Weak convergence rates for a full implicit scheme of stochastic Cahn-Hilliard equation with additive noise*

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Abstract

The aim of this study is the weak convergence rate of a temporal and spatial discretization scheme for stochastic Cahn–Hilliard equation with additive noise, where the spectral Galerkin method is used in space and the backward Euler scheme is used in time. The presence of the unbounded operator in front of the nonlinear term and the lack of the associated Kolmogorov equations make the error analysis much more challenging and demanding. To overcome these difficulties, we further exploit a novel approach proposed in [7] and combine it with Malliavin calculus to obtain an improved weak rate of convergence, in comparison with the corresponding strong convergence rates. The techniques used here are quite general and hence have the potential to be applied to other non-Markovian equations. As a byproduct the rate of the strong error can also be easily obtained.

AMS subject classification: 60H35, 60H15, 65C30

Key Words: stochastic Cahn-Hilliard equation, weak convergence rate, spectral Galerkin method, backward Euler method, Malliavin calculus.

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1 Introduction

The main objective of this article is to numerically investigate the following stochastic Cahn–Hilliard equation perturbed by additive noise,

\begin{equation}
\begin{aligned}
\text{d}u - \Delta w \text{d}t &= \text{d}W, & \text{in } & \mathcal{D} \times (0, T], \\
\text{d}w &= -\Delta u + f(u), & \text{in } & \mathcal{D} \times (0, T], \\
\frac{\partial u}{\partial n} &= \frac{\partial w}{\partial n} = 0, & \text{in } & \partial \mathcal{D} \times (0, T], \\
u(0, x) &= u_0, & \text{in } & \mathcal{D},
\end{aligned}
\end{equation}

where \( \mathcal{D} = (0, 1) \), \( 0 < T < \infty \), \( f(u) = u^3 - u \), \( u \in \mathbb{R} \) and \( \frac{\partial}{\partial n} \) denotes the outward normal derivative on \( \partial \mathcal{D} \). Throughout this paper, \( \{W(t)\}_{t \geq 0} \) is a \( Q \)-Wiener process with respect to a normal filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\), to be specified in Assumption 1.3. The Cahn–Hilliard equation is of fundamental importance in various applications, such as, the complicated phase separation and coarsening phenomena in a melted alloy [4, 6], spinodal decomposition for binary mixture [5], the diffusive process of populations and an oil film spreading over a solid surface [10]. Our motivating example arises from a simplified mesoscopic physical model for phase separation. In spite of an overwhelming activity of numerical SPDE under global Lipschitz condition and a fast increasing number of studies on stochastic Allen-Cahn equation, the stochastic Cahn-Hilliard equation, however, is in its beginning, especially from the numerical point of view.

In order to follow the semigroup framework in [17], (1.2) can be rewritten as an abstract stochastic evolution equation on some Hilbert space

\begin{equation}
\begin{aligned}
\text{d}X(t) + A(AX(t) + F(X(t))) \text{d}t &= \text{d}W(t), & t \in (0, T], \\
X(0) &= X_0.
\end{aligned}
\end{equation}

The framework for this equation is described as follows. Let \( H := L^2(\mathcal{D}, \mathbb{R}) \) be the Hilbert space with the usual scalar product \( \langle f, g \rangle = \int_0^1 f(x)g(x)dx \), the corresponding norm is denoted by \( \| \cdot \| \).

We also denote \( \|f\|_p := \left( \int_0^1 |f(x)|^p dx \right)^{1/p} \) so that \( \|f\|_2 = \|f\| \). Let \( \hat{H} := \{v \in H : \int_{\mathcal{D}} vd x = 0\} \) be a subspace of \( H \). We make the following assumptions on the linear operator \( A \), the nonlinear term \( F \), the noise process \( W(t) \) and the initial data \( X_0 \) appeared in the equation.

**Assumption 1.1.** \(-A \) is the Neumann Laplacian defined by \(-Au = \Delta u\) with \( u \in \text{dom}(A) := \{v \in H^2(\mathcal{D}) \cap \hat{H} : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \mathcal{D}\} \).

**Assumption 1.2.** \( F : L^6(\mathcal{D}, \mathbb{R}) \to H \) is the Nemytskii operator given by

\( F(v)(x) = f(v(x)) = v^3(x) - v(x), \ x \in \mathcal{D}, v \in L^6(\mathcal{D}, \mathbb{R}). \) (1.3)

**Assumption 1.3.** \( \{W(t)\}_{t \in [0, T]} \) is a \( \hat{H} \)-valued \( Q \)-Wiener process with the covariance operator \( Q \) satisfying

\( \text{Tr}(Q) < \infty. \) (1.4)

**Assumption 1.4.** The initial value \( X_0 : \mathcal{D} \to \mathbb{R} \) is deterministic and satisfies the following regularity,

\( \|X_0\|_4 < \infty. \) (1.5)
Under the above assumptions, a mild solution of (1.2) is given by

$$X(t) = E(t)X_0 - \int_0^t E(t-s)APF(X(s))\,ds + \int_0^t E(t-s)\,dW(s), \quad t \in [0, T],$$

(1.6)

where $E(t)$ denotes the analytic semigroup generated by $-A^2$ and the generalized orthogonal projector $P : L^1(\mathbb{D}, \mathbb{R}) \to \mathcal{H}$ is defined by $Pv = v - \int_P v\,dx$. We refer the readers to [2, 8, 11, 13, 16, 18, 25] for the existence and uniqueness of the mild solution for the equation. Since the exact solutions are rarely known explicitly, numerical simulations are often used to investigate the behavior of the solutions. In this work, we choose the spatial semi-discretization by spectral Galerkin approximation, i.e., projecting the equation on vector space $H^N := \text{span}\{e_0, e_1, e_2, \cdots, e_N\}$, the subspace spanned by the first $N + 1$ eigenvectors of $A$. The corresponding approximated equation is in the form

$$dX^N(t) + A(A X^N(t) + P_N F(X^N(t)))dt = P_N dW(t), \quad t \in (0, T]; \quad X^N(0) = P_N X_0,$$

(1.7)

where $X^N(t) \in H^N$ and $P_N$ is the projection operator to $H^N$. This Galerkin approximation equation is further discretized in time by the backward Euler scheme, described as

$$X_{t_m}^{M,N} - X_{t_{m-1}}^{M,N} + \tau A^2 X_{t_m}^{M,N} + \tau P_N AF(X_{t_m}^{M,N}) = P_N \Delta W_m, \quad m \in \{1, 2, \cdots, M\},$$

(1.8)

where $\Delta W_m := W(t_m) - W(t_{m-1})$, $\tau = \frac{T}{M}$ is the time stepsize and $t_m = m\tau$. The main result of this work is to obtain the following weak rate of convergence for the above space-time discretization scheme: For any $\epsilon > 0$,

$$|\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X_T^{M,N})]| \leq C \left( \lambda_N^{-\frac{3}{4} + \epsilon} + \tau^{\frac{3}{4} - \epsilon} \right),$$

(1.9)

where and throughout this article, $C$ denotes a generic nonnegative constant that is independent of the discretization parameters and may change from line to line, $\Phi \in C^4_b$, the space of all the space of all not necessarily bounded mappings from $H$ to $\mathbb{R}$ that have continuous and bounded Fréchet derivatives up to order 2.

The idea for the error analysis goes as follows. We first separate the error into two parts, the spatial error and the temporal error:

$$\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X_T^{M,N})] = (\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))]) + (\mathbb{E}[\Phi(X^N(T))] - \mathbb{E}[\Phi(X_T^{M,N})]).$$

(1.10)

To simplify the notation, we often write $O_t$ for $\int_0^t E(t-r)\,dW(r)$ and $O_t^N := P_N O_t$. By introducing two auxiliary processes $\bar{X}(t) := X(t) - O_t$ and $\bar{X}^N(t) := X^N(t) - O_t^N$, we further split the spatial error into two terms:

$$\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))] = (\mathbb{E}[\Phi(\bar{X}(T) + O_T)] - \mathbb{E}[\Phi(\bar{X}^N(T) + O_T)])$$

$$+ (\mathbb{E}[\Phi(\bar{X}^N(T) + O_T)] - \mathbb{E}[\Phi(\bar{X}^N(T) + O_T^N)]).$$

(1.11)
To proceed further, one relies on a Taylor expansion of the test function $\Phi$. For the first item, the key argument is then the strong error between $\bar{X}(T)$ and $\bar{X}(T)$,

$$
|\mathbb{E}[\Phi(\bar{X}(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\bar{X}(T) + \mathcal{O}_T)]| \\
\leq C|\mathbb{E}\int_0^1 \Phi'(X(t) + \lambda(\bar{X}(T) - \bar{X}(T))) (\bar{X}(T) - \bar{X}(T)) d\lambda| \\
\leq C\|\bar{X}(T) - P_N\bar{X}(T)\|_{L^2(\Omega,H)} + C\|P_N\bar{X}(T) - \bar{X}(T)\|_{L^2(\Omega,H)}.
$$

(1.12)

The above first term can be easily controlled owing to the higher spatial regularity of the process $\bar{X}(T)$, in the absence of the stochastic convolution. The second term $e(t) := P_N\bar{X}(t) - \bar{X}(T)$, satisfying the following random PDE,

$$
\frac{d}{dt}e(t) + A^2e(t) + P_NAP[F(X(t)) - F(X(t))] = 0, \quad e(0) = 0,
$$

(1.13)

must be carefully treated due to the presence of the unbounded operator $A$ before the nonlinear term $F$. We shall derive $\left\|\int_0^t \|e(s)\|^2 ds\right\|_{L^p(\Omega,\mathbb{R})} \leq C\lambda^{-3+\varepsilon}$ first and then make full use of the monotonicity of the nonlinear term $F$ and the regularity properties of $X(T)$, $X(T)$ and $\mathcal{O}_t$. Subsequently, we turn attention to the remaining term in (1.11). Applying the Taylor expansion gives

$$
|\mathbb{E}[\Phi(\bar{X}(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\bar{X}(T) + \mathcal{O}_T)]| \\
\leq \left|\mathbb{E}[\Phi'(X(T))(\mathcal{O}_T - \mathcal{O}_T^N)]\right| \\
+ \left|\mathbb{E}\left[\int_0^1 \Phi''(X(T) + \lambda(\mathcal{O}_T - \mathcal{O}_T^N))(\mathcal{O}_T - \mathcal{O}_T^N, \mathcal{O}_T - \mathcal{O}_T^N)(1 - \lambda)d\lambda\right]\right|.
$$

(1.14)

The Malliavin integration by parts technique is the key ingredient to deal with the first term (c.f. 3.39). Since $\Phi \in C^2$, the second term can be easily estimated to get the rates twice as high as the strong convergence rates. It is now easy to explain why the weak rate of convergence is expected to be higher than strong convergence rate. As a by-product of the weak error analysis, one can easily obtain the rate of the strong error, $\|X(t) - X(t)\|_{L^2(\Omega,H)} \leq \|\bar{X}(t) - \bar{X}(t)\|_{L^2(\Omega,H)} + \|\mathcal{O}_t - \mathcal{O}_t^N\|_{L^2(\Omega,H)}$, which is lower than the weak error, due to the presence of the second error. The basic idea to estimate of temporal error is the same as that of the spatial error by essentially exploiting the discrete analogue of the arguments. The main point is that error must be uniform on $N$ regardless of the spatial discretization.

Having sketched the central ideas for the proof of our main result, we now review some relevant results in the literature. For the linearized stochastic Cahn-Hilliard equations we refer to [3][20][23] for some strong convergence rate result for the finite element method. The strong convergence of the finite element method combined with time discretization under spatial regular noise was studied in [19][22] but with no rates were obtained. Meanwhile, [26] obtains strong convergence rates of the mixed finite element methods for the Cahn–Hilliard–Cook equation by using a priori strong moment bounds of the numerical approximations. For unbounded noise diffusion, the existence and regularity of solution has been investigated in [2][11] and the absolute continuity has been studied in [1][13]. Recently, the strong convergence rate of the spatial spectral Galerkin method and the temporal implicit Euler method for the stochastic Cahn-Hilliard equation was...
obtained in [14]. But no weak rate was addressed in this paper. In fact, for weak convergence analysis in the non-globally Lipschitz setting, we are only aware of the four papers [3, 7, 12, 15] concerning the stochastic Allen-Cahn equations and [21] for the linearized Cahn-Hilliard-Cook equation. To the best of our knowledge, the weak convergence rates of numerical method for the stochastic Cahn-Hilliard equation are absent. Inspired by our recent work [7], revealing the weak convergence rate of numerical approximation for another equation, stochastic Allen-Cahn equations without using the associated Kolmogorov equation, this paper fills a gap by deriving the rate of weak convergence of spatial spectral Galerkin method and temporal backward Euler method for stochastic Cahn-Hilliard equation. It is worthwhile to point out that issues arisen from the presence of the unbounded operator $A$ in front of the nonlinear term $F$ make the weak error analysis for stochastic Cahn-Hilliard equation much more sophisticated. To be more specific, in addition to the difficulty in the weak analysis mentioned earlier, the estimate of the Malliavin derivative during the spatial discretization process is also completely different, much more efforts is needed (c.f. Proposition 3.3).

The outline of the article is as follows. In the next section, we present some preliminaries, including the well-posedness and regularity of the mild solution and give a brief introduction to Malliavin calculus. Section 3 is devoted to the weak analysis of the spectral Galerkin method and Section 4 is concerned with the weak convergence rates of full discrete approximation.

2 Preliminaries

In this section, the basic setting, well-posedness and regularity of the model and a simple introduction of the Malliavin calculus are given.

2.1 Mathematical setting

Given two real separable Hilbert spaces $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$, $\mathcal{L}(U, H)$ stands for the space of all bounded linear operators from $U$ to $H$ endowed with the usual operator norm $\|\cdot\|_{\mathcal{L}(U,H)}$ and $\mathcal{L}_2(U, H)(\subset \mathcal{L}(U, H))$ denotes the space of all Hilbert-Schmidt operators from $U$ to $H$. To simplify the notation, we often write $\mathcal{L}(H)$ and $\mathcal{L}_2(H)$ (or $\mathcal{L}_2$ for short) instead of $\mathcal{L}(H, H)$ and $\mathcal{L}_2(H, H)$, respectively. It is well-known that $\mathcal{L}_2(U, H)$ is a Hilbert space equipped with the inner product and norm,

$$\langle T_1, T_2 \rangle_{\mathcal{L}_2(U,H)} = \sum_{i \in \mathbb{N}^+} \langle T_1 \phi_i, T_2 \phi_i \rangle, \quad \|T\|_{\mathcal{L}_2(U,H)} = \left( \sum_{i \in \mathbb{N}^+} \|T \phi_i\|^2 \right)^{\frac{1}{2}}, \quad (2.1)$$

where $\{\phi_i\}$ is an orthonormal basis of $U$. (2.1) is independent of the choice of orthonormal basis. If $T \in \mathcal{L}_2(U, H)$ and $L \in \mathcal{L}(H, U)$, then $TL \in \mathcal{L}_2(H)$ and $LT \in \mathcal{L}_2(U)$. Furthermore,

$$\|TL\|_{\mathcal{L}_2(H)} \leq \|T\|_{\mathcal{L}_2(U,H)}\|L\|_{\mathcal{L}(H,U)}, \quad \|LT\|_{\mathcal{L}_2(U)} \leq \|T\|_{\mathcal{L}_2(U,H)}\|L\|_{\mathcal{L}(H,U)}. \quad (2.2)$$

Also, if $T_1, T_2 \in \mathcal{L}_2(U, H)$ then

$$\langle T_1, T_2 \rangle_{\mathcal{L}_2(U,H)} \leq \|T_1\|_{\mathcal{L}_2(U,H)}\|T_2\|_{\mathcal{L}_2(U,H)}. \quad (2.3)$$
Now let $H = L^2(\mathcal{D}, \mathbb{R})$, where $\mathcal{D} = (0, 1)$ and $\hat{H} = \{v \in H : \langle v, 1 \rangle = 0 \}$. Additionally, for
simplicity of presentation we denote $V := C(\mathcal{D}, \mathbb{R})$ the Banach space of all continuous functions from $\mathcal{D}$ to $\mathbb{R}$ with supremum norm. We define $P : L^1(\mathcal{D}, \mathbb{R}) \to \hat{H}$ the generalized orthogonal projection defined by $Pv = v - \int_\mathcal{D} v \, dx$ and then $(I - P)v = \int_\mathcal{D} v \, dx$ is the average of $v$. Here and below, we denote $L^r(\mathcal{D}, \mathbb{R}) = \{ f : \mathcal{D} \to \mathbb{R}, \int_\mathcal{D} |f(x)|^r \, dx < \infty \}$. Sometimes we write it as $L^r(\mathcal{D})$ for short.

It is easy to check that $\hat{H}$ is a positive definite, self-adjoint and unbounded linear operator on $\hat{H}$ with compact inverse. Meanwhile, when extended to $H$ as $Av = APv$ for any $v \in H$, there exists a family of eigenpair $\{e_j, \lambda_j\}_{j \in \mathbb{N}}$ such that

$$Ae_j = \lambda_j e_j \quad \text{and} \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \quad \text{with} \quad \lambda_j \sim j^2 \to \infty,$$

where $e_0 = 1$ and $\{e_j, j = 1, \cdots \}$ forms an orthonormal basis of $\hat{H}$. Straightforward applications of the spectral theory yields the fractional powers of $A$ on $\hat{H}$, e.g., $A^\alpha v = \sum_{j=1}^\infty \lambda_j^\alpha \langle v, e_j \rangle e_j$, $\alpha \in \mathbb{R}$, $v \in \hat{H}$. The space $\hat{H}^\alpha := \text{dom}(A^\alpha)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_\alpha$ and the associated norm $| \cdot |_\alpha := \| A^\alpha \cdot \|$ given by

$$\langle v, w \rangle_\alpha = \sum_{j=1}^{\infty} \lambda_j^\alpha \langle v, e_j \rangle \langle w, e_j \rangle, \quad |v|_\alpha = \left( \sum_{j=1}^{\infty} \lambda_j^\alpha |\langle v, e_j \rangle|^2 \right)^{\frac{1}{2}}, \quad \alpha \in \mathbb{R}.$$ 

For $\alpha \geq 0$ we also define

$$|v|_\alpha = \left( |v|_\alpha^2 + |\langle v, e_0 \rangle|^2 \right)^{\frac{1}{2}}, \quad \text{if} \ v \in H$$

and the corresponding spaces are

$$\hat{H}^\alpha := \{ v \in \hat{H} : |v|_\alpha < \infty \}, \quad H^\alpha := \{ v \in H : |v|_\alpha < \infty \}.$$ 

A basic fact shows that for integer $\alpha \geq 0$, the norm $\| \cdot \|_\alpha$ is equivalent on $H^\alpha$ to the standard Sobolev norm $\| \cdot \|_{H^\alpha(\mathcal{D})}$ and $H^\alpha(\mathcal{D}), \alpha \in \mathbb{N}, \alpha \geq 2$ is an algebra. This means that there is a $C = C(\alpha)$ such that, for any $f, g \in H^\alpha$,

$$\|fg\|_{H^\alpha(\mathcal{D})} \leq C\|f\|_{H^\alpha(\mathcal{D})}\|g\|_{H^\alpha(\mathcal{D})} \leq C\|f\|_\alpha\|g\|_\alpha.$$ 

We recall that the operator $-A^2$ generates an analytic semigroup $E(t) = e^{-tA^2}$ on $H$ due to (2.4) and we have

$$E(t)v = e^{-tA^2}v = Pe^{-tA^2}v + (I - P)v, \quad v \in H.$$ 

At this point, we discuss the properties of the analytic semigroup $E(t)$ with aid of the eigenbasis of $A$ and Parseval’s identity,

$$\|A^\mu E(t)\|_{\mathcal{L}(H)} \leq Ct^{-\frac{\mu}{2}}, \quad t > 0, \mu \geq 0,$$

$$\|A^{-\nu}(I - E(t))\|_{\mathcal{L}(H)} \leq Ct^{-\nu}, \quad t \geq 0, \nu \in [0, 2],$$

$$\int_{t_1}^{t_2} \|A^\theta E(s)v\|^2 \, ds \leq C|t_2 - t_1|^{-\theta}\|v\|^2, \quad \forall v \in H, \theta \in [0, 1],$$

$$\|A^{2\rho} \int_{t_1}^{t_2} E(t_2 - \sigma)v \, d\sigma\| \leq C|t_2 - t_1|^{-\rho}\|v\|, \quad \forall v \in H, \rho \in [0, 1].$$
From Assumption \[1.2\] for \(v, \zeta, \zeta_1, \zeta_2 \in L^6(\mathcal{D}, \mathbb{R})\), we have
\[
(F'(v)(\zeta))(x) = f'(v(x))\zeta(x) = (3v^2(x) - 1)\zeta(x), \quad x \in \mathcal{D},
\]
\[
(F''(v)(\zeta_1, \zeta_2))(x) = f''(v(x))\zeta_1(x)\zeta_2(x) = 6v(x)\zeta_1(x)\zeta_2(x), \quad x \in \mathcal{D}.
\] (2.14)

Consequently, there exists a constant \(C\) such that
\[
-\langle F(u) - F(v), u - v \rangle \leq \|u - v\|^2, \quad u, v \in L^6(\mathcal{D}, \mathbb{R}),
\] (2.15)
\[
\|F'(v)u\| \leq C(1 + \|v\|_1^2)\|u\|, \quad u \in V, v \in L^6(\mathcal{D}, \mathbb{R}),
\] (2.16)
\[
\|F(u) - F(v)\| \leq C(1 + \|u\|_V^2 + \|v\|_V^2)\|u - v\|, \quad u, v \in V.
\] (2.17)

### 2.2 Well-posedness and regularity results of the model

Following the semigroup approach, Assumptions \[1.1, 1.2, 1.3\] are sufficient to establish well-posedness and spatio-temporal regularity of the mild solution of \[1.2\], the relevant results are stated below. For more details, one can refer to [25, Theorem 3.5].

Before stating the theorem, similar to the proof in [9, Theorem 2], we give the following spatio-temporal regularity result of stochastic convolution \(\int_0^t E(t-s) dW(s)\). To simplify the notation, we often write
\[
\mathcal{O}_t := \int_0^t E(t-s) dW(s).
\] (2.18)

**Lemma 2.1.** Suppose Assumptions \[1.1, 1.3\] hold, then the stochastic convolution \(\mathcal{O}_t\) satisfies
\[
\mathbb{E}\left[\sup_{t \in [0,T]} \|\mathcal{O}_t\|_2^2\right] < \infty,
\] (2.19)
and for \(\alpha \in [0, 2]\),
\[
\|\mathcal{O}_t - \mathcal{O}_s\|_{L^p(\Omega; H^\alpha)} \leq C|t - s|^\frac{2-\alpha}{4}.
\] (2.20)

The following theorem states the well-posedness and spatio-temporal regularities of the mild solution of stochastic Cahn-Hilliard equation.

**Theorem 2.2** (Well-posedness and regularity of the mild solution). Under Assumptions \[1.1, 1.4\], there is a unique mild solution of \[1.2\] given by
\[
X(t) = E(t)X_0 - \int_0^t E(t-s)APF(X(s)) ds + \int_0^t E(t-s) dW(s), \quad t \in [0, T].
\] (2.21)

Furthermore, for \(p \geq 1\),
\[
\sup_{t \in [0,T]} \|X(t)\|_{L^p(\Omega; H^2)} < \infty.
\] (2.22)

Moreover, for any \(\alpha \in [0, 2]\) and \(0 \leq s < t \leq T\),
\[
\|X(t) - X(s)\|_{L^p(\Omega; H^\alpha)} \leq C(t-s)^\frac{2-\alpha}{4}.
\] (2.23)

**Corollary 2.3.** The spatial regularity of \(X(T)\) shown in \[2.22\] together with \[2.8\] implies
\[
\sup_{t \in [0,T]} \|F(X(t))\|_{L^p(\Omega; H^2)} < \infty.
\] (2.24)
2.3 Introduction to Malliavin calculus

A brief introduction to Malliavin calculus is given in this subsection, one can refer to the classical monograph [24] for more details. Define a Hilbert space $U_0 = Q^{1/2}_\mathbb{H}(H)$ with inner product $\langle u, v \rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle$. Let $\mathcal{I} : L^2([0, T], U_0) \to L^2(\Omega, \mathbb{R})$ be an isonormal process for any deterministic mapping $\phi \in L^2([0, T], U_0)$, then $\mathcal{I}(\phi)$ is centered Gaussian with the covariance structure

$$
\mathbb{E}[\mathcal{I}(\phi_1)\mathcal{I}(\phi_2)] = \langle \phi_1, \phi_2 \rangle_{L^2([0, T], U_0)}, \quad \phi_1, \phi_2 \in L^2([0, T], U_0).
$$

(2.25)

By $C^\infty_p(\mathbb{R}^M, \mathbb{R})$ we denote the space of all $C^\infty$-mappings with polynomial growth. Then, the family of all smooth $H$-valued cylindrical random variables is given as

$$
\mathcal{S}(H) = \left\{ G = \sum_{i=1}^N g_i(\mathcal{I}(\phi_1), \ldots, \mathcal{I}(\phi_M))h_i : g_i \in C^\infty_p(\mathbb{R}^M, \mathbb{R}) \right\}.
$$

(2.26)

Next, the action of the Malliavin derivative on $G \in \mathcal{S}(H)$ reads as,

$$
\mathcal{D}_t G := \sum_{i=1}^N \sum_{j=1}^M \partial_j g_i(\mathcal{I}(\phi_1), \ldots, \mathcal{I}(\phi_M))h_i \otimes \phi_j(t),
$$

(2.27)

where $h_i \otimes \phi_j(t)$ denotes the tensor product, that is, for $1 \leq j \leq M$ and $1 \leq i \leq N$,

$$
(h_i \otimes \phi_j(t))(u) = \langle \phi_j(t), u \rangle_{U_0} h_i \in H, \quad \forall \ u \in U_0, \ h_i \in H, \ t \in [0, T].
$$

(2.28)

If $G$ is $\mathcal{F}_t$-measurable, then $\mathcal{D}_s G = 0$ for $s > t$. The derivative operator $\mathcal{D}_t$ is known to be closable, we then define $\mathbb{D}^{1,2}(H)$ as the closure of $\mathcal{S}(H)$ with respect to the norm

$$
\|G\|_{\mathbb{D}^{1,2}(H)} = \left( \mathbb{E}[\|G\|^2] + \mathbb{E} \int_0^T \|\mathcal{D}_t G\|_{\mathcal{L}_2(U_0, H)}^2 dt \right)^{1/2}.
$$

(2.29)

We are now prepared to give the Malliavin integration by parts formula as follows, for any $G \in \mathbb{D}^{1,2}(H)$ and adapted process $\Psi \in L^2([0, T], \mathcal{L}_2(U_0; H))$,

$$
\mathbb{E}\left[ \left\langle \int_0^T \Psi(t) dW(t), G \right\rangle \right] = \mathbb{E} \int_0^T \langle \Psi(t), \mathcal{D}_t G \rangle_{\mathcal{L}_2(U_0, H)} dt,
$$

(2.30)

where the stochastic integral is Itô integral. For brevity, we write $\langle \mathcal{D}_s G, u \rangle = \mathcal{D}_s^u G$ to represent the derivative in the direction $u \in U_0$. The Malliavin derivative acts on the Itô integral $\int_0^t \Psi(r) dW(r)$ satisfying for all $u \in U_0$,

$$
\mathcal{D}_s^u \int_0^t \Psi(r) dW(r) = \int_0^t \mathcal{D}_s^u \Psi(r) dW(r) + \Psi(s)u, \quad 0 \leq s \leq t \leq T.
$$

(2.31)

Given another separable Hilbert space $\mathcal{H}$, if $\sigma \in C^1_b(H, \mathcal{H})$ and $G \in \mathbb{D}^{1,2}(H)$, then $\sigma(G) \in \mathbb{D}^{1,2}(\mathcal{H})$ and the chain rule of the Malliavin derivative holds as $\mathcal{D}_t^\sigma(\sigma(G)) = \sigma'(G) \cdot \mathcal{D}_t^u G$.
3 Weak convergence rate of the spectral Galerkin method

This section is devoted to the weak analysis of the spatial spectral Galerkin discretization. In the beginning, we define a finite dimension subspace of $H$ spanned by $N + 1$ first eigenvectors of the dominant linear operator $A$, i.e., $H_N = \text{span}\{e_0, e_1, \cdots, e_N\}$ and define the projection operator $P_N$ by $P_N x = \sum_{i=0}^{N} \langle x, e_i \rangle e_i$ for $\forall x \in H^\beta, \beta \geq -2$. $I \in L(H)$ is used to denote the identity mapping on $H$. Therefore, it can be easily checked that $A$ and $P_N$ commute with each other and

$$\| (P_N - I)x \| \leq C \lambda_{N}^{-\beta} |x|_\beta, \quad \forall \beta \geq 0. \quad (3.1)$$

Next, applying the spectral Galerkin method to (1.2) results in the finite dimension stochastic differential equation yields

$$dX_N(t) + A^2 X_N(t) + P_N APF(X_N(t))dt = P_N dW(t), \quad t \in (0, T]; \quad X_N(0) = P_N X_0, \quad (3.2)$$

whose unique mild solution is an adapted process satisfying

$$X_N(t) = E(t)P_N X_0 - \int_0^t E(t-s)P_N APF(X_N(s))ds + \int_0^t E(t-s)P_N dW(s). \quad (3.3)$$

Similarly, we use $\mathcal{O}_t^N$ to represent $\int_0^t E(t-s)P_N dW(s)$, which enjoys the following regularity,

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0,T]} \| \mathcal{O}_t^N \|_2^p \right] < \infty, \quad (3.4)$$

and for $\alpha \in [0, 2]$,

$$\sup_{N \in \mathbb{N}} \| \mathcal{O}_t^N - \mathcal{O}_s^N \|_{L^p(\Omega, H^\alpha)} \leq C|t - s|^{\frac{2-\alpha}{4}}. \quad (3.5)$$

The proof of the following a priori estimate is referred to [25, Lemma 3.4].

**Theorem 3.1** (Spatio-temporal regularity of spatial semi-discretization). *If Assumptions 1.1-1.4 are satisfied, then the mild solution of spatial approximation problem (3.3), admits the following moment bounds,

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0,T]} \| X_N(t) \|_{L^p(D,R)}^p \right] < \infty \quad \forall \ p \geq 1. \quad (3.6)$$

Additionally, we have the spatio-temporal regularity as follows,

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \| X_N(t) \|_{L^p(\Omega, H^2)} < \infty. \quad (3.7)$$

Moreover, for any $\alpha \in [0, 2]$ and $0 \leq s < t \leq T$,

$$\sup_{N \in \mathbb{N}} \| X_N(t) - X_N(s) \|_{L^p(\Omega, H^\alpha)} \leq C(t - s)^{\frac{2-\alpha}{4}}. \quad (3.8)$$

**Corollary 3.2.** *The spatial regularity of $X(T)$ shown in (3.7) together with (2.8) implies*

$$\sup_{t \in [0,T]} \| F(X_N(t)) \|_{L^p(\Omega, H^2)} < \infty. \quad (3.9)$$
Before studying the weak convergence rate, we first show that \( X^N(t) \) is differentiable in Malliavin sense.

**Proposition 3.3 (Boundedness of the Malliavin derivative).** Let Assumptions 1.1-1.4 hold. Then the Malliavin derivative of \( X^N(t) \) satisfies

\[
\mathbb{E} [\| \mathcal{D}_s^y X^N(t) \|^2 ] \leq C \|y\|^2, \quad 0 \leq s < t \leq T. \tag{3.10}
\]

**Proof.** Taking the Malliavin derivative on the equation (3.3) in the direction \( y \in U_0 \) and using the chain rule yield that for \( 0 \leq s \leq t \leq T \),

\[
\mathcal{D}_s^y X^N(t) = E(t - s)P_Ny - \int_s^t E(t - r)P_NAPF'(X^N(r))\mathcal{D}_s^y X^N(r)dr. \tag{3.11}
\]

Thus, \( \mathcal{D}_s^y X^N(t) \) is differentiable in time and satisfies

\[
\frac{d\mathcal{D}_s^y X^N(t)}{dt} + A^2 \mathcal{D}_s^y X^N(t) + P_NAPF'(X^N(t))\mathcal{D}_s^y X^N(t) = 0. \tag{3.12}
\]

Multiplying \( A^{-1}\mathcal{D}_s^y X^N(t) \) on both sides infers

\[
\left\langle \frac{d\mathcal{D}_s^y X^N(t)}{dt}, A^{-1}\mathcal{D}_s^y X^N(t) \right\rangle + \left\langle A^2 \mathcal{D}_s^y X^N(t), A^{-1}\mathcal{D}_s^y X^N(t) \right\rangle + \left\langle P_NAPF'(X^N(t))\mathcal{D}_s^y X^N(t), A^{-1}\mathcal{D}_s^y X^N(t) \right\rangle = 0. \tag{3.13}
\]

Next, after integrating (3.13) over \([s, t]\), we use definition of \( F \) and Hölder’s inequality to deduce

\[
\| \mathcal{D}_s^y X^N(t) \|_{-1}^2 = \|y\|_{-1}^2 - 2 \int_s^t \| \mathcal{D}_s^y X^N(r) \|_1^2 dr - 2 \int_s^t \left\langle F'(X^N(r))\mathcal{D}_s^y X^N(r), \mathcal{D}_s^y X^N(r) \right\rangle dr
\]

\[
= \|y\|_{-1}^2 - 2 \int_s^t \| \mathcal{D}_s^y X^N(r) \|_1^2 dr + 2 \int_s^t \left\langle A^\frac{1}{2} \mathcal{D}_s^y X^N(r), A^{-\frac{1}{2}} \mathcal{D}_s^y X^N(r) \right\rangle dr
\]

\[
- 6 \int_s^t \left\langle (X^N(r))^2 \mathcal{D}_s^y X^N(r), \mathcal{D}_s^y X^N(r) \right\rangle dr
\]

\[
\leq \|y\|_{-1}^2 - \int_s^t \| \mathcal{D}_s^y X^N(r) \|_1^2 dr + \int_s^t \| \mathcal{D}_s^y X^N(r) \|_{-1}^2 dr.
\tag{3.14}
\]

Hence, by Gronwall’s inequality we have

\[
\| \mathcal{D}_s^y X^N(t) \|_{-1}^2 \leq C \|y\|_{-1}^2. \tag{3.15}
\]

Therefore,

\[
\int_s^t \| \mathcal{D}_s^y X^N(r) \|_1^2 dr \leq C \|y\|_{-1}^2. \tag{3.16}
\]

In the sequel, taking inner product of (3.12) by \( \mathcal{D}_s^y X^N(t) \) gives

\[
\left\langle \frac{d\mathcal{D}_s^y X^N(t)}{dt}, \mathcal{D}_s^y X^N(t) \right\rangle + \left\langle A^2 \mathcal{D}_s^y X^N(t), \mathcal{D}_s^y X^N(t) \right\rangle + \left\langle P_NAPF'(X^N(t))\mathcal{D}_s^y X^N(t), \mathcal{D}_s^y X^N(t) \right\rangle = 0. \tag{3.17}
\]
Similarly, taking integration and using the definition of \( F \) and the Sobolev embedding inequality, one finds that
\[
\| D^\theta_s X^N(t) \|^2 = \| y \|^2 - 2 \int_s^t \| A D^\theta_s X^N(r) \|^2 dr - 2 \int_s^t \langle F'(X^N(r)) D^\theta_s X^N(r), A D^\theta_s X^N(r) \rangle dr
\]
\[
\leq \| y \|^2 - 2 \int_s^t \| A D^\theta_s X^N(r) \|^2 dr + 2 \int_s^t \| A D^\theta_s X^N(r) \|^2 dr
\]
\[
+ \frac{1}{2} \int_s^t \| F'(X^N(r)) D^\theta_s X^N(r) \|^2 dr
\]
\[
\leq \| y \|^2 + C( \sup_{r \in [s,t]} \| X^N(r) \|_{L^6}^4 + 1 ) \int_s^t \| D^\theta_s X^N(r) \|^2 dr
\]
\[
\leq \| y \|^2 + C \| y \|^2 ( \sup_{r \in [s,t]} \| X^N(r) \|_{L^6}^4 + 1 ).
\]

This together with (3.6) finishes the proof. \( \Box \)

Subsequently, we collect some useful results on the nonlinear term \( F \).

**Lemma 3.4.** Let \( F : L^6(\mathcal{I}, \mathbb{R}) \to H \) be the Nemytskii operator defined in Assumption 1.2. Then it holds
\[
\| F'(x) y \|_1 \leq C(1 + \| x \|_1^2) \| y \|_1
\]
and for any \( \theta \in (0, 1) \) and \( \eta \geq 1, \)
\[
\| F'(\varsigma) \psi \|_{-\eta} \leq C(1 + \max\{\| \varsigma \|_V, \| \varsigma \|_\theta \}^2) \| \psi \|_{-\theta}, \quad \forall \varsigma \in V \cap \dot{H}^\theta, \psi \in V.
\]

Now, we are well prepared to give the following theorem about the weak error rate for spatial semi-discretization.

**Theorem 3.5** (Weak convergence rates of the spatial approximation). Let \( X(T) \) and \( X^N(T) \) be the solution of problem (1.2) and spatial semi-discretization (3.2), given by (2.21) and (3.3), respectively. Let Assumptions 1.3-1.4 hold. Then for \( \Phi \in C^0 \), and for any sufficiently small \( \epsilon > 0 \), there exists a constant \( C_{\Phi, \epsilon} \) such that
\[
|\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))]| \leq C_{\Phi, \epsilon} \lambda^{-\frac{1}{2} + \epsilon}.
\]

**Proof.** Define two processes \( \tilde{X}(t) = X(t) - \mathcal{O}_t \) and \( \tilde{X}^N(t) = X^N(t) - \mathcal{O}_t^N \). Then we can separate the error \( \mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))] \) into two terms as
\[
\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))] = \left( \mathbb{E}[\Phi(\tilde{X}(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T)] \right)
\]
\[
+ \left( \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T^N)] \right)
\]
\[
=: I_1 + I_2.
\]
To estimate $I_1$, it suffices to consider the strong convergence between $\bar{X}(T)$ and $\bar{X}^N(T)$ due to $\Phi \in C_b^2$. To be specific, by the Taylor expansion and triangle inequality we have

$$
|I_1| = \left| \mathbb{E}\left[ \Phi(\bar{X}(T) + \mathcal{O}_T) \right] - \mathbb{E}\left[ \Phi(\bar{X}^N(T) + \mathcal{O}_T) \right] \right| \leq \left| \mathbb{E}[\|\bar{X}(T) - \bar{X}^N(T)\|] \right| \leq \|\bar{X}(T) - P_N\bar{X}(T)\|_{L^2(\Omega, H)} + \|P_N\bar{X}(T) - \bar{X}^N(T)\|_{L^2(\Omega, H)}. \tag{3.23}
$$

The regularity of $\bar{X}(T)$, in the absence of the stochastic convolution in $X(T)$, can be improved by (2.10) and (2.24),

$$
\|A^{2-\varepsilon}\bar{X}(T)\|_{L^p(\Omega, H)} \leq \|A^{2-\varepsilon}E(T)X_0\|_{L^p(\Omega, H)} + \left\| \int_0^T A^{2-\varepsilon}E(T-s)APF(X(s)) \, ds \right\|_{L^p(\Omega, H)}
$$

$$
\leq C\|X_0\|_4 + \int_0^T (T-s)^{-1+\frac{\varepsilon}{4}} ds \sup_{s \in [0, T]} \|F(X(s))\|_{L^p(\Omega, H^2)} < \infty. \tag{3.24}
$$

Using this regularity of $\bar{X}(T)$ and (3.1), we can infer that

$$
\|\bar{X}(T) - P_N\bar{X}(T)\|_{L^p(\Omega, H)} = \|(I - P_N)A^{-2+\varepsilon}A^{2-\varepsilon}\bar{X}(T)\|_{L^p(\Omega, H)} \leq C\lambda_N^{-2+\varepsilon}. \tag{3.25}
$$

Next, we denote $e(t) = P_N\bar{X}(t) - \bar{X}^N(t)$ for simplicity of presentation, which satisfies

$$
\frac{d}{dt}e(t) + A^2e(t) + P_NAP\left[ F(\bar{X}(t) + \mathcal{O}_t) - F(\bar{X}^N(t) + \mathcal{O}^N_t) \right] = 0. \tag{3.26}
$$

Taking inner product on both sides with $A^{-1}e(s)$ in $\dot{H}$ and using (2.15), (2.17), Hölder’s inequality, Sobolev embedding inequality and Lemma 3.4 give, for sufficiently small $\varepsilon > 0$,

$$
\frac{1}{2}\frac{d}{ds}\|e(s)\|_{-1}^2 + \|e(s)\|_1^2 = -\langle e(s), F(\bar{X}(s) + \mathcal{O}_s) - F(P_N\bar{X}(s) + \mathcal{O}_s) \rangle
$$

$$
- \langle e(s), F(P_N\bar{X}(s) + \mathcal{O}_s) - F(\bar{X}^N(s) + \mathcal{O}_s) \rangle
$$

$$
- \langle e(s), F(\bar{X}^N(t) + \mathcal{O}_s) - F(\bar{X}^N(s) + \mathcal{O}^N_s) \rangle
$$

$$
\leq \frac{3}{2}\|e(s)\|_{-1}^2 + \frac{1}{2}\|F(\bar{X}(s) + \mathcal{O}_s) - F(P_N\bar{X}(s) + \mathcal{O}_s)\|_1^2
$$

$$
+ \frac{1}{4}\|e(s)\|_{-1}^2 + \|F(\bar{X}^N(s) + \mathcal{O}_s) - F(\bar{X}^N(s) + \mathcal{O}^N_s)\|_{-1}^2
$$

$$
\leq \frac{3}{4}\|e(s)\|_{-1}^2 + \frac{3}{8}\|e(s)\|_{-1}^2 + C\|\bar{X}(s) - P_N\bar{X}(s)\|^2(1 + \|\bar{X}(s)\|^4 + \|\mathcal{O}_s\|^4)
$$

$$
+ C\|\mathcal{O}_s - \mathcal{O}^N_s\|_{-1+\varepsilon}^2(1 + \|\bar{X}^N(s)\|^2 + \|\mathcal{O}_s\|^4). \tag{3.27}
$$

By the Gronwall inequality, we deduce

$$
\|e(T)\|_{-1}^2 + \int_0^T \|e(s)\|_{-1}^2 \, ds \leq C\int_0^T \|\bar{X}(s) - P_N\bar{X}(s)\|^2(1 + \|\bar{X}(s)\|^4 + \|\mathcal{O}_s\|^4) \, ds
$$

$$
+ C\int_0^T \|\mathcal{O}_s - \mathcal{O}^N_s\|_{-1+\varepsilon}^2(1 + \|\bar{X}^N(s)\|^2 + \|\mathcal{O}_s\|^4) \, ds. \tag{3.28}
$$
By aid of the regularity of $X(T)$ and $X^N(T)$, \( [3.25] \), Hölder’s inequality and the fact
\[
\|O_s - O_s^N\|_{L^p(\Omega, H^{-1+\epsilon})} \leq C \|(I - P_N) A^{-\frac{1+\epsilon}{2}} O_s\|_{L^p(\Omega, H)} \leq C \lambda_N^{-\frac{3+\epsilon}{2}}, \tag{3.29}
\]
one can find that
\[
\left\| \int_0^T \|e(s)\|^2_1 ds \right\|_{L^p(\Omega, \mathbb{R})} \leq C \int_0^T \|\tilde{X}(s) - P_N \tilde{X}(s)\|_{L^p(\Omega, H)}^2 ds
\]
\[
+ C \int_0^T \|O_s - O_s^N\|_{L^p(\Omega, H^{-1+2\epsilon})}^2 ds
\tag{3.30}
\leq C \lambda_N^{-4+2\epsilon} + C \lambda_N^{-3+2\epsilon}
\]
\[
\leq C \lambda_N^{-3+\epsilon}.
\]
Based on these estimates, we are now ready to bound the error $\|e(T)\|_{L^p(\Omega, H)}$, which can be split into three parts:
\[
\|e(T)\|_{L^p(\Omega, H)} = \left\| \int_0^T E(T - s) P_N aP(F(X(s)) - F(X^N(s))) ds \right\|_{L^p(\Omega, H)}
\]
\[
\leq \left\| \int_0^T E(T - s) P_N aP(F(\tilde{X}(s) + O_s) - F(P_N \tilde{X}(s) + O_s)) ds \right\|_{L^p(\Omega, H)}
\]
\[
+ \left\| \int_0^T E(T - s) P_N aP(F(P_N \tilde{X}(s) + O_s) - F(\tilde{X}^N(s) + O_s)) ds \right\|_{L^p(\Omega, H)}
\]
\[
+ \left\| \int_0^T E(T - s) P_N aP(F(\tilde{X}^N(s) + O_s) - F(\tilde{X}^N(s) + O^N_s)) ds \right\|_{L^p(\Omega, H)}
\tag{3.31}
\]
\[
=: e_1(T) + e_2(T) + e_3(T).
\]
Again, by (2.10), (2.17), (3.25), Sobolev embedding inequality and the regularity of $X(t)$, we have
\[
e_1(T) = \left\| \int_0^T E(T - s) P_N aP(F(\tilde{X}(s) + O_s) - F(P_N \tilde{X}(s) + O_s)) ds \right\|_{L^p(\Omega, H)}
\]
\[
\leq C \int_0^T (T - s)^{-\frac{1}{2}} \|\tilde{X}(s) - P_N \tilde{X}(s)\|_{L^2(\Omega, H^2)} ds
\tag{3.32}
\]
\[
(1 + \sup_{s \in [0,T]} \|\tilde{X}(s)\|_{L^2(\Omega, H^2)}^2 + \sup_{s \in [0,T]} \|O_s\|_{L^2(\Omega, H^2)}^2)
\]
\[
\leq C \lambda_N^{-2+\epsilon}.
\]
From (2.10), (3.19) in Lemma 3.4 Hölder’s inequality, (3.30) and regularity of $X(t)$ and $X^N(t)$,
it follows that for small $\epsilon > 0$,

$$
e_2(T) \leq C \left\| \int_0^T (T-s)^{-\frac{3}{4}} \left\| F(P_N \bar{X}(s) + \mathcal{O}_s) - F(\bar{X}^N(s) + \mathcal{O}_s) \right\|_1 ds \right\|_{L^p(\Omega, \mathbb{R})}
\leq C \left\| \int_0^T (T-s)^{-\frac{3}{4}} \|e(s)\|_1 (1 + \|\bar{X}(s)\|_2^2 + \|\bar{X}^N(s)\|_2^2 + \|\mathcal{O}_s\|_2^2) ds \right\|_{L^p(\Omega, \mathbb{R})}
\leq C \left\| \int_0^T \|e(s)\|^2_1 ds \right\|_{L^p(\Omega, \mathbb{R})} \left\| \int_0^T (T-s)^{-\frac{3}{4}} (1 + \|\bar{X}(s)\|_2^4 + \|\bar{X}^N(s)\|_2^4 + \|\mathcal{O}_s\|_2^4) ds \right\|^{\frac{1}{2}}_{L^p(\Omega, \mathbb{R})}
\leq C \lambda_n^{-\frac{3}{2} + \epsilon}.
$$

Similarly to the estimate of (3.32) with (3.20) and (3.29) instead, we obtain

$$
e_3(T) = \left\| \int_0^T E(T-s) P_N A^{\frac{3}{4}} A^{-\frac{1}{2}} P(F(X^N(s) + \mathcal{O}_s) - F(X^N(s) + \mathcal{O}_s^N)) ds \right\|_{L^p(\Omega, \mathbb{H})}
\leq C \int_0^T (T-s)^{-\frac{1}{4}} \left\| \mathcal{O}_s - \mathcal{O}_s^N \right\|_{L^{2p}(\Omega, H^{-1/2+\epsilon})} ds
\leq C \lambda_n^{-\frac{1}{2} + \epsilon}.
$$

The estimates for $e_1(T)$, $e_2(T)$ and $e_3(T)$ together yield

$$\|P_N \bar{X}(T) - \bar{X}^N(T)\|_{L^2(\Omega, \mathbb{H})} \leq C \lambda_n^{-\frac{3}{2} + \epsilon}.
$$

This combined with (3.25) yields

$$|I_1| \leq C \lambda_n^{-\frac{3}{2} + \epsilon}.
$$

Next, we bound $|I_2|$ by the second-order Taylor expansion,

$$|I_2| = \left| \mathbb{E} \left( \Phi'(X^N(T))(\mathcal{O}_T - \mathcal{O}_T^N) \right)
+ \int_0^1 \Phi''(X^N(T) + \lambda(\mathcal{O}_T - \mathcal{O}_T^N))(\mathcal{O}_T - \mathcal{O}_T^N, \mathcal{O}_T - \mathcal{O}_T^N)(1 - \lambda) d\lambda \right| 
\leq \left| \mathbb{E} \left( \Phi'(X^N(T))(I - P_N)\mathcal{O}_T \right) \right| + C \mathbb{E} \left[ \|\mathcal{O}_T - \mathcal{O}_T^N\|^2 \right].
$$

The second term can be easily estimated by utilizing (2.19) and (3.1) as follows

$$\mathbb{E} \left[ \|\mathcal{O}_T - \mathcal{O}_T^N\|^2 \right] = \mathbb{E} \left[ \|(I - P_N)\mathcal{O}_T\|^2 \right] \leq C \lambda_n^{-2}.
$$
For the first term, Proposition 3.3, the Malliavin integration by parts formula (2.30), the chain rule of the Malliavin derivative, (2.10), (3.1) and (1.4) enable us to obtain

\[ \left| \mathbb{E} \left[ \Phi'(X^N(T))(I - P_N) \mathcal{O}_T \right] \right| = \mathbb{E} \int_0^T \left( (I - P_N) E(T - s), \mathcal{D}_s \Phi'(X^N(T)) \right) \mathcal{L}_2^0 \, ds \]
\[ \leq \mathbb{E} \int_0^T \left\| (I - P_N) E(T - s) \right\|_{\mathcal{L}_2^0} \left\| \Phi''(X^N(T)) \right\|_{\mathcal{L}(H)} \left\| \mathcal{D}_s X^N(T) \right\|_{\mathcal{L}_2^0} \, ds \]
\[ \leq C \lambda_N^{-\frac{3}{2}} \int_0^T (T - s)^{-\frac{3}{2}} \, ds \]
\[ \leq C \lambda_N^{-\frac{3}{2}}. \]

Hence, the proof is complete. \(\square\)

4 Weak convergence rate of the backward Euler method

Based on the spatial spectral Galerkin discretization (3.2), this section concerns with the convergence analysis for a spatio-temporal full discretization, performed by a backward Euler method in time. We divide the interval \([0, T]\) into \(M\) equidistant subintervals with the time step-size \(\tau = \frac{T}{M}\) and denote the nodes \(t_m = m\tau\) for \(m \in \{1, \ldots, M\}\) where \(M \in \mathbb{N}^+\). Then, we propose the fully discrete numerical scheme in the following form,

\[ X_{t_m}^{M,N} = X_{t_m}^{M,N} + \tau A^2 X_{t_m}^{M,N} + \tau P_N AF(X_{t_m}^{M,N}) = P_N \Delta W_m, \quad X_0^{M,N} = P_N X_0, \quad (4.1) \]

where \(\Delta W_m := W(t_m) - W(t_{m-1})\) for short. By introducing a family of operators \(\{E_{\tau,N}^m\}_{m=1}^M:\)

\[ E_{\tau,N}^m v = (I + \tau A^2)^{-m} P_N v, \quad \forall \, v \in H, \quad (4.2) \]

we can rewrite the fully discrete scheme as

\[ X_{t_m}^{M,N} = E_{\tau,N}^m P_N X_0 - \tau \sum_{j=1}^m E_{\tau,N}^{m-j+1} P_N AF(X_{t_j}^{M,N}) + \sum_{j=1}^m E_{\tau,N}^{m-j+1} P_N \Delta W_j. \quad (4.3) \]

It is easy to check that the operator \(E_{\tau,N}^m\) satisfies

\[ \|A^\mu E_{\tau,N}^m v\| \leq C \tau^{-\frac{\mu}{2}} \|v\|, \quad \forall \, v \in H, \quad m \in \{1, 2, \cdots, M\}. \quad (4.4) \]

By a slight modification of the proof in [25, Theorem 4.1], a priori moment bounds for the fully discrete approximation can be derived.

**Theorem 4.1.** Suppose Assumptions 1.1-1.4 hold, then we have for \(N, M \in \mathbb{N}\) and \(\forall \, p \geq 1,\)

\[ \sup_{m \in \{0, 1, \cdots, M\}} \mathbb{E} \left[ \|X_{t_m}^{M,N}\|_2^p \right] < \infty. \quad (4.5) \]
Before starting the main theorem, we need some estimates between \( E(t)P_N \) and \( E^m_{\tau,N} \). For the sake of simplicity, we denote
\[
[s] := \max\{0, \tau, \cdots, m\tau, \cdots\} \cap [0, s], \quad \lfloor s \rfloor := \min\{0, \tau, \cdots, m\tau, \cdots\} \cap [s, T] \text{ and } [s] = \frac{s}{\tau},
\] and the fully discrete approximation operator
\[
\Psi^M_N(t) := E(t)P_N - E^k_{\tau,N}, \quad t \in [t_{k-1}, t_k), \quad k \in \{1, 2, \cdots, M\}.
\]

The following lemma of the fully discrete approximation operator plays a pivotal role in the weak convergence analysis.

**Lemma 4.2.** Under Assumption 1.1, we have the following statements.

(i) Let \( \rho \in [0, 4] \), there exists a constant \( C \) such that
\[
\|\Psi^M_N(t)u\| \leq C t^{-\frac{\rho}{2}} \|u\|_{-\rho}.
\] (4.8)

(ii) Let \( \beta \in [0, 4] \), there exists a constant \( C \) such that
\[
\|\Psi^M_N(t)u\| \leq C \tau^\beta \|u\|_{\beta}.
\] (4.9)

(iii) Let \( \alpha \in [0, 2] \), there exists a constant \( C \) such that
\[
\|\Psi^M_N(t)u\| \leq C \tau^{\frac{\alpha}{2}} t^{-1} \|u\|_{-\alpha}.
\] (4.10)

(iv) Let \( \mu \in [2, 4] \), there exists a constant \( C \) such that
\[
\|\Psi^M_N(t)u\| \leq C \tau^{\frac{\mu-2}{2}} t^{-\frac{\mu}{2}} \|u\|_{\mu-4}.
\] (4.11)

(v) Let \( \nu \in [0, 4] \), there exists a constant \( C \) such that
\[
\left( \int_0^T \|\Psi^M_N(s)u\|^2 \text{d}s \right)^{\frac{1}{2}} \leq C \tau^{\frac{\nu}{2}} \|v\|_{\nu-2}.
\] (4.12)

*Proof.* Elementary fact in [23, Lemma 5.3] yields the above (i), (ii), (iii) and (v). We only need to prove (iv). For \( \mu = 2 \), it is a consequence of (i) with \( \rho = 2 \) and for \( \mu = 4 \), it is a consequence of (iii) with \( \alpha = 0 \). By a standard interpolation argument we finish the proof. \( \square \)

For clarity of exposition, we write
\[
O^M_N = \sum_{j=1}^M E^M_{\tau,N}-j+1 P_N \Delta W_j = \int_0^T E^M_{\tau,N}[s] \text{d}W(s).
\]

The next lemma gives an estimate between \( O^N_T \) and \( O^{M,N}_T \).
Lemma 4.3. Under Assumptions 1.1 and 1.3, we have for \( \beta \in [0, 2] \) and \( p \geq 1 \),

\[
\sup_{m \in \{1, 2, \ldots, M\}} \| O_{t_m}^{M,N} - O_{t_m}^N \|_{L^p(\Omega, H_{-\beta})} \leq C \tau^{\frac{2+\beta}{4}}.
\] (4.13)

Proof. The Burkholder-Davis-Gundy inequality and (v) in Lemma 4.2 with \( \nu = 2 + \beta \) yield

\[
\| O_{t_m}^{M,N} - O_{t_m}^N \|_{L^p(\Omega, H_{-\beta})} \leq C \left( \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \| \Psi_{t}^{M,N}(t_m - s) A^{-\beta} Q_s^\frac{1}{2} \|_{L^2}^2 \text{d}s \right)^{\frac{1}{2}}
\]

\[
\leq C \tau^{\frac{2+\beta}{4}} \| Q^\frac{1}{2} \|_{L^2}
\]

\[
\leq C \tau^{\frac{2+\beta}{4}}.
\] (4.14)

This finishes the proof. \( \square \)

The next theorem states the weak convergence rates of the temporal semi-discretization.

Theorem 4.4. Suppose Assumptions 1.1-1.4 are satisfied. Let \( X^N(T) \) and \( X_T^{M,N} \) be given by (3.3) and (4.3), respectively. Then, we have for \( \Phi \in C^2_b \) and sufficiently small \( \epsilon > 0 \),

\[
|E[\Phi(X^N(T))] - E[\Phi(X_T^{M,N})]| \leq C \tau^{\frac{3}{4} - \epsilon}.
\] (4.15)

Proof. We define \( \bar{X}_T^{M,N} = X_T^{M,N} - O_T^{M,N} \) and then separate the error into two terms:

\[
E[\Phi(X^N(T))] - E[\Phi(X_T^{M,N})] = \left( E[\Phi(\bar{X}_T^{N}(T)) + O_T^{N}] - E[\Phi(\bar{X}_T^{N}(T) + O_T^{M,N})] \right)
\]

\[
+ \left( E[\Phi(\bar{X}_T^{N}(T) + O_T^{M,N})] - E[\Phi(\bar{X}_T^{M,N} + O_T^{M,N})] \right)
\]

\[=: K_1 + K_2 \] (4.16)

As before, the estimate of \( K_1 \) relies on the second-order Taylor expansion and the Malliavin integration by parts formula,

\[
|K_1| = \left| E[\Phi'(X^N(T)) + O_T^{N}] - E[\Phi'(X^N(T) + O_T^{M,N})] \right|
\]

\[
\leq \left| E\left[ \Phi'(X^N(T))(O_T^{M,N} - O_T^{N}) \right] \right|
\]

\[
+ \int_0^1 \Phi''(X^N(T) + \lambda(O_T^{M,N} - O_T^{N}))(O_T^{M,N} - O_T^{N}, O_T^{M,N} - O_T^{N})(1 - \lambda) \text{d}\lambda \right|
\]

\[
\leq \left| E\left[ \Phi'(X^N(T))(O_T^{M,N} - O_T^{N}) \right] \right| + CE\left[ \| O_T^{M,N} - O_T^{N} \|^2 \right]
\] (4.17)

The desired bound of the second term \( E\left[ \| O_T^{M,N} - O_T^{N} \|^2 \right] \) can be directly obtained by Lemma 4.3.
with \( \beta = 0 \). Thus, we only need to treat the first term as follows.

\[
\left| \mathbb{E} \left[ \Phi'(X^N(T)) \left( \mathcal{O}^M,N_T - \mathcal{O}^N_T \right) \right] \right|
\]

\[
= \left| \mathbb{E} \int_0^T \left\langle E_N(T - s) - E^{M-[s]}_{\tau,N}, \mathcal{D}_s \Phi'(X^N(T)) \right\rangle \, ds \right|
\]

\[
\leq \mathbb{E} \int_0^T \left\| E_N(T - s) - E^{M-[s]}_{\tau,N} \right\|_{L^2} \left\| \Phi''(X^N(T)) \mathcal{D}_s X^N(T) \right\|_{L^2} \, ds
\]

\[
\leq C \mathbb{E} \int_0^T \left\| (E_N(T - s) - E^{M-[s]}_{\tau,N}) \right\|_{L(H)} \left\| \Phi''(X^N(T)) \right\|_{L(H)} \left\| \mathcal{D}_s X^N(T) \right\|_{L(H)} \, ds
\]

\[
\leq C t^{1-\epsilon} \int_0^T (T - s)^{-1+\frac{3}{2}} \, ds
\]

\[
\leq C t^{1-\epsilon},
\]

where (2.30), (1.4), Proposition 3.3 and (iv) in Lemma 4.2 with \( \mu = 4 - 2\epsilon \) were used.

Next, it suffices to bound \( \| \bar{X}^N(T) - \bar{X}^M_N \|_{L^2(\Omega, H)} \) to estimate \( K_2 \). To this end, we introduce an auxiliary process \( Y^M_{t_m} \):

\[
Y^M_{t_m} = E^m_{\tau,N} P_N X_0 - \tau \sum_{j=1}^m E^{m-j+1}_{\tau,N} APF(X^N(t_j)) + \sum_{j=1}^m E^{m-j+1}_{\tau,N} P_N \Delta W_j,
\]

and define \( \bar{Y}^M_{t_m} = Y^M_{t_m} - \mathcal{O}^M_{t_m} \). A standard argument gives

\[
\| Y^M_{t_m} \|_{L^p(\Omega, H^2)} < \infty.
\]

Subsequently, by triangle inequality,

\[
\| \bar{X}^N(T) - \bar{X}^M_N \|_{L^2(\Omega, H)} \leq \| \bar{X}^N(T) - \bar{Y}^M_N \|_{L^2(\Omega, H)} + \| \bar{Y}^M_N - \bar{X}^M_N \|_{L^2(\Omega, H)}.
\]

The error term \( \| \bar{X}^N(T) - \bar{Y}^M_N \|_{L^p(\Omega, H)} \) can be further divided into three terms,

\[
\| \bar{X}^N(T) - \bar{Y}^M_N \|_{L^p(\Omega, H)} = \left\| (E(T) P_N - E^M_{\tau,N}) X_0 - \left( \int_0^T E(T - s) P_N APF(X^N(s)) \, ds 
\right.ight.
\]

\[
- \tau \sum_{j=1}^M E^{M-j+1}_{\tau,N} APF(X^N(t_j)) \right\|_{L^p(\Omega, H)}
\]

\[
\leq \left\| (E(T) P_N - E^M_{\tau,N}) X_0 \right\|_{L^p(\Omega, H)}
\]

\[
+ \left\| \int_0^T (E(T - s) P_N - E^{M-[s]}_{\tau,N}) APF(X^N(s)) \, ds \right\|_{L^p(\Omega, H)}
\]

\[
+ \left\| \int_0^T E^{M-[s]}_{\tau,N} APF(X^N(s)) - F(X^N([s])) \, ds \right\|_{L^p(\Omega, H)}
\]

\[
=: K_{21} + K_{22} + K_{23}.
\]
By (ii) of Lemma 4.2 with $\beta = 4$ and Assumption 1.4, we deduce
\[ K_{21} \leq C \tau \|X_0\|_4 \leq C \tau. \quad (4.23) \]

Concerning the term $K_{22}$, by applying (3.9) and (iv) of Lemma 4.2 with $\mu = \frac{7}{2}$, we observe that
\[ K_{22} \leq \int_0^T \|(E(T - s)P_N - E_{r,N}^{M-[s]}APF(X^N(s)))\|_{L^p(\Omega, H)} \, ds \]
\[ \leq C \tau \frac{3}{2} \int_0^T (T - s)^{-\frac{7}{2}} ds \cdot \sup_{s \in [0,T]} \|F(X^N(s))\|_{L^{2p}(\Omega, H^2)} \]
\[ \leq C \tau \frac{3}{4}. \quad (4.24) \]

To handle $K_{23}$, we decompose it into four terms with the aid of the Taylor formula and the mild form satisfied by $X^N(t)$,
\[ K_{23} \leq \left\| \int_0^T E_{r,N}^{M-[s]} AP \left( F'(X^N(s))(E([s] - s) - I)X^N(s) \right) ds \right\|_{L^p(\Omega, H)} \]
\[ + \left\| \int_0^T E_{r,N}^{M-[s]} AP \left( F'(X^N(s)) \int_s^{[s]} E([s] - r)P_N APF(X^N(r))dr \right) ds \right\|_{L^p(\Omega, H)} \]
\[ + \left\| \int_0^T E_{r,N}^{M-[s]} AP \left( F'(X^N(s)) \int_s^{[s]} E([s] - r)P_N dW(r) \right) ds \right\|_{L^p(\Omega, H)} \]
\[ + \left\| \int_0^T E_{r,N}^{M-[s]} AP \left( \int_0^1 F''(X^N(s) + \lambda(X^N([s]) - X^N(s))) \right) \right. \]
\[ \left. \left( X^N([s]) - X^N(s), X^N([s]) - X^N(s) \right) (1 - \lambda)d\lambda \right) ds \right\|_{L^p(\Omega, H)} \]
\[ =: K_{231} + K_{232} + K_{233} + K_{234}. \]

The smoothness property of $E_{r,N}^m$ in (4.4), (2.11), (3.20) and the regularity of $X^N(t)$ lead to
\[ K_{231} = \left\| \int_0^T E_{r,N}^{M-[s]} A^{\frac{3}{2}} A^{-\frac{1}{2}} P \left( F'(X^N(s))(E([s] - s) - I)X^N(s) \right) ds \right\|_{L^p(\Omega, H)} \]
\[ \leq C \int_0^T (T - [s])^{-\frac{3}{2}} \left( 1 + \|X^N(s)\|_2^2 \right) \cdot \|(E([s] - s) - I)X^N(s)\|_{-1+2\epsilon} L^p(\Omega, \mathbb{R}) \, ds \]
\[ \leq C \tau \frac{3}{4} \epsilon. \quad (4.26) \]
Following similar approach as above and utilizing (3.9) yield

\[
K_{232} = \left\| \int_0^T E_{r,N}^{M-[s]} A P F(X^N(s)) \int_s^{[s]} E([s] - r) P_N A P F(X^N(r)) dr \right\|_{L^p(\Omega, H)}
\leq C \int_0^T (T - [s])^{-\frac{3}{2}} \left\| (1 + \|X^N(s)\|_2^2) \times \int_s^{[s]} \|E([s] - r) P_N A P F(X^N(r))\|_{-1+2\epsilon} \right\|_{L^p(\Omega, \mathbb{R})} ds
\leq C \tau \int_0^T (T - [s])^{-\frac{3}{2}} \|F(X^N(r))\|_{L^2(\Omega, H^2)} ds
\leq C \tau.
\]  

(4.27)

From the stochastic Fubini theorem, the Burkholder–Davis–Gundy inequality and the Hölder inequality, it follows that

\[
K_{233} = \left\| \sum_{j=1}^M \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} 1_{[s,t_j]}(r) E_{r,N}^{M-[s]} A P F'(X^N(s)) E([s] - r) P_N dW(r) ds \right\|_{L^p(\Omega, H)}
= \left\| \sum_{j=1}^M \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} 1_{[s,t_j]}(r) E_{r,N}^{M-[s]} A P F'(X^N(s)) E([s] - r) P_N ds dW(r) \right\|_{L^p(\Omega, H)}
\leq \left( \sum_{j=1}^M \int_{t_{j-1}}^{t_j} \right\| \int_{t_{j-1}}^{t_j} 1_{[s,t_j]}(r) E_{r,N}^{M-[s]} A P F'(X^N(s)) E([s] - r) \right\|_{L^p(\Omega, L^2)}^2 ds dr \right)^{\frac{1}{2}}
\leq C \tau^\frac{1}{2} \cdot \left( \sum_{j=1}^M \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \|E_{r,N}^{M-[s]} A^\frac{1}{2} A^\frac{1}{2} P F'(X^N(s)) E([s] - r)\|_{L^p(\Omega, L^2)}^2 ds dr \right)^{\frac{1}{2}}
\leq C \tau^\frac{1}{2} \cdot \left( \sum_{j=1}^M \int_{t_{j-1}}^{t_j} (T - [s])^{-\frac{3}{2}} (1 + \|X^N(s)\|_{L^2(\Omega, H^2)}^2) ([s] - r)^{-\frac{3}{2}} ds dr \right)^{\frac{1}{2}}
\leq C \tau^\frac{1}{2} \cdot \left( \sum_{j=1}^M \int_{t_{j-1}}^{t_j} (T - [s])^{-\frac{3}{2}} \int_{t_{j-1}}^{t_j} ([s] - r)^{-\frac{1}{2}} dr \right)^{\frac{1}{2}}
\leq C \tau^\frac{3}{4}.
\]  

(4.28)

Owing to Hölder’s inequality, the Sobolev embedding theorem $H^\delta \subset V$ for $\delta > \frac{1}{2}$ and Theorem
we obtain
\[
K_{234} = \left\| \int_0^T E_{\tau,N}^{M-[s]} A P \left( \int_0^1 F''(X^N(s) + \lambda(X^N([s]) - X^N(s))) \right) (X^N([s]) - X^N(s), X^N([s]) - X^N(s))(1 - \lambda) d\lambda \right\|_{L^p(\Omega,H)}
\]
\[
\leq C \int_0^T (T - [s])^{-\frac{d}{2}} \|X^N([s]) - X^N(s)\|^2_{L^p(\Omega,H)} \left( 1 + \sup_{s \in [0,T]} \|X^N(s)\|_{L^2(\Omega,V)} \right) ds
\]
\[
\leq C \tau.
\]

Therefore, gathering the above estimates together gives
\[
\|\tilde{X}^N(T) - \tilde{Y}^M_N\|_{L^p(\Omega,H)} \leq C \tau^{\frac{1}{2} - \epsilon}.
\]
Finally, we turn our attention to the estimate of the error term \(\|\tilde{Y}^M_N - \tilde{X}^M_N\|_{L^2(\Omega,H)}\), which satisfies
\[
\tilde{Y}^M_N - \tilde{X}^M_N + \tau \sum_{j=1}^M E_{\tau,N}^{M-j+1} P_N A P(F(X^N(t_j)) - F(X^M_{t_j})) = 0
\]
Denote \(e^{M,N}_t := \tilde{Y}^M_N - \tilde{X}^M_N\). we reformulate (4.31) as
\[
e^{M,N}_t - e^{M,N}_{t-1} + \tau A^2 e^{M,N}_t = \tau P_N A P(F(X^M_{t,m}) - \tau P_N A P(F(X^N(t_m))).
\]
Multiplying both sides by \(A^{-1} e^{M,N}_t\) shows
\[
\langle e^{M,N}_t - e^{M,N}_{t-1}, A^{-1} e^{M,N}_t \rangle + \tau \langle A^2 e^{M,N}_t, A^{-1} e^{M,N}_t \rangle
\]
\[
= \tau \langle F(X^M_{t,m}) - F(X^N(t_m)), e^{M,N}_t \rangle
\]
\[
= \tau \langle F(X^M_{t,m} + \mathcal{O}^{M,N}_{t,m}) - F(X^N(t_m) + \mathcal{O}^{N}_{t,m}), e^{M,N}_t \rangle
\]
\[
= \tau \langle F(X^M_{t,m} + \mathcal{O}^{M,N}_{t,m}) - F(\tilde{Y}^M_{t_m} + \mathcal{O}^{M,N}_{t_m}), e^{M,N}_t \rangle
\]
\[
+ \tau \langle F(X^N(t_m) + \mathcal{O}^{N}_{t,m}) - F(\tilde{X}^N(t_m) + \mathcal{O}^{N}_{t,m}), e^{M,N}_t \rangle.
\]

Thus, using similar approach as in (3.27) and by \(\langle e^{M,N}_t - e^{M,N}_{t-1}, A^{-1} e^{M,N}_t \rangle \geq \frac{1}{2} (\|e^{M,N}_t\|^2 - \|e^{M,N}_{t-1}\|^2)\), we further obtain
\[
\frac{1}{2} (\|e^{M,N}_{t,m}\|^2 - \|e^{M,N}_{t,m-1}\|^2) + \tau \|e^{M,N}_{t,m}\|^2 \leq \frac{3}{4} \tau \|e^{M,N}_{t,m}\|^2 + \frac{9}{8} \tau \|e^{M,N}_{t,m-1}\|^2
\]
\[
+ C \tau \|F(Y^{M,N}_{t_m} + \mathcal{O}^{M,N}_{t_m}) - F(\tilde{X}^N(t_m) + \mathcal{O}^{N}_{t_m})\|^2
\]
\[
+ C \tau \|F(X^N(t_m) + \mathcal{O}^{N}_{t,m}) - F(\tilde{X}^N(t_m) + \mathcal{O}^{N}_{t_m})\|^2.
\]
By iteration in $m$ and by the Gronwall inequality, we obtain
\[
\|e_{m}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2} + \frac{1}{2} \tau \sum_{j=1}^{m} \|e_{t_{j}}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2} \leq C \tau \sum_{j=1}^{m} \left( \|F(Y_{t_{m}}^{M,N} + O_{t_{m}}^{M,N}) - F(X_{m}^{N}(t_{m}) + O_{t_{m}}^{M,N})\|^{2} + \|F(X_{m}^{N}(t_{m}) + O_{t_{m}}^{M,N}) - F(X_{m}^{N}(t_{m}) + O_{t_{m}}^{N})\|_{L^{p}(\Omega;\mathbb{R})}^{2} \right).
\]
Taking expectation and employing (2.17), (3.20), (4.30), Lemma 4.3 and regularity of $Y_{t_{m}}^{M,N}$ and $X_{m}^{N}(t)$ result in
\[
\left\|\tau \sum_{j=1}^{m} \|e_{t_{j}}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2}\right\|_{L^{p}(\Omega;\mathbb{R})} \leq C \tau \sum_{j=1}^{m} \left( \|Y_{t_{m}}^{M,N} - X_{m}^{N}(t_{m})\|_{L^{p}(\Omega;\mathbb{R})}^{2} (1 + \|Y_{t_{m}}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2} + \|Y_{m}^{N}(t_{m})\|_{L^{p}(\Omega;\mathbb{R})}^{2} + \|O_{t_{m}}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2}) \right)
\]
\[
+ C \tau \sum_{j=1}^{m} \left( \|O_{t_{m}}^{M,N} - O_{t_{m}}^{N}\|_{L^{p}(\Omega;\mathbb{R})}^{2} (1 + \|Y_{m}^{N}(t_{m})\|_{L^{p}(\Omega;\mathbb{R})}^{2} + \|O_{t_{m}}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}^{2} + \|O_{t_{m}}^{N}\|_{L^{p}(\Omega;\mathbb{R})}^{2}) \right)
\leq C \tau^{\frac{1}{2}} - 2\epsilon.
\]
Furthermore, we split $\|e_{t}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})}$ into three parts,
\[
\|e_{t}^{M,N}\|_{L^{p}(\Omega;\mathbb{R})} \leq \tau \left\| \sum_{j=1}^{M} E_{\tau,N}^{M-j+1} P_{N} \text{AP} \left( F(X_{m}^{N}(t_{j})) - F(X_{j}^{M,N}) \right) \right\|_{L^{p}(\Omega;\mathbb{R})}
\]
\[
\leq \tau \left\| \sum_{j=1}^{M} E_{\tau,N}^{M-j+1} P_{N} \text{AP} \left( F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{N}) - F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{N}) \right) \right\|_{L^{p}(\Omega;\mathbb{R})}
\]
\[
+ \tau \left\| \sum_{j=1}^{M} E_{\tau,N}^{M-j+1} P_{N} \text{AP} \left( F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{N}) - F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{N}) \right) \right\|_{L^{p}(\Omega;\mathbb{R})}
\]
\[
+ \tau \left\| \sum_{j=1}^{M} E_{\tau,N}^{M-j+1} P_{N} \text{AP} \left( F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{N}) - F(X_{m}^{N}(t_{j}) + O_{t_{j}}^{M,N}) \right) \right\|_{L^{p}(\Omega;\mathbb{R})}
\]
\[
=: \text{Err}_1 + \text{Err}_2 + \text{Err}_3.
\]
Taking (4.4), (2.17), (4.30), Hölder’s inequality and moment boundedness of $Y_{t_{m}}^{M,N}$ and $X_{m}^{N}(t)$ into
account, we arrive at
\[
Err_1 = \tau \sum_{j=1}^{M} \left\| E_{\tau,N}^{\tau,M-j+1} P_N A P \left( F(\bar{X}_N(t_j) + O_{t_j}^N) - F(\bar{Y}_t^{M,N} + O_{t_j}^N) \right) \right\|_{L^p(\Omega)} \\
\leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{-\frac{1}{2}} \left\| \bar{X}_N(t_j) - \bar{Y}_t^{M,N} \right\|_{L^2(\Omega)} \\
\left(1 + \left\| \bar{X}_N(t_j) \right\|_{L^2(\Omega)}^2 \right) \right) + \left\| O_{t_j}^N \right\|_{L^2(\Omega)}^2 \bigg) \\
\leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{-\frac{1}{2}} t_{M-j+1}^{-\frac{3}{2}} - \epsilon \\
\leq C \tau \frac{3}{2} - \epsilon.
\]

Analogously to the above estimate but now with (3.20) instead, we derive
\[
Err_2 = \tau \sum_{j=1}^{M} \left\| E_{\tau,N}^{\tau,M-j+1} P_N A P \left( F(\bar{Y}_t^{M,N} + O_{t_j}^N) - F(\bar{Y}_t^{M,N} + O_{t_j}^N) \right) \right\|_{L^p(\Omega)} \\
\leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{-\frac{3}{2}} \left\| O_{t_j}^N - O_{t_j}^{M,N} \right\|_{L^2(\Omega)} \\
\left(1 + \left\| O_{t_j}^N \right\|_{L^2(\Omega)}^2 \right) + \left\| O_{t_j}^{M,N} \right\|_{L^2(\Omega)}^2 \bigg) \\
\leq C \tau \frac{3}{2} - \epsilon.
\]

At last, combining (3.19), Hölder’s inequality, (4.36) and regularity of \(Y_t^{M,N}\) and \(X_t^{M,N}\) leads to
\[
Err_3 = \tau \sum_{j=1}^{M} \left\| E_{\tau,N}^{\tau,M-j+1} P_N A P \left( F(\bar{Y}_t^{M,N} + O_{t_j}^N) - F(\bar{X}_t^{M,N} + O_{t_j}^N) \right) \right\|_{L^p(\Omega)} \\
\leq C \left\| \tau \sum_{j=1}^{M} t_{M-j+1}^{-\frac{1}{2}} \left\| F(\bar{Y}_t^{M,N} + O_{t_j}^N) - F(\bar{X}_t^{M,N} + O_{t_j}^N) \right\|_1 \right\|_{L^p(\Omega)} \\
\leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{-\frac{1}{2}} \left\| e_{t_j}^{M,N} \right\|_1 \left(1 + \left\| \bar{Y}_t^{M,N} \right\|_2^2 + \left\| \bar{X}_t^{M,N} \right\|_2^2 \bigg) \\
\leq C \tau \sum_{j=1}^{M} \left\| e_{t_j}^{M,N} \right\|_1 \left(1 + \left\| \bar{Y}_t^{M,N} \right\|_2^2 + \left\| \bar{X}_t^{M,N} \right\|_2^2 \bigg) \\
\leq C \tau \frac{3}{2} - \epsilon.
\]

This completes the proof of this theorem. \(\square\)
Remark 4.5. As a by-product of the weak error analysis, one can easily obtain the rate of the strong error,

$$\|X(t) - X_t^{M,N}\|_{L^p(\Omega; H)} \leq \|\tilde{X}(t) - \tilde{X}_t^{M,N}\|_{L^p(\Omega; H)} + \|O_t - O_t^{M,N}\|_{L^p(\Omega; H)},$$

(4.41)

which is lower than the weak error, due to the presence of the second error. This is also a contribution of our work.

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