Cosmological renormalization of model parameters in the second-order perturbation theory

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It is shown that the serious problem on the cosmological tension between the direct measurements of the Hubble constant at present and the constant derived from the Planck measurements of the CMB anisotropies can be solved by considering the renormalized model parameters. They are deduced by taking the spatial average of second-order perturbations in the flat Λ-CDM model, which includes random adiabatic fluctuations.

1. Introduction

High precision cosmology has started with the measurements of fluctuations in the cosmic microwave background radiation (CMB) by WMAP[1] and Planck[2, 3] collaboration. Their studies have been useful to determine the Hubble constant and the cosmological parameters. However, it has been found that there is a tension between the Hubble constant ($H_0$) due to the Planck measurements of the CMB anisotropies and that due to the direct measurements.

Due to the Planck measurements[2, 3], we have the Hubble constant

$$H_0 = 67.3 \text{ km s}^{-1}\text{Mpc}^{-1}. \quad (1)$$

On the other hand, there are many direct measurements[4], and the three examples of direct measurements give us[5–7]

$$H_0 = 73.8, \ 74.3, \ \text{and} \ 78.7 \text{ km s}^{-1}\text{Mpc}^{-1}. \quad (2)$$

These data show that there is a large discrepancy of $9.7\% \sim 16.9\%$. To solve this problem, various mechanisms have been proposed, such as models of decaying dark matter[8], a local void model[9], and the dark radiation[4]. In this paper, it is shown that the tension can be solved by considering the renormalized model parameters which are deduced by taking the spatial average of second-order perturbations in the flat Λ-CDM model, which includes random adiabatic fluctuations.

In Sect. 2, we show the general-relativistic second-order perturbation theory in the flat Λ-CDM model, which was derived by the present author[10]. In Sect. 3, we derive spatial averages of second-order density and metric perturbations, and in Sect. 4, we define the renormalized Hubble constant due to the average second-order metric perturbations, and show that it is consistent with the measured Hubble constants and their various observed values correspond to the different upper limits of wave-numbers of perturbations which can be included in the renormalized perturbations. Other renormalized model parameters are also derived due to the average of second-order density perturbations. In Sect. 5, the
renormalization in the past is described and in Sect. 6, the concluding remarks are given. In Appendix A, we show the definition of various quantities included in the expressions for the second-order metric perturbations. In Appendix B, we show the model parameters corresponding to Hubble constants in Eq. (2).

2. Background and the perturbation theory

The background universe is expressed by a spatially flat model with the line-element

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \]  

(3)

where the Greek and Roman letters denote 0, 1, 2, 3 and 1, 2, 3, respectively. The conformal time \( \eta(= x^0) \) is related to the cosmic time \( t \) by \( dt = a(\eta)d\eta \).

In the comoving coordinates, the velocity vector and energy-momentum tensor of pressureless matter are

\[ u^0 = 1/a, \quad u^i = 0 \]  

(4) and

\[ T^0_0 = -\rho, \quad T^0_i = 0, \quad T^i_j = 0, \]  

(5)

where \( \rho \) is the matter density.

From the Einstein equations, we obtain

\[ \rho a^2 = 3(a'/a)^2 - \Lambda a^2, \]  

(6) and

\[ \rho a^3 = \rho_0, \]  

(7)

where a prime denotes \( \partial/\partial \eta \), \( \Lambda \) is the cosmological constant, and \( \rho_0 \) is an integration constant, and we use the units \( 8\pi G = c = 1 \) for the gravitational constant \( G \) and the light velocity \( c \).

The Hubble parameter \( H \) is defined as

\[ H = \dot{a}/a = a'/a^2. \]  

(8)

Eq. (6) gives

\[ H^2 = \frac{1}{3}(\rho + \Lambda), \]  

(9) which is also expressed as

\[ H^2 = H^2_0 \left( \Omega_M a^{-3} + \Omega_\Lambda \right), \]  

(10)

where \( H_0 \) is \( H \) at the present epoch \( t_0 \) and \( a_0 \equiv a(t_0) = 1 \), and

\[ \Omega_M = \frac{8\pi G \rho_0}{3H^2_0} = \frac{1}{3} + \frac{\rho_0}{H^2_0} \quad \text{and} \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2_0} = \frac{1}{3} \frac{\Lambda}{H^2_0}. \]  

(11)

In this paper we adopt the following background values:

\[ H_0 = 67.3 \text{ km s}^{-1}\text{Mpc}^{-1} \text{ and } (\Omega_M, \Omega_\Lambda) = (0.22, 0.78). \]  

(12)

The significance of these values will be explained later.

Next let us show the first-order density perturbations. The perturbations of metric, matter density and velocity are represented by \( \delta_1 g_{\mu\nu} \equiv h_{\mu\nu} \), \( \delta_1 \rho \), and \( \delta_1 u^\mu \). When we adopt the
synchronous coordinates (useful in the pressureless case), the metric perturbations satisfy the condition

\[ h_{00} = 0 \quad \text{and} \quad h_{0i} = 0. \]  (13)

The first-order perturbations are classified into the growing case and the decaying case. Both cases are found in the previous paper [10]. Here we show only those in the growing case, which are used in this paper:

\[ h^j_i = \delta^j_i F + P(\eta)F^j_i, \]
\[ \delta u^i_0 = 0, \quad \delta u^i = 0, \]  (14)
\[ \delta \rho/\rho = \frac{1}{\rho a^2} \left( \frac{a'}{a} P' - 1 \right) \Delta F, \]  (15)

where \( F \) is an arbitrary function of spatial coordinates \( x^1, x^2, \) and \( x^3 \), \( \Delta \equiv \nabla^2 \), \( h^j_i = g^j_l h^l_i \) and \( P(\eta) \) satisfies

\[ P'' + \frac{2a'}{a} P' - 1 = 0. \]

Its solution is expressed as

\[ P = \int_0^\eta a^{-2}(\eta') \int_0^{\eta'} a^{2}(\eta''). \]  (16)

After a partial integration, we obtain

\[ (H_0)^2 P = -\frac{2}{3\Omega_M} a^{-3/2} \sqrt{\Omega_M + \Omega_\Lambda a^3} \int_0^a \int_0^{b^3/2/\sqrt{\Omega_M + \Omega_\Lambda b^3}} \int_0^b \int_0^{1/2/\sqrt{\Omega_M + \Omega_\Lambda b^3}}. \]  (17)

The three-dimensional covariant derivatives \( |i \) are defined in the space with metric \( dl^2 = \delta_{ij} dx^i dx^j \) and their suffices are raised and lowered using \( \delta_{ij} \), so that their derivatives are equal to partial derivatives, i.e. \( F^j_i = F_{ij} \), where \( F, i \equiv \partial F/\partial x^i \).

The second-order perturbations were derived in the previous paper [10]. This is a simple extension of my paper [11] which derived the second-order perturbations in the case of zero \( \Lambda \) in the Lifshitz formalism with iterative second-order terms. The results in the latter paper were later derived independently by Russ et al. [12] and by Matarrese et al. [13] in the different formalisms, and the validity of this theory has been confirmed. Here let us show their components \( \delta_2 g_{\mu\nu}(\equiv \ell_{\mu\nu}), \delta_2 \rho, \) and \( \delta_2 u^\mu \) in the case of nonzero \( \Lambda \), where the total perturbations are

\[ \delta g_{\mu\nu} = h_{\mu\nu} + \ell_{\mu\nu}, \]
\[ \delta u^\mu = \delta u^\mu + \delta u^\mu, \]
\[ \delta \rho/\rho = \delta \rho/\rho + \delta \rho/\rho. \]  (18)

Here assuming the synchronous gauge condition, we have

\[ \ell_{00} = 0 \quad \text{and} \quad \ell_{0i} = 0. \]  (19)
The perturbations in the growing case are expressed as
\[ \ell^i_j = P(\eta)L^i_j + P^2(\eta)M^i_j + Q(\eta)N^i_j + C^i_j, \]  
(20)
where \( N^i_j = \delta^i_i N^i_i = N_{ij} \) and \( Q(\eta) \) satisfies
\[ Q'' + \frac{2a'}{a}Q' = P - \frac{5}{2}(P')^2. \]  
(21)
The expressions of other quantities \( L^i_j, M^i_j, N^i_j, \) and \( C^i_j \) are shown in Appendix A.

The velocity and density perturbations are found to be
\[ \frac{1}{2} \delta u^0 = 0, \quad \frac{1}{2} \delta u^i = 0 \]  
(22)
and
\[ \frac{1}{2} \frac{\delta \rho}{\rho} = \frac{1}{2a^2} \left\{ 1 \left[ (1 - \frac{a'}{a})P'(3F_i F_j + 8F_i \Delta F) + \frac{1}{2}P[\Delta F^2 + F_{kl} F_{kl}] \right] \\
+ \frac{1}{4} \left[ (P')^2 \frac{2a'}{7a}Q' \right] (\Delta F)^2 \right\}. \]  
(23)
The gauge used here is not only synchronous, but also comoving (cf. Eqs. (14) and (22)). The above perturbations are therefore physical density perturbations which are measured by comoving observers.

It is to be noticed that the present general-relativistic gravitational equations are nonlinear and applicable also in the super-horizon case, in contrast to linear gravitational equations in the Newtonian treatment [14, 15], and so the cosmological result in the following sections cannot be derived in the Newtonian treatment, because of their difference.

It is discussed in Sect. 6 what we should do, in order to obtain the consistency between the general-relativistic treatment and the Newtonian cosmology.

3. Average second-order perturbations

We consider random perturbations given by
\[ F(k) = \int dk \alpha(k) e^{ikx}, \]  
(24)
where \( \alpha(k) \) is a random variable and the average of \( F \) expressed as \( \langle F \rangle \) vanishes. Here we assume the average process with a power spectrum \( P_F(k) \), given by
\[ \langle \alpha(k)\alpha(k') \rangle = (2\pi)^{-2} P_F(k) \delta(k + k'). \]  
(25)
Here we have
\[ \langle \delta_1 \rho/\rho \rangle = 0 \]  
(26)
for the first-order density perturbation. For the second-order perturbations, we have
\[ F_i F_j = -\int\int dkdk' \langle \alpha(k)\alpha(k') \rangle kk' e^{i(k+k')x}, \]  
\[ F \Delta F = -\int\int dkdk' \langle \alpha(k)\alpha(k') \rangle k^2 e^{i(k+k')x}, \]  
(27)
\[ \text{and} \quad \frac{1}{2} \frac{\delta \rho}{\rho} = \frac{1}{2a^2} \left\{ 1 \left[ (1 - \frac{a'}{a})P'(3F_i F_j + 8F_i \Delta F) + \frac{1}{2}P[\Delta F^2 + F_{kl} F_{kl}] \right] \\
+ \frac{1}{4} \left[ (P')^2 \frac{2a'}{7a}Q' \right] (\Delta F)^2 \right\}. \]  
(23)
so that we obtain
\[ \langle F_i F_i \rangle = \langle F \Delta F \rangle = (2\pi)^{-2} \int dk \ k^2 P_F(k). \]  

Similarly, we have
\[ F_{ij} F_{ij} = \int \int dk dk' \langle \alpha(k) \alpha(k') \rangle (kk')^2 e^{i(k+k')x}, \]
\[ (\Delta F)^2 = \int \int dk dk' \langle \alpha(k) \alpha(k') \rangle k^2(k')^2 e^{i(k+k')x}, \]
so that we obtain
\[ \langle F_{ij} F_{ij} \rangle = \langle (\Delta F)^2 \rangle = (2\pi)^{-2} \int dk \ k^4 P_F(k). \]  

3.1. Second-order density perturbations

It follows therefore from Eq. (23) that
\[ \langle \delta \rho / \rho \rangle = \frac{1 - \frac{a'}{a} P'}{2\rho a^2} (2\pi)^{-2} \int dk k^2 P_F(k) + P \int dk k^4 P_F(k). \]  

Here \( F \) is related to the curvature fluctuation \( R \) by \( F = 2R \), and so we have the relation
\[ P_F(k) = 4 P_R(k), \]
where \( P_R \) is expressed using the power spectrum[16, 17] as
\[ P_R = 2\pi^2 P_{R0} k^{-3}(k/k_eq)^{n-1} T_s^2(k/k_eq) \]
and \( P_{R0} = 2.2 \times 10^{-9} \) according to the result of Planck measurements.[2, 3] The transfer function \( T_s(x) \) is expressed as a function of \( x = k/k_eq \), where
\[ k_eq (\equiv a_eq H_eq) = 219 (\Omega_M h) H_0 = 32.4 \ h_0. \]
Here \( H_0 (\equiv 100h) \) is the present background Hubble constant, \( (a_eq, H_eq) \) is \( (a, H) \) at the epoch of equal energy density, and \( (\Omega_M, h) = (0.22, 0.673) \) (given in Eq. (12)).

Moreover, we assume \( n = 1 \) here and in the following. Then we obtain for arbitrary \( a \)
\[ \langle \delta \rho / \rho \rangle = \frac{4\pi}{3} \frac{32.4^4 P_{R0}}{(\Omega_M/a + \Omega_\Lambda a^2)} \left[ \frac{1 - Y(a)}{\Omega_M/a + \Omega_\Lambda a^2} \right] \left[ \frac{11}{2} 32A^{-2} A + Z(a) B \right], \]
where \( \tilde{\rho} \equiv \rho + \Lambda \), and \( A \) and \( B \) are expressed as
\[ A \equiv \int_{x_{min}}^{x_{max}} \ dx \ x \ T_s^2(x), \quad B \equiv \int_{x_{min}}^{x_{max}} \ dx \ x^3 \ T_s^2(x) \]
using the transfer function \( T_s(x) \) for the interval \( (x_{max}, x_{min}) \). Here we have
\[ Y(a) \equiv \frac{a'}{a} P', \quad Z(a) \equiv (H_0)^2 P, \]
and using Eq.(17), we obtain
\[ Y(a) = a^{-5/2}(\Omega_M + \Omega_\Lambda a^3)^{1/2} I(a), \quad Z(a) = \frac{2}{3\Omega_M} a[1 - Y(a)], \]
where
\[ I(a) \equiv \int_0^a \ db \ b^3/(\Omega_M + \Omega_\Lambda b^3)^{1/2}. \]
3.2. Second-order metric perturbations

The second-order perturbation of the scale factor $\delta_2 a$ can be derived as follows using the metric second-order perturbations $l_{ij}$, which are given in Eqs.(20) and (21), and in Appendix A. The averaging of second-order metric perturbations leads to

$$\delta_2(a^2) = \frac{1}{3} \langle l_{im}^m \rangle,$$  \hspace{1cm} (40)

where we have

$$\langle l_{ij} \rangle = P(\eta)\langle L_{ij} \rangle + P^2(\eta)\langle M_{ij} \rangle + Q(\eta)\langle N_{ij} \rangle + \langle C_{ij} \rangle.$$  \hspace{1cm} (41)

Since $L_{ij}^i = L_{ij}$, $M_{ij}^i = M_{ij}$ and $L_{ii}^i = -\frac{1}{2} \left[ 2F\Delta F + \frac{3}{2} F_i F_i \right]$, $M_{ii}^i = -\frac{1}{28} \left[ 10(F_{,ii})^2 - 3(\Delta F)^2 \right]$, $\Delta N = -\frac{1}{28} \left[ (\Delta F)^2 - F_{,kl} F_{,kl} \right]$, $\Box C_{ii} = 0$, we obtain

$$\langle L_{ij}^i \rangle = -\frac{7}{4} \langle F\Delta F \rangle,$$

$$\langle M_{ij}^i \rangle = -\frac{1}{4} \langle (\Delta F)^2 \rangle,$$

$$\langle \Delta N \rangle = \langle C_{ii}^i \rangle = 0.$$  \hspace{1cm} (42)

Then we have using Eqs.(28) and (30)

$$\langle l_{ii} \rangle = \langle l_{ij}^i \rangle = -2\pi 32.4^4 P_{R0} Z(a) [7 \times 32.4^{-2} A + Z(a)B],$$  \hspace{1cm} (44)

where $A, B$ and $Z(a)$ are given by Eqs.(36) and (37).

The line-element can be expressed as

$$ds^2 = -dt^2 + a_{rem}(t)^2 [(dx^1)^2 + (dx^2)^2 + (dx^3)^2],$$  \hspace{1cm} (45)

where the renormalized scale-factor is defined by

$$a_{rem}(t)^2 = a^2 + \delta_2(a^2) = a^2 + \frac{1}{3} \langle l_{ij}^i \rangle,$$  \hspace{1cm} (46)

and the relative difference of scale-factors is given by

$$\xi \equiv a_{rem}(t)/a(t) - 1 = \left[ 1 + \frac{1}{3} \langle l_{ij}^i \rangle/a^2 \right]^{1/2} - 1.$$  \hspace{1cm} (47)

The renormalized redshift $z_{rem}$ corresponding to an arbitrary time $t$ is defined using the scale-factor $a_{rem}$ as

$$1 + z_{rem} \equiv a_{rem}(t_0)/a_{rem}(t) = \frac{1 + \xi(t_0)}{1 + \xi(t)} (1 + z),$$  \hspace{1cm} (48)

where $t_0$ denotes the present epoch, the background redshift $z$ is $1/a - 1$, and $\xi(t)$ is defined by Eq.(47).
The square of the background Hubble parameter $H$ is $(\dot{a}/a)^2$ and its perturbation is given by
\[ \delta_2(H^2) = \delta_2(\dot{a}^2)/a^2 - (\dot{\delta}_2 a^2)/a^4, \]
so that
\[ \frac{\delta_2(H^2)}{H^2} = \frac{2}{3} \left[ \frac{\langle \delta^2 \rangle}{2a^2} - \langle \delta \dot{\delta} \rangle / a^4 \right]. \]
From Eqs.(44) and (50), we obtain
\[ \frac{\delta_2(H^2)}{H^2} = \frac{4\pi}{3} 32.4^4 \mathcal{P}_{R0} \frac{1}{a^2} \left[ T(32.4)^{-2} \left( Z - \frac{Y(a)a}{2(\Omega_M + \Omega_M a^3)} \right) A + Z \left( \frac{1}{\Omega_M + \Omega_M a^3} \right) B \right]. \]
where $Y(a)$ and $Z(a)$ are given by Eq.(38).

### 3.3. Average perturbations of model parameters

In this paper we assume the simplest transfer function (BBKS) for cold matter, adiabatic fluctuations, given by [18–20]
\[ T_s(x) = \frac{\ln(1 + 0.171x)}{0.171x} [1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4]^{-1/4}. \]
This function has the peak around $x \simeq 1$, so that the upper and lower limits ($x_{\text{max}}$ and $x_{\text{min}}$) in the integrals $A$ and $B$ in Eq. (35) should have the values such as $x_{\text{max}} \sim 6 (> 1)$ and $x_{\text{min}} \sim 0.01 (\ll 1)$. Here $A$ and $B$ depend sensitively on $x_{\text{max}}$, but not on $x_{\text{min}}$.

In order to find the best value of $x_{\text{max}}$, we derive a length $L_{\text{max}}$ corresponding to $k_{\text{max}}$. Using Eq.(34), we have
\[ L_{\text{max}} \equiv 2\pi/k_{\text{max}} = 102/h \text{ Mpc} \]
for $x_{\text{max}} (\equiv k_{\text{max}}/k_{\text{eq}}) = 5.7$. This $L_{\text{max}}$ corresponds to the cosmological distance, over which the smooth observations on cosmological scales may be possible.

So we adopt $x_{\text{max}} = 5.7$ and $x_{\text{min}} = 0.1$. Then we obtain
\[ A = 2.22, \quad B = 20.95, \]
and
\[ \langle \delta \rho / \hat{\rho} \rangle = 0.121 \quad \text{and} \quad \frac{\delta_2(H^2)}{H^2} = 0.210 \]
at the present epoch ($a = 1$).

In similar cases with $B \approx 10A$, the terms with $A$ in Eqs. (35) and (51) are negligible, and then we have
\[ \langle \delta \rho / \hat{\rho} \rangle \simeq \frac{8\pi}{9} 32.4^4 \mathcal{P}_{R0} \frac{1}{a^2} \left[ \frac{1}{\Omega_M + \Omega_M a^3} \right] B, \]
and
\[ \frac{\delta_2(H^2)}{H^2} \simeq \frac{4\pi}{3} 32.4^4 \mathcal{P}_{R0} \frac{1}{a^2} Z(a) \left[ Z(a) - \frac{Y(a)a}{\Omega_M + \Omega_M a^3} \right] B. \]

By the way, we derive $(\delta_1 \rho / \hat{\rho})^2$ to estimate the dispersion of $\langle \delta_2 \rho / \hat{\rho} \rangle$:
\[ \langle (\delta \rho / \hat{\rho})^2 \rangle = \left[ \frac{1 - Y(a)}{\rho a^2} \right]^2 (2\pi)^{-2} \int dk k^4 \mathcal{P}_F(k), \]
so that we obtain
\[ \langle (\delta \rho / \hat{\rho})^2 \rangle \simeq \frac{8\pi}{9} 32.4^4 \mathcal{P}_{R0} \left[ \frac{1}{\Omega_M + \Omega_M a^3} \right]^2 B \simeq \langle \delta \rho / \hat{\rho} \rangle \frac{\Omega_M}{\Omega_M + \Omega_M a^3}. \]
4. Renormalization of model parameters

Now let us consider the renormalization of the background density and the Hubble constant. Since \( \langle \delta^2 \rho/\rho \rangle \) is spatially constant, we may assume that it is a part of the background density. Here we regard

\[
\rho_{\text{rem}} \equiv \rho + \langle \delta \rho \rangle
\]

as the renormalized background density.

Similarly \( \langle \delta^2 H^2 \rangle \) is spatially constant, and so we can regard

\[
H_{\text{rem}} \equiv \left[ H^2 + \langle \delta^2 (H^2) \rangle \right]^{1/2}
\]

as the renormalized background Hubble parameter.

For \( \langle \delta^2 H^2 \rangle / H^2 \) in Eq. (55) and \( H = H_0 \) in Eq. (12), we obtain at present epoch

\[
H_{\text{rem}} = 74.0 \text{ km s}^{-1} \text{ Mpc}^{-1},
\]

which is equal approximately to the measured Hubble constants.[6, 7] It is found, therefore, that the renormalized Hubble constant \( H_{\text{rem}} \) may be consistent with the directly measured Hubble constants.

The model parameters \( \Omega_M \) and \( \Omega_{\Lambda} \) describe the evolution of the background universe. But since our real universe is described using the renormalized quantities \( H_{\text{rem}} \) and \( \rho + \langle \delta \rho \rangle \) in the place of the background Hubble constant \( H \) and \( \rho \), we may obtain the following new set of model parameters :

\[
(\Omega_M)_{\text{rem}} = \Omega_M \frac{1 + \langle \delta^2 \rho/\rho \rangle}{1 + \langle \delta^2 \rho/\bar{\rho} \rangle}
\]

and

\[
(\Omega_{\Lambda})_{\text{rem}} = \Omega_{\Lambda} \frac{1}{1 + \langle \delta^2 \rho/\bar{\rho} \rangle}.
\]

Using the background model parameters Eq.(12) and \( \langle \delta^2 \rho/\bar{\rho} \rangle \) in Eq. (55), we obtain at the present epoch

\[
(\Omega_M)_{\text{rem}} = 0.305 \quad \text{and} \quad (\Omega_{\Lambda})_{\text{rem}} = 0.695.
\]

The recent observations of the redshift-magnitude relation[21] include many supernova with redshifts \( z = 0.1 - 1.0 \), so that the present Hubble constant \( (H_{\text{rem}}) \) is used, and \( ((\Omega_M)_{\text{rem}}, (\Omega_{\Lambda})_{\text{rem}}) \) (which are consistent with Eq.(65)) are obtained.

As for the observations of baryon acoustic oscillations of CMB (Planck[2, 3]) and large-scale galactic correlations[22, 23], we use the angular distance in the late time model, so that the derived model parameters are not \( (\Omega_M, \Omega_{\Lambda}) \), but \( ((\Omega_M)_{\text{rem}}, (\Omega_{\Lambda})_{\text{rem}}) \), which are given by Eq.(65). On the other hand, the scale of acoustic oscillations is determined at the recombination epoch with \( a(\approx 10^{-3}) \) and so the Hubble constant is given by \( H_0 \) (\( \approx H_{\text{rem}} \) at the recombination epoch), but not \( H_{\text{rem}} \) at the present epoch. So the above renormalized model parameters are consistent with the cosmological observations. [2, 3, 22, 23]

The relative difference of scale-factors \( \xi \) (in Eq.(47)) is \(-0.097\) at present epoch. Moreover, the present values of \( \langle \delta^2 \rho/\bar{\rho} \rangle \) and \( ((\Omega_M)_{\text{rem}}, (\Omega_{\Lambda})_{\text{rem}}) \) in the case when \( H_{\text{rem}} \) is 73.8, 74.3 or 78.7 are shown in Appendix B.
Fig. 1: The second-order perturbations $\langle \delta^2 \rho / \bar{\rho} \rangle$ is expressed as a function of $a$. The scale factor $a$ has 1 at the present epoch.

5. Renormalization of model parameters in the past

In the previous section, we treated the quantities at the present epoch ($a = 1$). Here we consider the quantities at the epochs of $a < 1$. First we calculate $\langle \delta^2 \rho / \bar{\rho} \rangle$ for $a < 1$ using Eq.(35) for $x_{max} = 5.7$ and $x_{min} = 0.01$. Its dependence on $a$ is shown in Fig.1. It is found that $\langle \delta^2 \rho / \bar{\rho} \rangle$ has a peak at around $a \sim 0.65$, but $\langle \delta^2 \rho / \rho \rangle$ increases monotonically, and that $\langle \delta^2 \rho / \bar{\rho} \rangle$ reduces to 0 in the limit of $a \to 0$.

Using Eqs. (51), (61), (63), and (64), moreover, we obtain $H_{rem}$ and $((\Omega_M)_{rem}, (\Omega_\Lambda)_{rem})$ in the past. They are shown in Fig. 2 and Fig. 3. It is found that $H_{rem}$ reduces to $H_0$ (in Eq. (12)), and $((\Omega_M)_{rem}, (\Omega_\Lambda)_{rem})$ reduces to $(\Omega_M, \Omega_\Lambda)$ in the limit of $a \to 0$.

In Fig. 4, the relative difference of scale-factors $\xi$ (in Eq.(47)) is shown and $\xi$ is $-0.097 \sim -0.195$ for $a = 1 \sim 0$, respectively. It is found from Eq.(48) that $z_{rem}$ is larger than $z$. 
Fig. 2: The renormalized Hubble constant $H_{rem}$ is expressed as a function of $a$. The scale factor $a$ has 1 at the present epoch.

6. Concluding remarks

It was found in this paper that the random adiabatic fluctuations bring a kind of energy density which has an influence upon the dynamics of the universe. For its derivation, the nonlinearity of general-relativistic perturbation theory was important.

As $k$ increases in the region of $x \equiv k/k_{eq} > 1$, the amplitude of perturbations decreases rapidly, and the frequency of perturbed objects is so small that they cannot be renormalized as part of the background matter density. The upper limit of $x$ for renormalized perturbations is $x_{\text{max}} (\approx 6)$. Because of their small frequency corresponding to large $k$, the value of $x_{\text{max}}$ has large fluctuations, and this may be the origin of the directional fluctuations of $x_{\text{max}}$ and the measured Hubble constant.

The background model parameters in Eq.(12) are rather different from the renormalized parameters in Eqs.(62) and (65). We should notice that the observed model parameters are the latter ones. The Hubble constant $H_0$ in the Planck measurements (Eq.(1)) is the
Fig. 3: The renormalized density parameter $(\Omega_M)_{rem}$ is expressed as a function of $a$. $(\Omega_\Lambda)_{rem} = 1 - (\Omega_M)_{rem}$. The scale factor $a$ has 1 at the present epoch.

renormalized Hubble constant measured at the early stage ($a \ll 1$), which is approximately equal to the background Hubble constant ($H_0$).

In this paper we adopted tentatively the background model parameters (in Eq. (12)) and the value of $x_{max}$. Their values should be selected so that they may reflect best the real observations of model parameters.

Large-scale perturbations such as $x \equiv k/k_{eq} = 0.01 \sim 6$, which were treated in this paper (cf. Eq.(53)) cross the horizon in the course of their evolution, so that taking the general-relativistic effect into consideration is indispensable for their dynamical analyses, which are not only linear but also on second-order.

In the Newtonian theory, the terms representing the gravitational strength $\xi \equiv GM/(c^2 R)$ are taken only linearly into account, assuming that it is extremely small, where $M$ and $R$ are characteristic mass and length of dynamical objects. In the cosmological
Fig. 4: The relative difference of scale-factors $\xi(t) \equiv a_{\text{rem}}(t)/a(t) - 1$ is expressed as a function of $a$. The scale factor $a$ has 1 at the present epoch.

\[ a_{\text{rem}}/a - 1 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ -0.18 \quad -0.16 \quad -0.14 \quad -0.12 \quad -0.10 \]

In circumstances, however, we have

\[ \xi \sim G\frac{\rho}{H^2} \cdot (\delta \rho/\rho) \sim \delta \rho/\rho, \tag{66} \]

where $H$ is the Hubble parameter, $R \sim c/H$, and $M \sim \rho R^3$. When $|\delta \rho/\rho|$ is not so small, we cannot neglect the nonzero mean of $\xi^2$, which should be treated in the post-Newtonian approximation. In the general-relativistic treatment of second-order perturbations, this nonzero mean is automatically taken into account. In the linear level, we have $\langle \xi \rangle = 0$, but in the second-order, $\langle \xi^2 \rangle \sim \langle (\delta \rho/\rho)^2 \rangle \neq 0$. Moreover, the necessity of considering nonlinear $\xi$ comes also from the energy-momentum conservation law. In order that the Newtonian theory is compatible with the energy-momentum conservation in the same way as in the general-relativistic cosmology, we must add a nonzero term $(\delta \rho/\rho)^2$ to the second-order density perturbation, so as to recover the post-Newtonian terms with $\xi^2$. By this addition, the Newtonian theory may be consistent with the general-relativistic cosmology. The correspondence of general-relativistic approach and Newtonian approach has been studied by
Matarrese et al.[24] The discussions about it in detail are beyond the scope of the present paper.

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A. Quantities in the second-order metric perturbation
The quantity $Q$ in the second-order metric perturbation is expressed as

$$Q = \int_0^\eta d\eta' a^{-2(\eta')} \int_0^{\eta'} d\eta'' a^2(\eta'') \left[ P(\eta'') - \frac{5}{2}(P'(\eta''))^2 \right].$$

(A1)

In the case $\Lambda = 0$, we have $P - (5/2)(P')^2 = 0$ because of $a \propto \eta^2$ and $P = \eta^2 / 10$, so that $Q$ vanishes. The functions $L^j_i$ and $M^j_i$ are defined by

$$L^j_i = \frac{1}{12} \left[ -3F_{i,j} - 2F \cdot F_{,ij} + \frac{1}{2}\delta_{ij} F_{,j} F_{,i} \right],$$

$$M^j_i = \frac{1}{28} \left\{ 19F_{i,il} F_{,jl} - 12F_{,ij} \Delta F - 3\delta_{ij} \left[ F_{,kl} F_{,kl} - (\Delta F)^2 \right] \right\},$$

(A2)

and $N$ is defined by

$$\Delta N = \frac{1}{28} \left[ (\Delta F)^2 - F_{,kl} F_{,kl} \right].$$

(A3)

The last term $C^j_i$ satisfies the wave equation

$$\Box C^j_i = \frac{3}{14}(P/a)^2 G^j_i + \frac{1}{7} \left[ P - \frac{5}{2}(P')^2 \right] \tilde{G}^j_i,$$

(A4)

where the operator $\Box$ is defined by

$$\Box \phi \equiv g^{\mu\nu} \phi_{,\mu\nu} = -a^{-2} \left( \partial^2 / \partial \eta^2 + 2a'/a \partial / \partial \eta - \Delta \right) \phi$$

(A5)

for an arbitrary function $\phi$ by use of the four-dimensional covariant derivative $;,$ and $G^j_i$ and $\tilde{G}^j_i$ are expressed as

$$G^j_i \equiv \Delta(F_{,ij} \Delta F - F_{,jl} F_{,jli}) + (F_{,ij} F_{,kl} - F_{,ik} F_{,jli})_{,kl} - \frac{1}{2}\delta_{ij} \Delta[(\Delta F)^2 - F_{,kl} F_{,kl}],$$

$$\tilde{G}^j_i \equiv F_{,ij} \Delta F - F_{,il} F_{,jl} - \frac{1}{4}\delta_{ij} [(\Delta F)^2 - F_{,kl} F_{,kl}] - 7N_{,ij}.$$  

(A6)

These functions satisfy the traceless and transverse relations

$$G^j_i = 0, \quad G^j_{i,l} = 0,$$

$$\tilde{G}^j_i = 0, \quad \tilde{G}^j_{i,l} = 0,$$

(A7)

so that $C^j_i$ also satisfies

$$C^j_i = 0, \quad C^j_{i,l} = 0.$$

(A8)

This means that $C^j_i$ represents the second-order gravitational radiation emitted by first-order density perturbations. The solution of the above inhomogeneous wave equation (Eq.(A4)) can be represented in an explicit form using the retarded Green function for the operator $\Box.$
B. Case when $H_{\text{rem}}$ is 73.8, 74.3, or 78.7

If we assume that the three values of Hubble constant due to direct measurements (= 73.8, 74.3, 78.7) are $H_{\text{rem}}$, we obtain

$$\langle \frac{\delta \rho}{\bar{\rho}} \rangle = 0.117, 0.126, 0.213,$$

respectively, and

$$\langle (\frac{\delta \rho}{\bar{\rho}})^2 \rangle^{1/2} = 0.160, 0.166, 0.216.$$  \hspace{1cm} (B1)

Moreover, corresponding to $H_{\text{rem}} = 73.8, 74.3, 78.7$ and $H = H_0$ in Eq. (12), it is found for a fixed $x_{\text{min}} = 0.01$ using Eq. (35) that

$$x_{\text{max}} = 5.6, 5.8, 7.4,$$

$$A = 2.20, 2.25, 2.59, \quad B = 20.14, 21.78, 36.65,$$  \hspace{1cm} (B2)

respectively. The values of $A$ and $B$ depend rather sharply on $x_{\text{max}}$, but not on $x_{\text{min}}$. It is found therefore that the directly observed Hubble constants appear, corresponding to various values of $x_{\text{max}}$ ($\approx 6$). These values of $x_{\text{max}}$ may depend on the direction in which we measure the Hubble constant.

For $H_{\text{rem}} = 73.8, 74.3, 78.7$, moreover, we obtain

$$\left( \Omega_{M} \right)_{\text{rem}} = (0.302, 0.307, 0.357) \quad \text{and} \quad \left( \Omega_{\Lambda} \right)_{\text{rem}} = (0.698, 0.693, 0.643),$$  \hspace{1cm} (B3)

respectively.

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