Localizing gravity on thick branes: a solution for massive KK modes of the Schrödinger equation

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Abstract. We generate scalar thick brane configurations in a 5D Riemannian space time which describes gravity coupled to a self–interacting scalar field. We also show that 4D gravity can be localized on a thick brane which does not necessarily respect $Z_2$–symmetry, generalizing several previous models based on the Randall–Sundrum system and avoiding the restriction to orbifold geometries as well as the introduction of the branes in the action by hand. We begin by obtaining a smooth brane configuration that preserves 4D Poincaré invariance and violates reflection symmetry along the fifth dimension. The extra dimension can have either compact or extended topology, depending on the values of the parameters of the solution. In the non–compact case, our field configuration represents a thick brane with positive energy density centered at $y = c_2$, whereas in the compact case we get pairs of thick branes. We recast as well the wave equations of the transverse traceless modes of the linear fluctuations of the classical solution into a Schrödinger’s equation form with a volcano potential of finite bottom. We solve Schrödinger equation for the massless zero mode $m^2 = 0$ and obtain a single bound wave function which represents a stable 4D graviton and is free of tachyonic modes with $m^2 < 0$. We also get a continuum spectrum of Kaluza–Klein (KK) states with $m^2 > 0$ that are suppressed at $y = c_2$ and turn asymptotically into plane waves. We found a particular case in which the Schrödinger equation can be solved for all $m^2 > 0$, giving us the opportunity of studying analytically the massive modes of the spectrum of KK excitations, a rare fact when considering thick brane configurations.

1. Introduction, setup and solution
The fact that we can live in a multidimensional space time with infinite extra dimensions turns out to be compatible with present time gravitational experiments. In such scenarios, gravity is essentially 4–dimensional from the point of view of an observer located at a 3–brane in which matter is confined, however, the world can be higher dimensional with extended extra dimensions and gravity can propagate in all of them. Multidimensional space times with large extra dimensions turned out to be very useful when addressing several problems of high energy physics like the cosmological constant, dark matter and the mass hierarchy problem [1]–[4] as well as the recent non–supersymmetric string model realization of the Standard Model at low energy with no extra massless matter fields [5]. The striking success of these higher dimensional
scenarios gave rise to several generalizations, including scalar thick brane configurations [6]–[8]. These field configurations were generalized in the framework of Weyl geometries for $Z_2$-symmetric manifolds in [9]. Moreover, localization of 4D gravity on thick branes that break reflection symmetry was presented in [10]–[11].

In this paper we begin by considering a 5–dimensional action which describes gravity coupled to a bulk scalar field. In this framework we obtain smooth brane solutions which respect 4D Poincaré invariance, do not necessarily respect reflection $Z_2$–symmetry and allow for both compact and non–compact manifolds in the extra dimension. The structure of these brane configurations depends on the topology of the extra dimension and the value of the parameter $p(\xi)$. We further investigate linear fluctuations of the metric around the classical background solution. We show that 4D gravity can be described in our setup since the analog quantum mechanical problem, a Schrödinger equation with a volcano potential, for the transverse traceless sector of the fluctuations of the metric yields a continuum and gapless spectrum of KK states with a stable zero mode that corresponds to the 4D graviton. Finally, we consider a particular case in which the Schrödinger equation can be completely integrated for all the massive modes of KK states and present analytical expressions for them.

Let us start by considering the following 5–dimensional Riemannian action

$$S_5 = \int_{M_5} \frac{d^5 x \sqrt{|g|}}{16 \pi G_5} [R_5 + 3 \xi (\nabla \omega)^2 + 6U(\omega)],$$

where $\omega$ is a bulk scalar field, $\xi$ is an arbitrary coupling parameter, and $U(\omega)$ is a self–interacting potential for the scalar field.

We shall consider solutions which respect 4–dimensional Poincaré invariance with the following line element

$$ds_5^2 = e^{2A(y)} \eta_{mn} dx^m dx^n + e^{\omega(y)} dy^2,$$

where $2\sigma(y) = 2A(y) + \omega(y)$ depends only on the extra coordinate $y$, and $m, n = 0, 1, 2, 3$. We shall call $e^{2A(y)}$ the warp factor of the metric.

Further, by following [9] we introduce the new variables $X = \omega'$ and $Y = 2A'$ and get the following pair of coupled field equations from the action (1)

$$X' + 2Y X + \frac{3}{2} X^2 = \frac{1}{\xi} \frac{dU}{d\omega} e^\omega,$$

$$Y' + 2Y^2 + \frac{3}{2} XY = \left( \frac{1}{\xi} \frac{dU}{d\omega} + 4U \right) e^\omega.$$  

(3)

In general, it is not a trivial task to fully integrate these field equations. However, it is straightforward to construct several particular solutions to them after some simplifications and, in special cases, these solutions lead to expressions of the dynamical variables that can be treated analytically in closed form, an important advantage from the physical point of view.

The system of equations (3) can be easily solved if one uses the condition $X = kY$, where $k$ is an arbitrary parameter which is not allowed to adopt the value $k = -1$ because the system would become incompatible. This condition restricts the self–interacting potential to adopt the form

$$U = \lambda e^{\frac{4k}{1+k} \omega}$$

and the field equations (3) reduce to the following differential equation

$$Y' + \frac{4 + 3k}{2} Y^2 = \frac{4\lambda}{1+k} e^{(1+ \frac{4k}{1+k}) \omega}. $$

(4)
In [9]–[10] this equation was solved by choosing the parameter \( \xi = -(1 + k)/(4k) \), while in [11] another solution was obtained by fixing the parameter \( k = -4/3 \). In this paper we shall continue to consider the latter particular case. Thus, after imposing the condition \( k = -4/3 \), the second term in the left hand side of the equation (4) vanishes, yielding the following differential equation

\[
\omega'' - 16\lambda e^{p\omega} = 0,
\]

(5)

where \( p = 1 + 16\xi \).

One can solve this equation for \( \omega \) and, by further integrating the relation \( 2A' = -3\omega'/4 \), one gets the solution

\[
\omega = -\frac{2}{p} \ln \left[ \frac{\sqrt{-8\lambda p}}{c_1} \cosh (c_1(y - c_2)) \right],
\]

\[
e^{2A} = \left[ \frac{\sqrt{-8\lambda p}}{c_1} \cosh (c_1(y - c_2)) \right]^{\frac{3}{2p}},
\]

(6)

where \( c_1 \) and \( c_2 \) are integration constants.

For \( p < 0 \), this solution represents a localized object which does not necessarily preserve the reflection symmetry \((y \to -y)\) along the fifth dimension and breaks it through non–trivial values of the shift parameter \( c_2 \). Thus, the 5–dimensional space time is not restricted to be an orbifold geometry as in the Randall–Sundrum (RS) case, allowing for a more general type of manifolds. The topology of the extra coordinate can be compact or extended depending on the signs of the constants \( p \) and \( \lambda \), and the real or imaginary character of the parameter \( c_1 \). This constant characterizes the width of the warp factor \( \Delta \sim 1/c_1 \).

Let us consider the cases of physical interest:

A) \( \lambda > 0, \ p < 0, \ c_1 > 0 \). In this case the domain of the fifth coordinate is infinite \(-\infty < y < \infty\) and we have a non–compact manifold in the extra dimension. In this case the warp factor is located around the point \( y = c_2 \) and represents a smooth localized function of width \( \Delta \) which reproduces the metric of the RS model in the thin brane limit, namely, when \( c_1 \to \infty, \ p \to -\infty \) keeping \( c_1/p = \beta \) finite. In [9]–[11] these smooth configurations were physically interpreted as thick branes in the framework of Weyl geometry.

B) \( \lambda > 0, \ p > 0, \ c_1 = iq_1 \). In this case we have a compact extra dimension with \(-\pi \leq q_1(y - c_2) \leq \pi \). The corresponding expressions for the warp factor and the scalar field are the following

\[
e^{2A(y)} = \left[ \frac{\sqrt{8\lambda p}}{q_1} \cos (q_1(y - c_2)) \right]^{\frac{3}{2p}},
\]

\[
\omega = -\frac{2}{p} \ln \left[ \frac{\sqrt{8\lambda p}}{q_1} \cos (q_1(y - c_2)) \right].
\]

(7)

In [11] it was shown that this solution describes, in fact, a pair of thick branes located in two different disconnected regions of the manifold due to the fact that the 5–dimensional curvature scalar is singular at the points \( y = \pm \frac{\pi}{2q_1} + c_2 \):

\[
R_5 = \frac{14q_1^2}{p} \left[ \frac{\sqrt{8\lambda p}}{q_1} \cos (q_1(y - c_2)) \right]^{-2/p} \left[ 1 + \frac{8p - 27}{8p} \tan^2 (q_1(y - c_2)) \right],
\]

(8)

where plausibly we have null scalar energy densities.

Other cases of physical interest are equivalent or contained in cases A) and B).
2. Fluctuations of the classical background

Let us turn to study the metric fluctuations $h_{mn}$ of the interval (2) given by the perturbed line element

$$ds^2 = e^{2\sigma(y)}[\eta_{mn} + h_{mn}(x, y)]dx^m dx^n + e^{\omega(y)} dy^2.$$  (9)

In the general case, one cannot avoid considering fluctuations of the scalar field when treating fluctuations of the background metric since they are coupled, however, in [6] it was shown that the transverse traceless modes of the background fluctuations decouple from the scalar sector. As a result, these modes can be approached analytically in closed form.

In order to apply this method, we first perform the coordinate transformation

$$dz = e^{-A} dy.$$  (10)

This change of variable leads to a conformally flat metric and, hence, the transverse traceless modes of the metric fluctuations $h^T_{mn}$ obey the following wave equation

$$(\partial_z^2 + 3\sigma' \partial_z + \Box^y) h^T_{mn} = 0.$$  (11)

It is easy to see that this equation supports a massless and normalizable 4D graviton given by $h^T_{mn} = C_{mn} e^{ipx}$, where $C_{mn}$ are constant parameters and $p^2 = m^2 = 0$.

By following [3] we shall recast equation (11) into a Schrödinger’s equation form. In order to accomplish this aim, we adopt the following ansatz for the transverse traceless modes of the metric fluctuations

$$h^T_{mn} = e^{ipx} e^{-3\sigma/2} \Psi_{mn}(z)$$

and get the equation

$$[\partial_z^2 - V(z) + m^2] \Psi = 0,$$  (12)

where we have dropped the subscripts in $\Psi$, $m$ is the mass of the KK excitation, and the quantum mechanical potential is completely defined by the curvature of the manifold and reads

$$V(z) = \frac{3}{2} \partial_z^2 \sigma + \frac{9}{4} (\partial_z \sigma)^2.$$  (13)

In the non-compact case A) we have found two particular cases ($p = -1/4$ and $p = -3/4$) for which we can invert the coordinate transformation (10) and explicitly express the variable $y$ in terms of the coordinate $z$. For the case $p = -3/4$ the functions $A(z)$ and $\sigma(z)$ adopt the form

$$A(z) = \frac{7}{6} \ln \left\{ c_1^2 / \left[ c_1^2 (z - z_0)^2 + 6\lambda \right] \right\}$$

and

$$\sigma(z) = \frac{7}{6} \ln \left\{ c_1^2 / \left[ c_1^2 (z - z_0)^2 + 6\lambda \right] \right\}.$$  

The expression for $\sigma(z)$ yields the following explicit formula for the analog quantum mechanical potential

$$V(z) = \frac{21 c_1^2 \left[ 3c_1^2 (z - z_0)^2 - 4\lambda \right]}{4 \left[ c_1^2 (z - z_0)^2 + 6\lambda \right]^2}.$$  (14)

In the framework of the Schrödinger equation, the spectrum of eigenvalues $m^2$ parameterizes the spectrum of graviton masses that a 4-dimensional observer located at $z_0$ sees. This equation


can be solved for the massless zero mode $m^2 = 0$ and the only normalizable eigenfunction adopts the form

$$\Psi_0 = q \left[ c_1^2(z - z_0)^2 + 6\lambda \right]^{-7/4},$$

where $q$ is a normalization constant. Thus, this function constitutes the lowest energy eigenfunction of the Schrödinger equation (12) since it has no zeros. This fact allows for the existence of a 4D graviton with no tachyonic instabilities from transverse traceless modes with $m^2 < 0$. In addition to this massless mode, there exists a tower of higher KK modes with a continuum spectrum of positive masses $m^2 > 0$.

A similar situation takes place in the compact case B). Remarkably, the coordinate transformation $dz = e^{-A}dy$ can be inverted for $p = 3/8$ yielding

$$\cos[q_1(y - c_2)] = \pm q_1/\sqrt{q_1^2 + 9\lambda^2(z - z_0)^2},$$

i.e., decompactifying the fifth dimension and pushing to infinity the singularities. This mathematical fact implies that we actually have two disconnected regions in the manifold: the region $-\pi/2 \leq q_1(y - c_2) \leq \pi/2$ is separated from the region $\pi/2 \leq q_1(y - c_2) \leq 3\pi/2$ (since we can shift the domain of the compact dimension to $-\pi/2 \leq q_1(y - c_2) \leq 3\pi/2$) by the physical singularities located at $y = \pm \pi q_1/2 c_2$ (recall that the curvature scalar is singular at these points). Each one of these regions leads to

$$A(z) = \ln \left\{ 3\lambda/[q_1^2 + 9\lambda^2(z - z_0)^2] \right\}$$

and

$$\sigma(z) = \frac{7}{3} \ln \left\{ 3\lambda/[q_1^2 + 9\lambda^2(z - z_0)^2] \right\}.$$

Thus, the analog quantum mechanical potential takes the form

$$V(z) = \frac{63\lambda^2 [72\lambda^2(z - z_0)^2 - q_1^2]}{[q_1^2 + 9\lambda^2(z - z_0)^2]^2} \quad (15)$$

and the wave function corresponding to the zero mode reads

$$\Psi_0 = k \left[ q_1^2 + 9\lambda^2(z - z_0)^2 \right]^{-7/2},$$

where $k$ is a constant.

Both potentials (14) and (15) have the volcano form: a well of finite bottom and positive barriers at each side that vanish asymptotically. The corresponding wave functions of the massless zero modes represent smooth lumps localized around the point $z_0$. These facts imply that we have only one gravitational bound state (the massless one) and a continuous and gapless spectrum of massive KK states with $m^2 > 0$ in both cases A) and B).

Thus, we have obtained smooth brane generalizations of the RS model with no reflection symmetry imposed in which the 4D effective theory possesses an energy spectrum quite similar to the spectrum of the thin wall case, in particular, 4D gravity turns out to be localized at a certain value of the fifth dimension in both cases A) and B).

Finally we would like to present the particular case $p = 3/4$ in which the coordinate transformation (10) can be inverted as well yielding the equality

$$\cos[q_1(y - y_0)] = \text{sech} \left[ \sqrt{6\lambda}(z - z_0) \right].$$
Here we have again two disconnected regions in the manifold due to the presence of physical singularities at $y = \pm \frac{\pi}{2q_1} + c_2$. In each one of these regions we perform the change of variable (10) and we observe that this is equivalent to decompactifying one more time the fifth dimension and pushing the singularities to spatial infinity. Thus we get the following expressions for the warp functions

$$A(z) = \ln \left\{ \frac{\sqrt{6} \lambda}{q_1} \operatorname{sech} \left[ \sqrt{6} \lambda (z - z_0) \right] \right\}$$

and

$$\sigma(z) = \frac{7}{3} \ln \left\{ \frac{\sqrt{6} \lambda}{q_1} \operatorname{sech} \left[ \sqrt{6} \lambda (z - z_0) \right] \right\}.$$ 

Consequently, the analog quantum mechanical potential takes the following form

$$V(z) = \frac{21 \lambda}{2} \left\{ 5 \tanh^2 \left[ \sqrt{6} \lambda (z - z_0) \right] - 2 \right\}.$$  (16)

It turns out that with this potential, the Schrödinger equation can be integrated for both the massless and the massive modes of the KK excitations. For the zero mode we get the following normalizable wave function

$$\Psi_0 = C_0 \operatorname{sech}^{7/4} \left[ \sqrt{6} \lambda (z - z_0) \right],$$  (17)

where $C_0$ is a constant parameter.

On the other side, for the massive modes of the KK excitations we get their expression in terms of the associated Legendre functions of first and second kind:

$$\Psi = C_1 P_{7/2}^{\sqrt{147} \lambda - 2m^2/\sqrt{12} \lambda} \left[ \tanh \left( \sqrt{6} \lambda (z - z_0) \right) \right] +$$

$$C_2 Q_{7/2}^{\sqrt{147} \lambda - 2m^2/\sqrt{12} \lambda} \left[ \tanh \left( \sqrt{6} \lambda (z - z_0) \right) \right],$$  (18)

where $C_1$ and $C_2$ are integration constants.

This remarkable fact gives us the possibility of studying analytically the massive modes of the spectrum of KK excitations, a scarce phenomenon when considering smooth brane configurations.

3. Concluding Remarks

We considered the generation of scalar thick brane configurations in a Riemannian manifold. We obtained a solution which preserves 4D Poincaré invariance and, in particular, represents a smooth localized function characterized by the width parameter $\Delta \sim 1/c_1$ and the constant $c_2$ which breaks the $\mathbb{Z}_2$–symmetry along the extra dimension: both of these parameters are integration constants of the relevant field equation [5]. Thus, our field configurations correspond to smooth generalizations of the RS model which do not restrict the 5–dimensional space time to be an orbifold geometry, a fact that can be useful in approaching several issues like the cosmological constant problem, black hole physics and holographic ideas, where there is a relationship between the position in the extra dimension and the mass scale.

We encountered two different cases regarding the topology of the extra dimension: an extended and a compact one. In the non–compact case A), the warp factor reproduces the
metric of the RS model in the thin brane limit, even if the matter content of the theory does not correspond to the same brane configuration. In the compact case B) the situation is different: we have several pairs of thick brane configurations disconnected by physical singularities. The structure of these branes depends on the value of the parameter $p(\xi)$. In two special cases ($p = 3/8$ and $p = 3/4$) we managed to invert the coordinate transformation (10) which makes the metric conformally flat, decompactifies the fifth dimension and simultaneously pushes the singularities of the manifold to infinity.

We wrote the wave equations of the transverse traceless modes of the linear fluctuations of the classical background into a Schrödinger’s equation form for both cases A) and B). In general, the analog quantum mechanical potential involved in it represents a volcano potential with finite bottom: a negative well located between two finite positive barriers that vanish when $z \to \pm \infty$. It turned out that for the massless zero modes ($m^2 = 0$) the Schrödinger equation can be solved in both cases. As a result of this fact, in each case we obtained an analytic expression for the lowest energy eigenfunction of the Schrödinger equation which represents a single bound state and allows for the existence of a stable 4D graviton since there are no tachyonic modes with $m^2 < 0$. Apart from these massless states, we also got a continuum and gapless spectrum of massive KK modes with positive $m^2 > 0$ that are suppressed at $y = c_2$ and turn asymptotically into continuum plane waves in both cases A) and B), as in [4], [6] and [10].

The shape of the analog quantum mechanical potential and the localization of 4D gravity on smooth branes with a continuum and gapless spectrum of massive KK modes are quite similar to those obtained by [4], [6] and [7].

We finally found that for the particular case ($p = 3/4$) the Schrödinger equation can be integrated for both the massless and the massive modes of the KK excitations. On one hand, for the zero mode we get a normalizable wave function that represents again a single bound state interpreted as a stable 4D graviton since it has no tachyonic modes with negative squared mass $m^2 < 0$. On the other hand, for the massive modes of the KK excitations we get expressions in terms of associated Legendre functions of first and second kind, a remarkable fact that open to us the possibility of analytically investigating the massive spectrum of KK states, a rather scarce phenomenon when studying thick brane configurations. We are currently analyzing the normalizability of the massive modes of the KK excitations (see [12]–[14]) and hope to present our results in the near future.

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