A derivation of sharp Moser-Trudinger-Onofri inequality from fractional Sobolev inequality

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Abstract

We derive sharp Moser-Trudinger-Onofri inequalities on standard $n$-sphere and $2n+1$ CR sphere as the limit of sharp fractional Sobolev inequalities for all $n \geq 1$, respectively. On 2-sphere and 4-sphere, this was established recently by [Chang, S.-Y. Alice; Wang, F.: J. Funct. Anal. 274 (2018), no. 4, 1177–1201.]. Our proof is elementary and much simple.

1 Introduction

In [17], E. Onofri proved the sharp Moser-Trudinger inequality on unit 2-sphere

$$\ln \int_{S^2} e^{2w} \, d\mu_{g_0} \leq \int_{S^2} |\nabla w|^2 \, d\mu_{g_0} + 2 \int_{S^2} w \, d\mu_{g_0} \quad \text{for } w \in W^{1,2}(S^2),$$

where $g_0$ is the standard metric and $\int_{S^2} d\mu_{g_0} = \frac{1}{|S^2|} \int_{S^2} d\mu_{g_0}$. Onofri’s proof based on a version of Moser-Trudinger inequality due to T. Aubin [1] which holds under the additional constraint $\int_{S^2} e^{2w} x \, d\mu_{g_0} = 0$, $x \in \mathbb{R}^3$; see C. Gui and A. Moradifam [15] for the proof of sharp form of Aubin’s inequality which was conjectured by S.-Y. Chang and P. Yang [11]. Till now, there have been several different proofs of the Moser-Trudinger-Onofri inequality. A collection of them can be found in the survey J. Dolbeault, M. J. Esteban, and G. Jankowiak [13]. In [18], Y. Rubinstein gave a Kähler geometry proof of the sharp inequality and obtained an optimal extension of it to higher dimensional Kähler-Einstein manifolds. Rubinstein’s proof based on earlier results of W. Ding and G. Tian [12] and G. Tian [19]. On general dimensional spheres $S^n$, Moser-Trudinger-Onofri inequality was established by T. Branson, S.-Y. Chang and P. Yang [4] and W. Beckner [2] for $n = 4$, and by [2] for all $n \geq 1$ based on the fundamental paper of Lieb [16].

Recently, S.-Y. Chang and F. Wang [10] derived the sharp Moser-Trudinger-Onofri inequality on 2 and 4 spheres by ‘differentiating’ the sharp fractional power Sobolev inequality at the endpoint, which was motivated by a dimensional continuation argument of T. Branson. To justify the differentiation, they used the extension formula of nonlocal conformally invariant operators, which was first introduced by L. Caffarelli and L. Silvestre [6] on Euclidean spaces, and later generalized to operators defined on the boundary of conformally compact Einstein manifolds by

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S.-Y. Chang and M. González [9], and J. Case and S.-Y. Chang [8]. On the dual side, E. Carlen and M. Loss [7] differentiated the sharp Hardy-Littlewood-Sobolev at the endpoint to obtain the sharp logarithmic HLS, which in turn implies the sharp Moser-Trudinger-Onofri inequality. In the final remark of [10], they commented that it is plausible that their arguments can be applied to other dimensions, but the arguments would become increasingly delicate when $n$ is large.

In this paper, we find a very simple way to compute the differentiation which is universal for the dimensions. Consequently, we derive sharp Moser-Trudinger-Onofri inequality as the limit case of the fractional power Sobolev inequalities on $\mathbb{S}^n$ for all $n \geq 1$.

This argument works in the CR setting, too. In this situation, a sharp Moser-Trudinger-Onofri inequality on CR sphere $\mathbb{S}^{2n+1}$ was discovered by T. Branson, L. Fontana and C. Morpurgo [5] after introducing the $A_Q'$ operator of order $Q = 2n + 2$. On the other hand, R. Frank and E. Lieb [14] proved a sharp fractional Sobolev inequality as a corollary of their sharp HLS inequality. [14] also proved the limiting cases of HLS by differentiating HLS at the endpoints; see Corollary 2.4 and Corollary 2.5. We derive the Moser-Trudinger-Onofri inequality of [5] by differentiating the sharp fractional Sobolev inequalities of [14] at the larger endpoint. The way of differentiation is slightly different. We need to split the best constant of fractional Sobolev inequality and rearrange them into both sides.

In the next section, we extend [10] to all dimensions $n \geq 1$ by a different approach. In section 3, we prove the analogue in the CR spheres setting.

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## 2 Standard spheres setting

Let $n \geq 1$, $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit $n$-dimensional sphere. For $0 < \gamma < n/2$, let

$$P_\gamma = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(n - 1\right)^2},$$

where $\Delta_{g_0}$ is the Laplace-Beltrami operator on $\mathbb{S}^n$ with respect to the standard induced metric $g_0$ from $\mathbb{R}^{n+1}$; see T. Branson [3]. Let $Y^{(k)}$ be a spherical harmonic of degree $k \geq 0$. Then we have

$$B\left(Y^{(k)}\right) = \left(k + \frac{n-1}{2}\right) Y^{(k)} \quad \text{and} \quad P_\gamma\left(Y^{(k)}\right) = \frac{\Gamma\left(k + \frac{n}{2} + \gamma\right)}{\Gamma\left(k + \frac{n}{2} - \gamma\right)} Y^{(k)}.$$

(1)

Let

$$P_{n/2} = \begin{cases} \prod_{k=0}^{n/2} (-\Delta_{g_0} + k(n - k - 1)) & \text{for even } n, \\ (-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2)^{1/2} \prod_{k=0}^{n-3} (-\Delta_{g_0} + k(n - k - 1)) & \text{for odd } n. \end{cases}$$

The sharp Sobolev inequality on $\mathbb{S}^n$ asserts that

$$Y(n, \gamma) \left(\int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\gamma}} \, d\mu_{g_0}\right)^{\frac{n-2\gamma}{n}} \leq \int_{\mathbb{S}^n} vP_\gamma(v) \, d\mu_{g_0} \quad \text{for } v \in C^\infty(\mathbb{S}^n),$$

(2)
where $Y(n, \gamma) := \frac{\Gamma\left(\frac{n+\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)}$ and $\int_{S^n} d\mu_{g_0} = \frac{1}{|S^n|} \int_{S^n} d\mu_{g_0}$. The sharp Moser-Trudinger-Onofri inequality asserts that

$$\frac{2(n-1)!}{n} \ln \int_{S^n} e^{nw} d\mu_{g_0} \leq \int_{S^n} wP_{n/2}w + 2(n-1)!w d\mu_{g_0} \quad \text{for } w \in C^\infty(S^n).$$

(3)

See W. Beckner [2] for the proofs of the both inequalities. Recently, S.-Y. Chang and F. Wang [10] studied the limit of (2) when $n = 2$ and $n = 4$. By using the extension formula of fractional Laplacian and proper choosing defining functions, they derived (3) from (2) in these two dimensions.

As $n = 2$ and $n = 4$ in [10], we have

**Proposition 1.** For any $w \in C^\infty(S^n)$, let $v = e^{(\frac{n}{2}-\gamma)w}$. Denote

$$LHS_\gamma := \frac{4}{(n-2\gamma)^2} Y(n, \gamma) \left[ \left( \int_{S^n} |v|^{\frac{2n}{n-2\gamma}} d\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - \int_{S^n} |v|^2 d\mu_{g_0} \right]$$

and

$$RHS_\gamma = \frac{4}{(n-2\gamma)^2} \left[ \int_{S^n} vP_{\gamma}(v) d\mu_{g_0} - Y(n, \gamma) \int_{S^n} |v|^2 d\mu_{g_0} \right].$$

Then

$$\lim_{\gamma \to n/2} LHS_\gamma = \frac{2(n-1)!}{n} \ln \int_{S^n} e^{n(w-\bar{w})} d\mu_{g_0}$$

(4)

and

$$\lim_{\gamma \to n/2} RHS_\gamma = \int_{S^n} wP_{n/2}w d\mu_{g_0},$$

(5)

where $\bar{w}$ is the average of $w$ over $S^n$.

Consequently, we immediately have

**Theorem 2.** We can derive the sharp Moser-Trudinger-Onofri inequality (3) from the sharp Sobolev inequality (2) by sending $\gamma \to \frac{n}{2}$.

**Proof of Proposition 1.** The proof of (4) essentially follows from the proof of Lemma 3.1 of [10]. Note that

$$\left( \int_{S^n} e^{nw} d\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - \int_{S^n} e^{(n-2\gamma)w} d\mu_{g_0} = \left( \int_{S^n} e^{nw} d\mu_{g_0} \right)^{\frac{n-2\gamma}{n}} - 1 - \int_{S^n} (e^{(n-2\gamma)w} - 1) d\mu_{g_0}.$$

Then by L'Hôpital's rule

$$\lim_{\gamma \to n/2} LHS_\gamma = 2\Gamma(n) \left( \frac{1}{n} \ln \int_{S^n} e^{nw} d\mu_{g_0} - \int_{S^n} w d\mu_{g_0} \right)$$

$$= \frac{2(n-1)!}{n} \ln \int_{S^n} e^{n(w-\bar{w})} d\mu_{g_0}.$$
Therefore, (4) is proved.

To prove (5), in terms of Taylor expansion of the exponential function we let

\[ v = e^{\frac{n-2\gamma}{2}w} = 1 + \left(\frac{n}{2} - \gamma\right)w + (n-2\gamma)^2 f, \]

where \( f = \frac{1}{8}w^2 \int_0^1 (1-s)e^{\frac{n-2\gamma}{2}ws} \, ds \in C^\infty(S^n) \) is uniformly bounded in \( C^{2n} \) norm as \( \gamma \to n/2 \). Then we see that

\[
\begin{align*}
\int_{\mathbb{S}^n} v P_\gamma(v) \, d\mu_{g_0} & = \int_{\mathbb{S}^n} (1 + (\frac{n}{2} - \gamma)w + (n-2\gamma)^2 f)(P_\gamma(1) + (\frac{n}{2} - \gamma)P_\gamma(w) + (n-2\gamma)^2 P_\gamma(f)) \, d\mu_{g_0} \\
& = \int_{\mathbb{S}^n} (Y(n, \gamma) + (n-2\gamma)Y(n, \gamma)w + 2(n-2\gamma)^2 Y(n, \gamma)f + (\frac{n}{2} - \gamma)^2 wP_\gamma w) \, d\mu_{g_0} \\
& \quad + O((n-2\gamma)^3),
\end{align*}
\]

where we have used the self-adjointness of \( P_\gamma \) and \( P_\gamma(1) = Y(n, \gamma) \), and that

\[
Y(n, \gamma)\int_{\mathbb{S}^n} |v|^2 \, d\mu_{g_0} = Y(n, \gamma)\int_{\mathbb{S}^n} (1 + 2(\frac{n}{2} - \gamma)w + 2(n-2\gamma)^2 f + (\frac{n}{2} - \gamma)^2 w^2 + O((n-2\gamma)^3)) \, d\mu_{g_0}.
\]

It follows that

\[
\begin{align*}
\int_{\mathbb{S}^n} v P_\gamma(v) \, d\mu_{g_0} - Y(n, \gamma)\int_{\mathbb{S}^n} |v|^2 \, d\mu_{g_0} & = (\frac{n}{2} - \gamma)^2 \int_{\mathbb{S}^n} wP_\gamma w \, d\mu_{g_0} + O((n-2\gamma)^3).
\end{align*}
\]

Let \( w = \sum_{k=0}^\infty Y^{(k)} \), where \( Y^{(k)} \) are spherical harmonics of degree \( k \). Hence,

\[
\begin{align*}
\int_{\mathbb{S}^n} w P_\gamma w \, d\mu_{g_0} & = \sum_{k=0}^\infty \frac{\Gamma(k + \frac{n}{2} + \gamma)}{\Gamma(k + \frac{n}{2} - \gamma)} \int_{\mathbb{S}^n} |Y^{(k)}|^2 \, d\mu_{g_0} \\
& \quad \to \sum_{k=1}^\infty \frac{\Gamma(k + n)}{\Gamma(k)} \int_{\mathbb{S}^n} |Y^{(k)}|^2 \, d\mu_{g_0} = \int_{\mathbb{S}^n} w P_{n/2} w \, d\mu_{g_0},
\end{align*}
\]

as \( \gamma \to \frac{n}{2} \), where we have used (1) in the first identity, the definition of \( P_{n/2} \) in the second one and have used the smoothness of \( w \) to ensure the convergence. Therefore, (5) follows.

Proposition 1 is proved.

\[ \square \]

3 CR spheres setting

Following T. Branson, L. Fontana and C. Morpurgo [5], we let \( \mathcal{H}_{j,k} \) be the space of harmonic polynomials of bidegree \((j,k)\) on CR sphere \( S^{2n+1} \), \( j, k = 0, 1, \ldots \); such spaces make up for the
standard decomposition of $L^2$ into $U(n + 1)$-invariant and irreducible subspaces, where $n \geq 1$. For $0 < d < Q := 2n + 2$, let $\mathcal{A}_d$ be the intertwining operator of order $d$ on CR sphere $\mathbb{S}^{2n+1}$, characterized by

$$\mathcal{A}_d Y^{(j,k)} = \lambda_j(d) \lambda_k(d) Y^{(j,k)}, \quad \lambda_j(d) = \frac{\Gamma(j + \frac{Q+d}{2})}{\Gamma(j + \frac{Q-d}{4})}$$

for every $Y^{(j,k)} \in \mathcal{H}_{j,k}$. When $d = 2$, it gives the CR invariant sub-Laplacian. Let

$$\mathcal{P} := \bigoplus_{j>0} \mathcal{H}_{j,0} \bigoplus \mathcal{H}_{0,j} \bigoplus \mathcal{H}_{0,0}.$$ 

Let $\mathcal{A}'_Q$ be the operator acting on CR-pluriharmonic functions as

$$\mathcal{A}'_Q F = \Pi_{\ell=0}^{n} \left( \frac{2}{n} \mathcal{L} + \ell \right) F = \lim_{d \to Q} \frac{1}{\lambda_0(d)} \mathcal{A}_d F, \quad \forall \ F \in C^\infty(\mathbb{S}^{2n+1}) \cap \mathcal{P},$$

where $\mathcal{L}$ is the sub-Laplacian operator. See Proposition 1.2 of [5].

The sharp Moser-Trudinger-Onofri inequality on CR $\mathbb{S}^{2n+1}$ proved by [5] asserts that

$$\frac{n!}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{QF} \leq \int_{\mathbb{S}^{2n+1}} FA'_Q F + \frac{n!}{Q} \int_{\mathbb{S}^{2n+1}} F \quad \text{for} \ F \in C^\infty(\mathbb{S}^{2n+1}) \cap \mathcal{P}. \quad (8)$$

(It is called Beckner-Onofri inequality in [5].) By duality, the sharp Hardy-Littlewood-Sobolev inequality on CR $\mathbb{S}^{2n+1}$ due to R. Frank and E. Lieb [14] yields that

$$\lambda_0(d)^2 \left( \int_{\mathbb{S}^{2n+1}} |v|^{\frac{2Q}{Q-d}} \right)^{\frac{Q-d}{Q}} \leq \int_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) \quad \text{for} \ v \in C^\infty(\mathbb{S}^{2n+1}). \quad (9)$$

**Proposition 3.** For any $F \in C^\infty(\mathbb{S}^{2n+1}) \cap \mathcal{P}$, let $v = e^{\frac{Q-d}{Q}F}$. Denote

$$LHS_d := \frac{4}{(Q-d)^2} \lambda_0(d) \left[ \left( \int_{\mathbb{S}^{2n+1}} |v|^{\frac{2Q}{Q-d}} \right)^{\frac{Q-d}{Q}} - \int_{\mathbb{S}^{2n+1}} |v|^2 \right]$$

and

$$RHS_d = \frac{4}{(Q-d)^2} \lambda_0(d)^{-1} \left[ \int_{\mathbb{S}^{2n+1}} v \mathcal{A}_d(v) - \lambda_0(d)^2 \int_{\mathbb{S}^{2n+1}} |v|^2 \right].$$

Then

$$\lim_{d \to Q} LHS_d = \frac{n!}{Q} \ln \int_{\mathbb{S}^{2n+1}} e^{Q(F-\bar{F})} \quad (10)$$

and

$$\lim_{d \to Q} RHS_d = \int_{\mathbb{S}^{2n+1}} FA'_Q F, \quad (11)$$

where $\bar{F}$ is the average of $F$ over $\mathbb{S}^{2n+1}$.
Proof. Note that
\[
\left( \int_{S^{2n+1}} \frac{v^{Q-d}}{|v|^2} \right)^{\frac{Q-d}{Q}} \frac{Q-d}{Q} - \int_{S^{2n+1}} |v|^2
\]
\[
= \left( \int_{S^{2n+1}} e^{QF} \right)^{\frac{Q-d}{Q}} - 1 - \int_{S^{2n+1}} (e^{(Q-d)F} - 1).
\]

Then by L’Hôpital’s rule
\[
\lim_{d \to n/2} LHS_d = \Gamma(n+1) \left( \frac{1}{Q} \ln \int_{S^{2n+1}} e^{QF} - \int_{S^{2n+1}} F \right)
\]
\[
= \frac{n!}{Q} \ln \int_{S^{2n+1}} e^{Q(F-F)}.
\]
Therefore, (10) is proved.

To prove (11), in terms of Taylor expansion of the exponential function we let
\[
v = e^{\frac{Q-d}{2}F} = 1 + \frac{1}{2}(Q - d)F + (Q - d)^2 f,
\]
where \( f = \frac{1}{8} F^2 \int_0^1 (1-s)e^{\frac{Q-d}{2}Fs} ds \in C^\infty(S^{2n+1}) \) is uniformly bounded in \( C^{4n} \) norm as \( d \to Q \).
Then we see that
\[
\int_{S^{2n+1}} v A_d(v)
\]
\[
= \int_{S^{2n+1}} (1 + \frac{1}{2}(Q - d)F + (Q - d)^2 f)(A_d(1) + \frac{1}{2}(Q - d)A_d(F) + (Q - d)^2 A_d(f))
\]
\[
= \int_{S^{2n+1}} \left( \lambda_0(d)^2 + (Q - d)\lambda_0(d)^2 F + 2(Q - d)^2 \lambda_0(d)^2 f
\]
\[
+ \frac{1}{4}(Q - d)^2 F A_d F + (Q - d)^3 f A_d F \right) + O((Q - d)^4),
\]
where we have used the self-adjointness of \( A_d \) and \( A_d(1) = \lambda_0(d)^2 \), and that
\[
\lambda_0(d)^2 \int_{S^{2n+1}} |v|^2
\]
\[
= \lambda_0(d)^2 \int_{S^{2n+1}} \left( 1 + (Q - d)F + 2(Q - d)^2 f + \frac{1}{2}(Q - d)^2 F^2 + O((Q - d)^3) \right).
\]
It follows that
\[
\int_{S^{2n+1}} v A_d(v) - \lambda_0(d)^2 \int_{S^{2n+1}} |v|^2
\]
\[
= \frac{1}{4}(Q - d)^2 \int_{S^{2n+1}} (F A_d F + 4(Q - d) f A_d F) + O((Q - d)^4).
\]
By (7), (11) follows immediately.
Therefore, Proposition 3 is proved.
Similarly, we immediately obtain

**Theorem 4.** We can derive the sharp Moser-Trudinger-Onofri inequality (8) from the sharp Sobolev inequality (9) by sending $d \to Q$.

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