Annular structures in perturbed low mass disc-shaped gaseous nebulae II: general and polytropic models

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Abstract This is the second of two papers where we study additional analytical solutions of a bidimensional low mass gaseous disc rotating around a central mass and submitted to small radial perturbations. In a first Paper, hydrodynamics equations were solved for the equilibrium and perturbed configurations and a wave-like equation for the gas perturbed specific mass was deduced and solved analytically for several cases of exponents of the power law distributions of the unperturbed specific mass and sound speed. In this paper, two other general cases of exponents, including a polytropic case, are solved analytically for small frequencies of the perturbations. Similar conclusions to the ones of Paper I are found, namely that the maxima of the gas perturbed specific mass are exponentially spaced and that their distance ratio is a constant, function of disc characteristics and of the perturbations frequency. Gaseous annular structures would eventually be formed in the disc by inward and outward gas flows from zones of minima toward zones of maxima of perturbed specific mass.

Keywords: Interdisciplinary astronomy, Astrophysical fluid dynamics, Hydrodynamics, Protoplanetary nebulae

1 Introduction

In a previous paper ([1] hereafter referred to as Paper 1), we presented analytical solutions of perturbations propagating in a differentially rotating, axisymmetric, thin gaseous nebular disc, undergoing polytropic transformations of index $\gamma$. Viscous and magnetohydrodynamic forces were neglected in the disc. The specific mass and sound speed in the disc at equilibrium had power law distributions in the radial distance $r$, respectively $\rho \sim r^d$ and $c \sim r^{s/2}$. Searching for solutions yielding annular structures to appear in the disc when submitted to small radial periodic perturbations, hydrodynamics equations were solved analytically for the equilibrium and perturbed configurations, for two particular cases ($d = 0$ and $d < 2(2\gamma - 1)$; $s = 2$) and for a general case ($d = (s - 2)$; $s < 2$) for small frequencies. In each case, the maxima of perturbed specific mass were found to be exponentially spaced and their distance ratio $\beta$ was found to be a constant, depending on characteristics of the disc (and on ORCID 0000-0003-4884-3827
the perturbations frequency for the first two cases). Inward and outward flows of gas appeared with negative and positive radial velocities between minima and maxima of gas perturbed specific mass, leading the nebular gas eventually to accumulate in the zones of maxima of perturbed specific mass. We present here analytical solutions for two other general models for small frequencies of the periodic perturbations. In section 2, the disc model notations and the equations deduced in Paper 1 are recalled. The general case for $d = (s - 2)/2$ is solved in section 3. We present in section 4 a method to solve a second general model for $d = s/\gamma - 1$, called the "polytropic model", and a complete solution is found for the particular values $\gamma = 3/2$, $d = -3$ and $s = -1$. We do not know of any previous similar general analytical resolutions. The conclusions are drawn in section 5 and are similar to the ones of Paper 1. Both papers are reworked excerpts of [2].

2 Model notations

In Paper 1 [1], we considered a gaseous disc of mass $M_d$ and of specific mass $\rho_0 = \rho_r R^d$ with a sound speed $c_0 = c_r R^{d/2}$, rotating at a circular velocity $v_0 = v_0(R)$ around a central mass $M^\ast (M^\ast >> M_d)$, $R = (r/r_c)$ is a dimensionless radial distance and the indexes 0 and c denote equilibrium characteristics and reference characteristics at the inner edge of the disc. Allowing for small radial periodic perturbations of circular frequency $\omega$ to appear, a wave-like equation was deduced for the spatial term $D$ of the gas perturbed specific mass, with the prime sign $'' = \partial / \partial R$,

$$D'' + \left(2s + 1 - \frac{d + s}{\gamma}\right) \frac{D'}{R} + \left(B^2 R^{d+2-s} + \omega^2 A^2 R^{2-s} + s \left(s - \frac{d + s}{\gamma}\right)\right) \frac{D}{R^2} = 0$$

(1)

where

$$A^2 = \frac{r_c^2}{c_r^2}; \quad B^2 = \frac{4\pi G \rho_c r_c^2}{c_r^2}$$

are constants. The spatial terms of the associated perturbed radial velocity $U$ and specific mass flux radial momentum $\Phi$ were found in function of $D$

$$U (R) = -\kappa r_c \rho_r^{-2} \int D (R) R dR$$

$$\Phi (R) = r_c \rho_r R^{d+1} U (R) = -\kappa r_c^2 \int D (R) R dR$$

(2)

(3)

Two boundary conditions were defined by, first, the gas perturbed specific mass matching at the disc inner edge, for $R = 1$, a certain value independent from disc physical characteristics (see Paper 1) and, second, decreasing perturbed specific mass for increasing $R$, vanishing far away from the central mass.

3 Solutions for $d = (s - 2)/2$

Searching for solutions yielding annular structures to appear in the disc, we consider a fourth case where the exponents $d$ and $s$ are linked by the relation $d = (s - 2)/2$. 

$$d = \frac{s - 2}{2}$$

$$s = \frac{3}{2}, \quad d = -3, \quad s = -1$$

$$A^2 = \frac{r_c^2}{c_r^2}; \quad B^2 = \frac{4\pi G \rho_c r_c^2}{c_r^2}$$

(1)

Two boundary conditions were defined by, first, the gas perturbed specific mass matching at the disc inner edge, for $R = 1$, a certain value independent from disc physical characteristics (see Paper 1) and, second, decreasing perturbed specific mass for increasing $R$, vanishing far away from the central mass.
The equation (1) reads

\[
D'' + \left( 4d + 5 - \frac{3d + 2}{\gamma} \right) \frac{D'}{R} + \left( \omega^2 A^2 R^{-2d} + B^2 R^{-d} + 2(d + 1) \left( 2(d + 1) - \left( \frac{3d + 2}{\gamma} \right) \right) \right) \frac{D}{R^2} = 0
\]  

(4)

Substituting the variable \(R\) for \(z = j \left( \frac{2}{d} \omega A R^{-d} \right)\) with \(j = \sqrt{-1}\), yields a confluent hypergeometric equation

\[
z^2 \frac{\partial^2 D}{\partial z^2} - \frac{1}{d} \left( 3d + 4 - \left( \frac{3d + 2}{\gamma} \right) \right) z \frac{\partial D}{\partial z} - \left( \frac{z^2}{4} + j \frac{B^2}{2d\omega A} z - 2 \left( \frac{d + 1}{d^2} \right) \left( 2(d + 1) - \left( \frac{3d + 2}{\gamma} \right) \right) \right) D = 0
\]  

(5)

With \(d \neq -2 / (3 + i\gamma)\) for all integers \(i\), (5) has a complex solution

\[
D_C = R^{-\frac{i}{2} \left( 3d + 4 - \left( \frac{3d + 2}{\gamma} \right) \right)} \exp \left( \frac{-z}{2} \right) \left( K_1 z^\frac{1}{2} \left( 1 + \left( \frac{3d + 2}{\gamma} \right) \right) \phi_1 + K_2 z^\frac{1}{2} \left( 1 - \left( \frac{3d + 2}{\gamma} \right) \right) \phi_2 \right)
\]  

(6)

with \(K_1\) and \(K_2\) constants and where \(\phi_1 = \phi(a_1, b_1, z)\) and \(\phi_2 = \phi(a_2, b_2, z)\) are Kummer confluent hypergeometric function of arguments

\[
a_1 = \frac{1}{2} \left( 1 + \frac{3d + 2}{d\gamma} \right) + j y ; \quad b_1 = 1 + \frac{3d + 2}{d\gamma}
\]

\[
a_2 = \frac{1}{2} \left( 1 - \frac{3d + 2}{d\gamma} \right) + j y ; \quad b_1 = 1 - \frac{3d + 2}{d\gamma}
\]

where

\[
y = \frac{B^2}{d\omega A \left( 3d + 2 - (3d + 4) \right)}
\]

For \(b_1\) and \(b_2\) non null and different from negative integers, the Kummer function \(\phi\) expands for both sets of arguments as

\[
\phi(a, b, z) = \Gamma(\nu) \left( \frac{z}{4} \right)^{-\nu} \exp \left( \frac{z}{2} \right) \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu)_k (b - 2a)_k}{k! (b)_k} I_{\nu+k} \left( \frac{z}{2} \right)
\]

(7)

where \(\Gamma\) is the Legendre Gamma function, \((x)_k\) are Pochhammer polynomials,

\[
(x)_k = \prod_{q=0}^{k-1} (x + q) ; \quad (x)_0 = 1
\]
and $I_{\nu+k}$ is the complex valued hyperbolic Bessel function of order $(\nu + k)$, with $\nu$ either of

$$
\nu_1 = \left( b_1 - a_1 - \frac{1}{2} \right) = \frac{3d + 2}{2d\gamma} - jy
$$
$$
\nu_2 = \left( b_2 - a_2 - \frac{1}{2} \right) = -\frac{3d + 2}{2d\gamma} + jy
$$

Developing the complex coefficients $H_k$ of $I_{\nu+k}$ in (7) as in Appendix A (5), $D_C$ (6) becomes

$$
D_C = CR^{-2(d+1) + \left( \frac{3d+2}{\gamma} \right) \frac{z}{2} jy} \left( K_1 C_1 \sum_{k=0}^{\infty} \left( H_1 k I_{\nu_1+k} \left( \frac{z}{2} \right) \right) 
+ K_2 C_2 \sum_{k=0}^{\infty} \left( H_2 k I_{\nu_2+k} \left( \frac{z}{2} \right) \right) \right) 
$$

(8)

where

$$
C = \sqrt{\omega A d} \exp \left( \frac{\pi}{4} \right) ; \quad C_1 = 2^{1+ \frac{3d+2}{\gamma}} \Gamma (\nu_1) ; \quad C_2 = 2^{1- \frac{3d+2}{\gamma}} \Gamma (\nu_2)
$$

are complex constants. Simple analytical expressions of zeros and extrema of the real part of $D_C$ (8) were not found. However, for small arguments $z$, $|z|/2 \ll 1$, i.e. for small frequencies

$$
\omega \ll |d| \frac{C_C}{R} R^{-|d|}
$$

(9)

where vertical bars denote the absolute value, the terms other than the first in the convergent series of (7) can be neglected. By the multiplication theorem [7], the hyperbolic Bessel function reduces then to

$$
I_{\nu} \left( \frac{z}{2} \right) \approx \left( \frac{z}{4} \right)^{-\nu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{m! n! F(\nu - m + n + 1)} = \left( \frac{z}{4} \right)^{-\nu} \frac{G(\nu)}{\nu F(\nu)}
$$

(10)

where terms of second order were neglected in front of unity and where $G(\nu)$ is a complex function developed in Appendix B (5). The relation (7) reads now

$$
\phi (a, b, z) = \frac{G(\nu)}{\nu} \left( \frac{z}{4} \right)^{-2\nu} \exp \left( \frac{z}{2} \right)
$$

(11)

The complex solution (8), written for the variable $R$, becomes

$$
D_C = \left( K_1 A R^{-2(d+1) + \frac{3d+2}{\gamma}} + K_2 M R^{-2(d+1)} \right) \exp \left( 2y \ln \left( \frac{2}{|d| |\omega A R^{-d}|} \right) \right)
$$

(12)

where $A = |A| \exp (j\lambda)$ and $M = |M| \exp (j\mu)$ are complex constants depending on $y$, $\nu$ and $G(\nu)$ (see Appendix B). The real part of (12) reads as the sum of two terms

$$
D_R = K_1 |A| R^{-2(d+1) + \frac{3d+2}{\gamma}} \cos \left( 2y \ln \left( \frac{2}{|d| |\omega A R^{-d}|} \right) + \lambda \right)
+ K_2 |M| R^{-2(d+1)} \cos \left( 2y \ln \left( \frac{2}{|d| |\omega A R^{-d}|} \right) + \mu \right)
$$

(13)
Due to the second boundary condition (decrease of $D$ for increasing $R$), either the first or the second or both terms of (13) should be considered for the general solution, depending on the respective values of $d$ and $\gamma$ as indicated in Table 1.

Without loss of generality in the resolution, we consider from now on only the case $s < 0$ and $d < -1$, yielding $K_2 = 0$ in (13), the other constant $K_1$ being fully determined by the first boundary condition.

The perturbed radial velocity $U$ and the specific mass flux momentum $\Phi$ read, from (2) and (3),

$$U = -\kappa K_1 C_4 \frac{r_c}{\rho_c} R^{-(3d+3) \frac{3d+2}{2}} \sin \left(2y \ln \left(\frac{2}{|d|} \omega AR^{-d}\right) + \lambda + \zeta \right)$$

$$\Phi = -\kappa K_1 C_3 r_c R^{-2d+ \frac{3d+2}{2}} \sin \left(2y \ln \left(\frac{2}{|d|} \omega AR^{-d}\right) + \lambda + \zeta \right)$$

with

$$C_4 = \frac{|A|}{\sqrt{\left(\frac{3d+2}{2} - 2d\right)^2 + (-2dy)^2}} ; \quad \zeta = \arctan \left(\frac{\frac{3d+2}{2} - 2d}{-2dy}\right)$$

The extrema (minima and maxima) of $D_R$ are found from

$$D_R' = -K_1 C_4 R^{-2(d+3) \frac{3d+2}{2}} \sin \left(2y \ln \left(\frac{2}{|d|} \omega AR^{-d}\right) + \lambda + \xi \right) = 0$$

with

$$C_4 = |A| \sqrt{\left(\frac{3d+2}{2} - 2(d+1)\right)^2 + (2dy)^2} ; \quad \xi = \arctan \left(\frac{\frac{3d+2}{2} - 2(d+1)}{2dy}\right)$$

The zeros of $D_R$ (13), $U$ (14), $\Phi$ (15) and $D_R'$ (16) read in this fourth case

$$R = \alpha_4 \left(\beta_4^2\right)^n ; \quad \alpha_4 = \left(\frac{2}{|d|} \omega A\right)^{\frac{1}{4}} \exp \left(\frac{\lambda + \varphi_4}{2 |d| y}\right) ; \quad \beta_4 = \exp \left(\frac{\pi}{2 |d| y}\right)$$

$n$ being non-negative integers and $\varphi_4 = (\pi/2)$ for $D_R$, $\varphi_4 = \zeta$ for $U$ and $\Phi$, and $\varphi_4 = \xi$ for $D_R'$. Provided that $\omega A$ is small enough, within the condition (9), one has $y >> 1$, yielding $\zeta << 1$ and $\xi << 1$. The initial phase between $D$ and $U$ is $(\pi/2) - \zeta \approx (\pi/2)$, while the initial phase between $U$ (or $\Phi$) and $D'$ is $(\xi - \zeta) \approx 0$.

The distances ratio of two successive maxima of $D$ is

$$\beta_4 = \left(\beta_4^2\right)^2 \exp \left(\frac{3d + 4 - \frac{3d+2}{2}}{4\pi G \rho_c r_c}\right)$$

| $d$ | $\gamma < \frac{2}{3}$ | $\gamma = \frac{2}{3}$ | $\gamma > \frac{2}{3}$ |
|-----|----------------|----------------|----------------|
| $d < \frac{2}{3}$ | $K_1 = 0$ | $K_1 = 0$ | Not Applicable |
| $d \geq \frac{2}{3} - \frac{1}{2}$ | $K_1 = 0$ | $K_1 = 0$ | Not Applicable |
| $d \leq \frac{2}{3} - \frac{1}{2}$ | $R_2 = 0$ | Not Applicable | $R_2 = 0$ |
| $d > \frac{2}{3} - \frac{1}{2}$ | $R_2 = 0$ | No decrease | $R_2 = 0$ |

Table 1 Terms of solution $D$ (13) for respective values of $d$ and $\gamma$
which is a real constant depending on the perturbations circular frequency $\omega$ and the disc reference characteristics. The period of the perturbations must be larger than a minimum value

$$P_m = \frac{2\pi c_c}{|d| c_c} R_{max}$$  \hspace{1cm} (19)$$
deduced from the condition (9) applied to the whole range of radial distances of the disc ($R_{max}$ is the ratio of the outer and inner radii of the disc).

### 4 Solution for the polytropic case

#### 4.1 General formulation

In the previous section and in Paper 1, we considered the exponents $d$ and $s$ taking particular values or linked by non-causal relations. However, a relation between $d$ and $s$ can be found if one considers that the specific mass and sound speed are fully governed by polytropic processes in the disc. Considering the two polytropic relations between the pressure $p$, the specific mass $\rho$ and the sound speed $c$

$$p = \frac{e^2 \rho}{\gamma} ; \ pp^{-\gamma} = \text{constant}$$  \hspace{1cm} (20)$$
one has successively, with the power law radial distributions $\rho_0 = \rho_c R^d$ and $c_0^2 = c_c^2 R^s$,

$$e^2 \rho^{1-\gamma} = e_c^2 \rho_c^{1-\gamma} R^{d(1-\gamma)} = \text{constant}$$  \hspace{1cm} (21)$$
$$s + d (1 - \gamma) = 0 \ or \ \gamma = 1 + \frac{s}{d}$$  \hspace{1cm} (22)$$

Replacing $\gamma$ for $s$ and $d$ in the equation (1) yields

$$D'' + (2s + 1 - d) \frac{D'}{R} + \left( B^2 R^{d+s} - \omega^2 A^2 R^{2-s} + s(s-d) \right) \frac{D}{R^2} = 0$$  \hspace{1cm} (23)$$
which becomes a differential Schrödinger type equation by posing

$$D = \frac{Y}{\sqrt{R^{2s+1-d}}}$$
yielding

$$Y'' + \left( B^2 R^{d-s} + \omega^2 A^2 R^{-s} - \left( \frac{d^2 - 1}{4} \right) R^{-2} \right) Y = 0$$  \hspace{1cm} (24)$$
An approximate solution to this equation can be found by the Wentzel-Kramers-Brillouin (WKB) theory [8]. Considering the case of small frequencies such as

$$\omega A << B^2 \ or \ \omega << 4\pi c_c \frac{\rho_c r_c}{c_c}$$  \hspace{1cm} (25)$$
one poses $\epsilon = \omega A B^{-2}$ with $\epsilon << 1$. The equation (24) reads then

$$\epsilon^2 Y'' = \left( -\omega^2 A^2 \epsilon^2 R^{-s} - \omega A \epsilon R^{d-s} + \epsilon^2 \left( \frac{d^2 - 1}{4} \right) R^{-2} \right) Y = Q(R) Y$$  \hspace{1cm} (26)$$
Let us consider the three following functions of $R$

\[ S_0 (R) = \int_1^R \sqrt{Q(x)} \, dx; \quad S_1 (R) = - \frac{\ln (Q(R))}{4}; \quad S_2 (R) = \int_1^R \left( \frac{QQ'' - \frac{3}{4} (Q')^2}{8Q^2} \right) \, dx \]

(27)

where the first two functions are referred to respectively as the eikonal function and the transport function and where $Q' = \frac{dQ(x)}{dx}$.

If $Q(R) \neq 0$ in the range of interest of $R$ (i.e., $1 \leq R \leq R_{\text{max}}$) and under the conditions

\[ S_1 (R) << \frac{S_0 (R)}{\epsilon}; \quad \epsilon S_2 (R) << S_1 (R); \quad \epsilon S_2 (R) << 1 \]

(28)

the leading orders in the WKB physical optics approximation to the exact solutions in $Y$ and $D$ reads generally

\[ Y (R) = K_3 \exp \left( \frac{S_0 (1, R)}{\epsilon} + S_1 (R) \right) + K_4 \exp \left( - \frac{S_0 (1, R)}{\epsilon} + S_1 (R) \right) \]

(29)

\[ D (R) = R^{-(2s+1-d)/2} \left( K_3 \exp \left( \frac{S_0 (1, R)}{\epsilon} + S_1 (R) \right) \right) + K_4 \exp \left( - \frac{S_0 (1, R)}{\epsilon} + S_1 (R) \right) \]

(30)

with $K_3$ and $K_4$ constants determined by the boundary conditions and where $S_0 (1, R)$ is the eikonal function on the interval $[1, R]$. Strictly speaking, the above equality sign should be replaced by an asymptotic equality sign. The eikonal function $S_0 (1, R)$ reads

\[ S_0 (1, R) = \int_1^R \sqrt{Q(x)} \, dx = j\omega A \epsilon \int_1^R x^{-s} + \frac{1}{\omega A \epsilon} x^{d-s} - \left( \frac{d^2 - 1}{4\omega^2 A^2} \right) x^{-2} \, dx \]

\[ = j\omega A \epsilon \int_1^R \sqrt{T(x)} \, dx \]

(31)

with

\[ T(x) = x^{2-s} + \frac{1}{\omega A \epsilon} x^{2+d-s} - \left( \frac{d^2 - 1}{4\omega^2 A^2} \right) \]

(32)

The integral (31) has to be evaluated for specific values of $d$ and $s$. This evaluation involves most of the time elliptic integrals, which makes it uneasy.

4.2 Solution for $\gamma = 3/2, d = -2$ and $s = -1$

In most nebula models, the gas specific mass and sound speed are decreasing outward from the central body, with the exponents $d$ and $s$ taking negative values and $s$ usually in the order of or close to $-1$. We consider here the particular polytropic case with $s = -1$ and $d = -2$, yielding $\gamma = 3/2$. Cases for other values of $d$, $s$ and $\gamma$ can be solved similarly. The integral (31) reads

\[ S_0 (1, R) = j \left( \frac{\omega A \epsilon}{3} I_1 + \frac{2}{3} I_2 - \frac{3\epsilon}{4\omega A} I_3 \right) \]

(33)
The cubic trinomial \( T(x) \) has a single real root and, neglecting terms in \( \epsilon^2 \) and of higher order, it becomes

\[
T(x) = \left( x - \frac{3\epsilon}{4\omega A} \right) \left( x^2 + \frac{3\epsilon}{4\omega A}x + \frac{1}{\omega A\epsilon} \right)
\]  

(35)

Posing

\[
C_5 = \sqrt{\frac{1}{\omega A\epsilon} + \frac{9}{8} \left( \frac{\epsilon}{\omega A} \right)^2}
\]

one substitutes \( \left( x - \frac{3\epsilon}{4\omega A} \right) \) for \( C_5z \) in integral \( I_2 \) and for \( C_5z^2 \) in integral \( I_3 \) in (34).

The integrals in (34) are evaluated under the two following conditions

\[
\frac{\omega A}{B} R << 1 \quad ; \quad B^2 R >> \frac{3}{4}
\]

(36)

in the range \( 1 \leq R \leq R_{\text{max}} \), showing also that the real root of the trinomial \( T(x) \) is outside the range of interest of \( R \), fulfilling the condition \( Q(R) \neq 0 \). The other conditions of application of the WKB theory are verified in Appendix C.

Under the above two conditions, \( I_2 \) and \( I_3 \) become

\[
I_2 = \frac{1}{\sqrt{C_2}} (F(\varphi(R), k) - F(\varphi(1), k))
\]

(37)

\[
I_3 = \frac{1}{(C_2)^3} \left( (F(\varphi(R), k) - 2E(\varphi(R), k)) - (F(\varphi(1), k) - 2E(\varphi(1), k)) \right)
\]

\[
- \frac{2}{C_2} \left( \frac{1}{R - \frac{3\epsilon}{4\omega A}} + C_2 \sqrt{\frac{R^2 + \frac{3\epsilon}{4\omega A}R + \frac{1}{\omega A\epsilon}}{R - \frac{3\epsilon}{4\omega A}}} \right)
\]

\[- \left( \frac{1}{1 - \frac{3\epsilon}{4\omega A}} + \frac{1}{C_2} \right) \right)
\]

\[
(38)
\]

where \( F \) and \( E \) are the incomplete elliptic integrals of the first and second kinds of argument and modulus

\[
\varphi(R) = 2 \arctan \left( \frac{1}{C_2} \sqrt{R - \frac{3\epsilon}{4\omega A}} \right) 
\]

\[
\approx 2 \arctan \left( \frac{\omega A}{B} R \right)
\]

(39)

\[
k = \sqrt{\frac{1}{2} - \frac{9\epsilon}{16\omega A\sqrt{C_2}}} \approx \frac{1}{\sqrt{2}}
\]

(40)

where the conditions \( \[40\] \) were used, yielding also \( C_2 = \frac{B}{A} \). Replacing in \( \[39\] \) and in \( \[34\] \) yields eventually

\[
S_0(1, R) = j\epsilon (s_{0R} - s_{01})
\]

(41)
\[ s_{0R} = \left( \frac{2}{3} \sqrt{\frac{B^3}{\omega A}} - \frac{3}{4} \sqrt{\frac{\omega A}{B^3}} \right) F(\varphi(R), k) + \frac{3}{2} \sqrt{\frac{\omega A}{B^3}} E(\varphi(R), k) \]
\[ + \sqrt{\left(\frac{\omega A}{B^3}\right)^2 R^3 + B^2 R - \frac{3}{4} \left( \frac{2 + \frac{2}{\omega A B^3}}{(B^2 R - \frac{3}{4}) \left( B^2 R - \frac{3}{4} + \frac{B^3}{B^3 N} \right)} \right)} \]  

(42) and a similar relation for \( s_{01} \) with \( R = 1 \).

The incomplete elliptic integrals \( F \) and \( E \) are evaluated after an ascending Landen transformation, yielding the new argument and modulus and the transformed expressions of \( F \) and \( E \) to be

\[ \varphi_t = \frac{1}{2} \left( \varphi + \arcsin (k \sin \varphi) \right) \approx \frac{1}{2} \arcsin \left( 2 \left( 1 + \frac{1}{\sqrt{B^3}} \right) \sqrt{\frac{\omega A}{B^3}} R \right) \]  

(43)
\[ k_t = \frac{2\sqrt{k}}{1+k} \approx \frac{2^{5/2}}{1 + \sqrt{2}} \approx 0.9852 \]  

(44)
\[ F(\varphi_t(R), k_t) = \frac{2}{1+k} F(\varphi_t(R), k_t) \]  

(45)
\[ E(\varphi_t(R), k_t) = (1+k) E(\varphi_t(R), k_t) + (1-k) F(\varphi_t(R), k_t) \]
\[ - \frac{(1+k) \tan \varphi}{2 + \frac{\sec \varphi (1 + \frac{3}{2} \sqrt{1-k^2})}{\cos \varphi + \frac{1}{2} \sqrt{1-k^2} \sin^2 \varphi}} \]  

(46)

As \( k_t \) is close to unity, one can use the expansions

\[ F(\varphi_t(R), k_t) = \frac{2}{\pi} K' \ln \left( \tan \left( \frac{\varphi_t}{2} + \frac{\pi}{4} \right) \right) - \sin \varphi_t \sec^2 \varphi_t \left( a_0 - \frac{2}{3} a_1 \tan^2 \varphi_t + \ldots \right) \]  

(47)
\[ E(\varphi_t(R), k_t) = \frac{2}{\pi} E' \ln \left( \tan \left( \frac{\varphi_t}{2} + \frac{\pi}{4} \right) \right) + \sin \varphi_t \sec^2 \varphi_t \left( b_0 - \frac{2}{3} b_1 \tan^2 \varphi_t + \ldots \right) \]  

(48)

where

\[ K' = K \sqrt{1-k_t^2} \; ; \; E' = E \sqrt{1-k_t^2} \]

are the complete integrals of the first and second kinds and \( a_0, a_1, \ldots, b_0, b_1, \ldots \) are decreasing coefficients, functions of \( k_t \). As \( k_t \approx 0.9852, K' \approx E' \approx \pi/2 \) within \( 7.5 \times 10^{-3}, a_0 \approx 7.5 \times 10^{-3}, a_1 \approx 10^{-4}, b_0 \approx -7.5 \times 10^{-3}, \ldots \), yielding

\[ F(\varphi_t(R), k_t) \approx E(\varphi_t(R), k_t) \approx \ln \left( \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B^3} R} \right) \]  

(49)

Replacing in (43), (45) and (42) and using conditions (30) to neglect small terms, it yields

\[ s_{0R} = W \ln \left( \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B^3} R} \right) + \frac{2}{3} \sqrt{(\omega A)^2 R^3 + B^2 R - \frac{3}{4}} \]  

(50)
and similarly for $s_{01}$ for $R = 1$, with $W = W(\omega, A, B)$

$$W = \frac{8}{3} \left( 1 - \frac{1}{\sqrt{2}} \right) \sqrt{\frac{B^3}{\omega A}} + 3 \sqrt{\frac{\omega A}{2B^3}} \approx \frac{8}{3} \left( 1 - \frac{1}{\sqrt{2}} \right) \sqrt{\frac{B^3}{\omega A}}$$

(51)

under the conditions (36).

The complex perturbed specific mass $D_C$ (30) and its real part read

$$D_C = \frac{B}{\sqrt{3j\omega A}(\omega A)^2 R^3 + B^2 R - \frac{3}{4}} \left( K_3 \exp (j (s_0 R - s_{01})) + K_4 \exp (-j (s_0 R - s_{01})) \right)$$

(52)

$$D_R = K_5 \frac{B}{\sqrt{\omega A}(\omega A)^2 R^3 + B^2 R - \frac{3}{4}} \cos (s_0 R - s_{01} - \kappa)$$

(53)

with

$$K_5 = \sqrt{K_3^2 + K_4^2} ; \kappa = \arctan \left( \frac{K_3 - K_4}{K_3 + K_4} \right)$$

The extrema of $D_R$ are solutions of

$$D'_R = -K_5 C_6 \frac{1}{\sqrt{B^2 R^3 + B^2 R - \frac{3}{4}}} \sin (s_0 R - s_{01} - \kappa + \tau) = 0$$

(54)

with

$$C_6 = \frac{4}{3} \left( 1 - \frac{1}{\sqrt{2}} \right) \frac{B^2}{\omega A} ; \tau = \arctan \left( \frac{3}{8} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B^3}} \right)$$

where the conditions (36) were used to neglect small terms. The zeros and extrema of $D_R$ are given by

$$\frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B}} R \exp \left( \frac{2}{3} \sqrt{\frac{(\omega A)^2 R^3 + B^2 R - \frac{3}{4}}{W}} \right)$$

$$= \exp \left( \frac{s_0 + \varphi_5 + \kappa}{W} \right) \exp \left( \frac{\pi n}{W} \right)$$

(55)

where $n$ are non-negative integers and $\varphi_5 = \pi/2$ for $D_R$ and $\varphi_5 = -\tau$ for $D'_R$. The exponential term in the above left hand side part reduces to

$$\exp \left( \frac{2}{3} \sqrt{\frac{(\omega A)^2 R^3 + B^2 R - \frac{3}{4}}{W}} \right) \approx \exp \left( \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B}} R \right)$$

(56)

under the conditions (36), and it can be neglected when multiplied by its argument, small by (36). The zeros and extrema of $D_R$ are then given in good approximation by

$$R = \alpha_5 (\beta_5^n) ; \alpha_5 = 16 \left( \frac{3}{2} - \frac{1}{\sqrt{2}} \right) \frac{B}{\omega A} \exp \left( \frac{2(s_0 + \varphi_5 + \kappa)}{W} \right) ; \beta_5 = \exp \left( \frac{2\pi}{W} \right)$$

(57)

The perturbed radial velocity $U$ and the specific mass flux radial momentum $\Phi$ are found from (2) and (3), with (53). However, their evaluation requires the resolution of
a new elliptic integral. To avoid this and as there are no zeros due to the transport function \( S_1(R) \) (in the 4-th root of the trinomial term in \( R \)) in (53), we evaluate \( U \) and \( \Phi \) by neglecting \( S_1(R) \). This is the geometrical optics approximation, which gives the most rapidly varying component (controlling factor) of the leading behaviour of the exact solution. In the geometrical optics (g.o.) approximation, the real part of the complex perturbed specific mass (53) reduces then to

\[
D_{R \text{ g.o.}} = K_5 \frac{B}{\sqrt{\omega A}} \cos (s_{0R} - s_{01} - \kappa)
\]

Under the conditions (50), the term \( s_{0R} \) can be written approximately

\[
s_{0R} \approx W \left( \ln \left( \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B} R} \right) + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B} R} \right)
\]

as was already done in (56). The radial velocity \( U \) and the specific mass flux radial momentum \( \Phi \) read then in the geometrical optics approximation

\[
U_{g.o.} = -\kappa K_5 C_7 \frac{r_c}{c} R^3 \sin (s_{0R} - s_{01} - \kappa + \sigma)
\]

\[
\Phi_{g.o.} = -\kappa K_5 C_7 r_c^2 R^2 \sin (s_{0R} - s_{01} - \kappa + \sigma)
\]

with

\[
C_7 = \frac{2B}{\sqrt{\omega A (W^2 + 16)}} ; \quad \sigma = \arctan \left( 3 \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{\omega A}{B^3}} \right)
\]

showing that their zeros are given like the zeros and extrema of \( D_R \) with \( \beta_5 \) (57) and \( \varphi_5 = -\sigma + \alpha_5 \) (57).

By the condition (25), one has \( \sigma << 1 \) and \( \tau << 1 \). The initial phase between \( D \) and \( U \) (or \( \Phi \)) is \( (\pi/2) - \sigma \approx \pi/2 \), while the initial phase between \( U \) (or \( \Phi \)) and \( D' \) is \( (\tau - \sigma) \approx 0 \).

Let us note that, in the physical optics approximation, retaining the term \( S_1(R) \) in \( D_R \) would change the amplitudes of \( U \) and \( \Phi \) (by the addition of decreasing terms in \( R \) in front of the sin function) and it would change the coefficient of \( \sqrt{\omega A / B^3} \) and add negligible terms in the argument of the arctan of \( \sigma \). But the distance ratio of two successive zeros of \( U \) (or \( \Phi \)) is unaffected and is still given by \( \beta_5 \) (57) in both WKB approximations.

The distances ratio of two successive maxima of \( D \) is given by

\[
\beta_5 = (\beta_5')^2
\]

with

\[
\beta_5 = \exp \left( \frac{\pi}{3} \frac{1 - \frac{1}{\sqrt{2}}}{4 \sqrt{2} (4\pi G \rho_c)^{\frac{3}{2}} \frac{r_c}{c} + \frac{3}{4} \frac{\sqrt{2}}{2} (4\pi G \rho_c)^{-\frac{3}{4}} \frac{c}{r_c} } \right)
\]
\[
\approx \exp \left( \frac{3\pi}{3} \frac{1 - \frac{1}{\sqrt{2}}}{r_c (4\pi G \rho_c)^{\frac{3}{2}}} \right)
\]

where the approximated value of \( W \) (51) is used in (63). The distances ratio \( \beta_5 \) is a constant, function of the perturbations circular frequency \( \omega \) and of the disc reference
The period of the small perturbations must be larger than a minimum value $P_m$, which is the greatest of the two values that can be deduced from the two conditions (25) and (36) on $\omega$, applied to the whole range of radial distances up to $R_{\text{max}}$, yielding

$$P_m = \frac{c_c}{2G\rho_c r_c} \quad \text{or} \quad P_m = \sqrt{\frac{\pi}{G\rho_c}} R_{\text{max}}$$  \hspace{1cm} (64)

5 Conclusions

We have extended the resolution of the wave-like equation of perturbed specific mass deduced in Paper 1 to two other general cases. The solution for the "polytropic case" could not be solved generally as one must choose particular values of $d$ and $s$, fixing the value of the polytropic index $\gamma$. However, a solution was found in the WKB physical-optics approximation for an important particular case ($\gamma = 3/2$ with $d = -2$ and $s = -1$).

For the two above cases, conclusions similar to the ones of Paper 1 are reached concerning the functions $D$, $D'$, $U$ and $\Phi$, namely that, first, $D$ has a sign opposite to the signs of $D'$, $U$ and $\Phi$; second, the functions $D'$, $U$ and $\Phi$ are in phase and have an initial phase difference of approximately $\pi/2$ with respect to the function $D$; third, the zeros of $U$ corresponds to the extrema of $D$ and vice-versa; and finally, for increasing $R$, the functions $U$ and $\Phi$ are positive (respectively negative) between successive minima and maxima (respectively successive maxima and minima) of $D$. This situation yields radial outward flows of gas between successive minima and maxima of $D$ and radial inward flows of gas between successive maxima and minima of $D$, that would eventually form annular structures of gas, with axial radii corresponding to the distances of maxima of the gas perturbed specific mass. Furthermore, the maxima of the gas perturbed specific mass are found to be exponentially spaced for the two cases and their distances ratios are constants depending on discs characteristics and on the circular frequency of the perturbations. These results can be applied to protoplanetary and proto-satellite discs.

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Appendix A

The $k$-th complex coefficient of the hyperbolic Bessel functions in the series of (27) reads

$$H_k = \frac{(-1)^k (2\nu)_k (b - 2a)_k}{k! (b)_k}$$  \hspace{1cm} (65)

where $(x)_k$ are Pochhammer polynomials. This expression can be written $H_k = H_{Rk} + jH_{Ik}$ by posing $a = a_R + ja_I$ and by developing the Pochhammer polynomials, yielding

$$H_{Rk} = T_k (P_{Rk} Q_{Rk} - P_{Ik} Q_{Ik}) ; \quad H_{Ik} = T_k (P_{Rk} Q_{Ik} - P_{Ik} Q_{Rk})$$  \hspace{1cm} (66)
with

\[
P_{lk} = \sum_{m=0}^{L_k} \left( \sum_{q=2m}^{k} (-1)^{m+q} \binom{q}{2m} S_k(q) a_R^{q-2m} a_I^{2m} \right)
\]

(67)

\[
Q_{lk} = \sum_{m=0}^{L_k} (-1)^m |S_k(2m)| a_I^{2m}
\]

(68)

\[
P_{Ik} = \sum_{m=0}^{L_I} \left( \sum_{q=2m+1}^{k} (-1)^{m+q} \binom{q}{2m+1} S_k(q) a_R^{q-(2m+1)} a_I^{2m+1} \right)
\]

(69)

\[
Q_{Ik} = \sum_{m=0}^{L_I} (-1)^m |S_k(2m+1)| a_I^{2m+1}
\]

(70)

\[
T_k = \frac{(-1)^k}{k! \sum_{m=0}^{k} |S_k(m)| b^m}
\]

(71)

where \( \binom{q}{2m} \) are the binomial coefficients, \( |S_k(m)| \) is the absolute value of the Stirling numbers of the first kind and \( L_R = L_I = k/2 \) for \( k \) even and \( L_R = (k - 1)/2, L_I = (k + 1)/2 \) for \( k \) odd.

Appendix B

Writing \( \nu = \nu_R + j\nu_I \), with

\[
\nu_R = \frac{3d + 2}{2d\gamma} ; \quad \nu_I = -y = \frac{-B^2}{\omega A a (\frac{3d + 2}{\gamma}) - (3d + 4)}
\]

(72)

the term \( G(\nu) \) in (11) can be written \( G(\nu) = G_R + jG_I \) with, for \( \nu_I \),

\[
G_{1R} = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{n=0}^{m} P_n + \sum_{n=m+1}^{\infty} \frac{T_n}{C_n} \right) ; G_{1I} = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{n=0}^{m} Q_n + \sum_{n=m+1}^{\infty} \frac{W_n}{C_n} \right)
\]

(73)

\[
P_n = \frac{1}{n!} \sum_{p=0}^{L_I} \sum_{q=2p}^{m-n} (-1)^{p+n} \binom{q}{2p} (S_{m-n}(q)) \nu_R^{q-2p} \nu_I^{2p}
\]

(74)

\[
Q_n = \frac{1}{n!} \sum_{p=0}^{L_I} \sum_{q=2p+1}^{m-n} (-1)^{p+n} \binom{q}{2p+1} (S_{m-n}(q)) \nu_R^{q-2p-1} \nu_I^{2p+1}
\]

(75)

\[
T_n = \sum_{p=0}^{L_I} \sum_{q=2p}^{m-n} (-1)^{m+p+q} \binom{q}{2p} (S_{m-n}(q)) (\nu_R + 1)^{q-2p} \nu_I^{2p}
\]

(76)

\[
W_n = \sum_{p=0}^{L_I} \sum_{q=2p+1}^{m-n} (-1)^{m+p+q+1} \binom{q}{2p+1} (S_{m-n}(q)) (\nu_R + 1)^{q-2p-1} \nu_I^{2p+1}
\]

(77)

\[
C_n = n! \left( T_n^2 + W_n^2 \right)
\]

(78)
with \( L_1 = L_2 = -L_3 = L_4 = (m - n) / 2 \) for \((m - n)\) even and \( L_1 = -L_4 = (m - n - 1) / 2 \) and \( L_2 = -L_3 = (m - n + 1) / 2 \) for \((m - n)\) odd.

Similar relations are found for \( \nu_1 \) replacing \( \nu_R \) by \(-\nu_R \).

The complex constants in (12) read \( A = |A| \exp (j\lambda) \) and \( M = |M| \exp (j\mu) \) with

\[
|A| = 2^2 \left( \frac{x}{\gamma d} \right) \sqrt{\left( \frac{2}{|d|} \omega A \right)^{1-\left( \frac{4d+2}{2\gamma d} \right)}} \exp (-\pi y) \frac{|G_1|}{|\nu_1|} \tag{79}
\]

\[
\lambda = \frac{\pi}{4} \left( 1 - \frac{3d+2}{\gamma d} \right) - 4y \ln 2 + \arg (G_1) - \arg (\nu_1) \tag{80}
\]

\[
|M| = 2^{-2} \left( \frac{x}{\gamma d} \right) \sqrt{\left( \frac{2}{|d|} \omega A \right)^{1+\left( \frac{4d+2}{2\gamma d} \right)}} \exp (-\pi y) \frac{|G_2|}{|\nu_2|} \tag{81}
\]

\[
\mu = \frac{\pi}{4} \left( 1 + \frac{3d+2}{\gamma d} \right) - 4y \ln 2 + \arg (G_2) - \arg (\nu_2) \tag{82}
\]

where \( |G| \) and \( \arg (G) \) are the modulus and the argument of the complex valued function \( G (\nu) \).

### Appendix C

We verify the conditions (28) of application of the WKB physical optics approximation for the moduli

\[
|S_1 (R)| < < \left| \frac{S_0 (R)}{\epsilon} \right| ; |\epsilon S_2 (R)| < < |S_1 (R)| ; |\epsilon S_2 (R)| < < 1 \tag{83}
\]

with

\[
\left| \frac{S_0 (R)}{\epsilon} \right| = W \ln \left( 2 \left( 1 + \frac{1}{\sqrt{2}} \right) \right) \sqrt{\frac{\omega A}{B}} R + \frac{2}{3} \sqrt{(\omega A)^2 R^3 + B^2 R - \frac{3}{4}} \tag{84}
\]

\[
|S_1 (R)| = \frac{1}{4} \pi^2 + \left( \ln \left( \frac{\omega A}{RB^2} \right)^2 \left( (\omega A)^2 R^3 + B^2 R - \frac{3}{4} \right) \right)^2 \tag{85}
\]

\[
|\epsilon S_2 (R)| = \frac{1}{32} \int_0^R \left( 4 \left( -2B^2 x + \frac{9}{4} \right) (\omega A)^2 x^3 + B^2 x - \frac{3}{4} \right) x \left( (\omega A)^2 x^3 + B^2 x - \frac{3}{4} \right)^2 dx + 5 \left( (\omega A)^2 x^3 + B^2 x \right)^2 + 15 (\omega A)^2 x^3 - B^2 x + \frac{3}{4} \right) x \left( (\omega A)^2 x^3 + B^2 x - \frac{3}{4} \right)^2 dx \tag{86}
\]

and \( W \) is given by (31). The first condition (33) reads

\[
\left| \frac{S_0 (R)}{\epsilon} \right| - |S_1 (R)| > > 0 \tag{87}
\]

One easily verifies that the left hand side of the inequality tends towards positive infinity either when taking the limit for \( R \to +\infty \) with \( \epsilon \) constant or when taking the
limit for $\epsilon \to 0$ (or $\omega \to 0$) with $R$ constant. The verification of the second and third conditions implies the solution of the uneasy elliptic integral (86). One can get some insights into the verification of these two conditions without solving (86), although, strictly speaking, this method is not exactly rigorous. Looking at the behaviour of the dominant terms, we take the limit for $\epsilon \to 0$ (or $\omega \to 0$) under the integral sign, which yields

$$|\epsilon S_2| \approx \frac{1}{8} \int^R \left( \frac{-2B^2}{\sqrt{(B^2x - \frac{3}{4})^3}} + \frac{3}{4} \frac{1}{x\sqrt{(B^2x - \frac{3}{4})^3}} + \frac{5}{4} \frac{B^4x}{\sqrt{(B^2x - \frac{3}{4})^5}} \right) dx \quad (88)$$

which solves easily in

$$|\epsilon S_2| \approx \frac{1}{16} \left( \frac{5}{4} \frac{1}{\sqrt{(B^2R - \frac{3}{4})^3}} + \frac{1}{\sqrt{B^2R - \frac{3}{4}}} - \frac{1}{2}\sqrt{3} \arctan \left( \frac{1}{\sqrt{\frac{3}{4}B^2R - 1}} \right) \right) \quad (89)$$

The second condition, for $\epsilon \to 0$ (or $\omega \to 0$), is always satisfied, provided that $B^2R \neq 3/4$ for all $R$, as $|S_1| \to +\infty$. The third condition is also satisfied provided that $B^2R > > 3/4$ for all $R$, which is the condition (83). Taking now the limit for $R \to +\infty$ in (89) yields that $|\epsilon S_2| \to 0$, which satisfies both the second and third conditions (83).

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