A Mechanism for Ferrimagnetism and Incommensurability in One-Dimensional Systems

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In this paper I discuss a mechanism for ferrimagnetism in (1+1)-dimensions. The mechanism is related to a special class of interactions described by operators with non-zero Lorentz spin. Such operators are present in such problems as the problem of tunneling between Luttinger liquids and the problem of frustrated spin ladder. Exact solutions are presented for a representative class of models possessing a continuous isotopic symmetry. It is shown that the interactions (i) dynamically generate static oscillations with the wave vector dependent on the coupling constant, (ii) give rise to spontaneous breaking of this symmetry at $T = 0$ accompanied by generation of the magnetic moment and appearance of gapless modes with a non-relativistic (ferromagnetic) dispersion $E \sim k^2$, (iii) generate massive (roton) modes.

I. INTRODUCTION

Though general stability of critical points is determined by scaling dimensions of the perturbing operators, to determine the ultimate destination of the renormalization group (RG) flow is much more difficult problem. Availability of non-perturbative methods makes this task easier in (1+1)- and two dimensions.

In (1+1)- or two dimensions critical points possess conformal symmetry with the Lorentz (O(2)) symmetry being its part. If the relevant perturbation preserves the Lorentz symmetry, the most it can do is to open a gap in a part of the spectrum. That is exactly what happens in all known solvable examples (for instance, in the sine-Gordon model).

It may happen however, that the perturbation violates the Lorentz symmetry. Such perturbations certainly make sense in condensed matter physics where preservation of the Lorentz symmetry is not required. There are quite a few models of condensed matter physics which can be treated as critical models perturbed by relevant Lorentz-symmetry-breaking operators. These operators have non-zero Lorentz spin. Since in (1+1)- or two dimensions the Lorentz symmetry at criticality is extended to conformal symmetry, Lorentz spin is also called conformal spin. At criticality where all two-point correlation functions follow power law behaviour, an operator $\hat{O}(z, \bar{z})$ is characterized by its scaling dimension $d$ and conformal (or Lorentz) spin $S$ defined as

$$< \hat{O}(z_1, \bar{z}_1)\hat{O}(z_2, \bar{z}_2) > = A z_{12}^{-(d+S)} \bar{z}_{12}^{-(d-S)}$$

where $z, \bar{z}$ are holomorphic coordinates defined as $z = \tau + ix, \bar{z} = \tau - ix$ (in (1+1)-dimensions $\tau$ is Matsubara time). Since only bosonic operators can appear as perturbations, possible Lorentz spins are integer. Furthermore, since in unitary theories conformal dimensions are positive ($d \pm S \geq 0$) and $d \leq 2$ for relevant operators, this leaves us with only one choice: $S = \pm 1$.

The simplest example of $S = \pm 1$ operator is a conserved charge. At criticality such perturbation generates incommensurability. As an illustration one can consider a change of the chemical potential for massless Dirac fermions:

$$A = \int d\tau dx[R^+ \partial R + L^+ \partial L - \mu (R^+ R + L^+ L)]$$

This perturbation is removed by a simple transformation of the fields

$$R \rightarrow e^{i\mu x} R, \ L \rightarrow e^{-i\mu x} L$$

1Stricktly speaking, the Lorentz symmetry is violated in the models with complex isotopic symmetry, where different sectors of the spectrum may have different velocities. However, this does not influence the argument made in the text.
and thus leads to a shift in the corresponding Fermi momentum and generates static oscillations with the new wave vector whose magnitude depends on the value of $\mu$. The perturbed theory remains critical in the infrared (IR).

The situation becomes less clear when the perturbation is not a constant of motion. Such non-trivial perturbation corresponds to a non-holomorphic operator, that is, in the notations of Eq. (1), an operator with $d \neq \pm S$.

A well studied example of $S = \pm 1$ perturbation is the the so-called $Z_N$ ‘chiral clock’ model of statistical physics (see [2] for review). In fact, this model is not too good for statistical physics purposes since its Boltzmann weights are not positively defined. However, the (1+1)-dimensional version of this model makes perfect sense and, as was demonstrated in [2], describes a tunneling between two spinless Luttinger liquids. It was established that for the chiral clock model the relevant $S = \pm 1$ perturbations generate incommensurability.

An immediate generalization of the problem of tunneling includes spin. The tunneling between two spin-1/2 Luttinger liquids is described by the Hamiltonian

$$H = H_{LL}^{(1)} + H_{LL}^{(2)} + \int dx (T_+ + T_-),$$

$$T_+ = t(R_1^\dagger R_2) + H.c., \quad T_- = t(L_1^\dagger L_2) + H.c.)$$

where $R_{1,2}$ and $L_{1,2}$ are the right- and the left-moving fermions on chains 1 and 2; $H_{LL}$ describe interacting electrons on the individual chains. Due to the intra-chain interaction the fermionic operators acquire non-trivial scaling dimensions such that

$$< T_\pm(\tau, x) T_\mp(0, 0) > \sim \frac{1}{(\tau \pm ix)^2} \frac{1}{(\tau^2 + x^2)^{\theta}}$$

where $\theta$ depends on the interaction (here I do not discuss the effects related to difference between the charge $v_c$ and the spin $v_s$ velocities of the excitations and set $v_c = v_s = 1$). According to definition [2] operators $T_\pm$ carry conformal (Lorentz) spin $S = \pm 1$.

Another problem much discussed in the literature is the problem of frustrated spin-1/2 two-leg Heisenberg ladder (alias the zig-zag ladder). In the decoupled limit, two $S=1/2$ chains represent an $SU(2)_1 \times SU(2)_2$ WZNW theory. Each $SU(2)_2$ WZNW model has its matrix field $\hat{g}_i(x)$ ($i = 1, 2$) $= e_i(x) + i\vec{n}_1 \cdot \vec{\sigma}$, $(n_{1,2}$ are staggered magnetizations of chains 1 and 2 and $e_{1,2}(x)$ are the staggered energy density operators). For the frustrated ladder the interchain interaction contains is dominated by the so-called twist term [3]. The most general $SU(2)$-invariant form of this term is

$$O_{\text{twist}} = A\vec{n}_1 \cdot \partial_x \vec{n}_2 + B\epsilon_1 \partial_x \epsilon_2 + [1 \leftrightarrow 2]$$

In the Heisenberg zigzag ladder the bare value of $B$ is zero but it is generated in the course of RG (due to the fusion of the $A$-operator with scalar marginal ones), where the leading interchain interaction gives the following contribution to the action density [3]. For a single chain the two-point function of the operators $\epsilon, \epsilon$ decays as

$$< \vec{n}(\tau, x) \vec{n}(0, 0) >= < \epsilon(\tau, x) \epsilon(0, 0) > \sim (x^2 + \tau^2)^{-1/2}$$

which means that

$$< O(\tau, x) O(0, 0) > \sim (x^2 + \tau^2)^{-1/2}$$

According to definition [3], this operator is a sum of two operators with $d = 2, S = \pm 1$.

Using the procedure suggested in [3], one can reformulate the zig-zag ladder model as a model of four Majorana fermions with the Lagrangian density given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{twist}},$$

$$\mathcal{L}_0 = \sum_{a=0}^3 \left[ \frac{1}{2} \chi_a (\partial_\tau - i \partial_x) \chi_a + \frac{1}{2} \bar{\chi}_a (\partial_\tau + i \partial_x) \bar{\chi}_a \right]$$

$$\mathcal{L}_{\text{twist}} = \{ g_1 \chi_1 \chi_2 \chi_3 \chi_0 + g_2 [\chi_0 \chi_1 \chi_2 \chi_3 + \chi_3 \chi_0 \chi_1 \chi_2 + \chi_2 \chi_3 \chi_0 \chi_1] + [\chi \leftrightarrow \bar{\chi}] \}$$

where $g_1, g_2$ are linear combinations of $A, B$. In the limit $A = B$ we get the perfect $O(4)$ symmetry $g_1 = g_2$, with a suitable choice of $A, B$ one can get $g_2 = 0$ which corresponds to the model I consider later in the paper.

Operators with non-zero conformal spin may also appear in models describing tunneling between edges of incompressible Fractional Quantum Hall (FQH) states [3].
Since operators with non-zero conformal spin break Lorentz symmetry, it lifts the restrictions on the low energy spectrum. With the Lorentz symmetry being preserved, one has two choices: either to have a linear spectrum or to have a spectral gap. Without Lorentz symmetry one can imagine a quadratic gapless spectrum which would be naturally associated with a ferromagnetic Goldstone mode. Thus, if a model in question has a continuous isotopic symmetry, the presence of \( S = \pm 1 \) perturbation may lead to a spontaneously breaking of this symmetry at \( T = 0 \) with a formation of a net 'magnetic' moment and the above-mentioned quadratic spectrum.

In all examples discussed above (except the one involving the edge states) the perturbations with \( S = \pm 1 \) appear together. This introduces an additional complication because a fusion of two operators with opposite Lorentz spins generates under RG a relevant operator with zero spin. However, in a certain range of parameters of model (5), this secondary flow does not catch up with the flow of the original coupling constant (for model (7) this is always the case).

In the subsequent sections I discuss a series of exactly solvable models which realize the symmetry breaking scenario outlined above. These models are related to Wess-Zumino-Novikov-Witten (WZNW) model. The latter model together with the related lattice models has been used extensively to address the problems in magnetism (see, for example, \([11], [12], [13], [14]\)). Some of these lattice models are rather similar to the zig-zag spin ladder \([10]\).

The particular model I discuss is described by the following action:

\[
A = W^* + h\bar{a} \int d^2x : J^a(x)\Phi_a(x) : \tag{11}
\]

where \( W^* \) is the action of the critical Wess-Zumino-Novikov-Witten (WZNW) model on the SU(2) group with level \( k \geq 2 \), \( \Phi_a(x) \) are the primary field from the adjoint representation, \( J^a(x) \) are the left Kac-Moody currents and \( h_a \) is a constant vector. The restriction on the value of \( k \) is related to the fact that for \( k = 1 \) field \( \Phi_{a\bar{a}} \) does not exist. The suggested solution can be easily generalized for other symmetry groups and coset models.

To make the discussion self-contained, I recall several basic facts about WZNW models. A WZNW model describes a matrix field \( g(x) \) defined on a group \( G \) whose dynamics is governed by the action

\[
W(g) = \frac{1}{16\pi} \int d^2x \text{Tr}(\partial_\mu g^{-1}\partial_\mu g) + k\Gamma[g] \tag{12}
\]

\[
\Gamma[g] = -\frac{i}{24\pi} \int_0^\infty d\xi \int d^2x e^{\alpha\beta\gamma} \text{Tr}(g^{-1}\partial_\mu gg^{-1}\partial_\gamma gg^{-1}\partial_\lambda g)
\]

The action contains two parameters - the coupling constant \( \lambda \) and integer number \( k \). The model has a global \( G_R \times G_L \) symmetry being invariant under the transformations

\[
g(x) \rightarrow V g(x)U \tag{13}
\]

Hence there are two conserved charges associated with left and right global shifts of the matrix field \( g \). In the perturbed model \([11]\) the right symmetry is broken, but the left symmetry is not. As we shall see below, this symmetry is broken spontaneously at zero temperature.

Model \([12]\) has a stable critical point; the critical value of the coupling constant is \( \lambda^* = k^{-1} \). In what follows I shall distinguish between WZNW model as such and critical WZNW model described by action \([12]\) with \( \lambda = k^{-1} \).

At the critical point WZNW models possess a higher symmetry: its right (left) currents become holomorphic (antiholomorphic):

\[
\bar{J} = \frac{k}{2\pi} g^{-1}\bar{\partial}g, \quad \partial\bar{J} = 0,
\]

\[
J = -\frac{k}{2\pi} g\partial g^{-1}, \quad \bar{\partial}J = 0 \tag{14}
\]

(where \( \partial = \partial_z, \quad \bar{\partial} = \partial_{\bar{z}}; z = \tau + ix, \quad \bar{z} = \tau - ix \)) and satisfy the Kac-Moody algebra:
\[ [J^a(x), J^b(y)] = \frac{i k}{4\pi} \delta^{ab} \delta'(x-y) + if^{abc} J^c(y) \delta(x-y) \]  
(15)

The critical WZNW model can be conveniently described using the Hamiltonian formalism:

\[
\hat{H} = \frac{2\pi}{k + c_v} \sum_{a=1}^D \int dx [ : J^a(x) J^a(x) : + : \bar{J}^a(x) \bar{J}^a(x) : ]
\]

where \( c_v \) is the quadratic Casimir in the adjoint representation. For the SU(2) group \( c_v = 2 \).

At criticality any field which is local in \( g \) can be represented as a linear combination of mutually local operators composing a basis in the operator space. This basis of fields contains the primary fields \( \Phi^G \). The primary fields transform under accordance with the above discussion, I will keep only the leading contribution in irrelevant operator responsible for this flow is critical point, but the entire RG flow towards it. As was shown in [17], in the vicinity of the critical point the leading terminates after certain number of fusions depending on the value of \( k \).

To make the discussion more concrete, I concentrate on the SU(2) group. In this case the primary fields are adjoint one. The maximal spin one can achieve by fusion of the fields from the fundamental representation. However, this process terminates after certain number of fusions depending on the value of \( k \).

III. BETHE ANSATZ SOLUTION

To solve model (11) I use the Bethe ansatz solution of WZNW model [16]. This solution describes not just the critical point, but the entire RG flow towards it. As was shown in [17], in the vicinity of the critical point the leading irrelevant operator responsible for this flow is

\[
V = \lambda : \bar{J}_a J_a(x) \Phi_{a,a}(x) : \]  
(16)

In the presence of the right magnetic field \( H_R \) this operator generates perturbation (11). Indeed, this field is coupled only to the currents of right chirality \( J_a \). In a finite magnetic field these currents acquire a finite expectation value

\[
<J_{\bar{b}} > = \delta^{ab} (H_a^{(R)}/2\pi) \]  
(17)

and therefore in the leading order in \( H \) the irrelevant operator (16) is transformed into a relevant perturbation (11) with

\[
h^a = \lambda (H_a^{(R)}/2\pi) \]  
(18)

Thermodynamic Bethe Ansatz (TBA) equations for SU(2) WZNW model of level \( k \) in magnetic field have the following form [10]:

\[
\epsilon_n(v) = Ts \ln[1 + e^{(n+1)v/\ell_s}] [1 + e^{(n-1)v/\ell_s}] - m\delta_{n,0} e^{\pi v/2} - m\delta_{n,k} e^{-\pi v/2}, \]  
(19)

where \( n = -\infty, ..., \infty \). \( H_R \) and \( H_L \) are right and left ‘magnetic’ fields corresponding to the two conserved charges. In accordance with the above discussion, I will keep only the leading contribution in \( H_R = H \) and keep \( H_L \) infinitesimal.

The free energy is given by

\[
F/L = -mT \int dv (e^{\pi v/2} \delta_{n,0} + \delta_{n,k} e^{-\pi v/2}) \ln[1 + e^{\epsilon_n(v)/\ell_s}] \]  
(20)

\( L \) is the system’s size and

\[
s \ast f(v) = \int_{-\infty}^{\infty} du \frac{f(u)}{4 \cosh[\pi(v - u)/2]} \]
We first study these equations at $T = 0$. In this case all $\epsilon_n$ with negative $n$ are of order of $nH$, $\epsilon_0(v)$ is positive at $v \to -\infty$ and changes its sign at $B \sim -\ln(m/H)$, $\epsilon_k$ with positive $n$ are also positive except of $\epsilon_k(v)$ which changes its sign from negative to positive at some value $Q > 0$. The ground state energy can be expressed in terms of $\epsilon_0$ and $\epsilon_k$ which satisfy the following integral equations:

$$\int_Q^\infty duD(v-u)\epsilon_k(u) = -me^{-\pi v/2} + \int_{-\infty}^{-B} duK(v-u)\epsilon_0(u)$$

(21)

$$\int_{-\infty}^{-B} duD(v-u)\epsilon_0(u) = -me^{\pi v/2} + H + \int_Q^\infty duK(v-u)\epsilon_k(u)$$

(22)

where the Fourier transforms of the kernels are

$$D(\omega) = \frac{\tanh \omega k|\omega|}{2 \sinh k\omega}, \quad K(\omega) = \frac{\tanh \omega}{2 \sinh k\omega}$$

(23)

The limits $B$ and $Q$ are determined by the conditions $\epsilon_0(-B) = 0, \epsilon_k(Q) = 0$.

To determine the spectrum, we also need to know the distribution densities $\rho_0, \rho_k$ which satisfy similar equations:

$$\int_Q^\infty duD(v-u)\rho_k(u) = \frac{m}{4}e^{-\pi v/2} + \int_{-\infty}^{-B} duK(v-u)\rho_0(u)$$

(24)

$$\int_{-\infty}^{-B} duD(v-u)\rho_0(u) = \frac{m}{4}e^{\pi v/2} + \int_Q^\infty duK(v-u)\rho_k(u)$$

(25)

These distribution densities determine the conserved charges (right and left `magnetic’ moments):

$$Q_L = \frac{1}{2} \int_Q^\infty du\rho_k(u), \quad Q_R = \frac{1}{2} \int_{-\infty}^{-B} du\rho_0(u)$$

(26)

Since we are interested only in the leading asymptotics in $H$, we can neglect $\epsilon_k$ in Eq. (22) (and respectively $\rho_k$ in Eq. (25)) and solve these equations as Wiener-Hopf ones. Eqs. (21, 24) in this case also become Wiener-Hopf equations. The case $k = 2$ is somewhat special and will be treated separately. For $k > 2$ the solutions at large $Q, B$ (small $H/m$) are given by

$$\epsilon_0(v) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega(v+B)} \epsilon_0^{(+)}(\omega), \quad \epsilon_0^{(+)}(\omega) = \frac{\pi H}{i\omega(-\pi + 2i\omega)G^{(+)}(\omega)G^{(+)0}},$$

(27)

$$\epsilon_k(v) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega(v-Q)} \epsilon_k^{(+)}(\omega), \quad \epsilon_k^{(+)}(\omega) = \frac{\pi (k-2)me^{-\pi Q/2}}{(\pi - 2i\omega)(\pi - ik\omega)G^{(+)}(\omega)G^{(-)}(-i\pi/2)}$$

(28)

$$\rho_k^{(+)}(\omega) = \epsilon_k^{(+)}(\omega)\frac{(\pi - ik\omega)}{2\pi(k-2)}$$

(29)

where

$$m \exp(-\pi B/2) = H \frac{G^{(+)}(i\pi/2)}{G^{(+)}(0)}$$

(30)

$$\exp(-\pi Q/2) = \beta_k(H/m)^{(k+2)/(k-2)}$$

(31)

$$\beta_k = \left\{ \frac{k \tan(\pi/k)}{2\pi(k+2)G^{(+)}(i\pi/k)^2} \right\}^{k/(k-2)} \left[ \frac{G^{(+)}(i\pi/2)}{G^{(+)}(0)} \right]^{(k+2)/(k-2)}$$

and

$$G^{(-)}(\omega) = \left( \frac{i\omega k + 0}{\pi e} \right)^{-\frac{i\omega k}{\pi e}} \frac{\Gamma(1 + i\omega/\pi)\Gamma(\frac{1}{2} + i\omega/\pi)}{\sqrt{2\pi k\Gamma(1 + i\omega/2)}}$$

(32)

with $G^{(-)}(\omega) = G^{(+)}(-\omega)$ and $D(\omega) = G^{(-)}(\omega)G^{(+)}(\omega)$.  

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To obtain the modified TBA equations describing model (11) I simply replace $\epsilon_0$ in Eq.(13) by its zero temperature approximate value (27). The result is

$$\epsilon_n(v) = T_s \ln[1 + e^{\epsilon_{n+1}(v)/T}] [1 + e^{\epsilon_{n-1}(v)/T}] + \delta_{n,1} s * \epsilon_0(v) - m \delta_{n,k} e^{-\pi v^2/2}, \quad (n = 1, \ldots)$$

(33)

$$\lim_{n \to \infty} \frac{\epsilon_n}{n} = H_L$$

The free energy is then given by

$$F/L = -mT \int \! dv \pi v/2 \ln[1 + e^{\epsilon_k(v)/T}]$$

(34)

Let us return to $T = 0$ solution. From equations (28, 29) we can extract the following information. First of all, I observe that $\epsilon_k$ approaches zero at two points: at $v = Q$ and $v \to \infty$. The latter point I identify with zero momentum; then point $v = Q$ corresponds to the wave vector

$$\delta P = 2\pi \int_Q^\infty \! dv \rho_k(v) = 2\pi \rho_k^{(+)}(0) = m \gamma_k (H/m)^{(k+2)/(k-2)}$$

(35)

where

$$\gamma_k = \frac{\beta_k}{G^{(+)}(i\pi/2)G^{(+)}(0)}$$

This incommensurate wave vector scales with $H$ exactly as one expects it to scale (that is $\delta P \sim H^{1/(2-d)}$), taking into account that the scaling dimension of the perturbation in Eq.(11) is $d = 1 + 4/(k+2)$. The excitation spectrum in the vicinity of $v = Q$ is linear; the velocity is given by

$$V_R = \frac{1}{2\pi \rho_k(Q)} \frac{\partial \epsilon_k(v)}{\partial v} \bigg|_{v=Q} \frac{(k-2)}{k} V_L$$

(36)

where $V_L$ is the velocity of the left-moving particles (I have been working in the system of units where $V_L = 1$).

The fact that $\epsilon_k$ approaches zero also at $v \to \infty$ means that there are soft modes at zero momentum. Since according to Eqs.(28, 29) $\epsilon_k(\omega), \rho_k(\omega)$ behave as $|\omega|$ at small $\omega$, their real space asymptotics at $v \to \infty$ are $\epsilon_k \sim \rho_k \sim -v^{-2}$. Since the momentum is given by

$$P(v) = 2\pi \int_v^\infty \! dv \rho_k(v)$$

(37)

this means that the spectrum is quadratic. Using this formula and Eq.(29) we find the dispersion law:

$$\epsilon(P) = \frac{(k-2)}{k} \frac{P^2}{\delta P}, \quad (P << \delta P)$$

(38)

The maximum of the energy is reached at $P \sim \delta P$ and is of order of $\delta P$.

For $k = 2$ the formulae for the velocity and the dispersion law should be modified:

$$Q = 2\pi/g + \frac{1}{\pi} \ln(4g), \quad g = |HG^{(+)}(i\pi/2)/m|^2$$

(39)

$$V_R/V_L = \frac{1}{\pi^2} g \sim H^2$$

(40)

$$\delta P = \frac{m}{H|G^{(+)}(i\pi/2)|^2} \exp\{ -[\pi m/HG^{(+)}(i\pi/2)]^2 \}$$

(41)

Taking $T \to 0$ limit in Eqs.(33) I obtain the following expressions for the energies:

$$\epsilon_n(\omega) = \frac{\sinh n\omega}{\sinh k\omega} e^{i\omega Q} \epsilon_k^{(+)}(\omega) + \frac{\sinh(k-n)\omega}{\sinh k\omega} e^{-i\omega B} \epsilon_0^{(+)}(-\omega), \quad (n < k)$$

(42)

$$\epsilon_{n+k}(\omega) = e^{-n|\omega|} \epsilon_k^{(+)}(\omega)e^{i\omega Q}$$

(43)
Let us study asymptotics of these energy functions. In real space we have

\[ \epsilon_{n+k}(v) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n}{(v-u)^2 + n^2} \epsilon_k^{(+)}(u) \quad (44) \]

From this expression it follows that \( \epsilon_{n+k}(v) \) decay as \( v^{-2} \) on both infinities. On the other hand, \( \epsilon_n(v) \) with \( n < k \) decay as \( v^{-2} \) only at \( v > Q \). At \( v < Q \) they have roton-like minima. For \( Q - v >> 1 \) I obtain from Eq. (42):

\[ \epsilon_n(v) \approx k^{-1} \sin(\pi n/k) \left[ \epsilon_k^+(i\pi/k) e^{\pi(v-Q)/k} + \epsilon_0^+(i\pi/k) e^{-\pi(v+B)/k} \right] 
   = 2k^{-1} \sin(\pi n/k) [\epsilon_k^+(i\pi/k) \epsilon_0^+(i\pi/k)]^{1/2} e^{-\pi(Q+B)/2k} \cosh[\pi(v-v_0)/k] \quad (45) \]

which corresponds to the relativistic spectrum with spectral gaps for \( k \neq 2 \) \((n = 1, \ldots, k-1)\) given by

\[ M_n = M_0 \sin(\pi n/k), \quad M_0 = \left[ \frac{2(k-2) \cot(\pi/k)}{\pi k(k+2)} \right]^{1/2} \delta P \quad (46) \]

For \( k = 2 \) the gap is exponential in \( H^{-2} \). The energy minimum for \( \epsilon_n \) with \( n = 1, \ldots, k-1 \) occurs at

\[ v_0 = Q - \frac{k}{2\pi} \ln \left\{ \frac{(k-2) \tan(\pi/k)}{4\pi(k+2)(k+1)^2(i\pi/k)^2} \right\} \quad (47) \]

This means two things. The first one is that the massive modes are centered at \textit{incommensurate} wave vector. The second is that in the rapidity space they are very close to \( Q \).

**IV. BETHE ANSATZ DERIVATION OF THE LOW ENERGY EFFECTIVE ACTION**

Thus we have the following regions in rapidity space where low-energy excitations are located:

(i) at \( v >> Q \) there are gapless modes \( \epsilon_n \) \((n = 1, \ldots)\) with the spectrum \( \sim v^{-2} \);

(ii) at \( v << Q \) there are gapless modes \( \epsilon_n \) \((n = k + 1, \ldots)\) with the spectrum \( \sim v^{-2} \);

(iii) at \( v = Q \) there is a gapless mode \( \epsilon_k \) with the spectrum \( \sim (v-Q) \);

(iv) at \( v \approx Q \) there are massive modes \( \epsilon_n \) \((n = 1, \ldots, k-1)\). The massive modes can be treated as low energy excitations only for \( k >> 1 \) when their masses are much smaller than \( \delta P \).

In the momentum space the spectrum in sectors (i),(ii) is quadratic \( \epsilon(P) \sim P^2 \) (see Figs. 1,2), the spectrum in sector (iii) is linear \( \epsilon(P) \sim |P - P(Q)| \) and the spectrum in sector (iv) is massive relativistic (see Fig.3).
Let us first consider the case of moderate $k$ when one does not need to consider roton modes. Then the low energy sector is described by the truly gapless modes (i - iii). Since their positions in rapidity space are well separated from each other and the integration kernels in Eqs. (33) decay exponentially, one can derive separate sets of TBA equations for each mode. Comparing these equations with TBA equations for known integrable models one can deduce the effective action for the low energy sector.

Let us derive TBA equations for sector (i). In this sector all energies are positive. Therefore it is convenient to have TBA equations in such a form where the integral kernels act on functions $\ln(1 + e^{-\eta_n/T})$. Such equations would provide a ready $T = 0$ limit. To get such form of TBA. The resulting equations read

$$T \ln[1 + e^{\eta_n(v)/T}] - A_{nm} \ast T \ln[1 + e^{-\tau_m(v)/T}] = A_{n,k}^{-1} \ast A_{n,k} \ast \epsilon_k^{(0)}(v) + n H_L, \quad n = 1, \ldots, \quad v \gg Q$$

(48)

where $\epsilon_k^{(0)}(v)$ is the solution at $T = 0$ and is given by Eq.(28). The kernels have the following standard Fourier transforms:

$$A_{nm}(\omega) = \coth |\omega| [\exp(-|n - m|\omega) - \exp(-|n + m|\omega)]$$

(49)

These equations are valid for temperatures $T << \delta P$ - the maximum value of $\epsilon_k^{(0)}(v)$. In this region These equations coincide with the right chiral sector of the spin-(k/2) integrable ferromagnet. The latter ones can be extracted from TBA equations obtained in [18]. In a similar fashion I obtain TBA for sector (ii):

$$T \ln[1 + e^{\eta_n+k(v)/T}] - A_{nm} \ast T \ln[1 + e^{-\tau_{m+k}(v)/T}] = a_n \ast \epsilon_k^{(0)} + n H_L, \quad n = 1, \ldots, \quad v << Q$$

(50)

These equations describe the left chiral sector of the spin-1/2 integrable ferromagnet where the dispersion law is $\epsilon(k) \sim k^2 \theta(k)$. Therefore the parity in the ferromagnetic sector is broken.

Another soft mode in the right chiral sector has the linear spectrum with velocity $[26]$ and is centered at the the momentum $\delta P$. Using the standard manipulations with TBA equations one can find its contribution to the specific heat and establish that this mode carries central charge 1. Therefore it is described by the non-chiral Gaussian model.
V. CONNECTION TO THE ZIG-ZAG CHAIN AT $K = 2$

For $k = 2$ model (11) simplifies considerably. In this case the critical WZNW action is equivalent to the theory of three massless Majorana fermions $\chi_a, \bar{\chi}_a$ ($a = 1, 2, 3$) and the adjoint operator and the currents are given by

$$\Phi_{ab} = \chi_a \bar{\chi}_b, \quad J^a = \frac{i}{2} \epsilon^{abc} \chi_b \chi_c.$$  

(51)

Therefore for $k = 2$ one can rewrite the perturbed action (11) solely in terms of Majorana fermions:

$$A = \int d^2x \left[ \frac{1}{2} \chi_a (\partial_\tau - i \partial_x) \chi_a + \frac{1}{2} \bar{\chi}_a (\partial_\tau + i \partial_x) \bar{\chi}_a + h_a \chi_1 \chi_2 \chi_3 \bar{\chi}_a \right]$$

(52)

In this form the model closely resembles the model for the zig-zag ladder (10) (see also [10]) and att $g_2 = 0$ it even coincides with model (11) exactly solvable by the Bethe ansatz.

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