Dynamics of vortex penetration, jumpwise instabilities and nonlinear surface resistance of type-II superconductors in strong rf fields

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We consider nonlinear dynamics of a single vortex in a superconductor in a strong rf magnetic field $B_0 \sin \omega t$. Using the London theory, we calculate the dissipated power $Q(B_0, \omega)$, and the transient time scales of vortex motion for the linear Bardeen-Stephen viscous drag force, which results in unphysically high vortex velocities during vortex penetration through the oscillating surface barrier. It is shown that penetration of a single vortex through the ac surface barrier always involves penetration of an antivortex and the subsequent annihilation of the vortex antivortex pairs. Using the nonlinear Larkin-Ovchinnikov (LO) viscous drag force at higher vortex velocities $v(t)$ results in a jump-wise vortex penetration through the surface barrier and a significant increase of the dissipated power. We calculate the effect of dissipation on nonlinear vortex viscosity $\eta(v)$ and the rf vortex dynamics and show that it can also result in the LO-type behavior, instabilities, and thermal localization of penetrating vortex channels. We propose a thermal feedback model of $\eta(v)$, which not only results in the LO dependence of $\eta(v)$ for a steady-state motion, but also takes into account retardation of temperature field around rapidly accelerating vortex, and a long-range interaction with the surface. We also address the effect of pinning on the nonlinear rf vortex dynamics and the effect of trapped magnetic flux on the surface resistance $R_s$ calculated as a function or rf frequency and field. It is shown that trapped flux can result in a temperature-independent residual resistance $R_i$ at low $T$, and a hysteretic low-field dependence of $R_i(B_0)$, which can decrease as $B_0$ is increased, reaching a minimum at $B_0$ much smaller than the thermodynamic critical field $B_c$. We propose that cycling of rf field can reduce $R_i$ due to rf annealing of magnetic flux which is pumped out by rf field from the thin surface layer of the order of the London penetration depth.

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I. INTRODUCTION

The behavior of superconductors in strong rf fields involves many complex mechanisms related to a nonlinear electromagnetic response of nonequilibrium quasiparticles, pairbreaking suppression of superconducting gap $\Delta$ and penetration of vortices at higher rf amplitudes\textsuperscript{1,2}. The physics behind the nonlinear rf response has recently attracted much attention due to the development of a new generation of high-performance superconducting Nb cavities for particle accelerators, in which the peak surface GHz fields $B(t) = B_0 \sin \omega t$ close to the thermodynamic critical field $B_c$ were reached at a very high quality factor $\sim 10^9 - 10^{11}$ characteristic of the Meissner state\textsuperscript{3,4}. At such strong rf fields the peak surface current density $B_0/\mu_0 \lambda$ approaches the depairing current density $J_d$ at which the Meissner state becomes unstable with respect to avalanche penetration of vortices once the instantaneous rf field $B(t) = B_0 \sin \omega t$ exceeds the superheating field $B_s \approx B_c$. In turn, penetration of vortices causes a sharp increase in the surface resistance $R_s$.

As far as the very high quality factors are concerned, of particular interest is the behavior of $R_s$ in $s$-wave superconductors at low temperatures $T < T_c$ and frequencies $\omega \ll \Delta$, for which the rf field cannot break the Cooper pairs, and the very low Meissner surface resistance $R_s \propto (\omega^2 \Delta/T) \exp(-\Delta/T)$ is due to an exponentially small density of thermally-activated quasiparticles (unlike the power-law dependence $R_s(T) \approx R_i + CT^n$ due to nodal quasiparticles in d-wave superconductors\textsuperscript{5,6,7}). In this case penetration of even few vortices driven by extremely high rf currents densities $J \sim J_d$ can produce strong energy dissipation comparable to that in the Meissner state, which in turn, can trigger thermomagnetic flux avalanches and the superconductivity breakdown. It is therefore important to understand mechanisms, which control dynamics of single vortex penetration under strong rf fields. Yet, the rf field onset of vortex penetration $B_{i,v}$, and the dissipated power $Q$ as functions of $B_0$ and $\omega$, and the relation between $B_c$ and the thermodynamic $B_c$ and the lower critical field $B_{i,2}$ are still not well understood. These problems include complex kinetics of the emergence of the vortex core at the surface, and the subsequent nonlinear large-amplitude oscillation of the vortex at the surface driven by strong rf currents much higher than depinning critical current density. This situation cannot be described by a well developed linear electrodynamics of a superconductor in the pinned mixed state weakly deformed by rf currents\textsuperscript{8,9,10,11,12}. Some issues of nonlinear vortex dynamics in ramping magnetic fields have been addressed in extensive numerical simulations of the time-dependent Ginzburg-Landau (TDGL) equations\textsuperscript{3,13,14,15,16} valid at $T \approx T_c$, and molecular dynamics simulations\textsuperscript{17}. However, few experimental and theoretical results on vortices driven by very strong rf currents at low temperatures have been published in the literature.

In this paper we address nonlinear rf dynamics of a sin-
ingle vortex moving in and out of a type-II superconductor through an oscillating magnetic surface barrier locally weakened by a surface defect. We show that already this basic problem involves a rich physics, since even weak Meissner fields $B_0 \ll B_c$ can drive the vortex with velocities $v(t)$ so high that the linear Bardeen-Stephen viscous drag model becomes inadequate. As a result, the vortex velocity $v(t)$ can exceed the sound velocity, causing the Cherenkov generation of hypersound\cite{18,19}. Moreover, $v(t)$ can exceed the critical velocity $v_0$, above which the vortex drag coefficient $\eta(v)$ decreases as $v$ increases, and the viscous drag force $f_v = v\eta'(v)$ reaches maximum at the critical velocity $v_0$, resulting in the jumpwise Larkin-Ovchinnikov (LO) instability\cite{20,21}. The LO instability has been extensively investigated by dc transport measurements\cite{22,23,24,25,26,27,28} on both low-$T_c$ and high-$T_c$ superconductors for which $v_0 \approx 1 - 10$ km/s have been typically observed at low $T$ and $B$. Single-vortex dynamics under strong rf field also involves annihilation of vortex-antivortex pairs, and a cascade of single, double and multiple vortex penetrations. Competition of the rf driving force, image attraction to the surface, and the viscous drag force results in a strong dependence of the dissipated power $Q$ on the rf amplitude and frequency. Very high vortex velocities achieved at fields $B_0 \approx B_c$ required to break the surface barrier make it possible to probe the behavior of vortices under extreme conditions, for which the Lorentz driving force approaches its ultimate depairing limit. Because of strong heating effects, these conditions are hard to reproduce in dc transport experiments (except in high-power pulse measurements\cite{29,30}).

The importance of heating effects for transport instabilities in superconductors at low temperatures is well known\cite{21,31,32,33}. In this paper we show that heating is a key limiting factor for the high-field surface resistance at $T \ll T_c$ as well, even for single vortices driven by strong rf Meissner currents. In particular, viscous vortex dynamics coupled with electron overheating can result in the LO-type behavior of $\eta(v)$, thermal rf breakdown, long-range interaction (on scales much greater than the London penetration depth) between a vortex and the surface and between vortices themselves.

The paper is organized as follows. In section II we establish the main parameters of interest by considering penetration and dissipation of a single vortex over the oscillating surface barrier in type-II superconductors described by the dynamic equation, in which the linear Bardeen-Stephen viscous drag force is balanced by the Lorentz driving force and the image attraction force at the surface in the London theory. Even in this basic model rf vortex dynamics always involves annihilation of vortex-antivortex pairs for $B_0 \approx B_c$ close to the penetration field $B_c$ and a strong dependence of the dissipated power $Q(B_0, \omega)$ on the rf frequency and amplitude. In section-III we show that the Bardeen-Stephen model actually has a very limited applicability because vortices breaking through the surface barrier reach supersonic velocity so the dependence of the viscous drag coefficient $\eta(v)$ must be taken into account. In this case vortex dynamics becomes strongly coupled with nonequilibrium overheating of the vortex core, resulting in jumpwise penetration of single vortices through the surface barrier, and significant increase in $Q$. In section IV we consider the effect of pinning on rf surface resistance. In particular, we show that trapped vortices can result in a temperature independent, field-hysteretic residual resistance, which can decrease as the rf field increases. Pinned vortices can also produce hotspots, which ignite thermal rf breakdown. Section V is devoted to dissipation around hotspots and their nonlinear contribution to the global surface resistance. The thermal breakdown of the Meissner state ignited by vortex hotspots is addressed. Section VI concludes with a discussion of the results.

II. PENETRATION OF A VORTEX OVER THE OSCILLATING SURFACE BARRIER

A. Dynamic equations and time scales

Penetration of vortices in a superconductor is controlled by the Bean-Livingston surface barrier, which results from a competition between the Meissner screening currents pushing the vortex in a superconductor and the attraction force between a vortex and the surface\cite{34}. This surface barrier oscillates under the rf field, so motion of a vortex in and out of a superconductor is described by a dynamic equation. We consider here a type-II superconductor in the London theory, assuming that the rf field $B(t) = B_0 \sin \omega t$ of amplitude $B_0$ and frequency $\omega$ is applied parallel to the flat surface at $x > 0$ as shown in Fig. 1. Then the equation of motion for a single vortex driven by the rf Meissner balanced by the image attraction force and the viscous drag force takes the form

$$\eta_0 \ddot{u} = \frac{\phi_0 B_0}{\mu_0 \lambda} e^{-u/\lambda} \sin \omega t - \frac{\phi_0^2}{2 \pi \mu_0 \lambda^3} K_1 \left[ \frac{2}{\lambda} \sqrt{u^2 + \xi_s^2} \right],$$

where $u(t)$ is the distance of the vortex core from the surface, $\lambda$ is the London penetration depth, $\eta_0 = \phi_0 B_{c2}/\rho_n$ is the Bardeen-Stephen vortex viscosity, $\rho_n$ is the normal state resistivity, $\phi_0$ is the magnetic flux quantum,
$B_{c2} = \phi_0/2\pi\xi^2$ is the upper critical field, $K_1(x)$ is the modified Bessel function. Here we introduce the local coherence length $\xi$ at the point of the vortex entry, which provides the cutoff in the London theory. For $u < \xi$, the last term in Eq. (1) gives a constant force of vortex attraction to the surface due to the formation of a "core string" of depressed order parameter revealed by computer simulations of the GL equations. In this work we treat the emergence of the vortex phenomenologically, assuming that it first appears in a small defect region at the surface. For the results presented below, the actual nature of the defect is not important as long as the defect size is much smaller than $\lambda$, and the local $\xi$ is larger than the bulk coherence length $\xi$. The vortex penetrates at the field $B(t) > B_v$ for which the local surface barrier disappears because the peak Meissner force $\phi_0 B_0/\mu_0 \lambda$ exceeds the maximum attraction force to the surface $\phi_0^2 K_1(2\xi)/2\mu_0 \lambda^3$. For $\xi \ll \lambda$, we can expand $K_1(\xi) \approx 1/x$, and obtain

$$B_v = \phi_0/4\pi\lambda\xi \approx 0.71 B_v,$$

which basically defines $\xi$, in terms of the observed local penetration field $B_v$, which has been calculated for different types of surface defects. We assume that there is a distribution of sparse small regions with reduced local $B_v$ on the surface where vortices first enter. Penetration of straight vortices can only be initiated by linear defects (for example, dislocations or grain boundaries) parallel to the vortex line. For more common 3D surface defects, such as precipitates or local variation of chemical composition, a vortex first emerges as a semi-loop, which then expands as illustrated by Fig. 2. The initial penetration of a curved vortex in Fig. 2 can hardly be described by Eq. (1) for a straight vortex parallel to the surface. However, the circular vortex semi-loop very quickly evolves into a loop strongly elongated along the surface because of the gradient of the Meissner current $J(x) = (B_0/\mu_0 \lambda) \exp(-u/\lambda)$ and the LO instability, which effectively straightens the vortex due to jump-wise lateral propagation of the loop, as shown below. Thus, Eq. (1) can be used after a short transient time, which is still much smaller than the rf period.

Eq. (2) gives $B_v$ close to the superheating field $B_s = 0.745 B_v$, above which the Meissner state in extreme type-II superconductors with $\kappa = \lambda/\xi \gg 1$ becomes absolutely unstable with respect to weak periodic perturbations of the order parameter as the Meissner current density at the surface $B_v/\mu_0 \lambda$ exceeds the local depairing current density $J_d$. For $B_0 > B_v$, a vortex moves in and out the superconductor under the action of the rf field. Since Eq. (1) can only be used on the scales $u(t) > \xi$, we neglect here a possible dependence of $\eta_0$ on $u$, although this dependence can occur if a long-range interaction of the vortex core with the surface due to nonequilibrium effects is taken into account, as shown below.

To estimate the scale of vortex oscillations and maximum velocities, we first disregard the image attraction force, which becomes negligible at distances $u > \lambda$. Then the solution of Eq. (1) takes the form

$$u(t) = \lambda \ln \left[ 1 + \frac{\phi_0 B_0}{\lambda^2 \omega \eta_0 \mu_0}(\cos \omega t_0 - \cos \omega t) \right],$$

where $t_0 = \sin^{-1}(B_v/B_0)/\omega$ is the time of vortex entry. The maximum vortex penetration depth $u_m$ corresponds to $\cos \omega t = -1$, whence

$$u_m = \lambda \ln \left[ 1 + \frac{\phi_0}{\lambda^2 \omega \eta_0 \mu_0}(\sqrt{B_0^2 - B_v^2} + B_v) \right]$$

Here $u_m$ depends logarithmically on the rf field and frequency. From Eq. (4) we estimate the time $\tau$ for the vortex to move by the distance $\approx \lambda$ from the surface. For GHz frequencies and the materials parameters of Nb and Nb$_3$Sn, $\tau$ turns out to be much shorter than the rf period so $\cos \omega (t_0 + \tau) \approx \cos \omega t_0 - \omega \tau \sin \omega t_0$, hence

$$\tau = \frac{\mu_0 \lambda^3 \eta}{\phi_0 B_v} \approx \frac{2\mu_0 \lambda^3}{\rho \eta \xi}.$$
positive cycle, except that once the antivortex reaches \( u_c \) on the way out, it creates a vortex at the surface, both annihilating at \( u_a \). This process repeats periodically.

Eq. (1) therefore describes vortex penetration and exit until the Meissner current density plus the current density of the outgoing vortex being at \( x = u_c \), reaches \(-B_v/\mu_0\lambda\) at the surface at the time \( t = t_c \) defined by:

\[
B_0|\sin \omega t_c| + \frac{\phi_0}{\pi \lambda^2} K_1 \left( \frac{u_c}{\lambda} \right) = B_v
\]

The second term in the l.h.s. of Eq. (6) is twice the current density of the vortex at the distance \( u_c \) in an infinite sample because the outgoing vortex and its antivortex image contribute equally to \( J(0,t) \). For \( t > t_c \), the vortex with the coordinate \( u_+(t) \) and the antivortex with the coordinate \( u_-(t) \) move toward each other, as described by the following equations:

\[
\eta_0 u_+ = \frac{\phi_0 B_0}{\mu_0 \lambda} e^{-\frac{u_c}{\lambda} \sin \omega t} - \frac{\phi_0^2}{2\pi \mu_0 \lambda^3} \left[ K_1 \left( \frac{2u_c}{\lambda} \right) + K_1 \left( \frac{u_+ - u_-}{\lambda} \right) - K_1 \left( \frac{u_+ + u_-}{\lambda} \right) \right],
\]

\[
\eta_0 u_- = \frac{\phi_0 B_0}{\mu_0 \lambda} e^{-\frac{u_c}{\lambda} \sin \omega t} - \frac{\phi_0^2}{2\pi \mu_0 \lambda^3} \left[ K_1 \left( \frac{2u_c}{\lambda} \right) u^2 + \xi^2 \right] - K_1 \left( \frac{u_+ - u_-}{\lambda} \right) - K_1 \left( \frac{u_+ + u_-}{\lambda} \right) \right]
\]

These equations reflect the balance of interaction forces between the vortex and antivortex and their corresponding images similar to those in Fig. 1. The first term in the r.h.s. of Eq. (8) has the minus sign because the Meissner current drives the antivortex in the opposite direction as compared to the vortex. The initial conditions for Eqs. (7) and (8) are: \( u_+(t_c) = u_c, u_-(t_c) = 0 \); and the condition \( u_+(t_a) = u_-(t_a) = u_a \) defines the annihilation distance \( u_a \) and time \( t_a \).

Dynamics of vortex penetration and annihilation is illustrated by Fig. 3 where the vortex penetration depth is \( u_m \simeq 3\lambda \), the critical distance is \( u_c \simeq 2\lambda \), and the vortex-antivortex annihilation occurs at \( u_a \simeq \lambda \). For \( \omega \tau \ll 1 \), the vortex first accelerates rapidly, penetrating by the distance \( \simeq \lambda \) during a time \( \sim \tau \), and then slowly turns around during the time of the order of the rf period and annihilates in a short time \( \sim \tau \).

The above results are limited to the field region \( B_v < B_0 < B_s \) where single vortices penetrate independently through regions where the Bean-Livingston barrier is locally suppressed by surface defects separated by distances \( \lambda \). The case \( B_0 > B_v \) corresponds to a global pair-breaking instability causing multi-vortex avalanche penetration. Yet, even for \( B_v < B_0 < B_s \), a multi-vortex chain penetration is possible. Indeed, penetration of a single vortex for \( B(t) > B_v \) suppresses the local pair-breaking instability at \( x = 0 \). However, as \( B(t) \) increases, the Meissner current density increases, while the counterflow of surface current density at \( x = 0 \) from the vortex decreases as it moves further away from the surface. As a result, \( J(0,t) \) can again reach \( J_s \), causing penetration of the second vortex at \( t = t_2 \) when the first vortex is located at \( x = u_2 \). The condition of the second vortex penetration is similar to Eq. (9),

\[
B_2 \sin \omega t_2 - \frac{\phi_0}{\pi \lambda^2} K_1 \left( \frac{u_2}{\lambda} \right) = B_v
\]

(8) except for the minus sign in the l.h.s. Eqs. (1) and (9) define the critical rf amplitude \( B_2 \) below which only the single vortex penetration occurs. Shown in Fig. 4 is the curve \( B_2(\omega) \) obtained by the numerical solution of Eqs. (1) and (9) for Nb3Sn. These results can be described well by the power law dependence

\[
B_2 = \left[ 1 + p \Omega^\alpha \right] B_v, \quad \Omega = 2\mu_0 \lambda^2 \omega/\rho_n,
\]

(10) where \( \Omega = \omega \tau/\kappa \), \( \alpha = 0.73 \) and \( p = 0.23 \). For \( \Omega \ll 1 \), the field \( B_2 \) is close to \( B_v \), however, dissipation produced by penetrating vortices can significantly reduce both \( B_v \) and \( B_2 \) (see below).
for both vortex and antivortex cycles, where we took ac-
surface, contributions to any closed vortex trajectory, which starts and ends at the slow (Ω = 0.2) field,

\[ \Delta G = \frac{\pi}{\mu_0} \left[ \int_{t_0}^{t_e} \dot{u}_t^2 \, dt + \int_{t_0}^{t_0} \dot{\mu}_t^2 + \ddot{u}_0^2 \right] dt \] \hspace{1cm} (11)

where \( u(t) \) is the solution of Eq. 11, which describes dynamics of a single vortex driven by rf field until \( t = t_e \) when the antivortex appears. The second integral in Eq. 11 is due to the collapse of the vortex-antivortex pair described by Eqs. 7 and 8. For a quasistatic field, \( Q \) can be obtained from the change of the vortex thermodynamic potential \( G(u) \)

\[
G(u) = \frac{\phi_0}{\mu_0} \left[ B_{c1} - B - Be^{-u/\lambda} - \frac{\phi_0}{4\pi\lambda^2} K_0 \left( \frac{2u}{\lambda} \right) \right] \hspace{1cm} (12)
\]

where \( B_{c1} = \epsilon \mu_0 / \phi_0 \) is the lower critical field and \( \epsilon \) is the vortex self-energy. If the ac field \( B(t) \) varies very slowly (\( \Omega \ll 1 \)), the dissipated energy equals the sum of \( \Delta G_+ = G(0) - G(u_m) \) during the positive half-cycle and \( \Delta G_- = G(u_m) - G(0) \) during the negative half-cycle. For any closed vortex trajectory, which starts and ends at the surface, contributions to \( Q \) due to vortex self-energy and the work \( \oint F_i(u) \, dt \) of the potential image force \( F_i(u) \) vanish. Thus, \( Q \) is only determined by the work of the driving Lorentz force, \( \simeq (2\phi_0 B_0 / \mu_0) [1 - \exp(-u_m / \lambda)] \), for both vortex and antivortex cycles, where we took account of the fact that the main contribution to \( Q \) comes from the initial acceleration of the vortex during the time \( \sim \tau \) when the field \( B(t) \) is close to \( B_c \). Neglecting \( \exp(-u_m / \lambda) \ll 1 \), we have

\[
Q = 2\omega \phi_0 B_0 / \pi \mu_0, \hspace{1cm} \Omega \to 0 \hspace{1cm} (13)
\]

The field region of the single-vortex penetration \( B_0 < B_0 < B_0 < B_2 \) defined by Eq. 10 shrinks as the frequency decreases. In this narrow field region the effect of vortex viscosity can radically change the dependence of \( Q \) on \( B_0 \) and \( \omega \). Shown in Fig. 5 are the results of numerical solution of Eqs. 9, 10, and 11 for different frequencies and the GL parameters \( \kappa = \lambda / \xi \). These data are described well by the following formula

\[
Q = \frac{2\omega \phi_0 B_0}{\pi \mu_0} \left( \frac{B_0^2 - B^2}{B_0^2} \right)^{\omega \tau_2}, \hspace{1cm} \tau_2 = 3.98\tau_{\kappa}^{2/3} \hspace{1cm} (14)
\]

which reduces to Eq. 13 for \( \omega \to 0 \). As follows from Fig. 5 and Eq. 14, the power \( Q \) decreases as \( \omega \) increases because of retardation effects due to vortex viscosity during the short fraction of the rf period in which \( B(t) > B_c \).

### III. INSTABILITIES AT HIGH RF FIELDS

Once the field \( B(t) \) exceeds \( B_c \), the vortex rapidly accelerates reaching the maximum velocity \( v_m \simeq \lambda / \tau \):

\[
v_m = \rho_i / 2 \mu_0 \lambda \hspace{1cm} (15)
\]

Eq. 15 gives \( v_m \simeq 30 \text{ km/s} \) for \( \text{Nb}_3\text{Sn} \) and \( v_m \simeq 10 \text{ km/s} \) for Nb. Not only are the so-obtained values of \( v_m \) much higher than the velocity of sound, they may even exceed the critical BCS pairbreaking velocity.

\[
v_D = \frac{\Delta}{m v_F} = \frac{h}{\pi m \xi}, \hspace{1cm} (16)
\]

where \( \xi = h v_F / \pi \Delta \), \( v_F \) is the Fermi velocity and \( m \) is the electron mass. Indeed, taking \( \xi = 40 \text{ nm} \) and the free electron mass \( m \), we obtain \( v_D \simeq 0.8 \text{ km/s} < v_m \) for Nb, and \( v_D \simeq 10 \text{ km/s} < v_m \) for \( \xi = 3 \text{ nm} \) in \( \text{Nb}_3\text{Sn} \). Here we use the Bardeen-Stephen model for qualitative estimates only, ignoring many still not well understood mechanisms essential at low temperatures, for example the effect of quantized electron states in the core and the core shrinkage due to the Kramer-Pesch effect\( ^{19} \), resulting in the factor \( \sim \ln(T_c / T) \) in the Bardeen-Stephen formula\( ^{26,27,28} \). Yet, for strong rf fields, \( B_0 \sim B_c \), the linear viscous drag force derived for small vortex velocities, becomes inadequate. It was first predicted theoretically\( ^{29} \) and observed in many experiments\( ^{22,23,24,25,26,27} \), that the dependence of \( \eta \) on \( v \) at high vortex velocities results in a nonmonotonic viscous drag force \( f_v = \eta(v) \) and jump-wise instabilities.

#### A. Instabilities of viscous flux flow

A nonlinear viscous drag force was first calculated by Larkin and Ovchinnikov (LO)\( ^{30} \), who showed that nonequilibrium effects in the vortex core decrease the drag coefficient \( \eta \) as \( v \) increases:

\[
\eta(v) = \frac{\eta_0}{1 + v^2 / v_0^2}, \hspace{1cm} (17)
\]
where \( \eta_0 \) is the Bardeen-Stephen viscosity. The critical velocity \( v_0 \) in the dirty limit is given by:

\[
v_0 \simeq 0.6 \left( \frac{\ell_i v_F}{\tau_e} \right)^{1/2} \left( 1 - \frac{T}{T_c} \right)^{1/4}
\]  

(18)

Here, \( \ell_i \) is the mean free path due to impurities, and \( \tau_e(T) \) is the quasiparticle energy relaxation time. Eq. (17) results in a nonmonotonic dependence of viscous drag force, \( f_v = v\eta(v) \) on the vortex velocity:

\[
f_v(v) = \frac{\eta_1 v}{1 + v^2/v_0^2} + \eta_2 v,
\]  

(19)

Here, following the LO approach, we use two effective viscosities \( \eta_1 \) and \( \eta_2 \), where \( \eta_2 \) phenomenologically takes into account the transition to the normal state as \( v \) reaches the pairbreaking velocity \( v_\Delta \). For \( \eta_1 < 8\eta_2 \), the force \( f_v(v) \) always increases as \( v \) increases, but for \( \eta_1 > 8\eta_2 \) the N-shaped dependence \( f_v(v) \) develops, as shown in Fig. 6. For \( \eta_2 = 0 \), and \( \eta_1 = \eta_0 \), the drag force reaches the maximum value \( F_m = \eta_0 v_0/2 \) at \( v = v_0 \).

The LO instability was originally associated with acceleration of normal quasiparticles in the vortex core by electric field, which can increase their energy above \( \Delta \). In this case quasiparticles can escape the core if the diffusion length \( L_D = (D\tau_e)^{1/2} \) exceeds the core size. The resulting quasiparticle depletion in the core reduces the core size and the vortex viscosity according to Eqs. (17) and (18). However, Eq. (17) is actually more general and may result from several different mechanisms. In particular, the velocity dependence (17) can result from coupling of the vortex motion with other diffusion process, including quasiparticle or temperature diffusion around the moving vortex core. For example, the electron overheating of the core can lead to Eq. (17) as follows.

The power \( \eta v^2 \) generated by the viscous drag increases the electron temperature in the core \( T_m \), reducing the vortex viscosity \( \eta(T_m) \propto B_{c2}(T_m) \). Linearizing \( \eta(T_m) \simeq \eta_0(1 - T_m/T_c) \), we write the thermal balance condition

\[
(T_m - T_0)g = \eta_0(1 - T_m/T_c)v^2,
\]  

(20)

where \( g \sim \pi k/\ln(L_\theta/L_\xi) \) defines the heat flux from the core due to the thermal conductivity \( k(T) \), where the thermal length \( L_\theta \) is of the order of the film thickness \( d \) for ideal cooling, and \( L_\xi \sim \sqrt{\xi\lambda} \) is the length related to the amplitude of vortex penetration. This logarithmic factor will be calculated in the next section in more detail; here we just use \( \ln(L_\theta/L_\xi) \sim \ln(d/\sqrt{\xi\lambda}) \) for qualitative estimates. Solving Eq. (20) for \( T_m \) results in the velocity-dependent \( \eta(T_0) \) of the LO form:

\[
\eta = \eta_0(1 - T_0/T_c)/(1 + \eta_0 v^2/g),
\]  

(21)

\[
\eta_0 = (g/\eta_0)^{1/2} \sim (\pi k_B v^2/\xi^2)^{1/2} \sim (\pi k_B v^2/\xi^2)^{1/2}
\]  

(22)

Substituting here the low-T quasiparticle thermal conductivity \( k \sim k_n(\Delta/T)\exp(-\Delta/T) \), and using the Wiedemann-Franz law \( k_n\rho_n = (\pi k_B v^2/\xi^2)^{1/2} \), we write the thermal balance condition

\[
\tau_e \sim \frac{h T}{\Delta^2} \ln \left( \frac{L_\theta}{L_\xi} \right) \exp \left( \frac{\Delta}{T} \right)
\]  

(23)

exhibits the exponential temperature dependence similar to the energy relaxation time \( \tau_e \) between quasiparticles and phonons\(^{19}\) in the LO theory. However, the exponential dependence of \( \tau_e(T) \) in the thermal model is cut off at lower \( T \) where \( k \) is limited by phonons. To estimate \( v_0 \), we take \( \rho_n = 0.2\mu 0 \), \( k = 10^{-2} \text{W/mK} \), \( B_{c2} = 23 \text{T} \) for Nb\(_3\)Sn at low temperatures\(^{45}\) and \( L_\xi \sim \sqrt{\xi\lambda} \sim 16 \text{ nm} \), \( L_\theta \sim d \sim 1 \text{ mm} \), \( \ln(L_\theta/L_\xi) \sim 11 \). For these parameters, Eq. (22) gives \( v_0 \sim 0.1 \text{km/s} \), much smaller than the estimates for \( v_\Delta \) and \( v_m \). Thus, overheating does result in the same Eq. (17), although in this case vortex core expands as it becomes warmer at higher velocities\(^{32,33}\), in contrast to the LO core shrinkage. Moreover, the critical velocity \( v_0 \) defined by Eq. (22)

remains constant at \( T_c \), unlike vanishing \( v_0(T_0) \) for the LO mechanism, which dominates at \( T \approx T_c \). However, for \( T \to T_c \), both Eq. (18) and (22) predict the critical velocity \( v_0(T) \) to exceed the linear viscous drag-limited velocity \( v_m(T) \approx (1 - T/T_c)^{1/2} \) given by Eq. (15).

As a result, Eq. (1) adequately describes vortex dynamics at strong fields \( B_0 \approx B_{c2} \) and temperatures close to \( T_c \).

To evaluate the overheating mechanism in more detail, we assume that \( \eta(T_m) \) depends on a local electron temperature \( T_m(t) \) in the vortex core. The distribution of \( T(r, t) \) around a moving vortex is described by a thermal diffusion equation,

\[
\dot{C}T = k\nabla^2 T - \alpha(T - T_0) + \eta(T_m)\psi^2(t)f(x - u(t), y),
\]  

(24)

which, after re-definition of the coefficients, can be reduced to the same mathematical form as the diffusion
equation for nonequilibrium quasiparticles. Here $C$ is the heat capacity, $k$ is the thermal conductivity, $u(t)$ is the coordinate of the vortex core moving with the velocity $v(t)$, and the function $f(x,y)$ accounts for the finite core size, so that $\int f(r)d^2r = 1$. The term $\alpha(T - T_0)$ describes heat exchange with the environment. For example, $\alpha = h/d$ in a thin film of thickness $d$ where $h$ is the Kapitza conductance at the sample surface, and $T_0$ is the bath temperature. For electron overheating, the parameter $\alpha = C/\tau_r$ describes heat exchange between electrons and the lattice, where $C$ is the electron specific heat and $\tau_r$ is the time of inelastic scattering of quasiparticles in the vortex core on phonons.\textsuperscript{33,49,50,51}

The last term in the r.h.s. of Eq. (24) describes dissipation in the vortex core proportional to the viscosity $\eta(v, T_m)$ taken at the local core temperature $T_m(t)$, which, in turn, depends on $v(t)$. The core form factor $f(r)$ is modeled by the Gaussian function $f(r) = \pi^{-\frac{1}{2}}e^{-r^2/\xi^2}$, where the core radius $\xi$ can be smaller than $\xi_c$ at $T < T_c$ due to the Kramer-Pesch effect\textsuperscript{22,24} (the solutions of Eq. (24) depend weakly on $\xi$). We also consider weak overheating for which dependencies of $C$, $k$ and $\alpha$ on $T$ can be neglected. As shown below, $T$ can be regarded as either the electron or the lattice temperature, depending on the time scale of the vortex dynamics involved.

The solution of Eq. (24) given in Appendix A, results in the following integral equation for the temperature $T_m(t)$ in the vortex core moving with a time-dependent velocity $v(t)$ near the surface:

$$T_m(t) = T_0 + \frac{1}{\pi k} \int_0^\infty dt' q(t-t')e^{-t'/t_0} \times$$

$$\{ \exp \left[ -\frac{|u(t) - \eta(T_m(t), v(t))v^2(t)|^2}{4t' + t_s} \right] \pm$$

$$\exp \left[ -\frac{|u(t) + \eta(T_m(t), v(t))v^2(t)|^2}{4t' + t_s} \right] \}, \quad (25)$$

where $q(t) = \eta(T_m(t), v(t))v^2(t)$ is the time-dependent power generated by the moving vortex, $D = k/C$ is the thermal diffusivity, $t_s = \xi^2/D$ is the diffusion time across the vortex core, and $t_0 = C/\alpha$ is the electron energy relaxation time. The second term in the parenthesis describes the effect of the surface: the plus sign corresponds to the thermally-insulated surface, $\partial_x T(x,t)|_{x=0} = 0$, and the minus sign corresponds to the ideal cooling, $T(0,t) = T_0$. Here we do not consider microscopic thermal gradients inside the vortex core\textsuperscript{52,53} assuming that their effect is included in the bare $\eta$.

The integral Eq. (25) takes into account retardation effects due to diffusive redistribution of $T(r, t)$ around an accelerating vortex, so $T_m(t)$ depends on the vortex velocity $v(t-t')$ at earlier times. The effect of the surface makes $T_m(t)$ dependent on the vortex coordinate $u(t)$ as well. Eq. (25) simplifies considerably if $v(t)$ varies slowly over the relaxation time $t_0$, and $t_s \ll t_0$, and $u(t) > \xi_c$. Then $q(t-t')$ can be taken out of the integral at $t' = 0$, and Eq. (25) yields the following equation for the local temperature difference $\delta T_m = T_m - T_0$:

$$\delta T_m = \frac{\eta(T_m)v^2}{2\pi k} \left[ \ln \frac{L_\theta}{\xi} \pm K_0 \left( \frac{2\mu}{L_\theta} \right) \right] \quad (26)$$

Here $\xi_c = \xi L_\theta/2 \approx 0.67\xi_c$, and $\gamma = 0.577$ is the Euler constant.\textsuperscript{22,54} The second term in the parenthesis decreases exponentially for $u > L_\theta$, and logarithmically, $K_0(z) \approx \ln(2/z) - \gamma$ for $z \ll 1$. In this case the expression in the parenthesis reduces to $\ln(L_\theta/\xi) + \ln(L_\theta/\dot{u})$, where $\dot{u} = u/e_\gamma/2$. Taking the characteristic amplitude of vortex penetration $\dot{u} \sim \lambda$, we can present the logarithmic part in the form $2\ln(L_\theta/\xi) + \ln(L_\theta/\dot{u})$, where $L_\xi = (u/\dot{u})^{1/2} \sim \sqrt{\xi/\lambda}$ was used before to obtain Eq. (17) in the thermal model. The weak logarithmic dependence of $\eta$ on $L_\xi$ and $L_\theta$ makes Eq. (22) nearly insensitive to the details of heat transfer and the behavior of $u(t)$.

In the other limiting case of very rapid variation of $v(t)$, the vortex reaches the critical velocity $v_0$ and then jumps by the distance $\delta u$, dissipating the energy $W$ during the short time $\delta t$. Then $q(t) = v_0 W/\delta t$ and Eq. (25) results in the following implicit equation for $\delta T_m(t)$ at $t > \delta t$:

$$\delta T_m(t) = \frac{\eta(T_m)v^2}{2\pi k} \ln \frac{L_\theta}{\xi} + \frac{W(T_m)}{4\pi kt} e^{-\delta u^2/4Dt-t/t_0}, \quad (27)$$

which describes a temperature spike in the core followed by relaxation of $\delta T_m(t)$, as shown in Fig. 7. Here the first term in the r.h.s. gives $\delta T_m$ before the jump, and the effect of the surface is neglected.

Next we consider the steady-state temperature field $T(r)$ averaged over rf oscillations, where $T(r)$ is determined by the balance of the vortex heat source and thermal diffusion. Solution of the thermal diffusion equation in Appendix A yields the following distribution of $\delta T(r) = T(r) - T_0$ from a heat source localized at the thermally-insulated surface ($x = 0$) of a slab of thickness $d$, ideally cooled from the other side, $\delta T(d) = 0$:

$$\delta T(r) = \frac{1}{2\pi k} \int_0^d q(x') \ln \frac{\cosh \frac{\pi x'}{2d} + \cos \frac{\pi(x+x')}{2d}}{\cosh \frac{\pi x'}{2d} - \cos \frac{\pi(x+x')}{2d}} \, dx' \quad (28)$$

FIG. 7: Temperature spike in the vortex core after the jump described by Eq. (27) for $t_0 = 10u_d$, $\delta u = \delta u^0/4D$, $\delta T_0 = Wd/\pi k\delta u^2$, and $T_d = T_0 + \eta_0 v^3_0\ln(L_\theta/\xi)/2\pi k$. 

\textsuperscript{22,24}
Here \( q(x) \) is the power density averaged over the rf period. On scales greater than the size of the heat source, \( \delta T(r) \) depends only on the total power \( Q = \int_0^\infty q(x) dx \):

\[
\delta T(r) = \frac{Q}{2\pi k} \ln \frac{\cosh(\pi y/2d) + \cos(\pi x/2d)}{\cosh(\pi y/2d) - \cos(\pi x/2d)},
\]

Eq. (29) reduces to Eq. (26) with \( u \sim r_0 \) and \( L_\theta \sim d \).

The physical meaning of \( T \) in the above formulas depends on the relevant vortex time scales. For example, for the supersond vortex penetration or vortex jumps on the time scale much shorter than the electron-phonon energy relaxation time, the quasiparticle are not in equilibrium with the lattice, and \( T(r,t) \) in Eq. (27) can be regarded as an electron core temperature. However, steady-state vortex oscillations in the rf field generate a dc power, which must be transferred to the coolant through phonons. In this case Eqs. (28) describes the lattice temperature distribution around a vortex if the phonon mean free path is shorter than the film thickness. Thus, the vortex oscillates in a "warm tunnel" with the lattice temperature \( \delta T(r) \) shown in Fig. 8, but in addition to that the vortex core gets overheated with respect to the lattice during short periods of rapid acceleration, jumps or annihilation with antivortices, as described before.

**B. Jumpwise vortex penetration**

For the LO vortex drag coefficient \( \eta(v) \) given by Eq. (17), the equation of motion becomes

\[
\frac{\eta_0 v}{1 + \alpha^2 v^2} = \frac{\phi_0 B_0}{\mu_0 \lambda} e^{-u/\lambda} \sin \omega t - \frac{\phi_0^2}{2\pi \mu_0 \lambda^3 K_1} \left( \frac{2}{\sqrt{u^2 + \xi_s^2}} \right).
\]

The nonmonotonic velocity dependence of the viscous drag force in the l.h.s. of Eq. (32) qualitatively changes vortex dynamics as \( v(t) \) exceeds the critical value \( v_0 \) for which the viscous force reaches the maximum \( F_m = \eta_0 v_0/2 \). Indeed, the differential equation for \( u(t) \) has the form \( \eta_0 u/\alpha^2 \eta_0 = F = F(u,t) \), where \( F \) is the net electromagnetic force given by the r.h.s. of Eq. (32). We can introduce the ratio \( P \) of the maximum Lorentz driving force at \( B_0 = B_v \) to the maximum viscous force:

\[
P = \frac{2\phi_0 B_v}{\mu_0 \lambda \eta_0 v_0}.
\]

As shown above, the Bardeen-Stephen viscous flow results in unphysically high vortex velocities, indicating that \( P \gg 1 \) and the dependence of \( \eta \) on \( v \) must be taken into account. However, in this case there are regions at the surface where \( F(u,t) \) exceeds \( F_m \) as shown in Fig. 9. In these regions the force balance Eq. (32) cannot be satisfied and the vortex jumps to the place where \( F(x) \leq F_m \). To see how it happens, we present the quadratic equation Eq. (32) for \( u \) in the form

\[
\dot{u} = v_\pm(F) = \frac{v_0 F_m}{F(u,t)} \left[ 1 \pm \sqrt{1 - \frac{F^2(u,t)}{F_m^2}} \right],
\]

where \( F_m = \eta_0 v_0/2 \). For \( v < v_0 \) and \( F < F_m \), Eq. (34) with the minus sign in the brackets reduces to Eq. (1). Penetration of the vortex at \( B(t) > B_v \) is therefore described by the first order differential equation \( \dot{u} = v_-(F) \), which is well defined only if \( F(u,t) \leq F_m \), otherwise the driving force exceeds the maximum friction force, and the square root in Eq. (34) becomes imaginary. Vortex dynamics in this case can be understood from Fig. 9, which shows an instantaneous profile of \( F(u,t) \) for \( B(t) > B_v \). Here a vortex enters the sample with zero
The vortex first moves from \( u = 0 \) to \( u = u_1 \), then jumps from point 1 to point 2 after which it moves continuously as described by Eq. (32) until the next jump and annihilation with the antivortex on the way back.

As mentioned above, the LO instability facilitates a quick evolution of the vortex semi-loop originating at a 3D surface defect into a straight vortex parallel to the surface. Indeed, as evident from Fig. 2, the vortex propagation velocity is maximum for the segments of the semi-loop perpendicular to the surface because they are driven by the continuously increasing maximum Lorentz force \( F_L = B(t)\phi_0/\mu_0\lambda \), while experiencing no counterbalancing image forces. As a result, the LO instability first occurs for the perpendicular vortex segments, causing them to jump along the surface to the place where the viscous force is able to balance the Lorentz force. However, unlike the parallel vortex segments whose jump distance \( \sim \lambda \) perpendicular to the surface is limited by the London screening, the jump length along the surface of a semi-infinite sample is infinite because \( F_L \) remains constant. Thus, the vortex semi-loop turns into a straight vortex in a jumpwise manner when the lateral velocity of the perpendicular vortex segments reach ±\( v_0 \).

Several points should be made regarding the jumpwise vortex dynamics. First, the vortex trajectory \( u(t) \) comprised of the jumps connected by smooth parts described by the equation \( \dot{u} = v_0(F) \) occurs only if \( \dot{F}(u_j, t_j) < 0 \) at the jump points where \( v(t_j) = \pm v_0 \). However, at higher frequencies, or as the vortex overheating is taken into account, there are situations when \( \dot{F}(u_j, t_j) > 0 \). In this case the vortex velocity after the jump exceeds \( v_0 \) and the smooth parts of \( u(t) \) are described by both branches \( \dot{u} = v_\pm(F) \), as shown in Appendix B.

The second point is that, for the overdamped dynamics described by Eq. (32), the jumps occur instantaneously unless the second ascending branch due to \( n_2 \) term in Eq. (19) is taken into account. However, this branch in the LO model corresponds to very high velocities \( \sim v_\Delta \), for which the adequate theory of the nonequilibrium vortex core structure and the vortex drag force is lacking. In our phenomenological London approach we assumed that the vortex jumps to the nearest point \( u_2 \) where the friction force is able to balance the driving force \( F(u_2) = F_m \). However, the instantaneous LO dependence \( \eta(v) \) does not include retardation effects due to finite relaxation times of the superconducting order parameter, or diffusive redistribution of nonequilibrium quasiparticles or temperature around a rapidly accelerating/decelerating
vortex core. These effects are taken into account by the integral Eq. (25) for the core temperature $T_m(t)$, which shows that the vortex jump time and length are affected by intrinsic dynamics of $\eta$. Thus, there is a diffusion time scale $\delta t \sim \delta u^2 / D$ for the vortex jump by the distance $\delta u$, where $D$ equals either $k/C$ in the model or the quasiparticle diffusivity in the LO theory. In the thermal model this estimate gives $\delta t \sim 4 \times 10^{-2} \text{s}$ if we take $\delta u \simeq \lambda = 65 \text{nm}$, $\kappa \simeq 0.1 \text{W/mK}$ and $C \simeq 100 \text{J/m}^3\text{K}$ for Nb, or even much shorter time for Nb, for which $\lambda = 40 \text{nm}$ and $\kappa \simeq 10 \text{W/mK}$. The so-estimated $\delta t$ is smaller than the inverse gap frequency, indicating that once the overheated core gets in the region where the lattice temperature can be close to the lattice temperature if $\tau\omega \simeq 1$, the quasiparticle diffusivity in the LO theory. In the thermal model $\delta t$ may be limited by much slower phonon irradiation from the overheated core. At the same time, the electron temperature relaxation time $\tau_e$ outside the core results from a slow phonon-mediated recombination of quasiparticles:

$$\tau_e \simeq \tau_0 \left( \frac{\Delta}{T} \right)^{1/2} \exp \left( \frac{\Delta}{T} \right),$$

(36)

which yields $\tau_e \sim 30 \text{ ns}$ much longer than the thermal diffusion time $\delta t$ for Nb at $2 \text{K}$. For $\omega \tau_e \ll 1$, the quasiparticles are overheated with respect to the lattice, in a highly inhomogeneous way according to the distribution of the lattice temperature $T(\mathbf{r})$ shown in Fig. 8. In this case the condition $\omega \tau_e(\mathbf{r}) > 1$ of the electron overheating can locally be satisfied in colder regions away from the core, but near the vortex core the electron temperature can be close to the lattice temperature if $\omega \tau_e(\mathbf{r}) < 1$ because of higher $T(\mathbf{r})$, which greatly accelerates the energy exchange between electron and phonons. For example, for Nb at $T_0 = 2 \text{K}$, the local increase of the lattice temperature to $T = 4 \text{K}$ yields $	au_e(T) \simeq \tau_e(T_0) \exp(-\Delta/T_0 + \Delta/T)$, giving $\tau_e \sim 0.3 \text{ ns}$, and $\omega \tau_e \sim 1$ at $1 - 2 \text{ GHz}$.

Another contribution to $\delta t$ comes from a finite vortex mass $M$. In the Suhl model $M = 2mk_F/\pi^3$ is due to localized electrons in the vortex core, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave vector. The jump by $\lambda$ due to the driving force $F = \phi_0 B_v/\mu_0\lambda$ takes the time $\delta t$ set by the Newton law $2\lambda/\delta t^2 \simeq F$. Using $B_v = \phi_0/4\pi \lambda$, $\lambda^2 = \mu_0 m/\pi e^2$, $\xi = h v_F/\pi \Delta$, we obtain

$$\delta t \simeq \frac{4h}{\Delta} \kappa^{1/2},$$

(37)

A more accurate account of quantized levels in the vortex core or lattice deformation around the moving vortex can increase the vortex mass, thus further increasing the jump time $\delta t$. Yet, although vortex jumps are quantified by the multiple relaxation times discussed above, they seem to occur much faster than the rf periods we are dealing with in this work.

Power $Q$ dissipated with the account of the jumpwise vortex instabilities can be written in the form

$$Q = \frac{\omega}{\pi} \left( \oint \eta(\dot{u})\dot{u}^2 dt + \oint \eta(\dot{u})(\dot{u}_1^2 + \dot{u}_2^2) dt + \sum_m \left[ G(t_m, u_{m+}) - G(t_m, u_{m-}) \right] \right),$$

(38)

where the integrals are taken over all smooth parts of the vortex trajectory $u(t)$, including the vortex-antivortex annihilation parts, like in Eq. (11). The last term in Eq. (38) is the sum of energies released during all vortex jumps at $t = t_m$, from $u = u_{m-}$ to $u = u_{m+}$, where the instantaneous free energy $G(u, t)$ is given by Eq. (12). Shown in Fig. 11 is the second vortex penetration field $B_2$ calculated by solving Eqs. (32), (35), and (38) numerically for the parameters of Nb. Here $B_2$ cannot be fit with a power-law similar to Eq. (10).

The LO instability makes the behavior of $Q(B_0, \omega)$ more complicated as compared to $Q(B_0)$ described by Eq. (13). As shown in Fig. 12, there are three distinct field regions of very different vortex dynamics. The pure Bardeen-Stephen dynamics like that shown in Fig. 3 is limited to a very narrow region of $B_0$ close to $B_v$ (labeled by $a$ in the inset). In this case the vortex penetration depth $u_m$ turns out to be smaller than $\xi$, indicating that the London theory combined with the Bardeen-Stephen drag cannot give a self-consistent description of vortex dynamics at low temperatures. However, Eq. (11) adequately describes rf vortex dynamics at higher $T$ close to $T_c$ where the LO instability is irrelevant because the critical velocity $\nu_0 \propto (1 - T/T_c)^{1/4}$ becomes larger than $v_m \propto (1 - T/T_c)^{1/2}$.

The parts of the $Q(B_0)$ curves labeled $a$ in Fig. 12 correspond to an intermediate case, for which the most part of $u(t)$, including the initial acceleration of the vortex, reaching the maximum penetration depth $u_m$, and turning back, does not involve the LO instability. However, as the velocity of the exiting vortex exceeds $-v_0$, it jumps.
to the surface and disappears, significantly increasing the dissipated power $Q$. Further increase of $B_0$ corresponds to the parts of $Q(B_0)$ curves labeled c, for which the LO instabilities occur both on the penetration and the exit parts of $u(t)$, like those in Fig. 10. In this case $Q(B_0)$ jumps up to a much higher level $Q \sim Q_0$ until the second penetration field $B_2$ is reached. Here the behavior of $Q(B_0)$ can also depend on $\omega$: for lower frequency (curve 1), $Q(B_0)$ increases weakly between $0.4 < \epsilon(B_0) < 0.8$, but for $\epsilon = (B_0 - B_1)/(B_2 - B_1) > 0.8$, the power $Q(B_0)$ jumps down. This behavior reflects the change in the vortex dynamics: for $0.4 < \epsilon < 0.8$, the vortex jumps out of the sample before the antivortex enters, while for $\epsilon > 0.8$, the jump of the vortex toward the surface is accompanied by the penetration of the antivortex and their annihilation, like that in Fig. 10. For higher frequencies, this change in the vortex dynamics formally occurs only for $B_0 > B_2$, so the down step in $Q(B_0)$ does not show up in curve 2 in Fig. 12.

D. Thermal self-localization of vortex penetration

Local temperature increase around oscillating vortex reduces both critical fields $\tilde{B}_v = B_v(T_m)$ and $\tilde{B}_2 = B_2(T_m)$ as compared to their isothermal values $B_v(T_0)$ and $B_2(T_0)$. To evaluate this effect we combine Eq. (11) for $Q$ and Eq. (14) for $T_m$ and obtain for $\omega \tau_2 \ll 1$:

$$\delta T_m \simeq \frac{2\omega \phi_0 B_v}{\pi^2 \mu_0 k \ln(4d/\pi r_0)},$$

(39)

Next we linearize $\tilde{B}_v \simeq B_v - |\partial_x B_v(T_0)| \delta T_m$ with respect to $\delta T_m$, where $B'_v = \partial_x B_v(T_0)$. Then the effective field $\tilde{B}_v$ and $\tilde{B}_2$ take the form:

$$\tilde{B}_v = (1 - b)B_v, \quad \tilde{B}_2 = (1 - b)B_2,$$

(40)

$$b = \frac{2\omega \phi_0 |B'_2|}{\pi^2 \mu_0 k \ln(4d/\pi r_0)}$$

Thus, both $B_2$ and $B_c$ are reduced by dissipation, which only produces constant shifts of local $B_v$ values but does not change their initial distribution. For Nb, taking $k = 10$ W/mK, $B'_v \sim B_v/T_c$, $B_v = 0.15$ T, $\omega/2\pi = 2$ GHz, $\ln(4d/\pi r_0) \simeq 10$, as before, we obtain a small value $b \sim 3 \times 10^{-4}$, for which the shift of $B_v$ is negligible. However, for Nb$_3$Sn, with $\kappa \sim 10^{-2}$ W/mK, we get a much higher value $b \sim 0.3$, indicating that dissipation can significantly reduce $B_v$ and $B_2$, expanding the field region $B_v < B_2$, of individual vortex penetration.

The condition of the single-vortex penetration $B_v < B_0 < B_2$ implies that the local value of $B_2$ is smaller than the uniform superheating field $B_s$. Multiple vortex penetration for $B_0 > B_2$ causes strong dissipation, further decreasing $B_2$ and resulting in avalanche-type dendritic vortex penetration.$^{60,61}$ Such thermo-magnetic dendritic flux avalanches have been observed in both low-$T_c$ and high-$T_c$ superconductors.$^{62,63,64,65,66,67}$ Notice that the superfrost vortex penetration through the surface barrier due to the jumpwise LO vortex instability may pertain to the supersonic vortex velocities observed for dendritic vortex penetration in YBa$_2$Cu$_3$O$_{7-\delta}$ and YNi$_2$B$_2$C films$^{62,67}$.

Temperature distribution$^{2f}$ results in the long-range dc repulsion force $f_T(L) = -s^*T$ between two oscillating vortices spaced by $L$. Here $s^*(T)$ is the vortex transport entropy responsible for thermomagnetic effects in the mixed state.$^{68,69}$ In thick films $d > \lambda$, vortices are localized at the surface, so to calculate the thermal force $f_T(L)$, we put $x = 0$ in Eq. (24) and obtain

$$f_T(L) = \frac{s^*Q}{2dk \sinh(\pi L/2d)}$$

(42)

This long-range force on the scale much greater than $\lambda$ results in repulsion of neighboring vortex penetration channels, facilitating bending instability and dendritic branching of the multi-vortex tracks.

IV. EFFECT OF PINNING

A. Trapping rf vortices at strong fields, $B_0 \sim B_v$

At the initial stage of vortex penetration, $B(t) \simeq B_v$, the driving force $F_L$ is much stronger than typical pinning forces by materials defects. However, as the vortex moves deeper in the sample, the force $F_L(u) \propto \exp(-u/\lambda)\sin \omega t$ decreases exponentially, so pinning becomes more effective if the vortex trajectory passes a pin aligned with the
The coordinate $y$ along the surface, and the last term is the sum of the elementary pinning forces $f_p(x, y)$. For $\ell > \lambda$, the dispersive vortex line tension $\epsilon$ reduces to the vortex self energy $\epsilon = \phi_0 B_m / \mu_0$ per unit length.

As evident from Fig. 13, the magnetic attraction to the surface makes the pinned vortex not straight even at zero rf field. As a result, there is a minimum trapping distance $d_m$, so that only vortices spaced by $u > d_m$ can be pinned. Vortices spaced by $u < d_m$ are unstable and annihilate at the surface, since the image attraction prevails over pinning. For weak identical pins, $d_m$ can be evaluated from the force balance equation:

$$\frac{\phi_0^2}{2\pi \mu_0 \lambda^3} K_1 \left( \frac{2d_m}{\lambda} \right) = \frac{f_p}{\ell},$$  \hspace{1cm} (45)

where $f_p$ is the maximum elementary pinning force. The vortex segments between the pins bow out toward the surface, but for vortices in the trapped flux zone $x > d_m$, the curvature of $u(y)$ is weak, and the image attraction force $F_i = (\phi_0^2 / 2\pi \mu_0 \lambda^3) K_1 (2u/\lambda)$ is nearly uniform. In this case the equilibrium shape of the vortex segment between the pins is determined by the equation $e(u_0 = F_i$ with $u_0(\pm \ell / 2) = d$, which yields the parabolic profile:

$$u_0(y) = d - u_0\left( 1 - \frac{4\mu^2}{\ell^2} \right),$$  \hspace{1cm} (46)

$$u_m = \frac{\phi_0^2 \ell^2}{16\pi \mu_0 \epsilon \lambda^3} K_1 \left( \frac{2d_m}{\lambda} \right) = 8 \kappa \ln \kappa.$$  \hspace{1cm} (47)

Here $\epsilon = \phi_0^2 (\ln \kappa + c_v) / 2\pi \mu_0 \lambda^2$ where the constant $c_v \approx 0.5$ accounts for the vortex core contribution. It is convenient to use the effective $\kappa \approx 1.65 \kappa$ defined by $\ln \kappa = \ln \kappa + c_v$. Eqs. (46) and (47) correspond to $\ell > \lambda$, so the condition that $u_m \ll \lambda$ is provided by $d_m > \lambda$, as follows from Eq. (45). For denser pins, $\ell < \lambda$, the nonlocal expression for $\epsilon$ should be used, in which case $\ln \kappa$ in Eq. (47) is to be replaced by $\ln (\ell / \kappa)$.

To calculate the power $Q_o$ dissipated by the pinned vortex under the weak ($B_0 \ll B_o$) rf field, we seek the solution of Eq. (45) in the form $u(y, t) = u_0(y) + \delta u(y, t)$. Here the Fourier component $\delta u_m(y) = \int \delta u(y, t) e^{-i\omega t} dt$ of the oscillating vortex displacement $\delta u(y, t)$ satisfies the linearized equation

$$i\omega \eta_0 \delta u_m = \epsilon \delta u_m'' + f_o,$$  \hspace{1cm} (48)

where $f_o = \phi_0 B_0 \exp(-d/\lambda) / \mu_0 \lambda$. In Eq. (48) we neglect the image force $\propto \exp(-d/\lambda) \delta u$, which is much smaller than the first elastic term in the r.h.s. for weak pinning and $d > d_m$ defined by Eq. (45). The solution of Eq. (48) with the boundary condition of the fixed vortex ends at the pins, $\delta u_m(\pm \ell / 2)$, has the form

$$\delta u_m = \frac{f_o}{i\omega \eta_0} \left( 1 - \frac{\cos[(1-i)\Omega_p^2/y/\ell]}{\cos((1-i)\Omega_p^2/2)} \right),$$  \hspace{1cm} (49)

$$\Omega_p = \omega \tau_p, \quad \tau_p = \eta_0 \ell^2 / 2\epsilon.$$  \hspace{1cm} (50)
Here we introduced the pinning relaxation time constant \( \tau_p \) and the dimensionless frequency \( \Omega_p \). The dissipated power per unit vortex length,

\[
Q_v = \frac{\eta_0 \omega^2}{2 \ell} \int_{-\ell/2}^{\ell/2} \delta u_\omega(y)^2 \, dy
\]

is given by Eq. \((51)\). A straightforward integration then yields

\[
Q_v = \frac{f_0^2}{2 \eta_0} \Gamma_\omega(\sqrt{\Omega_p}),
\]

\[
\Gamma_\omega(z) = 1 - \frac{\sin z + \sin z}{z (\cosh z + \cos z)}
\]

which has been often observed on many superconductors at low temperatures. In this case even a few pinned vortices can result in \( R_t \) comparable to the exponentially small \( R_{bc}(T) \). This scenario was first suggested by Rabinowitz who modeled pinning by a phenomenological spring constant and did not consider the critical depinning spacing \( d_m \) due vortex attraction to the surface. The account of a more realistic discrete pin structure in Fig. 13, and the gradient of the Lorentz force changes the frequency dependence of \( R_t \) and results in new effects discussed below.

2. \( R_t \) increases significantly as the superconductor gets dirtier. To evaluate this effect, we make a rough estimate Eq. \((52)\) gives the frequency-independent limit, \( Q_{vm} \to f_0^2/2 \eta_0 \) inversely proportional to \( \eta_0 \). However, for \( \omega \tau_p \ll 1 \), we obtain the quadratic frequency dependence, \( Q_v = f_0^2 \eta_0 \ell^4 \omega^2/240 \pi \), proportional to \( \eta_0 \).

The vortex rf power is \( Q_v/a = B^2_0/R_t/2 \mu_0^2 \) per unit surface area region in the additional surface resistance \( R_t \):

\[
R_t = \frac{\phi_0^2 (e^{-2d/\lambda} \lambda)}{\lambda^2 \eta_0 a} \Gamma_\omega(\sqrt{\Omega_p}),
\]

where \( a \) is a mean spacing between pinned vortices and \( \langle \cdot \rangle \) means averaging over the vortex positions \( d \) in the direction perpendicular to the surface. Since these positions must satisfy the stability condition \( d > d_m \), the main contribution to \( R_t \) comes from vortices in the critical belt \( d_m < d < d_m + \lambda \) where \( e^{-2d/\lambda} \) is maximum. For \( \omega \tau_p \ll 1 \), Eq. \((54)\) simplifies to

\[
R_t = \frac{\pi \mu_0^2 \omega^2 \eta_0}{60 \eta_0 a \ln^2 \kappa} \Gamma_\omega(\sqrt{\Omega_p}),
\]

where we used the Bardeen-Stephen \( \eta_0 \) and took \( \epsilon = \phi_0^2 \ln \kappa/2 \pi \mu_0 \lambda^2 \) for \( \lambda > \lambda \ln \kappa \to \ln(\ell/\xi) \) for \( \xi < \ell < \lambda \). These results show that:

1. For \( \omega \tau_p \ll 1 \), the frequency dependence of \( R_t \) is similar to the BCS surface resistance, \( R_{bc}(\omega) \propto \omega^2 \exp(-\Delta/\omega) \) at \( T \ll T_c \). However, unlike \( R_{bc}(T) \), the vortex contribution \( R_t \) remains finite at \( T \to 0 \), so trapped vortices can contribute to the non-BCS excess surface resistance, which has been often observed on many superconductors.

2. \( R_t \) increases significantly as the superconductor gets dirtier. To evaluate this effect, we make a rough estimate Eq. \((52)\) gives the frequency-independent limit, \( Q_{vm} \to f_0^2/2 \eta_0 \) inversely proportional to \( \eta_0 \). However, for \( \omega \tau_p \ll 1 \), we obtain the quadratic frequency dependence, \( Q_v = f_0^2 \eta_0 \ell^4 \omega^2/240 \pi \), proportional to \( \eta_0 \).

The vortex rf power is \( Q_v/a = B^2_0/R_t/2 \mu_0^2 \) per unit surface area region in the additional surface resistance \( R_t \):

\[
R_t = \frac{\phi_0^2 (e^{-2d/\lambda} \lambda)}{\lambda^2 \eta_0 a} \Gamma_\omega(\sqrt{\Omega_p}),
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where \( a \) is a mean spacing between pinned vortices and \( \langle \cdot \rangle \) means averaging over the vortex positions \( d \) in the direction perpendicular to the surface. Since these positions must satisfy the stability condition \( d > d_m \), the main contribution to \( R_t \) comes from vortices in the critical belt \( d_m < d < d_m + \lambda \) where \( e^{-2d/\lambda} \) is maximum. For \( \omega \tau_p \ll 1 \), Eq. \((54)\) simplifies to

\[
R_t = \frac{\pi \mu_0^2 \omega^2 \eta_0}{60 \eta_0 a \ln^2 \kappa} \Gamma_\omega(\sqrt{\Omega_p}),
\]
where $z = \ell/\ell_0$, $\Omega_p = \omega_0\ell_0^2/2\ell$, and $R_B = \phi_0^2(\epsilon^{-2d/\lambda})/\lambda^2\eta a$ is the high-frequency limit of $R_i(\omega)$. The resulting behavior of $R_i(\omega)$ is shown in Fig. 14: for low frequencies, the contribution of weakly pinned large-length segments makes $R_i(\omega)$ higher than $R_i(\omega)$, while for high frequencies, the contribution of strongly pinned small segments makes $R_i(\omega)$ lower than $R_i(\omega)$. The overall behavior of $R_i(\omega)$ resembles the power law dependence $R_i \propto \omega^{\beta}$ with $\beta \approx 0.5 - 0.7$, which has been observed in $\text{Ph}^{34}$ and $\text{Nb}^{41}$ at 0.1-1 GHz.

To estimate the pinning time constant $\tau_p$, we first evaluate the mean length of the vortex segment $\ell_0$. This can be done from an estimate of the single-vortex pinning force balance $f_p \sim J_c\phi_0\ell_0/\mu_0$, which express $\ell_0$ in terms of the depinning critical current density $J_c$. For core pinning, $f_p \propto \zeta B_0^2\pi \xi^2/\mu_0$, where $\zeta$ accounts for the change in the condensation energy by the pin due to variation of $\delta T_c/T_c$ and the mean-free path $\ell_0$. Hence,

$$\ell_0 \approx \frac{\zeta \phi_0}{8\pi \mu_0 \lambda^2 J_c}$$

Taking $\lambda \approx 40\text{nm}$ and $J_c = 10^8 \text{A/m}^2$ for Nb at $T < T_c$ yields $\ell_0 \approx 10^4\zeta \lambda$. The pinning relaxation time $\tau_p$ can then be obtained from Eqs. (20) and (59):

$$\tau_p \approx \frac{\zeta^2 \phi_0^2}{128\pi^2 \mu_0 \rho_n \lambda^2 \xi^2 J_c^2 \ln \kappa}$$

Taking $\lambda \approx 40\text{nm}$, $J_c = 10^8 \text{A/m}^2$, and $\rho_n = 10^{-9}\Omega \text{m}$ for Nb, we obtain $\tau_p[\text{s}] \approx 10^{-8} \zeta^2$. From the rf measurements of pinning relaxation time $\sim 10^{-3} \text{s}$ in Nb$^{34}$ we then deduce $\zeta \approx 0.1$ and $\ell_0 \approx 10^2\lambda \approx 4\mu_m$. Here $\tau_p$ is rather sensitive to the value of $\zeta$ determined by details of the order parameter suppression at the pin.

It is instructive to compare $Q$ from an oscillating vortex with $Q$ due to the rf electric field $E_i \approx B_0\omega\exp(-d/\lambda)$ induced in the fixed normal vortex core. In the latter case the power $Q_v \propto \pi \xi^2 E_i^2/\rho_n$ gives the surface resistance $R_s \propto \pi \omega \xi^2 \lambda^3 \exp(-2d/\lambda)/\rho_n^2 \kappa^4 a n$, which is by the factor $\kappa^{-4} \ln^2 \kappa \ll 1$ smaller than $R_i$ given by Eq. (55) for an oscillating vortex segment. Thus, for type-II superconductors with $\kappa > 1$ considered in this paper, the inductive contribution $^{70}$ is negligible.

C. Low-field nonlinear surface resistance and rf annealing of trapped magnetic flux

So far we have considered $R_i$ independent of the rf field. However, because $R_i$ is mostly determined by pinned vortices in the critical belt $d_m < d < d_m + \lambda$, the excess resistance $R_i$ may become dependent on weak rf field due to an increase of the critical distance $d_m$ as $B_0$ increases. This effect is evident from Fig. 13, which shows that, because both the image attraction force and the Meissner rf force increase as the vortex moves closer to the surface during the negative rf cycle, the vortex becomes asymmetric and shifted toward the surface. Thus, some of pinned vortices at $d \approx d_m$ can be pushed out of the sample by rf field, which means that $d_m$ effectively increases as $B_0$ increases.

To calculate the effect of the rf field on the mean attraction force $f(d)$ between a vortex and the surface, we average Eq. (41) in small vortex vibrations $\delta u(y, \zeta)$:

$$f = -\frac{\phi_0^2 K_1(2d/\lambda)}{2\pi \mu_0 \lambda^3} - \frac{f_p}{\lambda^2} \int_{-\ell/2}^{\ell/2} dy \sin \omega t \delta u(y).$$

In addition to the first static term in the r.h.s., $f$ contains the rf contribution, in which $\langle \delta u \rangle$ means time averaging over the rf period. Using $\langle \sin \omega t \delta u \rangle = \Re(\delta u \omega)/2$ and Eq. (49) for the Fourier component $\delta u \omega(y)$, we calculate the integral in Eq. (61) as follows:

$$\Re \int_{-\ell/2}^{\ell/2} dy \delta u = -\frac{f_p \Re}{\eta a \Omega_p} \frac{\tan[(1 - i)\Omega_p^2/2]}{1 + i}$$

Taking the real part of this expression, we reduce Eq. (61) to the following equation for $d_m$:

$$f_p = \frac{\phi_0^2 K_1(2d_m/\lambda)}{2\pi \mu_0 \lambda^3} + \frac{\pi \Omega_p^2 B_0^2 \kappa^4 a n}{2\mu_0 \lambda \ln \kappa} \exp(-2d_m/\lambda),$$

which defines the critical pinning depth $d_m$ as a function of the rf amplitude and frequency. For $\tau_p < 1$ and $2d_m > \lambda$, we can use $\Gamma_{\omega}(0) \approx 1$, and $K_1(z) \approx (\pi/2\kappa)^{1/2} \exp(-x)$, in which case Eq. (63) becomes

$$\frac{f_p}{\ell} = \frac{\phi_0^2}{4\pi \mu_0 \lambda^3} \sqrt{\frac{\pi \lambda}{d_m}} \left(1 + \frac{B_0^2}{B_0^2}ight) e^{-2d_m/\lambda}$$

$$B_\phi = \frac{\phi_0}{\pi \lambda \ln \kappa \left(2\Gamma(\Omega_p^2/2)\right)^{1/4}} \left(\frac{\pi \lambda}{d_m}\right)^{1/4}$$

Eq. (65) differs from the static Eq. (45) by the factor $(1 + B_0^2/B_0^2)$, which becomes essential at rather low fields $B_0 \ll B_c$. Substituting $e^{-2d_m/\lambda}$ from Eq. (65) into Eq. (66), we obtain the field dependence of $R_i$:

$$R_i(B_0) = \frac{R_i(0)}{1 + B_0^2/B_0^2}$$

where the zero-field $R_i(0)$ is given by Eq. (55). The field $B_\phi(\omega) = B_\phi(0)/\sqrt{6\Gamma_{\omega}}$ increases as $\omega$ increases, approaching $B_\phi(\omega) = B_{\phi}(0)\Omega_p^{3/4}/\sqrt{6}$ for $\omega \tau_p \gg 1$.

The decrease of $R_i(B_0)$ results from the field-induced shift of the critical belt $d_m < d < d_m + \lambda$ away from the surface, where the screening of the rf Meissner currents reduces vortex dissipation. As a result, the rf field irreversibly pumps parallel vortices out of a superconductor, resulting in a rf “annealing” of the field-cooled trapped magnetic flux. The field dependence of $R_i(B_0)$
is therefore hysteretic: if \( B_0 \) is first increased to a maximum value \( B_{0\text{mix}} \) and then decreased back to zero, \( R_\text{s}(B) \) on the ascending branch decreases according to Eq. (67) as \( B \) causes the vortex field \( B \) to turn, the dependencies of \( R_\text{s} \) on the ascending branch stays at the maximum \( B_0 \) reached on the ascending branch.

As illustrated by Fig. 15, the rf field cycling could reduce the surface resistance by irreversibly pumping a fraction of trapped flux out of the sample. The rf flux annealing considered in this paper is somewhat analogous to a directional motion of magnetic flux induced by transport ac current, resulting in a dc voltage on a superconductor.

\section{V. NONLINEAR HOTSPOTS IN THE SURFACE RESISTANCE}

Localized dissipation due to vortex penetration or oscillation of pinned vortices in thick films produces a long range temperature distribution, which spreads out on the scale \( 2d/\pi \), much greater than \( \lambda \) (see Fig. 8). Even if these temperature variations are weak, \( \delta T(r) = T(r) - T_0 \ll T_0 \), they can nevertheless produce strong variations in the surface resistance \( R_\text{s} \) of the surrounding areas, resulting in nonlinear contributions to \( R_\text{s} \) with very different field and frequency dependencies than \( R_{\text{bcs}}(T, \omega) \). This effect comes from the exponential temperature dependence of the BCS surface resistance,

\begin{equation}
R_{\text{bcs}}(r) \propto \frac{\omega^2}{T} \exp \left[ -\frac{\Delta}{T} + \frac{\delta T(r) \Delta}{T_0^2} \right],
\end{equation}

so that even weak variations \( \delta T(r) \ll T_0 \) can produce strong variations in \( R_{\text{bcs}}(r) \) at low temperatures, \( T_0^2 \ll \delta T \Delta \). Indeed, substituting the surface temperature distribution \( \delta T(0, y) = (Q/\pi k) \rho(y) \gamma y/4d \) from Eq. (29) at \( x = 0 \) into Eq. (71) we obtain:

\begin{equation}
R_\sigma(y) = R_{\text{bcs}}(0, \omega) \coth \left( \frac{\pi y}{4d} \right),
\end{equation}

\begin{equation}
\sigma(B_0, 0, \omega) = Q(B_0, 0, \omega) \Delta(T_0)/\pi k(T_0) T_0^2
\end{equation}

on the scales \( |y| > r_0 \) greater than the size \( r_0 \) of the heat source. Here the exponent \( \sigma \) is proportional to the dissipated power \( Q \), which depends on both \( B_0 \) and \( \omega \). For example, \( Q \propto (B_0^2 - B_x^2)^{\gamma} \) near the onset of the single vortex penetration (see Eq. (14), or \( Q \propto B_0^2 \Gamma(\omega) \)) for a pinned vortex near the surface (see Eq. (52)). In turn, the dependences of \( \sigma \) on \( B_0 \) and \( \omega \) result in a nonlinear contribution to the global surface resistance from sparse "hotspots" of size of the film thickness around much smaller heat sources. These hotspots contributions can have very different dependencies on \( B_0 \) and \( \omega \) as compared to the field-independent \( R_{\text{bcs}} \propto \omega^2 \). For \( \sigma < 1 \), the total excess resistance \( \delta R_\text{s} \) is insensitive to the power distribution \( q(x) \), and can be obtained by integrating Eq. (72) using a new variable \( \varphi = \tan^2(\pi y/4d) \):

\begin{equation}
\delta R_\text{s} = \int_0^{\infty} \left[ 4d/T_0 \right] \left[ \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1 - \sigma}{2} \right) \right] R_{\text{bcs}},
\end{equation}

where \( \psi(x) = d \ln \Gamma(x)/dx \), and \( \Gamma(x) \) is the gamma-function. For \( \sigma \ll 1 \), the expression in the brackets reduces to \( \pi^2 \sigma^2/4 \), thus

\begin{equation}
\delta R_\text{s} = \pi \sigma^2 (B_0, \omega, T_0) R_{\text{bcs}}, \quad \sigma \ll 1.
\end{equation}
If $Q$ is due to pinned vortices, the correction $\delta R_s$ from weak hotspots with $\sigma \ll 1$ is quadratic in $B_0$ and proportional to $\omega^4$ for low frequencies $\omega T_p \ll 1$ [see Eq. (50)]. As $\sigma \rightarrow 1$, the function in the brackets in Eq. (74) diverges logarithmically, indicating that the spatial distribution of the power density $q(x)$, which cuts off the logarithmic divergence in $T(x,y)$ in Eqs. (28) should be taken into account.

In the crossover region $\sigma \sim 1$, the behavior of $\delta R(B_0, \omega)$ is sensitive to the details of the power density distribution $q(x)$, but in the limiting case $\sigma \gg 1$ the main contribution to $\delta R$ comes from the hottest region near the heat source for which $\delta T(y)$ is given by Eq. (30). Substituting Eq. (30) into Eq. (71), we obtain

$$\frac{\delta R_s}{R_{\text{bcs}}} \approx \int_{-\infty}^{\infty} dy \left[ \frac{16d^2}{\sigma^2 (y^2 + r_0^2)} \right]^{\sigma/2} = r_0 \left( \frac{4d}{\pi r_0} \right) I_\sigma, \quad (76)$$

where $I_\sigma = \sqrt{\pi} \Gamma[(\sigma - 1)/2]/\Gamma(\sigma/2) \approx (2\pi/\sigma)^{1/2}$ for $\sigma \gg 1$. The behavior of $\delta R_s$ for $\sigma > 1$ changes radically as compared to $\sigma < 1$: instead of relatively weak power-law dependencies of $\delta R_s$ on $B_0$ and $\omega$ for $\sigma < 1$, Eq. (76) predicts much stronger exponential field and frequency dependencies of $\delta R_s$ for $\sigma > 1$, $r_0 \ll d$. The case of strong dissipation $\sigma > 1$ can result from vortex penetration amplified by grain boundaries, surface topography, local enhancements of $R_{\text{bcs}}$ due to impurity segregation etc.

The mean surface resistance $\bar{R}_s$ averaged over all hotspot contributions is given by

$$\bar{R}_s = R_{\text{bcs}} + R_t + \sum_n \delta R_s(r_n)/A. \quad (77)$$

Here the averaged residual resistance $\bar{R}_t$ results from either pinned vortices or other mechanisms, the last term in the r.h.s. is due to the effect of vortex dissipation on the BCS resistance, $r_n$ are the coordinates of the sparse (thermally noninteracting) hotspots, and $A$ is the surface area exposed to the rf field. As shown above, $\delta R_s$ can have very different temperature and frequency dependencies as compared to $R_{\text{bcs}}$, so the hotspot contribution can strongly affect the dependencies of global surface resistance $\bar{R}_s$ of $\omega$ and $T$, particularly at low temperatures, where $\bar{R}_{\text{bcs}}$ is exponentially small. Moreover, the last 2 terms in the r.h.s. of Eq. (77) can bring about a strong dependence of $\bar{R}_s$ on the rf amplitude, which can be well below the field $\sim B_c T/T_c$ of intrinsic nonlinearities of the BCS surface resistance due to the Doppler shift of quasiparticle energies.

A. Thermal instabilities ignited by hotspots

In the previous section we considered the rf power as a function on the bath temperature $T_0$. However, $Q$ is actually determined by the local temperature $T_m$, which should be calculated self-consistently from the heat balance condition. We consider the case, for which the mean spacing between hotspots $L_i$ is shorter than the thermal length $L_\theta = (dk/\alpha_k)^{1/2}$, over which $\delta T(r)$ decays away from a single hotspot. Here $\alpha_k = k_0\alpha_K/(\alpha_k + k)$ is the effective thermal resistance across the film which accounts for the resistance $d/k$ due to thermal conductivity plus the interface thermal resistance $1/\alpha_K$, where $\alpha_K$ is the Kapitza heat transfer coefficient. For $L_i \ll L_\theta$, thermal fields of hotspots overlap and the temperature $T_m$ along the surface becomes uniform. In this case the thermal balance equation takes the form

$$(T_m - T_0)\alpha_0 = \frac{B_0^2}{2\mu_0^2}[R_i + R_{\text{bcs}}(T_m)], \quad (78)$$

which determines self-consistently both the rf dissipated power and the maximum temperature $T_m$ as functions of $B_0$ and $\omega$. It is convenient to express $B_0(T_m)$ as a function of $T_m$ from Eq. (78):

$$B_0^2 = \frac{2\mu_0^2(T_m - T_0)\alpha_0}{R_i + R_0 \exp[(T_m - T_0)\Delta/T_0^2]} \quad (79)$$

Here we took into account the most essential exponential temperature dependence of the BCS surface resistance, where $R_0 = R_{\text{bcs}}(T_0)$, and the residual resistance $R_i$ due to trapped vortices is assumed temperature-independent for $T \ll T_c$. The function $B_0(T_m)$ has a maximum at:

$$T_m - T_0 = \frac{T^2_0}{\Delta} + \frac{B_0^2 R_i}{2\mu_0^2 \alpha_0}, \quad (80)$$

giving the critical overheating $T_m - T_0 \ll T_0$ above which a thermal instability develops. From Eqs. (80) and (79), we obtain the equation for the maximum field $B_p$:

$$\frac{R_0 B_p^2 \Delta}{2\mu_0^2 T^2 \alpha_0} \exp\left( \frac{R_i B_p^2 \Delta}{2\mu_0^2 T^2 \alpha_0} \right) = 1 \quad (81)$$

The thermal balance Eq. (78) has solutions $T_m(B_0)$ only if the rf amplitude is below the breakdown field $B_p$. For $B_0 > B_p$, the thermal runaway occurs because the heat generation grows faster than the heat flux to the coolant as $T_m$ increases. This situation is analogous to combustion in chemical systems or thermal quench in semiconductors, normal metals or superconductors.

For $R_i \ll R_0$, the exponential term in Eq. (81) can be neglected and the breakdown field is given by

$$B_p = \mu_0 \left( \frac{2\alpha_0 T_0^2}{R_0 e \Delta} \right)^{1/2} \quad (82)$$

The temperature dependence of $B_p(T_0)$ can be obtained taking $R_0(T_0) = R_{\text{no}}(\Delta/T_0)^{\alpha} \exp(-\Delta/T_0)$, $\alpha_0 = \alpha/\Delta T_0^4$, whence $B_p(T_0) \propto T_0^{(s+3)/2} \exp(-\Delta/2T_0)$ reaches a minimum at $T_{\text{min}} = \Delta/(s + 3)$ and increases as $T_0$ decreases below $T_{\text{min}}$. However, for lower temperatures, $R_{\text{bcs}}(T_0)$ becomes smaller than $R_i$ in which case the exponential term in Eq. (81) dominates, and $B_p(T_0) \sim$
The temperature dependence of $B_p(T_0)$ calculated from Eq. (81) for $R_{m,s}(T_0) = R_{n0}(\Delta/T_0) \exp(-\Delta/T_0)$, $R_i/eR_{n0} = 2 \times 10^{-3}$, $\alpha_0 = \alpha T_0$, $s = 3$, and $B_{i0}^2 = 2\mu_0^2/\lambda T_0^{1/2}/R_{n0}$.

\[ \mu_0(2\alpha T_0^2/R_0 \Delta)_{1/2} \propto T_0^{1+s/2} \]

decreases as $T_0$ decreases. The behavior of $B_p(T_0)$ is shown in Fig. 16. The maximum in $B_p(T)$ at the optimum temperature $T_{\text{max}}$ separates the regimes controlled by the BCS surface resistance at $T_0 > T_{\text{max}}$ and by hotspots due to frozen flux or other mechanisms of residual resistance at $T_0 < T_{\text{min}}$.

For Nb, ($\Delta \approx 18$K), the optimum temperature in Fig. 16 corresponds to $T_{\text{max}} \approx 2$K, while for Nb$_3$Sn ($\Delta \approx 36$K), $T_{\text{max}} \approx 4$K, if $R_i/R_{n0}$ is the same for both materials.

The above consideration based on the linear BCS surface resistance assumes that the breakdown field $B_p$ is much smaller than the field $\simeq T B_e/T_c$ at which the intrinsic nonlinearities in $R_s$ become important. A significant increase of the isothermal $R_s(B_0)$ due to these nonlinearities can strongly affect the thermal breakdown\textsuperscript{26}, limiting $B_p$ by the thermodynamic critical field $B_c$.

### VI. DISCUSSION

The results presented above show that the breakdown of the Meissner state by strong rf fields involves supersonic vortex penetration through the surface barrier weakened by defects, the jumpwise LO-type instability and high dissipation even for single vortices. Such dissipation results in thermal retardation effects and hotspots igniting the explosive thermal instability due to the exponential temperature dependence of the surface resistance. These effects are precursors for the avalanche vortex penetration and dendritic thermo-magnetic instabilities\textsuperscript{60,61}.

At the onset of vortex penetration pinning forces are much weaker than the driving forces of the rf Meissner currents. Yet, as the vortex moves away from the surface by the distance $\sim \lambda$, the Lorentz force decreases exponentially, so the vortex can be trapped by pinning centers. Such vortices trapped in the thin surface layer of rf field penetration during breaking through the surface barrier or field cooling of the sample can result in a temperature-independent residual surface resistance. However, because pinning centers are distributed randomly, the rf power dissipated by pinned vortices $Q(\omega)$ varies very strongly because of the exponential sensitivity of $Q \propto \exp(-2\mu/\lambda)$ to the vortex position. This effect results in hotspots of vortex dissipation, which peaks for vortices spaced from the surface by distances close to the minimum distance $d_m$, for which pinning forces can prevent vortex annihilation at the surface at low fields $B_0 \ll B_c$. The field dependence of $d_m$ causes rf flux annealing in which vortices are irreversibly pushed out from the surface layer. This effect results in a nonlinear hysteretic dependence of $R_i(B)$ at low fields, $B_0 \ll B_c$, which may pertain to the puzzling decrease of the surface resistance at low fields $B_0 \sim 3 - 20$mT, which has been often observed on Nb\textsuperscript{73,74} and other superconductors\textsuperscript{79,80}. Yet, Eq. (67) describes well the field dependence $R_i \propto 1/B_0^2$ observed on Nb cavities\textsuperscript{74}.

Besides the field and frequency dependencies of $R_i(B)$, another manifestation of the vortex pinning mechanism is the hysteretic behavior shown in Fig. 14 due to the rf annealing of the trapped flux. However, this mechanism caused by the gradient of the Lorentz force is only effective for the vortex segments parallel to the surface and does not affect vortex segments perpendicular to the surface. The segments of pinned vortices perpendicular to the surface generally give a field-independent contribution to $R_i$, however if these segments belong to the vortex semi-loop trapped at the surface, the rf Lorentz force gradient acting on the parallel component of the loop can eventually push the whole loop toward the surface where it shrinks and annihilates. In this case the rf annealing decreases $R_\mu$, eliminating some of the hotspots caused by trapped vortices. This is illustrated by Fig. 17, which shows 3 different type of vortices trapped at the surface. Vortex 1 cannot be pushed out by the rf field because only a small segment of it $\sim \lambda$ is exposed to the rf Lorentz force, while the rest part is pinned in the bulk. Vortex semi-loop 2 can be pushed out by the rf field, as discussed above. Vortex 3 has a parallel segment, which however cannot annihilate at the surface because it is held back by other pinned segments which are beyond the surface layer of the rf field penetration.

Penetration and trapping of even single vortices at low temperatures can significantly increase the exponentially small $R_s$, which in turn, decreases the thermal breakdown field $B_p$. For example, flux trapped during field cooling in the Earth magnetic field $B_i \approx 40\mu$mT, corresponds to the mean intervortex spacing $a = \langle \phi_0/B_i \rangle^{1/2} \approx 7\mu$m. To estimate $R_i$ for such trapped flux, we use Eq. (74) for strong pinning at $\omega \tau_p < 1$, taking $\langle \exp(-2d/\lambda) \rangle \sim \lambda/a$ and $d_m \sim \lambda$. Hence,

\[ R_i \approx \frac{\omega^2 \tau_p^2}{30} R_i(\infty), \quad R_i(\infty) = \frac{\rho_n B_i}{\lambda B_{c2}} \]

where $R_i(\infty)$ is the high frequency limit of $R_i(\omega)$. For Nb, taking $B_{c2} = 400$ mT, $\lambda = 40$ nm, $\rho_n = 10^{-9}$\Omega m,
FIG. 17: Vortices (shown as dashed lines) trapped near the surface by pinning centers (black dots).

and $B_i = 40 \mu T$, we obtain $R_i(\infty) = 2.5 \mu \Omega$, much higher than the typical values $R_i \sim 10 n\Omega$ observed on high-purity Nb. For Nb$_3$Sn, with $\rho_n = 0.2 \mu \Omega m$, $\lambda \approx 90$ nm, we obtain $R_i(\infty) \approx 3.9 \mu \Omega$. As follows from Eq. (83), pinning reduces $R_i$ by the factor $(\omega \tau_p)^2/30$, so $R_i \approx 10 n\Omega$ corresponds to $(\omega \tau_p)^2 \sim 0.1$, or $\tau_p \approx 0.5$ ns for $\omega/2\pi = 1$ GHz. By contrast, the pinning time constant $\tau_p \approx 10^{-8}$ s measured by Pioszyk et al. seems to indicate that their Nb sample was in the weak pinning limit, $(\omega \tau_p)^2 \gg 1$, for which $R_i \approx R_i(\infty)$ is in agreement with the measured $R_i(0.5\Omega e) \approx 2\mu \Omega$ and the estimate from Eq. (83).

Introducing dense pinning structures in the surface layer of the rf field penetration can therefore impede vortex oscillations and significantly reduce the part of $R_i$ caused by trapped flux, particularly for vortex loops like 1 and 3 in Fig. 17 which are not affected by the rf flux annealing. Because pinning is only effective if $\omega \tau_p < 1$, decreasing $\tau_p$ in Eq. (90) implies reducing the pin spacing. At the same time, a more effective rf flux annealing requires both weak and dense pins (small $f_p$ and $\ell$ in Eq. (90)). Overall, $R_i$ can be reduced by decreasing the relaxation time constant $\tau_p$, which can be done not only by decreasing the pin spacing $\ell$ but also by optimizing the mean-free path at the surface. Indeed, Eq. (91) shows that $\tau_p$ is larger for higher-$\kappa$ superconductors because of the softening of the vortex line tension $\xi$, although this effect can be offset by a higher normal resistivity. For example, the ratio $\tau_{p1}/\tau_{p2} = \kappa_2^2/\rho_{n2}/\kappa_2^2/\rho_{n1} \propto \rho_{n1}/\rho_{n2}$ for two different materials 1 and 2 (or two different mean free paths $\ell_i$) indicates that pinning becomes less effective for a dirtier surface. Furthermore, comparing Nb with $\kappa_1 = 1$, $\rho_{n1} = 10^{-9}\Omega m$ and Nb$_3$Sn with $\kappa_2 = 30$ and $\rho_{n1} = 0.2 \mu \Omega m$, we obtain $\tau_{pNb}/\tau_{pNb3Sn} \sim 1/5$. Thus, reduction of $R_i$ by pinning turns out to be somewhat more effective in Nb, although this conclusion can be strongly affected by impurities, as discussed above.

The results of this work show that reducing vortex dissipation is an important problem in achieving ultimate pairbreaking breakdown fields in superconductors. In particular, a significant progress has been made in increasing $R_s$ and $B_p$ by low-temperature annealing of Nb cavities which enables tuning the impurity concentration, nanoscale oxide layers and hotspot distribution on Nb surface. Another possibility in raising the ultimate breakdown fields is to use thin film superconductor-insulator-superconducting (SIS) multilayer coating with high-$B_c$ films of thickness $d < \lambda'$ to significantly increase $B_{c1}$ and delay the field onset of vortex penetration.

Moreover, the SIS coating may suppress the LO instability by decreasing the vortex flight time through the film and providing strong pinning due to magnetic interaction of the vortex with the film surfaces. The SIS multilayer coating of Nb cavities may enable increasing the ultimate breakdown fields $B_p$ above $B_{cN}^{-}$ by taking advantage of A15 superconductors or MgB$_2$ with $B_c > B_{cN}^{-}$ and potentially lower $R_{bcs}$.

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APPENDIX A: TEMPERATURE OF A MOVING VORTEX

Eq. (24) can be written in the dimensionless form:

$$\dot{\theta} = \nabla^2 \theta - \beta(\theta_m) s^2(t) f[x - u(t), y],$$

(A1)

where $\theta = (T - T_0)/(T_c - T_0)$, and the time, coordinates and vortex velocity are normalized by the thermal scales $t_0 = C/\alpha$, $L_0 = (k/\alpha)^{1/2}$ and $v_0 = L_0/t_0$, respectively, $\beta(\theta_m) = \gamma_0(\theta_m)v_0^2/\beta(T_c - T_0)k$, and $s(t) = v/v_0$ is the dimensionless vortex velocity. The Fourier transform of Eq. (A1) results in the following equation for the Fourier components, $\theta_p(t) = \int \theta(x, y, t) \exp(-i p \mathbf{r}) d^2 \mathbf{r}$:

$$\dot{\theta}_p + (1 + p^2)\theta_p = f_p g(t)e^{-i p \cdot u(t)}$$

(A2)

where $g(t) = \beta[\theta_m(t)]s^2(t)$. The solution of Eq. (A2) is:

$$\theta_p(t) = f_p \int_0^\infty e^{-(1+p^2)t' - i p \cdot u(t')} g(t - t') dt'$$

(A3)

For steady-state vortex oscillations, the rf field was turned on at $t_i = -\infty$, and $v(t)$ in Eq. (A3) accounts for all oscillations preceding the time $t$. For $f(r) = \pi^{-1/2}\exp(-r^2/\xi_1^2)$, and $f_p = \exp(-p^2\xi_1^2/4)$, the inverse Fourier transform of Eq. (A3) gives

$$\theta(r, t) = \frac{1}{\pi} \int_0^\infty dt' g(t - t') \frac{e^{-i x t' + (t - t')^2 + y^2/(4t')}}{4t' + \xi^2}$$

(A4)

Eq. (A4) was obtained for an infinite sample. For a thermally-insulated or ideally cooled surface at $x = 0$,
Eq. (A3) can be modified using the method of images:

\[
\theta(r, t) = \frac{1}{\pi} \int_0^\infty dt' g(t - t') \times e^{-t' - \frac{x^2}{4r'^2}} \left[ e^{-\frac{(x - u(t'))^2}{4r'^2}} + e^{-\frac{(x + u(t') - x_0)^2}{4r'^2}} \right],
\]

(A5)

where the plus and minus signs in the parenthesis correspond to the Dirichlet [\(\partial_x \theta(0, y, t) = 0\)] and the Neumann [\(\theta(0, x, t) = 0\)] boundary conditions at the surface, respectively. The core size is typically much smaller than \(L_0\), so the dynamic equation for the core temperature \(\theta_m(t)\) can be obtained from the self-consistency condition \(\theta_m(t) = \theta(u(t), 0, t)\), resulting in Eq. (25).

Next we consider the steady-state temperature distribution \(T(x, y)\) averaged over high-frequency vortex oscillations, at \(2\pi\omega t_0 \gg 1\). In this case \(T(x, y)\) satisfies the static thermal diffusion equation

\[
k\nabla^2 T + q(x) \delta(y) = 0
\]

(A6)

where \(q(x)\) is the mean power density distribution along the vortex trajectory. We solve Eq. (A6) for a film of thickness \(d\) with the boundary conditions: \(\partial_x T(x, y) = 0\) on the thermally-insulated surface \(x = 0\) where vortex dissipation is localized, and \(T(x, y) = T_0\) at the opposite surface, \(x = d\) kept at \(T_0\). By symmetry, this geometry has the same \(T(r)\) as in a film of thickness \(2d\) with isothermal boundary conditions \(T(\pm d, y) = T_0\), and the heat source in the middle at \(x \approx 0\). In this case Eq. (A6) can be solved using the Green function

\[
G(r, r') = \frac{1}{4\pi k} \ln \frac{\cosh \frac{\pi(y - y')}{2d} + \cos \frac{\pi(x + x')}{2d}}{\cosh \frac{\pi(y - y')}{2d} - \cos \frac{\pi(x + x')}{2d},}
\]

(A7)

which gives the distribution of \(\delta T(r) = T(r) - T_0\):

\[
\delta T(r) = \frac{1}{2\pi k} \int_0^d q(x') \ln \frac{\cosh \frac{\pi y}{2d} + \cos \frac{\pi(x + x')}{2d}}{\cosh \frac{\pi y}{2d} - \cos \frac{\pi(x + x')}{2d}} dx'.
\]

(A8)

If the length of dissipation source \(\sim r_0\) is much smaller than the film thickness, \(\delta T(r)\) around the source at \((x^2, y^2) \ll d^2\) reduces to

\[
\delta T(r) = \frac{1}{2\pi k} \int_0^\infty q(x') \ln \frac{16d^2}{\pi^2 y^2 + (x - x')^2} dx'.
\]

(A9)

Next, we take a rectangular approximation \(q(x) = q_0\) for \(x < r_0\) and \(q(x) = 0\) for \(x > r_0\), where \(r_0\) is a characteristic size of the dissipation source, so that \(q(r_0) = Q\) gives the total power \(Q\). In this case the distribution of \(\delta T(0, y)\) along the surface becomes

\[
\delta T(y) = \frac{q_0}{2\pi k} \int_0^{r_0} \ln \frac{16d^2}{\pi^2 y^2 + x^2} dx',
\]

(A10)

which yields after integration:

\[
\delta T(y) = \frac{Q}{2\pi k} \left[ \ln \frac{16d^2}{\pi^2 y^2 + r_0^2} + 2 - \frac{2y}{r_0} \tan^{-1} \frac{r_0}{y} \right]
\]

(A11)

The maximum \(\delta T_m = \delta T(0, 0)\) is given by:

\[
\delta T_m = \frac{Q}{\pi k} \left( \frac{4d}{\pi r_0} + 1 \right)
\]

(A12)

APPENDIX B: MULTIVALUED FRICTION FORCE

Eq. (32) describes an overdamped vortex driven by the force \(F(u, t)\) balanced by a nonlinear friction force \(\eta v/(1 + v^2/v_0^2)\). The force balance equation has either two or no roots, as shown in Fig. 17. The velocity \(v_\pm(F)\) for the left intersection point vanishes at \(F = 0\) and increases up to \(v_0\) as \(F\) increases. The velocity for the right intersection point \(v_+(F)\) decreases as \(F\) increases. If \(F < F_m\) the branch \(v_-(F)\) describes all smooth parts of the vortex trajectory and also provides \(v(0) = 0\) for the initial condition \(u(0) = 0\). For \(F > F_m\), the vortex jumps from \(x = u_1\) to the point \(x = u_2\) where friction is able to balance the drive. Here \(u_{1,2}\) are defined by the condition \(F(u_{1,2}, t_1) = F_m\). The branch \(v_-(F)\) describes all smooth parts of \(u(t)\) provided that \(F(u_{1,2}, t_1) < 0\), so that \(F(t)\) always decreases below \(F_m\) after the jump.

For high rf frequencies or strong vortex core overheating, there are situations, for which \(\hat{F} > 0\) after the jump. For example, if \(u > \lambda\), the term with the \(K(x)\) in \(F\) can be neglected, and \(\hat{F} = v_0 \partial_x F + \partial_t F\) becomes:

\[
\hat{F} = (-v_0/\lambda + \omega \cot \omega t) F
\]

(B1)

which tends to become positive at higher frequencies. In this case the vortex cannot jump to the point where \(F(u, t) = F_m\), since the friction force cannot balance \(F\) if \(\hat{F} > 0\), because \(F(u, t)\) keeps increasing above \(F_m\) so \(v(t)\) becomes greater than \(v_0\). Thus, for \(\hat{F} > 0\), friction can only stop the vortex jump if \(v > v_0\), thus we have to consider the branch \(v_+(F)\) as well. We should therefore construct the trajectory, for which the vortex first jumps at \(t = t_1\) to the new point \(x = u_1\) and acquires the velocity \(v_1 > v_0\). After that \(v(t)\) continuously decreases from
$v_i > v_0$ at $t = t_i$, to $v_0$ at $t = t_0$, as described by the first order differential equation $\dot{u} = v_+ [F(u, t)]$. Here $u_0$ and $t_0$ are determined by the equations:

$$F[u_0, t_0] = F_m, \quad v_0 \partial_t F + \partial_t F = 0,$$

which state that the $\dot{F}$ should change sign as the vortex reaches the maximum friction force at the critical velocity $v = 0$. Indeed, if $\dot{F}$ changes sign at any point of the descending branch $v_+(F)$, the vortex cannot not reach $v_0$, so $v(t)$ passes through a minimum at some $v > v_0$ and then starts accelerating continuously. On the other hand if the vortex reaches $v_0$ at $\dot{F} > 0$, the jump instability occurs. Therefore Eq. \[B2\] provides the only way for a stable switch from the descending branch of $v_+(F)$ for $t_i < t < t_0$, to the ascending branch of $v_+(F)$. Once $t_0$ and $u_0$ are found from Eq. \[B2\], the coordinate of the jump, $u_i$, can be calculated for the vortex going backward in time, taking the initial condition $u = u_0$, and $\dot{F} = F_m$ and then, solving the equation $\dot{u} = v_+ [F(t_0 - t)]$ with the "negative time" $t = t_0 - t_i$, from $t = 0$ to $t = t_0 - t_i.$
\[ F(u,t)/F_0 = \frac{F(u)}{F_m} \]