Partial Wave Decomposition in Friedrichs Model With Self-interacting Continua

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We consider the nonrelativistic model of coupling bare discrete states with continuum states in which the continuum states can have interactions among themselves. By partial-wave decomposition and constraint to the conserved angular momentum eigenstates, the model can be reduced to Friedrichs-like model with additional interactions between the continua. If a kind of factorizable form factor is chosen, the model can be exactly solvable, that is, the generalized discrete eigenstates including bound states, virtual states, and resonances, can all be represented using the original bare states, and so do the in-state and out-state. The exact S matrix is thus obtained. We then discuss the behaviors of the dynamically generated S-wave and P-wave discrete states as the coupling is varying when there is only one self-interacting bare continuum state. We find that even when the potential is repulsive there could also be resonances and virtual states. In the P-wave cases with attractive interaction, we find that when there is a near-threshold bound state, there will always be an accompanying virtual state and we also give a more general argument of this effect.
Resonance phenomena appear in most areas in modern physics, such as in optics, atomic physics, condensed matter, and particle physics. Especially, more and more resonances were found in hadron physics, from low energy light $\sigma$, $\kappa$ resonances to heavy quarkonium-like resonances found in recent years. Historically, the theoretical understanding of the resonant state dates back to the description of the nuclear alpha decay by Gamow using eigenstates with complex energy eigenvalues, which is also called Gamow state. The Gamow state can not be represented as a vector in the Hilbert space since it is a generalized eigenstate of the full selfadjoint Hamiltonian with a complex eigenvalue. The mathematical description of the Gamow state needs an extension of the Hilbert space to the rigged Hilbert space, which is composed of a Gel’fand triple $\Omega \subset \mathcal{H} \subset \Omega^\times$, where $\mathcal{H}$ is the usual Hilbert space of the normalizable states, $\Omega$ is a nuclear space which is dense in $\mathcal{H}$, and $\Omega^\times$ is the space of the anti-linear continuous functionals on the nuclear space. The Gamow state should be in the larger $\Omega^\times$. The descriptions of in-state and out-state are using different Rigged Hilbert spaces, $\Omega_+ \subset \mathcal{H} \subset \Omega_0^\times$ where the subscript “+” denotes the out-state space and + denote the in-state space. For further detailed discussion on the mathematical foundation, the readers are referred to [1,2].

Friedrichs model[1] is a solvable model which demonstrates the generation of the Gamow state, in which a discrete bare state is coupled to a continuum state. When the energy of the discrete state is above the threshold of the continuum state, the discrete state will move to the second Riemann sheet of complex energy plane of $S$ matrix and becomes a resonant state, i.e., a Gamow state, whose wave function can be solved. In fact, this model also demonstrates the transition between bound states, virtual states, and resonances when the couplings are changed. The in-states and the out-states wave function as the energy eigenstates of the full Hamiltonian can also be obtained. In nonrelativistic theory, the model can either be formulated in three-dimensional momentum space such as in [3] or in a special partial-wave channel with only one continuum. However, even in nonrelativistic theories, including spins of the continuum states, there could be different continua with different spin configurations and the same total angular momentum that can enter the interactions. We will see later that, from the full Hamiltonian in momentum representation of the Hamiltonian, after partial wave decomposition, the Hamiltonian will be reduced to the multi-continuum Friedrichs-like model. In fact, including more than one continua without the interaction between the continua, the Friedrichs model can also be solved [4]. The Friedrichs model can also produce the dynamically generated states which do not originate from the discrete state [5,6]. Thus, Friedrichs model provide a model to describe the observed bound states or resonances, in which the origin of the state could be investigated.

In recent years, more and more heavy quarkonium-like states and possible exotic states were observed in the experiments, such as $X(3872)$, $D_s^*(2317)$, $Z_c$’s, and $Z_b$’s [8], which can not be explained satisfactorily by the conventional “quenched” quark model such as the well-known Godfrey-Isgur model [10]. Efforts are made to understand the possible mechanisms of generating these states. Take the enigmatic $X(3872)$ for example. The $X(3872)$ state can be regarded as being generated by coupling a bare $\chi_{c1}(2P)$ state, the state in the “quenched” potential model, to the $DD^*$ continuum in a unified mechanism in which other charmonium-like states above the open-flavor thresholds can also be described [11,13]. This picture is also supported by a refined analysis of $B \rightarrow KJ/\psi \pi^+ \pi^-$ and $B \rightarrow KDD^*$ [14]. Alternatively, in [15,16], a model was proposed to study relation of the wave function of resonance states and the scattering amplitude, and the method was used to discuss $X(3872)$. The model contains no bare discrete state, and has only the continuum interactions and the form factor is assumed to be factorized. The $S$ matrix and the resonance or bound state wave function was obtained by solving Lippmann-Schwinger equation in momentum representation following [17,18]. In Ref. [19], the authors generalized this method to including also the bare discrete states, but only $S$-wave processes are considered. In fact, we will show that after the partial wave decomposition, this model can be reduced to generalized Friedrichs-like model which includes one or more discrete states and also interactions between continuum states. If the partial wave form factor in this model can be separated to two factors, like in [15,19,22], this kind of generalized Friedrichs-like model can also be solved, that is, the eigenstates, including in-states, out-states, and the discrete eigenstates of the full partial wave Hamiltonian can be obtained by directly solving the eigenstate equation. Hence the exact partial-wave $S$ matrix in this model can be obtained in this way. Thus, all the partial waves can be dealt with in a similar fashion, the only differences are in the form factors which should be postulated in different models. The discussions on the compositeness and elementaryness in [17,19] can be generalized to different partial waves.

As examples, we also studied the behavior of the dynamically generated $S$-wave and $P$-wave states as the coupling varies using a kind of exponential form factor in the Friedrichs-like model with only one self-interacting continuum state and without any discrete bare state. If this potential is repulsive, there could still be resonances in the $S$-wave channel and virtual states in the $P$-wave channel. For the attractive potential, in the $S$-wave channel, there could be a bound state for large coupling and as the coupling becomes weaker, it will become a virtual state. In the $P$-wave, the attractive potential will generate both a bound state and a virtual state for large coupling, and as the coupling is turning down, the two states move through the threshold and become a pair of resonant state poles. As the coupling tends to 0, the poles all move to the negative infinity.
The paper is organized as follows: In Sect. III the partial wave analysis in the non-relativistic model is performed and the most general model is discussed. In Sect. IIII the solution to the generalized Friedrichs-like model with a kind of factorizable form factors is given. The wave function of the discrete eigenstates, in-states, and out-states are obtained and thus the partial-wave S matrix is obtained. In Sect. IV we will discuss the dynamically generated discrete states using an example form factor. Sect. V is the conclusion and discussion.

II. NONRELATIVISTIC PARTIAL WAVE DECOMPOSITION

In this section we will make clear the connection between the momentum space and the partial wave expansion of the Hamiltonian, and show that the Hamiltonian in terms of the angular momentum eigenstates will be reduced to the Friedrichs-like models.

Suppose a discrete state |0; l, m⟩ with spin l, coupled with a continuum composed of two-particle state |p; SS⟩ with the c.m. momentum p for each particle and total spin S. In the non-relativistic theory, the free Hamiltonian in the c.m. frame can be expressed as

\[ H_0 = M_0 \sum_{l_z} |0; l_z⟩⟨0; l_z| + \sum_{S_z} \int d^3 p \omega |p; SS⟩⟨p; SS|, \]  

where \( M_0 \) is the rest energy of the discrete state and \( \omega = M_{th} + \frac{p^2}{2m} \) is the energy of the continuum state in the c.m. frame, \( M_{th} \) being the threshold energy of the two-particle states and \( m \) being the reduced mass in the c.m. frame. The normalizations and completeness relations for these states are

\[ |0; l_z⟩⟨0; l_z| = δ_{l,l_z}, \quad ⟨p; SS|p; SS⟩′ = δ^3(p - p')δ_{S,S'}, \]  

\[ \sum_{S_z} \int d^3 p |p; S_z⟩⟨p; S_z| + \sum_{l_z} |0; l_z⟩⟨0; l_z| = 1 \]  

The plane wave state |p; SS⟩ can be decomposed into different partial waves

\[ |p; SS⟩ = \sum_{LM_L} i^{L} Y_{LM_L}^M(\hat{p}) |p; LM_L, SS⟩ = \sum_{JM, LM_L} i^{L} Y_{LM_L}^M(\hat{p}) C_{LM_L,SS}^{JM} |p; JM; LS⟩ \]  

where \( LM_L \) in |p; LM_L, SS⟩ denotes orbital angular momentum quantum numbers, \( Y_{LM_L}^M(\hat{p}) \) is the spherical harmonic function with \( \hat{p} \) the direction of \( p \), and \( p \) denotes the absolute value of the momentum \( p \). |p; JM; LS⟩ is the eigenstate of the total angular momentum with \( JM \) denoting the quantum numbers of the total angular momentum. The free Hamiltonian can be expressed in the angular momentum representation as

\[ H_0 = M_0 \sum_{l_z} |0; l_z⟩⟨0; l_z| + \sum_{J,M; LM_L} \int d^3 p \omega |p; JM; LS⟩⟨p; JM; LS|, \]  

There is no direct interaction of the discrete state with itself, i.e., \( ⟨0; l_z|V|0; l_z⟩ = 0 \), since it can be absorbed into the rest energy term. The interaction between the discrete states and the continuum states is spherically symmetric and the matrix elements of the interaction potentials are supposed to be

\[ ⟨0; l_z|V|p; SS⟩ = \sum_{LM_L} i^{L} \bar{g}_L(p^2) C_{LM_L,SS}^{JM} Y_{LM_L}^M(\hat{p}) \]  

where \( C_{LM_L,SS}^{JM} \) is the Clebsch-Gordan coefficient. Thus the interaction term in the Hamiltonian can be expressed as

\[ H_{01} = \sum_{S_z, l_z} \int d^3 p |0; l_z⟩⟨0; l_z|V|p; SS⟩⟨p; SS| + h.c. = \sum_{L,M} \int d^3 p \bar{g}_L(p^2) ⟨0; l_z|V|0; l_z⟩⟨p; SS| + h.c. \]  

\[ = \sum_{L,M} \int \mu p d \omega \bar{g}_L(p^2) ⟨0; l_z|V|0; l_z⟩⟨p; SS| + h.c. \]  

Since the total angular momentum and the z component are supposed to be conserved, there is no crossing terms between states with different such quantum numbers, and we can restrict to the subspace with fixed total angular momentum \( L \) and its z-component \( l_z \). One can redefine the state and the form factor to be

\[ |\omega, L⟩ = \sqrt{\mu p} |p; SS⟩, \quad |0⟩ = |0; l_z⟩, \quad g_L(\omega) = \sqrt{\mu p} \bar{g}_L(p^2), \]
and then the orthogonal condition reads
\[ \langle \omega, L | \omega', L' \rangle = \delta(\omega - \omega')\delta_{LL'} . \] (10)

The interaction Hamiltonian between the discrete state and the continuum in this subspace now becomes
\[ H_{01} = \sum_L \int d\omega g_L(\omega) |\omega, L\rangle + h.c. \] (11)

There could also be the direct interaction between the continuum two-particle states, which is supposed to conserve only the total angular momentum \( J^2 \) and \( J_z \). To be more general, we consider the interaction between two kinds of continuum with spins \( S_1 \) and \( S_2 \).
\[ \langle p' S_2 S_2 | V | p S_1 S_1 \rangle = \sum_{JM L_1 L_2 L_1' L_2'} (-i)^{L_2'} Y_{L_2}^{M_2'}(p')C^{JM}_{L_2 M_2 L_2'}S_2 S_2 i^{L_1'} Y_{L_1}^{M_1'}(p)C^{JM}_{L_1 M_1 L_1'}S_1 S_1 \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} (p'^2, p^2) \] (12)

where \( \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} (p'^2, p^2) = \langle p' JM; L_2' S_2 | V | p JM; L_1' S_1 \rangle \), \( JM \) being the quantum numbers for the total angular momentum and its \( z \)-components, and \( L_1' \) and \( L_2' \) being the quantum numbers for orbital angular momenta. The function \( \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} \) should be decreasing sufficiently fast as \( p, p' \rightarrow \infty \), and behave as \( p L_1 p' L_2 \) at \( p, p' \rightarrow 0 \) limit. We will see later that this threshold behavior is consistent with the one for the scattering amplitude. Note that these momenta denote the momenta of the free states which are not the eigenstate of the full Hamiltonian. Therefore, this interaction does not mean the non-conservation of the energy. The real eigenstates of the full Hamiltonian are in-states and out-states which asymptotically tend to the free states in the \( t \rightarrow \pm \infty \) limit when they feel no interaction. The \( S \) matrix still conserves the energy. Now, the interaction Hamiltonian between continuum states can then be expressed as
\[ H_{21} = \sum_{S_2, S_1} \sum_{JM L_2 S_2} \int d^3 p d^3 p' \langle p' S_2 S_2 | V | p S_1 S_1 \rangle \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} (p'^2, p^2) |p' JM; L_2 S_2\rangle |p JM; L_1 S_1\rangle + h.c. \] (13)

\[ H_{21} = \sum_{JM L_2 S_2} \sum_{L_1 S_1} \int d\omega' d\omega \mu_{1,2} \int d\omega' p' \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} (p'^2, p^2) |p' JM; L_2 S_2\rangle |p JM; L_1 S_1\rangle + h.c. \] (14)

We have changed the integration variable from the momentum to the c.m. energy \( \omega \), and \( \mu_{1,2} \) are the corresponding reduced masses. Since the interaction is supposed not to mix states with different \( JM \), we can restrict in a subspace with fixed \( JM \), and redefine
\[ |\omega, L_i; i \rangle = \sqrt{\mu_i p} |p; JM; L_i S_i \rangle \text{, for } i = 1, 2, \] (15)
\[ \mathcal{J}^{21}_{L_2 L_1} (\omega', \omega) = \sqrt{\mu_{1,2} p' p} \mathcal{J}^{JM}_{L_2 S_2 L_1' S_1} (p'^2, p^2) . \] (16)

Then, the interaction term \( H_{21} \) for \( JM \) channel is recast into
\[ H_{21}^{JM} = \sum_{L_2, L_1} \int d\omega' d\omega \mathcal{J}^{21}_{L_2 L_1} (\omega', \omega) |\omega', L_2 \rangle \langle \omega, L_1 | + h.c. \] (17)

For the model with only one continuum, there is only one self-interaction of the continuum which can be obtained just by setting the continuum state to the one defined in (15) in above equation and the 1, 2 denoting different continuum can be omitted. Thus, the full Hamiltonian for the \( JM \) channel can be expressed as
\[ H = M_0 |0\rangle \langle 0| + \sum_{L} \int d\omega \omega |\omega, L\rangle \langle \omega, L | + \sum_{L} d\omega g_L(\omega) |\omega, L\rangle + h.c. + \sum_{L_1, L_1'} \int d\omega' d\omega \mathcal{J}^{21}_{L_1 L_1'} (\omega', \omega) |\omega', L_1 \rangle \langle \omega, L_1 | + h.c. \] (18)

One can generalize this model to more than one discrete states \( |j\rangle \), \( j = 1, 2, \ldots \) and more continuum states. One can also regard the continuum states with different \( (L_i, i) \) combination as different states and label them using sequential
integers 1, 2, ... and allow $M_{i,th}$ to have degenerate energies. The general Hamiltonian can be expressed as

$$H = \sum_{j=1}^{D} M_j |j\rangle \langle j| + \sum_{i=1}^{C} \int_{M_{i,th}}^{\infty} d\omega \omega |\omega; i\rangle \langle \omega; i|$$  \hspace{1cm} (19)

$$+ \sum_{i_2, i_1} \int_{M_{i_1,th}} \cdots \int_{M_{i_2,th}} d\omega \int_{M_{i_2,th}} \cdots \int_{M_{i_1,th}} d\omega f_{i_2,i_1}(\omega', \omega)|\omega'; i_2\rangle \langle \omega; i_1| + h.c.$$  \hspace{1cm} (20)

$$+ \sum_{j=1}^{D} \sum_{i=1}^{C} \int_{M_{i,th}} d\omega g_{ji}(\omega)|j\rangle \langle \omega; i| + h.c.$$  \hspace{1cm} (21)

where $D$ discrete states and $C$ continuum states are assumed. This is the most general Friedrichs-like model with interactions among continuum states and discrete states.

### III. Solution to a Special Kind of Friedrichs-Like Model with Interacting Continua

For general form factors of the discrete-continuum and continuum-continuum interactions, the Friedrichs-like model is not solvable. However, if we take the form factors as in [15, 19–22],

$$g_{ij}(\omega) = u_{ij} f_j^*(\omega), f_{ij}(\omega', \omega) = v_{ij} f_j'(\omega') f_i^*(\omega)$$  \hspace{1cm} (22)

in which $u_{ij}$ and $v_{ij}$ are constants and the form factor $f_j(\omega)$ comes always with the $j$th continuum state, the Friedrichs-like model is then solvable. In this case, the Hamiltonian can be expressed as

$$H = \sum_{i=1}^{D} M_i |i\rangle \langle i| + \sum_{i=1}^{C} \int_{a_i}^{\infty} d\omega \omega |\omega; i\rangle \langle \omega; i|$$  \hspace{1cm} (23)

$$+ \sum_{i,j=1}^{C} v_{ij} \left( \int_{a_i}^{\infty} d\omega f_i(\omega)|\omega; i\rangle \langle \omega; j| \right)$$  \hspace{1cm} (24)

$$+ \sum_{j=1}^{D} \sum_{i=1}^{C} \left[ u_{ji}^* |j\rangle \langle \omega; i| + u_{ji} \left( \int_{a_i}^{\infty} d\omega f_i(\omega)|\omega; i\rangle \langle \omega; i| \right) \right]$$  \hspace{1cm} (25)

where $v_{ij} = v_{ji}^*$ can be seen from the hermitian of the Hamiltonian.

The eigenstate of the Hamiltonian with eigenvalue $E$ can be expanded using the discrete states and the continuum states

$$|\Psi(E)\rangle = \sum_{i=1}^{D} \alpha_i(E) |i\rangle + \sum_{i=1}^{C} \int d\omega \psi_i(E, \omega) |\omega; i\rangle$$  \hspace{1cm} (26)

From the eigenstate equation, one finds equations

$$(M_j - E)\alpha_j(E) + \sum_{i=1}^{C} u_{ij}^* \int_{a_i}^{\infty} d\omega f_j^*(\omega) \psi_i(E, \omega) = 0$$  \hspace{1cm} (27)

$$\sum_{j=1}^{D} \alpha_j(E) u_{ji} f_i(\omega) + (\omega - E) \psi_i(E, \omega) + \sum_{j=1}^{C} v_{ij} A_j(E) f_i(\omega) = 0$$  \hspace{1cm} (28)

where we have defined $A_j(E) = \int_{a_j}^{\infty} d\omega f_j^*(\omega) \psi_j(E, \omega)$. If the eigenvalue $E \notin [a_i, \infty)$ for $i = 1, \cdots, C$, we can obtain

$$\alpha_j(E) = -\frac{1}{M_j - E} \sum_{i=1}^{C} u_{ji}^* A_i(E)$$  \hspace{1cm} (29)

$$\sum_{j} V_{ij}(E) f_i(\omega) A_j(E) + (\omega - E) \psi_i(E, \omega) = 0$$  \hspace{1cm} (30)
where \( V_{ij} \equiv v_{ij} - \sum_{l=1}^{D} \left( \frac{u_{ij} u_{k}}{M_l - E} \right) \) is defined. Multiplying Eq. (30) with \( f_1^*(\omega) \) and integrating w.r.t. \( \omega \), one finds the equation for \( A_i \)

\[
\sum_{j=1}^{C} M_{ji} A_i = \sum_{j=1}^{C} (\delta_{ji} + G_j(E) V_{ji}) A_i = 0, \tag{31}
\]

where \( G_j(E) \equiv \int_{\alpha_j} d\omega \frac{|f_j(\omega)|^2}{\omega - E} \), \( M_{ji} \equiv \delta_{ji} + G_j(E) V_{ji} \). To have non-zero \( A_i \) solution,

\[
\det M = \det [\delta_{ji} + G_j(E) V_{ji}(E)] = 0 \tag{32}
\]

must be satisfied. Thus, the solutions \( \tilde{E}_i \) to Eq. (32) will be the discrete eigenvalues for the full Hamiltonian and the eigenvectors \( A_i(\tilde{E}_k) \) at these eigenvalues can be solved with the normalization undetermined, where the subscript \( k \) denotes the different eigenvalues. The number of solutions may be more than the original bare discrete states. The extra states may be generated from the singularities of the form factors \([7, 8]\) or by coupled channel effects which cause shadow poles \([8, 23]\). Since \( M^*_{ij}(E) = M_{ji}(E^*) \) and \( (\det M(E))^* = \det M(E^*) \), the solution should be symmetric w.r.t. the real axis as expected. Then from Eqs. (29) and (30), we obtain the discrete eigenstate

\[
|\Psi(\tilde{E}_k)\rangle = \sum_{i=1}^{C} A_i(\tilde{E}_k) \left( -\sum_{l=1}^{D} \frac{u_{il}^*}{M_l - \tilde{E}_k} |l\rangle + G_i^{-1}(\tilde{E}_k) \int_{\alpha_i} d\omega \frac{f_i(\omega)}{\omega - \tilde{E}_k} |\omega; i\rangle \right), \tag{33}
\]

If the eigenvalue is real \( E_B \) below the threshold on the first sheet, the state represents a bound state. As what was done in \([19]\), one can define elementariness \( Z_i \) and compositeness \( X_j \)

\[
Z_i = |\alpha_i(E_B)|^2 = \frac{\sum_{k=1}^{C} A_i(\tilde{E}_k) u_{ik}^* u_{lj} A_j^*}{(M_l - E_B)^2} \tag{34}
\]

\[
X_i = \int d\omega |\psi_i(E_B, \omega)|^2 = G_i'(E_B) \sum_{j,j'} A_j A_j^* V_{ij} V_{ij} \tag{35}
\]

and the normalization of \( A_i \) can be fixed by requiring \( \langle \Psi(E) | \Psi(E) \rangle = 1 \), i.e.

\[
\langle \Psi(E) | \Psi(E) \rangle = \sum_{ij} A_i^* (E_B) V_{ij}^*(E_B) A_j(E_B) + \sum_{i,j} A_j A_j^* \sum_{i} V_{ij}^* V_{ij} G_i'(E_B) = \sum_{i} Z_i + \sum_{i} X_i = 1 \tag{36}
\]

where the prime in \( G_i' \) and \( V_{ij}' \) means the derivative. The first term \( Z = \sum Z_i \) is just the total elementariness and the second term \( X = \sum X_j \) is just the total compositeness defined in Ref. [19]. Using the eigenvalue equation (31), the compositeness and elementariness can also be written as

\[
X_i = -|A_i(E_B)|^2 (G_i(E_B)^{-1})', \quad Z_i = \sum_{ij} c_{ij} G_j^* V_{ij}'(E_B) G_i c_i, \tag{37}
\]

where \( c_{ij} \equiv \sum_k V_{ik} A_{jk} = -G_i^{-1}(E_B) A_i(E_B) \) is used which is just the same definition as in [19].

If the eigenvalue is not real, it should not be on the physical sheet which is required by causality and should be symmetric with respect to the real axis as explained above. The integral in (33) should be analytically continued to the sheet on which the solution \( \tilde{E}_k \) lies which can be achieved by deforming the integral path as did in \([8, 23]\). The corresponding state is also represented as in (33). There is also the left eigenstate with the same eigenvalue,

\[
\langle \tilde{\Psi}(\tilde{E}_k) | = \sum_{i=1}^{D} \tilde{\alpha}_i(\tilde{E}_k) |i\rangle + \sum_{i=1}^{C} \int d\omega \tilde{\psi}_i(\tilde{E}_k, \omega) |\omega; i\rangle \tag{38}
\]

where

\[
\tilde{\alpha}_j(\tilde{E}_k) = -\frac{1}{M_j - \tilde{E}_k} \sum_{i=1}^{C} u_{ij} \tilde{A}_i^*(\tilde{E}_k) \tag{39}
\]

\[
\tilde{\psi}_i(\tilde{E}_k, \omega) = -\frac{1}{\omega - \tilde{E}_k} \sum_{j} \tilde{A}_j^*(\tilde{E}_k) V_{ji}(\tilde{E}_k) f_i^*(\omega), \tag{40}
\]
and the normalization can be fixed by
\[ \langle \tilde{\Psi}(E_k)|\tilde{\Psi}(E_k) \rangle = \sum_{ij} A_i^*(E_k^*) V_{ij} (E_k) \delta_{ij} A_j(E_k) \]
\[ = \sum_{ij} c_i(E_k^*) G_j(E_k)V_{ij} G_j(E_k) + \delta_{ij} G_i^2(E_k) c_j = \sum_I Z_I + \sum_i X_i = 1. \] (42)

Here, \( Z_I \) and \( X_i \) are not real any more, and thus can not have probability explanation. However, the author of [19] also propose them to denote the compositeness and the elementariness of the resonance. In [24], another way to describe the elementariness and compositeness of resonances was proposed in which the two quantities can be real.

We now come to the continuum eigenstates of the full Hamiltonian. There are \( C \) continuum eigenstates for the free Hamiltonian, and we expect that there are also \( C \) continuum eigenstates which reduce to the free eigenstates as the couplings are turned off. The \( k \)th eigenstates can still be expanded using the free states as in Eq. (26)

\[ |\Psi^{(k)}(E)\rangle = \sum_{i=1}^D \alpha_i^{(k)}(E)|i\rangle + \sum_{i=1}^C \int d\omega \psi_i^{(k)}(E,\omega)|\omega;i\rangle \] (43)

and equations similar to Eq. (28) can also be obtained, with superscript \( (k) \) added to \( \alpha \) and \( \psi \)

\[ \langle M_j - E \rangle \alpha_j^{(k)}(E) + \sum_{i=1}^C u_{ji}^* \int_a^\infty d\omega f_i^*(\omega)\psi_i^{(k)}(E,\omega) = 0 \] (44)

\[ \sum_{j=1}^D \alpha_j^{(k)}(E)u_{ji}(\omega) + (\omega - E)\psi_i^{(k)}(E,\omega) + \sum_{j=1}^C v_{ij} A_j^{(k)}(E) f_i(\omega) = 0 \] (45)

The continuum eigenvalue corresponding to the \( k \)th continuum lies above the \( k \)th threshold, i.e. \( x > a_k \) and is real. Since the state reduces to the \( k \)th state when the couplings are turned off, there should be a delta function in the \( \psi_i^{(k)} \)

\[ \alpha_j^{(k)}(E) = -\frac{1}{M_j - E} \sum_{i=1}^C u_{ji}^* A_i^{(k)}(E) \]

\[ \psi_i^{(k)}(E,\omega) = -\frac{1}{\omega - E \pm i\epsilon} \sum_{j} V_{ij}(E) f_i(\omega) A_j^{(k)}(E) + \delta_{ik} \gamma_k(E) \delta(E - \omega) \] (47)

Thus, from the second equation we have

\[ \sum_j M_{\pm,ij}(E) A_j^{(k)}(E) = \sum_j (\delta_{ij} + G_{\pm,i}(E)V_{ij}(E))A_j^{(k)}(E) = \delta_{ik} f_k^*(E) \gamma_k(E) \] (48)

where \( G_{\pm,i} \equiv -\int_a^\infty d\omega \frac{|f_i(\omega)|^2}{E - \omega \pm i\epsilon} \). We can define a matrix \( G_{\pm} = \text{diag}\{G_{\pm,1},G_{\pm,2},\cdots,G_{\pm,C}\} \), and then in matrix form, \( M_{\pm} = 1 + G_{\pm} V \). Then \( A_j^{(k)}(E) \) can be solved

\[ A_j^{(k)}(E) = (M_{\pm}^{-1})_{jk} f_k^*(E) \gamma_k(E) \] (49)

( no sum for \( k \) ) and \( \psi_{\pm,i}(E,\omega) \) and \( \alpha^{(k)}(E) \) can be obtained

\[ \psi_{\pm,i}^{(k)}(E,\omega) = \gamma_k \left[ \frac{f_i(\omega)f_k^*(E)}{E - \omega \pm i\epsilon} \sum_j V_{ij}(E)(M_{\pm}^{-1})_{jk} + \delta_{ik} \delta(E - \omega) \right], \] (50)

\[ \alpha_{\pm}^{(k)}(E) = -\frac{\gamma_k(E)f_k^*(E)}{M_l - E} \sum_{i=1}^C u_{li}^*(M_{\pm}^{-1})_{ik}. \] (51)

Thus the continuum state can be expressed as

\[ |\Psi_{\pm}^{(k)}(E)\rangle = \gamma_k(E) \left[ |E;k\rangle - f_k^*(E) \sum_{j=1}^C (M_{\pm}^{-1})_{jk} \left( -\sum_{i=1}^C V_{ij} \int d\omega \frac{f_i(\omega)}{E - \omega \pm i\epsilon} |\omega;i\rangle + \sum_{i=1}^D \frac{u_{ij}^*}{M_l - E} |l\rangle \right) \right] \] (52)
\[ |\Psi^{(k)}(E)\rangle = |\text{in-state}\rangle, \quad |\Psi^{(k)}(E)\rangle = |\text{out-state}\rangle. \]

If \( \gamma_k(E) = 1 \), the continuum states can be normalized as
\[ \langle \Psi^{(i)}_+(E')|\Psi^{(k)}(E)\rangle = \delta_{jk}\delta(E - E'). \]

The partial-wave \( S \) matrix can then be obtained by the inner product of the in-state and the out-state,
\[ S_{k',k}(E',E) = \langle \Psi^{(k')}_-(E')|\Psi^{(k)}(E)\rangle = \delta(E' - E) - 2\pi i \delta(E' - E) f_{k'}(E) f^*_k(E) (V^{-1} + G_+)^{-1}_{k'k} \]  

The threshold behavior of the partial-wave amplitude is correct due to our requirement of the threshold behavior of the form factors. The overall \( \delta \) function means the energy conservation. It is also easy to check that the \( S \) matrix is unitary. If there is no discrete states, the \( S \) matrix reduces to the one discussed in [12].

As an example, if there is only one discrete state, \( D = 1 \), and no interaction between the continuum \( \nu_{ij} = 0 \), we define \( g_i = u_{1i} \) and \( V_{ij}(E) = -\frac{g_i g_j^*}{M_1 - E} \) and then
\[ (M^{-1}(E))_{ij} = (1 + G_{\pm}(E)V(E))^{-1} = \delta_{ij} - \frac{g_i g_j^*}{\eta_{\pm}(E)}, \]  

where \( \eta_{\pm}(E) = E - M_1 + \sum_{i=1}^{C} |g_i|^2 G_{\pm,i} \). Thus
\[ \sum_j V_{ij}(E)(M^{-1}_{\pm})_{jk} = -\frac{g_i g_j^*}{\eta_{\pm}(E)}, \quad \sum_j (M^{-1}_{\pm})_{jk} \frac{g_j^*}{M_1 - E} = -\frac{g_i^*}{\eta_{\pm}(E)} \]  

The continuum states and the discrete states can be reduced to
\[ |\Psi^{(k)}(E)\rangle = |E; k\rangle + \frac{g_i^* f_i(E)}{\eta_{\pm}(E)} |1\rangle + \sum_i \int_{a_i} d\omega \frac{f_i(\omega)}{E - \omega + i\epsilon} |\omega; i\rangle, \]  
\[ |\Psi(E_k)\rangle = - \sum_{i=1}^{C} \frac{A_i g_i^*}{M_1 - E_k} |1\rangle - \sum_i A_i g_i \int_{a_i}^{\infty} d\omega \frac{f_i(\omega)}{\omega - E_k} |\omega; i\rangle. \]  

The eigenvalues of the discrete states \( \tilde{E}_k \) are determined by
\[ \det M = -\frac{\eta(\tilde{E}_k)}{M_1 - \tilde{E}_k} = 0, \]  

and the eigenvalue equation for \( A_i \) is
\[ \sum_j (\delta_{ij} - \frac{G_{ij} g_j^*}{M_1 - E_k}) A_j = 0. \]  

from which we can see that \( \frac{A_i}{g_i} = \sum_j \frac{g_j^* A_j}{M_1 - E_k} \) is a constant independent of \( i \). Thus, the normalized discrete state is
\[ |\Psi(\tilde{E}_k)\rangle = N \left( |1\rangle - \sum_i g_i \int_{a_i} d\omega \frac{f_i(\omega)}{\omega - \tilde{E}_k} |\omega; i\rangle \right) \]  
\[ N = \frac{1}{\eta(\tilde{E}_k)^{1/2}} \]  

These results are the same as was obtained in [3].

IV. DYNAMICALLY GENERATED STATES

Another interesting case is that when there is no discrete state, only dynamically generated states may appear, which could be resonances, bound states, or virtual states. It is instructive to study the different pole trajectories of this kind of states in different partial waves as the coupling varies. The similar pole trajectory properties when a discrete state is coupled with a continuum for \( S \)-wave are studied in [25]. A comparison of the pole trajectories between the cases with and without the discrete state coupling to the continuum for higher partial waves is also studied in [26] using two specific models.
For simplicity, we consider only one continuum here. The Hamiltonian is

$$H = \int_a^\infty d\omega \omega |\omega| \langle \omega | \pm \lambda^2 \int_a^\infty d\omega' \int_a^\infty d\omega f(\omega)f^*(\omega')|\omega\rangle \langle \omega'|$$

where $f(\omega) = \langle \omega - a | |(l + 1/2)2 \exp\{-|\omega - a|/(2\Lambda)\}\rangle$ and the reduced mass $\mu = 1$ to make all quantities dimensionless.

We first look at the $S$-wave. Thus, $G$ function can be analytically continued on the first and second sheet as

$$G_S(E) = \int_a^\infty \frac{d\omega (\omega - a)^{1/2}}{\omega - E} \frac{\exp\{-|\omega - a|/(2\Lambda)\}}{\sqrt{\pi} - e^{(a-E)/\Lambda} \pi \sqrt{a - E}(1 - \text{erf}(\sqrt{a - E}/\Lambda))},$$

$$G_S^{II}(E) = G_S(E) + iF_S^{II}(E)$$

$$= \int_a^\infty \frac{d\omega (\omega - a)^{1/2}}{\omega - E} \frac{\exp\{-|\omega - a|/(2\Lambda)\}}{\pi \sqrt{a - E}(1 - \text{erf}(\sqrt{a - E}/\Lambda))},$$

$$= \sqrt{\pi} + e^{(a-E)/\Lambda} \pi \sqrt{a - E}(1 - \text{erf}(\sqrt{a - E}/\Lambda)),$$

where $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$, $F_S(x) = |f(x)|^2$, and $F_S^{II}(x) = -(\omega - a)^{1/2} \exp\{-|\omega - a|/(2\Lambda)\}$ is the analytic continuation of $F_S$ to the second sheet.

From Eq. (64), we can see that $M_+ = 0$ cannot have solutions on the first sheet, since the integral is either complex on the complex plane or positive below the threshold on the real axis. So there could be no bound state for this case. However, there may be resonances or virtual states on the second sheet. From (65), since the phase of $(E - a)^{1/2}$ at $E < a$ is $\pi/2$, and the second term is positive, the equation could not have solution for $E < a$. By numerical experiments, we find that there is a pair of resonance poles on the second sheet. As the coupling is turning down, the poles are moving towards the negative infinity on the complex plane. See Fig. 1 for illustration. However, this resonance is a little farther away compared with the $\Lambda$, and may not be physically meaningful.

However, $M_- = 0$ can have solutions on the first sheet real axis. In fact, by numerical experiments, there is a bound state pole on the first sheet when the coupling is large enough. As the coupling is turning down, the bound state moves through the threshold to the second sheet real axis below the threshold, becoming a virtual state. As the coupling continues turning down further, the virtual state moves towards the negative infinity. See Fig. 2 for an illustration. This situation is reminiscent of the deuteron and its virtual state partner. In the nucleon-nucleon scattering, in the spin-triplet channel there is a deuteron bound state for a stronger coupling, while in the spin-singlet channel with a weaker interaction, a virtual state is generated and contributes a large scattering length.

For $P$-wave, we can do the same thing. The analytically continued $G$ function on the first sheet and second sheet
The existence of the accompanying virtual state of a bound state in $P$-wave can be understood as follows. That there is a bound state means that $M_{\pm}(E_B) = 1 - \lambda^2 G_P(E_B) = 0$ where $E_B$ is the energy eigenvalue corresponding to the bound state. Since $G_P(E)$ is a monotonically increasing function below the threshold $a$, for $E < E_B$, $\lambda^2 G_P(E) < 1$ and for $E > E_B$, $\lambda^2 G_P(E) > 1$. On the second sheet, $M_{I}^{II}(E) = M_{-}(E) - 2\pi i \lambda^2 F_{II}^{II}(E)$, where $-2\pi i \lambda^2 F_{II}^{II}(E)$ is a monotonically decreasing function, and for $E \leq a$, $-2\pi i \lambda^2 F_{II}^{II}(E) \geq 0$. Thus, $M_{I}^{II}(E_B) > 0$ and $M_{I}^{II}(a) < 0$, and from continuity, there must be a solution to $M_{I}^{II} = 0$ between $E_B$ and $a$. The property near the threshold is determined by the $(E - a)^{3/2}$ factor and the positivity of the form factor $|f(\omega)|^2/(E - a)^{3/2}$ in the $F_{II}^{II}$. In general, $|f(\omega)|^2/(E - a)^{3/2}$ at the $E = a$ limit should be a positive value and would not change sign on the two sides of $E = a$, since otherwise, $|f(\omega)|^2$ would behave according to a different power of $E - a$ at the threshold. So, if a dynamically generated bound state is found to be very near the threshold there would also be an accompanied virtual state. In this case, it may not be distinguished with the more fundamental bound state generated from the discrete bare state, which is also accompanied with a virtual state [8]. The only difference is that in the small coupling limit

\[ G_P(E) = \int_a^\infty \frac{d\omega}{\omega - E} \frac{(\omega - a)^{3/2} \exp(\frac{-(\omega - a)}{2\Lambda})}{\omega - E} = \frac{1}{2} \sqrt{\pi \lambda (\Lambda - 2(a - E)) + e^{(a - E)/\Lambda} \pi (a - E)^{3/2} (1 - \exp(\frac{\sqrt{a - E}}{\Lambda}))}, \]  

\[ G_P^{I} = G_P(E) + 2\pi i F_{II}^{II}(E) = \int_a^\infty \frac{d\omega}{\omega - E} \frac{(\omega - a)^{3/2} \exp(\frac{-(\omega - a)}{2\Lambda})}{\omega - E} = \frac{1}{2} \sqrt{\pi \lambda (\Lambda - 2(a - E)) + e^{(a - E)/\Lambda} \pi (a - E)^{3/2} (1 - \exp(\frac{\sqrt{a - E}}{\Lambda}))}, \]  

where $F_{P}(x) = |f_P(x)|^2$, and $F_{II}^{II}(\omega) = -(\omega - a)^{3/2} \exp(\frac{-(\omega - a)}{2\Lambda})$ is its analytic continuation to the second sheet. The $M_{\pm} = 0$ still does not have bound state solution since $G(E)$ is still positive on the first sheet below the threshold. However, since the phase of $(E - a)^{3/2}$ term in the second term of Eq. (71) is $-e^{i\pi/2}$ for $E < a$ and the second term is negative, there is a virtual state solution and it will move to the negative infinity as the coupling is turning off. This is because, the range of the first term in Eq. (71) is bounded by $(0, G_P(a))$ and the second term is monotonically increasing and unbounded below. As $\lambda$ is decreasing the second term in Eq. (71) will become important. But as the coupling grows larger, the virtual state can not go through the threshold to the first sheet since we know that there can not be a bound state solution for the $M_{\pm} = 0$ on the first sheet. Thus there must be a limiting point of the virtual state as the coupling goes to positive infinity. This point is determined by $G_{II}^{I}(E) = 0$ which is independent of $\lambda$. See Fig. 3 for an illustration.

The solutions to $M_{-} = 0$ include one bound state and one virtual state for large coupling and as the coupling decreases the two solutions are moving through the threshold and becoming a pair of resonance poles on the second sheet, and then will move towards the negative infinity. See Fig. 4 for an illustration. If the coupling is increased to infinity, the bound state will move to the negative infinity, and the virtual state will approach a limiting point on the second sheet, which is determined by $G_{II}^{I} = 0$ the same as the previous case.

The existence of the accompanying virtual state of a bound state in $P$-wave bound state is found to be very near the threshold there would also be an accompanied virtual state. In this case, it may not be distinguished with the more fundamental bound state generated from the discrete bare state, which is also accompanied with a virtual state [8]. The only difference is that in the small coupling limit
FIG. 3. Dynamical generated $P$-wave virtual state pole on the second Riemann sheet moves towards negative infinity. $a = 0.5$, $\Lambda = 2$.

FIG. 4. Dynamically generated $P$-wave bound state pole on the first Riemann sheet and virtual state pole on the second Riemann sheet merge together at the threshold becoming a pair of resonance poles on the second sheet as the coupling decreases. $a = 0.5$, $\Lambda = 2$.

The fundamental bound state goes to the bare state, while the dynamically generated bound state will move to the second sheet and combines with the virtual-state pole to form a pair of resonance poles, and then the resonance poles move to the negative infinity.

V. CONCLUSION AND OUTLOOK

In this paper, we have studied the model which contains several discrete states and continuum states, in which the interactions between discrete states and continua, and the interaction between continua are included. We made the partial wave decomposition of the model, and showed that confined to a specific partial wave, it reduces to the Friedrichs-like model which include also the interaction between continua. If the form factors in each partial wave can be factorized as assumed in [15, 19–22], the model can be rigorously solved and the discrete eigenstates, the in-states, the out-states, and the exact $S$ matrix can be obtained subsequently.

As an example, we studied the behavior of the dynamically generated states in non-relativistic $S$-wave and $P$-wave cases with only one self-interacting continuum using a kind of exponential form factor. This is a kind of typical form factor used in the phenomenological analysis. In each case, there could be two kinds of interactions, one with a repulsive potential with plus sign before the interaction term and the other with an attractive potential with a minus sign. The $S$-wave case with a repulsive interaction has only second sheet resonances farther away compared to the cut-off and may not be of much physical meaning. However, for the attractive potential, when the coupling is strong, there could be a bound state for large couplings and when the coupling decreases, the bound state will move through the threshold to the second sheet becoming a virtual state. For the $P$-wave case with the repulsive potential, there is only one virtual state and as the coupling is turning off the pole moves to the negative infinity. For the attractive
potential, there could be a bound state and a virtual state for larger couplings, and when the coupling is decreasing, they move to the threshold and then form a pair of resonance poles on the second sheet. We also give an argument that, in the $P$-wave, since the threshold behavior of the form factor is determined by the $p^3$ times a positive factor, for any potential satisfying this condition, a bound state near the threshold will be accompanied with a virtual state. It is a requirement of the threshold behavior.

Thus, a general nonrelativistic framework to discuss interaction between the discrete states and the continuum is laid down. However, a remaining problem is to generalize it to relativistic cases suitable for particle physics application. In fact, the essential problem for the consistent relativistic generalization is that one must deal with the negative frequency modes in relativistic theory. It is not easy to include them in the Hamiltonian in the formalism used in present paper. There are different attempts of relativistic generalizations of the Friedrichs model. One typical kind of relativistic generalization has been developed by the school of Brussels, in [27, 28], in which a kind of bilocal field is used to represent the continuum two-particle state, which is not suitable for particle physics application. Another typical relativistic field theory generalization is discussed in [29], in which only a subset of interactions are included in the model. Both these generalizations utilise the field theory language and provide a clue for further work. The other problem one must face in applying this model in different physical situations is the determination of the form factors. The partial-wave form factor in Eq. (16) should behave as $p^{L_1+1/2}p'^{L_2+1/2}$ in the limit of $p, p' \to 0$ but also should be converged to zero sufficiently fast as $p, p' \to \infty$ in order for the integral to be well-defined. In different process, how to obtain a reasonable form factor from more fundamental model such as QCD is a challenging task. We have shown that when the form factor can be factorized, the model can be solved. In this case, the solution to this model is equivalent to summing over all the bubble-chain diagrams in the field theory language, similar to the situation in [30]. The form factor in reality may not be factorizable. Whether there could be other form factors which make the model solvable is another research direction.

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