DECOMPOSITION OF EUCLIDEAN NEARLY KÄHLER
SUBMANIFOLDS

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Abstract. We study the foliation space of complex and invariant (by torsion of intrinsic Hermitian connection) umbilic distribution on an isometric immersion from a nearly Kähler manifold $M$ into the Euclidean space. Under suitable conditions this leaf space is nearly Kähler and $M$ can be decomposed into a product of this leaf space and a 6-dimensional locally homogeneous nearly Kähler manifold.

1. Introduction

Gray in his study of weak holonomy groups in 70’s introduced nearly Kähler manifolds and obtained certain relations for the Riemannian curvature operator of such manifolds. These identities are slightly more complicated but resembling the corresponding formulas for the Riemannian curvature operator of Kähler manifolds. By using these identities he got many interesting results about the geometry and topology of these manifolds[8, 9, 10]. He also found a large class of homogeneous nearly Kähler manifolds using 3-symmetric space, introduced by Gray and Wolf [20, 21]. A 3-symmetric (semi)-Riemannian space is naturally reducible if and only if it is nearly Kähler with the canonical almost complex structure [9]. Gray propounded this conjecture “every nearly Kähler homogenous manifold is a 3-symmetric space equipped with its canonical almost complex structure”.

A nearly Kähler manifold can be described as an almost Hermitian manifold such that the torsion of intrinsic Hermitian connection is totally skew-symmetric [8]. In this point of view, nearly Kähler manifolds are interesting objects in string theory [6]. In 2002, Nagy proved that every simply connected complete nearly Kähler manifold is isometric to a Riemannian product space $M_1 \times \cdots \times M_k$ where $M_i$’s are nearly Kähler and belong to the following list:

(i) naturally reduce 3-symmetric spaces (these homogeneous spaces are divided into four separated classes [21]),
(ii) twistor spaces over positive Kähler-quaternion manifolds,
(iii) six dimensional nearly Kähler manifolds.

By this decomposition, Gray conjecture converts to this question: “Is it true that every 6-dimensional, complete, homogeneous nearly Kähler manifold is a 3-symmetric space?” Butruille in 2008 [2] showed that there exist only four complete, homogeneous 6-dimensional nearly Kähler manifolds (up to homothety and covering space) and all of them are 3-symmetric:

\[ S^6 = \frac{G_2}{SU(3)}, \quad S^3 \times S^3 = \frac{SU(2) \times SU(2)}{\langle 1 \rangle}, \]

\[ \mathbb{CP}^3 = \frac{Sp(2)}{SU(2) \cdot U(1)}, \quad \mathbb{RP}^3 = \frac{SU(3)}{U(1) \times U(1)}. \]

The only known examples of 6-dimensional nearly Kähler manifolds are above four manifolds. An important question in nearly Kähler geometry is the fundamental explanation of lack of such manifolds or difficulties of introducing non-homogeneous examples. This question can be formulated in Butruille conjecture “every complete nearly Kähler manifold is a 3-symmetric space”.

This conjecture can be separated by Nagy’s decomposition into two conjectures:

**Problem 1.1.** Every complete (compact) 6-dimensional nearly Kähler manifold is homogenous.

**Problem 1.2.** Every positive quaternion-Kähler manifold is a Wolf space.

These statements motivated us to study isometric immersions \( f : M^{2n} \rightarrow \mathbb{Q}^{2n+p} \) from a nearly Kähler manifold into a space form (especially the Euclidean space). We introduced in [12] an umbilic distribution which is complex and invariant by the torsion of intrinsic Hermitian connection and showed that this distribution is integrable and each leaf of the generated foliation is a 6-dimensional locally homogeneous nearly Kähler submanifold. In [13] by description of this foliation, we parameterized the isometric immersion \( f \) and produced some examples of nearly Kähler submanifolds in the standard space forms with arbitrary co-dimension \( p \).

In this article, we study the leaf space of the umbilic distribution which is complex and invariant by the torsion of intrinsic Hermitian connection. We investigate the existence of decompositions like that of Nagy in the Euclidean submanifolds case. We show that under suitable conditions there is a product decomposition such that the 6-dimensional nearly Kähler terms are (locally) homogeneous.

2. Preliminaries

A smooth manifold \( M \) is called almost complex if there exists \((1,1)\) tensor field \( J \) on \( M \) such that \( J^2 = -Id \). A Riemannian manifold \((M,g)\) with almost complex
structure $J$ is called almost Hermitian if $g(JX, JY) = g(X, Y)$ for all vector fields $X$ and $Y$ on $M$.

Gray and Hervella [11] classified almost Hermitian manifolds into sixteen classes. One of the most important classes in this classification is the class of Kähler manifolds. An almost Hermitian manifold $(M, g, J)$ is a Kähler manifold if $\nabla J = 0$ where $\nabla$ is the Levi-Civita connection of $g$ and nearly Kähler if $(\nabla_X J)X = 0$ for all vector field $X$ on $M$. Every Kähler manifold is nearly Kähler but the converse is not true. Non-Kähler nearly Kähler manifolds are called strictly nearly Kähler. There is no strictly nearly Kähler manifold in dimension less than six.

the canonical Hermitian connection on an almost Hermitian manifold defined by

$$\bar{\nabla}XY = \nabla_XY + \frac{1}{2}(\nabla_X J)Y$$

It's easy to see that $\bar{\nabla}$ is the unique linear connection on $M$ such that $\bar{\nabla}g = 0$ (it is a metric connection) and $\bar{\nabla}J = 0$ (it is a Hermitian connection).

**Proposition 2.1.** [2] Let $M$ be an almost Hermitian manifold. The following conditions are equivalent and define a nearly Kähler manifold:

1. The torsion $T(X, Y) = (\nabla_X J)Y$ of $\bar{\nabla}$ is totally skew-symmetric (equivalently, the tensor $T(X, Y, Z) = g(T(X, Y), Z)$ is skew-symmetric),
2. $(\nabla_X J)X = 0$ for all $X \in TM$,
3. $\nabla_X \omega = \frac{1}{2} i_X d\omega$ for all $X \in TM$ where $\omega(X, Y) = g(X, JY)$ is the Kähler 2-form on $M$,
4. $d\omega$ is of type $(0, 3) + (3, 0)$ and the Nijenhuis tensor $N$ is totally skew-symmetric.

**Proposition 2.2.** [4] For a nearly Kähler manifold, The torsion of the intrinsic Hermitian connection is totally skew-symmetric and parallel, that is $\bar{\nabla}\eta = 0$, where $\eta(X) = \frac{1}{2} J \circ (\nabla_X J)$. This is equivalent to $\bar{\nabla}\bar{\nabla}\omega = 0$ or $\bar{\nabla}d\omega = 0$.

**Lemma 2.3.** [8] For a nearly Kähler manifold $(M, g, J)$,

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \quad (\nabla_{JX} J)Y = (\nabla_X J)Y,$$

$$J(\nabla_X J)Y = -(\nabla_X J)JY = -(\nabla_{JX} J)Y, \quad g(\nabla_X Y, X) = g(\nabla_X JY, JX),$$

$$2g((\nabla_{W,X} J)Y, Z) = -\sigma_{X,Y,Z}g((\nabla_{JW} J)X, (\nabla_{JY} J)Z).$$

Gray used the following relations between the torsion of the intrinsic Hermitian connection and Riemannian curvature of nearly Kähler manifolds [10]. These formulas resemble the corresponding formulas for Kähler manifolds.
\begin{align*}
\langle R_{X,Y}X, Y \rangle - \langle R_{X,Y}JX, JY \rangle &= \|\nabla_X J Y\|^2 \\
\langle RW,X,Y,Z \rangle - \langle RW,X JY, JZ \rangle &= \langle (\nabla_W J X, (\nabla_Y J) Z \rangle \\
\langle RW,X,Y,Z \rangle &= \langle R_{JW,JX}JY, JZ \rangle \\
2g((\nabla_{W,X}^2 J Y, Z) &= \sigma_{X,Y,Z} g(R_{W,X} Y, Z).
\end{align*}

The star version of Ricci tensor of metric $g$ is defined by
\begin{equation*}
\langle Ric^*(X), Y \rangle = \frac{1}{2} \sum_{i=1}^{n} R(X, JY, e_i, Je_i),
\end{equation*}
where $R$ is the Riemannian curvature of $(M, g)$ and $\{e_i\}$ is a local frame field.

The difference tensor $r$ between $Ric^*$ and Ricci tensor is described by the following formula [16]:
\begin{equation*}
\langle r_{X,Y} \rangle = \sum_{i=1}^{n} \langle (\nabla e_i J) X, (\nabla e_i J) Y \rangle.
\end{equation*}

It is easy to see that $r$ is symmetric, positive and commutes with $J$. The tensor $r$ has strong geometric properties, e.g., Gray in [10] proved that
\begin{equation*}
2\langle (\nabla r)_{Y, Z} \rangle = \langle r(Y, JZ) \rangle + \langle r(Y), (\nabla X J) Z \rangle
\end{equation*}
in other words, $r$ is $\nabla$-parallel (i.e $\nabla r = 0$).

A nearly Kähler manifold is strictly nearly Kähler when the kernel of $r$ vanishes. This is equivalent to triviality of the distribution $x \mapsto \{-X \in T_x M | T(X, Y) = 0, \forall Y \in T_x M\}$.

**Proposition 2.4.** [16] Let $(M, g, J)$ be a complete nearly Kähler manifold. Then $M$ can be decomposed as a Riemannain product $M_1 \times M_2$ where $M_1$ is a Kähler manifold and $M_2$ is a strictly nearly Kähler manifold.

On nearly Kähler manifold, tensors
\begin{equation*}
A(X, Y, Z) = \langle (\nabla X J) Y, Z \rangle, \quad B(X, Y, Z) = \langle (\nabla X J) Y, JZ \rangle
\end{equation*}
are skew-symmetric and have type $(3,0) + (0,3)$ as (real) 3-forms.

We need the following classical relation between the covariant derivative of the almost complex structure $J$ and its Nijenhuis tensor $N$ which is proved by a straightforward computation using
\begin{equation*}
4N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]
\end{equation*}
and the anti-symmetry of the above tensors $A$ and $B$. 
Lemma 2.5. For every nearly Kähler manifold \((M, g, J)\) we have
\[ N(X, Y) = J(\nabla_X J)Y. \]

In lower dimensions, nearly Kähler manifolds are mainly determined. If \(M\) is a nearly Kähler with \(\dim M \leq 4\), then \(M\) is Kähler. If \(\dim M = 6\), then we have the following result.

Proposition 2.6. \([8, 10, 19]\) Let \((M, g, J)\) be a 6-dimensional, strict, nearly Kähler manifold. Then

1. \(\nabla J\) has constant type, that is
   \[\|((\nabla_X J)Y)\|^2 = \frac{S}{30}(\|X\|^2\|Y\|^2 - g(X, Y)^2 - g(JX, Y)^2),\]
2. the first Chern class of \((M, J)\) vanishes,
3. \(M\) is an Einstein manifold with
   \[\text{Ricc} = \frac{S}{6}g, \quad \text{Ricc}^* = \frac{S}{30}g.\]

Moreover if the tensor \(\nabla J\) has constant type \(\alpha\) then \(\dim M = 6\) and \(\alpha = \frac{S}{30}\) where \(S\) is the scalar curvature.

The next lemma follows immediately.

Lemma 2.7. For vector fields \(W, X, Y\) and \(Z\) we have
\[ g(\nabla_W J)X, (\nabla_Y J)Z) = \frac{S}{30}(g(W, Y)g(X, Z) - g(W, Z)g(X, Y) - g(W, JY)g(X, JZ) + g(W, JZ)g(X, JY)), \]
and
\[ g((\nabla_W \nabla_Z J)X, Y) = \frac{S}{30}\{g(W, Z)g(JX, Y) - g(W, X)g(JZ, Y) + g(W, Y)g(JZ, X)\}, \]
also
\[ \Sigma g(Je_i, e_j)R(e_i, e_j, X, Y) = -\frac{S}{15}g(Jx, Y), \]
\[ \Sigma g((\nabla_X J)e_i, e_j)R(e_i, e_j, Y, Z) = -\frac{S}{30}g((\nabla_X J)Y, Z), \]
where \(\{e_i\}\) is a local orthonormal frame field on \(M\).

Lemma 2.8. \([14]\) Let \(X, Y\) be two vector fields on \(M\) then \(T(X, Y)\) is an orthogonal vector field to \(X, JX, Y\) and \(JY\).

This lemma leads us to define a suitable local frame on a 6-dimensional nearly Kähler manifold which is particularly convenient for local calculations. Suppose that \(e_1, e_2\) be two orthogonal local vector fields on \(M\) and define
\[ e_3 = T(e_1, e_2) = (\nabla e_1, J)e_2 \quad e_4 = Je_1 \quad e_5 = Je_2 \quad e_6 = Je_3 \]
Therefore \(\{e_i, Je_i\}_{i=1,\ldots,3}\) is a local orthogonal frame on \(M\). \([14]\)
**Definition 2.9.** [12] Let \( f : M^{2n}(\langle \rangle, J) \rightarrow Q_c^{2n+p} \) be an isometric immersion from a nearly Kähler manifold into a space form with second fundamental form \( \alpha \) and \( 0 \neq \eta \in T^\perp fM \) is a non-zero normal vector field on \( M \). The umblic distribution of \( f \) defined by \( x \mapsto \Delta_x \) where

\[
\Delta_x = \{ X \in T_x M \mid \forall Y \in T_x M \; \alpha(X, Y) = \langle X, Y \rangle \eta \}
\]

complexification of this distribution is described by \( \Delta_x \cap \Delta'_x = \Delta_x \cap J\Delta_x \) where

\[
\Delta'_x = \{ X \in T_x M \mid \forall Y \in T_x M \; \alpha(JX, Y) + \alpha(X, JY) = 0 \}
\]

Now we put

\[
\Delta''_x = \{ X \in T_x M \mid \forall Y \in T_x M \; \alpha(T(X, Y), Z) + \alpha(X, T(Y, Z)) = 0 \}
\]

and define by \( D_x = \Delta_x \cap \Delta'_x \cap \Delta''_x \) the umblic distribution which is complex and invariant by the torsion of intrinsic Hermitian connection.

It is easy to see that

\[
D_x = \{ X \in T_x M \mid X, JX \in \Delta_x, \forall Y \in T_x M \; T(X, Y) \in \Delta_x \}
\]

**Theorem 2.10.** [12] Let \( f : M^{2n} \rightarrow Q_c^{2n+p} \) be an isometric immersion from a complete, simply connected strictly nearly Kähler manifold into a space form of constant curvature \( c \), then there is an involute umbilic complex foliation on \( M \) invariant by the torsion of the intrinsic Hermitian connection whose leaves are 6-nearly Kähler locally homogeneous manifolds (each leaf is an Amrose-Singer manifold). Moreover, each leaf coincides with a 6-dimensional nearly Kähler factor appearing in the Nagy decomposition.

**Remark 2.11.** In proposition 3.3 we described the structure of umblic distribution which is complex and invariant by the torsion of intrinsic Hermitian connection and we got \( 0 \neq \eta \in T^\perp fM \). Also we proved that in Nagy decomposition, a 6-dimensional term appears if and only if there exists \( 0 \neq \eta \in T^\perp fM \) such that \( D_x \) is a non-zero distribution. In the next section we prove more properties of \( \eta \) and leaves of the foliation generated by \( D \).

### 3. Main results

**Theorem 3.1.** Each leaf \( N \) of the complex and invariant umbilic foliation in \( M \) is minimal and the second fundamental form \( \beta \) of \( N \) satisfies \( \beta(X, JY) = J\beta(X, Y) \). Also \( \eta = 3H \) where \( H \) is the mean curvature of \( N^6 \hookrightarrow M^{2n} \rightarrow Q_c^{2n+p} \) and \( c + ||\eta|| = \frac{S}{6} \) where \( S \) is the scalar curvature of \( N \) which is constant.

**Proof.** Let \( \beta \) be the second fundamental form \( N^6 \hookrightarrow M^{2n} \). The distribution \( D \) is invariant under torsion of intrinsic Hermitian connection therefore

\[
T(X, Y) = (\nabla_X J)Y \in T_x N \quad (X, Y \in T_x N)
\]
and

\[(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y = \nabla'_X JY - J\nabla'_X Y + \beta(X, JY) - J\beta(X, Y) \in T_x N\]

where \(\nabla'\) is the Levi-Civita connection on \(N\). Therefore \(\beta(X, JY) - J\beta(X, Y) = 0\).

If \(\{e_i, Je_i\}\) is an orthonormal local frame on \(N\) then

\[H = \sum_{i=1}^{3} \beta(e_i, e_i) + \beta(Je_i, Je_i) = \sum_{i=1}^{3} \beta(e_i, e_i) - \beta(e_i, e_i) = 0\]

where \(H\) is the mean curvature of \(N^6 \hookrightarrow M^{2n}\). Therefore \(N\) is minimal in \(M\).

We show that \(N\) has constant type \(c + \langle \eta, \eta \rangle\). Using formulas in lemma 2.3 and Guass equation for submanifold \(M\) in the space form \(Q\) we have

\[\|T^N(X, Y)\|^2 = \|T^M(X, Y)\|^2 = \|\nabla_X JY\|^2 = -\langle R_{X,Y}X, Y \rangle + \langle R_{X,Y}JX, JY \rangle = \langle \alpha(X,Y), \alpha(Y,X) \rangle - \langle \alpha(X,X), \alpha(Y,Y) \rangle \]

\[c\langle X,Y \rangle \langle X,Y \rangle - \langle X,X \rangle \langle Y,Y \rangle - \langle \alpha(X,JY), \alpha(Y,X) \rangle + \langle \alpha(X,X), \alpha(Y,JY) \rangle \]

\[c\langle X,JY \rangle \langle X,JX \rangle - \langle X,JX \rangle \langle JY,JX \rangle \]

\[= (\|\eta\|^2 + c)(\langle X,Y \rangle^2 + \langle X,X \rangle \langle Y,Y \rangle + \langle X,JY \rangle \langle JX \rangle) \]

\[= (\|\eta\|^2 + c)(\|X\|^2 \|Y\|^2 - \langle X,Y \rangle^2 - \langle JX,JY \rangle^2),\]

for all \(X, Y \in \mathcal{X}(N)\). Hence by Proposition 2.6 \(N\) is 6-dimensional manifold and \(c + \|\eta\|^2 = \frac{S}{36}\) where \(S\) is the scalar curvature of \(N\). Also \(\eta\) has constant length because \(N\) is Einstein. By the definition of the tangent bundle \(TN\) at each point we have \(H = \eta\) where \(H\) is the mean curvature vector field of \(N\) as a submanifold of \(Q\).  

A complex and invariant umbilic foliation may be trivial. But in the next proposition we show that if a 6-dimensional factor in the Nagy decomposition appears then there exist \(\eta \in \Gamma(T^4_j M)\) such that the complex and invariant umbilic foliation defined by \(\eta\) is non-trivial. Proof of this claim needs next lemma.

**Lemma 3.2.** Let \((M,g,J)\) be a nearly Kähler manifold and \(N\) be an almost Hermitian embedded submanifold of \(M\) with the second fundamental form \(\beta\), then for all vector fields \(X, Y\) on \(N\) we have \(\beta(X,JY) = J\beta(X,Y)\).

**Proof.** Denote by \(T^M\) and \(T^N\) the torsion tensors of the canonical Hermitian connection on \(M\) and \(N\), respectively. We have

\[(3.1) \quad T^M(X,Y) = T^N(X,Y) + \beta(X,JY) - J\beta(X,Y) \quad (X,Y \in \mathcal{X}(N))\]
The torsion of the canonical Hermirion connection is skew-symmetric and $\beta$ is symmetric, hence
\[
J\beta(X,Y) - \beta(X,JY) = \beta(JX,Y) - J\beta(X,Y)
\]
\[
\Rightarrow 2J\beta(X,Y) = \beta(JX,Y) + \beta(X,JY).
\]
Therefore
\[
2J\beta(JX,JY) = -\beta(JX,Y) - \beta(X,JY) = -2J\beta(X,Y)
\]
that’s $\beta(JX,JY) = -\beta(X,Y)$.

Also $\beta(J(JX),JY) = -\beta(JX,Y)$, hence $\beta(X,JY) = \beta(JX,Y)$, and thus $J\beta(X,Y) = \beta(X,JY) = \beta(JX,Y)$.

□

There is a short proof of the above lemma when $N$ is nearly Kähler. By lemma 2.5 we have
\[
JT^M(X,Y) = NJ(X,Y) = JT^N(X,Y)
\]
and the result follows from relation 3.1.

**Proposition 3.3.** Let $f : (M^{2n}, g, J) \longrightarrow Q^{2n+p}_{\epsilon}$ be an isometric immersion from a nearly Kähler manifold into a space form. If $M$ has an embedded 6-dimensional strictly nearly Kähler submanifold $N$ then there exists $\eta \in \Gamma(T^\perp_f N)$ such that $N$ is locally isometric with a leaf of the complex and invariant umbilic foliation in direction $\eta$. In particular, $N$ is locally homogeneous.

**Proof.** Let $\beta$ be the second fundamental form of $N$ as a submanifold of $M$. By lemma 3.2 for all $X, Y \in \mathcal{X}(N)$ we have $\beta(X,Y) = J\beta(X,Y)$. Now consider $N$ as a submanifold of $Q^{2n+p}_{\epsilon}$ (by $N \longrightarrow M \longrightarrow Q$) and let $H$ be the mean curvature of $N$ in $Q$. We choose $\eta$ parallel to $H = \sum_{i=1}^3 \alpha(e_i, e_i) + \alpha(Je_i, Je_i)$ such that $c+||\eta||^2 = \frac{S}{30}$, where $\{e_i, Je_i\}_{i=1,2,3}$ is an adapted frame by lemma 2.5 for $N$, $S$ is the scalar curvature of $N$ and $\alpha$ is the second fundamental form of $M$ as a submanifold of $Q$ via $f$. Therefore $\eta \in \Gamma(T^\perp_f N)$ and for all $X, Y \in \mathcal{X}(N)$, $T^M(X,Y) = T^N(X,Y)$ where $T^N$ and $T^M$ are the torsion tensors of the canonical Hermitian connections on $N$ and $M$, respectively. Next, $\nabla J$ has type $\frac{S}{30}$ on $N$ so by a computation as in the proof of theorem 3.1 we have $\alpha(X,Y) = \langle X,Y \rangle \eta$, for all $X, Y \in \mathcal{X}(N)$. Now the result follows by theorem 2.10.

□

**Remark 3.4.** If there is a 6-dimensional factor $N$ in the Nagy decomposition of a nearly Kähler manifold $M$ then $N$ is an embedded 6-dimensional strictly nearly Kähler submanifold of $M$ and by following proposition there exist $\eta \in \Gamma(T^\perp_f N)$ such that the complex and invariant umbilic distribution in direction $\eta$ is non-trivial and $N$ is locally isometric with a leaf of the foliation and so it is locally homogeneous.
Proof. If a leaf $N$ of our foliation is Kähler, then $T^N(X, Y) = T^M(X, Y) = (\nabla_X J)Y$ and

$$\|\nabla_X J Y\|^2 = -\langle R_{X,Y}X,Y \rangle + \langle R_{X,Y}JX, JY \rangle$$

where $\alpha$ is the second fundamental form of $f$. Hence $\|\eta\|^2 + c = 0$. Now if $c = 0$ then the space form is isometric with the Euclidean space and $\eta = 0$ and $\alpha(X, Y) = 0$, for all $X, Y \in T_N(N)$. By using the above relation again, we have $T(X, Y) = 0$, for $X \in T_x N$, $Y \in T_x M$, which means that the tensor $r$ on $M$ has non-zero kernel and this is a contradiction, because $M$ is strictly nearly Kähler. If $c \neq 0$, we consider $N$ as a submanifold of $\mathbb{Q}$ (by $N \hookrightarrow M \longrightarrow \mathbb{Q}$). By the Guass equation and equality $\alpha(X, Y) = \langle Y, X \rangle \eta$, for $X, Y \in \mathcal{X}(N)$.

$$K(e_i, e_j) + K(e_i, Je_j) = \langle \beta(e_i, e_j), \beta(e_i, e_j) \rangle - \langle \beta(e_i, e_i), \beta(e_j, e_j) \rangle$$

(3.2) 

$$+ \langle \beta(e_i, Je_j), \beta(e_i, Je_j) \rangle - \langle \beta(e_i, e_i), \beta(Je_j, Je_j) \rangle = 2\|\beta(e_i, e_j)\|^2 \geq 0,$$

where $\beta$ is the second fundamental form of $N$ as a submanifold of $M$ and $\{e_i, Je_i\}_{i=1, \ldots, 3}$ is an adapted frame. By lemma 3.2 and similar computations as above, we have

$$K(e_i, Je_j) = K(Je_i, e_j), \quad K(e_i, e_j) = K(Je_i, Je_j), \quad K(e_i, Je_i) = \|\beta(e_i, e_i)\|^2 \geq 0.$$

Hence the scalar curvature $S$ of $N$ must be non-negative,

$$S = \sum_{i=1}^{3} K(e_i, e_j) + K(e_i, Je_j) + K(Je_i, Je_j) \geq 0,$$

But by theorem 7 in [5], the scalar curvature of an isometrically immersed Kähler submanifold of a space form satisfies the following inequality

$$S \leq 2(2n)^2(c + \|H\|^2),$$
hence the scalar curvature of $N$ must be non-positive by theorem 3.1 and lemma 3.2. Thus $S = 0$. Hence $\beta \equiv 0$ by (3.2) and $N$ must be locally isometric to an Euclidean affine space. This contradicts the completeness of $M$. □

An open connected subset $U \subseteq M$ is saturated if each leaf of the complex and invariant umbilic foliation in $U$ is maximal in $M$. If $M$ is complete we may put $U = M$. We consider the quotient space $V = U/D$ of the leaves in $U$ (each leaf is an equivalence class) with the projection map $\pi : U^{2n} \longrightarrow V^{2n-6} = U^{2n}/D$. In general, $V$ is not a manifold. It could fail to be Hausdorff and it may be a $V$ or $QF$-manifold. But if each leaf of the complex and invariant umbilic foliation in $U^{2n}$ is complete then $V$ becomes a manifold [13]. In this case, $V$ is called the foliation space of the complex and invariant umbilic foliation.

In [13] we equipped $V$ with a suitable metric and complex structure such that $V$ becomes a quasi-Kähler manifold and we used this in parametrization of Euclidean nearly Kähler submanifolds. In the next theorem we introduce a metric and an almost complex structure on $V$ such that it becomes a nearly Kähler manifold.

Remark 3.6. In the next theorem we will use Rummler-Sullivan criterion: there exist on a compact manifold $M$ a suitable metric such that the leaves defined on $M$ are minimal if and only if there exist an $m$-form $\chi$ positive on the leaves and relativity closed such that $d\chi(X_1, \ldots, X_m, Y) = 0$ where $X_1, \ldots, X_m$ are tangent to the foliation [17, 18].

Theorem 3.7. Let $f : M^{2n} \longrightarrow \mathbb{R}^{2n+p}$ be an isometric immersion from a nearly Kähler manifold into the Euclidean space. If each leaf of the complex and invariant umbilic foliation is complete then the leaf space is an almost complex manifold whose top cohomological group is non-trivial. If $V$ (the leaf space) is compact and $M$ is complete then by choosing a suitable metric on $V$, the projection map $\pi : M \longrightarrow V$ is a Riemannian submersion such that $\pi \circ J^M = J^V \circ M$, that is $\pi$ is an almost Hermitian submersion. In particular, $V$ is a nearly Kähler manifold.

Proof. Each leaf is minimal so if $\nu$ denotes the metric volume form of a leaf then it defines a non-trivial class in the basic cohomology $H(M/D = V)$. Indeed, let $\chi$ be the $m$-form given by Rummler-Sullivan criterion and suppose that $\nu = d\tau$ where $\tau \in \Omega^{2n-7}(V)$ then

$$\chi \wedge \nu = \chi \wedge d\tau = (-1)^m \{d(\chi \wedge \tau) - d\chi \wedge \tau\}$$

But $d\chi \wedge \tau = 0$ because $\chi$ is relativity closed. Therefore $\chi \wedge \nu$ is exact and this is a contradiction, because $\chi \wedge \nu$ is a volume form of the complete nearly Kähler manifold $M$ (note that $M$ is compact and orientable).

Now $V$ is invariant by the almost complex structure $J$ so it will be an almost
complex manifold. By the structure of the foliation space $V$ and using the projection map $\pi$ we may put a metric on $V$, inherited from metric from $M$, such that $\pi$ becomes a Riemannian submersion, preserving the almost complex structures, i.e., $\pi_* J^M = J^V \circ \pi_*$. Therefore $\pi$ is an almost Hermitian submersion. Finally we show that $V$ is a nearly Kähler manifold. If $X, Y$ are vector fields on $M$ which are $\pi$-related to $X', Y'$ on $V$ then
\[
\omega^M(X, Y) = g^M(X, JY) = g^V(X', JY') \circ \pi = (\pi^* \omega^V)(X, Y).
\]
Therefore, on the horizontal distribution $\omega^M$ and $d\omega^M$ coincide with $\pi^* \omega^V$ and $\pi^*(d\omega^V)$, respectively. Now $h((\nabla_X J)Y')$ is a vector field $\pi$-related to $(\nabla_{X'} J)Y'$ by proposition 3.5 of [4] where $h$ denote the horizontal component of a vector field that tangent to $M$, and we know that $\pi^*$ is a linear isomorphism on invariant differential forms so by proposition 2.4 we conclude that $V$ is a nearly Kähler manifold. □

Next we need some facts about Riemannain submersions:

**Definition 3.8.** Let $(M, g, J)$ and $(B, g', J')$ be almost Hermitian manifolds. A Riemannain submersion $\pi : M \rightarrow B$ is called an almost Hermitian submersion if it is an almost complex map i.e., $\pi_* J = J' \circ \pi_*$. An almost Hermitian submersion is a nearly Kähler submersion if the total space is nearly Kähler.

The tangent bundle of $M$ can be decomposed as the Whitney sum of vertical distribution $V = Ker \pi_*$ and the complementary orthogonal metric $H$ (the horizontal distribution). Denote by $v$ and $h$ the projection on vertical and horizontal distributions, respectively. The O'Neill tensors $B$ and $A$ are defined by
\[
B_X Y = h(\nabla_v X v Y) + v(\nabla_v X h Y),
A_X Y = v(\nabla_h X h Y) + h(\nabla_h X v Y).
\]
Here $B$ acts on each fiber (each leaf in our case) as the second fundamental form and $A$ measures the distance of horizontal distribution from integrability.

. With the same argument as in the proposition 1 in [4] we can prove the following.

**Theorem 3.9.** Let $f : M^{2n} \rightarrow \mathbb{R}^{2n+p}$ be an isometric immersion from a nearly Kähler manifold into the Euclidean space. the orthogonal complement of the complex and invariant umbilic distribution with respect to a Riemannian metric is an integrable distribution and each integral submanifold of this distribution is totally geodesic in $M$.

**Proof.** It is enough to show that the O’Neill tensor $A$ on $M$ vanishes. Let $X$ be a vector field and $V$ be a vertical field on $M$. Since
\[
(3.3) \quad h(\nabla_V X) = h(\nabla_X V) = A_X V
\]
we have $\nabla_V X = B_V X + A_X V$. Moreover, $B_V JX = JB_V X$ and we have $T(V, X) = \nabla_V JX - J\nabla_V X = A_X JV - JA_X V$. In [4] it was proved that for a quasi-Kähler submersion,

$$A_JX = -J \circ A_X = A_X \circ J.$$  

(3.4)

Thus

$$T(V, X) = \nabla_V JX - J\nabla_V X = A_JX V - JA_X V = -2JA_X V,$$

using (3.3) and (3.4) we have

$$h(T(V, X)) = h(\nabla_V JX - h(J\nabla_V X) = A_JX V - JA_X V = -2JA_X V,$$

but on nearly Kähler manifolds the torsion of intrinsic Hermitian connection $T$ is skew-symmetric that is, $T(X, V) = -T(V, X)$ and by (3.5) and (3.6) we conclude that $JA_X V = 0$, that is, $A = 0$. □

**Corollary 3.10.** Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be an isometric immersion from a complete, simply connected nearly Kähler manifold into the Euclidean space. There exist two integrable distributions $D, D^\perp$ on manifold $M$ such that leaves of the generated foliations by $D$ and $D^\perp$ are minimal and totally geodesic, respectively. Moreover, if for each $X \in D^\perp_x$ and $U \in D_x$ we have

$$R(X, JX, U, JU) = 0,$$

then locally $M$ is a product of the foliation space of complex and invariant umbilic foliation and a 6-dimensional homogeneous nearly Kähler manifold which is isometric with the corresponding factor in the Nagy decomposition.

**Proof.** This is proved like proposition 3.21 in [4]. □

**Remark 3.11.** Note that the 6-dimensional factor in the Nagy decomposition is not necessarily homogeneous, therefore this is a positive answer to problem 1.1 of the introduction under the assumption of Corollary 3.10.

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