Mean ergodic composition operators on spaces of holomorphic functions on a Banach space

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Abstract

We study mean ergodic composition operators on infinite dimensional spaces of holomorphic functions of different types when defined on the unit ball of a Banach or a Hilbert space: that of all holomorphic functions, that of holomorphic functions of bounded type and that of bounded holomorphic functions. Several examples in the different settings are given.

1 Introduction

If $X$ and $Y$ are Banach spaces and $U \subseteq X$ is open, then a function $f : U \to Y$ is holomorphic if it is Fréchet differentiable at every point of $U$. If the open unit ball $B_X$ of $X$ satisfies $B_X \subseteq U$ and $\varphi : B_X \to B_X$ is a holomorphic self-map on $B_X$, the associated composition operator is defined by $C_\varphi(f) := f \circ \varphi$. The function $\varphi$ is called symbol of the composition operator. When $Y = \mathbb{C}$ and $U = B_X$, the space of holomorphic functions $f : B_X \to \mathbb{C}$ is simply denoted by $H(B_X)$. Our aim is to study the power boundedness and (uniform) mean ergodicity of the composition operator $C_\varphi : H(B_X) \to H(B_X)$ in terms of the properties of the symbol $\varphi$ when $H(B_X)$ is equipped with its natural topology, the compact-open topology, and also when $H(B_X)$ is replaced by the space of holomorphic functions of bounded type $H_b(B_X)$ or that of bounded holomorphic functions $H^\infty(B_X)$. We study also the case when $X$ is a Hilbert space for each of the settings considered above.

Several authors have studied different properties of composition operators on spaces of holomorphic functions on the unit ball of a Banach space. See, for instance, [1, 14, 15, 17] and the references therein. However, it seems that there is no previous literature about the dynamics of such operators. The present work can be considered a sequel of [21] by the same authors, where we study some dynamical properties (especially mean ergodicity) of composition operators in spaces of homogeneous polynomials. As in [21], the motivation and inspiration of our investigation comes from several previous works, as [7], where the authors characterise those composition operators $C_\varphi : H(U) \to H(U)$ which are power bounded, where $H(U)$ is the space of holomorphic functions on a connected domain of holomorphy $U$ of $\mathbb{C}^d$. It was proved in [7] that $C_\varphi$ is power bounded if and only if it is (uniformly) mean ergodic, and this happens if and only if the symbol $\varphi$ has stable orbits. On the other hand, if the domain is the unit disc, it was characterised in [3] when $C_\varphi$ is mean ergodic or uniformly mean ergodic on the disc algebra or on the space...
of bounded holomorphic functions in terms of the asymptotic behaviour of the symbol. Power boundedness and (uniform) mean ergodicity of weighted composition operators on the space of holomorphic functions on the unit disc was analysed in [4] in terms of the symbol and the multiplier. In [22] power boundedness and mean ergodicity for (weighted) composition operators on function spaces defined by local properties was studied in a very general framework which extends previous work. In particular, it permits to characterise (uniform) mean ergodicity for composition operators on a large class of function spaces which are Fréchet-Montel spaces when equipped with the compact-open topology. Here, the results of [22] do not apply since \( H(B_X), H_b(B_X) \) or \( H^\infty(B_X) \) are not Fréchet-Montel spaces.

The paper is organised as follows. In Section 2 we give some basic definitions and fix the notation used throughout the paper. Moreover, we recall a specific result for Hilbert spaces which is useful along the text. In Section 3 we analyse some properties of stable and \( B_X \)-stable orbits. In Section 4 we study the mean ergodicity of the composition operator in the space of holomorphic functions on the unit ball of a Banach space. In Section 5 we consider the same problem for holomorphic functions of bounded type, while in Section 6 we consider the space of bounded holomorphic functions. In each section we treat the Hilbert-space case also.

## 2 Preliminaries

All along this paper \( E \) will always denote a locally convex Hausdorff space. The set of continuous seminorms on \( E \) is denoted by \( \Gamma_E \) and \( L(E) \) is the space of continuous linear maps \( T : E \to E \). We denote \( T^0 = \text{id} \) (the identity), \( T^1 = T \) and, for \( n \in \mathbb{N} \), we write \( T^n = T^{n-1} \circ T \) (that is, the \( n \)-th composition of \( T \) with itself). With this notation, the \( n \)-the Cesàro mean of the sequence \( (T^k)_{k=0}^\infty \) is defined as

\[
T_{[n]} := \frac{1}{n} \sum_{k=0}^{n-1} T^k.
\]

The operator \( T \) is said to be \textit{power bounded} if \( \{T^n : n \geq 0\} \) is equicontinuous. It is called \textit{mean ergodic} if there is \( L \in L(E) \) such that \( (T_{[n]} x)_n \) is convergent (in \( E \)) to \( L x \) for every \( x \in E \). It is \textit{uniformly mean ergodic} if \( (T_{[n]})_n \) converges uniformly on the bounded subsets of \( E \) (we will refer to the topology so defined as \textit{the topology of bounded convergence of} \( L(E) \)). Finally, we say that \( T \) is \textit{topologizable} if for each \( q \in \Gamma_E \) there exists a sequence \( (a_n)_{n \in \mathbb{N}} \) of positive numbers and \( p \in \Gamma_E \) such that

\[
q(a_n T^n x) \leq p(x),
\]

for all \( x \in E \) and all \( n \in \mathbb{N} \) (see [5, 32]).

Also \( X \) will always denote a Banach space and \( H \) a Hilbert space. We write \( E', X' \) and \( H' \) for the corresponding dual spaces. The set \( B_X \) is the open unit ball in \( X \). On \( B_H \) (recall that \( H \) is a Hilbert space) there is a group of automorphisms that, in some sense, plays the role of Möbius transforms in the unit disc. We give here the definition and a basic property that we use later.
From [30, Proposition 1] we know that, given \( a \in B_H \), the linear operator \( \gamma_a : H \to H \) defined by

\[
\gamma_a(x) := \frac{1}{1 + v(a)} a \langle x, a \rangle + v(a)x,
\]

where \( v(a) = \sqrt{1 - \|a\|^2} \), satisfies \( \| \gamma_a(x) \| \leq \| x \| \) for all \( x \in H \) and \( \gamma_a(a) = a \). Once we have this, for each \( a \in B_H \) we can define an automorphism \( \alpha_a : B_H \to B_H \) by doing

\[
\alpha_a(x) = \gamma_a \left( \frac{a - x}{1 - \langle x, a \rangle} \right).
\]  

(2)

This satisfies \( \alpha_a(0) = a \), \( \alpha_a(a) = 0 \), and \( \alpha_a^{-1} = \alpha_a \) (the first two follow by direct computation, and the third one proceeding as in [30, Proposition 1]). The following result follows from [30, (9')]; we include a proof for the sake of completeness.

**Lemma 2.1.** For each \( 0 < r < 1 \) there is \( 0 < \rho < 1 \) such that

\[
\alpha_a(rB) \subseteq \rho B,
\]

for every \( a \in rB_H \).

**Proof.** For \( x \in B_H \) with \( \| x \| < r \), we put \( y := \alpha_a(x) \). Straightforward (though long) computation (see [30, (2)]) yields

\[
1 - \| y \|^2 = \frac{(1 - \| a \|^2)(1 - \| x \|^2)}{1 - \langle x, a \rangle^2}.
\]

Since

\[
|1 - \langle x, a \rangle|^2 \leq (1 + \| x \|\| a \|)^2 \leq (1 + r)^2,
\]

we deduce \( 1 - \| y \|^2 \geq (1 - r^2)^2(1 + r)^{-2} = (1 - r)^2 \), which gives the conclusion for \( \rho := \sqrt{1 - (1 - r)^2} \).

A mapping \( P : X \to Y \) between two Banach spaces \( X \) and \( Y \) is a (continuous) \( m \)-homogeneous polynomial if there is a continuous \( m \)-linear mapping \( L : X \times \cdots \times X \to Y \) so that \( P(x) = L(x, \ldots, x) \) for every \( x \in X \). We write \( \mathcal{P}(mX) \) for the space of all \( m \)-homogeneous polynomials \( P : X \to \mathbb{C} \), which endowed with the norm \( \| P \| = \sup_{\| x \| \leq 1} \| P(x) \| \) is a Banach space.

We refer the reader to [26, 27] for general theory of functional analysis and Banach space theory, to [10, 12, 29] for the theory of holomorphic functions on Banach spaces and to [2, 19] for topics related with linear dynamics.

## 3 Stable and \( B_X \)-stable orbits

Given an open set \( U \subseteq X \), following [7] a self map \( f : U \to U \) is said to have stable orbits if for every compact subset \( K \) of \( U \) there is a compact subset \( L \subset U \) such that \( f^n(K) \subset L \) for every \( n \in \mathbb{N} \) or equivalently, if \( \bigcup_{n=0}^{\infty} f^n(K) \) is compact in \( U \) for every compact set \( K \subseteq U \). This property was already used in [7] or [4] to characterise power boundedness and/or mean ergodicity of weighted composition operators.
We introduce now a sort of ‘bounded type’ counterpart. A set \( A \subseteq U \) is \( U \)-bounded if it is bounded and has positive distance to the boundary of \( U \) (whenever \( U = X \), the notions of ‘bounded’ and ‘\( X \)-bounded’ coincide). Then we say that \( f \) has \( U \)-stable orbits if for every \( U \)-bounded set \( A \subset U \) there is a \( U \)-bounded set \( L \subset U \) such that \( f^n(A) \subseteq L \) for every \( n \in \mathbb{N} \) (equivalently, \( \bigcup_{n=0}^{\infty} f^n(A) \) is \( U \)-bounded for every \( U \)-bounded set \( A \subseteq U \)).

**Remark 3.1.** The orbit \( \{f^n(x) : n \in \mathbb{N}\} \) of each point \( x \in U \) is relatively compact if \( f \) has stable orbits and \( U \)-bounded if \( f \) has \( U \)-stable orbits.

The notion of a function having \( B_X \)-stable orbits (we only deal with the case \( U = B_X \)) seems to be new. However, it is not hard to find functions with this property. In fact, the following well known version of the Schwarz lemma gives immediate examples.

**Lemma 3.2.** Let \( \varphi : B_X \to B_X \) be holomorphic so that \( \varphi(0) = 0 \). Then \( \|\varphi(x)\| \leq \|x\| \) for every \( x \in B_X \).

**Proof.** It is enough to apply the classical Schwarz lemma to the family of functions
\[
\left\{ \left[ \lambda \in \mathbb{D} \mapsto x^* (\varphi(\lambda x/\|x\|)) \right] : x^* \in X^*, \|x^*\| \leq 1, 0 < \|x\| < 1 \right\}.
\]

\( \square \)

**Proposition 3.3.** Let \( \varphi : B_X \to B_X \) be a holomorphic mapping such that \( \varphi(0) = 0 \), then \( \varphi \) has \( B_X \)-stable orbits.

**Proof.** Lemma 3.2 clearly implies \( \|\varphi^n(x)\| \leq \|x\| \) for all \( n \in \mathbb{N} \) and all \( x \in B_X \) and, therefore, for each \( 0 < r < 1 \) we have
\[
\varphi^n(rB_X) \subseteq rB_X,
\]
for all \( n \in \mathbb{N} \). This gives the claim. \( \square \)

As a consequence, every continuous homogeneous polynomial \( P : X \to X \) (in particular every linear operator) with \( \|P\| \leq 1 \) has \( B_X \)-stable orbits.

**Example 3.4.** If \( X \) is either \( c_0 \) or \( \ell_p \) with \( 1 \leq p \leq \infty \) we consider the forward and backward shifts operators \( F, B : X \to X \) defined as
\[
F(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \text{and} \quad B(x_1, x_2, \ldots) = (x_2, x_3, \ldots).
\]

Both are linear and clearly have norm less or equal 1, hence have \( B_X \)-stable orbits. It is not difficult to see that \( B \) has stable orbits (just using the characterisation of compact sets in \( c_0 \) or in \( \ell_p \); see for instance \([11, p. 6]\)). For the forward shift, however, we have that the set
\[
\left\{ F^n \left( \frac{e_1}{2} \right) : n \in \mathbb{N} \right\} = \left\{ \frac{e_n}{2} : n > 1 \right\}
\]
is not relatively compact and, by Remark 3.1, \( F \) does not have stable orbits.

We may also consider the mapping \( \phi : B_X \to B_X \) defined as
\[
\phi(x_1, x_2, \ldots) = \left( \frac{x_1 + 1}{2}, 0, 0, \ldots \right).
\]
Note that $\phi^n(0) = \left( \sum_{i=1}^n \frac{1}{2^i}, 0, 0, \ldots \right)$ and, therefore,
\[
\lim_{n \to \infty} \|\phi^n(0)\| = \lim_{n \to \infty} \sum_{i=1}^n \frac{1}{2^i} = 1.
\]
Hence $\phi$ has neither stable nor $B_{c_0}$-stable orbits.

We do not know so far whether or not having stable orbits implies having $B_X$-stable orbit. However, if $T : X \to X$ is continuous and linear and has stable orbits, then it is power bounded (because $\{T^n x\}_n$ is bounded for every $x \in X$), and a simple computation shows that, then $T$ has $X$-stable orbits.

### 3.1 The Hilbert-space case

If $H$ is a Hilbert space, for each $a \in B_H$ the automorphism $\alpha_a : B_H \to B_H$ defined in (2) satisfies $\alpha_a^{-1} = \alpha_a$. Hence
\[
\bigcup_{n=0}^{\infty} \alpha_a^n(A) = A \cup \alpha_a(A),
\]
for every $A \subseteq B_H$. If $A$ is compact, $\alpha_a(A)$ is again compact, and if $A$ is $B_H$-bounded, by Lemma 2.1, so also is $\alpha_a(A)$. This shows that $\alpha_a$ has both stable and $B_H$-stable orbits.

Using these automorphisms, in the case of Hilbert spaces we can extend Proposition 3.3 showing that every holomorphic function with a fixed point has $B_H$-stable orbits.

**Lemma 3.5.** If $\varphi : B_H \to B_H$ has stable orbits (respectively $B_H$-stable orbits), then the mapping $\psi = \alpha_a \circ \varphi \circ \alpha_a$ has stable orbits (respectively $B_H$-stable orbits) for every $a \in B_H$.

**Proof.** If $K \subseteq B_H$ is compact, then $\alpha_a(K)$ is compact and, having $\varphi$ stable orbits, we can find a compact set $L \subseteq B_H$ so that $\varphi^n(\alpha_a(K)) \subseteq L$ for each $n \in \mathbb{N}$. Then $\alpha_a(L) \subseteq B_H$ is compact and $\alpha_a(\varphi^n(\alpha_a(K))) \subseteq \alpha_a(L)$. Since $\psi^n = \alpha_a \circ \varphi^n \circ \alpha_a$ (because $\alpha_a^2 = \text{id}$), $\psi$ has stable orbits.

The argument if $\varphi$ has $B_H$-stable orbits is exactly the same, using that by Lemma 2.1 $\alpha_a(A)$ is $B_H$-bounded for every $B_H$-bounded $A$. \qed

**Proposition 3.6.** Let $\varphi : B_H \to B_H$ be a holomorphic mapping with a fixed point. Then $\varphi$ has $B_H$-stable orbits.

**Proof.** Take $a \in B_H$ with $\varphi(a) = a$. The holomorphic function $\overline{\varphi} = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H$ satisfies $\overline{\varphi}(0) = \alpha_a(\varphi(\alpha_a(0))) = \alpha_a(\varphi(a)) = \alpha_a(a) = 0$. Then, by Proposition 3.3 the function $\overline{\varphi}$ has $B_H$-stable orbits, and Lemma 3.5 gives the conclusion. \qed

### 4 The space of holomorphic functions

Given a Banach space $X$, we define $H(B_X)$ as the space of all holomorphic functions $f : B_X \to \mathbb{C}$, endowed with the topology $\tau_0$ of uniform convergence on compact sets. This is a locally convex Hausdorff space.
Remark 4.1. If \( \varphi : B_X \to B_X \) is holomorphic, then the composition operator \( C_\varphi : H(B_X) \to H(B_X) \) is clearly well defined (and continuous). On the other hand, if \( C_\varphi \) is well defined, then \( x' \circ \varphi \) is holomorphic for every \( x' \in X' \) and, by Dunford’s theorem (see e.g. [10, Theorem 15.45]), \( \varphi \) is holomorphic. Then, there is no restriction to assume that \( \varphi \) is holomorphic.

Stable orbits of the symbol is the property that characterises the power boundedness of the composition operator.

Theorem 4.2. Let \( \varphi : B_X \to B_X \) be holomorphic. The following assertions are equivalent:

(a) \( \varphi \) has stable orbits on \( B_X \).

(b) \( C_\varphi : H(B_X) \to H(B_X) \) is power bounded.

(c) \( \left( \frac{1}{n} C_\varphi^n \right)_n \) is equicontinuous in \( L(H(B_X)) \).

(d) \( C_\varphi : H(B_X) \to H(B_X) \) is topologizable.

Proof. (a) \( \Rightarrow \) (b) If \( \varphi \) has stable orbits, given a compact set \( K \subseteq B_X \) there is a compact set \( L \subseteq B_X \) such that \( \varphi^n(K) \subseteq L \) for every \( n \in \mathbb{N} \). Hence

\[
\sup_{x \in K} |C_\varphi^n(f)(x)| = \sup_{x \in K} |f(\varphi^n(x))| \leq \sup_{x \in L} |f(x)|,
\]

for all \( f \in H(B_X) \) and \( n \in \mathbb{N} \). So the sequence \( (C_\varphi^n)_n \) is equicontinuous and \( C_\varphi \) is power bounded.

(b) \( \Rightarrow \) (c) Suppose now that \( C_\varphi \) is power bounded, then for each compact set \( K \subseteq B_X \) we can find \( c > 0 \) and a compact set \( L \subseteq B \) so that

\[
\sup_{x \in K} |C_\varphi^n(f)(x)| \leq c \sup_{x \in L} |f(x)|
\]

for every \( f \in H(B_X) \) and \( n \in \mathbb{N} \). This obviously implies

\[
\sup_{x \in K} \left| \frac{1}{n} C_\varphi^n(f)(x) \right| \leq c \sup_{x \in L} |f(x)|
\]

for every \( f \) and \( n \), and \( \left( \frac{1}{n} C_\varphi^n \right)_n \) is equicontinuous.

(c) \( \Rightarrow \) (d) follows just taking \( a_n = \frac{1}{cn} \) in (1).

(d) \( \Rightarrow \) (a) Fix some compact set \( K \subseteq B_X \). Since \( C_\varphi \) is topologizable, we can find some compact set \( W \subseteq X \), and \( (a_n)_{n \in \mathbb{N}} \) with \( a_n > 0 \) such that,

\[
\sup_{x \in K} |f(\varphi^n(x))| \leq \frac{1}{a_n} \sup_{x \in W} |f(x)|,
\]

(5)

for all \( f \in H(B_X) \) and \( n \in \mathbb{N} \). By [29, Corollary 10.7 and Theorem 11.4] the set \( L = \hat{W}_{H(B_X)} \) (recall (6)) is compact and contains \( W \). We see that \( \varphi^n(K) \subseteq L \) for every \( n \). Suppose that this is not the case and take \( x_0 \in K \) and \( n_0 \in \mathbb{N} \) so that \( \varphi^{n_0}(x_0) \notin L \). Then
there is \( f \in H(B_X) \) such that \( |f(\varphi^{n_0}(x_0))| > \sup_{y \in W} |f(y)| \), and there exists \( m \in \mathbb{N} \) such that the function \( g = \frac{f^{m}}{f(\varphi^{n_0}(x_0))^{m}} \in H(B_X) \) satisfies
\[
1 = |g(\varphi^{n_0}(x_0))| > \frac{1}{a_{n_0}} \sup_{y \in W} |g(y)|.
\]

But this contradicts (5), which shows that indeed \( \varphi^n(K) \subseteq L \) for all \( n \in \mathbb{N} \). 

**Proposition 4.3.** Let \( \varphi : B_X \to B_X \) be holomorphic. If \( C_\varphi : H(B_X) \to H(B_X) \) is power bounded, then it is also uniformly mean ergodic.

**Proof.** From [29, Proposition 9.16] we know that every bounded subset of \( H(B_X) \) is relatively compact, therefore \( H(B_X) \) is semi-Montel and, in particular, semi-reflexive. Then, as a consequence of [6, p. 917] (see also [21, Proposition 3.1]) we have that every power bounded operator is uniformly mean ergodic. \( \square \)

5 The space of holomorphic functions of bounded type

If \( X \) and \( Y \) are Banach spaces and \( U \subseteq X \) and \( V \subseteq Y \) are open sets, a function \( f : U \to V \) is of bounded type if it sends \( U \)-bounded sets to \( V \)-bounded sets. We consider the space \( H_b(B_X) \) of all holomorphic functions \( f : B_X \to \mathbb{C} \) of bounded type, endowed with the topology \( \tau_b \) of uniform convergence on \( B_X \)-bounded sets. This is a Fréchet space.

If \( \varphi : B_X \to B_X \) is holomorphic of bounded type, then clearly \( C_\varphi : H_b(B_X) \to H_b(B_X) \) is well defined. On the other hand, we observe that \( X' \subseteq H_b(B_X) \) because every functional is trivially Fréchet differentiable. In fact, \( X' \) is a complemented subspace of \( H_b(B_X) \), as we explain below in the proof of Proposition 5.4. So, if the composition operator is well defined (as a self map on \( H_b(B_X) \)), then the argument in Remark 4.1 shows that \( \varphi \) has to be holomorphic. Furthermore, [18, Proposition 3] shows that \( \varphi \) is of bounded type.

Our first goal in this section is to characterise the power boundedness of composition operators on \( H_b(B_X) \). Following [31] and [9], given a family \( \mathcal{F} \) of \( \mathbb{C} \)-valued holomorphic functions defined on an open set \( U \), the \( \mathcal{F} \)-hull of \( A \subseteq U \) is denoted
\[
\widehat{A}_\mathcal{F} = \{ x \in U : |f(x)| \leq \sup_{y \in A} |f(y)|, \text{ for all } f \in \mathcal{F} \}.
\]

**Lemma 5.1.** If \( U \) is an absolutely convex open set on a Banach space \( X \), then \( \widehat{A}_{H_b(U)} \) is \( U \)-bounded for every \( U \)-bounded set \( A \).

**Proof.** The polar set of \( A \) is a subset of \( X' \) and it is contained in \( H_b(U) \). Then a straightforward computation shows that \( \widehat{A}_{H_b(U)} \) is contained in the bipolar of \( A \), which by the Bipolar Theorem coincides with \( \overline{\text{co}}(A) \) (the closure of the absolutely convex hull of \( A \)). Since \( U \) is absolutely convex, [8, Remark, p. 527] gives that \( \overline{\text{co}}(A) \) is \( U \)-bounded, which completes the proof. \( \square \)

With exactly the same proof as in Theorem 4.2, replacing ‘compact’ by ‘\( B_X \)-bounded’ we have the following.
Theorem 5.2. Let $\varphi : B_X \to B_X$ be a holomorphic mapping. The following assertions are equivalent

(a) $\varphi$ has $B_X$-stable orbits.
(b) $C_\varphi : H_b(B_X) \to H_b(B_X)$ is power bounded.
(c) $\left(\frac{1}{n} C^n_\varphi \right)_n$ is equicontinuous in $L(H_b(B_X))$.
(d) $C_\varphi : H_b(B_X) \to H_b(B_X)$ is topologizable.

We now show that in this case every mean ergodic composition operator is power bounded, and there are power bounded operators that are not mean ergodic.

Proposition 5.3. Let $\varphi : B_X \to B_X$ a holomorphic mapping. If $C_\varphi : H_b(B_X) \to H_b(B_X)$ is mean ergodic, then $C_\varphi$ is power bounded.

Proof. The mean ergodicity immediately gives that the sequence $\left(\frac{1}{n} C^n_\varphi \right)_n$ tends to zero (pointwise), so it is pointwise bounded. Since $H_b(B_X)$ is barrelled (because it is a Fréchet space), it is also equicontinuous on $H_b(B_X)$. This, in view of Theorem 5.2, gives the conclusion.

We want to find now composition operators that are power bounded but not mean ergodic. The shifts defined in (4) provide us with such examples.

Proposition 5.4. The composition operators $C_B : H_b(B_{c_0}) \to H_b(B_{c_0})$ and $C_F : H_b(B_{c_1}) \to H_b(B_{c_1})$ are power bounded but not mean ergodic.

Proof. We already noted in Example 3.4 that $B$ has $B_{c_0}$-stable orbits which, in view of Theorem 5.2, shows that $C_B$ is power bounded.

We now see that $C_B$ is not mean ergodic. We begin by observing that $H_b(B_X)$ contains a complemented copy of $X'$ for every Banach space $X$. Indeed, given a holomorphic $f : B_X \to \mathbb{C}$, we denote its differential at 0 (that belongs to $X'$) by $df(0)$. Then a simple computation shows that the mappings $P : H_b(B_X) \to X'$ and $J : X' \to H_b(B_X)$ defined by $P(f) = df(0)$ and $J(u) = u|_{B_X}$ give our claim.

We consider now the restriction of $C_B$ to $J(\ell_1)$ (recall that $c'_0 = \ell_1$) and we have, for each $u \in \ell_1$ and $x \in c_0$,

$$\langle C_B u, x \rangle = u(B(x)) = \langle u, B(x) \rangle = \langle (u_1, u_2, u_3, \ldots), (x_2, x_3, x_4, \ldots) \rangle$$

$$= u_1 x_2 + u_2 x_3 + u_3 x_4 + \cdots = \langle (0, u_1, u_2, u_3, \ldots), (x_1, x_2, x_3, x_4, \ldots) \rangle = \langle Fu, x \rangle.$$

Thus $F = P \circ C_B \circ J$, which is not mean ergodic on $\ell_1$ (see, for instance, [6]). This implies that $C_B$ is not mean ergodic on $H_b(B_{c_0})$.

For the forward shift, Example 3.4 showed that $F$ has $B_{\ell_1}$-stable orbits. Essentially the same argument as before shows that the restriction of $C_F$ to $\ell'_1 = \ell_\infty$ is the backward shift $B$, which is not mean ergodic. This yields the conclusion. \qed

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We look now for sufficient conditions for a given power bounded composition operator to be mean ergodic (and up to some point, even to reverse the implication in Proposition 5.3). Before we need the following lemma. The argument of the proof is essentially the one in [24, Chapter 2, Theorem 1.1] (see also [6, page 908]); since our setting is slightly different we sketch the proof here for the sake of completeness.

**Lemma 5.5.** Let $E$ be a locally convex Hausdorff space, $T \in L(E)$ be power bounded and $x \in E$. If $y \in E$ is a $\sigma(E, E')$-cluster point of $(T^n x)_{n \in \mathbb{N}}$, then $\lim_{n \to \infty} T^n x = y$.

**Proof.** Fix $p \in \Gamma_E$. Since $T$ is power bounded we can find $q \in \Gamma_E$ so that $p(T^n z) \leq q(z)$ for every $z \in E$. If $y$ is a $\sigma(E, E')$-cluster point of $(T^n x)_{n \in \mathbb{N}}$ then it belongs to the $\sigma(E, E')$-closure of the set which (note that $(T^n x)_{n \in \mathbb{N}} \subseteq \operatorname{co}(T^n x)_{n \in \mathbb{N}_0}$) is contained in the $\sigma(E, E')$-closure of $\operatorname{co}(T^n x)_{n \in \mathbb{N}_0}$. But as a consequence of the Hahn-Banach theorem, for convex sets the $\sigma(E, E')$-closure coincides with the closure. So, $y \in \overline{\operatorname{co}(T^n x)_{n \in \mathbb{N}_0}}$, and for a given $\varepsilon > 0$ we can find $z \in \operatorname{co}(T^n x)_{n \in \mathbb{N}_0}$ so that $q(z - y) < \varepsilon$, so that we can write

$$p(y - T^n x) \leq p(y - T^n z) + p(T^n z - T^n x).$$

(7)

Note that $z = \sum_{k=0}^{m} \lambda_k T^k x$ for some $0 \leq \lambda_k \leq 1$ with $\sum_{k=0}^{m} \lambda_k = 1$. We define $S = \sum_{k=0}^{m} \lambda_k T^k \in L(E)$. Hence

$$p(T^n S x - T^n x) \leq \sum_{k=0}^{m} \lambda_k p(T^n T^k x - T^n x).$$

If $n \geq m \geq k$, we have

$$p(T^n T^k x - T^n x) \leq \frac{1}{n} \left( \sum_{j=0}^{k-1} p(T^j x) + \sum_{j=n}^{n+k-1} p(T^j x) \right) \leq \frac{2k}{n} q(x).$$

(8)

With this we can estimate the second addend in the right-hand term of (7). In order to control the first one it is enough to see that $y = Ty$ since, if this is the case then $y = T^n y$ and $p(y - T^n z) \leq q(y - z)$. Given $x' \in E'$, we have

$$|\langle y - Ty, x' \rangle| = |\langle y - T^n x, x' \rangle| + |\langle T^n x - TT^n x, x' \rangle| + |\langle Ty - TT^n x, x' \rangle|$$

$$\leq |\langle y - T^n x, x' \rangle| + |\langle T^n x - TT^n x, x' \rangle| + |\langle y - T^n x, T x' \rangle|.$$

The first and third term tend to 0 because $y$ is a $\sigma(E, E')$-cluster point. On the other hand, (8) implies that $T^n x - TT^n x$ tends to 0 as $m \to \infty$ and, then, so also does the second term. This shows that $y = Ty$ and completes the proof. □

We denote by $\mathcal{P}(X)$ the algebra of all continuous polynomials on $X$ (these are finite sums of homogeneous polynomials), and by $\sigma(X, \mathcal{P}(X))$ the coarsest topology making all $P \in \mathcal{P}(X)$ continuous. This is a Hausdorff topology, satisfies $\| \cdot \| \geq \sigma(X, \mathcal{P}(X)) \geq \sigma(X, X^*)$ and the concepts of (relatively) countably compact subset, (relatively) sequentially compact subset and (relatively) compact subset all agree with respect to this topology.

**Proposition 5.6.** Let $\varphi : B_X \to B_X$ be holomorphic, having $B_X$-stable orbits and such that $\varphi(A)$ is relatively $\sigma(X, \mathcal{P}(X))$-compact for every $B_X$-bounded set $A$. Then $C_\varphi : H_b(B_X) \to H_b(B_X)$ is mean ergodic.
Proof. From Theorem 5.2 we have that the composition operator $C_{\varphi}$ is power bounded, and the equicontinuity of $(C_{\varphi})_{n \in \mathbb{N}}$ gives that the set $\{(C_{\varphi})_{[n]}(f) : n \in \mathbb{N}\}$ is bounded for every $f \in H_b(B_X)$. Now, by [16, Theorem 2.9] $C_{\varphi}$ maps bounded sets of $H_b(B_X)$ into relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact sets, so for every $f \in H_b(B_X)$ the set

$$C_{\varphi}\{(C_{\varphi})_{[n]}(f) \}_{n} = \left\{ \frac{1}{n} \sum_{k=1}^{n} C_{\varphi}^{k}(f) : n \in \mathbb{N} \right\}$$

is relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact and, therefore it has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point. Our aim now is to see that $\{(C_{\varphi})_{[n]}(f) \}_{n} \subset (C_{\varphi})_{[n]}(B_X)$ has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point which, using Lemma 5.5, implies that the sequence $(\{(C_{\varphi})_{[n]}(f) \}_{n})_{n \in \mathbb{N}}$ converges in $H_b(B_X)$ for all $f \in H_b(B_X)$, and $C_{\varphi}$ is mean ergodic.

Note that

$$(C_{\varphi})_{[n]}(f) = \frac{1}{n}(f - C_{\varphi}^{n}(f)) + \frac{1}{n} \sum_{k=1}^{n} C_{\varphi}^{k}(f)$$

for every $n$. The fact that $C_{\varphi}$ is power bounded implies that $(\frac{1}{n}(id(f) - C_{\varphi}^{n}(f)))_{n \in \mathbb{N}}$ tends to 0 as $n \to \infty$, and this gives that $\{(C_{\varphi})_{[n]}(f) \}_{n}$ has a $\sigma(H_b(B_X), H_b(B_X)')$-cluster point, as we wanted. \qed

Corollary 5.7. Let $X$ be a Banach space such that every $B_X$-bounded set is relatively $\sigma(X, \mathcal{P}(X))$-compact. Then $C_{\varphi} : H_b(B_X) \to H_b(B_X)$ is power bounded if and only if $C_{\varphi}$ is mean ergodic.

An example of a Banach space which satisfies such a property is the Tsirelson space $T^*$: it is known that $T^*$ is reflexive and the polynomials on $T^*$ are weakly sequentially continuous [12, p. 121]. Hence, any sequence in the unit ball of $T^*$ has a weakly convergent subsequence, which converges in the topology $\sigma(T^*, \mathcal{P}(T^*))$. Since $\sigma(T^*, \mathcal{P}(T^*))$ is angelic [15, p. 150], the unit ball is also relatively $\sigma(T^*, \mathcal{P}(T^*))$-compact.

We find now conditions to ensure that a given composition operator is uniformly mean ergodic. Here $C_0$ denotes the composition operator defined by the constant function 0 (i.e. $C_0(f) = f(0)$ for every $f$).

Theorem 5.8. Let $\varphi : B_X \to B_X$ be holomorphic so that for every $0 < t < 1$ there exists $0 < \rho < t$ such that

$$\varphi(tB_X) \subseteq \rho B_X.$$  (9)

Then

$$C_{\varphi^n} \to C_0,$$

in the topology of bounded convergence on $H_b(B_X)$. In particular,

$$(C_{\varphi})_{[n]} \to C_0,$$  (10)

in the topology of bounded convergence on $H_b(B_X)$ and $C_{\varphi} : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic.
Proof. Fix some $0 < t < 1$. First of all, (9) implies, on the one hand, that $\varphi^n(tB_X) \subseteq \rho B_X$ for every $n \in \mathbb{N}$ and, on the other hand, that $\varphi(0) = 0$. We can then apply Lemma 3.2 to the function $[x \sim \frac{1}{t}\varphi(tx)]$ and get

$$\|\varphi^n(x)\| \leq \left(\frac{\rho}{t}\right)^n \|x\|,$$

(11)

for every $x \in tB_X$ and $n \in \mathbb{N}$. Now, given $f \in H_0(B_X)$, we obviously have $\|f \circ \varphi^n\|_{tB_X} \leq \|f\|_{tB_X}$ for every $n \in \mathbb{N}$. We define $g : B_X \rightarrow \mathbb{D}$ by $g(x) = \frac{1}{2\|f\|_{tB_X}}(f(\varphi(tx)) - f(0))$. This is clearly holomorphic and satisfies $g(0) = 0$. Then we can apply Lemma 3.2 to $g$ and (11) to obtain

$$\|C^n_{\varphi}(f) - C_0(f)\|_{tB_X} = \sup_{x \in tB_X} |f(\varphi^n(x)) - f(0)| \leq 2\|f\|_{tB_X} \sup_{x \in tB_X} \|\varphi^{-1}(x)\| \leq 2\|f\|_{tB_X} \left(\frac{\rho}{t}\right)^{n-1}.$$

This implies, for every $0 < t < 1$ and every bounded set $A \subseteq H_0(B_X)$,

$$\lim_{n \rightarrow \infty} \sup_{f \in A} \sup_{x \in tB_X} |C^n_{\varphi}(f)(x) - f(0)| = 0.$$ 

Hence, $C^n_{\varphi} \rightarrow C_0$ in the topology of bounded convergence. Once we have this, (10) is a straightforward consequence.

**Remark 5.9.** If $\varphi : B_X \rightarrow B_X$ is holomorphic and satisfies

$$\varphi(B_X) \subseteq r B_X \text{ for some } 0 < r < 1 \text{ and } \varphi(0) = 0,$$

(12)

then, applying Lemma 3.2 to the function $[x \sim \frac{1}{r}\varphi(x)]$ we get $\|\varphi(x)\| \leq r \|x\|$ for every $x \in B_X$, and this implies that $\varphi$ satisfies (9) with $\rho = tr$.

There are, however, functions satisfying (9) but not (12). To see this just consider the restriction to $B_X$ of any $m$-homogeneous polynomial (for $m > 1$) $P : X \rightarrow X$ with $\|P\| \leq 1$. For a fixed $0 < t < 1$ take any $0 < \varepsilon < t - t^m$ and note that

$$\|P(tx)\| \leq t^m \|x\|^m \leq (t - \varepsilon) \|x\|,$$

for every $x \in B_X$. That is, every homogeneous polynomial with norm $\leq 1$ satisfies (9). If $\|P\| = 1$ and attains its norm (that is, there is $x_0$ with $\|x_0\| = 1$ so that $\|P(x_0)\| = \|P\|$) then

$$\|P((1 - \frac{1}{n})x_0)\| = \left(1 - \frac{1}{n}\right)^m,$$

and there is no $0 < r < 1$ so that $P(B_X) \subseteq r B_X$. For a concrete example of such a polynomial just consider the 2-homogeneous one $P : \ell_2 \rightarrow \ell_2$ given by $P((x_n)_n) = (x_n^2)_n$ (in this case one can take $x_0 = e_1$).

In particular, we have that, if $m > 1$ and $P$ is an $m$-homogeneous polynomial with $\|P\| \leq 1$, then $C_P : H_0(B_X) \rightarrow H_0(B_X)$ is uniformly mean ergodic. For $m = 1$, that is, for linear operators, this property does not hold, as Proposition 5.13 shows.

One may also ask if in (12) we can drop the condition on the fixed point and still get (9) just assuming that $\varphi(B_X) \subseteq r B_X$ for some $0 < r < 1$. But this is not the case: fix some $x_0 \in B_X$ and consider the constant function $\varphi(x) = x_0$ for every $x \in B_X$. 

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5.1 Example of a composition operator which is mean ergodic but not uniformly mean ergodic in $H_b(B_{c_0})$

The following result is well known [23, §39.4(1), p. 138].

**Lemma 5.10.** Let $(T_n)_n$ be a sequence of equicontinuous operators on a locally convex space $E$. If $(T_n)$ is pointwise convergent to a continuous operator $T$ on some dense set $D \subseteq E$, then $(T_n)_n$ is pointwise convergent to $T$ in $E$.

We also need the following property [10, Theorem 15.60].

**Theorem 5.11.** For each $m \in \mathbb{N}$, the set $A_m := \{x^\alpha : |\alpha| = m\}$ of monomials generates a dense subspace of $\mathcal{P}(m_{c_0})$.

**Remark 5.12.** Since $B_{c_0}$ is a balanced set, the polynomials are dense on $H_b(B_{c_0})$. Therefore, by Theorem 5.11 we have that the set

$$\text{span}\{x^\alpha : \alpha \in \mathbb{N}^0\}$$

is dense on $H_b(B_{c_0})$.

**Proposition 5.13.** Let $F : B_{c_0} \to B_{c_0}$ be the forward shift. The composition operator $C_F : H_b(B_{c_0}) \to H_b(B_{c_0})$ is mean ergodic but not uniformly mean ergodic.

**Proof.** First, we see that $C_F$ is mean ergodic. We follow a similar scheme to that in [3, Theorem 2.2], using that span$\{x^\alpha : \alpha \in \mathbb{N}^{(n)}\}$ is a dense subspace of $H_b(B_{c_0})$ (Remark 5.12) together with Lemma 5.10. Since $C_F$ is power bounded on $H_b(B_{c_0})$ (because it has $B_{c_0}$-stable orbits), $(C_F^n)_n$ is equicontinuous. Therefore, $((C_F)_n(n))_n$ is also equicontinuous on $H_b(B_{c_0})$. Since $C_F(1) = 1 = C_0(1)$ for any constant mapping (this is in fact true for any composition operator), it remains to see that $(((C_F)_n(h)))_n$ $\tau_b$-converges to $C_0(h)$ for every $h \in A_m$ and $m > 0$ (on these cases $C_0(h) = 0$). For $h(x) = x^\alpha$ with $|\alpha| = m$, we define $n_h = \max\{j \in \mathbb{N} : (\alpha)_j \neq 0\}$ which is a finite number. Observe that $C^n_F(h) = C^n_F(x^\alpha) = (F^n(x))^\alpha = 0$ for all $n \geq n_h$, and the claim follows.

As in the proof of Proposition 5.4, one can see that $P \circ C_F \circ J = B$, where $B : \ell_1 \to \ell_1$ is the backward shift (recall (4)). If $C_F$ were uniformly mean ergodic on $H_b(B_{c_0})$, then $B : \ell_1 \to \ell_1$ would be uniformly mean ergodic, but this is not the case. Indeed, since $B^j x$ tends to 0 in $\ell_1$ for all $x \in \ell_1$, the only possible value for the limit projection of $\frac{1}{N} \sum_{j=0}^{N-1} B^j$ is 0. But, for each $N \in \mathbb{N}$, we have

$$\sup_{\|x\| \leq 1} \left\| \frac{1}{N} \sum_{j=0}^{N-1} B^j(x) \right\|_{\ell_1} \geq \frac{1}{N} \left\| \sum_{j=0}^{N-1} B^j(e_N) \right\|_{\ell_1} = \frac{1}{N} \left\| (1, (N), 1, 0, \ldots) \right\|_{\ell_1} = 1.$$

And it is not true that

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{j=0}^{N-1} B^j \right\| = 0.$$

$\square$
5.2 The Hilbert-space case

Let us go back to (12) for a moment. If we only assume \( \varphi(B_X) \subseteq rB_X \), the Earle-Hamilton fixed point theorem [13] implies that there exists a unique \( a \in B_X \) such that \( \varphi(a) = a \). It is then natural to ask if this is enough to ensure that the composition operator is uniformly mean ergodic. If we restrict ourselves to Hilbert spaces \( H \) we can say something in this respect. We need the following lemma.

**Lemma 5.14.** Let \( \varphi : B_H \to B_H \) be holomorphic so that \( C_{\varphi^n} \to C_0 \) in the topology of bounded convergence of \( L(H_b(B_H)) \). Then for every \( a \in B_H \) the mapping \( \overline{\varphi} = \alpha_a \circ \varphi \circ \alpha_a \) satisfies that \( C_{\overline{\varphi}} \to C_a \) in the topology of bounded convergence of \( L(H_b(B_H)) \).

**Proof.** Since both \( \varphi \) and \( \alpha_a \) are of bounded type (see Lemma 2.1), the composition \( \alpha_a \circ \varphi \circ \alpha_a \) is of bounded type and \( C_{\overline{\varphi}} : H_b(B_H) \to H_b(B_H) \) is well defined. Observe now that \( \overline{\varphi}^n = \alpha_a \circ \varphi^n \circ \alpha_a \) for all \( n \in \mathbb{N} \) since \( \alpha_a^{-1} = \alpha_a \). Then

\[
C_{\overline{\varphi}^n} = C_{\alpha_a \circ \varphi^n \circ \alpha_a} = C_{\alpha_a} \circ C_{\varphi^n} \circ C_{\alpha_a} = C_{\alpha_a} \circ C_0 \circ C_{\alpha_a} = C_{\alpha_a} \circ C_{\alpha_a}(0) = C_{\alpha_a} \circ C_a = C_a.
\]

\( \square \)

**Proposition 5.15.** Let \( \varphi : B_H \to B_H \) be holomorphic such that

\[
\varphi(B_H) \subseteq rB_H \quad \text{for some} \quad 0 < r < 1.
\]

Then, for the unique \( a \in B_H \) such that \( \varphi(a) = a \) we have \( C_{\varphi^n} \to C_a \) in the topology of bounded convergence of \( L(H_b(B_H)) \). In particular \( (C_\varphi)_n \to C_a \), and \( C_{\varphi} : H_b(B_H) \to H_b(B_H) \) is uniformly mean ergodic.

**Proof.** Define \( \phi = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H \), which clearly satisfies \( \phi(0) = 0 \). Also,

\[
\phi(B_H) = (\alpha_a \circ \varphi \circ \alpha_a)(B_H) = (\alpha_a \circ \varphi)(B_H) \subseteq \alpha_a(rB_H),
\]

and using Lemma 2.1 we can find some \( 0 < \varepsilon < 1 \) so that

\[
\phi(B_H) \subseteq (1 - \varepsilon)B_H.
\]

Then \( \phi \) satisfies (12) and, by Theorem 5.8, \( C_{\varphi^n} \to C_0 \). Since \( \varphi = \alpha_a \circ \phi \circ \alpha_a \) (because \( \alpha_a^{-1} = \alpha_a \)), Lemma 5.14 yields the claim. \( \square \)

We can find uniformly mean ergodic composition operators with symbols that do not satisfy neither (9) nor (13). To see this it is enough to consider \( \alpha_a : B_H \to B_H \). We observe that

\[
C_{\alpha_a}^n = \begin{cases} C_{\alpha_a} & \text{if } n \text{ is odd}, \\ C_{\mathrm{id}_{B_H}} & \text{if } n \text{ is even}. \end{cases}
\]

Then, for each \( k \in \mathbb{N} \) we have

\[
(C_{\alpha_a})_{[2k-1]} = \frac{1}{2k-1} \sum_{n=0}^{2k-1} C_{\alpha_a}^n = \frac{k}{2k-1} \left( C_{\alpha_a} + \mathrm{id}_{H_b(B_H)} \right),
\]

and

\[
(C_{\alpha_a})_{[2k]} = \frac{1}{2k} \sum_{n=0}^{2k} C_{\alpha_a}^n = \frac{1}{2} \left( C_{\alpha_a} + \mathrm{id}_{H_b(B_H)} \right) + \frac{1}{2k} \mathrm{id}_{H_b(B_H)}.
\]

Then \( \lim_{n \to \infty} (C_{\alpha_a})_{[n]} = \frac{1}{2} \left( C_{\alpha_a} + \mathrm{id}_{H_b(B_H)} \right) \) in the topology of bounded convergence of \( L(H_b(B_H)) \), and \( C_{\varphi} : H_b(B_H) \to H_b(B_H) \) is uniformly mean ergodic.
6 The space of bounded holomorphic functions

We consider now the space $H^\infty(B_X)$ of all holomorphic functions $f : B_X \to \mathbb{C}$ that are bounded. With the norm $\|f\|_\infty = \sup_{x \in B_X} |f(x)|$ it becomes a Banach space. We look at composition operators $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$. If $\varphi : B_X \to B_X$, then

$$\|C_\varphi^n(f)\|_\infty = \sup_{x \in B_X} |C_\varphi^n(f)(x)| = \sup_{x \in B_X} |\varphi^n(x)| \leq \sup_{x \in B_X} |f(x)| = \|f\|_\infty,$$

and $\|C_\varphi^n\| \leq 1$ for all $n \in \mathbb{N}$. Hence every $C_\varphi$ that is well defined on $H^\infty(B_X)$ is power bounded. Since $(X', \| \cdot \|) = (X', \tau_b)$, the dual space $X'$ is also complemented in $H^\infty(B_X)$, and the same arguments as in Proposition 5.4 give examples of composition operators $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$ which are not mean ergodic. However, $X'$ is in general not complemented in $H(B_X)$ since $(X', \| \cdot \|) \neq (X', \tau_0)$ and these arguments do not work for $H(B_X)$.

We give now conditions on the symbol to define a uniformly mean ergodic composition operator on $H^\infty(B_X)$.

**Proposition 6.1.** Let $\varphi : B_X \to B_X$ be holomorphic such that $\varphi(B_X) \subseteq rB_X$ for some $0 < r < 1$ and $\varphi(0) = 0$. Then

$$C_\varphi^n \to C_0,$$

in the norm operator topology of $L(H^\infty(B_X))$. In particular, $(C_\varphi)_n \to C_0$, and $C_\varphi : H^\infty(B_X) \to H^\infty(B_X)$ is uniformly mean ergodic.

**Proof.** Take some $f \in H^\infty(B_X)$ with $\|f\|_\infty \leq 1$. Defining $g : B_X \to \mathbb{D}$ by $g(x) = \frac{1}{2}(f(x) - f(0))$ and using Lemma 3.2 we get

$$|f(x) - f(0)| \leq 2\|x\|$$

for every $x \in B_X$. Proceeding as in (11), we get that $\|\varphi^n(x)\| \leq r^n\|x\|$ for every $x \in B_X$ and $n \in \mathbb{N}$. This yields

$$|f(\varphi^n(x)) - f(0)| \leq 2\|\varphi^n(x)\| \leq 2r^n\|x\|.$$ 

Therefore

$$\|C_\varphi^n - C_0\|_{L(H^\infty(B))} = \sup_{\|f\|_\infty \leq 1} \sup_{x \in B_X} |f(\varphi^n(x)) - f(0)| \leq 2 \sup_{x \in B_X} \|\varphi^n(x)\| \leq 2r^n,$$

which gives the claim.

We observe that the hypothesis in Proposition 6.1 is exactly the same one as (12) in Remark 5.9. One can ask if the result also holds assuming instead (9). This is not the case. We already saw in Remark 5.9 that the mapping $P : B_\ell_2 \to B_\ell_2$ given by $P((x_n)_n) = (x_n^2)_n$ satisfies (9). Then, by Theorem 5.8 the Cesàro means of $C_P$ converge to $C_0$. Hence, $C_P : H_b(B_\ell_2) \to H_b(B_\ell_2)$ is uniformly mean ergodic.

However, the operator $C_P : H^\infty(B_\ell_2) \to H^\infty(B_\ell_2)$ is not even mean ergodic. Notice that $H^\infty(B_\ell_2) \subseteq H_b(B_\ell_2)$ and $\tau_b$ is weaker than the norm topology. Then if $C_P$ were mean ergodic, $(C_P^n(f))_n$ should converge in norm to $C_0(f)$ for every $f \in H^\infty(B_\ell_2)$. Take
\[ f \in H^\infty(B_{\ell_2}) \text{ given by } f((x_n)_n) = x_1 \text{ and consider } z_m = (1 - \frac{1}{m})e_1 \in B_{\ell_2} \text{ for each } m \in \mathbb{N}. \]

Then \( P^k(z_m) = (1 - \frac{1}{m})^k e_1 \) for every \( k \) and

\[
(C_P)[n](f)(z_m) - C_0(f)(z_m) = \frac{1}{n} \sum_{k=0}^{n-1} f((1 - \frac{1}{m})^k e_1) - f(0) = \frac{1}{n} \sum_{k=0}^{n-1} (1 - \frac{1}{m})^{2k}.
\]

Thus

\[
\sup_{x \in B_{\ell_2}} |(C_P)[n](f)(x) - C_0(f)(x)| \geq \sup_{m \in \mathbb{N}} \frac{1}{n} \sum_{k=0}^{n-1} (1 - \frac{1}{m})^{2k} = 1,
\]

and \((C_P^n(f))_n\) does not converge in norm to \( C_0(f) \). This finally shows that \( C_P : H^\infty(B_{\ell_2}) \to H^\infty(B_{\ell_2}) \) is not mean ergodic.

The same argument as in Lemma 5.14 and Proposition 5.15 shows the following.

**Proposition 6.2.** Let \( \varphi : B_H \to B_H \) be analytic such that \( \varphi(B_H) \subseteq rB_H \) for some \( 0 < r < 1 \). Then, for the unique \( a \in B \) such that \( \varphi(a) = a \) we have that \( C_{\varphi^n} \to C_a \) in the norm of \( L(H^\infty(B_H)) \). In particular \( (C_{\varphi})[n] \to C_a \) and \( C_{\varphi} \) is uniformly mean ergodic.

We have formulated some results (Propositions 3.6, 5.15 and 6.2) for the open unit ball of a Hilbert space. The key element for the proofs of these is the existence of a family of biholomorphic automorphisms on the ball (as in (2)) satisfying (3). Hilbert spaces are not the only examples of such a situation. In every \( C^* \)-algebra, for example, also such a family of automorphisms can be defined. In fact, there is a wider class of Banach spaces, known as \( JB^* \)-triples, that also have this property: if \( X \) is a \( JB^* \)-triple, then there is a family of biholomorphic automorphisms \( \{\alpha_a\}_{a \in B_X} \) on \( B_X \) satisfying \( \alpha_a(0) = a, \alpha_a(a) = 0, \alpha_a^{-1} = \alpha_a \). The class of \( JB^* \)-triples includes Hilbert spaces and \( C^* \)-algebras, but also wider classes such as \( J^* \)-algebras (closed subspaces of the space of operators between two Hilbert spaces \( L(H_1, H_2) \) which are closed under \( T \sim TT^*T \), being \( T^* \) the adjoint of \( T \)); the interested reader may find more information on the subject in [20, 28]. Moreover, these automorphisms satisfy the corresponding analogue of Lemma 2.1 [25, Lemma 1]. So, the aforementioned results remain valid if \( B_H \) is replaced by the open unit ball \( B_X \) of a \( JB^* \)-triple (in particular a \( C^* \)-algebra) \( X \).

Finally, we observe that some questions remained open. It would be interesting to find examples of the following situations:

(a) A composition operator on \( H(B_X) \) which is mean ergodic but not uniformly mean ergodic.

(b) A composition operator on \( H(B_X) \) which is mean ergodic but not power bounded.

(c) A composition operator on \( H^\infty(B_X) \) which is mean ergodic but not uniformly mean ergodic.

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