HOLOMORPHIC BUNDLES ON THE BLOWN-UP PLANE AND THE BAR CONSTRUCTION

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Abstract. Let $X_q$ denote a simply connected positive definite four-manifold with $b_2 = q$, let $\mathcal{M}_k X_q$ denote the moduli space of based $SU(r)$ instantons on $X_q$ with second Chern class $c_2 = k$ and let $\mathcal{M}_k X_q = \coprod_{k} \mathcal{M}_k$. We show that, for $k = 1, 2$ we have homotopy equivalences between $\mathcal{M}_k(X_p \# X_q)$, the degree $k$ component of $\text{Bar}(\mathcal{M}_k X_p, \mathcal{M}_k S^4, \mathcal{M}_k X_q)$ and the degree $k$ component of $\text{Bar}(\mathcal{M}_k S^4, (\mathcal{M}_k S^4)^{p+q}, (\mathcal{M}_k S^2)^{p+q})$. A similar result holds in the limit when $c_2 \rightarrow \infty$: we have $\mathcal{M}_\infty(X_p \# X_q) \simeq \text{Bar}(\mathcal{M}_\infty X_p, \mathcal{M}_\infty S^4, \mathcal{M}_\infty X_q) \simeq \text{Bar}(\mathcal{M}_\infty S^4, (\mathcal{M}_\infty S^4)^{p+q}, (\mathcal{M}_\infty S^2)^{p+q})$.

1. Introduction

In this paper we will study the moduli space of holomorphic bundles over a rational surface with vanishing first Chern class, trivialized on a rational curve. Through the Kobayashi-Hitchin correspondence, this space is isomorphic to the moduli space of based instantons over a positive definite simply connected closed four-manifold (see [3], [9]). Let $\mathcal{M}_k X_q$ be the moduli space of based instantons over a positive definite simply connected closed rational surface with vanishing first Chern class, trivialized on a rational curve.

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Theorem 1.1. Let $I = \{x_1, \ldots, x_q\} \subset \mathbb{C}^2$. Then:

1. The map $\mathbb{R} \times \prod_{x_i} \mathcal{M}_{x_i} \rightarrow \mathcal{M}_I$ induces a map $\pi_{1, I} : \mathcal{M}_J \rightarrow \mathcal{M}_I$. Write $\mathcal{M}_J = \prod_{x_i} \mathcal{M}_{x_i}$ where $\mathcal{M}_{x_i}$ denotes the component with second Chern class $k$. For $k = 1, 2$ we obtain a description of the moduli space $\mathcal{M}_I$ in terms of the moduli spaces $\mathcal{M}_0$ and $\mathcal{M}_{x_i}$, with $x_i \in I$: Let $\mathcal{M}_{x_i} = \mathcal{M}_{x_i}$ for $k \leq 2$ and $\mathcal{M}_{x_i} = \mathcal{M}_{x_i}$ for $k > 2$, and let $\mathcal{M}_I = \prod_{x_i} \mathcal{M}_{x_i}$. Then, using results in [13] we construct degree preserving maps $\mathbb{H} : \mathcal{M}_I \times \prod_{x_i} \mathcal{M}_{x_i} \rightarrow \mathcal{M}_I$ for any disjoint finite sets $I_1, \ldots, I_n \subset \mathbb{C}^2$. These maps are compatible with pullback and give rise, in a standard way, to bar constructions $\text{Bar}(\mathcal{M}_I, \mathcal{M}_0, \mathcal{M}_J)$ and $\text{Bar}(\mathcal{M}_0, \mathcal{M}_I, \mathcal{M}_I, \mathcal{M}_x)$ (where $I = \{x_1, \ldots, x_q\}$).

2. If $I = J \cup K$, with $J \cap K = \emptyset$, then the map $\mathbb{R} \times \prod_{x_i} \mathcal{M}_{x_i} \rightarrow \mathcal{M}_I$ induces a map $\text{Bar}(\mathcal{M}_I, \mathcal{M}_0, \mathcal{M}_0, \mathcal{M}_x)$ which is a homotopy equivalence in the degree $k = 1, 2$ components.

There is an analogous result to Theorem 1.1 in the limit when $k \rightarrow \infty$. In [16], Taubes described, for $k' > k$, gluing maps $\mathcal{M}_{k', k} \rightarrow \mathcal{M}_{k, k}$ and showed that, in the limit when $k \rightarrow \infty$, there are homotopy equivalences $\mathcal{M}_{k, \infty} \simeq \text{Map}_*(\#_{n} \mathbb{P}^2, BSU(r))$. In particular, $\mathcal{M}_{k, \infty} \simeq \Omega^4 BSU(r)$. 

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Theorem 1.2. The map #_{n}P^2 \to (\bigvee_{n}P^2) \vee S^4, obtained by pinching n copies of $S^3$ induces a map

$$\text{Bar}\left(\mathcal{M}_{\emptyset, \infty}^{\emptyset}, \prod_{x \in I} \mathcal{M}_{\emptyset, \infty}^{\emptyset}, \prod_{x \in I} \mathcal{M}_{c, \infty}^{c}\right) \to \mathcal{M}_{I, \infty}^{r}$$

which is a homotopy equivalence.

Direct sum with a trivial rank $r' - r$ bundle induces a map $\mathcal{M}_{I}^{r} \to \mathcal{M}_{I}^{r'}$ and we let $\mathcal{M}_{I}^{r} = \text{colim} \mathcal{M}_{I}^{r}$. In [7], [12], [1] it was shown that we have homotopy equivalences $\mathcal{M}_{I}^{\infty} \simeq \prod_{k} BU(k)$ and $\mathcal{M}_{I}^{\infty} \simeq \prod_{k} BU(k) \times BU(k)$ (with $x \in \mathbb{C}^2$). Combining the maps $\pi_{I,I}$ with Whitney sum allows us to define, for each $I \subset \mathbb{C}^2$, a bar construction, which we denote by:

$$\|B_I\| = \text{Bar}\left(\|M_\emptyset\|, \prod_{x \in I} \|M_\emptyset\|, \prod_{x \in I} \|M_x\|\right),$$

and a map: $h_I: \|B_I\| \to M_I$. The second Chern class of the bundles gives a grading of the spaces $\|B_I\|$ and we write $\|B_{I,k}\|, h_{I,k}$ for the degree $k$ components. For the $k = 1, 2$ components, in the limit when $r \to \infty$ the maps $\|B_I\|$ become Whitney sum so Theorem 1.2 implies that $h_{I,1}$ and $h_{I,2}$ are homotopy equivalences.

Conjecture 1.3. The map $h_I: \|B_I\| \to M_I$ is a homotopy equivalence.

For each $k$, it is enough to check the conjecture for finite sets $I$ with $\# I \leq k$:

Theorem 1.4. If, for every finite set $J \subset I$ with $\# J \leq k$, the map $h_{I,k}$ is a homotopy equivalence, then $h_{I,k}$ is a homotopy equivalence.

The paper is organized as follows: In section 2 we describe the moduli spaces $\mathcal{M}_{I}^{r}$ and show that in the limit when $r \to \infty$ they have the structure of $E_{\infty}$-spaces. In sections 3 and 4 we describe the bar constructions in the limit when $r \to \infty$ and show that, for disjoint finite sets $I$ and $J$ we have (see Theorem 1.4):

$$\text{Bar}(\|B_I\|, \|B_J\|) \simeq \|B_{I,J}\|.$$  \hspace{1cm} (1)

Assuming Conjecture 1.3 holds, it will follow that $\mathcal{M}_{I,J} \simeq \text{Bar}(\mathcal{M}_I, \mathcal{M}_J)$. Also, from the finite rank version of equation (1) we see that parts (1) and (2) of Theorem 1.4 are equivalent. In section 5 we prove Theorem 1.4. This theorem is a consequence of the following fact: $\mathcal{M}_{I,k}$ is the colimit of $\mathcal{M}_{J,k}$ taken over the subsets $J \subset I$ with $\# J \leq k$. Theorem 1.4 together with equation (1), imply Theorem 1.4 for $k = 1$. In section 6 we prove Theorem 1.4 for $k = 2$ and use it to show that $h_{I,2}$ is a homotopy equivalence. In section 7 we prove Theorem 1.2. In the appendix we prove some results needed in section 6 using the monad descriptions of holomorphic bundles introduced in [5], [6].

2. Moduli spaces of holomorphic bundles

In this section we give the moduli space of holomorphic bundles the structure of an algebra over the linear isometries operad.

Definition 2.1. Let $V$ be a complex hermitian vector space of dimension $r$ and let $E \to \mathbb{P}_I^2$ be a rank $r$ smooth complex vector bundle with first Chern class $c_1(E) = 0$. A holomorphic structure on $E$ is a semi-connection $\partial_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the integrability condition $\partial_E^2 = 0$. Let $C(I, E, V)$ be the space of pairs $(\partial_E, \phi)$ where $\partial_E$ is a holomorphic structure on $E$ holomorphically trivial on $L_\infty$.
and φ : E|_{L\infty} → V × L\infty is a holomorphic trivialization. We define \( M(I, E, V) = C(I, E, V)/\text{Aut}(E) \).

In [8], it was shown that:

**Proposition 2.1.** The group Aut(E) of smooth bundle automorphisms of E acts freely on C(I, E, V) and the quotient has the structure of a finite dimensional Hausdorff complex analytic space.

**Proposition 2.2.** Let \( E_1, E_2 \to \tilde{\mathbb{P}}^2 \) be two isomorphic smooth complex vector bundles. Then there is a canonical isomorphism \( M(I, E_1, V) \cong M(I, E_2, V) \).

**Proof.** Given an isomorphism \( \psi : E_1 \to E_2 \) define a map \( \psi_\ast : C(I, E_1, V) \to C(I, E_2, V) \) by \( \psi_\ast(\partial, \phi) = (\psi \circ \partial \circ \psi^{-1}, \phi \circ \psi^{-1}) \). This map descends to the quotient to give a homeomorphism \( M(I, E_1, V) \to M(I, E_2, V) \) which is independent of the choice of isomorphism \( \psi \). □

Since the isomorphism class of \( E \) is completely determined by \( c_2(E) = k \) and \( \text{rk} E = \dim V \), we will use the notation \( M(I, E, V) = M^V_{I,k} \).

**Definition 2.2.** Let \( M^V_I = \prod_{k=0}^\infty M^V_{I,k} \). We also define the following maps:

1. Let \( \omega : C(I, E_1, V_1) \times C(I, E_2, V_2) \to C(I, E_1 \oplus E_2, V_1 \oplus V_2) \) be the map defined by \( \omega((\partial_1, \phi_1), (\partial_2, \phi_2)) = (\partial_1 + \partial_2, \phi_1 + \phi_2) \). This map descends to the quotient to give a map \( \omega : M^V_I \times M^V_J \to M^V_{I \oplus J} \).

2. Given finite sets \( J \subset I \subset \mathbb{C}^2 \), let \( \pi_{\ast,J} : \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2_J \) be the blowup of \( \tilde{\mathbb{P}}^2 \) along \( I \setminus J \). Then pullback of holomorphic bundles induces a map \( \pi_{\ast,J}^\ast : C(J, E, V) \to C(I, \pi_{\ast,J}^\ast E, V) \). This map descends to the quotient to give a map \( \pi_{\ast,J}^\ast : M^V_I \to M^V_J \).

3. Let \( \alpha : V \to W \) be an isometry. Let \( \epsilon_\alpha \) be the trivial bundle over \( \tilde{\mathbb{P}}^2 \) with fiber \( \alpha(V)^\perp \subset W \) and denote by \( \bar{\partial} \) the canonical holomorphic structure on \( \epsilon_\alpha \). We define the map \( C\alpha : C(I, E, V) \to C(I, E \oplus \epsilon_\alpha, W) \) by sending \( (\bar{\partial}_E, \phi) \) to \( (\bar{\partial}_E + \bar{\partial}, (\alpha \circ \phi) \oplus 1) \). This map descends to the quotient to give a map \( M\alpha : M^V_I \to M^V_J \).

**Lemma 2.3.** The map \( M\alpha \) is a closed embedding.

**Proof.** It follows easily from the monad description of \( M^V_I \) in [11]. □

**Lemma 2.4.** The assignement \( \alpha \mapsto M\alpha \) defines a continuous map between the space of linear isometries \( \mathcal{L}(V, W) \) from \( V \) to \( W \) and the space of maps \( \text{Map}(M^V_I, M^W_J) \).

**Proof.** We divide the proof into two steps:

1. When \( \dim V = \dim W \), \( C\alpha \) is the map \( (\bar{\partial}_E, \phi) \mapsto (\bar{\partial}_E, \alpha \circ \phi) \) and the result is clear.

2. We assume \( \dim W > \dim V \). Fix \( \alpha_0 \in \mathcal{L}(V, W) \); we will show continuity at \( \alpha_0 \). Let \( V_0 = \alpha_0(V)^\perp \) and let \( \rho : \mathcal{L}(V \oplus V_0, W) \to \mathcal{L}(V, W) \) be the principal bundle map adjoint to the canonical inclusion \( i : V \to V \oplus V_0 \). Let \( \theta \) be a local section of \( \rho \) on a neighbourhood \( U \) of \( \alpha_0 \). Given \( \alpha \in U \), the restriction of \( \theta(\alpha) \) to \( V_0 \) gives an isomorphism \( \theta(\alpha)|_{V_0} : V_0 \to \alpha(V)^\perp \) which induces an
isomorphism of holomorphic bundles $\psi_\alpha : E \oplus \epsilon_{\alpha_0} \to E \oplus \epsilon_\alpha$. We have the commutative diagram:

$$
\begin{array}{ccc}
C(I, E, V) & \xrightarrow{C_\theta} & C(I, E \oplus \epsilon_{\alpha_0}, V \oplus V_0) \\
\downarrow C_\alpha & & \downarrow C_\theta(\alpha) \\
C(I, E \oplus \epsilon_\alpha, W) & \xrightarrow{\psi_\alpha} & C(I, E \oplus \epsilon_{\alpha_0}, W)
\end{array}
$$

and hence, the map $\alpha \mapsto M_I$ is given, on the neighbourhood $U$ of $\alpha_0$, by the composition

$$
\mathcal{L}(V, W) \xrightarrow{\theta} \mathcal{L}(V \oplus V_0, W) \xrightarrow{M_I} \text{Map}(\mathcal{M}_I^{V \oplus V_0}, \mathcal{M}_I^W) \\
\xrightarrow{\text{Map}(\mathcal{M}_I^V, \mathcal{M}_I^W)} \text{Map}(\mathcal{M}_I^V, \mathcal{M}_I^W)
$$

where the last map is induced by the canonical inclusion $i : V \to V \oplus V_0$. The result then follows because, by Step 1, the middle map $M_I$ is continuous since $\dim(V \oplus V_0) = \dim W$.

Using the terminology of [11], the pair $(\mathcal{M}_I, \omega)$ is an $L^*$-functor. That is:

**Proposition 2.5.** Let $L^*$ denote the graded category whose objects are the finite dimensional complex hermitian vector spaces and whose morphisms are the linear isometries. Let $\oplus : L^* \times L^* \to L^*$ be the direct sum functor. Then the assignements $V \mapsto \mathcal{M}_I^V$ and $\alpha \mapsto \mathcal{M}_I^\alpha$ define a continuous functor $\mathcal{M}_I$ from $L^*$ to $\text{Top}$ and $\omega : \mathcal{M}_I \times \mathcal{M}_I \to \mathcal{M}_I \circ \oplus$ is a natural transformation satisfying $\omega(x, *) = x$, where $* \in \mathcal{M}_I$ is the basepoint.

**Proof.** We’ve already shown in Lemma 2.4 that the functor is continuous. The other statements are straightforward. \qed

**Proposition 2.6.** Let $L$ denote the category whose objects are the finite or countably infinite hermitian vector spaces and whose morphisms are the isometries. Then $\mathcal{M}_I$ extends to a functor $\mathcal{M}_I : L \to \text{Top}$ and $\omega$ extends to a natural transformation $\mathcal{M}_I \circ \oplus$.

**Proof.** We can extend $\mathcal{M}_I$ to infinite dimensional vector spaces $W$ by letting $\mathcal{M}_I^W = \text{colim} \mathcal{M}_I^V$ where the colimit is taken over the finite dimensional subspaces $V \subset W$. See [11]. \qed

Now let $\mathbb{H}$ be a countably infinite complex hermitian vector space which we call the universe.

**Proposition 2.7.** Let $\mathcal{L}^{\mathbb{H}}$ be the complex linear isometries operad over $\mathbb{H}$. Then $\omega$ induces an $\mathcal{L}^{\mathbb{H}}$-algebra structure on $\mathcal{M}_I^{\mathbb{H}}$. Furthermore, a morphism of universes $\alpha : \mathbb{H}_1 \to \mathbb{H}_2$ induces a map of $\mathcal{L}^{\mathbb{H}}$-algebras $\mathcal{M}_I^{\mathbb{H}_1} \to \mathcal{M}_I^{\mathbb{H}_2}$ which is a homeomorphism when $\alpha$ is an isomorphism.

**Proof.** See [11]. \qed

For $J \subset I$, the map $\pi^*_J$ passes to the colimit to give a map $\pi^*_J : \mathcal{M}_I^{\mathbb{H}} \to \mathcal{M}_J^{\mathbb{H}}$.

**Proposition 2.8.** Let $\mathcal{C}$ be the category of finite subsets of $\mathbb{C}^2$ with morphisms the inclusions. Then the assignements $I \mapsto \mathcal{M}_I^{\mathbb{H}}$ and $(J \subset I) \mapsto \pi^*_J$ define a functor between $\mathcal{C}$ and the category of $\mathcal{L}^{\mathbb{H}}$-spaces.
Then we can easily check that \( \pi^*_I \) is a natural transformation between the functors \( \mathcal{M}_I \) and \( \mathcal{M}_J \) which commutes with \( \omega \). The result follows. \( \square \)

**Remark 2.3.** For \( J \subset I \), \( \mathcal{M}^E_I \) is a module over the \( \mathcal{L}^E \)-algebra \( \mathcal{M}^E_J \). The structure is induced by the map \( \pi^*_I : \mathcal{M}^E_I \rightarrow \mathcal{M}^E_J \) and the \( \mathcal{L}^E \)-algebra structure on \( \mathcal{M}^E_J \).

### 3. The bar construction

Let \( \mathcal{P} \) be an \( E_\infty \) operad and let \( A \) be a \( \mathcal{P} \)-algebra. We begin by generalizing the notion of a module \( M \) over \( A \). We start with a module \( \mathcal{P}_M \) over \( \mathcal{P} \) with the spaces \( \mathcal{P}_M(n) \) modeled on \( \operatorname{Map}(M \times A^n, M) \cong \operatorname{Map}(A^n, \operatorname{End}(M)) \). Besides the usual maps \( \circ_i : \mathcal{P}_M(k) \times \mathcal{P}(j) \rightarrow \mathcal{P}_M(k+j) \), composition on \( \operatorname{End}(M) \) gives \( \mathcal{P}_M \) the structure of a graded associative monoid.

**Definition 3.1.** Let \( \mathcal{P} \) be an \( E_\infty \) operad with composition \( \gamma(\theta; \theta_1, \ldots, \theta_k) = \theta \circ (\theta_1, \ldots, \theta_k) \) and unit \( 1 \in \mathcal{P}(1) \). Then:

1. A monoidal module \( \mathcal{P}_M \) over \( \mathcal{P} \) is an \( E_\infty \) module over \( \mathcal{P} \) such that \( \bigoplus_n \mathcal{P}_M(n) \) is a graded associative monoid with unit \( 1_M \in \mathcal{P}_M(0) \); that is, we have continuous maps

\[
\gamma_M : \mathcal{P}_M(k) \times \mathcal{P}_M(j_k) \times \mathcal{P}_M(j_1) \times \cdots \times \mathcal{P}_M(j_0) \rightarrow \mathcal{P}_M(j_0 + \cdots + j_k),
\]

which we represented by \( \gamma_M(\theta; \theta_0, \theta_1, \ldots, \theta_k) = \theta \circ_M (\theta_0, \theta_1, \ldots, \theta_k) \), satisfying the associativity relation

\[
\theta \circ_M (\theta_0 \circ_M (\theta_{00}, \theta_{01}, \ldots, \theta_{0k_0}), \theta_1 \circ_M (\theta_{11}, \ldots, \theta_{1k_1}), \ldots, \theta_j \circ_M (\theta_{jk_j}))
\]

and the unit relation \( 1_M = \theta \circ_M (1, 1, \ldots, 1) = \theta \).

A morphism of operads \( (\psi_M, \psi) : (\mathcal{P}_M, \mathcal{P}) \rightarrow (\mathcal{P}_M', \mathcal{P}) \) is a collection of continuous maps \( \mathcal{P}_M(k) \rightarrow \mathcal{P}_M'(k) \) and \( \mathcal{P}_M(k) \rightarrow \mathcal{P}_M(k) \) such that \( \psi \) is a morphism of operads and \( (\psi_M \theta, \varphi_M \theta, \ldots, \varphi_M \theta) = \psi_M (\theta \circ_M (\theta_0, \theta_1, \ldots, \theta_k)) \).

2. A \( \mathcal{P}_M \)-module over a \( \mathcal{P} \)-algebra \( A \) is a topological space \( M \) together with maps \( \mathcal{P}_M(k) \times M \times A^k \rightarrow M \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
(M \times A^{k_0}) \times A^{k_1} \times \cdots \times A^{k_k} = M \times A^k & \xrightarrow{\theta_0 \times \theta_1 \cdots \times \theta_k} & M \\
\downarrow \theta_0 \times \theta_1 \cdots \times \theta_k & & \downarrow \theta_0(\theta_0, \theta_1, \ldots, \theta_k) \\
M & \xrightarrow{\theta} & M
\end{array}
\]

A morphism of pairs \( (f_M, f) : (M_1, A_1) \rightarrow (M_2, A_2) \) is a pair of maps \( (f_M, f) \) such that \( f : A_1 \rightarrow A_2 \) is a map of \( \mathcal{P} \)-algebras and \( f_M : M_1 \rightarrow M_2 \) is a continuous map such that, for any \( \theta \in \mathcal{P}_M(k) \) we have \( f_M (\theta(m, a_1, \ldots, a_k)) = \theta(f_M(m), f(a_1), \ldots, f(a_k)) \).

We are interested in the following examples:

**Example 3.2.** Let \( \mathcal{P} \) be an \( E_\infty \)-operad and, for each \( n \geq 0 \), let \( \mathcal{P}_+ (n) = \mathcal{P}(n+1) \). Then \( \mathcal{P}_+ \) is a monoidal module over \( \mathcal{P} \) and the \( \mathcal{P}_+ \)-modules over a \( \mathcal{P} \)-algebra \( A \) are the modules over \( A \).
Example 3.3. Given countably infinite complex hermitian vector spaces $V$ and $W$, let $\mathcal{L}$ be the linear isometries operad over $V$ and let $\mathcal{L}_{M}(n) = \mathcal{L}(W \oplus V^n, W)$. Then $\mathcal{L}_{M}$ is a monoidal module over $\mathcal{L}$. Notice that, when $V = W$, we have $\mathcal{L}_{M} = \mathcal{L}_{+}$.

We now define the bar construction.

Definition 3.4. (1) Given an $E_{\infty}$ operad $\mathcal{P}$ and monoidal modules $\mathcal{P}_{L}, \mathcal{P}_{R}$ over $\mathcal{P}$, we call $(\mathcal{P}_{L}, \mathcal{P}, \mathcal{P}_{R})$ an operad triple. A morphism of triples $(\mathcal{P}_{L1}, \mathcal{P}_{1}, \mathcal{P}_{R1}) \to (\mathcal{P}_{L2}, \mathcal{P}_{2}, \mathcal{P}_{R2})$ is a triple $(\psi_{L}, \psi, \psi_{R})$ such that $(\psi_{L}, \psi): (\mathcal{P}_{L1}, \mathcal{P}_{1}) \to (\mathcal{P}_{L2}, \mathcal{P}_{2})$ and $(\psi_{R}, \psi): (\mathcal{P}_{R1}, \mathcal{P}_{1}) \to (\mathcal{P}_{R2}, \mathcal{P}_{2})$ are morphisms of pairs.

(2) Let $\Delta$ be the simplicial category. Given an operad triple $\mathcal{P} = (\mathcal{P}_{L}, \mathcal{P}, \mathcal{P}_{R})$, let $\Delta(\mathcal{P})$ be the category with the same objects as $\Delta$ and whose morphisms are defined as follows: For each morphism $\mu \in \Delta(m, n)$ let

$$\Delta(\mathcal{P})(\mu) = \mathcal{P}_{L}(\mu_{0}) \times \prod_{\alpha = 1}^{m} \mathcal{P}(\mu_{\alpha} - \mu_{\alpha-1}) \times \mathcal{P}_{R}(n - \mu_{m});$$

Then, the space of morphisms is defined to be

$$\Delta(\mathcal{P})(m, n) = \coprod_{\mu \in \Delta(m, n)} \Delta(\mathcal{P})(\mu).$$

Composition of morphisms: $\Delta(\mathcal{P})(\mu) \times \Delta(\mathcal{P})(\nu) \to \Delta(\mathcal{P})(\nu \circ \mu)$ is done using the operad data:

$$\gamma_{L}: \mathcal{P}_{L}(\mu_{0}) \times \mathcal{P}_{L}(\nu_{0}) \times \prod_{\beta = 1}^{\mu_{0}} \mathcal{P}(\nu_{\beta} - \nu_{\beta-1}) \to \mathcal{P}_{L}(\nu_{\mu_{0}});$$

$$\gamma: \mathcal{P}(\mu_{\alpha} - \mu_{\alpha-1}) \times \prod_{\beta = \mu_{\alpha-1} + 1}^{\mu_{\alpha}} \mathcal{P}(\nu_{\beta} - \nu_{\beta-1}) \to \mathcal{P}(\nu_{\mu_{\alpha}} - \nu_{\mu_{\alpha-1}});$$

$$\gamma_{R}: \mathcal{P}_{R}(n - \mu_{m}) \times \mathcal{P}_{R}(p - \nu_{n}) \times \prod_{\beta = \mu_{m} + 1}^{n} \mathcal{P}(\nu_{\beta} - \nu_{\beta-1}) \to \mathcal{P}_{R}(p - \nu_{\mu_{m}}).$$

From the associativity of the operad data it is straightforward to prove that this composition law is associative.

(3) Given spaces $X_{L}, X$ and $X_{R}$, we say $(X_{L}, X, X_{R})$ is a $\mathcal{P}$-triple if $X$ is a $\mathcal{P}$-algebra and $X_{L}, X_{R}$ are respectively $\mathcal{P}_{L}$ and $\mathcal{P}_{R}$-modules over $X$. A morphism of $\mathcal{P}$-triples $(X_{L1}, X_{1}, X_{R1}) \to (X_{L2}, X_{2}, X_{R2})$ is a triple $(f_{L}, f, f_{R})$ where $(f_{L}, f): (X_{L1}, X_{1}) \to (X_{L2}, X_{2})$ and $(f_{R}, f): (X_{R1}, X_{1}) \to (X_{R2}, X_{2})$ are maps of pairs.

(4) Given a $\mathcal{P}$-triple $X = (X_{L}, X, X_{R})$, let $\mathfrak{B}(X): \Delta(\mathcal{P})_{op} \to \text{Top}$ be the functor defined on objects by the assignment

$$n \mapsto X_{L} \times X^{n} \times X_{R}$$

and defined on morphisms as follows: Given $\mu \in \Delta(m, n)$ and $f \in \Delta(\mathcal{P})(\mu)$ we can write

$$X_{L} \times X^{n} \times X_{R} = (X_{L} \times X^{\mu_{0}}) \times \left( \prod_{\alpha = 1}^{n} X^{\mu_{\alpha} - \mu_{\alpha-1}} \right) \times (X^{n-\mu_{m}} \times X_{R}).$$
Then the maps
\[ \mathcal{P}_L(\mu_0) \times X_L \times X^{\mu_0} \to X_L \]
\[ \mathcal{P}(\mu_0 - \mu_{a-1}) \times X^{\mu_a - \mu_{a-1}} \to X \]
\[ \mathcal{P}_R(n - \mu_m) \times X^{n - \mu_m} \times X_R \to X_R \]
induce the desired map \( X_L \times X^n \times X_R \to X_L \times X^n \times X_R \).

(5) We define the bar construction by taking the homotopy colimit:
\[ \text{Bar}(X_L, X, X_R) = \| \mathcal{B}(X) \| = \text{hocolim}_{\Delta(\mathcal{P})^m} \mathcal{B}(X). \]

### 3.1. Maps from the bar construction

We now wish to define maps \( \| \mathcal{B}(X) \| \to Y \) for some space \( Y \). The idea is to replace \( Y \) with a homotopically equivalent space. We will need, for each \( k \geq 0 \), spaces \( \mathcal{P}(k) \) modeled on \( \text{Map}(X_L \times X^k \times X_R, Y) \).

**Definition 3.5.** Let \( \mathcal{P} = (\mathcal{P}_L, \mathcal{P}, \mathcal{P}_R) \) be an operad triple. A module \( \widetilde{\mathcal{P}} \) over \( \mathcal{P} \) is a collection of contractible based spaces \( \mathcal{P}(n) \), one for each \( n \geq 0 \), together with continuous maps
\[ \gamma : \mathcal{P}(k) \times \mathcal{P}_L(j_L) \times \mathcal{P}_R(j_R) \to \mathcal{P}(j) \]  
for \( j = j_L + j_1 + \ldots + j_k + j_R \)

which we represent by \( \gamma(\theta; \theta_L, \theta_1, \ldots, \theta_k, \theta_R) \), satisfying the associativity relation
\[ \tilde{\theta}(\theta_L \circ_{M} (\theta_{L0} \ldots, \theta_{L_k}), \ldots, \theta_R) = \tilde{\theta}(\theta_{L0} \ldots, \theta_{L_k}, \theta_{R0} \ldots, \theta_{R_k}) \]

We call \((\mathcal{P}, \mathcal{P})\) an operad 4-tuple. Given a \( \mathcal{P} \)-triple \( X = (X_L, X, X_R) \) and a space \( Y \), we say that \( (X; Y) \) is a \( (\mathcal{P}; Y) \) 4-tuple if there are, for each \( k \), maps \( \mathcal{P}(k) \times X_L \times X \times X_R \to Y \) such that the following diagram commutes:
\[ \begin{array}{ccc}
\mathcal{P}(k) \times X_L \times X \times X_R & \xrightarrow{\hat{\gamma}(\theta_L, \theta_1, \ldots, \theta_k, \theta_R)} & Y \\
\mathcal{P}(k) \times X_L \times X \times X_R & \xrightarrow{\gamma(\theta_L, \theta_1, \ldots, \theta_k, \theta_R)} & Y \\
\end{array} \]

We are interested in the following example:

**Example 3.6.** Given countably infinite complex hermitian vector spaces \( V, W_1, W_2 \) and \( U \), let \( \mathcal{L} \) be the linear isometries operad over \( V \) and let \( \mathcal{L}_i(n) = \mathcal{L}(W_i \oplus V^n, W_i) \), for \( i = 1, 2 \) (see Example 3.3). Also let \( \mathcal{L}(n) = \mathcal{L}(W_1 \oplus V^n \oplus W_2, U) \). Then \((\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2)\) is an operad 4-tuple. Note that, when \( W_2 = V \) and \( W_1 = U \), we have \( \mathcal{L}_2 = \mathcal{L}_+ \) and \( \mathcal{L} = \mathcal{L}_1^+ \).

We now define a space homotopically equivalent to \( Y \).

**Definition 3.7.** Let \( \tilde{\Delta} \) denote the category whose objects are the sets \([n] = \{0, 1, \ldots, n\} \subset \mathbb{Z}\) plus the empty set and whose morphisms are the order preserving maps. We denote the emptyset by \([-1] \in \tilde{\Delta} \). Let \( \Delta \subset \tilde{\Delta} \) be the simplicial category.
(1) Given an operad 4-tuple \((\mathcal{P}; \tilde{\mathcal{P}}) = (\mathcal{P}_L, \mathcal{P}_R; \tilde{\mathcal{P}})\), we define a category \(\tilde{\Delta}(\mathcal{P}; \tilde{\mathcal{P}})\) equivalent to \(\tilde{\Delta}\) as follows: For \(m, n \neq [-1]\), the spaces of morphisms coincide with those of \(\Delta(\mathcal{P})\). For \(m = -1\) we let \(\tilde{\Delta}(\mathcal{P}; \tilde{\mathcal{P}})(-1, n) = \tilde{\mathcal{P}}(n)\). Given \(\mu \in \Delta(m, n)\), composition of morphisms
\[
\tilde{\Delta}(\mathcal{P}; \tilde{\mathcal{P}})(-1, m) \times \Delta(\mathcal{P})(\mu) \to \tilde{\Delta}(\mathcal{P}; \tilde{\mathcal{P}})(-1, n)
\]
is done using the operad data:
\[
\tilde{\gamma}: \tilde{\mathcal{P}}(m) \times \mathcal{P}_L(\mu_0) \times \prod_{a=1}^{m} \mathcal{P}(\mu_a - \mu_{a-1}) \times \mathcal{P}_R(n - \mu_m) \to \tilde{\mathcal{P}}(n).
\]

(2) Given a \((\mathcal{P}; \tilde{\mathcal{P}})\) 4-tuple of spaces \((X, \tilde{X}) = (X_L, X, X_R; \tilde{Y})\), let \(\mathfrak{B}(X, \tilde{X}) = \tilde{\Delta}(\mathcal{P}; \tilde{\mathcal{P}})^{op} \to \text{Top}\) be the functor extending \(\mathfrak{B}(X)\), sending the object \([-1]\) to \(Y\) and defined on morphisms by the map
\[
\tilde{\mathcal{P}}(n) \times X_L \times X^n \times X_R \to Y.
\]
We let \(\|\mathfrak{B}(X, \tilde{X})\| = \text{hcolim}\limits_{\Delta(\mathcal{P}; \tilde{\mathcal{P}})} \mathfrak{B}(X, \tilde{X})\).

**Proposition 3.1.** Let \((\mathcal{P}; \tilde{\mathcal{P}}) = (\mathcal{P}_L, \mathcal{P}_R; \tilde{\mathcal{P}})\) be an operad 4-tuple and let \((X, \tilde{X}) = (X_L, X, X_R; \tilde{Y})\) be a \((\mathcal{P}; \tilde{\mathcal{P}})\) 4-tuple of spaces. Then the inclusion \([-1] \to \tilde{\Delta}\) induces a map \(Y \to \|\mathfrak{B}(X, \tilde{X})\|\) which is a homotopy equivalence.

**Proof.** It is enough to observe that \([-1]\) is an initial object. \(\square\)

**Definition 3.8.** We represent by \(h_X, \tilde{h}_X: \|\mathfrak{B}(X)\| \to Y\) the map in \(\text{hTop}\) induced by the diagram:
\[
\|\mathfrak{B}(X, X, X_R)\| \xrightarrow{\tilde{h}_X} \|\mathfrak{B}(X, X, X_R; \tilde{Y})\| \xleftarrow{h_X} Y.
\]

**Proposition 3.2.** Let \(\mathcal{P}_L\) be a monoidal module over an operad \(\mathcal{P}\) and let \(X_L\) be a \(\mathcal{P}_L\)-module over a \(\mathcal{P}\)-algebra \(\tilde{X}\). Let \(\mathcal{P}_{L+}(n) = \mathcal{P}_L(n+1)\). Then \((\mathcal{P}_L, \mathcal{P}_R; \mathcal{P}_{L+})\) is an operad 4-tuple, \((X_L, X, X; X_L)\) is a \((\mathcal{P}_L, \mathcal{P}_R; \mathcal{P}_{L+})\) 4-tuple and the map \(h_{X, X}\): \(\|\mathfrak{B}(X, X; X)\| \to X_L\) is a homotopy equivalence.

Before we prove Proposition 3.2 we need to prove some lemmas. The identity \(X_L \times X^n \times X = X_L \times X^{n+1}\) leads us to make the following definition:

**Definition 3.9.** We represent by \(\Delta\) and \(\tilde{\Delta}\) the categories whose objects are the same as the objects of \(\Delta\) (with \(\emptyset = [-1]\)) and such that \(\Delta(m, n) \subset \Delta(m+1, n+1)\) is the set of order preserving maps \(\mu: \{0, \ldots, m+1\} \to \{0, \ldots, n+1\}\) with \(\mu(m+1) = n+1\), and \(\Delta(m, n)\) is the set of order preserving maps \(\tilde{\mu}: \{-1, 0, \ldots, m+1\} \to \{-1, 0, \ldots, n+1\}\) with \(\tilde{\mu}(m+1) = n+1\) and \(\tilde{\mu}(-1) = -1\).

**Remark 3.10.** The categories \(\Delta\) and \(\tilde{\Delta}\) are subcategories of \(\Delta\) since we can extend any morphism \([m] \to [n]\) uniquely to a morphism \([m+1] \to [n+1]\) by sending \(m+1\) to \(n+1\). In a similar way, \(\Delta\) is a subcategory of \(\tilde{\Delta}\). Also observe that, for \(m \neq -1\), restriction gives an isomorphism of sets \(\Delta(m, n) \cong \Delta(m+1, n+1)\).

**Lemma 3.3.** The object \([-1]\) is an initial (and final) object of both \(\Delta\) and \(\tilde{\Delta}\) and the inclusion functors \(F: \Delta^{op} \to \Delta^{op}\) and \(F: \tilde{\Delta}^{op} \to \tilde{\Delta}^{op}\) are cofinal.
Definition 3.11. We denote by $\overline{\Delta}(\mathcal{P}_L, \mathcal{P})$ the topological category equivalent to $\Delta$ whose morphisms are defined as follows: for each $\mu \in \Delta(m+1, n+1)$ let

$$\overline{\Delta}_\mu(\mathcal{P}_L, \mathcal{P}) = \mathcal{P}_L(\mu_0) \times \prod_{\alpha=1}^{m+1} \mathcal{P}(\mu_\alpha - \mu_{\alpha-1})$$

and define $\overline{\Delta}(\mathcal{P}_L, \mathcal{P})(m,n) = \coprod_{\mu \in \Delta(m,n)} \overline{\Delta}_\mu(\mathcal{P}_L, \mathcal{P})$.

Notice that, for $\mu_m \leq n$, we have $\mathcal{P}_+(n-\mu_m) = \mathcal{P}(\mu_{m+1}-\mu_m)$ hence $\Delta(\mathcal{P}_L, \mathcal{P}, \mathcal{P}_+)$ and $\Delta(\mathcal{P}_L, \mathcal{P}; \mathcal{P}_+, \mathcal{P}_{L+})$ are canonically subcategories of $\overline{\Delta}(\mathcal{P}_L, \mathcal{P})$.

Lemma 3.4. The functor $\overline{\mathcal{B}}(X_L, X, X; X_L)$ can be extended to a functor $\overline{\mathcal{B}}(X_L, X) : \overline{\Delta}(\mathcal{P}_L, \mathcal{P})^{op} \to \text{Top}$.

Proof. Given $\mu \in \Delta(m+1, n+1)$ with $\mu_{m+1} = n+1$ the map

$$X_L \times X^{n+1} = X_L \times X^{\mu_0} \times \prod_{\alpha=1}^{m+1} X^{\mu_\alpha - \mu_{\alpha-1}} \to X_L \times X^{m+1}$$

is induced by the maps

$$\mathcal{P}_L(\mu_0) \times X_L \times X^{\mu_0} \to X_L$$

$$\mathcal{P}(\mu_\alpha - \mu_{\alpha-1}) \times X^{\mu_\alpha - \mu_{\alpha-1}} \to X$$

Proposition 3.2 is now a direct consequence of the following lemma:

Lemma 3.5. We have a commutative diagram

$$\begin{array}{ccc}
\|\mathcal{B}(X_L, X, X)\| & \longrightarrow & \|\overline{\mathcal{B}}(X_L, X)\|
\downarrow & & \downarrow [-1]
\|\overline{\mathcal{B}}(X_L, X; X_L)\| & \longrightarrow & X_L
\end{array}$$

where every map is a homotopy equivalence.

Proof. It is enough to show that the inclusion functor $\Delta^{op}(\mathcal{P}_L, \mathcal{P}, \mathcal{P}_+) \to \overline{\Delta}^{op}(\mathcal{P}_L, \mathcal{P})$ and the functors $\ast \to \overline{\Delta}^{op}(\mathcal{P}_L, \mathcal{P}; \mathcal{P}_{L+})$ and $\ast \to \overline{\Delta}^{op}(\mathcal{P}_L, \mathcal{P})$ which send $\ast$ to
[-1] are cofinal. This follows from the commutative diagrams

\[
\begin{array}{ccc}
\Delta^{op}(\mathcal{P}_L, \mathcal{P}_+; \mathcal{P}_L) & \xrightarrow{\alpha} & \Delta^{op} \\
\downarrow & & \downarrow \\
\Delta^{op}(\mathcal{P}_L, \mathcal{P}) & \xrightarrow{\alpha} & \Delta^{op}
\end{array}
\]

(and a similar diagram for \(\Delta^{op}(\mathcal{P}_L, \mathcal{P}_+; \mathcal{P}_L)\)) where the horizontal maps are equivalences of categories and the right vertical map is cofinal.

\(\square\)

4. The space \(\|\mathfrak{B}_I\|\)

Recall that \(\mathcal{L}^{\mathbb{H}}\) denotes the linear isometries operad on a countably infinite complex hermitian vector space \(\mathbb{H}\), which we call a universe.

**Definition 4.1.** Given a finite set \(I \subset \mathbb{C}^2\) let \(\mathbb{H}^I = \bigoplus_{x \in I} \mathbb{H}\) and, for each non-negative integer \(n\), let

\[
\mathcal{L}^{\mathbb{H}}_I(n) = \mathcal{L}(\mathbb{H} \oplus (\mathbb{H}^I)^n, \mathbb{H}), \quad \mathcal{L}^{\mathbb{H}, I}(n) = \prod_{x \in I} \mathcal{L}^{\mathbb{H}}(n).
\]

Also, let \(\mathcal{L}^{\mathbb{H}}_+ (n) = \mathcal{L}^{\mathbb{H}}_I(n + 1)\) and \(\mathcal{L}^{\mathbb{H}, I}_+ (n) = \mathcal{L}^{\mathbb{H}, I}_I(n + 1)\).

In Example 3.9 we observed that \((\mathcal{L}^{\mathbb{H}}_I, \mathcal{L}^{\mathbb{H}, I}_I, \mathcal{L}^{\mathbb{H}}_+; \mathcal{L}^{\mathbb{H}, I}_+I)\) is an operad 4-tuple. Since \(\mathcal{L}^{\mathbb{H}, I}\) sits inside \(\mathcal{L}^{\mathbb{H}}\) as the subspace of bloc diagonal matrices, \((\mathcal{L}^{\mathbb{H}}_I, \mathcal{L}^{\mathbb{H}, I}_I, \mathcal{L}^{\mathbb{H}, I}_+; \mathcal{L}^{\mathbb{H}, I}_+I)\) is also an operad 4-tuple.

Now we give \((\mathcal{M}_0, \prod_{x \in I} \mathcal{M}_0, \prod_{x \in I} \mathcal{M}_x, \mathfrak{M}_I)\) the structure of a \((\mathcal{L}^{\mathbb{H}}_I, \mathcal{L}^{\mathbb{H}, I}_I, \mathcal{L}^{\mathbb{H}, I}_+; \mathcal{L}^{\mathbb{H}, I}_+I)\)-4-tuple. The spaces \(\prod_{x \in I} \mathcal{M}_0\) and \(\prod_{x \in I} \mathcal{M}_x\) are \(\mathcal{L}^{\mathbb{H}, I}\)-algebras and the pullback maps \(\pi_{0,x} : \mathcal{M}_0 \to \mathcal{M}_x\) make \(\prod \mathcal{M}_x\) into a \(\mathcal{L}^{\mathbb{H}, I}_+\)-module over \(\prod \mathcal{M}_0\). Also, \(\mathcal{M}_0\) is a \(\mathcal{L}^{\mathbb{H}}\)-module over \(\prod \mathcal{M}_0\). Finally, given \(f \in \mathcal{L}^{\mathbb{H}}_I(n)\) we have a map:

\[
\mathcal{M}_0 \times (\prod \mathcal{M}_0)^n \times (\prod \mathcal{M}_0) \xrightarrow{\pi} \mathfrak{M}_I \times (\prod \mathcal{M}_I)^n \times (\prod \mathcal{M}_I) = \mathfrak{M}_I \times (\prod \mathcal{M}_I)^{n + 1} \xrightarrow{f} \mathfrak{M}_I.
\]

**Definition 4.2.** We write:

\[
\overline{\Delta}^{\mathbb{H}}_I = \overline{\Delta}(\mathcal{L}^{\mathbb{H}}_I, \mathcal{L}^{\mathbb{H}, I}_I, \mathcal{L}^{\mathbb{H}, I}_+; \mathcal{L}^{\mathbb{H}, I}_+I); \quad \overline{\mathfrak{B}}^{\mathbb{H}}_I = \overline{\mathfrak{B}}(\mathcal{M}_0, \prod_{x \in I} \mathcal{M}_0, \prod_{x \in I} \mathcal{M}_x, \mathfrak{M}_I);
\]

\[
\Delta^{\mathbb{H}}_I = \Delta(\mathcal{L}^{\mathbb{H}}_I, \mathcal{L}^{\mathbb{H}, I}_I, \mathcal{L}^{\mathbb{H}, I}_+; \mathcal{L}^{\mathbb{H}, I}_+I); \quad \mathfrak{B}^{\mathbb{H}}_I = \mathfrak{B}(\mathcal{M}_0, \prod_{x \in I} \mathcal{M}_0, \prod_{x \in I} \mathcal{M}_x).
\]

We represent the map of Definition 4.3 by \(h_I : \|\mathfrak{B}^{\mathbb{H}}_I\| \to \|\mathcal{M}_0\|^I\).

Now Proposition 4.2 tells us that:

**Proposition 4.1.** The maps \(h_0, h_\times\) are homotopy equivalences.

Given finite sets \(J \subset I \subset \mathbb{C}^2\), the projection \(\mathcal{L}^{\mathbb{H}, I}(n) \to \mathcal{L}^{\mathbb{H}, J}(n)\) and the map \(\mathcal{L}^{\mathbb{H}}_I(n) \to \mathcal{L}^{\mathbb{H}}_J(n)\) adjoint to the inclusion \(\mathbb{H} \oplus (\mathbb{H}^I)^n \to \mathbb{H} \oplus (\mathbb{H}^J)^n\) induce equivalences of categories \(\Delta_i : \Delta^{\mathbb{H}}_I \to \Delta^{\mathbb{H}}_J\) and \(\Delta_\times : \Delta^{\mathbb{H}}_I \to \Delta^{\mathbb{H}}_{J}\), and hence homotopy equivalences \(\Delta_i : \|\Delta^{\mathbb{H}}_I\mathfrak{B}_J\| \to \|\mathfrak{B}_J\|\) and \(\Delta_\times : \|\Delta^{\mathbb{H}}_I\mathfrak{B}_J\| \to \|\mathfrak{B}_J\|\). The inclusions of based spaces:

\[
\prod_{x \in J} \mathcal{M}_0 \to \mathcal{M}_0, \quad \prod_{x \in J} \mathcal{M}_x \to \mathcal{M}_x, \quad \prod_{x \in J} \mathcal{M}_x \to \prod_{x \in J} \mathcal{M}_x,
\]
together with the pullback map $\pi^*_{J,I}: \mathcal{M}_J \to \mathcal{M}_I$ induce maps $\tilde{\mathcal{B}}_i: \|\tilde{\Delta}^*_{J,I}\|\to\|\tilde{\mathcal{B}}_i\|$ and $\mathcal{B}_i: \|\Delta^*_{J,I}\|\to\|\mathcal{B}_i\|$. 

**Proposition 4.2.** Let $\mathcal{C}$ denote the category of finite subsets of $\mathbb{C}^2$ with morphisms the inclusions. Given a morphism $i: J \to I$, let $\|\Delta_i\|^{-1}$, $\|\tilde{\Delta}_i\|^{-1}$ denote the homotopy inverses of the maps $\|\Delta_i\|$ and $\|\tilde{\Delta}_i\|$ in the homotopy category $h\text{Top}$. Then:

1. The assignements $I \mapsto \|\mathcal{B}_i\|$ and $(i: J \to I) \mapsto \|\mathcal{B}_i\| \circ \|\Delta_i\|^{-1}$ define a functor $\|\mathcal{B}\|: \mathcal{C} \to h\text{Top}$. 

2. The assignements $I \mapsto \|\tilde{\mathcal{B}}_i\|$ and $(i: J \to I) \mapsto \|\tilde{\mathcal{B}}_i\| \circ \|\tilde{\Delta}_i\|^{-1}$ define a functor $\|\tilde{\mathcal{B}}\|: \mathcal{C} \to h\text{Top}$. 

3. The assignements $I \mapsto \mathcal{M}_I$ and $(i: J \to I) \mapsto \pi^*_{J,I}$ define a functor $\mathcal{M}: \mathcal{C} \to h\text{Top}$ and the maps $h_i: \|\mathcal{B}_i\| \to \|\tilde{\mathcal{B}}_i\|$ define a natural transformation between the functors $\|\mathcal{B}\|, \|\tilde{\mathcal{B}}\| : \mathcal{C} \to h\text{Top}$. 

**Proof.** Given finite sets $I, J, K \subset \mathbb{C}^2$ and inclusions $i: I \to J$ and $j: J \to K$, we need to show that 

$$
\|\mathcal{B}_{joi}\| \circ \|\Delta_{joi}\|^{-1} = (\|\mathcal{B}_j\| \circ \|\Delta_j\|^{-1}) \circ (\|\mathcal{B}_i\| \circ \|\Delta_i\|^{-1}).
$$

We have $\Delta_{joi} = \Delta_i \circ \Delta_j$ and, for each non-negative integer $n$, we have $\mathcal{B}_{i0j}(n) = \mathcal{B}_i(n) \circ \mathcal{B}_j(n)$. We then have a commutative diagram:

```
\[
\begin{array}{ccc}
\text{hcolim} \mathcal{B}_I & \xrightarrow{\Delta_i} & \text{hcolim} \Delta^* \mathcal{B}_I \\
& \xleftarrow{\Delta_{joi}} & \text{hcolim} \Delta^* \mathcal{B}_J \\
& \xrightarrow{\Delta_j} & \text{hcolim} \Delta^* \mathcal{B}_J
\end{array}
\]
```

which concludes the proof of (1). The proof of (2) is completely analogous. To prove (3) it is enough to observe that, given $i: J \to I$, we have a commutative diagram:

```
\[
\begin{array}{ccc}
\|\mathcal{B}_I\| & \xrightarrow{\text{hcolim}} & \|\tilde{\mathcal{B}}_i\| \\
& \xleftarrow{\|\Delta^*_{J,I}\|} & \|\tilde{\mathcal{B}}_i\| \\
& & \|\mathcal{B}_J\|
\end{array}
\]
```

**Remark 4.3.** If we consider only subsets of a fixed $I$ we can get a functor to Top instead of $h\text{Top}$. For each $J \subset I$ we replace the functors $\mathcal{B}_j$ by the functors $\Delta^*_{J,I}$: $\Delta_I \to \text{Top}$ where $j: J \to I$ is the unique morphism, and given a morphism $i: J_1 \to J_2$, the natural transformation $\mathcal{B}_i: \Delta^*_{J_1} \to \Delta^*_{J_2}$ induces a natural transformation $\|\mathcal{B}_i\|: \|\Delta^*_{J_1}\| \to \|\Delta^*_{J_2}\|$.
The functor is defined.

Objects are the subsets of $I$, geometric realization gives a functor $C_I \to \text{Top}$. We will come back to this construction in section 5.

We will now show that $\|\mathfrak{B}_f\|$ and $\|\mathfrak{B}_I\|$ are $\mathcal{L}_+$-modules over $\mathfrak{M}_0$.

**Proposition 4.3.** An isometry $\alpha : \mathbb{H}_1 \to \mathbb{H}_2$ induces homotopy equivalences $\|\mathfrak{B}_f^{\mathbb{H}_1}\| \simeq \|\mathfrak{B}_f^{\mathbb{H}_2}\|$ and $\|\mathfrak{B}_I^{\mathbb{H}_1}\| \simeq \|\mathfrak{B}_I^{\mathbb{H}_2}\|$ which are homeomorphisms if $\alpha$ is an isomorphism.

**Proof.** The isomorphism $\alpha$ induces a map of operads $\mathcal{L}^{\mathbb{H}_1} \to \mathcal{L}^{\mathbb{H}_2}$ and, for any $J \in \mathfrak{C}$, homotopy equivalences $\mathfrak{M}_I^{\mathbb{H}_1} \to \mathfrak{M}_I^{\mathbb{H}_2}$. Thus, we get an equivalence of categories $\Delta_\alpha : \Delta_I^{\mathbb{H}_1} \to \Delta_I^{\mathbb{H}_2}$ and a weak equivalence of functors $\mathfrak{B}_\alpha : \Delta_\alpha^* \mathfrak{B}_I^{\mathbb{H}_2} \to \mathfrak{B}_I^{\mathbb{H}_1}$. Thus we have homotopy equivalences

$$\|\mathfrak{B}_f^{\mathbb{H}_1}\| \xrightarrow{\|\mathfrak{B}_\alpha\|} \|\Delta_\alpha^* \mathfrak{B}_I^{\mathbb{H}_1}\| \xrightarrow{\|\Delta_\alpha\|} \|\mathfrak{B}_f^{\mathbb{H}_2}\|$$

which concludes the proof for $\mathfrak{B}_f$. The proof for $\tilde{\mathfrak{B}}_f$ is completely analogous. □

Given universes $\mathbb{H}_0, \mathbb{H}_1$, we have canonical operad maps $i_j : \mathcal{L}^{\mathbb{H}_j} \to \mathcal{L}^{\mathbb{H}_0 \otimes \mathbb{H}_1}$ (with $j = 0, 1$): the map $i_0$ maps $f \in \mathcal{L}^{\mathbb{H}_0}(n)$ to the isometry $\alpha \otimes 1 : (\mathbb{H}_0^n) \otimes \mathbb{H}_1 \to \mathbb{H}_0 \otimes \mathbb{H}_1$ and similarly for $i_1$. If $X$ is a $\mathcal{L}^{\mathbb{H}_0 \otimes \mathbb{H}_1}$-space, then each $i_j$ gives $X$ the structure of a $\mathcal{L}^{\mathbb{H}_j}$-space.

**Proposition 4.4.** Let $\mathbb{H}_0, \mathbb{H}_1$ be universes, let $\mathbb{H} = \mathbb{H}_0 \otimes \mathbb{H}_1$ and let $i_j : \mathcal{L}^{\mathbb{H}_j} \to \mathcal{L}^{\mathbb{H}}$ be the canonical map of operads. Then $\|i_j^* \mathfrak{B}_I^{\mathbb{H}}\|$ and $\|i_j^* \mathfrak{B}_f^{\mathbb{H}}\|$ are $\mathcal{L}_+^{\mathbb{H}_j}$-modules over the $\mathcal{L}_+^{\mathbb{H}_0}$-algebra $\mathfrak{M}_0^{\mathbb{H}}$.

**Proof.** Given $f \in \mathcal{L}^{\mathbb{H}_j}(n)$ the map $(\mathfrak{M}_0^{\mathbb{H}})^n \times \|i_j^* \mathfrak{B}_I^{\mathbb{H}}\| \to \|i_j^* \mathfrak{B}_f^{\mathbb{H}}\|$ will be defined by the natural transformation $(\mathfrak{M}_0^{\mathbb{H}})^n \times i_j^* \mathfrak{B}_I^{\mathbb{H}} \to i_j^* \mathfrak{B}_f^{\mathbb{H}}$ given by the maps:

$$(\mathfrak{M}_0^{\mathbb{H}})^n \times \mathfrak{B}_I^{\mathbb{H}}(m) = (\mathfrak{M}_0^{\mathbb{H}})^{n+1} \times \left( \prod \mathfrak{M}_0^{\mathbb{H}} \right)^{m} \times \left( \prod \mathfrak{M}_0^{\mathbb{H}} \right) \xrightarrow{f} \mathfrak{M}_f^{\mathbb{H}} \times \left( \prod \mathfrak{M}_0^{\mathbb{H}} \right)^{m} \times \left( \prod \mathfrak{M}_0^{\mathbb{H}} \right) = \mathfrak{B}_f^{\mathbb{H}}(m) \quad (m \neq -1)$$

$$(\mathfrak{M}_0^{\mathbb{H}})^n \times \mathfrak{B}_I^{\mathbb{H}}(-1) = (\mathfrak{M}_0^{\mathbb{H}})^{n} \times \mathfrak{M}_f \xrightarrow{\pi_\delta, i} (\mathfrak{M}_I^{\mathbb{H}})^{n+1} \xrightarrow{f} \mathfrak{M}_f^{\mathbb{H}} \xrightarrow{\pi_\delta} \mathfrak{B}_f^{\mathbb{H}}(-1)$$

The fact that this is a natural transformation follows from the commutativity of the following diagram, where $f \in \mathcal{L}^{\mathbb{H}_j}(n+1)$ and $g \in \mathcal{L}^{\mathbb{H}_j}(k+1)$:

$$\begin{array}{ccc}
(\mathfrak{M}_0^{\mathbb{H}})^n \times \mathfrak{M}_f^{\mathbb{H}} \times (\mathfrak{M}_0^{\mathbb{H}})^k & \xrightarrow{\alpha(f) \times 1} & (\mathfrak{M}_0^{\mathbb{H}})^n \times (\mathfrak{M}_0^{\mathbb{H}})^k \\
\downarrow 1 \times i_i(g) & & \downarrow i_i(g) \\
(\mathfrak{M}_0^{\mathbb{H}})^n \times \mathfrak{M}_f^{\mathbb{H}} & \xrightarrow{\alpha(f)} & \mathfrak{M}_f^{\mathbb{H}}
\end{array}$$

The structure on $(\|i_j^* \mathfrak{B}_I^{\mathbb{H}}\|, \mathfrak{M}_0^{\mathbb{H}})$ is obtained by restriction of the structure on $(\|i_j^* \mathfrak{B}_f^{\mathbb{H}}\|, \mathfrak{M}_0^{\mathbb{H}})$. □

In the next theorem we use the notation $\mathfrak{B}^{\mathbb{H}}(\cdots)$ to indicate in which universe the functor is defined.
**Theorem 4.5.** Let $I, J \subset \mathbb{C}^2$ be finite disjoint sets. Fix universes $\aleph_I, \aleph_0$ and let $H = \aleph_I \otimes \aleph_0$. Let $i : \mathcal{L}_{\aleph_I} \to \mathcal{L}_H$ be the canonical operad map. Then, on $h\text{Top}$, we have a commutative diagram:

$$
\begin{array}{c}
\|B^{H^s}(\|i^*B^H_I\|, \|B^H_J\|)\| \xrightarrow{\simeq} \|B^{H^s}(M^H_I, M^H_J)\| \quad \xrightarrow{\simeq} \|B^{H^s}(M^H_I, M^H_J, M^H_J)\| \\
\|B^H_{I,J}\| \xrightarrow{h_{I,J}} M^H_{I,J}
\end{array}
$$

where the left vertical map is a homotopy equivalence and the top horizontal map is the map induced by $h_I$ and $h_J$.

**Proof.** The proof is essentially the observation that

$$
\text{Bar}(\text{Bar}(\prod_{x \in I} M_x, \prod_{x \in I} M_0, M_0), M_0, \text{Bar}(M_0, \prod_{x \in I} M_0, \prod_{x \in I} M_x)) \simeq \text{Bar}(\prod_{x \in I,J} M_x, \prod_{x \in I,J} M_0, M_0)
$$

The strategy of the proof is to define functors $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ in such a way that we get a commutative diagram in $\text{hTop}$:

$$
\begin{array}{c}
\|B(\|B_I\|, M_0, \|B_J\|)\| \xrightarrow{g_0} \|\mathfrak{F}_0\| \xrightarrow{\simeq} \|\mathfrak{F}_1\| \xrightarrow{\simeq} \|\mathfrak{F}_1\| \xrightarrow{d \simeq} \|\mathfrak{F}_1\| \\
\|B(\|\mathfrak{F}_I\|, M_0, \|\mathfrak{F}_J\|)\| \xrightarrow{g_2} \|\mathfrak{F}_2\| \xrightarrow{\simeq} \|\mathfrak{F}_3\| \xrightarrow{\simeq} \|\mathfrak{F}_3\| \xrightarrow{d \simeq} \|\mathfrak{F}_3\| \\
\|B(M_I, M_0, M_J)\| \xrightarrow{g_3} \|\mathfrak{F}_4\| \xrightarrow{\simeq} \|\mathfrak{F}_4\| \xrightarrow{\simeq} \|\mathfrak{F}_4\| \xrightarrow{\simeq} \|\mathfrak{F}_4\|
\end{array}
$$

The result will then immediately follow.

We begin by defining a category $C_2$ topologically equivalent to $\Delta \times \Delta \times \Delta$ and a functor $\mathfrak{F}_2 : C_2^{\text{op}} \to \text{Top}$. The objects of $C_2$ are the triples of integers $(n_I, n, n_J)$ with $n_I, n_J \geq -1$ and $n \geq 0$ and, on objects, $\mathfrak{F}(n_I, n, n_J) = \mathfrak{B}_2(n_I) \times (\mathfrak{B}_2)^n \times \mathfrak{B}_2(n_J)$.

Given morphisms $\mu_I \in \Delta(m_I, n_I), \mu \in \Delta(m, n)$ and $\mu_J \in \Delta(m_J, n_J)$, we let

$$
C_2(\mu_I, \mu, \mu_J)
= \left( \prod_{a=1}^{m_I+1} \mathcal{L}(\mu_{I,a} - \mu_{I,a-1}) \right) \times \mathcal{L}(1 + \mu_0 + \mu_{I,0}) \times \left( \prod_{a=1}^{m} \mathcal{L}(\mu_a - \mu_{a-1}) \right)
\times \mathcal{L}(1 + n - \mu_m + \mu_{J,0}) \times \left( \prod_{a=1}^{m_J+1} \mathcal{L}(\mu_{J,a} - \mu_{J,a-1}) \right),
$$

where the left vertical map is a homotopy equivalence and the top horizontal map is the map induced by $h_I$ and $h_J$. The result will then immediately follow.
The functor $\mathfrak{F}$ and define the morphisms in $C$ by

$$C_2((m_1, m_2), (n_1, n_2)) = \prod_{\mu_1, \mu_2} C_2(\mu_1, \mu_2).$$

The functor $\mathfrak{F}$ is defined on morphisms in the obvious way. Let $C_0 \subset C_2$ be the full subcategory whose objects are the triples of non-negative integers. We define the functor $\mathfrak{F}_0$ as the restriction of $\mathfrak{F}$ to $C_0^{op}$; then the inclusion $C_0 \subset C_2$ induces a map $\mathfrak{F}_0 \to \mathfrak{F}_2$.

We now define the usual equivalences $g_0$ and $g_2$. Let $\Delta_{H} = \Delta(L_{+}^{H}, L_{+}^{H}, L_{+}^{H})$ and consider the functor $F: \Delta_{H} \times \Delta_{I} \times \Delta_{I} \to C_2$ which is the identity on objects and is induced on morphisms by the canonical maps $i: L_{+}^{H} \to L_{+}^{H}$ and $i_0: L_{+}^{H} \to L_{+}^{H}$, and the maps $L^{H}(1 + a) \times L^{H}(1 + b) \to L^{H}(1 + a + b)$ which we now define: using matrix notation, the image of a pair of isometries

$$[f_0, g_0] : H_0 \oplus H_0 \to H_0, \quad [g_{I,J}, h_{I,J}] : H_{I,J} \oplus H_{I,J} \to H_{I,J}$$

is the isometry

$$[f_0 \otimes 1, g_0 \otimes g_{I,J}, 1 \otimes h_{I,J}] : H_0 \oplus H_0 \oplus H_0 \to H.$$

Now, a direct verification shows that

$$h\text{colim} \mathfrak{F}_0 = \mathfrak{B}^{H_0}([i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}]),$$

and since $F$ is an equivalence of categories, we get

$$h\text{colim} \mathfrak{F}_2 \simeq \mathfrak{F}_2$$

and

$$h\text{colim} \mathfrak{F}_2 \simeq \mathfrak{F}_2$$

Let $1_{h_0} F^* \mathfrak{F}_2 = \text{Bar}(F^* \mathfrak{F}_2, \Delta_{H} \times \Delta_{I} \times \Delta_{I}, \Delta_{H} \times \Delta_{I} \times \Delta_{I}, \Delta_{I} \times \Delta_{I})$ denote Segal’s pushdown. Then we have a commutative diagram in $\text{Top}$:

$$\begin{array}{ccc}
\mathfrak{B}^{H_0}([i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}]) & \simeq & h\text{colim} \mathfrak{F}_2 \\
\|i_0\| & \Rightarrow & 1_{h_0} F^* \mathfrak{F}_2 \\
\mathfrak{B}^{H_0}([i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}], [i^*\mathfrak{B}^{H_0}]) & \simeq & h\text{colim} \mathfrak{F}_2 \\
\end{array}$$

where the vertical maps are induced by the functor $\Delta_{H} \to C_2$ which sends $n$ to $(-1, n, -1)$. The map $g_0$ is constructed in a completely analogous way.

We now define the functors $\mathfrak{F}_3, \mathfrak{H}_3$ and $\mathfrak{I}_3$. Let $\mathcal{C}_3$ be the category topologically equivalent to $\Delta \times \Delta \times \Delta$ (see Definition [33]) with objects the triples $(m_1, m_2, m_3)$ of integers with $m_1, m_2, m_3 \geq -1$. The category $\mathcal{C}_3$ coincides with $C_2$ when $m \neq -1$. 


Given morphisms $\mu_l \in \tilde{\Delta}(m_l, n_l)$, $\mu \in \tilde{\Delta}(-1, n)$ and $\mu_J \in \tilde{\Delta}(m_J, n_J)$, we let
\[
\tilde{C}_3(\mu_l, \mu_J) = \left( \prod_{\alpha=1}^{m_l+1} \mathcal{L}(\mu_{l,\alpha} - \mu_{l,\alpha-1}) \right) \times \mathcal{L}(2+n+\mu_{I,0}+\mu_{J,0}) \times \left( \prod_{\alpha=1}^{m_J+1} \mathcal{L}(\mu_{J,\alpha} - \mu_{J,\alpha-1}) \right)
\]
and define $\tilde{C}_3((m_l, -1, m_J), (n_l, n, n_J)) = \prod \tilde{C}_3(\mu_l, \mu_J)$. We define the categories $\tilde{C}_1$ and $C_3$ by the pullback diagrams

\[
\[
\begin{array}{c}
\tilde{C}_1 \\ \downarrow \\
\tilde{\Delta}_{IJ} \\
\end{array} \quad \begin{array}{c}
\Delta \times \Delta \times \Delta \\
\downarrow \\
\tilde{\Delta} \times \tilde{\Delta} \times \tilde{\Delta} \\
\end{array} \quad \begin{array}{c}
C_3 \\ \downarrow \\
\tilde{\Delta}_{IJ} \\
\end{array}
\]
\]

and we let $C_1 = \tilde{C}_1 \cap C_3$. We will now construct functors $\tilde{\mathfrak{f}}_1 : \tilde{C}_1^{\text{op}} \to \text{Top}$ and $\mathfrak{f}_3 : C_3^{\text{op}} \to \text{Top}$. On objects:

\[
\mathfrak{f}_3(n_l, -1, n_J) = \begin{cases}
\{ \mathfrak{B}_{l,I}(n_l) \times (\prod_{l \in J} \mathfrak{m}_{l,J}^{\mathfrak{B}})^{n_J} \times (\prod_{l \in J} \mathfrak{B}_{l,J}^{\mathfrak{g}})^{n_J} \}, & \text{if } n_J \neq -1; \\
(\prod_{l \in I} \mathfrak{B}_{l,I}^{\mathfrak{g}}) \times (\prod_{l \in J} \mathfrak{B}_{l,J}^{\mathfrak{g}}) \times \mathfrak{B}_{I,J}^{\mathfrak{g}}(n_J), & \text{if } n_J \neq -1;
\end{cases}
\]

and $\mathfrak{f}_3(-1, -1, -1) = \mathfrak{B}_{I,J}^{\mathfrak{g}}$; the functor $\tilde{\mathfrak{f}}_1$ coincides with $\mathfrak{f}_1$ (on objects) whenever it is defined. The functors are defined on morphisms on the usual way. We also define $\tilde{\mathfrak{f}}_1$ as the restriction of $\mathfrak{f}_3$ to $C_1^{\text{op}}$ (which coincides with the restriction of $\tilde{\mathfrak{f}}_1$ to $C_1^{\text{op}}$). Since, by Lemma 4.1, the inclusions $\Delta \to \tilde{\Delta}$ and $\tilde{\Delta} \to \Delta$ are cofinal, it follows that the inclusions $C_0 \to \tilde{C}_1$ and $C_1 \to \tilde{C}_1$ are also cofinal and hence the maps $\|\tilde{\mathfrak{f}}_0\| \to \|\tilde{\mathfrak{f}}_1\|$ and $\|\mathfrak{f}_3\| \to \|\tilde{\mathfrak{f}}_1\|$ are homotopy equivalences.

We now define a diagonal functor $d : \Delta_{IJ} \to C_3$, given on objects by $n \mapsto (n, -1, n)$; to define $d$ on morphisms just observe that, for any $\mu \in \tilde{\Delta}(m, n)$, the spaces of morphisms $\Delta_{IJ}(\mu)$ and $C_3(\mu, 1, \mu)$ are canonicly homeomorphic. Now, direct inspection shows that $\mathfrak{B}_{I,J} = d^* \mathfrak{f}_3$. Restricting $d$ we get a functor $d : \Delta_{IJ} \to \tilde{\mathfrak{f}}_1$ and also have $\mathfrak{B}_{I,J} = d^* \tilde{\mathfrak{f}}_1$. We claim that the map $\|d^* \tilde{\mathfrak{f}}_3\| \xrightarrow{\text{diag}} \|\tilde{\mathfrak{f}}_1\|$ induced by $d$ and the inclusion $\tilde{\mathfrak{f}}_1 \to \tilde{\mathfrak{f}}_1$ is a homotopy equivalence. This will follow from the commutative diagram:

\[
\[
\begin{array}{c}
\Delta_{IJ} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Delta \quad \Delta \quad \Delta \quad \Delta \\
\end{array} \quad \begin{array}{c}
\Delta \times \Delta \\
\downarrow \quad \downarrow \quad \downarrow \\
\Delta \times \Delta \\
\end{array} \quad \begin{array}{c}
\tilde{\mathfrak{f}}_1 \\
\downarrow \\
\text{Top} \\
\end{array}
\]
\]

Since the bottom arrows are cofinal and the vertical arrows are equivalences, it follows that $d$ is cofinal and hence $\|d^* \tilde{\mathfrak{f}}_3\| \xrightarrow{\text{diag}} \|\tilde{\mathfrak{f}}_1\|$ is a homotopy equivalence. □

5. Proof of Theorem 1.4

For each $n \in \tilde{\Delta}_{J}$ and $J \in \mathfrak{C}$, the topological space $\mathfrak{B}_J(n)$ is naturally graded as a product of graded spaces, and given a morphism $f \in \tilde{\Delta}(m, n)$, the induced map $\mathfrak{B}_J(n) \to \mathfrak{B}_J(m)$ preserves the grading. Denote by $\mathfrak{B}_{J,k} : \tilde{\Delta}_J \to \text{Top}$ the functor
obtained by taking the degree $k$ component of $\mathcal{B}_J$. The objective of this section is to prove Theorem 1.4.

**Theorem 1.4.** If, for every $J \subset I$ with $\# J \leq k$, the map $h_{I,k}$ is a homotopy equivalence, then $h_{I,k}$ is a homotopy equivalence.

We first need the following result, which was proven in [13]:

**Lemma 5.1.** Let $\mathcal{C}_{I,k}$ be the full subcategory of $\mathcal{C}$ whose objects are the subsets $J \subset I$ with $\# J \leq k$ and let $\mathcal{M}_k: \mathcal{C} \to \text{Top}$ denote the degree $k$ component of the functor $\mathcal{M}$. Then the restriction of $\mathcal{M}_k$ to $\mathcal{C}_{I,k}$ is homeomorphic to the nerve of an open cover of $\mathcal{M}_{I,k}$.

We now turn to the proof of Theorem 1.4.

**Proof.** It is enough to show that the map $\| \mathcal{B}_{I,k} \| \to \| \mathcal{B}_{I,k} \|$ is a homotopy equivalence. For each morphism $j: J \to I$, it will be convenient to replace the functor $\mathcal{B}_J$ with the functor $\Delta_J^*: \mathcal{B}_J: \Delta_I \to \text{Top}$. Let $\Delta^*_J \mathcal{B}_k: \Delta_I^* \mathcal{C}_{I,k} \to \text{Top}$ be the functor defined on objects by $\Delta^*_J \mathcal{B}_k(n,J) = \Delta^*_J \mathcal{B}_{I,k}(n)$; given morphisms $i: (J_1,j_1) \to (J_2,j_2)$ and $f \in \Delta_I(m,n)$, we define $\Delta^*_J \mathcal{B}(f,i)$ by the commutative diagram:

\[
\begin{array}{ccc}
\Delta^*_J \mathcal{B}_{I,k}(n) & \xrightarrow{f} & \Delta^*_J \mathcal{B}_{I,k}(m) \\
i & \downarrow & \downarrow i \\
\Delta^*_J \mathcal{B}_{J,k}(n) & \xrightarrow{f} & \Delta^*_J \mathcal{B}_{J,k}(m)
\end{array}
\]

We define the functor $\Delta^*_J \mathcal{B}: \Delta_I^* \mathcal{C}_{I,k} \to \text{Top}$ by restricting $\Delta^*_J \mathcal{B}_k$. We claim that the maps

(2) \[ \mathcal{hcolim} \left( \frac{\mathcal{B}_k}{\Delta_I^* \mathcal{C}_{I,k}} \right) \cong \mathcal{hcolim} \frac{\mathcal{B}_k}{\Delta^*_J \mathcal{B}_k}, \]

(2) \[ \mathcal{hcolim} \left( \frac{\mathcal{B}_k}{\Delta_I^* \mathcal{C}_{I,k}} \right) \cong \mathcal{hcolim} \frac{\mathcal{B}_k}{\Delta^*_J \mathcal{B}_k} \]

induced by the maps $\Delta^*_J \mathcal{B}_{I,k} \to \mathcal{B}_{I,k}$ are homotopical equivalences; the theorem will follow since we then have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{hcolim} \frac{\mathcal{B}_k}{\Delta_I^* \mathcal{C}_{I,k}} & \rightarrow & \mathcal{hcolim} \frac{\mathcal{B}_k}{\Delta^*_J \mathcal{B}_k} \\
\downarrow \cong & \downarrow \cong \\
\| \mathcal{B}_{I,k} \| & \rightarrow & \| \mathcal{B}_{I,k} \|
\end{array}
\]

and by hypothesis the top horizontal map is a homotopy equivalence. We first prove that the map in equation (2) is a homotopy equivalence. It is enough to show that $\mathcal{hcolim}_{\Delta_I^* \mathcal{C}_{I,k}} \Delta^*_J \mathcal{B}_k \cong \mathcal{B}_{I,k}$ which we now prove. Let $\mathbb{Z}_k \subset \mathbb{Z} \times \mathbb{Z}^{(n+1)}$ be the subset of tuples of non-negative integers whose sum is $k$. We write an element $k \in \mathbb{Z}_k$ as $k = (k_0, (k_{\alpha x})_{\alpha=0,\ldots,n})$. Then, for $n \geq 0$ we have $\mathcal{B}_{I,k}(n) = \prod_{k} \mathcal{B}_{I,k}(n)$ where

\[
\mathcal{B}_{I,k}(n) = \mathcal{M}_{k_0,k_0} \times \left( \prod_{x \in I} \mathcal{M}_{0,k_{\alpha x}} \right) \times \left( \prod_{x \in I} \mathcal{M}_{x,k_{\alpha x}} \right).
\]
Let \( \text{supp} \ k \subset I \) be the set of points such that there is an \( \alpha \) for which \( k_{\alpha x} > 0 \). If we let
\[
\mathcal{B}_{j,k}(n) = \begin{cases} 
\mathcal{B}_k(n), & \text{if } \text{supp} \ k \subset J; \\
\emptyset, & \text{if } \text{supp} \ k \not\subset J,
\end{cases}
\]
then \( \mathcal{B}_{J,k}(n) \cong \coprod_k \mathcal{B}_{j,k}(n) \) and under this isomorphism the map \( \mathcal{B}_{J,k}(n) \to \mathcal{B}_{I,k}(n) \) corresponds to inclusion. If \( \mathcal{E}_{I,k} \subset \mathcal{E}_{J,k} \) denotes the full subcategory whose objects \( J \) satisfy \( \text{supp} \ k \subset J \), then
\[
\left( \text{hcolim} \Delta^* \mathcal{B}_k \right)(n) \cong \prod_{k \in \mathbb{Z}_+} \text{hcolim} \mathcal{B}_{j,k}(n) \cong \coprod_k B \mathcal{E}_{I,k} \times \mathcal{B}_{J,k}(n) \cong \mathcal{B}_{I,k}(n)
\]
because \( \mathcal{E}_{I,k} \) has an initial element, namely: \( \text{supp} \ k \). To complete the proof we need to show that the map in equation (21) is a homotopy equivalence. We just need to show that \( \left( \text{hcolim} \mathcal{E}_{I,k} \Delta^* \mathcal{B}_k \right)(-1) \cong \mathcal{B}_{I,k}(-1) \). This immediately follows from Lemma 5.1.

6. The case \( k = 2 \)

6.1. Proof of Theorem 1.1 Let \( V \) be a finite dimensional complex hermitian vector space. Let \( M^V_I = \coprod_k M^V_{I,k} \) where \( M^V_{I,k} = M^V_{I \cup J} \) for \( k \leq 2 \) and \( M^V_{I,k} = * \) for \( k > 2 \).

Proposition 6.1. Let \( I_1, \ldots, I_n \subset \mathbb{C}^2 \) be finite disjoint sets and write \( I = (I_1, \ldots, I_k) \) and \( I = \bigcup_i I_i \). Then there are maps \( \boxtimes_I : M^V_{I_1} \times \cdots \times M^V_{I_n} \to M^V_I \) such that:

1. Given any \( J = (J_1, \ldots, J_k) \) with \( J_i \subset I_i \), for any \( i \), we have \( \pi_{I,J} \circ \boxtimes_I = \boxtimes_J \circ (\pi_{J_1,I_1} \times \cdots \times \pi_{J_k,I_k}) \) (where \( J = \bigcup_i J_i \)).
2. For any \( f \in \mathcal{L}^I(m) \) we have a commutative diagram:
\[
\begin{array}{ccc}
\prod M^V_{I_i} & \boxtimes_I & M^V_I \\
	imes f & \downarrow & \downarrow f \\
\prod M^V_{I_i} & \boxtimes_I & M^V_I
\end{array}
\]
3. Let \( i : V \to V^m \) be inclusion onto the \( i \)-th component, let \( \omega \) be the map induced by Whitney sum. Then we have a commutative diagram:
\[
\begin{array}{ccc}
\prod M^V_{I_i} & \xrightarrow{\omega} & M^V_{I_i} \\
\downarrow \Pi i & & \downarrow \Pi i \\
\prod M^V_{I_i} & \xrightarrow{\omega} & M^V_{I_i}
\end{array}
\]

Proof. To define \( \boxtimes_I \) it is enough to construct maps \( \mathcal{M}^V_{I_1,1} \times \mathcal{M}^V_{I_2,1} \to \mathcal{M}^V_{I_1 \cup I_2,2} \) and using Lemma 5.1 we can reduce to the case where \( I_1 \) and \( I_2 \) are either empty or have only one element. The proposition now follows from Proposition 5.1 in the appendix.

For each finite set \( I \subset \mathbb{C}^2 \) we have a simplicial space \( B^V_I : \Delta^\text{op} \to \text{Top} \) with \( B^V_I(n) = M^V_{I_0} \times (\prod_{x \in I} M^V_x)^n \times (\prod_{x \in I} M^V_x) \) and a map from its geometric realization: \( h : |B^V_I| \to M^V_I \) induced by the maps of Proposition 6.1. We can now prove Theorem 1.1.
Theorem 1.1. Let $I = \{x_1, \ldots, x_q\} \subset \mathbb{C}^2$. Then:

1. The map $\mathbb{P}_{\emptyset, x_1, \ldots, x_q} : M^V_\emptyset \times (\prod_i M^V_{x_i}) \to M^V_I$ induces a map $h_M : |B^V_I| \to M^V_I$ which is a homotopy equivalence in the $k = 1, 2$ components.

2. If $I = J \cup K$, with $J \cap K = \emptyset$, then the map $\mathbb{P}_{J,K} : M^V_J \times M^V_K \to M^V_I$ induces

Proof. Assume $I = J \cup K$ with $J \cap K = \emptyset$. Then, by the same arguments as in the proof of Theorem 1.3 we have a commutative diagram:

$$
\begin{array}{ccc}
\text{Bar}(|B^V_J|, M^V_\emptyset, |B^V_K|) & \xrightarrow{\simeq} & \text{Bar}(M^V_J, M^V_\emptyset, M^V_K) \\
|B^V_I| & \xrightarrow{\simeq} & M^V_I
\end{array}
$$

where the left vertical map is a homotopy equivalence. It follows that part (2) of the theorem is a consequence of part (1), which we now prove. Let $M^V_k : \mathcal{C} \to \text{Top}$ be the functor defined on objects by $M^V_k(I) = M^V_{I,k}$ and defined on morphisms by pullback. Also, let $B^V_k : \mathcal{C} \times \Delta^{op} \to \text{Top}$ be the functor defined on objects by $B^V_k(n, J) = B^V_I(n)$. The same arguments as in the proof of Theorem 1.4 show that we have a commutative diagram:

$$
\begin{array}{ccc}
hcolim_{I,k} B^V_{I,k} & \xrightarrow{\simeq} & \colim_{I,k} M^V_{I,k} \\
|B^V_{I,k}| & \xrightarrow{\simeq} & M^V_{I,k}
\end{array}
$$

Thus, for $k = 1$ we only need to consider the trivial case when $I = \{x\}$, and for $k = 2$, we only need to consider the case when $I = \{x, y\} \subset \mathbb{C}^2$ which we now analyze. Now consider diagram (3) in the case where $J = \{x\}$ and $K = \{y\}$ and hence $I = \{x, y\}$. The maps $|B^V_{I,k}| \to M^V_I$ are trivially homotopy equivalences hence the top horizontal map is a homotopy equivalence. Thus we only have to show that the right vertical map in diagram (3) is a homotopy equivalence. The maps $\pi^*_{x,y}$, $\pi^*_{y,x}$ and the maps of Proposition 6.1.1 are open embeddings (see [13]) and their images form an open cover of $\mathcal{O}_{I,2}$ (see Proposition A.3 and [13], section 4). It is then a direct verification that the simplicial space $B^V_{I,2}$ is homeomorphic to the nerve of this open cover. Is follows that we have a homotopy equivalence $|B^V_{I,2}| \simeq M^V_{I,2}$, which concludes the proof of the theorem.

Taking the limit when $\dim V \to \infty$ we can now prove:

Theorem 6.2. Let $\mathbb{H}$ be a countably infinite complex hermitian vector space. Then the map $h_{I,2} : \|\mathbb{H}_{I,2}\| \to \mathcal{O}_{I,2}$ is a homotopy equivalence.

Proof. The maps in Proposition 6.1.1 pass to the colimit to define maps $\mathbb{H}_I : M^H_I \times \cdots \times M^H_{I,n} \to M^H_{I,2}$. We need to see how these maps are related with the action of the linear isometries operad $\mathcal{L}^H$. Let $i_{\alpha} : \mathbb{H} \to \mathbb{H}^n$ be inclusion onto the $\alpha$-th component. Let $\mathcal{P}$ be the operad where $\mathcal{P}(n)$ is the space of complex linear maps $f : \mathbb{H}^n \to \mathbb{H}$ such that $f \circ i_{\alpha}$ is an isometry for all $\alpha = 1, \ldots, n$, with operad data
given by composition. Clearly $\mathcal{P}$ contains the linear isometries operad $\mathcal{L}$. We fix basepoints $* \in \mathcal{P}(n)$ given in matrix notation by

$$* = [\begin{array}{ccc} 1 & \cdots & 1 \end{array}] : \mathbb{H}^n \to \mathbb{H}.$$ 

For any $f \in \mathcal{P}(n)$ and each $\alpha = 1, \ldots, n$, let $f_\alpha = f \circ \iota_\alpha$. Given finite disjoint sets $J_1, \ldots, J_n$ we define a map $\boxplus_{J,f}$ as the composition:

$$(4) \quad \boxplus_{J,f} : \prod_{\alpha=1}^n M_{J_\alpha}^\mathbb{H} \prod_{\alpha=1}^n M_{J_\alpha}^\mathbb{H} \to \prod_{\alpha=1}^n M_{J_\alpha}^\mathbb{H}.$$ 

Note that, for $f = *$, we have $\boxplus_{J,f} = \boxplus_J$. The maps $\boxplus_{J,f}$ are compatible with the operad data in the following sense: Let $j_1, \ldots, j_n$ be non-negative integers and let $j = \sum j_i$; let $f \in \mathcal{P}(n)$ and $g_i \in \mathcal{P}(j_i)$; consider finite disjoint sets $J_\alpha$, with $\alpha = 1, \ldots, j$ and for each $i = 1, \ldots, n$, let $s_i = j_1 + \cdots + j_i$ and let:

$$J_i = (J_{s_i-1+1}, \ldots, J_{s_i}), \quad K_i = \bigcup_{\alpha=s_{i-1}+1}^{s_i} J_\alpha \quad (i = 1, \ldots, n);$$

also, let:

$$J = (J_1, \ldots, J_j), \quad K = (K_1, \ldots, K_n), \quad \sum_{i=1}^n K_i = \bigcup_{\alpha=1}^n J_\alpha.$$

Then we have:

$$(5) \quad \boxplus_{J,f \circ \prod_{i} g_i} = \boxplus_{K,f} \circ \prod_{i=1}^n \boxplus_{J_i,g_i}.$$ 

To prove this last statement, let $\iota_\alpha : \mathbb{H} \to \mathbb{H}^{j_i}$ (with $\alpha = 1, \ldots, j$) be inclusion into the $\alpha - s_{\alpha - 1}$ component and let $f_i = f \circ \iota_i$ and $g_i = g_i \circ \iota_\alpha$. Equation (5) then follows from the commutativity of the following diagram (the lower triangle is commutative by Proposition 6.1):

Let $q = \# I$, let $\mathcal{P}_I(n) = \mathcal{P}(1+q n)$, $\mathcal{P}_I^I(n) = \prod I \mathcal{P}(n)$ and $\mathcal{P}_I^+ = \mathcal{P}_I^I(n+1)$ (compare with Definition 4.11). Then we can define a functor $\mathcal{F}_I : \Delta(\mathcal{P}_I, \mathcal{P}_I^+, \mathcal{P}_I^I) \to \text{Top}$ by letting $\mathcal{F}_I(n) = M_{\mathbb{H}}^{\mathcal{P}_I(n)} \times (\prod M_{\mathbb{H}}^{\mathcal{P}_I} )^{n} \times (\prod M_{\mathbb{H}}^{\mathcal{P}_I^I})$ and defining the morphisms using the maps $\boxplus_{J,f}$ in equation (4). The inclusion of operads $\mathcal{L}_{\mathbb{H}} \subset \mathcal{P}$ induces an equivalence of categories $\Delta_I \to \Delta(\mathcal{P}_I, \mathcal{P}_I^+, \mathcal{P}_I^I)$. For $f \in \mathcal{L}_{\mathbb{H}}(n) \subset \mathcal{P}(n)$ we have
the commutative diagram (see Proposition [6.1]):

\[
\begin{array}{c}
\prod_{\alpha=1}^{n} M_{f_\alpha}^{\#} \xrightarrow{\prod f} \prod_{\alpha=1}^{n} M_{f_\alpha}^{\#} \\
\oplus \circ \circ \overset{*}{\xrightarrow{\prod}} \\
M_{f}^{\#} \xrightarrow{f} M_{f}^{\#}
\end{array}
\]

so the restriction of \( F_I \) to \( \Delta_I \) is precisely \( \mathfrak{B}_I \). To finish the proof we observe that the inclusion of the base point in \( \mathcal{P}(n) \) induces an equivalence of categories \( \Delta \rightarrow \Delta(\mathcal{P}_I, \mathcal{P}_I^n, \mathcal{P}_I^+) \) and the restriction of \( F_I \) to \( \Delta \) equals the simplicial space \( B_I \). The functor \( F_I \) extends to a functor \( \tilde{F}_I : \Delta(\mathcal{P}_I, \mathcal{P}_I^n, \mathcal{P}_I^+) \rightarrow \mathcal{P}(n) \) and \( B_I \) extends to a functor \( B_I : \Delta \rightarrow \mathcal{P}(n) \) by letting \( B_I(-1) = B_I(-1) = \mathfrak{M}_I \), and we have a commutative diagram

\[
\begin{array}{c}
\mathfrak{B}_I \xrightarrow{\sim} \tilde{F}_I \xrightarrow{\sim} B_I \\
\mathfrak{B}_I \xrightarrow{\sim} \tilde{F}_I \xrightarrow{\sim} B_I
\end{array}
\]

which completes the proof.

7. The Limit When \( k \rightarrow \infty \)

In this section we will prove Theorem [1.2]. The moduli space \( \mathfrak{M}_k \) over the blowup of \( \mathbb{P}^2 \) at \( n \) points is isomorphic to the moduli space \( \mathfrak{M}_{\text{qst}} \) of based instantons over a connected sum \( \#_n \mathbb{P}^2 \) of \( n \) copies of \( \mathbb{P}^2 \) (see [3], [9]). In [15], [16] Taubes introduced, for \( k' > k \), a map \( \mathfrak{M}_k \rightarrow \mathfrak{M}_{k'} \) and showed that, by taking the colimit when \( k \rightarrow \infty \), we get a homotopy equivalence \( \mathfrak{M}_{\text{qst}} \simeq \text{Map}_* \left( \#_n \mathbb{P}^2, BSU(V) \right) \).

In this section \( I \) denotes the unit interval: \( I = [0, 1] \subset \mathbb{R} \). Given a based topological space \( (X, *) \) let \( \mathcal{M}_* X \) represent the space of compactly supported maps \( f : [0, +\infty) \times I^3 \rightarrow X \) such that \( f(t, x) = * \) whenever \( x \in \partial I^3 \).

1. Using Moore loops, we identify \( \Omega^4 X \) with the subspace of maps \( f \in \mathcal{M}_* X \) such that \( f(0, x) = * \) for any \( x \in I^3 \).

2. Let \( H : I^3 \rightarrow S^2 \) be the composition of the projection \( I^3 \rightarrow I^3 / \partial I^3 \) with the Hopf map. We identify \( \text{Map}_* (\mathbb{P}^2, X) \) with the subspace of maps \( f \in \mathcal{M}_* X \) whose restriction to \( 0 \times I^3 \) factors through \( H \). Restriction to \( 0 \times I^3 \) induces a map \( \rho : \text{Map}_* (\mathbb{P}^2, X) \rightarrow \Omega^2 X \).

3. Let \( x = (x_1, x_2, x_3) \in I^3 \). We identify \( \text{Map}_* (\#_n \mathbb{P}^2, X) \) with the subspace of maps \( f \in \mathcal{M}_* X \) such that:

   (a) \( f(0, i/n, x_2, x_3) = * \) for \( i = 0, \ldots, n \) and any \( x_2, x_3 \in I \);

   (b) for each \( i = 1, \ldots, n \), the restriction of \( f \) to \( 0 \times [(i-1)/n, i/n] \times I^2 \) factors through \( H : I^3 \rightarrow S^2 \).

We have a map \( \rho = (\rho_1, \ldots, \rho_n) : \text{Map}_* (\#_n \mathbb{P}^2, X) \rightarrow (\Omega^2 X)^n \) whose components \( \rho_i \) are induced by restriction to \( 0 \times [(i-1)/n, i/n] \times I^2 \), for \( i = 1, \ldots, n \).

We will now give \( (\Omega^4 X)^n \) the structure of an associative monoid and define actions of \( (\Omega^4 X)^n \) on \( (\text{Map}_* (\mathbb{P}^2, X))^n \) and \( \Omega^4 X \).
(4) First we define a right action of $\Omega^4 X$ on $\mathcal{M}_s X$. Given a function $f \in \mathcal{M}_s X$ we let $s_f$ be the infimum of the set of $t \in [0, +\infty)$ such that $f(s, x) = *$ for any $s \geq t$ and any $x \in F^3$. Then, given $g \in \Omega^4 X$ we define $f \cdot g \in \mathcal{M}_s X$ by:

$$f \cdot g(t, x) = \begin{cases} f(t, x), & \text{if } t \leq s_f; \\ g(t - s_f, x), & \text{if } t > s_f. \end{cases}$$

Clearly, $(f \cdot g_1) \cdot g_2 = f \cdot (g_1 \cdot g_2)$. This action preserves the subspaces $\Omega^4 X$, $\text{Map}_s(\mathbb{P}^2, X)$ and $\text{Map}_s(\#_n \mathbb{P}^2, X)$. Since $(\mathcal{M}_s X)^n \cong \mathcal{M}_s (X^n)$, we get an associative product on $(\Omega^4 X)^n$ and a right action of $(\Omega^4 X)^n$ on $(\mathcal{M}_s X)^n$.

(5) We now define a map $\omega$: $(\mathcal{M}_s X)^n \rightarrow \mathcal{M}_s X$ by concatenation in the second variable. Given $f = (f_1, \ldots, f_n) \in (\mathcal{M}_s X)^n$, for each $i = 1, \ldots, n$ and $x_1 \in [(i-1)/n, i/n]$, we let $\omega(f)(t, x_1, x_2, x_3) = f_i(t, nx_1 - i + 1, x_2, x_3)$. Then, given $f \in (\mathcal{M}_s X)^n$ and $g \in (\Omega^4 X)^n$, we have $\omega(f \cdot g) = \omega(f) \cdot \omega(g)$. The map $\omega$ restricts to define maps $(\Omega^4 X)^n \rightarrow \Omega^4 X$ and $\text{Map}_s(\mathbb{P}^2, X)^n \rightarrow \text{Map}_s(\#_n \mathbb{P}^2, X)$. We define a left action of $(\Omega^4 X)^n$ on $\Omega^4 X$ by $g \cdot f = \omega(g) \cdot f$.

Consider the bar construction $\text{Bar}((\text{Map}_s(\mathbb{P}^2, X))^n, (\Omega^4 X)^n, \Omega^4 X)$ induced by the actions in (4) and (5).

**Theorem 1.2** The maps $\text{Bar}((\text{Map}_s(\mathbb{P}^2, X))^n, (\Omega^4 X)^n, \Omega^4 X)$ sending $(f, f_1, \ldots, f_k, h)$ to $\omega(f \cdot f_1 \cdots f_k) \cdot h$ induce a map

$h: \text{Bar}((\text{Map}_s(\mathbb{P}^2, X))^n, (\Omega^4 X)^n, \Omega^4 X) \rightarrow \text{Map}_s(\#_n \mathbb{P}^2, X)$

which is a homotopy equivalence.

**Proof.** We have a commutative diagram:

$$
\begin{array}{ccc}
\text{Bar}((\Omega^4 X)^n, (\Omega^4 X)^n, \Omega^4 X) & \xrightarrow{\rho} & (\Omega^4 X)^n \\
\downarrow{\mu} & & \downarrow{\mu} \\
\text{Bar}((\text{Map}_s(\mathbb{P}^2, X))^n, (\Omega^4 X)^n, \Omega^4 X) & \xrightarrow{h} & \text{Map}_s(\#_n \mathbb{P}^2, X)
\end{array}
$$

The maps $\iota$ are inclusions and the maps $\rho, \rho$ are the ones defined above in [2] and [3]. The right vertical maps are induced by the cofibration $S^2 \rightarrow \mathbb{P}^2 \rightarrow S^1$, and hence they form a fibration. Since the top row is a homotopy equivalence, to finish the proof we only have to show that the left vertical maps form a fibration sequence. The map $\rho = \rho \circ h$ is induced by the restriction map $\rho: \text{Map}_s(\mathbb{P}^2, X) \rightarrow \Omega^2 X$ and hence, from Corollary 11.6 in [10] we see that the homotopy fiber of $\rho \circ h$ is $\text{Bar}(F^n, (\Omega^4 X)^n, \Omega^4 X)$, where $F$ is the homotopy fiber of the map $\text{Map}_s(\mathbb{P}^2, X) \rightarrow \Omega^2 X$. Since the inclusion $\Omega^4 X \rightarrow F$ is a homotopic equivalence, it follows from Proposition A.1 in [14] that the map

$\text{Bar}((\Omega^4 X)^n, (\Omega^4 X)^n, \Omega^4 X) \rightarrow \text{Bar}(F^n, (\Omega^4 X)^n, \Omega^4 X)$

is a homotopic equivalence. This concludes the proof. \qed
Appendix A. Monads

In this appendix we describe the maps introduced in Proposition 6.1 and prove their properties. Fix \( I = \{x, y\} \subset \mathbb{C}^2 \). Let \( V \) be a finite dimensional complex hermitian vector space. We will need the monad description of the moduli spaces \( M_{\text{reg}}^{V,k} \) and \( M_{\text{reg}}^{V,k} \equiv M_{\text{reg}}^{V,k} \), introduced in [5], [6], which we briefly review here.

Let \( W_0, W_1 \) be complex vector spaces of dimension \( k \). Let \( R \) be the space of 4-tuples \( (a_1, a_2, b, c) \) where \( a_i \in \text{End}(W_i), b \in \text{Hom}(V, W_1) \) and \( c \in \text{Hom}(W_1, V) \), obeying the integrability condition \( |a_1| + |a_2| + b + c = 0 \). Let \( R' \) be the space of 5-tuples \( (a_1', a_2', b', c', d') \) where \( a_1' \in \text{Hom}(W_1, W_0), b' \in \text{Hom}(V, W_0) \) and \( c' \in \text{Hom}(W_1, V) \), such that \( a_1'(W_1) + a_2'(W_1) + b'(V) = 0 \), obeying the integrability condition \( a_1'a_2 - a_2a_1 + b'c' = 0 \). The groups \( GL(W_1) \) and \( GL(W_0) \times GL(W_1) \) act by composition on \( R \) and \( R' \), respectively. A 4-tuple \( (a_1, a_2, b, c) \in R \) is called nondegenerate if, for any subspace \( U_i \subset W_i \), we have:

\[
\text{Im } b \subset U_1 \text{ and } a_i(U_i) \subset U_1 \text{ for } (i = 1, 2) \Rightarrow U_1 = W_1, \\
U_1 \subset \ker c \text{ and } a_i(U_i) \subset U_1 \text{ for } (i = 1, 2) \Rightarrow U_1 = \emptyset.
\]

A 5-tuple \( (a_1', a_2', b', c', d') \in R' \) is called nondegenerate if, for any subspaces \( U_0 \subset W_0 \) and \( U_1 \subset W_1 \) such that \( \dim U_0 = \dim U_1 \), we have:

\[
\text{Im } b' \subset U_0 \text{ and } d'(U_0) \subset U_1 \text{ and } a_i'(U_i) \subset U_0 \text{ for } (i = 1, 2) \Rightarrow U_1 = W_1, \text{ for } (i = 1, 2), \\
U_1 \subset \ker d' \text{ and } d'(U_0) \subset U_1 \text{ and } a_i'(U_i) \subset U_0 \text{ for } (i = 1, 2) \Rightarrow U_1 = \emptyset.
\]

Let \( R_{\text{reg}}, R'_{\text{reg}} \) denote the subspaces of nondegenerate configurations.

Theorem (Donaldson [5], King [6]). The actions of \( GL(W_1) \) and \( GL(W_0) \times GL(W_1) \) on \( R_{\text{reg}} \) and \( R'_{\text{reg}} \) respectively are free and we have isomorphisms

\[
R_{\text{reg}}/GL(W_1) \cong M_{\text{reg}}^{V,k}, \quad R'_{\text{reg}}/(GL(W_0) \times GL(W_1)) \cong M_{\text{reg}}^{V,k} \cong M_{\text{reg}}^{V,k}.
\]

Furthermore, the algebraic quotients \( R_{\text{reg}}/GL(W_1), R'_{\text{reg}}/(GL(W_0) \times GL(W_1)) \) are isomorphic to the Donaldson-Uhlenbeck completions \( \overline{M}_{\text{reg}}^{V,k}, \overline{M}_{\text{reg}}^{V,k} \) of \( M_{\text{reg}}^{V,k} \) respectively.

To define the maps of Proposition 6.1 it will be convenient to replace the spaces \( M_{\text{reg}}^{V,k}, M_{\text{reg}}^{V,k} \) by the following homeomorphic subspaces: fix a positive real number \( \delta < \|x - y\| \) and let

\[
M_{x,1}^{V,\delta} = M_{y,1}^{V,\delta} = \left\{ [a_1', a_2', d', b', c'] \in M_{x,1}^{V} : |d'a'_1| < \delta \right\}, \\
M_{0,1}^{V,\delta} = \left\{ [a_1, a_2, b, c] \in M_{0,1}^{V} : |a_1| < \delta \right\}.
\]

Proposition A.1. There are homeomorphisms \( M_{0,1}^{V,\delta} \cong M_{0,1}^{V} \) and \( M_{x,1}^{V,\delta} \cong M_{x,1}^{V} \).

Proof. It is clear that \( M_{0,1}^{V,\delta} \cong M_{0,1}^{V} \). Let \( \phi: [0, +\infty) \to [0, \delta] \) be given by \( \phi(r) = \delta/(1 + r) \) and let \( \psi: [0, \delta] \to [0, +\infty) \) be given by \( \psi(r) = 1/(\delta - r) \). Then \( z\phi(|z|) : \mathbb{C} \to B_\delta(0) \) is an homeomorphism with inverse \( z\phi(|z|) \). It follows that the map \( [a_1, a_2, d, b, c] \mapsto [\phi(da_1)]a_1, a_2, d, b, c \) is a homeomorphism \( M_{x,1}^{V} \to M_{x,1}^{V,\delta} \) with inverse \( [a_1, a_2, d, b, c] \mapsto [\psi(da_1)]a_1, a_2, d, b, c \).

Proposition A.2. Let \( V_1, V_2 \) be finite dimensional complex hermitian vector spaces and let \( J \subset \mathbb{C}^2 \) be either \( \emptyset \) or \( \{x\} \) (analogous results are valid for \( J = \{y\} \)).
(1) Given a linear isometry $\alpha : V_1 \rightarrow V_2$, with dual $\alpha^* : V_2 \rightarrow V_1$, the induced map $\alpha : M^V_{\delta} \rightarrow M^V_{\delta}$ is given by

$$\begin{align*}
[a_1, a_2, b, c] &\mapsto [a_1, a_2, b \circ \alpha^*, \alpha \circ c] \quad (J = \emptyset) \\
[a_1, a_2, d, b, c] &\mapsto [a_1, a_2, d \circ \alpha^*, \alpha \circ c] \quad (J = \{x\})
\end{align*}$$

In particular, the map $\alpha$ takes the subspaces $M^V_{\delta} \subset M^V_{\delta}$ to $M^V_{\delta}$.

(2) Whitney sum $\omega : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{\delta}$ is induced by direct sum:

$$\begin{align*}
\omega([a_1, a_2, b, c], [a_1', a_2', b', c']) &\mapsto [a_1 \oplus a_1', a_2 \oplus a_2', b \oplus b', c \oplus c'] \quad J = \emptyset \\
\omega([a_1, a_2, d, b, c], [a_1', a_2', d', b', c']) &\mapsto [a_1 \oplus a_1', a_2 \oplus a_2', d \oplus d', b \oplus b', c \oplus c'] \quad J = \{x\}
\end{align*}$$

(3) The pullback map $\pi^{V}_{\delta} : M^V_{\delta} \rightarrow M^V_{\{x\}}$ is given as follows: fix any isomorphism $d : W_0 \rightarrow W_1$; then

$$\pi^{V}_{\delta}(a_1, a_2, b, c) = [d^{-1}a_1, d^{-1}a_2, d, d^{-1}b, c].$$

In particular, the map $\alpha$ takes the subspace $M^V_{\delta} \subset M^V_{\delta}$ to $M^V_{\delta}$.

Proof. Statements (1) and (2) easily follow from the way a holomorphic bundle is construct from a 5-tuple $(a_1, a_2, d, b, c)$ (see [3] or [2], Theorem 3.2). For statement (3) see [2], Lemma 4.1. \hfill \square

As it would be expected, the pullback map does not depend on the choice of isomorphism $d$. We will usually identify $W_0$ with $W_1$ so that we can take $d = 1$.

**Definition A.1.** Fix points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{C}^2$. Let $\boxtimes_{\delta, \emptyset} : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{\delta}$ be the map given by

$$[a_1', a_2', b', c'] \boxtimes_{\delta, \emptyset} [a_1'', a_2'', b'', c''] = \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \oplus [a_1', a_2', b', c'] \boxplus [a_1'', a_2'', b'', c'']$$

Let $\boxtimes_{x, \emptyset} : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{x, 2}$ be the map given by (where $z = y - x$):

$$[a_1', a_2', b', c'] \boxplus [a_1'', a_2'', b'', c''] = \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \oplus [a_1', a_2', b', c'] \boxplus [a_1'', a_2'', b'', c'']$$

We also define $\boxtimes_{\emptyset, y} : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{y, 2}$ by setting $\boxtimes_{\emptyset, y} = \boxtimes_{-x, \emptyset}$.

A straightforward computation shows that $\pi^{V}_{\emptyset, x} \circ \boxtimes_{\emptyset, \emptyset} = \boxtimes_{x, \emptyset} \circ (\pi^{V}_{\emptyset, x} \times 1)$ and similarly for $\boxtimes_{\emptyset, y}$. We now show that there is a map $\boxtimes_{x, y} : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{x, 2}$ which extends $\boxtimes_{x, \emptyset}, \boxtimes_{\emptyset, y}$ in the following sense:

**Proposition A.3.** There is an open embedding $\boxtimes_{x, y} : M^V_{\delta} \times M^V_{\delta} \rightarrow M^V_{x, 2}$ such that, for any $m_x \in M^V_{\delta}$, $m_y \in M^V_{\delta}$ and $m_y \in M^V_{\delta}$ we have $m_x \boxtimes_{x, y} (\pi^{V}_{\emptyset, y} m_y) = \pi^{V}_{\emptyset, x} (m_x \boxtimes_{x, \emptyset} m_y)$ and $\pi^{V}_{\emptyset, y} m_y = \pi^{V}_{\emptyset, x} (m_x \boxtimes_{x, \emptyset} m_y)$. 
Proof. The proof follows the same lines as the proof of proposition 4.9 in [13]. We
sketch the proof here, referring to [13] for more details. The maps \( \oplus_{x,y}, \oplus_{y,0} \) can be
extended to the Donaldson-Uhlenbeck completion \( \mathcal{M}_V \) of the moduli spaces.
The same argument as in the proof of Proposition 4.5 in [13] shows that these extended
maps are embeddings. Let \( \pi_x : \mathbb{P}^2_x \to \mathbb{P}^2, \pi_{y,y} : \mathbb{P}^2_y \to \mathbb{P}^2 \) be the blowup at \( y \), and
let \( \pi_{y,I} : \mathbb{P}^2_I \to \mathbb{P}^2, \pi_{y,x} : \mathbb{P}^2_x \to \mathbb{P}^2 \) be the blowup at \( x \). Taking the direct image of
the bundles we get maps
\[
(\pi_{x,I})_* : \mathcal{M}^V_{I,2} \to \mathcal{M}^V_{x,2} \quad \text{and} \quad (\pi_{y,I})_* : \mathcal{M}^V_{y,2} \to \mathcal{M}^V_{y,2}.
\]
Given \( m_x \in \mathcal{M}^V_{x,1}, m_y \in \mathcal{M}^V_{y,1} \), we define \( m_x \oplus_{x,y} m_y \) as the unique solution of the
system of equations
\[
(\pi_{x,I})_*(m_x \oplus_{x,y} m_y) = m_x \oplus_{x,0} (\pi_{0,y})_* m_y
\]
\[
(\pi_{y,I})_*(m_x \oplus_{x,y} m_y) = (\pi_{0,x})_* m_x \oplus_{0,y} m_y.
\]
If \( m_y \in \text{Im} \pi_{y,I} \) or \( m_x \in \text{Im} \pi_{x,y}^* \), we can solve the equations and get
\[
m_x \oplus_{x,y} \pi_{0,y}^* m = \pi_{x,I}^*(m_x \oplus_{x,y} m), \quad \pi_{0,x}^* m \oplus_{x,y} m_y = \pi_{y,I}^*(m \oplus_{x,y} m_y).
\]
Otherwise, \( (\pi_{x,I})_*(m_x \oplus_{x,y} m_y) \in \mathcal{M}^V_{x,2} \) is the ideal instanton determined by \( m_x \in \mathcal{M}^V_{x,1} \)
and a delta at \( y \), and \( (\pi_{y,I})_*(m_x \oplus_{x,y} m_y) \in \mathcal{M}^V_{y,2} \) is the ideal instanton determined by \( m_y \in \mathcal{M}^V_{y,1} \)
and a delta at \( x \). By proposition 4.3 in [13], this completely determines \( m_x \oplus_{x,y} m_y \). Proposition 4.8 in [13]
shows that the image of \( \oplus_{x,y} \) is open. Continuity of \( \oplus_{x,y} \) is proven exactly as in [13], Proposition 4.9. Continuity
of the inverse follows easily from the fact that \( \oplus_{x,0} \) and \( \oplus_{0,y} \) are embeddings. \( \square \)

Proposition A.4. Let \( I \subset \{x,y\} \), let \( I_1 = I - \{y\} \) and \( I_2 = I - \{x\} \), and let \( I = (I_1,I_2) \). Let \( V, V' \) be finite dimensional complex hermitian vector spaces.

1. Given finite sets \( J_1 \subset I_1 \) and \( J_2 \subset I_2 \), we have:
\[
\pi^*_{J_1} \circ \oplus_J = \oplus_{I_1} \circ (\pi^*_{J_1,I_1} \times \pi^*_{J_2,I_2})
\]
(\( \oplus_J = (J_1,J_2) \) and \( J = J_1 \cup J_2 \)).

2. Given a linear isometry \( \alpha : V \to V' \), the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{M}^V_{J_1,1} \times \mathcal{M}^V_{J_2,1} & \overset{\oplus_{I_1}}{\longrightarrow} & \mathcal{M}^V_{I_1,2} \\
\downarrow{\alpha \times \alpha} & & \downarrow{\alpha} \\
\mathcal{M}^{V',\delta}_{J_1,1} \times \mathcal{M}^{V',\delta}_{J_2,1} & \overset{\oplus_{I_1}}{\longrightarrow} & \mathcal{M}^{V',\delta}_{I_1,2}
\end{array}
\]

3. Let \( \iota : V \to V \oplus V' \), \( \iota' : V' \to V \oplus V' \) be the canonical inclusions. Then the
Whitney sum map \( \omega \) equals the composition:
\[
\mathcal{M}^V_{I_1,1} \times \mathcal{M}^{V',\delta}_{I_2,1} \overset{\iota \times \iota'}{\longrightarrow} \mathcal{M}^{V \oplus V',\delta}_{I_1,1} \times \mathcal{M}^{V \oplus V',\delta}_{I_2,1} \overset{\oplus_{I_1}}{\longrightarrow} \mathcal{M}^{V \oplus V'}_{I_1,2}
\]

Proof. For \( I = \emptyset, \{x\} \) or \( \{y\} \) the proof is a direct computation. The results extend
to the case where \( I = \{x,y\} \) by continuity, since \( \pi^*_{0,x} \mathcal{M}^{V,\delta}_{0,1} \) and \( \pi^*_{0,y} \mathcal{M}^{V,\delta}_{0,1} \) are dense
in \( \mathcal{M}^{V,\delta}_{x,1} \) and \( \mathcal{M}^{V,\delta}_{y,1} \), respectively. \( \square \)
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