The Brezis-Nirenberg problem on non-contractible bounded domains of $\mathbb{R}^3$

Mohammed ALDAWOOD$^a$, Cheikh Birahim NDIAYE$^b$

$^a,^b$ Department of Mathematics Howard University
Annex 3, Graduate School of Arts and Sciences, # 217
DC 20059 Washington, USA.

Abstract

In this paper, we study the Brezis-Nirenberg problem on bounded smooth domains of $\mathbb{R}^3$. Using the algebraic topological argument of Bahri-Coron$^2$ as implemented in $^6$ combined with the Brendle$^4$-Schoen$^8$’s bubble construction, we solve the problem for non-contractible bounded smooth domains.

Key Words: Barycenter technique, PS-sequences, Self-action estimate, Inter-action estimate.

AMS subject classification: 53C21, 35C60, 58J60, 55N10.

1 Introduction and statement of the results

In their seminal paper $^5$, Brezis and Nirenberg initiated the study of nonlinear elliptic equations of the form

$$\begin{cases}
-\Delta u + qu = u^{\frac{n+2}{n-2}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $\Omega$ is a bounded and smooth domain of $\mathbb{R}^n$, $n \geq 3$ and $q$ is a bounded and smooth function defined on $\Omega$. In this paper, we revisit the Boundary Value problem (BVP) (1) in the 3-dimensional case, namely when $n = 3$. Thus, we will be dealing with the BVP

$$\begin{cases}
-\Delta u + qu = u^5 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^3$. It is well known that a necessary condition for the existence of positive solution to (2) is that the first eigenvalue of $-\Delta + q$ under zero Dirichlet boundary condition is positive, see $^3$. Moreover, we will assume that $-\Delta + q$ under zero Dirichlet boundary condition verifies the strong maximum principle. Hence the BVP (2) has a variational structure, since thanks to the strong maximum principle and standard elliptic regularity theory solutions of (2) can be found by looking at critical points of the Brezis-Nirenberg functional

$$J_q(u) := \frac{\langle u, u \rangle_q}{\langle \int_{\Omega} u^6 \, dx \rangle_2}, \quad u \in H^1_0(\Omega) := \{ u \in H^1_0(\Omega) : \, \, u \geq 0 \text{ and } u \neq 0 \},$$

where

$$\langle u, u \rangle_q = \int_{\Omega} (|\nabla u|^2 + qu^2) \, dx$$

E-mail addresses: cheikh.ndiaye@howard.edu, mohammed.aldawood@bison.howard.edu

C. B. Ndiaye was partially supported by NSF grant DMS-2000164.
and \( H^1_0(\Omega) \) is the usual Sobolev space of functions which are \( L^2 \)-integrable on \( \Omega \) together with their first derivatives and with zero trace on \( \partial \Omega \).

Existence of solutions under a Positive Mass type assumption has been obtained in an unpublished work by McLeod as discussed in the work of Brezis[3]. In this work, we use the Barycenter technique of Bahri-Coron[2] to remove the Positive Mass type assumption of McLeod and replace it by the non-contractibility of the domain. Precisely, we prove the following theorem.

**Theorem 1.1.** Assuming that \( \Omega \subset \mathbb{R}^3 \) is a non-contractible bounded and smooth domain, \( q \) is a smooth and bounded function defined on \( \Omega \), the first eigenvalue of the operator \(-\Delta + q\) under zero Dirichlet boundary condition on \( \partial \Omega \) is positive, \(-\Delta + q\) under zero Dirichlet boundary condition on \( \partial \Omega \) verifies the strong maximum principle and the Green's function \( G \) of \(-\Delta + q\) under zero Dirichlet boundary condition on \( \partial \Omega \) defined by (11) is positive, then the BVP (2) has a least one solution.

To prove Theorem 1.1, we will use the Algebraic topogical argument of Bahri-Coron[2] which is possible since as already observed by McLeod (see [3]), the problem under study is a Global one (for the definition of "Global" for Yamabe type problems, see [7]). Indeed, as in [7], we will follow the scheme of the Barycenter technique as performed in the work [6] of the second author and Mayer. One of the main difficulty with respect to the works [6], and [7] is the presence of the linear term "qu" and the lack of conformal invariance. Such a difficulty has already been encountered by Bahri-Brezis[1] on closed Riemannian manifolds. To deal with such a difficulty, Bahri-Brezis[1] have used the bubble construction of Bahri-Coron[2] recalling that their scheme of the Barycenter technique follows the original one of Bahri-Coron[2]. However, here we use the Brendle[4]-Schoen[5]’s bubble construction and have to deal with that difficulty in a way different from the work of Bahri-Brezis[1].

### 2 Notations and preliminaries

In this section, we fix some notation and discuss some preliminaries. We start with fixing some notation. \( \mathbb{N} \) denotes the set of non-negative integers and \( \mathbb{N}^* \) denotes set of the positive integers. For \( a \in \mathbb{R}^3 \) and \( \delta > 0 \), \( B_a(\delta) = B(a, \delta) \) denotes the Euclidean Ball with radius \( \delta \) centered at \( a \). \( 1_A \) denotes the characteristic function of \( A \). \( \nabla \) denotes the Euclidean gradient and \( \Delta \) denotes the Euclidean Laplacian.

All integrations are with respect to \( dx \) the standard Lebesgue measure on \( \mathbb{R}^3 \) with \( x = (x_1, x_2, x_3) \) the standard coordinate system of \( \mathbb{R}^3 \). \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \) respectively denote the standard norm and scalar product on \( \mathbb{R}^3 \). We also use \( | \cdot | \) to denote the absolute value on \( \mathbb{R} \). For \( E \subset \mathbb{R}^3 \) and \( p \in \mathbb{N}^* \), \( L^p(E) \) denotes the usual Lebesgue space of order \( p \) with respect to \( dx \). For \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) a smooth function, \( \chi' \) and \( \chi'' \) respectively denote the first derivative and second derivative of \( \chi \). For \( a \in K \subset \Omega \), \( K \) compact, and \( 0 \leq d_1 < d_2 \leq \infty \), we set \( \{ d_1 \leq |x - a| \leq d_2 \} = \{ x \in \Omega : d_1 \leq |x - a| \leq d_2 \} \). To simplify notation, we write \( d_1 \leq |x-a| \leq d_2 \) instead of \( \{ d_1 \leq |x-a| \leq d_2 \} \) if there is no possible confusion. Similarly, for \( 0 \leq d_1 < d_2 \leq \infty \), we set \( \{ d_1 \leq |y| \leq d_2 \} = \{ y \in \mathbb{R}^3 : d_1 \leq |y| \leq d_2 \} \). To simplify notation, we write \( d_1 \leq |y| \leq d_2 \) instead of \( \{ d_1 \leq |y| \leq d_2 \} \) if there is no possible confusion.

Next, we introduce the standard bubbles of the variational problem under study. For \( a \in \mathbb{R}^3 \) and \( \lambda > 0 \), we denote by \( \delta_{a, \lambda} \) the standard bubble on \( \mathbb{R}^3 \), namely

\[
\delta_{a, \lambda}(x) = c_0 \left( \frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^3,
\]

where \( c_0 > 0 \) is such that \( \delta_{a, \lambda} \) satisfies

\[
-\Delta \delta_{a, \lambda} = \delta_{a, \lambda}^5 \quad \text{on} \quad \mathbb{R}^3.
\]

We have also the following relations

\[
\int_{\mathbb{R}^3} |\nabla \delta_{a, \lambda}|^2 = \int_{\mathbb{R}^3} \delta_{a, \lambda}^6 = \int_{\mathbb{R}^3} |\nabla \delta_{0, 1}|^2 = \int_{\mathbb{R}^3} \delta_{0, 1}^6
\]

and

\[
S = \frac{\int_{\mathbb{R}^3} |\nabla \delta_{a, \lambda}|^2}{(\int_{\mathbb{R}^3} \delta_{a, \lambda}^6)^{\frac{1}{2}}},
\]

where

\[
S = \inf_{u \in D^1(\mathbb{R}^3), u \neq 0} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} u^6)^{\frac{1}{2}}}
\]

(9)
with
\[ D^1(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \} . \]

We set
\[ c_3 = \int_{\mathbb{R}^3} \left( \frac{1}{1 + |y|^2} \right)^{\frac{3}{2}} . \]

For \( a \in \Omega \), let \( G(a, x) \) be the unique solution of (see [3])
\[
\begin{cases}
-\Delta G(a, x) + qG(a, x) = 4\pi \delta_a(x), & x \in \Omega \\
G(a, x) = 0, & x \in \partial\Omega.
\end{cases}
\]

\( G(a, x) \) satisfies the following estimates
\[ |G(a, x) - \frac{1}{|x - a|}| \leq C \text{ for } x \neq a \in \Omega, \]
and
\[ \left| \nabla \left( G(a, x) - \frac{1}{|x - a|} \right) \right| \leq \frac{C}{|x - a|} \text{ for } x \neq a \in \Omega. \]

Moreover, under the assumption of Theorem [11], we have \( G > 0 \) in \( \Omega \).

Now, let \( \chi : \mathbb{R} \to [0, 1] \) be a smooth cut-off function satisfying
\[ \chi(t) = \begin{cases} 
1 & \text{if } t \leq 1 \\
0 & \text{if } t \geq 2.
\end{cases} \]

Using \( \chi \), for \( a \in \Omega \), and \( \delta > 0 \) and small, we define
\[ \chi_\delta^a(x) = \chi \left( \frac{|x - a|}{\delta} \right), \quad x \in \Omega. \]

Moreover, using \( \chi_\delta^a \) and the Green’s function \( G(a, \cdot) \), we define the Brendle [4]-Schoen [8]’s bubble
\[ u_{a, \lambda, \delta} = \chi_\delta^a \delta_{a, \lambda} + (1 - \chi_\delta^a) \frac{c_0}{\sqrt{\lambda}} G(a, x). \]

For \( K \subset \Omega \) compact, we set
\[ g_0 = g_0^K := \frac{\text{dis}(K, \partial\Omega)}{4} > 0. \]

Thus, for \( \forall a \in K \) and \( \forall 0 < 2\delta < g_0 \) we have
\[ u_{a, \lambda} := u_{a, \lambda, \delta} \in H^1_0(\Omega), \quad u_{a, \lambda} > 0 \text{ in } \Omega. \]

For \( a_i, a_j \in \Omega \) and \( \lambda_i, \lambda_j > 0 \), we define
\[ \varepsilon_{ij} = \left[ \frac{1}{\sqrt{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_j \lambda_i \lambda G^{-2}(a_i, a_j)} \right]^{\frac{1}{2}}. \]

Moreover, for \( a_i, a_j \in K \), \( 0 < 2\delta < g_0 \), and \( \lambda_i, \lambda_j > 0 \), we define
\[ \varepsilon_{ij} = \int_{\Omega} u_{a_i, \lambda_i, a_j, \lambda_j} \]
and
\[ \varepsilon_{ij} = \int_{\Omega} (-\Delta + q) u_{a_i, \lambda_i, a_j, \lambda_j} . \]

Using (6) and (11), we estimate the deficit of \( u_{a, \lambda} \) being a solution of BVP (2).

**Lemma 2.1.** Let \( K \subset \Omega \) be compact and \( \theta > 0 \) be small. Then there exists \( C > 0 \) such that \( \forall a \in K \), \( \forall 0 < 2\delta < g_0 \) and \( \forall 0 < \frac{\delta}{\delta^2} \leq \theta \delta \), we have
\[ |\Delta u_{a, \lambda} + q u_{a, \lambda} - u_{a, \lambda}^5| \leq C \left[ \frac{1}{\delta^2 \sqrt{\lambda}} 1_{\delta \leq |x - a| \leq 2\delta} + \delta_{a, \lambda} 1_{|x - a| \leq 2\delta} + \delta_{a, \lambda}^5 1_{|x - a| \geq 4\delta} \right], \]
where \( g_0 \) is as in (17).
This implies (for $x = a$) we get

\[
J = \chi_\delta = \chi_\delta(a) x - a + \frac{c_0}{\sqrt{\lambda|x - a|}} + \frac{c_0}{\sqrt{\lambda|x - a|}} - \frac{G_a}{\sqrt{\lambda}}.
\]

This implies

\[
(-\Delta + q) u_{\alpha, \lambda} = (-\Delta + q) \left( \chi_\delta \left( \frac{\delta_{\alpha, \lambda} - G_a}{\sqrt{\lambda}} \right) + \frac{(-\Delta + q) \tilde{G}_a}{\sqrt{\lambda}} \right).
\]

Clearly the lemma is true for $x = a$. Now, since for $x \neq a$, we have $(-\Delta + q) \tilde{G}_a = 0$, then for $x \neq a$ we get

\[
(-\Delta + q) u_{\alpha, \lambda} = -\Delta \chi_\delta \left( \delta_{\alpha, \lambda} - \frac{G_a}{\sqrt{\lambda}} \right) - 2 \nabla \chi_\delta \nabla \left( \delta_{\alpha, \lambda} - \frac{G_a}{\sqrt{\lambda}} \right) - \chi_\delta \Delta \delta_{\alpha, \lambda} + q \chi_\delta \delta_{\alpha, \lambda}.
\]

This implies (for $x \neq a$)

\[
(-\Delta + q) u_{\alpha, \lambda} - u_{\alpha, \lambda}^5 = \sum_{i=1}^{4} J_i
\]

with

\[
J_1 = -\Delta \chi_\delta \left( \delta_{\alpha, \lambda} - \frac{G_a}{\sqrt{\lambda}} \right)
\]

\[
J_2 = -2 \left( \nabla \chi_\delta, \nabla \left( \delta_{\alpha, \lambda} - \frac{G_a}{\sqrt{\lambda}} \right) \right)
\]

\[
J_3 = q \chi_\delta \delta_{\alpha, \lambda}
\]

\[
J_4 = -\chi_\delta \Delta \delta_{\alpha, \lambda} - u_{\alpha, \lambda}^5.
\]

Now, we are going to estimate separately each $J_i$'s. For $J_1$, we first write

\[
J_1 = \Delta \chi_\delta \left[ \delta_{\alpha, \lambda} - \frac{c_0}{\sqrt{\lambda|x - a|}} + \frac{c_0}{\sqrt{\lambda|x - a|}} - \frac{G_a}{\sqrt{\lambda}} \right].
\]  
\text{(22)}

Next, using (5) and (12), we derive

\[
\left| \delta_{\alpha, \lambda} - \frac{c_0}{\sqrt{\lambda|x - a|}} \right| \leq \frac{C}{\sqrt{\lambda}}
\]  
\text{(23)}

and

\[
\left| \frac{c_0}{\sqrt{\lambda|x - a|}} - \frac{G_a}{\sqrt{\lambda}} \right| \leq \frac{C}{\sqrt{\lambda}}
\]  
\text{(24)}

For $\Delta \chi_\delta$, we have

\[
\nabla \chi_\delta = \chi' \left( \frac{|x - a|}{\delta} \right) \frac{(x - a)}{\delta|x - a|}
\]  
\text{(25)}

This implies

\[
\Delta \chi_\delta = \chi'' \left( \frac{|x - a|}{\delta} \right) \frac{1}{\delta^2} + 2 \chi' \left( \frac{|x - a|}{\delta} \right) \frac{1}{\delta|x - a|}
\]  
\text{(26)}

Thus, recalling the definition of $\chi$ (see (14)), we have (26) implies

\[
|\Delta \chi_\delta| \leq \frac{C}{\delta^2} 1_{\{\delta \leq |x - a| \leq 2\delta}\}}.
\]  
\text{(27)}

Hence, combining (22), (23), (24), and (27), we get

\[
|J_1| \leq \frac{C}{\delta^2 \sqrt{\lambda}} 1_{\{\delta \leq |x - a| \leq 2\delta}\}}.
\]  
\text{(28)}

To estimate $J_2$, we first write

\[
J_2 = -2 \left( \nabla \chi_\delta, \nabla \left[ \delta_{\alpha, \lambda} - \frac{c_0}{\sqrt{\lambda|x - a|}} + \frac{c_0}{\sqrt{\lambda|x - a|}} - \frac{G_a}{\sqrt{\lambda}} \right] \right).
\]  
\text{(29)}
Next, using (5) and (13), we derive
\[
\left| \nabla \left[ \delta_{a,\lambda} - \frac{c_0}{\sqrt{\lambda|x-a|}} \right] \right| \leq \frac{C}{\sqrt{\lambda|x-a|}},
\]
and
\[
\left| \nabla \left[ \frac{c_0}{\sqrt{\lambda|x-a|}} - G_a \right] \right| \leq \frac{C}{\sqrt{\lambda|x-a|}}.
\]
On the other hand, using (25) and recalling (14), we derive
\[
\left| \nabla \chi_{\delta} \right| \leq \frac{C}{\lambda} \delta \{ \delta \leq |x-a| \leq 2\delta \}.
\]
Hence, combining (29)-(32), we get
\[
|J_2| \leq \frac{C}{\delta^2 \sqrt{\lambda}} 1_{\{ \delta \leq |x-a| \leq 2\delta \}}.
\]
For \( J_3 \), since \( q \) is bounded then using (14) and (15), we clearly obtain
\[
|J_3| \leq C \delta_{a,\lambda} 1_{\{ |x-a| \in 2\delta \}}.
\]
Finally to estimate \( J_4 \), we observe that for \( |x-a| \leq \delta \),
\[
\chi_{\delta}(x) = 1.
\]
Thus
\[
J_4 = -\chi_{\delta} \Delta \delta_{a,\lambda} - u^5_{a,\lambda} = -\Delta \delta_{a,\lambda} - \delta_{a,\lambda} = 0
\]
on \( \{ |x-a| \leq \delta \} \). On the other hand on \( \{ |x-a| > \delta \} \), we clearly have
\[
|u_{a,\lambda}| \leq C \delta_{a,\lambda}.
\]
Therefore, (35) and (36) imply
\[
|J_4| \leq \delta^5_{a,\lambda} 1_{\{ |x-a| \geq \delta \}}.
\]
Hence, the result follows from (28), (33), (34), and (37).

3 PS-sequences and Deformation lemma

In this section, we recall the analysis of Palais-Smale (PS) sequence for \( J_q \) defined by (3), see \[3\]. We also introduce the neighborhood of potential critical points at infinity of \( J_q \) and the associated selection maps. As in other applications of the Barycenter technique of Bahri-Coron\[2\], we also recall the associated Deformation lemma.

Lemma 3.1. Suppose that \((u_k) \subset H^1_0(\Omega)\) is a PS-sequence for \( J_q \), that is \( \nabla J_q(u_k) \rightarrow 0 \) and \( J_q(u_k) \rightarrow c \) up to a subsequence, and \( \int_\Omega u^6_k = c^2 \), then up to a subsequence, we have there exists \( u_\infty \geq 0 \), an integer \( p \geq 0 \), a sequence of points \( a_{i,k} \in \Omega \), \( i = 1, \cdots, p \), and a sequence of positive numbers \( \lambda_{i,k} \), \( i = 1, \cdots, p \), such that
1) \( -\Delta u_\infty + q u_\infty = u^5_\infty \).
2) \( \| u_k - u_\infty - \sum_{i=1}^p u_{a_{i,k},\lambda_{i,k}} \|_q \rightarrow 0 \).
3) \( J_q(u_k)^\frac{q}{2} \rightarrow J_q(u_\infty)^\frac{q}{2} + pS^\frac{q}{2} \).
4) For \( i \neq j = 1, \cdots, p \),
\[
\frac{\lambda_{j,k}}{\lambda_{i,k}} + \frac{\lambda_{j,k}}{\lambda_{i,k}} + \lambda_{i,k} \lambda_{j,k} G^{-2}(a_{i,k}, a_{j,k}) \rightarrow +\infty
\]
For $i = 1, \ldots, p$, 
\[ \lambda_i, \text{dist}(a_i, \partial \Omega) \rightarrow +\infty, \]
where $\| \cdot \|_q$ is the norm associated to the scalar product $\langle \cdot, \cdot \rangle_q$ defined by (4).

To introduce the neighborhoods of potential critical points at infinity of $J_q$, we first fix
\[ \varepsilon_0 > 0 \quad \text{and} \quad \varepsilon_0 \simeq 0. \quad (38) \]
Furthermore, we choose
\[ \nu_0 > 1 \quad \text{and} \quad \nu_0 \simeq 1. \quad (39) \]
Then for $p \in \mathbb{N}^*$, and $0 < \varepsilon \leq \varepsilon_0$, we define $V(p, \varepsilon)$ the $(p, \varepsilon)$-neighborhood of potential critical points at infinity of $J_q$ by
\[ V(p, \varepsilon) := \{ u \in H_0^1(\Omega) : \exists a_1, \ldots, a_p \in \Omega, \quad \alpha_1, \ldots, \alpha_p > 0, \quad \lambda_1, \ldots, \lambda_p > 0, \]
\[ \lambda_i \geq \frac{1}{\varepsilon} \quad \text{for} \quad i = 1 \ldots, p, \quad \lambda_i, \text{dist}(a_i, \partial \Omega) \geq \frac{1}{\varepsilon} \quad \text{for} \quad i = 1 \ldots, p, \]
\[ \| u - \sum_{i=1}^{p} \alpha_i u_{a_i, \lambda_i} \|_{q} \leq \varepsilon, \quad \frac{\alpha_i}{\alpha_j} \leq \nu_0 \quad \text{and} \quad \varepsilon_{i,j} \leq \varepsilon \quad \text{for} \quad i \neq j = 1, \ldots, p \}. \]

Concerning the sets $V(p, \varepsilon)$, for every $p \in \mathbb{N}^*$ there exists $0 < \varepsilon_p \leq \varepsilon_0$ such that for every $0 < \varepsilon \leq \varepsilon_p$, we have
\[ \forall u \in V(p, \varepsilon) \quad \text{the minimization problem} \quad \min_{B^p_\varepsilon} \| u - \sum_{i=1}^{p} \alpha_i u_{a_i, \lambda_i} \|_{q} \]
\[ \text{has a solution} \quad (\tilde{\alpha}, A, \tilde{\lambda}) \in B^p_\varepsilon, \quad \text{which is unique up to permutations}, \]
where $B^p_\varepsilon$ is defined as
\[ B^p_\varepsilon := \{ (\tilde{\alpha} = (\alpha_1, \ldots, \alpha_p), A = (a_1, \ldots, a_p), \tilde{\lambda} = (\lambda_1, \ldots, \lambda_p)) \in \mathbb{R}^p_+ \times \Omega^p \times (0, +\infty)^p \]
\[ \lambda_i \geq \frac{1}{\varepsilon}, \lambda_i, \text{dist}(a_i, \partial \Omega) \geq \frac{1}{\varepsilon}, i = 1, \ldots, p, \quad \frac{\alpha_i}{\alpha_j} \leq \nu_0 \quad \text{and} \quad \varepsilon_{i,j} \leq \varepsilon, i \neq j = 1, \ldots, p \}. \]

We define the selection map $s_p$ via
\[ s_p : V(p, \varepsilon) \rightarrow (\Omega)^p / \sigma_p : u \rightarrow s_p(u) = A \quad \text{and} \quad A \ \text{is given by (40)}. \]

To state the Deformation Lemma needed for the application of the algebraic topological argument of Bahri-Coron\cite{2}, we first set
\[ W_p := \{ u \in J_q(u) \leq (p + 1)^{\frac{p}{2}} \delta \mathcal{S} \}, \quad (41) \]
for $p \in \mathbb{N}$.

As in \cite{2, 6, 7}, we have Lemma 3.1 implies the following Deformation lemma.

**Lemma 3.2.** Assuming that $J_q$ has no critical points, then for every $p \in \mathbb{N}^*$, up to taking $\varepsilon_p$ given by (40) smaller, we have that for every $0 < \varepsilon \leq \varepsilon_p$, the topological pair $(W_p, W_{p-1})$ retracts by deformation onto $(W_{p-1} \cup A_p, W_{p-1})$ with $V(p, \varepsilon) \subset A_p \subset V(p, \varepsilon)$ where $0 < \varepsilon < \frac{\varepsilon_p}{2}$ is a very small positive real number and depends on $\varepsilon$.

## 4 Self-action estimates

In this section, we derive some sharp estimates needed for application of the Barycenter technique of Bahri-Coron\cite{2}.

For the numerator of $J_q$, we have.

**Lemma 4.1.** Assuming that $K \subset \Omega$ is compact and $\theta > 0$ is small, then there exists $C > 0$ such that $\forall a \in K$, $\forall \theta > 2\delta < \theta_0$, and $\forall \theta < \frac{1}{\lambda^2 \delta^2}$, we have
\[ \int_\Omega (-\Delta + q) u_{a, \lambda} \leq \int_\Omega u_{a, \lambda}^p + C \left( 1 + \frac{1}{\lambda^2 \delta^2} \right). \]
Proof. Setting
\[ I = \int_{\Omega} (-\Delta + q) u_{a,\lambda} u_{a,\lambda}, \]
we get
\[ I = \int_{\Omega} u_{a,\lambda}^6 + \int_{\Omega} [(-\Delta + q) u_{a,\lambda} - u_{a,\lambda}^5] u_{a,\lambda}. \]

To continue, let us estimate \( I_1 \). Using Lemma 2.1, we get
\[ |I_1| \leq \int_{\Omega} \left| (-\Delta + q) u_{a,\lambda} - u_{a,\lambda}^5 \right| u_{a,\lambda} \]
\[ \leq \frac{C}{\delta^2 \sqrt{\lambda}} \int_{\Omega} u_{a,\lambda} 1_{\{\delta \leq |x-a| \leq 2\delta\}} \]
\[ + C \int_{\Omega} \delta_{a,\lambda} u_{a,\lambda} 1_{\{|x-a| \leq 2\delta\}} \]
\[ + C \int_{\Omega} \delta_{a,\lambda}^5 u_{a,\lambda} 1_{\{|x-a| \geq \delta\}}. \]

We are going to estimate the three parts of the right hand side the latter formula. For the first term, we have
\[ \int_{\Omega} u_{a,\lambda} 1_{\{\delta \leq |x-a| \leq 2\delta\}} \leq C \int_{\delta \leq |x-a| \leq 2\delta} \left[ \frac{\lambda}{1 + \lambda^2 |x-a|^2} \right]^{\frac{1}{2}} \]
\[ \leq C \int_{\delta \leq |x-a| \leq 2\delta} \frac{1}{\sqrt{\lambda} |x-a|} \]
\[ \leq C \frac{\delta^2}{\sqrt{\lambda}} \int_{\delta}^{2\delta} r \, dr \]
\[ \leq C \frac{\delta^2}{\sqrt{\lambda}}. \]

For the second term, we obtain
\[ \int_{\Omega} \delta_{a,\lambda} u_{a,\lambda} 1_{\{|x-a| \leq 2\delta\}} \leq C \int_{|x-a| \leq 2\delta} \left[ \frac{\lambda}{1 + \lambda^2 |x-a|^2} \right] \]
\[ \leq \frac{1}{\lambda} \int_{0}^{2\delta} dr \]
\[ \leq C \frac{\delta}{\lambda}. \]

Finally for the last term, we get
\[ \int_{\Omega} \delta_{a,\lambda}^5 u_{a,\lambda} 1_{\{|x-a| \geq \delta\}} \leq C \int_{|x-a| \geq \delta} \delta_{a,\lambda}^6 \]
\[ \leq C \int_{|x-a| \geq \delta} \left[ \frac{\lambda}{1 + \lambda^2 |x-a|^2} \right]^3 \]
\[ \leq C \frac{\lambda^3}{\lambda^3} \int_{|x-a| \geq \delta} \frac{1}{|x-a|^6} \]
\[ \leq C \frac{\lambda^3}{\lambda^3} \int_{\delta}^{+\infty} r^{-4} \, dr \]
\[ \leq C \frac{\lambda^3}{\lambda^3}. \]

Thus, collecting all we have
\[ |I_1| \leq C \left[ \frac{1}{\lambda} + \frac{\delta}{\lambda} + \frac{1}{\lambda^3 \delta^4} \right]. \]

Hence, we obtain
\[ \int_{\Omega} (-\Delta + q) u_{a,\lambda} u_{a,\lambda} \leq \int_{\Omega} u_{a,\lambda}^6 + \frac{C}{\lambda} \left( 1 + \delta + \frac{1}{\lambda^3 \delta^4} \right), \]
thereby ending the proof. \( \blacksquare \)

For the denominator of \( J_q \), we have
Lemma 4.2. Assuming that \( K \subset \Omega \) is compact and \( \theta > 0 \) is small, then there exists \( C > 0 \) such that \( \forall a \in K, \forall 0 < 2\delta < \theta_0, \) and \( \forall 0 < \frac{1}{\lambda} \leq \theta \delta, \) we have

\[
\int_{\Omega} u_{a,\lambda}^6 = \int_{\mathbb{R}^3} \delta_{a,\lambda}^6 + O \left( \frac{1}{\lambda^3 \delta^3} \right).
\]

**Proof.** We have

\[
\int_{\Omega} u_{a,\lambda}^6 = \int_{|x-a| \leq \delta} u_{a,\lambda}^6 + \int_{\delta < |x-a| \leq 2\delta} u_{a,\lambda}^6 + \int_{|x-a| > 2\delta} u_{a,\lambda}^6.
\]

Now, we estimate each term of the right hand side of the latter formula. For the first term, we obtain

\[
\int_{|x-a| \leq \delta} u_{a,\lambda}^6 = \int_{|x-a| \leq \delta} \delta_{a,\lambda}^6 = \int_{\mathbb{R}^3} \delta_{a,\lambda}^6 - \int_{|x-a| > \delta} \delta_{a,\lambda}^6 = \int_{\mathbb{R}^3} \delta_{a,\lambda}^6 + O \left( \frac{1}{\lambda^3 \delta^3} \right).
\]

For the second term, we derive

\[
\int_{\delta < |x-a| \leq 2\delta} u_{a,\lambda}^6 \leq C \int_{\delta < |x-a| \leq 2\delta} \left( \frac{\lambda}{1 + \lambda^2 |x-a|^2} \right)^3 \leq \frac{C}{\lambda^3} \int_{\delta}^{2\delta} r^{-4} \, dr \leq \frac{C}{\lambda^3 \delta^3}.
\]

For the last term, using (12) we get

\[
\int_{|x-a| \geq 2\delta} u_{a,\lambda}^6 = \int_{|x-a| \geq 2\delta} \left( \frac{1}{\sqrt{\lambda} G_a} \right)^6 = \frac{C}{\lambda^3} \int_{|x-a| \geq 2\delta} G_a^6 \leq \frac{C}{\lambda^3} \int_{|x-a| \geq 2\delta} \frac{1}{|x-a|^6} \leq \frac{C}{\lambda^3 \delta^4}.
\]

Therefore, we have

\[
\int_{\Omega} u_{a,\lambda}^6 = \int_{\mathbb{R}^3} \delta_{a,\lambda}^6 + O \left( \frac{1}{\lambda^3 \delta^3} \right).
\]

Finally, we derive the \( J_q \)-energy estimate of \( u_{a,\lambda} \) needed for the application of the Barycenter technique of Bahri-Coron[2].

**Corollary 4.3.** Assuming that \( K \subset \Omega \) is compact and \( \theta > 0 \) is small, then there exists \( C > 0 \) such that \( \forall a \in K, \forall 0 < 2\delta < \theta_0, \) and \( \forall 0 < \frac{1}{\lambda} \leq \theta \delta, \) we have

\[
J_q(u_{a,\lambda}) \leq S \left( 1 + C \left[ \frac{1}{\lambda} + \frac{\delta}{\lambda} + \frac{1}{\delta^3 \lambda^3} \right] \right).
\]

**Proof.** It follows from the properties of \( \delta_{a,\lambda} \) (see (17)-(19)), Lemma 4.1 and Lemma 4.2.

5 Interaction estimates

In this section, we derive sharp inter-action estimates needed for the algebraic topological argument for existence. Recalling (20) and (21), we start with the following one relating \( e_{ij} \) and \( e_{ji} \).
Lemma 5.1. Assuming that \( K \subseteq \Omega \) is compact and \( \theta > 0 \) is small, then there exists \( C > 0 \) such that for all \( a_i, a_j \in K \), \( \forall 0 < 2\delta < \varrho_0 \), and \( \forall 0 < \frac{1}{\lambda_j}, \frac{1}{\lambda_j} \leq \theta \delta \), we have

\[
\int_{\Omega} u_{a_i, \lambda_i} \left| (-\Delta + q) u_{a_j, \lambda_j} - u_{a_j, \lambda_j}^5 \right| \leq C \left( \delta + \frac{1}{\lambda_j^2 \delta^2} \right) \left( \frac{\lambda_j}{\lambda_j} + \lambda_j |a_i - a_j|^2 \right)^{\frac{1}{2}}.
\]

**Proof.** Using Lemma [2.1], we have

\[
\frac{1}{\lambda_j^2 |x-a_j|^2} \leq \frac{C}{\lambda_j^2} \delta \leq \frac{\sqrt{2}}{C^2} \delta_{a_j, \lambda_j}.
\]

This implies

\[
\frac{1}{\lambda_j^2} \leq \frac{\sqrt{2}}{C^2} \delta_{a_j, \lambda_j}.
\]

Thus, we get

\[
L_j \leq C \left( 1 + \frac{1}{\delta} \right) \delta_{a_j, \lambda_j} 1_{|x-a_j| \leq 4\delta} + C \delta_{a_j, \lambda_j} 1_{|x-a_j| \geq \frac{\delta}{8}}.
\]

where \( L_j \) is as in (42). Hence, we obtain

\[
\int_{\Omega} u_{a_i, \lambda_i} L_j \leq C \left( 1 + \frac{1}{\delta} \right) \int_{|x-a_j| \leq 4\delta} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}} I_1
\]

\[
+ C \int_{|x-a_j| \geq \frac{\delta}{8}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}} I_2.
\]

Now, we estimate \( I_1 \) as follows.

\[
I_1 = \int_{|x-a_j| \leq 4\delta} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}}
\]

\[
= \int_{(2|x-a_i| \leq \frac{1}{\delta} + |a_i - a_j|) \cap \{|x-a_j| \leq 4\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}} I_1^1
\]

\[
+ \int_{(2|x-a_i| > \frac{1}{\delta} + |a_i - a_j|) \cap \{|x-a_j| \leq 4\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}} I_1^2.
\]

To continue, we first estimate \( I_1^1 \). Indeed, using triangle inequality we have

\[
I_1^1 \leq C \int_{|x-a_i| \leq 8\delta} \left( \frac{\lambda_j}{1 + \lambda_j^2 |a_i - a_j|^2} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_i|^2} \right)^{\frac{1}{2}}
\]

\[
\leq C \frac{\sqrt{\lambda_j}}{(1 + \lambda_j^2 |a_i - a_j|^2)^{\frac{1}{2}}} \int_{|x-a_i| \leq 8\delta} \frac{1}{|x-a_i|}.
\]

This implies

\[
I_1^1 = O \left( \delta \left( \frac{\lambda_i}{\lambda_j} + \lambda_i |a_i - a_j|^2 \right)^{\frac{1}{2}} \right).
\]
For $I_1^2$, we derive
\[
I_1^2 \leq C \int_{|x-a_j| \leq \frac{4}{5}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^\frac{2}{5} \left( \frac{\lambda_i}{1 + \lambda_i^2 |a_i-a_j|^2} \right)^\frac{2}{5} \left( \lambda_j \right)^\frac{1}{5}.
\]

Thus for $I_2^2$, we obtain
\[
I_2^2 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i-a_j|^2 \right)^\frac{2}{5} \right). \tag{45}
\]

Hence, combining (43) and (45), we get
\[
I_1 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i-a_j|^2 \right)^\frac{2}{5} \right). \tag{46}
\]

Next, let us estimate $I_2$. For this, we first write
\[
I_2 = \int_{|x-a_j| \geq \frac{4}{5}} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^\frac{2}{5} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x-a_j|^2} \right)^\frac{2}{5}.
\]

Setting $D = \{2|x-a_i| \leq \frac{4}{5} + |a_i-a_j| \cap \{|x-a_j| \geq \frac{4}{5} \}$, we estimate $I_2^2$ as follows
\[
I_2^2 \leq C \int_D \left( \frac{1}{1 + \lambda_j^2 |a_i-a_j|^2} \right)^\frac{2}{5} \left( \frac{\lambda_i}{1 + \lambda_i^2 |a_i-a_j|^2} \right)^\frac{2}{5} \left( \frac{1}{1 + \lambda_i^2 |a_i-a_j|^2} \right).
\]

This implies
\[
I_1^2 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i-a_j|^2 \right)^\frac{2}{5} \right). \tag{47}
\]

Next, we estimate $I_2^2$ as follows
\[
I_2^2 \leq C \int_{|x-a_j| \geq \frac{4}{5}} \left( \frac{1}{1 + \lambda_j^2 |a_i-a_j|^2} \right)^\frac{2}{5} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^\frac{2}{5} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x-a_j|^2} \right)^\frac{1}{5}.
\]
This gives
\[ I_2^2 = O \left( \frac{1}{\lambda^2 \delta^2} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{1}{2}} \right). \] (48)

Therefore, using (47) and (48), we obtain
\[ I_2 = O \left( \frac{1}{\lambda^2 \delta^2} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{1}{2}} \right). \] (49)

Hence, combining (43), (67) and (49), we get
\[
\int_{\Omega} u_{a_i, \lambda_i} L_j \leq C \( \delta + \frac{1}{\lambda^2 \delta^2} \) \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{1}{2}},
\]
thereby ending the proof of the lemma. \( \square \)

Clearly Lemma 5.1 implies the following sharp interaction-estimate relating \( \epsilon_{ij}, \epsilon_{ij}, \) and \( \epsilon_{ij} \) (for their definitions, see (19) and (21)).

**Corollary 5.2.** Assuming that \( K \subset \Omega \) is compact, \( \theta > 0 \) is small, and \( \mu_0 > 0 \) is small, then \( \forall a_i, a_j \in K, \quad \forall 0 < 2 \delta < \theta_0, \) and \( \forall 0 < \frac{\lambda_i}{\lambda_j} \leq \frac{\lambda_j}{\lambda_i} \leq \theta \delta \) such that \( \epsilon_{ij} \leq \mu_0, \) we have
\[ \epsilon_{ij} = \epsilon_{ij} + O \left( \delta + \frac{1}{\lambda^2 \delta^2} \right) \epsilon_{ij}. \]

The next lemma provides a sharp interaction estimate relating \( \epsilon_{ji} \) and \( \epsilon_{ij} \).

**Lemma 5.3.** Assuming that \( K \subset \Omega \) is compact, \( \theta > 0 \) is small, and \( \mu_0 > 0 \) is small, then \( \forall a_i, a_j \in K, \quad \forall 0 < 2 \delta < \theta_0, \) and \( \forall 0 < \frac{\lambda_i}{\lambda_j} \leq \frac{\lambda_j}{\lambda_i} \leq \theta \delta \) such that \( \epsilon_{ij} \leq \mu_0, \) we have
\[ \epsilon_{ji} = c_0^2 \epsilon_{ij} \left[ (1 + O(\delta + \frac{1}{\lambda^2 \delta^2})) (1 + O(\epsilon_{ij}^2 (\delta^{-2} + \log \delta^{-1}))) + O \left( \epsilon_{ij}^2 \frac{1}{\delta^2} \right) \right], \]
where \( c_0 \) is as in (50) and \( c_3 \) is as in (19).

**Proof.** By definition, we have
\[ u_{a_i, \lambda_i} = \chi_{\delta a_i, \lambda_i} (1 - \chi_{\delta}) \frac{c_0}{\sqrt{\lambda}} G_{a_i}, \]
with \( \chi_{\delta} := \chi_{\delta}^{a_i}. \) On the other hand, by definition of the standard bubble \( \delta_{a_i, \lambda_i}, \) we have
\[ \chi_{\delta} \delta_{a_i, \lambda_i} = c_0 \chi_{\delta} \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2} \frac{|x - a_i|^2}{\delta^2}} \right]^\frac{1}{2}. \]

Next, for \( |x - a_i| \leq 2 \delta, \) we have
\[
1 + \lambda_i^2 G_{a_i}^{-2} \frac{|x - a_i|^2}{G_{a_i}^2} = 1 + \lambda_i^2 G_{a_i}^{-2} (1 + O(\delta))
= 1 + \lambda_i^2 G_{a_i}^{-2} + O \left( \lambda_i^2 G_{a_i}^{-2} \delta \right)
= (1 + \lambda_i^2 G_{a_i}^{-2}) \left[ 1 + O \left( \frac{\lambda_i^2 G_{a_i}^{-2} \delta}{1 + \lambda_i^2 G_{a_i}^{-2}} \right) \right]
= (1 + \lambda_i^2 G_{a_i}^{-2}) [1 + O(\delta)].
\]
So, for \( \chi_{\delta} \delta_{a_i, \lambda_i}, \) we have
\[ \chi_{\delta} \delta_{a_i, \lambda_i} = c_0 \chi_{\delta} \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2} [1 + O(\delta)]} \right]^\frac{1}{2} = c_0 \chi_{\delta} [1 + O(\delta)] \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^\frac{1}{2}. \] (50)

We have also
\[ c_0 (1 - \chi_{\delta}) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^\frac{1}{2} = (1 - \chi_{\delta}) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} \left[ \frac{1}{1 + \lambda_i^2 G_{a_i}^{-2}} \right]^\frac{1}{2}. \]
Since on \( \{|x - a_i| \geq \delta\} \), we have
\[
\frac{1}{1 + \lambda_i^2 G_{a_i}^2} = 1 + O\left(\frac{G_{a_i}^2}{\lambda_i^2}\right) = 1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right),
\]
then we get
\[
c_0 (1 - \chi_\delta) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}} = (1 - \chi_\delta) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} \left( 1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right).
\]
This implies
\[
(1 - \chi_\delta) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i} = c_0 (1 - \chi_\delta) \left[ \frac{\lambda}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}} \left( 1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right).
\]
Thus, combining (50) and (51), we get
\[
u_{a_i, \lambda_i} = c_0 \left[ (1 + O(\delta)) \chi_\delta + (1 - \chi_\delta) \left( 1 + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right) \right] \left[ \frac{\lambda}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}}.
\]
Hence, we obtain
\[
u_{a_i, \lambda_i} = c_0 \left[ 1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \left[ \frac{\lambda}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}}.
\]
Now, we are going to use (52) to achieve our goal. First of all, we write
\[
\int_{\Omega} \nu_{a_i, \lambda_i} = \int_{B_{(a_i, \delta)}} \nu_{a_i, \lambda_i} + \int_{\Omega - B_{(a_i, \delta)}} \nu_{a_i, \lambda_i}.
\]
For the second term in the right hand side of the latter formula, we have
\[
\int_{\Omega - B_{(a_i, \delta)}} \nu_{a_i, \lambda_i} \leq C \int_{\Omega - B_{(a_i, \delta)}} \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \nu_{a_i, \lambda_i}
\]
\[
\leq C \left( \frac{1}{\lambda_i^2 \delta^2} \right)^{\frac{i}{2}} \left[ \int_{\Omega - (B_{(a_i, \delta)} \cup B_{(a_i, \delta)})} \nu_{a_i, \lambda_i} + \int_{(\Omega - B_{(a_i, \delta)}) \cap B_{(a_i, \delta)}} \nu_{a_i, \lambda_i} \right]
\]
\[
\leq C \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \left[ \frac{1}{\sqrt{\lambda_i}} + C \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \int_{B_{(a_i, \delta)}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x - a_i|^2} \right)^{\frac{i}{2}} \right]
\]
\[
\leq C \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \left[ \frac{1}{\sqrt{\lambda_i}} + C\delta^2 \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \right]
\]
\[
\leq C \left( \frac{1}{\lambda_i} \right)^{\frac{i}{2}} \left( \frac{1}{\delta} \right)^{\frac{i}{2}} \left[ \frac{1}{\sqrt{\lambda_i}} (1 + \delta^3) \right].
\]
Thus, we get
\[
\int_{\Omega} \nu_{a_i, \lambda_i} = \int_{B_{(a_i, \delta)}} \nu_{a_i, \lambda_i} + O\left( \frac{1}{\lambda_i^2 \sqrt{\lambda_i} \delta^6} \right).
\]
For the first term in the right hand side of (53) the latter formula, using (52) we have
\[
\int_{B_{(a_i, \delta)}} \nu_{a_i, \lambda_i} = c_0^6 \int_{B_{(a_i, \delta)}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x - a_i|^2} \right)^{\frac{i}{2}} \left[ 1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}}
\]
\[
= c_0^6 \left[ 1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \int_{B_{(a_i, \delta)}} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x - a_i|^2} \right)^{\frac{i}{2}} \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}}
\]
\[
= c_0^6 \left[ 1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \int_{B_{(0, \lambda_i, \delta)}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{i}{2}} \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2} \right]^{\frac{i}{2}}
\]
\[
= c_0^6 \left[ 1 + O(\delta) + O\left(\frac{1}{\lambda_i^2 \delta^2}\right) \right] \int_{B_{(0, \lambda_i, \delta)}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{i}{2}} \left[ \frac{\lambda_i}{1 + \lambda_i \lambda_j G_{a_i}^2} \right]^{\frac{i}{2}}.
\]
Recalling that $\lambda_i \leq \lambda_j$, then for $\varepsilon_{ij} \sim 0$, we have

1) Either $\varepsilon_{ij}^2 \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$,

2) or $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$.

To continue, let

$$\mathcal{A} = \left\{ \left\frac{y}{\lambda_j} \right| \leq \epsilon G_{a_i}^{-1}(a_j) \right\} \cap B(0, \delta \lambda_j) \cup \left\{ \left|\frac{y}{\lambda_j} \right| \leq \epsilon \right\} \cap B(0, \delta \lambda_j),$$

with $\epsilon > 0$ very small. Then by Taylor expansion on $\mathcal{A}$, we have

$$\left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{2}} = \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{2}} + \left( \frac{1}{2} \nabla G_{a_i}^{-2}(a_j) \lambda_i y \right) \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{2}} + O \left( \left| \frac{\lambda_i}{\lambda_j} \right| |y|^2 \right) \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{2}}.$$ 

Thus, we have

$$\int_{B(a_i, \delta)} u_{a_i, \lambda_j}^2 u_{a_i, \lambda_i} = c_0^2 \left[ 1 + O(\delta) + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right] \left( \sum_{i=1}^4 I_i \right),$$

with

$$I_1 = \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}},$$

$$I_2 = \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \nabla G_{a_i}^{-2}(a_j) \lambda_i y \right],$$

$$I_3 = \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} O \left( \left| \frac{\lambda_i}{\lambda_j} \right| |y|^2 \right),$$

and

$$I_4 = \int_{B(0, \lambda_j) \setminus \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{2}}.$$ 

Now, let us estimate $I_1$. We have

$$I_1 = \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{2}} \left[ c_3 + \int_{\mathbb{R}^3 \setminus \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \right],$$

where $c_3$ is as in (10). On the other hand, we have

$$\int_{\mathbb{R}^3 \setminus \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^3 \setminus B(0, \delta \lambda_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} + \int_{\mathbb{R}^3 \setminus B(0, \lambda_j \delta G_{a_i}^{-1}(a_i, a_j))} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}}$$

if $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, and

$$\int_{\mathbb{R}^3 \setminus \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^3 \setminus B(0, \delta \lambda_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} + \int_{\mathbb{R}^3 \setminus \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}}$$

if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$. We have

$$\int_{\mathbb{R}^3 \setminus B(0, \delta \lambda_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} = O \left( \frac{1}{\lambda_j^2 \delta^2} \right).$$

Moreover, if $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, then

$$\int_{\mathbb{R}^3 \setminus B(0, \lambda_j \delta G_{a_i}^{-1}(a_j))} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} = O \left( \frac{1}{\lambda_j^2 \delta^2 G_{a_i}^{-2}(a_j)} \right) = O \left( \frac{1}{\lambda_j \lambda_i G_{a_i}^{-2}(a_j)} \right) = O \left( \epsilon_{ij}^2 \right).$$
Furthermore if $\varepsilon_{ij}^{-2} \sim \frac{1}{\lambda_i}$, then
\[
\int_{B(0, \frac{\lambda_i}{\varepsilon_{ij}^2})} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \, dy = O \left( \varepsilon_{ij}^2 \right).
\]
This implies
\[
\int_{A} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} = O \left( \varepsilon_{ij}^2 + \frac{1}{\lambda_i^2} \right) = O \left( \varepsilon_{ij}^2 + \varepsilon_{ij}^2 \frac{1}{\delta^2} \right) = O \left( \varepsilon_{ij}^2 \frac{1}{\delta^2} \right).
\]
Thus, we get
\[
I_1 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_j}^{-1}(a_j) \right]^{\frac{1}{2}} \left[ c_3 + O \left( \varepsilon_{ij}^2 \frac{1}{\delta^2} \right) \right]
\]
\[= \varepsilon_{ij} \left( 1 + a_{c_{ij}}(1) \right) \left[ c_3 + O \left( \varepsilon_{ij}^2 \frac{1}{\delta^2} \right) \right].\]

Hence, we obtain
\[
I_1 = c_3 \varepsilon_{ij} \left[ 1 + a_{c_{ij}}(1) + O \left( \varepsilon_{ij}^2 \frac{1}{\delta^2} \right) \right].
\]

By symmetry, we have
\[
I_2 = 0.
\]

Next, to estimate $I_4$ we first observe that
\[
\int_A \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \leq \int_{B(0, \lambda_j G_{a_j}^{-1}(a_j)) - B(0, 1)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} + \int_{B(0, \lambda_j G_{a_j}^{-1}(a_j)) - B(0, 1)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} + O(1)
\]
\[= O \left( \log \left( \lambda_j G_{a_j}^{-1}(a_j) \right) + \log \left( \frac{\lambda_j}{\lambda_i} \right) \right) + O(1).
\]
Thus, we have
\[
I_3 = \varepsilon_{ij}^3 \left( \frac{\lambda_j}{\lambda_i} \right) \left( 1 + o_{c_{ij}}(1) \right) \left[ O \left( \log \left( \lambda_j G_{a_j}^{-1}(a_j) \right) + \log \left( \frac{\lambda_j}{\lambda_i} \right) \right) + O(1) \right]
\]
\[= \varepsilon_{ij}^3 \left( 1 + o_{c_{ij}}(1) \right) \left[ O \left( \log \left( \lambda_i \lambda_j G_{a_j}^{-2}(a_j) \right) + \log \left( \frac{\lambda_j}{\lambda_i} \right) \right) + O(1) \right].
\]

Hence, we obtain
\[
I_3 = O \left( \varepsilon_{ij}^3 \log \left( \varepsilon_{ij}^{-1} \right) \right).
\]

Finally, we estimate $I_4$ as follows. If $\varepsilon_{ij}^{-2} \sim \frac{1}{\lambda_i}$, then
\[
I_4 \leq C \varepsilon_{ij} \int_{B(0, \lambda_i \delta) - A} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \leq C \varepsilon_{ij} \left( \frac{\lambda_i}{\lambda_j} \right)^{\frac{1}{2}} \leq C \varepsilon_{ij}.
\]
If $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_j}^{-2}(a_j)$, then we argue as follows. In case $|a_i - a_j| \geq 2\delta$, since
\[
G_{a_j} \left( \frac{y}{\lambda_j} + a_j \right) \leq C \delta^{-1}
\]
for $y \in B(0, \lambda_j \delta)$, then we have
\[
I_4 \leq C \frac{\int_{B(0, \lambda_j \delta) - A} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} 1}{\sqrt{\lambda_i \lambda_j} \delta}
\]
\[\leq \frac{C}{\sqrt{\lambda_i \lambda_j} \delta} \left( \frac{1}{\lambda_j^2 G_{a_j}^{-2}(a_j)} \right)
\]
\[\leq \frac{C}{\lambda_i \lambda_j G_{a_j}^{-1}(a_j) \sqrt{\lambda_i \lambda_j G_{a_j}^{-1}(a_j)} \delta}
\]
\[\leq C \varepsilon_{ij}^2 \varepsilon_{ij} \frac{1}{\delta}.
\]
Thus, when $|a_i - a_j| \geq 2\delta$ we have

$$I_4 = O\left(\varepsilon_i^3 \frac{1}{\delta}\right).$$

(59)

In case $|a_i - a_j| < 2\delta$, we first observe that

$$B(0, \lambda_j \delta) \setminus A \subset A_1 \cup A_2$$

with

$$A_1 = \left\{ \varepsilon \lambda_j G_{a_i}^{-1}(a_j) \leq |y| \leq E \lambda_j G_{a_i}^{-1}(a_j) \right\}$$

and

$$A_2 = \left\{ E \lambda_j G_{a_i}^{-1}(a_j) \leq |y| \leq \lambda_j \delta \right\},$$

where $0 < \varepsilon < E$. Thus, we have

$$I_4 \leq I_4^1 + I_4^2,$$

with

$$I_4^1 = \int_{A_1} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{\varepsilon}}$$

and

$$I_4^2 = \int_{A_2} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{\varepsilon}}.$$

We estimate $I_4^1$ as follows:

$$I_4^1 \leq C \left[ 1 + \lambda_j^2 G_{a_i}^{-2}(a_j) \right]^{\frac{5}{8}} \int_{|y| \leq E \lambda_j G_{a_i}^{-1}(a_j)} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{\varepsilon}}$$

$$\leq C \left[ 1 + \lambda_j^2 G_{a_i}^{-2}(a_j) \right]^{\frac{5}{8}} \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{3}{8}} \int_{|y| \leq E \lambda_j G_{a_i}^{-1}(a_j)} \left[ 1 + \lambda_i^2 \left( \frac{y}{\lambda_j} + a_i - a_j \right)^2 \right]^{\frac{1}{\varepsilon}}$$

$$\leq C \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{5}{8}} \left( \lambda_i^2 G_{a_i}^{-2}(a_j) \right)$$

$$\leq C \varepsilon_i^5 \left( \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right),$$

where $E$ is a positive constant. So we obtain

$$I_4^1 = O \left( \varepsilon_i^3 \right).$$

(61)

For $I_4^2$, we have

$$I_4^2 = \int_{B_3} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{y}{\lambda_j} + a_j \right) \right]^{\frac{1}{\varepsilon}}$$

$$\leq C \int_{|y| \geq E \lambda_j G_{a_i}^{-1}(a_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{5}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{\varepsilon}}$$

$$\leq C \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right]^{\frac{1}{\varepsilon}} \left( \frac{1}{\lambda_j^2 G_{a_i}^{-2}(a_j)} \right).$$

This implies

$$I_4^2 = O \left( \varepsilon_i^3 \right).$$

(62)

Thus, combining (60)-(62), we have that if $|a_i - a_j| < 2\delta$, then

$$I_4 = O \left( \varepsilon_i^3 \right).$$

(63)

Now, using (61) and (63), we infer that in case $\varepsilon_i^{-2} \simeq \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$,

$$I_4 = O \left( \varepsilon_i^3 \frac{1}{\delta} \right).$$

(64)
Finally combining (53) and (63), we get

\[ I_4 = O \left( \varepsilon_{ij}^3 \frac{1}{\delta^6} \right). \]

Collecting all we get

\[ \int_{B(a_j, \lambda_j, \delta)} u_{a_j, \lambda_j}^{\beta} u_{a_i, \lambda_i} = \varepsilon_{ij}^5 \left[ 1 + O \left( \delta + \frac{1}{\lambda_i^2 \delta^2} \right) \right] [c_3 \varepsilon_{ij} (1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij}^2 (\delta^{-2} + \log \varepsilon_{ij}^{-1})))]. \] (65)

Therefore using (53) and (65), we arrive to

\[ \int_{\Omega} u_{a_j, \lambda_j}^{\beta} u_{a_i, \lambda_i} = \varepsilon_{ij}^5 \left[ 1 + O \left( \delta + \frac{1}{\lambda_j^2 \delta^2} \right) \right] [c_3 \varepsilon_{ij} (1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij}^2 (\delta^{-2} + \log \varepsilon_{ij}^{-1})))]
+ O \left( \frac{1}{\lambda_j^2 \sqrt{\lambda_i} \delta^6} \right). \]

Thus, we have

\[ \int_{\Omega} u_{a_j, \lambda_j}^{\beta} u_{a_i, \lambda_i} = \varepsilon_{ij}^5 \left[ 1 + O \left( \delta + \frac{1}{\lambda_j^2 \delta^2} \right) \right] [c_3 \varepsilon_{ij} (1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij}^2 (\delta^{-2} + \log \varepsilon_{ij}^{-1})))]
+ O \left( \varepsilon_{ij}^3 \frac{1}{\delta^6} \right). \]

Therefore, we obtain

\[ \int_{\Omega} u_{a_j, \lambda_j}^{\beta} u_{a_i, \lambda_i} = \varepsilon_{ij}^5 [c_3 \varepsilon_{ij} \left( 1 + O \left( \delta + \frac{1}{\lambda_j^2 \delta^2} \right) \right) (1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij}^2 (\delta^{-2} + \log \varepsilon_{ij}^{-1})))]
+ O \left( \varepsilon_{ij}^3 \frac{1}{\delta^6} \right). \] (66)

Hence the result follows from (20) and (66). \( \blacksquare \)

Clearly switching the index \( i \) and \( j \) in Lemma 5.3, we have the following corollary which is equivalent to Lemma 5.3. We decide to present the following corollary, because its form suits more our presentation of the Barycenter technique of Bahri-Coron[2] which follows the works [6] and [7].

**Corollary 5.4.** Assuming that \( K \subset \Omega \) is compact, \( \theta > 0 \) is small, and \( \mu_0 \) is small, then \( \forall a_i, a_j \in K \), \( \forall 0 < 2\delta < \varrho_0 \), and \( \forall 0 < \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \leq \theta \delta \) such that \( \varepsilon_{ij} \leq \mu_0 \), we have

\[ \varepsilon_{ij} = c_5 \varepsilon_{ij} \left( 1 + O \left( \delta + \frac{1}{\lambda_j^2 \delta^2} \right) \right) (1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij}^2 (\delta^{-2} + \log \varepsilon_{ij}^{-1}))) + O \left( \varepsilon_{ij}^3 \frac{1}{\delta^6} \right). \]

We present now some sharp high-order inter-action estimates needed for the application of the algebraic topological argument for existence. We start with the following balanced high-order inter-action estimate.

**Lemma 5.5.** Assuming that \( K \subset \Omega \) is compact, \( \theta > 0 \) is small, and \( \mu_0 \) is small, then \( \forall a_i, a_j \in K \), \( \forall 0 < 2\delta < \varrho_0 \), and \( \forall 0 < \frac{1}{\lambda_j} \leq \theta \delta \) such that \( \varepsilon_{ij} \leq \mu_0 \), we have

\[ \int_{\Omega} u_{a_i, \lambda_i}^{\beta} u_{a_j, \lambda_j}^{\beta} = O \left( \varepsilon_{ij}^3 \delta^{-6} \log \left( \frac{1}{\varepsilon_{ij} \delta} \right) \right). \]

**Proof.** By symmetry, we can assume without loss of generality (w.l.o.g) that \( \lambda_j \leq \lambda_i \). Thus we have

1. Either \( \varepsilon_{ij}^2 \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \)
2. Or \( \varepsilon_{ij}^2 \sim \lambda_j \lambda_i \).

16
Now, if $|a_i - a_j| \geq 2\delta$, then we have

$$I = \int_{\Omega} u^3_{a_i,A_i} u^3_{a_j,A_j}$$

$$\leq \int_{B(a_i,\delta)} \left( \frac{\lambda_i}{1 + \lambda_i^3|x - a_i|^2} \right)^{\frac{2}{3}} \left( \frac{\lambda_j}{1 + \lambda_j^3|x - a_j|^2} \right)^{\frac{2}{3}}$$

$$+ \frac{C}{\lambda^2_3 \delta^3} \int_{B(a_j,\delta)} \left( \frac{\lambda_j}{1 + \lambda_j^3|x - a_j|^2} \right)^{\frac{2}{3}} + \frac{C}{\delta^6} \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}}$$

$$\leq \int_{B(0,\lambda_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{2}{3}} \left( \frac{1}{1 + \lambda_i \lambda_j G_{a_i}^{-2} (a_i + \frac{x}{\delta^3})} \right)^{\frac{2}{3}}$$

$$I_1 \leq \int_{B(0,\lambda_j)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{2}{3}} + \frac{C}{\delta^6} \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}}$$

Now, we estimate $I_1$ as follows.

If $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$, then we get

$$I_1 \leq C \varepsilon_{ij}^3 \left[ \log(\lambda_i \delta) + C \right].$$

So, for $I$ we have

$$I \leq C \varepsilon_{ij}^3 \left[ \log(\lambda_i \delta) + C \right] + C \varepsilon_{ij}^3 \log(\lambda_i \lambda_j)$$

$$\leq C \varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-2} G_{a_i}^2 (a_j))$$

$$= O \left( \frac{\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1} \delta^{-1})}{\delta^6} \right).$$

If $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2} (a_j)$, then we get

$$I_1 \leq C \varepsilon_{ij}^3 \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}} \left[ \log(\lambda_i \delta) + C \right].$$

So, for $I$ we have

$$I \leq C \varepsilon_{ij}^3 \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}} \log(\lambda_i \lambda_j).$$

This implies

$$I \leq C \varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-2} G_{a_i}^2 (a_j)).$$

Hence, for $|a_i - a_j| \geq 2\delta$, we obtain

$$I = O \left( \frac{\varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1} \delta^{-1})}{\delta^6} \right).$$

(68)

On the other hand, arguing as above, if $|a_i - a_j| < 2\delta$ then we have also

$$I \leq I_1 + \frac{C}{\delta^6} \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}} \log(\lambda_i) + \frac{C}{\delta^6} \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}}$$

$$\leq I_1 + \frac{C}{\delta^6} \left( \frac{1}{\lambda_i \lambda_j} \right)^{\frac{2}{3}} \log(\lambda_i \lambda_j),$$

where $I_1$ is as in (67). Thus, if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$ then

$$I \leq I_1 + \frac{C}{\delta^6} \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{2}{3}} \left[ \log \left( \frac{\lambda_i \delta}{\lambda_j} \right) + \log(\lambda_j^2) \right].$$
This implies
\[ I \leq I_1 + \frac{C}{\delta^3} \varepsilon_i \log(\varepsilon_i^{-1}). \]

Next, if \( \varepsilon_{ij}^2 \simeq \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \) then we get
\[
I \leq I_1 + \frac{C}{\delta^3} \left( \frac{1}{\lambda_i \lambda_j G_{a_i}^{-2}(a_j)} \right)^{\frac{1}{2}} \left[ \log(\lambda_i \lambda_j G_{a_i}^{-2}(a_j) + \log(G_{a_i}^2(a_j))) \right] G_{a_i}^{-3}(a_j)
\]
\[
\leq I_1 + \frac{C}{\delta^3} \varepsilon_i \log(\varepsilon_i^{-1}).
\]

Now, to continue, we are going to estimate \( I_1 \). For this, we start by defining the following sets:

\[ A_1 = \left\{ \left| y \right| \leq \varepsilon \lambda \sqrt{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \right\} \]

\[ A_2 = \left\{ \varepsilon \lambda \sqrt{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \leq \left| y \right| \leq E \varepsilon \lambda \sqrt{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \right\} \]

\[ A_3 = \left\{ E \varepsilon \lambda \sqrt{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j^2}} \leq \left| y \right| \leq \frac{4 \varepsilon \lambda}{\lambda_j} \right\} \]

with \( 0 < \varepsilon < E < \infty \). Clearly by the definition of \( I_1 \) (see (67)), we have
\[
I_1 \leq \int_{A_1} L_{i,j} + \int_{A_2} L_{i,j} + \int_{A_3} L_{i,j},
\]
where
\[
L_{i,j} = \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda_j^2 + \varepsilon \lambda \lambda_j G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2}} \right)^{\frac{1}{2}}.
\]

For \( \int_{A_1} L_{i,j} \), we have
\[
\int_{A_1} L_{i,j} \leq C \varepsilon_i^3 \int_{A_1} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda_j^2 + \varepsilon \lambda \lambda_j G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2}} \right)^{\frac{1}{2}}
\]
\[
\leq C \varepsilon_i^3 \log \left( \frac{\lambda_i}{\lambda_j} \right) \varepsilon_i \log(\varepsilon_i^{-1}).
\]

For \( \int_{A_2} L_{i,j} \), we have
\[
\int_{A_2} L_{i,j} \leq C \left( \frac{1}{\lambda_j^2 + \varepsilon \lambda \lambda_j G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2}} \right)^{\frac{1}{2}} \int_{A_2} \left( \frac{1}{\lambda_j^2 + \varepsilon \lambda \lambda_j G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2}} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{1}{2}} \varepsilon_i^3 \int_{|y| \leq E \lambda \lambda_j} G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2} \left( \frac{1}{\lambda_j^2 + \lambda_j |y + \lambda_j(a_i - a_j)|^2} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{1}{2}} \varepsilon_i^3 \int_{|y| \leq E \lambda \lambda_j} G_{a_j}^{-2}(a_j) + \frac{1}{\lambda_j^2} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}}
\]
\[
\leq C \varepsilon_i^3 \log(\varepsilon_i^{-1}).
\]
Therefore, we have

\[ I_1 \leq C \varepsilon_{ij}^3 \log(\varepsilon_{ij}^{-1}). \]

This implies for \(|a_i - a_j| < 2\delta\), we have

\[ I = O \left( \frac{\varepsilon_{ij}^3}{\delta^6} \log(\varepsilon_{ij}^{-1}) \right). \]

Hence, combining with the estimate for \(|a_i - a_j| \geq 2\delta\) (see (68)), we have

\[ \int_\Omega u_{a_i, \lambda_i}^3 u_{a_j, \lambda_j}^3 = O \left( \frac{\varepsilon_{ij}^3}{\delta^6} \log(\varepsilon_{ij}^{-1} \delta^{-1}) \right). \]

Finally, we present a sharp unbalanced high-order inter-action estimate needed for the application of the Barycenter technique of Bahri-Coron[2].

**Lemma 5.6.** Assuming that \( K \subset \Omega \) is compact, \( \theta > 0 \) is small, and \( \mu_0 \) is small, then \( \forall a_i, a_j \in K \), \( \forall 0 < 2\delta < \varrho_0 \), and \( \forall 0 < \frac{\lambda_i}{\Lambda_j} \leq 2\delta \) such that \( \varepsilon_{ij} \leq \mu_0 \), we have

\[ \int_\Omega u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta = O \left( \frac{\varepsilon_{ij}^3}{\delta^6} \right). \]

where \( \alpha + \beta = 6 \) and \( \alpha > 3 > \beta > 1 \).

**Proof.** Let \( \tilde{\alpha} = \frac{1}{2}\alpha \) and \( \tilde{\beta} = \frac{1}{2}\beta \). Then we have \( \tilde{\alpha} + \tilde{\beta} = 3 \). Now, since \( \lambda_j \leq \lambda_i \), then we have

1) Either \( \varepsilon_{ij} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \)

2) Or \( \varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\Lambda_j} \).  

To continue, we write

\[ \int_\Omega u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta = \int_{B_{a_i}(\delta)} u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta + \int_{\Omega - B_{a_i}(\delta)} u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta \]

and estimate \( I_1 \) and \( I_2 \). For \( I_2 \), we have

\[ I_2 = \int_{\Omega - B_{a_i}(\delta) \cap B_{a_j}(\delta)} u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta + \int_{\Omega - (B_{a_i}(\delta) \cup B_{a_j}(\delta))} u_{a_i, \lambda_i}^\alpha u_{a_j, \lambda_j}^\beta \]

\[ \leq C \int_{\Omega - B_{a_i}(\delta) \cap B_{a_j}(\delta)} \left( \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right)^\tilde{\alpha} \left( \frac{\lambda_j}{1 + \lambda_j^2 |x - a_j|^2} \right)^\tilde{\beta} \]

\[ + C \int_{\Omega - (B_{a_i}(\delta) \cup B_{a_j}(\delta))} \left( \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right)^\tilde{\alpha} \left( \frac{\lambda_j}{1 + \lambda_j^2 G_{a_j}^2(x)} \right)^\tilde{\beta} \]

\[ \leq \frac{C}{\lambda_i^3 \lambda_j^{3-\tilde{\beta}}} \int_{B_{a_i}(\delta)} \left( \frac{1}{1 + |y|} \right)^\tilde{\beta} + \frac{C}{\lambda_i^3 \lambda_j^{3-\tilde{\beta}}} \int_{B_{a_j}(\delta)} \left( \frac{1}{1 + |y|} \right)^\tilde{\beta} \]

\[ \leq C \frac{1}{\lambda_i^3 \lambda_j^{3-\tilde{\beta}}} \left( \lambda_j \right)^{2\tilde{\beta} - 3} + C \frac{1}{\lambda_i^3 \lambda_j^{3-\tilde{\beta}}} \left( \lambda_i \right)^{2\tilde{\beta} - 3} \]
Thus, we have for $I_2$

$$I_2 \leq \frac{C}{\lambda_i^\alpha \lambda_j^\beta \delta^\gamma}$$

(69)

Next, for $I_1$ we have

$$I_1 = \int_{B_{a_i}(\delta)} \left( \frac{\lambda_i}{1 + \lambda_i^2 |x - a_i|^2} \right)^\alpha \left( \frac{\lambda_j}{1 + \lambda_j^2 G_{a_j}^{-2}(x)} \right)^\beta$$

$$= \int_{B_{0}(\lambda, \delta)} \left( \frac{1}{1 + |y|^2} \right)^\alpha \left( \frac{1}{\lambda_i \lambda_j} + \lambda_i \lambda_j G_{a_j}^{-2} \left( a_i + \frac{x}{\lambda_i} \right) \right)^\beta .$$

Thus, if $\varepsilon_{ij}^{-2} \sim \frac{a}{\lambda_i}$ then

$$I_1 \leq C \varepsilon_{ij}^{2\beta} \left[ \frac{1}{\lambda_i \delta} \right]^{2\alpha - 3} + C \leq C \varepsilon_{ij}^\beta .$$

If $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_j}^{-2}(a_j)$ and $|a_i - a_j| \geq 2\delta$, then we have

$$I_1 \leq C \left[ \frac{1}{\lambda_i \lambda_j \delta^2} \right]^\beta \left[ \frac{1}{\lambda_i \delta} \right]^{2\alpha - 3} + C \leq C \frac{1}{\delta^2} \left[ \frac{1}{\lambda_i \lambda_j \delta^2} \right]^\beta \left[ \frac{1}{\lambda_i \lambda_j G_{a_j}^{-2}(a_j)} \right]^{\frac{1}{2}} \beta$$

$$\leq C \frac{1}{\delta^2} \varepsilon_{ij}^\beta .$$

Now, if $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_j}^{-2}(a_j)$ and $|a_i - a_j| < 2\delta$, then we get

$$I_1 \leq C \int_{B_{0}(\lambda, \delta)} \left( \frac{1}{1 + |y|^2} \right)^\alpha \left[ \frac{1}{\lambda_i \lambda_j} + \frac{1}{|a_i + \frac{x}{\lambda_i} - a_j|^2} \right]^\beta .$$

Next, we define

$$B = \left\{ \frac{1}{2} |a_i - a_j| \leq \frac{|y|}{\lambda_i} \leq 2 |a_i - a_j| \right\}$$

and have

$$I_1 \leq C \int_B \left( \frac{1}{1 + |y|^2} \right)^\alpha \left[ \frac{1}{\lambda_i \lambda_j} + \frac{1}{|a_i + \frac{x}{\lambda_i} - a_j|^2} \right]^\beta$$

$$+ C \int_{B_0(\lambda, \delta) - B} \left( \frac{1}{1 + |y|^2} \right)^\alpha \left[ \frac{1}{\lambda_i \lambda_j} + \frac{1}{|a_i + \frac{x}{\lambda_i} - a_j|^2} \right]^\beta .$$

For the second term, we have

$$\int_{B_0(\lambda, \delta) - B} \left( \frac{1}{1 + |y|^2} \right)^\alpha \left[ \frac{1}{\lambda_i \lambda_j} + \frac{1}{|a_i + \frac{x}{\lambda_i} - a_j|^2} \right]^\beta \leq C \varepsilon_{ij}^{\beta} \left[ \frac{1}{\lambda_i \delta} \right]^{\alpha - 3} + C$$

$$\leq C \varepsilon_{ij}^\beta .$$
Thus, we have for

\[
\int_B \left( \frac{1}{1 + |y|^2} \right)^{\hat{a}} \left[ \frac{1}{\lambda_i \lambda_j |a_i + \frac{x_i}{\lambda_i} - a_j|} \right]^{\hat{b}} \leq C \left( \frac{1}{1 + \lambda^2_i |a_i - a_j|^2} \right)^{\hat{a}} \int_{|y| \leq 2\lambda_i |a_i - a_j|} \left[ \frac{1}{\lambda_i^2 + \frac{1}{\lambda_i^2} |y + \lambda_i (a_i - a_j)|^2} \right]^{\hat{b}} \leq C \left( \frac{1}{1 + \lambda^2_i |a_i - a_j|^2} \right)^{\hat{a}} \int_{|z| \leq 4\lambda_i |a_i - a_j|} \left[ \frac{1}{1 + |z|^2} \right]^{\hat{b}}.
\]

If \( \lambda_j |a_i - a_j| \) is bounded, then we get

\[
I_1 \leq C \left( \frac{1}{\lambda_i^2 + \lambda_i \lambda_j |a_i - a_j|^2} \right)^{\frac{\alpha}{2}} \leq C \varepsilon_{ij}.
\]

If \( \lambda_j |a_i - a_j| \) is unbounded, then we get

\[
I_1 \leq C \left( \frac{1}{\lambda_i^2 + \lambda_i \lambda_j |a_i - a_j|^2} \right)^{\frac{\alpha}{2}} (\lambda_j |a_i - a_j|)^{3 - 2\hat{b}} \leq C \left( \frac{1}{1 + \lambda^2_i |a_i - a_j|^2} \right)^{\hat{a} + \frac{\alpha}{2} - \frac{\hat{b}}{2}} \left( \frac{\lambda_i}{\lambda_j} \right)^{\hat{b}} \leq C \left( \frac{1}{\lambda_i^2 + \lambda_i \lambda_j |a_i - a_j|^2} \right)^{\hat{b} - \frac{\alpha}{2}} \leq C \varepsilon_{ij}.
\]

Thus, we have for \( I_1 \)

\[
I_1 \leq C \varepsilon_{ij} \varepsilon_{ij}.
\]

On the other hand, using the estimate for \( I_2 \) (see (69)), we have

\[
I_2 = O \left( \frac{\varepsilon_{ij}}{\delta^6} \right).
\]

Hence, combining (70) and (71), we have

\[
\int_{\Omega} u_{\alpha_i, \lambda_i}^\alpha u_{\alpha_j, \lambda_j}^\alpha = O \left( \frac{\varepsilon_{ij}}{\delta^6} \right).
\]

\section{Algebraic topological argument}

In this section, we present the algebraic topological argument for existence. We start by fixing some notation from algebraic topology. For a topological space \( Z \), \( H_*(Z) \) denotes the singular homology of \( Z \) with \( \mathbb{Z}_2 \) coefficients. If \( Y \) is a subspace of \( Z \), then \( H_*(Z, Y) \) stands for the relative homology with \( \mathbb{Z}_2 \) coefficients of the topological pair \( (Z, Y) \). For a map \( f : Z \to Y \) with \( Z \) and \( Y \) topological spaces, \( f_* \) denotes the induced map in homology. If \( f : (Z, Y) \to (W, X) \) is a map with \( (Z, Y) \) and \( (W, X) \) topological pairs, then \( f_* \) denotes the induced map in relative homology. Furthermore, we discuss some algebraic topological tools needed for our application of the Barycentre technique of Bahri-Coron\cite{2}. We start with the following observation. Since \( \Omega \) is a smooth domain of \( \mathbb{R}^3 \) which is non-contractible,
then there exists $n \in \{1, 2\}$ such that $H_n(\Omega)$ is not trivial, see [2] (see page 1 just after Theorem 1). Hence, as in [2] (see beginning of page 263), we have there exists $M$ a smooth compact connected $n$-dimensional manifold without boundary and a continuous map
\[ h : M \rightarrow \Omega \]
(72)
such that if we denote by $[M]$ the class of orientation (modulo 2) of $M$, then $h_*([M]) \neq 0$. Moreover, we have clearly the existence of a compact smooth manifold with boundary $K_0$ such that
\[ h(M) \subset K_0 \subset \Omega. \]
(73)

We recall the space of formal barycenter of $M$ that we need for our Barycenter technique for existence. For $p \in \mathbb{N}^*$, the set of formal barycenters of $M$ of order $p$ is defined as
\[ B_p(M) = \{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} : \alpha_i \in M, \alpha_i \geq 0, \ i = 1, \cdots, p, \ \sum_{i=1}^{p} \alpha_i = 1 \}, \quad B_0(M) = \emptyset, \]
(74)
where $\delta_{a}$ for $a \in M$ is the Dirac measure at $a$. we have the existence of $Z_2$ orientation classes (see [2])
\[ w_p \in H_{np-1}(B_p(M), B_{p-1}(M)), \quad p \in \mathbb{N}^*. \]
(75)

Now to continue, we fix $\delta$ small such that $0 < 2\delta \leq \epsilon_0$ where $\epsilon_0$ as in (17) with $K$ is replaced by $K_0$ and $K_0$ is given by (22). Moreover, we choose $\theta_0 > 0$ and small. After this, we let $\lambda$ varies such that $0 < \frac{1}{\lambda} \leq \theta_0 \delta$ and associate for every $p \in \mathbb{N}^*$ the map
\[ f_p(\lambda) : B_p(M) \rightarrow \Omega^{1+}(\Omega) \]
defined by the formula
\[ f_p(\lambda)(\sigma) = \sum_{i=1}^{p} \alpha_i u_{h(a_i), \lambda}, \quad \sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i}, \]
where $h$ is as in (72) and $u_{h(a_i), \lambda}$ as (18) (with $a$ replaced by $h(a_i)$).

As in Proposition 3.1 in [6] and Proposition 6.3 in [7], using Corollary 5.2 Corollary 5.3 Corollary 4.3 Lemma 5.5, and Lemma 5.6, we have the following multiple-bubble estimate.

**Proposition 6.1.** There exist $\tilde{C}_0 > 0$ and $\tilde{\epsilon}_0 > 0$ such that for every $p \in \mathbb{N}^*$, $p \geq 2$ and every $0 < \epsilon \leq \epsilon_0$, there exists $\lambda_p := \lambda_p(\epsilon)$ such that for every $\lambda \geq \lambda_p$ and for every $\sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(M)$, we have

1. If $\sum_{i \neq j} \epsilon_{i,j} > \epsilon$ or there exist $i_0 \neq j_0$ such that $\frac{\alpha_{i_0}}{\alpha_{j_0}} > \nu_0$, then
\[ J_q(f_p(\lambda)(\sigma)) \leq p^2 S. \]
2. If $\sum_{i \neq j} \epsilon_{i,j} \leq \epsilon$ and for every $i \neq j$ we have $\frac{\alpha_i}{\alpha_j} \leq \nu_0$, then
\[ J_q(f_p(\lambda)(\sigma)) \leq p^2 S \left( 1 + \frac{\tilde{C}_0}{\lambda} - \epsilon_0 (p-1) \right), \]

where $\epsilon_{i,j}$ is as in (19) with $(a_i, a_j)$ replaced by $(h(a_i), h(a_j))$ and $\lambda_i = \lambda_j = \lambda$, $\epsilon_0$ is as in (58) and $\nu_0$ is as in (59).

As in Lemma 4.2 in [6] and Lemma 6.4 in [7], we have the selection map $s_1$ (see (40)), Lemma 5.2 and Corollary 4.3 imply the following topological result.

**Lemma 6.2.** Assuming that $J_q$ has no critical points, then there exists $\tilde{\lambda}_1 > 0$ such that for every $\lambda \geq \tilde{\lambda}_1$,
\[ f_1(\lambda) : (B_1(M), B_0(M)) \rightarrow (W_1, W_0) \]
is well defined and satisfies
\[ (f_1(\lambda))_* (w_1) \neq 0 \text{ in } H_n(W_1, W_0). \]

As in Lemma 4.3 in [6] and Lemma 6.5 in [7], we have the selection map $s_p$ (see (40)), Lemma 5.2 and Proposition 6.1 imply the following recursive topological result.
Lemma 6.3. Assuming that $J_q$ has no critical points, then there exists $\tilde{\lambda}_p > 0$ such that for every $\lambda \geq \tilde{\lambda}_p$,

$$f_{p+1}(\lambda) : (B_{p+1}(M), B_p(M)) \rightarrow (W_{p+1}, W_p)$$

and

$$f_p(\lambda) : (B_p(M), B_{p-1}(M)) \rightarrow (W_p, W_{p-1})$$

are well defined and satisfy

$$(f_p(\lambda)_* w_p) \neq 0 \text{ in } H_{np-1}(W_p, W_{p-1})$$

implies

$$(f_{p+1}(\lambda)_* w_{p+1}) \neq 0 \text{ in } H_{n(p+1)-1}(W_{p+1}, W_p).$$

Finally, as in Corollary 3.3 in [6] and Lemma 6.6 in [7], we clearly have that Proposition 6.1 implies the following result.

Lemma 6.4. Setting

$$\tilde{p}_0 := 1 + \frac{C_0}{c_0} + 2$$

with $C_0$ and $c_0$ as in Proposition 6.1 and recalling (41), we have there exists $\tilde{\lambda}_{p_0} > 0$ such that for every $\lambda \geq \tilde{\lambda}_{p_0}$,

$$f_{p_0}(\lambda)(B_{p_0}(M)) \subset W_{p_0-1}.$$

Proof of Theorem 1.1

As in [6] and [7], the theorem follows by a contradiction argument from Lemma 6.2 - Lemma 6.4.

References

[1] Bahri A., Brezis H., Non-Linear Elliptic Equations on closed Riemannian Manifolds with the Sobolev Critical Exponent, Progress in Nonlinear Differential Equations, Volume 20. Topics in Geometry: In Memory of Joseph D’Arti.

[2] Bahri A., Coron J.M., On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math. 41-3, 253-294 (1988).

[3] Brezis H., Elliptic equations with limiting Sobolev exponent—the impact of topology, Frontiers of the mathematical sciences: 1985 (New York, 1985). Comm. Pure Appl. Math. 39 (1986).

[4] Brendle S., Convergence of the Yamabe flow for arbitrary initial energy, J. Diff. Geom. 69 (2005), 217-278.

[5] Brezis H., Nirenberg L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math., 36(4):437-477, 1983.

[6] Mayer M., Ndiaye C. B., Barycenter technique and the Riemann mapping problem of Cherrier-Escobar, J. Differential Geom, 107, no. 3, pp 519–560, 2017.

[7] Ndiaye C. B., Sire Y., Sun L., Uniformizations Theorems: Between Yamabe and Paneitz, Pacific Journal of Mathematics, 314-1 (2021), 115–159. DOI 10.2140/pjm.2021.314.115, Preprint arXiv:1911.02680.

[8] Schoen R., Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. 20, No. 2, 479-495 (1984).