Homogeneous finitely presented monoids of linear growth

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Abstract. If a finitely generated monoid $M$ is defined by a finite number of degree-preserving relations, then it has linear growth if and only if it can be decomposed into a finite disjoint union of subsets (which we call “sandwiches”) of the form $a\langle w \rangle b$ where $a, b, w \in M$ and $\langle w \rangle$ denotes the monogenic semigroup generated by $w$. Moreover, the decomposition can be chosen in such a way that the sandwiches are either singletons or “free” ones (meaning that all elements $aw^n b$ in each sandwich are pairwise different). So, the minimal number of free sandwiches in such a decompositions becomes a new numerical invariant of a homogeneous (and conjecturally, non-homogeneous) finitely presented monoid of linear growth.

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If a semigroup is a disjoint union of a finite number of free monogenic subsemigroups, then it is finitely presented and residually finite [1] and has linear growth [2]. It is easy to see that the reverse implication does not hold. For example, the monoid with zero $M = \langle x, y | xy = 0, xx = 0 \rangle$ is finitely presented with monomial relations (hence, residually finite) and has linear growth. However, $M$ cannot be represented as a finite union of monogenic semigroups since it contains an infinite set $\{y^n x | n \geq 0 \}$ of nilpotent elements.

Let us call a monoid $S$ homogeneous if its relations are degree-preserving with respect to some weight function, that is, for some set of generators $X$ of $S$ there is a function $d : X \to \mathbb{Z}_{>0}$ such that all relations of $S$ have either the form $w = 0$ (if $S$ contains zero) or $w = u$ with $d(w) = d(u)$, where for a word $w = x_1 \ldots x_k$ (resp., for a word $u$) on the generators we define $d(w)$ to be the sum $d(x_1) + \cdots + d(x_k)$. In particular, any monoid defined by the

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relations of the form $u = 0$ or $u = w$ where the words $u$ and $w$ have the same length is homogeneous with $d(x) = 1$ for all $x \in X$.

Given three elements $a, b$ and $w$ of a semigroup $S$, we call the subset $a\langle w \rangle b = \{aw^n b|n \geq 0\}$ sandwich. For example, each singleton $\{a\}$ is the sandwich $a\langle 1 \rangle 1$. A sandwich $a\langle w \rangle b$ is called free if its elements $aw^n b$ are pairwise different for all $n \geq 0$. For example, in free monoids all sandwiches containing two or more elements are free.

**Theorem 1.** Suppose that a monoid $S$ is homogeneous and finitely presented. Then the following conditions are equivalent.

(i) $S$ has at most linear growth;
(ii) $S$ is a finite union of sandwiches;
(iii) $S$ is a union of a finite subset and a finite disjoint union of free sandwiches.

We refer to the last decomposition as **sandwich decomposition**. For example, a sandwich decomposition of the above monoid $M$ consists of the finite set $\{0, 1\}$ and two free sandwiches $1(y)x$ and $1(y)1 = \langle y \rangle$.

**Proof.** The implication (ii)$\implies$(i) is straightforward since in each sandwich $a\langle w \rangle b$ the number of words $u$ of length $\text{len}(u)$ is not greater than

$$\frac{n - \text{len}(a) - \text{len}(b)}{\text{len}(w)} = O(n).$$

The implication (iii)$\implies$(ii) is trivial since any finite set is a finite union of singletons which are trivial sandwiches.

To complete the prove, let us prove the implications (i)$\implies$(ii) and (i)&(ii)$\implies$(iii). Let $A = F_2 S$ be the semigroup algebra (with common zero, if $S$ contains zero) over the two-element filed. It is $\mathbb{Z}$-graded connected, finitely presented, and has linear growth. By [6, Theorem 3.1], it is automaton in the sense of Ufnarovski with respect to any homogeneous finite set of generators. In particular, $A$ is automaton in the sense of Ufnarovski with respect to a minimal set of generators of $S$. Then the set of normal words in $A$ form a regular language. Now, the theorem follows from a theorem by Paun and Salomaa [5, Theorem 3.3] which describes slender regular languages.

Let us give also another proof which does not use methods of the theory of finite automata. By [6, Corollary 2.3], it follows that there exists a finite generating set $X$ of $S$ containing the unit of $S$ and a subset $Q \subset X \times X \times X$ such that the set

$$Y = \{aw^n b|n \geq 0, (a, b, w) \in Q\}$$

form a linear basis of $A$ (moreover, it is the set of normal words of $A$). It follows that either $S = Y \cup \{0\}$ (if $S$ contains zero) or $S = Y$. Since $Y$ is the union of sandwiches $a\langle w \rangle b$ for $(a, b, w) \in Q$, we get the implication (i)$\implies$(ii).

It remains to show that the set of words $Y$ is a finite disjoint union of sandwiches (since $Y$ is a subset of the free monoid $\langle X \rangle$, all these sandwiches are either free or singletons). To apply the induction argument, it is sufficient to use the next lemma. \qed
Lemma 2. Suppose that a subset $Z$ of a free monoid is a finite union of sandwiches. Then $Z$ is decomposable into a finite disjoint union of sandwiches.

Proof. Let $Z = \bigcup_{i=1}^{s} U_i$ be a decomposition of $Z$ into a union of $s$ sandwiches.

First let first consider the case $s = 2$. Let $U_1 = U = a\langle w \rangle b$ and $U_2 = U' = a'\langle w' \rangle b'$. We will show that the sets $U \cap U'$, $U \cup U'$, $U \setminus U'$ and $U' \setminus U$ are decomposed as the finite disjoint union of sandwiches.

If the intersection $I = U \cap U'$ is finite, then it is a disjoint union of singletons $\{a, b, a', b'\}$. Moreover, in this case the set $U \setminus I$ (respectively, $U' \setminus I$) is a union of a finite number of singletons and the subset $aw^m\langle w \rangle b$ for some $m \geq 0$ (resp., $a'w^n\langle w' \rangle b'$ for some $n \geq 0$). So, $U \cup U'$ admits the desired decomposition.

Suppose now that $I$ is infinite. Then the two-sided infinite words $w^\infty$ and $w'^\infty$ coincide. It is sufficient to prove our claim for the sets $aw^M\langle w \rangle b$ and $a'w^N\langle w' \rangle b'$ for all sufficiently large $M, N$ in place of $U$ and $U'$ respectively. Then up to a cyclic permutation of letters in $w$ and $w'$ (and possible change of the words $a, a', b, b'$), one can assume that there exist $m, n, p, q$ such that $w^m = w^n$ and $aw^p = a'w'^q$.

Now, if $T$ is one of the sets $I$ and $U \setminus I$, then $T$ is periodic in the following sense: for large enough $t$ we have $aw^tb \in T \iff aw^{t+m}b \in T$. Then $T = \{aw^{t+m}b \mid m \in \mathbb{Z}_+, t_0 \in S \}$ where $S$ is some finite set of nonnegative integers. It follows that $T$ is a disjoint union of a finite collection of sandwiches of the form $aw^t\langle w^m \rangle b$. Analogously, the set $U' \setminus I$ is a finite disjoint union of sandwiches of the form $a'w'^t\langle w'^m \rangle b'$. So, the set $U \cup U' = I \cup (U' \setminus I) \cup (U' \setminus I)$ admits the desired decomposition as well.

Now, for $s > 2$ we proceed by the induction. If $Z' = \bigcup_{i=1}^{s-1} U_i$ is decomposable into a disjoint union $\bigcup_{j=1}^{N} T_j$ with $T_j = p_j\langle q_j \rangle r_j$, then

$$Z = Z' \cup U_s = \bigcup_{j=1}^{N} (T_j \cup U_s),$$

where the sets $T_j \cup U_s$ admit the desired decomposition by the $s = 2$ case. 

Remark 3. Note that each finitely generated semigroup of linear growth is a finite union of sandwiches [4, Theorem 4.2] (see also [3, Proposition 2.174b]).

However, for monoids with infinite set of defining relations the conclusion of Theorem [I] may fail (so that the union is not disjoint).

For example, consider a monoid

$$N = \langle a, w, b \mid ba = 0, bw = 0, wa = 0, a^2 = 0, b^2 = 0, aw^{t^2}b = 0 \text{ for } t \geq 0 \rangle.$$  

Then the number $c_n$ of nonzero words of length $n \geq 2$ in $N$ is equal to 3 if $n = 2 + t^2$ for some $t \geq 0$ and 4 otherwise (these are the words $w^n, aw^{n-1}, w^{n-1}b$, and $aw^{n-2}b$). It follows that $N$ cannot be presented as a disjoint union of subsets of the desired form since the sequence $\{c_n\}_{n \geq 0}$ is not a sum of a finite number of arithmetic progressions.

If $S$ is a finitely presented monoid of linear growth (not necessary homogeneous), we do not know whether there it is a finite disjoint union of
free sandwiches and singletons. Ufnarovski [7, 5.10] conjectured that each finitely presented algebra of linear growth (in particular, the algebra $F_2S$) is automaton. This conjecture fails for homogeneous algebras over some infinite fields and holds for homogeneous algebras over finite fields [6]. Note that if the algebra $F_2S$ is automaton with respect to some ordering of the monomials on a finite set of generators of $S$, then $S$ is a finite disjoint union of sandwiches and singletons by the same arguments as above. So, we can formulate a weaker (in a sense) version of Ufnarovski’s conjecture.

**Conjecture 4.** Each finitely presented monoid $S$ of linear growth is a finite disjoint union of free sandwiches and a finite set.

Now we can introduce a new invariant for finitely generated monoids. Given such a monoid $S$, let $\gamma(S)$ be the minimal number $M$ such that $S$ is the disjoint union of $M$ free sandwiches and a finite set. In particular, for a finite monoid $S$ we have $\gamma(S) = 0$. If there is no such finite decompositions, we put for $\gamma(S) = \infty$. So, Theorem 1 and Conjecture 4 simply mean that $\gamma(S) < \infty$ if $S$ is a homogeneous (respectively, arbitrary) finitely presented monoid of linear growth.

**Proposition 5.** Let $S$ be a homogeneous monoid such that $\gamma(S) = 1$. Then $S$ is the union of a free monogenic monoid and a finite set.

Note that the above monoid $M$ (which is homogeneous of linear growth) with $\gamma(M) = 2$ cannot be decomposed into a finite union of monogenic semigroups and a finite set (again because $M$ contains an infinite subset $1(y)x$ of nilpotent elements).

**Proof.** Let $S$ be the disjoint union of a finite set $Y$ and a free sandwich $Z = a\langle w\rangle b$. For $m >> 0$, the set $S_m$ of elements of the degree $m$ in $S$ is either the singleton $\{aw^k b\}$ (if $k = (m − d(a) − d(b))/d(w)$ is integer) or empty. Since the element $w^t$ is nonzero for all $t \geq 0$, for $m = td(w)$ with $t >> 0$ this set $S_m$ contains $w^t$. So, $S_{td(w)}$ is non-empty for all $t >> 0$, so that $d(a) + d(b) = sd(w)$ for some integer $s$. We conclude that for each $m >> 0$ the set $S_m$ is non-empty if and only if $m−sd(w)/d(w)$ is an integer, or $m = td(w)$ for some integer $t$. In the last case, we have $S_m = \{w^t\}$, so that $S$ is the union of the free monogenic monoid $\langle w\rangle$ and a finite set. 

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