A New Algorithmic Scheme for Computing Characteristic Sets

Meng Jin

LMIB–School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

Xiaoliang Li

LMIB–School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

Dongming Wang

Laboratoire d’Informatique de Paris 6, CNRS – Université Pierre et Marie Curie, 4 place Jussieu – BP 169, 75252 Paris cedex 05, France

Abstract

Ritt-Wu’s algorithm of characteristic sets is the most representative for triangularizing sets of multivariate polynomials. Pseudo-division is the main operation used in this algorithm. In this paper we present a new algorithmic scheme for computing generalized characteristic sets by introducing other admissible reductions than pseudo-division. A concrete subalgorithm is designed to triangularize polynomial sets using selected admissible reductions and several effective elimination strategies and to replace the algorithm of basic sets (used in Ritt-Wu’s algorithm). The proposed algorithm has been implemented and experimental results show that it performs better than Ritt-Wu’s algorithm in terms of computing time and simplicity of output for a number of non-trivial test examples.

Key words: characteristic set; elimination; reduction; subresultant; triangular set.

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Email addresses: jinmeng101@gmail.com (Meng Jin), xiaoliangbuaa@gmail.com (Xiaoliang Li),

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1. Introduction

For solving systems of multivariate polynomial equations there are mainly three elimination approaches based on resultants, triangular sets, and Gröbner bases (see, e.g., Kapur and Lakshman 1992 and Wang 2001b). One of the best known concepts for triangular sets is characteristic set, which was introduced first by Ritt (1951) for prime ideals. Since 1980, W.-T. Wu has considerably developed Ritt’s theory and method of characteristic sets by removing irreducibility requirements and designing efficient algorithms for zero decomposition of arbitrary polynomial systems. Ritt-Wu’s method has been improved and extended by a number of researchers and has been successfully applied to many problems in science and engineering (see, e.g., Wu and Gao 2007).

To speed up the computation of characteristic sets, Chou (1988) and later Chou and Gao (1991) introduced the notions of W-prem and W-characteristic set. Characteristic sets and their complexity were studied by Gallo and Mishra (1991a,b) and new algorithms for computing characteristic sets with simple exponential sequential and polynomial parallel time complexities were presented. Wang (1992) proposed an effective strategy to improve Ritt-Wu’s algorithm of characteristic sets. Gao and Chou (1992) presented a method based on Ritt-Wu’s algorithm to deal with parametric polynomial systems. Certain properties about ascending chains were studied and used to enhance the efficiency of Ritt-Wu’s algorithm in Gao and Chou (1993). A complete implementation of Ritt-Wu’s method in the Maple system was reported in Wang (1995). Wang (2001a) also generalized Ritt-Wu’s algorithm by means of one-step pseudo-reduction with strategies for the selection of reductends and optimal reductors. Li (2006) described a modified Ritt-Wu algorithm that can avoid redundant decompositions.

Besides Ritt-Wu’s, there are many other efficient methods for decomposing systems of multivariate polynomials. Kalkbrener (1993) introduced the notion of regular chain and presented a method for decomposing any algebraic variety into unmixed-dimensional components represented by regular chains. Regular chain was also defined independently by Yang and Zhang (1994) under the name of normal ascending set. Lazard (1991) introduced the concept of normalized triangular set (which is a special regular chain) and provided a method for uniquely decomposing the zero set of any polynomial set into regular zero sets of normalized triangular sets. Another method of triangular decomposition was proposed by Wang (1993). Aubry et al. (1999) studied the relationship between several different notions of triangular sets used in these methods and proved the equivalence of four notions (including regular chain). Comprehensive investigations on triangular sets and triangularizing algorithms were also carried out in Hubert (2003a). The reader may refer to Szanto (1999); Aubry and Moreno Maza (1999); Moreno Maza (2000); Wang (2000, 2001b); Chen and Moreno Maza (2011) and references therein for other related work on triangular decomposition of polynomial systems. Extensions of the theories and methods in the algebraic case to the differential, difference, and other cases may be found in Boulier et al. (1993); Li and Wang (1994); Hubert (2000); Chen and Gao (2003); Hubert (2003b); Boulier et al. (2009); Gao et al. (2009); Boulier et al. (2010) and references given therein.

Pseudo-division is the main operation used in most of the above-mentioned methods. In the process of computing a characteristic set, one needs to construct polynomial remainder sequences (PRSs). The coefficients of intermediate polynomials in a PRS may
swell dramatically. As pointed out by [Collins 1967], the coefficients of polynomials in an Euclidean PRS (computed using pseudo-division) may grow exponentially. An exponential upper bound is given in [Knuth 1997, (27) in 4.6.1]. There are several other PRSs, such as subresultant PRS in which the coefficients of polynomials can be determined by using certain matrices (see [von zur Gathen and Lucking 2003] and [Mishra 1993]) and swell much more slowly than those in an Euclidean PRS. Li [1989] was the first who studied superfluous factors appearing in the computation of characteristic sets.

In this paper we follow the work of [Wang 2001a] to present a new algorithmic scheme for computing generalized characteristic sets efficiently. To compute characteristic sets, one can replace pseudo-division by one-step pseudo-reduction, but this does not enhance the efficiency very much because extraneous factors for the pseudo-rests may be created (see the analysis in [Sasaki and Furukawa 1984] and [Wang 2001a]). Our intention here is to design a reduction mechanism that can take advantage of the structure and properties of the polynomials under consideration and thus provide more flexibility for reduction strategies. A polynomial set which generates the same ideal as the input polynomial set is maintained, leading to the concept of generalized characteristic set. This set may contain polynomials of degrees smaller than the degrees of those in the input set and therefore may take less computing time for pseudo-reduction to 0. By introducing admissible reductions other than pseudo-division, it is possible to control the swell of coefficients of intermediate polynomials and to compute with smaller polynomials. Several strategies for concrete admissible reductions are adopted in a sample algorithm of the new scheme. Two versions of the sample algorithm have been implemented and compared with three algorithms implemented in Maple: two in the Epsilon package and one in the Groebner package. More details about the outputs of these algorithms for some examples are given in the appendix.

The paper is structured as follows. Section 2 consists mainly of terminologies and notations about characteristic sets. In Section 3, we describe the main algorithmic scheme for computing generalized characteristic sets. A concrete subalgorithm is designed to triangularize polynomial sets by means of admissible reductions and effective elimination strategies and to replace the algorithm of basic sets (used in Ritt-Wu’s algorithm). The termination and correctness of these algorithms are proved. In Section 4, discussions on concrete admissible reductions are given. Section 5 provides details for the sample subalgorithm. Section 6 contains an illustrative example and some experimental results, showing that our proposed algorithm performs better than Ritt-Wu’s in terms of efficiency and simplicity of output for a number of non-trivial test examples.

2. Preliminaries

Let $\mathcal{K}$ be any field. The notation $\mathcal{K}[x]$ or $\mathcal{K}[x_1, \ldots, x_n]$ stands for the ring of polynomials over $\mathcal{K}$ in the variables $x_1, \ldots, x_n$. In what follows, we always assume that the variables are ordered as $x_1 < \cdots < x_n$.

Let $F$ be an arbitrary non-zero polynomial in $\mathcal{K}[x]$. We use $\deg(F, x_k)$ and $\lc(F, x_k)$ to denote the degree and the leading coefficient of $F$ with respect to (w.r.t.) the variable $x_k$ respectively. The biggest variable that effectively appears in $F$ is called the leading variable of $F$ and denoted by $\lv(F)$. Moreover, $\ini(F) := \lc(F, \lv(F))$ and $\ldeg(F) := \deg(F, \lv(F))$ are called the initial and the leading degree of $F$ respectively. For any non-constant polynomial $F$, the index $c$ of its leading variable $x_c$ is called the class of $F$ and denoted by $\cls(F)$. The class of any non-zero constant is set to be 0.
**Definition 1.** An ordered polynomial set \([T_1, \ldots, T_r] \subseteq \mathbb{K}[x] \setminus \mathbb{K}\) is called a triangular set in \(\mathbb{K}[x]\) if \(\text{lv}(T_1) < \cdots < \text{lv}(T_r)\).

**Definition 2.** Let \(P\) and \(Q\) be two polynomials in \(\mathbb{K}[x]\). We say that \(P\) is reduced w.r.t. \(Q\) if \(\text{deg}(P, \text{lv}(Q)) < \text{ldeg}(Q)\); otherwise, \(P\) is reducible w.r.t. \(Q\).

Furthermore, let \(T = [T_1, \ldots, T_r]\) be a triangular set in \(\mathbb{K}[x]\). \(F\) is reduced w.r.t. \(T\) if for every \(i = 1, \ldots, r\), \(F\) is reduced w.r.t. \(T_i\); otherwise, \(F\) is reducible w.r.t. \(T\). The triangular set \(T\) is called an auto-reduced (or initial-reduced) set if for every \(i = 2, \ldots, r\), \(T_i\) (or \(\text{ini}(T_i)\)) is reduced w.r.t. \([T_1, \ldots, T_{i-1}]\).

**Definition 3.** Let \(P\) and \(Q\) be non-zero polynomials in \(\mathbb{K}[x]\). \(P\) is said to have a lower ranking than \(Q\), denoted as \(P \prec Q\), if \(\text{cls}(P) < \text{cls}(Q)\), or \(\text{cls}(P) = \text{cls}(Q)\) and \(\text{ldeg}(P) < \text{ldeg}(Q)\). In this case, we also say that \(Q\) has a higher ranking than \(P\) and denote it as \(Q \succ P\).

If neither \(P \prec Q\) nor \(Q \succ P\) holds, then we say that \(P\) and \(Q\) have the same ranking and write \(P \asymp Q\). Moreover, \(P \succ Q\) or \(P \asymp Q\) is denoted as \(P \succeq Q\), and \(P \preceq Q\) can be similarly defined.

**Definition 4.** Let \(T = [T_1, \ldots, T_r]\) and \(S = [S_1, \ldots, S_s]\) be two triangular sets in \(\mathbb{K}[x]\). \(T\) is said to have a lower ranking than \(S\), denoted as \(T \prec S\), if

(a) there exists an \(i \leq \min(r, s)\) such that \(T_1 \sim T_1, \ldots, T_{i-1} \sim S_{i-1}\) and \(T_i \prec S_i\); or
(b) \(r > s\) and \(T_1 \sim S_1, \ldots, T_r \sim S_s\).

In this case, we also say that \(S\) has a higher ranking than \(T\) and denote it as \(S \succ T\).

If neither \(T \succ S\) nor \(T \prec S\) holds, then \(T\) and \(S\) are said to have the same ranking and denoted as \(T \asymp S\). In this case, one can easily know that \(r = s\) and \(T_1 \sim S_1, \ldots, T_r \sim S_r\).

The order \(\succeq\) defined above is a partial order and can be used to order triangular sets. The following proposition indicates that any non-empty set of triangular sets has a minimal element.

**Proposition 5** \([\text{Wang \text{1985}}]\). For any triangular set sequence \(T_1 \preceq T_2 \preceq T_3 \preceq \cdots\), there exists an integer \(m\) such that for any \(i \geq m\), \(T_i \asymp T_m\).

An (initial-reduced) auto-reduced set in \(\mathbb{K}[x]\) is also called a (weak) non-contradictory ascending set, while \([a] \ (a \in \mathbb{K} \setminus \{0\})\) is called a (weak) contradictory ascending set. Both of them are jointly called (weak) ascending sets.

Let \(P\) be a non-empty polynomial set in \(\mathbb{K}[x]\) and \(\Omega\) denote the set of all the ascending sets that are contained in \(P\); \(\Omega\) is not empty because the set of any single polynomial in \(P\) is an ascending set by definition. By Proposition 5, \(\Omega\) has a minimal element, which is called a basic set of \(P\). This set can be computed, e.g., by the algorithm \text{BasSet} described in \([\text{Wang \text{2001b}}]\).

**Definition 6.** Let \(P\) be a non-empty polynomial set \(K[x]\). An ascending set \(\mathcal{M}\) is called a medial set of \(P\), if \(\mathcal{M} \subseteq \langle P \rangle\) and \(\mathcal{M}\) has ranking not higher than the ranking of any basic set of \(P\).

**Proposition 7** \([\text{Wang \text{2001b}}]\). Let \(P\) be a non-empty polynomial set in \(\mathbb{K}[x]\), \(\mathcal{M} = [M_1, \ldots, M_r]\) a medial set of \(P\) with \(M_1 \notin \mathbb{K}\), and \(M\) a non-zero polynomial reduced w.r.t. \(M\). Then for any medial set \(\mathcal{M}'\) of \(P' = P \cup M \cup \{M\}\), we have \(\mathcal{M}' \prec \mathcal{M}\).
For any $F, G \in K[x]$ with $G \neq 0$, let $\text{prem}(F, G, x^k)$ and $\text{pquo}(F, G, x^k)$ denote respectively the pseudo-remainder and the pseudo-quotient of $F$ w.r.t. $G$ in $x^k$. For any triangular set $T = [T_1, \ldots, T_r]$ and polynomial set $P \subseteq K[x]$, define $\text{prem}(F, T) := \text{prem}(\cdots \text{prem}(F, T_r, \text{lv}(T_r)), \ldots, T_1, \text{lv}(T_1))$ and $\text{prem}(P, T) := \{\text{prem}(P, T) : P \in P\}$.

**Definition 8.** A (weak) ascending set $C$ in $K[x]$ is called a (weak) characteristic set of a non-empty polynomial set $P \subseteq K[x]$ if $C \subseteq \langle P \rangle$ and $\text{prem}(P, C) = \{0\}$.

For characteristic sets, the following property is fundamental.

**Proposition 9** (Wu 1986). Let $C = [C_1, \ldots, C_r]$ be any characteristic set of a polynomial set $P \subseteq K[x]$ and

$$
I_i = \text{ini}(C_i), \quad P_i = P \cup \{I_i\}, \quad I = \{I_1, \ldots, I_r\}.
$$

Then

$$
\text{Zero}(C/I) \subseteq \text{Zero}(P) \subseteq \text{Zero}(C),
$$

$$
\text{Zero}(P) = \text{Zero}(C/I) \cup \bigcup_{i=1}^r \text{Zero}(P_i).
$$

(1)

3. **Algorithmic Scheme for Computing Generalized Characteristic Sets**

In this section we generalize the concept of characteristic set and give a proposition to show that the generalized concept preserves the basic property (Proposition 9) of the original one. For computing generalized characteristic sets, we propose a new algorithmic scheme, in which the set of reductions, the termination condition, and strategies for finding reduction polynomials can be viewed as placeholders. The scheme will be instantiated as concrete algorithms in Section 5.

**Definition 10.** For any non-empty polynomial set $P \subseteq K[x]$, an ascending set $C \subseteq K[x]$ is called a generalized characteristic set of $P$ if

(a) $C \subseteq \langle P \rangle$;

(b) there exists a polynomial set $Q \subseteq K[x]$ such that $\langle Q \rangle = \langle P \rangle$ and $\text{prem}(Q, C) = \{0\}$.

It is easy to see that characteristic sets are indeed a special kind of generalized characteristic sets. The zero relations in (1) also hold for generalized characteristic sets.

**Proposition 11.** Let $C = [C_1, \ldots, C_r]$ be any generalized characteristic set of a polynomial set $P \subseteq K[x]$ and

$$
I_i = \text{ini}(C_i), \quad P_i = P \cup \{I_i\}, \quad I = \{I_1, \ldots, I_r\}.
$$

Then the zero relations in (1) hold.

**Proof.** By Definition 10, there exists a $Q \subseteq K[x]$ such that $\langle P \rangle = \langle Q \rangle$ and $\text{prem}(Q, C) = \{0\}$. It is easy to know that $C$ is a characteristic set of $Q$. Bearing in mind Proposition 9 and $\text{Zero}(P) = \text{Zero}(Q)$, one sees clearly the truth of the proposition. □
Input: \( \mathcal{F} \) — non-empty polynomial set in \( K[x] \); 
\( \mathcal{D} \) — set of admissible reductions in \( K[x] \).

Output: \( \mathcal{A} \) — generalized characteristic set of \( \mathcal{F} \).

0.1 \( \mathcal{G} := \mathcal{F} \); \( \mathcal{R} := \mathcal{F} \);

0.2 while \( \mathcal{R} \neq \emptyset \) do

0.3 \( [\mathcal{A}, \mathcal{B}] := \text{MedSet}(\mathcal{G}, \mathcal{D}) \);

0.4 if \( \mathcal{A} \) is a contradictory ascending set then

0.5 \( \mathcal{R} := \emptyset \);

0.6 else

0.7 \( \mathcal{R} := \text{prem}(\mathcal{B} \setminus \mathcal{A}, \mathcal{A}) \setminus \{0\} \);

0.8 \( \mathcal{G} := \mathcal{G} \cup \mathcal{A} \cup \mathcal{R} \);

0.9 end

0.10 end

Now we can describe the main algorithm \textbf{NewCharSet} (Algorithm 1), which computes a generalized characteristic set of any given non-empty polynomial set in \( K[x] \).

Algorithm 1 is similar to the algorithm \textbf{GenCharSet} presented in Wang (2001b). The difference is that \textbf{MedSet} here outputs not only a medial set \( \mathcal{G} \) of the polynomial set \( \mathcal{G} \), but also another basis \( \mathcal{B} \) of the ideal \( \langle \mathcal{G} \rangle \). Properly storing in \( \mathcal{B} \) the information from computing \( \mathcal{A} \), one may check more efficiently whether \( \mathcal{A} \) is a characteristic set, which usually takes a large proportion of the total computing time. Before describing the concrete steps of \textbf{MedSet}, we need some additional definitions such as admissible reduction.

**Definition 12.** Let \( P, Q \in K[x] \setminus \{0\} \) and \( \prec_{\text{lex}} \) be the lexicographic order in \( K[x] \) such that \( x_1 < \cdots < x_n \). Denote \( P < Q \) or \( Q > P \) if

(a) \( \text{ht}(P) \prec_{\text{lex}} \text{ht}(Q) \), or
(b) \( \text{ht}(P) = \text{ht}(Q) \) and \( P - \text{hm}(P) < Q - \text{hm}(Q) \),

where \( \text{ht}(P) \) and \( \text{hm}(P) \) stand for the heading term and the heading monomial of \( P \) under \( \prec_{\text{lex}} \) respectively. Set \( 0 < P \) for any \( P \in K[x] \setminus \{0\} \).

If neither \( P < Q \) nor \( P > Q \), we write \( P \approx Q \). The notation \( P \preceq Q \) means that \( P > Q \) or \( P \approx Q \), and \( P \precsim Q \) is similarly defined.

The partial order \( \prec \) is a refinement of \( \prec \) (Definition 3). More precisely, for any \( P, Q \in K[x] \setminus \{0\} \), \( P \prec Q \) implies that \( P < Q \). On the contrary, \( P < Q \) implies only \( P \preceq Q \), and \( P \prec Q \) does not necessarily hold. Consider for example the polynomials \( P = y^3 + x \) and \( Q = y^3 + x^2 \) in \( K[x, y] \). By definition, \( P < Q \) and \( P \sim Q \).

**Lemma 13.** For any polynomial sequence \( P_1 \preceq P_2 \preceq P_3 \preceq \cdots \), there exists an integer \( m \) such that for any \( i \geq m \), \( P_i \approx P_m \).
Proof. From \( P_1 \preceq P_2 \preceq P_3 \preceq \cdots \), one can obtain the sequence \( \text{ht}(P_1) \geq \text{ht}(P_2) \geq \text{ht}(P_3) \geq \cdots \). In view of the property of term orders (see, e.g., Cox et al. 1997 Lemma 2 in Chapter 2), there exists an integer \( s \) such that for any \( i \geq s \), \( \text{ht}(P_i) = \text{ht}(P_s) \).

Let \( P'_1 = P_i - \text{hm}(P_i) \). We have a new sequence \( P'_s \preceq P'_{s+1} \preceq P'_{s+2} \preceq \cdots \). Similarly, there exists an integer \( t \geq s \) such that for any \( i \geq t \), \( \text{ht}(P'_t) = \text{ht}(P'_1) \). The rest can be done in the same manner. As the number of terms of any polynomial is limited, we will finally obtain an integer \( m \) such that for any \( i \geq m \), all \( P_i \) have the same set of terms, which means \( P_i \approx P_m \). □

Let \( \mathcal{D} \) be an operation with two polynomials \( P, Q \in \mathcal{K}[x] \setminus \mathcal{K} \) as its input and an ordered set of two polynomials in \( \mathcal{K}[x] \) as its output, and the latter is denoted by \( \text{Rem}(P, Q, \mathcal{D}) \).

The definition of admissible reduction, which is somewhat abstract, is given below. In the next section, we will discuss a number of concrete reductions used in our experiments.

**Definition 14.** Let \([ R_1, R_2 ] = \text{Rem}(P, Q, \mathcal{D})\) for any \( P, Q \in \mathcal{K}[x] \setminus \mathcal{K} \). The operation \( \mathcal{D} \) is called an admissible reduction (or reduction for short if there is no confusion) in \( \mathcal{K}[x] \) if \( R_1, R_2 \in (P, Q) \).

Suppose that \( \mathcal{D} \) is an admissible reduction in \( \mathcal{K}[x] \). We say that \( P \) is \( \mathcal{D} \)-reducible w.r.t. \( Q \) if \( P < R_1 \) and \( Q \preceq R_2 \); otherwise, \( P \) is \( \mathcal{D} \)-reduced w.r.t. \( Q \). Moreover, \( P \) and \( Q \) are called the reductend and the reductor respectively, and \( \text{Rem}(P, Q, \mathcal{D}) \) the \( \mathcal{D} \) reduction-rest of \( P \) by \( Q \).

The subalgorithm \textbf{MedSet} is given as Algorithm 2, where \textbf{cond} can be replaced by any Boolean expression that guarantees the termination of the \textbf{while} loop.

The operation \textbf{Find3R}(\mathcal{A}, \mathcal{D}) \) chooses polynomials \( P \) (reductend) and \( Q \) (reductor) from \( \mathcal{A} \) and \( \mathcal{D} \) (reduction) from \( \mathcal{D} \) such that \( P \) is \( \mathcal{D} \)-reducible w.r.t. \( Q \). If \textbf{Find3R}(\mathcal{A}, \mathcal{D}) \) finds appropriate \( P, Q, \mathcal{D} \), then it returns this triple; otherwise, it returns \([ \]].

The function \textbf{RemCh} is an extension of \textbf{Rem}. \( \textbf{RemCh}(P, Q, \mathcal{D}) \) returns not only the \( \mathcal{D} \) reduction-rest \( R \) of \( P \) by \( Q \), but also a Boolean value \( b \). When \( b \) is \textbf{true}, the reduction-rest \( R \) must satisfy the condition \( P, Q \in \langle R \rangle \). The computation of \( b \) depends on the concrete admissible reduction \( \mathcal{D} \), which will be discussed in the next section. The \textbf{if} block (lines 0.10–0.12) is designed to store information acquired in the reduction process, which may be used by \textbf{NewCharSet} to check more efficiently whether the output \( \mathcal{M} \) is a characteristic set.

In algorithm \textbf{MedSet}, one may view \textbf{cond}, \( \mathcal{D} \) and \textbf{Find3R} as placeholders, which will be instantiated in Section 5. The correctness and termination of Algorithms 2 and 1 are proved as follows.

Proof. (Algorithm 2) Correctness. For the statement \( [ R, b ] := \textbf{RemCh}(P, Q, \mathcal{D}) \) in line 0.4, we have \( R \subseteq \langle P, Q \rangle \) by Definition 14. It is thus easy to verify that \( \mathcal{A} \subseteq \langle \mathcal{F} \rangle \) always holds during the running of the algorithm, which means \( \mathcal{M} \subseteq \langle \mathcal{F} \rangle \). Obviously, the ranking of \( \mathcal{M} \) is not higher than the ranking of any basic set of \( \mathcal{F} \); hence \( \mathcal{M} \) is a medial set of \( \mathcal{F} \) by definition.

Let \( \mathcal{G}_i \) be the initial value of \( \mathcal{G} \) in the \( i \)th \textbf{while} loop. Obviously, \( \langle \mathcal{G}_1 \rangle = \langle \mathcal{F} \rangle \). Suppose that \( \langle \mathcal{G}_i \rangle = \langle \mathcal{F} \rangle \), and we assert that \( \langle \mathcal{G}_{i+1} \rangle = \langle \mathcal{F} \rangle \) as follows.

Consider the statement \( [ R, b ] := \textbf{RemCh}(P, Q, \mathcal{D}) \) in line 0.4 of the \( i \)th \textbf{while} loop. If the Boolean expression \( (P, Q \in \mathcal{G} \text{ and } b) \) equals \textbf{false}, then \( \mathcal{G}_{i+1} = \mathcal{G}_i \). Hence \( \langle \mathcal{G}_{i+1} \rangle = \langle \mathcal{G}_i \rangle = \langle \mathcal{F} \rangle \).
Input: $F$ — non-empty polynomial set in $K[x]$; 
$D$ — set of admissible reductions in $K[x]$.

Output: $M$ — medial set of $F$; 
$G$ — polynomial set satisfying $\langle G \rangle = \langle F \rangle$.

0.1 $A := F$; $G := F$;
0.2 while $\text{cond}$ do
0.3 $[P, Q, D] := \text{Find3R}(A, D)$;
0.4 $[R, b] := \text{RemCh}(P, Q, D)$;
0.5 if $R$ contains a non-zero constant then
0.6 $A := \{1\}$; $G := \{1\}$; break;
0.7 else
0.8 $A := A \setminus \{P, Q\} \cup R \setminus \{0\}$;
0.9 end
0.10 if $P, Q \in G$ and $b$ then
0.11 $G := G \setminus \{P, Q\} \cup P$;
0.12 end
0.13 end
0.14 $M := \text{BasSet}(A \cup F)$;

$\langle F \rangle$. Now assume that $(P, Q \in G$ and $b)$ equals true. Then line 0.11 is executed, and we have $G_{i+1} = G_i \setminus \{P, Q\} \cup P$. As $D$ is an admissible reduction, we have $R \subseteq \langle P, Q \rangle$. By the assumption on $b$ returned by RemCh, one has $P, Q \in \langle R \rangle$. Thus $\langle P, Q \rangle = \langle R \rangle$, which implies that $\langle G_{i+1} \rangle = \langle G_i \setminus \{P, Q\} \cup R \rangle$. From the above, $\langle G_{i+1} \rangle = \langle F \rangle$, so $\langle G \rangle = \langle F \rangle$ always holds.

Termination. It is obvious by the assumption on $\text{cond}$. □

Proof. (Algorithm 1) Correctness. From the properties of MedSet’s output, we know that $\langle B \rangle = \langle G \rangle$ in line 0.3, and $A$ is a medial set of $G$. It follows that $A \subseteq \langle G \rangle$. According to the pseudo-division formula, for any polynomial $R$ in $R$ of line 0.7, there exist $B \in B \setminus A$ and $G_i \in K[x]$ such that 

$\prod \text{ini}(A_i)^{s_i} \cdot B = \sum G_i A_i + R,$

where each $s_i$ is a non-negative integer and $A_i \in A$. Hence $R \subseteq \langle G \rangle$ always holds during the running of the algorithm. Thus the ideal generated by $G$ remains the same after the assignment in line 0.8, which means that $\langle G \rangle = \langle F \rangle$ always holds.

On the other hand, prem($G, A$) = $\{0\}$ when the while loop terminates. Hence $A$ is a generalized characteristic set of $F$ by Definition 10.

Termination. We use $A_i$ and $G_i$ to denote the values of $A$ and $G$ respectively in the $i$th while loop after executing line 0.3. Recalling the properties of MedSet, we know that
each \( \mathcal{A}_i \) is a medial set of \( \mathcal{G}_i \). By Proposition 7, one can obtain the sequence \( \mathcal{A}_1 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \cdots \) of triangular sets, which should be finite by Proposition 5. Thus the algorithm terminates. \( \square \)

4. Concrete Admissible Reductions

In this section we introduce and discuss several concrete admissible reductions.

\( \mathcal{D}_{\text{UG}} \) (univariate GCD reduction): define \( \text{Rem}(P, Q, \mathcal{D}_{\text{UG}}) := \)

\[
\begin{cases}
[0, \gcd(P, Q, x_q)], & \text{if } P, Q \text{ are univariate polynomials in } x_q; \\
[P, Q], & \text{otherwise.}
\end{cases}
\]

In the above definition, \( x_q \) is some variable in \( \mathcal{A} \). It is easy to verify that \( \mathcal{D}_{\text{UG}} \) is an admissible reduction, and \( P \) is \( \mathcal{D}_{\text{UG}}\)-reducible w.r.t. \( Q \) if and only if \( P, Q \) are univariate polynomials in the same variable.

Pseudo-division, used frequently in triangular decomposition, is also an admissible reduction.

\( \mathcal{D}_P \) (pseudo-division reduction): define \( \text{Rem}(P, Q, \mathcal{D}_P) := [\text{prem}(P, Q, \text{lv}(Q)), Q] \).

We have the following pseudo-division formula

\[
\text{ini}(Q)^s P = \text{pquo}(P, Q, \text{lv}(Q)) Q + \text{prem}(P, Q, \text{lv}(Q)),
\]

where \( s \) is a non-negative integer. Thus \( \text{Rem}(P, Q, \mathcal{D}_P) \subseteq (P, Q) \), which means that \( \mathcal{D}_P \) is an admissible reduction in \( K[x] \).

Proposition 15. \( P \) is \( \mathcal{D}_P\)-reducible w.r.t. \( Q \) if and only if \( P \) is reducible w.r.t. \( Q \).

Proof. Let \( x_q = \text{lv}(Q) \).

(\( \Rightarrow \)) Suppose that \( P \) is reduced w.r.t. \( Q \). According to the definition of pseudo-division, we have \( \text{prem}(P, Q, x_q) = P \). Thus \( P \) is \( \mathcal{D}_P\)-reducible w.r.t. \( Q \).

(\( \Leftarrow \)) Suppose that \( P \) is reducible w.r.t. \( Q \). Then either

\[
\text{prem}(P, Q, x_q) = 0 \quad \text{or} \quad \text{deg}([\text{prem}(P, Q, x_q), x_q]) < \text{ldeg}(Q).
\]

If \( \text{prem}(P, Q, x_q) = 0 \), then \( P \) is obviously \( \mathcal{D}_P\)-reducible w.r.t. \( Q \). Now assume that \( \text{prem}(P, Q, x_q) \neq 0 \). By analyzing the algorithm of pseudo-division [see, e.g., Mishra 1993, PseudoDivisionRec], one may find that for any \( i (q < i \leq n) \), \( \text{deg}(\text{prem}(P, Q, x_q), x_i) \leq \text{deg}(P, x_i) \). As \( P \) is reducible w.r.t. \( Q \),

\[
\text{deg}(\text{prem}(P, Q, x_q), x_q) < \text{ldeg}(Q) \leq \text{deg}(P, x_q).
\]

Hence \( \text{prem}(P, Q, x_q) < P \). It follows that \( P \) is \( \mathcal{D}_P\)-reducible w.r.t. \( Q \). \( \square \)

Pseudo-division can be done step by step using the following operation.

Definition 16. Let \( P, Q \in K[x] \setminus K \) with \( x_q = \text{lv}(Q) \), \( I = \text{ini}(Q) \), and \( J = \text{lc}(P, x_q) \) and suppose that \( P \) is reducible w.r.t. \( Q \). Then we can perform the following one-step pseudo-division:

\[
R := FP - GQ x_q^{\text{deg}(P, x_q) - \text{ldeg}(Q)},
\]

where \( F = \text{lcm}(I, J)/J \) and \( G = \text{lcm}(I, J)/I \). \( R \) is called the one-step pseudo-remainder of \( P \) w.r.t. \( Q \) and denoted by \( \text{stprem}(P, Q) \).
One can see that pseudo-division is recursive application of one-step pseudo-divisions and thus may immediately lead to big superfluous factors of the pseudo-remainder. In contrast, it is easier to control the selection and size of polynomials for one-step pseudo-division, which may result in smaller reduction-rests when combined with other reductions.

\( \mathcal{D}_{\text{SP}} \) (one-step pseudo-division reduction): define \( \text{Rem}(P, Q, \mathcal{D}_{\text{SP}}) := \)

\[
\begin{cases}
\text{stprem}(P, Q), & \text{if } P \text{ is reducible w.r.t. } Q; \\
[P, Q], & \text{otherwise}.
\end{cases}
\]

If \( P \) is reducible w.r.t. \( Q \), then \( \text{stprem}(P, Q) \in \langle P, Q \rangle \) by (3), and hence \( \text{Rem}(P, Q, \mathcal{D}_{\text{SP}}) = \langle P, Q \rangle \); otherwise, \( \text{Rem}(P, Q, \mathcal{D}_{\text{SP}}) = [P, Q] \subseteq \langle P, Q \rangle \) is obvious. Therefore \( \mathcal{D}_{\text{SP}} \) is also an admissible reduction by definition.

**Proposition 17.** \( P \) is \( \mathcal{D}_{\text{SP}} \)-reducible w.r.t. \( Q \) if and only if \( P \) is reducible w.r.t. \( Q \).

**Proof.** (\( \Rightarrow \)) If \( P \) is reduced w.r.t. \( Q \), then \( \text{Rem}(P, Q, \mathcal{D}_{\text{SP}}) = [P, Q] \), and hence \( P \) is \( \mathcal{D}_{\text{SP}} \)-reduced w.r.t. \( Q \).

(\( \Leftarrow \)) Suppose that \( P \) is reducible w.r.t. \( Q \). Then \( \text{Rem}(P, Q, \mathcal{D}_{\text{SP}}) = \text{stprem}(P, Q, Q) \).

If \( \text{stprem}(P, Q) = 0 \), then \( P \) is obviously \( \mathcal{D}_{\text{SP}} \)-reducible w.r.t. \( Q \). Now assume that \( \text{stprem}(P, Q) \neq 0 \) and consider (3). We know that

\[ F = \text{lcm}(I, J)/J = I/\gcd(I, J) \in K[x_1, \ldots, x_{q-1}]. \]

Thus for any \( i \) \((q < i \leq n)\), \( \deg(FP, x_i) = \deg(P, x_i) \). Furthermore,

\[ \deg(GQ, x_q^{\deg(P, x_i) - \deg(Q, x_i)}x_i) = \deg(G, x_i) = \deg(J/\gcd(I, J), x_i) \leq \deg(P, x_i). \]

Hence \( \deg(\text{stprem}(P, Q, x_i) \leq \deg(P, x_i) \). Moreover, we have

\[ \deg(FP, x_q) = \deg(GQ, x_q^{\deg(P, x_i) - \deg(Q, x_i)}, x_q), \]

which implies that \( \deg(\text{stprem}(P, Q, x_q) < \deg(P, x_q) \). It follows that \( \text{stprem}(P, Q) < P \), so \( P \) is \( \mathcal{D}_{\text{SP}} \)-reducible w.r.t. \( Q \).  

The division operation, which is used in the computation of Gröbner bases, can also be viewed as an admissible reduction. Instead of introducing the division reduction directly, we discuss the one-step division reduction first and then use it to recursively define the former.

\( \mathcal{D}_{\text{SD}} \) (one-step division reduction): define \( \text{Rem}(P, Q, \mathcal{D}_{\text{SD}}) := \)

\[
\begin{cases}
[P - M/\text{lht}(Q) \cdot Q, Q], & \text{if there exists a monomial } M \text{ of } P \text{ such that } \text{lht}(Q) \mid M; \\
[P, Q], & \text{otherwise}.
\end{cases}
\]

It is easy to verify that \( \mathcal{D}_{\text{SD}} \) is an admissible reduction, and \( P \) is \( \mathcal{D}_{\text{SD}} \)-reducible w.r.t. \( Q \) if and only if there exists a monomial of \( P \) which can be divided by \( \text{lht}(Q) \).

Bearing in mind the relation between \( \mathcal{D}_{\text{SP}} \) and \( \mathcal{D}_{\text{P}} \), we can define the division reduction \( \mathcal{D}_{\text{D}} \) as follows. First, set \( R_0 := P \) and \( S_0 := Q \). Then recursively compute \( [R_{i+1}, S_{i+1}] := \text{Rem}(R_i, S_i, \mathcal{D}_{\text{SD}}) \), until an integer \( m \) is found such that \( R_{m+1} = R_m \).

Define \( \text{Rem}(P, Q, \mathcal{D}_{\text{D}}) := [R_m, S_m] \), where \( S_m = Q \). Similar to \( \mathcal{D}_{\text{SD}} \), \( \mathcal{D}_{\text{D}} \) is an admissible
reduction, and \( P \) is \( D \)-reducible w.r.t. \( Q \) if and only if there exists a monomial of \( P \) which can be divided by \( \text{ht}(Q) \).

By computing subresultant PRS, one can also design a useful admissible reduction. We give the definition here and the reader may refer to \textit{von zur Gathen and L"{u}cking (2003)} for more details. Suppose that \( \text{lv}(P) = \text{lv}(Q) \), \( \text{ldeg}(P) \geq \text{ldeg}(Q) \) and treat \( P, Q \) as univariate polynomials in \( x_q = \text{lv}(Q) \) with coefficients in \( K[x_1, \ldots, x_q-1] \). Define the subresultant PRS of \( P \) and \( Q \) w.r.t. \( x_q \) to be

\[
P_1 := P, \quad P_2 := Q, \quad P_{i+2} := \frac{\text{prem}(P_i, P_{i+1}, x_q)}{Q_{i+2}}, \quad 1 \leq i \leq r - 2,
\]

where

\[
d_i := \text{deg}(P_i, x_q), \quad i = 1, \ldots, r,
Q_3 := (-1)^{d_1 - d_2 + 1}, \quad H_3 := -1,
Q_i := -\text{lc}(P_{i-2}, x_q)H_i^{d_{i-2} - d_i - 1}, \quad i = 4, \ldots, r,
H_i := (-\text{lc}(P_{i-2}, x_q))^{d_{i-3} - d_i - 2}H_i^{d_{i-1} - d_{i-3} + d_i - 2}, \quad i = 4, \ldots, r,
\]

and \( \text{prem}(P_{r-1}, P_r, x_q) = 0 \). The well-known Subresultant Chain Theorem (Mishra 1993, Theorem 7.9.1) indicates the relation between the subresultant PRS and the Euclidean PRS of \( P \) and \( Q \); \( P_i \) is similar to the element of the same degree in the Euclidean PRS. Furthermore, the former may have smaller superfluous factors.

We use \( \text{res}(P, Q, x_q) \) to denote the resultant of \( P \) and \( Q \) w.r.t. \( x_q \). If \( \text{res}(P, Q, x_q) = 0 \), then \( P_r \) is a greatest common divisor of \( P \) and \( Q \) w.r.t. \( x_q \) (Mishra 1993, Corollary 7.7.9). If \( \text{res}(P, Q, x_q) \neq 0 \), then \( \text{cls}(P_r) < \text{cls}(P) \) and \( \text{cls}(P_{r-1}) = \text{cls}(P) \). Thus \( P_{r-1} \) is a greatest common divisor of \( P \) and \( Q \) w.r.t. \( x_q \) under the condition \( P_r = 1 \). On the other hand, under the condition \( P_r \neq 0 \), \( P_r \) would be a greatest common divisor of \( P \) and \( Q \). Hence we can introduce the following reduction.

\( D_{SC} \) (subresultant PRS reduction): define \( \text{Rem}(P, Q, D_{SC}) := \)

\[
\begin{cases}
[0, P_r], & \text{if } \text{lv}(P) = \text{lv}(Q), \text{ldeg}(P) \geq \text{ldeg}(Q) \text{ and } \text{res}(P, Q, \text{lv}(Q)) = 0;

[P_r, P_{r-1}], & \text{if } \text{lv}(P) = \text{lv}(Q), \text{ldeg}(P) \geq \text{ldeg}(Q) \text{ and } \text{res}(P, Q, \text{lv}(Q)) \neq 0;

[P, Q], & \text{otherwise};
\end{cases}
\]

Example 4.1. Let \( P = a x^2 + b x + c \) and \( Q = dx + e \). Then the subresultant PRS of \( P \) and \( Q \) in \( x \) is \( a x^2 + b x + c, dx + e, d^2 c - e d b + a e^2 \). According to the definition of \( D_{SC} \), \( \text{Rem}(P, Q, D_{SC}) = [d^2 c - e d b + a e^2, dx + e] \).

According to Mishra (1993, Lemma 7.7.4), there exist \( A_j, B_j \in K[x] \) such that \( A_j P + B_j Q = P_j \), i.e., \( P_j \in (P, Q) \). Thus \( D_{SC} \) is an admissible reduction. In addition, it is easy to verify that \( P \) is \( D_{SC} \)-reducible w.r.t. \( Q \) if and only if \( \text{lv}(P) = \text{lv}(Q) \) and \( \text{ldeg}(P) \geq \text{ldeg}(Q) \). If \( \text{ldeg}(P) < \text{ldeg}(Q) \), then consider \( \text{Rem}(Q, P, D_{SC}) \).

The standard reduction used in Ritt-Wu’s algorithm CharSet and its variants is pseudo-division only, which may lead to superfluous factors and the swell of polynomial coefficients. This is one of the main causes that degrades the performance of Ritt-Wu’s

\footnote{An accurate description requires the use of evaluation homomorphism, see Mishra (1993, Section 7.3.1).}
algorithm in many cases. In the algorithm \textbf{NewCharSet} (line 0.7) other kinds of admissible reductions may also be used. The condition of admissible reductions is quite weak, which makes our algorithm clearly more flexible than other existing ones for computing characteristic sets.

The remaining problem is the computation of \( b \) in the step \([R, b] := \text{RemCh}(P, Q, \mathcal{D})\) of Algorithm 2, where \( P \) is \( \mathcal{D} \)-reducible w.r.t. \( Q \).

Let \([R_1, R_2] = R\). For \( \mathcal{D}_{UG} \), we have \( R_1 = 0 \) and \( R_2 = \gcd(P, Q, x_q) \), which implies that \( (P, Q) \subseteq (R_1, R_2) \). For \( \mathcal{D}_{SD} \) and \( \mathcal{D}_D \), \( R_2 = Q \) and the formula \( R_1 = P - AQ \) always holds, where \( A \) is a polynomial in \( \mathcal{K}[x] \). Thus it is obvious that \( P \in (R_1, Q) = (R_1, R_2) \) and \( Q \in (R_1, R_2) \). Hence for any of \( \mathcal{D}_{UG}, \mathcal{D}_{SD}, \) and \( \mathcal{D}_D \), we can let \( \text{RemCh} \) return \textbf{true} as the value of \( b \).

For \( \mathcal{D}_P \), recall the pseudo-division formula (2). Provided that \( \text{ini}(Q)^* \) is a non-zero constant, we also have \( P \in (R_1, Q) = (R_1, R_2) \). Thus for the pseudo-division reduction, we set \( b \) to be the value of the Boolean expression \( (\text{ini}(Q)^* \in \mathcal{K} \text{ or } s = 0) \). Similarly, consider (3) for \( \mathcal{D}_{SP} \) and set \( b \) to be the value of the Boolean expression \( (F \in \mathcal{K}) \).

Finally for \( \mathcal{D}_{SC} \), set \( b = \text{false} \) permanently.

5. A Sample Subalgorithm

In this section we provide details about the algorithm \textbf{MedSet}. The assignment of the Boolean expression \( \text{cond} \) is first considered and the termination of Algorithm 2 guaranteed by \( \text{cond} \) is proved. Then the basic idea and a sample algorithm for the operation \textbf{Find3R} are presented. We conclude this section with discussions about other possible strategies for \textbf{Find3R}.

There are several ways to assign \( \text{cond} \). For example, one can set \( \text{cond} = \text{false} \). In this case, the while loop in \textbf{MedSet} never starts and \textbf{NewCharSet} is identical to Ritt-Wu’s algorithm. Another example is to set a counter \( i \) of the while loop and assign an inequality to \( \text{cond} \), say \( i \leq 50 \); then the while loop will run 50 times. The termination of \textbf{MedSet} in both cases is obvious. In our implementation, we let \( \text{cond} \) be \textbf{Find3R}(\( A, \mathcal{D} \)) \( \neq [\,] \), which means that the while loop repeats until there is no triple \([P, Q, \mathcal{D}] (P, Q \in A, P \neq Q, \mathcal{D} \in \mathcal{D}) \) such that \( P \) is \( \mathcal{D} \)-reducible w.r.t. \( Q \). For this case, the termination of \textbf{MedSet} is proved as follows.

For any polynomial \( R_1 \) in \( \mathcal{F} \), if we perform \([R_2, Q_2] := \text{Rem}(R_1, Q_1, \mathcal{D})\), where \( Q_1 \) is chosen by \textbf{Find3R} such that \( R_1 \) is \( \mathcal{D} \)-reducible w.r.t. \( Q_1 \), then \( R_2 \) can be viewed as the successor of \( R_1 \) with \( R_1 > R_2 \); if we perform \([P_2, R_2] := \text{Rem}(P_1, R_1, \mathcal{D}')\), where \( P_1 \) is chosen by \textbf{Find3R} such that \( P_1 \) is \( \mathcal{D}' \)-reducible w.r.t. \( R_1 \), then \( R_2 \) can be viewed as the successor of \( R_1 \) with \( R_1 \gg R_2 \). As the algorithm runs, the successor \( R_3 \) of \( R_2 \) and then the successor \( R_4 \) of \( R_3 \) may be produced in the same way. Thus one may obtain a polynomial sequence \( R_1 \gg R_2 \gg R_3 \gg \cdots \). By Lemma 13, there exists an integer \( m \) such that \( R_m \approx R_{m+1} \approx \cdots \). As this is the case for every polynomial in \( \mathcal{F} \), the operation \textbf{Find3R} will find no triple in the end. Hence the algorithm terminates.

Now we explain the basic idea of the algorithm \textbf{Find3R}. The set of selected admissible reductions is \( \mathcal{D} = \{\mathcal{D}_{UG}, \mathcal{D}_{SD}, \mathcal{D}_{SC}, \mathcal{D}_{SP}\} \). There may exist several triples and/or several admissible reductions between a pair of polynomials. We would like to select better

\[ \mathcal{D}_P \text{ and } \mathcal{D}_D \text{ are omitted because they can be treated as recursion of } \mathcal{D}_{SD} \text{ and } \mathcal{D}_{SP} \text{ respectively.} \]
triples first. Therefore, a way to measure the goodness of triples should be defined. In addition, an order of the admissible reductions in $\mathfrak{D}$ is also needed. We give a sample order on triples as follows.

First of all, it is natural that a triple with $\mathfrak{D}_{\text{UG}}$ is better than other triples without it. In fact, several univariate pseudo-remainders may be produced in line 1.7 of the algorithm NewCharSet. It is easier to deal with the $\mathfrak{D}_{\text{UG}}$ reduction-rest of such pseudo-remainders in the next while loop than to deal with them immediately. Next, a triple with $\mathfrak{D}$ should be better than a triple with $\mathfrak{D}_{\text{SP}}$ or $\mathfrak{D}_{\text{SC}}$. There are two reasons for this.

1. For two triples $[P, Q, \mathfrak{D}_{\text{SD}}]$ and $[P, Q, \mathfrak{D}_{\text{SP}}]$, $P$ is both $\mathfrak{D}_{\text{SD}}$-reducible and $\mathfrak{D}_{\text{SP}}$-reducible w.r.t. $Q$. It is easy to see that computing the $\mathfrak{D}_{\text{SD}}$ reduction-rest $R$ of $P$ by $Q$ requires less multiplications than computing the $\mathfrak{D}_{\text{SP}}$ reduction-rest $R'$ and in general $R$ is of smaller size than $R'$. Therefore, $[P, Q, \mathfrak{D}_{\text{SD}}]$ is better than $[P, Q, \mathfrak{D}_{\text{SP}}]$.

2. For two triples $[P, Q, \mathfrak{D}_{\text{SD}}]$ and $[P', Q', \mathfrak{D}_{\text{SP}}]$ with $P \neq P'$ or $Q \neq Q'$, the former is better because $P$ is in $\langle Q, R \rangle$, while in general $P'$ is not in $\langle Q', R' \rangle$, where $R$ and $R'$ are the $\mathfrak{D}_{\text{SD}}$ reduction-rest of $P$ by $Q$ and the $\mathfrak{D}_{\text{SP}}$ reduction-rest of $P'$ by $Q'$ respectively.

Then, $[P, Q, \mathfrak{D}_{\text{SC}}]$ may be better than $[P, Q, \mathfrak{D}_{\text{SP}}]$ because the reduction-rest of the former generally involves coefficients of smaller size than that of the latter. Finally, a triple $[P, Q, \mathfrak{D}]$ might be better than $[P', Q', \mathfrak{D}] (\mathfrak{D} \in \mathfrak{D})$ if $P > P'$ or $Q < Q'$. This observation is based on the way of reduction in Ritt-Wu’s algorithm: the polynomial of maximal class and maximal degree is first reduced by a polynomial of minimal degree. To summarize, we provide the formal definition of an order on the triples: $[P, Q, \mathfrak{D}] < [P', Q', \mathfrak{D}']$ if $\mathfrak{D} < \mathfrak{D}'$, or $\mathfrak{D} = \mathfrak{D}'$ and $P < P'$, or $\mathfrak{D} = \mathfrak{D}'$ and $P \approx P'$ and $Q > Q'$. The admissible reductions in $\mathfrak{D}$ are ordered as $\mathfrak{D}_{\text{UG}} > \mathfrak{D}_{\text{SD}} > \mathfrak{D}_{\text{SC}} > \mathfrak{D}_{\text{SP}}$.

Taking as input a polynomial set $\mathcal{F}$ and a set $\mathfrak{D}$ of selected admissible reductions under the above order, Find3R will do the following steps to select a better triple or output $\emptyset$ if there is none.

1. Check if there exist triples $[P, Q, \mathfrak{D}_{\text{UG}}]$ ($P, Q \in \mathcal{F}, P \neq Q$). If there is only one, then return it; if there are many, then return the one in which $P$ has maximal class and maximal leading degree and $Q$ has fewest terms and minimal leading degree.

2. Sort the polynomials in $\mathcal{F}$ increasingly w.r.t. the partial order $\prec$. Start with a polynomial $P$ of highest order and check if there exists any polynomial $Q$ such that $P$ is $\mathfrak{D}_{\text{SD}}$-reducible w.r.t. $Q$. If there is only one $Q$, then return $[P, Q, \mathfrak{D}_{\text{SD}}]$; if there are many, then select $Q$ from $\{F \in \mathcal{F} : P$ is $\mathfrak{D}_{\text{SD}}$-reducible w.r.t. $F\}$ which has fewest terms and minimal leading degree and return $[P, Q, \mathfrak{D}_{\text{SD}}]$; otherwise, consider another $P$ of lower order and check again. If there is no such triple, then go to the next step.

3. Start with a polynomial $P$ of highest order again and check if there exists any $Q$ such that $P$ is $\mathfrak{D}_{\text{SC}}$-reducible w.r.t. $Q$. The process is similar to step 2.

4. Start with a polynomial $P$ of highest order again and check if there exists any $Q$ such that $P$ is $\mathfrak{D}_{\text{SP}}$-reducible w.r.t. $Q$. The process is similar to step 2.

5. If there is no triple at all, then output $\emptyset$. 

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Input: $\mathcal{F}$ — set of polynomials in $\mathbb{K}[x]$;

$\mathcal{D} = \{\mathcal{D}_{\text{UG}}, \mathcal{D}_{\text{SD}}, \mathcal{D}_{\text{SC}}, \mathcal{D}_{\text{SP}}\}$.

Output: $\emptyset$ or $[P, Q, \mathcal{D}]$ such that $P$ is $\mathcal{D}$-reducible w.r.t. $Q$.

0.1 $S := \emptyset$;
0.2 for $i = n$ to 1 while $|S| < 2$ do
0.3 $\quad S := \{P \in \mathcal{F} : P \in \mathbb{K}[x_i]\}$;
0.4 end
0.5 if $\exists P, Q \in S, P \neq Q$, such that $P$ is $\mathcal{D}_{\text{UG}}$-reducible w.r.t. $Q$ then
0.6 $\quad$ return $[P, Q, \mathcal{D}_{\text{UG}}]$ such that $P \in S$ has maximal degree, $Q \in S \setminus \{P\}$ has fewest terms and minimal degree, and $P$ is $\mathcal{D}_{\text{UG}}$-reducible w.r.t. $Q$;
0.7 end
0.8 $[P_1, \ldots, P_r] := \mathcal{F}$ (with $P_i$ sorted increasingly w.r.t. $<$);
0.9 for $i = r$ to 2 do
0.10 $\quad Q := \{Q \in \mathcal{F} \setminus \{P_i\} : P_i$ is $\mathcal{D}_{\text{SD}}$-reducible w.r.t. $Q\}$;
0.11 $\quad$ if $Q \neq \emptyset$ then
0.12 $\quad \quad$ choose $Q \in Q$ with fewest terms and minimal leading degree;
0.13 $\quad \quad$ return $[P_i, Q, \mathcal{D}_{\text{SD}}]$;
0.14 $\quad$ end
0.15 end
0.16 for $i = r$ to 2 do
0.17 $\quad Q := \{Q \in \mathcal{F} \setminus \{P_i\} : P_i$ is $\mathcal{D}_{\text{SC}}$-reducible w.r.t. $Q\}$;
0.18 $\quad$ if $Q \neq \emptyset$ then
0.19 $\quad \quad$ choose $Q \in Q$ with fewest terms and minimal leading degree;
0.20 $\quad \quad$ return $[P_i, Q, \mathcal{D}_{\text{SC}}]$;
0.21 $\quad$ end
0.22 end
0.23 for $i = r$ to 2 do
0.24 $\quad Q := \{Q \in \mathcal{F} \setminus \{P_i\} : P_i$ is $\mathcal{D}_{\text{SP}}$-reducible w.r.t. $Q\}$;
0.25 $\quad$ if $Q \neq \emptyset$ then
0.26 $\quad \quad$ choose $Q \in Q$ with fewest terms and minimal leading degree;
0.27 $\quad \quad$ return $[P_i, Q, \mathcal{D}_{\text{SP}}]$;
0.28 $\quad$ end
0.29 end
0.30 $\emptyset$;
The above process is described formally as Algorithm 3. Its correctness follows from the above analysis and its termination is obvious. If “$P_i$ is $D_{SP}$-reducible w.r.t. $Q$” in line 3.24 is replaced by “ini($P_i$) is $D_{SP}$-reducible w.r.t. $Q$”, then MedSet computes a weak-ascending set and therefore NewCharSet outputs a weak-characteristic set.

Note that the design of Find3R is flexible and can be made more technical and comprehensive. There are several ways to improve Find3R. One may introduce other admissible reductions by computing Bézout resultants or Gröbner bases, or using other techniques (e.g., from linear algebra). If the input set of Find3R contains polynomials of special form or structure, then new reduction strategies may be adopted. For example, if there is one univariate polynomial of low degree or few (say one or two) terms, then one should use this polynomial first to reduce (and simplify) other polynomials. The ordering used in line 3.8 is crucial for the efficiency of MedSet. One may sort polynomials w.r.t. different orderings depending on the admissible reductions.

6. Example and Preliminary Experiments

In this section we present first an illustrative example to compare the outputs of different algorithms and to show how the algorithm NewCharSet works and then our experimental results on the performance of NewCharSet. In what follows, an index tuple $[[\deg(P, x_1), \ldots, \deg(P, x_n)], \text{nops}(P), \text{hm}(P), m]$ is used to characterize a polynomial $P \in K[x] \setminus K$, where nops($P$) denotes the number of terms of (expanded) $P$, hm($P$) the heading monomial of $P$, and $m$ the maximal number of digits of the integer coefficients of $P$.

Example 6.1 [Wang 2004, Epsilon-A14]. Let $\mathcal{F} = \{F_1, F_2, F_3\} \subseteq \mathbb{Q}[w, x, y, z]$, where

$F_1 = x^2 + y^2 + z^2 - w^2, \quad F_2 = xy + z^2 - 1, \quad F_3 = xyz - x^2 - y^2 - z + 1,$

with $w < x < y < z$. The output of CharSet [3] consists of three polynomials with index tuples

$$[[520, 42, 0, 0], 5450, w^{508}x^{42}, 212],$$
$$[[6, 30, 1, 0], 93, x^{29}y, 3],$$
$$[[520, 40, 0, 1], 10390, w^{510}x^{40}z, 212]],$$

while the output of NewCharSet also consists of three polynomials yet with index tuples

$$[[8, 12, 0, 0], 23, x^{12}, 2], \quad [[4, 6, 1, 0], 12, w^2x^3y, 1], \quad [[4, 6, 0, 1], 17, x^6z, 1].$$

The Gröbner basis of $\mathcal{F}$ w.r.t. the lexicographic order (determined by $w < x < y < z$) contains 9 polynomials, of which one of the biggest has index tuple $[[12, 11, 1, 1], 41, yz, 5]$.

Now let us show how NewCharSet works. For brevity, the admissible reductions $D_{SD}$ and $D_{SP}$ are renamed $D_D$ and $D_P$ respectively. After the initialization of $\mathcal{G}$ and $\mathcal{A}$ in line 1.1 of NewCharSet, the first while loop (line 1.2) starts and MedSet($\mathcal{F}, D$) is called. Line 2.1 initializes the values of $\mathcal{A}$ and $\mathcal{G}$. Then it is checked whether the Boolean expression

$$\text{Corresponding to the Maple command charset}(\mathcal{F}, [w, x, y, z]) \text{ with the Epsilon package [Wang 2004].}$$
Find3R(\(A, \mathcal{D}\)) \neq [] is true or false. One sees clearly that \(F_3 < F_2 < F_1\) and there are three triples
\([F_2, F_3, \mathcal{D}], [F_1, F_3, \mathcal{D}], [F_1, F_2, \mathcal{D}]\).
Therefore the first while loop (line 2.2) starts. Find3R(\(A, \mathcal{D}\)) outputs the third triple and RemCh(\(F_1, F_2, \mathcal{D}\)) then returns \text{Rem}(F_1, F_2, \mathcal{D}), true). Since the reduction-rest contains no constant, \(A\) and \(G\) are updated and the second while loop (line 2.2) starts. After the second while loop, \(G\) is updated to be
\[\{y^2 - xy + x^2 - w^2 + 1, xyz - z - xy - w^2 + 2, z^2 + xy - 1\}\] (6)
and remains unchanged afterwards. After the 7th while loop, \(A\) is updated to be an ascending set with index tuples of its polynomials shown in (5) and Find3R(\(A, \mathcal{D}\)) = []. So MedSet returns a basic set of \(A \cup F\) (which is \(A\)) as well as (6). It can be verified by line 1.7 that the pseudo-remainders of the polynomials in (6) w.r.t. \(A\) are all zero. Note that it is easier to compute \text{prem}(G \setminus A, A) than \text{prem}(F \setminus A, A) because one polynomial in (6) is of smaller class than the polynomials in \(F\). Therefore, \(A\) with the index tuples in (5) for its three polynomials is a characteristic set of \(F\).

We have implemented the algorithm NewCharSet described in Section 3 using the Epsilon library. There are two versions of it: NewCharSetw outputs weak-characteristic sets and NewCharSet outputs characteristic sets. We have compared the performances of NewCharSet, NewCharSetw, CharSet, and CharSetw\(^4\) as well as the Gröbner basis algorithm Groebner\(^5\). The timings (in CPU seconds) given in Table 1 are for our implementation running in Maple 14 under Windows 7 on a laptop Intel Core 2 Duo T6670 Processor 2.00 GHz with 3 GB of memory. The asterisk * indicates that the computation is out of memory. Table 2 shows the size of the test examples (including the number of variables and the total degree of each polynomial) and the number of polynomials in the output of each algorithm. The entry [3, 3, 4] in the table means that example 1 consists of three polynomials whose total degree are 3, 3, 4 respectively. The test examples are taken from \url{http://www.symbolicdata.org/} and \url{http://www-salsa.lip6.fr/~wang/epsilon/}. It is easy to see that NewCharSet is more efficient than CharSet and CharSetw for most of the examples and is comparable with Groebner.

Now let us comment on the empirical results shown in Table 1. For examples 3, 5, 8, 10, 15, and 16, NewCharSet is slower than CharSet, and for examples 9, 13, 17, and 18, neither of them can get any result within 1000 seconds. For the 8 other examples (notably, examples 4 and 14), NewCharSet is always faster than CharSet. Among the first four algorithms, NewCharSetw is the most efficient (for all but one test example, i.e., example 15). There are five examples for which CharSet cannot get any result within 1000 seconds. The Groebner algorithm is slower than other algorithms for examples 1, 3, 7, 14, and 15, but considerably faster for examples 9 and 18. There are two examples for which Groebner cannot complete the computation due to the lack of memory.

There are several reasons for NewCharSet to be more efficient than CharSet. First of all, it is the polynomials in the output \(G\) of MedSet (instead of \(F\)) whose pseudo-remainders

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\(^4\) By CharSet and CharSetw we mean the algorithms implemented in Epsilon which adopted optional strategies from [Wang 1992, 2001a] for speeding up the computation of characteristic sets rather than Ritt-Wu’s original algorithm.

\(^5\) Corresponding to the Maple command Groebner[Basis](\(F, \text{plex}(x_1, \ldots, x_1)\)).
Table 1. Timings (in seconds) for five algorithms

| No | Name               | CharSetw | CharSet | NewCharSetw | NewCharSet | Groebner |
|----|--------------------|----------|---------|-------------|------------|----------|
| 1  | DiscrC2            | 0.016    | 0.016   | 0.016       | 0.016      | >1000    |
| 2  | Epsilon-A14        | >1000    | 16.302  | 0.016       | 0.031      | 0.156    |
| 3  | Chou156-1          | 0.218    | 0.093   | 0.047       | 0.156      | 19.812   |
| 4  | Trinks1            | >1000    | >1000   | 0.063       | 0.063      | 0.047    |
| 5  | ZeroDim14          | 34.398   | 0.795   | 0.156       | 7.909      | 0.172    |
| 6  | Epsilon-A16        | 761.145  | 8.128   | 0.390       | 3.073      | 0.266    |
| 7  | Schiele1           | 1.576    | 1.482   | 0.406       | 0.375      | 5.491    |
| 8  | Epsilon-A25        | 0.172    | 0.078   | 0.546       | 7.301      | 1.029    |
| 9  | Cyclic5            | >1000    | >1000   | 0.577       | >1000      | 0.062    |
| 10 | Fee1               | 676.061  | 2.777   | 1.326       | 5.273      | 0.156    |
| 11 | Weispfenning94     | >1000    | 33.680  | 2.340       | 7.238      | 0.921    |
| 12 | Steidel2           | >1000    | 49.312  | 5.475       | 15.678     | 0.905    |
| 13 | Epsilon-A30        | >1000    | >1000   | 7.941       | >1000      | 2.450    |
| 14 | Fateman            | >1000    | >1000   | 9.797       | 24.429     | 411.546  |
| 15 | Sym3-5             | 23.868   | 1.623   | 10.733      | 10.483     | 35.865   |
| 16 | Epsilon-A3         | 160.150  | 67.174  | 17.035      | 261.192    | *        |
| 17 | Wu90               | >1000    | >1000   | 44.835      | >1000      | *        |
| 18 | Cyclic6            | >1000    | >1000   | >1000       | >1000      | 0.374    |

w.r.t. the output (weak-) medial set $M$ of MedSet are computed. This may reduce the cost of zero pseudo-remainder verification in most cases because the polynomials in $G$ are “closer” (than those in $F$) to the polynomials in $M$. For example, if $G$ in MedSet is fixed to be $F$, then NewCharSet takes 42.307 seconds (vs 15.678 seconds when $G$ remains updated) for example 12 and gets no result within 500 seconds (vs 24.429 seconds when $G$ remains updated) for example 14. However, keeping $G$ updated may also lower the efficiency of the algorithm for some examples (including three of our test examples). How to choose $G$ to improve the efficiency is a question that remains for further investigation.

The use of admissible reductions allows us to effectively control the swell of polynomial coefficients as well as degrees. As we have seen, the maximal numbers of digits of coefficients in (4) are much bigger than those in (5). More comparisons are given in Table 6 in the appendix. The ordering $<$ used in line 3.8 can also be replaced by other orderings. According to our experiments, there may be fewer while loops in NewCharSet.
Table 2. Numbers of polynomials in the outputs of five algorithms

| No | No of vars | Total degree list | CharSetw | CharSet | NewCharSetw | NewCharSet | Groebner |
|----|------------|-------------------|----------|---------|-------------|------------|----------|
| 1  | 12         | [3, 3, 4]         | 3        | 3       | 3           | 3          |          |
| 2  | 4          | [1, 2, 3, 4]      | 3        | 3       | 3           | 3          | 9        |
| 3  | 4          | [2, 2, 2, 2]      | 4        | 4       | 4           | 4          | 49       |
| 4  | 6          | [1, 1, 2, 2, 2, 3]| 6        | 6       | 6           | 6          |          |
| 5  | 4          | [1, 2, 3, 4]      | 4        | 4       | 4           | 4          | 4        |
| 6  | 4          | [4, 3, 2, 4]      | 4        | 4       | 4           | 4          | 4        |
| 7  | 2          | [6, 11]           | 2        | 2       | 2           | 2          | 3        |
| 8  | 8          | [2, 2, 2, 2]      | 4        | 4       | 4           | 4          | 21       |
| 9  | 5          | [1, 2, 3, 4, 5]   | 5        |         |             |            | 11       |
| 10 | 4          | [2, 3, 4, 4]      | 4        | 4       | 4           | 4          | 5        |
| 11 | 3          | [4, 5, 5]         | 3        | 3       | 3           | 3          |          |
| 12 | 3          | [4, 5, 5]         | 3        | 3       | 3           | 3          |          |
| 13 | 10         | [3, 3, 3, 3, 3, 3, 3, 3, 3, 3] | 10       | 10      |              |            |          |
| 14 | 4          | [3, 5, 5]         | 3        | 3       | 3           | 3          | 108      |
| 15 | 3          | [6, 6, 6]         | 3        | 3       | 3           | 3          |          |
| 16 | 4          | [4, 4, 4, 5]      | 4        | 4       | 4           | 4          |          |
| 17 | 4          | [3, 3, 3, 4]      |          | 4       |              |            |          |
| 18 | 6          | [1, 2, 3, 4, 5, 6]|          |         |              |            | 17       |

if polynomials are sorted w.r.t. the lexicographic order of their degree tuples, i.e., \( P \) is lower than \( Q \) if

\[
[\deg(P, x_1), \ldots, \deg(P, x_n)] < \text{lex} [\deg(Q, x_1), \ldots, \deg(Q, x_n)].
\]

For example, let \( P = yz + x^2z + 1 \) and \( Q = xz + z + y^2 \) with \( x < y < z \); then \( P > Q \) according to Definition 12. However, the above ordering indicates that \( P \) is lower than \( Q \). One can see that the degree tuples of \( P \) and \( Q \) are \([2, 1, 1]\) and \([1, 2, 1]\) respectively, so \([2, 1, 1] < \text{lex} [1, 2, 1]\). Note that \( x < y < z \) is used instead of \( x > y > z \), refer to [Cox et al., 1997] Chapter 2, Definition 3 (Lexicographic Order) for the difference.

In the case where the polynomials in the input set have many (say more than 6) variables, Groebner needs to deal with many intermediate polynomials (see examples 3, 14, 16, and 17 in Table 2) and CharSet and CharSetw need to deal with polynomials of complex initials and high degrees, while NewCharSetw can partially avoid such problems.
Finally, we point out that admissible reductions may also be incorporated into other algorithms of triangular decomposition, which would create plenty of room for improvements on all such algorithms.

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Appendix

In this appendix we provide more details about the outputs of the five algorithms in comparison, including degree tuples, numbers of terms, head monomials, and maximal digits of coefficients of the output polynomials, for some (but not all, due to space limitation) of the examples.

Table 3 collects the degree tuples of the polynomials in the output of each algorithm. Let us explain the meanings of the entries in this table using an example: the entry in the second row and the third column containing three tuples implies that the output of CharSet for example 2 consists of three polynomials, say \(F, G, H\) with \(F < G < H\), the degrees of \(F\) in \(w, x, y, z\) (\(w < x < y < z\)) are 520, 42, 0, 0 respectively. One can see that for most examples, the output polynomials of NewCharSetw and NewCharSet are of lower degrees than those of CharSetw and CharSet. The entry in the second row and the last column contains \ldots\, which means that there are more elements.

Statistical data about the numbers of terms of expanded polynomials in the output of each algorithm are given in Table 4. The entry 6, 6, 6 in the second column and the second row indicates that for example 1, CharSetw outputs a set of three polynomials, which consists of 6, 6, 6 terms respectively. For example 3 there are four polynomials in 7 variables. The output of any of the first four algorithms contains four polynomials, and for NewCharSet some of the polynomials are of degree \(> 100\). This may explain why NewCharSet is slower than CharSet. Similar situations occur for examples 8 and 16. For examples 2, 6, and 12, the polynomials in the output of NewCharSet have fewer terms than those in the output of CharSet, so NewCharSet is faster. For most examples, the polynomials contained in the output of NewCharSetw have fewer terms, so NewCharSetw is faster than CharSetw, CharSet, and NewCharSet. For examples 3, 14, and 15, Groebner
outputs more polynomials of more terms than \texttt{NewCharSet}. It is therefore slower than \texttt{NewCharSet}.

Table 5 displays the head terms of the polynomials in the output of each algorithm. Because most of the initials are quite complicated, we have reproduced only the head terms instead of the initials. Table 6 shows the maximal digits of coefficients of the polynomials in the output of each algorithm. One sees clearly that the polynomials in the outputs of \texttt{NewCharSet} and \texttt{NewCharSetw} have smaller coefficients.
### Table 4. Numbers of terms of polynomials in the outputs of five algorithms

| No | CharSetw | CharSet | NewCharSetw | NewCharSet | Groebner                  |
|----|----------|---------|-------------|------------|---------------------------|
| 1  | 6, 6, 6  | 6, 6, 8 | 6, 6, 6     | 6, 6, 8    |                           |
| 2  | 5450, 93, 10390 | 23, 12, 5 | 23, 12, 17 | 23, 45, 35, 39, 5, 30, 31, 41, 3 |
| 3  | 18, 26, 38, 2 | 18, 26, 38, 33 | 18, 26, 5, 2 | 18, 26, 114, 108 | 10, 8, 5, 17, 25, 9,... |
| 4  | 3, 4, 4, 6, 6, 4 | 3, 4, 4, 4, 4 | 3, 3, 3, 3, 3 |
| 5  | 41, 80, 80, 4 | 74, 107, 142, 143 | 37, 51, 12, 4 | 37, 51, 60, 60 | 25, 25, 25 |
| 6  | 143, 284, 284, 286 | 165, 288, 327, 328 | 25, 50, 8, 5 | 25, 50, 48, 50 | 25, 25, 25 |
| 7  | 292, 522 | 292, 522 | 88, 158     | 88, 158    | 88, 393, 498              |
| 8  | 128, 21, 35, 4 | 89, 21, 35, 35 | 253, 116, 7, 4 | 3217, 116, 152, 15087 | 89, 220, 119, 150,... |
| 9  | 28, 55, 8, 10, 5 | 4, 8, 10, 5, 12, 13, 5 |
| 10 | 57, 112, 112, 112 | 103, 153, 180, 204 | 38, 137, 45, 6 | 60, 118, 118, 118 | 25, 26, 27, 25, 27 |
| 11 | 445, 792, 887 | 70, 140, 8 | 54, 140, 41 | 54, 55, 55 |
| 12 | 663, 1188, 1324 | 108, 198, 13 | 81, 162, 162 | 53, 55, 55 |
| 13 | 3752, 29, 29, 12, 8, 6, 8, 6, 7, 6 | 172, 11, 10, 11, 11, 10 |
| 14 | 87, 199, 16 | 76, 152, 150 | 45, 48, 48, 48,... |
| 15 | 139, 153, 3 | 139, 153, 152 | 90, 88, 3 | 90, 88, 88 | 90, 142, 142 |
| 16 | 3123, 210, 8, 547 | 1984, 210, 378, 547 | 20933, 210, 8, 6 | 8169, 210, 378, 547 |
| 17 | 49226, 305, 8, 6 |

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### Table 5. Head terms of polynomials in the outputs of five algorithms

| No | CharSetw | CharSet | NewCharSetw | NewCharSet | Groebner |
|----|----------|---------|-------------|------------|----------|
| 1  | $f, y^2 i, y^2 j$ | $f, y^2 i, y^2 j$ | $f, y^2 i, y^2 j$ | $f, y^2 i, y^2 j$ | $x^{12}, w^{14} y, x^{12}, w^{14} y,$ |
|    | $w^{508} x^{42}, x^{29} y,$ | $x^{12}, w^{2} x^{3} y,$ | $x^{12}, w^{2} x^{3} y,$ | $x^{12}, w^{14} y,$ | $w^{10} xy, . . . |
| 2  | $x^{40}, x^{30} x_2,$ | $x^{73}, x^{24} x_2,$ | $x^{36}, x^{24} x_2,$ | $x^{30}, x^{24} x_2,$ | $x^{14}, x_2, x_3, x_4$ |
|    | $x^{39} x_3, x_4$ | $x^{70} x_3, x^{70} x_4$ | $x^{70} x_3, x^{29} x_4$ | $x^{70} x_3, x^{29} x_4$ | $x^{14}, x_2, x_3, x_4$ |
| 5  | $d^{146}, d^{144} p,$ | $d^{168}, d^{146} p,$ | $d^{28}, d^{26} p,$ | $d^{27}, d^{26} p,$ | $d^{24}, p, c, q$ |
|    | $d^{145} c, d^{145} q$ | $d^{166} c, d^{166} q$ | $p^{3} c, pq$ | $p^{2} c, d^{26} q$ | $d^{26} c, d^{26} q$ |
| 8  | $d^{2}, t^{5} z,$ | $t^{6} d^{2}, t^{6} z,$ | $t^{12} d^{2}, t^{6} d z,$ | $t^{10} a^{5} b^{2} z^{2}, t^{6} d^{2}, t^{6} d z,$ | $t^{6} d^{2}, c^{2} d^{4} z,$ |
|    | $t^{6} y, t x$ | $t^{6} y, t^{6} x$ | $z y, y x$ | $t^{7} a d y, t^{6} a^{3} b^{2} c^{4} d^{11} x,$ | $t^{2} d^{2}, t^{2} b c^{3} z,$ . . . |
| 10 | $q^{50}, q^{55} c,$ | $q^{104}, q^{78} c,$ | $q^{73}, q^{62} c,$ | $q^{60}, q^{59} c,$ | $q^{24}, q c,$ |
|    | $q^{50} n, q^{55} d$ | $q^{91} p, q^{103} d$ | $q^{4} c^{2} p, q p d$ | $q^{59} p, q^{59} d$ | $c^{3}, p, d$ |
| 12 | $x^{675}, x^{608} y,$ | $x^{112}, x^{102} y,$ | $x^{81}, x^{80} y,$ | $x^{54}, y, z$ | $x^{141}, y, z$ |
|    | $x^{674} z$ | $x^{3} y^{5} z$ | $x^{80} z$ | $x^{131} z$ |
| 15 | $x^{153}, x^{120} y, z$ | $x^{158}, x^{120} y,$ | $x^{161}, x^{128} y,$ | $x^{161}, x^{128} y,$ | $x^{141}, y, z$ |
|    | $x^{125} z$ | $x^{5} z$ | $x^{131} z$ |
| 16 | $x^{16}, x^{7} t_{1}, x_{72},$ | $x^{14}, x^{7} t_{1}, x_{72},$ | $x^{16}, x^{7} t_{1}, x_{72},$ | $x^{14}, x^{7} t_{1}, x_{72},$ | $x^{14}, x^{7} t_{1}, x_{72},$ |
|    | $x^{2} t_{1}, x_{73}, x_{71}, x_{80}$ | $x^{2} t_{1}, x_{73}, x_{71}, x_{80}$ | $x^{2} t_{1}, x_{73}, x_{71}, x_{80}$ | $x^{2} t_{1}, x_{73}, x_{71}, x_{80}$ | $x^{2} t_{1}, x_{73}, x_{71}, x_{80}$ |
| 17 | $x_{1}^{2}, x_{1}^{2}, x_{2},$ | $x_{1}^{2}, x_{3}, x_{4}$ |
| No | CharSet | CharSetw | NewCharSet | NewCharSetw | Groebner |
|----|---------|----------|-----------|-------------|----------|
| 1  | 1, 1, 1 | 1, 1, 1  | 1, 1, 1   | 1, 1, 1   |          |
| 2  | 2, 2, 2 | 2, 2, 2  | 2, 2, 1, 1| 2, 2, 4, 4| 1, 1, 1, …|
| 3  | 1, 1, 1 | 1, 1, 1  | 1, 1, 1   | 1, 1, 1   |          |
| 4  | 212, 3, 212 | 2, 1, 1 | 2, 1, 1   | 2, 1, 1, 1| 2, 3, 3, …|
| 5  | 41, 110, 146 | 71, 55, 70, 70 | 20, 27, 4, 1 | 20, 28, 215, 216 | 20, 257, 258, 245 |
| 6  | 158, 164, 776, 468 | 184, 164, 188, 189 | 39, 36, 2, 1 | 18, 22, 614, 298 | 18, 266, 271, 266 |
| 7  | 11, 11 | 11, 11 | 6, 7 | 6, 6 | 6, 15, 17 |
| 8  | 2, 2, 2 | 2, 2, 2 | 5, 4, 1 | 6, 2, 9 | 2, 3, 3, … |
| 9  | 14, 21, 1, 1, 1 | 3, 6, 6, 1, 6 | 3, 6, 6, 1, 6 | 2, 3, 6, 6, 1, 6 |
| 10 | 36, 123, 330, 383 | 67, 50, 60, 288 | 20, 160, 9, 1 | 72, 264, 603, 904 | 15, 163, 186, 143, 182 |
| 11 | 94, 85, 97 | 18, 17, 1 | 8, 15, 298 | 8, 202, 202 |          |
| 12 | 89, 78, 99 | 16, 14, 1 | 12, 22, 364 | 9, 193, 194 |          |
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