National Taras Shevchenko University of Kyiv

Set-membership state estimation framework for uncertain linear differential-algebraic equations

Sergey Zhuk

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Abstract. We investigate a state estimation problem for the dynamical system described by uncertain linear operator equation in Hilbert space. The uncertainty is supposed to admit a set-membership description. We present explicit expressions for linear minimax estimation and error provided that any pair of uncertain parameters belongs to the quadratic bounding set. We introduce a new notion of minimax directional observability and index of non-causality for linear noncausal DAEs. Application of these notions to the state estimation problem for linear uncertain noncausal DAEs allows to derive new minimax recursive estimator for both continuous and discrete time. We illustrate the benefits of non-causality of the plant applying our approach to scalar nonlinear set-membership state estimation problem. Numerical example is presented.

Key words. set-membership state estimation, minimax, uncertain linear equation, DAE, descriptor systems, implicit systems, Kalman filter.

1 Introduction and problem statement

The applications of differential-algebraic equations (DAEs or descriptor systems) in economics, demography, mechanics and engineering are well known [1]. This in turns motivates researchers to investigate DAEs from the mathematical point of view [2]. Here we focus on a design of state estimation algorithm for uncertain linear non-causal DAE.

The most common approach to DAEs investigation is to reduce it to some canonical form which in turn is equal to some normal ODE. In particular, one of the basic results of the algebraic theory of regular linear DAEs with constant matrices\(^1\) was introduced in [3]: if the linear DAE with constant
matrices

\[
F \dot{x} = C x + B f
\]

is well defined \((\det[\lambda F - C] \neq 0)\) then for all initial values \(x(t_0) = x_0\) there exists the unique solution \(x(\cdot)\) provided that \(f(\cdot)\) is sufficiently smooth. The index \(s\) of the pencil \(F, C\) is said to be an index of linear DAE (1). One can reduce (1) to the ODE via change of coordinates so that the pencil \(F, C\) brings into canonical form \([4]\) and differentiating exactly \(s\) times provided that \(f\) is sufficiently smooth. In such a way one can derive an analogue of the celebrated Cauchy formula for the linear regular DAEs with constant matrices. This result is generalized to variable coefficients by means of a standard canonical form (SCF): in \([5]\) it was shown that (1) with analytical \(F, C, B\) is solvable (i.e. for every sufficiently smooth \(f\) there exists at least one continuously differentiable solution to (1) provided \(F, C\) to be sufficiently smooth) if there exists the SCF for (1). Note that in this case rank \(F(t)\) changes only at finite number of points from within any compact \([t_0, T]\).

In \([6]\) it was noted that not all solvable DAEs can be put into SCF and the solvable DAE is equal to some differential-algebraic equation in the canonical form which generalize SCF. In this respect we say that DAE is causal if it can be reduced – at least locally in nonlinear case – into normal ODE. The geometry of the reduction procedure for nonlinear causal DAEs \(F(x, \dot{x}) = 0\) was investigated in \([7, 8]\), where the index of DAE was defined as a smallest natural \(s\) so that the sequence of the constraint manifolds \([7]\]

\[
M_k := TW_{k-1} \cap M_{k-1}, M_0 := \{(x, p) : F(x, p) = 0\},
\]

\[
W_0 := \{x \in \mathbb{R}^n : (x, p) \in M_0\}
\]

becomes stationary for \(k > s\). This coincides with the definition of the index of linear DAE. Further discussion of the DAEs solvability theory and related topics is presented in \([1, 2]\).

The noncausal DAE differs radically from the causal one. For instance, consider

\[
\frac{d}{dt} \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + f(t), \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t_0) = x_0
\]

(2)

Let \(x_2(\cdot) \in L_2(t_0, T), f(\cdot) \in L_1(t_0, T)\) and \(x_0 \in \mathbb{R}\). By definition put

\[
x_1(t) := \exp(c_1(t - t_0))x_0 + \int_{t_0}^t \exp(c_1(t - s))c_2x_2(s) + f(s)ds
\]
It is clear that any solution of (2) is given by the formula \( t \mapsto (x_1(t), x_2(t))^T \). According to a behavioral approach [9] one can think about \( x_2 \) as an input or as a part of the system state representing uncertain inner disturbance generated by the plant itself. In order to clarify this ambiguity we shall give an exact definition of the DAEs solution accepted in this paper. According to [10] \( x(\cdot) \) is said to be a solution of

\[
\frac{d}{dt} Fx(t) = C(t)x(t) + f(t)
\]

with initial condition \( Fx(t_0) = 0 \) if \( Fx(\cdot) \) is totally continuous function, \( x(\cdot) \) satisfies (3) almost everywhere and \( Fx(t_0) = 0 \) holds. This definition allows to properly define the adjoint system. Also it guarantees that the linear mapping induced by (3) is closed in corresponding Hilbert space [10]. Note that this doesn’t hold for (1). Another useful application of the introduced solution is in the control theory. In [11] authors discuss difficulties arising while applying of proportional feedback \( f = Kx \) to the (1): even well defined DAE (\( \det(sF-C) \neq 0 \)) may become singular (\( \det(sF-C-BK) \equiv 0 \)). In [12] a properly stated leading term \( A(t) \frac{d}{dt} F(t)x \) is used in order to give a feedback solution to LQ-control problem with DAE constraints. This generalizes the definition of DAE solution [10] to the case of variable matrices.

Recently solvability conditions for abstract semi-linear non-causal DAE has been studied in [13] assuming that the pencil \( sF - C \) is singular, \( F, C \) are closed linear mappings in abstract Banach space. Properties of the solutions of noncausal implicit differential equation with special structure were discussed in [14]. Note that non-causal DAEs are not just a ”pure” mathematical structure which is suitable for solving control or observation problems only – some potential applications of non-causal DAEs was briefly discussed in [15].

A state estimation framework for linear dynamic models has several widely-used approaches: \( H_2/H_\infty \) filtering and set-membership state estimation. \( H_2 \)-estimators like Kalman or Wiener filters (also known as minimum variance filters [16]) give estimations of the system state with minimum error variance. These filters require an exact model of signal generating process and full information about a statistical nature of noise sources. Recently, the \( H_2 \)-estimation for linear DAEs has been studied in [17]. Authors derive a so-called ”3-block” form for the optimal filter and a corresponding 3-block Riccati equation using the maximum likelihood approach. The obtained recursion is stated in terms of a block matrix pseudoinverse. In [18] the filter
Recursion is represented in terms of a deterministic data fitting problem solution. Authors introduce an explicit form of the 3-block matrix pseudoinverse for a descriptor model with special structure, so that the form of obtained in [18] filter coincides with presented in [17]. A brief overview of steady-state $H_\infty$-estimators is presented in [19]. Optimal $H_\infty$ estimators minimize the 2-induced norm of the operator that maps unknown disturbances with finite energy to filtered errors [20]. In literature it is common to construct suboptimal estimators [21] that guarantee aforementioned norm to be less then a prescribed performance level $\gamma$. Note that $H_\infty$ estimators are certain Krein space $H_2$ filters [22]. Krein space approach was used in [23] for risk-sensitive filtering in linear time-invariant (LTI) descriptor models with regular matrix pencil under stochastic noise. A linear matrix inequality approach was used in [24] in order to construct reduced order $H_\infty$-filter for LTI DAE with regular matrix pencil. An up to date description of the state of the art is to be found at [25].

In the sequel we focus on the following problem: given some element (for instance measurements of the system output) $y$ from some functional space one needs to estimate the expression $\ell(\theta)$ provided that $g(\theta) = 0$. This problem becomes non-trivial if the latter equation has more than one solution and the equality $y = C(\theta)$ holds. In this case the estimation problem may be reformulated as follows: given $y = C(\theta), \theta \in \Theta, y \in Y$ one needs to find the estimation $\hat{\ell}(\theta)$ of the expression $\ell(\theta)$ provided that $g(\theta) = 0$ and $C(\cdot), \ell(\cdot)$ are given functions. Note that $\hat{\ell}(\theta) := \ell(\hat{\theta})$ if the equation $y = C(\theta)$ has the unique solution $\hat{\theta}$.

The estimation problem is said to be linear if $\Theta, Y$ are linear spaces and $C(\cdot), \ell(\cdot)$ are linear mappings. It is a common case when

$$C(\theta) = H\varphi + D\eta, g(\theta) = L\varphi + Bf,$$

where $\theta = (x, f, \eta) \subset X \times F \times Y$, $H, D, L, B$ are linear mappings. The linear estimation problem is said to be uncertain if $D \neq 0$, $L$ and $B$ are non-trivial or if $B = 0$ and $N(L) = \{\varphi : L\varphi = 0\} \neq \{0\}$. Note that the choice of the solution method depends on the “type of uncertainty”: if $f, \eta$ denote realizations of random elements then it’s natural to apply probability methods. This requires an a priori knowledge of distribution characteristics of the random elements. In the sequel we assume that there is uncertainty
in (*) if distributions of random elements or some deterministic parameters of the system are partially unknown. It is natural to choose the estimation from some class to be optimal in the sense of the given criteria. According to this the linear uncertain estimation problem is said to be minimax if the class of estimations $\hat{\ell}(\varphi)$ is restricted to all linear functions $(u, y) + c$ of $y$ and the criteria is set to be the minimum of the worst-case error. A description of the state of the art in the theory of linear uncertain minimax estimation problems with special $\ell, L, H, B, D$ in special spaces is to be found at [26, 27, 28, 29, 30, 31, 32, 33, 34].

1.0.1 Author’s own research activities

Classical theory of uncertain estimation problems [26]-[29] works well when the linear mapping $L$ in (*) has bounded inverse. One of the author’s theoretical achievements is the extension of the linear minimax estimation theory to abstract equations with closed linear non-injective mapping [35, 36, 37, 38] in Hilbert space. This extension is based on the general duality principle asserting that the linear minimax estimation problem is equal to some control problem with convex non-smooth cost and linear constraints provided that uncertain parameters belong to closed bounded convex sets in corresponding Hilbert spaces [39, 40, 36]. Note, that these results were previously obtained for the finite dimensional space [41, 42, 43, 44, 44, 45].

In order to apply the abstract theory to DAEs the sufficient conditions on DAEs matrices were introduced asserting that the linear mapping induced by the noncausal DAE is closed and has closed range [10]. Also a generalization of the integration by parts formula and the necessary and sufficient conditions of solvability of DAE in the form (3) is presented in [10]. The solvability condition is obtained via application of Tikhonov regularization approach. With help of these results new notions of the minimax directional observability and index of causality for discrete time [46] and continuous time linear non-causal DAEs [47] were introduced. The minimax directional observability provides a qualitative description of DAEs singularity with a respect to the given observations. Using this notion author developed several representations of the minimax estimation [48, 49, 50]. The final result is an algorithm which allows to compute the minimax estimation of the linear noncausal DAE state in the real time [47]. The structure of this algorithm coincides with celebrated Kalman filter recursions for normal linear ODEs with continuous time. Similar results were obtained for linear noncausal
DAEs with discrete time [51, 52, 53, 54, 55, 56, 46]. In [46] the author gives a complete solution to the problem of recursive implementation of the minimax a-posteriori estimation (similar to posed in [28]) for the linear non-causal DAEs with discrete time.

Also the theory developed in [36] was applied to the state estimation of the solutions of finite-dimensional linear boundary-value problems [57] and the minimax mean-square estimations of trends [58].

Notation. \( c(G, \cdot) = \sup\{ (z, f), f \in G \} \), 
\( \delta(G, x) = 0 \) if \( x \in G \) and \( +\infty \) otherwise,
\( \text{dom} f = \{ x \in H : f(x) < \infty \} \),
\( f^*(x^*) = \sup_x \{ (x^*, x) - f(x) \} \), \( (L^*c)(u) = \inf\{ c(G, z), L^*z = u \} \),
\( (fL)(x) = f(Lx) \), \( (L^*c)(u) = \inf\{ c(G, z), L^*z = u \} \),
\( c f = f^{**} \),
\( \text{Arginf}_u f(u) \) denotes the set of minimum points of \( f \),
\( \partial f(x) \) denotes the sub-differential of \( f \) at \( x \),
\( (\cdot, \cdot) \) denotes the inner product in Hilbert space,
\( S > 0 \) means \( (Sx, x) > 0 \) for all \( x \),
\( L^* \) denotes adjoint operator,
\( P_{L^*} \) denotes the orthogonal projector onto \( R(L^*) \),
\( R(L), N(L) \) and \( \mathcal{D}(L) \) denote the range, the null-space and the domain of the linear mapping \( L \),
\( F' \) denotes transposed matrix,
\( F^+ \) denotes pseudoinverse matrix,
\( E \) denotes the identity matrix,
\( \text{diag}(A_1 \ldots A_n) \) denotes diagonal matrix with \( A_i, \ i = 1, n \) on its diagonal,
\( \overline{G} \) denotes the closure of the set \( G \),
\( [x_1, \ldots, x_n] \) denotes an element of the Cartesian product \( H_1 \times \cdots \times H_n \) of Hilbert spaces \( H_i, i = 1, n \),
\( \mathbb{R}^n \) denotes \( n \)-dimensional arithmetic Hilbert space,
\( C^{m \times n}(t_0, T) \) denotes the space of all continuous on \( (t_0, T) \) functions with values in \( \mathbb{R}^{m \times n} \),
\( \mathbb{L}_2(t_0, T) \) denotes the space of all measurable functions with finite integral \( \int_{t_0}^T f^2 dt \),
\( \mathbb{W}_2^m(t_0, T) \) denotes the space of all absolutely continuous functions with derivative from \( \mathbb{L}_2(t_0, T) \),
\( M\xi \) denotes expected value of the random vector \( \xi \).
2 Linear uncertain estimation problem

In this section we present the main result of the paper [36]. All proofs are given in [36].

Suppose that \( L\varphi \in \mathcal{G} \) and

\[ y = H\varphi + \eta \]  

(4)

The mappings \( L, H \) and the set \( \mathcal{G} \) are supposed to be given. The element \( \eta \) is uncertain. Our aim is to solve the inverse problem: to construct the operator mapping the given \( y \) into the estimation \( \hat{\ell}(\varphi) \) of expression \( \ell(\varphi) \) and to calculate the estimation error \( \sigma \). Now let us introduce some definitions.

The operator \( L : \mathcal{H} \mapsto \mathcal{F} \) is assumed to be closed. Its domain \( D(L) \) is supposed to be a dense subset of the Hilbert space \( \mathcal{H} \), \( H \in \mathcal{L}(\mathcal{H}, \mathcal{Y}) \). Note that the condition \( L\varphi \in \mathcal{G} \) is equal to the following

\[ L\varphi = f, \]  

(5)

where \( f \) is uncertain and belongs to the given subset \( \mathcal{G} \) of the Hilbert space \( \mathcal{F} \). In the sequel \( \eta \) is supposed to be a random \( \mathcal{Y} \)-valued vector with zero mean so that its correlation \( R_\eta \in \mathcal{R} \), where \( \mathcal{R} \) is some subset of \( \mathcal{L}(\mathcal{Y}, \mathcal{Y}) \). Also we deal with deterministic \( \eta \) so that \((f, \eta) \in \mathcal{G}\), where \( \mathcal{G} \) is some subset of \( \mathcal{F} \times \mathcal{Y} \). Note that the realization of \( y \) depends on \( \eta, H \) and \( f \). Also it depends on elements of \( N(L) = \{ \varphi \in \mathcal{D}(L) : L\varphi = 0 \} \) so that \( y = H(\varphi_0 + \varphi) + \eta \), where \( \varphi_0 \) may be thought as inner noise in the state model (5).

Let \( \ell(\varphi) = (\ell, \varphi), \hat{\ell}(\varphi) = (u, y) + c \). Since \( L, H \) are not supposed to have a bounded inverse mappings the \( \ell(\varphi) \) and \( \hat{\ell}(\varphi) \) are not stable with a respect to small deviations in \( f, \eta \). Also \( f, \eta \) are supposed to be uncertain. Therefore we use the minimax design in order to construct the estimation.

**Definition 1.** The function \( \hat{\ell}(\varphi) = (\hat{u}, \cdot) + \hat{c} \) is called the a priori minimax mean-squared estimation iff \( \sigma(\ell, \hat{u}) = \inf_{u,c} \sigma(\ell, u) \) where

\[ \sigma(\ell, u) := \sup_{L\varphi \in \mathcal{G}, R_\eta \in \mathcal{R}} M(\ell(\varphi) - \hat{\ell}(\varphi))^2 \]  

(6)

The number \( \hat{\sigma}(\ell) = \sigma^{1/2}(\ell, \hat{u}) \) is said to be the minimax mean-squared error in the direction \( \ell \).
On the other hand the a posteriori estimation describes the evolution of the central point of the system reachability set
\[(L\varphi, y - H\varphi) \in \mathcal{G}\]
consistent with measured output \(y\) \([26, 28, 27]\). Note that the condition \((L\varphi, y - H\varphi) \in \mathcal{G}\) holds if \(\|y\| < C\) for some real \(C\). But it doesn’t hold in our assumptions if \(\eta\) is random since \(\|R\eta\| < c\) doesn’t imply \(\|y\| < C\) for realizations of \(\eta\). Therefore \(\eta\) is supposed to be deterministic.

**Definition 2.** The set
\[\mathcal{X}_y = \{\varphi \in \Phi(L) : (L\varphi, y - H\varphi) \in \mathcal{G}\}\]
is called an a posteriori set. The vector \(\hat{\varphi}\) is said to be minimax a posteriori estimation of \(\varphi\) in the direction \(\ell\) (\(\ell\)-minimax estimation) iff
\[\hat{d}(\ell) := \inf_{\varphi \in \mathcal{X}_y} \sup_{\psi \in \mathcal{X}_y} |(\ell, \varphi) - (\ell, \psi)| = \sup_{\psi \in \mathcal{X}_y} |(\ell, \hat{\varphi}) - (\ell, \psi)|\]
The expression \(\hat{d}(\ell)\) is called the minimax a posteriori error in the direction \(\ell\) (\(\ell\)-minimax error).

In the sequel the minimax mean-squared a priori estimation (error) is referred as minimax estimation (error).

**Proposition 1.** Assume that \(\mathcal{G}, \mathcal{R}\) are convex bounded closed subsets of \(\mathcal{F}, \mathcal{L}(\mathcal{Y}, \mathcal{Y})\) respectively. For the given \(\ell \in \mathcal{H}\) the minimax error \(\hat{\sigma}(\ell)\) is finite iff
\[\ell - H^*u \in \text{dom cl}(L^*c) \cap (-1)\text{dom cl}(L^*c)\]  
for some \(u \in \mathcal{Y}\). Under this condition
\[\sigma(\ell, u) = \sup_{R\eta \in \mathcal{R}} (R\eta u, u) + \frac{1}{4}[\text{cl}(L^*c)(-\ell + H^*u) + \text{cl}(L^*c)(-\ell + H^*u)]^2\]  
where
\[R(L^*) \subset \text{dom cl}(L^*c) \subset \overline{R(L^*)}\]
If \(\text{Arginf}_u \sigma(\ell, u) \neq \emptyset\), then \(\hat{\ell}(\varphi) = (\hat{u}, y) + \hat{c}\), where
\[\hat{u} \in \text{Arginf}_u \sigma(\ell, u)\]
and
\[ \hat{c} = \frac{1}{2}(\text{cl}(L^*c)(\ell - H^*\hat{u}) - \text{cl}(L^*c)(-\ell + H^*\hat{u})) \]

**Theorem 1.** Suppose that \( G \) is convex bounded closed balanced set and \( 0 \in \text{int} \ G \). Also assume that
\[ \eta \in \{ \eta : M(\eta, \eta) \leq 1 \} \]
Then for the given \( \ell \in \mathcal{H} \) the minimax estimation \( \hat{\sigma}(\ell) \) is finite iff \( \ell - H^*u \in R(L^*) \) for some \( u \in \mathcal{Y} \). Under this condition there exists a unique minimax estimation \( \hat{u} \) and
\[ \sigma(\ell, \hat{u}) = \min_u \sigma(\ell, u), \]
\[ \sigma(\ell, u) = (u, u) + \min_z \{ c^2(G, z), L^*z = \ell - H^*u \} \]  
(9)

If \( R(L), H(N(L)) \) are closed sets then \( \hat{u} \) is determined by the following conditions
\[ \hat{u} - H\hat{p}_0 \in H(\partial I_2(H^*\hat{u})), L\hat{p}_0 = 0, \]
\[ I_2(w) = \min_z \{ c^2(G, z), L^*z = P_L(\ell - w) \}, \]  
(10)

**Corollary 1.** Let
\[ \mathcal{G} = \{ f \in \mathcal{F} : (f, f) \leq 1 \}, \eta \in \{ \eta : M(\eta, \eta) \leq 1 \}, \]
and suppose that
1) \( R(L), H(N(L)) \) are closed sets;
2) \( R(T) = \{ [Lx, Hx], x \in \mathcal{D}(L) \} \) is closed set.
Then the unique minimax estimation \( \hat{u} \) is given by \( \hat{u} = H\hat{p} \) provided that \( \ell \in R(L^*) + R(H^*) \), \( \hat{p} \) obeys
\[ L^*\hat{z} = \ell - H^*H\hat{p}, \]
\[ L\hat{p} = \hat{z} \]  
(11)
The minimax error is given by the following expression
\[ \hat{\sigma}(\ell) = (\ell, \hat{p})^{\frac{1}{2}} \]
Corollary 2. Assume that linear mappings $L : \mathcal{H} \mapsto \mathcal{F}$, $H \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ obey 1) or 2) (Cor. 1). Then (11) has a solution $\hat{z} \in \mathcal{D}(L^*)$, $\hat{p} \in \mathcal{D}(L)$ iff $\ell = L^*z + H^*u$ for some $z \in \mathcal{D}(L^*)$, $u \in \mathcal{Y}$.

Corollary 3. Under conditions of Cor. 1 for any $\ell \in R(L^*) + R(H^*)$ and some realization of $y(\cdot)$ we have $(\hat{u}, y) = (\ell, \hat{\phi})$, where $\hat{\phi}$ obeys
\begin{equation}
L^*\hat{q} = H^*(y - H\hat{\phi}),
L\hat{\phi} = \hat{q} \tag{12}
\end{equation}

Consider an a posteriori estimation.

Proposition 2. Let $\mathcal{G}$ be a convex closed bounded subset of $\mathcal{Y} \times \mathcal{F}$. Then
\begin{equation}
R(L^*) + R(H^*) \subset \text{dom} c(\mathcal{X}_y, \cdot) \cap (\text{-1})\text{dom} c(\mathcal{X}_y, \cdot) \subset R(L^*) + R(H^*) \tag{13}
\end{equation}
The minimax a posteriori error in the direction $\ell$ is finite iff $\ell \in \text{dom} c(\mathcal{X}_y, \cdot) \cap (\text{-1})\text{dom} c(\mathcal{X}_y, \cdot)$ and
\begin{equation}
(\ell, \hat{\phi}) = \frac{1}{2}(c(\mathcal{X}_y, \ell) - c(\mathcal{X}_y, -\ell)),
\hat{d}(\ell) = \frac{1}{2}(c(\mathcal{X}_y, \ell) + c(\mathcal{X}_y, -\ell)) \tag{14}
\end{equation}

Theorem 2. Let
\begin{equation}
\mathcal{G} = \{(f, \eta) : \|f\|^2 + \|\eta\|^2 \leq 1\},
\end{equation}
and assume that 1) or 2) from Corollary 1 holds. The minimax a posteriori estimation $\hat{\phi}$ obeys
\begin{equation}
L^*\hat{q} = H^*(y - H\hat{\phi}),
L\hat{\phi} = \hat{q} \tag{15}
\end{equation}
iff $\ell \in R(L^*) + R(H^*)$. The estimation error is given by
\begin{equation}
\hat{d}(\ell) = (1 - (y, y - H\hat{\phi}))^{\frac{1}{2}}\hat{\sigma}(\ell) \tag{16}
\end{equation}

Corollary 4. Assume that the conditions of Theorem 2 are fulfilled and $\ell(\hat{\phi}) = (\ell, \hat{\phi})$ for any $\ell$, where $\hat{\phi}$ obeys (15). Then
\begin{align*}
\inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\| &= \sup_{x \in \mathcal{X}_y} \|\hat{\phi} - x\| = (1 - (y, y - H\hat{\phi}))^{\frac{1}{2}}\max_{\|\ell\|=1} \hat{\sigma}(\ell) \tag{17}
\end{align*}
In order to apply these results for linear DAEs we investigate some properties of the linear mapping induced by DAE [10]. Let

\[ L\varphi(t) = \left[ \frac{d}{dt} F\varphi(t) - C(t)\varphi(t), F\varphi(t_0) \right] \]

and set

\[ \mathcal{D}(L) = W_F := \{ \varphi(\cdot) \in L_2(t_0, T) : t \mapsto F\varphi(t) \in W_2(t_0, T) \} \]

It is clear that \( L\varphi(t) = [f(t), f_0] \) is equal to

\[ \frac{d}{dt} F\varphi(t) = C(t)\varphi(t) + f(t), F\varphi(t_0) = f_0 \]

Next proposition describes the adjoint \( L^* \).

**Theorem 3.** If \( x(\cdot) \in W_F, z \in W_{F'} \) then

\[ \int_{t_0}^T \left( \frac{d}{dt} Fx(t), z(t) \right) + \left( \frac{d}{dt} F'z(t), x(t) \right) dt = (Fx(T), F't + F'z(T)) - (Fx(t_0), F't + F'z(t_0)) \]  \hspace{1cm} (17)

\( L \) is closed linear mapping and its adjoint \( L^* : L_2(t_0, T) \times \mathbb{R}^n \to L_2(t_0, T) \) is defined as follows

\[ L^*(z, z_0)(t) = -\frac{d}{dt} F'z(t) - C'(t)z(t), \]

\[ \mathcal{D}(L^*) = \{ (z, F't + F'z(t_0) + d) : z \in W_{F'}, F'z(T) = 0, F'd = 0 \} \]

Note that \( R(L) \) is not necessary close. A sufficient condition for \( R(L) \) to be close is introduced in the next theorem assuming\(^3\) that

\[ F = \left( \begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right), C = \left( \begin{array}{cc} C_1 & C_2 \\ C_3 & C_4 \end{array} \right) \]

**Theorem 4.** If

\[ \sup_{1 > \varepsilon > -1} \| Q(\varepsilon) C_2' \|_{mod} < +\infty, Q(\varepsilon) := (\varepsilon^2 E + C_4'C_4)^{-1}, \| F \|_{mod} := \sum_{i,j} |F_{ij}| \]

then \( R(L) \) is closed.

Next subsections demonstrate the application of the above theory to the linear estimation problem for linear DAEs.

\[^3\text{This assumption holds for any linear DAE with constant matrices.}\]
2.1 DAEs with continuous time

In this subsection we present the main result of preprint [47] – linear reduced order minimax filter for linear noncausal DAEs with continuous time. All proofs are given in [47].

Consider a pair of systems

\[
\begin{align*}
\frac{d}{dt}Fx(t) &= C(t)x(t) + f(t), \quad Fx(t_0) = 0, \\
y(t) &= H(t)x(t) + \eta(t), \quad t \in [t_0, T],
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\), \(f(t) \in \mathbb{R}^m\), \(y(t) \in \mathbb{R}^p\), \(\eta(t) \in \mathbb{R}^p\) represent the state, input, measurement output and measurement noise respectively, \(F \in \mathbb{R}^{m \times n}\), \(t \mapsto C(t) \in \mathbb{C}^{m \times n}(t_0, T)\), \(f(\cdot) \in L_2(t_0, T), t \mapsto H(t) \in \mathbb{C}^{p \times n}(t_0, T), t_0, T \in \mathbb{R}\).

According to [10] we say that \(x(\cdot)\) is a solution of (18) if \(Fx(\cdot) \in W_{m,2}(t_0, T)\), the derivative of \(Fx(\cdot)\) coincides with the right side of (18) almost everywhere and \(Fx(t_0) = 0\) holds.

In the sequel we assume that \(\eta(\cdot)\) is a realization of the random process \(\eta\) with zero mean satisfying \(\eta \in W = \{\eta : M \int_{t_0}^{T} (R(t)\eta(t), \eta(t)) \leq 1\}\) (19)

and \(f(\cdot) \in G = \{f(\cdot) : \int_{t_0}^{T} (Q(t)f(t), f(t)) \leq 1\}\),

where \(Q(t) \in \mathbb{R}^{m \times m}\), \(Q = Q' > 0\), \(R(t) \in \mathbb{R}^{p \times p}\), \(R' = R > 0\) and \(Q(t), R(t), R^{-1}(t), Q^{-1}(t)\) are continuous functions of \(t\) on \([t_0, T]\).

Suppose \(y(t)\) is observed in (18) for some \(x(\cdot), f \in G\) and \(\eta\). Our aim here is to construct an algorithm\(^4\) giving online estimation of the linear function \(x(\cdot) \mapsto (\ell, Fx(T))\) on the basis of the measured on \([t_0, T]\) realization of the output \(y(t)\). With this purpose we introduce a notion of the linear minimax estimation [36].

**Definition 1.** The function \(\hat{u}(y) = \int_{t_0}^{T}(\hat{u}(t), y(t))dt\) is called minimax mean-squared a priori estimation if

\[\inf_{\hat{u}} \sigma(u) = \sigma(\hat{u})\]

\(^4\)In literature it is common to refer to this algorithm as filter [59]
where \( \sigma(u) = \sup_{x(\cdot), f(\cdot), \eta} M[(\ell, F x(T)) - u(y)]^2 \) is a maximum estimation error for \( u(\cdot) \). The number \( \hat{\sigma} = \sigma(\hat{u}) \) is called a minimax mean-squared a priori error. The state \( x(t) \) is said to be minimax observable in the direction \( \ell \) iff \( \hat{\sigma} < +\infty \).

The minimax directional observability differs from the classical observability property in the following way. If system state \( x(s) \) is minimax observable in the direction \( \ell \) then the projection of the reachability set (consistent with some realization of \( y(t), t_0 \leq t \leq s \)) onto direction \( \ell \) is expected to be \([-\hat{\sigma}, \hat{\sigma}]\), where \( \hat{\sigma} \) denotes the minimax estimation error. The expected estimation error varies in \([0, \hat{\sigma}]\) and depends on the noise realization, initial condition and input. If \( x(s) \) is unobservable in the minimax sense for \( \ell \) then the minimax estimation is set to zero and \( \hat{\sigma} = +\infty \). This means that the structure of the measurements do not provide any information about \((\ell, x(s))\). Therefore the minimax directional observability provides a qualitative description of DAEs singularity with a respect to the given observations. In particular, regular DAE is observable in the minimax sense for any direction in contrast to the classical observability.

**Remark 1.** This definition generalizes the notion of linear minimax a priori estimation introduced in [27]. Here we follow a common way of [27] deriving the minimax estimation: first step is to describe a dual control problem, next step is to solve it and the last step is to derive a minimax filter.

Assume that \( u(\cdot) \in L_2(t_0, T), \ell \in \mathbb{R}^m \) and \( z(\cdot) \) obeys DAE

\[
\frac{d}{dt} F' z(t) = -C'(t) z(t) + H'(t) u(t), \quad F' z(T) = F' \ell \tag{20}
\]

Let \( v(\cdot) \) denotes any solution of homogeneous DAE (20). Next proposition gives a generalization of the celebrated Kalman duality principle [59].

**Proposition 1.** The minimax estimation error

\[
\sigma(u) = \sup_{x(\cdot), f(\cdot), \eta} M[(\ell, F x(T)) - u(y)]^2 \to \inf_u
\]

is finite iff (20) has a solution \( z(\cdot) \). The minimax estimation problem \( \sigma(u) \to \inf_u \) is equal to the following optimal control problem

\[
I(u) = \min_v \left\{ \int_{t_0}^{T} (Q^{-1}(z-v), z-v) dt \right\} + \int_{t_0}^{T} (R^{-1} u, u) dt \to \min_u, \tag{21}
\]

provided that \( z(\cdot) \) is some solution of (20).
Proposition 1 states that minimax estimation problem is equal to some optimal control problem for appropriate $\ell$ which is called dual control problem. In the next proposition we introduce a representation for the minimax estimation and error.

**Proposition 2.** Let $p(\cdot)$ denotes some solution of the two-point boundary value problem

\begin{align}
\frac{d}{dt}Fx(t) &= C(t)x(t) + Q^{-1}(t)z(t), \quad Fx(t_0) = 0, \tag{22} \\
\frac{d}{dt}F'z(t) &= -C'(t)z(t) + H'(t)R(t)H(t)p(t), \quad F'z(T) = F'\ell
\end{align}

Then minimax estimation $\hat{u}$ is given by $\hat{u} = RHp$, the minimax error is represented as $\hat{\sigma} = (\ell, Fp(T))$.

It is known that (18) may be converted into SVD coordinate system [60] so that without loss of generality we assume that

$$F = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad Q(t) = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \quad R(t) = \begin{pmatrix} R_1 & R_3 \\ R_2 & R_4 \end{pmatrix}, \quad S(t) = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

where $S = H'RH$. By definition, put $A(t) = C_1 - Q_2Q_4^{-1}C_3 - (C_2 - Q_2Q_4^{-1}C_4)\tilde{S}_4^+(S_3 + C_4Q_4^{-1}C_3)$, $M(t) = S_1 + C_3Q_4^{-1}C_3 - (S_2Q_2Q_4^{-1}C_4)\tilde{S}_4^+(S_3 + C_4Q_4^{-1}C_3)$, $\tilde{C} = \tilde{S}_4^+((C_2 - Q_2Q_4^{-1}C_4)^T - (S_3 + C_4Q_4^{-1}C_3)K)$, $G(t) = Q_1 - Q_2Q_4^{-1}Q_3 + (C_2 - Q_2Q_4^{-1}C_4)\tilde{S}_4^+(C_2 - Q_2Q_4^{-1}C_4)^T$.

**Theorem 5.** Assume that $t \mapsto \tilde{S}_4^+(t) = (S_4(t) + C_4(t)Q_4^{-1}(t)C_3(t))$ is measurable matrix-valued function. For any $\ell \in \mathbb{R}^n$ the minimax estimation of the inner product $(\ell, Fx(T))$ is given by

$$\widehat{(\ell, Fx(T))} = (\ell_1, \hat{x}(T))$$

where $\hat{x}$ is the solution of the initial-value problem

\begin{align}
\frac{d}{dt}\hat{x} &= (A(t) - K(t)M(t))\hat{x} + K(t)[E, \tilde{C}]H'Ry(t), \quad \hat{x}(t_0) = 0, \tag{23} \\
\dot{K} &= AK + KA' + KMK - G, \quad K(t_0) = 0
\end{align}

The minimax estimation error is given by $\hat{\sigma} = (\ell_1, K(T)\ell_1)$, where $\ell$ is splitted into $(\ell_1, \ell_2)$ according to the block structure of $F$. 
2.2 DAEs with discrete time

In this subsection we present the main result of the preprint [46] – linear recursive minimax filter for linear noncausal DAE with discrete time. All proofs are given in [46].

Consider the model

\[ F_{k+1}x_{k+1} - C_kx_k = f_k, F_0x_0 = q, \]  \hspace{1cm} (24)  
\[ y_k = H_kx_k + g_k, k = 0, 1, \ldots \]  \hspace{1cm} (25)

where \( x_k \in \mathbb{R}^n \), \( f_k \in \mathbb{R}^m \), \( y_k, g_k \in \mathbb{R}^p \) represent the state, input, measurement output and measurement noise respectively, \( F_k, C_k \in \mathbb{R}^{m \times n}, H_k \in \mathbb{R}^{p \times n} \) and initial state \( x_0 \) belongs to the affine set \( \{ x : F_0x = q \}, q \in \mathbb{R}^m \). In what follows we assume that

\[ \xi \in \mathcal{G} = \{ \xi : \Psi_n(\xi_n) \leq 1, \forall n \in \mathbb{N} \} \]  \hspace{1cm} (26)

where \( \xi = [q, \{ f_s \}, \{ g_s \}], \{ f_s \} = [f_0, f_2, \ldots], \xi_k = [q, \{ f_s \}^k_0, \{ g_s \}^k_0], \{ f_s \}^k = [f_0 \ldots f_k] \) is the projection of \( \{ f_s \} \) onto linear span of \( e_1 \ldots e_{k+1}, e_1 = [1, 0 \ldots] \), \( \Psi_n(\xi_n) = (S_q, q) + \sum_{0}^{n-1} (S_{f_s}, f_s) + (R_{g_s}, g_s), S, S_k \in \mathbb{R}^{m \times m}, R_k \in \mathbb{R}^{p \times p} \) are symmetric and positive-definite.

Suppose that \( y_k^* \) is being observed in (25) with \( x_k = x_k^* \) and \( g_k = g_k^* \) provided that \( x_k^* \) is derived from (24) with \( f_k = f_k^*, q = q^* \) and \( \xi^* = [q^*, \{ f_s^* \}, \{ g_s^* \}] \in \mathcal{G} \). Our aim here is to describe the evolution in \( \tau \) of the \( \ell \)-minimax estimation \( \mathbf{T}_\tau \) of the state \( x_k^* \) at instant \( k = \tau \) along with error \( \hat{\rho}(\ell, \tau) \) through dynamic recurrence-type relation and to efficiently describe the structure of the minimax observable subspace \( \mathcal{L}(\tau) \). For this purpose we shall apply the theory developed in the previous section.

**Definition 2.** The set \( \mathcal{L}(\tau) = \{ \ell : \hat{\rho}(\ell, \tau) < \infty \} \) is called a minimax observable subspace for the model (24) at the instant \( k = \tau \). Its co-dimension \( I_{\tau} = n - \text{rank}Q_\tau \) is called an index of non-causality of the model (24).

**Theorem 6.** The minimax observable subspace for the model (24) is given by \( \mathcal{L}(\tau) = \{ \ell : P_\tau^+P_\ell = \ell \} \) and

\[ X(\tau) = \{ x \in \mathbb{R}^n : (P_\tau(x - \hat{x}_\tau), x - \hat{x}_\tau) \leq \hat{\beta}_\tau \} \]  \hspace{1cm} (27)

where \( \hat{x}_\tau = P_\tau^+r_\tau, \hat{\beta}_\tau := 1 - \alpha_\tau + (P_\tau\hat{x}_\tau, \hat{x}_\tau) \)

\[ P_k = H_k^TR_kH_k + F_k'[S_{k-1} - S_{k-1}C_{k-1}B_{k-1}^+C_{k-1}'S_{k-1}']F_k, \]
\[ P_0 = F_0'SF_0 + H_0'R_0H_0, B_k = P_k + C_k'S_kC_k, \]
\[ \alpha_i = \alpha_{i-1} + (R_i y_i, y_i) - (B_{i-1}^+ r_{i-1}, r_{i-1}), \alpha_0 = (R_0 y_0, y_0) \] and
\[ r_k = F'_k S_{k-1} C_{k-1} B_{k-1}^+ r_{k-1} + H'_k R_k y_k, r_0 = H'_0 R_0 y_0 \]

If \( \ell \in \mathcal{L}(\tau) \) then \( \hat{x}_\tau \) is \( \ell \)-minimax estimation of \( x^*_\tau \) and \( \hat{\rho}(\ell, \tau) = \beta_\tau^\frac{1}{2} (P^*_\tau \ell, \ell)^{\frac{1}{2}} \).

**Example.** Consider the following filtration problem: given measurements \( y_k \) one needs to construct the estimation \( \hat{x}_\tau \) of the state \( x_k \) at instant \( k = \tau \) and to describe the estimation error provided that
\[ p \rightarrow v_k(p) \] is some real-valued function and uncertain scalar parameters \( q, w_k \) and \( f_k \) are restricted by the inequality
\[ Sq^2 + \sum_{k=0}^{\tau-1} R_k w_k^2 + S \tau f_k^2 \leq 1, S, R_k, S_k > 0 \]

Let us show how one can construct \( \hat{x}_\tau \) by means of the set-membership state estimation approach for linear non-causal descriptor systems described in this section. Let \( z_k = [z_{1,k}, z_{2,k}] \) obeys
\[ F z_{k+1} = C_k z_k + f_k, F z_0 = q, a_k = H_k z_k + w_k \]
where \( F = (1, 0), C_k = (c_k, 1), H_k = (h_k, 0) \). Note that for any real \( z_{2,k} \) there exists exactly one \( z_{1,k} \) so that \( z_k \) obeys the first equation in (30).

We shall apply Theorem 6 in order to construct the \( \ell \)-minimax estimation of \( z_\tau \). Using definitions of \( P_k, r_k \) one obtains \( P_0 = \begin{pmatrix} q_0 & 0 \\ 0 & 0 \end{pmatrix}, r_0 = R_0 H'_0 a_0 \), where \( q_0 = S + R_0 \) so that \( B_0^+ = \begin{pmatrix} \frac{1}{q_0} & -\frac{c}{q_0} \\ -\frac{c}{q_0} & \frac{1}{q_0} + \frac{1}{S_0} \end{pmatrix} \) and therefore \( S_0 C_0 B_0^+ = (0, 1), S_0 C_0 B_0^+ C_0' = 1 \) so that \( P_1 = R_1 H'_1 H_k \) and \( r_1 = R_1 H'_1 a_1 \). It’s easy to prove by induction that \( P_k = R_k H'_k H_k \) and \( r_k = R_k H'_k a_k \). Theorem 6 implies: \( \hat{z}_\tau := P^*_\tau r_\tau \) represents the \( \ell \)-minimax estimation for \( \ell \in \mathcal{L}(\tau) \), where \( \mathcal{L}(\tau) = \{ \lambda e_1, e_1 = [1, 0], \lambda \in \mathbb{R} \} \) if \( h_\tau \neq 0 \) and \( \mathcal{L}(\tau) = \{ 0 \} \) otherwise. Hence if \( \ell = [0, \ell] \) then the \( \ell \)-minimax error is infinite. This fact reflects the non-causality of the model (30): since the a posteriori set of (30) is a shift of the convex set \( \mathcal{Y}(0) = \{ [z_0 \ldots z_\tau] \} \):
\[
\| (F H_k) z_0 \|^2 + \sum_{0}^{\tau-1} \| (F H_{k+1} - C_k) [z_{k+1}] \|^2 \leq \beta, \beta > 0
\]
its cross section $Z(\tau)$ at the instant $k = \tau$ is a shift of $P_\tau(\mathcal{G}(0))$. Thus $Z(\tau)$ is convex and unbounded implying that it recedes to infinity \cite{61,8} in the directions $\ell \notin \mathcal{L}(\tau)$. If $h_\tau \neq 0$ and $\ell = [l,0]$ then the $\ell$-minimax estimation $\hat{z}_\tau$ obeys

$$
(l, \hat{z}_\tau) = \frac{l}{h_\tau}a_\tau = l(z^1_\tau + \frac{w_\tau}{h_\tau}), (l, \hat{z}_\tau - z_\tau)^2 \leq \frac{l^2}{R_\tau h^2_\tau} \tag{31}
$$

since $w^2_\tau \leq R^{-1}_\tau$ due to (28). Let $y^*_k, x^*_k$ denote the realization of output $y_k$ and state $x_k$ derived from (28) with $f_k = f^*_k$, $q = q^*$, $w_k = w^*_k$ restricted by (29). Let $z^*_k = [z^*_{1,k}, z^*_{2,k}]$, $a^*_k$ be derived from (30) provided that $f_k = f^*_k$, $z^*_{2,k} := v_k(z^*_{1,k})$, $q = q^*$ and $w_k = w^*_k$. By direct calculation $z^*_{1,k} = x^*_k$ and $a^*_k = y^*_k$ so that $lx^*_\tau = (l, z^*_\tau)$ with $\ell = [l,0]$. Hence

$$
(lx^*_\tau - (l, \hat{z}_\tau))^2 = \left(\frac{lw^*_\tau}{h_\tau}\right)^2 \leq \frac{l^2}{R_\tau h^2_\tau}
$$

due to (31). Thus $\tau \mapsto (l, \hat{z}_\tau)$ gives the online estimation of $\tau \mapsto lx^*_\tau$ with worst-case error $\frac{l^2}{R_\tau h^2_\tau}$.

**Minimax estimator and $H_2/H_\infty$ filters.** In \cite{62} a connection between set-membership state estimation and $H_\infty$ approach is described for linear causal DAEs. The authors note that the notion of informational state ($X(\tau)$ in our notation) is shown to be intrinsic for both approaches: the mathematical relations between informational states of $H_\infty$ and set-membership state estimation are described in \cite{62, Lemma 6.2}. Comparisons of set-membership estimators with $H_\infty$ and other widely used filters for linear DAEs are presented in \cite{63} provided that $F_k \equiv E$.

In \cite{18} authors recover Kalman’s recursion to LTV DAE from a deterministic least square fitting problem over the entire trajectory: if $\text{rank} F_{H_k} \equiv n$ then the optimal estimation $\hat{x}_{i|k}$ can be found from

$$
\hat{x}_{i|k} = P_{k|k}F'_{k,A_{k-1}C_{k-1}}\hat{x}_{k-1|k-1} + P_{k|k}H'_{k}R_{k}y_k,
$$

$$
\hat{x}_{0|0} = P_{0|0}H'_{0}R_{0}y_0, A^{-1}_{k} = (S^{-1}_{k} + C_{k}P_{k|k}C_{k}'),
$$

$$
P_{k|k}^{-1} = F'_{k,A_{k-1}F_{k} + H'_{k}R_{k}H_{k}}, P_{0|0}^{-1} = F'_{0}SF_{0} + H'_{0}R_{0}H_{0}
$$

**Corollary 5.** Let $r_0 = F'_{0}S q + H'_{0}R_{0}y_0$. If $\text{rank} \left[ F_{H_k}^{-} \right] \equiv n$ then $I_k = 0$ and $P_{k}^+ r_k = \hat{x}_{k|k}$.
3 Numerical example

Let us show how to use the minimax estimation in the infinite-horizon setting. Consider the following DAE

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0
\end{bmatrix}
\begin{bmatrix} x_{1,k+1} \\
x_{2,k+1} \\
x_{3,k+1}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{10} & \frac{1}{2} & 0 \\
0 & \frac{1}{10} & 0
\end{bmatrix}
\begin{bmatrix} x_{1,k} \\
x_{2,k} \\
x_{3,k}
\end{bmatrix} +
\begin{bmatrix} f_{1,k} \\
f_{2,k}
\end{bmatrix}
\]

(32)

\[
\begin{bmatrix}
y_{1,k} \\
y_{2,k} \\
y_{3,k} \\
y_{4,k}
\end{bmatrix} =
\begin{bmatrix}
h_{1,k} & h_{2,k} & 0 & 0 \\
h_{4,k} & h_{5,k} & 0 & 0 \\
h_{8,k} & 0.005 & h_{3,k} & 0 \\
h_{6,k} & h_{7,k} & 0 & 0
\end{bmatrix}
\begin{bmatrix} x_{1,k} \\
x_{2,k} \\
x_{3,k}
\end{bmatrix} +
\begin{bmatrix} q_{1,k} \\
q_{2,k} \\
q_{3,k} \\
q_{4,k}
\end{bmatrix}
\]

(33)

where \( h_{1,k} = \frac{6k}{10}, h_{1,0} = \frac{6}{10}, h_{3,k} = 150k \) if \( k \) is odd and 0 otherwise; \( h_{4,k} = 100k, h_{4,0} = 1000, h_{2,k} = k, h_{2,0} = \frac{96}{100}, h_{5,k} = \frac{k}{100}, h_{5,0} = 2\frac{2}{10}, h_{6,k} = 0.05, h_{6,0} = 0, h_{7,k} = 10k, h_{7,0} = 0, h_{8,k} = 0, h_{8,0} = 1. \) Also we set \( x_{1,0} = 1, x_{2,0} = -3 \) and suppose that

\[
(Sq, q) + \sum_{0}^{\infty} (R_k g_k, g_k) + (S_k f_k, f_k) \leq 1
\]

where \( R_k = \frac{1}{k+1} \) diag\( \{ \frac{1}{11}, \frac{1}{22}, \frac{1}{33}, \frac{1}{44} \} \), \( S_k = \) diag\( \{ \frac{1}{35(k+1)}, \frac{1}{70(k+1)} \} \), \( S = \) diag\( \{ \frac{1}{66}, \frac{1}{120} \} \).

Note that rank \( H_{2k+1} \) = 2 and \( I_{2k-1} = 1 \) so that \( N_Q = \{ \ell : Q_N^+ Q_N \ell = 0 \} \) is nontrivial: \( \ell = (0, 0, 1) \) belongs to \( N_Q \). Theorem 6 implies \( \ell (\mathcal{Q}_{2k+1}^+) = (-\infty, +\infty) \) so that the a-posteriori minimax error in the direction \( \ell \) is infinite. Thus the estimation error is unbounded in general case. Really

\[
|\ell, Q_{2k+1}^+ r_{2k+1} - x_{2k+1}| = |x_{3,2k+1}|
\]

Note, that any function \( k \mapsto x_{3,k} \) satisfies (32). In this sense model (32)–(33) is non-causal. Since the estimation error in the direction \( \ell \) coincides with \( x_{3,k} \), its evolution is unpredictable for odd \( k \). In this sense the subspace \{\( k \in \mathbb{R} \} \) in the system state space is not observable for odd \( k \). On the other hand \( I_{2k} = 0 \). Thus \( R(Q_{2k}) = \mathbb{R}^3 \) and the system state space is observable in any direction \( \ell = \mathbb{R}^3 \) due to Theorem 6.

The dynamics of \( x_{i,k}, \hat{x}_{i,k}, |x_{i,k} - \hat{x}_{i,k}|, i = 1,3 \) and minimax error is illustrated by figures 1–2. Note, that Figure 2 demonstrates a singular case: for even \( k \) minimax estimation and error vanish but for odd \( k \) they gives nontrivial approximation. Thus one can observe some kind of oscillation of the estimation curve: \( \hat{x}_{3,2k} \) is near \( x_{3,2k} \) and \( \hat{x}_{3,2k+1} = 0 \). Note, that although
minimax error in the direction $\ell = (0, 0, 1)$ is infinite but $(Q_{2k+1}^\dagger \ell, \ell) = 0$. Thus the corresponding minimax error curve gives an upper bound of the $|x_{3,k} - \hat{x}_{3,k}|$ for odd $k$ and vanishes for even $k$.

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Figure 1: $N = 40$, state $x_{2,k}$ (solid), estimation $\hat{x}_{2,k}$ (dashed); real error $|x_{2,k} - \hat{x}_{2,k}|$ (dashed), minimax error (solid).
Figure 2: $N = 40$, state $x_{3,k}$ (solid), estimation $\hat{x}_{3,k}$ (dashed); real error $|x_{3,k} - \hat{x}_{3,k}|$ (dashed), minimax error (solid).