On tree form-factors in (supersymmetric) Yang-Mills theory

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ITEP-TH-47/98

Abstract

Perturbiner, that is, the solution of field equations which is a generating function for tree form-factors in $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory, is studied in the framework of twistor formulation of the $N = 3$ superfield equations. In the case, when all one-particle asymptotic states belong to the same type of $N = 3$ supermultiplets (without any restriction on kinematics), the solution is described very explicitly. It happens to be a natural supersymmetrization of the self-dual perturbiner in non-supersymmetric Yang-Mills theory, designed to describe the Parke-Taylor amplitudes. In the general case, we reduce the problem to a neatly formulated algebraic geometry problem (see Eqs (70), (71), (72)) and propose an iterative algorithm for solving it, however we have not been able to find a closed-form solution. Solution of this problem would, of course, produce a description of all tree form-factors in non-supersymmetric Yang-Mills theory as well. In this context, the $N = 3$ superfield formalism may be considered as a convenient way to describe a solution of the non-supersymmetric Yang-Mills theory, very much in the spirit of works by E. Witten and by J. Isenberg, P. B. Yasskin and P. S. Green.
1 Introduction

It is well known that multi-particle amplitudes, even in the tree approximation, are generically out of reach by means of available field theoretical methods, though, there is a lot of efforts and achievements in this direction (see [3]-[29] and other references to the paper [3]). In principle, the problem can be considered as a purely technical one, since the \(n\)-particle tree amplitude is, according to the text-book rules, represented as a sum of a number of rational functions of momenta of the particle (a term of the sum is a contribution of a Feynman diagram, and the amplitude is a sum of contributions of a number of diagrams). However, the number of terms grows enormously with the number of particles, so that the total expression becomes untreatable when the number of particles becomes bigger than, say, 9, to say nothing about arbitrary \(n\). On the other side, sometimes a final expression for the amplitude is essentially simpler than the intermediate ones. There are known cases when cancellations among contributions of different Feynman diagrams are just wonderful [13]-[17], [23]. These cases are, essentially, scalar field amplitudes with most of the external particles at threshold [15]-[17], [23]. If one is optimistic concerning possible cancellations in more general cases, say, in the case of Yang-Mills amplitudes with arbitrary helicities, one should look for a way to avoid those intermediate steps. A possible idea is to use the classical field equations since the tree amplitudes can, of course, be obtained from a classical solution of the field equations. This approach has been discussed in the classical text-books [30], [31], and it has recently resurrected in the literature. The threshold amplitudes in scalar field theories were obtained from spatially uniform classical solutions, thus, the field equations reduced to ordinary differential equations [13]-[17], [24]. The like-helicity Yang-Mills amplitudes were related to solutions of the self-duality equations in [24], [25].

From our subjective point of view, one of the most interesting by-products of the above developments was the idea of perturbiner, or ptb-solution, [26]-[29].

To define the perturbiner, one first fixes a solution of linearized field equations (which are assumed to describe asymptotic one-particle states) of the type of

\[
\phi^{(1)} = \sum_{J=1}^{L} a_J \epsilon_J t_J e^{kjx} = \sum_{J=1}^{L} \epsilon_J \hat{\mathcal{E}}
\]

where \(x\) stands for a space-time coordinate, \(k_J\) stands for a momentum of the \(J\)-th particle, \(\epsilon_J\) stands for a polarization of the \(J\)-th particle, \(t_J\) stands for a “polarization” of the \(J\)-th particle in the internal space (e.g. \(t_J\) is a generator of the color group), \(a_J\) is a symbol of annihilation/creation operator,

\[
\hat{\mathcal{E}}_J = t_J \mathcal{E}, \quad \mathcal{E} = a_J e^{kjx}
\]

The perturbiner is a complex solution of the field equations which is a formal power series in the “harmonics” \(\mathcal{E}_J, J = 1, \ldots, L\) Eq. (1)), the first order term of the series being just the solution Eq. (1).

Notice that \(x\)-dependence of the perturbiner comes only via monomials in \(\mathcal{E}_J, J = 1, \ldots, L\), on which differential operators entering the field equations act as algebraic operators, and existence/uniqueness \(^1\) of the perturbiner normally takes place provided the

\(^1\)In gauge theories the uniqueness is, of course, modulo gauge transformations
set of momenta, \( k_J, J = 1, \ldots, L \), is non-resonant, that is, provided none of linear combinations of \( k_J, J = 1, \ldots, L \) with positive integer coefficients including more than one momentum gets to the mass shell. The physical meaning of the perturbiner is that its expansion in powers of symbols \( a_J, J = 1, \ldots, L \) generates tree form-factors in the theory,

\[
\phi^{ptb}(x, \{k\}, \{a\}) = \sum_{l=1}^{L} \sum_{\{J\}} a_{J_1} \cdots a_{J_l} < k_{J_1}, \ldots, k_{J_l} | \phi(x) | 0 >_{\text{tree}}
\]

(3)

Notice that all the external mass-shell particles are arbitrarily considered as the outcomes. In principle, those with negative frequency should be considered as out-states while those with positive frequency should be considered as in-states but at tree level analytical continuation from negative to positive frequency is trivial, so we do not distinguish them. The form-factors \( < k_{J_1}, \ldots, k_{J_l} | \phi(x) | 0 >_{\text{tree}} \) in Eq.(3) are in the coordinate representation. The monomials in the harmonics \( E_J, J = 1, \ldots, L \) will produce the momentum conservation \( \delta \)-functions after transformation to the momentum space.

It is very convenient to add to the above definition of the perturbiner the requirement of nilpotency of the symbols \( a_J, J = 1, \ldots, L \), that is

\[
a_J^2 = 0.2
\]

(4)

In terms of the form-factors the nilpotency means that form-factors with identical asymptotic states will not appear in the expansion of the perturbiner in powers of \( a_J, J = 1, \ldots, L \) (see Eq.(3)). Clearly, it does not assume any loss of generality if the perturbiner is known for an arbitrary number \( L \) of the asymptotic one-particle states, since the form-factors with identical asymptotic states can be obtained from those with all asymptotic states different. It is clear that under the nilpotency assumption the perturbiner is, in fact, not the power series but just a polynomial in (nilpotent) harmonics \( E_J, J = 1, \ldots, L \) Eq.(2). Actually, for massless particles, the nilpotency assumption is necessary for the non-resonantness condition, because any multiple of a light-like momentum is again a light-like momentum, and the nilpotency makes these multiples irrelevant.

Anywhere below we shall assume the nilpotency condition Eq.(4).

We note that textbooks (see, e.g., [30], [31]) offer another definition of solution of field equations, generating tree amplitudes in the theory (they use the so-called Feynman asymptotic condition to define it).

Our definition above proved to be very convenient, in particular, it happened to be very conveniently compatible with the twistor description of solutions for the gauge self-duality equation ([26]) and for the gravitational self-duality equations ([27]) and for the gauge-gravitational self-duality equation ([28]). The traditional finite-action and reality conditions are substituted in the case of perturbiner by the condition of analyticity in the harmonics \( E_J, J = 1, \ldots, L \). Eqs.(4),(2). It is worth to explain that the perturbiner obeys the self-duality equations instead of the full equations, such as Yang-Mills equations or Einstein equations, when all polarizations entering Eq.(4) describe the same helicity state. Self-dual perturbiner generates only like-helicity form-factors. In this way the so-called Parke-Taylor amplitudes [13], [14] are very unusually described in terms of meromorphic functions on an auxiliary \( CP^1 \) space.

\footnote{This condition has nothing to do with the statistics of the particles considered. Say, for bosons the symbols still commute, while for fermions - anticommute.}
In this paper we describe perturbiner in $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory. $N = 3$ supersymmetric Yang-Mills is equivalent to $N = 4$ super Yang-Mills but $N = 3$ superfield formalism is more naturally combined with twistors \[1\], that is why we follow $N = 3$ notation. $N = 4$ super Yang-Mills multiplet consists of the following particles:

\[
\begin{align*}
1 \times 1 & \text{ (positive helicity gluon)} \\
4 \times 1/2 & \text{ (positive helicity gluinos)} \\
6 \times 0 & \text{ (scalars)} \\
4 \times -1/2 & \text{ (negative helicity gluinos)} \\
1 \times -1 & \text{ (negative helicity gluon)}
\end{align*}
\]

This multiplet decomposes into two $N = 3$ multiplets as follows

\[
\begin{align*}
1 \times 1 & \\
3 \times 1/2 & \\
3 \times 0 & \\
1 \times -1/2 & \\
1 \times 1/2 & \\
3 \times 0 & \\
3 \times -1/2 & \\
1 \times -1 &
\end{align*}
\]

It occurs that if one includes only states from one type of the $N = 3$ multiplets, say, the one from the table (6) (in arbitrary kinematics), that is all plane waves in Eq.(1) belong to the same type of the $N = 3$ multiplets (with arbitrary momenta $k_J$, $J = 1, \ldots, L$), the solution is obtained from the (non-supersymmetric) self-dual perturbiner \[2\] by substituting the harmonics $E$ Eq.(2) with their supersymmetric extensions $S$ Eq.(51). Such solution will be called chiral $N = 3$ perturbiner. If one includes both types of the $N = 3$ multiplets the problem becomes much more complicated.\[3\] In this case we describe a twistor reformulation of the problem, we show how it can be solved iteratively, but we have not been able to find a closed-form solution of the problem. Nevertheless, the problem is reduced to a neatly formulated algebraic geometry problem, Eqs.(70), (71),(72), and we feel that the complete solution might be somewhere nearby and we, perhaps, just do not know an appropriate mathematics to describe it.

The rest of the paper is organized as follows. In section 2 we, for the purpose of closeness of this paper, remind construction of non-supersymmetric self-dual perturbiner \[2\]. A key point is a sort of Riemann-Hilbert problem Eq.(51), which is solved upon introducing the so-called “color ordering” (see Eq.(2) and explanations about it). Interestingly, the same solution Eq.(53) of the same problem Eq.(51) was shown \[2\] to generate tree

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\[3\] This is not surprising because all tree form-factors in non-supersymmetric Yang-Mills are contained among the $N = 4$ form-factors (when all external particles are gluons, the fermions and the charged scalars can appear only in loops). Actually, all $N = 3$ machinery can be viewed on as a convenient way to describe solutions of ordinary Yang-Mills equations, very much in the spirit of \[1\] (see, also, \[4\]).

\[4\] A solution, similar to our self-dual perturbiner, has been discussed in \[32\].
form-factors in sin(h)-Gordon theory. In section 3 we remind $N = 3$ super Yang-Mill equations and construct the plane wave solution of the linearized field equations. In section 4 we describe the chiral $N = 3$ perturbiner. In section 5 the generic (non-chiral) perturbiner is considered.

2 Non-supersymmetric self-dual perturbiner [26]

We adopt the spinor notation, so that the connection-form has two indices, $A_\alpha\dot{\alpha}$, $\alpha = 1,2 \dot{\alpha} = \dot{1},\dot{2}$, so has the space-time derivative, $\partial_\alpha = \partial_{\alpha\dot{\alpha}}$, and the connection itself, $\nabla_\alpha = \partial_\alpha + A_\alpha\dot{\alpha}$. The curvature form,

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = [\nabla_\alpha, \nabla_\beta]\quad (8)$$

has four indices and, being antisymmetric with respect to transposition $\alpha\dot{\alpha} \leftrightarrow \beta\dot{\beta}$, decomposes as follows

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta}\quad (9)$$

where $\varepsilon_{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}$ are the standard antisymmetric tensors and the fields $F_{\dot{\alpha}\dot{\beta}}, F_{\alpha\beta}$ are symmetric with respect to transposition of indices. The first term in the r.h.s. of Eq.(9) is the self-dual part of the curvature, the second - antiself-dual (the metric in this notation is $g_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}$). Then, the self-duality equation is

$$F_{\alpha\beta} = 0\quad (10)$$

The (anti)symmetry equation is very well known to be sufficient for the Yang-Mills equation to be satisfied.

Construction of the perturbiner (see Introduction) starts with picking up a solution of the linearized (“free”) field equation. Linearized version of the self-duality equation reads

$$\left(\partial_\alpha A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}} A_\alpha\dot{\alpha}\right)_{\text{symmetrized in } \alpha,\beta} = 0\quad (11)$$

For a plane wave solution (see Eq.(11)),

$$A_\alpha\dot{\alpha} = \epsilon_{\alpha\dot{\alpha}} te^{k_{\beta\dot{\beta}} x^{\beta\dot{\beta}}}\quad (12)$$

Eq.(11) gives

$$\left(k_{\alpha\dot{\alpha}}\epsilon_{\beta\dot{\beta}} - k_{\beta\dot{\beta}}\epsilon_{\alpha\dot{\alpha}}\right)_{\text{symmetrized in } \alpha,\beta} = 0\quad (13)$$

which, in turn, gives that

$$k_{\alpha\dot{\alpha}} = \epsilon_{\alpha\dot{\alpha}} \tilde{\alpha}_{\dot{\alpha}}$$

$$\epsilon^{(+)\dot{\alpha}} = \frac{q_{\alpha}\tilde{\alpha}_{\dot{\alpha}}}{(\tilde{\alpha}, q)}\quad (14)$$

Eqs.(14) mean that $k_{\alpha\dot{\alpha}}$ and $\epsilon_{\alpha\dot{\alpha}}$ are both light-like (remind that the metric is $g_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}$), moreover, the dotted spinor in decomposition of $k_{\alpha\dot{\alpha}}$ and $\epsilon_{\alpha\dot{\alpha}}$ is the same. $q_{\alpha}$ is an arbitrary (reference) spinor defined up to (linearized) gauge equivalence,

$$q_{\alpha} \sim q_{\alpha} + \text{const} \cdot \tilde{\alpha}_{\alpha}\quad (15)$$
The factor \((\alpha, q)\) is introduced for normalization. The brackets \((p, q)\) here and below are defined as
\[
(p, q) = p^\alpha q_\alpha = \varepsilon_{\alpha\beta} p^\alpha q^\beta
\]  
(indices are raised and lowered with the \(\varepsilon\)-symbols). What concerns to the normalization, antiself-dual plane wave would have a polarization
\[
\epsilon^{(-)}_{\alpha\dot{\alpha}} = \frac{\alpha_\alpha q_\dot{\alpha}}{\langle \alpha, q \rangle}
\]  
and
\[
\epsilon^{(+)}_{\alpha\dot{\alpha}} \epsilon^{(-)\alpha\dot{\alpha}} = -1
\]
Polarizations Eqs. (14) and (17) can be seen to define positive and negative helicity states, correspondingly. Notice that the spinors \(\alpha_\alpha, \tilde{\alpha}_\dot{\alpha}\) entering \(k_{\alpha\dot{\alpha}}\) Eq.(14) can be considered as independent since we are anyway looking for a complex solution. At the end, in computation of probabilities, the reality condition
\[
\tilde{\alpha}_\dot{\alpha} = \sqrt{-1} \alpha^*_\alpha
\]
should be imposed.

So, the appropriate solution of the linearized field equations reads
\[
A^{(1)}_{\alpha\dot{\alpha}} = \sum_{J=1}^{L} \frac{q^J_{\dot{\alpha}} \tilde{\alpha}^J_{\alpha}}{\langle \alpha^J, q^J \rangle} \hat{E}^J
\]
where, as in Eq.(2),
\[
\hat{E}^J = t^J E, \ E = a^J e^{k^J x}
\]
where \(x^{\alpha\dot{\alpha}}\) stands for a space-time coordinate, \(k^J_{\alpha\dot{\alpha}}\) stands for a momentum of the \(J\)-th particle, \(t^J\) stands for a generator of the color group, \(a^J\) is a symbol of annihilation/creation operator of the \(J\)-th particle (obeying the nilpotency condition Eq.(4)).

We are now going to use the twistor construction [33] to describe solutions of the (nonlinear) self-duality equation (10).

Introduce a couple of complex numbers, \(\rho^\alpha, \alpha = 1, 2\). \(\rho^\alpha\) will also be referred below as the auxiliary spinor. \(\rho^\alpha, \alpha = 1, 2\) can be considered as homogeneous coordinates on a \(CP^1\) space. Contracting undotted indices of the curvature form \(F_{\alpha\dot{\alpha}\beta\dot{\beta}}\) Eq.(8) with \(\rho^\alpha\)'s one automatically picks up antiself-dual part of it (see Eq.(9)) (because the self-dual part is antisymmetric in the undotted indices). Hence, the self-duality equation is equivalent to a sort of zero-curvature condition
\[
[\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}] = 0 \text{ at any } \rho^\alpha, \alpha = 1, 2
\]
where \(\nabla_{\dot{\alpha}} = \rho^\alpha \nabla_{\alpha\dot{\alpha}}\). Thus, if one introduces
\[
A_{\dot{\alpha}} = \rho^\alpha A_{\alpha\dot{\alpha}},
\]
any self-dual connection form can be (locally) represented as
\[
A_{\dot{\alpha}} = g^{-1} \partial_{\dot{\alpha}} g
\]
where \(g\) is a group valued function of \(\rho\) and \(x\) and \(\partial_{\dot{\alpha}} = \rho^\alpha \partial_{\alpha\dot{\alpha}}\). All the non-triviality of the self-duality equation is now encoded in the condition that \(g\) must depend on \(\rho\) in such
a way that $A_\alpha$ is a polynomial of degree 1 in $\rho$, as in Eq.\((23)\). If $g$ is $\rho$-independent, it is a pure gauge transformation, as it is seen from Eq.\((24)\).

The above condition on $\rho$-dependence of $g$ is equivalent to condition that $g$ is a homogeneous meromorphic function of $\rho$ of degree 0 such that $A_\alpha$ from Eq.\((24)\) is a homogeneous holomorphic function of $\rho$ of degree 1 (a homogeneous holomorphic function of $\rho$ of degree 1 is necessary just linear in $\rho$, as in Eq.\((23)\)). Notice, that nontrivial (not a pure gauge) $g$ necessary has singularities in $\rho$, since if it is regular homogeneous of degree 0, then it is just $\rho$-independent, that is, a pure gauge.

All above is about $\rho$-dependence of $g$. In the case of perturbiner, the group-valued function $g$, like the connection-form $A_{a\hat{a}}$ (see definition of the perturbiner in the Introduction), must be polynomial in harmonics $\mathcal{E}_J, J = 1, \ldots, L$, Eq.\((21)\). First order terms of this polynomial,

$$g^{(1)}_{ptb}(\rho, \{\mathcal{E}\}) = \sum_{J=1}^L g_J(\rho) \hat{\mathcal{E}}_J$$  \hspace{1cm} (25)

are fixed by those in $A_{a\hat{a}}$, Eq.\((24)\). Expanding Eq.\((24)\) up to first order in the harmonic $\mathcal{E}_J$ and using Eq.\((21)\) one obtains

$$g_J(\rho) = \frac{(\rho, \hat{\mathcal{E}}_J)}{(\rho, \hat{\epsilon}^J)}$$  \hspace{1cm} (26)

Eqs.\((25),(26)\) define the first order terms in expansion of $g$ in powers of the harmonics $\mathcal{E}_J, J = 1, \ldots, L$.

Thus, in terms of $g$, our problem is as follows. We must find polynomial $g_{ptb}$,

$$g_{ptb}(\rho, \{\mathcal{E}\}) = 1 + g^{(1)}_{ptb}(\rho, \{\mathcal{E}\}) + \text{higher order terms in powers of } \mathcal{E}'s$$  \hspace{1cm} (27)

which is a rational function of $\rho^\alpha$ of degree 0, such that $A_\alpha$ from Eq.\((24)\) is regular.

Notice that $g^{(1)}_{ptb}$ has first order poles at $\rho^\alpha = \epsilon^\alpha_j, J = 1, \ldots, L$ (due to factors $(\rho, \hat{\epsilon}^J) = \epsilon_{\alpha\beta} \rho^\alpha \epsilon^\beta$ in denominators, see Eq.\((26)\)). We remind that $\epsilon^\alpha_j$ is a spinors which appears in decomposition of the corresponding four momentum $k_{a\hat{a}}$, Eq.\((14)\). Importantly, a singular part of complete $g_{ptb}$ (not only of $g^{(1)}_{ptb}$) at $\rho^\alpha = \epsilon^\alpha_j$ is necessary proportional to the harmonic $\mathcal{E}_J$ (that is because $g_{ptb}$ at $\mathcal{E}_J = 0$ does not contain any information about the $J$-th particle; form-factors including the $J$-th particle will not be generated by the perturbiner at $\mathcal{E}_J = 0$). Then, taking into account the nilpotency Eq.\((4)\), $\mathcal{E}^2_J = 0$, one can show that $g_{ptb}$ may have only simple pole at $\rho^\alpha = \epsilon^\alpha_j$ for $A_\alpha$ from Eq.\((24)\) to be regular at this point. Moreover, residue of $g_{ptb}$ at this point must obey the condition

$$\left(\partial_\alpha (\text{res}|_{\rho=\epsilon^\alpha_j} g \cdot g^{-1})\right)|_{\rho=\epsilon^\alpha_j} = 0,$$  \hspace{1cm} (28)

and, according to the rules of the game (see definition of the perturbiner in the Introduction), Eq.\((28)\) must be solved in the form of a polynomial in the harmonics $\mathcal{E}_J, J = 1, \ldots, L$. Clearly, the unique (up to a color Lee algebra valued constant) solution of Eq.\((28)\) reads

$$\text{res}|_{\rho=\epsilon^\alpha_j} g \cdot g^{-1}|_{\rho=\epsilon^\alpha_j} = \text{const}_J \cdot \mathcal{E}_J$$  \hspace{1cm} (29)

(recall, that $\partial_\alpha |_{\rho=\epsilon^\alpha_j} = \epsilon^\alpha_j \partial_{a\hat{a}}$). $\text{const}$ entering Eq.\((29)\) is constant in the sense that it is $\mathcal{E}$-independent. It can be found by putting all the harmonics $\mathcal{E}$ but $\mathcal{E}_J$ to 0. Then, using Eqs.\((25),(26)\), one finds

$$\text{res}|_{\rho=\epsilon^\alpha_j} g_J t_J = \text{const}_J$$  \hspace{1cm} (30)
with “one-particle” $g_J$ from Eq. (29).

Eqs. (29), (30) are seen to be equivalent to the condition that

$$(1 - g_J \tilde{\mathcal{E}}_J)g_{ptb} \text{ is regular at } \rho^\alpha = \alpha_J^\alpha, J = 1, \ldots, L$$

(with $g_J$ from Eq. (24)). The condition, expressed by Eq. (24), defines $g_{ptb}$ uniquely modulo gauge transformations (that is, modulo multiplication by a $\rho$-independent matrix from the right).

To conveniently represent the solution of Eq. (31) let us assume for a moment that the generators $t_J, J = 1, \ldots, L$ defining color states of the gluons (see Eq. (20)) belong to a free associative algebra (that is, there is no relation between them but the associativity relation $(t_J, t_J) t_{J_3} = t_J (t_J, t_{J_3})$). This means that all monomials of the type of $\tilde{\mathcal{E}}_J \tilde{\mathcal{E}}_{J_2} \ldots \tilde{\mathcal{E}}_{J_L}$ ($\tilde{\mathcal{E}}_J$ as in Eq. (21)) are linearly independent. $g_{ptb}$ is then uniquely represented as

$$g_{ptb}(\rho, \{ \mathcal{E} \}) = 1 + \sum_J g_J(\rho) \tilde{\mathcal{E}}_J + \sum_{J_1, J_2} g_{J_1, J_2}(\rho) \tilde{\mathcal{E}}_{J_1} \tilde{\mathcal{E}}_{J_2} + \ldots$$  \hspace{1cm} (32)

Eq. (31) is then easily solved for the coefficients $g_{J_1, J_2, \ldots, J_L}$,

$$g_{J_1, J_2, \ldots, J_L}(\rho) = \frac{(\rho, q^{J_1}) (\alpha^{J_1}, q^{J_2})(\alpha^{J_2}, q^{J_3}) \ldots (\alpha^{J_{L-1}}, q^{J_L})}{(\rho, \alpha^{J_1}) (\alpha^{J_1}, \alpha^{J_2})(\alpha^{J_2}, \alpha^{J_3}) \ldots (\alpha^{J_{L-1}}, \alpha^{J_L})}$$

\hspace{1cm} (33)

Eqs. (32), (33) is a solution of the problem Eq. (31). Of course, it remains to be a solution if ones introduces back the relations between the color group generators $t_J, J = 1, \ldots, L$. Any other solution is obtained from Eqs. (32), (33) multiplying it by a $\rho$-independent matrix from the right. \footnote{There is a minor subtlety at this point. When one specifies $t_J, J = 1, \ldots, L$ in Eqs. (32), (33) to be matrices belonging to a gauge Lie algebra, $g_{ptb}$ defined by Eqs. (32), (33) will not necessarily belong to the corresponding gauge group, only to $GL(\ast)$ instead. It will however be gauge equivalent (over $GL(\ast)$) to a matrix from the gauge group.}

The connection-form, $A_{\alpha \alpha}^{ptb}$, is obtained from $g_{ptb}$, Eqs. (32), (33), via Eqs. (28), (24). One can do it by a straightforward computation. One can also simplify the computation noticing that by construction $A_{\alpha \alpha}^{ptb} = g_{ptb}^{-1} \partial_{\alpha} g_{ptb}$ is linear in $\rho^\alpha$, $\alpha = 1, 2$. Hence $A_{\alpha \alpha}^{ptb}$ can be found as $\rho$-derivative of $A_{\alpha}^{ptb}$ taken at any value of $\rho$. Choosing all $q$’s in Eq. (33) equal each other and equal to a spinor $q$ (recall that $q$’s are defined up to the gauge freedom Eq. (15)) we find $A_{\alpha \alpha}^{ptb}$ as $\rho$-derivative of $A_{\alpha}^{ptb}$ at $\rho^\alpha = q^\alpha$. Since $g_{ptb}|_{\rho^\alpha=q^\alpha} = 1$ (see Eq. (33)), the computation becomes really easy, and one finds

$$A_{\alpha \alpha}^{ptb} = \sum_{J=1}^{L} A^{J}_{\alpha \alpha} \tilde{\mathcal{E}}_J + \sum_{J_1 J_2} A^{J_1 J_2}_{\alpha \alpha} \tilde{\mathcal{E}}_{J_1} \tilde{\mathcal{E}}_{J_2} + \ldots$$

\hspace{1cm} (34)

where $k_{\alpha \alpha}^{J_1 J_2 J_3} = k_{\alpha \alpha}^{J_1} + k_{\alpha \alpha}^{J_2} + \ldots + k_{\alpha \alpha}^{J_L}$. One can see that the above choice of $q$’s corresponds to the Lorentz gauge. Thus $A_{\alpha \alpha}^{ptb}$ from Eqs. (34) is a generating function (in the sense of Eq. (3)) for tree form-factors, or currents introduced in [14], in the Lorentz gauge.

Using $g_{ptb}$ Eqs. (33) one easily obtains the prominent Parke-Taylor amplitudes [13]. This is done in [24] and we shall not repeat it here.
3 N=3 supersymmetric perturbiner; preliminaries

N = 3 and N = 4 Yang-Mills theories are equivalent but N=3 formalism is more naturally combined with the twisters \cite{1} that is why we adopt N = 3 notation. N = 3 super-space is parametrized by commuting coordinates \(x^{\alpha \dot{\alpha}}\) and by the anticommuting ones \(\theta^{\alpha j}, \bar{\theta}^{\dot{\alpha} j}\). \(\alpha = 1, 2; \dot{\alpha} = 1, 2\) are the Lorentz indices, they can be lowered and raised with the antisymmetric \(\varepsilon\)-tensors, \(j = 1, 2, 3\) is an isotopic index. Supercharges act in the super-space as

\[
Q_{\alpha j} = \frac{\partial}{\theta^{\alpha j}} - \frac{1}{2} \bar{\theta}^{\dot{\alpha} j} \partial_{\alpha \dot{\alpha}} \\
\bar{Q}^{\dot{\alpha} j} = \frac{\partial}{\bar{\theta}^{\dot{\alpha} j}} - \frac{1}{2} \theta^{\alpha j} \partial_{\alpha \dot{\alpha}}
\]  

(35)

Introduce super-covariant derivatives,

\[
D_{\alpha j} = \frac{\partial}{\theta^{\alpha j}} + \frac{1}{2} \bar{\theta}^{\dot{\alpha} j} \partial_{\alpha \dot{\alpha}} \\
\bar{D}^{\dot{\alpha} j} = \frac{\partial}{\bar{\theta}^{\dot{\alpha} j}} + \frac{1}{2} \theta^{\alpha j} \partial_{\alpha \dot{\alpha}}
\]  

(36)

Introduce also super-connections

\[
\nabla_{\alpha j} = D_{\alpha j} + A_{\alpha j} \\
\nabla^{j}_{\dot{\alpha}} = \bar{D}^{j}_{\dot{\alpha}} + \bar{A}^{j}_{\dot{\alpha}}
\]  

(37)

where \(A_{\alpha j}\) and \(\bar{A}^{j}_{\dot{\alpha}}\) are superfields.

N=3 supersymmetric Yang-Mills equations can be represented as \cite{1}, \cite{34}-\cite{36}

\[
(\nabla_{\alpha j} \nabla_{\beta l} + \nabla_{\beta l} \nabla_{\alpha j}) \mid \text{symmetrized in } \alpha, \beta = 0 \\
(\nabla^{j}_{\dot{\alpha}} \nabla^{l}_{\dot{\beta}} + \nabla^{l}_{\dot{\beta}} \nabla^{j}_{\dot{\alpha}}) \mid \text{symmetrized in } \dot{\alpha}, \dot{\beta} = 0 \\
\nabla_{\alpha j} \nabla^{j}_{\dot{\alpha}} + \nabla^{j}_{\dot{\alpha}} \nabla_{\alpha j} = \delta_{\alpha j} \nabla_{\alpha \beta}
\]  

(38)

(only traceless part of the last equation is, in fact, equation on \(A_{\alpha j}\) and \(\bar{A}^{j}_{\dot{\alpha}}\), the rest is definition of connection \(\nabla_{\alpha \dot{\alpha}} = \partial_{\alpha \dot{\alpha}} + A_{\alpha \dot{\alpha}}\)).

Linearization of Eqs.\cite{35} reads

\[
(D_{\alpha j} A_{\beta l} + D_{\beta l} A_{\alpha j}) \mid \text{symmetrized in } \alpha, \beta = 0 \\
(\bar{D}^{\dot{\alpha} j} \bar{A}^{\dot{\beta} l} + \bar{D}^{\dot{\beta} l} \bar{A}^{\dot{\alpha} j}) \mid \text{symmetrized in } \dot{\alpha}, \dot{\beta} = 0 \\
D_{\alpha j} \bar{A}^{\dot{\alpha} j} + \bar{D}^{\dot{\alpha} j} A_{\alpha j} = \delta^{j}_{\dot{\alpha}} A_{\alpha \beta}
\]  

(39)

As discussed in the Introduction, \(N = 4\) multiplet splits into two \(N = 3\) multiplets, see tables (5),(6),(7).

We first describe plane waves corresponding to the highest helicity states in each multiplet, that is the positive helicity gluon and \(+\frac{1}{2}\) singlet gluino. These plane waves are

\[
A_{\alpha j} = \frac{q_{\alpha}(\theta j, \bar{\alpha} t) e^{k_{\alpha \alpha} g^{\alpha \dot{\alpha}}}}{(\bar{\alpha}, q)} \bar{A}^{\dot{\alpha} j} = 0
\]  

(40)
A_{\alpha j} = 0, \quad \bar{A}_{j}^{\dot{\alpha}} = 0\), and \(\theta^{\alpha j} \bar{g}^{\dot{\alpha}}\) is a chiral coordinate, defined so that

\[ \bar{D}_{l} y^{\alpha \dot{\alpha}} = 0, \quad (42) \]

the bracket \((a, b)\), as before, stands for contraction of \(a\) and \(b\) with the \(\varepsilon\)-tensor, Eq.(16), \(\varepsilon^{jlm}\) is the totally antisymmetric tensor in the isotopic space, \(k^{\alpha \dot{\alpha}} = \varepsilon^{\alpha} \varepsilon^{\dot{\alpha}}\) is a (light-like) four-momentum, \(t\) is a gauge Lie algebra generator, \(q_{\alpha}\) and \(\bar{q}_{\dot{\alpha}}\) are the reference spinors, they are defined modulo gauge equivalence

\[ q_{\alpha} \sim q_{\alpha} + \text{const} \cdot \varepsilon_{\alpha}, \quad \bar{q}_{\dot{\alpha}} \sim \bar{q}_{\dot{\alpha}} + \text{const} \cdot \bar{\varepsilon}_{\dot{\alpha}}. \quad (43) \]

One can check by a straightforward computation that the plane waves, Eqs.(40), (41) go through the linearized field equations Eqs.(39). One can also check that

\[ Q_{\alpha j} A_{l}^{\beta l} = 0 \]
\[ Q_{\alpha j} \bar{A}_{l}^{\dot{\beta}} = 0 \quad (44) \]

and

\[ (\bar{Q}^{j}, \bar{\varepsilon}) A_{l}^{\beta l} = 0 \]
\[ (\bar{Q}^{j}, \bar{\varepsilon}) \bar{A}_{l}^{\dot{\beta}} = 0 \quad (45) \]

on the both states Eqs.(40) and Eqs.(41). Eqs.(44) means that these states are highest states in the multiplets, while Eqs.(43) mean that half of the supercharges act trivially on the whole multiplets (since these states are massless).

Acting with the \(\bar{Q}\) charges on the states Eqs.(40) and Eqs.(41) one can obtain plane waves corresponding to all states in the tables Eqs.(6) and Eqs.(7). We shall, however do a bit differently. Instead of plane waves of the type of \(e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j}}\) we shall use plane waves of the type of

\[ e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j} + \chi_{j}^{l}} \]
\[ \chi_{l}, l = 1, 2, 3 \text{ are Grassmann variables; they are super-partners of momentu} \]

which are proper states for the supercharges \(Q_{\alpha j}\),

\[ Q_{j l} e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j}} = \varepsilon_{\alpha} \chi_{l} e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j}} \quad (47) \]

where \(\chi_{l}, l = 1, 2, 3\) are Grassmann variables; they are super-partners of momentum \(k_{\alpha \dot{\alpha}}\), so they will be referred to as momenintino.

Now the multiplets Eqs.(6) and Eqs.(7) can be organized as

\[ A_{\alpha j} = \frac{q_{\alpha}}{(\varepsilon, q)} ((\bar{\theta} j, \bar{\varepsilon}) + \chi_{j}) t e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j}} , \quad \bar{A}_{j}^{\dot{\alpha}} = 0 \quad (48) \]

(positive helicity gluon)

\[ A_{\alpha j} = 0, \quad \bar{A}_{j}^{\dot{\alpha}} = \frac{1}{2} \varepsilon^{jlm} ((\bar{\theta} j, \bar{\varepsilon}) + \chi_{l}) ((\bar{\theta} m, \bar{\varepsilon}) + \chi_{m}) t e^{k_{\alpha \dot{\alpha}} y^{\alpha \dot{\alpha}} + (\theta^{j}, \omega) x_{j}} \quad (49) \]
Various members of the multiplets Eqs.(6) and Eqs.(7) arise as coefficients in the Taylor expansion of Eqs.(48) and Eqs.(49) in powers of $\chi_j$. For example, the plane wave solution corresponding to the negative helicity gluon state arises as coefficient at $\frac{1}{3!}\varepsilon^{jlm}\chi_j\chi_l\chi_m$ in the expansion of Eq.(49).

4 Chiral N=3 supersymmetric perturbiner

In this section we construct perturbiner which generates only form-factors with all asymptotic states belonging to the same type of the $N=3$ super-multiplets, say, the one from the table (6) (no kinematical restrictions are assumed, that is, the set of momenta is arbitrary). According to the definition of the perturbiner (see Introduction), one first picks up a solution of the linearized field equations (39) in the form of a superposition of plane waves of the type of Eq.(48),

$$A_{\alpha j}^{(1)} = \sum_{J=1}^{L} \frac{q_J(A_J,\bar{\chi}_J) + \chi_J}{(w^J,q^J)} \tilde{S}_J, \quad \tilde{A}_{\alpha}^J = 0 \quad (50)$$

where

$$\tilde{S}_J = t_J S, \quad S = a_J e^{k_{\alpha\beta} y^{\alpha}(\theta^\beta,\bar{w}^J)} \chi_J^J, \quad (51)$$

$a^J$ is a commuting nilpotent (see Eq.(4)) symbol of annihilation/creation operator of the $J$-th particle, $\chi_J^J$ stands for the momentino of the $J$-th particle (see Eq(47)), other notations are as in Eqs.(40),(41).

Then one looks for a solution of the field equations (38), which is polynomial in the super-harmonics $S_J, J = 1, \ldots, L$ Eq.(51), and whose first order term is as in Eq.(50).

To describe this solution, introduce again an auxiliary $CP^1$ space with homogeneous coordinates $\rho^\alpha, \alpha = 1, 2$. Introduce, also, $D_j, \nabla_j$ and $A_j$ as

$$D_j = \rho^\alpha D_{\alpha j}$$
$$\nabla_j = \rho^\alpha \nabla_{\alpha j}$$
$$A_j = \rho^\alpha A_{\alpha j} \quad (52)$$

Then, as in the self-dual case (section 2), first equation of Eqs.(38) is represented as a zero-curvature condition by contracting its Lorentz indices with $\rho$'s. Hence, the first equation is (locally) solved as

$$A_j = g^{-1} D_j g \quad (53)$$

where (superfield) $g$ on r.h.s. is (similarly to Eq.(24)) a group valued rational homogeneous function of $\rho^\alpha, \alpha = 1, 2$ of degree 0, such that (superfield) $A_j$ is a regular homogeneous function of $\rho^\alpha, \alpha = 1, 2$ of degree 0 of degree 1. The rest equations of Eqs.(38) are solved provided

$$\tilde{D}_j^J g = 0 \quad (54)$$

According to the rules of the game, $g^{sptb}$ is sought for as a polynomial in the super-harmonics $S_J, J = 1, \ldots, L$ Eq.(51), first order term of the polynomial being defined by the one in $A_{\alpha j}$ Eq.(50) via Eq.(53). All steps in constructing such $g^{sptb}$ are parallel to the ones in section 2. Moreover, amusingly, the resulting $g^{sptb}$ is given by the same formulae as $g^{ptb}$ Eq.(32),(33) with the substitution

$$\hat{E} \rightarrow \hat{S} \quad (55)$$
(Eq. (54) is satisfied because $\bar{D} j \hat{S} = 0$)

Clearly, $g^{sptb}$ has the same singularities in the auxiliary $CP^1$ space as the non-supersymmetric self-dual perturbiner $g^{ptb}$, Eq.(32), (33), namely, it has simple poles at $\rho^J = \alpha^j_J, J = 1, \ldots, L$ where $\alpha^j_J$ is the spinor appearing in decomposition of momentum $k_{a\dot{a}}^J$ of the $J$-th particle, (see Eq.(14)).

Finally, the generating functions for form-factors of the superfields $A_{\alpha j}, \bar{A}_{\dot{\alpha}}^j$ are obtained from Eqs.(53) (computation is parallel to the one in section 2),

$$A_{\alpha j}^{sptb} = \sum_{J=1}^{L} A_{\alpha j}^J \hat{S}_J + \sum J_1 J_2 A_{\alpha j}^{J_1 J_2} \hat{S}_{J_1 J_2} + \ldots; \bar{A}_{\dot{\alpha}}^{sptb} = 0$$

$$A^{J_1 \ldots J_M}_{\alpha j} = - \frac{\alpha^j_J (\bar{\alpha}^\dot{\alpha} J_1 \cdots J_M + \bar{\theta}^\dot{\alpha} k_{a \dot{a}}^{J_1 \cdots J_M})}{(\alpha^j_J, \bar{q}^J)(\alpha^J_M, q_J)} \frac{1}{(\alpha^j_1, \bar{q}^j_2)(\alpha^j_2, \bar{q}^j_3) \ldots (\alpha^j_{M-1}, \bar{q}^j_M)}$$

(56)

where $[\alpha^j_J^{J_1 \cdots J_M} = \alpha^j_{\beta} \chi^J_{\beta} + \ldots + \alpha^j_{\beta} \chi^{J_M}_{\beta}$.}

5 Generic (nonchiral) N=3 perturbiner

Nonchiral $N = 3$ supersymmetric perturbiner is a generating function for all tree form-factors in the $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory. According to the rules of the game, the solution of the linearized field equations must include now both types of harmonics, Eqs.(48), (49),

$$A_{\alpha j}^{(1)} = \sum_{J=1}^{L} q^J_J (\bar{\theta}^J_j, \bar{\alpha}^J_j) + \chi^J_j t \hat{S}_J$$

$$\bar{A}_{\dot{\alpha}}^{(1) j} = \sum_{J=1}^{L} b_J q_J (\bar{\theta}^J_l, \bar{\alpha}^J_l) + \chi^J_l t \hat{S}_J$$

(57)

where $b_J$ is an anticommuting nilpotent symbol, all other notations are the same as in Eq.(74).

The nonchiral perturbiner is a solution of the field equations Eqs.(53), polynomial in the harmonics $S_j, J = 1, \ldots, L$, Eq.(54), first order term of the polynomial being just $A^{(1)}$ as in Eq.(50). We again use a twistor construction [1] to describe solutions of Eqs.(58). To this end, introduce two couples of complex numbers, $\rho^\alpha, \alpha = 1, 2, \bar{\rho}^{\dot{\alpha}}, \dot{\alpha} = 1, 2$ which can be viewed on as homogeneous coordinates on $CP^1 \times CP^1$ space. Contracting all Lorentz indices in Eqs.(58) with these $\rho$’s, $\bar{\rho}$’s one again obtains a sort of zero-curvature condition

$$\nabla_j \nabla_l + \nabla_l \nabla_j = 0$$

$$\bar{\nabla}^j \bar{\nabla}^l + \bar{\nabla}^l \bar{\nabla}^j = 0$$

$$\nabla_j \bar{\nabla}^l + \bar{\nabla}^l \nabla_j = \delta_j^l \nabla$$

(58)

where

$$\nabla_j = \rho^\alpha \nabla_{\alpha j}$$

$$\bar{\nabla}^j = \bar{\rho}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha} j}$$

$$\nabla = \rho^\alpha \bar{\rho}^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}}$$

(59)
Clearly, Eqs. (38) and Eqs. (58) are equivalent provided Eqs. (58) are solved identically in \( \rho, \bar{\rho} \). Eqs. (58) are locally solved as

\[
A_j = g^{-1} D_j g \\
\bar{A}^j = g^{-1} \bar{D}^j g
\]

(60)

where

\[
A_j = \rho^\alpha A_{\alpha j} \\
D_j = \rho^\alpha D_{\alpha j} \\
\bar{A}^j = \bar{\rho}^{\dot{\alpha}} \bar{A}_{\dot{\alpha} j} \\
\bar{D}^j = \bar{\rho}^{\dot{\alpha}} \bar{D}_{\dot{\alpha} j}
\]

(61)

and \( \rho, \bar{\rho} \) dependence of \( g \) must be such that \( A_j, \bar{A}^j \) are just linear in \( \rho, \bar{\rho} \), as in Eqs. (61).

More definitely, (superfield) \( g \) is a meromorphic homogeneous function of \( \rho, \bar{\rho} \) of degree \( (0, 0) \) such that (superfield) \( A_j \) Eqs. (61) is a holomorphic homogeneous function of \( \rho, \bar{\rho} \) of degree \( (1, 0) \) and (superfield) \( \bar{A}^j \) Eqs. (61) is a holomorphic homogeneous function of \( \rho, \bar{\rho} \) of degree \( (0, 1) \), as in Eqs. (61).

Again, in the case of perturbiner, \( g^{ncstpb} \) must be polynomial in the super-harmonics \( S_J, J = 1, \ldots, L \), Eq. (52). First order terms of this polynomial,

\[
g^{(1)}_{ncstpb}(\rho, \bar{\rho}, \{S\}) = \sum_{J=1}^{L} g^{ncstpb}_J(\rho, \bar{\rho}) \hat{S}_J
\]

are fixed by those in \( A \)'s Eqs. (57) via Eqs. (60) (analogously to Eqs. (25), Eqs. (26)). Expanding Eq. (60) up to first order in the harmonic \( S_J \) and using Eq. (57), Eq. (62) one obtains

\[
g^{ncstpb}_J(\rho, \bar{\rho}) = \frac{(\rho, q^J)}{(\rho, \bar{\omega}^J)(\bar{\omega}^J, q^J)} + b_J \frac{(\bar{\rho}, \bar{q}^J)}{(\bar{\rho}, \bar{\omega}^J)(\bar{\omega}^J, \bar{q}^J)} \\
\frac{1}{3!} \varepsilon^{jlm}((\bar{\theta}_l, \bar{\omega}^J) + \chi^J_l)((\bar{\theta}_m, \bar{\omega}^J) + \chi^J_m)
\]

(63)

Thus, according to Eqs. (52), Eq. (53), the first order terms in \( g^{ncstpb} \) have simple poles at surfaces

\[
\rho_\alpha = \bar{\omega}_\alpha^J, \; J = 1, \ldots, L
\]

(64)

and at surfaces

\[
\bar{\rho}_{\dot{\alpha}} = \bar{\omega}_{\dot{\alpha}}^J, \; J = 1, \ldots, L
\]

(65)

in the \( CP^1 \times CP^1 \) space parametrized by \( \rho, \bar{\rho} \). An analysis shows that for \( A, \bar{A} \) from Eqs. (60) to be regular, the higher order terms in the polynomial \( g^{ncstpb} \) may have only simple poles at the surfaces Eqs. (54), (55) and also at the surfaces \( C_M \) in \( CP^1 \times CP^1 \)

\[
C_M: \{ \rho^\alpha \bar{\rho}^{\dot{\alpha}} k^M_{\alpha \dot{\alpha}} = 0 \}
\]

(66)

where \( k^M_{\alpha \dot{\alpha}} = \sum_{J \in \mathcal{M}} k^J_{\alpha \dot{\alpha}} \), and \( \mathcal{M} \) is a subset of the set \( J = 1, \ldots, L \). That is, any linear combination of momenta of the asymptotic states included in the perturbiner defines a surface in \( CP^1 \times CP^1 \) at which \( g^{ncstpb} \) has a simple pole. Notice that due to the non-resonantness condition (see Introduction) the surfaces \( C_M \) Eq. (16) with \( \mathcal{M} \) including more
than one element never reduces to the ones of the type of Eqs. (64), (65). Furthermore, the regularity of a set of polynomials (see Eq. (32) and explanations about it),

\[ D_j (\text{res}|_{C_M} g_{\text{nscptb}} \cdot g_{\text{nscptb}}^{-1}) |_{C_M} = 0 \]

\[ \tilde{D}_j (\text{res}|_{C_M} g_{\text{nscptb}} \cdot g_{\text{nscptb}}^{-1}) |_{C_M} = 0 \]

(67)

where \( D_j \), \( \tilde{D}_j \) are as in Eqs. (61) and notation \( |_{C_M} \) in Eqs. (67) indicates restriction at the surface \( C_M \) Eq. (53). It is again convenient to use the expansion of \( g_{\text{nscptb}} \) in the color ordered monomials (see Eq. (32) and explanations about it),

\[ g_{\text{nscptb}}(\rho, \bar{\rho}, \{S\}) = 1 + \sum_j g_{\text{nscptb}}^j (\rho\bar{\rho}) \hat{S}_j + \sum_{J_1, J_2} g_{\text{nscptb}}^{J_1J_2} (\rho, \bar{\rho}) \hat{S}_{J_1} \hat{S}_{J_2} + \ldots \]

(68)

The singularity structure of \( g_{\text{nscptb}} \) Eqs. (60), (67) dictates the following form of coefficients \( g_{\text{nscptb}}^{J_1J_2,...,J_M} (\rho, \bar{\rho}) \),

\[ g_{J_1J_2,...,J_M}^{\text{nscptb}}(\rho, \bar{\rho}) = \frac{P^{J_1J_2,...,J_M}(\rho, \bar{\rho}, \bar{\theta})}{(\rho^\alpha \bar{\rho}^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_1J_2}) (\rho^\alpha \bar{\rho}^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_1J_2,J_3}) \ldots (\rho^\alpha \bar{\rho}^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_1J_2,...,J_M})} \]

(69)

where, we remind, \( k_{\alpha\bar{\alpha}}^{J_1J_2,...,J_M} = k_{\alpha\bar{\alpha}}^{J_1} + k_{\alpha\bar{\alpha}}^{J_2} + \ldots + k_{\alpha\bar{\alpha}}^{J_M} \), and \( P^{J_1J_2,...,J_M}(\rho, \bar{\rho}, \bar{\theta}) \) is a polynomial in \( \rho, \bar{\rho} \) of degree \((M, M)\) (so, that \( g_{J_1J_2,...,J_M}^{\text{nscptb}}(\rho, \bar{\rho}) \) is a rational function of \( \rho, \bar{\rho} \) of degree \((0, 0)\)). We have explicitly indicated only \( \rho, \bar{\rho} \) and \( \bar{\theta} \) dependence of these polynomials. They, of course, depend on quantum numbers of \( J_1 \)-th, \( J_2 \)-th, \ldots, \( J_M \)-th particles, such as momenta, momentinos etc.

Now, the problem of constructing \( g_{\text{nscptb}} \), and thus, the problem of computing all tree form-factors in (supersymmetric) Yang-Mills theory, will be formulated as a problem of constructing a set of polynomials \( P^{J_1J_2,...,J_M}(\rho, \bar{\rho}, \bar{\theta}), \ M = 1, \ldots, L \). From Eqs. (67), (68) and (53), crucially using the nilpotency, \( S^2 = 0 \), and linear independence of various color ordered products of \( S \)'s, one obtains

\[ \left( \partial^{\alpha}(\bar{\alpha}_{\alpha} \chi_{\bar{j}}^{J_1J_2,...,J_M} + \bar{\theta}^{\alpha} k_{\alpha\bar{\alpha}}^{J_1J_2,...,J_M} P^{J_1J_2,...,J_M}) \right) |_{C_{J_1J_2,...,J_M}} = 0 \]

\[ \left( \bar{\rho}^{\bar{\alpha}} \frac{\partial}{\partial \bar{\rho}^{\bar{\alpha}}} P^{J_1J_2,...,J_M} \right) |_{C_{J_1J_2,...,J_M}} = 0 \]

(70)

\[ P^{J_1J_2,...,J_M} |_{C_{J_1J_2,...,J_M}} = P^{J_1J_2,...,J_1} |_{C_{J_1J_2,...,J_1}} P^{J_{I+1}J_{I+2},...,J_M} |_{C_{J_1J_2,...,J_1}}, \ I < M, \]

(71)

and from Eqs. (53) one sees that

\[ P^J = \frac{(\rho, q^J)(\bar{\rho}, \bar{q}^J)}{(\bar{\alpha}^{\bar{J}}, q^J)} + b^J \frac{(\bar{\rho}, \bar{q}^J)(\rho, \bar{\alpha}^{\bar{J}})}{(\bar{\alpha}^{\bar{J}}, \bar{q}^J)} \]

\[ \frac{1}{3!} \epsilon^{jm}((\bar{\theta}^J, \bar{\alpha}^{\bar{J}}) + \chi^J_J)((\bar{\theta}^J, \bar{\alpha}^{\bar{J}}) + \chi^J_J)((\bar{\theta}^J, \bar{\alpha}^{\bar{J}}) + \chi^J_J) \]

(72)

We remind that \( P^{J_1J_2,...,J_M}(\rho, \bar{\rho}, \bar{\theta}), \ M = 1, \ldots, L \) are homogeneous polynomials on \( CP^1 \times CP^1 \) space, parametrized by \( \rho^\alpha, \bar{\rho}^{\bar{\alpha}}, \alpha = 1, 2, \bar{\alpha} = 1, \bar{2}, \) of degree \((M, M)\). \( k_{\alpha\bar{\alpha}}^{J_1J_2,...,J_L} = k_{\alpha\bar{\alpha}}^{J_1J_2} + k_{\alpha\bar{\alpha}}^{J_2J_3} + \ldots + k_{\alpha\bar{\alpha}}^{J_M} \), \( k_{\alpha\bar{\alpha}}^{J_1J_2J_3} = \bar{\alpha}_{\beta} \alpha_{\bar{\beta}} \bar{\alpha}_{\bar{\beta}} \) is momentum of the \( J \)-th particle, \( \bar{\alpha}_{\beta} \alpha_{\bar{\beta}} \bar{\alpha}_{\bar{\beta}} \chi^J_J \). Notation \( P|_C \) means that a polynomial \( P \) is restricted on a surface.
C. $C_{J_1J_2...J_M}$ are surfaces in $CP^1 \times CP^1$ defined by Eqs. (70). Due to the non-resonantness condition (that a sum of the type $k_{\alpha}^{J_1} + k_{\alpha}^{J_2} + \ldots + k_{\alpha}^{J_N}$ is never light-like when there is more than one term) only surfaces $C_{J}$ are reducible, as in Eqs. (63). Notice that the surfaces $C_{J_1J_2...J_l}$ and $C_{J_{l+1}J_2...J_M}$, $I < L$ intersect at two points in $CP^1 \times CP^1$, defined by the system

\begin{align*}
\rho^\alpha \bar{\rho}^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_1J_2...J_l} &= 0, \\
\rho^\alpha \bar{\rho}^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_{l+1}J_2...J_M} &= 0
\end{align*}

(73)

and at every one of these intersection points three surfaces meet - $C_{J_1J_2...J_l}$, $C_{J_{l+1}J_2...J_M}$ and $C_{J_1J_2...J_M}$. The bracket of the type of (a,b) has been defined in Eq. (16). $J^\alpha_j$, $\bar{J}_j^\alpha$, $\alpha = 1, 2; \bar{\alpha} = \bar{1}, \bar{2}$, $j = 1, 2, 3$ are Grassmann variables. $\chi^j_I$, momentino of the $J$-th particle, has been introduced in Eqs. (16), (17). $\alpha$ and $\bar{\alpha}$ are the reference spinors, they are defined modulo gauge equivalence $q_{\alpha} \sim q_{\alpha} + const \cdot \bar{\alpha}$, $\bar{q}_{\bar{\alpha}} \sim \bar{q}_{\bar{\alpha}} + const \cdot \bar{\alpha}$ (one may use this freedom in solving the problem Eqs. (70), (71), (72)).

We shall now explain how this problem, Eqs. (70), (71), (72), can be solved iteratively. Namely, we explain how to find polynomial $P^{J_1J_2...J_I}$, provided all polynomials $P^{J_1J_2...J_I}$, $I < L$ are known (in this sense Eq. (72) is a first step of the iteration).

Consider, first, the polynomials restricted on “their own” surfaces, that is, introduce

$$
\tilde{P}^{J_1J_2...J_l} = P^{J_1J_2...J_l}|_{C_{J_1J_2...J_l}},
$$

(74)

$\tilde{P}^{J_1J_2...J_l}$ is a homogeneous polynomial on $C_{J_1J_2...J_l}$ of degree $2L$, and hence it has $2L + 1$ degrees of freedom.

Equations (71) on the restricted polynomials reduce to

$$
\tilde{P}^{J_1J_2...J_l}|_{Z_i(C_{J_1J_2...J_l} \cap C_{J_{l+1}...J_M})} = P^{J_1J_2...J_l}|_{Z_i(C_{J_1J_2...J_l} \cap C_{J_{l+1}...J_M})}
$$

(75)

where $Z_i(C_{J_1J_2...J_l} \cap C_{J_{l+1}...J_M})$, $i = 1, 2, 3$ stand for the intersection points of the surfaces $C_{J_1J_2...J_l}$ and $C_{J_{l+1}...J_M}$, Eq. (73). Thus Eq. (73) defines polynomial $\tilde{P}^{J_1J_2...J_l}$ at these $2(L - 1)$ intersection points.

Equations (70) tell that the polynomial $\tilde{P}^{J_1J_2...J_l}$ must be of the form

$$
\tilde{P}^{J_1J_2...J_l}(\rho, \bar{\rho}, \theta) = \frac{1}{3!} \varepsilon^j m \left( \rho^\alpha ([\alpha_\alpha^j J_1...J_M] + \bar{\theta}_j^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{J_1J_2...J_M}) \right) \\
\left( \rho^\alpha ([\alpha_\alpha^j M_1...J_M] + \bar{\theta}_j^{\bar{\alpha}} k_{\alpha\bar{\alpha}}^{M_1J_2...J_M}) \right)
$$

(76)

where $R^{J_1J_2...J_l}(\rho, \bar{\rho})|_{C_{J_1J_2...J_l}}$ is a polynomial on the surface $C_{J_1J_2...J_l}$ of degree $2L - 3$. Thus, the number of degrees of freedom in the polynomial $R^{J_1J_2...J_l}(\rho, \bar{\rho})|_{C_{J_1J_2...J_l}}$, and so in the polynomial $\tilde{P}^{J_1J_2...J_l}$, is $2L - 2$, that is, precisely the number of points at which the polynomial is defined according to Eqs. (73)! (Eqs. (76) and (73) are compatible due to nilpotency of $\theta^{\alpha j}$, $\bar{\theta}_j^{\bar{\alpha}}$ and $\chi^j_I$).

Once restriction of the polynomial $P^{J_1J_2...J_l}$ on the surface $C_{J_1J_2...J_l}$ is found and restrictions of the polynomial $P^{J_1J_2...J_l}$ on the surfaces $C_{J_1J_2...J_M}$, $M < L$ are known due to equations (71), the polynomial $P^{J_1J_2...J_l}$ is fixed modulo a polynomial which is
zero at surfaces $C_{J_1 J_2 \ldots J_M}$, $M = 1, \ldots, L$, that is precisely modulo a polynomial in the denominator of $g_{n^{expb}}$, Eq.(24), that is, modulo the gauge freedom.

The described iterative procedure can, in principle, be used as an alternative to the usual perturbation theory, and it might even be the more economical alternative, but we shall not try here to prove its efficiency. Instead we would like to express our hope that Eqs.(70), (71), (72) can be, in some sense, solved completely. By the way, Eqs.(71), (72) allow such a complete solution up to a freedom which is to be fixed by Eqs.(70) (or Eqs.(76)), namely

$$P_{J_1 J_2 \ldots J_L} = \det \begin{pmatrix}
P_{J_1} & Q_{J_1 J_2} & Q_{J_1 J_2 J_3} & \ldots & Q_{J_1 \ldots J_L} \\
\rho^\alpha \bar{\rho}^{\dot{\alpha}} k_{\dot{\alpha} \alpha}^{J_1} & P_{J_2} & Q_{J_2 J_3} & \ldots & Q_{J_2 \ldots J_L} \\
0 & \rho^\alpha \bar{\rho}^{\dot{\alpha}} k_{\dot{\alpha} \alpha}^{J_1 J_2} & P_{J_3} & \ldots & Q_{J_3 \ldots J_L} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & P_{J_L}
\end{pmatrix}$$

Each entry of the $L \times L$ matrix in the above equation is a homogeneous polynomial on the $CP^1 \times CP^1$ space of degree $(1, 1)$. $Q$’s in the upper triangular part of the matrix represent the freedom which is to be fixed by Eqs.(70) (or Eqs.(76)). Unfortunately, we have not been able to implement these equations to fix this freedom.

**Acknowledgments**

I benefited a lot from discussions with A.Rosly whom I am very much obliged to. This work was supported by INTAS grant 97-0103.

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