Aspects of generic entanglement

Patrick Hayden,1,2,∗ Debbie W. Leung,1,† and Andreas Winter3,‡

1Institute for Quantum Information, Caltech 107–81, Pasadena, CA 91125, USA
2Department of Computer Science, McGill University, Montreal, Quebec, Canada H3A 2A7
3Department of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom
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We study entanglement and other correlation properties of random states in high-dimensional bipartite systems. These correlations are quantified by parameters that are subject to the “concentration of measure” phenomenon, meaning that on a large-probability set these parameters are close to their expectation. For the entropy of entanglement, this has the counterintuitive consequence that there exist large subspaces in which all pure states are close to maximally entangled. This, in turn, implies the existence of mixed states with entanglement of formation near that of a maximally entangled state, but with negligible quantum mutual information and, therefore, negligible distillable entanglement, secret key, and common randomness. It also implies a very strong locking effect for the entanglement of formation: its value can jump from maximal to near zero by tracing over a number of qubits negligible compared to the size of total system. Furthermore, such properties are generic. Similar phenomena are observed for random multiparty states, leading us to speculate on the possibility that the theory of entanglement is much simplified when restricted to asymptotically generic states. Further consequences of our results include a complete derandomization of the protocol for universal superdense coding of quantum states.

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I. INTRODUCTION

The subject of correlations between quantum systems can be bewildering. Beyond the simplest example, manipulation of pure bipartite states, very little is known. Exotic examples have implied that the rules governing interconversion of quantum states are often counterintuitive. The complexity of the subject is also manifested in the difficulties one encounters when attempting to quantify entanglement. Even in the bipartite, asymptotic case, there are many different mixed-state “entanglement measures,” most of which are poorly understood, both individually and in relation to each other; see [3, 5, 12, 21, 24, 51] and references therein.

One of the most striking features of asymptotic entanglement manipulations is irreversibility. Even in the limit of large number of copies, some states cost more EPR pairs to create than can be distilled from them. The corresponding entanglement measures, known as the entanglement cost ($E_c$) [21] and the entanglement of distillation ($E_d$) [5], are therefore different. In particular, for some “bound entangled” states [28], it has been shown that $E_d$ is zero while $E_c$ is not [51, 52].

Another intriguing issue in the study of entanglement is whether the entanglement cost of a state is equal to a much simpler measure, the entanglement of formation ($E_f$) [3]. If equality holds, the study of entanglement can be simplified significantly, while inequality implies the advantage of more collective strategies in the asymptotic preparation of quantum states. This problem has

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∗Electronic address: patrick@cs.mcgill.ca
†Electronic address: wcleung@cs.caltech.edu
‡Electronic address: a.j.winter@bris.ac.uk
recently been connected to other important additivity conjectures in quantum information theory. (See, for example, [45].)

While the general theory of entanglement appears to be very complicated, a much simplified theory may exist for generic quantum states if, in some appropriate regime, most states behave similarly. In particular, irreversibility has only been demonstrated for some carefully constructed states and it is natural to wonder whether it is the exception or the rule. Here, we investigate possible simplifications of the entanglement properties of quantum states in large systems.

Considerable effort has been devoted to understanding the average properties of quantum states. For example, the expected entropy [15, 33, 54, 42, 44] and purity [34, 56, 59] of reduced states for random pure quantum states have been calculated. In the case of mixed states, various distributions have been proposed (see [59] and references therein) and the likelihood of separable (i.e., \(E_f = 0\)) and bound entangled states have been studied [30, 47, 57, 58].

The present paper is a further step in the direction of a simplified theory of entanglement for generic states. We draw random pure states from the uniform (unitarily invariant) distribution and mixed states by tracing over part of a random pure state on an extended system. (Note that the induced distribution depends on the dimension of the system that was traced out.) We find that random pure states are extremely likely to have near-maximal entanglement, in fact, so likely that, with high probability, a random subspace of dimension close to the total dimension contains only near-maximally entangled states. These findings imply that random mixed states of up to almost full rank can have entanglement of formation close to maximal; at the same time distillable entanglement, secret key and common randomness can all be bounded by much smaller quantities. In fact, for a wide range of parameters, these random mixed states will not be one-copy pseudo-distillable, and will have arbitrarily small one-way distillable entanglement, secret key, and common randomness. Thus, near-maximal irreversibility is generic, unless our states turn out to be counterexamples to the additivity conjecture with near-maximal violation.

By building on the results for the bipartite case, we can make similar high-probability statements about many properties of random multiparty states as well. We find, for example, that a typical such state has near-maximal distillable entanglement between any two parties, provided the other parties are allowed to participate in the distillation protocol.

Finally, the existence of large subspaces containing only near-maximally entangled states has applications to the study of quantum communication, not just the study of correlations. In particular, it implies that the protocol for superdense coding of arbitrary 2l-qubit states using \(l\) ebits and \(l + o(l)\) qubits of communication [20] can be completely derandomized. The original construction consumed \(l + o(l)\) shared random bits in addition to the other resources.

**Guide to the paper** In Section III we introduce the pure and mixed state distributions we will be investigating. We discuss our basic techniques in Section III. These consist of an elementary discretization procedure, which “counts” the number of points in a geometrical manifold (of states, subspaces, etc), and explicit inequalities for the concentration of measure phenomenon in functions on high-dimensional spheres. Our main result, proving that random subspaces are likely to contain only near-maximally entangled states, appears in Section IV. We then study various entanglement, secret key and other correlation quantities in Section V. Some preliminary results on generic multiparty entanglement appear in Section VI. Finally, we show how our results de-randomize superdense coding in Section VII.

**Notation** We use the following conventions throughout the paper. \(\log\) and \(\exp\) are always taken base two. Unless otherwise stated, a “state” can be pure or mixed. The symbol for a state (such as \(\varphi\) or \(\rho\)) also denotes its density matrix. We will make an explicit indication when referring to a pure state. The density matrix \(|\varphi\rangle \langle \varphi|\) of the pure state \(|\varphi\rangle\) will frequently be written simply as
\( \varphi. \) \( B(\mathbb{C}^d) \) will be used to denote the set of linear transformations from \( \mathbb{C}^d \) to itself and \( \mathbb{U}(d) \subset B(\mathbb{C}^d) \) the unitary group on \( \mathbb{C}^d \). \( \mathbb{E}X \) refers to the expectation value of the random variable \( X \) and \( m(X) \) a median for \( X \). Quantum systems under consideration will be denoted \( A, B, \ldots \) and are freely associated with their Hilbert spaces, whose (finite) dimensions are denoted \( d_A, d_B, \) etc. In a bipartite system, when speaking of a “maximally entangled state”, we refer to a pure state whose nonzero Schmidt coefficients are all equal to the inverse of the smaller of the two dimensions. We use \( S(\rho) = -\mathrm{Tr} \rho \log \rho \) to refer to the von Neumann entropy of a density matrix \( \rho \), \( S(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \) to denote the quantum mutual information of a bipartite state \( \rho_{AB} \) and \( D(\rho \parallel \sigma) = \mathrm{Tr} \rho (\log \rho - \log \sigma) \) the relative entropy of the states \( \rho \) and \( \sigma \).

\[ F(\rho, \sigma) = (\mathrm{Tr} \sqrt{\rho^{1/2} \sigma^{1/2}})^2 \] is the Uhlmann fidelity, again between two states \( \rho \) and \( \sigma \).

### II. RANDOM STATES AND SUBSPACES

We are going to consider the state of large bipartite (and later multipartite) quantum systems under a random selection. We think of the pure or mixed state as being drawn at random from an ensemble. For pure states, there is a unique “uniform” distribution that is unitarily invariant. It is induced by the Haar measure on the unitary group by acting on an arbitrarily chosen generating vector.

**Definition II.1** A random pure state \( \varphi \) is any random variable distributed according to the unitarily invariant probability measure on the pure states \( \mathcal{P}(A) \) of the system \( A \). We formally express this by writing \( \varphi \in_R \mathcal{P}(A) \). (It is frequently convenient to choose a vector representative in \( A \) for the state \( \varphi \). When doing so, we will indicate this by using the notation \( |\varphi \rangle \).)

Similarly, there is also a unique, uniform distribution for subspaces that is unitarily invariant.

**Definition II.2** A random subspace \( S \) of dimension \( s \) is any random variable distributed according to the unitarily invariant measure on the \( s \)-dimensional subspaces of \( A \), the Grassmannian \( \mathcal{G}_s(A) \) (see, for example, [13]). We express this using the notation \( S \in_R \mathcal{G}_s(A) \). Note that \( \mathcal{G}_1(A) \) is naturally isomorphic to \( \mathcal{P}(A) \).

For mixed states, unitary invariance does not uniquely specify a probability measure. Instead, we follow an old proposal to induce probability measures on mixed states [4, 18, 59] by partial tracing.

**Definition II.3** For a system \( A \) and an integer \( s \geq 1 \), consider the distribution on the mixed states \( \mathcal{S}(A) \) of \( A \) induced by the partial trace over the second factor from the uniform distribution on pure states of \( A \otimes \mathbb{C}^s \). Any random variable \( \rho \) distributed as such will be called a rank-\( s \) random state; formally, \( \rho \in_R \mathcal{S}_s(A) \). Note that the rank of \( \rho \) is equal to \( \min(s, d_A) \) with probability 1. Also, \( \mathcal{P}(A) = \mathcal{S}_1(A) \).

These distributions on states have previously received considerable interest; so much indeed that the expectation values of several quantities of interest are known either exactly or to good approximations. It is clear that the average of any random rank-\( s \) state is the maximally mixed state, \( \frac{1}{d_A} \mathbb{1} \). We will also make explicit use of the average entropy of a subsystem, which was conjectured in [52] and proved in [13, 42, 44]:
Lemma II.4 Let $|\varphi\rangle$ be chosen according to the unitarily invariant measure on a bipartite system $A \otimes B$ with local dimensions $d_A \leq d_B$, i.e. $\varphi \in_R \mathcal{P}(A \otimes B)$. Then 

$$\mathbb{E} S(\varphi_A) = \frac{1}{\ln 2} \left( \sum_{j = d_B}^{d_A d_B} \frac{1}{j} - \frac{d_A - 1}{2d_B} \right) > \log d_A - \frac{1}{2}\beta,$$

where $\beta = \frac{1}{\ln 2} \frac{d_A}{d_B}$. □

The inequality can be demonstrated by making use of the estimate \[\frac{1}{2(d + 1)} < \sum_{j=1}^{d} \frac{1}{j} - \ln d - \gamma < \frac{1}{2d},\]

where $\gamma \approx 0.577$ is Euler’s constant.

In the following, we will identify the large probability behavior of functions such as $S(\varphi_A)$. It turns out that the probability in question is often exponentially close to 1 in some parameter $k$; that is, for sufficiently large $k$, the probability is at least $1 - \exp[-k/\text{polylog}(k)]$. We shall in this case adopt the expression that the behavior in question is $k$-likely. In some cases we won’t specify $k$ and will simply speak of likely behavior.

III. CONCENTRATION OF MEASURE

It is a striking yet elementary fact that the uniform measure on the $k$-sphere, $S^k$, concentrates very strongly about any equator as $k$ gets large; indeed, any polar cap strictly smaller than a hemisphere has relative volume exponentially small in $k$. This simple observation implies a similar result for the value of any slowly varying function on the sphere, which we can understand as a random variable induced by the sphere’s uniform measure: namely, it will take values close to the average except for a set of volume exponentially small in $k$. Levy’s Lemma (Lemma III.1 below) rigorously formalizes this idea: “slow variation” is encoded as a bound on the Lipschitz constant of the function (essentially the maximum gradient), and “close to the average” is modelled as a small but finite deviation. Given only these data, Levy’s Lemma gives an explicit exponential probability bound on the set of “large deviation”. Since pure quantum states in $d$ dimensions can be represented as $2d$-dimensional real unit vectors, the above observations on spheres ensure that as the dimension of a quantum system becomes large it comes to make sense to discuss typical behavior of random states, in the sense that for many properties of interest, almost all quantum states behave in essentially the same way.

The analysis leading to the various results in this paper will revolve around the concentration of the spectrum of the reduced density matrix of a bipartite system when both subsystems are large. This in turns implies many important concentration effects. One example is the concentration of the entropy of the reduced density matrix (or the entanglement between the two systems). Concentration effects for the maximum and minimum eigenvalues also imply tight bounds on the reduced density matrix itself and the values of various projections.

Our method of demonstrating generic properties is always to prove that the opposite is an unlikely event. We then rewrite the “bad event” as a union of “elementary bad events” on a net of states; the cardinality of the net is then bounded. In most cases, the cardinality of the net is exponentially large in the dimension parameter, while the “elementary bad event” has an exponentially small probability, due to some measure concentration. The probability of the bad event
is thus bounded by the product of these exponentially large and exponentially small quantities and our goal is to make it (exponentially) less than 1.

Because we strive for explicit probability and dimension bounds, the expressions in our theorems and some of the estimates may appear clumsy at first sight. It is in the nature of the problem (and partly of our method), however, that the crucial quantities are always composed of (a) a dimension parameter, which dominates, (b) a logarithmic factor, (c) a factor quantifying the allowable size of deviations from the average and (d) an absolute constant. Our obsession with explicit exponents throughout the paper is needed, since the exponentially large net size usually allows little optimization, and everything depends on the achievable strength of measure concentration.

In the rest of this section, we list a number of basic tools including concentration effects and net constructions. Readers who are specifically interested in the correlation properties of random states could read the statements (without the proofs) of Levy’s Lemma, the concentration of entropy and the existence of small nets, and move directly to the next section, referring back to the rest of the tools as necessary rather than trying to absorb them all beforehand.

**Lemma III.1 (Levy’s Lemma; see [35], Appendix IV, and [32])** Let \( f : \mathbb{S}^k \to \mathbb{R} \) be a function with Lipschitz constant \( \eta \) (with respect to the Euclidean norm) and a point \( X \in \mathbb{S}^k \) be chosen uniformly at random. Then

1. \( \Pr \{ f(X) - \mathbb{E} f \leq \pm \alpha \} \leq 2 \exp \left( -C_1(k+1)\alpha^2/\eta^2 \right) \) and
2. \( \Pr \{ f(X) - m(f) \leq \pm \alpha \} \leq \exp \left( -C_2(k-1)\alpha^2/\eta^2 \right) \)

for absolute constants \( C_i > 0 \) that may be chosen as \( C_1 = (9\pi^3 \ln 2)^{-1} \) and \( C_2 = (2\pi^2 \ln 2)^{-1} \). (\( \mathbb{E} f \) is the mean value of \( f \), \( m(f) \) a median for \( f \).) \( \square \)

We are going to apply Levy’s Lemma to the entropy of the reduced state of a randomly chosen pure state \( \varphi \) in a bipartite system \( A \otimes B \), i.e., \( f(\| \varphi \|) = S(\varphi_A) \). Note that \( k = 2d_Ad_B - 1 \), and all that remains is to bound the Lipschitz constant.

**Lemma III.2** The Lipschitz constant \( \eta \) of \( S(\varphi_A) \) is upper bounded by \( \sqrt{8} \log d_A \), for \( d_A \geq 3 \).

**Proof** We first consider the Lipschitz constant of the function \( g(\varphi) = H(M(\varphi_A)) \), where \( M \) is any fixed complete von Neumann measurement and \( H \) is the Shannon entropy. Let \( \| \varphi \| = \sum_{jk} \varphi_{jk} |e_j\rangle_A |f_k\rangle_B \) in terms of some orthonormal bases \( \{ |e_j\rangle_A \} \) for \( A \) and \( \{ |f_k\rangle_B \} \) for \( B \). By unitary invariance, we may assume that \( M_j = |e_j\rangle |e_j\rangle_A \). Therefore, if we define

\[
p(j|\varphi) = A(e_j|\varphi_A|e_j) = \sum_k |\varphi_{jk}|^2,
\]

then

\[
g(\varphi) = H(M(\varphi_A)) = -\sum_j p(j|\varphi) \log p(j|\varphi).
\]

An elementary calculation yields

\[
\eta^2 = \sup_{|\varphi| \leq 1} \nabla g \cdot \nabla g = \sum_{jk} \frac{4|\varphi_{jk}|^2}{(\ln 2)^2} [1 + \ln p(j|\varphi)]^2
\leq \frac{4}{(\ln 2)^2} [1 + \sum_j p(j|\varphi)(\ln p(j|\varphi))^2]
\leq \frac{4}{(\ln 2)^2} [1 + (\ln d_A)^2] \leq 8(\log d_A)^2,
\]
where the second inequality can be shown to hold for \( d_A \geq 3 \) using Lagrange multipliers.

Using the above bound, the Lipschitz constant for the von Neumann entropy \( S(\varphi_A) \) can be controlled as follows. Consider any two unit vectors \( |\varphi\rangle \) and \( |\psi\rangle \), and without loss of generality assume \( S(\varphi_A) \leq S(\psi_A) \). If we choose the measurement \( M \) to be along the eigenbasis of \( \varphi_A \), \( H(M(\varphi_A)) = S(\varphi_A) \) and we have \[53\]

\[
S(\psi_A) - S(\varphi_A) \leq H(M(\psi_A)) - H(M(\varphi_A)) \leq \eta \| |\psi\rangle - |\varphi\rangle \|_2.
\]

Thus, the Lipschitz constant for \( S(\varphi_A) \) is bounded by that of \( H(M(\varphi_A)) \) and we are done. \( \square \)

**Theorem III.3 (Concentration of entropy)** Let \( \varphi \in_R \mathcal{P}(A \otimes B) \) be a random state on \( A \otimes B \), with \( d_B \geq d_A \geq 3 \). Then

\[
\Pr\{ S(\varphi_A) < \log d_A - \alpha - \beta \} \leq \exp\left( -\frac{(d_A d_B - 1)C_3 \alpha^2}{(\log d_A)^2} \right),
\]

where \( \beta = \frac{1}{\ln 2} \frac{d_A}{d_B} \) is as in Lemma II.4 and \( C_3 = (8\pi^2 \ln 2)^{-1} \).

**Proof** As suggested earlier, we choose \( f(\varphi) = S(\varphi_A) \). We could use Lemma III.1.1 directly but will get better constants with a bit more work. We need to relate the median of \( f \) to the mean, which is known. Choose a subset \( X \) of the unit ball of \( A \otimes B \) having relative volume \( 1/2 \) and such that \( |\varphi\rangle \in X \) implies that \( S(\varphi_A) \leq m(f) \). Then

\[
\log d_A - \frac{1}{2} \beta \leq \mathbb{E} f = \int_X S(\varphi_A) d\varphi + \int_{\bar{X}} S(\varphi_A) d\varphi \\
\leq \frac{1}{2} m(f) + \frac{1}{2} \log d_A.
\]

Therefore, \( m(f) \geq \log d_A - \beta \) and the result follows by combining Lemmas III.1.2 and III.2 \( \square \)

This statement ensures that with overwhelming probability, a random pure state is almost maximally entangled. The exceptional set has measure exponentially small in a quantity essentially proportional to the total dimension. We will see in the next section that the strength of this concentration gives a whole large subspace of such states.

Whenever the reduced density matrix \( \varphi_A \) has near-maximal entropy, it is also close to the maximally mixed state \( \frac{1}{d_A} \mathbb{I} \). Sometimes, however, we want an even stronger estimate. The following bound is from Appendix A of [20].

**Lemma III.4 (Concentration of reduced density matrices)** For \( \varphi \in_R \mathcal{P}(A \otimes B), \) and \( 0 < \epsilon \leq 1, \)

\[
\Pr\left\{ \lambda_{\max}(\varphi_A) > \frac{1}{d_A} + \frac{\epsilon}{d_A} \right\} \leq \left( \frac{10d_A}{\epsilon} \right)^{2d_A} \exp\left( -d_B \frac{\epsilon^2}{14 \ln 2} \right),
\]

and

\[
\Pr\left\{ \lambda_{\min}(\varphi_A) < \frac{1}{d_A} - \frac{\epsilon}{d_A} \right\} \leq \left( \frac{10d_A}{\epsilon} \right)^{2d_A} \exp\left( -d_B \frac{\epsilon^2}{14 \ln 2} \right),
\]

where \( \lambda_{\max} \) and \( \lambda_{\min} \) denote the maximal and minimal nonzero eigenvalues of \( \varphi_A \), respectively.
This lemma says that the reduced state on $A$ of a random state in a bipartite system will be close to maximally mixed in the sense that all its eigenvalues cluster around $1/d_A$, if $d_B$ is a large enough multiple of $d_A \log d_A/\epsilon^2$. In fact, when $\varphi_A$ is not in the exceptional set in Lemma III.4

\[(1 - \epsilon) \frac{1}{d_A} \mathbb{1} \leq \varphi_A \leq (1 + \epsilon) \frac{1}{d_A} \mathbb{1}.\]  

The reduced state on $B$, $\varphi_B$, has the same spectrum as $\varphi_A$, and therefore $\varphi_B$ will also be close to maximally mixed on its (uniformly random) supporting subspace in a similar way.

Note that Eq. (1) is a statement of the concentration of the density matrix $\varphi_A$ itself, and is generally stronger than just a bound on the von Neumann entropy as in Theorem III.3. The price paid in Lemma III.4 is a lesser degree of concentration. The main tools in proving Lemma III.4 also differ from that of Theorem III.3. We now state these tools, and we will use them later in the paper. There are two essential ingredients. The first is the following concentration bound, which is a slight strengthening of Lemma II.3 from [22], which is in turn based on Cramér’s Theorem (see e.g. [7]):

**Lemma III.5 (Concentration of projector overlaps)** For $S \in_R \mathcal{G}_s(A)$, $P$ the projector onto $S$, $Q$ a fixed projector of rank $q$ in $A$, and $0 \leq \epsilon \leq 1$,

\[
\Pr \left\{ \operatorname{Tr} PQ > (1 + \epsilon) \frac{qs}{d_A} \right\} \leq \exp \left( -qs \frac{-\epsilon - \ln(1 + \epsilon)}{\ln 2} \right) \leq \exp \left( -qs \frac{\epsilon^2}{6 \ln 2} \right),
\]

\[
\Pr \left\{ \operatorname{Tr} PQ < (1 - \epsilon) \frac{qs}{d_A} \right\} \leq \exp \left( -qs \frac{-\epsilon - \ln(1 - \epsilon)}{\ln 2} \right) \leq \exp \left( -qs \frac{\epsilon^2}{6 \ln 2} \right).
\]

**Proof** The case $s = 1$ is, in fact, a special case of [22]’s Lemma II.3. To extend to $s > 1$, let $|\varphi_{AB}\rangle \in_R \mathcal{P}(A \otimes B)$, where $\dim B = s$. Writing $\varphi_A = \sum_i \lambda_i |e_i\rangle \langle e_i|$ in its eigenbasis, averages over $\varphi_A$ can be replaced by averaging over the independent random variables $\{\lambda_i\}$ and $\{|e_i\rangle\}$. We can then use the convexity of the exponential function to develop an inequality of moment generating functions. If $t \geq 0$, then

\[
\mathbb{E}_{\varphi_{AB}} \exp[ts \operatorname{Tr} \varphi_{AB}(Q \otimes \mathbb{1}_B)] = \mathbb{E}_{\varphi_A} \exp[ts \operatorname{Tr} \varphi_A Q]
\]

\[
= \mathbb{E}_{\{|e_i\rangle\}} \mathbb{E}_{\{\lambda_j\}} \exp \left[ ts \operatorname{Tr} \left( \sum_i \lambda_i |e_i\rangle \langle e_i|Q \right) \right]
\]

\[
\geq \mathbb{E}_{\{|e_i\rangle\}} \exp \left[ ts \operatorname{Tr} \left( \sum_i \mathbb{E}_{\{\lambda_j\}} \lambda_i |e_i\rangle \langle e_i|Q \right) \right]
\]

\[
= \mathbb{E}_{\{|e_i\rangle\}} \exp \left[ t \operatorname{Tr} \left( \sum_i |e_i\rangle \langle e_i|Q \right) \right]
\]

\[
= \mathbb{E}_S \exp[t \operatorname{Tr} PQ].
\]

Here we have used that $\mathbb{E}_{\{\lambda_j\}} \lambda_i = 1/s$, which follows from the permutation invariance of the eigenvalue distribution. Recall next from the proof of Lemma II.3 in [22] that the inequalities (2) and (3) for $s = 1$ themselves come from exploiting the moment generating function, in particular, applying the general upper bound

\[
\Pr \{ R > a \} \leq \mathbb{E}_R \exp(tR) \exp(-ta)
\]

for a random variable $R$ and $t \geq 0$. Since the left hand side of Eq. (4) is the moment generating function when $s = 1$ for the larger system $AB$ with a projector of rank $qs$, up to normalization,
and Eq. 5 the moment generating function for \( \text{Tr } PQ \), the inequality reduces the proof for \( s > 1 \) to the \( s = 1 \) case. \( \square \)

The second tool is the existence of “small” fine nets in state space, Lemma II.4 of [22].

**Lemma III.6 (Existence of small nets)** For \( 0 < \epsilon < 1 \) and \( \dim \mathcal{H} = d \) there exists a set \( \mathcal{N} \) of pure states in \( \mathcal{H} \) with \( |\mathcal{N}| \leq (5/\epsilon)^{2d} \), such that for every pure state \( |\varphi\rangle \in \mathcal{H} \) there exists \( |\tilde{\varphi}\rangle \in \mathcal{N} \) with \( \|\varphi - |\tilde{\varphi}\rangle\|_2 \leq \epsilon/2 \) and \( \|\varphi - |\tilde{\varphi}\rangle\|_1 \leq \epsilon \). (We call such a set an \( \epsilon \)-net.) \( \square \)

The following is a useful generalization of Lemma III.6 to bipartite pure states with bounded Schmidt rank.

**Lemma III.7** For \( 0 < \epsilon < 1 \), the set of pure states of Schmidt rank \( k \) in \( A \otimes B \) (with dimensions \( d_A \) and \( d_B \)) has an \( \epsilon \)-net \( \mathcal{N} \) of size \( |\mathcal{N}| \leq (10/\epsilon)^{2k(d_A + d_B)} \).

**Proof** For any Schmidt rank \( k \) state \( |\varphi\rangle \), there exists \( U \in \mathbb{U}(d) \) such that \( |\varphi_u\rangle := (\mathbb{1} \otimes U)|\varphi\rangle \in A \otimes \mathbb{C}^k \). Consider an \( \epsilon/4 \)-net for the Hilbert space norm \( \| \cdot \|_2 \) on \( A \otimes \mathbb{C}^k \), and let \( |\tilde{\varphi}_u\rangle \) be a net point for \( |\varphi_u\rangle \), with Schmidt decomposition \( |\tilde{\varphi}_u\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle \). Then, \( (\mathbb{1} \otimes U^\dagger)|\tilde{\varphi}_u\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes (U^\dagger |f_i\rangle) \) is within a distance \( \epsilon/4 \) of \( |\varphi\rangle \) in \( \| \cdot \|_2 \). Now, take an \( \epsilon/4 \)-net for \( \| \cdot \|_2 \) on \( B \) and let \( | f_i \rangle \) be the net point for \( U^\dagger | f_i \rangle \). It is straightforward to check that \( |\tilde{\varphi} := \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |\tilde{f}_i\rangle \) is within \( \epsilon/4 \) of \( (\mathbb{1} \otimes U^\dagger)|\tilde{\varphi}\rangle \) in \( \| \cdot \|_2 \). By the triangle inequality, \( |\tilde{\varphi}\rangle \) is within \( \epsilon/2 \) of \( |\varphi\rangle \) in \( \| \cdot \|_2 \) and within \( \epsilon \) in the trace norm \( \| \cdot \|_1 \). Altogether, the total number of net points is \( (10/\epsilon)^{2k(d_A + d_B)} \), as claimed. \( \square \)

We end the section with another Lipschitz constant estimate that will be useful when studying the separability of random states.

**Lemma III.8** Let \( |\varphi\rangle \in A \otimes B. \) Then the Lipschitz constant of the function \( f(\varphi) = \sqrt{\text{Tr } \varphi_A} \) is upper bounded by \( 2 \).

**Proof** Choose a basis \( \{|e_i\rangle |f_j\rangle\} \) of \( A \otimes B \), and let \( |\varphi\rangle = \sum_{ij} \varphi_{ij} |e_i\rangle |f_j\rangle \). Since \( \text{Tr } \varphi_A^2 \) is nonincreasing under dephasing,

\[
\begin{align*}
f(\varphi)^2 &= \text{Tr } \rho^2 \geq \sum_i (e_i |\varphi_A| e_i)^2 = \sum_i \left( \sum_j |\varphi_{ij}|^2 \right)^2 =: \tilde{f}(\varphi)^2
\end{align*}
\]

with equality if \( \{|e_i\rangle\} \) are the eigenvectors of \( \varphi_A \). The Lipschitz constant of \( \tilde{f}(\varphi) \) is easily seen to be bounded by \( 2 \): simply calculate the length of the gradient and use standard inequalities. We now apply the trick that proves Lemma III.2. For any two pure states \( \varphi \) and \( \psi \) with \( \tilde{f}(\varphi) \geq f(\psi) \), choose \( \{|e_i\rangle\} \) to be the eigenbasis of \( \varphi \). Then, \( \tilde{f}(\varphi) = f(\varphi), \tilde{f}(\psi) \leq f(\psi) \), and

\[
f(\varphi) - f(\psi) \leq \tilde{f}(\varphi) - \tilde{f}(\psi) \leq 2\|\varphi - |\psi\rangle \|_2.
\]

\( \square \)

The following sections will demonstrate the power of the above basic concentration statements and net construction for the understanding of generic entanglement.

**IV. Maximally Entangled Subspaces**

In this section, we put together the insights from the previous section to show, that a large subspace of appropriate dimension, chosen at random, will with high probability contain only
near-maximally entangled states. The relationship between concentration of measure and statements about large subspaces play an important role in [35]. The reader is also encouraged to compare our result with Theorem 3.19 of [32].

**Theorem IV.1 (Entangled subspaces)** Let $A$ and $B$ be quantum systems of dimension $d_A$ and $d_B$, respectively, for $d_B \geq d_A \geq 3$. Let $0 < \alpha < \log d_A$. Then there exists a subspace $S \subset A \otimes B$ of dimension

$$s = \left\lfloor d_A d_B - \frac{\Gamma \alpha^{2.5}}{\ln d_A^{2.5}} \right\rfloor$$

such that all states $|\varphi\rangle \in S$ have entanglement at least

$$E(\varphi) = S(\varphi_A) \geq \log d_A - \alpha - \beta,$$

where $\beta = \frac{1}{\ln \frac{d_A}{d_B}}$ is as in Lemma III.6 and $\Gamma$ is an absolute constant which may be chosen to be $1/1753$. In fact, the probability that a random subspace of dimension $s$ will not have this property is bounded above by

$$\left( \frac{15 \log d_A}{\alpha} \right)^{2s} \exp \left(-\frac{(d_A d_B - 1) \alpha^2}{32 \pi^2 \ln 2 \log d_A^2} \right).$$

**Proof** Let $S$ be a random subspace of $A \otimes B$ of dimension $s$. Let $\mathcal{N}_S$ be an $\epsilon$-net for states on $S$, for $\epsilon = \alpha/(\sqrt{8 \log d_A})$. In fact, since we may think of $S$ as $US_0$, with a fixed subspace $S_0$ and a Haar-distributed unitary $U$, we can fix the net $\mathcal{N}_S$ on $S_0$ and let $\mathcal{N}_S = U \mathcal{N}_S$, where $\mathcal{N}_S$ is chosen using Lemma II.6. Given $|\varphi\rangle \in S$, we can choose $|\tilde{\varphi}\rangle \in \mathcal{N}_S$ such that $\| |\varphi\rangle - |\tilde{\varphi}\rangle \| \leq \epsilon/2$. By the Lipschitz estimate, Lemma II.2, this implies that $|S(\varphi_A) - S(\tilde{\varphi}_A)| \leq \alpha/2$. We can then estimate

$$\Pr \left\{ \inf_{|\varphi\rangle \in S} S(\varphi_A) < \log d_A - \alpha - \beta \right\} \leq \Pr \left\{ \min_{|\tilde{\varphi}\rangle \in \mathcal{N}_S} S(\tilde{\varphi}_A) < \log d_A - \alpha/2 - \beta \right\} \leq |\mathcal{N}_S| \Pr \left\{ S(\varphi_A) < \log d_A - \alpha/2 - \beta \right\} \leq \left( \frac{15 \log d_A}{\alpha} \right)^{2s} \exp \left(-\frac{(d_A d_B - 1) \alpha^2}{32 \pi^2 \ln 2 \log d_A^2} \right).$$

This proves the upper bound on the probability that the randomly selected subspace $S$ will not satisfy the large entanglement requirement. If this is smaller than 1, a subspace with the stated properties exists; this can be secured by requiring

$$s < \frac{(d_A d_B - 1) \alpha^2}{438 \log d_A^2 \log (15 \log d_A / \alpha)}.$$ 

A less tight but simpler expression can be obtained. By using $\log x \leq \sqrt{\frac{16}{15}x}$ for $x \geq 15$, we get

$$\log(15 \log d_A / \alpha) \leq 4 \sqrt{\frac{\log d_A}{\alpha}},$$

because $\alpha < \log d_A$. Now, if we are to replace the denominator in Eq. (11) by this new expression and still aim to make a non-vacuous statement (i.e. that $s \geq 2$), then, keeping in mind that $\alpha < \log d_A$, we find that $d_A d_B \geq 3505$, so that $d_A d_B - 1 \geq \frac{3504}{3505} d_A d_B$, leading to Eq. (7).

If $\alpha + \beta$ is small, we can obtain another useful characterization of all the states in $S$: that they are all close to maximally entangled states.

**Corollary IV.2** Every pure state $|\varphi\rangle \in S$ constructed in Theorem IV.1 is close to a maximally entangled state $|\Phi\rangle \in A \otimes B$:

$$F(\varphi, \Phi) \geq 1 - \sqrt{2(\alpha + \beta)}, \quad \| \varphi - \Phi \|_1 \leq \sqrt{16(\alpha + \beta)}.$$
Proof The relative entropy between $\varphi_A$ and the maximally mixed state is given by

$$D\left(\varphi_A \parallel \frac{1}{d_A}1_{d_A}\right) = \log d_A - E(\varphi) \leq \alpha + \beta.$$ 

Hence, by Pinsker’s inequality (see [37, 43]), $\|\varphi_A - \frac{1}{d_A}1_{d_A}\|_1 \leq \sqrt{2(\alpha + \beta)}$. Using a well-known relation between trace distance and fidelity [16], we obtain $D(\varphi_A, \frac{1}{d_A}1_{d_A}) \leq \alpha + \beta$. By Uhlmann’s Theorem [29, 48] this means that $\varphi$ is indeed close to a purification of the maximally mixed state, i.e., a maximally entangled state: there exists a maximally entangled state $\Phi$ such that $F(\varphi, \Phi) \geq 1 - \sqrt{2(\alpha + \beta)}$, and hence, invoking [16] once more, $\|\varphi - \Phi\|_1 \leq \sqrt{16(\alpha + \beta)}$. □

V. CORRELATION MEASURES FOR RANDOM STATES

In this section, we consider correlation properties of rank-$s$ random states with distributions induced by partial tracing (see Definition II.3). Our study was motivated by some surprising properties of the maximally mixed states on the random subspaces discussed in the previous section. Since the spectrum of a rank-$s$ random state is likely to be almost flat, the two types of mixed states are very similar asymptotically, at least for the purposes of our investigation. Thus, after a full discussion on the asymptotic correlation properties of rank-$s$ random states, we derive, as corollaries, asymptotic correlation properties of maximally mixed states on random subspaces.

A. Some measures of correlation for quantum states

Consider interconversions between copies of some state $\sigma_{AB}$ and EPR pairs by local operations and (two-way) classical communications (LOCC) in the limit of many copies. The number of EPR pairs needed per copy of $\sigma_{AB}$ created is defined to be the entanglement cost [21],

$$E_c(\sigma_{AB}) = \lim_{n \to \infty} \frac{1}{n} E_f(\sigma_{AB}^\otimes n),$$

where

$$E_f(\sigma_{AB}) = \min_{\sum p_i |\varphi_i| = \sigma_{AB}} \sum_i p_i S(\varphi_i^A)$$

is the entanglement of formation [3]. $\sigma_{AB}$ is said to be separable if $E_f(\sigma_{AB}) = 0$. It is proved in [17] that any $d$-dimensional state $\sigma$ is separable if

$$\text{Tr} \sigma^2 \leq 1/(d - 1).$$

(13)

The number of EPR pairs that can be extracted per copy of $\sigma_{AB}$ is given by the entanglement of distillation, $E_d$ [3]. One can also quantify the amount of secret key $K(\sigma)$ distillable against an eavesdropper holding the purification of the state (see [8, 9, 11] and references therein), and the distillable “common randomness” $\text{CR}(\sigma)$ [2, 10] (discounting at the end of the protocol the amount of communication used). When the communication is restricted to one direction, say, from $A$ to $B$, we can define the corresponding distillable correlations $E_d^\rightarrow(\sigma), K^\rightarrow(\sigma)$, and $\text{CR}^\rightarrow(\sigma)$. In particular, it is proved in [10] that

$$\text{CR}^\rightarrow(\sigma) = \lim_{n \to \infty} \frac{1}{n} I^\rightarrow(\sigma^\otimes n),$$

(14)

where $I^\rightarrow(\sigma)$ is the maximum Holevo quantity [24] of the reduced ensemble of states in $B$ induced by a local measurement in $A$, if the initial state is $\sigma$. One can also formally define quantum mutual information

$$S(A : B)_\sigma = S(\sigma_A) + S(\sigma_B) - S(\sigma_{AB}),$$

(15)
by analogy to Shannon’s classical quantity. The various measures are related by many known inequalities:

\[
E_d(\sigma) \leq E_c(\sigma) \leq E_f(\sigma),
\]

\[
E_d(\sigma) \leq \frac{1}{2} S(A : B)_{\sigma},
\]

\[
E_d(\sigma) \leq K(\sigma) \leq CR(\sigma) \leq S(A : B)_{\sigma},
\]

\[
E_d^*(\sigma) \leq K^*(\sigma) \leq CR^*(\sigma),
\]

as well as the trivial bounds \(E_d^*(\sigma) \leq E_d(\sigma)\) etc. Most of these inequalities follow directly from the operational definitions. Equation (17) was proved in [6], exploiting the fact that the right hand side is an upper bound on the “squashed entanglement.” The rightmost inequality in Eq. (18) can easily be proved by generalizing the classical case [2].

As alluded to earlier, our investigation of correlation for random states was motivated by thinking about states on a maximally entangled subspace \(S\) produced by Theorem IV.1. Any state on \(S\), pure or mixed, has entanglement of formation at least \(\log d_A - \alpha - \beta\). Meanwhile, for the maximally mixed state \(\rho_{AB} = \int_S |\varphi\rangle \langle \varphi| \ d\varphi\) on \(S\), \(S(\rho_{AB}) = \log s\) can be very high. Taking \(\alpha \leq 1\) and \(s\) equal to the value given by Eq. (7) leads to a strong upper bound on the mutual information:

\[
S(A : B)_{\rho} \leq 2.5 \log \log d_A - \log (\Gamma^{2.5}) + 1.
\]

It follows from Eqs. (17–19) that all of \(E_d(\rho), K(\rho), CR(\rho)\) are small. In particular, \(E_f(\rho) \gg E_d(\rho)\), so that either \(E_f(\rho) \gg E_c(\rho)\) or \(E_c(\rho) \gg E_d(\rho)\). In the first case, \(\rho_{AB}\) is a (rather drastic) counterexample to the additivity conjecture for the entanglement of formation: \(\forall \sigma \ E_f(\sigma_{AB})^{2n} = nE_f(\sigma_{AB})\) (see, for example, [45]). In the second case, the preparation of \(\rho_{AB}\) is near-maximally irreversible, making it a kind of entanglement black hole; preparing \(\rho_{AB}\) requires nearly as much entanglement as the most highly entangled state even though no useful entanglement can be extracted from it. Moreover, for most values of \(s\), the gap \(E_f(\rho) \gg E_d(\rho)\) is generic.

Each of these states \(\rho_{AB}\) also provides an example of a quantum state that is more “entangled” than it is “correlated”— a hitherto unseen effect. At the very least, this reveals that “dividing” the correlations of a quantum state into entanglement and classical parts is problematic, since here we find a measure of entanglement that can exceed the combined quantum and classical correlations. It is interesting to note, however, that if one replaces the entanglement of formation by operational measures related to the entanglement of distillation, this decomposition of correlation into quantum and classical parts becomes possible, as demonstrated in [38].

**B. Analysis of correlation measures for rank-3 random states**

Throughout this section we select random states according to the prescription \(\rho \in_R S_s(A \otimes B)\), with \(d_B \geq d_A \geq 3\). Since the statements of the theorems are rather technical, we will begin by sketching a rough outline of the results to come. We will frequently need to make statements conditioned on the additivity conjecture for \(E_f\) or, equivalently, \(I^\rightarrow\) (see [51] for the equivalence). To simplify the discussion, from now on we will indicate that a statement is true conditioned on the conjecture by marking it with the symbol \(*\), either as a superscript at the end of a sentence or above a mathematical symbol: \(\leq^*\), for example. As a start, we confirm the gap between \(E_f(\rho)\) and \(S(A : B)_{\rho}\) that was discussed earlier for random maximally mixed states. Our findings are summarized in the Table [1].
Properties of high-rank random states:

\[ \epsilon > 0, \quad d_A = d, \quad d_B = d \log d, \quad s = d^2/(\log d)^6 \]

| Correlation Measure | Value | Likelihood |
|---------------------|-------|------------|
| \( E_f \) (\( \geq E_c \)) | \( \geq \log d - \epsilon \) | \( d^2 \)-likely |
| \( S(A : B) \) (\( \geq CR \geq K, \geq 2E_d \)) | \( \leq 7 \log \log d \) | \( d^4 \)-likely |
| \( I^\to \) (\( \geq CR \to K \to E_d^\to \)) | \( \leq \epsilon \) | \( d^2 \)-likely |

TABLE I: Properties of high-rank random states. For the specific choice of parameters made here, the gap between the entanglement of formation and the measures of distillable correlation is basically as large as is consistent with the entropy scale of the system.

In fact, we are able to determine much more. Other than when \( s \) is almost exactly equal to \( d_Ad_B \), we can compute excellent approximations to both \( E_f \) and \( I^\to \). Assuming the additivity conjecture, that is sufficient to calculate the entanglement cost and one-way distillable entanglement of rank-\( s \) random quantum states. Figure 1 illustrates the situation when \( d_B = d_A = d \) becomes large; it plots the normalized entanglement of formation, squashed entanglement and coherent information (see [9, 11] for the relevant lower bound) against the normalized entropy, which is essentially the rank, of the likely random states from \( S_s(A \otimes B) \).

FIG. 1: Illustration of the asymptotic (\( d \to \infty \)) behavior of entanglement \( E \) versus rank \( s \) of random states in \( \mathbb{C}^d \otimes \mathbb{C}^d \), with all quantities normalized over \( \log d \), the entropy scale of the system. The solid line is the entanglement of formation, dropping sharply from 1 to 0 at the threshold \( \frac{\log s}{\log d} \sim 2 \). The dotted line is the upper bound on distillable entanglement from Theorem [V.1] and Eq. (17), and the circled line is a lower bound on the one-way distillability via the hashing inequality: \( E_d^\to (\rho_{AB}) \geq S(\rho_B) - S(\rho_{AB}) \) [9, 11]. Finally, the dashed line is the one-way distillable common randomness* from Theorem [V.2]. Hence, the dashed line also represents the one-way distillable entanglement.*
Theorem V.1 Let $\rho \in R S_s(A \otimes B)$, with $d_B \geq d_A \geq 3$, and $0 < \alpha < \log d_A$. Then:

1. If $s < d_A d_B (\log d_A)^{-2.5} \Gamma \alpha^{2.5}$, then it is $d_A d_B$-likely that $E_f(\rho_{AB}) \geq \log d_A - \alpha - \beta$.

The parameters $\alpha, \beta, \Gamma$ are the same as in Theorem IV.1.

2. If $s > d_A d_B (\log d_A)^{-2} (6 \log d_B - 4 \log \epsilon) \frac{14 \ln 2}{\epsilon^2}$, then it is $s$-likely that $E_f(\rho) \leq \epsilon$.

3. If $s > 6(d_A d_B)^2$, then it is $\frac{s}{(d_A d_B)^2}$-likely that $\rho$ is separable.

4. If $s < d_A d_B$, it is $s d_A d_B$-likely that $S(A : B)_\rho \leq \log d_A + \log d_B - \log s + \alpha + \beta_1$ for $\beta_1 = \frac{1}{\ln 2} \frac{d d_A}{d_A d_B}$.

If $s > d_A d_B$, it is $s d_A d_B$-likely that $S(A : B)_\rho \leq \alpha + \beta_2$ for $\beta_2 = \frac{1}{\ln 2} \frac{d d_A}{s}$.

5. $E_d(\rho), K^\rightarrow(\rho), CR^\rightarrow(\rho), E_d(\rho), K(\rho), CR(\rho)$ share the same upper bound as $S(A : B)_\rho$, due to Eqs. (17)–(19).

Bounds on the probabilities $P_E$ of the various exceptional sets are given in the proof.

Proof

1. By the uniqueness of the unitarily invariant measure on the Grassmannian, the support of $\rho$ is a random $s$-dimensional subspace $S$. Since $s$ satisfies the condition of Eq. (7) in Theorem IV.1, the claim follows from Eq. (12), with $P_E$ given in Eq. (9).

2. We apply Lemma III.4 to $(A \otimes B) \otimes \mathbb{C}^s$ and choose $\epsilon'$ so that $(1 - \epsilon') \frac{1}{d_A d_B} \leq \rho \leq (1 + \epsilon') \frac{1}{d_A d_B}$ is $s$-likely, in which case $\rho = (1 - \epsilon') \frac{1}{d_A d_B} + \epsilon' \rho'$, for some state $\rho'$.

If we choose $\epsilon' = \frac{\epsilon}{\log d_A}$, then by the convexity of $E_f$, $E_f(\rho) \leq \epsilon' E_f(\rho') \leq \epsilon$, and $P_E \leq 2(10 d_A d_B (\log d_A)^2/\epsilon^2)^2 d_A d_B \exp\left(-\frac{s}{(\log d_A)^2} \frac{\epsilon^2}{14 \ln 2}\right)$.

3. Setting $d = d_A d_B$, we shall bound $\text{Tr} \rho^2$ by $\frac{1}{d} + \frac{1}{d^2}$ and use Levy’s Lemma to estimate the probability that this occurs; then by Eq. (13) we are done. This requires Lemma III.8 for the upper bound of 2 on the Lipschitz constant of $f(\rho) = \sqrt{\text{Tr} \rho^2}$, and a result from [34] which says $\mathbb{E} \text{Tr} \rho^2 = \frac{d + s}{d + s + 1}$. Noting that $\text{Tr} \rho^2 \geq 1/d$, an argument as in the proof of Theorem III.3 then implies that the median $m(\text{Tr} \rho^2) \leq 1/d + 2/s$. Then, by the definition of the median, $m(f) = \sqrt{m(\text{Tr} \rho^2)} \leq \sqrt{1/d + 2/s} \leq 1/\sqrt{d} + \sqrt{d}/s$. Since, on the other hand, $\sqrt{1/d + 1/d^2} \geq \sqrt{1/d + 1/(3d^{3/2})}$, and, assuming $s \geq 6d^2$, Lemma III.1 yields

$$P_E \leq \Pr \left\{ f > \sqrt{\frac{1}{d} + \frac{1}{d^2}} \right\} \leq \Pr \left\{ f > m(f) + \frac{1}{6d^{3/2}} \right\} \leq \exp\left(-\frac{ds - 1}{493d^5}\right),$$

which is less than one.

4. Using $S(\rho_A) \leq \log d_A$ and $S(\rho_B) \leq \log d_B$, it will be sufficient to lower bound $S(\rho_{AB})$. If $s < d_A d_B$, apply Theorem III.3 to the bipartite system $\mathbb{C}^s \otimes A \otimes B$, $S(\rho) \geq \log s - \alpha - \beta_1$ with $P_E \leq \exp\left(-(sd_A d_B - 1)C_3 \alpha^2/\log(s)^2\right)$. If $s > d_A d_B$, the same theorem gives $S(\rho) \geq \log(d_A d_B) - \alpha - \beta_2$, this time with $P_E \leq \exp\left(-(sd_A d_B - 1)C_3 \alpha^2/\log(d_A d_B)^2\right)$. Note that the concentration effect is achieved via a large $d_A d_B$, and very little is required of $s$. 

Theorem V.2 Let $\rho \in R \mathcal{S}_s(A \otimes B)$ be a random state of rank $s$. Then, for $\epsilon \leq 1/3$ and $d_B \geq d_A$,
\[
\Pr \{ I^\rightarrow(\rho) > 5\epsilon \} \leq 2 \left( \frac{20d_B^2}{\epsilon} \right)^{4d_B} \exp \left( -\frac{5\epsilon^2}{17} \right) .
\]
Thus it is likely that $I^\rightarrow(\rho)$ is asymptotically vanishing as long as the rank $s$ of $\rho$ is sufficiently bigger than $d_B \log d_B$. Eqs. (14) and (19) therefore imply the same vanishing upper bound on $E_d^\rightarrow(\rho)$, $K^\rightarrow(\rho)$ and $CR^\rightarrow(\rho)$.

On the other hand, for $s/\epsilon \leq d_B \leq s\epsilon d_A$, we have
\[
\Pr \{ \left| I^\rightarrow(\rho) - (\log d_B - \log s) \right| > 2\epsilon \} \leq 2 \left( \frac{15 \log s}{\epsilon} \right)^{2d_A} \exp \left( -\frac{(sd_B - 1)\epsilon^2}{32\pi^2 \ln 2(\log s)^2} \right) .
\]
In other words, as long as the above constraints and $d_A < s d_B \frac{\Gamma(2.5)}{(\log s)^{2.5}}$ hold, it is $sd_B$-likely that $I^\rightarrow(\rho) = \log d_B - \log s \pm 2\epsilon$.

Proof We start with the explicit expression for $I^\rightarrow(\rho)$ proved in [10]:
\[
I^\rightarrow(\rho) := S(\rho_B) - \min_{\text{POVM on } A} \sum_i \text{Tr}(\rho_A M_i) S \left( \frac{\text{Tr}_A(\rho(M_i \otimes \mathbb{I}))}{\text{Tr}(\rho_A M_i)} \right) \tag{21}
\]
where the right hand side is the Holevo quantity [24] on Bob’s reduced ensemble of states labeled by the measurement outcome. Without loss of generality, all $M_i$ are of rank one. For the first part of the theorem, we will show that it is likely that for every rank-one projector $P$ acting on $A$, the corresponding projected state on $B$,
\[
\sigma = \frac{\text{Tr}_A(\rho(P \otimes \mathbb{I}))}{\text{Tr}(\rho(P \otimes \mathbb{I}))} \tag{22}
\]
is close to maximally mixed, so that for every POVM on $A$, the difference in Eq. (21) is small.
Since \( \rho = \text{Tr}_C \varphi \) for \( \varphi \in_R A \otimes B \otimes \mathbb{C}^s \), we have, for rank-one projectors \( P \) and \( Q \), \( \text{Tr}(\rho(P \otimes Q)) = \text{Tr}(\varphi(P \otimes Q \otimes 1)) \), so Lemma III.5 gives us

\[
\Pr \left\{ \left| \text{Tr}(\rho(P \otimes Q)) - \frac{1}{d_A d_B} \right| > \frac{\epsilon/2}{d_A d_B} \right\} \leq 2 \exp \left( -\frac{s\epsilon^2}{17} \right).
\]

(23)

Now, Lemma III.6 gives us \( \mathcal{T}_{d_A d_B} \)-nets for the pure states (rank one projectors) on \( A \) and \( B \), of cardinality \( (20d_A d_B/\epsilon)^{2d_A} \) and \( (20d_A d_B/\epsilon)^{2d_B} \), respectively. Hence, by the union bound and triangle inequality,

\[
\Pr \left\{ \exists P, Q \mid \left| \text{Tr}(\rho(P \otimes Q)) - \frac{1}{d_A d_B} \right| > \frac{\epsilon}{d_A d_B} \right\} \leq 2 \left( \frac{20d_A d_B}{\epsilon} \right)^{2(d_A + d_B)} \exp \left( -\frac{s\epsilon^2}{17} \right) .
\]

If this event does not occur, then for every rank-one projector \( P \),

\[
(1 - \epsilon) \frac{1}{d_B} \leq \text{Tr}_A(\rho(P \otimes 1)) \leq (1 + \epsilon) \frac{1}{d_B} ,
\]

and the post-measurement state \( \sigma \) as defined in Eq. (22) satisfies

\[
(1 - 3\epsilon) \frac{1}{d_B} \leq \sigma \leq (1 + 3\epsilon) \frac{1}{d_B} ,
\]

which in turn easily implies

\[
S(\sigma) \geq \log d_B - \frac{3\epsilon}{\ln 2} \geq \log d_B - 5\epsilon ,
\]

by the operator monotonicity of \( \log \). Putting this, using \( d_B \geq d_A \) and substituting \( S(\rho_B) \leq \log d_B \) into Eq. (21) completes the argument.

For the second statement, we will use an alternative argument based on the entangled subspaces of Theorem IV.1. To begin with, there exists a \( \varphi \in_R \mathcal{P}(A \otimes B \otimes \mathbb{C}^s) \) such that \( \rho = \text{Tr}_C \varphi \). Hence, Theorem III.3 informs us that

\[
\Pr \left\{ S(\rho_B) < \log d_B - \epsilon - \frac{1}{\ln 2} \frac{d_B}{sd_A} \right\} \leq \exp \left( -\frac{(sd_A d_B - 1)C_3 \epsilon^2}{(\log d_B)^2} \right).
\]

(24)

On the other hand, consider the post-measurement state \( \sigma \) on \( B \) as in Eq. (22) — it can clearly be written as the corresponding post-measurement (pure) state \( \psi \) on \( B \otimes \mathbb{C}^s \), reduced to \( B \); \( \sigma = \psi_B \), and \( S(\sigma) = E(\psi) \). But \( |\psi\rangle \) lies in the supporting subspace of \( \text{Tr}_A \varphi \), which is a random subspace of dimension \( d_A \) in \( B \otimes \mathbb{C}^s \). Hence we can apply Theorem IV.1 telling us

\[
\Pr \left\{ \exists \text{post-meas. state}, \; S(\sigma) < \log s - \epsilon - \frac{1}{\ln 2} \frac{s}{sd_B} \right\} \leq \left( \frac{15 \log s}{\epsilon} \right)^{2d_A} \exp \left( -\frac{(sd_B - 1)\epsilon^2}{32\pi^2 \ln 2 (\log s)^2} \right).
\]

Since this dominates the bound in Eq. (24), we will be done if we just insert our entropy bounds, \( \log d_B - \epsilon - \frac{1}{\ln 2} \frac{d_B}{sd_A} \leq S(\rho_B) \leq \log d_B \) and \( \log s - \epsilon - \frac{1}{\ln 2} \frac{s}{sd_B} \leq S(\sigma) \leq \log s \), into Eq. (21) and respect the dimension constraints we inherit.

We finish this subsection by considering a more qualitative aspect of entanglement of a state \( \rho \) on \( A \otimes B \), one-copy (pseudo)-distillability, meaning that there exist two-dimensional projectors \( P \) and \( Q \) on \( A \) and \( B \) respectively such that

\[
\sigma = \frac{(P \otimes Q)\rho(P \otimes Q)}{\text{Tr}(P \otimes Q)\rho}
\]

(25)
has partial transpose that is not positive semidefinite (NPT). The motivation is that in this case \( \sigma \) is effectively a two-qubit state, and \( \sigma \) is distillable if it is NPT, and separable if it is not (PPT). Furthermore, \( \rho \) is distillable if and only if \( \rho \otimes n \) is one-copy distillable for some \( n \).

**Theorem V.3** Let \( \rho \in \mathcal{S}_s(A \otimes B) \) be a random state of rank \( s \) with \( d_B \geq d_A \). Then,

\[
\Pr \{ \rho \text{ one-copy distillable} \} \leq 2(10d_B)^{16d_B} \exp \left( -s/600 \ln 2 \right)
\]

In particular, once \( s > 7000d_B \log(10d_B) \), \( \rho \) is likely to be one-copy undistillable.

**Proof** We will show that \( \forall P \otimes Q, \sigma \) in Eq. (25) is likely to be separable, using a characterization from [12] that \( \sigma \) is separable if \( \| \sigma - \frac{1}{d_A} \|_\infty \leq \frac{1}{8} \).

To show the above, fix any Schmidt-rank two state \( |\psi\rangle \) on \( A \otimes B \). Since \( \text{Tr}(\rho \psi) = \text{Tr}_{C^*}(\varphi)(\rho \otimes 1) \) for \( \varphi \in \mathcal{P}(A \otimes B \otimes \mathbb{C}^*) \), Lemma III.5 yields

\[
\Pr \left\{ \left| \text{Tr} \frac{\rho \psi - \frac{1}{d_A} \rho}{d_A d_B} \right| > \frac{\delta}{d_A d_B} \right\} \leq 2 \exp \left( -s \frac{\delta^2}{6 \ln 2} \right).
\]

By the triangle inequality and the union bound over a \( \frac{\delta}{d_A d_B} \)-net \( \mathcal{N} \) for Schmidt-rank two states in \( A \otimes B \) (Lemma III.7), with \( |\mathcal{N}| \leq (10d_A d_B / \delta)^{4(d_A + d_B)} \), we obtain, putting \( \delta = 1/10 \) and using \( d_B \geq d_A \),

\[
\Pr \left\{ \exists \psi \text{ of Schmidt-rank 2} \left| \text{Tr} \frac{\rho \psi - \frac{1}{d_A d_B}}{d_A d_B} \right| > \frac{2\delta}{d_A d_B} \right\} \leq 2 \left( \frac{10d_B^2}{\delta} \right)^{8d_B} \exp \left( -s \frac{\delta^2}{6 \ln 2} \right).
\]

If for all Schmidt-rank two states \( \varphi \), \( \left| \text{Tr} \frac{\rho \psi - \frac{1}{d_A d_B}}{d_A d_B} \right| \leq \frac{2\delta}{d_A d_B} \), then for all rank two projectors \( P, Q \) and for any state \( |\xi\rangle \) in the support of \( P \otimes Q \),

\[
\frac{1 - 2\delta}{d_A d_B} \langle \xi | (P \otimes Q) | \xi \rangle \leq \langle \xi | (P \otimes Q) \rho (P \otimes Q) | \xi \rangle \leq \frac{1 + 2\delta}{d_A d_B} \langle \xi | (P \otimes Q) | \xi \rangle,
\]

and therefore

\[
\frac{1 - 2\delta}{d_A d_B} P \otimes Q \leq (P \otimes Q) \rho (P \otimes Q) \leq \frac{1 + 2\delta}{d_A d_B} P \otimes Q.
\]

and \( \frac{4(1 - 2\delta)}{d_A d_B} \leq \text{Tr}((P \otimes Q) \rho) \leq \frac{4(1 + 2\delta)}{d_A d_B} \). Thus, for \( \sigma \) defined in Eq. (25)

\[
1 - \frac{2\delta}{1 + 2\delta} \cdot \frac{1}{4} P \otimes Q \leq \sigma \leq \frac{1 + 2\delta}{1 - 2\delta} \cdot \frac{1}{4} P \otimes Q.
\]

The choice \( \delta \leq 1/10 \) will secure that \( \| \sigma - \frac{1}{d_A} P \otimes Q \|_\infty \leq 1/8 \) and we are done.

**Remark** We began our study of correlation by considering the entanglement of formation and mutual information for the maximally mixed state \( \rho_{AB} \) on a random subspace \( S \) of dimension \( s \), before moving on to study mixed states with the measure induced by tracing over part of a random pure state. To end, we note that both Theorems V.2 and V.3 apply unaltered if \( \rho_{AB} \) is chosen as a random maximally mixed state instead of according to \( \mathcal{S}_s(A \otimes B) \). Not even the proofs need to change: the crucial applications of Lemma III.5 in Eqs. (25) and (26) give exactly the same estimates for the new distribution.
VI. MULTIPARTY ENTANGLEMENT

With a little more work, and building upon the results obtained so far, we can learn a good deal about the entanglement properties of generic random multipartite states. To that end, let \( \varphi \in_{R} \mathcal{P}((\mathbb{C}^{d})^n) \); we could easily allow for different local dimensions but that would only result in more cumbersome notation. Also, let us label the \( n \) subsystems by numbers \( 1, \ldots, n \). A subset of the parties is given the name \( X \subset \{1, \ldots, n\} \), and its complement \( \overline{X} = \{1, \ldots, n\} \setminus X \). Each \( X \) thus defines a bipartite cut, and we will freely call the cut \( X \) as well. Let \( \varphi_{X} = \text{Tr}_{\overline{X}} \varphi \) denote the state reduced to the systems in \( X \).

The questions we address here are the following:

1. Entropy of entanglement across any bipartite cut.
2. Entanglement of formation and separability of reduced states on an arbitrary set of \( k < n \) parties.
3. Distillability of maximal entanglement between arbitrary pairs of parties by LOCC between all parties.

For the first two questions, either the local dimension \( d \) or the number of parties \( n \) can be treated as the asymptotic parameter; the important thing, in fact, is that the combination \( d^n \) become large.

For the distillability question, however, our results will only be valid for large \( d \).

**Corollary VI.1** Let \( \varphi \in_{R} \mathcal{P}((\mathbb{C}^{d})^n) \) be a random state, and \( \alpha > 0 \). Then

\[
\Pr\{ \exists X \ E(\varphi_{X,\overline{X}}) = S(\varphi_{X}) < x \log d - \alpha - \beta_{X} \} \leq 2^{n-1} \exp\left(-\frac{(d^n - 1)C_{3}\alpha^2}{n^2(\log d)^2}\right),
\]

(27)

where \( x = \min(|X|, |\overline{X}|) \), \( C_{3} = (8\pi \ln 2)^{-1} \) is the same as in Theorem III.3 and \( \beta_{X} = \frac{1}{\ln 2} d^{2x-n} \).

In other words, it is \( d^n \)-likely that \( \varphi \) is highly entangled across any bipartite cut and almost maximally entangled across any cut such that \( |X| \neq n/2 \).

**Proof** This follows immediately from Theorem III.3 and the union bound on all \( 2^{n-1} \) cuts \( X \) with \( x \leq n/2 \). The parameter \( \beta_{X} \) is just \( \beta \) in Theorem III.3 with the proper dimensions.

Note that we cover the case \( d = 2 \), too, since there the Lipschitz constant can be bounded by \( \sqrt{8\log 3} \) and in Eq. (27) we have substituted the much larger \( \sqrt{n \log d} \). \( \Box \)

**Corollary VI.2** Let \( \varphi \in_{R} \mathcal{P}((\mathbb{C}^{d})^n) \) be a random state, and consider arbitrary \( X \subset \{1, \ldots, n\} \) of cardinality \( x \) and arbitrary cuts within \( X \) into disjoint subsets, \( X_1, X_2 \) of sizes \( x_1 \leq x_2 \). Then there exist absolute numerical constants \( M_{1} \) and \( M_{2} \) such that

1. If \( \frac{x}{n/2} + 1 - M_{1} \frac{\log d}{\log \alpha} \log \frac{n \log d}{\alpha} > 0 \), \( \alpha < 1 \) and \( \beta = \frac{1}{2\ln 2} d^{x_1-x_2} \), it is \( d^{\epsilon} \)-likely that for all \( X_1, \)
   \[ E_{f}(\varphi_{X}) \geq x_1 \log d - \alpha - \beta. \]

2. If \( \frac{x}{n/2} - M_{2} \frac{\log d}{\log \epsilon} > 0 \) and \( \epsilon > 0 \), it is \( d^{n-x} \)-likely that for all \( X_1, \)
   \[ E_{f}(\varphi_{X}) \leq \epsilon. \]

3. If \( \frac{x}{n/3} - \frac{1}{\log d} \), it is \( d^{n-3x} \)-likely that \( \varphi_{X} \) is separable (as a multiparty state of \( x \) parties).
Proof For each $X$, the claims are simply Parts 1-3 of Theorem VI.1 applied to $\rho$ with total dimension $d^x$ and with rank $d^{n-x}$. The worst case $X$ is taken care of by a union bound over all $X$ and all possible cuts, of which there are at most $3^n$ in total.

Note that our proof of Part 3 in Theorem VI.1 actually shows separability for every decomposition of the system into arbitrary subsystems, because it uses only the bound on the purity and the result of [17] to that effect.

Remark Observe that the thresholds for the group sizes become, for fixed $n$ and $d \to \infty$, $n/2$, $n/2$ and $n/3$. The findings of Corollary VI.2 should be compared to numerical investigations reported in [30]; there the threshold $n/3$ was argued heuristically based on the knowledge of the expectation of $\text{Tr} \rho^2$ and the postulate that it would exhibit measure concentration. Interestingly, the numerical studies indicate that the reduced state already becomes PPT at $x \sim n/2$.

Corollary VI.3 With $n$ fixed, consider the following one-shot protocol for distilling entanglement between an arbitrary pair chosen from among $n d$-dimensional systems:

Let $\{|e_j\rangle\}$ be an agreed-upon local basis for each party, and let $X$ denote the chosen pair. Each party $i \in X$ measures in his local basis $\{|e_j\rangle\}$ and sends the result $j_i$ to $X$. Let $\varphi_{X,J}$ denote the resulting pure state in $X$, where $J$ is one of the $d^{n-2}$ possible measurement outcomes.

Then it is likely that $\forall X,J$, $E(\varphi_{X,J}) \geq \log d - \frac{1}{n^2} - \alpha$. In other words, there is one protocol which allows any pair of parties to distill almost $\log d$ ebits between them.

Proof Note that $\varphi_{X,J} \in R P(X)$. The claim then follows from Theorem III.3 and the union bound.

Remark The yield of the above distillation protocol is a nearly maximally entangled state between the members of the pair. This feature of generic multiparty entanglement is also shared by the cluster state of Briegel and Raussendorf [5]: in the language of their paper, random multiparty pure states are likely to have maximally persistent entanglement and to be maximally connected, modulo the fact that the state distilled in Corollary VI.3 is not exactly a maximally entangled state.

Along the same lines, when $n$ is fixed and $d$ is large, the protocol presented here can be used to distill arbitrary pairwise entanglement, which in turn allows any arbitrary pure state between the $n$ parties to be prepared. The efficiency, however, could be very poor.

It is clear that there are innumerable other entanglement parameters one could investigate for $\varphi \in R P((C^d)^{\otimes n})$. The question of identifying the maximal yield for states other than bipartite maximally entangled states seems to be particularly interesting given the difficulty inherent in studying such questions for non-generic states.

VII. DERANDOMIZATION OF SUPERDENSE CODING

Superdense coding of quantum states was introduced in [20]; there it was shown that, in the large-dimensional asymptotics, the state of two qubits can be communicated exactly with high probability using one ebit of entanglement and one transmitted qubit provided the sender has full knowledge of the communicated state. (This is known as the visible scenario.) However, the protocol in [20] also requires one shared bit of randomness per two qubits communicated.

Theorem VI.1 suggests an alternative protocol that does not require shared randomness: Let the sender and receiver possess systems $B$ and $A$ initially. (Note that this convention is opposite to common usage, but has the advantage that $d_B \geq d_A$ in accord with the rest of this paper.) Let
\( \alpha, \beta, \text{ and } \Gamma \) be as defined in Theorem IV.1, \( d_B = d_A (\log d_A)^{2.5} \Gamma^{-1} \alpha^{-2.5} \), so that a subspace \( S \) as described in Corollary IV.2 can be chosen with \( s = d_A^2 \). Here, \( \beta < \alpha \), so that for every \( |\varphi\rangle \in S \) there exists a maximally entangled state \( |\Phi\rangle \in A \otimes B \) with \( |\langle \varphi | \Phi \rangle|^2 \geq 1 - 2\sqrt{\alpha} \). Starting from a fixed maximally entangled state \( \Phi_0 \) on \( A \otimes B \), the sender can prepare any quantum state \( |\varphi\rangle \in S \) of \( \log s = 2 \log d_A \) qubits in the receiver’s laboratory by applying a unitary transformation \( U \) to \( B \) such that \( |\Phi\rangle = (\mathbb{1} \otimes U)|\Phi_0\rangle \) and sends his system to the receiver, who projects the state into the subspace \( S \) (and substitutes an arbitrary state if the projection fails). It is evident that this protocol achieves what we aimed for.

**Theorem VII.1** Asymptotically, \( 2 \log d_A \) qubits can be communicated visibly by using \( \log d_A \) ebits and \( \log d_B = \log d_A + 2.5 \log \log d_A - \log(\Gamma \alpha^{2.5}) \) qubits of communication. The fidelity is \( \geq 1 - 2\sqrt{\alpha} \). \( \square \)

Note a technical point, however: we pay a certain price for not having to spend shared randomness. The protocol of [20] produces an exact copy of the target state when it succeeds, which occurs with high probability. The protocol we propose here always succeeds, but is not guaranteed to be exact. While the distinction is unimportant in practice because the fidelity in our protocol can be made arbitrarily high, the probabilistic-exact formulation is nonetheless the stronger criterion from a theoretical point of view. We do not know if the small sacrifice of fidelity is essential for the derandomization or if, instead, a derandomized probabilistic-exact protocol exists.

**VIII. DISCUSSION**

We have seen that exponentially tight measure concentration, along with careful attention to the achievable exponents, leads to many interesting statements about the ubiquity, in composite systems, of subspaces and states with rather extreme properties. Specifically, many natural entanglement quantities are amenable to techniques from the theory; we found that there abound large subspaces containing only almost maximally entangled states, whereas states supported on such subspaces can be shown to yield almost no distillable correlation in the form of entanglement, secret key or common randomness. In fact, in sharp contrast to the difficulty one encounters for specific examples, our techniques yield very good approximations to the values of these correlation quantities for generic random states.

Figure 1 collects many of our results on correlation measures. Perhaps its most striking feature is the gap between a random state’s entanglement of formation and its distillable correlation as the rank of the random state approaches the total dimension. In that regime, the gap is as large as it would be between a maximally entangled state and a product state. Thus, strong irreversibility of entanglement, quantified as a gap between preparation cost and distillability, seems to be generic in large systems. While the conclusion relies on the assumption that the entanglement of formation is additive, the only way to evade it would be for additivity to fail very drastically for random states.

We have also begun exploring the effects of measure concentration in multipartite systems: once again, the states seem to behave in quite unexpected, even counterintuitive, but ultimately rather uniform ways. Random pure states, for example, almost always have near-maximal distillable entanglement between any pair of parties, provided all other parties are allowed to participate in the distillation protocol. For an \( n \)-party state, we can also identify \( x = n/2 \) as the point at which the state of a subset of \( x \) parties transitions from having near maximal entanglement of formation to near-zero entanglement of formation. Also, below \( x = n/3 \), the state becomes separable, confirming numerical evidence and heuristic reasoning from [30].
The large subspace of almost maximally entangled states mentioned earlier also has a constructive consequence: it allows us to get rid of the shared randomness in previous protocols for “superdense coding of quantum states”. The result presented here, moreover, can be considerably strengthened: optimal protocols for superdense coding of entangled quantum states are presented in [1].

Our work leaves open a number of questions, many of which we’ve mentioned in the course of our presentation. We collect here some of those we find most interesting:

1. There are some precedents in the literature for our results on entangled subspaces. If one relaxes the condition on the subspace, asking only that it contain no product states, as opposed to exclusively maximally entangled states, then the dimension of $S$ can be improved; Parthasarathy recently demonstrated that $S$ could be taken to be of dimension $d_A d_B - d_A - d_B + 1$, and that this is maximal [40]. For the sake of comparison, by taking $\alpha = \frac{1}{2} \log d_A$ in Theorem [IV.1] we find that there exists a subspace $S$ of dimension $\left\lfloor \frac{d_A d_B}{9917} \right\rfloor$, all of whose states have entanglement at least $\frac{1}{2} \log d_A - \beta$. While the gap between the two results is small if measured in qubits, it is still significant in absolute terms. It is, therefore, natural to ask how entangled the states of $S$ can be while still attaining Parthasarathy’s bound.

2. The techniques used here are inadequate for exploring the transition from near-maximal to near-zero entanglement of formation in rank-$s$ random states. What is the behavior of $E_f$ in the transition region?

3. How much can be said about the additivity conjecture for random quantum states? The results in this paper, for example, can be used to show that a random pure state of a sufficiently high-dimensional four-party system will not violate the superadditivity conjecture for the entanglement of formation. That, however, is insufficient to rescue us from the conditional nature of our conclusions about $E_c$ and $E_{d\rightarrow}$ based on results for $E_f$ and $I_{d\rightarrow}$.

4. Theorems [V.2 and V.3] can be interpreted as evidence that, when $s \gg d_B \log d_B$, rank-$s$ random states on $A \otimes B$ are actually undistillable. This would be very interesting to decide, as these same random states are likely to have near-maximal entanglement of formation, so being simultaneously undistillable would make them extreme examples of bound entanglement.

5. While we have studied the mixed-state measures induced by taking the partial trace over a larger system, there are other proposals for measures on the set of mixed states. Are our results still true, for example, if one substitutes the Bures measure [18, 46] for our choice?

6. How does one construct random states? Are there physical processes that will naturally produce states of the type we have studied here? One possibility for engineering them would be to use the pseudorandom unitaries of [14]. To what extent will the deviation from the true Haar measure affect our conclusions [54]?

There is no question that random entangled states are far easier to understand than all entangled states. While here we have focussed primarily on entanglement measures, it could even be the case that the theory of interconversions undergoes a similar drastic simplification. Perhaps equivalence via LOCC for random states can be completely resolved, up to the inevitable exceptional sets, a speculative note on which we would like to end this paper.
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