COUNTEREXAMPLES TO HEDETNIEMI’S CONJECTURE

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Abstract. The chromatic number of $G \times H$ can be less than the minimum of the chromatic numbers of finite simple graphs $G$ and $H$.

The tensor product $G \times H$ of finite simple graphs $G$ and $H$ has vertex set $V(G) \times V(H)$, and pairs $(g, h)$ and $(g', h')$ are adjacent if and only if $\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$. One can easily see that $\chi(G \times H) \leq \chi(G)$ because a proper coloring $\Psi$ of the graph $G$ can be lifted to the coloring $(g, h) \mapsto \Psi(g)$ of $G \times H$. Similarly, a proper coloring of $H$ leads to a proper coloring of $G \times H$ with the same number of colors, so we get

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}.$$  \hfill (H)

The classical conjecture of S. T. Hedetniemi \cite{hedetniemi1973} posited the equality for all $G$ and $H$. More than 50 years have passed since the conjecture appeared, and it keeps attracting serious attention of researchers working in graph theory and combinatorics; we mention four exhaustive survey papers \cite{golumbic2002, hugon2013, randerath2003, zhou2013} for more detailed information on the topic. Here, we briefly recall that Hedetniemi’s conjecture was proved in many special cases, including graphs with chromatic number at most four \cite{chung1983}, graphs containing large cliques \cite{chung1989, faudree1991}, circular graphs and products of cycles \cite{fouquet1985}, and Kneser graphs and hypergraphs \cite{fouquet1987}. The conjecture gave an impetus to the study of multiplicative graphs, which remains remarkably active and important in its own right \cite{chung1983, battaglia2002, kubjas2018}. A generalization of Hedetniemi’s conjecture to fractional chromatic numbers turned out to be true \cite{cohen2007}, but the version with directed graphs is false \cite{fleischner2002}, as well as the one with infinite chromatic numbers \cite{cohen2002, fleischner2002}. We show that the inequality (H) can be strict for finite simple graphs.

A standard tool in the study of Hedetniemi’s conjecture is the concept of the \emph{exponential graph} as introduced in \cite{chung1983}. Let $c$ be a positive integer, and let $\Gamma$ be a finite graph that we allow to contain loops; the graph $E_c(\Gamma)$ has all mappings $V(\Gamma) \to \{1, \ldots, c\}$ as vertices, and two distinct mappings $\varphi, \psi$ are adjacent if, and only if, the condition $\varphi(x) \neq \psi(y)$ holds whenever $\{x, y\} \in E(\Gamma)$. The relevance of $E_c(\Gamma)$ to the problem is easy to see because the graph $\Gamma \times E_c(\Gamma)$ has the proper $c$-coloring $(h, \psi) \mapsto \psi(h)$. The idea of our approach lies in the fact that the proper $c$-colorings of $E_c(\Gamma)$ become quite well-behaved if the graph $\Gamma$ is fixed and $c$ gets large; let us proceed to technical details and exact statements. A basic result in \cite{chung1983} tells that the constant mappings form a $c$-clique in $E_c(\Gamma)$, which means that these mappings get different colors in a proper $c$-coloring. So a relabeling of colors can turn any proper $c$-coloring $\Psi : E_c(\Gamma) \to \{1, \ldots, c\}$ into a \emph{suited} one, in which a color $i$ is assigned to the constant mapping sending every vertex of $\Gamma$ to $i$.

**Observation 1.** If $\Psi$ is a suited proper $c$-coloring of $E_c(\Gamma)$, then $\Psi(\varphi) \in \text{Im} \varphi$. 


Proof. The mapping $\varphi$ is adjacent to the constant mapping $\{v \rightarrow j\}$ for any $j$ not in $\text{Im} \varphi$, so $\varphi$ cannot get colored with such a $j$. \hfill $\square$

Claim 2. Consider a graph $\Gamma$ with $|V(G)| = n$ and a suited proper $c$-coloring $\Psi$ of $\mathcal{E}_c(\Gamma)$. Then there is a vertex $v = v(\Psi)$ of $\Gamma$ such that $c - \sqrt[3]{n^2c^{n-1}}$ color classes $\Psi^{-1}(b)$ are $v$-robust, which means that, for any $\varphi_b \in \Psi^{-1}(b)$, there is a $w \in N(v)$ satisfying $\varphi_b(w) = b$, where $N(v)$ stands for the closed neighborhood of $v$ in $\Gamma$.

Proof. For any color $b$ and any vertex $u \in V(\Gamma)$, we define $I(u, b)$ as the set of all $\varphi \in \Psi^{-1}(b)$ that satisfy $\varphi(u) = b$. According to Observation 1 we can find $\psi_{ub} \in I(u, b)$ for all $u \in \Gamma$, and $\psi_{ub}$ is an only vertex coloring $\Psi$ of $\Gamma$. For any $w \in N(u)$ satisfying $\varphi(w) = b$, if $\psi_{ub}$ is adjacent to the constant mapping $\psi_{ub}$, then $\psi_{ub}$ cannot equal those colors $b$, then we are done. Conversely, we can define more than $n^3c^{n-1}$ mappings $\varphi : V(\Gamma) \rightarrow \{1, \ldots, c\}$ for which the value of $\varphi$ on a vertex $w$ does not equal those colors $b$ for which $I(w, b)$ is large. None of these mappings belongs to a large class $I(u, b)$, but the non-large classes are too small to cover all of them. \hfill $\square$

Now we are ready to proceed with counterexamples. For a simple graph $G$, we define the graph $\Gamma_G$ by adding the loops to all the vertices, and the strong product $G \boxtimes K_q$ as the graph with vertex set $V(G) \times \{1, \ldots, q\}$ and edges between $(u, i)$ and $(v, j)$ when, and only when, $(u, v) \in E(G)$ or $(u = v) & (i \neq j)$.

Claim 3. Let $G$ be a finite simple graph with finite girth $\geq 6$. Then, for sufficiently large $q$, one has $\chi(\mathcal{E}_c(G \boxtimes K_q)) > c$ with $c = \lceil 3.1q \rceil$.

Proof. The restriction of a suited proper coloring $\Lambda : \mathcal{E}_c(G \boxtimes K_q) \rightarrow \{1, \ldots, c\}$ to the mappings that are constant on the cliques $\{g\} \times K_q \subset G \boxtimes K_q$ is a proper coloring $\Psi : \mathcal{E}_c(\Gamma_G) \rightarrow \{1, \ldots, c\}$ up to the identification of every such clique with $g$. We find a vertex $v = v(\Psi) \in V(G)$ as in Claim 2 and define the clique $\mathcal{M} = \{\mu_{q+1}, \ldots, \mu_c\}$ in $\mathcal{E}_c(G \boxtimes K_q)$ by setting, for all $i \in \{1, \ldots, q\}$ and $t \in \{q + 1, \ldots, c\}$,

\begin{align*}
(1.1) & \quad \mu_t(g, i) = i \quad \text{for all} \quad g \in V(G) \text{ satisfying } \text{dist}(v, g) \in \{0, 2\}, \\
(1.2) & \quad \mu_t(g, i) = q + i \quad \text{for all} \quad g \in V(G) \text{ satisfying } \text{dist}(v, g) = 1, \\
(1.3) & \quad \mu_t(g, i) = t \quad \text{for all} \quad g \in V(G) \text{ satisfying } \text{dist}(v, g) \geq 3.
\end{align*}

Due to the assumption on the girth of $G$, no pair of vertices defined in (1.1) and (1.2) can be adjacent in $G \boxtimes K_q$ and monochromatic at the same time; the condition (1.3) uses different colors for different $t$, and these colors are also different from those of the neighboring vertices dealt with in (1.1). Therefore, $\mathcal{M}$ is indeed a clique and requires $c - q \geq 2.1q$ colors. Using the pigeonhole principle, one finds a $\tau \in \{q+1, \ldots, c\}$ such that $\Lambda(\mu_\tau) \notin \{1, \ldots, 2q\}$, and due to Observation 1 we have $\tau = \Lambda(\mu_c)$. Further, it is only $o(q)$ classes that are not $v$-robust with respect to $\Psi$ in the terminology of Claim 2 so we can find a $v$-robust class $\sigma \notin \{1, \ldots, 2q, \tau\}$. Finally, we note that the mapping $\nu : G \boxtimes K_q \rightarrow \{1, \ldots, c\}$ defined as, for all $i$,
(2.1) \( \nu(g, i) = \tau \) for all \( g \in V(G) \) in the closed neighborhood \( N(v) \),
(2.2) \( \nu(g, i) = \sigma \) for all \( g \in V(G) \) satisfying \( \text{dist}(v, g) \geq 2 \),
is adjacent to \( \mu_{\tau} \) in \( E_c(G \boxtimes K_q) \). Since \( \sigma \) is \( \nu \)-robust, we cannot have \( \Lambda(\nu) = \sigma \) by Lemma 2, but rather we have \( \Lambda(\nu) = \tau \) according to Observation 1. So we have

\[
\Lambda(\nu) = \Lambda(\mu_{\tau}),
\]
which is a contradiction. \( \square \)

The classical paper [4] proves the existence of graphs with arbitrarily large girth and fractional chromatic number; so we can find a graph \( G \) of girth at least 6 that satisfies \( \chi_f(G) > 3 \). We set \( c = \lceil 3.1q \rceil \) and pass to sufficiently large \( q \); we immediately get \( \chi(G \boxtimes K_q) \geq q \cdot \chi_f(G) > c \) and also \( \chi(E_c(G \boxtimes K_q)) > c \) by Claim 3.

The equality

\[
\chi((G \boxtimes K_q) \times E_c(G \boxtimes K_q)) = c
\]
follows by standard theory [3] as the mapping \((u, \varphi) \rightarrow \varphi(u)\) is a proper \( c \)-coloring of any graph of the form \( \Gamma \times E_c(\Gamma) \).

References

[1] S. Burr, P. Erdős and L. Lovász, On graphs of Ramsey type, *Ars Combin.* 1 (1976) 167-190.
[2] D. Duffus, B. Sands, R. E. Woodrow, On the chromatic number of the product of graphs, *J. Graph Theor.* 9 (1985) 487-495.
[3] M. El-Zahar, N. Sauer, The chromatic number of the product of two 4-chromatic graphs is 4, *Combinatorica* 5 (1985) 121-126.
[4] P. Erdős, Graph theory and probability, *Canadian J. Math.* 11 (1959) 34-38.
[5] R. Häggkvist, P. Hell, D. J. Miller, V. Neumann Lara, On multiplicative graphs and the product conjecture, *Combinatorica* 8 (1988) 63-74.
[6] H. Hajiabolhassan, F. Meunier, Hedetniemi’s conjecture for Kneser hypergraphs, *J. Combin. Theory A* 143 (2016) 42-55.
[7] A. Hajnal, The chromatic number of the product of two \( K_1 \)-chromatic graphs can be countable, *Combinatorica* 5 (1985) 137-140.
[8] S. Hedetniemi, Homomorphisms of graphs and automata, Technical Report 03105-44-T, University of Michigan, 1966.
[9] S. Klavžar, Coloring graph products—a survey, *Discrete Math.* 155 (1996) 135-145.
[10] S. Poljak, V. Rödl, On the arc-chromatic number of a digraph, *J. Combin. Theory B* 31 (1981) 190-198.
[11] N. Sauer, Hedetniemi’s conjecture—a survey, *Discrete Math.* 229 (2001) 261-292.
[12] C. Tardif, Multiplicative graphs and semi-lattice endomorphisms in the category of graphs, *J. Combin. Theory B* 95 (2006) 338-345.
[13] C. Tardif, Hedetniemi’s conjecture, 40 years later, *Graph Theory Notes NY* 54 (2008) 46-57.
[14] A. Rinot, Hedetniemi’s conjecture for uncountable graphs, *JEMS* 19 (2016) 285-298.
[15] M. Valencia-Pabon, J.-C. Vera, Independence and coloring properties of direct products of some vertex-transitive graphs, *Discrete Math.* 306 (2006) 2275-2281.
[16] E. Welzl, Symmetric graphs and interpretations, *J. Combin. Theory B* 37 (1984) 235-244.
[17] M. Wrochna, On inverse powers of graphs and topological implications of Hedetniemi’s conjecture, *J. Combin. Theory B* (2019).
[18] X. Zhu, A survey on Hedetniemi’s conjecture, *Taiwan. J. Math.* 2 (1998) 1-24.
[19] X. Zhu, The fractional version of Hedetniemi’s conjecture is true, *European J. Combin.* 32 (2011) 1168-1175.

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