THE SUPPORT THEOREM FOR THE COMPLEX RADON TRANSFORM OF DISTRIBUTIONS

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Abstract. The complex Radon transform \( \hat{F} \) of a rapidly decreasing distribution \( F \in \mathcal{O}_c'(\mathbb{C}^n) \) is considered. A compact set \( K \subset \mathbb{C}^n \) is called linearly convex if the set \( \mathbb{C}^n \setminus K \) is a union of complex hyperplanes. Let \( \hat{K} \) denote the set of complex hyperplanes which meet \( K \). The main result of the paper establishes the conditions on a linearly convex compact \( K \) under which the support theorem for the complex Radon transform is true: from the relation \( \text{supp}(\hat{F}) \subset \hat{K} \) it follows that \( F \in \mathcal{O}_c'(\mathbb{C}^n) \) is compactly supported and \( \text{supp}(F) \subset K \).

If \( f \) is the function defined on \( \mathbb{R}^n (\mathbb{C}^n) \), the classical real (complex) Radon transform \( Rf \) of \( f \) is the function defined on hyperplanes; the value of \( Rf \) at a given hyperplane is the integral of \( f \) over that hyperplane. For the theory of the Radon transform we refer to J. Radon \( [10] \), F. John \( [6], [7] \), I.M. Gel'fand, M.I. Graev, and N.Ya. Vilenkin \( [1] \), S. Helgason \( [2], [3] \), D. Ludwig \( [8] \), A. Hertle \( [4] \). One of the basic results on the classical Radon transform is Helgason’s support theorem \( [2] \): A rapidly decreasing function must vanish outside a ball if its real Radon transform does. This theorem holds for every convex compact set in \( \mathbb{R}^n \) and remains valid for rapidly decreasing distributions \( [4] \).

In the present paper we prove the support theorem for the complex Radon transform of distributions.

Notations. For \( z, w \in \mathbb{C}^n \) we write \( \langle z, w \rangle = \sum z_j w_j \). \( B^n(z, R) := \{ w \in \mathbb{C}^n \mid |w - z| < R \} \) denotes the euclidean ball of center \( z \) and radius \( r \) in \( \mathbb{C}^n \). If \( X \) is a set, we denote by \( \overline{X} \) the closure of \( X \). The standard Lebesgue measure in \( \mathbb{C}^n \) is \( d\omega_{2n} \). \( S^{2n-1} \) denotes the unit sphere in \( \mathbb{C}^n \), and \( d\sigma \) is the area element on \( S^{2n-1} \). For \( n \)-tuples \( p = (p_1, p_2, \ldots, p_n) \) and \( q = (q_1, q_2, \ldots, q_n) \) of non-negative integers, we denote by \( \partial^p \bar{\partial}^q \) the partial derivative

\[
\frac{\partial^{|p|+|q|}}{\partial z_1^{p_1} \ldots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \ldots \partial \bar{z}_n^{q_n}}
\]

of order \( |p| + |q| = p_1 + \ldots + p_n + q_1 + \ldots + q_n \). Similarly, for \( z = (z_1, \ldots, z_n) \) we write \( z^p = z_1^{p_1} \ldots z_n^{p_n} \), \( \bar{z}^q = z_1^{q_1} \ldots z_n^{q_n} \). For a domain \( \Omega \subset \mathbb{C}^n \), we denote by \( \mathcal{S}(\Omega), \mathcal{D}(\Omega), \) and \( \mathcal{E}(\Omega) \) the spaces of rapidly decreasing \( C^\infty \) functions, \( C^\infty \) functions with compact support, and \( C^\infty \) functions, respectively. The dual spaces \( \mathcal{S}'(\Omega), \mathcal{D}'(\Omega), \) and \( \mathcal{E}'(\Omega) \) are the spaces of tempered distributions, distributions, and distributions with compact support, respectively.

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If $\varphi \in \mathcal{S}(\mathbb{C}^n)$, the standard complex Radon transform of $\varphi$ (denoted by $\hat{\varphi}$) is defined by

$$
\hat{\varphi}(\xi, s) = \frac{1}{|\xi|^2} \int_{\langle z, \xi \rangle = s} \varphi(z) \, d\lambda(z),
$$

where $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$, and $d\lambda(z)$ is the area element on the hyperplane $\{z : \langle z, \xi \rangle = s\}$. For a set $A \subset \mathbb{C}^n$, we denote by $\hat{A}$ the set of all $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$ such that the hyperplane $\{z : \langle z, \xi \rangle = s\}$ meets $A$. A set $A \subset \mathbb{C}^n$ is called linearly convex if, for every $w \notin A$, there is a complex hyperplane $\{z : \langle z, \xi \rangle = s\}$ which contains $w$ and does not meet $A$ (see Martineau \[9\]).

We use the approach of Gel’fand et al. \[1\] to introduce the complex Radon transform of distributions. Let $X = S^{2n-1} \times \mathbb{C}$, and let $\mathcal{E}(X)$ be the set of complex-valued functions $\varphi(w, s)$ on $S^{2n-1} \times \mathbb{C}$ which satisfy the following conditions:

(a) Functions $\varphi(w, s)$ are infinitely differentiable with respect to $s$.
(b) For all $p, q \geq 0$ the derivatives

$$
\frac{\partial^{p+q} \varphi(w, s)}{\partial s^p \partial \bar{s}^q}
$$

are continuous on $S^{2n-1} \times \mathbb{C}$.
(c) $\varphi(we^{i\theta}, se^{i\theta}) = \varphi(w, s)$ for all $\theta \in [0, 2\pi]$.

We give $\mathcal{E}(X)$ the topology defined by the system of seminorms

$$
q_k(f) = \max_{w,s} \max_{1 \leq k \leq k} \max_{|w| \leq 2^{n-1}} \left| \frac{\partial^{k_1+k_2} f(w, s)}{\partial s^{k_1} \partial \bar{s}^{k_2}} \right|.
$$

By $\mathcal{D}(X)$ we denote the space of all compactly supported functions in $\mathcal{E}(X)$. We give $\mathcal{D}(X)$ the standard topology of the inductive limit of the spaces

$$
\mathcal{D}_m = \{ \varphi \in \mathcal{E}(X) : \text{supp}(\varphi) \subset S^{2n-1} \times \{|s| \leq m\} \}.
$$

Let $R\mathcal{D}(X)$ be the subspace of $\mathcal{D}(X)$ formed by the Radon transforms $\hat{\varphi}$ of functions in $\mathcal{D}(\mathbb{C}^n)$ (the equality $\hat{\varphi}(we^{i\theta}, se^{i\theta}) \equiv \hat{\varphi}(we^{i\theta}, se^{i\theta})$ follows for $\varphi \in \mathcal{D}(\mathbb{C}^n)$ from the definition of $\hat{\varphi}$). Similarly, we define the subspace $R\mathcal{D}(X)$ of $\mathcal{S}(\mathbb{C}^n)$.

The dual Radon transform is the operator $R^* : \mathcal{E}(X) \to \mathcal{E}(\mathbb{C}^n)$ given by

$$
[R^*(f)](z) = \int_{S^{2n-1}} f(w, \langle z, w \rangle) \, d\sigma(w).
$$

It is easy to see that the operator $R^*$ is continuous. It follows from the definition of the Radon transform that

$$
\int_{\mathbb{C}^n} [R^*(f)](z) \varphi(z) \, d\omega_{2n}(z) = \int_{\mathbb{C}^n} \int_{S^{2n-1}} f(w, s) \hat{\varphi}(w, s) \, d\sigma(w) \, d\omega_2(s)
$$

for every function $\varphi \in \mathcal{D}(\mathbb{C}^n)$.

Let $M_{\varphi}$ be the subspace of $\mathcal{D}(X)$ formed by the functions

$$
\psi(w, s) = \frac{\partial^{2n-2} \hat{\varphi}(w, s)}{\partial s^{n-1} \partial \bar{s}^{n-1}}, \quad \hat{\varphi} \in R\mathcal{D}(X).
$$

We give $M_{\varphi}$ the topology induced from $\mathcal{D}(X)$. 
Definition 1. Let $F \in \mathcal{D}'$. The Radon transform $RF$ of $F$ is the functional on $M_\mathcal{D}$ given by

$$
\langle RF, \psi \rangle = \langle F, R^* \psi \rangle.
$$

For every function $\varphi \in \mathcal{S}(\mathbb{C}^n)$ the following inversion formula holds [11 p. 118]:

$$
\varphi(z) = (-1)^{n-1}c_nR^* \left( \frac{\partial^{2n-2}\hat{\varphi}(w,s)}{\partial s^{n-1}\partial s^{n-1}} \right),
$$

where $\hat{\varphi}(w,s)$ is the Radon transform of $\varphi$, and $c_n > 0$. It follows from the inversion formula (5) that for each function $\psi \in M_\mathcal{D}$ the function $R^*(\psi)(z)$ belongs to $\mathcal{D}(\mathbb{C}^n)$. Therefore the functional $RF$ is well defined.

Definition 2. We say that the Radon transform $RF$ of a distribution $F \in \mathcal{D}'$ is defined as a distribution if the functional $RF$ given by (4) can be extended to a continuous functional on $\mathcal{D}(X)$.

It has been shown in [4] that there are distributions in $\mathbb{R}^n$ whose real Radon transforms are not defined as distributions. It is natural to suppose that there are such examples in the case of the Complex Radon transform. If the distribution $F$ is given by the function $f(z) \in \mathcal{S}(\mathbb{C}^n)$, then it follows from (5) and (2) that the Radon transform $RF$ is defined as a distribution and it is given by the function $\hat{f}(w, s)$.

We denote by $\mathcal{O}'_C(\mathbb{C}^n)$ the space of rapidly decreasing distributions [5 p. 419]. A distribution $T \in \mathcal{D}'(\mathbb{C}^n)$ belongs to $\mathcal{O}'_C(\mathbb{C}^n)$ if and only if for every $k \in \mathbb{Z}$ the distribution $(1 + |x|^2)^k T$ is integrable; i.e.,

$$
(1 + |x|^2)^k T = \sum_{|p|+|q| \leq m(k)} \partial^p \partial^q \mu_{pq}(k),
$$

where $m(k) \in \mathbb{N}$ and $\{\mu_{pq}\}(k)$ is a finite family of bounded measures on $\mathbb{C}^n$. In particular, every distribution with compact support is rapidly decreasing.

Let $T \in \mathcal{O}'_C(\mathbb{C}^n)$. We show that equality (4) defines the extension of the Radon transform $RT$ to a continuous linear functional on $\mathcal{D}(X)$. Let $h(w, s) \in \mathcal{D}(X)$ be such that $|h(w, s)| \leq 1$. There is $R > 0$ such that $h(w, s) = 0$ for $|s| \geq R$, and we have

$$
||R^* h|| (z) \leq \int_{S^{2n-1}} |h(w, \langle z, w \rangle)| d\sigma(w) \leq \int_{|\langle z, w \rangle| \leq R} d\sigma(w) \leq d_n \max \left(1, \frac{R^2}{|z|^2} \right),
$$

where $d_n > 0$. Suppose that the sequence $\{h_N(w, s)\}$ in $\mathcal{D}(X)$ converges to 0. Then, for every multi-indices $p$ and $q$, we have

$$
\partial^p \partial^q [R^* (h_N)] (z) = \int_{S^{2n-1}} \frac{\partial^{|p|+|q|}}{\partial s^{|p|} \partial s^{|q|}} h_N(w, \langle z, w \rangle) w^p \bar{w}^q d\sigma(w).
$$

There exists $R > 0$ such that $\text{supp}(h_N) \subset S^{2n-1} \times \{s : |s| \leq R\}$ for all $N$. Then it follows from (7) and (8) that

$$
|\partial^p \partial^q [R^* (h_N)] (z)| \leq d_n \max \left(1, \frac{R^2}{|z|^2} \right) \max_{w,s} \left| \frac{\partial^{|p|+|q|}}{\partial s^{|p|} \partial s^{|q|}} h_N(w, s) \right|.
$$
This means that the functions $[R^*(h_N)](z)$, together with derivatives of all orders, vanish at infinity. By the definition of the topology of $\mathcal{D}(X)$ we have

(10) $$\lim_{N \to \infty} \max_{w,s} \left| \frac{\partial^{\lfloor |p|+|q| \rfloor} \partial_s^{|p|} \partial_{\bar{s}}^{|q|}}{\partial_s^{|p|} \partial_{\bar{s}}^{|q|}} h_N(w, s) \right| = 0.$$  

We set $k = 0$ in (9). Then we obtain from (6) and (11) that

$$\langle RT, h_N \rangle = \langle T, [R^* h_N] \rangle = \sum_{|p|+|q| \leq m} (-1)^{|p|+|q|} \int_{\mathbb{C}^n} \rho^p \varrho^q [R^* h_N](z) d\mu_{pq}(z).$$

Since the measures $\mu_{pq}$ are bounded, it follows from (9) and (10) that $\langle RT, h_N \rangle \to 0$ as $N \to \infty$. Thus, for every $T \in \mathcal{E}^r_C(\mathbb{C}^n)$, the functional $RT$ is well-defined and continuous on $\mathcal{D}(X)$.

**Theorem 1.** Let $T \in \mathcal{E}_C^r(\mathbb{C}^n)$ and let $K \subset \mathbb{C}^n$ be a linearly convex compact set. Suppose that for every $z \notin K$ there exists a hyperplane $P = \{\lambda : \langle \lambda, w_0 \rangle = s_0\}$ satisfying the following conditions:

(i) $P$ contains $z$.
(ii) $P$ does not meet $K$.
(iii) The set $\mathbb{C} \setminus K_{w_0}$ is connected, where $K_{w_0} = \{\langle \lambda, w_0 \rangle \}_{\lambda \in K}$ is the projection of $K$ on $w_0$. Then $T$ has support in $K$ if and only if its Radon transform $RT$ has support in $\hat{K}$.

**Remark.** Theorem 1 was proved by the author in the special case in which the distribution $T$ is given by a compactly supported continuous function [12]. The proof of Theorem 1 is based on the properties of the convolution of $T$ and smooth compactly supported functions. As in the proof of the similar theorem for the real Radon transform and convex compact sets [4], the proof of Theorem 1 can be easily reduced to the case of regular distributions if for small enough $\varepsilon > 0$ the set

$$K_\varepsilon = \bigcup_{z \in K} B^n(z, \varepsilon)$$

also satisfies the conditions (i)-(iii). It should be noted that, in contrast to the case of convex compacts, there are examples of compact sets $K$ satisfying (i)-(iii) such that the set $K_\varepsilon$ does not satisfy the condition (iii) for every $\varepsilon > 0$. Since it has been shown in [12] that assumption (iii) in Theorem 1 is essential, Theorem 1 is not a simple consequence of the result of [12].

**Proof of Theorem 1.** Suppose that $T \in \mathcal{E}_C^r(\mathbb{C}^n)$ has support in $K$. Then $T \in \mathcal{E}(\mathbb{C}^n)$. Let $h(w, s) \in \mathcal{D}(X)$ be such that $\text{supp}(h) \subset X \setminus \hat{K}$. If $z \in K$, then the point $(w, \langle z, w \rangle)$ belongs to $\hat{K}$ for every $w \in S^{2n-1}$. Therefore the functions

$$[R^* h](z) = \int_{S^{2n-1}} h(w, \langle z, w \rangle) d\sigma(w),$$

$$\partial^p \bar{\partial}^q [R^* h](z) = \int_{S^{2n-1}} \frac{\partial^{\lfloor |p|+|q| \rfloor} \partial_s^{|p|} \partial_{\bar{s}}^{|q|}}{\partial_s^{|p|} \partial_{\bar{s}}^{|q|}} h(w, \langle z, w \rangle) w^p \bar{w}^q d\sigma(w)$$

vanish on $K$. So $[R^* h](z)$ is an infinitely differentiable function which, together with derivatives of all orders, vanishes on the support of the distribution $T$. Then we have
Thus, for each \( h \in \mathcal{D}(X) \) with \( \text{supp}(h) \subset X \setminus \hat{K} \) we have \( \langle RT, h \rangle = \langle T, [R^* h] \rangle = 0 \). This means that \( \text{supp}(RT) \subset \hat{K} \).

Before proving the second statement of Theorem 1 we have to show that the dual Radon transform and the convolution operation commute:

**Lemma 1.** Let \( \varphi(z) \in \mathcal{D}(\mathbb{C}^n) \). Then for every \( \psi(w, s) \in \mathcal{E}(X) \) the following formula holds:

\[
\varphi \ast [R^* \psi] = R^* [\hat{\varphi} \ast_s \psi],
\]

where \( \hat{\varphi}(w, s) \) is the Radon transform of \( \varphi \), and \( \ast_s \) denotes the convolution with respect to the second variable \( s \).

**Proof.** For every function \( \alpha(z) \in \mathcal{D}(\mathbb{C}^n) \) we have

\[
\int_{\mathbb{C}^n} (\varphi \ast [R^* \psi])(z) \alpha(z) d\omega_{2n}(z) = \int_{\mathbb{C}^n} [R^* \psi](z) (\alpha \ast \varphi_1)(z) d\omega_{2n}(z),
\]

where \( \varphi_1(z) = \varphi(-z) \). Let \( J \) be the integral on the right-hand side of (11). It follows from (2) that

\[
J = \int_{S^{2n-1} \times \mathbb{C}} \psi(w, s) \tilde{\alpha} \ast \tilde{\varphi}_1(w, s) d\sigma(w) d\omega_{2}(s),
\]

where \( \tilde{\alpha} \ast \tilde{\varphi}_1(w, s) \) is the Radon transform of the convolution \( \tilde{\alpha} \ast \tilde{\varphi} \). We have \( \text{p.p. 116-117} \)

\[
\tilde{\alpha} \ast \tilde{\varphi}_1(w, s) = (\tilde{\alpha} \ast_s \tilde{\varphi}_1)(w, s), \quad \tilde{\varphi}_1(w, s) = \tilde{\varphi}(w, -s) = \tilde{\varphi}(w, s).
\]

Then

\[
J = \int_{S^{2n-1} \times \mathbb{C}} \psi(w, s) (\tilde{\alpha} \ast_s \tilde{\varphi}_1)(w, s) d\sigma(w) d\omega_{2}(s) = \int_{S^{2n-1} \times \mathbb{C}} (\psi \ast_s \tilde{\varphi})(w, s) \tilde{\alpha}(w, s) d\sigma(w) d\omega_{2}(s).
\]

In view of (2), we have

\[
J = \int_{\mathbb{C}^n} R^* [\varphi \ast_s \psi](z) \alpha(z) d\omega_{2n}(z).
\]

Then it follows from (11) that

\[
\int_{\mathbb{C}^n} \{ (\varphi \ast [R^* \psi])(z) - R^* [\varphi \ast_s \psi](z) \} \alpha(z) d\omega_{2n}(z) = 0
\]

for every \( \alpha(z) \in \mathcal{D}(\mathbb{C}^n) \). Therefore \( (\varphi \ast [R^* \psi])(z) \equiv R^* [\varphi \ast_s \psi](z) \). The lemma is proved.

Now suppose that the support of the Radon transform \( RT \) of a distribution \( T \in \mathcal{E}'_C(\mathbb{C}^n) \) is contained in \( \hat{K} \). Let \( \{ \alpha_m(z) \}_{m=1}^{\infty} \) be a sequence of smooth functions on \( \mathbb{C}^n \) with \( \text{supp}(\alpha_m) \subset \{ z : |z| \leq 1/m \} \) that converges in the space of measures to the delta function at the origin. We assume that the functions \( \alpha_m(z) \) are even, i.e., \( \alpha_m(-z) = \alpha_m(z) \). We set \( T_m = T \ast \alpha_m \). Then the function \( T_m(z) \) belongs to \( \mathcal{S}(\mathbb{C}^n) \) \( \text{p. 244} \), and \( T_m \to T \) in \( \mathcal{E}'_C(\mathbb{C}^n) \) \( \text{p. 244} \). Denote by \( K_m \) the compact set

\[
K_m = \bigcup_{z \in K} B^n(z, 1/m).
\]
Let $\hat{T}_m(w,s)$ be the Radon transform of $T_m(z)$. We show that $\text{supp}(\hat{T}_m) \subset \hat{K}_m$. The hyperplane $\{z : \langle z, w \rangle = s\}$ meets $K_m$ if and only if there are $z' \in K$, $z'' \in B^n(0,1/m)$ such that $\langle z', w \rangle = s - \langle z'', w \rangle$. Therefore

$$
\hat{K}_m = \bigcup \limits_{(w,s) \in K} \left( \{w\} \times \bar{B}^1(s,1/m) \right).
$$

Let $h(w,s) \in \mathcal{G}(S^{2n-1} \times \mathbb{C})$ be such that $\text{supp}(h) \cap \hat{K}_m = \emptyset$. Since the functions $\alpha_m$ are even, it follows from (4) that

$$
\langle RT_m, h \rangle = \langle T_m, R^*(h) \rangle = \langle T*\alpha_m, R^*(h) \rangle = \langle T, \alpha_m * R^*(h) \rangle.
$$

Then by Lemma 1, we have $\langle T, \alpha_m * R^*(h) \rangle = \langle T, R^*(\hat{\alpha}_m * s h) \rangle$. Then

$$
\langle RT_m, h \rangle = \langle T, R^*(\hat{\alpha}_m * s h) \rangle = \langle RT, \hat{\alpha}_m * s h \rangle.
$$

We claim that $\hat{K} \cap \text{supp}(\hat{\alpha}_m * s h) = \emptyset$. Indeed, suppose that $(w_0, s_0) \in \hat{K} \cap \text{supp}(\hat{\alpha}_m * s h)$. This implies (since $\hat{\alpha}_m(w,s) = 0$ for $|s| \geq 1/m$) that for some $s_1 \in \bar{B}^1(0,1/m)$ we have $(w_0, s_0 + s_1) \in \text{supp}(h)$. By (12) we also have $(w_0, s_0 + s_1) \in \hat{K}_m$, which contradicts that $\text{supp}(h) \cap \hat{K}_m = \emptyset$. Therefore $\hat{K} \cap \text{supp}(\hat{\alpha}_m * s h) = \emptyset$, and it follows from (13) (since $\text{supp}(RT) \subset \hat{K}$) that $\langle RT_m, h \rangle = 0$. Therefore

$$
\text{supp}(RT_m) \subset \hat{K}_m.
$$

As remarked above, the functions $T_m(z)$ belong to $\mathcal{S}(\mathbb{C}^n)$. Then the distributions $RT_m$ are given by the Radon transforms $\hat{T}_m(w,s)$ of functions $T_m(z)$.

In view of (12), there exist $R > 0$ such that for all $m$ the sets $\hat{K}_m$ are contained in the set $\{(w,s) : |s| \leq R\}$. Let $R_{\mathbb{R}}T_m(w,t)$ be the real Radon transform of $T_m(z)$, that is

$$
R_{\mathbb{R}}T_m(w,t) = \int \limits_{\text{Re}(z,w) = t} T_m(z) d\lambda(z),
$$

where $d\lambda(z)$ is the area element on the real hyperplane $\{z : \text{Re}(z, \bar{w}) = t\}$. Then we have

$$
R_{\mathbb{R}}T_m(w,t) = \int \limits_{-\infty}^{\infty} \hat{T}_m(\bar{w}, t + ix) d\mu_x.
$$

Since $\hat{K}_m \subset \{(w,s) : |s| \leq R\}$, it follows from (14) that $R_{\mathbb{R}}T_m(w,t) = 0$ for $|t| \geq R$. Then by the Helgason’s support theorem, the supports of the functions $T_m(z)$ are compact.

To complete the proof of Theorem 1 we need the following lemma:

**Lemma 2.** Under the hypotheses and notation of Theorem 1, there exist, for every $z_0 \not\in K$, a neighborhood $V_{z_0}$ and $\delta > 0$ such that the functions $T_m(z)$ vanish on $V_{z_0}$ for $m \geq 1/\delta$.

**Proof.** Fix $z_0 \not\in K$. Then there exists a point $(w_0, s_0) \in S^{2n-1} \times \mathbb{C}$ such that $\{z : \langle z, w_0 \rangle = s_0\} \cap K = \emptyset$, $\langle z_0, w_0 \rangle = s_0$ and the set $\mathbb{C} \setminus \{(z, w_0)\}_{z \in K}$ is connected. Then $(w_0, \langle z_0, w_0 \rangle) \not\in \hat{K}$. We set

$$
A = \left\{ s \in \mathbb{C} \mid (w_0, s) \in \hat{K} \right\}, \quad A_m = \left\{ s \in \mathbb{C} \mid (w_0, s) \in \hat{K}_m \right\}.
$$
It follows from (12) that
\[ A_m = \bigcup_{s \in A} B^1(s, 1/m). \]

By definition of \( \hat{K} \), for every \( s \in A \) there exists \( z \in K \) such that \( \langle z, w_0 \rangle = s \). Then \( A = \{ \langle z, w_0 \rangle \}_{z \in K} \). Similarly \( A_m = \{ \langle z, w_0 \rangle \}_{z \in K_m} \). Since the sets \( K \) and \( K_m \) are compact, it follows that the sets \( A \) and \( A_m \) are also compact. For some \( R > 0 \) we have \( A \cup A_m \subset \bar{B}^1(0, R) \). Since \( \langle z_0, w_0 \rangle \notin A \), there is \( \gamma > 0 \) such that \( \langle z_0 + \lambda, w_0 \rangle \notin A \) for every \( \lambda \in B^n(0, \gamma) \). Hence the convex compact set \( \Gamma_1 = \{ \langle z, w_0 \rangle, z \in B^n(z_0, \gamma) \} \) and the set \( A \) do not intersect. Fix \( s_1 \in \{ s \in \mathbb{C} : |s| > R \} \). Then \( s_1 \in \mathbb{C} \setminus A \). Since the set \( \mathbb{C} \setminus A \) is connected, there exists a broken line \( \Gamma_2 \subset \mathbb{C} \setminus A \) joining \( s_1 \) to the point \( \langle z_0, w_0 \rangle \). Thus \( (\Gamma_1 \cup \Gamma_2) \cap A = \emptyset \). Then, since the sets \( \Gamma_1 \cup \Gamma_2 \) and \( A \) are compact, there exists \( \delta \in (0, 1) \) such that for all \( m \geq 1/\delta \) we have
\[ \{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\} \cap \{A + B^1(0, 1/m)\} = \emptyset, \]
that is \( (\Gamma_1 \cup \Gamma_2) + B^1(0, \delta) \cap A_m = \emptyset \). Put
\[ D = \{ s \in \mathbb{C} : |s| > R \} \cup \{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\} \]

By construction \( D \) is a connected unbounded open set containing the point \( \langle z_0 + \lambda, w_0 \rangle \) for every \( \lambda \in B^n(0, \gamma) \). We have by the definition of the sets \( A_m \) that \( D \times \{ w_0 \} \cap K_m = \emptyset \) for \( m \geq 1/\delta \). Then it follows from (14) that \( D \times \{ w_0 \} \) \cap supp(\( \hat{T}_m \)) = \emptyset for \( m \geq 1/\delta \). Since the supports of \( T_m \) are compact, it follows from [12, Thm. 2] that for every \( \lambda \in B^n(0, \gamma) \) and \( m \geq 1/\delta \) the functions \( T_m(z) \) vanish on the hyperplane \( \{ z : \langle z, w_0 \rangle = \langle z_0 + \lambda, w_0 \rangle \} \). Then, for every \( z \in B^n(z_0, \gamma) \) and \( m \geq 1/\delta \), we have \( T_m(z) = 0 \). The lemma is proved.

As mentioned above, \( T_m \rightarrow T \) in \( \mathcal{O}'(\mathbb{C}^n) \). This means that
\[ \lim_{m \to \infty} \langle T_m, \varphi \rangle = \langle T, \varphi \rangle, \quad \forall \varphi \in \mathcal{O}'(\mathbb{C}^n), \]
where \( \mathcal{O}'(\mathbb{C}^n) \) is the space of all infinitely differentiable functions \( f \) on \( \mathbb{C}^n \) for which there exist an integer \( k \) such that \( (1 + |x|^2)^k \partial^p \bar{\partial}^q f(z) \) vanishes at infinity for all \( p, q \) [5, p. 173].

Since \( \mathcal{D}'(\mathbb{C}^n) \subset \mathcal{O}'(\mathbb{C}^n) \), formula (15) holds for every \( \varphi \in \mathcal{D}'(\mathbb{C}^n) \). Let \( \varphi \in \mathcal{D}'(\mathbb{C}^n) \) be such that \( \text{supp}(\varphi) \cap K = \emptyset \). By Lemma 2 for every \( z \in \text{supp}\varphi \) there are \( \delta(z) > 0 \) and a ball \( B^n(z, \gamma(z)) \) such that \( T_m(z) = 0 \) on \( B^n(z, \gamma(z)) \) for \( m \geq 1/\delta(z) \). Since the support of \( \varphi \) is compact, it can be covered by a finite union of balls \( B^n(z_k, \gamma(z_k)) \), where \( k = 1, 2, \ldots, N \). Setting \( \delta_0 = \min\{ \delta(z_k), 1 \leq k \leq N \} \), we have \( T_m(z) = 0 \) for \( z \in \text{supp}(\varphi) \) and \( m \geq 1/\delta_0 \). Then it follows from (15) that
\[ \langle T, \varphi \rangle = \lim_{m \to \infty} \langle T_m, \varphi \rangle = 0. \]

Since \( \varphi \in \mathcal{D}(\mathbb{C}^n) \) is an arbitrary function such that \( \text{supp}(\varphi) \cap K = \emptyset \), we have \( \text{supp}(T) \subset K \). The theorem is proved.

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