Abstract: Generally, the linear topological spaces successfully generate Tychonoff product topology in lower dimensions. This paper proposes the construction and analysis of a multidimensional topological space based on the Cartesian product of complex and real spaces in continua. The geometry of the resulting space includes a real plane with planar rotational symmetry. The basis of topological space contains cylindrical open sets. The projection of a cylindrically symmetric continuous function in the topological space onto a complex planar subspace maintains surjectivity. The proposed construction shows that there are two projective topological subspaces admitting non-uniform scaling, where the complex subspace scales at a higher order than the real subspace generating a quasinormed space. Furthermore, the space can be equipped with commutative and finite translations on complex and real subspaces. The complex subspace containing the origin of real subspace supports associativity under finite translation and multiplication operations in a combination. The analysis of the formation of a multidimensional topological group in the space requires first-order translation in complex subspace, where the identity element is located on real plane in the space. Moreover, the complex translation of identity element is restricted within the corresponding real plane. The topological projections support additive group structures in real one-dimensional as well as two-dimensional complex subspaces. Furthermore, a multiplicative group is formed in the real projective space. The topological properties, such as the compactness and homeomorphism of subspaces under various combinations of projections and translations, are analyzed. It is considered that the complex subspace is holomorphic in nature.

Keywords: topological spaces; quasinorm; norm; projections; topological group

MSC: 54B10; 54F05; 54F65

1. Introduction

The topological spaces have a wide variety of structures and associated properties. In view of category theory, a finite number of elementary axioms is sufficient to formulate the category of topological spaces as well as continuous mappings [1]. The topological spaces can be classified based on a set of properties, such as separability and path-connectedness. Interestingly, the concept of separable connectedness generalizes the path-connectedness property of topological spaces [2]. The two interesting varieties of spaces are complex spaces and normed linear spaces, which admit topological as well as algebraic structures. In the following subsections (Sections 1.1 and 1.2), brief descriptions about the topology of complex analytic spaces and normed linear spaces are presented. Next, the motivation for this work and the contributions made in this paper are given in Section 1.3.
1.1. Topology of Complex Analytic Spaces

In general, the topology of complex analytic spaces involves hypersurface structures and complex manifolds. The topology of hypersurface $X$ in a complex projective space is algebraically constructed considering that the homotopy is complementary type with respect to the corresponding hypersurface $X$ [3]. As a result, the topological complex hypersurface becomes decomposable into multiple differentiable manifolds. Thus, a decomposable topological hypersurface can be reconstructed as a sum of a set of differentiable manifolds $M$ with $k > 1$ copies of $S^n \times S^n$ product of $n$-spheres. It is shown that there are a variety of $C^1$ functions $f : CP_{n+1} \rightarrow [0,1]$ in the topological hypersurface [7]. Interestingly, such function exists in topological hypersurface if the complex surface is nonsingular in nature indicating that $\frac{\partial f}{\partial z}$ at a point $p$ does not simultaneously vanish on $C^{n+2}\setminus\{0\}$. In general, the topological manifold in a 2-dimensional normal and complex topological space is nonsingular in nature [4]. A singular point $z_p \in C$ in the complex hypersurface is a point such that $z_p = 0$.

The Brauer groups exist in complex analytic topological spaces and the equality between Brauer groups in view of geometry is evaluated in [5]. If the complex analytic topological space $X = Y^{an}$ is compact, then the canonical map between Brauer groups $f : Br(Y) \rightarrow Br(X)$ is invertible. The construction of the complex analytic surface considers that the topological space is convex and holomorphic.

1.2. Topology of Normed Linear Spaces

The structures of normed real and complex spaces determine the properties of stable null subspaces and the possibility of existence of convex polygons in a real plane. In the normed complex vector space, the equations of undetermined systems with real coefficients generate sparse solutions considering $l_1$ norm [6]. This gives rise to the varieties of null space properties and corresponding stability conditions. In the $\mathbb{R}^{m \times n}$ real space, the null space properties are equivalent to the properties of complex null space.

In view of general topology, the varieties of continuity and connectedness of a topological hyperspace can be formulated by employing the appropriate selection functions. For example, if $X$ is a topological space and $F(X)$ is a set of all closed subsets of $X$, then $f : E \subset F(X) \rightarrow X$ is called a selection function if $\forall A \in E, f(A) \in A$ condition is maintained in the hyperspace [7]. The continuity of such a selection function is expressed in terms of Vietoris topology [7]. As a result, the topological space becomes weakly ordered if the space is linearly ordered by the relation $<\delta$ such that $\forall x \in X, (\rightarrow \infty, x), (x, +\infty)$ are open and the selection function is continuous in the corresponding weakly ordered space. In a relatively different approach, the set-valued mapping is employed to establish various algebraic structures in the topological spaces [8]. It is shown that if $Y$ is a topological vector space (metrizable) and the topological space $X$ is arbitrary in nature, then the space of upper semi-continuous compact maps $M(X,Y)$ is a linear space. The conditions for the formation of a quasilinear space can also be derived from the corresponding linear constructions. The topological quasilinear space is a topological vector space, which is partially ordered under the relation $\leq$ and it follows the axioms of a normed vector space [9].

In the linear normed topological space, the addition and scalar multiplications are continuous in nature [10]. Moreover, if the linear normed topological space is Hausdorff, then the topology in it can be defined by means of a set of open neighbourhoods of origin [11]. Interestingly, the seminorm of linear topological space can include locally convex sets and the seminorm of a point $x_0$ in the space maintains the condition that $\|x_0\| \geq \frac{\|x_0\|_2 + \|x_0\|_1}{2}$. Note that it is possible to form a topological space of transformations from a space $X$, to another space, $Y$ [12]. The resulting topologized transformation space admits $k$-topology. The normed real operator spaces $X, Y$ can be complexified to generate operator spaces $XC, YC$ in the complex space while preserving isometry. Interestingly, the normed contractions maintain isometry such that, if $TR : X \rightarrow Y$ and $TC : XC \rightarrow YC$ are real and complex contractions, respectively, then $\|TR\|_{cb} = \|TC\|_{cb}$ [13]. Note that in this case the contraction is given as $TC(z \in C) = TR(x) + iT_R(y)$, where $z = x + iy$. Furthermore, the convex linear, as well as normed topological space, is integrable, which indicates that the space is well behaved [14,15].
1.3. Motivation and Contributions

In general, the lower dimensional topological spaces are considered to be relatively homogeneous and possibly uniform in nature. The homogeneity and uniformness of topological spaces are relaxed in higher dimensions. However, the geometry of constructions and topological analysis of multidimensional topological spaces are generally complex. For example, in complex-geometric settings of higher dimensional smooth manifolds, the proper group actions can be explicitly determined if, and only if, the group is compact [16]. The complexity of the structure of space is further enhanced if the space is not uniform, which would result in interesting analytical properties. This paper proposes the construction and analysis of a multidimensional topological space based on the product of complex space and real space in continua. The resulting space generates a multidimensional product topological space which is not a completely uniform space. Thus, the interesting and motivating questions are: what is the structure and topological basis of such space? What are the properties of symmetry and topological mapping of such space? What are the conditions for establishing a group algebraic structure in such space? These questions are addressed in this paper in relative details.

The main contributions of this paper can be summarized as follows. The proposed topological construction supports the formation of a quasinormed topological space admitting non-uniform scaling of a point in the space. Interestingly, in view of geometry the basis of the topological space is a set of open cylinders. Moreover, the resulting non-uniform topological space is holomorphic in nature maintaining continuity in respective subspaces under projections. A detailed analysis illustrates that the proposed topological space supports multidimensional group algebraic structure under the hybrid and commutative translation operations involving component wise translations in real as well as complex subspaces. Furthermore, the existence of an associative topological subspace passing through real origin in the space is identified with respect to the algebraic operation of multiplicative translation of a point in the complex subspace (i.e., respective complex plane). The main results are presented as a set of theorems and topological analysis. The algebraic standpoints are considered whenever required.

Rest of the paper is organized as follows. Section 2 presents preliminary concepts in the domain. Section 3 presents a set of definitions. Section 4 presents the main results. Finally, Section 5 concludes the paper.

2. Preliminary Concepts

In this section, a set of basic definitions and preliminary concepts are presented to establish the notions about topological spaces, topological vector spaces and hypersurfaces. The representation of preliminary concepts aims to enhance the self-containment of the paper to the readers. Let $C$, $R$ and $Z$ be the sets of complex numbers, real numbers and integers, respectively. The algebraic structure $G = (C, +)$ is a group, (II) if $a, b \in R, v \in X$ then $a \cdot (b \cdot v) = (a \cdot b) \cdot v$, (III) in $V = (X, +, )$ space the multiplicative identity is maintained as $\exists 1 \in X$ such that $1 \cdot v = v = v \cdot 1$ and, (IV) if $u, v \in X$ then $a \cdot (u + v) = a \cdot u + a \cdot v$ and $(a + b) \cdot v = a \cdot v + b \cdot v$. The topological vector space over $V = (X, +, )$ and a field $F$ is defined as a vector space satisfying the continuity of the following functions: (I) $V_X : X \times X \to X$ and, (II) $V_F : F \times X \to X$. A topological vector space is a normed space if $\forall v \in X, \|v\|_q$ computes a $q$-norm (in general, $q \in Z^+$) maintaining the axioms of a norm function. If $P_X(X)$ denotes the power set of $X$, then the structure $(X, \tau_X \subseteq P_X(X))$ is called a topological space if the following axioms are satisfied by it: (I) $[\phi, X] \subseteq \tau_X$, (II) $[A_i : i \in Z^+] \subseteq \tau_X \Rightarrow [\bigcup_{i \in Z^+} A_i] \in \tau_X$ and, (III) $[A_i : i \in Z^+] \subseteq \tau_X \Rightarrow [(A_i \cap A_k) \in \tau_X]$. The topological spaces $(X, \tau_X), (Y, \tau_Y)$ are homeomorphic if the function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous and $f^{-1}$ is also continuous. A function $g(x_1, x_2, \ldots, x_n)$ is called a symmetric function if $g(\langle x_i \rangle) = g(\langle x_k \rangle)$ where $\langle x_i \rangle$ denotes a permutation of variables and $\langle x_i \rangle \neq \langle x_k \rangle$ if $i \neq k$. 
3. The Multidimensional (C,R) Space

Let C be a set of complex numbers with Gauss origin given by $z_0 = (0, 0)$, and R be a set of extended real numbers (i.e., $R = [-\infty, +\infty]$). The space $X = C \times R$ is a multidimensional (C,R) space, where the origin of the space is denoted as $x_0 = (z_0, 0)$. An arbitrary point in (C,R) space is represented by $x_p = (z_p, r_p)$ and the corresponding topological (C,R) space is denoted by $(X, \tau_X)$. Note that, geometrically, the space is equipped with a $R^2$ plane, where the two real dimensions are interchangeable, resulting in planar rotational symmetry in the corresponding subspace. However, the complex dimension is distinct and fixed in the space. The details about the construction of basis elements of topological (C,R) space, the nature of subspaces and the topological projections are presented in the following subsections. First, we present the definitions of projections, associated norms under projections and the definition of quasinorm in a topological (C,R) space. In this paper, the $q$-norm in a topological projective space is denoted as $|||z|||_q$ considering that the norm is computed after projecting the point or the corresponding subspaces where $a \equiv C$ or $a \equiv R$. Additionally, the notation denoting $q$-quasinorm ($|||z|||_{C,R,p}$) includes the nature of projected subspaces (such as real subspace and complex plane), which is different from a norm. The notation $\text{hom}(A, B)$ signifies that A is homeomorphic to B.

3.1. Topological Projections in (C,R) Space

Let $x_p \in X$ be a point in topological (C,R) space $(X, \tau_X)$. The topological projections of $x_p$ are given by

$$
\begin{align*}
\pi_C : X &\to C, \\
\pi_C(x_p) &= z_p, \\
\pi_R : X &\to R, \\
\pi_R(x_p) &= r_p.
\end{align*}
$$

(1)

Note that if $A \subset X$ is open in $(X, \tau_X)$, then $\pi_C(A) \subset C$ and $\pi_R(A) \subset R$ are also open subspaces in complex and real subspaces, respectively.

3.2. Non-Uniform Scaling in (C,R) Space

Consider a point $x_p \in X$ in the topological (C,R) space $(X, \tau_X)$. If $\lambda \in R$ and $n, q \in Z^+$ are real and positive integers, respectively, then the non-uniform scaling of $x_p$ in (C,R) topological space is defined as

$$
\lambda^n x_p = (\lambda^{nq} z_p, \lambda^n r_p).
$$

(2)

The non-uniform scaling extends the complex plane at $q$-order higher as compared to the real subspace in one dimension for a given $n > 0$.

The topological (C,R) space can be equipped with a suitable norm function, resulting in the formation of a multidimensional normed topological space under projection. Interestingly, the non-uniform scaling preserves the properties of the quasinorm of topological (C,R) space, which is presented later as a proposition (Proposition 1).
3.3. Projective Norms

Let \( x_p \in X \) be a point in topological \((C, R)\) space \((X, \tau_X)\). The projective norms of \( x_p \) are a set of norms in the corresponding projective topological subspaces, which are defined as

\[
\| x_p \|_{C,R}^q = \| \pi_C(x_p) \|_q, \\
\| x_p \|_{R}^q = \| \pi_R(x_p) \|_q.
\]  

(3)

The above definition represents the generalized formulation of norms in projective topological subspaces. The quasinorm in topological subspaces is defined in later subsections considering non-uniform scaling. Note that according to the standard forms, \( \| x_p \|_{q=2} = \| z_p \|_2 \) holds in the complex subspace and \( \| x_p \|_{R}^q = \| r_p \|_2 \) holds in the real subspace, which is a Euclidean distance of \( x_p \) in \( R \) from \( x_0 = (z_0, 0) \) under projection.

3.4. Quasinorm in Topological \((C, R)\) Space

First, we define the 2-quasinorm of a point in the topological \((C, R)\) space, and next we extend it to \( q \)-quasinorm. We also prove that it is indeed a quasinorm under non-uniform translation of a point within the space. The 2-quasinorm of a point \( x_p \in X \) in a multidimensional topological \((C, R)\) space is defined as

\[
\| x_p \|_{C,R} = \sqrt{\| x_p \|_{C}^2 + \| x_p \|_{R}^2}.
\]  

(4)

**Proposition 1.** The \( \| \lambda^n x_p \|_{C,R}^q \) is a \( q \)-quasinorm \((q \in Z^+)\) in the topological \((C, R)\) space under non-uniform scaling.

**Proof.** We first prove that \( \| x_p \|_{C,R}^q \) for \( q = 2 \) is strictly positive. Note that the 2-quasinorm \( \| x_p \|_{C,R}^2 = 0 \) if and only if \( \| x_p \|_{C}^2 = 0 \) and \( \| x_p \|_{R}^2 = 0 \) in the projective topological subspaces. Moreover, the 2-quasinorm \( \| x_p \|_{C,R}^2 \geq 0 \) in every case, because \( \| x_p \|_{C}^2 \geq 0 \) and \( \| x_p \|_{R}^2 \geq 0 \) by following the properties of a norm function. The above condition is maintained for \( q \geq 2 \) within the topological space. Let \( \lambda \in R \) and \( n, q \in Z^+ \) be real and positive integers. This leads to the following derivation

\[
\| \lambda^n x_p \|_{C,R}^q = \| (\lambda^n z_p, \lambda^n r_p) \|_q,
\]

\[
= \lambda^n \sqrt{\| x_p \|_{C}^q + \| x_p \|_{R}^q},
\]

\[
= \lambda^n \sqrt{\| x_p \|_{C}^q + \| x_p \|_{R}^q}.
\]

Moreover, if \( x_1, x_2 \in X \) are two distinct points in the topological \((C, R)\) space, then, by following the properties of algebraic addition in 2D (ordered pairs), one can conclude \((z_1, r_1) + (z_2, r_2) = (z_1 + z_2, r_1 + r_2)\). Thus, evidently, further computation results in

\[
\| (z_1 + z_2, r_1 + r_2) \|_{C,R}^q = \sqrt{\| z_1 \|_q^2 + \| r_1 \|_q^2}^q, \;
\]

\[
\| (z_n, r_n) \|_{C,R}^q = \sqrt{\| z_n \|_q^2 + \| r_n \|_q^2}^q, \; n \in Z^+.
\]  

(5)

This leads to the observation that \( \exists M \in R^+ \) such that \( \| (z_1 + z_2, r_1 + r_2) \|_{C,R}^q \leq M(\| (z_1, r_1) \|_{C,R}^q + \| (z_2, r_2) \|_{C,R}^q) \).

Hence, the \( \| \lambda^n x_p \|_{C,R}^q \) is indeed a \( q \)-quasinorm of a point in the topological \((C, R)\) space under non-uniform scaling. \( \square \)
3.5. Open Cylindrical Basis

Let \((X, \tau_X)\) be a multidimensional topological \((C, R)\) space. The topological basis of \((X, \tau_X)\) is defined as \(B_\varepsilon = \{b_i : i \in Z^+\}\), such that \(b_i = D_{\varepsilon_i} \times I\) for some \(\varepsilon \in R^+, I = (a, b) \subset R\), where the open disk is given as \(D_{\varepsilon_i} = \{z_m \in C : ||z_m - z_i||_2 < \varepsilon, z_i \in C\}\) and \(||\sup(I) - \inf(I)|| \leq \varepsilon\). Evidently, a basis element \(b_i\) is an open cylinder in the multidimensional topological \((C, R)\) space, considering that the interval \(I\) is open in \(R\).

Remark 1. This indicates that if \(Y \subset X\) is a topological subspace such that \(Y \in \tau_X\) then \(\bigcup_{i=1}^k b_i \subseteq Y\) in \((X, \tau_X)\) for some \(k \in Z^+\). Moreover, if \([b_1, b_k] \subset B_\varepsilon\) then \(\exists b_k \in B_\varepsilon\) within the space such that it is open and \(b_k = b_i \cap b_k \neq \phi\) maintains the properties of topological basis.

3.6. Finite Translation in Topological \((C, R)\) Space

Let \((X, \tau_X)\) be a multidimensional topological \((C, R)\) space with open cylindrical basis given by \(B_\varepsilon\). Let \(T_C : X \rightarrow X\) and \(T_R : X \rightarrow X\) be two translations in the complex subspace and real subspace, respectively. The translations are admitted and finite if the following conditions are maintained

\[
\begin{align*}
T_R(x_p) &= (z_p, \beta + r_p), \beta \in R, \\
\|\pi_R(T_R(x_p))\|_q < +\infty, \\
T_C(x_p) &= (z_m, r_p), z_m \neq z_p, \\
\|T_C(x_p)\|_{C,R}\|_q < +\infty. 
\end{align*}
\]

The above-mentioned definition indicates that the translation is considered to be finite if the norm and quasinorm of translations are bounded in each translated and projected subspace. Moreover, if the translation is in the complex plane, then the quasinorm of all points in a translated topological subspace is considered. Otherwise, the norm of a translated real subspace is derived from the points in the corresponding projection.

Remark 2. Interestingly, a composite finite translation can be prepared from the individual translations within subspaces, which is algebraically given as \(T_C T_R\). It is to note that composite translation is commutative in nature indicating that \(T_C T_R = T_R T_C\). Moreover, the composite translation is finite if the following condition is maintained within the translated subspace: \(\|T_C T_R\|_{C,R}\|_q < +\infty\).

Suppose \(D_{z_m} \subset C\) is an open \(\varepsilon\)-disk centered at \(z_m \in C\) within the topological space \((X, \tau_X)\). The symmetry property of a continuous function within the \((C, R)\) subspace can be formulated as defined below.

3.7. Cylindrical Symmetry of Continuous Function

A function \(f : [0, 1] \rightarrow (D_{z_m} \times R)\) is called point-wise cylindrically symmetric in \((C, R)\) space with respect to \((D_{z_m} \times R) \subset X\) if \(\exists a, b \in (0, 1)\) such that \((\pi_C \circ f) (a) = (\pi_C \circ f) (b)\) and \([a \prec b] \Rightarrow [(\pi_R \circ f) (a) < R < (\pi_R \circ f) (b)]\). The function is strictly cylindrically symmetric if \(\|f\|_{[a, b]} = \|f\|_{[a, b]}\). Note that the concept of cylindrical symmetry of a function in \((C, R)\) space is different from the symmetry of a function in the Cartesian product space. The symmetry property of a continuous function in \((C, R)\) space plays an important role in determining the nature of projection functions within cylindrical subspaces. It is shown through analysis given in later sections in this paper that the cylindrical symmetry of a continuous function within a subspace induces surjection in the projective space.
4. Main Results

This section presents the analytical and structural properties of the topological \((C, R)\) space in two parts. The main results are comprised of algebraic structures of the space in view of group theory, and the topological properties of the space in view of analysis. The algebraic properties are analyzed by considering the construction of topological groups within the space. Next, the topological properties of the space are analyzed in regard to general topology. In this paper, the compactibility analysis of a space or a subspace considers open spaces or subspaces which have finite topological subcovers. If such subcovers do not exist, then the spaces or subspaces under consideration are not compactible in nature.

4.1. Topological Group in \((C, R)\) Space

In this section, a detailed analysis is presented to construct a topological group \(G = (X, \ast)\) in the multidimensional topological \((C, R)\) space \((X, \tau_X)\) under finite composite translations. Let the results of finite translations of \(x_m = (\alpha_m e^{i\theta_m}, r_m)\) expressed in Euler form for \(z_m \in C\) be given as 

\[ T_C(x_m) = x_m^T = (z_m^T, r_m) = (\alpha_m e^{i\theta_m}, r_m) \quad \text{and} \quad T_R(x_m) = (z_m, r_m + \beta) = (\alpha_m e^{i\theta_m}, r_m + \beta). \]

The corresponding group operation is algebraically formulated as

\[ \ast \equiv (T_C)(+T_R), \]
\[ \{x_m, x_n\} \subset X, \]
\[ x_m \ast x_n = (\alpha_m e^{i\theta_m}, r_m) \ast (\alpha_n e^{i\theta_n}, r_n), \]
\[ (\alpha_m e^{i\theta_m}, r_m) \ast (\alpha_n e^{i\theta_n}, r_n) = (\alpha_m \alpha_n e^{i(\theta_m + \theta_n)}, r_m + r_n + \beta). \]  

Note that the group operation is presented by considering the Euler (i.e., polar) representation of complex numbers and the finite translations within the space, along with two arithmetic group operations (i.e., addition with translation in real subspace and multiplication with translation in complex plane). First, we derive a set of algebraic conditions that must be satisfied to construct a topological group in \((X, \tau_X)\).

4.1.1. Condition for Associative \(\ast\)

If \(G = (X, \ast)\) is a topological group, then the group operation must be associative. This leads to the following derivation

\[ (z_n, r_n) \ast ((z_p, r_p) \ast (z_m, r_m)) = ((z_n, r_n) \ast (z_p, r_p)) \ast (z_m, r_m), \]
\[ (z_p, r_p) \ast (z_m, r_m) = (z_p z_m^T, r_p + r_m + \beta), \]
\[ (z_p, r_p) \ast ((z_p, r_p) \ast (z_m, r_m)) = (z_p z_p^T z_m^T, r_n + r_p + r_m + 2\beta), \]
\[ (z_n, r_n) \ast ((z_p, r_p) \ast (z_m, r_m)) = (z_n z_p^T z_m^T, r_n + r_p + r_m + 2\beta). \]

This further indicates that the following algebraic condition must be satisfied to maintain associativity of group operation in the space

\[ z_m^TT = z_m^T. \]  

The algebraic condition leads to the conclusion that \(T_C(z_m) = T_C(z_m)\) within the topological \((C, R)\) space. Thus, the successive translations on the complex plane under topological group operation in the space are invariant to the first translation, maintaining associativity.
4.1.2. Condition for Identity Element

The condition for existence of identity element \( x_e = (z_e, r_e) \) in topological \((C, R)\) space can be stated as: \((z_n, r_n) \ast (z_e, r_e) = (z_e, r_e) \ast (z_n, r_n) = (z_n, r_n)\). This leads to the following algebraic conditions

\[
\begin{align*}
    z_n z_{eT} &= z_e z_{nT} = z_n, \\
    r_e &= -\beta. 
\end{align*}
\]

(9)

Hence, if we consider \( z_e^T = 1 \) in the topological space, then the identity element can be represented as \( x_e = (z_n / z_{nT}, -\beta) \). However, the identity element is unique for a topological group \( G = (X, \ast) \). This leads to the following condition to be maintained by \( x_e \in X \)

\[
    z_n / z_{nT} = z_m / z_{mT} = \ldots = z_p / z_{pT} = a \in R. 
\]

(10)

Thus, the identity element of \( G = (X, \ast) \) can be represented as \( x_e = (a, -\beta) \), such that \( T_C(x_e) = (1, -\beta) \). Furthermore, in the space, the following condition should be maintained by finite complex translation: \( T_C(z_n) = a^{-1}z_n, a \neq 0 \).

**Remark 3.** Note that the finite complex translation and the identity element maintain the condition that \( a \in (-\infty, +\infty) \setminus \{0\} \) within the topological \((C, R)\) space.

4.1.3. Existence of Inverse Element

Let \( x_{-n} \in X \) be the unique inverse of \( x_n \in X \) such that \( x_{-n} \ast x_n = x_n \ast x_{-n} = x_e \) within the space. This leads to the following conditions to be maintained by \( G = (X, \ast) \) in the topological \((C, R)\) space

\[
\begin{align*}
    z_n z_{-nT} &= z_e z_{nT} = a, \\
    r_n + r_{-n} &= -2\beta. 
\end{align*}
\]

(11)

Hence, the inverse element can be represented as: \( x_{-n} = (a/T_C(z_n), -(2\beta + r_n)) \).

Although the construction of a group in topological \((C, R)\) space under composite translations and algebraic operations is complex in nature, however, the projections of the space easily admit varieties of additive as well as multiplicative group structures as presented in the next theorem.

**Theorem 1.** In topological \((C, R)\) space \( G_R = (\pi_R(X), \ast) \) is a group in one dimension, where \( \ast \in \{+, \cdot\} \).

**Proof.** Let \((X, \tau_X)\) be a topological \((C, R)\) space and the algebraic operation is given by \( \ast \equiv + \) in \( G_R = (\pi_R(X), \ast) \). As \( \pi_R(X) \subseteq R \), so \( G_R = (\pi_R(X), +) \) is an additive group in \( R \). Similarly, if \( \ast \equiv \cdot \), then \( G_R = (\pi_R(X), \cdot) \) is a multiplicative group in \( R \) in one dimension. □

This indicates that the projection of topological \((C, R)\) space on real line always prepares the additive or multiplicative group varieties in standard forms. A similar observation can be made in a complex plane under addition operation in projective space, as explained in the next lemma.

**Lemma 1.** In topological \((C, R)\) space \( G_C = (\pi_C(X), +) \) is a group in two dimensions.

**Proof.** The proof is relatively straightforward in nature if we consider that the identity element is \( z_0 \in C \). Recall that, in this case, the following condition is valid: \( \pi_C(X) \subseteq C \), which is a two-dimensional complex plane. Considering addition operation, \( \forall z_i, z_k \in C \), it is true that \( z_i + z_k = z_k + z_i \) and \( z_i + z_i \in C \). Moreover, \( \forall z_i \in C \) the identity element maintains that \( z_i + z_0 = z_0 + z_i = z_i \). Furthermore, \( \forall z_j \in C, 3 - z_j \in C \) such that \( z_i + (-z_i) = -z_i + z_i = z_0 \). This results in the formation of \( G_C = (\pi_C(X), +) \) group structure. □
Interestingly, the preparation of an associative topological subspace can be formulated if we consider a special case, where the finite translation in real subspace $T_R$ is multiplicative in nature such that $T_R(x_p) = (z_m, pr_p)$ where $p \in R$. Note that in this case, $T_C T_R = T_C$ if $x_p \in C \times [0]$. This observation leads to the following theorem.

**Theorem 2.** If $D_z = C \times [0]$ is a subspace in $(X, \tau_X)$ and $T_R(x_p) = (z_p, pr_p)$ then $A_D = (D_z, \tau_C)$ is an associative topological subspace, where $(x_m \in D_z)(T_C)(x_p \in D_z) = (z_m z_p e^{i \theta_T}, 0)$ and $0 \leq \theta_T \leq 2\pi$.

**Proof.** Let $D_z = C \times [0]$ be a complex open $\epsilon$-disk (i.e., topological subspace) in $(X, \tau_X)$. If $T_R(x_p \in D_z) = (z_p, pr_p)$, then $T_C T_R = T_C$ in $D_z$ because $T_C(x_p) = (z_p e^{i \theta_T}, 0)\), where $0 \leq \theta_T \leq 2\pi$. Suppose, we consider an arbitrary subset given by $\{x_p, x_m, x_n\} \subset D_z$ and an algebraic operation $(T_C)$ such that $(x_m \in D_z)(T_C)(x_p \in D_z) = (z_m z_p e^{i \theta_T}, 0)$. This leads to the following derivation

$$
(x_m(T_C)(x_p(T_C)x_n) = (x_m(T_C)x_p(T_C)x_n) = (z_m z_p z_n e^{i 2 \theta_T}, 0).
$$

Hence, the topological subspace $D_z = C \times [0]$ maintains $x_m(T_C)(x_p(T_C)x_n) = (x_m(T_C)x_p)(T_C)x_n$ resulting in the formation of an associative subspace under algebraic operation $(T_C)$. □

The algebraic constructions and analysis of a topological group in the $(C, R)$ space indicate that a set of specific conditions are required to be maintained within the space. Moreover, if the finite translation is associated with multiplication under projection on a real subspace, then the complex plane $P_{(C,0)} \subset X$ in the topological space is an associative non-compact subspace if $x_0 \in P_{(C,0)}$. In this case, the finite real translations are invariant in nature. The analysis of topological properties of the $(C, R)$ space is illustrated in the following subsection.

**4.2. Analysis of Topological Properties**

It was noted earlier that there is a non-compact and associative subspace within the topological $(C, R)$ space under specific multiplicative finite translations. The question is: what are the topological properties of complete space in terms of continuity and compactness? These properties are analyzed and presented in the form of a set of theorems. First, we show that the space preserves the standard topological notion of continuity under projections.

**Theorem 3.** If $f : [0, 1] \to (Y \subset X)$ in $(X, \tau_X)$ is continuous then $(\pi_C \circ f) : [0, 1] \to (A \subset C)$ and $(\pi_R \circ f) : [0, 1] \to (B \subset R)$ are also continuous.

**Proof.** Let $(X, \tau_X)$ be a topological $(C, R)$ space and a subspace is denoted by $Y \subset X$. Suppose $f : [0, 1] \to (Y \subset X)$ is a continuous function within the space. Thus, the function maintains the property that $\forall E \in \tau_Y$ open set $\exists f^{-1}(E) \subset [0, 1]$ open in $R^+$ and $\forall x \in [0, 1], \|f(x)\|_{C,R} \in [0, +\infty)$. Accordingly, it follows that under the corresponding projections $\pi_C(E) \subset C$ and $\pi_R(E) \subset R$ are open subspaces. However, there is $(\pi_C \circ f) : [0, 1] \to (A \subset C)$ such that $(\pi_C \circ f)(K) \subseteq \pi_C(E)$, where $K \subseteq f^{-1}(E)$ and $\pi_C(E) \subset A$. Therefore, it can be concluded that $\forall E \in \pi_C(Y)$ open in $C$, it is true that $\exists K \subset [0, 1]$ open in $R^+$ such that $(f^{-1} \circ \pi^{-1})(E) = K$. Hence, the function $(\pi_C \circ f) : [0, 1] \to (A \subset C)$ is continuous in $C$. The similar proof can be extended for $(\pi_R \circ f) : [0, 1] \to (B \subset R)$ for establishing the continuity in $R$. □

**Remark 4.** It can be concluded from the above theorem that if $(\pi_C \circ f) : [0, 1] \to (A \subset C)$ is a continuous closed curve, then there is a surjective $h_C : C \to A$, such that $h_C(\gamma)$ is holomorphic in $A$ with $\oint_{\gamma \subset C}(h_C(z)dz = 0$ (here, $\gamma$ is a set of points forming a close loop). Note that it is considered as $h_C \equiv (\pi_C \circ f)$ within the space. Similarly, there is a surjective $h_R : R \to (B \subset R)$ such that $h_R \equiv (\pi_R \circ f)$ and $h_R(R) \subset (-\infty, +\infty)$. 
Once the nature of continuity is analyzed, the next important topological notion is the compactness of a subspace. The following theorem illustrates that the compactness property varies in a topological \((C, R)\) space depending on the projections within subspaces. In other words, the compactness of a subspace in a topological \((C, R)\) space is not always preserved.

**Theorem 4.** If \(D_{zm} \subset C\) is an open \(\varepsilon\)-disk centered at \(z_m \in C\) and \(f : [0, 1] \to (D_{zm} \times R)\) is continuous in \((X, \tau_X)\) then \((\pi_C \circ f)([0, 1])\) is compactible and \((\pi_R \circ f)([0, 1])\) is not compactible.

**Proof.** Let \((X, \tau_X)\) be a topological \((C, R)\) space such that \(z_m \in C\) and \(D_{zm} \subset C\) is an open \(\varepsilon\)-disk in a complex plane centered at \(z_m\), where \((D_{zm} \times R) \in \tau_X\). Let \(f : [0, 1] \to (D_{zm} \times R)\) be a continuous function. Thus, the respective subspaces under projection, given by \((\pi_C \circ f)([0, 1]) = A\) and \((\pi_R \circ f)([0, 1]) = B\), are continuous in nature. However, this indicates that for all \(a \in [0, 1]\), \((\pi_C \circ f)(a) \subset D_{zm}\) and the holomorphic condition is maintained by closed loop integral around point \(h_C(z)\) given by \(\oint_{\gamma \subset A} h_C(z)dz = z_k\oint_{\gamma \subset A} dz = 0\), where \(h_C : C \to A, z_k = h_C(z)\) and, \(h_C \equiv (\pi_C \circ f)\). Hence, the closed subspace represented by \(\overline{D_{zm}} = D_{zm} \cup \partial D_{zm}\) is a closed disk in complex plane, which is compact and holomorphic. However, the other projection in real subspace given by \(B \subseteq R\) is open, because \(B \subseteq [-\infty, +\infty]\). Hence, the subspace \(B \subseteq R\) is not compactible under projection. \(\square\)

Interestingly, the symmetry of a continuous function within a cylindrical subspace affects the nature of projection on a complex plane. The following theorem illustrates that continuous and symmetric function within a cylindrical subspace induces surjective projection with respect to a complex plane.

**Theorem 5.** If \(f : [0, 1] \to (D_{zm} \times R)\) is a continuous and point-wise cylindrically symmetric function, then \((\pi_C \circ f) : [0, 1] \to C\) is a surjection.

**Proof.** Let \(f : [0, 1] \to (D_{zm} \times R)\) be a continuous function in the corresponding topological subspace in the \((C, R)\) space \((X, \tau_X)\). If the function is cylindrically symmetric with respect to open disk \(D_{zm} \subseteq C\) in \((D_{zm} \times R) \subseteq X\), then \(\exists a, b \in (0, 1)\) such that \((\pi_R \circ f)(a) < (\pi_R \circ f)(b)\), where \(a < b\). Moreover, according to the definition of point-wise cylindrical symmetry of the function, one can conclude that \((\pi_R \circ f)(a) < r < (\pi_R \circ f)(b)\). However, due to point-wise cylindrical symmetry of the function with respect to \((D_{zm} \times r) \subseteq X\), it maintains the condition that \((\pi_C \circ f)(a) = (\pi_C \circ f)(b)\) within the cylindrical subspace. Hence, the projection maintains the condition that \(\exists z_k \in C\), such that \((\pi_C \circ f)(a) = (\pi_C \circ f)(b) = z_k\), which is a surjection in \(C\). \(\square\)

**Corollary 1.** It can be concluded from the above theorem that \((\pi_C \circ f) : [0, 1] \to (D_{zm} \subset C)\) is also a surjection within the cylindrical topological subspace. Moreover, if \((\pi_C \circ f)([0, 1]) \subset D_{zm}\) and \(h_C : (A \subset D_{zm}) \to (A \subset D_{zm})\) such that \(h_C \equiv (\pi_C \circ f)\), then \(h_C(A) = id_C(A)\). Note that \(id_C(.)\) is a complex identity function. The proof is relatively straightforward to derive from the conditions of equivalence relation.

The locally homeomorphic subspaces under translation within the topological \((C, R)\) space preserve the functional translation between the subspaces. The local homeomorphism between subspaces indicates that both the open subspaces are locally compactible. This property is illustrated in the next theorem.

**Theorem 6.** In the \((X, \tau_X)\) topological \((C, R)\) space if \(\text{hom}(D_{z_1} \times R, D_{z_2} \times R)\) and \(T_C(\overline{D_{z_1}}) = \overline{D_{z_2}}\) conditions are maintained in disjoint compact subspaces, then \(T_C(h_1([0, 1])) = h_2([0, 1])\), where \(h_1 : [0, 1] \to (D_{z_1} \subset C)\) and \(h_2 : [0, 1] \to (D_{z_2} \subset C)\).

**Proof.** Let \((X, \tau_X)\) be a topological \((C, R)\) space such that \(D_{z_1} \times R \in \tau_X, D_{z_2} \times R \in \tau_X\) are two disjoint compactible subspaces (i.e., \(D_{z_1} \cap D_{z_2} = \emptyset\) and each are holomorphic subspaces with
Theorem 7. If $g : \mathbb{D} \to \mathbb{R} \times \mathbb{R}$, then $f : \mathbb{D} \times \mathbb{R} \to \mathbb{D} \times \mathbb{R}$ and $f^{-1} : \mathbb{D} \times \mathbb{R} \to \mathbb{D} \times \mathbb{R}$ are both continuous. Suppose $U_\alpha \subset \mathbb{D} \times \mathbb{R}$ is an open neighbourhood of $x \in \mathbb{D} \times \mathbb{R}$. Thus, $\exists y \in \mathbb{D} \times \mathbb{R}$ such that $f(U_\alpha) \subset U_y$ where $U_y$ is an open neighbourhood of $y = f(x)$ in $\mathbb{D} \times \mathbb{R}$. This indicates that if $T_C(D_\alpha) = D_\beta$, then $T_C((\pi_C \circ f)(U_\alpha)) = (\pi_C \circ f)(U_y)$, where $U_\alpha \subset \mathbb{D} \times \mathbb{R}$ and $U_y \subset \mathbb{D} \times \mathbb{R}$. Furthermore, if $h_1 : [0,1] \to (D_\alpha \subset \mathbb{C})$ and $h_2 : [0,1] \to (D_\beta \subset \mathbb{C})$ are two continuous functions in respective subspaces, then $\forall a \in h_1([0,1]), \forall \beta \in h_2([0,1])$, the finite translation maintains $T_C(U_\alpha \subset D_\alpha) = (U_\beta \subset D_\beta)$. Hence, it concludes that $T_C(\cup_{a \in [0,1]} h_1(a)) = \cup_{b \in [0,1]} h_2(b)$, indicating $T_C(h_1([0,1])) = h_2([0,1])$. □

The results presented in the above theorem can be extended as a generalization to the separable topological $(C,R)$ spaces $(X, \tau_X)$ and $(Y, \tau_Y)$. This generalization is presented in the next corollary.

Corollary 2. Suppose there is a continuous function $w : (X, \tau_X) \to (Y, \tau_Y)$ between the two topological $(C,R)$ spaces. One can conclude that $(X, \tau_X)$ and $(Y, \tau_Y)$ are homeomorphic if $(\pi_C \circ w^{-1})(\cdot)$ and $(\pi_R \circ w^{-1})(\cdot)$ are continuous in $C$ and $R$, respectively.

The compactibility of a real valued function with a domain in topological $(C,R)$ space can be analyzed if the function is considered to be composed of two other real valued functions in the respective subspaces under projections. The topological analysis shows that the diameter of the set resulting from the union of two individual subsets generated in real subspaces under projections should be finite. This property is presented in the following theorem.

Theorem 7. If $g_z : C \to \mathbb{R}$ and $g_r : R \to \mathbb{R}$ are two functions such that $f : (X, \tau_X) \to \mathbb{R}$ is given by $f(x_p \in X) = (g_z \circ \pi_C)(x_p) + (g_r \circ \pi_R)(x_p)$, then $f(Y \subset X)$ is compactible if $0 \leq \text{diam}(g_z(C) \cup g_r(R)) < +\infty$.

Proof. Let $(X, \tau_X)$ be a topological $(C,R)$ space and $f : (X, \tau_X) \to \mathbb{R}$ be a real valued function such that $f(x_p \in X) = (g_z \circ \pi_C)(x_p) + (g_r \circ \pi_R)(x_p)$ under projections. Note that $g_z : C \to \mathbb{R}$ and $g_r : R \to \mathbb{R}$ are two real valued functions. Suppose $A \subseteq R$ is a subspace such that $A = g_z(C) \cup g_r(R)$. If $Y \subset X$ is a topological subspace, then $f(Y) \subseteq R$, where $f(x_p \in Y) = (g_z \circ \pi_C)(x_p \in Y) + (g_r \circ \pi_R)(x_p \in Y)$. Thus, if $\exists x_p \in Y$ such that $(g_z \circ \pi_C)(x_p) \in [-\infty, +\infty]$, then $\text{diam}(A) \to +\infty$. Similarly, if $(g_r \circ \pi_R)(x_p) \in [-\infty, +\infty]$, then $\text{diam}(A) \to +\infty$. This indicates that if $0 \leq \text{diam}(g_z(C) \cup g_r(R)) < +\infty$, then $A \subset (-\infty, +\infty)$. Thus, if $f(Y) \subseteq B$ and $B = \{a_z : a_z = (a_z = A _z \subset A) + (a_z = A_r \subset A)\}$ is finite such that $A_z \cap A_r = \phi$, then $A_z \subset (-\infty, +\infty)$, $A_r \subset (-\infty, +\infty)$. Otherwise, if $A_z \cap A_r \neq \phi$, then $f(Y) \subset (-\infty, +\infty)$ in the real subspace, which is a finite subspace. Hence, $f(Y \subset X)$ is compactible if $0 \leq \text{diam}(g_z(C) \cup g_r(R)) < +\infty$. □

Lemma 2. If $Y = T_C T_B(B \subset X)$ is a translated subspace and $f : (Y \subset X) \to z_0 \times H$ is a continuous projection of the topological subspace under translation, then $f(Y)$ is compact if $H \subset R$ is closed.

Proof. Let $B \subset X$ and $Y \subset X$ be two topological $(C,R)$ subspaces such that $B \cap Y = \phi$ and $Y = T_C T_B(B \subset X)$. Recall that, in general, the $T_C T_B = T_B T_C$ condition is maintained within the space where the translations are finite in nature. Thus, the translations maintain $||x_p \in B||_{C,R} < +\infty$ and $\forall x_p \in Y, ||x_p||_{C,R} < +\infty$. Moreover, if $f : (Y \subset X) \to z_0 \times H$ is a continuous function, then $\forall (z_0, r) \in z_0 \times H, ||(z_0, r)||_{C,R} < +\infty$, where $H \subset R$. Hence, it can be concluded that $r \in (-\infty, +\infty)$ and if $H \subset R$ is closed, then $f(Y)$ is compact. □

The above lemma provides an interesting property of the topological $(C,R)$ space. It illustrates that the compactness of a continuous projective function under combined translation on a subspace passing through the Gauss origin is dependent on the nature of the corresponding real projective
5. Conclusions

The topological properties of a space, such as separability, linearity and compactness, depend on the underlying structure of the space. The conditions for the existence of a group in a topological space are determined by the nature of algebraic operations in the space along with geometry. This paper proposes the construction and analysis of a non-uniformly scalable multidimensional topological space, which is hybrid in nature. The basis elements of the proposed multidimensional topological space are cylindrical open sets and the space admits planar symmetry in a 2D real subspace under full circular rotation with respect to origin. The non-uniform scaling of a point in complex and real projective subspaces results in the formation of a quasinormed topological space. However, the corresponding topological projections on real and complex subspaces are normed subspaces. It is shown that the topological space can be equipped with two varieties of finite translations, which form the commutative composite translations in a combination. The composite translations, along with addition and multiplication operations, support the existence of a group structure in the topological space if the identity element is located on the real plane and its translation is restricted in the plane. The associativity of group operation requires the invariant complex translation of a point at first order. The projective subspaces admit standard additive and multiplicative groups depending on the type of projective spaces. There is a specific algebraic combination of complex translation and multiplication operations in the space, which prepares an associative complex subspace containing the real origin. The projective topological spaces are continuous, however, the compactness of projective subspaces varies. The construction and analysis consider that the complex subspace is continuous and free from any poles. The geometric viewpoints of the proposed topological space may find applications in several areas, such as digital imaging, shape analysis in computational fluid dynamics and digital geometric modeling.

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