TRAVELING WAVES FOR IN-SITU COMBUSTION IN POROUS MEDIA

ONDAS VIAJANTES PARA COMBUSTÃO IN-SITU EM MEIOS POROSOS

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Abstract: This work presents a mathematical model describing the in-situ combustion process, which can be used in enhanced oil recovery. The hyperbolic part of the system was solved as a Riemann Problem. Necessary conditions for the existence of a traveling wave solution were verified. Furthermore, theoretical results are verified by using numerical simulations.

Keywords: Combustion. Air injection. Porous media. Traveling waves.

Resumo: Este trabalho apresenta um modelo matemático aplicável no processo de combustão in-situ, o qual é utilizado na recuperação avançada de petróleo. Analisamos o caso hiperbólico via Problema de Riemann e encontramos condições necessárias para a existência de solução do tipo onda viajante para o sistema com termo fonte. Além disso, realizamos simulações numéricas para verificar os resultados teóricos obtidos.

Palavras-chave: Combustão. Injeção de ar. Meios porosos. Ondas viajantes.

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1 INTRODUCTION

During the exploration of oil reservoirs, occurs a gradual decay of the pressure gradient between the injection and producing wells. It is caused, in part, by the capillary and viscous forces, which are opposed to the flow, in addition to the depletion of the natural reservoir pressure by the primary production mechanism. Several artificial oil recovery techniques, called advanced or tertiary, are developed to reduce the impact of natural production decline.

Within the techniques of medium and high viscosity oil recovery, thermal methods consist of increasing the oil temperature, decreasing its viscosity. Therefore, there is an increase in flow and recovery. One of the techniques is the in-situ combustion, which consists of the injection of air, oxygen, or enriched air (with oxygen or nitrogen) into the reservoir. In this way, combustion and other chemical reactions are generated in addition to the release of heat. Heat is transported to the front of the combustion zone, reducing oil viscosity and separating light oil components. Carbon dioxide created during combustion helps the recovery by increasing the pressure in the reservoir and, by dissolving in the oil, contributes to the reduction of its viscosity. For more details see (LAKE, 1989; PRATS, 1982).

Previous works (CHAPIRO; SOUZA, 2016; CHAPIRO; MARCHESIN; SCHECTER, 2014) studied models of solid gas combustion (filtration combustion) as one-dimensional flow, with the reaction rate described by the first-order mass action law combined with the Arrhenius law. In such cases, the oil is so viscous that it can be considered a solid fuel. In the present physical model formulation, we consider a long cylinder of porous rock, thermally insulated throughout the lateral area, containing oil instead of solid fuel. Diffusion is neglected following some studies, which show that simplified models neglecting diffusion still present qualitatively acceptable solutions (CHAPIRO; MARCHESIN, 2008). The flow in the porous medium is governed by a system of balance laws corresponding to fuel density, oxygen density, and enthalpy. In summary, the idea of this work is to analyze the solution in the form of a traveling wave for a simple model describing in-situ combustion, considering oil burning instead of solid fuel, one-dimensional flow, and the neglecting the diffusion effect.

This paper is divided into seven sections. Section 2 provides a brief summary of the Fractional Flow Theory, addressing relative velocities of the two-phase fluid and explaining some simplifications used further. In Section 3, the physical model is presented with the simplifications necessary to obtain its dimensionless form. Section 4 provides an analysis of the hyperbolic case as a Riemann problem. In Section 5, we show the necessary conditions for the existence of the solution in the form of a traveling wave. Section 6 presents numerical simulations of the model and compares them against theoretical results. Finally, Section 7 presents conclusions and discusses its physical meaning.
2 FRACTIONAL FLOW THEORY

One way to model the fluid injection and oil production is through the Buckley-Leverett theory (BUCKLEY; LEVERETT, 1942), which allows the analysis of unidimensional and two-phase flow in a porous medium. According to this model, after the fluid injection, the oil from the reservoir is displaced; however, its saturation is never zero, since a portion of residual oil is always kept in pores. Likewise, after the injection, the fluid cannot be completely removed, so that its saturation will never be zero. Thus, the interval in which the relative saturation of the injected fluid varies is \[ S_{fc}, 1 - S_{or} \], where \( S_{fc} \) is the residual fluid saturation and \( S_{or} \) is the residual oil saturation. In this case, \( S_o \) and \( S_g \) are oil and gas saturation, respectively. It will be considered that \( S_o \) and \( S_g \) are normalized saturations, that is, they belong to the interval [0, 1] and \( S_g = 1 - S_o \). The fractional flow of a phase is defined as the flow of this phase in relation to the total flow of fluids being produced. Thus, the functions that describe fractional flows of oil and gas can be defined as:

\[
f_o = \frac{\lambda_o}{(\lambda_o + \lambda_g)}, \quad f_g = \frac{\lambda_g}{(\lambda_o + \lambda_g)},
\]

where \( \lambda_o \) and \( \lambda_g \) are oil and gas mobilities. Note that mobility depends on the relative permeability, commonly described by the Brooks-Corey model (BROOKS; COREY, 1964). In this case, \( f_o \) and \( f_g \) are functions of saturation. For purposes of simplifying the model, constant relative permeabilities and therefore constant functions will be considered, maintaining the relationship \( f_o + f_g = 1 \). Thus, the Darcy velocities of oil and gas phases are given by

\[
u_o = u_f o, \quad u_g = u_f g,
\]

are also constants and thus \( u_o + u_g = u \), where \( u \) is the Darcy velocity of the resulting fluid, composed of oil and gas.

3 MATHEMATICAL MODELING OF THE IN-SITU COMBUSTION

The model proposed in this paper is similar to the model proposed in (AKKUTLU; YORTOS, 2003). In this case, it will be considered the burning of oil instead of solid fuel, disregarding the effects of diffusion. It is also assumed that the flow is one-dimensional, with air (oxygen) injected to the left of a porous rock cylinder containing a non-reactive gas and a mobile fuel.

Some additional simplifications have been adopted. The molar density of gas \( \rho_g \) is assumed constant and approximately equal to the oxygen molar density \( \rho_o \). In the heat balance
equation, the term that accompanies the partial derivative with respect to \(x\) is 
\[ (c_\rho S_\rho u_o + c_g \rho g S g u g)(T - T_{res}), \]
where \(T\) and \(T_{res}\) represent local and initial temperatures of the reservoir, 
\(c_\rho\) and \(c_g\) - thermal capacities of oil and gas, \(C_m\) - thermal capacity of the porous medium. 
However, the thermal capacity of oil is much higher than the thermal capacity of gas, as well as the density. 
Then, the term \(c_g \rho g S g u g\) is considered negligible compared to \(c_\rho \rho o S o u o\) and, therefore, the equation is as in (3). 
It was also assumed that porosity (\(\varphi\)) and Darcy’s speed (\(u\)) are constant. 
The model with time coordinate \(t\) and space coordinate \(x\) is composed of the 
equations of heat balance (3), molar balance for oxygen (4) and molar balance for mobile fuel (5), is given by:

\[
C_m \partial_t T + \varphi \partial_x (c_\rho \rho o S o u o (T - T_{res})) = Q_r Y S_g \rho g S g u g W_r \omega_f \varphi, 
\]
(3)

\[
\varphi \partial_t (Y S_g \rho g) + \varphi \partial_x (Y S_g \rho g u g) = -\omega_o \varphi \rho g Y S_g W_r \rho o, 
\]
(4)

\[
\varphi \partial_t (\rho o S o) + \varphi \partial_x (\rho o S o u o) = -\omega_f \varphi S_g \rho g S g \rho o Y W_r. 
\]
(5)

In the reaction, \(\omega_f\) moles of fuel react with \(\omega_o\) moles of oxygen generating \(\omega_g\) moles of gaseous products and solid waste products, which are neglected in this model. 
For simplicity, consider the case \(\omega_o = \omega_f = \omega_g = 1\), such as in the reaction \(C + O_2 \rightleftharpoons CO_2\). 
The reaction rate is proportional to \(W_r\), given by the Arrhenius’ Law:

\[
W_r = k_p \exp((-E_r)/(RT)), 
\]
(6)
where \(k_p\) is a pre-exponential factor, \(E_r\) is the activation energy, \(R\) is the gas constant.

### 3.1 Constant Parameters

Parameter values describing oil properties vary significantly from one reservoir to another due to the change in the oil composition. 
Here we adopt values related to mineral oil because it possesses more consistent parameter values. 
Thermal capacity (\(c_o\)) and average molar density (\(\rho_o\)) were taken from (OLIVEIRA; GOUVEIA, 2006; HARTWIG, 2002). 
As for the reservoir, it will be considered a porous rock block with length \(L = 20\) cm (typical value for laboratory experiments) with a mean pressure gradient \(\nabla P = 10^6 Pa\) and permeability \(k = 10\text{ mD}\).

To determine the fluxes \(f_o\) and \(f_g\) (constants for this model), consider oil and gas viscosities given by 
\(\mu_o = 4.05 \times 10^{-5} \text{Pa} \cdot \text{s}\) and \(\mu_g = 2.49 \times 10^{-5} \text{Pa} \cdot \text{s}\), according to (PEGORARO, 2012). 
Thus, relative permeabilities are approximated as \(k_{ro} = 0.0109\text{ D}\) and \(k_{rg} = 0.3817\text{ D}\), 
which allow the calculation of relative mobilities:

\[
\lambda_o = k k_{ro}/(\mu_o) = 2.69 \text{ D}^2/\text{Pa} \cdot \text{s}, \quad \lambda_g = k k_{rg}/(\mu_g) = 153.29 \text{ D}^2/\text{Pa} \cdot \text{s}. 
\]
(7)
Using (1), we have \( f_o = 0.0172 \) and \( f_g = 0.9828 \), keeping the relation \( f_o + f_g = 1 \).

The Darcy velocity is estimated using:

\[
u = -(k\mu)/(L\nabla P),
\]

where \( k \) is the permeability of the porous rock, \( \mu \) is the viscosity of the fluid, \( \nabla P \) is the pressure gradient and \( L \) is the length of the porous medium. Considering that the average viscosity of the mixture is approximately equal to that of the gas \( (\mu \approx \mu_g) \), for (8), we obtain \( u = 2.03 \times 10^{-3} \text{m/s} \). Using (2), the relative Darcy velocities are \( u_o = 3.49 \times 10^{-5} \text{m/s} \) and \( u_g = 2.00 \times 10^{-3} \text{m/s} \). Under the previous considerations and \( \text{(AKKUTLU; YORTOS, 2003); (CHAPIRO, 2009; QUISPE, 2018)} \), the parameter values are summarized in Table 1.

### Table 1: Dimensional parameters for in-situ combustion and their typical values.

| Symbol | Physical quantity | Valor | Unidade |
|--------|-------------------|-------|---------|
| \( T_{res} \) | Initial reservoir temperature | 273 | [K] |
| \( C_m \) | Volumetric heat capacity of PM | \( 2 \times 10^6 \) | [J/m³K] |
| \( Q_r \) | Enthalpy of immobile fuel at \( T_{res} \) | \( 4 \times 10^5 \) | [J/mole] |
| \( E_r \) | Activation energy | 58000 | [J/mole] |
| \( k_p \) | Pre-exponential factor | 500 | [1/s] |
| \( R \) | Gas constant | 8.314 | [J/(mole K)] |
| \( \varphi \) | Porosity | 0.3 | [-] |
| \( \rho_{res}^{m} \) | Initial fuel molar density | 372 | [mole/m³] |
| \( Y_{inj} \) | Concentration of injected oxygen | 1.0 | [-] |
| \( u \) | Darcy velocity of oil and gas mixture | \( 2.03 \times 10^{-3} \) | [m/s] |
| \( \mu_o \) | Average oil viscosity | \( 4.05 \times 10^{-5} \) | [Pa · s] |
| \( \rho_o \) | Average molar density of oil | 1366,88 | [mole/m³] |
| \( c_o \) | Molar heat capacity of oil | 801.6 | [J/mole K] |
| \( u_o \) | Darcy velocity relative to oil | \( 3.49 \times 10^{-5} \) | [m/s] |
| \( \mu_g \) | Average gas viscosity | \( 2.49 \times 10^{-5} \) | [Pa · s] |
| \( \rho_g \) | Average molar density of gas | 45 | [mole/m³] |
| \( c_g \) | Molar heat capacity of gas | 27.42 | [J/mole K] |
| \( u_g \) | Darcy velocity relative to gas | \( 2.00 \times 10^{-3} \) | [m/s] |

#### 3.2 Non-dimensional Model

To perform the model’s analysis, it is useful to rewrite it in non-dimensional form, preventing operations without physical meaning and assisting numerical simulations. Next some dependent and independent dimensionless variables (denoted with bars) are introduced
as fractions of the dimensional and reference quantities (later denoted by stars):

\[
\ddot{t} = \frac{t}{t^*}, \quad \ddot{x} = \frac{x}{x^*}, \quad \ddot{\theta} = \frac{T - T_{res}}{T^*}, \quad t^* = 10^2 \frac{1}{k_p}, \quad x^* = \frac{\rho_g u}{\rho_{res} k_p}, \quad T^* = T_{res}.
\] (9)

Eq. (3), Eq. (4) and Eq. (5) are rewritten as follows:

\[
\frac{\partial}{\partial \ddot{t}} \ddot{\theta} + a_1 \frac{\partial}{\partial \ddot{x}} (S_o f_o \ddot{\theta}) = b_1 Y S_g S_o \Phi,
\] (10)

\[
\frac{\partial}{\partial \ddot{t}} (Y S_g) + a_2 \frac{\partial}{\partial \ddot{x}} (Y S_g f_g) = b_2 Y S_g S_o \Phi,
\] (11)

\[
\frac{\partial}{\partial \ddot{t}} (S_o) + a_3 \frac{\partial}{\partial \ddot{x}} (S_o f_o) = b_3 Y S_g S_o \Phi,
\] (12)

where:

\[
\Phi = t^* W_r = t^* k_p \exp \left( \frac{-E_r}{R(\theta + 1)T_{res}} \right) = t^* k_p \exp \left( \frac{-\epsilon}{(\theta + 1)T_{res}} \right)
\] (13)

and \( \epsilon = \frac{E_r}{R} \). The constants are given by:

\[
a_1 = \frac{t^*}{x^*} \frac{\varphi c_o \rho_o}{C_m}, \quad a_2 = \frac{t^*}{x^*} \frac{\varphi c_o \rho_o}{C_m}, \quad a_3 = \frac{t^*}{x^*} \frac{\varphi c_o \rho_o}{C_m}, \quad b_1 = \frac{1}{T^*} \frac{Q_f \rho_g \rho_o \omega_f \varphi}{C_m}, \quad b_2 = -\omega_o \rho_o, \quad b_3 = -\omega_f \rho_g.
\] (17)

Introducing relative oxygen saturation \( S_y = Y(1 - S_o) \), Eq. (14)-Eq. (16) are simplified as follows:

\[
[\theta, S_y, S_o]^T + [a_1 \theta S_o, a_2 S_y, a_3 S_o]^T = [0, 0, 0]^T.
\] (18)

4 STUDY OF THE HYPERBOLIC PART

In this section, the hyperbolic part of the equations describing the model is studied, i.e., with the source term (right side of the equations) equal to zero. This analysis is useful because outside the combustion zone, the value of the source terms are insignificant. As in the section “Fractional Flow Theory”, \( f_o \) and \( f_g \) are considered constants. For simplicity we remove the bars and rewrite the dimensionless equations:

\[
\frac{\partial}{\partial \ddot{t}} \theta + a_1 \frac{\partial}{\partial \ddot{x}} (S_o \theta) = 0,
\] (14)

\[
\frac{\partial}{\partial \ddot{t}} [Y(1 - S_o)] + a_2 \frac{\partial}{\partial \ddot{x}} [Y(1 - S_o)] = 0,
\] (15)

\[
\frac{\partial}{\partial \ddot{t}} S_o + a_3 \frac{\partial}{\partial \ddot{x}} S_o = 0.
\] (16)

where:

\[
a_1 = \frac{t^*}{x^*} \left( \frac{\varphi c_o \rho_o}{C_m} u \right) f_o, \quad a_2 = \frac{t^*}{x^*} \left( 1 - f_o \right) u, \quad a_3 = \frac{t^*}{x^*} \left( 1 - f_o \right) f_o.
\]

Using the values from Table 1, the dimensionless constants are:

\[
a_1 = 2.34, \quad a_2 = 814.45, \quad a_3 = 14.21, \quad b_1 = 13.52, \quad b_2 = -1366.88, \quad b_3 = -45.00.
\] (17)

Introducing relative oxygen saturation \( S_y = Y(1 - S_o) \), Eq. (14)-Eq. (16) are simplified as follows:

\[
[\theta, S_y, S_o]^T + [a_1 \theta S_o, a_2 S_y, a_3 S_o]^T = [0, 0, 0]^T.
\] (18)
Notice that this equation is in the form: \( U_t + [f(U)]_x = 0 \), where \( U = [\theta, S_y, S_o]^T \).

The eigenvalues of the matrix \( f'(U) \) are \( \lambda_1 = a_1 S_o, \lambda_2 = a_2 \) and \( \lambda_3 = a_3 \). The eigenvectors associated with eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are:

\[
\vec{r}_1 = \begin{bmatrix} 1, 0, 0 \end{bmatrix}^T, \quad \vec{r}_2 = \begin{bmatrix} 0, 1, 0 \end{bmatrix}^T, \quad \vec{r}_3 = \begin{bmatrix} \frac{-a_1 \theta}{a_1 S_o - a_3}, 0, 1 \end{bmatrix}^T. \tag{19}
\]

With this information, we can investigate each of the \( p \)-fields:

- \( \nabla \lambda_1(U) \cdot \vec{r}_1(U) = (0, 0, a_1) \cdot (1, 0, 0) = 0, \quad \forall U \in R^3 \),
- \( \nabla \lambda_2(U) \cdot \vec{r}_2(U) = (0, 0, 0) \cdot (0, 1, 0) = 0, \quad \forall U \in R^3 \),
- \( \nabla \lambda_3(U) \cdot \vec{r}_3(U) = (0, 0, 0) \cdot \left( \frac{-a_1 \theta}{a_1 S_o - a_3}, 0, 1 \right) = 0, \quad \forall U \in R^3 \).

We conclude that all fields are Linearly Degenerate, indicating that the constant states will be connected by contact discontinuities, whose velocities are the eigenvalues \( \lambda_i \).

### 4.1 Hugoniot Locus

A discontinuity (shock) \((U_l, U_r)\) can be a solution of the Riemann Problem if it satisfies the Rankine-Hugoniot Condition \( f(U_l) - f(U_r) = s(U_l - U_r) \) for some velocity \( s \), see (LEVEQUE, 1992) for more details.

In order to obtain such solution for the system, Hugoniot Locus will be studied for the initial states \( U_l \) and \( U_r \). By definition, the Hugoniot Locus of a point \( \hat{U} \) is given by:

\[
H(\hat{U}) = \left\{ \hat{U} \in R^3 : \exists s \in R : f(\hat{U}) - f(\hat{U}) = s(\hat{U} - \hat{U}) \right\}, \tag{20}
\]

where \( s \) is the jump velocity established by the Rankine-Hugoniot Condition.

Using (18), we obtain:

\[
\begin{align*}
a_1(\hat{\theta} \hat{S}_o - \hat{\theta} \hat{S}_o) &= s(\hat{\theta} - \hat{\theta}), \tag{21} \\
a_2(\hat{S}_y - \hat{S}_y) &= s(\hat{S}_y - \hat{S}_y), \tag{22} \\
a_3(\hat{S}_o - \hat{S}_o) &= s(\hat{S}_o - \hat{S}_o). \tag{23}
\end{align*}
\]

System (21)-(23) will be analyzed for two different cases:

- Case 1: \( \hat{\theta} \neq \hat{\theta}, \hat{S}_y \neq \hat{S}_y \) and \( \hat{S}_o \neq \hat{S}_o \).
From the Eq. (21), we obtain:

\[ s = a_1(\hat{\theta}\hat{S}_o - \hat{\theta}\hat{S}_o)/(\hat{\theta} - \hat{\theta}), \]  

(24)

replacing (24) into (22), we obtain:

\[ a_2 = a_1(\hat{\theta}\hat{S}_o - \hat{\theta}\hat{S}_o)/(\hat{\theta} - \hat{\theta}) \Rightarrow \hat{S}_o = a_2((\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o)/(a_1\hat{\theta}) \quad (\hat{\theta} \neq 0). \]

(25)

Replacing (25) into (24), after some simplifications, we find \( s = a_2 \). Replacing (24) into (23) results in:

\[ a_3 = a_1(\hat{\theta}\hat{S}_o - \hat{\theta}\hat{S}_o)/(\hat{\theta} - \hat{\theta}) \Rightarrow \hat{S}_o = a_3((\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o)/(a_1\hat{\theta}). \]

(26)

Replacing (26) into (24), after some simplifications, results in \( s = a_3 \).

- Case 2: \( \hat{\theta} \neq \theta, \hat{S}_y \neq \hat{S}_y \) and \( \hat{S}_o = \hat{S}_o \).

From the Eq. (21), we obtain:

\[ s = a_1(\hat{\theta}\hat{S}_o - \hat{\theta}\hat{S}_o)/(\hat{\theta} - \hat{\theta}) \Rightarrow s = a_1\hat{S}_o. \]

(27)

The other cases lead to same results as those found and also to the identity equations. Thus we do not consider them separately. The three shock curves are:

\[
\begin{align*}
\hat{S}_o &= (a_2(\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o)/(a_1\hat{\theta}), \quad s = a_2, \\
\hat{S}_o &= (a_3(\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o)/(a_1\hat{\theta}), \quad s = a_3, \\
\hat{S}_o &= \hat{S}_o, \quad s = a_1\hat{S}_o.
\end{align*}
\]

(28-30)

Note that the propagation speeds of the contact discontinuities coincide with the eigenvalues. Therefore, the Hugoniot Locus of \( \hat{U} \) denoted by \( H(\hat{U}) \), is given by:

\[
\left\{ (\hat{\theta}, \hat{S}_y, \hat{S}_o) : \hat{S}_o = \frac{a_2(\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o}{a_1\hat{\theta}} \text{ or } \hat{S}_o = \frac{a_3(\hat{\theta} - \hat{\theta}) + a_1\hat{\theta}\hat{S}_o}{a_1\hat{\theta}} \text{ or } \hat{S}_o = \hat{S}_o \right\}.
\]

(31)

Substituting the values of the constants given in (17) in the speeds of the system (28)-(30), we have that the velocities of gas, oil and thermal wave are, respectively: \( s_g = 814.45 \), \( s_o = 14.21 \), \( s_t = 2.34\hat{S}_o \).
5 TRAVELING WAVES

In this section, we look for the model’s solution in the form of a traveling wave. It is equivalent to search a solution that travels at a constant speed \( c \) (here we consider \( c > 0 \), i.e., the same direction as the fluid) with the same profile. It is also equivalent to search for stationary solution in traveling coordinates \( (\xi, t) \), where \( \xi = x - ct \). Mathematically, the traveling wave solution is an orbit connecting left equilibrium to right equilibrium as \( \alpha \)-limit and \( \omega \)-limit, see (CHAPIRO; MARCHESIN; SCHECTER, 2014; VOLPERT; VOLPERT; VOLPERT, 1994) for more details. One of the necessary conditions for the existence of such orbit is the suitable dimension of these equilibria. Obtaining these dimensions is the main goal of this section.

Considering the constant flux functions in the travelling wave variables, the system formed by equations (10)-(12) is rewritten as:

\[
\begin{align*}
(S_o a_1 - c) d\xi \theta + a_1 \theta d\xi S_o &= b_1 S_y S_o \Phi, \\
(-c + a_2) d\xi S_y &= b_2 S_y S_o \Phi, \\
(-c + a_3) d\xi S_o &= b_3 S_y S_o \Phi.
\end{align*}
\]

To continue the calculations, we assume hypotheses \( c \neq a_1 S_o, c \neq a_2 \) and \( c \neq a_3 \), resulting in the following system of Ordinary Differential Equations (ODEs) (for simplicity, we denote derivative with respect to \( \xi \) by primes):

\[
\begin{align*}
\theta' &= \frac{(b_1(-c + a_3) - b_3 a_1 \theta) S_y S_o \Phi}{(a_3 - c)(S_o a_1 - c)}, \\
S'_y &= \frac{b_2 S_y S_o \Phi}{-c + a_2}, \\
S'_o &= \frac{b_3 S_y S_o \Phi}{-c + a_3}.
\end{align*}
\]

Isolating \( S_o \) in Eq. (35) and replacing the result into (34), we obtain:

\[
-c b_3 S_y + a_2 b_3 S_y + c b_2 S_o - a_3 b_2 S_o = \text{constant.}
\]

As both left \((\theta^L, S^L_y, S^L_o) = (0, 1, 0)\) and right \((\theta^R, S^R_y, S^R_o) = (0, 0, 1)\) equilibria are stationary solutions, applying the limit \((\xi \rightarrow \pm \infty)\) and using the boundary conditions we can find the traveling wave speed:

\[
c = \frac{a_2 b_3 [S^R_y - S^L_y] + a_3 b_2 [S^L_o - S^R_o]}{(b_2 S^L_y - b_3 S^L_o) - (b_2 S^R_o - b_3 S^R_y)} = \frac{a_2 b_3 + a_3 b_2}{(b_2 + b_3)}.
\]
Replacing the constant values given in (17) into Eq. (37), the speed of traveling wave is $c = 39.72$. On the other hand, from Eq. (36), we obtain:

$$S_y = ((a_3b_2 - cb_2)S_o + cb_2 - a_3b_2)/(a_2b_3 - cb_3) \Rightarrow S_y = 1 - S_o. \quad (38)$$

As $S_y$ is the oxygen saturation, the previous relation results in $Y = 1$. This physically inaccurate approximation is due to the simplicity of the system. The study of a more realistic model will remain for future work. When replacing $S_y$, found in (33) to (35), yields:

$$\theta' = \frac{(b_1(-c + a_3) - b_3a_1\theta)(1 - S_o)S_o\Phi}{(a_3 - c)(S_oa_1 - c)},$$

$$S'_o = \frac{b_3(1 - S_o)S_o\Phi}{-c + a_3}. \quad (39)$$

Therefore, the Jacobian matrix $J$ of the system (39) is given by:

$$J = \begin{bmatrix}
\frac{(S_o(1 - S_o))(\epsilon G - a_1b_3(1 + \theta)^2)}{K(1 + \theta)^2} & \frac{G(2cS_o - a_1S_o^2 - c)}{b_3(1 - 2S_o)} \\
\frac{\epsilon b_3[1 - S_o]S_o}{(1 + \theta)^2} & \frac{K^2}{b_3(1 - 2S_o)}
\end{bmatrix}, \quad (40)$$

where $H = \epsilon/(\theta + 1)^2$, $K = (S_oa_1 - c)$, $M = (a_3 - c)$, $G = a_3b_1 - b_1c - b_3a_1\theta$.

At the equilibrium $(\theta^L, S_o^L) = (0, 0)$ and $(\theta^R, S_o^R) = (0, 1)$, the Jacobian matrix (40) becomes:

$$J^L = \begin{bmatrix}
0 & (-cG\Phi)/(c^2M) \\
0 & (b_3\Phi)/M
\end{bmatrix}, \quad J^R = \begin{bmatrix}
0 & (G\Phi(2c - a_1 - c))/(M(a_1 - c)^2) \\
0 & -(b_3\Phi)/M
\end{bmatrix}, \quad (41)$$

where the eigenvalues are $\lambda^L_1 = 0$ and $\lambda^L_2 = b_3\Phi/M > 0$ and $\lambda^R_1 = 0$ and $\lambda^R_2 = -(b_3\Phi)/M < 0$.

Note that both equilibria possess eigenvalues equal to zero, classified as nodes. Rigorous demonstration of the existence of an orbit connecting these equilibria stays outside of the scope of this work.

### 6 NUMERICAL SOLUTION

In order to obtain solution profiles we use Finite Difference Methods (STRIKWERDA, 2004), implemented in MATLAB. We use the notation $f(x_m, t_n) = f_m^n$ for all functions evaluated at $(x_m, t_n)$. To perform the discretization, the following approximations for the
spacial and temporal derivatives will be adopted:

\[
\begin{align*}
\partial_t(\theta(x_m, t_n)) &= (\theta_m^{n+1} - \theta_m^n)/(\Delta t), \\
\partial_x(F(\theta(x_m, t_n))) &= (\theta_m^{n+1} - \theta_m^{n-1})/(2\Delta x).
\end{align*}
\]

(Forward Euler) (42)

(43)

Using (42) and (43) in System (10)-(12) and denoting \( \Psi_m^n = S_y^m \Theta_m^n \Phi_m^n \), we obtain:

\[
\begin{align*}
\theta_m^{n+1} &= \theta_m^n - \alpha_1[(S_o\theta)_m^{n+1} - (S_o\theta)_m^n] + \alpha_2 \Psi_m^n, \\
S_y^{n+1} &= S_y^n - \beta_1[S_{ym+1}^n - S_{ym-1}^n] + \beta_2 \Psi_m^n, \\
S_o^{n+1} &= S_o^n - \gamma_1[S_{om+1}^n - S_{om-1}^n] + \gamma_2 \Psi_m^n,
\end{align*}
\]

(44)

(45)

(46)

where \( \alpha_1 = a_1(\Delta t)/(2\Delta x) \), \( \alpha_2 = b_1\Delta t \), \( \beta_1 = a_2(\Delta t)/(2\Delta x) \), \( \beta_2 = b_2\Delta t \), \( \gamma_1 = a_3(\Delta t)/(2\Delta x) \), \( \gamma_2 = b_3\Delta t \). To perform the simulations, we use the following boundary conditions. Dirichlet boundary condition on left:

\[
\begin{align*}
\theta(0, t) &= 0, \\
S_y(0, t) &= 1, \quad t \geq 0, \\
S_o(0, t) &= 0.
\end{align*}
\]

(47)

Neumann boundary condition on right:

\[
\partial_x \theta(0.04, t) = \partial_x S_y(0.04, t) = \partial_x S_o(0.04, t) = 0.
\]

(48)

To control instabilities, the diffusive terms were added: \( (\partial^2 \theta)/(\partial x^2) \) to Eq. (10), \( (\partial^2 S_y)/(\partial x^2) \) to Eq. (11) and \( (\partial^2 S_o)/(\partial x^2) \) to Eq. (12). The discretization used is:

\[
\begin{align*}
\theta_m^{n+1} &= \theta_m^n - \alpha_1[(S_o\theta)_m^{n+1} - (S_o\theta)_m^n] + \alpha_2 \Psi_m^n + \mu[\theta_m^n - 2\theta_m^n + \theta_m^{n-1}]L, \\
S_y^{n+1} &= S_y^n - \beta_1[S_{ym+1}^n - S_{ym-1}^n] + \beta_2 \Psi_m^n + \mu[S_{ym+1}^n - 2S_{ym}^n + S_{ym-1}^n]L, \\
S_o^{n+1} &= S_o^n - \gamma_1[S_{om+1}^n - S_{om-1}^n] + \gamma_2 \Psi_m^n + \mu[S_{om+1}^n - 2S_{om}^n + S_{om-1}^n]L,
\end{align*}
\]

(49)

where \( \mu = (\Delta t)/(\Delta x^2) \) and \( 0 \leq L \leq 1 \) is a constant that define the diffusion added to the system. The constants adopted in the implemented code were: \( \Delta t = 1.2 \cdot 10^{-8}, \Delta x = 10^{-4}, \)

\( a_1 = 2.34, a_2 = 814.45, a_3 = 7.1, b_1 = 135.2, b_2 = -1366.88, b_3 = -45, E_t = 58000, \)

\( R = 8.314, T_{res} = 273, t^* = 10^7, L = 0.02, M = 401 \) (points quantity in \( x \)), \( T = 120000 \) (points quantity in time). The simulation results are shown in Fig. 1.
Using the combustion wave velocity value and velocities of the gas, oil and thermal waves, considering $S_o = 0.5$ for simplicity, the wave sequence in space $x \times t$ is represented in Fig. 2. Note that the thermal wave is the fastest, followed by oil contact wave, combustion wave, and, finally, contact gas wave.

Figure 2: Graph of thermal (blue line), oil (yellow line), combustion (purple line) and gas (red line) waves. X horizontal axis and t vertical axis.

7 CONCLUSIONS

This paper proposes a simplified model of the in-situ combustion and discusses several simplifications that can be adopted. The solution of the Riemann problem was obtained as a sequence of three contact waves and one traveling wave. Numerical simulations have shown profiles compatible with the theoretical analysis.

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REFERENCES

AKKUTLU, I. Y.; YORTOS, Y. C. The dynamics of in-situ combustion fronts in porous media. *Combustion and Flame*, Elsevier, v. 134, n. 3, p. 229–247, 2003.

BROOKS, R. H.; COREY, A. T. Hydraulic properties of porous media and their relation to drainage design. *Transactions of the ASAE*, American Society of Agricultural and Biological Engineers, v. 7, n. 1, p. 26–0028, 1964.

BUCKLEY, S. E.; LEVERETT, M. Mechanism of fluid displacement in sands. *Transactions of the AIME*, Society of Petroleum Engineers, v. 146, n. 01, p. 107–116, 1942.

CHAPIRO, G. *Gas-solid combustion in insulated porous media*. Tese (Doutorado) — IMPA, 2009.

CHAPIRO, G.; MARCHESIN, D. Non-diffusive combustion waves in insulated porous media. *Brazilian Journal of Petroleum and Gas*, v. 2, n. 2, 2008.

CHAPIRO, G.; MARCHESIN, D.; SCHECTER, S. Combustion waves and riemann solutions in light porous foam. *Journal of Hyperbolic Differential Equations*, World Scientific, v. 11, n. 02, p. 295–328, 2014.

CHAPIRO, G.; SOUZA, A. J. Asymptotic approximation for counterflow combustion in porous media. *Applicable Analysis*, Taylor and Francis, v. 95, n. 1, p. 63–77, 2016.

HARTWIG, A. White mineral oil, pharmaceutical [mak value documentation, 2015]. *The MAK-Collection for Occupational Health and Safety: Annual Thresholds and Classifications for the Workplace*, Wiley Online Library, v. 2, n. 3, p. 1177–1191, 2002.

LAKE, L. W. *Enhanced oil recovery*. [S.l.]: Old Tappan, NJ; Prentice Hall Inc., 1989.

LEVEQUE, J. R. *Numerical methods for conservation laws*. [S.l.]: Springer, 1992. v. 132.

OLIVEIRA, J. C.; GOUVEIA, A. Specific heat capacity of some mineral, synthetic and semi-synthetic automotive lubricant oils after thermal degradation. *Chemical Technology*, An Indian journal, v. 1, n. 1, p. 8–13, 2006.

PEGORARO, R. T. *Escoamento Trifásico em Meios Porosos: Permeabilidade Relativa Óleo-Gás-Agua*. Tese (Doutorado) — MS Thesis, UFRJ, Rio de Janeiro, Brazil, 98pp, 2012.

PRATS, M. *Thermal recovery*. [S.l.]: SPE of AIME, New York, NY, 1982.

QUISPE, R. V. *Ondas viajantes para combustão in-situ com efeito de perdas térmicas em meios porosos*. Tese (Doutorado) — Universidade Federal de Juiz de Fora (UFJF), 2018.

STRIKWERDA, J. C. *Finite difference schemes and partial differential equations*. [S.l.:] Siam, 2004. v. 88.
