On the two-dimensional Coulomb-like potential with a central point interaction

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Abstract
In the first part of the paper, we introduce the Hamiltonian \( -\Delta - \frac{Z}{\sqrt{x^2 + y^2}} \), \( Z > 0 \), as a self-adjoint operator in \( L^2(\mathbb{R}^2) \). A general central point interaction combined with the two-dimensional Coulomb-like potential is constructed and the properties of the resulting one-parameter family of Hamiltonians are studied in detail. The construction is also reformulated in the momentum representation and a relation between the coordinate and the momentum representation is derived. In the second part of the paper, we prove that the two-dimensional Coulomb-like Hamiltonian can be derived as a norm resolvent limit of the Hamiltonian of a Hydrogen atom in a planar slab as the width of the slab tends to zero.

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1. Introduction

In this paper we discuss, in the framework of nonrelativistic quantum mechanics, two subjects related to the two-dimensional Coulomb-like potential in the plane. In the first part, section 2, we re-examine a two-dimensional hydrogen atom. This is to say that we consider a quantum model in the plane with the attractive potential

\[ V(x, y) = -\frac{Z}{\varrho}, \quad \varrho = \sqrt{x^2 + y^2}, \tag{1} \]

which we call the two-dimensional Coulomb (or hydrogenic) potential. This model has already been studied from various points of view in the physical literature. A detailed analysis of this system is given in [21], see also [16]. The corresponding Green’s function is constructed in [13], though with some minor misprints. More mathematically oriented questions, such as the
proper definition of the Hamiltonian as a self-adjoint operator, are briefly discussed in the recent paper [10]. Note that some interest to this type of models also comes from semiconductor physics. In a semiconductor quantum well under illumination, excited electrons and holes are essentially confined to the plane and interact via a mutual Coulomb interaction which results in the creation of electron–hole bound states, known as excitons [16]. Moreover, as shown in the second part of the paper, section 3, the two-dimensional hydrogenic Hamiltonian can be viewed as an approximation of the Hamiltonian of a hydrogen atom in a thin planar layer.

Let us point out, however, that if a hydrogen atom is supposed to be two-dimensional in the strict sense, i.e. all fields including electromagnetic fields, the angular momentum and the spin are confined to the plane, then (1) is no longer eligible to as the two-dimensional Coulomb potential. Indeed, the Coulomb law may be derived from the first Maxwell equation (Gauss’s law for electrostatics) stating that \( \text{div} \mathbf{E} = \sigma \) where \( \sigma \) stands for the planar charge density, \( E_z = 0 \), and the electric field is supposed to be rotationally symmetric. Integration of this equation over a disk of radius \( \varrho \) together with the application of Green’s theorem leads to the choice of the potential in the form

\[
V(x, y) = \text{const} \ln(\varrho).
\]

The Schrödinger equation for this potential is studied in [4].

One of the goals of the present paper is to describe a central point interaction combined with the two-dimensional Coulomb-like potential and to study its basic properties. The construction of point interactions based on the theory of self-adjoint extensions is now pretty well established. To the best of our knowledge, however, the two-dimensional Coulomb-like potential is not yet discussed in the literature, including the well-known monograph [2] where only the one- and three-dimensional cases are considered. On the other hand, there exists a general theoretical background for the construction of self-adjoint extensions with singular boundary conditions, as described in [9], which is directly applicable to our model.

Along with the construction of point interactions in the coordinate representation, and this is the standard way how to proceed, we discuss the construction also in the momentum representation. Moreover, we derive an explicit relation between the two representations. This correspondence is based on the Whittaker integral transformation whose integral kernel is a properly normalized generalized eigenfunction of the two-dimensional Coulomb-like Hamiltonian depending on the spectral parameter. Remarkably, this integral transformation has been studied in the mathematical literature quite recently [3, 14]. On this point we refer, first of all, to [11] where the unitarity of the eigenfunction expansion is proven for a much more general class of Schrödinger operators on a half-line.

In the second part of the paper, in section 3, we study the hydrogen atom in a thin planar layer of width \( a \), called \( \Omega_a \). In our model, we confine the atom to the slab by imposing the Dirichlet boundary condition on the parallel boundary planes. Our main goal in this section is to show that the resulting Hamiltonian in \( L^2(\Omega_a) \), called \( H^a \), is well approximated in a convenient way by the two-dimensional Coulomb-like Hamiltonian as the width of the layer approaches zero. The method we use is strongly motivated by the paper of Brummelhuis and Duclos [7]. Firstly, we apply the projection on the first transversal mode getting this way the so-called effective Hamiltonian in \( L^2(\mathbb{R}^2) \). Then, in subsection 3.3, we show that the norm resolvent limit of the effective Hamiltonian, as \( a \to 0^+ \), exactly equals the two-dimensional hydrogenic Hamiltonian plus the energy of the lowest transversal mode. As a next step we prove, in subsection 3.4, that the full Hamiltonian \( H^a \) is well approximated by the effective Hamiltonian, again in the norm resolvent sense. Since the spectrum of \( H^a \) is known explicitly one can use this approximation to derive, with the help of standard perturbation methods, asymptotic formulas for the eigenvalues of \( H^a \) though we do not go into details at this point.

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2. Two-dimensional Coulomb-like potential with a central point interaction

2.1. The coordinate representation

Let \(-\Delta\) be the free Hamiltonian in \(L^2(\mathbb{R}^2, dx \, dy)\). It is known that the Coulomb-like potential \((x^2 + y^2)^{-1/2}\) in the plane is \((-\Delta)\) form bounded with relative bound being zero. This is a consequence of the Kato inequality; the proof is given in [6] but see also [12], where even a more general case is treated. In more detail, the inequality claims that
\[
\frac{1}{\sqrt{x^2 + y^2}} \leq \frac{\Gamma(1/4)}{4\pi^2} \sqrt{-\Delta}.
\] (2)

Suppose \(Z > 0\). By the KLMN theorem [17, theorem X.17], the operator
\[
H_C = -\Delta - \frac{Z}{\sqrt{x^2 + y^2}} \quad \text{(the form sum)}
\] (3)
is self-adjoint. The form domain of the Hamiltonian \(H_C\) coincides with that of the free Hamiltonian, i.e. with the first Sobolev space. In particular, \(\text{Dom}(H_C) \subset \mathcal{H}^1(\mathbb{R}^2)\). Note that the same conclusions can also be deduced from the results in [19, chapter XIII.11]. The operator \(H_C\) has been studied quite intensively in the physical literature (see, for example, [13, 16, 21]).

To introduce a central point interaction let us consider the densely defined symmetric operator
\[
\hat{H} = -\Delta - \frac{Z}{\sqrt{x^2 + y^2}}, \quad \text{Dom}(\hat{H}) = C_0^\infty(\mathbb{R}^2 \setminus \{0\}).
\]
Denote by \(H_{\text{min}}\) the closure of \(\hat{H}\). Then \(H_C\) is exactly the Friedrichs extension of \(H_{\text{min}}\). As usual, passing to polar coordinates \((\varrho, \varphi)\) the Hilbert space naturally decomposes into the polar coordinates \((\varrho, \varphi)\),
\[
L^2(\mathbb{R}^2, dx \, dy) = \bigoplus_{m=-\infty}^{\infty} L^2(\mathbb{R}^+, \varrho \, d\varrho) \otimes \mathbb{C} e^{im\varphi},
\]
and the operator \(H_{\text{min}}\) decomposes correspondingly,
\[
H_{\text{min}} = \bigoplus_{m=-\infty}^{\infty} H_{\text{min},m} \otimes 1,
\]
where \(H_{\text{min},m}\) is the closure of the operator
\[
\hat{H}_m = -\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \frac{m^2}{\varrho^2} - \frac{Z}{\varrho}, \quad \text{Dom}(\hat{H}_m) = C_0^\infty(\mathbb{R}^+).
\]

Also put \(H_{\text{max},m} = H_{\text{min},m}'\). For the maximal operator one has [20, chapter 8]
\[
\text{Dom}(H_{\text{max},m}) = \{ f \in L^2(\mathbb{R}^+, \varrho \, d\varrho); f, f' \in AC_{\text{loc}}(\mathbb{R}^+), L_m f \in L^2(\mathbb{R}^+, \varrho d\varrho) \},
\]
with
\[
L_m = -\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \frac{m^2}{\varrho^2} - \frac{Z}{\varrho}.
\]

If \(f \in \text{Dom}(H_{\text{max},m})\), then \(H_{\text{max},m} f = L_m f\).

For \(z \in \mathbb{C} \setminus \mathbb{R}\) and \(m \in \mathbb{Z}\) consider the equation \((L_m - z) f = 0\). Two independent solutions are expressible in terms of the Whittaker functions, namely
\[
\varrho^{-1/2} M_{Z/(2\sqrt{-z}),m}(2\sqrt{-z}\varrho) \quad \text{and} \quad \varrho^{-1/2} W_{Z/(2\sqrt{-z}),m}(2\sqrt{-z}\varrho).
\] (4)
with \( \text{Re} \sqrt{-z} > 0 \). From the asymptotic expansions it follows (see, for instance, [1]) that the former function in (4) is square integrable at zero but is not square integrable at infinity while the latter one is square integrable at infinity but is not square integrable at zero, except for the case \( m = 0 \). Thus for \( m \neq 0 \), the operators \( H_{\min,m} = H_{\max,m} = H_m \) are self-adjoint while for \( m = 0 \), \( H_{\min,0} \) has deficiency indices \((1, 1)\). For a wide class of Schrödinger operators, including our case as well, an explicit construction of all self-adjoint extensions defined by boundary conditions can be found in [9].

**Proposition 1.** All self-adjoint extensions of \( H_{\min,0} \) are \( H_0(\kappa) \subseteq H_{\max,0} \). \( \kappa \in \mathbb{R} \cup \{ \infty \} \), with the domains

\[
\text{Dom}(H_0(\kappa)) = \{ f \in \text{Dom}(H_{\max,0}); f_1 = \kappa f_0 \},
\]

where the boundary values \( f_0, f_1 \) are defined by

\[
f_0 = \lim_{\varrho \to 0+} (-\ln \varrho)^{-1} f(\varrho), \quad f_1 = \lim_{\varrho \to 0+} (f(\varrho) + f_0 \ln \varrho). \tag{5}
\]

The self-adjoint extension \( H_0(\infty) \) determined by the boundary condition \( f_0 = 0 \) coincides with the Friedrichs extension of \( H_{\min,0} \).

All self-adjoint extensions \( H(\kappa) \) of \( H_{\min} \) are again labeled by \( \kappa \in \mathbb{R} \cup \{ \infty \} \) and are equal to

\[
H(\kappa) = \bigoplus_{m=-\infty}^{-1} H_m \oplus H_0(\kappa) \oplus \bigoplus_{m=1}^{\infty} H_m.
\]

In particular, \( H(\infty) \) coincides with \( H_C \).

**Proposition 2.** For the essential spectrum one has \( \sigma_{\text{ess}}(H_C) = [0, \infty) \) and, more generally, \( \sigma_{\text{ess}}(H(\kappa)) = [0, \infty) \) for all \( \kappa \in \mathbb{R} \cup \{ \infty \} \).

**Proof.** Let us introduce (temporarily) the functions

\[
U_1(x, y) = -\frac{Z}{\sqrt{x^2 + y^2 + 1}}, \quad U(x, y) = -\frac{Z}{\sqrt{x^2 + y^2}} + \frac{Z}{\sqrt{x^2 + y^2 + 1}}.
\]

and denote by \( U_1 \) and \( U \) the corresponding multiplication operators. Put \( A = -\Delta + U_1 \). Since \( U_1(x, y) \) is bounded and tends to zero at infinity one knows that \( \sigma_{\text{ess}}(A) = [0, \infty) \) (see, for instance, [5, theorem 4.1]). Note that, by the closed graph theorem, \((A + k)^{-1/2}(-\Delta + 1)^{-1/2}\) is bounded for \( k > 0 \) sufficiently large. Moreover, \( U \) is a relatively form-bounded perturbation of \( A \) and \( H_C \) equals the form sum \( A + U \). It is shown below, in the proof of lemma 5, that the operator \((-\Delta + 1)^{-1/2}U(-\Delta + 1)^{-1/2}\) is Hilbert–Schmidt and hence compact. Consequently, \( U \) is a relatively form-compact perturbation of \( A \) and, by the results in [19, chapter XIII.4] related to the Weyl theorem, \( \sigma_{\text{ess}}(H_C) = \sigma_{\text{ess}}(A) \). To extend the equality to all \( H(\kappa) \) it suffices to observe that, by the Krein formula, the resolvent of \( H(\kappa) \) is a rank-1 perturbation of the resolvent of \( H(\infty) \).

By the general theory of Sturm–Liouville operators, the resolvent kernels \( \mathcal{G}_m(z; \varrho, \varrho') \) of the partial Hamiltonians \( H_0(\infty) \), if \( m = 0 \) and \( H_m = H_{\min,m} = H_{\max,m} \) if \( m \neq 0 \), are equal to

\[
\mathcal{G}_m(z; \varrho, \varrho') = \frac{1}{2(2|m|)1/2\sqrt{-z} \sqrt{\varrho \varrho'}} \Gamma \left( \frac{1}{2} + |m| - \frac{Z}{2\sqrt{-z}} \right) \times M_{Z/2, \sqrt{\varrho} \varrho, |m|} (2\sqrt{-z})^{|m|} W_{Z/2, \sqrt{\varrho} \varrho, |m|} (2\sqrt{-z} \varrho').
\]

Here \( \varrho_<, \varrho_> \) denote the smaller and the greater out of \( \varrho, \varrho' \), respectively.
Green’s function $G_0^\kappa(z; \varrho, \varrho')$ for the Hamiltonian $H_0(\kappa)$, $\kappa \in \mathbb{R}$, can be constructed using the Krein resolvent formula that guarantees the existence of a function $\phi^\kappa(z)$ such that

$$G_0^\kappa(z; \varrho, \varrho') = G_0(z; \varrho, \varrho') + \frac{\phi^\kappa(z)}{\sqrt{\varrho'}} W_{Z/2\sqrt{-z\varrho}}(2\sqrt{-z\varrho}) W_{Z/2\sqrt{-z\varrho}}(2\sqrt{-z\varrho'}).$$

with $z \in \mathbb{C}\setminus\mathbb{R}$. Since the integral kernel must satisfy the same boundary condition as that defining $H_0(\kappa)$ we have

$$\phi^\kappa(z) = \frac{1}{2\sqrt{-z}} \Gamma\left(1 - \frac{Z}{2\sqrt{-z}}\right)^2 \left(2\nu + \ln(2\sqrt{-z}) + \Psi\left(1 - \frac{Z}{2\sqrt{-z}}\right) + \kappa\right)^{-1}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is the polygamma function and $\gamma = -\Gamma'(1)$ is the Euler constant.

Green’s function of $H(\kappa)$, $\kappa \in \mathbb{R} \cup \{\infty\}$, expressed in polar coordinates, equals

$$G^\kappa(z; \varrho, \varphi, \varrho', \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} G_m(z; \varrho, \varrho') e^{im(\phi-\phi')}$$

$$+ \frac{\phi^\kappa(z)}{2\pi \sqrt{\varrho'}} W_{Z/2\sqrt{\varrho}}(2\sqrt{-z\varrho}) W_{Z/2\sqrt{\varrho}}(2\sqrt{-z\varrho'}).$$

The point spectrum of $H_C$ equals the union of the point spectra of $H_m$, $m \in \mathbb{Z}$ (with $H_0 \equiv H_0(\infty)$). The eigenvalues of $H_C$ jointly with eigenfunctions are computed in [21] and correspond to the poles of the respective Green’s functions. Thus we recall that all eigenvalues of $H_m$, $m \in \mathbb{Z}$, are simple and are equal to

$$\lambda_{m,n} = -\frac{Z^2}{m^2 + 2n + 1}, \quad n \in \mathbb{Z}_+,$$

(here $\mathbb{Z}_+ = \{n \in \mathbb{Z}; n \geq 0\}$). Denote by $N = |m| + n + 1$ the principal quantum number and put $\lambda_N = \lambda_{m,n}$ for $|m| + n = N - 1$, i.e. $\lambda_N = -Z^2/(2N-1)^2$, $N \in \mathbb{N}$. Then the multiplicity of $\lambda_N$ in the spectrum of $H_C$ is $2N-1$. The corresponding normalized eigenfunctions are

$$\psi_{m,n}(\varrho, \varphi) = \left(\frac{n!}{(2\pi(n+2|m|)!}\right)^{1/2} \frac{2Z}{(2|m| + 2n + 1)^{3/2}} \frac{2Z\varrho}{(2|m| + 2n + 1)} e^{-Z\varrho/(2|m|+2n+1)} e^{im\varphi},$$

where $L_n^{(2|m|)}$ stands for the associated Laguerre polynomial.

Using a similar reasoning as in [8] one concludes that the point spectrum of $H(\kappa)$, $\kappa \in \mathbb{R}$, contains the eigenvalues $\lambda_N$ with multiplicities $2(N-1)$ (hence $\lambda_1$ is missing). In fact, the point spectrum of $H_0$ is simple and is formed by the eigenvalues $\lambda_N$, $N \geq |m| + 1$. If a point interaction is switched on, then the spectrum of the component $H_0$ is deformed while the point spectra of the components $H_m$, $m \neq 0$, remain untouched. On the other hand, if $\kappa \in \mathbb{R}$, then additional eigenvalues emerge in the spectrum of $H(\kappa)$, the so-called point levels. They are simple and negative. Let us denote them in ascending order by $\epsilon_j(Z; \kappa)$, $j = \mathbb{Z}_+$.

From the general theory concerned with Friedrichs extensions [17, theorem X.23] and location of discrete spectra of self-adjoint extensions [20, chapter 8.3] one deduces that the points levels are located as follows:

$$\epsilon_0(Z; \kappa) < \lambda_1 < \epsilon_1(Z; \kappa) < \lambda_2 < \epsilon_2(Z; \kappa) < \lambda_3 < \cdots < 0.$$

Using the substitution

$$\epsilon_j(Z; \kappa) = -Z^2 k_j(\kappa)^2, \quad k_0 = \kappa + \ln Z,$$
one finds from (6) and (7) that the equation on point levels takes the form
\[ 2\gamma + \ln(2k) + \Psi\left(\frac{1}{2} - \frac{1}{2k}\right) + \kappa_0 = 0, \] (10)
with the unknown \( k = k_j(\kappa) > 0 \). This implies the scaling property
\[ \epsilon_j(Z; \kappa) = Z\epsilon_j(1; \kappa + \ln Z), \quad j = 0, 1, 2, \ldots. \]
By an elementary analysis of equation (10) one can show that the functions \( \epsilon_j(Z; \kappa) \) are strictly increasing in the parameter \( \kappa \in \mathbb{R} \), and one has the asymptotic formulas
\[ \epsilon_j(Z; \kappa) = -\frac{Z^2}{(2j+1)^2} - \frac{4Z^2}{(2j+1)^3\kappa} + O(\kappa^{-2}) \quad \text{as} \quad \kappa \to +\infty, \]
for all \( j \geq 0 \), and
\[ \epsilon_0(Z; \kappa) = -4e^{-2\gamma - 2\kappa} + O(e^{-\kappa}) \quad \text{as} \quad \kappa \to -\infty, \]
\[ \epsilon_j(Z; \kappa) = -\frac{Z^2}{(2j-1)^2} - \frac{4Z^2}{(2j-1)^3\kappa} + O(\kappa^{-2}) \quad \text{as} \quad \kappa \to -\infty, \quad j \geq 1. \]
Figure 1 depicts several first-point levels as functions of \( \kappa \) for \( Z = 1 \).

Finally, note that from the form of Green’s function one can also derive normalized eigenfunctions corresponding to the point levels, namely
\[ \eta_j(\kappa; \varrho, \varphi) = \sqrt{\frac{Z}{2\pi \varrho}} k_j(\kappa) \left( \frac{k_j(\kappa)}{2} + \frac{1}{2} \Psi'\left(\frac{1}{2} - \frac{1}{2k_j(\kappa)}\right)\right)^{-1/2} \Gamma\left(\frac{1}{2} - \frac{1}{2k_j(\kappa)}\right) \]
\[ \times W_{1/(2k_j(\kappa))} \left( 2k_j(\kappa)Z\varrho \right), \]
where \( k_j(\kappa) = (-\epsilon_j(Z; \kappa))^{1/2}/Z, \quad j = 0, 1, 2, \ldots. \)

2.2. The momentum representation

The normalized generalized eigenfunctions for \( H_m, \ m \in \mathbb{Z} \) (with \( H_0 \equiv H_0(\infty) \)), are known including the correct normalization [21]. One has, with \( k > 0, \)
\[ \psi_m(k, \varrho) = \frac{1}{(2|m|)!} \left( \frac{2}{1 + e^{-\pi Z/k}} \right)^{1/2|m|-1} \prod_{s=0}^{1/2|m|-1} \left( s + \frac{1}{2} \right) \left( \frac{Z^2}{4k^2} \right)^{1/2} \frac{i^m}{\sqrt{2ik\varrho}} M_{1/(2k), |m|}(2ik\varrho). \]
In a comparatively recent paper [11] a large class of Schrödinger operators on a half-line with strongly singular potentials is studied, with the results being directly applicable to the operators \( H_m, m \in \mathbb{Z} \). In that paper, a measure on the dual space is constructed with the help of the associated Titchmarsh–Weyl \( m \)-function, and unitarity of the eigenfunction expansion, involving both proper and generalized eigenfunctions is proven (one can also consult paper [14] which appeared later and covers a less general class of potentials but with our example still being included). As a consequence one deduces that the integral transform

\[
\mathcal{F}_m : L^2(\mathbb{R}_+, \varnothing, d\varnothing) \to L^2(\mathbb{R}_+, k \, d\varnothing), \quad \mathcal{F}_m[f](k) = \int_0^\infty \psi_m(k, \varnothing) f(\varnothing) \varnothing \, d\varnothing
\]

is a well-defined bounded operator. Denote by \( \mathcal{H}_{m,pp} \) the closure of the subspace in \( L^2(\mathbb{R}_+, \varnothing, d\varnothing) \) spanned by the eigenfunctions \( \psi_{m,n}(\varnothing), n \in \mathbb{Z}_+ \), and by \( \mathcal{H}_{m,ac} \) its orthogonal complement. Then the kernel of \( \mathcal{F}_m \) equals \( \mathcal{H}_{m,pp} \), and the restriction

\[
\mathcal{F}_{m,ac} := \mathcal{F}_m|_{\mathcal{H}_{m,ac}} : \mathcal{H}_{m,ac} \to L^2(\mathbb{R}_+, k \, d\varnothing)
\]

is a unitary mapping. Moreover, \( \mathcal{F}_{m,ac} \) transforms \( H_m|_{\mathcal{H}_{m,ac}} \) into the multiplication operator by the function \( k^2 \) acting in \( L^2(\mathbb{R}_+, k \, d\varnothing) \). It follows that the essential spectrum of \( H_m \) is in fact absolutely continuous. The same is also true for all \( H_0(k), k \in \mathbb{R} \).

**Proposition 3.** For all \( m \in \mathbb{Z} \) one has \( \sigma_{ess}(H_m) = \sigma_{ac}(H_m) = [0, \infty) \) and \( \sigma_{pp}(H_m) = \emptyset \) (with the only accumulation point being just 0). Similarly, for all \( \kappa \in \mathbb{R} \) one has \( \sigma_{ess}(H_0(\kappa)) = \sigma_{ac}(H_0(\kappa)) = [0, \infty) \) and \( \sigma_{pp}(H_0(\kappa)) = [\kappa j(\kappa); j \in \mathbb{Z}_+] \). Moreover, the spectra of \( H_m, m \in \mathbb{Z} \) and \( H_0(\kappa), \kappa \in \mathbb{R} \), are simple. In particular, the singular continuous spectra of \( H_m \) and \( H_0(\kappa) \) are empty.

The transformation inverse to \( \mathcal{F}_{m,ac} \) is

\[
\mathcal{F}_{m,ac}^{-1} : L^2(\mathbb{R}_+, k \, d\varnothing) \to \mathcal{H}_{m,ac}, \quad \mathcal{F}_{m,ac}^{-1}[g](\varnothing) = \int_0^\infty \psi_m(k, \varnothing) g(k) \, d\varnothing.
\]

Thus, one concludes that \( H_m \) in \( L^2(\mathbb{R}_+, \varnothing, d\varnothing) \) is unitarily equivalent to \( \hat{H}_m \) in

\[
\hat{\mathcal{H}}_m = \ell^2(\mathbb{Z}_+) \oplus L^2(\mathbb{R}_+, k \, d\varnothing).
\]

\( \text{Dom}(\hat{H}_m) \) is formed by those \( \hat{f} = \{\hat{f}_n\}_{n=0}^\infty + \hat{f}(k) \in \hat{\mathcal{H}}_m \) for which \( k^2 \hat{f}(k) \in L^2(\mathbb{R}_+, k \, d\varnothing) \). If \( \hat{f} \in \text{Dom}(\hat{H}_m) \), then

\[
\hat{H}_m \hat{f} = \{\lambda_{m,n} \hat{f}_n\}_{n=0}^\infty + k^2 \hat{f}(k).
\]

The unitary mapping \( \hat{\mathcal{H}}_m \to L^2(\mathbb{R}_+, \varnothing, d\varnothing) : \hat{f} \mapsto f \) is given by

\[
f(\varnothing) = \sum_{n=0}^\infty \hat{f}_n \psi_{m,n}(\varnothing) + \int_0^\infty \psi_m(k, \varnothing) \hat{f}(k) \, d\varnothing.
\]

Conversely, \( \hat{f}_n = (\psi_{m,n}, f) \), \( \hat{f}(k) = \mathcal{F}_m[f](k) \). One has

\[
\|\hat{f}\|^2 = \sum_{n=0}^\infty |\hat{f}_n|^2 + \int_0^\infty |\hat{f}(k)|^2 k \, d\varnothing = \int_0^\infty |f(\varnothing)|^2 \varnothing \, d\varnothing = \|f\|^2.
\]

One can use the momentum representation for an alternative and equivalent construction of point interactions. It again turns out that a nontrivial result can be derived only in the sector \( m = 0 \) to which we confine our attention. A symmetric restriction \( A \) of \( \hat{H}_0 \) is obtained by requiring that \( f(0) = 0 \) if \( \hat{f} \in \text{Dom} A \subset \text{Dom} \hat{H}_0 \). More details follow. From now on we omit, in the notation, the hat over elements \( f \in \hat{\mathcal{H}}_0 \).
Let us denote the normalization factor of generalized eigenfunctions as

\[ N(k) = \left( \frac{2}{1 + e^{-z/k}} \right)^{1/2}, \quad k > 0. \]

For \( g \in \mathcal{H}_0 \) such that \( g(k) \in L^1(\mathbb{R}_+, k \, dk) \) put

\[ S(g) = \sum_{n=0}^{\infty} \frac{2Z}{(2n+1)^{3/2}} \xi_n + \int_0^\infty N(k)g(k)k \, dk. \]

For \( \xi \in \mathbb{C} \) and \( f \in \mathcal{H}_0 \) such that \( f(k) - \xi N(k)/(k^2 + Z^2) \in L^1(\mathbb{R}_+, k \, dk) \) put

\[ S(\xi, f) = \sum_{n=0}^{\infty} \frac{2Z}{(2n+1)^{3/2}} f_n + \int_0^\infty N(k) \left( f(k) - \frac{\xi N(k)}{k^2 + Z^2} \right) k \, dk. \]

Clearly, if \( \xi \) exists, then it is unambiguously determined by \( f \), and \( S(\xi, f) = S(0, g) \). Observe that \( \forall g \in \text{Dom}(\mathcal{H}_0), \ g(k) \in L^1(\mathbb{R}_+, k \, dk) \), and one has \( \tilde{g}(0) = S(g) \) where

\[ \tilde{g}(0) = \sum_{n=0}^{\infty} g_n \psi_{0,n}(\xi) + \int_0^\infty \psi_0(k, \xi) g(k)k \, dk. \tag{11} \]

One defines \( A \subset \mathcal{H}_0 \) by

\[ \text{Dom}(A) = \{g \in \text{Dom}(\mathcal{H}_0); \ S(g) = 0\}. \]

It is not difficult to check that \( f \in \text{Dom}(A^*) \) if and only if there exist (necessarily unique) \( \xi \in \mathbb{C} \) and \( \eta \in \mathcal{H}_0 \) such that for all \( n \in \mathbb{Z}_+ \), and almost all \( k > 0 \),

\[ \lambda_{0,n} f_n = \eta_n + \frac{2Z \xi}{(2n+1)^{3/2}}, \quad k^2 f(k) = \eta(k) + \xi N(k). \]

In that case, \( A^* f = \eta \). Note that if \( f \in \text{Dom}(A^*) \) and \( \xi, \eta \) are as above, then

\[ f(k) - \frac{\xi N(k)}{k^2 + Z^2} = \frac{Z^2 f(k)}{k^2 + Z^2} + \frac{\eta(k)}{k^2 + Z^2} \in L^1(\mathbb{R}_+, k \, dk). \]

Let us now discuss self-adjoint extensions of \( A \). The deficiency indices of \( A \) are \((1, 1)\). For \( z \in \mathbb{C} \setminus \mathbb{R}, \ \ker(A^* - z) = \mathbb{C} f_z \) where

\[ \forall n \in \mathbb{Z}_+, \quad (f_z)_n = \frac{2Z}{(2n+1)^{3/2}(\lambda_{0,n} - z)}; \quad \forall k > 0, \quad f_z(k) = \frac{N(k)}{k^2 - z}. \tag{12} \]

For the computational convenience the spectral parameter is chosen to be \( z = iZ^2/2 \). For \( e^{iz} \in \mathbb{T}^1 \) let \( A_u \) be the self-adjoint extension of \( A \) defined by

\[ \text{Dom}(A_u) = \text{Dom}(A) + \mathbb{C}(f_z + e^{iz} f_z), \]

\[ A_u(g + t(f_z + e^{iz} f_z)) = Ag + t(z f_z + z e^{iz} f_z). \]

If

\[ f = g + t(f_z + e^{iz} f_z) \in \text{Dom}(A_u), \quad \text{with} \quad g \in \text{Dom}(A), \quad t \in \mathbb{C}, \tag{13} \]

then there exists \( \xi \in \mathbb{C} \) (necessarily unique) such that

\[ f(k) - \frac{\xi N(k)}{k^2 + Z^2} \in L^1(\mathbb{R}_+, k \, dk), \]

namely \( \xi = t(1 + e^{iz}) \). Furthermore, one has

\[ S(\xi, f) = t(S(1, f_z) + e^{iz} S(1, f_z)). \]
A straightforward computation gives

$$S(1, f_c) = - \operatorname{Re} \left( \psi \left( \frac{1}{2} \right) \right) - \gamma - \frac{7}{2} \ln(2) + i \frac{\pi}{4} + i \frac{\pi}{2} \coth \left( \frac{\pi}{2} \right).$$

The computation is based on the following identities: for $a \not\in 2\mathbb{Z} + 1$,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+1-a)} = \frac{1}{2a} \left( \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1-a}{2} \right) \right),$$

and, for $a \not\in (-\infty, 0]$,

$$\int_{0}^{\infty} \frac{y}{(1 + e^{\pi y})(y^2 + a)} \, dy = -\frac{1}{4} \ln(4a) + \psi(\sqrt{a}) - \frac{1}{2} \psi \left( \frac{\sqrt{a}}{2} \right).$$

Note that $S(1, f_c) = S(1, f_0)$. Put

$$\lambda = \frac{1}{1 + e^{\pi i}} (S(1, f_c) + e^{\pi i} S(1, f_0))$$

$$= -\operatorname{Re} \left( \psi \left( \frac{1}{2} \right) \right) - \gamma - \frac{7}{2} \ln(2) + \left( 1 + \frac{\pi}{4} + \frac{\pi}{2} \coth \left( \frac{\pi}{2} \right) \right) \tan \left( \frac{\alpha}{2} \right).$$

Still assuming (13) one has $S(\xi, f) = \xi \kappa$. Let us redenote $\alpha = \hat{H}_0(\kappa)$.

One concludes that the one-parameter family of all self-adjoint extensions of $A$ is $\hat{H}_0(\kappa)$, $\kappa \in \mathbb{R} \cup \{ \infty \}$. A vector $f \in \mathcal{H}_0$ belongs to $\text{Dom}(\hat{H}_0(\kappa))$ iff there exists $\xi \in \mathbb{C}$ (necessarily unique) such that

$$f(k) - \frac{\xi N(k)}{k^2 + Z^2} \in L^1(\mathbb{R}_+, k \, dk), \quad k^2 f(k) - \frac{1}{\kappa} S(\xi, f) N(k) \in L^2(\mathbb{R}_+, k \, dk).$$

Then

$$\hat{H}_0(\kappa) f = \left\{ \lambda_{0,n} f_n - \frac{2Z\xi}{(2n+1)^{3/2}} \right\}_{n=0}^{\infty} + \left( k^2 f(k) - \frac{1}{\kappa} S(\xi, f) N(k) \right).$$

In addition one has $S(\xi, f)/\kappa = \xi$. Clearly, $\hat{H}_0(\infty) = \hat{H}_0$.

Let us check the point spectrum of $\hat{H}_0(\kappa)$, $\kappa \in \mathbb{R}$. Suppose $0 \neq f \in \mathcal{H}_0$ and $\lambda \in \mathbb{R}$ fulfill $\hat{H}_0(\kappa) f = \lambda f$. This means that

$$\lambda_{0,n} f_n - \frac{2Z\xi}{(2n+1)^{3/2}} = \lambda f_n \quad \text{for} \quad n \in \mathbb{Z}_+, \quad k^2 f(k) - \frac{\xi}{\kappa} N(k) = \lambda f(k) \quad \text{for} \quad k > 0.$$  

Clearly, $\lambda$ must be negative since otherwise $f(k) = N(k)/(k^2 - \lambda)$ would not be $L^2$ integrable.

Furthermore, the point spectrum of $\hat{H}_0(\kappa)$ is disjoint with the point spectrum of $\hat{H}_0$. In fact, suppose $\lambda = \lambda_{0,p}$ for some $p \in \mathbb{Z}_+$. Then from the first equation in (17), with $n = p$, it follows that $\xi = 0$. Moreover, (17) implies $f_n = 0$ for $n \neq p$, and $f(k) = 0$ for $k > 0$.

Necessarily, $f_p \neq 0$. Then

$$S(\xi, f) = 2Z(2p+1)^{-3/2} f_p \neq 0$$

and the second condition in (16) is not satisfied, which is a contradiction.

Suppose $\lambda < 0$ and $\lambda \neq \lambda_{0,n}, \forall n$, is an eigenvalue. Then there exists one independent eigenvector $f$ corresponding to $\lambda$ for which one can put $\xi = 1$:

$$f_n = \frac{2Z}{(2n+1)^{3/2}(\lambda_{0,n} - \lambda)} \quad \text{for} \quad n \in \mathbb{Z}_+, \quad f(k) = \frac{N(k)}{k^2 - \lambda} \quad \text{for} \quad k > 0.$$  

The eigenvalue equation reads $S(1, f) = \kappa$, with $f$ given in (18), i.e.

$$\sum_{n=0}^{\infty} \frac{4Z^2}{(2n+1)^{3/2}(\lambda_{0,n} - \lambda)} + \int_{0}^{\infty} \frac{2}{1 + e^{-Z/k}} \left( \frac{1}{k^2 - \lambda} - \frac{1}{k^2 + Z^2} \right) k \, dk = \kappa.$$
One can get rid of the parameter $Z$ using the substitution $\lambda = -Z^2/x^2$. With the help of (14) and (15) one finds that $\lambda = -Z^2/x^2$ is an eigenvalue iff $x$ solves the equation

$$\pi \tan \left( \frac{\pi}{2} x \right) + \ln(x) - \Psi \left( \frac{1+x}{2} \right) - \gamma - 4 \ln(2) = \hat{k}. \quad (19)$$

2.3. A relation between the two representations

We wish to compare the operators $\hat{H}_0(\hat{\kappa})$ and $H_0(\kappa)$. The domain of the latter Hamiltonian in the coordinate representation is given by a boundary condition at the origin. So we have to determine the asymptotic behavior of $\hat{g}(\rho)$ as $\rho \to 0$ for an arbitrary $g \in \text{Dom } \hat{H}_0(\hat{\kappa})$, with $\hat{g}$ being given in (11).

As a first step we find a relation between the basis function $f_z$ of the deficiency subspace given in (12) and $\hat{f}_z(\rho)$, a basis function of the deficiency subspace in the coordinate representation. To simplify the notation let us temporarily set $Z = 1$. We put

$$\hat{f}_z(\rho) = \frac{1}{\sqrt{\rho}} W_{1/2, \sqrt{-z}, 0}(2\sqrt{-z}\rho), \quad z \in \mathbb{C}\setminus[0, +\infty).$$

This can be rewritten in terms of confluent hypergeometric functions:

$$\hat{f}_z(\rho) = \sqrt{2(-z)^{1/4}} e^{-\sqrt{-z}\rho} U\left( \frac{1}{2}, \frac{1}{2\sqrt{-z}}, 1, 2\sqrt{-z}\rho \right).$$

One knows that

$$\sum_{n=0}^{\infty} |f_n|^2 + \int_0^{\infty} |f_z(k)|^2 k \, dk = C(z) |\hat{f}_z(\rho)|^2 \rho \, d\rho.$$

By unitarity

$$\sum_{n=0}^{\infty} |(f_n)|^2 + \int_0^{\infty} |f_z(k)|^2 k \, dk = |C(z)|^2 \int_0^{\infty} |\hat{f}_z(\rho)|^2 \rho \, d\rho.$$

Suppose $z < -1$. In that case

$$\int_0^{\infty} \hat{f}_z(\rho)^2 \rho \, d\rho = \frac{2\sqrt{-z} + \Psi\left( \frac{1}{2} - \frac{1}{2\sqrt{-z}} \right)}{2(-z) \Gamma\left( \frac{1}{2} - \frac{1}{2\sqrt{-z}} \right)}.$$

Furthermore,

$$\sum_{n=0}^{\infty} |f_n|^2 = \frac{1}{4(-z)^{3/2}} \left( \Psi\left( \frac{1}{2} - \frac{1}{2\sqrt{-z}} \right) - \Psi\left( \frac{1}{2} + \frac{1}{2\sqrt{-z}} \right) \right).$$

Using the identity

$$\int_0^{\infty} \frac{1}{\cosh(\pi x/2)^2(x^2 + a^2)} \, dx = \frac{1}{\pi a} \Psi\left( \frac{1+a}{2} \right)$$

one finds that

$$\int_0^{\infty} f_z(k)^2 k \, dk = -\frac{1}{2z} + \frac{1}{4(-z)^{3/2}} \Psi\left( \frac{1}{2} + \frac{1}{2\sqrt{-z}} \right).$$
Finally one arrives at the equality
\[
\sum_{n=0}^{\infty} (f \ast n) \psi_{0,n}(q) + \int_{0}^{\infty} f(k) \psi_{0}(k, q) \frac{N(k)}{k^2 + Z^2} k \, dk = \frac{1}{(-z)^{1/4}} \Gamma \left( \frac{1}{2} \right) \frac{1}{\sqrt{2\rho}} \exp \left( \sqrt{2\rho} \right) .
\] (20)

Using a simple scaling one can return back to a general parameter \( Z > 0 \). Considering the limit \( z \to -1 \) in (20) one derives the asymptotic formula
\[
\int_{0}^{\infty} \psi_{0}(k, q) \frac{N(k)}{k^2 + Z^2} k \, dk = -\ln(Z \rho) - \gamma + 3 \ln(2) + O(\ln(q)) \quad \text{as} \quad \rho \to 0 .
\]
Suppose \( f \in \text{Dom}(\hat{H}_0(\kappa)) \). Then (\( \xi \in \mathbb{C} \) is introduced in the definition of \( \text{Dom}(\hat{H}_0(\kappa)) \))
\[
\hat{f}(\rho) = \sum_{n=0}^{\infty} f_n \psi_{0,n}(\rho) + \int_{0}^{\infty} \psi_{0}(k, \rho) \left( f(k) - \frac{\xi N(k)}{k^2 + Z^2} \right) k \, dk + \xi \int_{0}^{\infty} \psi_{0}(k, \rho) \frac{N(k)}{k^2 + Z^2} k \, dk .
\]
Hence,
\[
\hat{f}(\rho) = S(\xi, f) + \xi(-\ln(Z \rho) - \gamma + 3 \ln(2)) + o(1) \quad \text{as} \quad \rho \to 0 .
\]
Recall that \( S(\xi, f) = \xi \kappa \). One concludes that
\[
\forall f \in \text{Dom}(\hat{H}_0(\kappa)), \quad \hat{f}(\rho) = \xi(-\ln(Z \rho) + \kappa - \gamma + 3 \ln(2)) + o(1) \quad \text{as} \quad \rho \to 0 .
\] (21)

Since the domain of \( H_0(\kappa) \) is determined by the asymptotic behavior at \( \rho = 0 \),
\[
\hat{f}(\rho) = -\kappa_0 \ln(\rho) + \alpha_1 + o(1) \quad \text{as} \quad \rho \to 0 , \quad \text{where} \quad \kappa_1 = \kappa \kappa_0 ;
\] (22)
one finds, by comparing (21) and (22), that the operators \( H_0(\kappa) \) and \( \hat{H}_0(\kappa) \) are unitarily equivalent if
\[
\kappa = \kappa - \ln(Z) - \gamma + 3 \ln(2) .
\]

3. A hydrogen atom in a thin layer

3.1. Notation

We wish to discuss a model describing a hydrogen atom or a hydrogen-like ion confined to an infinite planar slab \( \Omega_a \) of width \( a \). Thus, we denote
\[
\Omega_a = \mathbb{R}^2 \times \left( -\frac{a}{2}, \frac{a}{2} \right) \subset \mathbb{R}^3 .
\]
Our goal is to consider the limit when the width \( a \) tends to zero. Let us first introduce the notation and recall some related results.

For \( \Omega \subset \mathbb{R}^n \), an nonempty open set with a Lipschitz continuous boundary on each component, denote by \( \mathcal{H}^m(\Omega) \) the \( m \)th Sobolev space and by \( \mathcal{H}^m_0(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in \( \mathcal{H}^m(\Omega) \). One has a natural isometric embedding \( \mathcal{H}^m_0(\Omega) \subset \mathcal{H}^m(\mathbb{R}^n) \). Furthermore, \( \mathcal{D}^{1,2}(\mathbb{R}^n) \) denotes the completion of \( C^\infty_0(\mathbb{R}^n) \) with respect to the norm
\[
\| u \|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{1/2} .
\]
In this case one has a continuous embedding \( \mathcal{H}^1(\mathbb{R}^n) \subset \mathcal{D}^{1,2}(\mathbb{R}^n) \). Also recall that the Dirichlet Laplacian \( -\Delta_D \) is the unique self-adjoint operator associated with the closed positive form
with the effective potential is defined by $\int \langle \nabla q(f, g) \rangle_{\Omega}$. The form representation theorem implies that $\Delta_D = \mathcal{H}^1_0(\Omega)$.

Below we employ the Hardy inequality in $\mathbb{R}^3$ which states that for any $u \in D^{1,2}(\mathbb{R}^3)$

$$\frac{1}{4} \int_{\mathbb{R}^3} |u(x)|^2 \frac{dx}{|x|^2} \leq \int_{\mathbb{R}^3} |\nabla u(x)|^2 \, dx.$$  \hspace{1cm} (23)

The Hardy inequality is extended to domains with boundaries in [22] where one can find additional references. In the case of the Dirichlet boundary condition, however, one can simply make use of the chain of embeddings $\mathcal{H}^1_0(\Omega) \subset \mathcal{H}^1(\mathbb{R}^3) \subset D^{1,2}(\mathbb{R}^3)$. Hence, inequality (23) holds for any $u \in \mathcal{H}^1_0(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is still supposed to have the above stated properties.

In our model we introduce the Hamiltonian

$$H^a = -\Delta_D - \frac{Z}{r},$$ \hspace{1cm} (24)

with $r = \sqrt{x^2 + y^2 + z^2}$ and $Z > 0$, in the Hilbert space $L^2(\Omega_a)$. To see that a self-adjoint operator is well defined by relation (24) it suffices to show that the potential $\frac{1}{r}$ is bounded with a relative bound less than 1 (or even 0) and to refer to the Kato–Rellich theorem.

In fact, by the Hardy inequality (23), the estimate

$$\|r^{-1}\psi\|^2 \leq 4\|\nabla \psi\|^2 = 4\langle \psi, -\Delta_D \psi \rangle \leq 2\left(\epsilon^2 + \Delta_D \psi\right) \leq 2\left(\epsilon^2 + \Delta_D \psi\right)$$

holds for all $\psi \in \Delta_D$ and $\epsilon > 0$. Thus, one has

$$\text{Dom } H^a = \text{Dom}(-\Delta_D) = \mathcal{H}^1_0(\Omega_a) \cap \mathcal{H}^2(\Omega_a), \hspace{1cm} Q(H^a) = \mathcal{H}^1_0(\Omega_a)$$

(here $Q(A)$ stands for the form domain of $A$).

Using the scaling $x \rightarrow Zx$ one can readily see that $H^2_0$ (the Hamiltonian $H^a$ for a given constant $Z$) is unitarily equivalent to $Z^2 H^2_{0,1}$. This is why we can set, without loss of generality, $Z = 1$, and this is what we do in the remainder of the paper.

3.2. The effective Hamiltonian

The operator $-\Delta_D$ can be decomposed with respect to the basis in $L^2((-a/2, a/2), dz)$ formed by the transversal modes,

$$-\Delta_D = \bigoplus_{n=1}^{\infty} \left(-\Delta_{x,y} + E_n^a\right) \otimes \langle \chi_n^a, \cdot \rangle \chi_n^a,$$

with

$$E_n^a = \frac{n^2 \pi^2}{a^2}, \hspace{1cm} \chi_n^a(z) = \sqrt{\frac{1}{n}} \left\{ \begin{array}{ll}
\cos(n \pi z/a) & \text{if } n \text{ is odd} \\
\sin(n \pi z/a) & \text{if } n \text{ is even}.
\end{array} \right.$$  \hspace{1cm} n \in \mathbb{N}_0

Here $-\Delta_{x,y}$ is the free Hamiltonian in $L^2(\mathbb{R}^2)$. Put

$$P_n^a = 1 \otimes \langle \chi_n^a, \cdot \rangle \chi_n^a, \hspace{1cm} n \in \mathbb{N}.$$  \hspace{1cm}

Using the projection on the lowest transversal mode we define the effective Hamiltonian

$$H_n^a = P_n^a H^a P_n^a.$$  \hspace{1cm}

This Hamiltonian may be regarded as an operator on $L^2(\mathbb{R}^2)$,

$$H_n^a = -\Delta_{x,y} + E_1^a - V_n^a(q).$$  \hspace{1cm} (25)

where the effective potential is defined by

$$V_n^a(q) = 2 \int_{-a/2}^{a/2} \frac{\cos^2(\pi z/a)}{\sqrt{q^2 + z^2}} \, dz.$$ \hspace{1cm} (26)
and $\varrho = \sqrt{x^2 + y^2}$. Note that $0 < V^d_{\text{eff}}(\varrho) < 1/\varrho$ for all $\varrho > 0$. Hence, if the Coulomb-like potential is $(-\Delta)$ form bounded with relative bound zero than the same is true for $V^d_{\text{eff}}$. Thus, the rhs in (25) makes sense as a form sum and $Q(H^d_{\text{eff}}) = \mathcal{H}^1(\mathbb{R}^2)$. Moreover,
\[ -1 + E^d_1 \leq H_C + E^d_1 \leq n_{\text{eff}}, \tag{27} \]
with $H_C$ being defined in (3) (also denoted by $H(\infty)$ in the previous section).

It is even true that $V^a_{\text{eff}}$ is $(-\Delta)$ bounded with relative bound zero. In fact, recall that for any $\alpha > 0$ there is $\beta$ such that
\[ \forall f \in \mathcal{H}^2(\mathbb{R}^2), \quad \|f\| \leq \alpha \|\Delta f\| + \beta \|f\|, \tag{28} \]
with $\|\cdot\|$ being the $L^2$ norm, see [17, theorem IX.28]. Moreover, one observes that
\[ V^a_{\text{eff}}(\varrho) = -\frac{4}{a} \ln(\varrho) + O(1) \quad \text{as} \quad \varrho \to 0^+, \]
and $V^a_{\text{eff}}(\varrho)$ decays like $1/\varrho$ at infinity. Hence $V^a_{\text{eff}}$, regarded as a function on $\mathbb{R}^2$, is square integrable at the origin and tends to zero at infinity. It follows that for every $\epsilon > 0$ there exists a decomposition
\[ V^a_{\text{eff}} = V_0 + V_1, \quad \text{with} \quad V_0 \in L^\infty(\mathbb{R}^2), \quad V_1 \in L^2(\mathbb{R}^2), \tag{29} \]
such that $\|V_0\|_\infty < \epsilon$. Combining (28) and (29) one finds that, for all $f \in \mathcal{H}^2(\mathbb{R}^2)$,
\[ \|V^a_{\text{eff}} f\| \leq \|V_0\|_\infty \|f\| + \|V_1\| \|f\| \leq \alpha \|V_0\| \|\Delta f\| + (\beta \|V_1\| + \|V_0\|) \|f\|. \]
This shows the relative boundedness and thus one can apply the Kato–Rellich theorem. In particular, $\text{Dom} H^a_{\text{eff}}$ coincides with $\text{Dom}(-\Delta) = \mathcal{H}^2(\mathbb{R}^2)$. Moreover, the existence of decomposition (29) implies that $\sigma_{\text{ess}}(H^a_{\text{eff}}) = [E^a_1, \infty)$, see [19, theorem XIII.15].

3.3. The limit of the effective Hamiltonian for small $a$

Here we show that the Hamiltonian $H^a_{\text{eff}} - E^a_1$ converges to the two-dimensional hydrogenic Hamiltonian $H_C$ in the norm resolvent sense as $a \to 0^+$.

**Lemma 4.** One has $\|(-\Delta + 2)^{1/2}(H_C + 2)^{-1/2}\| \leq C_1$ where
\[ C_1 = \frac{1}{8\pi^2} \left( \Gamma \left( \frac{1}{4} \right)^4 + \sqrt{\Gamma \left( \frac{1}{4} \right)^8 + 64\pi^4} \right). \tag{30} \]

**Proof.** Put $L = (-\Delta + 2)^{1/2}(H_C + 2)^{-1/2}$. Then $L$ is bounded by the closed graph theorem but one can derive an upper bound explicitly with the help of the Kato inequality (2). Since
\[ \langle \psi, (-\Delta + 2)^{-1/4} \varrho^{-1} (-\Delta + 2)^{-1/4} \psi \rangle \leq \frac{\Gamma \left( \frac{1}{2} \right)^4}{4\pi^2} \||(-\Delta)^{1/4}(-\Delta + 2)^{-1/4}\psi\|^2 \leq \frac{\Gamma \left( \frac{1}{2} \right)^4}{4\pi^2} \|\psi\|^2 \]
one has
\[ L^1 L = 1 + (H_C + 2)^{-1/2} \frac{1}{\varrho} (H_C + 2)^{-1/2} \leq 1 + \frac{\Gamma \left( \frac{1}{2} \right)^4}{4\pi^2} (H_C + 2)^{-1/2}(-\Delta + 2)^{1/2}(H_C + 2)^{-1/2}. \]
It follows that
\[ \|L\psi\|^2 \leq \|\psi\|^2 + \frac{\Gamma \left( \frac{1}{2} \right)^4}{4\pi^2} \|(H_C + 2)^{-1/2}\|\|\psi\|\|L\psi\|. \]
For $\|(H_C + 2)^{-1/2}\| \leq 1$ we get
\[ \|L\|^2 \leq 1 + \frac{\Gamma \left( \frac{1}{2} \right)^4}{4\pi^2} \|L\|. \]
Consequently, $\|L\| \leq C_1$. \hfill $\square$
Lemma 5. Suppose $W \in L^1(\mathbb{R}_+, d\varrho)$ and $0 \leq W \leq 1$. Put
\begin{equation}
V^a(\varrho) = \frac{1}{\varrho} \left( 1 - W \left( \frac{\varrho}{a} \right) \right), \quad a > 0.
\end{equation}
Then for any $a$, $0 < a < 1/2$, one has
\begin{align*}
&\|(-\Delta + 2)^{-1/2} (\varrho^{-1} - V^a) (-\Delta + 2)^{-1/2}\|^2 \\
&\leq 12a^2 \ln^2(a) \left( \int_{\mathbb{R}_+} W(\varrho) \, d\varrho \right)^2 + 32a^2 \int_{\mathbb{R}_+} W(\varrho) \, d\varrho.
\end{align*}

Proof. Put
\begin{equation}
T_a = (\varrho^{-1} - V^a)^{1/2} (-\Delta + 1)^{-1/2}.
\end{equation}
Then
\begin{equation}
T_a^\dagger T_a = (-\Delta + 1)^{-1/2} (\varrho^{-1} - V^a) (-\Delta + 1)^{-1/2}
\end{equation}
and $\|T_a^\dagger T_a\| = \|T_a T_a^\dagger\|$. Let us estimate the Hilbert–Schmidt norm of $T_a T_a^\dagger$. The integral kernel of $T_a T_a^\dagger$ is
\begin{equation}
K(x_1, x_2) = \frac{1}{2\pi} \sqrt{\frac{1}{\varrho_1}} W \left( \frac{\varrho_1}{a} \right) K_0(|x_1 - x_2|) \sqrt{\frac{1}{\varrho_2}} W \left( \frac{\varrho_2}{a} \right),
\end{equation}
where $x_i = \varrho_i (\cos \varphi_i, \sin \varphi_i)$. Since the modified Bessel function $K_0$ is positive and strictly decreasing on $\mathbb{R}_+$, we get
\begin{equation}
\|T_a T_a^\dagger\|_{HS}^2 \leq I (\mathbb{R}_+ \times \mathbb{R}_+)
\end{equation}
where the symbol $I(M)$, $M \subset \mathbb{R}_+ \times \mathbb{R}_+$ measurable, is defined by
\begin{equation}
I(M) = \int_M W \left( \frac{\varrho_1}{a} \right) K_0(|\varrho_1 - \varrho_2|)^2 W \left( \frac{\varrho_2}{a} \right) \, d\varrho_1 \, d\varrho_2.
\end{equation}
For $0 < a < 1/2$ and $|\varrho_1 - \varrho_2| > a$ one has [1]
\begin{align*}
K_0(|\varrho_1 - \varrho_2|) &< K_0(a) < \left( \ln \left( \frac{2}{a} \right) - \gamma \right) I_0(a) + \frac{1}{2} I_0(\sqrt{2}a) - \frac{1}{2} \\
&< \left( \ln \left( \frac{2}{a} \right) - \gamma \right) I_0 \left( \frac{1}{2} \right) + \frac{1}{2} I_0 \left( \frac{1}{\sqrt{2}} \right) - \frac{1}{2} < -2 \ln(a).
\end{align*}
Consequently,
\begin{equation}
I(|\varrho_1 - \varrho_2| > a) \leq 4a^2 \ln^2(a) \left( \int_{\mathbb{R}_+} W(\varrho) \, d\varrho \right)^2.
\end{equation}
If $|\varrho_1 - \varrho_2| < a < 1/2$, then $K_0(a|\varrho_1 - \varrho_2|) < -2 \ln|\varrho_1 - \varrho_2|$. Moreover, for $0 \leq W \leq 1$,
\begin{equation}
\int_{|\varrho_1 - \varrho_2| < 1} W(\varrho_1) \ln^2(|\varrho_1 - \varrho_2|) \, d\varrho_1 \leq \int_{|v| < 1} \ln^2|v| \, dv = 4.
\end{equation}
It follows that
\begin{align*}
I(|\varrho_1 - \varrho_2| < a) &\leq 4 \int_{|\varrho_1 - \varrho_2| < a} W \left( \frac{\varrho_1}{a} \right) \ln^2(|\varrho_1 - \varrho_2|) W \left( \frac{\varrho_2}{a} \right) \, d\varrho_1 \, d\varrho_2 \\
&\leq 8a^2 \int_{|\varrho_1 - \varrho_2| < 1} W(\varrho_1)(\ln^2(a) + \ln^2|\varrho_1 - \varrho_2|) W(\varrho_2) \, d\varrho_1 \, d\varrho_2 \\
&\leq 8a^2 \ln^2(a) \left( \int_{\mathbb{R}_+} W(\varrho) \, d\varrho \right)^2 + 32a^2 \int_{\mathbb{R}_+} W(\varrho) \, d\varrho.
\end{align*}
We conclude that
\[ \| T_{\alpha} T_{\alpha}' \|_{\text{HS}}^2 \leq 12 a^2 \ln^2(a) \left( \int_{\mathbb{R}_+} W(\varrho) \, d\varrho \right)^2 + 32 a^2 \int_{\mathbb{R}_+} W(\varrho) \, d\varrho. \]

By the functional calculus \( \|(-\Delta + 2)^{-1/2}(-\Delta + 1)^{1/2}\| = 1 \), and this completes the proof. 

\[ \Box \]

Lemma 6. Suppose \( W \in L^1(\mathbb{R}_+, d\varrho) \) and \( W(\varrho) \geq 0 \). Let \( V^\alpha(\varrho) \) be defined as in (31). Then
\[ \|(-\Delta + 2)^{-1/2}(\varrho^{-1} - V^\alpha)(\Delta + 2)^{-1/2}\| \geq \frac{1}{2} \left( \int_0^R W(\varrho) \, d\varrho \right)^2 a \ln\left( \frac{1}{aR} \right) \]
whenever \( R > 1 \) and \( a > 0 \).

Proof. We again use definition (33). Chose \( R > 1 \) and truncate \( \tilde{W}(\varrho) = W(\varrho)\vartheta(R - \varrho) \) where \( \vartheta(x) \) is the Heaviside step function (the characteristic function of the positive half-line). If \( f \in L^2(\mathbb{R}^2, dx) \), \( f \neq 0 \), then \( \| T_{\alpha} T_{\alpha}' \| \geq \| f \| \). We choose
\[ f(\varrho) = \left[ \frac{1}{|\varrho|} \tilde{W} \left( \frac{|\varrho|}{a} \right) \right]^{1/2}. \]

Then
\[ \| f \|^2 = 2\pi a \int_0^R W(\varrho) \, d\varrho \]
and
\[ |\langle f, T_{\alpha} T_{\alpha}' f \rangle| = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_0(|\varrho_1 - \varrho_2|) \frac{1}{|\varrho_1|} \tilde{W} \left( \frac{|\varrho_1|}{a} \right) \frac{1}{|\varrho_2|} \tilde{W} \left( \frac{|\varrho_2|}{a} \right) \, d\varrho_1 \, d\varrho_2 \]
\[ = \frac{1}{2\pi} \int_{\mathbb{R}_+ \times S^1} \int_{\mathbb{R}_+ \times S^1} K_0 \left( |\varrho_1^2 + \varrho_2^2 - 2\varrho_1 \varrho_2 \cos(\varphi_1 - \varphi_2)|^{1/2} \right) \]
\[ \times \tilde{W} \left( \frac{\varrho_1}{a} \right) \tilde{W} \left( \frac{\varrho_2}{a} \right) \, d\varrho_1 \, d\varphi_1 \, d\varrho_2 \, d\varphi_2. \]

Recall that, by formula 11.4.44 in [1],
\[ K_0 \left( |\varrho_1^2 + \varrho_2^2 - 2\varrho_1 \varrho_2 \cos \varphi|^{1/2} \right) = \int_0^\infty J_0 \left( |\varrho_1^2 + \varrho_2^2 - 2\varrho_1 \varrho_2 \cos \varphi|^{1/2} \right) \frac{t}{t^2 + 1} \, dt. \]

Integrating Graf’s addition formula for Bessel functions we obtain
\[ \int_0^{2\pi} J_0 \left( |\varrho_1^2 + \varrho_2^2 - 2\varrho_1 \varrho_2 \cos \varphi|^{1/2} \right) \, d\varphi = J_0(\varrho_1 t) J_0(\varrho_2 t). \]

For
\[ \int_0^\infty J_0(\varrho_1 t) J_0(\varrho_2 t) \frac{t}{t^2 + 1} \, dt = I_0(\varrho_1) K_0(\varrho_2) \]
we conclude that
\[ \frac{1}{2\pi} \int_0^{2\pi} K_0 \left( |\varrho_1^2 + \varrho_2^2 - 2\varrho_1 \varrho_2 \cos \varphi|^{1/2} \right) \, d\varphi = I_0(\varrho_1) K_0(\varrho_2). \]

Also recall that \( I_0(\varrho) \geq 1 \) and \( K_0(\varrho) \geq \ln(2/\varrho) - \gamma \geq \ln(1/\varrho) \).
For any $a > 0$ we get
\[
\langle f, T_a T_a^* f \rangle = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} I_0(\varrho) K_0(\varrho) \tilde{W} \left( \frac{\varrho_1}{a} \right) \tilde{W} \left( \frac{\varrho_2}{a} \right) d\varrho_1 d\varrho_2 \\
\geq 4\pi a^2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \ln \left( \frac{1}{a\varrho_1} \right) W(\varrho_1) d\varrho_1 \right) W(\varrho_2) d\varrho_2 \\
\geq 2\pi a^2 |\ln(a)| \left( \int_{\mathbb{R}} W(\varrho) d\varrho \right)^2 - 2\pi a^2 \ln(R) \left( \int_{\mathbb{R}} W(\varrho) d\varrho \right)^2.
\]
Finally note that $\|(-\Delta + 1)^{-1/2}(-\Delta + 2)^{1/2}\| = \sqrt{2}$. The lemma follows. □

Remark 7. Note that the effective potential has the scaling property
\[
V_{\text{eff}}^a(\varrho) = \frac{1}{a} V_{\text{eff}}^1 \left( \frac{\varrho}{a} \right).
\]
Also recall that $0 < V_{\text{eff}}^a(\varrho) < 1/\varrho$. If we put
\[
W(\varrho) = 1 - \varrho V_{\text{eff}}^1(\varrho), \quad (34)
\]
then $0 < W(\varrho) < 1$ and
\[
\frac{1}{\varrho} \left( 1 - W \left( \frac{\varrho}{a} \right) \right) = V_{\text{eff}}^a(\varrho).
\]
Moreover, from (26) one gets
\[
W(\varrho) = 2 \int_{-1/2}^{1/2} \left( 1 - \frac{\varrho}{\sqrt{\varrho^2 + z^2}} \right) \cos^2(\pi z)dz \leq \frac{1}{\varrho} \int_{-1/2}^{1/2} z^2 \cos^2(\pi z)dz.
\]
Hence, $W \in L^2(\mathbb{R}_+, d\varrho)$. Thus, all assumptions of lemmas 5 and 6 are fulfilled. In the course of the proofs of these lemmas we have shown that there exist constants $0 < C_1 < C_2$ such that for all sufficiently small $a > 0$,
\[
C_1 |\ln(a)| < \|(-\Delta + 1)^{-1/2}(\varrho^{-1} - V_{\text{eff}}^a)(-\Delta + 1)^{-1/2}\| < C_2 |\ln(a)|.
\]

Further we need an estimate formulated in the following lemma which is easy to see and is in fact a standard result (see for instance, [18, chapter XI]).

Lemma 8. Assume that $A$ is semibounded, $A^{-1}$ exists and is bounded, $C$ is self-adjoint and $A$ form bounded. If
\[
a = \| |C|^{1/2} |A|^{-1/2} \| < 1,
\]
then $(A + C)^{-1}$ exists, is bounded and
\[
\| (A + C)^{-1} - A^{-1} \| \leq a^2 \| A^{-1} \| / (1 - a^2).
\]

Theorem 9. For every $\xi \in \text{Res}(H_C) \cap \mathbb{R}$ there exists $a_0(\xi) > 0$ such that for all $a$, $0 < a < a_0(\xi)$, one has $\xi \in \text{Res} \left( H_a^a - E_a^a \right)$ and
\[
\| (H_a^a - E_a^a - \xi)^{-1} - (H_C - \xi)^{-1} \| \leq \frac{2C_1 C_\Pi}{d_C(\xi)} \max \left\{ 1, \frac{2}{d_C(\xi)} \right\} a |\ln(a)|,
\]
where $d_C(\xi) = \text{dist}(\xi, \sigma(H_C))$, $C_1$ is given in (30) and
\[
C_\Pi = \frac{\sqrt{3}}{2} \left( 1 - \frac{4}{\pi^2} \right) \sqrt{1 + \frac{32\pi^2}{3(\pi^2 - 4) \ln^2(2)}}.
\]
(35)
Proof. One can apply lemma 8 with $A = H_C - \xi$, $C = q^{-1} - V_{\text{eff}}^a$. Then

$$A + C = H_{\text{eff}}^a - E_1^a - \xi,$$

$$\|A^{-1}\| = 1/dC(\xi),$$

and one has

$$\alpha^2 = \|I(C)\|^{-1} \|A\|^{-1} = \left\|H_C - \xi\right\|^{-1} \left\|q^{-1} - V_{\text{eff}}^a\right\| \left\|H_C - \xi\right\|^{-1} \left\|q^{-1} - V_{\text{eff}}^a\right\|

\leq \|(H_C + 2)^{1/2}H_C - \xi\|^{-1} \|q^{-1} - V_{\text{eff}}^a\|

\times \left\|(-\Delta + 2)^{1/2}(-\Delta + 2)\right\|^{-1} \left\|q^{-1} - V_{\text{eff}}^a\right\|^{-1} \left\|\left(-\Delta + 2\right)^{-1/2}\left(-\Delta + 2\right)^{-1/2}\right\|.

By lemma 4, $\|(-\Delta + 2)^{1/2}(-\Delta + 2)^{-1/2}\| \leq C_1$. Furthermore,

$$\|(H_C + 2)^{1/2}H_C - \xi\|^{-1} \leq \max\left\{1, \frac{2}{dC(\xi)}\right\}.$$

Finally, according to remark 7, for the same $W$ as given in (34) one has $V^a = V_{\text{eff}}^a$. In that case,

$$\int_0^{\infty} W(q) dq = 2 \int_{-1/2}^{1/2} \left[\int_0^{\infty} \left(1 - \frac{q}{\sqrt{q^2 + z^2}}\right) dq\right] \cos^2(\pi z) dz = \frac{1}{4} - \frac{1}{\pi^2}.$$

Suppose $0 < a < 1/2$. Recalling lemma 5 one derives the estimate

$$\|(H_C + 2)^{1/2}(q^{-1} - V^a)(-\Delta + 2)^{1/2}\| \leq C_H a |n a|,$$

with $C_H$ given in (35). Hence,

$$\alpha^2 \leq C_H^2 \max\left\{1, \frac{2}{dC(\xi)}\right\} a |n a|.$$

Now it is clear that for any $\xi \in \text{Res}(H_C) \cap \mathbb{R}$ one can find $a_0(\xi) > 0$ such that for all $a$, $0 < a < a_0(\xi)$, one has $\alpha^2 \leq 1/2$. Then $(H_{\text{eff}}^a - E_1^a - \xi)^{-1}$ exists, is bounded and

$$\|(H_{\text{eff}}^a - E_1^a - \xi)^{-1} - (H_C - \xi)^{-1}\| \leq \frac{\alpha^2}{dC(\xi)(1 - \alpha^2)} \leq \frac{2\alpha^2}{dC(\xi)}.$$

This proves the theorem.

3.4. Approximation of the full Hamiltonian by the effective Hamiltonian

In this section we show that the effective Hamiltonian $H_{\text{eff}}^a$ tends to the full Hamiltonian $H^a$ in the norm resolvent sense as $a \to 0^+$. To this end, we decompose the Hilbert space $L^2(\Omega_a)$ into an orthogonal sum determined by the projection onto the first transversal mode and the projection on all remaining higher order modes.

To simplify notation we write $P^a$ instead of $P_1^a$. In this subsection we denote the Coulomb potential $-1/r$ in the slab as $V$. Put

$$Q^a = 1 - P^a, \quad H_0^a = Q^a H^a Q^a, \quad R_1^a(\xi) = (H_{\text{eff}}^a - \xi)^{-1}.$$

Furthermore, we denote

$$\omega^a(\xi) = P^a V Q^a R_1^a(\xi) Q^a V P^a, \quad R_{\omega}^a(\xi) = (H_{\text{eff}}^a - \omega^a(\xi) - \xi)^{-1}.$$

If convenient, $\omega^a$ may be regarded as an operator in $L^2(\mathbb{R}^2)$. During some manipulations the dependence of operators on the spectral parameter $\xi$ will not be indicated explicitly.

With respect to the decomposition $L^2(\Omega_a) = \text{Ran} P^a \oplus \text{Ran} Q^a$ one can write

$$H^a = \left(\begin{array}{cc} H_{\text{eff}}^a & P^a H^a Q^a \\ Q^a H^a P^a & H_\perp^a \end{array}\right).$$
As is well known and easy to verify, if $A$, $B$, $C$, $D$ are bounded operators between appropriate Hilbert spaces, then the following formula for inversion of the operator matrix,

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = 
\begin{pmatrix}
W^{-1} & -W^{-1}BD^{-1} \\
-D^{-1}CW^{-1} & D^{-1} + D^{-1}CW^{-1}BD^{-1}
\end{pmatrix},
W = A - BD^{-1}C,
$$

holds true provided $D^{-1}$ and $W^{-1}$ exist and are bounded. This way one obtains a formula for the resolvent of $H^a$,

$$(H^a - \xi)^{-1} = 
\begin{pmatrix}
R^\text{eff} & -R^\text{eff} P^a V Q^a R^\text{eff} \\
-R^\text{eff} Q^a V P^a R^\text{eff} & R^\text{eff} + R^\text{eff} Q^a V P^a R^\text{eff} P^a V Q^a R^\text{eff}
\end{pmatrix}. \tag{36}
$$

Note that $P^a H^a Q^a = P^a V Q^a$, $Q^a H^a P^a = Q^a V P^a$.

**Proposition 10.** Let $\xi < E^a_1$ and $\xi \notin \sigma(H^a_{\text{eff}} - \mathcal{W}^{\text{eff}}(\xi))$. Then for all $a > 0$ sufficiently small one has $\xi \in \text{Res}(H^a)$ and

$$
\left\| (H^a - \xi)^{-1} - R^\text{eff}(\xi) \right\| \leq \frac{8a}{3\pi d^\text{eff}(\xi)} \left(1 + \frac{8a}{3\pi}\right) + \frac{2a^2}{3\pi^2}, \tag{37}
$$

where

$$
d^\text{eff}(\xi) = \text{dist}(\xi, \sigma(H^a - \mathcal{W}^{\text{eff}}(\xi))).
$$

**Proof.** This proof is inspired by the proof of theorem 3.1 in [7]. Note that

$$
\left\| \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \right\|^2 = \|AA^\dagger\| = \|A\|^2.
$$

Thus, from formula (36) one derives the estimate

$$
\left\| (H^a - \xi)^{-1} - R^\text{eff}(\xi) \right\| \leq \left\| R^\text{eff} P^a V Q^a R^\text{eff} \right\| + \left\| R^\text{eff} Q^a V P^a R^\text{eff} \right\| + \| R^\text{eff} \|
$$

$$
\leq \frac{1}{d^\text{eff}} \left\| V Q^a R^\text{eff} \right\| + \left\| V Q^a R^\text{eff} \right\| + \left\| V Q^a R^\text{eff} \right\|
$$

\begin{equation}
\leq \frac{1}{d^\text{eff}} \left\| V Q^a R^\text{eff} \right\| + \left\| V Q^a R^\text{eff} \right\| + \left\| V Q^a R^\text{eff} \right\|. \tag{38}
\end{equation}

To complete the proof one has to estimate $\| R^\text{eff} \|$ and $\| V Q^a R^\text{eff} \|$.

Let us denote (in this proof)

$$
T^\perp = Q^a (-\Delta_D) Q^a, \quad R_0 = (T^\perp - \xi)^{-1}.
$$

Since $\xi < E^a_1 = \pi^2/a^2$ and

$$
T^\perp = Q^a (-\Delta_{x,y} \otimes 1) Q^a + Q^a (-1 \otimes \hat{a}^2) Q^a \geq E^a_2 = 4\pi^2/a^2
$$

one has

$$
0 \leq R_0 \leq (E^a_2 - E^a_1)^{-1} = \frac{a^2}{3\pi^2}, \quad \xi R_0 \leq \frac{1}{3}.
$$

Further let us estimate $\| V Q^a R^\text{eff} \| = \| R^\text{eff} V Q^a V^2 Q^a R^\text{eff} \|^{1/2}$. By the Hardy inequality (23),

$$
R^\text{eff} V Q^a V^2 Q^a R^\text{eff} \leq 4 R^\text{eff} T^\perp R^\text{eff} = 4(Q^a + \xi R_0) \leq \frac{16}{3}.
$$

Hence,

$$
\| V Q^a R^\text{eff} \| \leq \frac{4}{\sqrt{3}}.
$$

Moreover,

$$
(R^\text{eff} V Q^a R^\text{eff})^2 = R^\text{eff} V Q^a R^\text{eff} Q^a R^\text{eff} V Q^a R^\text{eff} \leq \frac{a^2}{3\pi^2} R^\text{eff} V Q^a V^2 Q^a R^\text{eff} \leq \frac{a^2}{3\pi^2} R^\text{eff} V Q^a V^2 Q^a R^\text{eff}.
$$
and so
\[ \| R_0^{1/2} Q^a V Q^a R_0^{1/2} \| \leq \frac{4a}{3\pi}. \]  

(39)

Put \( a_H = 3\pi/4 \). If \( a < a_H \), then by (39) and the resolvent formula
\[ R^a_\perp(\xi) = (T_\perp + Q^a V Q^a - \xi)^{-1} = R_0^{1/2} (1 + R_0^{1/2} Q^a V Q^a R_0^{1/2})^{-1} R_0^{1/2}; \]

(40)

one has \( \xi \in \text{Res}(H^a) \) and the resolvent \( R^a_\perp(\xi) \) is positive. Moreover,
\[ \| R^a_\perp(\xi) \| \leq \| R_0 \| \frac{1}{1 - \| R_0^{1/2} Q^a V Q^a R_0^{1/2} \|}. \]

For \( a < a_H/2 \),
\[ \| R^a_\perp \| \leq 2 \| R_0 \| \leq \frac{2a^2}{3\pi^2}. \]

(41)

From (40) it follows that
\[ \| V Q^a R^a_\perp Q^a V \| \leq \frac{\| V Q^a R_0^{1/2} \|^2}{1 - \| R_0^{1/2} Q^a V Q^a R_0^{1/2} \|} \]

and this implies, again for \( a < a_H/2 \),
\[ \| V Q^a R^a_\perp \| \leq \| V Q^a (R^a_\perp)^{1/2} \| (R^a_\perp)^{1/2} \leq \frac{\| V Q^a R_0^{1/2} \| \| R^a_\perp \|^{1/2}}{1 - \| R_0^{1/2} Q^a V Q^a R_0^{1/2} \|} \leq \frac{8a}{3\pi}. \]

(42)

Finally we conclude that (38), (41) and (42) imply (37). \( \square \)

Lemma 11. If \( \xi < E_1^a \) and \( a > 0 \) is sufficiently small, then \( \mathcal{W}^a(\xi) \) is positive and
\[ \| (\Delta + 2)^{-1/2} \mathcal{W}^a(\xi)(\Delta + 2)^{-1/2} \| \leq \frac{\Gamma(1/4)^4}{6\sqrt{2}\pi^3} a, \]

(43)

where \( -\Delta \) is the free Hamiltonian in \( L^2(\mathbb{R}^2) \).

Proof. In the course of proof of proposition 10, while discussing formula (40), it is shown that if \( \xi < E_1^a \) and \( a > 0 \) is sufficiently small, then \( R^a_\perp(\xi) \) is positive and so is \( \mathcal{W}^a(\xi) \). Using (41) we get
\[
0 \leq \mathcal{W}^a = P^a V Q^a R^a_\perp Q^a V P^a \leq \frac{2a^2}{3\pi^2} \int_0^{a/2} \cos^2(\pi z/a) \frac{dz}{\varrho^2 + z^2} \leq \frac{8a}{3\pi^2} \int_0^{a/2} \frac{1}{\varrho^2 + z^2} dz = \frac{4a}{3\pi}. 
\]

Recalling the Kato inequality (2) one finds that
\[ (-\Delta + 2)^{-1/2} \mathcal{W}^a (-\Delta + 2)^{-1/2} \leq \frac{4a}{3\pi} (-\Delta + 2)^{-1/2} \left( -\Delta + 2 \right)^{-1/2}. \]

The lemma readily follows. \( \square \)

Lemma 12. If \( \mu \leq E_1^a - 2 \), then
\[ \| (\Delta + 2)^{1/2} (H_{\text{eff}}^a - \mu)^{-1/2} \| \leq C_1, \]

(44)

with \( C_1 \) being given in (30).
Proof. Since \(0 \leq V'_{\text{eff}}(\varrho) \leq 1/\varrho\) the Kato inequality (2) implies
\[
V^n_{\text{eff}}(\varrho) \leq \frac{\Gamma(1/4)^2}{4\pi^2} \sqrt{-\Delta}.
\]
Note that, in virtue of (27), \(0 < (H^n_{\text{eff}} - \mu)^{-1} \leq 1\) if \(\mu \leq E_n^1 - 2\). Now, to show (44), one can repeat the proof of lemma 4 word by word while replacing \(1/\varrho\) by \(V'_{\text{eff}}(\varrho)\) and \(H_C + 2\) by \(H^n_{\text{eff}} - \mu\).

Proposition 13. Suppose that \(\xi \in \text{Res}(H^n_{\text{eff}}) \cap \mathbb{R}\). If
\[
a < \frac{1}{2C_{\text{III}}} \min \left\{ 1, \frac{d_{\text{eff}}(\xi)}{2} \right\},
\]
where \(d_{\text{eff}}(\xi) = \text{dist}(\xi, \sigma(H^n_{\text{eff}}))\).
\[
C_{\text{III}} = \frac{c_1^2 \Gamma(1/4)^2}{6\sqrt{2}\pi^3},
\]
and \(c_1\) is defined in (30), then \(\xi \notin \sigma(H^n_{\text{eff}} - \mathcal{W}^n(\xi))\) and
\[
\| R^n_{\text{eff}}(\xi) - (H^n_{\text{eff}} - \xi)^{-1} \| \leq \frac{2C_{\text{III}}}{d_{\text{eff}}(\xi)} \max \left\{ 1, \frac{2}{d_{\text{eff}}(\xi)} \right\} a.
\]

Proof. Apply lemma 8 with \(A = H^n_{\text{eff}} - \xi\), \(C = -\mathcal{W}^n(\xi)\). In view of lemma 11, one observes that \(\xi < E_1^n\) and \(\mathcal{W}^n(\xi)\) is positive provided \(\xi \in \text{Res}(H^n_{\text{eff}}) \cap \mathbb{R}\) and \(a > 0\) is sufficiently small. By lemma 8 if
\[
\alpha = \left\| \mathcal{W}^n(\xi)^{1/2} [H^n_{\text{eff}} - \xi]^{-1/2} \right\| < 1,
\]
then \(\xi \notin \sigma(H^n_{\text{eff}} - \mathcal{W}^n(\xi))\) and
\[
\| R^n_{\text{eff}}(\xi) - (H^n_{\text{eff}} - \xi)^{-1} \| \leq \frac{\alpha^2}{d_{\text{eff}}(\xi)(1 - \alpha^2)}.
\]

If \(\mu < E_1^n - 1\), then, according to (27), \(H^n_{\text{eff}} - \mu\) is positive. One has
\[
\alpha^2 = \left\| [H^n_{\text{eff}} - \xi]^{-1/2} \mathcal{W}^n [H^n_{\text{eff}} - \xi]^{-1/2} \right\|
\leq \left\| (\Delta + 2)^{-1/2} \mathcal{W}^n (\Delta + 2)^{-1/2} \right\| \left\| (\Delta + 2)^{1/2} (H^n_{\text{eff}} - \mu)^{-1/2} \right\|^2
\times \left\| (H^n_{\text{eff}} - \mu)^{1/2} [H^n_{\text{eff}} - \xi]^{-1/2} \right\|^2.
\]

Observe that
\[
\left\| (H^n_{\text{eff}} - \mu)^{1/2} [H^n_{\text{eff}} - \xi]^{-1/2} \right\|^2 = \sup_{x \in \sigma(H^n_{\text{eff}})} \frac{x - \mu}{|x - \xi|} \leq \max \left\{ 1, \frac{E_1^n - \mu}{d_{\text{eff}}(\xi)} \right\}.
\]
Set \(\mu = E_1^n - 2\). Then (49) jointly with (43) and (44) imply
\[
\alpha^2 \leq C_{\text{III}} \max \left\{ 1, \frac{2}{d_{\text{eff}}(\xi)} \right\} a.
\]
If condition (45) is satisfied, then \(\alpha^2 < 1/2\) and (47) follows from (48).

Remark 14. Under the assumptions of proposition 13, \(\alpha\) in (48) fulfills \(\alpha^2 < 1/2\) and so
\[
\| R^n_{\text{eff}}(\xi) - (H^n_{\text{eff}} - \xi)^{-1} \| \leq \| (H^n_{\text{eff}} - \xi)^{-1} \|,
\]
whence
\[
\| R^n_{\text{eff}}(\xi) \| \leq 2 \| (H^n_{\text{eff}} - \xi)^{-1} \|.
\]
This means that
\[
\frac{1}{d_{\text{eff}}^{\gamma}((\xi))} \leq \frac{2}{d_{\text{eff}}((\xi))}.
\] (50)

Similarly, under the assumptions of theorem 9,
\[
\frac{1}{d_{\text{eff}}((\xi + E_1^a))} \leq \frac{2}{d_{C}(\xi)}.
\] (51)

**Theorem 15.** Assume that \( \xi \in \text{Res}(H_{\text{eff}}^{a}) \cap \mathbb{R} \) and \( a > 0 \) fulfills (45). Then \( \xi \in \text{Res}(H^{a}) \) and
\[
\left\| (H^{a} - \xi)^{-1} - (H_{\text{eff}}^{a} - \xi)^{-1} \right\| \leq \left( \frac{8}{\pi} + \max \left\{ 1, \frac{2}{d_{\text{eff}}((\xi))} \right\} C_{\text{III}} \right) \frac{2a}{d_{\text{eff}}((\xi))} + \frac{2a^2}{3\pi^2}
\]
where \( C_{\text{III}} \) is given in (46).

**Proof.** If \( \xi \in \text{Res}(H_{\text{eff}}^{a}) \cap \mathbb{R} \), then \( \xi < E_{1}^{a} \). Furthermore, from proposition 13 it follows that \( \xi \notin \sigma(H_{\text{eff}}^{a} - W^{a}(\xi)) \). Also observe that, by the fact that \( R_{1}^{a}(\xi) \geq 0 \) for any \( \xi < E_{1}^{a} \), one has \( H_{\text{eff}}^{a} > E_{1}^{a} \). Moreover, it can be directly verified that \( 1/C_{\text{III}} < 3\pi/2 \) and so (45) implies \( a < a_{H} = 3\pi/4 \) (see the proof of proposition 10). We conclude that under the assumptions of proposition 13, the assumptions of proposition 10 are fulfilled too. Thus, we have arrived at the estimates
\[
\left\| (H^{a} - \xi)^{-1} - (H_{\text{eff}}^{a} - \xi)^{-1} \right\| \leq \left( \frac{8}{\pi} + \max \left\{ 1, \frac{2}{d_{\text{eff}}((\xi))} \right\} C_{\text{III}} \right) \frac{2a}{d_{\text{eff}}((\xi))} + \frac{2a^2}{3\pi^2},
\]
where we have used (50). \( \square \)

Finally, let us note that by combining theorems 9 and 15 one can show that a hydrogen atom in a very thin planar layer is well approximated, in the norm resolvent sense, by the Coulomb-like potential in the plane.

**Theorem 16.** Let \( \xi \in \text{Res}(H_{C} + E_{1}^{a}) \) be such that \( -3 + E_{1}^{a} < \xi < E_{1}^{a} \), and \( a > 0 \) fulfill
\[
a < \min \left\{ a_{0}, \frac{d_{C}(\xi - E_{1}^{a})}{8C_{\text{III}}} \right\},
\]
where \( a_{0} \) is determined by the equation
\[
\frac{2C_{1}^{2}C_{H}}{d_{C}(\xi - E_{1}^{a})} \ln a_{0} = \frac{1}{2}
\]
with \( C_{1} \) and \( C_{H} \) being defined in (30) and (35), respectively. Then \( \xi \in \text{Res}(H^{a}) \) and
\[
\left\| (H^{a} - \xi)^{-1} - (H_{C} + E_{1}^{a} - \xi)^{-1} \right\| \leq \frac{4C_{1}^{2}C_{H}}{d_{C}(\xi - E_{1}^{a})^{3}} a | \ln a | + \frac{20C_{\text{III}}}{d_{C}(\xi - E_{1}^{a})^{2}} a + \frac{2}{3\pi^{2}} a^{2},
\]
with \( C_{\text{III}} \) being defined in (46).

**Proof.** First apply theorem 9, with \( \xi - E_{1}^{a} \) being substituted for \( \xi \). By the above bound on \( a \), the assumptions of the theorem are fulfilled. Since \( -3 + E_{1}^{a} < \xi < E_{1}^{a} \) implies \( d_{C}(\xi - E_{1}^{a}) < 2 \), it follows that \( \xi \in \text{Res}(H_{\text{eff}}^{a}) \) and
\[
\left\| (H_{\text{eff}}^{a} - \xi)^{-1} - (H_{C} + E_{1}^{a} - \xi)^{-1} \right\| \leq \frac{4C_{1}^{2}C_{H}}{d_{C}(\xi - E_{1}^{a})^{3}} a | \ln a |.
\]
According to (51), $d_C(\xi - E_1^a) \leq 2d_{\text{eff}}(\xi)$, and this jointly with the choice of $\xi$ implies (45). Hence the assumptions of theorem 15 are fulfilled, too. Thus, $\xi \in \text{Res}(H^a)$ and we can estimate
\[ \| (H^a - \xi)^{-1} - (H_C + E_1^a - \xi)^{-1} \| + \| (H^a_C - \xi)^{-1} - (H_C + E_1^a - \xi)^{-1} \| \leq \frac{4a}{d_C(\xi - E_1^a)} \left( \frac{8}{\pi} + \frac{4C_{\text{III}}}{d_C(\xi - E_1^a)} \right) \cdot \frac{2a^2}{3\pi^2} + \frac{4C_I^2C_{\text{II}}}{d_C(\xi - E_1^a)^2} |\ln a| . \]

Observing that
\[ \frac{8}{\pi} \leq \frac{C_{\text{III}}}{2} \leq \frac{C_{\text{III}}}{d_C(\xi - E_1^a)} \]
one arrives at (52).

\[ \square \]

Since the spectrum of $H_C$ is known explicitly one can use theorem 16 to localize the point spectrum of the full Hamiltonian $H^a$ with the help of fairly standard perturbation methods [15]. We do not pursue this problem here, however.

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