Effective action for quantum Hall skyrmions.

A. Teufel and H. Walliser

Fachbereich Physik, Universität Siegen, D57068 Siegen, Germany

Abstract

Recently, an $O(3)$ type of effective action was presented for the quantum Hall ferromagnet, which accounts for bag formation observed in microscopic Hartree-Fock calculations. We apply this action in the soliton sector and compare with Hartree-Fock results. We find good agreement over the whole parameter range where skyrmions exist. The standard minimal $O(3)$ model cannot explain the bag and has further shortcomings connected with that fact.

PACS 73.43.-f, 73.21.-b, 12.39.Dc
Keywords: skyrmions, $O(3)$-model

Various experiments \cite{1, 2} have revealed evidence that quantum Hall ferromagnets exhibit charged excitations that are topologically stable spin–textures named skyrmions. In a recent revised Hartree–Fock (HF) calculation \cite{3} based on the formulation by Fertig et. al. \cite{4, 5}, it was found that the magnetization is not a unit vector, instead its length is substantially reduced in regions with non–zero excess charge (bag formation). This phenomenon is intuitively understood, because electrons in quantum Hall systems are mobile such that in regions with reduced electron density also the magnetization is expected to be reduced. In this way, the charged spin–texture is intimately connected with a bag which accommodates the excess charge.

There has been a lot of effort to derive an effective field theory for the quantum Hall ferromagnet \cite{6, 7, 8, 9, 10, 12, 13}, but all these approaches restrict the magnetization to a unit vector field. It was quite recently, that an effective $O(3)$ type of effective action was proposed in terms of a gradient expansion \cite{3}, which actually accounts for bag formation. With lengths measured in units of the magnetic length $\ell = \sqrt{hc/eB}$, and energies in units of the Coulomb energy $e^2/\varepsilon \ell$, the full energy functional is given by

\[
E[\hat{m}] = \frac{\rho_s}{2} \int d^2r \left[ \partial_i \hat{m} \partial_i \hat{m} - \frac{1}{4} (\partial_i \hat{m} \partial_i \hat{m})^2 \right] \\
+ \frac{1}{2} \int \int d^2r_1 d^2r_2 \rho^C(r_1) V(r_1 - r_2) \rho^C(r_2) \\
+ \frac{g}{4\pi} \int d^2r \left[ 1 - \left(1 - \frac{1}{4} \partial_i \hat{m} \partial_i \hat{m} \right) \hat{m}_3 \right] + \mathcal{O}(6),
\]
where $\rho_s$ represents the spin stiffness, e.g. $4\pi\rho_s = \frac{1}{4}\sqrt{\pi/2}$ for the Coulomb interaction. Then, for a given electron–electron interaction there is only one dimensionless parameter in the theory, namely, the effective Zeeman coupling $g = g_s\mu_B B/(e^2/\ell)$.

In addition to the non–linear sigma, the interaction and Zeeman terms of the standard minimal $O(3)$ model [6, 7, 11, 12, 13], there appears a symmetric fourth order term and a correction to the Zeeman term. The most important amendment is that the magnetization is no longer directly identified with the $O(3)$ unit vector field, instead it is related to it in a nontrivial way

$$m = \sigma \cdot \hat{m} = \left(1 - \frac{1}{4}\partial_i\hat{m}\partial_i\hat{m}\right) \cdot \hat{m} + O(4). \quad (2)$$

This relation holds independently from the electron–electron interaction employed and reflects the fact that the bag is always present, for all soliton sizes. With the physical charge density replaced by the topological density,

$$\rho^C = -\frac{1}{8\pi}\epsilon_{ij}\hat{m}(\partial_i\hat{m} \times \partial_j\hat{m}) + O(4), \quad (3)$$

it was argued in Ref. [3], that the energy functional (4) is unique up to $O(4)$ in the gradient expansion and linear terms in the Zeeman coupling $g$. For short ranged forces it was explicitly shown that this energy functional gives the correct results for extended spin–textures. In the following we will investigate the case of the Coulomb interaction.

The classical equations of motion following from the energy functional (4) are solved by the hedgehog ansatz

$$\hat{m} = \begin{pmatrix} \sin F(r) \cos \varphi \\ \sin F(r) \sin \varphi \\ \cos F(r) \end{pmatrix}, \quad (4)$$

where the angle function $F(r)$ determines the orientation of the unit vector $O(3)$ field. With this ansatz the azimuthal angle may be integrated,

$$E[F] = \frac{\rho_s}{2} \int d^2r \left[ F'^2 + \frac{\sin^2 F}{r^2} - \frac{1}{4}\left(F''^2 + \frac{\sin^2 F}{r^2}\right)^2 \right] + \frac{1}{\pi} \int d^2r_1 \rho^C(r_1) \int d^2r_2 \rho^C(r_2) \frac{1}{r_>^2} K\left(\frac{r_<}{r_>}\right)$$

$$\quad + \frac{g}{4\pi} \int d^2r \left[ 1 - \cos F + \frac{1}{4}\left(F'^2 + \frac{\sin^2 F}{r^2}\right) \cos F \right]. \quad (5)$$

Here, $K(x)$ represents a complete elliptic integral of the first kind [13] with $r_> = \max(r_1, r_2)$ and $r_< = \min(r_1, r_2)$ respectively. The integro–differential equation
for the angle function is derived straightforwardly and solved with the appropriate boundary conditions using familiar relaxation methods [11, 12].

In the following we compare results of (i) the microscopic Hartree-Fock theory (HF), (ii) the effective field theory (EFT) defined by the energy functional (1) together with relation (2), and (iii) the standard minimal field theory (MFT). In Fig.1 we show the dependence of the soliton energy on the Zeeman coupling. For extremely small \( g \), i.e. large solitons, all three curves approach the Bogomol’nyi bound, \( \frac{1}{4}\sqrt{\pi/2} \), of the famous Belavin–Polyakov (BP) soliton [15]. With increasing

\[
\begin{align*}
\text{Figure 1: Skyrmion energies for the Coulomb interaction as a function of the Zeeman coupling strength calculated with HF (full line), EFT (dashed) and MFT (dotted). In the limit } g \to 0 \text{ the Bogomol’nyi bound is approached.}
\end{align*}
\]

\( g \) then, the minimal \( O(3) \) model starts to overestimate the soliton energy. In the HF calculation there exists a critical Zeeman coupling \( g_c \approx 0.053 \) beyond which solitons cease to exist. Although we should not expect the gradient expansion to be reliable for too small solitons, the energy functional (1) reflects this fact only at a somewhat smaller value \( g_c \approx 0.049 \): the attractive symmetric fourth order term is strengthened and finally destroys the soliton. In contrast, the minimal \( O(3) \) model possesses stable soliton solutions for all Zeeman couplings.

Next, we are going to compare the profiles \( F(r) \) and \( \sigma(r) \), and the charge and spin densities

\[
\sigma(r) = 1 - \frac{1}{4} \left( F'^2(r) + \frac{\sin^2 F(r)}{r^2} \right),
\]

(6)
\[ \rho^C(r) = -\frac{F'(r) \sin F(r)}{4\pi r}, \quad (7) \]
\[ m_3(r) = \sigma(r) \cos F(r) \]  

for selected Zeeman couplings i.e. \( g = 0.01 \) (larger soliton) and \( g = 0.03 \) (smaller soliton). As expected, larger deviations are observed of course for the smaller soliton, where the gradient expansion converges more slowly. Generally, the energy

Figure 2: Angle profiles for Zeeman couplings \( g = 0.01 \) and \( g = 0.03 \) versus the radius in magnetic lengths. Solid, dashed and dotted lines refer to HF, EFT and MFT.

Figure 3: Bag profiles for Zeeman couplings \( g = 0.01 \) and \( g = 0.03 \) versus the radius in magnetic lengths. Solid, dashed and dotted lines refer to HF, EFT and MFT. There exists no bag in the minimal field theory. Functional (4) leads to a much better agreement with the microscopic HF as compared to the minimal field theory. In particular, this improvement is noticed for the bag shape functions depicted in Fig. 3: there is no bag in the MFT and accordingly the corresponding profile is 1. From (8) it follows then immediately that the spin density starts always at \(-1\) at the origin (Fig. 5). For the spin densities of EFT
Figure 4: Charge densities for Zeeman couplings $g = 0.01$ and $g = 0.03$ versus the radius in magnetic lengths. Solid, dashed and dotted lines refer to HF, EFT and MFT.

Figure 5: Spin densities for Zeeman couplings $g = 0.01$ and $g = 0.03$ versus the radius in magnetic lengths. Solid, dashed and dotted lines refer to HF, EFT and MFT. The spin density of the minimal field theory is confined to $-1$ at the origin.

and HF there exists no such restriction. Finally, we discuss the number of reversed spins defined as the spatial integral over the spin density

$$K = \frac{1}{4\pi} \int d^2 r \left( 1 - m_3 \right) - \frac{1}{2}. \quad (9)$$

The value $1/2$ is subtracted for convenience such that the quasihole has $K = 0$. The dependence on the Zeeman coupling (Fig. 6) behaves similar to that of the soliton energy (Fig. 1) for the three models. With increasing coupling strength, the minimal $O(3)$ model starts to overestimate the number of spin–flips. The EFT stays close to the HF result practically over the whole region where stable solitons exist. Since the Coulomb interaction considered here represents the extreme case of a long ranged
force and the EFT becomes exact for short ranged forces, we may infer that this agreement is only improved when more realistic interactions, which for example take the finite thickness of the layer into account [5, 16, 17], are considered.

Concluding we may say that the energy functional (1) gives a good description of skyrmions in the soliton sector. The present investigation particularly supports our previous statement that the $O(4)$ terms in the gradient expansion are unique. Compared to the minimal $O(3)$ model this is considered a considerable improvement. Through the skyrmion–skyrmion and skyrmion–antiskyrmion interactions, the additional terms in the effective action modify also the properties of multi–skyrmion systems such as skyrmion gas and lattice [18, 19, 20].

Bag formation will also affect the time–derivative part of the effective action. This is interesting and should be investigated. Furthermore, it is desirable to derive an effective field theory which takes the scalar field as dynamical field seriously into account, without resorting to the gradient expansion.

References

[1] A. Schmeller, J.P. Eisenstein, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 75, 4290 (1995).
[2] S.P. Shukla, M. Shayegan, S.R. Parihar, S.A. Lyon, N.R. Cooper, and A.A Kiselev, Phys. Rev. B 61, 4469 (2000).

[3] H. Walliser, Phys. Rev. B 63, 075310 (2001).

[4] H.A. Fertig, L. Brey, R. Côté, and A.H. MacDonald, Phys. Rev. B 50, 11018 (1994).

[5] H.A. Fertig, L. Brey, R. Côté, A.H. MacDonald, A. Karlhede, and S.L. Sondhi, Phys. Rev. B 55, 10671 (1997).

[6] E. Fradkin, *Field Theory of Condensed Matter Systems*, (Addison–Wesley, Redwood City, CA, 1991).

[7] S.L. Sondhi, A. Karlhede, S.A. Kivelson and E.H. Rezayi, Phys. Rev. B 47, 16419 (1993).

[8] K. Moon, H. Mori, K. Yang, S.M. Gervin, and A.H. MacDonald, Phys. Rev. B 51, 5138 (1995).

[9] Y.A. Bychkov, T. Maniv, and I.D. Vagner, Phys. Rev. B 53, 10148 (1996).

[10] W. Apel and Y.A. Bychkov, Phys. Rev. Lett. 78, 2188 (1997).

[11] M. Abolfath, J.J. Palacios, H.A. Fertig, S.M. Girvin and A.H. MacDonald, Phys. Rev. B 56, 6795 (1997).

[12] K. Moon and K. Mullen, Phys. Rev. B 57, 14833 (1998).

[13] S.M. Girvin, *Lectures, delivered at Ecole d’Ete Les Houches*, July 1998, cond-mat/9907002.

[14] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).

[15] A.A. Belavin and A.M. Polyakov, JETP Lett. 22, 245 (1975).

[16] N.R. Cooper, Phys. Rev. B 55, 1934 (1997).

[17] V. Melik-Alaverdian, N.E. Bonesteel, and G.Ortiz, Phys. Rev. B 60, R8501 (1999).

[18] R. Côté, A.H. MacDonald, L. Brey, H.A. Fertig, S.M. Girvin, and H.T.C. Stoof, Phys. Rev. Lett. 78, 4825 (1997).

[19] Yu.V. Nazarov and A.V. Khaetskii,, Phys. Rev. Lett. 80, 576 (1998).

[20] K. Moon and K. Mullen, Phys. Rev. Lett. 84, 975 (2000).