Abstract

The bandit problem with graph feedback, proposed in [Mannor and Shamir, NeurIPS 2011], is modeled by a directed graph $G = (V, E)$ where $V$ is the collection of bandit arms, and once an arm is triggered, all its incident arms are observed. A fundamental question is how the structure of the graph affects the min-max regret. We propose the notions of the fractional weak domination number $\delta^*$ and the $k$-packing independence number capturing upper bound and lower bound for the regret respectively. We show that the two notions are inherently connected via aligning them with the linear program of the weakly dominating set and its dual — the fractional vertex packing set respectively. Based on this connection, we utilize the strong duality theorem to prove a general regret upper bound $O(\left(\delta^* \log |V| \right)^{\frac{1}{3}} T^{\frac{2}{3}})$ and a lower bound $\Omega\left(\left(\frac{\delta^*}{\alpha} \right)^{\frac{1}{3}} T^{\frac{2}{3}}\right)$ where $\alpha$ is the integrality gap of the dual linear program. Therefore, our bounds are tight up to a $(\log |V|)^{\frac{1}{3}}$ factor on graphs with bounded integrality gap for the vertex packing problem including trees and graphs with bounded degree. Moreover, we show that for several special families of graphs, we can get rid of the $(\log |V|)^{\frac{1}{3}}$ factor and establish optimal regret.

1 Introduction

The multi-armed bandit is an extensively studied problem in reinforcement learning. Imagining a player facing an $n$-armed bandit, each time the player pulls one of the $n$ arms and incurs a loss. At the end of each round, the player receives some feedback and tries to make a better choice in the next round. The expected regret is defined by the difference between the player’s cumulative losses and cumulative losses of the single best arm during $T$ rounds. In this article, we assume the loss at each round is given in an adversarial fashion. This is called the adversarial bandit in the literature. The difficulty of the adversarial bandit problem is usually measured by the min-max regret which is the expected regret of the best strategy against the worst possible loss sequence.

Player’s strategy depends on how the feedback is given at each round. One simple type of feedback is called full feedback where the player can observe all arm’s losses after playing an arm. An important problem studied in this model is online learning with experts [14, 17]. Another extreme, introduced in [8], is the vanilla bandit feedback where the player can only observe the loss of the arm he/she just pulled. Optimal bounds for the regret, either in $n$ or in $T$, are known for both types of feedback.

The work of [24] initialized the study on the generalization of the above two extremes, that is, the feedback consists of the losses of a collection of arms. This type of feedback can be naturally described by a feedback graph $G$ where the vertex set is $[n]$ and a directed edge $(i, j)$ means pulling
the arm \( i \) can observe the loss of arm \( j \). Therefore, the “full feedback” means that \( G \) is a clique with self-loops, and the “vanilla bandit feedback” means that \( G \) consists of \( n \) disjoint self-loops.

A natural yet challenging question is how the graph structure affects the min-max regret. The work of [1] systematically investigated the question and proved tight regret bounds in terms of the time horizon \( T \). They show that, if the graph is “strongly observable”, the regret is \( \Theta(T^{3/2}) \); if the graph is “weakly observable”, the regret is \( \Theta(T^{3/2}) \); and if the graph is “non-observable”, the regret is \( \Theta(T) \). Here the notions of “strongly observable”, “weakly observable” and “non-observable” roughly indicate the connectivity of the feedback graph and will be formally defined in Section 2. However, unlike the case of “full feedback” or “vanilla bandit feedback”, the dependency of the regret on \( n \), or more generally on the structure of the graph, is still not well understood. For example, for “weakly observable” graphs, an upper bound and a lower bound of the regret in terms of the weak domination number \( \delta(G) \) were proved in [1], but a large gap exists between the two. This suggests that the weak domination number might not be the correct parameter to characterize the regret.

We make progress on this problem for “weakly observable” graphs. This family of graphs is general enough to encode almost all feedback patterns of bandits. We introduce the notions of the fractional weak domination number \( \delta^*(G) \), the \( k \)-packing independence number and provide evidence to show that they are the correct graph parameters. The two parameters are closely related and help us to improve the upper bound and lower bound respectively. As the name indicated, \( \delta^*(G) \) is the fractional version of \( \delta(G) \), namely the optimum of the linear relaxation of the integer program for the weakly dominating set. We observe that this graph parameter has already been used in an algorithm for “strongly observable” graphs in [3], where it functioned differently. In the following, when the algorithm is clear from the context, we use \( R(G, T) \) to denote the regret of the algorithm on the instance \( G \) in \( T \) rounds.

Our main algorithmic result is:

**Theorem 1.** There exists an algorithm such that for any weakly observable graph, any time horizon \( T \geq n^3 \log(n)/\delta^*^2(G) \), its regret satisfies \( R(G, T) = O\left( (\delta^*(G) \log n)^{1/2} T^{3/2} \right) \).

Note that the regret of the algorithm in [1] satisfies \( R(G, T) = O\left( (\delta(G) \log n)^{1/2} T^{3/2} \right) \). The fractional weak domination number \( \delta^* \) is always no larger than \( \delta \), and the gaps between the two can be as large as \( \Theta(\log n) \). We will give an explicit example in Section 4.3 in which the gap matters and our algorithm is optimal. Theorem 1 can be seamlessly extended to more general time-varying graphs and probabilistic graphs. The formal definitions of these models are in Appendix E.

On the other hand, we investigate graph structures that can be used to fool algorithms. We say a set \( S \) of vertices is a \( k \)-packing independent set if \( S \) is an independent set and any vertex has at most \( k \) out-neighbors in \( S \). We prove the following lower bound:

**Theorem 2.** Let \( G = (V, E) \) be a directed graph. If \( G \) contains a \( k \)-packing independent set \( S \) with \( |S| \geq 2 \), then for any randomized algorithm and any time horizon \( T \), there exists a sequence of loss functions such that the expected regret is \( \Omega\left( \max \left\{ \frac{|S|}{T^{1/3}}, \log |S| \right\} T^{1/3} \right) \).

For every \( k \in \mathbb{N} \), we use \( \zeta_k \), the \( k \)-packing independence number, to denote the size of the maximum \( k \)-packing independent set. To prove Theorem 2, we reduce the problem of minimizing regret to statistical hypothesis testing for which powerful tools from information theory can help.

We can use Theorem 2 to strengthen lower bounds in [1]. Besides, we show that large \( \delta^* \) usually implies large \( \zeta_1 \) via studying the linear programming dual of fractional weakly dominating sets and applying a novel rounding procedure. This is also one of our main technical contributions. Combinatorially, the dual linear program is to find the maximum fractional vertex packing set in the graph. Specifically, we can establish lower bounds in terms of \( \delta^* \) by applying Theorem 2:

**Theorem 3.** If \( G \) is weakly observable, then for any algorithm and any sufficiently large time horizon \( T \in \mathbb{N} \), there exists a sequence of loss functions such that \( R(G, T) = \Omega\left( \left( \frac{\delta^*}{\alpha} \right)^{1/3} T^{2/3} \right) \), where \( \alpha \) is the integrality gap of the linear program for vertex packing.

Clearly the exact lower bound is determined by the integrality gap of a certain linear program. In general graphs, we have a universal upper bound \( \alpha = O(n/\delta^* \log n) \). For concrete instances, we can
obtain clearer and tighter bounds on \( \alpha \). For example, the linear program has a constant integrality gap \( \alpha \) on graphs of bounded degree.

**Corollary 4.** Let \( \Delta \in \mathbb{N} \) be a constant and \( \mathcal{G}_\Delta \) be the family of graphs with maximum in-degree \( \Delta \). Then for every weakly observable \( G = (V, E) \in \mathcal{G}_\Delta \), any algorithm and any sufficiently large time horizon \( T \in \mathbb{N} \), there exists a sequence of loss functions such that \( R(G, T) = \Omega((\delta^*)^{1/3} \cdot T^{2/3}) \).

We also show that for 1-degenerate directed graphs (formally defined in Section 2.1), the integrality gap is 1. This family of graphs includes trees and directed cycles. As a consequence, we have

**Corollary 5.** Let \( G \) be a 1-degenerate weakly observable graph. Then for any algorithm and any sufficiently large time horizon \( T \in \mathbb{N} \), there exists a sequence of loss functions such that \( R(G, T) = \Omega((\delta^*)^{1/3} \cdot T^{2/3}) \).

**Comparison of previous results and our results**

In Table 1, we compare our new upper bounds, lower bounds and their gap with previous best results.

| Graph Type | Previous best results [1] | This work |
|------------|--------------------------|-----------|
|            | Min-max regret | Gap | Min-max regret | Gap |
| General graphs | \( O \left( (\delta \log n)^{1/3} \cdot T^{2/3} \right) \) | \( \Omega \left( \max \left\{ \left( \frac{\delta}{\log^2 n} \right)^{1/3}, 1 \right\} \cdot T^{2/3} \right) \) | See discussion below | \( O \left( (\delta^* \log n)^{1/3} \cdot T^{2/3} \right) \) | \( \Omega \left( \max \left\{ \left( \frac{\delta^*}{\alpha} \right)^{1/3}, 1 \right\} \cdot T^{2/3} \right) \) | See discussion below |
|            | for \( \delta = \log^2 n \): \( \Omega \left( T^{2/3} \right) \) | \( O \left( \log n \right) \) | \( O \left( (\delta^* \log n)^{1/3} \cdot T^{2/3} \right) \) | \( \Omega \left( \left( \frac{\delta^*}{\alpha} \right)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( \frac{\log n}{\log \log n} \right) \) |
| Trees / Bounded in-degree | Same as general graphs | \( O \left( \log n \right) \) | \( O \left( (\delta^* \log n)^{1/3} \cdot T^{2/3} \right) \) | \( \Omega \left( \left( \frac{\delta^*}{\alpha} \right)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \right) \) |
| Complete bipartite graphs | \( O \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( \Omega \left( T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \right) \) | \( \Theta \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( 1 \right) \) |
| Orthogonal relation on \( \mathbb{F}_2^k \) | \( O \left( (\log n)^{2/3} \cdot T^{2/3} \right) \) | \( \Omega \left( T^{2/3} \right) \) | \( O \left( (\log n)^{2/3} \right) \) | \( \Theta \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( (\log n)^{1/3} \cdot T^{2/3} \right) \) | \( O \left( 1 \right) \) |

**Discussion.** In general, our upper bound is never worse than the previous one since \( \delta^* \leq \delta \). Our lower bound is not directly comparable to the previously known lower bound as they are stated in terms of different parameters. In fact, we can not find an instance such that our lower bound \( \Omega(\max \{1, (\delta^*/\alpha)^{1/3}\}) \) is worse than the previous lower bound \( \Omega(\max \{1, (\delta^*/(\log n)^{2/3})^{1/3}\}) \) and there are instances on which our bound outperforms. The two key quantities, namely the integrality gap \( \frac{\delta^*}{\alpha} \) of the primal linear programming and the integrality gap \( \alpha \) of the dual linear programming, seem to be correlated in a graph. The relation between the two bounds is worth further investigation.

**Related work**

The multi-armed bandit problem originated from the sequential decision making under uncertainty studied in [34, 6] and the adversarial bandit is a natural variant introduced by [7]. The work of [24] introduced the graph feedback model with a self-loop on each vertex in order to interpolate between the full feedback and bandit feedback settings. This model has been extensively studied in the work of [24, 4, 20, 3]. The work of [1] removed the self-loop assumption and considered generalized constrained graphs. They gave a full characterization of the mini-max regret in terms of the time horizon \( T \). In contrast to fixed graph feedback, recent work of [20, 15, 2, 32] considered the time-varying graphs. Another line of recent work in [22, 23, 3] is to study random graphs, or the graphs with probabilistic feedback.

Most algorithms for adversarial bandits are derived from the EXP3 algorithm, e.g. [9, 29]. However, even for the vanilla multi-armed bandit problem, a direct application of EXP3 can only get an
upper bound of $O(\sqrt{n \log n} \cdot T)$ [9], which is suboptimal. In fact, the EXP3 is a special case of online stochastic mirror descent (OSMD) algorithm when using negentropy function as the potential function. OSMD was developed by [26] and [27] for online optimization. By choosing a more sophisticated potential function, namely the 1/2-Tsallis entropy function [33], OSMD can achieve the tight bound $\Theta(\sqrt{nT})$ [35].

The idea of using domination number or related parameters to study the feedback graph appeared many times in literature, e.g. [1, 12, 13, 4, 3]. The work of [3] used the idea of the fractional dominating function. OSMD was developed by [26] and [27] for online optimization. By choosing a more convenient in the context. With this in mind, we have the inner product $\langle \ell, x \rangle \triangleq \sum_{i \in [n]} \ell(i) \cdot x(i)$ for every $x \in \mathbb{R}^n$.

2 Preliminaries

In this section, we formally describe the problem setting of bandits with graph feedback and introduce notations, definitions and propositions that will be used later.

Let $n \in \mathbb{N}$. We will use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Let $x \in \mathbb{R}^n$ be an $n$-dimensional vector. For every $i \in [n]$, we use $x(i)$ to denote the value on the $i$th-coordinate. We use $\{e_1, \ldots, e_n\}$ to denote the standard basis of $\mathbb{R}^n$. That is, $e_i \in \mathbb{R}^n$ is the vector such that $e_i(j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$ for every $j \in [n]$. For every $n \in \mathbb{N}$, we define $\Delta_n \triangleq \{x \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n x(i) = 1\}$ as the $n$-dimensional probability simplex. Clearly $\Delta_n$ is convex and every $x \in \Delta_n$ can be viewed as a distribution on $[n]$. Throughout the article, sometimes we will view a function $\ell : [n] \rightarrow \mathbb{R}$ equivalently as a vector in $\mathbb{R}^n$, depending on which form is more convenient in the context. With this in mind, we have the inner product $\langle \ell, x \rangle \triangleq \sum_{i \in [n]} \ell(i) \cdot x(i)$ for every $x \in \mathbb{R}^n$.

2.1 Graphs

In this article, we use $G = (V, E)$ to denote a directed graph with possible self-loops but no multiple edges. Therefore each $(u, v) \in E$ indicates a directed edge from $u$ to $v$ in $G$. If we say a graph $G = (V, E)$ is undirected, we view each undirected edge $\{u, v\} \in E$ as two directed edges $(u, v)$ and $(v, u)$. In the following, we assume $|V| \geq 2$ unless otherwise specified. For any $S \subseteq V$, $G[S]$ is the subgraph of $G$ induced by $S$. For every $v \in V$, we define $N_{\text{in}}(v) = \{u \in V : (u, v) \in E\}$ and $N_{\text{out}}(v) = \{u \in V : (v, u) \in E\}$ as the set of in-neighbors and out-neighbors of $v$ respectively. We also call $|N_{\text{in}}(v)|$ and $|N_{\text{out}}(v)|$ the in-degree and out-degree of $v$ respectively. A set $S \subseteq V$ is an independent set if there is no $u, v \in S$ such that $(u, v) \in E$. Note that we do not consider an isolated vertex with a self-loop as an independent set.

A vertex $v \in V$ is strongly observable if $(v, v) \in E$ or $\forall u \in V \setminus v, (u, v) \in E$. A vertex $v \in V$ is non-observable if $N_{\text{in}}(v) = \emptyset$. A directed graph $G$ is called strongly observable if each vertex of $G$ is strongly observable. It is called non-observable if it contains at least one non-observable vertex. The graph is called weakly observable if it is neither strong observable nor non-observable.

We say a directed graph $G$ is 1-degenerate if one can iteratively apply the following two operations in arbitrary orders on $G$ to get an empty graph: • Pick a vertex with in-degree one and remove the in-edge; • Pick a vertex with in-degree zero and out-degree at most one, and remove both the vertex and the out-edge. Typical 1-degenerate graphs include trees (directed or undirected) and directed cycles.

Let $U = \{i \in V : i \notin N_{\text{in}}(i)\}$ denote the set of vertices without self-loops. Consider the following linear program defined on $G$. We will call the linear program $(\mathcal{P})$

\[
\begin{align*}
\text{minimize} \quad & \sum_{i \in V} x_i; \\
\text{subject to} \quad & \sum_{i \in N_{\text{in}}(j)} x_i \geq 1, \quad \forall j \in U; \\
& 0 \leq x_i \leq 1, \quad \forall i \in V.
\end{align*}
\]
We use $\delta^*(G)$ to denote the optimum of the above linear program. We call $\delta^*(G)$ the fractional weak domination number of $G$. The dual of the linear program is

$$\text{maximize } \sum_{j \in U} y_j \quad \text{subject to } \sum_{j \in N_{\text{out}}(i) \cap U} y_j \leq 1, \forall i \in V; \quad 0 \leq y_j \leq 1, \forall j \in U. \quad (\mathcal{D})$$

We call this linear program $(\mathcal{D})$. We use $\zeta^*(G)$ to denote the optimum of the dual. We call $\zeta^*(G)$ the fractional vertex packing number of $G$. Then it follows from the strong duality theorem (see e.g. [11]) of linear programs that $\delta^*(G) = \zeta^*(G)$.

We remark that in [1], the weakly (integral) dominating set was defined to dominate all “weakly observable vertices” instead of “vertices without self-loops”. These two definitions are all equivalent for all results in this article. See Appendix A for more explanations on this.

2.2 Bandits

Let $G = (V, E)$ be a directed graph where $V = [n]$ is the collection of bandit arms. Let $T \in \mathbb{N}$ be the time horizon which is known beforehand. The bandit problem is an online-decision game running for $T$ rounds. The player designs an algorithm $\mathcal{A}$ with the following behavior in each round $t$ of the game: • The algorithm $\mathcal{A}$ chooses an arm $A_t \in [n]$; • An adversary privately provides a loss function $\ell_t : [n] \to [0, 1]$ and $\mathcal{A}$ pays a loss $\ell_t(A_t)$; • The algorithm receives feedback $\{\ell_t(j) : j \in N_{\text{out}}(A_t)\}$.

The expected regret of the algorithm $\mathcal{A}$ against a specific loss sequence $\ell^* = \{\ell_1, \ldots, \ell_T\}$ is defined by $R(G, T, \mathcal{A}, \ell^*) = \mathbb{E}\left[\sum_{t=1}^T \ell_t(A_t)\right] - \min_{\mathcal{A} \in [n]} \sum_{t=1}^T \ell_t(a)$. Note that we look at the expectation of the algorithm since $\mathcal{A}$ might be randomized and it is not hard to see that randomization is necessary to guarantee $o(T)$ regret due to the adversarial nature of the loss sequence. The purpose of the problem is to design an algorithm performing well against the worst loss sequence, namely determining the min-max regret $R(G, T) \triangleq \inf_{\mathcal{A}} \sup_{\ell^*} R(G, T, \mathcal{A}, \ell^*)$.

There is another model called stochastic bandits in which the loss function at each round is not adversarially chosen but sampled from a fixed distribution. It is clear that this model is not harder than the one introduced above in the sense that any algorithm performing well in the adversarial setting also performs well in the stochastic setting. Therefore, we will construct instances of stochastic bandits to derive lower bounds in Section 4.

2.3 Optimization

Our upper bound is obtained via the online mirror descent algorithm. In this section, we collect a minimal set of terminologies to understand the algorithm. More details about the approach and its application to online decision-making can be found in e.g. [28].

Let $C \subseteq \mathbb{R}^n$. We use $\text{int}(C)$ to denote the interior $C$. For a convex function $\Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, $\text{dom}(\Psi) \triangleq \{x : \Psi(x) < \infty\}$ is the domain of $\Psi$. Assume $\Psi$ is differentiable in its domain. For every $x, y \in \text{dom}(\Psi)$, $D_\Psi(x, y) \triangleq \Psi(x) - \Psi(y) - \langle x - y, \nabla \Psi(y) \rangle \geq 0$ is the Bregman divergence between $x$ and $y$ with respect to the convex function $\Psi$. The diameter of $C$ with respect to $\Psi$ is $D_\Psi(C) \triangleq \sup_{x, y \in C} \{\Psi(x) - \Psi(y)\}$. Let $A \in \mathbb{R}^{n \times n}$ be a semi-definite positive matrix and $x \in \mathbb{R}^n$ be a vector. We define $\|x\|_A \triangleq \sqrt{x^T A x}$ as the norm of $x$ with respect to $A$.

3 The algorithm

In this section, we design an algorithm to achieve the upper bound in Theorem 1. The proof is in Appendix B.

3.1 Online stochastic mirror descent (OSMD)

Our algorithm is based on the Online Stochastic Mirror Descent framework that has been widely used for bandit problems in various settings. Assuming the set of arms is $[n]$, possibly with many additional structures, a typical OSMD algorithm usually consists of following steps:
Online Stochastic Mirror Descent with Exploration

Although the framework of OSMD is standard, there are several key ingredients left for the algorithm designer to follow: • How to construct the distribution \( \tilde{X}_t \) from \( X_t \)? • How to pick the potential function \( \Psi \)? In general, filling these blanks heavily relies on the problem structure and sometimes requires ingenious construction to achieve low regret. We will first describe our algorithm and then explain our choices in Section 3.2.

### 3.2 The algorithm for bandits with graph feedback

Let \( G = (V, E) \) be the input directed graph with \( V = [n] \). A few offline preprocessing steps are required before the online part of the algorithm. We first solve the linear program (\( \mathcal{P} \)) to get an optimal solution \( \{x_i^*\}_{i \in [n]} \). Recall \( \delta^*(G) = \sum_{i \in [n]} x^*_i \) is the fractional weak domination number of \( G \). Define a distribution \( u \in \Delta_n \) by normalizing \( \{x^*_i\}_{i \in [n]} \); i.e., we let \( u(i) = \frac{x^*_i}{\sum_{j \in [n]} x^*_j} \) for all \( i \in [n] \). The distribution \( u \) will be the exploration distribution whose function will be explained later. Define parameters \( \gamma = \left( \frac{\delta^*(G) \log n}{2} \right)^{1/3} \), \( \eta = \frac{\gamma^2}{\delta^*(G)} \); where \( \gamma \) is the exploration rate and \( \eta \) is the step size in the mirror descent. Finally, we let the potential function \( \Psi : \mathbb{R}_+^n \to \mathbb{R} \) be \( x \in \mathbb{R}_+^n \mapsto \sum_{i=1}^n x(i) \log x(i) \) with the convention that \( 0 \cdot \log 0 = 0 \). When restricted to \( \Delta_n \), \( \Psi(x) \) is the negative entropy of the distribution \( x \).

**Algorithm 1:** Online Stochastic Mirror Descent with Exploration

```plaintext
begin
    \( X_1 \leftarrow \text{arg min}_{a \in \Delta_n} \Psi(a) \);
    for \( t = 1, 2, \ldots, T \) do
        \( \tilde{X}_t \leftarrow (1 - \gamma) \cdot X_t + \gamma \cdot u ; \)
        /* use \( u \) to explore with rate \( \gamma \). */
        Play \( A_t \sim \tilde{X}_t \) and observe \( \ell_t(j) \) for all \( j \in N_{out}(A_t) \);
        /* If \( j \not\in N_{out}(A_t) \), the value of \( \ell_t(j) \) is unset. */
        \( \forall j \in [n] : \ell_t(j) \leftarrow \frac{1}{\sum_{i \in N(j)} \Delta(i)} \cdot \ell_t(j) ; \)
        /* For \( j \not\in N_{out}(A_t) \), \( \ell_t(j) = 0 \). */
        \( X_{t+1} \leftarrow \text{arg min}_{x \in \Delta_n} \eta(x, \hat{\ell}_t) + B_\Psi(x, X_t) ; \)
        /* The update rule of mirror descent w.r.t. \( \Psi \). */
end
```

Clearly our algorithm implements OSMD framework by specializing the three ingredients mentioned in Section 3.1. • We choose \( \tilde{X}_t = (1 - \gamma) \cdot X_t + \gamma \cdot u \). This means that our algorithm basically follows \( X_t \) but with a certain probability \( \gamma \) to explore the arms according to \( u \). The reason for doing this is to guarantee that each arm has some not-so-small chance to be observed. It will be clear from the analysis of OSMD that the performance of the algorithm depends on the variance of \( \hat{\ell}_t \), and a lower bound for each \( \tilde{X}_t(i) \) implies an upper bound on the variance. On the other hand, we cannot choose \( \gamma \) too large since it is \( X_t \) who contains information on which arm is good, and our probability to follow \( X_t \) is \( 1 - \gamma \). Therefore, our choice of \( \gamma \) is optimized with respect to the trade-off between
We first use Theorem 2 to recover and strengthen lower bounds in [1]. Let us delay the proof of Theorem 2 to Appendix D and discuss some of its consequences in the following structural lemma was proved in [1].

\[ \text{Definition 6. Let } G(V, E) \text{ be a directed graph and } k \in \mathbb{N}. \text{ A set } S \subseteq V \text{ is a } k\text{-packing independent set of } G \text{ if (1) } S \text{ is an independent set; (2) For any } v \in V, \text{ it holds that } |N_{\text{out}}(v) \cap S| \leq k. \]

Intuitively, if a graph contains a large \( k \)-packing independent set \( S \), then one can construct a hard instance as follows: • All arms in \( V - S \) are bad, say with loss 1; • All arms in \( S \) have loss \( \text{Ber}(\frac{1}{2}) \) except a special one with loss \( \text{Ber}(\frac{1}{2} - \varepsilon) \). Then any algorithm with low regret must successfully identify the special arm from \( S \) without observing arms in \( S \) much (since each observation of arms in \( S \) comes from pulling \( V - S \), which costs a penalty at least \( \frac{1}{2} \) in the regret), and we can tweak the parameters to make this impossible. A similar idea already appeared in [1]. However, we will formally identify the problem of minimizing regret on this family of instances with the problem of best arm identification. Therefore, stronger lower bounds \( \Omega \left( \max \left\{ \frac{|S|}{n}, \log |S| \right\} \frac{4}{3} \cdot T^{\frac{2}{3}} \right) \) in Theorem 2 can be obtained using tools from information theory.

Remark. If the maximum independent set of the graph \( G \) is of size one and \( G \) is “weakly observable”, then it has been shown in [1] that \( R(G, T) = \Omega \left( T^{\frac{2}{3}} \right) \) for any algorithm. If the graph has no independent set, which means each vertex contains a self-loop, then the graph is “strongly observable” and its regret can be \( O(T^{\frac{1}{3}}) \). In particular, the problem of vanilla \( n \)-armed bandits falls into this case.

We delay the proof of Theorem 2 to Appendix D and discuss some of its consequences in the remaining of this section. We recover and strengthen the lower bound based on the (integral) domination number of [1] in Section 4.1. Then we prove Theorem 3 in Section 4.2 and discuss some of its useful corollaries. Finally, we discuss in Section 4.3 when our lower bounds are optimal.

Our main technical contribution here is that we relate \( \delta^* \) to the lower bound as well. This is achieved via applying the strong duality theorem of linear programming and using a new rounding method to construct hard instances from fractional solutions of the dual linear programming. This approach towards the lower bounds is much cleaner and in many cases stronger than previous ones in [1]. To the best of our knowledge, the method is new to the community of bandits algorithms.

### 4 Lower bounds

In this section, we prove several lower bounds for the regret in terms of different graph parameters. All the lower bounds obtained in this section are based on a meta lower bound (Theorem 2) via the notion of \( k \)-packing independent set.

#### 4.1 Lower bound via the (Integral) weak domination number

We first use Theorem 2 to recover and strengthen lower bounds in [1]. Let \( G = (V, E) \) be a directed graph and \( U \subseteq V \) be the set of vertices without self-loops.

The weakly dominating set of \( U \) is a set \( S \subseteq V \) such that for every \( u \in U \), there exists some \( v \in S \) with \( (v, u) \in E \). The weak domination number, denoted by \( \delta(G) \), is the size of the minimum weakly dominating set of \( U \). In fact, \( \delta(G) \) is the optimum of the integral restriction of the linear program \( (\mathcal{P}) \) in Section 2.1:

\[
\text{minimize } \sum_{i \in V} x_i; \quad \text{subject to } \sum_{i \in N_{\text{in}}(j)} x_i \geq 1, \forall j \in U; \quad x_i \in \{0, 1\}, \forall i \in V. \quad (\mathcal{P}')
\]

The following structural lemma was proved in [1]

**Lemma 7.** The graph \( G \) contains a \((\log n)\)-packing independent set \( S \) of size at least \( \frac{\delta(G)}{50 \log n} \).
Applying Lemma 7 to Theorem 2, we obtain

**Theorem 8.** If $G$ is weakly observable, then for any algorithm and any sufficiently large time horizon $T \in \mathbb{N}$, there exists a sequence of loss functions such that $R(G,T) = \Omega \left( \max \left\{ \frac{\delta(G)}{\log^2 n}, \log \left( \frac{\delta(G)}{\log n} \right) \right\}^{1/3} \cdot T^{2/3} \right)$.

Note that the bound in [1] is $R(G,T) = \Omega \left( \max \left\{ \frac{\delta(G)}{\log^2 n}, 1 \right\}^{1/3} \cdot T^{2/3} \right)$. Theorem 8 outperforms when $\omega(\log n) < \delta(G) < o(\log^2 n \cdot \log \log n)$.

### 4.2 Lower bound via the linear program dual

In this section, we use Theorem 2 to derive lower bounds in terms of $\delta^*(G)$. Recall the linear program $(\mathcal{P})$ in Section 2.1 whose optimum is $\zeta^*(G) = \delta^*(G)$ by the strong duality theorem of linear programming. Consider its integral restriction $(\mathcal{P}')$:

$$
\text{maximize } \sum_{j \in U} y_j \text{ subject to } \sum_{j \in N_{\text{out}}(i) \cap U} y_j \leq 1, \forall i \in V; \quad y_j \in \{0,1\}, \forall j \in U.
$$

For every feasible solution $\{\hat{y}_j\}_{j \in U}$ of $(\mathcal{P}')$, the set $S \triangleq \{j \in U : \hat{y}_j = 1\}$ is called a vertex packing set on $U$. It enjoys the property that for every $i \in V$, $|N_{\text{out}}(i) \cap S| \leq 1$.

Let $\zeta(G)$ be the optimum of $(\mathcal{P}')$, namely the size of the maximum vertex packing set on $U$. Let $\alpha \triangleq \frac{\zeta^*(G)}{\zeta(G)}$ be the integrality gap of $(\mathcal{P})$. In the following, we will write $\delta^*, \delta, \zeta^*, \zeta$ instead of $\delta^*(G), \delta(G), \zeta^*(G), \zeta(G)$ respectively if the graph $G$ is clear from the context.

We can use a greedy algorithm to find an 1-packing independent set of size at least $\frac{|S|}{\alpha}$ in $S$ and then Theorem 3 follows from Theorem 2. Theorem 3 is less informative if we do not know how large the integrality gap $\alpha$ is. On the other hand, the integrality gap of linear programs for packing programs has been well-studied in the literature of approximation algorithms. The following bound due to [31] is tight for general graphs.

**Lemma 9 ([31]).** For any directed graph $G$, the integrality gap $\alpha = O\left(\frac{n}{\delta^*}\right)$.

**Corollary 10.** If $G$ is weakly observable, then for any algorithm and any sufficiently large time horizon $T \in \mathbb{N}$, there exists a sequence of loss functions such that $R(G,T) = \Omega \left( \left( \frac{\delta^*}{n} \right)^{\frac{3}{2}} \cdot T^{\frac{2}{3}} \right)$.

The bound for the integrality gap in Lemma 9 holds for any graphs. It can be greatly improved for special graphs.

An interesting family of graphs with small $\alpha$ is those bounded degree graphs. If the in-degree of every vertex in $U$ is bounded, we have the following bound for the integrality gap:

**Lemma 11 ([10]).** If the in-degree of every vertex in $U$ is bounded by a constant $\Delta$, then the integrality gap $\alpha \leq 8\Delta$.

**Corollary 12.** Let $\Delta \in \mathbb{N}$ be a constant and $G_\Delta$ be the family of graphs with maximum in-degree $\Delta$. Then for every weakly observable $G = (V,E) \in G_\Delta$, any algorithm and any sufficiently large time horizon $T \in \mathbb{N}$, there exists a sequence of loss functions such that $R(G,T) = \Omega((\delta^*)^{\frac{3}{2}} \cdot T^{\frac{2}{3}})$.

Next, we show the integrality gap of another broad family of graphs, the 1-degenerate graphs, is 1. The family of 1-degenerate graphs was defined in Section 2.1. Graphs including trees (both directed and undirected), directed cycles belong to this family. The proof of the following lemma is in Appendix C.2.

**Lemma 13.** For any 1-degenerate directed graph, the integrality gap $\alpha = 1$.

**Corollary 14.** Let $G$ be a 1-degenerate weakly observable graph. Then for any algorithm and any sufficiently large time horizon $T \in \mathbb{N}$, there exists a sequence of loss functions such that $R(G,T) = \Omega((\delta^*)^{\frac{3}{2}} \cdot T^{\frac{2}{3}})$.
We obtained in Theorem 1 that \( R(G, T) = O\left( (\delta^* \log n)^{\frac{1}{q}} T^\frac{2}{3} \right) \), and therefore our lower bound is tight up to a factor of \((\log n)^{\frac{1}{3}}\) on 1-degenerate graphs and graphs of bounded degree.

### 4.3 Instances with optimal regret

In this section, we will examine several families of graphs in which the optimal regret bound can be obtained using tools developed in this article.

#### 4.3.1 Complete bipartite graphs

Let \( G = (V_1 \cup V_2, E) \) be an undirected complete bipartite graph with \( n = |V_1| + |V_2| \). Clearly \( \delta^* = \delta = 2 \). Therefore both our Theorem 1 and the algorithm in [1] satisfy \( R(G, T) = O\left( (\log n)^{\frac{1}{3}} \cdot T^\frac{2}{3} \right) \).

Assuming without loss of generality that \( |V_1| \geq |V_2| \), then \( V_1 \) is a \( |V_1| \)-packing independent set of size at least \( \frac{n}{2} \). Therefore it follows from Theorem 2 that any algorithm satisfies \( R(G, T) = \Omega\left( (\log n)^{\frac{1}{3}} \cdot T^\frac{2}{3} \right) \). Note that the lower bound in [1] is \( \Omega\left( T^\frac{2}{3} \right) \) for this instance.

#### 4.3.2 Orthogonal relation on \( \mathbb{F}_2^k \)

Our algorithm outperforms the one in [1] when \( \delta^* \ll \delta \). Let us now construct a family of graphs where \( \frac{\delta^*}{\delta} = \Theta(\log n) \).

Let \( \mathbb{F}_2 \) be the finite field with two elements and \( k \in \mathbb{N} \) be sufficiently large. The vertex set of the undirected graph \( G = (V_1 \cup V_2, E) \) consists of two disjoint parts \( V_1 \) and \( V_2 \) where \( V_1 \) and \( V_2 \) are both isomorphic to \( \mathbb{F}_2^k \setminus \{0\} \). Therefore we can write \( V_1 = \{ x_\alpha \}_{\alpha \in \mathbb{F}_2^k \setminus \{0\}} \) and \( V_2 = \{ y_\beta \}_{\beta \in \mathbb{F}_2^k \setminus \{0\}} \).

The set of edges \( E \) is as follows:
- \( E \) is the edge set such that \( G[V_1] \) is a clique and \( G[V_2] \) is an independent set;
- For every \( x_\alpha \in V_1 \) and \( y_\beta \in V_2 \), \( \{ x_\alpha, y_\beta \} \in E \) iff \( \langle \alpha, \beta \rangle = \sum_{i=1}^{k} \alpha(i) \cdot \beta(i) = 1 \), where all multiplications and summations are in \( \mathbb{F}_2 \).

We will show that the upper bound and the lower bound of the regret for this instance proved in [1] based on \( \delta = O\left( (\log n)^{\frac{1}{3}} \cdot T^\frac{2}{3} \right) \) and \( \Omega\left( T^\frac{2}{3} \right) \) respectively. However, we achieve the optimal \( \Theta\left( (\log n)^{\frac{1}{3}} \cdot T^\frac{2}{3} \right) \) regret. Thus we conclude that both our new upper bound and lower bound are crucial for the optimal regret on this family of instances. The details can be found in Appendix C.3

5 Conclusion

In this article, we introduced the notions of fractional weak domination number and \( k \)-packing independence number respectively to prove new upper bounds and lower bounds for the regret of bandits with graph feedback. Our results implied optimal regret on several families of graphs. Although there are still some gaps in general, we believe that these two notions are the correct graph parameters to characterize the complexity of the problem. We now list a few interesting problems worth future investigation.

- Let \( G \) be an \( n \)-vertex undirected cycle. What is the regret on this instance? Theorem 1 implies an upper bound \( O\left( (n \log n)^{\frac{1}{3}} T^\frac{2}{3} \right) \) and Theorem 2 implies a lower bound \( \Omega\left( n^3 T^\frac{2}{3} \right) \).

- The lower bound \( \Omega\left( (\frac{\delta^*}{\delta})^{\frac{1}{3}} \cdot T^\frac{2}{3} \right) \) in Theorem 3 for general graphs is not satisfactory. The lower bound is proved via the construction of a 1-packing independent set. This construction did not release the full power of Theorem 2 as the lower bound in the theorem applies for any \( k \)-packing independent set. It is still possible to construct larger \( k \)-packing independent sets via rounding the linear program \( \mathcal{Q} \) to some “less integral” solution.

- Is Theorem 2 tight? In fact, the bound for BESTARMID, which is constructed to prove Theorem 2 in the full version of the paper, is tight since matching upper bound exists. Therefore, one needs new constructions of hard instances to improve Theorem 2, if possible.
Acknowledgements

This work was partly supported by National Key Research and Development Program of China (2020AAA0107600), National Natural Science Foundation of China Grant (62006151, 62076161, 61802069, 61902241), Shanghai Sailing Program, and Shanghai Science and Technology Commission Grant (17JC1420200).

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A Two definitions of weak dominating set

In [1], the weakly (integral) dominating set was defined to dominate all “weakly observable vertices” instead of “vertices without self-loops”. This was indeed a flaw in the paper as in some extreme cases, the set may fail to dominate vertices in \( U \) (the set of vertices without self-loops) that are “strongly observable”. Therefore we ask for the set to dominate \( U \). Nevertheless the two definitions of domination number, both integral and fractional, differ by at most one and do not affect the asymptotic bounds.

Here we give an example to explain the difference between the definition of the weak domination number \( \delta \) in [1] and in this article. To ease the presentation, we call them \( \delta \) and \( \delta' \) respectively. We show that \( \delta = \delta' \) if \( \delta \geq 2 \), but it is possible that \( \delta = 1 \) and \( \delta' = 2 \).

![Figure 1: An example of \( \delta = 1, \delta' = 2 \)](image)

It is clear that \( \delta = 1 \) and \( \delta' = 2 \) in Figure 1 since the vertex \( C \) is strongly observable and in [1], the vertex \( C \) does not need to be dominated by a weak dominating set. Therefore, the minimum weak dominating set is \( \{ C \} \). However, in the proof of [1, Theorem 2] for the weakly observable graphs, they assumed that every vertex without a self-loop is dominated by the weakly dominating set. This is not true following their definition since the vertex \( C \), although strongly observable, is not dominated by itself and thus the lower bound on the probability that \( C \) is observed fails.

Hence we ask for the set to dominate the set of vertices without self-loops, namely \( U = \{ C, D \} \). The proof can then go through, and the only difference is that \( \delta' \) becomes two. It is also clear when \( \delta \geq 2 \), this situation will not occur as every strongly observable vertex without a self-loop can be dominated by any vertex other than itself.

B Proof of Theorem 1

Since our algorithm only deviates from the standard OSMD algorithm by incorporating an additional exploration term \( \gamma \cdot u \), the regret naturally consists of two parts: the standard OSMD regret and the amount introduced by the additional exploration.

Fix a sequence of loss functions \( \ell_1, \ldots, \ell_T \) and let \( a^* = \arg \min_{a \in [n]} \sum_{t=1}^T \ell_t(a) \) be the optimal arm.

**Lemma 15.** For any time horizon \( T \in \mathbb{N} \), the Algorithm 1 satisfies

\[
R(G,T) \leq \sum_{t=1}^T \mathbb{E} \left[ (X_t - e_{a^*}, \hat{\ell}_t) \right] + \gamma T.
\]

**Proof.** For every \( t = 1, 2, \ldots, T \), let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the random variables appeared at and before the \( t \)-th round. Define \( \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t] \), then

\[
R(G,T) = \mathbb{E} \left[ \sum_{t=1}^T \ell_t(A_t) \right] - \sum_{t=1}^T \ell_t(a^*) = \sum_{t=1}^T \mathbb{E}[\mathbb{E}_{t-1}[\ell_t(A_t)]] - \sum_{t=1}^T \mathbb{E}[\ell_t(a^*)].
\]
Since $\tilde{X}_t$ is $F_{t-1}$-measurable and $A_t \sim \tilde{X}_t$, we have
\[
R(G, T) = \sum_{t=1}^{T} \mathbb{E} [E_{t-1} [\ell_t(A_t)]] - \sum_{t=1}^{T} \mathbb{E} [\ell_t(a^*)]
\]
\[
= \sum_{t=1}^{T} \mathbb{E} [\langle \tilde{X}_t, \ell_t \rangle] - \sum_{t=1}^{T} \langle \epsilon_{a^*}, \ell_t \rangle \leq \sum_{t=1}^{T} \mathbb{E} [\langle X_t + \gamma \cdot u, \ell_t \rangle] - \sum_{t=1}^{T} \langle \epsilon_{a^*}, \ell_t \rangle
\]
\[
= \sum_{t=1}^{T} \mathbb{E} [\langle X_t - \epsilon_{a^*}, \ell_t \rangle] + \sum_{t=1}^{T} \gamma \cdot \langle u, \ell_t \rangle \leq \sum_{t=1}^{T} \mathbb{E} [\langle X_t - \epsilon_{a^*}, \hat{\ell}_t \rangle] + \gamma \cdot T,
\]
where in the last inequality we used the facts that $\hat{\ell}_t$ is an unbiased estimator for $\ell_t$ and $\langle u, \ell_t \rangle \leq 1$.

Expanding the first term, we have the following result.

**Lemma 16.**
\[
R(G, T) \leq \frac{D_{\Psi}(\Delta_n)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \mathbb{E}_{A_t \sim \tilde{X}_t} \left[ \sup_{z \in [Y_t, X_t]} \left\| \hat{\ell}_t \right\|^2_{(\nabla \psi(z))^{-1}} \right] + \gamma T,
\]
where $Y_t = \arg\min_{y \in \text{int}(\text{dom}(\psi))} (\eta y, \hat{\ell}_t) + B_\psi(y, X_t)$.

Lemma 16 is a consequence of Lemma 15 and an upper bound for $\sum_{t=1}^{T} \mathbb{E} [\langle X_t - \epsilon_{a^*}, \hat{\ell}_t \rangle]$. The latter is a standard regret bound for OSMD and a proof can be found in, e.g. [35] and references therein.

We are now ready to prove Theorem 1. It is proved by plugging our choice of potential function into the bound of Lemma 16.

**Theorem 1.** There exists an algorithm such that for any weakly observable graph, any time horizon $T \geq n^3 \log(n) / \delta^*(G)$, its regret satisfies $R(G, T) = O \left( (\delta^*(G) \log n)^{\frac{1}{3}} T^{\frac{2}{3}} \right)$.

**Proof.** Recall that we choose $\Psi(x) = \sum_{i \in [n]} x(i) \log x(i)$ for every $x \in \Delta_n$. Direct calculation yields $D_{\Psi}(\Delta_n) \leq \log n$, $(\nabla \psi(z))^{-1} = \text{diag}(z(1), \ldots, z(n))$ and $Y_t(i) = X_t(i) \cdot e^{-\eta \hat{\ell}_t(i)} \leq X_t(i)$ for all $t$ and $i \in [n]$. Therefore
\[
\mathbb{E}_{A_t \sim \tilde{X}_t} \left[ \sup_{z \in [X_t, Y_t]} \left\| \hat{\ell}_t \right\|^2_{(\nabla \psi(z))^{-1}} \right] = \mathbb{E}_{A_t \sim \tilde{X}_t} \left[ \sup_{z \in [X_t, Y_t]} \sum_{i=1}^{n} 1 [i \in N_{\text{out}}(A_t)]^2 \cdot (\sum_{j \in N_{\text{in}}(i)} X_t(j))^{-1} \cdot \hat{\ell}_t(i)^2 \cdot z(i) \right]
\]
\[
\leq \mathbb{E} \left[ \sum_{i=1}^{n} \frac{X_t(i)}{\sum_{j \in N_{\text{in}}(i)} X_t(j)} \right].
\]
It remains to lower bound $\sum_{j \in N_{\text{in}}(i)} \tilde{X}_t(j)$ for every $i \in [n]$, which is the probability that the arm $i$ is observed at the $t$-th round. We require that the probability is not too small compared to $X_t(i)$ for every $i \in [n]$. Recall $U = \{ i \notin N_{\text{in}}(i) \}$ denotes the set of vertices without self-loops. Then for every $i \notin U$, the self-loop on $i$ guarantees that the chance for $i$ to be observed is comparable to $X_t(i)$. On the other hand, if $i \in U$, we use our additional exploration term $\gamma \cdot u$ to lower bound the probability.

It is clear that $\gamma \leq \frac{1}{2}$ by our choice of $\gamma$ and $T$. So the contribution of vertices in $V \setminus U$ is
\[
\sum_{i \notin U} \frac{X_t(i)}{\sum_{j \in N_{\text{in}}(i)} X_t(j)} = \sum_{i \notin U} \frac{X_t(i)}{\sum_{j \in N_{\text{in}}(i)} (1 - \gamma) \cdot X_t(j) + \gamma \cdot u(j)} \leq \sum_{i \notin U} \frac{1}{1 - \gamma} \leq 2n.
\]
(1)

The contribution of vertices in $U$ is
\[
\sum_{i \in U} \frac{X_t(i)}{\sum_{j \in N_{\text{in}}(i)} X_t(j)} \leq \sum_{i \in U} \gamma \sum_{j \in N_{\text{in}}(i)} \frac{X_t(i)}{u(j)} \leq \sum_{i \in U} \frac{X_t(i) \cdot \delta^*(G)}{\gamma} \leq \frac{\delta^*(G)}{\gamma},
\]
(2)
where (\(\lor\)) is due to the first constraint of the linear program \(D\) and our definition of \(u\).

Combining (1) and (2), we obtain
\[
R(G,T) \leq \frac{\log n}{\eta} + \eta n T + \frac{\eta \delta^*(G) T}{2\gamma} + \gamma T. \tag{3}
\]

The theorem follows by plugging in our parameters \(\gamma = \left(\frac{\delta^*(G) \log n}{\eta}\right)^{1/3}\), \(\eta = \frac{\gamma^2}{T \alpha}\) and assuming \(T \geq n^3 \log n / \delta^*(G)^2\).

\[\blacksquare\]

C Proof of lower bounds

C.1 Proof of Theorem 3

**Theorem 3.** If \(G\) is weakly observable, then for any algorithm and any sufficiently large time horizon \(T \in \mathbb{N}\), there exists a sequence of loss functions such that \(R(G,T) = \Omega\left(\left(\frac{\delta^*}{\alpha}\right)^{1/3} \cdot T^{2/3}\right)\), where \(\alpha\) is the integrality gap of the linear program for vertex packing.

**Proof.** Since \(\delta^* = \zeta^* = \alpha \cdot \zeta\), the bound in Theorem 3 is equivalent to \(R(G,T) = \Omega\left(\left(\frac{\zeta^*}{\alpha}\right)^{1/3} \cdot T^{2/3}\right)\).

We will prove that \(G\) contains a 1-packing independent set of size \(\Theta(\zeta)\), then the theorem follows from Theorem 2.

Let \(\{y_j\}_{j \in U}\) be the optimal solution of the integral program \(D\). Let \(S^\dagger \triangleq \{j \in U : y_j^\dagger = 1\}\) be the corresponding vertex packing set. Then clearly \(\zeta = |S^\dagger|\). We will show that there exists a 1-packing independent set \(H \subseteq S^\dagger\) with \(|H| \geq |S^\dagger|/3\).

We use the following greedy strategy to construct \(H\).

- **Initialization**: Set \(H = \emptyset\) and \(S' = S^\dagger\).
- **Update**: While \(S' \neq \emptyset\): Pick a vertex \(v \in S'\) with minimum \(|N_{in}(v) \cap S'|\); Set \(H \leftarrow H \cup \{v\}\); \(S' \leftarrow S' \setminus (N_{in}(v) \cup \{v\} \cup N_{out}(v))\).

First of all, the set \(H\) constructed above must be an independent set as whenever we add some vertex \(v\) into \(H\), we remove all its incident vertices, both its in-neighbors and out-neighbors, from \(S'\). Clearly each \(S'\) after removing these vertices is still a vertex packing set. Therefore, every \(v \in S'\) has out-degree at most one in \(G[S']\). This implies that the vertex \(v \in S'\) with minimum \(|N_{in}(v) \cap S'|\), or equivalently minimum in-degree in \(G[S']\), satisfies \(|N_{in}(v) \cap S'| \leq 1\). So the update step \(S' \leftarrow S' \setminus (N_{in}(v) \cup \{v\} \cup N_{out}(v))\) removes at most three vertices from \(S'\). This concludes that \(H\) is a 1-packing independent set of size at least \(|S^\dagger|/3\).

\[\blacksquare\]

C.2 Proof of Lemma 13

**Lemma 13.** For any 1-degenerate directed graph, the integrality gap \(\alpha = 1\).

**Proof.** Let \(G = (V,E)\) be a 1-degenerate directed graph. We show that we can construct a vertex packing set \(S\) with \(|S| \geq \sum_{j \in U} y_j\) for any feasible solution \(\{y_j\}_{j \in U}\) of \(D\). The lemma follows by applying this to the optimal solution.

We use \(y_S \in \{0,1\}^U\) to denote the indicator vector of \(S\). So we have for every \(i \in U\), \(y_S(i) = 1 \iff i \in S\). The construction is to apply the following greedy strategy to determining \(y_S\) until the graph is empty:

- Pick a vertex \(i \in G\) with in-degree one. Let \((j,i)\) be the unique in-edge of \(i\). Remove the edge \((j,i)\) from \(E\). If \(i \in U\) and the value of \(y_S(i)\) is not determined, then (1) set \(y_S(i) = 1\); (2) for all \(k \in U \setminus \{i\}\) such \((j,k) \in E\), set \(y_S(k) = 0\).
- Pick a vertex with in-degree zero and out-degree at most one, and remove both the vertex and the out-edge. Keep doing so until no such vertex exists in \(G\).
It is clear that the above algorithm terminates at an empty $G$ since all operations to the graph coincide with ones defining 1-degeneration. We only need to verify that

1. $y_S$ is a feasible solution of $\mathcal{D}$; and
2. $\sum_{j \in U} y_S(j) \geq \sum_{j \in U} \hat{y}_j$ for any feasible solution $\{\hat{y}_j\}_{j \in U}$.

Let us first verify (1). The vector $y_S$ can become infeasible only when we set some $y_S(i) = 1$. Note that this happens only when the in-degree of $i$ is one, or equivalent there is only a unique edge $(j, i)$ pointing to $i$. We do not violate the constraint on $j$ as we set all $y_S(k) = 0$ for $k \in U \setminus \{i\}$ and $(j, k) \in E$. It is still possible that there exists some other $j' \in V$ such that the edge $(j', i)$ exists but has been removed. Since the value of $y_S(i)$ is not determined before, this happens only if $j'$ has out-degree one at the beginning, and so the constraint on $j'$ cannot be violated either.

To see (2), we assume the value on each $j \in U$ is $\hat{y}_j$ at the beginning. Our procedure to construct $y_S$ can be equivalently viewed as a process to change each $\hat{y}_j$ to either 0 or 1. That is, after we set the value of some $y_S(j)$ to 0 or 1, we change $\hat{y}_j$ to the same value. It is easy to verify that during the process, we never decrease $\sum_{j \in U} \hat{y}_j$. At last, $y_S(j) = \hat{y}_j$ for all $j \in U$ and some optimal solution $\{\hat{y}_j\}_{j \in U}$, and (2) is verified.

C.3 Proof for special graphs in Section 4.3.2

Let $n = |V_1| + |V_2| = 2^{k+1} - 2$. It is clear that the degree of each vertex $y_\beta \in V_2$ is $2^{k-1} = \frac{n+2}{4}$. A moment’s reflection should convince you that $\delta^*(G) \leq 2$ as we can put a fraction of $\frac{4}{n+2}$ on each $x_\alpha \in V_1$.

Therefore it follows from Theorem 1 that our algorithm has regret $O \left( (\log n)^{\frac{3}{4}} \cdot T^{\frac{3}{4}} \right)$ on this family of instances. Moreover, $V_2$ is a $\frac{n+2}{4}$-packing independent set of size $\frac{n}{2}$. It follows from Theorem 2 that any algorithm has regret $\Omega \left( (\log n)^{\frac{3}{4}} \cdot T^{\frac{3}{4}} \right)$.

Finally, we remark that $\delta(G) = k = \log_2 \left( \frac{n+2}{4} \right)$. To see this, we first observe that the minimum dominating set of the graph must reside in $V_1$ since $G[V_2]$ is an independent set. Then we show any $S \subseteq V_1$ with $|S| \leq k - 1$ cannot dominate all vertices in $V_2$. Assume $S = \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{k-1}}\}$. In fact, a vertex $y_\beta \in V_2$ is dominated by $S$ iff $\langle \alpha_i, \beta \rangle = 1$ for some $i \in [k - 1]$. In other words, if we view each $\alpha_i$ as a column vector in $\mathbb{F}_2^n$ and define the matrix $A = [\alpha_1, \alpha_2, \ldots, \alpha_{k-1}]$, then $y_\beta$ is dominated by $S$ iff $A^T \beta \neq 0$. However, the dimension of $A^T$ is at most $k - 1$ and therefore by the rank-nullity theorem that its null space is of dimension at least one. This means that there exists a certain $\beta' \in V_2$ with $A^T \beta' = 0$. So $y_{\beta'}$ is not dominated.

The above fact implies that the upper bound and the lower bound of the regret for this instance proved in [1] based on $\delta$ are $O \left( (\log n)^{\frac{3}{4}} \cdot T^{\frac{3}{4}} \right)$ and $\Omega \left( T^{\frac{3}{4}} \right)$ respectively.

D Proof of Theorem 2

Theorem 2. Let $G = (V, E)$ be a directed graph. If $G$ contains a $k$-packing independent set $S$ with $|S| \geq 2$, then for any randomized algorithm and any time horizon $T$, there exists a sequence of loss functions such that the expected regret is $\Omega \left( \max \left\{ \frac{|S|}{k}, \log |S| \right\} \cdot T^{\frac{3}{4}} \right)$.

Our strategy to prove Theorem 2 is to reduce the problem of minimizing regret to the problem of best arm identification. We first define the problem and discuss its complexity in Appendix D.2 with the help of Appendix D.1. Then we construct the reduction and prove Theorem 2 in Appendix D.3.

D.1 Information Theory

We borrow tools from information theory to establish lower bounds. More details can be found in the standard textbook [16] on the topic. To ease the notation, each “log” appeared in the article
without subscript is of base $e$. We fix a probability space $(Ω, F, Pr)$ and let $X, Y : Ω → U$ be discrete-valued random variables for a finite set $U$.

The entropy of $X$ is $H(X) \triangleq -\sum_{x \in U} Pr[X = x] \cdot \log Pr[X = x]$. The conditional information $H(X|Y) \triangleq -\sum_{x,y \in U} Pr[X = x, Y = y] \log Pr[X = x | Y = y]$. The mutual information between $X$ and $Y$ is $I(X; Y) \triangleq \sum_{x,y \in U} Pr[X = x, Y = y] \cdot \log \frac{Pr[X = x, Y = y]}{Pr[X = x] Pr[Y = y]}$. It is a basic fact that $H(X) = H(X|Y) + I(X; Y)$. Suppose we have another random variable $Z$, then $I(X; Y|Z) \triangleq H(X|Z) - H(X|Y, Z)$.

Suppose $Z : Ω → W$ is a random variable correlated to $X$ and one needs to guess the value of $X$ via observing $Z$. The Fano’s inequality reveals the inherent difficulty of this task:

**Lemma 17 (Fano’s inequality [18]).** For any function $f : W → U$, it holds that

$$Pr[f(Z) \neq X] ≥ \frac{H(X) - I(X; Z) - \log 2}{\log |U|}.$$  

If we assume $Y = (Y_1, \ldots, Y_n)$ is a vector of random variables such that $\{Y_i\}_{i \in [n]}$ are mutually independent conditional on $X$, then we have the following lemma of tensorization of mutual information:

**Lemma 18 (Tensorization of Mutual Information).** If $Y = (Y_1, \ldots, Y_n)$ and random variables $\{Y_i\}_{i \in [n]}$ are mutually independent conditional on $X$, then

$$I(X; Y) ≤ \sum_{i=1}^{n} I(X; Y_i).$$

**Proof.** By the chain rule of the mutual information,

$$I(X; Y) = \sum_{i=1}^{n} I(X; Y_i | Y_1, \ldots, Y_{i-1}).$$

For every $i \in [n]$, we have

$$I(X; Y_i | Y_1, \ldots, Y_{i-1}) = H(Y_i | Y_1, \ldots, Y_{i-1}) - H(Y_i | X, Y_1, \ldots, Y_{i-1}) \leq H(Y_i) - H(Y_i | X) = I(X; Y_i),$$

where we use the fact that $H(Y_i | X, Y_1, \ldots, Y_{i-1}) = H(Y_i | X)$ due to the conditional mutual independence.

### D.2 Best arm identification

The problem of best arm identification is formally defined as follows.

| BEST ARM IDENTIFICATION (BESTARMID) |
|-------------------------------------|
| **Input:** An instance of $n$ stochastic arms where the loss of the $i$-th arm follows $Ber(p_i)$. Each pull of arm $i$ receives a loss $\sim Ber(p_i)$ independently. |
| **Problem:** Determine the arm $i$ with minimum $p_i$ via pulling arms. |

Therefore, an instance of BESTARMID is specified by a vector $p = (p_1, \ldots, p_n) \in [0, 1]^n$. The goal is to design a strategy to find the arm $i$ with minimum $p_i$ via pulling arms. We call the arm with minimum $p_i$ the best arm. We want to minimize the number of pulls in total and the main result of this section is to provide lower bounds for this task: For some collection of vectors $p$, if the total number of pulls is below some threshold, then any algorithm cannot find the best arm for all instances with high probability.

In the following, we may abuse notations and simply say a vector $p \in [0, 1]^n$ is an instance of BESTARMID. Now for every $j \in [n]$, we define an instance $p^{(j)} = (p_1^{(j)}, \ldots, p_n^{(j)}) \in \mathbb{R}^n$ as...
Armed with Lemma 20, we now prove Lemma 19. For the convenience of the reduction in Lemma 19, we also define $p^{(0)}$ as an $n$-dimensional all-one vector.

There are several ways to explore the $n$ arms in order to determine the one with minimum mean. We first consider the most general strategy: In each round, the player can pick an arbitrary subset $S \subseteq [n]$ and pull the arms therein. The game proceeds for $T$ rounds and then the player needs to determine the best arm with collected information. Note that in each round, the exploring set $S$ may adaptively depend on previous information.

Now we fix such a (possibly randomized) strategy and denote it by $B$. For every $j \in [n]$ and $i \in [n]$, we use $N_{i}^{(j)}$ to denote the number of times that the $i$-th arm is pulled when we run $B$ on the instance $p^{(j)}$. Let $N^{(j)} = \sum_{i \in [n]} N_{i}^{(j)}$. Note that all these numbers can be random variables where the randomness comes from coins tossed in $B$.

**Lemma 19.** Assume $\varepsilon < 0.125$ and $n$ is sufficiently large. If for every $j \in [n]$, the algorithm $B$ can correctly identify the best arm in $p^{(j)}$ with probability at least 0.999 and outputs any arm for $p^{(0)}$, then for some $j \in \{0, 1, \ldots, n\}$, $\mathbb{E}\left[N^{(j)}\right] \geq \frac{Cn}{\varepsilon^2}$, where $C > 0$ is a universal constant.

Our proof of Lemma 19 is based on a reduction from a similar problem studied in [25], in which the following instances of BESTARMID have been considered:

- $q^{(0)} = (q_{0}^{(0)}, \ldots, q_{n}^{(0)}) \in \mathbb{R}^{n+1}$ where for every $i \in \{0, 1, \ldots, n\}$, $q_{i}^{(0)} = \begin{cases} \frac{1}{2}, & i = 0; \\ \frac{1}{2} - \varepsilon, & i \neq 0. \end{cases}$
- $\forall j \in [n]: q^{(j)} = (q_{0}^{(j)}, \ldots, q_{n}^{(j)}) \in \mathbb{R}^{n+1}$ where for every $i \in \{0, 1, \ldots, n\}$, $q_{i}^{(j)} = \begin{cases} \frac{1}{2}, & i = 0; \\ \frac{1}{2} - \varepsilon, & i = j; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$

Let us fix a strategy $B'$ for this collection of instances. Similarly define quantities $N^{(j)'}_{i}$ and $N^{(j)'}_{i}$ for $i, j = 0, 1, \ldots, n$ as we did for $B$ above. The proof of [25, Theorem 1] implicitly established the following:

**Lemma 20.** Assume $\varepsilon < 0.125$. If for every $j = 0, 1, \ldots, n$, the algorithm $B'$ can correctly identify the best arm in $q^{(j)}$ with probability at least 0.996, then

$$\mathbb{E}\left[N^{(0)'}\right] \geq \frac{C'n}{\varepsilon^2},$$

where $C'$ is a universal constant.

Armed with Lemma 20, we now prove Lemma 19.

**Proof of Lemma 19.** Assuming for the sake of contradiction that Lemma 19 does not hold, we now describe an algorithm $B'$ who can correctly identify the best arm in $q^{(j)}$ with probability at least 0.999 for every $j \in \{0, 1, \ldots, n\}$ and $\mathbb{E}\left[N^{(j)'}\right] < \frac{C'n}{\varepsilon^2}$ for sufficiently large $n$.

Since Lemma 19 is false, we have a promised good algorithm $B$ with $C = \frac{C'}{10}$. Given any instance $q^{(j)}$ with $j \in \{0, 1\}$, we first use $B$ to identify the best arm $i^*$ among arms in $\{1, 2, \ldots, n\}$. This step succeeds with probability 0.999. Then we are left to compare arm $i^*$ with arm 0. We pull each of the two for $K$ times and output the one with minimum practical mean. By Chernoff bound, this approach can successfully identify the best of the two with probability 0.999 when $K = \frac{C'n}{\varepsilon^2}$ for some universal constant $C'' > 0$. Therefore we have $\mathbb{E}\left[N^{(j)'}\right] < \frac{C'n}{\varepsilon^2} + \frac{C''n}{\varepsilon^2} \leq \frac{C'n}{\varepsilon^2}$ for sufficiently large $n$ and we can identify the best arm with probability at least 0.998 > 0.996 by the union bound. ■

Lemma 19 is quite general in the sense that it applies to any algorithm for BESTARMID. If we restrict our algorithm for BESTARMID to some special family of strategies, then a stronger lower bound can be obtained.
Consider the following algorithm $'C$: In every round, the player pulls each arm once. After $T$ rounds (so each arm has been pulled $T$ times), the player determines the best arm via the collected information. Note that we do not restrict how the player determines the best arm after collecting information for $T$ rounds, his/her strategy can be either deterministic or randomized. We prove a lower bound for $T$:

**Lemma 21.** If for every $j \in [n]$, the algorithm $'C$ can correctly identify the best arm in $p^{(j)}$ with probability at least 0.5, then $T \geq \frac{\log(n/4)}{16\varepsilon^2}$.

Note that if we apply Lemma 19 to $'C$, we can only get $T = \Omega\left(\frac{1}{\varepsilon^2}\right)$. The reason that we can obtain a stronger lower bound is the non-adaptive nature of $'C$.

**Proof of Lemma 21.** As a randomized algorithm can be viewed as a distribution of deterministic ones, we only need to prove the lower bound for deterministic algorithms. We prove the contrapositive of the lemma for a deterministic $'C$. Assume $T < \frac{\log(n/4)}{16\varepsilon^2}$. We let $W = (w_{ij})_{i \in [n]} \in [0, 1]^{n \times T}$ be a random matrix where $w_{ij}$ is the loss of the $i$-th arm during the $j$-th pull. Our task is to study the following stochastic process, which is called **hypothesis testing** in statistics:

- Pick $J \in [n]$ uniformly at random.
- Use $p^{(J)}$ to generate the matrix $W$.
- Apply $'C$ on $W$ to obtain $\hat{J} = 'C(W)$.

It then follows from Fano’s inequality (Lemma 17) that

$$\Pr\left[ \hat{J} \neq J \right] \geq \frac{H(J) - I(J; W) - \log 2}{\log n} = 1 - \frac{I(J; W) + \log 2}{\log n}. \quad (4)$$

It remains to upper bound $I(J; W)$. To ease the presentation, we write $W = [w^{(1)}, w^{(2)}, \ldots, w^{(T)}]$ where each $w^{(j)} = (w^{(j)}_1, w^{(j)}_2, \ldots, w^{(j)}_n)^T$ for $j \in [T]$ is an $n$-dimensional column vector. It is clear that these column vectors are mutually independent conditional on $J$. Moreover, for each $j \in [T]$, entries in $w^{(j)}$ are mutually independent conditional on $J$ as well. Therefore, it follows from Lemma 18 that

$$I(J; W) \leq \sum_{j \in [T]} I(J; w^{(j)}) \leq \sum_{j \in [T]} \sum_{i \in [n]} I(J; w^{(j)}_i) = nT \cdot I(J; w^{(1)}_1) \leq 8\varepsilon^2 T, \quad (5)$$

where the last inequality is from a direct calculation of $I(J; w^{(1)}_1)$.

Combining (4) and (5), we obtain

$$\Pr\left[ \hat{J} \neq J \right] \geq 1 - \frac{8\varepsilon^2 T + \log 2}{\log n} > \frac{1}{2},$$

which is a contradiction. \hfill \qed

### D.3 From BestArmID to regret

Let $G = (V, E)$ be a directed graph containing a $k$-packing independent set $S$ with $|S| \geq 2$. We assume without loss of generality that $S = \{1, 2, \ldots, |S|\}$. We construct a collection of stochastic bandit instances $\{I^{(j)}\}_{j \in |S|}$ with feedback graph $G$ as follows: For every $t \in [T]$ and $j \in |S|$, we use $\ell_t^{(j)}$ to denote the loss function of $I^{(j)}$ at round $t$. Let $\varepsilon \in (0, 1)$ be a parameter. Then

- For all $i \not\in S$, $\ell_t^{(j)}(i) = 1$;
- For $i = j$, $\ell_t^{(j)}(i) \sim \text{Ber}\left(\frac{1}{2} - \varepsilon\right)$;
- For all $i \in S \setminus \{j\}$, $\ell_t^{(j)}(i) \sim \text{Ber}\left(\frac{1}{2}\right)$. 


Given an algorithm \( \mathcal{A} \), a time horizon \( T \) and \( j \in [n] \), we use \( R(\mathcal{A}, j, T) \) to denote the expected regret after \( T \) rounds when applying \( \mathcal{A} \) on \( I^{(j)} \).

We show that for any \( j \in [|S|] \), if an algorithm \( \mathcal{A} \) has low expected regret on \( I^{(j)} \), then one can turn it into another algorithm \( \hat{\mathcal{A}} \) who can identify the best arm \( j \) among \( S \) without exploring \( S \) much.

**Lemma 22.** Let \( T \in \mathbb{N} \) be the time horizon, \( \delta \in (0,1) \) be a parameter. Let \( C = \min \{ C, 1 \} \) where \( C \) is the constant in Lemma 19. Assuming there exists an algorithm \( \mathcal{A} \) such that \( R(\mathcal{A}, j, T) \leq \frac{C \delta \varepsilon T}{4} \) for every \( j \in [|S|] \), then there exists an algorithm \( \hat{\mathcal{A}} \) satisfying for every \( j \in [|S|] \), if we apply \( \hat{\mathcal{A}} \) on \( I^{(j)} \) for \( T \) rounds, then

- \( \hat{\mathcal{A}} \) output \( j \) with probability at least \( 1 - \delta \);
- arms in \( V \setminus S \) are pulled at most \( \frac{C \delta \varepsilon T}{2} \) times in total.

**Proof.** The algorithm \( \hat{\mathcal{A}} \) simply simulates \( \mathcal{A} \) for \( T \) rounds and outputs the mostly-pulled arm, breaking ties arbitrarily.

- If the mostly-pulled arm is not \( j \), then suboptimal arms must be pulled at least \( T/2 \) times. These pulls contribute at least \( \frac{C \delta \varepsilon T}{2} > \frac{C \delta \varepsilon T}{4} \) expected regret.
- If arms in \( V \setminus S \) are pulled more than \( \frac{C \delta \varepsilon T}{2} \) times in total, then these pulls already contribute more than \( \frac{C \delta \varepsilon T}{4} \) regret.

Using the reduction in Lemma 22, we can prove Theorem 2 via lower bounds for BESTARMID.

**Proof of Theorem 2.** Assume both \( |S| \) and \( T \) are sufficiently large.

We first establish the \( \Omega \left( \left( \frac{|S|}{T} \right)^{1/4} \cdot T^{1/4} \right) \) lower bound. Choose \( \varepsilon = \left( \frac{|S|}{T} \right)^{1/4} \) and \( \delta = 0.001 \). Suppose there exists an algorithm \( \mathcal{A} \) such that \( R(\mathcal{A}, j, T) \leq \frac{T}{2000} \left( \frac{|S|}{T} \right)^{1/4} \cdot T^{1/4} \) for every \( j \in [|S|] \). Then by Lemma 22, we can find an algorithm \( \hat{\mathcal{A}} \) that can correctly identify the best arm with probability at least 0.999, and observe arms in \( S \) for at most \( \frac{T}{2000} |S|^{1/4} k^{1/4} \cdot T^{1/4} \) times (Since each pull of \( V \setminus S \) observes at most \( k \) arms in \( S \)). This contradicts Lemma 19.

Then we establish the \( \Omega \left( (\log |S|)^{1/4} \cdot T^{1/4} \right) \) lower bound, which needs more effort. Choose \( \varepsilon = \left( \frac{\log |S|}{T} \right)^{1/4} \) and \( \delta = 0.001 \). Similar to above, suppose there exists an algorithm \( \mathcal{A} \) such that \( R(\mathcal{A}, j, T) \leq \frac{T}{2000} (\log |S|)^{1/4} \cdot T^{1/4} \) for every \( j \in [|S|] \). Then by Lemma 22, we can find an algorithm \( \hat{\mathcal{A}} \) that can correctly identify the best arm with probability at least 0.999, and pull arms in \( V \setminus S \) for at most \( \frac{T}{2000} (\log |S|)^{1/4} \cdot T^{1/4} \) times in total.

Note that each pull of an arm \( v \in V \setminus S \) in \( \mathcal{A} \) observes arms in a subset \( S' \subseteq S \). The key observation is that we can assume without the loss of generality that \( S' = S \), since this assumption only increases the power of the algorithm. This assumption can make our algorithm non-adaptive.

We rigorously prove this using a coupling argument. Assume we design an algorithm \( \hat{\mathcal{B}} \) in which each pull of an arm in \( V \setminus S \) can observe all arms in \( S \). We show that \( \hat{\mathcal{B}} \) can perfectly simulate \( \hat{\mathcal{A}} \), and therefore \( \hat{\mathcal{B}} \) is more powerful. So if we have a lower bound for \( \hat{\mathcal{B}} \), it is automatically a lower bound for \( \hat{\mathcal{A}} \). Note that the behavior of an algorithm for BESTARMID in each round is determined by the information collected and coins tossed so far. If we apply both \( \hat{\mathcal{A}} \) and \( \hat{\mathcal{B}} \) to some \( I^{(j)} \) and assume (1) both algorithms use the same source of randomness; and (2) all the loss vectors \( \ell_i^{(j)} \) use the same source of randomness, then the information collected by \( \hat{\mathcal{A}} \) at any time is a subset of that of \( \hat{\mathcal{B}} \). Therefore we can use \( \hat{\mathcal{B}} \) to simulate \( \hat{\mathcal{A}} \) and outputs the same as \( \hat{\mathcal{A}} \).

Based on this observation, we use the non-adaptive nature of \( \hat{\mathcal{B}} \) and apply Lemma 21 to get an immediate contradiction.
E Time-varying graphs

In this section, we will consider time-varying graphs. Instead of a fixed graph $G = (V, E)$ for all $T$ rounds, for each round $t \in [T]$, we have a graph $G_t = (V, E_t)$. We slightly abuse notations here and let $G = (G_t)_{t \in [T]}$ and $R(G, T)$ denote the min-max regret for time-varying graphs $G$.

**Corollary 23.** The min-max regret for sequential time-varying graphs $G = (G_t)_{t \in [T]}$ satisfies

$$R(G, T) = O \left( \left( \delta^*(G) \log n \right)^{\frac{3}{2}} T^{\frac{5}{2}} \right),$$

where $\delta^* \triangleq \frac{\sum_{t=1}^{T} \delta^*(G_t)}{T}$ is the average fractional weak domination number for graphs $G$ in $T$ rounds.

**Proof.** We can slightly modify Algorithm 1: In each round $t$, we use $G_t$ as an input instance to calculate $\hat{\ell}_t$ and $u_t$. Then the left proof is totally similar to the proof of Theorem 1 except for replacing (3) with $\text{Regret} \leq \frac{\log n}{\eta} + \eta nT + \frac{\eta \sum_{t=1}^{T} \delta^*(G_t)}{2\gamma} + \gamma T = \frac{\log n}{\eta} + \eta nT + \frac{\eta \delta^* T}{2\gamma} + \gamma T$. ■

A similar case is the probabilistic graph model. A probabilistic graph can be denoted as a triple $G = (V, E, P)$ where $P : E \rightarrow (0, 1)$ assigns a triggering probability for each edge in $E$. In each round $t$, a realization of $G$ is a graph $G_t = (V, E_t)$ where $E_t = \{ e \in E : O_{te} = 1 \}$ and here $O_{te}$ is an independent Bernoulli random variable with mean $P(e)$. By abuse of notations, we denote by $G = (G_t)_{t \in [T]}$ the sequential realizations of $G$. Define $R(G, T) = \inf_{\text{alg}} \sup_{P} E \left[ \sum_{t=1}^{T} \ell_t(A_t) - \ell_t(a^*) \right]$ as the min-max regret for the probabilistic graph $G$ and here the randomness comes from the algorithm and sequential graphs $G$.

**Corollary 24.** The min-max regret for the probabilistic graph $G$ satisfies

$$R(G, T) = O \left( \left( \mathbb{E} [\delta^*(G_1)] \log n \right)^{\frac{3}{2}} T^{\frac{5}{2}} \right),$$

**Proof.**

$$R(G, T) = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(A_t) - \ell_t(a^*) \right] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(A_t) - \ell_t(a^*) \mid G \right] \right] \leq O \left( \left( \mathbb{E} [\delta^*(G_1)] \log n \right)^{\frac{3}{2}} T^{\frac{5}{2}} \right),$$

where $(\hat{\circ})$ follows from Corollary 23 and $(\clubsuit)$ comes from the Jensen inequality and the independence of $(G_t)_{t \in [T]}$. ■

F Numerical experiments

According to Theorem 1, our algorithm will outperform Exp3.G in [1] when $\delta^* \ll \delta$. Therefore, we run our experiments on graphs $G = (V_1 \cup V_2, E)$ with orthogonal relation on $\mathbb{F}_2^k$.

We choose $k = 2, 3, 4, 5$ and set $T = 20 \times n^3 \log(n) / \delta^2(G)$, $\gamma = \left( \frac{\delta^*(G) \log n}{T} \right)^{1/3}$, and $\eta = \frac{\gamma^2}{\delta^*(G)}$ for our algorithm. The similar parameters are set for Exp3.G by replacing $\delta^*$ with $\delta$. Our adversary is nonoblivious which means the loss vector $\ell_t$ is allowed to depend on $F_{t-1}$. Concretely, for each time $t$, the adversary will see the distribution $X_t$ and find the vertex $i$ in $V_2$ with the minimum $X_t(i)$. Then the adversary provides $i$ with loss 0 and loss 1 for all other vertices.

We can immediately see from Figure 2 that our algorithm always outperforms Exp3.G when $t$ is large compared to $T$.

According to Appendix C.3, $\delta^* = \frac{\delta^2}{\delta - 1}$ and $\delta = k$. Table 2 shows the experimental ratio of our algorithm’s regret to Exp3.G’s in the terminating time $T$. The ratio is positively linearly related to $k$, which supports our theory result $\frac{\delta}{\delta^*}$ to some extent.
Figure 2: Regret comparison for different sizes of graphs

Table 2: Regret ratio of two algorithms

| $k$ | 2    | 3    | 4    | 5    |
|-----|------|------|------|------|
| regret of Exp3.G | 724.6 | 5656.3 | 36461.5 | 157909.5 |
| regret of Algorithm 1 | 417.9 | 2443.0 | 11115.1 | 53305.8 |
| ratio | 1.73 | 2.32 | 3.28 | 2.96 |