Entanglement Detection via Direct-Sum Majorization Uncertainty Relations

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(Dated: October 1, 2018)

In this paper we investigate the relationship between direct-sum majorization formulation of uncertainty relations and entanglement, for the case of two and many observables. Our primary results are entanglement detection methods based on direct-sum majorization uncertainty relations. These nonlinear detectors provide a set of necessary conditions for detecting entanglement whose number grows with the dimension of the state being detected.

I. INTRODUCTION

Uncertainty relations form a central part of quantum mechanics. They impose fundamental limitations on our ability to simultaneously predict the outcomes of non-commuting observables. Different approaches have been proposed to quantify these relations. The original formulation is given by Heisenberg [1] in terms of standard deviations for momentum and position operators. His result is then generalized to two arbitrary observables [2]. Later it is recognized that one can express uncertainty relations in terms of entropies [3, 4]. In this approach, entropy functions like Shannon and Rényi entropies are used to quantify uncertainty (Ref. [5] is a nice survey on this topic). However, entropies are by no reason the most adequate to use. With this motivation, majorization is used to study uncertainty relations [6]. This line of research is further investigated in [7, 10].

Entanglement is another appealing feature of quantum mechanics and has been extensively investigated in the past decades [11]. Entangled states play important roles in quantum information processing, such as quantum teleportation [12] and dense coding [13]. Deciding whether a given quantum state is entangled is a key problem of quantum information theory and known to be computationally intractable in general [14]. Therefore, computationally tractable necessary conditions for entanglement detection, which provide a partial solution, have been the subject of active research in recent years [15].

Refs. [16, 17] present several methods for detecting entanglement via variance based uncertainty relations. Similar methods have been designed using entropy based uncertainty relations [18, 19]. One may wonder whether there exists a relationship between the majorization based uncertainty relation and entanglement. The answer is affirmative. In [20], the author applies the tensor-product majorization formulation of uncertainty to the problem of entanglement detection. In this paper we use the direct-sum majorization uncertainty relation, developed in [10], to design an entanglement detection method. As the direct-sum majorization bound has analytical solution while the tensor-product majorization bound does not, our direct-sum majorization based detection method is more practical than the tensor-product majorization based method.

The rest of this paper is organized as follows. In Sec. II we establish the notation and briefly review the direct-sum majorization formulation of uncertainty. In Sec. III we present our central result — an entanglement detection method based on the direct-sum majorization uncertainty. In Sec. IV we generalize our result to the case of many observables. We conclude in Sec. V. Some proofs are given in the Appendix.

II. DIRECT-SUM MAJORIZATION UNCERTAINTY RELATIONS

This section presents a basic review of the majorization theory and the formulation of direct-sum majorization approach to uncertainty relations.

A. Majorization

Let $\mathbb{R}_+ = [0, \infty)$ be the set of non-negative real numbers. $\mathbb{R}_+^d = \{ (p_1, \cdots, p_d) : p_i \in \mathbb{R}_+ \}$ be the set of $d$-dimensional real vectors with non-negative components. We denote by $p \in \mathbb{R}_+^d$ a $d$-dimensional vector and by $p_i$ the $i$-th element of $p$. For any vector $p \in \mathbb{R}_+^d$, let $p^\downarrow$ be the vector obtained from $p$ by arranging the components of the latter in descending order. Given two vectors $p, q \in \mathbb{R}_+^d$, $p$ is said to be majorized by $q$ and written $p \prec q$ if

$$\forall k \in [d - 1], \sum_{i=1}^{k} p_i^\downarrow \leq \sum_{i=1}^{k} q_i^\downarrow, \quad \text{and} \quad \sum_{i=1}^{d} p_i^\downarrow = \sum_{i=1}^{d} q_i^\downarrow,$$

where $[d] = \{1, \cdots, d\}$. Intuitively, $p \prec q$ means that the sum of largest $k$ components of $p$ is no larger than the sum of $k$ largest components of $q$. The majorization order is a partial order, i.e., not every two vectors are comparable under majorization. When studying majorization among two vectors of different dimensions, we...
append 0(s) to the vector with smaller dimension so that two vectors have the same dimension.

A related concept is the supremum of a set of $N$ vectors, defined as the vector that majorizes every element of the set and, is majorized by any vector that has the same property. We now briefly describe how to construct the supremum vector, more details can be found in [10]. Let $S = \{p^{(1)}, \ldots, p^{(N)} : p^{(n)} \in \mathbb{R}^+_0\}$ a set of $N$ vectors. To construct the supremum for $S$, we define a $(d + 1)$-dimensional vector $\Omega$ with components $\Omega_k = 0$, $\forall k \in [d]$, where $d = \min\{\text{dim}(\rho) : \rho \in \mathcal{D}(\mathcal{H})\}$, $D$ is the set of quantum states in the $d$-dimensional Hilbert space. Denote by $\omega^s$ the vector that majorizes every element of the set $S$ in descending order, say, $\omega^{(s)} = (\omega_1^{(s)}, \ldots, \omega_d^{(s)})$ and may, therefore, fail to be majorized by other vectors with the same property. In such case, we must perform a “flattening” process. This process starts with $\omega^s$ obtained in Eq. (1), and for each pair of components violating the descending order, say, $\omega_k^{s} < \omega_{k+1}^{s}$, replaces the pair by their mean such that the updated two elements are $\omega_k^{\text{sup}} = \omega_{k+1}^{\text{sup}} = (\omega_k^{s} + \omega_{k+1}^{s})/2$. This process continues until a descending vector corresponding to the supremum is obtained.

The construction given in Eq. (1) guarantees that $\omega^{\text{sup}}$ majorizes every element of the set $S$, but $\omega^{\text{sup}}$ does not necessarily appear in a descending order and may, therefore, fail to be majorized by other vectors with the same property. In such case, we must perform a “flattening” process. This process starts with $\omega^s$ obtained in Eq. (1), and for each pair of components violating the descending order, say, $\omega_k^{s} < \omega_{k+1}^{s}$, replaces the pair by their mean such that the updated two elements are $\omega_k^{\text{sup}} = \omega_{k+1}^{\text{sup}} = (\omega_k^{s} + \omega_{k+1}^{s})/2$. This process continues until a descending vector corresponding to the supremum is obtained.

**B. Direct-sum majorization uncertainty**

We now briefly introduce the uncertainty relation characterized by direct-sum majorization relation. We remark that the results summarized here is originally presented in [10].

Let $\mathcal{H}$ be a $d$-dimensional Hilbert space. Denote by $\mathcal{D}(\mathcal{H})$ the set of quantum states in $\mathcal{H}$. Let $X$ and $Z$ be two rank-one projective observables, and $\rho$ be a state on $\mathcal{H}$. Assume the spectral decompositions of $X$ and $Z$ are given by

$$X = \sum_{i=1}^{d} \alpha_i \lvert x_i \rangle \langle x_i \rvert, \quad Z = \sum_{j=1}^{d} \beta_j \lvert z_j \rangle \langle z_j \rvert,$$

where $\{\lvert x_i \rangle\}$ and $\{\lvert z_j \rangle\}$ are the eigenstates of $X$ and $Z$, respectively. By measuring $\rho$, $X$ induces a probability distribution given by

$$p(X|\rho) = (p_1, \ldots, p_d), \quad p_i = \langle x_i | \rho | x_i \rangle.$$

Similarly, $Z$ induces a probability distribution given by

$$q(Z|\rho) = (q_1, \ldots, q_d), \quad q_j = \langle z_j | \rho | z_j \rangle.$$

We are interested in the uncertainty relation induced by these two observables. In [10], the direct-sum majorization approach is used to is to characterize the uncertainty about $p(X|\rho)$ and $q(Z|\rho)$:

$$\forall \rho \in \mathcal{D}(\mathcal{H}), \quad p(X|\rho) \perp q(Z|\rho) \prec \omega^{X \otimes Z},$$

where $\omega^{X \otimes Z}$ is a $2d$-dimensional vector independent of $\rho$ which can be explicitly calculated from observables $X$ and $Z$. Intuitively, $\omega^{X \otimes Z}$ is the supremum vector of the set

$$\mathcal{S} = \{p(X|\rho) \perp q(Z|\rho) : \rho \in \mathcal{D}(\mathcal{H})\}.$$

We now show how to compute $\omega^{X \otimes Z}$ analytically. From the definitions of $p$ and $q$, we can see that only the eigenstates of $X$ and $Z$ matter. We define a $d \times d$ unitary operator $U$ whose elements are given by $U_{ij} = \langle x_i | z_j \rangle$. $U$ is known as the overlapping matrix as it characterizes the overlap of the two orthonormal bases. For each $k \in [d]$, let $\text{SUB}(U, k)$ be the set of submatrices of class $k$ of $U$ defined as

$$\text{SUB}(U, k) = \left\{ M : M \text{ is a submatrix of } U \text{ satisfying } \sharp \text{col}(M) + \sharp \text{row}(M) = k + 1 \right\}.$$

The symbols $\sharp \text{col}(M)$ and $\sharp \text{row}(M)$ denote the number of columns and rows of matrix $M$, respectively. Based on the concept of submatrices, we define the following set of coefficients, which is important in computing $\omega^{X \otimes Z}$:

$$s_k = \max \left\{ \| M \|_\infty : M \in \text{SUB}(U, k) \right\},$$

where $\| M \|_\infty$ is the operator norm of $M$, and the maximum is optimized over all submatrices of class $k$. By construction we have $c_1 = s_1 \leq \cdots \leq s_d = 1$. In [10] it is proved that $\omega^{X \otimes Z} = \{1\} \oplus s$, where $s$ is given by

$$s = (s_1, s_2 - s_1, \ldots, s_d - s_{d-1}, 0, \cdots, 0).$$

We append $d - 10$s to make $\omega^{X \otimes Z}$ a $2d$-dimensional vector. We remark that the vector $s$ is not necessarily sorted in descending order, but we can use the “flattening” process described in Sec. II A to make it descending ordered. In words, the direct-sum majorization uncertainty relation can be summarized in the following theorem.

**Theorem 1** [10]. Let $X$ and $Z$ be two rank-one projective observables on $\mathcal{H}$ whose corresponding overlapping matrix is $U$. For any state $\rho \in \mathcal{D}(\mathcal{H})$, it holds that $p(X|\rho) \perp q(Z|\rho) \prec \omega^{X \otimes Z} = \{1\} \oplus s$.

**III. ENTANGLEMENT DETECTION**

An entanglement detector decides whether a given bipartite state is separable by providing a condition that
is satisfied by all separable states, and if violated, witnesses entanglement. In this section, we design a detection method based on the direct-sum majorization bound described in Sec. [11]. As majorization relations, our detector actually provides a set of conditions whose numbers will grow with the dimension of the state. We first describe a majorization bound for all separable states. Then we show how this bound serves as a detector. In the end, we illustrate by some examples how well the detector works.

A. Majorization bounds

If an observable \( X \) is degenerate, the definition of \( p(X|\rho) \), given in Eq. (2), is not unique, since the spectral decomposition is not unique. By combining eigenstates with the same eigenvalue, however, there exists a unique spectral decomposition of the form \( \rho = \sum \lambda_i P_i \), with \( \lambda_i \neq \lambda_{i'} \) for \( i \neq i' \) and \( P_i \) are orthogonal projectors of maximal rank. Under this convention, we define for degenerate observable \( X \) the distribution \( p_i = \text{Tr} \{ P_i \rho \} \). Our entanglement detection method relies on the degeneracy properties of the product observables on bipartite systems. It is possible that for two non-commuting observables \( X_A \) and \( X_B \), their product \( X_A \otimes X_B \) is degenerate. Consequently, it may happen that \( X_A \otimes X_B \) and \( Z_A \otimes Z_B \) have a common eigenstate, and this eigenstate is an entangled pure state. In such cases, the probabilities \( p(X_A \otimes X_B|\rho) \) and \( p(Z_A \otimes Z_B|\rho) \) will reflect the stated difference and may be capable of detecting entanglement. As an example, consider the Pauli Z operator \( \sigma_z \) on system \( A \) and \( B \). The product observable on \( A \otimes B \) is given by \( \sigma_z \otimes \sigma_z \). The spectral decomposition of \( \sigma_z \otimes \sigma_z \) is (under our convention)

\[
\sigma_z \otimes \sigma_z = (|00\rangle \langle 00| + |11\rangle \langle 11|) - (|01\rangle \langle 01| + |10\rangle \langle 10|),
\]

Similarly, we have \( \sigma_x \otimes \sigma_x = P_+ - P_- \), where \( P_+ = |++\rangle \langle ++| + |--\rangle \langle --| \) and \( P_- = |+-\rangle \langle +| - |+-\rangle \langle -| \). There exists no state \( \rho_A \) that can result in certain outcomes for both \( \sigma_x \otimes \sigma_x \) and \( \sigma_z \otimes \sigma_z \), because they do not commute. But there do exist an entangled state \( |\psi\rangle \) that can give certain outcomes for both \( \sigma_x \otimes \sigma_x \) and \( \sigma_z \otimes \sigma_z \), as they commute. By the Schmidt decomposition, they can be expressed in the same eigenbases which are possibly entangled.

Let \( X_A \) and \( X_B \) be two full rank observables on \( A \) and \( B \), respectively. Assume their spectral decompositions are given by

\[
X_A = \sum_{i=1}^{d} \alpha_i |x^A_i \rangle \langle x^A_i|, \quad X_B = \sum_{i=1}^{d} \beta_i |x^B_i \rangle \langle x^B_i|. \]

Performing the product observable \( X_A \otimes X_B \) on a bipartite state \( \rho_{AB} \), we obtain a joint distribution

\[
p(i, j) = \langle x^A_i | x^B_j | \rho | x^A_i \rangle \langle x^B_i|.
\]

As \( X_A \otimes X_B \) might be degenerate, some elements \( p(i, j) \) are grouped together since they belong to the same eigenvalue. We denote by \( p(X_A \otimes X_B|\rho) \) the joint distribution after grouping. If we perform local observables, we obtain marginal distributions \( p(X_A|\rho_A) \) and \( p(X_B|\rho_B) \). It is proved in [22] that the joint distribution of a product state is majorized by the distribution of its marginal.

Lemma 2 (22), Lemma 1. Let \( \rho = \rho_A \otimes \rho_B \) be a product state and let \( X_A \) and \( X_B \) be two observables on \( A \) and \( B \), respectively. Then

\[
p(X_A \otimes X_B|\rho) \prec p(X_A|\rho_A), \quad p(X_A \otimes X_B|\rho) \prec p(X_B|\rho_B).
\]

Intuitively, this is because for the product observable \( X_A \otimes X_B \), its eigenstates are possibly entangled, and thus product state gives uncertain outcomes, however it is possible that the reduced state gives certain outcome for the corresponding local observable.

Now we consider the effect of several product observables. Let \( X_A \) and \( Z_A \) be two observables on \( A \), \( X_B \) and \( Z_B \) be two observables on \( B \), respectively. For arbitrary product state \( \rho = \rho_A \otimes \rho_B \), we obtain from Lemma 2 that

\[
p(X_A \otimes X_B|\rho) \prec p(Z_A|\rho_A), \quad p(Z_A \otimes Z_B|\rho) \prec p(Z_A|\rho_A).
\]

As the direct-sum operation preserves the majorization order [23], we have

\[
p(X_A \otimes X_B|\rho_A \otimes \rho_B) \otimes p(Z_A \otimes Z_B|\rho_A \otimes \rho_B) \prec p(X_A|\rho_A) \otimes p(Z_A|\rho_A).
\]

The RHS of Eq. (1) is the direct-sum of two distributions. By the virtue of Thm. 1 it holds that

\[
\forall \rho_A \in \mathcal{D}(H_A), \quad p(X_A|\rho_A) \otimes p(Z_A|\rho_A) \prec \omega^{X_A \otimes Z_A}.
\]

Combining Eq. (1) and Eq. (7), we reach the following statement for arbitrary product states \( \rho_A \otimes \rho_B \), one has

\[
p(X_A \otimes X_B|\rho_A \otimes \rho_B) \otimes p(Z_A \otimes Z_B|\rho_A \otimes \rho_B) \prec \omega^{X_A \otimes Z_A}.
\]

The majorization relation derived in Eq. (5) holds for product states. Now we show that this relation actually holds for arbitrary separable states. We are actually interested in the optimal state that majorizes all possible probability distributions \( p(X_A \otimes X_B|\rho) \otimes p(Z_A \otimes Z_B|\rho) \) induced by performing \( X_A \otimes X_B \) and \( Z_A \otimes Z_B \) on separable states. Such a state can be defined as

\[
\omega^{X_A \otimes X_B} \otimes (Z_A \otimes Z_B) := \sup \left\{ p(X_A \otimes X_B|\rho) \right. \}
\]

\[
\left. \oplus p(Z_A \otimes Z_B|\rho) : \rho \in \text{SEP}(H_A;H_B) \right\},
\]

where \( \text{SEP}(H_A;H_B) \) is the set of separable states of bipartite space \( H_A \otimes H_B \). In Appx. 4 we prove that \( \omega^{X_A \otimes X_B} \otimes (Z_A \otimes Z_B) \) can be achieved among pure product
states, and thus we reduce the optimization over all separable states required in Eq. (10) to the optimization over all pure product states:
\[
\omega_{\text{SEP}}^{(X_A X_B) \otimes (Z_A Z_B)} = \sup \left\{ p(X_A \otimes X_B | \phi) \right\} 
+ p(Z_A \otimes Z_B | \phi) : \phi = |\phi_A\rangle \otimes |\phi_B\rangle \langle \phi_B| \right\}.
\]

For an arbitrary separable state (be it pure or not) \(\rho_{AB}\), it then holds that
\[
p(X_A \otimes X_B | \rho) + p(Z_A \otimes Z_B | \rho) < \omega_{\text{SEP}}^{(X_A X_B) \otimes (Z_A Z_B)} < \omega_{\text{SEP}}^{X_A \otimes Z_A}.
\]

The first inequality follows from the definition of \(\omega_{\text{SEP}}\), while the second inequality follows from the fact that each element of \(\omega_{\text{SEP}}\) is achieved by some pure product state, and which in turn be majorized by \(\omega_{X_A \otimes Z_A}\) as proved in Eq. (11). To summarize, we have the following theorem.

**Theorem 3.** Let \(X_A \otimes X_B\) and \(Z_A \otimes Z_B\) be two product observables. For arbitrary separable state \(\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)\), it holds that
\[
p(X_A \otimes X_B | \rho) + p(Z_A \otimes Z_B | \rho) < \omega_{X_A \otimes Z_A},
\]
where \(\omega_{X_A \otimes Z_A}\) is defined in Thm. 1. Similarly, one has
\[
p(X_A \otimes X_B | \rho) + p(Z_A \otimes Z_B | \rho) < \omega_{X_B \otimes Z_B}.
\]

**B. The detection framework**

Thm. 3 states that \(\omega_{X_A \otimes Z_A}\) is a necessary condition for separability and its violation signals the existence of entanglement. This statement provides an operational method of entanglement detection. Given a bipartite state \(\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)\), we first calculate the direct-sum probability distribution \(p(X_A \otimes X_B | \rho) \otimes p(Z_A \otimes Z_B | \rho)\) induced by the product observables \(X_A \otimes X_B\) and \(Z_A \otimes Z_B\). Then we investigate the majorization relation between it and \(\omega_{X_A \otimes Z_A}\). If \(\omega_{X_A \otimes Z_A}\) does not majorize the direct-sum distribution, then we conclude that \(\rho\) is entangled. However, if \(\omega_{X_A \otimes Z_A}\) majorizes the distribution, we can say nothing about \(\rho\): it can be separable, it can also be entangled.

The proposed method is a collection of linear detectors. Indeed, Thm. 3 states the following fact. For arbitrary \(k \in [2d]\), one has
\[
\frac{k}{\sum_{i=1}^{k} \left\{ p(X_A \otimes X_B | \rho) \right\} ^i} \leq \sum_{i=1}^{k} \left\{ \omega_{X_B \otimes Z_B} \right\} ^i.
\]

As the first and the last \(d\) inequalities are trivial, we have \(d - 1\) effective inequalities in total, thus \(d - 1\) linear detectors. States that violate any of these inequalities will necessarily be entangled.

**IV. ENTANGLEMENT DETECTION VIA MANY OBSERVABLES**

The entanglement detection method described in Sec. II makes use of two incompatible observables on each part. In this section, we generalize this method to the case of many incompatible observables.

Tensor-product majorization based uncertainty relations for many observables was first studied in [24]. Here we show their results can be extended to the direct-sum majorization based uncertainty relations. Let \(\rho\) be a quantum state and \((X(l))\}_{l \in [L]}\) be a set of \(N\) observables on \(\mathcal{H}\), where \([L] = \{1, \ldots, L\}\). Assume the spectral decomposition of \((X(l))\) is given by
\[
X(l) = \sum_{i=1}^{d} \alpha_i(x_i^{(l)})^i(x_i^{(l)}),
\]
where \(\{|x_i^{(l)}\}\) are the eigenstates of \((X(l))\). By measuring \(\rho\), \((X(l))\) induces a probability distribution given by
\[
p\left(X(l) | \rho\right) = (p_1, \ldots, p_d), \quad p_i = |\langle x_i | \rho | x_i \rangle|.
\]

The direct-sum majorization based uncertainty relations for this set of observables has the following form:
\[
\forall \rho \in \mathcal{D}(\mathcal{H}), \quad \sum_{l=1}^{L} p\left(X(l) | \rho\right) < \omega_{l=l}^{X(l)},
\]
where \(\omega\) is a \(Nd\)-dimensional vector independent of \(\rho\) which can be explicitly calculated from observables \((X(l))\).

To compute \(\omega\), we define the following coefficients
\[
s_k = \max_{\sum_{l=1}^{L} S_l = k} \lambda_1\left[U(S_1, \ldots, S_L)\right],
\]
where \(\lambda_1(A)\) denotes the maximal singular value of \(A\), and the terms \(S_l, U(S_1, \ldots, S_L)\) are defined in [23]. The main differences between our definition of \(s_k\) in Eq. (10) and the \(s_k\) defined in Eq. 15 of [24] lie in that

1. In our definition \(S_l \geq 0\); while in their definition, \(S_l\) is strictly positive.

2. In our definition the optimization is over all \(\{S_l\}\) such that \(\sum_{l=1}^{L} S_l = k\); while in their definition, the optimization is over all \(\{S_x\}\) such that \(\sum_{x=1}^{L} S_l = k + L - 1\).

These two differences guarantee that we can use \(s_k\) to give upper bounds on the sum of the first \(d\) terms of \(\omega\). With coefficients \(\{s_k\}\), we can derive a direct-sum majorization bound for many observables.

**Lemma 4.** Let \((X(l))\) be a set of \(L\) observables on \(\mathcal{H}\). For any state \(\rho\) in \(\mathcal{D}(\mathcal{H})\), it holds that
\[
\sum_{l=1}^{L} p\left(X(l) | \rho\right) < \omega_{l=1}^{X(l)} = (s_1, s_2 - s_1, \ldots, L - s_L, 0, \ldots, 0),
\]
where \(s_{a+1}\) is the first component such that \(s_{a+1} = L\).
The proof of Lemma 5 is almost the same as the proof of Theorem 1 in [24], with the modified definition of $s_k$ substituted. Lemma 5 is a generalization of Thm. 4 to the case of many observables. In Sec. III we showed how Thm. 4 is used to construct an entanglement detector. We can also use Lemma 4 to design entanglement detectors, with the help of many observables.

**Lemma 5.** Let $\{X_A^{(l)}\}$ and $\{X_B^{(l)}\}$ be two sets of $N$ observables on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For arbitrary separable state $\rho_{AB}$, it holds that

$$\bigoplus_{l=1}^L p \left( X_A^{(l)} \otimes X_B^{(l)} | \rho \right) \prec \omega^{\oplus_{l=1}^L X_A^{(l)}}.$$

Similarly, one has

$$\bigoplus_{l=1}^L p \left( X_A^{(l)} \otimes X_B^{(l)} | \rho \right) \prec \omega^{\oplus_{l=1}^L X_B^{(l)}}.$$

The proof of Lemma 5 is similar to that of Thm. 4. Lemma 5 provides an operational method of entanglement detection, using many observables. Given a bipartite state $\rho_{AB}$, we first calculate the probability distributions $p \left( X_A^{(l)} \otimes X_B^{(l)} | \rho \right)$ induced by the product observables $X_A^{(l)} \otimes X_B^{(l)}$. This can be done by sampling from the source multiple times and gather the statistics. Then we investigate the majorization relation between it and $\omega$. If $\omega$ does not majorize the direct-sum distribution, then we conclude that $\rho_{AB}$ is entangled.

**V. CONCLUSIONS**

In this paper, we have studied the relationship between direct-sum majorization formulation of uncertainty relations and entanglement, for the case of two and many observables. We have designed entanglement detection methods based on such a formulation. Our nonlinear detectors are inherently stronger than similar scalar conditions as they are equivalent to and imply infinite classes of such scalar criteria. Our measurement-based entanglement detection methods are of practical importance, as they are experimental friendly and relatively easy to implement. We hope the results presented here can stimulate further investigations on the relations among uncertainty relations, majorization, and entanglement.

**Acknowledgments.** This work is supported by the National Natural Science Foundation of China (Grant No. 61300050) and the Chinese National Natural Science Foundation of Innovation Team (Grant No. 61321491).

**Appendix A: Bounds are found on pure product states**

Our task here is to establish the fact that direct-sum majorization induced bound $\omega_{\text{SEP}}$ (defined in Eq. 4) can be achieved among pure product states. Let $\mu_l$ be the $l$-th component of $\omega_{\text{SEP}}$. Assume w.l.o.g. that $\mu_l$ is achieved by the separable state $\tilde{\rho} = \sum_k \lambda_k |\phi_k^A\rangle \langle \phi_k^B| \otimes |\phi_k^B\rangle \langle \phi_k^B|$, where $\{|\phi_k^A\rangle\}$ and $\{|\phi_k^B\rangle\}$ are orthonormal bases of $A$ and $B$, respectively. Denote by $I$ ($J$) be subsets of distinct index pairs from $[d] \times [d]$, and by $|I|$ ($|J|$) the size (number of elements) of $I$ ($J$). We assume the two probability sequences achieving $\mu_l$ are given by $I$ and $J$ satisfying $|I| + |J| = l$. That is,

$$\mu_l = \sum_{(i,j)\in I} p(i,j) + \sum_{(m,n)\in J} q(m,n),$$

where $p$ and $q$ are the joint distributions given by product observable $X_A \otimes X_B$ and $Z_A \otimes Z_B$, respectively. From the linearity of the trace function, we have

$$p(i,j) = \langle x_i^A x_j^B | \tilde{\rho} | x_i^A x_j^B \rangle = \sum_k \lambda_k |\langle x_i^A x_j^B | \phi_k^A \phi_k^B|\rangle|^2,$$

$$q(m,n) = \langle z_m^A z_n^B | \tilde{\rho} | z_m^A z_n^B \rangle = \sum_k \lambda_k |\langle z_m^A z_n^B | \phi_k^A \phi_k^B|\rangle|^2.$$

Thus

$$\mu_l = \sum_{(i,j)\in I} p(i,j) + \sum_{(m,n)\in J} q(m,n) = \sum_k \lambda_k \left( \sum_{(i,j)\in I} |\langle x_i^A x_j^B | \phi_k^A \phi_k^B|\rangle|^2 + \sum_{(m,n)\in J} |\langle z_m^A z_n^B | \phi_k^A \phi_k^B|\rangle|^2 \right).$$

That is to say, if $\tilde{\rho}$ achieves $\mu_l$, then $\tilde{\rho}$ must be a pure product state, otherwise we can find a pure state which gives larger $\mu_l$ by simply choosing the eigenstate of $\tilde{\rho}$ with the largest eigenvalue.
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