Regular holonomic $\mathcal{D}[[\hbar]]$-modules

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Abstract

We describe the category of regular holonomic modules over the ring $\mathcal{D}[[\hbar]]$ of linear differential operators with a formal parameter $\hbar$. In particular, we establish the Riemann-Hilbert correspondence and discuss the additional $t$-structure related to $\hbar$-torsion.

Introduction

On a complex manifold $X$, we will be interested in the study of holonomic modules over the ring $\mathcal{D}_X[[\hbar]]$ of differential operators with a formal parameter $\hbar$. Such modules naturally appear when studying deformation quantization modules (DQ-modules) along a smooth Lagrangian submanifold of a complex symplectic manifold (see [11, Chapter 7]).

In this paper, after recalling the tools from loc. cit. that we shall use, we explain some basic notions of $\mathcal{D}_X[[\hbar]]$-modules theory. For example, it follows easily from general results on modules over $\mathbb{C}[[\hbar]]$-algebras that given two holonomic $\mathcal{D}_X[[\hbar]]$-modules $\mathcal{M}$ and $\mathcal{N}$, the complex $R\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{N})$ is constructible over $\mathbb{C}[[\hbar]]$ and the microsupport of the solution complex $R\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{O}_X[[\hbar]])$ coincides with the characteristic variety of $\mathcal{M}$.

Then we establish our main result, the Riemann-Hilbert correspondence for regular holonomic $\mathcal{D}_X[[\hbar]]$-modules, an $\hbar$-variant of Kashiwara’s classical theorem. In other words, we show that the solution functor with values in $\mathcal{O}_X[[\hbar]]$ induces an equivalence between the derived category of regular holonomic $\mathcal{D}_X[[\hbar]]$-modules and that of constructible sheaves over $\mathbb{C}[[\hbar]]$. A quasi-inverse is obtained by constructing the “sheaf” of holomorphic functions with temperate growth and a formal parameter $\hbar$ in the subanalytic site. This needs some care since the literature on this subject is written in the framework of sheaves over a field and does not immediately apply to the ring $\mathbb{C}[[\hbar]]$. 
We also discuss the $t$-structure related to $h$-torsion. Indeed, as we work over the ring $\mathbb{C}[[h]]$ and not over a field, the derived category of holonomic $\mathcal{D}_X[[h]]$-modules (or, equivalently, that of constructible sheaves over $\mathbb{C}[[h]]$) has an additional $t$-structure related to $h$-torsion. We will show how the duality functor interchanges it with the natural $t$-structure.

Finally, we describe some natural links between the ring $\mathcal{D}_X[[h]]$ and deformation quantization algebras, as mentioned above.

**Notations and conventions**

We shall mainly follow the notations of [10]. In particular, if $\mathcal{C}$ is an abelian category, we denote by $\mathcal{D}(\mathcal{C})$ the derived category of $\mathcal{C}$ and by $\mathcal{D}^*(\mathcal{C})$ ($* = +, -, b$) the full triangulated subcategory consisting of objects with bounded from below (resp. bounded from above, resp. bounded) cohomology.

For a sheaf of rings $\mathcal{R}$ on a topological space, or more generally a site, we denote by $\text{Mod}(\mathcal{R})$ the category of left $\mathcal{R}$-modules and we write $\mathcal{D}^*(\mathcal{R})$ instead of $\mathcal{D}^*(\text{Mod}(\mathcal{R}))$ ($* = \emptyset, +, -, b$). We denote by $\text{Mod}_{\text{coh}}(\mathcal{R})$ the full abelian subcategory of $\text{Mod}(\mathcal{R})$ of coherent objects, and by $\mathcal{D}_{\text{coh}}^b(\mathcal{R})$ the full triangulated subcategory of $\mathcal{D}^b(\mathcal{R})$ of objects with coherent cohomology groups.

If $R$ is a ring (a sheaf of rings over a point), we write for short $\mathcal{D}_{\text{coh}}^b(R)$ instead of $\mathcal{D}_{\text{coh}}^b(\text{Mod}(R))$.

1 \ Formal deformations (after [11])

We review here some definitions and results from [11] that we shall use in this paper.

**Modules over $\mathbb{Z}[h]$-algebras**

One says that a $\mathbb{Z}[h]$-module $\mathcal{M}$ has no $h$-torsion if $\mathcal{M} \to \mathcal{M}$ is injective and one says that $\mathcal{M}$ is $h$-complete if $\mathcal{M} \to \lim_{\mathcal{M}/h^n}$ is an isomorphism.

Let $\mathcal{R}$ be a $\mathbb{Z}[h]$-algebra, and assume that $\mathcal{R}$ has no $h$-torsion. One sets

$$\mathcal{R}^{\text{loc}} := \mathbb{Z}[h, h^{-1}] \otimes_{\mathbb{Z}[h]} \mathcal{R}, \quad \mathcal{R}_0 := \mathcal{R}/h\mathcal{R},$$
and considers the functors
\[(\cdots)^{\text{loc}}: \operatorname{Mod}(\mathcal{R}) \to \operatorname{Mod}(\mathcal{R}^{\text{loc}}), \quad \mathcal{M} \mapsto \mathcal{M}^{\text{loc}} := \mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M},\]
\[\text{gr}_h: D(\mathcal{R}) \to D(\mathcal{R}_0), \quad \mathcal{M} \mapsto \text{gr}_h(\mathcal{M}) := \mathcal{R}_0^{\mathbb{L}} \otimes_{\mathcal{R}} \mathcal{M}.\]

Note that \((\cdots)^{\text{loc}}\) is exact and that for \(\mathcal{M}, \mathcal{N} \in D^b(\mathcal{R})\) and \(\mathcal{P} \in D^b(\mathcal{R}^{\text{op}})\) one has isomorphisms:

\[
\begin{align*}
(1.1) \quad & \quad \text{gr}_h(\mathcal{P}^{\mathbb{L}} \otimes_{\mathcal{R}} \mathcal{M}) \simeq \text{gr}_h \mathcal{P}^{\mathbb{L}} \otimes_{\mathcal{R}_0} \text{gr}_h \mathcal{M}, \\
(1.2) \quad & \quad \text{gr}_h(\mathcal{R} \mathcal{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N})) \simeq \mathcal{R} \mathcal{H}om_{\mathcal{R}_0}(\text{gr}_h \mathcal{M}, \text{gr}_h \mathcal{N}).
\end{align*}
\]

### Cohomologically \(h\)-complete sheaves

**Definition 1.1.** One says that an object \(\mathcal{M}\) of \(D(\mathcal{R})\) is cohomologically \(h\)-complete if \(\mathcal{R} \mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M}, \mathcal{M}) = 0\).

Hence, the full subcategory of cohomologically \(h\)-complete objects is triangulated. In fact, it is the right orthogonal to the full subcategory \(D(\mathcal{R}^{\text{loc}})\) of \(D(\mathcal{R})\).

Remark that \(\mathcal{M} \in D(\mathcal{R})\) is cohomologically \(h\)-complete if and only if its image in \(D(\mathbb{Z}[\mathcal{X}[h]])\) is cohomologically \(h\)-complete.

**Proposition 1.2.** Let \(\mathcal{M} \in D(\mathcal{R})\). Then \(\mathcal{M}\) is cohomologically \(h\)-complete if and only if

\[\lim_{\substack{\text{U}\in\mathcal{X} \ni x \text{ open}} \mathbb{E}xt^j_{\mathbb{Z}[h]}(\mathbb{Z}[h, h^{-1}], H^i(U; \mathcal{M}))} = 0,\]

for any \(x \in X\), any integer \(i \in \mathbb{Z}\) and any \(j = 0, 1\). Here, \(U\) ranges over an open neighborhood system of \(x\).

**Corollary 1.3.** Let \(\mathcal{M} \in \operatorname{Mod}(\mathcal{R})\). Assume that \(\mathcal{M}\) has no \(h\)-torsion, is \(h\)-complete and there exists a base \(\mathcal{B}\) of open subsets such that \(H^i(U; \mathcal{M}) = 0\) for any \(i > 0\) and any \(U \in \mathcal{B}\). Then \(\mathcal{M}\) is cohomologically \(h\)-complete.

The functor \(\text{gr}_h\) is conservative on the category of cohomologically \(h\)-complete objects:

**Proposition 1.4.** Let \(\mathcal{M} \in D(\mathcal{R})\) be a cohomologically \(h\)-complete object. If \(\text{gr}_h(\mathcal{M}) = 0\), then \(\mathcal{M} = 0\).
Proposition 1.5. Assume that $M \in D(\mathcal{R})$ is cohomologically $\hbar$-complete. Then $R\mathcal{H}om(\mathcal{N}, M) \in D(\mathbb{Z}_X[\hbar])$ is cohomologically $\hbar$-complete for any $\mathcal{N} \in D(\mathcal{R})$.

Proposition 1.6. Let $f: X \rightarrow Y$ be a continuous map, and $M \in D(\mathbb{Z}_X[\hbar])$. If $M$ is cohomologically $\hbar$-complete, then so is $Rf_*M$.

Reductions to $\hbar = 0$

Now we assume that $X$ is a Hausdorff locally compact topological space.

By a basis $\mathcal{B}$ of compact subsets of $X$, we mean a family of compact subsets such that for any $x \in X$ and any open neighborhood $U$ of $x$, there exists $K \in \mathcal{B}$ such that $x \in \text{Int}(K) \subset U$.

Let $\mathcal{A}$ be a $\mathbb{Z}[\hbar]$-algebra, and recall that we set $\mathcal{A}_0 = \mathcal{A} / \hbar \mathcal{A}$. Consider the following conditions:

(i) $\mathcal{A}$ has no $\hbar$-torsion and is $\hbar$-complete,

(ii) $\mathcal{A}_0$ is a left Noetherian ring,

(iii) there exists a basis $\mathcal{B}$ of compact subsets of $X$ and a prestack $U \mapsto \text{Mod}_{\text{gd}}(\mathcal{A}_0|U)$ ($U$ open in $X$) such that

(a) for any $K \in \mathcal{B}$ and an open subset $U$ such that $K \subset U$, there exists $K' \in \mathcal{B}$ such that $K \subset \text{Int}(K') \subset K' \subset U$,

(b) $U \mapsto \text{Mod}_{\text{gd}}(\mathcal{A}_0|U)$ is a full subprestack of $U \mapsto \text{Mod}_{\text{coh}}(\mathcal{A}_0|U)$,

(c) for any $K \in \mathcal{B}$, any open set $U$ containing $K$, any $\mathcal{M} \in \text{Mod}_{\text{gd}}(\mathcal{A}_0|U)$ and any $j > 0$, one has $H^j(K; \mathcal{M}) = 0$,

(d) for an open subset $U$ and $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|U)$, if $\mathcal{M}|V$ belongs to $\text{Mod}_{\text{gd}}(\mathcal{A}_0|V)$ for any relatively compact open subset $V$ of $U$, then $\mathcal{M}$ belongs to $\text{Mod}_{\text{gd}}(\mathcal{A}_0|U)$,

(e) for any $U$ open in $X$, $\text{Mod}_{\text{gd}}(\mathcal{A}_0|U)$ is stable by subobjects, quotients and extensions in $\text{Mod}_{\text{coh}}(\mathcal{A}_0|U)$,

(f) for any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathcal{M}|U_i \in \text{Mod}_{\text{gd}}(\mathcal{A}_0|U_i)$,

(g) $\mathcal{A}_0 \in \text{Mod}_{\text{gd}}(\mathcal{A}_0)$,

(iii') there exists a basis $\mathcal{B}$ of open subsets of $X$ such that for any $U \in \mathcal{B}$, any $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|U)$ and any $j > 0$, one has $H^j(U; \mathcal{M}) = 0$. 

4
We will suppose that $\mathcal{A}$ and $\mathcal{A}_0$ satisfy either Assumption 1.7 or Assumption 1.8.

**Assumption 1.7.** $\mathcal{A}$ and $\mathcal{A}_0$ satisfy conditions (i), (ii) and (iii) above.

**Assumption 1.8.** $\mathcal{A}$ and $\mathcal{A}_0$ satisfy conditions (i), (ii) and (iii)' above.

**Theorem 1.9.** (i) $\mathcal{A}$ is a left Noetherian ring.

(ii) Any coherent $\mathcal{A}$-module $\mathcal{M}$ is $\mathfrak{h}$-complete.

(iii) Let $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{A})$. Then $\mathcal{M}$ is cohomologically $\mathfrak{h}$-complete.

**Corollary 1.10.** The functor $\text{gr}_\mathfrak{h} : D^b_{\text{coh}}(\mathcal{A}) \to D^b_{\text{coh}}(\mathcal{A}_0)$ is conservative.

**Theorem 1.11.** Let $\mathcal{M} \in D^+(\mathcal{A})$ and assume:

(a) $\mathcal{M}$ is cohomologically $\mathfrak{h}$-complete,

(b) $\text{gr}_\mathfrak{h}(\mathcal{M}) \in D^+_{\text{coh}}(\mathcal{A}_0)$.

Then, $\mathcal{M} \in D^+_{\text{coh}}(\mathcal{A})$ and for all $i \in \mathbb{Z}$ we have the isomorphism

$$H^i(\mathcal{M}) \cong \lim_{\leftarrow n} H^i(\mathcal{A} / \mathfrak{h}^n \mathcal{A} \otimes^L_\mathcal{A} \mathcal{M}).$$

**Theorem 1.12.** Assume that $\mathcal{A}_0^{\text{op}} = \mathcal{A}^{\text{op}} / \mathfrak{h} \mathcal{A}^{\text{op}}$ is a Noetherian ring and the flabby dimension of $X$ is finite. Let $\mathcal{M}$ be an $\mathcal{A}$-module. Assume the following conditions:

(a) $\mathcal{M}$ has no $\mathfrak{h}$-torsion,

(b) $\mathcal{M}$ is cohomologically $\mathfrak{h}$-complete,

(c) $\mathcal{M} / \mathfrak{h} \mathcal{M}$ is a flat $\mathcal{A}_0$-module.

Then $\mathcal{M}$ is a flat $\mathcal{A}$-module.

If moreover $\mathcal{M} / \mathfrak{h} \mathcal{M}$ is a faithfully flat $\mathcal{A}_0$-module, then $\mathcal{M}$ is a faithfully flat $\mathcal{A}$-module.

**Theorem 1.13.** Let $d \in \mathbb{N}$. Assume that $\mathcal{A}_0$ is $d$-syzygic, i.e., that any coherent $\mathcal{A}_0$-module locally admits a projective resolution of length $\leq d$ by free $\mathcal{A}_0$-modules of finite rank. Then
(a) $\mathcal{A}$ is $(d+1)$-syzygic.

(b) Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{A}$-modules concentrated in degrees $[a,b]$ and with coherent cohomology groups. Then, locally there exists a quasi-isomorphism $\mathcal{L}^\bullet \to \mathcal{M}^\bullet$ where $\mathcal{L}^\bullet$ is a complex of free $\mathcal{A}$-modules of finite rank concentrated in degrees $[a-d-1,b]$.

**Proposition 1.14.** Let $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{A})$ and let $a \in \mathbb{Z}$. The conditions below are equivalent:

(i) $H^a(\text{gr}_h(\mathcal{M})) \simeq 0$,

(ii) $H^a(\mathcal{M}) \simeq 0$ and $H^{a+1}(\mathcal{M})$ has no $h$-torsion.

**Cohomologically $h$-complete sheaves on real manifolds**

Let now $X$ be a real analytic manifold. Recall from [7] that the microsupport of $F \in D^b(Z_X)$ is a closed involutive subset of the cotangent bundle $T^*X$ denoted by $\text{SS}(F)$. The microsupport is additive on $D^b(Z_X)$ (cf Definition 3.3 (ii) below). Considering the distinguished triangle $F \xrightarrow{h} F \to \text{gr}_h F \xrightarrow{+1}$, one gets the estimate

\[(1.3) \quad \text{SS}(\text{gr}_h(F)) \subset \text{SS}(F).\]

Using Proposition 1.4 and 1.6, one easily proves:

**Proposition 1.15.** Let $F \in D^b(Z_X[h])$ and assume that $F$ is cohomologically $h$-complete. Then

\[(1.4) \quad \text{SS}(F) = \text{SS}(\text{gr}_h(F)).\]

For $\mathbb{K}$ a commutative unital Noetherian ring, one denotes by $\text{Mod}_{\mathbb{R},c}(\mathbb{K}_X)$ the full subcategory of $\text{Mod}(\mathbb{K}_X)$ consisting of $\mathbb{R}$-constructible sheaves and by $D^b_{\mathbb{R},c}(\mathbb{K}_X)$ the full triangulated subcategory of $D^b(\mathbb{K}_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology. In this paper, we shall mainly be interested with the case where $\mathbb{K}$ is either $\mathbb{C}$ or the ring of formal power series in an indeterminate $h$, that we denote by $\mathbb{C}^h := \mathbb{C}[[h]]$.

By Proposition 1.2 one has
Proposition 1.16. Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}^h_X)$. Then $F$ is cohomologically $h$-complete.

Corollary 1.17. The functor $\text{gr}_h : \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}^h_X) \to \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$ is conservative.

Corollary 1.18. For $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}^h_X)$, one has the equality

$$\text{SS}(\text{gr}_h(F)) = \text{SS}(F).$$

Proposition 1.19. For $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{C}^h_X)$ and $i \in \mathbb{Z}$ one has $\text{supp} H^i(F) \subset \text{supp} H^i(\text{gr}_h F)$. In particular if $H^i(\text{gr}_h F) = 0$ then $H^i(F) = 0$.

Proof. We apply Proposition 1.14 to $F_x$ for any $x \in X$. Q.E.D.

2 Formal extension

Let $X$ be a topological space, or more generally a site, and let $\mathcal{R}_0$ be a sheaf of rings on $X$. In this section, we let

$$\mathcal{R} := \mathcal{R}_0[[h]] = \prod_{n \geq 0} \mathcal{R}_0 h^n$$

be the formal extension of $\mathcal{R}_0$, whose sections on an open subset $U$ are formal series $r = \sum_{n=0}^{\infty} r_j h^n$, with $r_j \in \Gamma(U; \mathcal{R}_0)$. Consider the associated functor

$$\begin{align*}
(\cdot)^h : \text{Mod}(\mathcal{R}_0) &\to \text{Mod}(\mathcal{R}), \\
\mathcal{N} &\mapsto \mathcal{N}[[h]] = \lim_{\leftarrow n}(\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}),
\end{align*}$$

where $\mathcal{R}_n := \mathcal{R}/h^{n+1}\mathcal{R}$ is regarded as an $(\mathcal{R}, \mathcal{R}_0)$-bimodule. Since $\mathcal{R}_n$ is free of finite rank over $\mathcal{R}_0$, the functor $(\cdot)^h$ is left exact. We denote by $(\cdot)^{Rh}$ its right derived functor.

Proposition 2.1. For $\mathcal{N} \in \mathbf{D}^b(\mathcal{R}_0)$ one has

$$\mathcal{N}^{Rh} \simeq \mathcal{R}\text{Hom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/h\mathcal{R}, \mathcal{N}),$$

where $\mathcal{R}^{\text{loc}}/h\mathcal{R}$ is regarded as an $(\mathcal{R}_0, \mathcal{R})$-bimodule.

Proof. It is enough to prove that for $\mathcal{N} \in \text{Mod}(\mathcal{R}_0)$ one has

$$\mathcal{N}^h \simeq \mathcal{R}\text{Hom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/h\mathcal{R}, \mathcal{N}).$$

7
Let $\mathcal{R}_n^* = \text{Hom}_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0)$, regarded as an $(\mathcal{R}_0, \mathcal{R})$-bimodule. Then

$$\mathcal{N}^h = \lim_{\leftarrow n} (\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}) \simeq \text{Hom}_{\mathcal{R}_0}(\lim_{\leftarrow n} \mathcal{R}_n^*, \mathcal{N}).$$

Since

$$\mathcal{R}^{\text{loc}} / \hbar \mathcal{R} \simeq \lim_{\leftarrow n} (\hbar^{-n} \mathcal{R} / \hbar \mathcal{R}),$$

it is enough to prove that there is an isomorphism of $(\mathcal{R}_0, \mathcal{R})$-bimodules

$$\text{Hom}_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0) \simeq \hbar^{-n} \mathcal{R} / \hbar \mathcal{R}.$$

Recalling that $\mathcal{R}_n = \mathcal{R} / \hbar^{n+1} \mathcal{R}$, this follows from the pairing

$$(\mathcal{R} / \hbar^{n+1} \mathcal{R}) \otimes_{\mathcal{R}_0} (\hbar^{-n} \mathcal{R} / \hbar \mathcal{R}) \to \mathcal{R}_0, \quad f \otimes g \mapsto \text{Res}_{\mathcal{R}_0}(f g dh/\hbar).$$

Q.E.D.

Note that the isomorphism of $(\mathcal{R}, \mathcal{R}_0)$-bimodules

$$\mathcal{R} \simeq (\mathcal{R}_0)^h = \text{Hom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}} / \hbar \mathcal{R}, \mathcal{R}_0)$$

induces a natural morphism

(2.2) \hspace{1cm} \mathcal{R} \otimes_{\mathcal{R}_0} \mathcal{N} \to \mathcal{N}^{R\hbar}, \quad \text{for } \mathcal{N} \in \text{D}^b(\mathcal{R}_0).

**Proposition 2.2.** For $\mathcal{N} \in \text{D}^b(\mathcal{R}_0)$, its formal extension $\mathcal{N}^{R\hbar}$ is cohomologically $\hbar$-complete.

**Proof.** The statement follows from $(\mathcal{R}^{\text{loc}} / \hbar \mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}^{\text{loc}} \simeq 0$ and from the isomorphism

$$\text{RHom}_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}, \mathcal{N}^{R\hbar}) \simeq \text{RHom}_{\mathcal{R}_0}((\mathcal{R}^{\text{loc}} / \hbar \mathcal{R}) \otimes_{\mathcal{R}} \mathcal{R}^{\text{loc}}, \mathcal{N}).$$

Q.E.D.

**Lemma 2.3.** Assume that $\mathcal{R}_0$ is an $\mathcal{I}_0$-algebra, for $\mathcal{I}_0$ a commutative sheaf of rings, and let $\mathcal{I} = \mathcal{I}_0[[h]]$. For $\mathcal{M}, \mathcal{N} \in \text{D}^b(\mathcal{R}_0)$ we have an isomorphism in $\text{D}^b(\mathcal{I})$

$$\text{RHom}_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})^{R\hbar} \simeq \text{RHom}_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N}^{R\hbar}).$$
Proof. Noticing that $R_{\mathcal{L}/\hbar} \simeq R_0 \otimes_{\mathcal{I}/\hbar} (\mathcal{L}_{\mathcal{L}/\hbar})$ as $(R_0, \mathcal{I})$-bimodules, one has

$$R_{\text{Hom}}_{R_0} (M, N)^{R_{\mathcal{L}}} = R_{\text{Hom}}_{R_0} (\mathcal{I}_{\mathcal{L}/\hbar}, R_{\text{Hom}}_{R_0} (M, N))$$

$$\simeq R_{\text{Hom}}_{R_0} (\mathcal{I}_{\mathcal{L}/\hbar}, R_{\text{Hom}}_{R_0} (M, N))$$

$$\simeq R_{\text{Hom}}_{R_0} (M, R_{\text{Hom}}_{R_0} (\mathcal{I}_{\mathcal{L}/\hbar}, N))$$

$$= R_{\text{Hom}}_{R_0} (M, N^{R_{\mathcal{L}}}).$$

Q.E.D.

Lemma 2.4. Let $f: Y \to X$ be a morphism of sites, and assume that $\left(f^{-1} R_0\right)^{h} \simeq f^{-1} \mathcal{R}$. Then the functors $R_{f*}$ and $(\cdot)^{R_{\mathcal{L}}}$ commute, that is, for $\mathcal{P} \in D^{b}(f^{-1} R_0)$ we have $(R_{f*} \mathcal{P})^{R_{\mathcal{L}}} \simeq R_{f*} (\mathcal{P}^{R_{\mathcal{L}}})$ in $D^{b}(\mathcal{R})$.

Proof. One has the isomorphism

$$R_{f*} (\mathcal{P}^{R_{\mathcal{L}}}) = R_{f*} R_{\text{Hom}}_{f^{-1} R_0} (f^{-1} (\mathcal{I}_{\mathcal{L}/\hbar}), \mathcal{P})$$

$$\simeq R_{\text{Hom}}_{R_0} (\mathcal{I}_{\mathcal{L}/\hbar}, R_{f*} \mathcal{P})$$

$$= R_{f*} (\mathcal{P}^{R_{\mathcal{L}}}).$$

Q.E.D.

Proposition 2.5. Let $\mathcal{T}$ be either a basis of open subsets of the site $X$ or, assuming that $X$ is a locally compact topological space, a basis of compact subsets. Denote by $J_{\mathcal{T}}$ the full subcategory of $\text{Mod}(R_0)$ consisting of $\mathcal{T}$-acyclic objects, i.e., sheaves $\mathcal{N}$ for which $H^k(S; \mathcal{N}) = 0$ for all $k > 0$ and all $S \in \mathcal{T}$. Then $J_{\mathcal{T}}$ is injective with respect to the functor $(\cdot)^{h}$. In particular, for $\mathcal{N} \in J_{\mathcal{T}}$, we have $\mathcal{N}^{h} \simeq \mathcal{N}^{R_{\mathcal{L}}}$.}

Proof. (i) Since injective sheaves are $\mathcal{T}$-acyclic, $J_{\mathcal{T}}$ is cogenerating.

(ii) Consider an exact sequence $0 \to \mathcal{N}^{-} \to \mathcal{N} \to \mathcal{N}^{-h} \to 0$ in $\text{Mod}(R_0)$. Clearly, if both $\mathcal{N}^{-}$ and $\mathcal{N}$ belong to $J_{\mathcal{T}}$, then so does $\mathcal{N}^{-h}$.

(iii) Consider an exact sequence as in (ii) and assume that $\mathcal{N}^{-} \in J_{\mathcal{T}}$. We have to prove that $0 \to \mathcal{N}^{-h} \to \mathcal{N} \to \mathcal{N}^{-h} \to 0$ is exact. Since $(\cdot)^{h}$ is left exact, it is enough to prove that $\mathcal{N}^{-h} \to \mathcal{N}^{-h}$ is surjective. Noticing that $\mathcal{N}^{-h} \simeq \prod_{N} \mathcal{N}$ as $R_0$-modules, it is enough to prove that $\prod_{N} \mathcal{N} \to \prod_{N} \mathcal{N}^{-h}$ is surjective.

(iii)-(a) Assume that $\mathcal{T}$ is a basis of open subsets. Any open subset $U \subset X$ has a cover $\{U_i\}_{i \in I}$ by elements $U_i \in \mathcal{T}$. For any $i \in I$, the morphism
\( \mathcal{N}(U_i) \to \mathcal{N}''(U_i) \) is surjective. The result follows taking the product over \( \mathbb{N} \).

(iii)-(b) Assume that \( \mathcal{F} \) is a basis of compact subsets. For any \( K \in \mathcal{F} \), the morphism \( \mathcal{N}(K) \to \mathcal{N}''(K) \) is surjective. Hence, there exists a basis \( \mathcal{V} \) of open subsets such that for any \( x \in X \) and any \( V \ni x \in \mathcal{V} \), there exists \( V' \in \mathcal{V} \) with \( x \in V' \subset V \) and the image of \( \mathcal{N}(V') \to \mathcal{N}''(V') \) contains the image of \( \mathcal{N}''(V) \) in \( \mathcal{N}''(V') \). The result follows as in (iii)-(a) taking the product over \( \mathbb{N} \). Q.E.D.

**Corollary 2.6.** The following sheaves are acyclic for the functor \((\bullet)^h\):

(i) \( \mathbb{R}\text{-constructible sheaves of } \mathbb{C}\text{-vector spaces on a real analytic manifold } X \) (see [7, §8.4]),

(ii) coherent modules over the ring \( \mathcal{O}_X \) of holomorphic functions on a complex analytic manifold \( X \),

(iii) coherent modules over the ring \( \mathcal{D}_X \) of linear differential operators on a complex analytic manifold \( X \).

**Proof.** The statements follow by applying Proposition 2.5 for the following choices of \( \mathcal{F} \).

(i) Let \( F \) be an \( \mathbb{R}\text{-constructible sheaf}. \) Then for any \( x \in X \) one has \( F_x \cong \Gamma(U_x; F) \) for \( U_x \) in a fundamental system of open neighborhoods of \( x \). Take for \( \mathcal{F} \) the union of these fundamental systems.

(ii) Take for \( \mathcal{F} \) the family of open Stein subsets.

(iii) Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. The problem being local, we may assume that \( \mathcal{M} \) is endowed with a good filtration. Then take for \( \mathcal{F} \) the family of compact Stein subsets. Q.E.D.

**Example 2.7.** Let \( X = \mathbb{R} \), \( \mathcal{A}_0 = \mathcal{C}_X \), \( Z = \{1/n: n = 1, 2, \ldots \} \cup \{0\} \) and \( U = X \setminus Z \). One has the isomorphisms \( (\mathcal{C}^h)_X \cong (\mathcal{C}_X)^h \cong (\mathcal{C}_X)^{Rh} \) and \( (\mathcal{C}^h)_U \cong (\mathcal{C}_U)^h \). Considering the exact sequences

\[
0 \to (\mathcal{C}^h)_U \to (\mathcal{C}^h)_X \to (\mathcal{C}^h)_Z \to 0,
\]

\[
0 \to (\mathcal{C}_U)^h \to (\mathcal{C}_X)^h \to (\mathcal{C}_Z)^h \to H^1(\mathcal{C}_U)^{Rh} \to 0,
\]

we get \( H^1(\mathcal{C}_U)^{Rh} \cong (\mathcal{C}_Z)^h/(\mathcal{C}^h)_Z \), whose stalk at the origin does not vanish. Hence \( \mathcal{C}_U \) is not acyclic for the functor \((\bullet)^h\).
Assume now that
\[ \mathcal{A}_0 = \mathcal{B}_0 \quad \text{and} \quad \mathcal{A} = \mathcal{B}_0[[h]] \]
satisfy either Assumption 1.7 or Assumption 1.8 (where condition (i) is clear) and that \( \mathcal{A}_0 \) is syzygic. Note that by Proposition 2.5 one has \( \mathcal{A} \simeq (\mathcal{A}_0)^{Rh} \).

**Proposition 2.8.** For \( \mathcal{N} \in D^b_{coh}(\mathcal{A}_0) \):

(i) there is an isomorphism \( \mathcal{N}^{Rh} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{N} \) induced by (2.2),

(ii) there is an isomorphism \( \text{gr}_h(\mathcal{N}^h) \simeq \mathcal{N} \).

**Proof.** Since \( \mathcal{A}_0 \) is syzygic, we may locally represent \( \mathcal{N} \) by a bounded complex \( \mathcal{L}^\bullet \) of free \( \mathcal{A}_0 \)-modules of finite rank. Then (i) is obvious. As for (ii), both complexes are isomorphic to the mapping cone of \( h: (\mathcal{L}^\bullet)^h \to (\mathcal{L}^\bullet)^h \).

Q.E.D.

In particular, the functor \( (\bullet)^h \) is exact on \( \text{Mod}_{coh}(\mathcal{A}_0) \) and preserves coherence. One thus get a functor
\[ (\bullet)^{Rh}: D^b_{coh}(\mathcal{A}_0) \to D^b_{coh}(\mathcal{A}). \]

**The subanalytic site**

The subanalytic site associated to an analytic manifold \( X \) has been introduced and studied in [9, Chapter 7] (see also [13] for a detailed and systematic study as well as for complementary results). Denote by \( \text{Op}_X \) the category of open subsets of \( X \), the morphisms being the inclusion morphisms, and by \( \text{Op}_{X_{sa}} \) the full subcategory consisting of relatively compact subanalytic open subsets of \( X \). The site \( X_{sa} \) is the presite \( \text{Op}_{X_{sa}} \) endowed with the Grothendieck topology for which the coverings are those admitting a finite subcover. One calls \( X_{sa} \) the subanalytic site associated to \( X \). Denote by \( \rho: X \to X_{sa} \) the natural morphism of sites. Recall that the inverse image functors \( \rho^{-1} \), besides the usual right adjoint given by the direct image functor \( \rho_* \), admits a left adjoint denoted \( \rho_! \). Consider the diagram
\[
\begin{array}{cccc}
D^b(C_X) & \xrightarrow{R\rho_*} & D^b(C_{X_{sa}}) \\
\downarrow (\bullet)^{Rh} & & \downarrow (\bullet)^{Rh} \\
D^b(C_{hX}) & \xrightarrow{R\rho_*} & D^b(C_{hX_{sa}}).
\end{array}
\]
Lemma 2.9.  (i) The functors $\rho^{-1}$ and $(\cdot)^{Rh}$ commute, that is, for $G \in D^b(C_{X_{sa}})$ we have $(\rho^{-1}G)^{Rh} \simeq \rho^{-1}(G^{Rh})$ in $D^b(C^h_X)$.

(ii) The functors $R\rho_*$ and $(\cdot)^{Rh}$ commute, that is, for $F \in D^b(C_X)$ we have $(R\rho_*F)^{Rh} \simeq R\rho_*(F^{Rh})$ in $D^b(C^h_{X_{sa}})$.

Proof. (i) Since it admits a left adjoint, the functor $\rho^{-1}$ commutes with projective limits. It follows that for $G \in \text{Mod}(C_{X_{sa}})$ one has an isomorphism

$$\rho^{-1}(G^h) \to (\rho^{-1}G)^h.$$ 

To conclude, it remains to show that $(\rho^{-1}(\cdot))^{Rh}$ is the derived functor of $(\rho^{-1}(\cdot))^h$. Recall that an object $G$ of $\text{Mod}(C_{X_{sa}})$ is quasi-injective if the functor $\text{Hom}_{C_{X_{sa}}}((\cdot), G)$ is exact on the category $\text{Mod}_{\mathfrak{g}-c}(C_X)$. By a result of [13], if $G \in \text{Mod}(C_{X_{sa}})$ is quasi-injective, then $\rho^{-1}G$ is soft. Hence, $\rho^{-1}G$ is injective for the functor $(\cdot)^h$ by Proposition 2.5.

(ii) By (i) we can apply Lemma 2.4. Q.E.D.

3 $\mathcal{D}[[h]]$-modules and propagation

Let now $X$ be a complex analytic manifold of complex dimension $d_X$. As usual, denote by $C_X$ the constant sheaf with stalk $\mathbb{C}$, by $\mathcal{O}_X$ the structure sheaf and by $\mathcal{D}_X$ the ring of linear differential operators on $X$. We will use the notations

$$D' : D^b(C_X)^{op} \to D^b(C_X), \quad F \mapsto R\mathcal{H}om_{C_X}(F, C_X),$$
$$D : D^b_{\text{coh}}(\mathcal{D}_X)^{op} \to D^b_{\text{coh}}(\mathcal{D}_X), \quad M \mapsto R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \Omega_X^{-1})[d_X],$$
$$\text{Sol} : D^b_{\text{coh}}(\mathcal{D}_X)^{op} \to D^b(C_X), \quad M \mapsto R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X),$$
$$\text{DR} : D^b_{\text{coh}}(\mathcal{D}_X) \to D^b(C_X), \quad M \mapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M),$$

where $\Omega_X$ denotes the line bundle of holomorphic forms of maximal degree and $\Omega_X^{-1}$ the dual bundle.

As shown in Corollary 2.6, the sheaves $C_X, \mathcal{O}_X$ and $\mathcal{D}_X$ are all acyclic for the functor $(\cdot)^h$. We will be interested in the formal extensions

$$C^h_X = C_X[[h]], \quad \mathcal{O}^h_X = \mathcal{O}_X[[h]], \quad \mathcal{D}^h_X = \mathcal{D}_X[[h]].$$

In the sequel, we shall treat left $\mathcal{D}^h_X$-modules, but all results apply to right modules since the categories $\text{Mod}(\mathcal{D}^h_X)$ and $\text{Mod}(\mathcal{D}^{h,\text{op}}_X)$ are equivalent.
Proposition 3.1. The $\mathbb{C}_h$-algebras $\mathcal{D}_X^h$ and $\mathcal{D}_X^{h,\text{op}}$ satisfy Assumptions 1.7.

Proof. Assumption 1.7 hold for $\mathcal{A} = \mathcal{D}_X^h$, $\mathcal{A}_0 = \mathcal{D}_X$, $\text{Mod}_{\text{gd}}(\mathcal{A}_0|_U)$ the category of good $\mathcal{D}_U$-modules (see [5]) and for $\mathcal{B}$ the family of Stein compact subsets of $X$. Q.E.D.

In particular, by Theorem 1.11 one has that $\mathcal{D}_X^h$ is right and left Noetherian (and thus coherent). Moreover, by Theorem 1.13 any object of $\text{D}^{b}_{\text{coh}}(\mathcal{D}_X^h)$ can be locally represented by a bounded complex of free $\mathcal{D}_X^h$-modules of finite rank.

We will use the notations

$$
\begin{align*}
\mathcal{D}_h': \mathcal{D}^b_{\text{coh}}(\mathcal{C}_X^h)^{\text{op}} & \to \mathcal{D}^b(\mathcal{C}_X^h), \\
\mathcal{D}_h: \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^h)^{\text{op}} & \to \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^h), \\
\text{Sol}_h: \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^h)^{\text{op}} & \to \mathcal{D}^b(\mathcal{C}_X^h), \\
\text{DR}_h: \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X^h) & \to \mathcal{D}^b(\mathcal{C}_X^h),
\end{align*}
$$

By Proposition 2.8 and Lemma 2.3, for $\mathcal{N} \in \text{D}^{b}_{\text{coh}}(\mathcal{D}_X)$ one has

$$
\begin{align*}
\mathcal{N}^{\mathcal{R}h} & \simeq \mathcal{D}_X^h \otimes_{\mathcal{D}_X} \mathcal{N}, \\
\text{gr}_h(\mathcal{N}^{\mathcal{R}h}) & \simeq \mathcal{N}, \\
\text{Sol}_h(\mathcal{N}^{\mathcal{R}h}) & \simeq \text{Sol}(\mathcal{N})^{\mathcal{R}h}.
\end{align*}
$$

Definition 3.2. For $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$, denote by $\mathcal{M}_{h\text{-tor}}$ its submodule consisting of sections locally annihilated by some power of $h$ and set $\mathcal{M}_{h\text{-tf}} = \mathcal{M} / \mathcal{M}_{h\text{-tor}}$. We say that $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^h)$ is an $h$-torsion module if $\mathcal{M}_{h\text{-tor}} \to \mathcal{M}$ and that $\mathcal{M}$ has no $h$-torsion (or is $h$-torsion free) if $\mathcal{M} \to \mathcal{M}_{h\text{-tf}}$.

Denote by $n_{\mathcal{M}}$ the kernel of $h^{n+1}: \mathcal{M} \to \mathcal{M}$. Then $\mathcal{M}_{h\text{-tor}}$ is the sheaf associated with the increasing union of the $n_{\mathcal{M}}$’s. Hence, if $\mathcal{M}$ is coherent, the increasing family $\{n_{\mathcal{M}}\}_n$ is locally stationary and $\mathcal{M}_{h\text{-tor}}$ as well as $\mathcal{M}_{h\text{-tf}}$ are coherent.

Characteristic variety

Recall the following definition
Definition 3.3. (i) For \( \mathcal{C} \) an abelian category, a function \( c: \text{Ob}(\mathcal{C}) \to \text{Set} \) is called additive if \( c(M) = c(M') \cup c(M'') \) for any short exact sequence \( 0 \to M' \to M \to M'' \to 0 \).

(ii) For \( \mathcal{T} \) a triangulated category, a function \( c: \text{Ob}(\mathcal{T}) \to \text{Set} \) is called additive if \( c(M) = c(M[1]) \) and \( c(M) \subset c(M') \cup c(M'') \) for any distinguished triangle \( M' \to M \to M'' +1 \to 0 \).

Note that an additive function \( c \) on \( \mathcal{C} \) naturally extends to the derived category \( \text{D}(\mathcal{C}) \) by setting \( c(M) = \bigcup_i c(H^i(M)) \).

For \( \mathcal{N} \) a coherent \( \mathcal{D}_X \)-module, denote by \( \text{char}(\mathcal{N}) \) its characteristic variety, a closed involutive subvariety of the cotangent bundle \( T^*X \). The characteristic variety is additive on \( \text{Mod}_{\text{coh}}(\mathcal{D}_X) \). For \( \mathcal{N} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X) \) one sets \( \text{char}(\mathcal{N}) = \bigcup_i \text{char}(H^i(\mathcal{N})) \).

Definition 3.4. The characteristic variety of \( \mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_\hbar X) \) is defined by

\[
\text{char}_\hbar(\mathcal{M}) = \text{char}(\text{gr}_\hbar(\mathcal{M})).
\]

To \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_\hbar X) \) one associates the coherent \( \mathcal{D}_X \)-modules

\[
\begin{align*}
\mathcal{M}_0 &= \text{Ker}(h: \mathcal{M} \to \mathcal{M}) = H^{-1}(\text{gr}_\hbar(\mathcal{M})), \\
\mathcal{M}_0 &= \text{Coker}(h: \mathcal{M} \to \mathcal{M}) = H^0(\text{gr}_\hbar(\mathcal{M})).
\end{align*}
\]

Lemma 3.5. For \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_\hbar X) \) an \( \hbar \)-torsion module, one has

\[
\text{char}_\hbar(\mathcal{M}) = \text{char}((\mathcal{M}_0) = \text{char}(0\mathcal{M}).
\]

Proof. By definition, \( \text{char}_\hbar(\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}(\mathcal{M}_0) \). It is thus enough to prove the equality \( \text{char}(\mathcal{M}_0) = \text{char}(0\mathcal{M}) \).

Since the statement is local we may assume that \( h^N\mathcal{M} = 0 \) for some \( N \in \mathbb{N} \). We proceed by induction on \( N \).

For \( N = 1 \) we have \( \mathcal{M} \simeq \mathcal{M}_0 \simeq 0\mathcal{M} \), and the statement is obvious.

Assume that the statement has been proved for \( N - 1 \). The short exact sequence

\[
0 \to h\mathcal{M} \to \mathcal{M} \to \mathcal{M}_0 \to 0
\]

induces the distinguished triangle

\[
\text{gr}_\hbar h\mathcal{M} \to \text{gr}_\hbar \mathcal{M} \to \text{gr}_\hbar \mathcal{M}_0 \to 1.
\]
Noticing that $\mathcal{M}_0 \simeq (\mathcal{M}_0)_0 \simeq 0(\mathcal{M}_0)$, the associated long exact cohomology sequence gives

$$0 \to 0(\mathcal{M}) \to 0 \to \mathcal{M} \to (\mathcal{M}_0)_0 \to 0.$$ 

By inductive hypothesis we have $\text{char}(0(\mathcal{M})) = \text{char}((\mathcal{M}_0)_0)$, and we deduce $\text{char}(\mathcal{M}_0) = \text{char}(\mathcal{M}_0)$ by additivity of $\text{char}(\cdot)$. Q.E.D.

**Proposition 3.6.** (i) For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$ one has

$$\text{char}_h(\mathcal{M}) = \text{char}(\mathcal{M}_0).$$

(ii) The characteristic variety $\text{char}_h(\cdot)$ is additive both on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$ and on $\mathcal{D}_b(\mathcal{D}_X^h)$.

**Proof.** (i) As $\text{char}(\text{gr}_h \mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}(0(\mathcal{M}))$, it is enough to prove the inclusion

$$(3.7) \quad \text{char}(0(\mathcal{M})) \subset \text{char}(\mathcal{M}_0).$$

Consider the short exact sequence $0 \to \mathcal{M}_{\text{h-tor}} \to \mathcal{M} \to \mathcal{M}_{\text{h-tf}} \to 0$. Since $\mathcal{M}_{\text{h-tf}}$ has no $h$-torsion, $0(\mathcal{M}_{\text{h-tf}}) = 0$. The associated long exact cohomology sequence thus gives

$$0(\mathcal{M}_{\text{h-tor}}) \simeq 0(\mathcal{M}) \text{, } 0 \to (\mathcal{M}_{\text{h-tor}})_0 \to \mathcal{M}_0 \to (\mathcal{M}_{\text{h-tf}})_0 \to 0.$$ 

We deduce

$$\text{char}(0(\mathcal{M}) = \text{char}(\mathcal{M}_{\text{h-tor}})) = \text{char}((\mathcal{M}_{\text{h-tor}})_0) \subset \text{char}(\mathcal{M}_0),$$

where the second equality follows from Lemma 3.5.

(ii) It is enough to prove the additivity on $\text{Mod}_{\text{coh}}(\mathcal{D}_X^h)$, i.e. the equality

$$\text{char}_h(\mathcal{M}) = \text{char}_h(\mathcal{M}') \cup \text{char}_h(\mathcal{M}'')$$

for $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ a short exact sequence of coherent $\mathcal{D}_X^h$-modules.

The associated distinguished triangle $\text{gr}_h \mathcal{M}' \to \text{gr}_h \mathcal{M} \to \text{gr}_h \mathcal{M}'' +1$ induces the long exact cohomology sequence

$$0(\mathcal{M}'') \to (\mathcal{M}')_0 \to \mathcal{M}_0 \to (\mathcal{M}'')_0 \to 0.$$
By additivity of char(·), the exactness of this sequence at the first, second and third term from the right, respectively, gives:

\[
\text{char}_h(M'') \subset \text{char}_h(M), \\
\text{char}_h(M) \subset \text{char}_h(M') \cup \text{char}_h(M''), \\
\text{char}_h(M') \subset \text{char}_0(M'') \cup \text{char}_h(M).
\]

Finally, note that \(\text{char}_0(M'') \subset \text{char}_h(M'') \subset \text{char}_h(M).\) Q.E.D.

**Remark 3.7.** In view of Proposition 3.6 (i), in order to define the characteristic variety of a coherent \(D^h\)-module \(M\) one could avoid derived categories considering \(\text{char}(M_0)\) instead of \(\text{char}(\text{gr}_h(M))\). It is then natural to ask if these definitions are still compatible for \(M \in D^b_{\text{coh}}(D^h_X)\), i.e. to ask if the following equality holds

\[
\bigcup_i \text{char}(H^i(\text{gr}_h(M))) = \bigcup_i \text{char}((H^iM)_0).
\]

Let us prove it. By additivity of char(·), the short exact sequence

\[
0 \to (H^iM)_0 \to H^i(\text{gr}_h(M)) \to 0 \to (H^{i+1}M) \to 0
\]

from [11, Lemma 1.4.2] induces the estimates

\[
\text{char}((H^iM)_0) \subset \text{char}(H^i(\text{gr}_h(M))),
\]

\[
\text{char}(H^i(\text{gr}_h(M))) = \text{char}((H^iM)_0) \cup \text{char}_0(H^{i+1}M)).
\]

One concludes by noticing that (3.7) gives

\[
\text{char}_0(H^{i+1}M) \subset \text{char}_h(H^{i+1}M).
\]

**Proposition 3.8.** Let \(M \in \text{Mod}(D^h_X)\) be an \(h\)-torsion module. Then \(M\) is coherent as a \(D^h_X\)-module if and only if it is coherent as a \(D_X\)-module, and in this case one has \(\text{char}_h(M) = \text{char}(M)\).

**Proof.** As in the proof of Lemma 3.5 we assume that \(h^N M = 0\) for some \(N \in \mathbb{N}\). Since coherence is preserved by extension and since the characteristic varieties of \(D^h_X\)-modules and \(D_X\)-modules are additive, we can argue by induction on \(N\) using the exact sequence (3.6). We are thus reduced to the case \(N = 1\), where \(M = M_0\) and the statement becomes obvious. Q.E.D.

It follows from (3.2) that

**Proposition 3.9.** For \(N \in D^b_{\text{coh}}(D_X)\) one has \(\text{char}_h(N^h) = \text{char}(N)\).
**Holonomic modules**

Recall that a coherent $\mathcal{D}_X$-module (or an object of the derived category) is called holonomic if its characteristic variety is isotropic. We refer e.g. to [5, Chapter 5] for the notion of regularity.

**Definition 3.10.** We say that $\mathcal{M} \in \mathcal{D}_{coh}^b(\mathcal{D}_X^h)$ is holonomic, or regular holonomic, if so is $gr_h(\mathcal{M})$. We denote by $\mathcal{D}_{hol}^b(\mathcal{D}_X^h)$ the full triangulated subcategory of $\mathcal{D}_{coh}^b(\mathcal{D}_X^h)$ of holonomic objects and by $\mathcal{D}_{rh}^b(\mathcal{D}_X^h)$ the full triangulated subcategory of regular holonomic objects.

Note that a coherent $\mathcal{D}_X^h$-module is holonomic if and only if its characteristic variety is isotropic.

**Example 3.11.** Let $\mathcal{N}$ be a regular holonomic $\mathcal{D}_X$-module. Then
   (i) $\mathcal{N}$ itself, considered as a $\mathcal{D}_X^h$-module, is regular holonomic, as follows from the isomorphism $gr_h(\mathcal{N}) \simeq \mathcal{N} \oplus \mathcal{N}[1]$;
   (ii) $\mathcal{N}^h$ is a regular holonomic $\mathcal{D}_X^h$-module, as follows from the isomorphism $gr_h(\mathcal{N}) \simeq \mathcal{N}$. In particular, $\mathcal{O}_X^h$ is regular holonomic.

**Propagation**

Denote by $\mathcal{D}_{C-c}^b(\mathcal{C}_X^h)$ the full triangulated subcategory of $\mathcal{D}^b(\mathcal{C}_X^h)$ consisting of objects with $\mathbb{C}$-constructible cohomology over the ring $\mathbb{C}^h$.

**Theorem 3.12.** Let $\mathcal{M}, \mathcal{N} \in \mathcal{D}_{coh}^b(\mathcal{D}_X)$. Then

$$SS(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) = SS(R\mathcal{H}om_{\mathcal{O}_X}(gr_h(\mathcal{M}), gr_h(\mathcal{N}))).$$

If moreover $\mathcal{M}$ and $\mathcal{N}$ are holonomic, then $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is an object of $\mathcal{D}_{C-c}^b(\mathcal{C}_X^h)$.

**Proof.** Set $F = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. Then $F$ is cohomologically $h$-complete by Theorem 1.9 and Proposition 1.5. Hence $SS(F) = SS(gr_h(F))$ by Proposition 1.15. Moreover, the finiteness of the stalks $gr_h(F_x)$ over $\mathbb{C}$ implies the finiteness of $F_x$ over $\mathbb{C}^h$ by Theorem 1.11 applied with $X = \{pt\}$ and $\mathcal{A} = \mathbb{C}^h$.

Q.E.D.

Applying Theorem 3.12, and [7, Theorem 11.3.3], we get:
Corollary 3.13. Let $\mathcal{M} \in \mathcal{D}^b_{coh}(\mathcal{D}^h_X)$. Then

$$\mathrm{SS}(\mathrm{Sol}_h(\mathcal{M})) = \mathrm{SS}(\mathrm{DR}_h(\mathcal{M})) = \mathrm{char}_h(\mathcal{M}).$$

If moreover $\mathcal{M}$ is holonomic, then $\mathrm{Sol}_h(\mathcal{M})$ and $\mathrm{DR}_h(\mathcal{M})$ belong to $\mathcal{D}^b_{c,c}(\mathcal{C}^h_X)$.

Theorem 3.14. Let $\mathcal{M} \in \mathcal{D}^b_{hol}(\mathcal{D}^h_X)$. Then there is a natural isomorphism in $\mathcal{D}^b_{c,c}(\mathcal{C}^h_X)$

$$\mathrm{Sol}_h(\mathcal{M}) \simeq D'_h(\mathrm{DR}_h(\mathcal{M})).$$

Proof. The natural $\mathcal{C}^h$-linear morphism

$$\mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{O}^h_X, \mathcal{M}) \otimes_{\mathcal{C}^h_X} \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{M}, \mathcal{O}^h_X) \to \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{O}^h_X, \mathcal{O}^h_X) \simeq \mathcal{C}^h_X$$

induces the morphism in $\mathcal{D}^b_{c,c}(\mathcal{C}^h_X)$

$$\alpha: \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{M}, \mathcal{O}^h_X) \to D'_h(\mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{O}^h_X, \mathcal{M})).$$

(Note that, choosing $\mathcal{M} = \mathcal{D}^h_X$, this morphism defines the morphism $\mathcal{O}^h_X \to D'_h(\mathcal{O}^h_X[d_X])$.) The morphism (3.9) induces an isomorphism

$$\text{gr}_h(\alpha): \mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\text{gr}_h(\mathcal{M}), \mathcal{O}_X) \to D'(\mathcal{R}\mathcal{H}\text{om}_{\mathcal{D}^h_X}(\mathcal{O}_X, \text{gr}_h(\mathcal{M}))).$$

It is thus an isomorphism by Corollary 1.17.

Q.E.D.

4 Formal extension of tempered functions

Let us start by reviewing after [9, Chapter 7] the construction of the sheaves of tempered distributions and of $C^\infty$-functions with temperate growth on the subanalytic site.

Let $X$ be a real analytic manifold $X$. One says that a function $f \in \mathcal{C}^\infty_X(U)$ has polynomial growth at $p \in X$ if, for a local coordinate system $(x_1, \ldots, x_n)$ around $p$, there exist a sufficiently small compact neighborhood $K$ of $p$ and a positive integer $N$ such that

$$\sup_{x \in K \cap U} (\text{dist}(x, K \setminus U))^N |f(x)| < \infty.$$ 

18
One says that $f$ is \textit{tempered} at $p$ if all its derivatives are of polynomial growth at $p$. One says that $f$ is tempered if it is tempered at any point of $X$. One denotes by $\mathcal{C}^\infty_t(U)$ the $\mathbb{C}$-vector subspace of $\mathcal{C}^\infty(U)$ consisting of tempered functions. It then follows from a theorem of Lojasiewicz that $U \mapsto \mathcal{C}^\infty_t(U)$ ($U \in \text{Op}_{\text{Xsa}}$) is a sheaf on $\text{Xsa}$. We denote it by $\mathcal{C}^\infty_{\text{Xsa}}$ or simply $\mathcal{C}^\infty_{\text{X}}$ if there is no risk of confusion.

\textbf{Lemma 4.1.} One has $H^j(U; \mathcal{C}^\infty_{\text{X}}) = 0$ for any $U \in \text{Op}_{\text{Xsa}}$ and any $j > 0$.

This result is well-known (see [8, Chapter 1]), but we recall its proof for the reader’s convenience.

\textit{Proof.} Consider the full subcategory $\mathcal{J}$ of $\text{Mod}(\mathbb{C}_{\text{Xsa}})$ consisting of sheaves $F$ such that for any pair $U, V \in \text{Op}_{\text{Xsa}}$, the Mayer-Vietoris sequence

$$0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \rightarrow 0$$

is exact. Let us check that this category is injective with respect to the functor $\Gamma(U; \cdot)$. The only non obvious fact is that if $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is an exact sequence and that $F'$ belongs to $\mathcal{J}$, then $F(U) \rightarrow F''(U)$ is surjective. Let $t \in F''(U)$. There exist a finite covering $U = \bigcup_{i \in I} U_i$ and $s_i \in F(U_i)$ whose image in $F''(U_i)$ is $t|_{U_i}$. Then the proof goes by induction on the cardinal of $I$ using the property of $F'$ and standard arguments. To conclude, note that $\mathcal{C}^\infty_{\text{X}}$ belongs to $\mathcal{J}$ thanks to Lojaciewicz’s result (see [12]).

Q.E.D.

Let $\mathcal{D}b_X$ be the sheaf of distributions on $X$. For $U \in \text{Op}_{\text{Xsa}}$, denote by $\mathcal{D}b^\prime_X(U)$ the space of tempered distributions on $U$, defined by the exact sequence

$$0 \rightarrow \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \rightarrow \Gamma(X; \mathcal{D}b_X) \rightarrow \mathcal{D}b^\prime_X(U) \rightarrow 0.$$

Again, it follows from a theorem of Lojasiewicz that $U \mapsto \mathcal{D}b^\prime(U)$ is a sheaf on $\text{Xsa}$. We denote it by $\mathcal{D}b^\prime_{\text{Xsa}}$ or simply $\mathcal{D}b^\prime_X$ if there is no risk of confusion. The sheaf $\mathcal{D}b^\prime_X$ is quasi-injective, that is, the functor $\mathcal{H}om_{\mathcal{C}_{\text{Xsa}}} (\cdot, \mathcal{D}b^\prime_X)$ is exact in the category $\text{Mod}_{\text{g-c}}(\mathbb{C}_X)$. Moreover, for $U \in \text{Op}_{\text{Xsa}}$, $\mathcal{H}om_{\mathcal{C}_{\text{Xsa}}}(\mathbb{C}_U, \mathcal{D}b^\prime_X)$ is also quasi-injective and $R\mathcal{H}om_{\mathcal{C}_{\text{Xsa}}}(\mathbb{C}_U, \mathcal{D}b^\prime_X)$ is concentrated in degree 0. Note that the sheaf

$$\Gamma_{[U]} \mathcal{D}b_X := \rho^{-1} \mathcal{H}om_{\mathcal{C}_{\text{Xsa}}}(\mathbb{C}_U, \mathcal{D}b^\prime_X)$$

is a $\mathcal{C}^\infty_X$-module, so that in particular $R\Gamma(V; \Gamma_{[U]} \mathcal{D}b_X)$ is concentrated in degree 0 for $V \subset X$ an open subset.
Formal extensions

By Proposition 2.5 the sheaves $C_{X}^{\infty,t,h}$, $\mathcal{D}_{X}^{t,h}$ and $\Gamma_{[U]}\mathcal{D}_{X}$ are acyclic for the functor $(\cdot)^{h}$. We set

$$C_{X}^{\infty,t,h} := (\mathcal{C}_{X}^{\infty,t})^{h}, \quad \mathcal{D}_{X}^{t,h} := (\mathcal{D}_{X}^{t})^{h}, \quad \Gamma_{[U]}\mathcal{D}_{X}^{h} := (\Gamma_{[U]}\mathcal{D}_{X})^{h}.$$ 

Note that, by Lemmas 2.3 and 2.9,

$$\Gamma_{[U]}\mathcal{D}_{X}^{h} \simeq \rho^{-1}\mathcal{H}om_{C_{Xsa}}(\mathcal{C}_{U}, \mathcal{D}_{X}^{t,h}).$$

By Proposition 2.2 we get:

**Proposition 4.2.** The sheaves $C_{X}^{\infty,t,h}$, $\mathcal{D}_{X}^{t,h}$ and $\Gamma_{[U]}\mathcal{D}_{X}^{h}$ are cohomologically $h$-complete.

Now assume $X$ is a complex manifold. Denote by $\overline{X}$ the complex conjugate manifold and by $X^{R}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. One defines the sheaf (in fact, an object of the derived category) of tempered holomorphic functions by

$$\mathcal{O}_{X}^{t} := \mathcal{R}\mathcal{H}om_{\rho\mathcal{D}_{X}}(\rho_{\mathcal{O}_{X}}, \mathcal{C}_{X}^{\infty,t}) \simeq \mathcal{R}\mathcal{H}om_{\rho\mathcal{D}_{X}}(\rho_{\mathcal{O}_{X}}, \mathcal{D}_{X}^{t,h}).$$

Here and in the sequel, we write $\mathcal{C}_{X}^{\infty,t}$ and $\mathcal{D}_{X}^{t}$ instead of $\mathcal{C}_{X}^{\infty,t}$ and $\mathcal{D}_{X}^{t}$, respectively. We set

$$\mathcal{O}_{X}^{t,h} := (\mathcal{O}_{X}^{t})^{R_{h}},$$

a cohomologically $h$-complete object of $D^{b}(C_{Xsa})$. By Lemma 2.3,

$$\mathcal{O}_{X}^{t,h} \simeq \mathcal{R}\mathcal{H}om_{\rho\mathcal{D}_{X}}(\rho_{\mathcal{O}_{X}}, \mathcal{C}_{X}^{\infty,t,h}) \simeq \mathcal{R}\mathcal{H}om_{\rho\mathcal{D}_{X}}(\rho_{\mathcal{O}_{X}}, \mathcal{D}_{X}^{t,h}).$$

Note that $gr_{h}(\mathcal{O}_{X}^{t,h}) \simeq \mathcal{O}_{X}^{t}$ in $D^{b}(C_{Xsa})$.

5 Riemann-Hilbert correspondence

Let $X$ be a complex analytic manifold. Consider the functors

$$\mathcal{T}\mathcal{H}(\cdot): D^{b}_{C-c}(C_{X}) \to D^{b}_{rh}(\mathcal{D}_{X})^{op}, \quad F \mapsto \rho^{-1}\mathcal{R}\mathcal{H}om_{C_{Xsa}}(\rho_{\ast}F, \mathcal{O}_{X}^{t}),$$

$$\mathcal{T}\mathcal{H}_{h}(\cdot): D^{b}_{C-c}(C_{X}^{h}) \to D^{b}(\mathcal{D}_{X}^{h})^{op}, \quad F \mapsto \rho^{-1}\mathcal{R}\mathcal{H}om_{C_{Xsa}^{h}}(\rho_{\ast}F, \mathcal{O}_{X}^{t,h}).$$
The classical Riemann-Hilbert correspondence of Kashiwara [4] states that the functors Sol and TH are equivalences of categories between $\mathcal{D}^b_{\text{C-c}}(\mathcal{C}_X)$ and $\mathcal{D}^b_{\text{th}}(\mathcal{D}_X)^{\text{op}}$ quasi-inverse to each other. In order to obtain a similar statement for $\mathcal{C}_X$ and $\mathcal{D}_X$ replaced with $\mathcal{C}_X^h$ and $\mathcal{D}_X^h$, respectively, we start by establishing some lemmas.

**Lemma 5.1.** Let $\mathcal{M}, \mathcal{N} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X^h)$. The natural morphism in $\mathcal{D}^b_{\text{C-c}}(\mathcal{C}_X^h)$

$$R\mathcal{H}\text{om}_{\mathcal{D}_X^h}(\mathcal{M}, \mathcal{N}) \to R\mathcal{H}\text{om}_{\mathcal{C}_X^h}(\text{Sol}_h(\mathcal{N}), \text{Sol}_h(\mathcal{M}))$$

is an isomorphism.

*Proof.* Applying the functor $\text{gr}_h$ to this morphism, we get an isomorphism by the classical Riemann-Hilbert correspondence. Then the result follows from Corollary 1.17 and Theorem 3.12. Q.E.D.

Note that there is an isomorphism in $\mathcal{D}^b(\mathcal{D}_X)$

\begin{equation}
\text{gr}_h(\text{TH}_h(F)) \simeq \text{TH}_(\text{gr}_h(F)).
\end{equation}

**Lemma 5.2.** The functor $\text{TH}_h$ induces a functor

\begin{equation}
\text{TH}_h: \mathcal{D}^b_{\text{C-c}}(\mathcal{C}_X^h) \to \mathcal{D}^b_{\text{th}}(\mathcal{D}_X^h)^{\text{op}}.
\end{equation}

*Proof.* Let $F \in \mathcal{D}^b_{\text{C-c}}(\mathcal{C}_X^h)$. By (5.1) and the classical Riemann-Hilbert correspondence we know that $\text{gr}_h(\text{TH}_h(F))$ is regular holonomic, and in particular coherent. It is thus left to prove that $\text{TH}_h(F)$ is coherent. Note that our problem is of local nature.

We use the Dolbeault resolution of $\mathcal{O}_X^{t, h}$ with coefficients in $\mathcal{D}_X^{t, h}$ and we choose a resolution of $F$ as given in Proposition A.1 (i). We find that $\text{TH}_h(F)$ is isomorphic to a bounded complex $\mathcal{M}^*$, where the $\mathcal{M}^i$ are locally finite sums of sheaves of the type $\Gamma[U]\mathcal{D}_X^{t, h}$ with $U \in \text{Op}_{X_{\text{sa}}}$. It follows from Proposition 4.2 that $\text{TH}_h(F)$ is cohomologically $h$-complete, and we conclude by Theorem 1.11 with $\mathcal{A} = \mathcal{D}_X^h$. Q.E.D.

**Lemma 5.3.** We have $R\mathcal{H}\text{om}_{\rho_{\mathcal{D}_X^h}}(\rho, \mathcal{O}_X^h, \mathcal{O}_X^{t, h}) \simeq \mathcal{O}_{X_{\text{sa}}}^h$.

*Proof.* This isomorphism is given by the sequence

\begin{align*}
R\mathcal{H}\text{om}_{\rho_{\mathcal{D}_X^h}}(\rho, \mathcal{O}_X^h, \mathcal{O}_X^{t, h}) & \simeq R\mathcal{H}\text{om}_{\rho_{\mathcal{D}_X^h}}(\rho, \mathcal{O}_X^h, \mathcal{O}_X^{t, h})^R \\
& \simeq (\rho R\mathcal{H}\text{om}_{\rho_{\mathcal{D}_X^h}}(\mathcal{O}_X^h, \mathcal{O}_X^h))^R \simeq (\mathcal{C}_{X_{\text{sa}}}^h)^R \simeq \mathcal{C}_{X_{\text{sa}}}^h.
\end{align*}
where the first isomorphism is an extension of scalars, the second one is Lemma 2.3 and the third one is given by the adjunction between $\rho_!$ and $\rho^{-1}$.

Q.E.D.

**Theorem 5.4.** The functors $\text{Sol}_h$ and $\text{TH}_h$ are equivalences of categories between $\mathbf{D}^b_{\text{C-c}}(\mathbb{C}^h_X)$ and $\mathbf{D}^b_{\text{rh}}(\mathcal{D}^h_X)^{\text{op}}$ quasi-inverse to each other.

**Proof.** In view of Lemma 5.1, we know that the functor $\text{Sol}_h$ is fully faithful. It is then enough to show that $\text{Sol}_h(\text{TH}_h(F)) \simeq F$ for $F \in \mathbf{D}^b_{\text{C-c}}(\mathbb{C}^h_X)$. Since we already know by Lemma 5.2 that $\text{TH}_h(F)$ is holonomic, we may use (3.8).

We have the sequence of isomorphisms:

$$
\rho_*\mathcal{H}om_{\mathcal{O}_h}^h(\mathcal{O}^h_X, \text{TH}_h(F)) = \rho_*\mathcal{H}om_{\mathcal{O}_h}^h(\mathcal{O}^h_X, \rho^{-1}_!\mathcal{H}om_{\mathcal{C}^h_X}^h(\rho_*F, \mathcal{O}^h_X))
\simeq \mathcal{H}om_{\rho_!\mathcal{O}_h}^h(\rho_!\mathcal{O}^h_X, \mathcal{H}om_{\mathcal{C}^h_X}^h(\rho_*F, \mathcal{O}^h_X))
\simeq \mathcal{H}om_{\mathcal{C}^h_X}^h(\rho_*F, \mathcal{H}om_{\rho_!\mathcal{O}_h}^h(\rho_!\mathcal{O}^h_X, \mathcal{O}^h_X))
\simeq \mathcal{H}om_{\mathcal{C}^h_X}^h(\rho_*F, \mathcal{C}^h_X)
\simeq \rho_*\mathcal{D}^h_F,$$

where we have used the adjunction between $\rho_!$ and $\rho^{-1}$, the isomorphism of Lemma 5.3 and the commutation of $\rho_*$ with $\mathcal{H}om$. One concludes by recalling the isomorphism of functors $\rho^{-1}_*\rho_* \simeq \text{id}$. Q.E.D.

**t-structure**

Recall the definition of the middle perversity $t$-structure for complex constructible sheaves. Let $\mathbb{K}$ denote either the field $\mathbb{C}$ or the ring $\mathbb{C}^h$. For $F \in \mathbf{D}^b_{\text{C-c}}(\mathbb{K}_X)$, we have $F \in \mathbf{pD}^\leq_{\text{C-c}}(\mathbb{K}_X)$ if and only if

$$
\forall i \in \mathbb{Z} \quad \dim \text{supp} H^i(F) \leq d_X - i,
$$

and $F \in \mathbf{pD}^{\geq 0}_{\text{C-c}}(\mathbb{K}_X)$ if and only if, for any locally closed complex analytic subset $S \subset X$,

$$
H^i_S(F) = 0 \text{ for all } i < d_X - \text{dim}(S).
$$

With the above convention, the de Rham functor

$$
\text{DR} : \mathbf{D}^b_{\text{hol}}(\mathcal{D}_X) \to \mathbf{pD}^b_{\text{C-c}}(\mathbb{C}_X)
$$

is $t$-exact.
Theorem 5.5. The de Rham functor $\text{DR}_h: \mathcal{D}^b_{\text{hol}}(\mathcal{O}_X^h) \to \mathcal{D}^b_{\text{C}^c}(\mathcal{C}_X^h)$ is $t$-exact.

Proof. (i) Let $\mathcal{M} \in \mathcal{D}^\leq_{\text{hol}}(\mathcal{O}_X^h)$. Let us prove that $\text{DR}_h \mathcal{M} \in \mathcal{D}^\leq_{\text{C}^c}(\mathcal{C}_X^h)$. Since $\text{DR}_h \mathcal{M}$ is constructible, Proposition 1.19 shows that it is enough to check (5.3) for $\text{gr}_h(\text{DR}_h \mathcal{M}) \simeq \text{DR}(\text{gr}_h \mathcal{M})$. In other words, it is enough to check that $\text{DR}(\text{gr}_h \mathcal{M}) \in \mathcal{D}^\leq_{\text{C}^c}(\mathcal{C}_X^h)$. Since $\text{gr}_h \mathcal{M} \in \mathcal{D}^\leq_{\text{hol}}(\mathcal{O}_X)$, this result follows from the $t$-exactness of the functor $\text{DR}$.

(ii) Let $\mathcal{M} \in \mathcal{D}^\geq_{\text{hol}}(\mathcal{O}_X^h)$. Let us prove that $\text{DR}_h \mathcal{M} \in \mathcal{D}^\geq_{\text{C}^c}(\mathcal{C}_X^h)$. We set $N = (H^0 \mathcal{M})_{\text{tor}}$. We have a morphism $u: \mathcal{N} \to \mathcal{M}$ induced by $H^0 \mathcal{M} \to \mathcal{M}$ and we let $\mathcal{M}'$ be the mapping cone of $u$. We have a distinguished triangle

$$\text{DR}_h \mathcal{N} \to \text{DR}_h \mathcal{M} \to \text{DR}_h \mathcal{M}' \xrightarrow{+1}$$

so that it is enough to show that $\text{DR}_h \mathcal{N}$ and $\text{DR}_h \mathcal{M}'$ belong to $\mathcal{D}^\geq_{\text{C}^c}(\mathcal{C}_X^h)$.

(a) By Proposition 3.6 (ii) and Proposition 3.8, $\mathcal{N}$ is holonomic as a $\mathcal{O}_X$-module. Hence $\text{DR}_h \mathcal{N} \simeq \text{DR} \mathcal{N}$ is a perverse sheaf (over $\mathbb{C}$) and satisfies (5.4). Since (5.4) does not depend on the coefficient ring, $\text{DR}_h \mathcal{N} \in \mathcal{D}^\geq_{\text{C}^c}(\mathcal{C}_X^h)$.

(b) We note that $H^0 \mathcal{M}' \simeq (H^0 \mathcal{M})_{\text{tr}}$. Hence by Proposition 1.14, $\text{gr}_h \mathcal{M}' \in \mathcal{D}^\geq_{\text{hol}}(\mathcal{O}_X^h)$ and $\text{DR}(\text{gr}_h \mathcal{M}') \in \mathcal{D}^\geq_{\text{C}^c}(\mathcal{C}_X^h)$, that is, $\text{DR}(\text{gr}_h \mathcal{M}')$ satisfies (5.4). Let $S \subset X$ be a locally closed complex subanalytic subset. We have

$$\text{R}\Gamma_S(\text{DR}(\text{gr}_h \mathcal{M}')) \simeq \text{gr}_h(\text{R}\Gamma_S(\text{DR}_h \mathcal{M}'))$$

and it follows from Proposition 1.19 that $\text{DR}_h \mathcal{M}'$ also satisfies (5.4) and thus belongs to $\mathcal{D}^\geq_{\text{C}^c}(\mathcal{C}_X^h)$.

Q.E.D.

6 Duality and $h$-torsion

The duality functors $\mathcal{D}$ on $\mathcal{D}_{\text{hol}}(\mathcal{O}_X)$ and $\mathcal{D}'$ on $\mathcal{D}^b_{\text{C}^c}(\mathcal{C}_X)$ are $t$-exact. We will discuss here the finer $t$-structures needed in order to obtain a similar result when replacing $\mathcal{C}_X$ and $\mathcal{O}_X$ by their formal extensions $\mathcal{C}_X^h$ and $\mathcal{O}_X^h$.

Following [1, Chapter I.2], let us start by recalling some facts related to torsion pairs and $t$-structures. We need in particular Proposition 6.2 below, which can also be found in [2].
Definition 6.1. Let \( \mathcal{C} \) be an abelian category. A torsion pair on \( \mathcal{C} \) is a pair \((\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})\) of full subcategories such that

(i) for all objects \( T \) in \( \mathcal{C}_{\text{tor}} \) and \( F \) in \( \mathcal{C}_{\text{tf}} \), we have \( \text{Hom}_\mathcal{C}(T, F) = 0 \),

(ii) for any object \( M \) in \( \mathcal{C} \), there are objects \( M_{\text{tor}} \) in \( \mathcal{C}_{\text{tor}} \) and \( M_{\text{tf}} \) in \( \mathcal{C}_{\text{tf}} \) and a short exact sequence \( 0 \to M_{\text{tor}} \to M \to M_{\text{tf}} \to 0 \).

Proposition 6.2. Let \( \mathcal{D} \) be a triangulated category endowed with a t-structure \((pD^{\leq 0}, pD^{\geq 0})\). Let us denote its heart by \( \mathcal{C} \) and its cohomology functors by \( pH^i : \mathcal{D} \to \mathcal{C} \). Suppose that \( \mathcal{C} \) is endowed with a torsion pair \((\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})\). Then we can define a new t-structure \((\pi D^{\leq 0}, \pi D^{\geq 0})\) on \( \mathcal{D} \) by setting:

\[
\pi D^{\leq 0} = \{ M \in pD^{\leq 1} : pH^1(M) \in \mathcal{C}_{\text{tor}} \},
\]

\[
\pi D^{\geq 0} = \{ M \in pD^{\geq 0} : pH^0(M) \in \mathcal{C}_{\text{tf}} \}.
\]

With the notations of Definition 3.2, there is a natural torsion pair attached to \( \text{Mod}(\mathcal{D}^h_X) \) given by the full subcategories

\[
\text{Mod}(\mathcal{D}^h_X)_{h\text{-tor}} = \{ \mathcal{M} : \mathcal{M}_{h\text{-tor}} \sim \to \mathcal{M} \},
\]

\[
\text{Mod}(\mathcal{D}^h_X)_{h\text{-tf}} = \{ \mathcal{M} : \mathcal{M} \sim \to \mathcal{M}_{h\text{-tf}} \}.
\]

Definition 6.3. (a) We call the torsion pair on \( \text{Mod}(\mathcal{D}^h_X) \) defined above, the \( h \)-torsion pair.

(b) We denote by \( (D^{\leq 0}(\mathcal{D}^h_X), D^{\geq 0}(\mathcal{D}^h_X)) \) the natural t-structure on \( \mathcal{D}(\mathcal{D}^h_X) \).

(c) We denote by \( (tD^{\leq 0}(\mathcal{D}^h_X), tD^{\geq 0}(\mathcal{D}^h_X)) \) the t-structure on \( \mathcal{D}^h(\mathcal{D}^h_X) \) associated via Proposition 6.2 with the \( h \)-torsion pair on \( \text{Mod}(\mathcal{D}^h_X) \).

Proposition 1.14 implies the following equivalences for \( \mathcal{M} \in \mathcal{D}^b_{\text{coh}}(\mathcal{D}^h_X) \):

\[
(6.1) \quad \mathcal{M} \in tD^{\geq 0}(\mathcal{D}^h_X) \iff \text{gr}_h \mathcal{M} \in D^{\geq 0}(\mathcal{D}_X),
\]

\[
(6.2) \quad \mathcal{M} \in D^{\leq 0}(\mathcal{D}^h_X) \iff \text{gr}_h \mathcal{M} \in D^{\leq 0}(\mathcal{D}_X).
\]

Proposition 6.4. Let \( \mathcal{M} \) be a holonomic \( \mathcal{D}^h_X \)-module.

(i) If \( \mathcal{M} \) has no \( h \)-torsion, then \( \mathcal{D}_h \mathcal{M} \) is concentrated in degree 0 and has no \( h \)-torsion.

(ii) If \( \mathcal{M} \) is an \( h \)-torsion module, then \( \mathcal{D}_h \mathcal{M} \) is concentrated in degree 1 and is an \( h \)-torsion module.
Proof. By (1.2) we have $\text{gr}_h (\mathcal{D}_h\mathcal{M}) \simeq \mathcal{D}(\text{gr}_h \mathcal{M})$. Since $\text{gr}_h \mathcal{M}$ is concentrated in degrees 0 and $-1$, with holonomic cohomology, $\mathcal{D}(\text{gr}_h \mathcal{M})$ is concentrated in degrees 0 and 1. By Proposition 1.14, $\mathcal{D}_h\mathcal{M}$ itself is concentrated in degrees 0 and 1 and $H^0(\mathcal{D}_h\mathcal{M})$ has no $h$-torsion.

(i) The short exact sequence
\[ 0 \rightarrow \mathcal{M} \xrightarrow{h} \mathcal{M} \rightarrow \mathcal{M}/h\mathcal{M} \rightarrow 0 \]
induces the long exact sequence
\[ \cdots \rightarrow H^1(\mathcal{D}_h(\mathcal{M}/h\mathcal{M})) \rightarrow H^1(\mathcal{D}_h\mathcal{M}) \xrightarrow{h} H^1(\mathcal{D}_h\mathcal{M}) \rightarrow 0. \]
By Nakayama’s lemma $H^1(\mathcal{D}_h\mathcal{M}) = 0$ as required.

(ii) Since $\mathcal{M}$ is locally annihilated by some power of $h$, the cohomology groups $H^i(\mathcal{D}_h\mathcal{M})$ also are $h$-torsion modules. As $H^0(\mathcal{D}_h\mathcal{M})$ has no $h$-torsion, we get $H^0(\mathcal{D}_h\mathcal{M}) = 0$. Q.E.D.

Theorem 6.5. The duality functor $\mathcal{D}_h\colon \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X^h)^{\text{op}} \rightarrow t\mathcal{D}_{\text{hol}}^b(\mathcal{D}_X^h)$ is $t$-exact.

In other words, $\mathcal{D}_h$ interchanges $\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h)$ with $t\mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h)$ and $\mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h)$ with $t\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h)$.

Proof. (i) Let us first prove for $\mathcal{M} \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X^h)$:
\[ (6.3) \quad \mathcal{M} \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h) \iff \mathcal{D}_h(\mathcal{M}) \in t\mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h). \]
By (1.2) we have $\text{gr}_h(\mathcal{D}_h\mathcal{M}) \simeq \mathcal{D}(\text{gr}_h \mathcal{M})$ and we know that the analog of (6.3) holds true for $\mathcal{D}_X$-modules:
\[ \mathcal{N} \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X) \iff \mathcal{D}(\mathcal{N}) \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X). \]
Hence (6.3) follows easily from (6.1) and (6.2).

(ii) We recall the general fact for a $t$-structure $(D, D^{\leq 0}, D^{\geq 0})$ and $A \in D$:
\[ A \in D^{\leq 0} \iff \forall B \in D^{\geq 1} \text{ Hom}(A, B) = 0, \]
\[ A \in D^{\geq 0} \iff \forall B \in D^{\leq -1} \text{ Hom}(B, A) = 0. \]
Since $\mathcal{D}_h$ is an involutive equivalence of categories we deduce from (6.3) the dual statement:
\[ \mathcal{M} \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^h) \iff \mathcal{D}_h(\mathcal{M}) \in t\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^h). \]
Q.E.D.
Remark 6.6. The above result can be stated as follows in the language of quasi-abelian categories of [16]. We will follow the same notations as in [6, Chapter 2]. The category $\mathcal{C} = \text{Mod}(\mathcal{D}^h_X)_{\text{htf}}$ is quasi-abelian. Hence its derived category has a natural generalized $t$-structure $(\mathcal{D}^{\leq s}(\mathcal{C}), \mathcal{D}^{>s-1}(\mathcal{C}))_{s \in \frac{1}{2}\mathbb{Z}}$. Note that $\mathcal{D}^{[-1/2,0]}(\mathcal{C})$ is equivalent to $\text{Mod}(\mathcal{D}^h_X)$, and that $\mathcal{D}^{[0,1/2]}(\mathcal{C})$ is equivalent to the heart of $^t\mathcal{D}^b(\mathcal{D}^h_X)$. Then Theorem 6.5 states that the duality functor $\mathcal{D}_h$ is $t$-exact on $\mathcal{D}^b_{\text{hol}}(\mathcal{C})$.

Consider the full subcategories of $\text{Perv}(\mathcal{C}^h_X)$

$$\text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}} = \{F: \text{locally } h^N F = 0 \text{ for some } N \in \mathbb{N}\},$$

$$\text{Perv}(\mathcal{C}^h_X)_{\text{ht-tf}} = \{F: F \text{ has no non zero subobjects in } \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}}\}.$$

Lemma 6.7. (i) Let $F \in \text{Perv}(\mathcal{C}^h_X)$. Then the inductive system of sub perverse sheaves $\text{Ker}(h^n: F \to F)$ is locally stationary.

(ii) The pair $(\text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}}, \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tf}})$ is a torsion pair.

Proof. (i) Set $\mathcal{M} = \mathcal{D}_h \text{TH}_h(F)$. By the Riemann-Hilbert correspondence, one has $\text{Ker}(h^n: F \to F) \simeq \text{DR}_h(\text{Ker}(h^n: \mathcal{M} \to \mathcal{M}))$. Since $\mathcal{M}$ is coherent, the inductive system $\text{Ker}(h^n: \mathcal{M} \to \mathcal{M})$ is locally stationary. Hence so is the system $\text{Ker}(h^n: F \to F)$.

(ii) By (i) it makes to define for $F \in \text{Perv}(\mathcal{C}^h_X)$:

$$F_{\text{ht-tor}} = \bigcup_n \text{Ker}(h^n: F \to F), \quad F_{\text{ht-tf}} = F/F_{\text{ht-tor}}.$$

It is easy to check that $F_{\text{ht-tor}} \in \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}}$ and $F_{\text{ht-tf}} \in \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tf}}$. Then property (ii) in Definition 6.1 is clear. For property (i) let $u: F \to G$ be a morphism in $\text{Perv}(\mathcal{C}^h_X)$ with $F \in \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}}$ and $G \in \text{Perv}(\mathcal{C}^h_X)_{\text{ht-tf}}$. Then $\text{Im } u$ also is in $\text{Perv}(\mathcal{C}^h_X)_{\text{ht-tor}}$ and so it is zero by definition of $\text{Perv}(\mathcal{C}^h_X)_{\text{ht-tf}}$. Q.E.D.

Denote by $(^\pi \mathcal{D}^{\leq 0}_{C^h,c}(\mathcal{C}^h_X), ^\pi \mathcal{D}^{>0}_{C^h,c}(\mathcal{C}^h_X))$ the $t$-structure on $\mathcal{D}_{C^h,c}(\mathcal{C}^h_X)$ induced by the perversity $t$-structure and this torsion pair as in Proposition 6.2. We also set $^\pi \text{Perv}(\mathcal{C}^h_X) = ^\pi \mathcal{D}^{\leq 0}_{C^h,c}(\mathcal{C}^h_X) \cap ^\pi \mathcal{D}^{>0}_{C^h,c}(\mathcal{C}^h_X)$.

Corollary 6.8. There is a quasi-commutative diagram of $t$-exact functors
where the duality functors are equivalences of categories and the de Rham functors become equivalences when restricted to the subcategories of regular objects.

**Example 6.9.** Let \( X = \mathbb{C}, U = X \setminus \{0\} \) and denote by \( j: U \hookrightarrow X \) the embedding. Let \( L \) be the local system on \( U \) with stalk \( \mathbb{C} \) and monodromy \( 1 + \hbar \). The sheaf \( Rj_*L \simeq D'_h(j_!(D'_hL)) \) is perverse for both \( t \)-structures, as is the sheaf \( H^0(Rj_*L) = j_*L \simeq jL \). From the distinguished triangle \( j_*L \to Rj_*L \to \mathbb{C}\{0\}[-1] \to \), one gets the short exact sequences

\[
0 \to j_*L \to Rj_*L \to \mathbb{C}\{0\}[-1] \to 0 \quad \text{in} \ Perv(\mathbb{C}_X^h),
\]

\[
0 \to \mathbb{C}\{0\}[-2] \to j_*L \to Rj_*L \to 0 \quad \text{in} \ \pi \text{Perv}(\mathbb{C}_X^h).
\]

### 7 \( \mathcal{D}(\hbar) \)-modules

Denote by

\[
\mathbb{C}^{h,\text{loc}} := \mathbb{C}(\hbar) = \mathbb{C}[h^{-1}, \hbar]
\]

the field of Laurent series in \( \hbar \), that is the fraction field of \( \mathbb{C}^h \). Recall the exact functor

\[ (\bullet)^{\text{loc}}: \text{Mod}(\mathbb{C}_X^h) \to \text{Mod}(\mathbb{C}_X^{h,\text{loc}}), \quad F \mapsto \mathbb{C}_X^{h,\text{loc}} \otimes_{\mathbb{C}_X^h} F, \]

and note that by [7, Proposition 5.4.14] one has the estimate

\[ \text{SS}(F^{\text{loc}}) \subset \text{SS}(F). \]

For \( G \in \mathbb{D}^b(\mathbb{C}_X) \), we write \( G^{h,\text{loc}} \) instead of \( (G^h)^{\text{loc}} \). We will consider in particular

\[
\mathcal{O}_X^{h,\text{loc}} = \mathcal{O}_X((h)), \quad \mathcal{D}_X^{h,\text{loc}} = \mathcal{D}_X((h)).
\]

**Lemma 7.1.** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X^{h,\text{loc}} \)-module. Then \( \mathcal{M} \) is pseudo-coherent over \( \mathcal{D}_X^h \). In other word, if \( \mathcal{L} \subset \mathcal{M} \) is a finitely generated \( \mathcal{D}_X^h \)-module, then \( \mathcal{L} \) is \( \mathcal{D}_X^h \)-coherent.

**Proof.** The proof follows from [5, Appendix. A1]. Q.E.D.

**Definition 7.2.** A lattice \( \mathcal{L} \) of a coherent \( \mathcal{D}_X^{h,\text{loc}} \)-module \( \mathcal{M} \) is a coherent \( \mathcal{D}_X^h \)-submodule of \( \mathcal{M} \) which generates it.
Since \( \mathcal{M} \) has no \( h \)-torsion, any of its lattices has no \( h \)-torsion. In particular, one has \( \mathcal{M} \cong \mathcal{L}^{\text{loc}} \) and \( \text{gr}_h \mathcal{L} \cong \mathcal{L}_0 = \mathcal{L}/h\mathcal{L} \).

It follows from Lemma 7.1 that lattices locally exist: for a local system of generators \((m_1, \ldots, m_N)\) of \( \mathcal{M} \), define \( \mathcal{L} \) as the \( \mathcal{D}_X^h \)-submodule with the same generators.

**Lemma 7.3.** Let \( 0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0 \) be an exact sequence of coherent \( \mathcal{D}_X^{h,\text{loc}} \)-modules. Locally there exist lattices \( \mathcal{L}', \mathcal{L}, \mathcal{L}'' \) of \( \mathcal{M}', \mathcal{M}, \mathcal{M}'' \), respectively, inducing an exact sequence of \( \mathcal{D}_X^h \)-modules

\[
0 \rightarrow \mathcal{L}' \rightarrow \mathcal{L} \rightarrow \mathcal{L}'' \rightarrow 0.
\]

**Proof.** Let \( \mathcal{L} \) be a lattice of \( \mathcal{M} \) and let \( \mathcal{L}'' \) be its image in \( \mathcal{M}'' \). We set \( \mathcal{L}':= \mathcal{L} \cap \mathcal{M}' \). These sub-\( \mathcal{D}_X^h \)-modules give rise to an exact sequence.

Since \( \mathcal{L}' \) is a lattice of \( \mathcal{M}' \), being the kernel of a morphism \( \mathcal{L} \rightarrow \mathcal{L}'' \) between coherent \( \mathcal{D}_X^h \)-modules, \( \mathcal{L}' \) is coherent. To show that \( \mathcal{L}' \) generates \( \mathcal{M}' \), note that any \( m' \in \mathcal{M}' \subset \mathcal{M} \) may be written as \( m' = h^{-N}m \) for some \( N \geq 0 \) and \( m \in \mathcal{L} \). Hence \( m = h^N m' \in \mathcal{M}' \cap \mathcal{L} = \mathcal{L}' \). Q.E.D.

For an abelian category \( \mathcal{C} \), we denote by \( K(\mathcal{C}) \) its Grothendieck group. For an object \( M \) of \( \mathcal{C} \), we denote by \([M]\) its class in \( K(\mathcal{C}) \). We let \( \mathcal{K}(\mathcal{D}_X) \) be the sheaf on \( X \) associated to the presheaf

\[
U \mapsto K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|U)).
\]

We define \( \mathcal{K}(\mathcal{D}_X^{h,\text{loc}}) \) in the same way.

**Lemma 7.4.** Let \( \mathcal{L} \) be a coherent \( \mathcal{D}_X^h \)-module without \( h \)-torsion. Then, for any \( i > 0 \), the \( \mathcal{D}_X \)-module \( \mathcal{L}/h^i\mathcal{L} \) is coherent, and we have the equality

\[
[\mathcal{L}/h^i\mathcal{L}] = i \cdot [\text{gr}_h(\mathcal{L})]
\]

in \( K(\text{Mod}_{\text{coh}}(\mathcal{D}_X)) \).

**Proof.** Since the functor \((\cdot) \otimes_{\mathcal{C}} \mathbb{C}/h^i\mathbb{C}^h \) is right exact, \( \mathcal{L}/h^i\mathcal{L} \) is a coherent \( \mathcal{D}_X \)-module. Since \( \mathcal{L} \) has no \( h \)-torsion, multiplication by \( h^i \) induces an isomorphism \( \mathcal{L}/h\mathcal{L} \cong h^i\mathcal{L}/h^{i+1}\mathcal{L} \). We conclude by induction on \( i \) with the exact sequence

\[
0 \rightarrow h^i\mathcal{L}/h^{i+1}\mathcal{L} \rightarrow \mathcal{L}/h^{i+1}\mathcal{L} \rightarrow \mathcal{L}/h^i\mathcal{L} \rightarrow 0.
\]

Q.E.D.
Lemma 7.5. For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}})$, $U \subset X$ an open set and $\mathcal{L} \subset \mathcal{M}|_U$, the class $[\text{gr}_h(\mathcal{L})] \in K(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}))$ only depends on $\mathcal{M}$. This defines a morphism of abelian sheaves $\mathcal{K}(\mathcal{D}_X^{\text{h,loc}}) \to \mathcal{K}(\mathcal{D}_X)$.

Proof. (i) We first prove that $[\text{gr}_h(\mathcal{L})]$ only depends on $\mathcal{M}$. We consider another lattice $\mathcal{L}'$ of $\mathcal{M}|_U$. Since $\mathcal{L}$ is a $\mathcal{D}_X^{\text{h}}$-module of finite type, and $\mathcal{L}'$ generates $\mathcal{M}$, there exists $n > 1$ such that $\mathcal{L} \subset \mathcal{h}^{-n}\mathcal{L}'$. Similarly, there exists $m > 1$ with $\mathcal{L}' \subset \mathcal{h}^{-m}\mathcal{L}$, so that we have the inclusions

$$
\mathcal{h}^{m+n+2}\mathcal{L} \subset \mathcal{h}^{m+n+1}\mathcal{L} \subset \mathcal{h}^{m+1}\mathcal{L}' \subset \mathcal{h}^{m}\mathcal{L}' \subset \mathcal{L}.
$$

Any inclusion $A \subset B \subset C$ yields an identity $[C/A] = [C/B] + [B/A]$ in the Grothendieck group, and we obtain in particular:

$$
\begin{align*}
 [\mathcal{h}^m\mathcal{L}'/\mathcal{h}^{m+n+1}\mathcal{L}] &= [\mathcal{h}^m\mathcal{L}'/\mathcal{h}^{m+1}\mathcal{L}'] + [\mathcal{h}^{m+1}\mathcal{L}'/\mathcal{h}^{m+n+1}\mathcal{L}] \\
 [\mathcal{L}/\mathcal{h}^{m+n+1}\mathcal{L}] &= [\mathcal{L}/\mathcal{h}^{m+1}\mathcal{L}] + [\mathcal{h}^{m+1}\mathcal{L}'/\mathcal{h}^{m+n+1}\mathcal{L}] \\
 [\mathcal{L}/\mathcal{h}^{m+n+2}\mathcal{L}] &= [\mathcal{L}/\mathcal{h}^{m+1}\mathcal{L}] + [\mathcal{h}^{m+1}\mathcal{L}'/\mathcal{h}^{m+n+2}\mathcal{L}].
\end{align*}
$$

Since our modules have no $h$-torsion, we have isomorphisms of the type $h^k\mathcal{M}_1/h^k\mathcal{M}_2 \simeq \mathcal{M}_1/\mathcal{M}_2$. Then Lemma 7.4 and the above equalities give:

$$
\begin{align*}
 [\mathcal{L}'/\mathcal{h}^{n+1}\mathcal{L}] &= [\text{gr}_h(\mathcal{L}')] + [\mathcal{L}'/\mathcal{h}^n\mathcal{L}] \\
 (m + n + 1)[\text{gr}_h(\mathcal{L})] &= [\mathcal{L}/\mathcal{h}^{m+1}\mathcal{L}] + [\mathcal{L}'/\mathcal{h}^n\mathcal{L}] \\
 (m + n + 2)[\text{gr}_h(\mathcal{L})] &= [\mathcal{L}/\mathcal{h}^{m+1}\mathcal{L}'] + [\mathcal{L}'/\mathcal{h}^{n+1}\mathcal{L}].
\end{align*}
$$

A suitable combination of these lines gives $[\text{gr}_h(\mathcal{L})] = [\text{gr}_h(\mathcal{L}')]$, as desired.

(ii) Now we consider an open subset $V \subset X$ and $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}|_V)$. We choose an open covering $\{U_i\}_{i \in I}$ of $V$ such that for each $i \in I$, $\mathcal{M}|_{U_i}$ admits a lattice, say $\mathcal{L}_i$. We have seen that $[\text{gr}_h(\mathcal{L}_i)] \in K(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}|_{U_i}))$ only depends on $\mathcal{M}$. This implies that

$$
[\text{gr}_h(\mathcal{L}_i)]|_{U_{i,j}} = [\text{gr}_h(\mathcal{L}_j)]|_{U_{i,j}} \text{ in } K(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}|_{U_{i,j}})).
$$

Hence the $[\text{gr}_h(\mathcal{L}_i)]$’s define a section, say $c(\mathcal{M})$, of $\mathcal{K}(\mathcal{D}_X)$ over $V$. By Lemma 7.3, $c(\mathcal{M})$ only depends on the class $[\mathcal{M}]$ in $K(\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\text{h,loc}}|_V))$, and $\mathcal{M} \mapsto c(\mathcal{M})$ induces the morphism $\mathcal{K}(\mathcal{D}_X^{\text{h,loc}}) \to \mathcal{K}(\mathcal{D}_X)$. Q.E.D.

By Lemma 7.5, the following definition is well posed.
Definition 7.6. Let $\mathcal{M}$ be a coherent $\mathscr{D}_X^{h,\text{loc}}$-module. For $\mathcal{L} \in \text{Mod}_{\text{coh}}(\mathscr{D}_X^h)$ a (local) lattice, the characteristic variety of $\mathcal{M}$ is defined by

$$\text{char}_{h,\text{loc}}(\mathcal{M}) = \text{char}_h(\mathcal{L}).$$

For $\mathcal{M} \in D^b(\mathscr{D}_X^{h,\text{loc}})$, one sets $\text{char}_{h,\text{loc}}(\mathcal{M}) = \bigcup_j \text{char}_{h,\text{loc}}(H^j(\mathcal{M}))$.

Proposition 7.7. The characteristic variety $\text{char}_{h,\text{loc}}(\mathcal{M})$ is additive both on $\text{Mod}_{\text{coh}}(\mathscr{D}_X^{h,\text{loc}})$ and on $D^b(\mathscr{D}_X^{h,\text{loc}})$.

Proof. This follows from Proposition 3.6 (ii) and Lemma 7.3. Q.E.D.

Consider the functor

$$\text{Sol}_{h,\text{loc}}(\mathcal{M}) : D^b(\mathscr{D}_X^{h,\text{loc}})^{\text{op}} \to D^b(\mathbb{C}_X^{h,\text{loc}}), \; \mathcal{M} \mapsto \mathbb{R}\text{Hom}_{\mathscr{D}_X^{h,\text{loc}}}(\mathcal{M}, \mathcal{O}_X^{h,\text{loc}}).$$

Proposition 7.8. Let $\mathcal{M} \in D^b(\mathscr{D}_X^{h,\text{loc}})$. Then

$$\text{SS} \left( \text{Sol}_{h,\text{loc}}(\mathcal{M}) \right) \subset \text{char}_{h,\text{loc}}(\mathcal{M}).$$

Proof. By dévissage, we can assume that $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathscr{D}_X^{h,\text{loc}})$. Moreover, since the problem is local, we may assume that $\mathcal{M}$ admits a lattice $\mathcal{L}$.

One has the isomorphism $\text{Sol}_{h,\text{loc}}(\mathcal{M}) \simeq \mathbb{R}\text{Hom}_{\mathscr{D}_X^{h,\text{loc}}}(\mathcal{L}, \mathcal{O}_X^{h,\text{loc}})^{\text{op}}$ by extension of scalars. Taking a local resolution of $\mathcal{L}$ by free $\mathscr{D}_X^h$-modules of finite type, we deduce that $\text{Sol}_{h,\text{loc}}(\mathcal{M}) \simeq F^{\text{loc}}$ for $F = \text{Sol}_h(\mathcal{L})$. The statement follows by (7.2) and Corollary 3.13. Q.E.D.

One says that $\mathcal{M}$ is holonomic if its characteristic variety is isotropic.

Proposition 7.9. Let $\mathcal{M} \in D^b(\mathscr{D}_X^{h,\text{loc}})$. Then $\text{Sol}_{h,\text{loc}}(\mathcal{M}) \in D^b_{\text{C-c}}(\mathbb{C}_X^{h,\text{loc}})$.

Proof. By the same arguments and with the same notations as in the proof of Proposition 7.8, we reduce to the case $\text{Sol}_{h,\text{loc}}(\mathcal{M}) \simeq F^{\text{loc}}$, for $F = \text{Sol}_h(\mathcal{L})$ and $\mathcal{L}$ a lattice of $\mathcal{M} \in \text{Mod}_{\text{hol}}(\mathscr{D}_X^{h,\text{loc}})$. Hence $\mathcal{L}$ is a holonomic $\mathscr{D}_X^h$-module, and $F \in D^b_{\text{C-c}}(\mathbb{C}_X^{h})$. Q.E.D.

Remark 7.10. In general the functor

$$\text{Sol}_{h,\text{loc}} : D^b_{\text{hol}}(\mathscr{D}_X^{h,\text{loc}})^{\text{op}} \to D^b_{\text{C-c}}(\mathbb{C}_X^{h,\text{loc}})$$

30
is not locally essentially surjective. In fact, consider the quasi-commutative diagram of categories

\[
\begin{array}{ccc}
D^b_{hol}(\mathcal{D}_X^{\hbar})^{\text{op}} & \xrightarrow{\text{Sol}_h} & D^b_{\mathcal{C}-c}(\mathcal{C}_X^{\hbar}) \\
(\cdot)^{\text{loc}} \downarrow & & \downarrow (\cdot)^{\text{loc}} \\
D^b_{hol}(\mathcal{D}_X^{\hbar,\text{loc}})^{\text{op}} & \xrightarrow{\text{Sol}_{h,\text{loc}}} & D^b_{\mathcal{C}-c}(\mathcal{C}_X^{h,\text{loc}}).
\end{array}
\]

By the local existence of lattices the left vertical arrow is locally essentially surjective. If \(\text{Sol}_{h,\text{loc}}\) were also locally essentially surjective, so should be the right vertical arrow. The following example shows that it is not the case.

**Example 7.11.** Let \(X = \mathbb{C}\), \(U = X \setminus \{0\}\) and denote by \(j: U \hookrightarrow X\) the embedding. Set \(F = \text{R} j_! L\), where \(L\) is the local system on \(U\) with stalk \(\mathbb{C}^{h,\text{loc}}\) and monodromy \(h\). There is no \(F_0 \in D^b_{\mathcal{C}-c}(\mathbb{C}_X^{h})\) such that \(F \simeq (F_0)^{\text{loc}}\).

One can interpret this phenomenon by remarking that \(D^b_{hol}(\mathcal{D}_X^{h,\text{loc}})\) is equivalent to the localization of the category \(D^b_{\mathcal{C}-c}(\mathcal{D}_X^{h})\) with respect to the morphism \(h\), contrarily to the category \(D^b_{\mathcal{C}-c}(\mathbb{C}_X^{h,\text{loc}})\).

## 8 Links with deformation quantization

In this last section, we shall briefly explain how the study of deformation quantization algebras on complex symplectic manifolds is related to \(\mathcal{D}_X^{h}\). We follow the terminology of [11].

The cotangent bundle \(\mathfrak{X} = T^* X\) to the complex manifold \(X\) has a structure of a complex symplectic manifold and is endowed with the \(\mathbb{C}^{h}\)-algebra \(\hat{\mathcal{W}}_{\mathfrak{X}}\), a non homogeneous version of the algebra of microdifferential operators. Its subalgebra \(\mathcal{W}_X(0)\) of operators of order at most zero is a deformation quantization algebra. In a system \((x, u)\) of local symplectic coordinates, \(\mathcal{W}_X(0)\) is identified with the star algebra \((\mathcal{O}_X^{h}, \star)\) in which the star product is given by the Leibniz product:

\[f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{\left|\alpha\right|}}{\alpha!} (\partial_\alpha^u f)(\partial_\alpha x g), \quad \text{for } f, g \in \mathcal{O}_X.\]

(8.1)

In this section we will set for short \(\mathcal{A} := \mathcal{W}_X(0)\), so that \(\mathcal{A}^{\text{loc}} \simeq \mathcal{W}_X\). Note that \(\mathcal{A}\) satisfies Assumption 1.8.
Let us identify $X$ with the zero section of the cotangent bundle $\mathfrak{X}$. Recall that $X$ is a local model for any smooth Lagrangian submanifold of $\mathfrak{X}$, and that $\mathcal{O}_X^h$ is a local model of any simple $\mathcal{A}$-module along $X$. As $\mathcal{O}_X^h$ has both a $\mathcal{D}_X^h$-module and an $\mathcal{A}|_X$-module structure, there are morphisms of $\mathbb{C}^h$-algebras

\[
\mathcal{D}_X^h \to \mathcal{E}nd_{\mathbb{C}^h}(\mathcal{O}_X^h) \leftarrow \mathcal{A}|_X.
\]

**Lemma 8.1.** The morphisms in (8.2) are injective and induce an embedding $\mathcal{A}|_X \hookrightarrow \mathcal{D}_X^h$.

**Proof.** Since the problem is local, we may choose a local symplectic coordinate system $(x, u)$ on $\mathfrak{X}$ such that $X = \{u = 0\}$. Then $\mathcal{A}|_X$ is identified with $\mathcal{O}_X^h|_X$. As the action of $u_i$ on $\mathcal{O}_X^h$ is given by $\hbar \partial_x$, the morphism $\mathcal{A}|_X \to \mathcal{E}nd_{\mathbb{C}^h}(\mathcal{O}_X^h)$ factors through $\mathcal{D}_X^h$, and the induced morphism $\mathcal{A}|_X \to \mathcal{D}_X^h$ is described by

\[
\sum_{i \in \mathbb{N}} f_i(x, u)h^i \mapsto \sum_{j \in \mathbb{N}} \left( \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq j} \partial_{u}^\alpha f_j - |\alpha| \partial_x \right) h^j,
\]

which is clearly injective. Q.E.D.

Consider the following subsheaves of $\mathcal{D}_X^h$

\[
\mathcal{D}_X^{h,m} = \prod_{i \geq 0} \left( F_{i+m} \mathcal{D}_X \right) h^i, \quad \mathcal{D}_X^{h,f} = \bigcup_{m \geq 0} \mathcal{D}_X^{h,m}.
\]

Note that $\mathcal{D}_X^{h,0}$ and $\mathcal{D}_X^{h,f}$ are subalgebras of $\mathcal{D}_X^h$, that $\mathcal{D}_X^{h,0}$ is $\hbar$-complete while $\mathcal{D}_X^{h,f}$ is not and that $\mathcal{D}_X^{h,0,loc} \simeq \mathcal{D}_X^{h,f,loc}$. By (8.3), the image of $\mathcal{A}|_X$ in $\mathcal{D}_X^h$ is contained in $\mathcal{D}_X^{h,0}$. (The ring $\mathcal{D}_X^{h,0}$ should be compared with the ring $\mathfrak{B}_{X \times \mathbb{C}}$ of [14].)

**Remark 8.2.** More precisely, denote by $\mathcal{O}_X^h|_X \simeq (\mathcal{O}_X^h|_X)^h$ the formal restriction of $\mathcal{O}_X^h$ along the submanifold $X$. Then the star product in (8.1) extends to this sheaf, and (8.3) induces an isomorphism $(\mathcal{O}_X^h|_X, \ast) \simeq \mathcal{D}_X^{h,0}$.

Summarizing, one has the compatible embeddings of algebras

\[
\mathcal{A}^{loc}|_X \hookrightarrow \mathcal{D}_X^{h,0,loc} \simeq \mathcal{D}_X^{h,f,loc} \hookrightarrow \mathcal{D}_X^{h,loc}
\]

\[
\mathcal{A}|_X \hookrightarrow \mathcal{D}_X^{h,0} \hookrightarrow \mathcal{D}_X^{h,f} \hookrightarrow \mathcal{D}_X^h
\]

32
One has
\[ \text{gr}_h \mathcal{A}|_X \simeq \mathcal{O}_X|_X, \quad \text{gr}_h \mathcal{D}_X^{h,0} \simeq \mathcal{O}_X|_X, \quad \text{gr}_h \mathcal{D}_X^{h,f} \simeq \text{gr}_h \mathcal{D}_X^h \simeq \mathcal{D}_X. \]

**Proposition 8.3.**  
(i) The algebra \( \mathcal{D}_X^{h,0} \) is faithfully flat over \( \mathcal{A}|_X \).

(ii) The algebra \( \mathcal{D}_X^{h,\text{loc}} \) is flat over \( \mathcal{A}_{\text{loc}}|_X \).

**Proof.** (i) follows from Theorem 1.12.

(ii) follows from (i) and the isomorphism \( (\mathcal{D}_X^{h,0})_{\text{loc}} \simeq \mathcal{D}_X^{h,\text{loc}}. \) Q.E.D.

The next examples show that the scalar extension functor
\[ \text{Mod}_{\text{coh}}(\mathcal{D}_X^{h,0}) \to \text{Mod}_{\text{coh}}(\mathcal{D}_X^h) \]
is neither exact nor full.

**Example 8.4.** Let \( X = \mathbb{C}^2 \) with coordinates \((x, y)\). Then \( \hbar \partial_y \) is injective as an endomorphism of \( \mathcal{D}_X^{h,0}/\langle \hbar \partial_x \rangle \) but it is not injective as an endomorphism of \( \mathcal{D}_X^h/\langle \hbar \partial_x \rangle \), since \( \partial_x \) belongs to its kernel. This shows that \( \mathcal{D}_X^h \) is not flat over \( \mathcal{D}_X^{h,0} \).

**Example 8.5.** This example was communicated to us by Masaki Kashiwara. Let \( X = \mathbb{C} \) with coordinate \( x \), and denote by \((x, u)\) the symplectic coordinates on \( X = T^*\mathbb{C} \). Consider the cyclic \( \mathcal{A} \)-modules
\[ \mathcal{M} = \mathcal{A}/\langle x - u \rangle, \quad \mathcal{N} = \mathcal{A}/\langle x \rangle, \]
and their images in \( \text{Mod}(\mathcal{D}_X^h) \)
\[ \mathcal{M}' = \mathcal{D}_X^h/\langle x - \hbar \partial_x \rangle, \quad \mathcal{N}' = \mathcal{D}_X^h/\langle x \rangle. \]

As their supports in \( X \) differ, \( \mathcal{M} \) and \( \mathcal{N} \) are not isomorphic as \( \mathcal{A} \)-modules. On the other hand, in \( \mathcal{D}_X^h \) one has the relation
\[ (8.4) \quad x \cdot e^{\hbar \partial_x^2/2} = e^{\hbar \partial_x^2/2} \cdot (x - \hbar \partial_x), \]
and hence an isomorphism \( \mathcal{M}' \xrightarrow{\sim} \mathcal{N}' \) given by \([P] \mapsto [P \cdot e^{-\hbar \partial_x^2/2}]\). In fact, one checks that
\[ \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})|_X = 0, \quad \mathcal{H}om_{\mathcal{D}_X^h}(\mathcal{M}', \mathcal{N}') \simeq \mathbb{C}_X. \]

33
A Complements on constructible sheaves

Let us review some results, well-known from the specialists (see e.g., [15, Proposition 3.10]), but which are usually stated over a field, and we need to work here over the ring $\mathbb{C}^h$.

Let $K$ be a commutative unital Noetherian ring of finite global dimension. Assume that $K$ is syzygic, i.e. that any finitely generated $K$-module admits a finite projective resolution by finite free modules. (For our purposes we will either have $K = \mathbb{C}$ or $K = \mathbb{C}^h$).

Let $X$ be a real analytic manifold. Denote by $\text{Mod}_{R-c}(KX)$ the abelian category of $R$-constructible sheaves on $X$ and by $\text{D}^b_{R-c}(KX)$ the bounded derived category of sheaves of $K$-modules with $R$-constructible cohomology.

For the next two lemmas we recall some notations and results of [4, 7]. We consider a simplicial complex $S = (S, \Delta)$, with set of vertices $S$ and set of simplices $\Delta$. We let $|S|$ be the realization of $S$. Thus $|S|$ is the disjoint union of the realizations $|\sigma|$ of the simplices. For a simplex $\sigma \in \Delta$, the open set $U(\sigma)$ is defined in [7, (8.1.3)]. A sheaf $F$ of $K$-modules on $|S|$ is said weakly $S$-constructible if, $\forall \sigma \in \Delta, F|_\sigma$ is constant. An object $F \in \text{D}^b(K|S|)$ is said weakly $S$-constructible if its cohomology sheaves are so. If moreover, all stalks $F_x$ are perfect complexes, $F$ is said $S$-constructible. By [7, Proposition 8.1.4] we have isomorphisms, for a weakly $S$-constructible sheaf $F$ and for any $\sigma \in \Delta$ and $x \in |\sigma|:

\begin{align}
\Gamma(U(\sigma); F) & \sim \Gamma(|\sigma|; F) \sim F_x, \\
H^j(U(\sigma); F) & = H^j(|\sigma|; F) = 0, \quad \text{for } j \neq 0.
\end{align}

\[(A.1) \quad \Gamma(U(\sigma); F) \sim \Gamma(|\sigma|; F) \sim F_x,
\]

\[(A.2) \quad H^j(U(\sigma); F) = H^j(|\sigma|; F) = 0, \quad \text{for } j \neq 0.
\]

It follows that, for a weakly $S$-constructible $F \in \text{D}^b(K|S|)$, the natural morphisms of complexes of $K$-modules

\[(A.3) \quad \Gamma(U(\sigma); F) \rightarrow \Gamma(|\sigma|; F) \rightarrow F_x
\]

are quasi-isomorphisms.

For $U \subset X$ an open subset, we denote by $K_U := (K_X)_U$ the extension by 0 of the constant sheaf on $U$.

**Proposition A.1.** Let $F \in \text{D}^b_{R-c}(K_X)$. Then

(i) $F$ is isomorphic to a complex

\[0 \rightarrow \bigoplus_{i_a \in I_a} K_{U_a,i_a} \rightarrow \cdots \rightarrow \bigoplus_{i_b \in I_b} K_{U_b,i_b} \rightarrow 0,
\]

34
where the \( \{ U_{k,i} \}_{k,i} \)'s are locally finite families of relatively compact subanalytic open subsets of \( X \).

(ii) \( F \) is isomorphic to a complex

\[
0 \to \bigoplus_{i_a \in I_a} \Gamma_{V_{k,i_a}} K_X \to \cdots \to \bigoplus_{i_b \in I_b} \Gamma_{V_{k,i_b}} K_X \to 0,
\]

where the \( \{ V_{k,i} \}_{k,i} \)'s are locally finite families of relatively compact subanalytic open subsets of \( X \).

**Proof.** (i) By the triangulation theorem for subanalytic sets (see for example [7, Proposition 8.2.5]) we may assume that \( F \) is an \( S \)-constructible object in \( D^b(\mathbb{K}[S]) \) for some simplicial complex \( S = (S, \Delta) \). For \( i \) an integer, let \( \Delta_i \subset \Delta \) be the subset of simplices of dimension \( \leq i \) and set \( S_i = (S, \Delta_i) \).

We denote by \( \mathbb{K}^b(\mathbb{K}) \) (resp. \( \mathbb{K}^b(\mathbb{K}|S|) \)) the category of bounded complexes of \( \mathbb{K} \)-modules (resp. sheaves of \( \mathbb{K} \)-modules on \( |S| \)) with morphisms up to homotopy. We shall prove by induction on \( i \) that there exists a morphism \( u_i : G_i \to F \) in \( \mathbb{K}^b(\mathbb{K}|S|) \) such that:

(a) the \( G^k_i \) are finite direct sums of \( \mathbb{K}_{U(\sigma)} \)'s for some \( \sigma \in \Delta_i \),

(b) \( u_i|_{|S|} : G_i|_{|S|} \to F|_{|S|} \) is a quasi-isomorphism.

The desired result is obtained for \( i \) equal to the dimension of \( X \).

(i)-(1) For \( i = 0 \) we consider \( F|_{|S_0|} \simeq \bigoplus_{\sigma \in \Delta_0} F_\sigma \). The complexes \( \Gamma(U(\sigma); F) \), \( \sigma \in \Delta_0 \), have finite bounded cohomology by the quasi-isomorphisms (A.3). Hence we may choose bounded complexes of finite free \( \mathbb{K} \)-modules, \( R_{0,\sigma} \), and morphisms \( u_{0,\sigma} : R_{0,\sigma} \to \Gamma(U(\sigma); F) \) which are quasi-isomorphisms.

We have the natural isomorphism \( \Gamma(U(\sigma); F) \simeq a_* \mathcal{H}om_{\mathbb{K}^b(\mathbb{K}|S|)}(\mathbb{K}_{U(\sigma)}, F) \) in \( \mathbb{K}^b(\mathbb{K}) \), where \( a : |S| \to \text{pt} \) is the projection and \( \mathcal{H}om \) is the internal Hom functor. We deduce the adjunction formula, for \( R \in \mathbb{K}^b(\mathbb{K}) \), \( F \in \mathbb{K}^b(\mathbb{K}|S|) \):

\[
\text{Hom}_{\mathbb{K}^b(\mathbb{K})}(R, \Gamma(U(\sigma); F)) \simeq \text{Hom}_{\mathbb{K}^b(\mathbb{K}|S|)}(R_{U(\sigma)}, F).
\]

Hence the \( u_{0,\sigma} \) induce \( u_0 : G_0 := \bigoplus_{\sigma \in \Delta_0} (R_{0,\sigma})_U(\sigma) \to F \). By (A.3) \( (u_0)_x \) is a quasi-isomorphism for all \( x \in |S_0| \), so that \( u_0|_{|S_0|} \) also is a quasi-isomorphism, as required.
(i)-(2) We assume that \( u_i \) is built and let \( H_i = M(u_i)[-1] \) be the mapping cone of \( u_i \), shifted by \(-1\). By the distinguished triangle in \( \mathcal{K}^b(\mathbb{K}_S) \)

\[
(A.5) \quad H_i \xrightarrow{v_i} G_i \xrightarrow{u_i} F \xrightarrow{+1} \]

\( H_i \vert_{S_i} \) is quasi-isomorphic to 0. Hence \( \bigoplus_{\sigma \in \Delta_i \setminus \Delta_i} (H_i)_{\vert_{S_i}} \rightarrow H_i \vert_{S_i} \) is a quasi-isomorphism. As above we choose quasi-isomorphisms \( u_{i,\sigma} : R_{i+1,\sigma} \rightarrow \Gamma(U(\sigma); H_i), \sigma \in \Delta_i \setminus \Delta_i \), where the \( R_{i+1,\sigma} \) are bounded complexes of finite free \( \mathbb{K} \)-modules. By (A.4) again the \( u_{i,\sigma} \) induce a morphism in \( \mathcal{K}^b(\mathbb{K}_S) \)

\[
u_{i+1} : G'_{i+1} := \bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (R_{i+1,\sigma})_{\vert_{S_i}} \rightarrow H_i.
\]

For \( x \in |S_{i+1}| \setminus |S_i| \), \( (u'_{i+1})_x \) is a quasi-isomorphism by (A.3), and, for \( x \in |S_i| \), this is trivially true. Hence \( u'_{i+1} \vert_{S_i} \) is a quasi-isomorphism.

Now we let \( H_{i+1} \) and \( G_{i+1} \) be the mapping cones of \( u'_{i+1} \) and \( v_i \circ u'_{i+1} \), respectively. We have distinguished triangles in \( \mathcal{K}^b(\mathbb{K}_S) \)

\[
(A.6) \quad G'_{i+1} \xrightarrow{u'_{i+1}} H_i \rightarrow H_{i+1} \xrightarrow{+1}, \quad G'_{i+1} \xrightarrow{v_i \circ u'_{i+1}} G_i \rightarrow G_{i+1} \xrightarrow{+1}.
\]

By the contraction of the mapping cone, the definition of \( G'_{i+1} \) and the induction hypothesis, \( G_{i+1} \) satisfies property (a) above. The octahedral axiom applied to triangles (A.5) and (A.6) gives a morphism \( u_{i+1} : G_{i+1} \rightarrow F \) and a distinguished triangle \( H_{i+1} \rightarrow G_{i+1} \xrightarrow{u_{i+1}} F \xrightarrow{+1} \). By construction \( H_{i+1} \vert_{S_{i+1}} \) is quasi-isomorphic to 0 so that \( u_{i+1} \) satisfies property (b) above.

(ii) Consider the duality functor \( D'_{\mathbb{K}}(\cdot) = R\mathcal{H}om_{\mathbb{K}_X}(\cdot, \mathbb{K}_X) \). Set \( G = D'_{\mathbb{K}}(F) \), and represent it by a bounded complex as in (i). Since \( U_{k,i} \) corresponds to an open subset of the form \( U(\sigma) \) in \( |S| \), the sheaves \( \mathbb{K}_{U_{k,i}} \) are acyclic for the functor \( D'_{\mathbb{K}} \). Hence \( F \simeq D'_{\mathbb{K}}(G) \) can be represented as claimed.

Q.E.D.

**Lemma A.2.** Let \( F \rightarrow G \rightarrow 0 \) be an exact sequence in \( \text{Mod}_{\mathbb{K},c}(\mathbb{K}_X) \). Then for any relatively compact subanalytic open subset \( U \subset X \), there exists a finite covering \( U = \bigcup_{i \in I} U_i \) by subanalytic open subsets such that, for each \( i \in I \), the morphism \( F(U_i) \rightarrow G(U_i) \) is surjective.

**Proof.** As in the proof of Proposition A.1 we may assume that \( F \) and \( G \) are constructible sheaves on the realization of some finite simplicial complex \((S, \Delta)\). For \( \sigma \in \Delta \) the morphism \( \Gamma(U(\sigma); F) \rightarrow \Gamma(U(\sigma); G) \) is surjective, by (A.1). Since \( |S| \) is the finite union of the \( U(\sigma) \) this proves the lemma.

Q.E.D.
B Complements on subanalytic sheaves

We review here some well-known results (see [9, Chapter 7] and [13]) but which are usually stated over a field, and we need to work here over the ring \( \mathbb{C}^{\hbar} \).

Let \( \mathbb{K} \) be a commutative unital Noetherian ring of finite global dimension (for our purposes we will either have \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{K} = \mathbb{C}^{\hbar} \)). Let \( X \) be a real analytic manifold, and consider the natural morphism \( \rho: X \to X_{sa} \) to the associated subanalytic site.

**Lemma B.1.** The functor \( \rho_*: \text{Mod}_{\mathbb{R},c}(\mathbb{K}_X) \to \text{Mod}(\mathbb{K}_{X_{sa}}) \) is exact and \( \rho^{-1}\rho_* \) is isomorphic to the canonical functor \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_X) \to \text{Mod}(\mathbb{K}_X) \).

**Proof.** Being a direct image functor, \( \rho_* \) is left exact. It is right exact thanks to Lemma A.2. The composition \( \rho^{-1}\rho_* \) is isomorphic to the identity on \( \text{Mod}(\mathbb{K}_X) \) since the open sets of the site \( X_{sa} \) give a basis of the topology of \( X \). Q.E.D.

In the sequel, we denote by \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \) the image by the functor \( \rho_* \) of \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_X) \) in \( \text{Mod}(\mathbb{K}_{X_{sa}}) \). Hence \( \rho_* \) induces an equivalence of categories \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_X) \simeq \text{Mod}_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \). We also denote by \( \text{D}^b_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \) the full triangulated subcategory of \( \text{D}^b(\mathbb{K}_{X_{sa}}) \) consisting of objects with cohomology in \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \).

**Corollary B.2.** The subcategory \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \) of \( \text{Mod}(\mathbb{K}_{X_{sa}}) \) is thick.

**Proof.** Since \( \rho_* \) is fully faithful and exact, \( \text{Mod}_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \) is stable by kernel and cokernel. It remains to see that, for \( F, G \in \text{Mod}_{\mathbb{R},c}(\mathbb{K}_X) \)

\[
\text{Ext}^1_{\text{Mod}_{\mathbb{R},c}(\mathbb{K}_X)}(F, G) \simeq \text{Ext}^1_{\text{Mod}(\mathbb{K}_{X_{sa}})}(\rho_* F, \rho_* G).
\]

By [4] we know that the first \( \text{Ext}^1 \) may as well be computed in \( \text{Mod}(\mathbb{K}_X) \). We see that the functors \( \rho^{-1} \) and \( R\rho_* \) between \( \text{D}^b(\mathbb{K}_X) \) and \( \text{D}^b(\mathbb{K}_{X_{sa}}) \) are adjoint, and moreover \( \rho^{-1} R\rho_* \simeq \text{id} \). Thus, for \( F', G' \in \text{D}^b(\mathbb{K}_X) \) we have

\[
\text{Hom}_{\text{D}^b(\mathbb{K}_{X_{sa}})}(R\rho_* F', R\rho_* G') \simeq \text{Hom}_{\text{D}^b(\mathbb{K}_X)}(F', G'),
\]

and this gives the result. Q.E.D.

This corollary gives the equivalence \( \text{D}^b_{\mathbb{R},c}(\mathbb{K}_X) \simeq \text{D}^b_{\mathbb{R},c}(\mathbb{K}_{X_{sa}}) \), both categories being equivalent to \( \text{D}^b(\text{Mod}_{\mathbb{R},c}(\mathbb{K}_X)) \).
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