EIGENVALUE COINCIDENCES AND $K$-ORBITS, I

MARK COLARUSSO AND SAM EVENS

ABSTRACT. We study the variety $g(l)$ consisting of matrices $x \in gl(n, \mathbb{C})$ such that $x$ and its $n-1$ by $n-1$ cutoff $x_{n-1}$ share exactly $l$ eigenvalues, counted with multiplicity. We determine the irreducible components of $g(l)$ by using the orbits of $GL(n-1, \mathbb{C})$ on the flag variety $\mathcal{B}$ of $gl(n, \mathbb{C})$. More precisely, let $b \in \mathcal{B}$ be a Borel subalgebra such that the orbit $GL(n-1, \mathbb{C}) \cdot b$ in $\mathcal{B}$ has codimension $l$. Then we show that the set $Y_b := \{ Ad(g)(x) : x \in b \cap g(l), g \in GL(n-1, \mathbb{C}) \}$ is an irreducible component of $g(l)$, and every irreducible component of $g(l)$ is of the form $Y_b$, where $b$ lies in a $GL(n-1, \mathbb{C})$-orbit of codimension $l$. An important ingredient in our proof is the flatness of a variant of a morphism considered by Kostant and Wallach, and we prove this flatness assertion using ideas from symplectic geometry.

1. INTRODUCTION

Let $g := gl(n, \mathbb{C})$ be the Lie algebra of $n \times n$ complex matrices. For $x \in g$, let $x_{n-1} \in gl(n-1, \mathbb{C})$ be the upper left-hand $n-1$ by $n-1$ corner of the matrix $x$. For $0 \leq l \leq n-1$, we consider the subset $g(l)$ consisting of elements $x \in g$ such that $x$ and $x_{n-1}$ share exactly $l$ eigenvalues, counted with multiplicity. In this paper, we study the algebraic geometry of the set $g(l)$ using the orbits of $GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$ on the flag variety $\mathcal{B}$ of Borel subalgebras of $g$. In particular, we determine the irreducible components of $g(l)$ and use this to describe elements of $g(l)$ up to $GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$-conjugacy.

In more detail, let $G = GL(n, \mathbb{C})$ and let $\theta : G \to G$ be the involution $\theta(x) = d x d^{-1}$, where $d = \text{diag}[1, \ldots, 1, -1]$. Let $K := G^\theta = GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$. It is well-known that $K$ has exactly $n$ closed orbits on the flag variety $\mathcal{B}$, and each of these closed orbits is isomorphic to the flag variety $\mathcal{B}_{n-1}$ of Borel subalgebras of $gl(n-1, \mathbb{C})$. Further, there are finitely many $K$-orbits on $\mathcal{B}$, and for each of these $K$-orbits $Q$, we consider its length $l(Q) = \dim(Q) - \dim(\mathcal{B}_{n-1})$. It is elementary to verify that $0 \leq l(Q) \leq n-1$. For $Q = K \cdot b_Q$, we consider the $K$-saturation $Y_Q := Ad(K)b_Q$ of $b_Q$, which is independent of the choice of $b_Q \in Q$.

Theorem 1.1. The irreducible component decomposition of $g(l)$ is

$$g(l) = \bigcup_{l(Q) = n-1-l} Y_Q \cap g(l).$$

The proof uses several ingredients. The first is the flatness of a variant of a morphism studied by Kostant and Wallach [KW06], which implies that $g(l)$ is equidimensional. We
prove the flatness assertion using dimension estimates derived from symplectic geometry, but it also follows from results of Ovsienko and Futorny [Ovs03], [FO05]. The remaining ingredient is an explicit description of the $l + 1$ $K$-orbits $Q$ on $B$ with $l(Q) = n - 1 - l$, and the closely related study of $K$-orbits on generalized flag varieties $G/P$. Our theorem has the following consequence. Let $b_+$ denote the Borel subalgebra consisting of upper triangular matrices. For $i = 1, \ldots, n$, let $(i\, n)$ be the permutation matrix corresponding to the transposition interchanging $i$ and $n$, and let $b_i := \text{Ad}(i) b_+$.

**Corollary 1.2.** If $x \in \mathfrak{g}(l)$, then $x$ is $K$-conjugate to an element in one of $l + 1$ explicitly determined $\theta$-stable parabolic subalgebras. In particular, if $x \in \mathfrak{g}(n - 1)$, then $x$ is $K$-conjugate to an element of $b_i$, where $i = 1, \ldots, n$.

This paper is part of a series of papers on $K$-orbits on $B$ and the Gelfand-Zeitlin system. In [CE12], we used $K$-orbits to determine the so-called strongly regular elements in the nilfiber of the moment map of the Gelfand-Zeitlin system. These are matrices $x \in \mathfrak{g}$ such that $x_i$ is nilpotent for all $i = 1, \ldots, n$ with the added condition that the differentials of the Gelfand-Zeitlin functions are linearly independent at $x$. The strongly regular elements were first studied extensively in [KW06]. In later work, we will refine Corollary 1.2 to provide a standard form for all elements of $\mathfrak{g}(l)$. This uses $K$-orbits and a finer study of the algebraic geometry of the varieties $\mathfrak{g}(l)$. In particular, we will give a more conceptual proof of the main result from [Col11] and use $K$-orbits to describe the geometry of arbitrary fibers of the moment map for the Gelfand-Zeitlin system.

The work by the second author was partially supported by NSA grant H98230-11-1-0151. We would like to thank Adam Boocher and Claudia Polini for useful discussions.

2. Preliminaries

We show flatness of the partial Kostant-Wallach morphism and recall needed results concerning $K$-orbits on $B$.

2.1. The partial Kostant-Wallach map. For $x \in \mathfrak{g}$ and $i = 1, \ldots, n$, let $x_i \in \mathfrak{gl}(i, \mathbb{C})$ denote the upper left $i \times i$ corner of the matrix $x$. For any $y \in \mathfrak{gl}(i, \mathbb{C})$, let $tr(y)$ denote the trace of $y$. For $j = 1, \ldots, i$, let $f_{i,j}(x) = tr((x_i)^j)$, which is a homogeneous function of degree $j$ on $\mathfrak{g}$. The Gelfand-Zeitlin collection of functions is the set $J_{GZ} = \{f_{i,j}(x) : i = 1, \ldots, n, j = 1, \ldots, i\}$. The restriction of these functions to any regular adjoint orbit in $\mathfrak{g}$ produces an integrable system on the orbit $\mathfrak{g}(l)$ [KW06]. Let $\chi_{i,j} : \mathfrak{gl}(i, \mathbb{C}) \to \mathbb{C}$ be the function $\chi_{i,j}(y) = tr(y^j)$, so that $f_{i,j}(x) = \chi_{i,j}(x_i)$ and $\chi_i := (\chi_{i,1}, \ldots, \chi_{i,i})$ is the adjoint quotient for $\mathfrak{gl}(i, \mathbb{C})$. The Kostant-Wallach map is the morphism given by

$$
\Phi : \mathfrak{g} \to \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n; \Phi(x) = (\chi_1(x_1), \ldots, \chi_n(x)).
$$

We will also consider the partial Kostant-Wallach map given by the morphism

$$
\Phi_n : \mathfrak{g} \to \mathbb{C}^{n-1} \times \mathbb{C}^n; \Phi_n(x) = (\chi_{n-1}(x_{n-1}), \chi_n(x)).
$$
Note that
\begin{equation}
\Phi_n = pr \circ \Phi,
\end{equation}
where \( pr : \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n \to \mathbb{C}^{n-1} \times \mathbb{C}^n \) is projection on the last two factors.

**Remark 2.1.** By Theorem 0.1 of [KW06], the map \( \Phi \) is surjective, and it follows easily that \( \Phi_n \) is surjective.

We let \( I_n = \langle \{ f_{ij} \}_{i=n-1, n; j=1, \ldots, i} \rangle \) denote the ideal generated by the functions \( J_{GZ,n} := \{ f_{i,j} : i = n - 1, n; j = 1, \ldots, i \} \). We call the vanishing set \( V(I_n) \) the variety of partially strongly nilpotent matrices and denote it by \( SN_n \). Thus,
\begin{equation}
SN_n := \{ x \in g : x, x_{n-1} \text{ are nilpotent} \}.
\end{equation}

We let \( \Gamma_n := \mathbb{C}[\{ f_{ij} \}_{i=n-1, n; j=1, \ldots, i}] \) be the subring of regular functions on \( g \) generated by \( J_{GZ,n} \).

Recall that if \( Y \subset \mathbb{C}^m \) is a closed equidimensional subvariety of dimension \( m - d \), then \( Y \) is called a complete intersection if \( Y = V(f_1, \ldots, f_d) \) is the vanishing set of \( d \) functions.

**Theorem 2.2.** The variety of partially strongly nilpotent matrices \( SN_n \) is a complete intersection of dimension
\begin{equation}
d_n := n^2 - 2n + 1.
\end{equation}

Before proving Theorem 2.2 we show how it implies the flatness of the partial Kostant-Wallach map \( \Phi_n \).

**Proposition 2.3.**
\begin{enumerate}
\item For all \( c \in \mathbb{C}^{n-1} \times \mathbb{C}^n \), \( \dim(\Phi_n^{-1}(c)) = n^2 - 2n + 1 \). Thus, \( \Phi_n^{-1}(c) \) is a complete intersection.
\item The partial Kostant-Wallach map \( \Phi_n : g \to \mathbb{C}^{2n-1} \) is a flat morphism. Thus, \( \mathbb{C}[g] \) is flat over \( \Gamma_n \).
\end{enumerate}

**Proof.** For \( x \in g \), we let \( d_x \) be the maximum of the dimensions of irreducible components of \( \Phi_n^{-1}(\Phi_n(x)) \). For \( c \in \mathbb{C}^{n-1} \times \mathbb{C}^n \), each irreducible component of \( \Phi_n^{-1}(c) \) has dimension at least \( d_n \) since \( \Phi_n^{-1}(c) \) is defined by \( 2n - 1 \) equations in \( g \). Hence, \( d_x \geq d_n \). Since the functions \( f_{i,j} \) are homogeneous, it follows that scalar multiplication by \( \lambda \in \mathbb{C}^x \) induces an isomorphism \( \Phi_n^{-1}(\Phi_n(x)) \to \Phi_n^{-1}(\Phi_n(\lambda x)) \). It follows that \( d_x = d_{\lambda x} \). By upper semi-continuity of dimension (see for example, Proposition 4.4 of [Hum75]), the set of \( y \in g \) such that \( d_y \geq d \) is closed for each integer \( d \). It follows that \( d_0 \geq d_x \). By Theorem 2.2 \( d_0 = d_n \). The first assertion follows easily. The second assertion now follows by the corollary to Theorem 23.1 of [Mat86].

Q.E.D.

**Remark 2.4.** We note that Proposition 2.3 implies that \( \mathbb{C}[g] \) is free over \( \Gamma_n \). This follows from a result in commutative algebra. Let \( A = \oplus_{n \geq 0} A_n \) be a graded ring with \( A_0 = k \) a field and let \( M = \oplus_{n \geq 0} M_n \) be a graded \( A \)-module. The needed result asserts that \( M \) is flat over \( A \) if and only if \( M \) is free over \( A \). One direction of this assertion is obvious, and the
other direction may be proved using the same argument as in the proof of Proposition 20 on page 73 of [Ser00], which is the analogous assertion for finitely generated modules over local rings. In this context, the assumption that \( M \) is finitely generated over \( A \) is needed only to apply Nakayama’s lemma, but in our graded setting, Nakayama’s lemma (with ideal \( I = \oplus_{n \geq 0} A_n \)) does not require the module \( M \) to be finitely generated.

**Remark 2.5.** Let \( I = (J_{GZ}) \) be the ideal in \( \mathbb{C}[g] \) generated by the Gelfand-Zeitlin collection of functions \( J_{GZ} \), and let \( SN = V(I) \) be the strongly nilpotent matrices, i.e., \( SN = \{ x \in g : x_i \text{ is nilpotent for } i = 1, \ldots, n \} \). Ovsienko proves in \([Ovs03]\) that \( SN \) is a complete intersection, and results of Futorny and Ovsienko from \([FO05]\) show that Ovsienko’s theorem implies that \( C_{\text{SN}} \) is flat over \( \Gamma := \mathbb{C}[\{f_{ij}\}_{i=1,\ldots,n;j=1,\ldots,n}] \). It then follows easily that \( \mathbb{C}[g] \) is flat over \( \Gamma_n \), and hence that \( \Phi_n \) is flat. Although we could have simply cited the results of Futorny and Ovsienko to prove flatness of \( \Phi_n \), we prefer our approach, which we regard as more conceptual.

**Proof of Theorem 2.2.** Let \( \mathfrak{X} \) be an irreducible component of \( SN_n \). We observed in the proof of Proposition 2.3 that \( \dim \mathfrak{X} \geq d_n \). To show \( \dim \mathfrak{X} \leq d_n \), we consider a generalization of the Steinberg variety (see Section 3.3 of \([CG97]\)). We first recall a few facts about the cotangent bundle to the flag variety.

For the purposes of this proof, we denote the flag variety of \( \mathfrak{gl}(n, \mathbb{C}) \) by \( B_n \). We consider the form \( \langle \langle \cdot, \cdot \rangle \rangle \) on \( g \) given by \( \langle \langle x, y \rangle \rangle = \text{tr}(xy) \) for \( x, y \in g \). If \( b \in B_n \), the annihilator \( b^\perp \) of \( b \) with respect to the form \( \langle \langle \cdot, \cdot \rangle \rangle \) is \( n = [b, b] \). We can then identify \( T^*(B_n) \) with the closed subset of \( g \times B_n \) given by:

\[
T^*(B_n) = \{(x, b) : b \in B_n, x \in n\}.
\]

We let \( g_{n-1} = \mathfrak{gl}(n-1, \mathbb{C}) \) and view \( g_{n-1} \) as a subalgebra of \( g \) by embedding \( g_{n-1} \) in the top left-hand corner of \( g \). Since \( g \) is the direct sum \( g = g_{n-1} \oplus g_{n-1} \), the restriction of \( \langle \langle \cdot, \cdot \rangle \rangle \) to \( g_{n-1} \) is non-degenerate. For a Borel subalgebra \( b' \in B_{n-1} \), we let \( n' = [b', b'] \). We consider a closed subvariety \( Z \subset g \times B_n \times B_{n-1} \) defined as follows:

\[
(2.6) \quad Z = \{(x, b, b') : b \in B_n, b' \in B_{n-1} \text{ and } x \in n, x_{n-1} \in n'\}.
\]

Consider the morphism \( \mu : Z \to g \), where \( \mu(x, b, b') = x \). Since the varieties \( B_n \) and \( B_{n-1} \) are projective, the morphism \( \mu \) is proper.

We consider the closed embedding \( Z \hookrightarrow T^*(B_n) \times T^*(B_{n-1}) \cong T^*(B_n \times B_{n-1}) \) given by \( (x, b, b') \mapsto (x, -x_{n-1}, b, b') \). We denote the image of \( Z \) under this embedding by \( \tilde{Z} \subset T^*(B_n \times B_{n-1}) \). Let \( G_{n-1} \) be the closed subgroup of \( GL(n, \mathbb{C}) \) corresponding to \( g_{n-1} \). Then \( G_{n-1} \) acts diagonally on \( B_n \times B_{n-1} \) via \( k \cdot (b, b') = (k \cdot b, k \cdot b') \) for \( k \in G_{n-1} \). We claim \( \tilde{Z} \subset T^*(B_n \times B_{n-1}) \) is the union of conormal bundles to the \( G_{n-1} \)-diagonal orbits in \( B_n \times B_{n-1} \). Indeed, let \( (b, b') \in B_n \times B_{n-1} \), and let \( Q \) be its \( G_{n-1} \)-orbit. Then

\[
T_{(b,b')}^*(Q) = \text{span}\{(Y \mod b, Y \mod b') : Y \in g_{n-1}\}.
\]
Now let \( (\lambda_1, \lambda_2) \in (n, n') \) with \( (\lambda_1, \lambda_2) \in (T_Q^* (U_n \times U_{n-1})(b,b')) \), the fiber of the conormal bundle to \( Q \) in \( U_n \times U_{n-1} \) at the point \( (b,b') \). Then
\[
<< \lambda_1, Y >> + << \lambda_2, Y >> = 0 \text{ for all } Y \in g_{n-1}.
\]
Thus, \( \lambda_1 + \lambda_2 \in g_{n-1} \). But since \( \lambda_2 \in n' \subset g_{n-1} \), it follows that \( \lambda_2 = -(\lambda_1)_{n-1} \). Thus,
\[
T_Q^* (U_n \times U_{n-1}) = \{(\mu_1, b_1, -(\mu_1)_{n-1}, b_2), \mu_1 \in n_1, (\mu_1)_{n-1} \in n_2, \text{ where } (b_1, b_2) \in Q \}.
\]
We recall the well-known fact that there are only finitely many \( G_{n-1} \)-diagonal orbits in \( U_n \times U_{n-1} \), which follows from \([\text{VK78}], [\text{Bri87}], \) or in a more explicit form is proved in \([\text{Has04}]\). Therefore, the irreducible component decomposition of \( \tilde{Z} \) is:
\[
\tilde{Z} = \bigcup_i T_Q^* (U_n \times U_{n-1}) \subset T^* (U_n \times U_{n-1}),
\]
where \( i \) runs over the distinct \( G_{n-1} \)-diagonal orbits in \( U_n \times U_{n-1} \). Thus, \( \tilde{Z} \cong Z \) is a closed, equidimensional subvariety of dimension \( \dim Z = \frac{1}{2} (\dim T^* (U_n \times U_{n-1})) = d_n \).

Note that \( \mu : Z \to SN_n \) is surjective. Since \( \mu \) is proper, for every irreducible component \( X \subset SN_n \) of \( SN_n \), we see that
\[
(2.7) \quad X = \mu(Z_i)
\]
for some irreducible component \( Z_i \subset Z \). Since \( \dim Z_i = d_n \) and \( \dim X \geq d_n \), we conclude that \( \dim X = d_n \).

\[\text{Q.E.D.}\]

In Proposition 3.10, we will determine the irreducible components of \( SN_n \) explicitly.

2.2. \( K \)-orbits. We recall some basic facts about \( K \)-orbits on generalized flag varieties \( G/P \) (see \([\text{Mat79}], [\text{RS90}], [\text{MO90}], [\text{Yam97}], [\text{CE}]\) for more details).

By the general theory of orbits of symmetric subgroups on generalized flag varieties, \( K \) has finitely many orbits on \( U \). For this paper, it is useful to parametrize the orbits. To do this, we let \( B_+ \) be the upper triangular Borel subgroup of \( G \), and identify \( B \cong G/B_+ \) with the variety of flags in \( \mathbb{C}^n \). We use the following notation for flags in \( \mathbb{C}^n \). Let \( F = (V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_n = \mathbb{C}^n) \), be a flag in \( \mathbb{C}^n \), with \( \dim V_i = i \) and \( V_i = \text{span}\{v_1, \ldots, v_i\} \), with each \( v_j \in \mathbb{C}^n \). We will denote the flag \( F \) as follows:
\[
v_1 \subset v_2 \subset \cdots \subset v_i \subset v_{i+1} \subset \cdots \subset v_n.
\]
We denote the standard ordered basis of \( \mathbb{C}^n \) by \( \{e_1, \ldots, e_n\} \), and let \( E_{i,j} \in g \) be the matrix with 1 in the \((i,j)\)-entry and 0 elsewhere.
There are $n$ closed $K$-orbits on $\mathcal{B}$ (see Example 4.16 of [CE]), $Q_{i,i} = K \cdot b_{i,i}$ for $i = 1, \ldots, n$, where the Borel subalgebra $b_{i,i}$ is the stabilizer of the following flag in $\mathbb{C}^n$:

\begin{equation}
F_{i,i} = (e_1 \subset \cdots \subset e_{i-1} \subset e_i \subset \cdots \subset e_{n-1}).
\end{equation}

(2.8)

Note that if $i = n$, then the flag $F_{i,i}$ is the standard flag $F_+$:

\begin{equation}
F_+ = (e_1 \subset \cdots \subset e_n),
\end{equation}

(2.9)

and $b_{n,n} = b_+$ is the standard Borel subalgebra of $n \times n$ upper triangular matrices. It is not difficult to check that $K \cdot b_{i,i} = K \cdot \text{Ad}(i\,n)b_+$. If $i = 1$, then $K \cdot b_{1,1} = K \cdot b_-$, where $b_-$ is the Borel subalgebra of lower triangular matrices in $\mathfrak{g}$.

The non-closed $K$-orbits in $\mathcal{B}$ are the orbits $Q_{i,j} = K \cdot b_{i,j}$ for $1 \leq i < j \leq n$, where $b_{i,j}$ is the stabilizer of the flag in $\mathbb{C}^n$:

\begin{equation}
F_{i,j} = (e_1 \subset \cdots \subset e_i + e_n \subset e_{i+1} \subset \cdots \subset e_{j-1} \subset e_i \subset e_j \subset \cdots \subset e_{n-1}).
\end{equation}

(2.10)

There are $\binom{n}{2}$ such orbits (see Notation 4.23 and Example 4.31 of [CE]).

Let $w$ and $\sigma$ be the permutation matrices corresponding respectively to the cycles $(n\,n-1\,\cdots\,i)$ and $(i+1\,i+2\,\ldots\,j)$, and let $u_{\alpha_i}$ be the Cayley transform matrix such that

\begin{align*}
u_{\alpha_i}(e_i) &= e_i + e_{i+1}, \\
u_{\alpha_i}(e_{i+1}) &= -e_i + e_{i+1}, \\
u_{\alpha_i}(e_k) &= e_k, \quad k \neq i, i+1.
\end{align*}

For $1 \leq i \leq j \leq n$, we define:

\begin{equation}
v_{i,j} := \begin{cases} w & \text{if } i = j \\
w u_{\alpha_i} \sigma & \text{if } i \neq j
\end{cases}
\end{equation}

(2.11)

It is easy to verify that $v_{i,j}(F_+) = F_{i,j}$, and thus $\text{Ad}(v_{i,j})b_+ = b_{i,j}$ (see Example 4.30 of [CE]).

**Remark 2.6.** The length of the $K$-orbit $Q_{i,j}$ is $l(Q_{i,j}) = j - i$ for any $1 \leq i \leq j \leq n$ (see Example 4.30 of [CE]). For example, a $K$-orbit $Q_{i,j}$ is closed if and only if $Q = Q_{i,i}$ for some $i$. The $n-l$ orbits of length $l$ are $Q_{i,i+l}$, $i = 1, \ldots, n-l$.

For a parabolic subgroup $P$ of $G$ with Lie algebra $\mathfrak{p}$, we consider the generalized flag variety $G/P$, which we identify with parabolic subalgebras of type $\mathfrak{p}$ and with partial flags of type $\mathfrak{p}$. We will make use of the following notation for partial flags. Let

\[ \mathcal{P} = (V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_k = \mathbb{C}^n) \]

denote a $k$-step partial flag with $\dim V_j = i_j$ and $V_j = \text{span}\{v_1, \ldots, v_{i_j}\}$ for $j = 1, \ldots, k$.

Then we denote $\mathcal{P}$ as

\[ v_{i_1}, \ldots, v_{i_i} \subset v_{i_1+1}, \ldots, v_{i_2} \subset \cdots \subset v_{i_{k-1}+1}, \ldots, v_{i_k}. \]

In particular for $i \leq j$, we let $r_{i,j} \subset \mathfrak{g}$ denote the parabolic subalgebra which is the stabilizer of the $n-(j-i)$-step partial flag in $\mathbb{C}^n$

\begin{equation}
R_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \ldots, e_j \subset e_{j+1} \subset \cdots \subset e_n).
\end{equation}

(2.12)
It is easy to see that \( \mathfrak{r}_{i,j} \) is the standard parabolic subalgebra generated by the Borel subalgebra \( \mathfrak{b}_+ \) and the negative simple root spaces \( \mathfrak{g}_{-\alpha_i}, \mathfrak{g}_{-\alpha_{i+1}}, \ldots, \mathfrak{g}_{-\alpha_{j-1}} \). We note that \( \mathfrak{r}_{i,j} \) has Levi decomposition \( \mathfrak{r}_{i,j} = \mathfrak{m} + \mathfrak{n} \), with \( \mathfrak{m} \) consisting of block diagonal matrices of the form

\[
(2.13) \quad \mathfrak{m} = \mathfrak{gl}(1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(j + 1 - i, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1, \mathbb{C}).
\]

Let \( R_{i,j} \) be the parabolic subgroup of \( G \) with Lie algebra \( \mathfrak{r}_{i,j} \). Let \( \mathfrak{p}_{i,j} := \text{Ad}(v_{i,j})\mathfrak{r}_{i,j} \in G/R_{i,j} \), where \( v_{i,j} \) is defined in (2.11). Then \( \mathfrak{p}_{i,j} \) is the stabilizer of the partial flag

\[
(2.14) \quad \mathcal{P}_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \ldots, e_{j-1}, e_j \subset e_j \subset \cdots \subset e_{n-1}),
\]

and \( \mathfrak{p}_{i,j} \in G/R_{i,j} \) is a \( \theta \)-stable parabolic subalgebra of \( \mathfrak{g} \). Indeed, recall that \( \theta \) is given by conjugation by the diagonal matrix \( d = \text{diag}[1, \ldots, 1, -1] \). Clearly \( d(\mathcal{P}_{i,j}) = \mathcal{P}_{i,j} \), whence \( \mathfrak{p}_{i,j} \) is \( \theta \)-stable. Moreover, the parabolic subalgebra \( \mathfrak{p}_{i,j} \) has Levi decomposition \( \mathfrak{p}_{i,j} = \mathfrak{l} \oplus \mathfrak{u} \) where both \( \mathfrak{l} \) and \( \mathfrak{u} \) are \( \theta \)-stable and \( \mathfrak{l} \) is isomorphic to the Levi subalgebra in Equation (2.13). Since \( \mathfrak{p}_{i,j} \) is \( \theta \)-stable, it follows from Theorem 2 of [BH00] that the \( K \)-orbit \( Q_{\mathfrak{p}_{i,j}} = K \cdot \mathfrak{p}_{i,j} \) is closed in \( G/R_{i,j} \).

For a parabolic subgroup \( P \subset G \) with Lie algebra \( \mathfrak{p} \subset \mathfrak{g} \), consider the partial Grothendieck resolution \( \mathfrak{g}^P = \{ (x, \mathfrak{r}) \in \mathfrak{g} \times G/P \mid x \in \mathfrak{r} \} \), as well as the morphisms \( \mu : \mathfrak{g}^P \to \mathfrak{g}, \mu(x, \mathfrak{r}) = x \), and \( \pi : \mathfrak{g}^P \to G/P, \pi(x, \mathfrak{r}) = \mathfrak{r} \). Then \( \pi \) is a smooth morphism of relative dimension \( \dim \mathfrak{p} \) (for \( G/B \), see Section 3.1 of [CG97] and Proposition III.10.4 of [Har77], and the general case of \( G/P \) follows by the same argument). For \( \mathfrak{r} \in G/P \), let \( Q_\mathfrak{r} = K \cdot \mathfrak{r} \subset G/P \). Then \( \pi^{-1}(Q_\mathfrak{r}) \) has dimension \( \dim(Q_\mathfrak{r}) + \dim(\mathfrak{r}) \). It is well-known that \( \mu \) is proper and its restriction to \( \pi^{-1}(Q_\mathfrak{r}) \) generically has finite fibers (Proposition 3.1.34 and Example 3.1.35 of [CG97] for the case of \( G/B \), and again the general case has a similar proof).

**Notation 2.7.** For a parabolic subalgebra \( \mathfrak{r} \) with \( K \)-orbit \( Q_\mathfrak{r} \in G/P \), we consider the irreducible subset

\[
(2.15) \quad Y_\mathfrak{r} := \mu(\pi^{-1}(Q_\mathfrak{r})) = \text{Ad}(K)\mathfrak{r}.
\]

To emphasize the orbit \( Q_\mathfrak{r} \), we will also denote this set as

\[
(2.16) \quad Y_{Q_\mathfrak{r}} := Y_\mathfrak{r}.
\]

It follows from generic finiteness of \( \mu \) that \( Y_{Q_\mathfrak{r}} \) contains an open subset of dimension

\[
(2.17) \quad \dim(Y_{Q_\mathfrak{r}}) := \dim \pi^{-1}(Q_\mathfrak{r}) = \dim \mathfrak{r} + \dim(Q_\mathfrak{r}) = \dim \mathfrak{r} + \dim(\mathfrak{f}/\mathfrak{f} \cap \mathfrak{r}),
\]

where \( \mathfrak{f} = \text{Lie}(K) = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \).

**Remark 2.8.** Since \( \mu \) is proper, when \( Q_\mathfrak{r} = K \cdot \mathfrak{r} \) is closed in \( G/P \), then \( Y_{Q_\mathfrak{r}} \) is closed.

**Remark 2.9.** Note that

\[
\mathfrak{g} = \bigcup_{Q \subset G/P} Y_Q,
\]

is a partition of \( \mathfrak{g} \), where the union is taken over the finitely many \( K \)-orbits in \( G/P \).
Lemma 2.10. Let $Q \subset G/P$ be a $K$-orbit. Then

$$Y_Q = \bigcup_{Q' \subset Q} Y_{Q'}.$$  \hspace{3cm} (2.18)

Proof. Since $\pi$ is a smooth morphism, it is flat by Theorem III.10.2 of \textbf{Har77}. Thus, by Theorem VIII.4.1 of \textbf{Gro03}, $\pi^{-1}(Q) = \pi^{-1}(\overline{Q})$. The result follows since $\mu$ is proper.

Q.E.D.

2.3. Comparison of $K \cdot b_{i,j}$ and $K \cdot p_{i,j}$. We prove a technical result that will be needed to prove our main theorem.

Remark 2.11. Note that $b_{i,j} \subset p_{i,j}$ and when $i = j$, $p_{i,i}$ is the Borel subalgebra $b_{i,i}$. To check the first assertion, note that $b_{i,j} \subset r_{i,j}$ so that $b_{i,j} = \text{Ad}(v_{i,j})b_{i,j} \subset \text{Ad}(v_{i,j})r_{i,j} = p_{i,j}$. The second assertion is verified by noting that when $i = j$, the partial flag $P_{i,j}$ is the full flag $F_{i,i}$.

Proposition 2.12. Consider the $K$-orbits $Q_{i,j} = K \cdot b_{i,j} \subset B$ and $Q_{p_{i,j}} = K \cdot p_{i,j} \subset G/P_{i,j}$, with $1 \leq i \leq j \leq n$. Then $\dim(Y_{b_{i,j}}) = \dim(Y_{p_{i,j}})$ and $\overline{Y_{b_{i,j}}} = Y_{p_{i,j}}$.

Proof. By definitions and Remark 2.11, $Y_{b_{i,j}}$ is a constructible subset of $Y_{p_{i,j}}$. Since $Y_{p_{i,j}}$ is closed by Remark 2.8 and irreducible by construction, it suffices to show that $\dim(Y_{b_{i,j}}) = \dim(Y_{p_{i,j}})$.

We compute the dimension of $Y_{b_{i,j}}$ using Equation (2.17). Since $l(Q_{i,j}) = j - i$, it follows that $\dim Q_{i,j} = \dim B_{n-1} + j - i$. Since $\dim(B_{n-1}) = \binom{n-1}{2}$, Equation (2.17) then implies:

$$\dim Y_{b_{i,j}} = \dim b_{i,j} + \dim B_{n-1} + l(Q_{i,j}) = \binom{n+1}{2} + \binom{n-1}{2} + l(Q_{i,j})$$
$$= n^2 - n + 1 + j - i. \hspace{3cm} (2.19)$$

We now compute the dimension of $Y_{p_{i,j}}$. By Equation (2.17), it follows that

$$\dim Y_{p_{i,j}} = \dim p_{i,j} + \dim \mathfrak{k} - \dim(\mathfrak{k} \cap p_{i,j}). \hspace{3cm} (2.20)$$

Since both $\mathfrak{l}$ and $\mathfrak{u}$ are $\theta$-stable, it follows that $\dim \mathfrak{k} \cap p_{i,j} = \dim \mathfrak{k} \cap \mathfrak{l} + \dim \mathfrak{k} \cap \mathfrak{u}$. To compute these dimensions, it is convenient to use the following explicit matrix description of the parabolic subalgebra $p_{i,j}$, which follows from Equation (2.14).
Thus, we see that \( u \) (see Equation (2.13)). Thus, Equation (2.20) implies that

\[
\dim \mathfrak{k} = \dim Y_{x,\pi} = \dim \mathfrak{l} + (j - i + 1)^2 + n - j + i - 1 + \dim \mathfrak{u}.
\]

(see Equation (2.13)). Thus, Equation (2.20) implies that

\[
\dim Y_{x,\pi} = \dim \mathfrak{l} + (j - i + 1)^2 + n - j + i - 1 - (j - i)^2 - 1 = n^2 - n + 1 + j - i,
\]

which agrees with (2.19), and hence completes the proof.

Q.E.D.

Remark 2.13. It follows from Equation (2.21) that \((\mathfrak{p}_{i,j})_{n-1} := \pi_{n-1,\mathfrak{n}}(\mathfrak{p}_{i,j})\) is a parabolic subalgebra, where \(\pi_{n,n-1} : \mathfrak{g} \to \mathfrak{gl}(n-1, \mathbb{C})\) is the projection \(x \mapsto x_{n-1}\). Further, with \(l = j - i\), \((\mathfrak{p}_{i,j})_{n-1}\) has Levi decomposition \((\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}\) with \(\mathfrak{l}_{n-1} = \mathfrak{gl}(1, \mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l, \mathbb{C})\).

3. The varieties \(\mathfrak{g}(l)\)

In this section, we prove our main results.

For \(x \in \mathfrak{g}\), let \(\sigma(x) = \{\lambda_1, \ldots, \lambda_n\}\) denote its eigenvalues, where an eigenvalue \(\lambda\) is listed \(k\) times if it appears with multiplicity \(k\). Similarly, let \(\sigma(x_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}\) be the eigenvalues of \(x_{n-1} \in \mathfrak{gl}(n - 1, \mathbb{C})\), again listed with multiplicity. For \(i = n - 1, \ldots, 1\), let \(\mathfrak{h}_i \subset \mathfrak{g}_i := \mathfrak{gl}(i, \mathbb{C})\) be the standard Cartan subalgebra of diagonal matrices. We denote
elements of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ by $(x, y)$, with $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ the diagonal coordinates of $x$ and $y$. For $l = 0, \ldots, n-1$, we define

$$(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l) = \{(x, y) : \exists 1 \leq i_1 < \cdots < i_l \leq n-1 \text{ with } x_{i_j} = y_{k_j} \text{ for some } 1 \leq k_1, \ldots, k_l \leq n \text{ with } k_j \neq k_m\}.$$ 

Thus, \((\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)\) consists of elements of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ with at least $l$ coincidences in the spectrum of $x$ and $y$ counting repetitions. Note that $(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)$ is a closed subvariety of $\mathfrak{h}_{n-1} \times \mathfrak{h}_n$ and is equidimensional of codimension $l$.

Let $W_i = W_i(\mathfrak{g}_i, \mathfrak{h}_i)$ be the Weyl group of $\mathfrak{g}_i$. Then $W_{n-1} \times W_n$ acts on $(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)$. Consider the finite morphism $p : \mathfrak{h}_{n-1} \times \mathfrak{h}_n \to (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n)$. Let $F_i : \mathfrak{h}_i/W_i \to \mathbb{C}^l$ be the Chevalley isomorphism, and let

$$V^{n-1,n} := \mathbb{C}^{n-1} \times \mathbb{C}^n,$$ 

so that $F_{n-1} \times F_n : (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n) \to V^{n-1,n}$ is an isomorphism. The following varieties play a major role in our study of eigenvalue coincidences.

**Definition-Notation 3.1.** For $l = 0, \ldots, n-1$, we let

$$(3.1) \quad V^{n-1,n}(\geq l) := (F_{n-1} \times F_n)((\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq l)/(W_{n-1} \times W_n)), $$

$$(3.2) \quad V^{n-1,n}(l) := V^{n-1,n}(\geq l) \setminus V^{n-1,n}(\geq l+1).$$

For convenience, we let $V^{n-1,n}(n) = \emptyset$.

**Lemma 3.2.** The set $V^{n-1,n}(\geq l)$ is an irreducible closed subvariety of $V^{n-1,n}$ of dimension $2n - 1 - l$. Further, $V^{n-1,n}(l)$ is open and dense in $V^{n-1,n}(\geq l)$.

**Proof.** Indeed, the set $Y := \{(x, y) \in \mathfrak{h}_{n-1} \times \mathfrak{h}_n : x_i = y_i \text{ for } i = 1, \ldots, l\}$ is closed and irreducible of dimension $2n - 1 - l$. The first assertion follows since $(F_{n-1} \times F_n) \circ p$ is a finite morphism and $(F_{n-1} \times F_n) \circ p(Y) = V^{n-1,n}(\geq l)$. The last assertion of the lemma now follows from Equation (3.2).

Q.E.D.

**Definition 3.3.** We let

$$\mathfrak{g}(\geq l) := \Phi_n^{-1}(V^{n-1,n}(\geq l)).$$

**Remark 3.4.** Recall that the quotient morphism $p_i : \mathfrak{g}_i \to \mathfrak{g}_i/GL(i, \mathbb{C}) \cong \mathfrak{h}_i/W_i$ associates to $y \in \mathfrak{g}_i$ its spectrum $\sigma(y)$, and $(F_{n-1} \times F_n) \circ (p_{n-1} \times p_n) = \Phi_n$. It follows that $\mathfrak{g}(\geq l)$ consists of elements of $x$ with at least $l$ coincidences in the spectrum of $x$ and $x_{n-1}$, counted with multiplicity.

It is routine to check that

$$(3.3) \quad \mathfrak{g}(l) := \mathfrak{g}(\geq l) \setminus \mathfrak{g}(\geq l+1) = \Phi_n^{-1}(V^{n-1,n}(l))$$

consists of elements of $\mathfrak{g}$ with exactly $l$ coincidences in the spectrum of $x$ and $x_{n-1}$, counted with multiplicity.
Proposition 3.5.  
(1) The variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $n^2 - l$.  
(2) $\mathfrak{g}(\geq l) = \mathfrak{g}(l) = \bigcup_{k \geq l} \mathfrak{g}(k)$.

Proof. By Proposition 2.3, the morphism $\Phi_n$ is flat. By Proposition III.9.5 and Corollary III.9.6 of [Har77], the variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $\dim(V^{n-1,n}(\geq l)) + (n - 1)^2$, which gives the first assertion by Lemma 3.2. For the second assertion, by the flatness of $\Phi_n$, Theorem VIII.4.1 of [Gro03], and Lemma 3.2.

The remaining equality follows from definitions.

Q.E.D.

We now relate the partitions $\mathfrak{g} = \bigcup \mathfrak{g}(l)$ and $\mathfrak{g} = \bigcup_{Q \subseteq B} Y_Q$ (see Remark 2.9).

Theorem 3.6.  
(1) Consider the closed subvarieties $Y_{\mathfrak{p}_{i,j}}$ for $1 \leq i \leq j \leq n$, and let $l = j - i$. Then $Y_{\mathfrak{p}_{i,j}} \subseteq \mathfrak{g}(\geq n - 1 - l)$.

(2) In particular, if $Q \subseteq B$ is a $K$-orbit with $l(Q) = l$, then $Y_Q \subseteq \mathfrak{g}(\geq n - 1 - l)$.

Proof. The second statement of the theorem follows from the first statement using Remark 2.6 and Proposition 2.12.

We recall that $\Phi_n(x) = (\chi_{n-1}(x_{n-1}), \chi_n(x))$ where $\chi_i : \mathfrak{gl}(i, \mathbb{C}) \rightarrow \mathbb{C}^i$ is the adjoint quotient for $i = n - 1, n$. For $x \in \mathfrak{p}_{i,j}$, let $x_{l}$ be the projection of $x$ onto $l$ off of $u$. It is well-known that $\chi_n(x) = \chi_n(x_1)$. Using the identification $l \cong \mathfrak{gl}(1, \mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l+1, \mathbb{C})$, we decompose $x_1$ as $x_1 = x_{\mathfrak{gl}(1)^n-1-l} + x_{\mathfrak{gl}(l+1)}$, where $x_{\mathfrak{gl}(1)^n-1-l} \in \mathfrak{gl}(1, \mathbb{C})^{n-1-l}$ and $x_{\mathfrak{gl}(l+1)} \in \mathfrak{gl}(l+1, \mathbb{C})$. It follows that the coordinates of $x_{\mathfrak{gl}(1)^n-1-l}$ are in the spectrum of $x$ (see (2.71)).

Recall the projection $\pi_{n,n-1} : \mathfrak{g} \rightarrow \mathfrak{g}_{n-1}$, $\pi_{n,n-1}(x) = x_{n-1}$. Recall the Levi decomposition $(\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}$ of the parabolic subalgebra $(\mathfrak{p}_{i,j})_{n-1}$ of $\mathfrak{gl}(n - 1, \mathbb{C})$ from Remark 2.13, and recall that $\mathfrak{l}_{n-1} = \mathfrak{gl}(1, \mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l, \mathbb{C})$. Thus, $\chi_{n-1}(x_{n-1}) = \chi_{n-1}(x_{n-1})$. We use the decomposition $(x_{n-1})_{\mathfrak{l}_{n-1}} = x_{\mathfrak{gl}(1)^n-1-l} + \pi_{l+1,l}(x_{\mathfrak{gl}(l+1)})$, where $\pi_{l+1,l} : \mathfrak{gl}(l+1, \mathbb{C}) \rightarrow \mathfrak{gl}(l, \mathbb{C})$ is the usual projection. It now follows easily from Remark 3.4 that $\Phi_n(x) \in V^{n-1,n}(\geq n - 1 - l)$, since the coordinates of $x_{\mathfrak{gl}(1)^n-1-l}$ are eigenvalues both for $x$ and $x_{n-1}$.

Q.E.D.

We now recall and prove our main theorem.
**Theorem 3.7.** Consider the locally closed subvariety \( g(n - 1 - l) \) for \( l = 0, \ldots, n - 1 \). Then the decomposition

\[
(3.5) \quad g(n - 1 - l) = \bigcup_{l(Q) = l} Y_Q \cap g(n - 1 - l),
\]

is the irreducible component decomposition of the variety \( g(n - 1 - l) \), where the union is taken over all \( K \)-orbits \( Q \) of length \( l \) in \( \mathcal{B} \). (cf. Theorem (1.1)).

In fact, for \( 1 \leq i \leq j \leq n \) with \( j - i = l \), we have

\[
Y_{b_{i,j}} \cap g(n - 1 - l) = Y_{p_{i,j}} \cap g(n - 1 - l),
\]

so that

\[
(3.6) \quad g(n - 1 - l) = \bigcup_{j-i=l} Y_{p_{i,j}} \cap g(n - 1 - l).
\]

**Proof.** We first claim that if \( l(Q) = l \), then \( Y_Q \cap g(n - 1 - l) \) is non-empty. By Theorem 3.6 \( Y_Q \subset g(\geq n - 1 - l) \). Thus, if \( Y_Q \cap g(n - 1 - l) \) were empty, then \( Y_Q \subset g(\geq n - l) \). Hence, by part (1) of Proposition 3.5, \( \dim(Y_Q) \leq n^2 - n + l \). By Equation (2.19), \( \dim(Y_Q) = n^2 - n + l + 1 \). This contradiction verifies the claim.

It follows from Equation (3.3) that \( g(n - 1 - l) \) is open in \( g(\geq n - 1 - l) \). Thus, \( Y_Q \cap g(n - 1 - l) \) is a non-empty Zariski open subset of \( Y_Q \), which is irreducible since \( Y_Q \) is irreducible.

Now we claim that

\[
(3.7) \quad Y_Q \cap g(n - 1 - l) = \overline{Y_Q} \cap g(n - 1 - l),
\]

so that \( Y_Q \cap g(n - 1 - l) \) is closed in \( g(n - 1 - l) \). By Lemma (2.10) \( \overline{Y_Q} = \bigcup_{Q' \subset Q} Y_{Q'} \). Hence, if (3.7) were not an equality, there would be \( Q' \) with \( l(Q') < l(Q) \) and \( Y_{Q'} \cap g(n - 1 - l) \) nonempty. This contradicts Theorem 3.6 which asserts that \( Y_{Q'} \subset g(\geq n - l) \), and hence verifies the claim. It follows that \( Y_Q \cap g(n - 1 - l) \) is an irreducible, closed subvariety of \( g(n - 1 - l) \) of dimension \( \dim Y_Q = \dim g(n - 1 - l) \). Thus, \( Y_Q \cap g(n - 1 - l) \) is an irreducible component of \( g(n - 1 - l) \).

Since \( l(Q) = l \), Remark (2.6) implies that \( Q = Q_{i,j} \) for some \( i \leq j \) with \( j - i = l \). Then by Proposition (2.12) and Equation (3.7),

\[
(3.8) \quad Y_{b_{i,j}} \cap g(n - 1 - l) = Y_{p_{i,j}} \cap g(n - 1 - l).
\]

Let \( Z \) be an irreducible component of \( g(n - 1 - l) \). The proof will be complete once we show that \( Z = Y_{p_{i,j}} \cap g(n - 1 - l) \) for some \( i, j \) with \( j - i = l \). To do this, consider the nonempty open set

\[
U := \{ x \in g : x_{n-1} \text{ is regular semisimple} \}.
\]

Let \( \tilde{U}(n - 1 - l) := g(n - 1 - l) \cap U \).
Since $\Phi_n : g \to V^{n-1,n}$ is surjective (by Remark 2.1), it follows that $\widetilde{U}(n - 1 - l)$ is a nonempty Zariski open set of $g(n - 1 - l)$. By part (2) of Proposition 2.3 and Exercise III.9.1 of Har77, $\Phi_n(U) \subset V^{n-1,n}$ is open. Thus, $V^{n-1,n}(n - 1 - l) \setminus \Phi_n(U)$ is a proper, closed subvariety of $V^{n-1,n}(n - 1 - l)$ and therefore has positive codimension by Lemma 3.2. It follows by part (2) of Proposition 2.3 and Corollary III.9.6 of Har77 that $g(n - 1 - l) \setminus \widetilde{U}(n - 1 - l) = \Phi_n^{-1}(V^{n-1,n}(n - 1 - l) \setminus \Phi_n(U))$ is a proper, closed subvariety of $g(n - 1 - l)$ of positive codimension. Since $g(n - 1 - l)$ is equidimensional, it follows that $Z \cap \widetilde{U}(n - 1 - l)$ is nonempty. Thus, it suffices to show that

$$\widetilde{U}(n - 1 - l) \subset \bigcup_{j - i = l} Y_{p_{i,j}} \cap g(n - 1 - l). \quad (3.9)$$

To prove Equation (3.9), we consider the following subvariety of $\widetilde{U}(n - 1 - l)$:

$$\Xi = \{ x \in \widetilde{U}(n-1-l) : x_{n-1} = \text{diag}[h_1, \ldots, h_{n-1}], \text{ and } \sigma(x_{n-1}) \cap \sigma(x) = \{h_1, \ldots, h_{n-1-l}\}\} \quad (3.10)$$

It is easy to check that any element of $\widetilde{U}(n - 1 - l)$ is $K$-conjugate to an element in $\Xi$. By a linear algebra calculation from Proposition 5.9 of Col11, elements of $\Xi$ are matrices of the form

$$\begin{bmatrix}
h_1 & 0 & \cdots & 0 & y_1 \\
0 & h_2 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \cdots & \cdots & h_{n-1} & y_{n-1} \\
z_1 & \cdots & \cdots & z_{n-1} & w
\end{bmatrix}, \quad (3.11)$$

with $h_i \neq h_j$ for $i \neq j$ and satisfying the equations:

$$z_i y_i = 0 \text{ for } 1 \leq i \leq n - 1 - l$$

$$z_i y_i \in \mathbb{C}^\times \text{ for } n - l \leq i \leq n - 1. \quad (3.12)$$

Since the varieties $Y_{p_{i,j}} \cap g(n - 1 - l)$ are $K$-stable, it suffices to prove

$$\Xi \subset \bigcup_{j - i = l} Y_{p_{i,j}} \cap g(n - 1 - l). \quad (3.13)$$

To prove (3.13), we need to understand the irreducible components of $\Xi$. For $i = 1, \ldots, n - 1 - l$, we define an index $j_i$ which takes on two values $j_i = U$ ($U$ for upper) or $j_i = L$ ($L$ for lower). Consider the subvariety $\Xi_{j_1, \ldots, j_{n-1-l}} \subset \Xi$ defined by:

$$\Xi_{j_1, \ldots, j_{n-1-l}} := \{ x \in \Xi : z_i = 0 \text{ if } j_i = U, y_i = 0 \text{ if } j_i = L \}. \quad (3.14)$$
Then
\begin{equation}
(3.15) \quad \Xi = \bigcup_{j_i=U,L} \Xi_{j_1,\ldots,j_{n-1-l}}.
\end{equation}
is the irreducible component decomposition of \(\Xi\).

We now consider the irreducible variety \(\Xi_{j_1,\ldots,j_{n-1-l}}\). Suppose that for the subsequence \(1 \leq i_1 < \cdots < i_{k-1} \leq n-1-l\) we have \(j_{i_1} = j_{i_2} = \cdots = j_{i_{k-1}} = U\) and that for the complementary subsequence \(i_k < \cdots < i_{n-1-l}\) we have \(j_{i_k} = j_{i_{k+1}} = \cdots = j_{i_{n-1-l}} = L\). Then a simple computation with flags shows that elements of the variety \(\Xi_{j_1,\ldots,j_{n-1-l}}\) stabilize the \(n-l\)-step partial flag in \(\mathbb{C}^n\)
\begin{equation}
(3.16) \quad e_{i_1} \subset e_{i_2} \subset \cdots \subset e_{i_{k-1}} \subset e_{n-l}, \ldots, e_{n-1}, e_n \subset e_{i_k} \subset e_{i_{k+1}} \subset \cdots \subset e_{i_{n-1-l}}.
\end{equation}

(If \(l = 0\) the partial flag in \((3.16)\) is a full flag with \(e_n\) in the \(k\)-th position.) It is easy to see that there is an element of \(K\) that maps the partial flag in Equation \((3.16)\) to the partial flag \(P_{k,k+l}\) in Equation \((2.14)\):
\begin{equation}
(3.17) \quad P_{k,k+l} = (e_1 \subset e_2 \subset \cdots \subset e_{k-1} \subset e_{k+1}, \ldots, e_{k+l-1}, e_n \subset e_{k+l} \subset \cdots \subset e_{n-1}).
\end{equation}

(If \(l = 0\) the partial flag \(P_{k,k+l}\) is the full flag \(F_{k,k}\) (see Equation \((2.8)\)).) Thus, \(\Xi_{j_1,\ldots,j_{n-1-l}} \subset Y_{P_{k,k+l}} \cap g(n-1-l)\). Equation \((3.15)\) then implies that \(\Xi \subset \bigcup_{j-i=l} Y_{P_{i,j}} \cap g(n-1-l)\).

Q.E.D.

Using Theorem 3.7 we can obtain the irreducible component decomposition of the variety \(g(\geq n-1-l)\) for any \(l = 0, \ldots, n-1\).

**Corollary 3.8.** The irreducible component decomposition of the variety \(g(\geq n-1-l)\) is
\begin{equation}
(3.18) \quad g(\geq n-1-l) = \bigcup_{j-i=l} Y_{P_{i,j}} = \bigcup_{l(Q)=l} Y_{Q}.
\end{equation}

**Proof.** Taking Zariski closures in Equation \((3.16)\), we obtain
\begin{equation}
(3.19) \quad \overline{g(n-1-l)} = \bigcup_{j-i=l} Y_{P_{i,j}} \cap g(n-1-l)
\end{equation}
is the irreducible component decomposition of the variety \(\overline{g(n-1-l)}\). By Proposition 3.3 \(\overline{g(n-1-l)} = g(\geq n-1-l)\), and by Theorem 3.6 \(Y_{P_{i,j}} \subset g(\geq n-1-l)\). Hence \(Y_{P_{i,j}} \cap g(n-1-l)\) is Zariski open in the irreducible variety \(Y_{P_{i,j}}\), and is nonempty by Theorem 3.7. Therefore \(Y_{P_{i,j}} \cap g(n-1-l) = Y_{P_{i,j}}\). Equation \((3.18)\) now follows from Equation \((3.19)\) and Proposition 2.12.

Q.E.D.
Theorem 3.7 says something of particular interest to linear algebraists in the case where \( l = 0 \). It states that the variety \( \mathfrak{g}(n-1) \) consisting of elements \( x \in \mathfrak{g} \) where the number of coincidences in the spectrum between \( x_{n-1} \) and \( x \) is maximal can be described in terms of closed \( K \)-orbits on \( \mathfrak{b} \), which are the \( K \)-orbits \( Q \) with \( l(Q) = 0 \). It thus connects the most degenerate case of spectral coincidences to the simplest \( K \)-orbits on \( \mathfrak{b} \). More precisely, we have:

**Corollary 3.9.** The irreducible component decomposition of the variety \( \mathfrak{g}(n-1) \) is

\[
\mathfrak{g}(n-1) = \bigcup_{l(Q)=0} Y_Q.
\]

Using Corollary 3.9 and Theorem 2.2, we obtain a precise description of the irreducible components of the variety \( SN_n \) introduced in Equation (2.4).

**Proposition 3.10.** Let \( \mathfrak{b}_{i,i} \) be the Borel subalgebra of \( \mathfrak{g} \) which stabilizes the flag \( F_{i,i} \) in Equation (2.3) and let \( \mathfrak{n}_{i,i} = [\mathfrak{b}_{i,i}, \mathfrak{b}_{i,i}] \). The irreducible component decomposition of \( SN_n \) is given by:

\[
SN_n = \bigcup_{i=1}^{n} \operatorname{Ad}(K)\mathfrak{n}_{i,i},
\]

where \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \subset \mathfrak{g} \) denotes the \( K \)-saturation of \( \mathfrak{n}_{i,i} \) in \( \mathfrak{g} \).

**Proof.** We first show that \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) is an irreducible component of \( SN_n \) for \( i = 1, \ldots, n \). A simple computation using the flag \( F_{i,i} \) in Equation (2.3) shows that \( \mathfrak{n}_{i,i} \subset SN_n \). Since \( SN_n \) is \( K \)-stable, it follows that \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \subset SN_n \).

Recall the Grothendieck resolution \( \tilde{\mathfrak{g}} = \{(x, b) : x \in b\} \subset \mathfrak{g} \times \mathfrak{b} \) and the morphisms \( \pi : \tilde{\mathfrak{g}} \to \mathfrak{b} \), \( \pi(x, b) = b \) and \( \mu : \tilde{\mathfrak{g}} \to \mathfrak{g} \), \( \mu(x, b) = x \). Let \( Q_{i,i} = K \cdot \mathfrak{b}_{i,i} \subset \mathfrak{b} \) be the \( K \)-orbit through \( b_{i,i} \). Corollary 3.1.33 of [CG97] gives a \( G \)-equivariant isomorphism \( \tilde{\mathfrak{g}} \cong G \times_{\mathfrak{b}_{i,i}} \mathfrak{b}_{i,i} \). Under this isomorphism \( \pi^{-1}(Q_{i,i}) \) is identified with the closed subvariety \( K \times_{K\cap\mathfrak{b}_{i,i}} \mathfrak{n}_{i,i} \subset G \times_{\mathfrak{b}_{i,i}} \mathfrak{b}_{i,i} \). The closed subvariety \( K \times_{K\cap\mathfrak{b}_{i,i}} \mathfrak{n}_{i,i} \subset K \times_{K\cap\mathfrak{b}_{i,i}} \mathfrak{b}_{i,i} \) maps surjectively under \( \mu \) to \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \). Since \( \mu \) is proper, \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) is closed and irreducible. We also note that the restriction of \( \mu \) to \( K \times_{K\cap\mathfrak{b}_{i,i}} \mathfrak{n}_{i,i} \) generically has finite fibers (Proposition 3.2.14 of [CG97]). Thus, the same reasoning that we used in Equation (2.19) shows that

\[
\dim \operatorname{Ad}(K)\mathfrak{n}_{i,i} = \dim K \times_{K\cap\mathfrak{b}_{i,i}} \mathfrak{n}_{i,i} = \dim (Y_{Q_{i,i}}) - \operatorname{rk}(\mathfrak{g}) = d_n,
\]

where \( \operatorname{rk}(\mathfrak{g}) \) denotes the rank of \( \mathfrak{g} \). Thus, by Theorem 2.2 \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) is an irreducible component of \( SN_n \).

We now show that every irreducible component of \( SN_n \) is of the form \( \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) for some \( i = 1, \ldots, n \). It follows from definitions that \( SN_n \subset \mathfrak{g}(n-1) \cap \mathcal{N} \), where \( \mathcal{N} \subset \mathfrak{g} \) is the nilpotent cone in \( \mathfrak{g} \). Thus, if \( \mathfrak{X} \) is an irreducible component of \( SN_n \), then \( \mathfrak{X} \subset \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) by Corollary 3.9. But then \( \mathfrak{X} = \operatorname{Ad}(K)\mathfrak{n}_{i,i} \) by Equation (3.21) and Theorem 2.2.

Q.E.D.
We say that an element \( x \in g \) is \( n \)-strongly regular if the set
\[
dJZ_n(x) := \{ df_{i,j}(x) : i = n - 1, n; j = 1, \ldots, i \}
\]
is linearly independent in the cotangent space \( T^*_x(g) \) of \( g \) at \( x \). We view \( g_{n-1} \) as the top lefthand corner of \( g \). It follows from a well-known result of Kostant (Theorem 9 of [Kos63]) that \( x_i \in g_i \) is regular if and only if the set \( \{ df_{i,j}(x) : j = 1, \ldots, i \} \) is linearly independent. If \( x_i \in g_i \) is regular, and we identify \( T^*_x(g) = g^* \) with \( g \) using the trace form \( \langle x, y \rangle = tr(xy) \), then
\[
\text{span} \{ df_{i,j}(x) : j = 1, \ldots, i \} = zg_i(x_i),
\]
where \( zg_i(x_i) \) denotes the centralizer of \( x_i \) in \( g_i \). Thus, it follows that \( x \in g \) is \( n \)-strongly regular if and only if \( x \) satisfies the following two conditions:
\[
(1) \ x \in g \text{ and } x_{n-1} \in g_{n-1} \text{ are regular; and}
\]
\[
(2) \ zg_{n-1}(x_{n-1}) \cap zg(x) = 0.
\]

**Remark 3.11.** We claim that the ideal \( I_n \) is radical if and only if \( n \leq 2 \). The assertion is clear for \( n = 1 \), and we assume \( n \geq 2 \) in the sequel. Indeed, by Theorem 18.15(a) of [Eis95], the ideal \( I_n \) is radical if and only if the set \( dJZ_n \) is linearly independent on a dense open set of each irreducible component of \( SN_n = V(I_n) \). It follows that \( I_n \) is radical if and only if each irreducible component of \( SN_n \) contains \( n \)-strongly regular elements. Let \( n_+ = [b_+, b_+] \) and \( n_- = [b_-, b_-] \) be the strictly upper and lower triangular matrices, respectively. By Proposition 3.10 above, \( SN_n \) has exactly \( n \) irreducible components. It follows from the discussion after Equation (2.8) that two of them are \( K \cdot n_+ \) and \( K \cdot n_- \). By Proposition 3.10 of [CE12], the only irreducible components of \( SN_n \) which contain \( n \)-strongly regular elements are \( K \cdot n_+ \) and \( K \cdot n_- \). The claim now follows. See Remark 1.1 of [Ovs03] for a related observation, which follows also from the analysis proving our claim.

**References**

[BH00] Michel Brion and Aloysius G. Helminck, *On orbit closures of symmetric subgroups in flag varieties*, Canad. J. Math. 52 (2000), no. 2, 265–292.

[Bri87] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. 63 (1987), no. 2, 189–208.

[CE] Mark Colarusso and Sam Evens, *The Gelfand-Zeitlin integrable system and K-orbits on the flag variety*, to appear in: “Symmetry: Representation Theory and its Applications: In Honor of Nolan R. Wallach,” Progr. Math. Birkhauser, Boston.

[CE12] Mark Colarusso and Sam Evens, *K-orbits on the flag variety and strongly regular nilpotent matrices*, Selecta Math. (N.S.) 18 (2012), no. 1, 159–177.

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[Col11] Mark Colarusso, *The orbit structure of the Gelfand-Zeitlin group on n \times n matrices*, Pacific J. Math. 250 (2011), no. 1, 109–138.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.
[FO05] Vyacheslav Futorny and Serge Ovsienko, *Kostant’s theorem for special filtered algebras*, Bull. London Math. Soc. **37** (2005), no. 2, 187–199.

[Gro03] Alexander Grothendieck, *Revêtements Étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de Géométrie Algébrique du Bois Marie, 1960-1961, Augmenté de deux exposés de Michèle Raynaud. [With two exposés by Michèle Raynaud].

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Has04] Takashi Hashimoto, *Bn−1-orbits on the flag variety GLn/Bn*, Geom. Dedicata **105** (2004), 13–27.

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21.

[Kos63] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[KW06] Bertram Kostant and Nolan Wallach, *Gelfand-Zeitlin theory from the perspective of classical mechanics. I*, Studies in Lie theory, Progr. Math., vol. 243, Birkhäuser Boston, Boston, MA, 2006, pp. 319–364.

[Mat79] Toshihiko Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), no. 2, 331–357.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid.

[MÖ90] Toshihiko Matsuki and Toshio Oshima, *Embeddings of discrete series into principal series*, The orbit method in representation theory (Copenhagen, 1988), Progr. Math., vol. 82, Birkhäuser Boston, Boston, MA, 1990, pp. 147–175.

[Ovs03] Serge Ovsienko, *Strongly nilpotent matrices and Gelfand-Zetlin modules*, Linear Algebra Appl. **365** (2003), 349–367, Special issue on linear algebra methods in representation theory.

[RS90] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), no. 1-3, 389–436.

[Ser00] Jean-Pierre Serre, *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author.

[VK78] É. A. Vinberg and B. N. Kimel’fel’d, *Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 12–19, 96.

[Yam97] Atsuko Yamamoto, *Orbits in the flag variety and images of the moment map for classical groups. I*, Represent. Theory **1** (1997), 329–404 (electronic).

---

**Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI, 53201**

*E-mail address: colaruss@uwm.edu*

**Department of Mathematics, University of Notre Dame, Notre Dame, IN, 46556**

*E-mail address: sevens@nd.edu*