Dynamics of Weakly Localized Waves

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We develop a transport theory to describe the dynamics of (weakly) localized waves in a quasi-1D tube geometry both in reflection and in transmission. We compare our results to recent experiments with microwaves, and to other theories such as random matrix theory and supersymmetric theory.

Localization of waves has always been among the most difficult yet most fascinating topics in the study of wave propagation in disordered media. The first studies dealt with infinite media, showing that localization is always achieved in 1D, but that a minimum amount of disorder is required in dimensions larger than 2. In 3D the critical point is estimated by the Ioffe-Regel criterion \( kL \approx 1 \), with \( k \) the wavenumber and \( L \) the mean free path of the waves at a specified frequency. Later studies have considered localization in open media, and emphasized the ‘leakage’ through the boundaries — quantified by the conductance — as the basic localization parameter. The Thouless criterion states that ‘leaky’, extended states become localized when the ‘diusion parameter. The Thouless criterion states that ‘leaky’, extended states become localized when the ‘dimensionless conductance’ \( g = G/(2e^2/h) \) is of order one. Most recent studies, both in theory and experiment, have emphasized the giant fluctuations in transmission coefficients in the regime \( g \ll 1 \), confirming the fundamental importance of the Thouless conductance for all localization phenomena. In the diffuse regime \( (g \gg 1) \), apart from a factor of order unity, the dimensionless conductance \( g \) can be expressed as the ratio of the inverse microscopic level spacing, called the Heisenberg time \( t_H \), and the Thouless time \( t_D = (L + 2z_0)^2/\pi^2 D_B \) (with \( D_B \) the diffusion constant, \( L \) the size of the medium, and \( z_0 \approx t \) accounting for internal reflection).

A theory for ‘all localization’ does not exist. Important elements should be its capability to describe the transition from the diffuse to the localized regime, notably with regard to leakage and dynamics, and in all dimensions, and its flexibility to experimental details, such as internal reflection, anisotropic scattering and absorption. A very complete localization theory is random matrix theory. It can describe the transition from weak to strong localization, scaling, absorption, and fluctuations, and recently also dynamics but applies only for low-dimensional systems. Supersymmetric theory has also a very general range of applicability but does not always give the necessary physical insight to guide experiments. Finally, the self-consistent theory for localization holds in all dimensions, is able to describe critical behavior around the mobility edge, and has a clear generalization for dynamical problems. Its major disadvantages are that it applies only to the field correlation function and not to higher moment statistics, and its failure in the case of broken time-reversal invariance. It is valid on length scales larger than the mean free path, and — in the time domain — for times less than the Heisenberg time.

Leakage effects can be studied from the ‘leakage function’ \( L(\alpha) \) defined from the ensemble-averaged, time-dependent transmission (reflection) \( I_{T,R}(t) \) according to,

\[
I_{T,R}(t) = \int_0^\infty d\alpha \exp(-\alpha t) P_{T,R}(\alpha).
\]  

Supersymmetric theories have predicted strongly non-exponential decay in transmission, even for weakly localized waves \( (g \gg 1) \) and in quasi-1D typically of the kind \( \exp[-g \log^2(t/t_H)] \) beyond the Heisenberg time. It is in this regime that a modal picture is appropriate, like in chaotic cavities, and that \( P_{T,R}(\alpha) \) can be argued to equal the genuine distribution of resonant widths \( P(\Gamma) \) of the modes at small \( \Gamma \). \( P(\Gamma) \) has a log-normal behavior at very small \( \Gamma \) attributed to ‘prelocalized’ modes, which have become a central issue in the study of random lasers.

For times smaller than the Heisenberg time \( t_H \) supersymmetric theory predicts the transmission to decay like \( \exp[-t/t_D + (1/g\pi^2)^2 t_H^2] \). This would imply a narrow Gaussian distribution for \( P(\Gamma) \), centered around the average Thouless leakage \( 1/t_D \) with width \( \sim 1/\sqrt{g} \). A recent numerical simulation of wave dynamics in 2D disordered media has shown a similar, roughly quadratic increase of the logarithm of intensity. Chabanov, Zhang, and Genack recently studied weakly localized microwaves in quasi-1D at times scales up to the Heisenberg time, and observed a non-exponential transmission with time of the same type. Another interesting report — coming from random matrix theory, and first reported for purely 1D systems — is the \( 1/t^2 \) reflection coefficient for the semi-infinite quasi-1D tube, rather than the familiar \( 1/t^{3/2} \) decay expected from diffusion theory. This implies that in reflection \( P_R(\alpha) \propto \alpha \) for small \( \alpha \).

Transport theory ought to be valid for times less than the Heisenberg time, beyond which a modal picture takes over. The recent developments in theory and experiment call for a transport theory for the dynamics of (weakly)
localized waves, and notably for the leakage functions $P_{T,R}(\alpha)$ defined in Eq. 1. This is the subject of the present Letter. We will show that these functions are broadened by interference effects in a way compatible with observations and supersymmetric theory. We emphasize that for $\alpha$ larger than the inverse Heisenberg time (the typical level spacing) the equivalence between the leakage function $P_{T,R}(\alpha)$ and the resonant width distribution $P(\Gamma)$ is not established, and that $P_{T,R}(\alpha)$ sometimes takes negative values.

Constructive interferences can be included into transport theory using the self-consistent theory of localization. In finite, open media this requires the appearance of a dynamical, spatially dependent diffusion constant $D(r, \Omega)$, which can explain the observed rounding of coherent backscattering of light near the mobility edge $\beta\Delta$ as well as the non-Ohmic transmission $22$. We will here study the dynamics. Given a short release of energy at the source at $t = 0$, the central observable is the flux of ensemble-averaged photon energy $I(r, t)$ at position $r$ and at time $t$, with Fourier transform $I(r, \Omega)$ that we shall continue analytically in the whole complex plane. By causality is $I(r, \Omega)$ an analytic function in the upper complex sheet $\text{Im}\ \Omega > 0$. For positive times we can change the contour of the inverse Fourier transform with respect to frequency into the negative complex plane. If we assume that simple poles or branch cuts appear only along the negative imaginary axis, we find relation 11 with

$$P_{T,R}(\alpha) = -i \lim_{\epsilon \downarrow 0} [I_{T,R}(\Omega = -i \alpha + \epsilon) - I_{T,R}(\Omega = -i \alpha - \epsilon)] \quad (2)$$

In the normal diffuse regime only simple poles show up at $\Omega_0 = -i n^2/\tau_D$, and $P_{T,R}(\alpha)$ equals an infinite sum of Dirac delta distributions. Purely localized modes would show up as a contribution $\delta(\alpha)$ at zero leakage, but occur only in infinite or closed media. For an open quasi-1D system ($N \gg 1$ transverse modes, length $L \gg \ell$, classical diffusion constant $D_B = v_E L/3$, transport mean free path $\ell \gg$ wavelength, and the energy transport velocity $v_E$) the basic equation is the 1D dynamic diffusion equation for the intensity Green’s function $C(z, z', \Omega)$,

$$[-i \Omega - \partial_z D(z, \Omega) \partial_z] C(z, z', \Omega) = \delta(z - z'), \quad (3)$$

supplied by the self-consistency condition for the dynamic diffusivity imposed by reciprocity $27$,

$$\frac{1}{D(z, \Omega)} = \frac{1}{D_B} + \frac{2}{\xi} C(z, z, \Omega), \quad (4)$$

featuring the length scale $\xi = \frac{2}{\hbar} N \ell$. At the boundaries $z = 0, L$ we impose the usual radiative boundary conditions $C \equiv 0_{2}D[0/L, \Omega]/D_B \partial_z C = 0$, where $z_0 \sim \ell$ accounts for internal reflection. $I_{T,R}(\Omega)$ is related to $C$ through $I_{T,R}(\Omega) = \mp D(z = L/0, \Omega) \partial_z C(z = L/0, z' = \ell, \Omega)$. 

The stationary problem ($\Omega = 0$) can be solved analytically by the substitution $d\tau = dz/D(z, 0)$. This shows that for $L \gg \xi$ the average transmission decays as $\exp(-L/\xi)$ which identifies $\xi$ as the localization length. The diffuse regime $L \ll \xi$ has normal Ohmic transmission with conductance $g \simeq g_0 = \frac{2}{N} \ell/(L + 2z_0) \simeq 2 \xi/L$. These results basically agree with the ones obtained from the DMPK equation $7$ and supersymmetric theory $23$. Note that when $L \gg \xi$ it is important to discriminate between $g_0$ and the real conductance $g$ which can be much smaller by localization effects.

The solution for any complex-valued $\Omega$ has to be found numerically, by iteration. For $g_0 > 0.1$, we found satisfying and unique convergence for all $\Omega$ after $10–100$ iterations. We have evaluated the leakage function by solving Eqs. 3 and 4 for $\epsilon = 10^{-9}/\ell_D$ and $\epsilon = 10^{-10}/\ell_D$ and then using linear extrapolation to find the limit $\epsilon \downarrow 0$. We have also carefully checked the absence of singularities away from the negative imaginary axis. The time-dependent transmission $I_T(t)$ was then obtained from Eq. 11. Absorption can be added but this will just give rise to a trivial translation of the LF $P_{T}(\alpha)$ to higher values for $\alpha$. Following Chabanov, Zhang and Genack $17$, we shall interpret any non-exponential decay in terms of a time-dependent diffusion constant, in which case the transmis-
sion would decay as
\[
I_T(t) \sim \exp \left\{ -\frac{\pi^2}{(L + 2s_0)^2} \int_0^t dt' D(t') \right\} .
\] (5)

In Fig. 1 we have compared our calculation for \( D(t) \) to experimental results obtained for three different choices for the dimensionless conductance, corresponding to the samples A–C of Ref. [17]: \( g_0 = 9 \) (A), \( g_0 = 7.5 \) (B), and \( g_0 = 4 \) (C). These values are slightly larger than can be estimated from the data of Ref. [17]. Our transport theory describes the experimental results fairly well for all times below the Heisenberg time \( t_H \sim g_0 t_D \). The inset of Fig. 1 shows that the different branches of the leakage function \( P_T(\alpha) \) achieve a finite width, though all with finite support. Note that the second branch has a negative value. We have fitted the first, positive branch to a Gaussian distribution with the same average and the same variance, and studied their variation with \( g_0 \). We have noticed that this term increases the agreement. 

\[
D(t) = 1 + \frac{A}{g_0} - \frac{B}{g_0 t D} .
\] (6)

with \( A = 0.15 \) and \( B = 0.20 \). Supersymmetric theory [9] gives \( B = 2/\pi^2 \) for orthogonal symmetry, in good agreement with our value, but makes no report of \( A \). Yet, we have noticed that this term increases the agreement with experiment considerably. Our theory assigns no weight to \( P_T(\alpha) \) for values smaller than a certain threshold \( \alpha^* \approx (1/t_D)(1 - 0.8/\sqrt{g_0}) \), in strong disagreement with supersymmetric theory [11,12], which predicts a log-normal distribution for small \( \alpha \), caused by ‘prelocalized’ states that have localization lengths much smaller than the average localization length \( \xi \). Our transport theory is not valid when \( \alpha \) is small compared to the average level spacing. It is for this reason also that it can describe only the uninteresting short-time dynamics of transmission in the localized regime \( g < 1 \) where the Heisenberg time is smaller than the diffusion time.

We will finally study the dynamics in reflection, and apply the same procedure to calculate the leakage function \( P_R(\alpha) \). For \( g \gg 1 \) we find a series of clearly separated branches, all positive in sharp contrast to transmission, and again with width \( \sim 1/\sqrt{g} \). Their maxima typically vary as \( \sqrt{g} \) which generates the typically diffuse \( 1/t^{3/2} \) tail in the time domain. The threshold leakage rate \( \alpha^* \sim 1/t_D \) causes an exponential decay at times beyond the diffusion time \( t_D \).

As \( g \) decreases the different branches of \( P_R(\alpha) \) start to join when \( g_0 \approx 0.5 \). For \( g_0 \ll 1 \), the threshold leakage rate decreases exponentially with \( g_0 \). In \( \alpha^* \sim -1/g_0 \), and becomes very rapidly very small, implying the disappearance of exponential decay. We will consider a waveguide of length \( L \gg \xi \). We find that when \( L \gtrsim 20 \xi \), \( P_R(\alpha) \) has converged to its asymptotic limit at \( L \to \infty \). In this limit, \( g_0 = 0 \) and \( t_H = \infty \), and our theory applies at all times. The asymptotic \( P_R(\alpha) \) roughly has a square-root behavior that is taken over by a linear slope for small values of \( \alpha \) (see the inset of Fig. 2). The linear law gives rise to the tail \( I_R(t) \sim 1/t^2 \) in the time-domain, as can be seen in Fig. 2. This is consistent with the prediction of Titov and Beenakker using random matrix theory [18]. They have estimated the cross-over to occur at a time \( t \sim N^2 t_s \) (where \( t_s \) is the mean free time), again consistent with our findings. We conclude that this interesting dynamical cross-over is well captured by the self-consistent diffusion theory, which, in contrast to the method of Ref. [15], is not limited to the case of \( L \gg \xi \) and can be applied to a waveguide of any length and at any time below the Heisenberg time \( t_H \).

In conclusion, we have shown that the dynamics of (weak) localization both in transmission and in reflection of a quasi-1D waveguide can be described by a self-consistent diffusion equation. This theory is not valid beyond the Heisenberg time, and other methods such as proposed by supersymmetric \( \sigma \)-models have to be employed. Stimulated by recent accurate time-resolved experiments of strongly disordered 3D materials close to the mobility edge [24], a future challenge is the application of this theory to 2D and 3D systems.

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