Macroscopic quantum effects in capacitively- and inductively-coupled intrinsic Josephson junctions

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Abstract. A theory for macroscopic quantum tunneling (MQT) in intrinsic Josephson junction stacks is formulated. Both capacitive and inductive couplings between junctions are taken into account. We calculate the escape rate in the switching to the first resistive branch in the quantum regime. It is shown that the enhancement of the escape rate is caused mainly by the capacitive coupling between junctions in IJJ’s with small in-plane area of $\sim 1\mu m^2$.

1. Introduction
Macroscopic quantum tunneling (MQT) in intrinsic Josephson junction stacks (IJJ’s) has attracted a great interest since its discovery in Bi-2212 and YBCO IJJ’s[1, 2, 3, 4, 5, 6, 7]. Since IJJ’s are a nanoscale multi-junction system, the dynamics of the phase differences in IJJ’s is strongly affected by the coupling between junctions, which is well known in the classical regime[8, 9, 10, 11]. The quantum dynamics of the phase differences in IJJ’s in the low temperature region is also expected to depend on the coupling between junctions. In this paper we study the effect of both inductive and capacitive couplings between junctions on the quantum switching to the first resistive branch in IJJ’s, and construct a theory that can describe the macroscopic quantum effects in IJJ’s. We show that the escape rate in the quasi-classical regime in IJJ’s can be calculated in terms of the path-integral representation for the imaginary-time transition amplitude. A quantum description of a breathing mode that is localized on one of the stacked junctions is formulated on the basis of our quantum theory. The quantum tunneling of this mode is identified with MQT in the switching event from the zero-voltage state to the first resistive branch. The effects of inductive and capacitive couplings between junctions on the switching event is clarified in the quantum regime.

2. Formulation
The interaction between gauge-invariant phase differences in IJJ’s originates from the capacitive and inductive couplings between junctions. In the presence of these couplings the Josephson relations among the gauge-invariant phase difference $\theta_\ell$, voltage $V_\ell$ and magnetic flux $\Phi_\ell$ at $\ell$th junction site are generalized as $\hbar \frac{d}{dt} \theta_\ell = -\alpha V_{\ell+1} + (1 + 2\alpha) V_\ell - \alpha V_{\ell-1} \equiv \sum_m \Lambda_{\ell m} V_m$, and $\hbar \frac{d}{dt} \Phi_\ell = -\eta \Phi_{\ell+1} + (1 + 2\eta) \Phi_\ell - \eta \Phi_{\ell-1} \equiv \sum_m \Gamma_{\ell m} \Phi_m$, where $\alpha$ and $\eta$ are, respectively, the capacitive and inductive coupling constants defined as $\alpha = \epsilon \mu^2 / sd$ and $\eta = \lambda_{ab}^2 / sd$, where $\epsilon$ is the dielectric constant of the insulating layers, $\lambda_{ab}$ is the in-plane London penetration depth.
and \( s \) (\( d \)) is the width of superconducting (insulating) layers. In this paper we consider the finite system with an in-plane area, \( W = w_x w_y \), and restrict ourselves to the case where the in-plane spatial variation of \( \theta_i \) appears only along the \( x \)-direction, i.e., \( w_y \ll w_x \), for simplicity. In this case one can utilize the Maxwell equation,

\[
\partial_x \Phi_\ell = \frac{4\pi d}{c} j_c \sin \theta_\ell + \frac{\epsilon}{c} \partial_t V_\ell,
\]

where \( j_c \) is the Josephson critical current density per unit junction area. Then, from Eqs.(2), (2) and (1) it follows

\[
\frac{1}{\omega_{pl}^2} \sum_{m=1}^{N} \lambda_\ell^{-1} \partial_t^2 \theta_m - \lambda_\ell^2 \sum_{m=1}^{N} \Gamma_{\ell m}^{-1} \partial_x^2 \theta_m + \sin \theta_\ell = \frac{I}{j_c} \equiv \tilde{I},
\]

in the presence of a bias current \( I \), where \( \lambda_c \) is the c-axis penetration depth, \( \lambda_c^2 = \hbar c^2 / 4\pi e^* d j_c \), and \( \omega_{pl} \) is the Josephson plasma frequency, \( \omega_{pl} = c / \sqrt{\pi \lambda_c} \). In deriving Eq.(2) we assume \( w_x \ll \lambda_c \). One can construct a canonical theory which gives Eq.(2) as equation of motion. Note that Eq.(1) can be derived from the Hamiltonian,

\[
H = \sum_{\ell=1}^{N} \int_{x_0}^{x_w} dx \left\{ \frac{w_x E_c}{2h^2} \sum_{m=1}^{N} u_m(x) \lambda_{\ell m} u_m(x) + E_J \lambda_{\ell m} V(\theta_\ell) + \frac{\lambda_\ell^2}{2} \sum_{m=1}^{N} \partial_x \theta_\ell(x) \Gamma_{\ell m}^{-1} \partial_x \theta_m(x) \right\},
\]

where \( u_m(x) \) is the canonical momentum conjugate to the gauge-invariant phase difference \( \theta_\ell(x) \) and \( V(\theta_\ell) = 1 - \cos \theta_\ell(x) - i \theta_\ell(x) \). The parameters \( E_c \) and \( E_J \) in Eq.(2) are defined as \( E_c = \frac{4\pi e^* d j_c}{\omega_{pl}^2} = \frac{e^2}{\omega_{pl}^2} \), \( E_J = \frac{\hbar c}{w_x} \), where \( e^* \) is the charge of a Cooper pair. Note that \( C \) is equal to the capacitance of a single junction and the plasma frequency is related to these parameters as \( \omega_{pl}^2 = \sqrt{E_c E_J} \). The quantization of the intrinsic Josephson junction stacks can be performed by imposing the commutation relation, \( [\theta_\ell(x), u_m(x')] = i\hbar \delta_{\ell m}(x - x') \), for the canonical variables \( (\theta_\ell(x), u_m(x)) \). From eq.(3) one can derive the imaginary-time Lagrangian,

\[
\mathcal{L}(\Theta, \dot{\Theta}) = \sum_{\ell=1}^{N} \int_{x_0}^{x_w} dx \left\{ \frac{h^2}{2 E_c} \sum_{m=1}^{N} \partial_x \theta_\ell \Lambda_{\ell m}^{-1} \cdot \partial_x \theta_m + E_J \dot{V}(\theta_\ell) + \frac{\lambda_\ell^2}{2} \sum_{m=1}^{N} \partial_x \theta_\ell \cdot \Gamma_{\ell m}^{-1} \partial_x \theta_m \right\},
\]

with \( \tau \) being the imaginary time. In the absence of an external magnetic field one can introduce the Fourier series expansion in the \( x \)-direction for \( \theta_\ell(x) \). Assuming \( w_x \) is small, we use the expression in the single-harmonic approximation,

\[
\theta_\ell(x) = q_\ell + \sqrt{2} \xi_\ell \cos \left( \frac{\pi}{w_x} x \right),
\]

where the Fourier components \( q_\ell \) and \( \xi_\ell \) are regarded as the coordinates in the following calculations. Then, substituting Eq.(5) into Eq.(4), we have the Lagrangian in terms of \( q_\ell \) and \( \xi_\ell \) as

\[
\mathcal{L} = \sum_{\ell, m} \left\{ \frac{h^2}{2 E_c} \dot{q}_\ell \Lambda_{\ell m}^{-1} \dot{q}_m + E_J \dot{q}_\ell V(q_\ell) + \frac{h^2}{2 E_c} \xi_\ell \Lambda_{\ell m} \xi_m + \frac{E_J}{2} \left( \pi \lambda_\ell w_x \right)^2 \xi_\ell \Gamma_{\ell m}^{-1} \xi_m + \frac{\delta_{\ell m}}{2} E_J \cos q_\ell \xi_\ell^2 \right\},
\]

where \( \dot{q}_\ell \equiv dq_\ell / d\tau \) and \( \dot{\xi}_\ell \equiv d\xi_\ell / d\tau \). In deriving Eq.(9) we also assume that \( \xi_\ell \) is small.

Suppose that the \( k \)-th junction is switched to the resistive state and the other junctions still remain in the zero-voltage state. In this case one can utilize the approximation as \( V(q_\ell) \simeq \frac{1}{2} a_\ell q_\ell^2 \) with \( a_\ell = (1 - I^2)^{1/2} \) and \( \cos q_\ell \cdot \xi_\ell^2 \simeq \xi_\ell^2 \) for \( \ell \neq k \). Then, Eq.(6) can be rewritten as

\[
\mathcal{L} \simeq \frac{h^2}{2 E_c} \Lambda_{kk} \dot{q}_k^2 + E_J V(q_k) + \sum_{\ell \neq k} \sum_{m \neq k} \left\{ \frac{h^2}{2 E_c} \dot{q}_\ell \Lambda_{\ell m}^{-1} \dot{q}_m + \frac{1}{2} E_J a_\ell q_\ell^2 \delta_{\ell m} \right\} + \sum_{m \neq k} \frac{h^2}{2 E_c} \dot{q}_k \Lambda_{km}^{-1} \dot{q}_m.
\]
\[ \sum_{\ell} \sum_{m} \left\{ \frac{\hbar^2}{2E_c} \xi_{\ell} \Lambda_{\ell m}^{-1} \xi_m + \frac{E_c}{2} \left( \frac{\pi c}{w_x} \right)^2 \xi_{\ell} \Gamma_{\ell m}^{-1} \xi_m + \frac{1}{2} E_J \xi_{\ell}^2 \delta_{\ell m} \right\} - \frac{1}{2} E_J (1 - \cos q_k) \xi_k^2. \]  

Note that the terms containing \( q_k \) with \( \ell \neq k \), i.e., 3rd and 4th terms, in Eq. (7) are quadratic with respect to \( q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_N \). Then, these terms can be integrated out. In the case where \( \xi_\alpha = \alpha/(1 + 2\alpha) \) is small, which is valid in Bi-2212 IJJ’s, we obtain the approximate expression up to \( \mathcal{O}(\xi_{\alpha}^2) \) as

\[ \mathcal{L}_{\text{reno}} \simeq -\frac{1}{2} \frac{\hbar^2 (1 + 2\alpha)}{E_c} \sum_{i \neq k} (\Lambda_{ki}^{-1})^2 q_k^2. \]  

Since this term can be added to the kinetic energy term in Eq. (7), one may understand that the capacitive coupling between junctions causes mass renormalization as \( \Lambda_{kk}^{-1} \rightarrow m_\alpha = \Lambda_{kk}^{-1} - (1 + 2\alpha) \sum_{i \neq k} (\Lambda_{ki}^{-1})^2 \). This result indicates that the ‘mass of a particle’ on the \( k \)th junction is reduced by the capacitive coupling, that is, the quantum effect is expected to be enhanced by the capacitive coupling between junctions.

Let us next investigate the effect of inductive coupling between junctions. From Eq. (7) one understands that the variables \( \xi_\ell \)’s describe the oscillatory motion with frequencies bigger than the plasma frequency \( \omega_p = \sqrt{E_cE_J}/\hbar \). The motion of \( q_k \), which is assumed to describe the degree of freedom showing the quantum tunneling, is much more slower than the plasma mode.

On the basis of this observation we use the mean-field approximation as \( \xi_{\ell k}^2 \rightarrow \langle \xi_{\ell k}^2 \rangle \) in Eq. (7), where the mean value \( \langle \xi_{\ell k}^2 \rangle \) is calculated as

\[ \langle \xi_{\ell k}^2 \rangle = \left( \frac{E_c}{\hbar \omega_{pl}} \right) \frac{1}{N} \sum_{q=0}^{N-1} \frac{K_q}{1 + \left( \frac{\pi c}{w_x} q \right)^2}, \]  

in the \( N \)-junction system, where \( K_q = \sqrt{1 + \alpha(1 - \cos \frac{\pi q}{N})} \) and \( L_q = \sqrt{1 + \eta(1 - \cos \frac{\pi q}{N})} \). Then, from the above results, it follows the effective imaginary-time Lagrangian as

\[ \mathcal{L}_{\text{eff}} = \frac{\hbar^2}{2E_c} \sum_{i \neq k} (\Lambda_{ki}^{-1})^2 q_k^2 + E_J (1 - \cos q_k) - E_J \dot{I} q_k, \]  

with \( E_c^x = E_c (1 - \frac{1}{N} \langle \xi_{\ell k}^2 \rangle) \). From Eq. (10) one understands that the inductive coupling between junctions reduces the Josephson coupling energy. Let us now evaluate the mean value \( \langle \xi_{\ell k}^2 \rangle \). When the values of the material parameters are chosen as \( \alpha \sim 0.1, \lambda_J = \lambda_c/\sqrt{\eta} \sim 0.5\mu m \) and \( \omega_{pl}/2\pi \sim 200GHz \), which are suitable for Bi-2212 IJJ’s, we have \( \langle \xi_{\ell k}^2 \rangle \leq E_c/\hbar \omega_{pl} \sim 0.05 \) for \( W = w_x w_y \sim 1\mu m^2 \). In this case one finds \( 1 > E_J^x/E_J \geq 0.98 \), which indicates that the quantum fluctuation arising from the inductive coupling is very small in IJJ’S with a small in-plane area of \( W \sim 1\mu m^2 \).

Since the motion of the degree of freedom showing the MQT to the first resistive branche in the IJJ’s can be mapped onto that of a 1D quasi-classical particle, as seen in Eq. (10), the escape rate in the present system can be calculated, using the formula in the instanton approximation as follows,

\[ \Gamma = 12 \omega_J (\alpha) (\frac{3V_I}{2\pi \hbar \omega_J (\alpha)})^{1/2} \exp \left[ -\frac{36V_I}{5\hbar \omega_J (\alpha)} \right], \]  

where \( V_I = E_J (2a_I + \dot{I} [2 \sin^{-1} \dot{I} - \pi]) \), and \( \omega_J (\alpha) = \omega_{pl} \sqrt{a_I/m_\alpha} \).

In Fig. 1 we plot the escape rate (11) as a function of the bias current density for several values of \( \alpha \). As seen in this figure, the escape rate \( \Gamma \) shows strong dependence on the capacitive coupling between junctions and increases with \( \alpha \). From the analysis of the \( I - V \) characteristics.
one can get the value $\alpha \sim 0.1$ in Bi-2212 IJJ’s [9]. Then, our numerical results indicate that the enhancement due to the capacitive coupling cannot be neglected in Bi-2212 IJJ’s, as seen in the figure. We also mention that the bias current dependence of the escape rate for $\alpha = 0.1$ is quantitatively in good agreement with the experimental result given in [2]. Hence, one may conclude that the capacitive-coupled IJJ model well describes the phase dynamics of the IJJ stacks in both classical and quantum regimes.

In summary we have formulated a theory for the macroscopic quantum tunneling in IJJ’s. The multi-junction effects originating from the capacitive and inductive couplings between junctions has been clarified. We showed that our theory well explain the current dependence of the escape rate in the first switching in Bi-2212 IJJ’s.

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4. References

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