Equation of motion for multiqubit entanglement in multiple independent noisy channels

Zhong-Xiao Man\textsuperscript{1}, Yun-Jie Xia\textsuperscript{1} and Shao-Ming Fei\textsuperscript{2,3}

\textsuperscript{1} Shandong Provincial Key Laboratory of Laser Polarization and Information Technology, Department of Physics, Qufu Normal University, Qufu 273 165, People’s Republic of China
\textsuperscript{2} School of Mathematical Sciences, Capital Normal University, Beijing 100 048, People’s Republic of China
\textsuperscript{3} Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

E-mail: manzhongxiao@163.com, yjxia@mail.qfnu.edu.cn and feishm@mail.cnu.edu.cn

Received 29 September 2011, in final form 26 March 2012
Published 27 April 2012
Online at stacks.iop.org/JPhysA/45/195306

Abstract

We investigate the possibility and conditions to factorize the entanglement evolution of a multiqubit system passing through multi-sided noisy channels. By means of a lower bound of concurrence (LBC) as an entanglement measure, we derive an explicit formula of the LBC evolution of the $N$-qubit generalized Greenberger–Horne–Zeilinger (GGHZ) state under some typical noisy channels, based on which two kinds of factorizing conditions for the LBC evolution are presented. In this case, the time-dependent LBC can be determined by a product of initial LBC of the system and the LBC evolution of a maximally entangled GGHZ state under the same multi-sided noisy channels. We analyze the realistic situations where these two kinds of factorizing conditions can be satisfied. In addition, we also discuss the dependence of entanglement robustness on the number of qubits and that of the noisy channels.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.65.Yz

1. Introduction

Quantum entanglement, as a type of distributed or nonlocal coherence among several quantum subsystems, has been more and more recognized as an indispensable resource in realizing many intriguing quantum information processing and quantum computation applications [1]. Realistically, however, the unavoidable coupling of the entangled system to its environment will lead to a destruction of the necessary entanglement. Therefore, a deeper understanding of the entanglement dynamics is of great importance not only in the foundation of quantum mechanics [2], but also in the rapidly developing quantum technologies [1].

The usual way to study entanglement evolution is first to deduce the evolved state of the system and then to calculate its entanglement [3–10]. However, as far as multipartite systems (or higher dimensional systems) are concerned, it is generally very hard to solve the state...
equation, and therefore the evolution of entanglement can be determined only in very special cases. Different from the aforementioned state-evolution technique, it was found [11] for any two-qubit system with only one qubit being subjected to a noisy environment, i.e. under a one-sided noisy channel, that the evolution of the system’s entanglement in terms of concurrence [12] can be completely determined by the product of the system’s initial concurrence and the concurrence evolution of a maximally entangled state also under the one-sided channel. Through this channel-dependent technique one can characterize the entanglement dynamics under unknown channels by probing the entanglement evolution of a maximally entangled state alone without exploring the concrete action of the channel on all initial states [11]. The entanglement factorization law [11] has been experimentally verified in an amplitude decay channel [13], and in the combined channel of phase damping (PD) and amplitude decay [14]. The factorization law has been generalized to a finite-dimensional bipartite system for both initially pure state [15] and mixed state [16, 17], and to determine the evolution of the G concurrence [18] of two qubits [19]. In [20], the author studied the multipartite system with only one subsystem undergoing an arbitrary physical process and found that the entanglement evolution of the multipartite system can be determined uniquely by a single function of the quantum channel alone, irrespective of the number of qubits in the system. Therefore, in this case, the multipartite entanglement evolves exactly in the same way as the bipartite (two-qudit) entanglement [20].

In many practical situations, each particle of an entangled system is coupled with a local noisy environment; therefore, the factorization law of entanglement evolution under a one-sided noisy channel should be generalized to the case of multi-sided channels. It would be of great interest if the entanglement evolution under multi-sided noisy channels could be factorized as that in the one-sided channel. Unfortunately, however, even for the simplest case of the two-qubit entanglement under two-sided noisy channels, the factorization law [11] no longer holds except for some special forms of the two-qubit state under special environments [21, 22]. It was shown that [21] for the two-sided amplitude damping (AD) channels, the factorization law is valid only for the NOE (the number of excitations in the system is not more than 1) state. However, for the two-sided pure dephasing channel case, the factorization law is valid not only for the NOE state but also for all three basis states [22]. Therefore, a general factorization law of entanglement evolution does not hold for multi-sided channels. However, it is still useful to explore the factorization law for special forms of entangled states, especially the multipartite entanglement, under special noisy channels. This study may shed some light on the understanding of the entanglement dynamics of the multipartite system. In this work, focusing on the $N$-qubit generalized Greenberger–Horne–Zeilinger (GGHZ) state [23], we shall investigate the possibility and conditions to factorize the entanglement evolution of the multiqubit system under multi-sided noisy channels.

While studying the dynamics of the multipartite entanglement, one of the biggest obstacles is the lack of a computable entanglement measure. Most studies [24–28] on the entanglement dynamics of a multiqubit system are based on the strategy of bipartition and the measure of the bipartite entanglement. Another strategy one usually adopts is to generalize the bipartite entanglement measure directly to the multipartite case [29, 30]. However, the calculation for the mixed state of a multipartite system requires an optimization process, which is very difficult to solve exactly and can only be determined purely algebraically in the regime where the mixing is moderate [29]. To be able to assess the global entanglement of a multiqubit system, we adopt the concept of lower bound of concurrence (LBC) [31, 32]. Based on the derived formula of LBC evolution of an $N$-qubit system under some typical noisy channels, we find two kinds of sufficient conditions under which the factorization law holds, namely the time-dependent LBC can be determined by the product of the initial LBC of the system and
the LBC evolution of a maximally entangled \( N \)-qubit state under the same multi-sided noisy channels. Then we consider the realistic situations under which the two kinds of factorization conditions can be satisfied. The dependence of entanglement robustness on the number of qubits and that of the noisy channels is also discussed.

2. LBC evolution and the conditions for its factorization

2.1. Time-dependent LBC of an \( N \)-qubit system under multi-sided noisy channels

In this work, we adopt the LBC proposed in [32] to quantify the global entanglement of a multiqubit system, defined for an arbitrary \( N \)-qubit mixed state \( \rho \) as

\[
\mathcal{C}_N(\rho) = \sqrt{\frac{1}{2^N-1} \sum_{\mathcal{P}} \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} (\mathcal{C}_{l_1,l_2}^{P[N-k]}(\rho))^2},
\]

with

\[
\mathcal{C}_{l_1,l_2}^{P[N-k]}(\rho) = \max \left\{ 0, \sqrt{\lambda_{l_1,l_2}^{P[N-k]}(\rho)} - \sum_{i=1}^{k} \sqrt{\lambda_i^{P[N-k]}(\rho)} \right\}.
\]

In equations (1) and (2), \( k|N-k \) stands for the bipartitions with \( k \) qubits in one block and \( N-k \) ones in another. In view of the distinctions of the \( N \) qubits, we use \( \mathcal{P}[k|N-k] \) to specify a concrete combination of \( k \) and \( N-k \) qubits in constituting the bipartitions \( k|N-k \). Thus, \( \sum_{\mathcal{P}} \) stands for the summation over all possible concrete bipartitions of \( \mathcal{P}[k|N-k] \). In equation (2), \( \lambda_{l_1,l_2}^{P[N-k]} \) are the eigenvalues, in decreasing order, of the non-Hermitian matrix 

\[
\rho \left( L_{l_1}^k \otimes L_{l_2}^{N-k} \right) \rho^* \left( L_{l_1}^k \otimes L_{l_2}^{N-k} \right),
\]

with \( \{ L_{l_1}^k; l_1 = 1, 2, \ldots, k_1 \} \) and \( \{ L_{l_2}^{N-k}; l_2 = 1, 2, \ldots, k_2 \} \) being the generators of the groups \( \text{SO}(2^k) \) and \( \text{SO}(2^{N-k}) \) acting on the \( k \) and \( N-k \) qubits of a concrete bipartition \( \mathcal{P}[k|N-k] \). Surely, \( \mathcal{C}_N(\rho) > 0 \) signifies that \( \rho \) is entangled, and a separable state \( \rho \) always has \( \mathcal{C}_N(\rho) = 0 \). Yet, \( \mathcal{C}_N(\rho) = 0 \) does not necessarily imply separability of \( \rho \).

The LBC of three-qubit X states was analyzed in [33] to demonstrate when it goes to zero.

It is known that the forms of multiqubit entanglement are diverse. In this work, the consideration is restricted to the GGHZ state of \( N \) qubits in the form

\[
|\Psi_0\rangle_{1,2,\ldots,N} = \alpha |i_1,i_2,\ldots,i_N\rangle_{1,2,\ldots,N} + \beta |\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_N\rangle_{1,2,\ldots,N},
\]

where \( \alpha, \beta \in \mathbb{C} \) satisfying \( |\alpha|^2 + |\beta|^2 = 1 \), \( i_1, i_2, \ldots, i_N \in \{0, 1\} \) and \( \tilde{i}_1 = 1 \oplus i_1, \tilde{i}_2 = 1 \oplus i_2 \cdot \cdot \cdot , \tilde{i}_N = 1 \oplus i_N \), with \( \oplus \) being an addition mod 2. Consider that the \( N \) qubits are locally subjected to \( M \leq N \) independent noisy channels without mutual interactions between the qubits. The dynamics of the \( j \)th qubit is governed by a master equation that gives rise to a completely positive trace-preserving map (or channel), with \( \mathcal{E}_j \) describing the evolution as \( \rho_j(t) = \mathcal{E}_j(\rho_j(0)) \), where \( \rho_j(0) \) and \( \rho_j(t) \) are, respectively, the initial and evolved states of the \( j \)-th qubit. Under the actions of \( M \) independent noisy channels, the density matrix of the initial state (3) of the \( N \)-qubit system

\[
\rho_N(0) = |\Psi_0\rangle_{1,2,\ldots,N}\langle\Psi_0|
\]

\[
= |\alpha|^2 |i_1,i_2,\ldots,i_N\rangle_{1,2,\ldots,N}\langle i_1,i_2,\ldots,i_N| + |\beta|^2 |\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_N\rangle_{1,2,\ldots,N}\langle\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_N|
+ \alpha \beta^* |i_1,i_2,\ldots,i_N\rangle_{1,2,\ldots,N}\langle\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_N|
+ \alpha^* \beta |\tilde{i}_1,\tilde{i}_2,\ldots,\tilde{i}_N\rangle_{1,2,\ldots,N}\langle i_1,i_2,\ldots,i_N|
\]

(4)

will evolve into a mixed state \( \rho_N(t) \) given simply by the composition of the \( M \) individual maps:

\[
\rho_N(t) = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_M \rho_N(0),
\]

(5)
In this work, we shall consider several paradigmatic types of noisy channels, such as the AD, depolarization (D) and PD (dephasing) channels. Application of any one of these channels to the initial density operator \( \rho_N(0) \) multiplies its off-diagonal elements by a time-dependent factor and gives rise to new diagonal terms \([27]\). For convenience, in the representation spanned by the \( N \)-qubit product states \( \{|00\ldots00\rangle, |00\ldots01\rangle, \ldots, |11\ldots11\rangle \} \), we label the \( 2^N \) diagonal elements of \( \rho_N(t) \) as \( a_1(t), \ldots, a_m(t), \ldots, a_{2^N-1}(t), b_{2^N-1}(t), \ldots, b_m(t), \ldots, b_1(t) \) (from the top-left one to the lower-right one) and the two off-diagonal elements as \( d_m(t) \) and \( d'_m(t) \). Here, we specify the elements \( a_m(t) \) and \( b_m(t) \) as the time evolution of the initial nonzero diagonal (off-diagonal) elements of \( \rho_N(0) \). The other ones \( a_m(t) \) and \( b_m(t) \) (with \( m' \neq m \in \{1, 2, \ldots, 2^{N-1}\} \)) are the new diagonal elements derived in the time evolution. Also, note that the two diagonal terms of \( \rho_N(t) \) corresponding to the elements \( a_i(t) \) and \( b_i(t) \) \( (i = 1, 2, \ldots, 2^{N-1}) \) have completely opposite marginals for all the \( N \) individual qubits. As an example, suppose that the initial density operator of a three-qubit system is 

\[
\rho_3(0) = |\alpha|^2|000\rangle\langle000| + |\beta|^2|111\rangle\langle111| + |\alpha\beta|^2|001\rangle\langle001| + |\alpha^*\beta|^2|101\rangle\langle101| + a_1(t)|000\rangle\langle000| + a_2(t)|001\rangle\langle001| + a_3(t)|010\rangle\langle010| + a_4(t)|011\rangle\langle011| + b_1(t)|100\rangle\langle100| + b_2(t)|101\rangle\langle101| + b_3(t)|110\rangle\langle110| + b_4(t)|111\rangle\langle111| + d_1(t)|000\rangle\langle000| + d_2(t)|001\rangle\langle001| + d_3(t)|010\rangle\langle010| + d_4(t)|011\rangle\langle011| \]

In this case, the time evolution of the initial nonzero diagonal (off-diagonal) elements are obviously \( a_1(t) \) and \( b_1(t) \) \((d_1(t) \text{ and } d'_1(t))\). The other diagonal elements \( a_{m'}(t) \) and \( b_{m'}(t) \) with \( m' = 2, 3, 4 \) being the new diagonal elements derived in the time evolution. The two diagonal terms, such as \( |001\rangle\langle001| \) and \( |110\rangle\langle110| \) with the coefficients \( a_2(t) \) and \( b_2(t) \) (the same for the other three pairs of diagonal terms with the coefficients \( a_i(t) \) and \( b_i(t), i = 1, 3, 4 \)), involve completely opposite reduced states for all the three individual qubits.

By virtue of equation (2), we obtain the concurrence of a concrete bipartition \( P[k|N-k] \) of the evolved state \( \rho_N(t) \) as

\[
C^{P[k|N-k]}(\rho_N(t)) = 2 \max\{0, |d_{m}(t)| - \sqrt{a_{m}(t)b_{m}(t)}\},
\]

where \( d_{m}(t) \) is the time evolution of the initial off-diagonal element of the GGHZ state \( |4\rangle \), while \( a_{m}(t) \) and \( b_{m}(t) \) \((m' \neq m)\) are the diagonal elements derived in the evolution. The value of \( m' \in \{1, 2, \ldots, 2^{N-1}\} \) is determined by the definite bipartition \( P[k|N-k] \). By virtue of equations (1) and (6), the LBC of the \( N \)-qubit system can be expressed as

\[
\mathcal{L}_N(\rho_N(t)) = \left( \frac{1}{2^{N-1}-1} \sum_P |C^{P[k|N-k]}(\rho_N(t))|^2 \right)^{1/2}.
\]

So far, we have derived a general formula (7) of the time-dependent LBC of an \( N \)-qubit system under multi-sided noisy channels. It should be pointed out that we have not solved the state evolution equation of the qubit system in the sense that we do not know the explicit forms of the matrix elements of the evolved density operator \( \rho_N(t) \). Actually, we are aware of the structure of the evolved density operator \( \rho_N(t) \), since we are restricted to the GGHZ initial state of the \( N \) qubits and to the special noisy channels, which ensures the derivation of formula (7). Inspections of formula (7) show that to make the factorization law hold for the LBC evolution of the \( N \)-qubit system under multi-sided noisy channels, the associated bipartite concurrence \( C^{P[k|N-k]}(\rho_N(t)) \) (6) should satisfy either of the following two kinds of conditions.

2.2. The first kind of condition

An obvious situation under which the LBC (7) can be factorized is that the product of \( a_{m}(t) \) and \( b_{m}(t) \) in formula (6) is equal to zero, i.e. \( a_{m}(t)b_{m}(t) = 0 \forall m' \neq m \in \{1, 2, \ldots, 2^{N-1}\} \). In this
case, we have \( C^{P[k|N-k]}(\rho_N(t)) = 2|d_m(t)| \) which is independent of the concrete bipartition \( P[k|N - k] \). Accordingly, the LBC (7) of the N-qubit system is reduced to a simple form as
\[
\mathcal{C}_N(\rho_N(t)) = 2|d_m(t)|,
\]
in which \(|d_m(t)|\) is obviously the time evolution of the off-diagonal element \(|\alpha\beta|\) of the N-qubit system. That is, the LBC of the N-qubit system can be determined completely by the time evolution of the off-diagonal element \(|d_m(t)|\). As mentioned above, application of any of these noisy channels to the N-qubit system will multiply the off-diagonal element of \( \rho_N(0) \) (4) by a time-dependent factor, namely \( d_m(t) = d_m(0)D(t) \). Therefore, the LBC (8) can be further re-expressed as
\[
\mathcal{C}_N(\rho_N(t)) = \mathcal{C}_N(\rho_N(0))[D(t)],
\]
where \( \mathcal{C}_N(\rho_N(0)) = 2|d_m(0)| = 2|\alpha\beta| \) is the initial LBC of the GGHZ state (3). By virtue of equation (9), for the initially maximal entangled state of the N-qubit system with \( \mathcal{C}_N(\rho_N(0)) = 1 \), we have \( \mathcal{C}_N(\rho_N(t)) = |D(t)| \), which implies that \(|D(t)|\) is the LBC evolution of the maximally entangled state of the N-qubit system under the same multi-sided noisy channels. Therefore, the LBC evolution of the N-qubit system under multi-sided noisy channels can be determined by the product of the initial LBC and the LBC evolution of a maximally entangled state. Actually, as we shall show in the following section, the factor \( D(t) \) can be determined by the parameters of the independent noisy channels that act on the qubits. Of course, the factorization expression (equation (9)) is conditioned on the condition \( a_m(t)b_m(t) = 0 \) in (6). In the following section, we shall show that this condition can be satisfied in the AD and PD channels.

2.3. The second kind of condition

In some situations, the first kind of condition \( a_m(t)b_m(t) = 0 \ \forall m' \neq m \in \{1, 2, \ldots, 2^{N-1}\} \) may not hold and the LBC evolution cannot be factorized as the expression in equation (9). In this case, if \( a_m(t)b_m(t) \) can be decomposed to \(|d_m(0)||F_m(t)|\), the concurrence (6) of a definite bipartition \( P[k|N-k] \) will take the form
\[
C^{P[k|N-k]}(\rho_N(t)) = 2|d_m(0)|Q^{P[k|N-k]}(t),
\]
with \( Q^{P[k|N-k]}(t) = \max[0, |D(t)| - |F_m(t)|] \). By virtue of equation (7), the LBC of the N-qubit system is thus reduced to
\[
\mathcal{C}_N(\rho_N(t)) = \mathcal{C}_N(\rho_N(0)) \sqrt{\frac{1}{2^{N-1} - 1} \sum_P (Q^{P[k|N-k]}(t))^2},
\]
with \( \mathcal{C}_N(\rho_N(0)) = 2|d_m(0)| = 2|\alpha\beta| \) being the initial LBC of the GGHZ state (3). From equation (11), for \( \mathcal{C}_N(\rho_N(0)) = 1 \) corresponding to an initially maximal entangled state of the N-qubit system, we obtain its LBC evolution as \( \mathcal{C}_N(\rho_N(t)) = \sqrt{\frac{1}{2^{N-1} - 1} \sum_P (Q^{P[k|N-k]}(t))^2} \). Therefore, the LBC evolution of an arbitrary N-qubit system under multi-sided noisy channels can still be determined by the product of the initial LBC and the LBC evolution of a maximally entangled state. In the following section, we shall show that the LBC evolution of the N-qubit system under D channels can satisfy the above condition and exhibit the factorized form in (11).

3. The factorization LBC evolution in various noisy channels

We have shown theoretically that under certain conditions the LBC evolution of an N-qubit system under multi-sided noisy channels may be factorized in such a way that the LBC is...
determined by the product of the initial LBC and the LBC evolution of a maximally entangled state under the same multi-sided channels. Now, an immediate question arises: in what realistic situations can these conditions be satisfied? In the following, we provide an answer to this question by analyzing the evolution of the $N$-qubit system in the AD, D and PD channels, respectively.

### 3.1. Amplitude-damping channel

At first, we consider the AD channel which corresponds to the zero-temperature dissipative reservoir. The factorization law of a two-qubit system under a one-sided noisy channel [11] has been verified in the AD channel [13, 14]. The action of an AD channel $j$ for a qubit is described by a map $\mathcal{E}_j^{\text{AD}} : \rho(t) = \mathcal{E}_j^{\text{AD}} \rho(0)$, with $\rho(0)$ and $\rho(t)$ being the initial and evolved density matrices of the qubit. During the time evolution a qubit decays from its excited state $|1\rangle$ to the ground state $|0\rangle$ by emitting an excitation, with a probability $p_j(t) = 1 - \exp(-\Gamma_j t)$, where $\Gamma_j$ is the decay rate of the noisy channel. The action of $\mathcal{E}_j^{\text{AD}}$ on elements of the reduced density matrix of a qubit reads

$$
\mathcal{E}_j^{\text{AD}} : \begin{cases}
|0\rangle|0\rangle \to |0\rangle|0\rangle, \\
|0\rangle|1\rangle \to \sqrt{1 - p_j(t)}|0\rangle|0\rangle, \\
|1\rangle|0\rangle \to \sqrt{1 - p_j(t)}|1\rangle|0\rangle, \\
|1\rangle|1\rangle \to p_j(t)|0\rangle|0\rangle + (1 - p_j(t))|1\rangle|1\rangle.
\end{cases}
$$

(12)

Different forms of an initial GGHZ state $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$ and $|\psi_a\rangle = \alpha|001\rangle + \beta|110\rangle$ for a three-qubit system, will exhibit different dynamical evolution in the AD channels [28]. Therefore, we classify the initial GGHZ state of the $N$-qubit system into symmetrical and asymmetrical cases: in the symmetrical GGHZ state (13) all the $N$ qubits are equivalent with the same reduced states, while in the asymmetrical state (16) the $N$ qubits can be divided into two asymmetrical blocks with opposite reduced states.

The symmetrical GGHZ state can be expressed as

$$
|\Psi_1\rangle_{1,2,...,N} = \alpha |0, 0, \ldots, 0\rangle_{1,2,...,N} + \beta |1, 1, \ldots, 1\rangle_{1,2,...,N}.
$$

(13)

Without loss of generality, suppose the qubits $1, 2, \ldots, M (M \leq N)$ are locally coupled to $M$ independent and different AD channels, respectively. Under the actions of the $M$ AD channels, the initial density matrix $\rho_N^1(0) = |\Psi_1\rangle_{1,2,...,N} \langle \Psi_1|$ will map onto $\rho_N^1(t)$ which can be obtained as

$$
\rho_N^1(t) = \prod_{j=1}^{M} \left[ 1 - p_j(t) \right]^{\frac{1}{2}} (\alpha \beta^* |00\ldots 0\rangle_{1,2,...,N} \langle 1\ldots 1| + \alpha^* \beta |11\ldots 1\rangle_{1,2,...,N} \langle 0\ldots 0|)
$$

$$
+ \beta^2 \sum_{i_1,\ldots,i_M=0}^{1} q_1(t), \ldots, q_M(t) |i_1,\ldots,i_M, 1\ldots 1\rangle_{1,\ldots,M,1,\ldots,N} \langle i_1,\ldots,i_M, 1\ldots 1| + |\alpha|^2 |0\ldots 0\rangle_{1,\ldots,N} \langle 0\ldots 0|,
$$

(14)

where the coefficients $q_j(t) \equiv p_j(t)$ $\forall j = 0$ and $q_j(t) \equiv 1 - p_j(t)$ $\forall j = 1$ for $j = 1, 2, \ldots, M$.

Recall that we specify $a_m(t)$ and $b_m(t)$ ($d_m(t)$ and $d_m^*(t)$) as the time evolution of the nonzero diagonal (off-diagonal) elements of initial density operator $\rho_N(0)$, while $a_m(t)$ and $b_m(t)$ (with $m' \neq m$) are the derived diagonal elements in the evolution. From (14) we can see that the initial off-diagonal element $|d_m(0)| = |\alpha \beta|$ has evolved to $|d_m(t)| = |\alpha \beta| \prod_{j=1}^{M} [1 - p_j(t)]^{\frac{1}{2}}$, while the initial diagonal elements $a_m(0) = |\alpha|^2$ stay invariant and $b_m(0) = |\beta|^2$ has evolved to $b_m(t) = |\beta|^2 (1 - p_m(t)) \cdots (1 - p_M(t))$. To decide if the LBC of $\rho_N^1(t)$ can be factorized, we should check if the aforementioned condition $a_m(t)b_m(t) = 0 \forall m' \neq m \in \{1, 2, \ldots, 2^{N-1}\}$
The LBC evolution of a system with \( N > M \) qubits that are initially in the symmetrical GGHZ state (13) under actions of \( M \) independent AD channels. The parameter of the initial GGHZ state is \( |\alpha| = |\beta| = 1/\sqrt{2} \) and \( p \in [0, 1] \) for \( t \in [0, \infty] \).

Figure 1. The LBC evolution of a system with \( N > M \) qubits that are initially in the symmetrical GGHZ state (13) under actions of \( M \) independent AD channels. The parameter of the initial GGHZ state is \( |\alpha| = |\beta| = 1/\sqrt{2} \) and \( p \in [0, 1] \) for \( t \in [0, \infty] \).

From the second term of the right-hand side (RHS) of equation (14), we observe that if not all the \( N \) qubits are subjected to the AD channels, i.e. \( M < N \), all these diagonal terms have the same reduced density operators \(|1\rangle_{1} \cdots |1\rangle_{M+1} |1\rangle_{M+2} \cdots |1\rangle_{N}\) for the qubits \( M+1, M+2, \ldots, N \) that are not subjected to the noisy channels. In other words, the matrix \( \rho_{I}^{N}(t) \) (14) does not involve the derived diagonal terms whose reduced density operators for the qubits \( M+1, M+2, \ldots, N \) are \(|0\rangle_{M+1} |0\rangle_{M+2} \cdots |0\rangle_{N}\). Recall that the elements \( a_{mm}(t) \) and \( b_{mm}(t) \) are the derived diagonal terms that should possess opposite reduced density operators for all the \( N \) individual qubits; therefore, in the case of \( M < N \), we always have \( a_{mm}(t) b_{mm}(t) = 0 \). The LBC of the \( N \)-qubit system can thus be obtained by virtue of equation (9) as

\[
\mathcal{L}_{N}^{N}(\rho_{N}^{N}(t)) = 2|\alpha\beta| \prod_{j=1}^{M} (1 - p_{j}(t))^{2}.
\] (15)

Obviously, the \( 2|\alpha\beta| \) in (15) is the initial LBC of the GGHZ state (13), while the term \( \prod_{j=1}^{M} (1 - p_{j}(t))^{2} \) comprises the parameters of the \( M \) AD channels.

In contrast to the case of \( M < N \), if all the \( N \) qubits are coupled to the AD channels, i.e. \( M = N \), then all the \( 2^{N} \) diagonal elements of the states (14) are non-zero, which implies that the condition \( a_{mm}(t) b_{mm}(t) = 0 \) cannot be satisfied. Therefore, the LBC of the system cannot be factorized into the form (15). Actually, it has been shown [27] under the action of \( N \) independent AD channels, the \( N \)-qubit system in the GGHZ state (13) will suffer from the entanglement sudden death [7] when \( |\beta| > |\alpha| \). At the same time, the robustness of the \( N \)-qubit system, i.e. the time at which the entanglement become arbitrarily small, decreases with the increase of the number \( N \) of qubits [27]. Here, for the case of \( M < N \), we can see from the factorization expression (15) that the LBC is independent of the system size \( N \) and only related to the number \( M \) of the AD channels. In figure 1, we plot the evolution of the LBC under \( M = 1, \ldots, 5 \) identical AD channels for the systems with \( N > M \) qubits. For convenience, we have parameterized the time dependence in terms of \( p \equiv p(t) \) instead of \( t \) noting that \( p = 0 \) when \( t = 0 \) and \( p \to 1 \) when \( t \to \infty \), i.e. \( p \in [0, 1] \) for \( t \in [0, \infty] \). From
subjected to the independent AD channels, i.e. \( M \)

3.2. The depolarization channel

\( \rho \) can obtain the factorized form of LBC for the state \( \text{GGHZ states (16)} \) under the action of \( |\psi \rangle \). Note that there exist only two diagonal terms, i.e, \( \rho \) form in (18) still holds but the RHS of (18) should be replaced by \( 2 \) 

\[ \langle |\psi \rangle |_{1,2,..,N} = \alpha |0\ldots 0, 1\ldots 1 \rangle_{1,\ldots,n,n+1,\ldots,N} + \beta |1\ldots 1, 0\ldots 0 \rangle_{1,\ldots,n,n+1,\ldots,N}. \]

Suppose all the \( N \) qubits are coupled with their own AD channels. Then the initial density matrix \( \rho^\Pi(t) = |\psi^\Pi\rangle_{1,2,..,N} \langle \psi^\Pi| \) will map onto \( \rho^\Pi(t) \) which can be obtained as

\[
\rho^\Pi(t) = \prod_{j=1}^{N} (1 - p_j(t))^{1/2} (\alpha \beta^* |0\ldots 0, 1\ldots 1 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 1\ldots 1, 0\ldots 0 | + \text{h.c.})
\]

\[
+ |\alpha|^2 \sum_{i_{n+1},\ldots,i_N=0}^{1} q_{n+1}(t), \ldots, q_N(t) |0\ldots 0, 0, \ldots, 0, i_{n+1}, \ldots, i_N \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 0\ldots 0, 0, i_{n+1}, \ldots, i_N | 
\]

\[
+ |\beta|^2 \sum_{i_1,\ldots,i_n=0}^{1} q_1(t), \ldots, q_n(t) |i_1, \ldots, i_n, 0\ldots 0 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle i_1, \ldots, i_n, 0\ldots 0 |, \quad (17)
\]

where the coefficients \( q_j(t) \equiv p_j(t) \) \( \forall i_j = 0 \) and \( q_j(t) \equiv 1 - p_j(t) \) \( \forall i_j = 1 \) with \( j \in \{1,2,\ldots,N\} \). Here, we directly check if the first kind of condition \( a_m(t)b_{m'}(t) = 0 \) \( \forall m' \neq m \in \{1,2,\ldots,2^N-1\} \) can be satisfied by \( \rho^\Pi(t) \). From equation (17), we observe that there exist only two diagonal terms, i.e, \( |0\ldots 0, 1\ldots 1 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 0\ldots 0, 1\ldots 1 | \)

and \( |1\ldots 1, 0\ldots 0 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 1\ldots 1, 0\ldots 0 | \), which involve completely opposite reduced states for all the \( N \) individual qubits. However, these two terms are merely the nonzero diagonal terms of the initial density operator \( \rho^\Pi(0) \) rather than the derived terms in the evolution. That is, the derived diagonal terms with elements \( a_m(t) \) and \( b_{m'}(t) \), such as \( |0\ldots 0, 1\ldots 1 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 0\ldots 0, 1\ldots 1 | \)

and \( |1\ldots 1, 0\ldots 0 \rangle_{1,\ldots,n,n+1,\ldots,N} \langle 1\ldots 1, 0\ldots 0 | \), which involve completely opposite reduced states for all the \( N \) individual qubits, will not appear simultaneously (for the present example only the former term can appear while its counterpart does not). Therefore, the condition \( a_m(t)b_{m'}(t) = 0 \) can be satisfied for the \( N \)-qubit system in the GGHZ states (16) under the action of \( N \) independent AD channels. By virtue of (9), we can obtain the factorized form of LBC for the state \( \rho^\Pi(t) \) as

\[
\mathcal{L}_N(\rho^\Pi(t)) = 2|\alpha \beta| \prod_{j=1}^{N} (1 - p_j(t))^{1/2}. \quad (18)
\]

Obviously, \( 2|\alpha \beta| \) in (18) is the initial LBC of the state (16) while the term \( \prod_{j=1}^{N} (1 - p_j(t))^{1/2} \) reflects the actions of the \( N \) AD channels. Here we have considered all the \( N \) qubits are subjected to the independent AD channels, i.e. \( M = N \). For the case of \( M < N \), the factorized form in (18) still holds but the RHS of (18) should be replaced by \( 2|\alpha \beta| \prod_{j=1}^{M} (1 - p_j(t))^{1/2} \) implying the action of \( M \) AD channels.

3.2. The depolarization channel

The D channel \( j \) describes the situation in which a qubit remains untouched with probability \( 1 - p_j(t) \), or is depolarized (white noise), i.e. its state is taken to the maximally mixed state,
with the probability \( p_j(t) \). The action of the map \( \mathcal{E}_D^j \) of the D channel \( j \) on elements of the reduced density matrix of a qubit reads

\[
\mathcal{E}_D^j : \begin{cases}
|0\rangle|0\rangle \rightarrow (1 - \frac{p_j(t)}{2})|0\rangle|0\rangle + \frac{p_j(t)}{2}|1\rangle|1\rangle, \\
|1\rangle|1\rangle \rightarrow (1 - \frac{p_j(t)}{2})|1\rangle|1\rangle + \frac{p_j(t)}{2}|0\rangle|0\rangle, \\
|0\rangle|1\rangle \rightarrow (1 - p_j(t))|0\rangle|1\rangle, \\
|1\rangle|0\rangle \rightarrow (1 - p_j(t))|1\rangle|0\rangle.
\end{cases}
\]

(19)

Without loss of generality, we suppose the qubits 1, 2, ..., \( M \) are coupled to \( M (M \leq N) \) independent D channels, respectively. Then under the actions of the \( M \) different D channels the initial density matrix \( \rho_N(0) \) (4) of the \( N \)-qubit system will map onto \( \rho_N(t) \) which can be obtained as

\[
\rho_N^D(t) = \prod_{j=1}^{M} (1 - p_j(t))|\alpha\beta^\ast|_{i_1, \ldots, i_N} \langle i_1, \ldots, i_N| + \alpha^\ast\beta|_{i_1, \ldots, i_N} \langle i_1, \ldots, i_N| \langle i_1, \ldots, i_N|
\]

\[
+ |\alpha|^2 \sum_{i_1, \ldots, i_M = 0} \rho_1(t), \ldots, \rho_M(t) |i_1, \ldots, i_M, i_{M+1}, \ldots, i_N \rangle \langle i_1, \ldots, i_M, i_{M+1}, \ldots, i_N|
\]

\[
\times |i_1, \ldots, i_M, i_{M+1}, \ldots, i_N \rangle \langle i_1, \ldots, i_M, i_{M+1}, \ldots, i_N|
\]

\[
\times |i_1, \ldots, i_M, i_{M+1}, \ldots, i_N \rangle \langle i_1, \ldots, i_M, i_{M+1}, \ldots, i_N|
\]

\[
\times |i_1, \ldots, i_M, i_{M+1}, \ldots, i_N \rangle \langle i_1, \ldots, i_M, i_{M+1}, \ldots, i_N|
\]

(20)

where the coefficients \( \rho_j(t) = 1 - \frac{p_j(t)}{2} (\frac{p_j(t)}{2}) \) when \( i_j = i_j (i_j = i_j \oplus 1) \), \( \gamma_j(t) = 1 - \frac{p_j(t)}{2} (\frac{p_j(t)}{2}) \) when \( i_j = i_j (i_j = i_j \oplus 1) \) with \( j = 1, 2, \ldots, M \). Before discussing the general case, we first take a three-qubit system in the initial state \( \rho_3(0) = |\alpha|^2|000\rangle_{123} \langle 000| + |\beta|^2|111\rangle_{111} \langle 111| + \alpha\beta^\ast|000\rangle_{123} |111\rangle_{111} \langle 111| + \alpha^\ast\beta|111\rangle_{123} \langle 000| \) as an example and suppose only the former two qubits are subjected to the D channels. By virtue of equation (20), the evolved density operator reads

\[
\rho_3(t) = \prod_{j=1}^{2} (1 - p_j(t))|\alpha\beta^\ast|_{000} \langle 000| + |\alpha^\ast\beta|_{111} \langle 111|
\]

\[
+ |\alpha|^2 \left( 1 - \frac{p_1(t)}{2} \right) \left( 1 - \frac{p_2(t)}{2} \right) |000\rangle \langle 000| + \frac{p_1(t)}{2} \frac{p_2(t)}{2} |110\rangle \langle 110|
\]

\[
+ \left( 1 - \frac{p_1(t)}{2} \right) \frac{p_2(t)}{2} |010\rangle \langle 010| + \frac{p_1(t)}{2} \left( 1 - \frac{p_2(t)}{2} \right) |100\rangle \langle 100|
\]

\[
+ |\beta|^2 \left( 1 - \frac{p_1(t)}{2} \right) \left( 1 - \frac{p_2(t)}{2} \right) |111\rangle \langle 111| + \frac{p_1(t)}{2} \frac{p_2(t)}{2} |001\rangle \langle 001|
\]

\[
+ \left( 1 - \frac{p_1(t)}{2} \right) \frac{p_2(t)}{2} |101\rangle \langle 101| + \frac{p_1(t)}{2} \left( 1 - \frac{p_2(t)}{2} \right) |011\rangle \langle 011|
\]

(21)

From equation (21), we can see that any of the former four diagonal terms with a common coefficient \( |\alpha|^2 \) has a counterpart in the group of the latter four terms with a common coefficient \( |\beta|^2 \), such as \( |110\rangle \langle 110| \) and \( |001\rangle \langle 001| \), so that these two terms have completely opposite reduced states for all the three qubits. Therefore, the second factorization condition, i.e. \( \sqrt{a_m(t)}|\beta^\ast \rangle |\beta \rangle |F_m(t)\rangle \) can be decomposed to \( |a_m(t)| |\beta^\ast \rangle |\beta \rangle |F_m(t)\rangle \), is satisfied for the present instance and we have \( \sqrt{a_m(t)}|\beta^\ast \rangle |\beta \rangle = |\alpha\beta| |F_m(t)\rangle \) with \( F_m(t) = \frac{p_1(t) p_2(t)}{2}, (1 - \frac{p_1(t)}{2} \frac{p_2(t)}{2}) \) or \( \frac{p_1(t) p_2(t)}{2} (1 - \frac{p_1(t)}{2} \frac{p_2(t)}{2}) \) being determined by the concrete value of \( m \). This specific example can be generalized to the general case. Actually, for the evolved density operator (20), we
Figure 2. (a) The time-evolution of LBC for a four-qubit system in the GGHZ state (13) under $M = 1$ (solid line), $M = 2$ (dashed line) and $M = 3$ (dotted line) D channels. (b) The time-evolution of LBC for the system with $N = 4$ (solid line), $N = 3$ (dashed line) and $N = 2$ (dotted line) qubits under a single D channel. (c) The time-evolution of LBC for the system with $N = 4$ (solid line), $N = 3$ (dotted line) qubits under two D channels. The parameters used are $|\alpha| = |\beta| = 1/\sqrt{2}$ and $p \in [0, 1]$ for $t \in [0, \infty]$.

In the following, we make a brief discussion on the relations between the qubit LBC and the system size $N$ as well as the number of the noisy channels $M$. From (22), we conclude that in the time evolution of the $N$-qubit system, the LBC increases with the qubit number $N$ for the fixed number $M < N$ of D channels, while it decreases with the number $M$ for the fixed qubit number $N$. In figure 2(a), we have shown the evolution of the LBC of a four-qubit system in the GGHZ state (13) under $M = 1, 2, 3$ identical D channels, where we can see that the LBC decreases with an increase of $M$. Figure 2(b) shows the evolution of the LBC for systems with $N = 4, N = 3$ and $N = 2$ qubits under the one-sided D channel, from which we can see that the robustness of the LBC increases with $N$. In figure 2(c), we plot the evolution of the

\[
\mathcal{C}_N(E_N^D(t)) = 2|\alpha\beta| \sqrt{\frac{1}{2N-1} \sum_{P} (Q^P[k|N-k](t))^2},
\]

where $2|\alpha\beta|$ is the initial LBC of the GGHZ state (3) and $Q^P[k|N-k](t) = \max\{0, |\prod_{j=1}^{M} (1 - p_j(t))| - |\prod_{j=1}^{M} q_j(t)|\}$. As for the case of $M = N$, the second kind of condition no longer holds; therefore, the LBC evolution of the $N$-qubit system cannot be factorized as the form (22).
LBC for the system with $N = 4$ and $N = 3$ qubits under two identical D channels, which still shows that the robustness of the LBC increases with $N$. Here, we have parameterized the time dependence in terms of $p \equiv p(t)$ instead of $t$, noting that $p = 0$ when $t = 0$ and $p \to 1$ when $t \to \infty$, i.e. $p \in [0, 1]$ for $t \in [0, \infty]$.

3.3. The dephasing channel

The PD (or dephasing) channel represents the situation in which there is a loss of quantum coherence with the probability $p(t)$, but without any energy exchange. The action of the map $\mathcal{E}_{PD}$ of the PD channel $j$ on the density matrix of a qubit multiplies the off-diagonal elements by a factor $(1 - p(t))$ while keeping the diagonal elements invariant. Since the PD channels cannot lead to the population evolution of the $N$-qubit system, both the elements $a_{m'}(t)$ and $b_{m'}(t)$ representing the derived diagonal terms in the evolution are zero. This implies that the first kind of condition for the factorization of the LBC of the $N$-qubit system can always be satisfied when the $N$-qubit system is coupled with independent PD channels. Therefore, the LBC of the $N$-qubit system after evolution has the factorization form in equation (9).

4. Conclusion

In conclusion, using a lower bound of concurrence (LBC) as a measure of multiqubit entanglement, we have explored the possibility of generalizing the factorization law of two-qubit under the one-sided channel to multiqubit under multi-sided channels. Instead of a general answer to this issue, we have considered a special form of multiqubit entanglement, i.e. the GGHZ state (3) and three typical types of noisy channels, i.e. the amplitude-damping (AD), depolarizing (D) and phasing damping (PD) (dephasing) channels. The explicit formulae (7) for the evolution of LBC as well as the associated bipartite concurrence (6) for an arbitrary bipartition are derived, based on which we observe that under two kinds of conditions the LBC can be factorized. That is, the LBC evolution of the $N$-qubit system can be determined by the product of its initial LBC and the LBC evolution of a maximal entangled state under the same multi-sided noisy channels. We then study the realistic situations in which these two kinds of conditions can be satisfied. We have shown that (i) the $N$-qubit system in the symmetrical GGHZ state (13) can satisfy the first kind of factorization condition when the number $M$ of the independent AD channels is less than the number $N$ of qubits, while the $N$-qubit system in the asymmetrical GGHZ state (16) can always satisfy the first kind of condition irrespective of the number $M$ of the AD channels; (ii) in the D channels, the $N$-qubit system in the general GGHZ state (3) can satisfy the second factorization condition when the number $M$ of D channels is less than $N$; (iii) the PD channels can always lead to a factorization of the LBC of the $N$-qubit system in the GGHZ state (3) with the satisfaction of the first kind of condition. By virtue of the concise expressions of the LBC under different noisy channels, we have discussed the dependence of the entanglement robustness on the system size in terms of $N$ and number $M$ of the noisy channels. The GGHZ state, though only being a special class of various types of multipartite entanglement, has attracted extensive research works from different aspects, such as preparation [34], application [35], dynamics [24–27, 29, 28] and so on. Therefore, study of the factorization law of special multiqubit entanglement under some typical multi-sided noisy channels can not only deepen the understanding of multipartite entanglement dynamics but also facilitate the corresponding calculation.
Acknowledgments

This work was supported by National Natural Science Foundation of China under grant nos 10 947 006 and 61 178 012, the Specialized Research Fund for the Doctoral Program of Higher Education under grant no 20 093 705 110 001 and the Scientific Research Foundation of Qufu Normal University for Doctors.

References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Gisin N 1996 Phys. Lett. A 210 151
[3] Życzkowski K et al 2001 Phys. Rev. A 65 012101
[4] Dodd P J and Halliwell J J 2004 Phys. Rev. A 69 052105
[5] Roos C F et al 2004 Phys. Rev. Lett. 92 220402
[6] Carvalho A R R et al 2007 Phys. Rev. Lett. 98 190501
[7] Yu T and Eberly J H 2004 Phys. Rev. Lett. 93 140404
Yu T and Eberly J H 2006 Phys. Rev. Lett. 97 140403
Eberly J H and Yu T 2007 Science 316 555
[8] Santos M F et al 2006 Phys. Rev. A 73 040305
[9] Horodecki M, Shor P W and Ruskai M B 2003 Rev. Math. Phys. 15 629
[10] Bai Y K et al 2009 Phys. Rev. A 80 044301
[11] Konrad T, DeMelo F, Tiersch M, Kasztelan C, Aragao A and Buchleitner A 2008 Nature Phys. 4 99
[12] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[13] Farias L J, Latune C L, Walborn S P, Davidovich L and Ribeiro P H S 2009 Science 324 1414
[14] Xu J S, Li C F, Xu X Y, Shi C H, Zou X B and Guo G C 2004 Phys. Rev. Lett. 103 240502
[15] Li Z G, Fei S F, Wang Z D and Liu W M 2009 Phys. Rev. A 79 024303
[16] Li J G, Zou J and Shao B 2009 Phys. Rev. A 82 042318
[17] Yu C S, Yi X X and Song H S 2008 Phys. Rev. A 78 062330
[18] Gour G 2005 Phys. Rev. A 71 012318
Gour G 2005 Phys. Rev. A 72 042318
[19] Tiersch M, DeMelo F and Buchleitner A 2008 Phys. Rev. Lett. 101 170502
[20] Gour G 2010 Phys. Rev. Lett. 105 190504
[21] Li J G, Zou J and Shao B 2010 Phys. Rev. A 82 042318
[22] Li J G, Zou J and Shao B 2011 Phys. Lett. A 375 2300
[23] Greenberger D M, Horne M A and Zeilinger A 1989 Bell’s Theorem, Quantum Theory and Conceptions of the Universe (Dordrecht: Kluwer)
[24] Simon C and Kenne J 2002 Phys. Rev. A 65 052327
[25] Dür W and Briegel H J 2004 Phys. Rev. Lett. 92 180403
[26] Hein M, Dür W and Briegel H J 2005 Phys. Rev. A 71 032350
[27] Aolita L, Chaves R, Cavalcanti D, Acín A and Davidovich L 2008 Phys. Rev. Lett. 100 080501
[28] Man Z X, Xia Y J and An N B 2008 Phys. Rev. A 78 064301
[29] Carvalho A R R, Mintert F and Buchleitner A 2004 Phys. Rev. Lett. 93 230501
[30] Minter F, Kus M and Buchleitner A 2005 Phys. Rev. Lett. 95 240502
[31] Mintert F, Kus M and Buchleitner A 2004 Phys. Rev. Lett. 92 167902
Chen K, Albeverio S and Fei S M 2005 Phys. Rev. Lett. 95 210501
Breuer H P 2006 Phys. Rev. Lett. 97 080501
Ou Y C, Fan H and Fei S M 2008 Phys. Rev. A 78 012311
Aolita L, Buchleitner A and Mintert F 2008 Phys. Rev. A 78 022308
Schmid C et al 2008 Phys. Rev. Lett. 101 260505
[32] Li M, Fei S M and Wan Z X 2009 J. Phys. A: Math. Theor. 42 145303
[33] Weinstei Y S 2010 Phys. Rev. A 82 032326
[34] Leibfried D et al 2005 Nature 438 639
[35] Lu C Y et al 2007 Nature Phys. 3 91
[36] Hillery M, Buzek V and Berthiaume A 1999 Phys. Rev. A 59 1829
Man Z X, Xia J Y and An N B 2006 J. Phys. B: At. Mol. Opt. Phys. 39 3855
Man Z X, Xia J Y and An N B 2007 J. Phys. B: At. Mol. Opt. Phys. 40 1767