The energy-level statistics in the core of a vortex in a p-wave superconductor.

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(November 10, 1999)

In the presence of strong disorder, the statistics of quasiparticle levels in the core of a vortex in a two-dimensional p-wave superconductor belongs to the universality class B corresponding to the ensemble of orthogonal matrices in odd dimensions. This novel universality class appears as a consequence of the O(2) spin symmetry of p-wave pairing. It is preserved in the presence of random disorder, of electromagnetic vector potential, and of an admixture of the pairing of opposite chirality in the vortex core, but may be destroyed by spin-orbit coupling and by Zeeman splitting.

The indications of p-wave superconductivity in Sr$_2$RuO$_4$ [1] stimulated the study of exotic properties of p-wave superconductors. The order parameter in this compound is expected to be the same as in the A phase of $^3$He, $d_{\pm}(k) \propto \hat{\varepsilon}(k_x \pm i k_y)$. The direction of the vector $\hat{\varepsilon}$ of the triplet orientation is fixed by the anisotropy of Sr$_2$RuO$_4$ to be perpendicular to the Ru-O planes. Because of the strong anisotropy, one may consider a two-dimensional model as a starting approximation. In two dimensions, the p-wave superconducting gap does not have nodes, which resembles conventional superconductors. However, many differences appear in inhomogeneous setups including boundaries, vortices, and impurities. In particular, impurities generate bound states with circular currents [2]; similar subgap states appear at the boundary and at domain walls [3]; a single-quantum vortex possesses a zero-energy state of topological origin [4,5].

It is this last property that motivates the present work. The zero-energy state will be shown to survive a certain class of disordered perturbations, such as a random potential (modeling the impurities) or a random electromagnetic vector potential. If such a disorder is strong, the quasiparticle levels in the vortex core mix and, at the energy scale much smaller than the superconducting gap, they may be described by a random-matrix ensemble. In the present paper we identify the corresponding ensemble with that of orthogonal matrices in odd number of dimensions (class B in Cartan’s classification) [6]. Thus this example completes the list of universality classes corresponding to the eleven families of symmetric spaces (see Table 1). The three Wigner-Dyson universality classes correspond to $A$, $AI$, and $AII$ series (unitary, orthogonal, and symplectic, respectively) [7]. Three more classes ($AIII$, $BDI$, and $CII$) appear in systems with massless Dirac fermions, as a consequence of the chiral symmetry [8]. Finally, four more classes ($C$, $D$, $CI$, and $DIII$) were shown to describe mesoscopic superconducting systems, depending on the presence of the spin-rotation and time-reversal symmetries [9]. In the present paper we demonstrate that the remaining eleventh class $B$ appears in p-wave superconductors under topologically-nontrivial (vortex-type) boundary conditions responsible for the zero-energy level.

Let us start with briefly describing the properties of the ensemble of orthogonal matrices in odd dimensions (class B). The Lie algebra so($2N + 1$) consists of real antisymmetric ($2N + 1) \times (2N + 1)$ matrices. Its dimension is $\dim[\text{so}(2N + 1)] = N(2N + 1)$ and its rank $\text{rk}[\text{so}(2N + 1)] = N$. If a matrix $A$ belongs to so($2N + 1$), the Hermitian matrix $H = iA$ has one zero eigenvalue, and the remaining eigenvalues form pairs $(\omega_i, -\omega_i)$, $i = 1, \ldots, N$. At low energies we may, without loss of generality, assume the Gaussian probability distribution for the Hamiltonian $dP(H) = \exp(-\Tr H^2/2v^2)\prod dH_{ij}$, where $v$ is a large cut-off energy (in our problem, $v$ is of the order of the superconducting gap). Then the joint probability distribution for the (positive) eigenvalues $\omega_i$ is of the conventional form [10]:

$$dP\{\omega_i\} = |J\{\omega_i\}| \prod_{i=1}^{N} e^{-\omega_i^2/v^2} d\omega_i, \quad (1)$$

where $J\{\omega_i\}$ is the Jacobian of the diagonalization of the Hamiltonian,

$$|J\{\omega_i\}| = \prod_{i<j} |\omega_i^2 - \omega_j^2|^\beta \prod_{i=1}^{N} |\omega_i|^\alpha. \quad (2)$$

At energies much less than the cut-off energy $v$, the correlations of the quasiparticle levels $\omega_i$ are determined solely by the Jacobian $J\{\omega_i\}$ [the expression (2) follows from the explicit form of the roots $\xi_{(k)}$ of the Lie algebra so($2N + 1$) and from $|J\{\omega_i\}| = \prod_k |\sum_{i} \xi_{(k)} \omega_i|$; the values of $\alpha$ and $\beta$ may also be easily found from dimension counting]. For so($2N + 1$) the parameters of the level statistics are $\beta = 2$, $\alpha = 2$. The values of $\alpha$ and $\beta$ for the universality class
Next, we shall prove that a single-quantum vortex in a two-dimensional $p$-wave superconductor obeys the statistics of the $so(2N + 1)$ ensemble, provided the symmetry of the Hamiltonian preserves the zero-energy level. Consider the Bogoliubov-de-Gennes Hamiltonian

$$H = \sum_{\alpha} \psi_{\alpha}^\dagger \left[ \left( \frac{p - eA}{2m} \right)^2 + V(r) - \varepsilon_F \right] \psi_{\alpha} + \Psi_+^\dagger \left( \Delta_x * \frac{p_x}{k_F} + \Delta_y * \frac{p_y}{k_F} \right) \Psi_+^\dagger + \text{h.c.},$$  \hfill (3)

where $\psi_{\alpha}$ are the electron operators ($\alpha$ is the spin index), $V(r)$ is the external potential of impurities, $A(r)$ is the electromagnetic vector potential, $\Delta_x(r)$ and $\Delta_y(r)$ are the coordinate-dependent components of the superconducting gap. [In the bulk, the preferred superconducting order is one of the two chiral components $\eta_k = \Delta_x \pm \Delta_y$, but in inhomogeneous systems, such as a vortex core, an admixture of the opposite component is self-consistently generated \[10\]. We account for this effect by allowing the two independent order parameters $\Delta_x$ and $\Delta_y$.]

Star ($*$) denotes the symmetrized ordering of the gradients $p_{\mu}$ and the order parameters $\Delta_{\nu}$ [definition: $A * B \equiv (AB + BA)/2$]. At infinity, the order parameters impose the vortex boundary conditions:

$$\Delta_x(r \to \infty, \phi) = \Delta_0 e^{\pm i\phi}, \quad \Delta_y(r \to \infty, \phi) = i\Delta_0 e^{\pm i\phi},$$  \hfill (4)

where $r$ and $\phi$ are polar coordinates. Plus or minus signs in the exponent correspond to a positive or a negative single-quantum vortex. For an axially-symmetric vortex with the chirality of the order parameter non-self-consistently fixed (\[\Delta_y \equiv i\Delta_x\]), without the vector-potential $A(r)$ and without disorder $V(r)$, the low-lying eigenstates of the Hamiltonian \[3\] has been found by Kopnin and Salomaa \[4\]. The spectrum is

$$E_n = n\omega_0, \quad (p - \text{wave}), \quad n = 0, \pm 1, \pm 2, \ldots \hfill (5)$$

with $\omega_0 \sim \Delta^2/\varepsilon_F$. This result should be compared with the spectrum of the vortex core in a $s$-wave superconductor \[11\]:

$$E_n = \left( n + \frac{1}{2} \right) \omega_0, \quad (s - \text{wave}), \quad n = 0, \pm 1, \pm 2, \ldots \hfill (6)$$

The common feature of the spectra in the $s$-wave and $p$-wave cases is the symmetry about zero energy. If we interpret holes in the negative-energy levels as excitations with positive energies (and with the opposite spin), then this symmetry implies that the excitations are doubly degenerate in spin: to each spin-up excitation there corresponds a spin-down excitation at the same energy. For a $s$-wave vortex, this degeneracy is due to the full spin-rotation $SU(2)$ symmetry. The $p$-wave Hamiltonian \[3\] has a reduced spin symmetry. Namely, it has the symmetry group $O(2)$ generated by rotations about the $z$-axis ($\psi_{\uparrow} \mapsto e^{i\alpha} \psi_{\uparrow}, \psi_{\downarrow} \mapsto e^{-i\alpha} \psi_{\downarrow}$) and by the spin flip $\psi_{\uparrow} \mapsto \psi_{\downarrow}, \psi_{\downarrow} \mapsto \psi_{\uparrow}$. This non-abelian group causes the two-fold degeneracy of all levels (except for the zero-energy level(s) where the symmetry $O(2)$ may mix the creation and annihilation operators for the same state). This symmetry is crucial for our discussion. Note that we have not included in the Hamiltonian neither the spin-orbit term $(U_{SO} \cdot (\sigma \times p))$, nor the Zeeman splitting $H(r) \cdot \sigma$. Either of these terms would break the spin symmetry $O(2)$, which would eventually result in a different universality class of the disordered system (type $D$ with non-degenerate levels), if these terms are sufficiently strong.

The difference between the $s$- and $p$-wave vortices is the zero-energy level in the $p$-wave case. It has been shown by Volovik that this level has a topological nature \[3\]. Indeed, suppose we gradually increase disorder in the Hamiltonian \[3\]. The levels shift and mix, but the degeneracy of the levels remains the same as long as the symmetry $O(2)$ is preserved. The total number of levels remains odd, and therefore the zero-energy level cannot shift if the final Hamiltonian is a continuous deformation of the original one (without disorder), i.e. if the topological class of the boundary conditions \[3\] remains the same.

Now we proceed along the usual lines of the random-matrix approach. Let us take the most random distribution of Hamiltonians within a given symmetry class. The only symmetry of the Hamiltonian \[3\] is the spin symmetry $O(2)$. The time-reversal symmetry is already broken by the vortex and by the pairing, and therefore neither the vector potential $A(x)$ nor local deformations of $\Delta_{\mu}$ can reduce the symmetry of the Hamiltonian. When projected onto spin-up excitations $\gamma_{\uparrow}^\dagger = \int [u(r) \Psi_{\uparrow}^\dagger(r) + v(r) \Psi_{\downarrow}^\dagger(r)] d^2r$, the Hamiltonian for the two-component vector $(u, v)$ takes the form:
\[ H = \begin{pmatrix} \frac{-i\nabla - eA}{2m} & V(r) - \varepsilon_F \\ \frac{\Delta^*}{k_F} \ast (-i\nabla_x) + \frac{\Delta^*}{k_F} \ast (-i\nabla_y) & -\frac{[i\nabla + \varepsilon A]^2}{2m} + V(r) - \varepsilon_F \end{pmatrix}. \] (7)

In an arbitrary orthonormal basis of electronic states, this Hamiltonian may be written as a matrix

\[
H = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^* \end{pmatrix}.
\] (8)

From the hermiticity of the Hamiltonian, it follows that \( h^\dagger = h \). From the explicit form of the \( p \)-wave pairing, \( \Delta = -\Delta^T \) (it is here that the \( p \)-wave structure of the pairing is important; for \( s \)-wave pairing we would have \( \Delta = \Delta^T \) instead). These are the only restrictions on the Hamiltonian (8). If we define

\[
U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},
\] (9)

the restrictions on the Hamiltonian (8) are equivalent to the condition that the rotated matrix \( iU_0 H U_0^{-1} \) is real antisymmetric, i.e. it belongs to the Lie algebra \( so(M) \), where \( M \) is the dimension of the Hilbert space (the same rotation of the Hamiltonian was used in Ref. 8 to identify the \( D \) universality class).

The last step in our argument is to note that, under the vortex boundary conditions, the dimension of the Hamiltonian (8) is odd, not even (this may be difficult to visualize from the particle-hole representation (8), but easier from the rotated Hamiltonian \( U_0 H U_0^{-1} \)). Thus for a single-quantum vortex, we identify the space of the Hamiltonians with \( so(2N + 1) \).

A simple consequence of this result is the level distribution (2) with \( \alpha = \beta = 2 \). Besides the zero-energy level, this distribution is identical to that of class \( C \) realized in \( s \)-wave vortices [12,13]. This allows us to use the trick of mapping onto free fermions to compute any correlation function of the density of states (DOS) [8]. In particular, the average DOS is

\[
\langle \rho(\omega) \rangle = \frac{1}{\omega_0} - \frac{\sin(2\pi\omega/\omega_0)}{2\pi\omega} + \delta(\omega),
\] (10)

where \( \omega_0 \) is the average inter-level distance.

One more lesson from our analysis is the difference in the universality classes of \( p \)-wave mesoscopic systems from their \( s \)-wave analogues. In particular, a \( p \)-wave system \textit{without} a topological zero-energy level would belong to the universality class \( D \) (\( \beta = 2, \alpha = 0 \)), in contrast to the class \( C \) for its \( s \)-wave counterpart. A detectable physical consequence of such a difference is an increase of the average DOS near the Fermi energy (in class \( D \)) as opposed to a suppression of DOS near Fermi level in class \( C \) [in class \( B \), such a suppression is compensated by a \( \delta \)-function at zero].

In the present work we do not discuss the microscopic derivation of the level mixing. We assume “strong level mixing” which allows us to use the random-matrix approach. On the other hand, the symmetry of the \( p \)-wave pairing is known to provide certain “spectrum rigidity” suppressing shift and mixing of the low-lying levels by impurities [3]. Thus to drive the system into the regime of “strong level mixing” may require a stronger disorder than in a \( s \)-wave vortex. A microscopic picture of the crossover to the disordered regime in a \( s \)-wave case was developed in [14,15], and its extension to the \( p \)-wave case will be a subject of future studies. Furthermore, a disorder is known to suppress the \( p \)-wave superconductivity, and the possibility to reach the required level mixing before destroying superconductivity is not obvious. A coexistence of sufficiently strong disorder and \( p \)-wave superconductivity may possibly be achieved in alternative setups such as disordered normal-superconducting (\( p \)-wave) hybrid devices. Besides strong level mixing, the only requirements on the system to exhibit type \( B \) level statistics are the \( p \)-wave symmetry of the pairing and the topological zero-energy level.

One more approximation made in our model is neglecting the Zeeman splitting. This is a good approximation for strong type II superconductors (with \( \kappa = \lambda/\xi \gg 1 \)). In \( \text{Sr}_2\text{RuO}_4 \) the experiments indicate \( \kappa \sim 2.6 \) [3], which implies that the Zeeman level shift is of the order of the inter-level spacing \( \omega_0 \). However, in a clean vortex, this shift is approximately constant for the low-lying levels, due to the large coherence length (\( k_F\xi \gg 1 \)) and to a smooth magnetic-field profile \( H(r) \). It is likely that this property will hold even in the limit of strong disorder, then the overall level distribution will be simply shifted by a constant energy.

An important observation related to the Zeeman splitting of the energy levels is the \textit{fractional spin} \( 1/4 \) of the vortex in the ground state. Indeed, in the absence of the Zeeman field, the multi-particle ground state of the vortex is doubly degenerate, with the \( z \)-component of the spin in the two degenerate states differing by \( 1/2 \). The Zeeman field splits
the ground state as if the vortex were a particle with the \( z \)-component of the spin equal half the spin of the electron. Thus we conclude that the vortex has the \( z \)-component of the spin \( S_z = \pm 1/4 \). It would be interesting to understand possible physical implications of this effect.

Finally, for the convenience of the reader, we have gathered the information about all eleven symmetry classes in Table 1. This table is compiled from Refs. [6–9] and contains the dimensions and ranks of the symmetric spaces as well as the parameters \( \alpha \) and \( \beta \) of the joint probability distributions of the energy levels (they may be computed solely from the ranks and dimensions by a simple power counting). The present work provides the example of the physical system belonging to the class \( B \) (the last line of the table). Besides, one more symmetry subclass has not been studied so far in the context of mesoscopics: the odd-\( N \) subclass of \( \text{DIII} \). In the work of Altland and Zirnbauer [9], the even-\( N \) subclass of \( \text{DIII} \) is represented by a \( s \)-wave mesoscopic system with time-reversal symmetry and with broken spin symmetry. The novel odd-\( N \) subclass of \( \text{DIII} \) may occur as a topological modification of that construction or in a \( p \)-wave superconducting system without time-reversal-symmetry breaking, provided a zero mode is required by topology. Finding a physically plausible mesoscopic realization of the odd-\( N \) \( \text{DIII} \) subclass is an interesting problem in the framework of the symmetry classification of mesoscopic superconducting systems.

| Cartan class | Symmetric space | Dimension | Rank | \( \beta \) | \( \alpha \) | Remarks |
|--------------|----------------|-----------|------|----------|----------|---------|
| \( A \) [GUE] | \( SU(N) \) | \( N^2 - 1 \) | \( N - 1 \) | 2 | – | Wigner-Dyson [7] |
| \( A_1 \) [GOE] | \( SU(N)/SO(N) \) | \( (N-1)(N+2)/2 \) | \( N - 1 \) | 1 | – | |
| \( A_2 \) [GSE] | \( SU(2N)/Sp(N) \) | \( (N-1)(2N+1) \) | \( N - 1 \) | 4 | – | |
| \( A_{II} \) [chGUE] | \( SU(p+q)/SU(p) \times U(q) \) | \( 2pq \) | \( p \) | 2 | \( 1 + 2(q - p) \) | Chiral ensembles [8], \( p \leq q \) |
| \( BDI \) [chGOE] | \( SO(p+q)/SO(p) \times SO(q) \) | \( pq \) | \( p \) | 1 | \( q - p \) | \( (q - p) = \) number of zero modes |
| \( CII \) [chGSE] | \( Sp(p+q)/Sp(p) \times Sp(q) \) | \( 4pq \) | \( p \) | 4 | \( 3 + 4(q - p) \) | |
| \( C \) | \( Sp(N) \) | \( N(2N+1) \) | \( N \) | 2 | 2 | Altland-Zirnbauer [9] |
| \( D \) | \( SO(2N) \) | \( N(2N-1) \) | \( N \) | 2 | 0 | |
| \( CI \) | \( Sp(N)/U(N) \) | \( N(N+1) \) | \( N \) | 1 | 1 | |
| \( DIII \) | \( SO(2N)/U(N) \) | \( N(N+1) \) | \( N \) | 1 | \( \left[ \frac{N}{2} \right] \) | one zero mode \((\text{example unknown})\) |

\[ \left( \text{even } N \right) \]

\[ \left( \text{odd } N \right) \]

TABLE 1. Symmetric spaces and universality classes of random-matrix ensembles.

P.S. At the final stage of the preparation of this manuscript, the author has learned about the recent work of Bocquet, Serban, and Zirnbauer [16], where the type-\( B \) level statistics in \( p \)-wave vortices has been pointed out.

The author thanks M. V. Feigel’man for suggesting this problem, for fruitful discussions and for helpful comments on the manuscript. Useful discussions with G. Blatter, V. Geshkenbein, D. Gorokhov, R. Heeb, and M. Zhitomirsky are greatly acknowledged. The author thanks Swiss National Foundation for financial support.

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