RIESZ TRANSFORM CHARACTERIZATION OF HARDY SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH COMPACTLY SUPPORTED POTENTIALS

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Abstract. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$. We assume that $V$ is a nonnegative, compactly supported potential that belongs to $L^p(\mathbb{R}^d)$, for some $p > d/2$. Let $K_t$ be the semigroup generated by $-L$. We say that an $L^1(\mathbb{R}^d)$-function $f$ belongs to the Hardy space $H^1_L$ associated with $L$ if $\sup_{t > 0} |K_t f|$ belongs to $L^1(\mathbb{R}^d)$. We prove that $f \in H^1_L$ if and only if $R_j f \in L^1(\mathbb{R}^d)$ for $j = 1, ..., d$, where $R_j = \partial / \partial x_j \frac{1}{L^{1/2}}$ are the Riesz transforms associated with $L$.

1. Introduction.

Let

$$R_j f(x) = c_d \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \, dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} P_t(x-y) f(y) \, dy \frac{dt}{\sqrt{t}},$$

for $j = 1, 2, ..., d$, be the classical Riesz transforms on $\mathbb{R}^d$. Here and subsequently $P_t(x-y) = (4\pi t)^{-d/2} \exp \left(-\frac{|x-y|^2}{4t}\right)$ denotes the heat kernel. Clearly, for $f \in L^1(\mathbb{R}^d)$ the limits in (1.1) exist in the sense of distributions and define $R_j f$ as a distribution. It was proved in Fefferman and Stein [4] (see also [5]) that an $L^1(\mathbb{R}^d)$-function $f$ belongs to the classical Hardy space $H^1(\mathbb{R}^d)$ if and only if $R_j f \in L^1(\mathbb{R}^d)$ for $j = 1, ..., d$. Moreover,

$$\|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j f\|_{L^1(\mathbb{R}^d)},$$

defines one of the possible norms in $H^1(\mathbb{R}^d)$.

In this paper we consider a Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^d$, $d \geq 3$. We assume that $V$ is a nonnegative function, $\text{supp} \, V \subseteq B(0,1) = \{x \in \mathbb{R} : |x| < 1\}$, and $V \in L^p(\mathbb{R}^d)$ for some $p > d/2$. Let $K_t$ be the semigroup generated by $-L$. Since $V \geq 0$, by the Feynman-Kac formula, we have

$$0 \leq K_t(x,y) \leq P_t(x-y)$$

where $K_t(x,y)$ is the integral kernel of the semigroup $\{K_t\}_{t>0}$. Let

$$M f(x) = \sup_{t>0} |K_t f(x)|.$$
We say that an $L^1(\mathbb{R}^d)$-function $f$ belongs the Hardy space $H^1_L$ if
$$
\|f\|_{H^1_L} = \|\mathcal{M}f\|_{L^1(\mathbb{R}^d)} < \infty.
$$

For $j = 1, \ldots, d$ let us define the Riesz transforms $R_j$ associated with $L$ by setting

$$
R_j f = c_j \frac{\partial}{\partial x_j} L^{-1/2} f = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t f \frac{dt}{\sqrt{t}},
$$

where the limit is understood in the sense of distributions. The fact that for any $f \in L^1(\mathbb{R}^d)$ the operators

$$
R_j^2 f = \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t f \frac{dt}{\sqrt{t}}
$$

are well defined and the limit $\lim_{\varepsilon \to 0} R_j^2 f$ exists in the sense of distributions will be discussed below.

The main result of this paper is the following.

**Theorem 1.6.** Assume that $f \in L^1(\mathbb{R}^d)$. Then $f$ is in the Hardy space $H^1_L$ if and only if $R_j f \in L^1(\mathbb{R}^d)$ for every $j = 1, \ldots, d$. Moreover, there exists $C > 0$ such that

$$
C^{-1} \|f\|_{H^1_L} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{H^1_L}.
$$

The Hardy spaces $H^1_L$ associated with the Schrödinger operators $L$ with compactly supported potentials were studied in [2]. It was proved there that the elements of the space $H^1_L$ admit special atomic decompositions. Moreover, the space $H^1_L$ is isomorphic to the classical Hardy space $H^1(\mathbb{R}^d)$. To be more precise, let

$$
\Gamma(x, y) = \int_0^\infty K_t(x, y) \, dt, \quad \Gamma_0(x, y) = -\int_0^\infty P_t(x, y) \, dt,
$$

and denote

$$
L^{-1} f(x) = \int_{\mathbb{R}^d} \Gamma(x, y) f(y) \, dy, \quad \Delta^{-1} f(x) = \int_{\mathbb{R}^d} \Gamma_0(x, y) f(y) \, dy.
$$

The operators $(I - V\Delta^{-1})$ and $(I - VL^{-1})$ are bounded and invertible on $L^1(\mathbb{R}^d)$, and

$$
I = (I - V\Delta^{-1})(I - VL^{-1}) = (I - VL^{-1})(I - V\Delta^{-1}).
$$

Moreover, $(I - VL^{-1}) : H^1_L \to H^1(\mathbb{R}^d)$ is an isomorphism (whose inverse is $(I - V\Delta^{-1})$) and

$$
\| (I - VL^{-1}) f \|_{H^1(\mathbb{R}^d)} \simeq \|f\|_{H^1_L}
$$

for $f \in H^1_L$ (see [2, Corollary 3.17]).

The proof of the special atomic decompositions presented in [2] was based on the following identity

$$
P_t(I - VL^{-1}) = K_t - \int_0^t (P_t - P_{t-s}) VK_s \, ds, -\int_t^\infty P_t VK_s \, ds = K_t - W_t - Q_t,
$$

which comes from the perturbation formula

$$
P_t = K_t + \int_0^\infty P_{t-s} VK_s \, ds.
Proof. There exists $Q$ where

$$G(t) = K_t + \int_0^t P_{t-s} V K_s ds - P_t V L^{-1} = K_t + \tilde{W}_t - \tilde{Q}_t. \tag{1.10}$$

In the case of Schrödinger operators with potentials $V \geq 0$, $\mathcal{V} \not\equiv 0$, satisfying the reverse Hölder inequality with the exponent $d/2$ (which clearly implies supp $\mathcal{V} = \mathbb{R}^d$), Riesz transform characterizations of the relevant Hardy spaces $H^1_{\Delta+\mathcal{V}}$ were obtained in [1].

2. Auxiliary estimates

In this section we will use notation $f_t(x) = t^{-d/2} f(x/\sqrt{t})$.

For $f \in L^1(\mathbb{R}^d)$ and $0 < \varepsilon < 1$ we define the truncated Riesz transforms by setting

$$R^\varepsilon_j f = \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} K_{t}\frac{dt}{\sqrt{t}}, \quad R^\varepsilon_j f = \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} P_{t}\frac{dt}{\sqrt{t}}.$$

Denote

$$\begin{align*}
G(x,y) &= \int_0^\infty K_t(x,y)\frac{dt}{\sqrt{t}}, \\
G_0(x,y) &= \int_0^\infty P_t(x-y)\frac{dt}{\sqrt{t}}.
\end{align*}$$

Then $G(x,y) \leq G_0(x,y) = c|x - y|^{-d+1}$ and, consequently, for $\varphi$ from the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ we have

$$\lim_{\varepsilon \to 0} \langle R^\varepsilon_j f, \varphi \rangle = -\int G(x,y)f(y)\frac{\partial}{\partial x_j}\varphi(x)dydx.$$

Hence $R_j f$ is a well defined distribution and

$$|\langle R_j f, \varphi \rangle| \leq C\|f\|_{L^1(\mathbb{R}^d)} \left(\left\|\frac{\partial}{\partial x_j} \varphi\right\|_{L^1(\mathbb{R}^d)} + \left\|\frac{\partial}{\partial x_j} \varphi\right\|_{L^\infty}\right).$$

Using (1.9) and (1.10) we write

$$R^\varepsilon_j f = R^\varepsilon_j f = R^\varepsilon_j(I - V L^{-1}) f - \tilde{W}^\varepsilon_j f + \tilde{Q}^\varepsilon_j f + \tilde{W}^\varepsilon_j f + Q^\varepsilon_j f, \tag{2.1}$$

where $Q^\varepsilon_j, \tilde{Q}^\varepsilon_j, W^\varepsilon_j$, and $\tilde{W}^\varepsilon_j$ are operators with the following integral kernels

$$\begin{align*}
Q^\varepsilon_j(x,y) &= \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} Q_{t}(x,y)\frac{dt}{\sqrt{t}}, \\
\tilde{Q}^\varepsilon_j(x,y) &= \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} \tilde{Q}_{t}(x,y)\frac{dt}{\sqrt{t}}, \\
W^\varepsilon_j(x,y) &= \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} W_{t}(x,y)\frac{dt}{\sqrt{t}}, \\
\tilde{W}^\varepsilon_j(x,y) &= \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} \tilde{W}_{t}(x,y)\frac{dt}{\sqrt{t}}.
\end{align*}$$

We shall prove that $Q^\varepsilon_j, W^\varepsilon_j$, and $\tilde{W}^\varepsilon_j$ converge in the norm operator topology on $L^1(\mathbb{R}^d)$, while $\tilde{Q}^\varepsilon_j$ converges strongly on $L^1(\mathbb{R}^d)$ as $\varepsilon$ tends to 0.

Lemma 2.2. The operators $Q^\varepsilon_j$ converge in the norm operator topology on $L^1(\mathbb{R}^d)$ as $\varepsilon \to 0$.

Proof. There exists $\phi \in \mathcal{S}(\mathbb{R}^d), \phi \geq 0$, such that

$$\left|\frac{\partial}{\partial x_j} P_{t}(x-z)\right| \leq t^{-1/2}\phi_t(x-z). \tag{2.3}$$
On the other hand, by (1.3), $K_s(z, y) \leq Cs^{-d/2}$. Hence, for $0 < \varepsilon_2 < \varepsilon_1 < 1$, we have

$$
\int \left| Q_{j}^{x}(x, y) - Q_{j}^{x}(x, y) \right| dx \leq C \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{t}^{t} \int t^{-1/2} \phi_t(x - z)V(z)s^{-d/2}dzds\frac{dt}{\sqrt{t}}dx
$$

(2.4)

$$
\leq C \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} t^{-d/2}dt \cdot \|V\|_{L^1(\mathbb{R}^d)},
$$

which tends to zero uniformly with respect to $y$ as $\varepsilon_1, \varepsilon_2 \to 0$.

\[ \square \]

**Lemma 2.5.** The operators $W_{j}^{y}$ converge in the norm operator topology on $L^1(\mathbb{R}^d)$ as $\varepsilon \to 0$.

**Proof.** The proof borrows ideas from [2]. Let $0 < \varepsilon_2 < \varepsilon_1 < 1$. Then

\[
\int \left| W_{j}^{x}(x, y) - W_{j}^{x}(x, y) \right| dx \\
\leq \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{0}^{\varepsilon_1^{-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} (P_t(x - z) - P_{t-s}(x - z)) \right| V(z)K_s(z, y) dz ds \frac{dt}{\sqrt{t}} dx
\]

$$
+ \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{\varepsilon_1^{-1}}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} (P_t(x - z) - P_{t-s}(x - z)) \right| V(z)K_s(z, y) dz ds \frac{dt}{\sqrt{t}} dx
$$

= W'(y) + W''(y).

Observe that there exists $\phi \in S(\mathbb{R})$, $\phi \geq 0$, such that for $0 < s < t^{8/9},$

\[
\left| \frac{\partial}{\partial x_j} (P_t(x - z) - P_{t-s}(x - z)) \right| \leq s t^{-3/2} \phi_t(x - z).
\]

Therefore

$$
W'(y) \leq \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{0}^{\varepsilon_1^{-1}} \int t^{8/9} \int s t^{-2} \phi_t(x - z)V(z)K_s(z, y) dz ds dt dx
$$

(2.6)

$$
\leq \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} t^{-10/9} dt \cdot \int V(z)|z - y|^{2-d}dz \leq C_{\varepsilon_1}^{1/9}
$$

uniformly in $y$. The last inequality is a simple consequence of the H"older inequality and the assumption $p > d/2$.

For $t^{8/9} < s < t$ we have $K_s(z, y) \leq C s^{-d/2} \leq C t^{-d/9}$. Using (2.3) we get

$$
W''(y) \leq C \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{t^{8/9}}^{t} \int (t^{-1/2} \phi_t(x - z) + (t - s)^{-1/2} \phi_{t-s}(x - z))
$$

$$
\times V(z)t^{-d/9}dzds\frac{dt}{\sqrt{t}}dx
$$

(2.7)

$$
\leq C\|V\|_{L^1(\mathbb{R}^d)} \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} t^{-d/9}dt + C\|V\|_{L^1(\mathbb{R}^d)} \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} t^{-d/9-1/2} \int_{0}^{t}(t - s)^{-1/2}ds dt
$$

$$
\leq C_{\varepsilon_1}^{d/9-1}
$$

uniformly in $y$. Now the lemma follows from (2.6) and (2.7).
Lemma 2.8. There exists a limit of the operators \( \widetilde{W}^\varepsilon_j \) in the norm operator topology on \( L^1(\mathbb{R}^d) \) as \( \varepsilon \to 0 \).

Proof. Let \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \). Applying (2.3) we obtain

\[
\int_{\mathbb{R}^d} |\mathcal{W}^{\varepsilon_2}_j(x, y) - \mathcal{W}^{\varepsilon_1}_j(x, y)| \, dx
\]

\[\leq \int_{\mathbb{R}^d} \int_{\varepsilon_1}^{\varepsilon_2} \int_0^t \int_{\mathbb{R}^d} (t - s)^{-1/2} \phi_{t-s}(x - z) V(z) K_s(z, y) \, dz \, ds \, ds \, dt \, dx \]

\[= \int_{\mathbb{R}^d} \int_{\varepsilon_1}^{\varepsilon_2} \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\varepsilon_1}^{\varepsilon_2} \int_0^t \, dt \, dz \, ds \, dt = \mathcal{W}'(y) + \mathcal{W}''(y). \tag{2.9}\]

If \( 0 < s < t/2 \), then, of course, \( (t - s)^{-1/2} \leq Ct^{-1/2} \). Note that

\[
\int_0^t K_s(z, y) ds \leq C|z - y|^{2-d} \exp \left( -\frac{|z - y|^2}{8t} \right) \leq t\psi_t(z - y)
\]

for some \( \psi \in L^{p'}(\mathbb{R}^d) \), \( \psi \geq 0 \) (\( p' \) denotes the Hölder conjugate exponent to \( p \)). Hence

\[
\mathcal{W}'(y) \leq C \int_{\varepsilon_1}^{\varepsilon_2} \int_0^{t/2} \int_{\mathbb{R}^d} (t - s)^{-1/2} V(z) K_s(z, y) dz ds dt \leq C \int_{\varepsilon_1}^{\varepsilon_2} \int_{\mathbb{R}^d} V(z) \psi_t(z - y) dz dt
\]

\[\leq C \int_{\varepsilon_1}^{\varepsilon_2} \|V\|_p \|\psi_t\|_{p'} dt \leq C \int_{\varepsilon_1}^{\varepsilon_2} t^{-d/2p} \|\psi_t\|_{p'} dt \leq C \varepsilon_2^{1-d/2p} \tag{2.10}\]

uniformly in \( y \).

If \( t/2 \leq s \leq t \), then there exists \( \varphi \in S(\mathbb{R}^d) \), \( \varphi \geq 0 \), such that \( K_s(z, y) \leq \varphi_t(z - y) \). Therefore

\[
\mathcal{W}''(y) \leq C \int_{\varepsilon_1}^{\varepsilon_2} \int_0^{t/2} \int_{\mathbb{R}^d} (t - s)^{-1/2} V(z) K_s(z, y) dz ds dt \leq C \int_{\varepsilon_1}^{\varepsilon_2} \|V\|_p \|\varphi_t\|_{p'} dt \leq C \varepsilon_2^{1-d/2p} \tag{2.11}\]

uniformly in \( y \). Now the lemma is a consequence (2.10) – (2.11).

\[\square\]

Lemma 2.12. Assume that \( f \in L^1(\mathbb{R}^d) \). Then the limit \( F = \lim_{\varepsilon \to 0} \mathcal{Q}^\varepsilon f \) exists in the \( L^1(\mathbb{R}^d) \)-norm. Moreover, \( \|F\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)} \) with \( C \) independent of \( f \).

Proof. Of course, for any fixed \( y \in \mathbb{R}^d \), the function \( z \mapsto U(z, y) = V(z) \Gamma(z, y) \) is supported in the unit ball and \( \|U(z, y)\|_{L^r(\mathbb{R}^d)} \leq C_r \) for fixed \( r \in \left[ 1, \frac{d}{dp + d - 2p} \right] \) with \( C_r \) independent of \( y \).

The last statement follows from (1.3) and the Hölder inequality. Let

\[
H^\varepsilon_j(x, z) = \int_0^1 \frac{\partial}{\partial x_j} P_t(x - z) \, dt \sqrt{t}, \quad H^\varepsilon_j(x, z) = \int_{\mathbb{R}^d} H^\varepsilon_j(x, z) g(z) \, dz, \quad H^\varepsilon_j g(x) = \sup_{0<\varepsilon<1} |H^\varepsilon_j g(x)|.
\]

It follows from the theory of singular integral convolution operators (see, e.g., [3, Chapter 4]) that for \( 1 < r < \infty \) there exists \( C_r \) such that

\[\|H^\varepsilon_j g\|_{L^r(\mathbb{R}^d)} \leq C_r \|g\|_{L^r(\mathbb{R}^d)} \quad \text{for } g \in L^r(\mathbb{R}^d) \]

and \( \lim_{\varepsilon \to 0} H^\varepsilon_j g(x) = H_j g(x) \) a.e. and in \( L^r(\mathbb{R}^d) \)-norm.
Note that \( \tilde{Q}_j^\varepsilon(x,y) = H_j^\varepsilon U(\cdot,y)(x) \). Thus there exists a function \( \tilde{Q}_j(x,y) \) such that
\[
\lim_{\varepsilon \to 0} \tilde{Q}_j^\varepsilon(x,y) = \tilde{Q}_j(x,y) \text{ a.e. and}
\]
\[
\sup_y \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} |\tilde{Q}_j^\varepsilon(x,y)|^r \, dx \leq C_r' \text{ for } 1 < r < \frac{dp}{dp + d - 2p}.
\]
Since \( |H_j^\varepsilon(x,z)| \leq C_N |x-z|^{-N} \) for \(|x-z| > 1\),
\[
|\tilde{Q}_j^\varepsilon(x,y)| = \left| \int_{|z| \leq 1} H_j^\varepsilon(x,z) U(z,y) \, dz \right| \leq C_N |x|^{-N} \text{ for } |x| > 2.
\]
The Hölder inequality combined with (2.14) and (2.15) implies
\[
\sup_y \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} |\tilde{Q}_j^\varepsilon(x,y)| \, dx \leq C,
\]
(2.17)
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\tilde{Q}_j^\varepsilon(x,y) - \tilde{Q}_j(x,y)| \, dx = 0 \text{ for every } y.
\]
Now the lemma could be easily concluded from (2.16), (2.17), and Lebesgue’s dominated convergence theorem.

3. Proof of the main theorem

Recall that \((I - VL^{-1})\) is an isomorphism in \(L^1(\mathbb{R}^d)\). Consider \(f \in L^1(\mathbb{R}^d)\). Using (2.1) and lemmas 2.2, 2.5, 2.8, and 2.12 we get that \(R_j f\) belongs to \(L^1(\mathbb{R}^d)\) if and only if \(R_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d)\). Moreover,

\[
\|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j (I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)}.
\]

Applying the characterization of the classical Hardy space \(H^1(\mathbb{R}^d)\) by means of the Riesz transforms \(R_j\) (see (1.2)) and (1.8) we obtain the theorem.

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