Theoretical analysis of the 2D thermal cloaking problem

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Abstract. Coefficient inverse problems for the model of heat scattering with variable coefficients arising when developing technologies of design of thermal cloaking devices are considered. By the optimization method, these problems are reduced to respective control problems. The material parameters (radial and azimuthal conductivities) of the inhomogeneous anisotropic medium, filling the thermal cloak, play the role of control. The model of heat scattering acts as a functional restriction. A unique solvability of direct heat scattering problem in the Sobolev space is proved and the new estimates of solutions are established. Using these results, the solvability of control problem is proved and the optimality system is derived. Based on analysis of optimality system, the stability estimates of optimal solutions are established and efficient numerical algorithm of solving thermal cloaking problems is proposed.

1. Introduction

In recent years much attention has been paid to development of theoretical and experimental studies of problems of thermal cloaking of material bodies (see, e.g., [1-6] and references therein). The experimental studies in this domain are aimed at development of ways of manipulating heat fluxes using special new materials created for this purpose [1-3]. Theoretical analysis of thermal cloaking problems under some simplifying assumptions was performed in a number of papers and in particular in [5, 6]. The method proposed in [5, 6] is based on finding the exact solution of the respective direct heat transfer problem using the Fourier method and analyzing its dependence on the medium parameters. One should emphasize that this method is applicable only under fulfillment of severe constraints on the data, providing finding of exact or approximate solution of the heat transfer direct problem. In the general case, the study of inverse problems under consideration is possible only using efficient numerical algorithms based on previous theoretical analysis of respective inverse problems.

In this paper, we have the purpose to formulate exact statements of inverse problems for the stationary heat transfer model arising when designing thermal cloaking devices and to perform their theoretical analysis based on the optimization method of solving cloaking problems proposed in [7-12]. One can read about optimization method in detail in [13-15]. We mention papers [16, 17] and [18] where the optimization method is applied for solving related inverse problems arising while studying heat processes in Earth’s mantle or fluid flow processes in microturbine’s nozzles. We also mention papers [19, 20] devoted to studying invisibility cloaking problems in X-ray tomography.
2. Statement of direct scattering problem

As in [12], we begin with the introduction of externally applied thermal field $T_0$. Considering for the simplicity the case of two dimensions, we assume that applied field $T_0$ is created in rectangle $D = \{(x, y) : x < x_0, \ y < y_0\}$ (see figure 1a) by two vertical boundaries $\Gamma_1 : x = -x_0$ and $\Gamma_3 : x = x_0$ which are kept at temperatures $T_1$ and $T_3 < T_1$, respectively, while the top and bottom boundaries $\Gamma_2$ and $\Gamma_4$ are thermally insulated. Then applied field $T_0$ satisfies equation $k_0 \Delta T_0 = 0$ in $D$ and following boundary conditions:

$$T_0 \mid_{x=-x_0} = T_1, \quad T_0 \mid_{y=y_0} = T_3, \quad \partial T_0 / \partial y \mid_{y=\pm y_0} = 0.$$  \hfill (1)

Here $k_0$ is a thermal conductivity of homogeneous isotropic medium (background) filling $D$.

![Figure 1. The geometry of the problem: a – without cloak; b – for the case when cloak is placed in $D$.](image)

We assume further that material shell $(\Omega, k)$, which has the form of the ring $a < \rho < b$ in polar coordinates $(\rho, \theta)$ (see figure 1b) filled by anisotropic medium that is characterized by conductivity tensor $k$, is placed into $D$. Placing shell $(\Omega, k)$ into $D$ leads to appearance of the scattered thermal field in $\Omega \setminus D$, which can be determined by solving the direct thermal scattering problem. It consists in finding a triple of functions: $T_i$ in interior $\Omega_i$ of $\Omega$, $T_m$ - in $\Omega$ and $T_e$ - in exterior $\Omega_e$ of $\Omega$, satisfying equations

$$k_0 \Delta T_i = 0 \text{ in } \Omega_i, \quad \text{div}(k \text{grad} T_m) = 0 \text{ in } \Omega, \quad k_0 \Delta T_e = 0 \text{ in } \Omega_e, \quad \hfill (2)$$

and continuity conditions on internal $\Gamma_i$ and external $\Gamma_e$ components of boundary $\Gamma$ of $\Omega$. They have form

$$T_i = T_0, \quad k_0 \frac{\partial T_i}{\partial r} = k \frac{\partial T_m}{\partial r} \text{ at } r = a, \quad T_e = T_0, \quad k_0 \frac{\partial T_e}{\partial r} = k \frac{\partial T_m}{\partial r} \text{ at } r = b. \quad \hfill (3)$$

Let us define a weak solution of problem equations (1), (2) and (3). Preliminarily, we introduce as in [12] functional spaces $X \equiv H^1(D)$, $X_0 = \{\Phi \in X : \Phi \mid_{\Gamma_i} = \Phi \mid_{\Gamma_e} = 0\}$, $L^\infty(\Omega)$ and set $L^\infty_0(\Omega) := \{\lambda \in L^\infty(\Omega) : \lambda \geq \lambda_0 = \text{const} > 0\}$. We assume, similar to [6], that tensor $k$ is diagonal in
polar coordinates \((\rho, \theta)\) and its diagonal components are radial and azimuthal conductivities \(k_{\rho}\) and \(k_{\theta}\). Let us rewrite the second equation in (2) in polar coordinates as follows:

\[
\text{div}(k\text{grad}T) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho k_{\rho} \frac{\partial T}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \frac{k_{\theta}}{\rho} \frac{\partial T}{\partial \theta} \right). \tag{4}
\]

Now we multiply every equation in (2) by test function \(\Phi \in X_0\) integrate by part and add the results obtained. Using conditions in equation (3), we arrive at the following problem for finding triple \(X \in \Gamma \times X_0\):

\[
a(k;T,\Phi) := a_0(\rho;T,\Phi) + a_1(\lambda;T,\Phi) + a_2(\mu;T,\Phi) = 0 \quad \forall \Phi \in X_0, \quad T \big|_{\Gamma} = T_1, \quad T \big|_{\Gamma} = T_3. \tag{5}
\]

Here, \(k := (\lambda, \mu), \quad \lambda = k_{\rho}, \quad \mu = k_{\theta}, \quad a_0(\rho_0;\ldots), \quad a_1(\lambda;\ldots), \quad a_2(\mu;\ldots)\) are bilinear forms defined by

\[
a_0(\rho;T,\Phi) := \int_{\Omega} \nabla \cdot \nabla \Phi dx, \quad a_1(\lambda;T,\Phi) := \int_{\Omega} \left( \lambda \frac{\partial T}{\partial \rho} \frac{\partial \Phi}{\partial \rho} \right) d\rho d\theta, \tag{6}
\]

\[
a_2(\mu;T,\Phi) := \int_{\Omega} \left( \frac{\mu}{\rho^2} \frac{\partial T}{\partial \theta} \frac{\partial \Phi}{\partial \theta} \right) d\rho d\theta.
\]

We call triple \(T \in X\) satisfying equation (5), a weak solution of problem equations (2), (3). Based on the Lax-Milgram theorem, one can prove (see details in [12]) that under conditions \(k_{\rho} \in L^2(\Omega)\) and \(k_{\theta} \in L^2(\Omega)\) weak solution \(T = T(k_{\rho}, k_{\theta})\) of problem equations (1), (2), (3) exists. It is unique and, by maximum principle, the following estimate holds: \(T_3 \leq T(x, y) \leq T_1\) in \(D\). More precisely, the following theorem holds.

**Theorem 1.** Let \(T_1 \in H^{1/2}(\Gamma_1)\), \(T_3 \in H^{1/2}(\Gamma_3)\) be given functions. Then for any pair \(\lambda \in L^2(\Omega), \mu \in L^2(\Omega), \lambda_0 = \text{const} > 0, \mu_0 = \text{const} > 0\) direct problem equations (1), (2), (3) have unique weak solution \(T = (T_1, T_2, T_3) \in X\), which satisfies estimate

\[
\|T\|_X \leq C_{k,\mu} \left( \|T_1\|_{1/2, \Gamma_1} + \|T_3\|_{1/2, \Gamma_3} \right). \tag{7}
\]

If, moreover, \(\lambda \in K_1\) and \(\mu \in K_2\), where \(K_1 \subset H^{1/2}_0(\Omega), K_2 \subset H^{1/2}_0(\Omega)\) are bounded sets, then solution \(T\) satisfies estimate

\[
\|T\|_X \leq C_0 \left( \|T_1\|_{1/2, \Gamma_1} + \|T_3\|_{1/2, \Gamma_3} \right). \tag{7}
\]

Here, constant \(C_0\) does not depend on \(k = (\lambda, \mu)\).

3. **Statement of the inverse scattering problem. Using the optimization method.** **Main results**

We recall that our purpose is analysis of inverse problems arising, when developing technologies of creating thermal cloaking devices for manipulating heat flows. For this purpose, we apply the optimization method [13-15]. This approach is based on introducing the cost functional to be minimized, which adequately corresponds to the inverse problem of constructing the device for approximate cloaking. As a result, the initial cloaking problem is reduced to study of the respective control problem using the well known methods of extremum problems. As a cost functional we choose the following:
\[ I_i(T) = \left\| T - T^d \right\|^2_{L^2(Q)} = \int_Q (T - T^d)^2 \, dx. \]  

Here, function \( T^d \in L^2(Q) \) models the thermal field measured in some subset \( Q \subset D \) situated outside \( \Omega \). As controls we choose desired variable conductivity coefficients \( \lambda = k_\rho(x) \) and \( \mu = k_\sigma(x) \) of anisotropic inhomogeneous medium filling domain \( \Omega \).

In the particular case, when \( Q \subset \Omega^c \) and \( T^d = T_c \), functional \( I_i(T) \) has the sense of the \( L^2(\Omega) \) norm of external scattered thermal response \( T^{sc} \) in set \( Q \). Therefore, parameter \( k^{opt} = (\lambda^{opt}, \mu^{opt}) \equiv (k_\rho^{opt}, k_\sigma^{opt}) \), owing to which the minimum of functional \( I_i(T) \) is achieved, corresponds to the thermal cloaking problem. In the general case, when \( T^d = T_c + T_1^{sc} \) where \( T_1^{sc} \) is a thermal response of a certain object, minimizer \( k^{opt} \) of cost functional \( I_1 \) describes an approximate solution of the respective thermal illusion problem [12]. Finally, when norm \( \left\| T^d \right\|_Q \) is great enough, minimizer \( k^{opt} \) of functional \( I_i(T) \) corresponds to the problem of designing a heat concentrator [12].

We consider the following control problem:

\[ J(T, k) := \frac{\alpha_0}{2} I_0(T) + \frac{\alpha_1}{2} \| \lambda \|_{L^2(\Omega)}^2 + \frac{\alpha_2}{2} \| \mu \|_{L^2(\Omega)}^2 \rightarrow \inf, \quad G(T, k) = 0. \]  

Here, \( I(T) \) is a cost functional, \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are nonnegative parameters which serve to regulate the relative importance of each of the terms in equation (9), \( G(\cdot, k) \equiv G(\cdot, \lambda, \mu) : X \rightarrow X^*_\lambda \) is the operator of our direct problem, \( Z_{ad} = \{(T, \lambda, \mu) \in X \times K_1 \times K_2 : G(T, \lambda, \mu) = 0, \; I(T) < \infty \} \). It is assumed that controls \( \lambda \) and \( \mu \) can change in sets \( K_1, K_2 \) and the following condition holds:

(j) \( K_1 \subset H^s_{\lambda_0}(\Omega), \; K_2 \subset H^s_{\mu_0}(\Omega) \) are nonempty convex closed sets, where \( s > 3/2 \), \( \lambda_0 = \text{const} > 0 \), \( \mu_0 = \text{const} > 0 \); \( \alpha_0 > 0 \).

We apply the mathematical procedure, developed in [7-12], for studying control problems arising when optimization method is applied for solving cloaking problems. Based on this procedure, we can prove the following results.

**Theorem 2.** Let, under assumption (j), \( I : X \rightarrow R \) be a weakly lower semicontinuous functional and \( Z_{ad} \) be a nonempty set. Let \( \alpha_1 \geq 0, \; \alpha_2 \geq 0 \) and \( K_1, K_2 \) be bounded sets or \( \alpha_1 > 0, \; \alpha_2 > 0 \) and functional \( I(T) \) is bounded below. Then, control problem (9) has at least one solution \( (T, \lambda, \mu) \in X \times K_1 \times K_2 \).

**Theorem 3.** Let, under assumption (j), \( \alpha_1 > 0, \; \alpha_2 > 0 \) or \( \alpha_1 \geq 0, \; \alpha_2 \geq 0 \) and \( K_1, K_2 \) be bounded sets. Then control problem (9) for \( I = I_i(T) \) has at least one solution \( (T, \lambda, \mu) \in X \times K_1 \times K_2 \).

**Theorem 4.** Let, under assumption (j), pair \( (\hat{T}, \hat{k}) = (\hat{T}, \hat{\lambda}, \hat{\mu}) \in X \times K_1 \times K_2 \) be a solution of problem (9), where \( I = I_i(T) \). Then there exists unique Lagrange multiplier \( (R, \xi_1, \xi_2) \in X_\lambda^{1/2}(\Gamma_1)^* \times X^{1/2}(\Gamma_3)^* \) which is the solution of the adjoint problem in the form of the Euler-Lagrange equation:

\[ a(\hat{k}; \Psi, R) + \langle \xi_1, \Psi \rangle_{\Gamma_1} + \langle \xi_2, \Psi \rangle_{\Gamma_2} = -\alpha_0 (\hat{T}^d, \Psi) \quad \forall \Psi \in X \]  

and the following variational inequalities hold:
\[ \alpha_1(\hat{\lambda}, \lambda - \hat{\lambda})_{x,\Omega} + \alpha_2((\lambda - \hat{\lambda})\hat{\mathcal{T}}, R) \geq 0 \quad \forall \lambda \in K_1, \]

\[ \alpha_3(\hat{\mu}, \mu - \hat{\mu})_{x,\Omega} + \alpha_2((\mu - \hat{\mu})\hat{\mathcal{T}}, R) \geq 0 \quad \forall \mu \in K_2. \]

Direct problem equation (5), the Euler-Lagrange equation (10) which has the sense of the adjoint state \((R, \xi_1, \xi_2)\) and variational inequalities (11), (12) with respect to controls \(\hat{\lambda}, \hat{\mu}\) constitute the optimality system for control problem (9). Based on analysis of the optimality system, one can establish sufficient conditions on the data which provide the uniqueness and stability of solutions of particular control problem for \(I = I_1(T)\). More precisely, the following theorem holds.

**Theorem 5.** Let, in addition to assumption (j), \(K_1\) and \(K_2\) be bounded sets and let the triples \((T_k, \lambda_k, \mu_k)\) be solutions of control problem (9) for \(I = I_1(T)\) corresponding to given functions \(T_k^d \in L^2(Q), k = 1, 2\). Let the following conditions take place:

\[ \alpha_1(1 - \varepsilon) > 4 \alpha_0 C_0^2 M_T^0 M_T, \quad \alpha_2(1 - \varepsilon) > 4 \alpha_0 C_0^2 M_T^0 M_T, \quad \varepsilon \in (0, 1) \]

where \(\varepsilon\) is an arbitrary number,

\[ M_T := C_0 \left\{ \left\| T_1 \right\|_{L^2(\Omega_1)} + \left\| T_2 \right\|_{L^2(\Omega_2)} \right\}, \quad M_T^0 := M_T + \max \left\{ \left\| T_1^d \right\|_Q, \left\| T_2^d \right\|_Q \right\}. \]

Then the following stability estimates hold:

\[ \left\| T_1 - T_2 \right\|_Q \leq \left\| T_1^d - T_2^d \right\|_Q, \quad \left\| T_1 - T_2 \right\|_X \leq C_0 M_T \left( \sqrt{\alpha_0 / \varepsilon \alpha_1} + \sqrt{\alpha_0 / \varepsilon \alpha_2} \right) \left\| T_1^d - T_2^d \right\|_Q, \]

\[ \left\| \lambda_1 - \lambda_2 \right\|_{x,\Omega} \leq \sqrt{\alpha_0 / \varepsilon \alpha_1} \left\| T_1^d - T_2^d \right\|_Q, \quad \left\| \mu_1 - \mu_2 \right\|_{x,\Omega} \leq \sqrt{\alpha_0 / \varepsilon \alpha_2} \left\| T_1^d - T_2^d \right\|_Q. \]

**4. Numerical algorithms. Concluding remarks**

The optimality system having the form of equations (5), (10), (11), (12), derived above, can be used to design efficient numerical algorithms for solving control problem (9). The simplest one for \(I_1(T)\) can be obtained by applying the simple iteration method for solving the optimality system. The \(m\)-th iteration of this algorithm consists of finding unknown values \(T^m, \lambda^m, \mu^m\) and \(k^m = (\lambda^m, \mu^m), m = 0, 1, 2\ldots\) beginning with given initial values \(\lambda^0, \mu^0\) by sequentially solving following problems:

\[ a_0(k^m, T^m, \Phi) = 0 \quad \forall \Phi \in X_0, \quad a_0(k^m, \Psi, R^m) = -a_0(T^m - T^d, \Psi) \quad \forall \Psi \in X_0, \]

\[ \alpha_1(\lambda^m - \lambda, \lambda - \lambda^m)_{x,\Omega} + \alpha_2((\lambda - \lambda^m)T^m, R^m) \geq 0 \quad \forall \lambda \in K_1, \]

\[ \alpha_3(\mu^m - \mu, \mu - \mu^m)_{x,\Omega} + \alpha_2((\mu - \mu^m)T^m, R^m) \geq 0 \quad \forall \mu \in K_2. \]

For discretization and solving variational problems (17), one can use open source software freeFEM++ (www.freefem.org) based on using the finite element method. For discretization of variational inequalities (18), (19), it is comfortable to look for solutions \(\hat{\lambda}\) and \(\hat{\mu}\) as

\[ \lambda(x) = \sum_{j=1}^N \lambda_j \varphi_j(x), \quad \mu(x) = \sum_{k=1}^N \mu_k \varphi_k(x), \quad x \in \Omega. \]

Here, \(N\) is an integer, \(\varphi_j(x) \in H^s(\Omega)\) are nonnegative basis functions in \(H^s(\Omega)\) and \(\lambda_j \geq 0\) and \(\mu_k \geq 0\) are unknown coefficients.
5. Conclusion
In conclusion, we studied control problems for the 2D heat transfer model. These problems arise when optimization method is applied for solving cloaking problems for a respective thermal scattering model. Radial and azimuthal conductivities $\rho$ and $\theta$ of the inhomogeneous medium, filling the cloaking shell, play the role of controls. We studied some new properties of solutions of direct heat scattering problem, proved the solvability of control problems and derived the optimality system describing the necessary conditions of extremum. Basing on analysis of the optimality system, we established the uniqueness and stability estimates of optimal solutions. Besides, we proposed a numerical algorithm for solving our cloaking problem. We plan to devote a forthcoming paper to studying the properties of the algorithm and to analysis of results of numerical experiments performed using this algorithm.

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