A STUDY ON NEW RIGHT/LEFT INVERSES OF NONSQUARE POLYNOMIAL MATRICES

WOJCIECH P. HUNEK, KRZYSZTOF J. LATAWIEC

Institute of Control and Computer Engineering
Opole University of Technology, ul. Sosnkowskiego 31, 45–272 Opole, Poland
e-mail: {w.hunek, k.latawiec}@po.opole.pl

This paper presents several new results on the inversion of full normal rank nonsquare polynomial matrices. New analytical right/left inverses of polynomial matrices are introduced, including the so-called $\tau$-inverses, $\sigma$-inverses and, in particular, $S$-inverses, the latter providing the most general tool for the design of various polynomial matrix inverses. The application-oriented problem of selecting stable inverses is also solved. Applications in inverse-model control, in particular robust minimum variance control, are exploited, and possible applications in signal transmission/recovery in various types of MIMO channels are indicated.

Keywords: multivariable systems, right/left inverses of polynomial matrices, Smith factorization, minimum variance control.

1. Introduction

Whilst the task of the Moore–Penrose inversion of polynomial matrices (or rational matrices) has attracted considerable research interest (Ben-Israel and Greville, 2003; Karanpetakis and Tzekis, 2001; Kon’kova and KUBLANOvskaya, 1996; Stanimirović, 2003; Stanimirović and Petković, 2006; Varga, 2001; Vologiannidis and Karampetakis, 2004; Zhang, 1989), the problem of right/left inverting nonsquare (full normal rank) polynomial matrices has not been given proper attention by the academia. The suggested control applications (Baška and Dworak, 2006; 2007; Chen and Hara, 2001; Ferreira, 1988; Hautus and Heymann, 1983; Quadrat, 2004; Trentelman et al., 2001; Williams and Antsaklis, 1996) have not ended up with algorithms for obtaining right/left polynomial matrix inverses and their quantification. One possible reason could be an infinite number of solutions to the problem, the ambiguity impeding the general analytical outcome on the one hand and raising confusion with the selection of a ‘proper’ inverse on the other.

A common way out, not to say getting around the problem, has been the employment of the familiar minimum-norm right or least-squares left inverses. Those unique inverses are in fact ‘optimal’ in some sense, so under the lack of any ‘competitive’ inverses they could be thought of as the best ones. Such a minimum-norm/least-squares solution has been encountered in applications of right/left inverses of nonsquare polynomial matrices or nonsquare rational matrices, to mention control analysis and design problems (Kaczorek, 2005; Kac\-zorek et al., 2009; Latawiec, 1998; Williams and Antsaklis, 1996) as well as error-control coding (Fornasini and Pinto, 2004; Forney, 1991; Gluesing-Luerssen et al., 2006; Johannesson and Zigangirov, 1999; Lin and Costello, 2004; Moon, 2005) and perfect reconstruction filter banks (Bernardini and Rinaldo, 2006; Gan and Ling, 2008; Quevedo et al., 2009).

The employment of the minimum-norm right or least-squares left inverses has also been the authors’ first choice when solving the problem of the generation of the so-called ‘control zeros’ for a nonsquare LTI MIMO system, the zeros defined as poles of an inverse system or poles of a closed-loop Minimum Variance Control (MVC) system (Latawiec, 1998; Latawiec et al., 2000). The limited usefulness of minimum-norm right or least-squares left inverses, to be in the sequel called $T$-inverses, has soon brought us to the point where we have introduced the so-called $\tau$-inverses and $\sigma$-inverses of nonsquare polynomial matrices (Hunek, 2008; Latawiec, 2004; Latawiec et al., 2004; 2005).

Since in some applications it is welcome for an inverse polynomial matrix not to have any pole at all, we have offered pole-free right/left inverses of nonsquare polyno-
polynomial matrices (Hunek, 2008; 2009a; 2009b; Hunek et al., 2007; 2010). At last, we have presented a specific Smith factorization solution to the inverse problem for nonsquare polynomial matrices (Hunek, 2008; 2009a; Hunek et al., 2007).

In this paper, the Smith factorization approach is extended to finally obtain a new general class of inverses, valid for any nonsquare polynomial matrices and providing an arbitrary number of degrees of freedom in terms of a preselected number (and value) of the inverse’s zeros and poles, if any. For completeness, this paper recalls all the above mentioned (and presented mainly at conferences) new results in the inversion of nonsquare polynomial matrices. Applications of the results in process control technology have been reported (Hunek, 2008; 2009a; Hunek and Latawiec, 2009; Hunek et al., 2007; Latawiec, 2004; 2005; Latawiec et al., 2004; 2005) and are now being expanded (Hunek et al., 2010), whilst possible applications include, e.g., error coding control and perfect reconstruction filter banks seem forthcoming.

This paper is structured as follows. Having introduced the inversion problem for full normal rank polynomial matrices, system representations including the polynomial matrices to be inverted are reviewed in Section 2. Since our new concepts of polynomial matrix inversion have originated from closed-loop discrete-time MVC, this control strategy is recalled in Section 3. A fundamental idea behind the forthcoming introduction of new, MVC-related inverses of nonsquare polynomial matrices is illustrated in the instructive motivating example of Section 4. Analytical expressions for new polynomial matrices, including the so-called \( T \)-inverses and \( \sigma \)-inverses, in addition to the well-known but renamed \( T \)-inverse, are offered in Section 5. Control-related applications call for the selection of stable polynomial matrix inverses, which is covered in Sections 6 and 7. The Smith factorization approach of the latter is extended in Section 8 to culminate in the introduction of a new, general \( S \)-inverse of a nonsquare polynomial matrix. Actual and potential applications are indicated in Section 9. The discussion in Section 10 provides yet another justification for setting the inversion problem in the time-domain framework. Also, a series of open research problems are specified in that section. New results of the paper are summarized in the conclusions of Section 11.

2. System representations

We start from general system representations related to control applications of inverse polynomial matrices, including process control, error-control coding and perfect reconstruction filter banks.

Consider an \( n_x \)-input \( n_y \)-output Linear Time-Invariant (LTI) discrete-time system with the input \( u(t) \) and the output \( y(t) \), described by possibly rectangular transfer-function matrix \( G \in \mathbb{R}^{n_y \times n_u}(z) \) in the complex operator \( z \). The transfer function matrix can be represented in the Matrix Fraction Description (MFD) form \( G(z) = A^{-1}(z)B(z) \), where the left coprime polynomial matrices \( A \in \mathbb{R}^{n_u \times n_u}[z] \) and \( B \in \mathbb{R}^{n_y \times n_u}[z] \) can be given in the form \( A(z) = z^n I_{n_u} + \cdots + a_0 \) and \( B(z) = z^m b_0 + \cdots + b_m \), respectively, where \( n \) and \( m \) are the orders of the respective matrix polynomials and \( I_{n_u} \) is the identity \( n_u \)-matrix. An alternative MFD form \( G(z) = B(z)A^{-1}(z) \), involving right coprime \( A \in \mathbb{R}^{n_u \times n_u}[z] \) and \( B \in \mathbb{R}^{n_y \times n_u}[z] \), can also be tractable here but in a less convenient way (Latawiec, 1998).

Algorithms for the calculation of MFDs are known and software packages in Matlab’s Polynomial Toolbox® are available. Unless necessary, we will not discriminate between \( A(z^{-1}) = I_{n_u} + \cdots + a_0 z^{-n} \) and \( A(z) = z^n A(z^{-1}) \), nor between \( B(z^{-1}) = b_0 + \cdots + b_m z^{-m} \) and \( B(z) = z^m B(z^{-1}) \) with \( G(z) = A^{-1}(z)B(z) = z^{-d}A^{-1}(z)B(z^{-1}) \), where \( d = n - m \) is the time delay of the system. In the sequel, we will assume for clarity that \( B(z) \) is of full normal rank; a more general case of \( B(z) \) being of non-full normal rank can also be tractable (Latawiec, 1998).

Let us finally concentrate on the case when the normal rank of \( B(z) \) is \( n_y \) (‘symmetrical’ considerations can be made for the normal rank \( n_u \)). The first MFD form can be directly obtained from the AR(1)X/AR(1)MAX model of a system \( A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})v(t) \), where \( q^{-1} \) is the backward shift operator and \( v(t) \in \mathbb{R}^{n_v} \) is the uncorrelated zero-mean disturbance at (discrete) time \( t \). The pairs \( A \) and \( B \) as well as \( A \) and \( C \in \mathbb{R}^{n_y \times n_u}[z] \) are relatively prime polynomial matrices, with (stable) \( C(z^{-1}) = \xi_0 + \cdots + \xi_k z^{-k} \) and \( k \leq n \), and the \( D \) polynomial in the \( z^{-1} \)-domain is often equal to \( 1 - z^{-1} \) (or to unity in the discrete-time MVC considerations). In the sequel, we will also use the operator \( w = z^{-1} \) (or \( w = q^{-1} \), depending on the context), whose correspondence to the \( s \) operator for continuous-time systems has been pioneeringly explored by Hunek and Latawiec (2009).

The familiar Smith–McMillan form \( S_M(w) \) of \( G(w) = w^d A^{-1}(w)B(w) \) (as a special case of the MFD factorization (Desoer and Schulman, 1974)) is given by \( G(w) = U(w)S_M(w)V(w) \), where \( U \in \mathbb{R}^{n_y \times n_u}[w] \) and \( V \in \mathbb{R}^{n_u \times n_u}[w] \) are unimodular and the pencil \( S_M \in \mathbb{R}^{n_y \times n_u}(w) \) is of the form

\[
S_M(w) = \begin{bmatrix}
M_r & 0_{r \times (n_u-r)} \\
0_{(n_y-r) \times r} & 0_{(n_u-r) \times (n_u-r)}
\end{bmatrix},
\]

with \( M(w) = \text{diag}(\epsilon_1, \psi_1, \epsilon_2, \psi_2, \ldots, \epsilon_r, \psi_r) \), where \( \epsilon_i(w) \) and \( \psi_i(w), \ i = 1, \ldots, r \) (with \( r \) being the normal rank of \( G(w) \)), are monic coprime polynomials such that \( \epsilon_i(w) \) divides \( \epsilon_{i+1}(w), \ i = 1, \ldots, r-1 \) and \( \psi_i(w) \) divides \( \psi_{i+1}(w), \ i = 2, \ldots, r \). In particular, the Smith form
is given by the appropriate pencil $S(w)$, with $M(w) = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ often associated with Smith zeros or transmission zeros (Kaczorek et al., 2009; Tokarzewski, 2002, 2004). The polynomials $\varepsilon_i(w)$ are often called the invariant factors of $G(w)$ and their product $\varepsilon(w) = \Pi_i \varepsilon_i(w)$ is sometimes referred to as the zero polynomial of $G(w)$.

3. Closed-loop discrete-time minimum variance control

Our interest in minimum variance control results from the fact that it is a sort of inverse-model control, directly involving an inverse numerator matrix of the MFD system representation.

In the MVC framework, we consider the ARMAX system description

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})v(t). \tag{2}$$

For general purposes and duality with the continuous-time case, we use here the ARMAX model, even though it is well known that the $C(q^{-1})$ polynomial matrix of disturbance parameters is usually in control engineering practice unlikely to be effectively estimated (and is often used as a control design, observer polynomial matrix instead).

All the results to follow can be dualized for continuous-time systems described by a Laplace operator model analogous to Eqn. 2. This can be enabled owing to the unified, discrete-time/continuous-time MVC framework introduced by Hunek and Latawiec (2009).

Consider a right-invertible system ($n_y < n_u$) described by Eqn. 2 and assume that the observer (or disturbance-related) polynomial $C(q^{-1}) = \omega_0 + \omega_1 q^{-1} + \cdots + \omega_n q^{-n}$ has all its roots inside the unit disk. Then the general MVC law, minimizing the performance index

$$\min_{u(t)} E \{ [y(t + d) - y_{ref}(t + d)]^T [y(t + d) - y_{ref}(t + d)] \}, \tag{3}$$

where $y_{ref}(t + d)$ and $y(t + d) = \tilde{C}(q^{-1})\tilde{F}(q^{-1}) \cdot B(q^{-1})u(t) + \tilde{H}(q^{-1})y(t) + \tilde{L}(q^{-1})v(t)$ are the output reference/setpoint and the stochastic output predictor, respectively, is of the form (Hunek, 2008; Latawiec, 2004)

$$u(t) = R_R(q^{-1})\tilde{y}(t), \tag{4}$$

where

$$\tilde{y}(t) = \tilde{F}(q^{-1})^{-1} \left[ \tilde{C}(q^{-1})y_{ref}(t + d) - \tilde{H}(q^{-1})y(t) \right].$$

The appropriate polynomial ($n_y \times n_y$)-matrices $\tilde{F}(q^{-1}) = I_{n_y} + \tilde{F}_1 q^{-1} + \cdots + \tilde{F}_{d-1} q^{-d+1}$ and $\tilde{H}(q^{-1}) = L_0 + \tilde{L}_1 q^{-1} + \cdots + \tilde{L}_{n_y-1} q^{-n_y+1}$ are computed from the polynomial matrix identity (called the Diophantine equation)

$$\tilde{C}(q^{-1}) = \tilde{F}(q^{-1}) \tilde{A}(q^{-1}) + q^{-d} \tilde{H}(q^{-1}), \tag{5}$$

with

$$\tilde{C}(q^{-1}) \tilde{F}(q^{-1}) = \tilde{F}(q^{-1}) \tilde{C}(q^{-1}), \tag{6}$$

where $\tilde{F}(q^{-1}) = I_{n_y} + \tilde{F}_1 q^{-1} + \cdots + \tilde{F}_{d-1} q^{-d+1}$, $\tilde{C}(q^{-1}) = \tilde{C}_0 + \tilde{C}_1 q^{-1} + \cdots + \tilde{C}_n q^{-n}$. For right-invertible systems, the symbol $R_R(q^{-1})$ denotes, in general, an infinite number of right inverses of the numerator polynomial matrix $\tilde{F}(q^{-1})$.

Remark 1. The MVC problem reduces to the perfect control one when $v(t) = 0$ (with both control laws being identical) and specializes to the perfect regulation problem or to the output (predictor) zeroing one when $y_{ref} = 0$ and $v(t) = 0$.

Remark 2. Clearly, the interest in MVC is due to the fact that an inverse polynomial matrix $R_R(q^{-1})$ is involved here, with poles of the inverse constituting the so-called control zeros. Transmission zeros, if any, make a subset of the set of control zeros (Latawiec, 1998; Latawiec et al., 2000).

Remark 3. The above MVC result and all the results to follow can be dualized for left-invertible systems ($n_y > n_u$), with a left inverse of the appropriate matrix involved.

4. Motivating example

Consider a multivariable second-order system governed by the ARX model $y(t) + a_1 y(t - 1) + a_2 y(t - 2) = b_1 u(t - 1) + b_2 u(t - 2) + b_3 u(t - 3) + v(t)$, with the notation as in Section 2. Assume once again that the appropriate polynomial matrix $R_R(q^{-1})$ is of full normal rank $n_y$ and its (nonunique) right inverse is denoted by $R_R(q^{-1})$. Equating, in the standard perfect control manner, the (deterministic part of the) one-step output predictor to the reference/setpoint, we obtain

$$b_1 u(t) + b_2 u(t - 1) + b_3 u(t - 2) - a_1 y(t) - a_2 y(t - 1) = y_{ref}(t + 1). \tag{7}$$

On the one hand, Eqn. 7 immediately leads to the MV/perfect control law

$$u(t) = (b_1 + b_2 q^{-1} + b_3 q^{-2})^{-1} R_R(q^{-1})^{-1} v(t), \tag{8}$$

with $y(t) = y_{ref}(t + 1) + a_1 y(t) + a_2 y(t - 1)$. Equation 8 represents one set of solvers for Eqn. 7 for $u(t)$.

But on the other hand, assuming that $b_0$ is of full normal rank, Eqn. 7 can be given the form $u(t) = \cdots$
representing another set of solvers \( \{4\} \) of Eqn. \( \{7\} \).

Although both MVC laws \( \{5\} \) and \( \{6\} \) are derived from the same output predictor as in Eqn. \( \{7\} \), it is rather surprising that these laws are generally different and this is because \( B_{R1}(q^{-1}) = (b_0 + b_1 q^{-1} + b_2 q^{-2})^{-1}(b_0 + b_2 q^{-2})^{-1} - (b_0 + b_2 q^{-2})^{-1} R = B_{R1}(q^{-1}) \), in general. The difference results from specific properties of right inverses for polynomial matrices. Of course, both \( B(q^{-1}) B_{R1}(q^{-1}) = I_{n_u} \) and \( B(q^{-1}) B_{R2}(q^{-1}) = I_{n_u} \).

Observe that a solver to Eqn. \( \{7\} \) can be given yet another form, e.g., \( u(t) = (b_0 q^{-1})^{-1} [g(t) - b_2 u(t - 2)] \), which can be rewritten as \( u(t) = [I_{n_u} + (b_0 q^{-1}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} y(t) \) representing another set of solvers \( \{4\} \) of Eqn. \( \{7\} \) for \( u(t) \) and giving rise to the introduction of yet another inverse of \( B(q^{-1}) \), say \( B_{R3}(q^{-1}) = [I_{n_u} + (b_0 q^{-1}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R] \). A similar result can be obtained with the inverse \( B_{R4}(q^{-1}) = [I_{n_u} + (b_0 q^{-2}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R] \). But this is not the end. Another form of Eqn. \( \{7\} \) could be \( u(t) = \left[ (b_0 + b_1 q^{-1})^{-1} y(t) - b_2 u(t - 2) \right] \), resulting in one more set of solvers \( u(t) = \left[ I_{n_u} + (b_0 + b_1 q^{-1}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R \right] y(t) \), with another inverse of \( B(q^{-1}) \), say \( B_{R5}(q^{-1}) = [I_{n_u} + (b_0 + b_1 q^{-1}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R] \). But, by the same token, we could still introduce another set of solvers related to another inverse, say \( B_{R6}(q^{-1}) = [I_{n_u} + (b_0 + b_2 q^{-2}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R] \), and, at last, \( B_{R7}(q^{-1}) = [I_{n_u} + (b_0 + b_2 q^{-2}) R(b_0 + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R] \). Well, at last? Not quite because the last three inverses involve subinverses of (matrix) binomials, each of which can be presented in terms of two ‘elementary’ inverses involving monomials. The resulting inverses \( B_{R8}(q^{-1}) \) through \( B_{R13}(q^{-1}) \) are replete to Appendix A. Thus, for the above example, as many as 13 different types of right inverses can be involved in 13 various sets of solvers \( \{4\} \) of Eqn. \( \{7\} \) for \( u(t) \). And yet, each of these inverses \( B_{R1}(q^{-1}) \) through \( B_{R13}(q^{-1}) \) is nonunique due to the nonuniqueness of a right inverse.

In order to arrive at feasible analytical solutions, all the right inverses occurring in \( B_{R1}(q^{-1}) \) through \( B_{R13}(q^{-1}) \) are now specialized to what will be called the minimum-norm \( T \)-inverses and denoted as \( [5]^{R,T} \). Since the MV/perfect control law is clearly a time-domain equation, we shall use regular, rather than conjugate, transposes in the minimum norm \( T \)-inverse (Latawiec, 2004; Latawiec et al., 2005).

It is interesting to observe that all those inverses are associated with two classes of solvers for \( u(t) \) in the MVC problem, resulting from two different equations to be solved: \( B(q^{-1}) u(t) = y(t) \) and \( \{I_{n_u} + (b_0 + b_1 q^{-1} + b_2 q^{-2})^{-1} (b_0 + b_2 q^{-2})^{-1} R \} \).

\[ B(q^{-1}) u(t) = y(t) \] (10)

or

\[ \left\{ I_{n_u} + \left[ \beta(q^{-1}) \right]^{\text{inv}} \right\} \left[ B(q^{-1}) - \beta(q^{-1}) \right] u(t) = \left[ \beta(q^{-1}) \right]^{\text{inv}} y(t), \] (11)

where the inverse \( [\beta(q^{-1})]^{\text{inv}} \) is an appropriate generalization of a specific \( \beta(q^{-1}) \), depending on specific, rank-related properties of \( \beta(q^{-1}) \) (with, e.g., \( [\beta(q^{-1})]^{\text{inv}} = [\beta(q^{-1})]^{R} \) holding for a right-invertible \( \beta(q^{-1}) \)). Note that for \( \beta(q^{-1}) = \beta(q^{-1}) = B(q^{-1}) \).

Eqs. (10) and (11) are equivalent.

5.1. \( T \)-inverses. Based on the above considerations, the two general definitions below introduce various optimal inverses of the \( m \)-th order nonsquare polynomial matrix \( B(q^{-1}) \), which are associated with Class 1 optimal time-domain solvers for \( u(t) \) in the MVC problem (related to Eqn. (10)). The optimal, so-called \( T \)-inverses include regular (rather than conjugate) transposes of \( B(q^{-1}) \). Observe that these inverses are dimension preserving, i.e., not squaring the system down (Davison, 1983; Latawiec, 1998), the prerequisite aiming at protection from the reduction of the problem to the classical square MIMO one (with the standard transmission zeros).

**Definition 1.** Let the polynomial matrix \( B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_m q^{-m} \) be of full normal rank \( n_u \) (or \( n_u \)). The (unique) minimum-norm right (or the least-squares left) \( T \)-inverse of \( B(q^{-1}) \) is defined as \( B(q^{-1}) = B^{T}(q^{-1}) \left[ B(q^{-1}) B^{T}(q^{-1}) \right]^{-1} \) (or \( B^{-1}(q^{-1}) = B^{T}(q^{-1}) B(q^{-1})^{-1} B^{T}(q^{-1}) \)).
Definition 2. Let the polynomial matrix $B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_m q^{-m}$ of non-full normal rank $r$ be skeleton factorized as $B(q^{-1}) = C(q^{-1})D(q^{-1})$, where $\dim[B(q^{-1})] = n_y \times n_u$, $\dim[C(q^{-1})] = n_y \times r$, $\dim[D(q^{-1})] = r \times n_u$. The (unique) Moore–Penrose $T$-inverse of $B(q^{-1})$ is defined as $B^R(q^{-1}) = \{I_{n_u} + [\beta(q^{-1}) R \{B(q^{-1}) - \beta(q^{-1})\}]^{-1} [\beta(q^{-1}) R\}^{-1} \}$, in general. Nevertheless, based on the motivating example of Section 4, we can offer new general tools for the computation of all the $\tau$-inverses.

5.2.1. Algorithm and program for calculating $\tau$-inverses. Below is a new algorithm for calculating all the $\tau$-inverses. The combinatorics-based algorithm is very complicated but, surprisingly, it will self-verifyingly lead to a simple result of the forthcoming Theorem 1.

Algorithm 1. $\tau$-inverses algorithm.

STEP $k = 0 \rightarrow$ one $T$-inverse

\[
B^R(q^{-1}) = \{I_{n_u} + [\beta(q^{-1}) R \{B(q^{-1}) - \beta(q^{-1})\}]^{-1} [\beta(q^{-1}) R\}^{-1} \},
\]

(A.0)

with $l_0 = m + 1$, that is, $\beta_0(q^{-1}) = B(q^{-1})$. The index $l_0 = m + 1$ means that matrix parameters of the $m$-th order matrix polynomial $\beta_{m+1}(q^{-1})$ constitute $(m+1)$-combinations without repetition from an $(m+1)$-parameter set of the matrix polynomial $B(q^{-1})$, which makes the number of the combinations and thus the number of $T$-inverses equal to just one.

STEP $k = 1$

\[
B_i(q^{-1}) = \{I_{n_u} + [\beta_i(q^{-1}) R \{B(q^{-1}) - \beta_i(q^{-1})\}]^{-1} [\beta_i(q^{-1}) R\}^{-1} \},
\]

(A.1)

where matrix parameters of the polynomial matrix $\beta_i(q^{-1})$ are $l_i$-combinations without repetition from an $l_0$-parameter set of the matrix polynomial $\beta_i(q^{-1})$, and the superscript $i = k, \ldots, m$ stands for the $i$-th set of inverses calculated at Step $k$. The role of the index $i$ will be better understood from the proof of Theorem 1.

STEP $k = 2$

\[
B(q^{-1})^R = \{I_{n_u} + [\beta_i(q^{-1}) R \{B_i(q^{-1}) - \beta_i(q^{-1})\}]^{-1} [\beta_i(q^{-1}) R\}^{-1} \},
\]

(A.2)

with the notation quite similar to that for Step $k = 1$.

\[ \vdots \]

STEP $k$

\[
B(q^{-1})^R = \{I_{n_u} + [\beta_i(q^{-1}) R \{B_k(q^{-1}) - \beta_i(q^{-1})\}]^{-1} [\beta_i(q^{-1}) R\}^{-1} \},
\]
Remark 9. Although Theorem 1 has been presented by Latawiec (2004), it is not until this paper that an original, complete, formal proof has been provided.

Remark 9. The above total number of \( \tau \)-inverses shall be treated as the maximum number of \( \tau \)-inverses for a specific \( m \). In fact, in some cases \( \beta(q^{-1}) \) and/or \( \beta(q^{-1}) \) may appear nonfull normal rank (even though \( B(q^{-1}) \) is of full normal rank) so that the corresponding \( \tau \)-inverses do not exist. Exemplary maximum numbers of \( \tau \)-inverses are \( N_m = 13 \) for \( m = 2 \), \( N_m = 75 \) for \( m = 3 \) and \( N_m = 541 \) for \( m = 4 \).

Remark 10. It has been found in simulations (a rigorous mathematical confirmation seems unlikely) that for some, even ‘typical’, plants the properties of \( \tau \)-inverses of \( B(q^{-1}) \), including the \( T \)-inverse, may be unfavorable, in particular in terms of all unstable poles of \( B^{R}(q^{-1}) \) obtained. This may limit the applications of \( \tau \)-inverses.

5.3. \( \sigma \)-inverses. Let us proceed now to the most intriguing issue related to the family of inverses as in Eqn. (12). It is surprising that \( B(q^{-1})B^{R}(q^{-1}) = I_{n_{q}} \), with \( B^{R} \) as in Eqn. (12) and \( \beta(q^{-1}) = \beta(q^{-1}) \), even for arbitrary \( \beta(q^{-1}) \), that is, not related to \( B(q^{-1}) \) at all (but, of course, with adequate matrix dimensions). This way we arrive at the so-called \( \sigma \)-inverses, a number of which is infinite (in spite of the unique minimum-norm right \( T \)-inverse involved).

Definition 3. Let the polynomial matrix \( B(q^{-1}) = b_{0} + b_{1}q^{-1} + \ldots + b_{m}q^{-m} \). Then, a general \( \sigma \)-inverse of \( B(q^{-1}) \) can be defined as
\[
B^{R}(q^{-1}) = \{I_{n_{q}} + [\beta(q^{-1})]_{R}B(q^{-1}) - \beta(q^{-1})]\}^{-1}[\beta(q^{-1})]_{R},
\]
where \( z^s\beta(z^{-1}) = \beta(z) \in \mathbb{R}^{n_x \times n_{q}} [z] \) is arbitrary, including an arbitrary order \( s \), and \( [\beta(q^{-1})]_{R} \) stands for the \( T \)-inverse.

Unfortunately, rigorous formal proving that \( B(q^{-1})B^{R}(q^{-1}) = I_{n_{q}} \), with \( B^{R}(q^{-1}) \) as in Eqn. (14), is still an open problem. A missing part of the proof can be formulated as follows.

Conjecture 1. Let \( \beta(q^{-1}) \) be an arbitrary \( s \)-order matrix polynomial in \( q^{-1} \), with \( z^s\beta(z^{-1}) = \beta(z) \in \mathbb{R}^{n_x \times n_{q}} [z] \), and let \( \Phi(q^{-1}) = I_{n_{q}} + [\beta(q^{-1})]_{R}B(q^{-1}) - \beta(q^{-1}) \) be of full normal rank \( n_{q} \). Then
\[
[\beta(q^{-1})\Phi(q^{-1})]_{R} = \Phi(q^{-1})[\beta(q^{-1})]_{R}^{-1},
\]
Assuming that the above conjecture is true, the proof of \( B(q^{-1})B^{R}(q^{-1}) = I_{n_{q}} \), with \( B^{R}(q^{-1}) \) as in Eqn. (14), is immediate. In fact, \( \beta(q^{-1})\Phi(q^{-1}) = B(q^{-1}) \) whereas the right-hand side of Eqn. (15) is just the right-hand side of Eqn. (14). Now, omitting the subindex on the left-hand side of Eqn. (15) (in order to distinguish the \( \sigma \)-inverse from the \( T \)-inverse) would complete the proof.

Remark 11. We have verified Conjecture 1 and the identity \( B(q^{-1})B^{R}(q^{-1}) = I_{n_{q}} \), with \( B^{R}(q^{-1}) \) as in Eqn. (14), in a number of simulations, including MVC ones.
Remark 12. It is interesting to note here that, for arbitrary right-invertible \( \beta(q^{-1}) \) and arbitrary invertible \( \Phi(q^{-1}) \) of appropriate dimensions, we have that \( \Phi[q^{-1}]\beta(q^{-1}) \) of \( \Phi^{-1}(q^{-1}) \) of \( \Phi^{-1}(q^{-1}) \), but \( \beta[q^{-1}]\Phi(q^{-1}) \neq \Phi^{-1}(q^{-1})\beta(q^{-1}) \), in general. It is just for our specific \( \Phi(q^{-1}) = I_n + [\beta(q^{-1})][\Phi[q^{-1}]\beta(q^{-1})] \) that Eqn. (15) holds true, the intriguing property confirmed so far only in numerous simulations.

Even though the most general \( \sigma \)-inverses contain \( \tau \)-inverses, which in turn include the \( T \)-inverse, we discriminate between the three types of inverses of nonsquare polynomial matrices. Here \( \tau \)-inverses and \( \sigma \)-inverses generate what we call control zeros type 2 (Latawiec, 2004; Latawiec et al., 2004).

It is worth emphasizing that the formula (16) is a new, general, algorithmically very simple analytical expression for the calculation of right inverses for nonsquare polynomial matrices. Quite a similar formula can be given for left inverses. It is interesting to note how stimulating the MVC framework in the derivation of \( \tau \)- and \( \sigma \)-inverses has been (Latawiec, 2004; Latawiec et al., 2004).

Remark 13. Of course, the inverse as in Eqn. (16) can be rewritten in terms of \( B(z) = z^m \beta(z) \). We still prefer the form (14) associated with the time-domain MVC solution and the regular, rather than conjugate, transposes included in \( [\beta(q^{-1})]\)R \( 0_n \). Notwithstanding, the \( z \)-domain formulation is now used instructively below.

Example 1. Consider a specific matrix \( \beta(q^{-1}) \) corresponding to the matrix \( B \in \mathbb{R}^{2 \times 3} \) with \( b_{11}(z) = z^2 + 0.9z - 0.1, b_{12}(z) = z + 0.4, b_{13}(z) = z^2 + 0.1z - 0.02 \) and \( b_{21}(z) = z^2 + 0.4z - 0.05, b_{22}(z) = 1, b_{23}(z) = 2z^2 + 0.8z - 0.1 \). All the thirteen \( \tau \)-inverses (including the \( T \)-inverse) have unstable poles, that is, unstable control zeros, in addition to a stable transmission zero at \( z = 0.1 \). Selecting \( \beta = \beta \in \mathbb{R}^{2 \times 3} \) with \( \beta_{11}(z) = -0.0793, \beta_{12}(z) = 1.5352z - 0.6065, \beta_{13}(z) = -1.3474 \) and \( \beta_{21}(z) = 0.4694z - 0.9036, \beta_{22}(z) = 0.0359, \beta_{23}(z) = -0.6275z + 0.5354 \) yields the unstable \( \sigma \)-inverse \( B^{R}(z) \), whereas for \( \beta_{11}(z) = -0.0781, \beta_{12}(z) = 1.8148z - 0.6140, \beta_{13}(z) = -1.3928 \) and \( \beta_{21}(z) = 0.3931z - 0.6786, \beta_{22}(z) = 0.0332, \beta_{23}(z) = -0.8042z + 0.6203 \) we obtain the stable \( \sigma \)-inverse \( B^{R}(z) \). However, there are no formal tools for rigorous generalization of the latter, heuristic selection.

For a specific \( \beta(q^{-1}) \), we have computed an adequate \( \beta(q^{-1}) \) to obtain a stable (or pole-free) \( B^{R}(q^{-1}) \) by means of a standard, Matlab-based optimization procedure. (Note that, for space limitation reasons, we refrained from specifying the two \( \sigma \)-inverses \( B^{R}(z) \) in the above example as their six entries are up to the 5-th order, so that high-precision presentation would be necessary for accuracy reasons. Still, in Appendix C we specify control zeros, that is, poles of \( B^{R}(z) \), for the two cases.)

6. New approaches to stable design of inverse polynomial matrices

It is crucial in inverse-model control applications to be able to design stable inverses of the numerator matrix in the rational matrix description of an LTI MIMO system. Particular interest concerns the case of pole-free inverses, for which the control system is guaranteed to be asymptotically stable. Here we present two new approaches to the design of pole-free right inverses of a polynomial matrix \( \beta(q^{-1}) \).

6.1. Extreme Points and Extreme Directions (EPED) method. The method is recalled here for solving the linear matrix polynomial equation (Callier and Kreffer, 2005; Henrion, 1998; Kaczorek and Lopata, 2000)

\[
K(w)X(w) = P(w),
\]

where \( K(w) = K_0 + K_1w + \cdots + K_nw^n \) and \( P(w) = P_0 + P_1w + \cdots + P_{n_p}w^{n_p} \) are given \( m \times n \) and \( m \times p \) polynomial matrices in complex operator \( w \), respectively, and \( X(w) = X_0 + X_1w + \cdots + X_{n_X}w^{n_X} \) is an \( n \times p \) polynomial matrix to be found. By equating the powers of \( w \) in the formula (16), we obtain an equivalent linear system of equations

\[ KX = P, \]

where the real matrix

\[
\begin{bmatrix}
X_0 & X_1 & \cdots & X_{n_X}
\end{bmatrix} \in \mathbb{R}^{\hat{m} \times n},
\]

\[
P_0 & P_1 & \cdots & P_{n_p}
\end{bmatrix} \in \mathbb{R}^{\hat{m} \times p}.
\]

The problem of finding the matrix polynomial solution \( X(w) \) to Eqn. (16) has been reduced to finding the real matrix \( X \) of Eqn. (17) for given real matrices \( K \) and \( P \) as in (18) and (19). The matrix polynomial equation (16) has the solution for \( X(w) \) iff rank \( \begin{bmatrix} K & P \end{bmatrix} = \text{rank} \ K \). Using the Kronecker product, Eqn. (17) can be rewritten in the form

\[ AX = b, \]
where

\[ A = K \otimes P \in \mathbb{R}^{n \times n}, \]
\[ x = [\pi_1, \pi_2, \ldots, \pi_n]^T \in \mathbb{R}^n, \]
\[ b = [\pi_1, \pi_2, \ldots, \pi_n]^T \in \mathbb{R}^n, \]

with \( \pi = \tilde{n}_p \pi = \tilde{n}_p \), and \( \pi_i \) and \( \pi_j \) denote the \( i \)-th and \( j \)-th rows of \( K \) and \( P \), respectively.

Now, the problem of calculating the set of solutions to Eqs. (16) can be reduced to finding the set \( x \) satisfying Eqs. (20). Note that, if \( \tilde{n} \geq n \) and \( \text{rank} K = \tilde{n} \), the matrix \( A \) also has full row rank.

Let \( S = \{x : Ax = b\} \) be a non-empty set. A point \( x \) is an extreme point of \( S \) iff \( A \) can be decomposed into \( [B \ N] \) such that \( \text{det} B \neq 0 \) and \( x = [B^{-1}b] \). If \( \text{rank} A = \pi \), then \( S \) has at least one extreme point. The number of extreme points is less than or equal to \( \pi / (\pi - \pi)! \).

A vector \( d \) is an extreme direction of \( S \) iff \( A \) can be decomposed into \([a \ n]\) such that \( \text{det} B \neq 0 \) and \( d = [-B^{-1}a] \), where \( a_j \) is the \( i \)-th column of \( N \) and \( e_j \) is an \( \pi - \pi \) vector of zeros except for unity in position \( i \). The set \( S \) has at least one extreme direction if it is unbounded. The maximum number of extreme directions is bounded by \( \pi / (\pi - \pi)! \).

Let \( x_1, x_2, \ldots, x_k \) be the extreme points of \( S \) and \( d_1, d_2, \ldots, d_l \) the extreme directions of \( S \). Then every \( x \in S \) can be written as

\[ x = \sum_{j=1}^{k} \lambda_j x_j + \sum_{i=1}^{l} \mu_i d_i, \quad \sum_{j=1}^{k} \lambda_j = 1. \]

Let us now embed the EPED method in the framework of an inverse polynomial matrix \( B(z^{-1}) \).

**Theorem 2.** Let \( B(z^{-1}) = X(w) \), with \( w = z^{-1} \), be a solution of the linear matrix polynomial equation \( B(w)X(w) = I_n \). Then the necessary and sufficient condition for the existence of a solution \( X(w) \) by the EPED method is that the polynomial matrix \( B(z) \) has no transmission zeros.

**Proof.** It is well known that the necessary and sufficient condition for the existence of a solution of Eqn. (16) is that \( \text{rank} [K(w) P(w)] = \text{rank} K(w) \), where \( K(w) \) stands for normal rank. When translated to our polynomial matrix framework, the condition \( \text{rank} [B(w) I_n] = \text{rank} B(w) \) implies that the matrix \( B(w) \) has no transmission zeros.

The above essential constraint of the EPED method is revealed here for the first time.

The mathematically elegant EPED method provides a pole-free solution to the polynomial matrix problem, but it is computationally involving while its use is limited to systems with no transmission zeros. Therefore, in the next section we offer a much simpler method, which is valid for systems with transmission zeros as well. Even if the method is more effective, it is the EPED method which has turned our attention to pole-free design of inverse polynomial matrices.

### 6.2. Smith factorization approach

In an attempt to essentially reduce the computational burden of the EPED method, we introduce yet another effective (and much simpler) approach to stable design of inverse polynomial matrices. The applied approach is closely related to the Smith–McMillan theory (Bose and Mitra, 1978; Kalilath, 1980; Sonntag, 1980; Vardulakis, 1991).

Consider a right-invertible polynomial matrix \( B(z^{-1}) \) of dimension \( n_y \times n_u \). Set \( w = z^{-1} \) and factorize \( B(w) \) to the Smith form \( \bar{B}(w) = U(w)S(w)V(w) \), where \( U(w) \) and \( V(w) \) are (nonuniquely) unimodular matrices. Now, \( B^R(w) = V^{-1}(w)S^R(w)U^{-1}(w) \), with determinants of \( U(w) \) and \( V(w) \) being independent of \( w \), that is, with possible instability of an inverse polynomial matrix \( B^R(w) \) being related to \( S^R(w) \) only.

**Theorem 3.** Consider a right-invertible polynomial \( n_y \times n_u \) matrix \( B(z^{-1}) \). Use the Smith factorization and obtain the inverse polynomial matrix \( B^R(w) = V^{-1}(w)S^R(w)U^{-1}(w) \), with \( w = z^{-1} \) and \( U(w) \) and \( V(w) \) being unimodular. Then, applying the minimum-norm right \( T \)-inverse \( S^R_0(w) = S^T(w) [S(w)S^T(w)]^{-1} \) guarantees stable pole-free design of \( B^R(w) \) for \( B(w) \) without transmission zeros and stable design of \( B^R(w) \) for \( B(w) \) with stable transmission zeros.

**Proof.** Observe that performing the Smith factorization for \( B(w) \) one obtains \( B(w) = U(w)S(w)V(w) \), where \( U(w) \) and \( V(w) \) are unimodular. Now, \( B^R(w) = V^{-1}(w)S^R(w)U^{-1}(w) \), with determinants of \( U(w) \) and \( V(w) \) being independent of \( w \), that is with possible instability of an inverse polynomial matrix \( B^R(w) \) being related to \( S^R(w) \) only. Since, in general, \( S(w) = [\text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \quad 0_{n_y \times (n_u-n_y)}] = S_{12}(w)(\Sigma) \), where \( S_{12}(w) = [\text{diag}(\varepsilon_1, \ldots, \varepsilon_n)] \) includes transmission zeros and \( \Sigma = [I_{n_y} \quad 0_{n_y \times (n_u-n_y)}] \), we have \( B^R(w) = V^{-1}(w)S^R(w)U^{-1}(w) \). Now \( \Sigma^R = \Sigma^T[\Sigma^T \Sigma^T]^{-1} = [I_{n_y} \quad 0_{n_y \times (n_u-n_y)}] \) and the result follows immediately.

**Remark 14.** Obviously, the stability of an inverse polynomial matrix with respect to \( w \) is translated to the requirement for all its poles to lie outside the unit disk.

**Remark 15.** MVC applications of the above Theorems 2 and 3 are immediate.
7. Stable Smith factorization design of inverse polynomial matrices with arbitrary degrees of freedom

In the previous section, stable inverse polynomial matrix designs were obtained without any reference to a possible infinite number of degrees of freedom, which can be of interest in, e.g., control robustness considerations (Hunek, 2008; 2009a; 2009b; Hunek et al., 2007; Latawiec, 2004; 2005; Latawiec et al., 2004; 2005). Even though the unimodular matrices involved are nonunique, possible use of the resulting degrees of freedom is rather difficult due to the constraints imposed on matrix determinants. Here we present a simple Smith factorization approach to pole-free design of inverse polynomial matrices (or pole-free MVC design) with arbitrary degrees of freedom and controlled locations of (stable) zeros of $B^R(w)$. Recall that

$$B^R(w) = V^{-1}(w)S^R L^{-1}(w)U^{-1}(w). \quad (21)$$

With a specific form of $S = \begin{bmatrix} I_{n_y} & 0_{n_y \times (n_x-n_y)} \end{bmatrix}$, we can immediately offer its arbitrary right inverse,

$$S^R = S^R(w) = \begin{bmatrix} I_{n_y} & L(w) \end{bmatrix}. \quad (22)$$

where $L(w)$ is any polynomial matrix of the appropriate dimensions. A general form of that matrix can be $L(w) = \{l_{ij}(w)\}, i = 1, \ldots, n_x - n_y, j = 1, \ldots, n_y$, with $l_{ij}(w) = l_{ij}^{(0)} + l_{ij}^{(1)} w + \cdots + l_{ij}^{(m_{ij})} w^{m_{ij}}$, and $m_{ij}$ can be an arbitrary natural number selected by the designer.

A general selection of a ‘proper’ $L(w)$ to provide arbitrary zeros of $B^R(w)$ is a difficult, still open problem. In fact, the zeros of $B^R(w)$ result both from those of $L(w)$ and those of inverses of the two unimodular polynomial matrices involved. Specifically, the ‘zero matrix polynomial’ for $B^R(w)$ is $Z(w) = \text{adj } V(w)S^R(w)\text{adj } U(w)$, with $S^R(w) = \begin{bmatrix} I_{n_y} & L(w) \end{bmatrix}$, and $L(w)$ is a polynomial matrix as in Eqn. (22). (Note that $Z \in \mathbb{R}^{n_x \times n_y}[w]$.) The zeros of $Z(w)$ are the poles of a left inverse $Z^L(w)$, with the myriads of possible polynomial matrix inverses involved. Thus, the role of $L(w)$ in controlling the stability of zeros of $B^R(w)$ can be appreciated from what follows.

Example 2. Consider a specific matrix $B \in \mathbb{R}^{2 \times 3}(w)$ with

$$b_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 & -1 & -2 \\ -1 & 1 & 0.1 \end{bmatrix}$$

and

$$b_3 = \begin{bmatrix} 1 & 2 & 3 \\ -0.5 & -1.9 & -3 \end{bmatrix}.$$ 

Pursuing the inverse $B^R(w)$ according to Eqns. (21) and (22), we compare two cases: (i) $L(w) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and (ii) $L(w) = \begin{bmatrix} -1.5189 & 0.6388 \end{bmatrix}$. The matrices $Z(w)$ for the two cases are respectively $Z_1(w)$ and $Z_2(w)$, both relegated to Appendix D. We can employ various proposed left inverses to obtain $Z^R(w)$. However, we choose the simple $T$-inverse $Z_0^R(w) = [Z^T(w)Z(w)]^{-1}Z^T(w)$ as it will generate poles of $Z^L(w)$, that is, zeros of $B^R(w)$ to be controlled by means of $L(w)$. Now, in the first case, we can see that two out of eight zeros of $Z_1(w)$ are unstable, whereas for $Z_2(w)$ all the ten zeros are strictly stable (with all their modules with respect to $z = w^{-1}$ being lower than unity). It is worth mentioning that the parameters of $L(w)$ in the second case result from a numerical optimization procedure in which the sum of modules of all the zeros (in terms of $z$) is minimized with respect to the parameters $a$ and $b$ contained in $L(w) = \begin{bmatrix} a & b \end{bmatrix}$.

Remark 16. Note that the solution

$$S^R = S^R(w) = \begin{bmatrix} I_{n_y} \\ L(w) \end{bmatrix}$$

can as well be obtained by the EPED method, with $K(w) = S, X(w) = S^R$ and $P(w) = I_{n_y}$, although under a higher computational burden.

Remark 17. It is worth mentioning once more that all the above designs guarantee the stability of $B^R(z) = z^{-n}B^R(z^{-1})$ both in case of the lack of transmission zeros and under stable transmission zeros, with control zeros (other than transmission zeros) totally eliminated.

Remark 18. It is the right $T$-inverse applied to the matrix $S$ that allows eliminating control zeros according to Theorem 3. Applying some other inverses, that is, $\tau$- and $\sigma$-inverses, usually ends up with control zeros. However, our simulating experience shows that it is sometimes possible to choose a matrix polynomial $S(z^{-1})$ in Eqn. (14) such that pole-free or stable-pole $\sigma$-inverse(s) of $B(z^{-1})$ can be obtained. A general selection of such $\sigma$-inverses is a very difficult and still open problem. Therefore, it seems that an inverse of the $B(z^{-1})$ matrix, based on the Smith factorization as in Eqn. (21), can be more useful in stable design of inverse polynomial matrices.

Remark 19. The question arises whether in some control applications the pole-free, that is, control zero-free inverse polynomial matrix designs could be inferior to the synthesis allowing for (stable) poles of the control system, that is, (stable) control zeros selected to provide, e.g., a sort of robustness to MVC. Such a solution is presented in Section 8.

Remark 20. It is in general possible in the above stable Smith factorization design to select $L(w)$ as a (stable) rational matrix or, in a technically simpler way, as a matrix with all its elements being (stable) rational transfer functions, with an infinite number of degrees of freedom possible to be obtained from degree(s) of the transfer functions.
Then the poles of those transfer functions are the poles of an inverse polynomial matrix $B^R(w)$ or the poles of a closed-loop MVC system, that is, the control zeros. Alternatively, the control zeros (together with transmission zeros) can be generated as poles of $B^R(w)$ using Eqn. (14), with an infinite number of degrees of freedom obtained from parameter matrices and degree(s) of $B(w)$.

**Remark 21.** For high-order $B(w)$ and $L(w)$, the computation burden of the optimization procedure may become prohibitively high.

### 8. New inverse of a nonsquare polynomial matrix

Concluding the Smith factorization approach to the design of (stable) inverse polynomial matrices, we can offer yet another, general right inverse of a nonsquare polynomial matrix $B(z^{-1})$, which can be competitive to that of Eqn. (13) and which we call an $S$-inverse. The new result, following immediately from Theorem 3 and Section 7, is given in the form below.

**Corollary 1.** Consider a polynomial $n_y \times n_u$ matrix $B(w)$ of full normal rank $n_y$, with $w = z^{-1}$, under the Smith factorization $B(w) = U(w)S(w)V(w)$, where $U(w)$ and $V(w)$ are unimodular and $S(w) = \left[ \text{diag}(\varepsilon_1, \ldots, \varepsilon_{n_y}) \quad 0_{n_y \times (n_u - n_y)} \right]$. Then a general right inverse of $B(w)$ can be given as

$$B^R(w) = V^{-1}(w)S^R(w)S_{tz}^{-1}(w)U^{-1}(w), \tag{23}$$

where $S_{tz}(w) = \text{diag}(\varepsilon_1, \ldots, \varepsilon_{n_y})$, with $\varepsilon_i$ being the invariant factors, and

$$S^R(w) = \begin{bmatrix} I_{n_y} \\ L(w) \end{bmatrix},$$

where $L \in \mathbb{R}^{(n_u - n_y) \times n_y}(w)$ is an arbitrary rational matrix.

The above Corollary 1 is in fact a definition of new, general $S$-inverses of nonsquare polynomial matrices. The problem arises with the selection of $L(w)$ to provide arbitrary locations of poles and zeros of $B^R(w)$, which can be useful in inverse-model control designs (Hunek, 2009a; 2009b) and, possibly, in error-control coding (Bernardini and Rinaldo, 2006; Gan and Ling, 2008; Law et al., 2005; Moon, 2005), perfect precoding and equalization (Boche and Pohl, 2006; Guo and Levy, 2004; Kim and Park, 2004; Tidestav et al., 1999; Wahls and Boche, 2007; Wahls et al., 2009; Xia et al., 2001), perfect signal reconstruction, including perfect reconstruction filter banks (Bernardini and Rinaldo, 2006; Gan and Ling, 2008; Law et al., 2009; Quevedo et al., 2009; Vaidyanathan and Chen, 1995; Zhang and Makur, 2009) and perfect deconvolution (Inouye and Tanebe, 2000; Tuncan, 2004), also in the problem of image recovery (Castella and Pesquet, 2004; Harikumar and Bresler, 1999a; 1999b). On the other hand, the problem of perfect input reconstruction that has been solved employing the state-space approach (Edelmayer et al., 2004) could as well be tackled and resolved on the basis of the rational matrix system description.

It is striking how right/left inverses of polynomial matrices can contribute to the ‘perfectualization’ of signal transmission/reconstruction in various types of (generalized) MIMO channels. Although the perfect ‘behavior’ of MIMO channels is rather a theoretical, limiting property hardly attainable in the real, disturbance-corrupted world, inverse polynomial matrices can still be the foundation for application-oriented solutions. One example is robust MVC taking advantage of the ability of selecting inverse polynomial matrices with arbitrary poles and controlled zeros (Hunek, 2008; 2009b; Hunek et al., 2010).

**9. Actual and potential applications**

Actually, our interest in right/left inverses of polynomial matrices has evolved from MV/perfect control applications (Hunek and Latawiec, 2009; Latawiec, 1998; Latawiec et al., 2000) which are now being expanded towards robust MV/perfect control (Hunek, 2008; 2009a; Hunek et al., 2010; 2007; Latawiec, 2004; 2005; Latawiec et al., 2004; 2005). Quite similar families of applications include distortionless (or perfect) reception in error-control coding (Fornasini and Pinto, 2004; Forney, 1991; Gluesing-Luerssen et al., 2006; Johansson and Zigangirov, 1999; Johansson et al., 1998; Lin and Costello, 2004; Lu et al., 2005; Moon, 2005), perfect precoding and equalization (Boche and Pohl, 2006; Guo and Levy, 2004; Kim and Park, 2004; Tidestav et al., 1999; Wahls and Boche, 2007; Wahls et al., 2009; Xia et al., 2001), perfect signal reconstruction, including perfect reconstruction filter banks (Bernardini and Rinaldo, 2006; Gan and Ling, 2008; Law et al., 2009; Quevedo et al., 2009; Vaidyanathan and Chen, 1995; Zhang and Makur, 2009) and perfect deconvolution (Inouye and Tanebe, 2000; Tuncan, 2004), also in the problem of image recovery (Castella and Pesquet, 2004; Harikumar and Bresler, 1999a; 1999b). On the other hand, the problem of perfect input reconstruction that has been solved employing the state-space approach (Edelmayer et al., 2004) could as well be tackled and resolved on the basis of the rational matrix system description.
10. Discussion and open problems

One may argue that we should not use the term ‘minimum-norm’ \( T \)-inverse since, when solving the MVC equation (10) (or (11)) presented in the \( z \)-domain form

\[
B(z)U(z) = z^dY(z),
\]

with \( U(z) \) and \( Y(z) \) corresponding to \( u(t) \) and \( y(t) \), respectively, the actual minimum-norm solution would involve \( B^R(z) = B^*(z)B(z)B^*(z)^{-1} \), with \( \|U_*(z)\|^2 = U^*(z)U(z) \), where the asterisk denotes the conjugate transpose. In fact, \( \|U_*(z)\| \leq \|U_0(z)\| \), where \( \|U_0(z)\| \) is the norm involving the \( T \)-inverse, that is, the regular rather than conjugate transpose. Now, the term ‘minimum-norm’ solution/inverse should be reserved for the conjugate transpose-based norm. Similar should hold for the \( \sigma \)-inverse, that is, regular rather than conjugate transpose-based norm.

Firstly, the MVC Equation (10) (or (11)) is the \textit{time-domain} one. That means that a norm of a time-domain solver to this equation must use real-valued features, that is, regular rather than conjugate transposes. Thus, the minimum-norm or least-squares inverses must end up with the \( T \)-inverses, involving \( B^R(z) = B^*(z)B(z)^{-1}B^*(z) \). We will show that such an argument is not true and the \( z \)-domain approach is not proper for the solution of the inverse-model control problem considered.

Secondly, the \( z \)-domain solution to Eqn. (26) would not take us any further, in terms of the inability to transform it to the \( T \)-domain solution. The reason is obvious: the \( z \)-domain solution would involve \( z = \Re z - i \Im z \), in general.

To additionally exemplify the idea of the minimum-norm \( T \)-inverse/solution, consider the nonsquare full-rank MFD system specified in Appendix E. Choosing the \( \beta(z) = [I_{n_y}, 0_{n_y \times (n_u - n_y)}] \), we end up with the (dimension reducing or squaring down) \( \sigma \)-inverse

\[
B^R(z) = \begin{bmatrix}
\Psi_{n_y \times n_y} \\
0_{(n_u - n_y) \times n_y}
\end{bmatrix},
\]

with \( \Psi = \Psi(z) \) being a rational matrix, which implies that the associated system under MVC would be a squared-down \( (n_y \times n_y) \) control one. It might seem that in such a case the time-domain norm of the MV/perfect control signal could be lower than that for the minimum-norm \( T \)-solution. This cannot be true: the minimum-norm \( T \)-solution is really the minimum norm one, that is, \( \|u_0(t)\| < \|u_\Phi(t)\| \), where \( u_0(t) \) and \( u_\Phi(t) \) are the MV/perfect control signals involving the \( T \)-inverse and the specific \( \sigma \)-inverse of \( B(q^{-1}) \), respectively. The same can be verified in yet another interesting example with the \( 10 \times 1 \) MISO system specified in Appendix F and

\[
\beta(z) = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0
\end{bmatrix}
\]

producing the \( \sigma \)-inverse

\[
B^R(z) = \begin{bmatrix}
0 & \ldots & 0 & \Psi & 0 & \ldots & 0
\end{bmatrix}^T,
\]

that is, with the only nonzero entries at the \( i \)-th position, \( i = 1, \ldots, 10 \), and \( \Psi = \Psi(z) \) being a rational matrix. In fact, in all the squared-down SISO control systems obtained we have \( \|u_0(t)\| < \|u_\Phi(t)\| \) for each \( i = 1, \ldots, 10 \).

One may also argue that the notions ‘minimum-norm right inverse’ and ‘least-squares left inverse’ are justified for real-valued parameter matrices \( B^R = B^*(B^B)^{-1} \) and \( B^L = (B^B)^{-1}B^T \), respectively, so that transferring them to polynomial matrices (and rational matrices) introduces doubts and reservations. Although the transferring problem itself is indeed an open issue, we wish to stress once more that our solution is the time-domain one and there is no such transferring needed in our case at all. An attempt to find the minimum-norm solution in the Frobenius norm-involved polynomial matrix case has been presented by Kwan and Kok (2006), but it can by no means be applied to our time-domain formulated inverse problem.

It is worth emphasizing once more that, although our solution to the inverse-model control problem is time-domain, all the introduced polynomial matrix inverses can be used both in the time-domain-valued \( B^R(q^{-1}) \) form and the complex operator-associated version \( B^R(z) \). In other words, in our control synthesis problem we can operate either with \( B^R(q^{-1}) \) or \( B^R(z) \) or \( B^H(z) \), although with the associated time-domain norms involving regular transposes. Therefore, we have exchangeably operated with various arguments of polynomial/rational matrices.

The computation of right/left inverses for nonsquare polynomial matrices is a rather unexplored research area, so the new results obtained in the paper are accompanied with a series of open problems. In addition to a general open issue of possible transferring of minimum-norm/least-squares inverses of parameter matrices to the polynomial matrix case (which is of no interest in our case), there have been a number of detailed open problems scattered throughout the paper and specified below. Since \( \tau \)-inverses have appeared not to be of practical interest, we have skipped over the open issue of formal defining
of the inverses, which might deserve a research effort, at least from a theoretical point of view. Likewise, a theoretical analysis of unfavorable properties of τ-inverses, in terms of the appearance of stable/unstable poles, might be considered.

The new σ-inverses are definitely worth further research effort. Firstly, the difficult problem of proving Conjecture 1 presents a research challenge. Secondly, a general method for the selection of $\beta(q^{-1})$ to provide a stable σ-inverse is awaiting its developer. Similarly, we lack a general methodology for designing $S$-inverses with arbitrary zeros. Also, it might be interesting to analyze properties of σ-inverses when $\beta(q^{-1})$ is not of full normal rank, even though in those cases $B(q^{-1})B^H(q^{-1}) \neq I_{n_y}$, in general. Last but not least, it is of interest to seek a relation between the unique finite/infinite zero structure of the Moore–Penrose inverse of a nonsquare polynomial matrix and a finite/infinite zero structure of that matrix, the open issue brought to the authors’ attention by one of the anonymous reviewers. The above open problems are the subject of our current research works, in addition to seeking further application areas for nonsquare inverse polynomial matrices.

11. Conclusions

In this paper, new analytical expressions for various right/left inverses of full normal rank nonsquare polynomial matrices have been presented. The new inverses have been shown to originate from the minimum variance control strategy, the application that has stimulated research interest in this strictly mathematical field. In addition to the well-known but renamed $T$-inverses, three new classes of the inverses have been introduced, that is, $\tau$-inverses, $\sigma$-inverses and $S$-inverses. Also, the extreme points and extreme directions method has been adopted to the inverse polynomial matrix solution, but the approach has not been found promising here.

A number of the most algorithmically complicated $\tau$-inverses is finite so that their application is limited, even though nice, combinatorics-related, algorithmic and theoretical results have been achieved. A number of more general, algorithmically simple σ-inverses is infinite but, so far, they suffer from the lack of a general methodology for designing stable-pole polynomial matrix inverses, a crucial requirement in control-related applications. This disadvantage of σ-inverses is removed in the most general and algorithmically simple $S$-inverses, employing the Smith factorization approach.

In addition to ‘natural’ applications of the introduced polynomial matrix inverses in inverse-model control, in particular robust minimum variance control, possible applications in a variety of communication/vision ‘perfectualizing’ tasks have been indicated, with all of them still waiting for an extension of the up-to-date exploited minimum-norm/least-squares polynomial matrix inverses.

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Appendix A

Expanding $B^{R5}$ for $\beta(q^{-1}) = b_0 + b_2q^{-1}$ and

$$(b_0 + b_2q^{-1})^R = [I + (b_0)^Rb_2q^{-1}]^{-1}(b_0)^R,$$  

we arrive at

$$B^{R8}(q^{-1}) = \{I + \{I + (b_0)^Rb_2q^{-1}\}^{-1}(b_0)^R\} \times (b_2q^{-2})^{-1}\{I + (b_0)^Rb_2q^{-1}\}^{-1}(b_0)^R,$$

while for

$$(b_0 + b_2q^{-1})^R = [I + (b_0)^Rb_2q^{-1}]^{-1}(b_2q^{-1})^R$$

we have

$$B^{R9}(q^{-1}) = \{I + \{I + (b_0)^Rb_2q^{-2}\}^{-1}(b_0)^R\} \times (b_2q^{-2})^{-1}\{I + (b_0)^Rb_2q^{-1}\}^{-1} \times (b_0q^{-1})^R.$$  

By the same token, expanding $B^{R6}$ for $\beta(q^{-1}) = b_0 + b_2q^{-2}$ with

$$(b_0 + b_2q^{-2})^R = [I + (b_0)^Rb_2q^{-2}]^{-1}(b_0)^R$$

and

$$(b_0 + b_2q^{-2})^R = [I + (b_0)^Rb_2q^{-2}]^{-1}(b_2q^{-2})^R$$

we obtain

$$B^{R10}(q^{-1}) = \{I + \{I + (b_0)^Rb_2q^{-2}\}^{-1}(b_0)^R\} \times (b_0q^{-1})^{-1}\{I + (b_0)^Rb_2q^{-2}\}^{-1}(b_0)^R.$$  

and

$$B^{R11}(q^{-1}) = \{I + \{I + (b_0)^Rb_2q^{-2}\}^{-1}(b_0)^R\} \times \{I + (b_0)^Rb_2q^{-2}\}^{-1}(b_0q^{-2})^R,$$

respectively.

Finally, expanding $B^{R7}$ for $\beta(q^{-1}) = b_0q^{-1} + b_2q^{-2}$ with

$$(b_0q^{-1} + b_2q^{-2})^R = [I + (b_0q^{-1})^Rb_2q^{-2}]^{-1}(b_0q^{-1})^R$$

and

$$(b_0q^{-1} + b_2q^{-2})^R = [I + (b_2q^{-2})^Rb_0q^{-1}]^{-1}(b_2q^{-2})^R,$$

we have

$$B^{R12}(q^{-1}) = \{I + \{I + (b_0q^{-1})^Rb_2q^{-2}\}^{-1}(b_0q^{-1})^R\} \times (b_0)^{-1}\{I + (b_2q^{-2})^Rb_0q^{-1}\}^{-1}(b_2q^{-2})^R,$$

and

$$B^{R13}(q^{-1}) = \{I + \{I + (b_2q^{-2})^Rb_0q^{-1}\}^{-1}(b_2q^{-2})^R\} \times (b_0)^{-1}\{I + (b_2q^{-2})^Rb_0q^{-1}\}^{-1}(b_2q^{-2})^R,$$

respectively.

Note that all the above identity matrices shall be read as $I_n$.

Appendix B

Proof of Theorem 1. Table 1 gives the numbers of $\tau$-inverses calculated according to Algorithm 1 at each step $k = 1, \ldots, m$ and for each set $i = 1, \ldots, m$ of the inverses. Table 2 is a sample of Table 1 for $m = 4$. We use the standard notation for $C_i^k = k!/i!(k - i)!$ as the number of $i$-combinations without repetition from a $k$-element set.

It is essential that we calculate the total number $S$ of $\tau$-inverses as a sum of all the subsums counted by columns

$$S = 1 + \sum_{i=1}^{m} C_{m+1}^i \sum_{k=1}^{i} F_{k,i},$$  

with $C_{m+1}^i = 1$ being the number of the $T$-inverse. The constructions of the numbers of inverses for each set $i = 1, \ldots, m$ allows calculating the factor

$$\sum_{k=1}^{i} F_{k,i} = N_{i-1}, \quad i = 1, \ldots, m,$$

in a recurrent way. Specifically, for the column $i = 2$ we have $F_{12} + F_{22} = N_1 = 1 + C_2^1 N_0$, where $N_0 = 1$, and for the column $i = 3$ there is $F_{13} + F_{23} + F_{33} = N_2 = 1 + C_3^1 N_0 + C_3^2 N_1$.  

Table 1. Numbers of $r$-inverses for steps $k = 1, \ldots, m$ and inverse sets $i = 1, \ldots, m$.

|   | 0 | 1 | 2 | \cdots | $i$ | \cdots | $m - 1$ | $m$ |
|---|---|---|---|-------|-----|-------|-------|-----|
| 0 | $C_{m+1}^{m+1}$ | $C_{m+1}^1 F_{1,1}$ | $C_{m+1}^2 F_{1,2}$ | \cdots | $C_{m+1}^i F_{1,i}$ | \cdots | $C_{m+1}^{m-1} F_{1,m-1}$ | $C_{m+1}^m F_{1,m}$ |
| 1 | $F_{1,1} = 1$ | $F_{1,2} = 1$ | $F_{1,i} = 1$ | \cdots | $F_{1,m-1} = 1$ | $F_{1,m} = 1$ |
| 2 | $F_{2,2} = \sum_{j=1}^{i-1} C_{1}^j F_{1,j}$ | $F_{2,i} = \sum_{j=1}^{i-1} C_{1}^j F_{1,j}$ | \cdots | $F_{2,m-1} = \sum_{j=1}^{m-2} C_{1}^j F_{1,j}$ | $F_{2,m} = \sum_{j=1}^{m-1} C_{1}^j F_{1,j}$ |
| $k$ | $F_{k,i} = \sum_{j=k-1}^{i-1} C_{1}^j F_{k-1,j}$ | \cdots | $F_{k,m-1} = \sum_{j=k-1}^{m-2} C_{1}^j F_{k-1,j}$ | $F_{k,m} = \sum_{j=k-1}^{m-1} C_{1}^j F_{k-1,j}$ |
| $m - 1$ | $F_{m-1,m-1} = \sum_{j=m-2}^{m-1} C_{1}^j F_{m-2,j}$ | $F_{m-1,m} = \sum_{j=m-2}^{m-1} C_{1}^j F_{m-2,j}$ |
| $m$ | $F_{m,m} = \sum_{j=m-1}^{m-1} C_{1}^j F_{m-1,j}$ | \cdots | 0 | \cdots | 0 |
The general recurrence can be easily shown to be

\[ N_i = 1 + \sum_{j=1}^{i} C_{i+1}^j N_{j-1}, \quad i = 1, \ldots, m - 1. \]

The last recurrence is obviously included in Eqn. (27) with \( S = N_m \). Expanding \( C_{i+1}^j \) yields the final result (13).

### Appendix C

**Control zeros for \( B(z) \) as in Example 1.**

- unstable \( B^R(z) \): 0.1, 0.1063 ± 0.4429i, 0.0190 ± 1.8760i;
- stable \( B^R(z) \): 0.1, 0.2715 ± 0.5377i, 0.0032 ± 0.7353i.

### Appendix D

**Matrices \( Z_1(w) \) and \( Z_2(w) \) in Example 2.**

\[
Z_1(w) = \begin{bmatrix}
2 + 2.4w + 0.2w^2 + w^3 & 1.9 + 1.1w - 1.3w^2 + 2.1w^3 + 0.9w^4 + 0.062w^5 + 0.65w^6 \\
0.24 - 1.5w + 0.21w^2 - 0.69w^3 & 0.25 - 1.7w + w^2 - 0.61w^3 - 0.54w^4 + 0.17w^5 - 0.43w^6 \\
-0.48 + 1.2w + 0.34w^2 & -0.82 + 1.6w - 0.3w^2 - 0.57w^3 + 0.75w^4 + 0.22w^5 \\
1.8 + 0.87w - 2.8w^2 + 1.2w^3 - 1.3w^4 & \\
0.87 - 2.3w + 2.2w^2 - 1.2w^3 + 0.84w^4 & \\
-0.85 + 2.2w - w^2 - 0.42w^3 & \\
\ldots & 1.5 - 4w - 11w^2 + 4.2w^3 - 4.3w^4 - 1.1w^5 + 0.8w^6 - 0.79w^7 \\
\ldots & 2.4 - 5.1w + 8.3w^2 - 3w^3 + 2w^4 + 0.97w^5 - 0.8w^6 + 0.53w^7 \\
\ldots & -2.1 + 5.1w - 5.6w^2 - 1.5w^3 + 1.7w^4 - 0.63w^5 - 0.26w^6
\end{bmatrix}
\]

\[
Z_2(w) = \begin{bmatrix}
0.87 - 2.3w + 2.2w^2 - 1.2w^3 + 0.84w^4 & \ldots & \\
-0.85 + 2.2w - w^2 - 0.42w^3 & \ldots & \\
\ldots & 1.5 - 4w - 11w^2 + 4.2w^3 - 4.3w^4 - 1.1w^5 + 0.8w^6 - 0.79w^7 & \\
\ldots & 2.4 - 5.1w + 8.3w^2 - 3w^3 + 2w^4 + 0.97w^5 - 0.8w^6 + 0.53w^7 & \\
\ldots & -2.1 + 5.1w - 5.6w^2 - 1.5w^3 + 1.7w^4 - 0.63w^5 - 0.26w^6
\end{bmatrix}
\]
Appendix F

An example of a $10 \times 1$ MFD plant

$$G(z) = \frac{1}{z^3 + 0.5z^2 - 0.5z} \left[ 
\begin{array}{cccccccc}
10z^2 + 5z + 3 & z^2 - 0.5z + 0.3 & z^2 + 0.5z + 0.3 & 7z^2 + 0.5z - 0.13 & \\
\ldots & 17z^2 + 0.5z + 0.13 & z^2 + 0.5z + 0.1 & 1.1z^2 + 0.52z + 0.11 & \\
\ldots & -2z^2 + 0.5z + 0.4 & -2z^2 + z + 0.4 & -12z^2 + 5z + 4 & 
\end{array}
\right],$$

with the squaring down $\sigma$-inverse

$$B^R(z) = \begin{bmatrix}
0 & \ldots & 0 & \Psi & 0 & \ldots & 0
\end{bmatrix}^T,$$

produced by

$$\beta(z) = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0
\end{bmatrix}.$$