the integration of $\omega_X$ along $\text{Int } \Sigma_A$ vanishes. Indeed a fiberwise involution $\chi : \hat{B}_A \to \hat{B}_A$ defined by

$$\chi(v; x_1; x_4; x_5) := (v; x_1; 2x_1 v - x_4, 2x_1 v - x_5)$$

preserves the orientation of the fiber but $\chi^* \hat{\omega}_{X_A} = -\hat{\omega}_{X_A}$ (here we use that $\text{vol}_{S^{n-1}}$ is antisymmetric), hence we have $(\rho_A)_* \hat{\omega}_{X_A} = - (\rho_A)_* \hat{\omega}_{X_A}$. 

2.2. Nontrivalent cocycle. It is shown in [7] that, when $n > 3$, the induced map $I$ on cohomology restricted to the space of trivalent graph cocycles is injective. In [13], the author gave the first example of a nontrivalent graph cocycle $\Gamma$ (Figure 2.1) which also gives a nonzero class $[I(\Gamma)] \in H^{3n-8}_{DR}(K_n)$ when $n > 3$ is odd. In Figure 2.1 nontrivalent vertices and trivalent f-vertices are marked by $\times$ and $\bullet$, respectively, and other crossings are not vertices. Here we say an i-vertex $v$ is trivalent if there is exactly one edge emanating from $v$ other than the specified oriented line. Each edge $ij$ ($i < j$) is oriented so that $i$ is the initial vertex.

**Remark 2.2.** An analogous nontrivalent graph cocycle for the space of embeddings $S^1 \to \mathbb{R}^n$ for even $n \geq 4$ can be found in [12].

If $n = 3$, integrations along some hidden faces (called *anomalous contributions*) might survive, and hence the map $I$ might fail to be a cochain map. However, nonzero anomalous contributions arise from limited hidden faces.

**Theorem 2.3.** Let $X$ be a graph and $A \subset V(X) \cup \{\infty\}$ be such that $\Sigma_A$ is not principal. When $n = 3$, the integration of $\omega_X$ along $\Sigma_A$ can be nonzero only if the subgraph $X_A$ is trivalent.

Our main theorem is proved by using Theorem 2.3.

**Theorem 2.4.** $I(\Gamma) \in \Omega^I_{DR}(K_3)$ is a closed form.

**Proof.** We call the nine graphs in Figure 2.1 $\Gamma_1, \ldots, \Gamma_9$ respectively. The graphs $\Gamma_i$, $i \neq 3, 4, 9$, do not contain trivalent subgraphs $X_A$ satisfying the consecutive property (see the paragraph just before (2.1)). So $dI(\Gamma_i) = I(d\Gamma_i)$ for $i \neq 3, 4, 9$ by Theorem 2.3.

Possibly the integration of $\omega_{\Gamma_i}$ ($i = 3, 4, 9$) along $\Sigma_A$ ($A := \{2, \ldots, 5\}$) might survive, since the corresponding subgraph $X_A$ is trivalent. However, we can prove $(\rho_A)_* \hat{\omega}_{X_A} = 0$ (and hence $dI(\Gamma) = I(d\Gamma)$) as follows: $(\rho_A)_* \hat{\omega}_{X_A} = 0$ for $\Gamma_3$, $\Gamma_6$, $\Gamma_9$.
because there is a fiberwise free action of $\mathbb{R}_{>0}$ on $\hat{B}_A$ given by translations of $x_2$ and $x_4$ (see [17] Proposition 4.1) which preserves $\hat{\omega}_{X_A}$. Thus $(\rho_A)_*\hat{\omega}_{X_A} = 0$ by dimensional reason. The proof for $\Gamma_4$ has appeared in [2] page 5271; $\hat{\omega}_{X_A} = 0$ on $\hat{B}_A$ since the image of the Gauss map $\varphi : B_A \to (S^2)^3$ corresponding to three edges of $X_A$ is of positive codimension. As for $\Gamma_9$, $(\rho_A)_*\hat{\omega}_{X_A} = 0$ follows from $\deg(\rho_A)_*\hat{\omega}_{X_A} = 4$ which exceeds $\dim B_A$ (in fact $B_A = \{\ast\}$ in this case). □

Proof of Theorem 2.3. Let $A$ be a subset of $V(X)$ with $|A| \geq 3$ or $\infty \in A$, and $X_A$ is nontrivalent. We must show the vanishing of the integrations along the nonprincipal face $\Sigma_A$ of the fiber of $C_X \to K_3$. To do this it is enough to show $(\rho_A)_*\hat{\omega}_{X_A} = 0$. By dimensional arguments (see [7] (A.2)) the contributions of infinite faces vanish. So below we consider the hidden faces $\Sigma_A$ with $|A| \geq 3$.

If $X_A$ has a vertex of valence $\leq 2$, then $(\rho_A)_*\hat{\omega}_{X_A} = 0$ is proved by dimensional arguments or existence of a fiberwise symmetry of $B_A$ which reverses the orientation of the fiber of $\rho_A : \hat{B}_A \to B_A$ but preserves the integrand $\hat{\omega}_{X_A}$ (like $\chi$ from Example 2.1 see also [7] Lemmas A.7-A.9).

Next, consider the case that there is a vertex of $X_A$ of valence $\geq 4$. Let $e$, $s$ and $t$ be the numbers of the edges, the i-vertices and the f-vertices of $X_A$ respectively. Then $\deg\hat{\omega}_{X_A} = 2e$ and the dimension of the fiber of $\rho_A$ is $s + 3t - k$, where $k = 2$ or $4$ according to whether $s > 0$ or $s = 0$ (see [7] (A.1)). Thus $(\rho_A)_*\hat{\omega}_{X_A} \in \Omega^*_{DR}(B_A)$ is of degree $2e - s - 3t + k$. It is not difficult to see $2e - s - 3t > 0$ because at least one vertex of $X_A$ is of valence $\geq 4$. Hence $\deg(\rho_A)_*\hat{\omega}_{X_A}$ exceeds $B_A \ast 0$ or 2) and hence $(\rho_A)_*\hat{\omega}_{X_A} = 0$.

Thus only the integrations along $\Sigma_A$ with $X_A$ trivalent can survive. □

Remark 2.5. Every finite type invariant $v$ for long knots in $\mathbb{R}^3$ can be written as a sum of $I(\Gamma_v)$ ($\Gamma_v$ is a trivalent graph cocycle) and some “correction terms” which kill the contributions of hidden faces corresponding to trivalent subgraphs (see [1] [2] [11] [17]). So by Theorem 2.3 the problem whether $I : D^* \to \Omega^*_{DR}(K_3)$ is a cochain map or not is equivalent to the problem whether one can eliminate all the correction terms from integral expressions of finite type invariants. □

3. Evaluation on some cycles

Here we will show that $[I(\Gamma)] \in H^1_{DR}(K_3)$ restricted to some components of $K_3$ is not zero.

We introduce two assumptions to simplify computations.

Assumption 1. The support of (antisymmetric) $\text{vol}_{S^2}$ is contained in a sufficiently small neighborhood of the poles $(0, 0, \pm 1)$ as in [13]. So only the configurations with the images of the Gauss maps lying in a neighborhood of $(0, 0, \pm 1)$ can nontrivially contribute to various integrals below. Presumably $[I(\Gamma)] \in H^1_{DR}(K_3)$ may be independent of choices of $\text{vol}_{S^2}$ (see [7] Proposition 4.5).

Assumption 2. Every long knot in $\mathbb{R}^3$ is contained in $xy$-plane except for over-arc of each crossing, and each over-arc is in $\{0 \leq z \leq h\}$ for a sufficiently small $h > 0$ so that the projection onto $xy$-plane is a regular diagram of the long knot.

3.1. The Gramain cycle. For any $f \in K_3$, we denote by $K_3(f)$ the component of $K_3$ which contains $f$. Regarding $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and fixing $f$, we define the map $G_f : S^1 \to K_3(f)$, called the Gramain cycle, by $G_f(s)(t) := R(s)f(t)$, where $R(s) \in SO(3)$ is the rotation by the angle $s$ fixing “long axis” (the $x$-axis). $G_f$ generates an infinite cyclic subgroup of $\pi_1(K_3(f))$ if $f$ is nontrivial [9]. The homology class
[G_f] \in H_1(K_3(f)) is independent of the choice of f in the connected component; if \( f_t \in K_3 \) \((0 \leq t \leq 1)\) is an isotopy connecting \( f_0 \) and \( f_1 \), then \( G_{f_t} : [0, 1] \times S^1 \to K_3 \) gives a homotopy between \( G_{f_0} \) and \( G_{f_1} \). Therefore the Kronecker pairing gives an isometry invariant \( V(f) := \langle I(\Gamma), G_f \rangle \) for long knots.

**Theorem 3.1.** The invariant \( V \) is equal to Casson’s knot invariant \( v_2 \).

**Corollary 3.2.** \( \langle I(\Gamma), |K_3(f)| \rangle \in H^1_{DR}(K_3(f)) \) is not zero if \( v_2(f) \neq 0 \).

We will prove two statements which characterize Casson’s knot invariant: \( V \) is of finite type of order two and \( V(3_1) = 1 \), where \( 3_1 \) is the long trefoil knot. To do this, we will represent \( G_f \) using Browder operation, as in [13].

### 3.1.1. Little cubes action.

Let \( \bar{K}_n \) be the space of framed long knots in \( \mathbb{R}^n \) (embeddings \( \bar{f} : \mathbb{R}^1 \times D^{n-1} \to \mathbb{R}^n \) that are standard outside \([-1, 1] \times D^{n-1}\)). There is a homotopy equivalence \( \Phi : \bar{K}_3 \simeq K_3 \times \mathbb{Z} \) [3] that maps \( \bar{f} \) to the pair \((\bar{f}|_{\mathbb{R}^1 \times \{0,0\}}, \text{fr} \bar{f})\), where the framing number \( \text{fr} \bar{f} \) is defined as the linking number of \( \bar{f}|_{\mathbb{R}^1 \times \{0,0\}} \) with \( \bar{f}|_{\mathbb{R}^1 \times \{1,0\}} \). Since \( \text{fr} \bar{f} \) is additive under the connected sum, \( \Phi \) is a homotopy equivalence of \( H \)-spaces. In general, \( \bar{K}_n \simeq K_n \times \Omega SO(n-1) \) as \( H \)-spaces, where \( \Omega \) stands for the based loop space functor.

In [4] an action of the little two-cubes operad on the space \( \bar{K}_n \) was defined. Its second stage gives a map \( S^1 \times (\bar{K}_n)^2 \to \bar{K}_n \) up to homotopy, which is given as “shrinking one knot \( f \) and sliding it along another knot \( g \) by using the framing, and repeating the same procedure with \( f \) and \( g \) exchanged” (see [4, Figure 2]). Fixing a generator of \( H_1(S^1) \), we obtain the Browder operation \( \lambda : H_p(\bar{K}_n) \otimes H_q(\bar{K}_n) \to H_{p+q+1}(\bar{K}_n) \), which is a graded Lie bracket satisfying the Leibniz rule with respect to the product induced by the connected sum. The author proved in [13] that \( \langle I(\Gamma), r \cdot \lambda(c, v) \rangle = 1 \) when \( n > 3 \) is odd, where \( r : \bar{K}_n \to K_n \) is the forgetting map, \( c \in H_{n-3}(\bar{K}_n) \) comes from the space of framings, and \( v \in H_2(n-3)(\bar{K}_n) \) is the first nonzero class of \( K_n \) represented by a map \((S^{n-3})^\times 2 \to K_n \) (see below).

### 3.1.2. The case \( n = 3 \).

In [13] the assumption \( n > 3 \) was used only to deduce the closedness of \( I(\Gamma) \) from the results of [7]. The cycles \( e \) and \( v \) are defined even when \( n = 3 \):

- Under the homotopy equivalence \( \bar{K}_3 \simeq K_3 \times \mathbb{Z} \), the zero-cycle \( e \) is given by \((t, 1)\) where \( t \) is the trivial long knot \((t, t, 0, 0)\) for any \( t \in \mathbb{R}^1 \).
- The zero-cycle \( v = v(T) \) is given by \( \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 T_{\varepsilon_1, \varepsilon_2} \), where \( T = 3_1 \) and \( T_{\varepsilon_1, \varepsilon_2} \) is \( T \) with its crossing \( p_i \), for \( i = 1, 2 \) changed to be positive if \( \varepsilon_i = +1 \) and negative if \( \varepsilon_i = -1 \) (see Figure 3.1).

Notice that, for any \( f \in K_3 \) and any pair \((p_1, p_2)\) of its crossings, an analogous zero-cycle \( v = v(f; p_1, p_2) \) can be defined.
Regard \( f \in \mathcal{K}_3 \) as a zero-cycle of \( \bar{\mathcal{K}}_3 \) (with \( \text{fr}f = 0 \)) and consider \( r_s \lambda(e, f) \).

During a knot \( f \) “going through” \( e \), \( f \) rotates once around \( x \)-axis. Thus the one-cycle \( r_s \lambda(e, f) \) is homologous to the Gramain cycle \( G_f \). This leads us to the fact that, for \( v = v(f; p_1, p_2) \), the one-cycle \( r_s \lambda(e, v) \) is homologous to the sum \( \sum_{e_i = \pm 1} \varepsilon_1 \varepsilon_2 G_{f, e_i, e_2} \).

This is why we can apply the method in [13] to compute

\[
D^2V(f) := \sum_{e_j = \pm 1} \varepsilon_1 \varepsilon_2 V(f_{e_j, e_2}) = \sum_{e_j = \pm 1} \varepsilon_1 \varepsilon_2 (I(\Gamma), G_{f, e_1, e_2}) = (I(\Gamma), r_s \lambda(e, v(f))).
\]

Recall that our graph cocycle \( \Gamma \) is a sum of nine graphs \( \Gamma_1, \ldots, \Gamma_9 \) (see Figure 2.1). By Assumption 1, the integration \( (I(\Gamma_i), G_f) \) can be computed by “counting” the configurations with all the images of the Gauss maps corresponding to edges of \( \Gamma_i \) being around the poles of \( S^2 \). Lemma 3.4 below was proved in such a way in [13] when \( n > 3 \). Since \( [v(f)] \in H_0(\mathcal{K}_3(f)) \) is independent of small \( h > 0 \) (see Assumption 2), we may compute \( D^2V(f) \) in the limit \( h \to 0 \).

**Definition 3.3.** We say that the pair \((p_1, p_2)\) of crossings of \( f \) respects the diagram \( \square \circ \square \) if there exist \( t_1 < t_2 < t_3 < t_4 \) where \( f(t_1) \) and \( f(t_3) \) correspond to \( p_1 \), while \( f(t_2) \) and \( f(t_4) \) correspond to \( p_2 \). The notion of \((p_1, p_2)\) respecting \( \square \circ \square \) or \( \square \circ \square \) is defined analogously.

**Lemma 3.4** ([13]). Suppose that \((p_1, p_2)\) respects \( \square \circ \square \). Then, in the limit \( h \to 0 \), \( P_i(f) := \sum_{e_j = \pm 1} \varepsilon_1 \varepsilon_2 (I(\Gamma_i), G_{f, e_1, e_2}) \) converges to zero for \( i \neq 2 \), and \( P_2(f) \) converges to 1. Thus \( D^2V(f) = 1 \).

**Outline of proof.** Let \( \hat{\mathcal{C}}_{f, i} \to S^1 \) be the pullback of \( \mathcal{C}_{f, i} \to \mathcal{K}_3 \) via \( G_f \), and let \( \hat{G}_f : \hat{\mathcal{C}}_{f, i} \to \mathcal{C}_{f, i} \) be the lift of \( G_f \). By the properties of pullbacks and fiber-integrations,

\[
P_i(f) = \sum_{e_j = \pm 1} \varepsilon_1 \varepsilon_2 \int_{\hat{\mathcal{C}}_{f, i}} \hat{G}_{f, e_1, e_2}^* \omega_{\mathcal{C}_{f, i}}.
\]

Let \( t_1 < \cdots < t_4 \) be such that \( f(t_1) \) and \( f(t_3) \) correspond to \( p_1 \), while \( f(t_2) \) and \( f(t_4) \) correspond to \( p_2 \). Define the subspace \( \mathcal{C}_{f, i}' \subset \hat{\mathcal{C}}_{f, i} \) as consisting of \( (G_f(s); (x_j)) \) (\( s \in S^1 \)) such that, for each \( j = 1, 2 \), there is a pair \((l, m)\) of \( i \)-vertices of \( \Gamma_i \) such that \( x_j \) is on the over-arc of \( p_j \), \( x_m \) is on the under-arc of \( p_j \), and there is a sequence of edges in \( \Gamma_i \) from \( l \) to \( m \).

**First observation:** The integration over \( \hat{\mathcal{C}}_{f, i} \setminus \mathcal{C}_{f, i}' \) does not essentially contribute to \( P_i(f) \) in the limit \( h \to 0 \). This is because, over \( \hat{\mathcal{C}}_{f, i} \setminus \mathcal{C}_{f, i}' \), the integrals in (3.1) are well-defined and continuous even when \( h = 0 \) (\( p_j \) becomes a double point), so two terms in \( P_i(f) \) corresponding to \( \varepsilon_j = \pm 1 \) cancel each other. This implies \( \lim_{h \to 0} P_i(f) = 0 \) for \( i = 7, 8, 9 \), since \( \mathcal{C}_{f, i}' = \emptyset \) if \( \{i \text{-vertices}\} \leq 3 \).

**Second observation:** Consider the configurations \((x_i) \in \mathcal{C}_{f, i}' \) such that, for any pair \((l, m)\) of \( i \)-vertices of \( \Gamma_i \) with \( x_k \) on the over-arc of \( p_j \) and \( x_m \) on the under-arc of \( p_j \), all the points \( x_k \) (\( k \) is in a sequence in \( \Gamma_i \) from \( l \) to \( m \)) are near \( p_j \). Such configurations also do not essentially contribute to \( P_i(f) \) in the limit \( h \to 0 \), by the same reason as above. This implies \( \lim_{h \to 0} P_i(f) = 0 \) for \( i = 4, 5, 6 \); the configurations \((x_i) \in \mathcal{C}_{f, i}' \) (\( 4 \leq i \leq 6 \)) must be such that the point \( x_2 \in \mathbb{R}^3 \) (\( 1 \leq l \leq 4 \)) is near \( t_i \). By the second observation, the “free point” \( x_5 \) must be near \( p_1 \) or \( p_2 \). But then \( \omega_{\mathcal{C}_{f, i}} = 0 \), since at least one Gauss map \( \varphi_{i5} \) has its image outside the support of \( \text{vol}_{S^2} \) (see Assumption 1). Thus \( \lim_{h \to 0} P_i(f) = 0 \).
Finally consider $P_i(f)$ for $i = 1, 2, 3$. For $i = 1$ we have $\omega_{i1} = 0$ over $C_{i1}$, since the Gauss map corresponding to the edge 12 has its image outside of the support of $\text{vol}_{S^2}$. The same reasoning, using the loop edge 11, shows that $\omega_{12} = 0$ over $C_{12}$. Only $P_2(f)$ survives, since the configurations with $x_1$ near $t_1$, $x_2$ near $t_2$, $x_3$ and $x_4$ near $t_3$, and $x_5$ near $t_4$, contribute nontrivially to the integral (see [13] Lemma 4.6 for details).

\textbf{Remark 3.8.} $\langle t \rangle$ stands for the connected sum.

**Lemma 3.5.** If $(p_1, p_2)$ respects $\bigcirc\bigcirc$ or $\bigcirc\bigcirc$, then $D^2V(f) = 0$.

**Proof.** For $i = 4, \ldots, 9$, we see in the same way as in Lemma 3.4 that $P_i(f)$ approaches 0 as $h \to 0$. That $\lim_{h \to 0} P_i(f)$ for $i = 2, 3$ and the $\bigcirc\bigcirc$-case for $i = 1$ is proved by the first observation in the proof of Lemma 3.4.

In the $\bigcirc\bigcirc$-case for $P_i(f)$ over $C_{i1}$, only the configurations with $x_j$ near $t_j$, with $j = 1, 2, 3$, and $x_5$ near $t_4$ may essentially contribute to $P_i(f)$; in this case the edges 12 and 35 join the over/under arcs of $p_1$ and $p_2$ respectively. However, the Gauss map $\varphi_{14}$ cannot have its image in the support of $\text{vol}_{S^2}$, so $\omega_{11}$ vanishes. □

**Proof of Theorem 3.6.** For three crossings $(p_1, p_2, p_3)$ of $f \in K_3$, consider the third difference

$$D^3V(f) := \sum_{\epsilon_j = 1} \epsilon_1 \epsilon_2 \epsilon_3 V(f_{\epsilon_1, \epsilon_2, \epsilon_3}) = D^2V(g_{+1}) - D^2V(g_{-1})$$

where $g_{\pm 1} := f_{+1, +1, \pm 1}$ and $D^2V(g_{\pm 1})$ are taken with respect to $(p_1, p_2)$. Since the pair $(p_1, p_2)$ of $g_{+1}$ respects the same diagram as $(p_1, p_2)$ of $g_{-1}$, we have $D^2V(g_{+1}) = D^2V(g_{-1})$ by above Lemmas 3.4 and 3.5. Thus $D^3V = 0$ and hence $V$ is finite type of order two. Moreover $V(\iota) = 0$ for the trivial long knot $\iota$ since $K_3(\iota)$ is contractible [10]; therefore $G_3 \sim 0$, and $V(3) = 1$ by Lemma 3.4 and $V(\iota) = 0$. These properties uniquely characterize Casson’s knot invariant $v_2$. □

\textbf{3.2. The Browder operations.} We denote a framed long knot corresponding to $(f, k)$ under the equivalence $\tilde{K}_3 \simeq K_3 \times \mathbb{Z}$ by $f^k \in \tilde{K}_3$ (unique up to homotopy). As mentioned above, the Gramain cycle can be written as $[G_f] = [r_* \lambda (f^k, l^1)]$ ($k$ may be arbitrary). Below we will evaluate $I(\Gamma)$ on more general cycles $r_* \lambda (f^k, g^l)$ of $K_3$ for any nontrivial $f, g \in K_3$ and $k, l \in \mathbb{Z}$. This generalizes Theorem 3.1.

**Theorem 3.6.** We have $\langle I(\Gamma), r_* \lambda (f^k, g^l) \rangle = lv_2(f) + kv_2(g)$ for any $f, g \in K_3$ and $k, l \in \mathbb{Z}$.

**Corollary 3.7.** If at least one of $v_2(f)$ and $v_2(g)$ is not zero, then

$$[I(\Gamma)|_{\kappa_3(f^k g^l)}] \in H^3_{DR}(\kappa_3(f^k g^l)) \neq 0,$$

where $\sharp$ stands for the connected sum.

**Proof.** This is because $r_* \lambda (f^k, g^l)$ is a one-cycle of $\kappa_3(f^k g^l)$ for any $k, l \in \mathbb{Z}$. Since $v_2(f)$ or $v_2(g)$ is not zero, there exist some $k, l$ such that $lv_2(f) + kv_2(g) \neq 0$, so $\langle I(\Gamma), r_* \lambda (f^k, g^l) \rangle \neq 0$ by Theorem 3.6. □

**Remark 3.8.** If $v_2(f) = \mp v_2(g)$, then $v_2(f g^l) = 0$ since it is known that $v_2$ is additive under $\sharp$. Hence we cannot deduce $[I(\Gamma)|_{\kappa_3(f g^l)}] \neq 0$ from Corollary 3.2. Moreover if $v_2(f) = \mp v_2(g) \neq 0$, then Corollary 3.7 implies $[I(\Gamma)|_{\kappa_3(f g^l)}] \neq 0$. □
To prove Theorem 3.6 first we remark that $f^m \sim f^0 g^m$. Since $\lambda$ satisfies the Leibniz rule, $\lambda(f^k, g^l)$ is homologous to
\[
\lambda(f^0, g^0)z^{k+l} + \lambda(f^0, l^i)z^k + \lambda(k, l^i)z^l + \lambda(k, l^i)z^0 g^0.
\]
Since by definition $r_\ast \lambda(f^k, l^m) \sim m G_f$ ($k, m \in \mathbb{Z}$) and $G_i \sim 0$,
\[
(3.2) \quad r_\ast \lambda(f^k, g^l) \sim r_\ast \lambda(f^0, g^0) + l G_f z g + k f z g.
\]
Notice that $\zeta$ makes $\mathcal{K}_3$ an $H$-space and induces a coproduct $\Delta$ on $H^*_DR(\mathcal{K}_3)$.

**Lemma 3.9.** $\Delta([I(\Gamma)]) = 1 \otimes [I(\Gamma)] + [I(\Gamma)] \otimes 1 \in H^*_DR(\mathcal{K}_3)^{22}$.  

**Proof.** $\mathcal{D}$ also admits $\Delta$ defined as a “separation” of the graphs by removing a point from the specified oriented line (see [8, §3.2]). Theorem 6.3 of [8] shows, without using $n > 3$, that $(I \otimes I) \Delta(X) \Delta(I(X))$ if $X$ satisfies $dI(X) = I(\delta X)$.

As for our graphs in Figure 3.1 $\Delta \Gamma_i = 1 \otimes \Gamma_i + \Gamma_i \otimes 1$ and $\Delta(\Gamma_3 - \Gamma_4) = 1 \otimes (\Gamma_3 - \Gamma_4) + (\Gamma_3 - \Gamma_4) \otimes 1 + 1 \otimes 1 \otimes 1$, where $\Gamma'$ and $\Gamma''$ are as shown in Figure 3.2. Thus
\[
\Delta I(\Gamma') = 1 \otimes I(\Gamma') + I(\Gamma') \otimes 1 + I(\Gamma'') \otimes I(\Gamma').
\]
But in fact $\Gamma' = \delta \Gamma_0$ where $\Gamma_0 = \bigcirc$, and $I(\Gamma') = dI(\Gamma_0)$ since there is no hidden face in the boundary of the fiber of $\pi_{\Gamma_0}$.

By (3.2), Lemma 3.9 and Theorem 3.1
\[
\langle I(\Gamma), r_\ast \lambda(f^k, g^l) \rangle = \langle I(\Gamma), r_\ast \lambda(f^0, g^0) \rangle = l v_2(f) + k v_2(g).
\]
Thus it suffices to prove Theorem 3.6 in the case $k = l = 0$.

**Proof of Theorem 3.6.** Fix $g$ and regard $\langle I(\Gamma), r_\ast \lambda(f^0, g^0) \rangle$ as an invariant $V_0(f)$ of $f$. We choose two crossings $p_1$ and $p_2$ from the diagram of $f$ in $xy$-plane, and compute $D^2 V_0(f) := \sum_{x \in x_1} \langle I(\Gamma_i), r_\ast \lambda(f^0, x_2, g^0) \rangle$ in the limit $h \to 0$ as in (3.1) if this is zero for any $(p_1, p_2)$, then the arguments similar to that in the proof of Theorem 3.1 show that $V_0$ is of order two and takes the value zero for the trefoil knot, thus identically $V_0 = 0$ for any $g$. This will complete the proof.

We will compute each $P_i := \sum_{x \in x_1} \langle I(\Gamma_i), r_\ast \lambda(f^{x_1, s_2}, g^0) \rangle$ (1 $\leq$ $i$ $\leq$ 9) in the limit $h \to 0$. The two observations appearing in the proof of Lemma 3.4 allow us to conclude $P_i' \to 0$ for 4 $\leq$ $i$ $\leq$ 9 in the same way as before, so we compute $P_i'$ for $i = 1, 2, 3$ below. We may concentrate to the integration over $C_{T_1}$ by the first observation. Recall $C_{T_1} \subset S^1 \times \text{Conf}(\mathbb{R}^3, s) \times H^3(\mathbb{R}^3, t)$ by definition. We take $S^1$-parameter $\alpha \in S^1 = \mathbb{R}^1/2\pi \mathbb{Z}$ so that $g$ goes through $f$ during $0 \leq \alpha \leq \pi$, and $f$ goes through $g$ during $\pi \leq \alpha \leq 2\pi$.

First consider the integration over $0 \leq \alpha \leq \pi$. We may shrink $g$ sufficiently small. Then the sliding of $g$ through $f$ does not affect the integration, so almost all the integrations converge to zero for the same reasons as in Lemmas 3.4 and 3.5. Only the configurations $(x_i) \in C_{T_1}$ with $x_1$ and $x_2$ near $p_1$ may essentially contribute.
to $P'_i$ when $g$ comes around $p_1$; the form $\varphi_{12}^*\text{vol}_{S^2}$ may detect the knotting of $g$. However two terms for $\varepsilon_1 = \pm 1$ cancel each other.

Next consider the integration over $\pi \leq \alpha \leq 2\pi$. There may be two types of contributions to $P'_i$. One type comes from the configurations in which all the points on the knot concentrate in a neighborhood of $f$. Such a contribution depends only on the framing number $fr_g$ of $g$, not on the global knotting of $g$. Since $fr_g^0 = 0$ here, such configurations do not essentially contribute to $P'_i$.

The other possible contributions arise when $f$ comes near the crossings of $g$. For example, consider the case that $(p_1, p_2)$ respects $\begin{tikzpicture}[baseline=0pt,scale=0.5]
\draw[->,thick] (0,0) -- (1,1);
\draw[->,thick] (1,0) -- (0,1);
\draw[->,thick] (0.5,0.5) -- (0,0);
\end{tikzpicture}$. When $f$ comes near a crossing of $g$, a configuration $(x_1, \ldots, x_5) \in C_{\Gamma_1}$ as in Figure 3.3 is certainly in $C'_{\Gamma_1}$, so it may contribute to $P'_i$. However, such contributions converge to zero in the limit $h \to 0$, because $x_1$ cannot be near $p_1$ (see the second observation in the proof of Lemma 3.4). For $\Gamma_3$, we should take the configuration $(x_1, \ldots, x_5)$ with $x_j$ ($2 \leq j \leq 5$) near $t_{j-1}$ into account; but in this case the Gauss map $\varphi_{11}$ cannot have the image in the support of $\text{vol}_{S^2}$. In such ways we can check that all such contributions of $\Gamma_i$ ($i = 1, 2, 3$) can be arbitrarily small.

\section*{References}

[1] D. Altschuler and L. Freidel, \textit{Vassiliev knot invariants and Chern-Simons perturbation theory to all orders}, Comm. Math. Phys. 187 (1997), no. 2, 261–287.

[2] R. Bott and C. Taubes, \textit{On the self-linking of knots}, J. Math. Phys. 35 (1994), no. 10, 5247–5287.

[3] R. Budney, \textit{Topology of spaces of knots in dimension 3}, Proc. London Math. Soc. (2010) 101 (2), 477–496.

[4] R. Budney and F. R. Cohen, \textit{On the homology of the space of knots}, Geom. Topol. 13 (2009), 99–139.

[5] T. Kohno, \textit{Vassiliev invariants and de Rham complex on the space of knots}, Contemp. Math., vol. 179, pp. 123–138.

[6] R. Longoni, \textit{Nontrivial classes in $H^* (\text{Imb} (S^1, \mathbb{R}^n))$ from nontrivalent graph cocycles}, Int. J. Geom. Methods Mod. Phys. 1 (2004), no. 5, 639–650.

[7] K. Sakai, \textit{Nontrivalent graph cocycle and cohomology of the long knot space}, Algebr. Geom. Topol. 8 (2008), 1499–1522.
[14] V. Turchin, Calculating the First Nontrivial 1-Cocycle in the Space of Long Knots, Math. Notes 80 (2006), no. 1, 101–108.

[15] V. Vassiliev, Complements of discriminants of smooth maps: topology and applications, Trans. Math. Monographs, vol. 98, Amer. Math. Soc.

[16] , Combinatorial formulas for cohomology of knot spaces, Moscow Math. J. 1 (2001), no. 1, 91–123.

[17] I. Volić, A survey of Bott-Taubes integration, J. Knot Theory Ramifications 16 (2007), no. 1, 1–42.

Department of Mathematical Sciences, Shinshu University, 3-1-1 Asahi, Matsumoto, Nagano 390-8621, Japan
E-mail address: ksakai@math.shinshu-u.ac.jp
URL: http://math.shinshu-u.ac.jp/~ksakai/index.html