Asymptotics of Quasi-normal Modes for Multi-horizon Black Holes

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Abstract: The issue concerning rigorous methods recently developed in deriving the asymptotics of quasi-normal modes is revisited and applied to a generic non rotating multi-horizon black holes solution. Some examples are illustrated and the single horizon cases are also considered. As a result, the asymptotics for large angular momentum parameter is shown to depend on the difference between the maximal or Nariai black hole mass and the ordinary black hole mass. The extremal limit is also discussed and the exact evaluation of the quasi-normal frequencies related to the Nariai space-time is presented, as a consistent check of the general asymptotic formula.

1 Introduction

Recently there has been a renewed interest in a long standing issue in classical relativistic theory of gravitation: small perturbations or quasi-normal modes (QNMs) associated with static (eventually stationary) solutions of Einstein equation with spherical symmetry (for an introduction, see for example [1–3]). The interest is both at fundamental level and phenomenological one, in this last case through the generation and detection of gravitational radiation.

At the fundamental level, which we will mainly interested in, it has been recently conjectured a connection between the real part of the QNMs frequency and the level spacing of the black hole area spectrum [4] (see also [5] who considered the higher dimensional generalization along the line of [6] and [7]). The idea, which can be traced back to York’s paper [8], was that the QNMs are to be connected with the quantum spectrum of black hole excitations, which has to be discrete since the entropy is finite. In an asymptotically flat context, such conjectures have been used in the loop quantum gravity approach in order to uniquely fix [9] the Immirzi parameter, which otherwise, within that approach remains undetermined [10]. In the presence of negative cosmological constant, an asymptotically AdS context, a detailed analysis of the QNM spectrum for conformally coupled scalar waves has been given for Schwarzschild-Anti-de Sitter (SAdS) and the BTZ black holes [11,13], and later extended to topological black holes, as

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well [12]. Moreover, there is a connection with perturbations of CFT quasi-equilibrium states [14] and their relaxation times, via the AdS/CFT correspondence. As far as the asymptotic form of the QNMs frequencies in the SAdS background is concerned, a rather complete study can be found in [15], and for Reissner-Nordström AdS black holes in [16].

We recall that for 4-dimensional Schwarzschild black hole, the asymptotics for scalar perturbation in the limit of large damping\(^3\) reads [17, 18]:

\[
\omega_n \simeq 2\pi T_H \left[ i \left( n - \frac{1}{2} \right) + \frac{1}{2\pi} \ln 3 \right] + O[n^{-1/2}],
\]

where \(T_H\) is the Hawking temperature of the black hole. The analytic derivation of this result has been presented in [22–24]. Indeed, the factorization of the Hawking temperature seems to suggest the validity of the connection between the QNMs asymptotics and the physics of quantum black holes. This seems to hold true for a large class of space-time with a single horizon (see the recent papers [25–28] and references therein).

In this paper, we would investigate with a different but mathematically well established method, the asymptotic for QNMs associated with a generic multiple horizon D-dimensional black hole, in the limit of large angular momentum index \(l\). Working in arbitrary dimensions \((D > 3)\) may be justified by the phenomenological interest in extra spatial dimensions which has recently appeared in the literature, triggered by string theory considerations.

The method we will make use of, is based on the so called “complex scaling” or “dilatation analytic methods” (see for example [29–32] and original references therein). The methods, as we will see, works very well when one is dealing with multi-horizon black holes. For this reason, we will consider in detail the D-dimensional Schwarzschild-de Sitter black hole. Several papers have been appeared dealing with the same issue and making use of different approaches [33–39].

The method we will make use of does not work directly for asymptotically AdS black holes, but it is also appropriate for the asymptotically flat black holes, thus Schwarzschild and Reissner-Nordstrom black hole may be investigated.

The paper is organized as follow. In Section 2, the D-dimensional Schwarzschild-de Sitter black holes is revisited and the Nariai critical mass is introduced. In Section 3, the master equation associated with the gravitational perturbation as well as the related QNMs boundary conditions are presented. In Section 4, the analytic dilatation approach is briefly summarized. In Section 5, the asymptotics of QNMs frequencies are calculated for a generic non rotating black hole. In Section 6 and 7, the general formula is applied to the D-dimensional Schwarzschild and Schwarzschild-de Sitter cases. In Section 8, the exact formula of QNMs frequencies for the D-dimensional Nariai case is obtained and the consistency with the previous result checked. The Conclusion and an Appendix on the extremal limits will end the paper.

\section{D-dimensional Schwarzschild-de Sitter Black Holes}

We recall that the D-dimensional Einstein Eqs. in vacuum with cosmological constant are given by

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.1}
\]

\(A\) numerical investigation of the asymptotics of highly damped QNMs in Kerr space-time is given in [19–21].
A generic metric representing a non rotating black hole solution of the above equations, in the static coordinates (Schwarzschild gauge) reads
\[ ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2h_{ij}dx^idx^j, \] (2.2)
where the coordinates are labeled as \( x^\mu = (t, r, x^i), (i = 1, ..., D). \) The metric \( h_{ij} \) is a function of the coordinates \( x^i \) only, and we shall refer to this metric as the horizon metric. In more than 5-dimensions Einstein’s equations (2.1) do not imply that the horizon be a constant curvature sub-manifold, but they do imply that for positive \( \Lambda \) the horizon is compact. Hence we take it to be a closed orientable \( D - 2 \)-dimensional manifold. Black hole solutions may be defined by functions \( A(r) \) having simple and positive zeros. For metrics of the form (2.2), these zeros determine various horizons loci.

When the cosmological constant is absent or negative, typically one may have an interior Cauchy horizon but only one event horizon, defined by the largest positive root of the lapse function \( A(r) \), and the range of the allowed \( r \) is an infinite interval. Let us denote by \( r_H \) the horizon radius, such that \( A(r_H) = 0 \) with \( A'(r_H) \) not vanishing.

When the cosmological constant is positive, the case we are mainly interested in, there exists the possibility of multiple event horizons and the range of \( r \) may be a finite interval.

Near the horizon we have the expansion
\[ A(r) \simeq A'(r_H)(r - r_H) + ... \] (2.3)

If one considers the Euclidean version, \( t = -i\tau \), and this euclidean time is assumed to have a period \( \beta_H \), defining the new coordinates, \( \rho \) and \( \theta \), by means of
\[ r = r_H + \frac{A'(r_H)}{4} \rho^2, \quad \tau = \frac{2}{A'(r_H)} \theta, \] (2.4)
we can write
\[ ds^2 \simeq \rho^2d\theta^2 + d\rho^2 + r_H^2h_{ij}dx^idx^j. \] (2.5)

The first two terms define the metric of a flat 2-dimensional space in polar coordinates. The metric will be smooth, and the space isometric to a flat disk, if and only if the period of the angle is \( 2\pi \). This leads to
\[ \beta_H = \frac{4\pi}{|A'(r_H)|}. \] (2.6)
where \( \beta_H \) is now interpreted as inverse of the Hawking’s temperature. One often refers to this case as to the “on-shell black hole”.

The general case of a D-dimensional Schwarzschild-de Sitter black hole is described by the following lapse function,
\[ A(r) = 1 - \frac{c_DM}{r^{D-3}} - \frac{r^2}{L^2}, \] (2.7)
where \( c_D \) depends on the D-dimensional Newton constant \( G_D \) and reads
\[ c_D = \frac{16\pi G_D}{(D - 2)V_H}. \] (2.8)
the volume of the horizon, a \((D - 2)\)-dimensional sphere of unit radius, and \(L\) is the fundamental length scale associated with the positive cosmological constant according to

\[
\Lambda = \frac{(D - 1)(D - 2)}{2L^2}.
\]  

The parameter \(M\) may be identified with the mass of the black hole as an excitation over empty de Sitter space [40, 41], and satisfies the first law

\[
\delta M = \frac{\kappa H}{8\pi G_D} \delta A_H = -\frac{\kappa C}{8\pi G_D} \delta A_C
\]

where \(\kappa_C\) and \(A_C\) are the surface gravity and the area of the cosmological horizon, with radius \(r_C > r_H\). The inverse temperature of the black hole is a function of the horizon radius

\[
\beta_H = \frac{2\pi\kappa_H^{-1}}{L^2(D - 3) - (D - 1)r_H^2}.
\]  

The inverse temperature of the cosmological horizon is given by a similar expression

\[
\beta_C = \frac{2\pi\kappa_C^{-1}}{(D - 1)r_C^2 - L^2(D - 3)}.
\]

and away from the extremal case, one always has \(\beta_H < \beta_C\). Since the black holes admit a temperature, they possess an entropy, the Beckenstein-Hawking black hole entropy. From \(\beta_H\) and the expression of the temperature one derives immediately the area law for the event horizon.

Or we can write the first law of black hole thermodynamics as

\[
S_{BH} = \int \beta_H dM = \int \beta_H \frac{dM}{dr_H} dr_H. 
\]  

On the other hand, from \(A(r_H) = 0\), we have

\[
\frac{dM}{dr_H} = \frac{1}{c_D L^2} r_H^{D-4} \left[ (D - 3)L^2 - (D - 1)r_H^2 \right].
\]  

As a result, we get

\[
S_{BH} = \frac{4\pi}{c_D} \int r_H^{D-3} dr_H = \frac{4\pi}{(D - 2)c_D} r_H^{D-2} = \frac{V_H}{4G_D} r_H^{D-2}.
\]

The disadvantage of this method is that it is not obvious how to handle the constant of integration. One quarter of the area is also the entropy attributed to the cosmological horizon, but in this case it is much less obvious how this can be true. Nevertheless it has been shown [42] that the validity of the generalized second law would be seriously challenged, were it not for the geometric entropy of the cosmological horizon.

As an illustrative example, let us consider the 4-dimensional Schwarzschild-de Sitter case. Here, we have

\[
A(r) = 1 - \frac{2MG}{r} - \frac{\Lambda r^2}{3}.
\]

For

\[
0 < 1 - 9M^2G^2\Lambda < 1,
\]

\[\text{(2.10)}\]
there exist two real simple positive roots \( r_H \) and \( r_C > r_H \) of the Eq. \( A(r) = 0 \), and the static region is \( r_H < r < r_C \), \( r_H \) corresponds to the black hole event horizon and \( r_C \) to the cosmological horizon. The area law associated with the inner horizon is

\[
S_{BH} = \frac{4\pi r_H^2}{4G}.
\]

Furthermore, there exists a maximum allowed mass value given by

\[
M_N = \frac{L}{3\sqrt{3}G}.
\]

which represents the largest black hole one can have in de Sitter space. This picture carries over to the D-dimensional case. The lapse function and the Hawking’s temperature read

\[
T_H = \frac{L^2(D-3) - (D-1)r_H^2}{4\pi r_H L^2}.
\]

\[
A(r) = \frac{L^2 r^{D-3} - c_D M L^2 - r^{D-1}}{L^2 r^{D-3}}.
\]

The critical mass may be obtained when the event horizon radius coincides with the cosmological horizon, namely when there is a double zero in the lapse function, equivalent to the vanishing of \( T_H \). Thus, the critical radius \( r_N = r_H = r_C \) reads

\[
r_N = \left( \frac{D-3}{D-1} \right)^{\frac{2}{2}} L.
\]

and the critical mass

\[
M_N = \frac{2}{c_D(D-3)} \left( \frac{D-3}{D-1} \right)^{\frac{D-1}{2}} L^{D-3} = \frac{2}{(D-3) c_D L^2} r_N^{D-1}
\]

As before, for \( 0 < M < M_N \) there are two positive roots \( r_H \) and \( r_C > r_H \) of the Eq. \( A(r) = 0 \), and there is a static region \( r_H < r < r_C \). For \( M < 0 \) the solution is a naked singularity surrounded by a cosmological horizon. For \( M = M_N \) the two roots coalesce and we have a Nariai solution, the largest black hole one can have in de Sitter space. For \( M > M_N \) there are no positive real roots, \( A(r) < 0 \) and we have a truly naked singularity, with no horizon of any sort protecting it.

Let us briefly comment on the meaning of this critical mass. One can approach the extremal limit, where \( M = M_N \) and the original static region is lost, in at least two inequivalent ways\(^4\). There exists a smooth extremal limit with positive temperature, and this is the so called D-dimensional Nariai solution, a solution of Einstein equation with topology \( dS_2 \times S_{D-2} \) and product metric. We present a derivation of this result in the Appendix, along the lines of ref. [43]. And there is also a singular extremal solution with zero temperature which is in the same topological sector as the original black hole, and which may be the more natural ground state to consider. Very recently, a complete survey of these extremal solutions has been presented in [44].

\(^{4}\)That is, one obtains non diffeomorphic manifolds.
3 The Master Equations

It is well known that gravitational perturbations of a black hole are of special interest, since they are related to generation and detection of gravitational waves [1, 3]. One could also consider a massless scalar or vector field evolving in a fixed black hole background.

Within the class of black holes we are interested in, the horizon manifold is an higher dimensional sphere and spherical dimensional reduction is always possible. Thus, one is interested in the radial part of the perturbation equation. It is also well known that, instead of working with the original radial coordinate $r$, it is more convenient to introduce a new coordinate, called Regge-Wheeler or tortoise coordinate, denoted by $x$, defined by

$$ x = \int \frac{dr}{A(r)}, \quad \frac{dx}{dr} = A(r), $$ (3.1)

Since near the black hole horizon one has

$$ A(r) \simeq A'(r_H)(r - r_H), $$ (3.2)

it follows

$$ x \simeq \frac{1}{A'(r_H)} \ln(r - r_H), \quad r \simeq r_H + e^{A'(r_H)x}. $$ (3.3)

Thus, the tortoise mapping maps the horizon to $-\infty$ and, for $\Lambda = 0$, spatial infinity to infinity. With $r_C$ replacing $r_H$, Eq. (3.2) also holds near the cosmological horizon, but since in this case $A'(r_C) < 0$, $x$ maps the finite open interval $r_H < r < r_C$ of the static region into the whole real line, that is the horizon locus $r = r_C$ is mapped to $x = +\infty$.

The situation is different for asymptotically AdS black holes, which are relevant within the so called AdS/CFT correspondence. We may illustrate the issue in the simplest case, namely the BTZ 3-dimensional black hole. In the non rotating case, we have (here using units in which $8G = 1$)

$$ ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\theta^2, \quad A(r) = \frac{r^2}{L^2} - M. $$ (3.4)

The horizon radius is $r_H = L\sqrt{M}$ and Hawking temperature $T_H = \sqrt{M}2\pi L$. The relation between $r$ and the tortoise $x$ is known and reads

$$ r(x) = r_H \coth \frac{\pi r_H x}{L^2}. $$ (3.5)

Thus, the $r$ interval $(r_H, +\infty)$ is mapped to $(-\infty, 0)$. More generally, in AdS space the lapse function diverges as $\simeq r^2$ at infinity, making the integral (3.1) convergent. We will comment on the implication of this fact later.

The radial differential equation for the gravitational perturbations around the Schwarzschild original $D = 4$ black hole solution are well known and were derived by Regge and Wheeler and Zerilli.

The analog of Regge-Wheeler and Zerilli equations have been derived recently for D-dimensional Schwarzschild- de Sitter space-time and they are summarized by a master equation (see for example [45, 46]). This equation has the form

$$ \left[ -\frac{d^2}{dx^2} + V_l(r(x)) \right] \phi_l(x) = \omega^2 \phi_l(x), $$ (3.6)
where $\omega$ is the oscillation frequency of the perturbation, decomposed as

$$\phi_l(x)e^{-i\omega t}$$

and the potential $V_l$ depends on $l$, the index of spherical harmonics and on the tensorial nature of the perturbation. The general form is given in [45, 46]. For our purposes, we may write for vector and tensor perturbations respectively

$$V_{l,V}(r) = \frac{A(r)}{r^2} \left[ l(l+D-3) + W_V(r) \right], \quad (3.8)$$

$$V_{l,T}(r) = \frac{A(r)}{r^2} \left[ l(l+D-3) + W_T(r) \right], \quad (3.9)$$

where $W_V(r)$ and $W_T(r)$ are known functions independent on $l$. For $D = 4$, the vector contribution reduces to the axial or Regge-Wheeler contribution, while the tensor contribution is formally equal to the contribution associated with a scalar field. Finally, the so called “scalar” perturbation potential is very complicated. It is the one that reduces to the polar or Zerilli contribution in $D = 4$. Again, for our purposes, it will be sufficient to write it in a form valid for very large $l$, namely

$$V_{l,S}(r) = \frac{A(r)}{r^2} l(l+D-3) \left[ 1 + O\left(\frac{1}{l^2}\right) \right]. \quad (3.10)$$

The Sturm-Liouville problem associated with the QNMs consists in the eigenvalue equation in the real line plus the so called ingoing and outgoing boundary condition at the infinity, namely

$$\phi(x) \to e^{-i\omega x} \quad \text{as} \quad x \to -\infty, \quad (3.11)$$

$$\phi(x) \to e^{+i\omega x} \quad \text{as} \quad x \to \infty, \quad (3.12)$$

Such boundary conditions render the formally symmetric operator of Eq.(3.6) not self-adjoint. Thus, one has, in general, complex frequencies $\omega^2$. The real part of the QNMs frequencies are associated with the frequencies of the signal while the imaginary part is related its decay in time. As a consequence, these boundary condition render also highly non trivial the computation of the frequencies and sophisticated numeral techniques have been invented to deal with this problem [2, 3].

The situation is different for asymptotically AdS black holes and we recall that the $r$ interval $(r_H, +\infty)$ is mapped in $(-\infty, 0)$. For this reason, one is forced to impose a specific boundary condition (Dirichlet, for example [14], or vanishing flux [48]) at $x = 0$, namely at $r = \infty$. This is related to the well known fact that AdS is not a globally hyperbolic space-time.

We conclude this section with the useful analogy with the associated scattering problem. It was soon recognized [49] that the QNMs problem is very similar to the one related to the scattering resonances in (non relativistic) quantum mechanic. Within this approach, a rigorous treatment of the 4-dimensional Schwarzschild case has been presented in [50].
Since the master potential is (exponentially) vanishing at infinity of the real line, the asymptotically acceptable solution for scattering off the barrier from the right, may be written in the form
\[ \phi(x) \simeq C(\omega)e^{-i\omega x} \quad \text{as} \quad x \to -\infty, \] (3.13)
and
\[ \phi(x) \simeq E(\omega)e^{-i\omega x} + F(\omega)e^{+i\omega x} \quad \text{as} \quad x \to \infty. \] (3.14)
The scattering amplitude may be defined by
\[ S(\omega) = \frac{F(\omega)}{E(\omega)}. \] (3.15)
The ingoing boundary condition gives \( C(\omega) \) finite and not vanishing and the outgoing boundary condition gives \( E(\omega) = 0 \). It turns out that only complex \( \omega \) satisfy such condition, namely the complex frequencies of the QNMs are associated with the poles of the analytic continuation of the scattering matrix (see, for example [51]).

4 Analytic Dilatation Approach

We have recalled that the QNMs frequencies can be obtained by means of an analytic continuation in the variable \( \omega \). The analytic continuation will plays also an important role in the following technique we are going to briefly introduce.

In the investigation of the scattering resonances several approximation techniques have been developed. Fortunately, the one we will make use of, the so called analytic dilatation approach, can be used for a large class of black hole solutions.

Let us begin with the definition of QNMs which is useful for our purposes. First, if one identify the QNMs as scattering resonances associated with the Master operator
\[ L = -\frac{d^2}{dx^2} + V_l(r(x)). \] (4.1)
The QNMS frequencies are the poles of the meromorphic continuation for complex \( \omega \) of the associated resolvent \( (L_\omega - z)^{-1} \).

Let us introduce the scaled operator
\[ L_\theta = -e^{-2\theta} \frac{d^2}{dx^2} + V_l(e^{\theta} x), \] (4.2)
where \( \theta \) is a complex number such that \( 0 < \text{Im} \theta < \frac{\pi}{2} \). As a consequence \( L_\theta \) is not self-adjoint and the eigenvalue problem
\[ L_\theta \Psi(x) = \lambda \Psi(x), \] (4.3)
ammits solutions with complex \( \lambda \), and this complex eigenvalues are, of course, the poles of the associated resolvent \( (L_\theta - z)^{-1} \).

Now, for a class of so called dilatation analytic potentials and for \( 0 < \text{Im} \theta < \frac{\pi}{2} \) it follows that these complex eigenvalues coincide with the resonance (QNMS) frequencies related to the original \( V_l(x) \). This fact holds true as soon as the potential is exponentially vanishing at \( \pm \infty \).
As a result, for black holes solution with two horizons, the hypothesis are satisfied and we can look for the complex eigenvalue problem

\[ e^{-2\theta} \left[ -\frac{d^2}{dx^2} + e^{2\theta} V_l(e^\theta x) \right] \Psi(x) = \omega^2 \Psi(x). \quad (4.4) \]

In particular, we may select \( \theta = \frac{i \pi}{4} \). Thus we have to solve the eigenvalue equation associated with a non symmetric differential operator

\[ \left[ -\frac{d^2}{dx^2} + i V_l(\sqrt{ix}) \right] \Psi(x) = i \omega^2 \Psi(x). \quad (4.5) \]

Some remarks are in order. First, since the eigenfunctions \( \psi(x) \) are well behaved for large \( |x| \), then the original one cannot belong to the Hilbert space \( L^2(R) \) because of out-going and in-going boundary conditions. This fact should help the numerical calculations. In general, the explicit solutions of this equation are not explicitly known and one has to make use of approximation methods. The analytic method can also applied to the case of black hole solution with one horizon, provide they are asymptotically flat. In fact, in this case, the vanishing of the master potential for \( x \) going to \( \infty \) is still sufficient for the success of the approach.

5 Asymptotics of QNMs Frequencies

In general, the master potential as a function of \( x \) is non negative and admits a local maximum, which is in correspondence with the maximum attained by \( V_l \) as a function of \( r \). Furthermore, \( V_l \) consists of two parts, one depending on \( l \) and the other depending on the spin \( s \) of the perturbation.

In the large \( l \) limit, we have an universal dependence which is independent on the spin \( s \) and reads

\[ V_l(r) \simeq \frac{A(r)}{r^2} l(l + D - 3). \quad (5.1) \]

This term has an interesting meaning. In fact, if one investigates the equation of motion for a massless particle in the generic black hole background, it turns out that the classical potential has the form \( \frac{A(r)}{r^2} J^2 \), where \( J \) is proportional to the classical angular momentum. From now, on we will work in the limit of large \( l \). The maximum is reached for \( r_0 \) such that

\[ r_0 A'(r_0) = 2A(r_0). \quad (5.2) \]

In this limit, we may expand the potential around the maximum, thus

\[ V_l(r(x)) \simeq V_l(r_0) + \frac{1}{2} A^2(r_0) \frac{d^2}{dr^2} V_l(r_0)(x - x_0)^2 + \ldots \quad (5.3) \]

Putting \( y = x - x_0 \), the Sturm Liouville equation for the QNMs frequencies reads

\[ \left[ -\frac{d^2}{dy^2} - \Omega_l^2 y^2 \right] \Psi(y) \simeq (\omega^2 - V_l(r_0)) \Psi(y), \quad (5.4) \]

where we have put

\[ \Omega_l^2 = \frac{1}{2} A^2(r_0) \frac{d^2}{dr^2} V_l(r_0) \]
The corresponding equation for the dilated potential can be simply obtained by the substitution \( y \rightarrow \sqrt{i}y \). As a result, we arrived at

\[
\left[ -\frac{d^2}{dy^2} + \Omega^2 y^2 \right] \Psi(y) = i \left( \omega^2 - V_l(r_0) \right) \Psi(y) .
\] (5.5)

On the left hand side we have the harmonic oscillator operator. As a consequence, we have

\[
\omega^2_{l,n} \simeq V_l(r_0) - i(2n + 1)\Omega_l .
\] (5.6)

For large \( l \), the above expression can be written in the final form

\[
\omega_{l,n} \simeq \frac{\sqrt{A(r_0)}}{r_0} \left[ \pm \left( l + \frac{D - 3}{2} \right) - i(n + \frac{1}{2})\sqrt{|r_0 A''(r_0)|} - A(r_0) \right] .
\] (5.7)

Some remarks are in order. The QNMs frequencies occur symmetrically with respect the imaginary axis and have a finite dimensional multiplicity associated with (D-2)-dimensional spherical harmonics. For example, for \( D = 4 \), the multiplicity is \( 2l + 1 \). The formula derived above should be compared with an analog formula derived with the WKB methods (see [52]). The results are very similar, but the derivation is more simple and on a mathematically rigorous basis and gives, for large \( l \), the QNMs asymptotics in an explicit way. The asymptotic formula is also valid for asymptotically flat black holes.

It is possible to go beyond the harmonic or quadratic approximation, and to deal with anharmonic oscillator and related perturbation series (see [53]). For higher WKB order see, for example, [54].

### 6 D-dimensional Schwarzschild Case

In the following, we shall make applications of the general asymptotic formula we have derived. As first example, let us consider the pure D-dimensional Schwarzschild case.

Putting \( r_S^{D-3} = c_D M \), we have

\[
A(r) = 1 - \left( \frac{r_S}{r} \right)^{D-3} .
\] (6.1)

The horizon is \( r_H = r_S \) and the Hawking temperature \( T_H = \frac{D-3}{4\pi r_H} \). Furthermore, the maximum is attained at

\[
r_0 = \left( \frac{D - 1}{2} \right)^{1/(D-3)} ,
\]

and we have

\[
\omega_{l,n} \simeq \left( \frac{D - 3}{D - 1} \right)^{1/2} \frac{1}{r_0} \left[ \pm \left( l + \frac{D - 3}{2} \right) - i \left( n + \frac{1}{2} \right) (D - 3)^{1/2} \right] .
\] (6.3)

In terms of Hawking temperature we have (see the interesting paper [47] for related formulas in the case of a radially infalling particle)

\[
\omega_{l,n} \simeq \left( \frac{2^{D-2}}{(D - 1)^{D-3}} \right) \frac{2\pi}{\sqrt{D-3}} T_H \left[ \pm \left( l + \frac{D - 3}{2} \right) - i(n + \frac{1}{2})(D - 3)^{1/2} \right] .
\] (6.4)
In the pure 4-dimensional Schwarzschild, $A(r) = 1 - \frac{2GM}{r}$. Here $r_H = 2GM$, $r_0 = 3GM$, and one recovers [55, 56]

$$\omega_{l,n} \approx \frac{8\pi T_H}{3\sqrt{3}} \left[ \pm (l + \frac{1}{2}) - i(n + \frac{1}{2}) \right].$$

(6.5)

One can see that the asymptotics depends on the Hawking temperature. On dimensional grounds, this is quite natural, since the only length in the game is the horizon radius, which depends on the mass $M$ and on the Plank length via $G_D$.

### 7 D-dimensional Schwarzschild-de Sitter Case

As a second example, we consider the Schwarzschild-de Sitter D-dimensional case. We have

$$A(r) = 1 - \left( \frac{r_S}{r} \right)^{D-3} \frac{r^2}{L^2}.$$  

(7.1)

Typically we have two horizons, the event and the cosmological horizon and the associated Hawking temperatures. Furthermore, the maximum in the master potential is attained again at

$$r_0 = \left( \frac{D-1}{2} \right)^{\frac{1}{D-1}} r_S.$$  

(7.2)

Recalling the critical radius

$$r_N = \left( \frac{D-3}{D-1} \right)^{\frac{1}{2}} L,$$  

(7.3)

we have

$$\omega_{l,n} \approx \frac{1}{L} \left( \frac{r_N^2}{r_0^2} - 1 \right)^{1/2} \left[ \pm (l + \frac{D-3}{2}) - i(n + \frac{1}{2})(D-3)^{1/2} \right].$$  

(7.4)

This expression can be rewritten in terms of the Nariai critical mass $M_N$ and the mass $M$ of the black hole. One has

$$\omega_{l,n} \approx \frac{1}{L} \left( \frac{M}{M_N} \right)^{\frac{D-3}{2}} \left( \frac{M}{M_N} \right) \left[ \pm (l + \frac{D-3}{2}) - i(n + \frac{1}{2})(D-3)^{1/2} \right].$$  

(7.5)

For $D = 4$, we recover [31]

$$\omega_{l,n} \approx \frac{(1 - 9\Lambda M^2)^{1/2}}{3\sqrt{3}GM} \left[ \pm (l + \frac{1}{2}) - i(n + \frac{1}{2}) \right].$$  

(7.6)

Some remarks are in order. Here in the game enters several fundamental lengths and one can see the dependence on $L$ related to the cosmological constant and the ratio between the mass and the critical mass. In the extremal limit, namely when $M \to M_N$, the asymptotics for large
is formally vanishing. This has been recently observed in [35], and may be related to the fact that in this limit the temperature vanishes, and the resulting space is a natural ground state.

However, as already mentioned, there exists another extremal limit solution, namely the D-dimensional Nariai solution, which can be obtained making use of the extremal limit discussed in the Appendix. One can introduce a Nariai time \( t_1 = \varepsilon t \). Thus, the corresponding Nariai frequencies can be obtained by means of \( \omega = \frac{2}{D} \varepsilon \) in the limit \( \varepsilon \to 0 \). Here, \( \varepsilon \) is the extremal parameter. Let us investigate the extremal limit of the QNMs asymptotics (7.5). We have (see Appendix)

\[
M = M_N \left( 1 - k\varepsilon^2 \right).
\]  
(7.7)

As a result,

\[
\left( \frac{M}{M_N} \right)^{D-3} - 1 = \sqrt{\frac{2k}{D-3}} \varepsilon \left( 1 + O(\varepsilon^2) \right).
\]  
(7.8)

As a consequence, the asymptotic behaviour of the QNMs Nariai frequencies is predicted to be

\[
\omega_{n,l} \approx \frac{1}{L} \sqrt{\frac{2k}{D-3}} \left[ \pm (l + \frac{D-3}{2}) - i(n + \frac{1}{2}(D-3)) \right].
\]  
(7.9)

We will check this result in the next Section.

**8 The D-dimensional Nariai Case**

In this Section, we will compute exactly the QNMs frequencies associated with the D-dimensional Nariai solution. To begin with, by a simple rescaling of coordinates, it is allowed to set \( 2k = D-1 \) in the Nariai metric (10.15), so we may write

\[
d s^2 = - \left( 1 - \frac{(D-1)y^2}{L^2} \right) d t_1^2 + \frac{d y^2}{\left( 1 - \frac{(D-1)y^2}{L^2} \right)} + r_N^2 d S_{D-2}^2,
\]  
(8.1)

where \( r_N \) is given by (2.22). The manifold topology is \( d S_2 \times S_{D-2} \). Since \( d S_2 \) is not simply connected, this space-time has two horizons at

\[
y_H = \pm \frac{L}{\sqrt{D-1}},
\]

and

\[- \frac{L}{\sqrt{D-1}} < y < \frac{L}{\sqrt{D-1}}.
\]

\( y \) just covers one half of the two dimensional hyperboloid. To both horizons one may associate the same Hawking temperature

\[
T_H = \frac{\sqrt{D-1}}{2\pi L}.
\]

As already mentioned, the QNMs frequencies for this space-time can be analytically computed. In fact, the radial part of a scalar field or the tensor perturbation equation reads

\[
\left[ - \frac{d^2}{d x^2} + \frac{\lambda_n^2}{r_N} \left( 1 - (D-1)y^2(x)/L^2 \right) \right] \Psi(x) = \omega^2 \Psi(x).
\]  
(8.2)
where $\lambda_\alpha = l(l+D-3)$ are the eigenvalues related to the $(D-2)$-dimensional spherical harmonics and the tortoise coordinate is

$$x = \frac{1}{2\pi T_H} \ln \left( \frac{y_H + y}{y_H - y} \right).$$

(8.3)

As a result, $-\infty < x < \infty$ and

$$y = y_H \tanh(2\pi T_H x).$$

(8.4)

The radial master equation becomes

$$\left[ -\frac{d^2}{dx^2} + \frac{U_0}{\cosh^2 \gamma x} \right] \Psi(x) = \omega^2 \Psi(x),$$

(8.5)

where

$$U_0 = \frac{\lambda_\alpha^2}{r_N^2} = \frac{(D-1)\lambda_\alpha^2}{(D-3)L^2},$$

(8.6)

and

$$\gamma = \frac{\sqrt{2k}}{L} = 2\pi T_H.$$

(8.7)

This equation is formally equivalent to a one-dimensional Schrödinger equation with a Pöschl-Teller potential. This is an exact quantum mechanics potential and the resonances frequencies associated with it are well known (see for example, [56]).

The exact QNMS frequencies turn out to be

$$\omega_{n,\alpha} = \frac{\sqrt{D-1}}{L} \left( \pm \sqrt{\lambda_\alpha^2 - \frac{1}{4} - \frac{i(n + \frac{1}{2})}{(D-3)\left(\frac{1}{4} - \frac{i(n + \frac{1}{2})}{2} \right)} \right).$$

(8.8)

Recalling that $2k = D - 1$, in the large $\lambda_\alpha$ limit, this expression coincides with the one obtained previously. The above expression can be also rewritten in terms of the Hawking temperature

$$\omega_{n,\alpha} = 2\pi T_H \left( \pm \sqrt{\lambda_\alpha^2 - \frac{1}{4} - \frac{i(n + \frac{1}{2})}{2} \right).$$

(8.9)

9 Conclusions

The analytic dilatation method has been applied in order to derive the asymptotic for large angular momentum index of D-dimensional multi-horizon black holes. In general, the asymptotics depends on the black hole physical parameters and when $\Lambda = 0$, for dimensional reasons, the QNMs frequencies depends on the Hawking temperature and the Plank length, namely the Newton constant. When there are other physical parameters, like the cosmological constant $\Lambda$ or the charge $Q$, the dependence is more subtle and the supposed connection between the real part of the QNMs frequency and the area quantization of the black hole is much less transparent (see also [39]).

In the case of multiple horizon, we have derived a general asymptotic formula for the QNMs frequencies. This formula, making use of a suitable extremal limit, has permitted to derive an
asymptotics formula for the Nariai space-time, which has been confirmed by an exact calculation. As a by product, we have confirmed that the so called Pöschl-Teller approximation really is a near-horizon approximation, as stressed also recently by several authors [33, 35, 56]. The other extremal solution, corresponding to a zero temperature state, formally has vanishing QNMs asymptotics. One may wonder what are the implications of this fact. A naive conclusion might be that only a finite number of QNMs frequencies survive in this limiting case. However, if one takes care of the extremal limiting procedure, we have shown that there exist a QNMS asymptotics that coincides with the Nariai space-time. On the other hand, it has been conjectured that in de Sitter and asymptotically de Sitter manifolds, quantum gravity effects should be described within a finite dimensional Hilbert space, so that there could be the possibility to deal with a system having not only a finite number of degrees of freedom, but also a finite number of independent quantum states [63, 64]. Thus if QNMs prove to be related with the quantum spectrum of black holes, a finite dimensional Hilbert space is ruled out. If they are to be thought as collective excitations of a macroscopic collapsed object, then we see no contradictions with Banks proposal.

10 Appendix

In this Appendix, following [43], we shall study the extremal limit of a generic BH solution corresponding to a D-dimensional charged or neutral black hole depending on parameters as the mass $m$, charges $Q_i$ and the cosmological constant $\Lambda$, generally denoted by $g_i$. These extremal limits have been investigated in [57–62]. Recall that in the Schwarzschild static coordinates, we have

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2d\Sigma_{D-2}^2.$$ (10.1)

Here, $d\Sigma_{D-2}^2$ is the line element related to a constant curvature "horizon" (D-2)-dimensional manifold. The horizons are positive simple roots of the lapse function, i.e.

$$A(r_H) = 0, \quad A'(r_H) \neq 0.$$ (10.2)

The associated Hawking temperature is

$$\beta_H = \frac{4\pi}{A'(r_H)}.$$ (10.3)

In general, when the extremal solution exists, there exists a critical radius $r_c$ and we have

$$A(r_c) = 0, \quad A'(r_c) = 0, \quad A''(r_c) \neq 0.$$ (10.4)

Furthermore, there exists also a relationship between the parameters,

$$F(m, g_i) = 0.$$ (10.5)

When this condition is satisfied, it may happen that the original coordinates become inappropriate and the metric no more static. Typically, this happens in the presence of multiple horizons, namely $\Lambda > 0$ and when $A(r)$ has a local maximum, thus $A''(r_c) < 0$.

In order to investigate the extremal limit, it is convenient to introduce the non-extremal parameter $\epsilon$ and perform the following coordinate change

$$r = r_c + \epsilon r_1, \quad t = \frac{t_1}{\epsilon}.$$ (10.6)
and parametrize the non-extremal limit by means of

\[ F(M, g_i) = k\epsilon^2, \quad (10.7) \]

where the sign of constant \( k \) defines the physical range of the black hole parameters, namely the ones for which the horizon radius is non negative. In the near-extremal limit, we may make an expansion for \( \epsilon \) small. As a consequence

\[ A(r) = A(r_c) + A'(r_c) r_1 \epsilon + \frac{1}{2} A''(r_c) r_1^2 \epsilon^2 + O(\epsilon^3). \quad (10.8) \]

In general, when the parameters are near the extremal solution, one has

\[ A(r_c) = k_1 \epsilon^2, \quad A'(r_c) = k_2 \epsilon^2, \quad (10.9) \]

where \( k_i \) are known constants.

Thus, the metric in the extremal limit becomes

\[ ds^2 = -dt_1^2 (k_1 + \frac{1}{2} A''(r_c) r_1^2) + \frac{dr_1^2}{(k_1 + \frac{1}{2} A''(r_c) r_1^2)} + r_c^2 d\Sigma_{D-2}^2. \quad (10.10) \]

For \( \Lambda < 0 \) or vanishing, we may have simple horizons and asymptotically AdS or flat space-times. In these cases, it turns out that the constant \( k < 0 \) and \( A''(r_c) > 0 \), a local minimum. Thus, the extremal space-time is locally \( AdS_2 \times \Sigma_{D-2} \), where \( \Sigma_{D-2} \) is a compact constant curvature (horizon) manifold.

For \( \Lambda > 0 \), we have the possibility to deal with multiple horizons and asymptotically de Sitter space-times. In these cases, \( k > 0 \) and \( A''(r_c) < 0 \). As a result, the extremal space-time is locally \( dS_2 \times S_{D-2} \). Let us show this result. Let us consider the D-dimensional S-dS space. Recall

\[ A(r) = 1 - \frac{c_D M}{r^{D-3}} - \frac{L^2}{r^2}, \quad (10.11) \]

and \( r_c = \left( \frac{D-3}{D-1} \right)^{1/2} L \). Thus,

\[ A''(r_c) = -\frac{2(D-1)}{L^2}. \quad (10.12) \]

Furthermore, we have near the extremal case

\[ F = \left(1 - \frac{M}{M_c}\right) = k\epsilon^2 \quad (10.13) \]

with \( k > 0 \), since \( M < M_c \). We also have

\[ A(r_c) = \frac{2k}{D-1} \epsilon. \quad (10.14) \]

As a consequence, \( k_1 = \frac{2k}{D-1} \) and the extremal metric reads

\[ ds^2 = -\left(\frac{2k}{D-1} - d\frac{r_1^2}{L^2}\right) dt_1^2 + \frac{dr_1^2}{\left(\frac{2k}{D-1} - d\frac{r_1^2}{L^2}\right)} + \left(\frac{D-3}{D-1}\right) L^2 dS_{D-2}^2, \quad (10.15) \]

namely the D-dimensional Nariai space-time with topology \( dS_2 \times S_{D-2} \).
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