Surprising Aspects of Fluctuating “Pulled” Fronts

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Abstract. Recently it has been shown that when an equation that allows so-called pulled fronts in the mean-field limit is modelled with a stochastic model with a finite number $N$ of particles per correlation volume, the convergence to the speed $v^*$ for $N \to \infty$ is extremely slow — going only as $\ln^{-2} N$. However, this convergence is seen only for very high values of $N$, while there can be significant deviations from it when $N$ is not too large. Pulled fronts are fronts that propagate into an unstable state, and the asymptotic front speed is equal to the linear spreading speed $v^*$ of infinitesimal perturbations around the unstable state. In this paper, we consider front propagation in a simple stochastic lattice model. The microscopic picture of the front dynamics shows that for the description of the far tip of the front, one has to abandon the idea of a uniformly translating front solution. The lattice and finite particle effects lead to a “halt-and-go” type dynamics at the far tip of the front, while the average front behind it “crosses over” to a uniformly translating solution. In this formulation, the effect of stochasticity on the asymptotic front speed is coded in the probability distribution of the times required for the advancement of the “foremost occupied lattice site”. These probability distributions are obtained by matching the solution of the far tip with the uniformly translating solution behind in a mean-field type approximation, and the results for the probability distributions compare well to the results of stochastic numerical simulations. This approach allows one to deal with much smaller values of $N$ than it is required to have the $\ln^{-2} N$ asymptotics to be valid.

1. THE BASICS OF FRONT PROPAGATION: PULLED FRONTS

In pattern forming systems quite often situations occur where patches of different bulk phases occur which are separated by fronts or interfaces. In such cases, the relevant dynamics is usually dominated by the dynamics of these fronts. When the interface separates two thermodynamically stable phases, as in crystal-melt interfacial growth problems, the width of the interfacial zone is usually of atomic dimensions. For such systems, one often has to resort to a moving boundary description in which the boundary conditions at the interface are determined phenomenologically or by microscopic considerations. A question that naturally arises for such interfaces is the influence of stochastic fluctuations on the motion and scaling properties of such interfaces.

At the other extreme is a class of fronts which arise in systems that form patterns, and in which the occurrence of fronts or transition zones is fundamentally related to their nonequilibrium nature, as they do not connect two thermodynamic equilibrium phases

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which are separated by a first order phase transition. In such cases — for example, chemical fronts [1], the temperature and density transition zones in thermal plumes [2], the domain walls separating domains of different orientation in rotating Rayleigh-Bénard convection [3], or streamer fronts in discharges [4] — the fronts are relatively wide and therefore described by the same continuum equations that describe nonequilibrium bulk patterns. The lore in nonequilibrium pattern formation is that when the relevant length scales are large, (thermal) fluctuation effects are relatively small [5]. For this reason, the dynamics of many pattern forming systems can be understood in terms of the deterministic dynamics of the basic patterns and coherent structures. For fronts, the first questions to study are therefore properties like existence and speed of propagation of the front solutions of the deterministic equations. A class of fronts, for which these questions can be answered theoretically, are the so-called pulled fronts. Pulled fronts are the fronts that propagate into a linearly unstable state, and whose asymptotic front speed is equal to the linear spreading speed $v^*$ of infinitesimal perturbations around the unstable state [6, 7, 8]. The name pulled front refers to the picture that in the leading edge of these fronts, the perturbation around the unstable state grows and spreads with speed $v^*$, while the rest of the front gets “pulled along” by the leading edge. The intuitive idea is captured in Fig. 1, where we consider the (exaggerated) growth and spreading of an infinitesimal perturbation around the linearly unstable state. Three snapshots of this perturbation, taken at three different time instants in the laboratory frame are plotted in Fig. 1(a). The initial perturbation is chosen in such a way that it decays as $\exp[-\lambda^* x]$ for $x \to \infty$, where $\lambda^*$ is the exponent associated with $v^*$. The special status that the quantities $\lambda^*$ and $v^*$ have in the growth and spreading of this perturbation is coded in the fact that the $\exp[-\lambda^* x]$ decay of the perturbation for $x \to \infty$ remains preserved at all stages of its development. This fact is further illustrated in Fig. 1(b), where the same three snapshots [as in Fig. 1(a)] have been plotted in the comoving frame, moving with speed $v^*$ w.r.t. the laboratory frame. That the notion of the leading edge ahead of the front grows and spreads with speed $v^*$ and thereby pulls the rest of the front along with it is not merely an intuitive picture but can be turned into a mathematically precise analysis is illustrated by the recent derivation of exact results for the general power law convergence of the front speed to the asymptotic value $v^*$ [8]. On the other hand, fronts which propagate into a linearly unstable state and whose asymptotic speed is $> v^*$ are referred to as pushed, as it is the nonlinear growth in the region behind the leading edge that pushes their front speed to higher values. If the state is not linearly unstable, then $v^*$ is trivially zero; in such cases front propagation is always dominated by the nonlinear growth in

FIGURE 1. Illustration of $v^*$ as the linear spreading speed of infinitesimal perturbations around the unstable state.
the front region itself, and hence fronts in this case are in a sense pushed too. To obtain
the asymptotic speed of a pushed front, one has to solve the full nonlinear equation; in
general it is not possible to do so except for a special set of parameter values. To provide
a more quantitative flavour of how the growth and spreading of infinitesimal perturba-
tion around the unstable state play a very important role for pulled fronts, let us now
consider and examine an example of a deterministic equation that admits pulled fronts.
The equation that we choose for this purpose is the so-called Fisher equation, which was
at first used to model the spreading of advantageous genes in a population [9]. In this
model, the density of the advantageous genes is denoted by $\phi(x,t)$, whose dynamics is
described by means of the equation

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^n. \quad n > 1,$$

for example 2 or 3 (1)

Equation (1) has two stationary states, of which $\phi(x,t) = 0$ is (linearly) unstable and
$\phi(x,t) = 1$ is stable. Therefore, if the system is prepared in a way such that these two
states coexist in a certain region of space, then the stable state invades the unstable one
and propagates into it. An instantaneous configuration of the resulting front is shown in
Fig. 2.

To obtain the front solution admitted by Eq. (1), we rewrite it in a frame that moves
w.r.t. the laboratory frame at a constant speed $v$, and look for a stationary solution of $\phi$
in this comoving frame. In terms of the comoving co-ordinate $\xi = x - vt$, Eq. (1) can be
simply rewritten by means of a change of variables from $(x,t)$ to $(\xi,t)$, as

$$\frac{\partial \phi}{\partial t} - v \frac{\partial \phi}{\partial \xi} = D \frac{\partial^2 \phi}{\partial \xi^2} + \phi - \phi^n. \quad (2)$$

The crucial relevance of the growth and spreading of infinitesimal perturbations enters
naturally in this front solution, as the propagating infinitesimal perturbations around the
unstable state in the leading edge ahead of the front sets on the instability making way
for further growth. At the leading edge of the front, the $\phi$-values are very close to the
unstable state value, i.e., $\phi \ll 1$, and one can neglect the nonlinear term $\phi^n$ compared to
$\phi$ in Eq. (2). The stationary solution of the resulting linear equation can then be solved
by using $\phi \sim \exp[-\lambda \xi]$, yielding the following relation between $v$ and $\lambda$:

$$v(\lambda) = D \lambda + \frac{1}{\lambda}.$$  (3)

The curve for the dispersion relation between $v(\lambda)$ and $\lambda$ is schematically shown in
Fig. 3. It has a minimum at $\lambda^*$, and $v^* = v(\lambda^*) = 2\sqrt{D}$. The actual dispersion relation

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**FIGURE 2.** An instantaneous snapshot of the front configuration of Eq. (1).
depends on the model that one studies, but the U-shape is a characteristic of fronts that propagate into a linearly unstable state. Although Eq. (3) indicates that Eq. (1) has a front solution for all values of $\lambda$ (and correspondingly all possible front speeds), from which it might a priori seem that the quantities $\lambda^*$ and $v^*$ are not special in any way, the actual selection of the asymptotic front speed is obtained only after a proper stability analysis of the front profile in the comoving frame. Such a stability analysis yields the result that with the initial condition that $\phi(x,t)|_{t=0}$ that decays faster than $\exp[-\lambda^*x]$ for $x \to \infty$, the front speed converges uniformly\(^2\) to $v^*$ as [8, 10]

$$v(t) = v^* - \frac{3}{2\lambda^*} t + O(t^{-3/2}), \quad (4)$$

as the front shape relaxes to its asymptotic steady state configuration $\phi^*(x-v^*t)$.

2. FLUCTUATING “PULLED” FRONTS: A DIFFERENT PARADIGM

From the above perspective, it is maybe less of a surprise that the detailed questions concerning the stochastic properties of inherently nonequilibrium fronts have been addressed, to some extent, only relatively recently [11, 12, 13, 14, 15, 16, 17, 18], and that it has taken a while for researchers to become fully aware of the fact that the so-called pulled fronts [6, 7, 8, 19] that propagate into an unstable state, do not fit into the common mold: they have anomalous sensitivity to particle effects [14, 15, 16, 20, 21], and have been argued to display uncommon scaling behavior [18, 22, 23, 24, 25].

All these effects have one origin in common, namely the fact that the dynamics of pulled fronts, by its very nature, is not determined by the nonlinear front region itself, but by the region at the leading edge of the front, where deviations from the unstable state are small. To a large degree, this semi-infinite region alone determines the universal relaxation of the speed of a deterministic pulled front to its asymptotic value [7, 8, 14], as well as the anomalous scaling behavior of stochastic fronts [22, 23, 24] in continuum.

\(^2\) Uniform convergence means that the convergence behaviour (4) of the front speed is the same irrespective of the value of $\phi$ at which the speed is being measured.
equations with multiplicative noise. The crucial importance of the region, where the deviations from the unstable state are small, also implies that if one builds a lattice model version of a front propagating from a stable into a linearly unstable state, the front speed is surprisingly sensitive to the dynamics of the tip (the far end) of the front where only one or a few particles per lattice site are present. This is the main subject of this paper, and we will demonstrate the effect of discreteness on front propagation by means of considering a discrete particle model of Fisher equation (1).

Following Refs. [11, 26], the discrete particle model of Fisher equation (1) that we will consider in this paper is that of the reaction-diffusion system $X \leftrightarrow 2X$ on a lattice. In this system, an $X$ particle on any lattice site can diffuse to one of its nearest neighbour lattice sites with a diffusion rate $D$. The rate of the forward reaction $X \rightarrow 2X$ is $\frac{N}{2}$, while the rate of the backward reaction is normalized to unity. A forward reaction on a lattice site $k$ increases the number of $X$ particles on lattice site $k$ by one, and a backward $2X \rightarrow X$ reaction on lattice site $k$ reduces the number of $X$ particles on lattice site $k$ by one. For large $N$ values, the microscopic interaction indicates that an initial conglomeration of $X$ particles grows both in size and spread. The microscopic rules also dictate that the growth saturates when the number of particles on a lattice site reaches approximately $N$ — at that stage, on average, the amount of new $X$ particles generated due to the forward reaction equals the amount of $X$ particles annihilated due to the backward reaction.

The deterministic mean-field limit of the this reaction-diffusion system yields the equation [11, 26]

$$\frac{\partial \phi_k}{\partial t} = D \left[ \phi_{k+1} + \phi_{k-1} - 2\phi_k \right] + \phi_k - \phi_k^2. \quad (5)$$

Here, $\phi_k = \langle N_k \rangle / N$, where $\langle N_k \rangle$ is the conditionally (ensemble) averaged $3$ number of $X$ particles on the $k$-th lattice site. Equation (5) is the lattice version of Eq. (1), and it admits a pulled front solution, which can be obtained by using the uniformly translating solution $\phi_k(t) \equiv \phi(k - vt)$. From the analysis and discussion of the previous section, therefore, one can expect that the asymptotic front speed is $v^*$, where the corresponding front solution $\phi^*(k - v^*t)$ behaves $\sim \exp[-\lambda^* \xi]$ for $\xi \to \infty$. Here $\xi = k - v^*t$ and $\lambda^*$ and $v^*$ correspond to the minimum of the dispersion relation $v(\lambda)$ vs. $\lambda$ for Eq. (5) (similar to Fig. 3). In actuality, however, the asymptotic speed turns out to be $< v^*$. This was first observed numerically in Ref. [11].

The reason behind having an asymptotic speed $< v^*$ stems from the discreteness of the particles and the lattice effects, and is understood quite simply when one takes an instantaneous snapshot of the resulting front, moving from the left to the right, for one single realization at time $t \to \infty^4$. Such a snapshot is shown in Fig. 4. The important

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3 In view of the stochasticity in the system, one has to be careful about taking averages. A simple ensemble averaging does not yield a steady front shape due to diffusive wandering of individual front realizations. This notion is best understood by means of Fig. 5 of Ref. [11], which shows the ensemble average front shape by means of hashed lines. One needs to filter out the diffusive wandering of fronts before taking the average, and this is what is meant by conditional (ensemble) averaging. For a detailed discussion, see Sec. III.B of Ref. [26]

4 The limit $t \to \infty$ is necessary to ensure that the front relaxes to a steady shape in an average sense, so that there are no residual large scale transient effects.
aspect to notice in Fig. 4 is that unlike the semi-infinite region that the leading edge occupies for a deterministic front (as shown in Fig. 2), there exists a foremost occupied lattice site (f.o.l.s.) in this snapshot, on the right of which no lattice site has ever been occupied before. In the language of the mathematicians, this means that the actual front region has a compact support. The crucial role played by the discreteness effects of the particles and the lattice is reflected in the mechanism of the front propagation at the f.o.l.s., where the mechanism is not that of a uniform translation, but instead, is of **halt-and-go, which in this reaction-diffusion model is diffusion dominated**, as we demonstrate below. First of all, it is clear from the very nature of the reaction-diffusion equation that unless there is at least one X particle on a lattice site, the growth in the number of particles on that lattice site does not take place. This means that the position of the f.o.l.s. moves towards the right only when a particle from the f.o.l.s. makes a diffusive hop towards the right. The movement of the f.o.l.s. is therefore diffusion dominated, as opposed to being also driven by growth; i.e., the phenomenon of leading edge ahead of the front setting on the instability making way for further growth that takes place for a deterministic mean field equation of a pulled front, does not take place for the corresponding discrete particle realization — this makes the fronts made of discrete particles on a lattice “pulled” as opposed to being pulled. Secondly, these diffusive hops are not continuous in time, i.e., there is a finite time difference between any two successive forward movements of the f.o.l.s. (this is what is meant by halt-and-go; the f.o.l.s. halts for the times between the successive hops), and these time differences are stochastic in nature. We stress here that the halt-and-go mechanism of the dynamics of the f.o.l.s. is a generic consequence of having discrete particles on a discrete lattice; in the present model that we consider, the movement is diffusion dominated, but there can be other models where the f.o.l.s. moves by means of some other mechanism (see for example, clock model [20]).

The subtlety that the movement of the f.o.l.s. is not driven by growth for “pulled” fronts made of discrete particles on a lattice was first realized by Brunet and Derrida [14], who chose to implement it by having a growth cutoff at a $\phi_k$ value $1/N$ in the deterministic mean-field Eq. (5). The choice of $1/N$ was motivated by the fact that the number of particles on the f.o.l.s. is $\mathcal{O}(1)$, which corresponds to a field value $\phi_k \sim 1/N$. The uniformly translating solution of the corresponding mean-field front equation for
the reaction-diffusion process $X \leftrightarrow 2X$

$$\frac{\partial \phi_k}{\partial t} = D \left[ \phi_{k+1} + \phi_{k-1} - 2\phi_k \right] + \left[ \phi_k - \phi_k^2 \right] \Theta \left( \phi_k - \frac{1}{N} \right),$$  \hspace{1cm} (6)

where the $\Theta$ term is the heavyside theta function$^5$, yields an asymptotic speed $v_{\text{cutoff}}$, given by

$$v_{\text{cutoff}} = v^* - \frac{\pi^2 D \cosh \lambda^*}{\ln^2 N}.$$  \hspace{1cm} (7)

The essential correctness of Eq. (7), that the convergence of the asymptotic front speed to $v^*$ behaves $\sim 1/\ln^2 N$ has been observed elsewhere in systems of physical interest such as clock model [20], in field-theoretical approach [16], and as well as in studies on noisy front propagation in Fisher equation in the limit of “weak noise” by mathematicians [28]. However, it has also been noted by many theorists working in this field that one often has to go to very high values of $N$ to obtain the $1/\ln^2 N$ convergence of the asymptotic front speed to $v^*$. For large but more reasonable values of $N$, there are often significant deviations from Eq. (7) (see for example, Ref. [15]). A question which then naturally arises is the following: how does one bridge the gap between Eq. (7), which is valid for asymptotically large $N$, and the corresponding results for finite $N$?

Based on the discussion three paragraphs above, in what follows below, we will appeal to the reader for a more comprehensive description of fluctuating “pulled” fronts on a lattice that combines the stochastic halt-and-go nature of front propagation at the f.o.l.s. with a uniformly translating front solution few lattice sites behind the f.o.l.s. Such a description leads one to a concrete mathematical formalism developed elsewhere [26], of which we will provide only the flavour in this paper. This description and the associated formalism serves to bridge this gap between the results for asymptotically large $N$ and that for reasonably large values of $N$. Afterwards, we will argue why Eq. (7) is essentially correct for asymptotically large $N$ values. The mathematical formalism of Ref. [26], however, has open questions, which we will leave out for the next section.

We start with the fact that the very definition of the f.o.l.s. means that all the lattice sites on the right of it are empty (and they have never been occupied before). Naturally, a lattice site, which has never been occupied before attains the status of the f.o.l.s. as soon as one particle hops into it from the left. In reference to the lattice, the position of the f.o.l.s. remains fixed at this site for some time, i.e., after its creation, an f.o.l.s. remains the f.o.l.s. for some time. During this time, however, the number of particles on and behind the f.o.l.s. continues to grow. As the number of particles grows on the f.o.l.s., the chance of one of them making a diffusive hop on to the right also increases. At some instant, a particle from the f.o.l.s. hops over to the right: as a result of this hop, the position of the f.o.l.s. advances by one unit on the lattice, or, viewed from another angle, a new f.o.l.s. is created which is one lattice distance away on the right of the previous one. Microscopically, the selection process for the length of the time span

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$^5$ The “pulled” nature can be made mathematically very precise for Eq. (6), see for example Ref. [27].
between two consecutive f.o.l.s. creations is stochastic, and the inverse of the long time average of this time span defines the front speed. Simultaneously, the amount of growth of particle numbers on and behind the f.o.l.s. itself depends on the time span between two consecutive f.o.l.s. creations (the longer the time span, the longer the amount of growth). As a consequence, on average, the selection mechanism for the length of the time span between two consecutive f.o.l.s. movements, which determines the asymptotic front speed, is nonlinear.

This inherent nonlinearity makes the prediction of the asymptotic front speed difficult. One might recall the difficulties associated with the prediction of pushed fronts due to nonlinear terms in this context, although the nature of the nonlinearities in these two cases is completely different. In the case of pushed fronts, the asymptotic front speed is determined by the mean-field dynamics of the fronts, and the nonlinearities originate from the nonlinear growth terms in the partial differential equations that describe the mean-field dynamics. On the other hand, for fronts consisting of discrete particles on a discrete lattice, the corresponding mean-field growth terms are linear, but since the asymptotic front speed is determined from the probability distribution of the time span between two consecutive f.o.l.s. movements, on average, it is the relation between this probability distribution and the effect of the linear growth terms that the nonlinearities stem from. As one can now clearly see, Brunet and Derrida’s cutoff picture, as described above, does not take the stochasticity of the halt-and-go mechanism into account.

Our approach is to discuss a separate probabilistic theory for the hops to create the new f.o.l.s., and then to demonstrate that by matching the description of the behaviour in this region to the more standard one (of growth and roughly speaking, uniform translation) behind it, one obtains a consistent and more complete description of the stochastic and discreteness effects on front propagation. In the simplest approximation, the theory provides a very good fit to the data, but our approach can be systematically improved by incorporating the effect of fluctuations as well [26]. Besides providing insight into how a stochastic front propagates at the far tip of the leading edge, our analysis naturally leads to a more complete description that allows one to interpret the finite \( N \) corrections to the front speed for much smaller values of \( N \) than that are necessary to see the asymptotic result of Brunet and Derrida [14]. We stress here that such a procedure can be carried out for any fluctuating “pushed” front, although in the present context, we will confine ourselves only to the reaction-diffusion system \( X \leftrightarrow 2X \).

In the resulting mathematical formalism [26], we follow the movement of the f.o.l.s. of one single front realization over a long time at large times. Let us denote the \( j \) successive values of the duration of halts of the f.o.l.s. by \( \Delta t_1, \Delta t_2, \ldots, \Delta t_j \). One can then define the front speed as

\[
v_N = \lim_{j \to \infty} j \left[ j \sum_{j' = 1}^{j} \Delta t_j \right]^{-1}.
\] (8)

Put in a different way, if we denote the probability that a f.o.l.s. remains the f.o.l.s. for time \( \Delta t \) by \( \mathcal{P}(\Delta t) \), the asymptotic front speed, according to Eq. (8), is given by

\[
v_N = \left[ \int_0^{\infty} d(\Delta t) \Delta t \mathcal{P}(\Delta t) \right]^{-1}.
\] (9)
FIGURE 5. A snapshot of a realization of the reaction-diffusion system $X \leftrightarrow 2X$ (shown by the jaggered line) as $t \to \infty$, and how a theorist would picture such a front. In this picture, as the smooth line shows, a uniformly translating solution travelling with speed $v_N$, given by $\phi_k(t) = \phi^{(0)}(k - v_N t)$ and obeying Eq. (5), is suitable (in an average sense) everywhere but few lattice sites at the tip of the front leading up to the f.o.l.s. The region where such a description holds is denoted above by “deterministic region”. The “deterministic region” is further subdivided into two parts — in the “linear region”, the nonlinear term $[\phi^{(0)}]^2$ of Eq. (5) can be neglected. In the “nonlinear region”, however, all the terms of Eq. (5) have to be taken into account.

Our goal is to obtain a theoretical expression of $\mathcal{P}(\Delta t)$, but to do so, we draw the reader’s attention to Fig. 5. As described above, we assume that a uniformly translating (with speed $v_N$) deterministic mean-field description, given by $\phi_k(t) = \phi^{(0)}(k - v_N t)$, is valid everywhere but a few lattice sites leading up to the f.o.l.s. The region where such a description [namely that its dynamics is given by Eq. (5)] holds is denoted by “deterministic region” in Fig. 5. This region can be further subdivided into two parts, a “linear region” and a “nonlinear region”, where the nonlinear $[\phi^{(0)}]^2$ term can and cannot be neglected respectively. Although the amount of fluctuations in the number of particles on individual lattice sites at the strongly fluctuating tip region is of the same order of magnitude as the particle numbers themselves, following Ref. [26], we assume that the strongly fluctuating tip region can be modelled by a time dependent mean-field type description without uniform translation. In this description, at the tip region of the front, we express the front solution as $\phi_k(t) = \phi^{(0)}(k - v_N t) + \delta \phi_k(t)$. The quantities $\delta \phi_k(t)$ are non-zero due to the halt-and-go motion of the f.o.l.s., but at the boundary between the tip region and the “linear region” $\delta \phi_k(t)$ vanishes. As explained previously, to obtain the expression of $\mathcal{P}(\Delta t)$, one needs the expressions of $\phi^{(0)}(k - v_N t)$ and $\delta \phi_k(t)$. Moreover, the expression of $v_N$ itself is needed to solve for $\phi^{(0)}(k - v_N t)$, and $v_N$ can be determined only from $\mathcal{P}(\Delta t)$ as Eq. (9) shows. This indicates that the only way to obtain the expression of $\mathcal{P}(\Delta t)$ is to solve a whole system of equations self-consistently [26]. We also note here that in this self-consistent theory, there is an effective parameter.

This self-consistent set of equations are highly nonlinear and complicated, but due to the presence of the effective parameter in our theory, our procedure to obtain the $\mathcal{P}(\Delta t)$
is not predictive. However, the fact that it the theory generates a probability distribution that agrees so well with numerical simulations is indicative of the essential correctness of such a description [26] of a fluctuating “pulled” front. The self-consistent theoretical curves of $P(\Delta t) = \int_{\Delta t}^{\infty} dt' \mathcal{P}(t')$ for $D = 1$ and $N = 10^4, 10^2, 10^3$ and $10^5$ (in that order) are shown in Fig. 6. The corresponding numerical comparison of front speeds are shown in Table I. First, we notice that in the graphs of Fig. 6, the theoretical curves lie below the simulation histograms at $\Delta t \approx 2/v_N$ — this is an artifact of the matching that we had to do for the expressions of $P(\Delta t)$ below and above $\Delta t \sim 2/v_N$. This difference occurs due to certain fluctuation and correlation effects [26]. Secondly, the agreement between the simulation data and the self-consistent theory is not very good for $N = 10^5$ — at this value of $N$, the simulation gets very slow and one has to continuously remove particles from the saturation region of the front to gain program speed, which affects the $P(\Delta t)$ histograms significantly for large times.

| $N$   | $v_N$(simulation) | $v_N$(theoretical) | $v_N$[Eq. (7)] |
|-------|-------------------|-------------------|----------------|
| $10^2$| 1.778             | 1.808             | 1.465          |
| $10^3$| 1.901             | 1.899             | 1.803          |
| $10^4$| 1.964             | 1.988             | 1.925          |
| $10^5$| 2.001             | 2.057             | 1.976          |

Table I: Comparison of $v_N$ values, simulation, self consistent theory [26] [indicated by $v_N$(theoretical)], and that of Eq. (7).

In addition to such good agreements between our self-consistent theory and simulations for the $P(\Delta t)$ curves, a significant observation is the following: as the value of $N$ is increased in our self-consistent theory, we find that the quantity $\delta \phi_k / \phi_k^{(0)}$ values at
the tip of the front gradually reduces [26]. The stochastic halt-and-go character of the movement of the f.o.l.s., which is usually occupied by \( O(1) \) number of particles, however, continues to remain valid for any value of \( N \). This implies that for very large \( N \), one approaches the picture of a fluctuating “pulled” front, where a uniformly translating mean-field description (5) holds all the way up to one lattice site behind the f.o.l.s., while only the dynamics of the f.o.l.s. is a stochastic halt-and-go process. Such a simplified picture has been studied numerically in the last paper of Ref. [14].

We finally end this section with a short note arguing why the expression (7) is correct for asymptotically large \( N \). In fact, it can be understood very simply when one observes that the length of the “linear region” in Fig. 5 increases as \( \ln N \), and combines it with the expectation that the asymptotic front speed should be \( < v^* \) (as motivated in the origin of the terminology “pulled” front). For a deterministic pulled front equation, such as Eqs. (1) or (5), the length of the linear region is infinite, and in such cases, in the absence of any length scale, the stability criterion of the front solution in the comoving frame dictates that for the selected front speed, it is good enough to consider exponentially decaying solution \( \phi(\xi) \sim \exp[-\lambda \xi] \) with \( \lambda \equiv \text{Re}(\lambda) \). On the other hand, for “pulled fronts” composed of discrete particles on a lattice, the positivity of \( \phi(\xi) \) demands that one has to consider an oscillatory function of wavelength \( \pi/\ln N \) as a multiplicative factor to the exponentially decaying \( \exp[-\lambda \xi] \) profile for the front solution \( \phi(\xi) \) to obtain an asymptotic speed \( < v^* \). Just from this argument alone, it is possible to derive the asymptotic front speed of Eq. (7).

3. OUTLOOK AND UNSOLVED PROBLEMS

When an equation that allows so-called pulled fronts in the mean-field limit is modelled with a stochastic model with a finite number \( N \) of particles per correlation volume, the convergence to the speed \( v^* \) for asymptotically large \( N \), as obtained by Brunet and Derrida, behaves as \( \ln -2N \), and this behaviour is model independent. However, for large but more reasonable values of \( N \), there are significant deviations from their result, and these deviations stem from the complicated stochastic halt-and-go dynamics of the foremost occupied lattice site, where the actual microscopic rules of the system under consideration play a crucial role, and therefore the deviations from the results of Brunet and Derrida are model dependent.

The message of this paper is as follows: to obtain the deviations from the \( 1/\ln^2 N \) convergence to \( v^* \) one really has to focus at the stochastic halt-and-go movement of the foremost occupied lattice site. From the mean-field limit of this fluctuating front, we know that the tip region is very important for its dynamics; as a result, the fluctuating tip plays a very significant role in deciding the asymptotic front speed, in which two very important aspects come to play a role — discrete nature of particles and discrete nature of the lattice indices. In this paper, we have outlined the formulation of a self-consistent theory, which we developed in Ref. [26], to model the fluctuating tip by means of a mean-field type description. The mean-field type description of the fluctuating tip is then matched to a uniformly translating solution behind. Due to the presence of an effective parameter in this self-consistent theory, it is not predictive. How to obtain a
predictive theory for moderately large values of $N$, and demonstrate analytically how
the corresponding front speed approaches to the expression (7) still remain unsolved
problems. However, one has to remember that in actuality, the tip is strongly fluctuating
and there are time-correlation effects involved (for a more detailed discussion, see Sec.
IV.C of Ref. [26]). Any alternative predictive theory, that one might think at this point,
must be able to take these into account, in addition to the mean-field type self-consistent
theory of Ref. [26].

The prospect of such a theory however, looks grim at this point. Not only the problem
becomes highly nonlinear, but also one must realize that the fluctuations in the number
of X particles on the lattice sites at the tip of the front is of the same order as the number
of X particles in them ($\sim 1$), and there does not exist any small parameter that one can
do perturbation theory with.

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