Trap effects and continuum limit of the Hubbard model in the presence of a harmonic potential

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We give a prescription to perform the continuum limit of the $d$-dimensional Hubbard model in the presence of a harmonic trap at zero temperature. We perform the continuum limit at fixed number of particles. In $d \geq 3$ the lattice system of spin-1/2 particles is mapped into a non-interacting two-component Fermi gas in a harmonic trap. In $d = 1$ and $d = 2$ the particles with opposite spin interact via a Dirac delta interaction. We show that the properties of this continuum limit can be put in correspondence with those derived applying the Trap-Size scaling (TSS) formalism to the confined Hubbard model in the so called Dilute Regime (fixed number of particles and weak confinement). The correspondence in $d = 1$ and $d = 2$ has been tested comparing the numerical results obtained for lattice system with those of the continuum limit in the case of two-particle and in absence of spin-polarization ($N = 2, N^\uparrow = N^\downarrow = 1$).

I. INTRODUCTION

During the last few decades cold atoms systems have been extensively used in many fields of research and have played a key role in particular in the understanding of quantum many-body systems. All these results are related to the impressive progress that has been made in manipulating these many-body systems. Nowadays using Lasers it is possible to cool atoms down at very low temperature and to achieve regimes at which only fluctuations due to principles of quantum mechanics are relevant to study the many-body system behaviour. In addition, using Lasers it is possible to tune two-body interactions and moreover it is possible to realize spatially periodic potentials in which cooled neutral atoms congregate in structures resembling usual crystal lattices. These structures are called optical lattices. The main difference between optical and ordinary ladillattices is that all the lattice features are not fixed, but only depend on the experimental setup: tuning Laser wavelength and intensity and modifying the number of counter-propagating beams respectively, one can change the interatomic distance (lattice spacing), the interaction strength (lattice depth) and also the lattice geometry. An important feature of all the experiments with cold atoms on optical lattices is the presence of an external space-dependent confining potential. Such a kind of potential is usually harmonically-shaped and it directly couples with the particle density of the lattice system, leading to inhomogeneities which give rise to a wide variety interesting physical phenomena (for example the coexistence of many quantum-phases, the presence of an inhomogeneous Crossover, etc).

In a recent paper several of the most interesting features developed by lattice systems in the presence harmonic traps have been considered in details. There the authors use the Trap-Size Scaling (TSS) formalism and numerical techniques to investigate the properties developed in $d$-dimensional lattice system of spin-1/2 particles described by the Hubbard model in the presence of a harmonic trap (see next Section for details). While considering the one dimensional problem at very low temperature, fixed number of particles and weak confining potential, they argue about the existence of a one-to-one correspondence between the ground-state properties of the lattice problem and those of the Gaudin-Yang model in the presence of a harmonic trap.

In this paper we show that this correspondence can be analytically derived performing the proper continuum limit of the lattice problem. We do this for an arbitrary number of particles $N$ and in arbitrary dimension $d$. It turns out that this continuum limit strongly depends on the dimension $d$ and gives results coherent with those obtained by using the Renormalization-Group techniques in. The validity of this correspondence has been tested explicitly by considering what happens to the system in the simplest possible configuration, that is the unpolarized two-body problem ($N = 2$ and $N^\uparrow = N^\downarrow = 1$) where both analytical and numerical calculations can be carried out easily.

The remaining of the paper is organized as follows. In Sec. IV we introduce the hamiltonian model and we define also the regime of interest for the present work, that is the Dilute Regime. In Sec. III we report the ideas discussed in Ref. III. Here we show how it is possible to keep into account the effects introduced by an external confining potential within the TSS formalism. In Sec. IV we deduce the continuum limit of the Hubbard model in presence of a harmonic coupling and we discuss how this continuum limit can be put in correspondence with the TSS formalism. We refer to this correspondence by saying “the Correspondence Hypothesis”.

In Sec. V we test the correspondence hypothesis by considering the simplest non-trivial configuration for the system in analysis, that is the two-body unpolarized problem in $d = 1$ and $d = 2$. For $d \geq 3$ both the continuum limit and the trap-Size Scaling formalism prescribe a trivial behaviour of the lattice system, that is the behaviour of a two-component non-interacting
Fermi gas in a harmonic trap.
Finally, in Sec. [VI] we summarize our main results and draw our conclusions.

II. THE HAMILTONIAN MODEL & THE DILUTE REGIME

A system of spin-$\frac{1}{2}$ interacting fermions on a d-dimensional lattice can be described using the well-known Hubbard Hamiltonian

$$H_0 = -t \sum_{\langle x,y \rangle} \sum_{\sigma} (c_{x,\sigma}^\dagger c_{y,\sigma} + h.c.) + U \sum_x n_{x,\uparrow} n_{x,\downarrow}$$

where $x = (x_1, x_2, \cdots, x_d)$ denotes a site on the d-dimensional lattice, $t$ and $U$ are respectively the hopping and the on-site coupling constant, $\sigma = \uparrow, \downarrow$ is the spin label, $\langle \cdot, \cdot \rangle$ is the summation over first neighbour sites, $c_{x,\sigma}$ is the annihilation (creation) operator for a particle with spin $\sigma$ on the site $x$ and $n_{x,\sigma} = c_{x,\sigma}^\dagger c_{x,\sigma}$ is the number operator.

The presence of an isotropic harmonic potential coupled to the particle density can be taken into account by adding the following Hamiltonian term to the Hubbard model in Eq.(1)

$$H_t = \sum_{\sigma} \sum_x \frac{1}{2} v^2 ||x||^2 n_{x,\sigma}$$

where $v$ is the trap-intensity and $||x||^2 = \sum_{\alpha=1}^d x_{\alpha}^2$ quantifies the distance between the site $x$ and the centre of the lattice structure.

The total Hamiltonian now reads

$$H = H_0 + H_t$$

In the following sections we refer to the Hamiltonian in Eq.(3) by saying confined Hubbard model. The confining potential introduces inhomogeneities that can be characterized in terms a new characteristic length scale, $l$, called trap-size which is defined as follows

$$l = \frac{\sqrt{2l}}{v}$$

The definition in Eq.(4) naturally arises when trying to define the analogue of the thermodynamic limit in the presence of a harmonic confinement as discussed in Ref.[12] for a system of confined bosons.

In terms of the trap-size $l$ and the total mean-number of particles

$$N = \left\langle \sum_{x,\sigma} n_{x,\sigma} \right\rangle$$

it is possible to define two different regimes related to the ratio

$$\rho = \frac{N}{l^d}$$

which quantifies the mean-number of particles confined inside the d-dimensional “trap-volume” $l^d$. The main difference between these two regimes is the way in which the number of particles $N$ changes while increasing the trap-size $l$.

The first regime is called Trap Thermodynamic Limit (TTL). The TTL is the asymptotic regime obtained performing both the large $N$ and the large $l$ limits while keeping $\rho$ constant

$$\text{TTL : } N \to +\infty, l \to +\infty, \rho = \text{constant} \quad (7)$$

The other one is called Dilute Regime (DR). The DR, which is the regime of interest in the present work, is the asymptotic condition obtained performing the large $l$ limit at fixed number of particles $N$

$$\text{DR : } N = \text{constant}, l \to +\infty, \rho \to 0 \quad (8)$$

As discussed in details for the Bose-Hubbard model in [13] and for the fermionic case in [9], the properties shown by the lattice system in the two regimes introduced above can be studied by considering the large-$l$ behaviour of the Grand Canonical (GC) counterpart of the Hamiltonian in Eq.(3) within the Trap-Size Scaling (TSS) formalism (i.e. Renormalization Group formalism in the presence of spatial inhomogeneities). In the GC formalism one modifies Eq.(3) by adding a term proportional to the chemical potential $\mu$. The particular value of the parameter $\mu$ specifies completely the large-$l$ regime under study. In other words the choice of $\mu$ specifies the way in which the number of particles $N$ behaves as the trap-size $l$ increases.

In the following section we review how this correspondence can be used within the TSS formalism to characterize the DR properties of the hamiltonian model in Eq.(3).

III. THE TRAP-SIZE SCALING FORMALISM IN THE DILUTE REGIME

The GC counterpart of the hamiltonian model in Eq.(3) reads

$$H_{GCC} = H_0 + H_t + H_\mu, \quad (9)$$

where

$$H_\mu = -\mu \sum_{x,\sigma} n_{x,\sigma}. \quad (10)$$

As discussed in [9], in the DR the hamiltonian term $H_t$ can be treated as a weak perturbation to $H_{GCC} = H_0 + H_\mu$. This means that to characterize all the interesting features developed by the system due to the external confinement, that is the scaling properties of the correlation functions, one characterizes first the critical features of the hamiltonian without confinement and
then consider the effects introduced by the presence of the external potential, that is the existence of a finite correlation length related to the trap-size \( l \). The method described above is the basis of the TSS formalism, which has been extensively used during the last few decades to analyze the properties of confined lattice systems.

The critical properties of \( \mathcal{H}_{GC} \) can be deduced by studying the following Quantum Field Theory \[ 14 \] 15 (we set \( \hbar = 1, k_B = 1 \))

\[
Z = \int \mathcal{D}\psi^* \mathcal{D}\psi \exp \left( -\int_0^{1/T} d\tau d^d x \mathcal{L} \right),
\]

\[
\mathcal{L} = \sum_\sigma \left[ \frac{\psi_\sigma^*}{2} - \frac{1}{2M} \left| \nabla \psi_\sigma \right|^2 - \mu \left| \psi_\sigma \right|^2 \right] + \frac{u}{2} \psi_\sigma^* \psi_\sigma \psi_\sigma^* \psi_\sigma
\]

where \( T \) is the temperature of the system and \( \psi_\sigma = \psi_\sigma(x, \tau) \) and \( \psi_\sigma^* = \psi_\sigma^*(x, \tau) \) are the Grassmann fields associated to the creation and annihilation operators in Eq.\((10)\). \( M \) denotes the mass of the fields, \( u \) is the many-body interaction strength and \( \mu \) is the chemical potential.

The quantum theory in Eq.\((11)\) has a Gaussian fixed-point in correspondence of

\[
\mu = 0, \quad T = 0, \quad u = 0
\]

(12)

which corresponds to the following point in parameter space for the hamiltonian \( \mathcal{H}_{GC} \)

\[
\mu = -2t, \quad T = 0, \quad U = 0.
\]

(13)

This fixed-point prescribes the scaling behaviour for the theory in Eq.\((11)\) in correspondence of a Metal-Vacuum Quantum Phase transition (QPT). It is interesting to observe that this QPT is characterized by a low mean-occupation, which is physically consistent with the low mean-density condition which characterises the DR.

Since the fixed-point in Eq.\((12)\) is Gaussian, the scaling behaviour of all the variables entering in Eq.\((11)\) can be determined easily. Under a scaling transformation of the form

\[
x \rightarrow x' = \frac{x}{b}
\]

(14)

we have that the variables entering in the lagrangian \( \mathcal{L} \) transform this way

\[
\psi_\sigma \rightarrow \psi_\sigma' = b^{y_\psi} \psi_\sigma, \quad u \rightarrow u' = b^{y_u} u, \quad \mu \rightarrow \mu' = b^{y_\mu} \mu.
\]

(15)

where \( y_\psi, y_\mu \) and \( y_u \) are the Renormalization Group (RG) exponents associated respectively to the field \( \psi_\sigma \), to the chemical potential \( \mu \) and to the many-body interaction \( u \). By performing a simple dimensional analysis one finds the following values for the RG-exponents and for the dynamic critical exponent \( z \) \[ 14 \] 16

\[
y_\psi = \frac{d}{2}, \quad y_\mu = 2, \quad y_u = 2 - d
\]

(16)

\[
z = 2
\]

(17)

Let us now consider the effects of the confinement. In this framework the confining term can be introduced using the following contribution

\[
\mathcal{L}_v = \frac{1}{2} v^2 x^2 \sum_\sigma |\psi_\sigma|^2
\]

(18)

From Eq.\((10)\) and Eq.\((17)\), proceeding in the same way of Refs. \[ 9, 13, 17 \] the RG-exponent associated with the trap-intensity \( v \) is

\[
y_v = 2.
\]

(19)

The confining potential introduces a new characteristic length-scale \( \xi \) that is related to the trap-size by the following power-law dependence

\[
\xi \sim l^\theta,
\]

(20)

being

\[
\theta = 1/y_v = 1/2
\]

(21)

The presence of this characteristic length-scale modifies the correlation between particles, leading to a non-trivial scaling behaviour of all the expectation values called Trap-Size Scaling (TSS). Let us consider a generic operator \( \mathcal{O}(x; v, U) \) with RG-exponent \( y_o \) and let us consider also its n-points correlation function

\[
\mathcal{W}(x_1, \cdots, x_n; v, U) \equiv \langle \mathcal{O}(x_1; v, U), \cdots, \mathcal{O}(x_n; v, U) \rangle \]

(22)

where \( \langle \cdot \rangle \) denotes the expectation value on the ground-state configuration. We obtain the TSS behaviour of the correlation function in Eq.\((22)\) when considering a scaling transformation with parameter \( b = l^\theta \). In this case a scaling transformation of the lattice structure of the form \( x \rightarrow x' = x l^{-\theta} \) induces the following scaling transformation of \( \mathcal{W}(x_1, \cdots, x_n; v, U) \)

\[
\mathcal{W}(x_1, \cdots, x_n; v, U) = \mathcal{W}(x_1' \cdots, x_n'; v', U') =
\]

\[
= l^{-\theta y_W} \mathcal{W} \left( \frac{x_1}{l^\theta}, \cdots, \frac{x_n}{l^\theta}, 1, U l^{\theta y_u} \right) \equiv
\]

\[
= l^{-\theta y_W} \mathcal{F}(X_1, \cdots, X_n; U, r),
\]

(23)

where \( y_W = n y_O \) denotes the RG-exponent of the n-point correlation function in Eq.\((22)\), \( \mathcal{F}(X_1, \cdots, X_n; U, r) \) is the scaling function of Eq.\((22)\) and \( U_r \equiv U l^{\theta y_u} \). We have to stress that the equivalence in Eq.\((23)\) is true only in the large-\( l \) limit and in general for finite-\( l \) there are corrections which slightly modifies the scaling behaviour of the correlation functions.

For our purposes it is interesting to observe that the on-site coupling constant \( U \) shows a RG-behaviour which depends on the dimension \( d \). In \( d = 1 \), where the on-site coupling \( U \) is relevant, the scaling functions
are expected to depend strongly on this parameter. In $d = 2$, where $U$ is marginal, one expects to observe only a residual dependence of the scaling functions on the on-site interaction. For $d \geq 3$ instead, where this parameter becomes irrelevant, the TSS behaviour is expected to match the behaviour of a two-component Fermi gas in the presence of a harmonic trap.

The TSS Ansatz in Eq. (23) has been extensively tested and discussed in [9] for the one-dimensional case. Numerical results reported therein show that while performing a small-$a$ analysis of the trapped Hubbard model at fixed number of particles and fixed $l$, it is possible to introduce a continuous theory (Continuum limit) by means of which we can describe the properties developed by the lattice problem in the DR.

The Continuum limit of the lattice theory in the DR can be introduced by first performing a small-$a$ expansion of the Hamiltonian model in Eq. (9) in correspondence of $\mu = -2t$ which is the value prescribed by the RG analysis (see Eq. (13)) and then by taking the following limit

$$
H_c^{(d)} = \lim_{a \to 0} \frac{1}{a^2} \left[ H_{GCC} \right]_{\mu=-2t + t(d-1)N}^{\text{GR}}
$$

where $N = \sum_{x, \sigma} n_{x, \sigma}$. Here we consider the DR, but we expect this procedure to be quite general: if one is interested in the system properties in correspondence of an another large-$l$ regime, one has simply to choose properly the value of the chemical potential $\mu$ and to modify multiplicative constant in front of $N$ to cancel all the contributions proportional to the number of particles.

If one defines the following field operators at fixed $x = j a$ [18]

$$
\Psi_\sigma(x) = \lim_{a \to 0} \frac{C_{j, \sigma}}{a^{d/2}} \bigg|_{x = ja}, \quad \Psi_\sigma(x) = \lim_{a \to 0} \frac{C_{j, \sigma}}{a^{d/2}} \bigg|_{x = ja}
$$

the result of the limit in Eq. (24) is the following

$$
H_c^{(d)} = -t \sum_{\sigma} \int d^d x \nabla_\sigma \Phi_\sigma(x) + g_d \int d^d x \int d^d y \nabla_\sigma \Phi_\sigma(x) \delta(x-y) \nabla_\sigma \Phi_\sigma(y) +
$$

$$
+ \frac{v_c^2}{2} \sum_{\sigma} \int d^d x \left| \Phi_\sigma(x) \right|^2 \nabla_\sigma \Phi_\sigma(x),
$$

where $\nabla_\sigma$ is the Laplace operator in dimension $d$ and

$$
v_c = \frac{\hbar}{a^d}, \quad g_d = U a^{d-2}.
$$

If we now use the well-known relations between first and second quantization formalism [19], by setting

$$
t = \frac{\hbar^2}{2m}, \quad v_c^2 = \frac{m \omega^2}{a^d},
$$

the model in Eq. (26) is equivalent to following $d$-dimensional continuous Hamiltonian model

$$
H_c^{(d)} = \sum_{i=1}^N \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 x_i^2 \right) + g_d \sum_{i=1}^{N_\uparrow} \sum_{j=1}^{N_\downarrow} \delta(x_i - x_j).
$$

where $N_\sigma$ is the number of particles with spin polarization $\sigma (\sigma = \uparrow, \downarrow)$.

The hamiltonian in Eq. (29) describes the dynamics of a system of $N$ spin-$\frac{1}{2}$ fermions into an external harmonic potential. The presence of the local delta interaction depends on the dimension $d$. This fact is coherent with the

IV. CONTINUUM LIMIT OF THE CONFINED HUBBARD MODEL & TSS PROPERTIES

In this section we show how to introduce a continuous theory equivalent to the confined Hubbard model in the large-$l$ regimes discussed in the previous sections and we explicitly show how this limit is related to the results obtained within the TSS formalism.

A. The continuum limit in the presence of a harmonic trap

The idea at the basis of the method discussed here is quite simple and it is the following. As said before the presence of the confining potential introduces a new length scale $l$. To understand how these new length scale affects the system, we have to compare $l$ to the lattice spacing $a$, which is the intrinsic length scale of the lattice theory. In fact, the system properties do not depend on these two lengths separally, but they are determined by the ratio $a/l$. This consideration tells us that the case in which we send $l$ to infinity keeping fixed $a$ (which at fixed number of particles defines the DR) and the case in which we send $a$ to zero keeping constant the trap-size $l$ must show the same physical properties. Therefore, by performing a small-$a$ analysis of the trapped Hubbard model at fixed number of particles and fixed $l$, it is possible to introduce a continuous theory (Continuum limit)
results obtained by using the RG formalism. This becomes clearer if we explicitly consider \( g_d \) in \( d = 1, d = 2 \) and \( d \geq 3 \).

According to Eq. (30) we have that
\[
\begin{aligned}
g_1 &= U a^{-1}, \text{ in } d = 1 \\
g_2 &= U, \text{ in } d = 2 \\
g_d &= U a^{d-2}, \text{ in } d \geq 3.
\end{aligned}
\]

In \( d = 1 \), where the the on-site parameter is a relevant variable, we have that the continuum limit of the lattice model is a strong interacting many-body theory: for an arbitrary \( |U| \neq 0 \), it does not matter how small it is, the lattice theory is mapped into an interacting model with \( |g_1| \neq 0 \). This means that in \( d = 1 \) the properties of the continuum limit are strongly affected by the presence of an arbitrary small interparticle interaction in the lattice model. In \( d = 2 \), where the RG behaviour of the on-site parameter \( U \) is marginal, we have that \( g_2 = U \). This means that the continuum limit preserves the nature and the role played by the on-site coupling in the original theory. In \( d \geq 3 \) everything changes: since \( d - 2 > 0 \), the continuum limit in \( d \geq 3 \) is always a non-interacting theory, that is \( g_d \) is always zero. This is coherent with the irrelevant RG behaviour found for the \( U \) in dimensions higher than \( d = 2 \).

If we now use the definition of the trap-size \( l \) in Eq. (24) and Eqs. (27) and (28), we can “eliminate” the lattice spacing \( a \) and we can rephrase the correspondence between \( g_d \) and \( U \) in terms of the trap-size \( l \) and the others parameters entering in the continuum limit model. The relation is the following:
\[
g_d = U \left( \frac{\hbar}{m \omega} \right)^{\frac{d-2}{2}} l^{\frac{2-d}{2}}. \tag{31}
\]

If we neglect the \( d \geq 3 \) case that is trivial we have that
\[
g_1 = U_r \left( \frac{m \omega}{\hbar} \right)^{1/2}, \tag{32}
\]
and we have
\[
g_2 = U, \tag{33}
\]
where we used the definition of the rescaled coupling \( U_r = U l^{1/2} \).

The relation in Eq. (31) is the main result of this work. In the following we will essentially discuss the correspondence prescribed by the Eq. (31) by considering the scaling properties developed by the mean-occupation of the lattice problem.

B. The correspondence hypothesis: continuum limit & TSS formalism

To test the validity of the correspondence discussed above, the idea is to compare the TSS behaviour of a given observable \( \mathcal{O}(x_1, \cdots, x_n) \) obtained for the lattice problem with its analogue in first-quantization formalism, that is the mean-value of the corresponding operator on the state representing the ground-state configuration \( |\Psi_{GS}; g_d\rangle \).

In this work we consider several different one-point and two-point observables. In particular in \( d = 1 \) we consider the mean-occupation
\[
\rho(x; v, U) = \langle n_x \rangle = \sum_{\sigma=\uparrow, \downarrow} \langle n_{\sigma,x} \rangle, \tag{34}
\]
the double-occupancy
\[
d_0(x; v, U) = \langle n_{\uparrow,x} n_{\downarrow,x} \rangle, \tag{35}
\]
the pair-correlation
\[
P(x, y; v, U) = \langle C_{\uparrow,x} C_{\downarrow,y} + h.c. \rangle, \tag{36}
\]
the one-particle correlation
\[
C(x, y; v, U) = \sum_{\sigma=\uparrow, \downarrow} \langle C_{\sigma,x} C_{\sigma,y} + h.c. \rangle, \tag{37}
\]
and the two connected density-density correlations
\[
G(x, y; v, U) = \langle n_x n_y \rangle - \langle n_x \rangle \langle n_y \rangle, \tag{38}
\]
\[
M(x, y; v, U) = \langle n_{\uparrow,x} n_{\downarrow,x} \rangle \langle n_{\uparrow,y} \rangle. \tag{39}
\]

In \( d = 2 \) instead we only consider the scaling behavior of the mean-density.

According to the analysis performed in Ref. [9] the scaling behaviour of the observables introduced above under a scaling transformation of the lattice structure of the form \( x \rightarrow X = x/\ell^d \) can be cast in the following form
\[
\rho(x; v, U) \approx \ell^{-2d} \mathcal{R}(X; U_r), \tag{40}
\]
\[
d_0(x; v, U) \approx \ell^{-2d} \mathcal{D}(X; U_r), \tag{41}
\]
\[
P(x, y; v, U) \approx \ell^{-2d} \mathcal{P}(X, Y; U_r), \tag{42}
\]
\[
C(x, y; v, U) \approx \ell^{-d} \mathcal{C}(X, Y; U_r), \tag{43}
\]
\[
G(x, y; v, U) \approx \ell^{-2d} \mathcal{G}(X, Y; U_r), \tag{44}
\]
\[
M(x, y; v, U) \approx \ell^{-2d} \mathcal{M}(X, Y; U_r). \tag{45}
\]

In the DR and for a generic number of particles \( N = N_\uparrow + N_\downarrow \) we expect to have the following expressions for the scaling functions:
\[
\mathcal{R}(X, U_r) = \rho_{GS}(X; g_d), \tag{46}
\]
\[
\mathcal{D}(X, U_r) = d_0(X; g_d), \tag{47}
\]
\[
\mathcal{P}(X, Y; U_r) = P(X, Y; g_d), \tag{48}
\]
\[
\mathcal{C}(X, Y; U_r) = C(X, Y; g_d), \tag{49}
\]
\[
\mathcal{G}(X, Y; U_r) = G(X, Y; g_d), \tag{50}
\]
\[
\mathcal{M}(X, Y; U_r) = M(X, Y; g_d). \tag{51}
\]
\[ D(\mathbf{X}; U_r) = N_{i} \int \prod_{i=2}^{N_t-1} \prod_{j=2}^{N_i} dx_i dy_j \left| \Psi_{GS}(\mathbf{X}, \mathbf{x}_2, \cdots, \mathbf{x}_{N_t}, \mathbf{x}, \mathbf{y}_2, \cdots, \mathbf{y}_{N_i}; g_d) \right|^2, \]  

(47)

\[ C(\mathbf{X}, \mathbf{Y}; U_r) = N \int \prod_{i=2}^{N_t-1} \prod_{j=2}^{N_i} dx_i dy_j \left[ \Psi_{GS}^{*}(\mathbf{X}, \mathbf{x}_2, \cdots, \mathbf{x}_{N_t}, \mathbf{y}_1, \cdots; g_d) \Psi_{GS}(\mathbf{Y}, \mathbf{x}_2, \cdots, \mathbf{x}_{N_t}, \mathbf{y}_1, \cdots; g_d) + c.c. \right], \]  

(48)

\[ P(x, y; U_r) = N_{i} \int \prod_{i=2}^{N_t-1} \prod_{j=2}^{N_i} dx_i dy_j \left[ \Psi_{GS}^{*}(\mathbf{X}, \mathbf{x}_2, \cdots, \mathbf{x}_i, \mathbf{y}_1, \cdots; g_d) \Psi_{GS}(\mathbf{Y}, \mathbf{x}_2, \cdots, \mathbf{y}_i, \cdots; g_d) + c.c. \right], \]  

(49)

\[ G(x, y; U_r) = N(N - 1) \int \prod_{i=3}^{N_t} \prod_{j=1}^{N_i} dx_i dy_j \left| \Psi_{GS}(\mathbf{X}, \mathbf{y}_3, \cdots, \mathbf{x}_{N_t}, \mathbf{y}_1, \cdots; g_d) \right|^2 + \]  

\[ \delta(\mathbf{X} - \mathbf{Y}) \rho_{GS}(\mathbf{X}; g_d) - \rho_{GS}(\mathbf{X}; g_d) \rho_{GS}(\mathbf{Y}; g_d), \]  

(50)

\[ M(x, y; U_r) = N_{i} \int \prod_{i=2}^{N_t-1} \prod_{j=2}^{N_i} dx_i dy_j \left| \Psi_{GS}(\mathbf{X}, \mathbf{x}_2, \cdots, \mathbf{x}_i, \mathbf{y}_1, \cdots; g_d) \right|^2 + \]  

\[ \frac{N_{j} N_{i}}{N^2} \rho_{GS}(\mathbf{X}; g_d) \rho_{GS}(\mathbf{Y}; g_d), \]  

(51)

where \( \Psi_{GS}(\mathbf{x}_1, \cdots, \mathbf{x}_{N_t}, \mathbf{y}_1, \cdots; g_d) \) is the many-body ground-state wave-function and \( \rho_{GS}(\mathbf{x}; g_d) \) is the one-particle density function of the \( N \) body problem

\[ \rho_{GS}(\mathbf{x}; g_d) = N \prod_{i=2}^{N} \left| \Psi_{GS}(\mathbf{x}, \mathbf{x}_2 \cdots; g_d) \right|^2, \]  

(52)

with the parameters \( U_r \) and \( g_d \) related by Eq. 31.

The rest of this paper is devoted to the analysis of the equations Eq. 31 - Eq. 35.

V. THE TWO-BODY PROBLEM & THE CORRESPONDENCE HYPOTHESIS

In this section we discuss the correspondence hypothesis in \( d = 1 \) and \( d = 2 \) for the two-body unpolarized problem (\( N = 2 \) and \( N_1 = N_t = 1 \)). The one- and two-dimensional cases are the only non-trivial cases. Indeed, both the continuum limit procedure and the RG analysis reported above suggest that the interparticle interaction becomes negligible for \( d \geq 3 \), which means that the DR properties of the lattice problem in \( d \geq 3 \) are fully described by an hamiltonian of non-interacting fermions inside a harmonic trap.

A. The ground-state wave-function of the unpolarized problem in \( d = 1 \)

According to the analysis in Sec. IV, the DR properties of the one-dimensional confined Hubbard model at very low temperature are in one-to-one correspondence respectively with the ground state-properties of the following many-body hamiltonians

\[ H^{(1)} = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 x_i^2 \right) + g_1 \sum_{i=1}^{N} \sum_{j=1}^{N} \delta(x_i - x_j), \]  

(53)

which is the Gaudin-Yang model [5, 11] in the presence of an external harmonic potential. This model and its properties have been considered and studied in several papers, with many different purposes (see for example Refs. 20, 21). For a given number of particles \( N \), the ground-state wave-function of the Hamiltonian model in Eq. 53 can be found numerically for example by using methods Bethe-Ansatz formalism, DMRG or exact diagonalization. In our case, the explicit expression of all the eigenfunctions and their properties can be derived easily by using separation of variables and the results reported in Ref. 22. It turns out that for any value of the interaction strength \( g_1 \), the ground-state wave-function for the two-body unpolarized problem is always given by a \( S = 0 \) state, where \( S \) is the total spin of the two-body configuration. Therefore, the
Using adimensional variables, that is where the integration domain is the real axis. where the relation between the continuum limit of the lattice problem, that is the function associated to the ground-state wave-function of the two-body unpolarized problem we have that these two appear in the large function in Eq.(56) to be 

\[ O \]

where \( \Gamma(\frac{1}{2}, \frac{\nu}{2}) \) is a rescaled coupling

\[ \Gamma(\frac{1}{2}, \frac{\nu}{2}) = \alpha, \] (55)

where \( \Gamma(x) \) is the Euler Gamma function.

B. Trap-Size Scaling of the unpolarized problem in \( d = 1 \)

In \( d = 1 \) the TSS behavior of the mean density at fixed \( U_r \) is the following

\[ \sqrt{l} \rho(x, U) = \mathcal{R}(X = x/\sqrt{l}, U_r) + O(l^{-1}), \] (56)

where \( O(l^{-1}) \) indicates the finite \( l \) corrections which disappear in the large \( l \) limit [3].

The Correspondence Hypothesis prescribes the scaling function in Eq.(56) to be exactly the one-body density function associated to the ground-state wave-function of the continuum limit of the lattice problem, that is

\[ \mathcal{R}(X, U_r) = \rho_{GY}(X, g_1) \] (57)

where

\[ \rho_{GY}(X, g_1) = 2 \int dx |\Psi_{GS}(X, x, S = 0; g_1, m, \omega)|^2, \] (58)

where the integration domain is the real axis.

Using adimensional variables, that is

\[ \hbar = 1, \ t = 1, \ \lambda = 1 \] (59)

the relation between the \( g_1 \) and \( U_r \) in Eq.(62) becomes a relation between the parameter \( \alpha \) and \( U_r \). For the two-body unpolarized problem we have that these two parameters are related by the following law

\[ \alpha(U_r) = \frac{U_r}{2\sqrt{2}} \] (60)

As a first proof of the correspondence we extrapolate \( \mathcal{R}(X = 0, U_r) \) and we compare this value with \( \rho_{GY}(X = 0, \alpha(U_r)) \). The extrapolation of \( \mathcal{R}(X = 0, U_r) \) has been done by interpolation of the numerical results (exact diagonalization) obtained for finite values of the trap-size \( l \). According to the scaling relation in Eq.(50), the large \( l \) behaviour of \( \sqrt{l}\rho(x = 0, U_r) \) is the following

\[ \sqrt{l}\rho(x = 0, U_r) = a + bl^{-1} \] (61)

where the parameter \( a = \mathcal{R}(X = 0, U_r) \) and \( b \) quantifies the entity of finite \( l \) corrections. In Fig.4 and Fig.5 have been reported the interpolating curves respectively for the repulsive and the attractive regimes. The results of the extrapolation procedure have been reported in Table I.

In Fig.4 and in Fig.5 we report the numerical results for the scaling function \( \sqrt{l}\rho(x, U_r) \) for \( U_r = 100 \) and \( U_r = 10 \). In Fig.4 and in Fig.5 we report instead the numerical results for the scaling function \( \sqrt{l}\rho(x, U_r) \) for \( U_r = -100 \) and \( U_r = -10 \) respectively. In Figs. 3 4 5 and 6 we also report the corresponding one-body density function \( \rho_{GY}(x, \alpha(U_r)) \), where the values of \( \alpha(U_r) \) have been chosen using the relation in Eq. (60).

It is possible to observe that at increasing \( l \), for all the values of the rescaled coupling \( U_r \) considered, the data converge to the one-body density function \( \rho_{GY}(x, \alpha(U_r)) \). In addition, it is interesting to note that the TSS behaviour reported here for the repulsive case is coherent with the experimental results reported in Ref. [29] where a system of two spin-1/2 \( ^6 \)Li atoms in a harmonic trap has been considered. At increasing interaction strength the energy of the two-body ground-state increases and it reaches its maximum value when the coupling goes to \(+\infty\). In this infinitely repulsive regime the \( S = 0 \) and the \( S = 1 \) configurations become degenerate in energy and the system of two spin-1/2 particles becomes a system of two indistinguishable spinless fermions. This phenomenon, which is the analogue of the fermionization in one-dimensional systems of bosons [30], affects the one-body density. While increasing the interaction strength from zero, two peaks placed symmetrically around \( x = 0 \) appear in the one-body density and they become sharper as the coupling increases (compare Fig. 4 and Fig. 5).

\[ \begin{array}{lll}
U_r & \mathcal{R}(X = 0, U_r) & \rho_{GY}(X = 0, \alpha(U_r)) \\
100 & 0.58483303 & 0.58483303 \\
10  & 0.74938486 & 0.74938486 \\
-10 & 1.53935  & 1.53935 \\
-100 & 1.59513  & 1.59513 \\
\end{array} \]

TABLE I. Numerical estimates for the scaling function \( \mathcal{R}(X = 0, U_r) \) and exact numerical values for the density function \( \rho_{GY}(x = 0, \alpha(U_r)) \) in correspondence of different values of the parameter \( U_r \).

Let us now consider the TSS behaviour of the other ob-
observables listed in Sec. IV B. We study these observables for 
$U_r = -10$ and $U_r = 10$. For the two-body unpolarized case the scaling functions reported in Eq. (47) - Eq. (66) reduce respectively to the following expressions:

$$d_{GY}(x; g_1) = |\Psi_{GS}(x; x; g_1)|^2,$$

$$P_{GY}(x; g_1) = \Psi_{GS}(x; x; g_1) \Psi_{GS}(y; y; g_1) + c.c.,$$

$$C_{GY}(x; y; g_1) = \int dt [\Psi_{GS}(x; t; g_1) \Psi_{GS}(y; t; g_1) + \Psi_{GS}(y; t; g_1) \Psi_{GS}(x; t; g_1) + c.c.],$$

$$G_{GY}(x; y; g_1) = |\Psi_{GS}(x; x; g_1)|^2 + |\Psi_{GS}(y; x; g_1)|^2 + 2 \delta(x - y) \int dt \Psi_{GS}^*(x; t; g_1) \Psi_{GS}(y; t; g_1) - \rho_{GY}(x; g_1) \rho_{GY}(y; g_1),$$
different values of the trap-size clearly approach to the same scaling function. This function is the one prescribed by the continuum limit procedure discussed in Sec. (IV A). Therefore, the results shown in this section fully support the correspondence hypothesis in Eq. (57) and the correspondence between the two-body interaction strength prescribed by Eq. (60). In other words our results prove that there is a one-to-one correspondence between the low density TSS properties of the confined Hubbard model and those of the continuous theory deduced in the Sec. (IV A) in $d = 1$.

C. The ground-state wave-function of the unpolarized problem in $d = 2$

As discussed in Sec. (IV A) the continuum limit procedure prescribes that in $d = 2$ the confined-Hubbard model is in exact correspondence with the following
FIG. 7. TSS behaviour of the double-occupancy $l d_0(x, U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$), compared with the scaling function $d_{0;GY}(X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = -10$ (red dots).

FIG. 8. TSS behaviour of the double-occupancy $l d_0(x, U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$), compared with the scaling function $d_{0;GY}(X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = 10$ (red dots).

FIG. 9. TSS behaviour of the pair correlation $l P(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $P_{GY}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = -10$ (red dots).

FIG. 10. TSS behaviour of the pair correlation $l P(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $P_{GY}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = 10$ (red dots).

As for the $d = 1$ case, we want to determine the ground-state wave-function of this Hamiltonian. The main problem with this Hamiltonian is that the Dirac delta in $d = 2$ introduces divergences in wave functions and it is not possible to apply directly the method and the ideas used above for the $d = 1$ case. One has first to remove these divergencies. This can be done by regularizing the two-body interaction. This regularization can be achieved by replacing the contact interaction in Eq. (67) with the

$$H^{(2)}_c = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 x_i^2 \right) + g_2 \sum_{i=1}^{N} \sum_{j=1}^{N} \delta(x_i - x_j).$$
FIG. 11. TSS behaviour of the one-particle correlation $\sqrt{l} C(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $C_{\text{GY}}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = -10$ (red dots).

FIG. 12. TSS behaviour of the one-particle correlation $\sqrt{l} C(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $C_{\text{GY}}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = 10$ (red dots).

FIG. 13. TSS behaviour of the density-density correlation $l G(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $G_{\text{GY}}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = -10$ (red dots).

FIG. 14. TSS behaviour of the density-density correlation $l G(0, x; U_r)$ for different values of the trap-size $l$ ($l = 1000, 2000, 5000$) with a point fixed at the trap center, compared with the scaling function $G_{\text{GY}}(0, X; \alpha(U_r))$ of the two-body unpolarized problem for $U_r = 10$ (red dots).

The following contact interaction $^{[24, 28]}$

$$V(r) = g_2 \delta(r) \left[ 1 - \log(r) \frac{\partial}{\partial r} \right]$$

(68)

where $r = x_1 - x_2$ and $r = |r|$.

This version of the interaction term cancels the logarithmic divergence of the ground-state wave-function in correspondence of $r \to 0$ and it behaves exactly as the two-dimensional Dirac delta. As in $d = 1$ the ground-state wave-function is a $S = 0$ state and it has the following
expression
\[
\Psi(x_1, x_2, S = 0; \alpha) = \beta \exp \left( -\frac{x_1^2 + x_2^2}{2\lambda^2} \right) U \left( \nu - \frac{1}{2}, 1, \frac{(x_1 - x_2)^2}{2\lambda^2} \right),
\]
where \( \alpha = (g_2, m, \omega) \) is the set of parameters entering in Eq. (67), \( \beta \) is a normalization factor and \( U(a, b; x) \) is again a confluent hypergeometric function.

As for the \( d = 1 \) case, the parameter \( \nu \) in Eq. (69) is related to the presence of the contact interaction in the two-body problem and it is related to the interaction strength \( g_2 \) and all the other parameters entering in the Hamiltonian model by the following relation \[ \[ 27, 28 \]
\[
2\gamma - \log(2) + \psi \left( \frac{1}{2} - \frac{\nu}{2} \right) = \frac{8\pi\hbar\omega}{g_2},
\]
where \( \gamma \) is the Euler constant and \( \psi(x) \) the Digamma function \[ 24 \].

In the following section we use the wave-function in Eq. (69) to construct the one-body density function associated to the two-body problem. We compare this function to the numerical results obtained for the scaling function of the mean-density of the lattice problem to show that the correspondence holds also in \( d = 2 \) (at least in the repulsive case).

D. Trap-Size Scaling of the Ground-State Density and Analytical Solution for the unpolarized problem in \( d = 2 \)

In \( d = 2 \), as discussed in Sec. IV B, the TSS behaviour of the particle density under a scaling transformation of the lattice position \( x \) is the following
\[
l_\rho(x, U) = R^{(2d)}(X, U) + O(l^0),
\]
where \( R^{(2d)}(X, U) \) and \( O(l^0) \) denote respectively the scaling function of the \( d = 2 \) problem and the finite-\( l \) corrections.

As in the \( d = 1 \) case we compare the behaviour of rescaled density in Eq. (71) at increasing \( l \) with the one-body density associated to the ground-state wave-function reported in Eq. (69) assuming the correspondence relation reported in Eq. (33) to be true.

As in the \( d = 1 \) case we can introduce the one-body density function associated to the wave-function \( \Psi \) in Eq. (69)
\[
\rho(x_1, g_2) = 2 \int dx_2 |\Psi(x_1, x_2, S = 0; \alpha)|^2 = 2 \int dx_2 \beta^2 \exp \left[ -\frac{(x_1^2 + x_2^2)}{2\lambda^2} \right] \times U^2 \left( \frac{\nu - 1}{2}, 1, \frac{(x_1 - x_2)^2}{2\lambda^2} \right),
\]
where the integration domain is \( \mathbb{R}^2 \).

By inspection of Eq. (69), it is easy to see that the one-body density function is invariant under rotations of the vector \( x_1 \). This means that this one-body density is completely characterized by the knowledge of the \( \rho(x, g_2) \equiv \rho(x_1, g_2) \), with \( x_1 = (x, 0) \) and \( x \geq 0 \).

Using adimensional variables, that is
\[
h = 1, \quad t = 1, \quad \lambda = 1,
\]
the correspondence relation between the $g_2$ and the on-site coupling constant $U$ becomes

$$\alpha_{2d}(U) = \frac{U}{2},$$ \hspace{1cm} (74)

where $\alpha_{2d}$ is the adimensional analogue of the interaction strength $g_2$.

As in the $d = 1$ case, see Fig.17, while increasing $U$, the value of the rescaled density in $X = 0$ decreases with respect to the non-interacting case and in addition numerical results appear to converge to a non-trivial curve at increasing $l$. This curve depends on the particular value of the on-site coupling $U$.

Assuming the correspondence relation in Eq.(74) to be true we compared the numerical results obtained for $\rho(x, U)$ for the two values $U = 1$ and $U = 3$ with the profiles of the one-body density functions associated to the wave-function in Eq.(69) for the values of $\alpha_{2d}$ prescribed by Eq.(74). Results have been reported in Fig.18 and Fig.19.

Results reported in Fig.18 and in Fig.19 support the correspondence hypothesis for $U > 0$. Further studies are needed to prove explicitly the validity of the correspondence hypothesis for $U < 0$.

VI. SUMMARY AND CONCLUSIONS

In the present work we have considered the $d$-dimensional Hubbard model in the presence of an external harmonic confinement coupled to the particle density. In particular we have considered this system in the so called Dilute Regime (DR), which is the low-density regime achieved considering a fixed number of particles $N$ in the presence of a weak confinement.

The ground-state properties of the one-dimensional case in the presence of an harmonic confinement have been extensively studied in [9] using both Renormalization Group (RG) techniques and the Trap-Size Scaling formalism (TSS). In particular in [9], the authors argue that the leading TSS behaviour of the particle density reproduces the ground-state properties of a one-dimensional gas of spin-1/2 fermions described by the Gaudin-Yang model in the presence of an harmonic trap (GY model), with an interaction strength $g$ between particles with opposite polarization that is proportional to the variable $U_r = U_{1/2}$.

In the present work we have derived the continuous
theory which describes the TSS properties of the Hubbard model in the presence of an external harmonic confinement in any $d$. The expression of this continuous theory depends explicitly on the dimension $d$ of the lattice problem and its properties are in agreement with the RG analysis reported in [4].

In $d = 1$ the continuum limit of the lattice problem with $N$ particles is the GY model with the same number of particles. The correspondence hypothesis between the leading TSS behaviour of the particle density and the profiles associated to the one-body density function of the ground-state of the GY model has been tested comparing the numerical results obtained from the exact diagonalization of the lattice problem and the analytical solutions of the two-body unpolarized problem ($N = 2$ and $N_1 = N_2 = 1$). In particular we have found the analytic relation between on-site coupling $U$ of the Hubbard model and the interaction strength $g_1$ of the Gaudin-Yang model in presence of an external harmonic confinement. We have to stress that the relation which prescribes the correspondence between $g_1$ and $U$ does not depend on the number of particles $N$, so the its validity holds for any $N$ (see Eq. (33)).

In $d = 2$ the continuum limit of the lattice problem is an interacting theory with a local interaction between particles with opposite spin which is shaped as a Dirac delta. The main problem with interactions of this kind in $d = 2$ is that they have to be regularized to cancel the logarithmic divergences of the wave-functions. The regularization has been done following the same ideas reported in [28]. As in $d = 1$ we tested the correspondence hypothesis comparing numerical results and analytical results obtained for the ground-state configuration of the two-body unpolarized problem. Results reported support the correspondence between the DR properties of the two-dimensional confined Hubbard model and those of the continuum limit derived in Sec.IV in particular they confirm the validity of analytical relation reported in Eq. (33) between the on-site interaction $U$ and the coupling constant $g_2$ (at least for $U \geq 0$).

In $d \geq 3$ results obtained performing the continuum limit confirm that the leading TSS behaviour of the Hubbard model in the DR is described by a non-interacting theory as obtained in [4] using RG methods.

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