Existence of wave operators with time-dependent modifiers for the Schrödinger equations with long-range potentials on scattering manifolds

Shinichiro ITOZAKI*

January 4, 2012

Abstract

We construct time-dependent wave operators for Schrödinger equations with long-range potentials on a manifold $M$ with asymptotically conic structure. We use the two space scattering theory formalism, and a reference operator on a space of the form $\mathbb{R} \times \partial M$, where $\partial M$ is the boundary of $M$ at infinity. We construct exact solutions to the Hamilton-Jacobi equation on the reference system $\mathbb{R} \times \partial M$, and prove the existence of the modified wave operators.

1 Introduction

In this paper, we show the existence of wave operators for the Schrödinger equations with long-range potentials on scattering manifolds, which have asymptotically conic structure at infinity (see Melrose [16] about scattering manifolds). We employ the formulation of Ito-Nakamura [13], which uses the two-space scattering framework of Kato [14]. Following Hörmander [10] and Dereziński and Gérard [4], we construct exact solutions to the Hamilton-Jacobi equation and show the existence of the modified two-space wave operators using the stationary phase method.

Let $M$ be an $n$-dimensional smooth non-compact manifold such that $M$ is decomposed to $M_C \cup M_\infty$, where $M_C$ is relatively compact, and $M_\infty$ is diffeomorphic to $\mathbb{R}_+ \times \partial M$ with a compact manifold $\partial M$. We fix an identification map:

$$t : M_\infty \rightarrow \mathbb{R}_+ \times \partial M \ni (r, \theta).$$

We suppose $M_C \cap M_\infty \subset (0, 1/2) \times \partial M$ under this identification. We also suppose that $\partial M$ is equipped with a measure $H(\theta)d\theta$ where $H(\theta)$ is a smooth positive density.

Let $\{\phi_\lambda : U_\lambda \rightarrow \mathbb{R}^{n-1}\}, U_\lambda \subset \partial M$, be a local coordinate system of $\partial M$. We set $\{\tilde{\phi}_\lambda : \mathbb{R}_+ \times U_\lambda \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}\}$ to be a local coordinate system of $M_\infty \cong \mathbb{R}_+ \times \partial M$, and we denote $(r, \theta) \in$
$\mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in $M_\infty$. We suppose $G(x)$ is a smooth positive density on $M$ such that

$$G(x)dx = r^{n-1}H(\theta)d\theta drd\theta \text{ on } \left(\frac{1}{2}, \infty\right) \times \partial M \subset M_\infty,$$

and we set

$$\mathcal{H} = L^2(M, G(x)dx).$$

Let $P_0$ be a formally self-adjoint second order elliptic operator on $\mathcal{H}$ of the form:

$$P_0 = \frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \left(1 + a_1 \frac{a_2}{a_3} \right) \left(\begin{array}{c} \partial_r \\ \partial_\theta/r \end{array}\right) \text{ on } \tilde{M}_\infty = (1, \infty) \times \partial M$$

where $a_1, a_2$, and $a_3$ are real-valued smooth tensors.

**Assumption 1.** For any $l \in \mathbb{Z}_+, \alpha \in \mathbb{Z}^n_{+}$, there is $C_{l, \alpha}$ such that

$$|\partial_l^j \partial_\alpha \mu_j(r, \theta)| \leq C_{l, \alpha} r^{-\mu_j-l}$$

on $\tilde{M}_\infty$, where $\mu_j \geq 0$. Note that we use the coordinate system in $M_\infty$ described above.

We will construct a time-dependent scattering theory for $P_0 + V$ on $\mathcal{H}$ where $V$ is a potential.

**Definition 2.** Let $\mu_S > 0$. A finite rank differential operator $V^S$ of the form $V^S = \sum_{l, \alpha} V^S_{I, \alpha}(r, \theta) \partial_l^j \partial_\alpha \mu^j \theta$, $\partial_\alpha \mu^j \theta$ on $M_\infty$ is said to be a short range perturbation of $\mu_S$ type if for every $l, \alpha$ the coefficient $V^S_{I, \alpha}$ is a $L^2_{loc}$ tensor and satisfies

$$\int_{\mathbb{R}^+ \times U_1} |V^S_{I, \alpha}(x)|^2(r)^{-M} G(x)dx < \infty$$

for some $M$, and almost every $(\rho_0, \theta_0) \in \mathbb{R} \times \partial M$ has a neighborhood $\omega_{\rho_0, \theta_0}$ such that

$$\int_1^\infty \left(\int_{(\rho, \theta) \in \omega_{\rho_0, \theta_0}} |V^S_{I, \alpha}(\rho, \theta)|^2 d\rho H(\theta)d\theta\right)^{1/2} |\mu_S| d\mu < \infty.$$

Let $\mu_L > 0$. $V^L$ is called a long-range smooth potential if $V^L$ is a real-valued $C^\infty$ function with support in $\tilde{M}_\infty$, and satisfies for any indices $l, \alpha$,

$$|D_l^j D_\alpha^\mu V^L(r, \theta)| \leq C_{j, \alpha} r^{-\mu_l-j}.$$

A differential operator $V$ on $M$ is called an admissible long-range perturbation of $P_0$ if $V$ is of the form $V = V^S + V^L$ where $V^S$ is a short range perturbation of $\mu_S$ type and $V^L$ is a long-range smooth potential and

$$\varepsilon = \mu_1 = \mu_2 = \mu_L > 0, \quad \mu_3 = 0, \quad \mu_S = 1 - \varepsilon.$$

**Example 1.** If $V^S = V^S(r, \theta)$ is a multiplication operator and $|V^S(r, \theta)| \leq Cr^{-\eta}$, $\eta > 0$, then $V^S$ satisfies the short-range condition above.

If $V^S = \sum_{|\alpha|=1} V^S_{I, \alpha} \partial_\alpha \mu^j \theta$ and $|V^S_{I, \alpha}(r, \theta)| \leq Cr^{-\mu_l-\eta}$, $\eta > 0$, then $V^S$ satisfies the short-range condition above. As the order of the derivative with respect to $\theta$-variable increases, we need more rapid decay conditions on the coefficients.
Remark 2. If $V^S$ is a smooth function, then $P_0 + V$ is essentially self-adjoint. More generally, if $V^S$ is at most second-order differential operator with “small” smooth coefficients, then $V^S$ is $P_0$-bounded with relative bound less than one, and $P_0 + V$ is essentially self-adjoint. In this paper, we assume that $P_0 + V$ is essentially self-adjoint on suitable domains (see Theorem 3) and do not investigate the conditions of self-adjointness.

Remark 3. If we assume $\partial M$ is equipped with a positive $(2, 0)$-tensor $h = (h^{jk}(\theta))$, for some $\varepsilon > 0$, $|\partial^l M |\partial_r a_3(r, \theta) - h(\theta)| \leq C_l r^{-\varepsilon - l}$, and $V^S = 0$, then $P_0 + V$ has a self-adjoint extension $H$ and corresponds (via a unitary equivalence) to the Laplacian on Riemannian manifolds with asymptotically conic structure. Since $\varepsilon > 0$, our model includes the scattering metric of long-range type described in [12]. Thus our results are generalizations of [13].

We prepare a reference system as follows:

$$M_f = \mathbb{R} \times \partial M, \quad \mathcal{H}_f = L^2(M_f, H(\theta) dr d\theta), \quad P_f = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \text{ on } M_f$$

Note that $P_f$ is essentially self-adjoint on $C^\infty_0(M_f)$, and we denote the unique self-adjoint extension by the same symbol. Let $j(r) \in C^\infty(\mathbb{R})$ be a real-valued function such that $j(r) = 1$ if $r \geq 1$ and $j(r) = 0$ if $r \leq 1/2$. We define the identification operator $J : \mathcal{H}_f \to \mathcal{H}$ by

$$(J u)(r, \theta) = r^{-(n-1)/2} j(r) u(r, \theta) \text{ if } (r, \theta) \in M_\infty$$

and $J u(x) = 0$ if $x \notin M_\infty$, where $u \in \mathcal{H}_f$. We denote the Fourier transform with respect to $r$-variable by $\mathcal{F}$:

$$\mathcal{F} u(\rho, \theta) = \int_{-\infty}^{\infty} e^{-ir \rho} u(r, \theta) dr, \quad \text{for } u \in C^\infty_0(M_f).$$

We decompose the reference Hilbert space $\mathcal{H}_f$ as $\mathcal{H}_f = \mathcal{H}_f^+ \oplus \mathcal{H}_f^-$, where $\mathcal{H}_f^\pm$ are defined by

$$\mathcal{H}_f^+ = \{ u \in \mathcal{H} | \text{supp}(\mathcal{F} u) \subset [0, \infty) \times \partial M \},$$

$$\mathcal{H}_f^- = \{ u \in \mathcal{H} | \text{supp}(\mathcal{F} u) \subset (-\infty, 0] \times \partial M \}.$$ 

We use the following notation throughout the paper: For $x \in M$, we write

$$\langle x \rangle = \langle r \rangle = \begin{cases} 1 + r j(r) & \text{for } x \in M_\infty, \\ 1 & \text{for } x \in M_\varepsilon. \end{cases}$$

We state our main theorem.

**Theorem 3.** Let $V = V^L + V^S$ be an admissible long-range perturbation of $P_0$, and $V$ is symmetric on $I\mathcal{F}^{-1}C^\infty_0(M_f)$, and $P_0 + V$ has a self adjoint extension $H$. Let $S(t, \rho, \theta)$ be a solution
to the Hamilton-Jacobi equation which is constructed in Theorem 13. Then the modified wave operators

$$\Omega_{\pm} = s\lim_{t \to \pm\infty} e^{itH} e^{-iS(t, D_r, \theta)}$$

exist, and are partial isometries from $\mathcal{H}_{f}^{\pm}$ into $\mathcal{H}$ intertwining $H$ and $P_f$:

$$e^{isH} \Omega_{\pm} = \Omega_{\pm} e^{isP_f}.$$ 

We refer Reed and Simon [20], Dereziński and Gérard [4], and Yafaev [22] for general concepts of wave operators and scattering theory for Schrödinger equations. We here briefly review the history of wave operators. The concept of wave operator was introduced by Møller [17]. The existence of wave operators has long been studied (see Cook [2] and Kuroda [15]) for short range potentials, which decay faster than the Coulomb potential. For the Coulomb potential, it was proved by Dollard [5, 6] that the wave operators do not exist unless the definition is modified. Dollard introduced the concept of the modified wave operators $s\lim_{t \to \pm\infty} e^{itH} e^{-iS(t, D_r)}$. Buslaev-Mateev [1] showed the existence of modified wave operators by using stationary phase method and by employing an approximate solution to the Hamilton-Jacobi equation as a modifier function $S(t, \xi)$. Hörmander [10] constructed exact solutions to the Hamilton-Jacobi equation (see also [11] vol. IV).

The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese, Hislop and Perry [8, 9], and Donnelly [7] using the Mourre theory (see, the original paper Mourre [18], and Perry, Sigal, and Simon [19]). In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics (see [16] and references therein), and showed that the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions, is a Fourier integral operator. Vasy [21] studied Laplace operators on such manifolds with long-range potentials of Coulomb type decay ($|V(r, \theta)| \leq C r^{-1}$).

Ito and Nakamura [13] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [14] with a simple reference operator $D^2_r/2$ on a space of the form $\mathbb{R} \times \partial M$, where $\partial M$ is the boundary of the scattering manifold $M$.

We employ the formulation of Ito and Nakamura [13], and consider general long-range metric perturbations and potential perturbations. We assume that the scalar potential decay as $|V(r, \theta)| \leq C r^{-\varepsilon}, \varepsilon > 0$.

We make some remarks along with the outline of the proof. The time-dependent modifier function $S(t, \rho, \theta)$ is not uniquely determined. Our choice is a solution to the Hamilton-Jacobi equation on the reference manifold $\mathbb{R} \times \partial M$ with the long-range potential $V^L$:

$$h(\rho, \theta, \rho, -\frac{\partial S}{\partial \theta}) = \frac{\partial S}{\partial t},$$

$$h(r, \theta, \rho, \omega) = \frac{1}{2} \rho^2 + \frac{1}{r} a_1 \rho^2 + \frac{1}{r^2} a_2^j \rho \omega_j + \frac{1}{r^2} a_3^j k \omega_j \omega_k + V^L,$$

for large $t$ and for every $\rho$ in any fixed compact set of $\mathbb{R} \setminus \{0\}$, where $h$ is the corresponding classical Hamiltonian. We choose $\rho$ and $\theta$ as variables of $S$ because $\rho$ and $\theta$ components of the
classical trajectories have limits as \( t \) goes to infinity. The time-dependent modifier \( e^{-S(t,D_r,\theta)} \) is a Fourier multiplier in \( r \)-variable for each \( \theta \) and we only need to consider the 1-dimensional Fourier transform with respect to \( r \)-variable. We construct solutions to the Hamilton-Jacobi equation mainly following J. Dereziński and C. Gérard \[4\].

In Section 2.1 we consider the boundary value problem for Newton equation on \( \mathbb{R} \times \partial M \) with time-dependent slowly-decaying forces, which decay in time (Definition \[4\]). In Theorem 5, we construct solutions and show several estimates. We use an integral equation and Banach’s contraction mapping theorem (Proposition \[7\], refer Dereziński \[3\] and Section 1.5 of \[4\]). In the definition of slowly-decaying forces (Definition \[4\]) and the function spaces (Definition \[6\]), we assume different decaying rates on different variables \( r, \theta, \rho, \) and \( \omega \). These are efficiently used to show Proposition \[7\]. We observe that the classical trajectories will stay in outgoing (incoming) regions as \( t \to +\infty (-\infty) \).

In Section 2.2 we consider Newton equations with time-independent long-range forces which decay in space (Definition \[8\]) in appropriate outgoing (incoming) regions. By inserting time-dependent cut-off functions, we introduce an effective time-dependent force and reduce the time-independent problem to the time-dependent one (Theorem \[9\]). Our model (the Hamiltonian flow induced by the classical Hamiltonian) turns out to fit into this framework (Lemma \[11\]). These tricks are also used in \[4\] for Hamiltonians with long-range potentials on Euclidean spaces.

Finally, in Section 2.3 in Theorem \[13\] we construct exact solutions to the Hamilton-Jacobi equation, using the classical trajectories with their dependence on initial data. Here we use the idea by Hörmander \[10\], see also Section 2.7 of \[4\]. We show that these solutions with their derivatives satisfy “good estimates”, which are used to show the existence of the modifiers. Once we obtain a suitable modifier \( S(t, \rho, \theta) \), we can show the existence of modified wave operators through stationary phase method (Section 3).

Using the Cook-Kuroda method (see Cook \[2\], and Kuroda \[15\]) and 1-dimensional Fourier transform, we deduce the proof of the main theorem to estimates of the integral (Proposition \[14\]):

\[
\int \left[ h(r, \theta, \rho, -\frac{\partial S}{\partial \theta}(t, \rho, \theta)) - h\left(\frac{\partial S}{\partial \rho}(t, \rho, \theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t, \rho, \theta)\right)\right] e^{i\rho \rho - iS(t, \rho, \theta)} \hat{u}(\rho, \theta) d\rho.
\]

In Section 3, we apply the stationary phase method (Hörmander \[11\] Section 7.7). In the asymptotic expansion of the above integral, the terms in which \( h \) is not differentiated vanish since the equation \( r = \frac{\partial S}{\partial \rho} \) holds at the stationary points. To show the uniformly boundedness of constants which appear in the asymptotic expansions of the integral, we construct diffeomorphisms in small neighborhoods of the stationary points which transform the phase function into quadratic forms there (Lemma \[15\]). In the constructions of these diffeomorphisms, we use the estimates on the modifier function \( S \).

**Notations** We use the following notation throughout the paper. Let \( t \in \mathbb{R} \) and \( s \) be a parameter. We write \( f(t,s) \in g(s)O((t)^{-m}) \) if \( f(t,s) \leq C g(s) t^{-m} \) uniformly for \( t \) and \( s \). We denote \( f(t,s) \in g(s)o(t^0) \) if \( \lim_{t \to \infty} f(t,s)/g(s) = 0 \).

**Acknowledgments**

I would like to express my thank to my advisor, Professor Shu Nakamura for providing me various supports, comments, and encouragements.
2 Classical mechanics

In this section, we study classical trajectories and solutions to the Hamilton-Jacobi equation.

2.1 Classical trajectories with slowly-decaying time-dependent force

Let \((r, \theta, \varphi, \omega) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})\) and consider the Newton’s equation:

\[
(r, \dot{\theta}, \dot{\varphi}, \dot{\omega})(t) = (\dot{r} + F_r, \dot{\theta} + F_\theta, \dot{\varphi} + F_\varphi, \dot{\omega} + F_\omega)(t, (r, \theta, \varphi, \omega)(t))
\]  

(2)

where

\[
F = F(r, \theta, \varphi, \omega) = (F_r, F_\theta, F_\varphi, F_\omega)(t, r, \theta, \varphi, \omega)
\]

is a time-dependent force. Let \(\varepsilon > 0\) and \(\tilde{\varepsilon} = \frac{1}{2}\varepsilon\).

**Definition 4.** A time-dependent force \(F\) is said to be slow-decaying if \(F\) satisfies

\[
\sup_{(r, \theta, \varphi, \omega) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})} |\partial^l_r \partial^k_\theta \partial^\beta_\varphi \partial^\alpha_\omega F_r + \partial^l_r \partial^k_\theta \partial^\beta_\varphi \partial^\alpha_\omega F_\theta + \partial^l_r \partial^k_\theta \partial^\beta_\varphi \partial^\alpha_\omega F_\varphi + \partial^l_r \partial^k_\theta \partial^\beta_\varphi \partial^\alpha_\omega F_\omega(t)| \in O(\varepsilon^{-n, \alpha, \beta, \omega})
\]  

(3)

where

\[
\begin{align*}
n_r(l, \alpha, k, \beta) &= m(l, \alpha, k + 1, \beta), \quad n_\theta(l, \alpha, k, \beta) = m(l, \alpha, k, \beta + e_i), \\
n_\varphi(l, \alpha, k, \beta) &= m(l + 1, \alpha, k, \beta), \quad n_\omega(l, \alpha, k, \beta) = m(l, \alpha + e_i, k, \beta), \\
m(l, \alpha, 0, 0) &= l + e_i, \quad m(l, \alpha, 1, 0) = l + e, \quad m(l, \alpha, 2, 0) = l + \varepsilon, \\
m(l, \alpha, 0, e_i) &= l + 1 + \tilde{\varepsilon}, \quad m(l, \alpha, 1, e_i) = l + 1 + \varepsilon, \quad m(l, \alpha, 0, e_i + e_j) = l + 2, \\
m(l, \alpha, k, \beta) &= +\infty, \quad \text{if } \quad k + |\beta| \geq 3,
\end{align*}
\]

\(i, j = 1, \cdots, n - 1, \text{ and } e_i = (0, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}^{n-1}_+\) is the canonical unit vector, i.e., every component of \(e_i\) is 0 except \(i\)-th component.

In the next theorem, we show the unique existence of trajectories for the dynamics\(^2\) where the boundary conditions are the initial position and the final momentum.

**Theorem 5.** Assume that \(F\) is a time-dependent slowly-decaying force in the sense of Definition\(^4\) \(Then there exists \(T\) such that if \(T \leq t_1 < t_2 \leq \infty\) and \((r_i, \theta_f, \varphi_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})\), there exists a unique trajectory

\[
[t_1, t_2] \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\varphi}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \varphi_f, \omega_i)
\]

satisfying

\[
\begin{align*}
\partial_s (\tilde{r}, \tilde{\theta}, \tilde{\varphi}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \varphi_f, \omega_i) \\
= (\dot{r} + F_r, \dot{\theta} + F_\theta, \dot{\varphi} + F_\varphi, \dot{\omega} + F_\omega)(s, (\tilde{r}, \tilde{\theta}, \tilde{\varphi}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \varphi_f, \omega_i)), \\
(\tilde{r}, \tilde{\omega})(t_1, t_1, t_2, r_i, \theta_f, \varphi_f, \omega_i) = (r_i, \omega_i), \quad (\tilde{\theta}, \tilde{\varphi})(s, t_1, t_2, r_i, \theta_f, \varphi_f, \omega_i) = (\theta_f, \varphi_f).
\end{align*}
\]
We set \( r(s), \theta(s), p(s), \omega(s) \) by

\[
\begin{align*}
  r(s) &= \tilde{r}(s) - r_t - (s - t_1) \rho_f, \quad \theta(s) = \tilde{\theta}(s) - \theta_f, \\
  p(s) &= \tilde{p}(s) - \rho_f, \quad \omega(s) = \tilde{\omega}(s) - \omega_t.
\end{align*}
\]

Moreover the solution satisfies the following estimates uniformly for \( T \leq t_1 \leq s \leq t_2 \leq \infty, (r_t, \theta_f, \rho_f, \omega_t) \in T^* (\mathbb{R} \times \mathbb{R}^{n-1}) : \)

\[
\begin{align*}
  |r(s)| &\in o(s^0)|s - t_1|, \quad |\theta(s)| \in O(s^{-\varepsilon}), \\
  |p(s)| &\in O(s^{-\varepsilon}), \quad |\omega(s)| \in o(s^0)|s - t_1|^{1-\varepsilon}, \\
  \begin{pmatrix}
    \partial_r r(s) & \partial_\theta r(s) & \partial_p r(s) & \partial_\omega r(s) \\
    \partial_r \theta(s) & \partial_\theta \theta(s) & \partial_p \theta(s) & \partial_\omega \theta(s) \\
    \partial_r p(s) & \partial_\theta p(s) & \partial_p p(s) & \partial_\omega p(s) \\
    \partial_r \omega(s) & \partial_\theta \omega(s) & \partial_p \omega(s) & \partial_\omega \omega(s)
  \end{pmatrix}
  \in
  \begin{pmatrix}
    o(s^0)|s - t_1| & O(1) & O(s^{-\varepsilon}) \\
    O(s^{-\varepsilon}) & o(s^0)|s - t_1|^{1-\varepsilon}
  \end{pmatrix}
  \otimes
  (t_1^{-1-\varepsilon}, t_1^{-1-\varepsilon}, t_1^{-1-\varepsilon}, t_1^{-1-\varepsilon}),
\end{align*}
\]

\[
\begin{pmatrix}
  \partial_{r_i} \partial_\theta \partial_\beta \partial_\omega
  \begin{pmatrix}
    r \\
    \theta \\
    p \\
    \omega
  \end{pmatrix}
\end{pmatrix}
  \in
  \begin{pmatrix}
    o(s^0)|s - t_1| & O(1) & O(s^{-\varepsilon}) \\
    O(s^{-\varepsilon}) & o(s^0)|s - t_1|^{1-\varepsilon}
  \end{pmatrix}
  \cdot t_1^{-1-|\beta|}.
\]

Here \( \otimes \) is an outer product and \( [\big] \) means, for example, \( \partial_r r(s) \in o(s^0)|s - t_1|^{1-\varepsilon} \), and \( \partial_\theta \theta(s) \in O(1)t_1^{-\varepsilon} \).

A straightforward computation shows that \((r, \theta, p, \omega)(s)\) satisfies the following integral equation:

\[
(r, \theta, p, \omega)(s) = (P_r, P_\theta, P_p, P_\omega)(r, \theta, p, \omega)(s) :=
\]

\[
\begin{align*}
  \int_{t_1}^{t_2} \begin{pmatrix}
    \rho(u) + F_r(u, r_t + (u - t_1) \rho_f + r(u), \theta_f + \theta(u), \rho_f + \rho(u), \omega_t + \omega(u)) \\
    \rho(u) + F_\theta(u, r_t + (u - t_1) \rho_f + r(u), \theta_f + \theta(u), \rho_f + \rho(u), \omega_t + \omega(u)) \\
    \rho(u) + F_p(u, r_t + (u - t_1) \rho_f + r(u), \theta_f + \theta(u), \rho_f + \rho(u), \omega_t + \omega(u)) \\
    \rho(u) + F_\omega(u, r_t + (u - t_1) \rho_f + r(u), \theta_f + \theta(u), \rho_f + \rho(u), \omega_t + \omega(u))
  \end{pmatrix} du
\end{align*}
\]

where the map \( P = (P_r, P_\theta, P_p, P_\omega) \) depends on the parameters \( t_1, t_2, r_t, \theta_f, \rho_f, \omega_t \). We will apply the fixed point theorem to solve \((r, \theta, p, \omega)\). We define the Banach space on which the map \( P \) is defined as follows:

**Definition 6.** For \( m \geq 0 \), we define

\[
Z_T^m := \{ z \in C([T, \infty)) : \sup_{|t - T| < \infty} |z(t)| < \infty \}, \quad Z_{T, \infty}^m := \{ z \in Z_T^m : \lim_{t \to \infty} |z(t)| = 0 \}.
\]
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For \( m < 0 \), we define

\[
Z^m_T := \{ z \in C([T, \infty)) : \sup_{t \geq T} \frac{|z(t)|}{(t)^m} < \infty \}.
\]

We define

\[
Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} := \{(r, \theta, \rho, \omega) \in Z_{t_1,\infty}^1 \times Z_{t_1}^0 \times Z_{t_1}^{-\varepsilon} \times Z_{t_1,\infty}^{1-\varepsilon} \}.
\]

Then we have the following Proposition:

**Proposition 7.** For large enough \( T > 0 \), the map \( P \) is a contraction map on

\( Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} \) for any \( T \leq t_1 \leq t_2 \leq \infty, (r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) \). Indeed, for some constant \( c \) which does not depend on \( t_1, t_2, (r_i, \theta_f, \rho_f, \omega_i) \) but \( T \), we have

\[
\| \nabla x P(x) \|_{B(Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon})} < c < 1. \tag{9}
\]

**Proof.** We first note that \( P \) is well defined as a map of \( Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} \) into itself. Indeed, for example, if \( x = (r, \theta, \rho, \omega) \in Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} \),

\[
|P_r(x)(s)| \leq \int_{t_1}^{s} |(\rho(u) + F_r(u, r_i + (s-t_1)\rho_f + z(u), \theta_f + \theta_f(u), \rho_f + \rho(u), \omega + \omega(u)))| \, du
\]

\[
\leq \int_{t_1}^{s} |C(u)^{-\varepsilon} + C(u)^{-\varepsilon}| \, du,
\]

which implies \( P_r(x)(s) \in Z_{t_1,\infty}^1 \). Others are similar to prove.

Now we check that \( P \) is a contraction on \( Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} \). It suffices to show (9) for some constant \( c \) which does not depend on \( t_1, t_2, (r_i, \theta_f, \rho_f, \omega_i) \) but \( T \). Let \( v \in Z_{t_1,\infty}^1 \). Then

\[
|s-t_1|^{-1} (\nabla_x P_r(x)v)(s) \leq |s-t_1|^{-1} \int_{t_1}^{s} \| \nabla_r F_r(u, \cdot) \|_{\infty} |u-t_1| |v|_{Z_{t_1,\infty}} \, du
\]

\[
\leq \| v \|_{Z_{t_1,\infty}^1} \int_{t_1}^{s} |s-t_1|^{-1} |u-t_1| |u|^{-1-\varepsilon} \, du.
\]

If we let \( T \to \infty \), the right hand side goes to zero uniformly for \( T \leq t_1 \leq t_2 \leq \infty \). Moreover, the right hand side goes to zero as \( s \to \infty \). Hence taking \( T \) large enough, we may assure that

\[
\| \nabla_r P_r \|_{B(Z_{t_1,\infty}^1)} < c < 1
\]

for some constant \( c \) for any \( t \leq t_1 \leq t_2 \leq \infty \) and for any \((r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) \). In a similar way, we can show that for some large enough \( T \), (9) holds for any \( T \leq t_1 \leq t_2 \leq \infty \) and for any \((r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) \).

**Proof of Theorem 5.** The fixed point theorem together with Proposition 7 implies that there exists a unique solution \((r, \theta, \rho, \omega)(s) \in Z_{t_1}^{1,0,-\varepsilon,1-\varepsilon} \) for the integral equation (8) for each \( T \leq
Let us now prove (6). We use the identity

\[(I - \nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h_\gamma^r, h_\gamma^\theta, h_\gamma^\rho, h_\gamma^\omega)\]

\[= \left(\begin{array}{c}
\int_{t_1}^{s_1} (\nabla F_r)(u,y)\partial^\gamma(y-x)du \\
- \int_{t_1}^{s_1} (\nabla F_\theta)(u,y)\partial^\gamma(y-x)du \\
- \int_{t_1}^{s_1} (\nabla F_\rho)(u,y)\partial^\gamma(y-x)du \\
\int_{t_1}^{s_1} (\nabla F_\omega)(u,y)\partial^\gamma(y-x)du \\
\end{array}\right)
\]

where \(\partial^\gamma = \partial_r, \partial_\theta, \partial_\rho, \partial_\omega\), or \(\partial_\omega, x = (r, \theta, \rho, \omega)\) is the solution of (8), and \(y = (r_i + (u - t_1)\rho_f + \rho(u), \theta + \theta(u), \rho_f + \rho(u), \omega_t + \omega(u))\). By a straight computation we have

\[(h_{\partial_r}, h_{\partial_\theta}, h_{\partial_\rho}, h_{\partial_\omega}) \in \langle (t_1)^{1-\tilde{\epsilon}}, \langle t_1 \rangle^{-\tilde{\epsilon}}, \langle t_1 \rangle^{1-\tilde{\epsilon}}, \langle t_1 \rangle^{-1} \rangle Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}}.
\]

(9) implies that \(I - \nabla_x P(x)\) is invertible on \(Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}}\). Using (10), we get

\[(\partial_r x, \partial_\theta x, \partial_\rho x, \partial_\omega x) \in Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}},
\]

and

\[\|\partial_r x, \partial_\theta x, \partial_\rho x, \partial_\omega x\|_{Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}}} \in O\langle (t_1)^{1-\tilde{\epsilon}}, \langle t_1 \rangle^{-\tilde{\epsilon}}, \langle t_1 \rangle^{-1} \rangle,
\]

which implies (6).

Now we prove (7) by an induction. Assume that \(\partial^\gamma = \partial^l_r \partial^k_\theta \partial^p_\rho \partial^{\tilde{\omega}}_\omega, |l| + |k| + |p| + |\tilde{\omega}| = n \geq 2\), and (7) is true for \(l + |\alpha| + k + |\beta| \leq n - 1\). We use the identity

\[(I - \nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h_\gamma^r, h_\gamma^\theta, h_\gamma^\rho, h_\gamma^\omega)\]

\[= \left(\begin{array}{c}
\int_{t_1}^{s_1} \sum_{q \geq 2} (\nabla^q F_r)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\
- \int_{t_1}^{s_1} \sum_{q \geq 2} (\nabla^q F_\theta)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\
- \int_{t_1}^{s_1} \sum_{q \geq 2} (\nabla^q F_\rho)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\
\int_{t_1}^{s_1} \sum_{q \geq 2} (\nabla^q F_\omega)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\
\end{array}\right)
\]

where the sum is taken over \(\gamma = \sum_{p=1}^{q} \gamma_p, q \geq 2\). The induction hypothesis with a straight computation shows that

\[(h^\gamma) \in \langle (t_1)^{-1-|\beta|} \rangle Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}}.
\]

Thus we have

\[\|\partial^\gamma x\|_{Z_{t_1}^{1,0,-\tilde{\epsilon},1-\tilde{\epsilon}}} \in \langle (t_1)^{-1-|\beta|} \rangle,
\]

which implies (7).
2.2 Classical trajectories with long-range time-independent force

We denote the outgoing region by \( \Gamma_{R,U,J,Q}^{+,E} \):

\[
\Gamma_{R,U,J,Q}^{+,E} := \{(r, \theta, \rho, \omega) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : r > R, \theta \in U, \rho \in J, |\omega| \leq Qr^{1-E}\}
\]

for \( R > 0, U \subset \mathbb{R}^{n-1}, J \subset \mathbb{R}, Q > 0 \).

We now consider the dynamics with time-independent long-range forces.

**Definition 8.** A time-independent force \( F \) is said to be a long-range force if it satisfies

\[
\sup_{(r, \theta, \rho, \omega) \in \Gamma_{R,U,J,Q}^{+,E}} |\partial_r^\alpha \partial_\theta^\beta \partial_\rho^\gamma \partial_\omega^\delta (F_r, F_\theta, F_\rho, F_\omega)(r, \theta, \rho, \omega)| \leq O((R)^{-n_r,\theta,\rho,\omega}(l,\alpha,k,\beta))
\]

(12)

for any \( R > 0, U \subset \mathbb{R}^{n-1}, J \subset (0,\infty), Q > 0 \).

As in Theorem 8 we show the unique existence of trajectories for the dynamics where the boundary conditions are the initial position and the final momentum.

**Theorem 9.** Assume that \( F \) is a time-independent long-range force in the sense of Definition 8. Then for any open \( U \subset \bar{U} \subset \mathbb{R}^{n-1}, \) open \( J \subset \bar{J} \subset (0,\infty), \) and \( Q > 0, \) there exists \( R > 0 \) such that for any \( t \geq 0 \) and for any \((r_1, \theta_1, \rho_1, \omega_1) \in \Gamma_{R,U,J,Q}^{+,E}\) there exists a unique trajectory

\[
[0,t] \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_1,\theta_1,\rho_1,\omega_1)
\]

satisfying

\[
\partial_s(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_1,\theta_1,\rho_1,\omega_1) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_1,\theta_1,\rho_1,\omega_1))
\]

(13)

\[
(\tilde{r}, \tilde{\omega})(0,t,r_1,\theta_1,\rho_1,\omega_1) = (r_1, \omega_1), \quad (\tilde{\theta}, \tilde{\rho})(t,t,r_1,\theta_1,\rho_1,\omega_1) = (\theta_1, \rho_1),
\]

and the estimates

\[
|r(s)| \in O((s + \langle r_1 \rangle)^0)|s|, \quad |\theta(s)| \in O((s + \langle r_1 \rangle)^{-\tilde{E}}),
\]

(14)

\[
|\rho(s)| \in O((s + \langle r_1 \rangle)^{-\tilde{E}}), \quad |\omega(s)| \in O((s + \langle r_1 \rangle)^0)|s|^{1-\tilde{E}},
\]

and

\[
\tilde{\theta}(s,t,r_1,\theta_1,\rho_1,\omega_1) \in \bar{U}, \quad \tilde{\rho}(s,t,r_1,\theta_1,\rho_1,\omega_1) \in \bar{J}
\]

where

\[
\tilde{r}(s) = \tilde{r}(s) - r_1 - s\rho_f, \quad \tilde{\theta}(s) = \tilde{\theta}(s) - \theta_f
\]

\[
\tilde{\rho}(s) = \tilde{\rho}(s) - \rho_f, \quad \tilde{\omega}(s) = \tilde{\omega}(s) - \omega_f.
\]

Moreover the solution satisfies the following estimates uniformly for \( 0 \leq s \leq t \leq \infty \), \((r_1, \theta_1, \rho_1, \omega_1) \in \Gamma_{R,U,J,Q}^{+,E}\) :

\[
\begin{pmatrix}
\partial_s r(s) & \partial_s \theta(s) & \partial_s \rho(s) & \partial_s \omega(s) \\
\partial_\theta r(s) & \partial_\theta \theta(s) & \partial_\theta \rho(s) & \partial_\theta \omega(s) \\
\partial_\rho r(s) & \partial_\rho \theta(s) & \partial_\rho \rho(s) & \partial_\rho \omega(s) \\
\partial_\omega r(s) & \partial_\omega \theta(s) & \partial_\omega \rho(s) & \partial_\omega \omega(s)
\end{pmatrix}
\in O((s + \langle r_1 \rangle)^0)|s| \begin{pmatrix}
o((s + \langle r_1 \rangle)^0)|s| \\
o(1) \\
O((s + \langle r_1 \rangle)^{-\tilde{E}}) \\
o((s + \langle r_1 \rangle)^0)|s|^{1-\tilde{E}}
\end{pmatrix}
\otimes (\langle r_1 \rangle^{1-\tilde{E}}, \langle r_1 \rangle^{-\tilde{E}}, \langle r_1 \rangle^{1-\tilde{E}}, \langle r_1 \rangle^{-1}),
\]

(15)
\[ \partial_t^2 \partial_{\theta}^\alpha \partial_{\rho}^\beta \partial_{\omega}^{\gamma} \left( \begin{array}{c} \frac{\mathcal{L}}{\theta} \\ \frac{\mathcal{L}}{\rho} \\ \frac{\mathcal{L}}{\omega} \end{array} \right) \leq \begin{cases} \mathcal{O}(s + (r_i)^0) |s - t_1| & \text{if } \epsilon_0 > 0, \\ \mathcal{O}(1) & \text{if } \epsilon_0 = 0, \\ \mathcal{O}(s + (r_i))^{-\frac{1}{2}} & \text{if } \epsilon_0 < 0 \end{cases}. \]

**Proof.** There exists \( C_0 \) such that if \( \rho \in J \) and \( r > 0 \), then

\[ |r + (s - t_1)\rho| \geq C_0(|s - t_1| + r). \]  

(15)

We fix constants \( \epsilon_0, \bar{Q}, \epsilon_1 \) such that

\[ 0 < \epsilon_0 < C_0, \quad \bar{Q} \geq \frac{2Q}{C_0 - \epsilon}, \quad 0 < \epsilon_1 < \frac{1}{2} \bar{Q}(C_0 - \epsilon_0)^{1 - \epsilon}, \]

and introduce cut-off functions \( I_r, I_\theta, I_\rho, I_\omega \) as follows. We take \( I_r \in C^\infty(0, \infty) \) such that \( I_r = 1 \) on a neighborhood of \( \{ r > C_0 - \epsilon_0 \} \), \( I_\theta \in C^\infty(U, \tilde{U}^C) \) such that \( I_\theta = 1 \) on \( \tilde{U} \), \( I_\rho \in C^\infty(0, \infty) \) such that \( I_\rho = 1 \) on \( \tilde{J} \), and \( I_\omega \in C^\infty(0, \infty) \) such that \( I_\omega = 1 \) on a neighborhood of \( \{ \omega : |\omega| < \bar{Q} \} \). Using these cut-off functions, we define the effective time-dependent force \( F_e \) by

\[ F_e(t, r, \theta, \rho, \omega) = I_r(r, \theta, \rho, \omega)I_\theta(\theta)I_\rho(\rho)I_\omega(\omega)F(r, \theta, \rho, \omega). \]

It follows from (12) that \( F_e(t, r, \theta, \rho, \omega) \) is a slowly-decaying force in the sense of Definition 4. Therefore, we can find \( T \) such that the boundary value problem considered in Theorem 5 possesses a unique solution for any \( T \leq t_1 \leq t_2 \) and any \( r_i, \theta_f, \rho_f, \omega_i \). Let us denote this solution by

\[ (\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i). \]

By enlarging \( T \) if needed, we can guarantee that

\[ |\tilde{r}_e(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) - r_i - (s - t_1)\rho_f| \leq \epsilon_0|s - t_1|, \]

(16)

\[ |\tilde{\theta}_e(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) - \theta_f| \leq \text{dist}(U, \tilde{U}^C), \]

\[ |\tilde{\rho}_e(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) - \rho_f| \leq \text{dist}(J, \tilde{J}), \]

\[ |\tilde{\omega}_e(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) - \omega_i| \leq \epsilon_1|s - t_1|^{1 - \epsilon}. \]

(17)

We claim that if \( R = T(C_0 - \epsilon_0)/C_0 \) and \( (r_i, \theta_f, \rho_f, \omega_i) \in \Gamma_{R, U, J, \bar{Q}}^{t_1, t_2} \), then we can solve our boundary problem by setting

\[ (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, t, r, \theta_f, \rho_f, \omega_i) := (\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(s + r, s + r + t, r_i, \theta_f, \rho_f, \omega_i) \]

(18)

where \( r = |r_i|C_0/(C_0 - \epsilon) \). Indeed, from (15), (16), and (17) we see that

\[ |\tilde{r}_e(s, r, r + t, r_i, \theta_f, \rho_f, \omega_i)| \geq (C_0 - \epsilon_0)|s + r|, \]

\[ \tilde{\theta}_e(r + s, r + t, r_i, \theta_f, \rho_f, \omega_i) \in \tilde{U}, \]

\[ \tilde{\rho}_e(r + s, r + t, r_i, \theta_f, \rho_f, \omega_i) \in \tilde{J}. \]
and
\[|\tilde{\omega}(r + s, r + t, r_i, \theta_f, \rho_f, \omega_0)| \leq \varepsilon_1|s|^{1-\tilde{\varepsilon}} + \omega_i \leq \varepsilon_1|s|^{1-\tilde{\varepsilon}} + Qr_i^{1-\tilde{\varepsilon}} \]
\[\leq \varepsilon_1|s|^{1-\tilde{\varepsilon}} + Q\left(\frac{C_0 - \varepsilon_0}{C_0}\right)^{1-\tilde{\varepsilon}}r_i^{1-\tilde{\varepsilon}} \leq \tilde{Q}(C_0 - \varepsilon_0)^{1-\tilde{\varepsilon}}|s + r|^{1-\tilde{\varepsilon}} \]
\[\leq \tilde{Q}|\tilde{r}(r + s, r + t, r_i, \theta_f, \rho_f, \omega_0)|^{1-\tilde{\varepsilon}}.
\]

Hence we have
\[F_c(r + s, (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})|(r + s, r + t, r_i, \theta_f, \rho_f, \omega_0)) \]
\[= F((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})|(r + s, r + t, r_i, \theta_f, \rho_f, \omega_0)).
\]

Therefore the function (18) solves the boundary problem (13) with the initial time-independent force.

The estimates on \((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})|(s, t, r_i, \theta_f, \rho_f, \omega_0)\) are obtained directly from those of Theorem 5 using the identity (18) and replacing \(s, t_1, t_2\) there by \(s + \langle r_i \rangle, \langle r_i \rangle, t + \langle r_i \rangle\).

Finally, the uniqueness of the solution comes from the fact that any solution of (12) with (14) is also a solution of the problem considered in Theorem 5 for the force \(F_c(t, r, \theta, \rho, \omega)\) if time \(t\) is large enough.

Now we solve the dynamics with initial conditions.

**Theorem 10.** Assume \(F\) is a time-independent long-range force in the sense of Definition 8.
Then for any open \(U \subset \tilde{U} \subset \mathbb{R}^{n-1}\), open \(J \subset \tilde{J} \subset (0, \infty)\), and \(Q > 0\), there exists \(R > 0\) such that for any \((r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+, \tilde{\varepsilon}}_{R, \tilde{U}, \tilde{J}, Q} \) there exists a unique trajectory
\[\{0, \infty\} \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)
\]
satisfying
\[\partial_s(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)),
\]
\[\tilde{r}(s, r_0, \theta_0, \rho_0, \omega_0) = (r_0, \theta_0, \rho_0, \omega_0).
\]

Set
\[\mathcal{L}(s) = \tilde{r}(s) - r_0 - sp_0, \quad \Theta(s) = \tilde{\theta}(s) - \theta_0,
\]
\[\mathcal{P}(s) = \tilde{\rho}(s) - \rho_0, \quad \Omega(s) = \tilde{\omega}(s) - \omega_0.
\]

Moreover the solution satisfies the following estimates uniformly for \(0 \leq s \leq t \leq \infty, (r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+, \tilde{\varepsilon}}_{R, \tilde{U}, \tilde{J}, Q} :\)
\[\Theta(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{U}, \quad \mathcal{P}(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{J},
\]
Proof. Let \((\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})\) be the solutions in Theorem 9 with \(t = \infty\):

\[
[0, \infty] \ni s \mapsto (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i),
\]

\[
\partial_s \left( \bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega} \right)(s, \infty, r_i, \theta_f, \rho_f, \omega_i) = \left( \partial_s \bar{r}, \partial_s \bar{\theta}, \partial_s \bar{\rho}, \partial_s \bar{\omega} \right)(s, \infty, r_i, \theta_f, \rho_f, \omega_i),
\]

\[
(r, \omega)(0, \infty, r_i, \theta_f, \rho_f, \omega_i) = (r, \omega_i), \quad (\dot{\bar{r}}, \dot{\bar{\theta}}, \dot{\bar{\rho}}, \dot{\bar{\omega}})(0, \infty, r_i, \theta_f, \rho_f, \omega_i) = (\dot{r}, \dot{\omega}).
\]

Set

\[
(r_0, \theta_0, \rho_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) := (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(0, \infty, r_i, \theta_f, \rho_f, \omega_i).
\]

It is clear that

\[
(r_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i).
\]

Theorem 9 assures the following estimates:

\[
|\theta_0(r_i, \theta_f, \rho_f, \omega_i) - \theta_f| \in O((r_i)^{-\xi}), \quad |\rho_0(r_i, \theta_f, \rho_f, \omega_i) - \rho_f| \in O((r_i)^{-\xi}),
\]

\[
|\partial_s \theta_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_s \theta_f| \in O((r_i)^{-\xi}), \quad |\partial_s \rho_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_s \rho_f| \in O((r_i)^{-\xi}),
\]

\[
\begin{pmatrix}
\partial_r \theta_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_r \theta_f \\
\partial_r \rho_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_r \rho_f \\
\partial_{\theta} \theta_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_{\theta} \theta_f \\
\partial_{\theta} \rho_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_{\theta} \rho_f \\
\partial_{\rho} \theta_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_{\rho} \theta_f \\
\partial_{\rho} \rho_0(r_i, \theta_f, \rho_f, \omega_i) - \partial_{\rho} \rho_f
\end{pmatrix} \in \begin{pmatrix}
O(1) \\
O(1) \\
O(1) \\
O(1) \\
O(1) \\
O(1)
\end{pmatrix}
\]

By taking \(R\) large enough, we can assure that the map \((r_0, \theta_0, \rho_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i)\) is injective. Let \((r_i, \theta_i, \rho_i, \omega_i)(r_0, \theta_0, \rho_0, \omega_0)\) be the inverse function. We will show that

\[
(r, \omega)(s, r_0, \theta_0, \rho_0, \omega_0) := (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, (r_i, \theta_f, \rho_f, \omega_i), (r_0, \theta_0, \rho_0, \omega_0))
\]
gives the desired function. (19) implies

\[
\left( \frac{\partial r_0(\theta_f - \theta_0)}{\langle r_0 \rangle^{\varepsilon}}, \frac{\partial \rho_0(\rho_f - \rho_0)}{\langle r_0 \rangle^{\varepsilon}}, \frac{\partial \omega_0(\theta_f - \theta_0)}{\langle r_0 \rangle^{\varepsilon}}, \frac{\partial \rho_0(\rho_f - \rho_0)}{\langle r_0 \rangle^{\varepsilon}} \right) 
\in \left( \frac{O(1)}{\langle r_0 \rangle^{-\varepsilon}} \right) \otimes \left( \langle r_1 \rangle^{-1-\varepsilon}, \langle r_1 \rangle^{-\varepsilon}, \langle r_1 \rangle^{-\varepsilon}, \langle r_1 \rangle^{-1} \right). 
\]

(20)

Moreover, it is easy to see that

\[
\frac{\partial l}{\partial \theta_0} \frac{\partial \rho_0}{\partial \alpha} \frac{\partial \rho_0}{\partial \beta} \left( \frac{\theta_f - \theta_0}{\rho_f - \rho_0} \right) \in \left( \frac{O(\langle r_0 \rangle^{-l-|\beta|})}{O(\langle r_0 \rangle^{-l-|\beta|-\varepsilon})} \right). 
\]

(21)

(20) and (21) shows the desired estimates.

2.3 Solutions to the Hamilton-Jacobi equation

We state a lemma which relates the hamiltonian \( h \) with the time-independent force \( F \).

**Lemma 11.** Let

\[
h(r, \theta, \rho, \omega) = \frac{1}{2} \rho^2 + \tilde{h}(r, \theta, \rho, \omega) \\
\tilde{h}(r, \theta, \rho, \omega) = \frac{1}{2} a_1(r, \theta) \rho^2 + \frac{1}{r} a_j^j \rho \omega_j + \frac{1}{2r^2} a_{jk}^j \omega_j \omega_k + V_L(r, \theta),
\]

Assume

\[
|\partial^I_\alpha \partial^\beta_\theta \partial^k_\rho \partial^\beta_\omega (\tilde{h})| \leq C_{i, \alpha} r^{-|\mu| - l}, \\
|D^I_\alpha D^\beta_\theta V_L(r, \theta)| \leq C_J r^{-|\mu| - j},
\]

with

\[
\mu_1 = \mu_2 = \mu_L = \varepsilon > 0, \quad \mu_3 = 0.
\]

Then for any \( U \subset \mathbb{R}^{n-1}, J \in \mathbb{R}, \) and \( Q > 0, \)

\[
\sup_{(r, \theta, \rho, \omega) \in \Gamma_{R, U, J, Q}} |\partial^I_\alpha \partial^\beta_\theta \partial^k_\rho \partial^\beta_\omega (\tilde{h})| \in O(\langle R \rangle^{-m(1, \alpha, k, \beta)}). 
\]

(22)

This immediately implies that setting

\[
(F_r, F_\theta, F_\rho, F_\omega) = (\partial \rho \tilde{h}, \partial \omega \tilde{h}, -\partial \theta \tilde{h}, -\partial \rho \tilde{h}),
\]

(23)

we have (12), i.e., \( h \) defines a long-range time-independent force via (23).

Combining Theorem 10 and Lemma 11, we obtain solutions to the Hamilton-Jacobi equation:
Theorem 12. Let \( \tilde{h}, \hat{h} \) be as in Lemma 11. For any \( \tilde{U} \in U \in \mathbb{R}^{n-1}, \tilde{J} \in J \subset (0, \infty), C_{j, \alpha} > 0, \) there exists \( T > 0 \) such that if a smooth function \( \psi(\rho, \theta) \) defined on \( J \times U \) satisfies
\[
|\partial_t^{\alpha} \partial_\theta^\beta (\psi(\rho, \theta) - \frac{1}{2} s \rho^2)| \leq C_{j, \alpha} (s)^{1-\hat{\varepsilon}}
\] (24)
for some \( s > T, \) then there exists a unique function \( S(t, \rho, \theta) \) defined on a region \( \Theta \subset (0, \infty) \times \mathbb{R}^{n-1} \) (which will be defined in the proof), with \( \Theta \supset (0, \infty) \times \tilde{J} \times \tilde{U}, \) satisfying the Hamilton-Jacobi equation:
\[
(\partial_t S)(t, \rho, \theta) = h((\partial_\rho S)(\rho, \theta), \theta, \rho, -(\partial_\theta S)(\rho, \theta))
\]
with the initial value
\[
S(0, \rho, \theta) = \psi(\rho, \theta).
\]
Moreover the function \( S \) satisfies the following estimates:
\[
|\partial_t^{\alpha} \partial_\theta^\beta (S(t, \rho, \theta) - \frac{1}{2} s \rho^2)| \leq C_{j, \alpha} (t)^{1-\hat{\varepsilon}}.
\] (25)

Proof. Let
\[
[0, \infty) \ni t \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0)
\]
be the unique trajectory of the Hamilton equations with initial value problem as in the Theorem 10
\[
\partial_t (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0)),
\]
\[
(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(0, r_0, \theta_0, \rho_0, \omega_0) = (r_0, \theta_0, \rho_0, \omega_0),
\]
for \( (r_0, \theta_0, \rho_0, \omega_0) \in \Gamma_{R, U, J, Q}^+, \) where we took \( R > 0 \) such that
\[
\{(\partial_\rho \psi)(\rho_0, \theta_0), \theta_0, \rho_0, -(\partial_\theta \psi)(\rho_0, \theta_0) : (\theta_0, \rho_0) \in U \times J \} \subset \Gamma_{R, U, J, Q}^{+, \hat{\varepsilon}}.
\]
Set
\[
(r, \theta, \rho, \omega)(t; \rho_0, \theta_0) := (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, (\partial_\rho \psi)(\rho_0, \theta_0), \theta_0, \rho_0, -(\partial_\theta \psi)(\rho_0, \theta_0)).
\]
and consider the map
\[
(\rho_0, \theta_0) \mapsto (r, \theta, \rho, \omega)(t; \rho_0, \theta_0)
\] (26)
and its first derivatives. We set \( \Theta := \{(t, (\partial_\rho \psi)(\rho_0, \theta_0))(\rho_0, \theta_0) \in J \times U \}. \) By a straight computation, we obtain
\[
|\partial_t (\rho, \theta)(t, \rho_0, \theta_0)(\rho_0, \theta_0)| = C(s)^{-\hat{\varepsilon}}.
\]
where $C$ depends on $C_{j,\alpha}$ and not on the choice of $\psi$ as long as $\psi$ satisfies (24) for some $s$. We fix a large enough $T > 0$ so that for any $s > T$ we have

$$\left| \frac{\partial (\rho, \theta)}{\partial (\rho_0, \theta_0)}(t, \rho_0, \theta_0) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \ll 1.$$ 

Now (26) becomes an injective map for every $t > 0$. We denote its inverse by

$$\phi(\rho, \theta) \mapsto (\rho_0, \theta_0)(t; \rho, \theta).$$

Let

$$Q(t; \rho_0, \theta_0)$$

$$= \psi(\rho_0, \theta_0) + \int_0^t \left[ h((r, \theta, \rho, \omega)(u; \rho_0 \omega_0) + \langle r(t; \rho_0, \theta_0, (\partial_u \rho)(u; \rho, \theta)) - \langle \omega(t; \rho_0, \theta_0), (\partial_u \theta)(u; \rho, \theta) \rangle \right] du.$$ 

Then the function

$$S(t, \rho, \theta) = Q(t; \rho_0, \theta_0)(t; \rho, \theta)$$

defined on $\Theta$ is the desired solution to the Hamilton-Jacobi equation (see, for example, Appendix A.3). Moreover

$$(\partial_\rho S)(t, \rho, \theta) = r(t; \rho_0(t, \rho, \theta), \theta_0(t, \rho, \theta)),$$

$$-(\partial_\theta S)(t, \rho, \theta) = \omega(t; \rho_0(t, \rho, \theta), \theta_0(t, \rho, \theta)).$$

The derivatives of $S(t, \rho, \theta)$

$$\frac{\partial^2}{\partial \rho \partial \theta} (S(t, \rho, \theta) - \frac{1}{2} t \rho^2) = \frac{\partial^2}{\partial \rho \partial \theta} \tilde{h}(\partial_\rho S(t, \rho, \theta), \theta, \rho, -\partial_\theta S(t, \rho, \theta))$$

is a summation of the terms of the type

$$(\partial_\rho^l \partial_\theta^m \tilde{h} \partial_\rho^k \partial_\theta^l \partial_\rho^m \partial_\theta^n \partial_\rho S(t, \rho, \theta), \theta, \rho, -\partial_\theta S(t, \rho, \theta)) \times$$

$$\prod_{i=1}^l \partial_\rho^{k_i} \partial_\theta^{l_i} \partial_\rho^m \partial_\theta^n \partial_\rho S(t, \rho, \theta) \times \prod_{d=1}^{n-1} \prod_{j=1}^{m_d} \partial_\rho^{k_d} \partial_\theta^{l_d} (-\partial_\theta S)(t, \rho, \theta),$$

which belongs to $O((t)^{-m(\beta, k, n)+(1-\epsilon)\gamma}) < O((t)^{-\epsilon})$. This shows (25). 

Finally we extend $S(t, \rho, \theta)$ to a globally defined function on $\mathbb{R} \times (0, \infty) \times \partial M$ which satisfies the same kind of estimates locally.

**Theorem 13.** Let $h, \tilde{h}$ be as in Lemma 11 defined on $T^* \mathbb{R} \times T^* \partial M$. Then there exists a function $S(t, \rho, \theta)$ defined on $T^* \mathbb{R} \times T^* \partial M$ such that for every $J \in \mathbb{R} \setminus \{0\}$, there exists $T > 0$ such that the Hamilton-Jacobi equation:

$$(\partial_t S)(t, \rho, \theta) = h((\partial_\rho S)(\rho, \theta), \theta, \rho, -\partial_\theta S)(\rho, \theta))$$

is satisfied for $t > |T|$, $\rho \in J$, and $\theta \in \partial M$. Moreover the function $S$ satisfies the following estimates:

$$|\partial_\rho^j \partial_\theta^l (S(t, \rho, \theta) - \frac{1}{2} t \rho^2)| \leq \tilde{C}_{J, \alpha} (t)^{1-\epsilon}. \quad (28)$$

Proof. First note that since $\partial M$ is compact and the Hamilton-Jacobi equation is defined in a coordinate invariant manner, we can extend $U$ in the Theorem 12 to $\partial M$. It is sufficient to consider the case $J \subset (0, \infty)$ and $t > T$, since we can extend the function $S$ in a $C^\infty$-fashion.

Take a sequence of open sets in $(0, \infty)$ such that

$$J_0 \subset J_1 \subset J_2 \subset J_3 \subset \ldots, \quad \bigcup_{n=0}^{\infty} J_n = (0, \infty).$$

First we solve the Cauchy problem for the Hamilton-equation with initial data $S(t, \rho, \theta) = \frac{1}{2}t\rho^2$ when $\rho \in J_1, t = T_1 > 0$ for a big enough $T_1$ by Theorem 12 with $U$ replaced by $\partial M$. We denote the solution by $S_1$. We can assume that $S_1$ is defined on $(T_1, \infty) \times J_0 \times \partial M$.

Next we take $\chi_1 \in C_0^\infty(J_1)$ equal to 1 in a neighborhood of $\overline{J_0}$ (the closure of $J_0$). We solve the Cauchy Problem with initial data

$$S(t, \rho, \theta) = \chi_1 S_1 + (1 - \chi_1)\frac{1}{2}t\rho^2$$

when $\rho \in J_2, t = T_2$. By taking $T_2 > T_1$ large enough, the right hand side satisfies the conditions for $T_2$ in Theorem 12. So we can solve the Cauchy Problem for such $T_2$. We denote the solution by $S_2$.

Repeating this procedure, we obtain a sequence $S_n$ of functions and a sequence $T_1 < T_2 < \ldots$ such that $S_n$ is defined on $J_n \times [T_n, \infty) \times \partial M$,

$$S_{n+1} = S_n \text{ for } m \geq n + 1 \text{ on } J_n \times [T_m, \infty) \times \partial M,$$

and satisfies (25). Thus by extending in a $C^\infty$ fashion, we can construct a $C^\infty$ function $S$ which satisfies (27), and (25) for large enough $t$ and $\rho$ in any fixed compact subset of $(0, \infty)$.}

\section{Proof of Theorem 3}

In this section, we give the proof of Theorem 3. First we give the outline of the proof.

Outline of the Proof. We consider the $t \to +\infty$ case. By the density argument, it is sufficient to show the existence of the norm limit

$$\lim_{t \to +\infty} e^{itH} J e^{-iS(t, D_r, \theta)} u$$

for all $\hat{u} \in C_0^\infty((\mathbb{R} \setminus \{0\}) \times U_\lambda)$ for all $\lambda$. For such $u$, we have

$$\frac{1}{i} e^{-itH} \frac{\partial}{\partial t} [e^{itH} J e^{-iS(t, D_r, \theta)} u] = [HJ - J \frac{\partial S}{\partial t}(t, D_r, \theta)] e^{-iS(t, D_r, \theta)} u.$$

By the Cook-Kuroda method we only need to show that

$$\|[HJ - J \frac{\partial S}{\partial t}(t, D_r, \theta)] e^{-iS(t, D_r, \theta)} u\|_{L^1_t(1, \infty)} \in L^1_t.$$
We decompose
\[
[HJ - J \frac{\partial S}{\partial t}(t, D_r, \theta)] = [P_0 J - J P_0] + V S J + [V^L J - JV^L] + J [P_0 + V^L (r) - \frac{\partial S}{\partial t}(t, D_r, \theta)].
\]
The first three terms are essentially short range terms. It is easy to check
\[
||[P_0 J - J P_0] + V S J + [V^L J - JV^L] e^{-iS(t, D_r, \theta)} u||_{\mathcal{H}} \in L^1_t (1, \infty). \tag{29}
\]
We examine the last term:
\[
[P_0 + V^L (r) - (\partial_\rho S)(t, D_r, \theta)] e^{-iS(t, D_r, \theta)} u
\]
\[
= h'(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) e^{-iS(t, D_r, \theta)} u
\]
\[
- h((\partial_\rho S)(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) e^{-iS(t, D_r, \theta)} u
\]
\[
+ \frac{1}{r} a_2^\prime D_r + \frac{1}{2r^2} a_3^\prime \frac{\partial S}{\partial \theta}(t, D_r, \theta) e^{-iS(t, \rho, \theta)} (\partial_\theta u)
\]
\[
- \frac{1}{r} \frac{\partial S}{\partial \theta}(t, D_r, \theta) (\partial_\theta, \partial_\theta u)
\]
+ [ short range terms ].
\]
We apply the stationary phase method to the first two terms. Then the first terms which appear in the asymptotic expansion will vanish since the relation
\[
(\partial_\rho S)(t, \rho, \theta) = r
\]
gives the stationary point with respect to the \(\rho\)-variable. Therefore we obtain
\[
||[P_0 + V^L (r) - (\partial_\rho S)(t, D_r, \theta)] e^{-iS(t, D_r, \theta)} u||_{\mathcal{H}} \in L^1_t (1, \infty). \tag{30}
\]
We give a detailed proof of (29) and (30) in the remaining of this section.

First we consider the long-range term (30). The next proposition is our key estimate.

**Proposition 14.** Assume the assumptions of Theorem 3. Suppose \(u\) satisfies \(\hat{u} \in C^\infty_0 ([\mathbb{R} \setminus \{0\}] \times U_\lambda)\) and \(J \times U\) is a neighborhood of supp \(\hat{u}\). Then we have
\[
||[\hat{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))] e^{-iS(t, D_r, \theta)} u(r, \theta)|| \leq C t^{\frac{1}{2} - 1 - \varepsilon} \tag{31}
\]
for \((\frac{r}{t}, \theta) \in J \times U \subseteq (0, \infty) \times \partial M\), and
\[
||[\hat{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))] e^{-iS(t, D_r, \theta)} u(r, \theta)|| \leq C N (1 + |r| + |t|)^{-N} \tag{32}
\]
for any \(N\) and for \((\frac{r}{t}, \theta) \notin J \times U\).
Proof of (30). We fix a neighborhood \( J \times U \) of \( \text{supp} \hat{\alpha} \) which appears in Proposition 14. Then

\[
\int_1^\infty \|J|\left[\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))\right] e^{-iS(t, Dr_r, \theta)} u(t) \|_{\mathcal{W}} dt \\
= \int_1^\infty \left( \int_{\mathbb{R}+ \times \partial M} j(r) [\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))] e^{-iS(t, Dr_r, \theta)} u(r, \theta) \right]^2 drH(\theta) d\theta \frac{1}{2} dt \\
\leq \int_1^\infty \left( \int_{\mathbb{R}+ \times \partial M} j(r) [\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))] e^{-iS(t, Dr_r, \theta)} u(r, \theta) \right]^2 drH(\theta) d\theta \frac{1}{2} dt \\
+ \int_{\mathbb{R}+ \times \partial M} j(r) [\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))] e^{-iS(t, Dr_r, \theta)} u(r, \theta) \right]^2 drH(\theta) d\theta \frac{1}{2} dt.
\]

By (31), the first term is finite:

\[
\int_1^\infty \left( \int_{\mathbb{R}+ \times \partial M} j(r) [\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))] e^{-iS(t, Dr_r, \theta)} u(r, \theta) \right]^2 drH(\theta) d\theta \frac{1}{2} dt \\
\leq \int_1^\infty \left( \int_{\mathbb{R}+} |Ct^{-\frac{1}{2}-\frac{1}{2}-\varepsilon/2} |^2 t dR \right)^{\frac{1}{2}} dt \leq C \int_1^\infty t^{-1-\varepsilon} dt < \infty.
\]

By (32), the second term is also finite:

\[
\int_1^\infty \left( \int_{\mathbb{R}+ \times \partial M} j(r) [\hat{h}(r, \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta)) - \hat{h}(\frac{\partial S}{\partial \theta}(t, Dr_r, \theta), \theta, Dr_r, -\frac{\partial S}{\partial \theta}(t, Dr_r, \theta))] e^{-iS(t, Dr_r, \theta)} u(r, \theta) \right]^2 drH(\theta) d\theta \frac{1}{2} dt \\
\leq \int_1^\infty \left( \int_{\mathbb{R}+} C(1 + |r| + |t|)^{-N} dR \right)^{\frac{1}{2}} dt < \infty.
\]

Therefore

\[
\|J[V^L(r, Dr_r, \theta) - V^L(\frac{\partial W}{\partial \theta}(Dr_r, \theta, t), Dr_r, \theta)] e^{-iW(Dr_r, \theta, t)} u(t) \|_{\mathcal{W}} \in L^1_t(1, \infty).
\]

In order to prove Proposition 14 we prepare a lemma.

Lemma 15. Let \( S(t, \rho, \theta) \) satisfy the properties listed in Theorem 13. Set

\[
f_{r, \theta, t}(\rho) := \frac{1}{t}(\rho - S(t, \rho, \theta)).
\]
3 PROOF OF THEOREM

For \( \rho \) in any fixed compact subset of \( \mathbb{R} \setminus \{0\} \) and for large enough \( |t| \), there exists a function \( \Xi_{\theta,t}(r) \) which gives the critical point of \( f_{r,\theta,t}(\rho) \):

\[
\partial_\rho f_{r,\theta,t}(\rho) = 0 \iff \rho = \Xi_{\theta,t}(r).
\]

Set \( \Omega_d := (-d,d) \). Then there exist \( 0 < \tilde{d} < d \) and a function \( \Phi_{r,\theta,t} \in C^\infty(\Omega_d;\mathbb{R}) \) such that \( \Omega_{2\tilde{d}} \subset \Phi_{r,\theta,t}(\Omega_d) \). Setting

\[
\psi_{r,\theta,t}(y) := \Xi_{\theta,t}(r) + \Phi_{r,\theta,t}(y),
\]

\[
(f_{r,\theta,t} \circ \psi_{r,\theta,t})(y) = f_{r,\theta,t}(\Xi_{\theta,t}(r)) + \langle A_{r,\theta,t}, y \rangle / 2,
\]

where

\[
A_{r,\theta,t} = (\partial^2 f_{r,\theta,t})(\Xi_{\theta,t}(r)),
\]

we have

\[
|\partial^k \psi_{r,\theta,t}(0)| \leq Ct^{-\varepsilon} \quad (k \geq 2), \quad \partial_\psi \psi_{r,\theta,t}(0) = 1. \quad (33)
\]

Proof. We only consider the \( t > 0 \) and \( \rho > 0 \) case. First we prove that \( \Xi_{\theta,t}(r) \) is well-defined. Compute

\[
0 = \partial_\rho f_{r,\theta,t}(\rho) = \frac{1}{t} \left[ r - \frac{\partial S}{\partial \rho}(t,\rho,\theta) \right]
\]

We note that by (28),

\[
\left| \frac{1}{t} \frac{\partial^2 S}{\partial \rho^2}(t,\rho,\theta) - 1 \right| \leq Ct^{-\varepsilon}.
\]

This implies that \( \frac{1}{t} \frac{\partial^2 S}{\partial \rho^2} \) is monotonously increasing with respect to \( \rho \) for large enough \( t \). Thus there is a unique inverse function \( \Xi_{\theta,t}(r) \) such that

\[
(\partial_\rho f_{r,\theta,t})(\Xi_{\theta,t}(r)) = 0
\]

for large enough \( t \) and \( \frac{r}{t} \in J \), a fixed compact subset of \((0,\infty)\).

Now we construct \( \Phi_{r,\theta,t} \) and \( \psi_{r,\theta,t} \). We set

\[
A_{r,\theta,t} := \frac{1}{t} \frac{\partial^2 S}{\partial \rho^2}(t,\Xi_{\theta,t}(r),\theta) = - \frac{1}{t} \frac{\partial^2 S}{\partial \rho^2}(t,\Xi_{\theta,t}(r),\theta).
\]

Then (28) implies that

\[
|A_{r,\theta,t} + 1| \leq Ct^{-\varepsilon}.
\]

Hence we have \( A_{r,\theta,t} \to -1 \) uniformly for \( r/t \in J \). If we set

\[
g_{r,\theta,t}(\rho) := f_{r,\theta,t}(\Xi_{\theta,t}(r) + \rho),
\]

Then (28)
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Then
\[
g'_{r, \theta, t}(0) = f'_{r, \theta, t}(\Xi_{\theta, t}(r)) = 0, \quad g''_{r, \theta, t}(0) = f''_{r, \theta, t}(\Xi_{\theta, t}(r)) = A_{r, \theta, t},
\]
\[
g_{r, \theta, t}(\rho) - g_{r, \theta, t}(0) = \langle B_{r, \theta, t}(\rho) \rho, \rho \rangle / 2,
\]
where
\[
B_{r, \theta, t}(\rho) = 2 \int_0^1 g_{r, \theta, t}(s \rho)(1 - s) \, ds, \quad B_{r, \theta, t}(0) = A_{r, \theta, t}
\]
by Taylor’s formula. Now we compute
\[
|B_{r, \theta, t}(\rho) - A_{r, \theta, t}| = |B_{r, \theta, t}(\rho) - B_{r, \theta, t}(0)| = 2| \int_0^1 (g''_{r, \theta, t}(s \rho) - g''_{r, \theta, t}(0))(1 - s) \, ds |
\]
\[
\leq 2 \sup_{0 \leq s \leq 1} |g''_{r, \theta, t}(s \rho) - g''_{r, \theta, t}(0)|
\]
\[
\leq 2 \sup_{0 \leq s \leq 1} \left| \frac{\partial^2 S}{\partial \rho^2}(t, \Xi_{\theta, t}(r) + s \rho, \theta) - \frac{1}{t} \frac{\partial^2 S}{\partial \rho^2}(t, \Xi_{\theta, t}(r), \theta) \right|
\]
\[
\leq C t^{-\varepsilon} \to 0
\]
as \(t \to \infty\), uniformly for \(t, \rho \in J\) by (28). Hence by taking \(t\) sufficiently large, we may assume \(|B_{r, \theta, t}(\rho) - 1| < 1/2\) is uniformly very small. For such \(t, \rho\) and \(\rho\), we set
\[
X_{r, \theta, t}(\rho) := \sqrt{\frac{B_{r, \theta, t}(\rho)}{A_{r, \theta, t}}} \cdot \rho.
\]
Then we have
\[
g_{r, \theta, t}(\rho) - g_{r, \theta, t}(0) = \langle A_{r, \theta, t}X_{r, \theta, t}(\rho), X_{r, \theta, t}(\rho) \rangle / 2.
\]
Now we compute
\[
(\partial \rho X_{r, \theta, t})(\rho) = \left( \sqrt{\frac{B_{r, \theta, t}(\rho)}{A_{r, \theta, t}}} \right)' \cdot \rho + \frac{B_{r, \theta, t}(\rho)}{A_{r, \theta, t}} \cdot 1
\]
\[
= \frac{1}{\sqrt{A_{r, \theta, t}}} 2 B_{r, \theta, t}(\rho) \cdot B'_{r, \theta, t}(\rho) \cdot \rho + \frac{B_{r, \theta, t}(\rho)}{A_{r, \theta, t}} \cdot 1,
\]
\[
(\partial \rho B_{r, \theta, t})(\rho) = 2 \int_0^1 g'''_{r, \theta, t}(s \rho)s(1 - s) \, ds,
\]
\[
|g'''_{r, \theta, t}(s \rho)| = \left| - \frac{1}{t} (\partial^3 S)(t, \Xi_{\theta, t}(r) + s \rho, \theta) \right| \leq C t^{-\varepsilon},
\]
\[
|\partial \rho X_{r, \theta, t}(\rho) - 1| \leq C t^{-\varepsilon}.
\]
Hence for small enough \(d_0 > 0\) and for \(|\rho| \leq d_0\), we have \(|\partial \rho X_{r, \theta, t}(\rho) - 1|\) arbitrary small for all large enough \(t\), and \(X_{r, \theta, t} : \Omega_{d_0} \to X_{r, \theta, t}(\Omega_{d_0})\) is a \(C^\infty\)-diffeomorphism. We can pick \(d > 0\) such that, \(\Omega_d \subset X_{r, \theta, t}(\Omega_{d_0})\) for all \(r, \theta\), large enough \(t, \xi \in J\). Let \(\phi_{r, \theta, t}\) be the inverse map of
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Then we complete the proof of Lemma 15. The support of \( \hat{X}_{r, \theta, t} \) with domain \( \Omega_d \). Then we can also pick \( \tilde{d} > 0 \) such that \( \Omega_d \subset \phi_{r, \theta, \tilde{d}}(\Omega_d) \) for all \( r, \theta, \) large enough \( t, \tilde{t} \in J \). We note that

\[
g_{r, \theta, t} \circ \phi_{r, \theta, t}(y) - g_{r, \theta, t}(0) = \langle A_{r, \theta, t}X_{r, \theta, t} \circ \phi_{r, \theta, t}(y), X_{r, \theta, t} \circ \phi_{r, \theta, t}(y) \rangle / 2.
\]

We set

\[
\psi_{r, \theta, t}(y) = \phi_{r, \theta, t}(y) + \Xi_{r, t}(r).
\]

Then we have

\[
f_{r, \theta, t} \circ \psi_{r, \theta, t}(y) = f_{r, \theta, t}(\Xi_{r, t}(r)) + \langle A_{r, \theta, t}, y \rangle / 2.
\]

Lastly, we prove the estimates (33). For \( k \geq 1, \)

\[
|\partial^k_{\rho} B_{r, \theta, t}(\rho)| = 2\int^1_0 |g^{2+k}_{r, \theta, t}(s\rho) s^k(1 - s) ds| \leq 2 \sup |g^{2+k}_{r, \theta, t}(s\rho)|
\]

\[
\leq 2 \sup \int^1_0 |\partial^2_{\rho} B_{r, \theta, t}(\rho, \xi_{r, \phi, t}(r) + s\rho, \theta)| \leq Ct^{-\epsilon}
\]

by (28). We also have

\[
|\partial^k_{\rho} \sqrt{B_{r, \theta, t}(\rho)}| \leq Ct^{-\epsilon}.
\]

Therefore

\[
|\partial^k_{\rho} X_{r, \theta, t}(\rho)| \leq Ct^{-\epsilon} \quad (k \geq 2), \quad |\partial^k_{\rho} X_{r, \theta, t}(0) - 1| = Ct^{-\epsilon},
\]

and we have

\[
|\partial^k_{\rho} \psi_{r, \theta, t}(y)| = |\partial^k_{\rho} \phi_{r, \theta, t}(y)| \leq Ct^{-\epsilon} \quad (k \geq 2),
\]

\[
|\partial^k_{\rho} \psi_{r, \theta, t}(0) - 1| = |\partial^k_{\rho} \phi_{r, \theta, t}(0) - 1| \leq Ct^{-\epsilon}.
\]

Then we complete the proof of Lemma [15].

Proof of Proposition [14]. First we prove (31) for \( \tilde{t} \in J \). We fix \( \chi \in C^\infty_0(\mathbb{R}) \) such that \( \chi(x) = 1 \) if \( |x| \leq \frac{1}{2} \), and \( \chi(x) = 0 \) if \( |x| \geq 1 \). We split \( u \) into two terms depending on \( r, \theta, \) and \( t \):

\[
\hat{u}^c_{r, \theta, t}(\rho, \theta) = \hat{u}(\rho, \theta) \chi \left( \frac{\rho - \Xi_{r, \phi, t}(r)}{d} \right),
\]

\[
\hat{u}^d_{r, \theta, t}(\rho, \theta) = \hat{u}(\rho, \theta) \left[ 1 - \chi \left( \frac{\rho - \Xi_{r, \phi, t}(r)}{d} \right) \right]
\]

where we use notations defined in Lemma [15]. The support of \( \hat{u}^c_{r, \theta, t} \) is close to the critical point of \( rp - S(t, \rho, \theta) \), while that of \( \hat{u}^d_{r, \theta, t} \) is apart from it. Note that

\[
\text{supp} \hat{u}^c_{r, \theta, t} \subset \Xi_{r, \phi, t}(r) + \Omega_d \subset \text{Ran}(\psi_{r, \theta, t}).
\]
By a change of variables we have
\[
\tilde{h}(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) e^{-iS(t, D_r, \theta) \hat{u}_{r, \theta, t}(r, \theta)}
\]
\[
= \frac{1}{2\pi} \int_{\Omega_d} \tilde{h}(\frac{\partial S}{\partial \rho}(t, \rho, \theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t, \rho, \theta)) e^{ir \cdot \hat{u}_{r, \theta, t}(\rho, \theta)} d\rho
\]
\[
= \frac{1}{2\pi} \int_{\Omega_d} \tilde{h}(t, \psi_{r, \theta, t}(y), \theta, \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta)) \hat{u}_{r, \theta, t}(\psi_{r, \theta, t}(y), \theta)
\]
\[
\cdot J_{r, \theta, t}(y) e^{it \cdot \hat{u}_{r, \theta, t}(\xi_{\theta, t}(r))} e^{i \ell(A_{r, \theta, y})/2} dy
\]
where \( J_{r, \theta, t}(y) = |\psi_{r, \theta, t}(y)| \) is the Jacobian. Since
\[
|D^j_\rho \tilde{h}(t, \rho, \theta)| \leq Ct^{-e},
\]
we have
\[
|D^j_\rho \frac{\partial S}{\partial \rho}(t, \psi_{r, \theta, t}(y), \theta, \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta))| \leq Ct^{-e},
\]
\[
|\tilde{h}(r, \theta, \psi_{r, \theta, t}(y), \theta, \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta))| \leq Ct^{-e},
\]
\[
|D^j_\rho \hat{u}_{r, \theta, t}(\psi_{r, \theta, t}(y), \theta)| \leq C,
\]
\[
|D^j_\rho J_{r, \theta, t}(y)| \leq C,
\]
for \( y \in \Omega_d, t \in J \). Now we apply the stationary phase method (see Hörmander [11] Section 7.7) to the integral. In the asymptotic expansion of
\[
[\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) - \tilde{h}(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))]
\]
\[
\cdot e^{-iS(t, D_r, \theta) \hat{u}_{r, \theta, t}(r, \theta)}
\]
\[
= \frac{1}{2\pi} \int_{\Omega_d} [\tilde{h}(r, \theta, \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta))
\]
\[
- \tilde{h}(\frac{\partial S}{\partial \rho}(t, \psi_{r, \theta, t}(y), \theta, \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta))]
\]
\[
\cdot \hat{u}_{r, \theta, t}(\psi_{r, \theta, t}(y), \theta) \cdot J_{r, \theta, t}(y) e^{it \cdot \hat{u}_{r, \theta, t}(\xi_{\theta, t}(r))} e^{i \ell(A_{r, \theta, y})/2} dy,
\]
the terms in which \( \tilde{h} \) is not differentiated will vanish since
\[
\frac{\partial S}{\partial \rho}(t, \psi_{r, \theta, t}(0), \theta) = r.
\]
Especially, in the first step of the asymptotic expansion, we need to estimate only the remainder
terms. Therefore we have

\[
([\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) - \tilde{h}(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)))]
\cdot e^{-iS(t, D_r, \theta)} u_{r, \theta, t}^d (r, \theta)
\leq Ct^{-\frac{1}{2} - \varepsilon} \sup_{|k| \leq 3} ||D_k \tilde{h}(r, \theta, \psi_{r, \theta, t}(y), \frac{\partial S}{\partial \rho}(t, \psi_{r, \theta, t}(y), \theta))
- \tilde{h}(\frac{\partial S}{\partial \rho}(t, \psi_{r, \theta, t}(y), \theta), \psi_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, \psi_{r, \theta, t}(y), \theta)))].
\]

\[
\tilde{u}_{r, \theta, t}'(\psi_{r, \theta, t}(y), \theta) \cdot J_{r, \theta, t}(y) ||_{L^2} \leq Ct^{-\frac{1}{2} - \varepsilon}.
\]

We now consider \(u_{r, \theta, t}^d\) term.

\[
\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) e^{-iS(t, D_r, \theta)} u_{r, \theta, t}^d (r, \theta)
= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}} \frac{\partial S}{\partial \rho}(t, \rho, \theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t, \rho, \theta))e^{ir\rho - iS(t, \rho, \theta)} \tilde{u}_{r, \theta, t}^d (\rho, \theta) d\rho \right)
\]

We consider integration over \(\geq \Xi_{\theta, i} (r) + \frac{j}{d}\) only (the other part is similar to prove). (28) implies

\[
\partial_{\rho} f_{r, \theta, t}(\rho) \leq -C < 0,
|\partial_{\rho}^j f_{r, \theta, t}(\rho)| \leq Ct^{-\varepsilon}, j \geq 2
\]
in this region. Let \(y \mapsto h_{r, \theta, t}(y)\) be the inverse of \(\rho \mapsto f_{r, \theta, t}(\rho)\). Then

\[
|\partial_j h_{r, \theta, t}(y)| \leq C,
|\partial_{\rho}^j h_{r, \theta, t}(y)| \leq Ct^{-\varepsilon}, j \geq 2.
\]

By a change of the variables we obtain

\[
| \int_{\Xi_{\theta, i}(r) + \frac{j}{d}}^{\infty} \tilde{h}(\frac{\partial S}{\partial \rho}(t, \rho, \theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t, \rho, \theta)) e^{ir\rho - iS(t, \rho, \theta)} \tilde{u}_{r, \theta, t}^d (\rho, \theta) d\rho |
= | \int e^{ir\rho} \tilde{h}(\frac{\partial S}{\partial \rho}(t, h_{r, \theta, t}(y), \theta), \theta, h_{r, \theta, t}(y), -\frac{\partial S}{\partial \theta}(t, h_{r, \theta, t}(y), \theta)) \tilde{u}_{r, \theta, t}^d (h_{r, \theta, t}(y), \theta) |
| h_{r, \theta, t}'(y)| dy |
\leq Ct^{-N}.
\]

We can show the same kind of estimations for
\(\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) e^{-iS(t, D_r, \theta)} u_{r, \theta, t}^d (r, \theta)\). We have ended the proof of (31).
Now we show (32). (28) implies that there exists $\tilde{J}$ such that $\frac{1}{t} \frac{ds}{\partial \rho} \in \tilde{J} \subset J$ for large enough $t$. Thus the absolute value of the derivative of $\rho \mapsto (r \rho - S(t, \rho, \theta))/(r + |t|)$ is bounded below for $\xi \notin J$, large enough $t$, and $\rho \in \text{supp} \hat{u}$. Thus we can apply the non-stationary phase method and obtain (32). □

**Proof of (29), partial isometry, and intertwining property.** First we consider the short-range terms. On $\tilde{M}_0$,

$$P_0 J - JP_0 + V^L J - JV^L = O(r^{-\frac{m}{2}} r^{-1-\varepsilon}) \partial_r \partial_\theta + O(r^{-\frac{m}{2}} r^{-2}) \partial_\theta^2$$

$$+ \sum_j O(r^{-\frac{m}{2}} r^{-1-\varepsilon}) \partial_j^r.$$

These terms can be treated as a short range perturbation of $(1-\varepsilon) = \mu_S$ type. Hence on $\tilde{M}_0$, $P_0 J - JP_0 + V^L J - JV^L$ is a finite sum of terms of the form $v_{j,\alpha}(r, \theta) r^{-\frac{m}{2}} D_j^r \partial_\theta^j$ where $v_{j,\alpha}$ satisfy

$$\int_{\mathbb{R}^+ \times \mathbb{U}_J} |v_{j,\alpha}(x)|^2 (x)^{-M} G(x) dx < \infty,$$

$$\int_1^\infty \left( \int_{(\rho, \theta) \in J \times U} |v_{j,\alpha}(t \rho, \theta)|^2 d\rho d\theta \right)^{1/2} t^{(1-\varepsilon)} |\alpha| dt < \infty,$$

for some neighborhood $J \times U$ of almost every $(\rho_0, \theta_0) \in \mathbb{R} \times \partial M$. We assume $\text{supp} \hat{u} \subset J \times U$.

We consider the differential operators with respect to $\theta$-variable. $\partial_\theta e^{-iS(t, D_r, \theta)}$ yields $(\partial_\theta S)(t, D_r, \theta)$ terms which increase as $t^{1-\varepsilon}$. Hence, as in the long-range case, the inequalities (32) implies

$$|D_j^r \partial_\theta^j e^{-iS(t, D_r, \theta)} u(r, \theta)| = \frac{1}{2\pi} \int_{\mathbb{R}} \partial_\theta^j e^{i(r \rho - S(t, \rho, \theta))} \rho^j \hat{u}(\rho, \theta) d\rho \leq C t^{-\frac{1}{2} + |\alpha|(1-\varepsilon)} (34)$$

for $(\xi, \theta) \in J \times U$, and

$$|\partial_j^r \partial_\theta^j e^{-iS(t, D_r, \theta)} u(r, \theta)| \leq C_N(1+|r|+|t|)^{-N} (35)$$

for any $N$ for $(\xi, \theta) \notin J \times U$. Thus we obtain for such $v_{j,\alpha}$

$$\|v_{j,\alpha}(r, \theta) r^{-\frac{m}{2}} D_j^r \partial_\theta^j e^{-iS(t, D_r, \theta)} u\|_{\mathcal{H}} \in L^1_t(1, \infty),$$

which proves

$$\|[(P_0 J - JP_0) + V^L J + [V^L J - JV^L] e^{-iS(t, D_r, \theta)} u]\|_{\mathcal{H}} \in L^1_t(1, \infty).$$

We have proved the existence of the modified wave operators. (31), (32), (34), and (35) also show that $W_\pm$ are partial isometries from $\mathcal{H}_f$ into $\mathcal{H}_f$.

The intertwining property follows from

$$\text{s-lim}_{t \to +\infty} (e^{-iS(t+s, D_r, \theta)} - e^{-iS(D_r, \theta, s)} e^{-isP_f}) = 0 \quad (36)$$

which can be proved using (28) and the dominated convergence theorem. The proof of the theorem is complete. □
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