STRONG FELLER PROPERTY FOR SDES DRIVEN BY MULTIPlicative CYLINDRICAL STABLE NOISE

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Abstract. We consider the stochastic differential equation \( dX_t = A(X_{t-}) dZ_t, \) \( X_0 = x, \) driven by cylindrical \( \alpha \)-stable process \( Z_t \) in \( \mathbb{R}^d \), where \( \alpha \in (0, 1) \) and \( d \geq 2 \). We assume that the determinant of \( A(x) = (a_{ij}(x)) \) is bounded away from zero, and \( a_{ij}(x) \) are bounded and Lipschitz continuous. We show that for any fixed \( \gamma \in (0, \alpha) \) the semigroup \( P_t \) of the process \( X_t \) satisfies
\[
|P_t f(x) - P_t f(y)| \leq ct^{-\gamma/\alpha} |x - y|^\gamma \|f\|_\infty
\]
for arbitrary bounded Borel function \( f \). Our approach is based on Levi’s method.

1. Introduction

Let \( Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T \) be a cylindrical \( \alpha \)-stable process, that is \( Z_t^{(1)}, \ldots, Z_t^{(d)} \) are independent one-dimensional symmetric standard \( \alpha \)-stable processes of index \( \alpha \in (0, 1), d \in \mathbb{N}, d \geq 2 \). We consider the stochastic differential equations
\[
dX_t = A(X_{t-}) dZ_t, \quad X_0 = x \in \mathbb{R}^d,
\]
where \( A(x) = (a_{ij}(x)) \) is a \( d \times d \) matrix and there are constants \( \eta_1, \eta_2, \eta_3 > 0 \), such that for any \( x, y \in \mathbb{R}^d, i, j \in \{1, \ldots, d\} \)
\[
|a_{ij}(x)| \leq \eta_1, \quad \det(A(x)) \geq \eta_2, \quad |a_{ij}(x) - a_{ij}(y)| \leq \eta_3 |x - y|.
\]

It is well known that SDEs (1) has a unique strong solution \( X_t \), see e.g. [28, Theorem 34.7 and Corollary 35.3]. By [31, Corollary 3.3] \( X_t \) is a Feller process.

Let \( \mathbb{E}^x \) denote the expected value of the process \( X_t \) starting from \( x \) and \( \mathcal{B}_b(\mathbb{R}^d) \) denote the set of all Borel bounded functions \( f : \mathbb{R}^d \to \mathbb{R} \). For any \( t \geq 0, x \in \mathbb{R}^d \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \) we put
\[
P_t f(x) = \mathbb{E}^x f(X_t).
\]

The main result of this paper is the following theorem, which gives the strong Feller property of the semigroup \( P_t \).

Theorem 1.1. For any \( \gamma \in (0, \alpha), \tau > 0, t \in (0, \tau], x, y \in \mathbb{R}^d \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \) we have
\[
|P_t f(x) - P_t f(y)| \leq ct^{-\gamma/\alpha} |x - y|^\gamma \|f\|_\infty,
\]
where \( c \) depends on \( \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \gamma \).

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Strong Feller property for SDEs driven by additive cylindrical Lévy processes have been intensively studied recently (see e.g. [30, 36, 11]). The SDE (1) (with multiplicative noise) was studied by Bass and Chen in [1]. They proved existence and uniqueness of weak solutions of SDE (1) under very mild assumptions on matrices $A(x)$ (i.e. they assumed that $A(x)$ are continuous and bounded in $x$ and nondegenerate for each $x$). In [24] SDE (1) was studied for diagonal matrices $A(x)$, which diagonal coefficients are bounded away from zero, from infinity and Hölder continuous. Under these assumptions the corresponding transition density $p^A(t, x, y)$ was constructed and Hölder estimates $x \rightarrow p^A(t, x, y)$ were obtained. These estimates imply strong Feller property of the corresponding semigroup.

The case of non-diagonal matrices $A(x)$, treated in this paper, is much more difficult. The strong Feller property for semigroups generated by solutions to SDEs is often obtained by suitable versions of the Bismut-Elworthy-Li formula. We were not able to get such formula but we use instead Levi’s method to construct the semigroup $P_t$ and to obtain Theorem 1.1. However there are many problems in applying this method to the case of non-diagonal matrices $A(x)$. Therefore we had to introduce some new ideas. Below we briefly describe the main steps in our approach.

The first problem with Levi’s method in our case is that the standard approximation of the transition density (the so-called “frozen density”) does not have good integrability properties. To overcome this we truncate the Lévy measure of the process $Z_t$ in a convenient way. Then, using Levi’s method, we construct the transition density (denoted by $u(t, x, y)$) of the solution of (1) driven by this truncated process. As usual we represent $u(t, x, y)$ as a series $\sum_{n=0}^{\infty} q_n(t, x, y)$. Typically, in many papers using Levi’s method, the first step was to obtain precise bounds for $q_0(t, x, y)$ which allow to estimate $q_n(t, x, y)$ inductively point-wise. In our case it seems impossible to obtain such precise bounds, hence we prove (see Proposition 3.9) some different kind of results for $q_0(t, x, y)$, which are sufficient for our purposes. The main tools to prove Proposition 3.9 are Lemma 3.6 and the estimates (13). These key estimates (13) are proven using the techniques and results from [23], [22] and [33]. After constructing the transition density $u(t, x, y)$ we use the technique developed by Knopova and Kulik [19] to show that $u(t, x, y)$ satisfies the appropriate heat equation in the so-called approximate setting. In the next step we construct the semigroup $T_t$ for the solution of SDE (1) (driven by the not truncated process). Roughly speaking, this construction is based on adding long jumps to the truncated process. Next we show that $u(t, x) := T_t f(x)$ satisfies the appropriate heat equation in the approximate setting (see Lemma 4.18), which allows to prove that the constructed semigroup $T_t$ is in fact the semigroup $P_t$.

Our current technique is restricted to the case $\alpha \in (0, 1)$. The main difficulty for $\alpha \in [1, 2)$ is that in such case one has to effectively estimate the expression

$$p_y(t, x + a_i(x)w) + p_y(t, x - a_i(x)w) - 2p_y(t, x) \quad (7)$$

instead of

$$p_y(t, x + a_i(x)w) - p_y(t, x), \quad (8)$$

where $p_y(t, x)$ is the frozen density for the truncated process (see Section 3 for the precise definition of $p_y(t, x)$) and $a_i(x) = (a_{1i}(x), \ldots, a_{di}(x))$. Our crucial estimate (13) allows suitable estimate of (8) but fails to bound (7) in a way sufficient for our purpose.
It is worth mentioning that strong Feller property and gradient estimates for the semigroups associated to SDEs driven by Lévy processes in $\mathbb{R}^d$ with jumps, with absolutely continuous Lévy measures, have been studied for many years (see e.g. [34, 32, 25, 39, 35, 37]).

One may ask about further regularity properties of the semigroup $P_t$, in particular about boundedness of the operators $P_t : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$, which is related to the boundedness of the transition densities of $P_t$. It turns out that for some choices of matrices $A(x)$ (satisfying (2), (3), (4)) and for some $t > 0$ the operators $P_t : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ are not bounded (see Remark 4.23 and Remark 4.24). Nevertheless we have the following regularity result.

**Theorem 1.2.** For any $\gamma \in (0, \alpha/d)$, $\tau > 0$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have

$$|P_tf(x)| \leq ct^{-\gamma d/\alpha} \|f\|_{1-\gamma} \|f\|_{1},$$

where $c$ depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \gamma$.

The paper is organized as follows. In Section 2 we study properties of the transition density of a suitably truncated one-dimensional stable process. These properties are crucial in the sequel. In Section 3 we construct the transition density $u(t, x, y)$ of the solution of (1) driven by the truncated process. We also show that it satisfies the appropriate equation in the approximate setting. In Section 4 we construct the transition semigroup of the solution of (1). We also prove Theorems 1.1 and 1.2.

## 2. Preliminaries

All constants appearing in this paper are positive and finite. In the whole paper we fix $\tau > 0$, $\alpha \in (0, 1)$, $d \in \mathbb{N}$, $d \geq 2$, $\eta_1, \eta_2, \eta_3$ where $\eta_1, \eta_2, \eta_3$ appear in (2), (3) and (4). We adopt the convention that constants denoted by $c$ (or $c_1, c_2, \ldots$) may change their value from one use to the next. In the whole paper, unless explicitly stated otherwise, we understand that constants denoted by $c$ (or $c_1, c_2, \ldots$) depend on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$. We also understand that they may depend on the choice of the constants $\varepsilon$ and $\gamma$. We write $f(x) \approx g(x)$ for $x \in A$ if $f, g \geq 0$ on $A$ and there is a constant $c \geq 1$ such that $c^{-1}f(x) \leq g(x) \leq cf(x)$ for $x \in A$. The standard inner product for $x, y \in \mathbb{R}^d$ we denote by $xy$.

For any $t > 0$, $x \in \mathbb{R}^d$ we define the measure $\sigma_t(x, \cdot)$ by

$$\sigma_t(x, A) = \mathbb{P}^x(X_t \in A),$$

for any Borel set $A \subset \mathbb{R}^d$. $\mathbb{P}^x$ denotes the distribution of the process $X$ starting from $x \in \mathbb{R}^d$. For any $t > 0$, $x \in \mathbb{R}^d$ we have

$$P_tf(x) = \int_{\mathbb{R}^d} f(y)\sigma_t(x, dy), \quad f \in \mathbb{B}_b(\mathbb{R}^d).$$

It is well known that the density of the Lévy measure of the one-dimensional symmetric standard $\alpha$-stable process is given by $A_\alpha|x|^{-1-\alpha}$, where $A_\alpha = 2^{\alpha}\Gamma((1 + \alpha)/2)/\pi^{\alpha/2}\Gamma(-\alpha/2))$. In the sequel we will need to truncate this density. The truncated density will be denoted by $\mu^{(\delta)}(x)$. Let $\mu^{(\delta)} : \mathbb{R} \setminus \{0\} \to [0, \infty)$ where $\delta \in (0, 1)$. For $x \in (0, \delta]$ we put $\mu^{(\delta)}(x) = A_\alpha|x|^{-1-\alpha}$, for $x \in (\delta, 2\delta]$ we put $\mu^{(\delta)}(x) \in (0, A_\alpha|x|^{-1-\alpha})$ and for $x \geq 2\delta$ we put $\mu^{(\delta)}(x) = 0$. Moreover, $\mu^{(\delta)}$ is defined so that it is weakly decreasing, weakly convex and $C^1$ on $(0, \infty)$ and satisfies $\mu^{(\delta)}(-x) = \mu^{(\delta)}(x)$ for $x \in (0, \infty)$.
We also define
\[ S^{(\delta)} f(x) = \lim_{\xi \to 0^+} \int_{|w| > \xi} (f(x + w) - f(x)) \mu(\delta)(w) \, dw. \]

By \( g^{(\delta)}_t \) we denote the heat kernel corresponding to \( S^{(\delta)} \) that is
\[ \frac{\partial}{\partial t} g^{(\delta)}_t(x) = S^{(\delta)} g^{(\delta)}_t(x), \quad t > 0, x \in \mathbb{R}, \]
\[ \int_R g^{(\delta)}_t(x) \, dx = 1, \quad t > 0. \]

It is well known that \( g^{(\delta)}_t \) belongs to \( C^1((0, \infty)) \) as a function of \( t \) and belongs to \( C^2(\mathbb{R}) \) as a function of \( x \).

For any \( \varepsilon \in (0, 1], \tau > 0 \), \( t \in (0, \infty) \) and \( x \in \mathbb{R} \) we define
\[ h^{(\varepsilon)}_t(x) = \begin{cases} \frac{\varepsilon x}{|x|^{1+1/\alpha}} & \text{for } |x| < \varepsilon, \\ c_\varepsilon t^{1+(d-1)/\alpha} e^{-|x|} & \text{for } |x| \geq \varepsilon, \end{cases} \]
where \( c_\varepsilon = \frac{\varepsilon}{(1+1/\alpha)^{1+1/\alpha}} \). The constant \( c_\varepsilon \) is chosen so that for any \( t \in (0, \tau] \) the function \( x \to h^{(\varepsilon)}_t(x) \) is nonincreasing on \([0, \infty)\).

**Lemma 2.1.** For any \( \varepsilon \in (0, 1] \), there exist \( c \) such that for \( \delta = \varepsilon \min \left\{ \frac{\alpha}{\beta(n+d+2)}, \frac{1}{\beta(n+1)^2} \right\} \), and any \( t \in (0, \tau], x, y \in \mathbb{R} \), we have
\[ g^{(\delta)}_t(x) \leq c h^{(\varepsilon)}_t(|x|), \quad |g^{(\delta)}_t(x) - g^{(\delta)}_t(y)| \leq c |x - y| \left( \frac{h^{(\varepsilon)}_t(|x|)}{t^{1/\alpha} + |x|} + \frac{h^{(\varepsilon)}_t(|y|)}{t^{1/\alpha} + |y|} \right). \]

**Proof.** First we consider a general case of heat kernels \( g^{(\delta,n)} \) on \( \mathbb{R}^n \) for \( \delta \in (0, 1], n \in \{1, 2, \ldots\} \) such that
\[ g^{(\delta,n)}_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ixu} e^{-\Phi^{(n)}_\delta(u)} \, du, \]
where
\[ \Phi^{(n)}_\delta(u) = \int_{\mathbb{R}^n} (1 - \cos(uy)) \mu^{(\delta,n)}(y) \, dy, \quad u \in \mathbb{R}^n, \]
and \( \mu^{(\delta,n)}(y) = \mu^{(\delta,n)}(|y|) \) is isotropic unimodal Lévy density such that \( \mu^{(\delta,n)}(y) \approx |y|^{-n-\alpha} \) for \( |y| \leq \delta \), and \( \mu^{(\delta,n)} = 0 \) for \( |y| \geq 2\delta \). In the proof of this lemma we assume that constants \( c \) may additionally depend on \( n \). It follows from Lemma 1 in [3] that
\[ \frac{2}{n \pi^s} H(|u|) \leq \Phi^{(n)}_\delta(u) \leq 2H(|u|), \quad u \in \mathbb{R}^n, \]
where
\[ H(r) = \int_{\mathbb{R}^n} (1 \wedge r|y|)^2 \mu^{(\delta,n)}(y) \, dy, \quad r \geq 0, \]
hence we easily obtain
\[ \Phi^{(n)}_\delta(u) \geq c |u|^{2} \int_0^{\frac{1}{2}r^\delta} r^{1-\alpha} \, dr = c \frac{1}{\pi^s} \left((\delta|u|)^2 \wedge (\delta|u|)^\alpha \right), \quad u \in \mathbb{R}^n. \]
Similarly
\[ \Phi^{(n)}_\delta(u) \leq c \frac{1}{\pi^s} \left((\delta|u|)^2 \wedge (\delta|u|)^\alpha \right), \quad u \in \mathbb{R}^n. \]
In particular the symbol $\Phi_\delta^{(n)}$ has global weak lower scaling property with index $\alpha$ (see [3]). This yields, by Theorem 21 of [3],

$$g_t^{(\delta,n)}(x) \leq c \min\{(H^{-1}(1/t))^n, tH(1/|x|)|x|^{-n}\}, \quad t > 0, x \in \mathbb{R}^n.$$ 

Observing that $H(r) \leq cr^\alpha$ for $r \geq 0$ and $H^{-1}(1/t) \leq c_{1/t^\alpha}$, for $t \leq \tau$, we have

$$g_t^{(\delta,n)}(x) \leq c t^{1+\frac{\delta}{\alpha}} |x|^{-\delta} \quad x \in \mathbb{R}^n, t \in (0, \tau]. \quad (14)$$

Let $t \leq 1 \wedge \tau$. Using Lemma 4.2 from [33] we get

$$g_t^{(\delta,n)}(x) \leq e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)} g_t^{(\delta,n)}(0) = \left( \frac{m_0}{\delta} \right)^{\frac{|x|}{|x|}} g_t^{(\delta,n)}(0) \leq c t^{1+\frac{\delta}{\alpha}} e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)}, \quad |x| \geq \ell n \delta,$$

where $m_0 = \int_{\mathbb{R}^n} |y|^{2} \mu^{(\delta,n)}(y) \, dy \approx \delta^{-2\alpha}$. This yields

$$g_t^{(\delta,n)}(x) \leq c t^{1+\frac{\delta}{\alpha}} e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)},$$

provided $|x| \geq \max\{8\delta(1 + \frac{n+d-1}{\alpha}), \frac{\ell n \delta}{\delta}, t \}$. We observe that there exists $c_1 = c_1(\delta, \alpha, d, n)$ such that

$$e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)} \leq c_1 e^{-|x|/\alpha}, \quad x \in \mathbb{R}^n,$$

so we obtain

$$g_t^{(\delta,n)}(x) \leq c_2 t t^{1+\frac{\delta}{\alpha}} e^{-|x|/\alpha}, \quad |x| \geq \max\{8\delta(1 + \frac{n+d-1}{\alpha}), \frac{\ell n \delta}{\delta}, t \}, \quad (15)$$

with $c_2 = c_2(\delta, \alpha, d, n)$.

Let $1 \leq t \leq \tau$. Using again Lemma 4.2 from [33] we get

$$g_t^{(\delta,n)}(x) \leq e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)} g_t^{(\delta,n)}(0) \leq e^{-\frac{|x|}{8\pi |x|} \log \left( \frac{\delta|x|}{\delta n} \right)}, \quad |x| \geq \ell n \delta.$$

Let $\delta < \frac{\ell n \delta}{\delta (\alpha + d + |x|)}$. Now, (12) follows from (14), (15) and (16) with $n = 1$.

The function $\mu^{(\delta)}$ satisfies the assumption of Theorem 1.5 in [23] which yields that there exists a Lévy process $X_t^{(3)}$ in $\mathbb{R}^3$ with the characteristic exponent $\Phi^{(3)}_\delta(u) = \mathcal{P}_\delta^{(3)}(|u|), u \in \mathbb{R}^3$ and the radial, radially nonincreasing transition density $g_t^{(\delta,3)}(x) = g_t^{(\delta,3)}(|x|)$ satisfying

$$g_t^{(\delta,3)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} g_t^{(\delta,1)}(r), \quad r > 0.$$ 

Furthermore it follows from the proof of Theorem 1.5 in [23] that the Lévy measure of the process $X_t^{(3)}$ is given by $\mu^{(\delta,3)}(dx) = \mu^{(\delta,3)}(x) \, dx$, where $\mu^{(\delta,3)}(R) = \frac{-1}{2\pi R} \frac{d}{dR} \mu^{(\delta,3)}(R)$. In particular $\mu^{(\delta,3)}(R)$ is nonincreasing on $(0, \infty)$ and we have $\mu^{(\delta,3)}(x) = \frac{A_0}{2\pi(2+\alpha)} |x|^{-3-\alpha}$ for $|x| \leq \delta, \mu^{(\delta,3)}(x) \leq \frac{4}{2\pi(2+\alpha)} \delta^{3-\alpha},$ for $\delta \leq |x| \leq 2\delta,$ and $\mu^{(\delta,3)}(x) = 0$ for $|x| > 2\delta$.

By (14), (15) and (16), with $n = 3$, we obtain

$$g_t^{(\delta,3)}(x) \leq c t^{1+\frac{\delta}{\alpha}} e^{-|x|/\alpha}, \quad x \in \mathbb{R}^3, t \in (0, \tau]$$

and

$$g_t^{(\delta,3)}(x) \leq c t t^{1+\frac{\delta}{\alpha}} e^{-|x|/\alpha} \leq c t^{1+\frac{\delta}{\alpha}} e^{-|x|/\alpha}, \quad |x| > \max\{\varepsilon, c_3 t\}, t \in (0, \tau].$$
The above two inequalities yield
\[
g_t^{(6,3)}(x) \leq c \frac{h_t^{(e)}(|x|)}{(|x| + t^{1/\alpha})^2}, \quad x \in \mathbb{R}^3, \, t \in (0, \tau],
\]
and
\[
\left| \frac{d}{dr} g_t^{(6,1)}(r) \right| \leq c \frac{rh_t^{(e)}(r)}{(r + t^{1/\alpha})^2} \leq c \frac{h_t^{(e)}(r)}{r + t^{1/\alpha}}, \quad r > 0, \, t \in (0, \tau].
\]
Since \( h_t^{(e)} \) is nonincreasing, by the Lagrange theorem, we get
\[
|g_t^{(6,1)}(x) - g_t^{(6,1)}(y)| \leq c|x - y| \left( \frac{h_t^{(e)}(|x|)}{|x| + t^{1/\alpha}} + \frac{h_t^{(e)}(|y|)}{|y| + t^{1/\alpha}} \right), \quad x, y \in \mathbb{R}.
\]

\[\square\]

**Lemma 2.2.** Let \( \epsilon \in (0, 1] \). For any \( t \in (0, \tau], \, x, x' \in \mathbb{R} \) if \(|x - x'| \leq t^{1/\alpha} \) and \(|x - x'| \leq \epsilon/4 \) then
\[
h_t^{(e)}(x') \leq ch_t^{(e)}(x/2).
\]

**Proof.** Assume first that \(|x| \wedge |x'| \leq \epsilon/2\). Then by the definition of \( h_t^{(e)}(x) \) we have
\[
h_t^{(e)}(x) = \frac{t}{(t^{1/\alpha} + |x|)^{1+\alpha}} \geq \frac{t}{(t^{1/\alpha} + |x - x'| + |x'|)^{1+\alpha}} \geq \frac{t}{(2t^{1/\alpha} + 2|x'|)^{1+\alpha}} \geq ch_t^{(e)}(x').
\]
Assume now that \(|x| \wedge |x'| > \epsilon/2\). Then we have \(|x'| \geq |x| - |x' - x| \geq |x|/2 = |x|/2\). Hence \( h_t^{(e)}(x') \leq h_t^{(e)}(x/2)\). \[\square\]

**Lemma 2.3.** Let \( \epsilon \in (0, 1] \), \( \delta = \epsilon \min\{\frac{\alpha}{\min(\alpha+1, \delta)}, \frac{1}{4d \sqrt{\min(\alpha, \delta)^2}}\} \). For any \( t \in (0, \tau], \, x, x' \in \mathbb{R}^d \) if \(|x - x'| \leq t^{1/\alpha} \) and \(|x - x'| \leq \delta \) then
\[
\left| \prod_{i=1}^{d} g_t^{(\delta)}(x_i) - \prod_{i=1}^{d} g_t^{(\delta)}(x'_i) \right| \leq c \left( \prod_{i=1}^{d} h_t^{(e)}(x_i/2) \right) \left[ 1 \wedge \sum_{j=1}^{d} t^{-1/\alpha} |x_j - x'_j| \right]. \tag{17}
\]

**Proof.** By Lemma 2.1 we get
\[
\left| \prod_{i=1}^{d} g_t^{(\delta)}(x_i) - \prod_{i=1}^{d} g_t^{(\delta)}(x'_i) \right| \leq \sum_{j=1}^{d} \left| g_t^{(\delta)}(x_j) - g_t^{(\delta)}(x'_j) \right| \prod_{i \not= j, 1 \leq i \leq d} g_t^{(\delta)}(|x_i| \wedge |x'_i|) \leq c \left( \prod_{i=1}^{d} h_t^{(e)}(|x_i| \wedge |x'_i|) \right) \sum_{j=1}^{d} \frac{|x_j - x'_j|}{t^{1/\alpha}}.
\]
Clearly we have
\[
\left| \prod_{i=1}^{d} g_t^{(\delta)}(x_i) - \prod_{i=1}^{d} g_t^{(\delta)}(x'_i) \right| \leq \prod_{i=1}^{d} g_t^{(\delta)}(|x_i| \wedge |x'_i|). \]
Now the assertion follows from Lemmas 2.1 and 2.2. \[\square\]
Lemma 2.4. Let \( \varepsilon \in (0, 1] \), \( \delta = \varepsilon \min\left\{ \frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(1+\varepsilon)} \right\} \) and let \( a \in \mathbb{R} \) be such that \( |a| \leq \frac{\varepsilon}{4\delta} \). Then there exists \( c \) such that for any \( t \in (0, \tau] \), \( x \in \mathbb{R} \) we have

\[
\int_{\mathbb{R}} |g_t^{(\delta)}(x + aw) - g_t^{(\delta)}(x)| |\mu^{(\delta)}(w)| \, dw \leq \frac{c|a|^\alpha h_t^{(\varepsilon)} \left( \frac{|x|}{\alpha} \right)}{t^{\frac{1}{\alpha}}}. \tag{18}
\]

Proof. First we note that

\[
\int_{\mathbb{R}} |g_t^{(\delta)}(x + aw) - g_t^{(\delta)}(x)| |\mu^{(\delta)}(w)| \, dw \leq \int_{|w| < 2\delta} |g_t^{(\delta)}(x + aw) - g_t^{(\delta)}(x)| |A_\alpha| w|^{-\alpha} \, dw,
\]

and by the substitution \( s = aw \) we have

\[
\int_{|w| < 2\delta} |g_t^{(\delta)}(x + aw) - g_t^{(\delta)}(x)||w|^{-\alpha} \, dw = |a|^\alpha \int_{|s| < 2\delta|a|} |g_t^{(\delta)}(x + s) - g_t^{(\delta)}(x)||s|^{-\alpha} \, ds.
\]

Now we estimate the latter integral. Let

\[
\int_{|s| < 2\delta|a|} |g_t^{(\delta)}(x + s) - g_t^{(\delta)}(x)||s|^{-\alpha} \, ds = \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |g_t^{(\delta)}(x + s) - g_t^{(\delta)}(x)||s|^{-\alpha} \, ds
\]

\[
+ \int_{t^{1/\alpha} \wedge (2\delta|a|) \leq |s| < 2\delta|a|} |g_t^{(\delta)}(x + s) - g_t^{(\delta)}(x)||s|^{-\alpha} \, ds
\]

\[
=: I_1 + I_2.
\]

Using (13) we get

\[
I_1 \leq c \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |s|^{-\alpha} \left( \frac{h_t^{(\varepsilon)}(x + s)}{t^{1/\alpha} + |s|} + \frac{h_t^{(\varepsilon)}(x)}{t^{1/\alpha} + |x|} \right) \, ds
\]

\[
\leq ct^{-1/\alpha} \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |s|^{-\alpha} \left( h_t^{(\varepsilon)}(|x + s|) + h_t^{(\varepsilon)}(|x|) \right) \, ds.
\]

If \( |x| \geq 2t^{1/\alpha} \) then for \( |s| < t^{1/\alpha} \) we have \( h_t^{(\varepsilon)}(|x + s|) \leq h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right) \), since \( h_t^{(\varepsilon)} \) is nonincreasing. If \( |x| \leq 2t^{1/\alpha} \) and \( |s| < t^{1/\alpha} \) then \( h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right) \geq ct^{1/\alpha} \) and \( h_t^{(\varepsilon)}(|x + s|) \leq t^{1/\alpha} \), hence we obtain \( h_t^{(\varepsilon)}(|x + s|) \leq ch_t^{(\varepsilon)} \left( \frac{|x|}{2} \right) \). This yields

\[
I_1 \leq ct^{-1/\alpha} h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right) \int_{|s| < t^{1/\alpha}} |s|^{-\alpha} \, ds = \frac{c}{t} h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right).
\]

Now we estimate \( I_2 \). If \( t^{1/\alpha} > 2\delta|a| \) then \( I_2 = 0 \) so we assume that \( t^{1/\alpha} \leq 2\delta|a| \) and using (12) we obtain

\[
I_2 \leq c \int_{t^{1/\alpha} \leq |s| \leq 2\delta|a|} \left( h_t^{(\varepsilon)}(|x + s|) + h_t^{(\varepsilon)}(|x|) \right) |s|^{-\alpha} \, ds.
\]

Since we have

\[
\int_{|s| \geq t^{1/\alpha}} h_t^{(\varepsilon)}(|x|) |s|^{-\alpha} \, ds = \frac{c}{t} h_t^{(\varepsilon)}(|x|),
\]

we only need to estimate

\[
J = \int_{t^{1/\alpha} \leq |s| \leq 2\delta|a|} h_t^{(\varepsilon)}(|x + s|) |s|^{-\alpha} \, ds.
\]
Let \( g_t^{(\infty)} \) denote the transition density of the one-dimensional symmetric standard \( \alpha \)-stable process. It follows from [2] that \( g_t^{(\infty)}(x) \approx t^{\frac{1}{1+\alpha} \frac{1}{t t^{1/\alpha}}} \), hence \( h_t^{(\varepsilon)}(y) \leq c g_t^{(\infty)}(y) \), for all \( t \in (0, \tau], y \in \mathbb{R} \). Noting also that \( |s|^{-1-\alpha} \leq c \frac{d^{(\infty)}(s)}{c} \) for \( |s| \geq t^{1/\alpha} \), and using Chapman-Kolmogorov equation for \( g_t^{(\infty)} \) we get

\[
J \leq \int_{|s| \geq t^{1/\alpha}} h_t^{(\varepsilon)}(x + s)|s|^{-1-\alpha} \, ds \\
\leq \frac{c}{t} \int_{\mathbb{R}} g_t^{(\infty)}(x + s)g_t^{(\infty)}(s) \, ds = \frac{c}{t} g_{2t}^{(\infty)}(x),
\]

which yields \( J \leq \frac{ch_t^{(\varepsilon)}(x)}{t} \) for \( |x| \leq |\varepsilon| \).

Let now \( |x| \geq \varepsilon \geq 4\delta |a| \geq 2t^{1/\alpha} \). Then \( |x + s| \geq |x|/2 \) for \( s \leq 2\delta |a| \), and we obtain

\[
J \leq \int_{t^{1/\alpha} \leq |s| \leq 2\delta |a|} h_t^{(\varepsilon)}(|x + s|)|s|^{-1-\alpha} \, ds \leq h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right) \int_{t^{1/\alpha} \leq |s|} |s|^{-1-\alpha} \, ds \\
\leq \frac{c}{t} h_t^{(\varepsilon)} \left( \frac{|x|}{2} \right).
\]

\[\square\]

**Lemma 2.5.** For every \( \varepsilon \in (0, 1) \), \( \delta = \varepsilon \min\{\frac{\alpha}{8(\alpha+\delta+1)}, \frac{1}{4(3(\alpha+1))}\} \) and \( m, n \in \mathbb{N}, n \geq 2 \) there exists \( c = c(m, n, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta) \) such that for any \( t \in (0, \tau] \), \( x \in \mathbb{R} \) we have

\[
\left| \frac{d^m}{dx^m} g_t^{(\delta)}(x) \right| \leq ct^{-1/(1+m)/\alpha} (1 + |x|)^{-n}.
\]

**Proof.** In the proof we assume that constants \( c \) may additionally depend on \( m \) and \( n \).

We use Theorem 3 of [16]. Let \( f(s) = A_\alpha s^{-1-\alpha} \) for \( s \leq \delta \) and \( f(s) = A_\alpha \delta^{-1-\alpha} s^{-n} \) for \( s > \delta \). It is then obvious that the assumptions (1) and (2) of Theorem 3 in [16] hold and it follows that

\[
\left| \frac{d^m}{dx^m} g_t^{(\delta)}(x) \right| \leq ct^{-1/(1+m)/\alpha} \min \left\{ 1, t^{1+1/\alpha} f(|x|) + \left( 1 + \frac{|x|}{t^{1/\alpha}} \right)^{-n} \right\}, \quad x \in \mathbb{R}, t \in (0, \tau].
\]

Clearly, for \( |x| \geq 1 \) and \( t \in (0, \tau] \) we have \( f(|x|) \approx |x|^{-n} \) and \( \left( 1 + \frac{|x|}{t^{1/\alpha}} \right)^{-n} \leq c|x|^{-n} \).

This implies the assertion of the lemma. \[\square\]

**Lemma 2.6.** There is a constant \( C = C(\alpha) \) such that for \( a \geq 0 \), and any \( t > 0 \),

\[
\int_{\mathbb{R}} \left( \frac{a + |w||w|}{t^{1/\alpha}} \wedge 1 \right) \mu^{(\delta)}(w) \, dw \leq C t^{1/2} + a^\alpha.
\]

**Proof.** We have

\[
\left( \frac{a + |w||w|}{t^{1/\alpha}} \wedge 1 \right) \leq \left( \frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) + \left( \frac{w^2}{t^{1/\alpha}} \wedge 1 \right).
\]

Hence, using \( \mu^{(\delta)}(w) \leq \frac{c}{|w|^{1+\tau}} \), we obtain

\[
\int_{\mathbb{R}} \left( \frac{a + |w||w|}{t^{1/\alpha}} \wedge 1 \right) \mu^{(\delta)}(w) \, dw \leq c \int_{\mathbb{R}} \left( \frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} + c \int_{\mathbb{R}} \left( \frac{w^2}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}}.
\]
Proof. Let \( \beta \) estimated as follows

\[
\text{Since } M \text{ be represented as for any } y, \text{ we show that } \partial f \text{ which proves that all matrices from the set } B \text{ contains the ball } B(f(x), \beta^2/(2\eta)). \]

Next,

\[
\int_\mathbb{R} \left( \frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} = 2\int_0^{1/\alpha} \frac{aw}{t^{1/\alpha} w^{1+\alpha}} \frac{dw}{w^{1+\alpha}} + 2\int_{1/\alpha}^{\infty} \frac{dw}{w^{1+\alpha}} = \frac{a^\alpha}{t}.
\]

Similar calculations show that

\[
\int_\mathbb{R} \left( \frac{w^2}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} = ct^{-1/2}.
\]

In the sequel we will need a version of the inverse map theorem for a Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R}^n, n \in \mathbb{N} \). The corresponding theorem is the main result in [9], however it is not formulated in a suitable way for our purpose. Below, closely following the arguments from [9], we provide a version we need.

It is well known that \( y \) almost surely the Jacobi matrix \( \partial f(y) \) of \( f \) exists. For any \( y_0 \in \mathbb{R}^n \) we define (see Definition 1 in [9]) the generalized Jacobian denoted \( \partial f(y_0) \) as the convex hull of the set of matrices which can be obtained as limits of \( \partial f(y_n) \), when \( y_n \to y_0 \).

We denote by \( B(x, r) \) an open ball of the center \( x \in \mathbb{R}^n \) and radius \( r > 0 \). For any matrix \( M \) we denote by \( ||M||_{\infty} \) the maximum of its entries.

**Lemma 2.7.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a Lipschitz map and \( x \in \mathbb{R}^n \). Suppose that for any \( y \in \mathbb{R}^n \), the generalized Jacobian \( \partial f(y) \) consist of the matrices which can be represented as \( M(x) + R \), where matrices \( M(x), R \) satisfy the following conditions: there are positive \( \beta \) and \( \eta \) such that \( ||R||_{\infty} \leq \eta|y - x| \) and \( |vM(x)^T| \geq 2\beta \) for every \( v \in \mathbb{R}^n, |v| = 1 \). Then \( f \) is injective on \( B(x, \beta/(\eta)) \) and we have \( B(f(x), \beta^2/(2\eta)) \subseteq f(B(x, \beta/(\eta))) \).

**Proof.** Let \( v \) be an arbitrary unit vector in \( \mathbb{R}^n \). Let \( M = \partial f(y) \) and let \( z = vM(x)^T \). Since \( M^T = M(x)^T + R^T \) the scalar product of \( z \) and \( w = vM^T = z + vR^T \) can be estimated as follows

\[
zw = z(z + vR^T) = |z|^2 + z(vR^T) \geq |z|^2 - n\eta|z||x - y|.
\]

Next, taking \( w^* = z/|z| \) we have for \( |x - y| \leq \beta/(\eta) \),

\[
w^*(vM^T) \geq |z| - n\eta|x - y| \geq \beta.
\]

Using this fact we can apply Lemma 3 and Lemma 4 of [9] to claim that for every \( y_1, y_2 \in B(x, \beta/(\eta)) \) we have

\[
|f(y_1) - f(y_2)| \geq \beta|y_1 - y_2|,
\]

which shows that \( f \) is injective in a ball \( B(x, \beta/(\eta)) \). Next, by similar arguments, we show that

\[
|vM^T| \geq |vM(x)^T| - |vR^T| \geq 2\beta - n\eta|x - y| \geq \beta, \quad |y - x| \leq \beta/(\eta),
\]

which proves that all matrices from the set \( \partial f(y) \) are of full rank if \( |y - x| \leq \beta/(\eta) \).

Finally, we can apply Lemma 5 of [9] to show that the image of the ball \( B(x, \beta/(\eta)) \) contains the ball \( B(f(x), \beta^2/(2\eta)) \).
3. Construction and properties of the transition density of the solution of (1) driven by the truncated process

The approach in this section is based on Levi’s method (cf. [27, 12, 26]). This method was applied in the framework of pseudodifferential operators by Kochubei [20] to construct a fundamental solution to the related Cauchy problem as well as transition density for the corresponding Markov process. In recent years it was also used in several papers to study transition densities of Lévy-type processes see e.g. [17, 8, 15, 13, 5, 18, 19, 21]. Levi’s method was also used to study gradient and Schrödinger perturbations of fractional Laplacians see e.g. [4, 6, 38].

We first introduce the generator of the process $X_t$. We define $\mathcal{K}f(x)$ by the following formula

$$\mathcal{K}f(x) = A_\alpha \sum_{i=1}^d \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [f(x + a_i(x)w) - f(x)] \frac{dw}{|w|^{1+\alpha}},$$

for any Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ such that all the limits on the right hand side exist. Recall that $a_i(x) = (a_{i1}(x), \ldots, a_{id}(x))$. It is well known that $\mathcal{K}f(x)$ is well defined for any $f \in C^2_b(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$. By (1 Proposition 4.1) we know that if $f \in C^2_b(\mathbb{R}^d)$ then $f(X_t) - f(X_0) - \int_0^t \mathcal{K}f(X_s) \, ds$ is a martingale.

Let us fix $\varepsilon \in (0, 1]$ (it will be chosen later). Recall that for given $\varepsilon$ the constant $\delta$ is chosen according to Lemma 2.3. For such fixed $\varepsilon$, $\delta$ we abbreviate $\mu(x) = \mu^{(\delta)}(x)$, $G = G^{(\delta)}$, $g_t(x) = g^{(\delta)}_t(x)$, $h_t(x) = h^{(\varepsilon)}_t(x)$.

We divide $\mathcal{K}$ into two parts

$$\mathcal{K}f(x) = \mathcal{L}f(x) + \mathcal{R}f(x),$$

(19)

where

$$\mathcal{L}f(x) = \sum_{i=1}^d \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [f(x + a_i(x)w) - f(x)] \mu(w) \, dw,$$

for any Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ such that all the limits on the right hand side exist. Our first aim will be to construct the heat kernel $u(t, x, y)$ corresponding to the operator $\mathcal{L}$. This will be done by using the Levi’s method.

For each $z \in \mathbb{R}^d$ we introduce the “freezing” operator

$$\mathcal{L}^z f(x) = \sum_{i=1}^d \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [f(x + a_i(z)w) - f(x)] \mu(w) \, dw.$$

Let $G_t(x) = g_t(x_1) \ldots g_t(x_d)$ and $H_t(x) = h_t(x_1) \ldots h_t(x_d)$ for $t > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We also denote $B(x) = (b_{ij}(x)) = A^{-1}(x)$. Note that the coordinates of $B(x)$ satisfy conditions (2) and (1) with possibly different constants $\eta^*_i$ and $\eta^*_s$, but taking maximums we can assume that $\eta^*_i = \eta_i$ and $\eta^*_s = \eta_s$.

For any $y \in \mathbb{R}^d$, $i = 1, \ldots, d$ we put

$$b_i(y) = (b_{i1}(y), \ldots, b_{id}(y)).$$

We also denote $\|B\|_{\infty} = \max\{|b_{ij}| : i, j \in \{1, \ldots, d\}\}$.

For any $t > 0$, $x, y \in \mathbb{R}^d$ we define

$$p_y(t, x) = \det(B(y))G_t(x(B(y))^T) = \det(B(y))g_t(b_1(y)x) \ldots g_t(b_d(y)x).$$
It may be easily checked that for each fixed $y \in \mathbb{R}^d$ the function $p_y(t, x)$ is the heat kernel of $\mathcal{L}^y$ that is

$$\frac{\partial}{\partial t} p_y(t, x) = \mathcal{L}^y p_y(t, \cdot)(x), \quad t > 0, \ x \in \mathbb{R}^d,$$

$$\int_{\mathbb{R}^d} p_y(t, x) \, dx = 1, \quad t > 0.$$  

For any $t > 0, \ x, y \in \mathbb{R}^d$ we also define

$$r_y(t, x) = H_t(x(B(y))^T) = h_t(b_1(y)x) \ldots h_t(b_d(y)x).$$

For $x, y \in \mathbb{R}^d, \ t > 0$, let

$$q_0(t, x, y) = \mathcal{L}^x p_y(t, \cdot)(x - y) - \mathcal{L}^y p_y(t, \cdot)(x - y),$$

and for $n \in \mathbb{N}$ let

$$q_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z) q_{n-1}(s, z, y) \, dz \, ds. \quad (20)$$

For $x, y \in \mathbb{R}^d, \ t > 0$ we define

$$q(t, x, y) = \sum_{n=0}^{\infty} q_n(t, x, y)$$

and

$$u(t, x, y) = p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) \, dz \, ds. \quad (21)$$

In this section we will show that $q_n(t, x, y), \ q(t, x, y), \ u(t, x, y)$ are well defined and we will obtain estimates of these functions. First, we will get some simple properties of $p_y(t, x)$ and $r_y(t, x)$.

**Lemma 3.1.** Choose $\gamma \in (0, 1]$. For any $t \in (0, \tau], \ x, x', y \in \mathbb{R}^d$ we have

$$|p_y(t, x) - p_y(t, x')| \leq c(1 \wedge (t^{-\gamma/\alpha}|x - x'|^\gamma)) (r_y(t, x/2) + r_y(t, x'/2)).$$

**Proof.** Of course, we may assume that $\gamma = 1$. We have

$$p_y(t, x) - p_y(t, x') = \det(B(y)) \left( \prod_{i=1}^d g_{t_1}(b_i(y)x) - \prod_{i=1}^d g_{t_1}(b_i(y)x') \right)$$

If $|x - x'| \geq t^{1/\alpha}/\|B\|_\infty$ or $|x - x'| \geq \delta/\|B\|_\infty$ then the assertion clearly holds. So, we may assume that $|x - x'| \leq t^{1/\alpha}/\|B\|_\infty$ and $|x - x'| \leq \delta/\|B\|_\infty$. Then the assertion follows easily from Lemma 2.3.

For $x, y \in \mathbb{R}^d$ we have

$$|B(y)x|^2 = |xB(y)^T|^2 = (b_1(y)x)^2 + \ldots + (b_d(y)x)^2.$$

**Lemma 3.2.** For any $x, y \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$ we have

$$\max_{1 \leq i \leq d} (b_i(y)x)^2 \geq \frac{1}{\eta_1^4 d^3} |x|^2.$$
Proof. Indeed, for any \( u, x \) we have \( |uA(y)^T| \leq \eta_1 d|u| \). Setting \( u = xB(y)^T \) we obtain that
\[
|xB(y)^T| \geq \frac{1}{\eta_1 d}|x|.
\]
Since
\[
|xB(y)^T|^2 = |b_1(y)x|^2 + |b_2(y)x|^2 + \cdots + |b_d(y)x|^2,
\]
it follows that there is \( 1 \leq k \leq d \) such that \( |b_k(y)x| \geq \frac{1}{\eta_1 d} |x| \). \( \square \)

**Corollary 3.3.** Assume that \( \varepsilon \leq \frac{1}{\eta_1 d^{d/2}}. \) For any \( t \in (0, \tau + 1] \), \( x, y \in \mathbb{R}^d \), we have
\[
r_y(t, x - y) \leq c_1 t^{-d/\alpha} e^{-c|x-y|}.
\]
For any \( t \in (0, \tau + 1] \), \( x, y \in \mathbb{R}^d \), \( |x - y| \geq \varepsilon \eta_1 d^{d/2} \), we have
\[
r_y(t, x - y) \leq c_1 t^{-d/\alpha} e^{-c|x-y|}.
\]
Proof. For any \( t \in (0, \tau + 1] \), \( z \in \mathbb{R} \) by definition of \( h_t \) we have
\[
h_t(z) \leq c t^{-1/\alpha} e^{-c|z|}.
\]
Fix \( x, y \in \mathbb{R}^d \), \( t \in (0, \tau + 1] \). By Lemma 3.2 there exists \( i \in \{1, \ldots, d\} \) such that \( |b_i(y)(x-y)| \geq \frac{1}{\eta_1 d^{d/2}} |x-y| \). Using this and (22) we get (23). For any \( t \in (0, \tau + 1] \), \( z \in \mathbb{R} \), \( |z| \geq \varepsilon \) by definition of \( h_t \) we have
\[
h_t(z) \leq c t^{1+(d-1)/\alpha} e^{-c|z|}.
\]
If \( |x - y| \geq \varepsilon \eta_1 d^{d/2} \) then \( \frac{1}{\eta_1 d^{d/2}} |x - y| \geq \varepsilon \) hence, by the same arguments as above, we get (24). \( \square \)

Using the definition of \( p_y(t, x) \) and properties of \( g_t(x) \) we obtain the following regularity properties of \( p_y(t, x) \).

**Lemma 3.4.** The function \( (t, x, y) \rightarrow p_y(t, x) \) is continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \). The function \( t \rightarrow p_y(t, x) \) is in \( C^1((0, \infty)) \) for each fixed \( x, y \in \mathbb{R}^d \). The function \( x \rightarrow p_y(t, x) \) is in \( C^2(\mathbb{R}^d) \) for each fixed \( t > 0 \), \( y \in \mathbb{R}^d \).

**Lemma 3.5.** For any \( y \in \mathbb{R}^d \) we have
\[
\left| \frac{\partial}{\partial x_i} p_y(t, x - y) \right| \leq \frac{c}{t^{(d+1)/\alpha} (1 + |x - y|)^{d+1}}, \quad i \in \{1, \ldots, d\}, t \in (0, \tau], x \in \mathbb{R}^d,
\]
\[
\left| \frac{\partial^2}{\partial x_i \partial x_j} p_y(t, x - y) \right| \leq \frac{c}{t^{(d+2)/\alpha} (1 + |x - y|)^{d+1}}, \quad i, j \in \{1, \ldots, d\}, t \in (0, \tau], x \in \mathbb{R}^d.
\]
Proof. The estimates follow from Lemma 3.2 and the same arguments as in the proof of (22). \( \square \)

**Lemma 3.6.** Let \( b_i^*(x, y), x, y \in \mathbb{R}^d; i = 1, \ldots, d \), be real functions such that there are positive \( \eta_4, \eta_5 \) and
\[
|b_i^*(x, y)| \leq \eta_4, \quad x, y \in \mathbb{R}^d,
\]
\[
|b_i^*(x, y) - b_i^*(\overline{x}, \overline{y})| \leq \eta_5 (|x - \overline{x}| + |y - \overline{y}|), \quad x, y, \overline{x}, \overline{y} \in \mathbb{R}^d.
\]
Let, for fixed \( x \in \mathbb{R}^d \), \( \Psi_x \) be a map \( \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \) given by
\[
\Psi_x(w, y) = (w, \xi_1, \ldots, \xi_d) \in \mathbb{R}^{d+1}, \quad w \in \mathbb{R}, y \in \mathbb{R}^d,
\]
where \( \xi_i = b_i(y)(x - y) + b_i^*(x, y)w \).
There is a positive \( \varepsilon_0 = \varepsilon_0(\eta_1, \eta_3, \eta_4, \eta_5, d) \leq \frac{1}{2\eta_5} \) such that the map \( \Psi_x \) and its Jacobian denoted by \( J_{\Psi_x}(w, y) \) has the property
\[
|\Psi_x(w, y)| \leq 1,
\]
\[
2|\det B(y)| \geq |J_{\Psi_x}(w, y)| \geq (1/2)|\det B(y)|,
\]
for \( |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0, (w, y) \) almost surely. Moreover the map \( \Psi_x \) is injective on the set \( \{(w, y) \in \mathbb{R}^{d+1}; |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0\} \).

If, for fixed \( y \in \mathbb{R}^d \), \( \Phi_y \) be a map \( \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \) given by
\[
\Phi_y(w, x) = \Psi_x(w, y), \quad w \in \mathbb{R}, x \in \mathbb{R}^d,
\]
then the Jacobian of \( \Phi_y \) denoted by \( J_{\Phi_y}(w, x) \) has the property
\[
2|\det B(y)| \geq |J_{\Phi_y}(w, x)| \geq (1/2)|\det B(y)|,
\]
for \( |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0, (w, x) \) almost surely. Moreover the map \( \Phi_y \) is injective on the set \( \{(w, x) \in \mathbb{R}^{d+1}; |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0\} \).

Proof. In the proof we assume that constants \( c \) may additionally depend on \( \eta_4, \eta_5 \). We prove the statement for the map \( \Psi_x \), only. Since \( |\Psi_x(w, y)| \leq \sqrt{d}(1 + \eta_1 + \eta_4)(|w| + |x-y|) \) we have
\[
|\Psi_x(w, y)| \leq 1, \text{ if } |w| + |x-y| \leq \frac{1}{\sqrt{d}(1 + \eta_1 + \eta_4)}.
\]

Next, we observe that \( (w, y) \) almost surely
\[
\frac{\partial \xi_k}{\partial y_l} = -b_k(y) + (x-y) \cdot \frac{\partial b_k}{\partial y_l} + w \frac{\partial b^*_k}{\partial y_l}, \quad 1 \leq l, k \leq d.
\]
Since \( |(y-x) \cdot \frac{\partial b_k}{\partial y_l} + w \frac{\partial b^*_k}{\partial y_l}| \leq (\eta_3 + \eta_5)(|y-x| + |w|) \), it follows that
\[
J_{\Psi_x}(w, y) = (-1)^d \det B(y) + R(x, y, w), \quad |R(x, y, w)| \leq c(|y-x| + |w|),
\]
with \( c = c(d, \eta_1, \eta_3, \eta_5) \). Since \( |\det B(y)| = \frac{1}{|\det A(y)|} \geq \frac{1}{\det M} \), we have for sufficiently small \( \varepsilon_1 = \varepsilon_1(\eta_1, \eta_3, \eta_4, \eta_5, d) \), \( (w, y) \) almost surely
\[
J_{\Psi_x}(w, y) = (-1)^d \kappa(x, y, w)|\det B(y)|, \quad |y-x| \leq \varepsilon_1, \quad |w| \leq \varepsilon_1,
\]
where \( \frac{1}{2} \leq \kappa \leq 2 \).

Let \( J_{\Psi_x}(w, y) \) be the Jacobi matrix for the map \( \Psi_x \) which is defined \( (w, y) \) almost surely. Let \( \partial \Psi_x(w, y) \) denote the generalized Jacobian of \( \Psi_x \) at the point \( (w, y) \). Then from the form of \( J_{\Psi_x} \) it is clear that every matrix \( M \in \partial \Psi_x(w, y) \) can be written as
\[
M = \mathcal{B}(x) + \mathcal{R},
\]
where the coordinates \( \mathcal{B}_{kl}(x), 0 \leq k, l \leq d \) of the matrix \( \mathcal{B}(x) \) are
\[
\mathcal{B}_{kl}(x) = -b_{kl}(x), \quad k, l \geq 1,
\]
\[
\mathcal{B}_{00}(x) = 1; \quad \mathcal{B}_{0l}(x) = 0; \quad \mathcal{B}_{l0}(x) = b^*_l(x, x), \quad 1 \leq l \leq d,
\]
while all the entries of \( \mathcal{R} \) satisfy \( |\mathcal{R}_{kl}| \leq c\sqrt{|w|^2 + |x-y|^2} \) with \( c = c(\eta_3, \eta_5) \).

Now, for every \( (u, z), u \in \mathbb{R}, z \in \mathbb{R}^d: |u|^2 + |z|^2 = 1 \) we have
\[
|(u, z)\mathcal{B}(x)^T| \geq 2\beta > 0,
\]
with \( \beta = \beta(d, \eta_1, \eta_4) \). Since \( ||\mathcal{R}||_\infty \leq c\sqrt{|w|^2 + |x-y|^2} \) we can apply Lemma 2.7 with \( n = d+1 \) to show on the set \( \{(w, y); \sqrt{|w|^2 + |x-y|^2} \leq \beta/(c(d+1))\} \) the map...
\( \Psi_x \) is injective. This fact, combined with (27) and (28), completes the proof if we choose \( \varepsilon_0 = \varepsilon_1 \wedge \frac{1}{2} \left( \frac{2}{d+1} - \frac{d}{2} \right) \).

\[ \square \]

Remark 3.7. Let for \( x \in \mathbb{R}^d \), \( \tilde{\Psi}_x \) be the map \( \mathbb{R}^d \rightarrow \mathbb{R}^d \) given by
\[
\tilde{\Psi}_x(y) = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \quad y \in \mathbb{R}^d,
\]
where \( \xi_i = b_i(y)(x - y) \). Then using the same arguments as in the above proof we can find \( \varepsilon_0 \) such that all the assertions of Lemma 3.6 are true and additionally
\[
2|detB(y)| \geq |J_{\tilde{\Psi}_x}(y)| \geq (1/2)|detB(y)|,
\]
for \( |x - y| \leq \varepsilon_0 \), \( y \) almost surely. Moreover, the map \( \tilde{\Psi}_x \) is injective on \( B(x, \varepsilon_0) \). We can also find \( \delta_1 = \delta_1(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0 \) and \( \delta_2 = \delta_1(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0 \) such that the \( \tilde{\Psi}_x \) image of the ball \( B(x, \delta_1) \) contains \( B(0, \delta_2) \). To this end we apply the last assertion of Lemma 2.7.

Let \( b_i^*(x, y) \) be the functions introduced in Lemma 3.6. We will use the following abbreviations
\[
\begin{align*}
\xi_i &= B_i(x, y) = b_i(y)(x - y) = b_{i1}(y)(x_1 - y_1) + \ldots + b_{id}(y)(x_d - y_d), \\
b_i^* &= b_i^*(x, y), \\
b_{i0}^* &= b_i^*(x, x).
\end{align*}
\]

Let for \( l = 1, \ldots, d, \)
\[
A_l = A_l(x, y) = \int_{\mathbb{R}} \prod_{i \neq l} g_t(z_i + b_i^* w) |g_t(z_l + b_l^* w) - g_t(z_l + b_{l0}^* w)| \mu(w) dw.
\]

Corollary 3.8. Assume that \( 2\delta < \varepsilon_0 \), where \( \varepsilon_0 \) is from Lemma 3.6. With the assumptions of Lemma 3.6 we have for \( t \leq \tau, \)
\[
\int_{|y - x| \leq \varepsilon_0} A_l dy \leq ct^{-1/2}, \quad x \in \mathbb{R}^d,
\]
and
\[
\int_{|y - x| \leq \varepsilon_0} A_l dx \leq ct^{-1/2}, \quad y \in \mathbb{R}^d,
\]
where \( c = c(\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \varepsilon, \delta) \).

Proof. In the proof we assume that constants \( c \) may additionally depend on \( \eta_4, \eta_5 \). For \( x, y \in \mathbb{R}^d \) we get \( |b_i^* - b_{i0}^*| \leq \eta_5|x - y| \). Hence, from (13), we have for \( w \in \mathbb{R} \), \( 1 \leq l \leq d, \)
\[
|g_t(z_l + b_i^* w) - g_t(z_l + b_{l0}^* w)| \leq c \left( \frac{|b_i^* - b_{l0}^*| |w|}{t^{1/\alpha}} \wedge 1 \right) (h_t(z_l + b_i^* w) + h_t(z_l + b_{l0}^* w))
\]
\[
\leq c \left( \frac{|x - y| |w|}{t^{1/\alpha}} \wedge 1 \right) (g_t(z_l + b_i^* w) + g_t(z_l + b_{l0}^* w)).
\]

This implies that
\[
A_l \leq c(A_l^1 + A_l^2),
\]
where
\[
A_l^1 = \int_{\mathbb{R}} \left( \prod_{i=1}^d g_t(z_i + b_i^* w) \right) \left( \frac{|x - y| |w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw
\]
\[
A_l^2 = \int_{\mathbb{R}} \left( \prod_{i=1}^d g_t(z_i + b_i^* w) \right) \left( \frac{|x - y| |w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw
\]
and

\[
A_1^2 = \int_{\mathbb{R}} \left( \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \right) \left( \frac{|x - y|}{t^{1/\alpha}} \right)^{\alpha-1} \mu(w) \, dw,
\]

with \( \hat{b}_i = b^*_i, i \neq l \) and \( \hat{b}_1 = b^*_0 \). Note that the functions \( \hat{b}_i = \hat{b}_i(x, y) \) have the same properties as \( b^*_i \). To evaluate the integral \( \int_{|x-y| \leq \varepsilon_0} A_1^1 \, dy \) we introduce new variables in \( \mathbb{R}^{d+1} \), given by \((w, \xi) = \Psi_x(w, y)\), where \( \xi_i = z_i + b^*_i w, i = 1, \ldots, d \) (or \( \xi_i = z_i + \hat{b}_i w \) if \( A_2^2 \) is treated). Note that the vector \( \xi = (\xi_1, \ldots, \xi_d) \) can be written as

\[
\xi = (x - y) B(y)^T + w b^*,
\]

where \( b^* = (b_1^*, \ldots, b_d^*) \), hence

\[
(\xi - w b^*) A(y)^T = x - y.
\]

From this we infer that

\[
|w| |x - y| \leq c(|\xi| + |w|)|w|.
\]

Let \( Q_x = \{ (w, y) : |y - x| \leq \varepsilon_0, |w| \leq \varepsilon_0 \} \). Due to Lemma 3.6 almost surely on \( Q_x \), the absolute value of the Jacobian determinant of the map \( \Psi_x \) is bounded from below and above by two positive constants and \( \Psi_x \) is an injective transformation. Let \( V_x = \Psi_x(Q_x) \). Observing that the support of the measure \( \mu \) is contained in \([-\varepsilon_0, \varepsilon_0]\) and then applying the above change of variables, we have

\[
\int_{|y-x| \leq \varepsilon_0} A_1^1 \, dy \leq c \int_{|x-y| \leq \varepsilon_0} \int_{\mathbb{R}} \left( \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \right) \left( \frac{|\xi| + |w||w|}{t^{1/\alpha}} \right)^{\alpha-1} \mu(w) \, dw \, dy
\leq c \int_{|x-y| \leq \varepsilon_0} \int_{\mathbb{R}} \left( \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \right) \left( \frac{|\xi| + |w||w|}{t^{1/\alpha}} \right)^{\alpha-1} \\
\times \mu(w) |J_{\Psi_x}(w, y)| \, dw \, dy
= c \int_{V_x} \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \left( \frac{|\xi| + |w||w|}{t^{1/\alpha}} \right)^{\alpha-1} \mu(w) \, dw \, d\xi
\]

where the last equality follows from the general change of variable formula for injective Lipschitz maps (see e.g. [14, Theorem 3]). Since \( |\xi| \leq 1 \) for \((w, \xi) \in V_x \), we get

\[
\int_{|y-x| \leq \varepsilon_0} A_1^1 \, dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \int_{\mathbb{R}} \left( \frac{|\xi| + |w||w|}{t^{1/\alpha}} \right)^{\alpha-1} \mu(w) \, dw \, d\xi.
\]

Applying Lemma 2.6 we have

\[
\int_{\mathbb{R}} \left( \frac{|\xi| + |w||w|}{t^{1/\alpha}} \right)^{\alpha-1} \mu(w) \, dw \leq c \frac{t^{1/2} + |\xi|^\alpha}{t}.
\]

Finally,

\[
\int_{|y-x| \leq \varepsilon_0} A_1^1 \, dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^{d} g_t^{(\xi)}(\xi_i) \frac{t^{1/2} + |\xi|^\alpha}{t} \, d\xi \leq ct^{-1/2}.
\]
Similarly we obtain

\[ \int_{|y-x| \leq \varepsilon_0} A_1^2 dy \leq ct^{-1/2}, \]

which completes the proof of the first bound. To estimate \( \int_{|y-x| \leq \varepsilon_0} A_1 dx \) we proceed exactly in the same way. \( \square \)

For fixed \( l \in \{1, \ldots, d\} \) let us consider a family of functions \( b_i^*(x, y) = b_i(y) a_l(x), i \in \{1, \ldots, d\} \). They satisfy the conditions (25) and (26) with \( \eta_4 = d\eta_1^2 \) and \( \eta_5 = d\eta_1 \eta_3 \). Let \( \varepsilon_0 = \varepsilon_0(\eta_1, \eta_3, \eta_4, \eta_5, d) \) be as found in Lemma 3.6 and Remark 3.6. Finally we choose \( \varepsilon = \varepsilon(\eta_1, \eta_3, d) = \frac{\varepsilon_0}{4d^{1/2}(\eta_1 \eta_3)^2} \). From now on we keep \( \varepsilon_0, \varepsilon \) fixed as above. Recall that if we fixed \( \varepsilon \) we fix \( \delta \) according to Lemma 2.1

**Proposition 3.9.** For any \( x, y \in \mathbb{R}^d, t \in (0, \tau] \) we have

\[ |q_0(t, x, y)| \leq ct^{-(d-1)/\alpha} h_t\left( (\varepsilon/\varepsilon_0)|x-y|1_{|x| < \infty}(|x-y|) \right). \]  

(29)

In particular for \( x, y \in \mathbb{R}^d, t \in (0, \tau], \quad |y-x| \geq \varepsilon_0 \) we have

\[ |q_0(t, x, y)| \leq c e^{(-e/\varepsilon_0)|x-y|}. \]

For any \( t \in (0, \tau], \quad x \in \mathbb{R}^d \) we have

\[ \int_{\mathbb{R}^d} |q_0(t, x, y)| dy \leq ct^{-1/2}. \]  

(30)

For any \( t \in (0, \tau], \quad y \in \mathbb{R}^d \) we have

\[ \int_{\mathbb{R}^d} |q_0(t, y, x)| dx \leq ct^{-1/2}. \]  

(31)

**Proof.** We have

\[ q_0(t, x, y) = \sum_{i=1}^d \lim_{\zeta \to 0^+} \int_{|w| > \zeta} \left[ p_y(t, x - y + a_i(x) w) - p_y(t, x - y + a_i(y) w) \right] \mu(w) dw. \]

For \( i = 1, \ldots, d \) we put

\[ R_i = \lim_{\zeta \to 0^+} \int_{|w| > \zeta} \left[ p_y(t, x - y + a_i(x) w) - p_y(t, x - y + a_i(y) w) \right] \mu(w) dw. \]

(32)

We have \( q_0(t, x, y) = R_1 + \ldots + R_d. \) It is clear that it is enough to handle \( R_1 \) alone. Note that

\[ R_1 = \det(B(y)) \lim_{\zeta \to 0^+} \int_{|w| > \zeta} \left[ G_t \left( (x - y + w e_1(A(x))^T)(B(y))^T \right) - G_t \left( (x - y + w e_1(A(y))^T)(B(y))^T \right) \right] \mu(w) dw. \]

(33)

We will use the following abbreviations

\[ z_i = b_i(x, y) = b_i(y)(x - y) = b_{i1}(y)(x_1 - y_1) + \ldots + b_{id}(y)(x_d - y_d), \]

\[ k_i = b_{i1}(x, y) = b_i(y)a_1(x), \]

\[ k_{i0} = b_{i1}(x, y). \]

Note that \( k_{10} = 1 \) and \( k_{i0} = 0, \quad 2 \leq i \leq d. \)
We can rewrite (33) as

\[ R_1 = \det(B(y)) \lim_{\xi \to 0^+} \int_{|w|>\xi} \left[ \prod_{i=1}^{d} g_t(z_i + k_i w) - \prod_{i=1}^{d} g_t(z_i + k_0 w) \right] \mu(w) \, dw. \]

Hence,

\[ R_1 = \det(B(y))(I_1 + \ldots + I_d) = \det(B(y))(I'_1 + \ldots + I'_d), \]

where

\[ I_l = \int_{\mathbb{R}} \left( \prod_{i=1}^{l-1} g_t(z_i + k_i w) \right) \left[ g_t(z_l + k_l w) - g_t(z_l + k_0 w) \right] \left( \prod_{i=l+1}^{d} g_t(z_i) \right) \mu(w) \, dw, \]

\[ I'_l = \int_{\mathbb{R}} \left( \prod_{i=1}^{l-1} g_t(z_i + k_i w) \right) \left[ g_t(z_l + k_l w) - g_t(z_l + k_0 w) \right] \times \left( \prod_{i=l+1}^{d} g_t(z_i + k_i w) \right) \mu(w) \, dw \]

for \( l = 1, \ldots, d \) (with convention that \( \prod_{i=1}^{0} = 1 = \prod_{i=d+1}^{d} \)).

We start with the proof of the bound of \( q_0 \). We observe that \( |k_l|, |k_0| \leq d\eta_1^2 \leq \varepsilon/(4\delta), l \in \{1, \ldots, d\} \). By Lemmas 2.1 and 2.4, we obtain for \( l \leq d - 1 \),

\[ |I_l| \leq g_t(z_d)g_t^{d-2}(0) \int_{\mathbb{R}} |g_t(z_l + k_l w) - g_t(z_l + k_0 w)| \mu(w) \, dw \]

\[ \leq c(|k_l| + |k_0|)^\alpha g_t(z_d)g_t^{d-2}(0) \frac{h_t(0)}{t} \]

\[ \leq ct^{1-(d-1)/\alpha} h_t(z_d). \]

The same argument leads to

\[ |I_d| \leq g_t^{d-1}(0) \int_{\mathbb{R}} |g_t(z_d + k_d w) - g_t(z_d)| \mu(w) \, dw \]

\[ \leq c g_t^{d-1}(0) |k_d|^{\alpha} \frac{h_t(z_d/2)}{t} \]

\[ \leq ct^{-1-(d-1)/\alpha} h_t(z_d/2). \]

The above inequalities yield

\[ |R_1| \leq ct^{1-(d-1)/\alpha} h_t(z_d/2). \]

Since \( R_1 \) is invariant with respect to permutations of \( z_2, \ldots, z_d \) we infer that

\[ |R_1| \leq ct^{1-(d-1)/\alpha} \inf_{2 \leq i \leq d} h_t(z_i/2) = ct^{1-(d-1)/\alpha} h_t \left( \max_{2 \leq i \leq d} |z_i|/2 \right). \]

On the other hand, again by Lemma 2.4,

\[ |I'_1| \leq g_t^{d-1}(0) \int_{\mathbb{R}} |g_t(z_1 + k_1 w) - g_t(z_1 + w)| \mu(w) \, dw \]

\[ \leq c(|k_1| + 1)^\alpha g_t^{d-1}(0) \frac{h_t(z_1/2)}{t} \]

\[ \leq ct^{-1-(d-1)/\alpha} h_t(z_1/2). \]
For $l \geq 2$, a similar argument leads to
\[
|I_l| \leq \sup_{|w| \leq 2\delta} g_t(z_1 + w) g_t^{d-2}(0) \int_{\mathbb{R}} |g_t(z_1 + k_1w) - g_t(z_1)| \mu(w) \, dw \\
\leq c \sup_{|w| \leq 2\delta} g_t(z_1 + w) g_t^{d-2}(0) \frac{h_t(0)}{t} \\
\leq ct^{-(d-1)/\alpha} \sup_{|w| \leq 2\delta} g_t(z_1 + w).
\]

Observing that $|z_1 + w| \geq |z_1|/2$ for $|w| \leq 2\delta \leq 4\delta \leq |z_1|$ we conclude that
\[
|R_1| \leq ct^{-(d-1)/\alpha} h_t \left( \frac{|z_1|}{2} 1_{[4\delta, \infty)}(|z_1|) \right).
\]  
(35)

Combining (34) and (35) we arrive at
\[
|R_1| \leq ct^{-(d-1)/\alpha} h_t \left( \frac{\max_{1 \leq i \leq d} |z_i|}{2} 1_{[4\delta, \infty)}(\max_{1 \leq i \leq d} |z_i|) \right).
\]

By Lemma 3.2 $\max_{1 \leq i \leq d} |z_i| \geq \frac{1}{\eta_1 d^{3/2}} |x - y|$, and by the choice of $\varepsilon, \varepsilon_0, \delta$ we have $\frac{1}{\eta_1 d^{3/2}} \geq \varepsilon/\varepsilon_0$ and $4\delta \eta_1 d^{3/2} \leq \varepsilon_0$, hence
\[
|R_1| \leq ct^{-(d-1)/\alpha} h_t \left( \frac{|x - y|}{2 \eta_1 d^{3/2}} 1_{[4\delta, \infty)}(|x - y|) \right) \\
\leq ct^{-(d-1)/\alpha} h_t \left( \varepsilon/\varepsilon_0 \right) |x - y| 1_{[\varepsilon_0, \infty)}(|x - y|).
\]

Finally, for $|y - x| \geq \varepsilon_0$ we have
\[
|R_1| \leq ce^{-(\varepsilon/\varepsilon_0)|x-y|}.
\]

Next, we prove the bound of the integral (30). Let $x \in \mathbb{R}^d$ be fixed. Applying Lemma 3.8 with $b^*_i = k_i 1_{\{i \leq l\}}$ we have that
\[
\int_{|y-x| \leq \varepsilon_0} |I_1| \, dy \leq ct^{-1/2}.
\]

Hence
\[
\int_{|x-y| \leq \varepsilon_0} |g_0(t, x, y)| \, dy \leq ct^{-1/2}.
\]

For $|y - x| \geq \varepsilon_0$ we have $|g_0(t, x, y)| \leq ce^{-(\varepsilon/\varepsilon_0)|x-y|}$ which implies that
\[
\int_{|x-y| \geq \varepsilon_0} |g_0(t, x, y)| \, dy \leq c.
\]

This completes the proof of (30). The estimate (31) is proved exactly in the same way. \(\square\)

Using similar arguments as in the proof of Proposition 3.9 we obtain the following result.

**Proposition 3.10.** For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have
\[
\int_{\mathbb{R}^d} p_y(t, x - y) \, dy \leq c, \quad (36)
\]
\[
\int_{\mathbb{R}^d} r_y(t, (x - y)/2) \, dy \leq c. \quad (37)
\]
For any $\delta_1 > 0$,
\[
\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B^c(x, \delta_1)} p_y(t, x - y) \, dy = 0.
\]
(38)

We have
\[
\lim_{t \to 0^+} \int_{\mathbb{R}^d} p_y(t, x - y) \, dy = 1,
\]
(39)
uniformly with respect to $x \in \mathbb{R}^d$.

**Proof.** For fixed $x \in \mathbb{R}^d$ we introduce new variables $u = \tilde{\Psi}_x(y)$ given by
\[
u = (x - y)B(y)^T.
\]
Note that
\[
\frac{1}{d\eta_1} |x - y| \leq |u| = |(x - y)B(y)^T| \leq d\eta_1 |x - y|.
\]
(40)
For $r > 0$, let $V_x(r)$ be the $\tilde{\Psi}_x$ image of the ball $B(x, r)$. By Remark 3.7 we have almost surely
\[
|J_{\Psi_x}(y)| \geq (1/2) \det B(y) \geq c, \quad |y - x| \leq \varepsilon_0,
\]
and $\tilde{\Psi}_x$ is an injective map on $B(x, \varepsilon_0)$. Hence, for $0 < \delta_1 < \varepsilon_0$, by the change of variables formula (see e.g. [14, Theorem 3]), and then by (40) we obtain
\[
\int_{\delta_1 \leq |x - y| \leq \varepsilon_0} r_y(t, (x - y)/2) \, dy \leq c \int_{\delta_1 \leq |x - y| \leq \varepsilon_0} H_1(u/2) |J_{\Psi_x}(y)| \, dy
\]
\[
= c \int_{V_x(\varepsilon_0) \setminus V_x(\delta_1)} H_1(u/2) \, du
\]
\[
\leq c \int_{|u| \geq \frac{1}{2\eta_1}} H_1(u/2) \, du = I(t, \delta_1).
\]
It is clear that
\[
\lim_{\delta_1 \to 0^+} I(t, \delta_1) \leq c, \quad t \leq \tau.
\]
If $|x - y| \geq \varepsilon_0$ then $|x - y|/2 \geq \varepsilon_\eta_1 d^{1/2}$, hence, by (23), we obtain
\[
\int_{|x - y| \geq \varepsilon_0} r_y(t, (x - y)/2) \, dy \leq c_1 t \int_{|x - y| \geq \varepsilon_0} e^{-c|x - y|} \, dy = c_2 t.
\]
The last two inequalities prove that
\[
\sup_{x \in \mathbb{R}^d} \sup_{t \leq \tau} \int_{\mathbb{R}^d} r_y(t, (x - y)/2) \, dy < \infty.
\]
Noting that $\lim_{t \to 0^+} I(t, \delta_1) = 0$ we obtain
\[
\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|x - y| \geq \delta_1} r_y(t, (x - y)/2) \, dy = 0.
\]
Since $p_y(t, x - y) \leq c r_y(t, (x - y)/2)$ for $t \leq \tau, x, y \in \mathbb{R}^d$, the proof of [36, 37, 38] is completed.

Note that the coordinates of the matrix $B(y)$ have partial derivatives $y$ almost surely, bounded uniformly. We can calculate the absolute value of Jacobian determinant $J_{\Psi_x}(y)$, $y$ almost surely, as
\[
|J_{\Psi_x}(y)| = \det B(y) + R(x, y), \quad |R(x, y)| \leq c|y - x|.
\]
(41)
For any $t$

\[ \int_{|x-y| \leq \delta_1} p_y(t, x - y) \, dy = \int_{|x-y| \leq \delta_1} G_t(u) \det B(y) \, dy \]

\[ = \int_{|x-y| \leq \delta_1} G_t(u) \left| J_{\Psi_x}(y) \right| \, dy - \int_{|x-y| \leq \delta_1} G_t(u) R(x, y) \, dy \]

\[ = I_1 + I_2. \]

Applying (41), (40) and the change of variable formula we obtain

\[ |I_2| \leq c \int_{|x-y| \leq \delta_1} |x-y| G_t(u) \, dy \]

\[ \leq c \int_{|x-y| \leq \delta_1} |u| G_t(u) \left| J_{\Psi_x}(y) \right| \, dy \]

\[ = c \int_{V_x(\delta_1)} |u| G_t(u) \, du \]

\[ \leq c \int_{|u| \leq d \delta_1} |u| G_t(u) \, du \rightarrow 0, \text{ if } t \rightarrow 0^+. \]

Now we can pick, independently of $x$, positive $\delta_1$ and $\delta_2$ such that $B(0, \delta_2) \subset V_x(\delta_1)$ (see Remark 3.7). Applying again the change of variable formula we obtain

\[ I_1 = \int_{V_x(\delta_1)} G_t(u) \, du \geq \int_{|u| \leq \delta_2} G_t(u) \, du \rightarrow 1, \text{ if } t \rightarrow 0^+. \]

This completes the proof that uniformly with respect to $x$,

\[ \lim_{t \rightarrow 0^+} \int_{|x-y| \leq \delta_1} p_y(t, x - y) \, dy = 1, \]

which combined with (38) proves (39).

In the sequel we will use the following standard estimate. For any $\gamma \in (0, 1]$, $\theta_0 > 0$ there exists $c = c(\gamma, \theta_0)$ such that for any $\theta \geq \theta_0$, $t > 0$ we have

\[ \int_0^t (t-s)^{\gamma-1} s^{\theta-1} \, ds \leq \frac{c}{\theta^\gamma} t^{(\gamma-1)+\theta-1}. \]  

\( (42) \)

**Lemma 3.11.** For any $t > 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ the kernel $q_n(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we have

\[ \int_{\mathbb{R}^d} |q_n(t, x, y)| \, dy \leq \frac{c_1^{n+1} t^{(n+1)/2-1}}{\left(\frac{1}{n!}\right)^{1/2}}, \quad (43) \]

\[ \int_{\mathbb{R}^d} |q_n(t, y, x)| \, dy \leq \frac{c_1^{n+1} t^{(n+1)/2-1}}{\left(\frac{1}{n!}\right)^{1/2}}. \quad (44) \]

For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we have

\[ |q_n(t, x, y)| \leq c_1 \frac{c_2^n t^{n/2-1}}{\left(\frac{1}{n!}\right)^{1/2} t^d/\alpha}. \quad (45) \]

For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $|x - y| \geq n + 1$ we have

\[ |q_n(t, x, y)| \leq c_1 \frac{c_2^n t^{n/2}}{\left(\frac{1}{n!}\right)^{1/2}} e^{-\frac{|x-y|}{n+1}}. \quad (46) \]
where $\lambda = \varepsilon / \varepsilon_0$.

**Proof.** By Proposition 3.9 there is a constant $c^*$ such that for any $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$ we have

$$|q_0(t, x, y)| \leq c^* \frac{1}{t^{1+d/\alpha}},$$  \hspace{0.5cm} (47)

$$|q_0(t, x, y)| \leq c^* e^{-\lambda |x-y|}, |x-y| \geq 1.$$  \hspace{0.5cm} (48)

$$\int_{\mathbb{R}^d} |q_0(t, x, u)| \, du \leq c^* t^{-1/2},$$  \hspace{0.5cm} (49)

$$\int_{\mathbb{R}^d} |q_0(t, u, x)| \, du \leq c^* t^{-1/2}. $$  \hspace{0.5cm} (50)

It follows from (42) there is $p \geq 1$ such that for $n \in \mathbb{N}$,

$$\int_0^t (t-s)^{-1/2} s^{n/2-1/2} \, ds \leq \frac{p}{(n+1)^{1/2}} t^{n/2},$$

$$\int_{t/2}^t (t-s)^{-1/2} s^{n/2-1} \, ds \leq \frac{p}{(n+1)^{1/2}} t^{(n+1)/2-1},$$

$$\int_0^t (t-s)^{-1/2} s^{n/2} \, ds \leq \frac{p}{(n+1)^{1/2}} t^{(n+1)/2}. $$

We define $c_1 = pc^* \geq c^*$ and $c_2 = 2^{d/\alpha+1} c_1 (2+p) > c_1$.

We will prove (13), (14), (15) simultaneously by induction. They are true for $n = 0$ by (17), (18), (19) and the choice of $c_1$. Assume that (13), (14), (15) are true for $n \in \mathbb{N}$, we will show them for $n+1$. By the definition of $q_n(t, x, y)$ and the induction hypothesis we obtain

$$|q_{n+1}(t, x, y)| \leq c_1 c_2^{n+1} \frac{2^{d/\alpha+1}}{(n!)^{1/2} 2^{d/\alpha+1}} \int_0^t \int_{\mathbb{R}^d} |q_n(s, z, y)| \, dz \, ds$$

$$+ c_1 c_2^{n+1} \frac{2^{d/\alpha+1}}{(n!)^{1/2} 2^{d/\alpha+1}} \int_{t/2}^t \int_{\mathbb{R}^d} |q_0(t-s, x, z)| \, dz \, s^{n/2-1} \, ds$$

$$\leq c_1 c_2^{n+1} \frac{2^{d/\alpha+1}}{(n!)^{1/2} 2^{d/\alpha+1}} \int_0^t s^{(n+1)/2-1} \, ds$$

$$+ c_1 c_2^{n+1} \frac{2^{d/\alpha+1} c_1}{(n!)^{1/2} 2^{d/\alpha+1}} \int_{t/2}^t (t-s)^{-1/2} s^{n/2-1} \, ds$$

$$\leq c_1 c_2^{n+1} \frac{2^{d/\alpha+1} c_1}{(n!)^{1/2} 2^{d/\alpha+1}} \left( 2c_1 2^{d/\alpha+1} + c_1 2^{d/\alpha+1} p \right)$$

$$= c_1 c_2^{n+1} \frac{2^{d/\alpha+1} c_1}{(n!)^{1/2} 2^{d/\alpha+1}}.$$

Hence we get (15) for $n+1$. In particular this gives that the kernel $q_{n+1}(t, x, y)$ is well defined.
Proposition 3.12. For any $n$ which proves (43) for $n$, where $C$ which proves (46) for $a > 0$. There exists $t, x, y \in \mathbb{R}^d$. Using our induction hypothesis, (43) and (44) we get for $n + 1$. Similarly we get (44).

Now we will show (43). For $n = 0$ this follows from (48). Assume that (46) is true for $n \in \mathbb{N}$, we will show it for $n + 1$.

Using our induction hypothesis, (43) and (44) we get for $|x - y| \geq n + 2$

$$|q_{n+1}(t, x, y)| = \left| \int_0^t \int_{|x-z| \geq \frac{|x-y|}{n+2}} q_0(t - s, x, z) q_n(s, z, y) \, dz \, ds \right|$$

$$+ \int_0^t \int_{|x-z| \leq \frac{|x-y|}{n+2}} q_0(t - s, x, z) q_n(s, z, y) \, dz \, ds$$

$$\leq c_1 e^{-\frac{\lambda|x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_n(s, z, y)| \, dz \, ds$$

$$+ c_1 \frac{c_2}{(n)!^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_0(t - s, x, z)| \, dz \, ds s^{n/2} \, ds$$

$$\leq c_1 \frac{2c_1^{n+1} t^{(n+1)/2}}{(n+1)!^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}} + c_1 \frac{c_2 c_1^{n} t^{(n+1)/2} p}{((n+1))!^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}}$$

$$= (2c_1 c_1^{n+1} + c_2 c_1^{n}) \frac{t^{(n+1)/2}}{(n+1)!^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}},$$

which proves (46) for $n + 1$ since by the choice of constants $2c_1 c_1^{n+1} + c_2 c_1^{n} \leq c_1 c_1^{n+1}$. \qed

By standard estimates one easily gets

$$\sum_{n=k}^{\infty} \frac{C^n}{(n)!^{1/2}} \leq \frac{C^k}{(k)!^{1/2}} \sum_{n=k}^{\infty} \frac{C^{n-k}}{((n-k)!)^{1/2}} \leq C_1 e^{-k}, \quad k \in \mathbb{N}, \quad (51)$$

where $C_1$ depends on $C$.

Proposition 3.12. For any $t \in (0, \infty)$, $x, y \in \mathbb{R}^d$ the kernel $q(t, x, y)$ is well defined. For any $t \in (0, \tau)$, $x, y \in \mathbb{R}^d$ we have

$$|q(t, x, y)| \leq \frac{c}{t^{d/\alpha+1}} e^{-c_3 \sqrt{|x-y|}} \leq \frac{c}{t^{d/\alpha+1} (1 + |x-y|)^{d+1}}.$$

There exists $a > 0$ ($a$ depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$, $|x - y| \geq a$ we have

$$|q(t, x, y)| \leq c e^{-c_3 \sqrt{|x-y|}}.$$
For any \( t \in (0, \tau] \) and \( x \in \mathbb{R}^d \) we have

\[
\int_{\mathbb{R}^d} |q(t, x, y)| \, dy \leq ct^{-1/2}, \tag{52}
\]

\[
\int_{\mathbb{R}^d} |q(t, y, x)| \, dy \leq ct^{-1/2}. \tag{53}
\]

**Proof.** By (45) we clearly get \( \sum_{n=0}^{\infty} |q_n(t, x, y)| \leq ct^{-d/\alpha-1} \). This gives that \( q(t, x, y) \) is well defined and we have \( |q(t, x, y)| \leq ct^{-d/\alpha-1} \).

For \( |x-y| \geq 1 \) by (45), (46) and (51) we get

\[
|q(t, x, y)| = \left| \sum_{n=0}^{\left\lfloor |x-y|^{-1} \right\rfloor} q_n(t, x, y) + \sum_{n=\left\lfloor |x-y|^{-1} \right\rfloor + 1}^{\infty} q_n(t, x, y) \right|
\leq c_1 \sum_{n=0}^{\left\lfloor |x-y|^{-1} \right\rfloor} c_2^n t^{n/2} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=\left\lfloor |x-y|^{-1} \right\rfloor + 1}^{\infty} c_2^n t^{n/2} \left( \frac{n!}{(n!)^{1/2}} \right)^{d/\alpha+1}
\leq \frac{c}{\|d/\alpha+1\|} e^{-c_3 \sqrt{|x-y|}},
\]

where \([z]\) denotes the integer part of \([z]\). Take the smallest \( n_0 \in \mathbb{N} \) such that \( n_0/2 - 1 \geq d/\alpha \) and \( a = n_0^2 \). For \( \sqrt{|x-y|} \geq \sqrt{a} = n_0 \) we get

\[
|q(t, x, y)| \leq c_1 \sum_{n=0}^{\left\lfloor |x-y|^{-1} \right\rfloor} c_2^n t^{n/2} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=\left\lfloor |x-y|^{-1} \right\rfloor + 1}^{\infty} c_2^n t^{n/2} \left( \frac{n!}{(n!)^{1/2}} \right)^{d/\alpha+1}
\leq ce^{-c_3 \sqrt{|x-y|}}.
\]

(52) and (53) follows easily from (43) and (44).

By (21), Corollary 3.3, Proposition 3.10 and Proposition 3.12 we immediately obtain the following result.

**Corollary 3.13.** For any \( t \in (0, \infty), x, y \in \mathbb{R}^d \) the kernel \( u(t, x, y) \) is well defined. For any \( t \in (0, \tau], x, y \in \mathbb{R}^d \) we have

\[
|u(t, x, y)| \leq \frac{c}{\|d/\alpha\|} e^{-c_1 \sqrt{|x-y|}} \leq \frac{c}{\|d/\alpha\|} (1 + |x-y|)^{d+1}.
\]

There exists \( a > \varepsilon > 0 \) (\( \varepsilon \) depends on \( \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)) such that for any \( t \in (0, \tau], x, y \in \mathbb{R}^d, |x-y| \geq a \) we have

\[
|u(t, x, y)| \leq ce^{-c_2 \sqrt{|x-y|}}.
\]

For any \( t \in (0, \tau] \) and \( x \in \mathbb{R}^d \) we have

\[
\int_{\mathbb{R}^d} |u(t, x, y)| \, dy \leq c, \tag{54}
\]

\[
\int_{\mathbb{R}^d} |u(t, y, x)| \, dy \leq c. \tag{55}
\]
Proof. By Corollary 3.3, Proposition 3.10, (21), we only need to prove the corresponding bounds for

\[ I(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| \, dz \, ds. \]

For \(0 < s < t/2\) we have

\[ p_z(t-s, x-z) \leq \frac{c}{t^{d/\alpha}}, \]

and, by Proposition 3.12, for \(t/2 < s < t\),

\[ |q(s, z, y)| \leq \frac{c}{t^{d/\alpha + 1}}. \]

Hence,

\[
I(t, x, y) = \int_0^{t/2} \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| \, dz \, ds + \int_{t/2}^t \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| \, dz \, ds
\leq \frac{c}{t^{d/\alpha}} \int_0^{t/2} \int_{\mathbb{R}^d} |q(s, z, y)| \, dz \, ds + \frac{c}{t^{d/\alpha + 1}} \int_{t/2}^t \int_{\mathbb{R}^d} p_z(t-s, x-z) \, dz \, ds
\leq \frac{c}{t^{d/\alpha}}, \tag{56}
\]

where (53) and Proposition 3.10 were applied to estimate the integrals with respect to the space variable.

Let \(a\) the constant found in Proposition 3.12. Assume that \(|x-y| \geq 1 + a\). By Corollary 3.3 for \(0 < s < t\) we have

\[ p_z(t-s, x-z) \leq ce^{-c_1|x-y|}, \quad |x-z| > |x-y| > 1. \]

Proposition 3.12 implies that for \(0 < s < t\),

\[ |q(s, z, y)| \leq ce^{-c_1\sqrt{|x-y|}}, \quad |y-z| > |x-y| > a \]

Hence,

\[
I(t, x, y) \leq \int_0^t \int_{|x-z|>|x-y|} \ldots \, dz \, ds + \int_0^t \int_{|y-z|>|x-y|} \ldots \, dz \, ds
\leq ce^{-c_1|x-y|} \int_0^t \int_{\mathbb{R}^d} |q(s, z, y)| \, dz \, ds + ce^{-c_1\sqrt{|x-y|}} \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) \, dz \, ds
\leq ct^{1/2}e^{-c_1|x-y|} + cte^{-c_1\sqrt{|x-y|}}
\leq ce^{-c_1\sqrt{|x-y|}}. \tag{57}
\]

Combining (56) and (57) we obtain the desired pointwise estimates of \(u(t, x, y)\).

Next, (54) and (55) immediately follow from (52), (53) and Proposition 3.10. \(\square\)

For any \(\zeta > 0\) and \(x, y \in \mathbb{R}^d\) we put

\[
\mathcal{L}_\zeta f(x) = \sum_{i=1}^d \int_{|w|>\zeta} [f(x + a_i(x)w) - f(x)] \mu(w) \, dw,
\]
\[ \mathcal{L}_\xi f(x) = \sum_{i=1}^{d} \int_{|w|>\zeta} [f(x + a_i(y)w) - f(x)] \mu(w) dw. \]

**Lemma 3.14.** For any \( \xi \in (0, 1] \), \( \zeta > 0 \), \( x, y, v \in \mathbb{R}^d \) and \( t \in (\xi, \tau + \xi) \) we have
\[
\sum_{i=1}^{d} \int_{\mathbb{R}} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) dw \leq c(\xi)e^{-c|x-y|}, \quad (58) \\
\sum_{i=1}^{d} \int_{|w| \leq \zeta} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) dw \leq c(\xi) \zeta^{-\alpha}. \quad (59)
\]
where \( c(\xi) \) is a constant depending on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \xi, \varepsilon, \delta \).

**Proof.** We estimate the term for \( i = 1 \). By Lemma 3.1 for \( \gamma = 1 \) we get for \( w \in \mathbb{R} \)
\[
|p_y(t, x - y + a_1(v)w) - p_y(t, x - y)| \leq c t^{-1/\alpha} |w| \left( r_y \left( t, \frac{x - y}{2} \right) + r_y \left( t, \frac{x - y + a_1(v)w}{2} \right) \right).
\]
Recall that if \( |w| \geq 2\delta \) then \( \mu(w) = 0 \). So we may assume that \( |w| \leq 2\delta \). By Corollary 3.3 we get
\[
r_y \left( t, \frac{x - y}{2} \right) + r_y \left( t, \frac{x - y + a_1(v)w}{2} \right) \leq c_1 t^{-d/\alpha} e^{-c|x-y|}.
\]
Now (58) and (59) follow by the fact that \( \mu(w) \leq c_1|\mathbb{R}|\delta(w)|w|^{-1-\alpha}. \)

**Lemma 3.15.** Let \( \tau_2 > \tau_1 > 0 \) and assume that a function \( f_\xi(x) \) is bounded and uniformly continuous on \( [\tau_1, \tau_2] \times \mathbb{R}^d \). Then
\[
\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_y(\xi_1, x - y) f_\xi(y) dy - f_\xi(x) \right| \to 0 \quad \text{as} \quad \xi_1 \to 0^+.
\]

**Proof.** The lemma follows easily by Proposition 3.10.
and
\[ u_t(x, y) = p_y(t + \xi, x - y) + \varphi^\xi_y(t, x). \]

For any \( t \geq 0, \xi \in [0, 1], t + \xi > 0, x, y \in \mathbb{R}^d \), \( f \in \mathcal{B}_b(\mathbb{R}^d) \) we define
\[ \Phi_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} \varphi_y^\xi(t, x) f(y) \, dy, \]
\[ U_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} u_t^{(\xi)}(t, x, y) f(y) \, dy, \]
\[ \Phi_0 f(x) = 0, \quad U_0^{(0)} f(x) = U_0 f(x) = f(x). \]

By the same arguments as Corollary 3.13 we obtain the following result.

**Corollary 3.16.** For any \( t \in [0, \infty), \xi \in [0, 1], t + \xi > 0, x, y \in \mathbb{R}^d \) the kernel \( u^{(\xi)}(t, x, y) \) is well defined. For any \( t \in (0, \tau], \xi \in [0, 1], x, y \in \mathbb{R}^d \) we have
\[ |u^{(\xi)}(t, x, y)| \leq \frac{c}{(t + \xi)^{d/\alpha} (1 + |x - y|)^{d+1}}. \]

For any \( t \in (0, \tau], \xi \in [0, 1] \) and \( x \in \mathbb{R}^d \) we have
\[ \int_{\mathbb{R}^d} |u^{(\xi)}(t, x, y)| \, dy \leq c, \]
\[ \int_{\mathbb{R}^d} |u^{(\xi)}(t, y, x)| \, dy \leq c. \]

**Lemma 3.17.** Let \( f \in C_0(\mathbb{R}^d) \) and \( \tau \geq \tau_2 > \tau_1 > 0 \). Then \( Q_t f(x) \) as a function of \((t, x)\) is uniformly continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d \). We have \( \lim_{|x| \to \infty} Q_t f(x) = 0 \) uniformly in \( t \in [\tau_1, \tau_2] \). For each \( t > 0 \) we have \( Q_t f \in C_0(\mathbb{R}^d) \).

**Proof.** For any \( \zeta > 0, y \in \mathbb{R}^d \) by Lemma 3.4 we obtain that
\[(t, x) \to \mathcal{L}_c^\zeta p_y(t, \cdot)(x - y) - \mathcal{L}_c^y p_y(t, \cdot)(x - y)\]
is continuous on \((0, \infty) \times \mathbb{R}^d\). Using this and (59) we obtain that
\[(t, x) \to q_0(t, x, y)\]
is continuous on \((0, \infty) \times \mathbb{R}^d\).

By Proposition 3.9 we have
\[ |q_0(t, x, y)| \leq \frac{c}{t^{1+d/\alpha}} e^{-c_1 |x - y|}. \]

For any \( n \in \mathbb{N}, t > 0, x \in \mathbb{R}^d \) denote
\[ Q_{n, t} f(x) = \int_{\mathbb{R}^d} q_n(t, x, y) f(y) \, dy. \]

By (60), (61) and the dominated convergence theorem we obtain that \((t, x) \to Q_{0, t} f(x)\) is continuous on \((0, \infty) \times \mathbb{R}^d\). By Lemma 3.11 for any \( t \in (0, \tau], x \in \mathbb{R}^d, n \in \mathbb{N} \) we have
\[ |Q_{n, t} f(x)| \leq \frac{c_1^{n+1} t^{(n+1)/2} - 1}{(n!)^{1/2}} \| f \|_{\infty}. \]

Note that for any \( t > 0, x \in \mathbb{R}^d, n \in \mathbb{N}, n \geq 1 \) we have
\[ Q_{n, t} f(x) = \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z) Q_{n-1, s} f(z) \, dz \, ds. \]
For any \( \varepsilon_1 \in (0, \tau_1/2) \) using (61), (60) and (62) we obtain that
\[
(t, x) \to \int_0^{t-\varepsilon_1} \int_{\mathbb{R}^d} q_0(t-s, x, z)Q_{n-1,s}f(z) \, dz \, ds
\]
is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\). Note also that for any \( \varepsilon_1 \in (0, \tau_1/2), \, t \in [\tau_1, \tau_2], \, x \in \mathbb{R}^d, \, n \in \mathbb{N}, \, n \geq 1 \) we have by (30)
\[
\left| \int_{t-\varepsilon_1}^t \int_{\mathbb{R}^d} q_0(t-s, x, z)Q_{n-1,s}f(z) \, dz \, ds \right| \leq c\|f\|_\infty \int_{t-\varepsilon_1}^t (t-s)^{-1/2} s^{-1/2} \, ds 
\leq c\tau_1^{-1/2} \varepsilon_1^{1/2} \|f\|_\infty.
\]
This implies that \((t, x) \to Q_{t,f}(x)\) is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\). Using this and (62) we obtain that \((t, x) \to Q_tf(x) = \sum_{n=0}^{\infty} Q_{n,t}f(x)\) is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\).

By Proposition 3.12 we obtain that \(\lim_{|x| \to \infty} Q_tf(x) = 0\) uniformly in \(t \in [\tau_1, \tau_2]\). This implies the assertion of the lemma.

**Proposition 3.18.** Choose \( \gamma \in (0, \alpha) \). For any \( t \in (0, \tau], \, x, x' \in \mathbb{R}^d, \, f \in \mathcal{B}_b(\mathbb{R}^d) \) we have
\[
|U_tf(x) - U_tf(x')| \leq ct^{-\gamma/\alpha}|x - x'|^\gamma \|f\|_\infty.
\]

**Proof.** We have
\[
U_tf(x) - U_tf(x') = \int_{\mathbb{R}^d} (p_y(t, x-y) - p_y(t, x'-y))f(y) \, dy \\
+ \int_0^t \int_{\mathbb{R}^d} (p_z(t, s, x-z) - p_z(t, s, x'-z))Q_s f(z) \, dz \, ds \\
= I + II.
\]

By Lemma 3.1 and Proposition 3.10 we get
\[
|I| \leq c\|f\|_\infty |x-x'|^\gamma t^{-\gamma/\alpha} \int_{\mathbb{R}^d} (r_y(t, (x-y)/2) + r_y(t, (x'-y)/2)) \, dy \\
\leq c\|f\|_\infty |x-x'|^\gamma t^{-\gamma/\alpha}.
\]

By Lemma 3.1 and Propositions 3.10, 3.12 we obtain
\[
|II| \leq c\|f\|_\infty |x-x'|^\gamma \left[ \int_0^s (t-s)^{-\gamma/\alpha} r_z(t-s, (x-z)/2) \right] \\
+ \int_0^t (t-s)^{-\gamma/\alpha} s^{-1/2} ds \\
\leq c\|f\|_\infty |x-x'|^\gamma \left[ 1 - (t-s)^{-\gamma/\alpha} s^{-1/2} ight] \\
\leq c\|f\|_\infty |x-x'|^{1/2-\gamma/\alpha}.
\]

Note that by Lemma 3.14 for any \( \xi \in (0, 1], \, t \in [\xi, \tau + \xi], \, x, z \in \mathbb{R}^d \) we have
\[
\left| \frac{\partial p_z(t, x-z)}{t} \right| = |\mathcal{L}^z p_z(t, \cdot)(x-z)| \leq c(\xi) e^{-c|x-z|},
\] (63)
where \( c(\xi) \) is a constant depending on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta \).

The next lemma is similar to [17, Lemma 4.1].
Lemma 3.19. (i) For every \( f \in C_0(\mathbb{R}^d) \), \( \xi \in (0, 1] \) the function \( U_t^{(\xi)} f(x) \) belongs to \( C^1((0, \infty)) \) as a function of \( t \) and to \( C^2_0(\mathbb{R}^d) \) as a function of \( x \). Moreover we have

\[
\left| \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) \right| \leq c(\xi) \| f \|_{\infty},
\]

for each \( f \in C_0(\mathbb{R}^d) \), \( t \in (0, \tau] \), \( x \in \mathbb{R}^d \), \( \xi \in (0, 1] \), where \( c(\xi) \) depends on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \).

(ii) For every \( f \in C_0(\mathbb{R}^d) \) we have

\[
\lim_{t, \xi \to 0^+} \| U_t^{(\xi)} f - f \|_{\infty} = 0.
\]

(iii) For every \( f \in C_0(\mathbb{R}^d) \) we have

\[
U_t^{(\xi)} f(x) \to 0, \quad \text{as} \quad |x| \to \infty,
\]

uniformly in \( t \in [0, \tau] \), \( \xi \in [0, 1] \).

(iv) For every \( f \in C_0(\mathbb{R}^d) \) we have

\[
\| U_t^{(\xi)} f - U_t f \|_{\infty} \to 0, \quad \text{as} \quad \xi \to 0^+,
\]

uniformly in \( t \in [0, \tau] \).

Proof. (i) Let \( f \in C_0(\mathbb{R}^d) \), \( t \in (0, \tau] \), \( \xi \in (0, 1] \) and \( x \in \mathbb{R}^d \). We have

\[
\lim_{h \to 0^+} \frac{\Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x)}{h} = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^d} p_z(t + h - s + \xi, x - z) Q_s f(z) \, dz \, ds + \lim_{h \to 0^+} \int_0^t \int_{\mathbb{R}^d} \frac{p_z(t + h - s + \xi, x - z) - p_z(t - s + \xi, x - z)}{h} Q_s f(z) \, dz \, ds
\]

\[
= I + II.
\]

By Lemmas 3.3, 3.17, Corollary 3.3 and Proposition 3.12 we get

\[
I = \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) \, dz.
\]

By Lemma 3.3, the dominated convergence theorem, (63) and Proposition 3.12 we get

\[
II = \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) \, dz \, ds.
\]

By similar arguments we get the analogous result for \( \lim_{h \to 0^-} \left( \Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x) \right) / h \). By (63) we get

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} p_z(t + \xi, x - z) f(z) \, dz = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) \, dz.
\]
Hence we have
\[
\frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) \, dz \\
+ \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) \, dz \\
+ \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) \, dz \, ds
\] (65)

Using this, (64), Proposition 3.10 and 3.12 we obtain (64). We also obtain that for every \( f \in C_0(\mathbb{R}^d) \), \( \xi \in (0, 1] \) the function \( U_t^{(\xi)} f(x) \) belongs to \( C^1((0, \infty)) \) as a function of \( t \).

The fact that \( U_t^{(\xi)} f \in C_0^2(\mathbb{R}^d) \) for \( \xi \in (0, 1] \) follows by Lemmas 3.4, 3.5 Proposition 3.12 and Lemma 3.17

(ii) Fix \( f \in C_0(\mathbb{R}^d) \). For any \( \xi \in [0, 1] \), \( t \geq 0 \), \( t + \xi > 0 \), \( x \in \mathbb{R}^d \) we have
\[
U_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy + \Phi_t^{(\xi)} f(x).
\]

For any \( \xi \in [0, 1] \), \( t \in [0, \tau] \), \( \xi + t > 0 \), \( x \in \mathbb{R}^d \) by Proposition 3.12 and Proposition 3.10 we get
\[
\left| \Phi_t^{(\xi)} f(x) \right| \leq c\|f\|_{\infty} \int_0^t s^{-1/2} \, ds \leq c\|f\|_{\infty} t^{1/2}.
\] (66)

By (68), Proposition 3.10 and the fact that \( f \) is uniformly continuous on \( \mathbb{R}^d \) we obtain
\[
\lim_{t, \xi \to 0^+} \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy - f(x) = 0
\]
uniformly with respect to \( x \in \mathbb{R}^d \). This and (66) gives (ii).

(iii) This follows easily from (ii) and Corollary 3.16

(iv) Fix \( f \in C_0(\mathbb{R}^d) \). By Lemma 3.4, Corollary 3.3 and the dominated convergence theorem we obtain that
\[
(t, x) \to \int_{\mathbb{R}^d} p_y(t, x - y) f(y) \, dy
\]
is continuous on \( (0, \tau + 1] \times \mathbb{R}^d \). It follows that
\[
(\xi, t, x) \to \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy
\]
is continuous on \( [0, 1] \times (0, \tau] \times \mathbb{R}^d \). Using Lemma 3.4, Corollary 3.3, Proposition 3.12 and the dominated convergence theorem we obtain that for any \( s \in (0, \tau) \)
\[
(\xi, t, x) \to \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) \, dz
\]
is continuous on \( [0, 1] \times (s, \tau] \times \mathbb{R}^d \). Using this, Corollary 3.3, Proposition 3.12 and the dominated convergence theorem we obtain that
\[
(\xi, t, x) \to \Phi_t^{(\xi)} f(x) = \int_0^\tau 1_{(0,\tau)}(s) \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) \, dz \, ds
\]
is continuous on \( [0, 1] \times (0, \tau] \times \mathbb{R}^d \). Hence \( (\xi, t, x) \to U_t^{(\xi)} f(x) \) is continuous on \( [0, 1] \times (0, \tau] \times \mathbb{R}^d \). Using (66) we obtain that
\[
(\xi, t, x) \to (U_t^{(\xi)} f(x)) \text{ is continuous on } [0, 1] \times [0, \tau] \times \mathbb{R}^d.
\] (67)
This and (iii) implies (iv). □

By the same arguments as in the proof of Lemma 3.19 (iv) we obtain the following result.

**Lemma 3.20.** For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) the function \((t, x) \rightarrow U_t f(x)\) is continuous on \((0, \infty) \times \mathbb{R}^d\). For any \( \xi \in (0, 1], f \in \mathcal{B}_b(\mathbb{R}^d) \) the function \((t, x) \rightarrow U_t^{(\xi)} f(x)\) is continuous on \([0, \infty) \times \mathbb{R}^d\).

Heuristically, now our aim is to show that if \( \xi \) is small then \( \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x) \) is small. For any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d \) we put

\[
\Lambda_t^{(\xi)} f(x) = \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x).
\]

**Lemma 3.21.** \( \mathcal{L}(U_t^{(\xi)} f)(x) \) is well defined for every \( f \in C_0(\mathbb{R}^d), t \in (0, \tau], \xi \in (0, 1] \) and \( x \in \mathbb{R}^d \) and we have

\[
\Lambda_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} p_\xi(x, z)(x-z)Q_t f(z) \, dz - Q_{t+\xi} f(x) + \int_t^{t+\xi} \int_{\mathbb{R}^d} q_0(s-t+\xi, x, z)Q_s f(z) \, dz \, ds.
\]

Moreover we have

\[
\left| \mathcal{L}_{\xi}(U_t^{(\xi)} f)(x) \right| \leq c(\xi) \| f \|_\infty, \quad \xi > 0,
\]

\[
\left| \mathcal{L}(U_t^{(\xi)} f)(x) \right| \leq c(\xi) \| f \|_\infty.
\]

for each \( f \in C_0(\mathbb{R}^d), x \in \mathbb{R}^d, t \in (0, \tau], \xi \in (0, 1], \) where \( c(\xi) \) is a constant depending on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta \).

**Proof.** Let \( f \in C_0(\mathbb{R}^d), t \in (0, \tau], \xi \in (0, 1], x \in \mathbb{R}^d \) and \( \xi > 0 \). We have

\[
\mathcal{L}_{\xi}(U_t^{(\xi)} f)(x) = \int_{\mathbb{R}^d} \mathcal{L}_{\xi}^\varepsilon p_\varepsilon(t + \xi, \cdot)(x-z) f(z) \, dz + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_{\xi}^\varepsilon p_\varepsilon(t-s+\xi, \cdot)(x-z)Q_s f(z) \, dz \, ds.
\]

Using this, Lemma 3.14 and Proposition 3.12 we obtain (69). By (71), the dominated convergence theorem, Lemma 3.14 and Proposition 3.12 one gets

\[
\mathcal{L}(U_t^{(\xi)} f)(x) = \lim_{\xi \to 0^+} \mathcal{L}_{\xi}(U_t^{(\xi)} f)(x)
\]

\[
= \int_{\mathbb{R}^d} \mathcal{L}^\varepsilon p_\varepsilon(t + \xi, \cdot)(x-z) f(z) \, dz + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^\varepsilon p_\varepsilon(t-s+\xi, \cdot)(x-z)Q_s f(z) \, dz \, ds.
\]

Using this and again Lemma 3.14 and Proposition 3.12 we obtain (70).

Note that for \( s \in [0, t], z \in \mathbb{R}^d \) we have

\[
\frac{\partial}{\partial t} p_\varepsilon(t-s+\xi, x-z) - \mathcal{L}^\varepsilon p_\varepsilon(t-s+\xi, \cdot)(x-z) = -q_0(t-s+\xi, x, z).
\]
Using this, (65) and (72) we get

\[ \Lambda_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} p_\xi(t, x - z) Q_t f(z) \, dz \]

\[ - \int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) \, dz \]

\[ - \int_0^t \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) \, dz \, ds. \]  

(73)

For \( \xi \in (0, 1], t \in (0, \tau], x \in \mathbb{R}^d \) by the definition of \( q(t, x, y) \) we obtain

\[ \int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) \, dz = Q_{t+\xi} f(x) - \int_0^{t+\xi} \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) \, dz \, ds. \]

Using this and (73) we obtain (68).

The next lemma is similar to [19, Lemma 4.2].

**Lemma 3.22.** (i) For any \( f \in C_0(\mathbb{R}^d) \) we have

\[ \Lambda_t^{(\xi)} f(x) \to 0, \quad \text{as} \quad \xi \to 0^+, \]

uniformly in \((t, x) \in [\tau_1, \tau_2] \times \mathbb{R}^d\) for every \( \tau \geq \tau_2 > \tau_1 > 0 \). (ii) For any \( f \in C_0(\mathbb{R}^d) \) we have

\[ \int_0^t \Lambda_s^{(\xi)} f(x) \, ds \to 0, \quad \text{as} \quad \xi \to 0^+, \]  

(74)

uniformly in \((t, x) \in (0, \tau] \times \mathbb{R}^d\).

**Proof.** Let \( f \in C_0(\mathbb{R}^d) \) and \( 0 < \tau_1 < \tau_2 \leq \tau \). For any \( t > 0 \), \( x \in \mathbb{R}^d \), \( \xi \in (0, 1] \) we put

\[ \Lambda_t^{(\xi,1)} f(x) = \int_{\mathbb{R}^d} p_\xi(t, x - z) Q_t f(z) \, dz - Q_{t+\xi} f(x). \]

\[ \Lambda_t^{(\xi,2)} f(x) = \int_t^{t+\xi} \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) \, dz \, ds. \]

By Lemma 3.17 we get

\[ \sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} |Q_{t+\xi} f(x) - Q_t f(x)| \to 0 \quad \text{as} \quad \xi \to 0^+. \]

By Lemmas 3.15 and 3.17 we obtain

\[ \sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\xi(t, x - z) Q_t f(z) \, dz - Q_t f(x) \right| \to 0 \quad \text{as} \quad \xi \to 0^+. \]

This gives (i) for \( \Lambda_t^{(\xi,1)} f(x) \) instead of \( \Lambda_t^{(\xi)} f(x) \).

By Proposition 3.12 for any \( t \in (0, \tau], x \in \mathbb{R}^d, \xi \in (0, 1] \) we get

\[ \left| \Lambda_t^{(\xi,1)} f(x) \right| \leq c\|f\|_\infty t^{-1/2}. \]  

(75)

This allows to use the dominated convergence theorem in the integral (74) with \( \Lambda_t^{(\xi)} f(x) \) replaced by \( \Lambda_t^{(\xi,1)} f(x) \). So (ii) for \( \Lambda_t^{(\xi)} f(x) \) follows from (i) for \( \Lambda_t^{(\xi,1)} f(x) \).

For any \( t \in (0, \tau], x \in \mathbb{R}^d, \xi \in (0, 1] \) by Propositions 3.9 and 3.12 we get

\[ \left| \Lambda_t^{(\xi,2)} f(x) \right| \leq c\|f\|_\infty \int_t^{t+\xi} ((t - s + \xi)s)^{-1/2} \, ds. \]  

(76)
This implies (i) and (ii) for $A_t^{(0,2)}f(x)$.

**Lemma 3.23.** There exist $\varepsilon_1 \in (0, 1]$ and $t_1 \in (0, 1]$ such that for any $t \in (0, t_1]$, $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_1 t^{1/\alpha}$ we have

$$u(t, x, y) \geq c_1 t^{-d/\alpha}.$$ 

$\varepsilon_1$, $t_1$ depend on $\alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta$.

**Proof.** By the weak lower scaling property of the symbol $\Phi^{(1)}_\delta$ (see proof of Lemma 2.4 and by 3 formula (23)) we get that $g_t(0) \geq ct^{-1/\alpha}$. Using this and Lemma 3.21 there exist $\varepsilon_2 > 0$, $t_2 > 0$ such that for $|y| \leq \varepsilon_2 t^{1/\alpha}$, $t \leq t_2$ we have $g_t(y) \geq ct^{-1/\alpha}$. It follows that there exist $\varepsilon_3 > 0$, $t_3 > 0$ such that for any $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_3 t^{1/\alpha}$, $t \leq t_3$ we have $p_y(t, x - y) \geq ct^{-d/\alpha}$. By Lemma 3.12 and Proposition 3.10 for $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $a \in [1/2, 1)$ we get

$$|\varphi_y(t, x)| = \left| \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z)q(s, z, y)\ dz\ ds 
+ \int_t^{t_1} \int_{\mathbb{R}^d} p_z(t - s, x - z)q(s, z, y)\ dz\ ds \right|$$

$$\leq c(t - ta)^{-d/\alpha} \int_0^{ta} s^{-1/2} \ ds + c t^{-d/\alpha - 1} t(1 - a)$$

$$\leq c t^{-d/\alpha}(t^{1/2}(1 - a)^{-d/\alpha} + 1 - a).$$

By an appropriate choice of $a$ there exists $\varepsilon_1 > 0$ such that for any $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_1 t^{1/\alpha}$, $t \leq t_1$ we have

$$u(t, x, y) = p_y(t, x - y) - \varphi_y(t, x) \geq ct^{-d/\alpha} - c_2 t^{-d/\alpha}(t^{1/2}(1 - a)^{-d/\alpha} + 1 - a) \geq c_1 t^{-d/\alpha}. $$

4. **Construction and Properties of the Semigroup of $X_t$**

Let us introduce the following notation

$$\nu(x) = \frac{A_\alpha}{|x|^{1+\alpha}} - \mu(x), \quad x \in \mathbb{R},$$

$$\lambda = d \int_{\mathbb{R}} \nu(x)\ dx < \infty.$$ 

Note that by (19), for any $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, we have

$$\mathcal{R}f(x) = \sum_{i=1}^d \int_{\mathbb{R}} [f(x + a_i(x) w) - f(x)] \nu(w)\ dw.$$ 

We denote, for any $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathcal{N}f(x) = \sum_{i=1}^d \int_{\mathbb{R}} [f(x + a_i(x) w)] \nu(w)\ dw.$$ 

It is clear that

$$||\mathcal{N}f||_\infty \leq \lambda ||f||_\infty.$$

(77)
For any $t \geq 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $n \geq 1$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define
\[ \Psi_{0,t}f(x) = U_tf(x), \] \[ \Psi_{n,t}f(x) = \int_0^t U_{t-s}(N(\Psi_{n-1,s}f))(x) \, ds, \quad n \geq 1. \] (78)
(79)

For any $t \geq 0$, $\xi \in [0, 1]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define
\[ \Psi_{0,t}^{(\xi)}f(x) = U_t^{(\xi)}f(x), \] \[ \Psi_{n,t}^{(\xi)}f(x) = \int_0^t U_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)}f))(x) \, ds, \quad n \geq 1. \] (80)
(81)

We remark that
\[ \Psi_{n,t} = \Psi_{n,t}^{(0)}. \]

**Lemma 4.1.** $\Psi_{n,t}f(x)$ and $\Psi_{n,t}^{(\xi)}f(x)$ are well defined for any $t > 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $\xi \in [0, 1]$. For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we have
\[ |\Psi_{n,t}f(x)| \leq \frac{c_1^{n+1}t^n}{n!} \|f\|_{\infty}, \quad t \in (0, \tau], \] \[ |\Psi_{n,t}^{(\xi)}f(x)| \leq \frac{c_1^{n+1}t^n}{n!} \|f\|_{\infty}, \quad \xi \in (0, 1), \quad t \in [0, \tau]. \] (82)
(83)

**Proof.** We will only show the result for $\Psi_{n,t}f(x)$ using the induction. The proof for $\Psi_{n,t}^{(\xi)}f(x)$ is almost the same.

Let $c$ be the constant from (54) and put $c_1 = (\lambda \lor 1)c$. For $n = 0$ (82) follows from (54). Assume that (82) holds for $n \geq 0$, we will show it for $n + 1$. Indeed, applying (54) and (77), we get
\[ |\Psi_{n+1,t}f(x)| \leq \int_0^t \int_{\mathbb{R}^d} |u(t - s, x, z)| \, dz \frac{\lambda s^n n!}{(n + 1)!} \, ds \leq \frac{c_1^{n+2}t^{n+1}}{(n + 1)!}. \]

For any $x \in \mathbb{R}^d$ we define
\[ T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}f(x), \quad t > 0, \] \[ T_0 f(x) = f(x), \] \[ T_t^{(\xi)} f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)}f(x), \quad t \geq 0, \quad \xi \in [0, 1]. \]

Our ultimate aim will be to show that for any $t > 0$ we have $T_t = P_t$, where $P_t$ is given by (5).

By Lemma 4.1 we obtain

**Corollary 4.2.** $T_t f(x)$ and $T_t^{(\xi)} f(x)$ are well defined for any $t \geq 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\xi \in [0, 1]$ and for $t \in [0, \tau]$ we have $\max\{|T_t f(x)|, |T_t^{(\xi)} f(x)|\} \leq c \|f\|_{\infty}$.

Next, we obtain the following regularity results concerning operators $T_t$.

**Theorem 4.3.** For any $\gamma \in (0, \alpha/d)$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have
\[ |T_t f(x)| \leq ct^{-\alpha/d} \|f\|_{1-\gamma}^\gamma \|f\|_1^\gamma. \]
Lemma 4.6. Assume that \( f \in \mathcal{B}_b(\mathbb{R}^d) \). Put \( \beta = 1/(4(d/\alpha + 1)) \). For any \( t \in (0, \tau], \ x \in \mathbb{R}^d \), we have

\[
|Q_t f(x)| \leq \frac{c\|f\|_{\infty}}{t^{3/4}(\text{dist}(x, \text{supp}(f)) + 1)^{\beta}}.
\]
Proof. Let $t \in (0, \tau]$ be arbitrary. By Proposition 3.12 we get for $x \in \mathbb{R}^d$

$$|Q_t f(x)| \leq c t^{-1/2} \|f\|_\infty,$$

$$|Q_t f(x)| \leq \frac{c \|f\|_\infty}{t^{d/\alpha + 1} (\text{dist}(x, \text{supp}(f)) + 1)}.$$  

It follows that

$$|Q_t f(x)|^{1-\beta} \leq \frac{c (1-\beta) (1-\beta) \|f\|_\infty^{1-\beta} \leq c t^{-1/2} \|f\|_\infty^{1-\beta},$$

$$|Q_t f(x)|^\beta \leq \frac{c \|f\|_\infty^\beta}{t^{1/4} (\text{dist}(x, \text{supp}(f)) + 1)^\beta}.$$  

This implies the assertion of the lemma. \qed

**Lemma 4.7.** Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $\xi \in [0, 1]$, $t \in [0, \tau]$, $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$ then $|U_t^{(\xi)} f(x)| \leq \varepsilon_1 \|f\|_\infty$.

**Proof.** Let $\xi \in [0, 1]$, $t \in [0, \tau]$ be arbitrary. Assume that $\text{dist}(x, \text{supp}(f)) \geq 1$ and $t + \xi > 0$. By Lemmas 3.10, 4.6 we get

$$|\Phi^t_\xi f(x)| = \left| \int_0^t \int_{\mathbb{R}^d} p_z(t - s + \xi, x - z) f(z) \, dz \, ds \right| \leq \frac{c \|f\|_\infty}{(\text{dist}(x, \text{supp}(f)) + 1)^\beta} \int_0^t \frac{1}{s^{1/4}} \, ds.$$  

By Corollary 3.3 we get

$$\int_{\mathbb{R}^d} p_z(t + \xi, x - z) f(z) \, dz \leq c \|f\|_\infty e^{-c_1 \text{dist}(x, \text{supp}(f))}.$$  

This gives the assertion of the lemma. \qed

The proof of the next lemma is standard and it is omitted.

**Lemma 4.8.** Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that, for any $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$, then $|Nf(x)| \leq \varepsilon_1 \|f\|_\infty$.

**Lemma 4.9.** Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $\xi \in [0, 1]$, $t \in [0, \tau]$, $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$ then $\int_0^t |U_t^{(\xi)} (Nf)(x)| \, ds \leq \varepsilon_1 \|f\|_\infty$.

**Proof.** Let $\xi \in [0, 1]$, $t \in (0, \tau]$ be arbitrary. Choose $\varepsilon_1 > 0$. There exists a $r_1 > 0$ such that for any $g \in \mathcal{B}_b(\mathbb{R}^d)$ we have $|U_t^{(\xi)} g|_\infty \leq a \|g\|_\infty$, where $a$ depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$. By Lemma 4.8 there exists $r_1 \geq 1$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_1$, then

$$|Nf(x)| \leq \frac{\varepsilon_1 \|f\|_\infty}{2a \tau},$$  

where $r_1$ depends on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$. Put $A = \{x \in \mathbb{R}^d : \text{dist}(x, \text{supp}(f)) \geq r_1\}$.

We have, applying also (77),

$$|Nf(x)| = |Nf(x) 1_{A^c}(x)| + |Nf(x) 1_A(x)| \leq \lambda \|f\|_\infty 1_{A^c}(x) + \frac{\varepsilon_1 \|f\|_\infty}{2a \tau} 1_A(x).$$

Put $\tilde{f}(x) = \lambda \|f\|_\infty 1_{A^c}(x)$, $\tilde{f}(x) = \frac{\varepsilon_1 \|f\|_\infty}{2a \tau} 1_A(x)$.
We have
\[ \left| \int_0^t U_{t-s}^\xi(N(f))(x) \, ds \right| \leq \int_0^t U_{t-s}^\xi(\tilde{f})(x) \, ds + \int_0^t U_{t-s}^\xi(\check{f})(x) \, ds. \] (86)

For any \( x \in \mathbb{R}^d \) we get
\[ \int_0^t U_{t-s}^\xi(\tilde{f})(x) \, ds \leq \frac{\varepsilon_1 \| f \|_\infty}{2}. \] (87)

By Lemma 4.7 there exists \( r_2 \geq r_1 \) such that, if \( \text{dist}(x, \text{supp}(f)) \geq r_2 \) and \( s \in (0, t) \), then
\[ |U_{t-s}^\xi(\tilde{f})(x)| \leq \frac{\varepsilon_1 \| f \|_\infty}{2\tau}, \]
where \( r_2 \) depends on \( \varepsilon_1, r_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \). It follows that, if \( \text{dist}(x, \text{supp}(f)) \geq r_2 \), then
\[ \int_0^t U_{t-s}^\xi(\tilde{f})(x) \, ds \leq \frac{\varepsilon_1 \| f \|_\infty}{2}. \] (88)

Finally, (86) and (88) imply the assertion of the lemma. \( \square \)

**Lemma 4.10.** Assume that \( f \in \mathcal{B}_b(\mathbb{R}^d) \). For any \( \varepsilon_1 > 0 \) there exists \( r \geq 1 \) (depending on \( \varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)), such that for any \( \xi \in [0, 1] \), \( t \in [0, \tau] \), \( x \in \mathbb{R}^d \), if \( \text{dist}(x, \text{supp}(f)) \geq r \), then \( |T_t^{(\xi)} f(x)| \leq \sum_{n=0}^\infty |\Psi_{n,t}^{(\xi)} f(x)| \leq \varepsilon_1 \| f \|_\infty \).

**Proof.** Fix \( f \in \mathcal{B}_b(\mathbb{R}^d) \). Put
\[ M = \sup_{n \in \mathbb{N}} c_1^{n+1} \tau^n n!, \]
where \( c_1 \) is a constant from Lemma 4.11. Let \( \xi \in [0, 1] \), \( t \in [0, \tau] \) be arbitrary such that \( t + \xi > 0 \). Choose \( \varepsilon_2 > 0 \). By Lemma 4.11 there exists \( n_0 \) such that for any \( x \in \mathbb{R}^d \) we have
\[ \sum_{n=n_0}^\infty |\Psi_{n,t}^{(\xi)} f(x)| \leq \varepsilon_2 \| f \|_\infty. \] (89)

Put \( r_{-1} = 1 \). Now we will show that for any \( n \in \mathbb{N} \) there exists \( r_n \geq r_{n-1} \) such that, if \( \text{dist}(x, \text{supp}(f)) \geq r_n \), then
\[ |\Psi_{n,t}^{(\xi)} f(x)| \leq 2^n a^n \varepsilon_2 \| f \|_\infty, \] (90)
where \( a \geq 1 \) is a constant from Lemma 4.13 and \( r_n \) depends on \( r_{n-1}, \varepsilon_2, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \).

By Lemma 4.7 there exists \( r_0 \geq r_{-1} \) such that, if \( \text{dist}(x, \text{supp}(f)) \geq r_0 \), then
\[ |\Psi_{0,t}^{(\xi)} f(x)| = |U_{t}^{\xi} f(x)| \leq \varepsilon_2 \| f \|_\infty, \] (91)
where \( r_0 \) depends on \( \varepsilon_2, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \).

Assume that (90) holds for \( n \in \mathbb{N} \). We will show it for \( n+1 \). Put \( A_n = \{ x \in \mathbb{R}^d : \text{dist}(x, \text{supp}(f)) \geq r_n \} \). We have
\[ |\Psi_{n+1,t}^{(\xi)} f(x)| = |\Psi_{n+1,t}^{(\xi)} f(x) I_{A_n}(x) + \Psi_{n+1,t}^{(\xi)} f(x) I_{A_n^c}(x)| \leq M \| f \|_\infty I_{A_n^c}(x) + 2^n a^n \varepsilon_2 \| f \|_\infty I_{A_n}(x). \]

Put \( f_n(x) = M \| f \|_\infty I_{A_n^c}(x) \), \( \tilde{f}_n(x) = 2^n a^n \varepsilon_2 \| f \|_\infty I_{A_n^c}(x) \). We have
\[ \Psi_{n+1,t}^{(\xi)} f(x) = \int_0^t U_{t-s}^{\xi}(N(\Psi_{n+1,t}^{(\xi)} f))(x) \, ds. \]
Lemma 4.12. (i) For every $x \in \mathbb{R}^d$, we get
\[
\int_0^t U_{t-s}^{(\xi)}(N(f_n))(x) \, ds \leq a\|f_n\|_\infty \leq 2^n a^{n+1} \varepsilon_2 \|f\|_\infty.
\] (93)

By Lemma 4.9, there exists $r_{n+1} \geq r_n$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_{n+1}$, then
\[
\int_0^t U_{t-s}^{(\xi)}(N(f_n))(x) \, ds \leq \varepsilon_2 \|f\|_\infty,
\] (94)
where $r_{n+1}$ depends on $\varepsilon_2, r_n, \tau, a, d, \eta_1, \eta_2, \eta_3$.

By (92)(94) we obtain (90) for $n+1$. By (90), we obtain that, if $\text{dist}(x, \text{supp}(f)) \geq r_{n_0}$, then
\[
\sum_{n=0}^{n_0} \left| \Psi_{n+1}^{(\xi)} f(x) \right| \leq \varepsilon_2 \sum_{n=0}^{n_0} 2^n a^n \|f\|_\infty.
\]
Using this and (89) we get the assertion of the lemma. \qed

By Lemma 4.10 and Theorem 4.3 one easily obtains the following result.

Corollary 4.11. Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$, for any $n \in \mathbb{N}$, $n \geq 1$ we have $f_n \in \mathcal{B}_b(\mathbb{R}^d)$, $\sup_{n \in \mathbb{N}, n \geq 1} \|f_n\|_\infty < \infty$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for almost all $x \in \mathbb{R}^d$ with respect to the Lebesgue measure. Then, for any $t > 0$, $x \in \mathbb{R}^d$, we have $\lim_{n \to \infty} T_t f_n(x) = T_t f(x)$.

Lemma 4.12. (i) For every $f \in C_0(\mathbb{R}^d)$ we have
\[
\lim_{t, \xi \to 0^+} \|T_t^{(\xi)} f - f\|_\infty = 0.
\]

(ii) For every $f \in C_0(\mathbb{R}^d)$ we have
\[
T_t^{(\xi)} f(x) \to 0, \quad \text{as} \quad |x| \to \infty,
\]
uniformly in $t \in [0, \tau]$, $\xi \in [0, 1]$.

(iii) For every $f \in C_0(\mathbb{R}^d)$ we have
\[
\|T_t^{(\xi)} f - T_t f\|_\infty \to 0, \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in $t \in [0, \tau]$.

Proof. (i) This follows from Lemma 3.19(ii) and Lemma 4.1.

(ii) Note that $T_0^{(\xi)} f = U_0^{(\xi)}$, so (ii) for $t = 0$ follows from Lemma 3.19. So we may assume that $t > 0$. Let $t \in (0, \tau], \xi \in [0, 1]$ be arbitrary. By Lemma 4.2 we have
\[
\left\| T_t^{(\xi)} f \right\|_\infty \leq c_1 \|f\|_\infty.
\] (95)

Choose $\varepsilon_1 > 0$. Since $f \in C_0(\mathbb{R}^2)$ there exists $r_1 > 0$ such that if $|x| \geq r_1$ then $|f(x)| \leq \varepsilon_1/(2c_1)$, where $c_1$ is a constant from (95). Put $f_1(x) = f(x) 1_{B(0, r_1)}(x)$, $f_2(x) = f(x) 1_{B^c(0, r_1)}(x)$. By Lemma 4.11 there exists $r_2 > r_1$ such that, if $|x| \geq r_2$, then $|T_t^{(\xi)} f_1(x)| \leq \varepsilon_1/2$. Hence for any $|x| \geq r_2$ we have $|T_t^{(\xi)} f(x)| \leq |T_t^{(\xi)} f_1(x) + T_t^{(\xi)} f_2(x)| \leq \varepsilon_1/2 + (\varepsilon_1/(2c_1))c_1 = \varepsilon_1$. 


(iii) Let \( \xi \in (0, 1], n \in \mathbb{N}, n \geq 1, x \in \mathbb{R}^d, f \in C_0(\mathbb{R}^d) \). Note that for any \( t \in (0, \tau] \) we have

\[
\Psi_{n,t}^{(\xi)} f(x) = \int_0^t \int_{\mathbb{R}^d} p_y(t-s+\xi, x-y)N(\Psi_{n-1,s}^{(\xi)} f)(y) \, dy \, ds \\
+ \int_0^t \int_{\mathbb{R}^d} p_z(t-s-r+\xi, x-z)Q_r(N(\Psi_{n-1,s}^{(\xi)} f))(z) \, dz \, dr \, ds
\]

By the same arguments as in the proof of Lemma 3.19 (iv) we obtain that \( (\xi, t, x) \to \Psi_{n,t}^{(\xi)} f(x) \) is continuous on \( [0, 1] \times [0, \tau] \times \mathbb{R}^d \) for any \( n \in \mathbb{N}, n \geq 1 \). Using this, (67) and Lemma 4.1 we obtain that \( (\xi, t, x) \to T_t^{(\xi)} f(x) \) is continuous on \( [0, 1] \times [0, \tau] \times \mathbb{R}^d \). This and (ii) implies (iii). □

By the same arguments as in the proof of Lemma 3.19 (iv) we obtain the following result.

**Lemma 4.13.** For any \( f \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N} \), the function \( (t, x) \to \Psi_{n,t} f(x) \) is continuous on \( (0, \infty) \times \mathbb{R}^d \). For any \( \xi \in (0, 1], f \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N} \), the function \( (t, x) \to \Psi_{n,t}^{(\xi)} f(x) \) is continuous on \( [0, \infty) \times \mathbb{R}^d \).

**Lemma 4.14.** \( \frac{\partial}{\partial t} \left( \Psi_{n,t}^{(\xi)} f \right)(x), \mathcal{L} \left( \Psi_{n,t}^{(\xi)} f \right)(x) \) are well defined for any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d, n \in \mathbb{N}, n \geq 1 \) and \( f \in C_0(\mathbb{R}^d) \) and we have

\[
\frac{\partial}{\partial t} \left( \Psi_{n,t}^{(\xi)} f \right)(x) - \mathcal{L} \left( \Psi_{n,t}^{(\xi)} f \right)(x) = \int_{\mathbb{R}^d} p_x(\xi, x-z)N\left( \Psi_{n-1,t}^{(\xi)} f \right)(z) \, dz \\
+ \int_0^t \Lambda_{t-s}^{(\xi)} \left( N\left( \Psi_{n-1,s}^{(\xi)} f \right) \right)(x) \, ds.
\]

Moreover, \( \frac{\partial}{\partial t} \Psi_{n,t}^{(\xi)} f(x) \) is continuous as a function of \( t \) for \( t > 0 \).

**Proof.** Let \( \xi \in (0, 1], n \in \mathbb{N}, n \geq 1, x \in \mathbb{R}^d, f \in C_0(\mathbb{R}^d) \). Note that for any \( t \in (0, \tau], s \in (0, t) \), we have

\[
U_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)} f))(x) = \int_{\mathbb{R}^d} p_y(t-s+\xi, x-y)N(\Psi_{n-1,s}^{(\xi)} f)(y) \, dy \\
+ \int_0^{t-s} \int_{\mathbb{R}^d} p_z(t-s-r+\xi, x-z)Q_r(N(\Psi_{n-1,s}^{(\xi)} f))(z) \, dz \, dr.
\]

By similar arguments as in the proof of Lemma 3.19 (i) we obtain that \( \frac{\partial}{\partial t} U_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)} f))(x) \) is well defined and continuous as a function of \( t \) for \( t \in (s, \tau] \). Note that for any \( g \in C_0(\mathbb{R}^d) \), and \( t \geq 0 \) we have \( U_t^{(\xi)} g \in C_0(\mathbb{R}^d), N g \in C_0(\mathbb{R}^d) \). By (31), (64), Lemmas 3.20 4.13 3.19 (i) and standard arguments we get

\[
\frac{\partial}{\partial t} \left( \Psi_{n,t}^{(\xi)} f \right)(x) = U_0^{(\xi)} \left( N\left( \Psi_{n-1,t}^{(\xi)} f \right) \right)(x) + \int_0^t \frac{\partial}{\partial t} U_{t-s}^{(\xi)} \left( N\left( \Psi_{n-1,s}^{(\xi)} f \right) \right)(x) \, ds \\
= \int_{\mathbb{R}^d} p_x(\xi, x-z)N(\Psi_{n-1,t}^{(\xi)} f)(z) \, dz + \int_0^t \Lambda_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)} f))(x) \, ds \\
+ \int_0^t \mathcal{L}^x(U_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)} f)))(x) \, ds.
\]

This implies that \( \frac{\partial}{\partial t} \Psi_{n,t}^{(\xi)} f(x) \) is continuous as a function of \( t \) for \( t \in (0, \tau] \).
For any $\zeta > 0$ we have
\[
\mathcal{L}_\zeta \left( \Psi_{n,t}^{(1)} \right)(x) = \int_0^t \mathcal{L}_\zeta U_{t-s} \left( \mathcal{N} \left( \Psi_{n-1,s}^{(1)} \right) \right)(x) \, ds.
\]
By the dominated convergence theorem and (69) we obtain
\[
\mathcal{L} \left( \Psi_{n,t}^{(1)} \right)(x) = \lim_{\zeta \to 0^+} \mathcal{L}_\zeta \left( \Psi_{n,t}^{(1)} \right)(x) = \int_0^t \mathcal{L} U_{t-s} \left( \mathcal{N} \left( \Psi_{n-1,s}^{(1)} \right) \right)(x) \, ds.
\]
This and (97) gives the assertion of the lemma.

\[\square\]

**Lemma 4.15.** For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$, $k \in \mathbb{N}$ and $f \in C_0(\mathbb{R}^d)$ we have $\Psi_{k,t}^{(1)}(x) \in C^2(\mathbb{R}^d)$ (as a function of $x$) and
\[
\frac{\partial}{\partial x_i} \left( \sum_{n=0}^\infty \Psi_{n,t}^{(1)} \right)(x) = \sum_{n=0}^\infty \frac{\partial}{\partial x_i} \left( \Psi_{n,t}^{(1)} \right)(x),
\]
\[
\frac{\partial^2}{\partial x_i \partial x_j} \left( \sum_{n=0}^\infty \Psi_{n,t}^{(1)} \right)(x) = \sum_{n=0}^\infty \frac{\partial^2}{\partial x_i \partial x_j} \left( \Psi_{n,t}^{(1)} \right)(x),
\]
\[
\mathcal{L} \left( \sum_{n=0}^\infty \Psi_{n,t}^{(1)} \right)(x) = \sum_{n=0}^\infty \mathcal{L} \left( \Psi_{n,t}^{(1)} \right)(x),
\]
\[
\mathcal{N} \left( \sum_{n=0}^\infty \Psi_{n,t}^{(1)} \right)(x) = \sum_{n=0}^\infty \mathcal{N} \left( \Psi_{n,t}^{(1)} \right)(x).
\]

**Proof.** Fix $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$. By Lemma 4.14 we know that $t \to \frac{\partial}{\partial t} (\Psi_{n,t}^{(1)})(x)$ is continuous on $(0, \tau]$ for each fixed $n \in \mathbb{N}$, $x \in \mathbb{R}^d$. Using this, Lemma 4.11 and (64) we get (98).

Let $t \in (0, \tau]$, $n \in \mathbb{N}$, $i, j \in \{1, \ldots, d\}$. The fact that $\frac{\partial}{\partial x_i} (\Psi_{n,t}^{(1)})(x)$ is well defined and continuous as a function of $x \in \mathbb{R}^d$ follows from (96), Lemmas 3.3, 3.5 Proposition 3.12 and Lemma 3.19. By the above arguments we also get
\[
\left| \frac{\partial}{\partial x_i} (\Psi_{n,t}^{(1)})(x) \right| \leq c(\xi) \sup_{s \in (0,t]} \left\| \Psi_{n-1,s}^{(1)} \right\|_\infty, \quad n \geq 1.
\]
Using this, Lemma 3.19 and Lemma 4.1 we arrive at (99). By similar arguments we obtain that $\frac{\partial^2}{\partial x_i \partial x_j} (\Psi_{n,t}^{(1)})(x)$ is well defined and continuous as a function of $x \in \mathbb{R}^d$ and
\[
\left| \frac{\partial^2}{\partial x_i \partial x_j} (\Psi_{n,t}^{(1)})(x) \right| \leq c(\xi) \sup_{s \in (0,t]} \left\| \Psi_{n-1,s}^{(1)} \right\|_\infty, \quad n \geq 1.
\]
Using this and Lemma 4.1 we get (100).

For any $\zeta > 0$ we have
\[
\mathcal{L}_\zeta \left( \sum_{n=0}^\infty \Psi_{n,t}^{(1)} \right)(x) = \sum_{n=0}^\infty \mathcal{L}_\zeta \left( \Psi_{n,t}^{(1)} \right)(x)
\]
This, Lemma 4.1, (69) and (70) implies (101). (102) is easy.  

\[\square\]
Corollary 4.16. For every $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$ the function $T_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of $t$ and to $C_0^2(\mathbb{R}^d)$ as a function of $x$.

Proof. Fix $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$. The fact that $T_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of $t$ follows from (99), Lemma 4.14, Lemma 4.15 and Lemma 4.10. From the proof of Lemma 4.15 we know that for each $t > 0$, $n \in \mathbb{N}$, $i, j \in \{1, \ldots, d\}$ the function $\frac{\partial^2}{\partial x_i \partial x_j} \left( T_t^{(\xi)} f \right)(x)$ is continuous as a function of $x \in \mathbb{R}^d$. The fact that $T_t^{(\xi)} f(x)$ belongs to $C_0^2(\mathbb{R}^d)$ as a function of $x$ follows from (99), (103), Lemma 4.14 and Lemma 4.10.

Heuristically, now our aim is to show that if $\xi$ is small then $\frac{\partial^2}{\partial t} (T_t^{(\xi)} f)(x) - \mathcal{K}(T_t^{(\xi)} f)(x)$ is small. For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ let us denote

$$
\Upsilon_t^{(\xi)} f(x) = \frac{\partial}{\partial t} \left( T_t^{(\xi)} f \right)(x) - \mathcal{K} \left( T_t^{(\xi)} f \right)(x),
$$

$$
\Upsilon_t^{(\xi,1)} f(x) = e^{-\lambda \int_0^t N \left( \psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \psi_{n-1,t}^{(\xi)} \right)(x)},
$$

$$
\Upsilon_t^{(\xi,2)} f(x) = e^{-\lambda \int_0^t \Lambda^{(\xi)}(s) \left( N \left( \psi_{n-1,s}^{(\xi)} \right) \right) (x) ds}.
$$

By Lemma 4.11, 75, 76 and the boundedness of $N : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ the above series are convergent.

Lemma 4.17. For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ we have

$$
\Upsilon_t^{(\xi)} f(x) = e^{-\lambda \int_0^t N \left( \psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \psi_{n-1,t}^{(\xi)} \right)(x)},
$$

$$
\Upsilon_t^{(\xi,1)} f(x) = \Upsilon_t^{(\xi,1)} f(x) + \Upsilon_t^{(\xi,2)} f(x).
$$

Proof. By Lemmas 3.21, 4.14 and 4.15 we get

$$
\frac{\partial}{\partial t} \left( T_t^{(\xi)} f \right)(x) = -\lambda e^{-\lambda \int_0^t N \left( \psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \psi_{n-1,t}^{(\xi)} \right)(x)} + e^{-\lambda \int_0^t \Lambda^{(\xi)}(s) \left( N \left( \psi_{n-1,s}^{(\xi)} \right) \right) (x) ds}.
$$

Again by Lemma 4.15 this is equal to

$$
\mathcal{L} \left( e^{-\lambda \int_0^t N \left( \psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \psi_{n-1,t}^{(\xi)} \right)(x)} \right) - \lambda \left( e^{-\lambda \int_0^t N \left( \psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \psi_{n-1,t}^{(\xi)} \right)(x)} \right).
$$

Using the definition of $T_t^{(\xi)} f$ and noting that $N g(x) - \lambda g(x) = \mathcal{R} g(x)$ and $\mathcal{L} g(x) + \mathcal{R} g(x) = \mathcal{K} g(x)$ we obtain the assertion of the lemma.

Lemma 4.18. (i) For any $f \in C_0(\mathbb{R}^d)$ we have

$$
\Upsilon_t^{(\xi)} f(x) \to 0, \quad \xi \to 0^+,
$$

uniformly in $(t, x) \in [\tau_1, \tau] \times \mathbb{R}^d$ for every $\tau_1 \in (0, \tau)$.
(ii) For any \( f \in C_0(\mathbb{R}^d) \) we have
\[
\int_0^t T_s^{(\xi)} f(x) \, ds \to 0, \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in \((t, x) \in (0, \tau) \times \mathbb{R}^d\).

**Proof.** The lemma follows from Lemma 3.22, Proposition 3.10, Lemma 4.10, Lemma 4.11, (75), (76) and the boundedness of \( N : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \). \( \square \)

The next result (positive maximum principle) is based on the ideas from [19, Section 4.2]. Its proof is very similar to the proof of [19, Lemma 4.3] and it is omitted.

**Lemma 4.19.** Let us consider the function \( v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) and the family of functions \( v^{(\xi)} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}, \xi \in (0, 1] \). Assume that for each \( \xi \in (0, 1] \)
\[
\sup_{t \in [0, \tau], \mathbf{x} \in \mathbb{R}^d} \left| v^{(\xi)}(t, \mathbf{x}) \right| < \infty, \quad v^{(\xi)} \text{ is } C^1 \text{ in the first variable and } C^2 \text{ in the second variable.}
\]
We also assume that (for any \( \tau > 0 \))
\[
(i) \quad v^{(\xi)}(t, \mathbf{x}) \to v(t, \mathbf{x}) \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in \( t \in [0, \tau], \mathbf{x} \in \mathbb{R}^d \);
\[
(ii) \quad v^{(\xi)}(t, \mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty,
\]
uniformly in \( t \in [0, \tau], \xi \in (0, 1] \);
\[
(iii) \quad \text{for any } 0 < \tau_1 < \tau_2 \leq \tau, \quad \frac{\partial}{\partial t} v^{(\xi)}(t, \mathbf{x}) - \mathbf{K} v^{(\xi)}(t, \mathbf{x}) \to 0 \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in \( t \in [\tau_1, \tau_2], \mathbf{x} \in \mathbb{R}^d \);
\[
(iv) \quad v^{(\xi)}(t, \mathbf{x}) \to v(0, \mathbf{x}) \quad \text{as} \quad (\xi \to 0^+ \text{ and } t \to 0^+)\),
uniformly in \( \mathbf{x} \in \mathbb{R}^d \);
\[
(v) \quad \text{for any } \mathbf{x} \in \mathbb{R}^d, v(0, \mathbf{x}) \geq 0.
\]
Then for any \( t \geq 0, \mathbf{x} \in \mathbb{R}^d \) we have \( v(t, \mathbf{x}) \geq 0 \).

**Proposition 4.20.** \( T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d) \) is a linear, bounded operator for any \( t \in (0, \tau] \). For each \( t \in (0, \tau] \), \( f \in \mathcal{B}_b(\mathbb{R}^d) \) and \( R \geq 1 \) there exists a sequence \( f_k \in C_0(\mathbb{R}^d), k \in \mathbb{N} \) such that \( \lim_{k \to \infty} f_k(x) = f(x) \) for almost all \( x \in B(0, R) \); for any \( k \in \mathbb{N} \) we have \( \|f_k\|_\infty \leq \|f\|_\infty \) and for any \( x \in B(0, R) \) we have \( \lim_{k \to \infty} T_t f_k(x) = T_t f(x) \).

**Proof.** Fix \( t \in (0, \tau] \). The fact that \( T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d) \) is a linear, bounded operator follows by the definition of \( T_t \) and Lemma 4.11.

Fix \( f \in \mathcal{B}_b(\mathbb{R}^d), R \geq 1 \) and \( k \in \mathbb{N}, k \geq 1 \). By Lemma 4.10 there exists \( R_k \geq R \) such that for any \( x \in B(0, R) \) we have
\[
|T_t f 1_{B(0, R_k)}(x)| \leq \frac{1}{k}.
\]
(105)

Put \( g_{1,k}(x) = 1_{B(0, R_k)}(x) f(x), g_{2,k}(x) = 1_{B^c(0, R_k)}(x) f(x) \). By standard methods there exists \( f_k \in C_0(\mathbb{R}^d) \) such that
\[
\|f_k - g_{1,k}\|_1 \leq \frac{1}{k},
\]
and supp$(f_k) \subset B(0, R_k + 1)$, $\|f_k\|_\infty \leq \|f\|_\infty$. By Theorem 4.20 for any $x \in \mathbb{R}^d$, we have
\[
|T_t(f_k - g_{1,k})(x)| \leq \frac{c\|f\|_1^{1-\alpha/(2d)}}{k^\alpha/(2d)1/2}.
\]
This and (105) imply that for any $x \in B(0, R)$ we have $\lim_{k \to \infty} T_t f_k(x) = T_t f(x)$. We also have $\|f_k 1_{B(0,R)} - f 1_{B(0,R)}\|_1 \leq 1/k$. Hence, there exists a subsequence $k_m$ such that $\lim_{m \to \infty} T_{k_m}(x) = f(x)$ for almost all $x \in B(0, R)$.

Proposition 4.21. For any $t \in (0, \infty)$, $x \in \mathbb{R}^d$ and $f \in C_0^2(\mathbb{R}^d)$ we have
\[
T_t f(x) = f(x) + \int_0^t T_s(\mathcal{K}f)(x) \, ds. \tag{106}
\]

Proof. Step 1. $f \in C_0^2(\mathbb{R}^d)$.

For any $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put
\[
v(t, x) = T_t f(x) - f(x) - \int_0^t T_s(\mathcal{K}f)(x) \, ds,
\]
\[
v^{(\xi)}(t, x) = T_t^{(\xi)} f(x) - f(x) - \int_0^t T_s^{(\xi)}(\mathcal{K}f)(x) \, ds.
\]

Note that $\mathcal{K}f \in C_0(\mathbb{R}^d)$. By Lemmas 4.12, 4.18 and Corollary 4.16 we obtain that $v(t, x)$, $v^{(\xi)}(t, x)$ satisfy the assumptions of Lemma 4.19. Note that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$. The assertion of the proposition for $f \in C_0^2(\mathbb{R}^d)$ follows from Lemma 4.19.

Step 2. $f \in C_0^2(\mathbb{R}^d)$.

By standard methods there exists a sequence $f_n \in C_0^2(\mathbb{R}^d)$, $n = 1, 2, \ldots$ such that
\[
\sup_n \max_{i,j \in \{1, \ldots, d\}} \sup_{x \in \mathbb{R}^d} \left( |f_n(x)| + \left| \frac{\partial f_n}{\partial x_i}(x) \right| + \left| \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) \right| \right) < \infty.
\]
and for any $r \geq 1$ we have $\lim_{n \to \infty} (\sup_{|x| \leq r} |f_n(x) - f(x)|) = 0$. It follows that $\sup_{n \geq 1, x \in \mathbb{R}^d} |\mathcal{K}f_n(x)| < \infty$ and for each $x \in \mathbb{R}^d$ we have $\mathcal{K}f_n(x) \to \mathcal{K}f(x)$. By Corollary 4.11 it follows that for each $x \in \mathbb{R}^d$, $t > 0$ and $s \in (0, t]$ we have $T_{t,s} f_n(x) \to T_t f(x)$ and $T_s(\mathcal{K}f_n)(x) \to T_s(\mathcal{K}f)(x)$. By Step 1 and the dominated convergence theorem we obtain the assertion of the proposition.

The following result shows that $\{T_t\}$ is a Feller semigroup.

Theorem 4.22. We have

(i) $T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ for any $t \in (0, \infty)$,

(ii) $T_t f(x) \geq 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$,

(iii) $T_1 1_{\mathbb{R}^d}(x) = 1$ for any $t > 0$, $x \in \mathbb{R}^d$,

(iv) $T_{t+s} f(x) = T_t(T_s f)(x)$ for any $s, t > 0$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$,

(v) $\lim_{t \to 0^+} \|T_t f - f\|_\infty = 0$ for any $f \in C_0(\mathbb{R}^d)$.

(vi) there exists a nonnegative function $p(t, x, y)$ in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$; for each fixed $t > 0$, $x \in \mathbb{R}^d$ the function $y \to p(t, x, y)$ is Lebesgue measurable, $\int_{\mathbb{R}^d} p(t, x, y) \, dy = 1$ and $T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy$ for $f \in C_0(\mathbb{R}^d)$.

Proof. (i) This follows from Theorem 4.4 and Lemma 4.12 (ii).

(ii) Let $f \in C_0(\mathbb{R}^d)$ be such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$. For $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put $v(t, x) = T_t f(x)$, $v^{(\xi)}(t, x) = T_t^{(\xi)} f(x)$. By Lemmas 4.12, 4.18 and
Corollary 4.16 we obtain that $v(t, x)$, $v^{(i)}(t, x)$ satisfy the assumptions of Lemma 4.19. The assertion of Theorem 4.22 (ii) follows from Lemma 4.19.

(iii) The proof is very similar to the proof of [19, Lemma 4.5 b]. Let $f \in C_0^2(\mathbb{R}^2)$ be such that $f \equiv 1$ on $B(0, 1) \subset \mathbb{R}^d$ and let $f_n(x) = f(x/n)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$. For any $x \in \mathbb{R}^d$ we have $\lim_{n \to \infty} f_n(x) = 1$, $\lim_{n \to \infty} Kf_n(x) = 0$ and $\sup_{n \in \mathbb{N}, n \geq 1}(\|f_n\|_\infty \vee \|Kf_n\|_\infty) < \infty$. By Corollary 4.11 for any $s, t > 0$ and $x \in \mathbb{R}^d$, we get

$$\lim_{n \to \infty} T_{tf_n}(x) = T_{1R^d}(x), \quad \lim_{n \to \infty} T_s(Kf_n)(x) = 0. \quad (107)$$

Using (106) for $f_n$ and (107) we obtain (iii).

(iv) Let $f \in C_0(\mathbb{R}^d)$. For $s, t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put $v(t, x) = T_{t+s}f(x) - T_s(T_tf)(x)$, $v^{(i)}(t, x) = T^{(i)}_{t+s}f(x) - T^{(i)}_s(T_tf)(x)$. By Lemmas 4.12, 4.18 and Corollary 4.16 we obtain that $v(t, x)$, $v^{(i)}(t, x)$ satisfy the assumptions of Lemma 4.19. Note that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$. The assertion of Theorem 4.22 (iv) follows from Lemma 4.19.

(v) Choose $\varepsilon_1 > 0$. Since $f \in C_0(\mathbb{R}^d)$ there exists $\delta_1 > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad |x - y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon_1.$$ 

Fix arbitrary $x \in \mathbb{R}^d$, $t \in (0, \tau]$. Put $f_1(y) = 1_{B(x, \delta_1)}(y)(f(y) - f(x))$, $f_2(y) = 1_{B^c(x, \delta_1)}(y)(f(y) - f(x))$, $y \in \mathbb{R}^d$. By (iii) we have

$$T_tf(x) - f(x) = T_tf_1(x) + T_tf_2(x).$$

We also have

$$|T_tf_1(x)| < c\varepsilon_1,$$

$$|T_tf_2(x)| \leq 2\|f\|_\infty T_1 1_{B^c(x, \delta_1)}(x)$$

and

$$T_1 1_{B^c(x, \delta_1)}(x) = e^{-\lambda} \int_{B^c(x, \delta_1)} p_y(t, x - y) \, dy + e^{-\lambda} \Phi_1 1_{B^c(x, \delta_1)}(x)$$

$$+ e^{-\lambda} \sum_{n=1}^{\infty} \Psi_{n, t} 1_{B^c(x, \delta_1)}(x).$$

By Proposition 3.10 there exists $\tau_1 \in (0, \tau]$ such that

$$\forall t \in (0, \tau_1] \quad \int_{B^c(x, \delta_1)} p_y(t, x - y) \, dy < \varepsilon_1.$$

By Proposition 3.12 we obtain that

$$\forall t \in (0, \tau_1] \quad |\Phi_1 1_{B^c(x, \delta_1)}(x)| \leq c\tau_1^{1/2}.$$ 

By Lemma 4.1 we obtain that

$$\forall t \in (0, \tau_1] \quad \left|e^{-\lambda} \sum_{n=1}^{\infty} \Psi_{n, t} 1_{B^c(x, \delta_1)}(x)\right| \leq ct.$$ 

This implies (v).

(vi) This follows from (i), (ii), (iii) and Theorem 4.3. \hfill $\square$

We are now in a position to provide the proof of Theorems 1.1 and 1.2.
proof of Theorem 1.1 From Theorem 4.22 we conclude that there is a Feller process $\tilde{X}_t$ with the semigroup $T_t$ on $C_0(\mathbb{R}^d)$. Let $\tilde{P}^{x}$, $\tilde{E}^{x}$ be the distribution and expectation of the process $\tilde{X}_t$ starting from $x \in \mathbb{R}^d$.

By Theorem 4.22 (vi), Proposition 4.20 and Lemma 4.10 we get

$$\tilde{E}^{x} f(\tilde{X}_t) = T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy \quad f \in B_b(\mathbb{R}^d), \ t > 0, \ x \in \mathbb{R}^d. \quad (108)$$

By Proposition 4.21 for any function $f \in C^2_b(\mathbb{R}^d)$, the process

$$M_t^{\tilde{X}, f} = f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t K f(\tilde{X}_s) \, ds$$

is a $(\tilde{P}^{x}, F_t)$ martingale, where $F_t$ is a natural filtration. That is $\tilde{P}^{x}$ solves the martingale problem for $(X, C^2_b(\mathbb{R}^d))$. On the other hand, according to [1] Theorem 6.3, the unique solution $\tilde{X}$ to the stochastic equation [1] has the law which is the unique solution to the martingale problem for $(X, C^2_b(\mathbb{R}^d))$. Hence $\tilde{X}$ and $X$ have the same law so for any $t > 0$, $x \in \mathbb{R}^d$ and any Borel bounded set $A \subset \mathbb{R}^d$ we have

$$\sigma_t(x, A) = \int_A p(t, x, y) \, dy,$$

where $\sigma_t(x, A)$ is defined by (9). Using this, (10) and (108) we obtain

$$P_t f(x) = T_t f(x), \quad t > 0, \ x \in \mathbb{R}^d, f \in B_b(\mathbb{R}^d). \quad (109)$$

Now the assertion of Theorem 1.1 follows from Theorem 4.3 and (109).

proof of Theorem 1.2 The result follows from Theorem 4.3 and (109).

Remark 4.23. For any $\alpha \in (0, 1)$, $d \geq 2$ there exist $A(x)$ satisfying (2-4) and $t > 0$ such that $P_t : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is not bounded. For simplicity we will present an example for $d = 2$ but similar examples can be constructed for $d > 2$.

Proof. First we define $A(x_1, x_2)$. Let $\kappa(r) : [0, \infty) \to [0, \infty)$ be defined by $\kappa(r) = 0$ for $r \in [0, 1], \ k(r) = r - 1$ for $r \in (1, 1 + \pi/4]$, $\kappa(r) = \pi/4$ for $r > 1 + \pi/4$. It is easy to check that $\kappa(r) = ((r - 1) \vee 0) \wedge (\pi/4)$ and that it is a Lipschitz function. Now let us introduce standard polar coordinates $(r, \varphi)$, $r \in [0, \infty), \varphi \in [0, 2\pi)$ by $x_1 = r \cos \varphi, x_2 = r \sin \varphi$.

We put $A(x_1, x_2) = \hat{A}(r, \varphi) = \begin{bmatrix} \cos(\hat{\theta}(r, \varphi)) & -\sin(\hat{\theta}(r, \varphi)) \\ \sin(\hat{\theta}(r, \varphi)) & \cos(\hat{\theta}(r, \varphi)) \end{bmatrix}$, where $\hat{\theta}(r, \varphi)$ is defined in the following way. $\hat{\theta}(r, \varphi) = 0$ for $r \in [0, 1], \varphi \in [0, 2\pi), \hat{\theta}(r, \varphi) = \varphi$ for $r \in (1, 1 + \pi/4], \varphi \in [0, \kappa(r)], \hat{\theta}(r, \varphi) = 2\kappa(r) - \varphi$ for $r \in (1, 1 + \pi/4], \varphi \in (\kappa(r), 2\kappa(r))$, $\hat{\theta}(r, \varphi) = 0$ for $r \in (1, 1 + \pi/4], \varphi \in [2\kappa(r), 2\pi), \hat{\theta}(r, \varphi) = \varphi$ for $r > 1 + \pi/4, \varphi \in [0, \pi/4], \hat{\theta}(r, \varphi) = \pi/2 - \varphi$ for $r > 1 + \pi/4, \varphi \in (\pi/4, \pi/2], \hat{\theta}(r, \varphi) = 0$ for $r > 1 + \pi/4, \varphi \in (\pi/2, 2\pi)$.

One can check that $A(x_1, x_2) = \begin{bmatrix} \cos(\theta(x_1, x_2)) & -\sin(\theta(x_1, x_2)) \\ \sin(\theta(x_1, x_2)) & \cos(\theta(x_1, x_2)) \end{bmatrix}$, where $\theta(0, 0) = 0$ and for $(x_1, x_2) \neq (0, 0)$

$$\theta(x_1, x_2) = \kappa \left( \sqrt{x_1^2 + x_2^2} \right) - \left( \left( \text{Arg}(x_1 + ix_2) \vee 0 \right) \wedge \frac{\pi}{2} \right) - \kappa \left( \sqrt{x_1^2 + x_2^2} \right) \vee 0.$$

It is clear that $A(x)$ satisfies (2-4).
Put $D = B((3, 1), 1)$. Note that for any $x \in \mathbb{R}^2$ such that $x_2 \in [0, x_1]$ and $|x| \geq \pi/4 + 1$ we have $A(x) = A(x_1, x_2) = |x|^{-1} \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}$. In particular, this holds for $x \in D$.

For any $f \in \mathcal{B}_b(\mathbb{R}^d), t > 0, x \in \mathbb{R}^d$ we have $P_t f(x) = T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x)$. For our purposes it is enough to study $\Psi_{1,t}$. We have

$$
\Psi_{1,t} f(x) = \int_0^t U_{t-s} (N(U_s f))(x) \, ds = 
\sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^2} u(t-s, x, z) \int_{\mathbb{R}} \int_{\mathbb{R}^2} u(s, z + a_i(z) w, y) f(y) dy \nu(w) \, dw \, dz \, ds. \quad (110)
$$

By arguments similar to the proof of Theorem 4.22 one can show that for any $t > 0, x \in \mathbb{R}^d$ and almost all $y \in \mathbb{R}^d$ we have $u(t, x, y) \geq 0$ and for any $s, t > 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d)$ we have $U_{t+s} f(x) = U_t (U_s f)(x)$, (we omit the details here). Put

$$
u = \sum_{i=1}^1 u(t, x, y) = \sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^2} u(t-s, x, z) \int_{\mathbb{R}} \int_{\mathbb{R}^2} u(s, z + a_i(z) w, y) \nu(w) \, dw \, dz \, ds. \quad (111)
$$

By $(110)$ we have

$$\Psi_{1,t} f(x) = \int_{\mathbb{R}^2} u_1(t, x, y) f(y) \, dy.
$$

By Lemma 3.23 and the semigroup property of $U_t$ one can easily obtain that there exists $t_2 > 1$ such that for any $t \in [t_2 - 1, t_2]$

$$\int_D u(t, 0, z) \, dz \geq c. \quad (112)\n$$

Now our aim will be to estimate from below $u_1(t_2, 0, y)$ for $y$ which are sufficiently close to $0$. Let $c_1, \varepsilon_1, t_1$ be the constants from Lemma 8.23. First, note that for $z \in D$ we have $a_1(z) = (a_{11}(z), a_{21}(z)) = |z|^{-1} (z_1, z_2) = z/|z|$. Hence $|z + a_1(z) w| = |z| + w$ for any $z \in D, w \in \mathbb{R}$. It follows that for $z \in D$ we have

$$w \in (-|z| - \varepsilon_1 s^{1/\alpha}/2, -|z| + \varepsilon_1 s^{1/\alpha}/2) \iff |z + a_1(z) w| < \varepsilon_1 s^{1/\alpha}/2.
$$

Therefore, for $z \in D, |y| \leq \varepsilon_1 s^{1/\alpha}/2, w \in (-|z| - \varepsilon_1 s^{1/\alpha}/2, -|z| + \varepsilon_1 s^{1/\alpha}/2), w \in (-|z| - \varepsilon_1 s^{1/\alpha}/2, -|z| + \varepsilon_1 s^{1/\alpha}/2)$, we have $|z + a_1(z) w - y| \leq |z + a_1(z) w| + |y| \leq \varepsilon_1 s^{1/\alpha}$. Note also that

$$|y| \leq \varepsilon_1 s^{1/\alpha}/2 \iff (2|y|/\varepsilon_1)^\alpha \leq s.
$$

Hence, by Lemma 8.23, for $(2|y|/\varepsilon_1)^\alpha \leq t_1/2, s \in [(2|y|/\varepsilon_1)^\alpha, t_1], z \in D$ we have

$$\int_{-|z| + \varepsilon_1 s^{1/\alpha}/2}^{-|z| - \varepsilon_1 s^{1/\alpha}/2} u(s, z + a_1(z) w, y) \nu(w) \, dw \geq c \varepsilon_1 s^{1/\alpha} \frac{c_1 \varepsilon_1}{s^{2/\alpha}} = \frac{cc_1 \varepsilon_1}{s^{1/\alpha}}. \quad (113)
$$

Recall that $t_1 \in (0, 1]$ and $t_2 > 1 \geq t_1$. By $(112)$ for any $s \in (0, t_1]$ we have

$$\int_D u(t_2 - s, 0, z) \, dz \geq c. \quad (114)\n$$
Therefore, by (111), nonnegativity of \( u(\cdot, \cdot, \cdot) \), (113), (114) for \((2|y|/\varepsilon_1)^\alpha \leq t_1/2\) we have

\[
\begin{align*}
    u_1(t_2, 0, y) &\geq \int_{(2|y|/\varepsilon_1)^\alpha}^{t_1} \int_B u(t_2 - s, 0, z) \int_{-|z| - \varepsilon_1 s^{1/\alpha}/2}^{-|z| + \varepsilon_1 s^{1/\alpha}/2} u(s, z + a_1(z)w, y) \\
    &\times \nu(w) \, dw \, dz \, ds \\
    &\geq c \int_{(2|y|/\varepsilon_1)^\alpha}^{t_1} s^{-1/\alpha} \, ds \\
    &\geq c|y|^{\alpha-1}. \tag{115}
\end{align*}
\]

(One can show that in similar examples for \( d > 2 \) we have \( u_1(t_2, 0, y) \geq c|y|^{\alpha+1-d} \).)

Observe that \((2|y|/\varepsilon_1)^\alpha \leq t_1/2 \iff |y| \leq t_1^{1/\alpha} \varepsilon_1^{-1-\alpha/2}. \) For \( r \in \left(0, t_1^{1/\alpha} \varepsilon_1^{2-1-\alpha/2}\right) \) we get by (115)

\[
    T_{t_2}1_{{B(0,r)}}(0) \geq e^{-\lambda t_2} \Psi_{\frac{1}{1-\alpha}} 1_{{B(0,r)}}(0) = e^{-\lambda t_2} \int_{B(0,r)} u_1(t_2, 0, y) \, dy \geq cr^{\alpha+1}. \tag{116}
\]

By (109) we have \( T_l = P_l \). By Theorem 1.2 \( x \to P_l 1_{{B(0,r)}}(x) \) is continuous so \( \|P_l 1_{{B(0,r)}}\|_\infty \geq P_l 1_{{B(0,r)}}(0) \). Using this and (116) for \( r \in \left(0, t_1^{1/\alpha} \varepsilon_1^{2-1-\alpha/2}\right) \) we get

\[
    \frac{\|P_{t_2} 1_{{B(0,r)}}\|_\infty}{\|1_{{B(0,r)}}\|_1} \geq \frac{P_{t_2} 1_{{B(0,r)}}(0)}{\|1_{{B(0,r)}}\|_1} \geq cr^{\alpha+1},
\]

which implies the assertion of the remark. (One can show that in similar examples for \( d > 2 \) we have \( \|P_{t_2} 1_{{B(0,r)}}\|_\infty/\|1_{{B(0,r)}}\|_1 \geq cr^{\alpha+1-d} \).) \( \square \)

Remark 4.24. From Theorem 1.22 (vi) and (109) we infer that transition densities \( p(t, x, y) \) for \( X_t \) exist. We point out that the existence of transition densities is already well known, see (10). In the above example (in \( \mathbb{R}^2 \)) we showed that the transition density \( p(t, 0, y) \), for some \( t > 0 \), is an unbounded function. In fact, the following estimate holds almost surely

\[
    p(t, 0, y) \geq c|y|^{\alpha-1}, \quad |y| \leq \varepsilon_1,
\]

where \( c, \varepsilon_1 \) are some positive constants possibly dependent on \( t \).

Hence, we can not expect a general result saying that, with our assumptions, we have the standard estimates for \( p(t, x, y) \) of the form

\[
    p(t, x, y) \leq Ct^{-d/\alpha},
\]

as for example in the case of diagonal matrices (24), or matrices satisfying some further regularity assumptions (29). On the other hand, the assumption \( \alpha < 1 \) plays an important role (in \( \mathbb{R}^2 \)), since for \( \alpha > 1 \), by the results of (5), the transition density is bounded.

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