Estimating a discrete distribution subject to random left-truncation with an application to structured finance

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Abstract

Proper econometric analysis should be informed by data structure. Many forms of financial data are recorded in discrete-time and relate to products of a finite term. If the data comes from a financial trust, it will often be further subject to random left-truncation. While the literature for estimating a distribution function from left-truncated data is extensive, a thorough literature search reveals that the case of discrete data over a finite number of possible values has received little attention. A precise discrete framework and suitable sampling procedure for the Woodroofe-type estimator for discrete data over a finite number of possible values is therefore established. Subsequently, the resulting vector of hazard rate estimators is proved to be asymptotically normal with independent components. Asymptotic normality of the survival function estimator is then established. Sister results for the left-truncating random variable are also proved. Taken together, the resulting joint vector of hazard rate estimates for the lifetime and left-truncation random variables is proved to be the maximum likelihood estimate of the parameters of the conditional joint lifetime and left-truncation distribution given the lifetime has not been left-truncated. A hypothesis test for the shape of the distribution function based on our asymptotic results is derived. Such a test is useful to formally assess the plausibility of the stationarity assumption in length-biased sampling. The finite sample performance of the estimators is investigated in a simulation study. Applicability of the theoretical results in an econometric setting is demonstrated with a subset of data from the Mercedes-Benz 2017-A securitized bond.

Keywords: asset-backed security, asset-level disclosures, consumer lease securitization, product-limit estimator, reverse hazard rate, Reg AB II
1 Introduction

The current outstanding issuance of consumer auto lease asset-backed securities (ABS) in the United States is nearly $35 billion (SIFMA, 2022), and the recent implementation of Reg AB II (SEC, 2016) has made a glut of public asset level ABS data available to investors for the first time. While more transparency into the underlying assets is generally a benefit for investors, the data may be difficult to analyze. This is because the legal structure of an ABS trust, the terms of a standard consumer automobile lease contract, and the nature of a monthly due date creates a need to consider left-truncation, a finite time horizon, and discrete-time, respectively, in estimating the distribution of consumer lease lifetimes. We elaborate with a specific example from structured finance. Consider an automotive lease securitization, such as Mercedes-Benz (2017), in which consumer automotive lease contracts are pooled together into a trust. Standard automotive lease contracts have a fixed and known duration, such as 36 months, with required monthly payments. Further, the payment performance of the lessee will be reported monthly, so the observed survival times of the lease contracts will be discrete within the nonnegative integers, \( N \). Left-truncation occurs because only those leases that remain active long enough to be collected into the trust will be observable by the investor. In the literature of survival analysis, this is a form of bias under the general umbrella of delayed entry or length-biased sampling (e.g., Asgharian et al., 2002; De Uña-Álvarez, 2004; Asgharian and Wolfson, 2005; Huang and Qin, 2011).

To formalize, let \( X \) denote the random time of a lease contract termination (i.e., the lifetime or time-to-event random variable) and let \( T \) denote the random time of a lease contract origination. The context of our application naturally restricts \( X \) and \( T \) to a finite subset of consecutive integers. If \( \omega \in \mathbb{N} \) represents the age of the last lease termination in a sample, then \( X \leq \omega \). Since issuers of structured debt typically have a legal obligation to the trust to select lease contracts with a minimum history of on-time payments, the youngest lease in the trust will have a minimum age of \( \Delta \) as of the onset of the trust, where \( \Delta \in \mathbb{N} \). Hence, each lease will have a minimum survival time of \( \Delta + 1 \), and so \( \Delta + 1 \leq X \leq \omega \).
If \( m \in \mathbb{N} \) is the origination time of the youngest lease in the trust, then \( 1 \leq T \leq m \) and the trust starting time is \( m + \Delta \). For all practical purposes, \( m + \Delta \leq \omega \). The integers \( \Delta, m, \) and \( \omega \) are non-random and known as of the onset of the problem. Notably, if we define \( Y = m + \Delta + 1 - T \), then \( Y \) denotes a left-truncation random variable representing the minimum amount of time a lease must remain active to be observed in the trust. In other words, an investor will only observe those leases such that \( X \geq Y \). For completeness, \( \Delta + 1 \leq Y \leq \Delta + m \). We present a visualization of the connected random variables and timelines in Figure 1. Throughout, we assume \( X \) and \( T \) are independent (and therefore \( X \) and \( Y \) are independent).

\[
\begin{array}{c}
1 & \quad T & \quad m \\
\end{array}
\]

(a) Symbolically, \( T \) represents a random lease start time. Nonrandom time \( m \) is the origination time of the youngest lease in the trust as of the beginning of the trust observation window. Thus, \( 1 \leq T \leq m \), where \( T, m \in \mathbb{N} \).

\[
\begin{array}{c}
Y \\
\Delta + 1 & \quad m + \Delta + 1 - T & \quad \Delta + m \\
\end{array}
\]

(b) We call the time that the trust observation window begins \( \Delta + m \), and so nonrandom \( \Delta \) denotes the minimum age of a lease in the trust as of time \( \Delta + m \). Defining \( Y = m + \Delta + 1 - T \) with \( 1 \leq T \leq m \) implies \( \Delta + 1 \leq Y \leq \Delta + m \), where \( Y, m, \Delta, T \in \mathbb{N} \).

\[
\begin{array}{c}
Y & \quad X & \quad \omega \\
\end{array}
\]

(c) We only observe the random lease termination time, \( X \), if \( X \geq Y \). Nonrandom \( \omega \) represents the termination time of the lease with the longest active ongoing payments, and it coincides with the close of the trust observation window. Thus, \( \Delta + 1 \leq X \leq \omega \), where \( X, \Delta, \omega, Y \in \mathbb{N} \).

Figure 1: The connected discrete random variables \( T, Y, X \) and the associated finite timelines for left-truncated data from an auto lease securitization.

The classical problem of estimating a distribution function in the presence of random left-truncation has a sizable history in the statistical literature. Specifically, if we consider two independent positive random variables \( X \) and \( Y \) with distribution functions \( F \) and \( G \) such that we only observe the pairs \( (X,Y) \) for which \( Y \leq X \) and the pairs \( (X,Y) \) are
assumed to be independent and identically distributed (i.i.d.), it is not difficult to find many thorough studies (e.g., Lynden-Bell, 1971; Woodroofe, 1985; Wang et al., 1986; Keiding and Gill, 1990; Stute, 1993; He and Yang, 1998a). However, the nature of securitization data requires us to further assume that $X$ and $Y$ are nonnegative integer-valued random variables with a finite number of possible values (though the remaining assumptions of the classical problem remain valid).

To our surprise, a thorough literature review revealed that the case of discrete $F$ and $G$ have received little attention. Two seminal works in this field are Woodroofe (1985) and Wang et al. (1986). Woodroofe (1985) proves consistency results for the Lynden-Bell (1971) estimator and shows its weak convergence to a Gaussian process but left the exact form of the covariance structure of the limiting process undefined. In deriving the asymptotic results, Woodroofe (1985) assumes continuous distribution functions $F$ and $G$. Wang et al. (1986) extends the results of Woodroofe (1985) with a precise description of the asymptotic covariance structure. It is noteworthy that this structure is the analogue of the covariance structure of the Kaplan–Meier estimator. Wang et al. (1986) alludes to the idea that $F$ and $G$ need not be continuous in establishing strong consistency for the product limit estimator of $F$, but they assume continuity of $F$ and $G$ in working to define the covariance structure. Since Woodroofe (1985) and Wang et al. (1986), there has been many notable and significant contributions; interested readers may find a thorough literature review in Appendix A.

Typical approaches for avoiding an assumption of discrete-time may be problematic or inappropriate for consumer lease ABS data. First, one may be tempted to force an assumption of continuous $F$ and $G$, but this implies that ties are events with zero probability. Since there are likely many lease contracts with the same termination time, this creates an immediate complication. Second, one may treat the lease performance data as interval-censored, where the event is assumed to occur within an interval of time (i.e., a month) but the exact time within the interval is unknown. With lease contracts and loan contracts more generally, however, payments made prior to a due date are treated the same as payments made on the due
date (prepayments aside). In other words, a monthly payment was either received on-time or is delinquent. Thus, the ABS data is in actuality discrete with jumps at integer intervals; it is not a product of imprecise measurements. For similar reasons, even standard grouped survival data approaches (e.g., Prentice and Gloeckler, 1978) are not true representations of the failure time random variable for consumer lease data. One technical remark is that if the distribution function is known to contain jumps, but the location of such jumps is not known prior to performing the estimation, the analysis is subject to additional complications, as in Rabhi and Asgharian (2017). We are working over \( \mathbb{N} \) and thus may avoid this potentially cumbersome framework.

Overall, our contributions are thus. We fill the unexpected gap in the literature for the discrete case of \( X \) and \( Y \) for the Woodroofe-type estimator. The first main result is that the vector of these estimators in discrete-time over a finite number of possible values is asymptotically normal with a fully-specified diagonal covariance matrix. The second main result is that the vector of Woodroofe-type estimates is the MLE for the parameters of the discrete bivariate distribution of \((X, Y)\) given \( Y \leq X \). We also find these results have a significant application potential for the large fixed-income asset class of consumer lease ABS data. A detailed outline is as follows. In Section 2, we precisely define the joint conditional discrete sample space for \( X \) and \( Y \), the related discrete conditional bivariate probability mass function and its connection to \( F \) and \( G \) through the hazard and reverse hazard rates, respectively, and a suitable sampling procedure to mimic the realities of securitized trust data (this sampling process differs from Woodroofe (1985)). These preliminaries are necessary because the discrete case has not before received a rigorous treatment in the literature. Section 2 closes by presenting the estimators and the first major result: all together, these estimates are the MLE of the discrete conditional bivariate distribution. Section 3 provides the next set of major results in that we state the asymptotic normality and independence of the estimation vector of the hazard rates for \( F \) and its analog for \( G \); and, asymptotic normality of the estimator for the survival function of \( X \) and the estimator of the distribution
function of $Y$. In all cases, the diagonal covariance matrix is completely specified. We also derive a hypothesis test for the shape of the distribution function with applications to length-biased sampling. In Section 4, we experimentally validate the results in Section 3 with a simulation study. In Section 5, we apply our results to a sample of data from the Mercedes Benz 2017-A securitized bond (Mercedes-Benz, 2017). The paper closes with a brief discussion. Appendix A presents a thorough literature review, and Appendix B provides complete proofs of all major results.

2 Estimation

We begin by briefly reviewing notation and the identifiability results from Woodroofe (1985). Establishing the discrete sample space begins in the next section on recovery. This requires defining a discrete conditional bivariate probability mass function and related conditional distributions for discrete $X$ and $Y$. Because of the discrete nature of $X$ and $Y$, it is preferable to work in terms of the hazard rate of $X$ and the reverse hazard rate of $Y$ (for continuous $F$ and $G$, the cumulative hazard function works well). We then connect the hazard and reverse hazard rates to $F$ and $G$, respectively, in the discrete case. Estimators for both the hazard and reverse hazard rates are then formally defined in the context of sampling from a left-truncated population rather than left-truncating a joint random sample (a further distinction from Woodroofe (1985)). Finally, we formally state the result that the joint vector of estimates for the hazard and reverse hazard rates is a MLE for the discrete conditional bivariate probability distribution for $(X, Y)$ given $X \geq Y$.

2.1 Preliminaries

Working from the notation of Woodroofe (1985), let $F$ and $G$ be the distribution functions of non-negative independent random variables $X$ and $Y$, respectively. Let $H_*$ denote the joint distribution function of $X$ and $Y$ given $Y \leq X$, and let $F_*$ and $G_*$ denote the marginal
distributions functions given \( Y \leq X \) of \( X \) and \( Y \), respectively. That is,

\[
H_*(F, G, x, y) = \Pr(X \leq x, Y \leq y \mid X \geq Y),
\]
is the joint conditional distribution function with conditional marginal distributions \( F_* \) and \( G_* \). We include \( F \) and \( G \) within the definition of \( H_* \) to stress which \( F \) and \( G \) are employed to construct \( H_* \). For convenience, we may drop \( x \) and \( y \) from the notation for \( H_* \) when the meaning is clear or if the clarification is nonessential; i.e., \( H_*(F, G) \).

We now review key observations made by Woodroofe (1985). Define

\[
a_F = \inf\{z > 0 : F(z) > 0\} \geq 0,
\]

and

\[
b_F = \sup\{z > 0 : F(z) < 1\} \leq \infty.
\]

That is, \((a_F, b_F)\) is the interior of the convex support of \( F \) and similarly \((a_G, b_G)\) for \( G \). To avoid complete left-truncation and full data loss, we must have \( a_G < b_F \).

Next, we need to introduce two classes of distribution pairs \((F, G)\). The first class includes all pairs of \( F \) and \( G \) that allow the construction of the two-dimensional distribution \( H_* \),

\[
K = \{(F, G) : F(0) = 0 = G(0), \quad \Pr(Y \leq X) > 0\}.
\]

The second class includes those pairs \((F, G)\) that can be recovered from \( H_* \),

\[
K_0 = \{(F, G) \in K : a_G \leq a_F, \quad b_G \leq b_F\}.
\]

Woodroofe (1985) demonstrated in his Lemma 1 that if we take any \((F, G) \in K\) and let

\[
F_0 = \Pr(X \leq x \mid X \geq a_G) \quad \text{and} \quad G_0 = \Pr(Y \leq y \mid Y \leq b_F),
\]

then \((F_0, G_0) \in K_0\) and \(H_*(F_0, G_0) = H_*(F, G)\). This subtle but important result implies that, if given \( H_* \), we may
not be able to recover the pair \((F, G)\). This is because there is another pair, \((F_0, G_0)\), that gives us exactly the same \(H_*\). It is not surprising. For example, in the context of our motivating problem, we only observe \(X\) when it is equal or greater than \(\Delta + 1\). Hence, it is impossible to get any information on the distribution of \(X\) over values less than \(\Delta + 1\).

### 2.2 Recovery

Woodroofe (1985) shows in his Theorem 1 that if we restrict our construction of \(H_*\) to class \(K_0\), then this operation is “invertible”. More specifically, for every \(H\) based on some \((F, G) \in K\) there is only one pair \((F_0, G_0) \in K_0\) such that \(H_*(F_0, G_0) = H_*(F, G)\) and this pair is given by \(F_0\) and \(G_0\). Moreover, this theorem gives specific instructions on how to recover the cumulative hazard functions of \(F_0\) and \(G_0\) (and, therefore, \(F_0\) and \(G_0\) as well).

Once again, in the context of our example, we have \(F_0(x) = \Pr(\Delta + 1 \leq X \leq x)/\Pr(X \geq \Delta + 1)\) (that is, the range of \(F_0\) is \(\{\Delta + 1, \ldots, \omega\}\)) and \(G_0(y) = \Pr(Y \leq y) = G(y)\) because \(\Delta + m \leq \omega\) by assumption. The range of \(G_0\) is \(\{\Delta + 1, \ldots, \Delta + m\}\). Thus, from \(H_*\) based on the original \(F\) and \(G\), it is possible to recover \(G\) but only the \(F_0\) portion of \(F\).

We have discussed \(X\) and \(Y\) at length thus far, but we now do so with some additional precision. Specifically, let \(X \in \mathbb{N}\) and \(Y \in \mathbb{N}\) be independent random variables with ranges \(\{\Delta + 1, \ldots, \omega\}\) and \(\{\Delta + 1, \ldots, \Delta + m\}\), respectively. We will assume that \(\Pr(X = \Delta + 1), \Pr(Y = \Delta + 1), \Pr(X = \omega), \text{ and } \Pr(Y = \Delta + m)\) are strictly positive, and \(\Delta + m \leq \omega\).

Let \(A\) be a set of points on the plane \(\mathbb{N} \times \mathbb{N}\) with integer-valued coordinates \((u, v)\) such that \(u \in \{\Delta + 1, \ldots, \omega\}\), \(v \in \{\Delta + 1, \ldots, \Delta + m\}\), and \(v \leq u\). A visualization of \(A\) may be found in Figure 2.

Let

\[
    f(u) = \Pr(X = u), \quad g(v) = \Pr(Y = v), \quad \text{and} \quad \alpha = \Pr(Y \leq X).
\]

The bivariate distribution function \(H_*\) over the trapezoid \(A\) has probability mass function
Figure 2: The set of points on the plane with \((u, v) \in \mathbb{N}\) such that \(u \in \{\Delta + 1, \ldots, \omega\}\), \(v \in \{\Delta + 1, \ldots, \Delta + m\}\), and \(v \leq u\). The shaded region is the sample space of \(H_s\) and is denoted by trapezoid \(A\). Since \(X\) and \(Y\) are discrete, all of the probability is contained in masses on the discrete points within the shaded region. If we assume that \(\Pr(X = \Delta + 1)\), \(\Pr(Y = \Delta + 1)\), \(\Pr(X = \omega)\), and \(\Pr(Y = \Delta + m)\) are strictly positive, then the edges of \(A\) are identifiable.

(pmff)

\[
h_s(u, v) = \Pr(X = u, Y = v \mid Y \leq X) = \frac{f(u)g(v)}{\alpha}.
\]

This simple observation tells us that not every distribution over \(A\) can be a result of our left-truncation procedure. The marginal distributions of \(H_s\) are given by

\[
f_s(u) = \Pr(X = u \mid Y \leq X) = \sum_v h_s(u, v),
\]

and

\[
g_s(v) = \Pr(Y = v \mid Y \leq X) = \sum_u h_s(u, v).
\]

For our forthcoming results to be meaningful, it must be possible to express the pmf \(f\) (or \(g\)) in terms of the pmf \(h_s\). Notably, Woodroofe (1985) shows us by his Theorem 1 that we can
indeed do so by expressing the cumulative hazard rate function in terms of joint cumulative distribution function (cdf) $H_\ast$. Since we deal only with discrete random variables, however, it is more convenient to work with the hazard rate for $X$,

$$
\lambda(x) = \frac{\Pr(X = x)}{\Pr(X \geq x)},
$$

where $x \in \{\Delta + 1, \Delta + 2, \ldots, \omega\}$. One can show that

$$
\lambda(x) = \frac{f_\ast(x)}{C(x)},
$$

where

$$
C(x) = \Pr(Y \leq x \leq X \mid Y \leq X) = \sum_{v \leq x \leq u} h_\ast(u, v).
$$

Having $C(x)$ in the denominator is not a concern, because for any $x$,

$$
C(x) \geq h_\ast(\omega, \Delta + 1) = \frac{f(\omega)g(\Delta + 1)}{\alpha} > 0.
$$

The re-construction of the cdf $F$ from the hazard rate $\lambda$ is based on the following standard
result of survival analysis. For any integer $x$ such that $\Delta + 1 < x \leq \omega$,

$$
\prod_{\Delta + 1 \leq k < x} [1 - \lambda(k)] = \left[ \frac{\Pr(X \geq \Delta + 2)}{\Pr(X \geq \Delta + 1)} \right] \left[ \frac{\Pr(X \geq \Delta + 3)}{\Pr(X \geq \Delta + 2)} \right] \cdots \left[ \frac{\Pr(X \geq x)}{\Pr(X \geq x - 1)} \right]
$$

$$
= \Pr(X \geq x),
$$

(5)

with the convention that (5) is unity for $x \leq \Delta + 1$. Since $X$ is discrete, it is enough to know $F$ at the jump points.

In a similar fashion, one can derive an analog of formula (2) for what is sometimes known as the reverse hazard rate function (for a nice introduction, see Block et al., 1998). The reverse hazard rate is effectively analogous to the hazard rate in (2) but backwards-looking. That is, the reverse hazard rate is the probability of the event of interest occurring in the current interval, given we know the event of interest occurred prior to the current interval. Formally, the reverse hazard rate is defined as

$$
\beta(y) = \frac{\Pr(Y = y)}{\Pr(Y \leq y)} = \frac{g^*(y)}{C(y)},
$$

(6)

where $y \in \{\Delta + 1, \Delta + 2, \ldots, \Delta + m\}$. As a consequence, we get the following formula for the cdf $G$,

$$
\Pr(Y \leq y) = \prod_{\Delta + m \geq k > y} [1 - \beta(k)],
$$

(7)

where $\Delta + 1 \leq y \leq \Delta + m$.

### 2.3 Estimators

There are different ways to think about sampling in the case of left-truncation. For example, Woodroofe (1985) assumes that there is a population of $X$s and $Y$s, from which we take a sample of size $N$. Then we apply left-truncation to the sample, and this gives a sample of left-truncated pairs of random sample size $n$. Our thinking, however, is different. We
assume that there is the original population of $X$s and $Y$s. We apply left-truncation to the entire population to get a population of left-truncated pairs. Then we extract a sample of deterministic size $n$ from the left-truncated population. That is, our observations are directly from the distribution $H_*$. Given the practicalities of the securitization process, sampling from $H_*$ directly is more appropriate for our application than the assumed sampling process of Woodroofe (1985). Phrased differently, our sampling process effectively samples from the already left-truncated lease data within the trust rather than imagines we are able to sit with the ABS issuer and see loans that did not meet the minimum survival requirements to be included in the trust. A theoretical divergence with generally limited practical significance, but its importance is evident with ABS data.

Formally, let $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ be i.i.d. pairs of random variables with distribution $H_*$ (i.e., a sample from distribution $H_*$ on trapezoid $A$ of Figure 2). This is materially different than the sampling space of all possible target population pairs of $(X, Y)$ absent left-truncation. In other words, referring again to Figure 2, there is a bias from the left-truncation condition in that some pairs, such as $u = \Delta + 1$ and $v = \Delta + m$, are not observable. This distinction warrants emphasis because it is erroneous to assume both pairs $(X, Y)$ and $(X_i, Y_i)$, $1 \leq i \leq n$, share the same properties (e.g., while $X$ and $Y$ are assumed to be independent, $X_i$ and $Y_i$ clearly are not).

We desire to provide interval estimates for the hazard rates of $F_0$, the reverse hazard rates of $G$, and the cdfs $F_0$ and $G$ from the i.i.d. sample $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ with distribution $H_*$. Examination of (2) tells us that the hazard rate $\lambda(x)$ is a ratio of two probabilities of some events related to random variables $(X_i, Y_i)$, $1 \leq i \leq n$, and we have natural estimates of each probability within the ratio vis-à-vis the observed frequencies. This suggests the following estimator for the hazard rate,

$$
\hat{\lambda}_n(x) = \frac{\frac{1}{n} \sum_{i=1}^{n} 1_{X_i = x}}{\hat{C}_n(x)},
$$

(8)
where $1_{(.)}$ is the standard indicator function taking value 1 if statement $(\cdot)$ is true and 0 otherwise, and

$$
\hat{C}_n(x) = \frac{1}{n} \sum_{j=1}^{n} 1_{Y_j \leq x \leq X_j}.
$$

(9)

By employing the same method of (5), we immediately get an estimator for the cdf $F_0$,

$$
\hat{F}_n(x) = 1 - \prod_{\Delta + 1 \leq k \leq x} [1 - \hat{\lambda}_n(k)].
$$

(10)

In similar fashion we can produce the following estimator of the reverse hazard rate $\beta(y)$ of $Y$,

$$
\hat{\beta}_n(y) = \frac{\frac{1}{n} \sum_{i=1}^{n} 1_{Y_i = y}}{\hat{C}_n(y)},
$$

(11)

and the cdf $G$,

$$
\hat{G}_n(y) = \prod_{\Delta + m \geq k > y} [1 - \hat{\beta}_n(k)].
$$

(12)

It is theoretically satisfying that the estimators (10) and (12) coincide with the corresponding estimators (8) and (9) in Woodroofe (1985), despite the alternative constructive path our discrete data framework required. It is in proceeding to analyze the asymptotic properties of the estimators (8) and (11), however, that we can take advantage of the discrete structure of $h_*$ in (1) to directly address complications traditionally assumed away. For example, we can handle ties among the discrete sample space of $X$ and $Y$ in proving the vector of estimates (8) and (11) are together the maximum likelihood estimate (MLE) of the parameters of the conditional bivariate distribution $H_*$ (Woodroofe (1985) assumes no ties, for example). Furthermore, in deriving the estimator’s asymptotic properties, many authors (see Appendix A) assume continuous $F$ and $G$ to avoid convergence argument complications introduced by potentially unexpected discrete point masses in the distribution functions (for an example of the complications in trying to account for such point masses without a priori knowledge, see the change point analysis and proofs of Rabhi and Asgharian (2017)).

As alluded to in the previous paragraph, it is noteworthy that the joint vector of estimates
with components (8) and (11) can be shown to be the MLE of the parameters of the discrete conditional bivariate distribution $H_*$. That $H_*$ is a parametric distribution may not be obvious. To see this, observe that $h_*(u, v)$ defined in (1) is a function of the discrete mass probabilities $0 \leq f(u) \leq 1$, $\Delta + 1 \leq u \leq \omega$, and $0 \leq g(v) \leq 1$, $\Delta + 1 \leq u \leq \Delta + m$. Thus, the probabilities $f$ and $g$ are in actuality parameters (of which only $m + \omega - 2$ are free, as we require $\sum_u f(u) = \sum_v g(v) = 1$).

There also exists an equivalent one-to-one parameterization of $h_*(u, v)$ using the hazard rates $\lambda$ and $\beta$. Specifically, from (4) and (5),

$$f(u) = \lambda(u) \prod_{k=1}^{u-1} [1 - \lambda(k)], \quad \lambda(u) = \frac{f(u)}{1 - \sum_{k=1}^{u-1} f(k)}, \quad \Delta + 1 \leq u \leq \omega,$$

(13)

with the conventions $\prod_{k=1}^{0} [1 - \lambda(k)] = 1$ and $\sum_{k=1}^{0} f(k) = 0$. Similarly, from (6) and (7),

$$g(v) = \beta(v) \prod_{k=v+1}^{\Delta+m} [1 - \beta(k)], \quad \beta(v) = \frac{g(v)}{1 - \sum_{k=v+1}^{\Delta+m} g(k)}, \quad \Delta + 1 \leq v \leq \Delta + m,$$

(14)

with the conventions $\prod_{k=\Delta+m+1}^{\Delta+m} [1 - \beta(k)] = 1$ and $\sum_{k=\Delta+m+1}^{\Delta+m} g(k) = 0$. That there are still $m + \omega - 2$ free parameters is evident from the known hazard rate probabilities $\lambda(\omega) = \beta(\Delta + 1) = 1$. With this background, we formally state the MLE property of the joint vector of estimates with components (8) and (11) in terms of the parameters of $H_*$ in Theorem 2.1. The complete proof may be found in Appendix B.1, and we also present an outline of the proof immediately following Theorem 2.1, as it may be of interest to some readers.

**Theorem 2.1.** Define the discrete-mass trapezoid

$$\mathcal{A} = \{ (u, v) : \Delta + 1 \leq u \leq \omega, \Delta + 1 \leq v \leq \Delta + m, v \leq u \},$$

and consider the bivariate distribution $h_*(u, v)$ defined in (1) over $\mathcal{A}$. Let $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ be $n$ independent and identically distributed pairs of observations sampled from $h_*(u, v)$ such
that \( \hat{f}_{*,n}(u) = \frac{1}{n} \sum_{i=1}^{n} 1_{X_i = u} > 0 \) for \( u \in \mathcal{A} \), and \( \hat{g}_{*,n}(v) = \frac{1}{n} \sum_{i=1}^{n} 1_{Y_i = v} > 0 \) for \( v \in \mathcal{A} \).

Further define

\[
\hat{\Lambda}_n = (\hat{\lambda}_n(\Delta + 1), \ldots, \hat{\lambda}_n(\omega - 1), 1)^\top,
\]

and

\[
\hat{B}_n = (1, \hat{\beta}_n(\Delta + 2), \ldots, \hat{\beta}_n(\Delta + m))^\top,
\]

where \( \hat{\lambda}_n \) and \( \hat{\beta}_n \) follow from (8) and (11), respectively. Then the joint vector of estimates, \((\hat{\Lambda}_n, \hat{B}_n)^\top\), is a MLE for the parameters, \( \lambda(u), \beta(v) \), \( u, v \in \mathcal{A} \), of the bivariate distribution \( h_*(u,v) \).

**Proof Outline.** From the one-to-one correspondence of the two parameterizations of the distribution \( H_* \) and the invariance property of the MLE (e.g., Mukhopadhyay, 2000, Theorem 7.2.1, pg. 350), it is sufficient to find the MLE for the parameters \( f \) and \( g \) and demonstrate they are exactly equal to the estimates (8) and (11) in the same form as (13) and (14), respectively. It is preferable to define the likelihood in terms of the parameters \( f \) and \( g \) because the equivalent likelihood with a parameterization in terms of \( \lambda \) and \( \beta \) is cumbersome.

To maximize the likelihood, we restrict the parameter space of \( f \) and \( g \) to the convex set of all \( 0 < f, g < 1 \) such that \( \sum_u f(u) = \sum_v g(v) = 1 \). The convexity of this restricted parameter space in conjunction with the behavior of the loglikelihood on the boundary of the parameter space confirms that the maximum point must lie within the restricted parameter space. We next solve the system of partial derivatives with respect to each parameter \( f(u) \), \( g(v) \) \( u, v \in \mathcal{A} \), equated to zero sequentially to show the solution set admits only one solution, which must therefore be the global maximum and MLE. If we move sequentially from the minimum points of \( u \) and \( v \), we can show the MLE for each \( f \) is exactly of the form (13). The complete result for \( g \) then follows by solving the system of partial derivative equations equated to zero sequentially from the maximum points of \( u \) and \( v \) (i.e., symmetry).

\[\square\]

There are some related results. For example, Vardi (1982) finds the MLE in the situation
of a length-biased distribution, but the sampling procedure is not from \( h_\ast \). Instead, two independent samples are used, the latter of which is from a length-biased distribution. In the derivation of Woodroofe (1985), the sampling procedure used therein is not from \( h_\ast \) but instead the complete pairs \((X, Y)\), of which a random quantity are truncated (i.e., whenever \( Y > X \)). Further, the results are given assuming no ties among the left-truncation or lifetime distribution observations. Wang (1987) also assumes the same sampling procedure as Woodroofe (1985) (i.e., stopping time theory). Keiding and Gill (1990) assume throughout that \( \Pr(Y = X) = 0 \), which is also not necessary in our framework. Furthermore, as indicated in Appendix A, the discrete case is largely left unstudied. We thus find our proof of Theorem 2.1 to be the first direct proof in the literature that the estimation vector with components (8) and (11) is indeed the MLE for the parameters of the discrete conditional distribution \( H_\ast \).

3 Asymptotic Results

We now establish asymptotic normality of the hazard rate and reverse hazard rate estimators, along with the unanticipated result of independence. We also prove asymptotic normality of the survival function estimator for \( X \) and distribution function estimator for \( Y \). Finally, this section closes with a hypothesis test for the shape of \( G \). All corresponding proofs may be found in Appendix B. We set the stage with three comments as follows.

First, notice that throughout this section, as before, \( X \) and \( Y \) are positive discrete random variables with distribution functions \( F \) and \( G \), respectively, and \( \{(X_i, Y_i)\}_{1 \leq i \leq n} \) are independent and i.i.d. distributed pairs of random variables with distribution \( H_\ast \). More specifically, \( \{(X_i, Y_i)\}_{1 \leq i \leq n} \) are a random sample from a population with probability mass function \( h_\ast \) in Equation (1), spanning the finite set of points on the plane with integer-valued coordinates \((u, v)\) such that \( u \in \{\Delta + 1, \ldots, \omega\} \), \( v \in \{\Delta + 1, \ldots, \Delta + m\} \), \( \Delta + m \leq \omega \), and \( v \leq u \) (trapezoid \( A \) of Figure 2). Additionally, we will continue to assume that \( \Pr(X = \Delta + 1) \),
Pr(Y = Δ + 1), Pr(X = ω), and Pr(Y = Δ + m) are strictly positive. See Figure 2 as necessary.

Second, recall that we apply left-truncation to the entire population of X and Y, which yields a population of left-truncated pairs. From this left-truncated population, we draw a sample of deterministic size n. Therefore, in what follows, we investigate the limiting behavior as n → ∞.

Third and finally, to state our asymptotic results it is convenient to introduce the following notation. Let (X_i, Y_i), 1 ≤ i ≤ n be a sample pair from h*. Then we denote

\[ c(u, v) = \Pr(Y_i \leq u \leq X_i, Y_i \leq v \leq X_i) \]
\[ = \Pr(X \geq \max(u, v), Y \leq \min(u, v) \mid Y \leq X) \]
\[ = \sum_{y=\Delta+1}^{\min(u,v)} \sum_{x=\max(u,v)}^{L} h_{*}(x, y) \]
\[ = \frac{1}{\alpha} \Pr(Y \leq \min(u, v)) \Pr(X \geq \max(u, v)). \quad (17) \]

The various equation steps have been left as a form of summary of probability statements to show the connection between a single sampled pair (X_i, Y_i), 1 ≤ i ≤ n, from h*, the pair (X, Y) conditional on Y ≤ X, the pmf h* itself (i.e., Figure 2), and the importance of assuming independence between X and Y. Finally, we observe c(z, z) = C(z) and c(u, v) = c(v, u).

### 3.1 Hazard and Reverse Hazard Rate

We first inspect the estimator \( \hat{C}_n \) in the denominator of the Woodroofe-type estimator (8) with the multivariate Central Limit Theorem (CLT).

**Lemma 1 (\( \hat{C}_n \) Asymptotic Normality).** Define \( \hat{C}_n = (\hat{C}_n(\Delta + 1), \ldots, \hat{C}_n(\omega))^\top \). Then,

\[ \sqrt{n}(\hat{C}_n - C) \xrightarrow{L} N(0, \Sigma_c), \text{ as } n \to \infty, \]
where $C = (C(\Delta + 1), \ldots, C(\omega))^\top$ and $\Sigma_c$ is a covariance matrix $\|\sigma_{k',k}\|$ such that

$$
\sigma_{k',k} = \begin{cases} 
C(k)[1 - C(k)], & k' = k \\
c(k',k) - C(k')C(k), & k' \neq k
\end{cases},
$$

for $k', k = \Delta + 1, \Delta + 2, \ldots, \omega$.

**Lemma 2.** As $n \to \infty$, $\hat{C}_n \xrightarrow{p} C$.

The discrete nature of $X$ and $Y$ along with the finite sample space of trapezoid $A$ yields attractive mathematical conveniences, which lead themselves naturally to computational programming. The same is true for the Woodroofe-type estimator of the hazard rate $\lambda$, which we now examine.

**Theorem 3.1 (\hat{\Lambda}_n Asymptotic Normality).** For $\hat{\Lambda}_n$ defined in (15),

$$
\sqrt{n}(\hat{\Lambda}_n - \Lambda) \xrightarrow{d} N(0, \Sigma_f), \text{ as } n \to \infty,
$$

where $\Lambda = (\lambda(\Delta + 1), \lambda(\Delta + 2), \ldots, \lambda(\omega))^\top$ with $\lambda(z) = f_*(z)/C(z)$ and

$$
\Sigma_f = \text{diag}\left(\frac{f_*(\Delta + 1)c(\Delta + 1, \Delta + 2)}{C(\Delta + 1)^3}, \ldots, \frac{f_*(\omega - 1)c(\omega - 1, \omega)}{C(\omega - 1)^3}, 0\right). \quad (18)
$$

That is, the estimators $\hat{\lambda}_n(\Delta + 1), \ldots, \hat{\lambda}_n(\omega)$ are asymptotically normal and independent.

**Remark.** There is an alternative form of $\Sigma_f$ that may be preferable. Observe for $x \in \{\Delta + 1, \ldots, \omega\}$,

$$
\frac{f_*(x)c(x, x + 1)}{C(x)^3} = \frac{f_*(x) \alpha^{-1} \Pr(Y \leq x) \Pr(X \geq x + 1)}{C(x) \left[\alpha^{-1} \Pr(Y \leq x) \Pr(X \geq x)\right]^2}
= \frac{\lambda(x)^2[1 - \lambda(x)]}{f_*(x)}.
$$
Hence, alternatively,

$$
\Sigma_f = \text{diag} \left( \frac{\lambda(\Delta + 1)^2[1 - \lambda(\Delta + 1)]}{f_s(\Delta + 1)}, \ldots, \frac{\lambda(\omega - 1)^2[1 - \lambda(\omega - 1)]}{f_s(\omega - 1)}, 0 \right). \quad (19)
$$

Further, (18) and (19) are equivalent when the true quantities are replaced by their MLEs. That is, for \( x \in \{\Delta + 1, \ldots, \omega\} \) with

$$
\hat{f}_{s,n}(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{X_i = x}, \quad \text{and} \quad \hat{c}_n(x, x + 1) = \frac{1}{n} \sum_{i=1}^{n} 1_{Y_i \leq x, X_i \geq x + 1}, \quad (20)
$$

it is easy to show

$$
\frac{\hat{f}_{s,n}(x)\hat{c}_n(x, x + 1)}{C_n(x)^3} = \frac{\hat{\lambda}_n(x)^2[1 - \hat{\lambda}_n(x)]}{\hat{f}_{s,n}(x)}.
$$

We now state the following corollary without proof for completeness.

**Corollary 3.1.1.** As \( n \to \infty \), \( \hat{\Lambda}_n \overset{p}{\to} \Lambda \).

When estimating \( G \) is of interest, we may also obtain the sister statement for the reverse hazard rate \( \beta \) as follows.

**Theorem 3.2 (\( \hat{B}_n \) Asymptotic Normality).** For \( \hat{B}_n \) defined in (16),

$$
\sqrt{n}(\hat{B}_n - \mathbf{B}) \overset{c}{\to} N(0, \Sigma_g), \quad \text{as} \quad n \to \infty,
$$

where \( \mathbf{B} = (\beta(\Delta + 1), \beta(\Delta + 2), \ldots, \beta(\Delta + m))^\top \) with \( \beta(z) = g_s(z)/C(z) \) and

$$
\Sigma_g = \text{diag} \left( 0, \frac{g_s(\Delta + 2)c(\Delta + 1, \Delta + 2)}{C(\Delta + 2)^3}, \ldots, \frac{g_s(\Delta + m)c(\Delta + m - 1, \Delta + m)}{C(\Delta + m)^3} \right).
$$

That is, the estimators \( \hat{\beta}_n(\Delta + 1), \ldots, \hat{\beta}_n(\Delta + m) \) are asymptotically normal and independent.

**Remark.** One may also write

$$
\Sigma_g = \text{diag} \left( 0, \frac{\beta(\Delta + 2)^2[1 - \beta(\Delta + 2)]}{g_s(\Delta + 2)}, \ldots, \frac{\beta(\Delta + m)^2[1 - \beta(\Delta + m)]}{g_s(\Delta + m)} \right). \quad (21)
$$
We again state the following corollary without proof for completeness.

**Corollary 3.2.1.** As $n \to \infty$, $\hat{B}_n \xrightarrow{p} B$.

### 3.2 Survival and Distribution Function

For most analysts of survival data, the key quantity of interest is the survival function, $S(x) = 1 - F(x)$. From (8) and (10), we have the estimator

$$
\hat{S}_n(x) = \prod_{\Delta+1 \leq k \leq x} [1 - \hat{\lambda}_n(k)].
$$

(22)

Asymptotic normality also extends to (22), which we now show.

**Theorem 3.3** ($\hat{S}_n$ Asymptotic Normality). For the estimator $\hat{S}_n = (\hat{S}_n(\Delta + 1), \hat{S}_n(\Delta + 2), \ldots, \hat{S}_n(\omega))^\top$,

$$
\sqrt{n}(\hat{S}_n - S) \xrightarrow{L} N(0, RK\Sigma_f[RK]^\top), \text{ as } n \to \infty,
$$

where $S = (S(\Delta + 1), S(\Delta + 2), \ldots, S(\omega))^\top$, $\Sigma_f$ follows from Theorem 3.1,

$$
K = 
\begin{bmatrix}
-([1 - \lambda(\Delta + 1)]^{-1} & 0 & \ldots & 0 \\
-([1 - \lambda(\Delta + 1)]^{-1} & -([1 - \lambda(\Delta + 2)]^{-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-([1 - \lambda(\Delta + 1)]^{-1} & -([1 - \lambda(\Delta + 2)]^{-1} & \ldots & -([1 - \lambda(\omega)]^{-1}
\end{bmatrix}.
$$

and $R = \text{diag}(S(\Delta + 1), S(\Delta + 2), \ldots, S(\omega))$.

The sister theorem to estimator $G$ is as follows.

**Theorem 3.4** ($\hat{G}_n$ Asymptotic Normality). For the estimator $\hat{G}_n = (\hat{G}_n(\Delta + 1), \hat{G}_n(\Delta + 2), \ldots, \hat{G}_n(\Delta + m))^\top$

$$
\sqrt{n}(\hat{G}_n - G) \xrightarrow{L} N(0, WM\Sigma_g[WM]^\top), \text{ as } n \to \infty,
$$

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where \( G = (G(\Delta + 1), G(\Delta + 2), \ldots, G(\Delta + m))^T \), \( \Sigma_g \) follows from Theorem 3.2,

\[
M = \begin{bmatrix}
-\frac{1}{1 - \beta(\Delta + 1)} & -\frac{1}{1 - \beta(\Delta + 2)} & \cdots & -\frac{1}{1 - \beta(\Delta + m)} \\
0 & -\frac{1}{1 - \beta(\Delta + 2)} & \cdots & -\frac{1}{1 - \beta(\Delta + m)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\frac{1}{1 - \beta(\Delta + m)}
\end{bmatrix},
\]

and \( W = \text{diag}(G(\Delta + 1), G(\Delta + 2), \ldots, G(\Delta + m)) \).

### 3.3 Hypothesis Test

In many applications, it is desirable to test if the distribution of \( F \) or \( G \) corresponds to a known distribution. As one may anticipate, there is some history in the literature. For a starting point, we encourage the reader to review Hyde (1977) and the associated citations. For our purposes, we review a few notable examples. (For consistent terminology with the associated references within this paragraph, we will leave the form of truncation and censoring unspecified; e.g., simply “truncation” rather than left-truncation; ”censored” rather than right-censored). To begin, Guilbaud (1988) generalizes the ordinary Kolmogorov–Smirnov one-sample tests based on the product-limit estimator. The test we develop from Theorem 3.5 is more akin to a goodness-of-fit test, however. Mandel and Betensky (2007) is related, though they assume continuous \( F \) and \( G \) to introduce several goodness-of-fit tests for the truncation distribution. Similarly, Hwang and Wang (2008) assume the lifetime, truncation, and censoring random variables are continuous in proposing a chi-square test to test the hypothesis that the truncation distribution follows a parametric family. Further, the asymptotic properties of the nonparametric test of Ning et al. (2010) were derived assuming a continuous survival function. See also Moreira et al. (2014), in which goodness-of-fit tests are proposed for a semiparametric model under random double truncation. As there is no clear application to discrete \( F \) or discrete \( G \), we extend our results to propose a hypothesis testing procedure using a chi-square random variable. We state our results for the left-truncation
distribution $G$.

**Theorem 3.5.** Assume that $G$ follows a known distribution over the discrete points \{\(\Delta + 1, \ldots, \Delta + m\}\}. Then the test statistic

\[
\mathcal{Q}_G = \left[\sqrt{n}(\hat{\mathbf{B}}_n^* - \mathbf{B}^*)\right]^\top \left[\Sigma_g^*\right]^{-1} \left[\sqrt{n}(\hat{\mathbf{B}}_n^* - \mathbf{B}^*)\right] \overset{L}{\longrightarrow} \chi^2_q,
\]

where $\hat{\mathbf{B}}_n^* = (\hat{\beta}_n(\Delta + 2), \ldots, \hat{\beta}_n(\Delta + m))^\top$, $\mathbf{B}^* = (\beta(\Delta + 2), \ldots, \beta(\Delta + m))^\top$,

\[
\Sigma_g^* = \text{diag}\left(\frac{\beta(\Delta + 2)^2[1 - \beta(\Delta + 2)]}{g_*(\Delta + 2)}, \ldots, \frac{\beta(\Delta + m)^2[1 - \beta(\Delta + m)]}{g_*(\Delta + m)}\right),
\]

and $q = \text{card}\{\Delta+2, \ldots, \Delta+m\}$. The point $\Delta+1$ with the degenerate estimator $\hat{\beta}_n(\Delta+1) = 1$ and $\text{Var}[\hat{\beta}_n(\Delta + 1)] = 0$ is omitted from $\mathcal{Q}_G$.

Specifically, it is often of interest to test if $G$ follows a uniform distribution. This is an important assumption in the case length-biased sampling, see for instance Asgharian et al. (2002) and De Uña-Álvarez (2004). There is a long history of proposed methods for checking this assumption in the literature. For example, one stated use of the NPMLE for the left-truncated distribution derived by Wang (1991) is to informally check the validity of the stationarity assumption (i.e., $G$ coincides with a uniform distribution). Similarly, Asgharian et al. (2006) propose a graphical method to check the stationarity of the underlying incidence times. Addona and Wolfson (2006) propose a formal test for stationarity of the incidence rate, but they require a continuous truncation density (via Theorem 1 of Asgharian et al. (2006)). Our test, however, allows for exact $p$-value calculations and considers discrete $G$.

Formally, Corollary 3.5.1 may be used to test if the left-truncation random variable follows a discrete uniform distribution.

**Corollary 3.5.1.** Assuming the conditions and notation of Theorem 3.5, under the null
hypothesis that $G$ is a discrete uniform distribution over $\{\Delta+1, \ldots, \Delta+m\}$, the test statistic

$$Q_U = [\sqrt{n}(\hat{B}_n^* - B_U^*)]^T [\Sigma_{g,U}^*]^{-1} [\sqrt{n}(\hat{B}_n^* - B_U^*)] \overset{\mathcal{L}}{\rightarrow} \chi^2_q,$$

where $B_U^* = (1/2, 1/3, \ldots, 1/m)$, and

$$\Sigma_{g,U}^* = \text{diag}\left(\frac{1}{2^2} \frac{1 - 1/2}{\hat{g}_{*,n}(\Delta + 2)}, \ldots, \frac{1/m}{2^2} \frac{1 - 1/m}{\hat{g}_{*,n}(\Delta + m)}\right).$$

Consequently, for $H_0$ that $Y$ is discrete uniformly distributed and significance level $0 \leq \alpha \leq 1$, one can reject $H_0$ if $Q_U \leq \chi^2_{q,\alpha/2}$ or $Q_U \geq \chi^2_{q,1-\alpha/2}$, where $\chi^2_{q,\theta}$ is the $(100 \times \theta)$th $(0 < \theta < 1)$ percentile of a chi-square distribution with $q$ degrees of freedom. The accuracy of the asymptotic chi-square distribution was investigated for a discrete uniform $G$ in our simulation study. The empirical distribution of the test statistics matches closely to the limiting chi-square distribution, which we validated down to a sample size of $n = 500$.

4 Simulation Study

In this section, we examine the finite sample behavior of the estimation vectors $\hat{\Lambda}_n$ and $\hat{B}_n$. In addition to serving as an experimental verification of Theorems 3.1 and 3.2, our intention is to help practitioners estimate the minimum sample size of discrete-time left-truncated data needed to achieve a desired level of estimation accuracy. We proceed in two parts. First, for the purposes of illustration, we will consider a combination of classical distributions for the lifetime and left-truncation random variables. Second, with an eye towards our application, the section will close with a combination of lifetime and left-truncation random variables that is a closer approximation to those observed within structured finance (e.g., Section 5).

Assume first that $Y$ follows a discrete uniform distribution over $\mathcal{Y} = \{1, 2, \ldots, 10\}$, and that $X$ follows a truncated geometric distribution over $\mathcal{X} = \{1, 2, \ldots, 24\}$. Specifically, the
The pmf of $X$ is
\[
\Pr(X = x) = \begin{cases} 
  p(1-p)^{x-1}, & x = 1, 2, \ldots, 23, \\
  \sum_{x=24}^{\infty} p(1-p)^{x-1}, & x = 24, \\
  0, & \text{otherwise},
\end{cases}
\]
where $0 < p < 1$. From this, we may calculate many key quantities of interest. For example, with $p = 0.20$,
\[
\alpha = \Pr(Y \leq X) = \sum_{y=1}^{10} \Pr(Y = y) \Pr(X \geq y) = 0.4463,
\]
as well as the useful quantities (3), (17), (33), and (41). Notice here that $\Delta = 0$, $m = 10$, and $\omega = 24$.

We remark here on the behavior of (18) across various values of $p$. For larger values of $p$, the variance of $\hat{\lambda}_n$ for values of $X$ closer to 23 quickly explodes. This is not unexpected because, as $p$ increases, it becomes more and more unlikely to observe large values of $X$. On the other hand, for very small values of $p$ close to zero, the variance of $\hat{\lambda}_n$ for values of $X$ close to 1 is the largest and rapidly decreases until $X = 10$, the final possible left-truncation point. This suggests what we can already glean from a careful examination of (19): estimation accuracy of $\hat{\lambda}_n(x)$ is dependent on the quantities $\lambda(x)$ and $f_*(x)$.

In addition to demonstrating the asymptotic unbiasedness of the estimators (8) and (11), we will also compare the empirical covariance against the asymptotic covariance suggested by Theorems 3.1 and 3.2 by examining the resulting confidence interval estimates. As is standard practice, we will keep the intervals within the unit interval by constructing the confidence interval estimates on a log scale with the delta method (e.g., Mukhopadhyay, 2000, Theorem 5.3.5, pg. 261) and then transforming them back exponentially to the original scale. For example, the 95% confidence intervals for $\lambda(x)$, $x \in \{1, \ldots, 24\}$, are
\[
\exp\left\{ \ln \hat{\lambda}_n(x) \pm 1.96 \times \sqrt{\frac{1 - \hat{\lambda}_n(x)}{f_{*,n}(x) n}} \right\}.
\]
In our analysis summarized in the first two rows of Figure 3, we demonstrate the estimator’s asymptotic unbiasedness and normality (we assume \( p = 0.20 \) and consider 1,000 replicates). For the asymptotic unbiasedness, we plot the true hazard and reverse hazard rates (dashed black lines) against an average of the 1,000 estimated replicates (blue lines). The two are largely indistinguishable, especially as \( n \) increases. For the asymptotic normality and covariance structure specified within Theorems 3.1 and 3.2, we compare three quantities. The first quantity is the true 95% confidence interval, and it is represented by the blue ribbon. The second quantity is the average of the estimated confidence intervals derived from the simulated data, i.e., (24), over the 1,000 replicates. We represent this quantity by the red ribbon. The third quantity is the middle 95th empirical percentile of the 1,000 replicates, and it is represented by the purple ribbon. Once again, all closely agree, especially as \( n \) increases. We also found that the off-diagonal elements in the empirical covariance calculation across the 1,000 replicates of all estimators are each very close to zero, which is further experimental validation of independence. Readers interested in the full empirical covariance matrix comparison may contact the corresponding author for more details.

We next summarize the approximation accuracy across various sample sizes, and we observe it is a function of the underlying distribution. Intuitively, this can also be gleaned from (19); the variance of the estimator \( \hat{\lambda}_n \) is a function of the distributions of \( X \) and \( Y \). Hence, we see that a larger sample size is necessary to control the approximation accuracy towards the right tail of the distribution of \( X \), values of which occur with much smaller probability. We see minor tail failures of the approximation begin to materialize when \( n \) is as large as 1,000. On the other hand, the approximation for \( \hat{\beta}_n \) still works well for \( n = 1,000 \). See Figure 3 for details.

Table 1 summarizes the observed coverage probability over the 1,000 replicates for various sample sizes, \( n \). That is, the percentage of the 1,000 replicates of confidence intervals that contained the true value of \( \lambda(x), x \in \{1, \ldots, 24\} \) and \( \beta(y), y \in \{1, \ldots, 10\} \). We also track the number of replicates that did not return a valid estimate (i.e., we did not observe any samples
Figure 3: A simulation verification of Theorems 3.1 (i.e., $\lambda$, left-column) and 3.2 (i.e., $\beta$, right-column) for sample sizes of $n = 1,000$ and $n = 10,000$ with classical distributions (the lifetime distribution follows a truncated geometric distribution at $x = 24$, as defined in (23) with $p = 0.20$ and the left-truncation distribution follows a discrete uniform distribution over the integers $\{1, \ldots, 10\}$) and distributions more representative of an application to structured finance (compare Figures 4 and 5). Each chart presents a comparison of an average over all replicates of the estimate (blue lines) and true (dashed black lines), which are largely indistinguishable. Further the 95% true confidence intervals (blue ribbon), an average over all replicates of the 95% confidence intervals estimated from the empirical quantities (9) and (20) (red ribbon), and the middle 95th percentile (purple ribbon) of all the replicates each closely agree. The minor differences in the confidence intervals for the right tail of the lifetime distribution are eliminated as the sample size increases (row two versus row one). The only deterioration occurs in the very left tail of the lifetime distribution (bottom, left), which is a result of truncation causing very few simulated observations (the point estimate is still quite accurate). The results in the bottom row used a sample size of $n = 10,000$. All results used 1,000 replicates.

of $X$ or $Y$ at a particular value). Given these results, we recommend that a practitioner use judgment and available references to estimate the probability of less frequent observations. The smaller these probabilities, generally speaking, the larger the sample to ensure the
approximation works well. Alternatively, a practitioner may instead identify the values of $X$ or $Y$ that are of most interest. For example, the confidence interval approximation for $\hat{\lambda}_n$ still works very well for $X \leq 10$ when $n = 1,000$. More details may be found in Table 1.

Alternatively, if a known accurate estimate of $f_*$ and $\lambda$ is available, then determining the appropriate sample size is only a matter of selecting an approach to handle a simultaneous confidence region.

For the second part of our simulation study, we consider a combination of a lifetime random variable and a left-truncation random variable that is more representative of an application to structured finance; specifically, leases with an original termination schedule of 24 months. The probabilities are summarized in Figure 4. We can see the lifetime distribution obtains a peak near month 24, and the truncation distribution is not discrete uniform (compare with Figure 5). The bottom row of Figure 3 demonstrates experimental verification of Theorems 3.1 and 3.2 in this instance, as the true hazard rates and estimates overlap (asymptotic unbiasedness) and the true and empirical confidence intervals all closely agree (asymptotic normality and independence). The only deterioration in the estimator’s asymptotic performance occurs with the confidence intervals at the very left tail of the lifetime distribution, which is a direct result of the combined lifetime and left-truncation random variable probabilities causing heavy left-truncation. The sample size for each of the 1,000 replicates was $n = 10,000$.

5 Application

Recall the motivating example in Section 1. Here we apply the estimation and asymptotic results of earlier sections to a subset of auto lease securitization trust data. Specifically, we examine the Mercedes-Benz Auto Lease Trust (MBALT) 2017-A financial transaction (Mercedes-Benz, 2017). Detailed data and performance records are available at the individualized contract level from the Electronic Data Gathering, Analysis, and Retrieval (EDGAR)
Table 1: Coverage percentages (CP) of 95% confidence intervals under various sample sizes in the simulation study adjusted for the frequency of unrealized simulations (UR). Top: $\lambda_n(x)$ for $x \in \{1, \ldots, 24\}$; Bottom: $\beta_n(y)$, for $y \in \{1, \ldots, 10\}$.

| $n$    | $n = 250$ | $n = 500$ | $n = 750$ | $n = 1,000$ | $n = 10,000$ |
|--------|-----------|-----------|-----------|-------------|--------------|
| $x$    | CP        | UR        | CP        | UR          | CP           | UR          | CP        | UR          | CP           | UR          |
| 1      | 95.9      | 0         | 94.4      | 0           | 93.8         | 0           | 93.8      | 0           | 95.6         | 0           |
| 2      | 96.2      | 0         | 94.5      | 0           | 95.4         | 0           | 95.4      | 0           | 94.9         | 0           |
| 3      | 95.3      | 0         | 94.5      | 0           | 95.0         | 0           | 95.7      | 0           | 95.8         | 0           |
| 4      | 95.3      | 0         | 94.6      | 0           | 95.0         | 0           | 94.3      | 0           | 93.9         | 0           |
| 5      | 95.4      | 0         | 94.1      | 0           | 94.5         | 0           | 95.7      | 0           | 95.8         | 0           |
| 6      | 93.8      | 0         | 94.5      | 0           | 93.9         | 0           | 94.1      | 0           | 95.8         | 0           |
| 7      | 93.7      | 0         | 94.4      | 0           | 95.6         | 0           | 95.6      | 0           | 95.4         | 0           |
| 8      | 94.5      | 0         | 95.3      | 0           | 95.7         | 0           | 95.2      | 0           | 95.2         | 0           |
| 9      | 95.3      | 0         | 95.1      | 0           | 95.8         | 0           | 94.4      | 0           | 93.0         | 0           |
| 10     | 95.1      | 0         | 95.5      | 0           | 95.9         | 0           | 96.7      | 0           | 93.3         | 0           |
| 11     | 95.2      | 0         | 95.4      | 0           | 94.4         | 0           | 94.3      | 0           | 93.8         | 0           |
| 12     | 95.9      | 0         | 95.3      | 0           | 95.7         | 0           | 95.7      | 0           | 95.1         | 0           |
| 13     | 94.6      | 1         | 94.9      | 0           | 94.5         | 0           | 95.7      | 0           | 96.2         | 0           |
| 14     | 96.0      | 0         | 96.3      | 0           | 95.7         | 0           | 94.6      | 0           | 94.6         | 0           |
| 15     | 95.5      | 6         | 94.6      | 0           | 94.0         | 0           | 95.3      | 0           | 95.7         | 0           |
| 16     | 95.1      | 23        | 96.8      | 2           | 93.7         | 0           | 94.5      | 0           | 94.4         | 0           |
| 17     | 95.5      | 37        | 96.6      | 1           | 95.7         | 0           | 95.2      | 0           | 95.1         | 0           |
| 18     | 94.5      | 77        | 96.0      | 5           | 96.8         | 1           | 95.0      | 0           | 94.8         | 0           |
| 19     | 93.1      | 131       | 94.7      | 21          | 96.1         | 3           | 95.2      | 0           | 94.8         | 0           |
| 20     | 92.6      | 204       | 95.0      | 43          | 95.3         | 4           | 95.3      | 2           | 95.8         | 0           |
| 21     | 91.1      | 296       | 95.3      | 69          | 95.1         | 20          | 95.6      | 1           | 95.0         | 0           |
| 22     | 90.0      | 347       | 93.6      | 146         | 94.3         | 49          | 95.7      | 20          | 94.7         | 0           |
| 23     | 87.5      | 431       | 91.3      | 206         | 93.8         | 84          | 95.5      | 35          | 94.8         | 0           |

| $n$    | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1,000$ | $n = 10,000$ |
|--------|-----------|-----------|-----------|-------------|--------------|
| $y$    | CP        | UR        | CP        | UR          | CP           | UR          | CP        | UR          | CP           | UR          |
| 2      | 93.6      | 0         | 95.2      | 0           | 94.3         | 0           | 95.5      | 0           | 96.0         | 0           |
| 3      | 95.6      | 0         | 94.5      | 0           | 95.3         | 0           | 95.3      | 0           | 95.5         | 0           |
| 4      | 96.0      | 0         | 96.4      | 0           | 95.8         | 0           | 94.8      | 0           | 95.7         | 0           |
| 5      | 94.7      | 0         | 95.7      | 0           | 95.4         | 0           | 95.4      | 0           | 94.2         | 0           |
| 6      | 95.9      | 1         | 94.9      | 0           | 95.1         | 0           | 95.6      | 0           | 95.5         | 0           |
| 7      | 97.0      | 2         | 95.3      | 0           | 94.6         | 0           | 95.2      | 0           | 94.8         | 0           |
| 8      | 95.3      | 11        | 95.3      | 0           | 95.6         | 0           | 96.3      | 0           | 95.1         | 0           |
| 9      | 96.1      | 20        | 96.0      | 0           | 94.8         | 0           | 94.9      | 0           | 96.2         | 0           |
| 10     | 93.6      | 47        | 95.6      | 2           | 95.4         | 0           | 94.6      | 0           | 95.4         | 0           |
Figure 4: The lifetime distribution (black bars) and left-truncation distribution (blue bars) more representative of an application to structured finance (compare with Figure 5) used to produce an additional simulation verification of Theorems 3.1 and 3.2 (bottom row, Figure 3).

system, which is freely available to the public through the Securities and Exchange Commission (SEC, 2016). The MBALT 2017-A transaction had 56,402 lease contracts with original terms ranging from 24 to 60 months. For the purposes of illustration, we only consider ongoing lease contracts with an original termination schedule of 24 months. This reduced the sample to 866 lease contracts.

The MBALT 2017-A bond was placed in April of 2017. The transaction was paid in full and closed in August of 2019. Therefore, the observation window consisted of 28 months. Monthly loan performance information is available on EDGAR. Lease contracts must be delinquent no more than 30 days to be included in the securitization trust (Mercedes-Benz, 2017). Hence, the lease contracts are all active as of the onset of the transaction. At initialization, the oldest lease in the trust was 21 months old, and the youngest lease was
3 months old. Thus, to use our notation, \( \Delta = 3 \) and \( m = 18 \). Though each lease is scheduled to terminate after 24 months, lease contracts may terminate early through default or consumer option. Additionally, lease contracts may extend beyond 24 months due to missed payments or various extension clauses. Therefore, to estimate the time of a lease termination, we searched the data for three consecutive months of a zero payment. Once three consecutive zero payments were found, the month of lease termination was assigned to be the month of the first zero. For example, if a lease contract recorded a zero payment for months 11, 12, and 13, then month 11 was assumed to be the lease termination age. After performing this search, we identified eight contracts that did not terminate during the observation window and were thus right-censored. However, for simplicity, we assumed these eight leases all terminated as of the last observation month. (A related study, Lautier et al. (2022), generalizes the estimators of Section 2 to the case of right-censoring, see Section 6 for additional discussion.) The termination time of the oldest lease was 37 months, and so \( \omega = 37 \). Formally, then, \( Y \in \{4, \ldots, 22\} \) and \( X \in \{4, \ldots, 37\} \). (A minor comment here is that we began counting \( T \) at 0 within this application, which is why the maximum bound of \( Y \) extends to \( m + \Delta + 1 = 22 \).)

In Figure 5, we plot the estimated hazard rate for 24-month leases within the MBALT 2017-A transaction. Most of these leases have terminated at month 25, which we would expect for a pool of leases contractually designed to terminate after 24 payments. However, there are a few interesting observations. First, there is notable early lease termination activity beginning around lease age 20 months. Second, we have sporadic hazard rate behavior beyond lease age 25. Finally, the width of the 95% confidence band increases markedly beyond 25 months. The bands are quite narrow for leases that terminate prior to the original termination schedule of 24 months, however. Table 2 presents complete results for the estimated quantities \( \hat{f}_{s,n}, \hat{\lambda}_n, \hat{g}_{s,n}, \) and \( \hat{\beta}_n \), along with the standard errors for \( \hat{\lambda}_n \) and \( \hat{\beta}_n \).

Additionally, some practitioners may be more interested in estimating the left-truncation random variable, \( Y \). To this end, we present the estimated probability mass function for \( Y \)
in Figure 5. An interested investor could use this information to recover $T$, the distribution of lease origination times. Information about $T$ may be compared with economic trends or the “Selection of the Leases” section of Mercedes-Benz (2017), for example. Finally, it may be of interest to determine if the distribution of $Y$ is discrete uniform, particularly if one wishes to attempt to generalize these results into a length-biased model, such as with Asgharian et al. (2002) and De Uña-Álvarez (2004). Though it may be obvious from Figure 5 that $Y$ is not discrete uniform, we may also use Corollary 3.5.1 to calculate $Q_U = 1,530.6$. At $q = \text{card}\{5, \ldots, 22\} = 18$ degrees of freedom, this corresponds to a $p$-value of effectively zero. Hence, we reject the null hypothesis of a discrete uniform distribution for $Y$. Rejecting the null in this case implies utilizing a method to estimate a distribution function for $X$ that relies on the assumption that the left-truncation random variable is discrete uniform (i.e., stationarity), such as length-biased sampling, would be invalid for this application.

![Figure 5](image)

Figure 5: Summary plots for $\hat{g}_{*,n}$ (top) and $\hat{\lambda}_n$ plus estimated 95% confidence intervals (bottom) for a subset of 24-month leases from the MBALT 2017-A securitization.
6 Discussion

The estimates in Table 2 and Figure 5 may have important applications for practitioners to understand the behavior of consumer automobile leaseholders when given the option to terminate or extend a lease contract. Financially, risk professionals can use our estimation procedures to model the relationship between consumer lessee behavior and the credit risk of securitized bonds. Automobile manufacturers may also have an interest in our application in terms of modeling the relationship between profitability and the structure of a consumer lease contract. The connective thread of this manuscript is that the estimates of Section 5 were produced using the theoretical results of Sections 2 and 3.

To that end, this is to our knowledge the first thorough exposition of the case of data subject to random left-truncation in the case of discrete \( X \) and \( Y \) with finite support. We proved that the random estimation vectors \( \hat{\boldsymbol{A}}_n \) and \( \hat{\boldsymbol{B}}_n \) are together an MLE for the parameters of the conditional bivariate distribution \( H_\ast \) and asymptotically normal with independent components (i.e., a diagonal covariance matrix). Both results utilized an alternative sampling and left-truncation framework from Woodroofe (1985), which was necessary to appropriately mimic the practicalities of consumer ABS data. We also further proved asymptotic normality extends to the survival function estimator \( \hat{S}_n \) and the distribution function estimator \( \hat{G}_n \). The last main result of this work was to establish a hypothesis test to examine the shape of the distribution of \( G \), which has utility to formally test the stationarity assumption of the left-truncation distribution in length-biased sampling.

The practical realities of econometric data can inform statistical analysis, and we have identified a large group of securitized financial data that suggests the use of a survival analysis model adjusted for discrete-time data over a fixed time horizon subject to random left-truncation. However, many forms of economic or financial data fall into the same criteria studied herein. Indeed, payment history is often recorded on a periodic basis, such as monthly, quarterly, or annually. For example, a monthly frequency is common for insurance products and debt instruments, such as insurance premiums, credit card payments, mort-
Table 2: Estimated distributions for the MBALT 2017-A application: the lifetime of interest (lease terminations, $F_0$) and the left-truncation random variable, $G_0$.

| Age | $\hat{f}_{*,n}$ | $\hat{\lambda}_n$ | s.e.$[\hat{\lambda}_n]$ | Age | $\hat{g}_{*,n}$ | $\hat{\beta}_n$ | s.e.$[\hat{\beta}_n]$ |
|-----|-----------------|------------------|------------------------|-----|-----------------|-----------------|------------------------|
| 4   | 0               | 0                | 0                      | 4   | 0.057           | 1               | NA                     |
| 5   | 0               | 0                | 0                      | 5   | 0.042           | 0.424           | 1.577                  |
| 6   | 0               | 0                | 0                      | 6   | 0.048           | 0.331           | 1.229                  |
| 7   | 0               | 0                | 0                      | 7   | 0.033           | 0.186           | 0.917                  |
| 8   | 0.001           | 0.005            | 0.161                  | 8   | 0.030           | 0.143           | 0.763                  |
| 9   | 0               | 0                | 0                      | 9   | 0.023           | 0.100           | 0.621                  |
| 10  | 0               | 0                | 0                      | 10  | 0.039           | 0.145           | 0.675                  |
| 11  | 0.001           | 0.004            | 0.115                  | 11  | 0.024           | 0.082           | 0.505                  |
| 12  | 0.002           | 0.007            | 0.147                  | 12  | 0.031           | 0.096           | 0.516                  |
| 13  | 0.001           | 0.003            | 0.093                  | 13  | 0.043           | 0.117           | 0.531                  |
| 14  | 0.002           | 0.006            | 0.115                  | 14  | 0.053           | 0.127           | 0.515                  |
| 15  | 0.003           | 0.007            | 0.121                  | 15  | 0.069           | 0.143           | 0.502                  |
| 16  | 0.008           | 0.014            | 0.152                  | 16  | 0.107           | 0.182           | 0.503                  |
| 17  | 0.008           | 0.012            | 0.135                  | 17  | 0.082           | 0.124           | 0.404                  |
| 18  | 0.014           | 0.019            | 0.157                  | 18  | 0.087           | 0.117           | 0.373                  |
| 19  | 0.022           | 0.027            | 0.177                  | 19  | 0.097           | 0.118           | 0.355                  |
| 20  | 0.040           | 0.046            | 0.223                  | 20  | 0.080           | 0.090           | 0.305                  |
| 21  | 0.058           | 0.065            | 0.260                  | 21  | 0.053           | 0.059           | 0.250                  |
| 22  | 0.068           | 0.081            | 0.298                  | 22  | 0.001           | 0.001           | 0.041                  |
| 23  | 0.079           | 0.102            | 0.345                  | 23  |                 |                 |                        |
| 24  | 0.133           | 0.192            | 0.474                  | 24  |                 |                 |                        |
| 25  | 0.397           | 0.711            | 0.607                  | 25  |                 |                 |                        |
| 26  | 0.040           | 0.250            | 1.077                  | 26  |                 |                 |                        |
| 27  | 0.040           | 0.333            | 1.354                  | 27  |                 |                 |                        |
| 28  | 0.020           | 0.243            | 1.508                  | 28  |                 |                 |                        |
| 29  | 0.015           | 0.245            | 1.739                  | 29  |                 |                 |                        |
| 30  | 0.009           | 0.200            | 1.861                  | 30  |                 |                 |                        |
| 31  | 0.018           | 0.500            | 2.601                  | 31  |                 |                 |                        |
| 32  | 0.006           | 0.313            | 3.410                  | 32  |                 |                 |                        |
| 33  | 0.007           | 0.545            | 4.418                  | 33  |                 |                 |                        |
| 34  | 0.002           | 0.400            | 6.447                  | 34  |                 |                 |                        |
| 35  | 0.002           | 0.667            | 8.009                  | 35  |                 |                 |                        |
| 36  | 0               | 0                | 0                      | 36  |                 |                 |                        |
| 37  | 0.001           | 1                | NA                     | 37  |                 |                 |                        |
gages, auto loans, and so on. Further, many financial contracts typically have a fixed, finite term, such as any standard auto loan or term life insurance. Even whole life insurance, which is technically written with payments due in perpetuity is, in actuality, a fixed-length contract of unknown duration (one may comfortably cap assumed lifetimes at 130 years, for example). Our contributions to the asymptotic statistical properties of the discrete distribution function estimators can be applied to and further investigated in alternative applications, such as those of insurance, mortgages, and other debt instruments.

Looking ahead, many applications of estimating a lifetime distribution random variable from observed data will also be subject to the further incomplete data complication of right-censoring. Interested readers may find generalizations of select theoretical results within this paper to the case of both left-truncation and right-censoring in Lautier et al. (2022), which also includes an extended financial pricing model and application to securitization data that utilizes the associated distribution estimators. In addition, general empirical economic analysis may benefit from the introduction of explanatory variables or covariates, similar to the classical regression models for survival data (e.g., Klein and Moeschberger, 2003, Section 2.6) but appropriately calibrated for the discrete-time setting of Section 2. We leave this problem open to further research. In addition, it is of theoretical interest to consider a discrete lifetime distribution over countably infinite values (i.e., extending trapezoid $A$ in Figure 2 to the right indefinitely). Given our application to finite term financial products, however, we also leave this problem open to further research.

A Literature Review

The following is a chronological review of related literature to the seminal papers Woodroofe (1985) and Wang et al. (1986) regarding the problem of estimating a distribution function from left-truncated data. The distribution functions of the random variable of interest, $X$, and the left-truncation random variable, $Y$, are denoted by $F$ and $G$, respectively.
Chao and Lo (1988) further study the estimator of $F$ by expressing a hazard process as i.i.d. means of random variables and imposing the same conditions as Woodroofe (1985). The result is the ability to represent the difference of $F$ and its estimator as i.i.d. means of random variables to obtain weak convergence, including the associated covariance structures. Keiding and Gill (1990) reparametrize the left-truncation model as a three-state Markov process to invoke the statistical theory of counting processes by Aalen and Johansen (1978) to establish the nonparametric maximum likelihood estimator (NPMLE), consistency, asymptotic normality, and efficiency. Both papers derive results assuming continuity of $F$, however. Lai and Ying (1991) relax the continuity assumption of $F$ in using martingale integral representations and empirical process theory to prove uniform strong consistency and weak convergence results, though they modify the product-limit estimator in doing so.

Somewhat more recently, Gürler and Wang (1993) examine hazard functions and their derivatives for nonparametric kernel estimators. Similarly, they again assume continuity of $G$ in proving asymptotic normality. Stute (1993) derives an almost sure representation of the estimator for $F$ with weaker distributional assumptions than Woodroofe (1985) and improved error bounds. Chen et al. (1995) prove the Lynden-Bell (1971) estimator is uniformly strong consistent over the whole half line, a problem left open by Woodroofe (1985). Both papers assume continuity of $F$ and $G$ throughout. In part one of a two-part sequence, He and Yang (1998a) find a simpler representation for the estimator of the truncation probability to show strong consistency and asymptotic normality via an i.i.d. representation. While, these results are true for arbitrary $F$ and $G$, they do not consider the estimators for the distribution functions for $F$ and $G$. In part two, He and Yang (1998b) prove that the estimator for $F$ obeys the strong law of large numbers when estimating $F_0$ for arbitrary and not necessarily continuous $F$ (recall the distinction between $F$ and $F_0$ in Section 2). This relaxes the assumption of continuity but does not address asymptotic normality.

The classical problem of estimating $F$ from truncated data has by now become commonplace in textbooks (e.g., Karr, 1991; de la Peña and Giné, 1999; Owen, 2001; Hu, 2013), but
any extended treatment assumes continuity of \( F \) (e.g., de la Peña and Giné, 1999, §5.5.3).

Finally, we expanded our review to consider the random left-truncation model along with right-censoring. A seminal work in this field is Tsai et al. (1987), which gives asymptotic results when left-truncated data are also subject to right-censoring. Nonetheless, the authors also assume continuous \( F \). The continuity of \( F \) and \( G \) is assumed in related works (Uzongullari and Wang, 1992; Gijbels and Wang, 1993; Gürler, 1996; Zhou, 1996; Zhou and Yip, 1999; Asgharian and Wolfson, 2005; Huang and Qin, 2011).

\section*{B Complete Proofs}

\subsection*{B.1 Proof of Theorem 2.1}

\textit{Proof.} Without loss of generality, assume \( \Delta = 0 \). For convenience of notation, let \( f_u \equiv f(u) \), \( g_v \equiv g(v) \). Then, restating (1) in terms of the sampled pairs from \( h_\ast \), \( (X_i, Y_i) \), \( 1 \leq i \leq n \), we have

\[ h_\ast(u, v) = \Pr(X_i = u, Y_i = v) = \frac{f_u g_v}{\alpha}, \quad u, v \in \mathcal{A}, \]

with the accompanying extended definition

\[ \alpha = \Pr(Y \leq X) = \sum_{u=1}^{\omega} f_u \left( \sum_{v=1}^{\min(u, m)} g_v \right) = \sum_{v=1}^{m} g_v \left( \sum_{u=v}^{\omega} f_u \right). \quad (25) \]

Therefore, the quantities \( 0 < f_u < 1, \ u \in \mathcal{A} \), and \( 0 < g_v < 1, \ v \in \mathcal{A} \) are the parameters to be estimated. The shape of \( h_\ast \) over \( \mathcal{A} \) with this parametric interpretation continues to have complete flexibility, and so this is an alternative interpretation of a nonparametric estimation problem under the setting of Section 2. Since we are working with a probability space, we must have \( \sum_u f_u = \sum_v g_v = 1 \). This implies there are \( (\omega - 1) + (m - 1) \) free parameters.

Denoting \( \mathbf{f} = (f_1, \ldots, f_\omega)^\top \) and \( \mathbf{g} = (g_1, \ldots, g_m)^\top \), the likelihood and loglikelihood are
then,
\[
L(f, g | \{(X_i, Y_i)\}_{1 \leq i \leq n}) = \prod_{v=1}^{\omega} \prod_{u=v}^{\omega} \left[ \frac{f(u)g(v)}{\alpha} \right] \sum_{i=1}^{n} 1_{(X_i, Y_i) = (u, v)},
\]
and
\[
l(f, g) \equiv \frac{1}{n} \log L(f, g | \{(X_i, Y_i)\}_{1 \leq i \leq n}) = -\log \alpha + \sum_{v=1}^{\omega} \sum_{u=v}^{\omega} \hat{h}_{vu} \{\log f_u + \log g_v\}, \tag{26}
\]
where
\[
\hat{h}_{vu} = \frac{1}{n} \sum_{i=1}^{n} 1_{(X_i, Y_i) = (u, v)}.
\]

As is standard procedure, our goal is to maximize (26). There are two ways to formulate this problem. The first is as a constrained optimization. Specifically, the parameter space of \(f\) and \(g\) is the \(m \times \omega\) dimensional hypercube over the unit interval \(I = (0, 1)\), and we seek
\[
\left\{ \max_{f, g} l(f, g) : \sum_{u=1}^{\omega} f_u = 1; \sum_{v=1}^{m} g_v = 1; \quad f_u, g_v \in I \right\}.
\tag{27}
\]
That is, \(l(f, g) : (0, 1)^{m \times \omega} \mapsto \mathbb{R}\), subject to the constraints in (27). It is not straightforward to see that any solution will be a global maximum, however.

Alternatively, we can restrict the domain of \(l(f, g)\) to the convex set
\[
\Psi = \left\{ f_u, g_v \in I : \sum_{u=1}^{\omega} f_u = \sum_{v=1}^{m} g_v = 1 \right\}.
\]
To see that \(\Psi\) is convex, without loss of generality, let \(0 \leq \varphi \leq 1\) and suppose \(f_u^* = \varphi f'_u + (1 - \varphi) f''_u\) for \(f'_u, f''_u \in \Psi\) and \(u \in \mathcal{A}\). Then
\[
\sum_{u=1}^{\omega} f_u^* = \sum_{u=1}^{\omega} \{\varphi f'_u + (1 - \varphi) f''_u\}
= \varphi \sum_{u=1}^{\omega} f'_u + (1 - \varphi) \sum_{u=1}^{\omega} f''_u
= 1,
\]
and $f_u^* \in \Psi$. Thus, $l(f, g) : \Psi \mapsto \mathbb{R}$, and, from the convexity of $\Psi$, it is sufficient to claim we have found a global maximum if we can show $l(f, g)$ has only one stationary point that is not on the boundary of $\Psi$.

A point on the boundary of $\Psi$ implies that there exists at least one $f_u = 0$ or $g_v = 0$ for $u, v \in \mathcal{A}$. But, this immediately implies (26) explodes to negative infinity, (we assume here $\alpha > 0$ to avoid the degenerate case of complete data loss; see also the stricter conditions on $\hat{f}_{s,n}$ and $\hat{g}_{s,n}$ in the statement of Theorem 2.1). Hence, the maximum of (26) cannot lie on the boundary of $\Psi$, and, if we can show $l(f, g)$ has only one stationary point, we can be assured it is a global maximum (i.e., the MLE).

We now show the system of partial derivatives with respect to each parameter equated to zero has a single, unique solution. In the following, that $u, v \in \mathcal{A}$, i.e., $u, v \in \mathbb{N}$, is left assumed but will be dropped for ease of presentation. Observe first from (25),

$$\frac{\partial \alpha}{\partial f_u} = \frac{\min(u, m)}{\omega} \sum_{v=1}^{\omega} g_v,$$
and
$$\frac{\partial \alpha}{\partial g_v} = \sum_{u=v}^{\omega} f_u.$$

Hence,

$$\frac{\partial l(f, g)}{\partial g_v} = \frac{1}{g_v} \sum_{u=v}^{\omega} \hat{h}_{vu} - \frac{1}{\alpha} \frac{\partial \alpha}{\partial g_v} = 0, \quad 1 \leq v \leq m,$$

(28)

and

$$\frac{\partial l(f, g)}{\partial f_u} = \frac{1}{f_u} \sum_{v=1}^{\min(u, m)} \hat{h}_{vu} - \frac{1}{\alpha} \frac{\partial \alpha}{\partial f_u} = 0, \quad 1 \leq u \leq \omega.$$

(29)

The simultaneous solution to (28) and (29) may be determined sequentially. We proceed by mathematical induction. That is, for $v = 1$, with (28),

$$\frac{1}{g_1} \sum_{u=1}^{\omega} \hat{h}_{1u} - \frac{1}{\alpha} \sum_{u=1}^{\omega} f_u = 0 \implies \hat{g}_1 = \alpha \sum_{u=1}^{\omega} \hat{h}_{1u} = \alpha \hat{C}_n(1).$$
Thus, for \( u = 1 \), with (29),

\[
\frac{1}{f_1} \sum_{v=1}^{1} \hat{h}_{v1} - \frac{1}{\alpha} \sum_{v=1}^{1} \hat{g}_v = 0 \implies \hat{f}_1 = \frac{\hat{h}_{11}}{\hat{C}_n(1)} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{X_i=1} = \hat{\lambda}_n(1).
\]

Consider now \( u = 2 \) with (28),

\[
\frac{1}{g_2} \sum_{u=2}^{\omega} \hat{h}_{2u} - \frac{1}{\alpha} \sum_{u=2}^{\omega} f_u = 0 \implies \hat{g}_2 = \frac{\alpha \sum_{u=2}^{\omega} \hat{h}_{2u}}{1 - \hat{f}_1} = \frac{\alpha \sum_{u=2}^{\omega} \hat{h}_{2u}}{1 - \hat{\lambda}_n(1)}.
\]

Thus, for \( u = 2 \), with (29)

\[
0 = \frac{1}{f_2} \sum_{v=1}^{2} \hat{h}_{v2} - \frac{1}{\alpha} \sum_{v=1}^{2} \hat{g}_v
= \frac{1}{f_2} \sum_{v=1}^{2} \hat{h}_{v2} - \frac{1}{\alpha} \left[ \alpha \hat{C}_n(1) + \frac{\alpha \sum_{u=2}^{\omega} \hat{h}_{2u}}{1 - \hat{\lambda}_n(1)} \right]
= \frac{1}{f_2} \sum_{v=1}^{2} \hat{h}_{v2} - \left[ \hat{C}_n(1) - \sum_{i=1}^{n} \mathbf{1}_{X_i} + \sum_{u=2}^{\omega} \hat{h}_{2u} \right]
= \frac{1}{f_2} \sum_{v=1}^{2} \hat{h}_{v2} - \frac{\hat{C}_n(2)}{1 - \hat{\lambda}_n(1)}.
\]

That is,

\[
\hat{f}_2 = \hat{\lambda}_n(2)[1 - \hat{\lambda}_n(1)].
\]

Now assume the induction hypothesis for \( 1 \leq k < m \); i.e.,

\[
\hat{g}_k = \frac{\alpha \sum_{u=k}^{\omega} \hat{h}_{ku}}{1 - \sum_{j=1}^{k-1} \hat{f}_j}, \quad \text{and} \quad \hat{f}_k = \hat{\lambda}_n(k) \prod_{1 \leq j < k} [1 - \hat{\lambda}_n(j)],
\]

with the conventions \( \sum_{j=1}^{0} \hat{f}_j = 0 \) and \( \prod_{1 \leq j < 1} [1 - \hat{\lambda}_n(j)] = 1 \). Then by (28),

\[
\frac{1}{g_{k+1}} \sum_{u=k+1}^{\omega} \hat{h}_{k+1u} - \frac{1}{\alpha} \sum_{u=k+1}^{\omega} f_u \implies \hat{g}_{k+1} = \frac{\alpha \sum_{u=k+1}^{\omega} \hat{h}_{k+1u}}{1 - \sum_{j=1}^{k} \hat{f}_j}.
\]
But, for $1 \leq r \leq k$,

$$1 - \sum_{j=1}^{r} \hat{f}_j = 1 - \hat{\lambda}_n(1) - \hat{\lambda}_n(2)[1 - \hat{\lambda}_n(1)] - \cdots - \hat{\lambda}_n(r)[1 - \hat{\lambda}_n(r-1)] \cdots [1 - \hat{\lambda}_n(1)]$$

$$= \prod_{j=1}^{r} [1 - \hat{\lambda}_n(j)]. \quad (30)$$

Thus,

$$\hat{g}_{k+1} = \frac{\alpha \sum_{u=k+1}^{w} \hat{h}_{k+1u}}{\prod_{j=1}^{k} [1 - \hat{\lambda}_n(j)].}$$

Therefore, for $u = k+1$, with (29),

$$\frac{1}{f_{k+1}} \sum_{v=1}^{k+1} \hat{h}_{vk+1} - \frac{1}{\alpha} \sum_{v=1}^{k+1} \hat{g}_v = 0.$$

Further,

$$\sum_{v=1}^{k+1} \hat{g}_v = \alpha \hat{C}_n(1) + \frac{\alpha \sum_{u=2}^{w} \hat{h}_{2u}}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]} + \frac{\alpha \sum_{u=k}^{w} \hat{h}_{ku}}{\prod_{j=1}^{k} [1 - \hat{\lambda}_n(j)]}$$

$$= \frac{\alpha \hat{C}_n(2)}{1 - \hat{\lambda}_n(1)} + \frac{\alpha \sum_{u=3}^{w} \hat{h}_{3u}}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]} + \frac{\alpha \sum_{u=k}^{w} \hat{h}_{ku}}{\prod_{j=1}^{k} [1 - \hat{\lambda}_n(j)]}$$

$$= \frac{\alpha \hat{C}_n(3)}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]} + \frac{\alpha \sum_{u=k+1}^{w} \hat{h}_{k+1u}}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]}$$

$$= \frac{\alpha \hat{C}_n(k)}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]} + \frac{\alpha \sum_{u=k+1}^{w} \hat{h}_{k+1u}}{\prod_{j=1}^{k-1} [1 - \hat{\lambda}_n(j)]}$$

That is,

$$\hat{f}_{k+1} = \hat{\lambda}_n(k+1) \prod_{j=1}^{k} [1 - \hat{\lambda}_n(j)].$$

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Now, for $m < u \leq \omega$,

$$\frac{1}{f_u} \sum_{v=1}^{m} \hat{h}_{vu} - \frac{1}{\alpha} \sum_{v=1}^{m} \hat{g}_v = 0 \iff \hat{f}_u = \frac{1}{n} \sum_{i=1}^{n} X_i = u \frac{\prod_{j=1}^{m-1}[1 - \hat{\lambda}_n(j)]}{\hat{C}_n(m)}.$$

Hence,

$$\hat{f}_u = \frac{1}{n} \sum_{i=1}^{n} X_i = u \frac{\prod_{j=1}^{m-1}[1 - \hat{\lambda}_n(j)]}{\hat{C}_n(m)}$$

$$= \frac{1}{n} \sum_{i=1}^{n} 1_{X_i = u} \frac{\hat{C}_n(u)}{\hat{C}_n(u-1)} \ldots \frac{\hat{C}_n(m+1)}{\hat{C}_n(m)} \prod_{j=1}^{m-1}[1 - \hat{\lambda}_n(j)]$$

$$= \hat{\lambda}_n(u) \left[ \frac{\hat{C}_n(u-1)}{\hat{C}_n(u-1)} \ldots \frac{\hat{C}_n(m)}{\hat{C}_n(m)} \right] \prod_{j=1}^{m-1}[1 - \hat{\lambda}_n(j)]$$

$$= \hat{\lambda}_n(u) \prod_{j=1}^{u-1}[1 - \hat{\lambda}_n(j)].$$

Lastly, since $\hat{\lambda}_n(\omega) = 1$,

$$\sum_{u=1}^{\omega} \hat{f}_u = \sum_{u=1}^{\omega} \left( \hat{\lambda}_n(u) \prod_{j=1}^{u-1}[1 - \hat{\lambda}_n(j)] \right)$$

$$= \hat{\lambda}_n(1) + (1 - \hat{\lambda}_n(1)) \sum_{u=2}^{\omega} \left( \hat{\lambda}_n(u) \prod_{j=2}^{u-1}[1 - \hat{\lambda}_n(j)] \right)$$

$$= \hat{\lambda}_n(1) + (1 - \hat{\lambda}_n(1)) \cdot \cdots \cdot \left( 1 - \hat{\lambda}_n(\omega - 2) \right) \cdot \left[ \hat{\lambda}_n(\omega - 1) + 1 - \hat{\lambda}_n(\omega - 1) \right]$$

$$= \hat{\lambda}_n(1) + 1 - \hat{\lambda}_n(1)$$

$$= 1,$$

and the solution set $\hat{f}_u, u \in A$, is in $\Psi$ and omits only this single, unique solution. It is thus the global maximum of (26) and therefore the MLE. More specifically, we have found the MLE for the parameters $f_u, u \in A$, and they are of the form (13). Therefore, $\hat{A}_n$ is an MLE of $f_u$, for $u \in A$ by the invariance property of the MLE (e.g., Mukhopadhyay, 2000,
Theorem 7.2.1, pg. 350).

We can show \( \hat{B}_n \) is also a MLE for \( g_v, v \in \mathcal{A} \), by moving sequentially from the other direction; e.g., for \( m \leq k \leq \omega \), with (29),

\[
\frac{1}{f_k} \sum_{v=1}^{m} \hat{h}_{vk} - \frac{1}{\alpha} \sum_{v=1}^{m} g_v = 0 \implies \hat{f}_k = \alpha \sum_{v=1}^{m} \hat{h}_{vk},
\]

and thus, for \( v = m \)

\[
\frac{1}{g_m} \sum_{u=m}^{\omega} \hat{h}_{um} - \frac{1}{\alpha} \sum_{u=m}^{\omega} \hat{f}_u = 0 \implies \hat{g}_m = \frac{\alpha \sum_{u=m}^{\omega} \hat{h}_{um}}{\sum_{u=m}^{\omega} \hat{f}_u} = \frac{1}{n} \sum_{i=1}^{n} 1_{Y_i=m} \frac{\hat{C}_n(m)}{\hat{C}_n(m)} = \hat{\beta}_n(m).
\]

The remainder follows through symmetry. \( \square \)

### B.2 Proof of Lemma 1

**Proof.** Observe

\[
\hat{C}_n = \begin{bmatrix}
\frac{1}{n} \sum_{i=1}^{n} 1_{Y_i \leq \Delta+1 \leq X_i} \\
\vdots \\
\frac{1}{n} \sum_{i=1}^{n} 1_{Y_i \leq \omega \leq X_i}
\end{bmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} Y_{\Delta+1(i)} \\
\vdots \\
Y_{\omega(i)}
\end{bmatrix},
\]

where \( Y_{k(i)}, \Delta + 1 \leq k \leq \omega \) are i.i.d. Bernoulli random variables with probability of success given by \( \Pr(Y_i \leq k \leq X_i) = \Pr(Y \leq k \leq X \mid Y \leq X) = C(k) \) for \( k = \Delta + 1, \ldots, \omega \). Thus, \( \mathbb{E}[Y_{k(i)}] = C(k) \) and \( \text{Var}[Y_{k(i)}] = C(k)(1 - C(k)) \). Now, since

\[
1_{Y_i \leq k' \leq X_i} 1_{Y_i \leq k \leq X_i} = 1_{Y_i \leq \min(k', k), X_i \geq \max(k'k)},
\]

we have

\[
\mathbb{E}[Y_{k'(i)}Y_{k(i)}] = \mathbb{E}[1_{Y_i \leq \min(k', k), X_i \geq \max(k'k)}] = c(k', k),
\]

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for \( k' \), \( k = \Delta + 1, \ldots, \omega \). Thus,

\[
\text{Cov}[Y_{k'(i)}Y_{k(i)}] = E[Y_{k'(i)}Y_{k(i)}] - E[Y_{k'(i)}]E[Y_{k(i)}] \\
= c(k', k) - C(k')C(k).
\]

Recall that (32) reduces to \( C(k) \) when \( k' = k \). The result then follows by the multivariate Central Limit Theorem (CLT) (Lehmann and Casella, 1998, Theorem 8.21, pg. 61).

### B.3 Proof of Lemma 2

**Proof.** Applying the Weak Law of Large Numbers (Lehmann and Casella, 1998, Theorem 8.2, pg. 54-55) to (31) gives us the result.

### B.4 Proof of Theorem 3.1

For convenience of notation, let

\[
r(u,v) = \Pr(X_i = \max(u,v), Y_i \leq \min(u,v)) \\
= \Pr(X = \max(u,v), \min(u,v) \leq Y \mid Y \leq X) \\
= \sum_{y=\Delta+1}^{\min(u,v)} h(\max(u,v), y) \\
= \frac{1}{\alpha} \Pr(X = \max(u,v)) \Pr(Y \leq \min(u,v)). \tag{33}
\]

Notice \( r(z,z) = f_\alpha(z) \) and \( r(u,v) = r(v,u) \).
Proof. Recall (8)–(9) and observe

\[ \hat{\Lambda}_n - \Lambda = \begin{bmatrix} \hat{\lambda}_n(\Delta + 1) & \cdots & \hat{\lambda}_n(\omega) \\ \vdots & \ddots & \vdots \\ \hat{\lambda}_n(\omega) & \cdots & \lambda(\omega) \end{bmatrix} - \begin{bmatrix} \lambda(\Delta + 1) \\ \vdots \\ \lambda(\omega) \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \sum_{i=1}^{n} 1_{X_i = \Delta + 1} - f_*(\Delta + 1) \\ \vdots \\ \frac{1}{n} \sum_{i=1}^{n} 1_{X_i = \omega} - f_*(\omega) \end{bmatrix} \]

\[ = A_n \times \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} Z_{\Delta+1(i)} \\ \vdots \\ Z_{\omega(i)} \end{bmatrix}, \]

where, for \( \Delta + 1 \leq k \leq \omega, \)

\[ Z_{k(i)} = 1_{X_i = k} C(k) - 1_{Y_i \leq k \leq X_i} f_*(k), \]

and \( A_n = \text{diag}([\hat{\Lambda}_n(\Delta + 1)C(\Delta + 1)]^{-1}, \ldots, [\hat{\Lambda}_n(\omega)C(\omega)]^{-1}). \) That is,

\[ \hat{\Lambda}_n - \Lambda = A_n \times \frac{1}{n} \sum_{i=1}^{n} Z_{(i)}, \]

where \( Z_{(i)} = (Z_{\Delta+1(i)}, \ldots, Z_{\omega(i)})^\top, 1 \leq i \leq n \) are i.i.d. random vectors. We will also subsequently show that the components of random vector \( Z_{(i)} \) are uncorrelated.

More specifically, \( 1_{X_i = x} \) is a Bernoulli random variable with probability of success \( f_*(x) \) and, similarly, \( 1_{Y_i \leq x \leq X_i} \) is a Bernoulli random variable with probability of success \( C(x) \). Thus,

\[ \mathbb{E}[Z_{k(i)}] = f_*(k)C(k) - C(k)f_*(k) = 0. \]

Therefore,

\[ \text{Cov}[Z_{k(i)}Z_{k'(i)}] \quad (34) \]
We proceed to calculate $\text{Cov}[Z_{k(i)}Z_{k'(i)}]$ by cases.

Case 1: $k = k'$.

Notice $\mathbf{1}_{X_i=k}\mathbf{1}_{X_i=k'} = \mathbf{1}_{X_i=k}$ and $E[\mathbf{1}_{X_i=k}\mathbf{1}_{X_i=k'}] = f_*(k)$. Further,

$$\mathbf{1}_{X_i=k'}\mathbf{1}_{Y_i\leq k \leq X_i} = \mathbf{1}_{X_i=k}, \mathbf{1}_{Y_i\leq X_i} = \mathbf{1}_{X_i=k}.$$

Hence, $E[\mathbf{1}_{X_i=k}\mathbf{1}_{Y_i\leq k'}] = f_*(k)$. Also note that

$$\mathbf{1}_{Y_i\leq k \leq X_i}\mathbf{1}_{Y_i\leq k'} = \mathbf{1}_{Y_i\leq k},$$

and thus $E[\mathbf{1}_{Y_i\leq k}] = C(k)$. Replacing the expectations in (35) yields

$$\text{Cov}[Z_{k(i)}Z_{k'(i)}] = C(k)C(k')f_*(k) - f_*(k)C(k')f_*(k)$$

$$- C(k)f_*(k')f_*(k) + f_*(k)f_*(k')C(k)$$

$$= C(k)^2f_*(k) - 2f_*(k)^2C(k) + f_*(k)^2C(k)$$

$$= f_*(k)C(k)[C(k) - f_*(k)].$$

(36)

However,

$$C(k) - f_*(k) = \sum_{y=\Delta+1}^{k} \sum_{x=k}^{L} h_*(x, y) - \sum_{y=\Delta+1}^{k} h_*(k, y)$$

$$= \sum_{y=\Delta+1}^{k} \sum_{x=k+1}^{L} h_*(x, y)$$

$$= c(k, k + 1).$$

(37)
Replacing (37) in (36) and simplifying yields the diagonal matrix

\[ \mathbf{D} = \text{diag}(f_*(\Delta + 1)C(\Delta + 1)c(\Delta + 1, \Delta + 2), \ldots, f_*(\omega)C(\omega)c(\omega, \omega + 1)). \]

We emphasize here that \( c(\omega, \omega + 1) = 0 \).

Case 2: \( k \neq k' \).

Certainly, \( 1_{X_i = k}1_{X_i = k'} = 0 \) when \( k \neq k' \). Therefore,

\[ E[1_{X_i = k}1_{X_i = k'}] = 0. \tag{38} \]

Assume \( k < k' \) and notice \( 1_{X_i = k'}1_{Y_i \leq k \leq X_i} = 1_{X_i = k', Y_i \leq k \leq X_i} \). Thus, \( E[1_{X_i = k'}1_{Y_i \leq k \leq X_i}] = r(k', k) \). On the other hand, \( 1_{X_i = k}1_{Y_i \leq k' \leq X_i} = 1_{X_i = k, Y_i \leq k' \leq X_i} = 0 \) because \( \{X_i = k \cap k' \leq X_i \} = \emptyset \) when \( k < k' \). Now observe the symmetry between \( 1_{X_i = k}1_{Y_i \leq k' \leq X_i} \) and \( 1_{X_i = k'}1_{Y_i \leq k \leq X_i} \) to drop the assumption \( k < k' \) and more generally claim

\[ -f_*(k'C(k'))E[1_{X_i = k'}1_{Y_i \leq k \leq X_i}] - C(k)f_*(k')E[1_{X_i = k}1_{Y_i \leq k' \leq X_i}] \]

\[ = -r(k, k')f_*(\min(k, k'))C(\max(k, k')). \tag{39} \]

Further, \( 1_{Y_i \leq k \leq X_i}1_{Y_i \leq k' \leq X_i} = 1_{Y_i \leq k \leq X_i, Y_i \leq k' \leq X_i} = 1_{Y_i \leq \min(k, k'), X_i \geq \max(k, k')} \). Hence,

\[ E[1_{Y_i \leq k \leq X_i}1_{Y_i \leq k' \leq X_i}] = c(k, k'). \tag{40} \]

Replacing the expectations (38), (39), and (40) in (35) and simplifying yields

\[ E[Z_{k(i)}Z_{k'(i)}] = f_*(\min(k, k')) \times \{f_*(\max(k, k'))c(k, k') - r(k, k')C(\max(k, k'))\}. \]

But,

\[ f_*(\max(k, k'))c(k, k') \]
\[
= \frac{\Pr(X = \max(k, k'), Y \leq X)}{\alpha} \Pr(Y \leq \min(k, k')) \frac{\Pr(X \geq \max(k, k'))}{\alpha} \\
= \frac{\Pr(X = \max(k, k'))}{\alpha} \Pr(Y \leq \max(k, k')) \frac{\Pr(Y \leq \min(k, k'))}{\alpha} \Pr(X \geq \max(k, k')) \\
= \frac{\Pr(X = \max(k, k'))}{\alpha} \Pr(Y \leq \min(k, k')) \frac{\Pr(Y \leq \max(k, k'))}{\alpha} \Pr(X \geq \max(k, k')) \\
= r(k, k')C(\max(k, k')),
\]

and so (35) is zero whenever \( k \neq k' \). Now define
\[
\bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z(i),
\]

and use the multivariate CLT (Lehmann and Casella, 1998, Theorem 8.21, pg. 61) to claim
\[
\sqrt{n}[\bar{Z}_n - 0] \xrightarrow{\mathcal{L}} N(0, D), \text{ as } n \to \infty.
\]

Further note by Lemma 2,
\[
A_n \xrightarrow{P} V, \text{ as } n \to \infty
\]

where \( V = \text{diag}(C(\Delta + 1)^{-2}, \ldots, C(\omega)^{-2}) \). Therefore, by multivariate Slutsky’s Theorem (Lehmann, 1998, Theorem 5.1.6, pg. 283),
\[
\sqrt{n}[A_n \bar{Z}_n] \xrightarrow{\mathcal{L}} N(0, VDV^\top), \text{ as } n \to \infty.
\]

Finally, observe \( VDV^\top = \Sigma_f \) and \( A_n \bar{Z}_n = \hat{\Lambda}_n - \Lambda \) to complete the proof.

\( \square \)
B.5 Proof of Theorem 3.2

Proof. See the proof of Theorem 3.1, substituting $g_*$ for $f_*$ and adjusting the indicator logic as appropriate. It is useful to introduce similar notation to (33). That is,

$$s(u, v) = \Pr(Y_i = \min(u, v), X_i \geq \max(u, v))$$
$$= \frac{1}{\alpha} \Pr(Y = \min(u, v)) \Pr(X \geq \max(u, v)).$$  (41)

\[\square\]

B.6 Proof of Theorem 3.3

Proof. To motivate the demonstration, let $x \in \{\Delta + 1, \ldots, \omega\}$ and recall (5) to write,

$$S(x) = \prod_{z=\Delta+1}^{x} [1 - \lambda(z)].$$

Now consider the natural log,

$$\ln S(x) = \sum_{z=\Delta+1}^{x} \ln[1 - \lambda(z)].$$

Hence,

$$\sqrt{n}[\ln S_n(x) - \ln S(x)] = \sqrt{n} \left[ \sum_{z=\Delta+1}^{x} \ln \left( \frac{1 - \lambda_n(z)}{1 - \lambda(z)} \right) \right]$$
$$= \sqrt{n} \left[ \sum_{z=\Delta+1}^{x} \ln \left( 1 + \frac{\lambda(z) - \lambda_n(z)}{1 - \lambda(z)} \right) \right].$$

But $\ln(1 + x) = \sum_{n \geq 1}(-1)^{n+1}x^n/n$ and so

$$\sqrt{n}[\ln S_n(x) - \ln S(x)] = \sqrt{n} \left[ \sum_{z=\Delta+1}^{x} \left\{ \frac{\lambda(z) - \lambda_n(z)}{1 - \lambda(z)} - \frac{1}{2} \left[ \frac{\lambda(z) - \lambda_n(z)}{(1 - \lambda(z))^2} \right]^2 + \cdots \right\} \right]$$

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\[
= \sqrt{n} \left[ \sum_{z=\Delta+1}^{x} \frac{\lambda(z) - \lambda_n(z)}{1 - \lambda(z)} + O_p(|\lambda(z) - \lambda_n(z)|^2) \right] \\
= \sqrt{n} \left( - \sum_{z=\Delta+1}^{x} \frac{\lambda_n(z) - \lambda(z)}{1 - \lambda(z)} \right) + o_p(1), \tag{42}
\]

where (42) follows by Corollary 3.1.1 and Slutsky’s Theorem (Lehmann and Casella, 1998, Theorem 8.10, pg. 58). Now consider all \( x \in \{\Delta + 1, \ldots, \omega \} \) to write,

\[
\sqrt{n} \begin{bmatrix}
\left\{ \ln S_n(\Delta + 1) - \ln S(\Delta + 1) \right\} \\
\vdots \\
\left\{ \ln S_n(\omega) - \ln S(\omega) \right\}
\end{bmatrix} = K \times \sqrt{n}(\hat{\Lambda}_n - \Lambda) + o_p(1).
\]

Thus, by Theorem 3.1 and multivariate Slutsky’s Theorem (Lehmann, 1998, Theorem 5.1.6, pg. 283),

\[D \times \sqrt{n}(\hat{\Lambda}_n - \Lambda) + o_p(1) \xrightarrow{\mathcal{L}} N(0, K\Sigma_f K^\top), \text{ as } n \to \infty.\]

Finally, note \( S(x) = \exp\{\ln S(x)\} \) and apply the multivariate delta method (Lehmann and Casella, 1998, Theorem 8.22, pg. 61) to complete the proof.

\[\square\]

### B.7 Proof of Theorem 3.4

*Proof.* Recall (12) and see the proof of Theorem 3.3.

\[\square\]

### B.8 Proof of Theorem 3.5

*Proof.* Begin with Theorem 3.2 along with (21) and use the well-known multivariate normal results: (1) all subsets of multivariate normal random vectors have themselves a normal distribution (Ravishanker and Dey, 2002, Result 5.2.8, pg. 154) and (2) a centered and scaled quadratic form of a \( p \) dimensional multivariate normal random vector is a chi-squared
random variable with \( p \) degrees of freedom (Ravishanker and Dey, 2002, Result 5.3.3, pg. 167). The result then follows by the continuous mapping theorem (Lehmann and Casella, 1998, Corollary 8.11, pg. 58).

\[ \square \]

**B.9 Proof of Corollary 3.5.1**

*Proof.* By the Weak Law of Large Numbers (Lehmann and Casella, 1998, Theorem 8.2, pg. 54-55), \( \hat{g}_{*,n} \overset{P}{\rightarrow} g_* \). Further, if \( G \) is discrete uniform over \( \{ \Delta + 1, \ldots, \Delta + m \} \), then for \( y \in \{ \Delta + 1, \ldots, \Delta + m \} \)

\[
\beta(y) = \frac{\Pr(Y = y)}{\Pr(Y \leq y)} = \frac{1}{m} \cdot \frac{m}{y - (\Delta + 1) + 1} = \frac{1}{y - \Delta}.
\]

Finally, use the results of Theorem 3.5 substituting \( \beta(y) \) for \( y \in \{ \Delta + 2, \ldots, \Delta + m \} \) as appropriate along with multivariate Slutsky’s Theorem (Lehmann, 1998, Theorem 5.1.6, pg. 283) to complete the proof.

\[ \square \]

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