On an example of a system of differential equations that are integrated in Abelian functions

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Abstract. The short review of the theory of Abelian functions and its applications in mechanics and analytical theory of differential equations is given. We think that Abelian functions are the natural generalization of commonly used functions because if the general solution of the 2nd order differential equation depends algebraically on the constants of integration, then integrating this equation does not lead out of the realm of commonly used functions complemented by the Abelian functions (Painlevé theorem). We present a relatively simple example of a dynamical system that is integrated in Abelian integrals by “pairing” two copies of a hyperelliptic curve. Unfortunately, initially simple formulas unfold into very long ones. Apparently the theory of Abelian functions hasn’t been finished in the last century because without computer algebra systems it was impossible to complete the calculations to the end. All calculations presented in our report are performed in Sage.

1. Introduction
Abelian functions arose for the first time in the works of Jacobi as the natural generalization of elliptic functions [1]. Let $P$ be a polynomial from the ring $\mathbb{Q}[x]$ with simple roots in the field $\mathbb{C}$. If the polynomial has degree 1 or 2 and variables $x$ and $u$ are connected by the equation

$$\int \frac{dx}{\sqrt{P(x)}} = u,$$

then $x$ can be expressed as an entire elementary function of $u$. If the polynomial has degree 3 or 4 then $x$ can be expressed from equation (1) as a meromorphic function of $u$, this function isn’t elementary, but special function called elliptic function. In general case the finding of the dependence $x(u)$ is called the inversion of the integral (1).

If the polynomial has degree larger then 4 then the algebraic curve

$$C: \quad y^2 = P(x),$$

on a projective $xy$-plane is called hyperelliptic. We will consider integral the integral

$$u = \int \frac{dx}{\sqrt{P(x)}} = \int \frac{dx}{y}$$

as a function of point $(x, y)$ on the curve $C$ and and the finding of the dependence $x(u)$ is cal it the hyperelliptic integral. Jacobi wanted to solve the problem of the inversion for hyperelliptic integrals, that is to find of the dependence $x(u)$, but encountered on the following obstacle.
Theorem 1 (Jacobi) If \(\alpha\) and \(\beta\) are two roots of \(P\) then

\[
\omega = 2 \int_{\xi=\alpha}^{\beta} \frac{d\xi}{\eta}
\]

is a period of the hyperelliptic integral

\[
\int \frac{d\xi}{\eta}.
\]

Thus the number of independent periods of the hyperelliptic integral grows together with the degree of the polynomial \(P\).

If the polynomial has degree 3 or 4 then there are exactly two complex periods \(\omega_1\) and \(\omega_2\). There are meromorphic functions with such periods, for example, Weierstrass \(\wp\)-function

\[
\wp(u) = \frac{1}{z^2} + \sum_{m^2+n^2\neq 0} \left\{ \frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}.
\]

It is obvious that

\[
\wp \left( \int_{(a,b)}^{(x,y)} \frac{d\xi}{\sqrt{P(\xi)}} \right)
\]

is a single-valued function defined at all points of the elliptic curve (2). There are no singularities of the integral (1) on this curve thus the expression (3) is an algebraical function of the variable \(x\):

\[
\wp(u) = r(x).
\]

From this equation we can express \(x\) as algebraic function of \(\wp(u)\). For this reason the inversing of the elliptic integral in elliptic functions is possible.

However if degree of \(P\) is equal to 5 then it is not more possible to express \(x\) as a meromorphic function of \(u\) because there are no single-valued functions of one variable having 4 periods. Jacobi claimed that the problem of the inversion for hyperelliptic integrals isn’t solvable not only in meromorphic functions but even in any analytical expressions [1].

There are no single-valued functions of one variable having 4 periods, but there exist such functions among the functions of two variables. So Jacobi proposed the following generalization of the problem

\[
\left\{ \begin{array}{l}
\int_{(a_1,b_1)}^{(\xi_1,\eta_1)} \frac{d\xi}{\sqrt{P(\xi)}} + \int_{(a_2,b_2)}^{(\xi_2,\eta_2)} \frac{d\xi}{\sqrt{P(\xi)}} = u, \\
\int_{(a_1,b_1)}^{(\xi_1,\eta_1)} \frac{\xi d\xi}{\sqrt{P(\xi)}} + \int_{(a_2,b_2)}^{(\xi_2,\eta_2)} \frac{\xi d\xi}{\sqrt{P(\xi)}} = v.
\end{array} \right.
\]

(4)

Weierstrass [2] proved that rational symmetric functions of the variables \(\xi_1, \xi_2\) are really meromorphic functions of the variables \(u, v\), and they are called Abelian functions.

Theorem 2 (Weierstrass) The rational symmetric functions of two points \((\xi_1,\eta_1), (\xi_2,\eta_2)\) on the hyperelliptic curve (2) are meromorphic functions of the variables \(u, v\)
Weierstrass in his lectures in Berlin university [2] in 1870th years shown that the theory of \( \theta \)-series can be generalized on the case of several variables and that Abelian functions can be represented as ration of two \( \theta \)-series. In 1880th years it was clear that this theory is natural development of the theory of elliptic functions. In lectures of Weierstrass there is only the outline of the theory of the transcendental functions arising in the theory of Abelian functions.

Applications of hyperelliptic integrals in mechanics are obvious and numerous, we can see the integral (1) even in Philosophiae Naturalis Principia Mathematica of Newton. Abelian functions appear as a trick in the works of Jacobi and in the 1860s there was the theory of Abelian functions but there were no applications in physics.

Shortly after constructing the theory of Abelian functions, Weierstrass proposed to his students to find its applications in mechanics. H. Bruns investigated at this viewpoint the three bodies problem, he proved that this problem has no integrals which are expressed in algebraic functions but even in Abelian integrals [3, 4]. L. Königsberger and S.V. Kowalewski looked for the applications of abelian functions in mechanics with Weber forces [5] and in the theory of a gyroscope [6] respectively.

We can consider the equations of a gyroscope motion as the dynamical system

\[
\dot{x} = h(x, A, \ldots);
\]

the left side \( h \) depends of several parameters \( A, \ldots \). Mechanics of XVIII century found the condition on the parameters under which this system is integrated in elliptic functions. Kowalewski found the condition under which the dynamical system integrated in Abelian functions, with both arguments being polynomial functions \( t \), that is the solution of the dynamical system has the form

\[
x = A_l(f(t), g(t)), \tag{5}
\]

where \( A_l \) is an abelian function, \( f \) and \( g \) are univariable polynomials. One such case was successfully found by Kowalewski in the 1880s years and it is now known as the case of Kowalewski of a gyroscope motion [6]. Kowaliwski gyroscope really can be made thus at the end of the XIX century there is the example of real dynamical system integrating in Abelian functions. However it wasn’t clear when and why these functions appear in physical problems.

In the 1890s, Painlevé given some explanation of the reasons in the scope of his theory the transcendent functions [7]. He noticed that general solutions of differential equations solvable in commonly used functions are not only meromorphic functions of the independent variable, but also algebraic functions of the constants. Painlevé managed to invert this assertion for first- and second- order equations.

**Theorem 3 (Painlevé)** If the general solution \( y \) of a given second-order equation

\[
f(x, \dot{x}, \ddot{x}; t) = 0,
\]

depends algebraically on the integration constants, then this solution can be expressed algebraically in transcendental functions belonging to the following categories:

(i) abelian functions of the form \( A_l(u, v) \) and their derivatives with respect to \( u \) and \( v \) where \( u \) and \( v \) can be expressed in terms of \( t \) by the quadratures

\[
u = \int h(t)dt + C_1, \quad v = \int k(t)dt + C_2;
\]

(ii) the elliptic functions \( \wp(u + C) \) and \( \wp'(u + C) \) where \( u \) can be expressed in terms of \( t \) by the following quadrature:

\[
u = \int h(t)dt,
(iii) the solutions of the linear or Riccati equations.

Obviously, in case (i) we have solution in the form (5), which was postulated in the begin of
the investigation of Kowalewski.

There are nontrivial connections between Painlevé property, integrability of systems and
property of general solutions, described in the theorem [6]. Kowalewski’s gyroscope has all three
properties. However inverse square system of Calogero [8] has only two last properties. In other
hand the differential equation

$$y'' = 6y^2 + x$$

has only Painlevé property, but its general solution depends on the integration constants non
algebraically and thus it is non classical transcendental function know as Painlevé transcendents
[10]. Furthermore simplified equations for differential equations with Painlevé property can be
integrated in Abelian functions [9].

The theorem 3 is also important for computer algebra. The idea of solvability in finite terms
is one of those obscure notions that are commonly understood to be a historical convention
and are in essence the subject of a general agreement on what this term actually means. The
theorem 3 leads us to an unexpected conclusion: by fixing the algebraic properties of the general
solution one can derive a class of commonly used transcendental functions. In this viewpoint
Painlevé theory was given in [11].

2. Motivation

Mathematicians of 19th century considered the theory of abelian functions as the necessary
completion of mathematical analysis, but after WWI works in this theory have died away. So
Felix Klein wrote in 1926 [12]:

Als ich studierte, galten die Abelschen Funktionen (...) als der unbestrittene Gipfel
der Mathematik, und jeder von uns hatte den selbstverständlichen Ehrgeiz hier selbst
weiterzukommen. Und jetzt? Die junge Generation kennt die Abelschen Funktionen
kaum mehr.

Now there is a number of reference books on elliptic functions [13, 14], many systems of computer
algebra like Maple and Mathematica successfully apply the elliptic functions for symbolical
integration. At the same time there are no reference books and manuals, no packages in computer
algebra systems for the work with abelian functions. It should be noted that old textbook of
Baker [15] written before the publishing of Weierstrass lectures is still in the use.

For development of this theory we need test examples. As can be seen, the Kowalewski’s
gyroscope gives an illustration for the Painlevé theorem, but this illustration is extremely difficult
computationally. Now we want present a relatively simple example of a dynamical system that
is integrated in Abelian integrals by “pairing” two copies of a hyperelliptic curve. Unfortunately,
initially simple formulas unfold into very long ones. Apparently the theory of Abelian functions
did not evolve in the last century because without computer algebra systems it was impossible
to compute the calculations to the end.

3. Pairing two copies of a hyperelliptic curve

For a start we construct Abelian variety that is algebraic surface which can be parametrized
with the help of Abelian functions. We take two constant points \((a_1, b_1), (a_2, b_2)\) and two mobile
points \((\xi_1, \eta_1), (\xi_2, \eta_2)\) on the hyperelliptic curve

$$C: \quad \eta^2 = \xi(\xi - 1)(\xi - 2)(\xi - 3)(\xi - 4)$$
and define the correspondence between these points and two variables $u$ and $v$ by equation (4).
By theorem 2 symmetric functions

$$x = \xi_1 + \xi_2, \quad y = \xi_1 \xi_2, \quad z = \eta_1 + \eta_2$$

are Abelian functions of two parameters $u, v$.

These functions define an algebraic surface

$$S(x, y, z) = 0, \quad S \in \mathbb{Z}[x, y, z]$$

Eliminating the variables $\xi_1, \xi_2, \eta_1, \eta_2$ from system

$$x = \xi_1 + \xi_2, \quad y = \xi_1 \xi_2, \quad z = \eta_1 + \eta_2, \quad \eta_1^2 = P(\xi_1), \quad \eta_2^2 = P(\xi_2),$$

we can calculate the equation of this surface:

$$x^{10} - 20x^9 - 10x^8y + 170x^7 + 180x^7y + 35x^6y^2 - 800x^7 - 1360x^6y - 540x^5y^2$$

$$- 50x^4y^3 - 2x^5z^2 + 2723x^6y + 5600x^5y + 3400x^4y^2 + 600x^3y^3 + 25x^2y^4 + 20x^4z^2$$

$$+ 10x^3yz^2 - 3980x^5 - 13686x^y - 11100x^3y^2 - 2790x^2y^3 - 160xy^4 - 4y^5 - 70x^3z^2$$

$$- 80x^2y^2z^2 - 10x^2y^2z^2 + 4180x^4 + 20380x^3y + 19649x^2y^2 + 6000xy^3 + 280y^4 + 100xz^2$$

$$+ 2100xz^2 + 400z^2 + z^4 - 2400x^2 - 18400x^2y - 17840xy^2 - 5092y^3 - 48xz^2$$

$$- 200yz^2 + 576x^2 + 9600xy + 6720y^2 - 2304y = 0$$

This expression is very long thus it was never written out explicitly.

4. Cross-section $z = 0$

The most noticeable structures on the surface $S$ lies on cross-section by the plane $z = 0$. In general the cross-section of $S$ by the plane $z = a$ is irreducible plane curve of very large genus, however this cross-section is the union of two curves: the curve

$$Q : \quad x^4 - 10x^3 - 3x^2y + 35x^2 + 20xy + y^2 - 50x - 35y + 24 = 0$$

and the parabola $Q' : \quad 4y = x^2$. These curves on the surface $S$ have extremely small genus.

The curve $Q$ is elliptic curve and thus its genus is equal only 1. Points of the curve $Q$ are singularities of the surface $S$. We can check this in Sage by calculation of the radical ideal

$$\sqrt{\left( \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z} \right)} = (z, x^4 - 10x^3 - 3x^2y + 35x^2 + 20xy + y^2 - 50x - 35y + 24).$$

For interpretation of the second curve we return to variable $\xi_1, \ldots$. In these variables the equation $z = 0$ means that

$$\eta_1 = -\eta_2$$

and thus

$$P(\xi_1) = P(\xi_2).$$

So this cross-section consists of two lines, one of them is

$$\xi_1 = \xi_2, \quad \eta_1 = -\eta_2,$$

that is $Q'$. By Abel theorem two points $p, p' \in Q'$ correspondent to one value of the parameter $u$, e.g.

$$u_0 = \int_{(a_1,b_1)} \frac{d\xi}{\eta} + \int_{(a_2,b_2)} \frac{d\xi}{\eta} = \int_{(a_1,b_1)} \frac{d\xi}{\eta} + \int_{(a_2,b_2)} \frac{d\xi}{\eta}$$

and one value of the parameter $v$. So the correspondence between $u, v$ and points of $S$ is one-to-one almost everywhere, but $Q'$ is an exceptional curve [16].
5. Generator family on $S$

We can imagine the surface $S$, indicating a pencil of curves on $S$. If we fix the point $(\xi_2, \eta_2)$, but will move the point $(\xi_1, \eta_1)$ along the curve $C$, then formulas

$$x = \xi_1 + \xi_2, \quad y = \xi_1 \xi_2, \quad z = \eta_1 + \eta_2,$$

define a curve on $S$. So there is correspondence between points of $C$ and curves on $S$. We can describe these curves by explicit formulas

$$\begin{cases} x\xi_2 - \xi_2^2 - y = 0, \\ \xi_2^3 - 10\xi_2^2 + 35\xi_2^3 - 50\xi_2^2 - \eta_2^2 + 24\xi_2 = 0, \\ x^5 - 10x^4 - 5x^3y + 35x^3 + 40x^2y + 5xy^2 - 50x^2 - 105xy - 20y^2 - \\ -z^2 + 2zy\eta_2 - 2\eta_2^2 + 24x + 100y = 0 \end{cases}$$

This means that the curve which correspond to the point $(\xi_2, \pm \eta_2)$ lies in the plane

$$x\xi_2 - \xi_2^2 - y = 0.$$

Thus the surface $S$ is weaved by these plane curves (as hyperboloid is weaved by right lines). This curves form an irrational pencil on the surface [16].

6. Differentials

Any rational symmetric function of two points $(\xi_1, \eta_1), (\xi_2, \eta_2)$ is rational function on $S$ and vice versa. These functions are meromorphic with respect to the variables $u, v$.

On the surface $S$ there are two linear independent holomorphic differentials, namely

$$du = \frac{d\xi_1}{\eta_1} + \frac{d\xi_2}{\eta_2} = p_{11} dx + p_{12} dy,$$
$$dv = \frac{\xi_1 d\xi_1}{\eta_1} + \frac{\xi_2 d\xi_2}{\eta_2} = p_{21} dx + p_{22} dy,$$

For calculating of $p_{ij}$ it should be noted that

$$\begin{cases} dx = d\xi_1 + d\xi_2, \\ dy = \xi_1 d\xi_2 + \xi_2 d\xi_1, \end{cases}$$

so

$$du = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_1 dx - dy}{\eta_1} - \frac{\xi_2 dx - dy}{\eta_2} \right]$$

and

$$dv = \frac{1}{\xi_1 - \xi_2} \left[ \frac{\xi_1^2 dx - \xi_1 dy}{\eta_1} - \frac{\xi_2^2 dx - \xi_2 dy}{\eta_2} \right].$$

By elimination technique we can calculate in Sage the explicit expression for $p_{ij}$. For example $p_{11}$ is ratio of

$$-x^5 + 10x^4 + 3x^3y - 35x^3 - 20x^2y - xy^2 + 50x^2 + 35xy + z^2 - 24x$$

and

$$(-x^5 + 10x^4 + 5x^3y - 35x^3 - 40x^2y - 5xy^2 + 50x^2 + 105xy + 20y^2 + z^2 - 24x - 100y)z.$$

The restriction $du$ and $dv$ to the general curve $K$ on the surface $S$ give two linear independent holomorphic differentials for the curve $K$, thus genus of $K$ is more or equal to 2. This fact doesn’t contradict existence of the curves $Q$ and $Q'$. For example the numerator and the denominator of $p_{11}$ are equal to 0 on $Q$. 

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7. Dynamical system
Let $\text{Al}(u, v)$ be meromorphic function with 4 periods. Then

$$\text{Al}\left(\int p_{11}dx + p_{12}dy, \int p_{21}dx + p_{22}dy\right)$$

is rational function on the surface $S$.

Thus integration of dynamical system

$$\begin{cases}
 p_{11}(x, y, z)\dot{x} + p_{12}(x, y, z)\dot{y} = f(t) \\
 p_{21}(x, y, z)\dot{x} + p_{22}(x, y, z)\dot{y} = g(t)
\end{cases}$$

with arbitrary functions $f(t)$ and $g(t)$ introduces transcendental functions of the form

$$\text{Al}\left(\int f(t)dt, \int g(t)dt\right).$$

**Theorem 4** Integration of the dynamical system

$$\begin{cases}
 \dot{x} = \det \begin{pmatrix} f(t) & p_{12} \\ g(t) & p_{22} \end{pmatrix} / \det \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \\
 \dot{y} = \det \begin{pmatrix} p_{11} & f(t) \\ p_{21} & g(t) \end{pmatrix} / \det \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}
\end{cases}$$

on the surface $S$ introduces two expressions of the form

$$\text{Al}\left(\int f(t)dt, \int g(t)dt\right).$$

In mechanical problems the equation $S(x, y, z) = 0$ has the meaning of conservation law.

8. Conclusion
In last sections we have constructed the test-example of dynamical system integrating in Abelian functions and satisfying to the theorem 3 of Painlevé. For use in practice we need to solve the inverse problem. We have a system of ODEs on some surface, how we can understand that this system can be integrated in Abelian functions?

Unfortunately modern CAS (like Maple) can’t find:

(i) the group of automorphisms even for curves,
(ii) holomorphic differentials on the surface, in Maple we can do this only for curves, and
(iii) families of curves on the surface with some extremal properties like generator family.

Perhaps special curves $Q$ and $Q'$ are a key to identification at least in practice.

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