Abstract
The present work analyzes the distribution function of the finite scale local Lyapunov exponent of a pair of fluid particles trajectories in fully developed incompressible homogeneous isotropic turbulence. According to the hypothesis of fully developed chaos, this PDF is reasonably estimated by maximizing the entropy associated to such distribution, resulting in an uniform distribution function in a proper interval of variation of the local Lyapunov exponents. From this PDF, we determine the relationship between the average and maximum Lyapunov exponents and the longitudinal velocity correlation function. This link, which leads to the closure of von Kármán-Howarth and Corrsin equations, agrees with the relation obtained in the previous work [3], supporting the proposed PDF calculation, at least for the purposes of the energy cascade effect estimation. Furthermore, through the property that the Lyapunov vectors tend to align to the direction of the maximum growth rate of trajectories distance, we obtain the link between maximum and average Lyapunov exponents in line with the previous result.

Keywords: Corrsin equation, Fully developed chaos, Lyapunov exponent, Lyapunov vector, von Kármán-Howarth equation

1. Introduction
We analyze the distribution function of the finite scale local Lyapunov exponent of the fluid kinematic field in fully developed homogeneous isotropic turbulence for incompressible fluids. This exponent, defined as

$$\tilde{\lambda} \equiv \frac{d \ln \rho}{dt} = \frac{\xi \cdot \dot{\xi}}{\xi \cdot \xi}$$

is a quantity providing the instantaneous growth rate of the distance $\rho = \sqrt{\xi \cdot \xi}$ between the two fluid particles trajectories $x(t)$ and $y(t) = x(t) + \xi(t)$, where $\xi$ is the separation vector (finite scale Lyapunov vector). Because of non-smooth spatial variations of the velocity field, $\tilde{\lambda}$ can exhibit fluctuations of sizable amplitude with respect to its average value, thus $\tilde{\lambda}$ plays the role of a stochastic variable and will be distributed according to a certain PDF.

The present analysis is based on the property that the kinematics of a pair of fluid particles, characterized by $\xi$, is much faster and statistically independent with respect to the velocity field. This property, just discussed in [1, 2] for what concerns the closure of the von Kármán-Howarth and Corrsin equations [3, 1, 2], was previously supported by the arguments presented in Ref. [7] (and references therein), where the author observes that: i) the velocity fields $u(t, x)$ produce chaotic trajectories also for relatively simple mathematical structure of the right-hand sides $u(t, x)$, ii) the flows given by $u(t, x)$ stretch and fold continuously and rapidly causing an effective mixing of the particles trajectories.

Through the hypotheses of fully developed chaos and fluid incompressibility, we first estimate the interval of variation of $\tilde{\lambda}$. Thereafter, due to the fully developed chaos, we calculate the distribution function of $\tilde{\lambda}$ by maximizing the entropy associated to such PDF. As the consequence, $\tilde{\lambda}$ results to be uniformly distributed in such interval of
variations. In particular, we determine the link between average and maximum finite scale Lyapunov exponents, $\tilde{\lambda}$ and $\lambda$ respectively, resulting $\lambda = \lambda/2$. These exponents are so defined (see also the Appendix)

$$
\tilde{\lambda} = \langle \tilde{\lambda} \rangle
$$

$$
\lambda = \langle \lambda \rangle_{\xi \in \Omega}
$$

(2)

where $\langle \cdot \rangle$ and $\langle \cdot \rangle_{\xi \in \Omega}$ denote the average over the entire ensemble of $\xi$, and the average calculated on the ensemble of Lyapunov vectors where $\dot{\xi} \cdot \xi \geq 0$.

Through such PDF, $\lambda$ and $\lambda$ are expressed in terms of the longitudinal velocity correlation function $f = \langle u_r u_r \rangle / u^2$ where $u^2 = 1/3 \langle u \cdot u \rangle$, $u_r = u(t, x) \cdot r/r$ and $u_r' = u(t, x + r) \cdot r/r$. This relationship, which leads to the closure formulas of von Kármán–Howarth and Corrsin equations, coincides with that just presented in Ref. [3] where the author adopts only the average and maximum Lyapunov exponents, properly defined, without considering the distribution of the local exponents. Therefore, such PDF should adequately describe the statistics of $\tilde{\lambda}$, at least for what concerns the estimation of the energy cascade effects.

Finally, to obtain further confirmation of the previous result, we show that $\lambda = 2\tilde{\lambda}$ for incompressible fluid using the property that the Lyapunov vectors tend to align to the direction of the maximum growth rate of $\ln \rho$ [3].

2. Set of variations of finite scale local Lyapunov exponents in incompressible turbulence

This section proposes an analysis for estimating the range of variations of $\tilde{\lambda}$ based on the hypotheses of fluid incompressibility and fully developed chaos.

The present analysis starts from the consideration that the turbulent energy cascade is related to the fluid particles trajectories divergence, therefore the relative fluid kinematics plays an important role in the estimation of the properties of such energy cascade [3, 4]. The relative kinematics is expressed by finite scale Lyapunov vector $\xi$ which satisfies

$$
\dot{x} = u(t, x),
$$

$$
\dot{\xi} = u(t, x + \xi) - u(t, x)
$$

(3)

where $u = u(t, x)$ varies according to the Navier–Stokes equations, and $x(t)$ and $y(t) = x(t) + \xi(t)$ are two fluid particles trajectories. The local divergence between $x(t)$ and $y(t)$ is quantified by $\tilde{\lambda}$ which, due to the bifurcations of the kinematics field (kinematic bifurcations), exhibits oscillations of sizable amplitude with respect to its average value. These bifurcations continuously happen in those points of the physical space where

$$
\det (\nabla u(t, x)) = 0
$$

(4)

Observe that the kinematic bifurcations defined by Eq. (4) are not the Navier–Stokes equations bifurcations (dynamic bifurcation), but arise from these latter [4]. In fact, the Navier–Stokes bifurcations frequently occur determining continuously non–smooth spatial variations of $u(t, x)$ which in turn lead to the condition (4) in the several points of the space.

The definition of $\tilde{\lambda}$ given by Eq. (1) implies that $\xi$ can be locally expressed as

$$
\dot{\xi} = Q(t)\xi(0) \exp(\tilde{\lambda} t),
$$

(5)

where $\tilde{\lambda}$ plays the role of the stochastic variable and $Q(t)$ is a proper orthogonal matrix providing the orientation of $\xi$ in the inertial frame $\mathcal{R}$. Accordingly, $\dot{\xi}$ is

$$
\dot{\xi} = \tilde{\lambda} \xi + \omega \times \xi
$$

(6)

in which $\omega$ defines the angular velocity of $\xi$ with respect to $\mathcal{R}$. Therefore, the finite scale Lyapunov vectors $\xi_k$, $k = 1, 2, 3$ corresponding to three mutual orthogonal directions are locally represented by

$$
\xi_k = Q(t)\xi_k(0) \exp(\tilde{\lambda} t),
$$

$$
\dot{\xi}_k = \tilde{\lambda}_k \xi_k + \omega \times \xi_k,
$$

(7)

$k = 1, 2, 3.$
where ̃λ, k = 1, 2, 3 are the finite scale local Lyapunov exponents associated to the three directions. The classical local Lyapunov exponents ̃Λ, k = 1, 2, 3 are defined for ξk → 0, k = 1, 2, 3.

Now, in order to estimate the set of variations of ̃λ, observe that, due to fluid incompressibility, ∇·u ≡ 0, and the classical local exponents obey to the following condition
\[ \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 = 0, \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_3. \] (8)

In general, Eq.(8) does not hold for finite scale Lyapunov vectors. Nevertheless, Eq. (8) is valid for those finite scale local exponents for which ξi × ξj · ξk is locally preserved for i ≠ j, j ≠ k, k ≠ i, therefore, without lack of generality, such these exponents can be written in the form
\[ \tilde{\lambda}_1 = \lambda_m \cos(\varepsilon), \]
\[ \tilde{\lambda}_2 = \lambda_m \cos\left(\varepsilon + \frac{2}{3}\pi\right), \]
\[ \tilde{\lambda}_3 = \lambda_m \cos\left(\varepsilon + \frac{4}{3}\pi\right) \] (9)

where ε and λm are proper fluctuating quantities depending upon the velocity field, and
\[ \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 = 0, \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_3. \] (10)

Now, following the hypothesis of fully developed chaos, it is reasonable that ̃λ ranges in the set (λ0, λS), where λS > 0 assumes its maximum value compatible with Eq. (9), and λ0 is consequently calculated. This implies that λ0 = −λS/2, that is
\[ \tilde{\lambda} \in \left(-\frac{\lambda_S}{2}, \lambda_S\right), \]
\[ \lambda_S = \sup \{\lambda_m\} \] (11)

3. Incompressible fully developed turbulence

This section studies the distribution function of ̃λ, P̃ = P_̃(̃λ), in fully developed turbulence, where ̃λ ∈ (−λS/2, λS). P_̃ is formally given in terms of the PDF of the Lyapunov vectors, P, through the Frobenius Perron equation (see Appendix).

We will show that the proposed estimation of P_̃ leads to the following relations
\[ \langle \tilde{\lambda} \rangle = \frac{\lambda}{2}, \] (12)
\[ \langle \tilde{\lambda}^2 \rangle = \lambda^2. \] (13)

and to the link between ̃λ and the longitudinal velocity correlation function.

In order to determine P_̃, observe that the regime of fully developed turbulence corresponds to a situation of maximum chaos where the kinematic bifurcations cause a total loss of the initial condition data of ξ. Accordingly, it is reasonable that the entropy σ associated to P_̃ assumes its maximum value compatible with the condition that the integral of P_̃ over (−λS/2, λS) is equal to the unity, i.e.
\[ \sigma = - \int_{-\lambda_S/2}^{\lambda_S} P_\tilde{\lambda} \ln P_\tilde{\lambda} \, d\tilde{\lambda} = \max \] (14)
\[ \int_{-\lambda_S/2}^{\lambda_S} P_\lambda \, d\tilde{\lambda} = 1, \quad (15) \]

Equations (14) and (15) correspond to the following variational problem with fixed boundaries

\[ J = \int_{-\lambda_S/2}^{\lambda_S} \mathcal{V} \, d\tilde{\lambda} = \max, \quad (16) \]

where \( \mathcal{V} \) and \( \eta \) are, respectively, the lagrangian of the problem and the Lagrange multiplier associated to the condition (15). The maximum of \( \sigma \) is then calculated through steady condition for \( J \) (\( \delta J = 0 \)),

\[ \delta J = \int_{-\lambda_S/2}^{\lambda_S} \frac{\partial \mathcal{V}}{\partial P_\lambda} \delta P_\lambda \, d\tilde{\lambda} = 0, \quad \forall \, \delta P_\lambda, \quad \text{compatible with Eq. (15)} \quad (17) \]

Because of the arbitrary nature of \( \delta P_\lambda \), the maximum of \( J \) is achieved for

\[ \frac{\partial \mathcal{V}}{\partial P_\lambda} = 0, \quad (18) \]

that is \( \tilde{\lambda} \) is uniformly distributed in \((-\lambda_S/2, \lambda_S)\), being

\[ P_\lambda = \begin{cases} \frac{2}{3} \lambda_S, & \text{if } \tilde{\lambda} \in \left(-\frac{\lambda_S}{2}, \lambda_S\right) \\ 0 & \text{elsewhere} \end{cases} \quad (19) \]

Hence, \( \bar{\lambda} \) and \( \lambda \) are calculated in terms of \( P_\lambda \) with Eqs. (44) and (45) (see appendix)

\[ \lambda \equiv \langle \tilde{\lambda} \rangle = \frac{\lambda_S}{4} > 0, \quad (20) \]

\[ \lambda \equiv \langle \lambda \rangle_{\xi \leq 0} = \frac{\lambda_S}{2} = 2 \langle \tilde{\lambda} \rangle \]

Next, it is useful to calculate the mean square \( \langle \tilde{\lambda}^2 \rangle \)

\[ \langle \tilde{\lambda}^2 \rangle = \int_{-\lambda_S/2}^{\lambda_S} P_\lambda \tilde{\lambda}^2 \, d\tilde{\lambda} = \lambda^2. \quad (21) \]

According to Eq. (21), the mean square of \( \tilde{\lambda} \) equals the square of the average of \( \tilde{\lambda} \) calculated for \( \xi \cdot \xi \geq 0 \).

4. Lyapunov exponents in terms of longitudinal velocity correlation

The previous analysis allows to achieve the link between \( \lambda \) and the longitudinal velocity correlation function. In fact, the standard deviation of longitudinal velocity difference directly depends on \( u^2 \) and \( f \) according to

\[ \langle (u'_r - u_r)^2 \rangle = 2u^2 (1 - f(r)) \quad (22) \]

On the other hand, the Lyapunov theory gives the longitudinal velocity difference in terms of \( \tilde{\lambda} \)

\[ u'_r - u_r = \tilde{\lambda} \left( \xi \cdot \xi \right) \frac{\xi}{|\xi|} = r \quad (23) \]
Taking into account the isotropy and Eq. [21], \((u'_t - u_t)^2\) reads as
\[
\left\langle (u'_t - u_t)^2 \right\rangle \equiv \langle \xi^2 \rangle = \langle L^2 \rangle = \lambda^2 r^2,
\]
thus, \(\lambda = \lambda(r)\), \(\bar{\lambda} = \bar{\lambda}(r)\) depend upon the finite scale \(r\) through the longitudinal velocity correlation \(f\)
\[
\lambda(r) = \frac{\lambda(r)}{2} \equiv \frac{u}{r} \sqrt{\frac{1 - f(r)}{2}}
\]
Equation [25] leads to the closure of the von Kármán and Corrsin equations, and coincides with that given in Ref. [3]. Unlike Ref. [3], Eq. [25] is here achieved exploiting the shape of the distribution [19]. As the proposed analysis provides results which are in agreement with the previous ones, the hypothesis \(\sigma = \max\) seems to be an adequate assumption at least for the purposes of calculation of the energy cascade intensity.

5. Analysis through alignment property of \(\xi\)

An alternative way to achieve Eq. [12] consists in to exploit the fluid incompressibility, and the alignment property of \(\xi\), following which the Lyapunov vector tends to align to the direction of maximum growth rate of \(\rho\) [6]. In line with such alignment property, we consider a proper PDF \(P_\tau = P_\tau(t, x, \xi)\) arising from \(P(t, x, \xi)\) and exhibiting the same level of information of \(P\) at least of higher order terms, and which corresponds to a distribution of \(\xi\) such that \(\hat{\xi} \cdot \xi \geq 0\), i.e.
\[
\forall (x, \xi) \in \{x \times \{\xi\} \mid \hat{\xi} \cdot \xi \geq 0 \ P_\tau(t, x, \xi) > 0, \]
\[
\forall (x, \xi) \in \{x \times \{\xi\} \mid \hat{\xi} \cdot \xi < 0 \ P_\tau(t, x, \xi) = 0, \]
\[
\mathcal{H}(P_\tau) = \mathcal{H}(P) + \text{H.O.T.}
\]
where \(\mathcal{H}\) is the entropy associated to \(P\)
\[
\mathcal{H}(P) = - \int_x \int_\xi P \ln P dx d\xi
\]
Now, to estimate \(P_\tau\), we apply the mentioned alignment property of \(\xi\) [6]: in the absence of variations of \(x\), such alignment produces \(\hat{\xi} \cdot \xi \geq 0\), \(\forall (x, \xi) \in \{x \times \{\xi\}\) in the minimum time \(\tau = O(1/\lambda^2)\), where
\[
\Delta \xi = \hat{\xi}(t)\tau + O(\tau^2)
\]
is the changing of \(\xi\) calculated at \(x = \text{const}\). Hence, according to the Lyapunov theory, \(\ln \rho\) will exhibit variations that, in proximity of its average value \(\langle \ln \rho \rangle\), read as
\[
\frac{d \ln \rho}{dt} = \frac{\langle \ln \rho \rangle_{\tau+} - \ln \rho}{\tau}
\]
Therefore, we expect that the following PDF
\[
P_\tau(t, x, \xi) = P\left(t, x, \xi + \hat{\xi}(t)\tau + O(\tau^2)\right) = P + \nabla_{\xi} P \cdot \hat{\xi}\tau + O(\tau^2)
\]
represents a distribution function with \(\hat{\xi} \cdot \xi \geq 0\), where \(P = P(t, x, \xi)\) and \(\nabla_{\xi} P = \nabla_{\xi} P(t, x, \xi)\). Neglecting the higher order terms, we assume that
\[
P_\tau = \bar{P} + \nabla_{\xi} P \cdot \hat{\xi}\tau
\]
This PDF identically satisfies Eqs. [35] and [26]. In fact, for what regards Eq. [35], the integral over \(\{x \times \{\xi\}\) of the first term at the R.H.S. of Eq. [31] is equal to one, whereas the integral of the second one can be reduced to a proper
surface integral of $P$ over $\partial \{ \xi \}$ where $P \equiv 0$ through the second Green’s identity and fluid incompressibility, thus this latter identically vanishes. As far as Eq. (26) is concerned, the entropy $\mathcal{H}(P_\tau)$ is

$$\mathcal{H}(P_\tau) = -\int_x \int_\xi P_\tau \ln P_\tau \, dx \, d\xi$$

where the second addend at the R.H.S. of Eq. (32) identically vanishes, as, thanks to the fluid incompressibility, it can be reduced to be a surface integral of $P$ over $\partial \{ \xi \}$ where $P \equiv 0$. Therefore, the choice of $P_\tau$ given by Eq. (31) provides

$$\mathcal{H}(P_\tau) = \mathcal{H}(P) + O(\tau^2)$$

(33)

thus it is adequate to estimate $\lambda$. Accordingly, this latter is calculated as

$$\lambda = \int_x \int_\xi P_\tau(t, x, \xi) \frac{\dot{\xi} \cdot \xi}{\xi \cdot \xi} \, dx \, d\xi$$

(34)

Substituting Eq. (31) and (29) into Eq. (34), we have

$$\lambda = \langle \dot{\lambda} \rangle - \int_x \int_\xi \nabla_x P \cdot \dot{x} (\ln \rho - \langle \ln \rho \rangle) \, dx \, d\xi$$

(35)

Integrating by parts the second addend and taking into account the fluid incompressibility and the boundary conditions (41), we obtain

$$\int_x \int_\xi \nabla_x P \cdot \dot{x} (\ln \rho - \langle \ln \rho \rangle) \, dx \, d\xi = -\int_x \int_\xi P(t, x, \xi) \frac{\dot{\xi} \cdot \xi}{\xi \cdot \xi} \, dx \, d\xi \equiv \langle \dot{\lambda} \rangle$$

(36)

Hence

$$\lambda = 2 \langle \dot{\lambda} \rangle$$

(37)

in agreement with previous results.

6. Appendix: Lyapunov exponents and vectors distributions

In this appendix, the link between distribution functions of Lyapunov vectors and Lyapunov exponents is recalled. In order to express the distribution of $\dot{\lambda}$ in terms of the statistical properties of the solutions of Eq. (3), the distribution function of $x$ and $\xi$ is first considered.

$$P = P(t, x, \xi).$$

(38)

This PDF changes with the time according to the Liouville theorem [5] associated to Eqs. (1). This theorem, arising from the following relation

$$\int_x \int_\xi P \, dx \, d\xi = 1, \quad \forall t > 0,$$

(39)

and from Eqs. (1), provides the evolution equation of $P$ [5]

$$\frac{\partial P}{\partial t} + \nabla_x \cdot (P \dot{x}) + \nabla_\xi \cdot (P \dot{\xi}) = 0$$

(40)
where \( \nabla_x \cdot (\cdot) \) and \( \nabla_\xi \cdot (\cdot) \) denote the divergence of \((\cdot)\) defined in the spaces \([x]\) and \([\xi]\) respectively, and \(dx\) and \(d\xi\) are the elemental volumes in the corresponding spaces. Taking into account Eq. (39), and that the homogeneous isotropic turbulence is defined for unbounded fluid domains, \(P\) will satisfy the following boundary condition

\[
P = 0, \quad \forall (x, \xi) \in \partial (\{x\} \times \{\xi\}) \equiv \partial \{x\} \bigcup \partial \{\xi\}
\]

(41)

To describe the statistics of finite scale Lyapunov vectors and exponents for 3D flows, the initial condition of \(P\) is supposed to be of the kind

\[
P(0, x, \xi) = \frac{1}{\varphi} \delta \left( \frac{\xi - r}{\varphi} \right)
\]

(42)

where, due to homogeneous flow, \(\varphi = \text{const.}\) Accordingly, the statistical average of an integrable function of \(x\) and \(\xi\), say \(\zeta\), is calculated in terms of \(P\)

\[
\langle \zeta \rangle = \int_x \int_\xi P \cdot \zeta \, dx \, d\xi,
\]

(43)

In particular, average and maximum finite scale Lyapunov exponents are

\[
\langle \tilde{\lambda} \rangle = \int_x \int_\xi P(t, x, \xi) \frac{\dot{\xi} \cdot \xi}{\xi \cdot \xi} \, dx \, d\xi \equiv \int_{\lambda} P_\lambda(t, \lambda) \lambda \, d\lambda,
\]

(44)

\[
\lambda = \frac{\int_x \int_{\xi \geq 0} P(t, x, \xi) \frac{\dot{\xi} \cdot \xi}{\xi \cdot \xi} \, dx \, d\xi}{\int_x \int_{\xi \geq 0} P(t, x, \xi) \, dx \, d\xi} = \int_{\lambda \geq 0} \frac{P_\lambda(t, \lambda)}{\int_{\lambda} P_\lambda(t, \lambda) \, d\lambda} \lambda \, d\lambda
\]

(45)

where \(P_\lambda(t, \lambda)\) is the distribution function of \(\lambda\) calculated through \(P\), by means of the Frobenius Perron equation

\[
P_\lambda(t, \lambda) = \int_x \int_\xi P(t, x, \xi) \delta \left( \lambda - \frac{\dot{\xi} \cdot \xi}{\xi \cdot \xi} \right) \, dx \, d\xi,
\]

(46)

7. Conclusions

The distribution function of the finite scale local Lyapunov exponent of the kinematics field was established in homogeneous isotropic turbulence. Based on reasonable assumptions regarding the fully developed chaos and the fluid incompressibility, the shape of such distribution and the range of variations of \(\tilde{\lambda}\) are determined. This distribution results to be an uniform function in a proper interval of variations. The results arising from such PDF, in particular the link between \(\lambda\) and \(\tilde{\lambda}\) and the longitudinal velocity correlation, agree with those presented in Ref. [3], and this should support the proposed hypothesis for calculating this PDF. An alternative way to proof the relation between such Lyapunov exponents is also presented, which is based on the alignment property of the Lyapunov vectors.

8. Competing Interests

The author declares that there is no conflict of interests regarding the publication of this article.
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