Farrell–Jones spheres and inertia groups of complex projective spaces

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Abstract. We introduce and study a new class of homotopy spheres called Farrell–Jones spheres. Using Farrell–Jones sphere we construct examples of closed negatively curved manifolds \( M^{2n} \), where \( n = 7 \) or \( 8 \), which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds, thereby giving a partial answer to a question raised by C. S. Aravinda and F. T. Farrell. We show that every exotic sphere not bounding a spin manifold (Hitchin sphere) is a Farrell–Jones sphere. We also discuss the relationship between inertia groups of \( \mathbb{C} \mathbb{P}^n \) and Farrell–Jones spheres.

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1 Introduction

Let \( \Theta_m \) be the group of homotopy spheres defined by M. Kervaire and J. Milnor in [15].

Definition 1.1. We call \( \Sigma^{2n} \in \Theta_{2n} \) \((n \geq 4)\) a Farrell–Jones sphere if \( \mathbb{C} \mathbb{P}^n \# \Sigma^{2n} \) is not concordant to \( \mathbb{C} \mathbb{P}^n \).

The following theorem gives an equivalent definition of Farrell–Jones spheres, which we prove in Section 3:

Theorem 1.2. Let \( \Sigma^{2n} \) be an exotic sphere in \( \Theta_{2n} \) \((n \geq 4)\). Then \( \Sigma^{2n} \) is a Farrell–Jones sphere if and only if \( \mathbb{C} \mathbb{P}^n \# \Sigma^{2n} \) is not diffeomorphic to \( \mathbb{C} \mathbb{P}^n \).

By [10, Lemma 3.17], there exists a Farrell–Jones sphere \( \Sigma^m \in \Theta_m \) for all \( m = 8n + 2 \) \((n \geq 1)\) and for \( m = 8 \). Also we prove the following theorem in Section 3:

Theorem 1.3. The non-zero element of \( \Theta_{2n} \cong \mathbb{Z}_2 \) \((n = 7 \) or \( 8)\) is a Farrell–Jones sphere.
The study of Farrell–Jones spheres is motivated by the following result, which is a slight modification of [10, Theorem 3.20]:

**Theorem 1.4.** Let $M^{2n}$ be any closed complex hyperbolic manifold of complex dimension $n$. Let $\Sigma^{2n} \in \Theta_{2n}$ be a Farrell–Jones sphere. Given a positive real number $\epsilon$, there exists a finite sheeted cover $N^{2n}$ of $M^{2n}$ such that the following is true for any finite sheeted cover $N^{2n}$ of $N^{2n}$.

(i) The smooth manifold $N^{2n}$ is not diffeomorphic to $N^{2n} \# \Sigma^{2n}$.

(ii) The connected sum $N^{2n} \# \Sigma^{2n}$ supports a negatively curved Riemannian metric whose sectional curvatures all lie in the closed interval $[-4 - \epsilon, -1 + \epsilon]$.

The proof of the above Theorem 1.4 follows from [10, Corollary 3.14 and Proposition 3.19]. By using Theorem 1.3 and Theorem 1.4, we also construct in Section 2 examples of closed negatively curved manifolds $M^{2n}$, where $n = 7$ or 8, which are homeomorphic but not diffeomorphic to complex hyperbolic manifolds, thereby giving a partial answer to a question raised by C. S. Aravinda and F. T. Farrell [5].

Another source for Farrell–Jones spheres is the class of the so-called Hitchin spheres. In [12], Hitchin showed that if $\Sigma$ is a homotopy sphere with a metric of positive scalar curvature, then $\alpha(\Sigma) = 0$, where $\alpha : \Omega_{*}^{\text{spin}} \to \text{KO}_{*}$ is the ring homomorphism which associates to a spin bordism class the KO-valued index of the Dirac operator of a representative spin manifold. The following definition can be found in [19, Remark 3.4]:

**Definition 1.5.** An exotic sphere $\Sigma^m \in \Theta_m$ ($m \geq 1$) is called a Hitchin sphere if $\alpha(\Sigma^m) \neq 0$.

We prove the following theorem in Section 3:

**Theorem 1.6.** Every Hitchin $(8n + 2)$-sphere ($n \geq 1$) is a Farrell–Jones sphere.

Recall that the collection of homotopy spheres which admit an orientation preserving diffeomorphism $M \to M \# \Sigma$ form the inertia group of $M$, denoted by $I(M)$. There is a canonical topological identification $i : M \to M \# \Sigma$ which is the identity outside of the attaching region; the subset of the inertia group consisting of homotopy spheres that admit a diffeomorphism homotopic to $i$ is called the homotopy inertia group $I_h(M)$. Similarly, the concordance inertia group of $M^m$, $I_c(M^m) \subseteq \Theta_m$, consists of those homotopy spheres $\Sigma^m$ such that $M$ and $M \# \Sigma^m$ are concordant. By Theorem 1.2, we have that $\Sigma^{2n}$ is a Farrell–Jones sphere iff $\Sigma^{2n} \notin I(\mathbb{C}P^n)$ iff $\Sigma^{2n} \notin I_c(\mathbb{C}P^n)$ iff $\Sigma^{2n} \notin I_h(\mathbb{C}P^n)$. In Section 4, we discuss the group $I(\mathbb{C}P^{4n+1})$. 
2 Exotic smooth structures on complex hyperbolic manifolds

The negatively curved Riemannian symmetric spaces are of four types: \( \mathbb{R}H^n \), \( \mathbb{C}H^n \), \( \mathbb{H}H^n \) and \( \mathbb{O}H^2 \), where \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{H} \), \( \mathbb{O} \) denote the real, complex, quaternion and Cayley numbers, i.e., the four division algebras \( K \) over the real numbers whose dimensions over \( \mathbb{R} \) are \( d = 1, 2, 4 \) and \( 8 \) respectively. A Riemannian manifold \( M^{dn} \) is called a real, complex, quaternionic or Cayley hyperbolic manifold provided its universal cover is isometric to \( \mathbb{R}H^n \), \( \mathbb{C}H^n \), \( \mathbb{H}H^n \) and \( \mathbb{O}H^2 \), respectively. (Note that we need to consider only the cases \( n \geq 2 \) and when \( K = \mathbb{O} \), \( n = 2 \).)

In [5, p. 2], C. S. Aravinda and F. T. Farrell ask the following:

Question 2.1. For each division algebra \( K \) over the reals and each integer \( n \geq 2 \) (\( n = 2 \) when \( K = \mathbb{O} \)), does there exist a closed negatively curved Riemannian manifold \( M^{dn} \) (where \( d = \dim_{\mathbb{R}} K \)) which is homeomorphic but not diffeomorphic to a \( K \)-hyperbolic manifold.

Remark 2.2. For \( K = \mathbb{R} \) and \( n = 2, 3 \), this is impossible since homeomorphism implies diffeomorphism in these dimensions [17]. Also when \( K = \mathbb{R} \), it was shown in [11] that the answer is yes provided \( n \geq 6 \). When \( K = \mathbb{C} \), it was shown in [10] that the answer is yes for \( n = 4m + 1 \) for any integer \( m \geq 1 \) and for \( n = 4 \). When \( K = \mathbb{H} \), the answer is yes for \( n = 2, 4 \) and 5, see [5]. The answer to this question is yes for \( K = \mathbb{O} \) by [4] since only one dimension needs to be considered in this case. In this section, we consider the case \( K = \mathbb{C} \) and show that the answer is yes for \( n = 7, 8 \).

Since Borel [6] has constructed closed complex hyperbolic manifolds in every complex dimension \( m \geq 1 \) and by Theorem 1.3 and Theorem 1.4, we have the following result:

Theorem 2.3. Let \( n \) be either 7 or 8. Given any positive number \( \epsilon \in \mathbb{R} \), there exists a pair of closed negatively curved Riemannian manifolds \( M \) and \( N \) having the following properties:

(i) \( M \) is a complex \( n \)-dimensional hyperbolic manifold.

(ii) The sectional curvatures of \( N \) are all in the interval \( [-4 - \epsilon, -1 + \epsilon] \).

(iii) The manifolds \( M \) and \( N \) are homeomorphic but not diffeomorphic.

3 Farrell–Jones spheres and Hitchin sphere

In this section, we give proofs of the Theorems 1.2, 1.3 and 1.6.
Definition 3.1. Let $M$ be a topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a homeomorphism $f : N \to M$. Two such pairs $(N_1, f_1)$ and $(N_2, f_2)$ are concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to $f_1$, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

We recall some terminology from [15]:

Definition 3.2. (a) A homotopy $m$-sphere $\Sigma^m$ is an oriented smooth closed manifold homotopy equivalent to $S^m$.

(b) A homotopy $m$-sphere $\Sigma^m$ is said to be exotic if it is not diffeomorphic to $S^m$.

(c) Two homotopy $m$-spheres $\Sigma^m_1$ and $\Sigma^m_2$ are said to be equivalent if there exists an orientation preserving diffeomorphism $f : \Sigma^m_1 \to \Sigma^m_2$.

The set of equivalence classes of homotopy $m$-spheres is denoted by $\Theta_m$. The equivalence class of $\Sigma^m$ is denoted by $[\Sigma^m]$. When $m \geq 5$, $\Theta_m$ forms an abelian group with group operation given by the connected sum $\#$ and the zero element represented by the equivalence class of the round sphere $S^m$. M. Kervaire and J. Milnor [15] showed that each $\Theta_m$ is a finite group; in particular, $\Theta_8$, $\Theta_{14}$ and $\Theta_{16}$ are cyclic groups of order 2, $\Theta_{10}$ and $\Theta_{20}$ are cyclic groups of order 6 and 24 respectively and $\Theta_{18}$ is a group of order 16.

Start by noting that there is a homeomorphism $h : M^n \# \Sigma^n \to M^n$ ($n \geq 5$) which is the inclusion map outside of the homotopy sphere $\Sigma^n$ and well-defined up to topological concordance. We will denote the class of $(M^n \# \Sigma^n, h)$ in $\mathcal{C}(M)$ by $[M^n \# \Sigma^n]$. (Note that $[M^n \# S^n]$ is the class of $(M^n, \text{id}_{M^n})$.) Let $f_M : M^n \to S^n$ be a degree one map. Note that $f_M$ is well-defined up to homotopy. Composition with $f_M$ defines a homomorphism

$$f_M^* : [S^n, \text{Top}/O] \to [M^n, \text{Top}/O],$$

and in terms of the identifications

$$\Theta_n = [S^n, \text{Top}/O] \quad \text{and} \quad \mathcal{C}(M^n) = [M^n, \text{Top}/O]$$

given by [16, p. 25 and p. 194], $f_M^*$ becomes $[\Sigma^m] \mapsto [M^n \# \Sigma^m]$.

Definition 3.3. If $M$ is homotopy equivalent to $\mathbb{C}P^n$, we will call a generator of $H^2(M; \mathbb{Z})$ a c-orientation of $M$. 

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Hereafter $g$ denotes the conjugation map
\[(z_0, z_1, z_2, z_3, z_4, \ldots, z_n) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \ldots, \bar{z}_n)\]
(the complex conjugation) induces the diffeomorphism $g : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $g^*(c_1) = -c_1$ where $c_1$ is the $c$-orientation of $\mathbb{CP}^n$.

**Proof of Theorem 1.2.** Assume that $\Sigma^{2n}$ is a Farrell–Jones sphere. Suppose $\mathbb{CP}^n$ and $\mathbb{CP}^n \# \Sigma^{2n}$ are diffeomorphic. If $f : \mathbb{CP}^n \# \Sigma^{2n} \to \mathbb{CP}^n$ is a diffeomorphism, then $f$ induces an isomorphism on cohomology
\[f^* : H^*(\mathbb{CP}^n, \mathbb{Z}) \to H^*(\mathbb{CP}^n \# \Sigma^{2n}, \mathbb{Z})\]
such that $f^*(c_1) = \pm c_2$, where $c_1, c_2$ are $c$-orientations of $\mathbb{CP}^n, \mathbb{CP}^n \# \Sigma^{2n}$ respectively. If $f^*(c_1) = c_2$, then $f$ is a $c$-orientation preserving diffeomorphism. If $f^*(c_1) = -c_2$, then $g \circ f$ is a $c$-orientation preserving diffeomorphism, where $g : \mathbb{CP}^n \to \mathbb{CP}^n$ is the conjugation map. In both cases, we have that $\mathbb{CP}^n \# \Sigma^{2n}$ is $c$-orientation diffeomorphic to $\mathbb{CP}^n$. By [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is concordant to $\mathbb{CP}^n$. This is a contradiction since $\Sigma^{2n}$ is a Farrell–Jones sphere. Thus $\mathbb{CP}^n \# \Sigma^{2n}$ and $\mathbb{CP}^n$ are not diffeomorphic. Conversely, suppose $\mathbb{CP}^n \# \Sigma^{2n}$ and $\mathbb{CP}^n$ are not diffeomorphic. Then, by [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is not concordant to $\mathbb{CP}^n$. This shows that $\Sigma^{2n}$ is a Farrell–Jones sphere. This completes the proof of Theorem 1.2. \(\square\)

**Proof of Theorem 1.3.** Let $\Sigma^{2n}$ be the generator of $\Theta_{2n}$ (with $n = 7$ or 8). Suppose $\Sigma^{2n}$ is not a Farrell–Jones sphere. Then $\mathbb{CP}^n \# \Sigma^{2n}$ is concordant to $\mathbb{CP}^n$. By [18, Corollary 3, p. 97], $\mathbb{CP}^n \# \Sigma^{2n}$ is $c$-orientation diffeomorphic to $\mathbb{CP}^n$. Let $f : \mathbb{CP}^n \# \Sigma^{2n} \to \mathbb{CP}^n$ be a $c$-orientation diffeomorphism such that $f^*(c_1) = c_2$, where $c_1$ and $c_2$ are $c$-orientations of $\mathbb{CP}^n$ and $\mathbb{CP}^n \# \Sigma^{2n}$ respectively. Using properties of the cup product, we have $f^*(c_1^n) = c_2^n$. If $c_1 = c_2$ in $H^2(\mathbb{CP}^n, \mathbb{Z})$, then $f$ is an orientation preserving diffeomorphism. If $c_1 \neq c_2$ in $H^2(\mathbb{CP}^n, \mathbb{Z})$, then $g \circ f$ is an orientation preserving diffeomorphism with the property that $(g \circ f)^*(c_1) = f^*(g^*(c_1)) = -c_2 = c_1$, where $g : \mathbb{CP}^n \to \mathbb{CP}^n$ is the conjugation map. In both cases, we have that $\mathbb{CP}^n \# \Sigma^{2n}$ is an orientation preserving diffeomorphic to $\mathbb{CP}^n$. This is a contradiction because, by [13, Theorem 1], $\mathbb{CP}^n \# \Sigma^{2n}$ cannot be orientation preserving diffeomorphic to $\mathbb{CP}^n$. Thus $\Sigma^{2n}$ is a Farrell–Jones sphere. This completes the proof of Theorem 1.3. \(\square\)

Recall that the $\alpha$-invariant is the ring homomorphism $\alpha : \Omega_*^{\text{spin}} \to K_*O$ which associates to a spin bordism class the KO-valued index of the Dirac operator of a representative spin manifold. We also write $\alpha$ for the corresponding invariant on a framed bordism:

$$\alpha : \Omega_*^f \to \Omega_*^{\text{spin}} \to K_*O$$
Under the Pontryagin–Thom isomorphism $\Omega_*^f \cong \pi_*^s$, the $\alpha$-invariant has the following interpretation as Adams $d$-invariant $d_{\mathbb{R}} : \pi_r^s \to KO_*$, which was used already in [12, p. 44] and [9, Lemma 2.12].

**Lemma 3.4.** Under the Pontryagin–Thom isomorphism $\Omega_*^f \cong \pi_*^s$, the $\alpha$-invariant $\alpha : \Omega_{8n+2}^f \to KO_{8n+2}$ may be identified with $d_{\mathbb{R}} : \pi_{8n+2}^s \to KO_{8n+2}$.

We start by recalling some facts from smoothing theory [7], which were used already in [10, Lemma 3.17]. There are $H$-spaces $SF$, $F/O$ and $\text{Top}/O$ and $H$-space maps $\phi : SF \to F/O$, $\psi : \text{Top}/O \to F/O$ such that

$$\psi_* : \Theta_{8n+2} = \pi_{8n+2}(\text{Top}/O) \to \pi_{8n+2}(F/O)$$

(3.1)

is an isomorphism for $n \geq 1$. The homotopy groups of $SF$ are the stable homotopy groups of spheres $\pi_m^s$, i.e., $\pi_m(SF) = \pi_m^s$ for $m \geq 1$. For $n \geq 1$,

$$\phi_* : \pi_{8n+2}^s \to \pi_{8n+2}(F/O)$$

(3.2)

is an isomorphism. Since every homotopy sphere has a unique spin-structure, we obtain the $\alpha$-invariant on $\pi_{8n+2}^s \cong \pi_{8n+2}(F/O) \cong \Theta_{8n+2}$:

$$\alpha : \pi_{8n+2}^s \xrightarrow{\phi_*} \pi_{8n+2}(F/O) \xrightarrow{\psi_*^{-1}} \Theta_{8n+2} \xrightarrow{\Omega_{8n+2}^{\text{spin}}} \text{KO}_{8n+2},$$

where $\psi_*$ and $\phi_*$ are the isomorphisms as in equation (3.1) and (3.2) respectively.

Let $\text{Ker}(d_{\mathbb{R}})$ denote the kernel of the Adams $d$-invariant $d_{\mathbb{R}} : \pi_{8n+2}^s \to \mathbb{Z}_2$. By Lemma 3.4, $\text{Ker}(d_{\mathbb{R}})$ consists of framed manifolds which bound spin manifolds.

**Proof of Theorem 1.6.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
[S^m, \text{Top}/O] = \Theta_{2m} & \xrightarrow{\phi_*^m} & [\mathbb{C}P^m, \text{Top}/O] = \mathcal{C}(\mathbb{C}P^m) \\
\downarrow \psi_* & & \downarrow \psi_* \\
[S^m, F/O] & \xrightarrow{\phi_*^{\mathbb{C}P^m}} & [\mathbb{C}P^m, F/O] \\
\uparrow \phi_* & & \uparrow \phi_* \\
[S^m, SF] = \pi_{2m}^s & \xrightarrow{\phi_*^{\mathbb{C}P^m}} & [\mathbb{C}P^m, SF].
\end{array}
$$

(3.3)

In this diagram, $\phi_*$ and $\psi_*$ are induced by the $H$-space maps $\phi : SF \to F/O$ and $\psi : \text{Top}/O \to F/O$ respectively and the homomorphism

$$\phi_* : [\mathbb{C}P^m, SF] \to [\mathbb{C}P^m, F/O]$$
is monic for all $m \geq 1$ by a result of Brumfiel [8, p. 77]. Recall that the concordance class $[\mathbb{C}P^m \# \Sigma] \in [\mathbb{C}P^m, \text{Top}/O]$ of $\mathbb{C}P^m \# \Sigma$ is $f_{\mathbb{C}P^m}^*([\Sigma])$ when $m > 2$ and that $[\mathbb{C}P^m] = [\mathbb{C}P^m \# S^{2m}]$ is the zero element of this group.

Let $\Sigma^{8n+2} \in \Theta_{8n+2}$ be a Hitchin $(8n+2)$-sphere (with $n \geq 1$) and further let $\eta \in \pi^{8n+2} = [S^{8n+2}, SF]$ be such that

$$\psi_*^{-1}(\phi_*(\eta)) = \Sigma^{8n+2}.$$ 

Recall that $[X, SF]$ can be identified with the $0^{\text{th}}$ stable cohomotopy group $\pi^0(X)$. Let $h : S^{q+8n+2} \to S^q$ represent $\eta$. Since $\Sigma^{8n+2}$ is a Hitchin sphere and by Lemma 3.4, we have

$$0 \neq \alpha(\Sigma^{8n+2}) = d_{\mathbb{R}}(h) = h^* \in \text{Hom}(\text{KO}^q(S^q), \text{KO}^q(S^{q+8n+2})).$$

Also Adams and Walker [2] showed that $\Sigma^q f_{\mathbb{C}P^{4n+1}} : \Sigma^q \mathbb{C}P^{4n+1} \to S^{q+8n+2}$ induces a monomorphism on $\text{KO}^q(\cdot)$. Consequently the composite map

$$h \circ \Sigma^q f_{\mathbb{C}P^{4n+1}} : \Sigma^q \mathbb{C}P^{4n+1} \to S^q$$

induces a non-zero homomorphism on $\text{KO}^q(\cdot)$. This shows that

$$f_{\mathbb{C}P^{4n+1}}^*(\eta) = [h \circ \Sigma^q f_{\mathbb{C}P^{4n+1}}] \neq 0,$$

where

$$f_{\mathbb{C}P^{4n+1}}^* : [S^{8n+2}, SF] \to [\mathbb{C}P^{4n+1}, SF].$$

Since the homomorphism $\phi_* : [\mathbb{C}P^m, SF] \to [\mathbb{C}P^m, F/O]$ is monic, by the commutative diagram (3.3) where $m = 4n + 1$, we have

$$\psi_*(f_{\mathbb{C}P^{4n+1}}^*(\Sigma^{8n+2})) = \phi_*(f_{\mathbb{C}P^{4n+1}}^*(\eta)) \neq 0.$$ 

This implies that

$$f_{\mathbb{C}P^{4n+1}}^*(\Sigma^{8n+2}) \neq 0$$

and hence $\mathbb{C}P^{4n+1} \# \Sigma^{8n+2}$ is not concordant to $\mathbb{C}P^{4n+1}$. This shows that $\Sigma^{8n+2}$ is a Farrell–Jones sphere and this completes the proof of Theorem 1.6. □

**Remark 3.5.** (1) Let us note that the homotopy sphere $\Sigma^{8n+2} (n \geq 1)$ given by [10, Lemma 3.17] is the image of the Adams element $\mu_{8n+2}$ of order 2 under the composed isomorphism $\psi_*^{-1} \circ \phi_*$, where $\psi_*$ and $\phi_*$ are the isomorphisms as in equations (3.1) and (3.2) respectively (see [10, equation (3.17.4)]). By [1, Theorem 1.2] and Lemma 3.4, we have

$$d_{\mathbb{R}}(\mu_{8n+2}) = \alpha(\Sigma^{8n+2}) = 1.$$ 

This shows that $\Sigma^{8n+2}$ is a Hitchin sphere of order 2 in $\Theta_{8n+2}$. By Theorem 1.6, $\Sigma^{8n+2}$ is a Farrell–Jones sphere.
(2) Since $\Theta_{18} \cong \text{Ker}(\alpha) \oplus \mathbb{Z}_2$, where the $\alpha$-invariant $\alpha : \Theta_{18} \to \mathbb{Z}_2$ satisfies $\text{Ker}(\alpha) = \mathbb{Z}_8$ (see [9, p. 12]), this shows that there are exotic spheres of order $\neq 2$ in $\Theta_{18}$ which are not in the kernel of $\alpha$. This implies that there are Hitchin spheres of order $\neq 2$ in $\Theta_{18}$ which are all Farrell–Jones sphere by Theorem 1.6.

(3) In [3], Anderson, Brown and Peterson proved that one has $\alpha^\dagger m/\alpha^0$ iff $m = 8k + 1$ or $8k + 2$ iff $\Sigma^m$ is an exotic sphere not bounding a spin manifold, where $\alpha : \Theta_m \to \Omega^\text{spin}_m \to \text{KO} m$ is the $\alpha$-invariant. This implies that $\Sigma^m$ is a Hitchin sphere in $\Theta_m$ iff $\Sigma^m$ is an exotic sphere not bounding a spin manifold. By Theorem 1.6, every exotic sphere not bounding a spin manifold $\Sigma^{8n+2}$ in $\Theta_{8n+2}$ is a Farrell–Jones sphere.

(4) By [1, Theorem 7.2], $\Theta_{10} \cong \text{Ker}(d_{\mathbb{R}}) \oplus \mathbb{Z}_2$ such that $\text{Ker}(d_{\mathbb{R}}) = \mathbb{Z}_3$. If $\Sigma^{10}$ is a generator of $\text{Ker}(d_{\mathbb{R}})$, then $d_{\mathbb{R}}(\Sigma^{10}) = \alpha(\Sigma^{10}) = 0$. This shows that $\Sigma^{10}$ is not a Hitchin sphere. But, by [10, Lemma 3.17], $\Sigma^{10}$ is a Farrell–Jones sphere.

4 The inertia groups of complex projective spaces

In this section, we discuss the relationship between inertia groups of $\mathbb{CP}^n$ and Farrell–Jones spheres.

**Definition 4.1.** Let $M^m$ be a closed smooth, oriented $m$-dimensional manifold. Let $\Sigma \in \Theta_m$ and $g : S^{m-1} \to S^{m-1}$ be an orientation preserving diffeomorphism corresponding to $\Sigma$. Writing $M \# \Sigma$ as $(M^m \setminus \text{int}(\mathbb{D}^m)) \cup_g \mathbb{D}^m$, let $\iota : M \to M \# \Sigma$ denote the PL homeomorphism defined by $\iota|_{M \setminus \text{int}(\mathbb{D}^m)} = \text{id}$ and $\iota|_{\mathbb{D}^m} = Cg$, where $Cg : \mathbb{D}^m \to \mathbb{D}^m$ is the cone extension of $g$.

The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists an orientation preserving diffeomorphism $\phi : M \to M \# \Sigma$.

Define the homotopy inertia group $I_h(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to $\iota$.

Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to $M$. Clearly, $I_c(M) \subseteq I_h(M) \subseteq I(M)$.

Note that for $M = \mathbb{CP}^n$, Theorem 1.2 can be restated as:

**Theorem 4.2.** A sphere $\Sigma^{2n} \in \Theta_{2n}$ is a Farrell–Jones sphere iff $\Sigma^{2n} \notin I(\mathbb{CP}^n)$.

**Remark 4.3.** Since $I_c(\mathbb{CP}^n) \subseteq I_h(\mathbb{CP}^n) \subseteq I(\mathbb{CP}^n)$ and by Theorem 4.2, we have that $I_c(\mathbb{CP}^n) = I_h(\mathbb{CP}^n) = I(\mathbb{CP}^n)$.

The proof of Theorem 1.6 leads one to the following question.
Question 4.4. Let \( f : \mathbb{C}P^{4n+1} \to S^{8n+2} \) be any degree one map \((n \geq 1)\). Does there exist an element \( \eta \in \text{Ker}(d) \subset \pi^s_{8n+2} = \Theta_{8n+2} \) such that the following is true?

\((\ast)\) If any map \( h : S^{q+8n+2} \to S^{q} \) represents \( \eta \), then

\[ h \circ \Sigma^q f : \Sigma^q \mathbb{C}P^{4n+1} \to S^{q} \]

is not null homotopic.

Remark 4.5. (1) By [14, Lemma 9.1], \( I(\mathbb{C}P^{4n+1}) \subset \text{Ker}(d) \). If the answer to Question 4.4 is yes, then we have \( I(\mathbb{C}P^{4n+1}) \neq \text{Ker}(d) \), i.e., there exists an exotic sphere \( \Sigma \) bounding spin manifold in \( \Theta_{8n+2} \) such that \( \Sigma \notin I(\mathbb{C}P^{4n+1}) \). This can be seen as follows: Let \( \eta \in \text{Ker}(d) \) and let \( h : S^{q+8n+2} \to S^{q} \) represent \( \eta \) such that \( h \circ \Sigma^q f : \Sigma^q \mathbb{C}P^{4n+1} \to S^{q} \) is not null homotopic. This implies that

\[ f^*_{\mathbb{C}P^{4n+1}}(h) = [h \circ \Sigma^q f_{\mathbb{C}P^{4n+1}}] \neq 0, \]

where \( f^*_{\mathbb{C}P^{4n+1}} : \pi^0(S^{8n+2}) \to \pi^0(\mathbb{C}P^{4n+1}) \). A similar argument given in the proof of Theorem 1.6 using the commutative diagram (3.3) shows that there exists an exotic sphere \( \Sigma \in \Theta_{8n+2} \) such that \( \psi_\ast^{-1} \circ \phi_\ast(\eta) = \Sigma, d(\eta) = \alpha(\Sigma) = 0 \) and \( \mathbb{C}P^{4n+1} # \Sigma \) is not concordant to \( \mathbb{C}P^{4n+1} \), where \( \psi_\ast \) and \( \phi_\ast \) are the isomorphisms as in (3.1) and (3.2) respectively. This implies that \( \Sigma \) is a Farrell–Jones sphere such that \( \Sigma \in \text{Ker}(d) \). By Theorem 4.2, \( I(\mathbb{C}P^{4n+1}) \neq \text{Ker}(d) \).

(2) If all non-zero elements in \( \text{Ker}(d) \) satisfy the condition \((\ast)\) in Question 4.4, then, by the above remark (1), \( \Sigma \notin I(\mathbb{C}P^{4n+1}) \) for all exotic sphere \( \Sigma \in \text{Ker}(d) \) and hence \( I(\mathbb{C}P^{4n+1}) = 0 \).

Theorem 4.6. Let \( n \) be a positive integer such that \( \Theta_{8n+2} \) is a cyclic group of order 2. Then \( I(\mathbb{C}P^{4n+1}) = 0 \).

Proof. Let \( \Sigma^{8n+2} \) be the generator of \( \Theta_{8n+2} \cong \mathbb{Z}_2 \). Let

\[ \psi_\ast : \Theta_{8n+2} \to \pi_{8n+2}(F/O) \quad \text{and} \quad \phi_\ast : \pi^s_{8n+2} \to \pi_{8n+2}(F/O) \]

be the isomorphisms as in (3.1) and (3.2). By [1, Theorem 1.2], there exists an element \( \mu_{8n+2} \) of order 2 in \( \pi^s_{8n+2} \). This shows that

\[ \phi_\ast^{-1} \circ \psi_\ast(\Sigma^{8n+2}) = \mu_{8n+2}. \]

By [1, Theorem 1.2] and Lemma 3.4, \( d(\mu_{8n+2}) = \alpha(\Sigma^{8n+2}) = 1 \). This implies that \( \Sigma^{8n+2} \) is a Hitchin sphere. By Theorems 1.6 and 1.2, \( \mathbb{C}P^{4n+1} # \Sigma^{8n+2} \) is not diffeomorphic to \( \mathbb{C}P^{4n+1} \). This implies that \( I(\mathbb{C}P^{4n+1}) = 0 \). \( \square \)
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