Rotating Magnetic Solutions for 2+1D Einstein Maxwell Chern-Simons from Space-Time Duality

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ABSTRACT

It is studied a space-time duality that maps known static rotating electric solutions into static magnetic solutions. As an example this dualities are applied to known electric solutions in 2+1-dimensional Minkowski space-times within the framework of Einstein Maxwell Chern-Simons theory coupled to a dilaton-like scalar field. The magnetic solutions obtained have metric determinant $\sqrt{-g} \sim r^p$ for the range of the parameter $p \in ]-\infty, +\infty[/\{-1\}$ and are interpreted either as magnetic string-like configurations, configurations driven by an externally applied magnetic field or cosmological-like solutions with background magnetic fields.
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1 Introduction

The first studies on classical gravitational solutions in 2+1-dimensional space-times date back to 1984 and addressed Cosmological Einstein theories [1, 2]. Later developments addressed neutral solutions for Einstein theory (AdS BTZ black-hole) [3], Einstein Chern-Simons theory [4, 5] and the rotating BTZ black-hole [6, 7]. Following these developments charged solutions were studied for Einstein Maxwell Chern-Simons theory [8–12], Einstein Maxwell theory [13, 14], Dilaton Einstein Maxwell theories [15–22] and electric solutions of Einstein Maxwell Chern-Simons theory with a scalar field [23], as well as for Chern-Simons gravity [24–30].

In this work we build a space-time duality that allows to map static rotating electric and magnetic solutions into each other. As an example we apply this duality to the electric solutions computed in [23] obtaining new magnetic solutions that further extend the known existing solutions for Einstein Maxwell Chern-Simons theory coupled to a Dilaton-like scalar field. We recall that, when considering 2+1-dimensional gravitational solutions the inclusion of a scalar field is a natural extension of Einstein theory and it is justified by noting that a dimensional reduction from 3+1-dimensions generates such a scalar field, whether it is a Dilaton field [15, 16, 31] or obtained by gauging a higher dimensional symmetry [32, 33].

In addition when considering electromagnetic field solutions in 2+1-dimensions the Chern-Simons term [34, 35] is also a natural extension of Maxwell theory, at quantum level only the Maxwell Chern-Simons theory is consistent such that the Chern-Simons term is a quantum correction of the Maxwell theory [36–39].

As possible physical frameworks were such solutions may be relevant we note that 2+1-dimensional theories are often considered simpler laboratories for higher dimensional theories [7], higher dimensional examples with similar frameworks to the one discussed here are, inflationary models with exponential potentials [40], domain walls in 4+1-dimensions [41] and cosmological solutions in 4+1-dimensions [42, 43]. In addition often 3+1-dimensional systems exhibiting cylindrical symmetry are considered as effective 2+1-dimensional systems [32, 33, 44] as it is the example of cylindrical gravitational waves [45–49].

This work is organized as follows, in section 2 is discussed a web of space-time dualities that maps electric rotating gravitational solutions into magnetic rotating gravitational solutions. These dualities are a generalization of a similar duality originally considered in [4] and are available in the unpublished e-print [50]. In section 3 the space-time dualities are applied to the electric gravitational solutions computed in [23] such that new magnetic gravitational solutions are generated. Are also analysed the singularities, curvature, horizons, mass, angular momentum and magnetic flux for these magnetic configurations. In section 4 are summarized and discussed the results obtained, in particular are interpreted
either as magnetic string-like solutions, configurations driven by an external magnetic field or cosmological-like solutions. In addition in appendix A are re-derived directly from the equations of motion in the Cartan frame the solutions discussed in section 3 and in appendix B are listed, for particular cases not included in the main text, the expressions for the mass, angular momentum and magnetic flux.

2 Generating new solutions employing space-time dualities

In this section we describe a web of space-time dualities that map static rotating electric classical configurations into static rotating magnetic classical configurations based in a duality originally discussed in [4].

2.1 Original duality for rotating space-times

In [4,5] it was introduced a 2+1-dimensional space-time duality that maps between electric and magnetic charged static gravitational solutions. This duality consists of exchanging the role of the time coordinate \( t \) and the angular coordinate \( \varphi \). In the following we review the original construction and generalize it to stationary rotating space-times.

Let us take a 2+1-dimensional metric written in the standard ADM parameterisation [51,52] with ADM signature diag\((- , + , +)\) and the Maxwell Chern-Simons (MCS) Lagrangian

\[
\begin{align*}
ds^2 &= -f^2 dt^2 + dr^2 + h^2 (d\varphi + A dt)^2 , \\
\mathcal{L}^{\text{MCS}} &= F \wedge *F + A \wedge F .
\end{align*}
\]

In addition it is implicit that we are considering the Einstein action containing the standard curvature term. In the following we will always map the metric obtained by a duality transformation back to the above standard ADM parameterisation. Hence it is not required to explicitly consider the curvature definition to discuss the duality transformations. In addition we note that what follows also applies to pure Maxwell theory in a flat empty background (for which the Einstein action is trivially null) or any other theory containing form fields for which, in order to implement the following construction, it is enough to track how these fields transform under a given duality. We are further considering the following standard electric and magnetic field definitions

\[
\begin{align*}
E_s &= F_{tr} = \partial_t A_r - \partial_r A_t , \\
B_s &= F_{r\varphi} = \partial_r A_\varphi - \partial_\varphi A_r .
\end{align*}
\]

Here stared fields \( E_s \) and \( B_s \) stand for the electromagnetic fields in a specific coordinate
frame while non stared field \( E \) and \( B \) stand for the fields in the Cartan frame employed in appendix A to explicitly derive the classical solutions directly from the equations of motion.

In [4] are considered static non-rotating space-times with null shift function \( A = 0 \), for which a direct swapping of the coordinates \( t \leftrightarrow \varphi \), maintaining the metric ADM signature, corresponds to the gravitational fields map \( f \to ih \) and \( h \to if \) (i.e. \( f^2 \to -h^2 \) and \( h^2 \to -f^2 \)) and the respective electromagnetic fields map \( E \to -iB \) and \( B \to -iE \). In order to generalize this duality to rotating space-times with \( A \neq 0 \) we consider the Cartan triad [53]

\[
e^0 = f \, dt \quad , \quad e^1 = dr \quad , \quad e^2 = h(d\varphi + A dt) .
\]

In terms of this triad the duality simple exchanges \( e^0 \leftrightarrow e^2 \) and reads

\[
\begin{align*}
t &\to i\varphi \quad \Rightarrow \quad f &\to ih \quad , \quad E_* &\to -iB_* \\
\varphi &\to it \quad \Rightarrow \quad h &\to if \quad , \quad B_* &\to -iE_* .
\end{align*}
\]

The effect of this duality in the metric and the MCS Lagrangian (2.1) is

\[
\begin{align*}
ds^2 &\to d\tilde{s}^2 = -f^2(dt + \tilde{A} d\varphi)^2 + dr^2 + \tilde{h}^2d\varphi^2 \\
&= -f'^2dt^2 + dr^2 + h^2(d\varphi + A dt)^2 , \\
\mathcal{L}^{\text{MCS}} &\to \tilde{\mathcal{L}}^{\text{MCS}} = -\tilde{F} \wedge *\tilde{F} - \tilde{A} \wedge \tilde{F} \\
&= +F \wedge *F + A \wedge F .
\end{align*}
\]

The expressions in the second and fourth line correspond to, after applying the duality, re-writing the metric and the Lagrangian in the standard form given in (2.1) by defining new tilded fields. We note that this duality swaps the relative sign of the gauge sector with respect to the gravitational sector, hence interchanging standard gauge fields with ghost gauge fields (the gauge kinetic term swaps sign). We stress that this duality is a transformation of space-time coordinates, the gauge fields transform accordingly due to the swapping of coordinate indexes, however this is a consequence of the space-time duality, not a duality of the gauge fields.

In the following we discuss how to employ this duality in order to obtain new charged gravitational solutions from already known charged solutions. Let us assume that we have known solutions given by the fields \( \tilde{f} \), \( \tilde{h} \), \( \tilde{E}(\tilde{f}, \tilde{h}) \) and \( \tilde{B}(\tilde{f}, \tilde{h}) \). Then, from the above duality for the metric (2.5), we can compute the explicit map between the new tilded fields and the original ones. Explicitly the metric components corresponding to the line element \( d\tilde{s} \) written in terms of each set of fields are

\[
\begin{align*}
\tilde{g}_{00} &= -\tilde{f}^2 + \tilde{h}^2 A^2 \\
\tilde{g}_{11} &= 1 \\
\tilde{g}_{22} &= \tilde{h}^2 \\
\tilde{g}_{02} &= \tilde{h}^2 A \\
\tilde{g}_{00} &= -\tilde{f}^2 \\
\tilde{g}_{11} &= 1 \\
\tilde{g}_{22} &= \tilde{h}^2 - \tilde{f}^2 \tilde{A}^2 \\
\tilde{g}_{02} &= -\tilde{f}^2 \tilde{A}.
\end{align*}
\]

(2.6)
such that we obtain the following map for the new solutions obtained by applying the duality

\[
\begin{align*}
  f^2 &= \frac{\tilde{f}^2 \tilde{h}^2}{\tilde{h}^2 - \tilde{f}^2 A^2} \\
  h^2 &= \frac{\tilde{h}^2 - \tilde{f}^2 A^2}{\tilde{f}^2 \tilde{h}^2} \\
  A &= -\frac{\tilde{A} \tilde{f}^2}{\tilde{h}^2 - \tilde{f}^2 A^2}
\end{align*}
\]

Here the equalities in the argument of the electric and magnetic fields stand for the original known solutions \( \tilde{E}_*(\tilde{f}, \tilde{h}) \) and \( \tilde{B}_*(\tilde{f}, \tilde{h}) \) evaluated at \( \tilde{f} = f \) and \( \tilde{h} = h \). We stress again that this duality exchanges standard gauge fields with ghost gauge fields as given by the Lagrangian transformation in (2.5). In addition, depending of the specific solutions, the metric ADM signature can change under duality accordingly to the following cases

\[
\tilde{h}^2 - \tilde{f}^2 \tilde{A}^2 > 0 \implies \text{Map maintains metric ADM signature ,}
\]

\[
\tilde{h}^2 - \tilde{f}^2 \tilde{A}^2 < 0 \implies \text{Map changes metric ADM signature .}
\]  

(2.8)

This conclusion is directly drawn from the above map (2.7). When \( f^2 \) and \( h^2 \) change sign and further considering a redefinition of the radial coordinate \( r \rightarrow 1/r \) the metric ADM signature is \( \text{diag}(+, -, -) \) for the same region of space-time for which the original classical configuration (before the duality is applied) have metric ADM signature \( \text{diag}(-, +, +) \). Depending on the original solutions these relations can change over space-time, technically, this means that from a naked particle solution we may generate an horizon (which corresponds to the space-time line for which the new metric solutions swaps sign). However we may have a solution for which the duality swaps the metric ADM signature for all space-time, this is in principle not desirable since it may account for a change in signature convention instead of a new solution. In order to address these possibilities, or simply to obtain additional non-equivalent solutions, one may consider a second related duality that we address in the next section. Also depending on the reality conditions imposed on the solutions we may require this second duality in order to obtain valid solutions (we note that the duality (2.4) maps real electromagnetic fields into imaginary ones). We further remark that this discussion only applies to rotating spaces, by taking \( \tilde{A} \rightarrow 0 \) we get that \( \tilde{h}^2 > 0 \) (as long as \( \tilde{h} \) is real) and we retrieve the original duality of [4] such that the metric always maintains its signature under the duality.

### 2.2 Another possible duality for rotating space-times

There is yet another space-time duality for which the metric ADM signature signs change in the opposite way of the duality discussed in the previous section (2.8). Let us simply
consider the swapping of the time and angular coordinates obtaining the following map
\[
\begin{align*}
  t &\rightarrow \varphi \\
  \varphi &\rightarrow t
\end{align*}
\] \Rightarrow \begin{align*}
  f &\rightarrow h \\
  h &\rightarrow f
\end{align*}, \begin{align*}
  E_* &\rightarrow -B_* \\
  B_* &\rightarrow -E_* .
\end{align*}
(2.9)

The effect of the duality in the metric and in the Lagrangian (2.1) is
\[
ds^2 \rightarrow \hat{d}s^2 = \hat{f}^2(dt + \hat{A}d\varphi)^2 + \hat{d}r^2 - \hat{h}^2d\varphi^2 \\
= -f^2dt^2 + dr^2 + h^2(d\varphi + Adt)^2 ,
\]
(2.10)
\[
\mathcal{L}^{MCS} \rightarrow \hat{\mathcal{L}}^{MCS} = \hat{F} \wedge *\hat{F} + \hat{A} \wedge \hat{F} \\
= F \wedge *F + A \wedge F .
\]

The gauge sector does not swap relative sign, hence the nature of the gauge fields (being standard or ghost) is maintained by the duality. Again let us assume a known charged solution given by \( \hat{f}, \hat{h}, \hat{E}_*(\hat{f}, \hat{h}) \) and \( \hat{B}_*(\hat{f}, \hat{h}) \), such that from the above duality (2.10) and the metric components \( \hat{d}s^2 \)
\[
\begin{align*}
  \hat{g}_{00} &= -\hat{f}^2 + \hat{h}^2A^2 \\
  \hat{g}_{11} &= 1 \\
  \hat{g}_{22} &= \hat{h}^2 \\
  \hat{g}_{02} &= \hat{h}^2A
\end{align*}, \begin{align*}
  \hat{g}_{00} &= \hat{f}^2 \\
  \hat{g}_{11} &= 1 \\
  \hat{g}_{22} &= -\hat{h}^2 + \hat{f}^2A^2 \\
  \hat{g}_{02} &= \hat{f}^2A
\end{align*},
(2.11)

the explicit map between the fields can be computed
\[
\begin{align*}
  f^2 &= \frac{\hat{f}^2 \hat{h}^2}{-\hat{h}^2 + \hat{f}^2A^2} \\
  h^2 &= -\hat{h}^2 + \hat{f}^2A^2 \\
  A &= -\frac{\hat{A} \hat{f}^2}{-\hat{h}^2 + \hat{f}^2A^2}
\end{align*}, \begin{align*}
  E_* &= -\hat{B}_0 \left( \hat{f} = f, \hat{h} = h \right) \\
  B_* &= -\hat{E}_0 \left( \hat{f} = f, \hat{h} = h \right)
\end{align*}
(2.12)

The electric and magnetic field solutions are evaluated at \( \hat{f} = f \) and \( \hat{h} = h \) and under this duality we have the following behaviour for the metric ADM signature
\[
\hat{h}^2 - \hat{f}^2A^2 > 0 \ \Rightarrow \ \text{Map changes metric ADM signature ,}
\]
(2.13)
\[
\hat{h}^2 - \hat{f}^2A^2 < 0 \ \Rightarrow \ \text{Map maintains metric ADM signature .}
\]

Hence, the metric ADM signature changes in the opposite way under the dualities (2.4) and (2.9) such that both maps (2.7) and (2.12) are complementary and, depending on the specific classical solutions, may generate distinct new solutions.
2.3 Double wick rotation as a duality

Following the previous discussion it is straightforward to show that both dualities, as given by (2.4) and (2.9) can be related by a double Wick rotation

\[
\begin{align*}
  t &\rightarrow it \\
  \varphi &\rightarrow i\varphi
\end{align*}
\]

\[
\begin{align*}
  f &\rightarrow if \\
  h &\rightarrow ih
\end{align*}
\]

\[
\begin{align*}
  E_\ast &\rightarrow iE_\ast \\
  B_\ast &\rightarrow iB_\ast
\end{align*}
\]

The factor \(-h^2 + f^2 A^2 \rightarrow h^2 - f^2 A^2\) swaps sign as well as the Maxwell Chern-Simons Lagrangian \(F \wedge \ast F + A \wedge F \rightarrow -F \wedge \ast F - A \wedge F\). Thus we have obtained a web of dualities pictured in figure 1. In this way from a given electric or magnetic solution for the metric parameterization (2.1) one can obtain a new magnetic or electric solution (respectively) using the above dualities (2.4) or (2.9). The choice of the duality to be employed depends on the specific form of the known solutions such that the dual metric signature is set accordingly, either by condition (2.8) or (2.13). Also, from a given magnetic or electric solution with a given metric ADM signature, it is possible to obtain a new magnetic or electric solution with the opposite metric ADM signature. As already notice this last possibility is mostly useful for dressed solutions (containing an horizon), otherwise it may be interpreted as a change in convention for the metric signature instead of a duality between distinct solutions.

![Example for web of Dualities. The dualities can be considered in both directions. In the picture are expressed only the dualities for the directions of the arrows.](image)

Figure 1: Example for web of Dualities. The dualities can be considered in both directions. In the picture are expressed only the dualities for the directions of the arrows.
2.4 Generalization to higher dimensional space-times

The dualities discussed so far apply to planar 2+1-dimensional systems which are commonly interpreted as projected 3+1-dimensional systems and for which the magnetic field is a scalar (corresponding to \( B_{*(3D)} = F_{r\varphi} \), however we note that we have maintained the angular component of the electric field null, \( E^\varphi_{*(3D)} = F^{0\varphi} = 0 \). Due to the Einstein equations, if such component of the electric field is not null, we will need to further consider a non-null radial shift function for the metric [23] which does not constitute a physical gravitational degree of freedom in 2+1-dimensional gravity [53]. More generally, in 3+1-dimensions, we can assume non-projected cylindrical symmetry around the \( \theta \) direction such that there are 2 components of the magnetic field \( B^r_{*(4D)} \) and \( B^\theta_{*(4D)} \) as opposed to 2+1-dimensions where only one component exists, \( B_{*(3D)} = B^\theta_{*(4D)} \). Specifically, the electromagnetic fields for 3+1-dimensional systems with cylindrical symmetry are

\[
\begin{align*}
E^r_{*(4D)} &= F^{0r}, \\
E^\theta_{*(4D)} &= F^{0\theta}, \\
B^r_{*(4D)} &= F^\varphi_{r\theta} \\
B^\theta_{*(4D)} &= F^r_{r\varphi}.
\end{align*}
\]

Applied to these fields the duality (2.4) corresponds to the map

\[
\begin{align*}
E^r_{*(4D)} &\rightarrow -iB^\theta_{*(4D)}, \\
E^\theta_{*(4D)} &\rightarrow -iB^r_{*(4D)}, \\
B^r_{*(4D)} &\rightarrow -iE^\theta_{*(4D)}, \\
B^\theta_{*(4D)} &\rightarrow -iE^r_{*(4D)},
\end{align*}
\]

and the duality (2.9) corresponds to the map

\[
\begin{align*}
E^r_{*(4D)} &\rightarrow -B^\theta_{*(4D)}, \\
E^\theta_{*(4D)} &\rightarrow -B^r_{*(4D)}, \\
B^r_{*(4D)} &\rightarrow -E^\theta_{*(4D)}, \\
B^\theta_{*(4D)} &\rightarrow -E^r_{*(4D)},
\end{align*}
\]

such that these dualities are lifted from a planar system to a 3+1-dimensional system with cylindrical symmetry. As expected these sort of dualities are not possible for generic stationary solutions in 3+1-dimensions that do not possess cylindrical symmetry, the remaining field components \( E^r_{*(4D)} = F^{0r} \) and \( B^\varphi_{*(4D)} = -F^r_{r\theta} \) are maintained (up to sign changes) by the dualities such that the electric and magnetic fields are not mapped into each other.

A generalization is possible for \( N \)-form theories in higher dimensional space-times [56]. For instance in 5 + 1-dimensions, by considering the 2-form fields \( E^I_{*(4D)} = F^{0IJ} \) and \( B^I_{*(4D)} = F_{0IJ} \).
\[ \epsilon^{JKLM} F_{KLM} / 6, \] under the duality \((t \rightarrow ix^5, x^5 \rightarrow it)\) we obtain the map \(E_{s12}^\ast \leftrightarrow -iB_{s34}^\ast, E_{s13}^\ast \leftrightarrow -iB_{s24}^\ast, E_{s14}^\ast \leftrightarrow -iB_{s23}^\ast, E_{s23}^\ast \leftrightarrow -iB_{s14}^\ast, E_{s24}^\ast \leftrightarrow -iB_{s13}^\ast\) and \(E_{s34}^\ast \leftrightarrow -iB_{s12}^\ast.\)

As a final remark we note that there is no relation of these dualities with the usual electromagnetic duality of the gauge fields [54–58]. The original electromagnetic duality rotates the same components of the electric and magnetic fields into each other not mixing the space-time components of the fields.

### 3 Magnetic solutions for Minkowski space-time

In this section we derive explicit magnetic solutions for Einstein Maxwell Chern-Simons coupled to a scalar field employing the space-time duality developed in the previous section applied to the electric solutions computed in [23]. Hence we are considering the same Action of [23]

\[
S = \frac{1}{2\pi} \int_M d^3x \left\{ \sqrt{-\tilde{g}} \left[ e^{ab} \left( \tilde{R} + 2\lambda (\partial \phi)^2 \right) - e^{b\phi} \Lambda \right] 
+ \hat{\epsilon} \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right\}, \tag{3.1}
\]

where \(\hat{\epsilon} = \pm 1\) sets the relative sign between the gauge and gravitational sector and the remaining terms follow the conventions of [23] such that the metric has Minkowski ADM signature \(\text{diag}(-,+,+)+\) and we are employing natural units \(c_{\text{light}} = \hbar = 1\). We recall that \(\hat{\epsilon} = +1\) stands for a ghost gauge sector such that the gauge fields contribution to the total energy is negative while \(\hat{\epsilon} = -1\) stands for a standard gauge sector such that the gauge fields contribution to the total energy is positive [23, 51].

#### 3.1 Obtaining the solutions employing space-time duality

Directly applying the duality (2.4) to the electric solutions studied in [23] accounts for changing the metric parameterisation (2.1) under the map (2.5) such that we obtain the following metric parameterisation

\[
d\tilde{s}^2 = -\tilde{f}^2(dt + \tilde{A}d\phi)^2 + dr^2 + \tilde{h}^2 d\phi^2, \tag{3.2}
\]
and the magnetic solutions for the action (3.1) with $a = 0$, $c = -b/2$ and $\lambda \neq b^2/8$ [23]

$$\phi = -\frac{2}{b} \ln(C_\phi r)$$

$$\tilde{f} = C_f \sqrt{r}$$

$$\tilde{h} = C_h r^{p - \frac{3}{4}}$$

$$\tilde{A} = C_A r^{p - 1} + \theta$$

$$\tilde{B}_s = C_B r^{p - 2}$$

$$\tilde{A}_\varphi = \frac{C_B}{p - 1} r^{p - 1}$$

(3.3)

where $C_h, C_f, b$ and $\theta$ are free parameters and the constants $\lambda, C_\phi, C_A$ and $C_B$ have the following allowed values

$$\lambda = -\frac{b^2}{8p},$$

$$C_\phi = |m| \sqrt{\frac{1 - 6x}{1 - 3p}},$$

$$C_A = \frac{\text{sign} (m) C_h}{C_f (1 - p)} \sqrt{\frac{1 - 3p}{1 - 6x}},$$

$$C_B = \frac{C_h}{\sqrt{2|m|}} \sqrt{\frac{\epsilon (p - 4x + 6px)}{1 - 3p}} \left(\frac{1 - 3p}{1 - 6x}\right)^{\frac{3}{4}},$$

(3.4)

expressed in terms of a numerical parameter $p = p(x)$ and the ratio of the cosmological constant $\Lambda$ to the topological mass squared $m^2$

$$x = \frac{\Lambda}{m^2}.$$  

(3.5)

In the above expression for $C_B$ the factor of $(1 - 3p)$ was not simplified in order to maintain the factor inside of the square root explicitly positive. For completeness the equations of motion in the Cartan-frame for this metric parameterisation are also solved in Appendix A.

Imposing reality conditions for the solution constants there are four distinct allowed solutions depending on the parameter $\tilde{\epsilon} = \pm 1$, the range of values for the ratio $x$ and the
respective bounds on the parameter $p$

I. $\dot{\epsilon} = +1$, $x \in \left[0, \frac{1}{2}\right]$, 
$$p = -\frac{3x - \sqrt{x(2 - 3x)}}{1 - 6x} \in \left[0, \frac{1}{2}\right]$$

II. $\dot{\epsilon} = -1$, $x \in \left[0, \frac{1}{6}\right] \cup \left[\frac{1}{2}, \frac{2}{3}\right]$, 
$$p = -\frac{3x - \sqrt{x(2 - 3x)}}{1 - 6x} \in \left[0, \frac{1}{3}\right] \cup \left[\frac{1}{2}, \frac{2}{3}\right]$$

III. $\dot{\epsilon} = +1$, $x \in \left[0, \frac{1}{6}\right] \cup \left\{\frac{1}{14}\right\}$, 
$$p = -\frac{3x + \sqrt{x(2 - 3x)}}{1 - 6x} \in \left(-\infty, 0\right] \cup \left\{-1\right\}$$

IV. $\dot{\epsilon} = -1$, $x \in \left[\frac{1}{6}, \frac{2}{3}\right]$, 
$$p = -\frac{3x + \sqrt{x(2 - 3x)}}{1 - 6x} \in \left[\frac{2}{3}, +\infty\right].$$

The particular case $x = p = 0$ corresponds to $\lambda = 0$ and allows both for a limiting solution with $\tilde{B}_s = 0$, $m \neq 0$, $\phi \neq 0$ (a non-trivial dilaton field) and the trivial solution with $m = \Lambda = \phi = \tilde{B}_s = 0$ corresponding to empty flat Minkowski space-time. We note that the value of the cosmological constant is constrained by the mass $\Lambda < m^2$ (A.35) such that either of the limits $\Lambda \to 0$ or $m \to 0$ are equivalent to the limit $x \to 0$. In the following we consider that the particular case $x \to 0$ is retrieved by taking the limit $m \to 0$ such that this limiting solution corresponds to the trivial solution, empty flat Minkowski space-time. In solution I, the particular case $x = 1/6$ is well defined corresponding to the same solutions (3.3) with $p = 1/3$ however for solution II this value of the parameter does not allow for a real solution. Both in solution I and II the parameter value $x = p = 1/2$ is a well defined solution with null magnetic field, $C_B = 0$. In solution II and IV the parameter value $x = p = 2/3$ is also a well defined solution. In solution III the particular case $p = -1$ corresponding to $x = 1/14$ ($\lambda = b^2/8$) does not allow for solutions of the equations of motion, hence this value of the parameter is excluded. In solutions III and IV the value of the parameter $x = 1/6$ corresponds to $-\infty$ and $+\infty$, respectively. In addition, for solution IV, the particular case $x = 1/2$ corresponding to $p = 1$ has the solutions for $h, f, \tilde{B}$, and $\phi$ given in (3.3) and (3.4), however it has the particular solution for $A$

$$p = 1 \Rightarrow \tilde{A} = C_A \log(r) + \theta, \quad C_A = \frac{C_h \text{sign}(m)}{C_f}. \tag{3.7}$$

All the solutions presented correspond to positive cosmological constant and the solu-
tions I, II with \( x \in ]0, 1/6[ \) and III allow for the limiting solution corresponding to empty flat Minkowski space-time, \( x \to 0 \), while solution II with \( x \in [1/2, 2/3] \) and solution IV do not allow to obtain this limiting solution. For these solutions, the line element (3.2), re-written for the standard ADM parameterisation (2.1), is

\[
\begin{align*}
\tilde{ds}^2 &= -f^2dt^2 + dr^2 + h^2(d\varphi + Adt)^2 \\
f^2 &= \frac{C_f^2 r^{2p-1}}{1 - r^{2p-2} (\tilde{C}_A r^{p-1} + \tilde{\theta})^2} \\
h^2 &= C_h^2 r \left( 1 - r^{2p-2} \left( \tilde{C}_A r^{p-1} + \tilde{\theta} \right)^2 \right) \\
A &= -\frac{C_f r^{2p-2} (\tilde{C}_A r^{p-1} + \tilde{\theta})}{C_h \left( 1 - r^{2p-2} (\tilde{C}_A r^{p-1} + \tilde{\theta})^2 \right)}
\end{align*}
\]

This metric has determinant \( \sqrt{-\tilde{g}} = |C_f C_h| r^p \) and this parameterisation is obtained directly from the map (2.7) corresponding to the duality (2.4). The non-null metric components can be computed directly from this parameterisation as expressed in equation (2.6)

\[
\begin{align*}
\tilde{g}_{00} &= -C_f^2 r^{2p-1} , \\
\tilde{g}_{11} &= 1 , \\
\tilde{g}_{22} &= C_h^2 r \left( 1 - r^{2p-2} (\tilde{C}_A r^{p-1} + \tilde{\theta})^2 \right) , \\
\tilde{g}_{02} &= -C_f C_h r^{2p-1} (\tilde{C}_A r^{p-1} + \tilde{\theta}) .
\end{align*}
\]

In the above expressions we have replace the constants \( C_A \) (3.4) and \( \theta \) by the respective expressions multiplied by the ratio \( C_f/C_h \)

\[
\tilde{\theta} = \frac{C_f}{C_h} \theta , \quad \tilde{C}_A = \frac{C_f}{C_h} C_A = \frac{\text{sign}(m)}{(1-p) \sqrt{\frac{1-3p}{1-6x}}} .
\]

We note that for \( p < 1 \) the metric has ADM signature \( diag(-, +, +) \) corresponding to the chosen convention while for \( p > 1 \) the metric has ADM signature \( diag(+, +, -) \) such that further considering a radial coordinate transformation \( r \to 1/r \) it is obtained the metric ADM signature \( diag(+, -, -) \), hence corresponding to the opposite convention with respect to the originally chosen convention. At \( p = 1 \) the metric ADM signature depends on the sign of the factor \( (1 - (\tilde{C}_A + \tilde{\theta})^2) \), when this factor is positive it has ADM signature \( diag(-, +, +) \) and when this factor is negative (considering the coordinate transformation \( r \to 1/r \)) it has ADM signature \( diag(+, -, -) \). We recall that the coordinate transformation \( r \to 1/r \) implies exchanging the origin with spatial infinity \( r: 0 \leftrightarrow +\infty \). In addition, when horizons are present this swapping of signature is equivalent to swapping the exterior region with the interior region of the horizons.
For the solutions discussed here, the swapping of the metric ADM signature with respect to the chosen convention corresponds to solution IV with the parameter $x$ in the range $x \in [1/6, 1/2]$. Generally, for a given particular solution changing the metric ADM signature, the duality corresponding to a double Wick rotation of the coordinates $t$ and $\varphi$ (2.14) could generate new solutions which would maintain the metric ADM signature. However for the solutions just computed, when considering the reality conditions on the fields discussed in appendix 3.1, the duality (2.14) simply swaps the sign of $\hat{\epsilon}$ and the parameter $p$, hence no new solutions are obtained, instead solutions I and II are swapped with each other and solutions III and IV are swapped with each other.

### 3.2 Singularities and curvature analysis

In this section we analyse the space-time singularities, the existence of horizons and its location.

To analyse space-time singularities in 2+1-dimensions it is enough to analyse the contraction of the Ricci scalar $R_{\mu\nu}$ with itself [23]. For the solutions computed in the previous section this contraction is

$$R_{\mu\nu}R^{\mu\nu} = \frac{1}{4r^{12}} \left[ (3 - 16p + 34p^2 - 28p^3 + 8p^4) r^8 + 3\tilde{C}_A (p - 1)^4 r^{8p} - 2\tilde{C}_A^2 (p - 1)^2 (4p - 5) \times \right.$$

$$\times \left. (4p - 3) r^{4+4p} \right]. \tag{3.11}$$

For the particular case $p = 1$ we obtain $R_{\mu\nu}R^{\mu\nu} = 3/(2r^4)$, hence for $p \geq 1$ the dominant divergent term near $r = 0$ is proportional to $\sim 1/r^4$, while for $p < 1$ it is proportional to $1/r^{12-8p}$ such that we conclude that there is a space-time singularity at $r = 0$ for all values of $p$. In addition, for $p > 3/2$ corresponding to $x < 9/26$ in solution IV, spatial infinity is also a space-time singularity as the dominant divergent term is proportional to $\sim r^{8p-12}$.

As for the curvature it is

$$R = -\frac{(1 - 6p + 4p^2) r^4 + \tilde{C}_A (p - 1)^2 r^{4p}}{2r^6}. \tag{3.12}$$

For the particular case $p = 1$ the curvature is $R = 1/r^2$. Consistently with the singularity analysis discussed above, for $p < 3/2$ the curvature vanishes at spatial infinity, hence space-time is asymptotically flat, for $p = 3/2$ it converges to the positive constant $\tilde{C}_A^2/8$ and for $p > 3/2$ it diverges.

Depending on the values of $x$ the curvature is either always positive or exist regions where it is negative. For solution I and II, it is always positive for $x \geq (8 - 3\sqrt{5})/38$, while for $x < (8 - 3\sqrt{5})/38$ it is negative for $r > r_{0.1}$ having a negative minimum value at
\[ r = r_{\text{min, I}} > r_{0,1} \] and converging to 0 at spatial-infinity. Near the origin, for \( r < r_{0,1} \), it is positive. Here \( r_{0,1} \) and \( r_{\text{min, I}} \) are

\[
r_{0,1} = \left( 1 - 7x - 3\sqrt{x(2-3x)} \right)^{\frac{1}{4(p-1)}} \in ]1, +\infty[ ,
\]

\[
r_{\text{min, I}} = \left( \frac{3 - 45x + 102x^2 - (7 - 22x)\sqrt{x(2-3x)}}{9 - 26x} \right)^{\frac{1}{4(p-1)}} \in ]3^{1/4}, +\infty[ .
\]

(3.13)

For solution III the curvature is negative for \( r > r_{0,\text{III}} \) having a negative minimum value at \( r = r_{\text{min, III}} > r_{0,\text{III}} \), it converges to 0 at spatial-infinity and near the origin, for \( r < r_{0,\text{III}} \), it is positive. As for solution IV, \( x \in ]1/6, 9/26[ \) (\( x = 9/26 \) corresponds to \( p = 3/2 \)), the curvature is negative for \( r < r_{0,\text{III}} \) and it is positive for \( r > r_{0,\text{III}} \) diverging at spatial infinity, for \( x = 9/26 \) the curvature is negative for \( r < 2/\sqrt{13} \) and it is positive for \( r > 2/\sqrt{13} \) converging to 13/8 at spatial infinity, for \( x \in ]9/26, (8 + 3\sqrt{5})/38[ \) it is negative near the origin for \( r < r_{0,\text{III}} \) and it is positive for \( r > r_{0,\text{III}} \) converging to 0 at spatial infinity and it has a positive maximum value at \( r = r_{\text{min, III}} > r_{0,\text{III}} \), while for \( x \in ](8 + 3\sqrt{5})/38, 2/3[ \) the curvature is always positive converging to 0 at spatial infinity. Here \( r_{0,\text{III}} \) and \( r_{\text{min, III}} \) are

\[
r_{0,\text{III}} = \left( 1 - 7x + 3\sqrt{x(2-3x)} \right)^{\frac{1}{4(p-1)}}
\]

\[
eq \begin{cases} 
0.93, 1 & \text{for } x \in ]0, 1/6[ \\
0, 1.01 & \text{for } x \in ]1/6, 2/3[ 
\end{cases}
\]

\[
r_{\text{min, III}} = \left( \frac{3 - 45x + 102x^2 + (7 - 22x)\sqrt{x(2-3x)}}{9 - 26x} \right)^{\frac{1}{4(p-1)}}
\]

\[
eq \begin{cases} 
1, 3^{1/4} & \text{for } x \in ]0, 1/6[ \\
0, 1 & \text{for } x \in ]9/26, (8 + 3\sqrt{5})/38[ 
\end{cases}
\]

(3.14)

Hence, resuming the previous analysis, the curvature values for the several allowed solutions
are, for the several solutions discussed,

\[ I. \quad \dot{\epsilon} = +1, \]
\[
\begin{align*}
    x & \in \left[0, \frac{8 - 3\sqrt{5}}{38}\right], & R & \in ]R(r_{\text{min, I}}) < 0, +\infty[ \\
    x & \in \left[\frac{8 - 3\sqrt{5}}{38}, \frac{1}{2}\right], & R & \in ]0, +\infty[ \\
\end{align*}
\]

\[ II. \quad \dot{\epsilon} = -1, \]
\[
\begin{align*}
    x & \in \left[0, \frac{8 - 3\sqrt{5}}{38}\right], & R & \in ]R(r_{\text{min, I}}) < 0, +\infty[ \\
    x & \in \left[\frac{8 - 3\sqrt{5}}{38}, \frac{1}{6}\right] \cup \left[\frac{1}{2}, \frac{2}{3}\right], & R & \in ]0, +\infty[ \\
\end{align*}
\]

\[ III. \quad \dot{\epsilon} = +1, \]
\[
\begin{align*}
    x & \in \left[0, \frac{1}{6}\right] \setminus \left\{\frac{1}{14}\right\}, & R & \in ]R(r_{\text{min, III}}) < 0, +\infty[ \\
\end{align*}
\]

\[ IV. \quad \dot{\epsilon} = -1, \]
\[
\begin{align*}
    x & \in \left[\frac{1}{6}, \frac{9}{26}\right], & R & \in ]-\infty, +\infty[ \\
    x & = \frac{9}{26}, & R & \in ]-\infty, \frac{13}{8}[ \\
    x & \in \left[\frac{9}{26}, \frac{8 + 3\sqrt{5}}{38}\right], & R & \in ]-\infty, R(r_{\text{min, III}}) > 0[ \\
    x & \in \left[\frac{8 + 3\sqrt{5}}{38}, \frac{2}{3}\right], & R & \in ]0, +\infty[ \\
\end{align*}
\]

As for the nature of the space-time singularity we note that, independently of the value of the parameter \( p \), the maximum value of the coordinate \( \varphi \) diverges at the singularity \( r = 0 \) and it is finite up to spatial infinity being real outside the horizon (3.26) discussed in the next section. At spatial infinity it diverges for \( p \leq -1/4 \) and \( p > 3/2 \) and it is asymptotically null for \( p \in [1, 3/2] \) (being finite at \( p = 3/2 \)). As for the range of the parameter \( p \in ] -1/4, 1[ \) there is a specific frame for which the maximum value of the coordinate \( \varphi \) matches the usual relations corresponding to flat Minkowski space-time. Specifically, defining the 2-dimensional intrinsic metric \( \tilde{h}_{ij} = \text{diag}(1, h^2) \) corresponding to
metric $\tilde{g}_{\mu\nu}$ (3.9) and considering a rescaling of the radial coordinate

$$r = \tilde{r}^\xi \Rightarrow dr = \xi \tilde{r}^{\xi-1} d\tilde{r},$$

(3.16)

we obtain that the maximum value for the coordinate $\varphi$ is

$$\varphi_{\text{max}} = \frac{2\pi}{\sqrt{-|\tilde{g}_{\mu\nu}|}} \sqrt{\frac{h_{\varphi\varphi}}{h_{rr}}} = \frac{2\pi}{f \sqrt{h_{rr}}}$$

such that the following asymptotic expressions at spatial infinity are obtained

$$\sqrt{-|\tilde{g}_{\mu\nu}|} = |\xi C_f C_h| \tilde{r}^{-1+\xi(1+2p)},$$

$$\lim_{r \to \infty} \sqrt{|h_{ij}|} = |\xi C_h| \tilde{r}^{-1+\frac{3}{2}\xi},$$

$$\lim_{r \to \infty} \varphi_{\text{max}} = \frac{2\pi}{|\xi C_f|} \tilde{r}^{1-\xi(1+4p)}.$$

(3.17)

(3.18)

Setting $\xi = 2/(1+4p)$, $\varphi_{\text{max}}$ is asymptotically constant exactly matching $2\pi$ for $C_f = (1+4p)/2$. In addition we note that at spatial infinity both the space-time measure $\sqrt{-|\tilde{g}_{\mu\nu}|}$ and the space measure $\sqrt{|h_{ij}|}$ are, in this frame proportional to a positive exponent of $\tilde{r}$. Let us further note that the constant $C_f$ is interpreted as the velocity of light in vacuum and its value can be redefined by a re-scaling of the time coordinate $t$, hence there is some loss of generality when fixing the constant $C_f = (1+4p)/2$ (to ensure that $\lim_{\tilde{r} \to +\infty} \varphi = 2\pi$) as we are fixing the speed of light in a particular frame, hence we are generally leaving $C_f$ as a free constant. Resuming this discussion we conclude that the coordinate $\varphi$ can exactly match the angular coordinate for Minkowski empty flat space-time at spatial infinity for a particular frame only when

$$p \in \left[ \frac{1}{4}, \frac{1}{2} \right] \setminus \{0\},$$

(3.19)

When considering this constraint the range of the parameter $x$ for solution I is not affected, for solution II is reduced to $x \in [0,1/6]$, for solution III is reduced to $x \in [0,1/62]$ and solution IV is excluded. We further note that for this range only the space-time singularity at the origin exists (3.20) as $p < 3/2$ such that no singularity at spatial infinity is present.

For all values of $p$, $\varphi_{\text{max}}$ diverges at the singularity $r = 0$ and, for $p > 3/2$, $\varphi_{\text{max}}$ is null at the singularity $r \to +\infty$, hence we interpreted these singularities as a decompactification singularity and a conical singularity, respectively [23]

$$\forall_p, \lim_{r \to 0} \varphi_{\text{max}} = +\infty \Rightarrow r = 0 \text{ is a decompactification singularity}.$$

$$p > \frac{3}{2}, \lim_{r \to +\infty} \varphi_{\text{max}} = 0 \Rightarrow r \to +\infty \text{ is a conical singularity}.$$

(3.20)
Next we analyse the horizons for an external observer.

### 3.3 Horizons and photon topological mass

To analyse the existence of horizons the usual approach is to compute the geodesic motion of photons. From the point of view of an external observer the horizon corresponds to the spatial hyper-surface for which the photon freezes such that its geodesic equation is \( \dot{r} = 0 \). In [23] were computed the differential equations describing geodesic motion. For a particle with null angular momentum \( L = 0 \) we obtain

\[
\dot{r}_\kappa = \pm \frac{g_{22}}{\sqrt{-g \left( -g \frac{\kappa}{E^2} + g_{22} \right)}} \frac{|C_f|^p p^{\frac{1}{2}} \sqrt{1 + \frac{\kappa}{E^2} C_f^2 r^{2p-1} - r^{2p-2} (C_A r^{p-1} + \dot{\theta})^2}}{1 - r^{2p-2} (C_A r^{p-1} + \dot{\theta})^2}, \tag{3.21}
\]

\[
\dot{\phi}_\kappa = -\frac{g_{02}}{g_{22}} C_f C_h \frac{-r^{2p-2}(C_A r^{p-1} + \theta)}{1 - r^{2p-2}(C_A r^{p-1} + \theta)^2},
\]

where \( g = |g_{\mu\nu}| \) is the determinant of the metric, \( E \) the energy of the particle and \( \kappa = -1 \) for standard massive particles (corresponding to time-like trajectories), \( \kappa = 0 \) for photons or any other massless particles (corresponding to light-like trajectories) and \( \kappa = +1 \) for tachyons or other particles with imaginary energy eigenvalues (corresponding to space-like trajectories).

Generally the above equations are not solvable analytically. In the following we will analyse the zeros and divergences of the first equation for particles travelling towards the singularities which is enough to conclude whether a horizon exist or not. We further note that due to the Chern-Simons term the photon acquires a topological mass \( m \) such that its energy squared is \( E^2 = m^2 \) [35]. Specifically from the equation of motion for \( A_\mu \) we obtain [23]

\[
\partial_\alpha (\sqrt{-g} e^\phi F^{\alpha \mu}) + m e^{\mu\alpha\beta} F_{\alpha\beta} / 2 = 0
\]

such that computing the divergence of this equation, replacing itself in the resulting differential equation and using the definition of the dual field strength \( \star F = -\sqrt{-g} e^\phi e^{\mu\alpha\beta} F_{\alpha\beta} / (2\sqrt{-g} e^\phi) \) we obtain the photon propagation equation in dual form [35]

\[
(\Box - m^2) \star F^\mu = 0 , \quad \Box (\cdot) = \frac{1}{\sqrt{-g} e^\phi} \partial_\alpha (\sqrt{-g} e^{\phi} \partial^\alpha (\cdot)) , \tag{3.22}
\]

where \( \Box \) stands for the 2+1-dimensional Laplace operator for action (3.1) and the relative signs in this equation do depend on the metric signature convention. In particular we note that in flat space-time, for the convention adopted here, \( \eta_{\mu\nu} \sim \text{diag}(-,+,+) \) we consistently obtain \( (-\partial_0 \partial^0 + \partial_i \partial^i - m^2) \star F^\mu = 0 \) while for the opposite sign convention [35] \( \eta_{\mu\nu} \sim \text{diag}(+,−,−) \) we obtain \( (\Box + m^2) \star F^\mu = (\partial_0 \partial^0 - \partial_i \partial^i + m^2) \star F^\mu = 0 \) such that
both equations are the same up to an overall minus sign, corresponding to a photon with a standard (topological) mass $m$. Hence as extensively analysed in the literature we conclude that no massless photons exist for Maxwell Chern-Simons theories [34, 35].

It is straightforward to check that for all values of $p$ and $\kappa$, as we approach the singularity at $r = 0$, the velocity of any given particle vanishes

$$\lim_{r \to 0} \dot{r}_\kappa = 0 ,$$

(3.23)

while in this limit $\dot{\varphi}_\kappa$ is finite for $p = 1$ and null for all other values of $p$. This implies that the singularity is itself an horizon, hence it is not a naked singularity. However this result is not conclusive as for higher values of $r > 0$ there exists a divergence of $\dot{r}_\kappa$, specifically when the denominator of the first equation of (3.21) is null the particle velocity diverges. This divergence is located at the value of the radial coordinate $r = r_{\text{div}}$ obeying the equation

$$1 = \left( r_{\text{div}}^{p-1} (\tilde{C}_A r_{\text{div}}^{p-1} + \tilde{\theta}) \right)^2 .$$

(3.24)

This equation has one real positive solution $r_{\text{div}}$ for all values of $p$ and $\tilde{\theta}$. We recall that $\tilde{C}_A$ is not a free constant being expressed in equation (3.4) and (3.10) as a function of $x$ and $p = p(x)$. Specifically, one of the 4 solution $r_{\text{div}, \pm, \pm} = ((-\tilde{\theta} \pm \sqrt{\tilde{\theta}^2 \pm 4 \tilde{C}_A})/(2 \tilde{C}_A))^{1/\mp}$, is real and positive for all the allowed range of the parameters.

In addition to ensure that for $r > r_{\text{div}}$, the space-time has Minkowski signature $\text{diag}(-, +, +)$ and that $\dot{r}_\kappa$ describes the geodesic motion of a particle it is required that this quantity ($\dot{r}_\kappa$) be real valued and consistently have either positive sign for particles travelling away from the singularity either negative sign for particles travelling towards the singularity. These properties are obeyed as long as the factor $1 - \left( r_{\text{div}}^{p-1} (\tilde{C}_A r_{\text{div}}^{p-1} + \tilde{\theta}) \right)^2$ is real and positive for $r > r_{\text{div}}$. This statement is simply equivalent to the bound $p < 1$ such that the factor $\left( r_{\text{div}}^{p-1} (\tilde{C}_A r_{\text{div}}^{p-1} + \tilde{\theta}) \right)^2$ decreases with growing radial coordinate. Hence we obtain the bounds

$$\begin{cases} r > r_{\text{div}} \\ 1 > \left( r_{\text{div}}^{p-1} (\tilde{C}_A r_{\text{div}}^{p-1} + \tilde{\theta}) \right)^2 \iff p < 1 . \end{cases}$$

(3.25)

This bound, $p < 1$, is consistent with the analysis in the previous section.

For massless particles the velocity divergence in $\dot{r}_{\kappa=0}$ just analysed is outside any horizon. This is straightforwardly shown by noting that for $\kappa = 0$ the numerator of $\dot{r}_{\kappa=0}$ is the square root of its denominator (3.21) such that the only horizon is at $r = 0$ as already concluded (3.23). Classically there is no interpretation for a particle velocity divergence, however we note that upon path integral quantization this phenomena can be consistently described as a tunnelling effect, hence an instanton configuration [26]. We are not proceeding with this analysis here, instead let us note that from the photon equations of motion (3.22) the photon acquires a topological mass $m$ such that no massless photons
exist in the theory discussed here. Therefore, assuming that no massless particles exist in the theory let us analyse the photon geodesic motion with energy squared given by \( E^2 = m^2 \) and light-like trajectories \( (\kappa = -1) \). For this case we conclude that an horizon at the value of the radial coordinate for which the numerator of \( \dot{r}_{\kappa=-1} \) is null. Furthermore we note that, due to the denominator of \( \dot{r}_{\kappa=-1} \) being positive for \( r > r_{\text{div}} \) and the term \( \kappa C_f^2 / E^2 r^{2p-1} < 0 \) being negative for all values of \( r \), the value of the radial coordinate corresponding to the horizon \( r = r_H \) is greater than \( r_{\text{div}} \) (3.24)

\[
\begin{aligned}
\begin{cases}
p < 1 \\
p = 1 = \left( r_H^{p-1} (\tilde{C}_A r_H^{p-1} + \tilde{\theta}) \right)^2 + \frac{C_f^2}{m^2} r_H^{2p-1} \iff r_H > r_{\text{div}}.
\end{cases}
\end{aligned}
\]

Although the author failed to find a analytical solution for this equation the previous discussion is enough to conclude that for all allowed solutions and parameter ranges with \( p < 1 \) there exists an horizon for the value of the radial coordinate \( r_H \) given by this equation. Hence both the space-time singularity at \( r = 0 \) and the singularity in the particle velocity at \( r = r_{\text{div}} \) are inside the horizon and are not observable by an external observer. This is a valid statement both for photons (which are massive due to the Chern-Simons term) and for any other massive particles.

As for the particular case of solution IV with \( p > 1 \) we note that (further considering the redefinition \( r \rightarrow 1/r \)) the ADM signature of the metric for \( r > r_H \) (3.26) is \( \text{diag}(+, -, -) \), hence with the opposite sign of the original convention. Recalling that at the horizon the metric changes sign [51], this is simply interpreted as that the interior of the horizon for \( p > 1 \) corresponds to the region with \( r > r_H \), hence for an external observer in the region \( r \in [0, r_H] \) these solutions are interpreted as a dressed point-like singularity at \( r = 0 \) and an horizon at \( r = r_H \) such that \( r_{\text{div}} > r_H \) and the singularity at \( r = +\infty \) are within the region contained by the horizon \( (r > r_H) \).

Next we compute the mass, the magnetic flux and the angular momentum for the classical solutions obtained.

### 3.4 Mass, Angular Momentum and Magnetic Flux

In this section we derive and analyse the expressions for the mass, angular momentum and magnetic flux for the solutions computed (3.6). We postpone a interpretation of these results until the next section 4 where all the possible cases are gathered in table 1 and the results obtained are discussed.

We recall that there are several definitions of mass, namely in [23] it was computed the ADM mass [51,52,59] which for the metric parameterisation (2.1) is

\[
M_{\text{ADM}} = 2h' + 4\lambda \phi \phi' + 2\bar{c} e^{-b\phi/2} A'_{\phi} A'_{\psi} \bigg|_{r \rightarrow \delta_M} \bigg|_{r \rightarrow \infty},
\]

(3.27)
where $\delta_M$ is a cut-off near the singularity (of order of the Planck Length) introduced to regularize the singularity at the origin maintaining the mass value finite. However for the magnetic solutions (3.3) the value of the ADM mass is generally complex. We note that the ADM mass corresponds to the (classical) eigenvalue of the Hamiltonian constraint, hence, generally, aiming at the quantization of the gravitational sector of the theory. This is not the aiming of the present discussion. Instead of the ADM definition of mass we are taking a classical definition of mass that allows for real values to the solutions (3.3). The standard General Relativity definition of mass is the integral of the gravitational mass-energy density $\rho_g$. For a generic Einstein Tensor $G_{\mu\nu}$ the mass-energy density $\rho_g$ and pressure $p_g$ are [51]

$$\rho_g = G_{00} - p_g(1 - g_{00}) \ , \quad p_g = -\frac{G_{03}}{g_{03}},$$

such that the total mass and angular momentum are obtained by integrating these quantities over a spatial hyper-surface [51]

$$M = \int \sqrt{|h_{ij}|} \rho_g \, dx^2 \ , \quad S_z = \int \sqrt{|h_{ij}|} r \, p_g \, g^{03} \, dx^2 ,$$

where $|h_{ij}|$ stands for the determinant of the induced 2-dimensional spatial metric discussed in the previous section and we note that in 2+1-dimensions the only angular momentum component correspond to the 3+1-dimensional angular momentum along $z$ (from the definition $S_k = \int \epsilon_{kij} x^i T^{0j}$ [51] it is obtained that $S_r = S_\varphi = 0$).

For the action (3.1) there is also a contribution to the classical gravitational mass due to the dilaton-like scalar field $\phi$. This contribution can be read directly from the Einstein Equations [23]

$$G_{\mu\nu} + \lambda \partial_{\mu} \phi \partial_{\nu} \phi - \frac{\lambda}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} e^{b\phi} g_{\mu\nu} \Lambda = 2e^{-b\phi} T_{\mu\nu} ,$$

where we have taken in consideration the ansatz $a = 0$, $c = -b/2$ and the bare electromagnetic stress-energy tensor is $T_{\mu\nu} = \dot{\varepsilon} \left( F_{\mu\alpha} F^\alpha_{\mu} - g_{\mu\nu} F^2 / 4 \right)$. Hence we note that, for a classical configuration obeying these equations, the Einstein tensor contribution plus the scalar field contribution to the gravitational mass-energy density and pressure matches the respective electromagnetic quantities [51]

$$\rho_{grav} = \rho_g + \rho_\phi = \rho_{EM} \ , \quad p_{grav} = p_g + p_\phi = p_{EM} .$$

In the following we employ these definitions of gravitational energy-momentum density and pressure density to compute the respective total quantities. Noting that the only non-null component of the Maxwell tensor is $F_{r\varphi} = F_{12} = \tilde{B}_s$ (3.3) it is straightforward to obtain
the expressions for these quantities

\[ \rho_{grav} = \rho_{EM} = -\frac{\hat{\epsilon}}{2} g_{00} g^{11} g^{22} B_s^2 e^{-\frac{b}{2} \phi} - p_{EM} (1 - g_{00}) \]

\[ = -\frac{\hat{\epsilon} C^2_B C_\phi}{2 C_h^2} \ , \]

\[ p_{grav} = p_{EM} = \frac{\hat{\epsilon}}{2} g^{11} g^{22} B_s^2 e^{-\frac{b}{2} \phi} - b^2 \phi - p_{EM} (1 - g_{00}) \]

\[ = \frac{\hat{\epsilon} C^2_B C_\phi}{2 C_h^2} \ . \]  \quad (3.32)

These quantities are real valued for all the range of the parameter \( p \) and in the limit \( p \to 0 \) are consistently null, as already discussed the particular solution corresponding to \( p = 0 \) corresponds to Minkowski flat empty space-time. We also note that the equation of state for these solutions is a constant \( \omega_{grav} = \rho_{grav}/p_{grav} = -1 \). However, depending on the value of the parameter \( p \) they may have either a divergence at the origin \( r \to 0 \) (IR), either a divergence at spatial infinity \( r \to +\infty \) (UV) or both.

To regularize these divergences and allow for a simpler analysis of the total quantities we consider two cut-offs \( \delta_{IR} \) (lower cut-off) and \( \delta_{UV} \) (upper cut-off) which can be taken to 0 and \( +\infty \), respectively. Specifically for the mass \( M \) we obtain the following integral expression

\[ M = \int_{\delta_{IR}}^{\delta_{UV}} dr \int_{0}^{\varphi_{\text{max}}} d\varphi \sqrt{|h_{ij}|} \rho_{grav} \]

\[ = -\frac{\hat{\epsilon} C^2_B C_\phi \pi}{(C_f C_h)^2} \int_{\delta_{IR}}^{\delta_{UV}} dr r^{p-3} \left(1 - r^{2p-2}(\tilde{C}_A r^{p-1} + \tilde{\theta})^2\right) \quad \text{,} \]  \quad (3.33)

and for the angular momentum \( S_z \)

\[ S_z = \int_{\delta_{IR}}^{\delta_{UV}} dr \int_{0}^{\varphi_{\text{max}}} d\varphi \sqrt{|h_{ij}|} r p_{grav} g^{02} \]

\[ = -\frac{\hat{\epsilon} C^2_B C_\phi \pi}{(C_f C_h)^2} \int_{\delta_{IR}}^{\delta_{UV}} dr r^{p-3} \left(\tilde{C}_A r^{p-1} + \tilde{\theta}\right) \times \]

\[ \times \left(1 - r^{2p-2}(\tilde{C}_A r^{p-1} + \tilde{\theta})^2\right) \quad \text{.} \]  \quad (3.34)

We note that these quantities are evaluated in a 2-dimensional spatial hyper-plane, hence \( M \) has units of mass over length and \( S_z \) of mass such that when embedded into a 3-dimensional spatial manifold it is further required to integrated over the thickness of the 2-dimensional embedding along the orthogonal direction (\( z \)) to retrieve the standard 3-dimensional quantities with units of mass and angular momentum, respectively.
Evaluating the integral expression (3.33) for the Mass $M$ we obtain

$$p \neq -1, \frac{6}{5}, \frac{4}{3}, \frac{5}{4}, 2$$

$$M = -\hat{\epsilon} C_\pi^2 C_\phi \left( \frac{C_A^2}{6 - 5p} r^{5p-6} + \frac{2C_A \hat{\theta}}{5 - 4p} r^{4p-5} + \frac{\bar{\theta}^2}{4 - 3p} r^{3p-4} + \frac{1}{p-2} r^{p-2} \right) \delta_{IR}. \quad (3.35)$$

For $p = -1$ there are no allowed solution and the specific expressions for $p = 1, \frac{6}{5}, \frac{4}{3}, \frac{5}{4}, 2$ are listed in appendix B in equations (B.1–B.5). By direct inspection of the expressions for the mass it is straightforward to conclude that the divergence at the origin $r \to 0$ is present for $p \leq 2$ and that the divergence at spatial infinity $r \to +\infty$ is present for $p \geq \frac{6}{5}$. Hence, depending on the value of the parameter $p$, finite mass expressions $M$ can be evaluated by considering the following limits on $\delta_{IR}$ and $\delta_{UV}$

$$p \in \left[-\infty, \frac{6}{5}\right] \setminus \{-1, 0\} \Rightarrow \begin{cases} \delta_{IR} &\nRightarrow 0 \\ \delta_{UV} &\to +\infty \end{cases}$$

$$p \in \left[\frac{6}{5}, 2\right] \Rightarrow \begin{cases} \delta_{IR} &\nRightarrow 0 \\ \delta_{UV} &\nRightarrow +\infty \end{cases} \quad (3.36)$$

$$p \in \left[2, +\infty\right] \Rightarrow \begin{cases} \delta_{IR} &\Rightarrow 0 \\ \delta_{UV} &\nRightarrow +\infty \end{cases}$$

The first range for the parameter $p$ corresponds to solutions I, II, III and IV with $p \in \left[\frac{2}{3}, \frac{6}{5}\right]$ (3.6) while the second and third ranges correspond to solution IV.

As for the sign of the mass, for the range $p \in \left(-\infty, 1\right) \setminus \{-1, 0\}$ it has the opposite sign of $\hat{\epsilon}$, $M \sim -\hat{\epsilon}$ and for the range $p \in \left[1, +\infty\right]$ it has the same sign of $\hat{\epsilon}$, $M \sim \hat{\epsilon}$. We note that a negative mass is not unexpected since we are allowing for a gauge ghost sector, we recall that $\hat{\epsilon} = +1$ corresponds to a ghost gauge sector and that $\hat{\epsilon} = -1$ corresponds to a standard gauge sector. For the range $p \in \left(-\infty, 1\right]$, outside the horizon $\rho_{grav}$ has the opposite sign of $\hat{\epsilon}$ in accordance to whether the gauge sector is a ghost or a standard sector, however the predominant contribution to the value of the mass is within the horizon and the integrand in (3.33) changes sign at the horizon such that the total mass is actually positive when it is considered a ghost gauge sector and it is negative when a standard ghost gauge sector is considered. In the range $p \in \left[1, +\infty\right]$ the opposite behaviour is verified such that the total mass is negative when it is considered a ghost gauge sector and it is positive when a standard ghost gauge sector is considered. This is simply explained as due to the contribution of the scalar field to the total mass, its classical energy opposes the contribution from the standard gravitational sector.
Evaluating the integral expression (3.34) for the angular momentum $S_z$ we obtain

$$ p \neq -1, 1, \frac{7}{6}, \frac{5}{4}, \frac{3}{2}, 2 $$

$$ S_z = -\frac{\epsilon C_B^2 C_f \pi}{(C_f C_h)^2} \left( \frac{C_A^3}{7-6p} r^{6p-7} + \frac{3C_A^2 \hat{\theta}}{6-5p} r^{5p-6} + \frac{3C_A \hat{\theta}^2}{5-4p} r^{4p-5} + \frac{\hat{\theta}^3}{4-3p} r^{3p-4} + \frac{\hat{C}_A}{3-2p} r^{2p-3} + \frac{\hat{\theta}}{p-2} r^{p-2} \right) \delta_{UV} $$

The specific expressions for $p = -1, 7/6, 6/5, 5/4, 4/3, 3/2, 2$ are listed in appendix B in equations (B.6–B.12). By direct inspection of the expressions for the angular momentum it is straightforward to conclude that the divergence at the origin is present for $p \leq 2$ and that the divergence at spatial infinity is present for $p \geq 7/6$. Hence, depending on the value of the parameter $p$, finite angular momentum expressions $S_z$ can be evaluated by considering the following limits on $\delta_{IR}$ and $\delta_{UV}$

$$ p \in \left] -\infty, \frac{7}{6} \right[ \setminus \{ -1, 0 \} \Rightarrow \left\{ \begin{array}{l}
\delta_{IR} \not\to 0 \\
\delta_{UV} \to +\infty
\end{array} \right. $$

$$ p \in \left[ \frac{7}{6}, 2 \right] \Rightarrow \left\{ \begin{array}{l}
\delta_{IR} \not\to 0 \\
\delta_{UV} \not\to +\infty
\end{array} \right. $$

$$ p \in \left] 2, +\infty \right[ \Rightarrow \left\{ \begin{array}{l}
\delta_{IR} \to 0 \\
\delta_{UV} \not\to +\infty
\end{array} \right. $$

Similarly to the results obtained for the mass, the first range for the parameter $p$ corresponds to solutions I, II, III and IV with $p \in [2/3, 7/6]$ (3.6) while the second and third ranges correspond to solution IV.

As for the sign of the angular momentum $S_z$ we obtain that in the range $p \in [-\infty, 1/\{ -1, 0 \}$ it is $S_z \sim +\epsilon \text{sign}(\tilde{C}_A)$ which correspond to solution III in the range $p \in -\infty, 0/\{ -1 \}$, solution I and II in the range $p \in [0, 2/3]$ and solution IV in the range $p \in [2/3, 1]$. For all these cases $\tilde{C}_A \sim \text{sign}(m)$ such that the sign of the angular momentum is $S_z \sim +\epsilon \text{sign}(m)$.

For $p = 1$ we obtain that $S_z \sim -\epsilon \text{sign}(m)$. In the range $p \in [1, 1.2857]$ with $\hat{\theta} \neq 0$ it is $S_z \sim -\epsilon \text{sign}(\hat{\theta})$ corresponding to the solution IV. When $\tilde{\theta} = 0$, in the range $p \in [1, 5/4]$ it is $S_z \sim -\epsilon \text{sign}(\tilde{C}_A)$ for which $\tilde{C}_A \sim -\epsilon \text{sign}(m)$ such that $S_z \sim +\epsilon \text{sign}(m)$, for $p = 5/4$ it is $S_z \sim -\epsilon \text{sign}(\tilde{C}_A(1 - \tilde{C}_A))$ for which $\tilde{C}_A = -3\epsilon\text{sign}(m)$ such that $\tilde{C}_A(1 - \tilde{C}_A) = 30\epsilon\text{sign}(m)$, hence $S_z \sim -\epsilon \text{sign}(m)$ and in the range $p \in [5/4, 1.2857]$ it is $S_z \sim +\epsilon \text{sign}(\tilde{C}_A)$ for which $\tilde{C}_A = -\epsilon \text{sign}(m)$ such that $S_z \sim -\epsilon \text{sign}(m)$. In the range $p \in [1.2857, +\infty]$ it is $S_z \sim +\epsilon \text{sign}(\tilde{C}_A)$ corresponding to solution IV with $\tilde{C}_A \sim -\epsilon \text{sign}(m)$, hence we obtain $S_z \sim -\epsilon \text{sign}(m)$. 

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As for the magnetic flux we note that for action (3.1) the equations of motion are expressed in terms of the covariant electro-magnetic fields $B = \sqrt{-g}e^{\phi}B_*$ and $E = \sqrt{-g}e^{\phi}E_*$ instead of the bare electro-magnetic fields $\hat{B}_*$ and $\hat{E}_*$ [51] and that for stationary solutions (not depending explicitly on the time coordinate) the Bianchi identities for the Maxwell tensor can also be re-expressed with respect to these quantities. Hence the Maxwell equations are defined by the covariant fields $B$ and $E$ such that the measurable magnetic field is $B$ and its integral over the 2-dimensional manifold is

$$
\Phi_B = \int_{\delta IR}^{\delta UV} dr \int_0^{\varphi_{max}} d\varphi \sqrt{|h_{ij}|} \sqrt{-g}e^{\phi} \hat{B}_* 
= 2C_B C_\phi C_h^2 \pi \int_{\delta M}^{+\infty} dr r^p \left( 1 - r^{2p-2} \left( \tilde{C}_A r^{p-1} + \tilde{\theta} \right) \right).
$$

Evaluating this integral expression we obtain

$$
p \neq -1, 1, \frac{1}{3}, \frac{3}{5}, \frac{1}{2}, \frac{5}{3} \Rightarrow 
\Phi_B = 2C_B C_\phi C_h^2 \pi \left( \frac{1}{1 + p} r^{p+1} + \frac{\tilde{\theta}^2}{1 - 3p} r^{3p-1} + \frac{\tilde{C}_A}{1 - 2p} r^{4p-2} + \frac{\tilde{C}_A^2}{3 - 5p} r^{5p-3} \right) \delta_{IR}.
$$

The expressions for the particular values of $p = 1, 1/3, 3/5$ are listed in appendix B in equations (B.13–B.15). Again, depending on the value of the parameter $p$, finite magnetic flux expressions $\Phi_B$ can be evaluated by considering the following limits on $\delta_{IR}$ and $\delta_{UV}$

$$
p \in ]-\infty, -1[ \Rightarrow \begin{cases} 
\delta_{IR} \not\rightarrow 0 \\
\delta_{UV} \not\rightarrow +\infty
\end{cases}
$$

$$
p \in \left]-1, \frac{3}{5}\right]/\{0\} \Rightarrow \begin{cases} 
\delta_{IR} \not\rightarrow 0 \\
\delta_{UV} \not\rightarrow +\infty
\end{cases}
$$

$$
p \in \left]\frac{3}{5}, +\infty\right[ \Rightarrow \begin{cases} 
\delta_{IR} \rightarrow 0 \\
\delta_{UV} \not\rightarrow +\infty
\end{cases}
$$

The first range for the parameter $p$ corresponds to solution III, the second range to solution I, solution II with $p \in ]0, 1/3[ \cup ]1/2, 3/5]$ and solution III with $p \in ]-1, 0[$ while the third range corresponds to solution II with $p \in ]3/5, 2/3]$ and solution IV (3.6).

As for the sign of the magnetic flux $\Phi_B$ let us note that the sign of $C_\phi$ and $C_B$ are independent of the specific value of the parameter $p$. $C_\phi$ is always positive, however from the classical solutions of the equations of motion the sign of $C_B$ is arbitrary. This is simply understood by noting that, in the absence of an electric field the Einstein equations (A.10–A.13) only depend on the square of the magnetic field and that the Maxwell equations (A.8) and (A.9) with
null electric field $\tilde{E} = 0$ are invariant under a change of sign of the magnetic field $\tilde{B} \to -\tilde{B}$. Hence only solutions with both non-null electric and magnetic fields are actually sensitive to the relative electromagnetic fields direction (hence the polarization of the electromagnetic fields), both through the Maxwell equations and the '02' Einstein equation. For the specific expressions of the constants given in (3.4) the choice of the magnetic field sign can be selected by choosing the sign of the free constant $C_h$ which has no consequences at classical level, hence we will proceed our analysis leaving the sign of $C_B$ unspecified. In the range $p \in ]-\infty, 1/3[$ the magnetic flux sign is $\Phi_B \sim -\text{sign} (C_B)$, for $p = 1/3$ it is $\Phi_B \sim \text{sign} (C_B(1-\tilde{C}_A))$ corresponding to solution I for which $\tilde{C}_A = \sqrt{3}/2$ such that $\Phi_B \sim +\text{sign} (C_B)$, in the range $p \in ]1/3, 1[$ it is $\Phi_B \sim +\text{sign} (C_B)$ and in the range $p \in [1, +\infty[$ it is $\Phi_B \sim -\text{sign} (C_B)$. Next we gather all the results obtained for the solutions (3.3) and discuss possible interpretations for these configurations.

4 Discussion of results

4.1 Summary of results

In this work, based on the space-time duality (2.4) discussed in section 2 we have computed the classical solutions listed in equations (3.3-3.6) for the gravitational fields, a scalar field and the gauge fields of Einstein Maxwell Chern-Simons theory described by action (3.1) with a non-trivial magnetic field and null electric field. We have analysed the space-time singularities of such classical configurations and the curvature values in section 3.1; the existence of horizons taking in consideration that no massless photons exist in this theory due to the topological mass for the photon in section 3.3, concluding that a geodesic divergence is present in the interior of the horizon, hence not observable by an external observer; and in section 3.4 were derived the mass, angular momentum and magnetic flux for such configurations. We summarize all these results in table 1 as a function of the parameter $p \in ]-\infty, +\infty[/{\{-1\}}$.

In the first column of table 1 are listed the several ranges for the value of the parameter $p$, in the column labelled $\lim_{r \to +\infty} \varphi_{\text{max}}$ are listed the asymptotic finite values at spatial infinity of the maximum value for the coordinate $\varphi$ which simultaneously allow the space-time measure and space measure to have as the asymptotic leading term (also at spatial infinity) a positive exponent of the radial coordinate, in the columns labelled $M_{\text{div}}, S_{z,\text{div}}$ and $\Phi_{B,\text{div}}$ is listed whether the mass is divergent near the origin (IR divergence) or the mass is divergent at spatial infinity (UV divergence) in accordance to the results obtained in equations (3.36), (3.38) and (3.41), respectively, in the columns labelled $\text{sign} (M), \text{sign} (S_z)$ and $\text{sign} (\Phi_B)$ are listed the sign for these quantities evaluated from the respective expres-
sions (3.35), (3.37) and (3.40) as well as the particular cases listed in appendix B, in the column labelled \( \lim_{r \to +\infty} R \) are listed the asymptotic values of the curvature at spatial infinity obtained by inspection of the curvature (3.12) and summarized in (3.15), in the column labelled "Singularities" are listed the location of the space-time singularities obtained by inspection of the scalar invariant \( R_{\mu\nu}R^{\mu\nu} \) (3.11) and summarized in (3.20), in the column labelled "Horizons" it is listed whether the horizon at \( r = 0 \) and \( r = r_H \) (3.26) exists according to the discussion in section 3.3, in the column labelled "Signature" are listed the ADM signatures for the metric for values of the radial coordinate above the horizon \( r > r_H \) (3.26) obtained from inspection of the mapped gravitational fields \( f, h \) and \( A \) given in (3.8) corresponding to the standard ADM metric parameterisation (2.1) and finally in the last column labelled "Solution" are listed the correspondence to the solutions of type I, II, III and IV summarized in equation (3.6) for each of the ranges for the values of the parameter \( p \).
| $p$ | $\lim_{\ell \rightarrow \infty} \varphi_{\text{max}}$ | $M_{\text{div}}$ | $S_{\alpha, \text{div}}$ | $\Phi_{\beta, \text{div}}$ | $\text{sign}(M)$ | $\text{sign}(S)$ | $\text{sign}(\Phi)$ | $\lim_{\ell \rightarrow \infty} R$ | $\text{Singularities}$ | $\text{Horizon}$ | $\text{Signature, } r > r_H$ | Solution |
|-----|-----------------|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\epsilon \in [-\infty, -1]\) | $-$ | IR | IR | IR | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | III(Ghost) |
| \(\epsilon \in [-1, -\frac{1}{2}]\) | $-$ | IR | IR | IR/UV | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | III(Ghost) |
| \(\epsilon \in [-\frac{1}{2}, 0]\) | $\frac{1 + 4\psi}{C_f}$ | IR | IR | IR/UV | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | III(Ghost) |
| $= 0$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | empty flat Minkowski |
| \(\epsilon \in \left[0, \frac{1}{2}\right]\) | $\frac{1 + 4\psi}{C_f}$ | IR | IR | IR/UV | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | I(Ghost) and II |
| \(\epsilon \in \left[\frac{1}{2}, \frac{3}{4}\right]\) | $\frac{1 + 4\psi}{C_f}$ | IR | IR | IR/UV | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $+\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | I(Ghost) |
| $= \frac{3}{4}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $=$ | $=$ | $=$ | $=$ | $-$ | $-$ | empty flat Minkowski |
| \(\epsilon \in \left[\frac{3}{4}, 1\right]\) | $-$ | IR | IR | UV | $+\epsilon$ | $+\epsilon \cdot \text{sign}(m)$ | $+\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, +)$ | II |
| $= 1$ | $-$ | IR | IR | UV | $+\epsilon$ | $-\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in \left[1, \frac{7}{8}\right]\) | $-$ | IR | IR | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(\delta)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in \left[\frac{7}{8}, \frac{9}{10}\right]\) | $-$ | IR | IR | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(\delta)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in \left[\frac{9}{10}, 1.2857\right]\) | $-$ | IR/UV | IR/UV | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(\delta)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in [1.2857, \frac{9}{4}]\) | $-$ | IR/UV | IR/UV | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $0$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| $= \frac{9}{4}$ | $-$ | IR/UV | IR/UV | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $\frac{C^2}{\ell}$ | $r = 0$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in \left[\frac{9}{4}, 2\right]\) | $-$ | IR/UV | IR/UV | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $+\infty$ | $r = 0$, $+\infty$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |
| \(\epsilon \in [2, +\infty]\) | $-$ | UV | UV | UV | $-\epsilon$ | $-\epsilon \cdot \text{sign}(m)$ | $-\text{sign}(C_B)$ | $+\infty$ | $r = 0$, $+\infty$ | $3r = 0$, $3r > 0$ | $(-, +, -)$ | IV |

Table 1: Resume of solutions as a function of the parameter $p$. 
4.2 Conclusions

Given the solutions summarized in table 1 we proceed to interpret them physically. Of particular relevance are the divergences of the physical properties of the classical configurations, namely the total mass $M$, the total angular momentum $J_z$ and the total magnetic flux $\Phi_B$. A divergence near the space-time singularity (or singularities) is non uncommon in 2+1-dimensional space-times, this is mainly due to that a gravitational potential proportional to $\sim 1/r$ only in 3+1-dimensional space-times corresponds to a finite gravitational mass. Also we note that such a divergence near the singularity is usually associated with a breakdown of the theory such that a more complete theory is required. A simple regularization for the divergent quantities is to consider a lower cut-off $\delta_{IR}$ of the order of the Planck length near the singularity as was considered in [23].

As for configurations for which the total mass $M$, the total angular momentum $J_z$ and the total magnetic flux $\Phi_B$ are divergence when the integral of the respective densities is considered up to spatial infinity, let us note that considering a upper cut-off $\delta_{UV}$ for large values of the radial coordinate $r$ is simply interpreted as a description of a finite size system such that the cut-off $\delta_{UV}$ is interpreted as the maximum size of the system. Otherwise, for infinite size systems, it is not mandatory that these quantities be finite, instead they may be interpreted as cosmological-like solutions for 2+1-dimensional space-times as long as the respective densities are finite away from the singularity at the origin. Let us note that even for a uniformly distributed (meaning constant) mass-energy density in flat Minkowski space-time we would obtain a divergent total mass when integrating over all space up to spatial infinity.

Hence for the classical configurations discussed here, to regularize the divergence at the origin for $M$, $S_z$ or $\Phi_B$ we consider the lower cut-off $\delta_{IR}$ to be of the order of the Planck length $l_p$. To interpret the divergence and the respective upper cut-off $\delta_{UV}$ for large $r$ let us consider three possible cases:

- **string-like configurations**: a 2+1-dimensional point-like effective description of matter centred at the origin generating a magnetic field of finite flux. When embedded into a 3+1-dimensional space-time with cylindrical symmetry is interpreted as a magnetic string configuration. These configurations should also have a finite mass and finite angular momentum such that the upper cut-off $\delta_{UV}$ is not required;

- **configurations driven by an external magnetic field**: the upper cut-off $\delta_{UV}$ is justified by the finite range of the applied external field. Hence, from the point of view of 3+1-dimensions the magnetic field has cylindrical symmetric and is applied orthogonally to the planar system in the region $r < \delta_{UV}$;
• *cosmological-like solutions*: an infinite configuration with background magnetic fields such that are allowed total infinite magnetic flux, infinite mass and angular momentum as long as the respective densities are (locally) finite everywhere except at the space-time singularities.

By inspection of the table 1 we conclude that, considering only the cut-off δ_{IR} the solutions with a magnetic field generating a finite total flux, hence being interpreted as a magnetic string-like configuration in an infinite space-time are achievable only for the parameter range $p \in ]-\infty, -1[ \cap \mathbb{R}$ corresponding to solution III describing ghost gauge fields. For these configurations also the total mass and total angular momentum are finite. We remark that due to the particular value of the parameter $p = -1$ not allowing for a solution of the equations of motion, this configurations cannot be obtained from flat Minkowski space-time by continuously changing the parameter $p$.

As for the range $p \in ]-1, 1[$ (considering the lower cut-off δ_{IR}), $M$ and $J_z$ are finite. However, although the magnetic field $\mathcal{B}$ is finite, the total magnetic flux $\Phi_{\mathcal{B}}$ is divergent when integrating the magnetic field up to spatial infinity, hence these solutions can be interpreted either as driven by a cylindrical external magnetic field orthogonal to the planar system ranging from the origin up to the upper cut-off $r < \delta_{UV}$, either as a cosmological-like solution. In addition we note that, when considering an external magnetic field, the value of the field $\mathcal{B}$ is null for $p = 0$ and $p = 1/2$. Hence the solutions corresponding to these values of the parameters are interpreted as two possible backgrounds upon which the external magnetic field is applied to. Specifically $p = 0$ corresponds to empty flat Minkowski, such that when the magnetic field is turn on the solutions can be changed smoothly and continuously by varying the parameter $p$ (the variation of the field solutions with the parameter $p$ are continuous and their derivatives with respect to $p$ are also continuous) describing the deformation induced by the magnetic field, in the range $p \in ]-1/4, 0]$ corresponding to solution III (3.6) for ghost gauge fields, in the range $p \in ]0, 1/2[$ also for ghost gauge fields corresponding to solution I and in the range $p \in ]0, 1/3[$ for standard gauge fields corresponding to solution II. For $p = 1/2$ the background corresponds to a neutral dilatonic-like background and the solutions can be changed smoothly and continuously by varying the parameter $p$ in the range $p \in ]0, 1/2]$ describing ghost gauge fields corresponding to solution I and in the range $p \in ]1/2, 2/3]$ describing standard gauge fields corresponding to solution II. In the range $p \in ]2/3, 1[$ corresponding to solution IV describing standard gauge fields the solutions can also be changed smoothly and continuously by varying the parameter $p$, however when crossing the value $p = 2/3$ the derivative of the field solutions is not continuous such that this range cannot be obtained smoothly by varying the value of the parameter $p$ starting at any of the neutral backgrounds $p = 0$ or $p = 1/2$.

For values of the parameter $p \in [1, 3/2]$ corresponding to solution IV describing standard
gauge fields the metric ADM signature for values of the radial coordinate above the value of the radial coordinate of the horizon, \( r > r_H \), is the opposite to our original convention, while for the range \( r \in [0, r_H] \) the metric has the ADM signature \( \text{diag}(-, +, +) \) corresponding to the original convention. The interpretation for an external observer is that observable space-time is between \( r = 0 \) and the coordinate horizon \( r = r_H \) (3.26) such that \( r = 0 \) is a dressed singularity (\( r = 0 \) is both a singularity and an horizon) and a cosmological horizon exists at \( r = r_H \). In addition we note that the geodesics divergence analysed in section 3.3 located at \( r = r_{\text{div}} \) (3.24), is now beyond the cosmological horizon, specifically for \( p > 1 \) we obtain that \( r_H < r_{\text{div}} \). These configurations may be interpreted as cosmological-like configurations in 2+1-dimensions as the mass-energy density, the magnetic field and pressure are finite in between horizons.

As for the range \( p \in ]3/2, +\infty[ \) we obtain an exotic configuration for which space-time has two singularities at \( r = 0 \) and \( r = +\infty \). In particular for the range \( p \in ]2, +\infty[ \), \( M, S_z \) and \( \Phi_B \) have no divergence at the origin having only a divergence at spatial infinity. Hence by considering the map \( \hat{r} = 1/r \) we obtain, for our metric ADM signature convention, a magnetic string-like configuration for standard gauge fields with both a singularity at the origin within the horizon at \( \hat{r}_H > \hat{r}_{\text{div}} \) and a dressed singularity at spatial infinity (spatial infinity is itself both a singularity and an horizon).

We resume the main configuration types discussed in table 2.

| configuration type       | \( p \)               | solution  |
|-------------------------|-----------------------|-----------|
| string-like             | \( \in ]-\infty, -1[ \) | III (ghost) |
| driven by \( B_* \)     | \( \in ]-1, 1/2] \)    | I(ghost) and III(ghost) \( p = 0 \) \( \Leftrightarrow \) neutral background |
|                         | \( \in ]0, 1/2[ \)    | II \( p = 0 \) \( \Leftrightarrow \) neutral background |
|                         | \( \in ]1/2, 2/3[ \)   | II \( p = 1/2 \) \( \Leftrightarrow \) neutral background |
| cosmological-like       | \( \in ]2/3, 3/2[ \)   | IV        |

Table 2: Resume of discussed configuration types.

As a final remark we note that the magnetic string-like configuration corresponding to
solution III for the range of the parameter $p \in ]-\infty,-1[$ describing a ghost gauge sector suggests that, for extended gauge theories containing a ghost gauge sector coupled to magnetic charge [60, 61], similar magnetically charged solutions may be computed in 3+1-dimensions [62] and 2+1-dimensions [63].

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A Magnetic Solutions

For completeness, in this appendix we re-derive, directly from the equations of motion for action (3.1) in the Cartan frame, the solutions (3.3) obtained in the main text from space-time duality. In form notation the action (3.1) is

$$S = - \int_M \left\{ e^{\alpha \phi} \left[ \hat{R} \ast 1 + 2 \lambda d\phi \wedge *d\phi \right] - e^{b \phi} \Lambda \ast 1 + \hat{e} e^{c \phi} \left[ \hat{F} \wedge *\hat{F} + *J \wedge \hat{A} \right] + \hat{m} \frac{m}{2} \hat{A} \wedge \hat{F} \right\}$$

using the metric parameterisation (3.2)

$$d\tilde{s}^2 = -\tilde{f}^2 (dt + \tilde{A} d\varphi)^2 + dr^2 + \tilde{h}^2 d\varphi^2 .$$

The Cartan triad is then given by

$$e^0 = d\theta^0 = \tilde{f} (dt + \tilde{A} d\varphi) ,$$
$$e^1 = d\theta^1 = dr ,$$
$$e^2 = d\theta^2 = \tilde{h} d\varphi ,$$
$$e^0_0 = \tilde{f} , \ e^0_1 = 0 , \ e^0_2 = \tilde{f} \tilde{A} ,$$
$$e^1_0 = 0 , \ e^1_1 = 1 , \ e^1_2 = 0 ,$$
$$e^2_0 = 0 , \ e^2_1 = 0 , \ e^2_2 = \tilde{h} ,$$

such that the line element in the Cartan frame is

$$d\tilde{s}^2 = e^i e_i = \eta_{ij} d\theta^i d\theta^j = -(d\theta^0)^2 + (d\theta^1)^2 + (d\theta^2)^2 , \quad (A.2)$$

The electric field $\tilde{E}_s$ and magnetic field $\tilde{B}_s$ in the coordinate frame are given by

$$\tilde{E}_s = \tilde{E} \tilde{f} ,$$
$$\tilde{B}_s = \tilde{B} \tilde{h} - \tilde{E} \tilde{f} \tilde{A} , \quad (A.3)$$
where $\tilde{E}$ and $\hat{E}$ are the electromagnetic fields in the Cartan frame. We note that the metric parameterisation (3.2) allows for the electric field to be null both in the coordinate frame and in the Cartan frame, $\tilde{E} = 0 \Leftrightarrow \hat{E} = 0$. This parameterisation also allows for the Maxwell equations in the Cartan frame to have pure magnetic solutions as we will derive next.

Noting that
\begin{align*}
\text{de}^0 &= -\beta e^0 \wedge e^1 + \gamma e^1 \wedge e^2 , \\
\text{de}^2 &= \alpha e^1 \wedge e^2 , \\
\end{align*}
the Equations of motion, connections, curvature and remaining quantities depend only on the combinations
\begin{align*}
\alpha &= \tilde{h}' \tilde{h}, \\
\beta &= \tilde{f}' \tilde{f}, \\
\gamma &= \tilde{f} \tilde{A}' \tilde{h}.
\end{align*}

The non null connections in the Cartan frame are
\begin{align*}
\omega^{01}_{\text{0}0} = \omega^{10}_{\text{0}0} &= \beta , \\
\omega^{01}_{\text{0}2} = \omega^{10}_{\text{0}2} &= -\omega^{02}_{\text{2}0} = -\omega^{20}_{\text{0}1} = -\omega^{21}_{\text{1}0} = \gamma / 2 , \\
\omega^{12}_{\text{2}2} &= -\omega^{21}_{\text{1}2} = -\alpha ,
\end{align*}
and the Einstein and the energy-momentum tensor components are
\begin{align*}
\tilde{G}_{00} &= -\alpha^2 + 3\gamma^2/4 - \alpha' , \\
2\tilde{T}_{00} &= \hat{\epsilon} \left( \tilde{B}^2 + \tilde{E}^2 \right) , \\
\tilde{G}_{11} &= \alpha \beta + \gamma^2/4 , \\
2\tilde{T}_{11} &= \hat{\epsilon} \left( \tilde{B}^2 - \tilde{E}^2 \right) , \\
\tilde{G}_{22} &= \beta^2 + \gamma^2/4 + \beta' , \\
2\tilde{T}_{22} &= \hat{\epsilon} \left( \tilde{B}^2 + \tilde{E}^2 \right) , \\
\tilde{G}_{02} &= \beta \gamma + \gamma'/2 , \\
2\tilde{T}_{02} &= -2\hat{\epsilon} \tilde{B} \tilde{E} , \\
\Phi_{00} &= -a\phi'' + (\lambda/2 - a^2)(\phi')^2 , \\
\Phi_{11} &= \lambda/2(\phi')^2 , \\
\Phi_{22} &= a\phi'' - (\lambda/2 - a^2)(\phi')^2 .
\end{align*}

We note that under the duality (2.4) only the dilaton contribution to the energy-momentum tensor is invariant while the Maxwell energy-momentum tensor acquires a minus sign (this accounts to take $\hat{\epsilon} \to -\hat{\epsilon}$) and for the Einstein tensor the terms $\gamma^2/4$ and $3\gamma^2/4$ are swapped. For a direct comparison with the same tensor quantities for the standard metric ADM parameterisation (2.1) we refer the reader to the appendix of [23]). In the following we consider both cases $\hat{\epsilon} = +1$ and $\hat{\epsilon} = -1$.

The Maxwell Equations are
\begin{align*}
\tilde{B}' + \beta \tilde{B} + c \tilde{B} \phi' &= m \tilde{E} e^{-c\phi} ,
\end{align*}
\[ \tilde{E}' + \alpha \tilde{E} + c \tilde{E} \phi' + \gamma \tilde{B} = -m \tilde{B} e^{-c\phi}, \quad (A.9) \]

for pure magnetic solution \( \tilde{E} = \tilde{E}^* = 0 \) the Einstein equations are

\[ e^{a\phi} \left( \beta \gamma + \frac{\gamma'}{2} \right) = 0, \quad (A.10) \]

\[ e^{a\phi} \left[ \alpha^2 - \frac{3\gamma^2}{4} + \alpha' + a\phi'' + \left( a^2 - \frac{\lambda}{2} \right) (\phi')^2 \right] + \frac{1}{2} e^{b\phi} \Lambda = \hat{\epsilon} \tilde{B}^2 e^{c\phi}, \quad (A.11) \]

\[ e^{a\phi} \left[ \beta^2 + \frac{\gamma^2}{4} + \beta' + a\phi'' + \left( a^2 - \frac{\lambda}{2} \right) (\phi')^2 \right] + \frac{1}{2} e^{b\phi} \Lambda = -\hat{\epsilon} \tilde{B}^2 e^{c\phi}, \quad (A.12) \]

and the Dilaton equation is

\[ e^{a\phi} \left[ (4a^2 - \lambda) \phi'' + a \left( 4a^2 - 2\lambda \right) (\phi')^2 \right] + (3a - b) e^{b\phi} \Lambda = -\epsilon(a + c) \tilde{B}^2 e^{c\phi}. \quad (A.14) \]

From the second Maxwell Equation (A.9) we obtain

\[ \gamma = -m e^{-c\phi}. \quad (A.15) \]

Using (A.15) in (A.10) one obtains that \( \beta = c\phi'/2 \) such that

\[ \tilde{f} = c_f e^\tilde{\phi}, \quad (A.16) \]

where \( c_f \) is a free integration constant. From the first Maxwell Equation (A.8) with \( \tilde{E} = 0 \) we obtain

\[ \tilde{B} = \chi e^{-\frac{4}{2}c\phi}, \quad (A.17) \]

where \( \chi \) is an integration constant. The remain 3 Einstein (A.11-A.13) are

\[ a\phi'' + \left( a^2 - \frac{\lambda}{2} \right) (\phi')^2 + \alpha^2 + \alpha' - \frac{3m^2}{4} e^{-2c\phi} + \frac{1}{2} \Lambda e^{(b-a)\phi} = \hat{\epsilon} \chi^2 e^{(-a-2c)\phi}, \quad (A.18) \]

\[ (a + \frac{c}{2}) \phi'' + \left( a^2 - \frac{\lambda}{2} + \frac{c^2}{4} \right) (\phi')^2 + \frac{m^2}{4} e^{-2c\phi} + \frac{1}{2} \Lambda e^{(b-a)\phi} = -\hat{\epsilon} \chi^2 e^{(-a-2c)\phi}, \quad (A.19) \]

\[ \frac{\lambda}{2} (\phi')^2 + \frac{c}{2} \alpha \phi' + \frac{m^2}{4} e^{-2c\phi} + \frac{1}{2} \Lambda e^{(b-a)\phi} = -\hat{\epsilon} \chi^2 e^{(-a-2c)\phi}, \quad (A.20) \]

and Dilaton Equations (A.14) is

\[ (4a^2 - \lambda) \phi'' + a \left( 4a^2 - 2\lambda \right) (\phi')^2 + (3a - b) \Lambda e^{(b-a)\phi} = -\epsilon(a + c) \chi^2 e^{(-a-2c)\phi}. \quad (A.22) \]
We use the same ansatz of [23]

\begin{align*}
a &= 0, \\
c &= -\frac{b}{2}, \\
\lambda &\neq \frac{b^2}{8},
\end{align*}

where the particular case corresponding to \(b^2 = 8\lambda\) is excluded due to not admitting a solution for the above equations of motion. Given this ansatz we combine (A.19) with (A.22) obtaining

\begin{equation}
\phi' = \pm \sqrt{c_1} c^b \phi, \tag{A.24}
\end{equation}

such that the Dilaton is

\begin{equation}
\phi = -\frac{2}{b} \ln(c_\phi r), \tag{A.25}
\end{equation}

where

\begin{equation}
c_\phi = \frac{|b|}{2} \sqrt{c_1}, \quad c_1 = -2b^2(\hat{\epsilon}^2 \chi^2 + 2\Lambda) + 2\lambda(4\hat{\epsilon}^2 \chi^2 + 2\Lambda + m^2) \lambda(b^2 - 8\lambda). \tag{A.26}
\end{equation}

Imposing either of the equations (A.19) or (A.22) to be obeyed by this solution we obtain that

\begin{equation}
\chi^2 = -\hat{\epsilon} \frac{2\Lambda(b^2 + 12\lambda) + 4\lambda m^2}{b^2 + 24\lambda}, \tag{A.27}
\end{equation}

such that \(c_1\) is rewritten as

\begin{equation}
c_1 = 4 \frac{m^2 - 6\Lambda}{b^2 + 24\lambda}, \tag{A.28}
\end{equation}

and from (A.20) we obtain

\begin{equation}
\alpha = -\left(16\frac{\lambda}{b^2} + 1\right) \frac{1}{2} r. \tag{A.29}
\end{equation}

Therefore

\begin{equation}
\tilde{h} = c_h r^{-\frac{8\lambda}{b^2} - \frac{1}{2}}, \tag{A.30}
\end{equation}

and from (A.16)

\begin{equation}
\tilde{f} = c_f \sqrt{r}, \tag{A.31}
\end{equation}

where \(c_h\) and \(c_f\) are free constants. From (A.15) we obtain that

\begin{equation}
\tilde{A} = c_A r^{-\frac{8\lambda}{b^2} - 1} + c_{A_\infty}, \tag{A.32}
\end{equation}

where

\begin{equation}
c_A = \frac{m C_h}{C_f \left(\frac{8\lambda}{b^2} + 1\right)} \sqrt{\frac{1 + \frac{24\lambda}{b^2}}{m^2 - 6\Lambda}}. \tag{A.33}
\end{equation}

Replacing these solutions in (A.18) and demanding this equation to be obeyed we obtain that

\begin{equation}
\lambda_\pm = \frac{b^2}{8} \frac{3\Lambda \mp \sqrt{\Lambda(2m^2 - 3\Lambda)}}{m^2 - 6\Lambda}. \tag{A.34}
\end{equation}
It is further required to ensure that all these relations are possible for real valued constants, in particular that $c_1 > 0$ and $\chi^2 > 0$. We note that the condition $c_1 > 0$ is obeyed in the range $0 < \Lambda < m^2/3$ except for the particular case $\Lambda = m^2/6$ for which $c_1 = 0$. Then, imposing the condition $\chi^2 > 0$, we obtain the four possible solutions and respective bounds on the cosmological constant

\[
\begin{align*}
\{ \dot{\epsilon} = +1 \} & : \left\{ \begin{array}{l}
\chi^2 = \frac{1}{2} \left[ -\Lambda + \sqrt{\Lambda(2m^2 - 3\Lambda)} \right] \\
c_1 = \frac{4}{b^2} \left[ 3\Lambda + m^2 + \sqrt{\Lambda(2m^2 - 3\Lambda)} \right]
\end{array} \right. \\
0 < \Lambda < \frac{m^2}{2}
\end{align*}
\]

\[
\begin{align*}
\{ \dot{\epsilon} = -1 \} & : \left\{ \begin{array}{l}
\chi^2 = \frac{1}{2} \left[ -\Lambda - \sqrt{\Lambda(2m^2 - 3\Lambda)} \right] \\
c_1 = \frac{4}{b^2} \left[ 3\Lambda + m^2 + \sqrt{\Lambda(2m^2 - 3\Lambda)} \right]
\end{array} \right. \\
0 < \Lambda < \frac{m^2}{6} \lor \frac{m^2}{2} < \Lambda < \frac{2m^2}{3}
\end{align*}
\]

\[
\begin{align*}
\{ \dot{\epsilon} = +1 \} & : \left\{ \begin{array}{l}
\chi^2 = \frac{1}{2} \left[ \Lambda + \sqrt{\Lambda(2m^2 - 3\Lambda)} \right] \\
c_1 = \frac{4}{b^2} \left[ 3\Lambda + m^2 - \sqrt{\Lambda(2m^2 - 3\Lambda)} \right]
\end{array} \right. \\
0 < \Lambda < \frac{m^2}{6}
\end{align*}
\]

\[
\begin{align*}
\{ \dot{\epsilon} = -1 \} & : \left\{ \begin{array}{l}
\chi^2 = \frac{1}{2} \left[ \Lambda - \sqrt{\Lambda(2m^2 - 3\Lambda)} \right] \\
c_1 = \frac{4}{b^2} \left[ 3\Lambda + m^2 + \sqrt{\Lambda(2m^2 - 3\Lambda)} \right]
\end{array} \right. \\
m^2/6 < \Lambda < \frac{2m^2}{3}
\end{align*}
\]

B Expressions for $M$, $S_z$ and $\Phi_B$ for particular values of the parameter $p$

In this appendix are listed the explicit expressions for the mass $M$ (3.33), angular momentum $S_z$ (3.34) and magnetic flux $\Phi_B$ (3.39) for the particular values of the parameter $p$ not included in the expressions (3.35), (3.37) and (3.40).
Evaluating the integral expression for the mass $M$ for $p = 1$ with $A$ given in (3.7) we obtain

$$ p = 1, $$

$$ M = -\hat{e}C_B^2C_{\phi \pi} \left( r^{-1} \left( \bar{C}_A^2 - (\bar{C}_A + \bar{\theta}) \times \bar{C}_A + \bar{\theta} + 2\bar{C}_A \log(r)) - \bar{C}_A^2 \log(r^2) \right) \right)_{r=\delta_{IR}}, $$

for $p = 6/5$ evaluating (3.33) we obtain

$$ p = \frac{6}{5}, $$

$$ M = \hat{e}C_B^2C_{\phi \pi} \left( -\frac{5}{4} r^{-\frac{5}{4}} \left( 8\bar{C}_A \bar{\theta} r^{\frac{5}{4}} + 2\bar{\theta}^2 r^{\frac{5}{4}} - 1 \right) + \bar{C}_A^2 \log(r) \right)_{\delta_{IR}}, $$

for $p = 4/3$ we obtain

$$ p = \frac{4}{3}, $$

$$ M = \hat{e}C_B^2C_{\phi \pi} \left( 3r^{-\frac{3}{4}} \left( \bar{C}_A^2 r^{\frac{3}{4}} + 4\bar{C}_A \bar{\theta} r + 1 \right) + 2\bar{\theta}^2 \log(r) \right)_{\delta_{IR}}, $$

for $p = 5/4$ we obtain

$$ p = \frac{5}{4}, $$

$$ M = \frac{2\hat{e}C_B^2C_{\phi \pi}}{3|C_fC_h|} \left( r^{-\frac{5}{4}} \left( 6\bar{C}_A^2 r - 6\bar{\theta}^2 r^{\frac{5}{4}} + 2 \right) + 3\bar{C}_A \bar{\theta} \log(r) \right)_{\delta_{IR}}, $$

and for $p = 2$ we obtain

$$ p = 2 $$

$$ M = \frac{\hat{e}C_B^2C_{\phi \pi}}{12|C_fC_h|} \left( r^2 \left( 3\bar{C}_A^2 r^2 + 8\bar{C}_A \bar{\theta} r + 6\bar{\theta}^2 \right) -12 \log(r) \right)_{\delta_{IR}}. $$
Evaluating the integral expression for the angular momentum $S_z$ for $p = 1$ with $A$ given in (3.7) we obtain

$$p = 1,$$

$$S_z = \frac{\hat{e}C_B^2C_\phi\pi}{(C_fC_h)^2} \left( r^{-1} \left( (\tilde{C}_A + \tilde{\theta})^2 + \tilde{C}_A (5\tilde{C}_A + 3\tilde{\theta}) - \tilde{\theta} \right) + \tilde{C}_A \log(r) \left( 3(\tilde{C}_A + \tilde{\theta})^2 + 3\tilde{C}_A - 1 + \tilde{C}_A \log(r) \times \right. \right.$$

$$\left. \times (3(\tilde{C}_A + \tilde{\theta}) + \tilde{C}_A \log(r)) \right) \bigg|_{r=\delta_{IR}},$$

(B.6)

for $p = 7/6$ evaluating (3.34) we obtain

$$p = \frac{7}{6},$$

$$S_z = \frac{\hat{e}C_B^2C_\phi\pi}{(C_fC_h)^2} \left( -18\tilde{C}_A^2\tilde{\theta}r^{-\frac{7}{6}} - 9\tilde{C}_A\tilde{\theta}^2 r^{-\frac{1}{3}} - 2\tilde{\theta}^3 r^{-\frac{1}{2}} + \frac{3}{2}\tilde{C}_A r^{-\frac{1}{6}} + \frac{6}{5}\tilde{\theta} r^{-\frac{7}{6}} + \tilde{C}_A^3 \log(r) \right) \delta_{UV} \delta_{IR},$$

(B.7)

for $p = 6/5$ we obtain

$$p = \frac{6}{5},$$

$$S_z = \frac{5\hat{e}C_B^2C_\phi\pi}{12(C_fC_h)^2} \left( 12\tilde{C}_A r^{-\frac{1}{5}} - 36\tilde{C}_A\tilde{\theta}^2 r^{-\frac{1}{5}} + 4\tilde{C}_A r^{-\frac{7}{10}} + 3\tilde{\theta} r^{-\frac{4}{5}} - 6\tilde{\theta}^3 r^{-\frac{1}{5}} + \frac{36}{5}\tilde{C}_A^2 \tilde{\theta} \log(r) \right) \delta_{UV} \delta_{IR},$$

(B.8)

for $p = 5/4$ we obtain

$$p = \frac{5}{4},$$

$$S_z = \frac{\hat{e}C_B^2C_\phi\pi}{3(C_fC_h)^2} \left( 6\tilde{C}_A r^{-\frac{1}{4}} + 6\tilde{C}_A r^{-\frac{1}{2}} + 4\tilde{\theta} r^{-\frac{3}{4}} - 12\tilde{\theta}^3 r^{-\frac{1}{4}} + 36\tilde{C}_A^2 \tilde{\theta} r^{\frac{1}{4}} + 9\tilde{C}_A \tilde{\theta}^2 \log(r) \right) \delta_{UV} \delta_{IR},$$

(B.9)
and for $p = 4/3$ we obtain

$$\begin{align*}
p &= \frac{4}{3} , \\
S_z &= \frac{\hat{C}_B^2 C_\phi \pi}{2 (C_f C_h)^2} \left( 2 \tilde{C}_A^3 r + 18 \tilde{C}_A \tilde{\theta}^2 r^\frac{3}{2} + 6 \tilde{C}_A r^{-\frac{1}{2}} \\
&\quad + 3 \tilde{\theta} r^{-\frac{3}{2}} + 9 \tilde{C}_A \tilde{\theta} r^\frac{3}{2} + 2 \tilde{\theta}^3 \log(r) \right)_{\delta_{IR}}^{\delta_{UV}},
\end{align*}$$

(B.10)

for $p = 3/2$ we obtain

$$\begin{align*}
p &= \frac{3}{2} , \\
S_z &= \frac{\hat{C}_B^2 C_\phi \pi}{(C_f C_h)^2} \left( \frac{1}{2} \tilde{C}_A^3 r^2 + 3 \tilde{C}_A \tilde{\theta}^2 r + 2 \tilde{\theta} r^{-\frac{1}{2}} \\
&\quad + 2 \tilde{\theta}^3 r^\frac{1}{2} + 2 \tilde{C}_A \tilde{\theta} r^\frac{3}{2} - \tilde{C}_A \log(r) \right)_{\delta_{IR}}^{\delta_{UV}},
\end{align*}$$

(B.11)

for $p = 2$ we obtain

$$\begin{align*}
p &= 2 , \\
S_z &= \frac{\hat{C}_B^2 C_\phi \pi}{20 (C_f C_h)^2} \left( 4 \tilde{C}_A^3 r^5 + 20 \tilde{C}_A \tilde{\theta}^2 r^3 - 20 \tilde{C}_A r \\
&\quad + 10 \tilde{\theta}^3 r^2 + 15 \tilde{C}_A \tilde{\theta}^2 r^3 - \tilde{C}_A \log(r) \right)_{\delta_{IR}}^{\delta_{UV}}.
\end{align*}$$

(B.12)

Evaluating the integral expression for the magnetic flux $\Phi_B$ for $p = 1$ with $A$ given in (3.7) we obtain

$$\begin{align*}
p &= 1 , \\
\Phi_B &= \frac{2 \hat{C}_B^2 C_f^2 C_\phi \pi}{27} \left( r^3 \left( 9 - 2 \tilde{C}_A^2 + 6 \tilde{C}_A \tilde{\theta} - 9 \tilde{\theta}^2 \\
&\quad + 3 \tilde{C}_A \log(r) \left( 2 \tilde{C}_A - 6 \tilde{\theta} - 3 \tilde{C}_A \log(r) \right) \right) \right)_{r=\delta_{UV}}
\end{align*}$$

(B.13)

for $p = 1/3$ evaluating (3.39) we obtain

$$\begin{align*}
p &= \frac{1}{3} , \\
\Phi_B &= -C_B C_f C_h^2 \pi \left( -\frac{3}{2} r^\frac{5}{2} - 6 \tilde{\theta} \tilde{C}_A r^{-\frac{3}{2}} - \frac{3}{2} \tilde{C}_A^2 r^\frac{1}{2} \\
&\quad + 2 \tilde{\theta}^2 \log(r) \right)_{\delta_{IR}}^{\delta_{UV}},
\end{align*}$$

(B.14)
and for $p = 3/5$ we obtain

$$p = \frac{3}{5},$$

$$\Phi_B = - C_B C_\phi C_R^2 \pi \left( - \frac{5}{4} \dot{r}^2 + \frac{5}{2} \dot{\theta}^2 r^2 + 10 \dot{\theta} \dot{C}_A r \dot{\phi} + 2 \ddot{C}_A \log(r) \right) \delta_{UV}.$$  \hspace{1cm} (B.15)

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