THE COHOMOLOGY INVARIANT FOR CLASS DIII TOPOLOGICAL INSULATORS

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ABSTRACT. This work concerns with the description of the topological phases of band insulators of class DIII by using the equivariant cohomology. The main result is the definition of a cohomology class for general systems of class DIII which generalizes the well-known $\mathbb{Z}_2$-invariant given by the Teo-Kane formula in the one-dimension case. In the two-dimensional case this cohomology invariant allows a complete description of the strong and weak phases. The relation with the KR-theory, the Noether-Fredholm index and the classification of “Real” gerbes are also discussed.

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1. INTRODUCTION

This work concerns the classification of the protected phases of topological insulators of class DIII from the point of view of the equivariant cohomology theory. In this sense this work complements the research carried out by the same authors for classes AI, AII and AIII [DG1, DG2, DG3, DG4, DG6]. The literature on topological insulators is too vast to be reported here without crucial omissions. For this reason, and for the benefit of the reader, we will refer directly to the recent review [CTSR] which provides a global overview of the subject accompanied by a rich list of references. The works [RSFL, TK, BA], among others, deserve to be mentioned for the specific focus on systems of class DIII.

In order to present the logic and the main results of this work we need first to introduce some basic definition. An involutive space is a pair \((X, \tau)\) made of a compact Hausdorff space \(X\) endowed with a continuous involution \(\tau: X \to X, \tau \circ \tau = \text{Id}_X\). The fixed point set of \((X, \tau)\) is defined as \(X^\tau := \{x \in X \mid \tau(x) = x\}\). A special example is provided by the pair \((\mathbb{S}^1, \iota)\) made by the unit circle \(\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}\), parametrized by \(k \in [0, 2\pi)\), and endowed with the involution \(\iota\) defined by \(\iota(k) := -k\). In this specific case the fixed-point set is \((\mathbb{S}^1)^\iota = \{0, \pi\}\).

Let \(\mathcal{H}\) be a finite dimensional Hilbert space, \(\mathfrak{Her}(\mathcal{H})\) the set of self-adjoint (or Hermitian) operators on \(\mathcal{H}\) and \(\mathfrak{Her}(\mathcal{H})^\times\) the subset of invertible elements of \(\mathfrak{Her}(\mathcal{H})\). Let \(T: \mathcal{H} \to \mathcal{H}\) and \(C: \mathcal{H} \to \mathcal{H}\) be two anti-unitary maps that meet the conditions

\[
T^2 = -\mathbf{1}, \quad C^2 = +\mathbf{1}, \quad TC = -CT
\]

where \(\mathbf{1}\) is the identity operator on \(\mathcal{H}\). In the theory of topological insulators it is common to refer to \(T\) as time-reversal symmetry (TSR) and to \(C\) as particle-hole symmetry (PHS). The conditions on the square say that \(T\) is an odd symmetry while \(C\) is an even symmetry. The anti-commutation condition corresponds to fixing an arbitrary relative phase between \(T\) and \(C\). With this convention, the linear operator \(\chi := TC = -CT\), called chiral symmetry, satisfies

\[
\chi^2 = \mathbf{1}, \quad T\chi = -\chi T, \quad C\chi = -\chi C .
\]

The next definition is borrowed from the standard jargon for topological insulators [AZ, SRFL, Kit, RSFL].

**Definition 1.1** (Band insulator of class DIII). A **band insulator of class DIII** over the involutive space \((X, \tau)\) is a continuous map

\[
H: X \to \mathfrak{Her}(\mathcal{H})^\times
\]

such that

\[
\begin{align*}
C \, H(x) &= -H(\tau(x)) \, C, \\
T \, H(x) &= H(\tau(x)) \, T ,
\end{align*}
\]

\(\forall \, x \in X\).
Two band insulators of class DIII \( H \) and \( H' \) over the same involutive space \((X, \tau)\) and subjected to the same system of symmetries \( T, C \) are \textit{unitarily equivalent} if and only if there exists a continuous map \( V : X \to \mathbb{U}(\mathcal{H}) \) such that

\[
\begin{align*}
H'(x) &= V(x) \, H(x) \, V(x)^*, \\
C \, V(x) &= V(\tau(x)) \, C, \quad \forall \, x \in X, \\
T \, V(x) &= V(\tau(x)) \, T,
\end{align*}
\]

The unitary equivalence is called \textit{strong} if and only if \( V(\tau(x)) = V(x) \) for all \( x \in X \). When the involutive space is explicitly given by \((S^1, \iota)\), then one refers to a \textit{one-dimensional} (1D) band insulators of class DIII.

Definition 1.1 along with the relations (1.2) imply

\[
\chi_H(x) = -H(x) \, \chi, \quad \forall \, x \in X. \tag{1.5}
\]

An immediate consequence of this relation is that the dimension of the space \( \mathcal{H} \) must be even, \( \text{i.e.} \dim(\mathcal{H}) = 2m \) for some \( m \in \mathbb{N} \).

Let us focus for the moment on the one-dimensional case. Let \( H \) and \( H' \) be two 1-dimensional band insulators of class DIII and consider the \textit{spectrally flattened} band operators \( Q_H := H|H|^{-1} \) and \( Q_{H'} := H'|H'|^{-1} \). It is known that the triple \((S^1 \times \mathcal{H}, Q_H, Q_{H'})\) defines an element of the KR-group \( \text{KR}^{-3}(S^1, \iota) \simeq \mathbb{Z}_2 \) [Ati, Kit, FM] (see also Section 3.4). Thus, there exists a \( \mathbb{Z}_2 \)-invariant which classifies (the difference of) 1-dimensional band insulators of class DIII. The aim of this work is to interpret this invariant from a cohomology point of view, and possibly to generalize it to more generic situations.

The key (not new) observation is that as a consequence of the commutation relations between \( H \) and the symmetries \( T \) and \( C \), the topological information carried by \( H \) is completely encoded into a continuous map \( q_H : S^1 \to \mathbb{U}(m) \) such that

\[
q_H(\iota(k)) = -q_H(k)^t, \quad \forall \, k \in S^1, \tag{1.6}
\]

where the superscript \( t \) denotes the \textit{transpose} of the matrix \( q_H(k) \). There are at least two immediate ways in which the matrix \( q_H \) can produce a topological invariant.

(I) The map \( q_H \) can be interpreted as the \textit{sewing matrix} of a “Quaternionic” vector bundle [Kah, Dup, DG2, DG3, DG4]. More precisely, the map \( q_H \) endows the product bundle \( S^1 \times \mathbb{C}^m \) with the structure of a “Quaternionic” vector bundle \( \mathcal{E}_{q_H} \) explicitly given by

\[
(k, v) \mapsto (\iota(k), q_H(k) \cdot v), \quad \forall \, (k, v) \in S^1 \times \mathbb{C}^m. \tag{1.7}
\]

Unfortunately, it is known that “Quaternionic” vector bundles on \((S^1, \iota)\) are automatically trivial [DG2, Theorem 1.2]. Therefore, this is not this vector bundle the good candidate to detect the topology of 1-dimensional insulators of class DIII.
(II) In the presence of a fixed point \( k = \iota(k) \), the rank \( m \) of \( \mathbb{U}(m) \) is forced to be even, i.e. \( m = 2n \). In turn, the determinant of \( q_\mathcal{H} \) provides a map \( \det[q_\mathcal{H}] : \mathbb{S}^1 \to \mathbb{U}(1) \) which is invariant in the sense that

\[
\det[q_\mathcal{H}(\iota(k))] = \det[q_\mathcal{H}(k)], \quad \forall k \in \mathbb{S}^1.
\]

Since the homotopy classes of invariant maps on \([\mathbb{S}^1, \iota]\) are classified by the group \( H^1_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{Z}) \) (cf. Appendix A), one gets the topological invariant

\[
[\det[q_\mathcal{H}]] \in H^1_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{Z})
\]

associated to \( \mathcal{H} \). However, also in this case the invariant turns out to be trivial since \( H^1_{\mathbb{Z}_2}(\mathbb{S}^1, \mathbb{Z}) = 0 \) [DG1, Lemma 5.6]. The implication of this fact will be discussed in Remark 2.14.

Thus, to detect the class of \( \text{KR}^{-3}(\mathbb{S}^1, \iota) \) associated to a one-dimensional band insulator of class DIII we need to extract from \( q_\mathcal{H} \) some other type of topological invariant finer than (I) and (II). It turns out that a closer inspection to the triviality of the “Quaternionic” vector bundle \( \mathcal{E}_{q_\iota} \) allows to build a suitable non-trivial \( \mathbb{Z}/2 \)-invariant. The principal result of this work is the construction of such a topological invariant for band insulators of class DIII over a large class of involutive spaces \((X, \tau)\).

More concretely, given a sewing matrix \( q : X \to \mathbb{U}(2n) \) on the involutive space \((X, \tau)\) subject to certain conditions (cf. Assumption 2.5), one can associate to \( q \) a homotopy invariant

\[
\nu_q \in H^1(X, \mathbb{Z})/H^1_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1))\). \tag{1.8}
\]

In formula (1.8), \( H^1(X, \mathbb{Z}) \) is the first cohomology group of \( X \) with integer coefficients while \( H^1_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \) is the first equivariant cohomology group \( X \) with coefficients in the local system \( \mathbb{Z}(1) \) relative to the fixed point set \( X^\tau \) [DG2, Section 3.1]. In the case of the involutive circle \((\mathbb{S}^1, \iota)\), one has that

\[
H^1(\mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z} \cong H^1_{\mathbb{Z}_2}(\mathbb{S}^1|\mathbb{S}^1^\iota, \mathbb{Z}(1))\),
\]

but the quotient group in (1.8) turns out to be isomorphic to \( \mathbb{Z}_2 \) (cf. Lemma 3.1), showing that \( \nu_q \) is a \( \mathbb{Z}_2 \)-invariant as expected.

It is worth to point out, however, that the invariant \( \nu_q \) of the sewing matrix \( q \) cannot serve as an absolute invariant for band insulators of class DIII. This is a consequence of the fact that the way to identify the symmetries with a fixed standard form is not unique, and can be altered by an automorphism (cf. Proposition 2.1). Accordingly, the sewing matrix associated with \( \mathcal{H} \) is not unique, and if \( q_\mathcal{H} \) and \( q_\mathcal{H}' \) are two sewing matrices associated with the same \( \mathcal{H} \) one can have \( \nu_{q_\mathcal{H}} \neq \nu_{q_\mathcal{H}'} \) in general. In this sense, \( \nu_{q_\mathcal{H}} \) does not produce an invariant of the Hamiltonian \( \mathcal{H} \). However, if one fixes a way to identify the symmetries with the standard form, then the sewing matrix can be used to detect the difference of two band insulators \( \mathcal{H} \) and \( \mathcal{H}' \) of class DIII. In this sense the sewing matrix can be used to construct a relative invariant which provides a well-defined non-trivial
homomorphism
\[ \text{KR}^{-3}(X, \tau) \supseteq \ker(\kappa) \to H^1(X, \mathbb{Z})/\mathbb{Z}_2^1(X, X^\tau, \mathbb{Z}(1)) \]  
(1.9)
(cf. Theorem 2.11 for more details). In the one-dimensional case \((S^1, \iota)\) this construction provides a bijection \(\nu: \text{KR}^{-3}(S^1, \iota) \cong \mathbb{Z}_2\) (Proposition 3.15) and the \(\mathbb{Z}_2\)-value of the class \(\nu_q\) associated with the sewing matrix \(q\) can be computed with the well known Teo-Kane formula (Theorem 3.5) or as the \(\mathbb{Z}_2\)-index of an associated Toeplitz operator (Theorem 3.12).

**Structure of the paper.** In Section 2, we start by fixing the standard form of the symmetries of a band insulator of class DIII. Then, we introduce the invariant \(\nu_q\) of a sewing matrix \(q\) in a general setup. The construction of the homomorphism \(\nu\) in equation (1.9) and its relation with the KR-theory are also provided in this section. Section 3 is devoted to the study of the one-dimensional case. In this section we provide the equivalence of \(\nu_q\) with the well-known Teo-Kane formula \([\text{CTSR}, \text{eq.} (3.70)]\), we clarify the role of \(\nu_q\) as an obstruction class and we establish a bulk-edge-type correspondence as the coincidence of \(\nu_q\) with a \(\mathbb{Z}_2\)-index for Toeplitz operators. Finally, we will show that the homomorphism (1.9) amounts to an isomorphism in the classification of “Real” gerbes over \((S^1, \iota)\). In Section 4 we investigate the application of the invariant \(\nu_q\) for two-dimensional band insulators of class DIII over the sphere (Dirac case) and the torus (Bloch case). It turns out that \(\nu_q\) provides a complete description of the weak topological invariants in the two-dimensional case. Appendix A contains a brief summary of the relationship between cohomology and homotopy in the \(\mathbb{Z}_2\)-equivariant category and Appendix B deals with the notion of strong equivalence between “Quaternionic” structures.

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2. Construction of the \(\mathbb{Z}_2\)-invariant

2.1. Underlying “Quaternionic” structure. In view of condition (1.5), and the invertibility of \(H(x)\), we argued that band insulators of class DIII can be seated only in Hilbert spaces \(\mathcal{H}\) of even dimension. Moreover, the eigenspaces of the symmetry \(\chi\), determined by the range of the eigenprojections \(\chi_{\pm} := \frac{1}{2}(1 \pm \chi)\), must have same dimension. As a consequence there exists a (non unique) unitary isomorphism \(\mathcal{H} \cong \mathbb{C}^{2^m}\) which provides the matrix representation
\[ \chi = \begin{pmatrix} +1_m & 0 \\ 0 & -1_m \end{pmatrix} \]  
(2.1)
where \(1_m\) is the \(m \times m\) identity matrix. For the next result we need to introduce the operator \(K\) which implements the standard real structure on \(\mathbb{C}^m\). This is nothing more than the complex conjugation \(Kv := \overline{v}\) for every \(v \in \mathbb{C}^m\).
Proposition 2.1. (Standard representation) Let $H, T, C$ be the elements which describe a band insulator of class DIII over the involutive space $(X, \tau)$, according to Definition 1.1. Then, there exists a (non unique) unitary isomorphism $\mathcal{H} \simeq \mathbb{C}^{2m}$ which provides the matrix representation
\[
T = \begin{pmatrix}
0 & -K \\
+K & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & -K \\
-K & 0
\end{pmatrix}.
\] (2.2)

Moreover, the Hamiltonian $H : X \to \text{Herm}(\mathcal{H})^\times$ is represented by
\[
H(x) = \begin{pmatrix}
0 & h(x)^{-1} \\
h(x) & 0
\end{pmatrix},
\] (2.3)
where the map $h : X \to \mathfrak{gl}(m)$ meets the condition
\[
h(\tau(x)) = -K h(x)^{-1} K, \quad \forall x \in X. \tag{2.4}
\]

Proof. Under the unitary transformation $\mathcal{H} \simeq \mathbb{C}^{2m}$ which provides the representation (2.1), the operator $T$ takes the form
\[
T = U_T \begin{pmatrix}
K & 0 \\
0 & K
\end{pmatrix}
\] where $U_T$ is a unitary operator. Since the anti-linear part induced by $K$ commutes with $\chi$, it follows that the second constraint in (1.2) can be satisfied if and only if $\chi U_T = -U_T \chi$. However, this forces $U_T$ to be off-diagonal with respect to the diagonal representation of $\chi$, and in turn
\[
T = \begin{pmatrix}
0 & \tilde{U}_T K \\
\tilde{U}_T^* K & 0
\end{pmatrix}
\] with $\tilde{U}_T \in \mathbb{U}(m)$ a given $m \times m$ unitary matrix. The condition $T^2 = -I$ implies that $K \tilde{U}_T K = -\tilde{U}_T$. Therefore, one gets
\[
\left( \begin{pmatrix}
I_m & 0 \\
0 & \tilde{U}_T
\end{pmatrix} \right) \left( \begin{pmatrix}
0 & \tilde{U}_T K \\
\tilde{U}_T^* K & 0
\end{pmatrix} \right) \left( \begin{pmatrix}
I_m & 0 \\
0 & \tilde{U}_T^*
\end{pmatrix} \right) = \begin{pmatrix}
0 & -K \\
+K & 0
\end{pmatrix}
\]
and at the same time
\[
\left( \begin{pmatrix}
I_m & 0 \\
0 & \tilde{U}_T
\end{pmatrix} \right) \chi \left( \begin{pmatrix}
I_m & 0 \\
0 & \tilde{U}_T^*
\end{pmatrix} \right) = \chi.
\]
Then, up to a modification of the initial unitary transform $\mathcal{H} \simeq \mathbb{C}^{2m}$, one can represent $T$ as in (2.2) without affecting the representation (2.1). The representation of $C$ in (2.2) is obtained by the formula $C = \chi T$. The anti-commutation relation between $H(x)$ and $\chi$ implies that $H(x)$ must be off-diagonal. The invertibility of $H(x)$ for every $x \in X$ justifies the representation (2.3). Condition (2.4) follows from the first of (1.3). \qed

A straightforward adaptation of the arguments used in the proof of Proposition 2.1 provides the following result that will be used several times in the following.
Lemma 2.2. Let $V \in U(2n)$ be a unitary matrix which commutes with $\chi$ and $T$ as given by the representations (2.1) and (2.2), respectively. Then there exists a unitary matrix $\phi_V \in U(m)$ such that

$$V = \begin{pmatrix} \phi_V & 0 \\ 0 & \overline{\phi_V} \end{pmatrix}, \quad \overline{\phi_V} := K\phi_V K.$$  

Given the invertible matrix $H \in \mathfrak{Herm}(\mathcal{H})^\times$ one can define the flattened operator

$$Q_H := \frac{H}{|H|}, \quad |H| := \sqrt{H^*H}. \quad (2.5)$$

Evidently $Q_H \in U(H) \cap \mathfrak{Herm}(\mathcal{H})^\times$ is a unitary and self-adjoint operator, namely an involution $Q_H^2 = 1$. If $H : X \rightarrow \mathfrak{Herm}(\mathcal{H})^\times$ is a band insulator of class DIII, then $Q_H : X \rightarrow \mathfrak{Herm}(\mathcal{H})^\times$ is also a band insulator of class DIII, and Proposition 2.1 applies. In particular one has that

$$Q_H(x) = \begin{pmatrix} 0 & q_H(x)^{-1} \\ q_H(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & q_H(x)^* \\ q_H(x) & 0 \end{pmatrix}, \quad (2.6)$$

where the map $q_H : X \rightarrow U(n)$ meets the condition

$$q_H(\tau(x)) = -Kq_H(x)^*K = -q_H(x)^\tau. \quad (2.7)$$

The map $q_H$ allows to endow the trivial product bundle $X \times \mathbb{C}^m$ with the extra structure

$$\Theta_H : X \times \mathbb{C}^m \rightarrow X \times \mathbb{C}^m \text{ defined by } \Theta_H : (x, v) \mapsto (\tau(x), q_H(k) \cdot v), \quad \forall (x, v) \in X \times \mathbb{C}^m. \quad (2.8)$$

It turns out that $(X \times \mathbb{C}^m, \Theta_H)$ is a “Quaternionic” vector bundle in the sense of [Kah, Dup, DG2, DG3, DG4].

Definition 2.3 (Sewing matrix). Let $H : X \rightarrow \mathfrak{Herm}(\mathcal{H})^\times$ be a band insulator of class DIII over the involutive space $(X, \tau)$ according to Definition 1.1. Then $(X \times \mathbb{C}^m, \Theta_H)$ will be called the associated “Quaternionic” vector bundle, and $q_H$ the sewing matrix of the associated “Quaternionic” structure.

It is known that the existence of fixed points of the involutive space $(X, \tau)$ implies that the rank of the “Quaternionic” vector bundles over $(X, \tau)$ must be even [DG2, Proposition 2.1]. This fact has an immediate consequence:

Proposition 2.4. Let $H : X \rightarrow \mathfrak{Herm}(\mathcal{H})^\times$ be a band insulator of class DIII over the involutive space $(X, \tau)$, according to Definition 1.1. Assume that $X$ is path-connected and $X^\tau \neq \emptyset$. Then the associated “Quaternionic” vector bundle has even rank $m = 2n$ and $q_H : X \rightarrow U(2n)$. Moreover, the standard representation described in Proposition 2.1 is induced by a unitary isomorphism $\mathcal{H} \simeq \mathbb{C}^{4n}$. 

2.2. The topological invariant associated to the sewing matrix. In order to define a sufficiently good topological invariant for band insulators of class DIII we need to establish some working hypotheses.

Assumption 2.5. Let \((X, \tau)\) be an involution space and \(q : X \to \mathbb{U}(2n)\) a sewing matrix, i.e. a map such that
\[
q(\tau(x)) = -q(x)^t, \quad \forall x \in X.
\]
Let \(\mathcal{E}_q = X \times \mathbb{C}^{2n}\) be the “Quaternionic” vector bundle with “Quaternionic” \(\Theta_q\) structure induced by \(q\), i.e.
\[
\Theta_q : (x, v) \mapsto (\tau(x), q(k) \cdot v), \quad (x, v) \in X \times \mathbb{C}^{2n}.
\]
We will assume that:
\begin{enumerate}
  \item \(X\) is a path connected \(\mathbb{Z}_2\)-CW complex;
  \item The fixed point set is non-empty, i.e. \(X^\tau \neq \emptyset\);
  \item The FKMM invariant \(\kappa(\mathcal{E}_q) \in H^2_{\mathbb{Z}_2}(X; \mathbb{Z}(1))\) is trivial.
\end{enumerate}

For the definition and the properties of the FKMM invariant for “Quaternionic” vector bundles we refer to [DG2, DG3, DG4]. For more details on the relative cohomology groups \(H^\bullet_{\mathbb{Z}_2}(X; \mathbb{Z}(1))\) we will refer to [DG2, Section 3.1] and references therein. It is worth recalling that these groups fit into the long exact sequence for the pair \((X, X^\tau)\). The relevant part of the latter for the aims of this work reads
\[
0 \to H^1_{\mathbb{Z}_2}(X; \mathbb{Z}(1)) \to H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \to H^1_{\mathbb{Z}_2}(X^\tau, \mathbb{Z}(1)) \to \cdots
\]
\[
\cdots \to H^2_{\mathbb{Z}_2}(X; \mathbb{Z}(1)) \to H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \to \cdots
\]
where the first term of the sequence corresponds to \(H^0_{\mathbb{Z}_2}(X^\tau, \mathbb{Z}(1)) = 0\) [Gom]. As a matter of fact, the image of \(\kappa(\mathcal{E}_q)\) under \(H^2_{\mathbb{Z}_2}(X; \mathbb{Z}(1)) \to H^2_{\mathbb{Z}_2}(X^\tau, \mathbb{Z}(1))\) is the “Real” Chern class \(c^R_1(\text{det}(\mathcal{E}_q))\) that classifies the determinant “Real” line bundle \(\text{det}(\mathcal{E}_q)\) [Kah, DG1]. Therefore, \(\kappa(\mathcal{E}_q) = 0\) implies that the determinant line bundle \(\text{det}(\mathcal{E}_q)\) is trivial as “Real” line bundle. Let us also recall the isomorphisms \(H^1(X, \mathbb{Z}) \simeq [X, \mathbb{U}(1)]\) and \(H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \simeq [X, \mathbb{U}(1)]_{\mathbb{Z}_2}\), where the (standard) involution on \(\mathbb{U}(1)\) is given by the complex conjugation \(z \mapsto \overline{z}\) [Gom, Proposition A.2].

For the next result we need to introduce some notation. Let \(C(X, \mathbb{U}(1))\) be the set of continuous functions from \(X\) to \(\mathbb{U}(1)\) and \(q : X \to \mathbb{U}(2n)\) a sewing matrix as in Assumption 2.5. When \(x \in X^\tau\) is a fixed point under the involution, then \(q(x) = -q(x)^t\) is a skew-symmetric matrix and one can calculate the associated Pfaffian \(\text{Pf}[q(x)] \in \mathbb{U}(1)\). Let us define the subset
\[
C(X, \mathbb{U}(1))_q := \left\{ p : X \to \mathbb{U}(1) \left| \begin{array}{l}
\text{det}[q(x)] = p(\tau(x))p(x) \quad \forall x \in X, \\
p(x) = \text{Pf}[q(x)] \quad \forall x \in X^\tau
\end{array} \right. \right\}.
\]
Under the isomorphism $H^1(X, \mathbb{Z}) \simeq [X, U(1)]$ the subset $C(X, U(1))_q$ identifies a subset $H^1(X, \mathbb{Z})_q \subset H^1(X, \mathbb{Z})$ which coincides with the set of homotopy classes of maps in $C(X, U(1))_q$.

**Lemma 2.6.** Under Assumption 2.5, the following facts hold true:

1. The set $C(X, U(1))_q$ is non-empty, and is a torsor under the group of $\mathbb{Z}_2$-equivariant maps $r : X \to U(1)$ which take the value 1 on the fixed point set $X^\tau$.

2. The set $H^1(X, \mathbb{Z})_q$ is a torsor under the group $H^1_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1))$ with action induced by

\[ H^1_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1)) \to H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \to H^1(X, \mathbb{Z}) \supset H^1(X, \mathbb{Z})_q. \]

**Proof.** Let us start with (1). As pointed out above, the triviality of the FKMM invariant $\kappa(\mathcal{E}_q)$ implies the triviality of the determinant “Real” line bundle $\det(\mathcal{E}_q)$. Since the information on $\det(\mathcal{E}_q)$ is completely encoded in $\det[q] : X \to U(1)$, from the triviality of $\det(\mathcal{E}_q)$ one infers the existence of the a global trivialization $p : X \to U(1)$ which intertwines between the “Real” structure provided by $\det[q]$ and the trivial (or constant) “Real” structure. The latter fact is equivalent to the constraint $\det[q(x)] = p(\tau(x))p(x)$ for all $x \in X$. From [DG3, Proposition 2.10] one gets that the FKMM invariant $\kappa(\mathcal{E}_q) \in H^2_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1))$ coincides with the injective image under

\[ H^1_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1))/H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \to H^2_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1)) \]

doing the element whose representative is given by the map $X^\tau \ni x \mapsto p(x)/\text{Pf}[q(x)] \in \mathbb{Z}_2$ where $p$ is the global trivialization introduced above. Assumption 2.5 (c) says that $\kappa(\mathcal{E}_q)$ is trivial. Hence, one can choose $p$ so that $p(x) = \text{Pf}[q(x)]$ for every $x \in X^\tau$. This proves that the set $C(X, U(1))_q$ is non-empty. For a given pair $p', p \in C(X, U(1))_q$, let $r : X \to U(1)$ be the map $r(x) := p'(x)p(x)$. It turns out that $r(\tau(x)) = r(x)$ for every $x \in X$ and $r(x) = 1$ when $x \in X^\tau$. Hence $C(X, U(1))_q$ is a torsor as stated. For (2) it is enough to consider the homotopy classes of maps in (1). \qed

Let us recall that by definition a torsor coincides with the orbit generated by any of its points. We are now in position to introduce the main topological invariant of this work.

**Definition 2.7** (DIII-invariant). Under Assumption 2.5, one defines $\nu_q$ to be the orbit of the subset $H^1(X, \mathbb{Z})_q \subset H^1(X, \mathbb{Z})$ under $H^1_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1))$, i.e.

\[ \nu_q := [H^1(X, \mathbb{Z})_q] \in H^1(X, \mathbb{Z})/H^1_{\mathbb{Z}_2}(\mathbb{X}^\tau, \mathbb{Z}(1)). \]

Let us point out that the notation used in Definition 2.7 is a little redundant since $[H^1(X, \mathbb{Z})_q]$ and $H^1(X, \mathbb{Z})_q$ coincide as a set in view of the torsor property. Anyway, the notation proposed as the advantage of emphasizing that $\nu_q$ is an equivalence class (or an orbit). The topological properties of $\nu_q$ are specified in the following result.

**Theorem 2.8.** Under Assumption 2.5 the following facts hold true:
(1) \( \nu_q \) is an invariant of the homotopy class of \( q \).

(2) Let \( q_0 : X \to \mathbb{U}(2n) \) be the constant map given by
\[
q_0(x) = Q := \begin{pmatrix}
0 & -I_n \\
+I_n & 0
\end{pmatrix},
\]
then \( \nu_{q_0} = 0 \).

(3) Let \( q : X \to \mathbb{U}(2n) \) and \( q' : X \to \mathbb{U}(2n') \) be as in Assumption 2.5, and consider \( q \oplus q' : X \to \mathbb{U}(2n + 2n') \). Then
\[
\nu_{q \oplus q'} = \nu_q + \nu_{q'}
\]
as elements of the quotient group \( H^1(X, \mathbb{Z})/H^1_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \).

(4) Let \( q : X \to \mathbb{U}(2n) \) and \( q' : X \to \mathbb{U}(2n) \) be related by
\[
q'(x) = h(\tau(x))^\tau q(x) h(x)
\]
for a given map \( h : X \to \mathbb{U}(2n) \). Then,
\[
\nu_{q'} = \det[h] \cdot \nu_q = \left[ \det[h] \cdot H^1(X, \mathbb{Z}) \right]_q.
\]

**Proof.** (1) Let \( q \) and \( q' \) be sewing matrices satisfying Assumption 2.5. Suppose that these two matrices are connected by a homotopy \( \tilde{q} : X \times [0, 1] \to \mathbb{U}(2n) \) of sewing matrices. The map \( \tilde{q} \) defines the “Quaternionic” vector bundles \( \mathcal{E}_q \) over \( X \times [0, 1] \). By the homotopy property of “Quaternionic” vector bundles [DG2, Theorem 2.3], \( \mathcal{E}_q \) is isomorphic to the pull-back of \( \mathcal{E}_{q'} \) under the projection \( X \times [0, 1] \to X \). Hence, both the determinant “Real” line bundle of \( \mathcal{E}_q \) and the FKMM invariant of \( \mathcal{E}_q \) are trivial. As a result, \( C(X \times [0, 1], \mathbb{U}(1))_q \) is non-empty as a consequence of Lemma 2.6. Let us choose a \( \tilde{p} \in C(X \times [0, 1], \mathbb{U}(1))_q \). Such a \( \tilde{p} \) provides a homotopy between the representatives \( p := \tilde{p}|_{X \times \{0\}} \) and \( p' := \tilde{p}|_{X \times \{1\}} \) of \( \nu_q \) and \( \nu_{q'} \), respectively. Therefore, \( \nu_q = \nu_{q'} \).

(2) One can take \( p_0 \in C(X, \mathbb{U}(1))_q \) to be the constant map \( p_0(x) = 1 \). This map represents the trivial element \( 0 \in H^1(X, \mathbb{Z}) \), so that \( \nu_{q_0} = 0 \).

(3) If \( p \in C(X, \mathbb{U}(1))_q \) and \( p' \in C(X, \mathbb{U}(1))_{q'} \), then the map \( pp' : X \to \mathbb{U}(1) \) given by
\[
(pp')(x) := p(x)p'(x)
\]
belongs to \( C(X, \mathbb{U}(1))_{q \oplus q'} \). In the quotient group \( H^1(X, \mathbb{Z})/H^1_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \), the invariants \( \nu_q \) and \( \nu_{q'} \) are represented by the homotopy classes \( [p] \in H^1(X, \mathbb{Z}) \) and \( [p'] \in H^1(X, \mathbb{Z}) \), respectively. One has that \( [pp'] = [p] + [p'] \in H^1(X, \mathbb{Z}) \), so that \( \nu_{q \oplus q'} = \nu_q + \nu_{q'} \).

(4) If \( p \in C(X, \mathbb{U}(1))_q \), then one gets \( p' \in C(X, \mathbb{U}(1))_{q'} \) by setting
\[
p'(x) := \det[h(x)] p(x).
\]
Thus, the multiplication by \( \det[h] \) induces a transformation on \( H^1(X, \mathbb{Z}) \) which carries \( H^1(X, \mathbb{Z})_q \) to \( H^1(X, \mathbb{Z})_{q'} \). As a consequence, \( \nu_q \mapsto \nu_{q'} \) in the quotient group. \( \square \)


Remark 2.9. Let us recall that the for the equivariant cohomology the concept of reduced cohomology makes sense (see e.g. [DG1, Section 5.1]). Therefore, one has the usual decomposition

\[ H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \cong \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \oplus H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)), \]

where \( \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \) is the reduced cohomology group and \( H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \cong \mathbb{Z}_2 \). By means of a spectral sequence, one can show

\[ \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \cong H^1(X, \mathbb{Z})^\mathbb{Z}_2 := \{ f \in H^1(X; \mathbb{Z}) \mid \tau^*(f) = -f \}. \]

In particular, \( \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \) turns out to be a free abelian group. Since the generator of \( H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \subset H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \) can be represented by the constant map \( X \ni x \mapsto -1 \in \mathbb{Z} \), it follows from the exact sequence

\[ 0 \rightarrow H^1_{\mathbb{Z}_2}(X \times X, \mathbb{Z}(1)) \rightarrow H^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \oplus \mathbb{Z}_2 \]

that \( H^1_{\mathbb{Z}_2}(X \times X, \mathbb{Z}(1)) \) injects into the free part \( \tilde{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \).

2.3. The relation with the KR-theory. The KR-group \( KR^{-2}(X, \tau) \) of a space \( X \) with involution \( \tau \) may be regarded as a group which classifies the “difference” of two band insulators of class DIII. In this section we will relate \( KR^{-2}(X, \tau) \) with the invariant \( \nu_q \) of a sewing matrix \( q \), provided that certain assumptions are satisfied.

Let us start by recalling the formulation of the twisted equivariant K-theory [FM] (see also [DG5, Section 5] and [Gom2]). Let \( G \) be a finite group acting on a space \( X \) endowed with two homomorphisms \( \phi : G \rightarrow \mathbb{Z}_2 \) and \( c : G \rightarrow \mathbb{Z}_2 \) and a group 2-cocycle \( \sigma : G \times G \times X \rightarrow U(1) \), which satisfies

\[ \sigma(g_2, g_3, x)^{\phi(g_1)} \sigma(g_1 g_2, g_3, x)^{-1} \sigma(g_1, g_2 g_3, x) \sigma(g_1, g_2, g_3(x))^{-1} = 1 \]

for all \( g_1, g_2, g_3 \in G \) and all \( x \in X \). A \((\phi, \sigma, c)\)-twisted (ungraded) vector bundle with a \( Cl_{p,q}\)-action is a (finite-rank) Hermitian vector bundle \( \mathcal{E} \rightarrow X \) equipped with:

(a) An isometric bundle map \( \rho(g) : \mathcal{E} \rightarrow \mathcal{E} \) covering the action of \( g \in G \) on \( X \) for each \( g \in G \);

(b) Unitary maps \( \gamma_1, \ldots, \gamma_{p+q} \) on \( \mathcal{E} \), which satisfy the Clifford relations

\[ \gamma_i \gamma_j + \gamma_j \gamma_i = \begin{cases} 2 & \text{if } i = j = 1, \ldots, p, \\ -2 & \text{if } i = j = p + 1, \ldots, p + q, \\ 0 & \text{otherwise} \end{cases} \]

(c) The compatibility relations

\[ i \rho(g) = \phi(g) \rho(g) i, \]
\[ \rho(g) \rho(h) = \sigma(g, h) \rho(gh), \]
\[ \gamma_j \rho(g) = c(g) \rho(g) \gamma_j, \]

valid for every \( g, h \in G \) and \( j = 1, \ldots, p + q \).
A homomorphism from a twisted bundle \((\mathcal{E}, \rho, \gamma)\) to a twisted bundle \((\mathcal{E}', \rho', \gamma')\) is a map of complex vector bundles \(f: \mathcal{E} \to \mathcal{E}'\) such that

\[
f \circ \rho(g) = \rho'(g) \circ f, \quad f \circ \gamma_j = \gamma'_j \circ f,
\]
for every \(g \in G\) and every \(j = 1, \ldots, p + q\). The notion of isomorphism follows naturally.

Let \((\mathcal{E}, \rho, \gamma)\) be a twisted bundle with a Clifford action. An invertible Hermitian map \(H: \mathcal{E} \to \mathcal{E}\) is said to be compatible with the twisted action if it holds true that

\[
H \rho(g) = c(g) \rho(g) H, \quad H \gamma_j = -\gamma_j H.
\]
for every \(g \in G\) and every \(j = 1, \ldots, p + q\). If \(H\) and \(H'\) are invertible Hermitian maps on \((\mathcal{E}', \rho, \gamma)\) and \((\mathcal{E}', \rho', \gamma')\) respectively, such that \(f \circ H = H' \circ f\) for a given isomorphism \(f: \mathcal{E} \to \mathcal{E}'\), then \(H\) and \(H'\) will be regarded as isomorphic. Note that the self-adjoint involution \(Q_H \coloneqq H/|H|\) is nothing but a gradation, or a \(\mathbb{Z}_2\)-grading of the twisted bundle \(\mathcal{E}\).

Since \(Q_{H_1}\) and \(H\) are homotopic, we can generalize Karoubi’s formulation of twisted equivariant K-theory [DK, Ros, FHT, FM, Gom2] as follows:

**Definition 2.10** (Twisted equivariant K-theory). Let \(X, \phi, \sigma\) and \(c\) be as above.

(a) We define \(\Phi_M^{(\sigma, c)+(p, q)}(X)\) to be the monoid of isomorphism classes of triples \((\mathcal{E}, H_0, H_1)\) consisting of \((\phi, \sigma, c)\)-twisted bundles \(\mathcal{E}\) with \(\mathrm{Cl}_{p, q}\)-action and two invertible Hermitian maps \(H_0\) and \(H_1\) on \(\mathcal{E}\) compatible with the twisted actions.

(b) We define \(\Phi_N^{(\sigma, c)+(p, q)}(X)\) to be the submonoid of \(\Phi_M^{(\sigma, c)+(p, q)}(X)\) consisting of isomorphism classes of triples \((\mathcal{E}, H_0, H_1)\) such that \(H_0\) and \(H_1\) are homotopic within invertible Hermitian maps compatible with the twisted actions.

(c) We define \(\Phi K^{(\sigma, c)+(p, q)}(X) \coloneqq \Phi M^{(\sigma, c)+(p, q)}(X)/\Phi N^{(\sigma, c)+(p, q)}(X)\) to be the quotient monoid.

In view of the \((1, 1)\)-periodicity \(\Phi K^{(\sigma, c)+(p+1, q+1)}(X) \cong \Phi K^{(\sigma, c)+(p, q)}(X)\) [Gom2], we put

\[
\Phi K^{(\sigma, c)+(p, q)}(X) \coloneqq \Phi K^{(\sigma, c)+(p, q)}(X) - p(X).
\]

Let us now describe how from a band insulator \(H: X \to \mathcal{H} \text{erm}(\mathbb{C}^{2m})^\times\) of class DIII on the involutive space \((X, \tau)\) one gets an invertible Hermitian map on a twisted bundle. Consider the group \(G = \mathbb{Z}_2 = \{\pm 1\}\) as generated by \(\tau \equiv -1\), i.e., let \(G\) act on \(X\) through the involution. We choose \(\phi : G \to \mathbb{Z}_2\) to be the identity homomorphism \(\phi(\pm 1) = \pm 1\) and \(c : G \to \mathbb{Z}_2\) to be the trivial homomorphism \(c(\pm 1) = 0\). These choices can be shortly summarized by \(\phi = 1\) and \(c = 0\).

Let \(\sigma: G \times G \to \mathbb{U}(1)\) be the 2-cocycle (independent of the points of \(X\)) defined by

\[
\sigma(1, 1) = \sigma(-1, 1) = \sigma(1, -1) = 1, \quad \sigma(-1, -1) = -1.
\]

Let \(\mathcal{E} = X \times \mathbb{C}^{2m}\) be the product Hermitian vector bundle on \(X\) and define \(\rho(g): \mathcal{E} \to \mathcal{E}\) as follows:

\[
\rho(1) : (x, v) \mapsto (x, v), \quad \rho(-1) : (x, v) \mapsto (\tau(x), \overline{v})
\]
where $T$ denotes the TRS. We also define $\gamma : \mathcal{E} \to \mathcal{E}$ to be the unitary map defined by
\[
\gamma : (x, v) \mapsto (x, iv) .
\] (2.11)
where $\chi$ denotes the chiral symmetry. Then $(\mathcal{E}, \rho, \gamma)$ is a $(\phi, \sigma)$-twisted vector bundle with $\text{Cl}_{0,1}$-action. Finally, from $H : X \to \text{Sym}(\mathbb{C}^{2m})$ one gets the invertible Hermitian map $H : (x, v) \mapsto (x, H(x)v)$ on $\mathcal{E}$ compatible with the twisted action. As a result one has that differences of band insulators of class DIII provide elements of the twisted equivariant K-theory $\phi K_\sigma^0 \mathbb{Z}$, where we used the notation (2.9) and the reference to the trivial map $c = 0$ has been omitted. As a matter of fact a $\phi$-twisted vector bundle is nothing but a “Real” vector bundle, so that $\phi K_\sigma^0 \mathbb{Z} = KR_0(X, \tau)$ [Ati]. Since the cocycle $\sigma$ has the effect of the degree shift by $-4$ one gets
\[
\phi K_\sigma^0 \mathbb{Z} = KR^{-3}(X, \tau) .
\] (2.12)

The next result provides the link between the group $KR^{-3}(X, \tau)$ which classifies the (differences of) band insulators of class DIII and the invariant of sewing matrices described in Definition 2.7. For that we need to introduce the homomorphism
\[
\kappa : KR^{-3}(X, \tau) \longrightarrow H^1_\mathbb{Z}(X|X^\tau, \mathbb{Z}(1))
\] (2.13)
induced by the FKMM invariant [DG2, DG3, DG4]. More precisely, the homomorphism $\kappa$ assigns to a band insulator of class DIII, thought of as a class in $KR^{-3}(X, \tau)$, the FKMM invariant of the associated “Quaternionic” vector bundle as described in Definition 2.3.

**Theorem 2.11.** Let $(X, \tau)$ be a path connected finite $\mathbb{Z}_2$-CW complex which admits a fixed point. Then:

1. The map (2.13) induced by the FKMM invariant of sewing matrices provides a well-defined homomorphism;
2. The invariant of sewing matrices introduced in Definition 2.7 provides a well-defined homomorphism
\[
\nu : \text{Ker}(\kappa) \longrightarrow H^1(X, \mathbb{Z})/H^1_\mathbb{Z}(X|X^\tau, \mathbb{Z}(1))
\]

**Proof.** Let $(\mathcal{E}, H_0, H_1)$ be a triple representing an element of $KR^{-3}(X, \tau)$. In general, a twisted graded vector bundle on a compact space equivariant under a finite group admits a complementary twisted bundle $\mathcal{F}$ such that $\mathcal{E} \oplus \mathcal{F} \simeq X \times \mathcal{H}$ is isomorphic to a product bundle for some finite dimensional Hilbert space $\mathcal{H}$ and the twisted actions are independent of the points on $X$ (this fact is a straightforward generalization of a property of the usual equivariant vector bundles). Let $H_F : \mathcal{F} \to \mathcal{F}$ be an invertible Hermitian map. In view of the definition of $KR^{-3}(X, \tau)$ as a quotient monoid, $(\mathcal{E}, H_0, H_1)$ and $(\mathcal{E} \oplus \mathcal{F}, H_0 \oplus H_F, H_1 \oplus H_F)$ provide the same class in the K-theory. Thus, we can assume that $\mathcal{E} = X \times \mathcal{H}$ with twisted actions independent of points on $X$ since the beginning. Therefore, as shown in Proposition 2.1, we can represent the twisted actions on $X \times \mathcal{H}$ in a standard form, and express $H_0$ and $H_1$ in terms of sewing matrices $q_0 := q_{H_0}$ and
\[ q_1 := q_{H_1}, \text{ respectively.} \]

(1) With the argument above (and the identification $\mathcal{H} \equiv \mathbb{C}^{4n}$ in view of Proposition 2.4) one can specify the action of the map (2.13) as follows

\[ \kappa([X \times \mathbb{C}^{4n}, H_0, H_1]) := \kappa(\mathcal{E}_{q_0}) - \kappa(\mathcal{E}_{q_1}) \]  

(2.14)

where $\mathcal{E}_{q_i}$ is the “Quaternionic” vector bundle associated with the sewing matrix $q_i$, with $i = 0, 1$. Notice that this definition is independent of the way to identify the standard presentation. In fact if one changes the identification, one obtains new sewing matrices $q_i'$ which are related to $q_1$ via a continuous map $\phi : X \to \mathbb{U}(2n)$ according to equation (B.1). This map induces an isomorphism of the associated “Quaternionic” vector bundles $\mathcal{E}_{q_i'} \cong \mathcal{E}_{q_i}$. Since the FKMM invariant takes the same value on isomorphic “Quaternionic” bundles, it follows that equation (2.14) is insensitive on the way of fixing the standard presentation. Furthermore, if $H_0$ and $H_1$ are homotopic, then so are $q_0$ and $q_1$. In this case the homotopy invariance of “Quaternionic” bundles implies $\kappa(\mathcal{E}_{q_0}) = \kappa(\mathcal{E}_{q_1})$, and in turn one gets $\kappa([X \times \mathbb{C}^{4n}, H_0, H_1]) = 0$ for the trivial class. This fact, along with the additivity of the FKMM invariant under the direct sum of “Quaternionic” bundles, proves that (2.13) is a well-defined homomorphism.

(2) Let us start by showing that when $[X \times \mathbb{C}^{4n}, H_0, H_1] \in \text{Ker}(\kappa)$ then it is possible to choose the sewing matrices $q_i$ in such a way that $\kappa(\mathcal{E}_{q_i}) = 0$ for $i = 0, 1$. To see this observe that $[X \times \mathbb{C}^{4n}, H_0, H_1] \in \text{Ker}(\kappa)$ implies $\kappa(\mathcal{E}_{q_0}) = \kappa(\mathcal{E}_{q_1})$. As described in the proof of Lemma 2.6, the FKMM invariant $\kappa(\mathcal{E}_{q_i})$ can be represented by the map $X^\tau \ni x \mapsto p_1(x)/\text{Pf}[q_1(x)] \in \mathbb{Z}_2$ where $p_1 : X \to \mathbb{U}(1)$ is such that $\det[q_1(x)] = p_1(\tau(x))p_1(x)$ for all $x \in X$. Let $q_2 : X \to \mathbb{U}(2n)$ be the sewing matrix defined by $q_2(x) := q_0(x)^{-1}$. This provides $\text{Pf}[q_2(x)] = (-1)^n\text{Pf}[q_0(x)]^{-1}$ for all $x \in X^\tau$. If we define $p_2 : X \to \mathbb{U}(1)$ by $p_2(x) := (-1)^n p_0(x)^{-1}$ for all $x \in X$, then one gets $\det[q_2(x)] = p_2(\tau(x))p_2(x)$ for all $x \in X$ and the FKMM invariant $\kappa(\mathcal{E}_{q_2})$ is represented by the map

\[ X^\tau \ni x \mapsto \frac{p_2(x)}{\text{Pf}[q_2(x)]} = \left( \frac{p_0(x)}{\text{Pf}[q_0(x)]} \right)^{-1} \in \mathbb{Z}_2. \]

This means that $\kappa(\mathcal{E}_{q_2}) = -\kappa(\mathcal{E}_{q_0})$. Now, let us introduce the Hamiltonian $H_2 : X \to \text{Hom}(\mathbb{C}^{4n})^\times$ given by

\[ H_2(x) := \begin{pmatrix}
0 & q_2(x)^{-1} \\
q_2(x) & 0
\end{pmatrix}, \quad x \in X, \]

and the new triple $(X \times \mathcal{H}', H_0', H_1')$ defined by

\[ (X \times \mathcal{H}', H_0', H_1') := (X \times \mathbb{C}^{4n}, H_0, H_1) \oplus (X \times \mathbb{C}^{4n}, H_2, H_2) = (X \times (\mathbb{C}^{4n} \oplus \mathbb{C}^{4n}), H_0 \oplus H_2, H_1 \oplus H_2). \]
The classes \([X \times \mathbb{C}^{4n}, H_0, H_1]\) and \([X \times \mathcal{H}', H_0', H_1']\) represent the same element in \(\text{Ker}(\kappa)\) and
\[
\kappa(\varepsilon_{q_0}') = \kappa(\varepsilon_q + \varepsilon_{q_2}) = \kappa(\varepsilon_q) + \kappa(\varepsilon_{q_2}) = 0
\]
\[
\kappa(\varepsilon_{q_1}') = \kappa(\varepsilon_q + \varepsilon_{q_2}) = \kappa(\varepsilon_q) + \kappa(\varepsilon_{q_2}) = \kappa(\varepsilon_q) + \kappa(\varepsilon_{q_2}) = 0 .
\]
As a consequence, we can show that
\[
\text{Ker}(\kappa) = \phi N^\sigma_{\mathbb{Z}_2}((0,1)) / \left( \phi N^\sigma_{\mathbb{Z}_2}((0,1)) \cap \phi N^\sigma_{\mathbb{Z}_2}((0,1)) \right),
\]
where the submonoid \(\phi N^\sigma_{\mathbb{Z}_2}((0,1)) \subset \phi M^{\sigma(0,1)}_{\mathbb{Z}_2}(X)\) is defined by
\[
\phi N^\sigma_{\mathbb{Z}_2}((0,1)) := \left\{ (X \times \mathcal{H}, H_0, H_1) \in \phi M^{\sigma(0,1)}_{\mathbb{Z}_2}(X) \left| \kappa(E_q) = \kappa(E_{q_1}) = 0 \right. \right\} .
\]
Now, let us define the map \(\nu\) which associates to \((X \times \mathcal{H}, H_0, H_1) \in \phi N^\sigma_{\mathbb{Z}_2}((0,1))\) the difference of the invariants of the sewing matrices \(q_0\) and \(q_1\), i.e.
\[
\nu((X \times \mathcal{H}, H_0, H_1)) := \nu_{q_0} - \nu_{q_1} .
\] (2.15)
The rest of the proof consists in proving that the map \(\nu\) as defined by (2.15) provides a well-defined homomorphism on the group \(\text{Ker}(\kappa)\). First of all, let us notice that the sewing matrices \(q_0, q_1\) depend on the choice of the isomorphism \(X \times \mathcal{H} \simeq X \times \mathbb{C}^{4n}\) which makes the twisted actions in the standard form as discussed in Lemma 2.2. By combining the notation introduced in (2.10) and (2.11) with Proposition 2.1, one gets that the standard actions are given by
\[
\rho(-1) : (x, v) \mapsto (x, T_v) = \left( x, \begin{pmatrix} 0 & -i_{2n} \\ +i_{2n} & 0 \end{pmatrix} \right) v
\]
and
\[
\gamma : (x, v) \mapsto (x, i\chi v) = \left( x, \begin{pmatrix} +i_{2n} & 0 \\ 0 & -i_{2n} \end{pmatrix} \right) v .
\]
Moreover, every continuous map \(f : X \to \mathbb{U}(2n)\) provides the automorphism \(\psi_f\) of \(X \times \mathbb{C}^{4n}\) given by
\[
\psi_f : (x, v) \mapsto \left( x, \begin{pmatrix} f(x) & 0 \\ 0 & f(\tau(x)) \end{pmatrix} \right) v .
\]
Let \(\phi : X \times \mathcal{H} \cong X \times \mathbb{C}^{4n}\) and \(\phi' : X \times \mathcal{H} \cong X \times \mathbb{C}^{4n}\) be two standard representation intertwined by the automorphism \(\psi_f\), i.e. \(\phi = \psi_f \circ \phi'\). Let \(q_0, q_1\) be the sewing matrices associated to \(H_0\) and \(H_1\), respectively, via the isomorphism \(\phi\). Similarly, let \(q_0', q_1'\) be the related sewing matrices obtained via the isomorphism \(\phi'\). A straightforward computation shows that
\[
q_j'(x) = f(\tau(x))^5 q_j(x) f(x) , \quad j = 0, 1 .
\]
By using Theorem 2.8 (4) one gets the following fact: If \(\nu_{q_j}\) is represented by \(p_j \in C(X, \mathbb{U}(1))_{q_j}\), then \(\nu_{q_j'}\) is represented by \(p_j' = p_j \det[f]\). In terms of classes in \(H^1(X, \mathbb{Z}) = \)

[X, U(1)], one gets

\[ [p'_0] - [p'_1] = [p_0 \det[f]] - [p_1 \det[f]] = ([p_0] + [\det[f]]) - ([p_1] + [\det[f]]) = [p_0] - [p_1]. \]

Therefore \( \nu_{q'_0} - \nu_{q'_1} = \nu_{q_0} - \nu_{q_1} \) in \( H^1(X, \mathbb{Z})/H^1_{Z_2}(X|X^+, \mathbb{Z}(1)) \), and the map \( \nu \) defined by (2.15) turns out to be independent of the choice of the standard representation \( X \times \mathcal{H} \cong X \times \mathbb{C}^4 \). This also implies that \( \nu(X \times \mathcal{H}, H_0, H_1) = \nu(X \times \mathcal{H}', H'_0, H'_1) \) whenever the triples \( (X \times \mathcal{H}, H_0, H_1) \) and \( (X \times \mathcal{H}', H'_0, H'_1) \) are isomorphic. This leads to a map

\[ \nu : \Phi N_{Z_2}^{\sigma+,1}(X) \rightarrow H^1(X, \mathbb{Z})/H^1_{Z_2}(X|X^+, \mathbb{Z}(1)), \]

which is a monoid homomorphism by Theorem 2.8 (2) and (3). By Theorem 2.8 (1), the monoid homomorphism is trivial on the submonoid \( \Phi N_{Z_2}^{\sigma+,1}(X) \cap \Phi N_{Z_2}^{\sigma+,1}(X) \). Consequently, we get a well-defined homomorphism \( \nu \) on the group \( \ker(\kappa) \). Equation (2.12) completes the proof. \( \square \)

Remark 2.12. It is well known that the involutive space \( (S^1, \epsilon) \) admits only the trivial “Quaternionic” vector bundle [DG2, Theorem 1.2 (ii)]. As a consequence one gets that \( \ker(\kappa) = K^{-3}(S^1, \epsilon) \) in this special case. The properties of the homomorphism \( \nu \) on \( K^{-3}(S^1, \epsilon) \) will be studied in Proposition 3.15.

Remark 2.13. As pointed out in the proof of Theorem 2.11, the sewing matrix \( q \) depends on the way to express the twisted bundle \( X \times \mathcal{H} \) in a standard form. If we change the isomorphism by an automorphism associated to \( f : X \rightarrow U(m) \), then the sewing matrix \( q \) is changed into a sewing matrix \( q' \) and the relation between the two sewing matrices is expressed by the equation \( q'(x) = f(\tau(x))q(x)f(x) \). The difference \( \nu_q - \nu_{q'} \) is represented by \( [\det[f]] \) in the group \( H^1(X, \mathbb{Z})/H^1_{Z_2}(X|X^+, \mathbb{Z}(1)) \). Since \( \det : [X, U(m)] \rightarrow [X, U(1)] \) is surjective one has that all the elements in \( H^1(X, \mathbb{Z})/H^1_{Z_2}(X|X^+, \mathbb{Z}(1)) \) arise just by changing the isomorphism \( X \times \mathcal{H} \cong X \times \mathbb{C}^2 \). This means that we cannot define an absolute invariant of a band insulator \( H \) of class DIII by using the invariant \( \nu_q \) of its sewing matrix \( q \). Notice, however, that the invariant is able to detect the difference of two band insulators, as established in Theorem 2.11. In other words \( \nu_q \) works as relative invariant for band insulators of class DIII.

Remark 2.14. A map \( \phi : X \rightarrow U(1) \) over the involutive space \( (X, \tau) \) is called invariant if \( \phi = \phi \circ \tau \). The equivariant cohomology \( H^1_{Z_2}(X; \mathbb{Z}) \) is isomorphic to the group \( [X, U(1)]_{\text{inv}} \) of homotopy classes of invariant maps (see Appendix A). Thus, in general, \( [\det[q]] \in H^1_{Z_2}(X, \mathbb{Z}) \) is an invariant of the homotopy class of a sewing matrix \( q : X \rightarrow U(2n) \). By a proof similar to that of Theorem 2.11, one can show that this invariant defines a homomorphism

\[ \det : KR^{-3}(X, \tau) \rightarrow H^1_{Z_2}(X, \mathbb{Z}). \]

However, this invariant does not detect \( KR^{-3}(S^1, \epsilon) \cong Z_2 \) since \( H^1_{Z_2}(S^1, \mathbb{Z}) = 0 \). This observation is in agreement with the content item (II) presented in Section 1.
3. The one-dimensional case

In this section we will analyze in detail the case of one-dimensional band insulators of class DIII. According to Definition 1.1, the latter are band insulators of class DIII defined on the involutive space $(S^1, \iota)$. This peculiar case is relevant for its application to physical systems. In fact, in this section we will show the coincidence of the topological invariant $\nu_q$ described in Definition 2.7 with the topological invariant for topological insulators of class DIII usually used in the physical literature [RSFL, TK, BA, CTSR]. For the benefit of the reader we report here a table of some cohomology groups of the involutive space $(S^1, \iota)$ which will be useful in the following:

| $n$ | $H^n(S^1, \mathbb{Z})$ | $H^n_{\mathbb{Z}_2}(S^1, \mathbb{Z})$ | $H^n_{\mathbb{Z}_2}(S^1, \mathbb{Z}(1))$ | $H^n_{\mathbb{Z}_2}((S^1)^t, \mathbb{Z}(1))$ | $H^n_{\mathbb{Z}_2}(S^1, ((S^1)^t, \mathbb{Z}(1))$ |
|-----|------------------|------------------|------------------|------------------|------------------|
| 0   | $\mathbb{Z}$    | $\mathbb{Z}$    | $0$              | $0$              | $0$              |
| 1   | $\mathbb{Z}$    | $0$              | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}$    |
| 2   | $0$              | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $0$              | $0$              | $0$              |
| 3   | $0$              | $0$              | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ |

The proof of the results listed in the table can be found in [Gom, Lemma 2.12], in [DG1, Section 5.3] and in [DG3, Appendix B]. Let us also recall that $H^1(S^1, \mathbb{Z}) \simeq [S^1, U(1)]$ is generated by the identity map under the identification $U(1) \simeq S^1$. Similarly, in view of $H^1_{\mathbb{Z}_2}(S^1, \mathbb{Z}(1)) \simeq [S^1, U(1)]_{\mathbb{Z}_2}$ [Gom, Proposition A.2] one gets that the free subgroup $\mathbb{Z}$ is generated by the identity map under the identification of $U(1)$, endowed with the complex conjugation, with the involutive space $(S^1, \iota)$. The torsion subgroup $\mathbb{Z}_2$ is generated by the constant map with value $-1$. All “Real” and “Quaternionic” vector bundles on $(S^1, \iota)$ are trivial as proved in [DG1, Theorem 1.6] and [DG2, Theorem 1.2]. Finally, let us observe that $(S^1, \iota)$ satisfies the properties (a), (b) and (c) listed in Assumption 2.5.

3.1. The Teo-Kane formula. In this section we will provide an explicit formula to compute the invariant $\nu_q$ in the one-dimensional case. In particular we will establish the connection with the Teo-Kane formula, the latter being the topological invariant commonly used in the physical literature for one-dimensional for topological insulators of class DIII (see e. g. [TK, eq. (4.27)]) or [BA, eq. (8)]) or [CTSR, eq. (3.70)]).

Let us start with a couple of preliminary results about the topology of the maps which define the invariant $\nu_q$. The first result concerns with the analysis of the range of the invariant $\nu_q$.

**Lemma 3.1.** Consider the involutive space $(S^1, \iota)$. Let $\delta : C(S^1, U(1)) \rightarrow \mathbb{Z}_2$ be the map defined by $\delta(p) := (-1)^{\deg(p)}$ where $\deg(p)$ denotes the degree (or winding number) of $p$ as a map $p : S^1 \rightarrow U(1) \simeq S^1$. Then $\delta$ induces the isomorphism of groups

$$H^1(S^1, \mathbb{Z})/H^1_{\mathbb{Z}_2}(S^1, ((S^1)^t, \mathbb{Z}(1)) \simeq \mathbb{Z}_2.$$  

(3.1)

**Proof.** As anticipated in the introductory part of this section, $H^1_{\mathbb{Z}_2}(S^1, \mathbb{Z}(1)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ is generated by the identity map $S^1 \rightarrow U(1)$ and the constant map with value $-1$. From the
exact sequence [DG3, eq. (2.6)]

\[ 0 = H^0_{Z_2}((S^1)\uparrow, \mathbb{Z}(1)) \longrightarrow H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \longrightarrow H^2_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \]

one gets that the map \( r \), induced by the restriction \( S^1 \to (S^1)^\uparrow \), is surjective. Moreover, one infers that the injective image of \( H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \) under \( j \) is generated by the equivariant map \( e : \mathbb{S}^1 \to U(1) \) given by \( e(z) := e^{2\pi k} \). Since the map \( f : H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \to H^1(\mathbb{S}^1, \mathbb{Z}) \) which forgets the \( Z_2 \)-action is the projection which induces the bijection on the free part of the two groups, it follows that the image of

\[ H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \longrightarrow H^1(\mathbb{S}^1, \mathbb{Z}) \]

is generated by maps \( \mathbb{S}^1 \to U(1) \) with even degree. This proves (3.1) with the isomorphism given by the parity of the degree of the maps from \( \mathbb{S}^1 \) to \( U(1) \).

The next result provides a formula for the degree of the equivariant maps on the involutive space \((\mathbb{S}^1, \iota)\) with value in \( U(1) \). Let us emphasize that a map \( r : \mathbb{S}^1 \to U(1) \) is equivariant if \( r(\iota(k)) = \overline{r(k)} \) for every \( k \in \mathbb{S}^1 \).

**Lemma 3.2.** Let \( r : \mathbb{S}^1 \to U(1) \) be an equivariant map defined on the involutive space \((\mathbb{S}^1, \iota)\). Then, it holds true that

\[ (-1)^{\deg(r)} = \frac{r(\pi)}{r(0)}. \]

**Proof.** The equivariance condition implies that \( r(k) = \pm 1 \) at the fixed points \( k = 0, \pi \). In particular, the value at each fixed point is invariant under equivariant homotopy deformations of \( r \). Therefore, the assignment \( \Delta : r \mapsto r(\pi)/r(0) \) induces a well-defined homomorphism \( \Delta : H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \to \mathbb{Z}_2 \) (still denoted with the same symbol). By using the explicit basis of \( H^1_{Z_2}(\mathbb{S}^1, \mathbb{Z}(1)) \), one can verify the equality \( \Delta(r) = (-1)^{\deg(r)} \). \( \square \)

With the help of Lemma 3.1 and Lemma 3.2 we can provide a first formula for the invariant \( \nu_q \).

**Proposition 3.3** (The degree formula). Consider the involutive space \((\mathbb{S}^1, \iota)\) along with a map \( q : \mathbb{S}^1 \to U(2\pi) \) such that

\[ q(\iota(k)) = -q(k)^\uparrow, \quad \forall k \in \mathbb{S}^1. \]

Then, the topological invariant \( \nu_q \) of the map \( q \) described in Definition 2.7 can be computed with the formula

\[ \nu_q = (-1)^{\deg(p)} \in \mathbb{Z}_2, \]

independently of \( p \in C(\mathbb{S}^1, U(1))_q \).

**Proof.** First of all it is worth noting that all the conditions listed in Assumption 2.5 are satisfied and therefore the invariant \( \nu_q \) is well-defined. In view of Lemma 3.1 one gets
that
\[ \nu_q \in H^1(S^1, \mathbb{Z})/H^1_{2q}(S^1([S^1]^q), \mathbb{Z}(1)) \cong \mathbb{Z}_2. \]

Since the isomorphism with \( \mathbb{Z}_2 \) is realized by the parity of the degree, we get the expression \( \nu_q = (-1)^{\text{deg} p} \) in terms of \( p \in C(S^1, \mathbb{U}(1))_q \). Indeed, in view of Lemma 2.6 (1) we know that for every two \( p, p' \in C(S^1, \mathbb{U}(1))_q \) there is an equivariant map \( r : S^1 \to \mathbb{U}(1) \) such that \( p' = rp \) and \( r(0) = r(\tau) = 1 \). From Lemma 3.2 one infers that \( \text{deg}(r) \) is even and in turn \( \text{deg}(p) = \text{deg}(p') = \text{deg}(p) + \text{deg}(r) \) have the same parity. \( \square \)

The next result is preparatory for the proof of the Teo-Kane formula.

**Lemma 3.4.** Let \( q : S^1 \to \mathbb{U}(2n) \) be a map defined on the involutive space \((S^1, \iota)\) such that
\[ q(\iota(k)) = -q(k)^t, \quad \forall k \in S^1. \]
Then, there exists a map \( \tilde{q} : S^1 \times [0, 1] \to \mathbb{U}(2n) \) such that
\[ \tilde{q}(\iota(k), t) = -\tilde{q}(k, t)^t, \quad \tilde{q}(k, 0) = q(k), \quad \det[\tilde{q}(k, 1)] = 1, \]
for all \( k \in S^1 \) and \( t \in [0, 1] \).

**Proof.** As anticipated in item ii) of Section 1, \( \det[q] : S^1 \to \mathbb{U}(1) \) is an invariant map, i.e. \( \det[q(k)] = \det[q(\iota(k))] \) for every \( k \in S^1 \). Let \([S^1, \mathbb{U}(1)]_{\text{inv}} \) be the set of classes of homotopy equivalent invariant functions. Then, one has that \([S^1, \mathbb{U}(1)]_{\text{inv}} \cong H^1_{2q}(S^1, \mathbb{Z}) = 0 \) (cf. Appendix A). Therefore, there exists \( f : S^1 \times [0, 1] \to \mathbb{U}(1) \) such that \( f(k, t) = f(\iota(k), t), \quad f(k, 0) = 1 \) and \( f(k, 1) = \det[q(k)]^{-1} \) for every \( k \in S^1 \) and \( t \in [0, 1] \). The degree of the map \( k \mapsto f(k, t) \) is constant in \( t \) and therefore is trivial for every \( t \in [0, 1] \) (since it is trivial for \( t = 0 \)). This allows to define a square root
\[ g(k, t) := f(k, t)^{1/2} \]
consistently. This provides a continuous map \( g : S^1 \times [0, 1] \to \mathbb{U}(1) \) such that \( g(k, t) = g(\iota(k), t), \quad g(k, 0) = 1 \) and \( g(k, 1)^2 = \det[q(k)]^{-1} \) for every \( k \in S^1 \) and \( t \in [0, 1] \). Let us introduce the map \( \tilde{q} : S^1 \times [0, 1] \to \mathbb{U}(2n) \) defined by
\[ \tilde{q}(k, t) := \begin{pmatrix} g(k, t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cdots \end{pmatrix} q(k) \begin{pmatrix} g(k, t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cdots \end{pmatrix}. \]
(3.2)
By construction the map \( \tilde{q} \) meets the properties stated in the claim. \( \square \)

**Theorem 3.5** (The Teo-Kane formula). Let \( q : S^1 \to \mathbb{U}(2n) \) be a map defined on the involutive space \((S^1, \iota)\) such that
\[ q(\iota(k)) = -q(k)^t, \quad \forall k \in S^1. \]
Then, the invariant $\nu_q \in \mathbb{Z}_2$ admits the expression

$$
\nu_q = \frac{\text{Pf}[q(\pi)]}{\text{Pf}[q(0)]} \frac{\text{det}[q(0)]^{\frac{1}{2}}}{\text{det}[q(\pi)]^{\frac{1}{2}}}
$$

where the branch $k \mapsto \text{det}[q(k)]^{\frac{1}{2}}$ must be chosen continuously between the two fixed points $k = 0, \pi$.

Proof. Theorem 2.8 (1) ensures that the invariant $\nu_q$ only depends on the homotopy class of $q$. By combining this fact with Lemma 3.4 one gets that $\nu_q = \nu_{q'}$ where $q'(k) := \tilde{q}(k, 1)$ according to the notation of Lemma 3.4. The crucial property of $q'$ is that $\text{det}[q'(k)] = 1$ for every $k \in S^1$. From Proposition 3.3 we know that $\nu_{q'} = (-1)^{\text{deg}(\pi')}$, where $\pi' : S^1 \to U(1)$ is such that $1 = \pi'(\iota(k))\pi'(k)$ for all $k \in S^1$ and $\pi'(k) = \text{Pf}[q'(k)]$ for $k = 0, \pi$. Then, one has $\pi'(0)^2 = 1 = \pi'(\pi)^2$ so that $\pi'(\pi)/\pi'(0) \in \mathbb{Z}_2$. In view of Lemma 3.2, one has that

$$
\nu_{q'} = (-1)^{\text{deg}(\pi')} = \frac{\pi'(\pi)}{\pi'(0)} = \frac{\text{Pf}[q'(\pi)]}{\text{Pf}[q'(0)]}.
$$

From equation (3.2) which provides the link between $q$ and $q'$ and the well-known formula $\text{Pf}[BAB^t] = \text{det}[B]\text{Pf}[A]$ for generic matrices $A, B$ with $A$ skew-symmetric, one gets

$$
\text{Pf}[q'(k)] = g(k, 1) \text{Pf}[q(k)] = \frac{\text{Pf}[q(k)]}{\text{det}[q(k)]^{\frac{1}{2}}} , \quad k = 0, \pi.
$$

This concludes the proof. \hfill \square

For a concrete physical model of one-dimensional topological insulator of class DIII whose (non-trivial) topology is described by the Teo-Kane formula we refer to [TK, Section IV.C] or [CTSR, Section B.6.a]. The connection with the Kramers polarization is discussed in [BA].

3.2. The role as topological obstruction. In this section we will show that in dimension one the invariant $\nu_q$ can be interpreted as a topological obstruction. For this aim, we need to introduce the involutive space $(U(2n), \emptyset)$ given by the unitary group on $\mathbb{C}^{2n}$ endowed with the involution $0(U) := -U^t$. The fixed point set $U(2n)^0$ coincides with the set of skew-symmetric matrices on $\mathbb{C}^{2n}$. Let us introduce the standard symplectic matrix

$$
Q := \begin{pmatrix}
0 & -I_n \\
+I_n & 0
\end{pmatrix} \in U(2n)^0
$$

and the compact symplectic group $\text{Sp}(n) = \{ S \in U(2n) \mid S^t QS = Q \}$. One has that $\text{Pf}[Q] = (-1)^{\frac{n(n+1)}{2}}$. By considering the left action of $\text{Sp}(n)$ on $U(2n)$ one can define the quotient space $U(2n)/\text{Sp}(n)$. In view of the normal decomposition for skew-symmetric matrices, one gets the well-defined homeomorphism

$$
U(2n)/\text{Sp}(n) \ni [U] \longmapsto U^t Q U \in U(2n)^0.
$$
In particular, $\mathbb{U}(2n)^0$ can be seen as the base space of a principal $\text{Sp}(n)$-bundle.

Let us start with two preliminary results.

**Lemma 3.6.** Let $(X, \tau)$ be an involutive space with a fixed point $\ast \in X^\tau$. Let $q : X \to \mathbb{U}(2n)$ be a map satisfying

$$q(\tau(x)) = -q(x)^t, \quad \forall x \in X.$$  

Then there exists a map $\tilde{q} : X \times [0, 1] \to \mathbb{U}(2n)$ such that $\tilde{q}(\tau(x), t) = -\tilde{q}(x, t)^t$ and $\tilde{q}(x, 0) = q(x)$ for all $(x, t) \in X \times [0, 1]$ and $\tilde{q}(\ast, 1) = Q$.

**Proof.** At the fixed point $q(\ast) \in \mathbb{U}(2n)^0$ is a skew-symmetric unitary matrix. Let $U \in \mathbb{U}(2n)$ be a unitary matrix such that $U^t QU = q(\ast)$. Since $\mathbb{U}(2n)$ is path connected, there is a path $u : [0, 1] \to \mathbb{U}(2n)$ such that $u(0) = 1_{2n}$ and $u(1) = U^{-1}$. Now, let us define $\tilde{q} : X \times [0, 1] \to \mathbb{U}(2n)$ by

$$\tilde{q}(x, t) := u(t)^t q(x) u(t).$$

A straightforward check shows that this map has the prescribed properties. \qed

**Lemma 3.7.** The space

$$S\mathbb{U}(2n)^0 := \{ U \in \mathbb{U}(2n)^0 \mid \det[U] = 1 \}$$

has two connected components distinguished by their Pfaffians.

**Proof.** Under the homeomorphism $\mathbb{U}(2n)/\text{Sp}(n) \simeq \mathbb{U}(2n)^0$, one can identify the subspace $S\mathbb{U}(2n)^0 \subset \mathbb{U}(2n)^0$ with $\det^{-1}(1)/\text{Sp}(n) \sqcup \det^{-1}(-1)/\text{Sp}(n) \subset \mathbb{U}(2n)/\text{Sp}(n)$ where $\det^{-1}(\pm 1) := \{ U \in \mathbb{U}(2n) \mid \det[U] = \pm 1 \}$. In particular $\det^{-1}(1) = S\mathbb{U}(2n)$. Let

$$I := \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \in \mathbb{U}(2n).$$

The map $U \mapsto IU$ provides a homeomorphism $\det^{-1}(-1) \simeq \det^{-1}(+1)$. Thus, $S\mathbb{U}(2n)^0$ is homeomorphic to the disjoint union of two copies of $S\mathbb{U}(2n)/\text{Sp}(n)$, and in turn it has two connected components. From the form of the homeomorphism and the property of the Pfaffian one gets that $\text{Pf} : [U] \mapsto \text{Pf}[U^t QU] = \pm \text{Pf}[Q]$ for every $[U] \in \det^{-1}(\pm 1)/\text{Sp}(n)$. Therefore, the Pfaffian distinguishes the two connected components of $S\mathbb{U}(2n)^0$. \qed

The next result will shed light on the nature of the invariant $\nu_q$ as the topological obstruction for the existence of a homotopy transformation between two given sewing matrices.
Theorem 3.8 (Topological obstruction). Consider the involutive space \((S^1, \iota)\) and two maps \(q_j : S^1 \to U(n)\), \(j = 0, 1\), satisfying the sewing matrix condition
\[
q_j(\iota(k)) = -q_j(k)^t, \quad \forall k \in S^1, \quad j = 0, 1.
\]
The maps \(q_0, q_1\) are homotopy equivalent within the space of sewing matrices if and only if \(\nu_0 = \nu_1\).

Proof. Theorem 2.8 (1) provides the “only if” part of the claim. Therefore, we only need to show the “if” part. By Lemma 3.4 and Lemma 3.6, we can assume that \(\det[q_0(k)] = \det[q_1(k)] = 1\) for all \(k \in S^1\) and \(q_0(0) = q_1(0) = Q\). Under these assumptions, the coincidence of the invariants \(\nu_0 = \nu_1\) is equivalent to \(Pf[q_0(\pi)] = Pf[q_1(\pi)]\) in view of Theorem 3.5. We will proceed to build a homotopy of sewing matrices \(\tilde{q} : S^1 \times [0, 1] \to U(2n)\) such that \(\tilde{q}(k, j) = q_j(k)\) for \(j = 0, 1\) and \(k \in S^1\). We will do this in several steps:

1. As a first step let us choose a continuous map \(\tilde{Q} : [0, 1] \to SU(2n)^0 \subset U(2n), \quad \tilde{Q}(j) := q_j(\pi), \quad j = 0, 1\).

This is possible, because Lemma 3.7 establishes that \(SU(2n)^0\) has two connected components distinguished by the Pfaffian, and \(Pf[q_0(\pi)] = Pf[q_1(\pi)]\) by hypothesis.

2. As a second step, let us build the continuous map \(\bar{q} : \partial([0, \pi] \times [0, 1]) \to U(2n)\) defined by
\[
\bar{q}(k, t) := \begin{cases} 
q_0(0) = q_1(0) = Q & \text{if } k = 0, \quad 0 \leq t \leq 1, \\
q_0(k) & \text{if } 0 \leq k \leq \pi, \quad t = 0, \\
q_1(k) & \text{if } 0 \leq k \leq \pi, \quad t = 1, \\
\tilde{Q}(t) & \text{if } k = \pi, \quad 0 \leq t \leq 1.
\end{cases}
\]
The map above can be extended, without obstructions, to a continuous map \(\bar{q} : [0, \pi] \times [0, 1] \to U(2n)\) (still denoted with the same symbol). Indeed, by identifying \(\partial([0, \pi] \times [0, 1])\) with \(S^1\), one can think of \(\bar{q}\) as a loop in \(SU(2n)\). Since the fundamental group of \(SU(2n)\) is trivial, it is possible to get the invoked continuous extension.

3. The third step consists in a further extension \(\tilde{q} : [-\pi, \pi] \times [0, 1] \to U(2n)\) defined by
\[
\tilde{q}(k, t) := \begin{cases} 
\bar{q}(k, t) & \text{if } (k, t) \in [0, \pi] \times [0, 1], \\
-\tilde{q}(-k, t)^t & \text{if } (k, t) \in [-\pi, 0] \times [0, 1].
\end{cases}
\]
Since \(q_0(0) = q_1(0) = Q\) is skew-symmetric, \(\tilde{q}\) is a well-defined continuous map on \([-\pi, \pi] \times [0, 1]\). Moreover, since \(\tilde{q}(\pi, t) = \tilde{Q}(t)\) is chosen to be skew-symmetric for all \(t \in [0, 1]\), one gets that \(\tilde{q}(\pi, t) = \tilde{q}(-\pi, t)\) for all \(t \in [0, 1]\). This shows that \(\tilde{q}\) descends to a continuous map \(\tilde{q} : S^1 \times [0, 1] \to U(2n)\).

4. The fourth and final step consists in the direct check of the fact that the resulting map \(\tilde{q}\) provides the required homotopy of sewing matrices. \(\square\)
Remark 3.9 (A small generalization). It is likely that Theorem 3.8 could be generalized in the case of a 1-dimensional $\mathbb{Z}_2$-CW complex $(X, \tau)$ such that the fixed point set $X^\tau$ is a finite collection of points. In this case it is expected that two sewing matrices $q_0, q_1 : X \to \mathbb{U}(2n)$ are linked by a homotopy within the space of sewing matrices if and only if the following two conditions are satisfied:

1. $[\det(q_0)] = [\det(q_1)]$ in $H^1_{\mathbb{Z}_2}(X, \mathbb{Z}) \simeq [X, \mathbb{U}(1)]_{\text{inv}}$;
2. $\nu_{q_0} = \nu_{q_1}$ in $H^1(X, \mathbb{Z})/H^1_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1))$.

Note that in this case the invariants $\nu_{q_i}$ are well-defined since $H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) = 0$ as proved in [DG4, Proposition 4.1]. ▷

3.3. The role as an index. As an application of Theorem 3.8, we can establish an index-type theorem for the invariant for topological insulators of class DIII described in Definition 2.7. By this we mean that the quantity $\nu_q$, which is purely topological by definition, can be computed by an analytic expression. The latter is nothing more than a suitable index of an appropriate Fredholm operator.

In order to prove the main result of this section we need first to introduce some notation. Let $\text{Mat}_{2n}(\mathbb{C})$ be the $C^*$-algebra of $2n \times 2n$ matrices. Every continuous function $f : \mathbb{S}^1 \to \text{Mat}_{2n}(\mathbb{C})$ defines a bounded multiplication operator $M_f$ acting on the Hilbert space $L^2(\mathbb{S}^1) \otimes \mathbb{C}^{2n}$ according to the prescription

$$(M_f \psi)(k) := f(k)\psi(k), \quad \psi \in L^2(\mathbb{S}^1) \otimes \mathbb{C}^{2n}.$$ 

By construction the map $f \mapsto M_f$ is a homomorphism of $C^*$-algebras. In particular one has $M_f^* = M_f$. Using the standard complex conjugation on $\mathbb{C}^{2n}$, one can endow $L^2(\mathbb{S}^1) \otimes \mathbb{C}^{2n}$ with the real structure (or antiunitary involution) $K$ defined by $(K\psi)(k) := \overline{\psi(-k)}$. Given a (bounded) operator $A$ on $L^2(\mathbb{S}^1) \otimes \mathbb{C}^{2n}$, we define the complex conjugate of $A$ as $\overline{A} := KAK$, and the transpose of $A$ as $A^t := (\overline{A})^* = KA^*K$. The operator $A$ will be called real if $A = \overline{A}$, self-adjoint if $A = A^*$, complex symmetric if $A = A^t$, skew-adjoint if $A = -A^t$ and skew-complex symmetric if $A = -A^t$ [LZ, KKL].

The next result shows that the multiplication operators associated to sewing matrices are automatically skew-complex symmetric.

Lemma 3.10. Let $q : \mathbb{S}^1 \to \mathbb{U}(2n)$ be a map defined on the involutive space $(\mathbb{S}^1, \iota)$ which satisfies the sewing matrix condition

$$q(\iota(k)) = -q(k)^t, \quad \forall k \in \mathbb{S}^1.$$

Then, it holds true that

$$(M_q)^t = -M_q.$$ 

Proof. One has that

$$(KM_q^*K\psi)(k) = (M_{q^t}K\psi)(-k) = q(\iota(k))^* (K\psi)(-k) = -q(k)\psi(k) = -(M_q\psi)(k).$$
for every \( \psi \in L^2(S^1) \otimes \mathbb{C}^{2n} \). This completes the proof. \( \square \)

The Fourier modes \( \phi_m(k) := e^{imk} \), with \( m \in \mathbb{Z} \), provides a basis for \( L^2(S^1) \). Let \( \mathcal{H}_m := \mathbb{C}[\phi_m] \otimes \mathbb{C}^{2n} \) be the subspace of \( L^2(S^1) \otimes \mathbb{C}^{2n} \) generated by \( \phi_m \otimes \nu \) with \( \nu \in \mathbb{C}^{2n} \) and define the subspace of positive frequencies \( \mathcal{H}_+ := \bigoplus_{m \in \mathbb{N}} \mathcal{H}_m \). The associated orthogonal projection \( \Pi : L^2(S^1) \otimes \mathbb{C}^{2n} \to \mathcal{H}_+ \) is known as the *Hardy projection*. Its adjoint is given by the inclusion \( \Pi^* : \mathcal{H}_+ \to L^2(S^1) \otimes \mathbb{C}^{2n} \). Since \( K\phi_m = \phi_m \) for every \( m \in \mathbb{Z} \), it follows that \( \mathcal{H}_+ \) is preserved by the real structure \( K \). Therefore (with a little abuse of notation) one has that \( K\Pi = \Pi K \) and \( K\Pi^* = \Pi^* K \). The *Toeplitz operator* associated to the continuous function \( f : S^1 \to \text{Mat}_{2n}(\mathbb{C}) \) is the bounded operator \( T_f \) on \( \mathcal{H}_+ \) defined by the compression \( T_f := \Pi M_f \Pi^* \). As a matter of fact, if \( f \) vanishes nowhere, then \( T_f \) is a Fredholm operator [Mur, Corollary 3.5.12] with the associated Noether-Fredholm index [Mur, Section 1.4]

\[
\text{ind}(T_f) := \dim \ker(T_f) - \dim \ker(T_f^*) \in \mathbb{Z}.
\]

However, when \( T_f \) is skew-complex symmetric one automatically gets \( \text{ind}(T_f) = 0 \). In fact from \( (T_f)^* = -T_f \) one immediately infers the equality of the dimension of \( \ker(T_f) \) and \( \ker(T_f^*) \). Therefore, to capture the topology of skew-complex symmetric Toeplitz operators one needs a secondary index. Following [AS, DS, SB], we will consider the \( \mathbb{Z}_2 \)-(*Fredholm index*) defined by

\[
\text{ind}_{\mathbb{Z}_2}(T_f) := (-1)^{\dim \ker(T_f)} \in \mathbb{Z}_2.
\]  

This index turns out to be a deformation invariant for a skew-complex symmetric Fredholm operator. To justify the latter claim, let us consider the unitary operator on \( L^2(S^1) \otimes \mathbb{C}^{2n} \) defined by

\[
J := \mathbb{1}_{L^2} \otimes \begin{pmatrix} 0 & -\mathbb{1}_n \\ +\mathbb{1}_n & 0 \end{pmatrix}.
\]

One can check that \( J^{-1} = J^* = -J \), and \( K\Pi K = J \), namely \( J \) is *real skew-symmetric*. Moreover, \( J \) commutes with the Hardy projection \( \Pi \), and therefore it provides a real skew-symmetric operator also on the subspace \( \mathcal{H}_+ \). If \( A \) is skew-complex symmetric Fredholm operator then \( A' := JA \) satisfies the relation \( J^*(A')^*J = A' \), namely \( A' \) is an *odd symmetric operator* in the parlance of [DS, SB]. In view of the bijection induced by the mapping \( A \mapsto JA \), and the fact that the dimension of the kernels of \( A \) and \( JA \) are the same, we can use [SB, Theorem 2] to state:

**Lemma 3.11.** The set of skew-complex symmetric Toeplitz operators on the Hilbert space \( \mathcal{H}_+ \) is the disjoint union of two open and connected components labelled by \( \text{ind}_{\mathbb{Z}_2} \).

Therefore, as a consequence of Lemma 3.10 and Lemma 3.11, it turns out that the \( \mathbb{Z}_2 \)-index is the correct invariant to detect the topology of the Toeplitz operator \( T_q := \Pi M_q \Pi^* \) associated to a sewing matrix \( q : S^1 \to \mathbb{U}(2n) \). Since a homotopy of sewing matrices induces a homotopy of the associated Toeplitz operators in the space of skew-complex
symmetric Fredholm operators it follows that \( \text{ind}_{\mathbb{Z}_2}(T_q) \) provides a homotopy invariant of the sewing matrix \( q \). The next result shows that this invariant coincides with \( \nu_q \).

**Theorem 3.12 (Index theorem).** Let \( q : S^1 \to U(2n) \) be a map defined on the involutive space \((S^1, \iota)\) which satisfies the sewing matrix condition
\[
q(\iota(k)) = -q(k)^t, \quad \forall k \in S^1.
\]
Then, it holds true that
\[
\nu_q = \text{ind}_{\mathbb{Z}_2}(T_q).
\]
In particular, this implies that
\[
\frac{\text{Pf}[q(\pi)]}{\text{Pf}[q(0)]} \cdot \frac{\det[q(\pi)]^{\frac{1}{2}}}{\det[q(0)]^{\frac{1}{2}}} = (-1)^{\dim \ker(T_q)}.
\]
where the branch \( k \mapsto \det[q(k)]^{\frac{1}{2}} \) must be chosen continuously between the two fixed points \( k = 0, \pi \).

**Proof.** It is enough to check the coincidence of the invariants for a representative of each of the two homotopy classes of sewing matrices (cf. Theorem 3.8). One homotopy class is represented by the constant map \( q_+(k) = Q \) for all \( k \in S^1 \), with \( Q \) defined by (3.3). An immediate application of Theorem 3.5 provides \( \nu_{q_+} = +1 \). On the other hand, since \( M_{q_0} \) is the multiplication operator by a constant invertible matrix, it follows that \( T_{q_+} := \Pi M_{q_+} \Pi^* \) is invertible in \( \mathcal{H}_+ \). As a consequence the dimension of \( \ker(T_{q_+}) \) is zero, and in turn \( \text{ind}_{\mathbb{Z}_2}(T_{q_+}) = +1 \). This shows that \( \nu_{q_+} = \text{ind}_{\mathbb{Z}_2}(T_{q_+}) \). The second homotopy class can be represented by
\[
q_-(k) := \begin{pmatrix}
0 & +e^{i k} & 0 & 0 \\
-e^{-i k} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & +1 & 0
\end{pmatrix}, \quad k \in S^1.
\]
Indeed, from
\[
\text{Pf}[q_-(k)] = \text{Pf} \left[ \begin{pmatrix}
0 & +e^{i k} \\
-e^{-i k} & 0
\end{pmatrix} \right] \text{Pf} \left[ \begin{pmatrix}
0 & -1 \\
+1 & 0
\end{pmatrix} \right], \quad k = 0, \pi
\]
one gets that
\[
\text{Pf}[q_-(0)] = (-1)^{\frac{n(n-1)}{2}} = -\text{Pf}[q_-(\pi)],
\]
and in turn one has \( \nu_{q_-} = -1 \) as a consequence of Theorem 3.5 and the constancy of the determinant
\[
\det[q_-(k)] = \det \left[ \begin{pmatrix}
0 & -1 \\
+1 & 0
\end{pmatrix} \right] = (-1)^{n-1}, \quad \forall k \in S^1.
\]
To compute the kernel of $T_q$, let us consider the orthogonal decomposition $\mathcal{H}_+ = \mathcal{H}_{+,1} \oplus \mathcal{H}_{+,2}$ induced by the decomposition

$$\mathbb{C}^{2n} = \mathbb{C}[e_1, e_2] \oplus \mathbb{C}[e_3, \ldots, e_{2n}] \simeq \mathbb{C}^2 \oplus \mathbb{C}^{2(n-1)}$$

where $\{e_1, \ldots, e_{2n}\}$ denotes the canonical basis of $\mathbb{C}^{2n}$. Evidently $T_q$ is injective on $\mathcal{H}_{+,2}$. On the other hand, a generic element of $\psi \in \mathcal{H}_{+,1}$ can be represented in terms of the Fourier modes $\phi_m$ as

$$\psi = \left( \sum_{m \in \mathbb{N}} a_{m,1} \phi_m \otimes e_1 \right) + \left( \sum_{m \in \mathbb{N}} a_{m,2} \phi_m \otimes e_2 \right), \quad a_{m,\ell} \in \mathbb{C}, \quad \ell = 1, 2$$

and a direct computation provides

$$M_q \Pi^* \psi = \left( \sum_{m \in \mathbb{N}} -a_{m,1} \phi_{m-1} \otimes e_2 \right) + \left( \sum_{m \in \mathbb{N}} a_{m,2} \phi_{m+1} \otimes e_1 \right).$$

Therefore, one gets that the equation $T_q \psi$ has the non-trivial solution $\psi = a(\phi_1 \otimes e_1), \ a \in \mathbb{C}$, and as a consequence the dimension of $\text{Ker}(T_q)$ is one. This provides

$$\text{ind}_{\mathbb{Z}_2}(T_q) = -1$$

and in turn

$$\nu_q = \text{ind}_{\mathbb{Z}_2}(T_q).$$

This concludes the proof. □

Remark 3.13 (The bulk-edge correspondence). In the jargon of the theory of topological insulator an index formula of the type of Theorem 3.12 is referred as bulk-edge correspondence [PS]. In fact the topological invariant $\nu_q$ is derived by the data concerning the system in the bulk, while the index $\text{ind}_{\mathbb{Z}_2}(T_q)$ is computed by means of a Toeplitz operator which encodes the existence of an edge. ◀

Remark 3.14 (The interpretation in K-theory). By combining the map $A \mapsto IA$ introduced above with the argument used in the proof of [SB, Theorem 2] one obtain a homeomorphism between the set of skew-complex symmetric Toeplitz operators on the Hilbert space $\mathcal{H}_+$ and the classifying space $\mathcal{F}^2(\mathcal{H}_\mathbb{R})$ introduced in [AS]. As a consequence the K-group $\text{KR}^{-2}(\ast)$ can be identified with the set of homotopy class of skew-complex symmetric Fredholm operators, and the $\mathbb{Z}_2$-index (3.4) realizes the isomorphism $\text{KR}^{-2}(\ast) \simeq \mathbb{Z}_2$. The construction of the skew-complex symmetric Fredholm operator $T_q$ from $q$ extends to a homomorphism of KR-theory

$$T : \text{KR}^{-3}(S^1, \iota) \longrightarrow \text{KR}^{-2}(\ast),$$

which is bijective by the results obtained so far. Notice that the push-forward

$$\pi_* : \text{KR}^{-3}(S^1, \iota) \longrightarrow \text{KR}^{-2}(\ast)$$

along the projection $\pi : (S^1, \iota) \rightarrow \ast$ is also a bijection. Thus, we have $\pi_* = T$ which means that the construction with the Toeplitz operator realizes the push-forward. ◀

3.4. The relation with the KR-theory. In this section we will prove that in the one-dimensional case the homomorphism (1.9) is indeed an isomorphism. In Lemma 3.1 it
has been proved that
\[ H^1(S^1, \mathbb{Z})/H^1_{Z_2}(S^1|S^1|, Z(1)) \simeq Z_2. \]
On the other hand, by using formula (B.4) and Table B.1 in [DG1] one gets
\[ \text{KR}^{-3}(S^1, \iota) \simeq \text{KR}^{-3}(\ast) \oplus \text{KR}^{-2}(\ast) \]
\[ \simeq 0 \oplus Z_2 \simeq Z_2. \]

Theorem 2.11 and Remark 2.12 establish the existence of the well-defined homomorphism
\[ \nu : \text{KR}^{-3}(S^1, \iota) \longrightarrow H^1(S^1, \mathbb{Z})/H^1_{Z_2}(S^1|S^1|, Z(1)). \]

**Proposition 3.15.** The homomorphism (3.5) is an isomorphism.

**Proof.** From the computations above one has that the homomorphism (3.5) has the form
\[ \nu : Z_2 \rightarrow Z_2. \] Moreover, in the proof of Theorem 3.12 we construct two sewing matrices \( q \pm \) such that \( \nu q_\pm = \pm 1 \). Let \( [q_\pm] \in \text{KR}^{-3}(S^1, \iota) \) be the class of the band insulators represented by the sewing matrix \( q_\pm \). Since the homomorphism (3.5) corresponds to \( \nu : [q_\pm] = \nu q_\pm \) one gets the result. \( \square \)

### 3.5. The relation with the classification of “Real” gerbes

The aim of this section is to clarify the relation between the invariant \( \nu \) and a gerbe invariant introduced in [GT].

Let us first introduce the gerbe invariant in a way slightly different from the original definition in [GT]. First of all, let us recall that a gerbe \( \mathcal{G} \) over the space \( SU(2n) \) is classified by its Dixmier-Douady class
\[ DD[\mathcal{G}] \in H^3(SU(2n), \mathbb{Z}) \simeq Z. \]

The basic gerbe \( \mathcal{G} \) over \( SU(2n) \) is a representative of the class \( DD[\mathcal{G}] \simeq 1 \) which generates \( H^3(SU(2n), \mathbb{Z}) \). The basic gerbe \( \mathcal{G} \) admits only two “Real” structures (up to stable equivalences) over the involutive space \( (SU(2n), \theta) \). We will denote with \( \mathcal{G}_+ \) and \( \mathcal{G}_- \) the basic gerbe equipped with the two different “Real” structures. The two “Real” basic gerbes are classified by their “Real” Dixmier-Douady class
\[ DD_{R}[\mathcal{G}_\pm] \in H^3_{Z_2}(SU(2n), Z(1)) \]

and the two classes differ by an element of in \( H^3_{Z_2}(\ast, Z(1)) \subset H^3_{Z_2}(SU(2n), Z(1)) \), where \( \ast \in SU(2n) \) is any fixed point under the involution \( \theta \). As a result, we get a well-defined element
\[ DD_{R}[\mathcal{G}_\pm] \in H^3_{Z_2}(SU(2n), Z(1))/H^3_{Z_2}(\ast, Z(1)), \]

independently of the choice of the “Real” structures on \( \mathcal{G} \), as well as of the fixed point \( \ast \in SU(2n) \).

Let \( (X, \tau) \) be an involutive space which admits a fixed point \( \ast \in X \), and \( q : X \rightarrow SU(2n) \) an equivariant map. Such a map induces by pull-back the homomorphism
\[ q^* : H^3_{Z_2}(SU(2n), Z(1))/H^3_{Z_2}(\ast, Z(1)) \longrightarrow H^3_{Z_2}(X, Z(1))/H^3_{Z_2}(\ast, Z(1)). \]
The gerbe invariant of the map \( q : X \to SU(2n) \) is by definition the pull-back
\[
\dd_{R}[q] := q^* \bar{D}D_{R}[\mathcal{G}_\pm].
\]
This is an invariant of the equivariant homotopy class of \( q \). Therefore, \( \dd_{R} \) provides a well-defined map (still denoted with the same symbol)
\[
\dd_{R} : \left[ X, SU(2n) \right]_{Z_2} \longrightarrow H^3_{Z_2}(X, \mathbb{Z}(1))/H^3_{Z_2}(\ast, \mathbb{Z}(1)).
\]
The invariants \( \dd_{R}[q] \) and \( \nu_{q} \) take values in different groups, and therefore we cannot generally compare these two invariants. However, in the case of the involutive space \( (S^1, t) \), one has that
\[
H^3_{Z_2}(S^1, \mathbb{Z}(1))/H^3_{Z_2}(\ast, \mathbb{Z}(1)) \cong Z_2
\]
in view of
\[
H^3_{Z_2}(S^1, \mathbb{Z}(1)) \cong H^3_{Z_2}(S^1, \mathbb{Z}(1)) \oplus H^3_{Z_2}(\ast, \mathbb{Z}(1)) \cong Z_2 \oplus Z_2
\]
and \( H^3_{Z_2}(\ast, \mathbb{Z}(1)) \cong Z_2 \). This allows us to regard \( \dd_{R}[q] \in Z_2 \). Since also \( \nu_{q} \in Z_2 \) we can compare the two \( Z_2 \)-invariants \( \dd_{R}[q] \) and \( \nu_{q} \) for equivariant maps \( q : S^1 \to SU(2n) \) over the involutive circle \( (S^1, t) \).

**Theorem 3.16 (Gerbe invariant).** Let \( q : S^1 \to SU(2n) \) be an equivariant map defined on the involutive space \( (S^1, t) \) satisfying the sewing matrix condition
\[
q(t(k)) = -q(k)^t, \quad \forall \; k \in S^1.
\]
Then, it holds true that
\[
\nu_{q} = \dd_{R}[q].
\]

**Proof.** The proof is similar to that of Theorem 3.12. By Lemma 3.6 and Theorem 3.8, the invariant \( \nu \) induces a bijection \( [S^1, SU(2n)]_{Z_2} \cong Z_2 \). In the proof of Theorem 3.12, we constructed two equivariant maps \( q_{\pm} : S^1 \to SU(2n) \) such that \( \nu_{q_{\pm}} = \pm 1 \). Therefore, the claim is proved by showing that \( \dd_{R}[q_{\pm}] = \pm 1 \). For the computation, we make use of an expression of \( \dd_{R}[q_{\pm}] \in Z_2 \) in terms of the sign invariant of “Real” gerbes [GT, Definition 4.4]. In general, for a “Real” gerbe \( \mathcal{G} \) on an involutive space \( (X, \tau) \) and a path connected subspace \( Y = Y^\tau \subset X^\tau \) in the fixed point set, there is a well-defined sign \( \sigma(\mathcal{G}, Y) \in Z_2 \). Actually, this is the component in \( H^0(Y, Z_2) \cong Z_2 \) of the “Real” Dixmier-Douady class \( \bar{D}D_{R}(\mathcal{G}|_{Y}) \in H^3_{Z_2}(Y, \mathbb{Z}(1)) \) of the restricted gerbe \( \mathcal{G}|_{Y} \) with respect to the decomposition
\[
H^3_{Z_2}(Y, \mathbb{Z}(1)) \cong H^0(Y, Z_2) \oplus H^2(Y, Z_2)
\]
as shown in [GT, Lemma B.1]. The fixed point set of the involutive space \( (S^1, t) \) consists of the points \( k_+ = 0 \) and \( k_- = \pi \). Let \( \mathcal{G}_{\pm} \) be the two “Real” basic gerbes over \( SU(2n) \) and \( q^*\mathcal{G}_{\pm} \) the pull-back gerbes over \( S^1 \). In view of [GT, Corollary 4.7] one has that
\[
q^*\bar{D}D_{R}(\mathcal{G}_{\pm}) = \bar{D}D_{R}(q^*\mathcal{G}_{\pm}) = \sigma(q^*\mathcal{G}_{\pm}, [k_+]) \sigma(q^*\mathcal{G}_{\pm}, [k_-])
\]
\[
= \sigma(\mathcal{G}_{\pm}, [q(k_+)]) \sigma(\mathcal{G}_{\pm}, [q(k_-)])
\]
is computed as a product of signs. Since the right-hand side of the equation above is not affected to the passage to the quotient which defines the map $d d_R$, one gets

$$\sigma(G_{±}(q(k_{±})) \in \mathbb{Z}_2$$

where $G_{±}$ stands irrelevantly for $G_+$ or $G_-$. As seen in Lemma 3.7, the fixed point set of $(\text{SU}(2n), 0)$ has two connected components. As it is pointed out in [GT, Remark 5.7], the sign invariant of $G_{±}$ is constant on the connected components of $\text{SU}(2n)$ and differs on each of the two components. As a result, $d d_R[q] = +1$ if and only if $q(k_{±})$ are in the same connected component and $d d_R[q] = -1$ if and only if $q(k_{±})$ are in different connected components. This immediately proves that $d d_R[q_{±}] = \pm 1$ and the proof is over. □

4. THE TWO-DIMENSIONAL CASE

The analysis of physical systems shows that there are (at least) two interesting definitions for two-dimensional band insulators of class DIII which differ for the involutive base space. By borrowing the discussion in [DG1, Section 2], we can distinguish between the Dirac (or free) case and the Bloch (or periodic) case. In the Dirac case the involutive space is $(S^2, \iota)$ where

$$S^2 := \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}$$

is the two-dimensional sphere with involution given by

$$\iota(x_0, x_1, x_2) := (x_0, -x_1, -x_2), \quad \forall (x_0, x_1, x_2) \in S^2.$$ 

In the Bloch case the involutive space is $(T^2, \iota)$ where

$$T^2 := S^1 \times S^1 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$$

is the two-dimensional torus with involution given by

$$\iota(k_1, k_2) := (-k_1, -k_2), \quad \forall (k_1, k_2) \in T^2.$$ 

In short, one has that $(T^2, \iota) = (S^1, \iota) \times (S^1, \iota)$ is given by two copies of the involutive space for the one-dimensional case.

According to the terminology currently used in the theory of topological insulators, the topological phases detected in the Dirac case are called strong. Usually, the Bloch case present a richer family of topological phases. Some of these are inherited from the Dirac case via the pull-back of the equivariant map $\pi_0 : (T^2, \iota) \rightarrow (S^2, \iota)$ described in [DG2, eq. (4.14)] and are still called strong. The remaining topological phases are known as weak.

It is worth noting that both spaces $(S^2, \iota)$ and $(T^2, \iota)$ meet the properties (a) and (b) listed in Assumption 2.5. However, the isomorphisms

$$H_{S^2}(S^2|S^2, Z(1)) \simeq Z_2 \simeq H_{T^2}(T^2|T^2, Z(1)),$$
and the bijectivity of the FKMM invariant [DG2, Theorem 1.2], imply that the property (c) of Assumption 2.5 is not automatically guaranteed. This fact makes the analysis of the two-dimensional case slightly more complicated than the one-dimensional case.

4.1. **Strong invariant: the Dirac case.** Let us start by observing that for a two-dimensional sphere it holds true that $H^1(S^2, \mathbb{Z}) = 0$ which immediately implies that $H^1(S^2, \mathbb{Z})/H^1_S(S^2|S^2)^t, \mathbb{Z}(1)) = 0$.

Therefore, in the Dirac case the map $\nu$ in (1.9) amounts to the trivial homomorphism, and for this reason it is unable to capture the topology of the related band insulator of class DIII.

On the other hand, from equations (B.3) and (B.9) and Tables B.1 and B.3 in [DG1] one gets

$$KR^{-3}(S^2, \iota) \simeq \tilde{KR}^{-3}(S^2, \iota) \oplus KR^{-3}(\ast) \simeq \tilde{KR}^0(S^6, \iota) \oplus 0 \simeq \mathbb{Z}_2.$$  

The latter equation shows that there is a non-trivial topology for band insulators of class DIII over $(S^2, \iota)$.

The topology of $KR^{-3}(S^2, \iota)$ can be detected via the FKMM invariant by using the map $\kappa$ in (2.13). First of all let us observe that

$$\text{Vec}^{2n}_{\text{Q},0}(S^2, \iota) \simeq \text{Vec}^2_{\text{Q}}(S^2, \iota) \simeq H^2_{\mathbb{Z}_2}(S^2|S^2)^t, \mathbb{Z}(1)) \simeq \mathbb{Z}_2.$$  

The first isomorphism is a consequence of [DG2, Proposition 4.1] which provides the equality $\text{Vec}^{2n}_{\text{Q},0}(S^2, \iota) = \text{Vec}^2_{\text{Q}}(S^2, \iota)$ (i.e. every “Quaternionic” vector bundle over $(S^2, \iota)$ can be built over a product vector bundle) and [DG2, Corollary 2.1] which describes the stable range for low-dimensional “Quaternionic” vector bundles. The second isomorphism is provided by the FKMM invariant [DG2, Theorem 1.2]. With this information we can prove the following result:

**Proposition 4.1** (Two-dimensional strong invariant). The homomorphism

$$\kappa : KR^{-3}(S^2, \iota) \rightarrow H^2_{\mathbb{Z}_2}(S^2|S^2)^t, \mathbb{Z}(1))$$

described in Theorem 2.11 is indeed an isomorphism.

**Proof.** In view of Theorem 2.11 and the results showed above we are considering a well-defined homomorphism of the form $\kappa : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. Therefore, to prove the claim it is enough to show that $\kappa$ is surjective. Let us use the additive representation of $\mathbb{Z}_2 \simeq \mathbb{Z}/2\mathbb{Z}$. The two distinct classes in $\text{Vec}^2_{\text{Q}}(S^2, \iota) \simeq \mathbb{Z}_2$ can be represented by the “Quaternionic” vector bundles $E_{q_0}$ and $E_{q_1}$, obtained by endowing the product bundles $S^2 \times \mathbb{C}^2$ with two inequivalent “Quaternionic” structures $\Theta_{q_0}$ and $\Theta_{q_1}$ associated to two sewing matrices $q_0, q_1 : S^2 \rightarrow \mathcal{U}(2)$. Let $q_0$ be the constant map defined as in Theorem 2.8 (2). Then, $E_{q_0}$ turns out to be the trivial “Quaternionic” vector bundle with FKMM invariant $\kappa(E_{q_0}) = 0$. 
An explicit realization for the sewing matrix $q_1$ is described in [DG2, eq. (4.9)]. In this case $\delta_{q_1}$ provides a representative for the non trivial “Quaternionic” class and $\kappa(\delta_{q_1}) = 1$. Now, consider the Hamiltonians $H_i : S^2 \to \mathfrak{f} \text{erm}(\mathbb{C}^4)^\times$ given by

$$H_i(x) := \begin{pmatrix} 0 & q_i(x)^{-1} \\ q_i(x) & 0 \end{pmatrix}, \quad x \in S^2, \quad i = 0, 1.$$  

Then $[S^2 \times \mathbb{C}^4, H_1, H_0]$ is a well-defined class in $KR^{-3}(S^2, t)$ and by equation (2.14) one gets $\kappa([S^2 \times \mathbb{C}^4, H_1, H_0]) = 1$. This concludes the proof. \hfill \Box

Let $H : S^2 \to \mathfrak{f} \text{erm}(\mathbb{C}^4)^\times$ be a band insulator of class DIII over $(S^2, t)$ represented in a standard way. Denote with $H_0$ the trivial insulator constructed with the constant sewing matrix $q_0 : S^2 \to U(2n)$ described in item (2) of Theorem 2.8. Then

$$\nu^a(H) := \kappa([S^2 \times \mathbb{C}^4, H, H_0])$$

will be called the strong invariant of $H$. This is nothing more than the FKMM invariant $\kappa(\delta_{q_1})$ of the “Quaternionic” vector bundle associated to $H$ with the sewing matrix $q_1$. In view of Proposition 4.1 this invariant totally captures the topology of two-dimensional band insulators of class DIII in the Dirac case.

4.2. Weak-strong invariant: the Bloch case. In order to study the classification of two-dimensional band insulator $H$ of class DIII in the Bloch case we will specialize Theorem 2.11 for the involutive space $(T^2, t)$.

By using [DG1, eq. (B.7)] one gets

$$KR^{-3}(T^2, t) \simeq KR^{-3}(_) \oplus (KR^{-2}(_*))^{\oplus 2} \oplus KR^{-1}(_) \quad (4.1)$$

On the other hand, the product structure $(T^2, t) := (S^1, t) \times (S^1, t)$ and the use of the Künneth formula, provide

$$H^1(T^2, \mathbb{Z})/H^1_{Z_2}(T^2(T^2)^t, Z(1)) \simeq \left( H^1(S^1, \mathbb{Z})/H^1_{Z_2}(S^1(S^1)^t, Z(1)) \right)^{\oplus 2} \quad (4.2)$$

Finally, from [DG2, Theorem 1.2] one knows that

$$\text{Vec}_{Z_2}^2(T^2, t) \simeq \text{Vec}_{Z_2}^2(T^2, t) \simeq H^2_{Z_2}(S^1(S^1)^t, Z(1)) \simeq Z_2.$$ 

Lemma 4.2. The homomorphism

$$\kappa : KR^{-3}(T^2, t) \rightarrow H^2_{Z_2}(T^2(T^2)^t, Z(1))$$

described in Theorem 2.11 is surjective.

Proof. The proof follows exactly as in Proposition 4.1 by constructing a class $[T^2 \times \mathbb{C}^4, H_1, H_0]$ such that $\kappa([T^2 \times \mathbb{C}^4, H_1, H_0]) = 1$. In this case the Hamiltonians $H_i$
and $H'_0$ can be obtained from the Hamiltonians $H_1$ and $H_0$ introduced in the proof of Proposition 4.1 via the pull-back under the equivariant map $\pi_0 : (T^2, \iota) \to ([S^2, \iota]$ described in [DG2, eq. (4.14)]. More precisely one has that $H'_i(k) := H_i(\pi_0(k))$ for $i = 0, 1.$ This concludes the proof. 

Let $H : T^2 \to \mathfrak{h} \text{erm}(\mathbb{C}^{4n}) \times$ be a band insulator of class DIII over $(T^2, \iota)$ represented in a standard way. As in the Dirac case we can define the strong invariant
\[
\nu^s(H) := \kappa([T^2 \times \mathbb{C}^{4n}, H, H'_0])
\]
(4.3)
where $H_0$ is the trivial insulator constructed with the constant sewing matrix $q_0 : T^2 \to \mathbb{U}(2n)$ described in item (2) of Theorem 2.8. Since this invariant coincides with the FKMM invariant $\kappa(\iota, H)$ of the “Quaternionic” vector bundle associated to $H$ with the sewing matrix $q_{H_1},$ one can use the Fu–Kane–Mele index proved in [DG2, Theorem 4.2] to compute $\nu.$

**Proposition 4.3** (Fu–Kane–Mele formula for the strong invariant). The strong invariant $\nu^s$ defined by (4.3) can be computed by the formula
\[
\nu^s(H) = \prod_{k \in (T^2), i} \frac{\text{Pf}[q_{H_1}(k)]}{\text{det}[q_{H_1}(k)]^\frac{1}{2}} \in \{\pm 1\}
\]
(4.4)
where the branch $k \mapsto \text{det}[q_{H_1}(k)]^\frac{1}{2}$ must be chosen continuously and that $\nu^s$ must be understood as a map with values into the multiplicative group $\mathbb{Z}_2 \simeq \{\pm 1\}.$

**Proof.** According to [DG2, Theorem 4.2] one has that
\[
\nu^s(H) = \prod_{k \in (T^2), i} \frac{\text{Pf}[q_{H_1}(k)]}{p_{H_1}(k)}
\]
(4.5)
where $p_{H_1} : T^2 \to \mathbb{U}(1)$ meets $\text{det}[q_{H_1}(k)] = p_{H_1}(\iota(k))p_{H_1}(k)$ for all $k \in T^2.$ To obtain (4.4) from (4.5) we need to show that one can replace $p_{H_1}$ with a map $p'_{H_1} : T^2 \to \mathbb{U}(1)$ such that $\text{det}[q_{H_1}(k)] = p'_{H_1}(k)^2$ for all $k \in T^2.$ The following fact follows by observing that every continuous map $s : T^2 \to \mathbb{U}(1)$ such that $s(k)^2 = \text{det}[q_{H_1}(k)]$ automatically satisfies $s(k) = s(\iota(k))$ for all $k \in T^2.$ The key to the proof is that every continuous map $f : T^2 \to \mathbb{U}(1)$ can be uniquely expressed as
\[
f(k_1, k_2) = e^{i(\alpha + f(k_1, k_2) + n_1k_1 + n_2k_2)}, \quad (k_1, k_2) \in T^2
\]
where $n_1, n_2 \in \mathbb{Z}, \alpha \in \mathbb{R}/2\pi\mathbb{Z}$ and $\tilde{f} : T^2 \to \mathbb{R}$ is a continuous function such that $\tilde{f}(0, 0) = 0.$ The integers $n_1$ and $n_2$ provide the winding numbers of $f$ around the two standard homology cycles of $T^2$ and $\alpha$ is determined by $f(0, 0) = e^{i\alpha}.$ By applying the same representation to $\text{det}[q_{H_1}]$ one gets
\[
\text{det}[q_{H_1}(k_1, k_2)] = e^{i(\alpha + f_{H_1}(k_1, k_2) + n_1k_1 + n_2k_2)}, \quad (k_1, k_2) \in T^2
\]
By the invariance $\text{det}[q_{H_1}(k_1, k_2)] = \text{det}[q_{H_1}(-k_1, -k_2)]$, and the continuity of $f_{H_1}$ it follows that $n_1 = n_2 = 0$ and $f_{H_1}(k_1, k_2) = f_{H_1}(-k_1, -k_2).$ As a result, every continuous
function whose square coincides with \( \det[q_{H1}] \) must be of the form
\[
s(k_1, k_2) = \pm e^{i\left(\alpha_s + \frac{1}{2}f(k_1, k_2)\right)}
\]
with \( \alpha_s = \frac{\alpha_4}{2} \in \mathbb{R}/2\pi\mathbb{Z} \). In particular one gets that \( s(-k_1, -k_2) = s(k_1, k_2) \) as anticipated. \( \square \)

It is worth noting equation (4.4) agrees with formula \([\text{CTSR}, \text{eq. (3.69)}]\).

From Lemma 4.2 and (4.1) one infers that
\[
\text{Ker}(\kappa) \simeq \text{KR}^{-3}(T^2, t)/Z_2 = Z_2 \oplus Z_2 .
\]

Theorem 2.11 establishes the existence of the homomorphism \( \nu : \text{Ker}(\kappa) \to H^1(T^2, \mathbb{Z})/H^1_{Z_2}(\mathbb{T}^2, \mathbb{Z}(1)) \) (4.6)
which is of the form \( \nu : Z_2 \oplus Z_2 \to Z_2 \oplus Z_2 \) in view of (4.2). This homomorphism turns out to be surjective, and therefore bijective.

**Lemma 4.4.** The homomorphism (4.6) described in Theorem 2.11 is an isomorphism.

Lemma 4.4 follows as a special case of Proposition 4.5 below. For that, let us introduce the equivariant inclusions
\[
\phi_j : (S^1, t) \hookrightarrow (T^2, t) , \quad j = 1, 2
\]
defined by \( \phi_1(k) = (k, 1) \) and \( \phi_2(k) = (1, k) \). To every class \( [T^2 \times \mathbb{C}^2, H_0, H_1] \in \text{KR}^{-3}(T^2, t) \) we can associate via pull-back two classes
\[
[T^2 \times \mathbb{C}^2, H_0, H_1] : = [S^1 \times \mathbb{C}^2, \phi_j^*H_0, \phi_j^*H_1] \in \text{KR}^{-3}(S^1, t) , \quad j = 1, 2
\]
where \( \phi_j^*H_i(k) = H_i(\phi_j(k)) \) for all \( k \in S^1 \), \( j = 1, 2 \) and \( i = 0, 1 \). In view of Proposition 3.15 one can introduce the homomorphism
\[
\nu^w : \text{KR}^{-3}(T^2, t) \to \left(H^1(S^1, \mathbb{Z})/H^1_{Z_2}(S^1, \mathbb{Z}(1))\right)^{\oplus 2}
\]
given by
\[
\nu^w : [T^2 \times \mathbb{C}^2, H_0, H_1] \mapsto \left(\nu([T^2 \times \mathbb{C}^2, H_0, H_1]), \nu([T^2 \times \mathbb{C}^2, H_0, H_1]_{\mathbb{Z}})\right).
\]
Notice that \( \nu^w \) is well-defined on the whole group \( \text{KR}^{-3}(T^2, t) \) and not only on \( \text{Ker}(\kappa) \).

We are now in position to provide a complete classification of two-dimensional band insulators of class DIII in the Bloch case.

**Proposition 4.5** (Two-dimensional weak-strong invariant). The homomorphism \( \nu^w \) extends the homomorphism \( \nu \). Moreover the composition \( \underline{\nu} : = (\nu^w, \kappa) \) of the homomorphisms \( \nu^w \) and \( \kappa \) provides the isomorphism
\[
\underline{\nu} : \text{KR}^{-3}(T^2, t) \to H^1(T^2, \mathbb{Z})/H^1_{Z_2}(\mathbb{T}^2, \mathbb{Z}(1)) \oplus H^2_{Z_2}(\mathbb{T}^2, \mathbb{Z}(1)) .
\]

**Proof.** The proof is based on the direct calculations. For that let us construct four sewing matrices \( q_{w1}, q_{w2}, q_s, q_0 \) on \( T^2 \) with values on \( U(2) \) as described below. Let \( \pi_1 \) and
$\pi_2$ be the two canonical projections from $T^2 \simeq S^1 \times S^1$ to $S^1$. Then, $q_{w_1} := q_\cdot \circ \pi_j$, for $j = 1, 2$, where $q_\cdot : S^1 \to \mathbb{U}(2)$ is the sewing matrix introduced in the proof of Theorem 3.12. The sewing matrix $q_s := q_1 \circ \pi_0$ is the composition of the equivariant map $\pi_0 : T^2 \to S^2$ in the proof of Lemma 4.2 and the sewing matrix $q_1 : S^2 \to \mathbb{U}(2)$ such that $\kappa(q_1) = 1$ in Proposition 4.1. Finally, $q_0$ is the constant map as in Theorem 2.8 (2). These sewing matrices produce the following FKMM invariants:

$$\kappa(q_{w_1}) = +1, \quad \kappa(q_{w_2}) = +1, \quad \kappa(q_0) = +1, \quad \kappa(q_i) = -1.$$  

In addition, one has that

$$\nu_{q_{w_1} \circ \phi_1} = -1, \quad \nu_{q_{w_2} \circ \phi_1} = +1, \quad \nu_{q_0 \circ \phi_1} = +1, \quad \nu_{q_i \circ \phi_1} = +1,$$

$$\nu_{q_{w_1} \circ \phi_2} = +1, \quad \nu_{q_{w_2} \circ \phi_2} = -1, \quad \nu_{q_0 \circ \phi_2} = +1, \quad \nu_{q_i \circ \phi_2} = +1.$$  

In the above equations the computations for the cases $q_{w_1} \circ \phi_i = q_\cdot \circ (\pi_j \circ \phi_i)$ follows just by observing that $\pi_i \circ \phi_i$ is the identity map on $S^1$ when $i = j$, and the restriction to a fixed point if $i \neq j$. The cases $q_0 \circ \phi_i$ with $i = 1, 2$, are also trivial in view of the fact that $q_0$ is the constant map. For the last two cases $q_s \circ \phi_i$, with $i = 1, 2$, we notice that the equivariant map $\pi_0 : T^2 \to S^2$ carries the three fixed points $(0, 0), (0, \pi_1), (\pi, 0)$ to $(1, 0, 0)$ and the fixed point $(\pi, \pi)$ to $(-1, 0, 0)$. We also notice that $q_1$ takes values in $S \mathbb{U}(2) \subset \mathbb{U}(2)$. Since $q_s \circ \phi_i : S^1 \to S \mathbb{U}(2)$ carries the fixed points of $S^1$ to the same point of $S \mathbb{U}(2)$, Theorem 3.5 leads to the results above. Now, let us denote with $H_{w_1}, H_{w_2}, H_s$, and $H_0$ the band insulators of class DIII associated to $q_{w_1}, q_{w_2}, q_s$ and $q_0$, respectively. One gets that

$$\nu([T^2 \times \mathbb{C}^2, H_{w_1}, H_0]) = (-1, +1, +1),$$

$$\nu([T^2 \times \mathbb{C}^2, H_{w_2}, H_0]) = (+1, -1, +1),$$

$$\nu([T^2 \times \mathbb{C}^2, H_s, H_0]) = (+1, +1, -1),$$

so that $\nu$ is a surjection onto $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $KR^{-3}(T^2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the homomorphism $\nu$ must be an isomorphism. As a result, one obtains that $\text{Ker}(\kappa)$ is generated by $[T^2 \times \mathbb{C}^2, H_{w_i}, H_0]$, with $i = 1, 2$. This allows us to see that $\nu^{-1}$ extends $\nu$. \hfill $\square$

The novelty of Proposition 4.5 is that, in combination with Theorem 3.5, it provides a way to compute also the weak invariants

$$\nu^{-1}(H) := (\nu_{q_{w_1} \tau}, \nu_{q_{w_2} \tau})$$

of the two-dimensional band insulator $H$ of class DIII over $(T^2, \tau)$.

APPENDIX A. EQUIVARIANT COHOMOLOGY AND HOMOTOPY

Let $(X, \tau)$ be an involutive space such that $X$ has the structure of a (path connected) $\mathbb{Z}_2$-CW complex. A map $f : X \to \mathbb{U}(1)$ is called invariant if $f(x) = f(\tau(x))$ for every $x \in X$. Similarly, it is called equivariant if $f(x) = f(\tau(x))$ for every $x \in X$. Let $[X, \mathbb{U}(1)]$ be the set of classes of homotopy equivalent functions, $[X, \mathbb{U}(1)]_{\text{inv}}$ the set of
classes of homotopy equivalent invariant functions and \([X, \mathbb{U}(1)]\) the set of classes of homotopy equivalent equivariant functions. These three sets admit a representation in terms of cohomology groups. In fact, it holds true that

\[
\begin{align*}
[X, \mathbb{U}(1)] & \simeq H^1(X, \mathbb{Z}) \\
[X, \mathbb{U}(1)]_{\text{inv}} & \simeq H^1_{\text{Z}_2}(X, \mathbb{Z}) \\
[X, \mathbb{U}(1)]_{\text{Z}_2} & \simeq H^1_{\text{Z}_2}(X, \mathbb{Z}(1))
\end{align*}
\]

(A.1)

where \(H^\ast(X, \mathbb{Z})\) is the usual cohomology ring of \(X\) with coefficients in \(\mathbb{Z}\), and \(H^\ast_{\text{Z}_2}(X, \mathbb{Z})\) and \(H^\ast_{\text{Z}_2}(X, \mathbb{Z}(1))\) are the equivariant cohomology groups of \(X\) with system of coefficients in \(\mathbb{Z}\) and \(\mathbb{Z}(1)\), respectively. For a review of the equivariant cohomology we refer to [DG1, Section 5] and references therein. The first isomorphism in (A.1) is a classical result in algebraic topology and follows by observing that \(\mathbb{U}(1) = K(\mathbb{Z}, 1)\) is an Eilenberg-Maclane space and in turn \([X, K(\mathbb{Z}, 1)]\) is a representing space for the cohomology \(H^1(X, \mathbb{Z})\) in view of [Hat, Theorem 4.57]. The proof of the second isomorphism is provided in [GT, Lemma A.3], while the third isomorphism is proved in [Gom, Proposition A.2].

**Appendix B. Strong equivalence of “Quaternionic” structures**

Let \(\text{Vec}^n_{\Theta}(X, \tau)\) be the set of equivalence classes of “Quaternionic” vector bundles of rank \(n\) over the involutive space \((X, \tau)\). The subset \(\text{Vec}^n_{\Theta,0}(X, \tau) \subseteq \text{Vec}^n_{\Theta}(X, \tau)\) consists of the equivalence classes of “Quaternionic” vector bundles whose underlying (complex) bundle structure is trivial. Let \(\mathcal{E} := (X \times \mathbb{C}^n, \Theta)\) and \(\mathcal{E}' := (X \times \mathbb{C}^n, \Theta')\) be two “Quaternionic” vector bundles defined on the trivial product bundle \(X \times \mathbb{C}^n\). The two “Quaternionic” structures \(\Theta\) and \(\Theta'\) are defined by two sewing matrices \(q\) and \(q'\) according to (2.8). The isomorphism condition \(\mathcal{E} \simeq \mathcal{E}'\) is equivalent to the existence of a continuous intertwining map \(\phi : X \to \mathbb{U}(n)\) such that

\[
q'(x) = \phi(\tau(x)) q(x) \phi(x), \quad \forall x \in X. \quad \text{(intertwining)} \quad \text{(B.1)}
\]

In this case \(\mathcal{E}\) and \(\mathcal{E}'\) are representatives of the same equivalence class \([\mathcal{E}] \in \text{Vec}^n_{\Theta,0}(X, \tau)\).

**Definition B.1** (Strong isomorphism). We will say that the isomorphism between \(\mathcal{E}\) and \(\mathcal{E}'\) is **strong** if the intertwining map (B.1) meets the extra condition

\[
\phi(x) = \phi(\tau(x)), \quad \forall x \in X. \quad \text{(\(\tau\)-invariance)} \quad \text{(B.2)}
\]

In this case we will write \(\mathcal{E} \overset{\phi}{\simeq} \mathcal{E}'\).

The notion of strong isomorphism introduced with Definition B.1 provides a finer classification of \(\text{Vec}^n_{\Theta,0}(X, \tau)\). The following example should clarify the difference between the two notions of isomorphism.

**Example B.2.** It is known that in the case of the involutive space \((S^1, \iota)\) one has that \(\text{Vec}^2_{\Theta,0}(S^1, \iota) = \text{Vec}^2_{\Theta}(S^1, \iota) = 0\) [DG2, Proposition 2.6] meaning that every rank two
“Quaternionic” vector bundle over \((S^1, \iota)\) is isomorphic to the trivial “Quaternionic” vector bundle \(E_0 := (S^1 \times \mathbb{C}^2, \Theta_0)\) with trivial structure \(\Theta_0\) induced by the (constant) sewing matrix
\[
q_0(k) \equiv q_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \forall k \in S^1.
\]
Consider the “Quaternionic” vector bundle \(E_0' := (S^1 \times \mathbb{C}^2, \Theta_0')\) with trivial structure \(\Theta_0'\) induced by the (constant) sewing matrix
\[
q_0'(k) \equiv -q_0, \quad \forall k \in S^1.
\]
The (constant) intertwining map
\[
\phi_0(k) \equiv \phi_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \forall k \in S^1.
\]
provides the isomorphism between \(E_0\) and \(E_0'\). Moreover, since the map \(\phi_0\) is constant, hence \(\tau\)-invariant, one has that the “Quaternionic” vector bundles are strongly equivalent, i.e. \(E_0 \cong E_0'\). On the other hand, one can consider the “Quaternionic” vector bundle \(E_1 := (S^1 \times \mathbb{C}^2, \Theta_1)\) with “Quaternionic” structure \(\Theta_1\) induced by the (non constant) sewing matrix
\[
q_1(k) := \begin{pmatrix} \sin(k) & -\cos(k) \\ \cos(k) & \sin(k) \end{pmatrix}, \quad \forall k \in S^1.
\]
In this case the isomorphism \(E_1 \simeq E_0\) is induced by the intertwining map
\[
\phi_1(k) := e^{i\frac{k}{2}} \begin{pmatrix} \sin\left(\frac{k}{2}\right) & -\cos\left(\frac{k}{2}\right) \\ \cos\left(\frac{k}{2}\right) & \sin\left(\frac{k}{2}\right) \end{pmatrix}, \quad \forall k \in S^1.
\]
Since \(\phi_1(-k) \neq \phi_1(k)\) it follows that the “Quaternionic” structures \(\Theta_0\) and \(\Theta_1\) are only equivalent but not strongly equivalent, i.e. \(E_1 \not\cong E_0\). ▶

In the next result we will use the definitions of unitary equivalence introduced in Definition 1.1. Let us observe that if \(V\) is a unitary equivalence between two band insulators of class DIII then the equation
\[
\chi V(x) = V(x) \chi
\]
holds true, independently of the nature of \(V\).

**Proposition B.3.** Unitarily equivalent band insulators of class DIII define associated “Quaternionic” vector bundles which are isomorphic. When the unitary equivalence is strong then the associated “Quaternionic” vector bundles are strongly isomorphic.

**Proof.** Let \(H\) and \(H'\) be two unitarily equivalent band insulators of class DIII over \((X, \tau)\). The functional calculus provides \(Q_{H'}(x) = V(x)Q_H(x)V(x)^*\) where the operators \(Q_{H'}\)
and $Q_H$ are defined according to (2.5). According to Lemma 2.2, in the standard representation described in Preposition 2.1 the unitary $V(x)$ takes the form

$$V(x) = \begin{pmatrix} \phi_V(x) & 0 \\ 0 & \phi_V(\tau(x)) \end{pmatrix},$$

(B.3)

for a suitable map $\phi_V : X \to \mathbb{U}(n)$. Then, the unitary equivalence between $Q_H$ and $Q_{H'}$ translates into

$$q_{H'}(x) = \overline{\phi_V(\tau(x))} q_H(x) \phi_V(x)^* = [\phi_V(\tau(x))^*]^t q_H(x) \phi_V(x)^*$$

where the function $q_H$ and $q_{H'}$ are defined according to the off-diagonal decomposition (2.6). The latter equation shows that the “Quaternionic” vector bundles associated to $H$ and $H'$ are isomorphic. In the case of a strong unitary equivalence between $H$ and $H'$, the condition $V(\tau(x)) = V(x)$ translates into $\phi_V(\tau(x)) = \phi_V(x)$ and in turn in the strong isomorphism between the associated “Quaternionic” vector bundles.

□

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