CHARACTERIZATION OF SOFIC GROUPS AND EQUATIONS OVER GROUPS.

Lev Glebsky

Instituto de Investigación en Comunicación Óptica Universidad Autónoma de San Luis Potosí Av. Karakorum 1470, Lomas 4a 78210 San Luis Potosí, Mexico

glebsky@cactus.iico.uaslp.mx

Abstract

We give the following characterization of sofic (weakly sofic) groups: a group $G$ is sofic (weakly sofic) if and only if any system of equations solvable in any alternating group (any finite group) is solvable over $G$.

Keywords: sofic groups, equation over groups.

1 Introduction

We give the following characterization of sofic (weakly sofic) groups: a group $G$ is sofic (weakly sofic) if and only if any system of equations solvable in any alternating group (any finite group) is solvable over $G$, see Definition 4 for details. The “only if” part of the statement is almost trivial and an analogous statement is valid for hyperlinear and linear sofic groups. The “if” part of the statement is more interesting. Our approach is an elaboration of the one of [9]. Let $\langle \bar{a} \rangle$ be a free group and $N$ its normal subgroup. We include $\langle \bar{a} \rangle$ in a group $A$ defined in the section 3 in such a way, that $\langle \bar{a} \rangle/N$ is sofic if and only if $N = \langle \bar{a} \rangle \cap \langle \langle N \rangle \rangle_A$, where $\langle \langle N \rangle \rangle_A$ is the normal closure of $N$ in $A$.

We use following notations: $F < G$ – “$F$ is a subgroup of $G$”. $F \triangleleft G$ – “$F$ is a normal subgroup of $G$”. Let $X \subseteq F$ and $F < G$. $\langle X \rangle$ – “the subgroup generated by $X$”; $\langle \langle X \rangle \rangle_F$ – “the normal subgroup of $F$, generated by $X$”. For $\bar{a} = (a_1, a_2, \ldots, a_k)$ not in a group, $\langle \bar{a} \rangle$ denotes the free group (of rank $k$) freely generated by $\bar{a}$. As usual, $X \subseteq_{\text{fin}} Y$ means “$X$ is a finite subset of $Y$”.

2 Metric approximation of groups

Our exposition follows the lines of [10].

Definition 1. Let $G$ be a group. An invariant (pseudo) length function is a map $\| \cdot \|: G \to [0, \infty]$ such that

- $\| 1 \| = 0$
- $\| gh \| \leq \| g \| + \| h \|$
- $\| h^{-1}gh \| = \| g \|$ ($\| \cdot \|$ depends, in fact, on conjugacy class)
Definition 2. Let $\mathcal{K}$ be a class of groups with invariant length functions. (The elements of $\mathcal{K}$ are pairs: a group and a length function on it. All length functions we denote by $\| \cdot \|$, it should not lead us to misunderstanding.) We say that a group $G$ is approximable by $\mathcal{K}$ if

- there exists $\alpha : G \to \mathbb{R}$, $\alpha_1 = 0$ and $\alpha_g > 0$ for $g \neq 1$
- for any $\Phi \subset \text{fin} \ G$ for any $\epsilon > 0$ there exists a function $\phi : \Phi \to H \in \mathcal{K}$ such that
  - $\phi(1) = 1$
  - $\|\phi(g)\| \geq \alpha_g$ for any $g \in \Phi$
  - $\|\phi(gh)(\phi(g)\phi(h))^{-1}\| < \epsilon$ for any $g, h, gh \in \Phi$.

Let $\omega$ be a non-principal ultrafilter over $\mathbb{N}$ and $H_i \in \mathcal{K}$. Let $N \trianglelefteq \prod_{i=1}^{\infty} H_i$ be defined as

$$(h_1, h_2, \ldots) \in N \iff \lim_{\omega} \|h_i\| = 0.$$  

Denote $\prod_{\omega} H_i = \prod_{i} H_i / N$. The group $\prod_{\omega} H_i$ is called the metric ultraproduct of $H_i$ (with respect to $\| \cdot \|$ and $\omega$).

Proposition 1. $G$ is approximable by $\mathcal{K}$ if and only if there is an injection $G \hookrightarrow \prod_{\omega} H_n$ for some non principal ultrafilter $\omega$ and sequence $H_1, H_2, \ldots$ of groups in $\mathcal{K}$.

The following proposition is a modification of Proposition 1.7 of [16].

Proposition 2. Let $G$ be a group and $N \trianglelefteq G$. Suppose, that there is a sequence of homomorphisms $\phi_n : G \to H_n \in \mathcal{K}$ such that

$$\inf_{n \to \infty} \lim_{n \to \infty} \|\phi_n(x)\| > 0, \text{ for } x \in G \setminus N$$
$$\lim_{n \to \infty} \|\phi_n(x)\| = 0, \text{ for } x \in N.$$  

Then $G/N$ is approximable by $\mathcal{K}$.

We need the following classes $\mathcal{K}$

1. Class $\text{Alt}$ of all alternating groups with Hamming length functions $\| \cdot \|$, defined as follows. Let $A_n$ be the alternating group of $n$-element set $X$. For $g \in A_n$ define $\|g\| = \frac{|\{x \in X | g \neq x\}|}{|X|}$. Groups approximable by $\text{Alt}$ is said to be sofic, [13]. (The sofic groups are defined to be approximable by symmetric groups. But it is easy to see that symmetric and alternating groups approximate the same class of groups.)
2. Class $\text{Fin}$ of all finite groups, equipped with all possible invariant length functions. Groups approximable by $\text{Fin}$ are called weakly sofic, [9].

3. Class $\text{Uni}$ of $U(n, \mathbb{C})$, $n \in \mathbb{N}$. ($U(n, \mathbb{C})$ is the group of $n \times n$-unitary matrix over $\mathbb{C}$.) We equip $U(n, \mathbb{C})$ with $\|g\| = \sqrt{\frac{1}{n} \text{trace}(2 - g - g^*)}$. Groups approximable by $\text{Uni}$ are called hyperlinear, [13].

4. Class $\text{Lin}$ of $\text{GL}(n, \mathbb{C})$, $n \in \mathbb{N}$. ($\text{GL}(n, \mathbb{C})$ is the group of $n \times n$-invertable matrix over $\mathbb{C}$.) We equip $\text{GL}(n, \mathbb{C})$ with $\|g\| = \frac{1}{n} \text{rank}(1 - g)$. Groups approximable by $\text{Lin}$ are called linear sofic, [1].

The sofic and hyperlinear groups are studied rather extensively, see [13, 8]. It is an open question if all groups are sofic. Sofic groups are known to be approximable by all classes mentioned above. It is also known that linear sofic groups are weakly sofic, [1].

3 $F$-groups.

Definition 3. Let $F$ be a group. A group $G$ with an injective homomorphism $F \hookrightarrow G$ is said to be an $F$-group. One may think that $G$ has a fixed copy of $F < G$. Let $G_1$, $G_2$ be $F$-groups. A homomorphism $\phi : G_1 \rightarrow G_2$ is called an $F$-morphism if the restriction of $\phi$ on $F$ is the identity map ($\phi|_F = \text{Id}$).

This definition has an important use in algebraic geometry over groups, [2]. Clearly, $F$-groups form a category. For an $F$-group $G$ and $N \triangleleft F$ we denote $\bar{N}^G = F \cap \langle \langle N \rangle \rangle_G$. It is clear that $\bar{N}^G$ is a closure operator. The condition $N = \bar{N}^G$ is equivalent to the existence of the following commutative diagram with injective $\psi$:

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & G \\
\Bigg\downarrow & & \Bigg\downarrow \\
F/N & \xrightarrow{\psi} & Y 
\end{array}
\]

(the vertical arrow $F \rightarrow F/N$ is the natural homomorphism). Similarly, the reader may define $\bar{N}^G$ by a universal property of $F/\bar{N}^G$, but we need not it here.

Let $\langle \bar{a} \rangle$ be a (finitely generated) free group and $\mathcal{F} > \langle \bar{a} \rangle$ be its profinite closure, considered as an $\langle \bar{a} \rangle$-group. Let $N \triangleleft \langle \bar{a} \rangle$. The results of [9] say that $\langle \bar{a} \rangle/N$ is weakly sofic (w-sofic) if and only if $N = \bar{N}^\mathcal{F}$.

Let $A_n$ be an alternating group of $n$ elements. Consider the set of all homomorphisms $\phi : \langle \bar{a} \rangle \rightarrow A_n$, for all $n$. This set is countable and we may enumerate it, say $\phi_i : \langle \bar{a} \rangle \rightarrow A_n$. Let $\mathcal{A} = \prod_i A_{n_i}$ be unrestricted direct product. The expression $\phi_\infty(w) = (\phi_1(w), \phi_2(w), \ldots, \phi_j(w), \ldots)$ defines a homomorphism $\phi_\infty : \langle \bar{a} \rangle \rightarrow \mathcal{A}$. It is known that $\phi_\infty$ is an injection [11, 17]. So, $\mathcal{A}$ is an $\langle \bar{a} \rangle$-group. In Section 6 we show that $\langle \bar{a} \rangle/N$ is sofic if and only if $\bar{N}^\mathcal{A} = N$. Let us summarize the above discussion.
Theorem 3. Let $N \trianglelefteq \langle \bar{a} \rangle$ and $G = \langle \bar{a} \rangle / N$. Then $G$ is sofic if and only if $\bar{N}^A = N$; $G$ is $w$-sofic if and only if $\bar{N}^F = N$.

In the category of $F$-groups not for all pairs of objects $G_1$, $G_2$ there exists a morphism $G_1 \to G_2$. We write $G_1 \preceq_F G_2$ if there exists an $F$-morphism $\phi : G_1 \to G_2$.

Lemma 4. Let $H \preceq_F G$. Then $\bar{N}^H \subseteq \bar{N}^G$ for any $N \trianglelefteq F$.

Proof. Let $\phi : H \to G$ be an $F$-morphism. Then

$$\bar{N}^G = \langle \langle N \rangle \rangle_G \cup F \supseteq \langle \langle \phi(N) \rangle \rangle_{\phi(H)} \cap \phi(F) \supseteq \phi(\langle \langle N \rangle \rangle_H \cap F) = \bar{N}^H \quad \square$$

Let $G$ be an $\langle \bar{a} \rangle$-group. Denote

$$FP(G) = \{ H : H \text{ is a finitely presented } \langle \bar{a} \rangle \text{-group and } H \preceq G \},$$

here we call an $F$-group finitely presented if it is finitely presented as a group.

Lemma 5. Let $N \trianglelefteq \langle \bar{a} \rangle$. Then

$$\bar{N}^G = \bigcup_{H \in FP(G)} \bar{N}^H$$

Proof. Lemma 4 shows that $\bar{N}^G$ contains $\bigcup_{H \in FP(G)} \bar{N}^H$. Now suppose that $w \in \bar{N}^G$, then $w$ is a finite consequence of $N$ and relations of $G$, so there exists a finitely presented $\langle \bar{a} \rangle$-group $H \preceq G$ such that $w \in \bar{N}^H$. \square

4 Soficity and equations over groups

Let $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{x} = (x_1, \ldots, x_r)$ be symbols for constants and variables, $|\bar{a}| = k$, $|\bar{x}| = r$. Let $w_i \in \langle \bar{a}, \bar{x} \rangle$, $\bar{w} = (w_1, \ldots, w_n)$. Consider the system of equations $\bar{w} = 1$ ($w_1 = 1, \ldots, w_n = 1$). We consider only finite systems of equations.

Definition 4. We say that a system $\bar{w}(\bar{a}, \bar{x})$ is solvable in a group $G$ if

$$\forall \bar{a} \in G^{[\bar{a}]} \exists \bar{x} \in G^{[\bar{x}]} \bar{w}(\bar{a}, \bar{x}) = 1.$$

Let $\mathcal{K}$ be a class of groups. The system is called $\mathcal{K}$-system if it is solvable in all $G \in \mathcal{K}$. We use the classes defined above (Alt, Fin, Uni, Lin) forgetting their length functions.

We say that a system $\bar{w}(\bar{a}, \bar{x})$ is solvable over a group $G$ if there exists a group $H \supset G$ such that

$$\forall \bar{a} \in G^{[\bar{a}]} \exists \bar{x} \in H^{[\bar{x}]} \bar{w}(\bar{a}, \bar{x}) = 1.$$

If a system is solvable in $G$ then it is solvable in any factor of $G$. Also, any $\mathcal{K}$-system is solvable in any (restricted or unrestricted) direct product of groups from $\mathcal{K}$. With Proposition 1 it implies
Proposition 6. If $G$ is approximable by $K$ then any $K$-system is solvable over $G$.

In what follows we prove that for the case of sofic and weakly sofic groups the converse statement is also valid (Theorem 8). The proof is based on the following well-known relation of $\langle \bar{a} \rangle$-groups with solutions of equations over groups. First of all, a system $\bar{w}(\bar{h}, \bar{x})$ has a solution over $H$ if and only if the natural map $H \to (H * \langle \bar{x} \rangle)/\langle \langle \bar{w}(\bar{h}, \bar{x}) \rangle \rangle$ is an injection, see [14]. It follows that a system of equations $\bar{w}(\bar{a}, \bar{x})$ has a solution over $H = \langle \bar{a} \rangle N$ if and only if $N = \bar{N} G$ with $G = \langle \bar{a}, \bar{x} \mid \bar{w}(\bar{a}, \bar{x}) \rangle$. Here we suppose that $\bar{a} \in H^{[\bar{a}]}$ are images of the natural homomorphism $\langle \bar{a} \rangle \to H$ and $\bar{x}$ are variables.

Lemma 7. A finitely presented group $\langle \bar{a}, \bar{x} \mid \bar{w}(\bar{a}, \bar{x}) \rangle \in FP(A)$ if and only if $\bar{w}$ is an Alt-system.
A finitely presented group $\langle \bar{a}, \bar{x} \mid \bar{w}(\bar{a}, \bar{x}) \rangle \in FP(F)$ if and only if $\bar{w}$ is a Fin-system.

Proof. Let us prove the second statement of the lemma only. The proof of the first statement is similar. Let $G = \langle \bar{a}, \bar{x} \mid \bar{w} \rangle \in FP(F)$ and $\phi : G \to F$ be an $\langle \bar{a} \rangle$-morphism. Let $F$ be a finite group and $\bar{f} \in F^k$. There exists a homomorphism $\psi : F \to F$ such that $\psi(a_i) = f_i$. Consider the composition:

$$G \xrightarrow{\phi} F \xrightarrow{\psi} F.$$ 

The image $\psi(\phi(\bar{x}))$ is a solution of $\bar{w}$ in $F$ for $\bar{a} = \bar{f}$.

The other direction. Let $\bar{w}$ be a Fin-system and $G = \langle \bar{a}, \bar{x} \mid \bar{w} \rangle$. Let $F$ be a finite group and $\bar{f} \in F^k$. It follows that the map $\phi : \bar{a} \to \bar{f}$ may be extended to a homomorphism $\phi : G \to F$. So, any homomorphism from $\langle \bar{a} \rangle$ to a finite group factors through $G$, as it is shown in the following commutative diagram:

$$\begin{array}{ccc}
\langle \bar{a} \rangle & \to & G \\
& \searrow & \downarrow \\
& & F
\end{array}$$

It follows that there exists an $\langle \bar{a} \rangle$-morphism $G \to F$. It should be well known but let me sketch arguments for it. Consider profinite group $F$ as an inverse limit of finite groups. For any finite parts of the corresponding inverse system there exist compatible homomorphisms from $G$ by the above diagram. To show that there exist homomorphisms compatible with the whole inverse system one can use:

- There is only finite number of homomorphisms from $G$ to a finite group $F$ ($G$ is finitely generated)

- “The saturation principle for finite sets”: Let $X$ be a collection of finite sets, such that intersection of any finite subcollection of $X$ is nonempty. Then $\bigcap X \neq \emptyset$.

So, for any finite part of the inverse system there exists an extensible system of compatible homomorphisms from $G$. \qed
Lemma 7 and Theorem 3 imply

**Theorem 8.** A group $H$ is sofic (w-sofic) if and only if any Alt-system (Fin-system) has a solution over $H$.

**Proof.** We prove the w-sofic part of the theorem only. The “only if” part is Proposition 6.

The “if” part. Let $H = \langle \bar{a} \rangle / N$ be non weakly sofic. It follows that $N \neq N^F$. By Lemma 5 and Lemma 7 there exists a Fin-system $\bar{w}$ such that $N \neq N^G$ for $G = \langle \bar{a}, \bar{x} \mid \bar{w} \rangle$. But this means that the system $\bar{w}$ has no solution over $H$ for $\bar{a} \in H$).

5 Algebraic groups

Here we study algebraic groups over algebraically closed fields (AGCF). For example,

- $GL(n)$, the group of invertible $n \times n$-matrices over $\mathbb{C}$, it is defined by equation $x \det A = 1$;

- $O(n, \mathbb{C})$, the group of orthogonal $n \times n$-matrices over $\mathbb{C}$, defined by $AA^T = E$.

$U(n)$, the group of unitary matrices, is not AGCF because the equations $AA^* = E$ is over $\mathbb{R}$, not over $\mathbb{C}$.

**Lemma 9.** Any Fin-system is solvable in an AGCF group.

**Proof.** The proof is very similar to the proof of Malcev theorem in [5], see also [1]. We use the fact that, if a system $S$ of equations does not have a solution in an algebraically closed field then $S$ has no solutions in some finite field. (The algebraical closeness is essential here. For example, $x^2 + y^2 = -1$ has no solution in $\mathbb{R}$, but has a solution in any finite field.) Precisely, let $K$ be an algebraically closed field and $S \subset K[\bar{x}]$. The system $S$ has no solution in $K$ if and only if $1 \in I(S)$. This means that there exist $g_1, g_2, \ldots, g_r \in K[\bar{x}]$, and $h_1, h_2, \ldots, h_r \in S$ such that

$$1 = \sum_{i=1}^{r} g_i h_i. \quad (2)$$

Let $B \subset K$ be the set of coefficients of all $g_i$ and $h_i$. Consider the ring $R = \mathbb{Z}[B] \subset K$. Clearly, Eq. 2 holds in $R[\bar{x}]$ and no nontrivial homomorphism $R \rightarrow R/I$ sends 1 to 0. Now, there exists a maximal ideal $I$ of $R$, the corresponding field $R/I$ is finite [5], Let $\phi : R \rightarrow R/I$ be the natural morphism. Then the system $\phi(S)$ does not have solutions over $R/I$.

Let a system $\bar{w}(\bar{a}, \bar{x})$ has no solution in AGCF $G$ for some $\bar{a} \in G^{[\bar{a}]}$. Then the corresponding system of algebraic equations has no solution in $K$. Let $R, \phi, I$ be as defined above. We may assume that the entries of $\bar{a}$ and $\bar{a}^{-1}$ are in $R$. So, $\phi(\bar{a})$ (applied by entries) are invertable matrices over $R/I$. Particularly, it means that $\phi$ defines a homomorphism of a subgroup of $G$ to the corresponding algebraic groups over $R/I$. Then $\bar{w}(\phi(\bar{a}), \bar{x})$ has no solution in this algebraic group over $R/I$, which is finite. \(\square\)
The discussion after Definition 4 implies Corollary 10. Let $A$ be a factor of an (unrestricted) direct product of AGCF. Suppose that $G < A$. Then the group $G$ is w-sofic.

Particularly, this proves that, if a group has a metric approximation by some subclass of AGCF then it is w-sofic. It gives another proof that linear-sofic groups are w-sofic.

6 Proof of Theorem 3

It suffices to prove the sofic part of the theorem, as w-sofic part is proved in [9]. The “only if” part is easy. Indeed, let $\langle \bar{a} \rangle / N$ be sofic. Then there is inclusion $\langle \bar{a} \rangle / N \hookrightarrow \prod \omega A_i$, so we get the following commutative diagram:

\[ \begin{array}{c}
A \\
\downarrow \\
\langle \bar{a} \rangle \\
\downarrow \\
\langle \bar{a} \rangle / N \\
\downarrow \\
\prod_i A_i \\
\downarrow \\
\prod \omega A_i
\end{array} \]

The lift from $\phi$ to $\phi_1$ exists due to $\langle \bar{a} \rangle$ is free; the lift of $\phi_1$ to upper triangle exists by definition of $\mathcal{A}$. The external contour of the diagram is a specification of the diagram $(\prod)$, so $\bar{N}^A = N$.

The other direction. Suppose, that $N = \bar{N}^A$. It follows that $\langle \bar{a} \rangle / N$ is a subgroup of a factor of $\mathcal{A}$. So, it suffices to show the following lemma.

Lemma 11. All factors of $\mathcal{A}$ are sofic.

Proof. Recall, that $\mathcal{A} = \prod_j A_{n_j}$; $\cdot$ denotes the Hamming length functions on any $A_n$. Let $p_j: \mathcal{A} \to A_{n_j}$ be the natural projection, $\mathcal{N} \triangleleft \mathcal{A}$, $x \in \mathcal{A} \setminus \mathcal{N}$ and $\Phi \subseteq_{\text{fin}} \mathcal{N}$.

Claim 12.

$$\sup_j \min_{y \in \mathcal{N}} \frac{\| p_j(x) \|}{\| p_j(y) \|} = \infty$$

We prove it by contradiction. Suppose, that

$$\exists M > 0 \forall j \exists y \in \mathcal{N} \frac{\| p_j(x) \|}{\| p_j(y) \|} < M.$$  

For $z \in G$ let $[z] = \{ y^{-1}zy \mid y \in G \}$, the conjugacy class of $z$. It follows, see [7], that $p_j(x) \in \prod_{y \in \Phi} [p_j(y)]^{8M}$. So, $x \in \prod_{y \in \Phi} [y]^{8M} \subseteq \mathcal{N}$, a contradiction.
Claim 13. For any $\epsilon > 0$ there exists a homomorphism $\phi : A \to A_n$ such that $\|\phi(x)\| > 1/2$ and $\|\phi(y)\| < \epsilon$ for any $y \in \Phi$.

Let us stop for a moment to discuss simple constructions needed for the proof. The point is that a direct product of alternating groups $A_n \times A_m$ may be included in $A_{nm}$ as well as in $A_{n+m}$.

The first inclusion, $\Pi : A_n \times A_m \hookrightarrow A_{nm}$, is defined as follows. One may consider $A_{nm}$ as the alternating group on $X = \{1, \ldots, n\} \times \{1, \ldots, m\}$. Then for $\alpha \in A_n \times A_m$ and $x \in X$ we define $\Pi(\alpha)(x) = (\alpha_1(x), \alpha_2(x))$. A reasonable notation for $\Pi(\alpha)$ is $\alpha_1 \otimes \alpha_2$. It is easy to check that $1 - \|\alpha_1 \otimes \alpha_2\| = (1 - \|\alpha_1\|)(1 - \|\alpha_2\|)$.

The second inclusion, $\Sigma : A_n \times A_m \hookrightarrow A_{n+m}$, is defined the following way. Consider $A_{n+m}$ as the alternating group on $X = X_1 \cup X_2$, with $|X_1| = n$ and $|X_2| = m$. Consider $A_n$ ($A_m$) as the alternating group on $X_1$ ($X_2$). Then for $\alpha \in A_n \times A_m$ and $x \in X$ we define

$$\Sigma(\alpha)(x) = \begin{cases} \alpha_1(x), & \text{if } x \in X_1 \\ \alpha_2(x), & \text{if } x \in X_2 \end{cases}$$

A reasonable notation for $\Sigma(\alpha)$ is $\alpha_1 \oplus \alpha_2$. It is easy to check that

$$\|\alpha_1 \oplus \alpha_2\| = \frac{n}{n+m}\|\alpha_1\| + \frac{m}{n+m}\|\alpha_2\|.$$ 

In what follows we use the above defined notations not only for products of 2 alternating groups, but for products of several alternating groups. For example, $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k \in A_{n_1 n_2 \cdots n_k}$, where $\alpha_i \in A_{n_i}$.

Now, we return to the proof of Claim 13. Let

$$\frac{\|p_j(x)\|}{\max_{y \in \Phi} \|p_j(y)\|} = M_j,$$

Let $\delta_j = \max_{y \in \Phi} \|p_j(y)\|$. We chose $\phi = \phi^\otimes_{r} : A \to A_n$, $(\phi(z) = p_j(z) \otimes p_j(z) \otimes \ldots)$ for proper $j$ and $r$. The consideration above shows that $1 - \|\phi(z)\| = (1 - \|p_j(z)\|)^r$ (this is so called “duplication trick” [7]).

Suppose, that there is a subsequence $S$ of $j$ such that $M_j \to \infty$ and $p_j(x) = \delta_j M_j > \alpha$, for $j \in S$ and some $\alpha > 0$. Then one can take $r$ such that $(1 - \alpha)^r < 1/2$ and find large enough $j \in S$ to satisfy the claim.

Suppose, that there is a subsequence $S$ of $j$ such that $M_j \to \infty$ and $\delta_j M_j \to 0$ for $S$. Then $\frac{\log(1 - \delta_j M_j)}{\log(1 - \delta_j)} \approx M_j$. So, if $(1 - \delta_j M_j)^r \approx 1/2$ then $(1 - \delta_j)^r \approx (1/2)^{1/M_j} \to 1$. So, constructing $\phi$ with $j \in S$ large enough and $r = r_j$ we satisfy the claim.

Claim 14. Let $N \triangleleft A$ and $\Phi \subseteq \text{fin} A$. Then for any $\epsilon > 0$ there exists a homomorphism $\phi : A \to A_n$ such that $\|\phi(x)\| > \frac{1}{2}$ for $x \in \Phi \setminus N$ and $\|\phi(x)\| < \epsilon$ for $x \in \Phi \cap N$.

For each $x \in \Phi \setminus N$ we construct $\phi_x : A \to A_n$ of Claim 13 (with $\Phi = N \cap \Phi$). Moreover, considering $\phi^{\otimes k}_x : A \to A_{nk}$ and changing $kn \to n$ we may suppose that $n$ is the same for all $x \in \Phi \setminus N$. Now, let $\phi = \bigoplus_{x \in \Phi \setminus N} \phi_x : A \to A_{nm}$, where $m = |\Phi|$.

One can check that $\phi$ satisfies
• $\|\phi(x)\| > \frac{1}{2m}$, for $x \in \tilde{\Phi} \setminus \mathcal{N}$,

• $\|\phi(x)\| < \epsilon$, for $x \in \tilde{\Phi} \cap \mathcal{N}$.

The claim follows by the duplicating trick discussed in the proof of Claim 13.

Now, Claim 14 and Proposition 2 imply Lemma 11.

7 Concluding remarks and open questions.

Is the analogue of Theorem 3 valid for, say, hyperlinear groups? We may change approximations by $U(n)$ with the approximations by $PSU(n)$ and inject a free group into a direct product of sufficient copies of $PSU(n)$ (for all $n \in \mathbb{N}$). Then the “only if” analogue of Theorem 3 would be valid. Our proof of the theorem in the other direction is based on Lemma 11. Is the analogous statement valid for the hyperlinear case?

Open Question 15. Is a factor of a direct product of $PSU(n)$ (with different $n$) hyperlinear?

The main difficulty here is that the trace length function is not unique on $PSU(n)$, [15].

The Kervaire conjecture hypothesizes that any nonsingular system of equations is solvable over any group. A group $G$, over which any nonsingular system has a solution is called Kervaire. So, the Kervaire conjecture suggests that any group is Kervaire. Gerstenhaber and Rothaus show that any nonsingular system has a solution in a compact Lie group, [10]. It follows that hyperlinear groups are Kervaire, [13]. In order to understand better the relation of sofic and Kervaire properties one may try to study singular $Alt$-systems ($Fin$-systems). If, for example, one manage to show that all singular $Alt$-systems has a solution over any group, he proves that classes of Kervaire groups, sofic groups, and hyperlinear groups coincide. Unfortunately, the author has no idea about the structure of $Alt$-systems ($Fin$-systems). A system $\bar{w}(\bar{a}, \bar{x})$ is solvable in any group if and only it is solvable in the free group $\langle \bar{a} \rangle$. So, the systems, solvable in free groups, are noninteresting part of the set of $Fin$-systems ($Alt$-systems) There are examples of $Fin$-system that are not solvable in free groups, [6], [12]. But these examples are nonsingular and, moreover, obviously solvable over any group.

Open Question 16. Does there exist a singular $Fin$-system ($Alt$-system) unsolvable in free groups?

In general, what can be said about structure of $Fin$-systems ($Alt$-systems)?

In [14] Stallings in connection with the Kervaire conjecture gives the definition of normal convex subgroup: “Let $F < G$. $F$ is said to be normal convex in $G$ if $N^G = N$ for any $N \vartriangleleft F$.” So, the all groups are sofic (weakly sofic) if and only if $\langle a_1, a_2 \rangle$ is a normal convex subgroup of $\mathcal{A} (\mathcal{F})$. There are few works dealing with normal convexity directly [3, 4]. It is also related with embedding theorems and SQ-universality.
Acknowledgments. Some ideas of the work appear during the “Word maps and stability of representation” conference and the author visit to ESI, Vienna, April 2013. The visit was supported by CRSI22-130435 grant of SNSF. Also the work was partially supported by PROMEP grant UASLP-CA21.

The author thanks the organizers and participants of the conference, especially Goulhara Arzhantseva, Jakub Gismatullin, and Kate Juschenko for useful discussions. The author thanks Andreas Thom for pointing out an error in the previous version.

References

[1] G. Arzhantseva, L. Paunescu, Linear Sofic Groups an Algebras, preprint arXiv:1212.6780

[2] Baumslag, Gilbert; Myasnikov, Alexei; Remeslennikov, Vladimir. Algebraic geometry over groups. I. Algebraic sets and ideal theory. J. Algebra 219 (1999), no. 1, 1679.

[3] S. Brick, Normal-convexity and equations over groups, Inventiones math, 94, 81-104, (1988)

[4] S. Brick, Relative normal-convexity and amalgamations, Bull Australian Math Soc, 44, no 1,95-107, (1991)

[5] N.P. Brown and N. Ozawa, C -Algebras and Finite-Dimensional Approximations, Graduate Studies in Mathematics 88, American Mathematical Society, Providence, R.I., 2008.

[6] Thierry Coulbois and Anatole Khelif, Equations in Free Groups are not Finitely Approximable, Proceedings of the AMS 127, 4 (Apr., 1999), 963-965

[7] Elek, G., and Szabo, E. Hyperlinearity, essentially free actions and l2 - invariants. The sofic property. Math. Ann 332, 2 (2005), 421441.

[8] Valerio Capraro, Martino Lupini, Introduction to Sofic and Hyperlinear groups and Connes Embedding Conjecture, preprint arXiv:1309.2034

[9] Glebsky, Lev; Rivera, Luis Manuel, Sofic groups and profinite topology on free groups. J. Algebra 320 (2008), no. 9, 35123518

[10] M. Gerstenhaber and O.S. Rothaus, The solution of sets of equations in groups, Proc. Nat. Acad. Sci. USA 48(1962), 15311533.

[11] Robert A. Katz and Wilhelm Magnus. Residual properties of free groups. Communications on Pure and Applied Mathematics, 22:113, 1968.

[12] N. Nikolov and D. Segal, On finitely generated profinite groups, I: strong completeness and uniform bounds, Annals of Math. 165 (2007), 171238.
[13] Pestov, Vladimir G. Hyperlinear and sofic groups: a brief guide. *Bull. Symbolic Logic* **14** (2008), no. 4, 449-480.

[14] Stallings, J. Surfaces in three-manifolds and nonsingular equations in groups, *Math. Z.* **184** (1983), 1-17.

[15] Stolz, Abel; Thom, Andreas On the lattice of normal subgroups in ultraproducts of compact simple groups. *Proc. Lond. Math. Soc. (3)* **108** (2014), no. 1, 73-102.

[16] Thom, Andreas. About the metric approximation of Higman’s group. *J. Group Theory* **15** (2012), no. 2, 301-310.

[17] Henry Wilton, Alternating quotients of free groups, *Preprint* arXiv:1005.0015.