Diffusion in momentum space as a picture of second-order Fermi acceleration

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Abstract

Energetic particles in a turbulent medium can be subject to second-order Fermi acceleration due to scattering on moving plasma waves. This mechanism leads to growing particle momentum dispersion and, at the same time, increases the mean particle energy. In the most frequently met situations both processes can be represented by a single momentum diffusion term in the particle kinetic equation. In the present paper we discuss the conditions allowing the additional term for regular acceleration to arise. For forward-backward asymmetric scattering centres, besides the diffusive term one should explicitly consider the regular acceleration term in momentum space, which can consist of the first-order ($\propto V$), as well as the second-order ($\propto V^2$) part in the wave velocity $V$. We derive the condition for the scattering probability in the wave rest frame required for vanishing the regular acceleration term and provide a simple mechanical example illustrating the theoretical concepts. Finally, we address its possible role in cosmic ray acceleration processes.

Key words: acceleration mechanisms – Fermi acceleration – cosmic rays – non-linear waves

1 Introduction

The original theory conceived by Fermi [5] for the acceleration of cosmic rays considers charged particles scattered by magnetic clouds moving with speeds $V$ relative to the plasma rest frame. The resultant acceleration is represented by a systematic gain term. The following studies considered some particular acceleration scenarios involving large amplitude magnetohydrodynamic waves
and allowing for regular acceleration of cosmic ray particles (cf. Parker [7]). A general approach is based on the Fokker-Planck equation (cf. Parker & Tidman [8]; Clemmow & Dougherty [2]; Lacombe [6]; Blandford & Eichler [1]). In the recent years this approach supplemented with the quasi-linear derivation of transport coefficients for cosmic rays interacting with Alfvénic turbulence (e.g. Schlickeiser [9]) became an analytical basis for considering energetic particle transport phenomena in rarefied astrophysical plasmas. Within this approach the detailed balance principle applied to particle interaction with the field of magnetized plasma waves leads to the reduction of the general Fokker-Planck equation to the pure momentum diffusion equation. Let us note, however, that the principle of detailed balance which arises in a natural way in the kinetic description of gas of scattering particles is no longer a natural assumption for particles scattered on external ‘heavy’ scattering centres. In our opinion this till now neglected fact deserves more detailed study because of its potential importance for cosmic ray particle acceleration in turbulent media.

In the present work we consider detailed conditions for the angular scattering probability that is required for reduction of the Fokker-Planck equation describing particle momentum scattering into the momentum diffusion equation. The next section presents general derivation leading to such simplification. Then, in section 3, we present calculation of the Fokker-Planck coefficients for the scattering centres (below we call them simply ‘waves’) moving with velocities much smaller than the particle velocity. We show that the scattering probability distribution, which is - in the scattering wave rest frame - symmetric with respect to the interchange of particle momentum before and after scattering, is required to describe the process as a pure momentum diffusion. A simple mechanical model allowing the breaking this symmetry and yielding a regular acceleration (deceleration) term is discussed in section 4. In the discussion (section 5) we stress the importance of the possibility to have such regular term in the stochastic acceleration process for cosmic ray production in astrophysical sources.

2 The Fokker-Planck approach

As above mentioned the approach based on the Fokker-Planck equation is generally used to describe energetic particle interactions with magnetohydrodynamic turbulence. Following Blandford & Eichler [1] we write — in the phase space — the equation describing the evolution of the isotropic part of particle distribution function $f(r, p, t)$, due to momentum scattering as
\[ \frac{\partial f}{\partial t} = \nabla_p \cdot \left\{ -\frac{\langle \Delta p \rangle}{\Delta t} f + \frac{1}{2} \nabla_p \cdot \left[ \frac{\langle \Delta p \Delta p \rangle}{\Delta t} f \right] \right\} , \tag{2.1} \]

where the Fokker-Planck coefficients, the mean momentum gain rate \( \frac{\langle \Delta p \rangle}{\Delta t} \) represents the Fermi term and the rate for growing momentum dispersion is governed by the term \( \frac{\langle \Delta p \Delta p \rangle}{\Delta t} \); \( \nabla_p \equiv \partial / \partial p \). With a probability for changing a particle momentum in an individual scattering from \( p \) to \( p' \equiv p + \Delta p \) given by the function \( \Psi(p', p) \), with the normalization \( \int \Psi d\Delta p = 1 \), these coefficients are given as

\[ \frac{\langle \Delta p \rangle}{\Delta t} = \frac{1}{\Delta t} \int \Psi(p', p)(p' - p)d\Delta p \tag{2.2} \]

and

\[ \frac{\langle \Delta p \Delta p \rangle}{\Delta t} = \frac{1}{\Delta t} \int \Psi(p', p)(p' - p)^2 d\Delta p \tag{2.3} \]

where \( 1/\Delta t \) is the scattering frequency. If the principle of detailed balance holds, \( \Psi(p', p) = \Psi(p, p') \), we have (Blandford & Eichler [1]):

\[ \nabla_p \left\{ \frac{\langle \Delta p \rangle}{\Delta t} - \frac{1}{2} \nabla_p \cdot \frac{\langle \Delta p \Delta p \rangle}{\Delta t} \right\} = 0 . \tag{2.4} \]

The equilibrium momentum distribution can exist on the condition of the momentum-change velocity vanishing (i.e. a constant vector in parentheses in Eq. 2.4). Then, the Fokker-Planck coefficients relate as

\[ \frac{\langle \Delta p \rangle}{\Delta t} = \frac{1}{2} \nabla_p \cdot \frac{\langle \Delta p \Delta p \rangle}{\Delta t} \tag{2.5} \]

and Eq. (2.1) takes a simple form

\[ \frac{\partial f}{\partial t} = \nabla_p \cdot \{ D_{pp} \cdot \nabla_p f \} , \tag{2.6} \]

where one introduced the momentum diffusion tensor \( D_{pp} \equiv \langle \Delta p \Delta p \rangle / (2\Delta t) \). For the isotropic momentum diffusion \( D_{pp} = D_p I \), where \( D_p \) is the momentum diffusion coefficient and \( I \) is the diagonal unit matrix. Then Eq. (2.1) transforms into the form with the scalar momentum and the operator \( \partial / \partial p \) replacing the vector \( p \) and \( \nabla_p \), respectively, and, accordingly, Eq. (2.6) takes the form
\[
\frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left\{ p^2 D_p \frac{\partial f}{\partial p} \right\} . \tag{2.7}
\]

In the present work we consider detailed conditions required for reduction of Eq. (2.1) into (2.7).

3 Fokker-Planck coefficients and the condition for vanishing a regular acceleration term

Any wave damping or boosting process in rarefied astrophysical plasmas can be usually neglected for MHD waves at time scales relevant for single act of energetic particle interaction with a given wave. Due to this fact the electric field can be assumed to vanish in the wave rest frame (‘wave frame’) and the transport coefficients in the momentum space can be obtained using a straightforward procedure. Let us consider an individual wave (scattering centre) taken from the ensemble of waves with isotropic distribution of velocities. In the wave frame the scattering conserves the particle energy and can be described with the angular scattering cross-section. Thus, the energy gain of any individual particle can be derived from two Lorentz transformations, the transformation from the mean plasma rest frame (‘plasma frame’) to the wave frame before the scattering, and in the opposite direction after the scattering. To derive transport coefficients one simply has to calculate the change in momentum on scattering and to average it over incoming and outgoing pitch angles. This approach was proposed by Duffy [4] and the discussion in this section follows his work in essentials.

Let us take the \(z\)-axis of the considered reference frame along the mean background magnetic field. In the plasma frame we denote an \(X\) quantity before interaction and after interaction with a prime, \(X'\). In the wave frame an index \(w\) is added to respective quantities. Then, we denote with \(V\) a wave velocity assumed to be along the magnetic field\(^1\), with \(v\) being a particle velocity \((v \gg V)\), and with \(p\) and \(\mu\) a particle momentum and a pitch angle cosine, respectively (below, \(\mu\) is called shortly ‘a pitch angle’). The Lorentz transformation of particle momentum into the wave frame gives, to second order in \(V/v\) (as required by the Fokker–Planck approach),

\(^1\text{But the derivations are performed with all angles measured with respect to the direction of the scattering centre velocity and, in principle, has nothing to do with the magnetic field.}\)
\[ p_w = p \left[ 1 - \frac{\mu V}{v} + \frac{1}{2} \left( 1 - \mu^2 \right) \frac{V^2}{v^2} + \frac{\mu^2 V^2}{2 c^2} \right] \quad (3.1) \]

and, to the first order in \( V/v \),

\[ \mu_w = \mu - \left( 1 - \mu^2 \right) \frac{V}{v} \quad . \quad (3.2) \]

The particle momentum change \( \Delta p \equiv p' - p \) at an individual scattering transforming \( \mu \) into \( \mu' \) is

\[ \Delta p = \left[ (\mu' - \mu) \frac{V}{v} + \left( \mu'^2 - \mu^2 \right) \frac{V^2}{2 v^2} + (\mu' - \mu) \frac{V^2}{2 c^2} \right] p \quad (3.3) \]

where the terms of the higher-than-second order in \( V/v \) are neglected. Let us note that in the expression (3.3), there are no zero-order terms in wave velocity. Therefore, in order to derive \( \langle \Delta p \rangle \) and \( \langle (\Delta p)^2 \rangle \) valid up to the second-order in \( V/v \) one should derive the required weighting factors up to the first-order and to the zero-order, respectively. The factor which bears all physical information about the scattering is a probability for the particle pitch angle \( \mu \) to be transformed into \( \mu' \). The scattering process is described here in the wave frame, where we consider the probability \( P_{w}(p_w, \mu_w, \mu'_w) \) for a particle with momentum \( p_w \) to change \( \mu_w \) to \( \mu'_w \) during a time \( \Delta t_w = \Delta t + \text{terms } O(V^2/v^2) \). At a scattering the particle momentum \( p_w \) is assumed to be conserved. To the first order the scattering frequency in the plasma frame is proportional to the Lorentz factor

\[ \frac{\gamma_{w,v_w}}{\gamma v} = 1 - \frac{V}{v} \quad , \quad (3.6) \]

where \( \gamma \equiv \frac{1}{\sqrt{1 - v^2}} \). Thus, expressions for the Fokker-Planck coefficients valid to the second order in \( V/v \) are

\[ \langle \Delta p \rangle = \frac{1}{4} \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' \left( 1 - \mu \frac{V}{v} \right) P(p, \mu, \mu') \Delta p(p, \mu, \mu') \quad (3.8) \]

and

\[ \langle (\Delta p)^2 \rangle = \frac{1}{4} \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' P(p, \mu, \mu') \Delta p^2(p, \mu, \mu') \quad , \quad (3.9) \]
where the probability $P$ should be expressed with the use of $P_w$, by expanding it to the first-order in $V/v$. Let us note that the above coefficients are the ones for the isotropic part of the distribution function, denoted above as $f(p)$. In the wave frame, a number of particle transitions ($\mu_w \rightarrow \mu'_w$) per unit time is proportional to $P_w(p_w, \mu_w, \mu'_w)$ and to the slightly anisotropic particle density $f_w(p_w, \mu_w) = f(p)(d\mu/d\mu_w)$. Therefore, one can write the probability function in the plasma frame as

$$P(p, \mu, \mu') = P_w[p_w(p, \mu), \mu_w(\mu), \mu'_w(\mu')] \frac{d\mu'_w}{d\mu_w} \quad (3.10)$$

With the use of Eq-s (3.1,2) the integrand in (3.8) can be expanded to the first-order as

$$\left(1 - \mu \frac{V}{v}\right) P(p, \mu, \mu') = P_w[p_w(p, \mu), \mu_w(\mu), \mu'_w(\mu')] \left(1 - \mu \frac{V}{v} + 2\mu' \frac{V}{v}\right) - p\mu \frac{V}{v} \frac{\partial P_w(p, \mu, \mu')}{\partial p}$$

$$- (1 - \mu^2) \frac{V}{v} \frac{\partial P_w(p, \mu, \mu')}{\partial \mu} - (1 - \mu'^2) \frac{V}{v} \frac{\partial P_w(p, \mu, \mu')}{\partial \mu'} \quad (3.11)$$

In the integral in (3.9) we need analogous expansion to the zeroth-order in $V/v$, $P(p, \mu, \mu') = P_w(p, \mu, \mu')$, to obtain the momentum diffusion coefficient as

$$D_p = \frac{1}{\Delta t} \frac{p^2}{8} \left(\frac{V}{v}\right)^2 \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' (\mu' - \mu)^2 P_w(p, \mu, \mu') \quad (3.12)$$

In the above formula one can consider the transition probability in the wave rest frame, where the cosmic ray scattering can be described in the most simple way. For example, in the case of scattering on magnetohydrodynamic waves the electric field vanishes in such a frame leading to interactions conserving particle energies.

Let us consider the condition (2.5) required to reduce the acceleration process to the purely diffusive one (Eq. 2.6,8). With the scattering probability given in Eq. (3.11), one can derive the mean momentum change in Eq. (3.8). For a general form of $P_w$ the regular acceleration term should be appended to Eq. (2.7) besides the momentum diffusion term. Then, it takes a more general form.
\[ \frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left\{ p^2 D_{pp} \frac{\partial f}{\partial p} \right\} - \frac{1}{p^2} \frac{\partial}{\partial p} \left\{ p^2 \hat{p}_{\text{reg}} f \right\} , \quad (3.13) \]

where \( \hat{p}_{\text{reg}} \equiv < \Delta p >_{\text{reg}} / \Delta t \) is a regular momentum changing term, equivalent to the regular acceleration if \( \hat{p}_{\text{reg}} > 0 \). With Eq-s (2.5,3.8-9) one obtains

\[ < \Delta p >_{\text{reg}} = -\frac{1}{4} \left( \frac{V}{v} \right)^2 \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' (\mu' - \mu) P_w(p, \mu, \mu') \]

\[-\frac{1}{4} \left( \frac{V}{v} \right)^2 \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' (\mu'^2 - \mu^2) P_w(p, \mu, \mu') \]

\[ + \frac{1}{4} \left( \frac{V}{v} \right)^2 \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' (\mu' - \mu) \left[ (1 - \mu^2) \frac{\partial P_w(p, \mu, \mu')}{\partial \mu} + (1 - \mu'^2) \frac{\partial P_w(p, \mu, \mu')}{\partial \mu'} \right] \]

\[ + \frac{1}{8} p \left( \frac{V}{v} \right)^2 \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu' (\mu'^2 - \mu^2) \frac{\partial P_w(p, \mu, \mu')}{\partial p} . \quad (3.14) \]

The analytic condition for vanishing of the term (3.14) is the symmetry of the scattering probability \( P_w \) with respect to the interchange of \( \mu \) and \( \mu' \)

\[ P_w(p, \mu, \mu') = P_w(p, \mu', \mu) . \quad (3.15) \]

If the principle of detailed balance is satisfied, the above condition must also be held (in general, the opposite is not always true). Then, for two opposite scattering acts (\( \mathbf{p} \to \mathbf{p}' \)) and (\( \mathbf{p}' \to \mathbf{p} \)) the respective momentum gain and loss terms cancel each other exactly and any net energy change comes from different particle densities near \( \mathbf{p} \) and \( \mathbf{p}' \). In most cases of ordinary scattering processes the condition (3.15) is trivially satisfied and the momentum diffusion fully describes the acceleration process (cf. Drury [3]). In order to illustrate the physical reasons for non-vanishing regular term \( \hat{p}_{\text{reg}} \), in the following section a simple mechanical example with asymmetric particle scattering is presented.

4 An example with a non-vanishing regular acceleration term

As a simple illustration for the scattering process with the cross-section showing the forward-backward asymmetry we consider the elastically reflecting regular tetrahedrons (i.e. triangular pyramids with all 4 walls being the equal triangles), moving always with one of its bases directed forward and perpendicular to its direction of motion. Let us measure all angles with respect to the
base normal, pointing by definition into \( \mu = 1 \). We consider particle reflections in the tetrahedron rest frame. If we take a stream of particles hitting the base with \( \mu = -1 \), then the reflected particles will be characterized with pitch angle \( \mu' = 1 \). However, if a stream hitting the tetrahedron will have originally \( \mu = 1 \), then no one of particles reflected from the inclined walls of tetrahedron will be scattered into \( \mu' = -1 \). This simple example demonstrates the breaking of the condition (3.15). Next, let us derive the diffusion coefficient and, both, the diffusive and the regular acceleration term existing for such scattering centres. From numerical integration of Eq-s (3.12) and (3.14), with the reflection probability defined by the tetrahedron geometric shape, we obtain the following transport coefficients

\[
D_p = 0.0813 \left( \frac{V}{v} \right)^2 p^2 \quad (4.1)
\]

\[
\dot{p}_{\text{diff}} = 0.1626 \left( \frac{V}{v} \right)^2 p \quad (4.2)
\]

\[
\dot{p}_{\text{reg}} = 0.0043 \frac{V}{v} p + 0.0004 \left( \frac{V}{v} \right)^2 p, \quad (4.3)
\]

where we put \( \Delta t = 1 \). In expressions (3.14) and (4.3) for \( \dot{p}_{\text{reg}} \) one may note the existence of terms representing both the first-order and the second-order regular acceleration. The occurrence of the term linear in \( V/v \) is of particular interest in any application of the theory to particle acceleration. In our example, for small \( V/v \), this term can dominate over another terms.

5 Final remarks

In the present work we considered conditions required for reduction of the stochastic particle momentum scattering process into a pure momentum diffusion. Our considerations were intended to translate this rather general statement into the terms of scattering probability in particle pitch angle and transport coefficients derived to the second-order in \( V/v \), as required by the Fokker-Planck approach. The derived formula requires an equal probability in the wave frame to scatter a particle from the pitch angle \( \mu \) to \( \mu' \) and from \( \mu' \) to \( \mu \). It is equivalent, after averaging over the particle phase angle, to assumption of the detailed balance principle. We also derived an expression for a regular acceleration term occurring when the symmetry of the above probability is broken. In the previous section we illustrate with a simple example the conditions allowing for the regular acceleration term to exist. The resulting coefficient
$\dot{p}_{reg}$ can consist of the linear in the scattering centre velocity term as well as the second-order term proportional to $(V/v)^2$. In astrophysical conditions one usually has $V \ll c$. Thus the non-vanishing linear term could provide a fast way to accelerate particles in the first-order process, with the regular term up to three orders of magnitude larger than the diffusive term.

The importance of the present derivation depends on the possibility to occur conditions allowing to break the scattering symmetry in the wave frame (Ostrowski & Siemieniec-Oziębło, in preparation). In this respect one can consider regions of space with MHD turbulence influenced by the strong external driving force and leading to high-amplitude waves. Thus, particles can be scattered by non-linear asymmetric waves, allowing for different particle distributions resulting from head-on and tail-on collisions. We expect such conditions to occur near high Mach number shock fronts as well as in boundary regions of high velocity jets penetrating through the cold medium.

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