A BASIC FRAMEWORK FOR FIXED POINT THEOREMS: BALL SPACES AND SPHERICAL COMPLETENESS

HANNA ĆMIEL, FRANZ-VIKTOR KUHLMANN AND KATARZYNA KUHLMANN

Abstract. We systematically develop a general framework in which various notions of functions being contractive, as well as of spaces being complete, can be simultaneously encoded. Derived from the notions of ultrametric balls and spherical completeness, a very simple structure is obtained which allows this encoding. We give examples of generic fixed point theorems which then can be specialized to theorems in various applications which work with contracting functions and some sort of completeness property of the underlying space. As examples of such applications we discuss metric spaces, ultrametric spaces, ordered groups and fields, topological spaces, partially ordered sets and lattices. We characterize the particular properties of each of these cases and classify the strength of their completeness property. We discuss operations on the sets of balls to determine when they lead to larger sets; if so, then the properties of these larger sets of balls are determined. This process can lead to stronger completeness properties of the ball spaces, or to ball spaces of newly constructed structures, such as products. Further, the general framework makes it possible to transfer concepts and approaches from one application to the other; as examples we discuss theorems analogous to the Knaster–Tarski Fixed Point Theorem for lattices and theorems analogous to the Tychonoff Theorem for topological spaces. Finally, we present some generic multidimensional fixed point theorems and coincidence theorems.

Date: May 22, 2019.

1991 Mathematics Subject Classification. Primary 54A05, 54H25; Secondary 03E75, 06A05, 06A06, 06B23, 06B99, 06F20, 12J15, 12J20, 13A18, 47H09, 47H10, 54C10, 54C60, 54E50.

Key words and phrases. ball space, metric space, ultrametric space, topological space, partially ordered set, lattice, ordered abelian group, ordered field, spherically complete, complete, compact, fixed point theorem, multivalued fixed point theorem, coincidence theorem, Caristi–Kirk Fixed Point Theorem, Knaster–Tarski Theorem, Tychonoff Theorem.

The first two authors are supported by a Polish Opus grant 2017/25/B/ST1/01815.
## Contents

1. Introduction 3
2. Zorn’s Lemma in the context of ball spaces 10
   2.1. The case of $S_2$ ball spaces 11
   2.2. Posets of nests of balls 11
   2.3. Proof of the fixed point theorems 12
3. Some facts about the hierarchy of ball spaces 14
   3.1. A refinement of the hierarchy 14
   3.2. Connection with posets 15
   3.3. Singleton balls 16
   3.4. Ball spaces of ultrametric type 16
   3.5. Intersection closed ball spaces 16
4. Ball spaces and their properties in various applications 18
   4.1. Ultrametric spaces 18
   4.2. Metric spaces with metric balls 21
   4.3. Metric spaces with Caristi–Kirk balls or Oettli–Théra balls 24
   4.4. Ordered abelian groups and fields 28
   4.5. Topological spaces 31
   4.6. Partially ordered sets 34
   4.7. Lattices 38
5. $\mathbb{S}^*$ ball spaces 40
   5.1. Spherical closures in $\mathbb{S}^*$ ball spaces 41
   5.2. Ball spaces induced on subsets of $\mathbb{S}^*$ ball spaces 41
6. Set theoretic operations on ball spaces 43
   6.1. Subsets of ball spaces 43
   6.2. Unions of two ball spaces on the same set 44
   6.3. Closure under unions of balls 44
   6.4. Closure under nonempty intersections of balls 45
   6.5. Closure under finite unions and under intersections 47
6.6. The topology associated with an $S^*_i$ ball space 47
7. Shifting concepts between applications 48
   7.1. Knaster–Tarski type theorems 48
   7.2. Tychonoff type theorems 52
8. Other results related to fixed point theorems 56
   8.1. Multivalued fixed point theorems 57
   8.2. Coincidence theorems 58
References 59
1. Introduction

Fixed Point Theorems (FPTs) can be divided into two classes: those dealing with functions that are in some sense “contracting”, like Banach’s FPT and its ultrametric variant (cf. [22], [25]), and those that do not use this property (explicitly or implicitly), like Brouwer’s FPT. In this paper, we will be concerned with the first class.

Under which conditions do “contracting” functions have a fixed point? First of all, we have to say in which space we work, and we have to specify what we mean by “contracting”. These specifications will have to be complemented by a suitable condition on the space, in the sense that it is “rich” or “complete” enough to contain fixed points for all “contracting” functions.

In our paper [12], we have developed a general framework for this procedure. It presents a minimal setting in which the necessary conditions on the function and the space can be formulated. After having proved generic FPTs in this setting, they can then be adapted and interpreted in the settings of various applications. Let us briefly sketch the basics of this approach.

In all of the applications we will discuss in this paper, we look at functions $f$ on a set $X$ that has some additional structure. In order for FPTs to work, the set must have some completeness property under this structure. In all of the applications, the structure gives rise to collections $\mathcal{B}$ of distinguished subsets, explicitly or implicitly. If they are suitably chosen, then the required completeness property is equivalent to a completeness property of the collections $\mathcal{B}$, which we will introduce below. This will show that working with the set $X$ together with (one or more) collections $\mathcal{B}$ constitutes a unifying framework for FPTs. This is because they provide the minimal structure that allows the definition of completeness properties which encode the required completeness properties in each of the applications we will discuss.

A ball space $(X, \mathcal{B})$ consists of a nonempty set $X$ together with a nonempty set $\mathcal{B}$ of distinguished nonempty subsets $B$ of $X$. Note that $\mathcal{B}$, a subset of the power set $\mathcal{P}(X)$, is partially ordered by inclusion; we will write $(\mathcal{B}, \subseteq)$ when we refer to this partially ordered set (in short: poset). A nest of balls in $(X, \mathcal{B})$ is a nonempty totally ordered subset of $(\mathcal{B}, \subseteq)$. The basic completeness notion for ball spaces is inspired by the corresponding notion for ultrametric spaces: a ball space $(X, \mathcal{B})$ is called spherically complete if every nest of balls has a nonempty intersection.

We will now give examples of generic FPTs for ball spaces; they will be proved in Section 2.3. More such theorems and related results such
as coincidence theorems and so-called attractor theorems are presented in [12, 13, 14, 16]. In the present paper we will not discuss the uniqueness of fixed points; see the cited papers for this aspect. However, an exception will be made in Theorem 1.2 as this will be used later for an interesting comparison with a topological fixed point theorem proven in [29].

Given a function \( f \) on a set \( X \), we will call a subset \( S \) of \( X \) \( f \)-closed if
\[
f(S) \subseteq S.
\]
An \( f \)-closed set \( S \) will be called \( f \)-contracting if it satisfies
\[
f(S) \subset S
\]
unless it is a singleton. In the search for fixed points, it is a possible strategy to try to find \( f \)-closed singletons \( \{a\} \) because then the condition
\[
f(\{a\}) \subseteq \{a\}
\]
implies that \( f(a) = a \). The significance of this idea is particularly visible in the case of Caristi–Kirk and Oettli–Théra ball spaces discussed in Section 4.3.

**Theorem 1.1.** Take a spherically complete ball space \((X, B)\) and a function \( f : X \to X \). If each \( f \)-closed set contains an \( f \)-contracting ball, then \( f \) has a fixed point in each \( f \)-closed set.

The concept of ball spaces enables us to distinguish various levels of spherical completeness, which then helps to relax the conditions on the functions under consideration. On the one hand, we can specify what the intersection of a nest really is, apart from being nonempty. On the other hand, we can consider intersections of more general collections of balls than just nests. A **directed system of balls** is a nonempty collection of balls such that the intersection of any two balls in the collection contains a ball included in the collection. A **centered system of balls** is a nonempty collection of balls such that the intersection of any finite number of balls in the collection is nonempty. Note that every directed system is also a centered system (but in general, the converse is not true).

We introduce the following hierarchy of spherical completeness properties:

- \( S_1 \): The intersection of each nest in \((X, B)\) is nonempty.
- \( S_2 \): The intersection of each nest in \((X, B)\) contains a ball.
- \( S_3 \): The intersection of each nest in \((X, B)\) contains maximal balls.
- \( S_4 \): The intersection of each nest in \((X, B)\) contains a largest ball.
- \( S_5 \): The intersection of each nest in \((X, B)\) is a ball.
- \( S_i^d \): The same as \( S_i \), but with “directed system” in place of “nest”.
- \( S_i^c \): The same as \( S_i \), but with “centered system” in place of “nest”.

Note that \( S_1 \) is just the property of being spherically complete.
The strongest of these properties is $S_5^*$; we will abbreviate it as $S^*$ as it will play a central role, enabling us to prove useful results about several important ball spaces that have this property (it is the “star” among the above properties).

We have the following implications:

$$
S_1 \leftarrow S_1^d \leftarrow S_1^c \\
\uparrow \quad \uparrow \quad \uparrow \\
S_2 \leftarrow S_2^d \leftarrow S_2^c \\
\uparrow \quad \uparrow \quad \uparrow \\
S_3 \leftarrow S_3^d \leftarrow S_3^c \\
\uparrow \quad \uparrow \quad \uparrow \\
S_4 \leftarrow S_4^d \leftarrow S_4^c \\
\uparrow \quad \uparrow \quad \uparrow \\
S_5 \leftarrow S_5^d \leftarrow S_5^c = S^*
$$

The properties of the above hierarchy will be studied in detail in Section 3. In particular it is shown that under various conditions on the ball spaces certain properties in the hierarchy become equivalent. For completeness, a refinement of it will be discussed in Section 3.1. However, it will not be used further in the present paper.

We will now give an example showing how some of the above stronger notions of spherical completeness can be employed in general FPTs.

**Theorem 1.2.** Take an $S_5$ ball space $(X, B)$ and a function $f : X \to X$.

1) Assume that each $f$-closed ball is a singleton or contains a smaller $f$-closed ball. Then $f$ has a fixed point in each $f$-closed ball.

2) Assume that the image $f(B)$ of every $f$-closed ball $B \in B$ is an $f$-contracting ball. Then $f$ has a unique fixed point in each $f$-contracting ball. If in addition $X$ is an $f$-contracting ball, then $f$ has a unique fixed point.

**Theorem 1.3.** Take an $S_2$ ball space $(X, B)$ and a function $f : X \to X$. If every ball in $B$ contains a fixed point or a smaller ball, then $f$ has a fixed point in every ball.

We can get around asking that the ball space be $S_2$ by giving a condition on the intersection of nests; note that it is implicit in this condition that the ball space is spherically complete.

**Theorem 1.4.** Take a ball space $(X, B)$ and a function $f : X \to X$ such that the intersection of every nest of balls in $B$ contains a fixed
point or a smaller ball $B \in \mathcal{B}$. Then $f$ admits a fixed point in every ball of $\mathcal{B}$.

A condition like “contains a fixed point or a smaller ($f$-closed) ball” may appear a little unusual at first. However, a possible algorithm for finding fixed points should naturally be allowed to stop when it has found one, so from this point of view the condition is quite natural. We also sometimes use a condition like “each $f$-closed ball is a singleton or contains a smaller $f$-closed ball”. This implies “contains a fixed point or a smaller $f$-closed ball” because in an $f$-closed singleton $\{a\}$ the element $a$ must be a fixed point. But this condition is too strong: as we will see below, there are cases where finding a ball with a fixed point is easier and more natural than finding a singleton. One example are partially ordered sets where the balls are taken to be sets of the form $[a, \infty)$.

The assumptions of these theorems can be slightly relaxed by adapting them to the given function $f$. Instead of talking about the intersections of all nests of balls, we need information only about the intersections of nests of $f$-closed balls. Trivially, if $\emptyset \neq \mathcal{B}' \subseteq \mathcal{B}$, then also $(X, \mathcal{B}')$ is a ball space, and if $(X, \mathcal{B})$ is spherically complete, then so is $(X, \mathcal{B}')$. This flexibility of ball spaces appeared already implicitly in Theorem 1.2 where only $f$-closed balls are used; if nonempty, the subset of all $f$-closed balls is also a ball space, and it inherits important properties from the (possibly) larger ball space. Tailoring the assumptions on the ball space to the given function also comes in handy in the following refinement of Theorem 1.2. In its formulation, the condition “spherically complete” does not appear explicitly anymore, but is implicitly present for the ball space that is chosen in dependence on the function $f$.

**Theorem 1.5.** Take a function $f : X \to X$ and assume that there is a ball space $(X, \mathcal{B}^f)$ such that

1. each ball in $\mathcal{B}^f$ is $f$-closed,
2. the intersection of every nest of balls in $\mathcal{B}^f$ is a singleton or contains a smaller ball $B \in \mathcal{B}^f$.

Then $f$ admits a fixed point in every ball in $\mathcal{B}^f$.

At first glance, the conditions of these theorems may appear too strong, or somewhat unusual. But the reader should notice that their strength lies in the fact that we can freely choose the ball space. For example, it does not have to be a topology, and in fact, for essentially all of our applications it should not be. This makes it possible to even
choose the balls relative to the given function, which leads to results like the theorem above.

When uniqueness of fixed points is not required, then in certain settings (such as ultrametric spaces, see Section 4.1) the condition that a function be “contracting” on all of the space can often be relaxed to the conditions that the function just be “non-expanding” everywhere and “contracting” on orbits. Again, there is some room for relaxation, and this is why we will now introduce the following notion. For each \( i \in \mathbb{N} \), \( f^i \) will denote the \( i \)-th iteration of \( f \), that is, \( f^0 x = x \) and \( f^{i+1} x = f(f^i x) \). A function \( f \) will be called \textbf{ultimately contracting on orbits} if there is a function \( X \ni x \mapsto B_x \in \mathcal{B} \) such that for all \( x \in X \), the following conditions hold:

\begin{enumerate}[label=(SC\arabic*)]
  \item \( x \in B_x \).
  \item \( B_{f^i x} \subseteq B_x \), and if \( x \neq f x \), then \( B_{f^i x} \subseteq B_x \) for some \( i \geq 1 \).
\end{enumerate}

Note that (SC1) and (SC2) imply that \( f^i x \in B_x \) for all \( i \geq 0 \).

We will say that a nest \( \mathcal{N} \) of balls is an \textbf{\( f \)-nest} if \( \mathcal{N} = \{ B_x \mid x \in S \} \) for some set \( S \subseteq X \) that is closed under \( f \). Now we can state our sixth basic theorem:

\textbf{Theorem 1.6.} Take a function \( f \) on a ball space \( (X, \mathcal{B}) \) which is ultimately contracting on orbits. If for every \( f \)-nest \( \mathcal{N} \) in this ball space there is some \( z \in \bigcap \mathcal{N} \) such that \( B_z \subseteq \bigcap \mathcal{N} \), then for every \( x_0 \in X \), \( f \) has a fixed point in \( B_{x_0} \).

A particularly elegant version of this approach can be given in the case of Caristi–Kirk and Oettli–Théra ball spaces (see Theorem 4.12 in Section 4.3). These ball spaces are used in complete metric spaces. Usually, proofs of fixed point theorems in this setting work with Cauchy sequences, while the use of metric balls is inefficient and complicated. For this reason, a ball spaces approach to metric spaces may seem pointless at first glance. However, it has turned out that ball spaces made up of Caristi–Kirk or Oettli–Théra balls have a particularly strong property (cf. Proposition 4.11), which makes the ball space approach in this case exceptionally successful, as demonstrated in Section 4.3 and the papers [2, 14].

Interestingly, the exceptional strength of the Caristi–Kirk and Oettli–Théra ball spaces is shared by the ball space made up of the sets \([a, \infty)\) on partially ordered sets. It would be worthwhile to find more examples of such strong ball spaces.
The proofs of our generic fixed point theorems are based on Zorn’s Lemma. They will be given in Section 2 after first investigating the relation between partially ordered sets and ball spaces. In the present paper we are not interested in avoiding the use of the axiom of choice, nor is it our task to study its equivalence with certain fixed point theorems. For a detailed discussion of the case of Caristi–Kirk and Oettli–Théra ball spaces, see Remark 4.15.

After having provided the basic setting and results, the task of proving fixed point theorems is shifted to finding the suitable interpretations of the notion of “ball space” in the various applications, together with the suitable level(s) of spherical completeness. Here are a few examples which will be worked out in Section 4.

| spaces                        | balls                                      | completeness property          |
|-------------------------------|--------------------------------------------|--------------------------------|
| ultrametric spaces            | all closed ultrametric balls               | spherically complete          |
| metric spaces                 | metric balls with radii in suitable sets of positive real numbers | complete |
| totally ordered sets, ordered abelian groups and fields | all intervals $[a, b]$ with $a \leq b$ | symmetrically complete |
| topological spaces            | all nonempty closed sets                   | compact                        |
| posets                        | intervals $[a, \infty)$                   | inductively ordered           |
| metric spaces                 | Caristi–Kirk balls or Oettli–Théra balls | complete                      |

Here the last entry, the second one for metric spaces, is different from all the other ones. In all the other cases the table has to be read as saying that the completeness property of the given space is equivalent to the spherical completeness of one single associated ball space containing the indicated balls. But if we work with Caristi–Kirk balls or Oettli–Théra balls, then the completeness of the metric space is equivalent to the spherical completeness of a whole variety of Caristi–Kirk ball spaces or Oettli–Théra ball spaces that can be defined on it (see Section 4.3). While this may appear impracticable at first glance, it turns out that these types of balls offer a much better ball spaces approach to metric spaces than the metric balls, as noted above.
Not only the specialization of the general framework to particular applications is important. It is also fruitful to develop the abstract theory of ball spaces, in particular the behaviour of the various levels of spherical completeness in the hierarchy \( (1) \) under basic operations on ball spaces.

In Section 5 we study our strongest, the \( S^* \) ball spaces. Examples are the topological spaces, where we take the balls to be the nonempty closed sets. These ball spaces allow the definition of what we call spherical closures of subsets. These help us to deal with ball space structures induced on subsets of the set underlying the ball space.

In Section 6 we will consider set theoretic operations on ball spaces, such as adding unions or intersections of their balls. Products of ball spaces will be studied in Section 7.2. In the paper [1], we discuss a notion of continuity for functions between ball spaces, as well as quotient spaces and category theoretical aspects of ball spaces.

Further, the fact that a general framework links various quite different applications can help to transfer ideas, approaches and results from one to the other. For instance, the Knaster–Tarski Theorem in the theory of complete lattices presents a useful property of the set of fixed points: they form again a complete lattice. In Section 7.1, using our general framework and in particular the results from Section 5, we transfer this result to other applications, such as ultrametric and topological spaces. Similarly, in Section 7.2 the Tychonoff Theorem from topology is proven for ball spaces and then transferred to ultrametric spaces. To derive the topological Tychonoff Theorem from its ball spaces analogue, particular use is made of the results of Section 6.

Finally, the last section of our paper is devoted to a quick discussion of two types of theorems that are related to fixed point theorems (and in fact are generalizations, as fixed point theorems can be deduced from them). First, we will present generic multivalued fixed point theorems for ball spaces. Multivalued fixed point theorems consider functions \( F \) from a nonempty set \( X \) to its power set \( P(X) \) and ask for criteria that guarantee the existence of a fixed point \( x \in X \) in the sense that

\[
x \in F(x).
\]

Multivalued ultrametric fixed point theorems have been successfully applied in logic programming (see [26, 5]).

Second, we will present generic coincidence theorems for ball spaces. Coincidence theorems consider two or more functions \( f_1, \ldots, f_n \) from a nonempty set \( X \) to itself and ask for criteria that guarantee the
existence of a coincidence point \( x \in X \) in the sense that
\[
f_1(x) = \ldots = f_n(x)
\]
A number of coincidence theorems for ball spaces and ultrametric spaces have been proven in [16] (see also [24] for theorems on ultrametric spaces).

For both types of theorems we will use two approaches. Inspired by the theory of strongly contractive ball spaces which we will develop in connection with Caristi–Kirk and Oettli–Théra ball spaces in Section 4.3, we will first employ criteria for the existence of singleton balls with suitable properties. Thereafter, we will prove variants which work with minimal balls instead.

We hope that we have convinced the reader that the advantage of a general framework is (at least) threefold:

- provide generic proofs of results which then only have to be specialized to the various applications,
- exhibit the underlying principles that make the theorems in the various applications work,
- transfer concepts and results from one application to another.

2. Zorn’s Lemma in the context of ball spaces

Consider a poset \((T, <)\). By a \textbf{chain} in \(T\) we mean a nonempty totally ordered subset of \(T\). An element \(a \in T\) is said to be an upper bound of a subset \(S \subseteq T\) if \(b \leq a\) for all \(b \in S\). A poset is said to be \textbf{inductively ordered} if every chain has an upper bound.

Zorn’s Lemma states that every inductively ordered poset contains maximal elements. By restricting the assertion to the set of all elements in the chain and above it, we obtain the following more precise assertion:

\textbf{Lemma 2.1.} In an inductively ordered poset, every chain has an upper bound which is a maximal element in the poset.

\textbf{Corollary 2.2.} In an inductively ordered poset, every element lies below a maximal element.

Take a ball space \((X, \mathcal{B})\). If we order \(\mathcal{B}\) by setting \(B_1 < B_2\) if \(B_1 \supseteq B_2\), then we obtain a poset \((\mathcal{B}, <)\). Under this transformation, nests of balls in \(\mathcal{B}\) correspond to chains in the poset. A maximal element in the poset \((\mathcal{B}, <)\) is a \textbf{minimal ball}, i.e., a ball that does not contain any smaller ball.
2.1. The case of $S_2$ ball spaces.

The following observation is straightforward:

**Lemma 2.3.** The ball space $(X, \mathcal{B})$ is $S_2$ if and only if every chain in $(\mathcal{B}, <)$ has an upper bound.

From this fact, one easily deduces the following result.

**Proposition 2.4.** In an $S_2$ ball space, every ball and therefore also the intersection of every nest contains a minimal ball.

In view of Lemma 2.3 it is important to note that every $S_1$ ball space $(X, \mathcal{B})$ can easily be turned into an $S_2$ ball space by adding all singleton subsets of $X$: we define

$$\mathcal{B}_s := \mathcal{B} \cup \{\{a\} | a \in X\}.$$ 

The proof of the following result is straightforward.

**Lemma 2.5.** The ball space $(X, \mathcal{B}_s)$ is $S_2$ if and only if $(X, \mathcal{B})$ is $S_1$.

However, in many situations the point is exactly to prove that a given ball space admits singleton balls. This is in particular the case when we work with ball spaces that are adapted to a given function, as in Theorem 1.5. In such cases, instead of applying Zorn’s Lemma to chains of balls, one can work with chains of nests instead, as we will discuss in Section 2.2.

2.2. Posets of nests of balls.

A poset is called **chain complete** if every chain of elements has a least upper bound.

**Lemma 2.6.** For every ball space $(X, \mathcal{B})$, the set of all nests of balls, ordered by inclusion, is a chain complete poset.

Proof: The union over a chain of nests of balls is again a nest of balls, and it is the smallest nest that contains all nests in the chain. □

This shows that in particular every chain of nests that contains a given nest $\mathcal{N}_0$ has an upper bound. Hence Zorn’s Lemma shows:

**Corollary 2.7.** Every nest $\mathcal{N}_0$ of balls in a ball space is contained in a maximal nest.
2.3. Proof of the fixed point theorems.

Take a ball space \((X, \mathcal{B})\) and a function \(f : X \to X\). By \(\mathcal{B}^f\) we will denote the collection of all \(f\)-closed balls in \(\mathcal{B}\), provided there exist any. From Corollary 2.7 we infer that every nest in \((X, \mathcal{B})\) and every nest in \((X, \mathcal{B}^f)\) is contained in a maximal nest.

Under various conditions on \(f\) and on \((X, \mathcal{B})\) or \((X, \mathcal{B}^f)\), we have to make sure that the intersections of such nests contain a fixed point for \(f\). We observe:

a) If \(S\) is an \(f\)-closed set, then \(ff(S) \subseteq f(S)\) since \(f(S) \subseteq S\), hence \(f(S)\) is \(f\)-closed.

b) The intersection over any collection of \(f\)-closed sets is again an \(f\)-closed set.

Proof of Theorem 1.1: Take any \(f\)-closed set \(S\). By the assumption of the theorem we know that it contains an \(f\)-contracting ball \(B\). By definition, \(B\) is \(f\)-closed. By Corollary 2.7 there exists a maximal nest \(\mathcal{N}\) in the set \(\mathcal{B}^f\) of all \(f\)-closed balls in \(\mathcal{B}\) which contains the nest \(\{B\}\).

Then by b) above, \(\bigcap \mathcal{N}\) is an \(f\)-closed set. By assumption, it contains an \(f\)-contracting ball \(B'\). Suppose that \(B'\) is not a singleton. Then \(B'\) properly contains \(f(B')\), which by a) above is an \(f\)-closed set. Again by assumption, it contains an \(f\)-contracting and hence \(f\)-closed ball \(B''\). Since \(B'' \subseteq f(B') \subset B' \subseteq \bigcap \mathcal{N}\), we find that \(\mathcal{N} \cup \{B''\}\) is a larger nest than \(\mathcal{N}\), which contradicts the maximality of \(\mathcal{N}\). This proves that \(B'\) is an \(f\)-closed singleton contained in \(S\) and thus, \(S\) contains a fixed point. \(\square\)

Proof of Theorem 1.2: Assume that \((X, \mathcal{B})\) is an \(S_5\) ball space and that each \(f\)-closed ball is a singleton or contains a smaller \(f\)-closed ball. Take some \(f\)-closed ball \(B \in \mathcal{B}\).

1): As in the previous proof, choose a maximal nest \(\mathcal{N}\) in \(\mathcal{B}^f\) which contains the nest \(\{B\}\). Then \(\bigcap \mathcal{N}\) is an \(f\)-closed set. As \((X, \mathcal{B})\) is assumed to be an \(S_5\) ball space, it is also a ball, so \(\bigcap \mathcal{N} \in \mathcal{B}^f\). By the maximality of \(\mathcal{N}\), we have that \(\mathcal{N} \cup \{\bigcap \mathcal{N}\} = \mathcal{N}\), i.e., \(\bigcap \mathcal{N}\) is the smallest ball in \(\mathcal{N}\). It must be a singleton, because otherwise, it would contain a smaller \(f\)-closed ball giving rise to a nest properly containing \(\mathcal{N}\), which is impossible. Thus, \(\bigcap \mathcal{N}\) is an \(f\)-closed singleton contained in \(B\) and therefore, \(B\) contains a fixed point.

2): We will apply Theorem 2 of [12], which states that if \((Y, \mathcal{B}')\) is a ball space and \(f : Y \to Y\) such that \(Y\) is an \(f\)-contracting ball, the image \(f(B)\) of every \(f\)-contracting ball is again an \(f\)-contracting ball, and that so is the intersection of every nest of \(f\)-contracting balls, then \(f\) has a unique fixed point.
We fix an \( f \)−closed ball \( B \); if \( X \) is itself a ball, then we take \( B = X \). Then we set \( Y = B \) and take \( \mathcal{B}' \) to be the collection of all \( f \)−closed balls from \( \mathcal{B} \) that are contained in \( B \). Then the first two of the above conditions hold by our choice of \( Y \) and by our assumption that the image \( f(B) \) of every \( f \)−closed ball \( B \in \mathcal{B} \) is again an \( f \)−contracting ball.

To show that the third condition holds, take a nest of \( f \)−contracting balls. As \((X, \mathcal{B})\) is assumed to be an \( S_5 \) ball space, its intersection \( \bigcap \mathcal{N} \) is a ball. By our observation b) above, it is \( f \)−closed. Suppose that \( \bigcap \mathcal{N} \) were not \( f \)−contracting. Then \( f(\bigcap \mathcal{N}) = \bigcap \mathcal{N} \). But then by the assumption of our theorem, \( \bigcap \mathcal{N} \) is \( f \)−contracting. We have shown that the conditions of the theorem are satisfied, which yields the desired fixed point.

\[ \square \]

Proof of Theorem 1.3: Assume that \((X, \mathcal{B})\) is an \( S_2 \) ball space and that every ball in \( \mathcal{B} \) contains a fixed point or a smaller ball. Take a ball \( B \in \mathcal{B} \). By Proposition 2.4, \( B \) contains a minimal ball \( B_0 \). As \( B_0 \) cannot contain a smaller ball, it must contain a fixed point by assumption, which then is also an element of \( B \). \[ \square \]

Proof of Theorem 1.4: Assume that the intersection of every nest of balls in \((X, \mathcal{B})\) contains a fixed point or a smaller ball. Take a ball \( B \in \mathcal{B} \). As before, there exists a maximal nest \( \mathcal{N} \) in \( \mathcal{B} \) which contains the nest \( \{B\} \). Now \( \bigcap \mathcal{N} \) cannot contain a smaller ball since this would contradict the maximality of \( \mathcal{N} \). Hence by assumption, \( \bigcap \mathcal{N} \) and thus also \( B \) must contain a fixed point. \[ \square \]

Proof of Theorem 1.5: Assume that \( \mathcal{B}' \) is a ball space of \( f \)−closed balls and that the intersection of every nest of balls in \( \mathcal{B}' \) is a singleton or contains a smaller ball \( B \in \mathcal{B}' \). Take a ball \( B \in \mathcal{B}' \). As in the previous proofs, there exists a maximal nest \( \mathcal{N} \) in \( \mathcal{B}' \) which contains the nest \( \{B\} \). The intersection \( \bigcap \mathcal{N} \) cannot contain a smaller ball \( B' \in \mathcal{B}' \) since this would contradict the maximality of \( \mathcal{N} \). Hence by assumption, \( \bigcap \mathcal{N} \) must be a singleton. As it is also \( f \)−closed and contained in \( B \), we have proved that \( f \) has a fixed point in \( B \). \[ \square \]

Proof of Theorem 1.6: Take a function \( f \) on the ball space \((X, \mathcal{B})\) which is ultimately contracting on orbits. For every \( x \in X \), the set \( \{B_{f^i x} \mid i \geq 0\} \) is an \( f \)−nest. The set of all \( f \)−nests is partially ordered in the following way. If \( \mathcal{N}_1 = \{B_x \mid x \in S_1\} \) and \( \mathcal{N}_2 = \{B_x \mid x \in S_2\} \) are \( f \)−nests with \( S_1 \) and \( S_2 \) are closed under \( f \), then we define \( \mathcal{N}_1 \leq \mathcal{N}_2 \) if \( S_1 \subseteq S_2 \). Then the union over an ascending chain of \( f \)−nests is again an \( f \)−nest since the union over sets that are closed under \( f \) is again closed under \( f \). Hence by Corollary 2.2, for every \( x_0 \in X \) there is a maximal \( f \)−nest \( \mathcal{N} \) containing \( \{B_{f^i x_0} \mid i \geq 0\} \). By the assumption of
Theorem 1.6, there is some \( z \in \bigcap \mathcal{N} \) such that \( B_z \subseteq \bigcap \mathcal{N} \). We wish to show that \( z \) is a fixed point of \( f \). If \( z \neq fz \) would hold, then by (SC2), \( B_{fz} \subseteq B_z \subseteq \bigcap \mathcal{N} \) for some \( i \geq 1 \), and the \( f \)-nest \( \mathcal{N} \cup \{ B_{fkz} \mid k \in \mathbb{N} \} \) would properly contain \( \mathcal{N} \). But this would contradict the maximality of \( \mathcal{N} \). Hence, \( z \in \bigcap \mathcal{N} \subseteq B_{x_0} \) is a fixed point of \( f \). \( \square \)

3. SOME FACTS ABOUT THE HIERARCHY OF BALL SPACES

3.1. A refinement of the hierarchy.

By considering stronger properties of directed and centered systems of balls, we will now add further entries to the hierarchy (\( \mathbb{I} \)).

We will say that a centered system of balls is

- \( c' \) if the intersection of any finite number of balls in the system contains a ball,
- \( c'' \) if the intersection of any finite number of balls in the system contains a largest ball,
- \( c''' \) if the intersection of any finite number of balls in the collection is a ball.

We will say that a directed system of balls is

- \( d' \) if the intersection of any finite number of balls in the system contains a ball which is again in the system,
- \( d'' \) if the intersection of any finite number of balls in the system contains a largest ball which is again in the system,
- \( d''' \) if the intersection of any finite number of balls in the system is a ball which is again in the system.

For \( 1 \leq i \leq 5 \) we will say that a ball space is \( S_i^{d'} \) (or \( S_i^{d''} \), or \( S_i^{d'''} \)) if it satisfies the definition of \( S_i^d \) with “directed system” replaced by “\( d' \) directed system” (or “\( d'' \) directed system”, or “\( d''' \) directed system”, respectively). Again for \( 1 \leq i \leq 5 \), we will say that a ball space is \( S_i^{c'} \) (or \( S_i^{c''} \), or \( S_i^{c'''} \)) if it satisfies the definition of \( S_i^c \) with “centered system” replaced by “\( c' \) centered system” (or “\( c'' \) centered system”, or “\( c''' \) centered system”, respectively).

By induction one shows that in the above definitions for \( d' \) and \( d''' \), “any finite number of” can be replaced by “any two” without changing the meaning. In particular, every directed system of balls is \( d' \). We also note that every nest of balls is a \( d'' \) directed system of balls. This together with the obvious implications between the properties defined
above gives us the following refinement of each row of the hierarchy (1):

\[
\begin{align*}
S_i & \Leftarrow S_i^{\prime\prime} \Leftarrow S_i^{\prime\prime\prime} \Leftarrow S_i^f = S_i^d \\
\uparrow & \quad \uparrow \quad \uparrow \\
S_i^{\prime\prime} & \Leftarrow S_i^{\prime}\Leftarrow S_i^{\prime\prime} \Leftarrow S_i^c
\end{align*}
\]

for 1 \leq i \leq 5.

3.2. Connection with posets.

In a poset, a set \( S \) of elements is **bounded** if and only if it has an upper bound. A poset is **bounded complete** if every nonempty bounded set has a least upper bound. A **directed system** in a poset is a nonempty subset which contains an upper bound for any two of its elements. A poset is called **directed complete** if every directed system has a least upper bound. As every chain is a directed system, every directed complete poset is chain complete.

The proof of the following observations is straightforward:

**Proposition 3.1.**

1) A ball space \((X, \mathcal{B})\) is \( S_2 \) if and only if every chain in \((\mathcal{B}, <)\) has an upper bound.

2) A ball space \((X, \mathcal{B})\) is \( S_2^d \) if and only if every directed system in \((\mathcal{B}, <)\) has an upper bound.

3) A ball space \((X, \mathcal{B})\) is \( S_4 \) if and only if \((\mathcal{B}, <)\) is chain complete.

4) A ball space \((X, \mathcal{B})\) is \( S_4^d \) if and only if \((\mathcal{B}, <)\) is directed complete.

Let us point out that the intersection of a system of balls may not be itself a ball, even if it is nonempty (but if it is a ball, then it is clearly the largest ball contained in all of the balls in the system). For this reason, in general, the properties \( S_1, S_4, S_5 \) and \( S_2 \) cannot be translated into a corresponding property of \((\mathcal{B}, <)\). This shows that ball spaces have more expressive strength than the associated poset structures.

A proof of the following fact can be found in [4, p. 33]. See also [18] for generalizations.

**Proposition 3.2.** Every chain complete poset is directed complete.

This proposition together with Proposition 3.1 yields:

**Corollary 3.3.** Every \( S_4 \) ball space is an \( S_4^d \) ball space.

From Proposition 3.1 and Corollary 2.2 we obtain that for a \( S_2 \) ball space, every element of \((\mathcal{B}, <)\) lies below a maximal element. This proves:
Proposition 3.4. *In an $S_2$ ball space, every ball contains a minimal ball.*

In the next sections, we will give further criteria for the equivalence of various properties in the hierarchy.

3.3. Singleton balls.
In many applications (e.g. metric spaces, ultrametric spaces, $T_1$ topological spaces) the associated ball spaces have the property that singleton sets are balls. The following observation is straightforward:

**Proposition 3.5.** *For a ball space in which all singleton sets are balls, $S_1$ is equivalent to $S_2$, $S_1^d$ is equivalent to $S_2^d$, and $S_1^c$ is equivalent to $S_2^c$.*

3.4. Ball spaces of ultrametric type.
We will call a ball space $(X, B)$ of **ultrametric type** if any two balls in $B$ with nonempty intersection are comparable by inclusion. We will see in Section 3.3 (Proposition 3.6) that the ball spaces associated with classical ultrametric spaces are of ultrametric type.

**Proposition 3.6.** *In a ball space of ultrametric type, every centered system of balls is a nest. For such a ball space, $S_1$, $S_1^d$ and $S_1^c$ are equivalent, for each $1 \in \{1, \ldots, 5\}$. If in addition, in this ball space all singleton sets are balls, then $S_1$ is equivalent to $S_2^c$.*

Proof: The first assertion follows from the fact that in a ball space of ultrametric type, every two balls in a centered system have nonempty intersection and therefore are comparable by inclusion, so the system is a nest. From this, the second assertion follows immediately. The third assertion follows by way of Proposition 3.5. □

3.5. Intersection closed ball spaces.
A ball space $(X, B)$ will be called **finitely intersection closed** if $B$ is closed under nonempty intersections of any finite collection of balls, **chain intersection closed** or **nest intersection closed** if $B$ is closed under nonempty intersections of nests of balls, and **intersection closed** if $B$ is closed under nonempty intersections of arbitrary collections of balls.

We deduce from Proposition 3.6:

**Proposition 3.7.** *Every chain intersection closed ball space of ultrametric type is intersection closed.*
Proof: Every collection $\mathcal{C}$ of balls with nonempty intersection in an arbitrary ball space is a centered system. If the ball space is of ultrametric type, then by Proposition 3.6, $\mathcal{C}$ is a nest. If in addition the ball space is chain intersection closed, then the intersection $\bigcap \mathcal{C}$ is a ball. Hence under the assumptions of the proposition, the ball space is intersection closed. □

The proofs of the following two propositions are straightforward:

**Proposition 3.8.** Assume that the ball space $(X, \mathcal{B})$ is finitely intersection closed. Then by closing under finite intersections, every centered system of balls can be expanded to a directed system of balls which has the same intersection. Hence for a finitely intersection closed ball space, $S_i^d$ is equivalent to $S_i^c$, for $1 \leq i \leq 5$.

**Proposition 3.9.** For chain intersection closed ball spaces, the properties $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$ are equivalent.

As can be expected, the intersection closed ball spaces are the strongest when it comes to equivalence of the properties in the hierarchy.

**Theorem 3.10.** For an intersection closed ball space, $S_1$ is equivalent to $S^*$, so all properties in the hierarchy are equivalent.

Proof: Since $(X, \mathcal{B})$ is intersection closed, the intersection of any nest is a ball as soon as it is nonempty. This yields that $S_1$ is equivalent to $S_5$ and hence also to $S_2$, $S_3$ and $S_4$. The same holds if we replace “nest” by “directed system”, i.e., the $S_i$ by $S_i^d$, and if we replace “nest” by “centered system”, i.e., the $S_i$ by $S_i^c$.

In particular, $S_1$ implies $S_4$ and $S_5^c$ implies $S_5^d$. From Corollary 3.3 we know that $S_4$ implies $S_4^d$, and from Proposition 3.8 that $S_4^d$ implies $S_4^c$. Consequently, $S_1$ implies $S_5^c$, which shows that all properties in the hierarchy are equivalent. □

A **bounded system of balls** is a nonempty collection of balls whose intersection contains a ball. Note that a bounded system of balls is a centered system, but the converse is in general not true (not even a nest of balls is necessarily a bounded system if the ball space is not $S_2$).

**Lemma 3.11.** The poset $(\mathcal{B}, <)$ is bounded complete if and only if the intersection of every bounded system of balls in $(X, \mathcal{B})$ contains a largest ball. In an intersection closed ball space, the intersection of every bounded system of balls is a ball.
4. Ball spaces and their properties in various applications

In what follows, we will give the interpretation of various levels of spherical completeness in our applications of ball spaces.

4.1. Ultrametric spaces.

An ultrametric $u$ on a set $X$ is a function from $X \times X$ to a partially ordered set $\Gamma$ with smallest element 0, such that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $u(x, y) = 0$ if and only if $x = y$,
(U2) if $u(x, y) \leq \gamma$ and $u(y, z) \leq \gamma$, then $u(x, z) \leq \gamma$,
(U3) $u(x, y) = u(y, x)$ (symmetry).

The pair $(X, u)$ is called an ultrametric space. Condition (U2) is the ultrametric triangle law.

We set $uX := \{u(x, y) \mid x, y \in X\}$ and call it the value set of $(X, u)$. If $uX$ is totally ordered, we will call $(X, u)$ a classical ultrametric space; in this case, (U2) is equivalent to:

(UT) $u(x, z) \leq \max\{u(x, y), u(y, z)\}$.

We will now introduce four ways of deriving a ball space from an ultrametric space. A closed ultrametric ball is a set $B_\alpha(x) := \{y \in X \mid u(x, y) \leq \alpha\}$, where $x \in X$ and $\alpha \in \Gamma$. We obtain the ultrametric ball space $(X, B_u)$ from $(X, u)$ by taking $B$ to be the set of all such balls $B_\alpha(x)$.

It follows from symmetry and the ultrametric triangle law that every element in a ball is a center, meaning that

(B) $B_\alpha(x) = B_\alpha(y)$ if $y \in B_\alpha(x)$.

Further,

$$B_\beta(y) \subseteq B_\alpha(x) \quad \text{if} \quad y \in B_\alpha(x) \text{ and } \beta \leq \alpha .$$

If $B$ and $B'$ are any two ultrametric balls with nonempty intersection in a classical ultrametric space, then $B \subseteq B'$ or $B' \subseteq B$.

A problem with the ball $B_\alpha(x)$ can be that it may not contain any element $y$ such that $u(x, y) = \alpha$; if it does, it is called precise. It is therefore convenient to work with the precise balls of the form

$$B(x, y) := \{z \in X \mid u(x, z) \leq u(x, y)\} ,$$

where $x, y \in X$. We obtain the precise ultrametric ball space $(X, B_{[u]})$ from $(X, u)$ by taking $B$ to be the set of all such balls $B(x, y)$.

It follows from symmetry and the ultrametric triangle law that

$$B(x, y) = B(y, x)$$
and that
\( (5) \ B(t, z) \subseteq B(x, y) \) if and only if \( t \in B(x, y) \) and \( u(t, z) \leq u(x, y) \).

In particular,
\[
B(t, z) \subseteq B(x, y) \quad \text{if} \quad t, z \in B(x, y).
\]

Two elements \( \gamma \) and \( \delta \) of \( \Gamma \) are comparable if \( \gamma \leq \delta \) or \( \gamma \geq \delta \). Hence if \( u(x, y) \) and \( u(y, z) \) are comparable, then \( B(x, y) \subseteq B(y, z) \) or \( B(y, z) \subseteq B(x, y) \). If \( u(y, z) < u(x, y) \), then in addition, \( x \notin B(y, z) \) and thus, \( B(y, z) \nsubseteq B(x, y) \). We note:
\[
(6) \quad u(y, z) < u(x, y) \implies B(y, z) \subseteq B(x, y).
\]

From (4), we derive:

**Proposition 4.1.** In a classical ultrametric space \((X, u)\), any two balls with nonempty intersection are comparable by inclusion. Hence \((X, B_\{u\})\) and \((X, B_u)\) are ball spaces of ultrametric type.

We define \((X, u)\) to be spherically complete if its ultrametric ball space \((X, B_u)\) is \( S_1 \). For this definition, it actually makes no difference whether we work with \( B_u \) or \( B_{\{u\}} \):

**Proposition 4.2.** The classical ultrametric ball space \((X, B_u)\) is \( S_1 \) if and only if the precise ultrametric ball space \((X, B_{\{u\}})\) is.

Proof: Since \( B_{\{u\}} \subseteq B_u \), the implication “\( \Rightarrow \)” is clear. Now take a nest \( \mathcal{N} \) of balls in \( B_u \). We may assume that it does not contain a smallest ball since otherwise this ball equals the intersection over the nest, which consequently is nonempty. Further, there is a coinitial subnest \( (B_{\alpha_v}(x_v))_{\nu < \kappa} \) such that \( \kappa \) is an infinite limit ordinal and \( \mu < \nu < \kappa \) implies that \( B_{\alpha_\nu}(x_\nu) \subseteq B_{\alpha_\nu}(x_\mu) \). It follows that this subnest has the same intersection as \( \mathcal{N} \).

For every \( \nu < \kappa \), also \( \nu + 1 < \kappa \) and thus \( B_{\alpha_{\nu+1}}(x_{\nu+1}) \nsubseteq B_{\alpha_\nu}(x_\nu) \). Hence there is \( y_{\nu+1} \in B_{\alpha_\nu}(x_\nu) \setminus B_{\alpha_{\nu+1}}(x_{\nu+1}) \). It follows that
\[
u(x_{\nu+1}, y_{\nu+1}) > \alpha_{\nu+1}
\]
and from (4) we obtain that
\[
B_{\alpha_{\nu+1}}(x_{\nu+1}) \subseteq B_{u(x_{\nu+1}, y_{\nu+1})}(x_{\nu+1}) = B(x_{\nu+1}, y_{\nu+1}).
\]
Since \( x_{\nu+1}, y_{\nu+1} \in B_{\alpha_\nu}(x_\nu) \), we know that
\[
u(x_{\nu+1}, y_{\nu+1}) \leq \max\{u(x_{\nu+1}, x_\nu), u(x_\nu, y_{\nu+1})\} \leq \alpha_\nu,
\]
and again from (4) we obtain that
\[
B(x_{\nu+1}, y_{\nu+1}) = B_{u(x_{\nu+1}, y_{\nu+1})}(x_{\nu+1}) \subseteq B_{\alpha_\nu}(x_{\nu+1}) = B_{\alpha_\nu}(x_\nu).
\]
It follows that
\[ \bigcap N = \bigcap_{\nu < \kappa} B_{\alpha_{\nu}}(x_{\nu}) = \bigcap_{\nu < \kappa} B(x_{\nu+1}, y_{\nu+1}). \]

Consequently, if \( B_{[u]} \) is \( S_1 \), then this intersection is nonempty and we have proved that also \( B_u \) is \( S_1 \). \( \square \)

Since \( uX \) contains the smallest element 0 := \( u(x, x) \), \( B_u \) contains all singletons \( \{x\} = B_0(x) \). Therefore, each ultrametric ball space is already \( S_2 \) once it is \( S_1 \). The same is true for the precise ultrametric ball space \((X, B_{[u]})\) in place of \((X, B_u)\). However, these ball spaces will in general not be \( S_4 \) or \( S_5 \) because even if an intersection of a nest is nonempty, it will not necessarily be a ball of the form \( B_{\alpha}(x) \) or \( B(x, y) \), respectively.

If \((X, u)\) is a classical ultrametric space, then this problem can be remedied if we work with a larger set of ultrametric balls. Given \( x \in X \) and an initial segment \( S \neq \emptyset \) of \( uX \), we define:
\[ B_S(x) = \{ y \in X \mid u(x, y) \in S \}. \]

Setting
\[ B_{u^+} := \{ B_S(x) \mid x \in X \text{ and } S \text{ a nonempty initial segment of } uX \}, \]
we obtain what we will call the **full ultrametric ball space** \((X, B_{u^+})\).

Note that \( X = B_{uX}(x) \in B_{u^+} \). We leave it to the reader to prove:

\[ B_S(x) = \bigcup_{\alpha \in S} B_\alpha(x) = \bigcap_{\beta \geq S} B_\beta(x) \]

where \( \beta \geq S \) means that \( \beta \geq \gamma \) for all \( \gamma \in S \), and the intersection over an empty index set is taken to be \( X \).

We have that
\[ B_{[u]} \subseteq B_u \subseteq B_{u^+} \]
where the second inclusion holds because \( B_{\alpha}(x) = B_S(x) \) for the initial segment \( S = [0, \alpha] \) of \( uX \).

**Proposition 4.3.** Let \((X, u)\) be a classical ultrametric space. Then the following assertions hold.

1) The intersection over every nest of balls in \((X, B_{u^+})\) is equal to the intersection over a nest of balls in \((X, B_u)\) and therefore, \((X, B_{u^+})\) is chain intersection closed.

2) The ball space \((X, B_{u^+})\) is spherically complete if and only if \((X, B_u)\) is.

3) If the intersection of two balls in \( B_{u^+} \) is nonempty, then these two balls are comparable under inclusion.
Proof: Assertions 1) and 2) are proven in [10, Theorem 1.1]. In order to prove assertion 3), take two balls $B_1, B_2 \in \mathcal{B}_{u+}$. In view of (7) we can write them as

$$B_1 = \bigcap_{\beta \geq S_1} B_\beta(x) \quad \text{and} \quad B_2 = \bigcap_{\beta \geq S_2} B_\beta(y)$$

with $x, y \in X$ and nonempty initial segments $S_1, S_2$ of $uX$. Take an element $z \in B_1 \cap B_2$. Then by (3), $B_\beta(x) = B_\beta(z)$ for all $\beta \geq S_1$ and $B_\beta(y) = B_\beta(z)$ for all $\beta \geq S_2$. Hence if, say, $S_1 \subseteq S_2$, then

$$B_1 = \bigcap_{\beta \geq S_1} B_\beta(z) \subseteq \bigcap_{\beta \geq S_2} B_\beta(z) = B_2.$$

From part 3) of the proposition it follows that every centered system of balls in $\mathcal{B}_{u+}$ is already a nest. Therefore, from parts 1) and 2) of the proposition, we obtain:

**Theorem 4.4.** Let $(X, u)$ be a classical ultrametric space. Then the full ultrametric ball space $(X, \mathcal{B}_{u+})$ is intersection closed and all properties in the hierarchy are equivalent for $(X, \mathcal{B}_{u+})$.

If $(X, \mathcal{B}_u)$ is spherically complete, then $(X, \mathcal{B}_{u+})$ is an $S^*$ ball space.

By [10, Theorem 1.2], assertions 1) and 2) of Proposition 4.3 also hold for all ultrametric spaces $(X, u)$ with countable narrow value sets $uX$; the condition narrow means that all sets of mutually incomparable elements in $uX$ are finite. On the other hand, it is shown in [10] that the condition “narrow” cannot be dropped in this case. It is however an open question whether the condition “countable” can be dropped.

A large number of ultrametric fixed point and coincidence point theorems have been proven by S. Prieß-Crampe and P. Ribenboim (see e.g. [22, 23, 24, 25, 27]). Using ball spaces, some of them have been reproven and new ones have been proven in [12, 13, 16].

### 4.2. Metric spaces with metric balls.

In metric spaces $(X, d)$ we can consider the closed metric balls

$$B_\alpha(x) := \{y \in X \mid d(x, y) \leq \alpha\}$$

for $x \in X$ and $\alpha \in \mathbb{R}^\geq := \{r \in \mathbb{R} \mid r \geq 0\}$. We set

$$B_d := \{B_\alpha(x) \mid x \in X, \alpha \in \mathbb{R}^\geq\}.$$

The following theorem will be deduced from Theorem 4.6 below:
Theorem 4.5. If the ball space \((X, \mathcal{B}_d)\) is spherically complete, then \((X, d)\) is complete.

The converse is not true. Consider a rational function field \(k(x)\) together with the \(x\)-adic valuation \(v_x\). Choose an extension of \(v_x\) to a valuation \(v\) of the algebraic closure \(K_0\) of \(k(x)\). Then the value group is \(\mathbb{Q}\). An ultrametric in the sense of Section 4.1 is obtained by setting, for instance,

\[
u(a, b) := e^{-v(a-b)}.
\]

Take \((K, u)\) to be the completion of \((K_0, u)\). It can be shown that the balls

\[
B_{\alpha_i} \left( \sum_{j=1}^{i-1} x^{-\frac{1}{j}} \right) \quad \text{with} \quad \alpha_i = e^{\frac{1}{i}} \quad (2 \leq i \in \mathbb{N})
\]

have empty intersection in \(K\). Hence \((K, u)\) is not spherically complete, that is, the ultrametric ball space induced by \(u\) on \(K\) is not spherically complete. But this ultrametric is a complete metric.

Note that from Theorem 4.20 below it follows that the ball space \((X, \mathcal{B}_d)\) is spherically complete if every closed metric ball in \((X, d)\) is compact under the topology induced by \(d\), as the closed metric balls are closed in this topology.

In order to characterize complete metric spaces by spherical completeness, we have to choose smaller induced ball spaces. For any subset \(S\) of the set \(\mathbb{R}^>0\) of positive real numbers, we define:

\[
\mathcal{B}_S := \{B_r(x) \mid x \in X, r \in S\}.
\]

Theorem 4.6. The following assertions are equivalent:

a) \((X, d)\) is complete,

b) the ball space \((X, \mathcal{B}_S)\) is spherically complete for some \(S \subset \mathbb{R}^>0\) which admits 0 as its only accumulation point,

c) the ball space \((X, \mathcal{B}_S)\) is spherically complete for every \(S \subset \mathbb{R}^>0\) which admits 0 as its only accumulation point.

Proof: We note that every \(S \subset \mathbb{R}^>0\) which admits 0 as its only accumulation point is discretely ordered. Take a nest \(\mathcal{N}\) of balls in \(\mathcal{B}_S\). If \(\mathcal{N}\) contains a smallest ball, then this ball is equal to \(\bigcap \mathcal{N}\), which is hence nonempty. So one only has to consider nests without a smallest ball. If we take such a nest \(\mathcal{N} = \{B_{r_i}(x_i) \mid i \in I\}\) in \(\mathcal{B}_S\), then the set \(\{r_i \mid i \in I\} \subset S\) has no smallest element and therefore, 0 is a limit point also of this set.

a) \(\Rightarrow\) c): Assume that \((X, d)\) is complete and take a set \(S \subset \mathbb{R}^>0\) which admits 0 as its only accumulation point. This implies that \(S\) is
discretely ordered, hence every infinite descending chain in \( S \) with a maximal element can be indexed by the natural numbers.

Take any nest \( \mathcal{N} \) of closed metric balls in \( B_S \). If the nest contains a smallest ball, then its intersection is nonempty; so we assume that it does not. If \( B \in \mathcal{N} \), then \( \mathcal{N}_B := \{ B' \in \mathcal{N} \mid B' \subseteq B \} \) is a nest of balls with \( \bigcap \mathcal{N} = \bigcap \mathcal{N}_B \); therefore, we may assume from the start that \( \mathcal{N} \) contains a largest ball. Then the radii of the balls in \( \mathcal{N} \) form an infinite descending chain in \( S \) with a maximal element, and 0 is their unique accumulation point. Hence we can write \( \mathcal{N} = \{ B_r(x_i) \mid i \in \mathbb{N} \} \) with \( r_j < r_i \) for \( i < j \), and with \( \lim_{i \to \infty} r_i = 0 \).

For every \( i \in \mathbb{N} \) and all \( j \geq i \), the element \( x_j \) lies in \( B_{r_i}(x_i) \) and therefore satisfies \( d(x_i, x_j) \leq r_i \). This shows that \( (x_i)_{i \in \mathbb{N}} \) is a Cauchy sequence. Since \( (X, d) \) is complete, it has a limit \( x \) in \( X \). We have that \( d(x_i, x) \leq r_i \), so \( x \) lies in every ball \( B_{r_i}(x_i) \). This proves that the nest has nonempty intersection.

c) \( \Rightarrow \) b): Trivial.

b) \( \Rightarrow \) a): Assume that \( (X, B_S) \) is spherically complete. Take any Cauchy sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \). By our assumptions on \( S \), we can choose a sequence \( (s_i)_{i \in \mathbb{N}} \) in \( \{ s \in S \mid s < s_0 \} \) such that \( 0 < 2s_{i+1} \leq s_i \).

Now we will use induction on \( i \in \mathbb{N} \) to choose an increasing sequence \( (n_i)_{i \in \mathbb{N}} \) of natural numbers such that the balls \( B_i := B_{s_i}(x_{n_i}) \) form a nest.

Since \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence, we have that there is \( n_1 \) such that \( d(x_n, x_m) < s_2 \) for all \( n, m > n_1 \). Once we have chosen \( n_{i-1} \), we choose \( n_i > n_{i-1} \) such that \( d(x_n, x_m) < s_{i+1} \) for all \( n, m \geq n_i \). We show that the so obtained balls \( B_i \) form a nest. Take \( i \in \mathbb{N} \) and \( x \in B_{i+1} = B_{s_{i+1}}(x_{n_{i+1}}) \). This means that \( d(x_{n_{i+1}}, x) \leq s_{i+1} \). Since \( n_i, n_{i+1} \geq n_i \), we have that \( d(x_{n_i}, x_{n_{i+1}}) < s_{i+1} \). We compute:

\[
\begin{align*}
d(x_{n_i}, x) & \leq d(x_{n_i}, x_{n_{i+1}}) + d(x_{n_{i+1}}, x) \\
& \leq s_{i+1} + s_{i+1} = 2s_{i+1} \leq s_i
\end{align*}
\]

Thus \( x \in B_i \) and hence \( B_{i+1} \subseteq B_i \) for all \( i \in \mathbb{N} \). The intersection of this nest \( (B_i)_{i \in \mathbb{N}} \) contains some \( y \), by our assumption. We have that \( y \in B_i \) for all \( i \in \mathbb{N} \), which means that \( d(x_{n_i}, y) \leq s_i \). Since

\[
\lim_{i \to \infty} s_i = 0,
\]

we obtain that

\[
\lim_{i \to \infty} x_{n_i} = y,
\]

which proves that \( (X, d) \) is a complete metric space. \( \square \)
Proof of Theorem 4.5: Assume that \((X, B_d)\) is spherically complete. Then so is \((X, B')\) for every nonempty \(B' \subset B_d\). Taking \(B' = B_S\) with \(S\) as in Theorem 4.6, we obtain that \((X, d)\) is complete. □

Remark 4.7. Theorems 4.5 and 4.6 remain true if instead of the closed metric balls the open metric balls

\[ B_\alpha(x) := \{ y \in X \mid d(x, y) < \alpha \} \]

are used for the metric ball space.

4.3. Metric spaces with Caristi–Kirk balls or Oettli–Théra balls. Consider a metric space \((X, d)\). A function \(\varphi : X \to \mathbb{R}\) is lower semicontinuous if for every \(y \in X\),

\[ \liminf_{x \to y} \varphi(x) \geq \varphi(y). \]

If \(\varphi\) is lower semicontinuous and bounded from below, we call it a Caristi–Kirk function on \(X\). For a fixed Caristi–Kirk function \(\varphi\) we consider Caristi–Kirk balls of the form

\[ B_\varphi^\phi := \{ y \in X \mid d(x, y) \leq \varphi(x) - \varphi(y) \}, \quad x \in X, \]

and the corresponding Caristi–Kirk ball space \((X, B^\phi)\) given by

\[ B^\phi := \{ B_\varphi^\phi \mid x \in X \}. \]

These ball spaces and their underlying theory can be employed to prove the Caristi–Kirk Theorem in a simple manner (see below). We found the sets that we call Caristi–Kirk balls in a proof of the Caristi–Kirk Theorem given by J.-P. Penot in [20].

We say that a function \(\phi : X \times X \to (-\infty, +\infty]\) is an Oettli–Théra function on \(X\) if it satisfies the following conditions:

(a) \(\phi(x, \cdot) : X \to (-\infty, +\infty]\) is lower semicontinuous for all \(x \in X\);
(b) \(\phi(x, x) = 0\) for all \(x \in X\);
(c) \(\phi(x, y) \leq \phi(x, z) + \phi(z, y)\) for all \(x, y, z \in X\);
(d) there exists \(x_0 \in X\) such that \(\inf_{x \in X} \phi(x_0, x) > -\infty\).

This notion was, to our knowledge, first introduced by Oettli and Théra in [19]. An Oettli–Théra function \(\phi\) yields balls of the form

\[ B_\varphi^\phi := \{ y \in X \mid d(x, y) \leq -\phi(x, y) \}, \quad x \in X, \]

which will be called Oettli–Théra balls. If an element \(x_0\) satisfies condition (d) above, then we will call it an Oettli–Théra element for
In $X$, for a fixed Oettli–Théra element $x_0$ we define the associated Oettli–Théra ball space to be $(B_{x_0}^\phi, B_{x_0}^\phi)$, where

$$B_{x_0}^\phi := \{ B_x^\phi \mid x \in B_{x_0}^\phi \}.$$ 

We observe that for a given Caristi–Kirk function $\varphi : X \to \mathbb{R}$, the mapping

$$\phi(x, y) := \varphi(y) - \varphi(x)$$

is an Oettli–Théra function. Furthermore, every Caristi–Kirk ball is also an Oettli–Théra ball.

In general the balls defined above are not metric balls. However, when working in complete metric spaces they prove to be a more useful tool than metric balls. As observed in the previous section, the completeness of a metric space need not imply spherical completeness of the space of metric balls $(X, B_d)$. In case of Caristi–Kirk and Oettli–Théra balls, completeness turns out to be equivalent to spherical completeness, as shown in the following two propositions.

**Proposition 4.8.** Let $(X, d)$ be a metric space. Then the following assertions are equivalent:

a) The metric space $(X, d)$ is complete.

b) Every Caristi–Kirk ball space $(X, B_\varphi)$ is spherically complete.

c) For every continuous function $\varphi : X \to \mathbb{R}$ bounded from below, the Caristi–Kirk ball space $(X, B_\varphi)$ is spherically complete.

**Proposition 4.9.** A metric space $(X, d)$ is complete if and only if the Oettli–Théra ball space $(B_{x_0}^\phi, B_{x_0}^\phi)$ is spherically complete for every Oettli–Théra function $\phi$ on $X$ and every Oettli–Théra element $x_0$ for $\phi$ in $X$.

The proofs of Proposition 4.8 and Proposition 4.9 can be found in [14, Proposition 3] and [2], respectively.

To describe the properties of Caristi–Kirk and Oettli–Théra balls, we introduce the following notion. A ball space $(X, B)$ is a $B_x$–ball space if $B = \{ B_x \mid x \in X \}$. We call a $B_x$–ball space **contractive** if for every $x, y \in X$, the following conditions hold:

(C1) $x \in B_x$,

(C2) if $y \in B_x$ then $B_y \subseteq B_x$,

(C3) if $B_x$ is not a singleton, then there exists $y \in B_x$ such that $B_y \nsubseteq B_x$.

A $B_x$–ball space $(X, B)$ is **strongly contractive** if it satisfies (C1), (C2), and:

(C3s) if $y \in B_x \setminus \{ x \}$, then $B_y \nsubseteq B_x$.

The proofs of Proposition 4.8 and Proposition 4.9 can be found in [14, Proposition 3] and [2], respectively.

To describe the properties of Caristi–Kirk and Oettli–Théra balls, we introduce the following notion. A ball space $(X, B)$ is a $B_x$–ball space if $B = \{ B_x \mid x \in X \}$. We call a $B_x$–ball space **contractive** if for every $x, y \in X$, the following conditions hold:

(C1) $x \in B_x$,

(C2) if $y \in B_x$ then $B_y \subseteq B_x$,

(C3) if $B_x$ is not a singleton, then there exists $y \in B_x$ such that $B_y \nsubseteq B_x$.

A $B_x$–ball space $(X, B)$ is **strongly contractive** if it satisfies (C1), (C2), and:

(C3s) if $y \in B_x \setminus \{ x \}$, then $B_y \nsubseteq B_x$.
Then every strongly contractive ball space is contractive. On the other hand, it will turn out that condition (C1), while present in many applications, is not always necessary. Thus, we will call a \(B_x\)-ball space \((X,B)\) \textbf{weakly contractive} if it just satisfies (C2) and (C3).

The next proposition is proved in [2].

**Proposition 4.10.** Every Caristi–Kirk ball space \((X,B^\varphi)\) and every Oettli–Théra ball space \((B^\varphi_{x_0},B^\varphi_{x_0})\) is strongly contractive.

We will meet another strongly contractive ball space in the case of partially ordered sets; see Proposition 4.29.

In general, a strongly contractive ball space \((X,B)\) may not contain balls of the form \(\{x\}\) for every \(x \in X\). Then we cannot apply Lemma 2.5 to acquire the equivalence between properties \(S_1\) and \(S_2\). However, the following lemma yields the existence of a “sufficient” amount of singleton balls to obtain this equivalence.

**Proposition 4.11.** In a weakly contractive \(B_x\)-ball space, the intersection of a maximal nest of balls is a singleton ball if it is nonempty.

Proof: Let \(M\) be a maximal nest of balls and assume that \(a \in \bigcap M\) for some element \(a \in X\). Since \(a \in B\) for every ball \(B \in M\), we obtain from (C2) that \(B_a \subseteq B\) for every \(B \in M\) and thus \(B_a \subseteq \bigcap M\). This means that \(M \cup \{B_a\}\) is a nest of balls, so by maximality of \(M\) we have that \(B_a \in M\). Consequently, \(B_a = \bigcap M\). Suppose that \(B_a\) is not a singleton. Then by condition (C3) there is some element \(b\) such that \(B_b \subsetneq B_a\) whence \(B_b \notin M\). But then \(M \cup \{B_b\}\) is a nest which strictly contains \(M\). This contradiction to the maximality of \(M\) shows that \(B_a\) is a singleton. 

Since by Corollary 2.7 every nest is contained in a maximal nest, this proposition yields:

**Theorem 4.12.**

1) A weakly contractive \(B_x\)-ball space is \(S_1\) if and only if it is \(S_2\).

2) In every spherically complete weakly contractive \(B_x\)-ball space every ball \(B_x\) contains a singleton ball. If in addition the ball space satisfies (C1), then there exists \(a \in B_x\) such that \(B_a = \{a\}\).

3) If \((X,B)\) is a spherically complete contractive \(B_x\)-ball space and \(f : X \to X\) a function such that \(f(x) \in B_x\) for every \(x \in X\), then every ball \(B_x\) contains a fixed point of \(f\).

A version of part 3) of this theorem (with “contractive” replaced by “strongly contractive”) together with Propositions 4.8 and 4.10 is used
Theorem 4.13. Take a complete metric space \((X, d)\) and a lower semi-continuous function \(\varphi : X \to \mathbb{R}\) which is bounded from below. If a function \(f : X \to X\) satisfies the Caristi condition
\[
d(x, fx) \leq \varphi(x) - \varphi(fx),
\]
for all \(x \in X\), then \(f\) has a fixed point on \(X\).

Also in [2], the same tools (with Proposition 4.8 replaced by Proposition 4.9) are used to prove the following generalization:

Theorem 4.14. Take a complete metric space \((X, d)\) and \(\phi\) an Oettli-Théra function on \(X\). If a function \(f : X \to X\) satisfies
\[
d(x, fx) \leq -\phi(x, fx),
\]
for all \(x \in X\), then \(f\) has a fixed point on \(X\).

A variant of part 2) of Theorem 1.12 is used in [2] to give quick proofs of several theorems that are known to be equivalent to the Caristi–Kirk Fixed Point Theorem (see [19, 20, 21] for presentations of these equivalent results and generalizations). In the Introduction, we already pointed out the importance of \(f\)-closed singletons for the existence of fixed points of a given function \(f\). The Caristi condition on \(f\) in the Caristi–Kirk Fixed Point Theorem does not necessarily imply that every ball \(B_x\) is \(f\)-closed, but simply that \(f(x) \in B_x\). If \(B_x\) is a singleton, then this yields that \(x\) is a fixed point of \(f\). This proves assertion 3) of the theorem, which in turn provides a quick proof of the Caristi–Kirk Fixed Point Theorem. The weak condition that \(f(x) \in B_x\) together with the condition that the ball space be strongly contractive implies that (SC2) holds and enables us to drop the condition on \(f\)-nests in Theorem 1.6.

Remark 4.15. Assume that \((X, \mathcal{B})\) is a \(\mathcal{B}_x\)-ball space which satisfies (C1) and (C2). Then we can define a partial ordering on \(X\) by setting
\[
x \prec y :\iff B_y \subsetneq B_x.
\]
If \((X, \mathcal{B})\) is strongly contractive, then the function \(x \mapsto B_x\) is injective, and \(X\) together with the reverse of the partial order we have defined is order isomorphic to \(\mathcal{B}\) with inclusion, that is, the function \(x \mapsto B_x\) is an order isomorphism from \((X, \prec)\) onto \((\mathcal{B}, \subset)\) where the latter is defined as in the beginning of Section 2.

If the \(B_x\) are the Caristi–Kirk balls defined in (8), then we have that
\[
x \prec y \iff d(x, y) < \varphi(x) - \varphi(y),
\]
which means that $\prec$ is the Brønsted ordering on $X$. The Ekeland Variational Principle (cf. [2]) states that if the metric space is complete, then $(X, \prec)$ admits maximal elements, or in other words, $B$ admits minimal balls. The Brønsted ordering has been used in several different proofs of the Caristi–Kirk Fixed Point Theorem. However, at least in the proofs that also define and use the Caristi–Kirk balls (such as the one of Penot [20] in [14]), it makes more sense to use directly their natural partial ordering. But the main incentive to use the balls instead of the ordering is that it naturally subsumes the metric case in the framework of fixed point theorems in several other areas of mathematics which is provided by the general theory of ball spaces as laid out in the present paper (see also [12, 13, 16]).

It has been shown that the Ekeland Variational Principle can be proven in the Zermelo Fraenkel axiom system ZF plus the axiom of dependent choice DC which covers the usual mathematical induction (but not transfinite induction, which is equivalent to the full axiom of choice). Conversely, it has been shown in [3] that the Ekeland Variational Principle implies the axiom of dependent choice.

Several proofs have been provided for the Caristi–Kirk FPT that work in ZF+DC. Kozlowski has given a proof that is purely metric as defined in his paper [9], which implies that the proof works in ZF+DC. The proofs of Proposition 4.8 in [14] and of Proposition 4.9 in [2] are purely metric. The existence of singleton balls in Caristi–Kirk and Oettli-Théra ball spaces over complete metric spaces can also be shown directly by purely metric proofs and this result can be used to give quick proofs of many principles that are equivalent to the Caristi–Kirk FPT in ZF+DC (cf. [2]). However, in other settings it may not be possible to deduce the existence in ZF+DC, so then the axiom of choice is needed. Therefore, in view of the number of possible applications even beyond the scope as presented in this paper, we do not hesitate to use Zorn’s Lemma for the proof.

We should point out that proofs have been given that apparently prove the Caristi–Kirk FPT in ZF (see [17, 7]). This means that the Caristi–Kirk FPT and the Ekeland Variational Principle are equivalent in ZF+DC, but not in ZF. For the topic of axiomatic strength, see also the discussions in [6, 8, 9].

4.4. Ordered abelian groups and fields.
If $(K, <)$ is an ordered abelian group or ordered field, then the distance function $d(x, y) = |x - y|$ takes its values in the nonnegative part of $K$. But the ordering on $K$ may not be archimedean, that is, $(K, <)$
may not admit an order preserving embedding in the reals; in this case, 
The following fact is not hard to show. It was first proved in [28]
d is not a metric. Nevertheless, we can form a meaningful ball space
for ordered fields, and then for any ordered sets in [15]. By a cut in
associated with \((K, <)\) by taking \(B_{cb}\) to consist of all closed bounded
an ordered set \((I, <)\) we mean a partition \((C, D)\) of \(I\) such that \(c < d\)
intervals \([a, b]\) with \(a, b \in K\). More generally, the same can be done for
for all \(c \in C, d \in D\) and \(C, D\) are nonempty. The cofinality of a
totally ordered set is the least cardinality of all cofinal subsets, and the
cofinality of a totally ordered set is the cofinality of this set under
the reverse ordering.

Lemma 4.16. The ball space \((I, B_{cb})\) associated with the totally ordered
set \((I, <)\) is \(S_1\) if and only if every cut \((C, D)\) in \((I, <)\) is asymmetric,
that is, the cofinality of \(C\) is different from the coinitiality of \(D\).

Totally ordered sets and ordered abelian groups or fields whose cuts
are all asymmetric are called symmetrically complete. In [28] it has
been shown that arbitrarily large symmetrically complete ordered fields
exist. With a different construction idea, this has been generalized in
[15] to the case of ordered abelian groups and totally ordered sets, and
a characterization of symmetrically complete ordered abelian groups
and fields has been given.

In order to give an example of a fixed point theorem that can be
proven in this setting, it is enough to consider symmetrically complete
ordered abelian groups, as the additive group of a symmetrically com-
plete ordered field is a symmetrically complete ordered abelian group.
The following is Theorem 21 of [12] (see also [15]).

Theorem 4.17. Take an ordered abelian group \((G, <)\) and a function
\(f : G \to G\). Assume that every nonempty chain of closed bounded
intervals in \(G\) has nonempty intersection and that \(f\) has the following
properties:

1) \(f\) is nonexpanding:

\[|fx - fy| \leq |x - y| \text{ for all } x, y \in G,\]

2) \(f\) is contracting on orbits: there is a positive rational number \(\frac{m}{n} < 1\)
with \(m, n \in \mathbb{N}\) such that

\[n|fx - f^2x| \leq m|x - fx| \text{ for all } x \in G.\]

Then \(f\) has a fixed point.
As in the case of ultrametric spaces, all singletons in $B_{cb}$ are balls: $\{a\} = [a, a]$. So also here, $(I, B_{cb})$ is $S_2$ as soon as it is $S_1$. But again as in the case of ultrametric spaces, $S_2$ does not necessarily imply $S_5$ or even $S_3$. For example, consider a nonarchimedean ordered symmetrically complete field. The set of infinitesimals is the intersection of balls $[-a, a]$ where $a$ runs through all positive elements that are not infinitesimals. This intersection is not a ball, nor is there a largest ball contained in it.

Further, we note:

**Lemma 4.18.** Assume that $(I, <)$ is a totally ordered set and its associated ball space $(I, B_{cb})$ is an $S_1^d$ or $S_3$ ball space. Then $(I, <)$ is cut complete, that is, for every cut $(C, D)$ in $(I, <)$, $C$ has a largest or $D$ has a smallest element.

Proof: First assume that $(I, B_{cb})$ is an $S_1^d$ ball space, and take a cut $(C, D)$ in $I$. If $a, c \in C$ and $b, d \in D$, then $\max\{a, c\} \in C$ and $\min\{b, d\} \in D$ and $[a, b] \cap [c, d] = [\max\{a, c\}, \min\{b, d\}]$. This shows that

$$\{[c, d] \mid c \in C, b \in D\}$$

is a directed system in $B_{cb}$. Hence its intersection is nonempty; if $a$ is contained in this intersection, it must be the largest element of $C$ or the least element of $D$. Hence $(I, <)$ is cut complete.

Now assume that $(I, <)$ is not cut complete; we wish to show that $(I, B_{cb})$ is not an $S_3$ ball space. Take a cut $(C, D)$ in $I$ such that $C$ has no largest element and $D$ has no least element. Pick some $c \in C$. Then

$$\{[c, d] \mid d \in D\}$$

is a nest of balls in $(I, B_{cb})$. Its intersection is the set $\{a \in C \mid c \leq a\}$. Since $C$ has no largest element, this set does not contain a maximal ball. This shows that $(I, B_{cb})$ is not an $S_4$ ball space. \qed

It is a well known fact that the only cut complete densely ordered abelian group or ordered field is $\mathbb{R}$. So we have:

**Proposition 4.19.** The associated ball space of the reals is $S^*$. For all other densely ordered abelian groups and ordered fields the associated ball space can at best be $S_2$.

Proof: Take any centered system $\{[a_i, b_i] \mid i \in I\}$ of intervals in $\mathbb{R}$. We set $a := \sup_{i \in I} a_i$ and $b := \inf_{i \in I} b_i$. Then

$$\bigcap_{i \in I} [a_i, b_i] = [a, b].$$
We have to show that \([a, b] \neq \emptyset\), i.e., \(a \leq b\). Suppose that \(a > b\). Then there are \(i, j \in I\) such that \(a_i > b_j\). But by assumption, \([a_i, b_j] \cap [a_j, b_i] \neq \emptyset\), a contradiction. We have now proved that the associated ball space of the reals is \(S^*\).

The second assertion follows from Lemma 4.18.

4.5. **Topological spaces.**

If \(\mathcal{X}\) is a topological space on a set \(X\), we will take its associated ball space to be \((X, \mathcal{B})\) where \(\mathcal{B}\) consists of all nonempty closed sets. Since the intersections of arbitrary collections of closed sets are again closed, this ball space is intersection closed.

The following theorem shows how compact topological spaces are characterized by the properties of their associated ball spaces; note that we use “compact” in the sense of “quasi-compact”, that is, it does not imply the topology being Hausdorff.

**Theorem 4.20.** The following are equivalent for a topological space \(\mathcal{X}\):

a) \(\mathcal{X}\) is compact,

b) the nonempty closed sets in \(\mathcal{X}\) form an \(S_1\) ball space,

c) the nonempty closed sets in \(\mathcal{X}\) form an \(S^*\) ball space.

**Proof:** 
a) \(\Rightarrow\) b): Assume that \(\mathcal{X}\) is compact. Take a nest \((X_i)_{i \in I}\) of balls in \((X, \mathcal{B})\) and suppose that \(\bigcap_{i \in I} X_i = \emptyset\). Then \(\bigcup_{i \in I} X \setminus X_i = X\), so \(\{X \setminus X_i \mid i \in I\}\) is an open cover of \(\mathcal{X}\). It follows that there are \(i_1, \ldots, i_n \in I\) such that \(X \setminus X_{i_1} \cup \ldots \cup X \setminus X_{i_n} = X\), whence \(X_{i_1} \cap \ldots \cap X_{i_n} = \emptyset\). But since the \(X_i\) form a nest, this intersection equals the smallest of the \(X_j\), which is nonempty. This contradiction proves that the nonempty closed sets in \(\mathcal{X}\) form an \(S_1\) ball space.

b) \(\Rightarrow\) c): This follows from Theorem 3.10.

c) \(\Rightarrow\) a): Assume that the nonempty closed sets in \(\mathcal{X}\) form an \(S^*\) ball space. Take an open cover \(Y_i, i \in I\), of \(\mathcal{X}\). Since \(\bigcup_{i \in I} Y_i = X\), we have that \(\bigcap_{i \in I} X \setminus Y_i = \emptyset\). As the ball space is \(S^*\), this means that \(\{X \setminus Y_i \mid i \in I\}\) cannot be a centered system. Consequently, there are \(i_1, \ldots, i_n \in I\) such that \(X \setminus Y_{i_1} \cap \ldots \cap X \setminus Y_{i_n} = \emptyset\), whence \(Y_{i_1} \cup \ldots \cup Y_{i_n} = X\).

The following two topological fixed point theorems were proven in [12, Theorem 11]. We will give their proofs here as they illustrate applications of theorems 1.6 and 1.2.
Theorem 4.21. Take a compact space $X$ and a closed function $f : X \to X$. Assume that for every $x \in X$ with $fx \neq x$ there is a closed subset $B$ of $X$ such that $x \in B$ and $x \notin f(B) \subseteq B$. Then $f$ has a fixed point in $B$.

Proof: For every $x \in X$ we consider the following family of balls:

$$B_x := \{ B \mid B \text{ closed subset of } X, x \in B \text{ and } f(B) \subseteq B \}.$$  

(9) $B_x := \bigcap B_x$.

We see that $x \in B_x$ and that $f(B_x) \subseteq B_x$. Further, $B_x$ is closed, being the intersection of closed sets. This shows that $B_x$ is the smallest member of $B_x$.

For every $B \in B_x$ we have that $fx \in B$ and therefore, $B \in B_{fx}$. Hence we find that $B_{fx} \subseteq B_x$.

Assume that $fx \neq x$. Then by hypothesis, there is a closed set $B$ in $X$ such that $x \in B$ and $x \notin f(B) \subseteq B$. Since $f$ is a closed function, $f(B)$ is closed. Moreover, $f(f(B)) \subseteq f(B)$ and $fx \in f(B)$, so $f(B) \in B_{fx}$. Since $x \notin f(B)$, we conclude that $x \notin B_{fx}$, whence $B_{fx} \subset B_x$. We have now proved that $f$ is ultimately contracting on orbits. Further, $B \in B_x$, whence $B_x \subset B$, $f(B_x) \subset f(B)$ and therefore, $x \notin f(B_x)$. This shows that $B_x$ is the smallest of all closed sets $B$ in $X$ for which $x \in B$ and $x \notin f(B) \subseteq B$.

Take an $f$-nest $\mathcal{N}$ in $\{ B_x \mid x \in X \}$. Theorem 4.20 shows that $\bigcap \mathcal{N}$ is nonempty. Take any $z \in \bigcap \mathcal{N}$. Choose an arbitrary $B \in \mathcal{N}$. Then $z \in B$ and thus, $B \in B_x$. So we have that $B_x \subseteq B$. Therefore, $B_z \subseteq \bigcap \mathcal{N}$. Our theorem now follows from Theorem 1.6.

An interesting interpretation of the ball $B_x$ defined in (9) will be given in Remark 5.3 below.

Theorem 4.22. Take a compact space $X$ and a closed function $f : X \to X$.

1) If every nonempty closed and $f$-closed subset $B$ of $X$ contains a closed $f$-contracting subset, then $f$ has a fixed point in $X$.

2) If every nonempty closed and $f$-closed subset $B$ of $X$ is $f$-contracting, then $f$ has a unique fixed point in $X$.

3) Assume that $\mathcal{B}'$ is a set of closed subsets of $X$ such that $f(B) \in \mathcal{B}'$ for each $B \in \mathcal{B}'$ and $\mathcal{B}'$ is chain intersection closed. If every $f$-closed ball $B \in \mathcal{B}'$ is $f$-contracting, then $f$ has a unique fixed point in $X$. 
Proof: By Theorem 4.20 the ball space consisting of all nonempty closed subsets of the compact space $X$ is $S^*$, hence also $S_5$. We will now deduce our theorem from Theorem 1.2.

1): Assume that every nonempty closed and $f$-closed subset $B$ of $X$ contains a closed $f$-contracting subset, that is, every $f$-closed ball contains an $f$-contracting ball. We wish to show that each $f$-closed ball $B$ is a singleton or contains a smaller $f$-closed ball. By assumption, $B$ contains a closed $f$-contracting subset $B'$. We see that $B'$ is an $f$-closed ball. If $B' \neq B$, then we are done. Hence assume that $B' = B$, which yields that $B$ is $f$-contracting. If $B$ is a singleton, then we are done again. Otherwise, we must have that $f(B) \subsetneq B$. By observation a) at the start of Section 2.3, $f(B)$ is $f$-closed, and it is closed since $f$ is assumed to be a closed function. As the $f$-closed ball $f(B)$ is properly contained in $B$, we are done also in this final case. Thus part 1) of our theorem follows from part 1) of Theorem 1.2.

2): Now assume that every nonempty closed and $f$-closed subset $B$ of $X$ is $f$-contracting. We wish to show that the image $f(B)$ of every $f$-closed ball $B$ is an $f$-contracting ball. As before, $f(B)$ is closed and $f$-closed. Therefore, it is $f$-contracting by assumption. Thus part 2) of our theorem follows from part 2) of Theorem 1.2.

3): As mentioned in the beginning, the ball space of all nonempty closed subsets of $X$ is $S_5$, hence also $S_1$, and thus the same holds for $(X, B')$ (cf. Proposition 6.1 below). As $B'$ is chain intersection closed by assumption, it follows from Proposition 3.9 that $(X, B')$ is $S_5$. Take an $f$-closed ball $B \in B'$. As before, $f(B)$ is closed and $f$-closed. By our assumptions, we also have that $f(B) \in B'$ and then that $f(B)$ is $f$-contracting. Thus part 3) of our theorem follows from part 2) of Theorem 1.2, where $B$ is taken to be $B'$.

The condition that every $f$-closed ball is a singleton or contains a smaller $f$-closed ball appears to be quite strong. Yet there is a natural example in the setting of topological spaces where this condition is satisfied. In [29], Steprans, Watson and Just define the notion of $J$-contraction for functions $f : X \rightarrow X$ on topological spaces. We do not need the slightly lengthy definition here; instead, we use two important facts about $J$-contractions $f$ on a connected compact Hausdorff space $X$ which the authors prove in the cited paper:

(J1) If $B$ is a closed subset of $X$ with $f(B) \subseteq B$, then the restriction of $f$ to $B$ is also a $J$-contraction ([29, Proposition 1, p. 552]);

(J2) If $f$ is onto, then $|X| = 1$ ([29, Proposition 4, p. 554]).

The following is Theorem 4 of [29]:
Theorem 4.23. Take a connected compact Hausdorff space $X$ and a continuous $J$-contraction $f : X \to X$. Then $f$ has a unique fixed point.

We show how to deduce this theorem from part 3) of Theorem 4.22.

First, as $f$ is a continuous function on the compact Hausdorff space $X$, it is a closed function. We take $\mathcal{B}'$ to be the set of all closed connected subsets of $X$. Take any $B \in \mathcal{B}'$. As $f$ is a closed function, $f(B)$ is closed and since $B$ is connected and $f$ is continuous, $f(B)$ is also connected; hence $f(B) \in \mathcal{B}'$. Further, the intersection of any chain of closed connected subsets of $X$ is closed and connected, which shows that $\mathcal{B}'$ is intersection closed.

Finally, we have to show that every $f$-closed ball $B \in \mathcal{B}'$ is $f$-contracting. As $B$ is closed in $X$, it is also compact Hausdorff, and it is connected as it is a ball in $\mathcal{B}'$. By (J1), the restriction of $f$ to $B$ is also a $J$-contraction. Therefore, we can replace $X$ by $B$ and apply (J2) to find that if $f$ is onto then $B$ is a singleton, i.e., $B$ is $f$-contracting. Now Theorem 4.23 follows from part 3) of Theorem 4.22 as desired.

It should be noted that $J$-contractions appear in a natural way in the metric setting. The following results are Theorems 2 and 3 of [29]:

Theorem 4.24. Any contraction on a compact metric space is a $J$-contraction. Conversely, if $f$ is a $J$-contraction on a connected compact metrizable space $X$, then $X$ admits a metric under which $f$ is a contraction.

4.6. Partially ordered sets.

Take any nonempty partially ordered set $(T, \prec)$. We will associate with it two different ball spaces; first, the ball space of principal final segments, and then later the interval ball space.

The ball space $(T, \mathcal{B}_{pfs})$, where $\mathcal{B}_{pfs} := \{[a, \infty) \mid a \in T\}$ is the set of all principal final segments $[a, \infty) := \{c \in T \mid a \leq c\}$. The following proposition gives the interpretation of spherical completeness for this ball space:

Proposition 4.25. The following assertions are equivalent:

a) the poset $(T, \prec)$ is inductively ordered,

b) the ball space $(T, \mathcal{B}_{pfs})$ is spherically complete,

c) $(T, \mathcal{B}_{pfs})$ is an $S_2$ ball space.

Proof: We observe that $\{a_i \mid i \in I\}$ is a chain in $T$ if and only if $\mathcal{N} = ([a_i, \infty))_{i \in I}$ is a nest of balls in $\mathcal{B}_{pfs}$. 

a) ⇒ c): Take a nest \( N = ([a_i, \infty])_{i \in I} \). Since \((T, <)\) is inductively ordered, the chain \( \{a_i \mid i \in I\} \) admits an upper bound \( a \in T \). Then for all \( i \in I \), \( a_i \leq a \), whence \([a, \infty) \subseteq [a_i, \infty)\). Thus, \([a, \infty) \subseteq \bigcap N\), which proves that \((T, B_{pfs})\) is an \( S_2 \) ball space.

c) ⇒ b): This holds by the general properties of the hierarchy.

b) ⇒ a): Take a chain \( \{a_i \mid i \in I\} \) in \( T \). Since \((T, B_{pfs})\) is spherically complete, the intersection of the nest \( N = ([a_i, \infty])_{i \in I} \) is nonempty. If \( a \in \bigcap N \), then for all \( i \in I \), \( a \in [a_i, \infty) \), whence \( a_i \leq a \). Thus, \( a \) is an upper bound of \( \{a_i \mid i \in I\} \), which proves that \((T, <)\) is inductively ordered. □

We leave it to the reader to show that \((T, B_{pfs})\) is an \( S_3 \) (or \( S_3^3 \) or \( S_3^c \)) ball space if and only if every chain (or directed system, or centered system, respectively) has minimal upper bounds.

We will need the following observation:

Lemma 4.26. The equality

\[
[a, \infty) = \bigcap_{i \in I} [a_i, \infty)
\]

holds if and only if \( a = \sup_{i \in I} a_i \). Further, \( \bigcap_{i \in I} [a_i, \infty) \) is the (possibly empty) set of all upper bounds for \( \{a_i \mid i \in I\} \).

Proof: We have \( a \in \bigcap_{i \in I} [a_i, \infty) \) if and only if \( a \in [a_i, \infty) \) and hence \( a \geq a_i \) for all \( i \), which means that \( a \) is an upper bound for the \( a_i \). Hence, \( \bigcap_{i \in I} [a_i, \infty) \) is the set of all upper bounds of the \( a_i \), and this set is equal to \([a, \infty)\) if and only if \( a \) is the least upper bound. □

An element \( a \) in a poset is called top element if \( b \leq a \) for all elements \( b \) in the poset, and bottom element if \( b \geq a \) for all elements \( b \) in the poset. A top element is commonly denoted by \( \top \), and a bottom element by \( \bot \). A poset \((T, <)\) is an upper semilattice if every two elements in \( T \) have a least upper bound, and a complete upper semilattice if every nonempty set of elements in \( T \) has a least upper bound.

Proposition 4.27. 1) \((T, B_{pfs})\) is finitely intersection closed if and only if every nonempty finite bounded subset of \( T \) has a supremum.

2) \((T, B_{pfs})\) is intersection closed if and only if every nonempty bounded subset of \( T \) has a supremum, i.e., \((T, <)\) is bounded complete.

If in addition \((T, <)\) has a top element, then

3) \((T, B_{pfs})\) is finitely intersection closed if and only if \((T, <)\) is an upper semilattice,

4) \((T, B_{pfs})\) is intersection closed if and only if \((T, <)\) is a complete upper semilattice.
Proof: 1), 2): Assume that \((T, \mathcal{B}_{\text{pfs}})\) is (finitely) intersection closed and take a nonempty (finite) subset \(\{a_i \mid i \in I\}\) of \(T\). If this set is bounded, then \(\bigcap_{i \in I} [a_i, \infty)\) is nonempty, and thus by assumption it is equal to \([a, \infty)\) for some \(a \in T\). By Lemma 4.26 this implies that \(a = \sup_{i \in I} a_i\), showing that \(\{a_i \mid i \in I\}\) has a supremum.

Now assume that every nonempty (finite) bounded subset of \(T\) has a supremum. Take a nonempty (finite) set \(\{[a_i, \infty) \mid i \in I\}\) of balls in \(\mathcal{B}_{\text{pfs}}\) with nonempty intersection. Take \(b \in \bigcap_{i \in I} [a_i, \infty)\). Then \(b\) is an upper bound of \(\{a_i \mid i \in I\}\). By assumption, there exists \(a = \sup_{i \in I} a_i\) in \(T\). Again by Lemma 4.26 this implies that \(\bigcap_{i \in I} [a_i, \infty) = [a, \infty)\). Hence, \((T, \mathcal{B}_{\text{pfs}})\) is (finitely) intersection closed.

3) and 4) follow from 1) and 2), respectively, because if \((T, <)\) has a top element, then every nonempty subset is bounded. \(\Box\)

Now we can characterize chain complete and directed complete posets by properties from our hierarchy:

**Theorem 4.28.** Take a poset \((T, <)\). Then the following are equivalent:

a) \((T, <)\) is chain complete,

b) \((T, <)\) is directed complete,

c) \((T, \mathcal{B}_{\text{pfs}})\) is an \(S_5\) ball space,

d) \((T, \mathcal{B}_{\text{pfs}})\) is an \(S_d\) ball space.

If every finite bounded subset of \(T\) has a supremum, then the above properties are also equivalent to

e) \((T, \mathcal{B}_{\text{pfs}})\) is an \(S^*\) ball space.

Proof: The equality of assertions a) and b) follows from Proposition 3.2

b) \(\Rightarrow\) d): Assume that \((T, <)\) is directed complete and take a directed system \(S = \{[a_i, \infty) \mid i \in I\}\) in \(\mathcal{B}_{\text{pfs}}\). Then also \(\{a_i \mid i \in I\}\) is a directed system in \((T, <)\). By our assumption on \((T, <)\) it follows that \(\{a_i \mid i \in I\}\) has a supremum \(a\) in \(T\). By Lemma 4.26 \([a, \infty) = \bigcap_{i \in I} [a_i, \infty)\), which shows that the intersection of \(S\) is a ball.

d) \(\Rightarrow\) c) holds by the general properties of the hierarchy.

c) \(\Rightarrow\) a): Take a chain \(\{a_i \mid i \in I\}\) in \(T\). Since \((T, \mathcal{B}_{\text{pfs}})\) is an \(S_5\) ball space, the intersection of the nest \(N = ([a_i, \infty))_{i \in I}\) is a ball \([a, \infty)\). It follows by Lemma 4.26 that \(a\) is the least upper bound of the chain, which proves that \((T, <)\) is chain complete.
If every finite bounded subset of \( T \) has a supremum, then by part 1) of Proposition 4.27, \((T, B_{pfs})\) is finitely intersection closed, hence by Proposition 3.8, properties \( S_5^d \) and \( S^* \) are equivalent.

The ball space \((T, B_{pfs})\) shares an important property with Caristi–Kirk and Oettli–Théra ball spaces:

**Proposition 4.29.** The ball space \((T, B_{pfs})\) is strongly contractive.

**Proof:** We define
\[
B_x := [x, \infty) \in B_{pfs}.
\]
Then \( x \in B_x \) for every \( x \in T \). If \( y \in B_x \), then \( x \leq y \) and therefore \([y, \infty) \subseteq [x, \infty)\); if in addition \( x \neq y \), then \( x < y \) so that \( x \notin [y, \infty) \) and \([y, \infty) \subsetneq [x, \infty)\).

A function \( f \) on a poset \((T, <)\) is **increasing** if \( f(x) \geq x \) for all \( x \in T \). The following result is an immediate consequence of Zorn’s Lemma, but can also be seen as a corollary to Propositions 4.25 and 4.29 together with part 3) of Theorem 4.12:

**Theorem 4.30.** Every increasing function \( f : X \to X \) on an inductively ordered poset \((T, <)\) has a fixed point.

Note that this theorem implies the **Bourbaki-Witt Theorem**, which differs from it by assuming that every increasing chain in \((T, <)\) even has a least upper bound.

A function \( f \) on a poset \((T, <)\) is called **order preserving** if \( x \leq y \) implies \( fx \leq fy \). The following result is an easy consequence of Theorem 4.30:

**Theorem 4.31.** Take an order preserving function \( f : T \to T \) which contains at least one \( x \) such that \( fx \geq x \) (in particular, this holds when \((T, <)\) has a bottom element). Assume that \((T, <)\) is chain complete. Then \( f \) has a fixed point.

**Proof:** Take \( S := \{x \in T \mid fx \geq x\} \neq \emptyset \). Then also \( S \) is chain complete. Indeed, if \((x_i)_{i \in I}\) is a chain in \( S \), hence also in \( T \), then it has a least upper bound \( z \in T \). Since \( z \geq x_i \) and \( f \) is order preserving, we have that \( fz \geq f x_i \geq x_i \) for all \( i \in I \), so \( fz \) is also an upper bound for \((x_i)_{i \in I}\). Therefore, \( fz \geq z \), showing that \( fz \in S \). Now the existence of a fixed point follows from Theorem 4.30.

The second ball space we associate with posets will be particularly useful for the study of lattices. We define the **interval ball space** \((T, B_{iv})\) of the poset \((T, <)\) by taking \( B_{iv} \) to consist of all **closed intervals**, that is, sets of the form \([a, b] := \{c \in T \mid a \leq c \leq b\} \) for \( a, b \in T \)
with \(a \leq b\), or of the form \(\{c \in T \mid c \leq a\}\) or \(\{c \in T \mid a \leq c\}\) for \(a \in T\). Note that all closed intervals are of the form \([a, b]\) if and only if \(T\) has a top element \(\top\) and a bottom element \(\bot\). Even if \(T\) does not have these elements, we will still use the notation \([\bot, b]\) for \(\{c \in T \mid c \leq b\}\) and \([a, \top]\) for \(\{c \in T \mid a \leq c\}\). We also include \([\bot, \top]\) := \(T\) in \(\mathcal{B}_{iv}\). Hence, \(\mathcal{B}_{iv} = \{[a, b] \mid a \in T \cup \{\bot\}, b \in T \cup \{\top\}\}\).

**Lemma 4.32.** We have that \(a = \sup_{i \in I} a_i\) and \(b = \inf_{i \in I} b_i\) if and only if \([a, b] = \bigcap_{i \in I} [a_i, b_i]\).

Proof: We can write
\[
\bigcap_{i \in I} [a_i, b_i] = \bigcap_{i \in I} [a_i, \top] \cap [\bot, b_i] = \bigcap_{i \in I} [a_i, \top] \cap \bigcap_{i \in I} [\bot, b_i]
\]
Applying Lemma 4.26, we obtain that \([a, \top] = \bigcap_{i \in I} [a_i, \top]\) if and only if \(a = \sup_{i \in I} a_i\). Applying Lemma 4.26 to \(L\) with the reverse order, we obtain that \([\bot, b] = \bigcap_{i \in I} [\bot, b_i]\) if and only if \(b = \inf_{i \in I} b_i\). These two facts together yield the assertion of our lemma. \(\square\)

### 4.7. Lattices.

A **lattice** is a poset in which every two elements have a supremum (least upper bound) and an infimum (greatest lower bound). It then follows that all finite sets in a lattice \((L, <)\) have a supremum and an infimum. A **complete lattice** is a poset in which all nonempty sets have a supremum and an infimum. Lemma 4.32 implies:

**Proposition 4.33.** The ball space \((L, \mathcal{B}_{iv})\) associated to a lattice \((L, <)\) is finitely intersection closed. The ball space \((L, \mathcal{B}_{iv})\) associated to a complete lattice \((L, <)\) is intersection closed.

For a lattice \((L, <)\), we denote by \((L, >)\) the lattice endowed with the reverse order. We will now characterize complete lattices by properties from our hierarchy.

**Theorem 4.34.** For a poset \((L, <)\), the following assertions are equivalent.

a) \((L, <)\) is a complete lattice,

b) \((L, <)\) and \((L, >)\) are complete upper semilattices,

c) the principal final segments of \((L, <)\) and of \((L, >)\) form \(S^*\) ball spaces,
d) \((L, \mathcal{B}_v)\) is an \(S^*\) ball space and \((L, <)\) admits a top and a bottom element.

e) \((L, \mathcal{B}_v)\) is an \(S^*\) ball space and every finite set in \((L, <)\) has an upper and a lower bound.

Proof: The equivalence of a) and b) follows directly from the definitions. The equivalence of b) and c) follows from part 4) of Proposition 4.27.

a) \(\Rightarrow\) d): Assume that \((L, <)\) is a complete lattice. Then it admits a top element (supremum of all its elements) and a bottom element (infimum of all its elements). Take a centered system \(\{[a_i, b_i] \mid i \in I\}\) in \((L, \mathcal{B}_v)\). Then for all \(i, j \in I\), \([a_i, b_i] \cap [a_j, b_j] \neq \emptyset\), so \(a_i \leq b_j\). Hence every \(b_j\) is an upper bound of \(\{a_i \mid i \in I\}\), and every \(a_i\) is a lower bound of \(\{b_j \mid j \in I\}\). Set \(a := \sup_{i \in I} a_i\) and \(b := \inf_{i \in I} b_i\). It follows that also \(b\) is an upper bound of \(\{a_i \mid i \in I\}\), and \(a\) is a lower bound of \(\{b_j \mid j \in I\}\). Therefore, \(a, b \in [a_i, b_i]\) for all \(i\) and thus, \(\bigcap_{i \in I} [a_i, b_i] \neq \emptyset\). From Lemma 4.32 it follows that \(\bigcap_{i \in I} [a_i, b_i] = [a, b]\) and hence is a ball. We have proved that \((L, \mathcal{B}_v)\) is an \(S^*\) ball space.

d) \(\Rightarrow\) e): A top element is an upper bound and a bottom element a lower bound for every set of elements.

e) \(\Rightarrow\) a): Take a poset \((L, <)\) that satisfies the assumptions of e), and any subset \(S \subseteq L\). If \(S_0\) is a finite subset of \(S\), then it has an upper bound \(b\) by assumption. Hence the balls \([a, \top], a \in S_0\), have a nonempty intersection, as it contains \(b\). This shows that \(\{[a, \top] \mid a \in S\}\) is a centered system of balls. Since \((L, \mathcal{B}_v)\) is an \(S^*\) ball space, its intersection is a ball \([c, d]\), where we must have \(d = \top\). By Lemma 4.26 \(c\) is the supremum of \(S\).

Working with the reverse order, one similarly shows that \(S\) has an infimum since \((L, \mathcal{B}_v)\) is an \(S^*\) ball space. Hence, \((L, <)\) is a complete lattice.

For our next theorem, we will need one further lemma:

**Lemma 4.35.** For a lattice \((L, <)\), the following are equivalent:

a) \((L, <)\) is a complete lattice,

b) \((L, <)\) and \((L, >)\) are directed complete posets,

c) \((L, <)\) and \((L, >)\) are chain complete posets.

Proof: The implication a) \(\Rightarrow\) b) is trivial as every nonempty set in a complete lattice has a supremum and an infimum.

b) \(\Rightarrow\) a): Take a nonempty subset \(S\) of \(L\). Let \(S'\) be the closure of \(S\) under suprema and infima of arbitrary finite subsets of \(S\). Then \(S'\)
is a directed system in both \((L, <)\) and \((L, >)\). Hence by b), \(S'\) has an infimum \(a\) and a supremum \(b\). These are lower and upper bounds, respectively, for \(S\). Suppose there was an upper bound \(c < b\) for \(S\). Then there would be a supremum \(d\) of some finite subset of \(S\) such that \(d > c\). But as \(c\) is also an upper bound of this finite subset, we must have that \(d \leq c\). This contradiction shows that \(b\) is also the supremum of \(S\). Similarly, one shows that \(a\) is also the infimum of \(S\). This proves that \((L, <)\) is a complete lattice.

b) \(\iff\) c) follows from Proposition 3.2.

Now we can prove:

**Theorem 4.36.** For a lattice \((L, <)\), the following are equivalent:

a) \((L, <)\) is a complete lattice,

b) \((L, B_{iv})\) is an \(S_5\) ball space,

c) \((L, B_{iv})\) is an \(S^*\) ball space.

Proof: a) \(\Rightarrow\) c): This follows from Theorem 4.34.

c) \(\Rightarrow\) b) holds by the general properties of the hierarchy.

b) \(\Rightarrow\) a): By Lemma 4.35 it suffices to prove that \((L, <)\) and \((L, >)\) are chain complete posets. Take a chain \(\{a_i \mid i \in I\}\) in \((L, <)\). Then \(\{[a_i, \top] \mid i \in I\}\) is a nest of balls in \((L, B_{iv})\). Since \((L, B_{iv})\) is an \(S_5\) ball space, the intersection of this nest is a ball \([a, b]\) for some \(a, b \in L\); it must be of the form \([a, \top]\) since the intersection contains \(\top\). From Lemma 4.26 we infer that \(a = \sup_{i \in I} a_i\). This shows that \((L, <)\) is a chain complete poset. The proof for \((L, >)\) is similar.

An example for a fixed point theorem that holds in complete lattices is the Knaster-Tarski Theorem, which we will discuss in Section 7.1.

5. \(S^*\) Ball Spaces

Take a ball space \((X, \mathcal{B})\) and a subset \(Y \subseteq X\). If there is at least one ball \(B \in \mathcal{B}\) such that \(Y \cap B \neq \emptyset\), then with

\[
\mathcal{B} \cap Y := \{B \cap Y \mid B \in \mathcal{B}\} \setminus \{\emptyset\},
\]

\((Y, \mathcal{B} \cap Y)\) is a ball space. We will now study how \((Y, \mathcal{B} \cap Y)\) can inherit properties of spherical completeness from \((X, \mathcal{B})\). For this it is important to know whether nests of balls in \((Y, \mathcal{B} \cap Y)\) can be lifted to nests of balls in \((X, \mathcal{B})\). As we will show, this can be done in \(S^*\) ball spaces. We will then apply our results in Section 7.1 to prove a generic Knaster–Tarski theorem which generalizes the original Knaster–Tarski Theorem.
5.1. **Spherical closures in S* ball spaces.**

Throughout this and the next section, we consider an S* ball space \((X, \mathcal{B})\). As before, if \(f : X \to X\) is a function, then \(\mathcal{B}^f\) will denote the collection of all \(f\)-closed balls in \(\mathcal{B}\). The following is a simple but useful observation. It follows from the fact that the intersection over any collection of \(f\)-closed sets is again \(f\)-closed.

**Lemma 5.1.** Also \((X, \mathcal{B}^f)\) is an S* ball space, provided that \(\mathcal{B}^f \neq \emptyset\).

For every nonempty subset \(S\) of a ball in \(\mathcal{B}\), we define

\[
\text{scl}_\mathcal{B}(S) := \bigcap \{B \in \mathcal{B} : S \subseteq B\}
\]

and call it the *(spherical) closure* of \(S\) in \(\mathcal{B}\).

**Lemma 5.2.**
1) For every nonempty subset \(S\) of a ball in \(\mathcal{B}\), \(\text{scl}_\mathcal{B}(S)\) is the smallest ball in \(\mathcal{B}\) containing \(S\).
2) If \(f : X \to X\) is a function, then for every nonempty subset \(S\) of an \(f\)-closed ball in \(\mathcal{B}\), \(\text{scl}_{\mathcal{B}^f}(S)\) is the smallest \(f\)-closed ball containing \(S\).

**Proof:**
1) The collection of all balls containing the nonempty set \(S\) is a centered system. It is nonempty by our condition that \(S\) is a subset of a ball in \(\mathcal{B}\). The intersection of this system contains \(S\), and since \((X, \mathcal{B})\) is S*, it is a ball. As all balls containing \(S\) appear in the system, the intersection must be the smallest ball containing \(S\).

2) This follows from part 1) together with Lemma 5.1. \(\square\)

Note that if \(X \in \mathcal{B}\), then we can drop the condition that \(S\) is the subset of a ball (or an \(f\)-closed ball, respectively) in \(\mathcal{B}\).

**Remark 5.3.** The ball \(B_x\) defined in (9) in the proof of Theorem 4.21 is equal to \(\text{scl}_{\mathcal{B}^f}(\{x\})\), where \(\mathcal{B}^f\) is the set of all closed \(f\)-closed sets of the topological space under consideration.

The proof of the following observation is straightforward:

**Lemma 5.4.** If \(S \subseteq T\) are nonempty subsets of a ball in \(\mathcal{B}\), then \(\text{scl}_\mathcal{B}(S) \subseteq \text{scl}_\mathcal{B}(T)\).

5.2. **Ball spaces induced on subsets of S* ball spaces.**

Now we take \(Y \subseteq X\) and consider the ball space \((Y, \mathcal{B} \cap Y)\).
Lemma 5.5. 1) For each $B \in \mathcal{B} \cap Y$, 
$$\text{scl}_\mathcal{B}(B) \cap Y = B.$$ 

2) The assignment 
$$\mathcal{B} \cap Y \ni B \mapsto \text{scl}_\mathcal{B}(B)$$ 

preserves inclusion in the strong sense that 
$$B_1 \subset B_2 \iff \text{scl}_\mathcal{B}(B_1) \subset \text{scl}_\mathcal{B}(B_2).$$ 

3) If $(B_i)_{i \in I}$ is a centered system of balls in $(Y, \mathcal{B} \cap Y)$, then $(\text{scl}_\mathcal{B}(B_i))_{i \in I}$ is a centered system of balls in $(X, \mathcal{B})$ with 

$$\bigcap_{i \in I} B_i = \left( \bigcap_{i \in I} \text{scl}_\mathcal{B}(B_i) \right) \cap Y. \quad (10)$$ 

Proof: 1): It follows from the definition of $\text{scl}_\mathcal{B}(B)$ that $B \subseteq \text{scl}_\mathcal{B}(B)$, so $B \subseteq \text{scl}_\mathcal{B}(B) \cap Y$. Since $B \in \mathcal{B} \cap Y$, we can write $B = B' \cap Y$ for some $B' \in \mathcal{B}$. Since $\text{scl}_\mathcal{B}(B)$ is the smallest ball in $X$ containing $B$, it must be contained in $B'$ and therefore, $\text{scl}_\mathcal{B}(B) \cap Y \subseteq B' \cap Y = B$. 

2): In view of Lemma 5.4, it suffices to show that $B_1 \neq B_2$ implies $\text{scl}_\mathcal{B}(B_1) \neq \text{scl}_\mathcal{B}(B_2)$. But this is a consequence of part 1) of this lemma. 

3): Take a centered system of balls $(B_i)_{i \in I}$ in $(Y, \mathcal{B} \cap Y)$. Then $(\text{scl}_\mathcal{B}(B_i))_{i \in I}$ is a centered system of balls in $(X, \mathcal{B})$ since $B_i \cap \ldots \cap B_{i_n} \neq \emptyset$ implies that $\text{scl}_\mathcal{B}(B_i) \cap \ldots \cap \text{scl}_\mathcal{B}(B_{i_n}) \neq \emptyset$. By part 1), $B_i = \text{scl}_\mathcal{B}(B_i) \cap Y$, whence 

$$\bigcap_{i \in I} B_i = \bigcap_{i \in I} (\text{scl}_\mathcal{B}(B_i) \cap Y) = \left( \bigcap_{i \in I} \text{scl}_\mathcal{B}(B_i) \right) \cap Y. \quad \square$$ 

With the help of this lemma, we obtain: 

**Proposition 5.6.** Take an $S^*$ ball space $(X, \mathcal{B})$ and $Y \subset X$. Assume that $B \cap Y \neq \emptyset$ for every $B \in \mathcal{B}$. Then also $(Y, \mathcal{B} \cap Y)$ is an $S^*$ ball space. 

Proof: Take a centered system of balls $(B_i)_{i \in \mathbb{N}}$ in $(Y, \mathcal{B} \cap Y)$. Then by part 3) of Lemma 5.5, $(\text{scl}_\mathcal{B}(B_i))_{i \in \mathbb{N}}$ is a centered system of balls in $(X, \mathcal{B})$ with $\bigcap_{i \in I} B_i = (\bigcap_{i \in I} \text{scl}_\mathcal{B}(B_i)) \cap Y$. Since $(X, \mathcal{B})$ is assumed to be $S^*$, $\bigcap_{i \in I} \text{scl}_\mathcal{B}(B_i)$ is a ball in $\mathcal{B}$. Therefore, $\bigcap_{i \in I} B_i = (\bigcap_{i \in I} \text{scl}_\mathcal{B}(B_i)) \cap Y \neq \emptyset$ is a ball in $\mathcal{B} \cap Y. \quad \square$ 

In the special case considered in Section 7.1, the set $Y$ is taken to be the set $\text{Fix}(f)$ of fixed points of a given function $f : X \to X$. If $(X, \mathcal{B})$ is an $S^*$ ball space with $\mathcal{B}^f \neq \emptyset$ and every $f$-closed ball contains a fixed
point, then it follows from Lemma 5.1 together with Proposition 5.6 that also

\[(\text{Fix}(f), \mathcal{B}^f \cap \text{Fix}(f))\]
is an \(S^*\) ball space. However, we are more interested in the possibly finer ball space

\[(\text{Fix}(f), \mathcal{B} \cap \text{Fix}(f)).\]

The following proposition gives a criterion for the two ball spaces to be equal:

**Proposition 5.7.** Take an \(S^*\) ball space \((X, \mathcal{B})\) and a function \(f : X \rightarrow X\). If \(B_0 \in \mathcal{B} \cap \text{Fix}(f)\) is such that \(\text{scl}_{\mathcal{B}}(B_0)\) is \(f\)-closed, then

\[(11) \quad \text{scl}_{\mathcal{B}}(B_0) = \text{scl}_{\mathcal{B}^f}(B_0).\]

If this holds for every \(B_0 \in \mathcal{B} \cap \text{Fix}(f)\), then

\[(12) \quad \mathcal{B}^f \cap \text{Fix}(f) = \mathcal{B} \cap \text{Fix}(f).\]

**Proof:** Pick \(B_0 \in \mathcal{B} \cap \text{Fix}(f)\). By part 1) of Lemma 5.2, \(\text{scl}_{\mathcal{B}}(B_0)\) is the smallest of all balls in \(\mathcal{B}\) that contain \(B_0\). Consequently, if \(\text{scl}_{\mathcal{B}}(B_0)\) is \(f\)-closed, then it is also the smallest of all balls in \(\mathcal{B}^f\) that contain \(B_0\). Then by part 2) of Lemma 5.2 it must be equal to \(\text{scl}_{\mathcal{B}^f}(B_0)\).

Since \(B_0 = \text{scl}_{\mathcal{B}}(B_0) \cap \text{Fix}(f)\) by part 1) of Lemma 5.3 the equality implies that \(B_0 = \text{scl}_{\mathcal{B}^f}(B_0) \cap \text{Fix}(f) \in \mathcal{B}^f \cap \text{Fix}(f)\). If the equality \((11)\) holds for all \(B_0 \in \mathcal{B} \cap \text{Fix}(f)\), then this implies \((12)\). □

**Corollary 5.8.** Take an \(S^*\) ball space \((X, \mathcal{B})\) and a function \(f : X \rightarrow X\). Assume that \(f^{-1}(B) \in \mathcal{B}\) for every \(B \in \mathcal{B}\) that contains a fixed point. Then \((12)\) holds.

**Proof:** Pick \(B_0 \in \mathcal{B} \cap \text{Fix}(f)\). Since \(B := \text{scl}_{\mathcal{B}}(B_0) \in \mathcal{B}\), we have that \(f^{-1}(B) \in \mathcal{B}\). Since \(B_0\) consists of fixed points, it is contained in \(f^{-1}(B)\). As \(B\) is the smallest ball containing \(B_0\), it follows that \(B \subseteq f^{-1}(B)\) and thus \(f(B) \subseteq f(f^{-1}(B)) \subseteq B\). Hence \((11)\) holds, which by Proposition 5.7 implies that \((12)\) holds. □

### 6. Set theoretic operations on ball spaces

#### 6.1. Subsets of ball spaces.

**Proposition 6.1.** Take two ball spaces \((X, \mathcal{B}_1)\) and \((X, \mathcal{B}_2)\) on the same set \(X\) such that \(\mathcal{B}_1 \subseteq \mathcal{B}_2\). If \((X, \mathcal{B}_2)\) is \(S_1\) (or \(S^1_1\) or \(S^1_1\)), then also \((X, \mathcal{B}_1)\) is \(S_1\) (or \(S^1_1\) or \(S^1_1\), respectively). This does in general not hold for any other property in the hierarchy.
Proof: The first assertion holds since every nest (or directed system, or centered system) in $B_1$ is also a nest (or directed system, or centered system) in $B_2$. To prove the second assertion one constructs an $S^*$ ball space $(X, B_2)$ and a nest (or directed system, or centered system) $N$ such that the intersection $\bigcap N \in B_2$ does not lie in $N$. Then to obtain $B_1$ one removes all balls from $B_2$ that lie in $\bigcap N$. $\square$

6.2. Unions of two ball spaces on the same set.

The easy proof of the following proposition is left to the reader:

**Proposition 6.2.** If $(X, B_1)$ and $(X, B_2)$ are $S_1$ ball spaces on the same set $X$, then so is $(X, B_1 \cup B_2)$. The same holds with $S_2$ or $S_5$ in place of $S_1$, and for all properties in the hierarchy if $B_2$ is finite.

Note that the assertion may become false if we replace $S_1$ by $S_4$. Indeed, the intersection of a nest in $B_1$ may properly contain a largest ball which does not remain the largest ball contained in the intersection in $B_1 \cup B_2$.

It is also clear that in general infinite unions of $S_1$ ball spaces on the same set $X$ will not again be $S_1$. For instance, ball spaces with just one ball are always $S_1$; by a suitable infinite union of such spaces one can build nests with empty intersection.

For any ball space $(X, B)$, we define: the ball space $(X, \hat{B})$ by setting:

$$\hat{B} := B \cup \{X\}.$$  

Taking $B_1 = B$ and $B_2 = \{X\}$ in Proposition 6.2, we obtain:

**Corollary 6.3.** A ball space $(X, B)$ is $S_1$ if and only if $(X, \hat{B})$ is $S_1$. The same holds for all properties in the hierarchy in place of $S_1$.

6.3. Closure under unions of balls.

Take a ball space $(X, B)$. By $f$-$\text{un}(B)$ we denote the set of all unions of finitely many balls in $B$.

The following lemma is inspired by Alexander’s Subbase Theorem:

**Lemma 6.4.** If $S$ is a maximal centered system of balls in $f$-$\text{un}(B)$ (that is, no subset of $f$-$\text{un}(B)$ properly containing $S$ is a centered system), then there is a subset $S_0$ of $S$ which is a centered system in $B$ and has the same intersection as $S$.

Proof: It suffices to prove the following: if $B_1, \ldots, B_n \in B$ such that $B_1 \cup \ldots \cup B_n \in S$, then there is some $i \in \{1, \ldots, n\}$ such that $B_i \in S$. 

Suppose that $B_1, \ldots, B_n \in \mathcal{B}\setminus \mathcal{S}$. By the maximality of $\mathcal{S}$ this implies that for each $i \in \{1, \ldots, n\}$, $\mathcal{S} \cup \{B_i\}$ is not centered. This in turn means that there is a finite subset $\mathcal{S}_i$ of $\mathcal{S}$ such that $\bigcap \mathcal{S}_i \cap B_i = \emptyset$. But then $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ is a finite subset of $\mathcal{S}$ such that
\[
\bigcap (\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n) \cap (B_1 \cup \ldots \cup B_n) = \emptyset.
\]
This yields that $B_1 \cup \ldots \cup B_n \notin \mathcal{S}$, which proves our assertion. □

The centered systems of balls in a ball space form a poset under inclusion. Since the union of every chain of centered systems is again a centered system, this poset is chain complete. Hence by Corollary 2.2 every centered system is contained in a maximal centered system.

**Theorem 6.5.** If $(X, \mathcal{B})$ is an $S_c^1$ ball space, then so is $(X, \text{f-un}(\mathcal{B}))$.

**Proof:** Take a centered system $\mathcal{S}'$ of balls in $\text{f-un}(\mathcal{B})$. As shown before this theorem, there is a maximal centered system $\mathcal{S}$ of balls in $\text{f-un}(\mathcal{B})$ which contains $\mathcal{S}'$. By Lemma 6.4 there is a centered system $\mathcal{S}_0$ of balls in $\mathcal{B}$ such that $\bigcap \mathcal{S}_0 = \bigcap \mathcal{S} \subseteq \bigcap \mathcal{S}'$. Since $(X, \mathcal{B})$ is an $S_c^1$ ball space, we have that $\bigcap \mathcal{S}_0 \neq \emptyset$, which yields that $\bigcap \mathcal{S}' \neq \emptyset$. This proves that $(X, \text{f-un}(\mathcal{B}))$ is an $S_c^1$ ball space. □

In [1] the following result is proven:

**Theorem 6.6.** Take a symmetrically complete ordered field $K$ and $\mathcal{B}$ to be the set of all convex sets in $K$ that are finite unions of closed bounded intervals and ultrametric balls. Then $(K, \mathcal{B})$ is spherically complete.

6.4. **Closure under nonempty intersections of balls.**

Take a ball space $(X, \mathcal{B})$. We define:

(a) $\text{ic}(\mathcal{B})$ to be the set of all nonempty intersections of arbitrarily many balls in $\mathcal{B}$,
(b) $\text{fic}(\mathcal{B})$ to be the set of all nonempty intersections of finitely many balls in $\mathcal{B}$,
(c) $\text{ci}(\mathcal{B})$ to be the set of all nonempty intersections of nests in $\mathcal{B}$,
(d) $\text{di}(\mathcal{B})$ to be the set of all nonempty intersections of arbitrary directed systems of balls in $\mathcal{B}$.

Note that $(X, \mathcal{B})$ is intersection closed if and only if $\text{ic}(\mathcal{B}) = \mathcal{B}$, finitely intersection closed if and only if $\text{fic}(\mathcal{B}) = \mathcal{B}$, and chain intersection closed if and only if $\text{ci}(\mathcal{B}) = \mathcal{B}$. If $(X, \mathcal{B})$ is $S_5$, then $\text{ci}(\mathcal{B}) = \mathcal{B}$. If $(X, \mathcal{B})$ is $S_{\text{d}}^5$, then $\text{di}(\mathcal{B}) = \mathcal{B}$. If $(X, \mathcal{B})$ is $S^*$, then $\text{ic}(\mathcal{B}) = \mathcal{B}$ (because an arbitrary set of balls that has a nonempty intersection is automatically a centered system).
We note the following observation:

**Proposition 6.7.** Take an arbitrary ball space \((X, B)\). Then the ball space \((X, \text{ic}(B))\) is intersection closed, and the ball space \((X, \text{fic}(B))\) is finitely intersection closed.

**Proof:** Take balls \(B_i \in \text{ic}(B), i \in I\), and for every \(i \in I\), balls \(B_{i,j} \in B, j \in J_i\), such that \(B_i = \bigcap_{j \in J_i} B_{i,j}\). Then

\[
\bigcap_{i \in I} B_i = \bigcap_{i \in I, j \in J_i} B_{i,j} \in \text{ic}(B).
\]

If \(I\) is finite and \(B_i \in \text{fic}(B)\) for every \(i \in I\), then every \(J_i\) can be taken to be finite and thus the right hand side is a ball in \(\text{fic}(B)\).

In view of these facts, we call \((X, \text{ic}(B))\) the intersection closure of \((X, B)\), and \((X, \text{fic}(B))\) the finite intersection closure of \((X, B)\). If a chain intersection closed ball space \((X, B')\) is obtained from \((X, B)\) by a (possibly transfinite) iteration of the process of replacing \(B\) by \(\text{ci}(B)\), then we call \((X, B')\) a chain intersection closure of \((X, B)\).

Chain intersection closures are studied in [10] and conditions are given for \((X, \text{ci}(B))\) to be the chain intersection closure of \((X, B)\). As stated already in Section 4.1 (cf. Theorem 4.4), this holds for classical ultrametric spaces. By [10, Theorem 1.2], it also holds for ultrametric spaces with countable narrow value sets. Here is the essence of the cited Theorem 1.1:

**Theorem 6.8.** If \((X, B)\) is a ball space of ultrametric type, then \((X, \text{ci}(B))\) is its intersection closure.

**Proof:** By Proposition 3.6 every centered system of balls in \(B\) is a nest. Therefore, \(\text{ic}(B) = \text{ci}(B)\).

**Theorem 6.9.** If \((X, B)\) is an \(S^c_1\) ball space, then its intersection closure \((X, \text{ic}(B))\) is an \(S^*_1\) ball space.

**Proof:** Take a centered system \(\{B_i \mid i \in I\}\) in \((X, \text{ic}(B))\). Write \(B_i = \bigcap_{j \in J_i} B_{i,j}\) with \(B_{i,j} \in B\). Then \(\{B_{i,j} \mid i \in I, j \in J_i\}\) is a centered system in \((X, B)\): the intersection of finitely many balls \(B_{i_1,j_1}, \ldots, B_{i_n,j_n}\) contains the intersection \(B_{i_1,j_1} \cap \ldots \cap B_{i_n,j_n}\), which by assumption is non-empty. Since \((X, B)\) is \(S^c_1\), \(\bigcap_i B_i = \bigcap_{i,j} B_{i,j} \neq \emptyset\). This proves that \((X, \text{ic}(B))\) is an \(S^*_1\) ball space. Since \((X, \text{ic}(B))\) is intersection closed, Theorem 3.10 now shows that \((X, \text{ic}(B))\) is an \(S^*_1\) ball space.
6.5. Closure under finite unions and under intersections.

From Theorems 6.5 and 6.9 we obtain:

**Theorem 6.10.** Take any ball space \((X, \mathcal{B})\). If \(\mathcal{B}'\) is obtained from \(\mathcal{B}\) by first closing under finite unions and then under arbitrary nonempty intersections, then:

1) \(\mathcal{B}'\) is closed under finite unions,
2) \(\mathcal{B}'\) is intersection closed,
3) if in addition \((X, \mathcal{B})\) is a \(\mathbf{S}_1^i\) ball space, then \((X, \mathcal{B}')\) is an \(\mathbf{S}^*\) ball space.

**Proof:**

1): Take \(S_1, \ldots, S_n \subseteq \text{f-un}(\mathcal{B})\) such that \(\bigcap S_i \neq \emptyset\) for \(1 \leq i \leq n\). Then
\[
\bigcap S_1 \cup \cdots \cup \bigcap S_n = \bigcap \{B_1 \cup \cdots \cup B_n \mid B_i \in S_i \text{ for } 1 \leq i \leq n\}.
\]
Since \(B_i \in \text{f-un}(\mathcal{B})\) for \(1 \leq i \leq n\), we have that also \(B_1 \cup \cdots \cup B_n \in \text{f-un}(\mathcal{B})\). This implies that \(\bigcap S_1 \cup \cdots \cup \bigcap S_n \in \mathcal{B}'\).

2): Since \(\mathcal{B}'\) is an intersection closure, it is intersection closed.

3): By Theorems 6.5 and 6.9 \((X, \mathcal{B}')\) is an \(\mathbf{S}^*\) ball space. \(\square\)

6.6. The topology associated with an \(\mathbf{S}_1^i\) ball space.

Take an \(\mathbf{S}_1^i\) ball space \((X, \mathcal{B})\). Theorem 6.10 tells us that in a canonical way we can associate with it an \(\mathbf{S}^*\) ball space \((X, \mathcal{B}')\) which is closed under nonempty intersections and under finite unions. If we also add \(X\) and \(\emptyset\) to \(\mathcal{B}'\); then we obtain the collection of closed sets for a topology. Every topology on a nonempty set \(X\) can be obtained in this way by starting from the ball space associated with the topology.

**Theorem 6.11.** This associated topology is compact if and only if \((X, \mathcal{B})\) is an \(\mathbf{S}_1^i\) ball space.

**Proof:** One direction of the equivalence follows from Theorems 6.10 and 4.20. The other direction follows from Proposition 6.1. \(\square\)

**Example:** the \(p\)-adics.

The field \(\mathbb{Q}_p\) of \(p\)-adic numbers together with the \(p\)-adic valuation \(v_p\) is spherically complete. (This fact can be used to prove the original Hensel’s Lemma via the ultrametric fixed point theorem, see [22], or even better, via the ultrametric attractor theorem, see [11].) The associated ball space is a classical ultrametric ball space and hence of ultrametric type. It follows from Proposition 3.6 that it is an \(\mathbf{S}_2^i\) ball space. Hence by Theorem 6.11 the topology derived from this ball space is compact.
However, $\mathbb{Q}_p$ is known to be locally compact, but not compact under the topology induced by the $p$-adic metric. But this in this topology the ultrametric balls $B_\alpha(x)$ are basic open sets, whereas in the topology derived from the ultrametric ball space they are closed and their complements are the basic open sets. It follows that the balls $B_\alpha(x)$ are not open. It thus turns out that the usual $p$-adic topology on $\mathbb{Q}_p$ is strictly finer than the one we derived from the ultrametric ball space.

7. Shifting concepts between applications

7.1. Knaster–Tarski type theorems.
In 1927 B. Knaster and A. Tarski proved a set-theoretical fixed point theorem by which every function on the family of all subsets of a given set, which is increasing under inclusion, has at least one fixed point. In 1955 Tarski generalized the result to the lattice-theoretical fixed point theorem which is now known as the Knaster–Tarski Theorem (cf. [30, Theorem 1]). It states:

**Theorem 7.1.** Let $L$ be a complete lattice and $f : L \rightarrow L$ an order-preserving function. Then the set $\text{Fix}(f)$ of fixed points of $f$ in $L$ is also a complete lattice.

We prove an analogue for ball spaces $(X, B)$ with a function $f : X \rightarrow X$. As before, $B^f$ will denote the collection of all $f$-closed balls in $B$.

**Theorem 7.2.** Take an $S^*$ ball space $(X, B)$ and a function $f : X \rightarrow X$. Assume that $B$ contains an $f$-closed ball and every $f$-closed ball in $B$ contains a fixed point or a smaller ball. Then every $f$-closed ball in $B$ contains a fixed point, and $(\text{Fix}(f), B^f \cap \text{Fix}(f))$ is an $S^*$ ball space.

Proof: By Lemma 5.1 $(X, B^f)$ is an $S^*$ ball space. Hence it follows from our assumptions together with Theorem 1.3 that every $f$-closed ball $B$ in $B$ contains a fixed point, that is, $B \cap \text{Fix}(f) \neq \emptyset$. By Proposition 5.3 it follows that $(\text{Fix}(f), B^f \cap \text{Fix}(f))$ is an $S^*$ ball space. 

In what follows, we will discuss some applications.

The case of lattices.
We show how to derive Theorem 7.1 from Theorem 7.2. We take a complete lattice $(L, <)$. By Theorem 4.36 the associated ball space $(L, B_{iv})$ is $S^*$. Take an order-preserving function $f : L \rightarrow L$ and consider the set $B_{iv}^f$ of all $f$-closed balls in $B_{iv}$, that is, all $f$-closed intervals
Take an \( f \)-closed interval \([a, b]\). Since \( f \) is order preserving, it follows that \( a \leq f(a) \leq f(b) \leq b \). If \( f(a) = a \) or \( f(b) = b \), then the interval contains a fixed point. If \( f(a) \neq a \) or \( f(b) \neq b \), then \([f(a), f(b)]\) is an \( f \)-closed interval that is properly contained in \([a, b]\). We have shown that the assumptions of Theorem 7.2 hold, so we obtain that \((\text{Fix}(f), \mathcal{B}^f_{iv} \cap \text{Fix}(f))\) is an \( S^* \) ball space.

Next, we show that \( \mathcal{B}^f_{iv} \cap \text{Fix}(f) \) is exactly the set of all intervals \([a, b]_{\text{Fix}(f)}\) in the poset \( \text{Fix}(f) \). Indeed, if \( a, b \) are fixed points, then \([a, b]\) is an \( f \)-closed interval in \( L \) with \([a, b]_{\text{Fix}(f)} = [a, b] \cap \text{Fix}(f) \in \mathcal{B}^f_{iv} \cap \text{Fix}(f) \). Conversely, if \( S = B \cap \text{Fix}(f) \) for some \( B \in \mathcal{B}^f_{iv} \), then the spherical closure \( \text{scl}_{\mathcal{B}^f_{iv}}(S) \) of \( S \) in the ball space \((L, \mathcal{B}^f_{iv})\) is an \( f \)-closed interval \([a, b]\) in \( L \), and it is contained in \( B \). If \( a \) or \( b \) is not a fixed point, then \([f(a), f(b)]\) is an \( f \)-closed interval properly contained in \([a, b]\). But as it also contains the set \( S \) of fixed points, this is a contradiction to the definition of the spherical closure. Hence, \( a, b \) are fixed points. We have that \( S \subseteq [a, b] \cap \text{Fix}(f) = \text{scl}_{\mathcal{B}^f_{iv}}(S) \cap \text{Fix}(f) \subseteq B \cap \text{Fix}(f) = S \), so \( S = [a, b] \cap \text{Fix}(f) = [a, b]_{\text{Fix}(f)} \).

We have now shown that \((\text{Fix}(f), \{[a, b]_{\text{Fix}(f)} \mid a, b \in \text{Fix}(f)\})\) is an \( S^* \) ball space. Let us show that all finite sets \( S \) in \( \text{Fix}(f) \) have an upper and a lower bound. Since \( \top \in L \), the \( f \)-closed intervals \([a, \top]\), \( a \in S \), have a nonempty intersection. Since \((L, \mathcal{B}^f_{iv})\) is an \( S^* \) ball space, their intersection is again an \( f \)-closed ball. By Theorem 7.2 it contains a fixed point, which consequently is an upper bound for \( S \) in \( \text{Fix}(f) \). Similarly, one shows the existence of a lower bound. It now follows from Theorem 4.34 that \( \text{Fix}(f) \) is a complete lattice.

The ultrametric case.

Take a classical ultrametric space \((X, u)\) and a function \( f : X \to X \). Then \( f \) is called nonexpanding if \( u(fx, fy) \leq u(x, y) \) for all \( x, y \in X \). Further, \( f \) is called contracting on orbits if \( u(fx, f fx) < u(x, fx) \) for all \( x \in X \) such that \( x \neq fx \).

Now assume that \((X, u)\) is spherically complete. Then by Theorem 4.4, the full ultrametric ball space \((X, \mathcal{B}_{u+})\) is \( S^* \). Further, take a function \( f : X \to X \). We need the following auxiliary result:

**Lemma 7.3.** If \( f : X \to X \) is nonexpanding, then every ball \( B(x, fx) \) is \( f \)-closed, and the same holds for every \( B(x, y) \) where \( x \) is a fixed point of \( f \).
Proof: Take \( z \in B(x, fx) \). Then \( u(x, z) \leq u(x, fx) \) and since \( f \) is nonexpanding, \( u(fx, fz) \leq u(x, z) \leq u(x, fx) \). By the ultrametric triangle law, this yields that \( u(x, fz) \leq u(x, fx) \), whence \( fz \in B(x, fx) \).

Now assume that \( x \) is a fixed point of \( f \), and take \( z \in B(x, y) \). Then \( u(x, z) \leq u(x, y) \) and since \( f \) is nonexpanding, \( u(x, fz) = u(fx, fz) \leq u(x, z) \leq u(x, y) \), whence \( fz \in B(x, y) \). \( \square \)

Now assume that \( f \) is both nonexpanding and contracting on orbits. As before, we let \( B_{u+}^f \) denote the set of all \( f \)-closed balls in \( B_{u+} \). Lemma 7.3 shows that \( B_{u+}^f \) is nonempty, as it contains \( B(x, fx) \) for each \( x \in X \). Take \( B \in B_{u+}^f \) and \( x \in B \). Then also \( fx \in B \), hence \( B(x, fx) \subseteq B \). If \( x = fx \), then \( B \) contains a fixed point. Suppose that \( x \neq fx \). Since \( f \) is contracting on orbits, we then have that \( B(fx, ffx) \subseteq B(x, fx) \subseteq B \), hence Lemma 7.3 shows that \( B(fx, ffx) \) is an \( f \)-closed ball properly contained in \( B \). Therefore, from Lemma 5.1 and Theorem 7.2 we obtain that every \( f \)-closed ball in \( B_{u+}^f \) contains a fixed point and

\[
(\text{Fix}(f), B_{u+}^f \cap \text{Fix}(f))
\]

is an \( S' \) ball space.

We observe:

**Lemma 7.4.** The ball space \((\text{Fix}(f), B_{u+}^f \cap \text{Fix}(f))\) is equal to the full ball space of \((\text{Fix}(f), u)\).

**Proof:** For \( x, y \in \text{Fix}(f) \), denote by \( B_F(x, y) \) the smallest ball in \((\text{Fix}(f), u)\) that contains \( x \) and \( y \).

Take any ball \( B \in B_{u+}^f \) with \( B \cap \text{Fix}(f) \neq \emptyset \), and pick any element \( x \in B \cap \text{Fix}(f) \). Then

\[
B \cap \text{Fix}(f) = \bigcup \{ B_F(x, y) \mid y \in B \cap \text{Fix}(f) \}.
\]

This shows in particular that all balls in \( B_{u+}^f \cap \text{Fix}(f) \) are balls in the full ultrametric ball space of \((\text{Fix}(f), u)\).

For the converse, consider any ball \( B_F \) in the full ultrametric ball space of \((\text{Fix}(f), u)\) and pick some \( x \in B_F \). Then \( B_F \) can be written as

\[
B_F = \bigcup \{ B_F(x, y) \mid y \in B_F \} = \bigcup \{ B(x, y) \cap \text{Fix}(f) \mid y \in B_F \} = \text{Fix}(f) \cap \bigcup \{ B(x, y) \mid y \in B_F \}.
\]

The second assertion of Lemma 7.3 shows that each ultrametric ball \( B(x, y) \) is \( f \)-closed. Therefore, \( \bigcup \{ B(x, y) \mid y \in B_F \} \) is an \( f \)-closed ball in the full ultrametric ball space of \((X, u)\). Hence \( B_F \in B_{u+}^f \cap \text{Fix}(f) \). \( \square \)
In fact, we could also have used Proposition 5.7. Indeed, it can be seen from the second part of the above proof that the full ball space of \((\text{Fix}(f), u)\) is equal to \((\text{Fix}(f), \mathcal{B}_u \cap \text{Fix}(f))\). Further, if \(B_0 \in \mathcal{B}_{u+} \cap \text{Fix}(f)\) and \(x \in B_0\), then
\[
\text{scl}_{\mathcal{B}_{u+}}(B_0) = \bigcup \{B(x, y) \mid y \in \text{scl}_{\mathcal{B}_{u+}}(B_0)\}
\]
is a union of balls which by Lemma 7.3 are \(f\)-closed and is thus itself \(f\)-closed. This shows that the assumption of Proposition 5.7 is satisfied and consequently,
\[
\mathcal{B}_{u+} \cap \text{Fix}(f) = \mathcal{B}_{u+}^f \cap \text{Fix}(f).
\]

We have now proved that \(\text{Fix}(f) \neq \emptyset\) and the full ultrametric ball space of \((\text{Fix}(f), u)\) is \(S^*\). It follows that \((\text{Fix}(f), u)\) is spherically complete. So we obtain the following theorem:

**Theorem 7.5.** Take a spherically complete ultrametric space \((X, u)\) and a nonexpanding function \(f : X \to X\) which is contracting on orbits. Then every \(f\)-closed ultrametric ball contains a fixed point, \(\text{Fix}(f) \neq \emptyset\), and \((\text{Fix}(f), u)\) is again a spherically complete ultrametric space.

**The topological case.**

Take a compact topological space \(X\) and \((X, \mathcal{B})\) the associated ball space formed by the collection \(\mathcal{B}\) of all nonempty closed sets. If \(f : X \to X\) is any function, then \(\mathcal{B}^f\) can be taken as the set of all nonempty closed and \(f\)-closed sets of a (possibly coarser) topology, as arbitrary unions and intersections of \(f\)-closed sets are again \(f\)-closed. From Theorem 4.20, Lemma 5.1 and Theorem 7.2, we obtain:

**Theorem 7.6.** Take a compact topological space \(X\) and a function \(f : X \to X\). Assume that every nonempty closed, \(f\)-closed set contains a fixed point or a smaller, \(f\)-closed set. Then the topology on the set \(\text{Fix}(f)\) of fixed points of \(f\) having \(\mathcal{B}^f \cap \text{Fix}(f)\) as its collection of nonempty closed sets is itself compact.

As we are rather interested in the topology on \(\text{Fix}(f)\) induced by the original topology of \(X\), we ask for criteria on \(f\) which guarantee that \(\mathcal{B}^f \cap \text{Fix}(f) = \mathcal{B} \cap \text{Fix}(f)\). As an application of Corollary 5.8, we obtain:

**Corollary 7.7.** Take a compact topological space \(X\) and a continuous function \(f : X \to X\). Assume that every nonempty closed, \(f\)-closed set contains a fixed point or a smaller closed, \(f\)-closed set. Then the induced topology on the set \(\text{Fix}(f)\) of fixed points of \(f\) is itself compact.
7.2. Tychonoff type theorems.

In [1] the notion of a continuous function between two ball spaces is introduced. Further, it is shown that the category consisting of all ball spaces together with the continuous functions as morphisms allows products and coproducts. The products can be defined as follows.

Assume that \((X_j, B_j)_{j \in J}\) is a family of ball spaces. We set \(X = \prod_{j \in J} X_j\) and define the product \((X, B^\text{pr})\), where

\[
B^\text{pr} := \left\{ \prod_{i \in I} B_i \subseteq X \mid \text{for some } k \in I, \ B_k \in \hat{B}_k \text{ and } \forall i \neq k : B_i = X_i \right\}.
\]

Further, we define the topological product \((X, B^\text{tpr})_{j \in J}\) to be \((X, B^\text{tpr})\), where

\[
B^\text{tpr} := \left\{ \prod_{j \in J} B_j \mid \forall j \in J : B_j \in \hat{B}_j \text{ and } B_j = X_j \text{ for almost all } j \right\}.
\]

and the box product \((X, B^\text{bpr})_{j \in J}\) of the family to be \((X, B^\text{bpr})\), where

\[
B^\text{bpr} := \left\{ \prod_{j \in J} B_j \mid \forall j \in J : B_j \in B_j \right\}.
\]

Since the sets \(B_i\) are nonempty, it follows that \(B \neq \emptyset\), and as no ball in any \(B_i\) is empty, it follows that no ball in \(B^\text{pr}\), \(B^\text{tpr}\) and \(B^\text{bpr}\) is empty. Note that \(B^\text{pr} = \hat{B}^\text{pr}\) and \(B^\text{tpr} = \hat{B}^\text{tpr}\).

The box product \((X, \hat{B}^\text{bpr})_{j \in J}\) is equal to \((X, \hat{B}^\text{bpr})\), where

\[
\hat{B}^\text{bpr} := \left\{ \prod_{j \in J} B_j \mid \forall j \in J : B_j \in \hat{B}_j \right\}.
\]

Note that \(B^\text{tpr}\) and \(\hat{B}^\text{bpr}\) coincide when \(J\) is finite. We also see that in all cases, \(B^\text{pr} \subseteq B^\text{tpr} \subseteq \hat{B}^\text{bpr}\). Hence if \((X, B^\text{tpr})\) is spherically complete, then so is \((X, B^\text{tpr})\), and if \((X, \hat{B}^\text{bpr})\) is spherically complete, then so are \((X, B^\text{tpr})\) and \((X, B^\text{pr})\). The same holds with "\(S^\text{d}_1\)" and "\(S^\text{c}_1\)" in place of "\(S^\text{d}_1\)".

We leave the proof of the following observations to the reader:

**Proposition 7.8.** The following equations hold:

\[
\text{fic} \left( (X_j, B^\text{pr}_j)_{j \in J} \right) = (X_j, \text{fic}(B^\text{pr}_j))_{j \in J}
\]
\[
\text{ic} \left( (X_j, B^\text{pr}_j)_{j \in J} \right) = \text{ic} \left( (X_j, B^\text{tpr}_j)_{j \in J} \right) = (X_j, \text{ic}(B^\text{tpr}_j))_{j \in J}
\]
The following theorem presents our main results on the various products.

**Theorem 7.9.** The following assertions are equivalent:

a) the ball spaces \((X_j, B_j)\), \(j \in J\), are spherically complete,

b) their box product is spherically complete,

c) their topological product is spherically complete.

The same holds with “\(S^d_1\)” and “\(S^c_1\)” in place of “\(S^1\)”.

The equivalence of a) and b) also holds for all other properties in the hierarchy, in place of “\(S^1\)”.

**Proof:** Take ball spaces \((X_j, B_j)\), \(j \in J\), and in every \(B_j\) take a set of balls \(\{B_{i,j} \mid i \in I\}\). Then we have:

\[
\bigcap_{i \in I} \prod_{j \in J} B_{i,j} = \prod_{j \in J} \bigcap_{i \in I} B_{i,j}.
\]

If \(\mathcal{N} = (\prod_{j \in J} B_{i,j})_{i \in I}\) is a nest of balls in \((\prod_{j \in J} X_j, \hat{B}^{pr})\), then for every \(j \in J\), also \((B_{i,j})_{i \in I}\) must be a nest in \((X_j, \hat{B}_j)\).

a) \(\Rightarrow\) b): Assume that all ball spaces \((X_j, B_j)\), \(j \in J\), are spherically complete. Then for every \(j \in J\), \((B_{i,j})_{i \in I}\) has nonempty intersection. By (13) it follows that \(\bigcap \mathcal{N} \neq \emptyset\). This proves the implication a) \(\Rightarrow\) b).

b) \(\Rightarrow\) a): Assume that \((\prod_{j \in J} X_j, B^{bpr})\) is spherically complete. Take \(j_0 \in J\) and a nest of balls \(\mathcal{N} = (B_i)_{i \in I}\) in \((X_{j_0}, B_{j_0})\). For each \(i \in I\), set \(B_{i,j_0} = B_i\) and \(B_{i,j} = B_{0,j}\) for \(j \neq j_0\) where \(B_{0,j}\) is an arbitrary fixed ball in \(B_j\). Then \((\prod_{j \in J} B_{i,j})_{i \in I}\) is a nest in \((\prod_{j \in J} X_j, B^{bpr})\). By assumption,

\[
\emptyset \neq \bigcap_{i \in I} \prod_{j \in J} B_{i,j} = \left(\bigcap_{i \in I} B_i\right) \times \left(\prod_{j_0 \neq j \in J} B_{0,j}\right),
\]

whence \(\bigcap_{i \in I} B_i \neq \emptyset\). We have shown that for every \(j \in J\), \((X_j, B_j)\) is spherically complete. This proves the implication b) \(\Rightarrow\) a).

a) \(\Rightarrow\) c): Assume that all ball spaces \((X_j, B_j)\), \(j \in J\), are spherically complete. Then by Corollary 6.3, all ball spaces \((X_j, \hat{B}_j)\), \(j \in J\), are spherically complete. By the already proven implication a) \(\Rightarrow\) b), their box product \((X, \hat{B}^{pr})\) is spherically complete. By our remark before the theorem, \((X, B^{pr})\) is spherically complete, too.

c) \(\Rightarrow\) d): This has already been noted before the theorem.

d) \(\Rightarrow\) a): Same as the proof of b) \(\Rightarrow\) a), where we now take \(B_{0,j} = X_j\).
A similar proof of the equivalence of a) and b) also holds for all other properties in the hierarchy. For the properties $S_2$, $S_3$, $S_4$ and $S_5$, one uses the fact that by definition, $\prod_{j \in J} B_j$ is a ball in $\mathcal{B}^{\text{bpr}}$ if and only if every $B_j$ is a ball in $B_j$ and that

1) $\prod_{j \in J} B_j$ is a ball contained in $\prod_{j \in J} B_j$ if and only if every $B_j$ is a ball contained in $B_j$,

2) $\prod_{j \in J} B_j$ is a maximal ball contained in $\prod_{j \in J} B_j$ if and only if every $B_j$ is a maximal ball contained in $B_j$.

For the transfer of the other properties, one observes the following:

3) $\{ \prod_{j \in J} B_{i,j} \mid i \in I \}$ is a centered system if and only if all sets $\{ B_{i,j} \mid i \in I \}$, $j \in J$, are.

4) If $\{ \prod_{j \in J} B_{i,j} \mid i \in I \}$ is a directed system, then so are $\{ B_{i,j} \mid i \in I \}$ for all $j \in J$.

5) Fix $j_0 \in J$. If $\{ B_{i,j_0} \mid i \in I \}$ is a directed system, then so is $\{ \prod_{j \in J} B_{i,j} \mid i \in I \}$ when the balls are chosen as in the proof of b) $\Rightarrow$ a).

**Example 7.10.** There are $S^*$ ball spaces $(X_j, B_j)$, $j \in \mathbb{N}$, such that the ball space $(X, \mathcal{B}^{\text{bpr}})$ is not even $S_2$. Indeed, we choose a set $Y$ with at least two elements, and for every $j \in \mathbb{N}$ we take $X_j = Y$ and $B_j = \{ B \}$ with $\emptyset \neq B \neq Y$. Then trivially, all ball spaces $(X_j, B_j)$ are $S^*$. For all $i, j \in \mathbb{N}$, define

$$B_i := \underbrace{B \times B \times \ldots \times B}_{i \text{ times}} \times Y \times Y \times \ldots \in \mathcal{B}^{\text{bpr}}.$$ 

Then $\mathcal{N} = \{ B_i \mid i \in I \}$ is a nest of balls in $\mathcal{B}^{\text{bpr}}$, but the intersection $\bigcap \mathcal{N} = \prod_{j \in \mathbb{N}} B$ does not contain any ball in this ball space.

**Example 7.11.** There are $S^*$ ball spaces $(X_j, B_j)$, $j = 1, 2$, such that the ball space $(X, \mathcal{B}^{\text{bpr}})$ is not $S_3^\sharp$. Indeed, we choose again a set $Y$ with at least two elements and take $B_1 = B_2 = \{ B \}$ with $\emptyset \neq B \neq Y$. Then as in the previous example, $(X_j, B_j)$, $j = 1, 2$ are $S^*$ ball spaces. Further, $\mathcal{B}^{\text{bpr}} = \{ Y \times Y, B \times Y, Y \times B \}$, which is a centered system whose intersection does not contain any ball.

**The ultrametric case.**

If $(X_j, u_j)$, $j \in J$ are ultrametric spaces with value sets $u_j X_j = \{ u_j(a, b) \mid a, b \in X_j \}$, and if $B_j = B_{\gamma_j}(a_j)$ is an ultrametric ball in $(X_j, u_j)$ for each $j$, then

$$\prod_{j \in J} B_j = \{ (b_j)_{j \in J} \mid \forall j \in J : u_j(a_j, b_j) \leq \gamma_j \}.$$
This shows that the box product is the ultrametric ball space for the product ultrametric on $\prod_{j \in J} X_j$ which is defined as

$$u_{\text{prod}}((a_j)_{j \in J}, (b_j)_{j \in J}) = (u_j(a_j, b_j))_{j \in J} \in \prod_{j \in J} u_j X_j.$$ 

The latter is a poset, but in general not totally ordered, even if all $u_j X_j$ are totally ordered and even if $J$ is finite. So the product ultrametric is a natural example for an ultrametric with partially ordered value set.

If the index set $J$ is finite and all $u_j X_j$ are contained in some totally ordered set $\Gamma$ such that all of them have a common least element $0 \in \Gamma$, then we can define an ultrametric on the product $\prod_{j \in J} X_j$ which takes values in $\bigcup_{j \in J} u_j X_j \subseteq \Gamma$ as follows:

$$u_{\max}((a_j)_{j \in J}, (b_j)_{j \in J}) = \max_j u_j(a_j, b_j)$$

for all $(a_j)_{j \in J}, (b_j)_{j \in J} \in \prod_{j \in J} X_j$. We leave it to the reader to prove that this is indeed an ultrametric. The corresponding ultrametric balls are the sets of the form

$$\{(b_j)_{j \in J} \mid \forall j \in J : u_j(a_j, b_j) \leq \gamma\}$$

for some $(a_j)_{j \in J} \in \prod_{j \in J} X_j$ and $\gamma \in \bigcup_{j \in J} u_j X_j$. Now the value set is totally ordered. It turns out that the collection of balls so obtained is a (usually proper) subset of the full ultrametric ball space of the product ultrametric. Therefore, if all $(X_j, u_j)$ are spherically complete, then so is $(\prod_{j \in J} X_j, u_{\max})$ by Theorem 7.9 and Proposition 6.1.

**Theorem 7.12.** Take ultrametric spaces $(X_j, u_j)$, $j \in J$. Then the ultrametric space $(\prod_{j \in J} X_j, u_{\text{prod}})$ is spherically complete if and only if all $(X_j, u_j)$, $j \in J$, are spherically complete.

If the index set $J$ is finite and all $u_j X_j$ are contained in some totally ordered set $\Gamma$ such that all of them have a common least element, then the same also holds for $u_{\max}$ in place of $u_{\text{prod}}$.

Proof: As was remarked earlier, the ultrametric ball space of the product ultrametric is the box product of the ultrametric ball spaces of the ultrametric spaces $(X_j, u_j)$. Thus the first part of the theorem is a corollary to Theorem 7.9.

To prove the second part of the theorem, it suffices to prove the converse of the implication we have stated just before the theorem. Assume that the space $(\prod_{j \in J} X_j, u_{\max})$ is spherically complete and choose any $j_0 \in J$. Let $N_{j_0} = \{B_{\gamma_i}(a_{i,j_0}) \mid i \in I\}$ be a nest of balls in $(X_{j_0}, u_{j_0})$. Further, for every $j \in J \setminus \{j_0\}$ choose some element $a_j \in X_j$ and for
every $i \in I$ set $a_{i,j} := a_j$ and
\[
B_i := \{ (b_j)_{j \in J} \in \prod_{j \in J} X_j \mid u_{\max}((a_{i,j})_{j \in J}, (b_j)_{j \in J}) \leq \gamma_i \}
\]
\[
= \{ (b_j)_{j \in J} \in \prod_{j \in J} X_j \mid \forall j \in J : u_j(a_{i,j}, b_j) \leq \gamma_i \}.
\]
In order to show that $\mathcal{N} := \{ B_i \mid i \in I \}$ is a nest of balls in $(\prod_{j \in J} X_j, u_{\max})$, we have to show that any two balls $B_i, B_k, i, k \in I$, have nonempty intersection. Assume without loss of generality that $\gamma_i \leq \gamma_k$. As $\{ B_{\gamma_i}(a_{i,j_0}) \mid i \in I \}$ is a nest of balls, we have that $a_{i,j_0} \in B_{\gamma_k}(a_{k,j_0})$. It follows that $u_{j_0}(a_{k,j_0}, b_{j_0}) \leq \gamma_k$, and since $a_{i,j} = a_j = a_{k,j}$ for every $j \in J \setminus \{j_0\}$,
\[
(a_{i,j})_{j \in J} \in B_i \cap \{(b_j)_{j \in J} \in \prod_{j \in J} X_j \mid \forall j \in J : u_j(a_{k,j}, b_j) \leq \gamma_k \} = B_i \cap B_k.
\]
As $(\prod_{j \in J} X_j, u_{\max})$ is assumed to be spherically complete, there is some $(z_j)_{j \in J} \in \mathcal{N}$; it satisfies $u_j(a_{i,j}, z_j) \leq \gamma_i$ for all $i \in I$ and all $j \in J$. In particular, taking $j = j_0$, we find that $z_{j_0} \in B_{\gamma_i}(a_{i,j_0})$ for all $i \in I$ and thus, $z_{j_0} \in \bigcap \mathcal{N}_{j_0}$. \hfill \Box

### The topological case.
In which way does Tychonoff’s theorem follow from its analogue for ball spaces? The problem in the case of topological spaces is that the product ball space we have defined, while containing only closed sets of the product, does not contain all of them, as it is not necessarily closed under finite unions and arbitrary intersections. We have to close it under these operations.

If the topological spaces $X_i, i \in I$, are compact, then their associated ball spaces $(X_i, B_i)$ are $S^c_i$. By Theorem 7.9 their topological product is also $S^c_i$. The product topology of the topological spaces $X_i$ is the closure of $B^{pr}$ under finite unions and under arbitrary nonempty intersections, when $\emptyset$ and the whole space are adjoined. By Theorem 6.11 this topology is compact.

We have shown that Tychonoff’s Theorem follows from its ball spaces analogue.

### 8. Other results related to fixed point theorems
In this section, we will discuss two types of theorems that are related to fixed point theorems.
8.1. **Multivalued fixed point theorems.** We take a function $F$ from a nonempty set $X$ to its power set $\mathcal{P}(X)$ and ask for criteria that guarantee the existence of a fixed point $x \in X$ in the sense that

$$x \in F(x).$$

A very elegant approach to proving a generic multivalued fixed point theorem can be given by use of contractive ball spaces:

**Theorem 8.1.** Take a spherically complete contractive ball space $(X, B)$ and a function $F : X \rightarrow \mathcal{P}(X)$. Assume that

$$B_x \cap F(x) \neq \emptyset \text{ for all } x \in X.$$

Then $F$ admits a fixed point in $X$.

**Proof:** By part 2) of Theorem 4.12, $B$ contains a singleton ball $B_a = \{a\}$. Since by hypothesis $B_a \cap F(a) \neq \emptyset$, it follows that $a \in F(a)$. □

This theorem together with Proposition 4.9 and 4.10 can be used to prove the following result:

**Theorem 8.2.** Take a complete metric space $(X, d)$ and an Oettli-Théra function $\phi$ on $X$. If a function $F : X \rightarrow \mathcal{P}(X)$ satisfies

$$\forall x \in X \exists y \in F(x) : d(x, y) \leq -\phi(x, y),$$

then $F$ has a fixed point on $X$.

In [2] this theorem and its variants are proved using a version of part 2) of Theorem 4.12 together with Proposition 4.9.

The following is a slight generalization of Theorem 8.1, replacing the existence of singletons by that of minimal balls. Here again, as in Theorems 1.5 and 1.6, the general condition on the ball space is adapted to the given function: condition (C3) is replaced by a condition that depends on the function $F$.

**Theorem 8.3.** Take a nonempty set $X$ and a function $F : X \rightarrow \mathcal{P}(X)$. Assume that $(X, \{B_x \mid x \in X\})$ is a spherically complete ball space such that for all $x, y \in X$,

1) $x \in B_x$ and $B_x \cap F(x) \neq \emptyset$,
2) if $y \in B_x$, then $B_y \subseteq B_x$,
3) if $x \notin F(x)$, then there is some $z \in B_x$ such that $B_z \subsetneq B_x$.

Then $F$ admits a fixed point in $X$.

**Proof:** A straightforward adaptation of the proof of Proposition 4.11 shows that the intersection of a maximal nest of balls, if nonempty, must be a minimal ball $B_a$ which consequently must satisfy
\( a \in F(a) \). The assumption that the ball space is spherically complete guarantees that the intersection is nonempty.

8.2. Coincidence theorems. We take a nonempty set \( X \) and two or more functions \( f_1, \ldots, f_n : X \to X \) and ask for criteria that guarantee the existence of a coincidence point \( x \in X \) in the sense that

\[
(14) \quad f_1(x) = \ldots = f_n(x).
\]

In order to obtain a generic coincidence theorem for ball spaces, one can again use the idea of showing the existence of singleton balls with suitable properties.

**Theorem 8.4.** Take a spherically complete weakly contractive ball space \((X, B)\) and functions \( f_1, \ldots, f_n : X \to X \). Assume that

\[
f_1(a), \ldots, f_n(a) \in B_a \quad \text{for all} \quad x \in X.
\]

Then \( f_1, \ldots, f_n \) admit a coincidence point in \( X \).

**Proof:** By part 2) of Theorem 4.12, \( B \) contains a singleton ball \( B_a \). Since by hypothesis \( f_1(a), \ldots, f_n(a) \in B_a \), it follows that \( f_1(a) = \ldots = f_n(a) \).

As in the previous section, we prove a generalization that replaces the existence of singletons by that of minimal balls.

**Theorem 8.5.** Take a nonempty set \( X \) and functions \( f_1, \ldots, f_n : X \to X \). Assume that there is a \( B_x \)-ball space \( B \) on \( X \) such that \((X, B)\) is an \( S_2 \) ball space and for all \( x \in X \), if (14) does not hold, then there is some \( y \in X \) such that \( B_y \subseteq B_x \).

Then \( f_1, \ldots, f_n \) admit a coincidence point in \( X \).

**Proof:** Let \( M \) be a maximal nest of balls in \( B \) (it exists by Corollary 2.7). Since \((X, B)\) is an \( S_2 \) ball space, there is a ball \( B_x \subseteq \bigcap M \). This means that \( M \cup \{ B_x \} \) is a nest of balls, so by maximality of \( M \) we have that \( B_x \in M \). Consequently, \( B_x = \bigcap M \). Suppose that (14) does not hold. Then by hypothesis there is some element \( y \in X \) such that \( B_y \subseteq B_x \) whence \( B_y \notin M \). But then \( M \cup \{ B_y \} \) is a nest which strictly contains \( M \). This contradiction to the maximality of \( M \) shows that (14) must hold.

Let us note that condition (14) can be replaced by any other condition on \( x \). In this way, a generic theorem is obtained that is neither a fixed point theorem nor a coincidence theorem but can be specialized to such theorems. This idea has been exploited in [16].
REFERENCES

[1] Bartsch, R. – Kuhlmann, F.-V. – Kuhlmann, K.: Construction of ball spaces and the notion of continuity, submitted. arXiv:1810.09275
[2] Blaszkiewicz, P. – Ćmiel, H. – Linzi. A. – Szewczyk, P.: Caristi–Kirk and Oettli–Théra ball spaces, and applications, submitted. arXiv:1901.03853
[3] Brunner, N.: Topologische Maximalprinzipien (German) [Topological maximal principles], Z. Math. Logik Grundlag. Math. 33 (1987), 135–139
[4] Cohn, P. M.: Universal algebra, Harper and Row, New York, 1965
[5] Hitzler, P. – Seda, A. K.: The fixed-point theorems of Priess-Crampe and Ribenboim in logic programming, Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999), 219–235, Fields Inst. Commun. 32, Amer. Math. Soc., Providence, RI, 2002
[6] Jachymski, J. R.: Caristi’s fixed point theorem and selections of set-valued contractions, J. Math. Anal. Appl. 227 (1998), 55–67
[7] Jachymski, J. R.: Order-theoretic aspects of metric fixed point theory, Handbook of metric fixed point theory, 613641, Kluwer Acad. Publ., Dordrecht, 2001
[8] Kirk, W. A.: Metric fixed point theory: a brief retrospective, Fixed Point Theory Appl. 2015, article 215
[9] Kozlowski, W. M.: A purely metric proof of the Caristi fixed point theorem, Bull. Aust. Math. Soc. 95 (2017), 333–337
[10] Kubis, W. – Kuhlmann, F.-V.: Chain intersection closures, submitted, arXiv:1810.05832
[11] Kuhlmann, F.-V.: Maps on ultrametric spaces, Hensel’s Lemma, and differential equations over valued fields, Comm. in Alg. 39 (2011), 1730–1776
[12] Kuhlmann, F.-V. – Kuhlmann, K.: A common generalization of metric and ultrametric fixed point theorems, Forum Math. 27 (2015), 303–327; and: Correction to "A common generalization of metric, ultrametric and topological fixed point theorems", Forum Math. 27 (2015), 329–330; alternative corrected version available at: http://math.usask.ca/fvk/GENFPTAL.pdf
[13] Kuhlmann, F.-V. – Kuhlmann, K.: Fixed point theorems for spaces with a transitive relation, Fixed Point Theory 18 (2017), 663–672
[14] Kuhlmann, F.-V. – Kuhlmann, K. – Paulsen, M.: The Caristi–Kirk Fixed Point Theorem from the point of view of ball spaces, Journal of Fixed Point Theory and Applications, open access (2018)
[15] Kuhlmann, F.-V. – Kuhlmann, K. – Shelah, S.: Symmetrically Complete Ordered Sets, Abelian Groups and Fields, Israel J. Math. 208 (2015), 261–290
[16] Kuhlmann, F.-V. – Kuhlmann, K. – Sonallah, F.: Coincidence Point Theorems for Ball Spaces and Their Applications, to appear in: Ordered Algebraic Structures and Related Topics, CIRM, Luminy, France, October 12–16 2015, Contemporary Mathematics, AMS
[17] Mańka, R.: Some forms of the axiom of choice, Jbuch. Kurt-Gödel-Ges. 1988, 24–734
[18] Markowsky, G.: Chain-complete posets and directed sets with applications, Algebra Universalis 6 (1976), 53–68
[19] Oettli, W. – Théra, M.: Equivalents of Ekeland’s principle, Bull. Austral. Math. Soc. 48 (1993), 385–392
[20] Penot, J.-P.: *Fixed point theorems without convexity*, Analyse non convexe (Proc. Colloq., Pau, 1977). Bull. Soc. Math. France Mém. 60 (1979), 129–152

[21] Penot, J.-P.: *The drop theorem, the petal theorem and Ekeland’s variational principle*, Nonlinear Anal. 10 (1986), 813–822

[22] Prieß-Crampe, S.: *Der Banachsche Fixpunktsatz für ultrametrische Räume*, Results in Mathematics 18 (1990), 178–186

[23] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Points, Combs and Generalized Power Series*, Abh. Math. Sem. Hamburg 63 (1993), 227–244

[24] Prieß-Crampe, S. – Ribenboim, P.: *The Common Point Theorem for Ultrametric Spaces*, Geom.Ded. 72 (1998), 105–110

[25] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Point and Attractor Theorems for Ultrametric Spaces*, Forum Math. 12 (2000), 53–64

[26] Prieß-Crampe, S. – Ribenboim, P.: *Ultrametric spaces and logic programming*, J. Logic Programming 42 (2000), 59–70

[27] Prieß-Crampe, S. – Ribenboim, P.: *Ultrametric dynamics*, Illinois J. Math. 55 (2011), 287–303

[28] Shelah, S.: *Quite Complete Real Closed Fields*, Israel J. Math. 142 (2004), 261–272

[29] Steprāns, J. – Watson, S. – Just, W.: *A topological Banach fixed point theorem for compact Hausdorff spaces*, Canad. Bull. Math. 37(4) (1994), 552–555

[30] Tarski, A.: *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. 5 (1955), 285-309

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: Hanna.Cmiel@stud.usz.edu.pl

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: fvk@usz.edu.pl

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: Katarzyna.Kuhlmann@usz.edu.pl