SHELLABLE WEAKLY COMPACT SUBSETS OF $C[0, 1]$

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Abstract. We show that for every weakly compact subset $K$ of $C[0, 1]$ with finite Cantor-Bendixson rank, there is a reflexive Banach lattice $E$ and an operator $T : E \to C[0, 1]$ such that $K \subseteq T(B_E)$. On the other hand, we exhibit an example of a weakly compact set of $C[0, 1]$ homeomorphic to $\omega^\omega + 1$ for which such $T$ and $E$ cannot exist. This answers a question of M. Talagrand in the 80’s.

1. Introduction

In the celebrated paper [9], W. Davis, T. Figiel, W. Johnson and A. Pelczynski showed that every weakly compact operator between Banach spaces factors through a reflexive Banach space. This fact relies on a general construction, arising from interpolation theory, which shows that every weakly compact set in a Banach space is contained in the image of the unit ball by an operator defined on a reflexive Banach space. Analogous statements in the category of Banach lattices and Banach algebras have been considered respectively by C. Aliprantis and O. Burkinshaw in [2] and A. Blanco, S. Kaijser and T. J. Ransford in [7]. In the former paper, it was shown that under some extra assumptions, a weakly compact operator between Banach lattices can be factored through a reflexive Banach lattice. In [19], M. Talagrand showed that these extra assumptions cannot be completely removed.

In fact, let us say that a set $A$ in a Banach space $X$ is shellable by reflexive Banach lattices if there is a reflexive Banach lattice $E$ and an operator $T : E \to X$ such that $A \subseteq T(B_E)$, where $B_E$ denotes the unit ball of $E$. In [19], the author constructs a weakly compact set $K_T$ of continuous functions on the interval $[0, 1]$ which is not shellable by reflexive Banach lattices. Associated to this compact set $K_T$, one can consider a weakly compact operator $T : \ell_1 \to C[0, 1]$ which cannot be factored through a reflexive Banach lattice. The compact set $K_T$ is small, as it is homeomorphic to the ordinal $\omega^\omega + 1$. It was naturally asked in [19] what is the smallest ordinal $\alpha$ for which there is a weakly compact set $K \subset C[0, 1]$ homeomorphic to $\alpha$ and not shellable by reflexive Banach lattices. The main result of this note is that this ordinal is precisely $\omega^\omega + 1$.

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Our proof has two parts. For the lower bound, we will use crucially that when $K$ is a countable compact space with finite Cantor-Bendixson rank then the space $C(K)$ of continuous functions on $K$ has an unconditional basis, in fact it is isomorphic to $c_0$. This implies that when $K$ is a countable weakly-compact subset of $C[0,1]$, the natural evaluation operator $\xi \in C[0,1]^* \mapsto C(K), \xi(\mu)(p) := \int pd\mu, p \in K$, factors through a reflexive Banach lattice. From this, a simple duality argument then shows that $K$ is shellable by a reflexive Banach lattice.

For the upper bound we will construct a weakly compact set $K_\omega \subset C(\omega^\omega + 1)$ homeomorphic to $\omega^\omega$ which is not shellable by reflexive Banach lattices. It is well-known that on $L_1(\mu)$ spaces, weak-compactness, uniform integrability and the Banach-Saks property are equivalent. In [19], uniform integrability in combination with martingale techniques was heavily used. In this paper, instead of that we exploit the weak Banach-Saks property of $L_1$.

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [1, 13, 14]. Throughout, $B_X$ denotes the unit ball of a Banach space $X$, that is $B_X = \{ x \in X : \| x \| \leq 1 \}$. Also, an operator will always refer to a bounded and linear map. Recall that $A \subset X$ is a Banach-Saks set if every $(x_n) \subset A$ has a Cesaro convergent subsequence $(x_{n_k})$, that is the arithmetic means $\frac{1}{n} \sum_{k=1}^{n} x_{n_k}$ are convergent. Recall that a reflexive Banach lattice $E$ can always be represented as a space of measurable functions on some probability space $(\Omega, \Sigma, \mu)$ such that $L^\infty(\mu) \subset E \subset L^1(\mu)$, with $\| f \|_1 \leq \| f \|_E$ [14, Theorem 1.b.12]. In particular, $E^*$ is also a space of measurable functions on $(\Omega, \Sigma, \mu)$ and both $B_E$ and $B_{E^*}$ are equi-integrable sets in $L^1(\mu)$ (or equivalently, Banach-Saks sets). A subset of a Banach lattice $A \subset E$ is solid when $|x| \leq |y|$ with $y \in A$ implies that $x \in A$. Given a set $A \subset E$, we define its solid hull $\text{sol}(A) = \{ x \in E : |x| \leq |y| \text{ for some } y \in A \}$. The reader is referred to [3, 14] for further explanations concerning Banach lattices.

Recall the classical result by S. Mazurkiewicz and W. Sierpinski [10] that states that every countable compact space $K$ is homeomorphic to a unique ordinal number $\omega^{\alpha_K} \cdot n_k + 1$, $n_K \in \mathbb{N}$, with its order topology. We will use here the fact that when $K$ has finite Cantor-Bendixson index, that is when $\alpha_K$ is finite, then the corresponding space of continuous functions $C(K)$ has an unconditional basis, in fact, $C(K)$ is isomorphic to $c_0$. This is actually a characterization of those compact spaces (see [10, Theorem 4.5.2] for more details).

**Definition 2.1.** Let $Y$ be a Banach space, and $\mathcal{X}$ a collection of Banach spaces. A set $K \subset Y$ will be called $\mathcal{X}$-shellable (or shellable by $\mathcal{X}$) if there is a space $X$ in $\mathcal{X}$ and an operator $T : X \to Y$ such that $K \subset T(B_X)$. In this case, we will write $K \in \text{Sh}_\mathcal{X}(Y)$. 
We will denote by $\text{Sh}_X(Y)$ the family of subsets $K \subseteq Y$ which are $X$-shellable. In particular, let $\text{Sh}_R(Y)$ and $\text{Sh}_{RL}(Y)$ be the collections of subsets of $Y$ which are shellable by a reflexive Banach space and by a reflexive Banach lattice, respectively. The following is easy to prove.

**Proposition 2.1.**
(a) $\text{Sh}_X(Y)$ is closed under subsets and convex hulls.

(b) If $X$ is closed under finite direct sums, then $\text{Sh}_X(Y)$ is closed under finite unions.

(c) If $Y_0$ is a complemented subspace of $Y_1$, then $\text{Sh}_X(Y_0) = \{W \subseteq Y_0 : W \in \text{Sh}_X(Y_1)\}$. \hfill $\square$

We recall the main construction in [9], later adapted in [2] to the lattice setting (see also [3, Section 5.2]).

**Theorem 2.2.** Let $W$ be a convex, circled, bounded subset of a Banach space $X$. There is a Banach space $Y$, and a continuous linear injection $J : Y \to X$ such that $W \subseteq J(B_Y)$. In addition,

(a) $Y$ is reflexive if and only if $W$ is relatively weakly compact.

(b) If $X$ is a Banach lattice, and $W$ is solid, then $Y$ can be taken to be a Banach lattice.

It is not true in general that the solid hull of a weakly-compact set in a Banach lattice is weakly-compact. Nevertheless, if $X$ is a space with an unconditional basis, or if $X$ is a Banach lattice not containing $c_0$, then every weakly compact set has weakly compact solid hull (see [8] for more details). As a consequence we obtain the following.

**Corollary 2.3.** Let $W$ be a relatively weakly-compact subset of a Banach space $X$. Then

(a) $W$ is shellable by a reflexive Banach space.

(b) If $X$ is a Banach lattice and $W$ is solid, then $W$ is shellable by a reflexive Banach lattice.

(c) If $X$ has an unconditional basis, then $W$ is shellable by a reflexive Banach lattice.

(d) If $X$ is a Banach lattice that does not contain $c_0$, then $W$ is shellable by a reflexive Banach lattice. \hfill $\square$

3. Main results

3.1. Lower bound. Let us begin this section with the lower bound concerning Talagrand's problem:

**Theorem 3.1.** Every countable weakly-compact subset of $C[0,1]$ of finite Cantor-Bendixson rank is in $\text{Sh}_{RL}(C[0,1])$.

**Proof.** Let us consider a countable weakly-compact subset $K \subseteq C[0,1]$ with finite Cantor-Bendixson rank. Let

$$\phi : C[0,1]^* \to C(K)$$
given by $\phi(\mu)(f) = \int f d\mu$ for each $\mu \in C[0,1]^*$ and $f \in K \subset C[0,1]$. Clearly, $\phi$ defines a bounded linear operator with $\|\phi\| \leq \max\{\|f\| : f \in K\}$.

Let $L = \phi(B_{C[0,1]}^*)$, which is a weakly compact subset of $C(K)$. Since $K$ has finite Cantor-Bendixson rank, $C(K)$ has an unconditional basis $(e_n)$. Hence, by [3, Theorem 4.39], it follows that its solid hull (with respect to the unconditional basis)

$$\text{sol}(L) = \{ \sum_n a_n e_n \in C(K) : \text{there is } \sum_n b_n e_n \in L \text{ such that } |b_n| \leq |a_n| \text{ for all } n \}$$

is a weakly compact subset of $C(K)$. Hence, by [2, Theorem 2.2], $\phi$ factors through a reflexive Banach lattice $E$. More precisely, there is $\tilde{\phi} : C[0,1]^* \to E$ and an interval preserving lattice homomorphism $J : E \to C(K)$ (with respect to the lattice structure induced by the unconditional basis $(e_n)$ of $C(K)$) with $\text{sol}(L) \subset J(B_E)$ and $\|J\| \leq \|\phi\|$, such that the following diagram commutes:

$$\begin{array}{ccc}
\text{C}[0,1]^* & \xrightarrow{\phi} & \text{C}(K) \\
\downarrow{\tilde{\phi}} & & \downarrow{J} \\
E & & \\
\end{array}$$

Thus, we clearly have the dual diagram as follows:

$$\begin{array}{ccc}
\text{C}(K)^* & \xrightarrow{\phi^*} & \text{C}[0,1]^* \\
\downarrow{J^*} & & \downarrow{E^*} \\
\text{C}[0,1]^* & \xrightarrow{\tilde{\phi}^*} & \\
\end{array}$$

Let $j : C[0,1] \to C[0,1]^*$ denote the canonical embedding in the bidual. Note that for every $f \in K$ we have that $\phi^*(\delta_f) = j(f)$. Indeed, for every $\mu \in C[0,1]^*$ we have

$$\phi^*(\delta_f)(\mu) = \delta_f(\phi(\mu)) = \mu(f) = j(f)(\mu).$$

Now, since $\phi^*$ is $w^* - w$ continuous, we have that

$$\phi^*(B_{C(K)^*}) = \phi^*(\overline{\text{co}}^w(\{\pm \delta_f : f \in K\})) \subset \overline{\text{co}}^w(\{\pm \phi^*(\delta_f) : f \in K\}) \subset j(C[0,1]),$$

so we actually have $\phi^* : C(K)^* \to C[0,1]$. Let $F$ be the (reflexive) sublattice of $E^*$ generated by $\{J^*(\delta_f) : f \in K\}$. Let $T : F \to C[0,1]$, be the restriction of $\tilde{\phi}^*$ on $F$. Since for $f \in K$ we have that $J^*(\delta_f) \in \|\phi\|B_E$, and

$$T(J^*(\delta_f)) = \phi^*(\delta_f) = f,$$

we have that $K \subseteq \|\phi\|T(B_F)$ and the proof is finished. □
3.2. Upper bound. We continue by proving that the Cantor-Bendixson rank $\omega + 1$ is sharp. The main result is the following.

**Theorem 3.2.** There is a weakly-compact subset of $C[0,1]$ homeomorphic to $\omega^{\omega} + 1$ which is not shellable by reflexive Banach lattices.

In fact, we find a weakly compact subset of $C(\omega^{\omega} + 1)$ homeomorphic to $\omega^{\omega} + 1$ which is not shellable by reflexive lattices. This implies Theorem 3.2 since for any countable compact space $L$ its space of continuous functions $C(L)$ can be isometrically embedded in $C[0,1]$ in a complemented way (any such compact space is homeomorphic to a retract of $[0,1]^\mathbb{N}$). It is more convenient to work with the appropriate compact families of finite subsets of $\mathbb{N}$ rather than directly with ordinal numbers. Recall that a family $\mathcal{F}$ of finite subsets of an index set $I$ is considered as a topological space by identifying each element $s$ of $\mathcal{F}$ with its characteristic function, and then by considering the induced product topology on $2^I$. In particular, we will consider the Schreier family

\[
\mathcal{S} = \{s \subseteq \mathbb{N} : \#s \leq \min s\}.
\]

It is well-known that $\mathcal{S}$ is compact and homeomorphic to $\omega^{\omega} + 1$. The Schreier barrier $\mathcal{G}$ is the family of maximal elements in $\mathcal{S}$ or equivalently, the set of isolated points of $\mathcal{S}$. Explicitly,

\[
\mathcal{G} = \{s \subseteq \mathbb{N} : \#s = \min s\}.
\]

Since $\mathcal{S}$ is a scattered compact space, its set of isolated points $\mathcal{G}$ is dense in it. So it makes sense to consider the following continuous extension property. Let us call a sequence $(x_s)_{s \in \mathcal{S}}$ a weakly-convergent tree if the assignment $s \in \mathcal{S} \mapsto x_s \in E$ is continuous with respect to the weak topology in $E$.

**Lemma 3.3.** Given a reflexive Banach space $E$ and $(x_s)_{s \in \mathcal{G}}$ bounded in $E$, there is an infinite set $M \subseteq \mathbb{N}$ such that we can extend $(x_s)_{s \in \mathcal{S} \cap M}$ to be a weakly convergent tree.

**Proof.** First of all, we can assume that $(E, w)$ is metrizable with $d$. An induction argument on $n \in \mathbb{N}$ gives that for every $\varepsilon > 0$, $n \in \mathbb{N}$ and any infinite set $M \subseteq \mathbb{N}$, there exists an infinite set $N \subseteq M$ such that $(x_s)_{s \in [N]^{\leq n}}$ is a weakly convergent tree with

\[
\text{diam}_d((x_s)_{s \in [N]^{\leq n}}) < \varepsilon.
\]

Using this, one can find a sequence $(M_n)_{n \in \mathbb{N}}$ with $M_{n+1} \subseteq M_n \subseteq \mathbb{N}$, so that if $m_n = \min M_n$, then $(x_{\{m_n\} \cup s})_{s \in [M_n]^{\leq m_n-1}}$ is a weakly convergent tree with $d$-diameter smaller than $1/n$. Let $L \subseteq \{m_n\}_{n \in \mathbb{N}}$ be such that $x_m \to_{m \in L} x_\emptyset$. Then $(x_s)_{s \in \mathcal{S} \cap L}$ is a weakly convergent tree. \qed

The previous Lemma generalizes to uniform families with a similar proof (see [3] for details on uniform families).
We will also use the “square” of the Schreier family $S_2$. Recall that given two families $\mathcal{F}$ and $\mathcal{G}$ of finite subsets of $\mathbb{N}$ their product is

$$\mathcal{F} \otimes \mathcal{G} := \left\{ \bigcup_{i=1}^{n} s_i : s_1 < \cdots < s_n, s_i \in \mathcal{F} \text{ for } 1 \leq i \leq n \text{ and } \{\min s_i\}_i \in \mathcal{G} \right\},$$

where $s < t$ means $\max s < \min t$. Let $S_2 := S \otimes S$. Given $n \in \mathbb{N}$, let $\mathbb{N}_n := [n, \infty]$. Let $F_n := [\mathbb{N}_n]^{\leq n}$, $G_n := S \otimes [\mathbb{N}_n]^{\leq n}$, $F_\omega := S$ and $G_\omega := S_2$. Notice that $\bigcup_n F_n = F_\omega$ and $\bigcup_n G_n = G_\omega$. The families $F_\alpha$, $G_\alpha$, $\alpha \leq \omega$ are compact, hereditary and any restriction of them have rank $\alpha + 1$ and $\omega \cdot \alpha + 1$, respectively. It is easy to show from the definition of $S_2$ that each element $t \in S_2 = S \otimes S$ has a unique decomposition $s = s[0] \cup s[1] \cdots \cup s[n]$, where

(a) $s[0] < s[1] < \cdots < s[n]$,
(b) $\{\min s[i]\}_{i \leq n} \in S$,
(c) $s[0], \cdots, s[n-1] \in \mathcal{G}$ and $s[n] \in \mathcal{S}$.

**Definition 3.1.** Given $s = \{m_0 < \cdots < m_k\} \in S$ and $t = t[0] \cup \cdots \cup t[l] \in S_2$, let us denote

$$\langle s, t \rangle = \#(\{0 \leq i \leq \min\{k, l\} : m_i \in t[i]\}).$$

Let $\Theta : S \times S_2 \to \{0, 1\}$ be the mapping that to $(s, t) \in S \times S_2$ assigns

$$\Theta(s, t) := \langle s, t \rangle + 1 \quad \text{ mod } 2. \quad (1)$$

**Proposition 3.4.** The mapping $\Theta$ is coordinate-wise continuous.

**Proof.** Since

$$\Theta(s, t) = \frac{1}{2} \left( (-1)^{\langle s, t \rangle} + 1 \right),$$

we only need to prove that $\langle \cdot, \cdot \rangle$ is a coordinate-wise continuous mapping. Suppose that $s_n \to_n s$ in $\mathcal{S}$, and fix $t \in S_2$. Let $n_0$ be such that $s_n \cap t = s \cap t$ for every $n \geq n_0$. Then $\langle s_n, t \rangle = \langle s, t \rangle$ for $n \geq n_0$. Similarly one proves the continuity with respect to the second variable. \(\square\)

It is interesting to note that Talagrand’s compactum given in [19] can also be constructed as $\Theta_1(S_2)$ where $\Theta_1 : S_2 \to C(S)$ is the mapping $t \in S_2 \mapsto \Theta_1(t) := \Theta(\cdot, t) \in C(S)$. Similarly, let $\Theta_0 : S \to C(S_2)$ be the mapping $s \in S \mapsto \Theta_0(s) := \Theta(s, \cdot) \in C(S_2)$. Set

$$K_\alpha := \Theta_0(F_\alpha), \quad L_\alpha := \Theta_1(G_\alpha),$$

for $\alpha \leq \omega$.

**Proposition 3.5.** $K_\alpha$ and $L_\alpha$ have Cantor-Bendixson rank $\alpha + 1$ and $\omega \cdot \alpha + 1$, respectively.

**Proof.** Since $\Theta_0$ and $\Theta_1$ are continuous, it follows that $K_\alpha$ and $L_\alpha$ have Cantor-Bendixson index at most $\alpha + 1$ and $\omega \cdot \alpha + 1$, respectively. Since in addition, $\Theta_1$ is 1-1, it follows that $L_\alpha$ has rank exactly $\omega \cdot \alpha + 1$. 


On the other hand, let $M := \{2^n\}_{n \geq 0}$. Then the restriction of $\Theta_0$ to $F_\alpha \upharpoonright M := \{s \in F_\alpha : s \subseteq M\}$ is 1-1, which shows that the rank of $K_\alpha$ is $\alpha + 1$. Suppose that $s_0 \neq s_1$ are elements of $F_\alpha \upharpoonright M$, $s_0 = \{2^0 < \cdots < 2^{k-1} < 2^k < \cdots < 2^u\}$, and $s_1 = \{2^0 < \cdots < 2^{k-1} < 2^k < \cdots < 2^{j}\}$, with $i_k \neq j_k$, say $i_k < j_k$. For each $m \leq k$, let $t_m := [2^{im}, 2^{im+1} - 1] \in \mathcal{G}$. Then it follows that $2^{im} \in t_m$ for every $m \leq k$ and $2^{ik} \notin t_k$. If we set $t := \bigcup_{m \leq k} t_m \in \mathcal{S}_2$, then $(t_m)_{m \leq k}$ is the canonical decomposition of $t$, hence $(s_0, t) = (s_1, t) + 1$, so $\Theta(s_0, t) \neq \Theta(s_1, t)$. □

**Definition 3.2.** Let $E$ be a Banach space and let $\alpha \leq \omega$. By a $(\Theta, \alpha)$-embedding into $E \times E^*$ we mean mappings $\gamma_0 : F_\alpha \upharpoonright M \to E$ and $\gamma_1 : G_\alpha \upharpoonright M \to E^*$ for some infinite $M \subseteq \mathbb{N}$ such that

(a) $\gamma_0$ and $\gamma_1$ are continuous, when we consider $E$ with its weak-topology and $E^*$ with its weak*-topology, respectively.

(b) $\gamma_1(t)(\gamma_0(s)) = \Theta(s,t)$ for every $(s,t) \in F_\alpha \upharpoonright M \times G_\alpha \upharpoonright M$.

The diameter of a $(\Theta, \alpha)$-embedding is

$$D(\gamma_0, \gamma_1) = \sup\{\|\gamma_0(s)\| \|\gamma_1(t)\| : (s,t) \in F_\alpha \upharpoonright M \times G_\alpha \upharpoonright M\}.$$

**Theorem 3.6.** No reflexive Banach lattice admits a $(\Theta, \omega)$-embedding. In fact, if $(\gamma_0, \gamma_1)$ is a $(\Theta, n)$-embedding in a reflexive Banach lattice, then $D(\gamma_0, \gamma_1) \geq n$.

**Proof.** Suppose otherwise, and fix a $(\Theta, n)$-embedding $\gamma_0 : F_n \upharpoonright M \to E$ and $\gamma_1 : G_n \upharpoonright M \to E^*$. To simplify the notation, we assume without loss of generality that $M = \mathbb{N}$. For $s \in F_n$ let $x_s = \gamma_0(s) \in E$, and for $t \in G_n$ let $x_t^* = \gamma_1(t) \in E^*$.

**Claim 3.6.1.** There is a mapping $\Delta : [\mathbb{N}]^{\leq n} \to \mathcal{S} \cup \{\emptyset\}$ such that

(a) $\Delta(\emptyset) = \emptyset$, $\Delta(s) \neq \emptyset$ for every $\#s < n$, $s \neq \emptyset$, and for every $s < m_0 < m_1$,

$$n, \Delta(s) < \Delta(s \cup \{m_0\}) < \Delta(s \cup \{m_1\}).$$

(b) Let $y_\emptyset = x_\emptyset$ and $y_\emptyset^* = x_\emptyset^*$, and for each $s := \{m_1 < \cdots < m_k\}$, let

$$s_i := \{m_1, \ldots, m_i\} \text{ for } i = 1, \ldots, k,$$

$$y_s := \frac{1}{\prod_{i=1}^k \#(\Delta(s_i))} \sum_{(r_i)_{i=1}^k \in [\prod_{i=1}^k \Delta(s_i)]} x_{(r_i)_{i=1}^k}^*,$$

$$y_s^* := x_{\Delta(s_1) \cup \cdots \cup \Delta(s_k)}^*.$$

Then $y_s \in \text{conv}(\gamma_0(F_n))$ and $y_s^* \in \gamma_1(G_n)$, and for every $s \in F_n$ with $\#s < n$ and for every $m > s$,

$$\|y_{s \cup \{k\}}\| \to \|y_s\|,$$

$$|y_{s \cup \{m\}}^* (y_s - y_{s \cup \{m\}})| = 1,$$

$$y_\emptyset^*(y_\emptyset) = 1.$$
Proof of Claim: We define \( \Delta_k : [N]^{\leq k} \to \mathcal{S} \cup \{\emptyset\} \) with the properties (a), (b) above by induction on \( k \leq n \). If \( k = 0 \), \( \Delta_0(\emptyset) = \emptyset \). Suppose done for \( k \), and let us define \( \Delta_{k+1} : [N]^{\leq k+1} \to \mathcal{S} \cup \{\emptyset\} \) extending \( \Delta_k \): Fix \( s := \{m_1 < \cdots < m_k\} \), \( s_i := \{m_1, \ldots, m_i\} \), \( i = 1, \ldots, k \). Then by definition, \[ y_s = \frac{1}{\prod_{i=1}^k \#(\Delta(s_i))} \sum_{(r_i)_{i=1}^k \in \prod_{i=1}^k \Delta(s_i)} x_{\{r_i\}_{i=1}^k}. \tag{5} \]

We know that for every \( r_1 < \cdots < r_k \) one has that \( x_{\{r_1, \ldots, r_k, r\}} \xrightarrow{\text{weak}, L_1} x_{\{r_1, \ldots, r_k\}} \). So by the weak-Banach-Saks property of \( L_1 \mathbb{R} \), we can find for every \( m > m_k \) an element \( \bar{s}_m \in \mathcal{S} \) with \( \bar{s}_m > n \) such that

(i) \( \Delta(s) < \bar{s}_p < \bar{s}_q \) for \( s < p < q \).

(ii) For every \( (r_i)_{i=1}^k \in \prod_{i=1}^k \Delta(s_i) \), \( u := \{r_i\}_{i=1}^k \), one has that

\[ \frac{1}{\# \bar{s}_m} \sum_{r \in \bar{s}_m} x_{u \cup \{r\}} \xrightarrow{\|\cdot\|_1} m \ x_u \tag{6} \]

Let \( \Delta(s \cup \{m\}) := \bar{s}_m \) for every \( s < m \). Then it follows from (ii) that

\[ y_{s \cup \{m\}} = \frac{1}{\prod_{i=1}^k \#(\Delta(s_i))} \sum_{(r_i)_{i=1}^k \in \prod_{i=1}^k \Delta(s_i)} \frac{1}{\# \bar{s}_m} \sum_{r \in \bar{s}_m} x_{\{r_i\}_{i=1}^k \cup \{r\}} \xrightarrow{\|\cdot\|_1} m \ x_u \]

For (3): Let \( s < m, s = \{m_1 < \cdots < m_k\} \) with \( k < n \). Let \( s_i = \{m_1, \ldots, m_i\} \) for \( 1 \leq i \leq k \), and set

\[ L_0 := \prod_{i=1}^k \#(\Delta(s_i)), \]

\[ L_1 := \#(\Delta(s \cup \{m\})), \]

\[ L_2 := L_0 \cdot L_1, \]

\[ t_0 := \bigcup_{i=1}^k \Delta(s_i), \]

\[ t_1 := t_0 \cup \Delta(s \cup \{m\}). \]

Then

\[ |y_{s \cup \{m\}}(y_s - y_{s \cup \{m\}})| = |x_{t_0}^* (y_s - y_{s \cup \{m\}})| = \]

\[ = \frac{1}{L_2} \left| L_1 \cdot \sum_{v \in \prod_{i=1}^k \Delta(s_i)} \gamma_1(t_1) \left( \gamma_0(v) \right) - \sum_{w \in \left( \prod_{i=1}^k \Delta(s_i) \times \Delta(s \cup \{m\}) \right)} \gamma_1(t_1) \left( \gamma_0(w) \right) \right|. \]
Now, given $v \in \prod_{i=1}^k \Delta(s_i)$,

$$\gamma_1(t_1)(\gamma_0(v)) = \Theta(v, t_1) = \Theta(v, t_0) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Similarly, given $w \in \left( \prod_{i=1}^k \Delta(s_i) \right) \times \Delta(s \cup \{m\})$,

$$\langle \gamma_1(t_1)(\gamma_0(w)) = \Theta(w, t_1) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, we have

$$L_1 \cdot \sum_{v \in \prod_{i=1}^k \Delta(s_i)} \gamma_1(t_1)(\gamma_0(v)) - \sum_{w \in \left( \prod_{i=1}^k \Delta(s_i) \right) \times \Delta(s \cup \{m\})} \gamma_1(t_1)(\gamma_0(w)) = \begin{cases} L_1 \cdot L_0 & \text{if } k \text{ is even} \\ -L_2 & \text{if } k \text{ is odd.} \end{cases}$$

Hence,

$$|y_{s \cup \{m\}}^* (y_s - y_{s \cup \{m\}})| = 1.$$  

□

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. If $P(n)$ is a statement about $n \in \mathbb{N}$, we will write $\mathcal{U}nP(n)$ to denote that $\{n \in \mathbb{N} : P(n) \text{ holds} \} \in \mathcal{U}$. The following is a slight improvement of [19, Proposition 1].

**Claim 3.6.2.** Suppose that $\{h_n\}$ is a bounded sequence in $E^*$. Then for every $\varepsilon > 0$ there is $a > 0$ such that for every $x \in B_E$ one has that

$$\mathcal{U}n \int_{|h_n| > a} |h_n||x| < \varepsilon. \quad (7)$$

**Proof of Claim:** Suppose otherwise that there is some $\varepsilon > 0$ such that for every $k$ there is $x_k \in B_E$ with

$$\mathcal{U}n \int_{|h_n| > k} |h_n||x_k| > \varepsilon. \quad (8)$$

Let $L := \sup_n \|h_n\|_{E^*}$. Let $y$ be a weak-accumulation point of $\{|x_k|\}_k$. Since $E$ is order continuous we can find $d > 0$ such that $\|y - y \wedge d\|_E < \varepsilon/(3L)$; using the equi-integrability of $B_{E^*}$, there is $a > 0$ such that

$$\sup_n \int_{|h_n| > a} |h_n| < \frac{\varepsilon}{3d}. \quad (9)$$

Let $a < k_1 < \cdots < k_r$, and let $\{b_i\}_{i=1}^r$ be a convex combination such that

$$\|y - \sum_{i=1}^r b_i|x_{k_i}|\|_E < \frac{\varepsilon}{3L}. \quad (10)$$

Choose now $n$ such that

$$\min_{1 \leq i \leq r} \int_{|h_n| > k_i} |h_n||x_{k_i}| > \varepsilon. \quad (11)$$
Then,

\[
\min_{1 \leq i \leq r} \int_{|h_n| > k_i} |h_n| |x_{k_i}| \leq \int_{|h_n| > a} |h_n| \sum_{i=1}^{r} b_i |x_{k_i}| \leq \int_{|h_n| > a} |h_n| y + \int_{|h_n| > a} |h_n||y - \sum_{i=1}^{r} b_i |x_{k_i}| | \leq \\
\leq \int_{|h_n| > a} |h_n| (y \land d) + \int_{|h_n| > a} |h_n||y - y \land d| + \frac{\varepsilon}{3} \leq \\
\leq d \frac{\varepsilon}{3d} + \|h_n\|_{E^*} \|y - y \land d\|_{E} + \frac{\varepsilon}{3} = \varepsilon,
\]

a contradiction with (11).

Claim 3.6.3. For every \( \varepsilon > 0 \) and every \( 0 \leq i \leq n \) there is \( s \in [\mathbb{N}]^i \) such that

\[
\int |y_s^*||y_s| \geq 1 + i - \varepsilon(1 + 5i).
\]  

(12)

Proof of Claim: Fix \( \varepsilon > 0 \). Induction on \( 0 \leq i \leq n \). \( i = 0 \): we know that

\[
1 - \varepsilon < 1 = y_0^*(y_0) \leq \int |y_0^*||y_0|.
\]  

(13)

Suppose now that \( s \in [\mathbb{N}]^i \) is such that

\[
\int |y_s^*||y_s| > 1 + i - \varepsilon(1 + 5i).
\]  

(14)

We use the previous Claim 3.6.2 to find \( a > 0 \) such that

\[
\mathcal{U}m \int_{|y_{s \cup \{m\}}| > a} |y_{s \cup \{m\}}^*||y_s| < \varepsilon.
\]  

(15)

Since \( y_{s \cup \{m\}}^* \to_m y_s^* \), it follows that

\[
\mathcal{U}m \int |y_{s \cup \{m\}}^*||y_s| > 1 + i - \varepsilon(2 + 5i).
\]  

(16)

Combining (15) and (16) we obtain that

\[
\mathcal{U}m \int_{|y_{s \cup \{m\}}| \leq a} |y_{s \cup \{m\}}^*||y_s| > 1 + i - \varepsilon(3 + 5i).
\]  

(17)

Since \( y_{s \cup \{m\}} \to_{\|\cdot\|_{E^1}} y_s \), it follows from (17) that

\[
\mathcal{U}m \int_{|y_{s \cup \{m\}}| \leq a} |y_{s \cup \{m\}}^*||y_{s \cup \{m\}}| > 1 + i - \varepsilon(4 + 5i).
\]  

(18)
On the other hand, given $n$ we have from (3) that
\[
\int_{|y_{s∪\{m\}}|>a} |y^*_s| - |y_{s∪\{m\}}| \geq \int_{|y_{s∪\{m\}}|\leq a} |y^*_s| - |y_{s∪\{m\}}| - \int_{|y_{s∪\{m\}}|>a} |y^*_s| - |y_{s∪\{m\}}| -
\]
\[
- \int_{|y_{s∪\{m\}}|>a} |y^*_s| - |y_{s∪\{m\}}| \geq
\]
\[
\geq 1 - \int_{|y_{s∪\{m\}}|\leq a} |y^*_s| - |y_{s∪\{m\}}| - \int_{|y_{s∪\{m\}}|>a} |y^*_s| - |y_{s∪\{m\}}|.
\]
(19)

Since $y_{s∪\{m\}} \stackrel{\|\cdot\|_1}{\rightarrow} y_s$, it follows that
\[
\mathcal{U}_m \int_{|y_{s∪\{m\}}|\leq a} |y^*_s| - |y_{s∪\{m\}}| < \varepsilon.
\]
(20)

Combining (15) and (20) in (19) we obtain that
\[
\mathcal{U}_m \int_{|y_{s∪\{m\}}|>a} |y^*_s| - |y_{s∪\{m\}}| > 1 - 2\varepsilon.
\]
(21)

Combining (15) and (21), we obtain
\[
\mathcal{U}_m \int |y^*_s| - |y_{s∪\{m\}}| > 1 + i + 1 - \varepsilon(1 + 5(i + 1)).
\]
(22)

From this last claim, we can find $s \in [N]^n$ such that
\[
\int |y^*_s| y_s \geq n.
\]
(23)

Since, $y_s \in \text{conv}(\gamma_0(\mathcal{F}_n))$ and $y^*_s \in \gamma_1(\mathcal{G}_n)$, it follows that $D(\gamma_0, \gamma_1) \geq n$.
\[
\square
\]

**Proposition 3.7.** Let $E$ be a reflexive Banach space.

(a) Suppose $T : E \rightarrow C(S_2)$ is such that $K_\alpha \subseteq T(B_E)$. Then there is a $(\Theta, \alpha)$-embedding in $E \times E^*$ with diameter less or equal than $\|T\|$.

(b) Suppose $T : E \rightarrow C(S)$ is such that $L_\alpha \subseteq T(B_E)$. Then there is a $(\Theta, \alpha)$-embedding in $E \times E^*$ with diameter less or equal than $\|T\|$.

**Proof.** Suppose that $T : E \rightarrow C(S_2)$ is such that $K_\alpha \subseteq T(B_E)$. For each maximal $s \in \mathcal{F}_\alpha$ choose $x_s \in B_E$ such that $T(x_s) = \Theta_0(s)$. From Lemma 3.3 there is an infinite $M \subseteq \mathbb{N}$ such that $(\gamma_0(s))_{s \in \mathcal{F}_\alpha \upharpoonright M}$ is a weakly convergent tree. Since $T$ is weak-continuous, it follows that $T(\gamma_0(s)) = \Theta_0(s)$ for every $s \in \mathcal{F}_\alpha \upharpoonright M$. For each $t \in \mathcal{G}_\alpha$, let $\gamma_1(t) := T^*(\delta_t)$. Then for every $(s, t) \in \mathcal{F}_\alpha \upharpoonright M \times \mathcal{G}_\alpha \upharpoonright M$ one has that
\[
\gamma_1(t)(\gamma_0(s)) = T^*(\delta_t)(\gamma_0(s)) = \delta_t(T(\gamma_0(s))) = \delta_t(\Theta_0(s)) = \Theta(s, t).
\]
(24)

Notice that $\|\gamma_0(s)\| \|\gamma_1(t)\| \leq \|T^*\| = \|T\|$, thus $D(\gamma_0, \gamma_1) \leq \|T\|$.

\[
\square
\]
Now, if $T : E \to C(S)$ is such that $L_n \subseteq T(B_E)$, then arguing as above, one could find $M \subseteq \mathbb{N}$ and a continuous $\gamma_1 : G_\alpha \upharpoonright M \to E$ such that $T(\gamma_1(t)) = \Theta_1(t)$ for every $t \in G_\alpha \upharpoonright M$. Let $\gamma_0 : G_\alpha \upharpoonright M \to E^*$, $\gamma_0(s) := T^*(\delta_s)$ for every $s \in G_\alpha \upharpoonright M$. Again, $\gamma_1(t)(\gamma_0(s)) = \Theta(s, t)$, and we get $D(\gamma_0, \gamma_1) \leq \|T\|$. \hfill \Box

As a consequence we get the following:

**Corollary 3.8.** Neither $K_\omega$ nor $L_\omega$ are shellable by reflexive Banach lattices. In fact, if $T : E \to C[0, 1]$ is such that $K_n$ or $L_n \subseteq T(B_E)$, then $\|T\| \geq n$. \hfill \Box

Several important aspects of the geometric structure of $C(\Omega)$ spaces are closely related to operators $T : C(\Omega) \to C(\Omega)$ (see [12, 17] for recent and complete accounts on this relation). Note that the compact $K_\omega$ and $L_\omega$ constructed above are not contained in the image of the unit ball by any weakly compact operator $T : C[0, 1] \to C[0, 1]$. In fact, we have the following:

**Proposition 3.9.** Let $\Omega$ be a compact Hausdorff space and $K \subset C(\Omega)$ a weakly compact set such that $K \subseteq T(B_{C(\Omega)})$ for some weakly compact operator $T : C(\Omega) \to C(\Omega)$. Then $K$ is in $Sh_{RL}(C(\Omega))$.

**Proof.** Since $T$ is weakly compact, so is its adjoint. Now, $C(\Omega)^*$ does not contain $c_0$ so the solid hull of $T(B_{C(\Omega)^*})$ is weakly compact. Thus, we could factor $T^*$, and also $T$, through a reflexive Banach lattice. \hfill \Box

The converse however is not true. In fact, it is not even true that a subset of $C[0, 1]$ shellable by a reflexive lattice is contained in $T(B_{C[0, 1]})$ for a weak Banach-Saks operator $T : C[0, 1] \to C[0, 1]$. Recall that an operator is weak Banach-Saks when it sends weakly compact sets to Banach-Saks sets. Note that the Dunford-Pettis property of $C[0, 1]$ yields in particular that every weakly compact operator $T : C[0, 1] \to C[0, 1]$ is weak Banach-Saks. Now, suppose that $X$ is a copy of the Schreier space in $C[0, 1]$ and let $K = \{0\} \cup \{u_n\}_n$ where $(u_n)_n$ is the unit basis of $X$. By Theorem [3,1] we have that $K$ is in $Sh_{RL}(C[0, 1])$. If $T : C[0, 1] \to C[0, 1]$ is such that $K \subseteq T(B_{C[0, 1]})$, then one can prove that its Szlenk index (see [17]) is at least $\omega^2$. Hence, by a result of D. E. Alspach [4], $T$ fixes a copy of $C(S)$, hence $T$ cannot be weak Banach-Saks.

It was observed in [11] that Talagrand’s weakly compact set $K_T$ is a Banach-Saks set. Since every space with the Banach-Saks property is reflexive, as a consequence of [19] it follows that $K_T$ is not shellable by Banach lattices with the Banach-Saks property. In a similar spirit, one might wonder what is the smallest ordinal $\alpha$ such that there exists a Banach-Saks set $K \subseteq C[0, 1]$ homeomorphic to $\alpha$ which is not shellable by Banach lattices with the Banach-Saks property. It can be seen that the compact set $K_\omega$ constructed before fails the Banach-Saks property. However, in [15] an example was given of a Banach-Saks set whose convex hull is not Banach-Saks, so it is not even shellable by Banach spaces with the Banach-Saks property. Note this set is homeomorphic to $\omega + 1$. 


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