LOOPS IN SURFACES AND STAR-FILLINGS

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Abstract. We discuss a new approach to computing the standard algebraic operations on homotopy classes of loops in a surface: the homological intersection number, Goldman’s Lie bracket, and the author’s Lie cobracket. Our approach uses fillings of the surface by certain graphs.

1. Introduction

The homological intersection number of loops in an oriented surface $\Gamma$ is computed by deforming these loops into a transversal position and counting their intersections with signs determined by the orientation of $\Gamma$. This defines a skew-symmetric bilinear form $\cdot:\Gamma H_1(\Gamma) \times H_1(\Gamma) \rightarrow \mathbb{Z}$. A subtler way to count intersections of loops leads to Goldman’s Lie bracket in the module generated by free homotopy classes of loops in $\Gamma$, see [Go1], [Go2]. This bracket was complemented in [Tu1] by a Lie cobracket defined in terms of self-intersections of loops; for recent work on these operations see [AKKN1], [AKKN2], [Ha1], [Ha2], [Ka], [KK1], [KK2], [LS], [Ma]. For compact $\Gamma$, the duality isomorphism $H_1(\Gamma) \approx H_1(\Gamma, \partial \Gamma)$ carries the form $\cdot:\Gamma$ into the composition of the cup-product $\cup : H^1(\Gamma, \partial \Gamma) \times H^1(\Gamma, \partial \Gamma) \rightarrow H^2(\Gamma, \partial \Gamma)$ with the linear map $H^2(\Gamma, \partial \Gamma) \rightarrow \mathbb{Z}$ evaluating 2-cohomology classes on the fundamental class of $\Gamma$. So, the intersection number of two elements of $H_1(\Gamma)$ can be computed by taking the dual cohomology classes, representing them by 1-cocycles on a triangulation of $\Gamma$, evaluating the cup-product of these 1-cocycles on all triangles of the triangulation, and summing up the resulting values. This method allows us to compute the algebraic number of intersections of loops without ever considering the intersections themselves. The aim of this paper is to give similar computations of the Lie bracket and cobracket mentioned above. To do it, we switch from triangulations to a more flexible language of graphs in surfaces.

In the rest of the introduction we focus on the case of surfaces with non-void boundary. Let $\Gamma$ be a compact connected oriented surface with $\partial \Gamma \neq \emptyset$. By a star we mean an oriented graph formed by $n \geq 2$ vertices of degree 1 (the leaves), a vertex of degree $n$ (the center), and $n$ edges leading from the center to the leaves. The set of leaves of a star $s$ is denoted by $\partial s$. A star $s$ in $\Gamma$ is a star embedded in $\Gamma$ so that $s \cap \partial \Gamma = \partial s$. The orientation of $\Gamma$ at the center of $s$ determines a cyclic order in the set Edg($s$) of edges of $s$. For $e \in$ Edg($s$) we let $e^+ \in$ Edg($s$) be the next edge with respect to this order. We say that a loop in $\Gamma$ is s-generic if it misses the vertices of $s$, meets all edges of $s$ transversely, and never traverses a point of $s$ more than once. It is clear that any loop in $\Gamma$ can be made s-generic by a small deformation. Given an s-generic loop $a$ in $\Gamma$ and an edge $e \in$ Edg($s$) we let $a \cap e$ be the set of points of $e$ traversed by $a$. For $p \in a \cap e$, the intersection sign of $a$ and $e$ at $p$ is denoted by $\mu_p(a)$. The integer $a \cdot e = \sum_{p \in a \cap e} \mu_p(a)$ is the algebraic
number of intersections of $a$ and $e$. For $s$-generic loops $a, b$ in $\Gamma$, set
\[ a \cdot_s b = \sum_{e \in \text{Edg}(s)} ((a \cdot e)(b \cdot e^\perp) - (b \cdot e)(a \cdot e^\perp)). \]

**Theorem 1.1.** The map $(a, b) \mapsto a \cdot_s b$ defines a skew-symmetric bilinear form $\cdot_s : H_1(\Gamma) \times H_1(\Gamma) \to \mathbb{Z}$ depending only on the isotopy class of the star $s$ in $\Gamma$.

The idea behind this theorem is to view each pair of points $p \in a \cap e, q \in b \cap e^\perp$ as a pseudo-intersection of $a$ and $b$, and to consider the (skew-symmetrized) algebraic number of such pairs. To recover the standard homological intersection form $\cdot_\Gamma$ in $H_1(\Gamma)$, we need one more notion. A **star-filling** of $\Gamma$ is a finite family $F$ of disjoint stars in $\Gamma$ such that each component of the set $\Gamma \setminus \bigcup_{s \in F} s$ is a disk meeting $\partial \Gamma$ at one or two open segments. We explain in the body of the paper that $\Gamma$ has star-fillings. The following theorem computes the intersection form $\cdot_\Gamma$ in terms of star-fillings.

**Theorem 1.2.** For any star-filling $F$ of $\Gamma$, we have
\[
2 \cdot_\Gamma = \sum_{s \in F} \cdot_s : H_1(\Gamma) \times H_1(\Gamma) \to \mathbb{Z}.
\]

So, for any $x, y \in H_1(\Gamma)$, the integer $\sum_{s \in F} x \cdot_s y$ is even and is equal to $2 \cdot_\Gamma y$. As a consequence, the form $\cdot_\Gamma$ can be fully recovered from the forms $\{\cdot_s\}_{s \in F}$. Similar remarks apply to our computations of the bracket and cobracket below.

To state an analogue of Theorems 1.1 and 1.2 for Goldman’s Lie bracket, we first recall the definition of this bracket. Let $\mathcal{L} = \mathcal{L}(\Gamma)$ be the set of free homotopy classes of loops in $\Gamma$ and let $M = M(\Gamma)$ be the free abelian group with basis $\mathcal{L}$. For a loop $a$ in $\Gamma$, we let $\langle a \rangle \in \mathcal{L} \subset M$ be the free homotopy class of $a$. For any point $p \in \Gamma$ traversed by $a$ once, we let $a_p$ be the loop starting at $p$ and going along $a$ until the return to $p$. We say that a pair of loops $a, b$ in $\Gamma$ is **generic** if these loops are transversal and do not meet at self-intersections of $a$ or $b$. Then the (finite) set of intersection points of $a, b$ is denoted by $a \cap b$. For $p \in a \cap b$, we let $\varepsilon_p(a, b) = \pm 1$ be the intersection sign of $a$ and $b$ at $p$. Also, $a_pb_p$ stands for the product of the loops $a_p, b_p$ based at $p$. Goldman’s bracket is the bilinear form $[\cdot, \cdot] : M \times M \to M$ defined on the basis $\mathcal{L} \subset M$ by
\[
[(a), (b)]_\Gamma = \sum_{p \in a \cap b} \varepsilon_p(a, b)\langle a_pb_p \rangle
\]
for any generic pair of loops $a, b$ in $\Gamma$. The bracket $[\cdot, \cdot]_\Gamma$ is a well-defined homotopy lift of the form $\cdot_\Gamma$ in $H_1(\Gamma)$.

With any star $s$ in $\Gamma$ we now associate a bilinear map $[(\cdot), (\cdot)]_s : M \times M \to M$. For any $s$-generic loops $a, b$ in $\Gamma$ and any points $p, q \in s$ traversed respectively by $a, b$, we pick a path $c$ from $p$ to $q$ in $s$ and write $a_pb_q$ for the loop $a_pcb_qc^{-1}$. Clearly, the free homotopy class of this loop does not depend on the choice of $c$. Set
\[
[a, b]_s = \sum_{c \in \text{Edg}(s)} \left( \sum_{p \in a \cap c, q \in b \cap c^+} \mu_p(a) \mu_q(b) \langle a_pb_q \rangle - \sum_{p \in a \cap c^+, q \in b \cap c} \mu_p(a) \mu_q(b) \langle a_pb_q \rangle \right).
\]

**Theorem 1.3.** The map $(\langle a \rangle, \langle b \rangle) \mapsto [a, b]_s$ defines a skew-symmetric bilinear form $[\cdot, \cdot]_s : M \times M \to M$ depending only on the isotopy class of $s$ in $\Gamma$. For any star-filling $F$ of $\Gamma$, we have
\[
2 [\cdot, \cdot]_\Gamma = \sum_{s \in F} [\cdot, \cdot]_s.
\]
One may ask whether the bracket $[\cdot,\cdot]_s$ shares the fundamental properties of Goldman’s bracket, namely, whether it satisfies the Jacobi identity and induces Poisson brackets on the moduli spaces of $\Gamma$. In general, the answer to both questions is negative though some weaker results hold true, see [Tu2]. Note also that in analogy with Goldman’s bracket, Kawazumi and Kuno [KK1] defined an action of the Lie algebra $M$ on the modules generated by homotopy classes of paths in $\Gamma$ with endpoints in $\partial \Gamma$. The Kawazumi-Kuno action can be computed similarly to Theorem 1.3 in terms of star-fillings. The same methods work for the double brackets of surfaces defined in $[MT]$, this will be discussed elsewhere.

We next state our results concerning the Lie cobracket on loops in $\Gamma$, see [Tu1]. For any star $s$ in $\Gamma$ we define a linear map $\nu_s : M \to M \otimes M$ depending only on the isotopy class of $s$. For any $\Gamma$-generic loop $a$ in $\Gamma$ we define a skew-symmetric cobracket $\nu_T : M \to M \otimes M$ defined on the basis $\mathcal{L} \subset M$ by

$$\nu_T(\langle a \rangle) = \sum_{r \in \#a} \langle a^1_r \rangle_0 \otimes \langle a^2_r \rangle_0 - \langle a^2_r \rangle_0 \otimes \langle a^1_r \rangle_0$$

for any generic loop $a$ in $\Gamma$. The cobracket $\nu_T$ is skew-symmetric in the sense that its composition with the permutation in $M \otimes M$ is equal to $-\nu_T$.

For any star $s$ in $\Gamma$ we define a linear map $\nu_s : M \to M \otimes M$. Given an $s$-generic loop $a$ in $\Gamma$ and points $p_1, p_2$ of $s$ traversed by $a$, we write $a_{p_1, p_2}$ for the loop going from $p_1$ to $p_2$ along $a$ and then going back to $p_1$ along a path in $s$. The free homotopy class of this loop does not depend on the choice of the latter path. Set

$$\nu_s(\langle a \rangle) = \sum_{e \in \text{Edg}(s)} \sum_{p_1 \in a \cap e, p_2 \in a \cap e^+} \mu_{p_1}(a) \mu_{p_2}(a) \left( \langle a_{p_1, p_2} \rangle_0 \otimes \langle a_{p_2, p_1} \rangle_0 - \langle a_{p_2, p_1} \rangle_0 \otimes \langle a_{p_1, p_2} \rangle_0 \right).$$

**Theorem 1.4.** The map $\langle a \rangle \mapsto \nu_s(\langle a \rangle)$ defines a skew-symmetric cobracket $M \to M \otimes M$ depending only on the isotopy class of $s$ in $\Gamma$. For any star-filling $F$ of $\Gamma$, we have

$$2 \nu_T = \sum_{s \in F} \nu_s.$$  

The cobracket $\nu_T$ has a refinement depending on a framing of $\Gamma$, see [Tu1], Section 18.1 and [AKKN2]; it would be interesting to extend Theorem 1.4 to this refined cobracket.

Theorems 1.1, 1.3 are proved in Section 8 where we also discuss the case of closed surfaces. The proofs of Theorems 1.1, 1.4 are based on the theory of quasi-surfaces developed in Sections 2, 7.

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2. **Quasi-surfaces**
2.1. Basics. By a surface we mean a smooth 2-dimensional manifold with boundary. A quasi-surface is a topological space $X$ obtained by gluing an oriented surface $\Sigma$ to a topological space $Y$ along a continuous map $\alpha \to Y$ where $\alpha \subset \partial \Sigma$ is the union of a finite number of disjoint segments in $\partial \Sigma$. Clearly, $Y \subset X$ and $X \setminus Y = \Sigma \setminus \alpha$. We call $Y$ the singular core of $X$ and call $\Sigma$ the surface core of $X$. We fix a closed neighborhood of $\alpha$ in $\Sigma$ and identify it with $\alpha \times [-1,1]$ so that

$$\alpha = \alpha \times \{-1\} \quad \text{and} \quad \partial \Sigma \cap (\alpha \times [-1,1]) = \alpha \cup (\partial \alpha \times [-1,1]).$$

The surface

$$\Sigma' = \Sigma \setminus (\alpha \times [-1,0]) \subset \Sigma \setminus \alpha \subset X$$

is a copy of $\Sigma$ embedded in $X$. We provide $\Sigma'$ with the orientation induced from that of $\Sigma$ and call $\Sigma'$ the reduced surface core of $X$.

Set $\pi_0 = \pi_0(\alpha) = \pi_0(\alpha \times \{0\})$. For $k \in \pi_0$, we let $\alpha_k$ be the corresponding segment component of $\alpha \times \{0\} \subset \partial \Sigma' \subset X$. We call $\alpha_k$ the $k$-th gate of $X$. The gates $\{\alpha_k\}_{k \in \pi_0}$ separate $\Sigma' \subset X$ from the rest of $X$. A gate orientation of $X$ is an orientation of all gates. Gate orientations of $X$ canonically correspond to orientations of the 1-manifold $\alpha$. For a gate orientation $\omega$ of $X$, we let $\overline{\omega}$ be the gate orientation of $X$ opposite to $\omega$ on all gates.

We keep the notation $X,Y,\alpha,\Sigma,\Sigma',\{\alpha_k\}_k,\pi_0$ till the end of Section 7.

2.2. Loops in $X$. By a loop in $X$ we mean a continuous map $a : S^1 \to X$. A generic loop $a$ in $X$ is a loop in $X$ such that (i) all branches of $a$ in $\Sigma'$ are smooth immersions meeting $\partial \Sigma'$ transversely at a finite set of points lying in the interior of the gates, and (ii) all self-intersections of $a$ in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma') = \Sigma' \setminus \partial \Sigma'$. The set of self-intersections in $\Sigma'$ of a generic loop $a$ is denoted by $\# a$. This set is finite and lies in $\text{Int}(\Sigma')$. Using cylinder neighborhoods of the gates, it is easy to see that any loop in $X$ may be transformed into a generic loop by a small deformation.

More generally, a finite family of loops in $X$ is generic if these loops are generic and all their intersections in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma')$. In particular, these loops can not meet at the gates. As above, any finite family of loops in $X$ may be transformed into a generic family by a small deformation.

For a loop $a$ in $X$ and any point $p$ of $a(S^1)$ which is not a self-intersection of $a$ (i.e., which is traversed by $a$ only once), we let $a_p$ be the loop which starts at $p$ and goes along $a$ until coming back to $p$. For any $k \in \pi_0$, we set $a \cap \alpha_k = a(S^1) \cap \alpha_k$. If the loop $a$ is generic then it never traverses a point of $a \cap \alpha_k$ more than once and the set $a \cap \alpha_k$ is finite.

2.3. Local moves. We define six local moves $L_0-L_5$ on a generic loop $a$ in $X$ keeping its free homotopy class. The move $L_0$ is a deformation of $a$ in the class of generic loops. This move preserves the number $\text{card}(\# a)$. The moves $L_1-L_3$ modify $a$ in a small disk in $\text{Int}(\Sigma')$. The move $L_1$ adds a small curl to $a$ and increases $\text{card}(\# a)$ by 1. The move $L_2$ pushes a branch of $a$ across another branch of $a$ increasing $\text{card}(\# a)$ by 2. The move $L_3$ pushes a branch of $a$ across a double point of $a$ keeping $\text{card}(\# a)$. The moves $L_4,L_5$ modify $a$ in a neighborhood of a gate $\alpha_k$ in $X$. The move $L_4$ pushes a branch of $a$ across $\alpha_k$ increasing $\text{card}(a \cap \alpha_k)$ by 2 and keeping $\text{card}(\# a)$. The move $L_5$ pushes a double point of $a$ across $\alpha_k$ keeping $\text{card}(a \cap \alpha_k)$ and decreasing $\text{card}(\# a)$ by 1. We call the moves $L_0-L_5$ and
their inverses loop moves. It is clear that generic loops in $X$ are freely homotopic if and only if they can be related by a finite sequence of loop moves.

3. Homological intersection forms

We define homological intersection forms in $H_1(X)$. Here and below, by $H_1(X)$ we mean the 1-homology of the underlying topological space of the quasi-surface $X$.

3.1. The intersection form of $(X, \omega)$. Given a gate orientation $\omega$ of $X$, we define a bilinear form in $H_1(X)$ called the homological intersection form of the pair $(X, \omega)$. The idea is to properly position the loops in $X$ near the gates and then to count their algebraic number of intersections in $\Sigma'$. We begin with definitions.

For any points $p, q$ of a gate $\alpha_k$, we say that $p$ lies on the $\omega$-left of $q$ and write $p <_\omega q$ if $p \neq q$ and the $\omega$-orientation of $\alpha_k$ leads from $p$ to $q$. We say that an ordered pair of loops $a, b$ in $X$ is $\omega$-admissible if it is generic (in the sense of Section 2.2) and the crossings of $a$ with any gate lie on the $\omega$-left of the crossings of $b$ with this gate. Taking a generic pair of loops $a, b$ in $X$ and pushing the branches of $a$ crossing the gates to the $\omega$-left of the crossings of $b$ with the gates, we obtain an $\omega$-admissible pair of loops. Thus, any pair of loops in $X$ may be deformed into an $\omega$-admissible pair.

For each generic pair of loops $a, b$ in $X$, we consider the finite set

$$a \cap b = a(S^1) \cap b(S^1) \cap \Sigma' \subset \text{Int}(\Sigma').$$

For $r \in a \cap b$, set $\varepsilon_r(a, b) = 1$ if the tangent vectors of $a$ and $b$ at $r$ form a positive basis in the tangent space of $\Sigma'$ at $r$ and set $\varepsilon_r(a, b) = -1$ otherwise.

Lemma 3.1. For any $\omega$-admissible pair $a, b$ of loops in $X$, the integer

$$a \bullet_{X, \omega} b = \sum_{r \in a \cap b} \varepsilon_r(a, b)$$

depends only on the homology classes of $a, b$ in $H_1(X)$. The formula $(a, b) \mapsto a \bullet_{X, \omega} b$ defines a bilinear form

$$\bullet_{X, \omega} : H_1(X) \times H_1(X) \to \mathbb{Z}$$

Proof. For each $k \in \pi_0$, one endpoint of the gate $\alpha_k$ lies on the $\omega$-left of the other endpoint. Pick disjoint closed segments $\alpha_k^- \subset \alpha_k$ and $\alpha_k^+ \subset \alpha_k$ containing these two endpoints respectively. Clearly, $p <_\omega q$ for all $p \in \alpha_k^-$ and $q \in \alpha_k^+$. We say that a loop in $X$ is $\omega$-left (respectively, $\omega$-right) if it is generic and meets the gates of $X$ only at points of $\cup_k \alpha_k^-$ (respectively, of $\cup_k \alpha_k^+$). Given an $\omega$-admissible pair of loops $a, b$ in $X$, we can push the branches of $a$ crossing the gates to the left and push the branches of $b$ crossing the gates to the right without creating or destroying intersections between $a$ and $b$. Consequently, $a$ is homotopic (in fact, isotopic) to an $\omega$-left loop $a'$ and $b$ is homotopic to an $\omega$-right loop $b'$ such that $a \bullet_{X, \omega} b = a' \bullet_{X, \omega} b'$. Since $\alpha_k^-$ is a deformation retract of $\alpha_k$ for all $k$, any $\omega$-left loops homotopic in $X$ are homotopic in the class of $\omega$-left loops. Similarly, any $\omega$-right loops homotopic in $X$ are homotopic in the class of $\omega$-right loops. Such homotopies of the loops $a', b'$ expand as compositions of loop moves keeping $a', b'$ respectively $\omega$-left and $\omega$-right. The latter moves obviously preserve $a' \bullet_{X, \omega} b'$. Therefore the integer $a \bullet_{X, \omega} b = a' \bullet_{X, \omega} b'$ depends only on the (free) homotopy classes of $a, b$ in $X$. Moreover, since $a \bullet_{X, \omega} b$ depends linearly on $a$ and $b$, it depends only on the homology classes of $a, b$. This implies the claim of the lemma. \qed
We stress that the crossings in $X \setminus \Sigma'$ of an $\omega$-admissible pair of loops $a, b$ do not contribute to $a \bullet_{X,\omega} b$. Note also that for such $a, b$, the pair $b, a$ is $\overline{\omega}$-admissible. Using these pairs to compute $a \bullet_{X,\omega} b$ and $b \bullet_{X,\omega} a$, we obtain the same terms with opposite signs. Hence, for any $x, y \in H_1(X)$,

\[(3.1.2)\quad x \bullet_{X,\omega} y = -y \bullet_{X,\omega} x.\]

3.2. The intersection form of $X$. We state the main result of this section.

**Theorem 3.2.** The skew-symmetric bilinear form $\bullet_X : H_1(X) \times H_1(X) \to \mathbb{Z}$ defined by

\[x \bullet_X y = x \bullet_{X,\omega} y - y \bullet_{X,\omega} x\]

for all $x, y \in H_1(X)$ and a gate orientation $\omega$ of $X$ does not depend on $\omega$.

We will prove this theorem in Section 3.3. We call $\bullet_X$ the **homological intersection form** of $X$. Both $\bullet_{X,\omega}$ and $\bullet_X$ generalize the intersection form in the homology of $\Sigma'$: the value of $\bullet_{X,\omega}$ (respectively, of $\bullet_X$) on any pair of homology classes of loops in $\Sigma' \subset X$ is equal to the usual intersection number of these loops in $\Sigma'$ (respectively, twice this number). The image of the inclusion homomorphism $H_1(Y) \to H_1(X)$ annihilates both $\bullet_{X,\omega}$ and $\bullet_X$.

To prove Theorem 3.2 we study the dependence of $\bullet_{X,\omega}$ on $\omega$. We start with notation. For a generic loop $a$ in $X$, the sign $\varepsilon_p(a)$ of $a$ at a point $p \in a \cap \alpha_k$ is $+1$ if $a$ goes near $p$ from $X \setminus \Sigma'$ to $\text{Int}(\Sigma')$ and $-1$ otherwise. The linear map $v_k : H_1(X) \to \mathbb{Z}$ “dual” to the gate $\alpha_k$ carries the homology class of any generic loop $a$ to $\sum_{p \in a \cap \alpha_k} \varepsilon_p(a)$. For any loops $a, b$ in $X$, we define the set of triples

\[T(a, b) = \{(k, p, q) \mid k \in \pi_0, p \in a \cap \alpha_k, q \in b \cap \alpha_k, p \neq q\} .\]

Given a gate orientation $\omega$ of $X$, we set

\[T_\omega(a, b) = \{(k, p, q) \in T(a, b) \mid q < \omega p \} \subset T(a, b) .\]

For $k \in \pi_0$, we set $\varepsilon(\omega, k) = +1$ if the $\omega$-orientation of $\alpha_k$ is compatible with the orientation of $\Sigma'$, i.e., if the pair $(a \omega$-positive tangent vector of $\alpha_k \subset \partial \Sigma'$, a vector directed inside $\Sigma'$) is positively oriented in $\Sigma'$. Otherwise, $\varepsilon(\omega, k) = -1$.

**Lemma 3.3.** For any gate orientation $\omega$ and any $x, y \in H_1(X)$ represented by a generic pair of loops $a, b$ in $X$, we have

\[(3.2.1)\quad x \bullet_{X,\omega} y = \sum_{r \in a' \cap b'} \varepsilon_r(a, b) + \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b).\]

**Proof.** Consider an $\omega$-admissible pair of loops $a', b$ where $a'$ is obtained from $a$ by pushing its branches crossing the gates to the $\omega$-left of the branches of $b$ crossing the gates. This transformation modifies $a$ in a small neighborhood of the gates so that $a', b$ have the same intersections in $\Sigma'$ as $a, b$ plus one additional intersection $r = r(k, p, q) \in \Sigma'$ for each triple $(k, p, q) \in T_\omega(a, b)$. It is easy to check that $\varepsilon_r(a', b) = \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b)$. Consequently,

\[x \bullet_{X,\omega} y = a' \bullet_{X,\omega} b = \sum_{r \in a' \cap b'} \varepsilon_r(a', b) \]

\[= \sum_{r \in a \cap b'} \varepsilon_r(a, b) + \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b).\]

\[\square\]
at their intersections in $\Sigma'$ under all loop moves on $a, b$
the right-hand side of (4.1.1) is an algebraic sum of all possible ways to graft $\omega$
by an $\omega$
Pick a gate orientation $\omega$
the quasi-surface $X$
The Computation of $3.4.$

(4.1.1) $\]$
x, y
Formula (3.3.1) implies that $l$ for all $a$
The pair $a, b$ is $\omega$-admissible, the set $T_{\omega}(a, b)$ consists of all triples $(l, p, q)$ with $p \in a \cap \alpha_l, q \in b \cap \alpha_l$. Therefore
$x \bullet_{X, l} y = x \bullet_{X, \omega} y + \sum_{p \in a \cap \alpha_l, q \in b \cap \alpha_l} \varepsilon(l, q) \varepsilon_p(a) \varepsilon_q(b) \varepsilon(\omega, l) v_l(x) v_l(y).
$\]
$x \bullet_{X, \omega} y - \varepsilon(\omega, l) v_l(x) v_l(y).
Formula (3.3.1) implies that
$x \bullet_{X, l} y - y \bullet_{X, l} x = x \bullet_{X, \omega} y - y \bullet_{X, \omega} x$
for all $l \in \pi_0$. This implies the claim of the theorem.

3.4. Computation of $\bullet_{X, \omega}$. To compute $x \bullet_{X, \omega} y$ for $x, y \in H_1(X)$ we will use the following method. Pick a generic pair of loops $a, b$ in $X$ representing $x, y$. Let $\omega_0$ be the orientation of the gates induced by the orientation of $\Sigma' \subset \Sigma$ so that $\varepsilon(\omega_0, k) = 1$ for all $k \in \pi_0$. Lemma 3.3 implies that
$x \bullet_{X} y = x \bullet_{X, \omega_0} y - y \bullet_{X, \omega_0} x
= 2 \sum_{r \in a \cap b} \varepsilon_r(a, b) + \sum_{(k, p, q) \in T(a, b)} \delta(p, q) \varepsilon_p(a) \varepsilon_q(b)$
where $\delta(p, q) = 1$ for $q <_\omega p$ and $\delta(p, q) = -1$ for $p <_\omega q$. If $a \cap b = \emptyset$, then
(3.4.1) $x \bullet_{X} y = \sum_{(k, p, q) \in T(a, b)} \delta(p, q) \varepsilon_p(a) \varepsilon_q(b).
$

4. The intersection brackets

We define homotopy intersection brackets refining the homological forms above.

4.1. The brackets. Let $L = L(X)$ be the set of free homotopy classes of loops in the quasi-surface $X$ and let $M = M(X)$ be the free abelian group with basis $L$. Pick a gate orientation $\omega$ of $X$. By Section 3.3, any pair $x, y \in L$ can be represented by an $\omega$-admissible pair of loops $a, b$ in $X$. For a point $r \in a \cap b$, consider the loops $a_r, b_r$, which are reparameterizations of $a, b$ based at $r$. Consider the product loop $a_r b_r$ and set
(4.1.1) $[x, y]_{X, \omega} = \sum_{r \in a \cap b} \varepsilon_r(a, b) \langle a_r b_r \rangle \in M$
where for a loop $c$ in $X$, we let $\langle c \rangle \in L \subset M$ be its free homotopy class. The sum on the right-hand side of (4.1.1) is an algebraic sum of all possible ways to graft $a$ and $b$ at their intersections in $\Sigma'$. It is straightforward to see that this sum is preserved under all loop moves on $a, b$ keeping this pair $\omega$-admissible. Hence, $[x, y]_{X, \omega}$ does
not depend on the choice of \(a, b\) in the homotopy classes \(x, y\). Extending the map \( (x, y) \mapsto [x, y]_{X,\omega} \) by bilinearity, we obtain a bilinear bracket \([-,-]_{X,\omega}\) in \(M\). The proof of Formula (4.1.2) applies here and shows that for any \(x, y \in M\),

\[
[x, y]_{X,\omega} = -[y, x]_{X,\omega}.
\]

We now compute the bracket \([x, y]_{X,\omega}\) from an arbitrary generic pair of loops \(a, b\) representing \(x, y\). Note that for any points \(p \in a \cap \alpha_k\), \(q \in b \cap \alpha_k\) on the same gate, we can multiply the loops \(a_p, b_q\) based at \(p, q\) using a path connecting \(p, q\) in \(\alpha_k\). The product loop determines an element of \(L\) denoted \(\langle a_p b_q \rangle\).

**Lemma 4.1.** Let \(x, y \in L\) be represented by a generic pair of loops \(a, b\). Then

\[
[x, y]_{X,\omega} = \sum_{r \in a' b} \varepsilon_r(a, b) (a_r b_r) + \sum_{(k,p,q) \in T_L(a,b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle.
\]

The proof repeats the proof of Lemma 4.3 with obvious modifications. If \(a \cap b = \emptyset\), then (4.1.3) simplifies to

\[
[x, y]_{X,\omega} = \sum_{(k,p,q) \in T_L(a,b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle.
\]

**Theorem 4.2.** The skew-symmetric bracket \([-,-]_{X}\) in \(M\) defined by

\[
[x, y]_{X} = [x, y]_{X,\omega} - [y, x]_{X,\omega}
\]

for all \(x, y \in M\) and a gate orientation \(\omega\) of \(X\) does not depend on \(\omega\).

We prove this theorem in Section 4.3 using the content of Section 4.2. We call the bracket \([-,-]_{X}\) the homotopy intersection bracket of \(X\). Both brackets \([-,-]_{X,\omega}\) and \([-,-]_{X}\) generalize Goldman’s bracket (\([Go1],[Go2]\)):

the value of \([-,-]_{X,\omega}\) (respectively, \([-,-]_{X}\)) on any pair of free homotopy classes of loops in \(\Sigma' \subset X\) is equal to their Goldman’s bracket (respectively, twice this bracket). The free homotopy classes of loops lying in \(Y \subset X\) annihilate both \([-,-]_{X,\omega}\) and \([-,-]_{X}\).

### 4.2. The pairing \(\mu_k\).

For each \(k \in \pi_0\), we define a bilinear form \(\mu_k : M \times M \to M\) as follows. It suffices to define \(\mu_k(x, y)\) for all \(x, y \in L\). To this end, pick generic loops \(a, b\) representing \(x, y\), and set

\[
\mu_k(x, y) = \sum_{p \in a \cap \alpha_k; q \in b \cap \alpha_k} \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle \in M.
\]

**Lemma 4.3.** The vector \(\mu_k(x, y) \in M\) does not depend on the choice of generic loops \(a, b\) representing \(x, y\).

**Proof.** We need only to prove that \(\mu_k(x, y)\) is preserved under the loop moves on \(a, b\). For the moves \(L_0 - L_3\), this is clear from the definitions. A move \(L_4\) on \(a\) creates two additional points \(p', p'' \in a \cap \alpha_k\) for some \(k\) such that \(\varepsilon_{p'}(a) = -\varepsilon_{p''}(a)\). Then the expressions \(\varepsilon_{p'}(a) \varepsilon_q(b) \langle a_p b_q \rangle\) and \(\varepsilon_{p''}(a) \varepsilon_q(b) \langle a_{p'} b_q \rangle\) cancel each other for all \(q \in b \cap \alpha_k\). So, this move preserves \(\mu_k(x, y)\). The move \(L_5\) on \(a\) replaces two points \(p_1, p_2 \in a \cap \alpha_k\) by two points \(p'_1, p'_2\) such that \(\varepsilon_{p'_1}(a) = \varepsilon_{p_1}(a)\) and \(\langle a_{p'_1} b_q \rangle = \langle a_p b_q \rangle\) for \(i = 1, 2\) and any \(q \in b \cap \alpha_k\). So, this move preserves \(\mu_k(x, y)\). Similar computations show that the moves \(L_4, L_5\) on \(b\) preserve \(\mu_k(x, y)\). \(\square\)

**Lemma 4.3** shows that \(\mu_k(x, y)\) depends only on \(x, y\). The identity \(\langle a_p b_q \rangle = \langle b_q a_p \rangle\) implies that \(\mu_k(x, y) = \mu_k(y, x)\) for all \(x, y \in M\).
4.3. Proof of Theorem 4.2. It suffices to prove that

\[(4.3.1) \quad [x, y]_{X,l\omega} - [y, x]_{X,l\omega} = [x, y]_{X,\omega} - [y, x]_{X,\omega}\]

for all \(x, y \in L\) and \(l \in \pi_0\). Pick an \(\omega\)-admissible pair of loops \(a, b\) representing respectively \(x, y\). We compute \([x, y]_{X,\omega}\) from (4.1.1) and compute \([x, y]_{X,l\omega}\) applying (4.3.2) (with \(\omega\) replaced by \(l\omega\)) to the pair \(a, b\). The same argument as in the proof of Theorem 3.2 shows that

\[(4.3.2) \quad [x, y]_{X,l\omega} - [x, y]_{X,\omega} = \sum_{p \in a \cap \alpha_1, q \in b \cap \alpha_1} \varepsilon(l\omega, l) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle\]

Since the form \(\mu_1\) is symmetric, the expression \([x, y]_{X,l\omega} - [x, y]_{X,\omega}\) is symmetric in \(x, y\). This implies (4.3.1) and completes the proof of the theorem.

4.4. Computation of \([-\cdot, -]_X\). To compute \([x, y]_X\) for \(x, y \in L\) we will use a method parallel to the one in Section 3.3. Pick a generic pair of loops \(a, b\) in \(X\) representing \(x, y\). Let \(\omega_0\) be the orientation of the gates induced by the orientation of \(\Sigma' \subset \Sigma\). Lemma 4.1 implies that

\([x, y]_X = [x, y]_{X,\omega_0} - [y, x]_{X,\omega_0}\]

where \(\delta(p, q) = \pm 1\) is defined in Section 3.4. If \(a \cap b = \emptyset\), then

\[(4.4.1) \quad [x, y]_X = \sum_{(k, p, q) \in T(a, b)} \delta(p, q) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle,\]

4.5. Remarks. 1. Applying (4.3.2) consecutively to all \(l \in \pi_0\) and using (4.1.2), we obtain that for any \(x, y \in M\) and any gate orientation \(\omega\) of \(X\),

\([x, y]_{X,\omega} + [y, x]_{X,\omega} = \sum_{l \in \pi_0} \varepsilon(\omega, l) \mu_l(x, y)\]

Since \([x, y]_X = [x, y]_{X,\omega} - [y, x]_{X,\omega}\), we deduce that

\[2[x, y]_X = [x, y]_X + \sum_{l \in \pi_0} \varepsilon(\omega, l) \mu_l(x, y)\]

As a consequence, the form \(\bullet_{X,\omega}\) in \(H_1(X)\) may be computed from \(\bullet_X\) via

\[2x \bullet_{X,\omega} y = x \bullet_X y + \sum_{l \in \pi_0} \varepsilon(\omega, l) v_l(x) v_l(y)\]

for all \(x, y \in H_1(X)\).

2. Generally speaking, the brackets \([-\cdot, -]_{X,\omega}\) and \([-\cdot, -]_X\) do not satisfy the Jacobi identity. Their Jacobiators can be computed in terms of operations associated with the gates of \(X\). Similar results hold for the self-intersection cobrackets defined in the next section; this will be discussed in more detail elsewhere.

5. THE COBRACKETS

We define self-intersection cobrackets for loops in the quasi-surface \(X\).
5.1. The cobracket $\nu_{X,\omega}$. For a loop $a$ in $X$, we set $\langle a \rangle_0 = \langle a \rangle \in M = M(X)$ if $a$ is non-contractible and $\langle a \rangle_0 = 0 \in M$ if $a$ is contractible. A generic loop $a$ crosses each point $r \in \#a$ twice; we let $v_r^1, v_r^2$ be the tangent vectors of $a$ at $r$ numerated so that the pair $(v_r^1, v_r^2)$ is positively oriented. For $i = 1, 2$, let $a_i^a$ be the loop starting in $r$ and going along $a$ in the direction of the vector $v_r^i$ until the first return to $r$. Up to parametrization, $a = a_1^a a_2^a$ is the product of the loops $a_1^a, a_2^a$ based at $r$. Also, for any $k \in \pi_0$ and any distinct points $p_1, p_2 \in a \cap \alpha_k$ we define a loop $a_{p_1, p_2}$ in $X$ which goes from $p_1$ to $p_2$ along $a$ and then goes back to $p_1$ along the gate $\alpha_k$. Consider the set of ordered triples

\[
T(a) = \{(k \in \pi_0, p_1 \in a \cap \alpha_k, p_2 \in a \cap \alpha_k) \mid p_1 \neq p_2\}.
\]

We call a triple $(k, p_1, p_2) \in T(a)$ a chord of $a$ with endpoints $p_1, p_2$.

For a gate orientation $\omega$ of $X$, we let $T_\omega(a)$ be the set of chords $(k, p_1, p_2) \in T(a)$ such that $p_1 <_\omega p_2$. Set

\[
\nu_{X,\omega}(a) = \sum_{r \in \#a} (\langle a^1_r \rangle_0 \otimes \langle a^2_r \rangle_0 - \langle a^2_r \rangle_0 \otimes \langle a^1_r \rangle_0)
\]

\[
+ \sum_{(k, p_1, p_2) \in T_\omega(a)} \varepsilon(\omega, k) \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) \langle a_{p_1, p_2} \rangle_0 \otimes \langle a_{p_1, p_2} \rangle_0 \in M \otimes M.
\]

**Lemma 5.1.** The cobracket $\nu_{X,\omega}(a)$ is preserved under all loop moves on $a$.

**Proof.** The moves $L_0 - L_3$ proceed in $\Sigma'$ and are treated as in [Tu1]. The move $L_4$ pushes a branch of $a$ across a gate $\alpha_l$ for some $l \in \pi_0$ creating a loop $a'$ which has two additional crossings $q_1, q_2 \in a' \cap \alpha_l$ such that $q_1 <_\omega q_2$ and $\varepsilon(q_1)(a) = -\varepsilon(q_2)(a)$. The contributions to the cobracket of the self-intersection points and of the chords containing neither $q_1$ nor $q_2$ are the same before and after the deformation. The contributions to $\nu_{X,\omega}(a')$ of the chords containing exactly one of the points $q_1, q_2$ cancel each other. The chord $(l, q_1, q_2)$ of $a'$ contributes zero to $\nu_{X,\omega}(a')$ because at least one of the loops $a_{q_1, q_2}$ and $a_{q_2, q_1}$ is contractible. Therefore $\nu_{X,\omega}(a) = \nu_{X,\omega}(a')$.

We now prove the invariance of $\nu_{X,\omega}(a)$ under the move $L_5$ which pushes a self-crossing of $a$ in $\Sigma'$ across a gate, say $\alpha_l$, into $X \setminus \Sigma'$. Note that under the inversion of the orientation of $\Sigma$, the signs $\varepsilon(\omega, l)$ and the expression $\nu_{X,\omega}(a)$ are multiplied by $-1$. Therefore, inverting if necessary the given orientation of $\Sigma$, we can reduce the proof of the invariance of $\nu_{X,\omega}(a)$ under our move to the case where $\varepsilon(\omega, l) = +1$.

Assume that the move changes $a$ in a small disk $D$ by pushing a self-crossing $r_0 \in \#a$ from $D \cap \Sigma'$ to $D \setminus \Sigma'$. For $i = 1, 2$ set $\gamma_i = (a_{r_0}^i)_0 \in M$. The branches of $a$ meeting at $r_0$ intersect the gate $\alpha_l$ in two points lying in $D$. We label these points $q_1, q_2$ so that $q_1 <_\omega q_2$. By definition, $r_0$ contributes $\gamma_1 \otimes \gamma_2 - \gamma_2 \otimes \gamma_1$ to $\nu_{X,\omega}(a)$. The contribution of the chord $(l, q_1, q_2)$ to $\nu_{X,\omega}(a)$ also can be computed from the definitions: it is equal to $\gamma_2 \otimes \gamma_1$ if $\varepsilon(q_1)(a) = \varepsilon(q_2)(a)$ and to $-\gamma_1 \otimes \gamma_2$ otherwise. Thus the joint contribution of $r_0$ and $(l, q_1, q_2)$ to $\nu_{X,\omega}(a)$ is equal to $\gamma_1 \otimes \gamma_2$ if $\varepsilon(q_1)(a) = \varepsilon(q_2)(a)$ and to $-\gamma_2 \otimes \gamma_1$ otherwise. The loop, $a'$, produced by the move meets $D \cap \alpha_l$ in two points forming a chord of $a'$. A similar computation shows that the contribution of this chord to $\nu_{X,\omega}(a')$ also is $\gamma_1 \otimes \gamma_2$ if $\varepsilon(q_1)(a) = \varepsilon(q_2)(a)$ and $-\gamma_2 \otimes \gamma_1$ otherwise. All the other self-crossings and chords contribute the same expressions to $\nu_{X,\omega}(a)$ and $\nu_{X,\omega}(a')$. Therefore $\nu_{X,\omega}(a) = \nu_{X,\omega}(a')$. \hfill \Box

Lemma 5.1 implies that $\nu_{X,\omega}(a) \in M \otimes M$ depends only on the free homotopy class $(a)$ of $a$. The map $(a) \mapsto \nu_{X,\omega}(a) : L \to M^{\otimes 2}$ extends uniquely to a linear map $M \to M^{\otimes 2}$ denoted $\nu_{X,\omega}$. 
5.2. The cobracket $\nu_X$. We define a cobracket $\nu_X$ independent of $\omega$.

**Theorem 5.2.** Let $P$ be the linear automorphism of $M \otimes M$ carrying $x \otimes y$ to $y \otimes x$ for all $x, y \in M$. The skew-symmetric cobracket

$$\nu_X = \nu_{X,\omega} - P\nu_{X,\omega} : M \to M \otimes M$$

does not depend on the choice of $\omega$.

**Proof.** It suffices to prove that $\nu_{X,\omega} - P\nu_{X,\omega}$ is preserved when $\omega$ is replaced with $l\omega$ for $l \in \pi_0$. For any generic loop $a$ in $X$, we have

$$\nu_{X,\omega}(a) - \nu_{X,l\omega}(a) = \varepsilon(\omega, l) \sum_{(l,p_1,p_2) \in T_\omega(a)} \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) (a_{p_2,p_1})_0 \otimes (a_{p_1,p_2})_0$$

$$- \varepsilon(l\omega, l) \sum_{(l,p_1,p_2) \in T_{l\omega}(a)} \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) (a_{p_2,p_1})_0 \otimes (a_{p_1,p_2})_0.$$

Clearly, $\varepsilon(l\omega, l) = -\varepsilon(\omega, l)$. Also, the inclusion $(l,p_1,p_2) \in T_{l\omega}(a)$ holds if and only if $(l,p_2,p_1) \in T_\omega(a)$. Therefore

$$\nu_{X,\omega}(a) - \nu_{X,l\omega}(a) =$$

$$= \varepsilon(\omega, l) \sum_{(l,p_1,p_2) \in T_\omega(a)} \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) (a_{p_2,p_1})_0 \otimes (a_{p_1,p_2})_0 + (a_{p_1,p_2})_0 \otimes (a_{p_2,p_1})_0 \varepsilon(\omega, l).$$

The latter expression is, obviously, invariant under the transposition $P$. So,

$$\nu_{X,\omega}(a) - \nu_{X,l\omega}(a) = P\nu_{X,\omega}(a) - P\nu_{X,l\omega}(a)$$

or, equivalently,

$$\nu_{X,\omega}(a) - P\nu_{X,\omega}(a) = \nu_{X,l\omega}(a) - P\nu_{X,l\omega}(a).$$

□

We call $\nu_X$ the **self-intersection cobracket** of $X$. Both cobrackets $\nu_{X,\omega}$ and $\nu_X$ generalize the cobracket $\nu$ defined for loops in surfaces in $\Sigma' \subset X$ is equal to the value of $\nu$ on this class (respectively, twice that value). The free homotopy classes of loops lying in $Y \subset X$ are annihilated by both $\nu_{X,\omega}$ and $\nu_X$.

5.3. Computation of $\nu_X$. To compute $\nu_X$ we use a method parallel to the one used in Sections 4.3 and 4.4. Namely, for any generic loop $a$ in $X$, we have

$$\nu_X(\langle a \rangle) = 2 \sum_{r \in \#a} \langle a_r^1 \rangle_0 \otimes \langle a_r^2 \rangle_0 - \langle a_r^2 \rangle_0 \otimes \langle a_r^1 \rangle_0$$

$$+ \sum_{(k,p_1,p_2) \in T(a)} \delta(p_1,p_2) \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) (a_{p_2,p_1})_0 \otimes (a_{p_1,p_2})_0.$$

If $\#a = 0$, then

$$\nu_X(\langle a \rangle) = \sum_{(k,p_1,p_2) \in T(a)} \delta(p_1,p_2) \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) (a_{p_2,p_1})_0 \otimes (a_{p_1,p_2})_0.$$
5.4. Examples. We give two examples where the operations \( \bullet_X, [-, -]_X, \) and \( \nu_X \) vanish. Set \( I = [0, 1] \) and \( \Sigma = I^2 \) with an arbitrary orientation.

1. Let \( \alpha = I \times \{0\} \subset \partial \Sigma \) and \( \Sigma' = I \times [1/3, 1] \subset \Sigma \). Then \( X \) has only one gate \( I \times \{1/3\} \). It is clear that any loop in \( X \) can be deformed away from \( \Sigma' \). Therefore \( \bullet_{X, \omega} = 0, [-, -]_{X, \omega} = 0 \), and \( \nu_{X, \omega} = 0 \) for both gate orientations \( \omega \) of \( X \). Consequently, \( \bullet_X = 0, [-, -]_X = 0 \), and \( \nu_X = 0 \).

2. Let \( \alpha = I \times \{0, 1\} \subset \partial \Sigma \) and \( \Sigma' = I \times [1/3, 2/3] \subset \Sigma \). Then \( X \) has two gates \( \alpha_1 = I \times \{1/3\} \) and \( \alpha_2 = I \times \{2/3\} \). Let \( \omega \) be the orientation of \( \alpha_1, \alpha_2 \) induced by the orientation of \( I \) from 0 to 1. Any pair of free homotopy classes of loops in \( X \) can be represented by loops \( a, b \) such that \( a \) meets \( \Sigma' \) at several segments \( \{s\} \times [1/3, 2/3] \) with \( s \in [0, 1/3] \) and \( b \) meets \( \Sigma' \) at several segments \( \{t\} \times [1/3, 2/3] \) with \( t \in [2/3, 1] \). Then the pair \( a, b \) is \( \omega \)-admissible and \( a \cap b = \emptyset \). Hence, \( a \bullet_X b = 0 \). Consequently, \( \bullet_{X, \omega} = 0 \) and \( \bullet_X = 0 \). Similar arguments show that \( [-, -]_X = 0 \). Next, any free homotopy class of loops in \( X \) can be represented by a generic loop \( a \) which meets \( \Sigma' \) at the segments \( \{s\} \times [1/3, 2/3] \) where \( s \) runs over a finite set \( S \subset (0, 1) \). Clearly, \( \# a = 0 \). The set \( T_\omega(a) \) consists of the triples \( (k, s_1, s_2) \) with \( s_1, s_2 \in S \) and \( s_1 < s_2 \). For any such \( s_1, s_2 \), the triples \( (1, s_1, s_2) \) and \( (2, s_1, s_2) \) contribute opposite values to \( \nu_{X, \omega}(a) \) because \( \epsilon(\omega, 1) = -\epsilon(\omega, 2) \) while all other terms of these contributions are the same. Thus, \( \nu_{X, \omega}(a) = 0 \). Consequently, \( \nu_{X, \omega} = 0 \) and \( \nu_X = 0 \). Note that for the gate orientation \( \omega \) of \( X \) which directs one gate from 0 to 1 and the other gate from 1 to 0, the operations \( \bullet_{X, \omega}, [-, -]_{X, \omega}, \) and \( \nu_{X, \omega} \) may be non-zero.

6. Transformations of quasi-surfaces

We study two transformations of the quasi-surface \( X \): the transformation \( D \) (for disjoint unions) and the transformation \( C \) (for cuttings). Both \( D \) and \( C \) preserve the underlying topological space of \( X \) but change the structure of a quasi-surface.

6.1. The transformation \( D \). The transformation \( D \) applies when \( \Sigma = \bigsqcup_{j=1}^N \Sigma_j \) is a disjoint union of \( N \geq 2 \) oriented surfaces. For each \( j = 1, \ldots, N \), we let \( Y_j \) be the topological space obtained by gluing \( N - 1 \) surfaces \( \{\Sigma_j\}_{j \neq \ell} \) to \( Y \) along the maps \( \alpha \cap \partial \Sigma_j \to Y \) used in the definition of \( X \). The underlying topological space of \( X \) is obtained by gluing \( \Sigma_j \) to \( Y_j \) along the map \( \alpha \cap \partial \Sigma_j \to Y \subset Y_j \) used in the definition of \( X \). This turns the space in question into a quasi-surface, \( X_j \), with surface core \( \Sigma_j \) and singular core \( Y_j \). For the reduced surface core and the gates of \( X_j \) we take \( \Sigma'_j \cap \Sigma_j \) and the gates of \( X \) lying in \( \Sigma_j \).

Lemma 6.1. We have

\[
\bullet_X = \sum_{j=1}^N \bullet_{X_j} : H_1(X) \times H_1(X) \to \mathbb{Z},
\]

\[
[-, -]_X = \sum_{j=1}^N [-, -]_{X_j} : M \times M \to M,
\]

\[
\nu_X = \sum_{j=1}^N \nu_{X_j} : M \to M \otimes M.
\]
Proof. Note that each gate orientation $\omega$ of $X$ restricts to a gate orientation $\omega_j$ of $X_j$. Formulas (6.1.1)–(6.1.3) are direct consequences of the following stronger claim: for any $\omega$, we have

\begin{align}
\bullet_{X,\omega} &= \sum_{j=1}^{N} \bullet_{X_j,\omega_j} : H_1(X) \times H_1(X) \to \mathbb{Z}. \\
[-, -]_{X,\omega} &= \sum_{j=1}^{N} [-, -]_{X_j,\omega_j} : M \times M \to M, \\
\nu_{X,\omega} &= \sum_{j=1}^{N} \nu_{X_j,\omega_j} : M \to M \otimes M.
\end{align}

To prove (6.1.4), we pick an $\omega$-admissible pair $a, b$ of loops in $X$ representing $x, y \in H_1(X)$. Then $x \cdot_x y$ is the algebraic number of intersections of $a, b$ in $Y = \bigcup_j (\Sigma' \cap \Sigma_j)$. Also, for all $j$, the pair of loops $a, b$ is $\omega_j$-admissible in $X_j$ and $x \cdot_{X_j,\omega_j} y$ is the algebraic number of intersections of $a, b$ in $\Sigma' \cap \Sigma_j$. Hence,

\[ x \cdot_x y = \sum_{j=1}^{N} x \cdot_{X_j,\omega_j} y. \]

The proofs of (6.1.5), (6.1.6) are similar. \(\square\)

6.2. The transformation $C$. A submanifold $\beta$ of a manifold $N$ is said to be proper if $\beta \cap \partial N = \partial \beta$. The transformation $C$ applies when we are given a proper compact 1-dimensional submanifold $\beta$ of $\Sigma'$ whose components are segments disjoint from the gates of $X$ (which, recall, all lie in $\partial \Sigma'$). Cutting $\Sigma \supset \Sigma' \supset \beta$ along $\beta$, we obtain an oriented surface $\Sigma^\beta$. A copy of the 1-manifold $\alpha = \partial \Sigma \setminus \Sigma' \subset \partial \Sigma^\beta$ and is denoted $\alpha^\beta$. The 1-manifold $\beta$ gives rise to two copies of itself in $\partial \Sigma^\beta$. The underlying topological space of $X$ can be obtained by gluing the surface $\Sigma^\beta$ to the disjoint union $Y^\beta = Y \cup \beta$ along the map $\alpha^\beta = \alpha \to Y \subset Y^\beta$ used in the definition of $X$ and along the tautological identity maps of the copies of $\beta$ in $\partial \Sigma^\beta$ to $\beta \subset Y^\beta$. This turns the underlying topological space of $X$ into a quasi-surface $X^\beta$ with surface core $\Sigma^\beta$ and singular core $Y^\beta$.

Lemma 6.2. We have

\begin{align}
\bullet_X &= \bullet_{X^\beta} : H_1(X) \times H_1(X) \to \mathbb{Z}, \\
[-, -]_X &= [-, -]_{X^\beta} : M \times M \to M, \\
\nu_X &= \nu_{X^\beta} : M \to M \otimes M.
\end{align}

Proof. The gates of $X^\beta$ are the gates of $X$ and additional gates associated with the components $\{\beta_l\}$ of $\beta$. Namely, each $\beta_l$ gives rise to two gates of $X^\beta$ which are proper segments in $\Sigma'$ running “parallel” to $\beta_l$ on different sides of $\beta_l$ in $\Sigma'$. For each $l$, fix an orientation of $\beta_l$ and orient the associated gates so that they look in the same direction as $\beta$. Then every gate orientation $\omega$ of $X$ determines a gate orientation $\omega^\beta$ of $X^\beta$. Formulas (6.2.1)–(6.2.3) are consequences of the following stronger claim: for any $\omega$, we have

\[ \bullet_{X,\omega} = \bullet_{X^\beta,\omega^\beta} : H_1(X) \times H_1(X) \to \mathbb{Z}. \]
(6.2.5) \[-, -\]_{X, \omega} = \{-, -\}^{\beta} : M \times M \to M,

(6.2.6) \nu_{X, \omega} = \nu^{\beta} : M \to M \otimes M.

To prove (6.2.4), pick an \(\omega\)-admissible pair \(a, b\) of loops in \(X\) representing \(x, y \in H_1(X)\). Deforming if necessary \(a, b\) near \(\beta\) we can assume that \(a, b\) are transversal to \(\beta\), all crossings of \(a\) with \(\beta\) lie near the tails of the components, and all crossings of \(b\) with \(\beta\) lie near the heads of the components. Then the pair \(a, b\) is \(\omega\)-admissible. The integer \(x \bullet_{X, \omega} y\) is the algebraic number of intersections of \(a, b\) in \(\Sigma'\). Since all these intersections lie away from \(\beta\) and from the gates of \(X\), they bijectively correspond to the intersections of \(a, b\) in the reduced surface core of \(X^\beta\) (and have the same signs). Therefore \(x \bullet_{X, \omega} y = x \bullet_{X, \omega^\beta} y\). The proofs of the equalities (6.2.5) and (6.2.6) are similar.

\[\square\]

7. Stars in quasi-surfaces

We state and prove analogues of Theorems 1.1–1.4 for quasi-surfaces.

7.1. Stars in \(X\). By a star in the quasi-surface \(X\) we mean a star \(s\) embedded in the surface core \(\Sigma\) of \(X\) so that \(\partial s = s \cap \partial \Sigma \subset \partial \Sigma \setminus \alpha\). We derive from \(s\) a new structure of a quasi-surface in the underlying topological space of \(X\). Denote the number of leaves of \(s\) by \(|s|\). Pick a closed regular neighborhood \(V\) of \(s\) in \(\Sigma \setminus \alpha \subset X\) and provide \(V\) with orientation induced by that of \(\Sigma\). It is clear that \(V\) is a 2-disk whose boundary is formed by \(|s|\) disjoint segments in \(\partial \Sigma \setminus \alpha\) and \(|s|\) disjoint proper segments \(\beta^1_1, ..., \beta^{|s|}_{|s|}\) in \(\Sigma\). Set \(Y_s = X \setminus \overline{V} \subset X\) where the overline stands for the closure in the underlying topological space of \(X\). Taking \(V\) as the surface core, \(Y_s\) as the singular core and glueing \(V\) to \(Y_s\) along the inclusions \(\{\beta^i_1 \to Y_s\}^{|s|}_{i=1}\) we obtain a quasi-surface, \(X_s = X_s(s)\), with the same underlying topological space as \(X\). This allows us to consider the bilinear maps \(\bullet_s : H_1(X) \times H_1(X) \to \mathbb{Z}\), \([-,-]_s = [-,-]_X : M \times M \to M\)

and the linear map \(\nu_s = \nu_{X_s} : M \to M \otimes M\).

We next compute the form \(\bullet_s\) via intersections of loops with \(s\). As in Section 1 the orientation of \(\Sigma\) at the center of \(s\) determines a cyclic order in the set \(\text{Edg}(s)\) of edges of \(s\). For \(e \in \text{Edg}(s)\) we let \(e^+ \in \text{Edg}(s)\) be the next edge with respect to this order. We say that a loop in \(X\) is \(s\)-generic if it misses the vertices of \(s\), meets all edges of \(s\) transversely, and never traverses a point of \(s\) more than once. A family of loops in \(X\) is \(s\)-generic if these loops are \(s\)-generic and do not meet at points of \(s\). Given an \(s\)-generic loop \(a\) in \(X\), we let \(a \cap s\) be the set of points of \(s\) traversed by \(a\). For an edge \(e \in \text{Edg}(s)\), set \(a \cap e = (a \cap s) \cap e\). For \(p \in a \cap e\), the intersection sign of \(a\) and \(e\) at \(p\) is denoted \(\mu_p(a)\) (recall that the edges of \(s\) are directed from the center of \(s\) to the leaves). The integer \(a \cdot e = \sum_{p \in a \cap e} \mu_p(a)\) is the algebraic number of the intersections of \(a\) and \(e\).

Lemma 7.1. For any \(s\)-generic pair of loops \(a, b\) in \(X\) representing \(x, y \in H_1(X)\),

(7.1.1) \[x \bullet_s y = \sum_{e \in \text{Edg}(s)} ((a \cdot e)(b \cdot e^+) - (b \cdot e)(a \cdot e^+)).\]

\textbf{Proof.} Consider the quasi-surface \(X_s = X_s(s)\) derived as above from a closed regular neighborhood \(V\) of \(s\) in \(\Sigma \setminus \alpha\). For the reduced surface core of \(X_s\) we take a smaller closed regular neighborhood \(V' \subset V\) of \(s\). The gates of \(X_s\) are the segments
in \( \partial V' \) separating \( V' \) from the rest of \( V \). Moving along the circle \( \partial V' \) in the direction determined by the orientation of \( V' \) induced by that of \( \Sigma \), we meet all gates in a certain cyclic order. For a gate \( K \) of \( X_+ \), denote the next gate with respect to this cyclic order by \( K^+ \). Note that there is a unique edge \( e = e_K \) of \( s \) such that \( K \) is obtained by pushing the segment \( e \cup e^+ \) into \( \Sigma \setminus s \). Clearly, \( e_{K^+} = (e_K)^+ \).

To proceed, we select the closed regular neighborhood \( V' \subset V \) taking into account the loops \( a, b \). We say that \( V' \) is \( a\text{-adapted} \) if the set \( a(S^1) \cap V' \) is a disjoint union of proper segments \( \{ f_p \}_{p \in a \cap s} \) in \( V' \) such that \( p \in f_p \) for all \( p \in a \cap s \). We denote the endpoints of the segment \( f_p \) by \( p', p'' \) so that the pair (the orientation of \( f_p \) from \( p' \) to \( p'' \), the orientation of the edge of \( s \) containing \( p \)) determines the given orientation of \( \Sigma \) at \( p \). If \( V' \) is \( a\text{-adapted} \), then \( a \) has no self-intersections in \( V' \) and crosses the gates precisely at the points \( \{ p', p'' \}_{p \in a \cap s} \). The crossing signs of \( a \) with the gates (see Section 3.2) are computed by

\[
\varepsilon_{p'}(a) = \mu_p(a) \quad \text{and} \quad \varepsilon_{p''}(a) = -\mu_p(a)
\]

for all \( p \in a \cap s \). We select the neighborhood \( \varnothing' \subset V \) of \( s \) so small (= narrow), that it is \( a\text{-adapted} \), \( b\text{-adapted} \), and the loops \( a, b \) do not meet in \( \varnothing' \). Then for any gate \( K \) of \( X_+ \), we have

\[
a \cap K = \{ p' \mid p \in a \cap e_K \} \cup \{ p'' \mid p \in a \cap (e_K)^+ \}
\]

and similarly,

\[
b \cap K = \{ q' \mid q \in b \cap e_K \} \cup \{ q'' \mid q \in b \cap (e_K)^+ \}.
\]

We compute \( x \inpX \), \( y \) via (3.11). The sum on the right hand-side of (3.11) runs over all triples (a gate \( K \) of \( X_+ \), a point of \( a \cap K \), a point of \( b \cap K \)). By (7.1.3) and (7.1.4), the contribution of such triples with fixed \( K \) is the sum of the following four expressions:

\[
\sigma_K^1 = \sum_{p \in a \cap e_K, q \in b \cap e_K} \delta(p', q') \varepsilon_{p'}(a) \varepsilon_q(b),
\]
\[
\sigma_K^2 = \sum_{p \in a \cap e_K, q \in b \cap (e_K)^+} \delta(p', q'') \varepsilon_{p'}(a) \varepsilon_q(b),
\]
\[
\sigma_K^3 = \sum_{p \in a \cap (e_K)^+, q \in b \cap e_K} \delta(p'', q') \varepsilon_{p''}(a) \varepsilon_q(b),
\]
\[
\sigma_K^4 = \sum_{p \in a \cap (e_K)^+, q \in b \cap (e_K)^+} \delta(p'', q'') \varepsilon_{p''}(a) \varepsilon_q(b).
\]

To simplify these expressions we use the following notation: for distinct points \( p, q \) of an edge of \( s \), set \( \mu(p, q) = 1 \) if \( p \) lies between the center of \( s \) and \( q \) and set \( \mu(p, q) = -1 \) otherwise. For any \( e \in \Edg(s) \) and any points \( p \in a \cap e, q \in b \cap e \), we have \( \delta(p', q') = \mu(p, q) \) and \( \delta(p'', q'') = -\mu(p, q) \). Using this and (7.1.2), we get

\[
\sigma_K^1 = \sum_{p \in a \cap e_K, q \in b \cap e_K} \mu(p, q) \mu_p(a) \mu_q(b)
\]

and

\[
\sigma_K^4 = -\sum_{p \in a \cap (e_K)^+, q \in b \cap (e_K)^+} \mu(p, q) \mu_p(a) \mu_q(b).
\]
When $K$ runs over all gates, both $e_K$ and $(e_K)^+$ run over all edges of $s$. Therefore
\[ \sum_K (\sigma_K^1 + \sigma_K^2) = 0. \]
Observe next that $\delta(p', q'') = -1$ for all $p \in a \cap e_K$ and $q \in b \cap (e_K)^+$. Hence,
\[ \sigma_K^2 = \sum_{p \in a \cap e_K, q \in b \cap (e_K)^+} \mu_p(a) \mu_q(b) = (a \cdot e_K)(b \cdot (e_K)^+). \]
Similarly, $\delta(p'', q') = 1$ for all $p \in a \cap (e_K)^+, q \in b \cap e_K$ and so
\[ \sigma_K^3 = - \sum_{p \in a \cap (e_K)^+, q \in b \cap e_K} \mu_p(a) \mu_q(b) = -(a \cdot (e_K)^+)(b \cdot e_K). \]
Therefore
\[ x \cdot s = \sum_K (\sigma_K^1 + \sigma_K^2 + \sigma_K^3 + \sigma_K^4) = \sum_{e \in \text{Edg}(s)} \left( (a \cdot e)((b \cdot e^+)) - (b \cdot e)(a \cdot e^+) \right). \]

Given two $s$-generic loops $a, b$ in $X$ and points $p \in a \cap s, q \in b \cap s$, we pick a path $c$ from $p$ to $q$ in $s$ and write $a_p b_q$ for the loop $a_p c b_q c^{-1}$. Clearly, the free homotopy class of this loop in $X$ does not depend on the choice of $c$.

**Lemma 7.2.** For any $s$-generic loops $a, b$ in $X$, we have
\[ [(a), (b)]_s = \sum_{e \in \text{Edg}(s)} \left( \sum_{p \in a \cap e, q \in b \cap e^+} \mu_p(a) \mu_q(b) \langle a_p b_q \rangle - \sum_{p \in a \cap e^+, q \in b \cap e} \mu_p(a) \mu_q(b) \langle a_p b_q \rangle \right). \]

**Proof.** We use the same $V'$ as in the proof of Lemma 7.1 and apply Formula (4.4.1) to compute $[(a), (b)]_s$. The expressions $\sigma_K^1, \ldots, \sigma_K^4$ are replaced with the sums
\[ \sum_{p \in a \cap e_K, q \in b \cap e_K} \delta(p', q') \varepsilon_{p'}(a) \varepsilon_{q'}(b) \langle a_p b_q \rangle, \]
\[ \sum_{p \in a \cap e_K, q \in b \cap (e_K)^+} \delta(p', q'') \varepsilon_{p'}(a) \varepsilon_{q''}(b) \langle a_p b_q \rangle, \]
\[ \sum_{p \in a \cap (e_K)^+, q \in b \cap e_K} \delta(p'', q') \varepsilon_{p''}(a) \varepsilon_{q'}(b) \langle a_p b_q \rangle, \]
\[ \sum_{p \in a \cap (e_K)^+, q \in b \cap (e_K)^+} \delta(p'', q'') \varepsilon_{p''}(a) \varepsilon_{q''}(b) \langle a_p b_q \rangle. \]
The rest of the argument goes along the same lines as the proof of Lemma 7.1.

Given an $s$-generic loop $a$ in $X$ and points $p_1, p_2 \in a \cap s$, we write $a_{p_1, p_2}$ for the loop going from $p_1$ to $p_2$ along $a$ and then going back to $p_1$ along a path in $s$. Clearly, the free homotopy class of this loop does not depend on the choice of the latter path.

**Lemma 7.3.** For any $s$-generic loop $a$ in $X$, we have
\[ \nu_s([a]) = \sum_{e \in \text{Edg}(s)} \sum_{p_1 \in a \cap e, p_2 \in a \cap e^+} \mu_{p_1}(a) \mu_{p_2}(a) \left( \langle a_{p_1, p_2} \rangle_0 \otimes \langle a_{p_2, p_1} \rangle_0 - \langle a_{p_2, p_1} \rangle_0 \otimes \langle a_{p_1, p_2} \rangle_0 \right). \]
Proof. We use the same $V'$ as in the proof of Lemma 7.1 and apply Formula (6.3.1) to compute $\nu_\beta(\langle a \rangle)$. The rest of the argument uses the same ideas as the proof of Lemma 7.1. □

7.2. Star-fillings of $X$. A star-filling of $X$ is a finite family $F$ of disjoint stars in $X$ such that each component of the set $\Sigma \setminus \cup_{s \in F}s$ is a disk meeting $\partial\Sigma \setminus \alpha$ at one or two open segments.

Lemma 7.4. If the surface core $\Sigma$ of $X$ is compact and each component of $\Sigma$ has a non-void boundary then $X$ has a star-filling.

Proof. Cutting $\Sigma$ along a finite set of disjoint proper segments $\{\beta_i\}_i$ we can get a disjoint union of closed disks $\{D_j\}_j$. Pushing if necessary the endpoints of the segments $\{\beta_i\}_i$ along $\partial\Sigma$ we can ensure that $\beta_i \cap \alpha = \emptyset$ for all $i$. For each $j$, the circle $\partial D_j$ is formed by several, say, $n_j$ segments in $\partial\Sigma \setminus \alpha$ and the same number of segments which are either components of $\alpha$ or copies of some $\beta_i$'s. If $n_j \geq 2$, then we pick a star $s_j \subset D_j$ with center in $\text{Int}(D_j)$ and with leaves inside the above-mentioned $n_j$ segments in $\partial\Sigma \setminus \alpha$. Composing the inclusion $s_j \subset D_j$ with the natural embedding $D_j \hookrightarrow \Sigma$, we can view each $s_j$ as a star in $\Sigma$. Then the family $\{s_j|n_j \geq 2\}_j$ is a star-filling of $X$. □

Lemma 7.5. For any star-filling $F$ of $X$, we have

\begin{equation}
\bullet_X = \sum_{s \in F} \bullet_s : H_1(X) \times H_1(X) \to \mathbb{Z},
\end{equation}

\begin{equation}
[-, -]_X = \sum_{s \in F} [-, -]_s : M \times M \to M,
\end{equation}

\begin{equation}
\nu_X = \sum_{s \in F} \nu_s : M \to M \otimes M.
\end{equation}

Proof. We prove the first equality, the other two are proven similarly. Without loss of generality we can assume that all stars in the family $F$ lie in the reduced surface core $\Sigma' \subset \Sigma$. We choose closed regular neighborhoods $\{V^s \supset s\}_{s \in F}$ so that they are pairwise disjoint and lie in $\Sigma'$. For each $s \in F$, let $\beta_1^s, ..., \beta_{|s|}^s$ be the segments in $\partial V^s$ separating $V^s$ from the rest of $X$, cf. Section 7.1. Then the 1-manifold $\beta = \cup_{s \in F} \cup_{i=1}^{|s|} \beta_i^s$ satisfies the requirements needed to apply the transformation C to $X$. This transformation produces a quasi-surface $X^\beta$ having the same underlying topological space as $X$. By (6.2.1), $\bullet_X = \bullet_{X^\beta}$. Note that the surface core of $X^\beta$ is a disjoint union of the surfaces $\{V^s \supset s\}_{s \in F}$ and $\Sigma \setminus \cup_{s \in F} V^s$. By (6.1.1), the form $\bullet_{X^\beta}$ is the sum of the forms $\bullet$ associated with the connected components of the surface core of $X^\beta$. Each component $V^s \supset s$ contributes the form $\bullet_s$ to this sum. By the definition of a star-filling, all components of the surface $\Sigma \setminus \cup_{s \in F} V^s$ are homeomorphic to the quasi-surfaces described in Section 5.4. Therefore the associated pairings $\bullet$ are equal to zero and

\[\bullet_X = \bullet_{X^\beta} = \sum_{s \in F} \bullet_s : H_1(X) \times H_1(X) \to \mathbb{Z}.\]
8. Proof of Theorems 1.1–1.4 and the Case of Closed Surfaces

8.1. Proof of Theorems 1.1–1.4 Consider the quasi-surface $X = X(\Gamma)$ with surface core $\Gamma$, empty singular core, and $\alpha = \emptyset \subset \partial \Gamma$. (Alternatively, one can use in this proof a quasi-surface with the surface core $\Gamma$, a 1-point singular core, and a set $\alpha \subset \partial \Gamma$ consisting of a single segment.) Any star $s$ in $\Gamma$ is a star in $X$. Lemma 7.1 implies that the skew-symmetric bilinear form $\star_s$ in $H_1(\Gamma)$ satisfies the conditions of Theorem 1.1 and we set $\cdot_s = \star_s$. Note that $\cdot_s = 2 : \cdot_s$ as is clear from the remarks after the statement of Theorem 8.2. Therefore Theorem 1.2 is a direct consequence of Formula (7.2.1). Similarly, the first claim of Theorem 1.3 follows from Lemma 7.2 and the second claim of Theorem 1.3 follows from Formula (7.2.2). The claims of Theorem 1.4 follow respectively from Lemma 7.3 and Formula (7.2.3).

8.2. The Case of Closed Surfaces Each closed oriented surface $\Phi$ gives rise to the homological intersection form $\cdot : H_1(\Phi) \times H_1(\Phi) \to \mathbb{Z}$, to Goldman’s bracket $[-, -]_\Phi : M \times M \to M$, and to the cobracket $\nu_\Phi : M \to M \otimes M$. Here $M$ is the free abelian group whose basis is the set of free homotopy classes of loops in $\Phi$. The definitions of $[-, -]_\Phi$ and $\nu_\Phi$ repeat word for word the definitions given in the introduction in the case of surfaces with boundary. In this setting there seem to be no analogues of the maps $\cdot_s, [-, -]_s, \nu_s$ derived from stars. We compute the maps $\cdot_\Phi, [-, -]_\Phi, \nu_\Phi$ in terms of so-called filling graphs which we now define.

By a bipartite graph we mean a finite graph $G$ provided with a partition of its set of vertices into two disjoint subsets $B(G)$ and $R(G)$ whose elements are called respectively the blue vertices and the red vertices so that every edge of $G$ connects a blue vertex to a red vertex. Note that a bipartite graph can not have edges connecting a vertex to itself.

A filling graph of $\Phi$ is a bipartite graph $G$ embedded in $\Phi$ so that (i) all components of $\Phi \setminus G$ are open disks and (ii) going along the boundary of each of these disks one traverses at most 4 edges and 4 vertices of $G$. For example, any triangulation $\tau$ of $\Phi$ determines a filling graph of $\Phi$ formed by the vertices of $\tau$ (declared to be “blue”), the centers of the 2-simplices of $\tau$ (declared to be “red”), and the edges connecting the center of each 2-simplex of $\tau$ to the vertices of this 2-simplex. Exchanging the colors of the vertices, one derives from any filling graph of $\Phi$ the dual filling graph.

We show now how to calculate the maps $\cdot_\Phi, [-, -]_\Phi, \nu_\Phi$ via a filling graph $G \subset \Phi$. Note that the orientation of $\Phi$ at any blue vertex $v \in B(G)$ determines a cyclic order in the set Edg$_v$ of the edges of $G$ adjacent to $v$. For $e \in \text{Edg}_v$, we let $e^+ \in \text{Edg}_v$ be the next edge with respect to this order. The set Edg$(G)$ of all edges of $G$ is a disjoint union of the sets $\{\text{Edg}_v \mid v \in B(G)\}$. Therefore the map $e \mapsto e^+$ is a permutation in Edg$(G)$.

We say that a loop in $\Phi$ is $G$-generic if it misses the vertices of $G$, meets all edges of $G$ transversely, and never traverses a point of $G$ more than once. It is clear that any loop in $\Phi$ can be made $G$-generic by a small deformation. We orient each edge $e$ of $G$ from its blue vertex to its red vertex. For a $G$-generic loop $a$ in $\Phi$, set $a \cap e = a(S^1) \cap e$. The intersection sign of $a$ and $e$ at any point $p \in a \cap e$ is denoted by $\mu_p(a)$. Set $a \cdot e = \sum_{p \in a \cap e} \mu_p(a)$.
Proof. Since the loops \(a, b\) miss the vertices of \(G\), each red vertex \(w \in R(G)\) has a closed disk neighborhood \(D_w\) in \(\Phi \setminus (a(S^1) \cup b(S^1))\). We can assume that the disks \(\{D_w\}_w\) are disjoint and \(D_w \cap G \subset \Phi\) is a star with center \(w\) for all \(w\). Then

\[ \Gamma = \Phi \setminus \bigcup_{w \in R(G)} \text{Int}(D_w) \]

is a compact connected subsurface of \(\Phi\) which we provide with orientation induced by that of \(\Phi\). For every blue vertex \(v \in B(G)\), the set

\[ s_v = \Gamma \cap \bigcup_{e \in \text{Edg}_v} e \]

is a star in \(\Gamma\) with center \(v\). It is clear that the stars \(\{s_v \mid v \in R(G)\}\) are pairwise disjoint. The assumption that \(G\) is a filling graph of \(\Phi\) implies that this family of stars is a star filling of \(\Gamma\). Since \(a(S^1) \cup b(S^1) \subset \Gamma\), the loops \(a, b\) represent certain homology classes \(x', y' \in H_1(\Gamma)\). Theorems \ref{the:1.1} and \ref{the:1.2} imply that

\[ 2x' \cdot y' = \sum_{e \in \text{Edg}(G)} ((a \cdot e)(b \cdot e^+) - (b \cdot e)(a \cdot e^+) ). \]

(8.2.2) On the other hand, the inclusion homomorphism \(H_1(\Gamma) \to H_1(\Phi)\) carries \(x', y'\) respectively to \(x, y\). The usual definition of the intersection forms \(\cdot_T\) and \(\cdot_\Phi\) implies that \(x' \cdot_T y' = x \cdot_{\Phi} y\). Combining this with (8.2.2), we obtain (8.2.1). \(\square\)

To state similar results for the bracket \([-, -]_q\) in the module \(M\), we need more notation. For a loop \(a\) in \(\Phi\), we let \(\langle a \rangle \in M\) be the free homotopy class of \(a\). For a point \(p \in \Phi\) traversed by \(a\) only once, we let \(a_p\) be the loop in \(\Phi\) starting at \(p\) and going along \(a\) until the return to \(p\). We say that a pair of loops \(a, b\) in \(\Phi\) is \(G\)-generic if \(a, b\) are \(G\)-generic and do not meet at points of \(G\). Consider a \(G\)-generic pair of loops \(a, b\) and consider edges \(e, e'\) of \(G\) sharing the same blue vertex \(v\). For any points \(p \in a \cap e, q \in b \cap e'\), we let \(c = c_{p,q}\) be the path going from \(p\) to \(v\) along \(e\) and then going from \(v\) to \(q\) along \(e'\). We write \(a_p b_q\) for the loop \(a_p c_{p,q} b_q c^{-1}\).

Theorem 8.2. For any free homotopy classes \(x, y\) of loops in \(\Phi\) and any \(G\)-generic pair of loops \(a, b\) in \(\Phi\) representing \(x, y\), we have

\[ 2[x, y]_\Phi = \sum_{e \in \text{Edg}(G)} \left( \sum_{p \in a \cap e, q \in b \cap e^+} \mu_p(a) \mu_q(b) \langle a_p b_q \rangle - \sum_{p \in a \cap e^+, q \in b \cap e} \mu_p(a) \mu_q(b) \langle a_p b_q \rangle \right). \]

This theorem is deduced from Theorem \ref{the:1.8} using the construction introduced in the proof of Theorem \ref{the:8.1}.

We finally compute the cobracket \(\nu_\Phi : M \to M \otimes M\). For a loop \(a\) in \(\Phi\), we set \(\langle a \rangle_0 = \langle a \rangle \in M\) if \(a\) is non-contractible and \(\langle a \rangle_0 = 0 \in M\) if \(a\) is contractible. Consider a \(G\)-generic loop \(a\) in \(\Phi\) and edges \(e, e'\) of \(G\) sharing the same blue vertex \(v\). For points \(p \in a \cap e, p' \in a \cap e'\), we write \(a_{p,p'}\) for the loop going from \(p\) to \(p'\) along \(a\), then going to \(v\) along \(e'\), then going back to \(p\) along \(e\).
Theorem 8.3. For any free homotopy class $x$ of loops in $\Phi$ and any $G$-generic loop $a$ in $\Phi$ representing $x$, we have

$$2\nu(a) = \sum_{e \in Edg(G)} \sum_{p \in a[e], p' \in a[e]^+} \mu_p(a) \mu_{p'}(a) \left( (a_p,p')_0 \otimes (a_{p'},p)_0 - (a_{p'},p)_0 \otimes (a_{p},p')_0 \right).$$

This theorem is deduced from Theorem 1.4 using the construction in the proof of Theorem 8.1.

8.3. Example. We check Formula (8.2.1) in a case where all computations can be done explicitly. Set $I = [0,1]$. Identifying the opposite sides of the square $I^2$ via $(t,0) = (t,1)$ and $(0,t) = (1,t)$ for all $t \in I$ we obtain a 2-torus $T$. Let $p : I^2 \to T$ be the projection. The vertices $A_1 = (0,0), A_2 = (1,0), A_3 = (1,1), A_4 = (0,1)$ of $I^2$ project to the same point of $T$ denoted $A$. Let $O = (1/2,1/2)$ be the center of $I^2$. Consider the graph $G \subset T$ with two vertices $p(O)$, $A$ and four edges $\{e_i = p(OA_i)\}_{i=1}^4$ where $OA_i$ is the straight segment connecting $O$ and $A_i$ in $I^2$. We let $p(O)$ be the blue vertex of $G$ and let $A$ be the red vertex of $G$. This turns $G$ into a filling graph of $T$. Consider the homology classes $x, y \in H_1(T)$ represented respectively by the loops

$$a : S^1 \to T, e^{2\pi it} \mapsto p((t,1/3)) \quad \text{and} \quad b : S^1 \to T, e^{2\pi it} \mapsto p((1/4,t)).$$

These loops meet in the point $p((1/4,1/3))$. We orient $T$ so that $x \cdot_T y = +1$. Then

$$a \cdot e_1 = a \cdot e_2 = -1, \quad a \cdot e_3 = a \cdot e_4 = 0$$

and

$$b \cdot e_1 = b \cdot e_4 = 1, \quad b \cdot e_2 = b \cdot e_3 = 0.$$ 

Clearly, $(e_i)^+ = e_{i+1}$ for $i = 1, 2, 3$ and $(e_4)^+ = e_1$. Therefore

$$\sum_{e \in Edg(G)} ((a \cdot e)(b \cdot e^+) - (b \cdot e)(a \cdot e^+)) = -(b \cdot e_1)(a \cdot e_2) - (b \cdot e_4)(a \cdot e_1) = 2 = 2x \cdot_T y$$

which checks Formula (8.2.1) in this case.

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