Grand Unification with Higher Rank Product Groups

Erik Kramer\textsuperscript{1}

\textsuperscript{1}Santa Cruz Institute for Particle Physics
Santa Cruz CA 95064, USA
lunenor@physics.ucsc.edu

Abstract

Various ideas support the notion that the GUT gauge group might be a semi-simple direct-product group such as $SU(5) \times SU(5)$. The doublet-triplet splitting problem can be solved with a direct product group. String theory suggests that the GUT scale is a modulus. Requiring this rules out a single $SU(5)$ gauge group. A model with $SU(5) \times SU(5)$ gauge symmetry and the GUT scale as a modulus has been shown to exist. It is shown that extending these ideas to $SO(10) \times SO(10)$ cannot be done with the above requirement without unwanted massless modes at lower energy scales that spoil the unification of couplings. Therefore these two conditions highly constrain the class of possible GUT models.
I. INTRODUCTION

The idea of unifying the gauge interactions of the Standard Model in one gauge group has long been appealing to theorists, both for the aesthetic virtue of explaining Standard Model physics under one simple gauge group, and for the experimental predictions such unifications make, e.g. $\sin \theta_W$. One of the problems with implementing GUTs is keeping massless the color triplet partners to the Higgs doublet. As in [1, 2], one way to solve this problem is to introduce a second copy of the fundamental group and then introduce a discrete symmetry. Breaking the unified group can leave a combination of this symmetry and a gauge transformation unbroken, and this symmetry forbids masses for doublets while allowing triplet mass terms. Witten showed that this cannot be accomplished with a single group. Witten considered GUTs motivated by deconstruction of higher dimensional models. One can consider taking the extra dimension to be a lattice, rather than a continuum, of points. In particular, consider a lattice of only two points, with the fundamental groups on each point. This picture may possibly be generalized, but we restrict ourselves to the simplest picture.

Dine et al. point out in [3] that if one considers the construction of grand unified models in string theory that the fields required to break unification are approximately or exactly massless. The GUT scale is about two orders of magnitude below the Planck scale. Some adjoints will be massless at the Planck scale, and acquire very large VEVs at the GUT scale. These fields must have very flat potentials in order for this to happen. Even without looking at GUT models with a string theory bias, an approximately flat potential gives a natural way of obtaining the GUT scale from the Planck scale and the supersymmetry breaking scale, as we will argue. With models we describe we will not give any explanation of the relative values of these scales, however, we will outline how the ratio of scales might arise naturally.

We will show as well that flat directions in the GUT potential are difficult to realize with a single $SU(5)$ gauge group. Although we already know that we need more than just a simple gauge group to solve the problem of doublets and triplets, it is interesting to see that this is not the only reason to consider using semi-simple gauge groups.

We find that if we require a theory to have approximate flat directions (exact up to nonrenormalizable terms), forbid the input of explicit mass terms, and implement Witten’s ideas for solving the doublet triplet splitting problem with only discrete symmetries, that if our fundamental gauge group is $SO(10) \times SO(10)$ this cannot be done successfully. The fact that extending these ideas to the next simplest group is not possible suggests that the result with an $SU(5) \times SU(5)$ gauge group is unique in this capacity. Although not all possible gauge groups are eliminated at this juncture, it is quite clear that there exists a set of criterion that cannot be met by $SO(10) \times SO(10)$ and groups of which this is a subgroup. Thus, the criteria we described above are extremely selective.

II. MOTIVATIONS

An important feature of our models should be that the Higgs doublet does not acquire mass at the GUT scale, while the Higgs triplet acquires a large mass to suppress proton decay. As discussed by Witten [1], an unbroken discrete symmetry that is not a subgroup of hypercharge, under which the Higgs doublets and triplets transform differently, can explain the existence of massless doublets with massive triplets.
As shown by Witten, this is not possible with a single $SU(5)$, $SO(10)$ or $E_6$. Rather, he suggests the use of a semi-simple group, such as $SU(5) \times SU(5)$, and shows that in such models it is possible to use a discrete symmetry to allow triplet mass terms while forbidding doublet masses at the GUT scale. Let us review Witten’s argument. In a model with a single $SU(5)$, one might imagine putting the Higgs doublets in chiral superfields $H$ and $\bar{H}$ that transform as a 5 and a $\bar{5}$, respectively. These fields then contain color triplets $q$ and $\bar{q}$, and the doublets $h$ and $\bar{h}$. The triplets have couplings related by $SU(5)$ to the couplings required of the doublets to give mass to quarks and leptons. The couplings of the triplets mediate proton decay, and must therefore have masses close to the GUT scale for the proton lifetime to be long enough [1]. Introduce a discrete symmetry under which the $(q, h)$ transform as $(e^{i\alpha}, e^{i\beta})$ and the $(\bar{q}, \bar{h})$ transform as $(e^{i\tilde{\alpha}}, e^{i\tilde{\beta}})$. For $e^{i(\alpha+\tilde{\alpha})} = 1$ and $e^{i(\beta+\tilde{\beta})} \neq 1$ the doublet mass term is not allowed, and the triplets are allowed to gain mass. Hopefully, at the supersymmetry breaking scale this discrete symmetry is spontaneously broken so that the Higgs doublets can gain the appropriate mass for phenomenology.

Up to a gauge transformation, in $SU(5)$ a discrete symmetry of the low energy theory commutes with $SU(5)$ [1]. For example, this symmetry might be a combination of a discrete hypercharge transformation, and a discrete symmetry that commutes with $SU(5)$. Both doublet and triplet mass terms are invariant under gauge transformations, and a symmetry that commutes with $SU(5)$ will transform both mass terms in the same way, so they are either both forbidden or both allowed. By introducing a product gauge group, $SU(5) \times SU(5)$ for example, it is possible to have the doublet and triplet mass terms transform differently under a discrete symmetry, in particular by having the superfields that contain the higgs doublet and color triplet fields transform under different $SU(5)$’s.

It has been shown by Dine, Nir, and Shadmi [3] that it is possible with the group $SU(5) \times SU(5)$ and discrete symmetries to construct models that solve the doublet triplet splitting problem and have an approximately flat potential in the GUT breaking direction, and furthermore have no extra massless particles that might spoil the prediction of coupling constant unification. Models with exact or approximate flat directions in the symmetry breaking potential have the desireable feature that the value of $M_{GUT}$ need not be a fundamental scale, but rather can arise dynamically. As discussed in [3], models with approximate flat directions are those for which the symmetries forbid renormalizable operators. Suppose that the lowest dimensional operator in the superpotential contributing to the F-term potential is

$$W = \frac{1}{M_{Pl}^{n-3}} X^n$$

and that once supersymmetry is broken a small negative mass squared is generated, giving rise to a potential of the form

$$V = -m^2|X|^2 + \frac{1}{M_{Pl}^{2n-6}} |X|^{2n-2}.$$  

This gives a VEV

$$\langle X \rangle \sim \left( \frac{m}{M_{Pl}} \right)^{\frac{1}{n-2}} M_{Pl}.$$  

If $m$ is at the weak scale, $n$ around 10 will give a VEV on the order of $M_{GUT}$. In models with approximate flatness we therefore hope to have exact flatness at the level of renormalizable terms, and to have the lowest allowable nonrenormalizable terms be of mass dimension 10.
In the case of a model where exact flat directions can be achieved, the value of $M_{\text{GUT}}$ is fixed by supersymmetry breaking, or possibly by some other mechanism.

**III. SINGLE GAUGE GROUPS**

The use of product groups is further motivated by the difficulty of obtaining flat theories from a single gauge group. The problem lies in that we want a flat potential, but don’t want massless modes below the GUT scale that will spoil the unification of couplings. Therefore, we need to generate mass-terms with a non-trivial potential and VEVs. Ideally, the potential we write would be one with terms that are allowed by some discrete symmetries or discrete R-symmetries. Consider $SU(5)$ as the gauge group. We will use only adjoint representations for GUT scale fields, and we do not want to add any explicit mass terms. In this case our superpotential will be limited to terms involving three adjoints per term, that is, terms of the form

$$\lambda_{ijk} A_i A_j A_k,$$

for a general set of adjoint fields $A_i$. In general, if the fields are allowed to acquire VEVs proportional to the generator of the $U(1)$ hypercharge subgroup

$$\langle A_i \rangle = a_i \left( \begin{array}{c} -1/3 \\ 0 \\ 1/2 \end{array} \right),$$

we find that the conditions for a zero potential, setting the auxiliary $F^\dagger$ fields equal to zero, are generally equal in number to the number of fields. It is in principle possible to choose coefficients of the terms in the superpotential in such a way that one or more of the conditions is redundant, and in that case there would be a free parameter describing the space of VEVs, that is, there would be a flat direction. It would not be natural to expect such a special set of coefficients. Furthermore, one could imagine making one or more condition trivial, that is

$$F^\dagger_i = 0,$$

thereby removing one of the conditions, however, in this case the corresponding multiplet remains massless. Thus, with the general number of conditions to minimize the potential equal to the number of parameters, all of the VEVs are completely determined and there are no flat directions in a general potential that leaves no massless modes.

We will find that with a product group, and the addition of bifundamental representations, many of the flatness conditions will be trivial for appropriately parameterized VEVs, and so the VEVs will not be so highly constrained, and flat directions will be possible, as we shall see an explicit example of in the next section.

**IV. AN $SU(5) \times SU(5)$ EXAMPLE**

The ideas of Witten regarding product groups and the requirement of approximate flat directions in the superpotential using only discrete symmetries have been successfully combined in [3]. In this section we review one of their models, and in the next we attempt a generalization of this model to $SO(10) \times SO(10)$.

A symmetry that is a linear combination of an ordinary discrete symmetry that commutes with $SU(5)$ and a discretized gauge transformation in one of the $SU(5)$’s is what is necessary
to split triplets from doublets in an $SU(5) \times SU(5)$ theory. For a $Z_N$ symmetry with $N$ taken to be odd, an appropriate gauge symmetry is

$$g_1 = \begin{pmatrix}
\alpha^{-1} & \alpha^{-1} & \alpha^{-1} \\
\alpha^{-1} & \alpha^{-1} & \alpha^{-1} \\
\alpha^{N+3} & \alpha^{N+3} & \alpha^{N+3}
\end{pmatrix},$$

(7)

where $\alpha$ is an $N$’th root of unity. One might introduce then as GUT fields representations that transform as bifundamentals under the product group. In particular one might take the fields $\Phi_i, \tilde{\Phi}_j, i, j = 1, 2$ and take them to transform under the $Z_N$ symmetry

$$\Phi_1 \rightarrow \alpha \Phi_1, \tilde{\Phi}_1 \rightarrow \alpha^{-1} \tilde{\Phi}_1, \Phi_2 \rightarrow \alpha^{-\frac{N+3}{2}} \Phi_2, \tilde{\Phi}_2 \rightarrow \alpha^{\frac{N+3}{2}} \tilde{\Phi}_2.$$  

(8)

The combined symmetry, $Z_N$, of the discrete gauge transform and $Z_N$ is preserved by the following VEVs in the bifundamental fields

$$\langle \Phi_1 \rangle = \langle \tilde{\Phi}_1 \rangle = \begin{pmatrix} v_1 & v_1 & 0 \\ v_1 & v_1 & 0 \end{pmatrix}, \quad \langle \Phi_2 \rangle = \langle \tilde{\Phi}_2 \rangle = \begin{pmatrix} 0 & 0 & v_2 \\ 0 & 0 & v_2 \end{pmatrix}.$$  

(9)

There are components of the bifundamental fields that are not eaten by the Higgs’s mechanism and need to be made massive by the superpotential without breaking flatness or adding any explicit mass parameters. This can be done simply by adding three adjoints of the first $SU(5)$, $A_i = 1, 2, 3$ and a gauge singlet, $S$, with the following potential

$$W = \lambda_{12} \Phi_1 A_1 \tilde{\Phi}_2 + \lambda_{21} \Phi_2 A_2 \tilde{\Phi}_1 + \lambda_{11} \Phi_1 A_1 \tilde{\Phi}_1 + \lambda_{22} \Phi_2 A_2 \tilde{\Phi}_2 + \eta_{12} SA_1 A_2 + \eta_{33} SA_3 A_3.$$  

(10)

The gauge singlet in this model gains a VEV $\langle S \rangle = s$, which is another flat direction of the potential.

Flatness may be broken by terms that are not forbidden by symmetry. It is possible to forbid all such terms with a continuous, global symmetry. However, with discrete symmetries it is at best possible to have approximate flat directions. Adding a discrete $R$ symmetry, $Z_N^R$, and assigning charges as in Table I one finds that the lowest dimensional flatness breaking terms are

$$\frac{1}{M_{Pl}^2} S^4(\Phi_1^2 \tilde{\Phi}_2^2), \frac{1}{M_{Pl}^2}(\Phi_1^2 \tilde{\Phi}_2^2)^2.$$  

(11)

These are terms of dimension 9 and 10, respectively, and as was argued in Section II and originally in [3], if the singlet acquires a negative mass-squared in supersymmetry breaking, the VEVs of these fields will be fixed near $M_{GUT}$.

Now add the following Higgs fields, $h$ and $\bar{h}'$ that transform under the gauge group as $(5, 1)$ and $(1, 5)$, respectively. Also, give $h$ charge one under $Z_N^R$, and give $\bar{h}'$ $R$-charge zero, as indicated in the table. This will then allow the following term

$$W_1 = h \Phi_1 \bar{h}'.$$  

(12)

This will then give mass to the triplet fields, and leave the doublets massless. There are some problems with this, however, because as it stands this theory is not anomaly free. Adding another pair of Higgs fields transforming in the opposite $SU(5)$ fixes this, but adds an extra massless doublet. One might also imagine cancelling the anomaly with standard model matter fields. These issues are discussed in greater detail in [3].
We want to consider now to what extent the model described in the previous section can be generalized to other groups. Given the success of $SO(10)$ unification, $SO(10) \times SO(10)$ is a natural place to start. We first do this somewhat naively, but will find that the generalization does not carry over as well as we might hope. In $SO(10) \times SO(10)$, much as in $SU(5) \times SU(5)$, we should first see how it might be possible to implement Witten’s ideas for solving the doublet-triplet splitting with a discrete symmetry. Suppose one has two bifundamentals of $SO(10) \times SO(10)$, $\Phi_1, \Phi_2$. The unbroken discrete symmetry may be a linear combination of a discrete symmetry acting on these bifundamentals, and a discrete gauge transformation. Taking this gauge transformation to be a discrete hypercharge transformation in one of the $SO(10)$’s but not the other will forbid only doublet masses as prescribed by Witten. The $SU(5)$ subgroup of a single $SO(10)$ is generated by the following generator of $SO(10)$ in the fundamental representation, written as a direct product of a $5 \times 5$ space and a $2 \times 2$

$$A_5 \otimes I_{2 \times 2} + S_5 \otimes i\sigma_2,$$

(13)

where $A_5$ is an antisymmetric $5 \times 5$ matrix, and $S_5$ symmetric and traceless, $\sigma_2$ is the second Pauli matrix and $I_{2 \times 2}$ the identity matrix. Knowing this we can find out how hypercharge acts in the $SO(10)$ multiplets. The generator of hypercharge is realized in the above notation by setting:

$$S_5 = \begin{pmatrix} -1 & -1 & -1 & 3/2 & 3/2 \\ -1 & -1 & 3/2 & 3/2 \\ -1 & 3/2 & 3/2 \\ 3/2 & 3/2 \\
\end{pmatrix},$$

(14)

which generates in $SO(10)$ the group transformation in the fundamental representation:

$$g_h = \begin{pmatrix} \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^3 \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^3 \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^3 \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^3 \\ \alpha^{-2} & \alpha^{-2} & \alpha^{-2} & \alpha^3 \\
\end{pmatrix},$$

(15)

\begin{table}
\begin{center}
\begin{tabular}{|c|c|}
\hline

Field & $SU(5) \times SU(5) \times Z_N \times Z_2^R$ \\
\hline
$\Phi_1$ & $(5, 5, 1, 0)$ \\
$\Phi_1^\dagger$ & $(5, 5, N - 1, 0)$ \\
$\Phi_2$ & $(5, 5, (N - 3)/2, 3)$ \\
$\Phi_2^\dagger$ & $(5, 5, (N + 3)/2, 8)$ \\
$A_1$ & $(24, 1, (N - 5)/2, 4)$ \\
$A_1^\dagger$ & $(24, 1, (N + 5)/2, 9)$ \\
$A_2$ & $(24, 1, 0, 1)$ \\
$S$ & $(1, 1, 0, 10)$ \\
h & $(5, 1, 0, 1)$ \\
$\bar{h}'$ & $(1, 5, 0, 0)$ \\
\hline
\end{tabular}
\end{center}
\end{table}
where $\alpha$ is a general two dimensional rotation matrix. We are allowed to combine a discrete hypercharge transformation with a $Z_N$ symmetry for the unbroken symmetry. The general discrete form of the hypercharge transformation that we might wish to use is, taking $N$ to be odd (a similar expression may be derived for even $N$),

$$g_1 = \begin{pmatrix}
\alpha^{-1} & \alpha^{-1} \\
\alpha^{-1} & \alpha^{N+3} \\
\alpha^{N+3} & \alpha^{-2}
\end{pmatrix},$$

(16)

where $\alpha$ is now a $2 \times 2$ rotation matrix such that $\alpha^N = 1$. For the product group $SO(10) \times SO(10)$, the discrete hypercharge trasformation may be in either subgroup, and without loss of generality we may take it to be a subgroup of the right $SO(10)$. Now take the bifundamentals to transform under the $Z_N$ symmetry

$$\Phi_1 \rightarrow 1_{5 \times 5} \otimes \alpha \Phi_1, \quad \Phi_2 \rightarrow 1_{5 \times 5} \otimes \alpha^{-\frac{N+3}{2}} \Phi_2,$$

(17)

where this transformation is in general a ten by ten matrix, written as a direct product as before. Note that this discrete symmetry does not commute with $SO(10)$, but does commute with the $SU(5)$ subgroup generated by eqn. (13). The symmetry $Z_N'$ which is a combination of this discrete symmetry and the hypercharge transformation in eqn. (16) will solve the doublet-triplet splitting problem in the manner prescribed by Witten. This symmetry is respected by the following VEVs, written in block diagonal form, of the bifundamentals

$$\langle \Phi_1 \rangle = i \ast v_1 \begin{pmatrix}
\sigma_2 & \sigma_2 \\
\sigma_2 & 0 \\
0 & 0
\end{pmatrix} + u_1 \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},$$

(18)

$$\langle \Phi_2 \rangle = i \ast v_2 \begin{pmatrix}
0 & 0 \\
0 & \sigma_2 \\
\sigma_2 & 0
\end{pmatrix} + u_2 \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.$$
group elements of each simple subgroup, for generalized parameters in the above VEVs. This implies the existence of extra gauge symmetries leftover in models of this type, and so the symmetry breaking has not in fact broken the GUT group sufficiently, to the groups of the Standard Model. The existence of such off diagonal symmetries may be seen indirectly in the following model through analysis of the Goldstone modes

\[ W = \lambda_1 \Phi_1 S_1 \Phi_1 + \lambda_2 \Phi_2 S_1 \Phi_2 + \lambda_3 \Phi_1 S_2 \Phi_2 + \lambda_4 \Phi_1 A \Phi_2 + \lambda_5 X AA + \lambda_6 X S_1 S_1 + \lambda_7 X S_2 S_2. \]  

(19)

The \( \Phi \) fields are the bifundamentals with VEVs as already described. The \( X \) field is a gauge singlet, and in general may acquire a VEV. The \( S \)'s are symmetric representations of, say, the first \( SO(10) \), and \( A \) is in the adjoint representation of the same. Although this model may be looked at as a generalization of the \( SU(5) \times SU(5) \) model described earlier, the point of studying it here is to see the effects of unbroken off-diagonal symmetries. The bifundamentals branch into a symmetric plus an adjoint plus a singlet in going from the product group to a single \( SO(10) \). The adjoints branching out of the bifundamentals as well as the explicit adjoint field contain the \((3, 1)\) and \((\bar{3}, 1)\) representations under the \( SU(3) \times SU(2) \) subgroup. Putting in the VEVs one can compute the following mass matrix for these modes

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} \lambda_4 v_1 \\
0 & \frac{1}{2} \lambda_4 v_1 & 2 \lambda_5 x
\end{pmatrix},
\]

(20)

which has only one massless mode for nontrivial VEVs. However, breaking \( SO(10) \times SO(10) \) to \( SU(3) \times SU(2) \times U(1) \) predicts two massless Goldstone modes in the above mass matrix, assuming no other unbroken symmetries exist. Clearly, that is not the case, and there do exist other unbroken symmetries outside of the unbroken symmetry in the diagonal subgroup. It is interesting to note as well that if \( v_1 = 0 \) in the bifundamental VEV, the correct number of massless modes is once again present. It turns out that for that particular choice, and also \( v_2 = 0 \), there are no off-diagonal symmetries, which we will now see from a more detailed analysis of the symmetries.

To investigate more precisely what symmetries may be unbroken, consider the following VEV of a single bifundamental field, which leaves an unbroken \( SU(5) \times U(1) \) in the diagonal subgroup:

\[
\langle \Phi \rangle = i * v \begin{pmatrix}
\sigma_2 \\
\sigma_2 \\
\sigma_2 \\
\sigma_2
\end{pmatrix}
\]

(21)

there exists a transformation that preserves this VEV but transforms differently in each \( SO(10) \), that is it is not in the diagonal subgroup. In fact for group elements expressed in the fundamental representation of each group, they differ only by a sign. Written as a group element in one of the groups this is:

\[
g = O_5 \otimes \sigma_1,
\]

(22)

where \( O_5 \) is any 5 by 5 orthogonal matrix, and the direct product is taken with the Pauli matrix, and the group element acting on group indices transforming under the second \( SO(10) \) has the opposite sign when acting in the fundamental representation. That is

\[
g = (O_5 \otimes \sigma_1, -O_5 \otimes \sigma_1)
\]

(23)
in the bifundamental representation. A similar group element can be written using \( \sigma_3 \). Thus, there exists a subgroup of \( SO(10) \times SO(10) \) that is not broken and is not entirely in the diagonal subgroup.

In the case of two bifundamentals with the VEVs chosen as in eq. (18) to try to break to the standard model, basically the same problem occurs for general choices of the parameters, as we have already seen indirectly from the Goldstone modes in the model above in eq. (19). However, we will find that there is one acceptable choice of parameters for which there are no preserved off-diagonal symmetries. For simplicity, consider again just one bifundamental, but give it the following VEV:

\[
\Phi_0 = \langle \Phi \rangle = i \ast v \begin{pmatrix}
\sigma_2 & \sigma_2 \\
\sigma_2 & \sigma_2 \\
\sigma_2 & \sigma_2 \\
\end{pmatrix} + u \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}.
\]

Now we can consider how this transforms under an infinitesimal \( SO(10) \times SO(10) \) transformation. In particular, the infinitesimal generators of transformations by the left \( SO(10) \) can be written as

\[
\delta \Phi = [A_2^L \otimes 1_{2 \times 2} + S_2^I \otimes i \sigma_2 + A_1^L \otimes \sigma_1 + A_3^L \otimes \sigma_3] \Phi
\]

and similarly the infinitesimal transformation from the right is

\[
\delta \Phi = \Phi[-A_2^R \otimes 1_{2 \times 2} + S_2^I \otimes i \sigma_2 - A_1^R \otimes \sigma_1 - A_3^R \otimes \sigma_3].
\]

The notation here is that the \( A \)'s are antisymmetric five by five matrices, and the \( S \)'s are symmetric. The infinitesimal transformation is then the direct products of dimension five and two matrices. The minus signs in the transformation from the right reflect that this is the transpose of the fundamental transformation (since we compare group elements in the fundamental representation to see if we are in a diagonal subgroup, we cannot just take the signs away as a matter of convention without putting signs in somewhere else). A general transformation that combines left and right may be written in terms of commutators and anti-commutators. Furthermore, we are not interested in the first two parts of each of the above transformations, because they can only generate diagonal transformations that are in the diagonal \( SU(5) \times U(1) \) subgroup. The combined transformation with only the \( \sigma_1 \) and \( \sigma_3 \) parts can be written as follows

\[
\delta \Phi = [A_1 \otimes \sigma_1, \Phi] + \{ \tilde{A}_1 \otimes \sigma_1, \Phi \} + [A_3 \otimes \sigma_3, \Phi] + \{ \tilde{A}_3 \otimes \sigma_3, \Phi \},
\]

where the five by five antisymmetric matrices above are linear functions of \( A_1^R, A_1^L, A_3^R, \) and \( A_3^L \) in the original left and right transformations. For \( A_1 = (\frac{2u}{v^2}) \tilde{A}_3 \) and \( A_3 = -(\frac{2u}{v^2}) \tilde{A}_1 \) this preserves the above VEV: \( \langle \Phi \rangle = \Phi_0 \). We may rewrite this as follows

\[
\delta \Phi = (2u[A \otimes \sigma_1, \Phi] + v\{A \otimes \sigma_3, \Phi\}) + (2u[\tilde{A} \otimes \sigma_3, \Phi] - v\{\tilde{A} \otimes \sigma_1, \Phi\}).
\]

The transformation generated above is in general not in the diagonal subgroup of the dual group, with the exception that if \( v = 0 \) this transformation is just \( A_1^L = A_1^R \) and \( A_3^L = A_3^R \) in the notation of the fundamental, dual group. This latter case corresponds to merely having broken \( SO(10) \times SO(10) \) to a single, diagonal \( SO(10) \). If we now change our VEV
to the original form in eq. (18) and follow the same procedure as above we find that if $v_1 = v_2 = 0$ there are no off diagonal symmetries and the preserved diagonal subgroup is $SO(6) \times SO(4) \times U(1)$. This situation is somewhat analogous to what happens in single $SO(10)$ GUTs when a field transforming as a symmetric $(54)$ is used to break the symmetry \[19\]. Just as in the case of breaking with an adjoint field, an additional rank breaking sector is required when the GUT breaking field is a symmetric $(54)$. A rank breaking sector will also work for dual group models because the dual symmetry is already broken to a diagonal subgroup.

VI. RANK BREAKING AND FLATNESS

In single $SO(10)$ models a $16 + \overline{16}$ or $126 + \overline{126}$ sector can be used to break the rank of $SO(10)$ \[4\]. As was found above, a rank breaking sector consisting of the appropriate fields that transform under either group is what is necessary. For our models we require a flat potential for the fields that break the rank of $SO(10)$. Specifically, we require that no VEVs are fixed and no GUT scale masses are put in as input parameters. We need to satisfy these requirements, or do so approximately with possible nonrenormalizable terms breaking flatness as in the section above, to allow for a natural way of generating the GUT scale. We also need to make sure that all the fields are massive. It will be found, however, that it is impossible to construct a rank breaking sector satisfying these criteria. One other requirement that should be mentioned is that the rank breaking sector must couple to the rest of the theory involving bifundamentals, without spoiling their VEVs, in order to avoid light pseudogoldstone bosons.

To start off assume we use $16 + \overline{16}$ fields to break the rank, and we will see that the same issues will carry over to the $126 + \overline{126}$ case. The programme we are to follow then is to try to invent terms and add fields as needed to try to give mass to all the fields without fixing the $SU(5)$ singlet VEVs of the $16$s or using explicit mass terms (e.g. all masses come from VEVs, the fundamental theory has no mass terms). Specifically, we might look at the $16$ field’s branching rules into $SU(3) \times SU(2)$ from the standard model, without worrying about hypercharge.

\[16 = \overline{5} + 10 + 1 = (\overline{3}, 1) + (1, 2) + (3, 1) + (3, 2) + \text{Singlets} \] \hspace{1cm} (29)

Here we have first written the $SU(5)$ fields and then the branching rules for SM representations. The doublet fields are of particular interest, for the simple reason that making them massive is very difficult under the requirement of flat potentials and no explicit mass terms. Furthermore, in the Standard Model there are no doublets eaten by the Higgs mechanism, so it is a requirement that these doublets acquire masses from the VEVs.

Firstly, there does not appear to exist another representation of $SO(10)$ that contains a standard model doublet and can couple to the $16 + \overline{16}$. This assertion can be verified up to well studied representations using tables of branching rules, such as those in \[6\]. The Standard Model doublets generally branch out of the $5$ and $\overline{5}$ representations of $SU(5)$. The $120$ of $SO(10)$, for example, contains both a $5$ and a $\overline{5}$, but it is a three index antisymmetric tensor, and by itself cannot couple to the $16 + \overline{16}$. One can look at some higher representations and find similar results. It is rather difficult to find the right field because we are required to use a field that couples to both $16$ and $\overline{16}$ and we are restricted to products of three fields in the superpotential. In any case, if we restrict ourselves to representations commonly used
in $SO(10)$ model building, in particular the adjoint and symmetric tensor, then we know there is no field with doublets that we can couple to the $16 + \bar{16}$. As a result, the only way to make the doublets massive is to couple the 16's to some other field that acquires a VEV, and then hope the VEV generates sufficient mass terms. The 16 fields’ $F$-fields equal to zero require, however, that there is more than one coupling to these fields, because were there only one coupling, the $F$-fields require that the field coupled to has zero VEV. That is to say, a coupling of the form $\bar{C}AC$ with no other fields coupling to the $16 + \bar{16}$ fields implies that $A$ has zero VEV, or at the very least no VEV that could give mass to doublets.

The only remaining possibility is to add another field to couple to the 16s. This field is necessarily in a different representation of the symmetry group because otherwise we could redefine the fields in such a way that there is only one coupling, again. However, we will find that in general this will fix all the VEVs, in particular the 16s’ VEVs are constrained by the $F$-fields for each field they couple to. With two such constraints, the 16 VEV is fixed except for very special coincidences of coupling constants which could not be justified in a natural way, in the sense that there is no symmetry to justify such special choices of coefficients.

To see this, consider just two fields coupling to the 16’s, and these two couple only to each other. Then there are two constraints coming from setting the $F$”s equal to zero for the two added fields, and the constraint coming from the 16’s, which is generally constrained. Adding fields adds constraints and additional VEV parameters in equal number, generally, so the VEVs have fixed values in general.

The 126 also contains a single Standard Model doublet, and the issues with coupling to other fields are similar to the 16’s, so the same arguments generally hold for the 126 rank breaking fields.

One might also ask whether it might be possible to introduce some generalization of bifundamental fields, that is, fields that transform under both groups, but generalizing this beyond the fundamental representation of each group. The problem is, we still need the bifundamental fields we introduced before to break to $SO(6) \times SO(4) \times U(1)$. Once that is done all the fields transform in the diagonal subgroup, and we are left with essentially the same predicament.

VII. CONCLUSIONS

With the requirement of a flat potential so that $M_{GUT}$ is a modulus we have shown that models with $SO(10) \times SO(10)$ gauge group are impossible to construct. We also have shown that models with a single $SU(5)$ gauge group without input mass terms do not allow flat potentials, and this appears to be general to simple groups without exotic representations in the theory. This means that the GUT masses are still functions of parameters we put in by hand in theories with a simple gauge group. Product groups provide a natural way to solve the problem of splitting doublets from triplets, as well as in the case of $SU(5) \times SU(5)$ providing a way to break the unification group at the GUT scale that leaves the scale of this breaking a modulus. What we have essentially shown is that this set of ideas used successfully for $SU(5) \times SU(5)$ cannot be generalized to $SO(10) \times SO(10)$, nor groups containing $SO(10) \times SO(10)$ because ultimately these kinds of models would run into the
same problems with rank breaking.

[1] E. Witten [arXiv:hep-ph/0201018
[2] S. M. Barr Phys. Rev. D 55, 6775 (1997)
[3] M. Dine, Y. Nir and Y Shadmi Phys. Rev. D 66, 115001 (2002)
[4] H. Georgi “Lie Algebras in Particle Physics,” Reading, MA: Perseus Books (1999) (Frontiers In Physics) (Second Edition).
[5] J. Wess and J. Bagger Princeton, NJ: University Press (1992)
[6] R. Slansky Phys. Rept. 79, 1-128 (1981)
[7] K. S. Babu and S. M. Barr Phys. Rev. D 51, 2463 (1995)
[8] K. S. Babu and S. M. Barr Phys. Rev. D 48, 5354 (1993)
[9] K. S. Babu and S. M. Barr Phys. Rev. D 50, 3529 (1994)
[10] L. J. Hall and S. Raby Phys. Rev. D 51, 6524 (1995)
[11] D. Bailin and A. Love “Supersymmetric Gauge Field Theory and String Theory,” London, UK: Institute of Physics Publishing (1994) (Graduate Student Series In Physics).