Critical geometry of two-dimensional passive scalar turbulence

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Passive scalars advected by a magnetically driven two-dimensional turbulent flow are analyzed using methods of statistical topography. The passive tracer concentration is interpreted as the height of a random surface and the scaling properties of its contour loops are analyzed. Various exponents that describe the loop ensemble are measured and compared to a scaling theory. This leads to a geometrical criterion for the intermittency of scalar fluctuations.

New experimental techniques which provide complete real-space images of fluctuating fields are being developed throughout the physical sciences [1,2,3]. They provide data on spatial structures inaccessible to methods based on system-wide averages, or on measurements that probe the system at only a few points. Consequently, new measures designed to usefully exploit this wealth of information are of great practical importance. Developing such measures not only provides the experimentalist with new tools with which to analyze and classify correlations in image data, but may also inspire theories that can predict the correlation behavior.

Scalar turbulence, which deals with dispersion of a passive substance (pollutants, temperature, etc.) by a turbulent flow, provides an intriguing example of this paradigm. When describing the fluctuations of the tracer concentration, recently the focus has shifted from the so-called structure functions, which characterize the probability distribution function for the concentration difference between two points, to multi-point correlations which better take into account spatial information. This has lead to significant advances in turbulence research. In particular it has allowed for the identification of the source of intermittency of the scalar fluctuations, i.e., the breakdown of simple scaling as dictated by dimensional analysis [4]. This shift of perspective is nicely exemplified by a recent study of the Lagrangian evolution of three tracer particles [5]. It was shown that intermittency in the structure functions can be related to the evolution of the shape of the triangle defined by the three particles, while the overall size of the triangle evolved with a simple scaling law.

Here we develop a geometrical description of scalar fields dispersed by two-dimensional turbulence, based on their level sets. Unlike earlier studies of the fractal properties of the level set [1,2,3,4,5], our work focuses on distribution functions of level-set loops, in analogy with cluster distributions at a critical point. Similar ideas were exploited recently by Catrakis and Dimotakis [5] who introduced the joint distribution function of shape complexity and size to describe level-set islands and lakes obtained from images of jet turbulence. We define new measures for the random geometry defined by the scalar field and propose scaling relations between various exponents that characterize these measures. Checking for the validity of these scaling relations reveals interesting properties of the fluctuations of the scalar field, in particular it provides a new, geometrical way of identifying intermittency in experimental scalar turbulence data.

1. Scaling theory of contour loops

The scaling theory of contour loops of self-affine rough fields (defined below) was developed in Refs. [6] and [7]. Here we extend this theory to analyze the scaling properties of scalar fields advected by turbulent flows, which are typically intermittent, i.e., not self-affine. For this purpose we distinguish between scaling relations which are a consequence of the scale invariance of the loop ensemble only, and those which require the self-affine property for the concentration field as well. This allows us to test these two distinct hypothesis separately against experimental data.

For a random two-dimensional field $c(x)$, we define the contour loop ensemble as a union of level sets for different values of $c$. Each level set consists of contour loops which come in many shapes and sizes. The contour loop ensemble is characterized by the loop correlation function $G(r)$, which measures the probability that two points separated by $r$ are on the same contour loop, and the joint distribution function of loop lengths and radii $n(s, R)$. Here $n(s, R)dsdR$ is the number of loops, per unit area, which pass through a fixed point (say the origin), and whose length $s \in (s, s+ds)$, and radius $R \in (R, R+dR)$. The radius of a loop is defined by the side of the smallest square that covers it.

Assuming that the contour loop ensemble is self-similar, we can define geometrical exponents which express the scaling properties of the two measures, $G(r)$ and $n(s, R)$. The loop correlation function,
defines the loop correlation exponent \( x_l \), while the joint distribution function has a scaling form,

\[
n(s, R) \sim s^{1-\tau-1/D} f_n(s/R^D)
\]

and it defines the exponents \( D \) and \( \tau \). Both relations are expected to hold for large loops with radii much bigger than the lattice spacing (for image data, lattice spacing = pixel size). For a self-similar contour loop ensemble \( D \) is also the fractal dimension of large loops, not too be confused with the dimension of the whole level set studied previously \[9\].

These three geometrical exponents satisfy a scaling relation,

\[
D(3 - \tau) = 2 - 2x_l
\]

which is derived from a sum rule \[3\]. The sum rule equates the two ways of calculating the mean loop length in a finite region, one using \( G(r) \) and the other based on \( n(s, R) \). This scaling relation is a useful check on the validity of the assumption of scale invariance of the loop ensemble and in particular Eqs. \[1\] and \[2\].

For a random field which is rough \((0 \leq \alpha \leq 1)\) and self affine we can derive a second relation:

\[
D(\tau - 1) = 2 - \alpha
\]

where \( \alpha \) is the affine exponent. It is defined by the relation \( c(\mathbf{x}) \equiv b^{-\alpha} c(b \mathbf{x}) \), where \( \equiv \) means that the original field and its rescaled version are statistically equivalent; \( b > 1 \) is an arbitrary scale parameter. Eq. \[4\] is a kind of hyperscaling relation, and it follows from the fact that the number of large loops at a particular scale of observation grows with the scale to power \( \alpha \) \[3\]. (The usual hyperscaling relation, \( D(\tau - 1) = 2 \), say in 2D critical percolation, follows from the assumption that at every scale the number of large clusters is of order one, i.e., \( \alpha = 0 \).)

Furthermore, for self-affine rough fields the three geometrical exponents, \( \tau, D, \) and \( x_l \), depend on the value of \( \alpha \) only \[3\]. Namely, based on exact results derived in the two limiting cases \( \alpha = 0 \) and \( \alpha = 1 \), \( x_l = 1/2 \) independent of \( \alpha \) was conjectured \[3\]; this has been checked in numerical simulations \[8,10\]. Once this value of \( x_l \) is accepted, formulas for the other two geometrical exponents,

\[
D = \frac{3 - \alpha}{2}, \quad \tau - 1 = \frac{4 - 2\alpha}{3 - \alpha},
\]

follow from Eq. \[3\] and Eq. \[4\].

We can analyze the level sets of random two-dimensional fields based on the above scaling theory. The procedure consists of tracing out contour loops of \( c(\mathbf{x}) \).

Then for each loop its radius and length are recorded, as well as its contribution to the loop correlation function. The three geometrical exponents are measured by power-law fitting of \( G(r) \) (for \( x_l \)), the average loop length as a function of radius (for \( D \)), and the number of loops whose length is greater than \( s \) (for \( \tau - 2 \)). The last two measures are derived from \( n(s, R) \) by integration:

\[
\langle s \rangle (R) = \frac{\int_0^\infty ds \ n(s, R) s}{\int_0^\infty ds \ n(s, R)} \sim R^D
\]

\[
N_>(s) = \int_s^\infty ds' \int_0^\infty dR \ n(s', R) \sim s^{-(\tau-2)}.
\]

Given that we can reliably extract the three exponents (i.e. the data show at least a decade of scaling) we can test our assumptions about the loop ensemble by checking the validity of the various scaling relations. First, if the loop ensemble is self similar, Eq. \[3\] should hold. Furthermore, if it is self-affine, we expect both relations in Eq. \[4\] to hold. We check this by extracting \( \alpha \) from \( D \) and \( \tau \) separately and comparing them. For an intermittent field we do not expect the two values of \( \alpha \) to coincide.

b. Experimental data The contour loop analysis is performed on images of a fluorescent dye (passive scalar) advected by magnetically forced two-dimensional turbulent flows \[1,2\]. The flow is generated by running a current through a thin slab of salty water in the presence of a perpendicular magnetic field; the Lorentz force on the charged ions provides the stirring. The magnetic field is generated by an array of permanent magnets placed in proximity to the flow.

With a suitable arrangement of magnets and choice of driving current a turbulent velocity field is set up either in the regime of a direct enstrophy cascade \[1\] or an inverse energy cascade \[2\]. In both cases the advection of fluorescent dye was monitored using a CCD camera. The data we analyze below consists of two sets, one for each of the two flow regimes, of 10 images of the tracer field. These images were selected from the time interval during which stationarity of the tracer fluctuations was achieved. In this time interval the energy dissipation was observed to be roughly time-independent \[1\].

c. Results of the loop analysis The contour loop analysis for the two types of forcing was performed by measuring the loop correlation function, the average loop length as a function of the loop radius, and the distribution of loop lengths from the images provided by P. Tabiling’s group. Each image consists of a 512 \( \times \) 512 array of integers which represent the gray scale values of the intensity of the fluorescent dye, which is proportional to its concentration \( c(\mathbf{x}) \). The unit of length we use throughout is the lattice spacing which physically corresponds to the size of one pixel. For each image \( 10^4 \) points on the dual lattice were chosen at random and through each point a contour loop was constructed following the algorithm
described in [9]. For each loop we measured the length and radius, and its contribution to the loop correlation function.

To check the stationarity condition for the data from the two flow regimes we performed the loop analysis on each image separately. We found no discernible variation in time for the three measures $G(r)$, $\langle s \rangle (R)$, and $N_s(s)$. Therefore we combined the loop data from all 10 images to improve our statistics.

**FIG. 1** The top row gives the level set and loop analysis of the scalar data in the direct cascade while the bottom row figures corresponds to scalar data obtained from the inverse cascade. All lengths ($r$, $R$, and $s$) are reported in units of one pixel; $G(r)$ is not normalized.

i. **Direct cascade** – The loop correlation function shows the $1/r$ (i.e. $x_l = 1/2$) behavior characteristic of rough fields for separations $r < 100$ (see Fig. 1); in terms of the geometrical exponent $x_l$ we find from a linear least squares fit a value of $x_l = 0.495(10)$. The rather large (compared to the inverse cascade data) crossover region for $100 < r < 300$ which precedes the fall off at $r > 400$ (due to system size) might be indicative of an intermediate length scale at which the scaling of the loop ensemble changes. This is confirmed by the cumulative loop length distribution, $N_s(s)$.

The graph of the mean loop length as a function of the radius, Fig. 1, shows a scaling range for loop radii $3 < R < 300$. It leads to a fractal dimension for contour loops of $D = 1.20(3)$. The value of the exponent is estimated by comparing results from linear least-square fits to data within several different subintervals of the scaling range.

The cumulative loop length distribution graph, Fig. 1, shows scaling for $10 < s < 200$ after which there is a crossover to what looks like a power law with a smaller (more negative) exponent. The fall off at lengths $s > 7000$ occurs as the system size intervenes. The crossover occurs at length scales comparable to the effective dissipation scale which was identified from the scalar power spectrum in Ref. [11]. The exponent $\tau - 2 = 0.19(2)$ is extracted from the scaling range, using the same procedure (sub-interval fits) as for $D_f$.

Using the scaling relations Eq. (3), we extract the value of the affine exponent $\alpha = 0.60(6)$ from the measured $D$, and $\alpha = 0.53(8)$ from $\tau - 2$. The fact that they match within error bars indicates that the loop data is consistent with simple scaling of the tracer field below the effective dissipation scale.

Above the dissipation scale Batchelor (logarithmic) scaling is expected to hold [11]. This implies $\alpha = 0$ and is not well supported by the loop analysis. Namely, we would expect no crossover for $x_l$ which is universal,
and (from Eq. 3) a crossover of $D$ to higher and $\tau - 2$ to a lower value. This is qualitatively consistent with the loop data (accept for $x_l$) but the actual numbers are off. We conclude that if the Batchelor regime is reached it occurs for a narrow range of scales which is not resolved by the loop analysis.

Additional checks on the critical nature of the loop ensemble is provided by measuring the distribution of loop radii, $N_s(R)$. $N_s(R)$ is the number of loops whose radius is larger than $R$ and $N_s(R) \sim R^{-D(\tau - 2)}$ follows from Eq. (3). In the $N_s(R)$ plot we again see a crossover occurring at $R \approx 60$, which from the $\langle s \rangle (R)$ plot in Fig. 1 corresponds to $s \approx 300$, consistent with the $N_s(s)$ plot. For loops with $3 < R < 40$ we measure $D(\tau - 2) = 0.22(1)$ by fitting subintervals to a power law, while from the previous measurements of $D$ and $\tau - 2$, $D(\tau - 2) = 0.23(3)$ follows.

ii. Inverse cascade – Loop analysis of the inverse energy cascade images reveals a very different picture. The scaling is much better as no crossover scale is detected. The geometrical exponents can be extracted reliably from the data and they do not satisfy the scaling relations Eq. (3).

The loop correlation function decays with distance as a power law, for $r < 100$ (see Fig. 1); the measured value of the exponent $x_l = 0.485(5)$ is consistent with the universal value $x_l = 1/2$. The $\langle s \rangle (R)$ plot shows a scaling range for $10 < R < 100$, from which we extract $D = 1.30(1)$ (see Fig. 1). The $N_s(s)$ plot, Fig. 1, has a rather remarkable scaling range for loop lengths $10 < s < 3000$ from which the geometrical exponent $\tau - 2 = 0.273(4)$ is obtained. Assuming the validity of the affine scaling relations Eq. (3) the exponent $\alpha = 0.40(2)$ follows from the value of $D$, while $\alpha = 0.25(2)$ is the value implied by the measured $\tau - 2$. Clearly, these two values are different, thus signaling the breakdown of simple scaling. We conclude that the scalar field is intermittent.

To check that the loop ensemble is described by a joint distribution function that has a scaling form, Eq. (3), we check for the validity of the scaling relation Eq. (3). From the measured values of $D$ and $\tau - 2$ we have $D(\tau - 3) = 0.95(1)$ while we expect $2 - 2x_l = 1$ from the universal value $x_l = 1/2$; $2 - 2x_l = 1.03(1)$ is the value that follows from the $G(r)$ measure. Furthermore we measure the combination $D(\tau - 2)$ from $N_s(R)$. We find $D(\tau - 2) = 0.344(4)$ (scaling range $3 < R < 100$), while the measures used above yield $D(\tau - 2) = 0.355(8)$. Taken together these results give strong support to the assumed critical nature of the contour loop ensemble of the scalar data in the inverse energy cascade regime.

In conclusion we have shown that a scaling analysis of contour loops of a passive scalar field advected by turbulent flow is a useful geometrical measure for studying its fluctuations. The loop analysis successfully distinguishes between different flow regimes and indicates the breakdown of simple scaling in the case of the inverse cascade. Even though the scalar field is intermittent in this case, the loop ensemble is critical and characterized by two independent geometrical exponents ($D$ and $\tau$). The direct cascade data reveals an intermediate length scale which can be identified with an effective dissipation length. Below this dissipation scale simple scaling of the scalar field is favored by the loop analysis, while the data above this scale is not indicative of a Batchelor regime with logarithmic structure functions.

The loop analysis described here has an advantage over structure-function measurements as the geometrical exponents are sensitive to higher moments of the concentration difference and they capture the spatial structure of the fluctuations more fully by incorporating connectiveness properties. As three dimensional data becomes more available extensions of these ideas to contour surfaces will be a promising direction for future research.

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[1] P. Constant in, I. Procaccia, and K. R. Sreenivasan, Phys. Rev. Lett. 67, 1739 (1991).
[2] R. Ramshankar and J.P. Gollub, Phys. Fluids A 3, 1344 (1991).
[3] W.B. Wright, R. Budakian and S.J. Putterman, Phys. Rev. Lett. 76, 4528 (1996).
[4] B.I. Shraiman and E. Siggia, Nature 405, 639 (2000).
[5] A. Celani and M. Vergassola, preprint nlin.CD/0006009.
[6] O. Cardoso, B. Gluckmann, O. Parcollet, and P. Tabeling, Phys. Fluids 8, 209 (1996).
[7] H.J. Catrakis and P.E. Dimotakis, Phys. Rev. Lett. 77, 3795 (1996); 80, 968 (1998).
[8] M. B. Isichenko, Rev. Mod. Phys. 64, 961 (1992).
[9] J. Kondev and C.L. Henley, Phys. Rev. Lett. 74, 4580 (1995); J. Kondev, D.G. Salinas, and C.L. Henley, Phys. Rev. E 61, 104 (2000).
[10] C. Zeng, J. Kondev, D. McNamara, and A. A. Middleton, Phys. Rev. Lett. 80, 109 (1998).
[11] M.-C. Jullien, P. Castiglione, and P. Tabeling, Phys. Rev. Lett. 85, 3636 (2000).
[12] J. Paret and P. Tabeling, Phys. Rev. Lett. 79, 4162 (1997).
[13] G. Batchelor J. Fluid Mech. 5, 113 (1959).