EXISTENCE, UNIQUENESS AND STABILITY OF STEADY VORTEX RINGS OF SMALL CROSS-SECTION

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Abstract. This paper is concerned with steady vortex rings in an ideal fluid of uniform density, which are special global solutions of the three-dimensional incompressible Euler equation. We systematically establish the existence, uniqueness and nonlinear stability of steady vortex rings of small cross-section for which the potential vorticity is constant throughout the core. The proof is based on a combination of the Lyapunov–Schmidt reduction argument, the local Pohozaev identity technique and the variational method.

Keywords: The 3D Euler equation, Steady vortex rings, Existence, Uniqueness, Nonlinear stability

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1. Introduction and main results

The motion of particles in an ideal fluid in $\mathbb{R}^3$ is described by its velocity field $v(x, t)$ which satisfies the Euler equation

$$\begin{cases}
\partial_t v + (v \cdot \nabla)v = -\nabla P, \\
\nabla \cdot v = 0,
\end{cases} \quad (1.1)$$

for some pressure function $P(x, t)$. Corresponding to $v$ is its vorticity vector defined by $\omega := \nabla \times v$. Taking curl of the first equation in Euler equation (1.1), H. Helmholtz obtained the equation for vorticity

$$\begin{cases}
\partial_t \omega + (v \cdot \nabla)\omega = (\omega \cdot \nabla)\omega, \\
v = \nabla \times (-\Delta)^{-1}\omega.
\end{cases} \quad (1.2)$$

We refer to [13, 27] for more detail about this system.

We are interested in solutions of the Euler equation whose vorticities are large and uniformly concentrated near an evolving smooth curve embedded in entire $\mathbb{R}^3$. This type of solutions, vortex filaments, have been a subject of active studies for a long time. By the first Helmholtz theorem, in $\mathbb{R}^3$ a vortex must form a loop with compact support. The simplest vortex loop is a circular vortex ring, whose analysis traces back to the works of Helmholtz [23] in 1858 and Lord Kelvin [37] in 1867. Vortex rings are an intriguing marvel of fluid dynamics that can be easily observed experimentally, e.g. when smoke is ejected from a tube, a bubble rises in a liquid, or an ink is dropped in another fluid, and so on. We refer the reader to [1, 28, 35] for some good historical reviews of the achievements in experimental, analytical, and numerical studies of vortex rings.

Helmholtz detected that vortex rings have an approximately steady form and travel with a large constant velocity along the axis of the ring. In 1970, Fraenkel [19] (see also [20]) provided a first constructive proof for the existence of a vortex ring concentrated around a torus with fixed radius $r^*$ with a small, nearly singular cross-section $\varepsilon > 0$, traveling with constant speed $\sim |\ln \varepsilon|$, rigorously establishing the behavior predicted by Helmholtz (see, figure (1) (a), where the cross-section is depicted much ‘fatter’ than in reality, so as to show the streamline pattern clearly). Indeed, Lord Kelvin and Hicks showed that such a vortex ring would approximately move at the velocity (see [25, 37])

$$\frac{\kappa}{4\pi r^*} \left( \ln \frac{8r^*}{\varepsilon} - \frac{1}{4} \right), \quad (1.3)$$

where $\kappa$ denotes its circulation. Fraenkel’s result is consistent with the Kelvin–Hicks formula (1.3).

Roughly speaking, vortex rings can be characterized simply as an axi-symmetric flow with a (thin or fat) toroidal vortex tube. Here the word ‘toroidal’ means topologically equivalent to a torus. In the usual cylindrical coordinate frame \(\{e_r, e_\theta, e_z\}\), the velocity field $v$ of an axi-symmetric flow can be expressed in the following way

$$v = v^r(r, z)e_r + v^\theta(r, z)e_\theta + v^z(r, z)e_z.$$
The component $v^\theta$ in the $e_\theta$ direction is usually called the swirl velocity. If an axi-symmetric flow is non-swirling (i.e., $v^\theta \equiv 0$), then the vorticity admits its angular component $\omega^\theta$ only, namely, $\omega = \omega^\theta e_\theta$. Let $\zeta = \omega^\theta / r$ be the potential vorticity. Then the vorticity equation (1.2) is reduced to an active scalar equation for $\zeta$

$$\partial_t \zeta + \mathbf{v} \cdot \nabla \zeta = 0, \quad \mathbf{v} = \nabla \times (-\Delta)^{-1} (r \zeta).$$

(1.4)

We shall refer to an axi-symmetric non-swirling flow as ‘vortex ring’ if there is a toroidal region inside of which $\omega \neq 0$ (the core), and outside of which $\omega = 0$. By a steady vortex ring we mean a vortex ring that moves vertically at a constant speed forever without changing its shape or size. In other words, a steady vortex ring is of the form

$$\zeta(\mathbf{x},t) = \zeta(\mathbf{x} + t \mathbf{v}_\infty),$$

(1.5)

where $\mathbf{v}_\infty = -W e_z$ is a constant propagation speed. Substituting (1.5) into (1.4), we arrive at a stationary equation

$$(\mathbf{v}_\infty + \mathbf{v}) \cdot \nabla \zeta = 0, \quad \mathbf{v} = \nabla \times (-\Delta)^{-1} (r \zeta).$$

(1.6)

In 1894, Hill [24] found an explicit solution of (1.6) supported in a sphere (Hill’s spherical vortex, see, figure (1) (b)). In 1972, Norbury [31] provided a constructive proof for the existence of steady vortex rings with constant $\zeta$ that are close to Hill’s vortex but are homeomorphic to a solid torus; and he also presented some numerical results for the existence of a family of steady vortex rings of small cross-section [32]. General existence results of steady vortex rings with a given vorticity function was first established by Fraenkel–Berger [21] in 1974. Following these pioneering works, the existence and abundance of steady vortex rings has been rigorously established; see [2, 5, 7, 12, 16, 22, 28, 29, 40, 41] and the references therein.

Compared with the results on the existence, rather limited work has been done on the uniqueness of steady vortex rings. In 1986, Amick–Fraenkel [3] proved that Hill’s vortex is the unique solution when viewed in a natural weak formulation by the method of moving planes; and they (1988) [4] also established local uniqueness for Norbury’s nearly spherical vortex. However, to the best of our current knowledge, the uniqueness of steady vortex rings of small cross-section is still open. The first goal of this paper is to give an answer to this question.

The stability problem for steady flows are classical objects of study in fluid dynamics. Very recently, Choi [14] established the orbital stability of Hill’s vortex. We would like to mention that Hill’s vortex is not exactly a steady vortex ring since its vortex core is a ball, not a topological torus. It is still not clear whether some stable steady vortex rings exist. Recent numerical computations in [33] revealed that while ‘thin’ vortex rings remain neutrally stable to axi-symmetric perturbations, they become linearly unstable to such perturbations when they are sufficiently ‘fat’. By virtue of our local uniqueness result, we will establish orbital stability of a family of steady vortex rings of small cross-section, which is also the second main goal of this paper.
We shall focus on steady vortex rings for which $\zeta$ is a constant throughout the core. As remarked by Fraenkel [20], this simplest of all admissible vorticity distributions has been a favourite for over a century. Now, we turn to state our main results. To this end, we need to introduce some notation. We shall say that a scalar function $\vartheta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is axi-symmetric if it has the form of $\vartheta(x) = \vartheta(r, z)$, and a subset $\Omega \subset \mathbb{R}^3$ is axi-symmetric if its characteristic function $1_{\Omega}$ is axi-symmetric. The cross-section parameter $\sigma$ of an axi-symmetric set $\Omega \subset \mathbb{R}^3$ is defined by

$$\sigma(\Omega) := \frac{1}{2} \cdot \sup \{ \delta_z(x, y) \mid x, y \in \Omega \},$$

where the axisymmetric distance $\delta_z$ is given by

$$\delta_z(x, y) := \inf \{ |x - Q(y)| \mid Q \text{ is a rotation around } e_z \}.$$

Let $C_r = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = r^2, x_3 = 0 \}$ be a circle of radius $r$ on the plane perpendicular to $e_z$. For an axi-symmetric set $\Omega \subset \mathbb{R}^3$, we define the axisymmetric distance between $\Omega$ and $C_r$ as follows

$$\text{dist}_{C_r}(\Omega) = \sup_{x \in \Omega} \inf_{x' \in C_r} |x - x'|.$$
The circulation of a steady vortex ring $\zeta$ is given by
\[
\frac{1}{2\pi} \int_{\mathbb{R}^3} \zeta(x) \, dx.
\]
A steady vortex ring $\zeta$ is said to be centralized if $\zeta$ is symmetric non-increasing in $z$, namely,
\[
\zeta(r, z) = \zeta(r, -z), \quad \text{and} \quad \zeta(r, z) \text{ is a non-increasing function of } z \text{ for } z > 0, \text{ for each fixed } r > 0.
\]

Our first main result is on the existence of steady vortex rings of small cross-section for which $\zeta$ is constant throughout the core. The existence for such kind of solutions was proved in $[12, 20, 22]$ by different methods. However, we will construct steady vortex rings from a new perspective of Stokes stream function, which not only leads to a desired estimate for the cross-section, but also casts a profound light on our approach for uniqueness.

**Theorem 1.1 (Existence).** Let $\kappa$ and $W$ be two positive numbers. Then there exists a small number $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ there is a centralized steady vortex ring $\zeta_\varepsilon$ with fixed circulation $\kappa$ and translational velocity $W \ln \varepsilon \, e_z$. Moreover,

(i) $\zeta_\varepsilon = \varepsilon^{-2} 1_{\Omega_\varepsilon}$ for some axi-symmetric topological torus $\Omega_\varepsilon \subset \mathbb{R}^3$.

(ii) It holds $C_1 \varepsilon \leq \sigma(\Omega_\varepsilon) < C_2 \varepsilon$ for some constants $0 < C_1 < C_2$.

(iii) As $\varepsilon \to 0$, $\text{dist}_{C_r^*}(\Omega_\varepsilon) \to 0$ with $r^* := \kappa / 4\pi W$.

Our existence result is established by an improved Lyapunov–Schmidt reduction argument on planar vortex patch problem in $[9]$. Compared with the method taken in $[9]$, our approach in the present paper is the first time reduction argument being used to deal with a non-uniform elliptic operator. To obtain desired estimates, we use an equivalent integral formulation of the problem, and introduce a weighted $L^\infty$ norm to handle the degeneracy at infinity and singularity near $z$-axis. Another difficulty in our construction is the lack of compactness, which arises from whole-space $\mathbb{R}^3$. To overcome it, we will use a few techniques, so that versions of Ascoli–Arzelà theorem can be applied to recover the compactness.

There are similar existence results for different types of steady vortex rings in the works $[2, 7, 12, 17, 19, 20, 22]$. For instance, de Valeriola et al. $[17]$ constructed vortex rings with $C^{4,\alpha}$ regularity by mountain pass theorem, and recently Cao et al. $[12]$ studied desingularization of vortex rings by solving variational problems for the potential vorticity $\zeta$. However, in the absence of a comprehensive uniqueness theory, the corresponding relations between solutions with fixed vorticity distributions constructed by the various methods remains unclear. Our second main result is to address this question.

**Theorem 1.2 (Uniqueness).** Let $\kappa$ and $W$ be two positive numbers. Let $\{\zeta^{(1)}_\varepsilon\}_{\varepsilon > 0}$ and $\{\zeta^{(2)}_\varepsilon\}_{\varepsilon > 0}$ be two families of centralized steady vortex rings with fixed circulation $\kappa$ and translational velocity $W \ln \varepsilon \, e_z$. If, in addition,

(i) $\zeta^{(1)}_\varepsilon = \varepsilon^{-2} 1_{\Omega^{(1)}_\varepsilon}$ and $\zeta^{(2)}_\varepsilon = \varepsilon^{-2} 1_{\Omega^{(2)}_\varepsilon}$ for certain axi-symmetric topological tori $\Omega^{(1)}_\varepsilon$, $\Omega^{(2)}_\varepsilon \subset \mathbb{R}^3$. 


(ii) As \( \varepsilon \to 0 \), \( \sigma \left( \Omega^{(1)} \varepsilon \right) + \sigma \left( \Omega^{(2)} \varepsilon \right) \to 0 \).

(iii) There exists a \( \delta_0 > 0 \) such that \( \Omega^{(1)} \varepsilon \cup \Omega^{(2)} \varepsilon \subset \{ \mathbf{x} \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} \geq \delta_0 \} \) for all \( \varepsilon > 0 \).

Then there exists a small \( \varepsilon_0 > 0 \) such that \( \zeta^{(1)} \varepsilon \equiv \zeta^{(2)} \varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0] \).

To obtain the uniqueness, we first give a rough estimate for vortex rings by blow up analysis. Then we improve the estimate step by step, and obtain an accurate version of Kelvin–Hicks formula (1.3). Actually, our result is slightly stronger than Fraenkel’s in [19] by a careful study of vortex boundary and a bootstrap procedure. With a delicate estimate in hand, a local Pohozaev identity can be used to derive contradiction if there are two different vortex rings satisfying assumptions in Theorem 1.2. It is notable that the methods in [3, 4] depend strongly on specific distribution of vorticity in cross-section. While our method has much broader applicability, and provides a general approach for uniqueness of ‘thin’ vortex in axi-symmetry case.

Using the uniqueness result in Theorem 1.2, we can further show that the solutions constructed in Theorem 1.1 is orbitally stable in the Lyapunov sense. Recalling (1.4), for an axisymmetric flow without swirl, the vorticity equation (1.2) can be reduced to the active scalar equation for the potential vorticity \( \zeta = \omega / r \):

\[
\begin{align*}
\partial_t \zeta + \mathbf{v} \cdot \nabla \zeta &= 0, & \mathbf{x} \in \mathbb{R}^3, \; t > 0, \\
\mathbf{v} &= \nabla \times (-\Delta)^{-1} (r \zeta), & \mathbf{x} \in \mathbb{R}^3, \; t > 0, \\
\zeta |_{t=0} &= \zeta_0, & \mathbf{x} \in \mathbb{R}^3.
\end{align*}
\]

(1.7)

The existence and uniqueness of solutions \( \zeta(x, t) \) can be studied analogously as the two-dimensional case. We refer to [8, 14, 27, 30, 34, 39] for some discussion in this direction. Let \( BC([0, \infty); X) \) denote the space of all bounded continuous functions from \([0, \infty)\) into a Banach space \( X \). Define the weighted space \( L^1_w(\mathbb{R}^3) \) by \( L^1_w(\mathbb{R}^3) = \{ \varphi : \mathbb{R}^3 \to \mathbb{R} \) measurable \mid r^2 \varphi \in L^1(\mathbb{R}^3) \}. \) We introduce the kinetic energy of the fluid

\[
E[\zeta] := \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{v}(\mathbf{x})|^2 d\mathbf{x}, \quad \mathbf{v} = \nabla \times (-\Delta)^{-1} (r \zeta),
\]

and its impulse

\[
\mathcal{P}[\zeta] = \frac{1}{2} \int_{\mathbb{R}^3} r^2 \zeta(\mathbf{x}) d\mathbf{x} = \pi \int_0^\infty r^3 \zeta drdz.
\]

The following result has been established, see e.g. Lemma 3.4 in [14].

**Proposition 1.3.** For any non-negative axi-symmetric function \( \zeta_0 \in L^1 \cap L^\infty \cap L^1_w(\mathbb{R}^3) \) satisfying \( r\zeta_0 \in L^\infty(\mathbb{R}^3) \), there exists a unique weak solution \( \zeta \in BC([0, \infty); L^1 \cap L^\infty \cap L^1_w(\mathbb{R}^3)) \).
of \((1.7)\) for the initial data \(\zeta_0\) such that
\[
\zeta(\cdot, t) \geq 0 : \text{axi-symmetric,}
\]
\[
\|\zeta(\cdot, t)\|_{L^p(\mathbb{R}^3)} = \|\zeta_0\|_{L^p(\mathbb{R}^3)}, \quad 1 \leq p \leq \infty,
\]
\[
P[\zeta(\cdot, t)] = P[\zeta_0],
\]
\[
E[\zeta(\cdot, t)] = E[\zeta_0], \quad \text{for all } t > 0,
\]
and, for any \(0 < \nu_1 < \nu_2 < \infty\) and for each \(t > 0\),
\[
\int_{\{x \in \mathbb{R}^3 | \nu_1 < \zeta(x, t) < \nu_2\}} \zeta(x, t) dx = \int_{\{x \in \mathbb{R}^3 | \nu_1 < \zeta_0(x) < \nu_2\}} \zeta_0(x) dx.
\]

Our result on nonlinear orbital stability is as follows.

**Theorem 1.4 (Stability).** The steady vortex ring \(\zeta_\epsilon\) in Theorem 1.1 is stable up to translations in the following sense:

For any \(\eta > 0\), there exists \(\delta_1 > 0\) such that for any non-negative axi-symmetric function \(\zeta_0\) satisfying \(\zeta_0, r\zeta_0 \in L^\infty(\mathbb{R}^3)\) and
\[
\|\zeta_0 - \zeta_\epsilon\|_{L^1 \cap L^2(\mathbb{R}^3)} + \|r^2(\zeta_0 - \zeta_\epsilon)\|_{L^1(\mathbb{R}^3)} \leq \delta_1,
\]
the corresponding solution \(\zeta(x, t)\) of \((1.7)\) for the initial data \(\zeta_0\) satisfies
\[
\inf_{\tau \in \mathbb{R}} \{\|\zeta(\cdot - \tau e_z, t) - \zeta_\epsilon\|_{L^1 \cap L^2(\mathbb{R}^3)} + \|r^2(\zeta(\cdot - \tau e_z, t) - \zeta_\epsilon)\|_{L^1(\mathbb{R}^3)}\} \leq \eta
\]
for all \(t > 0\). Here, \(\| \cdot \|_{L^1 \cap L^2(\mathbb{R}^3)}\) means \(\| \cdot \|_{L^1(\mathbb{R}^3)} + \| \cdot \|_{L^2(\mathbb{R}^3)}\).

The paper is organized as follows. In Section 2, we construct vortex rings of small cross-section by a Lyapunov–Schmidt reduction argument. In Section 3, we study the asymptotic behavior of vortex rings carefully as its cross-section shrinks, and prove the uniqueness result in Theorem 1.2. The nonlinear orbital stability for vortex rings of small cross-section is proved in Section 4 based on variational method. In Appendix A and B, we discuss the symmetry and boundary shape of the cross-section. In Appendix C, we give several estimates for the local Pohozaev identity, which are used to prove uniqueness in Section 3.

2. Existence

2.1. Formulation of the problem. The main objective of this paper is to deal with steady vortex rings, which are actually traveling-wave solutions for \((1.7)\). Thanks to the continuity equation in \((1.1)\), we can find a Stokes stream function \(\Psi\) such that
\[
v = \frac{1}{r} \left( - \frac{\partial \Psi}{\partial z} e_r + \frac{\partial \Psi}{\partial r} e_z \right).
\]
In terms of the Stokes stream function $\Psi$, the problem of steady vortex rings can be reduced to a steady problem on the meridional half plane $\Pi = \{(r, z) \mid r > 0\}$ of the form:

\[
\begin{cases}
  L\Psi = 0, & \text{in } \Pi \setminus A, \\
  L\Psi = \lambda f_0(\Psi), & \text{in } A, \\
  \Psi(0, z) = -\mu \leq 0, & \\
  \Psi = 0, & \text{on } \partial A, \\
  \frac{1}{r} \frac{\partial \Psi}{\partial r} \to -\mathcal{W} \quad \text{and} \quad \frac{1}{r} \frac{\partial \Psi}{\partial z} \to 0, & \text{as } r^2 + z^2 \to \infty,
\end{cases}
\]

where

\[ L := -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2}{\partial z^2}. \]

Here the positive vorticity function $f_0$ and the vortex-strength parameter $\lambda > 0$ are prescribed; $A$ is the (a priori unknown) cross-section of the vortex ring; $\mu$ is called the flux constant measuring the flow rate between the $z$-axis and $\partial A$; The constant $\mathcal{W} > 0$ is the translational speed, and the condition (2.5) means that the limit of the velocity field $v$ at infinity is $-\mathcal{W} e_z$. For a detailed derivation of this system, we refer to [3, 14, 21] and the references therein.

By the maximum principle, we see that $\Psi > 0$ in $A$ and $\Psi < 0$ in $\Pi \setminus \bar{A}$. Therefore the cross-section $A$ is given by

\[ A = \{(r, z) \in \Pi \mid \Psi(r, z) > 0\}. \]

It is convenient to write

\[ \Psi(r, z) = \psi(r, z) - \frac{1}{2} \mathcal{W} r^2 - \mu, \]

where $\psi$ is the stream function due to vorticity. In addition, it is also convenient to define

\[ f(\tau) = \begin{cases} 0, & \tau \leq 0, \\
 f_0(\tau), & \tau > 0,
\end{cases} \]

so that $\lambda f(\Psi)$ is exactly the potential vorticity $\zeta$. We now can rewrite (2.1)-(2.5) as

\[
(\mathcal{P}) \quad \begin{cases}
  \mathcal{L}\psi = \lambda f(\psi - \frac{1}{2} \mathcal{W} r^2 - \mu), & \text{in } \Pi, \\
  \psi(0, z) = 0, & \\
  \psi, \quad |\nabla \psi|/r \to 0 \quad \text{as } r^2 + z^2 \to \infty.
\end{cases}
\]

In the following, we will focus on the construction of $\psi$ satisfying $(\mathcal{P})$.

In order to simplify notations, we will use

\[ \mathbb{R}^2_+ = \{x = (x_1, x_2) \mid x_1 > 0\} \]

to substitute the meridional half plane $\Pi$, and abbreviate the elliptic operator $L$ as

\[ \Delta^* := \frac{1}{x_1} \text{div} \left( \frac{1}{x_1} \nabla \right). \]
We will use \( \varepsilon := \lambda^{-1/2} \) as the parameter instead of \( \lambda \) in the rest of this paper. Since we are concerned with steady vortex rings for which \( \zeta \) is a constant throughout the core, we will choose the vorticity function \( f \) in (2.6) having the following form

\[
f(\tau) = \begin{cases} 
0, & \tau \leq 0, \\
1, & \tau > 0,
\end{cases}
\]

and the cross-section of the vortex ring is

\[
A_\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}_+^2 \mid \psi_\varepsilon - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon \right\}
\]

for some flux constant \( \mu_\varepsilon > 0 \). Here we let \( \Psi \) equal \( W \ln(1/\varepsilon) \) according to Kelvin–Hicks formula (1.4). The fact that \( \mu_\varepsilon > 0 \) means \( A_\varepsilon \) will not touch the \( x_2 \)-axis. Thus we can rewrite (\( \mathcal{P} \)) to

\[
\begin{align*}
-\varepsilon^2 \Delta^* \psi_\varepsilon &= 1\left\{ \psi_\varepsilon - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon \right\}, & \text{in } \mathbb{R}_+^2, \\
\psi_\varepsilon &= 0, & \text{on } x_1 = 0, \\
\psi_\varepsilon, \left| \nabla \psi_\varepsilon \right|/x_1 & \to 0, & \text{as } |\mathbf{x}| \to \infty.
\end{align*}
\]

(2.10)

Since the problem is invariant in \( x_2 \)-direction, we may assume

\[
\psi_\varepsilon(x_1, x_2) = \psi_\varepsilon(x_1, -x_2)
\]

(2.11)
due to the method of moving planes in Appendix A (see also Lemma 2.1 in [4]), which also means the steady vortex ring \( \zeta_\varepsilon \) corresponding to \( \psi_\varepsilon \) is centralized; see [4].

The existence result in Theorem 1.1 can be deduced from following proposition.

**Proposition 2.1.** For every \( \kappa > 0 \) and \( W > 0 \), there exists an \( \varepsilon_0 > 0 \) such that for each \( \varepsilon \in (0, \varepsilon_0] \), problem (2.10) has a solution \( \psi_\varepsilon \) satisfying (2.11). Moreover,

(i) The cross-section \( A_\varepsilon \) is a convex domain, and satisfies

\[
B(\sqrt{\pi} \varepsilon (1-L_1|\ln \varepsilon|) (z)) \subset A_\varepsilon \subset B(\sqrt{\pi} \varepsilon (1+L_2|\ln \varepsilon|) (z)),
\]

where \( L_1, L_2 \) are two positive constants independent of \( \varepsilon \), and \( z = (z_1, 0) \) is on \( x_1 \)-axis with the estimate

\[
z_1 - \frac{\kappa}{4\pi W} = O\left( \frac{1}{|\ln \varepsilon|} \right).
\]

(ii) As \( \varepsilon \to 0 \), it holds

\[
\kappa_\varepsilon := \varepsilon^{-2} \int_{A_\varepsilon} x_1 d\mathbf{x} \to \kappa.
\]

**Remark 2.2.** Notice that in Proposition 2.1, the circulation parameter \( \kappa_\varepsilon \) is not fixed, which only has the limiting behavior described in property (ii). To obtain a family of vortex rings with fixed circulation \( \kappa \) as in Theorem 1.1, we can rescale \( \psi_\varepsilon \) as follows

\[
\tilde{\psi}_\varepsilon(\mathbf{x}) := \frac{\kappa_\varepsilon^2}{\kappa^2} \cdot \psi_\varepsilon \left( \frac{\kappa}{\kappa_\varepsilon} \cdot \mathbf{x} \right).
\]
Then \( \tilde{\psi}_\varepsilon(x) \) is the solution to
\[
-\varepsilon^2 \Delta^* \tilde{\psi}_\varepsilon = 1 \{ \tilde{\psi}_\varepsilon - \frac{\mu_\varepsilon}{x_1^2} \ln \frac{1}{\varepsilon} \}
\]
where
\[
\bar{\varepsilon} = \frac{\kappa_\varepsilon}{\kappa_\varepsilon} \varepsilon, \quad \text{and} \quad \bar{\mu}_\varepsilon = \frac{\kappa_\varepsilon^2}{\kappa_\varepsilon^2} \mu_\varepsilon.
\]
It is easy to verify that
\[
\int_{\mathbb{R}^3_+} x_1 \{ \tilde{\psi}_\varepsilon - \frac{\mu_\varepsilon}{x_1^2} \ln \frac{1}{\varepsilon} \} dx = \kappa,
\]
and the vortex ring \( \bar{\zeta}_\varepsilon \) corresponding to \( \tilde{\psi}_\varepsilon \) satisfies all assumptions in Theorem 1.1.

For the study of steady vortex rings of small cross-section, our main tool is the Green’s representation of Stokes stream function \( \tilde{\psi}_\varepsilon \). To be more rigorous, \( \tilde{\psi}_\varepsilon \) satisfies the integral equation
\[
\tilde{\psi}_\varepsilon(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3_+} G_*(x, x') 1_{A_\varepsilon}(x') dx',
\]
where \( G_*(x, x') \) is the Green’s function for \(-\Delta^* \) with boundary condition in (2.10). Using Biot–Savart law in \( \mathbb{R}^3 \) and a coordinate transformation, we can derive an explicit formula of \( G_*(x, x') \) as
\[
G_*(x, x') = \frac{x_1 x_1'^2}{4\pi} \int_{-\pi}^{\pi} \frac{\cos \theta d\theta}{\left[ (x_2 - x_2')^2 + x_1^2 + x_1'^2 - 2x_1 x_1' \cos \theta \right]^{3/2}}.
\]
Then, denoting
\[
\rho(x, x') = \frac{(x_1 - x_1')^2 + (x_2 - x_2')^2}{x_1 x_1'},
\]
we have the following asymptotic estimates
\[
G_*(x, x') = \frac{x_1^{1/2} x_1'^{3/2}}{4\pi} \left( \ln \left( \frac{1}{\rho} \right) + 2 \ln 8 - 4 + O \left( \rho \ln \frac{1}{\rho} \right) \right), \quad \text{as } \rho \to 0,
\]
and
\[
G_*(x, x') = \frac{x_1^{1/2} x_1'^{3/2}}{4} \left( \frac{1}{\rho^{3/2}} + O(\rho^{-5/2}) \right), \quad \text{as } \rho \to \infty,
\]
which can be found in [18, 20, 25, 36]. Actually, the theory of elliptic integrals can be used to obtain a more precise expansion of \( G_* \) on \( \rho \).

To simplify integral equation (2.12), we let \( z = (z_1, 0) \) with \( z_1 > 0 \) determined later, and split \( G_* \) as
\[
G_*(x, x') = z_1^2 G(x, x') + H(x, x'),
\]
where
\[
G(x, x') = \frac{1}{4\pi} \ln \frac{(x_1 + x_1')^2 + (x_2 - x_2')^2}{(x_1 - x_1')^2 + (x_2 - x_2')^2},
\]
is the Green’s function for \(-\Delta \) in right half plane, and \( H(x, x') \) is a relatively regular function. By the definition of \( G_* \) and \( G \), it is obvious that \( H(x, z) \in C^\alpha(\mathbb{R}^3_+) \) for every
To make Rankine vortex, we can write \( V \) linearly dependent on \( \varepsilon \) in a neighborhood \( D \subset \mathbb{R}^2 \) of \( z \), namely, for any \( x^{(1)}, x^{(2)} \) in a neighborhood \( D \subset \mathbb{R}^2 \) of \( z \), there exists a constant \( C(D) \) such that

\[
|H(x^{(1)}, z) - H(x^{(2)}, z)| \leq C(D) \cdot |x^{(1)} - x^{(2)}|(1 + \ln |x^{(1)} - x^{(2)}|).
\]

Our construction is divided into several steps, which is known as the Lyapunov–Schmidt reduction. We will first give a series of approximate solutions of \( \psi_\varepsilon \), so that (2.10) is transformed to a semilinear problem on the error term \( \phi_\varepsilon \). Then, we establish the linear theory of corresponding projected problem. The existence and limit behavior of \( \psi_\varepsilon \) will be obtained by contraction mapping theorem and one-dimensional reduction in the last part of our proof.

2.2. Approximate solutions. To give suitable approximate solutions to (2.10) and (2.11), let us consider the following problem

\[
\begin{cases}
-\varepsilon^2 \Delta V_{z,\varepsilon}(x) = z_1^2 \mathbf{1}_{B_s(z)}, & \text{in } \mathbb{R}^2, \\
V_{z,\varepsilon}(x) = \frac{a}{2\pi} \ln \frac{1}{\varepsilon}, & \text{on } \partial B_s(z),
\end{cases}
\]

with \( z = (z_1, z_2) \in \mathbb{R}^2 \) and \( z_1 \neq 0 \), \( a \) is a parameter to be determined later, and \( s > 0 \) sufficiently small such that \( B_s(z) \cap \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \} = \emptyset \). Recalling the planar Rankine vortex, we can write \( V_{z,\varepsilon} \) explicitly as

\[
V_{z,\varepsilon}(x) = \begin{cases}
\frac{a}{2\pi} \ln \frac{1}{\varepsilon} + \frac{z_2^2}{4\varepsilon^2} (s^2 - |x - z|^2), & |x - z| \leq s, \\
\frac{a}{2\pi} \ln \frac{1}{\varepsilon}, & |x - z| \geq s.
\end{cases}
\] (2.16)

To make \( V_{z,\varepsilon} \) a \( C^1 \) function, we impose the gradient condition on \( \partial B_s(z) \)

\[
\mathcal{N} := \frac{a}{2\pi} \ln \frac{1}{\varepsilon} \cdot \frac{1}{s |\ln s|} = \frac{s}{2\varepsilon^2} \cdot z_1^2,
\] (2.17)

where \( \mathcal{N} \) is the value of \( |\nabla V_{z,\varepsilon}| \) at \( |x - z| = s \). From (2.17), we see that \( s \) is asymptotically linearly dependent on \( \varepsilon \) by

\[
s = \left( \sqrt{\frac{a}{\pi z_1^2}} + o_\varepsilon(1) \right) \varepsilon.
\]

In our construction, \( V_{z,\varepsilon}(x) \) will be used as the building block of approximate solutions. To further explain our strategy, for general \( x = (x_1, x_2) \in \mathbb{R}^2 \) we denote \( \bar{x} = (-x_1, x_2) \) as the reflection of \( x \) with respect to \( x_2 \)-axis, and let

\[
V_{z,\varepsilon}(x) := V_{z,\varepsilon}(x) - V_{z,\varepsilon}(x)
= \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}_+^2} \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}_+^2} \frac{z_1^2}{|x - x'|} 1_{B_s(z)}(x')d\mathbf{x}' - \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}_+^2} \frac{z_1^2}{|x - \bar{x}'|} 1_{B_s(z)}(x')d\mathbf{x}'
= \frac{z_1^2}{\varepsilon^2} \int_{\mathbb{R}_+^2} G(x, x')1_{B_s(z)}(x')d\mathbf{x}'
\]
be an approximation of singular part of $\psi_\varepsilon$, where $z = (z_1, 0)$ will be determined in the last part of construction (Note that we introduce a conjugate part $V_{z, \varepsilon}$ to obtain desired boundary condition). Then $\mathcal{V}_{z, \varepsilon}(x)$ is the unique solution to the following problem

\[
\begin{cases}
-\varepsilon^2 \Delta \mathcal{V}_{z, \varepsilon}(x) = z_1^2 1_{B_\varepsilon(z)}, & \text{on } \mathbb{R}^2_+,

\mathcal{V}_{z, \varepsilon} = 0, & \text{on } x_1 = 0,

\mathcal{V}_{z, \varepsilon}, \frac{|\nabla \mathcal{V}_{z, \varepsilon}|}{x_1} \to 0, & \text{as } |x| \to \infty.
\end{cases}
\]

To approximate the regular part of $\psi_\varepsilon$, let

\[
\mathcal{H}_{z, \varepsilon}(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} H(x, x') 1_{B_\varepsilon(z)}(x') dx'.
\]

According to the definition of $H(x, x')$, it is obvious that $\mathcal{H}_{z, \varepsilon}(x)$ solves

\[
\begin{cases}
-\varepsilon^2 \Delta^* (\mathcal{V}_{z, \varepsilon} + \mathcal{H}_{z, \varepsilon}) = z_1^2 1_{B_\varepsilon(z)}, & \text{on } \mathbb{R}^2_+,

\mathcal{H}_{z, \varepsilon} = 0, & \text{on } x_1 = 0,

\mathcal{H}_{z, \varepsilon}, \frac{|\nabla \mathcal{H}_{z, \varepsilon}|}{x_1} \to 0, & \text{as } |x| \to \infty.
\end{cases}
\]

Moreover, using the definition of $H(x, x')$ and standard elliptic estimates, we have

\[
\mathcal{H}_{z, \varepsilon}(x) - \frac{s^2 \pi}{\varepsilon^2} H(x, z) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} (H(x, x') - H(x, z)) 1_{B_\varepsilon(z)}(x') dx' = O(\varepsilon),
\]

and

\[
\partial_1 \mathcal{H}_{z, \varepsilon}(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} \partial_{x_1} H(x, x') 1_{B_\varepsilon(z)}(x') dx' = O(\varepsilon |\ln \varepsilon|).
\]

After all this preparation, we write a solution $\psi_\varepsilon$ to (2.10) as

\[
\psi_\varepsilon(x) = \mathcal{V}_{z, \varepsilon} + \mathcal{H}_{z, \varepsilon} + \phi_\varepsilon,
\]

where $\phi_\varepsilon(x)$ is a error term with boundary condition

\[
\begin{cases}
\phi_\varepsilon = 0, & \text{on } x_1 = 0,

\phi_\varepsilon, \frac{|\nabla \phi_\varepsilon|}{x_1} \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]

and symmetry condition

\[
\phi_\varepsilon(x_1, x_2) = \phi_\varepsilon(x_1, -x_2).
\]
Then we can derive the equation for $\phi_\varepsilon$ by direct computations

$$
0 = -x_1 \varepsilon^2 \Delta^* (V_{z,\varepsilon} + H_{z,\varepsilon} + \phi_\varepsilon) - x_1 \mathbf{1}_{\{\psi_{z} - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon\}}
$$

$$
= x_1 \left( -\varepsilon^2 \Delta^* (V_{z,\varepsilon} + H_{z,\varepsilon}) - \mathbf{1}_{\{V_{z,\varepsilon} > \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}} \right)
$$

$$
+ \varepsilon^2 \left( -x_1 \Delta^* \phi_\varepsilon - \frac{2}{s z_1} \phi_\varepsilon (s, \theta) \delta_{|x-z|=s} \right)
$$

$$
- \left( x_1 \mathbf{1}_{\{\psi_{z} - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon\}} - x_1 \mathbf{1}_{\{V_{z,\varepsilon} > \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}} - \frac{2}{s z_1} \phi_\varepsilon (s, \theta) \delta_{|x-z|=s} \right)
$$

$$
= \varepsilon^2 \mathbb{L}_\varepsilon \phi_\varepsilon - \varepsilon^2 R_\varepsilon (\phi_\varepsilon),
$$

where $\mathbb{L}_\varepsilon$ is a linear operator defined by

$$
\mathbb{L}_\varepsilon \phi = -x_1 \Delta^* \phi - \frac{2}{s z_1} \phi (s, \theta) \delta_{|x-z|=s},
$$

(2.18)

and

$$
R_\varepsilon (\phi) = \frac{1}{\varepsilon^2} \left( x_1 \mathbf{1}_{\{\psi_{z} - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon\}} - x_1 \mathbf{1}_{\{V_{z,\varepsilon} > \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}} - \frac{2}{s z_1} \phi (s, \theta) \delta_{|x-z|=s} \right)
$$

is the nonlinear perturbation.

To make $R_\varepsilon (\phi_\varepsilon)$ as small as possible, we are to take

$$
\mu_\varepsilon = \frac{z_1}{2\pi} \cdot \kappa \ln \frac{1}{\varepsilon} - \frac{W}{2} z_1^2 \ln \frac{1}{\varepsilon}
$$

and choose the parameter $a$ such that

$$
\frac{a}{2\pi} \ln \frac{1}{\varepsilon} = \mu_\varepsilon + \frac{W}{2} z_1^2 \ln \frac{1}{\varepsilon} - H_{z,\varepsilon} (z) + V_{z,\varepsilon} (z).
$$

(2.19)

For simplicity in further discussion, we will denote

$$
U_{z,\varepsilon} (x) = V_{z,\varepsilon} (x) + H_{z,\varepsilon} (x) - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} - \mu_\varepsilon.
$$

Problem (2.10) and (2.11) is then transformed into finding the pairs $(z, \phi_\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0$ sufficiently small, such that

$$
\left\{
\begin{array}{l}
\mathbb{L}_\varepsilon \phi_\varepsilon = R_\varepsilon (\phi_\varepsilon), \quad \text{in } \mathbb{R}_+^2, \\
\phi_\varepsilon = 0, \quad \text{on } x_1 = 0, \\
\phi_\varepsilon, \ |\nabla \phi_\varepsilon|/x_1 \to 0, \quad \text{as } |x| \to \infty.
\end{array}
\right.
$$

(2.20)
2.3. **The linear theory.** To solve (2.20) we need first to study the properties of linear operator \( L_\varepsilon \) and the corresponding projected problem. Fix a point \( z = (z_1, 0) \in \mathbb{R}^2 \) with \( z_1 \neq 0 \). Let \( K \) be the operator defined on the whole plane \( \mathbb{R}^2 \) by
\[
Kv := -\frac{1}{z_1} \Delta v - \varepsilon^{-2} z_1 1_{v > \frac{\varepsilon}{2\pi} \ln \frac{1}{\varepsilon}}(v > \frac{\varepsilon}{2\pi} \ln \frac{1}{\varepsilon}),
\]
where \( a \) is the same parameter as in approximate solutions. A direct calculation yields its linearized operator \( L_\varepsilon \) as
\[
L_\varepsilon \phi := -\frac{1}{z_1} \Delta \phi - \frac{2}{s z_1} \phi(s, \theta) \delta_{|x-z|=s},
\]
with \( \phi(s, \theta) = \phi(z_1 + s \cos \theta, s \sin \theta) \). In view of the nondegeneracy property for \( L \) in [9], we have
\[
\ker(L) = \text{span} \left\{ \frac{\partial V_{z,\varepsilon}}{\partial x_1}, \frac{\partial V_{z,\varepsilon}}{\partial x_2} \right\},
\]
where
\[
\frac{\partial V_{z,\varepsilon}}{\partial x_m} = \left\{ \begin{array}{ll}
-\frac{s^2}{2\pi |\ln \varepsilon|} (x_m - z_m), & |x - z| \leq s, \\
-\frac{s^2}{2\pi |\ln s|} \frac{x_m - z_m}{|x-z|^m}, & |x - z| \geq s.
\end{array} \right.
\]
Recall that \( L_\varepsilon \) is defined on \( \mathbb{R}_+^2 \) and \( \phi_\varepsilon \) is even symmetric with respect to \( x_1 \)-axis. When \( \varepsilon \) is chosen sufficiently small, the kernel of \( L \) can be approximated by
\[
Z_{z,\varepsilon} = \chi_\varepsilon \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1},
\]
where \( \chi_\varepsilon \) are smooth truncation functions satisfy
\[
\chi_\varepsilon(x) = \begin{cases}
1, & |x - z| \leq \delta_\varepsilon, \\
0, & |x - z| \geq 2\delta_\varepsilon.
\end{cases}
\]
for \( \delta_\varepsilon = \varepsilon |\ln \varepsilon| \). Moreover, we assume that \( \chi_\varepsilon \) are radially symmetric with respect to \( z \) and
\[
|\nabla \chi_\varepsilon| \leq \frac{2}{\delta_\varepsilon}, \quad |\nabla^2 \chi_\varepsilon| \leq \frac{2}{\delta_\varepsilon^2}.
\]

To solve (2.20), we will first consider the following projected problem
\[
\begin{cases}
L_\varepsilon \phi = h(x) - \Lambda x_1 \Delta^* Z_{z,\varepsilon}, & \text{in } \mathbb{R}_+^2, \\
\int_{\mathbb{R}_+^2} \frac{1}{x_1} \nabla \phi \cdot \nabla Z_{z,\varepsilon} dx = 0, \\
\phi = 0, & \text{on } x_1 = 0, \\
\phi, |\nabla \phi|/x_1 \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]
(2.23)
where \( \phi \) is even with respect to \( x_1 \)-axis, \( \text{supp } h \subset B_{2s}(z) \), and \( \Lambda \) is the projection coefficient such that
\[
\int_{\mathbb{R}_+^2} Z_{z,\varepsilon}(L_\varepsilon \phi - h + \Lambda x_1 \Delta^* Z_{z,\varepsilon}) dx = 0.
\]
Let
\[ \rho_1(x) := \frac{(1 + |x - z|^2)^{\frac{3}{2}}}{1 + x_1^2} \quad \text{and} \quad \rho_2(x) := \left( \frac{1}{x_1 + 1} \right). \]  
(2.24)

We define the weighted \( L^\infty \) norm of \( \phi \) by
\[ ||\phi||_* := \sup_{x \in \mathbb{R}^2_+} \rho_1(x) \rho_2(x) |\phi(x)|. \]  
(2.25)

We have a priori estimate for solutions of the projective problem (2.23).

**Lemma 2.3.** Assume that \( h \) satisfies \( \text{supp } h \subset B_2(z) \) and
\[ \varepsilon^{1-\frac{2}{p}} \|h\|_{W^{-1, p}(B_{Ls}(z))} < \infty \]
with \( p \in (2, +\infty) \), then there exists a small \( \varepsilon_0 > 0 \), a large constant \( L > 0 \) and a positive constant \( c_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \) and solution pair \( (\phi, \Lambda) \) to (2.23), one has
\[ ||\phi||_* + \varepsilon^{1-\frac{2}{p}} \|\nabla \phi\|_{L^p(B_{Ls}(z))} \leq c_0 \varepsilon^{1-\frac{2}{p}} \|h\|_{W^{-1, p}(B_{Ls}(z))}, \]  
(2.26)

and
\[ |\Lambda| \leq c_0 \varepsilon^{2-\frac{2}{p}} \|h\|_{W^{-1, p}(B_{Ls}(z))}. \]  
(2.27)

**Proof.** First we are to obtain an estimate for coefficient \( \Lambda \). To proceed an energy method, we multiply the first equation in (2.23) by \( Z_{z, \varepsilon} \). By integrations by parts we obtain
\[ \Lambda \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla Z_{z, \varepsilon} \cdot \nabla Z_{z, \varepsilon} \, dx = \int_{\mathbb{R}^2_+} Z_{z, \varepsilon} \nabla \varepsilon \phi \, dx - \int_{\mathbb{R}^2_+} Z_{z, \varepsilon} h \, dx. \]  
(2.28)

Recall the definition of \( Z_{z, \varepsilon} \). For the integral in the left hand side of (2.28), we have
\[ \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla \left( \chi_{\varepsilon} \cdot \frac{\partial V_{z, \varepsilon}}{\partial x_1} \right) \cdot \nabla \left( \chi_{\varepsilon} \cdot \frac{\partial V_{z, \varepsilon}}{\partial x_1} \right) \, dx \]
\[ = \int_{\mathbb{R}^2_+} \frac{\chi_{\varepsilon}^2}{z_1} \cdot \left( \nabla \frac{\partial V_{z, \varepsilon}}{\partial x_1} \right)^2 \, dx + \int_{\mathbb{R}^2_+} \frac{2\chi_{\varepsilon} \nabla \chi_{\varepsilon}}{z_1} \cdot \left( \nabla \frac{\partial V_{z, \varepsilon}}{\partial x_1} \right) \cdot \frac{\partial V_{z, \varepsilon}}{\partial x_1} \, dx \]
\[ + \int_{\mathbb{R}^2_+} \frac{(\nabla \chi_{\varepsilon})^2}{z_1} \cdot \left( \frac{\partial V_{z, \varepsilon}}{\partial x_1} \right)^2 \, dx + \frac{C}{\varepsilon^2} \cdot \delta_{\varepsilon} \]
\[ = \frac{C_Z}{\varepsilon^2} \cdot (1 + o_\varepsilon(1)), \]
where \( C_Z > 0 \) is some constant independent of \( \varepsilon \). We let \( \chi^*(x) \) be a smooth truncation function taking the value 1 in \( B_{2s}(z) \), and 0 in \( \mathbb{R}^2_+ \setminus B_{Ls}(z) \). Then it holds following
estimate
\[
\left\| \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \right\|_{L^p(B_{Ls}(z))} \\
\leq \left\| \nabla \chi^* \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right\|_{L^p(B_{Ls}(z))} + \left\| \chi^* \cdot \left( \nabla \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \right\|_{L^p(B_{Ls}(z))} \\
\leq C\varepsilon \left( \int_{2s}^{Ls} \frac{\tau}{\tau'} d\tau \right)^{\frac{1}{p}} + \frac{C}{\varepsilon^2} \left( \int_0^s \tau d\tau \right)^{\frac{1}{p}} + \left( \int_{s}^{Ls} \frac{\tau}{\tau^{2p'}} d\tau \right)^{\frac{1}{p}} \\
= C\varepsilon^{\frac{2}{p'} - 2}.
\]
Since \( \text{supp} \, h \subset B_{2s}(z) \), for the second term in the right hand side of (2.28), we have
\[
\left| \int_{\mathbb{R}^d_+} \chi^* \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \cdot h \, d\mathbf{x} \right| = \left| \int_{\mathbb{R}^d_+} \chi^* \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \cdot h \, d\mathbf{x} \right| \\
\leq C\|h\|_{W^{-1,p}(B_{Ls}(z))} \left\| \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \right\|_{L^p(B_{Ls}(z))} \\
\leq C\varepsilon^{\frac{2}{p'} - 2} \|h\|_{W^{-1,p}(B_{Ls}(z))},
\]
where a Poincaré inequality
\[
\left\| \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right\|_{L^p(B_{Ls}(z))} \leq C\varepsilon \left\| \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \right\|_{L^p(B_{Ls}(z))}
\]
is used. For the first term in the right hand side of (2.28), it holds
\[
\int_{\mathbb{R}^d_+} \chi^* \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \cdot \mathbb{L}_\varepsilon \phi \, d\mathbf{x} = \int_{\mathbb{R}^d_+} \phi \cdot \mathbb{L}_\varepsilon \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x} \\
= \int_{\mathbb{R}^d_+} \frac{1}{x_1} \nabla \phi \cdot \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x} - \frac{2}{s}\int_{|x-z|=s} \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \, d\mathbf{x} \\
= -\int_{\mathbb{R}^d_+} \phi \cdot \nabla \left( \frac{1}{x_1} \right) \cdot \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x} - \int_{\mathbb{R}^d_+} \phi \left( \frac{1}{x_1} - \frac{1}{z_1} \right) \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x} \\
- \int_{\mathbb{R}^d_+} \phi \cdot \left( \frac{2}{x_1} \frac{\partial V_{z,\varepsilon}}{\partial x_1} + \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x},
\]
where we have used the fact that \( \partial V_{z,\varepsilon}/\partial x_1 \) is in the kernel of \( \mathbb{L} \). Notice that for terms in above identity we have the following estimates
\[
\int_{\mathbb{R}^d_+} \left| \nabla \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \right| \, d\mathbf{x} \leq C |\ln \varepsilon|, \\
\int_{\mathbb{R}^d_+} \left| \frac{1}{x_1} - \frac{1}{z_1} \right| \Delta \left( \chi^* \phi \cdot \frac{\partial V_{z,\varepsilon}}{\partial x_1} \right) \, d\mathbf{x} \leq s \cdot 2\pi s \cdot \frac{1}{s^2} \leq C,
\]
\[
\int_{\mathbb{R}^2_+} \left| \nabla \chi_{\varepsilon} \cdot \left( \nabla \frac{\partial z_{\varepsilon}}{\partial x_1} \right) \right| \, d\mathbf{x} \leq \frac{C}{\delta_{\varepsilon}} \cdot \int_{\delta_{\varepsilon}}^{2\delta_{\varepsilon}} \frac{1}{\tau} \, d\tau \leq \frac{C}{\delta_{\varepsilon}},
\]
\[
\int_{\mathbb{R}^2_+} \left| (\Delta \chi_{\varepsilon}) \cdot \frac{\partial z_{\varepsilon}}{\partial x_1} \right| \, d\mathbf{x} \leq \frac{C}{\delta_{\varepsilon}^2} \cdot \int_{\delta_{\varepsilon}}^{2\delta_{\varepsilon}} \, d\tau \leq \frac{C}{\delta_{\varepsilon}}.
\]
As a result, it holds
\[
\int_{\mathbb{R}^2_+} \chi_{\varepsilon} \cdot \frac{\partial z_{\varepsilon}}{\partial x_1} \cdot \mathbb{I}_{\varepsilon} \, d\mathbf{x} \leq (|\ln \varepsilon| + \delta_{\varepsilon}^{-1}) \| \phi \|_{L^\infty(B_{2\delta_{\varepsilon}}(z))}
\]
\[
\leq (|\ln \varepsilon| + \delta_{\varepsilon}^{-1}) \| \phi \|_{*}.
\]
Then combining all above estimates for (2.28), we derive
\[
|\Lambda| \leq C \varepsilon^2 (|\ln \varepsilon| + \delta_{\varepsilon}^{-1}) \cdot \| \phi \|_{*} + C \varepsilon^\frac{2}{p} \| \mathbf{h} \|_{W^{-1,p}(B_{L_{\varepsilon}}(z))}.
\]  
By the explicit formulation of \(Z_{z,\varepsilon}\) in \(B_{L_{\varepsilon}}(z)\), it holds
\[
\| x_1 \Delta^* Z_{z,\varepsilon} \|_{W^{-1,p}(B_{L_{\varepsilon}}(z))} \leq C \| \nabla Z_{z,\varepsilon} \|_{L^p(B_{L_{\varepsilon}}(z))} = C \varepsilon^{\frac{2}{p} - 2}.
\]
So we finally deduce from (2.29) that
\[
\| \Lambda x_1 \Delta^* Z_{z,\varepsilon} \|_{W^{-1,p}(B_{L_{\varepsilon}}(z))} \leq C |\Lambda| \cdot \varepsilon^{\frac{2}{p} - 2}
\]
\[
= C \varepsilon^\frac{2}{p} (|\ln \varepsilon| + \delta_{\varepsilon}^{-1}) \cdot \| \phi \|_{*} + C \| \mathbf{h} \|_{W^{-1,p}(B_{L_{\varepsilon}}(z))}.
\]
Now we are to prove (2.26). Suppose not, then there exists a sequence \(\{\varepsilon_n\}\) tending to 0 and \(\phi_n\) such that
\[
\| \phi_n \|_{*} + \varepsilon_n^{-\frac{2}{p}} \| \nabla \phi_n \|_{B_{L_{\varepsilon}}(z))} = 1,
\]  
(2.30)
and
\[
\varepsilon_n^{-\frac{2}{p}} \| \mathbf{h} \|_{W^{-1,p}(B_{L_{\varepsilon}}(z)))} \leq \frac{1}{n}.
\]
Let
\[
-\text{div} \left( \frac{1}{x_1} \nabla \phi_n(x) \right) = \frac{2}{s_{\varepsilon_1}} \delta_{|x-z|=s} \phi_n(s, \theta) + \mathbf{h} - \Lambda x_1 \Delta^* Z_{z,\varepsilon}
\]
\[
= \frac{2}{s_{\varepsilon_1}} \delta_{|x-z|=s} \phi_n(s, \theta) + f_n
\]
with \(\text{supp} \, f_n \subset B_{2\delta_{\varepsilon_n}}(z)\). For a general function \(v\), we define its rescaled version centered at \(z\) as:
\[
\tilde{v}(y) := v(sy + z).
\]
Notice that parameter \(s\) also depends on \(\varepsilon_n\). Denoting \(D_n = \{y \mid sy + z \in \mathbb{R}^2_+\}\), then we obtain
\[
\int_{D_n} \frac{1}{sy_1 + z_1} \cdot \nabla \phi_n \cdot \nabla \varphi \, dy = 2 \int_{|y|=1} \frac{1}{z_1} \tilde{\phi}_n \varphi + (\tilde{f}_n, \varphi), \quad \forall \varphi \in C_0^\infty(D_n),
\]
where for each $p \in (2, \infty]$, it holds
\[
\|\tilde{f}_n\|_{W^{-1,p}(B_L(0))} \leq C\varepsilon_n^{1-\frac{2}{p}} \left(\varepsilon_n^{\frac{2}{p}}(|\ln \varepsilon_n| + \delta_n^{-1}) \cdot \|\phi_n\|_* + \|h\|_{W^{-1,p}(B_L(z))}\right) = o_n(1).
\]
Hence $\tilde{\phi}_n$ is bounded in $C^\alpha_{\text{loc}}(\mathbb{R}^2)$ for some $\alpha > 0$, and $\tilde{\phi}_n$ converges uniformly in any fixed compact set of $\mathbb{R}^2$ to $\phi^* \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, which satisfies
\[
-\Delta \phi^* = 2\phi^*(1, \theta)\delta_{|y|=1}, \quad \text{in } \mathbb{R}^2,
\]
and $\phi^*$ can be written as
\[
\phi^* = C_1 \frac{\partial w}{\partial y_1} + C_2 \frac{\partial w}{\partial y_2}
\]
with
\[
w(y) = \begin{cases} \frac{1}{4}(1 - |y|^2), & |y| \leq 1, \\ \frac{1}{2} \ln \frac{1}{|y|}, & |y| \geq 1. \end{cases}
\]
Since $\phi^*$ is even with respect to $x_1$-axis, it holds $C_2 = 0$. Then, from the second equation in (2.23), we have
\[
\int_{\mathbb{R}^2} \nabla\phi^* \cdot \nabla \frac{\partial w}{\partial x_1} = 0.
\]
Thus we get $C_1 = 0$, and $\phi_n \to 0$ in $B_{Ls}(z)$ as $n \to \infty$.

To derive the estimate for $||\phi_n||_*$, we will use a comparison principle. We see that $\phi_n$ satisfy
\[
\begin{cases} 
\phi_n(x) = 0, & \text{on } x_1 = 0, \\
\phi_n, |\nabla \phi_n|/x_1 \to 0, & \text{as } |x| \to \infty.
\end{cases}
\]
Moreover, $\phi_n \to 0$ in $B_{Ls}(z)$ as $n \to \infty$, and $x_1 \Delta^* \phi_n = 0$ in $\mathbb{R}^2_+ \setminus B_{Ls}(z)$. By letting
\[
\bar{\phi}_n(x) := ||\phi_n||_{L^\infty(B_{Ls}(z))} \cdot G_*(x, z),
\]
we have
\[
\begin{cases} 
\bar{\phi}_n - \phi_n \geq 0, & \text{on } x_1 = 0, \\
\bar{\phi}_n - \phi_n \geq 0, & \text{as } |x| \to \infty,
\end{cases}
\]
and
\[
x_1^2 \Delta^* \bar{\phi}_n - x_1^2 \Delta^* \phi_n = \Delta(\bar{\phi}_n - \phi_n) + x_1 \nabla \left(\frac{1}{x_1}\right) \cdot \nabla(\bar{\phi}_n - \phi_n) = 0, \quad \text{in } \mathbb{R}^2_+ \setminus B_{Ls}(z).
\]
Since $x_1 \nabla (1/x_1)$ is locally bounded on $\mathbb{R}^2_+ \setminus B_{Ls}(z)$, we can use the strong maximum principle to deduce $\phi_n \leq \bar{\phi}_n$ on $\mathbb{R}^2_+ \setminus B_{Ls}(z)$, and hence $|\phi_n| \leq \bar{\phi}_n$ on $\mathbb{R}^2_+ \setminus B_{Ls}(z)$. By the definition of $\bar{\phi}_n(x)$, we have actually shown that
\[
||\phi_n||_* \leq ||\phi_n||_{L^\infty(B_{Ls}(z))} = o_n(1).
\]
On the other hand, for any \( \tilde{\varphi} \in C^\infty_0(D_n) \) it holds
\[
\left| \int_{D_n} \frac{1}{s|y| + z_1} \cdot \nabla \tilde{\varphi}_n \cdot \nabla \tilde{\varphi} dy \right| = \left| 2 \int_{|y|=1} \frac{1}{z} \tilde{\varphi}_n \tilde{\varphi} + \langle \tilde{f}_n, \tilde{\varphi} \rangle \right|
= o_n(1) \cdot \| \tilde{\varphi} \|_{W^{1,1}(B_L(0))} + o_n(1) \cdot \| \tilde{\varphi} \|_{W^{1,\nu'}(B_L(0))}
= o_n(1) \cdot \left( \int_{B_L(0)} |\nabla \tilde{\varphi}|^\nu \right)^{\frac{1}{\nu}},
\]
which leads to
\[
\varepsilon^{1-\frac{2}{p}} \| \nabla \tilde{\varphi}_n \|_{L^p(B_L(0))} \leq C \| \nabla \tilde{\varphi}_n \|_{L^p(B_L(0))} = o_n(1).
\]
Combining (2.31) and (2.32), we get a contradiction to (2.30). Hence (2.26) holds, and (2.27) is a consequence of (2.26) and (2.29).

Using Lemma 2.3, we obtain the following result.

**Lemma 2.4.** Suppose that \( \text{supp } h \subset B_{2\varepsilon}(z) \) and
\[
\varepsilon^{1-\frac{2}{p}} \| h \|_{W^{-1,\nu}(B_L(\varepsilon))} < \infty
\]
with \( p \in (2, +\infty] \). Then there exists a small \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), (2.23) has a unique solution \( \phi_\varepsilon = T_\varepsilon h \), where \( T_\varepsilon \) is a linear operator of \( h \). Moreover, there exists a constant \( c_0 > 0 \) independent of \( \varepsilon \), such that
\[
\| \phi_\varepsilon \|_{\ast} + \varepsilon^{1-\frac{2}{p}} \| \nabla \phi_\varepsilon \|_{L^p(B_L(\varepsilon))} \leq c_0 \varepsilon^{1-\frac{2}{p}} \| h \|_{W^{-1,\nu}(B_L(\varepsilon))},
\]
where \( L > 0 \) is a large constant.

**Proof.** Let \( H_\varepsilon(\mathbb{R}^2_+) \) be the Hilbert space consists of functions satisfying the boundary condition
\[
\begin{aligned}
& u = 0, \quad \text{on } x_1 = 0, \\
& u, \ |\nabla u|/x_1 \rightarrow 0, \quad \text{as } |x| \rightarrow \infty,
\end{aligned}
\]
and endowed with the inner product
\[
[u, v]_{H_\varepsilon(\mathbb{R}^2_+)} = \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla u \cdot \nabla v dx.
\]
To yield the compactness of operator in \( \mathbb{R}^2_+ \), we also introduce another weighted \( L^\infty \) norm as
\[
\| \phi \|_{\ast, \nu} := \sup_{x \in \mathbb{R}^2} \rho_1(x)^{1-\nu} \rho_2(x)^{1-\nu} |\phi(x)|,
\]
where \( 0 < \nu < 1/4 \) is a small number, and \( \rho_1, \rho_2 \) are defined in (2.24). We introduce two spaces. The first one is
\[
E_\varepsilon := \left\{ u \in H_\varepsilon(\mathbb{R}^2_+) \mid \| u \|_{\ast, \nu} < \infty, \ u(x_1, x_2) = u(x_1, -x_2), \ \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla u \cdot \nabla Z_{2\varepsilon} dx = 0 \right\}
\]
with norm $|| \cdot ||_{*, \nu}$, and the second one is
\[
F_{\varepsilon} := \left\{ h^* \in W^{-p}(B_{L_2}(z)) \mid h^*(x_1, x_2) = h^*(x_1, -x_2), \int_{\mathbb{R}^2_+} Z_{z, \varepsilon} h^* d\mathbf{x} = 0 \right\}
\]
with $p \in (2, +\infty]$. Then for $\phi_{\varepsilon} \in E_{\varepsilon}$, problem (2.23) has an equivalent operation form
\[
\phi_{\varepsilon} = (-x_1 \Delta^*)^{-1} P_{\varepsilon} \left( \frac{1}{sz_1} \phi_{\varepsilon}(s, \varepsilon) \delta_{|x-z|=s} \right) + (-x_1 \Delta^*)^{-1} P_{\varepsilon} h
\]
\[
= K \phi_{\varepsilon} + (-x_1 \Delta^*)^{-1} P_{\varepsilon} h,
\]
where
\[
(-x_1 \Delta^*)^{-1} u := \int_{\mathbb{R}^2_+} G_{\varepsilon}(x, x') x_1^{p-1} u(x') d\mathbf{x},
\]
and $P_{\varepsilon}$ is the projection operator to $F_{\varepsilon}$. Since $Z_{z, \varepsilon}$ has a compact support due to the truncation (2.22), by the definition of $G_{\varepsilon}(x, x')$, we see that $K$ maps $E_{\varepsilon}$ to $E_{\varepsilon}$.

To show that $K$ is a compact operator, we let $K_n := \{ x \in \mathbb{R}^2 \mid 1/n < x_1 < n, |x_2| < n \}$ with $n \in N^*$. It is obvious that $K_n \to \mathbb{R}^2_+$ as $n \to +\infty$. Recall that the asymptotic estimate for the Green’s function $G_{\varepsilon}$ given in (2.14) and (2.15). For any small $\epsilon > 0$, we can find an $N$ sufficiently large such that if $n > N$, then it holds
\[
|\rho_1(x)^{1-\nu} \rho_2(x)^{1-\nu}|Ku(x)| < \epsilon, \quad u \in E_{\varepsilon}, \quad x \in \mathbb{R}^2_+ \setminus K_n.
\]
While for $x \in K_n$, standard elliptic estimates shows that the $C^{\alpha}$ norm of $Ku(x)$ is bounded, and hence $Ku(x)$ is uniformly bounded and equi-continuous in $K_n$. By the Ascoli–Arzela theorem, we conclude that $K$ is indeed a compact operator. It is also noteworthy that this approach of recovering compactness is generally applicable in ‘gluing method’, see [15, 16].

Using the Fredholm alternative, (2.23) has a unique solution if the homogeneous equation
\[
\phi_{\varepsilon} = K \phi_{\varepsilon}
\]
has only trivial solution in $E_{\varepsilon}$, which can be obtained from Lemma 2.3. Now we let
\[
T_{\varepsilon} := (\text{Id} - K)^{-1} (-x_1 \Delta^*)^{-1} P_{\varepsilon},
\]
and the estimate (2.33) holds by Lemma 2.3. The proof is thus complete. \hfill \Box

2.4. The reduction and one-dimensional problem. Recall that our aim is to solve (2.20). However, since the linear operator $\mathbb{L}_{\varepsilon}$ has a nontrivial kernel, we have to settle for second best, and first deal with the projective problem in the space $E_{\varepsilon}$. Using the linear operator $T_{\varepsilon}$ given in Lemma 2.4, we are to consider
\[
\phi_{\varepsilon} = T_{\varepsilon} R_{\varepsilon}(\phi_{\varepsilon}) \tag{2.34}
\]
with
\[
R_{\varepsilon}(\phi_{\varepsilon}) = \frac{1}{\varepsilon^2} \left( x_1 1_{\{\psi_{\varepsilon} - \frac{\ln h_0}{\varepsilon} \ln \frac{1}{\varepsilon} > \mu_{\varepsilon}\}} - x_1 1_{\{V_{z, \varepsilon} > \frac{1}{2\pi} \ln \frac{1}{z}\}} - \frac{2}{sz_1} \phi_{\varepsilon}(s, \theta) \delta_{|x-z|=s} \right)
\]
for each small $\varepsilon \in (0, \varepsilon_0]$. In the following lemma, we will give a delicate estimate for the error term $R_{\varepsilon}(\phi_\varepsilon)$, so that a contraction mapping theorem can be applied to obtain the existence of $\phi_\varepsilon$ in $E_{\varepsilon}$.

**Lemma 2.5.** There exists a small $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there is a unique solution $\phi_\varepsilon \in E_{\varepsilon}$ to (2.34), which satisfies

$$\|\phi_\varepsilon\|_s + \varepsilon^{1-\frac{2}{p}}\|\nabla \phi_\varepsilon\|_{L^p(B_{L_s}(z))} = O(\varepsilon \ln \varepsilon)$$

with the norm $\|\cdot\|_s$ defined in (2.12), $p \in (2, +\infty]$.

**Proof.** Denote $G_{\varepsilon} := T_{\varepsilon} R_{\varepsilon}$, and a neighborhood of origin in $E_{\varepsilon}$ as

$$B_{\varepsilon} := E_{\varepsilon} \cap \left\{ \phi \mid \|\phi\|_s + \varepsilon^{1-\frac{2}{p}}\|\nabla \phi\|_{L^p(B_{L_s}(z))} \leq \varepsilon \ln \varepsilon, \ p \in (2, \infty) \right\}.$$

We will show that $G_{\varepsilon}$ is a contraction map from $B_{\varepsilon}$ to $B_{\varepsilon}$, so that a unique fixed point $\phi_\varepsilon$ can be obtained by the contraction mapping theorem. Actually, letting $h = R_{\varepsilon}(\phi)$ for $\phi \in B_{\varepsilon}$, and noticing that $R_{\varepsilon}(\phi)$ satisfies assumptions for $h$ in Lemma 2.4 by Appendix B, we hence have

$$\|T_{\varepsilon} R_{\varepsilon}(\phi)\|_s + \varepsilon^{1-\frac{2}{p}}\|\nabla T_{\varepsilon} R_{\varepsilon}(\phi)\|_{L^p(B_{L_s}(z))} \leq c_0 \varepsilon^{1-\frac{2}{p}}\|R_{\varepsilon}(\phi)\|_{W^{-1,p}(B_{L_s}(z))}.$$

To begin with, we are to show that $G_{\varepsilon}$ maps $B_{\varepsilon}$ continuously into itself. We use $\bar{v}(y)$ to denote $v(sy + z)$. For each $\varphi \in C_0^\infty(B_{L_s}(z))$, in view of Lemma B.2 and Lemma B.3 in Appendix B, we have

$$\langle R_{\varepsilon}(\phi), \varphi \rangle = \frac{s^2}{\varepsilon^2} \int_{B_{L_s}(0)} (sy_1 + z_1) \left( 1_{\{\psi_\varepsilon - \frac{w_{\varepsilon}}{s} \ln |h_\varepsilon| > \mu_\varepsilon\}} - 1_{\{\bar{v}_{s,\varepsilon} > \frac{1}{sN} \ln \varepsilon\}} \right) \bar{\varphi} dy$$

$$- \frac{2}{z_1} \int_0^{2\pi} \bar{\varphi}(1, \theta) d\theta$$

$$= (1 + O(\varepsilon)) \cdot z_1 \cdot \frac{s^2}{\varepsilon^2} \int_0^{2\pi} \int_0^{1+t_\varepsilon + t_e, \theta} t \bar{\varphi}(t, \theta) dt d\theta - \frac{2}{z_1} \int_0^{2\pi} \bar{\varphi}(1, \theta) d\theta$$

$$= \frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} \int_0^{1+t_\varepsilon + t_e, \theta} t \bar{\varphi}(1, \theta) dt d\theta - \frac{2}{z_1} \int_0^{2\pi} \bar{\varphi}(1, \theta) d\theta$$

$$+ \frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} \int_0^{1+t_\varepsilon + t_e, \theta} t (\bar{\varphi}(t, \theta) - \bar{\varphi}(1, \theta)) dt d\theta + O(\varepsilon) \cdot \int_0^{2\pi} |\bar{\varphi}| d\theta$$

$$= \frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} \left( \frac{\bar{\varphi}(1, \theta)}{sN} + O(\varepsilon \ln \varepsilon) \right) \bar{\varphi}(1, \theta) d\theta + O(\varepsilon) \cdot \int_0^{2\pi} |\bar{\varphi}| d\theta$$

$$+ \frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} \int_0^{1+t_\varepsilon + t_e, \theta} t \left( \frac{\partial \bar{\varphi}(s, \theta)}{\partial s} ds dt d\theta - \frac{2}{z_1} \int_0^{2\pi} \bar{\varphi}(1, \theta) d\theta \right)$$

$$= \frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} |t + t_e, \theta| \int_0^{1+t_\varepsilon + t_e, \theta} \left| \frac{\partial \bar{\varphi}(s, \theta)}{\partial s} \right| ds dt d\theta + O(\varepsilon \ln \varepsilon) \cdot \||\bar{\varphi}||_{W^{1,p'}(B_{L_s}(0))}$$

$$= O(\varepsilon \ln \varepsilon) \cdot \||\bar{\varphi}||_{W^{1,p'}(B_{L_s}(0))}$$.
where we have used the definition of $N$ in (2.17). Thus we have

$$
\varepsilon^{1-\frac{2}{p}} ||R_\varepsilon(\phi)||_{W^{-1, p}(B_{L_s}(z))} = O(\varepsilon |\ln \varepsilon|),
$$

which yields

$$
||T_\varepsilon R_\varepsilon(\phi)||_* + \varepsilon^{1-\frac{2}{p}} ||\nabla T_\varepsilon R_\varepsilon(\phi)||_{L^p(B_{L_s}(z))} = O(\varepsilon |\ln \varepsilon|) < \varepsilon |\ln \varepsilon|^2
$$

by Lemma 2.4. Arguing in a same way, we can deduce

$$
\varepsilon ||\nabla \phi||_{L^\infty(B_{L_s}(z))} = O(\varepsilon |\ln \varepsilon|) < \varepsilon |\ln \varepsilon|^2
$$

from the estimate

$$
\varepsilon ||R_\varepsilon(\phi)||_{W^{-1, \infty}(B_{L_s}(z))} = O(\varepsilon |\ln \varepsilon|).
$$

Thus operator $G_\varepsilon$ indeed maps $B_\varepsilon$ to $B_\varepsilon$ continuously.

In the next step, we are to verify that $G_\varepsilon$ is a contraction mapping under the norm

$$
|| \cdot ||_{G_\varepsilon} = || \cdot ||_* + \varepsilon^{1-\frac{2}{p}} || \cdot ||_{W^{1, p}(B_{L_s}(z))}, \quad p \in (2, +\infty].
$$

We already know that $B_\varepsilon$ is close under this norm. Let $\phi_1$ and $\phi_2$ be two functions in $B_\varepsilon$. From Lemma 2.4, it holds

$$
||G_\varepsilon \phi_1 - G_\varepsilon \phi_2||_{G_\varepsilon} \leq C \varepsilon^{1-\frac{2}{p}} ||R_\varepsilon(\phi_1) - R_\varepsilon(\phi_2)||_{W^{-1, p}(B_{L_s}(z))}, \quad (2.36)
$$

where

$$
R_\varepsilon(\phi_1) - R_\varepsilon(\phi_2) = \frac{1}{\varepsilon^2} \left( x_1 1_{\{U_{s, \varepsilon} + \phi_1 > 0\}} - x_1 1_{\{U_{s, \varepsilon} + \phi_2 > 0\}} - \frac{2}{s^2} (\phi_1(s, \theta) - \phi_2(s, \theta)) \delta_{|x - z| = s} \right).
$$

For $m = 1, 2$, let

$$
S_{m1} := \{ y \mid U_{s, \varepsilon} + \phi_m > 0 \} \cap B_L(0),
$$

and

$$
S_{m2} := \{ y \mid U_{s, \varepsilon} + \phi_m < 0 \} \cap B_L(0).
$$

Then it holds

$$
1_{\{U_{s, \varepsilon} + \phi_1 > 0\}} - 1_{\{U_{s, \varepsilon} + \phi_2 > 0\}} = 0, \quad \text{in} \quad (S_{11} \cap S_{21}) \cup (S_{12} \cap S_{22}).
$$
According to Lemma B.3, for each $\tilde{\phi} \in C^0_0(B_L(0))$, we have

$$
\frac{s^2}{\varepsilon^2} \int_{B_L(0)} (\nabla \tilde{\phi} + \varepsilon^2 \nabla \phi) d\mathbf{y} = \frac{s^2}{\varepsilon^2} \int_{S_{12}} (\nabla \tilde{\phi} + \varepsilon^2 \nabla \phi) d\mathbf{y}
$$

Then it holds

$$
\int_{S_{12}} (\nabla \tilde{\phi} + \varepsilon^2 \nabla \phi) d\mathbf{y} = \frac{s^2}{\varepsilon^2} \int_{S_{12}} (\nabla \tilde{\phi} + \varepsilon^2 \nabla \phi) d\mathbf{y}
$$

where we have used the fact

$$
|t_{\tilde{\phi}_1} - t_{\tilde{\phi}_2}| \leq C\|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(B_L(0))}.
$$

To handle the first term in above identity, we let $\phi_* := \tilde{\phi}_1 - \tilde{\phi}_2$, and

$$
y_{\varepsilon,m} := (1 + t_{\varepsilon}(\theta) + t_{\varepsilon,\hat{m}}(\theta))(\cos \theta, \sin \theta)
$$

where $\varepsilon, m \Rightarrow \hat{m}$, and $\varepsilon \neq m$.

Then it holds

$$
\hat{U}_{z,1}(y_{\varepsilon,1}) - \hat{U}_{z,1}(y_{\varepsilon,2}) = \tilde{\phi}_2(y_{\varepsilon,2}) - \tilde{\phi}_1(y_{\varepsilon,1})
$$

$$
= \tilde{\phi}_2(y_{\varepsilon,2}) - \tilde{\phi}_1(y_{\varepsilon,1}) + \int_{1+\varepsilon+t_{\varepsilon,\hat{m}}(\theta)}^{1+\varepsilon+t_{\varepsilon,\hat{m}}(\theta)} \frac{\partial \tilde{\phi}_2(t, \theta)}{\partial t} dt
$$

$$
= \phi_*(1, \theta) + \int_{1}^{1+\varepsilon+t_{\varepsilon,\hat{m}}(\theta)} \frac{\partial \tilde{\phi}_2(t, \theta)}{\partial t} dt.
$$

By the expansion

$$
\hat{U}_{z,2}(y_{\varepsilon,1}) - \hat{U}_{z,2}(y_{\varepsilon,2}) = -\frac{1}{s^2N} \left( y_{\varepsilon,1} - y_{\varepsilon,2} \right) + O(\varepsilon |\ln \varepsilon|^2),
$$

we have

$$
t_{\varepsilon,\tilde{\phi}_1} - t_{\varepsilon,\tilde{\phi}_2} = |y_{\varepsilon,1} - y_{\varepsilon,2}|
$$

$$
= -sN(1 + o_1(1)) \cdot \left( \phi_*(1, \theta) + \int_{1}^{1+\varepsilon+t_{\varepsilon,\hat{m}}(\theta)} \frac{\partial \tilde{\phi}_2(t, \theta)}{\partial t} dt \right).
$$
Then using the definition of $\mathcal{N}$ in (2.17), one can deduce
\[
\frac{s^2}{\varepsilon^2} \int_0^{2\pi} (t,\phi_1 - t,\phi_2)(sy_1 + z_1)\tilde{\varphi}(1,\theta)d\theta = \frac{2}{z_1}(1 + o_\varepsilon(1)) \cdot \int_0^{2\pi} (\tilde{\phi}_1 - \tilde{\phi}_2)\tilde{\varphi}(1,\theta)d\theta
\]
\[
- \frac{2}{z_1}(1 + o_\varepsilon(1)) \cdot \left(\int_{1+\varepsilon+t,\phi_1}^{1+\varepsilon+t,\phi_2} \frac{\partial \tilde{\phi}_1(t,\theta)}{\partial t} dt + \int_{1+\varepsilon+t,\phi_2}^{1+\varepsilon+t,\phi_1} \frac{\partial \tilde{\phi}_2(t,\theta)}{\partial t} dt\right)
\]
\[
= \frac{2}{z_1} \int_0^{2\pi} (\tilde{\phi}_1 - \tilde{\phi}_2)\tilde{\varphi}(1,\theta)d\theta + o_\varepsilon(1) \cdot \|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(\mathcal{B}_\varepsilon(\mathcal{O}))}
\]
\[
+ \left(O\left(\varepsilon |\ln \varepsilon|^2\right) + \|\tilde{\phi}_2\|_{W^{1,p}(\mathcal{B}_\varepsilon(\mathcal{O}))}\right) \cdot \|\tilde{\phi}_2\|_{L^\infty(\mathcal{B}_\varepsilon(\mathcal{O}))} \cdot \|\tilde{\varphi}\|_{W^{1,p'}(\mathcal{B}_\varepsilon(\mathcal{O}))}.
\]
Finally, we conclude that
\[
\varepsilon^{1-\frac{2}{p}}\|R_\varepsilon(\phi_1) - R_\varepsilon(\phi_2)\|_{W^{-1,p}(\mathcal{B}_\varepsilon(\mathcal{O}))} = o_\varepsilon(1) \cdot \|\phi_1 - \phi_2\|_{\mathcal{G}_\varepsilon},
\]
which yields
\[
\|\mathcal{G}_\varepsilon \phi_1 - \mathcal{G}_\varepsilon \phi_2\|_{\mathcal{G}_\varepsilon} = o_\varepsilon(1) \cdot \|\phi_1 - \phi_2\|_{\mathcal{G}_\varepsilon}
\]
from (2.36). Hence we have shown that $\mathcal{G}_\varepsilon$ is a contraction map from $\mathcal{B}_\varepsilon$ into itself.

Using the contraction mapping theorem, we now can claim that there is a unique $\phi_\varepsilon \in \mathcal{B}_\varepsilon$ such that $\phi_\varepsilon = \mathcal{G}_\varepsilon \phi_\varepsilon$, which satisfies (2.35). Since $\|\phi_\varepsilon\|_{\mathcal{G}_\varepsilon}$ is bounded by a constant $C$ independent of $\varepsilon$, we conclude that $\phi_\varepsilon$ is continuous with respect to $\varepsilon$ in the norm $\|\cdot\|_{\mathcal{G}_\varepsilon}$. □

From the above lemma, the problem of solving (2.20) is now transformed into a one-dimensional problem: Finding the sufficient condition to ensure
\[
\Lambda = 0,
\]
which will also determine the location of $z = (z_1,0)$ as a crucial parameter in approximate solutions. In the next lemma, we will derive a condition equivalent to $\Lambda = 0$, which enables us to prove the existence of $\psi_\varepsilon$.

**Lemma 2.6.** If $z = (z_1,0)$ satisfies
\[
\varepsilon^2 \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla \psi_\varepsilon \cdot \nabla Z_{z_1,\varepsilon} d\mathbf{x} - \int_{A_\varepsilon} x_1 \cdot Z_{z_1,\varepsilon} d\mathbf{x} = 0,
\]
then $\psi_\varepsilon$ is a solution to (2.10) and (2.11).

**Proof.** If the assumption (2.37) holds true, from (2.20) we will have
\[
\varepsilon^2 \Lambda \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla Z_{z_1,\varepsilon} \cdot \nabla Z_{z_1,\varepsilon} d\mathbf{x} = 0.
\]
Proceeding as in the proof of Lemma 2.3, we deduce
\[
\varepsilon^2 \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla Z_{z_1,\varepsilon} \cdot \nabla Z_{z_1,\varepsilon} d\mathbf{x} = C_Z + o_\varepsilon(1).
\]
Hence it holds $\Lambda = 0$ when $\varepsilon$ is sufficiently small. This fact implies that $\psi_\varepsilon$ is a solution to (2.10) and (2.11). □
Taking advantage of the above characterization, we are now in the position to prove Proposition 2.1.

**Proof of Proposition 2.1:** We will show that condition (2.37) is equivalent to

\[ z_1 - \frac{\kappa}{4\pi W} = O \left( \frac{1}{|\ln \varepsilon|} \right). \]

Since \( \phi \in E_\varepsilon \), we have

\[ \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla \phi \cdot \nabla Z_{z,\varepsilon} \, dx = 0. \]

Hence it holds

\[ \varepsilon^2 \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla \psi \cdot \nabla Z_{z,\varepsilon} \, dx = \int_{A_\varepsilon} x_1 \cdot Z_{z,\varepsilon} \, dx \]

\[ = \varepsilon^2 \int_{\mathbb{R}^2_+} \frac{1}{x_1} \nabla (\mathcal{V}_{z,\varepsilon} + \mathcal{H}_{z,\varepsilon}) \cdot \nabla Z_{z,\varepsilon} \, dx - \int_{A_\varepsilon} x_1 \cdot Z_{z,\varepsilon} \, dx \]

\[ = \int_{B_{L_\varepsilon}(z)} x_1 (1_{\{V_{z,\varepsilon}>\frac{\alpha}{\pi} \ln \frac{1}{\varepsilon}\}} - 1_{\{\psi_\varepsilon-\frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}}) \cdot Z_{z,\varepsilon} \, dx. \]

By denoting

\[ \bar{Z}_{z,\varepsilon} = Z_{z,\varepsilon}(s y + z), \]

direct computation yields

\[ \|\bar{Z}_{z,\varepsilon}\|_{W^{1,p}(B_L(0))} = O(\varepsilon^{-1}). \]

Note that

\[ \int \frac{2}{z_1} \int_0^{2\pi} \frac{\phi_\varepsilon(1, \theta)}{\varepsilon} \bar{Z}_{z,\varepsilon} \, d\theta \]

\[ = \frac{1}{s y_1 + z_1} \cdot \nabla \phi_\varepsilon \cdot \nabla \bar{Z}_{z,\varepsilon} \, dx + O_\varepsilon(1) \cdot \left( \|\phi_\varepsilon\|_* + \varepsilon \|\nabla \phi_\varepsilon\|_{L^\infty(B_{L_\varepsilon}(z))} \right) \]

\[ = O_\varepsilon(1) \cdot \left( \|\phi_\varepsilon\|_* + \varepsilon \|\nabla \phi_\varepsilon\|_{L^\infty(B_{L_\varepsilon}(z))} \right), \]

due to the nondegeneracy property of operator \( \mathbb{L} \) defined in (2.21). Then, similar to the proof of Lemma 2.6, we can deduce

\[ \int_{B_{L_\varepsilon}(z)} x_1 (1_{\{V_{z,\varepsilon}>\frac{\alpha}{\pi} \ln \frac{1}{\varepsilon}\}} - 1_{\{\psi_\varepsilon-\frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}}) \cdot Z_{z,\varepsilon} \, dx \]

\[ = -\frac{s^2}{\varepsilon^2} \int_{B_L(0)} (s y_1 + z_1) \left( 1_{\{\psi_\varepsilon-\frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon}\}} - 1_{\{V_{z,\varepsilon}>\frac{\alpha}{\pi} \ln \frac{1}{\varepsilon}\}} \right) \bar{Z}_{z,\varepsilon} \, dy \]

\[ = -\frac{s^2}{\varepsilon^2} \cdot z_1 \int_0^{2\pi} \left( \frac{\phi(1, \theta)}{sN} + s \cos \theta \cdot \left( \frac{s^2}{4\varepsilon^2} \cdot z_1 \ln \frac{1}{\varepsilon} - W z_1 \ln \frac{1}{\varepsilon} \right) + o(\varepsilon) \right) \bar{Z}_{z,\varepsilon} \, d\theta + O_\varepsilon(1) \]

\[ = \frac{\pi}{2} \cdot \frac{s^4}{\varepsilon^4} \cdot z_1 \left( \frac{s^2}{4\varepsilon^2} \cdot z_1 \ln \frac{1}{\varepsilon} - W z_1 \ln \frac{1}{\varepsilon} \right) + O_\varepsilon(1). \]
Since it holds \( s^2 \pi z_1 / \varepsilon^2 = \kappa + O(1/|\ln \varepsilon|) \) by our choice of \( a \) in (2.19), condition (2.37) yields
\[
\frac{\kappa}{4\pi} - W z_1 = O \left( \frac{1}{|\ln \varepsilon|} \right).
\]
Then we can solve above equation on \( z_1 \) and obtain at least one \( z_1 \) satisfying (2.38). In view of Lemma 2.6, we obtain the existence of \( \psi_\varepsilon \) for every \( \varepsilon \in (0, \varepsilon_0] \). Moreover, the estimates for \( A_\varepsilon \) can be deduced from Lemma 2.5 and Appendix B. Thus we have completed the proof of Proposition 2.1. \( \square \)

3. Uniqueness

In this section, we will prove the local uniqueness of a vortex ring of small cross-section for which \( \zeta \) is constant throughout the core. Moreover, we assume the cross-section \( A_\varepsilon \) is simply-connected and has a positive distance from \( x_2 \)-axis, so that it is given by
\[
A_\varepsilon = \left\{ x \in \mathbb{R}_+^2 \mid \psi_\varepsilon - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon \right\},
\]
where \( \mu_\varepsilon > 0 \) have a positive lower bound independent of \( \varepsilon \). Using notations in Section 2, the Stokes stream function \( \psi_\varepsilon \) satisfies
\[
\begin{cases}
-\varepsilon^2 \Delta^* \psi_\varepsilon = 1_{A_\varepsilon}, & \text{in } \mathbb{R}_+^2, \\
\psi_\varepsilon = 0, & \text{on } x_1 = 0, \\
\psi_\varepsilon, |\nabla \psi_\varepsilon|/x_1 \to 0, & \text{as } |x| \to \infty.
\end{cases}
\]
To discuss the uniqueness of vortex rings of small cross-section, we will fix the circulation
\[
\kappa = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} x_1 dx,
\]
and the parameter \( W \) in translational velocity \( W \ln \varepsilon e_2 \). Since \( \psi_\varepsilon \) determines the vortex ring \( \zeta_\varepsilon \) absolutely, the uniqueness result in Theorem 1.2 can be concluded from following proposition.

**Proposition 3.1.** Let \( \kappa \) and \( W \) be two fixed positive constants. Suppose that the cross-section \( A_\varepsilon \) is simply-connected with a positive distance from \( x_2 \)-axis, and satisfies
\[
\text{diam } A_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0.
\]
Then for each \( \varepsilon \in (0, \varepsilon_0] \) with \( \varepsilon_0 > 0 \) sufficiently small, equation (3.1) together with (3.2) has a unique solution \( \psi_\varepsilon \) up to translations in the \( x_2 \)-direction.

To study the local behavior of \( \psi_\varepsilon \) near \( A_\varepsilon \), we denote
\[
\sigma_\varepsilon := \frac{1}{2} \text{diam } A_\varepsilon
\]
as the cross-section parameter. By our assumptions, it will hold \( \sigma_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Intuitively, the maximum point of \( \psi_\varepsilon \) in \( A_\varepsilon \) gives the exact location of cross-section. So we can choose a point \( p_\varepsilon \in A_\varepsilon \) satisfying

\[
\psi_\varepsilon(p_\varepsilon) = \max_{x \in A_\varepsilon} \psi_\varepsilon(x),
\]

which is always possible by maximum principle of \(-\Delta^*\). In view of Lemma A.1 in Appendix A, the set \( A_\varepsilon \) must be symmetric with respect to some horizontal line \( x_2 = h \). Using the translation invariance of (3.1) in \( x_2 \)-direction, we may always assume \( A_\varepsilon \) is even symmetric with respect to \( x_1 \)-axis (i.e. \( (x_1, x_2) \in A_\varepsilon \) if and only if \( (x_1, -x_2) \in A_\varepsilon \)). Then, by the integral equation

\[
\psi_\varepsilon(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} G_\varepsilon(x, x')1_{A_\varepsilon}(x')dx',
\]

we see that \( \psi_\varepsilon \) attains its maximum on \( x_1 \)-axis, and

\[
\psi_\varepsilon(x) - \frac{W}{2} \ln \frac{1}{\varepsilon} - x_1^2 < 0, \quad \text{as} \quad x_1 \to +\infty.
\]

Thus we may assume that \( p_\varepsilon = (p_\varepsilon, 0) \), and \( p_\varepsilon \) satisfies \( c_1 < p_\varepsilon < c_2 \), where \( c_1, c_2 \) are two positive constants.

Now, by letting \( z = (z_1, 0) \) with \( z_1 > 0 \), we decompose the Green’s function for \(-\Delta^*\) in boundary condition of (3.1) as

\[
G_\varepsilon(x, x') = z_1^2 G(x, x') + H(x, x'),
\]

where \( G(x, x') \) is the Green’s function of \(-\Delta\) on the half plane, and \( H(x, x') \) is the rest regular part. At this stage, we only assume \( |z_1 - p_\varepsilon| = o(\varepsilon) \). More accurate description of \( z \) will be given in second part of our proof.

Applying this decomposition of \( G_\varepsilon(x, x') \), we can split the stream function \( \psi_\varepsilon \) as \( \psi_{1, \varepsilon} + \psi_{2, \varepsilon} \), where

\[
\psi_{1, \varepsilon}(x) = \frac{z_1^2}{\varepsilon^2} \int_{\mathbb{R}^2_+} G(x, x')1_{A_\varepsilon}(x')dx',
\]

and

\[
\psi_{2, \varepsilon}(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} H(x, x')1_{A_\varepsilon}(x')dx'.
\]

According to (3.1), \( \psi_{1, \varepsilon}(x) \) solves the problem

\[
\begin{cases}
-\varepsilon^2 \Delta \psi_{1, \varepsilon}(x) = z_1^2 1_{A_\varepsilon}, & \text{in } \mathbb{R}^2_+, \\
\psi_{1, \varepsilon} = 0, & \text{on } x_1 = 0, \\
\psi_{1, \varepsilon}, \ |\nabla \psi_{1, \varepsilon}|/x_1 \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]

and \( \psi_{2, \varepsilon}(x) \) satisfies

\[
\begin{cases}
-\varepsilon^2 \Delta^* (\psi_{1, \varepsilon}(x) + \psi_{2, \varepsilon}(x)) = 1_{A_\varepsilon}, & \text{in } \mathbb{R}^2_+, \\
\psi_{2, \varepsilon} = 0, & \text{on } x_1 = 0, \\
\psi_{2, \varepsilon}, \ |\nabla \psi_{2, \varepsilon}|/x_1 \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]
We see that the above two equations constitute a coupled system of $\psi_1, \varepsilon$ and $\psi_2, \varepsilon$, which seems more complicated than (3.1). However, it should be noted that $\psi_1, \varepsilon$ is a solution to a semilinear Laplace equation. While $\psi_2, \varepsilon$ is a more regular function than $\psi_1, \varepsilon$ with the $L^\infty$ norm bounded independent of $\varepsilon$. These fine properties enable us to decouple $\psi_1, \varepsilon$ and $\psi_2, \varepsilon$ in the main order, and use the local Pohozaev identity in Appendix C to analyse the asymptotic behavior.

To prove the uniqueness, the key idea is to derive the main parts for $\psi_\varepsilon$ and $\nabla \psi_\varepsilon$ as precise as possible, which are to be obtained by several steps of approximation and bootstrap arguments. In this process, we also obtain a relationship of $\kappa, W, \sigma_\varepsilon$ and $z_1$, namely, an accurate version of Kelvin–Hicks formula (1.3).

**Proposition 3.2.** For steady vortex rings of small cross-section depicted in Proposition 3.1, the parameters $\kappa, W, \sigma_\varepsilon$ and $z_1$ satisfy

$$Wz_1 \ln \frac{1}{\varepsilon} = \frac{\kappa}{4\pi} \left( \ln \frac{8z_1}{\sigma_\varepsilon} - \frac{1}{4} \right) + O(\varepsilon^2 |\ln \varepsilon|), \quad \text{as } \varepsilon \to 0.$$

In [19], Fraenkel has obtained a slightly weaker form of the above estimate with the error term $O(\varepsilon^2 \ln |\varepsilon|^2)$. We reach a level of $O(\varepsilon^2 |\ln \varepsilon|)$ since a better $z$ is chosen to be the center of $V_{z,\varepsilon}$ in the approximate solution. Actually, if we replace $z$ with $p_\varepsilon$ in above formula, then the error term will be the same as [19].

Our approach for uniqueness is divided into several parts. In the first part of our proof, we give a coarse estimate for $\psi_\varepsilon$ and $A_\varepsilon$. Then we improve this estimate by constructing approximate solutions and deal with the error term carefully, which can be regarded as an inverse of Lyapunov–Schmidt reduction we have done in Section 2. The uniqueness for $\psi_\varepsilon$ is obtained by contradiction in the last part of this section.

3.1. Asymptotic estimates for vortex ring. The purpose of this part is to derive an asymptotic estimate for $\psi_\varepsilon$, and to obtain the following necessary condition on the location of $A_\varepsilon$, which is a coarse version of Kelvin–Hicks formula in Proposition 3.2.

**Proposition 3.3.** As $\varepsilon \to 0$, it holds

$$Wp_\varepsilon \ln \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \ln \frac{8p_\varepsilon}{\sigma_\varepsilon} + \frac{\kappa}{16\pi} = o(1).$$

To prove Proposition 3.3, we will begin with the estimate for $\psi_{1,\varepsilon}$ away from the cross-section $A_\varepsilon$. In the following, we always assume that $L > 0$ is a large constant.

**Lemma 3.4.** For every $x \in \mathbb{R}^2_+ \setminus \{x \mid \text{dist}(x, A_\varepsilon) \leq L\sigma_\varepsilon\}$, we have

$$\psi_{1,\varepsilon}(x) = \frac{\kappa}{2\pi} \cdot p_\varepsilon \ln \frac{|x - \tilde{p}_\varepsilon|}{|x - p_\varepsilon|} + O \left( \frac{\sigma_\varepsilon}{|x - p_\varepsilon|} \right),$$

and

$$\nabla \psi_{1,\varepsilon}(x) = -\frac{\kappa}{2\pi} \cdot p_\varepsilon \frac{x - p_\varepsilon}{|x - p_\varepsilon|^2} + \frac{\kappa}{2\pi} \cdot p_\varepsilon \frac{x - \tilde{p}_\varepsilon}{|x - \tilde{p}_\varepsilon|^2} + O \left( \frac{\sigma_\varepsilon}{|x - p_\varepsilon|^2} \right).$$
Proof. For every \( x \in \mathbb{R}^2_+ \setminus \{ x \mid \text{dist}(x, A_\varepsilon) \leq L\sigma_\varepsilon \} \), it holds \( x \notin A_\varepsilon \). Recall the notation \( \bar{x} = (-x_1, x_2) \). For each \( x' \in A_\varepsilon \) we have

\[
|x - x'| = |x - p_\varepsilon| - \left( \frac{x - p_\varepsilon}{|x - p_\varepsilon|}, x' - p_\varepsilon\right) + O\left( \frac{|x' - p_\varepsilon|^2}{|x - p_\varepsilon|} \right),
\]

and

\[
|x - \bar{x}'| = |x - \bar{p}_\varepsilon| - \left( \frac{x - \bar{p}_\varepsilon}{|x - \bar{p}_\varepsilon|}, \bar{x}' - \bar{p}_\varepsilon\right) + O\left( \frac{|x' - \bar{p}_\varepsilon|^2}{|x - \bar{p}_\varepsilon|} \right).
\]

Hence we deduce

\[
\psi_{1,\varepsilon}(x) = \frac{z_1^2}{2\pi \varepsilon^2} \int_{A_\varepsilon} \ln \frac{|x - \bar{x}'|}{|x - x'|} dx'
= \frac{\kappa}{2\pi} \cdot p_\varepsilon \ln \frac{|x - \bar{p}_\varepsilon|}{|x - p_\varepsilon|} + \frac{p_\varepsilon^2}{2\pi \varepsilon^2} \int_{A_\varepsilon} \ln \frac{|x - p_\varepsilon|}{|x - x'|} dx - \frac{p_\varepsilon^2}{2\pi \varepsilon^2} \int_{A_\varepsilon} \ln \frac{|x - \bar{p}_\varepsilon|}{|x - \bar{x}'|} dx' + O\left( \frac{\sigma_\varepsilon}{|x - p_\varepsilon|} \right),
\]

where we use the circulation constraint (3.2) and \( |x - p_\varepsilon| < |x - \bar{p}_\varepsilon| \). Similarly, from the relations

\[
\frac{x - p_\varepsilon}{|x - p_\varepsilon|^2} - \frac{x - x'}{|x - x'|^2} = O\left( \frac{\sigma_\varepsilon}{|x - p_\varepsilon|^2} \right),
\]

and

\[
\frac{x - \bar{p}_\varepsilon}{|x - \bar{p}_\varepsilon|^2} - \frac{x - \bar{x}'}{|x - \bar{x}'|^2} = O\left( \frac{\sigma_\varepsilon}{|x - \bar{p}_\varepsilon|^2} \right),
\]

we obtain

\[
\nabla \psi_{1,\varepsilon}(x) = -\frac{\kappa}{2\pi} \cdot p_\varepsilon \frac{x - p_\varepsilon}{|x - p_\varepsilon|^2} + \frac{\kappa}{2\pi} \cdot p_\varepsilon \frac{x - \bar{p}_\varepsilon}{|x - \bar{p}_\varepsilon|^2} + O\left( \frac{\sigma_\varepsilon}{|x - p_\varepsilon|^2} \right).
\]

Thus the proof is complete. \( \square \)

Compared with the main term \( \psi_{1,\varepsilon} \), the secondary term \( \psi_{2,\varepsilon} \) is more regular, as can be seen from the following estimate, and we can therefore obtain its estimates in the whole right half-plane.

**Lemma 3.5.** For \( x \in \mathbb{R}^2_+ \), it holds

\[
\psi_{2,\varepsilon}(x) = \frac{\kappa}{p_\varepsilon} H(x, z) + O(\sigma_\varepsilon |\ln \sigma_\varepsilon|).
\]

**Proof.** Using the definition of \( H(x, x') \) and standard elliptic estimate, it holds

\[
\psi_{2,\varepsilon}(x) - \frac{\kappa}{p_\varepsilon} H(x, z) \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2_+} \left( H(x, x') - H(x, z) \right) 1_{A_\varepsilon} dx' + O(\sigma_\varepsilon)
= O(\sigma_\varepsilon |\ln \sigma_\varepsilon|),
\]

which is the desired result. \( \square \)

Next we turn to study the local behavior of \( \psi_{1,\varepsilon} \) near \( p_\varepsilon \).
Proposition 3.6. \( \psi_{1,\varepsilon} \) has the following asymptotic behavior as \( \varepsilon \to 0 \),

\[
\psi_{1,\varepsilon}(x) = \frac{\sigma_{\varepsilon}^2}{\varepsilon^2} \cdot p_{\varepsilon}^2 \left( w \left( \frac{x - p_{\varepsilon}}{\sigma_{\varepsilon}} \right) + o_{\varepsilon}(1) \right) + \mu_{\varepsilon} + \frac{W}{2} p_{\varepsilon}^2 \ln \frac{1}{\varepsilon} - \frac{\kappa}{p_{\varepsilon}} H(p_{\varepsilon}, z), \quad x \in B_{L_{\sigma_{\varepsilon}}}(p_{\varepsilon}),
\]

\[
\frac{\kappa}{2\pi} \cdot p_{\varepsilon} \ln \left( \frac{1}{\sigma_{\varepsilon}} \right) - \frac{\kappa}{2\pi} \cdot p_{\varepsilon} \ln \frac{1}{2p_{\varepsilon}} + \frac{\kappa}{p_{\varepsilon}} H(p_{\varepsilon}, z) - \frac{W}{2} p_{\varepsilon}^2 \ln \frac{1}{\varepsilon} - \mu_{\varepsilon} = o_{\varepsilon}(1),
\]

and

\[
\frac{|A_{\varepsilon}|}{\sigma_{\varepsilon}^2} \to \pi,
\]

where

\[
w(y) = \begin{cases} 
\frac{1}{4}(1 - |y|^2), & |y| \leq 1, \\
\frac{1}{2} \ln \frac{1}{|y|}, & |y| \geq 1.
\end{cases}
\]

In order to show Proposition 3.6, we first prove the following lemma, which means the kinetic energy of the flow in vortex core is bounded.

Lemma 3.7. As \( \varepsilon \to 0 \), it holds

\[
\frac{1}{\varepsilon^2} \int_{A_{\varepsilon}} x_1 \left( \psi_{\varepsilon} - \frac{W}{2} \ln \frac{1}{\varepsilon} x_1^2 - \mu_{\varepsilon} \right)_+ \, dx = O_{\varepsilon}(1).
\]

Proof. We take \( \psi_+ = (\psi_{\varepsilon} - \frac{W}{2} \ln \frac{1}{\varepsilon} x_1^2 - \mu_{\varepsilon})_+ \) as the upper truncation of \( \psi_{\varepsilon} \). From equation (3.1), it holds

\[
\begin{cases} 
-\varepsilon^2 \Delta^* \psi_+(x) = 1_{A_{\varepsilon}}, \\
\psi_+(x) = 0, \quad \text{on } \partial A_{\varepsilon}.
\end{cases}
\]

Thus we can integrate by part to obtain

\[
\int_{A_{\varepsilon}} \frac{1}{x_1} |\nabla \psi_+|^2 \, dx = \frac{1}{\varepsilon^2} \int_{A_{\varepsilon}} x_1 \psi_+ \, dx \leq \frac{C|A_{\varepsilon}|^{1/2}}{\varepsilon^2} \left( \int_{A_{\varepsilon}} |\psi_+|^2 \, dx \right)^{1/2},
\]

where we use the restriction \( c_1 < p_{\varepsilon} < c_2 \). By Sobolev embedding, it holds

\[
\left( \int_{A_{\varepsilon}} |\psi_+|^2 \, dx \right)^{1/2} \leq C \int_{A_{\varepsilon}} |\nabla \psi_+| \, dx.
\]

Hence we deduce

\[
\left( \int_{A_{\varepsilon}} |\nabla \psi_+|^2 \, dx \right)^{1/2} \leq \frac{C|A_{\varepsilon}|^{1/2}}{\varepsilon^2} \left( \int_{A_{\varepsilon}} |\nabla \psi_+|^2 \, dx \right)^{1/2}.
\]

Using the circulation constraint (3.2), we finally obtain

\[
\frac{1}{\varepsilon^2} \int_{A_{\varepsilon}} x_1 \psi_+ \, dx = \int_{A_{\varepsilon}} \frac{1}{x_1} |\nabla \psi_+|^2 \, dx = O_{\varepsilon}(1),
\]

which is the estimate we need by the definition of \( \psi_+ \). \( \square \)
Now we introduce a scaling version of $\psi_{1,\varepsilon}$ by letting

$$w_{\varepsilon}(y) = \frac{1}{p_{\varepsilon}} \cdot \frac{\varepsilon^2}{\sigma_{\varepsilon}^2} \left( \psi_{1,\varepsilon}(\sigma_{\varepsilon} y + p_{\varepsilon}) + \frac{\kappa}{p_{\varepsilon}} H(p_{\varepsilon}, z) - \frac{W}{2} p_{\varepsilon}^2 \ln \frac{1}{\varepsilon} - \mu_{\varepsilon} \right),$$

so that $w_{\varepsilon}$ satisfies

$$- \Delta w_{\varepsilon} = 1_{\{w_{\varepsilon} > 0\}} + f(\sigma_{\varepsilon} y + p_{\varepsilon}, w_{\varepsilon}), \quad \text{in } \mathbb{R}^2,$$

with

$$f(x, w) := \frac{\varepsilon^2}{p_{\varepsilon}^2} \cdot 1_{\{\psi_{\varepsilon}(x) - W x_1^2 \ln \frac{1}{\varepsilon} - \beta_{\varepsilon} > 0\}} - 1_{\{w > 0\}},$$

and $w_{\varepsilon}(y) = O(\sigma_{\varepsilon} \ln \sigma_{\varepsilon})$, if $\sigma_{\varepsilon} y + p_{\varepsilon} \in \partial A_{\varepsilon}$.

Intuitively, the limiting equation for $w_{\varepsilon}$ as $\varepsilon \to 0$ is $- \Delta w = 1_{\{w > 0\}}$. To show the convergence, we are to give a uniform bound for $w_{\varepsilon}$ in $L^\infty$ norm.

**Lemma 3.8.** For any $R > 0$, there exists a constant $C_R > 0$ independent of $\varepsilon$ such that

$$\|w_{\varepsilon}\|_{L^\infty(B_R(0))} \leq C_R.$$

**Proof.** It follows from Lemma 3.7 and the assumption on $p_{\varepsilon}$ that

$$O_{\varepsilon}(1) = \frac{1}{\varepsilon^2} \int_{A_{\varepsilon}} x_1 \left( \psi_{\varepsilon} - \frac{W}{2} \ln \frac{1}{\varepsilon} x_1^2 - \mu_{\varepsilon} \right) \, dx$$

$$= \frac{\sigma_{\varepsilon}^4}{\varepsilon^4} \cdot (p_{\varepsilon}^3 + O(\sigma_{\varepsilon})) \cdot \int_{\mathbb{R}^2} (w_{\varepsilon})_+ \, dy + O(\sigma_{\varepsilon} \ln \sigma_{\varepsilon}).$$

Notice that $\kappa = \varepsilon^{-2} \cdot p_{\varepsilon} |A_{\varepsilon}| + o_{\varepsilon}(1) \leq C \varepsilon^{-2} \sigma_{\varepsilon}^2$. We deduce

$$\int_{\mathbb{R}^2} (w_{\varepsilon})_+ \, dy \leq C.$$

By Morse iteration, we then obtain

$$\|w_{\varepsilon}\|_{L^\infty(B_R(0))} \leq C.$$

To prove that the $L^\infty$ norm of $w_{\varepsilon}$ is bounded, we consider the following problem.

$$\begin{cases}
- \Delta w_1 = 1_{\{w_{\varepsilon} > 0\}} + f(\sigma_{\varepsilon} y + p_{\varepsilon}, w_{\varepsilon}), \quad \text{in } B_R(0), \\
w_1 = 0, \quad \text{on } \partial B_R(0).
\end{cases}$$

It is obvious that $|w_1| \leq C$. Let $w_2 := w_{\varepsilon} - w_1$. Since $\sup_{B_R(0)} w_{\varepsilon} \geq 0$, function $w_2$ is harmonic in $B_R(0)$ and satisfies

$$\sup_{B_R(0)} w_2 \geq \sup_{B_R(0)} w_{\varepsilon} - C \geq -C.$$

On the other hand, we infer from $\|w_{\varepsilon}\|_{L^\infty(B_R(0))} \leq C$ that

$$\sup_{B_R(0)} w_2 \leq \sup_{B_R(0)} w_{\varepsilon} + C \leq M,$$
for some constant $M$. Hence $M - w_2$ is a positive harmonic function. Using the Harnack inequality, we have
\[
\sup_{B_R(0)} (M - w_2) \leq L \inf_{B_R(0)} (M - w_2) \leq L(M + \sup_{B_R(0)} w_2) \leq C.
\]
Since $\sup_{B_R(0)} (M - w_2) = M - \inf_{B_R(0)} w_2$, we deduce
\[
\inf_{B_R(0)} w_2 \geq C,
\]
which implies the boundedness of $w_\varepsilon$. \[\square\]

The limiting function for $w_\varepsilon$ as $\varepsilon \to 0$ is established in the following lemma.

**Lemma 3.9.** As $\varepsilon \to 0$, it holds
\[
w_\varepsilon \to w
\]
in $C^1_{\text{loc}}(\mathbb{R}^2)$ for some radial function $w$.

**Proof.** For $y \in B_R(0) \setminus B_L(0)$, we infer from Lemma 3.4 and Lemma 3.5 that
\[
w_\varepsilon(y) = \frac{1}{p_\varepsilon^2} \cdot \frac{\varepsilon^2}{\sigma_\varepsilon^2} \cdot \left(\psi_{1,\varepsilon}(\sigma_\varepsilon y + p_\varepsilon) + \frac{\kappa}{p_\varepsilon} H(p_\varepsilon, z) - \frac{W}{2} p_\varepsilon^2 \ln \frac{1}{\varepsilon} - \mu_\varepsilon\right)
\]
\[
= \frac{|A_\varepsilon| \cdot (1 + O(\sigma_\varepsilon))}{\sigma_\varepsilon^2} \cdot \left(\frac{1}{2\pi} \ln \frac{1}{\sigma_\varepsilon y + \bar{p}_\varepsilon - p_\varepsilon}\right) - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y|} - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y + \bar{p}_\varepsilon - p_\varepsilon|}
\]
\[
+ \frac{1}{p_\varepsilon^2} H(p_\varepsilon, z) - \frac{W}{2\kappa} \cdot p_\varepsilon \ln \frac{1}{\varepsilon} - \frac{\mu_\varepsilon}{p_\varepsilon \kappa} + O\left(\frac{1}{\varepsilon}\right)
\]
\[
= \frac{|A_\varepsilon| \cdot (1 + O(\sigma_\varepsilon))}{\sigma_\varepsilon^2} \cdot \frac{1}{2\pi} \ln \frac{1}{|y|}
\]
\[
+ \frac{|A_\varepsilon| \cdot (1 + O(\sigma_\varepsilon))}{\sigma_\varepsilon^2} \cdot \left(\frac{1}{2\pi} \ln \frac{1}{\sigma_\varepsilon} - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y + \bar{p}_\varepsilon - p_\varepsilon|}\right) - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y|}
\]
\[
+ \frac{1}{p_\varepsilon^2} H(p_\varepsilon, z) - \frac{W}{2\kappa} \cdot p_\varepsilon \ln \frac{1}{\varepsilon} - \frac{\mu_\varepsilon}{p_\varepsilon \kappa} + O\left(\frac{1}{\varepsilon}\right).
\]
Since $|A_\varepsilon|/\sigma_\varepsilon^2 \leq C$ and $||w_\varepsilon||_{L^\infty(B_R(0))} \leq C_R$ by Lemma 3.8, we may assume
\[
|A_\varepsilon|/\sigma_\varepsilon^2 \to t,
\]
and
\[
\frac{|A_\varepsilon| \cdot (1 + O(\sigma_\varepsilon))}{\sigma_\varepsilon^2} \cdot \left(\frac{1}{2\pi} \ln \frac{1}{\sigma_\varepsilon} - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y + \bar{p}_\varepsilon - p_\varepsilon|}\right) - \frac{1}{2\pi} \ln \frac{1}{|\sigma_\varepsilon y|}
\]
\[
+ \frac{1}{p_\varepsilon^2} H(p_\varepsilon, z) - \frac{W}{2\kappa} \cdot p_\varepsilon \ln \frac{1}{\varepsilon} - \frac{\mu_\varepsilon}{p_\varepsilon \kappa} \to \tau,
\]
for some \( t \in [0, +\infty) \) and \( \tau \in (-\infty, +\infty) \). By (3.3), we may further assume that \( w_\varepsilon \to w \) in \( C^1_{\text{loc}}(\mathbb{R}^2) \) and \( w \) satisfies

\[
\begin{cases}
-\Delta w = 1_{\{w > 0\}}, & \text{in } B_R(0), \\
w = \frac{t}{2\pi} \ln \frac{1}{|y|} + \tau + O \left( \frac{1}{L} \right), & \text{in } B_R(0) \setminus B_L(0).
\end{cases}
\]

Moreover, \( w \) will satisfy the integral equation

\[ w(y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \left( \frac{1}{|y - y'|} \right) 1_{\{w > 0\}}(y') dy' + \tau. \]

Then the method of moving planes shows that \( w \) is radial and decreasing (See e.g. [38]), which completes the proof of this lemma. \( \square \)

**Proof of Proposition 3.6:** By the definition of \( \sigma_\varepsilon \), there exists a \( y_\varepsilon \) with \(|y_\varepsilon| = 1\) and \( \sigma_\varepsilon y_\varepsilon + p_\varepsilon \in \partial A_\varepsilon \). Thus it holds

\[
w(y) = \begin{cases}
\frac{1}{4}(1 - |y|^2), & |y| \leq 1, \\
\frac{1}{2} \ln \frac{1}{|y|}, & |y| \geq 1.
\end{cases}
\]

We further have that \( t = \pi \) and \( \tau + O(1/L) = 0 \). Since \( \tau \) is not dependent on \( L \), while \( O(1/L) \to 0 \) as \( L \to +\infty \), one must have \( \tau = 0 \) and \( O(1/L) = 0 \). The proof of Proposition 3.6 is hence complete. \( \square \)

**Proof of Proposition 3.3:** Now we can apply the local Pohozaev identity (C.1) in Appendix C to \( \psi_{1,\varepsilon} \) and obtain

\[
-\int_{\partial B_\delta(z)} \frac{\partial \psi_{1,\varepsilon}}{\partial \nu} \frac{\partial \psi_{1,\varepsilon}}{\partial x_1} dS + \frac{1}{2} \int_{\partial B_\delta(z)} |\nabla \psi_{1,\varepsilon}|^2 \nu_1 dS = -\frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} \partial_1 \psi_{2,\varepsilon}(x) \cdot 1_{A_\varepsilon}(x) dx + \frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} W x_1 \ln \frac{1}{\varepsilon} \cdot 1_{A_\varepsilon}(x) dx,
\]

where \( \delta \) is a small positive number. Since \(|A_\varepsilon|/\sigma_\varepsilon^2 \to \pi \) as \( \varepsilon \to 0 \) and \(|z_1 - p_\varepsilon| = o(\varepsilon)\), from the isoperimetric inequality, we see that \( A_\varepsilon \) tends to a disc with radius \( \sigma_\varepsilon \to s_0 := \left( \frac{\kappa}{z_1 \pi} \right)^{1/2} \) centered at \( z \), and \(|A_\varepsilon \Delta B_{s_0}(z)| = o(\varepsilon^2)\).

Using Lemma C.4, we have

\[
W p_\varepsilon \ln \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \ln \frac{8p_\varepsilon}{\sigma_\varepsilon} + \frac{\kappa}{16\pi} = o(\varepsilon^2).
\]

So we have finished the proof of Proposition 3.3. \( \square \)
3.2. Refined estimates and revised Kelvin–Hicks formula. For the uniqueness of \(\psi_{\varepsilon}\), we need to improve the results in Propositions 3.3 and 3.6. So we reconsider the problem (3.1)
\[
\begin{cases}
-\varepsilon^2 \Delta \psi_{\varepsilon} = 1_{\{\psi_{\varepsilon} - \frac{\psi_{\varepsilon}}{x_1^2} \ln \frac{1}{\varepsilon} > \mu_{\varepsilon}\}}, & \text{in } \mathbb{R}^2, \\
\psi_{\varepsilon} = 0, & \text{on } x_1 = 0, \\
\psi_{\varepsilon}, |\nabla \psi_{\varepsilon}|/x_1 \to 0, & \text{as } |x| \to \infty
\end{cases}
\]
together with circulation constraint (3.2)
\[
\frac{1}{\varepsilon^2} \int_{A_{\varepsilon}} x_1 dx = \kappa.
\]
To obtain a more accurate estimate for \(\psi_{\varepsilon}\), we will construct a series of approximate solutions \(\Phi_{z,\varepsilon}\), and calculate their differences with \(\psi_{\varepsilon}\). Let us recall the definition of functions \(V_{z,\varepsilon}, H_{z,\varepsilon}\), whose properties are discussed in the second part of Section 1. We choose the approximate solutions to (3.1) and (3.2) of the form
\[
\Phi_{z,\varepsilon}(x) = V_{z,\varepsilon}(x) + H_{z,\varepsilon}(x),
\]
where the parameters \(z, s\) and \(a\) in \(\Phi_{z,\varepsilon}(x)\) satisfy
\[
\partial_1 \Phi_{z,\varepsilon}(p_{\varepsilon}) = 0, \tag{3.4}
\]
\[
a \cdot \ln \frac{1}{\varepsilon} = \mu_{\varepsilon} + \frac{W}{2} z_1^2 \frac{1}{\varepsilon} - H_{z,\varepsilon}(z) + V_{z,\varepsilon}(z), \tag{3.5}
\]
and
\[
a \cdot \ln \frac{1}{\varepsilon} \cdot \frac{1}{s} \ln s = \frac{s}{2\varepsilon^2} \cdot z_1^2. \tag{3.6}
\]
As (2.17) in Section 2, here we also denote
\[
\mathcal{N} := \frac{a}{2\pi} \ln \frac{1}{\varepsilon} \cdot \frac{1}{s} \ln s = \frac{s}{2\varepsilon^2} \cdot z_1^2 \tag{3.7}
\]
as the value of \(|\nabla V_{z,\varepsilon}|\) at \(|x - z| = s\). Notice the first condition (3.4) is equivalent to
\[
\frac{|z_1 - p_{\varepsilon}|}{2\varepsilon} + O(\varepsilon) = \partial_1 V_{z,\varepsilon}(p_{\varepsilon}) - \partial_1 H_{z,\varepsilon}(p_{\varepsilon}) + O(\varepsilon),
\]
where the right hand side blows up at order \(O(|\ln \varepsilon|)\). By the asymptotic estimates given in Proposition 3.6, we then obtain
\[
|z_1 - p_{\varepsilon}| = O(\varepsilon^2 |\ln \varepsilon|),
\]
\[
a \cdot \ln \frac{1}{\varepsilon} = \mu_{\varepsilon} + \frac{W}{2} p_{\varepsilon}^2 \frac{1}{\varepsilon} + O_{\varepsilon}(1),
\]
and
\[
|\sigma_{\varepsilon} - s| = o(\varepsilon).
\]
The same as in Section 2, we also denote the difference of \(\psi_{\varepsilon}\) and \(\Phi_{z,\varepsilon}\) as the error term
\[
\phi_{\varepsilon}(x) := \psi_{\varepsilon}(x) - \Phi_{z,\varepsilon}(x).
\]
Hence our task in this part is to improve the estimate for \(\phi_{\varepsilon}\).
Recall the definition of $\| \cdot \|_*$ norm in (2.25). With the result in Proposition 3.6, we have the following lemma concerning $\phi_\varepsilon$.

**Lemma 3.10.** As $\varepsilon \to 0$, it holds

$$\| \phi_\varepsilon \|_* = o_\varepsilon(1).$$

**Proof.** In view of Proposition 3.6 and our assumptions (3.4)–(3.6), it is obvious that

$$\| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))} = o_\varepsilon(1)$$

for some $L > 0$ large.

While for those $x$ far away from $B_{Ls}(z)$, it holds

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^2} \int \mathbb{1}_{x_1}(x, x') (1 - \mathbb{1}_{B_s(z)}(x')) dx'.$$

Since

$$\frac{1}{\varepsilon^2} \| 1 - \mathbb{1}_{B_s(z)} \|_{L^1(B_{Ls}(z))} = o_\varepsilon(1),$$

we can use the expansion

$$\left( \frac{1}{x_1} + 1 \right) G_s(x, x') \leq C \cdot \frac{1 + x_1^2}{(1 + |x - z|^2)^{\frac{3}{2}}},$$

and Young inequality to derive

$$\| \phi_\varepsilon \|_* = o_\varepsilon(1),$$

which yields the conclusion. \qed

By a linearization procedure, we see that $\phi_\varepsilon$ satisfies the equation

$$\mathbb{L}_\varepsilon \phi_\varepsilon = R_\varepsilon(\phi_\varepsilon),$$

where $\mathbb{L}_\varepsilon$ is the linear operator defined by

$$\mathbb{L}_\varepsilon \phi_\varepsilon = -x_1 \Delta^* \phi_\varepsilon - \frac{2}{sz_1} \phi_\varepsilon(s, \theta) \delta_{|x-z|=s},$$

and

$$R_\varepsilon(\phi_\varepsilon) = \frac{1}{\varepsilon^2} \left( x_1 \mathbb{1}_{\{ \psi - \frac{w}{x_1^2} \ln \frac{1}{\varepsilon} \} > \mu_\varepsilon} - x_1 \mathbb{1}_{\{ V_{\varepsilon} > \frac{a}{x_1^2} \ln \frac{1}{\varepsilon} \}} - \frac{2}{sz_1} \phi_\varepsilon(s, \theta) \delta_{|x-z|=s} \right).$$

By Lemma B.4 in Appendix B, it holds

$$R_\varepsilon(\phi_\varepsilon) = 0, \quad \text{in } \left( \mathbb{R}_+^2 \setminus B_{2s}(z) \right) \cup B_{s/2}(z)$$

for some $L > 0$ large.

To derive a better estimate for $\phi_\varepsilon$, let us first establish the following lemma about the linear operator $\mathbb{L}_\varepsilon$. 

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Lemma 3.11. Suppose that \( \text{supp} \mathbb{L}_z \phi \subset B_{2\delta}(z) \). Then for any \( p \in (2, +\infty) \) and a constant \( c_0 > 0 \), there exists an \( \epsilon_0 > 0 \) small such that for any \( \epsilon \in (0, \epsilon_0] \), it holds
\[
\epsilon^{1-\frac{2}{p}} \|\mathbb{L}_z \phi\|_{W^{-1,p}(B_{Ls}(z))} + \epsilon^2 \|\mathbb{L}_z \phi\|_{L^\infty(B_{\delta/2}(z))} \geq c_0 \left( \epsilon^{1-\frac{2}{p}} \|\nabla \phi\|_{L^p(B_{Ls}(z))} + \|\phi\|_\ast \right)
\]
with \( L > 0 \) a large constant.

Proof. We will argue by contradiction. Suppose on the contrary that there exists \( \epsilon_n \to 0 \) such that \( \phi_n := \phi_{\epsilon_n} \) satisfies
\[
\epsilon_n^{1-\frac{2}{p}} \|\mathbb{L}_z \phi_n\|_{W^{-1,p}(B_{Ls}(z))} + \epsilon_n^2 \|\mathbb{L}_z \phi_n\|_{L^\infty(B_{\delta/2}(z))} \leq \frac{1}{n},
\]
and
\[
\epsilon_n^{1-\frac{2}{p}} \|\nabla \phi_n\|_{L^p(B_{Ls}(z))} + \|\phi_n\|_\ast = 1. \tag{3.8}
\]
By letting \( f_n = \mathbb{L}_z \phi_n \), we have
\[
-\Delta^* \phi_n = \frac{2}{s \delta_1} \phi_n(s,\theta) \delta_{|x-z|=s} + f_n.
\]
Here, we also denote \( \tilde{v}(y) := v(\delta y + z) \) for an arbitrary function. Then the above equation has a weak form
\[
\int_{\mathbb{R}^2_1} \frac{1}{sy_1 + z_1} \cdot \nabla \tilde{\phi}_n \cdot \nabla \varphi \, dy = 2 \int_{|y|=1} \frac{1}{\delta_1} \tilde{\phi}_n \varphi + \langle \tilde{f}_n, \varphi \rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).
\]
Since the right hand side of the equation is bounded in \( W_{1,\infty}(\mathbb{R}^2) \), \( \tilde{\phi}_n \) is bounded in \( W_{1,\infty}(\mathbb{R}^2) \) and hence bounded in \( C_0^\infty(\mathbb{R}^2) \) for some \( \alpha > 0 \) by Sobolev embedding. We may assume that \( \tilde{\phi}_n \) converges uniformly in any compact subset of \( \mathbb{R}^2 \) to \( \phi^* \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2) \), and the limiting function \( \phi^* \) satisfies
\[
-\Delta \phi^* = 2\phi^*(1,\theta) \delta_{|y|=1}, \quad \text{in } \mathbb{R}^2.
\]
Therefore, we conclude from the nondegeneracy of limiting operator and symmetry with respect to \( x_1 \)-axis that
\[
\phi^* = C_1 \cdot \frac{\partial w}{\partial y_1}
\]
with \( C_1 \) a constant, and
\[
w(y) = \begin{cases} \frac{1}{2} \left( 1 - |y|^2 \right), & |y| \leq 1, \\ \frac{1}{2} \ln \frac{1}{|y|}, & |y| \geq 1. \end{cases}
\]
On the other hand, since \( \epsilon_n^2 |f_n| \leq 1/n \) in \( B_{\delta/2}(z) \) and \( |\tilde{\phi}_n| \leq 1 \), we know that \( \tilde{\phi}_n \) is bounded in \( W_{1,\infty}(B_{1/4}(0)) \). Thus we may assume \( \phi_n \rightarrow \phi^* \) in \( C^1(B_{1/4}(0)) \). Since \( \partial^1 \tilde{\phi}_n(\frac{p_{\epsilon_n} - z}{s}) = s \partial^1 \phi_n(\frac{p_{\epsilon_n}}{s}) = 0 \) by (3.5) and \( \frac{p_{\epsilon_n} - z}{s} \to 0 \), it holds \( \partial_1 \phi^*(0) = 0 \). This implies \( C_1 = 0 \) and hence \( \phi^* \equiv 0 \).

Therefore, we have proved \( \phi_n = o_n(1) \) in \( B_{Ls}(z) \) for any \( L > 0 \) fixed. Then, using the strong maximum principle and a similar argument as in the proof of Lemma 2.3, we can derive
\[
||\phi_n||_\ast \leq C ||\phi_n||_{L^\infty(B_{Ls}(z))} = o_n(1). \tag{3.9}
\]
Now we turn to consider the norm $||\nabla \phi_\varepsilon||_{L^p(B_{Ls}(z))}$. For any $\tilde{\varphi} \in C^\infty_0(B_L(0))$, it holds

$$\left| \int_{B_n} \frac{1}{s y_1 + z_1} \cdot \nabla \tilde{\varphi}_n \cdot \nabla \tilde{\varphi} dy \right| = 2 \int_{|y| = 1} \frac{1}{z} \tilde{\varphi}_n \varphi + \langle \tilde{\varphi}_n, \tilde{\varphi} \rangle = o_n(1) \cdot ||\tilde{\varphi}||_{W^{1,1}(B_L(0))} + o_n(1) \cdot ||\tilde{\varphi}||_{W^{1,p^*}(B_L(0))}$$

(3.10)

Thus we have

$$\varepsilon^{1/2} - \frac{2}{p} ||\nabla \phi_\varepsilon||_{L^p(B_{Ls}(z))} \leq C ||\nabla \tilde{\varphi}_n||_{L^p(B_L(0))} = o_n(1).$$

We see that (3.9) and (3.10) is a contradiction to (3.8), and hence the proof of Lemma 3.11 is finished. \(\Box\)

Now we are in the position to improve the estimate for error term $\phi_\varepsilon$.

**Lemma 3.12.** For $p \in (2, +\infty]$ and $\varepsilon \in (0, \varepsilon_0]$ small, it holds

$$||\phi_\varepsilon||_s + \varepsilon^{1/2} - \frac{2}{p} ||\nabla \phi_\varepsilon||_{L^p(B_{Ls}(z))} = O_\varepsilon \left( s W(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_\varepsilon^{1/2 + 1/p} \right),$$

(3.11)

with $W(x)$ defined in (B.1) of Appendix B, and

$$\gamma_\varepsilon := ||\phi_\varepsilon||_{L^\infty(B_{Ls}(z))} + s W(s).$$

**Proof.** In view of Lemma 3.11, it is sufficient to verify that

$$\varepsilon^{1/2} - \frac{2}{p} ||R_\varepsilon(\phi_\varepsilon)||_{W^{-1,p}(B_{Ls}(z))} + \varepsilon^2 ||R_\varepsilon(\phi_\varepsilon)||_{L^\infty(B_{s/2}(z))}$$

$$= O_\varepsilon \left( s W(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_\varepsilon^{1/2 + 1/p} \right).$$

Notice that we have

$$R_\varepsilon(\phi_\varepsilon) \equiv 0, \text{ in } B_{s/2}(z).$$

So it remains to estimate $\varepsilon^{1/2} - \frac{2}{p} ||R_\varepsilon(\phi_\varepsilon)||_{W^{-1,p}(B_{Ls}(z))}$.

We will make an appropriate scaling, and use $\tilde{v}(y)$ to denote $v(s y + z)$. For each $\varphi \in C_0^1(B_{Ls}(z))$, we have

$$\langle R_\varepsilon(\phi_\varepsilon), \varphi \rangle = \frac{s^2}{\varepsilon^2} \int_{B_1(0)} (s y_1 + z_1) \left( 1_{\{\psi_\varepsilon \sim \frac{s}{\varepsilon} x_1 \ln \frac{1}{\varepsilon} > \mu_\varepsilon \}} - 1_{\{V_\varepsilon > \frac{s}{\varepsilon} \ln \frac{1}{\varepsilon} \}} \right) \tilde{\varphi} dy$$

$$- \int_{0}^{2\pi} \tilde{\varphi}(1, \theta) d\theta.$$
Denote $y_ε(θ) = (1 + t_ε + t_ε,φ_ε) \cos θ, (1 + t_ε + t_ε,φ_ε) \sin θ$ as the notations given in Lemma B.4. We deduce that

$$
\frac{s^2}{ε^2} \int_{B_L(0)} \left( sy_1 + z_1 \right) \left( 1_{(ψ_ε,ψ_ε^2 \ln \frac{1}{ε} > μ_ε)} - 1_{(V_ε,ψ_ε^2 \ln \frac{1}{ε})} \right) \tilde{φ} dy
$$

$$
= \frac{s^2}{ε^2} \int_{0}^{2π} \int_{1}^{1+t_ε+φ_ε} z_1 t_ε \tilde{φ}(t, θ) dt dθ + O(ε) \cdot \| t_ε + t_ε,φ_ε \| \| \tilde{φ} \| L^4(B_L(0))
$$

$$
= \frac{s^2}{ε^2} \int_{0}^{2π} \int_{1}^{1+t_ε+φ_ε} z_1 t_ε \tilde{φ}(1, θ) dt dθ + \frac{s^2}{ε^2} \int_{0}^{2π} \int_{1}^{1+t_ε+φ_ε} z_1 t_ε \tilde{φ}(t, θ) - \tilde{φ}(1, θ) dt dθ
$$

$$
+ O(ε) \cdot \| t_ε + t_ε,φ_ε \| \| \tilde{φ} \| W^{1,p'}(B_L(0))
$$

$$
= I_1 + I_2 + O_ε \left( ε^{\frac{1}{2} + \frac{1}{p'}} \right) \cdot \| \tilde{φ} \| W^{1,p'}(B_L(0))
$$

where we use Sobolev embedding and choose $q = \frac{2p'}{2 - p}$. It follows from Lemma 3.10 and Lemma B.4 that

$$
I_1 = \frac{s^2}{ε^2} \int_{0}^{2π} \int_{1}^{1+t_ε+φ_ε} z_1 t_ε \tilde{φ}(1, θ) dt dθ
$$

$$
= \frac{2}{z_1} \int_{0}^{2π} \left( \tilde{φ}_ε(y_ε(θ)) + O_ε \left( sW(s) + ε^2 | \ln ε | + \| \tilde{φ}_ε \| L^∞(B_L(0)) \right) \right) \tilde{φ}(1, θ) dθ
$$

$$
= \frac{2}{z_1} \int_{|y|=1} \tilde{φ}_ε \tilde{φ} dθ + \frac{2}{z_1} \int_{0}^{2π} \left( \tilde{φ}_ε(y_ε(θ)) - \tilde{φ}_ε(1, θ) \right) \tilde{φ} dθ
$$

$$
+ O_ε \left( sW(s) + ε^2 | \ln ε | + O_ε(1) \cdot \| \tilde{φ}_ε \| L^∞(B_L(0)) \right) \int_{|y|=1} \tilde{φ}(1, θ) dθ
$$

$$
= \frac{2}{z_1} \int_{|y|=1} \tilde{φ}_ε \tilde{φ} dθ + \frac{2}{z_1} \int_{0}^{2π} \int_{1}^{1+t_ε+φ_ε} \frac{∂\tilde{φ}_ε(s, θ)}{∂s} \tilde{φ} ds dθ
$$

$$
+ O_ε \left( sW(s) + ε^2 | \ln ε | + O_ε(1) \cdot \| \tilde{φ}_ε \| L^∞(B_L(0)) \right) \int_{|y|=1} \tilde{φ}(1, θ) dθ
$$

$$
= \frac{2}{z_1} \int_{|y|=1} \tilde{φ}_ε \tilde{φ} dθ
$$

$$
+ O_ε \left( sW(s) + ε^2 | \ln ε | + O_ε(1) \cdot \| \tilde{φ}_ε \| L^∞(B_L(0)) + O_ε(1) \cdot \| \tilde{φ}_ε \| L^p(B_L(0)) \right) \cdot \| \tilde{φ} \| W^{1,p'}(B_L(0))
$$
Using Lemma B.4, we can also deduce that
\[
I_2 = \frac{s^2}{\varepsilon^2} \int_0^{2\pi} \int_1^{1+t_\varepsilon + t_{\varepsilon, \tilde{\phi}_\varepsilon}} z_1 t (\tilde{\phi}(t, \theta) - \tilde{\phi}(1, \theta)) dtd\theta
\]
\[
= \frac{s^2}{\varepsilon^2} \int_0^{2\pi} \int_1^{1+t_\varepsilon + t_{\varepsilon, \tilde{\phi}_\varepsilon}} z_1 t \int_1^t \frac{\partial \tilde{\phi}(s, \theta)}{\partial s} ds dtd\theta
\]
\[
\leq \frac{s^2}{\varepsilon^2} \int_0^{2\pi} z_1 |t_\varepsilon(\theta) + t_{\varepsilon, \tilde{\phi}_\varepsilon}(\theta)| \int_1^{1+t_\varepsilon + t_{\varepsilon, \tilde{\phi}_\varepsilon}} |\frac{\partial \tilde{\phi}(s, \theta)}{\partial s}| ds d\theta
\]
\[
= O_\varepsilon \left( sW(s) + \varepsilon^2 |\ln \varepsilon| + O_\varepsilon(1) \cdot ||\tilde{\phi}_\varepsilon||_{L^\infty(B_L(0))} \right) \int_0^{2\pi} \int_1^{1+t_\varepsilon + t_{\varepsilon, \tilde{\phi}_\varepsilon}} |\frac{\partial \tilde{\phi}(s, \theta)}{\partial s}| ds d\theta
\]
\[
= o_\varepsilon(1) \cdot O_\varepsilon \left( sW(s) + \varepsilon^2 |\ln \varepsilon| + ||\tilde{\phi}_\varepsilon||_{L^\infty} \cdot ||\tilde{\phi}||_{W^{1,\gamma'}(B_L(0))} \right).
\]
Combining above estimates, we arrive at
\[
\langle \mathcal{R}_\varepsilon(\phi_\varepsilon), \varphi \rangle
\]
\[
= O_\varepsilon \left( sW(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_\varepsilon^{1+\frac{1}{p}} \right) \cdot ||\tilde{\phi}||_{W^{1,\gamma'}(B_L(0))}
\]
\[
+ o_\varepsilon(1) \cdot \left( ||\tilde{\phi}_\varepsilon||_{L^\infty(B_L(0))} + ||\nabla \tilde{\phi}_\varepsilon||_{L^p(B_L(0))} \right) \cdot ||\tilde{\phi}||_{W^{1,\gamma'}(B_L(0))},
\]
which implies
\[
\varepsilon^{1-\frac{2}{p}} ||\mathcal{R}_\varepsilon(\phi_\varepsilon)||_{W^{1,\gamma'}(B_L(z))}
\]
\[
= O_\varepsilon \left( sW(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_\varepsilon^{1+\frac{1}{p}} \right)
\]
\[
+ o_\varepsilon(1) \cdot \left( ||\phi_\varepsilon||_{L^\infty} + \varepsilon^{1-\frac{2}{p}} ||\nabla \phi_\varepsilon||_{L^p(B_L(z))} \right).
\]
Thus from the above discussion, we finally obtain
\[
||\phi_\varepsilon||_{L^\infty} + \varepsilon^{1-\frac{2}{p}} ||\nabla \phi_\varepsilon||_{L^p(B_L(z))}
\]
\[
= O_\varepsilon \left( sW(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_\varepsilon^{1+\frac{1}{p}} \right),
\]
which is exactly the result we desired. \(\square\)

With the refined estimate of \(\phi_\varepsilon\) in hand, we can improve the estimate for \(\hat{\Gamma}_{\varepsilon, \tilde{\phi}_\varepsilon}\) in Lemma B.4 as follows.

**Lemma 3.13.** The set
\[
\hat{\Gamma}_{\varepsilon, \tilde{\phi}_\varepsilon} := \left\{ y \mid \psi_\varepsilon(sy + z) - \frac{W}{2} (sy_1 + z_1)^2 \ln \frac{1}{\varepsilon} \cdot e_1 = \mu_\varepsilon \right\}
\]
is a continuous closed convex curve in $\mathbb{R}^2$, and for each $\theta \in (0, 2\pi]$, it holds
\[
\tilde{\Gamma}_{\varepsilon, \tilde{\phi}_\varepsilon} = (1 + t_\varepsilon(\theta) + t_{\varepsilon, \tilde{\phi}_\varepsilon}(\theta))(\cos \theta, \sin \theta)
= (\cos \theta, \sin \theta) + O_\varepsilon \left(sW(s) + \varepsilon \gamma_{\varepsilon}^{\frac{1}{2} + \frac{1}{p}}\right)
\]
with
\[
\gamma_{\varepsilon} = \|\phi_{\varepsilon}\|_{L^\infty(B_{Ls}(z))} + sW(s).
\]

Using a bootstrap method, we can further improve the estimate for $\phi_{\varepsilon}$ and $|A_\varepsilon \Delta B_{s_0}(z)|$ to our desired level.

**Lemma 3.14.** For $p \in (2, +\infty]$, it holds
\[
|||\phi_{\varepsilon}|||_* + \varepsilon^{1-\frac{2}{p}}|||\nabla \phi_{\varepsilon}|||_{L^p(B_{Ls}(z))} = O(\varepsilon^2 |\ln \varepsilon|).
\]
Moreover, we have
\[
|A_\varepsilon \Delta B_{s_0}(z)| = O(\varepsilon^4 |\ln \varepsilon|),
\]
and
\[
W(s) = O(\varepsilon^2 |\ln \varepsilon|).
\]

**Proof.** At the first stage, we have $W(s) = O(|\ln \varepsilon|)$ in hand by the definition of $W(x)$ in (B.1). Hence from (3.11), we can deduce
\[
|||\phi_{\varepsilon}|||_* + \varepsilon^{1-\frac{2}{p}}|||\nabla \phi_{\varepsilon}|||_{L^p(B_{Ls}(z))} = O(\varepsilon |\ln \varepsilon|).
\]

Note that $s_0 = (\frac{2}{1-\varepsilon})^{1/2}$. By the circulation constraint (3.2) and Lemma B.3, we have
\[
\frac{s_0^2}{\varepsilon^2} \cdot z_1 \pi = \frac{s^2}{2\varepsilon^2} \int_0^{2\pi} z_1 \left(1 + t_\varepsilon(\theta) + t_{\varepsilon, \tilde{\phi}_\varepsilon}(\theta)\right)^2 d\theta
+ \frac{s^3}{3\varepsilon^2} \int_0^{2\pi} \left(1 + t_\varepsilon(\theta) + t_{\varepsilon, \tilde{\phi}_\varepsilon}(\theta)\right)^3 \cos \theta d\theta
= \frac{s^2}{\varepsilon^2} \cdot z_1 \pi + O_\varepsilon \left(|t_\varepsilon(\theta) + t_{\varepsilon, \tilde{\phi}_\varepsilon}(\theta)|\right).
\]
Hence it holds
\[
\frac{|s_0 - s|}{\varepsilon} = O_\varepsilon \left(|||\phi_{\varepsilon}|||_{L^\infty(B_{Ls}(z))} + sW(s) + \varepsilon^2 |\ln \varepsilon|\right).
\]
Using Lemma 3.13, we then derive
\[
|A_\varepsilon \Delta B_{s_0}(z)| = O(\varepsilon^3 |\ln \varepsilon|).
\]
In view of Lemma C.4 in Appendix C, it holds
\[
W(s) = Wz_1 \ln \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \ln \frac{8z_1}{s_0} + \frac{\kappa}{16\pi}
+ O_\varepsilon \left(|||\phi_{\varepsilon}|||_{L^\infty(B_{Ls}(z))} + sW(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma_{\varepsilon}^{\frac{1}{2} + \frac{1}{p}}\right)
= O(\varepsilon |\ln \varepsilon|).
\]

(3.12)
So we have improved the estimate for \( \mathcal{W}(s) \) from \( O(|\ln \varepsilon|) \) to \( O(\varepsilon |\ln \varepsilon|) \).

In the second step, we combine above estimates with (3.11) to obtain
\[
\| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))} \leq \| \phi_\varepsilon \|_s = O\left(\varepsilon^2 |\ln \varepsilon| + \varepsilon \| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))}^{\frac{1}{p} + \frac{1}{q}}\right), \quad \forall p \in (2, +\infty].
\]
Now we claim
\[
\| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))} = O(\varepsilon^2 |\ln \varepsilon|). \tag{3.13}
\]
Suppose not. Then there exists a series \( \{\varepsilon_n\} \) tends to 0, such that \( \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))} > n\varepsilon_n^2 |\ln \varepsilon_n| \). Since it holds
\[
\varepsilon_n \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))}^{\frac{1}{p} + \frac{1}{q}} = \varepsilon_n \left( n\varepsilon_n^2 |\ln \varepsilon_n| \right)^{1 - \frac{1}{p}} \cdot \left( n\varepsilon_n^2 |\ln \varepsilon_n| \right)^{1 - \frac{1}{q}} \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))}^{\frac{1}{p} + \frac{1}{q}} \leq \varepsilon_n \left( n\varepsilon_n^2 |\ln \varepsilon_n| \right)^{1 - \frac{1}{q}} \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))},
\]
we can let \( p > 2 \) be sufficiently close to 2 and \( \varepsilon_n (n\varepsilon_n^2 |\ln \varepsilon_n|)^{\frac{1}{p} + \frac{1}{q}} = o_{\varepsilon_n}(1) \). According to (3.11), we have
\[
\| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))} = O(\varepsilon_n^2 |\ln \varepsilon_n|) + o_{\varepsilon_n}(1) \cdot \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))},
\]
which is a contradiction to \( \| \phi_{\varepsilon_n} \|_{L^\infty(B_{Ls}(z))} > n\varepsilon_n^2 |\ln \varepsilon_n| \), and verifies (3.13).

In the last step, we use (3.11) again, and improve the estimate for \( \phi_\varepsilon \)
\[
\| \phi_\varepsilon \|_s + \varepsilon^{1 - \frac{1}{q}} \| \nabla \phi_\varepsilon \|_{L^p(B_{Ls}(z))} = O\left(\varepsilon |\ln \varepsilon| + \varepsilon^2 |\ln \varepsilon| \right)^{\frac{1}{p} + \frac{1}{q}} = O(\varepsilon^2 |\ln \varepsilon|).
\]
Note that we have obtained \( \mathcal{W}(s) = O(\varepsilon |\ln \varepsilon|) \) in (3.12). Proceeding as the first step, we deduce
\[
|A_\varepsilon \Delta B_{s_0}(z)| = O(\varepsilon^4 |\ln \varepsilon|),
\]
and
\[
\mathcal{W}(s) = O(\varepsilon^2 |\ln \varepsilon|).
\]
Hence the proof is complete. \( \square \)

Now we can obtain the Kelvin–Hicks formula in Proposition 3.2.

**Proof of Proposition 3.2:** It holds \( |A_\varepsilon \Delta B_{s_0}(z)| = O(\varepsilon^4 |\ln \varepsilon|) \) by Lemma 3.14. Using Lemma C.4, we obtain
\[
W \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \ln \frac{8z_1}{s_0} + \frac{\kappa}{16\pi} = O(\varepsilon^2 |\ln \varepsilon|). \tag{3.14}
\]
On the other hand, we have
\[
\frac{|s_0 - s|}{\varepsilon} = O\varepsilon \left( \| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))} + s \mathcal{W}(s) + \varepsilon^2 |\ln \varepsilon| \right) = O(\varepsilon^2 |\ln \varepsilon|), \tag{3.15}
\]
and
\[
\frac{|s - \sigma_\varepsilon|}{\varepsilon} = O\varepsilon \left( \| \phi_\varepsilon \|_{L^\infty(B_{Ls}(z))} + s \mathcal{W}(s) + \varepsilon^2 |\ln \varepsilon| + \varepsilon \gamma^\frac{1}{p} \right) = O(\varepsilon^2 |\ln \varepsilon|)
\]
by Lemma 3.13. Thus we have verified Proposition 3.2. \( \square \)
3.3. The uniqueness. To show the uniqueness of \( \psi_\varepsilon \) satisfying (3.1) and (3.2), we first refine the estimate for the cross-section \( A_1 \). Notice that the value of \( s \) depends on \( \varepsilon \) and \( z_1 \) by (3.6). The following result is a direct consequence of Lemma 3.14 and Proposition 3.2.

**Lemma 3.15.** For each \( \varepsilon \in (0, \varepsilon_0] \) with \( \varepsilon_0 > 0 \) sufficiently small, let \( x^* \) be the only zero point of

\[
g(x) = Wx \ln \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \left( \ln \frac{8x}{s_0(x)} - \frac{1}{4} \right), \quad x > 0,
\]

with \( s_0(x) = (\frac{\varepsilon}{\pi x})^{1/2} \). Then we have

\[ |z_1 - x^*| = O(\varepsilon^2), \]

and

\[ s(z_1) = s(x^*) + O(\varepsilon^3 |\ln\varepsilon|). \]

**Proof.** Direct computation yields \( g'(x^*) = (W + o_{\varepsilon}(1)) \cdot |\ln\varepsilon| \). By (3.14), we have

\[ |z_1 - x^*| = O(\varepsilon^2). \]

To derive the estimate for \( s \), we can use the definition \( s_0(x) = (\frac{\varepsilon}{\pi x})^{1/2} \) and above estimate for \( z_1 \) to obtain

\[ s_0(z_1) = s_0(x^*) + O(\varepsilon^3). \]

Since \( |s(x) - s_0(x)| = O(\varepsilon^3 |\ln\varepsilon|) \) from (3.15), we then conclude

\[ s(z_1) = s(x^*) + O(\varepsilon^3 |\ln\varepsilon|) \]

by triangle inequality. \( \square \)

Suppose on the contrary there are two different \( \psi_\varepsilon^{(1)} \) and \( \psi_\varepsilon^{(2)} \) that are even symmetric with respect to \( x_1 \)-axis and solve (3.1) (3.2). Define

\[
\Theta_\varepsilon(x) := \frac{\psi_\varepsilon^{(1)}(x) - \psi_\varepsilon^{(2)}(x)}{||\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}||_{L^\infty(\mathbb{R}_+^2)}}.
\]

Then, \( \Theta_\varepsilon \) satisfies \( ||\Theta_\varepsilon||_{L^\infty(\mathbb{R}_+^2)} = 1 \) and

\[
\begin{cases}
-\varepsilon^2 x_1 \Delta^* \Theta_\varepsilon = f_\varepsilon(x), & \text{in } \mathbb{R}_+^2, \\
\Theta_\varepsilon = 0, & \text{on } x_1 = 0, \\
\Theta_\varepsilon, ||\nabla \Theta_\varepsilon||_{x_1} \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]

where

\[
f_\varepsilon(x) = \frac{x_1 \left( \frac{1}{\psi_\varepsilon^{(1)}} - \frac{1}{\psi_\varepsilon^{(2)}} \ln \frac{1}{\mu_\varepsilon^{(1)}} \right) - \frac{\varepsilon^2 ||\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}||_{L^\infty(\mathbb{R}_+^2)}}{\varepsilon^2 ||\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}||_{L^\infty(\mathbb{R}_+^2)}} \right)}{\varepsilon^2 ||\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}||_{L^\infty(\mathbb{R}_+^2)}}.
\]

We see that \( f_\varepsilon(x) = 0 \) in \( \mathbb{R}_+^2 \setminus B_{Ls^{(1)}}(z^{(1)}) \) for some large \( L > 0 \) due to Lemma 3.15.

In the following, we are to obtain a series of estimates for \( \Theta_\varepsilon \) and \( f_\varepsilon \). Then we will derive a contradiction by local Pohozaev identity whenever \( \psi_\varepsilon^{(1)} \neq \psi_\varepsilon^{(2)} \). For simplicity, we always use \( | \cdot |_\infty \) to denote \( || \cdot ||_{L^\infty(\mathbb{R}_+^2)} \), and abbreviate the parameters \( s^{(1)} \) as \( s \), \( z^{(1)} \) as \( z \).
Lemma 3.16. For $p \in (2, \infty]$ and any large $L > 0$, it holds
\[ ||s^2 f_\varepsilon(sy + z)||_{W^{-1,p}(B_L(0))} = O_\varepsilon(1). \]
Moreover, as $\varepsilon \to 0$, for all $\tilde{\varphi} \in C^\infty_0(\mathbb{R}^2)$ it holds
\[ \int_{\mathbb{R}^2} s^2 f_\varepsilon(sy + z) \tilde{\varphi} dy = \frac{2}{z_1} \int_{|y|=1} \left( b_\varepsilon \cdot \frac{\partial w}{\partial y_1} + O(\varepsilon) \right) \tilde{\varphi}, \]
where $b_\varepsilon$ is bounded independent of $\varepsilon$, and $w$ is defined by
\[ w(y) = \begin{cases} \frac{1}{2} (1 - |y|^2), & |y| \leq 1, \\ \frac{1}{2} \ln |y|, & |y| \geq 1. \end{cases} \]

Proof. Let \[ \tilde{\Gamma}_\varepsilon^{(i)} := \left\{ y \mid \psi_\varepsilon^{(i)}(sy + z^{(i)}) - \frac{W}{2} (sy_1 + z_1^{(i)})^2 \ln \frac{1}{\varepsilon} \cdot e_1 = \mu_\varepsilon^{(i)} \right\}, \quad i = 1, 2. \]
We take \[ y_\varepsilon^{(1)} = (1 + t_\varepsilon^{(1)}(\theta)) (\cos \theta, \sin \theta) \in \tilde{\Gamma}_\varepsilon^{(1)} \]
with $|t_\varepsilon^{(1)}(\theta)| = O(\varepsilon^2 |\ln \varepsilon|)$ by Lemma 3.14. Similarly, there is a $t_\varepsilon^{(2)}$ satisfying $|t_\varepsilon^{(2)}(\theta)| = O(\varepsilon^2 |\ln \varepsilon|)$ such that
\[ y_\varepsilon^{(2)} = (1 + t_\varepsilon^{(2)}(\theta)) (\cos \theta, \sin \theta) \in \tilde{\Gamma}_\varepsilon^{(2)}. \]
We will take $z^{(1)}$ and $z^{(2)}$ as a same point $z = z^{(1)}$ in the following. As a cost, this leads to some loss on the estimate of $t_\varepsilon^{(2)}(\theta)$: since $|z_1^{(i)} - x^*| = O(\varepsilon^2)$ from Lemma 3.15, we only have
\[ |t_\varepsilon^{(2)}(\theta)| = O(\varepsilon) \]
by letting $z^{(2)}$ coincide with $z^{(1)}$.
Using the definition of $\tilde{\Gamma}_\varepsilon^{(i)}$ and the estimate
\[ \mathcal{W}(s) = O(\varepsilon^2 |\ln \varepsilon|) \]
obtained from Lemma 3.14, we have
\[
\psi_\varepsilon^{(1)}(sy_\varepsilon^{(2)} + z) - \psi_\varepsilon^{(2)}(sy_\varepsilon^{(2)} + z) \\
= \psi_\varepsilon^{(1)}(sy_\varepsilon^{(2)} + z) - \psi_\varepsilon^{(1)}(sy_\varepsilon^{(1)} + z) + \psi_\varepsilon^{(1)}(sy_\varepsilon^{(1)} + z) - \psi_\varepsilon^{(2)}(sy_\varepsilon^{(2)} + z) \\
= \psi_\varepsilon^{(1)}(sy_\varepsilon^{(2)} + z) - \psi_\varepsilon^{(1)}(sy_\varepsilon^{(1)} + z) - (\mu_\varepsilon^{(2)} - \mu_\varepsilon^{(1)}) \\
- W \left( sy_\varepsilon^{(2)} + z_1 \right)^2 \ln \frac{1}{\varepsilon} + W \left( sy_\varepsilon^{(1)} + z_1 \right)^2 \ln \frac{1}{\varepsilon} \\
= (-s\mathcal{N} + O(\varepsilon^2 |\ln \varepsilon|)) (t_\varepsilon^{(2)}(\theta) - t_\varepsilon^{(1)}(\theta)) - (\mu_\varepsilon^{(2)} - \mu_\varepsilon^{(1)}),
\]
with
\[ \mathcal{N} = \frac{s}{2\varepsilon^2} \cdot z_1^2. \]
in (3.7) as the value of $|\nabla V_{x, \varepsilon}|$ at $|x - z| = s$. Thus it holds
\[
\begin{aligned}
t^{(2)}_{\varepsilon}(\theta) - t^{(1)}_{\varepsilon}(\theta) &= (-sN + O(\varepsilon^2|\ln\varepsilon|)) \times \left( \psi^{(1)}_{1, \varepsilon}(s\mathbf{y}^{(2)}_{\varepsilon} + z) - \psi^{(2)}_{1, \varepsilon}(s\mathbf{y}^{(1)}_{\varepsilon} + z) - (\mu^{(2)}_{\varepsilon} - \mu^{(2)}_{\varepsilon}) \right), \tag{3.16}
\end{aligned}
\]

On the other hand, the circulation constraint (3.2) yields
\[
\begin{aligned}
\kappa &= \frac{s^2}{2\varepsilon^2} \int_{0}^{2\pi} z_1 (1 + t^{(1)}_{\varepsilon}(\theta))^2 d\theta + \frac{s^3}{3\varepsilon^2} \int_{0}^{2\pi} (1 + t^{(1)}_{\varepsilon}(\theta))^3 \cos\theta d\theta \\
&= \frac{s^2}{2\varepsilon} \int_{0}^{2\pi} z_1 (1 + t^{(2)}_{\varepsilon}(\theta))^2 d\theta + \frac{s^3}{3\varepsilon^2} \int_{0}^{2\pi} (1 + t^{(2)}_{\varepsilon}(\theta))^3 \cos\theta d\theta,
\end{aligned}
\]

and hence
\[
\int_{0}^{2\pi} z_1 (t^{(2)}_{\varepsilon}(\theta) - t^{(1)}_{\varepsilon}(\theta)) \left( 1 + \frac{1}{2} t^{(1)}_{\varepsilon}(\theta) + \frac{1}{2} t^{(2)}_{\varepsilon}(\theta) + O(\varepsilon) \right) d\theta = 0.
\]

It follows that
\[
\begin{aligned}
\int_{0}^{2\pi} (sN + O(\varepsilon^2|\ln\varepsilon|)) \left( \psi^{(1)}_{1, \varepsilon}(s\mathbf{y}^{(2)}_{\varepsilon} + z) - \psi^{(2)}_{1, \varepsilon}(s\mathbf{y}^{(1)}_{\varepsilon} + z) \right) (2 + t^{(1)}_{\varepsilon}(\theta) + t^{(2)}_{\varepsilon}(\theta) + O(\varepsilon)) d\theta \\
= (\mu^{(1)}_{\varepsilon} - \mu^{(2)}_{\varepsilon}) \int_{0}^{2\pi} (sN + O(\varepsilon^2|\ln\varepsilon|)) (2 + t^{(1)}_{\varepsilon}(\theta) + t^{(2)}_{\varepsilon}(\theta) + O(\varepsilon)) d\theta,
\end{aligned}
\]

which implies
\[
\frac{|\mu^{(1)}_{\varepsilon} - \mu^{(2)}_{\varepsilon}|}{|\psi^{(1)}_{\varepsilon} - \psi^{(2)}_{\varepsilon}|_{\infty}} = O(1),
\]

and
\[
\frac{|t^{(2)}_{\varepsilon}(\theta) - t^{(1)}_{\varepsilon}(\theta)|}{|\psi^{(1)}_{\varepsilon} - \psi^{(2)}_{\varepsilon}|_{\infty}} = O(1)
\]

by (3.16).

We then define the normalized difference of $\psi^{(i)}_{\varepsilon} - \mu^{(i)}_{\varepsilon}$ as
\[
\Theta_{\varepsilon, \mu} := \frac{\psi^{(1)}_{\varepsilon} - \mu^{(1)}_{\varepsilon}}{|\psi^{(1)}_{\varepsilon} - \psi^{(2)}_{\varepsilon}|_{\infty}} - \frac{\psi^{(2)}_{\varepsilon} - \mu^{(2)}_{\varepsilon}}{|\psi^{(1)}_{\varepsilon} - \psi^{(2)}_{\varepsilon}|_{\infty}}.
\]

Recall that for a general function $v$, we denote $\tilde{v}(\mathbf{y}) = v(s\mathbf{y} + z)$, and $D_s = \{ \mathbf{y} \mid s\mathbf{y} + z \in \mathbb{R}^2_+ \}$. $\tilde{\Theta}_{\varepsilon, \mu}$ will satisfy the equation
\[
-\text{div} \left( \frac{\nabla \tilde{\Theta}_{\varepsilon, \mu}}{s \mathbf{y}_1 + z_1} \right) = \tilde{f}_{\varepsilon}(\mathbf{y}), \quad \text{in } D_s.
\]
For any $\varphi \in C_0^\infty(B_L(z))$ and $p' \in [1, 2)$, we have

\[
\int_{\mathbb{R}^2} s^2 f_\varepsilon(sy + z) \tilde{\varphi} dy = -s^2 \int_{|y|=1} \varepsilon^2 |\tilde{\psi}_\varepsilon^{(1)}(y, z) - \tilde{\psi}_\varepsilon^{(2)}(y, z)| |\tilde{\varphi}(y) - \tilde{\varphi}(y)| dy
\]

with $w$ defined in the statement of lemma, and

\[
b_\varepsilon = \left( \int_{B_L(0)} \tilde{\Theta}_{\varepsilon, \mu}(y) \cdot (-\Delta) \frac{\partial w}{\partial y_1} dy \right) \left( \int_{B_L(0)} \frac{\partial w}{\partial y_1} \cdot (-\Delta) \frac{\partial w}{\partial y_1} dy \right)^{-1}
\]
as the projection coefficient bounded independent of \( \varepsilon \). Then for any \( \varphi \in C^\infty_0(B_{L_2}(z)) \), \( \tilde{\Theta}_{\varepsilon, \mu}^* \) satisfies

\[
\int_{B_L(0)} \frac{1}{sy_1 + z_1} \cdot \nabla \tilde{\Theta}_{\varepsilon, \mu}^* \cdot \nabla \tilde{\varphi} dy - \frac{2}{z_1} \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu}^* \tilde{\varphi} = -b_\varepsilon \left( \int_{B_L(0)} \frac{1}{sy_1 + z_1} \cdot \nabla \frac{\partial w}{\partial y_1} \cdot \nabla \tilde{\varphi} dy - \frac{2}{z_1} \int_{|y|=1} \frac{\partial w}{\partial y_1} \tilde{\varphi} \right) + \left( \int_{B_L(0)} s^2 \tilde{f}_\varepsilon \tilde{\varphi} dy - \frac{2}{z_1} \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu} \tilde{\varphi} \right)
\]

Since the kernel of \( I \) we deduce \( I_1 = O(\varepsilon) \cdot \| \tilde{\varphi} \|_{W^{1, \nu'}}(B_L(0)) \). For the term \( I_2 \), using (3.16) and the estimate \( |\eta_{\varepsilon}(\theta)| = O(\varepsilon) \), we have

\[
I_2 = -\frac{s^2}{\varepsilon^2|\psi_{\varepsilon}^{(1)} - \psi_{\varepsilon}^{(2)}|_\infty} \int_0^{2\pi} \int_{1+t_\varepsilon^{(2)}}^{1+t_\varepsilon^{(1)}} (z_1 + t \cos \theta) t \tilde{\varphi}(t, \theta) dt d\theta - \frac{2}{z_1} \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu} \tilde{\varphi} \\
= -\frac{s^2}{\varepsilon^2|\psi_{\varepsilon}^{(1)} - \psi_{\varepsilon}^{(2)}|_\infty} \int_0^{2\pi} \int_{1+t_\varepsilon^{(2)}}^{1+t_\varepsilon^{(1)}} (z_1 + t \cos \theta) t (\tilde{\varphi}(t, \theta) - \tilde{\varphi}(1, \theta)) dt d\theta \\
- \frac{s^2}{\varepsilon^2|\psi_{\varepsilon}^{(1)} - \psi_{\varepsilon}^{(2)}|_\infty} \int_0^{2\pi} \int_{1+t_\varepsilon^{(2)}}^{1+t_\varepsilon^{(1)}} (z_1 + t \cos \theta) t \tilde{\varphi}(1, \theta) dt d\theta - \frac{2}{z_1} \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu} \tilde{\varphi} \\
= -\frac{s^2(1 + o_{\varepsilon}(1))}{\varepsilon^2|\psi_{\varepsilon}^{(1)} - \psi_{\varepsilon}^{(2)}|_\infty} \int_{1+t_\varepsilon^{(2)}}^{1+t_\varepsilon^{(1)}} z_1 t(t - 1) \nabla \tilde{\varphi}((1 + \sigma(t - 1))y) \cdot y d\sigma dt dy \\
+ \left( \frac{2}{z_1} + O(\varepsilon^2 |\ln \varepsilon|) \right) \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu} (1 + O_{\varepsilon}(t_\varepsilon^{(2)})) \tilde{\varphi} \\
- \frac{2}{z_1} \int_{|y|=1} \tilde{\Theta}_{\varepsilon, \mu} \tilde{\varphi} + O(\varepsilon) \cdot \| \tilde{\varphi} \|_{W^{1, \nu'}(B_L(0))} \\
= O(\varepsilon) \cdot \| \tilde{\varphi} \|_{W^{1, \nu'}(B_L(0))}.
\]

Actually, we can regard the left hand side of (3.18) as the weak form of linear operator

\[
\mathbb{L}_s^* v = -\text{div} \left( \frac{\nabla v}{sy_1 + z_1} \right) - \frac{2}{z_1} v(1, \theta) \delta_{|y|=1}
\]

acting on \( \tilde{\Theta}_{\varepsilon, \mu}^* \). Since both \( \tilde{\Theta}_{\varepsilon, \mu} \) and \( \tilde{\Theta}_{\varepsilon, \mu}^* \) are even with respect to \( x_1 \)-axis, the kernel of \( \mathbb{L}_s^* \) is then approximated by \( \partial w/\partial y_1 \). Consequently, if a function \( v^* \in W^{-1, \nu}(B_L(0)) \) with
\[ p \in (2, +\infty) \] satisfies orthogonality condition
\[
\int_{B_L(0)} v^* \cdot (-\Delta) \frac{\partial w}{\partial y_1} \, dy = 0,
\]
then it holds following local coercive estimate
\[
\|v^*\|_{L^\infty(B_L(0))} + \|\nabla v^*\|_{L^p(B_L(0))} \leq C\|L^*_p v^*\|_{W^{-1,p}(B_L(0))}, \quad \forall p \in (2, +\infty],
\]
which is verified in the proof of Lemma 2.3. Since \( \tilde{\Theta}_{\varepsilon, \mu}^* \) satisfy the orthogonality condition by projection (3.17), we deduce from the estimates for \( I_1, I_2 \) that
\[
\|\tilde{\Theta}_{\varepsilon, \mu}^*\|_{L^\infty(B_L(0))} + \|\nabla \tilde{\Theta}_{\varepsilon, \mu}^*\|_{L^p(B_L(0))} = O(\varepsilon), \quad \forall p \in (2, +\infty].
\]

Now we arrive at a conclusion: by the definition of \( \tilde{\Theta}_{\varepsilon, \mu}^* \) in (3.17), for each \( p \in (2, +\infty] \), it holds
\[
\tilde{\Theta}_{\varepsilon, \mu} = b_\varepsilon \frac{\partial w}{\partial y_1} + O(\varepsilon), \quad \text{in} \ W^{1,p}(B_L(0)),
\]
and for all \( \tilde{\varphi} \in C_0^\infty(\mathbb{R}^2) \), it holds
\[
\int_{\mathbb{R}^2} s^2 f_\varepsilon(sy + z) \tilde{\varphi} \, dy = \frac{2}{z_1} \int_{|y|=1} \left( b_\varepsilon \frac{\partial w}{\partial y_1} + O(\varepsilon) \right) \tilde{\varphi},
\]
where \( b_\varepsilon \) is bounded independent of \( \varepsilon \). So we have completed the proof of Lemma 3.16. \( \square \)

To make use of the local Pohozaev identity in Appendix C and obtain a contradiction, we let
\[
\xi_\varepsilon(x) := \frac{\psi_{1,\varepsilon}^{(1)}(x) - \psi_{1,\varepsilon}^{(2)}(x)}{|\psi_{1,\varepsilon}^{(1)} - \psi_{1,\varepsilon}^{(2)}|_\infty}
\]
be the normalized difference of \( \psi_{1,\varepsilon}^{(1)}(x) \) and \( \psi_{1,\varepsilon}^{(1)}(x) \). Then \( \xi_\varepsilon \) has the following integral representation
\[
\xi_\varepsilon = z_1^2 \int_{\mathbb{R}^2_+} G(x, x') \cdot x_1^{t-1} f_\varepsilon(x') \, dx'. \tag{3.19}
\]
By the asymptotic estimate for \( f_\varepsilon(sy + z) \) in Lemma 3.16, it holds
\[
\left| \frac{\psi_{2,\varepsilon}^{(1)}(x) - \psi_{2,\varepsilon}^{(2)}(x)}{|\psi_{2,\varepsilon}^{(1)} - \psi_{2,\varepsilon}^{(2)}|_\infty} \right| = \int_{\mathbb{R}^2_+} H(x, x') \cdot x_1^{t-1} f_\varepsilon(x') \, dx' = o_\varepsilon(1).
\]
So we see that \( \xi_\varepsilon \) is the main part in \( \Theta_\varepsilon \), and \( ||\xi_\varepsilon||_{L^\infty(\mathbb{R}^2_+)} = 1 - o_\varepsilon(1) \). To derive a contradiction and obtain uniqueness, we only have to show \( ||\xi_\varepsilon||_{L^\infty(\mathbb{R}^2_+)} = o_\varepsilon(1) \).

For the purpose of dealing with boundary terms in the local Pohozaev identity, we need the following lemma concerning the behavior of \( \xi_\varepsilon \) away from \( z \).
Lemma 3.17. For any large $L > 0$, it holds
\[
\xi_{\varepsilon}(x) = B_{\varepsilon} \cdot \frac{s z^2}{2\pi} \frac{x_1 - z_1}{|x - z|^2} + B_{\varepsilon} \cdot \frac{s z^2}{2\pi} \frac{x_1 + z_1}{|x + z|^2} + B_{\varepsilon} \cdot \frac{s z_2}{2\pi} \ln \frac{|x - \tilde{z}|}{|x - z|} + O(\varepsilon^2),
\]
(3.20)
in $C^1(\mathbb{R}^2_+ \setminus B_{\delta/2}(z))$, where $\delta > 0$ is the small constant in (C.1), and
\[
B_{\varepsilon} := \frac{1}{s} \int_{B_{2s}(z)} (x_1 - z_1) x_1^{-1} f_{\varepsilon}(x) dx
\]
is bounded independent of $\varepsilon$.

Proof. Since $\xi_{\varepsilon}$ is symmetric with respect to $x_1$-axis, for $x \in \mathbb{R}^2_+ \setminus B_{\delta/2}(z)$ we have
\[
\xi_{\varepsilon}(x) = \frac{z_1^2}{2\pi} \int_{\mathbb{R}^2_+} x_1^{-1} \ln \left( \frac{|x - x'|}{|x - x'|} \right) f_{\varepsilon}(x') dx' = \frac{z_1^2}{2\pi} \int_{B_{Ls}(x)} x_1^{-1} \ln \left( \frac{|x - x'|}{|x - x'|} \right) f_{\varepsilon}(x') dx'
\]
\[
= \frac{z_1^2}{2\pi} \ln \frac{1}{|x - z|} \int_{B_{Ls}(x)} f_{\varepsilon}(x') dx' + \frac{z_1^2}{2\pi} \int_{B_{Ls}(x)} x_1^{-1} \ln \left( \frac{|x - z|}{|x - x'|} \right) f_{\varepsilon}(x') dx'
\]
\[
- \frac{z_1^2}{2\pi} \ln \frac{1}{|x - z|} \int_{B_{Ls}(x)} f_{\varepsilon}(x') dx' - \frac{z_1^2}{2\pi} \int_{B_{Ls}(x)} x_1^{-1} \ln \left( \frac{|x - z'|}{|x - x'|} \right) f_{\varepsilon}(x') dx'
\]
\[
= \frac{z_1^2}{2\pi} \ln \frac{|x - z|}{|x - z|} \int_{B_{Ls}(x)} (x_1 - z_1) x_1^{-1} f_{\varepsilon}(x) dx
\]
\[
= \frac{z_1^2}{2\pi} \ln \frac{|x - z|}{|x - z|} \int_{B_{Ls}(x)} (x_1 - z_1) x_1^{-1} f_{\varepsilon}(x) dx
\]
Moreover, $B_{\varepsilon}$ is bounded independent of $\varepsilon$ since $||s^2 f_{\varepsilon}(s y + z)||_{W^{-1,p}(B_{Ls}(0))} = O_{\varepsilon}(1)$ for $p \in [2, \infty)$. Then we can verify (3.20) in $C^1(\mathbb{R}^2_+ \setminus B_{\delta/2}(z))$ by a same argument. □

If we apply (C.1) in Appendix C to $\psi_{1,\varepsilon}^{(1)}$ and $\psi_{1,\varepsilon}^{(2)}$ separately and calculate their difference, we can obtain the following local Pohozaev identity:
\[
- \int_{\partial B_{Ls}(x)} \frac{\partial \xi_{\varepsilon}}{\partial \nu} \frac{\partial \psi_{1,\varepsilon}^{(1)}}{\partial x_1} dS - \int_{\partial B_{Ls}(x)} \frac{\partial \psi_{1,\varepsilon}^{(2)}}{\partial \nu} \frac{\partial \xi_{\varepsilon}}{\partial x_1} dS + \frac{1}{2} \int_{\partial B_{Ls}(x)} \langle \nabla (\psi_{1,\varepsilon}^{(1)} + \psi_{1,\varepsilon}^{(2)}), \nabla \xi_{\varepsilon} \rangle \nu_1 dS
\]
\[
= - \frac{z_1^2}{\varepsilon^2 |\psi_{1,\varepsilon}^{(1)} - \psi_{1,\varepsilon}^{(2)}|_\infty} \int_{B_{Ls}(x)} \left( \partial_1^{(1)} \psi_{2,\varepsilon}^{(1)} \cdot 1_{A_{1}^{(1)}} - \partial_1^{(2)} \psi_{2,\varepsilon}^{(2)} \cdot 1_{A_{1}^{(2)}} \right) dx.
\]
(3.21)
The proof of the uniqueness of a vortex ring with small cross-section is based on a careful estimate for each term in (3.21).

Proof of Proposition 3.1: Using the asymptotic estimate for $\psi_{1,\varepsilon}$ in Lemma C.2 and $\xi_\varepsilon$ in Lemma 3.17, we see that

$$
\int_{\partial B_\delta(z)} \frac{\partial \xi_\varepsilon}{\partial \nu} \frac{\partial \psi_{1,\varepsilon}(1,\varepsilon)}{\partial x_1} dS + \int_{\partial B_\delta(z)} \frac{\partial \psi_{1,\varepsilon}(2,\varepsilon)}{\partial \nu} \frac{\partial \xi_\varepsilon}{\partial x_1} dS = \frac{1}{2} \int_{\partial B_\delta(z)} \langle \nabla (\psi_{1,\varepsilon}(1) + \psi_{1,\varepsilon}(2)), \nabla \xi_\varepsilon \rangle \nu_1 dS
$$

(3.22)

To deal with the right hand side of (3.21), we write

$$
\int_{\partial B_\delta(z)} \frac{z_1^2}{\varepsilon^2 |\nu_{1,\varepsilon} - |x - x'||} \int_{B_\delta(z)} \left( \partial_1 \psi_{2,\varepsilon}(1) - \partial_1 \psi_{2,\varepsilon}(2) \right) d\mathbf{x}
$$

$$
= \int_{\partial B_\delta(z)} \frac{z_1^2}{\varepsilon^2 |\nu_{1,\varepsilon} - |x - x'||} \int_{B_\delta(z)} \left( \partial_1 \psi_{2,\varepsilon}(1) - 1_{A_\varepsilon^{(1)}}(1_{A_\varepsilon^{(2)}}) + 1_{A_\varepsilon^{(2)}}(\partial_1 \psi_{2,\varepsilon} - \partial_1 \psi_{2,\varepsilon}(2)) \right) d\mathbf{x}
$$

$$
= G_1 + G_2,
$$

and

$$
G_1 = \int_{\partial B_\delta(z)} \frac{z_1^2}{\varepsilon^2} \frac{1}{f_{\varepsilon}(\mathbf{x})} \int_{B_\delta(z)} \partial_1 H(x, x') \cdot 1_{A_\varepsilon^{(1)}}(1_{A_\varepsilon^{(2)}}) d\mathbf{x} = G_{11} + G_{12} + G_{13} + G_{14},
$$

where

$$
G_{11} = \frac{z_1^2}{4\pi\varepsilon^2} \cdot \ln \left( \frac{1}{\varepsilon} \right) \int_{B_\delta(z)} \frac{1}{x_1} \int_{A_\varepsilon^{(1)}} x_1^{3/2} f_{\varepsilon}(\mathbf{x}) d\mathbf{x} d\mathbf{x}',
$$

$$
G_{12} = \frac{z_1^2}{4\pi\varepsilon^2} \cdot \int_{B_\delta(z)} \frac{1}{x_1} \int_{A_\varepsilon^{(1)}} x_1^{3/2} f_{\varepsilon}(\mathbf{x}) d\mathbf{x}' d\mathbf{x},
$$

$$
G_{13} = -\frac{z_1^2}{2\pi\varepsilon^2} \cdot \int_{B_\delta(z)} \frac{1}{x_1} \int_{A_\varepsilon^{(1)}} \left( \frac{1}{x_1^{1/2} x_1^{3/2} - x_1^{1/2} x_1^{3/2} - z_1^2} \right) \cdot \frac{x_1 - x_1'}{|x - x'|^2} d\mathbf{x}' d\mathbf{x},
$$

$$
G_{14} = O(\varepsilon) \cdot B_{\varepsilon} + O(\varepsilon^2).
$$
and $G_{14}$ a regular term. Using the circulation constraint (3.2) and Lemma 3.16, we have

$$G_{11} = \frac{z_1^2}{4\pi} \cdot \ln \left( \frac{1}{s} \right) \cdot \int_{B_3(z)} x_1^{-3/2} f_\varepsilon \cdot \frac{1}{s^2} \int_{\Omega_1^{(1)}} x_1' \left( z_1'^{1/2} + O(\varepsilon) \right) d\mathbf{x}' d\mathbf{x}$$

$$= \frac{\kappa z_1^2}{4\pi} \cdot \left( z_1^{1/2} + O(\varepsilon) \right) \cdot \ln \left( \frac{1}{s} \right) \cdot \int_{B_3(z)} x_1^{-3/2} f_\varepsilon(x) d\mathbf{x}$$

$$= \frac{\kappa z_1^2}{4\pi} \cdot \left( z_1^{1/2} + O(\varepsilon) \right) \cdot \ln \left( \frac{1}{s} \right) \cdot \int_{B_3(z)} f_\varepsilon \cdot \left( z_1^{-3/2} - \frac{3}{2z_1^{5/2}} \cdot (x_1 - z_1) + O(\varepsilon^2) \right) d\mathbf{x}$$

$$= \frac{\kappa z_1^2}{4\pi} \cdot \left( z_1^{1/2} + O(\varepsilon) \right) \cdot \ln \left( \frac{1}{s} \right) \cdot \int_{B_3(z)} \left( \frac{3}{2z_1^{5/2}} \cdot s y_1 + O(\varepsilon^2) \right) s^2 f_\varepsilon(sy + z) d\mathbf{y}$$

$$= \frac{\kappa z_1^2}{4\pi} \cdot \left( z_1^{1/2} + O(\varepsilon) \right) \cdot \ln \left( \frac{1}{s} \right) \cdot \int_{|y|=1} \left( \frac{3}{2z_1^{5/2}} \cdot s y_1 + O(\varepsilon^2) \right) \left( b_\varepsilon \cdot \frac{y_1}{z_1|y|^2} + O(\varepsilon) \right) d\mathbf{y}$$

$$= -\frac{3\kappa}{8z_1} \cdot b_\varepsilon s \ln \left( \frac{1}{s} \right) + O(\varepsilon).$$

For the term $G_{12}$, it holds

$$G_{12} = \frac{z_1^2}{4\pi \varepsilon^2} \int_{B_3(z)} \left( z_1^{-3/2} + O(\varepsilon) \right) f_\varepsilon \int_{A_1^{(1)}} \left( z_1^{3/2} + O(\varepsilon) \right) \ln \left( \frac{s}{|\mathbf{x} - \mathbf{x}'|} \right) d\mathbf{x}' d\mathbf{x}$$

$$= \frac{z_1^2}{4\pi \varepsilon^2} \int_{B_3(z)} f_\varepsilon \int_{B_3(z)} \ln \left( \frac{s}{|\mathbf{x} - \mathbf{x}'|} \right) d\mathbf{x}' d\mathbf{x} + O(\varepsilon)$$

$$= \frac{z_1^2 s^2}{4\pi \varepsilon^2} \int_{|y|=1} \left( b_\varepsilon \cdot \frac{y_1}{z_1|y|^2} + O(\varepsilon) \right) \left( \int_{B_1(0)} \ln \left( \frac{1}{|\mathbf{y} - \mathbf{y}'|} \right) d\mathbf{y}' \right) + O(\varepsilon)$$

$$= O(\varepsilon),$$

where we have used the formula of Rankine vortex

$$\frac{1}{2\pi} \int_{B_1(0)} \ln \left( \frac{1}{|\mathbf{y} - \mathbf{y}'|} \right) d\mathbf{y}' = \left\{ \begin{array}{ll} \frac{1}{2}(1 - |\mathbf{y}|^2), & |\mathbf{y}| \leq 1, \\ \frac{1}{2} \ln \frac{1}{|\mathbf{y}|}, & |\mathbf{y}| \geq 1. \end{array} \right.$$
is a bounded function even symmetric with respect to $y_1 = 0$. While $\partial w / \partial y_1$ is odd symmetric with respect to $y_1 = 0$. Hence it holds

$$G_{13} = -\frac{z_1^2 s^2}{4\pi \varepsilon^2} \int_{|y_1|=1} \left( \frac{2}{z_1} \cdot b_\varepsilon \cdot \frac{\partial w}{\partial y_1} + O(\varepsilon) \right) g(y) + O(\varepsilon) = O(\varepsilon).$$

For the regular term $G_{14}$, it is easy to verify that $G_{14} = O(\varepsilon)$. Summarizing all the estimates above, we get

$$G_1 = -\frac{3\kappa}{8z_1} \cdot b_\varepsilon s \ln \left( \frac{1}{s} \right) + O(\varepsilon). \quad (3.23)$$

Then we turn to deal with $G_2$. Using Fubini’s theorem, we have

$$G_2 = \frac{z_1^2}{\varepsilon^4 |\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}|_\infty} \int_{A_\varepsilon^{(1)}} \left( \int_{A_\varepsilon^{(2)}} \partial_{x_1} H(x, x') dx' - \int_{A_\varepsilon^{(2)}} \partial_{x_1} H(x, x') dx' \right) dx$$

$$= \frac{z_1^2}{\varepsilon^4 |\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}|_\infty} \int_{B_\varepsilon(z)} \left( 1_{A_\varepsilon^{(1)}} - 1_{A_\varepsilon^{(2)}} \right) \int_{A_\varepsilon^{(2)}} \partial_{x_1} H(x, x') dx' dx$$

$$= \frac{z_1^2}{\varepsilon^4 |\psi_\varepsilon^{(1)} - \psi_\varepsilon^{(2)}|_\infty} \int_{B_\varepsilon(z)} \left( 1_{A_\varepsilon^{(1)}} - 1_{A_\varepsilon^{(2)}} \right) \partial_1 \psi_\varepsilon^{(2)} \, dx.$$

Due to the dual formulation of $G_1$ and $G_2$, we claim

$$G_2 = -\frac{3\kappa}{8z_1} \cdot b_\varepsilon s \ln \left( \frac{1}{s} \right) + O(\varepsilon). \quad (3.24)$$

Now from (3.22) (3.23) (3.24), we have

$$\frac{3\kappa}{4z_1} \cdot b_\varepsilon s \ln \left( \frac{1}{s} \right) = O(\varepsilon). \quad (3.25)$$

Since $z_1$ is near $x^* > 0$ defined in Lemma 3.15, and $s \ln (1/s) = O(\varepsilon |\ln \varepsilon|)$, we can derive from (3.25) that

$$b_\varepsilon = O \left( \frac{1}{|\ln \varepsilon|} \right).$$

According to Lemma 3.16, we can also use the fact that for fixed $y \in \mathbb{R}^2$ it holds

$$\frac{1}{2\pi} \ln \left( \frac{1}{|y - \cdot|} \right) \in W^{1,p'}_{\text{loc}}(\mathbb{R}^2), \quad \forall p' \in [1, 2).$$
and deduce
\[
\tilde{\xi}_\varepsilon(y) = \frac{z_1}{2\pi} \int_{\mathbb{R}^2^+} \ln \left( \frac{1}{s|y - y'|} \right) \cdot \left( 1 - \frac{s y_1'}{z_1} \right) s^2 f_\varepsilon(s y' + z) dy' + O \left( \frac{1}{|\ln \varepsilon|} \right)
\]
\[
= \frac{1}{\pi} \int_{|y'|=1} \ln \left( \frac{1}{|y - y'|} \right) \cdot \left( 1 - \frac{s y_1'}{z_1} \right) \left( b_\varepsilon \cdot \frac{\partial w(y')}{\partial y_1} + O(\varepsilon) \right)
\]
\[
+ \frac{1}{\pi} \ln \left( \frac{1}{s} \right) \cdot \int_{|y'|=1} \left( b_\varepsilon \cdot \frac{\partial w(y')}{\partial y_1} + O(\varepsilon) \right) + O \left( \frac{1}{|\ln \varepsilon|} \right)
\]
\[
= O \left( \frac{1}{|\ln \varepsilon|} \right).
\]
Thus we conclude $||\xi_\varepsilon||_{L^\infty(\mathbb{R}^2_+)} = O(1/|\ln \varepsilon|)$, which is a contradiction to $||\xi_\varepsilon||_{L^\infty(\mathbb{R}^2_+)} = 1 - o_\varepsilon(1)$. By the discussion given before Lemma 3.17, we have verified the uniqueness of $\psi_\varepsilon$ for $\varepsilon > 0$ small, which means the vortex ring $\zeta_\varepsilon$ with assumptions in Proposition 3.1 is unique.

\section*{4. Stability}

In this section, we study nonlinear orbital stability of the steady vortex ring $\zeta_\varepsilon$ constructed in Theorem 1.1. We will provide the proof of Theorem 1.4. The key idea is to build a bridge between the existence result of [7, 12] based on variational method and the uniqueness result established in the preceding section in order to apply the concentration-compactness principle of Lions [26] to a maximizing sequence.

4.1. Variational setting. Let $\kappa$ and $W$ be as in Theorem 1.1. We now show that $\zeta_\varepsilon$ enjoys a variational characteristic. We set the space of admissible functions
\[
\mathcal{A}_\varepsilon := \{ \zeta \in L^\infty(\mathbb{R}^3) \mid \zeta : \text{axi-symmetric, } 0 \leq \zeta \leq 1/\varepsilon^2, ||\zeta||_{L^1(\mathbb{R}^3)} \leq 2\pi\kappa \}.
\]
We shall consider the maximization problem:
\[
\mathcal{E}_\varepsilon = \sup_{\zeta \in \mathcal{A}_\varepsilon} \left( E[\zeta] - W \ln \frac{1}{\varepsilon} \mathcal{P}[\zeta] \right).
\]
Denote by $\mathcal{S}_\varepsilon \subset \mathcal{A}_\varepsilon$ the set of maximizers of (4.1). Note that any $z$-directional translation of $\zeta \in \mathcal{S}_\varepsilon$ is still in $\mathcal{S}_\varepsilon$.

The following result is essentially contained in [7, 12].

\textbf{Proposition 4.1.} If $\varepsilon > 0$ is sufficiently small, then $\mathcal{S}_\varepsilon \neq \emptyset$ and each maximizer $\hat{\zeta}_\varepsilon \in \mathcal{S}_\varepsilon$ is a steady vortex ring with circulation $\kappa$ and translational velocity $W \ln \varepsilon \mathbf{e}_z$. Furthermore,
\begin{enumerate}
  \item $\hat{\zeta}_\varepsilon = \varepsilon^{-2} \mathbf{1}_{\hat{\Omega}_\varepsilon}$ for some axi-symmetric topological torus $\hat{\Omega}_\varepsilon \subset \mathbb{R}^3$.
  \item It holds $C_1 \varepsilon \leq \sigma \left( \hat{\Omega}_\varepsilon \right) < C_2 \varepsilon$ for some constants $0 < C_1 < C_2$.
  \item As $\varepsilon \to 0$, $\text{dist}_{C^{*}\varepsilon}(\hat{\Omega}_\varepsilon) \to 0$ with $r^* = \kappa/4\pi W$.
\end{enumerate}
If \( \zeta \in S_\varepsilon \) for \( \varepsilon > 0 \) small, then it must be symmetric with respect to some horizontal line \( x_2 = h \) by Steiner symmetrization, and it can be centralized by a unique translation in the \( z \)-direction that makes it a centralized steady vortex ring. We shall denote its centralized version by \( \zeta^# \). We also set \( S^\#_\varepsilon := \{ \zeta^# \mid \zeta \in S_\varepsilon \} \). In view of Theorem 1.2, we see that \( S^\#_\varepsilon = \{ \zeta_\varepsilon \} \) for all \( \varepsilon > 0 \) small.

The following elementary estimates can be found in [14] (see Lemma 2.3 in [14]).

**Lemma 4.2.** There exists a positive number \( C \) such that

\[
|E[\zeta]| \leq E[|\zeta|] \leq C \left( \| r^2 \zeta \|_{L^1(\mathbb{R}^3)} + \| \zeta \|_{L^1 \cap L^2(\mathbb{R}^3)} \right) \| r^2 \zeta^1/2 \|_{L^1(\mathbb{R}^3)}^1/2, \]
\[
|E[\zeta_1] - E[\zeta_2]| \leq C \left( \| r^2(\zeta_1 + \zeta_2) \|_{L^1(\mathbb{R}^3)} + \| \zeta_1 + \zeta_2 \|_{L^1 \cap L^2(\mathbb{R}^3)} \right) \times \| r^2(\zeta_1 - \zeta_2) \|_{L^1(\mathbb{R}^3)}^1/2 \| r^2(\zeta_1 - \zeta_2) \|_{L^1(\mathbb{R}^3)}^1/2,
\]

for any axi-symmetric \( \zeta, \zeta_1, \zeta_2 \in (L^1 \cap L^2 \cap L^1_w)(\mathbb{R}^3) \).

### 4.2. Reduction to absurdity

**Proof of Theorem 1.4:** We argue by contradiction. Suppose that there exist a positive number \( \eta_0 \), a sequence \( \{ \zeta_{0,n} \}_{n=1}^\infty \) of non-negative axi-symmetric functions, and a sequence \( \{ t_n \}_{n=1}^\infty \) of non-negative numbers such that, for each \( n \geq 1 \), we have \( \zeta_{0,n}, (r\zeta_{0,n}) \in L^\infty(\mathbb{R}^3) \),

\[
\| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1 \cap L^2(\mathbb{R}^3)} + \| r^2(\zeta_{0,n} - \zeta_\varepsilon) \|_{L^1(\mathbb{R}^3)} \leq \frac{1}{n^2},
\]

and

\[
\inf_{\tau \in \mathbb{R}} \| \zeta_n(\cdot - \tau e_2, t_n) - \zeta_\varepsilon \|_{L^1 \cap L^2(\mathbb{R}^3)} + \| r^2(\zeta_n(\cdot - \tau e_2, t_n) - \zeta_\varepsilon) \|_{L^1(\mathbb{R}^3)} \geq \eta_0,
\]

where \( \zeta_n(x, t) \) is the global-in-time weak solution of (1.7) for the initial data \( \zeta_{0,n} \) obtained by Proposition 1.3. Using Lemma 4.2, we get

\[
\lim_{n \to \infty} E[\zeta_{0,n}] = E[\zeta_\varepsilon].
\]

Thus, we have

\[
\lim_{n \to \infty} \mathcal{P}[\zeta_{0,n}] = \mathcal{P}[\zeta_\varepsilon], \quad \lim_{n \to \infty} E[\zeta_{0,n}] = E[\zeta_\varepsilon],
\]

\[
\lim_{n \to \infty} \| \zeta_{0,n} \|_{L^p(\mathbb{R}^3)} = \| \zeta_\varepsilon \|_{L^p(\mathbb{R}^3)}, \quad \forall 1 \leq p \leq 2.
\]

Let us write \( \zeta_n = \zeta_n(\cdot, t_n) \). By virtue of the conservations, we conclude that

\[
\lim_{n \to \infty} \mathcal{P}[\zeta_n] = \mathcal{P}[\zeta_\varepsilon], \quad \lim_{n \to \infty} E[\zeta_n] = E[\zeta_\varepsilon],
\]

\[
\lim_{n \to \infty} \| \zeta_n \|_{L^p(\mathbb{R}^3)} = \| \zeta_\varepsilon \|_{L^p(\mathbb{R}^3)}, \quad \forall 1 \leq p \leq 2.
\]

Note that

\[
\int_{\{ x \in \mathbb{R}^3 : |\zeta_n(x) - 1/\varepsilon^2| \geq 1/n \}} \zeta_n \, dx = \int_{\{ x \in \mathbb{R}^3 : |\zeta_{0,n}(x) - 1/\varepsilon^2| \geq 1/n \}} \zeta_{0,n} \, dx.
\]
Set \( D(n) := \{ x \in \mathbb{R}^3 \mid |\zeta_{0,n}(x) - 1/\varepsilon| \geq 1/n \} \) and \( Q := \text{supp} \zeta \). We check that
\[
\int_{D(n)} \zeta_{0,n} \, dx = \| \zeta_{0,n} \|_{L^1(D(n) \cap Q)} + \| \zeta_{0,n} \|_{L^1(D(n) \setminus Q)} \\
\leq \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(D(n) \cap Q)} + \| \zeta_\varepsilon \|_{L^1(D(n) \cap Q)} + \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(D(n) \setminus Q)} \\
\leq \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3)} + \| \zeta_\varepsilon \|_{L^1(D(n) \cap Q)} \\
\leq \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3)} + |D(n) \cap Q| \leq (n + 1) \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3)} \leq \frac{n + 1}{n^2} \to 0
\]
as \( n \to \infty \), where we used the fact that
\[
\frac{1}{n}|D(n) \cap Q| \leq \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(D(n) \cap Q)} \leq \| \zeta_{0,n} - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3)}.
\]
Set
\[
A_\varepsilon^* := \{ \zeta \in A_\varepsilon \mid \mathcal{P}[\zeta] = \mathcal{P}[\zeta_\varepsilon] \}.
\]
It is easy to see that
\[
E[\zeta_\varepsilon] = \max_{\zeta \in A_\varepsilon^*} E[\zeta] \quad \text{and} \quad S_\varepsilon = \{ \zeta \in A_\varepsilon^* \mid E[\zeta] = E[\zeta_\varepsilon] \}.
\]
Therefore, we can now use Theorem 3.1 in [14] as a consequence of the concentration-compactness principle to obtain a subsequence (still using the same index \( n \)) and \( \{ \tau_n \}_{n=1}^\infty \subset \mathbb{R} \) such that
\[
\| r^2 (\zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon) \|_{L^1(\mathbb{R}^3)} \to 0, \quad \text{as} \quad n \to \infty.
\]
Recalling (4.2), we can further deduce that
\[
\| \zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon \|_{L^2(\mathbb{R}^3)} \to 0, \quad \text{as} \quad n \to \infty.
\]
By Hölder’s inequality, we get
\[
\lim_{n \to \infty} \int_Q \zeta_n(x - \tau_n \varepsilon) \, dx = \int_Q \zeta_\varepsilon(x) \, dx,
\]
which implies
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3 \setminus Q} \zeta_n(x - \tau_n \varepsilon) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} \zeta_n(x - \tau_n \varepsilon) \, dx - \lim_{n \to \infty} \int_Q \zeta_n(x - \tau_n \varepsilon) \, dx = 0.
\]
It follows that
\[
\| \zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon \|_{L^1} = \| \zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon \|_{L^1(Q)} + \| \zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3 \setminus Q)}
\leq |Q|^{1/2} \| \zeta_n(\cdot - \tau_n \varepsilon) - \zeta_\varepsilon \|_{L^2(\mathbb{R}^3)} + \| \zeta_n(\cdot - \tau_n \varepsilon) \|_{L^1(\mathbb{R}^3 \setminus Q)} \to 0
\]
as \( n \to \infty \). In sum, we have
\[
0 = \lim_{n \to \infty} \| \zeta_n(\cdot - \tau_n \varepsilon, t_n) - \zeta_\varepsilon \|_{L^1(\mathbb{R}^3)} + \| r^2 (\zeta_n(\cdot - \tau_n \varepsilon, t_n) - \zeta_\varepsilon) \|_{L^1(\mathbb{R}^3)} \geq \eta_0 > 0,
\]
which is a contradiction. The proof is thus complete. \( \square \)
APPENDIX A. METHOD OF MOVING PLANES

In this appendix, we will show that the cross-section \(A_\varepsilon\) and Stokes stream function \(\psi_\varepsilon\) are symmetric with respect to the line \(\{x_2 = h\}\) for some \(h\) by the method of moving planes (see also Lemma 2.1 in [4]). Though the proof is almost the same as that of Proposition 4.1 in [11], we give it in detail here for readers’ convenience.

**Proposition A.1.** Suppose that a bounded set \(A\) with \(\bar{A} \subset \mathbb{R}^2_+\), satisfies

\[
A = \{x \in B_R(0) \cap \{x_1 > 0\} \mid \psi(x) + \frac{W}{2} x_1^2 > \mu\}
\]

for some constants \(W\) and \(\mu\). Moreover, \(\psi\) is the potential of \(A\) in the sense

\[
\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2_+} G_s(x, x') 1_A(x')d\mathbf{x}'.
\]

Then, \(A\) is symmetric with respect to the line \(\{x_2 = h\}\) for some \(h \in \mathbb{R}\).

**Proof.** To prove this proposition, the key observation is that \(G_s(x, x')\) is a strictly decreasing function of \(|x_2 - x'_2|^2\) for fixed \(x_1\) and \(x'_1\). Namely, for any fixed \(x_1\) and \(x'_1\), if we denote \(r_2 := |x_2 - x'_2|^2\), then we have \(G_s(x, x') = J_{x_1, x'_1}(r_2)\) for some strictly decreasing function \(J_{x_1, x'_1}(\cdot)\).

For \(-R < t < R\), define

\[
A_t := \{x \in A \mid x_2 < t\},\quad A_t^* := \{x \in \mathbb{R}^2 \mid (x_1, 2t - x_2) \in A_t\}.
\]

This is, \(A_t^*\) is the reflection of \(A_t\) with respect to the line \(x_2 = t\). Let \(d := \inf_{y \in A} y_2\). We will carry out the proof of Proposition A.1 by two steps.

**Step 1.** Let us first show that there exists \(\varepsilon > 0\) small enough such that, for any \(d < t \leq d + \varepsilon\),

\[
A_t^* \subset A.
\]

For any \(x \in \{x_2 = d\} \cap \bar{A}\), we compute

\[
\partial_{x_2} \psi(x) = \int_A 2\partial_{r_2} J_{x_1, x'_1}(|x_2 - x'_2|^2)(x_2 - x'_2)d\mathbf{x}' \geq c_0 > 0,
\]

for some constant \(c_0\) independent of \(x\). We define the set \(S_\varepsilon := \{x \in A \mid d < x_2 < d + \varepsilon\}\). Arguing by contradiction, we can show that \(\sup_{x \in S_\varepsilon} \text{dist}(x, \{x_2 = d\} \cap \bar{A}) \to 0\) as \(\varepsilon \to 0\). Then, by the \(C^1_{\text{loc}}\) continuity of \(\psi\) in \(\mathbb{R}^2_+\), there exists \(\varepsilon_1 > 0\) small such that \(\partial_{x_2} \psi(x) > c_0/2 > 0\) for all \(x \in S_\varepsilon\) whenever \(0 < \varepsilon < \varepsilon_1\). Since \(\psi \in C^1_{\text{loc}}(\mathbb{R}^2_+)\) by the regularity theory and \(A\) is far away from the boundary \(x_1 = 0\), for \(d < t < d + \varepsilon_1\), we have for all \(x \in A_t\),

\[
\psi(x_1, 2t - x_2) - \psi(x_1, x_2) = 2\partial_{x_2} \psi(x)(t - x_2) + O((t - x_2)^{1+\alpha})
\]

\[
\geq c_0(t - x_2) + O((t - x_2)^{1+\alpha}).
\]

Thus, there exists \(0 < \varepsilon_2 \leq \varepsilon_1\) small such that for any \(d < t < d + \varepsilon_2\), it holds

\[
\psi(x_1, 2t - x_2) - \psi(x_1, x_2) \geq 0, \quad \forall x \in A_t,
\]

which implies \(A_t^* \subset A\).
Step 2. We move the line continuously until its limiting position. Step 1 provides a starting point for us to move lines. Define the limiting position

\[ h := \sup \{ t \mid A_2^* \subset A, \ \forall d < t \leq t \}. \]

We will show that \( A \) is symmetric with respect to the line \( \{ x_2 = h \} \). In fact, we are going to prove that

\[ |N| = 0, \quad \text{for} \ N = A \setminus (A_h \cup A_h^*). \]

Suppose that \( |N| > 0 \), we will get a contradiction.

By step 1, we have \( d < h < \sup_{x \in A} x_2 \). By the definition of \( h \), we have \( A_h^* \subset \bar{A} \). We first claim that \( \partial A \cap \partial A_h^* \neq \emptyset \). Indeed, suppose on the contrary that \( A_h^* \cap A \). This means that \( A_h \) lies far away from the line \( \{ x_2 = h \} \) and the set \( A \) is divided into disjoint sets by \( \{ x_2 = h \} \). Then, it is easy to see that there exists a \( d < t < h \) such that \( A_h^* \cap A \), which contradicts the definition of \( h \). Therefore, we must have \( \partial A \cap \partial A_h^* \neq \emptyset \).

Suppose that there exists a point \( x^* \in \partial A \cap \partial A_h^* \) such that \( x^*_2 > h \). We write \( x = (x^*_1, 2h - x^*_2) \). Then, we calculate

\[ 0 = \psi(x) - \psi(x^*) = \int_N (G_+(x, x') - G_+(x^*, x')) \, dx' < 0, \]

if \( |N| > 0 \). Here, we have used the fact that \( |x_2 - x'_2| > |x^*_2 - x'_2| \) for any \( x' \in N \). This is a contradiction and thus we must have \( |N| = 0 \) in this case.

Now, we consider the remaining case, where for any \( x^* \in \partial A \cap \partial A_h^* \), it must hold \( x^*_2 = h \) and thus \( x = x^* \). However, for any \( x \in A \setminus \{ x_2 = h \} \), it holds

\[ \partial_{x_2} \psi(x) = \int_N 2 \partial_{x_2} J_{x_1, x'_1} (|x_2 - x'_2|^2)(x_2 - x'_2) \, dx' \geq c_0 > 0, \]

for some constant \( c_0 \) independent of \( x \) provided that \( |N| > 0 \). We can take \( \varepsilon_3 > 0 \) small such that \( \partial_{x_2} \psi(x) \geq c_0/2 > 0 \) for all \( x \) lies in the strip \( \{ x \in A \mid h - \varepsilon_3 < x_2 < h + \varepsilon_3 \} \). We denote \( A_{b,c}^* \) as the reflection of the set \( A_b \) with respect to line \( x_2 = c \) for any \( b, c \in \mathbb{R} \).

We first have \( \text{dist}(A_{h - \varepsilon_3}^*, \partial A) \geq c_{\varepsilon_3} \) for some constant \( c_{\varepsilon_3} > 0 \). Otherwise, we will obtain a point \( x^* \in \partial A_h^* \cap \partial A \) with \( x^*_2 \geq h + \varepsilon > h \), which has already been considered. Therefore, if we take \( \varepsilon_4 := \min \{ \varepsilon_3, c_{\varepsilon_3} \} \), then for all \( h < t < h + \varepsilon_4 \), it holds

\[ A_{h - \varepsilon_3}^{*, t} \subset A. \]

For \( x \) in the strip \( A \cap \{ h - \varepsilon_3 \leq x_2 < h \} \), we have

\[ \psi(x_1, 2t - x_2) - \psi(x_1, x_2) = 2 \partial_{x_2} \psi(x)(t - x_2) + O((t - x_2)^{1+\alpha}) \]

\[ \geq c_0 (t - x_2) + O((t - x_2)^{1+\alpha}). \]

Thus, there exists \( 0 < \varepsilon_5 \leq \varepsilon_4 \) such that for any \( h < t < h + \varepsilon_5 \), it holds

\[ \psi(x_1, 2t - x_2) - \psi(x_1, x_2) \geq 0, \quad \forall x \in A \cap \{ s - \varepsilon_3 \leq x_2 < t \}, \]

which implies \( A_h^* \subset A \). This contradicts the definition of \( h \) and hence we must have \( |N| = 0 \), which means that \( A \) is symmetric with respect to some line \( \{ x_2 = h \} \).
The proof is thus finished. □

APPENDIX B. ESSENTIAL ESTIMATES FOR THE FREE BOUNDARY

In this appendix, we will give some estimates and statements for free boundary $\partial A_\varepsilon$. For a general function $v$, we denote $\tilde{v}(y) = v(sy + z)$. In the following, we always assume that $L > 0$ is a large fixed constant. Recall that

$$U_{z,\varepsilon}(x) = V_{z,\varepsilon}(x) + H_{z,\varepsilon}(x) - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} - \mu_\varepsilon$$

with $V_{z,\varepsilon}$ and $H_{z,\varepsilon}$ being the same as defined in Section 2. To simplify notation we will write $U_{z,\varepsilon}, V_{z,\varepsilon}$ simply as $U_\varepsilon, V_\varepsilon$ respectively in the sequel.

For the variable $x > 0$, Let

$$W(x) = \frac{s^2}{4\varepsilon^2} \cdot z_1 \ln \frac{1}{s} - Wz_1 \ln \frac{1}{\varepsilon} + \frac{1}{8\varepsilon^2} \cdot z_1 \left\{ \begin{array}{ll}
(s^2 - x^2), & 0 < x < s \\
2 \ln(s/x), & x \geq s
\end{array} \right. + \frac{3}{16\varepsilon^2} \cdot z_1 \left\{ \begin{array}{ll}
2s^2 - x^2, & 0 < x < s \\
\frac{s^4}{x^2}, & x \geq s
\end{array} \right. (B.1)$$

Then we have the following estimate for $U_\varepsilon(x)$.

**Lemma B.1.** For every $y \in D_\varepsilon = \{ y \mid sy + z \in \mathbb{R}^2_+ \}$ bounded, it holds

$$\tilde{U}_\varepsilon(y) = \tilde{V}_\varepsilon(y) - \frac{a}{2\pi} \ln \frac{1}{\varepsilon} + sy_1 \cdot W(|sy|) + O(\varepsilon^2 |\ln \varepsilon|).$$

**Proof.** By the definition of $U_\varepsilon(x)$, it holds

$$U_\varepsilon(x) = \frac{1}{2\pi\varepsilon^2} \int_{B_s(z)} x_1^{1/2} x_1^{3/2} \ln \left( \frac{1}{|x - x'|} \right) dx' - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} - \mu_\varepsilon$$

$$+ \frac{1}{4\pi\varepsilon^2} \int_{B_s(z)} x_1^{1/2} x_1^{3/2} \left( \ln(x_1 x_1') + 2 \ln 8 - 4 + O \left( \rho \ln \frac{1}{\rho} \right) \right) dx'$$

$$= \frac{z_1^2}{2\pi\varepsilon^2} \int_{B_s(z)} \ln \left( \frac{1}{|x - x'|} \right) dx' + \frac{1}{2\pi\varepsilon^2} \int_{B_s(z)} (x_1^{1/2} x_1^{3/2} - z_1^2) \ln \left( \frac{1}{|x - x'|} \right) dx'$$

$$+ \frac{1}{4\pi\varepsilon^2} \int_{B_s(z)} x_1^{1/2} x_1^{3/2} \left( \ln(x_1 x_1') + 2 \ln 8 - 4 + O \left( \rho \ln \frac{1}{\rho} \right) \right) dx'$$

$$- \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} - \mu_\varepsilon.$$
with \( \rho \) defined in (2.13). According to Taylor’s formula, we have
\[
\frac{1}{2 \pi \varepsilon^2} \int_{B_s(z)} \left( x_1^{1/2} x_1^{3/2} - z_1^2 \right) \ln \left( \frac{1}{|x - x'|} \right) \, dx'
= \frac{1}{2 \pi \varepsilon^2} \int_{B_s(z)} \left( z_1^{1/2} + \frac{1}{2 s_1^{1/2}} (x_1 - z_1) + O(s^2) \right) \left( z_1^{3/2} + \frac{3 z_1^{1/2}}{2} (x_1' - z_1) + O(s^2) \right) - z_1^2 \right) \\
\times \ln \left( \frac{1}{|x - x'|} \right) \, dx'
= \frac{z_1}{2 \pi \varepsilon^2} \int_{B_s(z)} \left( \frac{x_1 - z_1}{2} + \frac{3 (x_1' - z_1)}{2} \right) \ln \left( \frac{1}{|x - x'|} \right) \, dx' + O(\varepsilon^2 |\ln \varepsilon|)
= \frac{s^2}{4 \varepsilon^2} \cdot z_1 (x_1 - z_1) \ln \frac{1}{s} + \frac{(x_1 - z_1)}{2 s_1^{1/2}} \cdot z_1 \left\{ \frac{(s^2 - |x - z|^2)}{8 \varepsilon^2}, \quad |x - z| < s \right\}
+ \frac{3 (x_1 - z_1)}{16 \varepsilon^2} \cdot z_1 \left\{ \frac{2 s^2 - |x - z|^2}{s}, \quad |x - z| < s \right\}
+ O(\varepsilon^2 |\ln \varepsilon|),
\]
where we have used the formula of planar Rankine vortex and integral
\[
\frac{1}{2 \pi} \int_{B_1(x)} y_1^3 \ln \frac{1}{|y - y'|} \, dy = \left\{ \begin{array}{ll}
\frac{y_1}{4} - \frac{|y|^2 y_1}{8}, & |y| < 1, \\
\frac{y_1}{8 |y|^2}, & |y| \geq 1.
\end{array} \right.
\]
Let
\[
\mathcal{R}(x) = \frac{1}{4 \pi \varepsilon^2} \int_{B_s(z)} x_1^{1/2} x_1^{3/2} \ln (x_1 x_1') + 2 \ln 8 - 4 + O(\rho \ln(1/\rho))) \, dx'
- \frac{W z_1}{2 s_1^{1/2}} \ln \frac{1}{\varepsilon} - \mu_1.
\]
By our choice of \( a \) in (2.19) and (3.5), it holds
\[
\mathcal{R}(x) = \mathcal{R}(z) + (x_1 - z_1) \cdot \partial_1 \mathcal{R}(z) + O(\varepsilon^2 |\ln \varepsilon|)
\]
with
\[
\mathcal{R}(z) = -\frac{a}{2 \pi} \ln \frac{1}{\varepsilon},
\]
and
\[
\partial_1 \mathcal{R}(z) = \frac{1}{4 \pi \varepsilon^2} \int_{B_s(z)} \left( \frac{x_1^{3/2}}{2 s_1^{1/2}} \ln (x_1 x_1') + 2 \ln 8 - 4 + \frac{x_1^{3/2}}{z_1^{1/2}} \right) \, dx' - W z_1 \ln \frac{1}{\varepsilon}
= \frac{s^2}{4 \varepsilon^2} \cdot z_1 (\ln 8 z_1 - 1) - W z_1 \ln \frac{1}{\varepsilon} + O(\varepsilon |\ln \varepsilon|).
\]
Combining all the facts above, we have
\[
U_\varepsilon(x) = V_\varepsilon(x) - \frac{a}{2 \pi} \ln \frac{1}{\varepsilon} + (x_1 - z_1) \cdot W(|x - z|) + O(\varepsilon^2 |\ln \varepsilon|).
\]
By letting \( x = s y + z \), the proof of Lemma B.1 is then complete.
We give an estimate for the level set of approximate solutions without error term $\phi$ in following lemma.

**Lemma B.2.** The set

$$\tilde{\Gamma}_\varepsilon := \{y \mid \tilde{U}_\varepsilon = 0\}$$

is a closed convex curve in $\mathbb{R}^2$, which can be rewritten as

$$\tilde{\Gamma}_\varepsilon = (1 + t_\varepsilon)(\cos \theta, \sin \theta)$$

$$= (\cos \theta, \sin \theta) + \frac{1}{N} \cdot \mathcal{W}(s) \cdot (\cos \theta, 0)$$

$$+ o_\varepsilon(\varepsilon \mathcal{W}(s)) + O(\varepsilon^2 |\ln \varepsilon|), \quad \theta \in (0, 2\pi]$$

with $||t_\varepsilon(\theta)||_{C^1((0,2\pi])} = O(\varepsilon |\ln \varepsilon|)$, and $N$ defined in (2.17). Moreover, it holds

$$\tilde{U}_\varepsilon((1 + t)(\cos \theta, \sin \theta)) \begin{cases} > 0, & t < t_\varepsilon(\theta), \\ < 0, & t > t_\varepsilon(\theta). \end{cases}$$

**Proof.** In view of lemma B.1, for every $y \in D_\varepsilon = \{y \mid sy + z \in \mathbb{R}^2_+\}$ bounded, it holds

$$\tilde{U}_\varepsilon(y) = \tilde{V}_\varepsilon(y) - \frac{a}{2\pi} \ln \frac{1}{\varepsilon} + sy_1 \cdot \mathcal{W}(|sy|) + O(\varepsilon^2 |\ln \varepsilon|).$$

Notice that

$$\tilde{V}_\varepsilon = \begin{cases} \frac{a}{2\pi} \ln \frac{1}{\varepsilon} + \frac{z^2 s^2}{4\varepsilon^2} (1 - |y|^2), & y \leq 1, \\ \frac{a}{2\pi} \ln \frac{1}{\varepsilon} \left(1 + \frac{|y|}{\ln s}\right), & y \geq 1, \end{cases}$$

and

$$s|\tilde{V}(y)| = O(\varepsilon |\ln \varepsilon|).$$

If $|y| < 1 - L_1\varepsilon |\ln \varepsilon|$ for some large $L_1 > 0$, then

$$\tilde{U}_\varepsilon \geq \frac{z^2 s^2}{4\varepsilon^2} (1 - |1 - L_1\varepsilon |\ln \varepsilon||^2) + O(\varepsilon |\ln \varepsilon|) > 0.$$
Hence it holds
\[ t_\varepsilon(\theta) = \frac{\cos \theta}{N} \cdot W(s) + o_\varepsilon(\varepsilon W(s)) + O(\varepsilon^2|\ln \varepsilon|), \]
and (B.2) is verified.

To obtain an estimate for \( t'_\varepsilon(\theta) \), we differentiate \( \tilde{U}_\varepsilon((1 + t_\varepsilon)(\cos \theta, \sin \theta)) = 0 \) with respect to \( \theta \) and derive
\[ \frac{\partial \tilde{U}_\varepsilon((1 + t_\varepsilon)(\cos \theta, \sin \theta))}{\partial \theta} = O(\varepsilon) \cdot |t'_\varepsilon(\theta)| + O(\varepsilon|\ln \varepsilon|). \]
Using the implicit function theorem again, we have
\[ \frac{\partial \tilde{U}_\varepsilon((1 + t_\varepsilon)(\cos \theta, \sin \theta))}{\partial \theta} = (sN + O(\varepsilon|\ln \varepsilon|)) \cdot t'_\varepsilon(\theta). \]
Thus we conclude that \(|t'_\varepsilon(\theta)| = O(\varepsilon|\ln \varepsilon|)\), and \( \tilde{\Gamma}_\varepsilon \) is a closed convex curve. \( \square \)

Thanks to the implicit function theorem, now we can estimate the free boundary \( \partial A_\varepsilon \).

**Lemma B.3.** Suppose that \( \tilde{\phi} \) is a function satisfying
\[ \|\nabla \tilde{\phi}\|_{L^\infty(B_L(0))} \leq \varepsilon|\ln \varepsilon|^2, \quad \|\tilde{\phi}\|_{L^\infty(B_L(0))} \leq \varepsilon|\ln \varepsilon|^2. \] (B.3)

Then the set
\[ \tilde{\Gamma}_{\varepsilon,\tilde{\phi}} := \{ y \mid \tilde{U}_\varepsilon + \tilde{\phi} = 0 \} \]
is a closed convex curve in \( \mathbb{R}^2 \), and
\[ \tilde{\Gamma}_{\varepsilon,\tilde{\phi}} = (1 + t_\varepsilon + t_{\varepsilon,\tilde{\phi}})(\cos \theta, \sin \theta) \]
\[ = \left(1 + \frac{1}{sN}\tilde{\phi}(\cos \theta, \sin \theta)\right)(\cos \theta, \sin \theta) + \frac{1}{N} \cdot W(s) \cdot (\cos \theta, 0) \] (B.4)
\[ + o_\varepsilon(sW(s) + \|\tilde{\phi}\|_{L^\infty(B_L(0))}) + O(\varepsilon^2|\ln \varepsilon|), \quad \theta \in (0, 2\pi] \]
for \( N \) defined in (2.17). Moreover, we have
\[ (\tilde{U}_\varepsilon + \tilde{\phi})((1 + t_\varepsilon + t)(\cos \theta, \sin \theta)) \begin{cases} > 0, & t < t_{\varepsilon,\tilde{\phi}}(\theta), \\ < 0, & t > t_{\varepsilon,\tilde{\phi}}(\theta), \end{cases} \]
and
\[ |\tilde{\Gamma}_{\varepsilon,\tilde{\phi}_1} - \tilde{\Gamma}_{\varepsilon,\tilde{\phi}_2}| = \left(\frac{1}{sN} + O(\varepsilon|\ln \varepsilon|)\right) \|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(B_L(0))} \] (B.5)
for functions \( \tilde{\phi}_1, \tilde{\phi}_2 \) satisfying (B.3).

**Proof.** From Lemma B.1, we have
\[ \tilde{U}_\varepsilon(y) + \tilde{\phi} = \tilde{V}_\varepsilon - \frac{a}{2\pi} \ln \frac{1}{\varepsilon} + sy \cdot W(|sy|) + \tilde{\phi} + O(\varepsilon^2|\ln \varepsilon|). \]
Hence it holds
\[ 1 + t_\varepsilon + t_{\varepsilon,\tilde{\phi}} \in (1 - L_1\varepsilon|\ln \varepsilon|^2, 1 + L_2\varepsilon|\ln \varepsilon|^2) \]
in a similar way as Lemma B.2. Using the fact
\[
\left. \frac{\partial \tilde{U}_\varepsilon + \partial \tilde{\phi}}{\partial t}((1 + t_\varepsilon)(\cos \theta, \sin \theta)) \right|_{t=0} = -s\mathcal{N} + O(\varepsilon |\ln \varepsilon|) < 0,
\]
we see that \(t_{\varepsilon, \tilde{\phi}}\) is unique, and \(\tilde{\Gamma}_{\varepsilon, \tilde{\phi}}\) is a continuous closed curve in \(\mathbb{R}^2\). Then we let
\[
y_\varepsilon = (1 + t_\varepsilon + t_{\varepsilon, \tilde{\phi}})(\cos \theta, \sin \theta) \in \tilde{\Gamma}_{\varepsilon, \tilde{\phi}}.
\]
By the implicit function theorem, it holds
\[
|y_\varepsilon| - 1 = \frac{\cos \theta \cdot s\mathcal{W}(s) + \tilde{\phi}(y_\varepsilon) + (t_\varepsilon + t_{\varepsilon, \tilde{\phi}}) \cdot O(\varepsilon) + O(\varepsilon^2 |\ln \varepsilon|)}{s\mathcal{N} + (t_\varepsilon + t_{\varepsilon, \tilde{\phi}}) \cdot O(1)}.
\]
While for \(\tilde{\phi}(y_\varepsilon)\), it holds
\[
|\tilde{\phi}(y_\varepsilon) - \tilde{\phi}(\cos \theta, \sin \theta)| \leq \|\nabla \tilde{\phi}\|_{L^\infty(B_L(0))} \cdot |t_\varepsilon(\theta)|,
\]
from which we can verify (B.4). Moreover, we can obtain \(|t_\varepsilon'(\theta) + t_{\varepsilon, \tilde{\phi}}'(\theta)| = O(\varepsilon |\ln \varepsilon|^2)\) as in Lemma B.2. So \(\tilde{\Gamma}_{\varepsilon, \tilde{\phi}}\) is also convex.

Denote \(y_{\varepsilon,m}\) as the coordinate corresponding to \(\tilde{\phi}_m\) (\(m = 1, 2\)). Then according to the definition of \(y_{\varepsilon,m}\), we have
\[
\tilde{U}_\varepsilon(y_{\varepsilon,1}) - \tilde{U}_\varepsilon(y_{\varepsilon,2}) = \tilde{\phi}_1(y_{\varepsilon,1}) - \tilde{\phi}_2(y_{\varepsilon,1}) + \tilde{\phi}_2(y_{\varepsilon,1}) - \tilde{\phi}_2(y_{\varepsilon,2}) = \|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(B_L(0))} + \|\nabla \tilde{\phi}\|_{L^\infty(B_L(0))} \cdot |y_{\varepsilon,1} - y_{\varepsilon,2}| = \|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(B_L(0))} + O(\varepsilon |\ln \varepsilon|^2) \cdot |y_{\varepsilon,1} - y_{\varepsilon,2}|.
\]
Since
\[
\left. \frac{\partial \tilde{U}_\varepsilon((1 + t_\varepsilon)(\cos \theta, \sin \theta))}{\partial t} \right|_{t=0} = -s\mathcal{N} + O(\varepsilon |\ln \varepsilon|),
\]
we conclude (B.5) and finish our proof. \(\square\)

In Section 3 in the proof of uniqueness of steady vortex rings, we have used a coarse version of Lemma B.3, which is summarized as follows. Since the proof is similar to Lemma B.3, we omit it here therefore.

**Lemma B.4.** Suppose that \(\tilde{\phi}\) is a function satisfying
\[
\|\nabla \tilde{\phi}\|_{L^\infty(B_L(0))} = o_\varepsilon(1), \quad \|\tilde{\phi}\|_{L^\infty(B_L(0))} = o_\varepsilon(1),
\]
and let
\[
\mathcal{W} = \|\tilde{\phi}\|_{L^\infty(B_L(0))} + s\mathcal{W}(s).
\]
Then the set
\[
\tilde{\Gamma}_{\varepsilon, \tilde{\phi}} := \{ y \mid \tilde{U}_\varepsilon + \tilde{\phi} = 0 \}.
\]
is a closed convex curve in \( \mathbb{R}^2 \), and
\[
\tilde{\Gamma}_{\varepsilon, \phi} = (1 + t_\varepsilon + t_{\varepsilon, \phi})(\cos \theta, \sin \theta)
\]
\[
= \left( 1 + \frac{1}{sN} \tilde{\phi}(\cos \theta, \sin \theta) \right) (\cos \theta, \sin \theta) + \frac{1}{N} \cdot W(s) \cdot (\cos \theta, 0)
\]
\[
+ o_\varepsilon(1) \cdot \gamma_\varepsilon + O(\varepsilon^2 |\ln |), \quad \theta \in (0, 2\pi]
\]
for \( N \) defined in (3.7).

### Appendix C. Estimates for the Pohozaev identity

This appendix is devoted to the proof of some facts and estimates that have been used in obtaining the uniqueness of steady vortex rings in Section 3. Suppose that \( u \in H^1(\mathbb{R}_+^2) \cap C^1(\mathbb{R}_+^2) \). Set
\[
F(x, u) := \int_0^u f(x, u)dt,
\]
where \( f(x, u) \) is continuous in \( x \), and nondecreasing with respect to \( u \). We have the following local Pohozaev identity, which corresponds to the translation transformation of semilinear elliptic equations.

**Lemma C.1.** Suppose that \( u \in H^1(\mathbb{R}_+^2) \cap C^1(\mathbb{R}_+^2) \) is a weak solution to
\[
- \Delta u = f(x, u), \quad \text{in } \mathbb{R}_+^2.
\]
Then for any bounded smooth domain \( D \subset \mathbb{R}_+^2 \), it holds
\[
\int_{\partial D} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial \nu} dS - \frac{1}{2} \int_{\partial D} |\nabla u|^2 \nu_i dS + \int_{\partial D} F(x, u) dS = \int_D F_{x_i}(x, u) dx, \quad i = 1, 2,
\]
with \( \nu \) the unit outward normal to the boundary \( \partial D \).

The proof of Lemma C.1 can be found in [10] (see Theorem 6.2.1 in [10]) together with an approximation procedure. In our case, we let the domain \( D \subset \mathbb{R}_+^2 \) be \( B_\delta(z) \) with a small constant \( \delta > 0 \), let the function \( u \) be \( \psi_{1, \varepsilon} \), and let the nonlinearity \( f \) be
\[
f(x, \psi_{1, \varepsilon}) = \frac{z_1^2}{\varepsilon^2} \cdot 1_{\{\psi_{1, \varepsilon} - \frac{W}{2}x_1^2 \ln \frac{1}{\varepsilon} > \mu_{\varepsilon} \}}.
\]
Thus the primitive function for \( f \) is
\[
F(x, \psi_{1, \varepsilon}) = \frac{z_1^2}{\varepsilon^2} \cdot \left( \psi_{1, \varepsilon} - \frac{W}{2}x_1^2 \ln \frac{1}{\varepsilon} - \mu_{\varepsilon} \right)_{+},
\]
and the local Pohozaev identity in Lemma C.1 with \( i = 1 \) turns to be
\[
- \int_{\partial B_\delta(z)} \frac{\partial \psi_{1, \varepsilon}}{\partial \nu} \frac{\partial \psi_{1, \varepsilon}}{\partial x_1} dS + \frac{1}{2} \int_{\partial B_\delta(z)} |\nabla \psi_{1, \varepsilon}|^2 \nu_1 dS
\]
\[
= - \frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} \partial_1 \psi_{2, \varepsilon}(x) \cdot 1_{A_{\varepsilon}}(x) d\mathbf{x} + \frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} W x_1 \ln \frac{1}{\varepsilon} \cdot 1_{A_{\varepsilon}}(x) d\mathbf{x}
\]  \hspace{1cm} (C.1)
with

\[ A_\varepsilon = \left\{ x \in \mathbb{R}^2_+ \mid \psi_\varepsilon - \frac{W}{2} x_1^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon \right\}. \]

According to the estimates obtained in Section 3, we see that \( A_\varepsilon \) is an area close to \( B_{s_0}(z) \) with

\[ s_0 = \sqrt{\varepsilon^2 \kappa / 2 n}. \]

By denoting the symmetry difference

\[ A_\varepsilon \Delta B_{s_0}(z) := (A_\varepsilon \setminus B_{s_0}(z)) \cup (B_{s_0}(z) \setminus A_\varepsilon), \]

and the error

\[ e_\varepsilon := |A_\varepsilon \Delta B_{s_0}(z)|, \]

we will proceed a series of lemma to compute each terms in (C.1).

**Lemma C.2.** For every \( x \in \mathbb{R}^2 \setminus \{ x \mid \text{dist}(x, A_\varepsilon) \leq L s_0 \} \), we have

\[ \psi_{1,\varepsilon}(x) = \frac{\kappa}{2 \pi} \cdot z_1 \ln \left| \frac{x - \bar{z}}{x - z} \right| + O \left( \frac{e_\varepsilon}{|x - z|} \right), \]

and

\[ \nabla \psi_{1,\varepsilon}(x) = -\frac{\kappa}{2 \pi} \cdot z_1 \frac{x - z}{|x - z|^2} + \frac{\kappa}{2 \pi} \cdot z_1 \frac{x - \bar{z}}{|x - \bar{z}|^2} + O \left( \frac{e_\varepsilon}{|x - z|^2} \right). \]

**Proof.** For each \( x \in \mathbb{R}^2 \setminus \{ x \mid \text{dist}(x, A_\varepsilon) \leq L s_0 \} \) with \( L > 0 \) large, it must hold \( x \notin \Omega_\varepsilon \). Then, using Taylor’s formula

\[ |x - x'| = |x - z| - \langle \frac{x - z}{|x - z|}, x' - z \rangle + O \left( \frac{|x' - z|^2}{|x - z|^2} \right), \quad \forall x' \in A_\varepsilon, \]

we obtain

\[ \psi_{1,\varepsilon}(x) = \frac{z_1^2}{2 \pi \varepsilon^2} \int_{A_\varepsilon} \ln \left| \frac{x - \bar{x'}}{|x - x'} \right| \, dx' \]

\[ = \frac{\kappa}{2 \pi} \cdot z_1 \ln \left| \frac{x - \bar{z}}{x - z} \right| + \frac{z_1^2}{2 \pi \varepsilon^2} \int_{A_\varepsilon} \ln \left| \frac{x - z}{x - x'} \right| \, dx' \]

\[ - \frac{z_1^2}{2 \pi \varepsilon^2} \int_{A_\varepsilon} \ln \left| \frac{x - \bar{z}}{x - z} \right| \, dx' + O \left( \frac{e_\varepsilon}{|x - z|^2} \right) \]

\[ = \frac{\kappa}{2 \pi} \cdot z_1 \ln \left| \frac{x - \bar{z}}{x - z} \right| - \frac{z_1^2}{2 \pi \varepsilon^2} \int_{A_\varepsilon} \frac{(x - z) \cdot (z - x')}{|x - z|^2} \, dx' \]

\[ + \frac{z_1^2}{2 \pi \varepsilon^2} \int_{A_\varepsilon} \frac{(x - z) \cdot (\bar{z} - \bar{x'})}{|x - z|^2} \, dx' + O \left( \frac{e_\varepsilon}{|x - z|^2} \right). \]

Using the odd symmetry, we have

\[ \int_{A_\varepsilon} \frac{(x - z) \cdot (\bar{z} - \bar{x'})}{|x - z|^2} \, dx' \]
Thus we will focus on the first term in the right hand side of (C.1). While, for the other terms, we can use a same argument to deduce

\[ \int_{A_{\varepsilon}} \frac{(x - z) \cdot (z - x')}{|x - z|^2} dA' = O \left( \frac{\varepsilon \cdot e_x}{|x - z|} \right). \]

Hence we have verified the first part of this lemma. The second part can be verified by similar procedure. □

Using Lemma C.2, we can compute the left hand side of (C.1) as follows.

**Lemma C.3.** It holds

\[ -\int_{\partial B_\delta(z)} \frac{\partial \psi_{1,\varepsilon}}{\partial \nu} \frac{\partial \psi_{1,\varepsilon}}{\partial x_1} dS + \frac{1}{2} \int_{\partial B_\delta(z)} |\nabla \psi_{1,\varepsilon}|^2 \nu_1 dS = \kappa \cdot \frac{s^2}{4\varepsilon^2} \cdot z_1^2 + O \left( \frac{e_x}{\varepsilon} \right). \]

**Proof.** Using the identity

\[ -\int_{\partial B_\delta(z)} G(x, x') \cdot G(x, x') \cdot \frac{\partial}{\partial x_1} dS + \frac{1}{2} \int_{\partial B_\delta(z)} |\nabla G(x, x')|^2 \nu_1 dS = -\partial_1 \left( \frac{1}{\pi} \ln \frac{1}{|x - z|} \right) \bigg|_{x = z}, \]

and the asymptotic estimate in Lemma C.2, this lemma can be verified by direct computation. □

Using the circulation constraint (3.2), it is obvious that

\[ \frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} W x_1 \ln \frac{1}{\varepsilon} \cdot 1_{A_{\varepsilon}}(x) dA' = \kappa \cdot W z_1^2 \frac{1}{\varepsilon}. \]  

(C.2)

Thus we will focus on the first term in the right hand side of (C.1) relevant to \( \partial_1 \psi_{2,\varepsilon} \).

**Lemma C.4.** It holds

\[ -\frac{z_1^2}{\varepsilon^2} \int_{B_\delta(z)} \partial_1 \psi_{2,\varepsilon} \cdot 1_{A_{\varepsilon}}(x) dA' = -\kappa \cdot \frac{s_0^2}{4\varepsilon^2} \cdot z_1^2 \left( \ln \frac{8s_1}{s_0} - \frac{5}{4} \right) + O \left( \frac{e_x}{\varepsilon^2} + \varepsilon^2 |\ln \varepsilon| \right). \]

**Proof.** By the definition of \( \partial_1 \psi_{2,\varepsilon} \), it holds

\[ \partial_1 \psi_{2,\varepsilon} = \frac{1}{\varepsilon^2} \int_{R_{\varepsilon}^3} \partial_1 H(x, x') 1_{A_{\varepsilon}}(x') dA', \]
where
\[
H(x, x') = \left( \frac{x_{1/2}^{3/2}}{2\pi} - \frac{z_1^{2}}{2\pi} \right) \ln \frac{1}{|x - x'|} + \frac{z_1^{2}}{2\pi} \ln \frac{1}{|x - x'|} + \frac{x_{1/2}^{3/2}}{4\pi} (\ln(x_1 x_1') + 2\ln 8 - 4 + \rho),
\]
with \( \rho = O(\rho \ln(1/\rho)) \) a regular remainder and \( \rho \) defined before (2.5). For simplicity, we let
\[
-z_1^{2} \int_{B_{s}(z)} \partial_1 \psi_{2, x}(x) \cdot 1_{A_{s}}(x) dx = I_{1} + I_{2} + I_{3} + I_{\rho},
\]
where
\[
I_{1} = -\frac{z_1^{2}}{4\pi\varepsilon^{4}} \int_{A_{s}} x_1^{-1/2} \int_{A_{s}} x_1^{3/2} \ln \left( \frac{1}{s_0} \right) d x' dx,
\]
\[
I_{2} = -\frac{z_1^{2}}{4\pi\varepsilon^{4}} \int_{A_{s}} x_1^{-1/2} \int_{A_{s}} x_1^{3/2} \ln \left( \frac{s_0}{|x - x'|} \right) d x' dx,
\]
\[
I_{3} = \frac{z_1^{2}}{2\pi\varepsilon^{4}} \int_{A_{s}} \int_{A_{s}} \left( x_{1/2}^{3/2} - z_1^{2} \right) \cdot \frac{x_1 - x_1'}{|x - x'|^{2}} d x' dx,
\]
and \( I_{\rho} \) the remaining regular terms.

Let us consider \( I_{1} \) first. Using Taylor’s expansion, \( I_{1} \) can be rewritten as
\[
I_{1} = -\frac{z_1^{2}}{4\pi\varepsilon^{4}} \cdot \ln \frac{1}{s} \cdot \int_{A_{s}} x_1 \left( z_1^{-3/2} - \frac{3}{2z_1^{3/2}} \cdot (x_1 - z_1) + O(|x_1 - z_1|^2) \right) dx
\times \int_{A_{s}} x_1' \left( z_1^{1/2} + \frac{1}{2z_1^{1/2}} \cdot (x_1' - z_1) + O(|x_1' - z_1|^2) \right) dx'.
\]

Then, we are to estimate each terms in the product. Using circulation constraint (3.2), we have
\[
\frac{z_1}{4\pi\varepsilon^{4}} \cdot \ln \frac{1}{s_0} \cdot \int_{A_{s}} x_1 dx \int_{A_{s}} x_1' dx' = \kappa \cdot \frac{s_0^{2}}{4\pi\varepsilon^{2}} \cdot z_1^{2} \ln \frac{1}{s_0}.
\]
By the odd symmetry of \( x_1 - z_1 \) on \( x_1 = z_1 \), it holds
\[
\frac{1}{\varepsilon^{2}} \int_{A_{s}} x_1 (x_1 - z_1) dx = \frac{1}{\varepsilon^{2}} \int_{A_{s}} x_1' (x_1' - z_1) dx
= \frac{1}{\varepsilon^{2}} \int_{B_{s_0}(z)} z_1 (x_1 - z_1) dx + \frac{1}{\varepsilon^{2}} \int_{B_{s_0}(z)} (x_1 - z_1)^2 dx
+ \frac{1}{\varepsilon^{2}} \left( \int_{A_{s}} x_1 (x_1 - z_1) dx - \int_{B_{s_0}(z)} x_1 (x_1 - z_1) dx \right)
= O(\varepsilon^{2}) + O \left( \frac{1}{\varepsilon} \right) \cdot |A_{s} \Delta B_{s_0}(z)| = O \left( \varepsilon^{2} + \frac{e_{s}}{\varepsilon} \right)
\]
Notice that the remaining terms in the product have a higher order on $\varepsilon$. Thus we have shown

$$I_1 = \kappa \cdot \frac{s_0^2}{4\varepsilon^2} \cdot \frac{z_1^2}{s_0} \cdot \ln \left( \frac{1}{\varepsilon^2} \right) + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right).$$

(C.3)

For the second term $I_2$, we also expand it as

$$I_2 = -\frac{z_1^3}{8\pi \varepsilon^4} \int_{B_{s_0}(z)} \int_{B_{s_0}(x)} x_1^{-1/2} \cdot (x_1 - z_1) + O(|x_1 - z_1|^2)$$

$$\times \int_{B_{s_0}(x)} \left( z_1^{3/2} + \frac{3z_1^{1/2}}{2} \cdot (x_1' - z_1) + O(|x_1' - z_1|^2) \right) \ln \left( \frac{s_0}{|x - x'|} \right) dx' dx.$$  

Using a similar method as we deal with $I_1$, it holds

$$I_2 = -\frac{z_1^3}{4\pi \varepsilon^4} \int_{B_{s_0}(z)} \int_{B_{s_0}(x)} x_1^{-1/2} \cdot (x_1 - z_1) + O(|x_1 - z_1|^2)$$

$$\times \left( z_1^{3/2} + \frac{3z_1^{1/2}}{2} \cdot (x_1' - z_1) + O(|x_1' - z_1|^2) \right) \cdot \ln \left( \frac{s_0}{|x - x'|} \right) dx' dx + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right)$$

(C.4)

Now we turn to $I_3$ and obtain

$$I_3 = \frac{z_1^3}{2\pi \varepsilon^4} \int_{B_{s_0}(z)} \int_{B_{s_0}(x)} \left( z_1^{1/2} + \frac{1}{2z_1^{1/2}} \cdot (x_1 - z_1) + O(|x_1 - z_1|^2) \right)$$

$$\times \left( z_1^{3/2} + \frac{3z_1^{1/2}}{2} \cdot (x_1' - z_1) + O(|x_1' - z_1|^2) \right) \cdot \frac{x_1 - x_1'}{|x - x'|^2} dx' dx$$

$$= \frac{z_1^3}{4\pi \varepsilon^4} \int_{B_{s_0}(z)} \int_{B_{s_0}(x)} (x_1 - z_1) + O(|x_1 - z_1|^2) \cdot \frac{x_1 - x_1'}{|x - x'|^2} dx' dx + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right)$$

(C.5)

$$= \frac{z_1^3}{2\pi \varepsilon^4} \int_{B_{s_0}(z)} \left( \frac{s_0^2}{4} - \frac{|x - z|^2(x_1 - z_1)}{8} \right) dx + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right)$$

$$= -\frac{z_1^3}{2\pi \varepsilon^4} \int_{B_{s_0}(z)} \left( \frac{s_0^2}{4} - \frac{|x - z|^2(x_1 - z_1)}{8} \right) dx + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right)$$

$$= -\kappa \cdot \frac{s_0^2}{8\varepsilon^2} \cdot z_1^2 + O \left( \frac{e_\varepsilon}{\varepsilon^2} \right).$$
For the last term $I_\rho$, it is easy to verify that

$$I_\rho = -\frac{z_1^2}{4\pi\varepsilon^2} \int_{A_\varepsilon} x_1^{-1/2} \int_{A_\varepsilon} \left( \frac{x_1^{3/2}}{2} \cdot (\ln(x_1'x_1) + 2 \ln 8 - 4) + x_1^{3/2} \right) dx' dx$$

$$+ \kappa \cdot \frac{s_0^2}{4\varepsilon^2} \cdot z_1^2 + O \left( \frac{e_\varepsilon}{\varepsilon^2} + \varepsilon^2|\ln\varepsilon| \right)$$

$$= -\kappa \cdot \frac{s_0^2}{4\varepsilon^2} \cdot z_1^2 \left( \ln \frac{8z_1}{s_0} - 2 \right) + O \left( \frac{e_\varepsilon}{\varepsilon^2} + \varepsilon^2|\ln\varepsilon| \right).$$

Combining (C.3) (C.4) (C.5) (C.6), we finally obtain

$$-\frac{1}{\varepsilon^2} \int_{B_\varepsilon(z)} x_1^2 \partial_1 \psi_2,\varepsilon(x) \cdot 1_{A_\varepsilon}(x) dx = -\kappa \cdot \frac{s_0^2}{4\varepsilon^2} \cdot z_1^2 \left( \ln \frac{8z_1}{s_0} - \frac{5}{4} \right) + O \left( \frac{e_\varepsilon}{\varepsilon^2} + \varepsilon^2|\ln\varepsilon| \right),$$

which is the desired result. $\square$

From (C.2), Lemma C.3 and Lemma C.4, we obtain a relation of $\kappa$, $W$, $s_0$ and $z_1$, which has been used to derive Kelvin–Hicks formula in Section 3. We summarize this result as follows.

**Lemma C.5.** It holds

$$Wz_1 \ln \frac{1}{\varepsilon} - \frac{\kappa}{4\pi} \ln \frac{8z_1}{s_0} + \frac{\kappa}{16\pi} = O \left( \frac{e_\varepsilon}{\varepsilon^2} + \varepsilon^2|\ln\varepsilon| \right).$$

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