COMPACTNESS OF ISO-RESONANT POTENTIALS FOR SCHRÖDINGER OPERATORS IN DIMENSIONS ONE AND THREE

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Abstract. We prove compactness of a restricted set of real-valued, compactly supported potentials $V$ for which the corresponding Schrödinger operators $H_V$ have the same resonances, including multiplicities. More specifically, let $B_R(0)$ be the ball of radius $R > 0$ about the origin in $\mathbb{R}^d$, for $d = 1, 3$. Let $I_R(V_0)$ be the set of real-valued potentials in $C^\infty_0(B_R(0); \mathbb{R})$ so that the corresponding Schrödinger operators have the same resonances, including multiplicities, as $H_{V_0}$. We prove that the set $I_R(V_0)$ is a compact subset of $C^\infty_0(B_R(0))$ in the $C^\infty$-topology. An extension to Sobolev spaces of less regular potentials is discussed.

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1. Statement of the problem and results

We are interested in the resonances of Schrödinger operators
\[ H_V = -\Delta + V, \]
acting on \( L^2(\mathbb{R}^d) \), for \( d = 1, 3 \), with compactly supported, real-valued potentials \( V \in C_0^\infty(\mathbb{R}^d) \). Resonances are the poles of the meromorphic continuation of the cut-off resolvent of \( H_V \) defined as follows. For any \( V \in C_0^\infty(\mathbb{R}^d) \), we denote by \( \chi_V \in C_0^\infty(\mathbb{R}^d) \) any smooth, real-valued, compactly supported function such that \( \chi_V V = V \). For \( \text{Im} \lambda > 0 \), the cut-off resolvent
\[ R_V(\lambda) := \chi_V(H_V - \lambda^2)^{-1}\chi_V \]
is meromorphic operator-valued function with at most finitely-many poles on the positive imaginary axis. These poles at \( i\lambda_j \), with \( \lambda_j > 0 \), are distinguished by the fact that \( -\lambda_j^2 \) is a negative eigenvalue of \( H_V \). This operator-valued function \( R_V(\lambda) \) admits a meromorphic continuation as a bounded operator on \( L^2(\mathbb{R}^d) \) across the real axis to the lower half-complex plain when \( d \geq 1 \) is odd. The poles of the continuation are the resonances of \( H_V \). They are independent of the choice of cut-off \( \chi_V \) satisfying the above conditions.

Our goal is to characterize the set of potentials so that the corresponding Schrödinger operators are iso-resonant, that is, they have the same resonances with the same multiplicities. However, if \( V \) and \( \tilde{V} \) are related by a Euclidean symmetry, the Schrödinger operators \( H_V \) and \( H_{\tilde{V}} \) are iso-resonant. Consequently, we must remove translational invariance from the problem to get a meaningful result. We fix a ball \( B_R(0) \) of finite radius \( R > 0 \) and consider a fixed potential \( V_0 \in C_0^\infty(\overline{B_R(0)}; \mathbb{R}) \). The iso-resonant set of \( V_0 \), denoted here by \( \mathcal{I}_R(V_0) \), is the set of all \( V \in C_0^\infty(\overline{B_R(0)}) \) so that \( H_V \) has the same resonances as \( H_{V_0} \), including multiplicities. This set is invariant under the action of the compact rotation group. Our main result is the following theorem.

**Theorem 1.** The set \( \mathcal{I}_R(V_0) \) of iso-resonant potentials to \( V_0 \in C_0^\infty(\overline{B_R(0)}) \) is compact in the \( C^\infty \)-topology in dimensions \( d = 1, 3 \).

Theorem 1 is the analog of the compactness result for iso-spectral potentials of Brüning [5]. He considered a compact Riemannian manifold \( (M, g) \) and Schrödinger operators \( H_V = -\Delta_g + V \), where \( -\Delta_g \geq 0 \) is the Laplace-Beltrami operator on \( M \) for metric \( g \), and the real-valued potential \( V \in C^\infty(M) \). These self-adjoint, lower semi-bounded operators \( H_V \) have only discrete spectrum accumulating at infinity. For a fixed metric \( g \), given \( V_0 \in C^\infty(M) \), the iso-spectral set of \( V_0 \) is the set of real-valued potentials \( V \in C^\infty(M) \) so that the corresponding Schrödinger operators have the same eigenvalues, including multiplicity, as \( H_{V_0} \). For \( \dim M = 2, 3 \), Brüning proved that the iso-spectral set of \( V_0 \) is compact in the \( C^\infty \)-topology. The restriction of the dimension comes
from use of the Sobolev Embedding Theorem, Theorem [19]. This result was improved and extended by Donnelly [10].

The existence of infinitely-many resonances for Schrödinger with nonzero, real-valued potentials \( V \in C_0^\infty(\mathbb{R}^d; \mathbb{R}) \) in odd dimensions greater than or equal to three was proven first by Melrose [18] using the Poisson formula and wave invariants discussed in section [3]. This result was extended for more general potentials, including super-exponentially decaying potentials of noncompact support, by Sá Barreto and Zworski [22, 23]. The existence of resonances for Schrödinger operators in even dimensions \( 4 \leq d \) was proven by Sá Barreto and Tang [21]. We note that these results prove the resonance-rigidity of the zero potential. Since \( H_0 = -\Delta \) has no resonances, the iso-resonance set of \( V_0 = 0 \) contains precisely one element among super-exponentially decaying real-valued potentials. This rigidity is also a consequence of the analysis of the heat invariants in section [3].

The present work focuses on real-valued potentials. Christiansen [6, 7] proved that there are many complex-valued potentials for which the Schrödinger operator \( H_V \) has no resonances. This implies that among complex-valued potentials, the iso-resonance set of \( V_0 = 0 \) is large. In fact, Christiansen [6] constructs a large family of nontrivial complex-valued potentials with no resonances.

In a recent work [25], Smith and Zworski relaxed the smoothness assumption and proved that a nontrivial real-valued potential in \( L_0^\infty(B_R(0); \mathbb{R}) \), for \( d \geq 3 \) odd, has at least one resonance. Furthermore, they proved that if two potentials \( V_j \in L_0^\infty(\mathbb{R}^d; \mathbb{R}) \), for \( j = 1, 2 \), are iso-resonant, then \( V_1 \in H^m(\mathbb{R}^d) \) if and only if \( V_2 \in H^m(\mathbb{R}^d) \), for any \( m \in \mathbb{N} \). We can then define iso-resonant classes for less regular potentials. If \( V_0 \in H^m(B_R(0); \mathbb{R}) \), for \( m \geq 2 \), then \( V_0 \in L_0^\infty(B_R(0); \mathbb{R}) \) since \( d \leq 3 \).

We define \( \mathcal{I}_R^m(V_0) \subset H^m(B_R(0)) \) as the iso-resonant class of \( V_0 \) in \( H^m(B_R(0)) \). Although the Poisson formula (3.7) holds for the wave trace if \( V \in L_0^\infty(\mathbb{R}^d) \), we do not know if a finite term small \( t \)-expansion holds for the wave trace as Smith and Zworski [25] proved for the heat trace (see section [6] for a discussion.) As a consequence, we must make an assumption:

\textbf{WTE.} For \( V \in H^m(B_R(0)) \), with \( m \geq 2 \) and \( d = 3 \), the wave trace admits a small \( t \) asymptotic expansion of the form

\[
\text{Tr}(W_V(t) - W_0(t)) \sim u_1(V)\delta(t) + \sum_{j=2}^{m+2} w_j(V)|t|^{2j-3} + r_{m+2}(t), \quad (1.1)
\]

where \( r_{m+2}(t) \in C^{2m+1}(\mathbb{R}) \).

We prove the following result on the compactness of less regular iso-resonant potentials.

\textbf{Proposition 2.} We suppose the dimension \( d = 1, 3 \).

(1) For any \( V_0 \in C_0^\infty(B_R(0); \mathbb{R}) \), if a potential \( V \in L_0^\infty(B_R(0); \mathbb{R}) \) is iso-resonant to \( V_0 \), then \( V \in C_0^\infty(B_R(0); \mathbb{R}) \). Hence, the iso-resonant set of \( V_0 \) in \( L_0^\infty(B_R(0); \mathbb{R}) \) is the same as the iso-resonant set of \( V_0 \) in \( C_0^\infty(B_R(0); \mathbb{R}) \).
If $V_0 \in H^m(B_R(0); \mathbb{R})$ for $m \geq 3$, $d = 3$, and (WTE) holds, then $\mathcal{I}_R^m(V_0)$ is compact in the $H^{m-3}$-topology.

There seem to be few results on the characterization of iso-resonant potentials. The one-dimensional problem was studied by Zworski [29]. He proved that if $V_0 \in L^\infty_0(\mathbb{R}; \mathbb{R})$ and is even, the poles of the $S$-matrix determine $V_0$ uniquely if $H_{V_0}$ does not have a half-bound state (a resonance) at zero. If $V_0$ has a half-bound state at zero, Zworski proved that there exists exactly one other potential $V_1 \neq V_0$ whose $S$-matrix has the same poles. The poles of the meromorphic continuation of the $S$-matrix are contained in the set of resolvent resonances defined here. Other results on the inverse problem for Schrödinger operators on $\mathbb{R}$ may be found, for example, in Korotyaev [17], Bledsoe [4], and Bennewitz, Brown, and Wei
card [3] and references therein.

Datchev and Herazi [9] considered the iso-resonant problem for smooth, positive, real, super-exponentially decaying radial potentials $V(x) = R(|x|)$ in dimension $d \geq 1$ odd in the semiclassical regime. Additionally, the potentials are monotonic with $R'(r = 0) = 0$ and at any other point where $R(r) = 0$. They proved that if two such potentials $V_0$ and $V$ have the same resonance set up to $o(h^2)$, then there exists a vector $x_0 \in \mathbb{R}^d$ so that $V(x) = V_0(x - x_0)$. This shows that the resonances uniquely determine, in the semiclassical sense, the radial potential up to translations.

There are some results on iso-resonant and iso-phasal metrics for the Laplace-Beltrami operator on various families of noncompact Riemannian manifolds. Hassell and Zworski [14] proved uniqueness for obstacle scattering in three-dimensions. If a bounded, connected obstacle $\Omega \subset \mathbb{R}^3$ has the same resonances as a ball of the same volume, then the region is such a ball. This proves resonance rigidity of the ball in $\mathbb{R}^3$.

We mention that Theorem 1 should hold for $d = 2$. However, the Poisson formula of Zworski [30] is not strong enough to allow us to prove that the wave invariants are constant across $\mathcal{I}_R(V_0)$ from the small $t$-asymptotics of the wave trace. In recent work [8], Christiansen proved (among other results) that the heat invariants are constant across the iso-resonant class $\mathcal{I}_R(V_0)$ for Schrödinger operators in even dimensions $d \geq 2$ using different methods. As a consequence, Christiansen proves [8, Theorem 1.3] compactness of the iso-resonant class $\mathcal{I}_R(V_0)$ for the $d = 2$ case.

1.1. Idea of the proof and contents. The idea of the proof of Theorem 1 is as follows. The Poisson formula for the regularized trace of the wave group shows that any potential $V \in \mathcal{I}_R(V_0)$ has the same regularized wave trace as $V_0$ since they have the same resonances, including multiplicities. From the small time asymptotic expansion of the regularized wave trace, it follows that the coefficients of powers of $t$ in the expansion are the same for any $V \in \mathcal{I}_R(V_0)$. The coefficients in the small $t$-expansion of the wave trace are known constant multiples of the heat coefficients occurring in the small time expansion of the regularized heat trace. These heat invariants are integrals of $V$ and its derivatives. Using the machinery developed by Br"uning [5] and by Donnelly [10] for the iso-spectral case, extended to our noncompact case, the equality of these coefficients across the iso-resonance set imply that families of Sobolev norms

\[ (2) \text{ If } V_0 \in H^m(B_R(0); \mathbb{R}) \text{ for } m \geq 3, d = 3, \text{ and (WTE) holds, then } \mathcal{I}_R^m(V_0) \text{ is compact in the } H^{m-3}\text{-topology.} \]
are uniformly bounded on the set $\mathcal{I}_R(V_0)$. We then prove that this implies compactness of $\mathcal{I}_R(V_0)$ in the $C^\infty$-topology using the uniform bounds on the Sobolev norms and the regularized determinant.

We review basic facts concerning Schrödinger operators and their resonances in section 2 

We define the regularized determinant and introduce the analytic functions whose zeros coincide with the resonances of $H_V$. We prove a continuity result for these functions. The Poisson formula is described in section 3. Compactness of $\mathcal{I}_R(V_0)$ for smooth potentials is proved in section 5. The case of less regular potentials is presented in section 6.

1.2. Notation. For an open subset $\Omega \subset \mathbb{R}^d$, with boundary satisfying the cone condition, see Theorem [19], we denote the associated Sobolev spaces by $H^{s,p}(\Omega)$ and their norms by $\| \cdot \|_{s,p}$. When $s = 0$, we write $H^{0,p}(\Omega)$ as $L^p(\Omega)$, and we write $\| \cdot \|_p$ for $\| \cdot \|_{0,p}$. When $p = 2$, we write $H^{s,2}(\Omega)$ as $H^s(\Omega)$, and we write the norm as $\| \cdot \|_{s,2}$. We also write $\| \cdot \|_\infty$ for the $L^\infty$-norm.

2. Resonances of Schrödinger Operators

In this section, we recall the characterization of the resonances of a Schrödinger operator $H_V$ as the zeros of a holomorphic function and present some continuity results for this functions with respect to the potential. The resonances of $H_V$ are defined via the meromorphic extension of the cut-off resolvent $\mathcal{R}_V(\lambda) = \chi_V(H_V - \lambda^2)^{-1}\chi_V$ of $H_V$. This truncated resolvent may be expressed in terms of the cut-off resolvent for the Laplacian $H_0 = -\Delta$, defined as $\mathcal{R}_0(\lambda) \equiv \chi_V(H_0 - \lambda^2)^{-1}\chi_V$. The cut-off resolvent $\mathcal{R}_0(\lambda)$ of the Laplacian $H_0 = -\Delta$ is a holomorphic, bounded, operator-valued function on $L^2(\mathbb{R}^d)$ for $\lambda \in \mathbb{C}$ for $d \geq 1$ odd, and for $\lambda \in \Lambda$, for $d \geq 4$ even. For $d = 2$, the cut-off free resolvent is holomorphic on $\Lambda \setminus \{0\}$, with a logarithmic singularity at $\lambda = 0$.

An application of the second resolvent formula allows us to write the cut-off perturbed resolvent $\mathcal{R}_V(\lambda)$ for $H_V$ as

$$\mathcal{R}_V(\lambda)(1 + V\mathcal{R}_0(\lambda)) = \mathcal{R}_0(\lambda). \tag{2.1}$$

The resolvent $R_V(\lambda)$ is analytic for $\text{Im} \lambda \gg 0$, and it is meromorphic for $\text{Im} \lambda > 0$ with at most finitely-many poles with finite multiplicities corresponding to the eigenvalues of $H_V$. Thanks to (2.1) and the meromorphic Fredholm Theorem (see, for example, [20, Theorem XIII.13]), the cut-off perturbed resolvent $\mathcal{R}_V(\lambda)$ extends as a meromorphic, bounded operator-valued function on $\mathbb{C}$ if $d \geq 1$ is odd, and onto the Riemann surface $\Lambda$ if $d \geq 4$ is even, with an additional logarithmic singularity at $\lambda = 0$ when $d = 2$. The poles of these continuations are the resonances of $H_V$. They are independent of the choice of $\chi_V$ satisfying the above conditions.

From (2.1), the resonances of the Schrödinger operator $H_V$ are the zeros of the meromorphic continuation of the bounded operator $1 + V\mathcal{R}_0(\lambda) = 1 + V\chi_V\mathcal{R}_0(\lambda)\chi_V$ to the lower-half complex plane, or the Riemann surface $\Lambda$, depending on the parity of the dimension. It is convenient to introduce a holomorphic function whose zeros correspond with these zeros. Since for $d \geq 2,$
the operator $K_V(\lambda) := VR_0(\lambda)$ is no longer trace class, it is necessary to use the $p$-regularized determinant (see, for example, [11, Appendix B.7]).

We denote the $p^{th}$ von Neumann-Schatten trace ideal of operators by $L_p$. A bounded operator $A \in L_p$ if the singular values $\mu_j(A)$ of $A$ satisfy $\sum_j \mu_j(A)^p < \infty$.

**Definition 3.** For any operator $A \in L_p$, with $p \geq 1$, we define the operator $R_p(A)$ by

\[
R_p(A) := (I + A) \exp \left( \sum_{i=1}^{p-1} \frac{(-A)^i}{i!} \right) - 1.
\]

(2.2)

It is easy to check that $R_p(A)$ is a trace class operator. As a consequence, the regularized determinant of $A$ may be defined using $R_p(A)$.

**Definition 4.** For any operator $A \in L_p$, with $p \geq 1$, the regularized $p$-determinant of $A$ is defined to be

\[
\det_p(I + A) := \det(I + R_p(A)) = \det \left[ (I + A) \exp \left( \sum_{j=1}^{p-1} \frac{(-A)^j}{j!} \right) \right].
\]

(2.3)

Applying these definitions to the resolvent of $H_0$, we have the basic result.

**Proposition 5.** Let $K_V(\lambda) := VR_0(\lambda)\chi_V$, on $L^2(\mathbb{R}^d)$, denote the extension of the bounded operator $VR_0(\lambda)\chi_V$ for $\text{Im} \lambda > 0$ to $\mathbb{C}$ or $\Lambda$, depending on the dimension, described above.

1. The operator $K_V(\lambda) := VR_0(\lambda)\chi_V$ belongs to the trace ideal $L_p$, with $p = 1$ when $d = 1$, $p = 2$ for $d = 2, 3$, and, in general, $K_V(\lambda) \in L_p$ for $p > (d + 1)/2$ when $d > 3$.

2. The regularized $p$-determinant of $K_V(\lambda)$,

\[
D_{p,V}(\lambda) := \det_p(I + K_V(\lambda)) = \det(I + R_p(K_V(\lambda)));
\]

extends to an entire function on $\mathbb{C}$ or $\Lambda$, depending on the parity of $d$.

3. The zeros of $D_{p,V}(\lambda)$ occur at values $\lambda$ for which $\lambda^2$ is an eigenvalue or resonance of $H_V$. Moreover, the multiplicities of the zeros of $D_{p,V}(\lambda)$ equal the (algebraic) multiplicities of the eigenvalues or resonances of $H_V$.

The proof of this standard theorem may be found in [11, section 3.4]. For simplicity, we will occasionally omit the index $p$ from the notation for the regularized determinant and simply write $D_V(\lambda)$.

The multiplicity $m_V(\lambda)$ of a pole of $VR_0(\lambda)$ at $\lambda$ is defined as follows (see, for example, [11, section 4.2]). The multiplicity of any nonzero resonance $\lambda_0$ of $R_V(\lambda)$ is given by

\[
m_V(\lambda_0) := \text{Rank} \left[ \int_{\gamma_0} R_V(\lambda) \, 2\lambda \, d\lambda \right],
\]

where $\gamma_0$ is a simple closed contour about $\lambda_0$ containing no other pole. The multiplicity at zero is slightly more complicated as there may be a resonance
and an eigenvalue at zero:

\[ m_V(0) := \frac{1}{2} \text{Rank} \left[ \int_{\gamma_0} R_V(\lambda) \, d\lambda \right] + \text{Rank} \left[ \int_{\gamma_0} R_V(\lambda) \, 2\lambda \, d\lambda \right]. \]

If the second integral vanishes, and the first one does not, there is only a resonance at zero energy. We can now define the resonance set of \( H_V \).

**Definition 6.** The resonance set \( \mathcal{R}(V) \) of \( H_V \) be the set of \( \lambda \in \mathbb{C} \) consisting of values \( i\lambda, \lambda > 0 \), so that \(-\lambda^2\) is an eigenvalue (necessarily nonpositive) of \( H_V \), and all \( \lambda \in \mathbb{C}^- \) that are poles of \( \mathcal{R}(\lambda) \), including multiplicities \( m_V(\lambda) \):

\[ \mathcal{R}(V) := \{ (\lambda, m_V(\lambda)) \mid m_V(\lambda) \neq 0 \}. \]

Let \( m_D(\lambda) \) be the order of zero of \( D_V(\lambda) \) at \( \lambda \). The equality of these two multiplicities is proved in Dyatlov and Zworski [11, Theorem 8.3.3]:

\[ m_V(\lambda) = m_D(\lambda). \]

This relation plays an important role in our compactness argument.

The continuity properties of \( D_V(\lambda) \) with respect to the potential \( V \) is important in order to prove the compactness of \( I_R(V_0) \).

**Proposition 7.** Suppose \( V_j \in L_0^\infty(B_R(0)) \) converges to \( V_\infty \) in the \( L^\infty \)-norm. Then the analytic functions \( D_{2,V_j}(\lambda) \) converge uniformly on compact subsets of \( \mathbb{C} \) to the analytic function \( D_{2,V_\infty}(\lambda) \).

**Proof.** 1. We give the proof for \( d = 3 \). Let \( \{V_j\} \subset L^\infty(B_R(0)) \) be a sequence of potentials converging in the \( L^\infty \)-topology to \( V_\infty \). We let

\[ \nu_0 := \max_j \{ \sup \|V_j\|_\infty, \|V_\infty\|_\infty \} \]

(2.4)

denote the uniform bound on the family of potentials. Let \( K_i(\lambda) := V_i R_0(\lambda) \chi_R \), where \( \chi_R \) is the characteristic function on \( B_R(0) \) and note that \( \chi_R V_i = V_i \), and similarly for \( K_\infty(\lambda) \). We also write \( K_0(\lambda) := \chi_R R_0(\lambda) \chi_R \). We note that these operators \( K_X(\lambda), X = 0, i, \infty, \) are in the Hilbert-Schmidt class for any \( \lambda \in \mathbb{C} \) and \( \|K_X(\lambda)\|_2 \leq C(\lambda) \). For \( d = 3 \), we recall that \( R_{2,V_j}(\lambda) := (I + K_i(\lambda)) e^{-K_i(\lambda)} - I \in I_1 \), so that by a standard trace ideal identity (see, for example [24, Theorem 3.4])

\[ |D_{V_j}(\lambda) - D_{V_\infty}(\lambda)| = \left| \det(I + R_{2,V_j}(\lambda)) - \det(I + R_{2,V_\infty}(\lambda)) \right| \]

\[ \leq \|R_{2,V_j}(\lambda) - R_{2,V_\infty}(\lambda)\|_1 \exp\{1 + \|R_{2,V_j}(\lambda)\|_1 + \|R_{2,V_\infty}(\lambda)\|_1\} \]

\[ \leq C_{\nu_0,\lambda} \|I + K_i(\lambda)e^{-K_i(\lambda)} - (I + K_\infty(\lambda))e^{-K_\infty(\lambda)}\|_1. \]

(2.5)

It follows from the convergence estimates proved below that the function

\[ C_{\nu_0,\lambda} := \exp\{1 + \|R_{2,V_j}(\lambda)\|_1 + \|R_{2,V_\infty}(\lambda)\|_1\} \]

is locally uniformly bounded in \( \lambda \) and depends only on \( \nu_0 \) defined in (2.4). Upon expanding the exponentials on the right side of (2.5), we have

\[ (I+K_i(\lambda))e^{-K_i(\lambda)} - (I+K_\infty(\lambda))e^{-K_\infty(\lambda)} = \sum_{m=2}^{\infty} \frac{(m-1)(-1)^m}{m!}[K_\infty^m(\lambda) - K_i^m(\lambda)]. \]

(2.6)
2. We consider the terms $K_i^m(\lambda) - K_1^m(\lambda)$ and factor the difference as

$$K_i^m(\lambda) - K_1^m(\lambda) = \sum_{\ell=1}^{m} K_i^\ell(\lambda)(K_i(\lambda) - K_1(\lambda))K_i^{m-\ell}(\lambda).$$

(2.7)

To compute the trace norm of the summands in (2.7), we consider two cases: $m \geq 3$ and $m = 2$. For $m \geq 3$ and $\ell \geq 3$, we estimate a typical term on the right in (2.7) as

$$\|K_i^{\ell}(\lambda)(K_i(\lambda) - K_1(\lambda))K_i^{m-\ell}(\lambda)\|_1 \leq \|V_i - V_\infty\|_\infty\|K_i^{\ell}(\lambda)\|_1\|K_0(\lambda)K_i^{m-\ell}(\lambda)\| \leq \|V_i - V_\infty\|_\infty\|K_i(\lambda)\|^{\ell}_1\|K_0(\lambda)\|\|K_i(\lambda)\|^{m-\ell}\|K_i^{m-\ell}(\lambda)\|.\] (2.8)

For $m \geq 3$ and $\ell = 2$, the bound is

$$\|V_i - V_\infty\|_\infty\|K_i(\lambda)\|\|K_0(\lambda)\|_2\|K_\infty(\lambda)\|_2\|K_\infty(\lambda)\|^{m-3},$$

and for $m \geq 3$ and $\ell = 1$, we obtain

$$\|V_i - V_\infty\|_\infty\|K_0(\lambda)\|\|K_0(\lambda)\|_2\|K_\infty(\lambda)\|_2\|K_\infty(\lambda)\|^{m-2}.$$ \[2.9\]

Similarly, for the case of $m = 2$, we find the bound of

$$\|V_i - V_\infty\|_\infty\|K_0(\lambda)\|_2 + \|K_\infty(\lambda)\|_2\|K_0(\lambda)\|_2.$$ \[2.9\]

As a consequence of (2.7)–(2.9), and using the uniform bound on $\|K_X(\lambda)\|$ given in (8.3), we obtain

$$\|K_i^m(\lambda) - K_1^m(\lambda)\|_1 \leq \sum_{\ell=1}^{m} \|K_i^\ell(\lambda)(K_i(\lambda) - K_1(\lambda))K_i^{m-\ell}(\lambda)\|_1 \leq \sum_{\ell=1}^{m} m\|V_i - V_\infty\|_\infty\|C\nu\| e^{-c(Im\lambda)m}.$$ \[2.10\]

3. Returning to the main estimate of the trace norm of (2.6), we obtain

$$\|(I + K_i(\lambda))e^{-K_i(\lambda)} - (I + K_1(\lambda))e^{-K_1(\lambda)}\|_1 \leq \sum_{m=2}^{\infty} \frac{(m-1)}{m!}\|K_i(\lambda)\|^{m} - (K_1(\lambda))^{m}\|_1 \leq \|V_i - V_\infty\|_\infty\left(\sum_{m=2}^{\infty} \frac{(m-1)e^{-c(Im\lambda)m}}{m!}\right) \leq C(\|\text{Im} \lambda\|, \nu_0)\|V_i - V_\infty\|_\infty.$$ \[2.11\]

The ratio test shows that the sum converges for all $\lambda \in \mathbb{C}$. The constant $C(\|\text{Im} \lambda\|, \nu_0) > 0$ is locally uniformly bounded on compact subsets of $\mathbb{C}$ and depends on the uniform bound $\nu_0$ (2.4) on family of potentials $V_j$. Since $\|V_i - V_\infty\|_\infty \to 0$, estimates (2.5) and (2.11) imply the locally uniform convergence of the functions $D_V(\lambda)$ to $D_{V_\infty}(\lambda).$ \[\Box\]
We will apply this proposition to a sequence of potentials in $C_0^\infty(\overline{B}_R(0); \mathbb{R})$ in the proof of Theorem 1 and to a sequence of potentials in $H^m(\overline{B}_R(0))$ for Proposition 2.

3. Poisson formula for the wave trace and the wave invariants

In this section, we recall the Poisson formula for the wave trace (see, for example [11, Part 1, chapter 3] or [18, chapter 4] and its connection with the resonances. We then derive a Gilkey-type formula for the wave invariants using a representation of the heat invariants due to Hitrik and Polterovich [15].

3.1. Poisson formula. The Poisson formula connects the regularized wave trace with the resonance set $R(V)$ of $H_V$. We briefly recall the formulation of the wave trace, for further details see, for example, [18, Chapter 4]. The operators $H_0$ and $H_V$ are associated with initial value problems for the wave equation on $\mathbb{R}^d$. We write $H_X$ for $H_0$ or $H_V$. Let $u(x, t)$ denote the solution of the initial-value problem for the wave equation:

$$\left( \partial_{tt} - H_X \right) u(x, t) = 0, \quad u(x, t = 0) = u_0(x), \quad \partial_t u(x, t)|_{t=0} = u_1(x). \quad (3.1)$$

Letting $w(x, t) := (u(x, t), (\partial_t u)(x, t))^T$, it is easy to check that $w(x, t)$ solves the system of equations

$$-i \frac{\partial}{\partial t} w(x, t) = L_X w(x, t), \quad (3.2)$$

with initial conditions $w(x, 0) = u_0(x), u_1(x))$. The operator $L_X$ is the matrix-valued operator on the Hilbert space $H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ given by

$$L_X := \begin{pmatrix} 0 & 1 \\ H_X & 0 \end{pmatrix}, \quad X = 0, V. \quad (3.3)$$

The wave group $W_X(t)$ associated with the Schrödinger operator $H_X$ is the matrix-valued unitary operator

$$W_X(t) := e^{itL_X}. \quad (3.4)$$

It implements the time evolution $w(x, t) = W_X(t) w(x, t = 0)$. The associated regularized wave trace is the distribution defined by

$$\text{Tr}(W_V(t) - W_0(t)) = 2\text{Tr}(\cos tH^{1/2}_V - \cos tH^{1/2}_0), \quad t \neq 0. \quad (3.5)$$

It is well-known that the regularized wave trace is a distribution, even in $t \neq 0$, whose explicit formula depends on the resonance set $R(V)$, the eigenvalues and resonances of $H_V$. We define the resonance set $R(V)$ of $H_V$ be the set of complex numbers consisting of values $i\lambda$, $\lambda > 0$, so that $-\lambda^2$ is an eigenvalue (necessarily nonpositive) of $H_V$, and all $\lambda \in \mathbb{C}^-$ that are poles of $R_V(\lambda)$, including multiplicities: $R(V) := \{(\lambda, m_V(\lambda)) \mid m_V(\lambda) \neq 0\}$. In the case of odd dimensions, the wave trace is related to the resonance set by the following Poisson formula.
Theorem 8. [18] Proposition 4.2] Let \( d \geq 3 \) be odd, and let \( \mathcal{R}(V) \) be the resonance set of \( H_V \). Let \( W_X(t) \) be the wave group for \( H_X \), for \( X = 0, V \). Then, as distributions,

\[
\text{Tr}(W_V(t) - W_0(t)) = \sum_{\lambda \in \mathcal{R}(V)} m_V(\lambda) e^{i|t|\lambda}, \; t \neq 0.
\]

The small time asymptotics of the regularized wave trace in odd dimensions \( d \geq 3 \) are given by

\[
\text{Tr}(W_V(t) - W_0(t)) \sim \frac{d + 1}{2} \sum_{j=1}^{d+1} w_j(V) D^{n-1-2j}\delta(t) + \sum_{j=\frac{d+1}{2}}^{N} w_j(V)|t|^{2j-d} + r_N(t), \quad (3.6)
\]

where \( r_N(t) \in C^{2N-d}(\mathbb{R}) \). We note that in dimension \( d = 3 \), the expansion of the wave trace has the form

\[
\text{Tr}(W_V(t) - W_0(t)) \sim w_1(V)\delta(t) + \sum_{j=2}^{N} w_j(V)|t|^{2j-3} + r_N(t). \quad (3.7)
\]

These formulas may be found in [18, Lemma 4.1].

It is known that the wave invariants \( w_j(V) \) are dimension-dependent multiples of the heat invariants associated to the pair \( (H_0, H_V) \), see, for example, [23, Proposition 2.3]. The heat invariants are the coefficients occurring in the small time asymptotic expansion of the regularized heat trace

\[
\text{Tr}(e^{-tH_V} - e^{-tH_0}) \sim \frac{1}{(4\pi t)^{\frac{d}{2}}} \sum_{j=1}^{\infty} c_j(V)t^j. \quad (3.8)
\]

Sá Barreto and Zworski [23] noted that the basic formula

\[
e^{-tx^2} = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int e^{-\frac{d^2}{4t} \cos(sx)}ds,
\]

may be used to express the regularized heat trace [18] in terms of an integral of the regularized wave trace. Substituting the asymptotic expansion [18] into this integral results in the following identities relating the heat invariants to the wave invariants:

\[
w_j(V) = \begin{cases} 
\frac{2^{d(1-d)+1}M_j}{N_j} c_j(V) & 1 \leq j \leq \frac{d-1}{2} \\
\frac{2^{d(1-d)+1}}{N_j} c_j(V) & j \geq \frac{d+1}{2}
\end{cases}
\]

\[
:= d_j c_j(V). \quad (3.9)
\]

The constants \( N_j \) and \( M_j \) are given by

\[
M_j = \left[ \left( \frac{d}{\theta} \right)^{d-1-2j} e^{-\theta^2} \right]_{\theta=0}, \quad 1 \leq j \leq \frac{d-1}{2},
\]

\[
N_j = \int e^{-\theta^2} \theta^{2j-d}d\theta, \quad j \geq \frac{d+1}{2}. \quad (3.10)
\]
We record here well-known \cite{18} section 4.1] expressions for the heat invariants:
\begin{align*}
c_1(V) &= \int_{\mathbb{R}^d} V(x) \, d^d x \\
c_2(V) &= \int_{\mathbb{R}^d} V(x)^2 \, d^d x \\
c_3(V) &= \int_{\mathbb{R}^d} \left[ V(x)^3 + \frac{1}{2} |\nabla V(x)|^2 \right] \, d^d x.
\end{align*} \tag{3.11}

3.2. Gilkey-type formulas for the wave invariants. In this section, we will prove explicit formulas for the heat invariants $c_j(V)$ in terms of the potential $V$ and its derivatives. These will yield formulas for the wave invariants from \eqref{3.9}. For Schrödinger operator on a compact manifold, as studied by Brüning \cite{5} and by Donnelly \cite{10}, the heat invariants $c_j(V)$, as defined in \eqref{3.9}, may be explicitly written in terms of the potential and the metric. Gilkey \cite{13} provided a general formula valid for compact manifolds similar to formula \eqref{3.14} for the heat invariants. In the noncompact case of $\mathbb{R}^d$, both Bañuelos and Sá Barreto \cite{2} and Hitrik and Polterovich \cite{15} proved formulas for the heat invariants $c_j(V)$. We show that formulas similar to those obtained by Gilkey hold for suitable potentials starting with the formulas derived by Hitrik and Polterovich \cite{15}. These authors proved that the heat invariants $c_j(V)$ are obtained by
\begin{equation}
c_j(V) = \int_{\mathbb{R}^d} c_j(x) \, d^d x,
\end{equation}
where the densities $c_j(x)$ are given by
\begin{equation}
c_j(x) = (-1)^j \sum_{k=0}^{j-1} c_{j,k} \frac{(H_V)_y^{k+j}(\|x - y\|^{2k})|_{x=y}}{4^k k!(k + j)!}, \tag{3.12}
\end{equation}
where $(H_V)_y$ denotes the Schrödinger operator $H_V$ in the $y$-variable. The numerical coefficients $c_{j,k}$ are given by
\begin{equation}
c_{j,k} = \left( j - 1 + \frac{d}{2} \right). \tag{3.13}
\end{equation}

Due to the relation between the heat and the wave invariants \eqref{3.9}, we have the following result. We define the index set of $k$-tuples $A_{j,k}$, for $j \geq 3$ and $3 \leq k \leq j$ as:
\begin{equation}
A_{j,k} = \left\{ \alpha = (\alpha^1, \ldots, \alpha^k) \mid \alpha^m = (\alpha^m_1, \ldots, \alpha^m_d) \in \mathbb{N}_0^d, \right. \left| \alpha^m \right| = \sum_{\ell=1}^d \alpha^m_\ell \leq j - k, \left. \sum_{m=1}^k \left| \alpha^m \right| = 2(j - k), \sum_{m=1}^k \alpha^m_\ell \in 2\mathbb{N}, \forall \ell = 1, \ldots, d. \right\}
\end{equation}

\textbf{Proposition 9.} Suppose $V \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$. Then the wave invariants $w_j(V)$, $j \geq 3$, appearing in the small time asymptotics of the wave trace \eqref{3.6}--\eqref{3.7},
are given by
\[ w_j(V) = d_j \int_{\mathbb{R}^d} |\nabla j^{-2} V|^2 + \sum_{\alpha \in A_{j,k}} c_{\alpha} \int_{\mathbb{R}^d} (D^{\alpha_1} V)(D^{\alpha_2} V) \cdots (D^{\alpha_k} V) \, d^d x, \]
where \( d_j \) is the constant defined in (3.9) and the index \( A_{j,k} \) is defined in (3.13).

As a consequence, a bound on the Sobolev norm of \( V \) is obtained from a rearrangement of (3.14). For each \( j \geq 3 \), we have the bound
\[ \|V\|_{j-2,2}^2 \leq C_j \left( 1 + \sum_{k=3}^j \sum_{\alpha \in A_{j,k}} \int_{\mathbb{R}^d} |(D^{\alpha_1} V)(D^{\alpha_2} V) \cdots (D^{\alpha_k} V)| \, d^d x \right), \]
where \( C_j > 0 \) depends on \( w_j(V) \). In our application, the constant is independent of our choice of \( V \) from the iso-resonant set \( \mathcal{I}_R(V_0) \), depending only on \( j \) and \( V_0 \). We will utilize this bound in section 5 to obtain uniform bounds on the Sobolev norms for \( \mathcal{I}_R(V_0) \).

4. Properties of the iso-resonant set: Uniform bounds on the Sobolev norms

We will successively bound the higher-order Sobolev norms by induction.

**Theorem 10.** Let us assume that \( \|V\|_{j-3,2} \leq C \), for some \( j \geq 3 \). For \( d = 1 \), there is a constant \( C > 0 \) so that
\[ \|V\|_{j-2,2}^2 \leq C; \]
whereas for \( d = 3 \), there is a constant \( C > 0 \) and \( \beta \) with \( 0 \leq \beta < 2 \) so that
\[ \|V\|_{j-2,2}^2 \leq C(1 + \|V\|_{j-2,2}^\beta) \]
Consequently, if \( \|V\|_{0,2} \) is uniformly bounded for \( V \in \mathcal{I}_R(V_0) \), then for each \( s \in \mathbb{N} \), there is a finite constant \( C_s(V_0) > 0 \), depending only on \( s \) and \( V_0 \), so that for all \( V \in \mathcal{I}_R(V_0) \), we have the uniform bound
\[ \|V\|_{s,2} \leq C_s(V_0). \]

The strategy of the proof of Theorem 10 is to show that each term
\[ \int_{\mathbb{R}^d} |(D^{\alpha_1} V)(D^{\alpha_2} V) \cdots (D^{\alpha_k} V)| \, d^d x \]
is bounded by a constant independent of our choice of \( V \in \mathcal{I}_R(V_0) \) (\( d = 1 \)) or by a multiple of \( 1 + \|V\|_{j-2,2}^\beta \) with \( \beta < 2 \) (\( d \geq 3 \)). Together these bounds yield a uniform bound on \( \|V\|_{j-2,2} \) for each \( j \). The details of this argument, similar to those in [5] and in Donnelly [10], are sketched in the appendix in section 7.

We mention, as in [5] and in [10], why the iteration procedure does not work for \( d \geq 4 \). The heat invariant \( c_3(V) \) is given by
\[ c_3(V) = \int_{\mathbb{R}^d} \left[ V(x)^3 + \frac{1}{2} \nabla V(x)^2 \right] \, d^d x. \]
We want to extract the $H^1$-bound on $V$ from this expression. In particular, We need to prove a bound of the form
\[
\left| \int_{\mathbb{R}^d} V(x)^3 \, d^d x \right| \leq C(\|V\|_2)\|V\|_{1,2}^\beta,
\]
for some $0 \leq \beta < 2$. Using the generalized Hölder inequality, we obtain
\[
\left| \int_{\mathbb{R}^d} V(x)^3 \, d^d x \right| \leq \|V\|_4^2 \|V\|_2.
\]

We recall from the Sobolev Embedding Theorem, Theorem 19, the inequality
\[
\|u\|_q \leq C\|u\|_{k,p},
\]
for indices
\[
\frac{1}{q} = \frac{1}{p} - \frac{k}{d}.
\]
We are interested in $p = 2$. In that case, for $k = 0$, we have $q = 2$ which is the $L^2$-norm. For $k = 1$, we obtain $q = 2d(d - 2)^{-1}$, so for $d = 3$, we obtain $q = 6$ whereas for $d = 4$, we obtain $q = 4$. Applying the interpolation result (20) in the case $d = 3$, this interpolation result has the form:
\[
\|V\|_4 \leq \|V\|_2^{\frac{1}{2}} \|V\|_6^{\frac{3}{2}}.
\]
As a consequence, substituting these bounds into the right side of (4.4), we obtain for $d = 3$:
\[
\left| \int_{\mathbb{R}^d} V(x)^3 \, d^d x \right| \leq \|V\|_2^2 \|V\|_2 \leq \|V\|_2 \|V\|_6^{\frac{3}{2}} \leq C\|V\|_2^\frac{3}{2} \|V\|_{1,2}^\frac{1}{2}.
\]

Since the exponent of $\|V\|_{1,2}$ is less than two, and the $L^2$-norm is constant across $\mathcal{I}_R(V_0)$, the term on the left of (4.7) can be absorbed into the left side of the inequality of (4.1). Repeating the same analysis for $d = 4$ (and similarly for $d \geq 5$), with the appropriate interpolation estimate (20), we find the bound
\[
\left| \int_{\mathbb{R}^d} V(x)^3 \, d^d x \right| \leq C\|V\|_2^\frac{3}{2} \|V\|_{1,2}^2,
\]
and this cannot be used in the inductive step because the exponent of the term $\|V\|_{1,2}$ is two and thus cannot be absorbed in the left side of (4.1).

5. Compactness of the iso-resonant set $\mathcal{I}_R(V_0)$

In this section, we combine the results of section 3 and 7 to prove the main Theorem 1. We fix a nontrivial, real-valued potential $V_0 \in C_0^\infty(B_R(0); \mathbb{R})$ and recall that $\mathcal{I}_R(V_0)$ is the set of similar potentials iso-resonant with $V_0$. It follows from the Poisson formula, Theorem 8 and the relation between the wave and the heat invariants (3.9), that for all $V \in \mathcal{I}_R(V_0)$, we have the equality
\[
c_j(V_0) = c_j(V), j \geq \frac{d+1}{2}, \; d \geq 3 \text{ odd}.
\]
For $d = 3$, it follows that the equality $c_2(V_0) = c_2(V)$ implies that

$$\|V\|_2 = \|V_0\|_2, \forall V \in I_R(V_0).$$

This provides the first step of the induction in Theorem 10 so we conclude that for each $s \in \mathbb{N}$, there is a finite constant $C_s(V_0) > 0$ so that $\|V\|_{s,2} \leq C_s(V_0)$ for all $V \in I_R(V_0)$.

It now remains to prove compactness. Let $C_s := \{C_s : s \in \mathbb{N}\}$ be any sequence of finite, nonnegative constants. We first consider a larger set $V_R(C_s) \subset C_0^\infty(B_R(0); \mathbb{R})$ consisting of all $V \in C_0^\infty(B_R(0); \mathbb{R})$ such that $\|V\|_{s,2} \leq C_s$, for all $s \in \mathbb{N}$. It is clear that $I_R(V_0) \subset V_R(C_s)$, for a suitable sequence $C_s$.

We recall the Fréchet metric on $C_0^\infty(\mathbb{R}^d)$ defined as follows.

**Definition 11.** Let $\alpha : \mathbb{N} \to \mathbb{N}^d$ be a bijective map so that $\alpha(i) = (\alpha(i)^1, \ldots, \alpha(i)^d) \in \mathbb{N}^d$. For any $V, W \in C_0^\infty(\mathbb{R}^d)$, we define the Fréchet metric by

$$\rho_F(V, W) = \sum_{i \in \mathbb{N}} 2^{-i} \frac{\|D^{\alpha(i)}(V - W)\|_2}{1 + \|D^{\alpha(i)}(V - W)\|_2}. \tag{5.2}$$

**Proposition 12.** The family $V_R(C_s) \subset C_0^\infty(B_R(0); \mathbb{R})$ is compact with respect to the Fréchet metric \textup{(5.2)}.

The proof follows from an application of the Ascoli Theorem \cite[Theorem 1.28]{19}. We begin with a lemma.

**Lemma 13.** The family $V_R(C_s)$ is equicontinuous in every derivative.

**Proof.** The family $V_R(C_s)$ is equicontinuous in each derivative if, for each $i \in \mathbb{N}$ and for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $V \in V_R(C_s)$, the condition $\|x - y\| \leq \delta$, for $x, y \in B_R(0)$, implies

$$|D^{\alpha(i)}V(x) - D^{\alpha(i)}V(y)| \leq C_i \delta,$$

for a constant $C_i$ depending only on $i$. From the Sobolev Embedding Theorem, Theorem 10 we have

$$\|V\|_\infty \leq \|V\|_{s,2} \text{ for } s > \frac{d}{2},$$

which implies

$$\|D^{\alpha(i)}V\|_\infty \leq \|V\|_{|\alpha(i)|+s,2}, \text{ for } s > \frac{d}{2}.$$

Since for all $V \in V_R(C_s)$, the $\|V\|_{s,2} \leq C_s$, for all $s \in \mathbb{N}$, we get a uniform bound on the $L^\infty$-norms of $D^{\alpha(i)}V$ which depends only on $|\alpha(i)|$. By the Mean Value Theorem for $x, y \in B_R(0)$, we have

$$|D^{\alpha(i)}V(x) - D^{\alpha(i)}V(y)| \leq \| \nabla(D^{\alpha(i)}V) \|_\infty \|x - y\| \leq C_{|\alpha(i)|+1+s}\|x - y\|,$$

for $s > \frac{d}{2}$ and where $C_{|\alpha(i)|+s}$ is uniform with respect to $V_R(C_s)$. This establishes the equicontinuity of the family $V_R(C_s)$ in the $\alpha(i)^{th}$ derivative for each $i \in \mathbb{N}$. \hfill \Box

**Proof of Proposition 12**
Proof. 1. Let \( \{V_j\} \subset \mathcal{V}_R(\mathcal{C}_s) \) be an arbitrary sequence. By equicontinuity, there exists a subsequence \( \{V_{j_k}\} \) that converges uniformly. By Lemma 13 for \( i \in \mathbb{N} \) and multi-index \( \alpha(i) \in \mathbb{N}^d \), the sequence \( \{D^{\alpha(i)}V_j\} \) is also equicontinuous and so has a uniformly convergent subsequence \( \{V_{j_m(\alpha(i))}\} \). So for each multi-index \( \alpha(i) \), there is a subsequence \( m \to j_m(\alpha(i)) \) so that \( \{D^{\alpha(i)}V_{j_m(\alpha(i))}\} \) converges uniformly. Diagonalizing the sequence of subsequences \( \{V_{j_m(\alpha(i))}\}_m \) yields the subsequence \( \{V_{j_m(\alpha(i))}\}_t \) such that it and all the subsequences \( \{D^{\alpha(i)}V_{j_m(\alpha(i))}\}_t \) converge uniformly on \( B_R(0) \). Hence the limit potential \( V \in \mathcal{C}_0^\infty(\overline{B_R}(0); \mathbb{R}) \).

2. We next prove that \( V_{j_m(\alpha(i))} \to V \) in the Fréchet metric defined in (5.2). Because the series on the right in (5.2) converges, given \( \epsilon > 0 \), there exists \( L > 0 \), independent of \( j \), so that

\[
\sum_{i=L+1}^{\infty} 2^{-i} \frac{\|D^{\alpha(i)}V_{j_m(\alpha(i))} - D^{\alpha(i)}V\|_2}{1 + \|D^{\alpha(i)}V_{j_m(\alpha(i))} - D^{\alpha(i)}V\|_2} < \frac{\epsilon}{2}.
\]

(5.3)

To control the sum up to \( L \), we note that for each multi-index \( \alpha(i) \), we can choose \( J(i) \) such that \( j > J(i) \) implies

\[
\|D^{\alpha(i)}V_{j_m(\alpha(i))} - D^{\alpha(i)}V\|_2 < \frac{2^{i-1}}{L + 1} \epsilon.
\]

Setting \( J = \max\{J(i) \mid i = 1, \ldots, L\} \), it follows that for \( j > J \)

\[
\sum_{i=0}^{L} 2^{-i} \frac{\|D^{\alpha(i)}V_{j_m(\alpha(i))} - D^{\alpha(i)}V\|_2}{1 + \|D^{\alpha(i)}V_{j_m(\alpha(i))} - D^{\alpha(i)}V\|_2} < \frac{\epsilon}{2}.
\]

(5.4)

Combining (5.3) and (5.4), we obtain \( \rho_F(V_{j_m(\alpha(i))}, V) < \epsilon \), for \( j > J \). We then note that subsequential limit point \( V \in \mathcal{C}_0^\infty(\overline{B_R}(0); \mathbb{R}) \) and that \( \|D^{\alpha(i)}V\|_2 < C_{\alpha(i)} \) for all multi-indices \( \alpha(i) \). So \( V \in \mathcal{V}_R(\mathcal{C}_s) \). This means that \( \{V_n\} \) has a convergent subsequence so \( \mathcal{V}_R(\mathcal{C}_s) \) is compact. \( \square \)

Lemma 14. Suppose \( \{V_j\} \subset \mathcal{I}_R(V_0) \) converges to \( V_\infty \) in the \( \mathcal{C}^\infty \)-Fréchet metric. Then \( D_{p,\infty}(\lambda) \) has the same zeros with the same orders as \( D_{p,V_0}(\lambda) \) for \( \lambda \in \mathbb{C} \) and \( p = 1 \) if \( d = 1 \) and \( p = 2 \) if \( d = 3 \). Consequently, the potential \( V_\infty \in \mathcal{I}_R(V_0) \). Thus, the set \( \mathcal{I}_R(V_0) \) is a closed subset of \( \mathcal{V}_R(\mathcal{C}_s) \) and hence compact.

Proof. The second heat invariants satisfy \( c_2(V_0) = c_2(V_\infty) = \int V_0^2 \neq 0 \) so as \( \lim c_2(V_j) = c_2(V_\infty) \), the limit potential \( V_\infty \) is not identically zero. Since \( V_\infty \in \mathcal{C}_0^\infty(\mathbb{R}^d; \mathbb{R}) \), the corresponding Schrödinger operator has infinitely many resonances [22]. This means that the analytic function \( D_{p,\infty}(\lambda) \) is not identically zero so we can apply the Hurwitz Theorem. The family of analytic functions \( \{D_{p,V}(\lambda)\} \) all have the same zeros, including order, as the function \( D_{p,V_0}(\lambda) \). If \( D_{p,\infty}(\lambda) \) did not have a zero at a zero of \( D_{p,V_0}(\lambda) \), it would contradict Hurwitz Theorem. Similarly the order of the zero has to be the same as the order of the zero of \( D_{p,V_0}(\lambda) \). Hence, the potential \( V_\infty \in \mathcal{I}_R(V_0) \). \( \square \)
6. Compactness of the iso-resonant set $\mathcal{T}_R^m(V_0)$

We discuss Proposition 2 concerning less regular potentials in this section. For dimension $d = 3$ and $V_0 \in H^m(\overline{B}_R(0); \mathbb{R})$, we let $\mathcal{T}_R^m(V_0)$ denote the set of real potentials in $H^m(\overline{B}_R(0); \mathbb{R})$ that are iso-resonant with $V_0$. The first statement part of Proposition 2 follows directly from [25, Theorem 1.2].

As for the second part, suppose that $V_0 \in H^m(\overline{B}_R(0))$, for $m \geq 3$. Then, the resonances determine the wave trace through the Poisson formula. Assumption (WTE) guarantees that the wave invariants satisfy $w_j(V) = w_j(V_0)$, for $j = 2, 3, \ldots, m + 2$, for all $V \in \mathcal{T}_R^m(V_0)$. Thus, for any $V \in \mathcal{T}_R^m(V_0)$, the heat invariants satisfy

$$c_j(V) = c_j(V_0), \quad j = 1, \ldots, m + 2.$$ (6.1)

It is a consequence of the formula (3.14), the identity (3.9), (6.1), and the induction of section 7, that for $j = 0, \ldots, m$, we have uniform bounds on the Sobolev norms across the iso-resonant class:

$$\|V\|_{j,2} \leq C_j, \quad \forall \ V \in \mathcal{T}_R^m(V_0).$$ (6.2)

As in the proof of equicontinuity in Lemma 13, the derivatives of $V$ are uniformly bounded in the $L^\infty$-norm up to and including order $m - 2$ for $d = 3$. Finally, the same argument shows uniform equicontinuity of the derivatives of the potentials up to order $m - 3$. The compactness of $\mathcal{T}_R^m(V_0) \subset H^{m-3}(\overline{B}_R(0))$ now follows by the same argument as in section 5.

7. Appendix: Proof of the uniform Sobolev bounds

In this appendix, we present the details of the uniform Sobolev bounds on the iso-resonant class of potentials $\mathcal{T}_R(V_0)$ presented in Theorem 10. We use standard notation $H^{s,p}(\Omega)$, for $\Omega \subset \mathbb{R}^d$, an open set (with a boundary satisfying the cone condition, see Theorem 19), for the space of functions with $s$-weak-derivatives in $L^p(\Omega)$. The norm is denoted by $\| \cdot \|_{s,p}$.

We recall from (3.13) the index set of $k$-tuples $\mathcal{A}_{j,k}$, defined for $j \geq 3$ and $3 \leq k \leq j$ by:

$$\mathcal{A}_{j,k} = \left\{ \alpha = (\alpha^1, \ldots, \alpha^k) \mid \alpha^m = (\alpha_1^m, \ldots, \alpha_d^m) \in \mathbb{N}_0^d, \quad |\alpha^m| \leq j - k, \quad \sum_{m=1}^k |\alpha^m| = 2(j - k), \quad \sum_{m=1}^k \alpha^m_\ell \in 2\mathbb{N}, \quad \forall \ \ell = 1, \ldots, d. \right\}. \quad (7.1)$$

From Proposition 9 and (3.15), the Sobolev norm $\|V\|_{j-2,2}^2$, for $j \geq 3$, is bounded above as

$$\|V\|_{j-2,2}^2 \leq C_j \left( 1 + \sum_{k=3}^j \sum_{\alpha \in \mathcal{A}_{j,k}} \int_{\mathbb{R}^d} |D^{\alpha^1}(V)D^{\alpha^2}(V) \cdots D^{\alpha^k}(V)| \, d^d x \right), \quad (7.2)$$

where the constant $C_j > 0$ is independent of our choice of $V$ from the iso-resonant set. Proceeding inductively, we must find an upper bound on the sum (7.2) for the $H^{j-2,2}$-norm of $V$ in terms of the lower Sobolev norms of $V$. 

7.1. The one-dimensional case. We first discuss the case $d = 1$ for which the bounds are easier to obtain. We begin with a simple and useful bound.

**Lemma 15.** For any $u \in C^1_0(\mathbb{R})$, we have
\[ \|u\|_{\infty} \leq C\|u\|_{1,2}, \]
where the constant depends on the support of $u$.

The first step of an inductive proof of the uniform Sobolev bounds is the following proposition.

**Proposition 16.** Let $d = 1, j \geq 3$, and suppose that $\|V\|_{j-3,2} \leq M$. We then have
\[ \int_{\mathbb{R}} |D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_k}(V)| \leq C_j, \quad (7.3) \]
where $C_j > 0$ depends on $M$ and $j$.

**Proof.** 1. Since we are in one dimension, we write $\alpha^m$ for $\alpha^{m_1}$. We use the bounds on the order of the $D^{\alpha_i}(V)$-terms to conclude: 1) since $k \geq 3$, we have $\alpha^m \leq j - 3$, 2) we have the constraint $\sum_{m=1}^{k} \alpha^m \leq 2(j - 3)$, and 3) there are at most two terms with order $(j - 3)$, since
\[ \sum_{m=1}^{k} \alpha^m = 2(j - k) \leq 2(j - 3). \]
These restrictions, together with Lemma 15, will then allow us to get the desired bounds as follows:

2. **Case 1:** We assume that the product contains no terms of $j - 3$. Then, using Lemma 15 for each $i$,
\[ |D^{\alpha_i}(V)| \leq C\|D^{\alpha_i}(V)\|_{1,2} \leq C\|V\|_{j-3,2}. \quad (7.4) \]
This gives
\[ \int_{\mathbb{R}} |D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_k}(V)| \leq C^k M^k \leq C^j M^j, \quad (7.5) \]
where we assume that $C, M \geq \max(1, |B_R(0)|)$.

3. **Case 2:** We assume that the product contains one term of order $j - 3$, say $\alpha^1$. Using the results from Case 1, together with the Hölder inequality, we have
\[ \int_{\mathbb{R}} |D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_k}(V)| \leq C^{k-1} M^{k-1} \int_{\mathbb{R}} |D^{\alpha_1}(V)| \leq C^{k-1} M^{k-1} |B_R(0)||V|_{j-3,2} \leq C^j M^j. \quad (7.6) \]

3. **Case 3:** We assume the product contains two terms of order $j - 3$. Using the results of Case 1 and the Hölder inequality, we obtain
\[ \int_{\mathbb{R}} |D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_k}(V)| \leq C^{k-2} M^{k-2} \int_{\mathbb{R}} |D^{\alpha_1}(V)D^{\alpha_2}(V)| \leq C^{k-2} M^{k-2} |V|_{j-3,2}^2 \leq C^j M^j. \quad (7.7) \]
\[ \square \]
It follows from the bound in Proposition 16 and (7.5)–(7.7) that
\[ \| V \|_{2,j-2}^2 \leq C_j, \] (7.8)
where the constant depends on \( M \). Consequently, the constant is uniform over all \( V \in \mathcal{I}_R(V_0) \).

7.2. The \( d \geq 3 \) dimensional case. For \( d \geq 3 \), we follow the idea of the proof given by Donnelly [10]. This requires reordering the \( D^{\alpha_i}(V) \) terms in the integral of (7.2) according to their order \( |\alpha| \). For fixed \( k \), we write the integrand of the integral in (7.2) as
\[ T = D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_{\ell}}(V)D^{\alpha_{\ell+1}}(V) \cdots D^{\alpha_k}(V), \] (7.9)
where the ordering is chosen such that
\[ \begin{align*}
i \leq \ell & \Rightarrow d > 2(j - |\alpha^i| - 3) \quad (7.10) \\
i > \ell & \Rightarrow d \leq 2(j - |\alpha^i| - 3). \quad (7.11)
\end{align*} \]
We will bound the integral in (7.2) using the generalized Hölder’s inequality (8.7) and the Sobolev Embedding Theorem, Theorem 19. The conditions on \( |\alpha^i| \) determine which case of the Sobolev Embedding Theorem, Theorem 19, for \( p = 2 \) and \( k = (j - |\alpha^i| - 3) \) is appropriate.

Proposition 17. [10, Lemma 4.6] If \( d \geq 3, j > \frac{d}{2} + 1 \), and \( \| V \|_{j-2,2} \leq C_1 \), then
\[ \int_{\mathbb{R}^d} |D^{\alpha_1}(V)D^{\alpha_2}(V) \cdots D^{\alpha_{\ell}}(V)D^{\alpha_{\ell+1}}(V) \cdots D^{\alpha_k}(V)| \leq C_2 \left( 1 + \| V \|_{j-2,2}^\beta \right), \] (7.12)
where \( \beta < 2 \) and \( C_2 \) depends on \( C_1 \).

Proof. 1. We will look at the possible values of \( \ell \) and for each case the general strategy will be to use the generalized Hölder’s inequality (8.7) to show
\[ \int_{\mathbb{R}^d} |T| \leq C \prod_{i=1}^k \| D^{\alpha_i}(V) \|_{r_i}, \] (7.13)
with \( \sum_{i=1}^k \frac{1}{r_i} = 1 \). We recall that the integrand \( T \) in (7.2) is ordered as in (7.9) with \( \ell \) determined as in (7.10). For the factors with \( i \geq \ell + 1 \), so that \( 2(j - |\alpha^i| - 3) > d \), we have
\[ \| D^{\alpha_i}(V) \|_{\infty} \leq C \| V \|_{j-3,2}, \] (7.14)
and when \( i \) is such that \( 2(j - |\alpha^i| - 3) = d \), for any \( r_i \) with \( 2 \leq r_i < \infty \), we have
\[ \| D^{\alpha_i}(V) \|_{r_i} \leq C \| V \|_{j-3,2}. \] (7.15)
These two inequalities result in the bound
\[ \int_{\mathbb{R}^d} |T| \leq C \prod_{i=1}^k \| D^{\alpha_i}(V) \|_{r_i} \leq \tilde{C} \| V \|_{j-3,2}^{\ell - \ell} \prod_{i=1}^k \| D^{\alpha_i}(V) \|_{r_i}. \] (7.16)
The remainder of the proof is devoted to showing that for \( 1 \leq i \leq \ell \), we can choose \( r_i \) in (7.13) in order to apply the appropriate Sobolev inequality in
Theorem (19). We first note that when \( \ell = 0 \) the estimate holds for \( \beta = 0 \) using the above method.

3. **Case 1:** For \( \ell = 1 \), we have \( d > 2(j - |\alpha^1| - 3) \), so setting

\[
r_1 = \frac{2d}{d - 2(j - |\alpha^1| - 3)}
\]

yields

\[
\|D^{\alpha^1}(V)\|_{r_1} \leq C\|D^{\alpha_i}(V)\|_{j-|\alpha^1|-3,2} \leq C\|V\|_{j-3,2},
\]

by the Sobolev inequality in Theorem (19). The only condition on \( j \) is \( 2 \leq \frac{2d}{d - 2(j - |\alpha^1| - 3)} \) or \( j - 3 \geq |\alpha^1| \) which is true for every \( \alpha^i \) and \( j \) as in (3.13). Since \( \frac{1}{r_1} \leq \frac{1}{2} \), we can choose the remaining \( r_i \)'s to meet the condition \( \sum_{i=1}^{k} \frac{1}{r_i} = 1 \). So for \( \ell = 1 \) we have the bound with \( \beta = 0 \).

4. **Case 2:** For \( \ell = 2 \), we argue as follows. If \( |\alpha^1| \) and \( |\alpha^2| \) are such that \( r_1 \) and \( r_2 \) (as chosen in case \( \ell = 1 \)) satisfy

\[
\frac{1}{r_1} + \frac{1}{r_2} < 1
\]

then we proceed as in the case \( \ell = 1 \) and apply the generalized Hölder’s inequality to get the result with \( \beta = 0 \). Now assume \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \). Since \( |\alpha^i| \leq j - 3 \), this implies \( |\alpha^1| = |\alpha^2| = j - 3 \) and thus \( r_1 = r_2 = 2 \). We may then apply the generalized Hölder’s inequality to get:

\[
\int_{\mathbb{R}^d} |T| \leq C\|D^{\alpha^1}(V)\|_{r_1+\varepsilon_i} \prod_{i=2}^{k} \|D^{\alpha^i}(V)\|_{r_i},
\]

where \( \varepsilon > 0 \) and \( r_i \) for \( i \geq 3 \) are chosen to satisfy the Hölder condition. Furthermore if we choose \( \varepsilon \) such that \( r_1 + \varepsilon < \frac{2d}{d - 2} \) then the general Sobolev inequality in Theorem (19) gives that

\[
\|D^{\alpha^1}(V)\|_{r_1+\varepsilon_i} \leq C_1\|D^{\alpha^1}(V)\|_{1,2} \leq C_2\|V\|_{j-2,2},
\]

so we get the result with \( \beta = 1 \).

5. **Case 3:** We now consider \( \ell \geq 3 \) and we suppose that \( d > 2(j - |\alpha^i| - 2) \) for, say, \( i = 1, 2 \). Let \( r_i \) be as in cases 1 and 2 and set

\[
s_i = \frac{2d}{d - 2(j - |\alpha^1| - 2)}.
\]

The standard \( L^p \) interpolation (4.6) estimate allows us to conclude that for any \( 0 < \varepsilon_i < 1 \), there exists a \( 0 < \beta_i < 1 \) such that

\[
\|D^{\alpha^i}(V)\|_{r_i+\varepsilon_i} \leq \|D^{\alpha^1}(V)\|_{r_i}^{\beta_i} \|D^{\alpha^i}(V)\|_{s_i}^{1-\beta_i},
\]

(7.17)
Using the generalized Hölder’s inequality we have

\[
\int_{\mathbb{R}^d} |T| \leq C \|D^{\alpha_1}(V)\|_{r_1+\varepsilon_1} \|D^{\alpha_2}(V)\|_{r_2+\varepsilon_2} \prod_{i=3}^{k} \|D^{\alpha_i}(V)\|_{r_i}
\]

\[
\leq \|D^{\alpha_1}(V)\|_{s_1}^{\beta_1} \|D^{\alpha_1}(V)\|_{s_1}^{1-\beta_1} \|D^{\alpha_2}(V)\|_{s_2}^{\beta_2} \|D^{\alpha_2}(V)\|_{s_2}^{1-\beta_2} \prod_{i=3}^{k} \|D^{\alpha_i}(V)\|_{j-|\alpha_i|-3,2}
\]

\[
\leq C \|V\|_{j-3,2}^{\beta_1} \|D^{\alpha_1}(V)\|_{s_1}^{1-\beta_1} \|V\|_{j-3}^{\beta_1} \|D^{\alpha_2}(V)\|_{s_2}^{1-\beta_2} \prod_{i=3}^{k} \|V\|_{j-3,2}
\]

\[
\leq C \|D^{\alpha_1}(V)\|_{s_1}^{1-\beta_1} \|D^{\alpha_2}(V)\|_{s_2}^{1-\beta_2}
\]

\[
\leq C \|V\|_{j-2,2}^{\beta},
\]

(7.18)

where \(\beta < 2\). The index \(r_i\) may be chosen arbitrarily for \(i > l\) and as in case 1 for \(i \leq l\). Consequently, in order to satisfy the Hölder condition we require:

\[
\frac{1}{r_1+\varepsilon_1} + \frac{1}{r_2+\varepsilon_2} + \sum_{i=3}^{l} \frac{1}{r_i} < 1. \tag{7.19}
\]

For sufficiently large \(\varepsilon_1, \varepsilon_2 > 1\), this is implied by

\[
\frac{1}{s_1} + \frac{1}{s_2} + \sum_{i=3}^{l} \frac{1}{r_i} < 1.
\]

Substituting for \(s_i\) and \(r_i\) gives

\[
\sum_{i=1}^{2} \frac{d-2(j-|\alpha_i|-2)}{2d} + \sum_{i=3}^{l} \frac{d-2(j-|\alpha_i|-3)}{2d} < 1, \tag{7.20}
\]

which may be rewritten as

\[
(d-2j-6)l + \sum_{i=1}^{l} |\alpha_i| < 2d + 4.
\]

Because \(\sum_{i=1}^{l} |\alpha_i| \leq 2(j-k)\), it is sufficient to show

\[
(d-2j-6)l + 4(j-k) < 2d + 4.
\]

Using assumption \(l \geq 3\) lets us rewrite the inequality as

\[
\frac{d}{2} + 3 - \frac{2k-4}{l-2} < j.
\]

Then \(k \geq l \geq 3\) gives \(\frac{2k-4}{l-2} \geq \frac{2k-4}{k-2} = 2\). Consequently, the condition on the indices for the generalized Hölder inequality (7.19) is satisfied provided

\[
\frac{d}{2} + 1 < j,
\]

which is an assumption on \(j\) and \(d \geq 3\) in Proposition [16].
6. Case 4: We now consider $\ell \geq 3$ and we suppose the complementary situation to case 3: $d \leq 2(j - |\alpha^i| - 2)$ for, say, $i = 1, 2$. For $2 \leq s < \infty$, we have the embedding

$$\|D^{\alpha^i}(V)\|_s \leq C\|D^{\alpha^i}(V)\|_{j - |\alpha^i|-2,2} \leq C\|V\|_{j,2}.$$

Then, standard $L^p$ interpolation given in Theorem 20, gives for $2 < t < s$

$$\|D^{\alpha^i}(V)\|_t \leq \|D^{\alpha^i}(V)\|_{s}^{\beta_i} \|D^{\alpha^i}(V)\|_{\frac{1}{2}}^{1-\beta_i}.$$

We may take $t$ to be arbitrarily large reducing the Hölder condition to

$$\sum_{i=3}^{\ell} \frac{1}{r_i} < 1.$$

If $\ell = 3$, the condition is met as $r_3 \geq 2$, so we assume $\ell \geq 4$. Substituting for $r_i$ and rewriting the inequality we get

$$(\ell - 2)(d - 2j + 6) + 2\sum_{i=3}^{\ell} |\alpha^i| < 2d.$$

Using the inequality $\sum_{i=3}^{\ell} |\alpha^i| \leq \sum_{i=1}^{k} |\alpha^i| \leq 2(j - k)$ gives the sufficient condition

$$(l - 2)(d - 2j + 6) + 4(j - k) < 2d,$$

which can be recast as

$$(l - 4)d + 6(l - 2) - 4k < (2l - 8)j.$$

If $\ell = 4$, then the inequality reduces to $12 - 4k < 0$ which always holds as $4 = \ell \leq k$. For $\ell \geq 5$, we rewrite the inequality as

$$\frac{d}{2} + \frac{3l - 4}{l - 4} + \frac{2}{l - 4} - \frac{2k}{l - 4} < j,$$

which reduces to

$$\frac{d}{2} + 3 - \frac{(2k - 6)}{\ell - 4} < j.$$

Since $k \geq \ell \geq 5$, it is sufficient for the following inequality to hold:

$$\frac{d}{2} + 3 - \frac{(2k - 6)}{k - 4} < j.$$

With $k \geq 5$, this is satisfied if

$$\frac{d}{2} + 1 - \frac{4}{k - 4} < j,$$

which is guaranteed by the hypothesis $\frac{d}{2} + 1 < j$. \qed

8. Appendix: Various estimates

We summarize various estimates necessary for the proofs. Complete descriptions for the material in sections 8.1 and 8.2 may be found in, for example, [11], and for section 8.3 in, for example, [12] and [1].
8.1. **Resolvent bounds.** The kernels of the resolvent \( R_0(\lambda) = (H_0 - \lambda^2)^{-1} \) in dimensions \( d = 1 \) and \( d = 3 \) are given by

\[
R_0(x - y; \lambda) = \frac{i}{2\lambda} e^{i\lambda\|x - y\|}, \quad d = 1, \tag{8.1}
\]

\[
R_0(x - y; \lambda) = \frac{1}{4\pi^2} e^{i\lambda\|x - y\|}/\|x - y\|, \quad d = 3. \tag{8.2}
\]

Let \( \rho \in L^\infty_0(\mathbb{R}^d) \) be a compactly supported function. be the characteristic function on a ball of radius \( R > 0 \). For all \( \lambda \in \mathbb{C}\setminus\{0\} \), the free resolvent in \( d = 1 \) satisfies the bound:

\[
\|\rho R_0(\lambda)\rho\| \leq C_\rho \frac{1}{|\lambda|} e^{-\alpha_\rho \text{Im}\lambda}. \tag{8.3}
\]

For all \( \lambda \in \mathbb{C} \), the free resolvent in \( d = 3 \) satisfies the bound:

\[
\|\rho R_0(\lambda)\rho\| \leq D_\rho e^{-\beta_\rho \text{Im}\lambda}. \tag{8.4}
\]

The finite positive constants \( C_\rho, D_\rho, \alpha_\rho, \beta_\rho > 0 \) depend on \( \rho \).

8.2. **Singular value estimates.** If \( \rho \geq 0 \) is a compactly supported \( C^2 \)-function, then

\[
\mu_j(\rho R_0(\lambda)\rho) \leq C \frac{1}{j^{\frac{1}{2}}}. \tag{8.5}
\]

Recall that for \( V \in C^\infty_0(B_R(0); \mathbb{R}) \), the operator \( K_V(\lambda) := VR_0(\lambda)\chi_R \), where \( \chi_R \) is the characteristic function on \( B_R(0) \). We then have

\[
\mu_j(K_V(\lambda)) \leq C(V) \frac{1}{j^{\frac{1}{2}}}, \tag{8.6}
\]

where \( C(V) \) depends only of \( \|V\|_\infty \) and is locally uniformly bounded in \( \lambda \). A consequence of (8.6) is that \( K_V(\lambda) \) is in the Hilbert Schmidt class.

8.3. **Inequalities.** Section 7 makes repeated use of the following results.

**Generalized Hölder Inequality**

**Theorem 18.** [12, Appendix B.2.g.] Let \( \Omega \subset \mathbb{R}^d \) be an open subset of \( \mathbb{R}^d \). For indices \( 1 \leq p_1, \ldots, p_m \leq \infty \) satisfying

\[
\sum_{j=1}^m \frac{1}{p_j} = 1,
\]

for any \( u_k \in L^{p_k}(\Omega) \), we have

\[
\int_{\Omega} |u_1 \cdots u_m| \leq \prod_{j=1}^m \|u_k\|_{p_j}. \tag{8.7}
\]

**General Sobolev Inequality**

The Sobolev Embedding Theorem and corresponding inequalities are used through section 7 and 6. The general theorem holds for domains \( \Omega \subset \mathbb{R}^d \) with
the cone conditions \[\text{Definition } 4.6\]. We state only the cases used in the present paper.

**Theorem 19.** [1, Theorem 4.12] [12, Theorem 6, section 5.6] Let \(\Omega \subset \mathbb{R}^d\) be an open set with the cone condition and let \(j \geq 0\) and \(m \geq 1\) be integers.

1. Let \(u \in H^{k,p}(\Omega)\). If \(0 < k < \frac{d}{p}\), then \(u \in L^q(\Omega)\), where the indices satisfy \(\frac{1}{q} = \frac{1}{p} - \frac{k}{d}\), and
   \[
   \|u\|_q \leq C\|u\|_{k,p},
   \]
   for a finite constant \(C > 0\) depending on \(k, p, d\) and \(\Omega\).

2. For any \(u \in H^{j+m,2}(\Omega)\), with \(m > \frac{d}{2}\), we have
   \[
   \|u\|_{j,q} \leq C\|u\|_{j+m,2},
   \]
   for \(2 \leq q \leq \infty\) and a finite constant \(C > 0\) depending on \(j, m, d\) and \(\Omega\).

**L**^p-interpolation Inequality

**Theorem 20.** [12, Appendix B.2.h.] Let \(\Omega \subset \mathbb{R}^d\) be an open bounded subset of \(\mathbb{R}^d\). If \(0 < \alpha < 1\) and \(1 \leq r < p < s\), for all \(u \in L^p(\Omega)\), we have
\[
\|u\|_p \leq \|u\|_r^{\alpha}\|u\|_s^{1-\alpha},
\]
provided the indices satisfy
\[
\frac{1}{p} = \frac{\alpha}{r} + \frac{1-\alpha}{s}.
\]

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