Improvements of the Hermite-Hadamard inequality for the simplex

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Abstract
In this study, the simplex whose vertices are barycenters of the given simplex facets plays an essential role. The article provides an extension of the Hermite-Hadamard inequality from the simplex barycenter to any point of the inscribed simplex except its vertices. A two-sided refinement of the generalized inequality is obtained in completion of this work.

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1 Introduction
A concise approach to the concept of affinity and convexity is as follows. Let \( X \) be a linear space over the field \( \mathbb{R} \). Let \( P_1, \ldots, P_m \in X \) be points, and let \( \lambda_1, \ldots, \lambda_m \in \mathbb{R} \) be coefficients. A linear combination

\[
\sum_{j=1}^{m} \lambda_j P_j
\]

is affine if \( \sum_{j=1}^{m} \lambda_j = 1 \). An affine combination is convex if all coefficients \( \lambda_j \) are nonnegative.

Let \( S \subseteq X \) be a set. The set containing all affine combinations of points of \( S \) is called the affine hull of the set \( S \), and it is denoted with \( \text{aff} \, S \). A set \( S \) is affine if \( S = \text{aff} \, S \). Using the adjective convex instead of affine, and the prefix \( \text{conv} \) instead of \( \text{aff} \), we obtain the characterization of the convex set.

A convex function \( f : \text{conv} \, S \rightarrow \mathbb{R} \) satisfies the Jensen inequality

\[
f \left( \sum_{j=1}^{m} \lambda_j P_j \right) \leq \sum_{j=1}^{m} \lambda_j f (P_j)
\]

for all convex combinations of points \( P_j \in S \). An affine function \( f : \text{aff} \, S \rightarrow \mathbb{R} \) satisfies the equality in equation (2) for all affine combinations of points \( P_j \in S \).

Throughout the paper, we use the \( n \)-dimensional space \( X = \mathbb{R}^n \) over the field \( \mathbb{R} \).

2 Convex functions on the simplex
The section is a review of the known results on the Hermite-Hadamard inequality for simplices, and it refers to its generic background. The main notification is concentrated in Lemma 2.1, which is also the generalization of the Hermite-Hadamard inequality.
Let $A_1, \ldots, A_n+1 \in \mathbb{R}^n$ be points so that the points $A_1 - A_n+1, \ldots, A_n - A_n+1$ are linearly independent. The convex hull of the points $A_i$ written in the form of $A_1 \cdots A_n+1$ is called the $n$-simplex in $\mathbb{R}^n$, and the points $A_i$ are called the vertices. So, we use the notation

$$A_1 \cdots A_n+1 = \text{conv}[A_1, \ldots, A_n+1].$$

(3)

The convex hull of $n$ vertices is called the facet or $(n-1)$-face of the given $n$-simplex.

The analytic presentation of points of an $n$-simplex $A= A_1 \cdots A_n+1$ in $\mathbb{R}^n$ arises from the $n$-volume by means of the Lebesgue measure or the Riemann integral. We will use the abbreviation $\text{vol}$ instead of $\text{vol}_n$.

Let $A \in \mathcal{A}$ be a point, and let $A_i$ be the convex hull of the set containing the point $A$ and vertices $A_j$ for $j \neq i$, formally as

$$A_i = \text{conv}[A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n+1].$$

(4)

Each $A_i$ is a facet or $n$-subsimplex of $\mathcal{A}$, so $\text{vol}(A_i) = 0$ or $0 < \text{vol}(A_i) \leq \text{vol}(A)$, respectively. The sets $A_i$ satisfy $\mathcal{A} = \bigcup_{i=1}^{n+1} A_i$ and $\text{vol}(A_i \cap A_j) = 0$ for $i \neq j$, and so it follows that $\text{vol}(A) = \sum_{i=1}^{n+1} \text{vol}(A_i)$.

The point $A$ can be uniquely represented as the convex combination of the vertices $A_i$ by means of

$$A = \sum_{i=1}^{n+1} \alpha_i A_i,$$

(5)

where we have the coefficients

$$\alpha_i = \frac{\text{vol}(A_i)}{\text{vol}(A)}.$$

(6)

If the point $A$ belongs to the interior of the $n$-simplex $\mathcal{A}$, then all sets $A_i$ are $n$-simplices, and consequently all coefficients $\alpha_i$ are positive. Furthermore, the reverse implications are valid.

If $\mu$ is a positive measure on $\mathbb{R}^n$, and if $S \subseteq \mathbb{R}^n$ is a measurable set such that $\mu(S) > 0$, then the integral mean point

$$S = \left( \frac{\int_S x_1 \, d\mu(x)}{\mu(S)}, \ldots, \frac{\int_S x_n \, d\mu(x)}{\mu(S)} \right)$$

(7)

is called the $\mu$-barycenter of the set $S$. In the above integrals, points $x \in S$ are used as $x = (x_1, \ldots, x_n)$. The $\mu$-barycenter $S$ belongs to the convex hull of $S$. When we use the Lebesgue measure, we say just barycenter. If $S$ is closed and convex, then a $\mu$-integrable continuous convex function $f : S \to \mathbb{R}$ satisfies the inequality

$$f \left( \frac{\int_S x_1 \, d\mu(x)}{\mu(S)}, \ldots, \frac{\int_S x_n \, d\mu(x)}{\mu(S)} \right) \leq \frac{\int_S f(x) \, d\mu(x)}{\mu(S)}$$

(8)

as a special case of Jensen’s inequality for multivariate convex functions; see the excellent McShane paper in [1]. If $f$ is affine, then the equality is valid in (8).
We consider a convex function \( f \) defined on the \( n \)-simplex \( A = A_1 \cdots A_{n+1} \). The following lemma presents a basic inequality for a convex function on the simplex, and it refers to the connection of the simplex barycenter with simplex vertices.

**Lemma 2.1** Let \( \mu \) be a positive measure on \( \mathbb{R}^n \). Let \( A = A_1 \cdots A_{n+1} \) be an \( n \)-simplex in \( \mathbb{R}^n \) such that \( \mu(A) > 0 \). Let \( A \) be the \( \mu \)-barycenter of \( A \), and let \( \sum_{i=1}^{n+1} \alpha_i A_i \) be its unique convex combination by means of

\[
A = \left( \frac{\int_A x_1 \ d\mu(x)}{\mu(A)}, \ldots, \frac{\int_A x_n \ d\mu(x)}{\mu(A)} \right) = \sum_{i=1}^{n+1} \alpha_i A_i. \tag{9}
\]

Then each convex function \( f : A \to \mathbb{R} \) satisfies the double inequality

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \frac{\int_A f(x) \ d\mu(x)}{\mu(A)} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{10}
\]

**Proof** We have three cases depending on the position of the \( \mu \)-barycenter \( A \) within the simplex \( A \).

If \( A \) is an interior point of \( A \), then we take a supporting hyperplane \( x_{n+1} = h_1(x) \) at the graph point \((A, f(A))\), and the secant hyperplane \( x_{n+1} = h_2(x) \) passing through the graph points \((A_1, f(A_1)), \ldots, (A_{n+1}, f(A_{n+1}))\). Using the affinity of the functions \( h_1 \) and \( h_2 \), we get

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) = h_1(A) = \frac{\int_A h_1(x) \ d\mu(x)}{\mu(A)} \leq \frac{\int_A f(x) \ d\mu(x)}{\mu(A)} \leq \frac{\int_A h_2(x) \ d\mu(x)}{\mu(A)} = h_2(A) = \sum_{i=1}^{n+1} \alpha_i h_2(A_i) = \sum_{i=1}^{n+1} \alpha_i f(A_i) \tag{11}
\]

because \( h_2(A_i) = f(A_i) \). So, formula (10) works for the interior point \( A \).

If \( A \) is a relative interior point of a certain \( k \)-face where \( 1 \leq k \leq n - 1 \), then we can apply the previous procedure to the respective \( k \)-simplex. For example, if \( A_1 \cdots A_{k+1} \) is the observed \( k \)-face, then the coefficients \( \alpha_1, \ldots, \alpha_{k+1} \) are positive, and the coefficients \( \alpha_{k+2}, \ldots, \alpha_{n+1} \) are equal to zero.

If \( A \) is a simplex vertex, suppose that \( A = A_1 \), then the trivial inequality \( f(A_1) \leq f(A_1) \leq f(A_1) \) represents formula (10).

More generally, if the \( \mu \)-barycenter \( A \) lies in the interior of \( A \), the inequality in formula (10) holds for all \( \mu \)-integrable functions \( f : A \to \mathbb{R} \) that admit a supporting hyperplane at \( A \), and satisfy the supporting-secant hyperplane inequality

\[
h_1(x) \leq f(x) \leq h_2(x) \tag{12}
\]

for every point \( x \) of the simplex \( A \).
Lemma 2.1 was obtained in [2], Corollary 1, the case $\alpha_i = 1/(n + 1)$ was obtained in [3], Theorem 2, and a similar result was obtained in [4], Theorem 2.4.

By applying the Lebesgue measure or the Riemann integral in Lemma 2.1, the condition in (9) gives the barycenter

$$A = \left( \frac{\int_A x_1 \, dx}{\text{vol}(A)}, \ldots, \frac{\int_A x_n \, dx}{\text{vol}(A)} \right) = \frac{\sum_{i=1}^{n+1} A_i}{n + 1},$$

and its use in formula (10) implies the Hermite-Hadamard inequality

$$f\left( \frac{\sum_{i=1}^{n+1} A_i}{n + 1} \right) \leq \frac{\int_A f(x) \, dx}{\text{vol}(A)} \leq \frac{\sum_{i=1}^{n+1} f(A_i)}{n + 1}. \quad (14)$$

The above inequality was introduced by Neuman in [5]. An approach to this inequality can be found in [6].

The discrete version of Lemma 2.1 contributes to the Jensen inequality on the simplex.

**Corollary 2.2** Let $A = A_1 \cdots A_{n+1}$ be an $n$-simplex in $\mathbb{R}^n$, and let $P_1, \ldots, P_m \in A$ be points. Let $A = \sum_{j=1}^{m} \lambda_j P_j$ be a convex combination of the points $P_j$, and let $\sum_{i=1}^{n+1} \alpha_i A_i$ be the unique convex combination of the vertices $A_i$ such that

$$A = \sum_{j=1}^{m} \lambda_j P_j = \sum_{i=1}^{n+1} \alpha_i A_i. \quad (15)$$

Then each convex function $f : A \to \mathbb{R}$ satisfies the double inequality

$$f\left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{j=1}^{m} \lambda_j f(P_j) \leq \sum_{i=1}^{n+1} \alpha_i f(A_i). \quad (16)$$

**Proof** The discrete measure $\mu$ concentrated at the points $P_j$ by the rule

$$\mu(P_j) = \lambda_j \quad (17)$$

can be utilized in Lemma 2.1 to obtain the discrete inequality in formula (16).

Putting $\sum_{j=1}^{m} \lambda_j P_j$ instead of $\sum_{i=1}^{n+1} \alpha_i A_i$, within the first term of formula (16), we obtain the Jensen inequality extended to the right.

Corollary 2.2 in the case $\alpha_i = 1/(n + 1)$ was obtained in [3], Corollary 4.

One of the most influential results of the theory of convex functions is the Jensen inequality (see [7] and [8]), and among the most beautiful results is certainly the Hermite-Hadamard inequality (see [9] and [10]). A significant generalization of the Jensen inequality for multivariate convex functions can be found in [1]. Improvements of the Hermite-Hadamard inequality for univariate convex functions were obtained in [11]. As for the Hermite-Hadamard inequality for multivariate convex functions, one may refer to [2, 4, 5, 12–16], and [17].
3 Main results
Throughout the section, we will use an $n$-simplex $A = A_1 \cdots A_{n+1}$ in the space $\mathbb{R}^n$, and its two $n$-subsimplices which will be denoted with $\mathcal{B}$ and $\mathcal{C}$.

Let $B_i$ stand for the barycenter of the facet of $A$ not containing the vertex $A_i$ by

$$B_i = \frac{\sum_{j \neq i}^{n+1} A_j}{n},$$

and let $\mathcal{B} = B_1 \cdots B_{n+1}$ be the $n$-simplex of the vertices $B_i$.

The simplices $\mathcal{A}$ and $\mathcal{B}$ in our three-dimensional space are tetrahedrons presented in Figure 1. Our aim is to extend the Hermite-Hadamard inequality to all points of the inscribed simplex $\mathcal{B}$ excepting its vertices. So, we focus on the non-peaked simplex $\mathcal{B}' = \mathcal{B} \setminus \{B_1, \ldots, B_{n+1}\}$.

Lemma 3.1 Let $A = A_1 \cdots A_{n+1}$ be an $n$-simplex in $\mathbb{R}^n$, and let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices $A_i$.

The point $A$ belongs to the $n$-simplex $\mathcal{B} = B_1 \cdots B_{n+1}$ if and only if the coefficients $\alpha_i$ satisfy $\alpha_i \leq 1/n$.

The point $A$ belongs to the non-peaked simplex $\mathcal{B}' = \mathcal{B} \setminus \{B_1, \ldots, B_{n+1}\}$ if and only if the coefficients $\alpha_i$ satisfy $0 < \alpha_i \leq 1/n$.

Proof The first statement, relating to the simplex $\mathcal{B}$, will be covered as usual by proving two directions.

Let us assume that the coefficients $\alpha_i$ satisfy the limitations $\alpha_i \leq 1/n$. Then the coefficients

$$\beta_i = 1 - n\alpha_i,$$

are nonnegative, and their sum is equal to 1. Since $\beta_i = 1 - \sum_{j=1}^{n+1} \beta_j$, the reverse connection

$$\alpha_i = \frac{\sum_{j=1}^{n+1} \beta_j}{n}$$

(19)

(20)
follows. The last of the convex combinations

\[
A = \sum_{i=1}^{n+1} \alpha_i A_i
\]

\[
= \sum_{i=1}^{n+1} \frac{\sum_{j \neq i}^{n+1} \beta_j}{n} A_i \equiv \sum_{i=1}^{n+1} \frac{\sum_{j \neq i}^{n+1} \beta_j A_i}{n}
\]

\[
= \sum_{i=1}^{n+1} \beta_i B_i
\]

(21)

confirms that the point \( A \) belongs to the simplex \( B \).

Let us assume that the point \( A \) belongs to the simplex \( B \). Then we have the convex combination \( A = \sum_{i=1}^{n+1} \lambda_i B_i \). Using equation (21) in the reverse direction, we get the convex combinations equality

\[
\sum_{i=1}^{n+1} \lambda_i B_i = \sum_{i=1}^{n+1} \alpha_i A_i
\]

(22)

with the coefficient connections \( \alpha_i = \sum_{j \neq i}^{n+1} \lambda_j / n \) from which we may conclude that \( \alpha_i \leq 1/n \).

The second statement, relating to the non-peaked simplex \( B' \), follows from the first statement and the convex combinations in formula (18) which uniquely represent the facet barycenters \( B_i \).

\( \square \)

We need another subsimplex of \( A \). Let \( A \) be a point belonging to the interior of \( A \). In this case, the sets \( A_i \) defined by formula (4) are \( n \)-simplices. Let \( C_i \) stand for the barycenter of the simplex \( A_i \) by means of

\[
C_i = \frac{A + \sum_{j \neq i}^{n+1} A_j}{n+1},
\]

(23)

and let \( C = C_1 \cdots C_{n+1} \) be the \( n \)-simplex of the vertices \( C_i \).

**Lemma 3.2** Let \( A = A_1 \cdots A_{n+1} \) be an \( n \)-simplex in \( \mathbb{R}^n \), and let \( A = \sum_{i=1}^{n+1} \alpha_i A_i \) be a convex combination of the vertices \( A_i \) with coefficients \( \alpha_i \) satisfying \( \alpha_i > 0 \).

The point \( A \) belongs to the non-peaked simplex \( C' = C \setminus \{ C_1, \ldots, C_{n+1} \} \) if and only if the coefficients \( \alpha_i \) satisfy the additional limitations \( \alpha_i \leq 1/n \).

**Proof** Suppose that the coefficients \( \alpha_i \) satisfy \( 0 < \alpha_i \leq 1/n \). Let \( \beta_i \) be the coefficients as in equation (19). Using the trivial equality \( A = A/(n+1) + nA/(n+1) \), and the coefficient connections of equation (20), we get

\[
A = \sum_{i=1}^{n+1} \alpha_i A_i = \frac{1}{n+1} A + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i A_i
\]

\[
= \sum_{i=1}^{n+1} \beta_i \frac{A}{n+1} + \sum_{i=1}^{n+1} \sum_{j \neq i}^{n+1} \beta_j \frac{A_j}{n+1}
\]
indicating that the point $A$ lies in the simplex $C$. To show that the convex combination 
\[
\sum_{i=1}^{n+1} \beta_i C_i = \sum_{i=1}^{n+1} \beta_i A_i + \sum_{i \neq j=1}^{n+1} \frac{\beta_i A_j}{n+1}
\]

\[
= \sum_{i=1}^{n+1} \beta_i A + \sum_{i \neq j=1}^{n+1} A_j = \sum_{i=1}^{n+1} \beta_i C_i
\]

(24)

The proof of the reverse implication goes exactly in the same way as in the proof of Lemma 3.1.

Each simplex $C$ is homothetic to the simplex $B$. Namely, combining equations (23) and (18), we can represent each vertex $C_i$ by the convex combination

\[
C_i = \frac{1}{n+1} A + \frac{n}{n+1} B_i.
\]

Then it follows that

\[
C_i - A = \frac{n}{n+1} (B_i - A),
\]

and using free vectors, we have the equality $\overrightarrow{AC_i} = (n/(n+1))\overrightarrow{AB_i}$. So, the simplices $C$ and $B$ are similar respecting the homothety with the center at $A$ and the coefficient $n/(n+1)$.

If $A \in B'$, then $C \subset B'$ by the convex combinations in formula (25). Combining Lemma 3.1 and Lemma 3.2, and applying Corollary 2.2 to the simplex inclusions $C \subset B$ and $B \subset A$, we get the Jensen type inequality as follows.

**Corollary 3.3** Let $A = A_1 \cdots A_{n+1}$ be an $n$-simplex in $\mathbb{R}^n$, let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices $A_i$ with coefficients $\alpha_i$ satisfying $0 < \alpha_i \leq 1/n$, and let $\beta_i = 1 - n\alpha_i$.

Then it follows that

\[
\sum_{i=1}^{n+1} \beta_i C_i = \sum_{i=1}^{n+1} \beta_i B_i = \sum_{i=1}^{n+1} \alpha_i A_i,
\]

(26)

and each convex function $f : A \to \mathbb{R}$ satisfies the double inequality

\[
\sum_{i=1}^{n+1} \beta_i f(C_i) \leq \sum_{i=1}^{n+1} \beta_i f(B_i) \leq \sum_{i=1}^{n+1} \alpha_i f(A_i).
\]

(27)

The point $A$ used in the previous corollary lies in the interior of the simplex $A$ because the coefficients $\alpha_i$ are positive. In that case, the sets $A_i$ are $n$-simplices, and they will be used in the main theorem that follows.

**Theorem 3.4** Let $A = A_1 \cdots A_{n+1}$ be an $n$-simplex in $\mathbb{R}^n$, let $A = \sum_{i=1}^{n+1} \alpha_i A_i$ be a convex combination of the vertices $A_i$ with coefficients $\alpha_i$ satisfying $0 < \alpha_i \leq 1/n$, and let $\beta_i = 1 - n\alpha_i$.

Let $A_i$ be the simplices defined by formula (4).
Then each convex function \( f : A \rightarrow \mathbb{R} \) satisfies the double inequality

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{i=1}^{n+1} \beta_i \int_A f(x) \, dx \leq \sum_{i=1}^{n+1} \alpha_i f(A_i).
\] (28)

**Proof** Using the convex combinations equality \( \sum_{i=1}^{n+1} \alpha_i A_i = \sum_{i=1}^{n+1} \beta_i C_i \), and applying the Jensen inequality to \( f(\sum_{i=1}^{n+1} \beta_i C_i) \), we get

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{i=1}^{n+1} \beta_i f(C_i) = \sum_{i=1}^{n+1} \beta_i \left( \frac{A + \sum_{i \neq j}^{n+1} A_j}{n+1} \right).
\]

Summing the products of the Hermite-Hadamard inequalities for the function \( f \) on the simplices \( A_i \) and the coefficients \( \beta_i \), it follows that

\[
\sum_{i=1}^{n+1} \beta_i f \left( \frac{A + \sum_{i \neq j}^{n+1} A_j}{n+1} \right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_A f(x) \, dx}{\text{vol}(A_i)} \leq \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i \neq j}^{n+1} f(A_j)}{n+1}.
\]

Repeating the procedure which was used for the derivation of formula (24), we obtain the series of equalities

\[
\sum_{i=1}^{n+1} \beta_i f(A) + \sum_{i \neq j}^{n+1} f(A_j) = \frac{1}{n+1} f(A) + \frac{n}{n+1} \sum_{i=1}^{n+1} \beta_i \frac{\sum_{i \neq j}^{n+1} f(A_j)}{n}
\]

\[
= \frac{1}{n+1} f(A) + \frac{n}{n+1} \sum_{i=1}^{n+1} \beta_i f(A_i)
\]

\[
= \frac{1}{n+1} \sum_{i=1}^{n+1} \alpha_i A_i + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i f(A_i).
\]

Finally, applying the Jensen inequality to \( f(\sum_{i=1}^{n+1} \alpha_i A_i) \), we get the last inequality

\[
\frac{1}{n+1} \sum_{i=1}^{n+1} \alpha_i A_i + \frac{n}{n+1} \sum_{i=1}^{n+1} \alpha_i f(A_i) \leq \sum_{i=1}^{n+1} \alpha_i f(A_i).
\]

Bringing together all of the above, we obtain the multiple inequality

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{i=1}^{n+1} \beta_i f \left( \frac{A + \sum_{i \neq j}^{n+1} A_j}{n+1} \right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_A f(x) \, dx}{\text{vol}(A_i)}
\]

\[
\leq \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i \neq j}^{n+1} f(A_j)}{n+1} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i),
\] (29)

of which the most important part is the double inequality in formula (28). \( \square \)

The inequality in formula (29) is a generalization and refinement of the Hermite-Hadamard inequality. Taking the coefficients \( \alpha_i = 1/(n+1) \), in which case \( \beta_i = 1/(n+1) \),
we realize the five terms inequality
\[
f\left(\frac{\sum_{i=1}^{n+1} A_i}{n+1}\right) \leq \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{A_i + (n + 2) \sum_{j \neq i}^{n+1} A_j}{(n+1)(n+1)}\right) \leq \frac{\int_A f(x) \, dx}{\text{vol}(A)}
\]
\[
\leq \frac{1}{n+1} f\left(\frac{\sum_{i=1}^{n+1} A_i}{n+1}\right) + \frac{n}{n+1} \sum_{i=1}^{n+1} f(A_i) \leq \frac{\sum_{i=1}^{n+1} f(A_i)}{n+1},
\]
(30)

where the second and fourth terms refine the Hermite-Hadamard inequality. The third term is generated from all of \(n+1\) simplices \(A_i\). In the present case, these simplices have the same volume equal to \(\text{vol}(A)/(n+1)\).

The inequality in formula (30) excepting the second term was obtained in [2], Theorem 2. Similar inequalities concerning the standard \(n\)-simplex were obtained in [5, 6] and [18]. Special refinements of the left and right-hand sides of the Hermite-Hadamard inequality were recently obtained in [19] and [20].

4 Generalization to the function barycenter

If \(\mu\) is a positive measure on \(\mathbb{R}^n\), if \(S \subseteq \mathbb{R}^n\) is a measurable set, and if \(g : S \to \mathbb{R}\) is a nonnegative integrable function such that \(\int_S g(x) \, d\mu(x) > 0\), then the integral mean point
\[
S = \left(\frac{\int_S x_1 g(x) \, d\mu(x)}{\int_S g(x) \, d\mu(x)}, \ldots, \frac{\int_S x_n g(x) \, d\mu(x)}{\int_S g(x) \, d\mu(x)}\right)
\]
(31)

can be called the \(\mu\)-barycenter of the function \(g\). It is about the following measure. Introducing the measure \(\nu\) as
\[
\nu(S) = \int_S g(x) \, d\mu(x),
\]
(32)

we get
\[
S = \left(\frac{\int_S x_1 \, d\nu(x)}{\nu(S)}, \ldots, \frac{\int_S x_n \, d\nu(x)}{\nu(S)}\right).
\]
(33)

Thus the \(\mu\)-barycenter of the function \(g\) coincides with the \(\nu\)-barycenter of its domain \(S\). So, the barycenter \(S\) belongs to the convex hull of the set \(S\). By using the unit function \(g(x) = 1\) in formula (31), it is reduced to formula (7).

Utilizing the function barycenter instead of the set barycenter, we have the following reformulation of Lemma 2.1.

Lemma 4.1 Let \(\mu\) be a positive measure on \(\mathbb{R}^n\). Let \(A = A_1 \cdots A_{n+1}\) be an \(n\)-simplex in \(\mathbb{R}^n\), and let \(g : A \to \mathbb{R}\) be a nonnegative integrable function such that \(\int_A g(x) \, d\mu(x) > 0\). Let \(A\) be the \(\mu\)-barycenter of \(g\), and let \(\sum_{i=1}^{n+1} \alpha_iA_i\) be its unique convex combination by means of
\[
A = \left(\frac{\int_A x_1 g(x) \, d\mu(x)}{\int_A g(x) \, d\mu(x)}, \ldots, \frac{\int_A x_n g(x) \, d\mu(x)}{\int_A g(x) \, d\mu(x)}\right) = \sum_{i=1}^{n+1} \alpha_iA_i.
\]
(34)
Then each convex function $f : \mathcal{A} \to \mathbb{R}$ satisfies the double inequality
\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \frac{\int_{\mathcal{A}} f(x) g(x) d\mu(x)}{\int_{\mathcal{A}} g(x) d\mu(x)} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i). \tag{35}
\]

The proof of Lemma 2.1 can be employed as the proof of Lemma 4.1 by using the measure $\nu$ in formula (32) or by utilizing the affinity of the hyperplanes $h_1$ and $h_2$ in the form of the equalities
\[
h_{1,2} \left( \frac{\int_{\mathcal{A}} x_1 g(x) d\mu(x)}{\int_{\mathcal{A}} g(x) d\mu(x)}, \ldots, \frac{\int_{\mathcal{A}} x_{n} g(x) d\mu(x)}{\int_{\mathcal{A}} g(x) d\mu(x)} \right) = \frac{\int_{\mathcal{A}} h_{1,2}(x) g(x) d\mu(x)}{\int_{\mathcal{A}} g(x) d\mu(x)}.	ag{36}\]

Lemma 4.1 is an extension of the Fejér inequality (see [21]) to multivariable convex functions. As regards univariable convex functions, using the Lebesgue measure on $\mathbb{R}$ and a closed interval as 1-simplex in Lemma 4.1, we get the following generalization of the Fejér inequality.

**Corollary 4.2** Let $[a, b]$ be a closed interval in $\mathbb{R}$, and let $g : [a, b] \to \mathbb{R}$ be a nonnegative integrable function such that $\int_a^b g(x) \, dx > 0$. Let $c$ be the barycenter of $g$, and let $\alpha a + \beta b$ be its unique convex combination by means of
\[
c = \frac{\int_a^b xg(x) \, dx}{\int_a^b g(x) \, dx} = \alpha a + \beta b. \tag{37}\]

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality
\[
f(\alpha a + \beta b) \leq \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \leq \alpha f(a) + \beta f(b). \tag{38}\]

Fejér used a nonnegative integrable function $g$ that is symmetric with respect to the midpoint $c = (a + b)/2$. Such a function satisfies $g(x) = g(2c - x)$, and therefore
\[
\int_a^b (x - c) g(x) \, dx = 0.
\]

As a consequence it follows that
\[
\frac{\int_a^b xg(x) \, dx}{\int_a^b g(x) \, dx} = \frac{\int_a^b (x - c) g(x) \, dx}{\int_a^b g(x) \, dx} + \frac{\int_a^b cg(x) \, dx}{\int_a^b g(x) \, dx} = \frac{a + b}{2},
\]
and formula (38) with $\alpha = \beta = 1/2$ turns into the Fejér inequality
\[
f \left( \frac{a + b}{2} \right) \leq \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \leq \frac{f(a) + f(b)}{2}. \tag{39}\]

Using the barycenters of the restrictions of $g$ onto simplices $\mathcal{A}_i$ in formula (4), we have the following generalization of Theorem 3.4.
Theorem 4.3 Let \( \mu \) be a positive measure on \( \mathbb{R}^n \). Let \( A = A_1 \cdots A_{n+1} \) be an \( n \)-simplex in \( \mathbb{R}^n \), let \( A = \sum_{i=1}^{n+1} a_i A_i \) be a convex combination of the vertices \( A_i \) with coefficients \( a_i \) satisfying \( 0 < a_i \leq 1/n \), and let \( \beta_i = 1 - na_i \). Let \( A_i \) be the simplices defined by formula (4), and let \( g_i : A_i \to \mathbb{R} \) be nonnegative integrable functions such that \( \int_{A_i} g_i(x) \, d\mu(x) > 0 \) and

\[
C_i = \left( \frac{\int_{A_i} x_1 g_i(x) \, d\mu(x)}{\int_{A_i} g_i(x) \, d\mu(x)}, \ldots, \frac{\int_{A_i} x_n g_i(x) \, d\mu(x)}{\int_{A_i} g_i(x) \, d\mu(x)} \right) = \frac{A + \sum_{i,j=1}^{n+1} A_j}{n+1}.
\]

Then each convex function \( f : A \to \mathbb{R} \) satisfies the double inequality

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_{A_i} f(x) g_i(x) \, d\mu(x)}{\int_{A_i} g_i(x) \, d\mu(x)} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i) \leq \frac{f(A) + \sum_{i,j=1}^{n+1} f(A_j)}{n+1}.
\]

Proof The first step of the proof is to apply Lemma 4.1 to the functions \( f \) and \( g_i \) on the simplex \( A_i \) in the way of

\[
f \left( \frac{A + \sum_{i,j=1}^{n+1} A_j}{n+1} \right) \leq \frac{\int_{A_i} f(x) g_i(x) \, d\mu(x)}{\int_{A_i} g_i(x) \, d\mu(x)} \leq \frac{f(A) + \sum_{i,j=1}^{n+1} f(A_j)}{n+1}.
\]

Summing the products of the above inequalities with the coefficients \( \beta_i \), we obtain the double inequality that may be combined with formula (29), and so we obtain the multiple inequality

\[
f \left( \sum_{i=1}^{n+1} \alpha_i A_i \right) \leq \sum_{i=1}^{n+1} \beta_i \left( \frac{A + \sum_{i,j=1}^{n+1} A_j}{n+1} \right) \leq \sum_{i=1}^{n+1} \beta_i \frac{\int_{A_i} f(x) g_i(x) \, d\mu(x)}{\int_{A_i} g_i(x) \, d\mu(x)}
\]

\[
\leq \sum_{i=1}^{n+1} \beta_i \frac{f(A) + \sum_{i,j=1}^{n+1} f(A_j)}{n+1} \leq \sum_{i=1}^{n+1} \alpha_i f(A_i)
\]

containing the double inequality in formula (41).

The conditions in formula (40) require that the \( \mu \)-barycenter of the function \( g_i \) coincides with the barycenter \( C_i = (A + \sum_{i,j=1}^{n+1} A_j)/(n+1) \) of the simplex \( A_i \).

Using the Lebesgue measure and functions \( g_i(x) = 1 \), the inequality in formula (42) reduces to the inequality in formula (29).

Competing interests
The author declares that he has no competing interests.

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