Quantum correlations preparation assisted by a steering Maxwell demon

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A Maxwell demon can reduce the entropy of a quantum system by performing measurements on its environment. This avoids the disturbance to average energy of the system. We study the preparations of quantum correlations from a thermal qubit and an auxiliary qubit, assisted by a demon who obtains information of the system qubit from measurements on its environment. The demon can affect the postmeasured states of system by choosing different measurements, which establishes the relationships between quantum steering and other correlations in the thermodynamic framework. We present the optimal protocols for creating mutual information, entanglement and Bell-nonlocality, the maximums of which depend monotonically upon the extractable work in a similar process. These maximal correlations are found to relate exactly to the steerable boundary of the system-environment state with maximally mixed marginals. We also present upper bounds of the prepared correlations by utilizing classical environment-system correlation, which can be regarded as steering-type inequalities bounding the correlations created with the aid of classical demons.

I. INTRODUCTION

The connection between thermodynamics and information provides a new angle of view to understand the physical world. In the history of this topic, the Maxwell’s demon, first introduced by Maxwell in 1871 [1], has played an important role. The Maxwell demon is a creature who can reduce the entropy of a system, by observing its microstates, without performing any work on it. Szilárd [2] presented a one-molecule heat engine assisted by a Maxwell demon measuring the (binary) position of the molecule. His model showed for the first time an explicit connection between information and physics that, one can extract work \( W = kT \log 2 \) from the one-molecule system at a temperature \( T \) by using 1 bit information acquired by the demon.

In the field of quantum thermodynamics [3, 4], many quantum versions of the Maxwell demon and Szilárd engine have been presented, to investigate the role of quantumness in thermodynamics and the interplay between quantum information and thermodynamics [5–15]. The definitions of these models rely on the division between the quantum and classical worlds. For instance, in Zurek’s division [6], a quantum demon is the one who can perform global measurements on composite systems, while a classical demon is local. On the other hand, quantum correlations in thermodynamics have gotten a lot of attention, as they are the most profound quantum features and deeply connected to quantum information. The thermodynamic cost and fundamental limitations for preparation of quantum correlations were studied under different conditions [16, 17]. The correlations in turn can be used to enhance the extraction of work [10, 11, 18–21]. The Maxwell demons and Szilárd engines often played key roles in these works, such as the studies of work deficit [18], discord [6] and steering heat engines [10, 11].

Measurements on a quantum system would in general disturb the expectation value of its energy. This actually provides a new paradigm in quantum thermodynamics in which measurement apparatuses are used to fuel engines [8, 22]. That is, Maxwell demons directly measuring quantum systems lack a basic feature of their classical counterparts: acquiring information but without affecting the energy. The difficulty was overcome in the version of quantum Szilárd engine presented by Beyer et al. [10], where a demon obtains the information of a system from measurements on its environment. Their approach connects the thermodynamic task of work extraction with the quantum steering, which is a kind of quantum correlation lying between Bell nonlocality and entanglement [23]. Here, the term steering, introduced by by Schrödinger[24], means that the demon can project the system into different states by choosing her measurements on the environment. Her ability of steering can be convincingly demonstrated, only when the postmeasured states of the system cannot be described by a local-hidden-state (LHS) model. In this case, the state of system and environment is said to have quantum steering from the environment to the system [23], and the demon is termed truly quantum by Beyer et al. [10].

In this work we investigate the process for creating quantum correlations from a thermal qubit and an auxiliary qubit assisted by a Maxwell demon measuring its environment. This is based on the consideration that the work output of the quantum engine desinged by Beyer et al. [10] could serve as the thermodynamic cost of correlations preparation. The information acquired by demon to hence the extractable work can certainly be directly used for creating correlations. Our processes connect the quantum steering with other correlations in the thermodynamic framework. One difference from the scheme in [10] is that, we allow operator to optimize the joint unitary transformations between the thermal qubit and the ancilla, while there are only two pairs of unitaries to choose from in the work extraction of Beyer et al. [10]. For an arbitrary set of observables on the environ-

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ment, we show that the maximums of quantum mutual
information, entanglement and Bell-nonlocality allowed
between the system and ancilla are all monotone increasing
functions of the average length of Bloch vectors in the
postmeasured states for the system qubit. Consequently,
they depend monotonically upon the extractable work.
When the dimension of the Hilbert space of the environ-
ment can be measured by the demon is 2, these maximal
correlations are related exactly to the steerable boundary
of the system-environment state with maximally mixed
marginals. We also present upper bounds of the pre-
pared correlations for unsteering demons. These can be
regarded as steering-type inequalities bounding the cor-
relations created with the aid of classical environment-
system correlation.

In the next section, we present the optimal protocols
of quantum correlations preparation (including the cases
without and with a demon), with brief introductions to
the correlation measures. In Sec. III, we deal with the
case that the demon performs measurements on a two-
dimensional Hilbert space of the environment, to show
the advantage of a quantum demon. Finally a summary
of our results and some outlooks are given in Sec. IV.

II. CORRELATIONS PREPARATION

Suppose that a system qubit $S$ is governed by the
Hamiltonian

$$H_s = -\frac{\omega}{2} \sigma_z.$$  

(1)

with $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ being the third Pauli operator.
Its thermal state is

$$\tau_s = \frac{e^{-\beta H}}{\mathcal{Z}} = \frac{1}{2}(\mathbb{I} + \eta \sigma_z),$$  

(2)

where $\mathcal{Z} = \text{Tr} e^{-\beta H} = 2 \cosh(\beta \omega/2)$ is the partition function,
$\beta = \frac{1}{kT}$ is the inverse temperature, and $\eta = \tanh(\beta \omega/2)$.

We introduce the two participants, Alice and Bob, in
our protocol. Alice is the Maxwell demon who can per-
form measurements on the environment $\mathcal{E}$ of $S$. Here, $\mathcal{E}$
should be understood as a part (a subsystem or a sub-
space of the Hilbert space) of the whole environment
of $S$, which Alice is able to measure. Bob is the operator
manipulating the system qubit and an auxiliary qubit $A$.
The task of Bob is to create quantum correlations from
his two qubits, in an uncorrelated initial state, by apply-
ing a global unitary on them. In this work, we study three
types of correlations between $S$ and $A$: total correlation
measured by quantum mutual information $I$ [25], entan-
glement measured by concurrence $C$ [26] and negativity
$N$ [27], and Bell-nonlocality measured by the maximal
violation of the Clauser-Horne-Shimony-Holt (CHSH) in-
equality $B$ [28].

A. Without the demon

Let us begin with the case without the help of Alice as
a preview. We consider an arbitrary initial state of the
system qubit

$$\rho_s = \frac{1}{2} (\mathbb{I} + \mathbf{r} \cdot \mathbf{\sigma}),$$  

(3)

with the Bloch vector $|\mathbf{r}| = r \in [0, 1]$ and $\mathbf{\sigma}$ being the
vector of Pauli matrices. Specifically, in the following,
we show all the maximal correlations of

$$\zeta_{sa} = U \rho_s \otimes \rho_a U^\dagger,$$  

(4)

among all the global unitaries $U$ and initial states of $A \rho_a$,
are monotonic increasing functions of $r$. To reach these
maximums, the initial state of $A$ can be chosen as $\rho_a =
|0\rangle\langle 0|$, and the global unitaries $U$ can be implemented
in two steps: (1) a local unitary diagonalizing $\rho_s$ into
$\frac{1}{2} (\mathbb{I} + r \sigma_z)$; (2) a global unitary $U_0$ such that
$U_0 |00\rangle = |\psi_+\rangle$ and $U_0 |01\rangle = |01\rangle$ for entanglement while
$U_0 |01\rangle = |\psi_-\rangle$ for mutual information and Bell-nonlocality. Here
$|\psi_{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ are two of the Bell states.

Mutual information.-- The quantum mutual informa-
tion of a bipartite state $\rho_{a\beta}$ is defined as

$$I(\rho_{a\beta}) = H(\rho_a) + H(\rho_\beta) - H(\rho_{a\beta}),$$  

(5)

where $\rho_a = \text{Tr}_\beta \rho_{a\beta}$ and $\rho_\beta = \text{Tr}_a \rho_{a\beta}$ are the reduced
states of subsystems $\alpha$ and $\beta$ respectively, and
$H(\rho) = -\text{Tr} (\rho \log \rho)$ is the von Neumann entropy of $\rho$.
It measures the total correlation between the two sub-
systems, which does not distinguish classical correlation
from quantum one [29].

The total entropy $H(\zeta_{sa}) = H(\rho_s) + H(\rho_a)$ is conserved
under the global unitary $U$, and the local entropies are
upper bounded by their dimension as $H(\zeta_a), H(\zeta_s) \leq
\log 2$. These lead to the maximum of the mutual informa-
tion as

$$\max_{\{U, \rho_s\}} I(\zeta_{sa}) = 2 \log 2 - H(\rho_s) = 2 \log 2 - h(r),$$  

(6)

where $h(r) = -\frac{1 + r}{2} \log \frac{1 + r}{2} - \frac{1 - r}{2} \log \frac{1 - r}{2}$. It can be
achieved by choosing $\rho_a = |0\rangle\langle 0|$ and transforming the
whole state into $\zeta_{sa} = \frac{1}{\sqrt{2}} |\psi_+\rangle \langle \psi_+| + \frac{1 - r}{\sqrt{2}} |\psi_-\rangle \langle \psi_-|$

Entanglement.-- Concurrence and negativity are the two
most widely used measures of entanglement in two-
qubit states. The former leads to a computable formula
for entanglement of formation in the two-qubit case [26].
The latter is closely related to partial transpose crite-
ron of entanglement [27]. Both of them do not increase
on average, which is the condition of convexity usually
satisfied by known entanglement measures [30]. Based
on the convexity, one can easily draw a conclusion that
the maximal concurrence and negativity can be simulta-

neously reached by choosing \( \rho_a = |0\rangle\langle 0| \). Without loss of generality, we suppose \( \rho_a = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| \) with \( p_{0,1} \in [0,1] \) and \( p_0 + p_1 = 1 \), and only show the computation procedure for \( C \) as

\[
C(\zeta_{sa}) \leq p_0 C(U\rho_s \otimes |0\rangle\langle 0|U^\dagger) + p_1 C(U\rho_s \otimes |1\rangle\langle 1|U^\dagger) \\
\leq p_0 \max_{U} C(U\rho_s \otimes |0\rangle\langle 0|U^\dagger) \\
+ p_1 \max_{U} C(U\rho_s \otimes |1\rangle\langle 1|U^\dagger) \\
= \max_{U} C(U\rho_s \otimes |0\rangle\langle 0|U^\dagger).
\]

Then, both of the measures of entanglement for the rank 2 state \( U\rho_s \otimes |0\rangle\langle 0|U^\dagger \) can be maximized among global unitaries \( U \) by \( \zeta_{sa} = U\rho_s \otimes |0\rangle\langle 0|U^\dagger = \frac{1+r^2}{4}\langle \psi_+|\psi_+ \rangle + \frac{1-r^2}{4}|0\rangle\langle 0| + \frac{r^2}{4}|1\rangle\langle 1| \) and

\[
\max_{U,\rho_s} C(\zeta_{sa}) = \frac{1}{2}(1+r),
\]

\[
\max_{U,\rho_s} \mathcal{N}(\zeta_{sa}) = \frac{1}{2}\left[\sqrt{2(1+r^2)} - 1 + r\right].
\]

It is interesting that, the final state \( \zeta_{sa} \) above has the minimal negativity for a fixed concurrence [33].

**Bell-nonlocality.**— Bell-nonlocality exists in the states whose outcomes of local measurements do not admit by any local-hidden-variable models, which can be witnessed by the violation of Bell-type inequalities. We adopt the maximal quantum violation of the CHSH inequality, \( B \), as the degree of Bell-nonlocality for two-qubit systems. For a two-qubit state \( \rho \) with spin correlation matrix \( T \), whose elements \( T_{ij} = \text{Tr}(\sigma_i \otimes \sigma_j \rho) \) \((i,j = 1,2,3)\), \( B(\rho) = 2\sqrt{t_1^2 + t_2^2} \) [34]. Here, \( t_1^2 \) and \( t_2^2 \in [0,1] \) are the two largest eigenvalues of \( TT^\dagger \). The amount \( B(\rho) > 2 \) demonstrates the Bell-nonlocality of \( \rho \).

In the region of nonlocality, for a fixed linear entropy \( S_L(\rho) = \frac{d}{2}(1 - \text{Tr} p^2) \), \( B(\rho) \) is maximized by the rank 2 states mixed by any two of the Bell states [35]. And, the maximal \( B \) decreases with \( S_L \). Therefore, for a given \( \rho_s \), the maximal Bell-nonlocality can be created is

\[
\max_{\{U,\rho_a\}} B(\zeta_{sa}) = 2\sqrt{(1+r^2)},
\]

which is reached by \( \rho_a = |0\rangle\langle 0| \) and \( \zeta_{sa} = \frac{1+r}{4}|\psi_+\rangle\langle\psi_+| + \frac{1-r}{4}|\psi_-\rangle\langle\psi_-| \).

**B. Assisted by the demon**

Now, we introduce the procedure for quantum correlations preparation assisted by the demon, which is shown in Fig. 1. Alice, the demon, and Bob share the whole state of \( E \) and \( S, \rho_{sa} \). The reduced state of \( S \) is the thermal state \( (2) \); i.e. \( \rho_s = \text{Tr}_{s\rho_{sa}} = \tau_s \). Suppose \( \{M^n, n = 1,2,\ldots\} \) is the set of observables Alice can measure on \( E \), and \( M^n_k \) with \( k = 0,1,\ldots \) denote the POVM elements of \( M^n \). Bob generates a value of \( n \) with a probability \( q_n \) and sends it to Alice. Then, Alice performs \( M^n \) on \( E \). The probability of the outcome \( k \) is \( p^n_{k} = \text{Tr}(\mathbb{1}_e \otimes M^n_k \rho_{sa}) \), and the corresponding collapsed state of the system is \( \rho^n_k = \text{Tr}_e(\mathbb{1}_e \otimes M^n_k \rho_{sa})/p^n_{k} \). Each \( M^n \) leads to a decomposition of the thermal state as \( \tau = \sum_k p^n_k \rho^n_k \). Alice informs Bob her outcome. According to the outcome, Bob performs a local unitary \( U^n_k \) on \( S \) and the auxiliary qubit \( A \) in his hands, which are in the initial state \( \tau_s \otimes \rho_a \). The final state of \( S \) and \( A \) prepared by Bob with the aid of Alice is

\[
\xi_{sa} = \sum_{n,k} q_n p^n_k \left(U^n_k \rho^n_k \otimes \rho_a U^n_k \right).
\]

Here, we assume that another observer, receiving the two-qubit state prepared by Bob, knows in advance the details of the procedure but is ignorant of the values of \( n \) and \( k \) in a specific run.

The above procedure can also act as the scenario of work extraction, in which \( A \) plays the role of a work storage system [10]. The maximum of average extractable work can be derived by optimizing each of the terms \( U^n_k \rho^n_k \otimes \rho_a U^n_k \) individually. In addition, the result of an global unitary, preserving total energy of \( S \) and \( A \), on \( S \) is always equivalent to a local rotation of its Bloch vector [36]. Hence, the maximal work can be directly obtained by rotating all the Bloch vectors \( \tau^n_k \) of the postmeasured states \( \rho^n_k \) into \( Z \) axis, as

\[
\max_{\{U^n_k, \rho_a\}} W(\xi_{sa}) = \frac{\omega}{2}(\bar{r} - \eta),
\]

where \( \bar{r} = \sum_{n,k} q_n p^n_k \tau^n_k \) is the average length of \( \tau^n_k \).

In the following, we show all the maximal prepared quantum correlations of \( \xi_{sa} \) are monotonic increasing functions of \( \bar{r} \). These maximas can be reached by opti-
mizing each of the terms $U_k^r \rho_k^r \otimes \rho_{k'} U_k^{r'}$ in the ways given in the above subsection. That it, one chooses $\rho_a = |0\rangle \langle 0|$ and $U_k^r$, and transforms $\rho_k^r \otimes |0\rangle \langle 0|$ into the mixtures of $|\psi_+\rangle$ and $|\psi_-\rangle$ for maximal $I$ and $B$ while into the mixtures of $|\psi_+\rangle$ and $|01\rangle$ for maximal entanglement. This is nontrivial, as these measures of quantum correlations are nonlinear functions of the state, while the inner energy (or work) is linear.

**Mutual information.** The minimum of the entropy for $\xi_{sa}$ can be achieved by setting: (1) $\rho_a = |0\rangle \langle 0|$; (2) the elements are diagonal in the same set of basis as $U_k^r \rho_k^r \otimes |0\rangle \langle 0| U_k^{r'} = \sum_i \lambda_i^{n,k} |\phi_i\rangle \langle \phi_i|$ with $\lambda_0^{n,k} \geq \lambda_1^{n,k}$. The first point can be easily proved by using the concavity property of the von Neumann entropy [25], as the calculation of concurrence in (7). Here, we omit the procedure for brevity. The second point can be derived based on the Lemma 1 in Appendix A. Namely, we set $X_k^a = q_n \rho_k^a (1 - U_k^r \rho_k^r \otimes |0\rangle \langle 0| U_k^{r'})$ and $X = \sum_{n,k} X_k^a = 1 - \xi_{sa}$. The von Neumann entropy for the two-qubit state can be written as

$$\mathcal{H}(\xi_{sa}) = 3 - \sum_{n=2}^{+\infty} \frac{1}{n(n-1)} \text{Tr}(X^n).$$

(13)

All the terms $\text{Tr}(X^n)$ can be simultaneously maximized by the above condition (2).

Then, the entropies of two reduced states of $\xi_{sa}$ can be maximized by transforming $|\phi_0\rangle$ and $|\phi_1\rangle$ into the two Bell states $|\phi_+\rangle$ and $|\phi_-\rangle$ without affecting entropy for $\xi_{sa}$. These lead to the maximal mutual information as

$$\max_{\{U_k^r, \rho_a\}} I(\xi_{sa}) = 2 \log 2 - h(\bar{r}).$$

(14)

**Entanglement.** One can still use the procedure in (7) to restrict in the initial state of $A$ as $\rho_a = |0\rangle \langle 0|$. The convexity of concurrence further gives

$$C(\xi_{sa}) \leq \sum_{n,k} q_n \rho_k^a C(U_k^r \rho_k^r \otimes |0\rangle \langle 0| U_k^{r'}).$$

(15)

When the states $U_k^r \rho_k^r \otimes |0\rangle \langle 0| U_k^{r'} = \frac{1 + q_n}{2} |\psi_+\rangle \langle \psi_+| + \frac{1 - q_n}{2} |\psi_-\rangle \langle \psi_-|$, each concurrence of the right hand side reaches its maximum, and meanwhile, the equality holds. Then, the maximum of concurrence is given by

$$\max_{\{U_k^r, \rho_a\}} C(\xi_{sa}) = \frac{1}{2} (1 + \bar{r}).$$

(16)

These choices simultaneously maximize the negativity as

$$\max_{\{U_k^r, \rho_a\}} N(\xi_{sa}) = \frac{1}{2} \left[ \sqrt{2(1 + \bar{r}^2)} - 1 + \bar{r} \right].$$

(17)

We give the details in Appendix B.

**Bell-nonlocality.** The optimization of the Bell-nonlocality measured by $\mathcal{B}$ can be solved by minimizing the linear entropy again. The minimum of $\mathcal{S}_L(\xi_{sa})$ is achieved under the same two conditions above for the von Neumann entropy, which can be proved by using a similar procedure. Then, the state $\xi_{sa}$ is rank 2. By transforming its eigenstates into $|\phi_+\rangle$ and $|\phi_-\rangle$, Bob obtains the maximal Bell-nonlocality assisted by the demon as

$$\max_{\{U_k^r, \rho_a\}} \mathcal{B}(\xi_{sa}) = 2 \sqrt{(1 + \bar{r}^2)}.$$

(18)

### III. CLASSICAL AND QUANTUM DEMONS

The maximal prepared correlations are monotonic increasing functions of the average length of the Bloch vectors in postmeasured states of the system qubit. Therefore, the length change of the Bloch vector, $\Delta r = \bar{r} - \eta$, measures the quantum correlations enhanced by the participation of the demon. It also indicates the corresponding relationships between the correlations and extractable work in (12). In this part, we adopt the increase in the created entanglement

$$\Delta C = \frac{1}{2} (\bar{r} - \eta),$$

(19)

which is proportional to $\Delta r$, as a figure of merit of the demon.

#### A. A two-dimensional environment

To succinctly show the different effects between a quantum demon and a classical one, we consider that the sub-system of the whole environment in Alice’s hands is two-dimensional. That is, $E$ is equivalent to a qubit. And, we focus on the case of Alice’s local von Neumann measurements. A general form of the initial state for $S$ and $E$ can be expressed as

$$\rho_{se} = \frac{1}{4} (\mathbb{1} + \eta \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma \cdot \vec{b} + \sum_{ij} T_{ij} \sigma_i \otimes \sigma_j),$$

(20)

where $\vec{b}$ is the Bloch vector on Alice’s side and $T$ is the $3 \times 3$ spin correlation matrix. An observable of Alice can be labelled by a unit vector as $M^{\vec{a}} = \vec{a} \cdot \vec{\sigma}$, and its elements are two projectors

$$M_k^{\vec{a}} = \frac{1}{2} (\mathbb{1} + k \vec{n} \cdot \vec{\sigma})$$

(21)

corresponding to the outcomes $k = \pm 1$. After her measurements, the system qubit is left in the unnormalized state

$$\rho_k^a = \frac{1}{2} \left[ \mathbb{1} + \frac{\eta \sigma_z + k (T \vec{a}) \cdot \vec{\sigma}}{1 + k \vec{b} \cdot \vec{n}} \right],$$

(22)

with the measurement probability $p_k^a = \frac{1}{2} (1 + k \vec{b} \cdot \vec{n})$. 

To show the advantage of a quantum Maxwell demon, she should have at least two observables. This is because the optimal results with one measurement $M^\mathbf{a}$ can always be achieved by the classically correlated state
\[
\rho_{sc} = \frac{1 + \eta}{2}|0\rangle\langle 0| \otimes M^\mathbf{a}_{+1} + \frac{1 - \eta}{2}|1\rangle\langle 1| \otimes M^\mathbf{a}_{-1}. \tag{23}
\]
In this case, the average length of the Bloch vectors of postmeasured states achieves $\mathbf{r} = 1$. For the case with two observables onto the two the general state (20), denoted by the two unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, chosen by Bob with equal probabilities, one can directly obtain the entanglement enhanced by Alice is
\[
\Delta C = -4\eta + \sum_{i=1,2; k = \pm 1} |\mathbf{n} + kT\mathbf{n}_i|, \tag{24}
\]
where $\mathbf{n} = (0, 0, \eta)$.

Although the final states in the work extraction studied by Beyer et al. [10] are also in the form of (11), the key difference between our procedure and theirs is that our joint unitaries are optimal to prepare quantum correlations (or to extract work), while the operations of Bob who handles the qubit work medium are restricted to two fixed pairs of unitaries. We adopt the state in the case study of Beyer et al. [10] as an example to show the variation caused by the difference, as well as the advantage of a quantum correlated state. That is
\[
\rho(p, \eta) = p|\Psi\rangle\langle\Psi| + (1 - p)\rho_{cl}, \tag{25}
\]
with the two components
\[
|\Psi\rangle = \sqrt{\frac{1 + \eta}{2}}|00\rangle + \sqrt{\frac{1 - \eta}{2}}|11\rangle
\]
and
\[
\rho_{cl} = \frac{1 + \eta}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1 - \eta}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|.
\]
The pure state $|\Psi\rangle$ is fully quantum. An arbitrary $M^\mathbf{a}$ can collapse the system into a pure state with a unit Bloch vector. The mixed state $\rho_{cl}$ is classically correlated. Only $M^\mathbf{a}$ $= \sigma_z$ leads to a unit Bloch vector. Choosing $M^\mathbf{a}_1 = \sigma_x$ and $M^\mathbf{a}_2 = \sigma_z$, one can directly derive the enhanced concurrence $\Delta C = |1 - 2\eta + \sqrt{\eta^2 + p^2(1 - \eta^2)}|/4$. We convert it into the work in (12) and plot it together with the extractable work in the model of Beyer et al. (with the choice $c = 1$) [10] in Fig. 2. For fixed $p$ and $\eta$, our scheme is always more powerful than the one in [10]. The amounts of extractable work in the two schemes increase with the proportion of the quantum correlated state. The two schemes tend to be same since the two pairs of unitaries chosen by Beyer et al. [10] are according to the postmeasured states of $|\Psi\rangle$.

![Extractable Work (enhanced concurrence) of the state $\rho(p, \eta)$ in two schemes.](image)

**FIG. 2:** Extractable work (enhanced concurrence) of the state $\rho(p, \eta)$ in two schemes. The solid lines are equivalent to our enhanced concurrence $\Delta C$, and the dashed lines are for the results in [10]. The parameter $\eta = 0.2$, $\eta = 0.5$, and $\eta = 0.8$ from top to bottom.

**B. Classical-quantum boundary**

The operator, Bob, is assumed to know the form of state $\rho_{sc}$ to perform the optimal operations. The triple $(\eta, T, \mathbf{b})$ is his a priori knowledge of the system qubit and its environment. Alice’s measurements and outcomes convert these information to the system $S$, as shown in the form of $\rho_{sc}^T$ in (22), which leads to the enhancement of created correlations.

The case with the bloch vectors $\eta = 0$ and $\mathbf{b} = 0$ is notable. That is, the system $S$ is in the high temperature limit $T \rightarrow +\infty$ and the local state of $E$ is completely unknown. Such a $\rho_{sc}$ is equivalent to a Bell diagonal state [37–39] under some local unitary transformations. When the measurement directions $\mathbf{n}$ are completely randomised on the unit sphere, the entanglement enhanced by Alice is given by
\[
\Delta C = \int \frac{1}{8\pi} |T\mathbf{n}|d\mathbf{n}, \tag{26}
\]
where the integral is over the unit sphere and $d\mathbf{n}$ is the surface element. A critical value, $\Delta C_c = 1/4$, corresponds exactly to the necessary and sufficient condition for steerability of the state $\rho_{se}$ [39–41]. Consequently, the amounts of the quantum correlations prepared with the assistance of the demon can serve as criterions and measures for steerability of the state $\rho_{sc}$ in this case.

We now revisit the situation with a general $\rho_{sc}$ and two observables to reveal the quantumness of the demon. The quantumness, characterized by Alice’s ability of steering, is demonstrated by the difference between the postmeasured states of $S$ corresponding different local measurements on $E$. Therefore, it is reasonable to assume that Bob requires the two measurement directions to satisfy $T\mathbf{n}_1 \perp T\mathbf{n}_2$, according to his a priori knowledge of the matrix $T$. That is, the changes of direction of the Bloch vector of $S$ affected by Alice’s measurements are perpen-
dicular to each other. Under this condition, the optimal two measurements of the $\rho(p, \eta)$ in Fig. 2 to create the maximal quantum correlations are exactly $M_{\tilde{n}_1} = \sigma_x$ and $M_{\tilde{n}_2} = \sigma_x$. Below we present an upper bound on the enhanced concurrence under this condition for classical (unsteerable) demons.

A LHS model admitted by a state $\rho_{se}$ can be identified with a hidden Bloch vector $\tilde{\lambda}$ with a distribution $\omega(\tilde{\lambda})$, and a function $f(\tilde{n}, \tilde{\lambda}) \in [-1, 1]$ of $\tilde{\lambda}$ and the measurement direction $\tilde{n}$ [39]. They satisfy

$$\int \omega(\tilde{\lambda}) (1 + \tilde{\lambda} \cdot \tilde{\sigma}) d\tilde{\lambda} = 1 + \eta \sigma_z,$$

$$\int \omega(\tilde{\lambda}) f(\tilde{n}, \tilde{\lambda}) (1 + \tilde{\lambda} \cdot \tilde{\sigma}) d\tilde{\lambda} = (\tilde{n} \cdot \tilde{b}) 1 + (T\tilde{n}) \cdot \tilde{\sigma},$$

(27)

where the integral is over the Bloch sphere and $d\tilde{\lambda}$ is the surface element. We denote $T\tilde{n}_1 = \tilde{\alpha}_1$ and $T\tilde{n}_2 = \tilde{\alpha}_2$. The average length of Bloch vectors in (24) satisfies

$$\bar{r} \leq \frac{1}{2} \left( \sqrt{\eta^2 + \alpha_1^2} + \sqrt{\eta^2 + \alpha_2^2} \right),$$

(28)

and the two changes

$$\alpha_i^2 = \int \omega(\tilde{\lambda}) f(\tilde{n}, \tilde{\lambda}) (\tilde{\lambda} \cdot \tilde{\alpha}_i) d\tilde{\lambda},$$

(29)

with $i = 1, 2$. The amount of $\alpha_i^2$ is upper bounded by $f(\tilde{n}, \tilde{\lambda}) = \text{sgn}(\tilde{\lambda} \cdot \tilde{\alpha}_i)$, where sgn is the sign function. Set the unit vectors $\tilde{e}_1 = \tilde{\alpha}_1 / \alpha_1$, $\tilde{e}_2 = \tilde{\alpha}_2 / \alpha_2$ and $\tilde{e}_3 = \tilde{e}_1 \times \tilde{e}_2$. Under such a set of bases, the integral above with $f(\tilde{n}, \tilde{\lambda}) = \text{sgn}(\tilde{\lambda} \cdot \tilde{\alpha}_i)$ is invariant to $\omega(\lambda_1, \lambda_2, \lambda_3) \rightarrow \omega(-\lambda_1, \lambda_2, \lambda_3)$ and $\omega(\lambda_1, \lambda_2, \lambda_3) \rightarrow \omega(\lambda_1, -\lambda_2, \lambda_3)$. Consequently, one can always maximize the right hand side of (28) among the distributions with the symmetries $\omega(\lambda_1, \lambda_2, \lambda_3) = \omega(-\lambda_1, \lambda_2, \lambda_3)$ and $\omega(1, \lambda_2, \lambda_3) = \omega(\lambda_1, -\lambda_2, \lambda_3)$. The two values of $\alpha_i^2$ are determined by $\tilde{q} = \int_{\lambda_1 \geq 0, \lambda_2 \geq 0} \omega(\tilde{\lambda}) \tilde{d}\tilde{\lambda}$. Then, an upper bound of the average length of Bloch vectors is given by $\tilde{q} = \left( \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0 \right)$ as $\bar{r} \leq \sqrt{\frac{1}{2} + \eta^2}$. Considering that a physical Bloch vector is not longer than 1, one can conclude that the entanglement enhanced by a classical demon is bounded by

$$\Delta C_{\text{el}} \leq \frac{1}{2} \left( -\eta + \min \left\{ \sqrt{\eta^2 + \frac{1}{2}}, 1 \right\} \right).$$

(30)

Comparing the bound to the enhanced concurrence for $\rho(p, \eta)$ shown in Fig. 2, one can draw the quantum region in the parameter space of $(p, \eta)$, or equivalently the space of $p$ and extractable work. We show the boundary in Fig. 3 together with the result for extractable work by the two pairs of unitaries in [10]. The boundaries indicate that, the quantumness in a pure state $|\Psi\rangle$ with a larger $\eta$ is more fragile under the mixture of classical state, although an arbitrary $|\Psi\rangle$ leads to $\bar{r} = 1$. To demonstrate the quantumness, the extractable work or enhanced entanglement increases with the proportion of classical state. And, the quantum region is narrowed in the scheme with the two pairs of non-optimal unitaries.

IV. SUMMARY

We studied the preparation of quantum correlations from a thermal qubit and an auxiliary qubit, assisted by a Maxwell demon who obtains information of the thermal qubit from measurements on its environment. These processes avoid the disturbance to average energy of the system by direct measurements, and establish the relationships between quantum steering and other correlations in the thermodynamic framework. We derived the optimal operations to create the maximal mutual information, entanglement and Bell-nonlocality. The maximums are monotonic increasing functions of the average length of the Bloch vectors in postmeasured states of the system qubit. A critical value of the average length naturally corresponds to the necessary and sufficient condition for steerability in the case with maximally mixed marginals. We also presented an upper bound of the average length for unsteerable environment-system correlation, which can be regarded as a steering-type inequality demonstrating the quantumness of the Maxwell demon.

It would be interesting to consider extensions of the current results in several directions. On the theoretical side, one can try to derive more general relationships between the preparation of quantum correlations and the quantum steering, especially in multipartite systems and in the processes with thermodynamic cycles. And, the result in the case with maximally mixed marginals suggests that, it is possible to find an operational interpretation in thermodynamic tasks of the necessary and sufficient conditions for steerability of general two-qubit states [42–44]. Experimentally, we hope that the processes studied
in this paper can be implemented in laboratories with the recent techniques developed in spin systems [11].

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Appendix A: Lemma 1

Lemma 1: Let \( \{X_j\} \) be a set of Hermitian semipositive definite operators in the \( d \)-dimension Hilbert space, with the spectral decomposition \( X_j = \sum_{i=1}^{d} \lambda_i^{(j)} |\phi_i^{(j)}\rangle \langle \phi_i^{(j)}| \) and \( \lambda_1^{(j)} \geq \lambda_2^{(j)} \cdots \geq \lambda_d^{(j)} \). For given spectrums \( \{\lambda_i^{(j)}\} \), the maximum of \( \text{Tr}(\sum_j X_j^n) \) (\( n = 2, 3, \ldots \)) occurs when the eigenstates satisfy \( |\langle \phi_i^{(j)}| \phi_i^{(j')}\rangle| = \delta_{i,i'} \).

Proof. For given spectrums \( \{\lambda_i^{(j)}\} \), the amount of \( \text{Tr}(\sum_j X_j^n) \) depends on the cross terms in the form \( \text{Tr}(X_j^n Y) \), where \( 1 \leq k_j \leq d - 1 \) and \( Y \) is a product of \( X_j \)'s with \( j' \neq j \) and \( \sum_j k_j = n \). The operator \( Y \) is semipositive definite. Suppose \( Y = \sum_{i=1}^{d} y_i |y_i\rangle \langle y_i| \) and \( y_1 \geq y_2 \cdots \geq y_d \). One can directly calculate and obtain

\[
\text{Tr}(X_j^n Y) = \sum_{i,i'} \lambda_i^{(j)} y_{i'} P_{ii'},
\]

where \( P_{ii'} = |\langle \phi_i^{(j)}| \phi_i^{(j')}\rangle|^2 \) is a doubly stochastic matrix. For given \( \lambda_i^{(j)} \) and \( y_i \), \( \text{Tr}(X_j^n Y) \) is maximized by \( P_{ii'} = \delta_{i,i'} \). For all the cross terms and all the choices of the subscript \( j \), the above maximization can be simultaneously achieved by \( |\langle \phi_i^{(j)}| \phi_i^{(j')}\rangle| = \delta_{i,i'} \).

Appendix B: Optimization of negativity

For given eigenvalues \( \{\lambda_i\} \) of two-qubit states in nonascending order, the maximal negativity is given by

\[
\mathcal{N}_{\text{max}} = \max\{0, \sqrt{(\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_4)^2 - \lambda_2 - \lambda_4}\}. \quad (B1)
\]

It is reached by

\[
\rho = \lambda_1 |\psi_+\rangle \langle \psi_+| + \lambda_2 |01\rangle \langle 01| + \lambda_3 |\psi_-\rangle \langle \psi_-| + \lambda_4 |10\rangle \langle 10|. \quad (B2)
\]

By using this result, one can derive the maximal negativity for a given the maximal eigenvalue, \( \lambda_1 \in [1/4, 1] \).

We define four lines on the plane of \( (\lambda_3, \lambda_4) \) as

(a) \( \lambda_4 = 0 \),
(b) \( \lambda_4 = \lambda_3 \),
(c) \( \lambda_4 = 1 - \lambda_1 - 2\lambda_3 \),
(d) \( \lambda_4 = 1 - 2\lambda_1 - \lambda_3 \).

They are equivalent to the four equals signs in \( 0 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \) in order. For a fixed value of \( \lambda_1 \), there are three different situations for the physical region on the plane of \( (\lambda_3, \lambda_4) \): (i) a triangle defined by lines (a), (b) and (c), when \( \lambda_1 \in [1/2, 1] \); (ii) a quadrilateral defined by lines (a), (b), (c) and (d), when \( \lambda_1 \in [1/3, 1/2] \); (iii) a triangle defined by lines (b), (c) and (d), when \( \lambda_1 \in [1/4, 1/3] \). Because of the convexity of negativity, the maximum occurs on the vertices of the triangles or quadrilateral. Calculating the value of \( \mathcal{N}_{\text{max}} \) on these vertices, one obtains

\[
\mathcal{N}_{\text{max}} = \max\{0, \sqrt{10\lambda_1^2 - 6\lambda_1 + 1 - \lambda_1},
\sqrt{2\lambda_1^2 - \lambda_1 + 1 + \lambda_1 - 1}\}. \quad (B3)
\]

It is a monotonic increasing function of \( \lambda_1 \).

The choice with \( \rho_a = |0\rangle \langle 0| \) and \( U_k^n \rho_b^n \otimes |0\rangle \langle 0| U_k^{-1} = \frac{1 + \sqrt{1 - 2\lambda_1^2}}{2} |\psi_+\rangle \langle \psi_+| + \frac{1 - \sqrt{1 - 2\lambda_1^2}}{2} |01\rangle \langle 01| \) maximizes the maximal eigenvalue of \( \xi_{ao} \), and meanwhile leads to \( \mathcal{N}(\xi_{ao}) = \mathcal{N}_{\text{max}} = \sqrt{2\lambda_1^2 - \lambda_1 + 1 + \lambda_1 - 1} \). Here \( \lambda_1 = \frac{1 + \epsilon}{2} \in [1/2, 1] \). Hence, this optimizes the amount of negativity.

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