Stationary Axisymmetric Configuration of the Resistive Thick Accretion Tori around a Schwarzschild Black Hole

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Abstract

We examine a thick accretion disc in the presence of external gravity and intrinsic dipolar magnetic field due to a non-rotating central object. In this paper, we generalize the Newtonian theory of stationary axisymmetric resistive tori of Tripathy, Prasanna & Das (1990) by including the fully general relativistic features. If we are to obtain the steady state configuration, we have to take into account the finite resistivity for the magnetofluid in order to avoid the piling up of the field lines anywhere in the accretion discs. The efficient value of conductivity must be much smaller than the classical conductivity to be astrophysically interesting. The accreting plasma in the presence of an external dipole magnetic field gives rise to a current in the azimuthal direction. The azimuthal current produced due to the motion of the magnetofluid modifies the magnetic field structure inside the disc and generates a poloidal magnetic field for the disc. The solutions we have found show that the radial inflow, pressure and density distributions are strongly modified by the electrical conductivity both in relativistic and Newtonian regimes. However, the range of conductivity coefficient is different for both regimes, as well as that of the angular momentum parameter and the radius of the innermost stable circular orbit. Furthermore, it is shown that the azimuthal velocity of the disc which is not dependent on conductivity is sub-Keplerian in all radial distances for both regimes. Owing to the presence of pressure gradient and magnetic forces. This work may also be important for the general relativistic computational magnetohydrodynamics that suffers from the lack of exact analytic solutions that are needed to test computer codes.

1 Introduction

After the discovery of quasars in 1963, black holes have attracted a lot of attention in high-energy astrophysics as energizers. Release of gravitational energy through the accretion of matter on to a massive black hole is widely regarded as the most plausible origin of the enormous luminosities of quasars. Spherical accretion on to massive black holes is inefficient in releasing this energy, and any accreting matter is likely to have the relative high angular momentum content. Hence, it is
common to invoke the presence of an accretion disc, that could be thin or thick depending upon their geometrical shapes.

The standard theory of thin accretion discs is mostly based on the fundamental paper of Shakura & Sunyaev (1973, hereafter SS73). Because of its relative simplicity and successful applicability, this theory has achieved the status of a textbook paradigm. It introduces a Keplerian rotating gas flow which is stationary, rotationally symmetric, geometrically thin and optically thick. This flow has subcritical accretion rate with vertical hydrostatic equilibrium, negligible pressure gradients and insignificant infall velocity. Moreover, it makes the physically reasonable and enormously productive Ansatz about the turbulent viscosity which is parametrized with an alpha parameter. The SS73 model has been found to be quite successful in reproducing at least the gross features of the observations (see e.g. Frank, King & Raine 2002).

Novikov & Thorne (1973) then included general relativity and extended the SS73 model to relativistic flows around the rotating black holes. Studies wishing to model more accurately phenomena beyond the SS73 model like time-dependent behavior, radiatively driven mass loss, various instabilities etc. have stretched the SS73 model over the range of its limits. For a review of such applications see e.g. Lin & Papaloizou (1996).

The common practice to treat accretion discs using the SS73 paradigm has always been accompanied by vertical averaging or in other word integrating the flow equations in the vertical direction (i.e. parallel to the rotation axis). Vertical averaging is a standard approximation in accretion disc theory since a long time ago, owing to the simplifications it introduces (Mineshige & Umemura 1997; Gammie & Popham 1998; Tsuribe 1999). The physical motivation behind this approximation is that the vertical thickness of the disc is usually much smaller than the local radius, so that the flow velocities are likely to be more or less independent of height. It is then reasonable to expect that very little is lost by integrating out the vertical coordinate and employing the vertical hydrostatic equilibrium.

In this paper we avoid the height-integration approximation and subsequently the vertical hydrostatic equilibrium. Instead, we set up the exact flow equations for steady axisymmetric flow in the $r\theta$ plane. Consequently, the resulting vertical pressure gradient may cause the disc to bulge and grow outwards from the equatorial plane, becoming a thick disc in which the vertical dimension of the disc is comparable with its radial size. For a geometrically thick accretion disc that the pressure forces play an essential role in its equilibrium structure, pressure provides a substantial support in the radial as well as vertical direction. As a result, gravity of the central star is no longer balanced by centrifugal force. Thus, the rotation law may be far from Keplerian distribution here.

The basic equations governing the motion of an axisymmetric stationary magnetofluid disc around a compact object in a curved space-time background are given by Prasanna, Tripathy & Das (1989, hereafter PTD89). The behaviour of a fluid in the presence of electromagnetic fields is governed to a large extent by the magnitude of the electrical conductivity. In most of the astrophysical phenomena, the ideal MHD approximation, in which the conductivity is actually assumed to be infinite, represents a very good approximation. In this case, the magnetic flux is conserved and the magnetic field is frozen in the fluid, being simply advected with it. Ideal MHD equations neglect any effect of resistivity on the dynamics. However, in cold, dense plasmas such as might be expected at the centres of protostellar discs (Stone et al. 2000; Fleming & Stone 2003), discs in dwarf nova systems (Gammie & Menou 1998), the ionization fraction may become so small that this approximation no longer holds and the conductivity must be assumed to be
finite. Moreover, in practice, even in the scenarios of hot plasmas like accretion discs around black holes (Kudoh & Kaburaki 1996), and in Galactic centre (Melia & Kowalenkov 2001; Kaburaki et al. 2010), there will be spatial regions where the electrical conductivity is finite too. In these instances, the resistive effects, most notably, magnetic reconnection will be expected to occur in reality. It will provide an important contribution to the energy losses from the system. Inclusion of a finite resistivity is particularly essential for a non-viscous disc to liberate gravitational energy.

The first analytical equilibrium solution including the conductivity for the plasma was obtained by Kaburaki (1986, 1987). Although his equilibrium solutions were not self-consistent, but he showed that similar to the standard viscous disc model of SS73, about one half of the gravitational energy is released in the magnetized disc through the Joule dissipation. As a consequence, the magnetic stress can take the place of viscous stress in the standard disc model, and extracts angular momentum from the disc.

In recent decades, plenty of studies on the dynamics of accretion discs have been carried out by applying some simplifying assumptions on the general equations derived by PTD89. The most usual of these assumptions are thin disc approximation (Prasanna & Bhaskaran 1989; Prasanna 1989; Bhaskaran & Prasanna 1990) and flow restriction only to the azimuthal component, with the inherent assumption that the radial and meridional components are negligible in comparison with the azimuthal one (Banerjee et al. 1997). Furthermore, Newtonian limit of relativistic equations of PTD89 for a thick resistive plasma disc surrounding a non-rotating black hole with an intrinsic dipolar magnetic field has been examined by Tripathy, Prasanna & Das (1990, hereafter TPD90). Despite the fact that for the first time, they took into consideration all three components of the flow velocity of matter, but yet another important aspect in thick accretion disc theory around the compact objects is left. Lacuna is the generalization of the Newtonian analysis of TPD90 to general relativistic formalism wherein the space-time curvature produced by the strong gravitational field of the central body introduces the new features.

In this paper we fill in this lacuna and follow the model drawn by TPD90, but put aside the Newtonian limit and work in the fully relativistic framework. We investigate the general relativistic effects by choosing the Schwarzschild geometry and ignore the self-gravity contribution of the disc. Further, we consider the central object to possess a poloidal magnetic field which has the usual dipolar form in the asymptotic limit of the Schwarzschild metric. We are not interested in outflowing motion from the surface of the disc. Consequently, neglecting the meridional flow (i.e. $V^\theta = 0$), seems an admissible approximation as a simplifying assumption in solving equations (Gu et al. 2009). Therefore, we proceed to study a pressure-supported, accreting, stationary, axisymmetric, conducting, magnetized thick disc around a compact object within the relativistic domain without shear viscosity and meridional flow.

The objective of this work is to attempt to understand the effect of electrical conductivity and the other free parameters on the dynamics of disc with approximations that allow the problem to be treated mainly by analytical methods as far as possible. A brief outline of this paper is as follows: In Section 2, we derive a set of dynamical basic equations in Schwarzschild background and discuss the magnetic field configuration. In Section 3, the analytical and numerical solutions of the relativistic disc are obtained and the effects of various parameters involved in the governing equations on these solutions are investigated. Newtonian limit of the relativistic solutions and comparison of these two sets of solutions are practiced in Section 4. Section 5 summarizes our conclusions.
2 GENERAL FORMULATION

2.1 Basic Equations

We are interested in relativistic magnetized flow accreted from the source of plasma around a non-rotating black hole in the form of a thick disc. Our desired magnetofluid disc is in stationary ($\partial_t \equiv 0$) and axisymmetric ($\partial_{\phi} \equiv 0$). Further, it is not massive in comparison with the central compact object. Hence, the self-gravity of the disc is considered to be negligible and the space-time structure supporting the disc is determined entirely by the central body. Besides, energy of the electromagnetic field is regarded to be negligible as compared to the energy associated with the mass of the central star. It means the electromagnetic fields do not influence the geometry, but they can be modified by the background geometry that is defined by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

The fundamental unit of length is $m = \frac{GM}{c^2}$, with $G$ as the universal gravitational constant, $M$ mass of the central object, and $c$ the speed of light. The Schwarzschild radial coordinate $r$ may be normalized with respect to $m$.

In general, motion of the magnetized plasma is described by three sets of general relativistic MHD equations. These are the energy-momentum conservation laws

$$T_{ij}^{\dot{;}j} = 0, \tag{1}$$

and Maxwell equations

$$F_{ij}^\dot{;}j = -\frac{4\pi}{c} J^i, \tag{2}$$

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0, \tag{3}$$

along with the generalized Ohm law

$$J^i = \sigma F^i_k u^k. \tag{4}$$

It is worth noting here that semicolon denotes a covariant derivative with respect to $x^j$. For a fluid endowed with a magnetic field, stress-energy tensor $T^{ij}$ is obtained by adding the energy-momentum tensor of the fluid

$$T_{F,\text{fluid}}^{ij} = \left(\rho + \frac{p}{c^2}\right) u^i u^j - \frac{p}{c^2} g^{ij},$$

to that of the electromagnetic field

$$T_{Em}^{ij} = -\frac{1}{4\pi c^2} \left(F_{ik} F^i_k - \frac{1}{4} g^{ij} F_{kl} F^{kl}\right),$$

as

$$T^{ij} = T_{F,\text{fluid}}^{ij} + T_{Em}^{ij}.$$
It consists of a perfect fluid with the rest-mass density $\rho$, the pressure $p$, the four-velocity $u^i$, and an electromagnetic field tensor $F^{ij}$ satisfying Maxwell equations. The other fluid variables are the electric four-current density $J$ and the electric conductivity $\sigma$ which is assumed constant for simplicity. We express the dynamical equations in terms of physical quantities by writing them in the orthonormal tetrad frame appropriate to the Schwarzschild metric

$$\lambda_i^{(a)} = \text{diag} \left[ \left( 1 - \frac{2m}{r} \right)^{-1/2}, \left( 1 - \frac{2m}{r} \right)^{1/2}, \frac{1}{r}, \frac{1}{r \sin \theta} \right],$$

satisfying

$$\lambda_i^{(a)} \lambda_j^{(b)} g_{ij} = \eta^{(a)(b)},$$

where $g_{ij}$ and $\eta^{(a)(b)}$ are the metric and Minkowski tensors, respectively. All global variables are then defined in local Lorentz frame as follows:

$$F^{(\alpha)(\beta)} = \lambda_i^{(\alpha)} \lambda_j^{(\beta)} F_{ij},$$

$$J^{(\alpha)} = \lambda_i^{(\alpha)} J^i,$$

$$E^{(\alpha)} = F^{(\alpha)(t)},$$

$$B^{(\alpha)} = \epsilon_{\alpha\beta\gamma} F^{(\beta)(\gamma)},$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. Using these definitions, one can express the spatial 3-velocity $V^\alpha$ defined through the relation $u^\alpha = \frac{c}{\epsilon} V^\alpha$, in terms of local Lorentz components as given by

$$V^{(r)} = \left( 1 - \frac{2m}{r} \right)^{-1} V^r,$$

$$V^{(\theta)} = r \left( 1 - \frac{2m}{r} \right)^{-1/2} V^\theta,$$

$$V^{(\varphi)} = r \sin \theta \left( 1 - \frac{2m}{r} \right)^{-1/2} V^\varphi.$$

There exists a minimal radius at which stable circular motion is still possible for the plasma orbiting the central black hole. It defines the so-called innermost stable circular orbit (ISCO) in a given background. For the Schwarzschild geometry, the radius of ISCO equals to $6m$ (Jefremov, Tsupko & Bisnovatyi 2015). Although, if the surface magnetic field of the central black hole is not too high, it can reach well within the usual $6m$ limit, almost up to $3m$ (Bhaskaran & Prasanna 1990). As a result, we presume the disc spreads in radial direction from $6m$ to $50m$ and in meridional direction has the angular thickness $\pi/3$ on either side of the equator.

To determine the configuration of magnetic field lines and the velocity of MHD flows streaming along each magnetic field line, we must solve the basic equations (1) - (4) self-consistently. We expand the equations in local Lorentz frame noting to the fact that the Roman indices run from 0 to 3 and the Greek ones run from 1 to 3. Besides, we adopt the standard convention for the
summation over repeated indices. The zeroth component of equation (1) is the continuity equation
\[
\left( \rho + \frac{p}{c^2} \right) \left\{ \frac{\partial V^{(r)}}{\partial r} + \frac{2}{r} V^{(r)} + \frac{1}{r} \left( 1 - \frac{2m}{r} \right)^{-1/2} \left[ \frac{\partial V^{(\theta)}}{\partial \theta} + \cot \theta \frac{\partial V^{(\varphi)}}{\partial \theta} \right] \right\} \\
+ V^{(r)} \frac{\partial}{\partial r} \left( \rho - \frac{p}{c^2} \right) + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{V^{(\theta)}}{r} \frac{\partial}{\partial \theta} \left( \rho - \frac{p}{c^2} \right) \\
+ \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{2}{c^2} \left[ B_{(\theta)} V^{(r)} J^{(\varphi)} - E_{(r)} V^{(r)} J^{(t)} - B_{(\varphi)} V^{(\theta)} J^{(r)} - E_{(\theta)} V^{(\theta)} J^{(t)} \right] = 0, \quad (5)
\]
and its spatial components provide the momentum equations
\[
\frac{\left( \rho + \frac{p}{c^2} \right)}{1 - V^{(r)^2}} \left[ V^{(r)} \frac{\partial V^{(r)}}{\partial r} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{V^{(\theta)}}{r} \frac{\partial V^{(r)}}{\partial \theta} \right] \\
+ \frac{mc^2}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1} \left\{ 1 - \frac{[V^{(r)}]^2}{c^2} \right\} \\
- \frac{1}{r} \left\{ [V^{(\theta)}]^2 + [V^{(\varphi)}]^2 \right\} + \frac{\partial p}{\partial r} \\
+ \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{c} \left[ E_{(r)} J^{(t)} - B_{(\theta)} J^{(\varphi)} \right] = 0, \quad (6)
\]
\[
\frac{\left( \rho + \frac{p}{c^2} \right)}{1 - V^{(r)^2}} \left[ V^{(r)} \frac{\partial V^{(\theta)}}{\partial r} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{V^{(\theta)}}{r} \frac{\partial V^{(\theta)}}{\partial \theta} \right] \\
+ \left( 1 - \frac{3m}{r} \right) \frac{V^{(r)} V^{(\theta)}}{r^2} - \cot \theta \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{[V^{(\varphi)}]^2}{r} \\
+ \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{r} \frac{\partial p}{\partial \theta} \\
+ \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{c} \left[ E_{(\theta)} J^{(t)} + B_{(\varphi)} J^{(r)} \right] = 0, \quad (7)
\]
\[
\frac{V^{(r)}}{r} \frac{\partial V^{(\varphi)}}{\partial r} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{V^{(\theta)}}{r} \frac{\partial V^{(\varphi)}}{\partial \theta} \\
+ \left( 1 - \frac{3m}{r} \right) \frac{V^{(r)} V^{(\varphi)}}{r^2} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \cot \theta \frac{V^{(\theta)} V^{(\varphi)}}{r} = 0. \quad (8)
\]
The Maxwell equations (2) and (3) can be expanded as

\[ \frac{\partial}{\partial \theta} \left( \sin \theta \, B_\phi(r) \right) = -\frac{4\pi}{c} r \sin \theta J^r, \]  

(9)

\[ \frac{\partial}{\partial r} \left[ r \left(1 - \frac{2m}{r} \right)^{1/2} B_\phi(r) \right] = \frac{4\pi}{c} r J^\theta, \]  

(10)

\[ \frac{\partial}{\partial r} \left[ r \left(1 - \frac{2m}{r} \right)^{1/2} B_\theta(r) \right] - \frac{\partial B_\phi(r)}{\partial \theta} = -\frac{4\pi}{c} r J^\phi, \]  

(11)

\[ \left(1 - \frac{2m}{r} \right)^{1/2} \sin \theta \frac{\partial}{\partial r} [r^2 E_r] + r \frac{\partial}{\partial \theta} [\sin \theta E_\theta] = -\frac{4\pi}{c} r^2 \sin \theta J^t, \]  

(12)

\[ \sin \theta \frac{\partial}{\partial r} [r^2 B_r] + r \left(1 - \frac{2m}{r} \right)^{-1/2} \frac{\partial}{\partial \theta} \left[ \sin \theta B_\theta \right] = 0, \]  

(13)

\[ \frac{\partial}{\partial r} \left[ r \left(1 - \frac{2m}{r} \right)^{1/2} E_\theta \right] - \frac{\partial E_r}{\partial \theta} = 0. \]  

(14)

At the first glance, the set of basic equations (5) - (14) seems rather complex to solve, and in general requires the use of plausible simplifying assumptions. As a consequence of axisymmetry of the problem, it looks reasonable to vanish the toroidal component of the electromagnetic field \((E_\phi = B_\phi = 0)\). This, when used in the Maxwell equations (9) and (10) leads to the omission of the poloidal current density \((J^r = J^\theta = 0)\). Afterwards, the Ohm law (equation 4) yields

\[ E_r = B_\theta \frac{V_\phi}{c}, \]  

(15)

\[ E_\theta = -B_r \frac{V_\phi}{c}, \]  

(16)

\[ J^\phi = -\frac{\sigma}{c} u^0 \left(1 - \frac{2m}{r} \right)^{1/2} \left[ B_\theta V^r - B_r V_\phi \right], \]  

(17)

\[ J^t = -\frac{\sigma}{c} u^0 \left(1 - \frac{2m}{r} \right)^{1/2} \left[ E_r V^r + E_\theta V_\phi \right] = \frac{J^\phi V^r}{c}. \]  

(18)
Defining the total derivative

\[ d \equiv V^{(r)} \frac{\partial}{\partial r} + \left( 1 - \frac{2m}{r} \right)^{-1/2} V^{(\theta)} \frac{\partial}{\partial \theta}, \]

we rewrite the equations (5) - (8) as

\[
\left( \rho + \frac{p}{c^2} \right) \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left[ r^2 \sin \theta V^{(r)} \right] + \frac{\partial}{\partial \theta} \left[ r \sin \left( 1 - \frac{2m}{r} \right)^{-1/2} V^{(\theta)} \right] \right\} + d \left( \rho - \frac{p}{c^2} \right) - \frac{2}{\sigma c^2 \mu^0} \left( 1 - \frac{2m}{r} \right)^{-1} \left( J^{(\varphi)} \right)^2 \left\{ 1 - \frac{\left( V^{(\varphi)} \right)^2}{c^2} \right\} = 0,
\]

\[
(19)
\]

\[
\left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{V^2}{c^2} \right)^{-1} \left\{ dV^{(r)} + \frac{mc^2}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1} \times \left\{ 1 - \frac{\left( V^{(r)} \right)^2}{c^2} \right\} \right\} - \frac{1}{r} \left\{ [V^{(\theta)}]^2 + [V^{(\varphi)}]^2 \right\} + \frac{\partial p}{\partial r} - \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{c} B^{(\theta)} J^{(\varphi)} \left\{ 1 - \frac{\left( V^{(\varphi)} \right)^2}{c^2} \right\} = 0,
\]

\[
(20)
\]

\[
\left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{V^2}{c^2} \right)^{-1} \left\{ dV^{(\theta)} + \frac{\left( 1 - \frac{3m}{r} \right) V^{(r)} V^{(\theta)}}{r} \right\} - \cot \theta \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{\left( V^{(\varphi)} \right)^2}{r} \right\} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{r} \frac{\partial p}{\partial \theta} \right\} + \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{1}{c} B^{(r)} J^{(\varphi)} \left\{ 1 - \frac{\left( V^{(\varphi)} \right)^2}{c^2} \right\} = 0,
\]

\[
(21)
\]

\[
dV^{(\varphi)} + \left( 1 - \frac{3m}{r} \right) \left( 1 - \frac{2m}{r} \right)^{-1} \frac{V^{(r)} V^{(\varphi)}}{r} + \cot \theta \left( 1 - \frac{2m}{r} \right)^{-1/2} \frac{V^{(\theta)} V^{(\varphi)}}{r} = 0.
\]

\[
(22)
\]

Equation (22) can be rewritten as

\[
d \left[ r \sin \theta \left( 1 - \frac{2m}{r} \right)^{-1/2} V^{(\varphi)} \right] = 0.
\]

\[
(23)
\]
Integrating equation (23), the azimuthal velocity is achieved

\[ V(\phi) = \frac{L}{r \sin \theta} \left( 1 - \frac{2m}{r} \right)^{1/2}, \] (24)

where \( L \) is a constant of integration and is defined as \( L = lcm \), wherein \( l \) is called the angular momentum parameter.

The poloidal component of the disc’s magnetic field is given by equations (13) and (14). Self-consistent solution of these two equations can be achieved by combining these equations together as

\[ B(r) \left( 1 - \frac{2m}{r} \right)^{-1/2} \left( 1 - \frac{3m}{r} \right) + B(\theta) \cos \theta \sin \theta = 0. \] (25)

One admissible solution set for the magnetic field that satisfy the equation (25) is given by

\[ B(r) = -B_1 r^{k-1} \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \sin^{k-1} \theta \cos \theta, \] (26)

\[ B(\theta) = B_1 r^{k-1} \left( 1 - \frac{3m}{r} \right) \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \sin^k \theta, \] (27)

wherein \( k \) and \( B_1 \) are constant. Relations (15) and (16) give the poloidal components of the electric field too

\[ E(r) = \frac{L}{c} B_1 r^{k-2} \left( 1 - \frac{3m}{r} \right) \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \sin^{k-1} \theta, \] (28)

\[ E(\theta) = \frac{L}{c} B_1 r^{k-2} \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \sin^{k-2} \theta \cos \theta. \] (29)

Current density components can be achieved by the other unused Maxwell equations (11) and (12)

\[ J(\varphi) = \frac{c}{4\pi} B_1 r^{k-2} \sin k \theta \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \times \left[ \left( 1 - \frac{3m}{r} \right) - k \frac{(1 - \frac{3m}{r})^2}{(1 - \frac{2m}{r})} + (1 - k) \cot^2 \theta \right], \] (30)

\[ J(t) = \frac{L}{4\pi} B_1 r^{k-3} \sin^{k-1} \theta \left( 1 - \frac{2m}{r} \right)^{-\frac{k}{2}} \times \left[ \left( 1 - \frac{3m}{r} \right) - k \frac{(1 - \frac{3m}{r})^2}{(1 - \frac{2m}{r})} + (1 - k) \cot^2 \theta \right]. \] (31)
As seen, two different definitions have been obtained for the current density \( J^\phi \) (equations 17 and 30). Evidently, they have to be consistent.

\[
\left(1 - \frac{3m}{r}\right) V^r + \left(1 - \frac{2m}{r}\right)^{1/2} \cot \theta V^\theta = \frac{c^2}{4\pi \sigma u_0 r} \times \\
\left[ k \left(1 - \frac{3m}{r}\right)^2 \left(1 - \frac{2m}{r}\right) - \left(1 - \frac{3m}{r}\right) + (k - 1) \cot^2 \theta \right]. \tag{32}
\]

In deriving this consistency equation, the components of the poloidal magnetic fields, equations (26) and (27), have been substituted. To sum up, the remaining basic equations that have not been solved yet, are listed as equations (19) - (21) and (32) including the undetermined physical variables (i.e. \( V^r, V^\theta, \rho \) and \( p \)).

### 2.2 Magnetic Field Configuration

As a matter of fact, the magnetic field in the MHD equations of the surrounding space of the central black hole consists generally of two parts:

\[ B = B^S + B^D. \]

\( B^S \) represents the external magnetic field generated by the current streaming outside the event horizon of the central black hole, while \( B^D \) is the disc field caused by the current flowing in the accretion disc. Roughly speaking, the region where \( |B^S| \geq |B^D| \) may be called the magnetosphere. Magnetodisc is also the region where \( |B^S| << |B^D| \). Its structure is maintained by the Lorentz force acting on the current there. Therefore, within the disc, \( B \) can be replaced by \( B^D \) with sufficient accuracy. Indeed, the magnetic field appearing in the equations of the previous sections is just the same as \( B^D \), that its superscript \( D \) has been dropped for simplicity. From now on, we use superscript only for the external field \( B^S \). Dipolar magnetic field is a proper model for the magnetosphere of a black hole that results from current rings exterior to the event horizon (Prasanna & Vishveshwara 1978; Takahashi & Koyama 2009).

\[
B^S_r = -\frac{3\mu}{4m^2} r^2 \left\{ \frac{2m}{r} \left(1 + \frac{m}{r}\right) + \ln \left(1 - \frac{2m}{r}\right) \right\} \sin \theta \cos \theta,
\]

\[
B^S_\theta = \frac{3\mu}{4m^2} \left\{ 1 + \left(1 - \frac{2m}{r}\right)^{-1} + \frac{r}{m} \ln \left(1 - \frac{2m}{r}\right) \right\} \sin^2 \theta,
\]

wherein, \( \mu \) is the dipole magnetic moment of the central star that may be expressed in terms of the surface magnetic field \( B_s \) and the radius \( R \) of the central object as \( \mu = B_s R^3 \).

To study the magnetic field configuration, the solutions of magnetic lines of force equations

\[
\frac{dr}{B_r} = \frac{r \, d\theta}{B_\theta} = \frac{r \sin \theta \, d\phi}{B_\phi},
\]

10
should be analyzed. In order to visualize the field line structure, it is useful to transform to a Cartesian frame through the usual relations \((X = r \sin \theta \cos \varphi, \ Y = r \sin \theta \sin \varphi, \ Z = r \cos \theta)\) and then to obtain the corresponding parametric equations that generate the curves both for the external magnetic field at infinity

\[
X = \frac{r_0}{\sin \theta}, \quad Z = \frac{r_0}{\sin^2 \theta} \cos \theta,
\]

and for the disc field

\[
X = r_0 \cos(\varphi_0 - \beta r_0 (k - 2) \cot \theta),
\quad Y = r_0 \sin(\varphi_0 - \beta r_0 (k - 2) \cot \theta),
\quad Z = r_0 \cot \theta,
\]

where \(r_0\) and \(\varphi_0\) are the constants of integration and \(\beta\) is the ratio of the toroidal magnetic field \(B_{\psi}(\varphi)\) to \(B_1\). However, in our model \(\beta = 0\), because of the absence of \(B_{\psi}(\varphi)\). Further, due to the azimuthal symmetry, \(\varphi_0\) can be set zero without any loss of generality. Inasmuch as the meridional structure of the disc is presumed to extend to about 60° on either side of the equatorial plane, necessity of the continuity of the magnetic field lines across the disc boundary surface (i.e. \(\theta = \pi/6\)) demands that

\[
(B^D)^2 \big|_{r=r_0, \ \theta=\pi/6} = (B^S)^2 \big|_{r=r_0, \ \theta=\pi/6}, \tag{33}
\]

where

\[
(B^D)^2 = (B^D)_{(r)}^2 + (B^D)_{(\theta)}^2,
\quad (B^S)^2 = \left[(B^S)_{(r)}^2 + (B^S)_{(\theta)}^2 \right]^2,
\]

\(r_0\) is the radius where two field lines connect together. Thus, with the foregoing matching condition (equation 33), one can express the constant \(B_1\) in terms of the determined constants as

\[
B_1 = \frac{3\sqrt{13} \mu}{8 m^2} \left(\frac{r_0}{2}\right)^{-k} r_0.
\]

Fig. 1 shows a typical profile of the magnetic field structure in the meridional plane without (Fig. ??a) and with the disc field (Fig. ??b). As pointed out by Ghosh & Lamb (1978), magnetic lines of force can penetrate the accretion disc owing to the presence of a finite conductivity. Because, the value of \(\sigma\) is a measure of the rate of slippage of field lines through the disc plasma. It is seen that, inside the disc (Fig. ??b), the penetrating field lines are pushed outward as parallel to the \(z\) axis. The constant field lines of the disc are connected with the undistorted external field lines at the surface of the disc and are continuous at the boundary surface as postulated earlier. It does not depend on the value of \(k\) as well.
3 Possible Equilibrium Solution

To simplify the appearance of the governing equations (19) - (22), we multiply equation (20) by $V^r$ and equation (21) by $V^\theta$ and equation (22) by $V^\phi$ and adding

$$
\left( \rho + \frac{p}{c^2} \right) \left[ \frac{d\left( \frac{V^2}{c^2} \right)}{1 - \frac{V^2}{c^2}} + V^r \frac{2m}{r} \right] + 2 d\left( \frac{p}{c^2} \right) = 
$$

$$
- \frac{2}{\sigma u_0 c^2} \left( 1 - \frac{2m}{r} \right)^{-1} J(\phi)^2 \left\{ 1 - \frac{[V(\phi)]^2}{c^2} \right\}, \quad (34)
$$

wherein the total velocity $V$ is defined as

$$
V^2 = \left[ V^r \right]^2 + \left[ V^\theta \right]^2 + \left[ V^\phi \right]^2.
$$

Continuity equation (19) can be simplified too

$$
\left( \rho + \frac{p}{c^2} \right) \nabla \cdot \vec{V} + d\left( \rho - \frac{p}{c^2} \right) = 
$$

$$
\frac{2}{\sigma u_0 c^2} \left( 1 - \frac{2m}{r} \right)^{-1} J(\phi)^2 \left\{ 1 - \frac{[V(\phi)]^2}{c^2} \right\}, \quad (35)
$$

with a new definition for total velocity as

$$
\vec{V} = V^r \hat{r} + V^\theta \hat{\theta} + V^\phi \hat{\phi}, \quad (36)
$$

Right-hand side of these latter two equations (34 and 35) are similar with opposite sign. It motivates us to add them and achieve a simplified equation in terms of the total derivative $d$ with vanished right-hand side

$$
d\ln \left[ \left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{V^2}{c^2} \right)^{-1} \right] + \nabla \cdot \vec{V} + d\ln \left( \rho + \frac{p}{c^2} \right) = 0.
$$

In this equation, all terms have been written in terms of the total derivative except $\nabla \cdot \vec{V}$. If this term can be written so, then this equation are integrated simply. To this aim, we assume $V^\theta = 0$, which is a reasonable approximation in the case of no outflow production from the disc’s surface (Gu et al. 2009). Accordingly,

$$
\nabla \cdot \vec{V} = d\ln \left[ r^2 V^r \right],
$$

and equation (36) is reduced to

$$
d\ln \left[ \left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{V^2}{c^2} \right)^{-1} r^2 V^r \right] = 0, \quad (37)
$$

that can integrate simply

$$
\left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{2m}{r} \right) \left( 1 - \frac{V^2}{c^2} \right)^{-1} r^2 V^r = C(\theta), \quad (38)
$$
where $C(\theta)$ is an arbitrary function of $\theta$. Indeed, it is the general relativistic definition of the mass accretion rate $\dot{M}$. Employing the assumption $V(\theta) = 0$ in equation (32), the radial inflow velocity is acquired as
\[
V^{(r)} = \frac{1 - [V(\phi)]^2}{\left(1 - \frac{2m}{r}\right)^{-1} + \left(\frac{\omega B}{4 \pi \sigma r}\right)^2 \frac{4 \pi \sigma}{r}}\frac{c^2}{r} B,
\]
where
\[
B = k \left(1 - \frac{3m}{r}\right) \left(1 - \frac{2m}{r}\right)^{-1} - 1 + (k - 1) \left(1 - \frac{3m}{r}\right)^{-1} \cot^2 \theta.
\]
Contrary to the azimuthal velocity (equation 24), the radial velocity is dependent on conductivity. Here, a discussion for probable values of the electrical conductivity of the magnetofluid comes up due to the condition $|V^{(r)}| < V^{(\phi)} < c$. We choose the value of conductivity in the interval
\[
2 \times 10^4 \frac{1}{s} \leq \sigma \leq 3 \times 10^6 \frac{1}{s}.
\]
This choice refers to the fact that below the lower bound, the condition $|V^{(r)}| < V^{(\phi)}$ is disturbed and above the upper bound, the radial velocity enters the Newtonian region (i.e. $|V^{(r)}| \ll c$). This range for conductivity is much smaller than the classical conductivity that is estimated by electron - proton Coulomb collisions in the Newtonian regime (Kudoh & Kaburaki 1996)
\[
\sigma_{\text{clas}} \approx 3 \times 10^{12} \left(\frac{T}{10^4 K}\right)^{3/2} \frac{1}{s}.
\]
Substituting
\[
\left(\rho + \frac{p}{c^2}\right) \left(1 - \frac{V^2}{c^2}\right)^{-1} = \frac{C(\theta)}{r^2 V^{(r)}} \left(1 - \frac{2m}{r}\right)^{-1},
\]
in the equations (20) and (21), the components of the pressure gradient are achieved as
\[
\frac{\partial p}{\partial r} = -DC(\theta) + E \equiv R(r, \theta), \quad (40)
\]
\[
\frac{\partial p}{\partial \theta} = FC(\theta) - H \equiv T(r, \theta), \quad (41)
\]
where
\[
D = \frac{1}{r^2 V^{(r)}} \left(1 - \frac{2m}{r}\right)^{-1} \left\{ \left[ V^{(r)} \frac{\partial V^{(r)}}{\partial r} + \frac{mc^2}{r^2} \times \left(1 - \frac{2m}{r}\right)^{-1} \left\{ 1 - \frac{[V^{(r)}]^2}{c^2} \right\} - \frac{[V^{(\phi)}]^2}{r} \right] \right\}, \quad (42)
\]
\[
E = \frac{1}{c} \left(1 - \frac{2m}{r}\right)^{-1/2} B_{\theta(\phi)} \left(1 - \frac{[V^{(\phi)}]^2}{c^2} \right), \quad (43)
\]
\[ F = \frac{1}{r^2 V(r)} \left( 1 - \frac{2m}{r} \right)^{-1} \cot \theta [V'(r)]^2, \quad (44) \]

\[ H = \frac{w}{c} B_{r(r)} J^{(\varphi)} \left\{ 1 - \frac{[V'(r)]^2}{c^2} \right\}, \quad (45) \]

and \( R \) and \( T \) are functions of \( r \) and \( \theta \). A necessary and sufficient condition for the existence of the solution of the above-mentioned set of partial differential equations (40) and (41), describing a distribution of pressure in the fluid, is given by the integrability condition

\[ \frac{\partial R}{\partial \theta} = \frac{\partial T}{\partial r}, \]

having the form

\[ D \frac{dC(\theta)}{d\theta} + \left[ \frac{\partial D}{\partial \theta} + \frac{\partial F}{\partial r} \right] C(\theta) = \frac{\partial E}{\partial \theta} + \frac{\partial H}{\partial r}. \quad (46) \]

This condition gives an ordinary differential equation for \( C(\theta) \) which can be solved numerically with an appropriate boundary condition. Now, with specified \( C(\theta) \), one may determine both the gas pressure as a function of \( r \) and \( \theta \) by integrating the equation (40) over the radial distance

\[ p(r, \theta) = p_0 - C(\theta) \int_{r_0=6m}^{r} D \, dr + \int_{r_0=6m}^{r} E \, dr, \quad (47) \]

and the gas density through the equation (38)

\[ \rho(r, \theta) = \frac{C(\theta)}{r^2 V(r)} \left( 1 - \frac{V^2}{c^2} \right) \left( 1 - \frac{2m}{r} \right)^{-1} - \frac{p}{c^2}. \quad (48) \]

Here, \( p_0 \) that has been appeared as an integration constant, is indeed the pressure of the ISCO \((r_0 = 6m)\). Since it is expected that both pressure and density to be positive throughout the disc, a natural restriction is exerted on \( p_0 \).

As accretion gives rise to radiation, the equilibrium configuration is provided only when at the inner layer, the hydrostatic gas pressure matches the radiation pressure \( p_R \), namely \( p_0 = p_R \). Under these circumstances, a subcritical regime (i.e. \( \dot{M} < \dot{M}_{cr} = 3 \times 10^{-8} \frac{M}{M_\odot \text{year}} \)) is preferred. Because, at a subcritical rate of flow of matter into the Roche lobe of a black hole, it may be assumed that most of the inflowing matter is accreted. However, at supercritical value of inflow, an effective outflow of matter may take place under the influence of radiation pressure. It disturbs the equilibrium configuration of the disc. As a result, to have an equilibrium structure, we choose the subcritical regime of accretion. Besides, due to no meridional flow approximation (i.e. \( V^{(\theta)} = 0 \)) employed in our calculations, which is requisite in the absence of outflows, this regime has been preferred.

At essentially subcritical fluxes \( \dot{M} = 10^{-12} - 10^{-10} \frac{M}{\text{year}} \), the luminosity of the disc is of the order of \( L = 10^{34} - 10^{36} \frac{\text{erg}}{\text{s}} \). Maximal surface temperatures are of the order of \( T_s = 3 \times 10^5 - 10^6 \text{K} \) in the inner regions of the disc where most of the energy is released. This energy is radiated mainly
in the ultraviolet and soft X-ray bands, which are inaccessible to direct observations. When the rate of accretion increases, the luminosity grows linearly and the effective temperature of radiation rises.

At fluxes $\dot{M} = 10^{-9} - 10^{-8} M_{\odot}$ year$^{-1}$, disc is found to be a powerful X-ray source with luminosity $L = 10^{37} - 10^{38}$ erg sec$^{-1}$ and an effective temperature of radiation $T = 10^7 - 10^8$ K. It radiates also in the optical and ultraviolet spectral bands (SS73).

With these observations, it seems suitable to choose $C(\theta = \pi/4) = -10^{-9} M_{\odot}$ year$^{-1}$ as a boundary condition for integrating the differential equation (46) and $T = 10^7$ K for the temperature of ISCO ($r_0 = 6m$). Thus, the radiation pressure in that layer is specified conveniently (i.e. $p_0 = \frac{1}{3} a T^4$, where the radiation constant $a = 7.565767 \times 10^{-16}$ J/m$^3$K$^4$).

Figs 2 and 3 give the vertical or meridional structure of all physical variables at some radial distances represented in legend. As has been deduced from equation (38), mass accretion rate is just dependent on $\theta$. Accordingly, a proper set of the free parameters must be chosen so that the mass accretion rate does not vary significantly with the radial distances (Fig. 2a). Mass accretion rate is negative as well as radial velocity. Their negativity indicates the inflow towards the central black hole. Both radial inflow (Figs 2a and b) and rotation (Fig. 2c) of the disc slow down from the disc surface ($\theta = \pi/6$) towards the equator ($\theta = \pi/2$). Pressure shows the similar behavior too. It falls from the surface towards the equator (Fig. 3a), whereas the density seems to remain nearly constant there (Fig. 3b). This constancy refers to the fact that the variation of density in radial direction is much larger than that in the meridional direction (Fig. 4d).

The radial behaviour of the physical variables is plotted in Fig. 4. Close to the black hole event horizon, the gas temperature and velocities become extremely high (Popham & Gammie 1998) and gradually fall off outwards. Fig. 4 confirms this result as well. It demonstrates that in radial direction, both radial (Fig. 4a) and rotational (Fig. 4b) velocities of the disc become faster inwards. Pressure and density are the descending functions of the radial distance as well (Figs 4c and d). The density falls off rapidly as $r$ increases whereas pressure tends to remain constant after an initial decrease.

The effect of free parameters $\sigma$ and $k$ on some impressible physical variables is investigated by plotting them as a function of $r$ for different values of $\sigma$ (Fig. 5) and $k$ (Fig. 6). Rotational velocity is not affected by these parameters. However, radial inflow velocity slows down with rising them (Figs 5a and 6a). For lower values of $\sigma$, the pressure decrease outward is minimal. As $\sigma$ becomes larger, this difference becomes appreciable and also the pressure drops off (Fig. 5b) contrary to the behavior of the density. The disc becomes denser with rising the conductivity (Fig. 5c) and free parameter $k$ (Fig. 6b).

Fig. 7 shows the meridional dependency of the pressure and density with respect to the angular momentum parameter $l$. As disc rotates faster, both pressure (Fig. 7a) and density (Fig. 7b) increase. However, ascending the pressure with $l$, is just significant on the surface layers and gradually this sensitivity diminishes towards the equator. Furthermore, higher the value of $l$, larger the variation of density with $\theta$.

Isodensity contours for different values of $\sigma$ and $k$ are plotted in Figs 8 and 9 respectively. For lower values of these parameters, the related isodensity contours occur in nearer radial distances to central star.

Meridional flow pattern for different values of $\sigma$ and $k$ is depicted in Figs 10 and 11 respectively.
The flow is represented by the arrows at a grid of points in the X-Z plane indicating the direction of streamlines. As mentioned before, the conductivity ascent leads to decelerate the radial inflow velocity, so that the rotation would be much faster than the radial inflow for large values of $\sigma$. We would also have a rotating non-accreting equilibrium configuration around the central star (Fig. 10). Opposing to this behaviour is observable in Fig. 11 with lowering $k$. Because in this case, radial inflow velocity becomes faster than rotational velocity in such a way that for $k = -3$, inflow occurs as pure free fall without any rotation.

4 Newtonian Limit and Comparison with the TPD90’s Solutions

To convince about the correctness of our solutions and also comparing with those of TPD90, one may employ the Newtonian limit (i.e. $m \ll 1$ and $V_c << 1$) on the equations (24) - (32) and acquires their Newtonian counterparts as follows:

Azimuthal velocity

$$V^{(\phi)} = \frac{\tilde{L}}{r \sin \theta}, \quad (49)$$

Components of electromagnetic field

$$B_{(r)} = -B_1 r^{k-1} \sin^{k-1} \theta \cos \theta, \quad (50)$$

$$B_{(\theta)} = B_1 r^{k-1} \sin^k \theta, \quad (51)$$

$$E_{(r)} = \frac{\tilde{L}}{c} B_1 r^{k-2} \sin^{k-1} \theta, \quad (52)$$

$$E_{(\theta)} = \frac{\tilde{L}}{c} B_1 r^{k-2} \sin^{k-2} \theta \cos \theta. \quad (53)$$

and current density

$$J^{(\phi)} = (1 - k) \frac{c}{4\pi} B_1 (r \sin \theta)^{k-2}, \quad (54)$$

$$J^{(t)} = (1 - k) \frac{\tilde{L}}{4\pi} B_1 (r \sin \theta)^{k-3}, \quad (55)$$

Consistency equation

$$V^{(r)} + \cot \theta V^{(\theta)} = (k - 1) \frac{c^2}{4\pi \sigma r \sin^2 \theta}. \quad (56)$$
According to TPD90’s notation, \( \tilde{L}^2 = l M G r_{in} \), wherein the inner radius locates in \( r_{in} = 15 \, m \). Tilde symbol is written over the Newtonian counterparts of the variables to distinguish them from the relativistic ones. Considering the condition \( V^{(\theta)} = 0 \), the radial inflow velocity is achieved too

\[
V^{(r)} = (k - 1) \frac{c^2}{4\pi \sigma \, r^2 \sin^2 \theta} \frac{1}{r^2}. \tag{57}
\]

As we know, \( V^{(r)} \) must be negative to indicate inflows. It exerts an upper bound on the integer \( k \) as \( k \leq 0 \). In addition, the condition \( |V^{(r)}| < c \) leads to

\[
\sigma > (1 - k) \frac{c}{4\pi \, r^2 \sin^2 \theta}. \tag{58}
\]

Newtonian counterpart of the equation (48) provides a relation for density as

\[
\rho = \frac{\dot{M}}{1 - k} \frac{4\pi \sigma}{c} \frac{c^2}{r^2} \sin^2 \theta. \tag{59}
\]

Although TPD90 assumes that the mass accretion rate \( \dot{C}(\theta) \) is constant and denotes it by \( \dot{M} \)

\[
\rho = \frac{\dot{M}}{1 - k} \frac{4\pi \sigma}{c} \frac{c^2}{r^2} \sin^2 \theta. \tag{58}
\]

Equations (42) - (45) in Newtonian limit are simplified as

\[
\tilde{D} = \frac{1}{r^2 |V^{(r)}|} \left[ V^{(r)} \frac{\partial V^{(r)}}{\partial r} + \frac{MG}{r^2} - \frac{[V^{(r)}]^2}{r} \right] = \frac{c^2}{4\pi \sigma} \frac{(k - 1)}{r^4 \sin^2 \theta} + \frac{4\pi \sigma}{c^2} \frac{1}{(1 - k)} \left[ \frac{MG}{r^3 \sin^2 \theta} - \frac{L^2}{r^4} \right],
\]

\[
\tilde{E} = \frac{1}{c} \frac{B^{(\theta)} J^{(\varphi)}}{r^2 V^{(r)}} = \frac{B_p^2}{4\pi} \frac{(1 - k)}{r^2} \frac{1}{k + 1} \frac{L^2}{r^3} \cot \theta = -\frac{\partial}{\partial \theta} \left( \frac{B_p^2}{8\pi} \right),
\]

\[
\tilde{F} = \frac{\cot \theta}{r^2 V^{(r)}} \left[ V^{(r)} \right]^2 = \frac{4\pi \sigma}{c^2} \frac{L^2 \cot \theta}{k + 1} = \frac{\cot \theta}{r^2 V^{(r)}},
\]

\[
\tilde{H} = \frac{c}{r} B^{(r)} J^{(\varphi)} = -\frac{B_p^2}{4\pi} \frac{(1 - k)}{r^2} \frac{1}{k + 1} \frac{L^2}{r^3} \sin^2 \theta \cos \theta = -\frac{\partial}{\partial \theta} \left( \frac{B_p^2}{8\pi} \right),
\]

where poloidal magnetic field \( B_p \) is defined as

\[
B_p^2 = B^{(r)}_p + B^{(\theta)}_p = B_p^2 (r \sin \theta)^{2k-2}.
\]
Under these circumstances, by introducing the effective pressure
\[ \bar{p} = p + \frac{B_p^2}{8\pi}, \]
equations (40) and (41) become more concise
\[ \frac{\partial \bar{p}}{\partial r} = -\hat{D} \dot{M} \equiv \hat{R}(r, \theta), \]  
\[ \frac{\partial \bar{p}}{\partial \theta} = \hat{F} \dot{M} \equiv \hat{T}(r, \theta). \]  
Integrating the equation (59) over the radial distance yields
\[ \bar{p}(r, \theta) = -\dot{M} \int_{r_n=15m}^r \hat{D} \, dr, \]
then, the gas pressure can be obtained
\[ p = p_0 + \left\{ \frac{2\pi \sigma GM \dot{M} \sin^2 \theta}{(1 - k)c^2} \right\} \left[ \frac{1}{3r^3} \left[ \frac{(1 - k)\dot{M}c^2}{4\pi \sigma \sin^2 \theta} + \frac{4\pi \sigma ML^2}{(1 - k)c^2} \right] \right\} \]
\[ - \frac{B_p^2}{8\pi}. \]
These solutions (equations 49 - 58 and 61) are in exact agreement with those obtained by TPD90. Now, it deserves to compare the relativistic and Newtonian solutions both qualitatively and quantitatively. In close vicinity of a black hole, the Newtonian description is only a rough approximation and it is natural to expect the effects of spacetime curvature there. Thus, employing the Newtonian approximation is just permissible on farther radial distances relative to central compact object. That is why the ISCO is placed farther for Newtonian regime. Another quantitative difference rests on the range of velocities (Figs 12a,b and 13a,b). It demands two different ranges for the free parameters \( \sigma \) and \( l \) for both regimes (Table 1). In Newtonian regime, this allowed range of free parameters results in a huge reduction in the density of the disc relative to the relativistic regime (Fig. 12c). Besides that the density has an obvious quantitative difference in both regimes, it behaves qualitatively different as well. The descending radial gradient density in relativistic regime is steeper than Newtonian regime (Fig. 13c). The pressure acts qualitatively different in both regimes as well as density (Figs 12d and 13d). In relativistic regime, pressure diminishes from the layer surface toward the equator in meridional direction. However, exactly contrary to this behaviour can be seen in Newtonian regime (Fig. 12d). Moreover, in relativistic regime, in radial direction, pressure decreases rapidly in the inner radial distances, then steadies itself and reaches an almost constant value as one goes outwards in the disc. Whereas in Newtonian regime, pressure falls off rapidly (Fig. 13d). Figs 14 and 15 have been depicted to illustrate the role of \( \sigma \) and \( k \) in Newtonian regime respectively. With respect to these free parameters, all impressible physical variables behave similarly in both regimes (Figs 14a,c and 15), except the pressure. In Newtonian regime, quite the opposite to the relativistic regime, pressure rises with an increase in conductivity (Fig. 14b).
Table 1: Range of free parameters in Newtonian and Relativistic Regimes.

| Parameter | Newtonian Regime | Relativistic Regime |
|-----------|-----------------|---------------------|
| $|\frac{1}{c}| << 1$ | $\sigma \gg \frac{(1 - k) l}{r \sin \theta} \approx 5 \times 10^4 \frac{1}{s}$ | $2 \times 10^4 \frac{1}{s} \leq \sigma \leq 3 \times 10^6 \frac{1}{s}$ |
| $l = 0.1$ | $0.1 \leq l \leq 1.5$ |
| $r_{in} = 15 m$ | $r_{ISCO} = 6 m$ |

5 Conclusion

In this paper, we find a self-consistent solution of fully relativistic equations for a thick disc around a compact object having the radial and azimuthal components of the flow velocity non-zero. In general, it is not easy to solve this set of coupled partial differential equations without invoking the simplifying assumptions. Mainly because these equations are so complicated and highly non-linear, especially when the disc is magnetized. We have included a finite conductivity for the plasma and have ignored the shear viscosity and self-gravity of the disc. Despite the great simplifications coming from these assumptions, the scenario is still physically reasonable and non-trivial. Few variety of degrees of freedom can be captured and the free parameters can be conveniently chosen in order to describe an astrophysically relevant situation.

Using Ohm law explicitly, we have derived a self-consistent equilibrium solution for a plasma disc in the presence of an external stellar dipolar magnetic field along with a self-consistently generated poloidal magnetic field of the disc. This class of solutions is somewhat a general relativistic generalization of the Newtonian solutions obtained by TPD90. The space-time curvature produced by the strong gravitational field of the central body modifies the magnetic fields and other physical variables relative to the Newtonian case. However, similar to Newtonian model of TPD90, the magnetic field lines inside the disc are constant in nature.

The meridional structure of our magnetized thick disc is mainly determined by the force balance of the vertical component of the plasma pressure gradient, magnetic and centrifugal forces rather than that of gravity and gas pressure like in the standard viscous disc model. The existence of such structure, in fact, encourages one to look for generalizations of the analysis to cases having meridional flow (i.e. $V^\theta \neq 0$). Of course, such more complicated choice requires more complicated calculations. This might be suggestive of generating the jets from the disc. Such analyses of the thick accretion disc dynamics are in progress, but are beyond the scope of the present paper.

Even if such resistive tori with foregoing particular characteristics do not exist in nature, the exact solutions presented here can still be useful for numerical general relativistic MHD, which has attracted a lot of interest (Koide, Shibata & Kudoh 1999; Komissarov 2004; Anton et al. 2006).

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