Vacuum Fluctuations Cannot Mimic a Cosmological Constant

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Abstract

When the vacuum fluctuation pressure is calculated directly from fundamental principles of quantum field theory, in the same manner as vacuum fluctuation energy density is commonly calculated, one finds it is not equal to the negative of the vacuum fluctuation energy density. Thus, vacuum fluctuations cannot manifest as a cosmological constant of any order.

1 Introduction

As summarized by Peebles and Ratra [1], Padmanabhan [2], and others, there are presently three overriding cosmologic issues involving phenomena for which no generally accepted theoretical solutions exist: 1) dark matter (non-baryonic, unseen “normal” matter), 2) dark energy (small positive cosmological constant or quintessence), and 3) a vanishing sum of zero-point energies.

We will not deal with 1) here. In [3], I show a possible mechanism for 3), somewhat related to ideas proposed by others [4] - [14]. In the course of investigating 2), many of these authors, as well as many others, such as [15] - [19], have surmised that the zero-point energy which arises from the so-called vacuum fluctuations leads to a cosmological constant, albeit one that, by the most straightforward quantum field theory (QFT) calculations, differs from the observed value by a factor of almost $10^{120}$ [15].

In this article, I show that the vacuum fluctuations, while they give rise to vacuum energy, do not give rise to the appropriate vacuum pressure needed to result in a cosmological constant of any magnitude.

2 Background

2.1 Lorentz Invariance of the Vacuum

The stress energy tensor for a perfect fluid, which the vacuum is assumed to be, has the form in the local rest mass frame of

$$\langle T^{\mu\nu} \rangle = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \quad (1)$$
Most consider that the stress-energy tensor for the vacuum should be Lorentz invariant, which means that it must take the form

\[
\langle T^\mu_\nu \rangle_{\text{vac}} = \rho_{\text{vac}} \eta^\mu_\nu = \rho_{\text{vac}} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

(2)

and hence,

\[
p_{\text{vac}} = -\rho_{\text{vac}}.
\]

(3)

Expressing the equation of state for any perfect fluid as

\[
p = w \rho,
\]

(4)

for the vacuum, we would have

\[
w_{\text{vac}} = -1.
\]

(5)

### 2.2 Effective Cosmological Constant

In the Einstein field equations, the cosmological constant of scalar value \( \Lambda \) must take the tensor form

\[
\Lambda \eta^\mu_\nu.
\]

(6)

Thus, from (1), any perfect fluid which might mimic a cosmological constant must have

\[
w_{\text{perf fluid CC}} = -1.
\]

(7)

### 2.3 Constant Vacuum Energy Density

It is well known [20], and derivable from the Einstein field equations with a Friedmann metric, that the rate of change of energy density for a perfect fluid in an expanding universe is

\[
\dot{\rho} = -3 (\rho + p) \frac{\dot{a}}{a},
\]

(8)

where \( a \) is the time dependent scale factor of the universe (the “radius” of the universe). One would expect the energy density of the vacuum to remain constant, and from (5), it can only do so if the vacuum pressure is equal in magnitude and opposite in sign from the vacuum density, i.e., as in (5).

### 2.4 Cosmological Observation

Recent observation [21] [22] indicates that

\[
w_{\text{vac}} = -1 \ (\pm 10\%) ,
\]

(9)

and the data suggests that this value has been constant, or very close to constant, for at least, a substantial part of the history of the universe. Thus, these observations support the existence of a cosmological constant. Further, the observations indicate it has a small, positive value, leading to accelerating expansion of the universe.
2.5 Kinds of Vacuum Energy

There are primarily two known ways in which the vacuum can possess energy, a) vacuum fluctuations and b) symmetry breaking remnants. Both have been proposed as possible sources for the observed cosmological constant.

2.5.1 Vacuum fluctuations

In second quantization, one takes the classical field theory Poisson brackets over into commutators. Thus, the field (e.g. $\phi$) and its conjugate momentum (e.g. $\pi$) do not commute, and this leads to a Hamiltonian operator with an infinite series of $\frac{1}{2}$ quanta terms. The vacuum expectation value for energy then becomes infinite (or finite, but extremely large, if a suitable energy cutoff, such as the Planck energy, is employed). [15]

2.5.2 Symmetry breaking remnants

In the typical Higgs symmetry breaking mechanism, particles gain mass as the universe cools, and the vacuum gains an energy density (the remnant). This is simply a constant term arising in the Hamiltonian density after the symmetry breaking, usually designated as $V$. This remnant energy density is typically on the order of the energy density of the mass-energy density in the universe at the time of the symmetry breaking. This is far smaller than that calculated for the vacuum fluctuations, yet far larger than that observed.

2.6 Conclusions for the Vacuum

Lorentz invariance, constancy of vacuum energy density, and observation all indicate a vacuum pressure which is negative and equal in magnitude to the vacuum energy density. Both vacuum fluctuations and symmetry breaking remnant energy densities are far larger than what is observed, but various corrective mechanisms have been proposed for both, which presumably might make either a candidate for the observed cosmological constant.

3 Determining Vacuum Pressure from Theory

3.1 The Common Approach

Many researchers [16] use vacuum energy density relations, expressed in the form of (8) or its sibling thermodynamic relation (derivable from (8)),

$$d(\rho V) = -pdV$$

(10)

where $V$ here is volume, to deduce (3). Further justification is derived from the Lorentz invariance logic of Section 2.1.

This is then used to justify that both

$$p_{\text{vac fluct}} = -\rho_{\text{vac fluct}}$$

(11)

and

$$p_{\text{sym remnant}} = -\rho_{\text{sym remnant}}$$

(12)
3.2 The Fundamental Theoretic Approach

For the symmetry breaking remnant, Peebles [23] takes a more fundamental approach, which does not assume that the vacuum energy densities we calculate from theory must, a priori, have $w = -1$.

Peebles starts with the stress-energy tensor for a quantum field and finds the energy density and the pressure in the vacuum from that field. From this, he proves that the symmetry breaking remnant, under the proper conditions, can give rise to a $w_{\text{sym remnant}} = -1$.

Critically, Peebles does not simply assume that pressure from a vacuum contribution must equal the negative of the energy density. From fundamental principles of QFT, he proves it.

3.3 Lack of Application of Fundamental Approach to Vacuum Fluctuations

No one known to this author applies Peebles fundamental theoretic approach to vacuum fluctuations, the other form of vacuum energy density. Further, the assumption that (11) holds is widespread, in fact dominant, in the literature.

In the next section, we apply the fundamental approach of Peebles to vacuum fluctuations, to determine the true relation between pressure and energy density of the vacuum for these fluctuations [24].

4 Determining Pressure of Vacuum Fluctuations

For simplicity, we restrict ourselves to the free real scalar field, with Lagrangian density

$$\mathcal{L}_\phi = \frac{1}{2} \left( \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi - m^2 \phi^2 \right).$$

It can be shown [23] that

$$T^0_0 = \rho_\phi = \frac{1}{2} \left( \left( \dot{\phi} \right)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right),$$

$$T^{11} = p_\phi = (\partial_1 \phi)^2 + \frac{1}{2} \left( \left( \dot{\phi} \right)^2 + (\nabla \phi)^2 - m^2 \phi^2 \right).$$

First, we review the well known calculation of (14), and then follow similar steps to determine (15).

4.1 Energy Density of Vacuum Fluctuations

The real scalar field and its derivatives are

$$\phi = \sum_k \frac{1}{\sqrt{2 V_s \omega_k}} \left( a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right),$$

$$\dot{\phi} = \sum_k \frac{i \omega_k}{\sqrt{2 V_s \omega_k}} \left( -a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right),$$

$$\phi_{,i} = \sum_k \frac{i k_i}{\sqrt{2 V_s \omega_k}} \left( a(k) e^{-ikx} - a^\dagger(k) e^{ikx} \right).$$
where $V_s$ is spatial volume. The first term on the RH of (14) is

$$\frac{1}{2} \dot{\phi} \dot{\phi} = \frac{1}{2} \left( \sum_{k} \frac{i \omega_k}{\sqrt{2V_s \omega_k}} (-a(k)e^{-ikx} + a^\dagger(k)e^{ikx}) \right) \left( \sum_{k'} \frac{i \omega_{k'}}{\sqrt{2V_s \omega_{k'}}} (-a(k')e^{-ik'x} + a^\dagger(k')e^{ik'x}) \right).$$

(19)

In

$$\langle \phi_k | \rho_\phi | \phi_k \rangle$$

(20)

where $\rho_\phi$ is an operator represented by (14), and $| \phi_k \rangle$ can be any state, all terms with $k \neq k'$ drop out, as do all terms in $a(k)a(k)$ and $a^\dagger(k)a^\dagger(k)$. (19) is part of (14), so using the commutation relations

$$\left[ a(k), a^\dagger(k') \right] = \delta_{kk'},$$

(21)

that part reduces to

$$\frac{1}{2} \dot{\phi} \dot{\phi} = \frac{1}{2} \sum_{k} \left( \frac{\omega_k}{V_s} \left( a(k)a^\dagger(k) + a^\dagger(k)a(k) \right) \right) = \frac{1}{2V_s} \sum_{k} \omega_k \left( a^\dagger(k)a(k) + \frac{1}{2} \right),$$

(22)

Similarly, in (14),

$$\frac{1}{2} (\nabla \phi)^2 \rightarrow \frac{1}{2V_s} \sum_{k} \left( \frac{|k|^2}{\omega_k} \left( a^\dagger(k)a(k) + \frac{1}{2} \right) \right)$$

(23)

$$\frac{1}{2} m^2 \phi^2 \rightarrow \frac{1}{2V_s} \sum_{k} \left( \frac{m^2}{\omega_k} \left( a^\dagger(k)a(k) + \frac{1}{2} \right) \right).$$

(24)

Since $(\omega_k)^2 = m^2 + |k|^2$, the above three relations summed reduce to the well-known operator form for (14)

$$\rho_\phi = \frac{1}{V_s} \sum_{k} \omega_k \left( a^\dagger(k)a(k) + \frac{1}{2} \right),$$

(25)

which has the non-zero vacuum expectation value (VEV)

$$\langle 0 | \rho_\phi | 0 \rangle = \langle \rho_\phi \rangle = \frac{1}{V_s} \sum_{k} \omega_k \left( a^\dagger(k)a(k) + \frac{1}{2} \right).$$

(26)

4.2 Pressure of the Vacuum Fluctuations

In similar fashion, the pressure operator of (15) reduces to

$$p_{1\phi} = \frac{1}{V_s} \sum_{k} \left( \frac{|k|^2}{\omega_k} \left( a^\dagger(k)a(k) + \frac{1}{2} \right) \right)$$

(27)

and therefore from (25) and (27)

$$\langle \rho_\phi \rangle \neq -\langle p_{1\phi} \rangle.$$ 

(28)

In particular, on average, with most $k$, $\omega_k \gg m$, one finds

$$k_1^2 = k_2^2 = k_3^2 = \frac{|k|^2}{3} = \frac{\omega_k^2 - m^2}{3} \approx \frac{\omega_k^2}{3},$$

(29)

Thus, we have, from (25) and (27),

$$p_{\phi \text{flucts}} \approx \frac{\rho_{\phi \text{flucts}}}{3} \quad w_{\phi \text{flucts}} \approx \frac{1}{3} \quad (\omega_k \gg m).$$

(30)
For lower energies, where $\omega_k \approx m$, \([29]\) becomes (using classical concepts with particle velocity $v$ for illustration)

$$k^2_1 = \frac{\omega^2_k - m^2}{3} = \frac{1}{3} \left( \frac{m^2}{1 - \omega^2} - m^2 \right) = \frac{1}{3} (2m) (K.E.) \ll \omega^2_k,$$

(31)

where the fourth part above is the standard “low” energy (still relativistic) case. For energies of this level, \([27]\) becomes much less than \([25]\), so effectively

$$p_{\phi \text{flucts}} \approx 0 \quad w_{\phi \text{flucts}} \approx 0 \quad (\omega_k \approx m).$$

(32)

Thus the equation of state \([4]\) for scalar vacuum fluctuations has a range for $w$ of

$$\frac{1}{3} \geq w_{\phi \text{flucts}} \geq 0.$$

(33)

This is not the $w = -1$ value of \([7]\), and thus these vacuum fluctuations cannot give rise to an effective cosmological constant, at any level of energy.

Similar results should be found for a complex scalar field, fermions, and spin 1 bosons.

5 Conclusion

When one calculates the vacuum fluctuations pressure from fundamental principles of QFT, in the same well accepted manner as the enormous vacuum fluctuation energy is calculated, one finds the pressure is not equal to the negative of the vacuum fluctuation energy density.

The common approach to vacuum fluctuation pressure determination comprises the following.

1. Assume vacuum energy density $\rho_{\text{vac}}$ must be Lorentz invariant, constant in time, and give rise (in principle) to a cosmological constant.

2. Assume the $1/2$ quanta energy summation in the VEV of the quantized Hamiltonian (equivalently, the time-time component of the quantized stress energy tensor), for example, $\rho_{\phi \text{flucts}}$, is vacuum energy obeying 1.

3. Conclude that one must then have vacuum fluctuation pressure $p_{\phi \text{flucts}} = -\rho_{\phi \text{flucts}}$ (i.e., $w_{\phi \text{flucts}} = -1$).

On the other hand, the fundamental theoretic approach, which is generally accepted as valid for determining $w$ for the symmetry breaking vacuum energy density remnant, applied to vacuum fluctuations, comprises the following.

1. Use the quantized stress-energy tensor for a given field to calculate $1/2$ quanta pressure, for example, $p_{\phi \text{flucts}}$.

2. Compare the value in 1 to the known $\rho_{\phi \text{flucts}}$.

3. Conclude that $0 < p_{\phi \text{flucts}} < \frac{\rho_{\phi \text{flucts}}}{3}$ (i.e., $w_{\phi \text{flucts}} \neq -1$).
The common approach seems to “put the cart before the horse” by encompassing *a priori* assumptions, whereas the fundamental theoretic approach starts from elemental principles, and does not. Thus, it is submitted that, contrary to common belief, vacuum fluctuations cannot qualify as a candidate for an effective cosmological constant (which requires \( w = -1 \)), regardless of order.

This can mean one of two things. For one, an undetermined symmetry \[25\] may exist that cancels out the \( \frac{1}{2} \) quanta contributions leaving a net vacuum fluctuation energy of zero (which is Lorentz invariant, constant, and has no cosmological implications). Alternatively, and less satisfying, the \( \frac{1}{2} \) quanta energy density may simply be neither Lorentz invariant nor constant in time.

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