Collapsing sequences of solutions to the Ricci flow on 3-manifolds with almost nonnegative curvature

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1 Introduction

We shall prove a general result about sequences of solutions to the Ricci flow on compact or complete noncompact 3-manifolds with locally uniformly bounded almost nonnegative sectional curvatures and diameters tending to infinity. It is known that such sequences occur when one dilates about a singularity of a solution to the Ricci flow on a 3-manifold. Our main result assumes collapse and is complementary to the injectivity radius estimate in [H95a, §25] and [CKL]. In particular, the bump-like point condition required in those papers is not assumed here; instead, we assume collapse, which rules out bump-like points. As an application of our result we give a generalization of Hamilton’s singularity theory in dimension 3 to classify collapsed singularity models arising from Type IIb (infinite time) singularities. From Perelman’s work [Per, §§4 and 7], such collapsed singularity models cannot occur as limits of dilations about finite time singularities. Besides obtaining a local injectivity radius estimate and ruling out the cigar as a limit for finite time singularities, Perelman has also enlarged the class of points and times about which one can obtain good limits. In view of Perelman’s improvements of Hamilton’s singularity theory, one expects to be able to combine Perelman’s ideas with Fukaya’s ideas; this is not discussed here.

Limits of collapsing 3-manifolds with lower curvature bounds are Alexandrov spaces with integer dimension 1 or 2 (we rule out dimension 0 by assuming the diameters tend to infinity). In the study of singularities, the sequences of solutions of the Ricci flow on 3-manifolds which arise have almost nonnegative sectional curvatures. This assumption, together with the smoothing properties of the Ricci flow (especially the strong maximum principle), put strong restrictions on the local geometries of the solutions in the sequence. In particular, the limit local covering geometries are locally the products of positively curved surfaces with the real line.

It is partly for the above reason that we shall be able to extract a virtual 2-dimensional limit solution of the Ricci flow. The reason we call this ‘limit’
solution ‘virtual’ is that it is not actually a limit of the sequence, but rather constructed from limits of local covers of the sequence. We expect that it is a limit of covers of exhaustions of the solutions in the sequence.

When the limit space is 2-dimensional, this allows us to extract a virtual solution to the Ricci flow when the actual limit is an orbifold which may not be a solution. The possible types of singularities of the orbifold are: $D^2/\mathbb{Z}_p$ with rotation action, $D^2/\mathbb{Z}_2$ with reflection action and $D^2/\mathbb{D}_{2p}$, where $D^2$ is the 2-disk and $\mathbb{D}_{2p}$ is the dihedral group of order $2p$ for some $p > 1$. We shall show that in this case the orbifold limit has at most 1 singular point of type $D^2/\mathbb{Z}_p$ or $D^2/\mathbb{D}_{2p}$, and in particular, it is a good orbifold with a finite cyclic or dihedral cover diffeomorphic to the plane.

The virtual limit associated to a 1-dimensional limit space is rotationally symmetric, complete, noncompact, with bounded positive curvature. The advantage of obtaining a 2-dimensional virtual limit as compared to a 1-dimensional actual limit is that 1-dimensional spaces have no intrinsic geometry except for distances and in particular have no nontrivial curvature.

Our construction of the virtual limit relies on Hamilton’s strong maximum principle for systems (see [H-86]), Fukaya’s local covering geometry theory (see [F]), and a Cheeger-Gromov type compactness theorem for the Ricci flow (see [H-95b] and [Gl]). The reader is also directed to [CM] for an application of Gromov-Hausdorff distance in the study of the Ricci flow.

Two abbreviations we shall commonly use are GH for Gromov-Hausdorff and RF for Ricci flow.

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2 The sequences of solutions with almost nonnegative sectional curvature

2.1 The definition of almost nonnegative sectional curvature

Let $\{(M^3_i, g_i(t), O_i) : t \in (\alpha, \omega)\}_{i \in \mathbb{N}}$, where $\alpha < 0$ and $\omega > 0$, be a sequence of orientable complete solutions to the RF with origins $O_i \in M^3_i$. Let $Rm_i(x, t)$ denote the Riemannian curvature operator of $g_i(t)$.

Definition 2 (i) The sequence $\{(M^3_i, g_i(t), O_i)\}$ is said to have **locally uniformly bounded geometry** (or **bounded geometry** for short) if for every closed subinterval $[\beta, \psi] \subset (\alpha, \omega)$, $p < \infty$, and $k \in \mathbb{N} \cup \{0\}$ there exists a constant $C = C(\beta, \psi, p, k) < \infty$ such that

$$\sup_{t \leq k, \, i \in \mathbb{N}} \left\{ \max_{B_{g_i(t)}(O_i, \rho)} \|\nabla Rm_i(x, t)\|_{g_i(t)} \right\} \leq C.$$
(ii) We say that the sequence \( \{(M^3_i, g_i(t), O_i)\} \) has **bounded diameters** if there exists a constant \( C < \infty \) such that
\[
diam(M^3_i, g_i(0)) \leq C, \text{ for all } i \in \mathbb{N}.
\]
If
\[
\lim_{i \to \infty} diam(M^3_i, g_i(0)) = \infty
\]
we say the sequence has **unbounded diameters**.

(iii) We say that the origins are **essential** if there exists \( c > 0 \) such that
\[
|R_{m_i}(O_i, 0)|_{g_i(0)} \geq c \text{ for all } i \in \mathbb{N}.
\]

Let \( \lambda_1(R_{m_i}) \leq \lambda_2(R_{m_i}) \leq \lambda_3(R_{m_i}) \) denote the eigenvalues of \( R_{m_i} \). Note that \( \lambda_k(R_{m_i}) \) is twice the sectional curvature.

**Definition 3**

(i) The sequence is said to have **almost nonnegative sectional curvatures** (or ANSC) if it has bounded geometry and for every \([\beta, \psi] \subset (\alpha, \omega)\) and \( \rho < \infty \), there exists \( \delta_i \searrow 0 \) such that
\[
\lambda_1(R_{m_i}) \geq -\delta_i \quad \text{on } B_{g_i(0)}(O_i, \rho) \times [\beta, \psi] \text{ for all } i \in \mathbb{N}.
\]

(ii) An **injectivity radius estimate holds for the origins** if there exists \( c > 0 \) such that
\[
|R_{m_i}(O_i, 0)|_{g_i(0)} \quad \text{inj}_{g_i(0)}(O_i)^2 \geq c \quad \text{for all } i \in \mathbb{N}.
\]

(iii) The sequence is said to have **collapsing origins** if
\[
|R_{m_i}(O_i, 0)|_{g_i(0)} \quad \text{inj}_{g_i(0)}(O_i)^2 \to 0 \quad \text{as } i \to \infty.
\]

By passing to a subsequence we may assume that one of the two alternatives (ii) or (iii) holds.

**Definition 4**

An ANSC sequence \( \{(M^3_i, g_i(t), O_i)\} \) has **bump-like origins** if there exists \( c > 0 \) such that
\[
\frac{\lambda_1(R_{m_i}) (O_i, 0)}{|R_{m_i}(O_i, 0)|_{g_i(0)}} \geq c
\]
for all \( i \in \mathbb{N} \). If \( \lim_{i \to \infty} \lambda_1(R_{m_i}) (O_i, 0) / |R_{m_i}(O_i, 0)|_{g_i(0)} = 0 \) then we say the origins are **split-like**.

Throughout this paper we shall assume that the sequence \( \{(M^3_i, g_i(t), O_i)\} \) has bounded geometry. In this section we further assume that the sequence has ANSC with essential origins.
2.2 The four types of sequences

We shall categorize the types of sequences by whether they have bump-like or split-like origins and whether the diameters are bounded or unbounded.

T1. The essential bump-like origins with bounded diameters case.
In this case we assume that the sequence \( \{(M^3_i, g_i(t), O_i)\} \) has essential bump-like origins and bounded diameter. Then a subsequence of \( M^3_i \) are diffeomorphic to spherical space forms.

Lemma 5 Let \( \{(M^3_i, g_i(t), O_i)\} \) be a sequence of solutions to the RF with ANSC, essential bump-like origins, and bounded diameters. Then there exists a subsequence such that \( g_i(0) \) has positive sectional curvature and \( M^3_i \) is diffeomorphic to a spherical space form for all \( i \).

Proof. It follows from the proof of Lemma 25.2 in [H-95a], using the bounded diameters assumption, that we get positive sectional curvature on all of \( M^3_i \) for a subsequence. The lemma now follows from Hamilton’s classification of compact 3-manifolds with positive Ricci curvature [H-82].

T2. The essential bump-like origins with unbounded diameters case - Hamilton’s injectivity radius estimate.
In this case we assume that the sequence \( \{(M^3_i, g_i(t), O_i)\} \) has essential bump-like origins and unbounded diameter. Then there is an injectivity radius estimate for a subsequence and hence there is a subsequence which converges to a complete solution diffeomorphic to \( \mathbb{R}^3 \) with positive sectional curvature.

Proposition 6 Let \( \{(M^3_i, g_i(t), O_i)\} \) be a sequence of solutions to the RF with ANSC, essential bump-like origins, and unbounded diameters. Then there exists a subsequence and a constant \( c > 0 \) such that

\[
\text{inj}_{g_i(0)}(O_i) \geq c.
\]

For the proof see [H-95a, §25] and [CKL].

T3. The essential split-like origins with bounded diameters case.
In this case we assume that the sequence \( \{(M^3_i, g_i(t), O_i)\} \) has essential split-like origins and bounded diameters. If the origins are not collapsing, then there is an injectivity radius estimate for a subsequence and hence there is a further subsequence which converges to a compact \( \{(M^3_\infty, g_\infty(t), O_\infty)\} \) with \( \lambda_1(Rm_{g_\infty(t)}) = \lambda_2(Rm_{g_\infty(t)}) = 0 \) and \( \lambda_3(Rm_{g_\infty(t)}) > 0 \). Such solutions are classified in [H-86] using the strong maximal principle; \( M^3_\infty \) is diffeomorphic to \( S^2 \times S^1 \) or the twisted product \( S^2 \# S^1 \). If the origins are collapsing there is a subsequence such that \( M^3_i \) are graph manifolds.

Lemma 7 Let \( \{(M^3_i, g_i(t), O_i)\} \) be a sequence of solutions to the RF with ANSC, essential collapsing origins, and bounded diameters. Then for any \( \varepsilon > 0 \) there exists a subsequence such that \( M^3_i \) is \( \varepsilon \)-collapsed for all \( i \), i.e. the supremum of the injectivity radii of \( M^3_i \) is at most \( \varepsilon \). When \( \varepsilon \) is small enough each \( M^3_i \) is diffeomorphic to a graph manifold.
Proof. By the injectivity radius decay estimate of [CGT] or [CLY] and the bounded diameter assumption, the sequence of Riemannian manifolds \( \{ (M_i^3, g_i (0)) \} \) collapses. That is, \( \max_{x \in M_i} \text{inj}_{g_i (0)} (x) \to 0 \) as \( i \to \infty \). The topological conclusion follows from Cheeger-Gromov theory (see [CG-86], [CG-90]), as there exists an F-structure on \( M_i \) for \( i \) large enough. For more details see the discussion in [Ri] p.548.

T4. The essential split-like origins with unbounded diameters case.
In this case we assume that the sequence \( \{ (M_i^3, g_i (t_i), O_i) \} \) has essential split-like origins and unbounded diameters. If the origins are not collapsing, then there is an injectivity radius estimate at \( O_i \) for a subsequence and there exists a subsequence which converges to a complete noncompact \( \{ (M_\infty^3, g_\infty (t), O_\infty) \} \) with \( \lambda_1 (Rm_{g_\infty (t)}) = \lambda_2 (Rm_{g_\infty (t)}) = 0 \) and \( \lambda_3 (Rm_{g_\infty (t)}) > 0 \). Such solutions can be classified using the strong maximal principle as in [H-86]: the universal cover of \( M_\infty^3 \) is diffeomorphic to \( \Sigma^2 \times \mathbb{R}^1 \) where \( \Sigma^2 \) is a complete surface with positive curvature.

The main focus of this paper is on the case when \( \{ (M_i^3, g_i (t_i), O_i) \} \) has ANSC, unbounded diameters, and essential collapsing split-like origins.

2.3 Dilations about sequences of points and times
In this subsection we show how ANSC sequences \( \{ (M_i^3, g_i (t_i), O_i) \} \) can be possibly obtained by dilation of the RF at singularities. Recall the best estimate of the sectional curvatures tending to nonnegative due to Hamilton [H-95a Theorem 4.1], which improves earlier estimates of [H-95a Theorem 24.4] and [L]

Proposition 8 Let \( (M^3, g(t)) , t \in [0, T) \) be a complete solution to the RF with time-dependent bounded curvature, i.e. \( \sup_{x \in M^3} |Rm (x, t)|_{g(t)} \leq C (t) \). If \( \lambda_1 \geq \lambda_1 (Rm) \geq -C_0 \) at time 0 for some \( C_0 > 0 \), then at any point and time where \( \lambda_1 < 0 \) we have

\[
R \geq -\lambda_1 \ln (-\lambda_1) + \ln (1 + C_0 t) - \ln C_0 - 3,
\]

where \( R \) is the scalar curvature.

Let \( (M^3, g(t)) , t \in [0, \infty) \), be a solution to the RF. Let \( (x_i, t_i) \) be a sequence of spacetime points and \( K_i \equiv |Rm (x_i, t_i)|_{g(t_i)} \). We say that \( (x_i, t_i) \) is dilatable if there exists \( \beta < 0 \) and \( \psi > 0 \) such that for every \( \rho < \infty \) there exists \( C < \infty \) such that

\[
|Rm (x, t)|_{g(t)} \leq CK_i
\]

for \( x \in B_{g(t_i)} (x_i, \rho/\sqrt{K_i}) \) and \( t \in [t_i + \beta/K_i, t_i + \psi/K_i] \). Let \( g_i (t) \equiv K_i g(t_i + t/K_i) \). If \( (x_i, t_i) \) is dilatable, then \( |Rm_i (x, t)|_{g_i(t)} \leq C \) on \( B_{g_i(0)} (x_i, \rho) \times [\beta, \psi] \).

A sequence \( (x_i, t_i) \) with \( t_i \to \infty \) is called Type III-like if there exists a constant \( C < \infty \) such that \( K_i \leq C/t_i \) for all \( i \in \mathbb{N} \). If \( \lim_{i \to \infty} t_i K_i = \infty \) the sequence \( (x_i, t_i) \) is said to be Type IIIb-like. Let \( \lambda_k (i) \equiv \lambda_k (Rm_i) (x_i, t_i) \) for \( k = 1, 2, 3 \). The sequence \( (x_i, t_i) \) has almost nonnegative sectional curvatures if \( \lambda_3 (i) > 0 \) for \( i \) large enough and \( \lim_{i \to \infty} \min \{ \lambda_1 (i), 0 \}/\lambda_3 (i) = 0 \).
Corollary 9 Let \( \{(x_i, t_i)\} \) be a dilatable and Type IIb-like sequence, then \( (x_i, t_i) \) has almost nonnegative sectional curvatures and the dilated solutions \( \{(M^3, g_i(t), x_i)\}, \)
t, \([\beta, \psi]\) have ANSC.

Proof. If \( \lambda_1(x, t_i) < 0 \) where \( x \in B_{g_i(t_i)}(x_i, \rho/\sqrt{K_i}) \), then by Proposition 8 and \( R(x, t_i) = \lambda_1(x, t_i) + \lambda_2(x, t_i) + \lambda_3(x, t_i) \),

\[
\lambda_3(x, t_i) \geq -\frac{1}{2} \lambda_1(x, t_i) \ln \left[ -\lambda_1(x, t_i) \left( C_0^{-1} + t_i \right) e^{-2} \right]. \tag{1}
\]

First consider \( x = x_i \). For any \( L \in (1, \infty) \), if \( -\lambda_1(i) > e^{2L+2} \left( C_0^{-1} + t_i \right)^{-1} \), then \( \lambda_3(i) / [-\lambda_1(i)] \geq L > 1 \) by (1). Note that this implies that \( \lambda_3(i) > \lambda_1'^2(i) \). If \( 0 \leq -\lambda_1(i) \leq e^{2L+2} \left( C_0^{-1} + t_i \right)^{-1} \) then, by the Type IIb assumption, for \( i \) large enough

\[
K_i = \left( \sum_{k=1}^{3} \lambda_k(i)^2 \right)^{1/2} \geq \sqrt{3} Le^{2L+2} \left( C_0^{-1} + t_i \right)^{-1} \geq -\sqrt{3} L \lambda_1(i). \tag{2}
\]

This implies (since \( L > 1 \) that \( \lambda_3(i) > \lambda_1'^2(i) > 2 \lambda_1^2(i) \). Therefore \( \lambda_3(i) > \lambda_1'^2(i) \) with no condition on \( \lambda_1(i) \), so \( \sqrt{3} \lambda_3(i) \geq K_i \geq \lambda_3(i) \). Hence, by (2), \( \sqrt{3} \lambda_3(i) \geq K_i \geq -\sqrt{3} L \lambda_1(i) \), or \( \lambda_3(i) / [-\lambda_1(i)] \geq L \). Since \( L \) is arbitrary, we have that \( (x_i, t_i) \) has almost nonnegative sectional curvatures.

Now consider any \( x \in B_{g_i(t_i)}(x_i, \rho/\sqrt{K_i}) \). We want to show that \( g_i(t) \) has ANSC, so we need to show that \( \lambda_1(g_i(t)) = \lambda_1(x, t) \), where \( \lambda_3(i) \) is comparable to \( K_i \), so it is sufficient to prove \( \lambda_1(x, t) / \lambda_3(i) \geq -\delta_i \). For any \( L \in (1, \infty) \), if \( -\lambda_1(x, t) > e^{2L+2} \left( C_0^{-1} + t \right)^{-1} \), then \( \lambda_3(x, t) / [-\lambda_1(x, t)] \geq L \) by (1). This implies \( \lambda_3(x, t) \) is comparable to \( K(x, t) \), i.e.

\[
\lambda_3(x, t) \leq K(x, t) \leq \sqrt{3} \lambda_3(x, t),
\]
as above. By the dilatable assumption \( C(\rho) K_i \geq K(x, t) \), so \( C(\rho) \sqrt{3} \lambda_3(i) \geq \lambda_3(x, t) \) and \( \lambda_3(i) / [-\lambda_1(x, t)] \geq (C(\rho) \sqrt{3})^{-1} L \).

Now suppose \( 0 \leq -\lambda_1(x, t) \leq e^{2L+2} \left( C_0^{-1} + t \right)^{-1} \). Since \( t \in [t_i + \beta/K_i, t_i + \psi/K_i] \) and \( \lim t_i K_i = \infty \), we get for all \( t \) that \( t/t_i \geq 1 + \frac{\beta}{\psi K_i} \geq 1/2 \) for \( i \) large enough. Because \( \sqrt{3} \lambda_3(i) \geq K_i \), we have, for \( i \) large enough (using the Type IIb assumption),

\[
\lambda_3(i) \geq \frac{1}{\sqrt{3} K_i} \geq Le^{2L+2} \left( C_0^{-1} + \frac{1}{2} t_i \right)^{-1} \geq Le^{2L+2} \left( C_0^{-1} + t \right)^{-1} \geq -L \lambda_1(x, t).
\]

We have proven that for any \( L \in (1, \infty) \) and for \( i \) large enough we have \( \lambda_3(i) / [-\lambda_1(x, t)] \geq C'(\rho)^{-1} L \). We conclude that \( g_i(t) \) has ANSC.

The above result is not true for Type III-like sequences of points; for example, constant negative sectional curvature solutions are of Type III.
3 Review of Fukaya’s local theory

One of the main tools we shall use is Fukaya’s local theory, which describes the local geometry of collapsed limits. In dimension 3 the types of the local geometries are quite limited. In this section when we use the ANSC assumption we shall make it explicit.

3.1 Fukaya’s main theorem

Recall Definition 0-4 in [F].

**Definition 10** We say that a metric space \((X, d)\) is **nice** if for every point \(p \in X\), there exists

1. a neighborhood \(U\) of \(p\) in \(X\), and a neighborhood \(V\) of \(\vec{0}\) in \(\mathbb{R}^m\) for some \(m \in \mathbb{N} \cup \{0\}\),
2. a compact Lie group \(\Gamma\) with a faithful representation of \(\Gamma\) into \(O(m, \mathbb{R})\) where \(\Gamma^0\) (the identity component of \(\Gamma\)) is isomorphic to a torus, and
3. a \(\Gamma\)-invariant Riemannian metric \(h\) on \(V\),

such that \((U, d|_U)\) is isometric to \((V, d_h)/\Gamma\), which is a metric space with distance function induced on the quotient \(V/\Gamma\) from \(h\) (so that, in particular, \(U\) is homeomorphic to \(V/\Gamma\)).

In [F] Fukaya proves that given a sequence of Riemannian manifolds with \(C^\infty\)-uniformly bounded geometry there is a subsequence which converges to a nice metric space. This implies that nice metric spaces are Lipschitz dense in the closure of Riemannian manifolds of dimension \(n\) with bounded curvature. We will outline Fukaya’s proof since we shall need elements of it in our classification of the local geometries. In addition, we shall clarify the exact versions of Fukaya’s results which we will need and generalize them to the case of solutions of the RF as done in [Gl].

3.2 Construction of the limit metric, local group \(G_{\infty}\)

Recall the definition of local groups from [PG 23D]; these are sometimes called pseudogroups. For example, a neighborhood of the identity in a Lie group is a local group. It will be important to consider local groups which are not connected.

Let \(\{(M^3_t, g_t(0), O_t) : t \in (\alpha, \omega)\}\) be a sequence of solutions of the RF. Suppose \((M^3_t, g_t(0), O_t)\) converges to the metric space \((X_\infty, d_\infty(0), O_\infty)\) in the pointed GH topology. Fix an \(\epsilon > 0\) and \(P_\infty \in X_\infty\) and let \(P_i \in M^3_I\) such that \(B^3_{g_t(0)}(P_i, 1 + \epsilon)\) converges to \(B^3_{d_\infty(0)}(P_\infty, 1 + \epsilon) \subset X_\infty\) in GH. Fix a frame \(F_i\) of \(T_{P_i}M^3_t\) orthonormal with respect to the metric \(g_t(0)\); these frames allows us

\[\text{We use the terminology nice instead of Fukaya’s smooth to distinguish it from } C^\infty.\]
to identify each unit ball in $T_{P_i} M^3_i$ centered at the origin in $T_{P_i} M^3_i$ with the Euclidean unit ball $B^3 (1) \subset \mathbb{R}^3$ centered at the origin 0. Consider the exponential map for $g_t(0)$ restricted to ball:

$$\exp_{P_i} \equiv \exp_{P_i}^{g_t(0)} : B^3(1 + \epsilon) \to M^3_i.$$ 

Assume that the sectional curvatures $K_{g_t(0)} \leq 1$ on the ball $B_{g_t(0)}(P_i, 1 + \epsilon)$ so that $\exp_{P_i}$ is a local diffeomorphism. We consider the pulled-back metrics $\tilde{g}_i(t) \equiv \exp_{P_i}^* g_t(t)$ on $B^3(1 + \epsilon)$ for $t \in (\alpha, \omega)$. By the Arzela-Ascoli theorem and the bounded geometry assumption, there exists a subsequence (we still denote the subsequence by $\tilde{g}_i(t)$ and continue to use this convention with further subsequences below) such that $\tilde{g}_i(t)$ converges in $C^\infty$ on $B^3(1)$ to a smooth solution $\tilde{g}_\infty(t)$ to the RF on $B^3(1)$. By a diagonalization argument we may assume that this convergence holds for all $P_\infty$ in a countable dense subset of $X_\infty$.

We shall consider the set of continuous maps $C^0(B^3(1/2), B^3(1))$ as a metric space with the metric

$$d_C(\gamma, \gamma') = \sup_{x \in B^3(1/2)} d_{3\infty}(0) \left| \gamma(x), \gamma'(x) \right|.$$ 

We define the sets $G_i$ which consist of local deck transformations of the local covering map $\exp_{P_i} : B^3(1) \to M_i$ as

$$G_i \equiv \{ \gamma \in C^0(B^3(1/2), B^3(1)) : \exp_{P_i} \circ \gamma = \exp_{P_i} \}.$$ 

Clearly each $G_i$ is a discrete local group of local isometries of the Riemannian manifold $(B^3(1), \tilde{g}_i(t))$ for each $t$. Gromov [Gr] calls this group the local fundamental pseudogroup and there is a geometric description of elements of $G_i$ in [CGT] §4.

We will define a limit group $G_\infty$. Since $G_i$ are local isometries of $(B^3(1), \tilde{g}_i(0))$ and $\tilde{g}_i(0)$ converge to $\tilde{g}_\infty(0)$ on $B^3(1)$ in $C^\infty$, for large $i$ each $G_i$ is a closed subset of the following set of quasi-isometries

$$L \equiv \left\{ \gamma \in C^0(B^3(1/2), B^3(1)) : \frac{1}{2} \leq \frac{d_{3\infty}(0) \left| \gamma(x), \gamma(y) \right|}{d_{3\infty}(0) \left| x, y \right|} \leq 2 \right\}.$$ 

By the Arzela-Ascoli theorem $L$ is compact. We consider the space $S$ of closed subsets of $L$ with the Hausdorff topology. $S$ is compact because $L$ is compact (see [BBI] Theorem 7.3.8); hence there is a subsequence of $\{ G_i \} \subset S$ which converges to a set $G_\infty \in S$.

The inclusion $G_\infty \subset C^0 \left(B^3(1/2), B^3(1)\right)$ defines a local action of $G_\infty$ on $B^3(1/2)$. Elements of $G_\infty$ are local isometries with respect to $\tilde{g}_\infty(0)$, that is, if $\gamma \in G_\infty$ then $\gamma^* \left( \tilde{g}_\infty(0) \right) = \tilde{g}_\infty(0) \left|_{B^3(1/2)} \right.$. The product operation in $G_i$ gives a product operation in the limit $G_\infty$ which makes it a local group.

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A quasi-isometry is a homeomorphism which distorts distances by a bounded amount.
$G_\infty$ is a Lie group germ acting smoothly [F Lemma 3.1]. A Lie group germ is a local group isomorphic to a neighborhood of the identity of a Lie group. Furthermore, $G_\infty$ is nilpotent and has a neighborhood of the identity where the exponential map of the Lie algebra is onto [F Lemma 4.1].

Let $H^0$ denote the identity component of a local group $H$. We have the following important consequence of the nilpotency of Lie$(G_\infty)$ together with dim $M_i = 3$. We shall use some results from [4].

Lemma 11 Suppose the GH limit $X_\infty$ of an essential ANSC sequence has Hausdorff dim $X_\infty \geq 1$, then $G^0_\infty$ is abelian.

Proof. It follows from the ANSC assumption and [4, 2] that $G_\infty$ acts isometrically on $D^2 (1/2) \times (-1/2, 1/2)$ with metric $2\delta(t) + du^2$. Let $Y$ be a vector field on $D^2 (1/2) \times (-1/2, 1/2)$ generated by the $G^0_\infty$ action. Then by Lemma 21 we have $Y = K + b\theta$ where $K$ is a Killing vector field on $D^2 (1/2)$. Let $Y_1$ and $Y_2$ be two such vector fields on $D^2 (1/2) \times (-1/2, 1/2)$, then $[Y_1, Y_2] = [K_1, K_2]$ and hence we get a Lie algebra $\text{Lie}_0 = \{K : K + b\theta \in \text{Lie}(G^0_\infty)\}$ of Killing vector fields on $D^2 (1/2)$. By Lemma 4.1 of [F], Lie$(G_\infty)$ = Lie$(G^0_\infty)$ is nilpotent, this implies that Lie algebra $\text{Lie}_0$ is nilpotent. Since $\text{Lie}_0$ consists of Killing vector fields on $D^2 (1/2)$ with positive curvature, $\text{Lie}_0$ must be one dimensional. Hence $G^0_\infty$ is an abelian local Lie group. ■

Remark 12 We think this can be proven without the ANSC assumption using the results of [4, §5].

Next we will prove that dim $[G_\infty (0)] \geq 1$ when dim $X_\infty \leq 2$, where $G_\infty (0) = \{\gamma (0) : \gamma \in G_\infty\}$. The fact that $G_\infty (0)$ is infinite is stated in [SY]. Let $\gamma_i$ be the minimizer of $d_{\gamma_i (0)} \left[ \gamma_i (0), 0 \right]$ over all $\gamma \in G_i$. This corresponds to the shortest geodesic loop in $(M^3_i, g_i (0))$ based at $P_i$.

Claim 13 Let $\gamma_i \in G_i$ as defined above, then for all $\varepsilon, 0 < \varepsilon < 1/4$, there exists $n_i$ such that $\gamma_i^{n_i}$ exists and $d_{\gamma_i (0)} \left[ \gamma_i^{n_i} (0), 0 \right] > \varepsilon$ and $d_{\gamma_i (0)} \left[ \gamma_i^{n_i-1} (0), 0 \right] \leq \varepsilon$

Proof. Suppose there was $\varepsilon > 0$ such that this is not true. Then $d_{\gamma_i (0)} \left[ \gamma_i^n (0), 0 \right] < \varepsilon$ for all $n > 0$. Hence $\gamma_i^n$ exists for all $n$. It follows from [CGT, Lemma 4.6] that for any $k$, $\gamma_i^k \neq id$. Since $G_i$ is a closed subset of the compact set $L$, there is a subsequence $n_j$ which converges to $\gamma_i^\infty \in G_i$. But then we can find $n_j$ such that $\gamma_i^{n_j}$ is arbitrarily close to $\gamma_i^\infty$. This means that $d_{\gamma_i (0)} \left[ (\gamma_i^\infty)^{-1} \gamma_i^{n_j} (0), 0 \right] \to 0$ as $j \to \infty$. Hence $(M^3_i, g_i (0))$ has arbitrarily small geodesic 1-gons based at $P_i$, a contradiction. ■

Lemma 14 Suppose the GH limit $X_\infty$ has (Hausdorff) dim $X_\infty \leq 2$. Then dim $[G_\infty (0)] \geq 1$. 9
Given a set \(\mathcal{X}\) decomposition of the actions are geodesics of both the pulled back metric and the Euclidean metric. This follows by differentiating the action of \(\Gamma\) since rays from the origin isometries on the Riemannian manifold \(\mathcal{X}\) since an isometry is uniquely determined by its derivative at one point.

The metric space \(\mathcal{X}\) of \(G\) with our construction of \(\Delta\).

Proof. We shall construct a one-dimensional path in \(G_\infty (\bar{0})\). Since \(\dim X_\infty \leq 2\) and \(d_{\bar{0},(0)} [\gamma_\iota (\bar{0}), \bar{0}] = 2 [\inj P, (g_i (0))]\), we must have \(d_{\bar{0},(0)} [\gamma_\iota (\bar{0}), \bar{0}] \to 0\) as \(i \to \infty\). By Claim 4 there exists a subsequence \(\{\gamma_{i_j}\}\) such that \(\gamma_{i_j} \to \gamma_\infty\) with \(\gamma_\infty (\bar{0}) = \bar{0}\) and there exist \(n_{i_j}\) such that \(\gamma_{i_j} \to \gamma'_\infty \in G_\infty\) as \(j \to \infty\) with \(\gamma'_\infty (\bar{0}) \neq \bar{0}\). We can now take the path \(\gamma_t = \lim_{j \to \infty} \gamma_{i_j}^{[n_{i_j}]},\) where \([q]\) is the greatest integer less than or equal to \(q\), for \(t \in [0, 1]\). This is a path from \(\gamma_\infty\) to \(\gamma'_\infty\), and hence \(\gamma_t (\bar{0})\) is a path from \(\bar{0}\) to \(\gamma'_\infty (\bar{0})\) inside the orbit \(G_\infty (\bar{0})\).

We define

\[
\Gamma \doteq \{ \gamma \in G_\infty : \gamma (\bar{0}) = \bar{0}\},
\]

the isotropy sub-local group of \(\bar{0}\) in \(G_\infty\). We have a faithful representation of \(\Gamma\) into \(O (3)\) with metric \(\tilde{g}_\infty (0)\) at \(\bar{0}\) defined by \(\gamma \mapsto \gamma_* (\bar{0}) : T_{\bar{0}} B^3 (1/4) \to T_{\bar{0}} B^3 (1/4)\). Actually, the elements of \(\Gamma\) are orientation preserving because \(M_i\) are orientable, so the representation is into \(SO (3)\). The representation is faithful since an isometry is uniquely determined by its derivative at one point.

The action can be thought of as the linear action of \(SO (3)\). That is, every element of \(\Gamma\) is the restriction to \(B^3 (1/2)\) of an element of \(SO (3)\). \(\Gamma\) acts by isometries on the Riemannian manifold \((B^3 (1), g_{\bar{0}})\), where \(g_{\bar{0}}\) is the Euclidean metric. This follows by differentiating the action of \(\Gamma\) since rays from the origin are geodesics of both the pulled back metric and the Euclidean metric.

### 3.3 Decomposition of the actions

Given a set \(G\) acting on a metric space \((X, d)\), for \(p \in X\) and \(\varepsilon > 0\) we define

\[
G (p, \varepsilon) \doteq \{ \gamma \in G : d (\gamma (p), p) < \varepsilon \}.
\]

For any \(\varepsilon_1 \in (0, 1/4]\), since \(G_i\) act on \(B^3 (1/2)\) as isometries, there is a well defined equivalence relation \(\sim\) on \(B^3 (\varepsilon_1)\) defined by \(x \sim y\) if and only if there exists a \(\gamma \in G_i (\bar{0}, 2\varepsilon_1)\) such that \(\gamma (x) = y\). Hence we have a quotient \(B^3 (\varepsilon_1) / G_i (\bar{0}, 2\varepsilon_1)\). Similarly we can define the quotient \(B^3 (\varepsilon_1) / G_\infty (\bar{0}, 2\varepsilon_1)\). The metric space \(B^3 (\varepsilon_1) / G_\infty (0)\) is isometric to \(B^3 (\varepsilon_1) / G_\infty (\bar{0}, 2\varepsilon_1)\) with the quotient distance induced by \(\tilde{g}_\infty (0)\), which can be seen using an equivariant version of pointed GH convergence (see [E] p. 10).

We would like to write the group as \(G_\infty (\bar{0}, 2\varepsilon_1) = \Gamma \Delta\) where \(\Delta\) is a subgroup of \(G_\infty (\bar{0}, 2\varepsilon_1)\) acting freely on \(B^3 (\varepsilon_1)\). Then \(B^3 (\varepsilon_1) / \Delta\) is a manifold (where we think of \(G_\infty\) acting on the left) and \(B^3 (\varepsilon_1) / \Gamma\) is isometric to \([B^3 (\varepsilon) / \Delta] / \Gamma\). For the rest of this subsection we assume that \(X_\infty\) is the GH limit of an essential ANSC sequence to ensure that \(G_\infty^0\) is abelian. We proceed with our construction of \(\Delta\).
Proof. Let \( \varepsilon > 0 \) small so that \( \exp : \text{Lie}(G_0) \cap B^k(\varepsilon_2) \to G_0^\infty \) is injective, then we can define the local group \( \Delta = \{ \exp v : v \in \mathfrak{h}' \cap B^k(\varepsilon_2) \} \).

We note that \( \Delta \) is a sub-local group of \( G_0^\infty \) and that for all \( \delta \in \Delta - \{id\} \), \( \delta(\vec{0}) \neq \vec{0} \). Since \( \text{Lie}(\Gamma^0) \) is abelian, \( \exp v (\exp v') = \exp (v + v') \) for \( v \in \text{Lie}(\Gamma^0) \) and \( v' \in \mathfrak{h}' \) so \( \Gamma^0 \Delta \) generates a neighborhood \( G_0^\infty(\vec{0}, 2\varepsilon_0) \) of the identity for some small \( \varepsilon_0 \in (0, 1/4] \). We have the following properties of \( \Delta \).

**Lemma 15** There exists \( \varepsilon_0 \in (0, 1/4] \) such that

1. \( \Gamma^0 \Delta = \Delta \Gamma^0 = G_0^\infty(\vec{0}, 2\varepsilon_0) \),
2. \( \Delta \cap \Gamma = \{id\} \), and
3. \( \Delta \) acts freely on \( B^3(\varepsilon_0) \).

**Proof.** 1 and 2 follow from the discussion above. To prove 3, suppose for every \( \varepsilon_1 > 0 \), there exists \( \delta \in \Delta \) and \( p \in B^3(\varepsilon_1) \) such that \( \delta(p) = p \). Then we can take a sequence \( v_i \in \mathfrak{h}' \) and \( p_i \in B^3(1/i) \) such that \( \exp v_i [p_i] = p_i \). There exists a subsequence \( i_j \) and \( n_j \in \mathbb{N} \) such that \( n_j v_i \to v_\infty \neq 0 \). Since \( p_i \to 0 \) and \( \exp (n_j v_i) [p_i] = \exp v_i (n_j) [p_i] = p_i \) we have \( \exp v_\infty[I] = \vec{0} \), so \( v_\infty \notin \mathfrak{h}' \). However, since \( \mathfrak{h}' \) is closed, \( v_\infty \notin \mathfrak{h}' \), a contradiction. □

Now we can decompose the entire action as follows.

**Lemma 16** There exist \( \varepsilon_0 \in (0, 1/4] \) and a manifold with Riemannian metrics \( (V^m, h(t)) \) such that \( \Gamma \) acts on \( (V^m, h(t)) \) isometrically and \( (V^m, h(t))/\Gamma = (B^3(\varepsilon_0), \tilde{g}_\infty(t))/G_0^\infty(\vec{0}, 2\varepsilon_0) \).

**Proof.** By construction \( G_0(\vec{0}, 2\varepsilon_0) \) preserves the metrics \( \tilde{g}_t(t) \) for all \( t \) and hence \( G_0^\infty(\vec{0}, 2\varepsilon_0) \) preserves the metrics \( \tilde{g}_\infty(t) \) for all \( t \). Since \( \Delta \) acts freely, \( V^m = B^3(\varepsilon_0)/\Delta \) is a manifold with quotient Riemannian metrics \( h(t) \) induced by the metrics \( \tilde{g}_\infty(t) \). We claim that \( \Delta \) is normal in \( G_0^\infty(\vec{0}, 2\varepsilon_0) \), this implies that there is an isometric action \( \Gamma \times V^m \to V^m \). The lemma follows from the fact that \( \Gamma \Delta = G_0^\infty(\vec{0}, 2\varepsilon_0) \). It is clear that \( \Gamma \subset O(m) \) where \( m = \dim V \) by looking at the derivatives of the action.

The claim can be proved from the fact that \( G_0^\infty \) acts isometrically on \( D^2(1/2) \times (-1/2, 1/2) \) with metric \( 2h_\infty(t) + du^2 \) (see [12]). □

We will call \( (V^m, \Gamma, G_0^\infty) \) the local model of some neighborhood of \( P_\infty \) in \( X_\infty \).
4 Classifying the limit space

4.1 General properties of GH limits of solutions

Let \( \{ (M_i^3, g_i(t), O_i) \} \) be a sequence with ANSC, essential collapsing split-like origins, and unbounded diameter. By the Gromov compactness theorem there exists a subsequence which converges in pointed GH distance to Alexandrov spaces \( (X_\infty(t), d_\infty(t), O_\infty) \). We can use the fact that solutions to the RF are uniformly bi-Lipschitz to each other to see that the metric spaces \( X_\infty(t) \) are topologically the same for all \( t \), and call that space \( X_\infty = X_\infty(0) \). Details are in \cite{G}. It follows from \cite{G} that \( (B^3(\varepsilon_0), \tilde{g}_\infty(t)) / G_\infty(\tilde{0}, 2\varepsilon_0) \) in \( \mathbb{S}^3 \) are isometric to metric spaces \( \left( B^{X_\infty}_{d_\infty(0)}(P_\infty, \varepsilon_0), d_\infty(t) \right) \) if \( \varepsilon_0 \) is taken small enough (independent of \( t \)).

\( (X_\infty, d_\infty(t), O_\infty) \) has nonnegative curvature by the following standard result: a GH limit of pointed Alexandrov spaces with curvature \( \geq k \) is itself a space of curvature \( \geq k \) \cite{BH} Proposition 10.7.1. We summarize in the following proposition.

Proposition 17 Let \( \{ (M_i^3, g_i(t), O_i) \} \) be a sequence with ANSC, essential collapsing split-like origins, and unbounded diameter. Then there is a subsequence which converges to an Alexandrov spaces \( (X_\infty, d_\infty(t), O_\infty) \) with nonnegative curvature whose dimension is 1 or 2.

4.2 Consequences of the strong maximum principle for RF on local covering geometries

The ANSC condition and Hamilton’s strong maximum principle for systems will restrict the limit local covering geometries of the sequence. When combined with low dimension of the sequence \( \{ (M_i^3, g_i(t), O_i) \} \), it restricts how the local Lie groups of isometries act.

Assume that all sectional curvatures \( K_{g_i(0)} \leq 1 \) on \( M_i^3 \). Choose \( P_i \) as in \cite{12} to be \( O_i \) and let \( \tilde{g}_O(t) \) denote the limit metric. Then the essential collapsing split-like origins imply that at origin \( O_\infty \) and \( t = 0 \), \( \lambda_1 (\text{Rm} \tilde{g}_O(0)(O_\infty)) = \lambda_2 (\text{Rm} \tilde{g}_O(0)(O_\infty)) = 0 \) and \( 0 < \lambda_3 (\text{Rm} \tilde{g}_O(0)(O_\infty)) \). By the strong maximum principle, the metrics \( \tilde{g}_O(t) \) locally split as the product of a surface metric with \( \mathbb{R} \). Moreover, the image of \( \text{Rm} \tilde{g}_O(0)(t) \) in \( \wedge^2 B^3(1) \) is 1-dimensional, independent of time, and invariant under parallel translation (see \cite{LSH} Theorem 8.3). There is a unit 1-form which is parallel and independent of time spanning the null space of \( \text{Re} \tilde{g}_O(0)(t) \) and perpendicular to the image of \( \text{Rm} \tilde{g}_O(0)(t) \).

By the deRham theorem, since \( B^3(1) \) is contractible, for all \( t \in (\alpha, \omega) \) there exists a solution \( \tilde{h}_{O_\infty}(t) \) to the RF on \( D^2(1/2) \) such that

\[
\iota_{O_\infty} : \left( D^2(1/2) \times (-1/2, 1/2), \tilde{h}_{O_\infty}(t) + du^2, (\tilde{0}, 0) \right) \cong \left( B^3(1), \tilde{g}_{O_\infty}(t), \tilde{0} \right)
\]

is an isometric embedding of the product of an evolving surface with an interval and \( \tilde{h}_{O_\infty}(0) \) is a metric in normal coordinates on the disk \( D^2(1/2) \).
The local covering geometry of any point \( P_\infty \in X_\infty \) also splits as the product of a surface and an interval. One way to see this is as follows. Let \( P_t \in M_t \) be a sequence of points such that \((M_t, d_{g_t}(t), P_t)\) converges to \((X_\infty, d_\infty(t), P_\infty)\). Let \( \gamma_t : [0, L_t] \rightarrow M_t \) be a minimal geodesic joining \( O_t \) to \( P_t \) with respect to the metric \( g_t \). \( \gamma_0 \) extends (uniquely) to a geodesic \( \tilde{\gamma}_0 : [-1, L_t + 1] \rightarrow M_t \). Consider the geodesic tube \( T_t : B^2(1) \times [-1, L_t + 1] \rightarrow M_t \) corresponding to \( \tilde{\gamma}_0 \) with the pulled-back metrics \( \bar{g}_t \). \( \bar{g}_t \) are solutions to the RF. By passing to a subsequence, we obtain a limit solution \((B^2(1) \times (-1, L_\infty + 1), \tilde{g}_\infty(t))\) with nonnegative sectional curvature, where \( L_\infty = \lim_{t \to \infty} L_t \). \( \tilde{g}_\infty(0) \) is the origin in \( B^2(1) \). By the strong maximum principle, \( \lambda_1(\text{Rm}_{\bar{g}_\infty(t)}) = 0 \) at \((\tilde{\gamma}_0, L_\infty)\); actually \( \lambda_1(\text{Rm}_{\bar{g}_\infty(t)}) = 0 \) for every point in \((B^2(1) \times (-1, L_\infty + 1))\). This implies that \( \lim_{t \to \infty} \lambda_1(\text{Rm}_{\bar{g}_t}(P_t)) = 0 \) for all \( t \) and hence \( \lambda_1(\text{Rm}_{\tilde{g}_\infty}(\tilde{P}_\infty)) = 0 \) where \( \tilde{P}_\infty \) is the origin in \( B^3(1) \) and \( \tilde{g}_\infty(t) \) is the limit metric coming from the local covering geometry construction around the \( P_t \). From this we conclude that the local covering geometry \( \tilde{g}_\infty(t) \) of \( P_\infty \in X_\infty \) has zero curvature and hence splits as a product of a surface and an interval, similarly to how it splits at \( O_\infty \).

Another way to prove \( \lambda_1(\text{Rm}_{\tilde{g}_\infty}(\tilde{P}_\infty)) = 0 \) is to assume, by contradiction, that there is a subsequence such that the \( P_t \)’s are bump-like. Then there exists a uniform injectivity radius estimate at \( P_t \). Since \( d_{g_t}(0, P_t, O_t) \) is uniformly bounded, this implies a uniform injectivity radius estimate at \( O_t \), which is a contradiction.

### 4.3 Killing vector fields on surfaces

We shall show that a local surface with a Killing vector field is locally a warped product. This clearly must be a classical fact but since we have not found a reference we include a sketch of the proof.

Let \((\Sigma^2, h)\) be a Riemannian surface (not necessarily complete) with a Killing vector field \( K \). Let \( J : T\Sigma^2 \rightarrow T\Sigma^2 \) be the complex structure, that is, rotation by 90° in the counterclockwise direction. Let \( x \in \Sigma^2 \) and define a smooth unit speed path \( \gamma : (r_0 - \varepsilon, r_0 + \varepsilon) \rightarrow \Sigma^2 \) by \( \gamma(r) = \frac{J(K)(\gamma(r))}{|J(K)(\gamma(r))|}, \gamma(r_0) = x \). Define also a 1-parameter family of smooth paths \( \beta_{\varepsilon} : (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \rightarrow \Sigma^2 \) by \( \beta_{\varepsilon}(\theta) = K(\beta_{\varepsilon}(\theta)), \beta_{\varepsilon}(\theta_0) = \gamma(r), \) and a 1-parameter family of smooth unit speed paths \( \gamma_{\theta} : (r_0 - \varepsilon, r_0 + \varepsilon) \rightarrow \Sigma^2 \) by \( \gamma_{\theta}(r) = \frac{J(K)(\gamma_{\theta}(r))}{|J(K)(\gamma_{\theta}(r))|}, \gamma_{\theta}(r_0) = \beta_{\varepsilon}(\theta). \) Note that \( \gamma_{\theta_0} = \gamma \).

#### Lemma 18

\[
\beta_{\varepsilon}(\theta) = \gamma_{\theta}(r).
\]

**Proof.** This follows from

\[
\begin{bmatrix} K, J(K) \\ J(K) \end{bmatrix} = \frac{1}{|J(K)|} \begin{bmatrix} K, J(K) \end{bmatrix} - \frac{1}{2} \frac{1}{|J(K)|^2} K |J(K)|^2 J(K) = 0
\]

13
since \([K, J(K)] = 0\) and \(K | J(K)|^2 = 0\). ■

Hence \((r, \theta)\) defines local coordinates on a neighborhood of \(x\). Define the function
\[
\begin{align*}
  f(r) &
  \doteq |K| (\beta_r)
\end{align*}
\]
where we are using the fact that \(|K|\) is constant on \(\beta_r\). The metric is given by
\[
\begin{align*}
  h &= \left\langle \frac{J(K)}{|J(K)|}, \frac{J(K)}{|J(K)|} \right\rangle dr^2 + 2 \left\langle K, \frac{J(K)}{|J(K)|} \right\rangle d\tau + \langle K, K \rangle d\theta^2 \\
  &= dr^2 + f(r)^2 d\theta^2.
\end{align*}
\]

We have proved:

**Lemma 19** (i) Given \(x \in \Sigma^2\), there exists a neighborhood \(U\) of \(x\) and local coordinates \(r\) and \(\theta\) on \(U\) such that
\[
\begin{align*}
  \frac{\partial}{\partial r} &= \frac{J(K)}{|J(K)|}, \quad \frac{\partial}{\partial \theta} = K, \\
  h &= dr^2 + f(r)^2 d\theta^2.
\end{align*}
\]

(ii) Let \((r_i, \theta_i), i = 1, 2\) be two coordinate systems of some neighborhood \(U\) of \(x\) and \(f_i(r_i)\) be the functions defining the metric in (i), then
\[
\begin{align*}
  r_2 &= r_1 + r_0, \quad \theta_2 = \theta_1 + \theta_0, \\
  f_2(r_2) &= f_1(r_2 - r_0)
\end{align*}
\]
where \(r_0 = r_2(x) - r_1(x)\) and \(\theta_0 = \theta_2(x) - \theta_1(x)\) are constants.

Next we prove a uniqueness theorem about Killing vector fields.

**Lemma 20** Let \((\Sigma^2, h)\) be a Riemannian surface with nonzero curvature everywhere. Suppose \(K_1\) and \(K_2\) are two Killing vector fields satisfying \([K_1, K_2] \equiv 0\). Then \(K_1\) and \(K_2\) are linearly dependent.

**Proof.** Using \([K_1, K_2] \equiv 0\) and the Killing vector field equation for \(K_i\)
\[
\langle \nabla_X K_i, Y \rangle + \langle \nabla_Y K_i, X \rangle = 0 \tag{3}
\]
we have \(\langle \nabla_{K_2} K_1, K_2 \rangle = 0\) and \(\langle \nabla_{K_2} K_1, K_1 \rangle = \langle \nabla_{K_1} K_2, K_1 \rangle = 0\). If \(K_1\) and \(K_2\) are linearly independent, then \(\nabla_{K_2} K_1 \equiv \nabla_{K_1} K_2 \equiv 0\). Also from \(\langle \nabla_{K_1} K_1, K_2 \rangle = - \langle \nabla_{K_2} K_1, K_1 \rangle = 0\) and \(\langle \nabla_{K_2} K_1, K_1 \rangle = 0\) we get \(\nabla_{K_1} K_1 \equiv 0\). Similarly we have \(\nabla_{K_2} K_2 \equiv 0\). Hence the metric is flat, a contradiction. ■

4.4 A canonical form for actions on a surface \(\times \mathbb{R}\)

Let \((\Sigma^2, h)\) be a Riemannian surface with positive curvature.
Lemma 21 Given an \( R_{\text{loc}} \) local group action of local isometries on \( \Sigma^2 \times R_{\text{loc}} \) with the product metric, there exist coordinates \((r, \theta)\) on \( \Sigma^2 \) such that we can write the action of \( \tau \in R_{\text{loc}} \) as
\[
\tau : (r, \theta, u) \mapsto (r, \theta + a\tau, u + b\tau) \tag{4}
\]
for some constant \( a, b \in \mathbb{R} \).

Proof. Let \((x, y)\) be coordinates on \( \Sigma^2 \). Denote the image of \((x, y, u)\) under action of \( \tau \) by \((x\tau, y\tau, u\tau)\). Since \((\Sigma^2, h)\) has positive curvature and \( \tau \) is a local isometry, the tangential map \( d\tau \) must be of the block diagonal form
\[
T_{(x,y)}\Sigma^2 \times T_uR_{\text{loc}} \rightarrow T_{(x\tau,y\tau)}\Sigma^2 \times T_{u\tau}R_{\text{loc}}.
\]
This implies that the functions \( x\tau(x,y,u) \) and \( y\tau(x,y,u) \) are independent of \( u \) and the function \( u\tau(x,y,u) = x\tau(x,y), y\tau(x,y,u) = y\tau(x,y) \), and \( u\tau(x,y,u) = u\tau(u) \).

The assumption that the action is locally isometric on \( \Sigma^2 \times R_{\text{loc}} \) implies that the action \( \tau: R_{\text{loc}} \rightarrow R_{\text{loc}} \) with \( \tau : u \mapsto u\tau(u) \) is an isometry and the action \( \tau: \Sigma^2 \rightarrow \Sigma^2 \) with \( \tau : (x, y) \mapsto (x\tau(x,y), y\tau(x,y)) \) is an isometry. Hence \( u\tau(u) = \pm u + b\tau \). If action \( \tau: \Sigma^2 \rightarrow \Sigma^2 \) is nontrivial, then by Lemma 19 there are coordinates \((r, \theta)\) on \( \Sigma^2 \) such that \( \tau : (r, \theta) \mapsto (r, \pm \theta + a\tau) \) for some \( a \neq 0 \). If the action \( \tau: \Sigma^2 \rightarrow \Sigma^2 \) is trivial, then any coordinates on \( \Sigma^2 \) will be fine with \( a = 0 \).

4.5 Listing of the possible local model data \((V^m, \Gamma, G_\infty)\)

In this section we assume that \( X_\infty \) is the GH limit of an essential ANSC sequence to ensure that \( G_0^\infty \) is abelian. We shall classify all possible local model data \((V^m, \Gamma, G_\infty)\) for a neighborhood of \( P_\infty \in X_\infty \).

The Hausdorff dimension of the limit \( X_\infty \) is the same everywhere \[BBI\, Theorem 10.6.1\] and must be an integer \[BBI\, Theorem 10.8.2\]. Hence \( \dim X_\infty \) is either 1 or 2. The dimension cannot be 0 since the diameters are unbounded and the dimension cannot be 3 since the origins are collapsing.

Recall the following elementary fact \[GB\, Theorem 2.2.1\].

**Lemma 22** The only discrete subgroups of \( O(2) \) are \( \mathbb{Z}_p \) and \( \mathbb{D}_{2p} = \mathbb{Z}_p \rtimes \mathbb{Z}_2 \), the dihedral group of order \( 2p \), and the only one dimensional subgroups are \( \text{SO}(2) \) and \( O(2) \).

We can now enumerate the possible models; we will use the notations from \[GB\]. It will be important to check that \( \Delta \) is normal in \( G_\infty^0 \), which we will check individually for each case.

**Case 0.** \((m = 0)\) This case cannot occur because of the unbounded diameters assumption.

**Case 1.** \((m = 1)\) It follows from the ANSC assumption and \[\S\,1.3.2\] and \[\S\,1.4\] that there is coordinate \((r, \theta, u)\) on \( D^2(1/2) \times (-1/2, 1/2) \) with metric \( \tilde{g}_\infty(t) = \)
\[dr^2 + f^2(r, t)d\theta^2 + du^2,\] such that \(\Delta \ni (\tau_1, \tau_2)\) acts by \((r, \theta, u) \to (r, \theta + \tau_1, u + \tau_2)\). \(\Gamma\) can be either

(i) \(\{0\}\), or
(ii) \(\mathbb{Z}_2\) acting by \(-1 : (r, \theta, u) \to (r, -\theta, -u)\), or
(iii) \(-1 : (r, \theta, u) \to (-r, \theta, -u)\) when \(f(r) = f(-r)\), or
(iv) \(-1 : (r, \theta, u) \to (-r, -\theta, u)\) when \(f(r) = f(-r)\).

In all cases \(\Delta\) is normal in \(G_\infty(\vec{0}, 2\vec{r}_0)\). This implies that \(P_\infty\) has a neighborhood homeomorphic to \((a, b)\), for some \(a < 0 < b\) in case (i) and (iii), and \(P_\infty\) has a neighborhood homeomorphic to \([0, b)\), for some \(0 < b\) in case (ii) and (iv). \(P_\infty\) corresponds to \(0\). \(G_\infty^0 = \mathbb{R}_{\text{loc}}^2\) and \(G_\infty = \mathbb{R}_{\text{loc}}^2\) in case (i) and \(G_\infty = \mathbb{R}_{\text{loc}}^2 \times \mathbb{Z}_2\) in case (iii), (iii) and (iv). We will rule out case (iii) and (iv) in the proof of Theorem 36.

**Case 2a.** \((m = 2, \dim X_\infty = 1)\) Then \(\dim \Gamma^0 = 1\) and the action generates a Killing vector field on \(D^2(1/2)\) which vanishes at origin. It follows from the ANSC assumption and §4.2 and 4.4 that there is coordinate \((r, \theta, u)\) on \(D^2(1/2) \times (-1/2, 1/2)\) with metric \(\tilde{g}_\infty(t) = dr^2 + f^2(r, t)d\theta^2 + du^2\) and \(f(0, t) = 0\), such that \(\Gamma^0 \ni \tau\) acts by \((r, \theta, u) \to (r, \theta + \tau, u)\). \(\Gamma\) can be either

(a) \(\Gamma = SO(2) = \Gamma^0\) or
(b) \(\Gamma = O(2) = \Gamma^0 \times \mathbb{Z}_2\) acting by \(-1 : (r, \theta, u) \to (r, -\theta, -u)\) for \(-1 \in \mathbb{Z}_2\).

In both cases \(\Delta\) is normal in \(G_\infty(\vec{0}, 2\vec{r}_0)\) and acts by \(\Delta \ni \tau : (r, \theta, u) \to (r, \theta + c_1\tau, u + c_2\tau)\) with two constants \(c_1\) and \(c_2\) \(\neq 0\). This implies that \(P_\infty\) has a neighborhood homeomorphic to \([0, b)\), for some \(b > 0\) and \(P_\infty\) corresponds to \(0\) in both cases. \(G^0_\infty = \mathbb{R}^2_{\text{loc}}\) in both case. \(G_\infty = \mathbb{R}_{\text{loc}}^2 \times SO(2)\) in case (2a)i and \(G_\infty = \mathbb{R}_{\text{loc}}^2 \times O(2)\) in case (2a)ii.

**Case 2b.** \((m = 2, \dim X_\infty = 2)\) Then \(\Gamma\) is discrete; since \(\Gamma^0 = \{id\}\) so \(\Delta\) is normal in \(G_\infty(\vec{0}, 2\vec{r}_0)\) and \(\Gamma \subset O(2)\). By the lemma, either

(a) \(\Gamma = \mathbb{Z}_p\) for some \(p \in \mathbb{N}\) or
(b) \(\Gamma = \mathbb{D}_{2p}\).

This implies that \(P_\infty\) has a neighborhood homeomorphic to (2b) \(D^2/\mathbb{Z}_p\) or (2bii) \(D^2/\mathbb{D}_{2p}\) where \(D^2\) is a two dimensional disk and \(\mathbb{Z}_2 < \mathbb{D}_{2p}\) acts by reflection, \(G^0_\infty = \mathbb{R}_{\text{loc}}^2\).

**Case 3.** \((m = 3)\) This case cannot occur. By Lemma 14 \(\dim \Delta \geq 1\), so \(m = \dim V = 3 - \dim \Delta \leq 2\). This is a contradiction.

We summarize the models in the following proposition.

**Proposition 23** The following tables give the complete list of the possibilities for local models of collapse in our situation (where \(p \in \mathbb{N}\)):

| \(m = 1\) | \(\Gamma\) | \(G^0_\infty\) | \(X_\infty\) locally |
|---|---|---|---|
| (i) | \{0\} | \(\mathbb{R}^2_{\text{loc}}\) | \((a, b)\) |
| (i)ii | \(\mathbb{Z}_2\) | \(\mathbb{R}^2_{\text{loc}}\) | \((a, b)\) |
| (i)iii | \(\mathbb{Z}_2\) | \(\mathbb{R}^2_{\text{loc}}\) | \([0, b)\) |
| (i)iv | \(\mathbb{Z}_2\) | \(\mathbb{R}^2_{\text{loc}}\) | \([0, b)\) |
and

| $m = 2$ | $\Gamma$ | $G^0_{\infty}$ | $X_\infty$ (locally) |
|---|---|---|---|
| (2ai) | $SO(2)$ | $\mathbb{R}^1_{\text{loc}} \times SO(2)$ | $[0, b)$ |
| (2a(ii) | $O(2)$ | $\mathbb{R}^1_{\text{loc}} \times SO(2)$ | $[0, b)$ |
| (2bi) | $\mathbb{Z}_p$ | $\mathbb{R}_{\text{loc}}$ | $D^2/\mathbb{Z}_p$ |
| (2bii) | $\mathbb{D}_{2p}$ | $\mathbb{R}_{\text{loc}}$ | $D^2/\mathbb{D}_{2p}$ |

5 Constructing the 2-dimensional virtual limit

In this section $(X_\infty, h_\infty(t), O_\infty)$ is the limit of a sequence $\{ (M^3_i, g_i(t), O_i) \}$ with ANSC, essential collapsing split-like origins, and unbounded diameters.

5.1 The 2-dimensional limit orbifold and its virtual limit

First we recall the following definition of orbifolds ($V$-manifolds) due to I. Satake (1956) and W. Thurston.

**Definition 24** An $n$-dimensional orbifold $X$ is a Hausdorff space $X$ together with a collection of pairs of open sets and finite groups $\{ (U_i, \Gamma_i) \}$ such that

1. $\{U_i\}$ is closed under finite intersections;
2. For each $i$, there is an open subset $\tilde{U}_i \subset \mathbb{R}^n$ such that $\Gamma_i$ acts on $\tilde{U}_i$ and $U_i \approx \tilde{U}_i/\Gamma_i$;
3. Whenever $U_i \subset U_j$ there is an injective homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ such that

$$\tilde{\phi}_{ij} (\gamma x) = f_{ij} (\gamma) \tilde{\phi}_{ij} (x)$$

for all $\gamma \in \Gamma_i$ and such that the appropriate diagram commutes. An orbifold is smooth if $\tilde{\phi}_{ij}$ are smooth maps.

Recall that a length space such that every point has a neighborhood isometric to a region with a Riemannian metric is a smooth Riemannian manifold (see, for instance, [BBI, §5.1]). We give an analogous result for a smooth orbifold.

**Proposition 25** If $X$ is a length space such that every point $p \in X$ has a neighborhood $U$ isometric to $\tilde{U}/\Gamma_p$ where $\tilde{U} \subset \mathbb{R}^n$ (with a smooth Riemannian metric) is a simply connected neighborhood of the origin $\tilde{0}$ and $\Gamma_p$ is a finite group acting effectively on $\tilde{U}$ such that

1. the isometries $\tilde{U}/\Gamma_p \to U_p$ take $\tilde{0}$ to $p$ and either
2a. the only fixed point of the action of $\Gamma_p$ is $\tilde{0}$ (if there is any), or
Then $X$, together with the open sets $\{(U_p, \Gamma_p)\}$ and all intersections of these open sets paired with the trivial group or $\mathbb{Z}_2$, is a smooth Riemannian orbifold.

**Proof.** Since the only fixed points of the group $\Gamma'_p$ actions are at the origin in $\hat{U}_p$, whenever there is an inclusion $U \hookrightarrow U'$, the group for $U$ is trivial or $\mathbb{Z}_2$. Furthermore, the lift $\hat{U} \to \hat{U}'$ is well defined. Since it is an isometry into the image, the map is smooth by Myers-Steenrod (see, for instance, [Pet, Theorem 9.1]), and hence the orbifold is smooth.  

**Lemma 26** The two dimensional limit $(X_{\infty}, d_{\infty}(t))$ is a smooth orbifold with singularities of types $D^2/\mathbb{Z}_p$ or $D^2/\mathbb{D}_2p$, where $D^2 \subset \mathbb{R}^2$ is a disk.

**Proof.** By Proposition 23 every point has a neighborhood isometric to $V/\Gamma$ where $V$ is a neighborhood of $0$ in $\mathbb{R}^2$ and $\Gamma$ is $\mathbb{Z}_p$ or $\mathbb{D}_2p$. By Proposition 25 this is a smooth orbifold with the stated singularity types.  

Since $X_{\infty}$ is an orbifold and the metric $d_{\infty}(t)$ comes from a Riemannian structure, we can write the limit as $(X_{\infty}, h_{\infty}(t))$ where $h_{\infty}(t)$ is the Riemannian metric on the orbifold. Next we will prove:

**Lemma 27** The two dimensional limit $(X_{\infty}, h_{\infty}(t))$ is a positively curved orbifold.

**Proof.** At each point $P_{\infty} \in X_{\infty}$ there is a neighborhood of $P_{\infty}$ isometric to a finite quotient of $(V, h_V(t)) = \left(\frac{D^2 (1/2) \times (-1/2, 1/2),^2 \tilde{h}_{\infty}(t) + du^2}{\Delta}\right)$, with $\Delta = \mathbb{R}_{\text{loc}}$ acting freely by isometries. By Lemma 22 there are coordinates $(r, \theta)$ on $D^2(1/2)$ such that the Killing vector field of the action of $\Delta$ is $K = (a\frac{\partial}{\partial r}, b\frac{\partial}{\partial \theta})$ and $^2\tilde{h}_{\infty}(0) = dr^2 + f_{P_{\infty}}(r)^2 d\theta^2$ has positive curvature. We consider two cases.

A. If $b \neq 0$, by a simple calculation using O’Neill’s formula for the submersion

$$D^2 (1/2) \times (-1/2, 1/2) \to \left(\frac{D^2 (1/2),^2 \tilde{h}_{\infty}(t)}{\Delta}\right) \times (-1/2, 1/2)$$

we conclude that the curvature of $(V, h_V(0))$ is strictly positive everywhere.

B. If $b = 0$, then $(V, h_V(0)) \cong \left(\left(\frac{D^2 (1/2),^2 \tilde{h}_{\infty}(t)}{\Delta}\right) \times (-1/2, 1/2)\right)$ has zero curvature everywhere.

Combining A and B we see that if we can find a point $P_{\infty} \in X_{\infty}$ such that $b \neq 0$ then for every point in $X_{\infty}$ we have $b \neq 0$. Hence $(X_{\infty}, h_{\infty}(0))$ has positive curvature and so does $(X_{\infty}, h_{\infty}(t))$. Otherwise, for every point in $X_{\infty}$ we have $b = 0$ and $(X_{\infty}, h_{\infty}(0))$ is flat. Since the $\Delta$ action is free, $f_{P_{\infty}}(r) > 0$ when $b = 0$. In the next two lemmas we will show by contradiction that $(X_{\infty}, h_{\infty}(0))$ cannot be flat. More precisely we will construct a complete metric on $(-\infty, \infty) \times S^1$ of positive curvature in Lemma 30 such a metric on
First we prove a general property of the limit metric \( \tilde{g}_{P_{\infty}}(t) \). Let \( \{(N^i_n, g_i, O_i)\} \) be a sequence of pointed complete \( n \)-dimensional Riemannian manifolds with bounded geometry. We assume that \( (N^i_n, g_i, O_i) \) converges to the metric space \( (X_\infty, d_\infty, O_\infty) \) in the pointed GH topology. Fix an \( \epsilon > 0 \) and let \( P_{\infty, \mu} \) and \( P_{\infty, \nu} \) be two points in \( X_\infty \).

By the definition of GH-convergence we can pass to a subsequence such that there exist maps \( \varphi_i : (N^i_n, d_{g_i}, O_i) \to (X_\infty, d_\infty, O_\infty) \) and \( \psi_i : (X_\infty, d_\infty, O_\infty) \to (N^i_n, d_{g_i}, O_i) \) which are \( 1/i \)-pointed GH approximations. Let \( P_{i, \mu} \doteq \psi_i(P_{\infty, \mu}) \) and \( P_{i, \nu} \doteq \psi_i(P_{\infty, \nu}) \). Fix orthonormal frames \( F_{i, \mu} \) of \( T_{P_{i, \mu}} N^i_n \) and \( F_{i, \nu} \) of \( T_{P_{i, \nu}} N^i_n \), and define the exponential maps restricted to balls:

\[
\begin{align*}
\exp_{P_{i, \mu}} : B^\mu_n (1 + \epsilon) &\to N^i_n, \\
\exp_{P_{i, \nu}} : B^\nu_n (1 + \epsilon) &\to N^i_n,
\end{align*}
\]

where \( B^\mu_n (1 + \epsilon) \) is a ball of radius \( 1 + \epsilon \) in \( \mathbb{R}^n \) and \( \mu \) is used to indicate the dependence on the sequence and the point in \( X_\infty \). Assume that the sectional curvatures \( K_{g_i} \leq 1 \) on the balls \( B_{g_i} (P_{i, \mu}, 1 + \epsilon) \) and \( B_{g_i} (P_{i, \nu}, 1 + \epsilon) \) so that \( \exp_{P_{i, \mu}} \) and \( \exp_{P_{i, \nu}} \) are local diffeomorphisms. We assume that the pulled-back metrics \( \tilde{g}_{i, \mu} \doteq \exp^*_P g_{i, \mu} \) converge to \( \tilde{g}_{\infty, \mu} \) on \( B^\mu_n (1) \) and \( \tilde{g}_{i, \nu} \doteq \exp^*_P g_{i, \nu} \) converges to \( \tilde{g}_{\infty, \nu} \) on \( B^\nu_n (1) \).

The next lemma shows how minimal geodesics allow us to identify parts of balls in the tangent space to create overlap maps. Consider a manifold \( (M^n, g) \) with pullback metrics with local covers \( \exp_{P_{i, \mu}} : \left( B^\mu_n (1), \tilde{g}_{i, \mu} \right) \to (M^n, P_{i, \mu}) \) and \( \exp_{P_{i, \nu}} : \left( B^\nu_n (1), \tilde{g}_{i, \nu} \right) \to (M^n, P_{i, \nu}) \) (say, assume \( K_g \leq 1 \)), and define \( \tilde{g}_\mu = \exp^*_{P_{i, \mu}} g \) and \( \tilde{g}_\nu = \exp^*_{P_{i, \nu}} g \).

**Lemma 28** Let \( \beta \) be a unit speed minimal geodesic joining \( P_{i, \mu} \) and \( P_{i, \nu} \) such that \( \beta \subset \exp_{P_{i, \mu}}(B^\mu_n (1)) \cup \exp_{P_{i, \nu}}(B^\nu_n (1)) \). Let \( Q \) be the midpoint of \( \beta \). Then for \( \delta < 1 - d_\infty(P_{i, \mu}, P_{i, \nu})/2 \) there exist \( Q_\mu \in B^\mu_n (1) \) and \( Q_\nu \in B^\nu_n (1) \) such that \( (B^n (Q_\mu, \delta), \tilde{g}_\mu) \) is isometric to \( (B^n (Q_\nu, \delta), \tilde{g}_\nu) \) (the balls are taken with respect to the given metric).

**Proof.** There is a unique lift \( \tilde{\beta}_\mu \) in \( B^\mu_n (1), \tilde{g}_\mu \) of \( \beta \) with \( \tilde{\beta}_\mu (0) = \tilde{0}_\mu \in B^\mu_n (1) \) and a unique lift \( \tilde{\beta}_\nu \) in \( B^\nu_n (1), \tilde{g}_\nu \) of \(-\beta \) (which is \( \beta \) with the time parameter reversed) with \( \tilde{\beta}_\nu (0) = \tilde{0}_\nu \in B^\nu_n (1) \). \( Q \) is lifted to \( Q_\mu \in B^\mu_n (1) \) by \( \tilde{\beta}_\mu \) and \( Q \) is lifted to \( Q_\nu \in B^\nu_n (1) \) by \( \tilde{\beta}_\nu \). Note that we have chosen \( \delta \) such that \( \exp_{P_{i, \mu}}(B^n (Q_\mu, \delta)) \) and \( \exp_{P_{i, \nu}}(B^n (Q_\nu, \delta)) \) are contained in that \( \exp_{P_{i, \mu}}(B^\mu_n (1)) \cap \exp_{P_{i, \nu}}(B^\nu_n (1)) \). Since \( \exp_{P_{i, \mu}} \) and \( \exp_{P_{i, \nu}} \) are local diffeomorphisms we can lift the exponential map \( \exp_{P_{i, \mu}} \) to a unique diffeomorphism \( \iota : B^n (Q_\mu, \delta) \to B^n (Q_\nu, \delta) \) which satisfies \( \iota(Q_\mu) = Q_\nu \). Moreover, the metrics are mapped isometrically. ■

We can now apply this to our setting.
Lemma 29 If $d_\infty(P_{\infty,\mu}, P_{\infty,\nu}) < 2$, then there are neighborhoods $B^n(Q_{\infty,\mu}, \delta) \subset (B^n(1), \tilde{g}_{\infty,\mu})$ and $B^n(Q_{\infty,\nu}, \delta) \subset (B^n(1), \tilde{g}_{\infty,\nu})$ that are isometric for some $\delta > 0$.

Proof. Choose a unit speed minimal geodesic $\beta_i$ joining $P_{i,\mu} = \beta_i(0)$ to $P_{i,\nu} = \beta_i(L_i)$. Let $Q_i$ be the midpoint of $\beta_i$ so that $r_i \geq d_{g_{\mu}}(Q_i, P_{i,\mu}) = d_{g_{\nu}}(Q_i, P_{i,\nu})$ satisfies $|2r_i - d_\infty(P_{\infty,\mu}, P_{\infty,\nu})| \leq 1/i$. Let $\delta = (2 - d_\infty(P_{\infty,\mu}, P_{\infty,\nu}))/3$. We have by Lemma 28 that there are isometries $i_\mu$ between $(B^n(Q_{i,\mu}, \delta), \tilde{g}_{i,\mu})$ and $(B^n(Q_{i,\nu}, \delta), \tilde{g}_{i,\nu})$ which satisfy $i_\mu(Q_{i,\mu}) = Q_{i,\nu}$. By passing to a subsequence, we get a limit isometry $i_\infty : (B^n(Q_{\infty,\mu}, \delta), \tilde{g}_{\infty,\mu}) \to (B^n(Q_{\infty,\nu}, \delta), \tilde{g}_{\infty,\nu})$.  

We now prove a gluing lemma. We use the notations in the proof of Lemma 29.

Lemma 30 Suppose that for all $P_{\infty} \in B_{d_\infty(0)}(O_{\infty}, a) \subset X_\infty$ there is a neighborhood of $P_{\infty}$ isometric to $\left(D^2(1/2) \times (-1/2, 1/2), 2\tilde{h}_{\infty}(t) + du^2\right) / G_{\infty,\mu}$, where $2\tilde{h}_{\infty}(t)$ has positive curvature, and that there is a free isometric action of $\Delta \subset G_{\infty,\mu}$ on $D^2(1/2)$. Then there is a metric on $(-a, a) \times S^1$ with positive curvature such that the diameter is at least $2a$.

Proof. For two different points $P_{\infty,\mu}$ and $P_{\infty,\nu}$ in $X_\infty$, by Lemma 29 the two metrics $2\tilde{h}_{\infty,\mu}(0) + du^2_\mu$ and $2\tilde{h}_{\infty,\nu}(0) + du^2_\nu$ are isometric over some balls. We can extend the isometry to domains which map to $B(P_{\infty,\mu}, 1) \cap B(P_{\infty,\nu}, 1) \subset X_\infty$ under the quotient map. Note that we have

$$B(P_{\infty,\mu}, 1) \supset \left(D^2(1/2) \times (-1/2, 1/2), 2\tilde{h}_{\infty,\mu}(0) + du^2_\mu\right) / G_{\infty,\mu},$$

$$B(P_{\infty,\nu}, 1) \supset \left(D^2(1/2) \times (-1/2, 1/2), 2\tilde{h}_{\infty,\nu}(0) + du^2_\nu\right) / G_{\infty,\nu},$$

and $G_{\infty,\mu}^0 = \Delta_\mu$, $G_{\infty,\nu}^0 = \Delta_\nu$. Since $\Delta_\mu$ and $\Delta_\nu$ give nontrivial Killing vector fields of $2\tilde{h}_{\infty,\mu}(0)$ and $2\tilde{h}_{\infty,\nu}(0)$ respectively, by Lemma 19(i) we can write

$$2\tilde{h}_{\infty,\mu}(0) = dr_\mu^2 + f_\mu(r_\mu)^2 d\theta_\mu^2,$$

$$2\tilde{h}_{\infty,\nu}(0) = dr_\nu^2 + f_\nu(r_\nu)^2 d\theta_\nu^2,$$

such that $f_\mu$ and $f_\nu$ are never zero and the Killing fields are proportional to $\frac{\partial}{\partial r_\mu}$ and $\frac{\partial}{\partial r_\nu}$. By Lemma 19(ii) different $f_\mu$ and $f_\nu$ can be pieced together to form $f$ which restricts to $f_\mu$ and $f_\nu$.

Fix a $P_{\infty} \in X_\infty$. We can define a curve in $X_\infty$ to be the image of the coordinate curve $\{(r, 0) \in V_{P_{\infty}}\}$. Then we can extend this curve from both ends. Let $P_{1,\mu}$ be an end point, then we can use the isometry in Lemma 29 to get a short curve which is part of $\{(r, \theta_0) \in V_{P_{1,\mu}}\}$ for some fixed $\theta_0$. Then we use the image of $\{(r, \theta_0) \in V_{P_{1,\mu}}\}$ to extend the image of curve $\{(r, 0) \in V_{P_{\infty}}\}$ in this way we get a curve $\gamma(s), s \in (-a, a)$ of length $2a$ in $X_\infty$.

We can then piece together $f_{\gamma(s_i)}(r)$ associated to different points $\gamma(s_i)$ to form a smooth function $f(r), r \in (-a, a)$. Then $dr^2 + f(r)^2 d\theta^2$ defines a
metric on \((r, \theta) \in (-a, a) \times S^1\) with positive curvature since \(f_{\gamma(s)}(r) > 0\) and \(dr^2 + f_{\gamma(s)}(r)^2 d\theta^2\) has positive curvature.

**Proposition 31** \(X_\infty\) has at most one singular point of type \(D^2/\mathbb{Z}_p\) or \(D^2/\mathbb{D}_p\) where \(p > 1\).

More generally, we have:

**Proposition 32** Let \((X^n, g)\) be a complete, noncompact Riemannian orbifold with singular points of type \(D^n/\Gamma'\) or \(D^n/(\Gamma' \times \mathbb{Z}_2)\), where the only fixed point of the action \(\Gamma'\) is \(0\), \(\mathbb{Z}_2\) acts by reflection with respect to a hyperplane, and \(\Gamma' < \Gamma' \times \mathbb{Z}_2\) is a normal subgroup. Suppose the sectional curvature \(K_g > 0\) on \(X^n\). Then \(X^n\) has at most one singular point with rank \(|\Gamma'| > 1\).

**Proof.** We will use \(d\) to denote the distance induced by \(g\). The proof is by contradiction. Suppose the proposition is not true; let \(p\) and \(q\) be two singular points, and let \(\gamma\) be a unit-speed minimal geodesic from \(p = \gamma(0)\) to \(q = \gamma(L)\).

Without loss of generality we may assume that \(\gamma(t)\) is a smooth point for all \(t \in (0, L)\) or of type \(D^n/\mathbb{Z}_2\) for all \(t \in (0, L)\), otherwise we may take two consecutive singular points on \(\gamma\) be to \(p\) and \(q\).

Let \(\rho > 0\) be a constant. There is an \(\varepsilon > 0\) depending on \(L + \rho\) and \(\rho\) so that all sectional curvatures \(\kappa\) on the ball of radius \(L + \rho\) centered \(p\) have \(\kappa \geq \varepsilon/\rho^2\). Let \(z\) be a point with \(d(p, z) \geq r_0\) where \(r_0\) will be chosen very large (to be specified later). The existence of \(z\) follows from the completeness and noncompactness of \(X\). Let \(\eta\) be a unit-speed minimal geodesic from \(z\) to \(\gamma([0, L])\), say \(\gamma(t_0) = \eta(0)\) and \(z = \eta(r + \rho)\) for some \(r > 0\).

We claim that \(\gamma'(t_0)\) is perpendicular to \(\eta'(0)\). Note that if \(\gamma(t_0)\) is a singular point the perpendicularity means that there is a lift \(\tilde{\gamma}'(t_0)\) of \(\gamma'(t_0)\) and a lift \(\tilde{\eta}'(0)\) of \(\eta'(0)\) in the uniformized tangent space \(\tilde{T}_{\gamma(t_0)}M\) of the tangent cone \(T_{\gamma(t_0)}M\) such that \(\tilde{\gamma}'(t_0)\) is perpendicular to \(\tilde{\eta}'(0)\). If \(t_0 \in (0, L)\) the claim is clearly true since otherwise we can move \(t\) to left or right of \(t_0\) to shorten \(\eta\). If \(t_0 = 0\) (a similar argument holds for \(t_0 = L\)), fix a lifting \(\tilde{\eta}'(0)\) of \(\eta'(0)\) and consider all the lifting of \(\gamma'(t_0)\) in \(\tilde{T}_{\gamma(t_0)}M\) moved by \(\Gamma'\). Since \(\gamma(t_0)\) is singular point with rank \(|\Gamma'| > 1\), the sum of these lifts is zero, hence at least one of the lifts must make an angle less than or equal to \(\pi/2\). But it cannot be less than \(\pi/2\), otherwise we can shorten the distance between \(z\) and \(\gamma([0, L])\). The claim is proved.

By our choice of \(\varepsilon\) all sectional curvatures are greater than or equal to \(\varepsilon/\rho^2\) along \(\eta\) for a distance \(\rho\) from \(\eta(0)\). Let \(\zeta, s \in (t_0 - \delta, t_0 + \delta)\) be minimal geodesics starting from \(\gamma(s)\) and going through \(z\). The standard computation shows the second variation of the length of \(\zeta\) at \(s = t_0\) is strictly negative. Indeed let \(Z = \gamma'(t_0)\) be the unit tangent vector to \(\gamma\) and extend \(Z\) to a vector field on \(\eta\) by parallel translation. Choose a function \(\varphi\) to be identically \(1\) within distance \(\rho\) of \(\eta(0)\) along \(\eta\) and then to drop linearly to zero with slope \(1/\rho\). The second variation of the length of \(\zeta\) in the direction \(\varphi Z\) is

\[
I(\varphi Z, \varphi Z) = \int \left[|D\varphi|^2 - \kappa \varphi^2\right] ds
\]

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where $\kappa = Rm(\eta', Z, \eta, Z)$ is the sectional curvature. Considering the separate contributions from the part of $\eta$ within $\rho$ of $\eta(0)$ and the part beyond we get

$$I(\varphi Z, \varphi Z) \leq -\varepsilon/\rho + 1/r.$$  

We can take $r_0 \leq r + \rho$ large, and hence $r$ large, enough such that

$$1/r \leq \varepsilon/2\rho$$  

and so we get

$$I(\varphi Z, \varphi Z) \leq -\varepsilon/2\rho.$$  

The second variation is strictly negative. The first variation of the length of $\gamma$ at $t$ is noncompact. We call such a solution the virtual limit, i.e., its universal cover is a smooth manifold. We have proved the following.

**Theorem 33** There is a universal cover $\left(\hat{X}_\infty, \hat{h}_\infty(t)\right)$ of the two dimensional limit $(X_\infty, d_\infty(t))$ which is a Riemannian manifold with positive curvature for each $t$ and is diffeomorphic to $\mathbb{R}^2$.

Recall that around any smooth point $P_\infty \in X_\infty$, we have a neighborhood $\hat{U}_{P_\infty}$ in $\hat{X}_\infty$ such that

$$\left(\hat{U}_{P_\infty}, \hat{h}_\infty(0)\right) \cong \left(D^2(1/2) \times (-1/2, 1/2), dr^2 + f_{P_\infty}(r)^2 d\theta^2 + du^2\right) / \Delta,$$  

where the Killing vector field of the $\Delta$ action is $K = (a \partial_r, b \partial_u)$ for some constants $a$ and $b \neq 0$ independent of the choice of $P_\infty$. Since by construction the action of $\Delta$ preserves the metric $2\hat{h}_\infty(t) + du^2$ on $D^2(1/2) \times (-1/2, 1/2)$ for all $t$, there is a function $k(t) > 0$ and functions $f_{P_\infty}(r, t)$ such that $k(0) = 1$, $f_{P_\infty}(r, 0) = f_{P_\infty}(r)$ and

$$\left(\hat{U}_{P_\infty}, \hat{h}_\infty(t)\right) \cong \left(D^2(1/2) \times (-1/2, 1/2), k(t)^2 dr^2 + f_{P_\infty}(r, t)^2 d\theta^2 + du^2\right) / \Delta.$$  

By the calculation in [CH] or [CGH], the quotient metric is

$$\hat{h}_\infty(t) = k(t)^2 dr^2 + \frac{f_{P_\infty}(r, t)^2}{1 + \left(\frac{a}{b}\right)^2 f_{P_\infty}(r, t)^2} d\theta^2.$$  

When $a \neq 0$, $\hat{h}_\infty(t)$ is not necessarily a solution of the RF.

However we can construct a solution of the RF on $\mathbb{R}^2$ by piecing together the RF solutions $k(t)^2 dr^2 + f_{P_\infty}(r, t)^2 d\theta^2$ when $a \neq 0$. The piecing together can be done with Lemma [30] we stop the extension in defining the curve $\gamma(s)$ when we get to $r_0$ and $P_\infty$ with $f_{P_\infty}(r_0, 0) = 0$. This can only happen on one end since $X_\infty$ is noncompact. We call such a solution the virtual limit associated with the two dimensional limit $(X_\infty, d_\infty(t))$. 

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Theorem 34 Suppose that the dimension of the limit \((X_\infty, d_\infty(t))\) is two and \(a \neq 0\) in the Killing vector field \(K = (a \frac{\partial}{\partial t}, b \frac{\partial}{\partial r})\) generated by \(\Delta\) which is defined in local Fukaya theory, then there is a rotationally symmetric surface \((\tilde{X}_\infty, \tilde{h}_\infty(t))\) associated to \((X_\infty, d_\infty(t))\) and \(\tilde{h}_\infty(t)\) is a rotationally symmetric solution of the RF with positive curvature. \(\tilde{X}_\infty\) is diffeomorphic to \(\mathbb{R}^2\).

5.2 The virtual limit associated with a 1-dimensional limit \(X_\infty\)

From Proposition 23 when the dimension of \(X_\infty\) is 1, \(X_\infty\) has a manifold structure, possibly with boundary. Since \(X_\infty\) has unbounded diameter, it is diffeomorphic to \(\mathbb{R}\) or \([0, \infty)\). When \(X_\infty \cong [0, \infty)\) there is a unique point \(P_\infty \in X_\infty\) which fits into case (1(iii)), (1(iv)), (2(ai) or (2(aii)), and all other points fit into case (1i) or (1ii). We will rule out \(X_\infty \cong \mathbb{R}\) next.

Proposition 35 \(X_\infty\) cannot be \(\mathbb{R}\).

Proof. If \(X_\infty \cong \mathbb{R}\), every point \(P_\infty \in X_\infty\) fits into case (1i) or (1ii). Then \(\Delta = \mathbb{R}^2_{\text{loc}}\) acts freely and locally isometrically on \(\left(D^2 (1/2) \times (-1/2, 1/2), 2 \tilde{h}_\infty(0) + du^2\right)\).

Note that \(\Delta\) cannot act trivially on the second factor \((-1/2, 1/2)\), otherwise we get a free action of \(\Delta\) by isometries on \(\left(D^2 (1/2), 2 \tilde{h}_\infty(0)\right)\), which by Lemma 20 would give us independent Killing fields. Hence using Lemma 19 we may assume that \(\mathbb{R}^2_{\text{loc}}\) acts freely on \(D^2 (1/2) \times (-1/2, 1/2)\) by \((r, \theta, u) \mapsto (r, \theta + a \tau, u + b \tau)\) and \(2 \tilde{h}_\infty(0) = dr^2 + f_{P_\infty}(r)^2 d\theta^2\). There is a subgroup of \(\mathbb{R}^2_{\text{loc}}\) which acts freely by \((r, \theta, u) \mapsto (r, \theta + a \tau, u)\). This implies \(f_{P_\infty}(r) > 0\).

Now we can construct a complete metric on \((-\infty, \infty) \times S^1\) with positive curvature by piecing together \(k(t)^2 dr^2 + f_{P_\infty}(r, t)^2 d\theta^2\) when \(a \neq 0\) via Lemma 31. Such metric on \((-\infty, \infty) \times S^1\) cannot exist as argued before. We obtain a contradiction and the proposition is proved.

When \(X_\infty \cong [0, \infty)\) we construct a 2-dimensional virtual limit. A simple construction follows. A more geometric construction is given in next section.

Theorem 36 There is a rotationally symmetric surface \((\tilde{X}_\infty, \tilde{h}_\infty(t))\) diffeomorphic to \(\mathbb{R}^2\) associated to \(X_\infty \cong ([0, \infty), h_\infty(t))\) and \(\tilde{h}_\infty(t)\) is a solution of the RF with positive curvature.

Proof. For every point \(P_\infty \in (0, \infty)\), we have \(2 \tilde{h}_\infty(t) = k(t)^2 dr^2 + f_{P_\infty}(r, t)^2 d\theta^2\).

We can construct a metric \(k(t)^2 dr^2 + f(r, t)^2 d\theta^2\) with positive curvature on \((0, \infty) \times S^1\) such that the projection \((0, \infty) \times S^1 \to (0, \infty) \cong X_\infty - \{O_\infty\}\) is a submersion. This can be done by piecing together using Lemma 30. To see what happens at \(0 \in X_\infty\), we choose \(P_\infty = 0\) and assume the corresponding coordinate \(r(P_\infty) = 0\). Then \(0\) fits into case (1(iii)), (1(iv)), (2(ai) or (2(aii)).

If \(0\) fits into either case (1(iii)) or (1(iv)), then \(f_{P_\infty}(0, 0) \neq 0\) from the fact that \(\Delta = \mathbb{R}^2_{\text{loc}}\) acts freely and \(f_{P_\infty}(-r, 0) = f_{P_\infty}(r, 0)\) from the \(\mathbb{Z}_2\) reflection.
We can extend the metric \( k(0)^2 \, dr^2 + f(r,0)^2 \, d\theta^2 \) using the \( \mathbb{Z}_2 \) reflection to get a complete metric on \( (-\infty,\infty) \times S^1 \) with positive curvature. Such metric on \( (-\infty,\infty) \times S^1 \) cannot exist as argued before. Hence case (1(iii)) or (1(iv)) can not happen.

If 0 fits into either case (2ai), (2a(ii)), then \( f_{P_\infty}(0,0) = 0 \) from the fact that the action of \( \Gamma \) has fixed point 0. Hence \( f(0,t) = 0 \) and \( k(t)^2 \, dr^2 + f(r,t)^2 \, d\theta^2 \) extend to a smooth metric on \( \left( (0,\infty) \times S^1 \right) / \left( \{0\} \times S^1 \right) \cong \mathbb{R}^2 \) which is our \( \mathcal{X}_\infty \). It is clear that \( k(t)^2 \, dr^2 + f(r,t)^2 \, d\theta^2 \) is a rotationally symmetric solution of the RF with positive curvature. ■

### 5.3 A geometric construction

We now give a geometric, but more technical description of how to construct the virtual limit of the one-dimensional \( \mathcal{X}_\infty \).

**Step 1**: Create framework. Consider the covering \( \{U_{\infty,k}\}_{k \in \mathbb{N}\cup\{0\}} \) of \( \mathcal{X}_\infty \) defined by \( U_{\infty,0} = [0,3) \) and \( U_{\infty,k} = (4k-3,4k+3) \) for \( k \in \mathbb{N} \). It is clear that \( U_{\infty,k} \cap U_{\infty,\ell} \neq \emptyset \) if and only if \( |k-\ell| \leq 1 \). Let \( P_{\infty,k} = 4k \), which are the centers of \( U_k \). A word about notation: \( B_{3,k}^i(3) \subset \mathbb{R}^3 \) all denote the same 3-ball of radius 3, where the indices \( i \in \mathbb{N} \cup \{\infty\} \) and \( k \in \mathbb{N}\cup\{0\} \) are just to remind the reader of the dependence on the term in the sequence and the point in \( \mathcal{X}_\infty \). We also have frames at \( P_{i,k} \), orthonormal with respect to \( g_i(0) \), which we use to identify the tangent spaces \( T_{P_{i,k}} M_i \) with \( \mathbb{R}^3 \).

**Step 2**: Glue open sets in the sequence. By the definition of GH convergence, passing to a subsequence there exist maps \( \varphi_i(t) : (M_i, d_{g_i(t)}, O_i) \to (X_\infty, d_\infty(t), O_\infty) \) and \( \psi_i(t) : (X_\infty, d_\infty(t), O_\infty) \to (M_i, d_{g_i(t)}, O_i) \) for each \( t \in [\beta,\psi] \) which are \( 1/i \)-pointed GH approximations. By choosing a countable dense set of times in \( [\beta,\psi] \) and passing to a subsequence we can find time-independent maps \( \varphi_i : (M_i, d_{g_i(t)}, O_i) \to (X_\infty, d_\infty(t), O_\infty) \) and \( \psi_i : (X_\infty, d_\infty(t), O_\infty) \to (M_i, d_{g_i(t)}, O_i) \) which are \( 1/i \)-pointed GH approximations for all \( t \in [\beta,\psi] \). Let \( \hat{P}_{i,k} \equiv \psi_i(P_{\infty,k}) \) for \( k < i/4 \). Choose a minimal geodesic \( \beta_{i,k} \) with respect to \( g_i(0) \) joining \( \hat{P}_{i,k} \) to \( P_{i,k+1} \) for \( k+1 < i/4 \). Let \( Q_{i,k} \) be the midpoint of \( \beta_{i,k} \) so that \( r_{i,k} \equiv d_{g_i(0)}(Q_{i,k}, P_{i,k}) = d_{g_i(0)}(Q_{i,k}, P_{i,k+1}) \) satisfies \( |2r_{i,k} - 4| < 1/i \). For \( i \) large enough we have \( B_{g_i(0)}(Q_{i,k},1/2) \subset B_{g_i(0)}(P_{i,k},3) \cap B_{g_i(0)}(P_{i,k+1},3). \)

For each \( i \in \mathbb{N} \) and \( t \in [\beta,\psi] \), let \( \hat{g}_{i,k}(t) = \exp_{\hat{P}_{i,k}} g_i(t) \) (recall the definition from \[\text{[5.2]}\]) and consider the 3-balls of radius three \( (\hat{B}_{i,k}(3), \hat{g}_{i,k}(t)) \) isometrically covering \( (B_{g_i(0)}(P_{i,k},3), g_i(t)) \) by the exponential map \( \exp_{\hat{P}_{i,k}} \) for each \( k \in \mathbb{N}\cup\{0\} \).

We can use \( \beta_{i,k} \) to identify \( (B_{i,k}(3), \hat{g}_{i,k}(0)) \) and \( (B_{i,k+1}(3), \hat{g}_{i,k+1}(0)) \) by an isometry \( \iota_{i,k} \) on the overlap regions (which we define below). Moreover, \( \iota_{i,k} \) is an isometry with respect to the metrics \( \hat{g}_{i,k}(t) \) and \( \hat{g}_{i,k+1}(t) \) for all \( t \in [\beta,\psi] \).

As in Lemma \[\text{[28]}\] there are balls around points \( \hat{Q}_{i,k} \) and \( Q_{i,k} \) respectively in \( B_{i,k}(3) \) and \( B_{i,k+1}(3) \) which are isometric and \( \exp_{\hat{P}_{i,k}} \hat{Q}_{i,k} = \exp_{P_{i,k+1}} Q_{i,k} = \).
Let $\tilde{U}_{i,k}$ be the connected component of

$$(\exp_{P_{i,k}})^{-1} \left( B_{g_i(0)}(P_{i,k},3) \cap B_{g_i(0)}(P_{i,k+1},3) \right)$$

containing $\tilde{Q}_{i,k}$ and $\tilde{\mathcal{U}}_{i,k}$ be the connected component of

$$(\exp_{P_{i,k+1}})^{-1} \left( B_{g_i(0)}(P_{i,k},3) \cap B_{g_i(0)}(P_{i,k+1},3) \right)$$

containing $\tilde{\mathcal{U}}_{i,k}$. By local covering space theory, there exists a unique diffeomorphism $i_{i,k} : (\tilde{U}_{i,k}, \tilde{Q}_{i,k}) \rightarrow (\tilde{\mathcal{U}}_{i,k}, \tilde{\mathcal{Q}}_{i,k})$ such that $\exp_{P_{i,k}} \big|_{\tilde{U}_{i,k}} = \exp_{P_{i,k+1}} \circ i_{i,k}$ which extends the isometry from Lemma 28 mentioned above. The map $i_{i,k}$ is an isometry from $\tilde{g}_{i,k}(t)$ to $\tilde{g}_{i,k+1}(t)$ for all $t \in [\beta, \psi]$. By passing to a subsequence, we get a limit isometry $\iota_{\infty,k}$ of the overlap regions $\tilde{U}_{\infty,k}$ of $(B_{\infty,k}(3), \tilde{g}_{\infty,k}(t))$ and $\tilde{\mathcal{U}}_{\infty,k}$ of $(B_{\infty,k+1}(3), \tilde{g}_{\infty,k+1}(t))$ for all $k \in \mathbb{N} \cup \{0\}$ and $t \in [\beta, \psi]$.

**Step 3:** Find good coordinates. From $B_{\infty,0}(3)$ we get rotationally symmetric metrics $h_{\infty,0}(t)$ on a 2-disk $\Sigma_{\infty,0}^2 = D^2(\tau_{\infty,0})$ for some $\tau_{\infty,0} \geq 2$. We obtain these metrics by considering the surface slice passing through $\tilde{0}$ (recall that metrically the ball is locally a surface product with a line). Alternatively, these metrics can be obtained from the isometric embedding of $D^2(\tau_{\infty,0}) \times (-\varepsilon, \varepsilon)$ into $(B_{\infty,0}(3), \tilde{g}_{\infty,k}(t))$ by quotienting out the interval direction. For $t = 0$, letting $\varepsilon \rightarrow 0$ allows us to take $\tau_{\infty,0} \rightarrow 3$. For $k \in \mathbb{N}$ from $B_{\infty,k}(3)$ we get metrics $\left( \Sigma_{\infty,k}^2, h_{\infty,k}(t) \right)$ which are locally warped products. Let $O_{\infty,k}$ denote the point in $\Sigma_{\infty,k}^2$ corresponding to $\tilde{0} \in B_{\infty,k}(3)$. Then $B_{O_{\infty,k}, r} \subset \Sigma_{\infty,k}^2$ is compact for all $r < 3$. The warped product metric $h_{\infty,1}(0)$ on $\Sigma_{\infty,k}^2$ induces oriented local coordinates $(r, \theta)$ such that $h_{\infty,k}(0) = dr^2 + f_k(r)^2 d\theta^2$. The coordinate $r$ is uniquely determined up an additive constant and $\theta$ is uniquely determined by $\theta(O_{\infty,k}) = 0$ by Lemma 19. For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(r, \theta) : (-3 + \varepsilon, 3 - \varepsilon) \times (-\delta, \delta) \rightarrow \Sigma_{\infty,k}^2$$

is an embedding.

**Step 4:** Form metrics on disks. This step corresponds to Lemma 20. We have constructed an overlap map from $\Sigma_{\infty,0}^2$ to $\Sigma_{\infty,1}^2$. Then we extend $S_{\infty,1}^2 = \Sigma_{\infty,0}^2 \cup \Sigma_{\infty,1}^2 / \sim$ (alternately we could have defined $S_{\infty,1}^2 = \Sigma_{\infty,0}^2 \cup \left( [-3 + \varepsilon, 3 - \varepsilon] \times (-\delta, \delta) \right) / \sim$), where the identification $\sim$ is defined by the isometry $i_{\infty,0}$, to rotationally symmetric metrics $f_{\infty,1}(t)$ on a disk $\Delta_{\infty,1}^2 = D^2(\tau_{\infty,1})$. One can see this explicitly in terms of the coordinates $(r_0, \theta_0)$ and $(r_1, \theta_1)$ on $\Sigma_{\infty,0}^2 \setminus \{O_{\infty,0}\}$ and $\Sigma_{\infty,1}^2$ with $\lim_{x \rightarrow O_{\infty,0}} r_0(x) = 0$ and $r_1(O_{\infty,1}) = 0$. In these coordinates, the isometry $i_{\infty,0}$ is given by $r_1 = r_0 - 4$ and $\theta_1 = c_1 \theta_0 + c_2$, for some $c_1, c_2 \in \mathbb{R}$, and maps $((-3 + \varepsilon, 3 - \varepsilon) \times (-\delta, \delta), h_{\infty,0}(t))$ to $((-3 + \varepsilon, 3 - \varepsilon) \times (-\delta, \delta), h_{\infty,1}(t))$. Under this isometry at $t = 0$ we have $f_1(r - 4) = f_0(r)$ for $r \in (1 + \varepsilon, 3 - \varepsilon)$. 


Define the smooth function \( f \) as
\[
 f(r) = \begin{cases} 
 f_0(r) & 0 \leq r \leq 2 \\
 f_1(r - 4) & 2 < r \leq 6.
\end{cases}
\]
This gives us a metric \( dr^2 + f(r)^2 d\theta^2 \) on \( D^2(6) \). We can also construct this space by starting on \( S^2_{\infty,1} \) and then using the symmetry to extend to \( D^2(6) \) via
\[
 D^2(6) = S^2_{\infty,1} \times S^1 / \sim,
\]
where \((r, \theta, \phi) \sim (r, \theta + a, \phi + a)\) for all \( a \in S^1 \) and the metric is extended so the translation of the circle (denoted by \( a \) above) is isometric. We can continue this process to \( D^2(k + 2) \) for all \( k \). This produces a complete metric \( dr^2 + f(r)^2 d\theta^2 \) on \( \mathbb{R}^2 \). This construction works for any \( t \). We take as the virtual limit the warped product metric \( \overline{h}(t) \) with \( \overline{h}(0) = dr^2 + f(r)^2 d\theta^2 \) on \( \mathbb{R}^2 \). Note that every point has a neighborhood isometric to \( \left( \Sigma^2_{\infty,k}, h_{\infty,k}(t) \right) \) for some \( k \geq 0 \). Note that after a translation of \( r_k \) and \( \theta_k \) and scaling of \( \theta_k \) the coordinates \( (r_k, \theta_k) \) define isometric immersions of \( \left( \Sigma^2_{\infty,k}, h_{\infty,k}(t) \right) \) into \( \left( \mathbb{R}^2, \overline{h}(t) \right) \).

6 Hamilton’s singularity theory in the collapsed Type IIb case

Let \( (M^3, g(t)) \), \( t \in [0, \infty) \), be a complete solution of the Ricci flow with bounded curvature forming a Type IIb singularity at time \( t \). That is, \( \sup_{M \times [0, \infty)} |Rm (x, t)| = \infty \). We follow §16 of [H95a] in choosing a sequence of points and times to dilate about. Let \( T_i \rightarrow \infty \) be any sequence of times. Given any sequence \( \varepsilon_i \rightarrow 0 \), choose \((x_i, t_i) \in M \times [0, T_i] \) so that \( t_i \rightarrow \infty \) and
\[
\frac{t_i}{T_i} \left| Rm (x_i, t_i) \right| \sup_{M \times [0, T_i]} \left\{ t \left( T_i - t \right) \left| Rm (x, t) \right| \right\} \geq 1 - \varepsilon_i.
\]
This can be done because
\[
\sup_{M \times [0, T_i]} \left\{ t \left( T_i - t \right) \left| Rm (x, t) \right| \right\} \geq \frac{T_i}{2} \sup_{M \times [0, T_i/2]} \left\{ t \left| Rm (x, t) \right| \right\},
\]
and the right side goes to infinity. Let \( K_i = \left| Rm (x_i, t_i) \right| \). The solution \( g_i(t) \propto K_i g(t_i + t/K_i) \) exists on the time interval \([-\alpha_i, \infty)\), where \( \alpha_i = t_i K_i \rightarrow \infty \) as \( i \rightarrow \infty \). Let \( \omega_i = (T_i - t_i) K_i \). Then \( \omega_i \rightarrow \infty \). We have
\[
\left| Rm \left[ g_i \right] (x, t) \right| \leq \frac{1}{1 - \varepsilon_i \alpha_i + t} \frac{\omega_i}{\omega_i - t}
\]
for all \( x \in M \) and \( t \in [-\alpha_i, \omega_i] \), as well as \( \left| Rm \left[ g_i \right] (x_i, 0) \right| = 1. \) Note that \( g_i(t) \) have ANSC by [2.3].

Now consider the sequence \( (M^3, g_i(t), x_i) \), \( t \in [-\alpha_i, \omega_i] \), and assume that the diameters tend to infinity and the sequence collapses. Recall that this
implies the origins are split-like. The virtual limit \( \left( \overset{\rightarrow}{X}_\infty, \overset{\rightarrow}{h}_\infty(t) \right) \) constructed is a complete solution on a surface diffeomorphic to \( \mathbb{R}^2 \) with positive, bounded curvature and exists for \( t \in (-\infty, \infty) \). The base point \( \overset{\rightarrow}{x}_\infty \in \overset{\rightarrow}{X}_\infty \) satisfies
\[
\left| Rm \left[ \overset{\rightarrow}{h}_\infty(t) \right] (\overset{\rightarrow}{x}_\infty, 0) \right| = \sup_{\overset{\rightarrow}{X}_\infty \times (-\infty, \infty)} \left| Rm \left[ \overset{\rightarrow}{h}_\infty(t) \right] (x, t) \right|.
\]
The reason is that any other point \( (\overset{\rightarrow}{y}_\infty, 0) \in \overset{\rightarrow}{X}_\infty \times (-\infty, \infty) \) corresponds to a sequence \( (y_i, 0) \in M \times [\alpha_i, \omega_i] \) endowed with the metric \( g_i(0) \). The curvatures at \( (y_i, 0) \) are almost smaller than the curvatures at \( (x_i, 0) \), and hence the curvatures of the limit of the covering geometries of the \( y_i \) are less than those of the \( x_i \). The curvature of the virtual limit is the same as the curvature in the surface direction of the limit of the covering geometries (which splits as a product), and hence the curvature at \( (\overset{\rightarrow}{y}_\infty, 0) \) is less than the curvature at \( (\overset{\rightarrow}{x}_\infty, 0) \).

Hence, by Hamilton’s result that eternal solutions are steady solitons \cite{H-93}, which uses his matrix Harnack estimate, \( \left( \overset{\rightarrow}{X}_\infty, \overset{\rightarrow}{h}_\infty(t) \right) \) is isometric to a cigar soliton solution. Here we have used the fact that since \( \overset{\rightarrow}{X}_\infty \) is 2-dimensional, the maximum of \( |Rm| \) being attained at \( (\overset{\rightarrow}{x}_\infty, 0) \) is the same as the maximum of \( R \) being attained at \( (\overset{\rightarrow}{x}_\infty, 0) \). This last condition is what is assumed in Hamilton’s eternal solutions result.

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