On Magnetic Inhibition Theory in Non-resistive Magnetohydrodynamic Fluids: Existence of Solutions in Some Classes of Large Data

Fei Jiang\textsuperscript{a,c}, Song Jiang\textsuperscript{b}

\textsuperscript{a}School of Mathematics and Statistics, Fuzhou University, Fuzhou, 350108, China.
\textsuperscript{b}Institute of Applied Physics and Computational Mathematics, Huayuan Road 6, Beijing 100088, China.
\textsuperscript{c}Key Laboratory of Operations Research and Control of Universities in Fujian, Fuzhou 350108, China.

Abstract

This paper is concerned with existence of solutions to the incompressible non-resistive viscous magnetohydrodynamic (MHD) equations with large initial perturbations in three-dimensional (3D) periodic domains (in Lagrangian coordinates). Motivated by the Diophantine condition imposed by the approximate theory of non-resistive MHD equations in \cite{5}, Chen–Zhang–Zhou in \cite{10} and the magnetic inhibition mechanism of Lagrangian coordinates version in our previous paper \cite{25}, we prove the existence of unique classical solutions under some class of large initial perturbations, where the intensity of impressive magnetic fields depends increasingly on the $H^{17} \times H^{21}$-norm of the initial perturbation of both the velocity and magnetic field. Our result not only mathematically verifies that magnetic fields prevent the singularity formation of solutions with large initial velocity in the viscous case, but also provide a starting point for the existence theory of large perturbation solutions of the 3D non-resistive viscous MHD equations. In addition, we further rigorously prove that, for large time or strong magnetic field, the MHD equations reduce to the corresponding linearized equations by providing the error estimates, which enjoy the types of algebraic decay with respect to the both of time and field intensity, between the solutions of both the nonlinear and linear equations.

Keywords: Incompressible MHD fluids; large initial data; algebraic time-decay; convergence in the field intensity.

2000 MSC: 35Q60, 35B10, 76E25.

1. Introduction

In this paper, we investigate the existence of the global (-in-time) solutions to the following system of equations of an incompressible non-resistive viscous magnetohydrodynamic (MHD) fluid with large initial perturbations:

\begin{align*}
\rho v_t + \rho v \cdot \nabla v + \nabla p - \mu \Delta v &= \lambda M \cdot \nabla M / 4\pi, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0,
\end{align*}

where the unknowns $v := v(x,t)$, $M := M(x,t)$ and $p := p(x,t)$ denote the velocity, magnetic field and the sum of both the magnetic and kinetic pressures of the MHD fluid, resp., and the three positive (physical) parameters $\rho$, $\mu$ and $\lambda$ stand for the density, shear viscosity coefficient...

Email addresses: jiangfei0591@163.com (Fei Jiang), jiang@iapcm.ac.cn (Song Jiang)
and permeability of vacuum, resp.. In particular, neglecting the shear viscosity coefficient, we have

\[
\begin{align*}
\rho v_t + \rho v \cdot \nabla v + \nabla p &= \lambda M \cdot \nabla M/4\pi, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0.
\end{align*}
\tag{1.2}
\]

Physicists pointed out that, in the nonlinear MHD system, a strong enough magnetic field will reduce the nonlinear interaction \cite{31} and inhibit the formation of strong gradients. This effect was also observed in direct numerical simulations of the ideal MHD system \eqref{1.2}, with periodic boundary conditions \cite{16}. In 1998, Bardos–Sulem–Sulem used the hyperbolicity of the two/three-dimensional (2/3D) idea MHD system \eqref{1.2} and gave a rigorous proof of the global well-posedness when the initial data \((v^0, M^0)\) is around the equilibrium state \((0, \bar{M})\) in the Hölder space \cite{5} (see \cite{6, 21, 43, 45} for the case of Sobolev spaces), where \(\bar{M}\) is often called the impressive magnetic field. In particular, the result of Bardos et.al. presents that the system \eqref{1.2} is global well-posed with large initial perturbation, when the impressive magnetic field is sufficiently large. We remark that such global stability result is not excepted for the 3D incompressible Euler equations, i.e., the magnetic field is absent in the system \eqref{1.2}. Interesting readers can refer to \cite{8, 35} and \cite{30, 50} for the singularity formation of solutions and the solutions with double-exponential growth in Euler equations, resp.. Bardos–Sulem–Sulem’s large perturbation result can be roughly described as follows:

**Assertion 1.1.** If

\[
\frac{\text{initial perturbation around } (0, \bar{M})}{\text{field intensity of } \bar{M}} \ll 1,
\]

then the idea incompressible MHD system \eqref{1.2} admits a global stability solution. Here and in what follows, \(A \ll B\) means that \(A\) is much smaller than \(B\).

We naturally make an association with the following well-known assertion in viscous (pure) fluids (see \cite{12, 17, 33} for examples):

**Assertion 1.2.** If

\[
\frac{\text{initial perturbation of velocity around zero}}{\text{value of viscosity } \mu} \ll 1,
\]

then Navier–Stokes equations (i.e. the equations in \eqref{1.1} with \(M = 0\)) admits a global stability solution.

The above two assertions present that magnetic fields can inhibit the singularity formation of solutions with large initial velocity as well as viscosity, though the physical mechanisms of inhibition/stabilizing are different. Based on the above two assertions, we easily further believe that such large perturbation result shall also exists for the hyperbolic-parabolic system \eqref{1.1}, that is (roughly speaking):

**Assertion 1.3.** If

\[
\frac{\text{initial perturbation around } (0, \bar{M})}{\text{field intensity of } \bar{M}} \ll 1, \tag{1.3}
\]

then the incompressible non-resistive viscous MHD system \eqref{1.1} admits a global stability solution.
Unfortunately, due to the parabolic structure of \((1.1)_1\), the hyperbolic method used by Bardos et al. cannot be applied to the system \((1.1)\). Nearly twenty years after the pioneering work of Bardos et al., Zhang developed new ideas based on the energy method and spectrum analysis, and successfully established the large perturbation result for the system \((1.1)\) under the 2D case [48]. Zhang’s result supports Assertion 1.3 at least for the 2D case. However the corresponding 3D case is still an open problem. At present, all 3D global well-posedness results without symmetry structure imposed on the initial data (see [32] for the axially symmetric solutions with large initial data) are concerned with the small perturbation solutions, i.e. if

\[
\text{initial perturbation around } (0, \bar{M}) \ll 1, \quad (1.4)
\]

then the system \((1.1)\) admits a global stability solution, see [1, 37, 46] and [40] for the Cauchy problem and the initial-boundary value problem of \((1.1)\), resp.. Interesting readers can further refer to [34, 38, 39, 47] and [8, 14, 15] for the 2D small perturbation results and the 3D local (-in-time) existence results of large initial perturbations, resp.. In this paper, we will develop some new ideas to further establish the existence of large perturbation solutions for the system \((1.1)\) in Lagrangian coordinates under a relatively small condition as \((1.3)\), see Remark 2.6 for further discussion. Our result provides a starting point for the existence theory of large perturbation solutions of the 3D non-resistive viscous MHD fluids. Next let us first recall a heuristic physical idea, which lights up the road to Assertion 1.3 for the 3D case.

It is well-known that we can think that the 3D non-resistive MHD fluid under rest state is made up of infinite (fluid) element lines which are parallel to the impressive field and can be regarded elastic strings under magnetic tension. Let us pick up a segment from an element line, and denote it by \(y^1y^2\), see the following first figure, in which \(F_{y^i}\) denotes the magnetic tension acting on the element endpoint \(y^i\) for \(i = 1, 2\), and is given by the formula (see [25] for the detailed derivation)

\[
F_{y^i} = \lambda \partial_{\bar{M}}^i \zeta(y^i, s) \cdot \nu_{y^i} \partial_{\bar{M}}^i \zeta(y^i, s) / 4\pi, \quad (1.5)
\]

where \(\zeta\) denotes the flow function (see (2.1) for the definition) and \(\nu_{y^i}\) represents the unit outer normal vector at element endpoint \(y^i\). Once we disturb the rest state, the segment \(y^1y^2\) will be bend, see the following second figure, in which \(F_r\) denotes the resultant force of \(F_{y^1}\) and \(F_{y^2}\). However the magnetic tension will straighten the bent element line, playing the role of resilience. Since the magnetic tension intensity is strictly increasing on the impressed field intensity by the formula \((1.5)\), we easily see that a strong enough magnetic field can inhibit the singularity formation, and even the flow instabilities [7, 22–24, 26, 28, 41]. Moreover, such inhibition mechanism is also useful for understanding the stabilizing effect of magnetic fields on the motion of resistive MHD fluids [18, 42, 49].

![Diagram](image-url)
Since the above inhibition mechanism is described in Lagrangian coordinates, this naturally motivates us to consider the large perturbation condition (1.3) in Lagrangian coordinates. Hence we write down the well-known energy law of (1.1) in Lagrangian coordinates:

\[
\rho \int_{\Omega} |u|^2 dy + \frac{\lambda \omega}{4\pi} \int_{\Omega} |\partial_\omega \eta|^2 dy + 2\mu \int_0^t \int_{\Omega} |\nabla_A u|^2 dyd\tau \\
= \rho \int_{\Omega} |u_0|^2 dy + \frac{\lambda \omega}{4\pi} \int_{\Omega} |\partial_\omega \eta_0|^2 dy =: I^0. \tag{1.6}
\]

We shall explain the notations appearing in the above identity. \(\Omega\) is the fluid domain. We have defined that

\[
\vec{M} := \omega \omega,
\]

where \(\omega\) and \(m\) denote the unit vector in \(\mathbb{R}^3\) and the field intensity of \(\vec{M}\), resp.. \(\eta := \zeta - y\) represents the deviation function of particles, and \(u\) is the velocity function in Lagrangian coordinates. The differential operator \(\nabla_A\) is given by (2.3). Here and in what follows, \(f^0\) always denotes the initial data of \(f\). We call \(I^0\) the initial mechanical energy \([25]\).

It is easy to see from (1.6) that

\[
\partial_\omega \eta \ll 1, \tag{1.8}
\]

if the initial mechanical energy \(I_0\) is given and satisfies

\[
\frac{\sqrt{I_0}}{\vec{\omega}} \ll 1. \tag{1.9}
\]

The above fact provides a new idea to prove the existence of (global) large perturbation solutions for the system (1.1). Motivated by (1.9) and the odevity conditions imposed by Pan–Zhou–Zhu in [37], for the 2D spatially periodic domain \(T^2\) and \(\omega = (0, 1)^T\), the authors renew to prove the existence of large perturbation solutions for the system (1.1) in Lagrangian coordinates \([27]\), where \(T := \mathbb{R} \setminus \mathbb{Z}\) and the initial data satisfies some relatively small condition similar to (1.3).

The significance of odevity condition lies in that the following important inverse relation can be further established for the 2D case:

\[
\nabla \eta \propto \vec{\omega}^{-1}. \tag{1.10}
\]

This relation intuitively reveals that the (nonlinear) solutions of (1.1) in Lagrangian coordinates can be approximated by the (linear) solutions of the corresponding linearized equations for \(\vec{\omega} \gg 1\), and thus we can expect to establish the existence of large perturbation solutions. However it seems that this idea by using odevity conditions fails to the corresponding 3D case, since the relation (1.10) for the 3D case can not be obtained.

Recently Chen–Zhang–Zhou made an important progress \([10]\). They observed that if \(\vec{M}\) satisfies the Diophantine condition:

\[
\text{there exists a constant } c_M > 0 \text{ such that } |\chi \cdot \vec{M}| \geq c_M |\chi|^3 \text{ for any } \chi \in \mathbb{Z}^3 \setminus \{0\}, \tag{1.11}
\]

where \(c_M\) depends on \(\vec{M}\) and \(\cdot\) denotes the inner product of two vectors, then one has the generalized Poincaré’s inequality \([33]\), which results into that the existence of small perturbation solutions for the system (1.1) can be directly proved without the help of any transformation of
functions. Motivated by the Chen–Zhang–Zhou’s work, we amazingly find that if \( \bar{M} = \omega \omega \) and the unit vector \( \omega \in \mathbb{R}^3 \) satisfies the Diophantine condition (please refer to Remark 2.1 for the existence)

\[
\exists \text{ a constant } c_\omega > 0, \text{ s.t. } |\chi \cdot \omega| \geq c_\omega |\chi|^{-3} \text{ for any } \chi \in \mathbb{Z}^3 \setminus \{0\},
\]

then the relation (1.10) also holds for the 3D case, and thus the existence of large perturbation solutions for the system (1.1) can be establish by directly using a multi-layers energy method, see Theorem 2.1 for the details.

We mention that the proof of our 3D result in Theorem 2.1 is very different to Zhang’s 2D one in [48]. In fact, Zhang first obtained a (linear) solution of linearized equations in Eulerian coordinates, then proved the existence of a small error solution between the both linear and nonlinear solutions for strong magnetic field, and finally got a large solution by adding the linear solution and the small error solution together. However the relation (1.10) for the 3D case allows us to directly establish the existence of solutions under some class of large initial perturbations by one-step procedure in this paper, rather than Zhang’s three-step procedure. In addition, we further rigorously prove that, for large time or strong magnetic field, the MHD equations reduce to the corresponding linearized equations. More precisely,

- the nonlinear interactions of the large perturbation solutions become asymptotically negligible as \( t \to \infty \) in Lagrangian coordinates (Such phenomenon had been verified by Bardos–Sulem–Sulem for the system (1.2) [3]);

- the difference between the both solutions of (1.1) in Lagrangian coordinates and the corresponding linearized system can be bounded from above by \( O(\varpi^{-1/2}) \) as \( \varpi \to \infty \).

please refer to Theorem 2.2 for details. Finally, we mention that the asymptotic behaviors of solutions with respect to other parameters, such as the Mach and Alfvén numbers, in MHD fluids have been also extensively investigated, see, for example, [11] and the references cited therein.

The rest of this paper is organized as follows: In Section 2 we introduce our main results including the existence of unique classical solutions with some class of large initial data to the 3D system (1.1) in a periodic domain in Lagrangian coordinates and the both convergence rates of the classical solutions for \( \varpi \to \infty \) and \( t \to \infty \), i.e., Theorems 2.1 and 2.2, the proofs of which are given in Sections 3–4 in sequence.

2. Main results

In this section we will state the main results in details. To begin with, we reformulate the equations (1.1) in Lagrangian coordinates. Recalling that we study (1.1) in a 3D periodic domain in this paper, we see, without loss of generality, that it suffices to consider the periodic domain \( \mathbb{T}^3 \) with \( \mathbb{T} := \mathbb{R}/\mathbb{Z} \).

2.1. Reformulation in Lagrangian coordinates

Let \( (v, \bar{M}) \) be the solution of the 3D system (1.1), and the flow map \( \zeta \) be the solution to

\[
\begin{cases}
\partial_t \zeta(y, t) = v(\zeta(y, t), t) & \text{in } \mathbb{T}^3 \times \mathbb{R}^+,

\zeta(y, 0) = \zeta^0(y) & \text{in } \mathbb{T}^3,
\end{cases}
\]

where \( \mathbb{R}^+ = (0, \infty) \), \( \zeta^0(y) \) satisfies \( \det \nabla \zeta^0 = 1 \) and “det” denotes the determinant.
Since $v$ is divergence-free, then

$$\det \nabla \zeta = 1 \quad (2.2)$$

as well as $\det \nabla \zeta^0 = 1$. Thus, we define

$$\mathcal{A}^T := (\nabla \zeta)^{-1} := (\partial_j \zeta_i)_{3 \times 3}^{-1},$$

where the superscript $T$ represents transposition.

Now, we introduce some differential operators involving $\mathcal{A}$, which will be used later. The differential operators $\nabla \mathcal{A}$, $\text{div} \mathcal{A}$ and $\Delta \mathcal{A}$ are defined by

$$\nabla \mathcal{A} f := (A_{lk} \partial_k f, A_{lk} \partial_k f, A_{lk} \partial_k f)^T, \quad (2.3)$$

$$\text{div} \mathcal{A} (X_1, X_2, X_3)^T := A_{lk} \partial_k X_l$$

and

$$\Delta \mathcal{A} f := \text{div} \mathcal{A} \nabla \mathcal{A} f$$

for a scalar function $f$ and a vector function $X := (X_1, X_2, X_3)^T$, where $A_{ij}$ denotes the $(i,j)$-th entry of the matrix $\mathcal{A}$. It should be remarked that we have used the Einstein convention of summation over repeated indices, and $\partial_k = \partial_{y_k}$. In addition, thanks to (2.2), we have

$$\partial_k A_{ik} = 0 \quad (2.4)$$

Let $\nu = \mu/\rho$ and

$$(u, B, q)(y, t) = (v, M, p/\rho)(\zeta(y, t), t) \text{ for } (y, t) \in T^3 \times \mathbb{R}^+.$$  

By virtue of the equations (1.1) and (2.1), the evolution equations for $(\zeta, u, q)$ in Lagrangian coordinates read as follows.

$$\begin{align*}
\zeta_t &= u, \\
u u_t + \nabla \mathcal{A} q - \nu \Delta \mathcal{A} u &= \lambda B \cdot \nabla \mathcal{A} B / 4 \pi \rho, \\
B_t - B \cdot \nabla \mathcal{A} B &= 0, \\
\text{div} \mathcal{A} u &= 0, \\
\text{div} \mathcal{A} B &= 0.
\end{align*} \quad (2.5)$$

We can derive from (2.5)$_3$ the differential version of magnetic flux conservation $[25]$:

$$\mathcal{A}_{jl} B_j = \mathcal{A}^0_{jl} B^0_j,$$

which yields

$$B = \nabla \zeta \mathcal{A}^T_0 B^0. \quad (2.6)$$

Here and in what follows, the notation $f_0$ (except for the notations $\mathcal{E}_0$, $\mathcal{D}_0$, $\varphi_0$ and $\psi_0$ in (2.12), (4.8) and (4.9)) as well as $f^0$ denotes the value of the function $f$ at $t = 0$. If we assume that the frozen condition holds, i.e.

$$\mathcal{A}^0_0 B^0 = \bar{M} \text{ (i.e., } B^0 = \partial_M \zeta^0), \quad (2.7)$$

where $\bar{M}$ is defined by (1.7), then (2.6) reduces to

$$B = \pi \partial_\omega \zeta. \quad (2.8)$$
Here we should point out that $B$ given by (2.8) automatically satisfies (2.5) and (2.5)$_5$. Moreover, from (2.8) we see that the magnetic tension in Lagrangian coordinates has the relation

$$B \cdot \nabla \omega B = \omega^2 \partial_\zeta \zeta.$$

Let $I$ denote a $3 \times 3$ identity matrix, $m^2 = \lambda \omega^2/4\pi\rho$ and $\eta = \zeta - y$. Consequently, under the assumption (2.7), the system (2.5) is equivalent to the following system of evolution equations for $(\eta, u, q)$:

$$\begin{cases}
\eta_t = u, \\
u_t + \nabla Aq - \nu \Delta Au = m^2 \partial_\omega^2 \eta, \\
\text{div}_A u = 0,
\end{cases}$$

(2.9)

where $B = m\partial_\omega (\eta + y)$ and $A = (\nabla \eta + I)^{-T}$.

For the well-posedness of (2.9) defined in $\mathbb{T}^3$, we impose the initial condition:

$$(\eta, u)|_{t=0} = (\eta^0, u^0) \text{ in } \mathbb{T}^3.$$

(2.10)

2.2. Notations

Before stating our main results, we introduce some notations which will be frequently used throughout this paper.

1. Basic notations: $\langle t \rangle := t + 1$, $I_T := (0, T)$ for $0 < T \leq \infty$, $\overline{I}_T$ is the closure of $I_T$ (in particular, $\overline{I}_\infty = \mathbb{R}_+ := [0, \infty)$), $\overline{A} := A - I$, $\Omega_T := \mathbb{T}^3 \times I_T$, $\tilde{f} := \int_{(-1,1)^3}$, $\langle w \rangle_{\mathbb{T}^3} := \int wdy$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ denotes the multi-index with respect to the variables $y$. In addition,

$$\sigma(s) := \begin{cases}
s & \text{for } s \geq 0; \\
0 & \text{for } s < 0.
\end{cases}$$

(2.11)

2. Simplified Banach spaces:

$$L^r := L^r(\mathbb{T}^3) = W^{0,r}(\mathbb{T}^3), \quad H^i := W^{i,2}(\mathbb{T}^3),$$

$$H^i_{\sigma} := \{u \in H^i \mid \text{div}u = 0\}, \quad H^{i+1} := \{\eta \in H^{i+1} \mid \det \nabla (\eta + y) = 1\},$$

$$U_T := \{u \in C^0(\overline{I}_T, H^{17}) \cap L^2(I_T, H^{18}) \mid \partial_t^i u \in L^2(I_T, L^2)\},$$

for $1 \leq i \leq 8$,

$$U_T := \{u \in U_T \mid \langle u \rangle_{\mathbb{T}^3} = 0\},$$

$$X := \{w \in X \cap L^2 \mid \langle w \rangle_{\mathbb{T}^3} = 0\},$$

where $X$ denotes a Banach space, $1 < r \leq \infty$ and $i \geq 0$ is an integer.

3. Simplified function classes:

$$H^{18}_* := \{\xi \in H^{18} \mid \xi(y) + y : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is a } C^{16} \text{-diffeomorphism mapping}\},$$

$$H^{18,*}_{\overline{T}_T} := \{\eta \in C^0(\overline{I}_T, H^{18}) \mid \eta(t) \in H^{18}_* \text{ for each } t \in \overline{I}_T\}.$$

4. Simplified norms and functionals: for integers $i \geq 0$,

$$\| \cdot \|_i := \| \cdot \|_{H^i(\mathbb{T}^3)}, \quad \| \nabla^i \cdot \|_0^2 := \sum_{|\alpha|=i} \| \partial^\alpha \cdot \|^2,$$

$$\mathcal{E}_i := \| (\nabla \xi, u, m\partial_\omega \eta) \|_i^2, \quad \mathcal{D}_i := \| (\nabla u, m\partial_\omega \eta) \|_i^2,$$

(2.12)

$$\mathcal{E}_H := \mathcal{E}_{16} + \| \eta \|_{18}^2 + m^{-2/3} \| (u, m\partial_\omega \eta) \|_{17}^2,$$

$$\mathcal{D}_H := \| (u, m\partial_\omega \eta) \|_{17}^2 + m^{-2/3} \| u \|_{18}^2.$$

(2.13)
In addition, we define the following two parameters, which depend on $m$ and the initial energy functionals:

$$\Xi := \sum_{i=0}^{3} (1 + m^{-2})^i \mathcal{E}_{4i}^0,$$

$$\vartheta = (\mathcal{E}_{12}^0)^{1/8} (1 + (\mathcal{E}_4^0)^{3/2}) \Xi^{3/8} + \mathcal{E}_6^0 + (\mathcal{E}_0^0)^2.$$

(5) General constants: $c$ and $c_i$ ($1 \leq i \leq 3$) denote constants, which depend on $\omega$ and $\nu$ at most (independent of $m$); moreover $c$ may vary from line to line, but $c_i$ are fixed. $c_0$ denotes a generic constant independent of any parameter. In addition, we use the notation $c(\chi_1, \ldots, \chi_n)$ to denote a generic constant, which only depends on the parameters $\chi_1, \ldots, \chi_n$. Finally, $A \lesssim_{c} B$, $A \lesssim B$ and $A \lesssim_{\chi_1, \ldots, \chi_n} B$ mean that $A \leq c_0 B$, $A \leq cB$ and $A \leq c(\chi_1, \ldots, \chi_n)B$, resp.

2.3. Existence of large perturbation solutions

Now we state the first result for the existence result of solutions with large initial perturbation in some class for the initial value problem (2.9)–(2.10).

**Theorem 2.1.** If the unit vector $\omega \in \mathbb{R}^3$ satisfy the Diophantine condition (1.12), then there are positive constants $c_1 \geq 4$, $c_2 > 0$ and a sufficiently small constant $c_3 \in (0, 1]$, such that for any $(\eta^0, u^0) \in (H_+^{18} \cap H_+^{18}) \times H_+^{17}$ and $m$ satisfying the incompressible condition $\text{div}_A u^0 = 0$ and the condition of strong magnetic field

$$m^{-1} \max \left\{ (c_1 \mathcal{E}_H^0 e^{c_2 \vartheta})^{1/2}, c_1 \mathcal{E}_H^0 e^{c_2 \vartheta} \right\} \leq c_3,$$

the initial value problem (2.9)–(2.10) admits a unique global classical solution $(\eta, u, q) \in H_+^{18, *} \times U_{\infty} \times C^0(\mathbb{R}_+^+, H_+^{17})$. Moreover, the solution $(\eta, u)$ enjoys, for any $t \geq 0$,

1. the decay-in-time of lower-order derivatives

$$\sum_{i=0}^{3} \left( (1 + m^{-2})^{i}(t)^{(3-i)} \mathcal{E}_{4i} + (1 + m^{-2})^{i} \int_0^t \langle \tau \rangle^{(3-i)} \mathcal{D}_{4i} d\tau \right) \lesssim \Xi.$$

2. the stability estimates

$$\mathcal{E}_i + \int_0^t \mathcal{D}_i d\tau \lesssim \mathcal{E}_i^0 \text{ for } 0 \leq i \leq 12,$$

$$\mathcal{E}_H \lesssim \mathcal{E}_H^0 e^{c_2 \vartheta},$$

$$\int_0^t \mathcal{D}_H d\tau \lesssim \mathcal{E}_H^0 (1 + \vartheta e^{c_2 \vartheta}).$$

In addition, $(\eta, u, q)$ satisfies the additional estimates (3.10), (3.12), (3.16) and

$$\sup_{0 \leq t \leq \infty} \|\eta\|_{15} \lesssim_0 1.$$

**Remark 2.1.** For any given $\tau > N - 1$ and $N \geq 2$, for almost all $\omega \in \mathbb{R}^N$, there exists a positive constant $c(\omega, N, \tau)$ such that

$$|\chi \cdot \omega| \geq c(\omega, N, \tau)|\chi|^{-\tau} \text{ for any } \omega \in \mathbb{Z}^N \setminus \{0\},$$

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Remark 2.5. Recalling the derivation of the decay-in-time (2.17), we easily see that, for any given \( \tau > N - 1 \) and \( N \geq 2 \), for almost all \( \omega \in \partial B \) with respect to the \((N - 1)\)-dimensional measure, where \( \partial B := \{ y \in \mathbb{R}^N \mid |y| = 1 \} \), there exists a constant \( c(\omega, N, \tau) \) such that
\[ |\chi \cdot \omega| \geq c(\omega, N, \tau)|\chi|^{-\tau} \text{ for any } \chi \in \mathbb{Z}^N \setminus \{0\}. \]
Hence \( \omega \) mentioned in the above Theorem 2.1 exists.

**Remark 2.2.** The direction condition for \( \omega \), such as the Diophantine condition, is necessary in Theorem 2.1. In fact, by the magnetic inhibition mechanism, we do not expect that Theorem 2.1 can be extended to the case \( \omega = e_i \), unless additional structural conditions are imposed. Here \( e_i \in \mathbb{R}^3 \) is the unit vector with the \( i \)-th component being 1.

**Remark 2.3.** We can easily construct a family of initial data \( (\eta^0, u^0) \) satisfying all the assumptions in Theorem 2.1, where \( \eta^0 \neq 0 \) and \( u^0 \neq 0 \). In fact, let
\[ \bar{\eta} = \bar{u} = (\sin x_1 \cos x_2 \cos x_3, \cos x_1 \sin x_2 \cos x_3, -2 \cos x_1 \cos x_2 \sin x_3). \]
Because of \( \text{div} \bar{\eta} = \text{div} \bar{u} = 0 \), for sufficiently small \( \varepsilon \), there exists a function pair \( (\eta^0, u^0) \) enjoying the form \( (\eta^0, u^0) = (\varepsilon \bar{\eta} + \varepsilon^2 \eta^r, \bar{u} + \varepsilon u^r) \), where \( (\eta^r, u^r) \in H^{18} \times H^{17} \) satisfies \( \|\eta^r\|_{18} + \|u^r\|_{17} \leq c_0 \),
\[
\begin{align*}
-\Delta \eta^r + \nabla \beta_1 &= 0, \\
\text{div} \eta^r &= \varepsilon^{-2} r_{\varepsilon \bar{\eta} + \varepsilon^2 \eta^r}, \\
(\eta^r)_{\mathbb{T}^3} &= 0
\end{align*}
\]
and
\[
\begin{align*}
-\Delta u^r + \nabla \beta_2 &= 0, \\
\text{div} u^r &= -\varepsilon^{-1} \text{div} \vec{A}^0 (\bar{u} + \varepsilon u^r), \\
(u^r)_{\mathbb{T}^3} &= 0,
\end{align*}
\]
where \( r_{\varepsilon \bar{\eta} + \varepsilon^2 \eta^r} \) is defined as \( r_\eta \) in (3.15) with \( \varepsilon \bar{\eta} + \varepsilon^2 \eta^r \) in place of \( \eta \) and \( \vec{A}^0 := (\nabla \eta^0 + I)^{-T} - I \). We refer the reader to [23, Proposition 5.1] for a proof. Moreover, it is easy to check that for sufficiently small \( \varepsilon \in (0, 1] \), \( (\eta^0, u^0) \) constructed above is non-zero, belongs to \( (H^{18}_1 \cap H^{18}) \times H^{17} \), and satisfies \( \text{div} \vec{A}^0 u^0 = 0 \). We further take \( m = \varepsilon^{-1} \) to immediately see that \( (\eta^0, u^0) \) and \( m \) satisfy (2.16) for sufficiently small \( \varepsilon \). Furthermore, \( \mathcal{E}^0_H \leq c_0 \) for some constant \( c_0 \) independent of \( \varepsilon \) and \( m \).

**Remark 2.4.** In the above theorem, we have assumed \( (\eta^0)_{\mathbb{T}^3} = (u^0)_{\mathbb{T}^3} = 0 \). If \( ((\eta^0)_{\mathbb{T}^3}, (u^0)_{\mathbb{T}^3}) \neq 0 \), we can define \( \bar{\eta}^0 := \eta^0 - (\eta^0)_{\mathbb{T}^3} \) and \( \bar{u}^0 := u^0 - (u^0)_{\mathbb{T}^3} \). Then, by virtue of Theorem 2.1, there exists a unique global classical solution \( (\bar{\eta}, \bar{u}, \bar{q}) \) to the initial value problem (2.9)–(2.10) with initial data \( (\eta^0, \bar{u}^0) \). It is easy to verify that \( (\eta, u, q) := (\bar{\eta} + t(u)_{\mathbb{T}^3} + (\eta^0)_{\mathbb{T}^3}, \bar{u} + (u)_{\mathbb{T}^3}, q) \) is just the unique classical solution of (2.9)–(2.10) with initial data \( (\eta^0, u^0) \).

**Remark 2.5.** Recalling the derivation of the decay-in-time (2.17), we easily see that, higher the regularity of the initial data is, quicker the decay-in-time of lower-order derivatives of solutions. Hence we also further establish a result of almost exponential decay-in-time as in [40].
Remark 2.6. Let \( m \geq 1 \), \( W = B - \bar{M} \), \( \Theta_i = \| u \|_i^2 + \| W \|_{i+4}^2 \) and \( \eta^0 \) further belongs to \( H^{21} \). It should be noted that the perturbation magnetic field \( W \) is equal to \( \varpi \partial_x \eta \) or \( 2m \sqrt{\pi \rho / \lambda} \partial_x \eta \), then we have

\[
\mathcal{E}_H^0 \lesssim_{\rho, \omega} \Theta_1^0 \quad \text{and} \quad \vartheta \leq c_4(\rho, \omega)(\sqrt{\Theta_{12}^0} + (\Theta_{12}^0)^2),
\]

where the constant \( c_4(\rho, \omega) \) depends on \( \rho \) and \( \omega \). Thus we easily see that, if

\[
\max \left\{ \sqrt{\mathcal{I}^0}, \mathcal{I}^0 \right\} \lesssim_{\vartheta} 1,
\]

(2.23)

where \( \mathcal{I}^0 := \Theta_{17}^0 e^{2c_4(\sqrt{\Theta_{12}^0 + (\Theta_{12}^0)^2})} \), then the condition of strong magnetic field (2.16) can be satisfied. Hence we have mathematically verified Assertion 1.3 in Lagrangian coordinates. Moreover we see from (2.23) that the field intensity \( \varpi \) increasingly depends on the \( H^{17} \times H^{21} \)-norm of the initial perturbation of both the velocity and magnetic field.

Remark 2.7. Recalling that the solution \( \eta \) in Theorem 2.1 satisfies

\[
\zeta := \eta(y, t) + y : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is a } C^{16} \text{ diffeomorphism mapping},
\]

(2.24)

we can easily obtain the existence result of large perturbation solutions for the original 3D system (1.1) by an inverse transformation of Lagrangian coordinates [29], and thus provides a mathematical evidence to support Assertion 1.3 for the 3D case. Interesting readers can refer to [27, Theorem 2.3] for writing down the existence result of large perturbation solutions of the 3D system (1.1).

Now we briefly describe the proof idea of Theorem 2.1. We rewrite (2.9) as a nonhomogeneous system:

\[
\begin{cases}
  u_t - \nu \Delta u - m^2 \partial_\omega^2 \eta = \mathfrak{N} , \\
  \text{div} u = -\text{div} \tilde{A} u ,
\end{cases}
\]

(2.25)

where \( \mathfrak{N} := \mathcal{N}^\nu - \nabla \tilde{A} q - \nabla q \), \( \mathcal{N}^\nu := \partial_i (N_{1,t}^{\nu}, N_{2,t}^{\nu}, N_{3,t}^{\nu})^T \) and \( N_{3,t}^{\nu} := \nu (\mathcal{A}_{kl} \tilde{A}_{km} + \tilde{A}_{ml}) \partial_m u_j \). It should be noted that the linearized pressure term \( \nabla q \) can be regarded as a nonlinear term under the sense of energy integrals.

If we ignore the nonlinear terms in (2.25), then by a method of \textit{a priori} estimates,

\[
\|(\nabla \eta, u, m \partial_\omega \eta)\|_{i+4} \lesssim \|(\nabla \eta^0, u^0, m \partial_\omega \eta^0)\|_{i+4}.
\]

(2.26)

Thus, exploiting the Diophantine condition (1.12) and the generalized Poincaré’s inequality (3.3), we obtain

\[
\| \nabla \eta \|_i \lesssim \|(\nabla \eta^0, u^0, m \partial_\omega \eta^0)\|_{i+4} / m.
\]

(2.27)

Motivated by the approximate theory of non-resistive MHD equations in [3, 48], we naturally guess that (2.27) shall hold in the nonlinear equations (2.25) for sufficiently large \( m \).

In turn, if (2.27) exists in the nonlinear equations (2.25), it is easy to see the system (2.25) can be approximated by the corresponding linearized system. Since the linear system admits a global solution, the nonlinear system (2.25) may also admit a global solution with large data under a strong magnetic field. Thus, motivated by (2.26), for given initial energy \( \mathcal{E}_i^0 \), we naturally expect to derive the \textit{a priori} estimate of \((\eta, u)\) like

\[
\|(\nabla \eta, u, m \partial_\omega \eta)\|_{17} \leq K / 2 \quad \text{for sufficiently large} \ m
\]
under the \textit{a priori} assumption
\[ \|(\nabla \eta, u, m \partial_\omega \eta)\|_{17} \leq K \text{ with } Km^{-1} \ll 1. \]

However, the nonlinear terms appearing in (2.25) destroy the above expectation. In fact, when we perform the estimates of highest-order derivatives, an integral term increasing on \( m \) appears, see the last term in (3.35). To balance the increasing term, we shall adjust the highest-order energy functional. This is the reason why the structures of the both energy and dissipation of highest-order in (2.13) are different to the ones of lower-order in (2.12). After this adjustment, we can use a three-energy method to conclude that there are constants \( K \) (increasingly depending on \( \vartheta \) and \( E_0^H \)) and \( \delta \), such that
\[ \sup_{0 \leq t \leq T} \left( \|\eta(t)\|_{18}^2 + m^{-2/3}\|u\|_{17}^2 + \|(u, m \partial_\omega \eta)(t)\|_{16}^2 \right) \leq K^2/4, \quad (2.28) \]
if
\[ \sup_{0 \leq t \leq T} (\|\eta(t)\|_{18}^2 + \|(u, m \partial_\omega \eta)(t)\|_{16}^2) \leq K^2 \text{ for any given } T > 0 \quad (2.29) \]
and
\[ \max\{K, K^2\}/m \in (0, \delta] \text{ with } \delta \ll 1. \quad (2.30) \]

The \textit{a priori} stability estimate (2.28), together with a local well-posedness result on (2.9)–(2.10), immediately yields Theorem 2.1. Here we explain how to perform the three-energy method, which had been widely used in the investigation of the problems involving wave phenomena, see [19, 20] for examples. Roughly speaking, we call (2.17), (2.18) with \( i = 12 \) and (2.19) the lower-order, higher-order and highest-order energy inequalities, resp. Under the assumptions (2.29) and (2.30), we first get the higher-order energy inequality, and then further obtain the lower-order energy inequality. Thanks to the decay-in-time of lower-order energy, finally we can close the highest-order energy inequality, see Section 3 for the detailed performance.

2.4. Vanishing phenomena of nonlinear interactions

Now we turn to mathematically stating the vanishing phenomena of nonlinear interactions with respect to \( t \) and \( m \).

**Theorem 2.2.** Let the global solution of (2.9)–(2.10) \((\eta, u, q)\) be given by Theorem 2.1.

1. Then, we can use the initial data of \((\eta, u)\) to construct a function pair \((\eta^r, u^r)\) such that the following linear pressureless initial value problem
\[ \left\{ \begin{array}{l}
\eta^L_t = u^L, \\
u_L - \nu \Delta u^L = m^2 \partial_\omega^2 \eta^L, \\
\text{div} u^L = 0, \\
(\eta^L, u^L)|_{t=0} = (\eta^0 + \eta^r, u^0 + u^r) \in H^{18}_0 \times H^{17}
\end{array} \right. \quad (2.31) \]

admits a unique classical solution \((\eta^L, u^L)\) \(\in C^0(\mathbb{R}_+; H^{18}_0) \times H^{17}_\infty\). Moreover, the linear solution \((\eta^L, u^L)\) enjoys the following estimates
\[ \sum_{i=0}^{3} \left( (1 + m^{-2})^i \langle t \rangle^{(3-i)} \mathcal{E}_{4i}^L + (1 + m^{-2})^i \int_0^t \langle t \rangle^{(3-i)} \mathcal{D}_{4i}^L \, d\tau \right) \lesssim \sum_{i=0}^{3} (1 + m^{-2})^i \mathcal{E}_{4i}^L|_{t=0}, \quad (2.32) \]
\[ \mathcal{E}_j^L + \int_0^t \mathcal{D}_j^L \, d\tau \lesssim \mathcal{E}_j^L|_{t=0}, \quad (2.33) \]
where $0 \leq j \leq 17$ and

$$
\mathcal{E}_\chi^L := \| (\nabla \eta, u, m \partial_\omega \eta) \|^2 \quad \text{and} \quad \mathcal{D}_\chi^L := \| (\nabla u, m \partial_\omega \eta) \|^2.
$$

In addition the function pair $(\eta^r, u^r)$ satisfies

$$
div(u^0 + u^r) = div(\eta^0 + \eta^r) = 0, \quad (2.34)
$$

$$
\| u^r \|_k \lesssim_0 \| \eta^0 \|_3 \| u^0 \|_k + \| \eta^0 \|_k \| u^0 \|_3, \quad (2.35)
$$

$$
\| \partial_\omega \eta^r \|_k \lesssim_0 \| \eta^0 \|_3 \| \partial_\omega \eta^0 \|_k + \| \partial_\omega \eta^0 \|_3 \| \eta^0 \|_k, \quad (2.36)
$$

where $1 \leq k \leq 17$ and $1 \leq l \leq 18$.

(2) Let $(\eta^d, u^d) = (\eta - \eta^L, u - u^L)$, then for any $t \geq 0$,

$$
\mathcal{E}_{12}^d + \int_0^t \mathcal{D}_{12}^d \, dt \lesssim m^{-1} \mathcal{E}_{17}^0 e^{c_2 \theta} \left( \sqrt{\Xi + \mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right), \quad (2.37)
$$

$$
\sum_{i=0}^{3} (1 + m^{-2})^i (t)^{(3-i)} \mathcal{E}_{4i}^d + (1 + m^{-2})^i \int_0^t \langle \tau \rangle^{(3-i)} \mathcal{D}_{4i}^d \, d\tau \lesssim m^{-1} \left( \mathcal{E}_{15}^0 \Xi + \Phi \right), \quad (2.38)
$$

where

$$
\Phi := \Xi \left( \sqrt{\mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right) \sum_{i=0}^{2} (1 + m^{-2})^i + \mathcal{E}_{17}^0 e^{c_2 \theta} \left( \sqrt{\Xi + \mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right) (1 + m^{-2})^3, \quad (2.39)
$$

$$
\mathcal{E}_\chi^d := \| (\nabla \eta^d, u^d, m \partial_\omega \eta^d) \|^2 \quad \text{and} \quad \mathcal{D}_\chi^d := \| (\nabla u^d, m \partial_\omega \eta^d) \|^2.
$$

**Remark 2.8.** Exploiting the interpolation inequality (3.5), we easily derive from (2.37) and (2.38) that, for any $0 \leq i \leq 12$,

$$
\mathcal{E}_i^d \lesssim m^{-1} \left( \mathcal{E}_{15}^0 \Xi + \Phi \right)^{(12-i)/12} \left( \mathcal{E}_{17}^0 e^{c_2 \theta} \left( \sqrt{\Xi + \mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right) \right)^{i/12}.
$$

In addition, for $m \geq 1$, the estimate (2.38) can be simplified as follows:

$$
\sum_{i=0}^{3} (t)^{(3-i)} \mathcal{E}_{4i}^d + \int_0^t \langle \tau \rangle^{(3-i)} \mathcal{D}_{4i}^d \, d\tau \lesssim m^{-1} \mathcal{E}_{17}^0 e^{c_2 \theta} \left( \sqrt{\mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right).
$$

**Remark 2.9.** The decay-in-time in (2.32) can be easily observed, since the both linear and nonlinear solutions enjoy the same decay-in-time, see (2.17) and (2.32). Noting that the inhomogeneous term $\mathcal{N}$ includes $\partial_\omega \eta_j$ (the term $\nabla q$ should be understood under the sense of the energy integrals) with $1 \leq i, j \leq 3$, and

$$
\| \nabla \eta \|_{L^\infty} \lesssim_0 \| \partial_\omega \eta \|_6 \leq K/2m \quad (2.40)
$$

by (2.28), (3.3) and (3.4), we formally see from the error equations in (4.5) the appearance of $m^{-1}$ in (2.37) and (2.38). The two estimates (2.37) and (2.38) present the physical phenomena of vanishing of nonlinear interactions for large time and strong magnetic field.

We can follow the idea of deriving the estimates (2.17) and (2.18) to establish Theorem 2.2, the proof of which will be presented in Section 4. Here we explain why we have to modify $(\eta^0, u^0)$ to be initial data of $(\eta^L, u^L)$ in (2.31)4.
Since the initial data \( u^L \mid_{t=0} \) is divergence-free, i.e. \( \text{div}(u^L \mid_{t=0}) = 0 \), we have to adjust the initial data \( u^0 \) as in (2.31).

If the initial data \( \eta^0 \) of \( \eta \) is directly used to be initial data of the corresponding linear problem, then we see that \( \text{div}\eta^L = \text{div}\eta^0 \), and a time-decay of \( \partial_\omega \eta^d \) in (2.38) can not be expected unless \( \text{div}\eta^0 = 0 \). Hence, we have to modify \( \eta^0 \) as in (2.31), so that the obtained new initial data “\( \eta^0 + \eta^r \)” is also divergence-free.

Finally we mention that recently some authors studied the case of inviscid, non-resistive MHD fluids with velocity damping, i.e., the viscosity term in the system (1.1) is replaced by the velocity damping term \( \kappa \rho v \) with \( \kappa \) being the damping coefficient [13, 27, 44]. Following the arguments of Theorems 2.1, 2.2 and [27, Theorem 2.5], we can extend our aforementioned results in Theorems 2.1 and 2.2 to the inviscid case with the damping term \( \kappa \rho v \), and show that for the inviscid case with damping, the decay in time is exponentially fast; while the convergence rate in \( m \) as \( m \to \infty \) of a classical solution of the original nonlinear system to the solution of the corresponding linear system is in the form of \( m^{-1} \), which is faster than \( m^{-1/2} \) in the viscous case.

3. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. First we derive some energy (\textit{a priori}) estimates for the solution \((\eta, u)\) of the initial value problem (2.9)–(2.10) under the \textit{a priori} assumption (2.29) associated with the relative smallness condition (2.30) in Subsection 3.1, then further establish the desired \textit{a priori} estimate (2.28) in Subsection 3.2, and finally, introduce a local well-posedness result for (2.9)–(2.10) and complete the proof of Theorem 2.1 by a standard continuity argument in Subsection 3.3. In the derivation of (2.28), we obtain the stability estimates (2.18)–(2.20) as by-products.

3.1. Energy estimates

Let \((\eta, u, q)\) be a solution of the initial value problem (2.9)–(2.10) defined on \( \Omega_T \) for any given \( T > 0 \), and the initial data \((\eta^0, u^0)\) is a non-zero function, belongs to \( H^{18} \times H^{17} \), and satisfies \( \text{div}_\omega u^0 = 0 \). Obviously \((\eta, u)\) automatically satisfies \((\eta)^3 = (u)^3 = 0 \) as well as the initial data \((\eta^0, u^0)\). We further assume that \((\eta, u, q)\) and \( K \) satisfy \((q)^3 = 0 \) and (2.29), where \( K \) will be defined by (3.61).

Before deriving the energy estimates for \((\eta, u)\) in Lemmas 3.4–3.5 we shall introduce some basic inequalities and establish some preliminary estimates for \((\eta, u)\) by the following three lemmas.

**Lemma 3.1.** We have the following basic inequalities:

1. For any \( s_1, s_2, s_3 \in \mathbb{R} \) satisfying \( s_1 \leq s_2 \leq s_3 \), and any positive constant \( a \), it holds that
   \[
   a^{s_2} \leq a^{s_1} + a^{s_3}.
   \]
   The above inequality is obvious by the monotonicity of \( a^s \) with respect to the variables \( s \).

2. Poincaré’s inequality [36]: for any given \( i \geq 0 \),
   \[
   \|f\|_i \lesssim \|\nabla^i f\|_0 \quad \text{for any } f \in H^i.
   \]

3. Generalized Poincaré’s inequalities [14]: if \( \omega \in \mathbb{R}^3 \) satisfies the Diophantine condition (1.12), then it holds that, for any given \( i \geq 0 \),
   \[
   \|f\|_i \lesssim_{\omega,i} \|\partial_\omega f\|_{i+3} \quad \text{for any } f \in H^{3+i} (\mathbb{T}^3).
   \]
(4) Interpolation inequalities for given $0 \leq j \leq i$,
\begin{align}
\|f\|_{L^\infty} & \lesssim_0 \|f\|_1^{1/2}\|f\|_2^{1/2} \quad \text{for any } f \in H^2, \quad (3.4) \\
\|f\|_j & \lesssim_{i,j} \|f\|_0^{-j/i}\|f\|_i^{j/i} \quad \text{for any } f \in H^i. \quad (3.5)
\end{align}

(5) Product estimates:\n\begin{align}
\|fg\|_i & \lesssim_0 \|f\|_1 \sqrt{\|g\|_i\|g\|_{i+1}} \quad \text{for any } (f, g) \in H^1 \times H^{i+1}, \quad (3.6) \\
\|fg\|_j & \lesssim_j \|f\|_{L^\infty}\|g\|_j + \|f\|_j\|g\|_{L^\infty} \quad \text{for any } f, g \in H^j \cap L^\infty, \quad (3.7)
\end{align}
where $0 \leq i \leq 1$ and $j \geq 0$ are given. In particular, the two estimates (3.4) and (3.7) imply that, for any given $i \geq 0$,
\begin{align}
\|fg\|_i & \lesssim_i \|f\|_2\|g\|_i + \|f\|_i\|g\|_2 \quad \text{for any } f, g \in H^1 \cap H^2. \quad (3.8)
\end{align}

Lemma 3.2. Let $\eta$ further satisfy
\begin{equation}
\sup_{0 \leq t \leq T} \|\eta\|_3 \lesssim_0 \delta \in (0, 1], \quad (3.9)
\end{equation}
then
\begin{align}
\|\tilde{A}\|_i & \lesssim_0 \|\eta\|_{i+1} \quad \text{for } 0 \leq i \leq 16, \quad (3.10) \\
\|A_t\|_i^2 & \lesssim_0 \|u\|_{i+1} + \begin{cases} 0 & \text{for } 0 \leq i \leq 2; \\ \|\eta\|_{i+1}\|u\|_3 & \text{for } 3 \leq i \leq 16 \end{cases} \quad (3.11)
\end{align}
and
\begin{equation}
\|(\partial_t^2 \text{div}\eta, \text{div} \tilde{A}\partial_t^2 \eta)\|_i^2 \lesssim_0 F_i, \quad (3.12)
\end{equation}
where
\begin{align}
F_i := \begin{cases} \|\partial_\omega \eta\|_2\|\partial_\omega \eta\|_3 + \|\eta\|_3\|\partial_t^2 \eta\|_2 & \text{for } i = 1; \\ \|\eta\|_3\|\partial_t^2 \eta\|_{i+1} + \|\partial_\omega \eta\|_3\|\partial_\omega \eta\|_{i+1} + \|\eta\|_{i+1}(\|\partial_t^2 \eta\|_3 + \|\partial_\omega \eta\|_3^2) & \text{for } 10 \leq i \leq 15. \end{cases}
\end{align}

Proof. Noting that $\det(\nabla \eta^0 + I) = 1$, then
\begin{equation}
\det(\nabla \eta + I) = 1. \quad (3.13)
\end{equation}
Thus it is easy to compute out that
\begin{equation}
\tilde{A} = \tilde{A}^L + \tilde{A}^N, \quad (3.14)
\end{equation}
where
\begin{equation}
\tilde{A}^L := \begin{pmatrix} \partial_2 \eta_2 + \partial_3 \eta_3 & -\partial_1 \eta_2 & -\partial_1 \eta_3 \\ -\partial_2 \eta_1 & \partial_1 \eta_1 + \partial_3 \eta_3 & -\partial_2 \eta_3 \\ -\partial_3 \eta_1 & -\partial_3 \eta_2 & \partial_1 \eta_1 + \partial_2 \eta_2 \end{pmatrix}
\end{equation}
and
\begin{equation}
\tilde{A}^N := \begin{pmatrix} \partial_2 \eta_2 \partial_3 \eta_3 - \partial_2 \eta_3 \partial_3 \eta_2 & \partial_1 \eta_3 \partial_3 \eta_2 - \partial_1 \eta_2 \partial_3 \eta_3 & \partial_1 \eta_2 \partial_2 \eta_3 - \partial_1 \eta_3 \partial_2 \eta_2 \\ \partial_3 \eta_2 \partial_3 \eta_3 - \partial_3 \eta_3 \partial_3 \eta_2 & \partial_1 \eta_3 \partial_3 \eta_3 - \partial_1 \eta_2 \partial_3 \eta_3 & \partial_1 \eta_2 \partial_2 \eta_3 - \partial_1 \eta_3 \partial_2 \eta_2 \\ \partial_2 \eta_1 \partial_3 \eta_2 - \partial_2 \eta_2 \partial_3 \eta_1 & \partial_1 \eta_2 \partial_3 \eta_2 - \partial_1 \eta_1 \partial_3 \eta_2 & \partial_1 \eta_1 \partial_2 \eta_2 - \partial_1 \eta_2 \partial_2 \eta_1 \end{pmatrix}.
\end{equation}
Applying \( \| \cdot \|_i \) and \( \| \partial_1 \cdot \|_i \) to the identity \((3.14)\), resp., and then using \((2.9)_1, (3.8)\) and \((3.9)\), we immediately get \((3.10)\) and \((3.11)\).

By \((3.14)\), we get
\[
\text{div} \eta = r_\eta,
\]
where \(r_\eta := r_2^\eta + r_3^\eta\) with
\[
r_2^\eta := \partial_2 \eta_1 \partial_1 \eta_2 + \partial_2 \eta_3 \partial_3 \eta_2 + \partial_3 \eta_1 \partial_1 \eta_3 - \partial_1 \eta_1 \partial_2 \eta_2 - \partial_1 \eta_1 \partial_3 \eta_3 - \partial_2 \eta_2 \partial_3 \eta_3,
\]
\[
r_3^\eta := \partial_1 \eta_1 (\partial_2 \eta_3 \partial_3 \eta_2 - \partial_2 \eta_2 \partial_3 \eta_3) + \partial_2 \eta_1 (\partial_1 \eta_2 \partial_3 \eta_3 - \partial_1 \eta_3 \partial_2 \eta_3) + \partial_3 \eta_1 (\partial_1 \eta_3 \partial_2 \eta_2 - \partial_1 \eta_2 \partial_3 \eta_3).
\]

Thanks to \((2.4)\), \((3.14)\) and \((3.15)\), we have
\[
\partial_\omega^2 \text{div} \eta = \partial_\omega^2 r_2^\eta + \partial_\omega^2 r_3^\eta
\]
and
\[
\text{div} \tilde{A} \partial_\omega^2 \eta = \tilde{A}^T : \partial_\omega^2 \nabla \eta + \tilde{A}^N : \partial_\omega^2 \nabla \eta.
\]

Applying \( \| \cdot \|_i \) to the two identities above, and then using \((3.6)\), \((3.8)\) and \((3.9)\), we immediately get \((3.12)\).

\[\Box\]

**Lemma 3.3.** Under the condition \((3.9)\) with sufficiently small \(\delta\), we have
\[
\|q\|_{i+2} \lesssim_0 \begin{cases} \|u\|_2 \|u\|_3 + m^2 F_1 & \text{for } i = 1; \\ \|u\|_2 (\|u\|_{i+2} + \|\eta\|_{i+2} \|u\|_3) + \|u\|_3 \|u\|_{i+1} + m^2 (F_i + \|\eta\|_{i+2} F_1) & \text{for } 10 \leq i \leq 15. \end{cases}
\]

**Proof.** (1) By \((2.4)\) and \((2.9)_3\), we have
\[
\text{div} A u_t = -\text{div} A u = -\text{div}(A_t^T u)
\]
and
\[
\text{div} A \nabla_A q = \text{div}(A^T \nabla_A q) = \text{div}(\nabla q + (\tilde{A}^T + \tilde{A}) \nabla q + \tilde{A}^T \nabla_A q).
\]

Let \(i = 1, 10, \ldots, 15\) and \(\alpha\) satisfy \(|\alpha| = i\). Applying \(\partial^\alpha\text{div} A\) to \((2.9)_2\), and then using \((3.17)\), \((3.18)\) and the fact “\(\text{div} A \Delta_A u = 0\)”, we obtain
\[
\Delta \partial^\alpha q = \partial^\alpha f,
\]
where
\[
f := \text{div}(A_t^T u) - \text{div}((\tilde{A} + \tilde{A}^T) \nabla q + \tilde{A}^T \nabla_A q) + m^2 (\partial_\omega^2 \text{div} \eta + \text{div} \tilde{A} \partial_\omega^2 \eta).
\]

Multiplying \((3.19)\) by \(\Delta \partial^\alpha q\) in \(L^2\), and then using regularity theory of elliptic equations, we get
\[
\|\partial^\alpha q\|_2^2 \lesssim_0 \|\Delta \partial^\alpha q\|_0^2 \lesssim_0 \int |\partial^\alpha f \partial^\alpha \Delta q| dy.
\]

(3.20)
The integral term on the right hand side can be bounded as follows:

\[
\int |\partial^\alpha f \Delta^\alpha q| dy \lesssim_0 |\|A_x\|_{i+1}||u||_2 + |\|A_{x}\|_2 u||_i + 1 + m^2(\|\partial^2_{\omega} \text{div} \eta + \text{div} \tilde{\omega} \partial^2_{\omega} \eta)||_i + (1 + |A_x||\|A_{x}\|_{i+1}q||_3 + |\|A_{x}\|_2 q||_{i+2})|q||_{i+2} \\
\lesssim_0 |\|u||_2 |\|A_x||_{i+1} + |\|u||_3|u||_{i+1} + m^2(\|\partial^2_{\omega} \text{div} \eta + \text{div} \tilde{\omega} \partial^2_{\omega} \eta)||_i + |\|\eta||_{i+2}||\|q||_3 + \delta|q||_{i+2})|q||_{i+2},
\]

(3.21)

where we have used Hölder’s inequality, (3.8) in the first inequality, and (3.9)–(3.11) in the last inequality.

Making use of the fact \((q)_T = 0\), Poincaré’s inequality (3.2), (3.9), (3.11) and (3.12) we get (3.16) from (3.20) and (3.21) for sufficiently small \(\delta\).

From now on, we further assume that \((\eta, u)\) satisfies (2.29) with (2.30), in which \(\delta \in (0, 1]\) is a sufficiently small constant. It should be noted that the smallness of \(\delta\) only depends on the parameter \(\nu\) and the known unit vector \(\omega\). Exploiting (3.3) and the interpolation inequality (3.5), we can derive from (2.29) and (2.30) that

\[
\sup_{0 \leq t \leq T} \|\eta\|_{15}^2 \lesssim_0 \sup_{0 \leq t \leq T} (\|\partial_\omega \eta\|_{16}|\eta||_{17}) \lesssim_0 K^2 m^{-1} \lesssim_0 \delta \in (0, 1],
\]

(3.22)

\[
\sup_{0 \leq t \leq T} (|\|\eta||_{13} + |\|\partial_\omega \eta||_{16})(|\|u||_{16} + |\eta||_{18}) \lesssim_0 K^2 m^{-1} \lesssim_0 \delta
\]

(3.23)

and

\[
\sup_{0 \leq t \leq T} (|\|\eta||_{13} + (|\|\eta||_{18} + |\|u||_{16}^{(2-a)/2} |\|m \partial_\omega \eta||_{16}^{a/2})/m) \lesssim_0 K m^{-1} \lesssim_0 \delta
\]

(3.24)

for any \(0 \leq a \leq 2\).

Now we proceed to derive some energy estimates for \((\eta, u)\).

**Lemma 3.4.** Under the conditions (2.29)–(2.30) with sufficiently small \(\delta\), we have

\[
\frac{d}{dt} \|\nabla^i (u, m \partial_\omega \eta)\|^2 + \nu \|u\|^2_{i+1} \leq \begin{cases}
\delta m^2 \|\partial_\omega \eta\|^2 & \text{for } 0 \leq i \leq 12;\\
\delta m^2 \|\partial_\omega \eta\|^2_{16} + \|\eta\|_{18}(\|\eta\|_{17}(|\|u||_{16}^2 + |\|u||_{13}^2) + m^2(\|\partial_\omega \eta\|_{13}|\|\partial_\omega \eta||_{16}|\|u||_{16})) & \text{for } i = 16;
\end{cases}
\]

(3.25)

\[
K^2 m(\|\partial_\omega \eta\|^2_{17} + \|\eta\|_{18}(|\|u||_{16}^2 + |\|u||_{13}^2) + m^2(\|\partial_\omega \eta\|_{13}|\|\partial_\omega \eta|_{16}|\|u||_{16})) & \text{for } i = 17.
\]

**Proof.** Let \(i = 0, \ldots, 12, 16, 17\), and \(\alpha\) satisfy \(|\alpha| = i\). If \(\alpha \neq 0\), then there exists a component, denoted by \(\alpha_j\), such that \(\alpha_j \neq 0\), and thus we further define \(\alpha^-\) and \(\alpha^+\) as follows:

\[
\alpha^-_k = \alpha^+_k = \alpha_k \text{ for } k \neq j, \quad \alpha^-_j = \alpha_j - 1 \text{ and } \alpha^+_j = \alpha_j + 1.
\]

(3.26)

Applying \(\partial^\alpha\) to (2.25) yields

\[
\partial^\alpha (u_t - \nu \Delta u - m^2 \partial^2_{\omega} \eta) = \partial^\alpha (\Lambda^\nu - \nabla \tilde{\omega} q - \nabla q).
\]

(3.27)
Multiply the above identity by $\partial^\alpha u$ in $L^2$, and then using the integral by parts and (2.25), we get

$$
\frac{1}{2} \frac{d}{dt} \|\partial^\alpha (u, m\partial_x \partial^\alpha \eta)\|_0^2 + \nu \|\nabla \partial^\alpha u\|_0^2 = I_1 + I_2,
$$

(3.28)

where $I_1 := \int \partial^\alpha \mathcal{N}^\gamma_{ij} \partial_t \partial^\alpha u_j dy$ and

$$
I_2 := -\int \partial^\alpha \text{div} \tilde{A} u \partial^\alpha q dy \begin{cases} -\int u \cdot \nabla \tilde{A} q dy & \text{for } \alpha = 0; \\
+ \int \partial^\alpha u \cdot \partial^\alpha \nabla \tilde{A} q dy & \text{for } \alpha \neq 0.
\end{cases}
$$

Exploiting Hölder’s inequality, (3.8), (3.10) and (3.24), we find that

$$
I_1 \lesssim \sum_{1 \leq i,j \leq 3} (\|\partial_t u_j\|_i \|\mathcal{N}^\gamma_{ij}\|_i) \lesssim \|u\|_{i+1} (1 + \|\tilde{A}\|_2) (\|\tilde{A}\|_2 \|u\|_{i+1} + \|\tilde{A}\|_i \|u\|_3)
\lesssim \|\eta\|_{i+1} \|u\|_3 \|u\|_{i+1} + \delta \|u\|_{i+1}^2.
$$

(3.29)

Similarly, the integral $I_2$ can be estimated as follows.

$$
I_2 \lesssim \|\tilde{A}\|_i \|u\|_3 \|q\|_i + \|u\|_{i+1} (\|\tilde{A}\|_{\sigma(i-1)} \|q\|_3 + \|\tilde{A}\|_2 \|q\|_i)
\lesssim \|\eta\|_{i+1} \|u\|_3 \|u\|_{i+1} + \|u\|_{i+1} (\|\eta\|_{\sigma(i-1)+1} \|q\|_3 + \|\eta\|_3 \|q\|_i),
$$

(3.30)

see (2.11) for the definition of $\sigma$.

(1) Case $0 \leq i \leq 3$. Exploiting (3.3) and the interpolation inequality (3.5), we further derive from (3.29) with $0 \leq i \leq 3$ that

$$
I_1 \lesssim \|\partial_\omega \eta\|_{i+1}^{1/2} \|\partial_\omega \eta\|_{i+1}^{1/2} \|u\|_2 \|u\|_3 \|u\|_{5-i} + \delta \|u\|_{i+1}^2.
$$

(3.31)

Similarly, thanks to (3.3), (3.16) with $i = 1$, we can further derive from (3.30) with $0 \leq i \leq 3$ that

$$
I_2 \lesssim (\|\eta\|_3 \|u\|_{i+1} + \|\eta\|_{i+1} \|u\|_3) (\|u\|_2 \|u\|_3 + m^2 (\|\partial_\omega \eta\|_2 \|\partial_\omega \eta\|_3 + \|\eta\|_3 \|\partial_\omega^2 \eta\|_2))
\lesssim \|\eta\|_4 \|u\|_2 \|u\|_{i+1} + m^2 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_7 \|u\|_4
\lesssim \|\eta\|_4 \|u\|_{10-2i} \|u\|_{i+1} + m^2 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_{16-2i} \|u\|_4.
$$

(3.32)

Putting the above two estimates into (3.28) for $|\alpha| = 1$, and then using Young inequality, (3.2) and (3.23), we get (3.25) with $0 \leq i \leq 3$ for sufficiently small $\delta$.

(2) Case $4 \leq i \leq 12$. We can derive from (3.30) with $4 \leq i \leq 12$ that

$$
I_2 \lesssim \|\eta\|_i \|u\|_{i+1} (\|u\|_2 \|u\|_3 + m^2 (\|\partial_\omega \eta\|_2 \|\partial_\omega \eta\|_3 + \|\eta\|_3 \|\partial_\omega^2 \eta\|_2))
+ (\|\eta\|_3 \|u\|_{i+1} + \|\eta\|_{i+1} \|u\|_3) (\|u\|_2 \|u\|_i + \|\eta\|_i \|u\|_3 + \|u\|_3 \|u\|_{i-1}
+ m^2 (\|\eta\|_3 \|\partial_\omega^2 \eta\|_{i-1} + \|\partial_\omega \eta\|_i \|\partial_\omega \eta\|_{i-1} + \|\eta\|_{i-1} \|\partial_\omega^2 \eta\|_3 + \|\partial_\omega \eta\|_3^2)
+ \|\eta\|_i (\|\partial_\omega \eta\|_2 \|\partial_\omega \eta\|_3 + \|\eta\|_3 \|\partial_\omega^2 \eta\|_2))
\lesssim \|\eta\|_{i+1} \|u\|_{i+1} (1 + \|\eta\|_i (\|u\|_3 \|u\|_i + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_i)
+ m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_{i+2} \|\partial_\omega \eta\|_{i+4} \|u\|_3.
$$

(3.33)
where we have used (3.16) with $i = 1, 10$, in the first inequality, and (3.3) in the last inequality.

Putting (3.29) with $4 \leq i \leq 12$ and the above estimate into (3.28) for $|\alpha| = i$, and then using (3.2), (3.22), (3.23) and Young’s inequality, we get (3.25) with $4 \leq i \leq 12$.

(3) **Case $i = 16$.**

We can derive from (3.30) with $i = 16$ that

$$I_2 \lesssim \|\eta\|_{16} \|u\|_{17} (\|u\|_2 \|u\|_3 + m^2 (\|\partial_\omega \eta\|_2 \|\partial_\omega \eta\|_3 + \|\eta\|_3 \|\partial_\omega^2 \eta\|_2))$$

$$+ (\|\eta\|_3 \|u\|_{17} + \|\eta\|_{17} \|u\|_3)(\|u\|_2 \|u\|_{16} + \|\eta\|_{16} \|u\|_3) + \|u\|_3 \|u\|_{15}$$

$$+ m^2 (\|\eta\|_3 \|\partial_\omega^2 \eta\|_{15} + \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_{15} + \|\eta\|_{15} (\|\partial_\omega^2 \eta\|_3 + \|\partial_\omega \eta\|_3))$$

$$\lesssim \|u\|_{17} (\|\eta\|_{17} (1 + \|\eta\|_3)(\|u\|_2^2 + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_6)$$

$$+ \|\eta\|_3 (\|u\|_3 \|u\|_{16} + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_{16}) + \|\eta\|_{17} (\|\eta\|_{16} \|u\|_3 (\|u\|_2 \|u\|_3$$

$$+ m^2 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_6) + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_{15} \|\eta\|_{17} \|u\|_3$$

$$\lesssim \delta (\|u\|_{17}^2 + m \|\partial_\omega \eta\|_{16} \|u\|_{17}) + \|\eta\|_{17} (\|u\|_2 \|u\|_3 + m^2 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_6)$$

$$+ m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_{17} \|u\|_{18} \|u\|_3)$$

$$\lesssim \|\eta\|_{17} (\|\partial_\omega \eta\|_2 \|\partial_\omega \eta\|_3 + \|\eta\|_3 \|\partial_\omega^2 \eta\|_2))$$

$$\lesssim \|u\|_{18} (\|\eta\|_{18} (1 + \|\eta\|_3)(\|u\|_2^2 + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_6)$$

$$+ \|\eta\|_3 (\|u\|_3 \|u\|_{17} + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_{17})) + \|\eta\|_{18} (\|\eta\|_{17} \|u\|_3 (\|u\|_2 \|u\|_3$$

$$+ m^2 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_6 + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_{17} \|u\|_3 + m^2 \|\partial_\omega \eta\|_{18} \|u\|_{16} \|u\|_3$$

$$\lesssim \delta (\|u\|_{18}^2 + K \sqrt{m \|\partial_\omega \eta\|_{17} \|u\|_{18} + \|\eta\|_{18} (\|u\|_2 \|u\|_3 + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_6)$$

$$+ m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_3 \|\partial_\omega \eta\|_6 + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_{17} \|u\|_3)$$

$$+ m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_{18} \|u\|_3$$

(3.35)

where we have used (3.16) with $i = 1, 15$, in the first inequality, (3.3) in the second inequality and (3.22), (3.23) the interpolation inequality in the last inequality.

Putting (3.29) with $i = 17$ and the above estimate into (3.28) for $|\alpha| = 17$, and then using (3.2) and Young’s inequality, we get (3.25) with $i = 17$. □
Lemma 3.5. Under the conditions (2.29)–(2.30) with sufficiently small \( \delta \), we have

\[
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^{i+1} \eta \|_0^2 + \sum_{|\alpha|=1} \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy \right) + \| m \partial_\omega \eta \|_i^2 \leq 2 \| u \|_i^2 + \left\{ \begin{array}{ll}
c\delta \| u \|_{i+1}^2 & \text{for } 0 \leq i \leq 12; \\
c\| \eta \|_{i+1}(\| \eta \|_{i+1}(m^2 \| \partial_\omega \eta \|_0^2) + \| u \|_3(1 + \| u \|_3)) & \text{for } i = 16; \\
\| \eta \|_{2/3}^2 \| \partial_\omega \eta \|_0^{1/3} \| u \|_{i+1} & \text{for } i = 17.
\end{array} \right.
\] (3.36)

Proof. Let \( i = |\alpha| = 0, \ldots, 12, 16 \) and 17. Multiplying (3.27) by \( \partial^\alpha \eta \) in \( L^2 \), we get that

\[
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^\alpha \eta \|_0^2 + \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy \right) + \| m \partial_\omega \partial^\alpha \eta \|_0^2 = \| \partial^\alpha u \|_0^2 + I_3 + I_4,
\] (3.37)

where \( I_3 := - \int \partial^\alpha N^{\rho \nu}_{j,i} \cdot \partial \partial^\alpha \eta \, dy \) and

\[
I_4 := \int \partial^\alpha q \partial^\alpha (r_2^\eta + r_3^\eta) \, dy \left\{ - \int \eta \cdot \nabla q \, dy \quad \text{for } \alpha = 0; \\
+ \int \partial^\alpha \eta \cdot \partial^\alpha \nabla \tilde{A} q \, dy \quad \text{for } \alpha \neq 0.
\right.
\]

with \( \alpha^- \) and \( \alpha^+ \) being defined by (3.26).

Following the arguments of (3.29) and (3.30) with slight modifications, we find that

\[
I_3 \leq \| \eta \|_3 \| \eta \|_{i+1} \| u \|_{i+1} + \left\{ \begin{array}{ll} 0 & \text{for } i = 0, 1; \\
\| \eta \|_{i+1}^2 \| u \|_3 & \text{for } i \geq 2,
\end{array} \right.
\] (3.38)

\[
I_4 \leq \| \eta \|_{i+1}(\| \partial_\omega \eta \|_3 \| u \|_{i+1} + \| \eta \|_{\sigma(i-1)+1} \| q \|_3).
\] (3.39)

(1) Case \( 0 \leq i \leq 3 \).

Exploiting (3.3), (5.5) and (3.16), we derive from (3.38) and (3.39) with \( i = 3 \) that

\[
I_3 \leq \| u \|_{i+1}(\| \partial_\omega \eta \|_6 \| \partial_\omega \eta \|_{i+4} + \| \partial_\omega \eta \|_{2i+4}) \\
\leq \| \partial_\omega \eta \|_{i+1}(\| \partial_\omega \eta \|_{10} + \| \partial_\omega \eta \|_{i+8})
\] (3.40)

and

\[
I_4 \leq \| \eta \|_3 \| \eta \|_4(\| u \|_2 \| u \|_3 + m^2(\| \partial_\omega \eta \|_2 \| \partial_\omega \eta \|_3 + \| \eta \|_3 \| \partial_\omega \eta \|_2)) \\
\leq \| \partial_\omega \eta \|_3^2 \| u \|_3^2 + m^2 \| \partial_\omega \eta \|_6 \| \partial_\omega \eta \|_3 \\
\leq \| \partial_\omega \eta \|_i^{2(3-i)/(10-2i)} \| \partial_\omega \eta \|_{10-i}^{2(7-i)/(10-2i)} \| u \|_i^{2(7-i)/(10-2i)} \| u \|_{10-i}^{2(3-i)/(10-2i)} \\
+ m^2 \| \partial_\omega \eta \|_i^{25/(12-i)} \| \partial_\omega \eta \|_{12}^{23/(4i)/(12-i)},
\]

where \( 0 < 2(3-i)(10-2i)^{-1} \leq 1 \) for \( 0 \leq i \leq 2 \) and \( 1 < (23-4i)/(12-i) < 2 \) for any \( 0 \leq i \leq 3 \). Inserting the above two estimates into (3.37) with \( i = 3 \), and then using (3.24) and Young’s inequality, we obtain (3.36) with \( 0 \leq i \leq 3 \).

(2) Case \( 4 \leq i \leq 12 \).
Similarly to (3.40), we have, for $4 \leq i \leq 12$,
\[
I_3 \lesssim \|\eta\|_{i+1}^2 \|u\|_{i+1} \lesssim \|\partial_\omega \eta\|_{i} \|\eta\|_{i+5} \|u\|_{i+1} \lesssim \delta m \|\partial_\omega \eta\|_{i} \|u\|_{i+1},
\]
where we have used (3.24) in the last inequality. Noting that
\[
\|\eta\|_{i} \|\partial_\omega \eta\|_{6} \lesssim \|\partial_\omega \eta\|_{i} \|\partial_\omega \eta\|_{9} \text{ for } i = 4, 5,
\]
thus making use of (3.3), (3.16) with $i = 1$, 10, (3.22), (3.23) and (3.42), we can derive from (3.39) that, for $4 \leq i \leq 12$,
\[
\begin{align*}
|I_4| & \lesssim \|\eta\|_{3} \|\eta\|_{i+1}(\|u\|_{2}(\|u\|_{i} + \|\eta\|_{i} \|u\|_{3}) + \|u\|_{3} \|u\|_{i-1} \\
& \quad + m^2(\|\eta\|_{3} \|\partial_\omega \eta\|_{15} + \|\partial_\omega \eta\|_{3} \|\partial_\omega \eta\|_{15} + \|\eta\|_{15}(\|\partial_\omega \eta\|_{3} + \|\partial_\omega \eta\|_{3}^2)) \\
& \quad + \|\eta\|_{16}(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \quad + \|\eta\|_{16}(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{i+1}(\|u\|_{2} \|u\|_{3} + \|u\|_{3} \|u\|_{i-1} \\
& \quad + m^2(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{i+1}(\|u\|_{2} \|u\|_{3} + \|u\|_{3} \|u\|_{i-1}) \\
& \lesssim \|\partial_\omega \eta\|_{i} \|u\|_{i} \|\eta\|_{i} \|\partial_\omega \eta\|_{i} \|u\|_{i} \\
& \lesssim \delta(\|u\|_{i}^2 + m^2 \|\partial_\omega \eta\|_{i}^2)
\end{align*}
\]
Inserting (3.41) and (3.43) into (3.37) with $i = 12$, and then using Young’s inequality, we obtain (3.36) with $4 \leq i \leq 12$.

(3) Case $i = 16$.

We can derive from (3.39) with $i = 16$ that
\[
\begin{align*}
|I_4| & \lesssim \|\eta\|_{3} \|\eta\|_{17}(\|u\|_{2}(\|u\|_{16} + \|\eta\|_{16} \|u\|_{3}) + \|u\|_{3} \|u\|_{15} \\
& \quad + m^2(\|\eta\|_{3} \|\partial_\omega \eta\|_{16} + \|\partial_\omega \eta\|_{3} \|\partial_\omega \eta\|_{16} + \|\eta\|_{16}(\|\partial_\omega \eta\|_{3} + \|\partial_\omega \eta\|_{3}^2)) \\
& \quad + \|\eta\|_{17}(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{17}(\|u\|_{3} \|u\|_{i-1} + m^2(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{i+1}(\|u\|_{2} \|u\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2}) \\
& \lesssim \delta(\|u\|_{i}^2 + m^2 \|\partial_\omega \eta\|_{i}^2)
\end{align*}
\]
where we have used (3.16) with $i = 1, 14$ in the first inequality, (3.3) in the second inequality and (3.22) and (3.23) in the last inequality. Inserting (3.38) with $i = 16$ and the above estimate into (3.37) for $|\alpha| = 16$ yields (3.36) with $i = 16$.

(4) Case $i = 17$.

We can derive from (3.39) with $i = 17$ that
\[
\begin{align*}
|I_4| & \lesssim \|\eta\|_{3} \|\eta\|_{18}(\|u\|_{2}(\|u\|_{17} + \|\eta\|_{17} \|u\|_{3}) + \|u\|_{3} \|u\|_{16} \\
& \quad + m^2(\|\eta\|_{3} \|\partial_\omega \eta\|_{17} + \|\partial_\omega \eta\|_{3} \|\partial_\omega \eta\|_{17} + \|\eta\|_{17}(\|\partial_\omega \eta\|_{3} + \|\partial_\omega \eta\|_{3}^2)) \\
& \quad + \|\eta\|_{18}(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{18}(\|u\|_{3} \|u\|_{17} + m^2(\|\partial_\omega \eta\|_{2} \|\partial_\omega \eta\|_{3} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2})) \\
& \lesssim \|\eta\|_{3} \|\eta\|_{18}(\|u\|_{3} \|u\|_{17} + \|\eta\|_{3} \|\partial_\omega \eta\|_{2}) \\
& \lesssim \delta(\|u\|_{i}^2 + m^2 \|\partial_\omega \eta\|_{i}^2)
\end{align*}
\]
where we have used (3.16) with $i = 1, 15$ in the first inequality, (3.3) in the second inequality, and (3.22) and (3.23) in the last inequality. Inserting (3.38) with $i = 17$ and the above estimate into (3.37) for $|\alpha| = 17$, and then using (3.3), we get (3.36) with $i = 17$. \qed
3.2. Stability estimates

With the energy estimates in Lemmas 3.4–3.5 in hand, we are in the position to establish the a priori estimate (2.28).

To begin with, we use (3.25) and (3.36) with $0 \leq i \leq 12$ to build the following $i$-th layer energy inequality:

$$\frac{d}{dt} \tilde{\mathcal{E}}_i + c \mathcal{D}_i \lesssim 0 \quad (3.46)$$

where

$$\tilde{\mathcal{E}}_i := c\|\nabla^i (u, m \partial \omega \eta)\|_0^2 + \nu \|\nabla^{i+1} \eta\|_0^2 + \sum_{|\alpha| = i} \int \partial^\alpha \eta \cdot \partial^\alpha u dy \quad (3.47)$$

and $\mathcal{E}_i$ satisfies

$$\mathcal{E}_i \lesssim \tilde{\mathcal{E}}_i \lesssim \mathcal{E}_i. \quad (3.48)$$

Integrating (3.46) over $(0, t)$ yields

$$\mathcal{E}_i + \int_0^t \mathcal{D}_i d\tau \lesssim \mathcal{E}_i^0. \quad (3.49)$$

By (3.3), we see that, for $0 \leq i \leq 12$,

$$\mathcal{E}_i \lesssim (1 + m^{-2}) \mathcal{D}_{i+4}. \quad (3.50)$$

Thus we further derive the following lower-order energy inequality from (3.46), (3.48) and (3.50):

$$\sum_{i=0}^3 \left( d_i (1 + m^{-2}) \frac{d}{dt} (t)^{(3-i)} \tilde{\mathcal{E}}_i + h_i (1 + m^{-2}) (t)^{(3-i)} \mathcal{D}_i \right) \lesssim 0 \quad (3.51)$$

for some constants $d_i$ and $h_i$ depending on $\nu$ and $\omega$. Integrating (3.51) over $(0, t)$, and then using (3.48), we get

$$\sum_{i=0}^3 \left( (1 + m^{-2}) (t)^{(3-i)} \mathcal{E}_i + (1 + m^{-2}) (t)^{(3-i)} \int_0^t (t)^{(3-i)} \mathcal{D}_i d\tau \right) \lesssim \Xi, \quad (3.52)$$

where $\Xi$ is defined by (2.14). Using the interpolation inequality, we can derive from (3.49) and (3.52) that, for $0 \leq i \leq 12$,

$$\mathcal{E}_i \lesssim \left( \mathcal{E}_{12}^0 \right)^{i/12} \left( (t)^{-3} \Xi \right)^{(12-i)/12}. \quad (3.53)$$

It is easy see from (3.25) and (3.36) that

$$\frac{d}{dt} \|\nabla^i (u, m \partial \omega \eta)\|_0^2 + \nu \|u\|_{i+1}^2$$

$$\lesssim \begin{cases} 
\delta \mathcal{D}_H + \mathcal{E}_H \mathcal{D}_6 (1 + \sqrt{\mathcal{E}_3} + \mathcal{E}_4) + \sqrt{\mathcal{E}_3 \mathcal{E}_4} \mathcal{D}_H \mathcal{D}_H & \text{for } i = 16; \\
K^2 m^{-1} \mathcal{D}_H + \mathcal{E}_H (\mathcal{D}_6 (m^{2/3} + K + K \sqrt{\mathcal{E}_4}) + K \sqrt{\mathcal{E}_3}) & \text{for } i = 17. 
\end{cases} \quad (3.54)$$
and
\[
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^{i+1} \eta \|^2 + \sum_{|\alpha|=i} \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy \right) + \| m \partial_\omega \eta \|^2 \
\leq 2 \| u \|^2 + c(\sqrt{E_H D_H (\sqrt{E_3} + E_3^{1/3} D_6^{1/6}) + E_H (\sqrt{E_3} + D_6)}) \text{ for } i = 16, 17, \tag{3.55}
\]

Noting that, by (3.1) and (2.30),
\[
\begin{align*}
K^2 & \lesssim_0 0 \delta^{5/3} \quad \text{and} \quad \frac{K}{m^{5/3}} \lesssim_0 0 \delta^{2/3}. \tag{3.56}
\end{align*}
\]

Exploiting (3.1), Poincaré’s inequality (3.2), (3.56) and Young’s inequality, we derive the following highest-order energy inequality from (3.54) and (3.55):
\[
\frac{d}{dt} \tilde{E}_H + c D_H \lesssim E_H (\sqrt{E_3} + E_3 E_4 + (1 + E_4) D_6), \tag{3.57}
\]

where
\[
\tilde{E}_H := c \| \nabla^{16} (u, m \partial_\omega \eta) \|^2 + m^{-2/3} \| \nabla^{17} (u, m \partial_\omega \eta) \|^2 \\
+ \frac{\nu}{2} \| \nabla^{17} (\eta, \nabla \eta) \|^2 + \sum_{16 \leq |\alpha| \leq 17} \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy,
\]

satisfying
\[
E_H \lesssim \tilde{E}_H \lesssim E_H. \tag{3.58}
\]

Applying Gronwall’s lemma to (3.57), and then using (3.49), (3.53) with \( i = 3 \) and (3.58), we arrive at that there exists a constant \( \delta_1 \in (0,1] \) such that for any \( \delta \leq \delta_1 \),
\[
E_H \lesssim E_H^0 e^\left( c_1 E_H^0 e^{c_2 \vartheta} \right) \lesssim c_1 E_H^0 e^{c_2 \vartheta} / 4 \text{ for any } 0 \leq t \leq T, \tag{3.59}
\]

where \( c_1 \geq 4 \) and \( \vartheta \) is defined by (2.15). In addition, thanks to (3.59), we further derive from (3.57) that
\[
\int_0^t D_H \, d\tau \lesssim E_H^0 (1 + e^{c_2 \vartheta}). \tag{3.60}
\]

Now we take
\[
K := \sqrt{c_1 E_H^0 e^{c_2 \vartheta}} > 0, \tag{3.61}
\]

we immediately obtain the desired a priori stability estimate (2.28) from (3.59) under the a priori assumption (2.29) with the relative smallness condition (2.30) for any \( \delta \leq \delta_1 \).

3.3. Proof of Theorem 2.1

We start with introducing a local (-in-time) well-posedness result for the initial value problem (2.9)–(2.10) and a result concerning diffeomorphism mappings.
Proposition 3.1. Let \((\eta^0, u^0) \in H^{18} \times H^{17}\) satisfy \(\| (\nabla \eta^0, u^0) \|_{17} \leq B\) and \(\text{div} \mathcal{A}^0 u^0 = 0\), where \(B\) is a positive constant, \(c^0 := \eta^0 + y\) and \(\mathcal{A}^0\) is defined by \(c^0\). Then there is a constant \(\delta_2 \in (0, 1]\), such that for any \((\eta^0, u^0)\) satisfying
\[
\| \nabla \eta^0 \|_{12} \leq \delta_2,
\] (3.62)
there exist a local existence time \(T > 0\) (depending possibly on \(B, \nu, m\) and \(\delta_2\)) and a unique local classical solution \((\eta, u, q) \in C^{\infty}(\overline{T_T} \times H^{18}) \times \mathcal{U}_T \times C^{\infty}(\overline{T_T} \times H^{17})\) to the initial value problem (2.9) -- (2.10), satisfying \(0 < \inf_{(y,t) \in \mathbb{R}^3 \times \overline{T_T}} \text{det}(\nabla \eta + I)\) and \(\sup_{t \in \overline{T_T}} \| \nabla \eta \|_{12} \leq 2\delta_2\). 

PROOF. Please refer to [27, Proposition 7.3]. □

Proposition 3.2. There is a positive constant \(\delta_3\), such that for any \(\varphi \in H^{18}\) satisfying \(\| \nabla \varphi \|_2 \leq \delta_3\), we have (after possibly being redefined on a set of zero measure) \(\text{det}(\nabla \varphi + I) > 1/2\) and
\[
\psi : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is a } C^{16} \text{ homeomorphism mapping},
\] (3.63)
where \(\psi := \varphi + y\).

PROOF. Please refer to [25, Lemma 4.2] for a detailed proof. □

With the a priori estimate (3.59) (under the conditions (2.29) and (2.30) with \(\delta \leq \delta_1\)), and Propositions 3.1, 3.2 in hand, we can easily establish Theorem 2.1. We briefly give the proof below.

Let \(m\) and \((\eta^0, u^0) \in (H^{18}_{11} \cap H_{11}^{18}) \times H^{17}\) satisfy
\[
\max\{K, K^2\}/m \leq \min\{\delta_1, \delta_2/c_{\omega, i3}, \delta_3/c_{\omega, i3}\} =: c_3 \leq 1,
\] (3.64)
where \(K\) is defined by (3.61), and the above two constants \(c_{\omega, i3}\) come from (3.3) with \(i = 13\). Then we see that \(\eta^0\) satisfies (3.62) by (3.3) and (3.64). Hence, by virtue of Proposition 3.1 there exists a unique local solution \((\eta, u, q)\) of (2.9) -- (2.10) with the maximal existence time \(T_{\max}\), satisfying

- for any \(T \in I_{T_{\max}}\), the solution \((\eta, u, q)\) belongs to \(C^{\infty}(\overline{T_T} \times H^{18}_{11}) \times \mathcal{U}_T \times C^{\infty}(\overline{T_T} \times H^{17})\) and
\[
\sup_{t \in \overline{T_T}} \| \nabla \eta \|_{12} \leq 2\delta_2;
\]

- \(\limsup_{t \to T_{\max}} \| \nabla \eta(t) \|_{12} > \delta_2\) or \(\limsup_{t \to T_{\max}} \| (\nabla \eta, u)(t) \|_{17} = \infty\), if \(T_{\max} < \infty\).

Let
\[
E(t) := \| \eta(t) \|_{18}^2 + m^{-2/3} \| u(t) \|_{17}^2 + \| m \partial_{\omega} \eta(t) \|_{16}^2,
\]
\[
T^* = \sup \{ T \in I_{T_{\max}} \mid E(t) \leq K^2 \text{ for any } t \leq T \}.
\]
Recalling the definition of \(K\) and the condition \(c_1 \geq 4\), we easily see that the definition of \(T^*\) makes sense and \(T^* > 0\). In addition, by (3.3), we have \(\| \nabla \eta \|_{12} \leq \delta_3\) for all \(t \in I_{T^*}\), then

\[\text{Here the uniqueness means that if there is another solution } (\tilde{\eta}, \tilde{u}, \tilde{q}) \in C^{\infty}(\overline{T_T} \times H^{18}) \times \mathcal{U}_T \times C^{\infty}(\overline{T_T} \times H^{17}) \text{ satisfying } 0 < \inf_{(y,t) \in \mathbb{R}^3 \times \overline{T_T}} \text{det}(\nabla \tilde{\eta} + I), \text{ then } (\tilde{\eta}, \tilde{u}, \tilde{q}) = (\eta, u, q) \text{ by virtue of the smallness condition } \sup_{t \in \overline{T_T}} \| \nabla \eta \|_{12} \leq 2\delta_2.\]
\( \eta(t) \in H^{18}_t \) for all \( t \in I_{T^*} \) by Proposition 3.2. Thus, to obtain the existence of a global solution, it suffices to verify \( T^* = \infty \). Now, we show this by contradiction.

Assume \( T^* < \infty \). Keeping in mind that \( T_{\max} \) denotes the maximal existence time and \( K/m \leq \delta_2/c_{\omega,13} \) by virtue of (3.64), we apply Proposition 3.1 to find that \( T_{\max} > T^* \) and

\[
E(T^*) = K^2. \tag{3.65}
\]

Since \( m^{-1} \max\{K, K^2\} \leq \delta_1 \) and sup_\( t \in T^* \) \( E(t) \leq K^2 \), we can still show that the solution \( (\eta, u) \) enjoys the stability estimate (3.59) with \( T^* \) in place of \( T \) by the regularity of \( (\eta, u, q) \). More precisely, we have sup_\( t \in T^* \) \( E(t) \leq K^2/4 \), which contradicts with (3.64). Hence, \( T^* = \infty \), and thus \( T_{\max} = \infty \).

The uniqueness of the global solutions is obvious due to the uniqueness of the local solutions in Proposition 3.1 and the fact sup_{\( T \geq 0 \)} \( \| \nabla \eta \|_2 \leq 2\delta_2 \). Finally, it is obvious that the global solution \( (\eta, u, q) \) enjoys the estimates (2.17), (2.21), (3.10)–(3.12) and (3.16) by recalling the derivation of \( \alpha \) priori energy estimates for \( (\eta, u) \). This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. Let \( (\eta^0, u^0) \) satisfy all the assumptions in Theorem 2.1 and \( (\eta, u, q) \) be the solution constructed by Theorem 2.1. Exploiting (2.16), (2.21) and (3.3), we have

\[
\| \eta^0 \|_{15} \lesssim 1 \quad \text{and} \quad \| \eta^0 \|_{13} \sqrt{\mathcal{C}_H} \lesssim 1. \tag{4.1}
\]

By the regularity theory of the Stokes problem, there exists a unique solution \( (\eta^r, u^r, Q_1, Q_2) \) satisfying

\[
\begin{align*}
-\Delta \eta^r + \nabla Q_1 &= 0, \\
\text{div} \eta^r &= -\partial_t \eta^0, \\
(\eta^r)_{r=1} &= 0
\end{align*}
\]

and

\[
\begin{align*}
-\Delta u^r + \nabla Q_2 &= 0, \\
\text{div} u^r &= \text{div} A_0 u^0, \\
(u^r)_{r=1} &= 0,
\end{align*}
\]

where \( A_0 := A^0 - I \). Moreover, \( (\eta^r, u^r) \in H^{18} \times H^{17} \) satisfies (2.35) and (2.36) by making use of the classical regularity theory of Stokes equations, the integral by parts, Poincaré’s inequality, Young’s inequality, (2.3), the interpolation inequality (3.3), (3.10) with \( t = 0 \), (3.12), (3.13), (3.15), (4.1) and the following identity

\[
\text{div} \eta^0 = \text{div} \left( -\eta^0_1 (\partial_2 \eta^0_2 + \partial_3 \eta^0_3) + \eta^0_1 (\partial_2 \eta^0_3 \partial_2 \eta^0_2 - \partial_2 \eta^0_2 \partial_3 \eta^0_3) + \eta^0_2 (\partial_1 \eta^0_2 \partial_3 \eta^0_3 - \partial_1 \eta^0_3 \partial_3 \eta^0_3) + \eta^0_3 (\partial_1 \eta^0_3 \partial_2 \eta^0_2 - \partial_1 \eta^0_2 \partial_2 \eta^0_3) \right).
\]

Moreover, using the interpolation inequality (3.5), we have

\[
\begin{align*}
\| \eta^r \|_{k+1} &\lesssim \| \eta^0 \|_3 \| \eta^0 \|_{k+1} \quad \text{for} \ 0 \leq k \leq 17, \\
\| \partial_\omega \eta^r \|_{k} &\lesssim \| \eta^0 \|_3 \| \partial_\omega \eta^0 \|_{k} + \begin{cases} \| \eta^0 \|_7 \| \partial_\omega \eta^0 \|_{k} & \text{for} \ 0 \leq k \leq 2; \\
\| \eta^0 \|_k \| \partial_\omega \eta^0 \|_{3} & \text{for} \ 3 \leq k \leq 17, \end{cases} \\
\| u^r \|_{k} &\lesssim \| \eta^0 \|_3 \| u^0 \|_{k} + \begin{cases} \sqrt{\| \partial_\omega \eta^0 \|_8 \| u^0 \|_6 \| \partial_\omega \eta^0 \|_{k} \| u^0 \|_{k}} & \text{for} \ 0 \leq k \leq 2; \\
\| \eta^0 \|_k \| u^0 \|_{3} & \text{for} \ 3 \leq k \leq 17. \end{cases}
\end{align*}
\]
Let $\eta^0 = \eta^0 + \eta^f$ and $\tilde{u}^0 = u^0 + u^f$. Thus, it is easy to see that $(\tilde{\eta}^0, \tilde{u}^0)$ belongs to $H^{18}_\sigma \times H^{17}_\sigma$. Therefore, there exists a unique global solution $(\eta^L, u^L, q^L) \in C^0(\mathbb{R}_+^*, H^{18}_\sigma) \times H^{\infty}_\sigma \times C^0(\mathbb{R}_+^*, H^{17}_\sigma)$ to the linearized problem (2.31) with the initial condition $(\eta^L, u^L)|_{t=0} = (\tilde{\eta}^0, \tilde{u}^0)$.

Similarly to (3.49) and (3.52), we easily see that the solution $(\eta^L, u^L)$ of the linearized problem (2.31) enjoys the estimates (2.32) and (2.33). Moreover, by (1.1) and (4.2), we can further derive form (2.32) and (2.33) that

$$
\mathcal{E}_j^L + \int_0^t \mathcal{D}_j^L d\tau \lesssim \mathcal{E}_0^L \text{ for any } 1 \leq j \leq 15
$$

and

$$
\sum_{i=0}^3 \left( (1 + m^{-2}) (t^{(3-i)} \mathcal{E}_{4i}^L + (1 + m^{-2}) \int_0^t t^{(3-i)} \mathcal{D}_{4i}^L d\tau) \right) \lesssim \Xi,
$$

where $\Xi$ is defined by (2.14).

Let $(\eta^d, u^d) = (\eta - \eta^L, u - u^L)$, then the error function $(\eta^d, u^d)$ satisfies

$$
\begin{align*}
\begin{cases}
\eta_t^d = u^d, \\
u \Delta u^d - m^2 \partial_y^2 \eta^d = \mathfrak{N}, \\
\text{div} u^d = -\text{div} \mathfrak{N}, \\
(\eta^d, u^d)|_{t=0} = (-\eta^f, u^f).
\end{cases}
\end{align*}
$$

It is easy to see from (4.5) that $(u^d)_{T^3} = (\eta^d)_{T^3} = 0$, $\text{div} \eta^d = \text{div} \eta$ for any $t > 0$, since $(\eta^f)_{T^3} = (u^f)_{T^3} = 0$ and $\text{div} \eta^L = 0$.

Recalling that $(\eta, u, q)$ is constructed by Theorem 2.1 the solution $(\eta, u, q)$ satisfies all the estimates in (2.21), (3.10)–(3.12) and (3.16). Hence, we can follow the same arguments as in the proof of Lemmas 3.4 and 3.5 with some modifications to deduce from (4.5) that

$$
\frac{d}{dt} \| \nabla (u^d, m \partial_y \eta^d) \|^2_i + \nu \| u^d \|^2_{i+1} \lesssim \varphi_i / m,
$$

and

$$
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^{i+1} \eta^d \|^2_0 + \sum_{|\alpha|=i} \int \partial^\alpha \eta^d \cdot \partial^\alpha u^d dy \right) + \| m \partial_y \eta^d \|^2_i \lesssim \| u^d \|^2_i + c \psi_i / m,
$$

where

$$
\varphi_i := m \| (u, u^L) \|_{i+1} (\| \partial_y \eta \|_6 \| u \|_{i+1} + \| \partial_y \eta \|_{i+4} \| u \|_3)
$$

$$
+ m \left\{ \| \partial_y \eta \|_7 \| (u, u^L) \|_4 (\| u \|^2_3 + m^2 \| \partial_y \eta \|_3 \| \partial_y \eta \|_6) \right\} + \| \partial_y \eta \|_{i+1} (\| u \|^2_3 + m^2 \| \partial_y \eta \|_3 \| \partial_y \eta \|_6)
$$

$$
\text{for } 0 \leq i \leq 3;
$$

and

$$
\psi_i := m \| (\eta, \eta^L) \|_{i+1} \left( \| \partial_y \eta \|_6 \| u \|_{i+1} + \| \partial_y \eta \|_{i+4} \| u \|_3 \right)
$$

$$
+ m \| (\eta, \eta^L) \|_{i+1} \left( \| \partial_y \eta \|_6 (\| u \|^2_3 + m^2 \| \partial_y \eta \|_3 \| \partial_y \eta \|_6) \right) \text{ for } 0 \leq i \leq 3;
$$

$$
+ m \| (\eta, \eta^L) \|_{i+1} \left( \| \partial_y \eta \|_6 (\| u \|^2_3 + m^2 \| \partial_y \eta \|_3 \| \partial_y \eta \|_6) \right) \text{ for } 4 \leq i \leq 12.
$$
Similarly to \((3.46)\), we further derive from \((1.6)\) and \((4.7)\) that
\[
\frac{d}{dt} \tilde{E}_i^d + cD^d_i \lesssim m^{-1}(\varphi_i + \psi_i) \quad \text{for } 0 \leq i \leq 12, \tag{4.10}
\]
where \(\tilde{E}_i^d\) is defined as \(\tilde{E}_i\) in \((3.47)\) with \((\eta^d, u^d)\) in place of \((\eta, u)\), and \(\tilde{E}_i^d\) satisfies
\[
\mathcal{E}_i^d \lesssim \tilde{E}_i^d \lesssim \mathcal{E}_i^d. \tag{4.11}
\]
In addition, by \((2.18), (2.19), (2.20)\) and \((4.3)\), it is easy to see that
\[
\int_0^t (\varphi_{12} + \psi_{12}) d\tau \lesssim m \int_0^t (\|u, u^L\|_{13}(\|\partial_\omega \eta\|_6\|u\|_{13} + \|\partial_\omega \eta\|_{16}\|u\|_3 + \|\partial_\omega \eta\|_{15}\|u\|_3^2 + m^2(\|\partial_\omega \eta\|_4\|\partial_\omega \eta\|_6) + \|\eta, \eta_1\|_{13}(\|\partial_\omega \eta\|_6\|u\|_{13} + \|\partial_\omega \eta\|_{16}\|u\|_3 + \|\partial_\omega \eta\|_6\|u\|_{12} + \|\partial_\omega \eta\|_{15}\|u\|_3^2 + m^2(\|\partial_\omega \eta\|_4\|\partial_\omega \eta\|_6))) d\tau
\lesssim \int_0^t \left( \sqrt{\mathcal{E}_{16} + \mathcal{E}_{16}^L} \left(1 + \sqrt{\mathcal{E}_{16}}\right)(D_{12} + D_{12}^L) + \sqrt{\mathcal{E}_3 \mathcal{E}_{16}(\mathcal{E}_{12} + \mathcal{E}_{12}^L)} \right) d\tau
\lesssim \mathcal{E}_{17}^0 e^{-c_2\theta} \left( \sqrt{\mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} + \sqrt{\Xi \mathcal{E}_{12}^0 e^{-c_2\theta}} \right). \tag{4.12}
\]
Thus, integrating \((4.10)\) over \((0, t)\), and then making use of \((4.1), (4.2), (4.11)\) and \((4.12)\), we easily get \((2.37)\).

Finally we derive \((2.38)\). Similarly to \((3.52)\), we derive the following inequality from \((3.3), (4.2), (4.10)\) and \((4.11)\):
\[
\sum_{i=0}^3 \left( (1 + m^{-2})^i (t)^{3-i} \mathcal{E}_{4i}^d + (1 + m^{-2})^i \int_0^t \langle \tau \rangle^{3-i} D_{4i}^d d\tau \right)
\lesssim m^{-1} \left( \mathcal{E}_{15}^0 + \sum_{i=0}^3 \int_0^t (1 + m^{-2})^i \langle \tau \rangle^{3-i}(\varphi_i + \psi_i) d\tau \right). \tag{4.13}
\]
Noting that \((\eta, u)\) satisfies \((3.03)\), thus making use of the interpolation inequality \((3.4), (2.17), (2.18), (4.3)\) and \((4.4)\), we have
\[
\int_0^t \langle \tau \rangle^3 (\varphi_0 + \psi_0) d\tau
\lesssim m \int_0^t \langle \tau \rangle^3 (\|u, u^L\|_1(\|\partial_\omega \eta\|_6\|u\|_1 + \|\partial_\omega \eta\|_4\|u\|_3) + \|\partial_\omega \eta\|_7\|u, u^L\|_4(\|u\|_3^2 + m^2(\|\partial_\omega \eta\|_3\|\partial_\omega \eta\|_6))) d\tau
\lesssim \int_0^t \langle \tau \rangle^3 (1 + \sqrt{\mathcal{E}_{12}}) \sqrt{(\mathcal{E}_0 + \mathcal{E}_{12}^L)(D_0 + D_0^L)(D_{12} + D_{12}^L)} d\tau
\lesssim \Xi \left( \sqrt{\mathcal{E}_{12}^0 + \mathcal{E}_{12}^0} \right).
\]
Similarly,

\[ \int_0^t \langle \tau \rangle^2 (\varphi_4 + \psi_4) d\tau \]

\[ \lesssim m \int_0^t \langle \tau \rangle^2 (\|u, u^L\|_5 (\|\partial_\omega \eta\|_6 \|u\|_5 + \|\partial_\omega \eta\|_8 \|u\|_3 \\
+ \|\partial_\omega \eta\|_s (\|u\|_3^2 + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_4) + \|(\eta, \eta^L)\|_5 (\|\partial_\omega \eta\|_6 \|u\|_5 \\
+ \|\partial_\omega \eta\|_s \|u\|_3 + \|\partial_\omega \eta\|_7 (\|u\|_3^2 + m^2 \|\partial_\omega \eta\|_4 \|\partial_\omega \eta\|_6))) d\tau \]

\[ \lesssim \int_0^t \langle \tau \rangle^2 (1 + \sqrt{E_{12}^0}) \sqrt{D_{12} (E_4 + E_4^L)(D_4 + D_4^L)} d\tau \]

\[ \lesssim \Xi \left( \sqrt{E_{12}^0 + E_{12}^0} \right) \]

and

\[ \int_0^t \langle \tau \rangle (\varphi_8 + \psi_8) d\tau \]

\[ \lesssim m \int_0^t \langle \tau \rangle (\|u, u^L\|_9 (\|\partial_\omega \eta\|_6 \|u\|_9 + \|\partial_\omega \eta\|_{12} \|u\|_3 + \|\partial_\omega \eta\|_{11} (\|u\|_3^2 \\
+ m^2 \|\partial_\omega \eta\|_s \|\partial_\omega \eta\|_6) + (\|u\|_3 \|u\|_8 + m^2 (\|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_8 \\
+ \|\partial_\omega \eta\|_s \|\partial_\omega \eta\|_6) (\|u, u^L\|_9 + \|\partial_\omega \eta\|_{12} (\|u, u^L\|_3) \\
+ \|(\eta, \eta^L)\|_9 (\|\partial_\omega \eta\|_6 \|u\|_9 + \|\partial_\omega \eta\|_{12} \|u\|_3 + \|\partial_\omega \eta\|_{11} (\|u\|_3^2 \\
+ m^2 \|\partial_\omega \eta\|_s \|\partial_\omega \eta\|_6 + \|\partial_\omega \eta\|_6 (\|u\|_3 \|u\|_8 + m^2 \|\partial_\omega \eta\|_6 \|\partial_\omega \eta\|_8))) d\tau \]

\[ \lesssim \int_0^t \langle \tau \rangle (1 + \sqrt{E_{12}^0}) \sqrt{D_{12} (E_8 + E_8^L)(D_8 + D_8^L)} d\tau \]

\[ \lesssim \Xi \left( \sqrt{E_{12}^0 + E_{12}^0} \right). \]

Putting the above three estimates and (4.12) together yields

\[ \sum_{i=0}^{3} \int_0^t (1 + m^{-2})^i \langle \tau \rangle^{3-i} (\varphi_i + \psi_i) d\tau \lesssim \Phi, \]

where \( \Phi \) is defined by (2.39). Inserting the above estimate into (4.13) yields (2.38). This completes the proof of Theorem 2.2.

**Acknowledgements.** The research of Fei Jiang was supported by NSFC (Grant Nos. 12022102 and the Natural Science Foundation of Fujian Province of China (2020J02013), and the research of Song Jiang by National Key R&D Program (2020YFA0712200), National Key Project (GJXM92579), and NSFC (Grant No. 11631008), the Sino-German Science Center (Grant No. GZ 1465) and the ISF–NSFC joint research program (Grant No. 11761141008).

**References**

[1] H. Abidi, P. Zhang, On the global solution of a 3-D MHD system with initial data near equilibrium, Comm. Pure Appl. Math. 70 (2017) 1509–1561.
[27] F. Jiang, S. Jiang, Asymptotic behaviors of global solutions to the two-dimensional non-resistive MHD equations with large initial perturbations, Adv. Math. 393 (2021) 108084.

[28] F. Jiang, S. Jiang, Y.J. Wang, On the Rayleigh–Taylor instability for the incompressible viscous magnetohydrodynamic equations, Comm. Partial Differential Equations 39 (2014) 399–438.

[29] F. Jiang, S. Jiang, W.C. Zhang, Instability of the abstract Rayleigh–Taylor problem and applications, Math. Models Methods Appl. Sci. 30 (2020) 2299–2388.

[30] A. Kiselev, V. Šverák, Small scale creation for solutions of the incompressible two-dimensional euler equation, Ann. of Math. 180 (2014) 1205–1220.

[31] R.H. Kraichnan, Inertial–range spectrum of hydromagnetic turbulence, Phys. Fluids 8 (1965) 1385–1387.

[32] Z. Lei, On axially symmetric incompressible magnetohydrodynamics in three dimensions, J. Differential Equations 259 (2015) 3202–3215.

[33] Z. Lei, F.H. Lin, Global mild solutions of Navier–Stokes equations, Comm. Pure Appl. Math. 64 (2011) 1297–1304.

[34] F.H. Lin, L. Xu, P. Zhang, Global small solutions of 2-D incompressible MHD system, J Differ. Equations 259 (2015) 5440–5485.

[35] G. Luo, T.Y. Hou, Potentially singular solutions of the 3D axisymmetric Euler equations, Proc. Nat. Acad. Sci. 111 (2014) 12968–12973.

[36] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow, Oxford University Press, USA, 2004.

[37] R.H. Pan, Y. Zhou, Y. Zhu, Global classical solutions of three dimensional viscous mhd system without magnetic diffusion on periodic boxes, Arch. Ration. Mech. Anal. 227 (2018) 637–662.

[38] X.X. Ren, J.H. Wu, Z.Y. Xiang, Z.F. Zhang, Global existence and decay of smooth solution for the 2-DMHD equations without magnetic diffusion, J. Funct. Anal. 267 (2014) 503–541.

[39] X.X. Ren, Z.Y. Xiang, Z.F. Zhang, Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain, Nonlinearity 29 (2016) 1257.

[40] Z. Tan, Y.J. Wang, Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems, SIAM J. Math. Anal. 50 (2018) 1432–1470.

[41] Y.J. Wang, Sharp nonlinear stability criterion of viscous non-resistive MHD internal waves in 3D, Arch. Rational Mech. Anal. 231 (2019) 1675–1743.

[42] Y.J. Wang, Z.P. Xin, Global well-posedness of free interface problems for the incompressible inviscid resistive MHD, To appear in Commun. Math. Phys., https://doi.org/10.1007/s00220-021-04235-3 (2021).

[43] D.Y. Wei, Z.F. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, Anal. PDE 10 (2017) 1361–1406.

[44] J.H. Wu, Y.F. Wu, X.J. Xu, Global small solution to the 2D MHD system with a velocity damping term, SIAM J. Math. Anal. 47 (2013) 2630–2656.

[45] L. Xu, On the ideal magnetohydrodynamics in three-dimensional thin domains: well-posedness and asymptotics, Arch. Rational Mech. Anal. 15 (2020) 1–70.

[46] L. Xu, P. Zhang, Global small solutions to three-dimensional incompressible magnetohydrodynamical system, SIAM J. Math. Anal. 47 (2015) 26–65.

[47] T. Zhang, An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, arXiv preprint arXiv:1404.5681 (2014).

[48] T. Zhang, Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field, J. Differential Equations 260 (2016) 5450–5480.

[49] Y. Zhou, Y. Zhu, Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain, J. Math. Phys. 59 (2018) 081505, 12 pp.

[50] A. Zlatoš, Exponential growth of the vorticity gradient for the euler equation on the torus, Adv. Math. 268 (2015) 396–403.