On Variants of Facility Location Problem with Outliers

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Abstract. In this work, we study the extension of two variants of the facility location problem (FL) to make them robust towards a few distantly located clients. First, $k$-facility location problem ($k$FL), a common generalization of FL and $k$ median problems, is a well studied problem in literature. In the second variant, lower bounded facility location (LBFL), we are given a bound on the minimum number of clients that an opened facility must serve. Lower bounds are required in many applications like profitability in commerce and load balancing in transportation problem. In both the cases, the cost of the solution may be increased grossly by a few distantly located clients, called the outliers. Thus, in this work, we extend $k$FL and LBFL to make them robust towards the outliers. For $k$FL with outliers ($k$FLO) we present the first (constant) factor approximation violating the cardinality requirement by $+1$. As a by-product, we also obtain the first approximation for FLO based on LP-rounding. For LBFL, we present a tri-criteria solution with a trade-off between the violations in lower bounds and the number of outliers. With a violation of $1/2$ in lower bounds, we get a violation of $2$ in outliers.

Keywords: Facility Location · Outliers · Approximation · Lower Bound · $k$-Facility Location · $k$-Median.

1 Introduction

Consider an e-retail company that wants to open warehouses in a city for home delivery of essential items. Each store has an associated opening cost depending on the location in the city. The aim of the company is to open these warehouses at locations such that the cost of opening the warehouses plus the cost servicing all the customers in the city from the nearest opened store is minimised. In literature, such problems are called facility location problems (FL) where warehouses are the facilities and customers are the clients. Formally, in FL we are given a set $\mathcal{F}$ of $n$ facilities and a set $\mathcal{C}$ of $m$ clients. Each facility $i \in \mathcal{F}$ has an opening cost $f_i$ and cost of servicing a client $j \in \mathcal{C}$ from a facility $i \in \mathcal{F}$ is $c(i, j)$ (we assume that the service costs are metric). The goal is to open a subset $\mathcal{F}' \subseteq \mathcal{F}$ of facilities such that the cost of opening the facilities and servicing the clients from the opened facilities is minimised. In a variant of FL, called $k$-facility location problem ($k$FL), we are given an additional bound $k$ on the maximum number...
of warehouses/facilities that can be opened i.e. $|F'| \leq k$. In our example this requirement may be imposed to maintain the budget constraints or to comply with government regulations. In another variant of the problem, we are required to serve some minimum number of customers/clients from an opened facility. Such a requirement is natural to ensure profitability in our example. This minimum requirement is captured as lower bounds in facility location problems. That is, in lower bounded FL (LBFL), we are also given a lower bound $L_i$ on the minimum number of clients that an opened facility $i$ must serve.

In the above scenarios, a few distant customers/clients can increase the cost of the solution disproportionately; such clients are called outliers. Problem of outliers was first introduced by Charikar et al. [3] for the facility location and the $k$-median problems. In this paper we extend $k$-facility location and lower bounded facility location to deal with the outliers and denote them by $kFLO$ and $LBFLO$ respectively. Since FL is well known to be NP-hard, NP-hardness of $kFLO$ and $LBFLO$ follows. We present the first (constant factor) approximation for $kFLO$ opening at most $k + 1$ facilities. In particular, we present the following result:

**Theorem 1.** There is a polynomial time algorithm that approximates $k$-facility location problem with outliers opening at most $(k + 1)$ facilities within 11 times the cost of the optimal solution.

Our result is obtained using LP rounding techniques. As a by product, we get first constant factor approximation for FLO using LP rounding techniques. FLO is shown to have an unbounded integrality gap [3] with solution to the standard LP. We get around this difficulty by guessing the most expensive facility opened in the optimal solution. In particular we get the following:

**Corollary 1.** There is a polynomial time algorithm that approximates facility location problem with outliers within 11 times the cost of the optimal solution.

We reduce $LBFLO$ to $FLO$ and use any algorithm to approximate $FLO$ to obtain a tri-criteria solution for the problem. To the best of our knowledge, no result is known for $LBFLO$ in literature. In particular, we present our result in Theorem 2 where a tri-criteria solution is defined as follows:

**Definition 1.** A tri-criteria solution for $LBFLO$ is an $(\alpha, \beta, \gamma)$- approximation solution $S$ that violates lower bounds by a factor of $\alpha$ and outliers by a factor of $\beta$ with cost no more than $\gamma OPT$ where $OPT$ denotes the cost of an optimal solution of the problem, $\alpha < 1$ and $\beta > 1$.

**Theorem 2.** A polynomial time $(\alpha, \frac{1}{1-\alpha}, \lambda(\frac{1+\alpha}{1-\alpha}))$-approximation can be obtained for $LBFLO$ problem where $\alpha = (0, 1)$ is a constant and $\lambda$ is an approximation factor for the FLO problem.

Theorem 2 presents a trade-off between the violations in the lower bounds and that in the number of outliers. Violation in outliers can be made arbitrarily small by choosing $\alpha$ close to 0. And, violation in lower bounds can be chosen close to 1.
at the cost of increased violation in the outliers. Similar result can be obtained for LBkFLO with +1 violation in cardinality using Theorem 1. The violation in the cardinality comes from that in kFLO.

**Our Techniques:** For kFLO, starting with an LP solution $\rho^* = \langle x^*, y^* \rangle$, we first eliminate the $x^*_{ij}$ variables and work with an auxiliary linear programming (ALP) relaxation involving only $y_i$ variables. This is achieved by converting $\rho^*$ into a complete solution in which either $x^*_{ij} = y^*_i$ or $x^*_{ij} = 0$. Using the ALP, we identify the set of facilities to open in our solution. ALP is solved using iterative rounding technique to give a pseudo-integral solution (a solution is said to be pseudo integral if there are at most two fractional facilities). We open both the facilities at +1 loss in cardinality and at a loss of factor 2 in the cost by guessing the maximum opening cost of a facility in the optimal. Once we identify the set of facilities to open, we can greedily assign the first $m - t$ clients in the increasing order of distance from the nearest opened facility. Thus, in the rest of the paper, we only focus on identifying the set of facilities to open.

For LBkFLO, we construct an instance $I'$ of FLO by ignoring the lower bounds and defining new facility opening cost for each $i \in F$. An approximate solution $AS'$ to $I'$ is obtained using any approximation algorithm for FLO. Facilities serving less than $\alpha L_i$ clients are closed and their clients are either reassigned to the other opened facilities or are made outliers. This leads to violation in outliers that is bounded by $1 - 1/\alpha$. Facility opening costs in $I'$ are defined to capture the cost of reassignments.

**Related Work:** The problems of facility location and $k$-median with outliers were first defined by Charikar et al. [3]. Both the problems were shown to have unbounded integrality gap [3] with their standard LPs. For FLO, they gave a $(3 + \epsilon)$-approximation using primal dual technique by guessing the most expensive facility opened by the optimal solution. For a special case of the problem with uniform facility opening costs and doubling metrics, Friggstad et al. [5] gave a PTAS using multiswap local search. For $k$MO, Charikar et al. [3] gave a $4(1 + 1/\epsilon)$-approximation with $(1 + \epsilon)$-factor violation in outliers. Using local search techniques, Friggstad et al. [5] gave $(3 + \epsilon)$ and $(1 + \epsilon)$-approximations with $(1 + \epsilon)$ violation in cardinality for general and doubling metric respectively. Chen [3] gave the first true constant factor approximation for the problem using a combination of local search and primal dual. Their approximation factor is large and it was improved to $(7.081 + \epsilon)$ by Krishnaswamy et al. [10] by strengthening the LP. They use iterative rounding framework and, their factor is the current best result for the problem.

Lower bounds in FL were introduced by Karger and Minkoff [9] and Guha et al. [6]. They independently gave constant factor approximations with violation in lower bounds. The first true constant factor(448) approximation was given by Zoya Svitkina [12] for uniform lower bounds. The factor was improved to 82.6 by Ahmadian and Swamy [1]. Shi Li [11] gave the first constant factor approximation for general lower bounds, with the constant being large (4000). Han et al. [7] studied the general lower bounded $k$-facility location (LBkFL) violating the
lower bounds. Same authors \cite{8} removed the violation in the lower bound for the \(k\)-Median problem.

The only work that deals with lower bound and outliers together is by Ahmadian and Swamy \cite{2}. They have given constant factor approximation for lower-bounded min-sum-of-radii with outliers and lower-bounded k-supplier with outliers problems using primal-dual technique.

**Organisation of the paper:** A constant factor approximation for \(k\)-FLO is given in Section 2 opening at most \((k + 1)\) facilities. In Section 3, the tri-criteria solution for LBFLO is presented. Finally we conclude with future scope in Section 4.

## 2 \((k + 1)\) solution for kFLO

The problem kFLO can be represented as the following integer program (IP):

\[
\text{Minimize } \text{Cost}_{k\text{FLO}}(x, y) = \sum_{j \in C} \sum_{i \in F} c(i, j)x_{ij} + \sum_{i \in F} f_i y_i
\]

subject to

\[
\sum_{i \in F} x_{ij} \leq 1 \quad \forall j \in C
\]

\[
x_{ij} \leq y_i \quad \forall i \in F, j \in C
\]

\[
\sum_{i \in F} y_i \leq k
\]

\[
\sum_{j \in C} \sum_{i \in F} x_{ij} \geq m - t
\]

\[
y_i, x_{ij} \in \{0, 1\}
\]

where variable \(y_i\) denotes whether facility \(i\) is open or not and \(x_{ij}\) indicates if client \(j\) is served by facility \(i\) or not. Constraints [1] ensure that the extent to which a client is served is no more than 1. Constraints [2] ensure that a client is assigned only to an open facility. Constraint [3] ensures that the total number of facilities opened are at most \(k\) and Constraint [4] ensures that total number of clients served are at least \(m - t\). LP-Relaxation of the problem is obtained by allowing the variables \(y_i, x_{ij}\) \(\in [0, 1]\). Let us call it \(LP\).

Let \(\rho^* = <x^*, y^*>\) denote the optimal solution of \(LP\) and \(LP_{opt}\) denote the cost of \(\rho^*\). A solution is said to be a complete solution either \(x_{ij}^* = y_i^*\) or \(x_{ij}^* = 0\), \(\forall i \in F\) and \(\forall j \in C\). We first eliminate \(x\) variables from our solution \(\rho^*\) by making it complete. This is achieved by standard technique of splitting the openings and making collocated copies of facilities. For every client \(j \in C\), we will define a bundle, \(F_j\) as the set of facilities that are serving \(j\) in our complete solution. Formally, \(F_j = \{i \in F : x_{ij}^* > 0\}\). Let \(r_{F_j} = \max_{i \in F_j} c(i, j)\) be the distance of farthest facility in \(F_j\) from \(j\). See Fig. 1(a). Note that the complete solution \(<x^*, y^*>\) satisfies the following property:

1. \(\sum_{i \in F_j} y_i^* \leq 1 \quad \forall j \in C\) as \(\sum_{i \in F_j} y_i^* = \sum_{i \in F} x_{ij}^* \leq 1\).
2. \(\sum_{i \in F} y_i^* \leq k\)
3. \(\sum_{j \in C} \sum_{i \in F_j} y_i^* \geq m - t\) as \(\sum_{i \in F_j} y_i^* = \sum_{i \in F} x_{ij}^*\) and \(\sum_{j \in C} \sum_{i \in F} x_{ij}^* \geq m - t\).
2.1 Auxiliary LP (ALP)

We first discretize our distances $c(i, j)$, by rounding them to the nearest power of 2. Let $c'(i, j) = 2^r$, where $r$ is smallest power of 2 such that $c(i, j) \leq 2^r$. See Fig. 1(b). Next, we identify a set $C_{full}$ of clients that are going to be served fully in our solution. Ideally, we would like to open at least one facility in $F_j$ for every $j \in C_{full}$. If all the $F_j$'s ($j \in C_{full}$) were pair-wise disjoint, an LP constraint like $\sum_{i \in F_j} w_i \geq 1$ for all $j \in C_{full}$, along with constraints 8(for partially served clients, say clients in $C_{part}$), 9(for cardinality) and 10(for outliers), is sufficient to get us a pseudo-integral solution. But this, in general, is not true. Thus we further identify a set $C^* \subseteq C_{full}$ so that we open one facility in $F_j$ for every $j \in C^*$ and (i) $F_j$'s ($j \in C^*$) are pair-wise disjoint (disjointness property) (ii) for every $j \in C_{full} \setminus C^*$, there is a close-by (within constant factor of $rF_j$ distance from $j$) client in $C^*$. On a close observation, we notice that instead of $F_j$'s, we are rather interested in smaller sets: let $rmax_j$ be the (rounded) distance of the farthest facility in $F_j$ serving $j$ in our solution and $T_j = \{i \in F_j : c'(i, j) \leq rmax_j\}$. Then we actually want $T_j$'s ($j \in C^*$) to be pair-wise disjoint. As the distances are discretized, we have that $rmax_j$ is either $rF_j$ or is $\leq rF_j/2$. Since we don’t know $rmax_j$, once a client is identified to be in $C_{full}$, we search for it by starting with $T_j = F_j$, $rT_j = rF_j$ and, shrinking it over iterations. Shrinking is done whenever, for $B_j = \{i \in F_j : c'(i, j) \leq rmax_j/2\}$, we obtain $\sum_{i \in B_j} w_i = 1$. Thus we add a constraint $\sum_{i \in B_j} w_i \leq 1$ in our ALP and arrive at the following auxiliary LP (ALP). Variable $w_i$ denotes whether facility $i$ is opened in the solution or not. Constraints (9) and (10) correspond to the requirements of cardinality and outliers. For $j \in C_{full}$, if the ALP doesn’t open a facility within $B_j$, it bounds the cost of sending $j$ up to a distance of $rT_j$.  

Fig. 1. (a) Set $F_j$ corresponding to a client $j$, (b) Discretization of distances.
We also prove that the cost of the solutions computed is non-increasing over iterations.

We next present an iterative rounding algorithm (IRA) for solving the ALP. In each iteration, the solution obtained in an iteration is feasible for the ALP of the next iteration. Such a \( T_j \) shrinks (see Fig. 2 for illustration).

The following lemma gives a feasible solution to ALP such that cost is bounded by LP optimal within a constant factor.

**Lemma 1.** A feasible solution \( w' \) can be obtained to the ALP such that \( \text{Cost}_{\text{ALP}}(w') \leq 2L_{\text{opt}} \).

**Proof.** Let \( w'_i = y^*_i \).

1. **Feasibility:** Constraints 6 and 7 hold vacuously as \( C_{\text{full}} \) and hence \( C^* \) are empty. Constraints 8, 9 and 10 hold by properties 1, 2 and 3 respectively.

2. **Cost Bound:** As \( T_j = F_j \), \( \text{Cost}_{\text{ALP}}(w'_i) = \sum_{j \in C} \sum_{i \in F_j} c'(i, j)x^*_i + \sum_{i \in F} f_i y^*_i \leq 2 \sum_{j \in C} \sum_{i \in F_j} c(i, j)x^*_i + \sum_{i \in F} f_i y^*_i = 2L_{\text{opt}}. \) The inequality follows as \( c'(i, j) \leq 2c(i, j) \) and \( x^*_i = 1 \).

\( \Box \)

### 2.2 Iterative Rounding

We next present an iterative rounding algorithm (IRA) for solving the ALP. In every iteration of IRA, we compute an extreme point solution \( w^* \) to ALP and check whether any of the constraints 7 or 8 has become tight. If a constraint corresponding to \( j \in C_{\text{part}} \) gets tight, we move the client to \( C_{\text{full}} \) and remove it from \( C_{\text{part}} \). We also update \( C^* \) so that disjointness property is satisfied. If a constraint corresponding to \( j \in C_{\text{full}} \) gets tight, we shrink \( T_j \) to \( B_j \); update \( B_j \) and \( C^* \) accordingly. The algorithm is formally stated in Algorithm 1.

Lemmas 2, 3 and 4 help us analyse our algorithm. Lemma 2 shows that the solution obtained in an iteration is feasible for the ALP of the next iteration. We also prove that the cost of the solutions computed is non-increasing over iterations.
Algorithm 1 Iterative Rounding Algorithm

1: \( C_{\text{full}} \leftarrow \emptyset, C_{\text{part}} \leftarrow C, C^* \leftarrow \emptyset, T_j = \mathcal{F}_j, r_{T_j} = r_{\mathcal{F}_j} \)
2: while true do
3:  Find an extreme point solution \( w^* \) to ALP
4:  if there exists some \( j \in C_{\text{part}} \) such that \( \sum_{i \in T_j} w_i^* = 1 \) then
5:    \( C_{\text{part}} \leftarrow C_{\text{part}} \setminus \{j\}, C_{\text{full}} \leftarrow C_{\text{full}} \cup \{j\}, B_j \leftarrow \{i \in T_j : c'(i, j) \leq \lfloor r_{T_j}/2 \rfloor \} \)
6:    process \(-C^*(j)\).
7:  end if
8:  if there exists \( j \in C_{\text{full}} \) such that \( \sum_{i \in B_j} w_i^* = 1 \) then
9:    \( T_j \leftarrow B_j, r_{T_j} = \lfloor r_{T_j}/2 \rfloor, B_j \leftarrow \{i \in T_j : c'(i, j) \leq \lfloor r_{T_j}/2 \rfloor \} \)
10:   process \(-C^*(j)\)
11: end if
12: end while
13: Return \( w^* \)

14: process \(-C^*(j)\)
15: if there exists \( j' \in C^* \) with \( r_{T_j} < r_{T_j} \) and \( T_j \cap T_{j'} \neq \emptyset \) then
16:  \( \text{resp}(j) = j' \), if there are more than one such \( j' \)'s, choose any arbitrarily.
17: else
18:  if \( j \in C^* \) then update \( r_{T_j} \) to its new value
19:     else Add \( j \) to \( C^* \) with \( r_{T_j} \) and \( \text{resp}(j) = j \).
20: Remove all \( j' \) from \( C^* \) for which \( r_{T_j} < r_{T_{j'}} \), and \( T_j \cap T_{j'} \neq \emptyset \), \( \text{resp}(j') = j \).
21: end if

Fig. 2. (a) Initially both \( j \) and \( j' \) are in \( C_{\text{full}} \). Suppose \( \sum_{i \in T_j} w_i^* = 1 \), then \( j \) is added to \( C_{\text{full}} \) and \( B_j \) is defined for \( j \). \( j \) is added to \( C^* \) as well. (b) Subsequently, suppose \( \sum_{i \in T_{j'}} w_i^* = 1 \), then \( j' \) is added to \( C_{\text{full}} \) and \( B_{j'} \) is defined for \( j' \). \( j' \) is added to \( C^* \) whereas \( j \) is removed from \( C^* \) because \( r_{T_j} < r_{T_{j'}} \) and \( T_j \cap T_{j'} \neq \emptyset \). Next, suppose \( \sum_{i \in B_j} w_i^* = 1 \) in a future iteration, then \( T_j \) and \( B_j \) shrink. \( j \) is added to \( C^* \) again and \( j' \) is removed from \( C^* \) because after shrinking \( r_{T_j} < r_{T_{j'}} \).
Lemma 2. Let ALP_t and ALP_{t+1} be the auxiliary LPs before and after iteration t of IRA. Let w^t be the extreme point solution obtained in t^{th} iteration. Then (i) w^t is a feasible solution to ALP_{t+1}, (ii) CostALP_{t+1}(w^t) ≤ CostALP_t(w^t) and hence CostALP_{t+1}(w^{t+1}) ≤ CostALP_t(w^t).

Proof. Note that the feasibility and the cost can change only when one of condition at step 4 or condition at step 6 of the algorithm is true.

(i) When one of constraints [8] corresponding to a client j becomes tight i.e. \( \sum_{i \in T_j} w^t = 1 \), we move client j from \( C_{part} \) to \( C_{full} \) and define the set \( B_j \). Thus, \( \sum_{i \in B_j} w^t \leq \sum_{i \in T_j} w^t = 1 \). Thus the new constraints added in constraints [7] and [5] (if j is added to \( C^* \)) are satisfied. Constraint [10] holds as \( |C_{full}| \) increases by 1 and \( \sum_{j \in C_{part}} \sum_{i \in T_j} w^t \) decreases by 1. There is no change in constraint [9]

Let one of the constraints [7] corresponding to a full client j becomes tight i.e. \( \sum_{i \in B_j} w^t = 1 \). Thus constraint [6] is satisfied if j is added to \( C^* \). (ii) shrink \( B_j \) to half its radius, thus \( \sum_{i \in B_j} w^t \leq \sum_{i \in T_j} w^t = 1 \). Thus constraint [7] corresponding to j continue to be satisfied with the shrunk \( B_j \). There is no change in constraints [9] and [10]

(iii) For a client j, let \( rF_j, B_j^t \) and \( T_j^t \) be the set \( rF_j, B_j \) and \( T_j \) corresponding to client j in ALP_t and \( rF_j^{t+1}, B_j^{t+1} \) and \( T_j^{t+1} \) be the respective values in ALP_{t+1}.

a. When \( T_j \) and \( B_j \) shrink because constraint [7] becomes tight for a client j. Cost paid by j in \( w^t \) in the t^{th} iteration = \( \sum_{i \in B_j^t} c'(i,j)w^t \) because \( \sum_{i \in B_j^t} w^t = 1 \) in the t^{th} iteration. Since \( B_j^t = T_j^{t+1} \), \( \sum_{i \in B_j^t} c'(i,j)w^t = \sum_{i \in T_j^{t+1} \cap T_j^{t+1}} c'(i,j)w^t + \sum_{i \in T_j^{t+1} \cap T_j^{t+1}} c'(i,j)w^t = \sum_{i \in B_j^{t+1}} c'(i,j)w^t + (1 - \sum_{i \in B_j^{t+1}} w^t)rF_j^{t+1} = \) Cost paid by j in \( w^t \) in the \((t+1)^{th}\) iteration. Thus change in cost is 0.

b. When a client j is moved from \( C_{part} \) to \( C_{full} \) because constraint [8] becomes tight. Cost paid by j in \( w^t \) in the t^{th} iteration = \( \sum_{i \in T_j^t} c'(i,j)w^t = \sum_{i \in T_j^t \cap T_j^t} c'(i,j)w^t + \sum_{i \in T_j^t \cap T_j^t} c'(i,j)w^t = \sum_{i \in B_j^{t+1}} c'(i,j)w^t + (1 - \sum_{i \in B_j^{t+1}} w^t)rT_j^{t+1} = \) Cost paid by j in \( w^t \) in the \((t+1)^{th}\) iteration. Thus change in cost is 0.

Thus we have, \( CostALP_{t+1}(w^{t+1}) \leq CostALP_{t+1}(w^t) = CostALP_t(w^t) \) where the first inequality follows because \( w^{t+1} \) is a feasible solution to \( ALP_{t+1} \). Hence, if \( n \) is the number of iterations of the IRA then \( CostALP_n(w^*) \leq CostALP_1(w^1) \leq CostALP(w^1) \leq 2LP_{opt} \) where the second last inequality follows as \( w^1 \) is an extreme point solution and \( w^* \) is a feasible solution for \( ALP = ALP_1 \), last inequality follows from Lemma 1.

Let \( w^* \) be the solution returned by the IRA, then \( w^* = w^n \). Lemma 3 establishes that at the end of our IRA, solution \( w^* \) is pseudo-integral.
**Lemma 3.** \( w^* \) returned by Algorithm 1 has at most two fractionally opened facilities.

**Proof.** At the termination of the algorithm constraints \([7]\) and \([8]\) will not be tight. Let \( n_f \) be the number of fractional variables at the end of the algorithm. Then there are exactly \( n_f \) number of independent tight constraints from \([6]\), \([9]\) and \([10]\). Let \( X \) be the number of tight constraints of type \([6]\). There must be at least 2 fractional variables corresponding to each of these constraints. Also, there must be at least 2 fractional variables corresponding to constraint \([6]\) different from those obtained constraints \([10]\). Thus, \( n_f \geq 2X + 2 \) i.e. \( X \leq n_f/2 - 1 \). Also, the number of tight constraints is at most \( X + 2 \) and hence is at most \( n_f/2 + 1 \) giving us \( n_f \leq n_f/2 + 1 \) or \( n_f \leq 2 \).

We open both the fractionally opened facilities at a loss of \(+ f_{max}\) in the facility opening cost where \( f_{max} \) is the guess of the most expensive facility opened by the optimal. In Lemma 4 we show that for a client \( j \) in \( C_{full} \setminus C^* \) there is some client in \( C^* \), that is close to \( j \), i.e. within \( 5rT_j \) distance of \( j \).

**Lemma 4.** At the conclusion of the algorithm, for every \( j \in C_{full} \), there exists at least 1 unit of open facilities within distance \( 5rT_j \) from \( j \). Formally, \( \sum_{i: c(i,j) \leq 5rT_j} w_i \geq 1 \).

**Proof.** Let \( j \in C_{full} \). If \( resp(j) = j \), then this means that \( j \) was added to \( C^* \) and was present in \( C^* \) at the end of the algorithm. Then, one unit is open in \( T_j \) i.e. within a distance of \( rT_j \) of \( j \).

If \( resp(j) = j' (\neq j) \) then \( j \) was either never added to \( C^* \) or was removed later. In either case responsibility of opening a facility in a close vicinity of \( j \) was taken by \( j' \). First we consider the case when \( j \) was added to \( C^* \) but removed later. Let \( j_0, j_1, \ldots, j_r \) be the sequence of clients such that \( resp(j_i) = j_{i-1}, i = 1 \ldots r, resp(j_0) = j_0 \) and \( j_r = j \). Since \( resp(j_0) = j_0 \), one unit is open in \( T_{j_0} \) i.e. within a distance of \( rT_{j_0} \) of \( j_0 \). Clearly, \( rT_{j_0} \leq rT_{j_0}/2 \). Thus \( rT_{j_0} \leq (1/2)^{r-i} rT_{j_i} \) for all \( i = 0 \ldots r-1 \). Thus, \( c'(j_r, j_0) \leq \sum_{i=1}^{r} c'(j_i, j_{i-1}) \leq \sum_{i=1}^{r} (rT_{j_i} + rT_{j_i-1}) \leq rT_{j_0} + 2 \sum_{i=1}^{r} rT_{j_i} \leq rT_{j_0} + 2 \sum_{i=1}^{r} (1/2)^{r-i} rT_{j_i} \). Thus one unit of facility is open within a distance of \( 2 \sum_{i=0}^{r-1} (1/2)^{r-i} rT_{j_i} + rT_{j_r} \), \( rT_{j_r} \leq 3rT_{j_r} \) from \( j \).

Next, let \( j \) was never added to \( C^* \). Then since \( resp(j) = j' (\neq j) \), \( j' \) was added to \( C^* \) at some point of time. Thus, from above one unit of facility is opened within distance \( 3rT_{j_r} \) of \( j' \). Also, \( c'(j, j') \leq rT_{j_r} \). Thus, one unit of facility is opened within distance \( 5rT_{j_r} \) of \( j \).

We run the algorithm for all the guesses of \( f_{max} \) and select the one with the minimum cost.

**Combining Everything:** Let \( \bar{w} \) be our final solution. \( \text{Cost}(\bar{w}) \leq \text{Cost}(w^*) + f_{max} \leq 5 \cdot \text{CostALP}(w^*) + f_{max} \leq 5 \cdot 2 \cdot \text{LP}_{\text{opt}} + f_{max} \leq 11 \text{OPT} \) where \( \text{OPT} \) is the cost of the optimal solution.
3 Tri-criteria for LBFLO

In this section, we present a tri-criteria solution for LBFLO problem with 
\( \alpha = (0,1) \)-factor violation in lower bound and at most \( \beta = (\frac{1}{1-\alpha}) \)-factor violation in outliers at \( (\frac{1}{1-\alpha}) \)-factor loss in cost where \( \lambda \) is approximation for FLO. Let \( I \) be an instance of LBFLO. For a facility \( i \), let \( \mathcal{N}_i \) be the set of \( \mathcal{L}_i \) nearest clients. We construct an instance \( I' \) of FLO with lower bounds ignored and facility costs updated as follows: if a facility \( i \) is opened in optimal solution of \( I \), then it pays at least \( \sum_{j \in \mathcal{N}_i} c(i,j) \) cost for serving \( \mathcal{N}_i \) clients. Therefore, \( f'(i) = f(i) + \delta \sum_{j \in \mathcal{N}_i} c(i,j) \) where \( \delta \) is a tunable parameter.

**Lemma 5.** Optimal solution of \( I' \) is bounded by \( (\delta + 1)\text{Cost}_I(O) \) where \( O \) is the optimal solution of \( I \).

**Proof.** Clearly \( O \) is a feasible solution for \( I' \). Thus, service cost is same as that in \( O \). And, \( \sum_{i \in O} f'(i) = \sum_{i \in O} [f(i) + \delta \sum_{j \in \mathcal{N}_i} c(i,j)] \leq \delta \text{Cost}_I(O) \). Therefore, \( \text{Cost}_{I'}(I') \leq (\delta + 1)\text{Cost}_I(O) \).

Once we have an instance \( I' \) of FLO, we use any algorithm for FLO to get a solution \( AS' \) to \( I' \) of cost no more than \( \lambda \text{Cost}_{I'}(O') \) where \( O' \) is the optimal solution of \( I' \) and \( \lambda \) is approximation solution for FLO. Note that a facility \( i \) opened in solution \( AS' \) might serve less than \( \alpha \mathcal{L}_i \) clients as we ignored the lower bounds in instance \( I' \). We close such facilities and do some reassignments to improve the violation in the lower bounds to \( \alpha \); in the process we make some violation in the number of outliers.

We convert the solution \( AS' \) to a solution \( AS \) of LBFLO. We close every facility \( i \) that is serving less than \( \alpha \mathcal{L}_i \) clients in \( AS' \) and either reassign its clients to other opened facilities or decide to leave them unserved. Cost of reassignment is charged to the facility opening costs of the closed facilities. Consider a facility \( i \) open in \( AS' \) that served less than \( \alpha \mathcal{L}_i \) clients. Let \( \mathcal{C}_i \) be the set of clients, in \( \mathcal{N}_i \), assigned to \( i \) in \( AS' \) and \( \overline{\mathcal{C}}_i \) be the remaining clients in \( \mathcal{N}_i \). Since \( i \) serves \( < \alpha \mathcal{L}_i \) clients, \( |\mathcal{C}_i| \geq (1-\alpha)\mathcal{L}_i \). Some of the clients in \( \mathcal{C}_i \) are outliers in \( AS' \) and some are assigned to other facilities. Let \( \mathcal{O}_i \) be the clients in \( \mathcal{N}_i \) that are outliers and \( \mathcal{R}_i \) be the clients in \( \mathcal{N}_i \) assigned to some other facilities. See Fig. 3(a). If \( \mathcal{R}_i \neq \phi \) then let \( j' \in \mathcal{R}_i \) be the nearest client to \( i \), then,

\[
\frac{[\sum_{j \in \mathcal{R}_i} c(i,j)]}{|\mathcal{R}_i|} \leq \sum_{j \in \mathcal{N}_i} c(i,j) \leq \sum_{j \in \mathcal{N}_i} c(i,j) |\mathcal{R}_i| |\mathcal{O}_i| \quad (12)
\]

Clients in \( \mathcal{C}_i \) are assigned to the facilities serving the clients in \( \mathcal{R}_i \) and are made outliers proportionally. That is, we assign \( \frac{|\mathcal{R}_i|}{|\mathcal{R}_i| + |\mathcal{O}_i|} |\mathcal{C}_i| \) clients in \( \mathcal{C}_i \) to the nearest facility \( i' \neq i \) opened in \( AS' \) and leave \( \frac{|\mathcal{O}_i|}{|\mathcal{R}_i| + |\mathcal{O}_i|} |\mathcal{C}_i| \) clients in \( \mathcal{C}_i \) unserved. If \( |\mathcal{R}_i| \neq 0 \), then the total cost of reassignment is \( \sum_{j \in \mathcal{C}_i} c(j,i') \leq \sum_{j \in \mathcal{C}_i} [c(i,j) + c(i,j') + c(j',i')](\text{by triangle inequality, see Fig. 3(b)}) \leq \sum_{j \in \mathcal{C}_i} c(i,j) + (\frac{|\mathcal{R}_i|}{|\mathcal{R}_i| + |\mathcal{O}_i|}|\mathcal{C}_i| \cdot 2c(i,j')) \)

(as \( j' \) was assigned to \( i' \) and not to \( i \) in \( AS' \)) \leq \sum_{j \in \mathcal{C}_i} c(i,j) + (\frac{|\mathcal{R}_i|}{|\mathcal{R}_i| + |\mathcal{O}_i|} \cdot 2c(i,j')).
Variants of Facility Location Problem with Outliers

Fig. 3. (a) Division of clients in $N_i$ for a facility $i$ opened in $AS$ (b) $c(j, i') \leq c(i, j) + c(i, j') + c(j', i')$

$$
\sum_{j \in N_i} c(i, j) \quad \text{(using (12))} \leq \sum_{j \in C_i} c(i, j) + \frac{2\alpha L_i}{(1 - \alpha) L_i} \sum_{i \in N_i} c(i, j) \quad \text{(As $|C_i| \leq \alpha L_i$ and $|R_i| + |O_i| \geq (1 - \alpha) L_i$)} \leq \sum_{j \in C_i} c(i, j) + f'(i) \quad \text{(for $\delta \geq \frac{2\alpha}{1 - \alpha}$).}
$$

Overall Cost Bound: It is easy to see that $Cost_I(AS) = Cost_I'(AS')$ as cost of solution $AS$ is sum of (i) the original connection cost which is equal to the connection cost of $AS'$, (ii) the additional cost of reassignment, which is paid in $AS'$ by facilities that are closed in $AS$ and, (iii) the facility cost of the remaining facilities. Thus, $Cost_I(AS) = Cost_I'(AS') \leq \lambda Cost_I'(O') \leq \lambda(1 + \delta) Cost_I(O) = \lambda \left( \frac{1 + \alpha}{1 - \alpha} \right) Cost_I(O)$ for $\delta = \frac{2\alpha}{1 - \alpha}$. Using $\lambda = (3 + \epsilon)$-approximation of Charikar et al. [3] for FLO, we get $(3 + \epsilon) \left( \frac{1 + \alpha}{1 - \alpha} \right)$ factor loss in cost for $\epsilon > 0$.

4 Conclusion and Future Scope

In this paper, we first presented a 11-factor approximation for $k$-facility location problem with outliers opening at most $k + 1$ facilities. This also gives us the first constant factor approximation for FLO using LP rounding techniques. Our result can be extended to knapsack median problem with outliers with $(1 + \epsilon)$ violation in budget using enumeration techniques.

We also gave a tri-criteria, $(\alpha, \frac{1}{1 - \epsilon}, (3 + \epsilon) \frac{1 + \alpha}{1 - \alpha})$-solution for general LB$k$FLO where $\alpha = (0, 1)$ and $\epsilon > 0$. It will be interesting and challenging to see if we can reduce the violation in outliers to $< 2$ maintaining $\alpha > 1/2$.

We believe that using pre-processing and strengthened LP techniques of Krishnaswamy et al. [10] we can get rid of the $+1$ violation in cardinality for $k$FLO. This will also directly extend our tri-criteria solution to lower bounded $k$-facility location problem with outliers (LB$k$FLO).

References

1. Ahmadian, S., Swamy, C.: Improved approximation guarantees for lower-bounded facility location. In: Erlebach, T., Persiano, G. (eds.) Approximation and Online Algorithms. pp. 257–271. Springer Berlin Heidelberg, Berlin, Heidelberg (2013)
2. Ahmadian, S., Swamy, C.: Approximation Algorithms for Clustering Problems with Lower Bounds and Outliers. In: Chatzigiannakis, I., Mitzenmacher, M., Rabani, Y., Sangiorgi, D. (eds.) 43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016). Leibniz International Proceedings in Informatics (LIPIcs), vol. 55, pp. 69:1–69:15. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2016). https://doi.org/10.4230/LIPIcs.ICALP.2016.69
3. Charikar, M., Khuller, S., Mount, D.M., Narasimhan, G.: Algorithms for facility location problems with outliers. In: Proceedings of the Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms. p. 642–651. SODA ’01, Society for Industrial and Applied Mathematics, USA (2001)
4. Chen, K.: A constant factor approximation algorithm for k-median clustering with outliers. In: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms. p. 826–835. SODA ’08, Society for Industrial and Applied Mathematics, USA (2008)
5. Friggstad, Z., Khodamoradi, K., Rezapour, M., Salavatipour, M.R.: Approximation schemes for clustering with outliers. ACM Trans. Algorithms 15(2), 26:1–26:26 (2019). https://doi.org/10.1145/3301446
6. Guha, S., Meyerson, A., Munagala, K.: Hierarchical placement and network design problems. In: Proceedings of the 41st Annual Symposium on Foundations of Computer Science. p. 603. FOCS ’00, IEEE Computer Society, USA (2000)
7. Han, L., Hao, C., Wu, C., Zhang, Z.: Approximation algorithms for the lower-bounded k-median and its generalizations. In: Kim, D., Uma, R.N., Cai, Z., Lee, D.H. (eds.) Computing and Combinatorics. pp. 627–639. Springer International Publishing, Cham (2020)
8. Han, L., Hao, C., Wu, C., Zhang, Z.: Approximation algorithms for the lower-bounded knapsack median problem. In: Zhang, Z., Li, W., Du, D.Z. (eds.) Algorithmic Aspects in Information and Management. pp. 119–130. Springer International Publishing, Cham (2020)
9. Karget, D.R., Minkoff, M.: Building steiner trees with incomplete global knowledge. In: Proceedings 41st Annual Symposium on Foundations of Computer Science. pp. 613–623 (2000). https://doi.org/10.1109/SFCS.2000.892329
10. Krishnaswamy, R., Li, S., Sandeep, S.: Constant approximation for k-median and k-means with outliers via iterative rounding. In: Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing. p. 646–659. STOC 2018, Association for Computing Machinery, New York, NY, USA (2018). https://doi.org/10.1145/3188745.3188882
11. Li, S.: On facility location with general lower bounds. In: Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms. p. 2279–2290. SODA ’19, Society for Industrial and Applied Mathematics, USA (2019)
12. Svitkina, Z.: Lower-bounded facility location. ACM Trans. Algorithms 6(4), (Sep 2010). https://doi.org/10.1145/1824777.1824789