On Lipschitz Semicontinuity Properties of Variational Systems with Application to Parametric Optimization

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Abstract In this paper, two properties of recognized interest in variational analysis, known as Lipschitz lower semicontinuity and calmness, are studied with reference to a general class of variational systems, i.e. to solution mappings to parameterized generalized equations. In the consideration of the metric nature of such properties, some related sufficient conditions are established, which are expressed via nondegeneracy conditions on derivative-like objects appropriate for a metric space analysis. For certain classes of generalized equations in Asplund spaces, it is shown how such conditions can be formulated by using the Fréchet coderivative of the field and the derivative of the base. Applications to the stability analysis of parametric constrained optimization problems are proposed.

Keywords Generalized equation · Implicit multifunction · Lipschitz lower semicontinuity · Calmness · Parametric constrained optimization · Strict outer slope · Fréchet subdifferential and coderivative

1 Introduction

When looking at several areas of pure and applied mathematics, the same dichotomy appears in a variety of situations: on the one hand, for treating a formalized problem as well as for understanding modeled phenomena, one has solve various kinds of equations; on the other hand, very often one experiences difficulties in finding an explicit solution, if any, for sufficiently general classes of equations. Such dichotomy
gains more evidence when parameters, alongside unknowns, enter the equations under examination. The presence of the parameters, nonetheless, is essential, in as much as it allows one to describe effects of errors and/or inaccuracies frequently arising in real-world measurements and transmissions of data. Moreover, parameters enable one to implement perturbation methods of analysis that sometimes can present interesting theoretical insights into the original problem. It is clear that, in the presence of parameters, the solution set associated with an equation becomes a (generally) set-valued mapping, which is expected to be only implicitly defined, due to the aforementioned computational difficulties. In such a case, even the solvability of an equation depends on parameters. As a result of such irreducible dichotomy, one is forced to indirectly study the features and behavior of the solution mapping, whose analytic form remains hidden, by performing proper inspections of the given equation data. Roughly speaking, this is the spirit essentially shared by many implicit function and multifunction theorems recently established in different areas of analysis (see [1–8], just to mention those works cited in the present paper for other purposes). The case of parameterized generalized equations in variational analysis makes no exception. The present paper is devoted to the study of special semicontinuity properties of mappings implicitly defined by parameterized generalized equations.

2 Problem Statement

Given a set-valued mapping \( F : P \times X \rightharpoonup Y \) and a function \( f : P \times X \rightarrow Y \), the following problem is meant by a parameterized generalized equation:

\[
(GE_p) \quad \text{find } x \in X \text{ such that } f(p,x) \in F(p,x).
\]

According to the nowadays vast literature devoted to such subject, parameterized generalized equations are formalized in several different fashions. For the purposes of the current research work, the form resulting from \((GE_p)\), adopted also in [8, 9], seems to be the appropriate one. After [6], the problem data \( f \) and \( F \) are sometimes referred to as the base and the field of \((GE_p)\), respectively. In \((GE_p)\), the variable denoted by \( x \) plays the role of the problem unknown (state), whereas \( p \) indicates a varying parameter. The related solution mapping \( G : P \rightharpoonup X \), implicitly defined by \((GE_p)\), namely the (generally) set-valued mapping

\[
G(p) = \{ x \in X : f(p,x) \in F(p,x) \},
\]

is often referred to as the variational system associated with \((GE_p)\). Notice that in the formulation considered above, both the base and the field do depend on the parameter.

Correspondingly with the specialized forms taken by the base and the field of \((GE_p)\), variational systems appear and show their relevance in a wide variety of contexts from mathematical programming, variational analysis, equilibrium and control theories. More precisely, they may formalize parametric constraint systems or

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1The terminology “generalized equation”, as well as the problem itself that it formalizes, was introduced by S.M. Robinson in [10].
the set of local/global solutions in parametric optimization problems. In adequately structured settings, variational conditions in the Robinson’s sense,\(^2\) that is, abstract problems able to include optimality conditions, variational inequalities and complementarity systems, are other examples of variational systems. Furthermore, even problems seemingly not having an extremal nature, such as the search for fixed points and equilibria, can be reduced to variational systems. The reader is referred to [14] for a rich account on possible applications of such a formalism.

The aim of the present paper, in accordance with the spirit of implicit multifunction theorems, is to contribute to the study of certain properties of variational systems associated with \((GE_p)\). Within the huge scientific production devoted to this theme, a major research line seems to concentrate on the analysis of such properties as Aubin continuity and metric regularity, the latter being related to the former through the inverse mapping (see, for instance, [3, 5, 8, 15–21]). Indeed, there exist many important findings about them and the achievements, accumulated on the subject during several decades, allow one to draw now a clear and comprehensive theoretical picture. Nonetheless, there are other forms of Lipschitzian behavior, whose study seems to be justified by not weaker motivations. Among them, Lipschitz lower semicontinuity and calmness are here considered.

Lipschitz lower semicontinuity describes in quantitative terms, by means of a Lipschitzian type metric estimation, the lower semicontinuity behavior of a given multifunction at a reference point. As such, it requires a metric structure on the domain and on the range space. If applied to variational systems, such a property not only ensures local solvability of the corresponding parameterized generalized equations, but also provides estimates of distances of their values from a certain solution, pertaining to a reference value of the parameter. In the more particular context of parametric constrained optimization, the validity of Lipschitz lower semicontinuity property of the feasible region mapping yields calmness from above behavior of the value function (see Sect. 3.1). Apart from being interesting in itself, Lipschitz lower semicontinuity may be exploited to characterize other stability properties of multifunctions. For instance, its occurrence at all points of the graph of a set-valued mapping, with fixed related constants, near a given point is known to be equivalent to the Aubin continuity at that point (see [4]). Again, a multifunction is Lipschitz lower semicontinuous at a given point and calm at all points of its graph near such a point iff it satisfies the Aubin property ibidem (see [4]). The latter characterization demonstrates also a possible employment of calmness, thereby introducing the second property under study.

Calmness is a property that relaxes at the same time the requirements of Aubin continuity and those of upper Lipschitz semicontinuity. Considered by several authors in different contexts and under various names (see [11, 22–24]), it turned out to capture a crucial behavior of multifunctions. It has been established to be equivalent, via the inverse mapping, to metric subregularity, a weaker “one-point” variant of metric regularity. As such, it was successfully introduced already in [25] for establishing optimality conditions. In more recent times, since the failure of metric regularity has

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\(^2\)The term “variational condition” for generalized equations can be traced back to [11]. In fact, it already appears in [12]. In [13], S.M. Robinson adopted such a term, thereby contributing to popularizing it.
been proved by B.S. Mordukhovich for major classes of variational systems (see [18, 19]), calmness is receiving an increasing interest. On the other hand, in the particular context of constraint systems analysis, it has been widely employed as a key condition to guarantee the occurrence of highly desirable phenomena: existence of local error bounds for the relations formalizing the constraints, viability of penalty function methods, validity of constraint qualifications (see [4, 26]). More precisely, the interplay of calmness and the Abadie constraint qualification in relation to Karush–Kuhn–Tucker points for mathematical programming problems has been well understood for many years. Furthermore, in [27] it has been shown how calmness/metric subregularity can be utilized to carry out a deep analysis of constraint qualifications for the basic rules of nonsmooth subdifferential calculus. Again, calmness plays a primary role in characterizing weak sharpness of local minimizers in optimization (see [28]).

Motivated by the aforementioned theoretical references and applicative instances, the study reported in the present paper deals mainly with conditions for detecting Lipschitz lower semicontinuity and calmness of variational systems associated with \((\mathcal{GE}_p)\). Roughly speaking, in both cases a basic sufficient condition is obtained according to the scheme summarized below:

| Calmness of the base \( f \) w.r.t. the parameter \( p \) at a given point \( \bar{x} \) uniformly in \( x \) near \( \bar{x} \) |
|---|
| Lipschitz lower semicontinuity of the field \( F \) upper Lipschitz continuity of the field \( F \) |
| \( \Rightarrow \) Lipschitz lower semicontinuity of \( G \) at \( (\bar{p}, \bar{x}) \) |
| \( \Rightarrow \) Calmness of \( G \) at \( (\bar{p}, \bar{x}) \) |
| \( \Rightarrow \) Appropriate nondegeneracy condition |

Since the natural environment where to conduct the analysis of both such properties are metric spaces, the appropriate nondegeneracy conditions appearing in the above scheme will be expressed in terms of positivity of certain derivative-like objects, whose relevance in variational analysis has been exemplarily illustrated in [29]. Criteria for Lipschitz lower semicontinuity and calmness properties of set-valued mappings have been studied by many authors (see, among the others, [4, 27]). For special classes of generalized equations, so far the main efforts have been concentrated on calmness (see [15, 30, 31]). To the author’s knowledge, a study considering both such properties in the case of variational systems defined by \((\mathcal{GE}_p)\) has not been undertaken, yet.

The material proposed in the rest of the paper is arranged in four main sections, whose contents are briefly outlined below. In Sect. 3, needed preliminaries from several topics of variational analysis are presented. The Lipschitzian type properties under examination are precisely stated and useful connections with other stability properties of multifunctions are recalled. In particular, in view of subsequent applications,
some consequences of such kind of properties on the value function associated with a family of parametric constrained optimization problems are discussed. The second part of this section is devoted to reviewing those tools and constructions of generalized differentiation, which are involved in the regularity conditions of the main conditions. In Sect. 4, a sufficient condition for the Lipschitz lower semicontinuity of variational systems is established in a purely metric setting. In Sect. 5, a similar condition for calmness of variational systems, complemented with an estimation of the related calmness modulus, is achieved. Such a result is then compared with a quite close very recent calmness condition, which considers the special case of parameterized generalized equations in Asplund spaces having constant (null) base. An analogous condition for generalized equations with a smooth base is also provided. In Sect. 6, the result formulated in Sect. 4 is applied to the study of parametric constrained optimization problems. More precisely, conditions able to guarantee the Lipschitz lower semicontinuity of the solution mapping are provided in different settings.

The notations employed throughout the paper are consistent with those used in [6, 14] and in a large part of the variational analysis literature. For the reader’s convenience, a list is provided below. Symbol \( \mathbb{R} \) stands for the real number field, while \( \mathbb{N} \) indicates the subset of all positive integers. \( (X,d) \) denotes a metric space. In such a setting, the distance of an element \( x \) from a set \( S \) is denoted by \( \text{dist}(x,S) \), \( B(x,r) := \{ z \in X : d(x,z) \leq r \} \) stands for the closed ball with centre at \( x \) and radius \( r \), and \( B(S,r) := \{ z \in X : \text{dist}(z,S) \leq r \} \) for the \( r \)-enlargement of \( S \subseteq X \), provided that \( r \geq 0 \). Given a subset \( S \), its indicator function evaluated at \( x \) is denoted by \( \mathbf{1}_S(x) \). By \( \text{int} S \) the topological interior of \( S \) is denoted. Given a function \( \phi : X \to \mathbb{R} \cup \{ \pm \infty \} \), \( \text{dom} \phi := \{ x \in X : |\phi(x)| < \infty \} \) denotes the domain of \( \phi \) and, given \( s \in \mathbb{R} \), \( \text{lev}_{s^+} \phi := \{ x \in X : \phi(x) > s \} \) the \( s \)-super level set of \( \phi \). Given a set-valued mapping (multifunction) \( \Phi : X \rightrightarrows Y \), \( \text{dom} \Phi := \{ x \in X : \Phi(x) \neq \emptyset \} \) and \( \text{gph} \Phi := \{(x,y) \in X \times Y : y \in \Phi(x)\} \) indicate the domain and the graph of \( \Phi \), respectively. Unless otherwise stated, all multifunctions will be assumed to take closed values. Whenever \( (X, \| \cdot \|) \) is a Banach space, \( X^* \) will denote its topological dual, with \( X \) and \( X^* \) being paired in duality by the bilinear form \( \langle \cdot , \cdot \rangle : X^* \times X \to \mathbb{R} \). The null element of a vector space is marked by \( 0 \), whereas the null element of its dual by \( 0^* \). The unit ball centred at \( 0^* \) is denoted by \( B^* \). Whenever \( \Lambda : X \to Y \) indicates a linear bounded operator between Banach spaces, \( \Lambda^* \) denotes its adjoint operator and \( \| \Lambda \|_\mathcal{L} \) its operator norm. The acronym l.s.c. will be used for short in place of lower semicontinuous, while u.s.c. in place of upper semicontinuous.

3 Variational Analysis Preliminaries

3.1 Lipschitzian Type Properties of Multifunctions

Let us start with recalling the basic properties that are the main theme of the paper.

**Definition 3.1** A set-valued mapping \( \Phi : P \rightrightarrows X \) between metric spaces is said to be **Lipschitz lower semicontinuous** at \( (\bar{p}, \bar{x}) \in \text{gph} \Phi \) iff there exist positive real con-
stents $\zeta$ and $l$ such that

$$\Phi(p) \cap B(\tilde{x}, l d(p, \tilde{p})) \neq \emptyset, \quad \forall p \in B(\tilde{p}, \zeta).$$

The following example demonstrates a situation in which, while lower semicontinuity takes place, Lipschitz lower semicontinuity fails to hold. It is aimed at clarifying the difference between the quantitative notion of lower semicontinuity appearing in Definition 3.1 and the merely topological one.

**Example 3.1** Let $P = X = \mathbb{R}$ be endowed with its usual Euclidean metric structure. Consider the epigraphical set-valued mapping $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ associated with the profile $\varphi(p) = \sqrt{|p|}$, that is,

$$\Phi(p) = \{x \in \mathbb{R} : x \geq \sqrt{|p|}\},$$

and the points $\tilde{p} = \tilde{x} = 0$. Mapping $\Phi$ is clearly lower semicontinuous at $(0, 0)$: for any $\epsilon > 0$, by taking $\delta_\epsilon \in [0, \epsilon^2]$, one has

$$\Phi(p) \cap B(0, \epsilon) \neq \emptyset, \quad \forall p \in B(0, \delta_\epsilon).$$

Nonetheless, $\Phi$ is not Lipschitz lower semicontinuous at $(0, 0)$. Indeed, for every $l > 0$ and $\zeta > 0$ one can find $p \in B(0, \zeta)$ such that $\sqrt{|p|} > l|p|$, and therefore, for such value of $p$, it results in

$$\Phi(p) \cap B(0, l|p|) = \emptyset.$$

Another notion related to stability properties of multifunctions is provided by the next definition.

**Definition 3.2** A set-valued mapping $\Phi : P \nrightarrow X$ between metric spaces is said to be **calm** at $(\tilde{p}, \tilde{x}) \in \text{gph} \Phi$ iff there exist positive real constants $\delta$, $\zeta$ and $\ell$ such that

$$\Phi(p) \cap B(\tilde{x}, \delta) \subseteq B(\Phi(\tilde{p}), \ell d(p, \tilde{p})), \quad \forall p \in B(\tilde{p}, \zeta),$$

or, equivalently,

$$\text{dist}(x, \Phi(\tilde{p})) \leq \ell d(p, \tilde{p}), \quad \forall x \in \Phi(p) \cap B(\tilde{x}, \delta), \forall p \in B(\tilde{p}, \zeta).$$

Any such constant $\ell$ is called a **calmness constant** for $\Phi$ at $(\tilde{p}, \tilde{x})$. The value

$$\text{clm} \Phi(\tilde{p}, \tilde{x}) := \inf\{\ell > 0 : \exists \delta, \zeta > 0 \text{ for which } (1) \text{ holds}\}$$

is called the **calmness modulus** of $\Phi$ at $(\tilde{p}, \tilde{x})$.

Calmness has strict connections with other well-known stability properties for multifunctions. One the one hand, it can be regarded as an “image localization” of the notion of upper Lipschitz continuity introduced by S.M. Robinson (see [9, 32]).
us recall that a set-valued mapping $\Phi : P \rightrightarrows X$ between metric spaces is called upper Lipschitz (alias outer Lipschitz continuous) at $\bar{p} \in \text{dom} \Phi$ iff there exist positive constants $\zeta$ and $\ell$ such that

$$\Phi(p) \subseteq B(\Phi(\bar{p}), \ell d(p, \bar{p})), \quad \forall p \in B(\bar{p}, \zeta).$$

(2)

Clearly, inclusion (2) makes (1) satisfied with any positive $\delta$. Nonetheless, note that, whereas upper Lipschitz continuity entails upper semicontinuity of a set-valued mapping, calmness does not.

On the other hand, calmness can be regarded as a “one-point” version of the Aubin continuity, known also under the name of pseudo-Lipschitz continuity, which postulates the existence of positive reals $\delta$, $\zeta$ and $\ell$ such that

$$\Phi(p) \cap B(\bar{x}, \delta) \subseteq B(\Phi(p'), \ell d(p, p')), \quad \forall p, p' \in B(\bar{p}, \zeta)$$

(see [4, 6, 7, 11]).

**Example 3.2** Let $P = X = \mathbb{R}$ be endowed with its usual metric structure. Consider the set-valued mapping $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\Phi(p) = \begin{cases} [0, +\infty) & \text{if } p = 0, \\ (-\infty, -1] \cup \{0\} & \text{otherwise.} \end{cases}$$

This mapping is calm at $(\bar{p}, \bar{x}) = (0, 0)$ because for any $\ell \geq 0$ it holds

$$\Phi(p) \cap B(0, \delta) = \{0\} \subseteq B([0, +\infty), \ell |p|) = [-\ell |p|, +\infty), \quad \forall p \in \mathbb{R},$$

provided that $\delta < 1$. Notice that $\text{clm} \Phi(0, 0) = 0$. Mapping $\Phi$ clearly fails to be upper Lipschitz at 0. The reader should notice as well that $\Phi$ fails also to be Aubin continuous at $(0, 0)$ because, for every $\ell \geq 0$ and positive $\delta$ and $\zeta$, there is $p \in B(0, \zeta)$ such that

$$[0, \delta] \not\subseteq [0, \ell |p|].$$

**Remark 3.1** (i) Note that Definition 3.2 implies that $\Phi(\bar{p}) \neq \emptyset$, while the set $\Phi(p) \cap B(\bar{x}, \delta)$ may happen to be empty even for $p$ near $\bar{p}$. Definition 3.1 entails, instead, the nonemptiness of the values taken by $\Phi$ near $\bar{p}$.

(ii) Whenever $\Phi$ is a single-valued mapping, inclusion (1) will take the form

$$d(\Phi(p), \Phi(\bar{p})) \leq \ell d(p, \bar{p}), \quad \forall p \in B(\bar{p}, \zeta) \text{ such that } \Phi(p) \in B(\Phi(\bar{p}), \delta),$$

so, like in the set-valued case, calmness as resulting from Definition 3.2 does not imply continuity. Nevertheless, in most of the literature devoted to variational analysis (see, for instance, [11, 33]), the limitation $\Phi(p) \in B(\Phi(\bar{p}), \delta)$ is dropped out in the case of functions. It is clear that, with this variant, calmness for single valued-mappings yields continuity and, actually, it becomes equivalent to upper Lipschitz continuity.

(iii) Through simple examples, it is readily shown that Lipschitz lower semicontinuity and calmness are properties independent of one another.
For scalar functions (i.e. when $X = \mathbb{R} \cup \{\pm \infty\}$), the notion of calmness has been conveniently split into two distinct ones. A function $\phi : P \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is said to be calm from below at $\bar{p} \in P$ iff $\bar{p} \in \text{dom} \phi$ and
\[
\liminf_{p \to \bar{p}} \frac{\phi(p) - \phi(\bar{p})}{d(p, \bar{p})} > -\infty.
\]
By replacing $\liminf_{p \to \bar{p}}$ and $-\infty$ with $\limsup_{p \to \bar{p}}$ and $+\infty$, respectively, and by reversing the above inequality, one obtains the calmness from above of $\phi$ at $\bar{p}$. Of course, $\phi$ is calm at $\bar{p}$ iff it is both calm from above and from below at the same point.

So far, the notion of calmness has been referred to mappings and functionals. The next definition enlightens the variational nature of calmness by referring it to perturbed optimization problems.

Let $\varphi : P \times X \rightarrow \mathbb{R}$ and $h : P \times X \rightarrow Y$ be given functions and let $C$ be a nonempty closed subset of $Y$. The basic format of the parametric optimization problems considered in what follows is defined as below:

\[
(P_p) \quad \min_{x \in X} \varphi(p, x) \quad \text{subject to} \quad h(p, x) \in C,
\]
where $R(p) = \{x \in X : h(p, x) \in C\}$ is the feasible region and $\varphi$ the objective functional. The associated optimal value (or marginal) function $\text{val}_P : P \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is
\[
\text{val}_P(p) := \inf_{x \in R(p)} \varphi(p, x).
\]

The associated global solution (minimizer) set, denoted by $\text{Argmin}_P : P \rightrightarrows X$, is the (generally) set-valued mapping
\[
\text{Argmin}_P(p) := \{x \in R(p) : \varphi(p, x) = \text{val}_P(p)\}.
\]

Observe that, in such a format, the class of problems $(P_p)$ needs no linear structure. However, for performing any quantitative analysis of it, the parameter space $P$ must be endowed at least with a metric structure. Owing to its abstract formalism, $(P_p)$ can cover large classes of constrained (and unconstrained) optimization problems. A particular case is the standard nonlinear programming problem: in such an event, $X$ becomes a finite-dimensional Euclidean space and the feasible region is given by the solution set of a system of finitely many equalities/inequalities, namely $R(p) = \{x \in X : h_i(p, x) = 0, i = 1, \ldots, m_1; h_i(p, x) \leq 0, i = m_1 + 1, \ldots, m_1 + m_2\}$.

**Definition 3.3** With reference to a parametric family of problems $(P_p)$, let $\bar{p} \in P$ and let $\bar{x} \in R(\bar{p})$ be a global solution for $(P_{\bar{p}})$. Problem $(P_{\bar{p}})$ is said to be calm at $\bar{x}$ iff there exists a positive constant $r > 0$ such that
\[
\inf_{p \in B(\bar{p}, r) \setminus \{\bar{p}\}} \inf_{x \in R(p) \cap B(\bar{x}, r)} \frac{\varphi(p, x) - \varphi(\bar{x}, \bar{p})}{d(p, \bar{p})} > -\infty.
\]
Calmness for parametric constrained optimization problems and calmness from below for functionals are readily linked via the value function, as stated in the following proposition.

**Proposition 3.1** Let \( \tilde{p} \in P \) be given and let \( \tilde{x} \in X \) be a global solution for problem \((P_{\tilde{p}})\). If the value function \( \text{val}_P : P \rightarrow \mathbb{R} \cup \{ \pm \infty \} \) associated with \((P_p)\) is calm from below at \( \tilde{p} \), then problem \((P_{\tilde{p}})\) is calm at \( \tilde{x} \).

**Proof** Notice that if \( \tilde{x} \in \text{Argmin}_{\tilde{p}}(P_{\tilde{p}}) \) then \( \tilde{p} \in \text{dom} \text{val}_P \). So it suffices to observe that for any \( p \in P \) and \( r > 0 \), according to the definition of \( \text{val}_P \), one has

\[
\inf_{x \in R(p) \cap B(\tilde{x}, r)} \frac{\varphi(p, x) - \varphi(\tilde{x}, \tilde{p})}{d(p, \tilde{p})} \geq \inf_{x \in R(p)} \frac{\varphi(p, x) - \varphi(\tilde{x}, \tilde{p})}{d(p, \tilde{p})} = \frac{\text{val}_P(p) - \text{val}_P(\tilde{p})}{d(p, \tilde{p})}
\]

and to recall Definition 3.3, along with the definition of calmness from below. \( \square \)

It is worth mentioning that, in the particular case in which \( X \) and \( Y \) are normed vector spaces and the problem data take the special forms

\[
\varphi(p, x) = \varphi(x) \quad \text{and} \quad h(p, x) = g(x) - p,
\]

the problem calmness of \((P_{\tilde{p}})\) at \( \tilde{x} \in \text{Argmin}_{\tilde{p}}(P_{\tilde{p}}) \) is known to characterize the existence of penalty functions (see [22]). The next proposition provides a sufficient condition for \( \text{val}_P \) to be calm from above.

**Proposition 3.2** With reference to a parametric class of problems \((P_p)\), let \( \tilde{p} \in P \) and \( \tilde{x} \in \text{Argmin}_{\tilde{p}}(P_{\tilde{p}}) \). If the set-valued mapping \( R : P \rightrightarrows X \) is Lipschitz l.s.c. at \((\tilde{p}, \tilde{x})\) and \( \varphi : P \times X \rightarrow \mathbb{R} \) is calm from above at \((\tilde{p}, \tilde{x})\), then the function \( \text{val}_P : P \rightarrow \mathbb{R} \cup \{ \pm \infty \} \) is calm from above at \( \tilde{p} \).

**Proof** Since \( \varphi \) is calm from above at \((\tilde{p}, \tilde{x})\), there exist positive \( \delta \) and \( \kappa \) such that, using the sum metric on the space \( P \times X \), it holds

\[
\varphi(p, x) - \varphi(\tilde{p}, \tilde{x}) \leq \kappa \left[ d(p, \tilde{p}) + d(x, \tilde{x}) \right], \quad \forall x \in B(\tilde{x}, \delta), \forall p \in B(\tilde{p}, \delta). \tag{3}
\]

By the Lipschitz lower semicontinuity of \( R \) at \((\tilde{p}, \tilde{x})\), one gets the existence of positive \( \zeta \) and \( l \) such that

\[
R(p) \cap B(\tilde{x}, ld(p, \tilde{p})) \neq \emptyset, \quad \forall p \in B(\tilde{p}, \zeta).
\]

This means that, if \( p \in B(\tilde{p}, \zeta) \), there exists \( x_p \in R(p) \) such that

\[
d(x_p, \tilde{x}) \leq ld(p, \tilde{p}).
\]

Notice that, without any loss of generality, it is possible to assume that \( \zeta < \delta / l \). Such an assumption implies that also \( x_p \in B(\tilde{x}, \delta) \). From the above estimation of \( d(x_p, \tilde{x}) \), by taking into account (3), one obtains

\[
\frac{\text{val}_P(p) - \text{val}_P(\tilde{p})}{d(p, \tilde{p})} \leq \frac{\varphi(p, x_p) - \varphi(\tilde{p}, \tilde{x})}{d(p, \tilde{p})} \leq \kappa \left[ d(p, \tilde{p}) + d(x_p, \tilde{x}) \right] d(p, \tilde{p}) \leq \kappa [1 + l], \quad \forall p \in B(\tilde{p}, \zeta) \setminus \{ \tilde{p} \}.
\]
Consequently, this allows one to conclude that
\[ \limsup_{p \to \bar{p}} \frac{\text{val}_p(p) - \text{val}_p(\bar{p})}{d(p, \bar{p})} \leq \kappa [1 + l] < +\infty. \]

A sufficient condition for \( \text{val}_P \) to be calm from below is established in the next proposition.

**Proposition 3.3** With reference to a parametric class of problems \( \{P_p\} \), let \( \bar{p} \in P \) and \( \bar{x} \in \text{Argmin}_P(\bar{p}) \). If the set-valued mapping \( R : P \rightrightarrows X \) is upper Lipschitz at \( \bar{p} \) and \( \varphi \) is Lipschitz continuous on \( P \times X \), then \( \text{val}_P : P \longrightarrow \mathbb{R} \cup \{ \pm \infty \} \) is calm from below at \( \bar{p} \) and problem \( \{P_{\bar{p}}\} \) is calm at \( \bar{x} \).

**Proof** The Lipschitz continuity of \( \varphi \) on \( P \times X \) ensures the existence of a positive \( \kappa \) such that, by equipping \( P \times X \) with the sum metric, it holds
\[ |\varphi(p, x_1) - \varphi(p, x_2)| \leq \kappa [d(x_1, x_2) + d(p_1, p_2)], \]
\[ \forall x_1, x_2 \in X, \forall p_1, p_2 \in P. \tag{4} \]
Since \( R \) has been supposed to be upper Lipschitz continuous at \( \bar{p} \), there are \( \ell \geq 0 \) and \( \tilde{\zeta} > 0 \) such that
\[ \text{dist}(x, R(\bar{p})) \leq \ell d(p, \bar{p}), \quad \forall x \in R(p), \forall p \in B(\bar{p}, \tilde{\zeta}). \]
The above inequality means that for every \( x \in R(p) \) there is \( z_x \in R(\bar{p}) \) satisfying the property
\[ d(z_x, x) \leq (\ell + 1)d(p, \bar{p}), \]
provided that \( p \in B(\bar{p}, \tilde{\zeta}) \). Therefore, by using the sum metric in \( P \times X \) and inequality (4), one obtains
\[ \varphi(p, x) \geq \varphi(\bar{p}, z_x) - \kappa [d(p, \bar{p}) + d(x, z_x)] \geq \varphi(\bar{p}, \bar{x}) - \kappa [\ell + 2] d(p, \bar{p}), \]
\[ \forall x \in R(p), \]
whence it follows
\[ \frac{\text{val}_P(p) - \text{val}_P(\bar{p})}{d(p, \bar{p})} = \inf_{x \in R(p)} \frac{\varphi(p, x) - \varphi(\bar{p}, \bar{x})}{d(p, \bar{p})} \geq -\kappa [\ell + 2], \]
\[ \forall p \in B(\bar{p}, \tilde{\zeta}) \setminus \{ \bar{p} \}. \]

The second assertion of the thesis becomes a straightforward consequence of the first one, in the light of Proposition 3.1.

If relaxing the assumption of Lipschitz upper semicontinuity on \( R \), using calmness instead, and the assumption of Lipschitz continuity on \( \varphi \), using its local counterpart, one can still obtain a calmness behavior of the value function, yet in a weaker form.
In order to introduce such a form, given \( \hat{x} \in X \), let us define the function \( \text{locval}_{P, \hat{x}} : P \times [0, +\infty[ \rightarrow \mathbb{R} \cup \{ \pm \infty \} \) as follows:

\[
\text{locval}_{P, \hat{x}} (p, r) := \inf_{x \in R(p) \cap B(\hat{x}, r)} \varphi(p, x).
\]

**Proposition 3.4** With reference to a parametric class of problems \( (P_p) \), let \( \bar{p} \in P \) and \( \bar{x} \) be a local minimizer for \( (P_{\bar{p}}) \). If the set-valued mapping \( R : P \rightrightarrows X \) is calm at \( (\bar{p}, \bar{x}) \) and \( \varphi \) is locally Lipschitz near \( (\bar{p}, \bar{x}) \), then there exists \( r_0 > 0 \) such that \( \text{locval}_{P, \bar{x}} (\cdot, r_0) \) is calm from below at \( \bar{p} \).

**Proof** Since \( \bar{x} \) is a local minimizer for \( (P_{\bar{p}}) \), for some \( \hat{\delta} > 0 \) one has

\[
\varphi(\bar{p}, x) \geq \varphi(\bar{p}, \bar{x}), \quad \forall x \in R(\bar{p}) \cap B(\bar{x}, \hat{\delta}).
\]

If \( \varphi \) is locally Lipschitz around \( (\bar{p}, \bar{x}) \), there exist positive reals \( \kappa, \tilde{\delta}, \) and \( \tilde{\zeta} \) such that

\[
|\varphi(p_1, x_1) - \varphi(p_2, x_2)| \leq \kappa [d(p_1, p_2) + d(x_1, x_2)],
\] \( \forall p_1, p_2 \in B(\bar{p}, \tilde{\zeta}), \forall x_1, x_2 \in B(\bar{x}, \tilde{\delta}). \) (5)

Since \( R \) is calm at \( (\bar{p}, \bar{x}) \), positive \( \ell, \delta \) and \( \zeta \) must exist such that

\[
\text{dist}(x, R(\bar{p})) \leq \ell d(p, \bar{p}), \quad \forall x \in R(p) \cap B(\bar{x}, \delta), \forall p \in B(\bar{p}, \zeta). \) (6)

Without loss of generality, it is possible to assume

\[
\delta < \min \left\{ \frac{\tilde{\delta}}{3}, \hat{\delta} \right\} \quad \text{and} \quad \zeta < \min \left\{ \frac{\tilde{\zeta}}{3(\ell + 1)}, \hat{\delta} \right\}.
\]

By inequality (6), one has that for every \( x \in R(p) \cap B(\bar{x}, \delta) \) a \( z_x \in R(\bar{p}) \) can be found such that

\[
d(z_x, x) \leq (\ell + 1)d(p, \bar{p}).
\]

Notice that, by virtue of the assumptions made on \( \delta \) and \( \zeta \), whenever \( p \in B(\bar{p}, \zeta) \) and \( x \in B(\bar{x}, \delta) \) it results in

\[
d(z_x, \bar{x}) \leq d(z_x, x) + d(x, \bar{x}) \leq (\ell + 1)\zeta + \delta < \frac{2}{3} \hat{\delta} < \tilde{\delta}.
\]

Thus, by exploiting inequality (5), one obtains

\[
\varphi(p, x) \geq \varphi(\bar{p}, z_x) - \kappa [d(p, \bar{p}) + d(x, z_x)] \geq \varphi(\bar{p}, \bar{x}) - \kappa [\ell + 2]d(p, \bar{p}),
\] \( \forall x \in R(p) \cap B(\bar{x}, \delta), \forall p \in B(\bar{p}, \zeta), \)

wherefrom it follows

\[
\frac{\text{locval}_{P, \bar{x}} (p, \delta) - \text{locval}_{P, \bar{x}} (\bar{p}, \delta)}{d(p, \bar{p})} \geq -\kappa [\ell + 2] > -\infty, \quad \forall p \in B(\bar{p}, \zeta) \setminus \{ \bar{p} \}.
\]

To complete the proof, it suffices to set \( r_0 = \delta \). \( \square \)
3.2 Tools of Generalized Differentiation

Let us start with introducing selected derivative-like tools, which enable one to carry out a local variational analysis in metric spaces. The basic one is the strong slope of a functional, originally proposed in [34] for quite different purposes. Given a function \( \varphi : X \rightarrow \mathbb{R} \cup \{ \pm \infty \} \) and an element \( \bar{x} \in \text{dom} \varphi \), the strong slope of \( \varphi \) at \( \bar{x} \) the real-extended value

\[
|\nabla \varphi|(\bar{x}) := \begin{cases} 
0 & \text{if } \bar{x} \text{ is a local minimizer for } \varphi, \\
\limsup_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x})}{d(x, \bar{x})} & \text{otherwise.}
\end{cases}
\]

is meant. The utility of its employment has been demonstrated in several circumstances (see, for instance, [35, 36]). In the present paper, the strong slope will be merely exploited as an element to construct a more robust object, considering “variational properties” of \( \varphi \) also at points of \( \text{lev}_{\varphi}(\bar{x}) \), near \( \bar{x} \). This object is the strict outer slope of \( \varphi \) at \( \bar{x} \), which is denoted by \( |\nabla \varphi|^>(\bar{x}) \) and defined by

\[
|\nabla \varphi|^>(\bar{x}) := \lim_{\epsilon \to 0^+} \inf \{ |\nabla \varphi|(x) : x \in B(\bar{x}, \epsilon), \varphi(x) < \varphi(x) \leq \varphi(\bar{x}) + \epsilon \}.
\]

The latter will be the key tool for formulating the main Lipschitz semicontinuity conditions proposed in paper (for a more general account on this slope and on other variations on this theme, the reader is referred to [29]).

In order to estimate the strict outer slope in the more structured setting of Banach spaces, one needs to deal with more refined generalized differentiation constructions, namely subdifferentials and coderivatives. Nonsmooth analysis abounds with notions of this kind. According to the features of the study exposed here, the following dual object, which is based on the Fréchet \( \epsilon \)-subdifferential and called the strict outer \( \epsilon \)-subdifferential, turns out to be useful

\[
\hat{\partial}^> \varphi(\bar{x}) := \bigcup \{ \hat{\partial} \varphi(x) : x \in B(\bar{x}, \epsilon), \varphi(x) < \varphi(x) \leq \varphi(\bar{x}) + \epsilon \}
\]

where, given \( \epsilon \geq 0 \), the set

\[
\hat{\partial} \varphi(\bar{x}) := \left\{ x^* \in X^* : \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} > -\epsilon \right\}
\]

denotes the Fréchet \( \epsilon \)-subdifferential (presubdifferential) of \( \varphi \) at \( \bar{x} \). For \( \epsilon = 0 \), one exactly gets the Fréchet (alias regular, according to [11]) subdifferential of \( \varphi \) at \( \bar{x} \) (in [1, 6, 7, 11, 14], the related theory, with examples and applications, is presented in full detail).

In a special class of Banach spaces, the next subdifferential construction, called the strict outer subdifferential slope of \( \varphi \) at \( \bar{x} \) and defined as

\[
|\bar{\partial} \varphi|^>(\bar{x}) := \lim_{\epsilon \to 0^+} \inf \{ \|x^*\| : x^* \in \hat{\partial}^> \varphi(\bar{x}) \},
\]

revealed to be a proper tool, in order to provide a characterization of the strict outer slope.
Some useful facts concerning the strict outer slope of a functional and its dual characterizations in terms of strict outer subdifferential slope are collected in the following remark. Their proofs in some cases are straightforward, in some other can be found in [7, 29, 36].

Remark 3.2  (i) Let \( \varphi : X \to \mathbb{R} \cup \{\pm \infty\} \) and \( \psi : X \to \mathbb{R} \) be given functions, with \( \psi \) locally Lipschitz near \( x \in \text{dom} \varphi \), having Lipschitz constant \( k \). Then one has

\[
|\nabla (\varphi + \psi) (x)| \geq |\nabla \varphi (x)| - k.
\]

Consequently, it follows

\[
|\nabla (\varphi + \psi) (x)|^\ominus (x) \geq |\nabla \varphi (x)|^\ominus (x) - k.
\]

(ii) If \( (X, \| \cdot \|) \) is an Asplund space and \( \varphi : X \to \mathbb{R} \cup \{\pm \infty\} \) is l.s.c. near \( \bar{x} \in \text{dom} \varphi \), the following exact estimation holds true

\[
|\nabla \varphi (\bar{x})| = |\partial \varphi (\bar{x})|.
\]

Let us recall that a Banach space is said to be Asplund provided that each of its separable subspaces admits separable dual. It is to be noted that, with the development of variational and extremal principles, alternative characterizations of the notion of Asplundity have been discovered, which are expressed in terms of dense Fréchet differentiability or dense Fréchet subdifferentiability properties of continuous convex or merely l.s.c. functions, respectively (see [6, 7]). Even though it excludes certain Banach spaces (for example, the space \( \ell_1 (\mathbb{N}) \)), the class of Asplund spaces is sufficiently rich for major applications. For instance, it includes all weakly compactly generated spaces, and hence all reflexive spaces.

Again, the above assumptions on \( X \) and \( \varphi \) enable one to exploit a simpler representation for the strict outer \( \epsilon \)-subdifferential as follows

\[
\hat{\partial}^\ominus \epsilon \varphi (\bar{x}) = \bigcup \{ \delta \varphi (x) : x \in B(\bar{x}, \epsilon), \varphi (x) < \varphi (\bar{x}) \leq \varphi (\bar{x}) + \epsilon \}.
\]

(iii) It is well known that, whenever \( \varphi \) is Fréchet differentiable at \( \bar{x} \), with derivative \( \widehat{D} \varphi (\bar{x}) \in X^* \), it results in

\[
|\nabla \varphi (\bar{x})| = \| \widehat{D} \varphi (\bar{x}) \|.
\]

Analogously, whenever \( \varphi \) is strictly differentiable at \( \bar{x} \), with strict derivative \( \overline{D} \varphi (\bar{x}) \in X^* \), it holds

\[
|\nabla \varphi (\bar{x})|^\ominus = \| \overline{D} \varphi (\bar{x}) \|.
\]

It is well known that for the Fréchet subdifferential a nonconvex counterpart of the Moreau–Rockafellar rule generally fails to hold. Nonetheless, in Asplund spaces such a rule can be restored in an approximate form, called fuzzy sum rule. In view of its employment in a subsequent section, it is recalled in the next lemma (see [1, 6, 7, 36]).
Lemma 3.1 Let \((X, \| \cdot \|)\) be an Asplund space, let \(\psi_1 : X \rightarrow \mathbb{R}\) and \(\psi_2 : X \rightarrow \mathbb{R} \cup \{\pm \infty\}\) be given functions, and let \(\bar{x} \in X\). Suppose \(\psi_1\) is Lipschitz continuous around \(\bar{x}\) and \(\psi_2\) is l.s.c. in a neighborhood of \(\bar{x} \in \text{dom} \, \psi_2\). Then, for any \(\eta > 0\) there exist \(x_1, x_2 \in B(\bar{x}, \eta)\) such that

\[
|\psi_i(x_i) - \psi_i(\bar{x})| \leq \eta, \quad i = 1, 2,
\]

and

\[
\hat{\partial}(\psi_1 + \psi_2)(\bar{x}) \subseteq \hat{\partial} \psi_1(x_1) + \hat{\partial} \psi_2(x_2) + \eta \mathbb{B}^*.
\]

When dealing with sets and set-valued mappings, one needs further elements of Fréchet nonsmooth calculus. Given a subset \(\Omega \subseteq X\) and \(\bar{x} \in \Omega\), the set of all the Fréchet normals to \(\Omega\) at \(\bar{x}\), denoted by

\[
\hat{\mathcal{N}}(\bar{x}, \Omega) := \left\{ x^* \in X^* : \limsup_{x \rightarrow \bar{x}} \inf_{\Omega} \langle x^*, x - \bar{x} \rangle / \|x - \bar{x}\| \leq 0 \right\},
\]

is called the Fréchet normal (alias, regular) cone to \(\Omega\) at \(\bar{x}\). The Fréchet normal cone to a set is linked with the Fréchet subdifferential via the set indicator function, as clarified by the equality

\[
\hat{\mathcal{N}}(\bar{x}, \Omega) = \hat{\partial} \iota(\cdot, \Omega)(\bar{x}), \quad \bar{x} \in X.
\]

In order to define derivatives of set-valued mappings, the graphical approach relying on the notion of coderivative, due to B.S. Mordukhovich (see [37]), is here adopted. Given a multifunction \(\Phi : X \rightrightarrows Y\) between Banach spaces and \((\bar{x}, \bar{y}) \in \text{gph} \, \Phi\), by the Fréchet coderivative of \(\Phi\) at \((\bar{x}, \bar{y})\) the set-valued mapping \(\hat{D}^* \Phi : Y^* \rightrightarrows X^*\), defined through the Fréchet normal cone as

\[
\hat{D}^* \Phi(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* : (x^*, -y^*) \in \hat{\mathcal{N}}((\bar{x}, \bar{y}), \text{gph} \, \Phi) \right\}, \quad y^* \in Y^*,
\]

is meant. Being a positively homogeneous mapping, one can consider its outer norm, i.e. the value

\[
\|\hat{D}^* \Phi(\bar{x}, \bar{y})\|_+ := \sup_{\|y^*\|=1} \sup \{ \|x^*\| : x^* \in \hat{D}^* \Phi(\bar{x}, \bar{y})(y^*) \},
\]

which will be useful in a subsequent section.

The next lemma, whose proof can be found, for instance, in [8], will be employed as well in the sequel.

Lemma 3.2 Let \(f : X \rightarrow Y\) and \(F : X \rightrightarrows 2^Y\) be mappings between metric spaces and let \(\bar{x} \in X\). If \(f\) is continuous at \(\bar{x}\) and \(F\) is u.s.c. at the same point, then the function \(d[f, F] : X \rightarrow [0, +\infty]\) defined by

\[
d[f, F](x) = \text{dist}(f(x), F(x))
\]

is l.s.c. at \(\bar{x}\).
4 Lipschitz Lower Semicontinuity of Variational Systems

In order to denote a function measuring displacements of the values of \( f \) from those of \( F \), the following shortened notation is introduced

\[ d[f, F](p, x) := \text{dist}(f(p, x), F(p, x)). \]

For the function \( d[f, F] : P \times X \rightarrow [0, +\infty] \), a partial version of the strict outer slope, which will be employed in the statement of the next result, is defined as follows:

\[
\left| \nabla_x d[f, F] \right|(p, x) := \begin{cases} 0 & \text{if } x \text{ is a local minimizer for } d[f, F](p, \cdot), \\ \limsup_{z \to x} \frac{d[f, F](p, x) - d[f, F](p, z)}{d(z, x)} & \text{otherwise.} \end{cases}
\]

One is now in a position to formulate the following sufficient condition for Lipschitz lower semicontinuity of variational systems, which is valid in metric spaces.

**Theorem 4.1** Let \( f : P \times X \rightarrow Y \) and \( F : P \times X \rightrightarrows Y \) be mappings defining a problem \((\mathcal{G}E_p)\), with solution mapping \( G : P \rightrightarrows X \). Let \( \bar{p} \in P \), \( \bar{x} \in G(\bar{p}) \) and \( \bar{y} = f(\bar{p}, \bar{x}) \). Suppose that:

(i) \((X, d)\) is metrically complete;

(ii) There exists \( \delta_1 > 0 \) such that, for every \( p \in B(\bar{p}, \delta_1) \), the mapping \( F(p, \cdot) : X \rightrightarrows Y \) is u.s.c. at each point of \( B(\bar{x}, \delta_1) \);

(iii) There exist \( \delta_2 > 0 \) and \( l_F > 0 \) such that

\[ F(p, \bar{x}) \cap B(\bar{y}, l_F d(p, \bar{p})) \neq \emptyset, \quad \forall p \in B(\bar{p}, \delta_2); \]

(iv) There exists \( \delta_3 > 0 \) such that, for every \( p \in B(\bar{p}, \delta_3) \), \( f(p, \cdot) \) is continuous at each point of \( B(\bar{x}, \delta_3) \);

(v) There exist \( \delta_4 > 0 \) and \( l_f > 0 \) such that

\[ d(f(p, \bar{x}), f(\bar{p}, \bar{x})) \leq l_f d(p, \bar{p}), \quad \forall p \in B(\bar{p}, \delta_4); \]

(vi) It holds

\[ \left| \nabla_x d[f, F] \right|^\sigma(\bar{p}, \bar{x}) > 0. \]

Then \( G \) is Lipschitz l.s.c. at \((\bar{p}, \bar{x})\).
Proof. Let \( \delta := \min\{\delta_1, \delta_2, \delta_3, \delta_4\} \), where each \( \delta_i \), with \( i = 1, 2, 3, 4 \), is as stated in hypotheses (ii), (iii), (iv), and (v), respectively. According to hypothesis (vi), it is possible to fix \( c \) and \( c' \) in such a way that
\[
0 < c < c' < \left\lfloor \nabla_x d[f, F] \right\rfloor((\bar{p}, \bar{x})).
\]

Corresponding to \( c' \), there exists \( \delta_* \in (0, \delta] \) such that
\[
\left\lfloor \nabla_x d[f, F] \right\rfloor((p, x) > c', \forall p \in B(\bar{p}, \delta_*), \forall x \in B(\bar{x}, \delta_*) \) with \( 0 < d[f, F](p, x) \leq \delta_* \).
\]

(7)

This amounts to saying that for every \((p, x) \in B(\bar{p}, \delta_*) \times B(\bar{x}, \delta_*) \), with \( f(p, x) \notin F(p, x) \) and \( d[f, F](p, x) \leq \delta_* \), and for every \( \eta > 0 \), an element \( x_\eta \in B(x, \eta) \) must exist such that
\[
\text{dist}(f(p, x_\eta), F(p, x_\eta)) > \text{dist}(f(p, x), F(p, x)) + c'd(x_\eta, x).
\]

Now, choose a positive \( \zeta_c \) in such a way that
\[
\zeta_c < \min\left\{ \frac{1}{2}, \frac{c}{2(l_F + l_f + 1)}, \frac{1}{l_F + l_f + 1} \right\} \delta_*,
\]

and fix \( p \in B(\bar{p}, \zeta_c) \setminus \{\bar{p}\} \). By virtue of hypothesis (iii), being \( \zeta_c < \delta_2 \), it holds
\[
F(p, \bar{x}) \cap B(\bar{y}, lF d(p, \bar{p})) \neq \emptyset.
\]

This means that there is \( v \in F(p, \bar{x}) \) such that
\[
d(v, \bar{y}) \leq l_F d(p, \bar{p}).
\]

(8)

Notice that, if \( \zeta_c < \delta_1 \) and \( \zeta_c < \delta_3 \), \( f(p, \cdot) \) is continuous on \( B(\bar{x}, \delta_*) \) as well as \( F(p, \cdot) \) is u.s.c. on the same subset. Thus, according to Lemma 3.2, if restricted to the complete metric space \( B(\bar{x}, \delta_*) \), the displacement function \( d[f, F](p, \cdot) \) turns out to be l.s.c.. Moreover, by virtue of hypothesis (v) and inequality (8), for \( \zeta_c < \delta_4 \), one has
\[
d[f, F](p, \bar{x}) = \inf_{w \in F(p, \bar{x})} d(f(p, \bar{x}), w)
\]
\[
\leq d(f(p, \bar{x}), \bar{y}) + d(\bar{y}, v) \leq (l_f + l_F) d(p, \bar{p})
\]
\[
\leq \inf_{x \in B(\bar{x}, \delta_*)} d[f, F](p, x) + (l_f + l_F)d(p, \bar{p}).
\]

By invoking the Ekeland variational principle, with \( \lambda = \frac{l_f + l_F}{c'} d(p, \bar{p}) \), one gets the existence of \( \hat{x} \in B(\bar{x}, \delta_*) \) such that
\[
d[f, F](p, \hat{x}) = \text{dist}(f(p, \hat{x}), F(p, \hat{x})) \leq d[f, F](p, \bar{x}) \leq (l_f + l_F) d(p, \bar{p}),
\]
\[
d(\hat{x}, \bar{y}) \leq \frac{l_f + l_F}{c'} d(p, \bar{p}) \leq \frac{l_f + l_F}{c'} \zeta_c < \frac{c}{2c'} \delta_* < \frac{\delta_*}{2},
\]

(9)  (10)
and
\[
d[f, F](p, \hat{x}) < d[f, F](p, x) + c'd(x, \hat{x}), \quad \forall x \in B(\bar{x}, \delta_*) \setminus \{\hat{x}\}. \tag{11}
\]
Inequality (11) allows one to say that
\[
f(p, \hat{x}) \in F(p, \hat{x}). \tag{12}
\]
Indeed, assume the contrary. Since in particular \(\hat{x} \in B(\bar{x}, \delta_*)\) and \(p \in B(\bar{p}, \delta_*)\), and, by the last inequality in (9), one has \(0 < d[f, F](p, \hat{x}) < \delta_*\), then one is enabled to apply what has been obtained as a consequence of (7). Accordingly, corresponding to \(\eta = \delta_*/2\), an element \(x_\eta \in B(\hat{x}, \delta_* / 2)\) must exist such that
\[
dist(f(p, \hat{x}), F(p, \hat{x})) > d(f(p, x_\eta), F(p, x_\eta)) + c'd(x_\eta, \hat{x}). \tag{13}
\]
Notice that, since
\[
d(x_\eta, \bar{x}) \leq d(x_\eta, \hat{x}) + d(\hat{x}, \bar{x}) < \frac{\delta_*}{2} + \frac{\delta_*}{2} = \delta_*
\]
x_\eta \in B(\bar{x}, \delta_*). In light of this, inequality (13) becomes inconsistent with (11), thereby validating inclusion (12). It follows that \(\hat{x} \in G(p)\). Thus, for \(\hat{x} \in B(\bar{x}, \frac{l_f + l_F}{c}d(p, \bar{p}))\) by virtue of (10), it has been shown that
\[
\hat{x} \in G(p) \cap B\left(\bar{x}, \frac{l_f + l_F}{c}d(p, \bar{p})\right) \neq \emptyset.
\]
The proof is now complete. \(\square\)

On the base of Remark 3.1(i), the reader should notice that Theorem 4.1 guarantees, in particular, nonempty-valuedness of variational systems near a point of interest. This fact seems to be notable in consideration of the bizarre geometry of solution mappings to parameterized generalized equations, whose graphs often exhibit “corner configurations”: near them \(G\) passes from multivaluedness for some values of \(p\), to emptiness for some other.

5 Calmness of Variational Systems

In order to study the calmness property of variational systems associated with \((GE_p)\), a further displacement function \(d[f(\bar{p}, \cdot), F(\bar{p}, \cdot)] : X \times Y \rightarrow \mathbb{R} \cup \{\pm \infty\}\), defined as
\[
d[f(\bar{p}, \cdot), F(\bar{p}, \cdot)](x, y) := d(f(\bar{p}, x), y) + \iota((x, y), \text{gph } F(\bar{p}, \cdot)),
\]
is exploited. It enters the statement of the next result.

**Theorem 5.1** Let \(f : P \times X \rightarrow Y\) and \(F : P \times X \rightrightarrows Y\) be mappings defining a problem \((GE_p)\), with solution mapping \(G : P \rightrightarrows X\). Let \(\bar{p} \in P\), \(\bar{x} \in G(\bar{p})\) and \(\bar{y} = f(\bar{p}, \bar{x})\). Suppose that:
(i) \((X, d)\) and \((Y, d)\) are metrically complete;
(ii) There exists \(\delta_1 > 0\) such that \([B(\bar{x}, \delta_1) \times B(\bar{y}, \delta_1)] \cap \text{gph } F(\bar{p}, \cdot)\) is a closed subset of \(X \times Y\);
(iii) There exist \(\delta_2 > 0\) and \(\ell_F > 0\) such that
\[F(p, x) \subseteq B\left(F(\bar{p}, x), \ell_F d(p, \bar{p})\right), \quad \forall x \in B(\bar{x}, \delta_2), \forall p \in B(\bar{p}, \delta_2);\]
(iv) There exist \(\delta_3 > 0\) such that \(f(\bar{p}, \cdot)\) is continuous at each point of \(B(\bar{x}, \delta_3)\);
(v) There exist \(\delta_4 > 0\) and \(\ell_f > 0\) such that
\[d\left(f(p, x), f(\bar{p}, x)\right) \leq \ell_f d(p, \bar{p}), \quad \forall x \in B(\bar{x}, \delta_4), \forall p \in B(\bar{p}, \delta_4);\]
(vi) It holds
\[\left|\nabla d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right]\right| > (\bar{x}, \bar{y}) > 0.\]

Then \(G\) is calm at \((\bar{p}, \bar{x})\) and the following estimation holds:
\[\text{clm } G(\bar{p}, \bar{x}) \leq \frac{\ell_f + \ell_F}{\left|\nabla d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right]\right| > (\bar{x}, \bar{y})}.\]

**Proof** To prove both assertions, it suffices to show that for every positive \(c\), with \(c < \left|\nabla d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right]\right| > (\bar{x}, \bar{y})\), there exists \(\zeta_c > 0\) with the property
\[G(p) \cap B(\bar{x}, \zeta_c) \subseteq B\left(G(\bar{p}), \frac{\ell_f + \ell_F}{c} d(p, \bar{p})\right), \quad \forall p \in B(\bar{p}, \zeta_c).\]  

(14)

Set \(\delta := \min\{\delta_1, \delta_2, \delta_3, \delta_4\}\). Then, by virtue of hypothesis (vi), there exists \(\delta_\ast \in ]0, \delta[\) such that
\[\left|\nabla d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right]\right|(x, y) > c', \quad \forall (x, y) \in \left[B(\bar{x}, \delta_\ast) \times B(\bar{y}, \delta_\ast)\right] \cap \text{gph } F(\bar{p}, \cdot)\text{ with } y \neq f(\bar{p}, x).\]

This means, in particular, that for every \((x, y) \in [B(\bar{x}, \delta_\ast) \times B(\bar{y}, \delta_\ast)] \cap \text{gph } F(\bar{p}, \cdot)\text{ with } y \neq f(\bar{p}, x)\) and \(\eta > 0\) there is \((x, y) \in [B(x, \eta) \times B(\bar{y}, \eta)] \cap \text{gph } F(\bar{p}, \cdot)\text{ such that}
\[d\left(f(\bar{p}, x), y\right) > d\left(f(\bar{p}, x, \eta), y, \eta\right) + c'd\left((x, y), (x, y)\right).\]

According to hypothesis (iv), corresponding to \(\delta_\ast/16\) there exists \(\tilde{\delta} > 0\) such that
\[d\left(f(\bar{p}, z), \bar{y}\right) < \frac{\delta_\ast}{16}, \quad \forall z \in B(\bar{x}, \tilde{\delta}).\]

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Now, take a positive $\zeta_c$ as follows

$$\zeta_c < \min \left\{ \frac{\delta^*}{16}, \frac{c \delta^*}{2(\ell_f + 2 \ell_F + 1)}, \frac{\delta^*}{16(\ell_F + 1)}, \frac{\delta^*}{16(\ell_f + 1)}, \frac{\delta^*}{\ell_f + 2 \ell_F + 1}, \delta \right\}$$

(15)

and fix $p \in B(\bar{p}, \zeta_c) \setminus \{\bar{p}\}$. If $G(p) \cap B(\bar{x}, \zeta_c) = \emptyset$, then inclusion (14) trivially holds true. Otherwise, take an arbitrary $z \in G(p) \cap B(\bar{x}, \zeta_c)$. Then

$$d(z, \bar{x}) \leq \zeta_c$$

and

$$f(p, z) \in F(p, z).$$

Notice that, due to inequality (15), $z \in B(\bar{x}, \delta_2)$ and $p \in B(\bar{p}, \delta_2)$ so, on the account of hypothesis (iii), one obtains from inclusion (16)

$$f(p, z) \in B\{F(\bar{p}, z), \ell_F d(p, \bar{p})\}.$$  

The above inclusion entails the existence of $w \in F(\bar{p}, z)$ such that

$$d\left(f(p, z), w\right) \leq (1 + \epsilon)\ell_F d(p, \bar{p}),$$

where $\epsilon$ can be assumed to fulfill the inequalities

$$0 < \epsilon < \min\left\{1, \frac{(c' - c)(\ell_f + \ell_F)}{c\ell_F}\right\}.$$  

(17)

Thus, in the light of inequality (15), it results in

$$d\left(f(p, z), w\right) < 2\ell_F \zeta_c < \frac{\delta^*}{8}.$$  

Since in particular $z \in B(\bar{x}, \delta_4)$ and $z \in B(\bar{x}, \tilde{\delta})$, it follows

$$d(w, \bar{y}) \leq d\left(w, f(p, z)\right) + d\left(f(p, z), f(\bar{p}, z)\right) + d\left(f(\bar{p}, z), \bar{y}\right)$$

$$< \frac{\delta^*}{8} + \frac{\delta^*}{16} + \frac{\delta^*}{16} = \frac{\delta^*}{4}$$

(18)

and hence, a fortiori, it turns out that $(z, w) \in B(\bar{x}, \delta_4) \times B(\bar{y}, \delta_4)$. Therefore, if restricting the function $d[f(\bar{p}, \cdot), F(\bar{p}, \cdot)]$ to the complete metric space $B(\bar{x}, \delta_4) \times B(\bar{y}, \delta_4)$, one obtains

$$d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right](z, w)$$

$$\leq d\left(f(\bar{p}, z), f(p, z)\right) + d\left(f(p, z), w\right) \leq \left(\ell_f + (1 + \epsilon)\ell_F\right)d(p, \bar{p})$$

$$\leq \inf_{(x, y) \in B(\bar{x}, \delta_4) \times B(\bar{y}, \delta_4)} d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right](x, y) + \left(\ell_f + (1 + \epsilon)\ell_F\right)d(p, \bar{p})$$
Consequently, inequality (21) takes the form
\[
\lambda = \frac{\ell_f + (1 + \epsilon)\ell_F}{c'} d(p, \bar{p}),
\]
there exists \((\hat{x}, \hat{y}) \in B(\bar{x}, \delta_n) \times B(\bar{y}, \delta_n)\) such that
\[
d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right](\hat{x}, \hat{y}) \leq \frac{\ell_f + (1 + \epsilon)\ell_F}{c'} d(p, \bar{p}) < \frac{\ell_f + 2\ell_F}{c'} < \frac{c\delta_n}{2c'} < \frac{\delta_n}{2},
\]
and
\[
d\left(f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right)(\hat{x}, \hat{y}) < \frac{\ell_f + (1 + \epsilon)\ell_F}{c'} d(p, \bar{p}) < \frac{\ell_f + 2\ell_F}{c'} < \frac{c\delta_n}{2c'} < \frac{\delta_n}{2},
\]
Notice that, since \(d(f(\bar{p}, \cdot), F(\bar{p}, \cdot))(\hat{x}, \hat{y}) = +\infty\) as a consequence of (19), it must be true that \(\hat{y} \in F(\bar{p}, \hat{x})\), and hence it results in
\[
d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right](\hat{x}, \hat{y}) = d(f(\bar{p}, \hat{x}), \hat{y}).
\]
Consequently, inequality (21) takes the form
\[
d\left(f(\bar{p}, \hat{x}), \hat{y}\right) = \frac{\ell_f + (1 + \epsilon)\ell_F}{c'} d(p, \bar{p}) < \frac{\ell_f + 2\ell_F}{c'} < \frac{c\delta_n}{2c'} < \frac{\delta_n}{2},
\]
Moreover, since \(\zeta_c < \frac{\delta_n}{\ell_f + 2\ell_F + 1}\), one finds
\[
d\left[f(\bar{p}, \cdot), F(\bar{p}, \cdot)\right](\hat{x}, \hat{y}) = d\left(f(\bar{p}, \hat{x}), \hat{y}\right) \leq \frac{\ell_f + (1 + \epsilon)\ell_F}{c'} d(p, \bar{p}) < \delta_n.
\]
The last inequalities lead to conclude that
\[
f(\bar{p}, \hat{x}) \in F(\bar{p}, \hat{x}).
\]
Indeed, assume ab absurdo that \(f(\bar{p}, \hat{x}) \notin F(\bar{p}, \hat{x})\) and hence \(\hat{y} \neq f(\bar{p}, \hat{x})\). Since, by inequalities (20) and (18),
\[
d(\hat{x}, \bar{x}) \leq d(\hat{x}, z) + d(z, \bar{x}) < \frac{\delta_n}{2} + \frac{\delta_n}{16} < \frac{3}{4}\delta_n
\]
and
\[
d(\hat{y}, \bar{y}) \leq d(\hat{y}, w) + d(w, \bar{y}) < \frac{\delta_n}{2} + \frac{\delta_n}{4} = \frac{3}{4}\delta_n,
\]
actually one has \((\hat{x}, \hat{y}) \in B(\bar{x}, \delta_\ast) \times B(\bar{y}, \delta_\ast)\). Moreover, \((\hat{x}, \hat{y}) \in gph F(\bar{p}, \cdot)\), so the consequence of hypothesis (vi) applies: corresponding to \(\eta = \delta_\ast / 4\) an element 
\((x_\eta, y_\eta) \in \left[ B(\hat{x}, \delta_\ast / 4) \times B(\hat{y}, \delta_\ast / 4) \right] \cap gph F(\bar{p}, \cdot)\) must exist such that
\[
d\left( f(\bar{p}, \hat{x}), \hat{y} \right) > d\left( f(\bar{p}, x_\eta), y_\eta \right) + c'd\left( (x_\eta, y_\eta), (\hat{x}, \hat{y}) \right),
\]
(24)
Since
\[
d(x_\eta, \bar{x}) \leq d(x_\eta, \hat{x}) + d(\hat{x}, \bar{x}) < \frac{3}{4} \delta_\ast = \delta_\ast
\]
and
\[
d(y_\eta, \bar{y}) \leq d(y_\eta, \hat{y}) + d(\hat{y}, \bar{y}) < \frac{3}{4} \delta_\ast = \delta_\ast,
\]
the existence of \((x_\eta, y_\eta) \in \left[ B(\bar{x}, \delta_\ast) \times B(\bar{y}, \delta_\ast) \right] \cap gph F(\bar{p}, \cdot)\) fulfilling inequality (24) clearly contradicts (22).
Thus, inclusion (23) means that \(\hat{x} \in G(\bar{p})\). The fact that, according to the first inequality in (20) and (17),
\[
d(z, \hat{x}) \leq \frac{\ell_F + (1 + \epsilon) \ell_F}{c'} d(p, \bar{p}) < \frac{\ell_F + \ell_F}{c} d(p, \bar{p})
\]
shows that \(z \in B(G(\bar{p}), \frac{\ell_F + \ell_F}{c} d(p, \bar{p}))\), so that inclusion (14) now appears to be satisfied. This completes the proof. □

It has been remarked by several authors (see, for instance, [26]) that existent conditions for calmness actually imply Aubin continuity. The following example shows that this is not true for the condition proposed in Theorem 5.1, which thereby turns out to be a specific tool for detecting calmness in circumstances where its stronger variant fails.

**Example 5.1** Let \(P = X = Y = \mathbb{R}\) be endowed with its usual metric structure. Consider the variational system associated with the parameterized generalized equation having base \(f \equiv 0\) and field \(F(p, x) = \{ y \in \mathbb{R} : |y| \geq |px| \}\). By explicitly resolving the inclusion \(0 \in F(p, x)\), one readily finds
\[
G(p) = \left\{ \begin{array}{ll}
\mathbb{R} & \text{if } p = 0, \\
\{0\} & \text{otherwise.}
\end{array} \right.
\]
So, choosing \(\bar{p} = \bar{x} = 0\) as reference values, one has \(\bar{x} \in G(\bar{p})\). One sees that \(gph F(0, \cdot) = \mathbb{R} \times \mathbb{R}\), which is a closed set. Besides, being polyhedral, the mapping \(F\) is upper Lipschitz (uniformly in \(x\)) at 0, with any \(\ell_F > 0\). Indeed, it trivially holds that
\[
F(p, x) \subseteq \mathbb{R} = B(\mathbb{R}, 0 \cdot |p|) = B(\mathbb{R}, \ell_F d(p, 0)), \quad \forall p \in \mathbb{R}, \forall x \in \mathbb{R}.
\]
Since, in the case under examination,
\[
d\left[ f(\bar{p}, \cdot), F(\bar{p}, \cdot) \right](x, y) = d(0, y) + l((x, y), gph F(0, \cdot)) = |y|,
\]
one obtains that \(0 < d[f(\bar{p}, \cdot), F(\bar{p}, \cdot)](x, y) \leq \delta\) iff \((x, y) \in B((0, 0), \delta) \setminus \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}\). As one checks at once, whenever \((x, y) \in B((0, 0), \delta) \setminus \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}\), it results in

\[
\left| \nabla d(0, F(0, \cdot))(x, y) \right| = 1,
\]

and therefore

\[
\left| \nabla d(0, F(0, \cdot)) \right|^{\triangleright} (0, 0) = 1 > 0.
\]

According to Theorem 5.1, the variational system \(G\) is calm at \((0, 0)\), with \(clm G(0, 0) = 0\), which one can verify by using the definition of \(G\) explicitly derived above. By a direct inspection of its graph, it is not difficult to see that \(G\) fails to be Aubin continuous at \((0, 0)\).

Whenever \(X = \mathbb{X}\) and \(Y = \mathbb{Y}\) are Asplund spaces, from Theorem 5.1, it is possible to derive an implicit multifunction theorem, establishing the calmness of variational systems associated with parameterized generalized equations with constant (null) base, i.e. having the special form

\[
(G \mathcal{E}^0_p)\quad 0 \in F(p, x),
\]

under a non-degeneracy condition on the Fréchet coderivative of \(F\). Such result seems to be rather close to the one achieved in [2] (see Theorem 3.1 therein). To see this fact in detail, given a solution \(\bar{x} \in G(\bar{p})\), with \(\bar{p} \in P\), let us set

\[
c[F(\bar{p}, \cdot)](\bar{x}, 0) := \lim_{\epsilon \to 0^+} \inf \{\|x^*\| : x^* \in \mathbb{D}^*F(\bar{p}, \cdot)(x, y)(y^*), x \in B(\bar{x}, \epsilon), y \in B(0, \epsilon) \setminus \{0\}, (x, y) \in gph F(\bar{p}, \cdot), \|y^*\| = 1\}.
\]

**Proposition 5.1** Let \(F : P \times \mathbb{X} \rightrightarrows \mathbb{Y}\) be a set-valued mapping defining a problem \((G \mathcal{E}^0_p)\), with solution mapping \(G : P \rightrightarrows \mathbb{X}\), and let \(\bar{x} \in G(\bar{p})\). Suppose that:

(i) \((\mathbb{X}, \|\cdot\|)\) and \((\mathbb{Y}, \|\cdot\|)\) are Asplund;

(ii) The graph of \(F(\bar{p}, \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}\) is closed in a neighborhood of \((\bar{x}, 0)\);

(iii) There exist \(\delta > 0\) and \(\ell > 0\) such that

\[
F(p, x) \subseteq B(F(\bar{p}, x), \ell d(p, \bar{p})), \quad \forall x \in B(\bar{x}, \delta), \forall p \in B(\bar{p}, \delta);
\]

(iv) It holds

\[
c[F(\bar{p}, \cdot)](\bar{x}, 0) > 0. \quad (25)
\]

Then, \(G\) is calm at \((\bar{p}, \bar{x})\).

**Proof** The thesis follows as a consequence of Theorem 5.1, upon having proved that condition (25) implies

\[
\nabla d[0, F(\bar{p}, \cdot)]^{\triangleright}(\bar{x}, 0) > 0.
\]
According to the definition of strict outer slope and to its subdifferential representation (see Remark 3.2(ii)), one has

\[
\nabla d\left[0, F(\bar{p}, \cdot) \right] > (\bar{x}, 0)
\]

\[
= \lim_{\epsilon \to 0^+} \inf \left\{ \| (x^*, y^*) \| : (x^*, y^*) \in \partial d\left[0, F(\bar{p}, \cdot) \right](x, y), (x, y) \in B(\bar{x}, \epsilon) \times B(0, \epsilon), 0 < \|y\| + \ell((x, y), gph F(\bar{p}, \cdot)) \leq \epsilon \right\}.
\]

By virtue of hypothesis (iv), it is possible to find \( c \in \mathbb{R} \) such that

\[
0 < c < \min\{1, c[F(\bar{p}, \cdot)](\bar{x}, 0)\}.
\]

By definition of \( c[F(\bar{p}, \cdot)](\bar{x}, 0) \), corresponding to \( c \) one can find \( \epsilon_c > 0 \) such that

\[
\inf\left\{ \|x^*\| : x^* \in \partial d F(\bar{\rho}, \cdot)(x, y)(y^*) \right\} \leq 1 > c.
\]

Now fix \((x, y) \in B(\bar{x}, \epsilon_c/2) \times B(0, \epsilon_c/2)\), with \(0 < \|y\| + \ell((x, y), gph F(\bar{p}, \cdot)) \leq \epsilon_c/2\). Notice that, from the last inequality, one can deduce that \((x, y) \in gph F(\bar{p}, \cdot)\).

Moreover, since

\[
d[0, F(\bar{p}, \cdot)](x, y) = \|y\| + \ell((x, y), gph F(\bar{p}, \cdot)),
\]

the function \( d[0, F(\bar{p}, \cdot)] : \mathbb{R} \times \mathbb{Y} \longrightarrow \mathbb{R} \cup \{\pm \infty\} \) is expressible as a sum of a Lipschitz continuous term and an addend which is l.s.c. in a neighborhood of \((\bar{x}, 0)\) as indicator of the locally closed set \( gph F(\bar{p}, \cdot) \) (recall hypothesis (ii)). Thus, it is possible to employ the fuzzy sum rule for estimating the Fréchet subdifferential of \( d[0, F(\bar{p}, \cdot)] \) at \((x, y)\). Taking \( \eta \) in such a way that

\[
0 < \eta < \min\left\{ \frac{(1-c)^2 c}{2}, \frac{\epsilon_c}{2}, \|y\| \right\},
\]

according to Lemma 3.1, for every \((x^*, y^*) \in \partial d[0, F(\bar{p}, \cdot)](x, y)\) there exist:

\((x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{Y}, \) with \(x_i \in B(x, \eta), y_i \in B(y, \eta), i = 1, 2,\)

\((x_1^*, y_1^*), (x_2^*, y_2^*) \in \mathbb{R} \times \mathbb{Y}, \) with \(x_1^* = 0^*, \|y_1^*\| = 1, \{y_1^*, y_1\} = \|y_1\|, \) and

\((x_2^*, y_2) \in \mathbb{R} \times \mathbb{Y}, \) with \(y_2 \in B(0, \eta), y_1 \in B(y, \eta), i = 1, 2, \)

such that

\[
\|x^* - x_2^*\| \leq \eta, \quad \|y^* - y_1^* - y_2\| \leq \eta,
\]

and

\[
0 < \|y\| - \|y_i\| + \ell((x_i, y_i), gph F(\bar{p}, \cdot))] \leq \eta, \quad i = 1, 2.
\]
Observe that by the last inequality it must be true that \((x_i, y_i) \in \text{gph } F(\bar{p}, \cdot), i = 1, 2\). Moreover, since \(\eta < \epsilon_c / 2\), one has

\[
d(x_i, \bar{x}) \leq d(x_i, x) + d(x, \bar{x}) \leq \frac{\epsilon_c}{2} + \frac{\epsilon_c}{2} = \epsilon_c, \quad i = 1, 2,
\]

\[
d(y_i, 0) \leq d(y_i, y) + d(y, 0) \leq \frac{\epsilon_c}{2} + \frac{\epsilon_c}{2} = \epsilon_c, \quad i = 1, 2.
\]

Let us estimate the norm of an arbitrary \((x^*, y^*) \in \hat{\partial}d[0, F(\bar{p}, \cdot)](x, y)\). If \(\|y^*\| \geq c\), then, by equipping the product space \(X^* \times Y^*\) with the norm

\[
\|(x^*, y^*)\| = \max \left\{ \frac{\|x^*\|}{(1 - c)^2}, \|y^*\| \right\},
\]

one finds

\[
\|(x^*, y^*)\| \geq \|y^*\| \geq c. \quad (29)
\]

Otherwise, if \(\|y^*\| < c\), by taking into account the second inequality in \((27)\), one obtains

\[
\|y_2^*\| \geq \|y^* - y_1^*\| - \eta \geq \|y^*\| - \|y_1^*\| - \eta = 1 - \|y^*\| - \eta > 1 - c - \eta > 1 - c - (1 - c)c = (1 - c)^2, \quad (30)
\]

as, for \(0 < c < 1\), one gets \(\eta < \frac{(1-c)^2c}{2} < (1-c)c\). The last inequalities enable one to set

\[
y_0^* := \frac{y_2^*}{\|y_2^*\|}, \quad x_0^* := \frac{x_2^*}{\|y_2^*\|}, \quad x_0 = x_2 \quad \text{and} \quad y_0 = y_2.
\]

Since \((x_2^*, -y_2^*) \in \hat{N}((x_2, y_2), \text{gph } F(\bar{p}, \cdot)),\) one has \(x_2^* \in \hat{D}^* F(\bar{p}, \cdot)(x_2, y_2)(y_2^*)\) and hence, by the positive homogeneity of the Fréchet coderivative, it results in

\[
x_0^* \in \hat{D}^* F(\bar{p}, \cdot)(x_0, y_0)(y_0^*), \quad \text{with } \|y_0^*\| = 1.
\]

Besides, on account of \((28)\) and of the choice of \(\eta\),

\[
\|y_0\| \geq \|y\| - \eta > 0
\]

and hence \(y_0 \neq 0\). With the above definitions, by virtue of the first inequality in \((27)\), one finds

\[
\|(x^*, y^*)\| \geq \frac{\|x^*\|}{(1 - c)^2} \geq \frac{\|x_2^*\| - \eta}{(1 - c)^2} \geq \frac{\|y_2^*\|\|x_0^*\| - \eta}{(1 - c)^2}.
\]

Therefore, since \(x_0 \in B(\bar{x}, \epsilon_c), y_0 \in B(0, \epsilon_c) \setminus \{0\}\) and \((x_0, y_0) \in \text{gph } F(\bar{p}, \cdot),\) in the light of inequality \((26)\), by recalling \((30)\), one obtains

\[
\|(x^*, y^*)\| \geq \frac{1}{(1 - c)^2} \left[ (1 - c)^2 c - \frac{(1 - c)^2 c}{2} \right] = \frac{c}{2}.
\]
The latter estimation of $\| (x^*, y^*) \|$, along with (29), leads to conclude that
\[
\nabla_d \left( 0, F(\bar{p}, \cdot) \right) \geq (\bar{x}, 0) \geq \frac{c}{2},
\]
thereby completing the proof. □

**Remark 5.1**

(i) The non-degeneracy condition appearing in Proposition 5.1 (hypothesis (iv)) requires computing Fréchet coderivatives of $F(\bar{p}, \cdot)$ at each point of a set like $[B(\bar{x}, \epsilon) \times (B(0, \epsilon) \setminus \{0\})] \cap \mathrm{gph} \ F(\bar{p}, \cdot)$. Such a drawback typically arises in many other regularity conditions. Nevertheless, it is worthwhile mentioning the possibility to pass to an one-point condition, by replacing the basic Fréchet coderivatives with a single limiting coderivative construction, as indicated, for instance, in [2] (where more details can be found). It has been mentioned that Proposition 5.1 provides a condition for calmness, which is rather close to a condition formulated for the same class of variational systems in [2] (see Theorem 3.1). More precisely, the constant expressing the non-degeneracy condition in that case, here denoted by $\tilde{c}[F(\bar{p}, \cdot)](\bar{x}, 0)$, includes in its definition the additional requirement $|\langle y^*, y \rangle - \| y^* \| | < \epsilon$ on all pairs $(y^*, x^*) \in \mathrm{gph} \tilde{D}^* F(\bar{p}, \cdot)(x, y)$ to be considered. Since for such pairs $\| y^* \| = 1$, this further requirement actually means $\| y^* \| - \epsilon < \langle y^*, y \rangle$. It follows that, in general, it holds
\[
\tilde{c}[F(\bar{p}, \cdot)](\bar{x}, 0) \geq c[F(\bar{p}, \cdot)](\bar{x}, 0),
\]
with the consequence that Theorem 3.1 in [2] seems to provide a sufficient condition for calmness of wider use. On the other hand, the approach proposed here, leading to Proposition 5.1 as a consequence of Theorem 5.1, reveals that the corresponding sufficient condition can be subsumed into a very general theory, whose validity covers also the setting of purely metric spaces.

(ii) A constant defined as $c[F(\bar{p}, \cdot)]$, apart from the presence of a parameter, appears already in [6] (see formula (4.2) therein). In that context, it is used to characterize the local covering/metric regularity of the set-valued mapping to which it refers, albeit for a different purpose. In fact, condition (25) provides a verifiable sufficient condition for the calmness of the solution mapping to $(\mathcal{GE}_p^0)$, which is only implicitly defined, in terms of problem data. Of course, condition (25) implies the local covering of the mapping $F(\bar{p}, \cdot) : X \rightharpoonup Y$ around $(\bar{x}, 0)$, but this fact does not imply in general the same property, or the Aubin continuity, of the solution mapping, as illustrated by Example 5.1. In that case indeed, even if $F(0, \cdot)$ is metrically regular around $(0, 0)$, nonetheless, since $G^{-1} = G$, the solution mapping fails to cover locally and to be Aubin continuous around the reference point.

In the case, in which a nonnull but smooth base term enters $(\mathcal{GE}_p)$, by strengthening the assumption on the space $(Y, \| \cdot \|)$, a further calmness condition can be formulated in the following way.

**Proposition 5.2** Let $f : P \times X \longrightarrow Y$ be a function and let $F : P \times X \rightharpoonup Y$ be a set-valued mapping defining a problem $(\mathcal{GE}_p)$, with solution mapping $G : P \rightharpoonup X$, and let $\bar{x} \in G(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x})$. Suppose that:
Therefore, both

\((\mathcal{X}, \| \cdot \|)\) is Asplund and \((\mathcal{Y}, \| \cdot \|)\) is a Fréchet smooth renormable Banach space;

(ii) The graph of \(F(\tilde{p}, \cdot) : \mathcal{X} \rightarrow \mathcal{Y}\) is closed in a neighborhood of \((\tilde{x}, \tilde{y})\);

(iii) \(f\) is calm with respect to \(p\) at \(\tilde{p}\), uniformly in \(x\) in a neighborhood of \(\tilde{x}\);

(iv) The function \(f(\tilde{p}, \cdot)\) is Fréchet differentiable in a neighborhood of \(\tilde{x}\);

(v) There exist \(\delta > 0\) and \(\ell > 0\) such that

\[
F(p, x) \subset B\left(F(\tilde{p}, x), \ell d(p, \tilde{p})\right), \quad \forall x \in B(\tilde{x}, \delta), \forall p \in B(\tilde{p}, \delta);
\]

(vi) There exist positive \(\gamma\) and \(\epsilon_\gamma\) such that

\[
\forall (x, y) \in B(\tilde{x}, \epsilon_\gamma) \times B(\tilde{y}, \epsilon_\gamma), \quad \text{with} \quad (x, y) \in gph F(\tilde{p}, \cdot), \quad f(\tilde{p}, x) \neq y, \quad \text{and}
\]

\[
\| f(\tilde{p}, x) - y \| \leq \epsilon_\gamma
\]

it holds

\[
\inf_{\| y^* \|=1} \left\| \hat{\partial} f(\tilde{p}, \cdot)(x)^* y^* \right\| > (1 + \gamma) \| \hat{\partial}^* F(\tilde{p}, \cdot)(x, y) \|_+ + \gamma. \tag{31}
\]

Then, \(G\) is calm at \((\tilde{p}, \tilde{x})\).

**Proof** Recall that, according to the Ekeland–Lebourg theorem, any Banach space admitting an equivalent Fréchet differentiable norm is an Asplund space (see [38]). Therefore, both \((\mathcal{X}, \| \cdot \|)\) and \((\mathcal{Y}, \| \cdot \|)\) are Asplund spaces. It is then possible to exploit once again the subdifferential representation of the strict outer slope mentioned in Remark 3.2(ii), according to which one has

\[
\begin{align*}
\left[ \nabla d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right] \right]^* & (\tilde{x}, \tilde{y}) \\
&= \left[ \partial d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right] \right]^* (\tilde{x}, \tilde{y}) \\
&= \lim_{\epsilon \to 0^+} \inf \left\{ \left\| (x^*, y^*) \right\| : (x^*, y^*) \in \hat{\partial} d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right](x, y), \right. \\
& \quad \left. (x, y) \in B(\tilde{x}, \epsilon) \times B(\tilde{y}, \epsilon), 0 < d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right] (x, y) \leq \epsilon \right\}.
\end{align*}
\]

Fix an arbitrary \((x, y) \in B(\tilde{x}, \epsilon_\gamma) \times B(\tilde{y}, \epsilon_\gamma)\), with \(0 < d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right] (x, y) \leq \epsilon_\gamma\). The finiteness of the value \(d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right] (x, y)\) implies that \((x, y) \in gph F(\tilde{p}, \cdot)\), whereas its positivity entails that \(f(\tilde{p}, x) \neq y\). Since the norm of \(\mathcal{Y}\) can be assumed to be Fréchet differentiable at \(f(\tilde{p}, x) - y\) by hypothesis (i), the function \((u, v) \mapsto \| f(\tilde{p}, u) - v \|\) is Fréchet differentiable at \((x, y)\). Indeed, without loss of generality, \(B(\tilde{x}, \epsilon_\gamma)\) can be assumed to lie within the neighborhood of differentiability of \(f(\tilde{p}, \cdot)\). According to known Fréchet subdifferential calculus rules, it holds

\[
\hat{\partial} d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right](x, y) = \hat{D} \| f(\tilde{p}, \cdot) - \cdot \| (x, y) + \hat{N}\left((x, y), gph F(\tilde{p}, \cdot)\right).
\]

Thus, if \((x^*, y^*) \in \hat{\partial} d\left[ f(\tilde{p}, \cdot), F(\tilde{p}, \cdot) \right](x, y)\), there exist

\[
v_1^* \in \mathcal{Y}^*, \quad \text{with} \quad \left\| v_1^* \right\| = 1 \quad \text{and} \quad \{ v_1^*, f(\tilde{p}, x) - y \} = \| f(\tilde{p}, x) - y \|
\]
and
\[(u_2^*, v_2^*) \in X^* \times Y^*, \quad \text{with } u_2^* \in \mathring{D}^* F(\bar{\rho}, \cdot)(x, y)(v_2^*)\]
such that
\[(x^*, y^*) = (\mathring{D} f(\bar{\rho}, \cdot)(x)^* v_1^* + u_2^*, -v_1^* - v_2^*).\]

One has now to verify that the norm of \((x^*, y^*)\) remains greater than a positive constant. Similarly as for the proof of the previous proposition, \(X^* \times Y^*\) is assumed to be equipped with the norm \(\| (x^*, y^*) \| = \max\{\|x^*\|, \|y^*\|\}\). In the case \(v_2^* = \mathbf{0}^*\), one readily finds
\[\| (x^*, y^*) \| \geq \| v_1^* \| = 1.\]

In the case \(v_2^* \neq \mathbf{0}^*\), let us distinguish two cases. If \(\|v_2^*\| > 1 + \gamma\), one obtains at once
\[\| (x^*, y^*) \| \geq \| v_2^* + v_1^* \| \geq \| v_2^* \| - \| v_1^* \| > \gamma.\]

Otherwise, if \(\|v_2^*\| \leq 1 + \gamma\), set
\[v_0^* := \frac{v_2^*}{\| v_2^* \|} \quad \text{and} \quad u_0^* := \frac{u_2^*}{\| v_2^* \|},\]
so that \(u_0^* \in \mathring{D}^* F(\bar{\rho}, \cdot)(x, y)(v_0^*)\). Since it holds
\[\| u_2^* \| = \| u_0^* \| \| v_2^* \| \leq \| \mathring{D}^* F(\bar{\rho}, \cdot)(x, y) \|_+ \| v_2^* \| \leq (1 + \gamma) \| \mathring{D}^* F(\bar{\rho}, \cdot)(x, y) \|_+ ,\]
then, as a consequence of condition (31), it results in
\[\| (x^*, y^*) \| \geq \| \mathring{D} f(\bar{\rho}, \cdot)(x)^* v_1^* + u_2^* \| \geq \| \mathring{D} f(\bar{\rho}, \cdot)(x)^* v_1^* \| - \| u_2^* \| \geq \inf_{\| v^* \| = 1} \| \mathring{D} f(\bar{\rho}, \cdot)(x)^* v^* \| - (1 + \gamma) \| \mathring{D}^* F(\bar{\rho}, \cdot)(x, y) \|_+ > \gamma.\]

Thus, in any case, one can conclude that
\[\inf\{\| (x^*, y^*) \| : (x^*, y^*) \in \partial d[ f(\bar{\rho}, \cdot), F(\bar{\rho}, \cdot)](x, y), (x, y) \in B(\bar{x}, \varepsilon_\gamma) \times B(\bar{y}, \varepsilon_\gamma), \quad 0 < d[ f(\bar{\rho}, \cdot), F(\bar{\rho}, \cdot)](x, y) \leq \varepsilon_\gamma \} > \min\{1, \gamma\}.\]

By virtue of the sufficient condition expressed by Theorem 5.1, the obtained results allow one to get the thesis. \(\square\)

### 6 Applications to Parametric Constrained Optimization

The stability analysis of parametric optimization problems is a well developed and active field of research, dealing with qualitative and quantitative investigations about the behavior of the optimal solution set and of the optimal value function in the presence of perturbations. One of the early monographs to be mentioned, entirely devoted
to a systematic presentation of this topic, is [39]. It was followed by a good amount of works (see, for example, [4, 40, 41] and the references therein).

Some results concerning the value function associated with a family of parametric constrained optimization problems have been already exposed in Sect. 3. In the current section, the analysis is focused on the quantitative stability behavior of the solution set mapping. Indeed, the first presented result provides a sufficient condition for the Lipschitz lower semicontinuity of such a set-valued mapping, which works in a purely metric space setting.

**Proposition 6.1** With reference to a family \((P_p)\) of perturbed problems, let \(\bar{p} \in P\) and \(\bar{x} \in \text{Argmin}_P(\bar{p})\). Suppose that:

(i) \((X, d)\) is complete;

(ii) For every \(p \in P\) near \(\bar{p}\), function \(x \mapsto \varphi(p, x)\) is continuous in a neighborhood of \(\bar{x}\);

(iii) The function \(h\) is locally Lipschitz around \((\bar{p}, \bar{x})\) with respect to \(x\), uniformly in \(p\), i.e. there exist positive \(\kappa\) and \(\delta_\kappa\) such that

\[
d(h(p, x_1), h(p, x_2)) \leq \kappa d(x_1, x_2), \quad \forall p \in B(\bar{p}, \delta_\kappa), \forall x_1, x_2 \in B(\bar{x}, \delta_\kappa);
\]

(iv) The functions \(p \mapsto \varphi(p, \bar{x})\), \(p \mapsto h(p, \bar{x})\) and \(\text{val}_P\) are calm at \(\bar{p}\);

(v) It holds

\[
|\nabla_x \varphi|_\infty (\bar{p}, \bar{x}) > \kappa.
\]

Then \(\bar{p} \in \text{int}(\text{dom \text{Argmin}_P})\) and \(\text{Argmin}_P\) is Lipschitz l.s.c. at \((\bar{p}, \bar{x})\).

**Proof** The first part of the thesis is obviously a consequence of the second one. The latter is readily proved by applying Theorem 4.1 to the generalized equation \((GE_p)\) defined by a base \(f : P \times X \rightarrow (\mathbb{R} \cup \{\pm \infty\}) \times Y\) given by

\[
f(p, x) = \left(\varphi(p, x) - \text{val}_P(p), h(p, x)\right)
\]

and by the constant field \(F : P \times X \rightrightarrows \mathbb{R} \cup \{\pm \infty\} \times Y\)

\[
F(p, x) = \{0\} \times C.
\]

In fact, the associated variational system coincides with the set-valued mapping \(\text{Argmin}_P : P \rightrightarrows 2^X\). Thus, one has to verify that, in the case under consideration, all hypotheses of Theorem 4.1 are fulfilled. Since \(F\) is a constant set-valued mapping and

\[
f(\bar{p}, \bar{x}) = (0, h(\bar{p}, \bar{x})) \in F(p, \bar{x}) = \{0\} \times C, \quad \forall p \in P,
\]

hypotheses (ii) and (iii) are evidently satisfied. Next, hypothesis (iv) is in force, because of the continuity of the functions \(x \mapsto \varphi(p, x)\) and of the (local Lipschitz) continuity of the functions \(x \mapsto h(p, x)\), around \(\bar{x}\), with \(p\) in a neighborhood of \(\bar{p}\). Again, calmness of \(f(\cdot, \bar{x})\) at \(\bar{p}\) (hypothesis (v)) follows from calmness of \(\varphi(\cdot, \bar{x})\), \(\text{val}_P\), and \(h(\cdot, \bar{x})\) at the same point. It remains to show that hypothesis (vi) is also
valid. To this aim, observe that, equipping the product space $\mathbb{R} \cup \{\pm \infty\} \times Y$ with the sum metric, one obtains
\[
d[f, F](p, x) = \text{dist}\left((\varphi(p, x) - \text{val}_P(p), h(p, x)), \{0\} \times C\right) \\
= \varphi(p, x) - \text{val}_P(p) + \text{dist}(h(p, x), C).
\] (32)

Since the function $x \mapsto h(p, x)$ is locally Lipschitz around $(\bar{p}, \bar{x})$ with constant $\kappa > 0$ and the function $y \mapsto \text{dist}(y, C)$ is Lipschitz continuous with constant 1 all over $Y$, their composition $x \mapsto \text{dist}(h(p, x), C)$ turns out to be locally Lipschitz with constant $\kappa$ around $\bar{x}$, uniformly in $p$ near $\bar{p}$. By recalling what has been observed in Remark 3.2, for $|\nabla_x \varphi| > \kappa$, one obtains from (32)
\[
|\nabla_x d[f, F]| > (\bar{p}, \bar{x}) \geq |\nabla_x \varphi| > (\bar{p}, \bar{x}) - \kappa > 0.
\]

This completes the proof. □

Relying on the nice properties enjoyed by the Fréchet subdifferential calculus in Asplund spaces, a stability result for perturbed nonsmooth optimization problems can be formulated as follows.

**Corollary 6.1** With reference to a family $(P_p)$ of perturbed problems, let $\bar{p} \in P$ and $\bar{x} \in \text{Argmin}_P(\bar{p})$. Suppose that $(X, \|\cdot\|)$ is Asplund and that hypotheses (ii), (iii) and (iv) of Proposition 6.1 are in force. If
\[
\lim_{\epsilon \to 0^+} \inf \|x^*\| : (p, x) \in B(\bar{p}, \epsilon) \times B(\bar{x}, \epsilon), \varphi(\bar{p}, \bar{x}) < \varphi(p, x) \leq \varphi(\bar{p}, \bar{x}) + \epsilon, \quad x^* \in \partial_x \varphi(p, x) > \kappa,
\]
where $\kappa$ is the Lipschitz constant as in hypothesis (iii), then $\bar{p} \in \text{int}(\text{dom} \text{Argmin}_P)$ and $\text{Argmin}_P$ is Lipschitz l.s.c. at $(\bar{p}, \bar{x})$.

**Proof** The thesis follows immediately, after having estimated $|\nabla_x \varphi| > (\bar{p}, \bar{x})$ in terms of the partial Fréchet subdifferentials of $\varphi$ near $(\bar{p}, \bar{x})$, as indicated in Remark 3.2(ii). □

On the base of simple representations of the strict outer slope enabled by the presence of differentiability, for optimization problems with smooth data it is possible to reformulate the above condition in general Banach spaces.

**Corollary 6.2** With reference to a family $(P_p)$ of perturbed problems, let $\bar{p} \in P$ and $\bar{x} \in \text{Argmin}_P(\bar{p})$. Suppose that:

(i) $(X, \|\cdot\|)$ is a Banach space;
(ii) For every $p$ near $\bar{p}$, each function $x \mapsto \varphi(p, x)$ is strictly differentiable at $\bar{x}$, with derivative $\overline{D}_x \varphi(p, \bar{x})$;
(iii) There exists $\delta > 0$ such that each function $x \mapsto h(p, x)$ is Gâteaux differentiable on $B(\bar{x}, \delta)$, for every $p$ near $\bar{p}$, with derivative $D_x h(p, x)$, and it holds

$$\|D_x \varphi(\bar{p}, \bar{x})\| > \sup_{(p,x) \in B(\bar{p}, \delta) \times B(\bar{x}, \delta)} \|D_x h(p, x)\| \mathcal{L};$$

(iv) The functions $p \mapsto \varphi(p, \bar{x})$, $p \mapsto h(p, \bar{x})$ and $\text{val}_P$ are calm at $\bar{p}$.

Then $\bar{p} \in \text{int}(\text{dom} \text{Argmin}_P)$ and $\text{Argmin}_P$ is Lipschitz l.s.c. at $(\bar{p}, \bar{x})$.

**Proof** Since the function $h(p, \cdot)$ is Gâteaux differentiable on the convex subset $B(\bar{x}, \delta)$, for each $p$ near $\bar{p}$, then as a consequence of the mean value theorem one has

$$\|h(p, x_1) - h(p, x_2)\| \leq \sup_{x \in B(\bar{x}, \delta)} \|D_x h(p, x)\| \mathcal{L} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in B(\bar{x}, \delta).$$

In other words, in light of hypothesis (iii), $h$ is locally Lipschitz with respect to $x$, uniformly in $p$, around $(\bar{p}, \bar{x})$, having a Lipschitz constant not greater than $\sup_{(p,x) \in B(\bar{p}, \delta) \times B(\bar{x}, \delta)} \|D_x h(p, x)\| \mathcal{L}$. By hypothesis (ii), the function $\varphi(p, \cdot)$ is continuous in a neighborhood of $\bar{x}$, for every $p$ near $\bar{p}$. Moreover, as observed in Remark 3.2(iii), it holds $|\nabla_x \varphi| > (\bar{p}, \bar{x}) = \|D_x \varphi(\bar{p}, \bar{x})\|. \quad \text{Thus, all its hypotheses being fulfilled, it remains to apply Proposition 6.1.}$

7 Conclusions

In this paper, verifiable sufficient conditions for the Lipschitz lower semicontinuity property and for the calmness property of a broad class of variational systems have been established, in terms of problem data. These conditions are expressed by non-degeneracy requirements, involving various elements of generalized differentiation, and they have been obtained by a variational technique, relying on the use of appropriate slopes of certain displacement functions. In the particular case of calmness, also an estimate from above of the corresponding modulus is provided. Moreover, it has been shown that the obtained condition is specific for calmness, in the sense that, in general, it does not imply Aubin continuity, in contrast to what happens for other existent conditions. A question, arising within the proposed analysis, is whether the presented conditions can be refined, in such a way to achieve complete characterizations of the Lipschitzian properties under consideration, while maintaining the same generality of the approach. Another question relates to the exact estimation of the calmness modulus, in terms of problem data. In the last section, an application of the paper findings to parametric constrained optimization has been considered. Advances in this research direction, carried out in the special case of quasidifferentiable extremum problems during the reviewing process of the present paper, can be found in [42].

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 Springer
References

1. Borwein, J.M., Zhu, Q.J.: Techniques of Variational Analysis. Springer, New York (2005)
2. Chuong, T.D., Kruger, A.Y., Yao, J.-C.: Calmness of efficient solution maps in parametric optimization. J. Glob. Optim. 51, 677–688 (2011)
3. Durea, M., Strugariu, R.: Openness stability and implicit multifunction theorems: applications to variational systems. Nonlinear Anal. 75, 1246–1259 (2012)
4. Klatte, D., Kummer, B.: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications. Kluwer Academic, Dordrecht (2002)
5. Lee, G.M., Tam, N.N., Yen, N.D.: Normal coderivative for multifunctions and implicit function theorems. J. Math. Anal. Appl. 11, 22 (2008)
6. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory. Springer, Berlin (2006)
7. Schirotzek, W.: Nonsmooth Analysis. Springer, Berlin (2007)
8. Uderzo, A.: On some regularity properties in Variational Analysis. Set-Valued Var. Anal. 17, 409–430 (2009)
9. Levy, A.B., Rockafellar, R.T.: Sensitivity analysis of solutions to generalized equations. Trans. Am. Math. Soc. 345, 661–671 (1994)
10. Robinson, S.M.: Generalized equations and their solutions, Part I: Basic theory. Math. Program. Stud. 10, 128–141 (1979)
11. Rockafellar, R.T., Wets, J.-B.: Variational Analysis. Springer, Berlin (1998)
12. Levy, A.B., Rockafellar, R.T.: Variational conditions and the proto-differentiation of partial subgradient mappings. Nonlinear Anal. 26, 1951–1964 (1996)
13. Robinson, S.M.: Variational conditions with smooth constraints: structure and analysis. Math. Program., Ser. B 97, 245–265 (2003)
14. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, II: Applications. Springer, Berlin (2006)
15. Aragón Artacho, F.J., Mordukhovich, B.S.: Metric regularity and Lipschitzian stability of parametric variational inequalities. Nonlinear Anal. 72, 1149–1170 (2010)
16. Aragón Artacho, F.J., Mordukhovich, B.S.: Enhanced metric regularity and Lipschitzian properties of variational systems. J. Glob. Optim. 50, 145–167 (2011)
17. Aubin, J.-P.: Lipschitz behavior of solutions to convex minimization problems. Math. Oper. Res. 9, 87–111 (1984)
18. Geremew, W., Mordukhovich, B.S., Nam, N.M.: Coderivative calculus and metric regularity for constraint and variational systems. Nonlinear Anal. 70, 529–552 (2009)
19. Mordukhovich, B.S.: Failure of metric regularity for major classes of variational system. Nonlinear Anal. 69, 918–924 (2008)
20. Mordukhovich, B.S., Nam, N.M.: Variational analysis of extended generalized equations via coderivative calculus in Asplund spaces. J. Math. Anal. 350, 663–679 (2009)
21. Ngai, H.V., Tron, N.H., Théra, M.: Implicit multifunction theorems in complete metric spaces. Math. Program., Ser. B 139, 301–326 (2013)
22. Burke, J.V.: Calmness and exact penalization. SIAM J. Control Optim. 29, 493–497 (1991)
23. Clarke, F.H.: A new approach to Lagrange multipliers. Math. Oper. Res. 1, 165–174 (1976)
24. Ye, J.J., Zhu, D.L.: Optimality conditions for bilevel programming problems. Optimization 33, 9–27 (1995). (with Erratum in Optimization 39, 361–366 (1997))
25. Ioffe, A.D.: Necessary and sufficient conditions for a local minimum. I: A reduction theorem and first order conditions. SIAM J. Control Optim. 17, 245–250 (1979)
26. Henrion, R., Outrata, J.V.: Calmness of constraint systems with applications. Math. Program., Ser. B 104, 437–464 (2005)
27. Ioffe, A.D., Outrata, J.: On metric and calmness qualification conditions in subdifferential calculus. Set-Valued Anal. 16, 199–227 (2008)
28. Studniarski, M., Ward, D.E.: Weak sharp minima: characterizations and sufficient conditions. SIAM J. Control Optim. 38, 219–236 (1999)
29. Fabian, M.J., Henrion, R., Kruger, A.Y., Outrata, J.: Error bounds: necessary and sufficient conditions. Set-Valued Var. Anal. 18, 121–149 (2010)
30. Zheng, X.Y., Ng, K.F.: Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces. SIAM J. Optim. 18, 437–460 (2007)
31. Zheng, X.Y., Ng, K.F.: Metric subregularity and calmness for nonconvex generalized equations in Banach spaces. SIAM J. Optim. 20, 2119–2136 (2010)
32. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Math. Program. Stud. 14, 206–214 (1981)
33. Levy, A.B.: Supercalm multifunctions for convergence analysis. Set-Valued Anal. 14, 249–261 (2006)
34. De Giorgi, E., Marino, A., Tosques, M.: Problems of evolution in metric spaces and maximal decreasing curves. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) 68, 180–187 (1980). In Italian
35. Azé, D., Corvellec, J.-N., Lucchetti, R.E.: Variational pairs and applications to stability in nonsmooth analysis. Nonlinear Anal. 49, 643–670 (2002)
36. Ioffe, A.D.: Metric regularity and subdifferential calculus. Russ. Math. Surv. 55, 501–558 (2000)
37. Mordukhovich, B.S.: Metric approximations and necessary optimality conditions for general classes of extremal problems. Sov. Math. Dokl. 22, 526–530 (1980)
38. Ekeland, I., Lebourg, G.: Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces. Trans. Am. Math. Soc. 224, 193–216 (1976)
39. Fiacco, A.: Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. Academic Press, New York (1983)
40. Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, New York (2000)
41. Yen, N.D.: Stability of the Solution Set of Perturbed Nonsmooth Inequality Systems and Applications. J. Optim. Theory Appl. 93, 199–225 (1997)
42. Uderzo, A.: On a quantitative semicontinuity property of variational systems with applications to perturbed quasidifferentiable optimization. In: Demjanov, V., Pardalos, P.M., Batsyn, M. (eds.) Constructive Nonsmooth Analysis and Related Topics. Springer, Berlin (2013). To appear