CALLIAS-TYPE OPERATORS IN VON NEUMANN ALGEBRAS

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Abstract. We study differential operators on complete Riemannian manifolds which act on sections of a bundle of finite type modules over a von Neumann algebra with a trace. We prove a relative index and a Callias-type index theorems for von Neumann indexes of such operators. We apply these results to obtain a version of Atiyah’s $L^2$-index theorem, which states that the index of a Callias-type operator on a non-compact manifold $M$ is equal to the $\Gamma$-index of its lift to a Galois cover of $M$. We also prove the cobordism invariance of the index of Callias-type operators. In particular, we give a new proof of the cobordism invariance of the von Neumann index of operators on compact manifolds.

1. Introduction

Index theory of operators acting on bundles of modules over a von Neumann algebra was initiated by Atiyah’s [3]. Atiyah studied the special case of a Galois cover of a compact manifold $M$ with a covering group $\Gamma$. More specifically, let $D$ be an elliptic operator acting on a finite dimensional vector bundle over a compact manifold $M$ and let $\tilde{M}$ be a Galois cover of $M$ with covering group $\Gamma$. We denote by $\mathcal{N}\Gamma$ the von Neumann algebra of $\Gamma$. The lift $\tilde{D}$ of $D$ to $\tilde{M}$ can be viewed as an operator on $M$ with values in a bundle of $\mathcal{N}\Gamma$-modules. Atiyah studied the $\Gamma$-index of this operator and showed that it is equal to the index of $D$. The index theorem for an operator acting on a general bundle of von Neumann modules over a compact manifold was obtained by Schick [36].

In this paper we investigate the von Neumann index of Callias-type operators on non-compact manifolds. A Callias-type operator on a complete Riemannian manifold is an operator of the form $B = D + \Phi$, where $D$ is a first order elliptic operator and $\Phi$ is a potential satisfying certain conditions. If $B$ acts on sections of a finite dimensional vector bundle, then the conditions on $\Phi$ guarantee that the operator $B$ is Fredholm. The study of such operators was initiated by Callias, [19], and further developed by many authors, cf., for example, [7], [17], [2], [18]. Several generalizations and new applications of the Callias-type index theorem were obtained recently, cf. [30], [21], [37], [31], [14].

The extension of the definition of Callias-type index to operators acting on sections of a bundle of modules over a von Neumann algebra was obtained in [13]. In this paper we obtain several properties of this index.

1.1. The relative index theorem. The relative index theorem of Gromov and Lawson [25] allows to use a “cut and paste” procedure to compute the index. This theorem was reformulated

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for Callias-type operators by Anghel [1] and Bunke [18]. Moreover, Bunke considered the case of C*-algebra valued differential operators invertible at infinity.

Our first result in this paper is a version of the relative index theorem for operators acting on bundles of modules over a von Neumann algebra with finite trace. Notice, that our result does not follow directly from [18], since it is not clear whether the von Neumann index descends from the $K$-theoretical index used by Bunke. In particular, it is not clear whether our operators satisfy the conditions required in [18].

1.2. The Callias-type index theorem. The Callias-type index theorem, [2], [18], states that the index of a Callias-type operator $B$ is equal to the index of a certain operator induced by the restriction of $B$ to a compact hypersurface. We show that this result continues to hold for the von Neumann index of operators acting on sections of a bundle of modules over a von Neumann algebra.

Our proof uses the relative index theorem and is similar to the proof in the case of a finite dimensional vector bundles in [2] and [18]. However, more care is needed due to the fact that the spectrum of $B$ is not discrete around 0, see Section 4.6 for details.

1.3. The cobordism invariance of the von Neumann index. The cobordism invariance of the index was used in the original proof of the index theorem [4] (see also [35]). Since then many generalizations and new analytic proofs of the cobordism invariance of the index appeared in the literature, see for example [27], [10], [8], [20], [22], [9], [28], [11]. The cobordism invariance of the index of Callias-type operators acting on finite dimensional vector bundles was recently obtained in [14].

In this paper we show that the cobordism invariance of the index can be obtained as an easy corollary of the Callias-type index theorem. By this method we prove the cobordism invariance of the von Neumann index both for Dirac-type operators on compact manifolds and for Callias-type operators on non-compact manifolds.

1.4. Atiyah’s type $L^2$-index theorem. Let $M$ be a complete Riemannian manifold and let $B$ be a Callias-type operator on $M$, acting on sections of a finite dimensional vector bundle $E \to M$. Let $\tilde{M}$ be a Galois cover of $M$ with covering group $\Gamma$. We denote by $\tilde{B}$ the lift of $B$ to $\tilde{M}$. Then $\tilde{B}$ is a $\Gamma$-equivariant operator and its $\Gamma$-index $\text{ind}_\Gamma \tilde{B}$ is well defined. We show that

\[ \text{ind}_\Gamma \tilde{B} = \text{ind} B. \]

For the case when the manifold $M$ is compact, this result was proven by Atiyah [3].

1.5. The paper is organized as follows: in Section 2 we formulate the main results of the paper. In Section 3 we prove the relative index theorem. Sections 4–6 are devoted to the proof of the Callias-type index theorem. In Section 7 we prove the cobordism invariance of the index. Finally in Section 8 we prove the Atiyah-type $L^2$-index theorem for Callias-type operators.

2. The main results

In this section we formulate the main results of the paper.
2.1. Callias-type operators. Let $A$ be a von Neumann algebra with a finite, faithful, normal trace $\tau : A \to \mathbb{C}$ and let $M$ be a complete Riemannian manifold. Assume $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded $A$-Hilbert bundle of finite type over $M$ (for this notion we refer to [36, Sections 8.1-8.3] and [13, Section 3.5]). Let

\[
D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}
\]

be an elliptic differential operator acting on smooth sections of $E$, where

\[
D^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^{\mp})
\]

are formally adjoint to each other. We make the following

**Assumption 2.2.** The principal symbol of $D$ is uniformly bounded from above, i.e. there exists a constant $b > 0$ such that

\[
\|\sigma(D)(x, \xi)\| \leq b|\xi|, \quad \text{for all } x \in M, \ \xi \in T^*_x M.
\]

Here $|\xi|$ denotes the length of $\xi$ defined by the Riemannian metric on $M$, $\sigma(D)(x, \xi) : E^+_x \to E^-_x$ is the leading symbol of $D$, and $\|\sigma(D)(x, \xi)\|$ is its operator norm.

By [13], the above assumption guarantees that the operator $D$ is essentially self-adjoint.

Let $F^+ : E^+ \to E^-$ be a morphism of $A$-Hilbert bundles and consider the odd self-adjoint endomorphism of $E$

\[
\mathcal{F} := \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix},
\]

where $F^- : E^- \to E^+$ is the formal adjoint of $F^+$.

**Definition 2.3.** The endomorphism $\mathcal{F}$ is called admissible if

(i) the anticommutator

\[
\{D, \mathcal{F}\} := D \circ \mathcal{F} + \mathcal{F} \circ D = \begin{pmatrix} D^- F^+ + F^- D^+ & 0 \\ 0 & D^+ F^- + F^+ D^- \end{pmatrix}
\]

is an operator of order 0;

(ii) there exist a constant $c > 0$ and a compact set $K \subset M$ such that

\[
\mathcal{F}^2 + \{D, \mathcal{F}\} \geq c, \quad \text{on } M \setminus K.
\]

In this case, we say that the compact set $K$ is an essential support of $\mathcal{F}$.

**Definition 2.4.** A Callias-type operator is an operator of the form

\[
B := D + \mathcal{F} = \begin{pmatrix} 0 & D^- + F^- \\ D^+ + F^+ & 0 \end{pmatrix},
\]

where $\mathcal{F}$ is an admissible endomorphism. We set $B^\pm := D^\pm + F^\pm$. 
2.5. The $\tau$-index of a Callias-type operator. We use the Riemannian metric on $M$ and the inner product on the fibers of $E$ to define the $A$-Hilbert space $L^2(M,E)$ of square-integrable sections of $E$ and we regard $B$ as an $A$-linear unbounded operator on $L^2(M,E)$. By [13, Theorem 2.3], $B$ is essentially self-adjoint with initial domain $C_0^\infty(M,E)$. We denote its closure by the same symbol $B$.

The trace $\tau : A \rightarrow \mathbb{C}$ on $A$ induces a dimension function

$$\dim_{\tau} : \text{Gr}(L^2(M,E)) \rightarrow [0, \infty),$$

where $\text{Gr}(L^2(M,E))$ denotes the set of closed $A$-invariant subspaces of $L^2(M,E)$: for the construction of the function $\dim_{\tau}$ out of the trace $\tau$ see [33, Sections 1.1.3 and 9.1.4]. By [13, Theorem 2.19], the operator $B$ is $\tau$-Fredholm (for the notion of closed $\tau$-Fredholm operator we refer to [15] and [16]). In particular, this means that the $A$-Hilbert spaces $\text{Ker} B^\pm$ have finite $\tau$-dimension and we define the $\tau$-index of $B$ by

$$\text{ind}_{\tau} B := \dim_{\tau} \ker B^+ - \dim_{\tau} \ker B^-.$$  \hspace{1cm} (2.4)

2.6. A relative index theorem. Suppose $E_j$, with $j = 0, 1$, are $\mathbb{Z}_2$-graded $A$-Hilbert bundles over complete Riemannian manifolds $M_j$ and $B_j = D_j + F_j$ are Callias-type operators. Suppose $M_j = W_j \cup_{N_j} V_j$ are partitions of $M_j$, where $N_j$ are compact hypersurfaces. We make the following

**Assumption 2.7.** There exist tubular neighborhoods $U(N_0)$, $U(N_1)$ respectively of $N_0$, $N_1$ and a diffeomorphism $\psi : U(N_0) \rightarrow U(N_1)$ such that:

(i) $\psi$ restricts to a diffeomorphism between $N_0$ and $N_1$;

(ii) there exists an isomorphism of $\mathbb{Z}_2$-graded $A$-Hilbert bundles $\Psi : E_0|_{U(N_0)} \rightarrow E_1|_{U(N_1)}$ covering $\psi$;

(iii) the restrictions $D_0|_{U(N_0)}$ and $D_1|_{U(N_1)}$ are conjugated through the isomorphism $\Psi$.

We cut $M_i$ along $N_i$ and use the map $\psi$ to glue the pieces together interchanging $V_0$ and $V_1$. In this way we obtain the manifolds

$$M_2 := W_0 \cup_{N} V_1, \quad M_3 := W_1 \cup_{N} V_0,$$

where $N \cong N_0 \cong N_1$. We refer to $M_2$ and $M_3$ as manifolds obtained from $M_0$ and $M_1$ by cutting and pasting.

We use the map $\Psi$ to cut the bundles $E_0$, $E_1$ at $N_0$, $N_1$ and glue the pieces together interchanging $E_0|_{V_0}$ and $E_1|_{V_1}$. With this procedure we obtain $\mathbb{Z}_2$-graded $A$-Hilbert bundles $E_2 \rightarrow M_2$ and $E_3 \rightarrow M_3$. Use Condition (iii) of Assumption 2.7 to construct an odd formally self-adjoint first order elliptic operator $D_2$ acting on smooth sections of $E_2$ satisfying Assumption 2.2 and such that $D_2|_{W_0} = D_0|_{W_0}$ and $D_2|_{V_1} = D_1|_{V_1}$. In the same fashion, define an elliptic operator $D_3$ acting on smooth sections of the bundle $E_3$.

Assume $F_0$ and $F_1$ are admissible endomorphisms respectively of the bundles $E_0$ and $E_1$. For $j = 0, 1$ we choose positive functions $\alpha_j, \beta_j \in C^\infty(M_j)$ such that:

(C.1) $\text{supp} \alpha_j \subset W_j \cup U(N_j)$ and $\text{supp} \beta_j \subset V_j \cup U(N_j);$
\((C.2)\) \(\alpha_j = 1\) on \(W_j \setminus U(N_j)\) and \(\beta_j = 1\) on \(V_j \setminus U(N_j)\);

\((C.3)\) \(\alpha_j^2 + \beta_j^2 = 1\).

By construction, \(E_2|_{W_0 \cup U(N_0)} \cong E_0|_{W_0 \cup U(N_0)}\) and \(E_2|_{V_1 \cup U(N_1)} \cong E_1|_{V_1 \cup U(N_1)}\). We use this identification and Condition \((C.1)\) to define the endomorphism \(F_2 := F_0 \alpha_0 + F_1 \beta_1\) of \(E_2\). In the same way we define the endomorphism \(F_3 := F_1 \alpha_1 + F_0 \beta_0\) of \(E_3\). Observe that \(F_2\) and \(F_3\) are admissible so that the operators \(B_2 := D_2 + F_2\) and \(B_3 := D_3 + F_3\) are of Callias-type. We often refer to \(B_2\) and \(B_3\) as operators obtained from \(B_0\) and \(B_1\) by cutting and pasting.

The first result of the paper is the following

**Theorem 2.8** (Relative index theorem in von Neumann setting).

\[ \text{ind}_\tau B_0 + \text{ind}_\tau B_1 = \text{ind}_\tau B_2 + \text{ind}_\tau B_3. \]

**Remark 2.9.** In the case \(A = \mathbb{C}\), this theorem was proved by Gromov and Lawson in [25]. A \(K\)-theoretical version has been proved by Bunke in [18]. Our formulation of the relative index theorem is close to this last one. The reason why we cannot apply directly [18, Theorem 1.2] is that it is not clear whether the numerical index that we use in the present paper descends from a \(K\)-theoretical one. In particular, it is not clear whether the operator \(B\) is invertible at infinity in the sense of Bunke, cf. the discussion on page 258 of [18].

**2.10. Ungraded bundles and Callias-type operators.** We now describe a class of Callias-type operators constructed out of an ungraded bundle. Let \(M\) be a complete odd-dimensional Riemannian manifold and let \(\Sigma \to M\) be an ungraded \(\mathcal{A}\)-Hilbert bundle over \(M\). Suppose \(D : C^\infty_c(M, \Sigma) \to C^\infty_c(M, \Sigma)\) is a formally self-adjoint first order elliptic operator satisfying Assumption 2.2. Let \(\Phi\) be a self-adjoint endomorphism of \(\Sigma\).

**Definition 2.11.** The endomorphism \(\Phi\) is said to be admissible for the pair \((\Sigma, D)\) if

(i) the commutator \([D, \Phi] := D\Phi - \Phi D\) is an endomorphism of \(\Sigma\);
(ii) there exist a constant \(d > 0\) and a compact set \(K \subset M\) such that

\[ \Phi^2(x) \geq d + \|[D, \Phi](x)\|, \quad x \in M \setminus K, \]

where \(\|[D, \Phi](x)\|\) denotes the operator norm of the bounded operator \([D, \Phi](x) : \Sigma_x \to \Sigma_x\). In this case we say that \(K\) is an essential support of \(\Phi\) with respect to the pair \((\Sigma, D)\) and that \((\Sigma, D, \Phi)\) is an admissible ungraded triple over \(M\).

Suppose the endomorphism \(\Phi\) is admissible and regard \(\Sigma \oplus \Sigma\) as a \(\mathbb{Z}_2\)-graded \(\mathcal{A}\)-Hilbert bundle over \(M\). Then the first order elliptic operator

\[ D := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \]

is odd with respect to this grading, formally self-adjoint and satisfies Assumption 2.2. We use the endomorphism \(\Phi\) to define the odd self-adjoint endomorphism of \(\Sigma \oplus \Sigma\)

\[ F_\Phi := i \begin{pmatrix} 0 & -\Phi \\ \Phi & 0 \end{pmatrix}. \]
Conditions (i) and (ii) of Definition 2.11 imply that $\mathcal{F}_\Phi$ is admissible according to Definition 2.3 and any essential support of $\Phi$ is also an essential support of $\mathcal{F}_\Phi$.

**Definition 2.12.** The operator

$$B_\Phi := D + \mathcal{F}_\Phi = \begin{pmatrix} 0 & D - i\Phi \\ D + i\Phi & 0 \end{pmatrix}$$

is called the Callias-type operator associated with the triple $(\Sigma, D, \Phi)$.

The equation (2.4) for the Callias index takes in this case the form

$$\text{ind}_\tau B_\Phi = \dim_\tau \ker(D + i\Phi) - \dim_\tau \ker(D - i\Phi).$$

(2.9)

In particular, we obtain

$$\text{ind}_\tau B_{-\Phi} = -\text{ind}_\tau B_\Phi.$$ (2.10)

Since the index of the Callias-type operator on a compact manifold is independent of $\Phi$ we obtain the following

**Proposition 2.13.** The index of a Callias-type operator associated with an ungraded triple $(\Sigma, D, \Phi)$ over a compact manifold is equal to zero.

**Remark 2.14.** The operator (2.8) is a Callias-type operator in the sense of Definition 2.4. Notice, however, that not every Callias-type operator is associated with some triple $(\Sigma, D, \Phi)$. In fact, an operator (2.3) is associated with such a triple if and only if there exists an isomorphism $E^+ \to E^-$ which commutes with the leading symbol of $D$ and anticommutes with the admissible endomorphism $\mathcal{F}$.

In the case $A = \mathbb{C}$, Callias-type operators associated with a triple $(\Sigma, D, \Phi)$ were extensively studied, cf. for example, [19], [17] and [2].

2.15. **Callias-type Theorem.** Let $M$ be a complete oriented $n$-dimensional Riemannian manifold. Let $\Sigma \to M$ be an ungraded Dirac $A$-Hilbert bundle over $M$. This means that $\Sigma$ is an $A$-Hilbert bundle of finite type endowed with a Clifford action $c : T^*M \to \text{End}_A(\Sigma)$ of the cotangent bundle and a metric connection $\nabla^\Sigma$ compatible with the inner product of the fibers and satisfying the Leibniz rule (for the theory of connections on $A$-Hilbert bundles we refer to [36 Sections 8.1-8.3]). When $A = \mathbb{C}$, our definition coincides with the classical notion of a Dirac bundle (see [32 Definition II.5.2]).

An operator $D : C^\infty_0(M, \Sigma) \to C^\infty(M, \Sigma)$ is called a (generalized) Dirac operator if

$$[D, f] = c(df) \quad \text{for all } f \in C^\infty(M).$$

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of the tangent bundle, and $(e^1, \ldots, e^n)$ be the dual basis of $T^*M$. By [5 §3.3], there exists $V \in \text{End}_A(\Sigma)$ such that

$$D = \sum_{i=1}^n c(e^i) \nabla^\Sigma_{e_i} + V.$$ (2.11)
In this situation we refer to \( D \) as the \textit{Dirac operator associated with the connection} \( \nabla^\Sigma \) \textit{and the potential} \( V \).

It follows from (2.11) that \( D \) satisfies Assumption 2.2. Suppose \( \Phi \in \text{End}_A(\Sigma) \) is admissible. Note that Condition (i) of Definition 2.3 is equivalent in this case to the condition that

\[
[c(\xi), \Phi] = 0, \quad \text{for all} \quad \xi \in T^*M. \tag{2.12}
\]

We denote by \( B_\Phi \) the Callias-type operator associated with the triple \( (\Sigma, D, \Phi) \).

Suppose that there is a partition \( M = M_- \cup N \cup M_+ \), where \( N = M_- \cap M_+ \) is a smooth compact hypersurface and \( M_- \) is a compact submanifold, whose interior contains an essential support of \( \Phi \). Let \( \Sigma_N \) be the restriction of \( \Sigma \) to \( N \subset M \). By Condition (ii) of Definition 2.11, zero is not in the spectrum of \( \Phi(x) \) for all \( x \in N \). Therefore

\[
\Sigma_N = \Sigma_{N+} \oplus \Sigma_{N-}, \tag{2.13}
\]

where the fiber of \( \Sigma_{N+} \) (resp. \( \Sigma_{N-} \)) over \( x \in N \) is the image of the spectral projection of \( \Phi_N(x) \) corresponding to the interval \((0, \infty)\) (resp. \((-\infty, 0)\)). By (2.12) the endomorphism \( \Phi \) commutes with the Clifford multiplication. Hence \( c(\xi) : \Sigma_{N\pm} \to \Sigma_{N\pm} \) for all \( \xi \in T^*M \). It follows that both bundles, \( \Sigma_{N+} \) and \( \Sigma_{N-} \), inherit the Clifford action of \( T^*M \).

In particular, the Clifford multiplication by the unit normal vector field pointing at the direction of \( M_+ \) defines an endomorphism \( \gamma : \Sigma_{N\pm} \to \Sigma_{N\pm} \). Since \( \gamma^2 = -1 \), the endomorphism \( \alpha := -i\gamma \) induces a grading

\[
\Sigma_{N\pm} = \Sigma_{N\pm+} \oplus \Sigma_{N\pm-}, \tag{2.14}
\]

where \( \Sigma_{N\pm} \) is the span of the eigenvectors of \( \alpha \) with eigenvalues \( \pm 1 \).

We use the Riemannian metric on \( M \) to identify \( T^*N \) with a subspace of \( T^*M \). Then the Clifford action of \( T^*N \) on \( \Sigma_{N\pm} \) is graded with respect to this grading, i.e. \( c(\xi) : \Sigma_\pm \to \Sigma_\pm \) for all \( \xi \in T^*N \).

Let \( \nabla^{\Sigma_N} \) be the connection on \( \Sigma_N \) obtained by restricting the connection on \( \Sigma \). It does not, in general, preserve decomposition (2.13). We define a connection \( \nabla^{\Sigma_N\pm} \) on the bundle \( \Sigma_{N\pm} \) by

\[
\nabla^{\Sigma_N\pm}s^\pm = \text{pr}_{\Sigma_{N\pm}}(\nabla^{\Sigma_N}s^\pm), \quad s^\pm \in C^\infty(N, \Sigma_{N\pm}), \tag{2.15}
\]

where \( \text{pr}_{\Sigma_{N\pm}} \) is the projection onto the bundle \( T^*N \otimes \Sigma_{N\pm} \). Since \( \gamma \) commutes with both, the connection \( \nabla^{\Sigma_N} \) and the projection \( \text{pr}_{\Sigma_{N\pm}} \), it also commutes with the connection \( \nabla^{\Sigma_{N\pm}} \). In other words, \( \nabla^{\Sigma_{N\pm}} \) is a graded connection with respect to the grading (2.14). By [1, Lemma 2.7] (see also Section 5.1), it is also compatible with the inner product of the fibers of \( \Sigma_{N\pm} \) and satisfies the Liebniz rule. Therefore, \( \Sigma_{N\pm} \) carries a \( \mathbb{Z}_2 \)-graded Dirac \( A \)-Hilbert bundle structure.

We denote by \( D_N := D_{N+} \oplus D_{N-} \) the Dirac operator on \( N \) associated with the connection \( \nabla^{\Sigma_{N+}} \oplus \nabla^{\Sigma_{N-}} \) and the zero potential. Then

\[
D_{N\pm} := \sum_{i=1}^{n-1} c(e^i) \nabla_{e_i}^{\Sigma_{N\pm}}, \tag{2.16}
\]
where \((e_1, \ldots, e_{n-1})\) is an orthonormal frame of \(TN\) and \((e^1, \ldots, e^{n-1})\) is the dual frame of \(T^*N\). The operators \(D_{N\pm}\) are odd with respect to the grading \((2.14)\), i.e. they have the form

\[
D_{N\pm} = \begin{pmatrix}
0 & D^-_{N\pm} \\
D^+_N & 0
\end{pmatrix},
\]

where \(D^+_N\) (respectively \(D^-_{N\pm}\)) is the restriction of \(D_{N\pm}\) to \(\Sigma^+\) (respectively \(\Sigma^-_{N\pm}\)). It is a classical fact that \(D_{N\pm}\) are \(\tau\)-Fredholm (see for instance [36, Section 7]) so that they have a well-defined \(\tau\)-index

\[
\text{ind}_\tau D_{N\pm} = \dim_\tau \ker D^+_N - \dim_\tau \ker D^-_{N\pm}.
\]

The next theorem is the first main result of this paper. It generalizes [2, Theorem 1.5] and [18, Theorem 2.9] to the von Neumann setting.

**Theorem 2.16** (Callias-type theorem in von Neumann setting).
\[
\text{ind}_\tau B \Phi = \text{ind}_\tau D_{N^+}.
\]

From this theorem and \((2.10)\) we immediately obtain

**Corollary 2.17.** \(\text{ind}_\tau B \Phi = - \text{ind}_\tau D_{N^-}\). Hence,
\[
\text{ind}_\tau D_{N^+} = - \text{ind}_\tau D_{N^-}.
\]

**Remark 2.18.** Equality \((2.18)\) might look a bit surprising, since at the first glance the bundles \(\Sigma^+\) and \(\Sigma^-\) might look completely unrelated. However, the fact that both operators, \(D_{N^+}\) and \(D_{N^-}\), are induced by the same operator \(D\) puts a strong relation between them. In fact, the direct sum \(D_{N^+} \oplus D_{N^-}\) is cobordant to \(0\) in the sense of \([10]\), and the cobordism is given by the operator \(D\). Thus the equality \((2.18)\) can be viewed as a version of the cobordism invariance of the index. In fact, it is, in a sense, equivalent to the cobordism invariance, as explained in the next subsection.

If \(\Phi\) is an admissible function for \(\Sigma\) then so is \(\lambda \Phi\) for all \(\lambda > 1\). As an immediate corollary of Theorem \(2.16\) we obtain the following

**Corollary 2.19.** The index \(\text{ind}_\tau B_{\lambda \Phi}\) is independent of \(\lambda \geq 1\).

**Remark 2.20.** If the endomorphism \(\Phi\) is bounded then the domain of the operator \(B_{\lambda \Phi}\) is independent of \(\lambda\) and the corollary follows directly from the stability of the \(\tau\)-index, cf. \([13\), Lemma 7.3]. However, if \(\Phi\) is not bounded the domain of \(B_{\lambda \Phi}\) depends on \(\lambda\) and the corollary is not a priori obvious.

2.21. **The cobordism invariance.** We now introduce a class of non-compact cobordisms similar to those considered in \([24, 26, 8, 12, 14]\). Then we show that the index is invariant under this type of cobordisms. To make the exposition simpler we will only discuss the null-cobordism (i.e. the cobordism between a manifold and the empty set). We show that the index of a null-cobordant operator vanishes. A standard argument, cf. for example \([14\) Remark 2.10], shows that this is equivalent to the cobordism invariance of the index.
Definition 2.22. Let $\Sigma \to M$ be a Dirac bundles over a complete oriented Riemannian manifold $M$. Let $D$ be a (generalized) Dirac operator on $\Sigma$.

A triple $(W, \Sigma, D)$, where $W$ is a complete manifold with boundary $\partial W \simeq M$, $\Sigma$ is a Dirac $A$-bundle over $W$, and $\overline{D}$ is a Dirac operator on $\Sigma$, is called a null-cobordism of $D$ if the following conditions are satisfied:

(i) There is an open neighborhood $U$ of $\partial W$ and a metric-preserving diffeomorphism
\[ \phi : M \times (-\epsilon, 0] \to U. \]
(2.19)

(ii) Let $\hat{\Sigma} \simeq \Sigma \times \mathbb{R}$ denote the lift of $\Sigma$ to $M \times \mathbb{R}$. Then the bundle $\hat{\Sigma} \oplus \hat{\Sigma}$ has a natural structure of a Dirac bundle over $M \times \mathbb{R}$, cf. Section 5.1. We assume that the restriction of $\Sigma$ to $M \times (-\epsilon, 0] \subset U$ is isomorphic to $\hat{\Sigma} \oplus \hat{\Sigma}$.

(iii) Let $t$ denote the coordinate on $\mathbb{R}$. Consider the operator
\[ \hat{D} := \begin{pmatrix} \frac{\partial}{\partial t} & D \\ D & -\frac{\partial}{\partial t} \end{pmatrix} : C^\infty_0(M \times \mathbb{R}, \hat{\Sigma} \oplus \hat{\Sigma}) \to C^\infty_0(M \times \mathbb{R}, \hat{\Sigma} \oplus \hat{\Sigma}). \]

Then the restriction of $\overline{D}$ to $M \times (-\epsilon, 0]$ is equal to the restriction of $\hat{D}$ to the same set.

Notice, that the notion of non-compact cobordism is not very useful by itself. For example for any triple $(M, \Sigma, D)$ the pair $(M \times [0, \infty), \Sigma \times [0, \infty), \overline{D})$ is a null-cobordism of $(M, \Sigma, D)$. However, if one considers cobordisms carrying some extra structure, like in the next definition, the notion of non-compact cobordism becomes useful and non-trivial.

Definition 2.23. Let $\Sigma \to M$ be a Dirac bundle over a complete oriented Riemannian manifolds $M$ and let $D$ be a (generalized) Dirac operator on $\Sigma$. Suppose $\Phi \in \text{End}_A(\Sigma)$ is an admissible endomorphism for $(\Sigma, D)$. Let $B_\Phi$ be the corresponding Callias-type operator. A null-cobordism of $B_\Phi$ consists of the following data:

(i) A null-cobordism $(W, \Sigma, D)$ of $D$;

(ii) A bundle map $\overline{\Phi} \in \text{End}_A(\Sigma)$, such that the restriction of $\overline{\Phi}$ to $M \subset \partial W$ is equal to
\[ \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \in \text{End}_A(\hat{\Sigma} \oplus \hat{\Sigma}) \]

and there exist a constant $d > 0$ and a compact set $K \subset W$ such that Inequality (2.5) holds for all $x \in W \setminus K$.

The operator $B_\Phi$ is called null-cobordant if there exists a null-cobordism of $B_\Phi$.

Theorem 2.24. The $\tau$-index of a null-cobordant Callias-type operator is equal to 0.

In the case when the manifold $M$ is compact the theorem implies the cobordism invariance of the $\tau$-index of Dirac operators on compact manifolds. In fact, as a part of the proof of Theorem 2.24 in Section 7, we give a new proof of the cobordism invariance of the $\tau$-index on compact manifolds.
Remark 2.25. In this paper we first prove the relative index and the Callias-type index theorems and then use them to prove the cobordism invariance of the index. The opposite order of arguments is also possible. In fact, for the case \(A = \mathbb{C}\), i.e. for operators acting on finite dimensional vector bundles, the direct proof of the cobordism invariance of the Callias index was given in [14]. The proof in [14] can be adapted to our current situation using a method similar to the one used in Section 4. Then, as it is explained in [14], the cobordism invariance of the index implies the relative index theorem.

2.26. The \(\Gamma\)-index Theorem. Let \(M\) be a complete oriented Riemannian manifold and let \(S \to M\) be an ungraded finite dimensional Dirac bundle. Let \(D\) be a Dirac operator on \(S\), cf. (2.11). Let \(\Phi\) be an admissible endomorphism of \(\Sigma\). We also suppose that \(\tilde{M}\) is a Galois cover of \(M\) with deck transformation group \(\Gamma\). The bundle \(S\) lifts to a bundle \(\tilde{S} \to \tilde{M}\) and the Callias-type operator \(B_{\Phi} := D + F_{\Phi}\) defined by (2.8) lifts to a \(\Gamma\)-equivariant first order elliptic operator \(\tilde{B}_{\Phi} := \tilde{D} + \tilde{F}_{\Phi}\) acting on smooth sections of \(\tilde{S} \oplus \tilde{S}\).

The \(\Gamma\)-action on \(C^\infty_c(\tilde{M}, \tilde{S})\) induces an action of \(\Gamma\) on the Hilbert space \(L^2(\tilde{M}, \tilde{S})\). Fix a closed \(\Gamma\)-invariant subspace \(L\) of \(L^2(\tilde{M}, \tilde{S})\). Denote by \(P_L\) the orthogonal projection onto \(L\) and by \(K_L(\cdot, \cdot)\) the Schwartz kernel of \(P_L\). For \(x \in \tilde{M}\), \(K_L(x, x)\) is an endomorphism of the finite dimensional complex vector space \(\tilde{S}_x\) and has a well-defined trace \(\text{tr} K_L(\cdot, \cdot) \in [0, \infty]\). The \(\Gamma\)-dimension of \(L\) is defined by the formula

\[
\text{dim}_\Gamma(L) := \int_\Omega \text{tr} K_L(x, x) \, d\tilde{\mu}(x), \tag{2.20}
\]

where \(\Omega \subset \tilde{M}\) is a fundamental domain for the action of \(\Gamma\) and \(d\tilde{\mu}\) is the positive \(\Gamma\)-invariant measure induced by the Riemannian metric on \(M\) lifted to \(\tilde{M}\). Equation (2.20) defines a dimension function on the set of closed \(\Gamma\)-invariant subspaces of \(L^2(\tilde{M}, \tilde{S})\). We say that \(L\) has a finite von Neumann dimension if \(\text{dim}_\Gamma(L) < \infty\).

In Section 8.4, we show that \(\tilde{B}_\Phi\) can be regarded as a Callias-type operator constructed out of a suitable ungraded Dirac \(\mathcal{N}\Gamma\)-bundle \(W\), where \(\mathcal{N}\Gamma\) is the group von Neumann algebra of \(\Gamma\) (see Section 8.4). As a consequence of [13, Theorem 2.3] we deduce the following

Lemma 2.27. The closed \(\Gamma\)-invariant subspaces \(\ker(\tilde{D} \pm i\tilde{\Phi})\subset L^2(\tilde{M}, \tilde{V}^\pm)\) have finite von Neumann dimension.

Define the \(\Gamma\)-index of \(\tilde{B}_\Phi\) by

\[
\text{ind}_\Gamma \tilde{B}_\Phi := \text{dim}_\Gamma \ker(\tilde{D} + i\tilde{\Phi}) - \text{dim}_\Gamma \ker(\tilde{D} - i\tilde{\Phi}).
\]

The second main result of this paper is the following generalization of Atiyah’s \(\Gamma\)-index theorem (cf. [3, Formula (1.1)]) to operators of Callias-type.

---

1In other words in this section we assume that \(A = \mathbb{C}\). We use the notation \(S\) (rather than \(\Sigma\)) for the Dirac bundle, to stress the difference from the other sections where \(\Sigma\) was a Hilbert \(A\)-bundle.
Theorem 2.28 (Γ-index theorem for Callias-type operators). Let $S$ be an ungraded Dirac bundle over a complete oriented Riemannian manifold $M$ and let $Φ$ be an admissible self-adjoint endomorphism of $S$. If $\tilde{M}$ is a Galois cover of $M$, then

$$\operatorname{ind}_Γ \tilde{B}_Φ = \operatorname{ind} B_Φ,$$

(2.21)

where $Γ$ is the group of the deck transformations of $\tilde{M} \to M$.

2.29. Idea of the proof. We prove Theorem 2.28 by reduction to the compact case. We choose a compact hypersurface $N$ lying outside of an essential support of $Φ$ and use Theorem 2.16 to construct a $\mathbb{Z}_2$-graded Dirac bundle $E \to N$ and a flat Dirac $\mathcal{N}$-bundle $W \to N$ such that

$$\operatorname{ind} B_Φ = \operatorname{ind} D, \quad \operatorname{ind}_Γ \tilde{B}_Φ = \operatorname{ind}_Γ τ_D W,$$

(2.22)

where $D$ (resp. $D_W$) is the graded Dirac operator associated with $E$ (resp. $E \otimes W$). The Atiah’s $L^2$-index theorem [3] (see also [36]) implies that

$$\operatorname{ind} D = \operatorname{ind}_Γ τ_D W.$$ 

(2.23)

Finally, formula (2.21) follows from (2.22) and (2.23).

3. The relative index theorem

In this section we prove Theorem 2.8. We adapt to the von Neumann setting the $K$-theoretical argument used by Bunke in [18].

3.1. Idea of the Proof. Let $M_j, E_j, B_j := D_j + F_j$ denote the same objects as in Section 2.6 and set $M := M_1 \sqcup M_2 \sqcup M_3 \sqcup M_4$. We use $E_1, E_2, E_3$ and $E_4$ to construct the bundle

$$E := E_1 \oplus E_2 \oplus E_3^{\text{op}} \oplus E_4^{\text{op}}$$

(3.1)

over $M$, where the superscript “$\text{op}$” means that we consider the opposite grading on the fibers of $E_3$ and $E_4$ (see also Section 3.2 for this notation). Consider the Callias-type operator

$$B := B_1 \oplus B_2 \oplus B_3 \oplus B_4$$

(3.2)

acting on smooth sections of $E$. Then

$$\operatorname{ind}_Γ B = \operatorname{ind}_Γ B_1 + \operatorname{ind}_Γ B_2 - \operatorname{ind}_Γ B_3 - \operatorname{ind}_Γ B_4.$$ 

(3.3)

Theorem 2.8 is then equivalent to the equality $\operatorname{ind}_Γ B = 0$.

For $j = 1, 2$, let $α_j, β_j \in C^∞(M_j)$ be the functions defined in Section 2.6. We use these functions to construct an odd self-adjoint $A$-equivariant bundle map $U \in C^∞(M, \operatorname{End}_A(E))$ with the following properties:

(U.1) $U^2 = \operatorname{id}_E$;

(U.2) the anticommutator $\{B, U\} := BU + UB$ is an even compactly supported endomorphism of the bundle $E$.  


By Condition (U.1), $U^{-1} = U$. Since the degree of $U$ with respect to the grading on $E$ is odd, i.e. $U : E^\pm \to E^\mp$, we have

$$\text{ind}_\tau UBU = - \text{ind}_\tau B. \tag{3.4}$$

From Condition (U.2) we deduce that $B + UBU$ is a compactly supported odd self-adjoint $A$-linear endomorphism of $E$. Hence, by [13, Theorem 2.21] we get

$$\text{ind}_\tau UBU = \text{ind}_\tau B. \tag{3.5}$$

By Equations (3.4) and (3.5) we finally deduce that $\text{ind}_\tau B = - \text{ind}_\tau B$, from which the thesis follows.

3.2. Construction of the operator $B$. Let us first introduce some notation. Given a $\mathbb{Z}_2$-graded $A$-module $Z = Z^+ \oplus Z^-$, we denote by $Z^{\text{op}}$ the $A$-module $Z$ endowed with the opposite grading, i.e. $(Z^{\text{op}})^\pm = Z^\mp$. The bundles $E_1 \to M_1$, $E_2 \to M_2$, $E_3^{\text{op}} \to M_3$ and $E_4^{\text{op}} \to M_4$ define a $\mathbb{Z}_2$-graded $A$-Hilbert bundle $E \to M$ through Formula (3.1). Observe that $C^\infty(M,E^\pm) = C^\infty(M_1,E_1^\pm) \oplus C^\infty(M_2,E_2^\pm) \oplus C^\infty(M_3,E_3^\mp) \oplus C^\infty(M_4,E_4^\mp)$.

With respect to this decomposition, define the odd formally self-adjoint elliptic differential operator

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

where

$$D^\pm = \begin{pmatrix} D_1^\pm & 0 & 0 & 0 \\ 0 & D_2^\pm & 0 & 0 \\ 0 & 0 & D_3^\mp & 0 \\ 0 & 0 & 0 & D_4^\mp \end{pmatrix} : C^\infty(M,E^\pm) \to C^\infty(M,E^\mp).$$

Here, $D_j^\pm = D_j|_{C^\infty(M,E^\pm)}$. Notice that the operator $D$ satisfies Assumption 2.2. Define the self-adjoint endomorphism of $E$

$$F := \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}, \tag{3.6}$$

where

$$F^\pm = \begin{pmatrix} F_1^\pm & 0 & 0 & 0 \\ 0 & F_2^\pm & 0 & 0 \\ 0 & 0 & F_3^\mp & 0 \\ 0 & 0 & 0 & F_4^\mp \end{pmatrix} : E^\pm \to E^\mp.$$ 

Observe that $F$ is admissible so that the operator

$$B := D + F$$

is of Callias-type. Denote by $H_j$ the $\mathbb{Z}_2$-graded $A$-Hilbert spaces $L^2(M_j,E_j)$ and set

$$H := H_1 \oplus H_2 \oplus H_3^{\text{op}} \oplus H_4^{\text{op}}. \tag{3.7}$$

Notice that $H$ coincides with the space of $L^2$-sections of the $A$-Hilbert bundle $E$. As in Section 2.1 we use [13] to conclude that the Callias-type operator $B$ as a closed odd self-adjoint $\tau$-Fredholm operator on $H$. Similarly, the Callias-type operators $B_j$ ($j = 1, \ldots, 4$) is a closed odd self-adjoint
We now assume that the functions $\alpha_j, \beta_j \in C^\infty(M_j)$ were defined in Section 2.6. The function $\alpha_1$ is smooth and supported in $W_1 \cup U(N_1)$. Since $E_1|_{W_1 \cup U(N_1)} \cong E_3|_{W_1 \cup U(N_1)}$, multiplication by $\alpha_1$ defines a bundle map $a : E_1 \to E_3$. In the same fashion, we construct bundle maps $b : E_1 \to E_4$, $c : E_2 \to E_3$, and $d : E_2 \to E_4$ by using respectively the functions $\beta_1, \beta_2$, and $\alpha_2$. Observe that we can also regard $\alpha_1$ as a smooth function on $M_3$ with support in $W_1 \cup U(N_1)$. By using again the isomorphism $E_3|_{W_1 \cup U(N_1)} \cong E_1|_{W_1 \cup U(N_1)}$, multiplication by $\alpha_1$ gives a bundle map $E_3 \to E_1$, that coincides with the map $a^*$ adjoint to $a$. In the same way we see that the maps $b^*, c^*$ and $d^*$ are multiplication respectively by the functions $\beta_1, \beta_2$, and $\alpha_2$. Condition (C.3) of Section 2.6 implies

$$aa^* + bb^* = 1, \quad cc^* + dd^* = 1. \tag{3.8}$$

We now assume that the functions $\alpha_j$ and $\beta_j$ are chosen in such a way that

$$\alpha_1|_{U(N_1)} = \alpha_2|_{U(N_2)}, \quad \beta_1|_{U(N_1)} = \beta_2|_{U(N_2)}, \tag{3.9}$$

where the neighborhoods $U(N_1)$ and $U(N_2)$ are identified through the isomorphism $\psi$ of Assumption 2.7. Define the odd $A$-linear self-adjoint operator

$\begin{bmatrix}
0 & 0 & -a^* & -b^* \\
0 & 0 & -c^* & d^* \\
a & c & 0 & 0 \\
b & -d & 0 & 0
\end{bmatrix} : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp).$

We remark that the operator $V$ is odd because we use the opposite grading on $E_3$ and $E_4$. Let $\epsilon$ be the grading operator on $E$, i.e., $\epsilon|_{E^\pm} = \pm 1$. Finally set

$$U := \epsilon V.$$

In the next two lemmas we show that the map $U$ has the required properties.

First, notice that since $V$ is odd with respect to the grading of $E$, we have

$$\{\epsilon, U\} = \epsilon V + V \epsilon = 0, \tag{3.10}$$

i.e., $U$ is also an odd operator.

**Lemma 3.4.** $U$ satisfies Condition (U.1).

**Proof.** Equations (3.8) and (3.9) together with Condition (C.1) of Section 2.6 imply

$$V \circ V = \begin{pmatrix}
-a_1^2 - \beta_1^2 & -\alpha_1 \beta_2 + \beta_1 \alpha_2 & 0 & 0 \\
-\beta_2 \alpha_1 + \alpha_2 \beta_1 & -\beta_2^2 - \alpha_2^2 & 0 & 0 \\
0 & 0 & -\alpha_1^2 - \beta_2^2 & -\alpha_1 \beta_1 + \beta_2 \alpha_2 \\
0 & 0 & -\beta_1 \alpha_1 + \alpha_2 \beta_2 & -\beta_1^2 - \alpha_2^2
\end{pmatrix} = -\text{id}_E.$$

Hence, $U^2 = \epsilon V \epsilon V = -V^2 = \text{id}_E$. \hfill \Box

**Lemma 3.5.** $U$ satisfies condition (U.2).


Proof. Since both $B$ and $U$ are odd differential operators, the commutator $\{B,U\}$ is a differential operator of even degree. It remains to show that $\{B,U\}$ is of order zero and compactly supported. For $s^+ \in C^\infty(M_1,E^+_1)$, we have

$$(UB + BU)(s^+) = aB_1^+s^+ - B_3^+ (as^+) = \alpha_1 (D_1^+ + F_1^+)(s^+) - (D_1^+ + F_1^+)(\alpha_1 s^+) = \alpha_1 D_1^+ s^+ - D_1^+ (\alpha_1 s^+) = [\alpha_1, D_1^+](s^+).$$

Conditions (C.1) and (C.2) of Section 2.6 imply that $[\alpha_1, D_1^+]$ is a bundle map supported in $U(N_1)$. Since the closure of $U(N_1)$ is a compact set, the previous calculation shows that $\{U, B\}|_{C^\infty(M_1,E_1^+)}$ is a compactly supported homomorphism $E_1^+ \to (E^3_3)^+$. A similar argument shows that the homomorphism $\{U, B\}|_{C^\infty(M_j,E_\pm_j)}$ is compactly supported for all $j$. □

3.6. Proof of Theorem 2.8. We need to show that Conditions (U.1) and (U.2) imply Equations (3.4) and (3.5).

We start with proving that Equation (3.4) follows from Condition (U.1). Since the functions $\alpha_j, \beta_j$ are uniformly bounded, $U$ defines an $A$-equivariant bounded operator on the $A$-Hilbert space $L^2(M,E)$, that we denote by the same symbol $U$. By Condition (U.1), $U$ is unitary. Since $B$ is $\tau$-Fredholm and $U$ is an $A$-equivariant isomorphism of $A$-Hilbert spaces, $UBU$ is $\tau$-Fredholm and the $\tau$-index of $UBU$ is well-defined. As a bundle map, $U$ is odd with respect to the grading of $E$, i.e. it is of the form

$$U = \begin{pmatrix} 0 & U^- \\ U^+ & 0 \end{pmatrix},$$

where $U^\pm : C^\infty(M,E^\pm) \to C^\infty(M,E^\mp)$. It follows that $(UBU)^\pm = U^\pm B^\mp U^\pm$. Since $U$ is an $A$-equivariant isomorphism, we deduce

$$\text{Ker}(UBU)^\pm \cong \text{Ker} B^\mp,$$

where the isomorphism is taken in the category of $A$-Hilbert spaces. Since $U$ is unitary, we use Borel functional calculus and from Equation (3.11) we deduce

$$\dim_\tau (UBU)^\pm = \dim_\tau B^\mp,$$

from which Equation (3.4) follows.

It remains to show that Equation (3.5) holds. By Conditions (U.1) and (U.2), the operator

$$B + UBU = BU^2 + UBU = \{B, U\}U$$

is a compactly supported odd self-adjoint $A$-linear endomorphism of the bundle $E$. It follows from [13 Theorem 2.21] that

$$\text{ind}_\tau B = \text{ind}_\tau (B - (B + UBU)) = \text{ind}_\tau (-UBU).$$

Finally, Equation (3.5) follows by noticing that $\text{ind}_\tau UBU = \text{ind}_\tau (-UBU)$. □
4. Analysis on the cylinder

The next three sections are devoted to the proof of Theorem 2.16. In this section we define a particular Callias-type operator acting on a cylinder $N \times \mathbb{R}$ with compact base $N$. We call this operator the model operator and show that Theorem 2.16 holds in this case. In Section 5 we consider the case of a manifold with cylindrical ends. We use the relative index theorem to reduce this case to the case of a cylinder $N \times \mathbb{R}$. Finally, in Section 6 we again use the relative index theorem to reduce the general case to the case of a manifold with cylindrical ends.

4.1. The model operator. Let $E_N = E_N^+ \oplus E_N^-$ be a $\mathbb{Z}_2$-graded Dirac $A$-Hilbert bundle over a compact oriented manifold $N$. Let $\nabla^E = \nabla^E_N^+ \oplus \nabla^E_N^-$ be the connection on $E_N$ and let $D_N$

$$D_N = \begin{pmatrix} 0 & D_N^- \\ D_N^+ & 0 \end{pmatrix},$$

be a Dirac operator associated with $\nabla^E_{N}$ and the zero potential, cf. Section 2.15.

It is a classical fact (see for instance [23]) that the operator $D_N$ is $\tau$-Fredholm and its $\tau$-index is defined through the formula:

$$\text{ind}_\tau D_N = \dim_\tau \ker D_N^+ - \dim_\tau \ker D_N^-.$$

Let $p : N \times \mathbb{R} \to N$ be the projection onto the first factor and denote by $\hat{E}_N$ the pull-back bundle $p^*E_N$. Then

$$\hat{E}_N = \hat{E}_N^+ \oplus \hat{E}_N^-,$$

where $\hat{E}_N^\pm := p^*E_N^\pm$. (4.1)

The bundle $\hat{E}_N$ has a natural Clifford action given by:

$$\hat{c}(x, t) = c(x) + \gamma t, \quad (x, t) \in T^*_N N \oplus \mathbb{R} = T^*_N \mathbb{R}, \quad (x, r) \in N \times \mathbb{R},$$

where $c$ is the Clifford action of $T^*N$ on $E_N$ and $\gamma|_{\hat{E}_N^\pm} = \pm i$. Notice, however, that this action does not preserve the grading (4.1).

Endowed with the pull-back connection $\nabla^\hat{E}_N$ induced by the connection on $E_N$, the bundle $\hat{E}_N$ becomes an ungraded Dirac $A$-Hilbert bundle. Let $\hat{D}_N$ denote the Dirac operator associated with the connection $\nabla^\hat{E}_N$ and the zero potential. With respect to the decomposition

$$L^2(N \times \mathbb{R}, \hat{E}) = L^2(E) \otimes L^2(\mathbb{R}).$$

$\hat{D}_N$ has the form

$$\hat{D}_N := D_N \otimes 1 + \gamma \otimes \partial_r,$$

where $\partial_r$ denotes the operator of derivation in the axial direction of the cylinder $N \times \mathbb{R}$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$h(r) = \begin{cases} -1, & r < R_1 \\ 1, & r > R_2 \end{cases}$$

for some constants $R_1 < R_2$. By a slight abuse of notation we will denote by $h$ also the induced function $h : N \times \mathbb{R} \to [-1, 1]$. Notice that the multiplication by $h$ is an admissible endomorphism of the ungraded Dirac $A$-Hilbert bundle $\hat{E}_N$ (see Definition 2.11).
Definition 4.2. The model operator $M$ associated with the pair $(N,E_N)$ is the Callias-type operator associated with the ungraded triple $(\tilde{E}_N,\tilde{D}_N,h)$.

Thus
\[
M := \begin{pmatrix}
0 & M_-
M_+ & 0
\end{pmatrix},
\tag{4.6}
\]
where $M_\pm := \tilde{D}_N \pm ih$.

4.3. The index of the model operator. As already seen in Section 2.1 $M$ is $\tau$-Fredholm and its $\tau$-index is given by
\[
\text{ind}_\tau M = \dim_\tau \ker M^+ - \dim_\tau \ker M^-.
\]

Our proof of Theorem 2.16 is based on the following

Proposition 4.4. $\text{ind}_\tau D_N = \text{ind}_\tau M$.

Remark 4.5. The definition of the operator $M$ depends on the choice of a function $h$ satisfying (4.5). Proposition 4.4 shows that the $\tau$-index of $M$ is independent of this choice. This fact justifies the notation used in Definition 4.2.

The rest of this section is occupied with the proof of Proposition 4.4.

4.6. Differences from the case $A = \mathbb{C}$. When $A = \mathbb{C}$, Proposition 4.4 was proven by Anghel in [2] by using a separation of variables argument. Anghel proved that for every $\lambda > 0$ there exists $\mu > 0$ such that
\[
\text{im} E_{(-\mu,\mu)}(M) \subseteq \text{im} E_{(-\lambda,\lambda)}(D_N) \otimes L^2(\mathbb{R}),
\tag{4.7}
\]
where $E_{(-\mu,\mu)}(M)$ is the spectral projection of $M$ relative to the interval $(-\mu,\mu)$ and $E_{(-\lambda,\lambda)}(D_N)$ is the spectral projection of $D_N$ relative to the interval $(-\lambda,\lambda)$. Since the operator $D_N$ is Fredholm, $0$ is not in the essential spectrum of $D_N$ and we can choose $\lambda_0$ such that
\[
\text{im} E_{(-\lambda_0,\lambda_0)}(D_N) = \ker D_N.
\]
It then follows from (4.7) that
\[
\ker M \subseteq \ker D_N \otimes L^2(\mathbb{R}).
\tag{4.8}
\]

In the case when $A$ is an arbitrary von Neumann algebra with a finite trace $\tau$, (4.3) and (4.7) still hold. However, in general the $\tau$-Fredholmness of $D_N$ doesn’t imply that $0$ is not in the essential spectrum of this operator and we might not be able to deduce (4.8) from (4.7). Instead we study the square of the model operator and use the method of [10, §3] (see also [8, §11.2]) to compute $\ker M = \ker M^2$. 
4.7. The square of the model operator. Our proof of Proposition 4.4 is based on the study of the operator
\[ M^2 = \begin{pmatrix} M_- M_+ & 0 \\ 0 & M_+ M_- \end{pmatrix}. \]
First we compute \( M_- M_+ \). Since \( \gamma \otimes \partial_r \) anticommutes with \( D_N \), multiplication by \( i\hbar \) commutes with \( D_N \), \( \gamma^2 = 1 \), and \( \partial_r \hbar - \hbar \partial_r = h' \), we obtain
\[ M_- M_+ = D_N^2 \otimes 1 - 1 \otimes \partial_r^2 + i\gamma \otimes h' + 1 \otimes h^2. \] (4.9)

Even though the model operator \( M \) is neither even, nor odd with respect to the grading (4.1), it follows from the equality (4.9) that \( M_- M_+ \) does preserve this grading. Moreover,
\[ M_- M_+|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} = D_N^2 \otimes 1 + 1 \otimes Q^\pm, \] (4.10)
where
\[ Q^\pm = -\partial_r^2 \mp h' + h^2 = (i\partial_r \mp i\hbar) (i\partial_r \pm i\hbar) \geq 0. \] (4.11)

Similarly,
\[ M_+ M_-|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} = D_N^2 \otimes 1 + 1 \otimes Q^\mp. \] (4.12)

4.8. The kernel of \( M_\pm \). Since the operators \( M_+ \) and \( M_- \) are adjoint of each other, we conclude that
\[ \ker M_+ = \ker M_- M_+, \quad \ker M_- = \ker M_+ M_- \] (4.13)
In particular, it follows from the discussion in Section 4.7 that
\[ \ker M_\pm = \ker M_-|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} \oplus \ker M_+|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)}. \]
Since the operators \( D_N^2 \) and \( Q^\pm \) are non-negative, we have
\[ \ker M_- M_+|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} = \ker D_N^2|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} \otimes \ker Q^\pm = \ker D_N^2 \otimes \ker Q^\pm; \]
\[ \ker M_+ M_-|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} = \ker D_N^2|_{C^\infty(N \times \mathbb{R}, \hat{E}_N^\pm)} \otimes \ker Q^\mp = \ker D_N^2 \otimes \ker Q^\mp. \] (4.14)
Thus to compute \( \ker M_\pm \) it remains to compute \( \ker Q^\pm \).

**Lemma 4.9.** \( \dim \ker Q^+ = 1 \) and \( \dim \ker Q^- = 0 \).

**Proof.** By (4.11), \( y(r) \in \ker Q^\pm \subset L^2(\mathbb{R}) \) if and only if
\[ y'(r) \pm h(r) y(r) = 0. \]
The lemma follows now from the fact that the solutions of the ODE \( y' + h y = 0 \) are square-integrable, whereas the solutions of the ODE \( y' - h y = 0 \) are not. \( \square \)

Using (4.13) and (4.14) we obtain the following corollary of Lemma 4.9

**Corollary 4.10.** The \( A \)-Hilbert spaces \( \ker M_\pm \) and \( \ker D_N^\pm \) are isomorphic. Similarly, the \( A \)-Hilbert spaces \( \ker M_- \) and \( \ker D_N^- \) are isomorphic. In particular,
\[ \dim_r \ker M_\pm = \dim_r D_N^\pm. \]
4.11. **Proof of Proposition 4.4.** By Corollary 4.10 we have

\[ \text{ind}_r M = \dim_r \ker M_+ - \dim_r \ker M_- = \dim_r \ker D^+_N - \dim_r \ker D^-_N = \text{ind}_r D_N. \]

□

5. **A manifold with cylindrical ends**

In this section we prove Theorem 2.16 for the special case of a manifold with cylindrical ends.

5.1. **A manifold with cylindrical ends.** We use the notation of Section 2.15. In addition we assume that \( M_+ = N \times [1, \infty) \) for some compact manifold \( N \). In other words, we assume that

\[ M = M_- \cup_N (N \times [1, \infty)), \tag{5.1} \]

where \( M_- \) is a compact manifold with boundary, whose interior contains an essential support of the potential \( \Phi \in \text{End}_A(\Sigma) \).

5.2. **A Dirac operator on the hypersurface** \( N \). We identify \( N \) with \( N \times \{1\} \subset M \). Let \( \Sigma_N \) and \( \Phi_N \) denote the restrictions of \( \Sigma \) and \( \Phi \) to \( N \cong N \times \{1\} \). Since an essential support of \( \Phi \) is contained in the interior of \( M_- \), the bundle map \( \Phi_N \) is non-degenerate. As in Section 2.15, we decompose

\[ \Sigma_N = \Sigma_{N+} \oplus \Sigma_{N-}, \tag{5.2} \]

where, for every \( x \in N \), \( \Sigma_{N+}(x) \) (respectively \( \Sigma_{N-}(x) \)) is the closed \( A \)-invariant subspace of \( \Sigma_N(x) \) obtained as the image of the spectral projection of \( \Phi_N(x) \) corresponding to the interval \((0, \infty)\) (resp. \((-\infty, 0))\). We denote by \( \Phi_{N \pm} \) the restrictions of \( \Phi_N \) to \( \Sigma_{N \pm} \). It follows from (2.12) that the grading (5.2) is preserved by Clifford multiplication, i.e.

\[ c(\xi) : \Sigma_{N \pm} \rightarrow \Sigma_{N \pm}, \quad \text{for all} \quad \xi \in T^*N. \tag{5.3} \]

The bundle \( \Sigma_{N+} \) plays in what follows the same role as the bundle \( E_N \) in Section 4. However, in general, the connection \( \nabla^{\Sigma_N} \), induced on \( \Sigma_N \) by \( \nabla^\Sigma \), does not preserve the grading (5.2). That is why we need to define new connections

\[ \nabla^{\Sigma_{N \pm}} := \text{pr}_{\Sigma_{N \pm}} \circ \nabla^\Sigma \]

on \( \Sigma_{N \pm} \), cf. (2.15). This makes the situation of this section slightly different from the one considered in Section 4. To apply the results of Section 4 we need to deform the connection \( \nabla^{\Sigma_N} \) so that it does preserve the grading. Lemma 5.8 guarantees that such a deformation exists and preserves the index.

Let \( D_N = D_{N+} \oplus D_{N-} \) be the Dirac operator on \( \Sigma_N \) associated to the connection \( \nabla^{\Sigma_{N+}} \oplus \nabla^{\Sigma_{N-}} \) and the zero potential, cf. (2.16).
5.3. A Dirac operator on the cylinder $N \times \mathbb{R}$. Let $p : N \times \mathbb{R} \to N$ be the projection and denote by
\[
\widehat{\Sigma}_N = \widehat{\Sigma}_{N+} \oplus \widehat{\Sigma}_{N-}, \quad \widehat{\Sigma}_{N\pm} := p^*\Sigma_{N\pm},
\] (5.4)
the pull-back bundle over $N \times \mathbb{R}$. Let $\nabla^{\widehat{\Sigma}_N}$ denote the connection on $\widehat{\Sigma}_N$ obtained by pulling back the connection $\nabla^{\Sigma_N}$. Notice that in general this connection does not preserve the grading (5.4).

Endowed with the connection $\nabla^{\widehat{\Sigma}_N}$ and the Clifford action (4.2) the bundle $\widehat{\Sigma}_N$ becomes a Dirac bundle. Let
\[
\hat{D}_N : C^\infty(N \times \mathbb{R}, \widehat{\Sigma}_{N+}) \to C^\infty(N \times \mathbb{R}, \widehat{\Sigma}_{N+})
\]
denote the Dirac operator on $\widehat{\Sigma}_N$ associated with the connection $\nabla^{\widehat{\Sigma}_N}$. Notice that in general this connection does not preserve the grading (5.4).

Remark 5.4. We warn the reader that the operator $\hat{D}_N$ introduced above is not related to $D_N$ by formula (5.4). This is because $D_N$ was defined using the connection $\nabla^{\Sigma_N}$.

This is one of the difficulties which arise in trying to use the results of Section 4 in the proof of Theorem 2.16. We address this problem by deforming the connection $\nabla^{\Sigma_N}$. This is one of the difficulties which arise in trying to use the results of Section 4 in the proof of Theorem 2.16. We address this problem by deforming the connection $\nabla^{\Sigma_N}$.

Finally, we denote by $\hat{\Phi}_N \in \text{End}_A(\widehat{\Sigma}_N)$ and $\hat{\Phi}_{N\pm} \in \text{End}_A(\widehat{\Sigma}_{N\pm})$ the lifts of $\Phi_N$ and $\Phi_{N\pm}$ to the cylinder $N \times \mathbb{R}$. Notice that $\hat{\Phi}_{N+}$ is a strictly positive operator, while $\hat{\Phi}_{N-}$ is a strictly negative operator.

5.5. A Callias-type theorem for a manifold with cylindrical ends. The main result of this section is the following special case of Theorem 2.16.

Proposition 5.6. In the situation of Theorem 2.16, suppose that $M = M_- \cup_N (N \times (1, \infty))$, where $M_-$ is a compact submanifold with boundary $\partial M_- \simeq N$. Assume that there exist $\epsilon > 0$ and an open neighborhood of $N$ in $M_-$ isometric to $N \times (1 - \epsilon, 1]$ such that $M \setminus (N \times (1 - \epsilon, \infty))$ is an essential support of $\Phi$. Suppose that the restriction $\Sigma|_{N \times (1 - \epsilon, \infty)}$ coincides with $\widehat{\Sigma}_{N+} \oplus \widehat{\Sigma}_{N-}$, the restriction $D|_{N \times (1 - \epsilon, \infty)}$ coincides with $\hat{D}_N$ and $\Phi|_{N \times (1 - \epsilon, \infty)} = \hat{\Phi}_{N+} + \hat{\Phi}_{N-}$. Then
\[
\text{ind}_\tau B_\Phi = \text{ind}_\tau D_{N+}. \quad (5.5)
\]

The rest of this section is occupied with the proof of Proposition 5.6.

5.7. Deformation of the data on the cylindrical end. In the next two lemmas we construct a deformation of the restriction of the connection $\nabla^{\Sigma}$ and the potential $\Phi$ to the cylindrical end $N \times (1 - \epsilon, \infty)$, and show that the $\tau$-index of $B_\Phi$ is preserved by these deformations.

In particular, in Lemma 5.8 we construct a new connection $\nabla^{\Sigma}$ on $\Sigma$. We denote by $D'$ the Dirac operator associated with the connection $\nabla^{\Sigma}$ and the zero potential, cf. Section 2.5. We also denote by $\nabla^{\Sigma \pm}$ the connection on $\Sigma_{N\pm}$ obtained by pulling back the connection $\nabla^{\Sigma_{N\pm}}$.

Lemma 5.8. Under the hypotheses of Proposition 5.6 there exists a connections $\nabla^{\Sigma}$ on $\Sigma$ and a number $\lambda \geq 1$, such that
(i) \( \nabla^\Sigma|_{N \times (1-\epsilon,\infty)} = \nabla^\Sigma N_+ \oplus \nabla^\Sigma N_- \)

(ii) \( \lambda \Phi \) is an admissible endomorphism for the pair \((\Sigma, D')\);

(iii) \( \text{ind}_\tau B_{\lambda \Phi} = \text{ind}_\tau B_{\Phi} \) for all \( t \in [0,1] \), where \( B'_{\lambda \Phi} \) denotes the Callias-type operator associated with the triple \((\Sigma, D', \lambda \Phi)\).

**Lemma 5.9.** Under the hypotheses of Proposition 5.6 assume that

\[
\nabla^\Sigma|_{N \times (1-\epsilon,\infty)} = \nabla^\Sigma N_+ \oplus \nabla^\Sigma N_-.
\]

Then there exists an admissible endomorphism \( \Phi' \) for \((\Sigma, D)\) such that

(i) The restriction of \( \Phi' \) to \( N \times (1-\epsilon,\infty) \) is the grading operator on \( \Sigma|_{N \times (1-\epsilon,\infty)} = \Sigma N_+ \oplus \Sigma N_- \), i.e. \( \Phi'|_{\Sigma N_\pm} = \pm 1 \);

(ii) \( \text{ind}_\tau B_{\Phi'} = \text{ind}_\tau B_{\Phi} \), where \( B_{\Phi'} \) denotes the Callias-type operator associated with the triple \((\Sigma, D, \Phi')\).

The proofs of these lemmas are presented at the end of this section, after we explained how the lemmas are used to prove Proposition 5.6.

**5.10. Sketch of the proof of Proposition 5.6.** If follows from Lemmas 5.8 and 5.9 that it is enough to prove the proposition for the case when the connection \( \nabla^\Sigma \) satisfies \( \nabla^\Sigma N = \nabla^\Sigma N_+ \oplus \nabla^\Sigma N_- \) and \( \Phi_N \) is the grading operator on \( \Sigma_N \).

Let \( M_1 = -M \) denote a copy of \( M \) with the opposite orientation. Then \( M_1 \) is naturally isomorphic to \((N \times (-\infty,1)) \cup_N (-M_-)\). The bundle \( \Sigma \) induces a Dirac bundle \( \Sigma_1 := -\Sigma \) on \( M_1 \), cf. Section 5.11. Let \( D_1 \) be the corresponding Dirac operator.

In Section 5.12 we construct a potential \( \Phi_1 \) on \( \Sigma_1 \) such that

(i) the restriction of the Callias-type operator \( B_{\Phi_1} \) associated with the triple \((\Sigma_1, D_1, \Phi_1)\) to the cylinder \( N \times (-\infty,1] \) is equal to \( M \oplus T \), where \( M \) is the model operator of Definition 4.2 and \( T^2 > 0 \);

(ii) \( \text{ind}_\tau B_{\Phi_1} = 0 \).

The restriction of all the data to neighborhoods of \( N \times \{1\} \) in \( M \) and \( M_1 \) coincide. Hence, we can apply the relative index theorem to compute

\[
\text{ind}_\tau B_{\Phi} = \text{ind}_\tau B_{\Phi} + \text{ind}_\tau B_{\Phi_1}.
\]

The cut and paste procedure of Section 2.6 applied to manifolds \( M \) and \( M_1 \) yields manifolds

\[
M_2 = N \times \mathbb{R} \quad \text{and} \quad M_3 = M_- \cup_N (-M_-).
\]

Let \( B_2 \) and \( B_3 \) be the Callias-type operators on \( M_2 \) and \( M_3 \) obtained from \( B_{\Phi} \) and \( B_{\Phi_1} \) by cutting and pasting. One readily sees that \( B_2 \) is equal to \( M \oplus T \), where \( M \) is the model operator of Definition 4.2 and \( T^2 > 0 \). Hence, \( \text{ind}_\tau B_2 = \text{ind}_\tau M \). Also \( B_3 \) is a Callias-type operator on a compact manifold \( M_3 \). Thus \( \text{ind}_\tau B_3 = 0 \) by Proposition 2.13. The relative index theorem and (5.7) imply that

\[
\text{ind}_\tau B_{\Phi} = \text{ind}_\tau B_2 + \text{ind}_\tau B_3 = \text{ind}_\tau M.
\]

Proposition 5.6 follows now from Proposition 4.4.
5.11. The manifold with the reversed orientation. Before presenting the details of the proof of Proposition 5.6, we introduce some additional notation.

For an oriented manifold \( W \) we denote by \(-W\) a copy of this manifold with the opposite orientation. If \( E \) is a Dirac bundle over \( W \), we denote by \(-E\) the same bundle viewed as a vector bundle over \(-W\), endowed with the opposite Clifford action. This means that a vector \( \xi \in T^*W \simeq T^*(-W) \) acts on \(-E\) by \( c(-\xi) \). The change of the Clifford action is needed because we reversed the orientation of \( M \) (for more details about this construction we refer to [18, Section 2.3.2] and [6, Chapter 9]).

From now on we assume that \( W = W_- \cup N \) \((N \times [1, \infty))\). Then there is a natural orientation preserving isometry \( \psi : -W \xrightarrow{\sim} (N \times (-\infty, 1]) \cup_N (-W_-) \), such that

\[
\psi(x) = x, \quad \text{if } x \in -W_-, \quad \psi(y, t) = (y, 2 - t), \quad \text{if } (y, t) \in N \times [1, \infty).
\]

To simplify the notation we will skip \( \psi \) from the notation and simply write

\[
-W = (N \times (-\infty, 1]) \cup_N (-W_-).
\]

Let \( E_N \) be a bundle over \( N \) and let \( \tilde{E}_N \) denote the pull-back of \( E_N \) by the projection map \( p : N \times \mathbb{R} \to N \). Suppose that \( E \) is a bundle over \( W = W_- \cup N \) \((N \times [1, \infty))\) whose restriction to \( N \times [1, \infty) \) coincides with the restriction of \( \tilde{E}_N \) to the same cylinder. Recall that \(-E\) is a bundle over the manifold \((5.9)\). One readily sees that

\[
-E \bigg|_{N \times (-\infty, 1]} \simeq \tilde{E}_N \bigg|_{N \times (-\infty, 1]}.
\]

5.12. Construction of a potential on \(-M\). Consider the manifold \( M_1 := -M \). Then by \((5.9)\) we have

\[
M_1 = (N \times (-\infty, 1]) \cup_N (-M_-).
\]

Consider the bundle \( \Sigma_1 = -\Sigma \) over \( M_1 \). By \((5.10)\) the restriction of \( \Sigma_1 \) to the cylinder part coincides with \( \Sigma_N \). In particular, this restriction has a natural grading \((5.4)\)

\[
\Sigma_N = \Sigma_{N+} \oplus \Sigma_{N-}.
\]

We denote by \( D_1 \) the Dirac operator associated with the connection on \( \Sigma_1 \).

Let \( h : \mathbb{R} \to [-1, 1] \) be a smooth function such that

\[
h(r) = \begin{cases} 
-1, & r < -1 \\
1, & r > 0 
\end{cases}, \quad r \in \mathbb{R}.
\]

By a slight abuse of notation we also denote by \( h \) the induced function \( h : N \times \mathbb{R} \to [-1, 1] \). Let \( \Phi_1 \) be the admissible endomorphism of \( \Sigma_1 \), which coincides with \( \Phi \) on \( (N \times (1 - \epsilon, 1]) \cup_N (-M_-) \) and such that

\[
\Phi_1 \bigg|_{N \times (-\infty, 1+\epsilon)} = \begin{pmatrix} h & 0 \\
0 & -1 \end{pmatrix}.
\]
Notice, that we can view the product $N \times (1 - \epsilon, 1 + \epsilon)$ as a subset of both manifolds $M$ and $M_1$. Then the restrictions of $\Phi$ and $\Phi_1$ to this subset coincide.

**Lemma 5.13.** Let $B_{\Phi_1}$ denote the Callias-type operator associated with the pair $(\Sigma_1, \Phi_1)$. Then

$$\text{ind}_\tau B_{\Phi_1} = 0.$$  

(5.12)

**Proof.** Consider a new potential $\Phi'_1 \equiv -\text{Id}$ on $\Sigma_1$ and denote by $B_{\Phi'_1}$ the corresponding Callias-type operator. Notice, that the bundle maps $\Phi_1$ and $\Phi'_1$ coincide outside of the compact set

$$(N \times [-1, 1]) \cup N (-M_-).$$

It follows now from Theorem 2.21 of [13] that

$$\text{ind}_\tau B_{\Phi_1} = \text{ind}_\tau B_{\Phi'_1}.$$  

(5.13)

Since

$$B_{\Phi'_1}^2 = \left( \begin{array}{cc} D^2_2 + 1 & 0 \\ 0 & D^2_1 + 1 \end{array} \right) > 0,$$

we conclude that $\text{ind}_\tau B_{\Phi'_1} = 0$. The lemma follows now from (5.13). \qed

**5.14. Proof of Proposition 5.6** By Lemmas $5.8$ and $5.9$ it is enough to prove the proposition for the case when $\nabla \Sigma_N = \nabla \Sigma_{N+} \oplus \nabla \Sigma_{N-}$ and $\Phi_N$ is the grading operator on $\Sigma_N$.

Let $M_1$, $\Sigma_1$, and $\Phi_1$ be as in the previous section. As we already noted, the restrictions of $\Phi$ and $\Phi_1$ to $N \times (1 - \epsilon, 1 + \epsilon)$ coincide. Thus we can apply the relative index theorem $2.8$ to the pair of manifolds $M$ and $M_1$. As a result of the cutting and pasting procedure used in this theorem we obtain two new manifolds

$$M_2 := N \times \mathbb{R}, \quad M_3 := M_- \cup_N (-M_-).$$  

(5.14)

Let $B_2$ and $B_3$ be the operators on $M_2$ and $M_3$ obtained from $B_{\Phi}$ and $B_{\Phi_1}$ by cutting and pasting. It follows immediately from our assumptions that $\Phi$ is the grading operator and from the construction of $\Phi_1$ that the restriction of $B_2$ to $\hat{\Sigma}_{N+} \oplus \hat{\Sigma}_{N+}$ is equal to the model operator $M$, while the restriction of $B_2$ to $\hat{\Sigma}_N- \oplus \hat{\Sigma}_N-$ is equal the Callias-type operator associated with the ungraded triple $(\hat{\Sigma}_N-, \hat{D}_{N-}, -1)$. In other words,

$$B_2|_{\hat{\Sigma}_N- \oplus \hat{\Sigma}_N-} = \left( \begin{array}{cc} 0 & \hat{D}_{N-} + i \\ \hat{D}_{N-} - i & 0 \end{array} \right).$$

Hence,

$$B_2^2|_{\hat{\Sigma}_N- \oplus \hat{\Sigma}_N-} = \left( \begin{array}{cc} \hat{D}_{N-}^2 + 1 & 0 \\ 0 & \hat{D}_{N-}^2 + 1 \end{array} \right) > 0.$$

We conclude that $\text{ind}_\tau B_2|_{\hat{\Sigma}_N- \oplus \hat{\Sigma}_N-} = 0$ and

$$\text{ind}_\tau B_2 = \text{ind}_\tau M.$$  

Furthermore, $B_3$ is a Callias-type operator on compact manifold $M_3$. Hence, $\text{ind}_\tau B_3 = 0$ by Proposition $2.13$. Applying the relative index theorem and using Lemma $5.13$ we now obtain

$$\text{ind}_\tau B_{\Phi} = \text{ind}_\tau B_{\Phi} + \text{ind}_\tau B_{\Phi_1} = \text{ind}_\tau B_2 + \text{ind}_\tau B_3 = \text{ind}_\tau M.$$
Proposition 5.6 follows now from Proposition 4.4. □

We now pass to the proofs of Lemmas 5.8 and 5.9.

5.15. A rescaling of the potential. Fix \( \lambda \geq 1 \). If the endomorphism \( \Phi \) is admissible, then also \( \lambda \Phi \) is admissible with the same essential support. Denote by \( B_{\lambda \Phi} \) the Callias-type operator associated with \((\Sigma, D, \lambda \Phi)\). The hypotheses of Proposition 5.6 imply that \( \Phi \) is constant in the axial direction over the cylindrical end. Therefore, \( \Phi \) is uniformly bounded and defines a bounded operator on \( L^2(M, \Sigma) \). It follows that \( \{ B_t \Phi \}_{1 \leq t \leq \lambda} \) is a continuous homotopy of \( \tau \)-Fredholm operators with fixed domain. By [13, Lemma 7.3], we deduce

\[
\text{ind}_\tau B_\Phi = \text{ind}_\tau B_{\lambda \Phi}.
\]

(5.15)

5.16. Proof of Lemma 5.8. By the hypotheses of Proposition 5.6

\[
\Sigma |_{N \times (1- \epsilon, \infty)} = \tilde{\Sigma}_N^+ \oplus \tilde{\Sigma}_N^-.
\]

With respect to this decomposition, we have

\[
D |_{N \times (1- \epsilon, \infty)} = \\
\begin{pmatrix}
\hat{D}_{N^+} & \hat{\pi}_- \circ \hat{D}_N \circ \hat{\pi}^+ \\
\hat{\pi}_+ \circ \hat{D}_N \circ \hat{\pi}^- & \hat{D}_{N^-}
\end{pmatrix},
\]

(5.16)

where \( \hat{\pi}_\pm \) are the projections onto \( \tilde{\Sigma}_N^\pm \).

Let \( \hat{\alpha} \in \text{End}_A(\tilde{\Sigma}_N) \) be the grading operator, i.e. \( \hat{\alpha}|_{\Sigma^\pm_N} = \pm 1 \). Then \( \hat{\pi}_\pm = \frac{1}{2} (I \pm \hat{\alpha}) \) and

\[
\hat{\pi}_\pm \circ \hat{D}_N \circ \hat{\pi}_\mp = \frac{1}{2} [\hat{D}_N, \hat{\alpha}] \circ \hat{\pi}_\pm,
\]

where \( \hat{\alpha} = \pm 1 \) on \( \tilde{\Sigma}_N^\pm \). It follows from (5.3) that \( \hat{\alpha} \) commutes with the Clifford multiplication. Hence, the commutator \([\hat{D}_N, \hat{\alpha}]\) is a zero-order differential operator, i.e. a bundle map. We conclude that \( \hat{\pi}_\pm \circ \hat{D}_N \circ \hat{\pi}_\mp \) are also bundle maps.

Pick a constant \( \epsilon_1 \) such that \( 0 < \epsilon_1 < \epsilon \) and let \( \Psi \) be a self-adjoint endomorphism of \( \Sigma \) such that \( \Psi = 0 \) off \( N \times (1- \epsilon, \infty) \) and

\[
\Psi |_{N \times (1- \epsilon_1, \infty)} := \\
\begin{pmatrix}
0 & \hat{\pi}_- \circ \hat{D}_N \circ \hat{\pi}^+ \\
\hat{\pi}_+ \circ \hat{D}_N \circ \hat{\pi}^- & 0
\end{pmatrix}.
\]

Since both \( \Psi \) and \( \Phi \) are uniformly bounded, the commutator \([\Psi, \Phi]\) is in \( L^\infty(M, \text{End}_A(\Sigma)) \). Since the restriction of \( D \) to \( N \times (1- \epsilon, \infty) \) is the lift of \( D_N \), the commutator \([D, \Phi]\) is also in \( L^\infty(M, \text{End}_A(\Sigma)) \). Hence, we can choose constants \( d > 0 \) and \( \lambda \geq 1 \) such that

\[
\lambda^2 \Phi^2(x) \geq d + \lambda (\|D, \Phi\|_\infty + \|\Psi, \Phi\|_\infty), \quad x \in N \times (1- \epsilon, \infty).
\]

(5.17)

Consider the family \( D_t := D - t \Psi \) of Dirac operators on \( \Sigma \). We claim that if \( \lambda \) satisfies (5.17) and \( t \in [0, 1] \), then \( \lambda \Phi \) is an admissible endomorphism for \((\Sigma, D_t)\). Indeed, since

\[
[D - t \Psi, \lambda \Phi] = \lambda ([D, \Phi] - t [\Psi, \Phi]),
\]
by using (5.17) we obtain
\[
\lambda^2 \Phi^2(x) \geq d + \lambda (\|D, \Phi\|_{\infty} + \|\Psi, \Phi\|_{\infty}) \\
\geq d + \lambda (\|D, \Phi\|_{\infty} + t [\Psi, \Phi]_{\infty}) \\
\geq d + \lambda [\|D\|_{\infty} - t [\Psi, \Phi]_{\infty}] \\
\geq d + \lambda [\|D - \lambda \Psi, \lambda \Phi\|_{\infty}],
\]
(5.18)
for every \(x \in N \times (1 - \epsilon, \infty)\).

Let \(B_{\lambda \Phi}^{t}\) denote the Callias-type operator associated with the triple \((\Sigma, D - t \Psi, \lambda \Phi)\). Since \(\Psi\) is uniformly bounded, the family of operators
\[
\{ B_{\lambda \Phi}^{t} : t \in [0, 1] \}
\]
is a continuous homotopy of \(\tau\)-Fredholm operators with fixed domain. By [13, Lemma 7.3], the \(\tau\)-index of \(B_{\lambda \Phi}^{t}\) is independent of \(t\). Using (5.15) we now deduce
\[
\text{ind}_{\tau} B_{\lambda \Phi} = \text{ind}_{\tau} B_{\lambda \Phi}^{0} = \text{ind}_{\tau} B_{\lambda \Phi}^{1}.
\]
(5.19)

Let \(\nabla_{N}^{\Sigma}\) be a connection on \(\Sigma\) satisfying condition (i) of the lemma. In other words we assume that the restriction of \(\nabla_{N}^{\Sigma}\) to \(N \times (1 - \epsilon, \infty)\) is equal to \(\nabla_{N}^{\Sigma} \oplus \nabla_{N}^{\Sigma}\). Let \(D'\) be the Dirac operator associated with the connection \(\nabla_{N}^{\Sigma}\) and the zero potential. Let \(B_{\lambda \Phi}^{t}\) denote the Callias-type operator associated with the triple \((\Sigma, D', \lambda \Phi)\). Then the restrictions of the operators \(B_{\lambda \Phi}^{t}\) and \(B_{\lambda \Phi}^{1}\) to the cylindrical part \(N \times (1 - \epsilon, \infty)\) coincide. Hence, \(\text{ind}_{\tau} B_{\lambda \Phi}^{t} = \text{ind}_{\tau} B_{\lambda \Phi}^{1}\) by Theorem 2.21 of [13]. The lemma follows now from (5.19).

\[\square\]

5.17. Proof of Lemma 5.9

By (5.6)
\[
D\Big|_{N \times (1 - \epsilon, \infty)} = \hat{D}_{N}^{+} \oplus \hat{D}_{N}^{-},
\]
where \(\hat{D}_{N}^{\pm}\) is the Dirac operator on \(\hat{\Sigma}_{N}^{\pm}\) associated with the connection \(\nabla_{N}^{\Sigma}\) and the zero potential.

Pick constants \(\epsilon_{1}, \epsilon_{2}\) such that \(0 < \epsilon_{1} < \epsilon_{2} < \epsilon\). Let \(f : (1 - \epsilon, \infty) \to [0, 1]\) be a smooth function such that \(f = 0\) on \((1 - \epsilon, 1 - \epsilon_{2})\) and \(f = 1\) on \((1 - \epsilon_{1}, \infty)\). Let \(\Phi'\) be a self-adjoint endomorphism of \(\Sigma\) such that \(\Phi' = 0\) off \(N \times (1 - \epsilon, \infty)\) and
\[
\Phi'(y, r) = \begin{pmatrix} f(r) & 0 \\ 0 & -f(r) \end{pmatrix}, \quad \text{for } (y, r) \in N \times (1 - \epsilon, \infty).
\]
Observe that \([D, \Phi'] = 0\) outside of \(N \times (1 - \epsilon, \infty)\) and
\[
[D, \Phi']|_{N \times (1 - \epsilon, \infty)} = \begin{pmatrix} \hat{D}_{N}^{+} & 0 \\ 0 & \hat{D}_{N}^{-} \end{pmatrix}, \quad \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} = \begin{pmatrix} [\hat{D}_{N}^{+}, f] & 0 \\ 0 & [f, \hat{D}_{N}^{-}] \end{pmatrix}.
\]
Hence, \([D, \Phi']\) is a zero-order differential operator. Thus \(\Phi'\) satisfies Condition (i) of Definition 2.11.
Furthermore,
\[ \Phi'|_{N \times (1-\epsilon_1,\infty)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [D, \Phi'] |_{N \times (1-\epsilon_1,\infty)} = 0. \quad (5.20) \]

Thus Condition (ii) of Definition 2.11 is also satisfied. We conclude that \( \Phi' \) is an admissible endomorphism for \((\Sigma, D)\) with essential support \( M \setminus (N \times (1-\epsilon, \infty)) \).

To prove the lemma it remains to show that
\[ \text{ind}_r B_{\Phi'} = \text{ind}_r B_{\Phi}, \quad (5.21) \]
where \( B_{\Phi'} \) is the Callias-type operator associated with \((\Sigma, D, \Phi')\). We prove this equality by considering the homotopy
\[ \Phi_t := t \Phi + (1-t) \Phi', \quad 0 \leq t \leq 1. \quad (5.22) \]
A subtlety here is that in general the endomorphism \( \Phi_t \) is not admissible. However, we show below that there exists \( \lambda \geq 1 \) such that the endomorphism \( \lambda \Phi_t \) is admissible for all \( t \in [0,1] \).

First, recall that in Proposition 5.6 we assumed that
\[ \Phi |_{N \times (1-\epsilon,\infty)} = \hat{\Phi}_N^+ \oplus \hat{\Phi}_N^-, \]
where \( \hat{\Phi}_N^+ = \hat{\Phi}|_{\Sigma^N_+} \) is a strictly positive operator and \( \hat{\Phi}_N^- = \hat{\Phi}|_{\Sigma^N_-} \) is a strictly negative operator. Since \( \hat{\Phi}_{N\pm}(x) \) are constant along the axial direction on the cylinder, there exists a constant \( d_1 > 0 \) such that \( \hat{\Phi}_N^+ \geq d_1 \) and \( \hat{\Phi}_N^- \leq -d_1 \). It follows that
\[ \Phi^2 |_{N \times (1-\epsilon,\infty)} = \begin{pmatrix} \hat{\Phi}_N^2 + & 0 \\ 0 & -\hat{\Phi}_N^- \end{pmatrix} \geq d_1^2. \quad (5.23) \]

Using (5.20) we also obtain
\[ \{ \Phi, \Phi' \} |_{N \times (1-\epsilon,\infty)} := (\Phi \circ \Phi' + \Phi' \circ \Phi) |_{N \times (1-\epsilon,\infty)} = 2 \begin{pmatrix} \hat{\Phi}_N^+ & 0 \\ 0 & -\hat{\Phi}_N^- \end{pmatrix} \geq 2 d_1. \quad (5.24) \]

Since \( \Phi' \) is constant in the axial direction over the cylindrical end, the commutator \([D, \Phi']\) is in \( L^\infty(M, \text{End}_A(\Sigma)) \). Set
\[ d_2 := \min\{d_1, d_1^2, 1\} \]
and pick constants \( d > 0, \lambda \geq 1 \) such that
\[ \lambda^2 d_2 \geq d + \lambda \| [D, \Phi'] \|_\infty. \quad (5.25) \]

We claim that \( \lambda \Phi_t \) is an admissible endomorphism for all \( t \in [0,1] \). Indeed, using (5.20), (5.24), (5.23), and the equality
\[ [D, \lambda \Phi_t] = \lambda \left( t[D, \Phi] + (1-t)[D, \Phi'] \right), \]
we obtain
\[ (\lambda \Phi_t)^2 (x) = t^2 \lambda^2 \Phi^2(x) + (1 - t)^2 \lambda^2 \Phi'^2(x) + 2t(1 - t)\lambda^2 \{\Phi, \Phi'\}(x) \]
\[ \geq t^2 \lambda^2 d_1^2 + (1 - t)^2 \lambda^2 + 2t(1 - t)\lambda^2 d_1 \]
\[ \geq (t^2 + (1 - t)^2 + 2t(1 - t)) \lambda^2 d_2 \]
\[ \geq d + \lambda \| [D, \Phi'] \|_\infty \geq d + \lambda \| [D, \lambda \Phi_t](x) \|, \]
for every \( x \in N \times (1 - \epsilon_1, \infty) \).

Let \( B_{\lambda \Phi_t} \) denote the Callias-type operator associated with \((\Sigma, D, \lambda \Phi_t)\). Since both endomorphisms, \( \Phi \) and \( \Phi' \), are constant in the axial direction, the family of endomorphisms \( \Phi_t \) is uniformly bounded and depends continuously on \( t \). By [13, Lemma 7.3], the index \( \text{ind}_\tau B_{\lambda \Phi_t} \) is independent of \( t \in [0, 1] \). Using (5.15) we now obtain
\[ \text{ind}_\tau B_{\Phi'} = \text{ind}_\tau B_{\lambda \Phi'} = \text{ind}_\tau B_{\lambda \Phi} = \text{ind}_\tau B_{\Phi}. \]

6. Proof of the Callias-type theorem in general case

In this section we conclude the proof of Theorem 2.16. We use the relative index theorem to reduce the computation of the index in general case to a computation on a manifold with cylindrical ends.

6.1. Notation. Throughout the section we use the notation of Section 2.15. In particular, \( M = M_- \cup N M_+ \), \( \Sigma \) is a Dirac bundle over \( M \), and \( D \) is a Dirac operator on \( \Sigma \) associated to a connection \( \nabla \Sigma \) and a potential \( V \).

Let \( \Sigma_N \) and \( \Phi_N \) be the restrictions of \( \Sigma \) and \( \Phi \) to \( N \). We denote by \( \hat{\Sigma}_N \) and \( \hat{\Phi}_N \) the lifts of \( \Sigma_N \) and \( \Phi_N \) to \( N \times \mathbb{R} \), cf. Section 5.3.

6.2. Sketch of the proof of Theorem 2.16. We first deform all the structures in a small neighborhood \( U(N) \subset M \) of \( N \) so that the following conditions hold:

(N.1) \( U(N) \) is isometrically diffeomorphic to \( N \times (-3\epsilon, 3\epsilon) \);

(N.2) the restrictions of \( \Sigma \) and \( \hat{\Sigma}_N \) to \( N \times (-\epsilon, \epsilon) \) are isomorphic as A-Hilbert bundles with connections;

(N.3) \( \Phi|_{N \times (-\epsilon, \epsilon)} = \hat{\Phi}_N|_{N \times (-\epsilon, \epsilon)} \);

(N.4) the potential \( V \) vanishes on \( N \times (-\epsilon, \epsilon) \);

(N.5) the essential support of the Callias-type operator associated to the new structures is still contained in the interior of \( M_- \).

By [13, Lemma 7.3], the \( \tau \)-index of a Callias-type operator does not change under such deformation of the data. Hence, it suffices to prove Theorem 2.16 in the case when the conditions (N.1)–(N.5) are satisfied, which we shall henceforth assume.

The rest of the proof is very similar to the proof of Proposition 5.6. In Section 5 we used the relative index theorem to reduce the computation of the index on a manifold with cylindrical ends to the computation of an index on a cylinder. Here in exactly the same way we use the
relative index theorem to reduce the computation of the index on $M$ to a computation of an index on a manifold with cylindrical ends.

The rest of this section is occupied with the details of the proof of Theorem 2.16.

6.3. Deformation of the metric. Before presenting the construction of the deformation of the data near $N$, discussed in the beginning of Section 6.2, let us recall how a Dirac-type operator changes under a conformal change of the Riemannian metric.

Let $g^M$ denote the Riemannian metric on $M$. Let $c : T^*M \to \text{End}_A(\Sigma)$, $\nabla^\Sigma$, and $\langle \cdot, \cdot \rangle$ denote respectively the Clifford action of the cotangent bundle, the metric connection, and the inner product on fibers of $\Sigma$.

Let $h : M \to \mathbb{R}$ be a smooth compactly supported function and define a new Riemannian metric $g^M_h := e^{-2h}g^M$. For a cotangent vector $\xi$ we denote by $|\xi|$ and $|\xi|_h$ its norms with respect to the metrics $g^M$ and $g^M_h$ respectively. Then $|\cdot|_h = e^h|\cdot|$.

In order to make $\Sigma$ a Dirac bundle over the Riemannian manifold $(M,g^M_h)$ we also need to change the other structures. The new Clifford action $c_h : T^*M \to \text{End}_A(\Sigma)$ is given by

$$c_h(\xi) := e^h c(\xi), \quad \xi \in T^*M.$$  \hspace{1cm} (6.1)

This formula was obtained by Hitchin [29, §1.4] (see also [32, §II.5]) for the case when $\Sigma$ is the bundle of spinors. The general case was treated in [34, §4].

We now set

$$D_h := e^h \left( D - \frac{n-1}{2} c(dh) \right).$$ \hspace{1cm} (6.2)

Then $D_h$ is the Dirac operator associated with connection $\nabla^\Sigma_h$ and potential $e^hV$.

More generally, let $\chi : M \to [0,1]$ be a smooth function, such that $\chi(x) \equiv 1$ for all $x \in M \setminus K$, where $K$ is an essential support of $\Phi$. We consider the Dirac operator $D_{h,\chi}$ associated with the connection $\nabla^\Sigma_h$ and the potential $\chi e^hV$. Then

$$D_{h,\chi} = D_h - (1 - \chi)e^hV.$$ \hspace{1cm} (6.3)

Notice, that since $h$ and $1 - \chi$ have compact support, any endomorphism $\Phi : \Sigma \to \Sigma$ which is admissible for $D$ is also admissible for $D_{h,\chi}$. We denote by $B_{\Phi,h,\chi}$ the Callias-type operator associated with the triple $(\Sigma_h,D_{h,\chi},\Phi)$. Since $B_\Phi$ and $B_{\Phi,h,\chi}$ coincide outside of a compact set,

$$\text{ind}_T B_{\Phi,h,\chi} = \text{ind}_T B_\Phi$$ \hspace{1cm} (6.4)
Proof. Let $\Sigma$ be a function and $H$ be a compact set $K_1 \subset K$ such that $\Phi^2(x) > d_1$ for all $x \notin K_1$. Then there exists a function $h \in C_0^\infty(M)$ such that $K_1$ is an essential support for $\Phi$ with respect to the pair $(\Sigma_h, D_{h,\chi})$.

Together with (6.4) the lemma suggests that for all index related questions we could define the essential support as the set $K_1$ such that $\Phi^2(x) > d_1$ for all $x \notin K_1$.

**Lemma 6.4.** Let $\Phi$ be a Callias-type operator associated with a triple $(\Sigma, D, \Phi)$. Assume that $K$ is an essential support of $\Phi$ with respect to the pair $(\Sigma, D)$ and that there exists a constant $d_1 > 0$ and a compact set $K_1 \subset K$ such that $\Phi^2(x) > d_1$ for all $x \notin K_1$. Then there exists a function $h \in C_0^\infty(M)$ such that $K_1$ is an essential support for $\Phi$ with respect to the pair $(\Sigma_h, D_{h,\chi})$.

By (2.12) $\Phi$ commutes with the Clifford multiplication. Hence, using (6.2) and (6.3) we obtain

$$[D_{h,\chi}, \Phi](x) = e^{h(x)} [D, \Phi](x) - (1 - \chi(x)) e^{h(x)} [V, \Phi](x), \quad x \in M,$$

and

$$\| [D_{h,\chi}, \Phi](x) \| = e^{h(x)} \| [D, \Phi](x) - (1 - \chi(x)) [V, \Phi](x) \| < \frac{d_1}{2}.$$ 

Therefore,

$$\Phi(x)^2 > d_1 > \frac{d_1}{2} + \| [D_{h,\chi}, \Phi](x) \|, \quad \text{for all } x \in K \setminus K_1. \quad (6.6)$$

Since $e^h \leq 1$ and $1 - \chi(x) = 0$ for $x \notin K$, we conclude from (6.5) that

$$\| [D_{h,\chi}, \Phi](x) \| \leq \| [D, \Phi](x) \|, \quad \text{for all } x \notin K.$$ 

Hence, by part (ii) of Definition 2.11 there exists $d > 0$ such that

$$\Phi(x)^2 > d + \| [D_{h,\chi}, \Phi](x) \|, \quad \text{for all } x \notin K. \quad (6.7)$$

Set $d_2 := \min\{\frac{d}{2}, d\}$. Then combining (6.6) and (6.7) we conclude that

$$\Phi(x)^2 > d_2 + \| [D_{h,\chi}, \Phi](x) \|, \quad \text{for all } x \notin K_1.$$

\[\square\]

**6.5. Deformation of the data in a neighborhood of $N$.** By [13] Chapter 9, we can deform the Riemannian metric on $M$, the Clifford action of $T^* M$ on $\Sigma$ and the connection on $\Sigma$ in a small neighborhood $U(N) \subset M$ of $N$ such that Conditions (N.1)–(N.3) of Section 6.2 are satisfied and $\Phi(x)^2 > 0$ for all $x \in (N \times (-\epsilon, 0]) \cup N M_+.$

Let $\chi : M \to [0, 1]$ be a smooth function such that $\chi(x) = 0$ for all $x \in N \times (-\epsilon, \epsilon) \subset U(N)$ and $\chi(x) = 1$ for all $x \notin N \times (-2\epsilon, 2\epsilon)$. We replace the potential $V$ with $\chi V$. Then Condition (N.4) of Section 6.2 is also satisfied.

It follows now from Lemma 6.4 that we can deform the structures in a small neighborhood of $N$ so that with respect to the new structures the essential support of $\Phi$ is contained in the interior of $M_\epsilon \setminus (N \times (-\epsilon, 0])$. Then Condition (N.5) of Section 6.2 is satisfied.
Since all our changes occurred only in a relatively compact neighborhood of \( N \), it follows from Theorem 2.21 of [13] that they don’t change the index of the associated Callias-type operator. Hence, it is enough to prove Theorem 2.16 for the case when Conditions (N.1)–(N.5) of Section 6.2 are satisfied, which we will henceforth assume.

6.6. Proof of Theorem 2.16. Let \( M_1 := N \times \mathbb{R} \) be the cylinder, and let \( B_1 = B_{\hat{\Phi}_N} \) be the Callias-type operator on \( M_1 \) associated with the triple \((\hat{\Sigma}_N, \hat{D}_N, \hat{\Phi}_N)\). Since the essential support of \( \hat{\Phi}_N \) is empty, \( B_1^2 > 0 \). Hence, \( \text{ind}_\tau B_1 = 0 \).

As in Section 5.6 we are going to cut and paste manifolds \( M \) and \( M_1 \) along \( N \) and use the relative index theorem. Notice, that we can do it, because in Section 6.5 we deformed all the data on the collar neighborhood of \( N \) in \( M \) so that now it matches the data on the cylinder \( M_1 \).

Applying the cut and paste procedure of Section 2.6 to manifolds \( M \) and \( M_1 \) and potentials \( \Phi \) and \( \hat{\Phi}_N \) we obtain manifolds \( M_2 = M_\_ \cup_N (N \times [0, \infty)) \) and \( M_3 := N \times (-\infty, 0] \cup_N M_\+ \), with potentials \( \Phi_2 \) and \( \Phi_3 \) respectively. Let \( B_{\Phi_2} \) and \( B_{\Phi_3} \) be the corresponding Callias-type operators.

The restriction of \( \Phi_2 \) to the cylindrical part \( N \times [0, \infty) \) is equal to \( \hat{\Phi}_N \). Similarly, the restriction of \( \Phi_3 \) to \( N \times (-\infty, 0] \) is equal to \( \hat{\Phi}_N \). Moreover, the essential support of \( \Phi_3 \) is empty. Hence, \( \text{ind}_\tau B_3 = 0 \). From Proposition 5.6 we obtain \( \text{ind}_\tau B_2 = \text{ind}_\tau D_{N^\+} \). From the relative index theorem 2.8 we now obtain

\[
\text{ind}_\tau B_{\Phi} = \text{ind}_\tau B_{\Phi} + \text{ind}_\tau B_1 = \text{ind}_\tau B_2 + \text{ind}_\tau B_3 = \text{ind}_\tau D_{N^\+}.
\]

7. Cobordism invariance of the \( \tau \)-index

In this section we prove Theorem 2.24 about the cobordism invariance of the \( \tau \)-index of a Callias-type operator. As a first step we give a new proof of the cobordism invariance of the index of Dirac operators on compact manifolds.

In this section we freely use the notation introduced in Sections 2.21 and 5.1–5.3.

7.1. Compact cobordisms. Let \( \Sigma \) be a Dirac \( A \)-bundle over a compact manifold \( M \), and let \( D \) be a Dirac operator on \( \Sigma \). We say that \( D \) is compactly null-cobordant if there exists a null-cobordism \((W, \hat{\Sigma}, \hat{D})\) of \( D \) with \( W \) a compact manifold with boundary.

Proposition 7.2. If \( D \) is compactly null-cobordant Dirac operator, then \( \text{ind}_\tau D = 0 \).

Proof. Consider the manifold \( W' := W \cup_{M} (M \times [0, \infty)) \). Let \( \Sigma' \) denote the Dirac bundle over \( W' \), whose restriction to \( W \) is equal to \( \hat{\Sigma} \) and whose restriction to the cylinder \( M \times [0, \infty) \) is equal to the bundle \( \hat{\Sigma} \). Let \( D' \) be the Dirac operator whose restriction to \( W \) is equal to \( \hat{D} \) and whose restriction to \( M \times [0, \infty) \) is equal to \( \hat{D} \).
For $\Phi' \in \text{End}_A(\Sigma')$ we denote by $B'_{\Phi'}$ be the Callias-type operator on $W'$ associated with the triple $(\Sigma', D', \Phi')$. Applying the Callias-type index theorem 2.16 to the operators $B'_{\text{Id}}$ and $B'_{-\text{Id}}$ we obtain

$$\text{ind}_r B'_{\text{Id}} = \text{ind}_r D, \quad \text{ind}_r B'_{-\text{Id}} = 0.$$ 

The proposition follows now from Corollary 2.17. □

### 7.3. Proof of Theorem 2.24

We now deduce the cobordism invariance of a Callias-type operator from Theorem 2.16 and Proposition 7.2.

Let $B_{\Phi}$ be the Callias-type operator associated with a triple $(\Sigma, D, \Phi)$ and let $(W, \Sigma, D, \Phi)$ be a null-cobordism of $B_{\Phi}$. Choose an open subset $\Omega \subset W$ with compact closure such that

1. $\Omega$ contains the essential support of $\Phi$;
2. the boundary $N := \partial \Omega$ of $\Omega$ is a smooth manifold which intersects $M = \partial W$ transversely.

Then $\Omega \cap \partial W$ is an open subset of $M = \partial W$ which contains an essential support of $\Phi$. Furthermore

$$N := N \cap \partial W = \partial (\Omega \cap M)$$

is a smooth compact hypersurface in $M$.

Let $\Sigma_N$ denote the restriction of $\Sigma$ to $N$. Then $\Sigma_N$ has a natural grading

$$\Sigma_N = \Sigma_{N+} \oplus \Sigma_{N-},$$

where the fiber of $\Sigma_{N+}$ (respectively $\Sigma_{N-}$) over $x \in N$ is the image of the spectral projection of $\Sigma_N$ corresponding to the interval $(0, \infty)$ (respectively $(-\infty, 0)$). We denote by $D_N$ the restriction of $\Sigma_N$ which preserves the grading (7.1). We denote by $D_{N+}$ the restriction of $D_N$ to $C^\infty_0 (N, \Sigma_{N+})$.

Let $D_{N+}$ be the operator induced on $N$ by $B_{\Phi}$, cf. Section 2.15. Then $(N, \Sigma_{N+}, D_{N+})$ is a null-cobordism of $D_{N+}$. Hence, it follows from Proposition 7.2 that $\text{ind}_r D_{N+} = 0$. We now use the Callias-type theorem 2.16 to obtain $\text{ind}_r B_{\Phi} = \text{ind}_r D_{N+} = 0$. □

### 8. The Γ-index Theorem

This section is devoted to the proof of the Γ-index theorem for Callias-type operators. In Section 8.1 we analyze Callias-type operators twisted by an $A$-Hilbert bundle of finite type. In section 8.4 we show that the Callias-type operators lifted to Galois covers can be interpreted using the twisted construction and deduce from this Lemma 2.27 and Theorem 2.28.

#### 8.1. Twisted Callias-type operators

Let $M, S, D$ and $\Phi$ be as in Section 2.26. We denote by $\nabla^S$ the connection on $S$. Recall from (2.11) that $D$ is the Dirac operator on $S$ given by

$$D = \sum_i c(e^i) \nabla^S_{e_i} + V,$$

where $V \in \text{End}_A(S)$ is a bundle map.
Suppose that $H$ is an $A$-Hilbert bundle of finite type endowed with a connection $\nabla^H$. Then the bundle $S \otimes H$ carries a Dirac $A$-Hilbert bundle structure with connection
\[ \nabla^{S \otimes H} := \nabla^S \otimes 1 + 1 \otimes \nabla^H. \]

We define a twisted Dirac operator on $S \otimes H$ by
\[ D_H = \sum_i c(e^i) \nabla^S_{e^i} \otimes (\nabla^H_{X^i} h) + V \otimes 1. \tag{8.2} \]

**Lemma 8.2.** The endomorphism
\[ \Phi_H := \Phi \otimes 1 \in \text{End}_A (S \otimes H), \]
is admissible for the pair $(S \otimes H, D_H)$.

**Proof.** Fix a trivializing neighborhood $U$ of $S \otimes H$ and local sections $s \in C^\infty(U, S|_U)$, $h \in C^\infty(U, H|_U)$. By (8.1) and (8.2) we have
\[ D_H(s \otimes h) = (Ds) \otimes h + \sum_i (c(X^i) s) \otimes (\nabla^H_{X^i} h), \]
where $\{X_i\}$ is a local orthonormal frame of $TM$ and $X^i$ is the dual frame of $T^*M$. It follows that
\[ [D_H, \Phi_H] (s \otimes h) = ([D, \Phi] s) \otimes h + \sum_i ([c(X_i), \Phi] s) \otimes (\nabla^H_{X^i} h). \tag{8.3} \]

By (2.12), the endomorphism $\Phi$ commutes with Clifford multiplication. Hence the second term on the right-hand side of (8.3) vanishes. Therefore,
\[ [D_H, \Phi_H] = [D, \Phi] \otimes 1. \]
It follows that $\Phi_H$ is admissible whenever $\Phi$ is admissible. \hfill \Box

Let $B_H^\tau$ denote the Callias-type operator associated with the triple $(S \otimes H, D_H, \Phi_H)$. When the connection $\nabla^H$ is flat, Theorem 2.16 allows to connect the indices $B_H^\tau$ and $B_\Phi$ in a particularly nice way.

**Theorem 8.3.** Let $S$ be an ungraded Dirac bundle over a complete odd-dimensional oriented Riemannian manifold $M$ and let $\Phi$ be an admissible self-adjoint endomorphism of $S$. Suppose $H \rightarrow M$ is a flat $A$-Hilbert bundle of finite type. Then
\[ \text{ind}_\tau B_\Phi^H = d \cdot \text{ind} B_\Phi, \]
where $d$ is the $\tau$-dimension of the typical fiber of $H$.

**Proof.** Choose a compact hypersurface $N \subset M$ such that $M = M_- \cup_N M_+$, where $M_-$ is compact and contains an essential support of both endomorphisms $\Phi$ and $\Phi_H$.

We apply the construction of Section 2.15 to construct the Dirac bundles $S_{N+}$ and $(S \otimes H)_{N+}$ over $N$. Let $D_{N+}$ and $D_{N+}^H$ the Dirac operators on $N$ defined as in (2.16). By Theorem 2.16 we have
\[ \text{ind}_\tau B_\Phi^H = \text{ind}_\tau D_{N+}^H, \quad \text{ind} B_\Phi = \text{ind} D_{N+}. \tag{8.4} \]
Let $H_N \to N$ denote the restriction of the flat bundle $H$ to $N$ and let $D_{N+}^{H_N}$ denote the Dirac operator $D_{N+}$ twisted with the bundle $H_N$. Observe that $(S \otimes H)_{N+} = S_{N+} \otimes H_N$ so that we can identify $D_{N+}^{H_N}$ with the operator $D_{N+}^{H_N}$. Therefore,

$$\text{ind}_\tau D_{N+}^{H_N} = \text{ind}_\tau D_{N+}^{H_N}.$$  \hspace{1cm} (8.5)

Finally, since $H_{N+}$ is a flat $A$-Hilbert bundle, from [36, Theorem 7.30 and Corollary 5.13] we get

$$\text{ind}_\tau D_{N+}^{H_N} = d \cdot \text{ind}(D_{N+}),$$  \hspace{1cm} (8.6)

where $d$ is the $\tau$-dimension of the typical fiber of $H_N$, that by definition of this bundle coincides with the typical fiber of $H$. The theorem follows now from equations (8.4), (8.5) and (8.6). \hspace{1.5cm} \square

### 8.4. Galois covers.

Suppose $\Gamma$ is a discrete group and denote by $l^2(\Gamma)$ the Hilbert space of complex valued square summable functions on $\Gamma$. We let $\Gamma$ act on the Hilbert space $l^2(\Gamma)$ by the right regular representation

$$(R_g f)(h) := f(h \cdot g), \quad g, h \in \Gamma, \quad f \in l^2(\Gamma).$$  \hspace{1cm} (8.7)

Observe that this action induces an action of the group algebra $\mathbb{C}\Gamma$ on $l^2(\Gamma)$, that coincides with the right convolution multiplication. Observe also that the operator $R_g$ defined by formula (8.7) is bounded and that $R_g^* = R_{g^{-1}}$. In this way we identify the group algebra $\mathbb{C}\Gamma$ with a $*$-subalgebra of $\mathcal{B}(l^2(\Gamma))$. The weak closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(l^2(\Gamma))$ is called the group von Neumann algebra of $\Gamma$ and is denoted by $\mathcal{N}\Gamma$. On this algebra we have the canonical faithful positive trace $\tau$ defined by

$$\tau(f) = \langle f(\delta_e), \delta_e \rangle_{l^2(\Gamma)}, \quad f \in \mathcal{N}\Gamma,$$  \hspace{1cm} (8.8)

where $\delta_e \in l^2(\Gamma)$ is by definition the characteristic function of the unit element.

Notice that the right $\Gamma$-action on $l^2(\Gamma)$ extends to a right $\mathcal{N}\Gamma$-action. In this way we endow the space $l^2(\Gamma)$ with a Hilbert $\mathcal{N}\Gamma$-space structure. We also let $\Gamma$ act on the Hilbert space $l^2(\Gamma)$ by the left regular representation

$$(L_g f)(h) := f(g^{-1} \cdot h), \quad g, h \in \Gamma, \quad f \in l^2(\Gamma).$$  \hspace{1cm} (8.9)

The right action of $\mathcal{N}\Gamma$ on $l^2(\Gamma)$ commutes with the left $\Gamma$-action. Therefore,

$$H_\Gamma := \widetilde{M} \times_\Gamma l^2(\Gamma)$$

is an $\mathcal{N}\Gamma$-Hilbert bundle of finite type on $M$.

It follows from (8.8) that the $\tau$-dimension of $l^2(\Gamma)$ is one. Hence, the $\tau$-dimension of the typical fiber of $H_\Gamma$ is also 1, i.e.

$$\text{dim}_\tau H_{\Gamma,x} = 1, \quad x \in M.$$  \hspace{1cm} (8.10)
8.5. **Lift of a Callias-type operator to a Galois cover.** Notice that, since $\Gamma$ is discrete, $H_\Gamma$ is endowed with a canonical flat connection $\nabla^{H_\Gamma}$. Let $D_{H_\Gamma}$ and $\Phi_{H_\Gamma}$ be the twisted Dirac operator and the endomorphism induced in $S \otimes H_\Gamma$ by $D$ and $\Phi$ as in Section 8.1.

Let $M$, $S$, $\Phi$, $B_\Phi$, $\tilde{M}$, $\tilde{S}$, $\tilde{\Phi}$, $\tilde{B}_\Phi$ be as in Section 2.26. Notice that

$$
\tilde{B}_\Phi = \begin{pmatrix} 0 & \tilde{D} - i\tilde{\Phi} \\ \bar{\tilde{D}} + i\tilde{\Phi} & 0 \end{pmatrix},
$$

where $\tilde{D}$ and $\tilde{\Phi}$ are the lifts of $D$ and $\Phi$ to the Galois cover. We want to compare the operator $\tilde{B}_\Phi$ with an operator $B_{H_\Gamma}^\tau$ given by the twisted construction of Section 8.1.

Observe that the action of $\Gamma$ on $L^2(\tilde{M}, \tilde{S})$ induces an $\mathcal{N}\Gamma$-Hilbert space structure on this space. Observe also that a closed subspace $L \subset L^2(\tilde{M}, \tilde{S})$ is $\Gamma$-invariant if and only if it is $\mathcal{N}\Gamma$-invariant and in this case the $\tau$-dimension and the $\Gamma$-dimension of $L$ coincide. Moreover, there is a $\mathcal{N}\Gamma$-Hilbert space isomorphism

$$
\mathcal{I} : L^2(\tilde{M}, \tilde{S}) \rightarrow L^2(M, S \otimes H_\Gamma)
$$

with the following properties:

(a) For every $s \in C_\infty^c(\tilde{M}, \tilde{S})$

$$
(\mathcal{I}s)(x) = \sum_{\gamma \in \Gamma} s(\gamma \cdot \tilde{x}) \otimes (\tilde{x}, \gamma), \quad x \in M,
$$

where $\tilde{x}$ is any lift of $x$ to $\tilde{M}$. Notice that in Equation (8.12) the fibers $\tilde{S}_{\gamma \tilde{x}}$ and $S_x$ are identified.

(b) The operators $\tilde{D}$ and $D_{H_\Gamma}$ are conjugated through $\mathcal{I}$. Here, $\tilde{D}$ is the lifting of $D$ to $\tilde{M}$ and $D_{H_\Gamma}$ is the Dirac operator associated with the twisted bundle $S \otimes H_\Gamma$: cf. Formula (8.2).

(c) If $L$ is a closed $\mathcal{N}\Gamma$-invariant subspace of $L^2(\tilde{M}, \tilde{S})$, then $\dim_L L = \dim_{\tau} \mathcal{I}(L)$.

(d) Suppose that

$$
\mathcal{D}_1 = \begin{pmatrix} 0 & D_1^- \\ D_1^+ & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D}_2 = \begin{pmatrix} 0 & D_2^- \\ D_2^+ & 0 \end{pmatrix}
$$

are odd formally self-adjoint $\mathcal{N}\Gamma$-equivariant differential operators acting respectively on $C_\infty^c(\tilde{M}, S \otimes \tilde{S})$ and $C_\infty^c(M, (S \oplus S) \otimes H)$. Suppose also that $\mathcal{I} \circ D_1^\pm = D_2^\pm \circ \mathcal{I}$ on $C_\infty^c(\tilde{M}, \tilde{S})$. Then $\mathcal{D}_1$ is essentially self-adjoint if and only if $\mathcal{D}_2$ is and $\mathcal{D}_1$ is $\tau$-Fredholm if and only if $\mathcal{D}_2$ is. In this case, it follows from (c) that $\text{ind}_\tau \mathcal{D}_1$ and $\text{ind}_\tau \mathcal{D}_2$ coincide.

For more details about the construction of the map $\mathcal{I}$ and its properties, we refer to [36, Section 7.5], where the case when $M$ is compact is treated. The case when $M$ is noncompact follows with minor modifications.

**Lemma 8.6.** The operator $\tilde{B}_\Phi$ is $\tau$-Fredholm and we have

$$
\text{ind}_\Gamma \tilde{B}_\Phi = \text{ind}_\tau B_{H_\Gamma}^\tau.
$$
Proof. By [13] (see also Section 2.1 of the present paper), the operator $B^H_\Phi$ is essentially self-adjoint and its closure is $\tau$-Fredholm. By points (b) and (d), to prove the lemma it suffices to show that $\Phi_H \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}$ on $C_c^\infty(\tilde{M}, \tilde{S})$. Fix $s \in C_c^\infty(\tilde{M}, \tilde{S})$. By point (a) we have
\[
(\mathcal{I} \circ \tilde{\Phi})(s)(x) = \sum_{\gamma \in \Gamma} \tilde{\Phi}(s(\gamma \cdot \tilde{x})) \otimes (\tilde{x}, \gamma) = \sum_{\gamma \in \Gamma} \tilde{\Phi}(s(\gamma \cdot \tilde{x})) \otimes (\tilde{x}, \gamma)
\]
where $\tilde{x}$ is any lift of $x$ to $\tilde{M}$. The proof is complete. \qed

8.7. Proof of Lemma 2.27. It follows from Lemma 8.6 and the fact that the $\tau$-dimension and the $\Gamma$-dimension coincide on closed $\mathcal{N}T$-invariant subspaces of $L^2(\tilde{M}, \tilde{S})$. \qed

8.8. Proof of Theorem 2.28. Since $H_\Gamma$ is a flat $\mathcal{N}T$-bundle, the thesis follows from (8.10), Lemma 8.6 and Theorem 8.3. \qed

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