ON A COUNTEREXAMPLE RELATED TO WEIGHTED WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS

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Abstract. We show that the Hilbert transform does not map $L^1(M_{\Phi}w)$ to $L^{1,\infty}(w)$ for every Young function $\Phi$ growing more slowly than $t \log \log(e^e + t)$. Our proof is based on a construction of M.C. Reguera and C. Thiele.

1. Introduction

Let $H$ be the Hilbert transform. One of open questions in the one-weighted theory of singular integrals is about the optimal Young function $\Phi$ for which the weak type inequality

$$w\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \leq \frac{c}{\lambda} \int_{\mathbb{R}} |f| M_{\Phi} w \, dx \quad (\lambda > 0)$$

holds for every weight (i.e., non-negative measurable function) $w$ and any $f \in L^1(M_{\Phi}w)$, where $M_{\Phi}$ is the Orlicz maximal operator defined by

$$M_{\Phi}f(x) = \sup_{I \ni x} \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_{I} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.$$

If $\Phi(t) = t$, then $M_{\Phi} = M$ is the standard Hardy-Littlewood maximal operator. If $\Phi(t) = t^r, r > 1$, denote $M_{\Phi}f = M_rf$.

C. Fefferman and E.M. Stein [6] showed that if $H$ is replaced by the maximal operator $M$, then the corresponding inequality holds with $\Phi(t) = t$. Next, A. Córdoba and C. Fefferman [11] proved (1.1) with $\Phi(t) = t^r, r > 1$. This result was improved by C. Pérez [8] who showed that (1.1) holds with $\Phi(t) = t \log^\varepsilon(e + t), \varepsilon > 0$ (see also [7] for a different proof of this result).

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Very recently, C. Domingo-Salazar, M.T. Lacey and G. Rey [5] obtained a further improvement; their result states that (1.1) holds whenever $\Phi$ satisfies

$$\int_1^{\infty} \frac{\Phi^{-1}(t)}{t^2 \log(e + t)} dt < \infty.$$  

For example, one can take $\Phi(t) = t \log \log (e^e + t), \alpha > 1$ or $\Phi(t) = t \log \log (e^e + t) \log \log \log (e^e + t)$ ($\alpha > 1$) etc.

A question whether (1.1) is true with $\Phi(t) = t$ has become known as the Muckenhoupt-Wheeden conjecture. This conjecture was disproved by M.C. Reguera and C. Thiele [10] (see also [9] and [2] for dyadic and multidimensional versions of this result).

Denote $\Psi(t) = t \log \log (e^e + t)$. It was conjectured in [7] that (1.1) holds with $\Phi = \Psi$. The above mentioned result in [5] establishes (1.1) for essentially every $\Phi$ growing faster than $\Psi$.

The main result of this note is the observation that the Reguera-Thiele example [10] actually shows that (1.1) does not hold for every $\Phi$ growing more slowly than $\Psi$.

**Theorem 1.1.** Let $\Phi$ be a Young function such that

$$\lim_{t \to \infty} \frac{\Phi(t)}{t \log \log (e^e + t)} = 0.$$  

Then for every $c > 0$, there exist $f, w$ and $\lambda > 0$ such that

$$w \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} > \frac{c}{\lambda} \int_{\mathbb{R}} |f| M_\Phi w \, dx.$$  

This theorem along with the main result in [5] emphasizes that the case of $\Phi = \Psi$ is critical for (1.1). However, the question whether (1.1) holds with $\Phi = \Psi$ remains open.

We mention briefly the main ideas of the Reguera-Thiele example [10] and, in parallel, our novel points. First, it was shown in [10] that given $k \in \mathbb{N}$ sufficiently large, there is a weight $w_k$ supported on $[0, 1]$ satisfying $Hw_k \geq ckw_k$ and $Mw_k \leq cw_k$ on some subset $E \subset [0, 1]$. In Section 2, we show that the latter “$A_1$ property” can be slightly improved until $M_r w_k \leq cw_k$ with $r > 1$ depending on $k$. The second ingredient in [10] was the extrapolation argument of D. Cruz-Uribe and C. Pérez [3]. This argument says that assuming (1.1) with $Mw$ on the right-hand side, one can deduce a certain weighted $L^2$ inequality for $H$. It is not clear how to extrapolate in a similar way starting with a general Orlicz maximal function $M_\Phi$ in (1.1). In Section 3, we obtain a substitute of the argument in [3] for $M_r w, r > 1$, instead of $Mw$. 
2. The Reguera-Thiele construction

We describe below the main parts of the example constructed by M.C. Reguera and C. Thiele [10].

An interval \( I \) of the form \([3^j n, 3^j (n + 1)), j, n \in \mathbb{Z}, \) is called a triadic interval.

Fix \( k \in \mathbb{N} \) large enough. Given a triadic interval \( I \subset [0, 1) \), denote \( I^\Delta = \frac{1}{3} I \), namely, \( I^\Delta \) is the interval with the same center as \( I \) and of one third of its length; further, denote by \( P(I) \) a triadic interval adjacent to \( I^\Delta \) and such that \( |P(I)| = \frac{1}{3^k} |I| \). Observe that \( P(I) \) can be situated either on the left or on the right of \( I^\Delta \), and we will return to this point a bit later.

Set now \( J^1 = [0, 1) \) and \( I_{1,1} = P(J^1) \). Next, we subdivide \( (J^1)^\Delta \) into \( 3^{k-1} \) triadic intervals of equal length, and denote them by \( J_m^2, m = 1, 2, \ldots, 3^{k-1} \). Set correspondingly \( I_{2,m} = P(J_m^2) \). Notice that \( |J_m^2| = \frac{1}{3^k} \) and \( |I_{2,m}| = \frac{1}{3^m} \) for \( m = 1, 2, \ldots, 3^{k-1} \). Observe also that the intervals \( I_{1,1} \) and \( I_{2,m} \) are pairwise disjoint.

Proceeding by induction, at \( l \)-th stage, we subdivide each interval \( (J_m^{l-1})^\Delta \) into \( 3^{k-1} \) triadic intervals of equal length, and denote all obtained intervals by \( J_m^l, m = 1, 2, \ldots, 3^{(k-1)(l-1)} \). Set \( I_{l,m} = P(I_m^l) \). Then \( |J_m^l| = \frac{1}{3^{(l-1)k}} \) and \( |I_{l,m}| = \frac{1}{3^m} \), and the intervals \( \{I_{l,m}\} \) are pairwise disjoint.

Denote by \( I_l \) and \( J_l \) the families of all intervals \( \{I_{l,m}\} \) and \( \{J_m^l\} \), respectively, and set \( \Omega_l = \bigcup_{I \in I_l} I \). Define the weight \( w_k \) such that \( w_k([0, 1]) = 1, w_k \) is a constant on \( \Omega_l \), and for every \( I \in I_l \) and \( J \in J_{l+1} \), \( w_k(I) = w_k(J) \) (we use the standard notation \( w_k(E) = \int_E w_k \).

It was proved in [10] that one can specify the situation of the intervals \( \{I_{l,m}\} \) such that if \( k > 3000 \) and \( x \in \cup_{l,m} I_{l,m}^\Delta \), then
\[
|H w_k(x)| \geq (k/3) w_k(x);
\] moreover,
\[
M w_k(x) \leq 7 w_k(x) \quad (x \in \cup_{l,m} I_{l,m}^\Delta),
\] irrespective of the precise configuration of \( \{I_{l,m}\} \).

We will show that the latter estimate can be improved by means of replacing \( M w_k \) on the left-hand side by a larger operator \( M_r w_k \) with \( r > 1 \) depending on \( k \). In order to do that, we need a more constructive description of \( w_k \).

Lemma 2.1. We have,
\[
w_k(x) = \sum_{i=1}^{\infty} \left( \frac{3^k}{3^{k-1} + 1} \right)^i \chi_{\Omega_i}(x).
\]
implies (2.4)

\[ w_k(J) = w_k(I) + w_k(J^\Delta) = w_k(I) + \sum_{J' \in J^\Delta, J' \subset J^\Delta} w_k(J'). \]

Let \( I' \in \mathcal{I}_{l-1} \). Then
\[ w_k(J) = w_k(I') = \alpha_{l-1}|I'| = \alpha_{l-1}|J|. \]
Similarly, \( w_k(J') = \alpha_l|J'| \), and also \( w_k(I) = \alpha_l|I| = \alpha_l \frac{|J|}{3^k} \). Hence, (2.3) implies
\[ \alpha_{l-1}|J| = \alpha_l \frac{|J|}{3^k} + \alpha_l \sum_{J' \in J^\Delta, J' \subset J^\Delta} |J'| = \alpha_l \frac{|J|}{3^k} + \alpha_l \frac{|J|}{3}. \]
From this, \( \alpha_l = \frac{3^k}{3k-1+1} \alpha_{l-1} \), and therefore \( \alpha_l = \left( \frac{3^k}{3k-1+1} \right)^l \gamma \) for some \( \gamma > 0 \).
From the condition \( w_k([0,1]) = 1 \), we obtain
\[ 1 = w_k([0,1]) = \gamma \sum_{l=1}^{\infty} \left( \frac{3^k}{3k-1+1} \right)^l |\Omega_l| \]
\[ = \gamma \sum_{l=1}^{\infty} \left( \frac{3^k}{3k-1+1} \right)^l \frac{3^{(k-1)(l-1)}}{3^k} = \gamma \frac{1}{3k-1} \sum_{l=1}^{\infty} \left( \frac{3^k-1}{3k-1+1} \right)^l = \gamma, \]
and therefore the lemma is proved. \( \square \)

**Lemma 2.2.** Let \( r = 1 + \frac{1}{3k+1} \). Then for every \( I \in \mathcal{I}_l \), \( l \in \mathbb{N} \), and for all \( x \in I^\Delta \),
\[ M_r w_k(x) \leq 21 w_k(x). \]

**Proof.** Let \( I \in \mathcal{I}_l \), and let \( x \in I^\Delta \). Take an arbitrary interval \( R \) containing \( x \). If \( R \subset I \), then
\[ \left( \frac{1}{|R|} \int_R w_k^r(y)dy \right)^{1/r} = \left( \frac{3^k}{3k-1+1} \right)^l = w_k(x). \]
Assume that \( R \not\subset I \). Then \( |R| \geq |I|/3 \). Denote by \( \mathcal{F} \) the family of all triadic intervals \( I' \subset [0,1) \) such that \( |I'| = |I| \) and \( I' \cap R \neq \emptyset \). There are at most two intervals \( I' \in \mathcal{F} \) not containing in \( R \), and therefore,
\[ \sum_{I' \in \mathcal{F}} |I'| \leq |R| + \sum_{I' \in \mathcal{F}: I' \not\subset R} |I'| \leq |R| + 2|I| \leq 7|R|. \]
We claim that if \( r = 1 + \frac{1}{3k+1} \), then for every \( I' \in \mathcal{F} \),
\[ \left( \frac{1}{|I'|} \int_{I'} w_k^r(y)dy \right)^{1/r} \leq 3 w_k(x). \]
This property would conclude the proof since then, by (2.4),
\[
\frac{1}{|R|} \int_R w_k^r(y)dy \leq \sum_{I' \in \mathcal{F}} \frac{1}{|I'|} \frac{1}{|R|} \int_{I'} w_k^r(y)dy \leq 7(3w_k(x))^r.
\]

To show (2.5), one can assume that \(I'\) has a non-empty intersection with the support of \(w_k\). If \(I' \neq J\) for some \(J \in \mathcal{J}_{l+1}\), then \(I' \subset L\), where \(L \in \mathcal{I}_\nu, \nu \leq l\), and hence
\[
\left( \frac{1}{|I'|} \int_{I'} w_k^r(y)dy \right)^{1/r} = \left( \frac{3^k}{3^{k-1} + 1} \right)^\nu \leq w_k(x).
\]

It remains to consider the case when \(I' = J\) for some \(J \in \mathcal{J}_{l+1}\). Using that for every \(j \geq l+1\), \(J \in \mathcal{J}_{l+1}\) contains \(3^{(k-1)(j-l-1)}\) intervals \(I \in \mathcal{I}_j\), we obtain
\[
\frac{1}{|I'|} \int_{I'} w_k^r(y)dy = 3^{l+1} \sum_{j=l+1}^{\infty} \sum_{I \in \mathcal{I}_j : I \subset I'} \int_I w_k^r(y)dy = \sum_{j=l+1}^{\infty} 3^{(k-1)(j-l-1)} 3^j (3^k)^{jr} = \frac{1}{3^{k-l}} \sum_{j=1}^{\infty} 3^{-j} \left( \frac{3^k}{3^{k-1} + 1} \right)^{(j+l)r}.
\]

Therefore,
\[
\frac{1}{|I'|} \int_{I'} w_k^r(y)dy = \frac{1}{3^{k-l}} \left( \sum_{j=1}^{\infty} 3^{-j} \left( \frac{3^k}{3^{k-1} + 1} \right)^{jr} \right) w_k(x)^r \leq \frac{1}{3^{k-l}} \frac{3}{3^{k-l} - (3^k/(3^{k-1} + 1))^r} w_k(x)^r,
\]
whenever \(\left( \frac{3^k}{3^{k-1} + 1} \right)^r < 3\).

If \(r = 1 + \frac{1}{3^{k+l}}\), then
\[
\left( \frac{3^k}{3^{k-1} + 1} \right)^{1+\frac{1}{3^{k+l}}} \leq \frac{3^k}{3^{k-1} + 1} \leq \left( 1 + \frac{1}{3^k} \right) \frac{3^k}{3^{k-1} + 1} = 3 - \frac{2}{3^{k-1} + 1},
\]
Hence
\[
\frac{1}{|I'|} \int_{I'} w_k^r(y)dy \leq \frac{3}{2} \frac{3^{k-1} + 1}{3^{k-1}} w_k(x)^r \leq 3w_k(x)^r,
\]
which completes the proof.
3. Extrapolation

Here we follow the extrapolation argument of D. Cruz-Uribe and C. Pérez [3], with some modifications.

**Lemma 3.1.** Assume that for every weight \( w \) and for all \( f \in L^1(M_r w) \),
\[
\| Hf \|_{L^{1,\infty}(w)} \leq A_r \| f \|_{L^1(M_r w)} \quad (1 < r < 2).
\]

Let \( \alpha_r = \frac{r}{2-r} \). There is \( c > 0 \) such that for any weight \( w \) supported in \([0,1]\) one has
\[
\int_0^1 \left( \frac{|Hw|}{(M_\alpha w)^{\alpha_r}} \right)^2 w^{\alpha r} dx \leq c A_r^2 \int_0^1 w dx \quad (1 < r < 2).
\]

**Proof.** Denote \( \beta_r = \frac{r(r-1)}{2-r} \). The numbers \( \alpha_r \) and \( \beta_r \) are chosen in such a way in order to satisfy \( \alpha_r - \beta_r = r \) and \( \alpha_r - \frac{2\beta_r}{r} = 1 \).

Let \( g \geq 0 \). Since
\[
1 \left| \int_I (gw)^r \right| = \left( \frac{1}{w^{\alpha_r}(I)} \int_I (g^{r/\beta_r} w^{\alpha_r}) \right) \frac{w^{\alpha_r}(I)}{|I|},
\]
we get
\[
(3.1) \quad M_r(gw)(x) \leq 2 \left( M^c_{w^{\alpha_r}}(g^{r/\beta_r})(x) M_{\alpha_r}(w)(x) \right)^{1/r},
\]
where \( M^c_v \) means the centered weighted maximal operator with respect to a weight \( v \).

Using the initial assumption on \( H \) along with (3.1), and applying Hölder’s inequality along with the boundedness of \( M^c_v \) on \( L^p(v), p > 1 \), we obtain
\[
\int_{\{|Hf| > 1\}} gw \leq A_r \| f \|_{L^1(M_r(gw))}
\]
\[
\leq 2A_r \int_{\mathbb{R}} \left( |f| M_{\alpha_r}(w)^{\alpha_r} \frac{1}{w^{\alpha_r/2}} \right) \left( M^c_{w^{\alpha_r}}(g^{r/\beta_r})^{1/2} w^{\alpha_r/2} \right) dx
\]
\[
\leq 2A_r \| f \|_{L^2(M_{\alpha_r} w)^{2\alpha_r}/w^{\alpha_r}} \| M^c_{w^{\alpha_r}}(g^{r/\beta_r})^{1/2} \| L^2(w^\alpha)
\]
\[
\leq c A_r \| f \|_{L^2(M_{\alpha_r} w)^{2\alpha_r}/w^{\alpha_r}} \| g \| L^2(w).
\]

Taking here the supremum over all \( g \geq 0 \) with \( \| g \| L^2(w) = 1 \) yields
\[
\| Hf \|_{L^{2,\infty}(w)} \leq c A_r \| f \|_{L^2(M_{\alpha_r} w)^{2\alpha_r}/w^{\alpha_r}}.
\]

By duality, the latter inequality is equivalent to
\[
\| Hf \|_{L^2(w^{\alpha_r}/(M_{\alpha_r} w)^{2\alpha_r})} \leq c A_r \| f/w \|_{L^{2,1}(w)},
\]
where \(L^{2,1}(w)\) is the weighted Lorentz space. It remains to take here \(f = w\) and use that
\[
\|\chi_{[0,1]}\|_{L^{2,1}(w)} = \int_0^{w([0,1])} t^{-1/2} dt = 2w([0,1])^{1/2}.
\]

\[
\square
\]

4. Proof of Theorem 1.1

Our goal is to use the extrapolation Lemma 3.1, assuming (1.1) with a general Orlicz maximal function \(M\Phi\). Hence, we need a relation between \(M\Phi\) and \(M_r\) with possibly good dependence of the corresponding constant on \(r\) when \(r \to 1\). Such a relation was recently obtained in [4] (see Lemma 6.2 and inequality (6.4) there). For the reader convenience we include a proof here.

**Lemma 4.1.** For all \(x \in \mathbb{R}\),
\[
M\Phi f(x) \leq \left(2 \sup_{t \geq \Phi^{1/2}(1/2)} \frac{\Phi(t)}{t^r}\right)^{1/r} M_r f(x) \quad (r > 1).
\]

**Proof.** For any interval \(I \subset \mathbb{R}\),
\[
\int_I \Phi \left(\frac{|f|}{\lambda}\right) = \int_{\{x \in I: |f| < \Phi^{-1}(1/2)\lambda\}} \Phi \left(\frac{|f|}{\lambda}\right) + \int_{\{x \in I: |f| \geq \Phi^{-1}(1/2)\lambda\}} \Phi \left(\frac{|f|}{\lambda}\right)
\leq \frac{|I|}{2} + c_r \int_I (|f|/\lambda)^r dx,
\]
where \(c_r = \sup_{t \geq \Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r}\). Therefore, setting \(\lambda_0 = \left(2c_r \int_I |f|^r\right)^{1/r}\), we obtain \(\frac{1}{|I|} \int_I \Phi(|f|/\lambda_0) dx \leq 1\), which proves (4.1). \(\square\)

It follows easily from (4.1) that
\[
M\Phi f(x) \leq c \left(\sup_{t \geq 1} \frac{\Phi(t)^{1/r}}{t}\right) M_r f(x) \quad (r > 1),
\]
where \(c\) may depend on \(\Phi\) but it does not depend on \(r\).

**Proof of Theorem 1.1**. Suppose, by contrary, that (1.1) holds. Then combining (4.2) with Lemma 3.1 we obtain
\[
\int_0^1 \left(\frac{|Hw|}{(M_\alpha, w)^{\alpha/r/r}}\right)^2 w^{\alpha} dx \leq c \left(\sup_{t \geq 1} \frac{\Phi(t)^{1/r}}{t}\right)^2 \int_0^1 w dx \quad (1 < r < 2).
\]
Set here $r = r_k = 1 + \frac{1}{2^{y+1}+1}$, and $w = w_k$ as constructed in Section 2. Then $\alpha_r = \frac{r_k}{2-r_k} = 1 + \frac{1}{3^{y+1}}$. Applying (2.1) along with Lemma 2.2 yields
\[
\int_0^1 \left( \frac{|Hw_k|}{(M_\alpha w_k)^{\alpha_r/r}} \right)^2 w_k^{\alpha_r/r} \, dx \geq \frac{k^2}{27^{1/2 - 1/r}} \int_{\cup \mathcal{J} \in \mathcal{J}_v} \int_{\mathcal{I} \in \mathcal{I}} \Delta w_k \geq \frac{k^2}{27^{1/2 - 1/r}} \int_0^1 w_k,
\]
and we obtain
\[
(4.3) \quad k \leq c \sup_{t \geq 1} \frac{\Phi(t)^{1/r}}{t}. \tag{4.3}
\]

It remains to estimate the right-hand side of (4.3). Write $\Phi(t) = t \log \log(e^t + t) \phi(t)$, where $\lim_{t \to \infty} \phi(t) = 0$. If $t > e^{r'}$, then
\[
\log \log t = \log(r') + \log \log t^{1/r'} \leq \log(r') + t^{1/r'},
\]
and hence
\[
\frac{\Phi(t)^{1/r}}{t} = \frac{\left( \log \log(e^t + t) \phi(t) \right)^{1/r}}{t^{1/r'}} \leq c \left( \log r' \right)^{1/r} \sup_{t \geq t'} \phi(t)^{1/r}.
\]

On the other hand, if $0 < \delta < 1$, then
\[
\sup_{1 \leq t \leq t'} \frac{\Phi(t)^{1/r}}{t} \leq \sup_{1 \leq t \leq e^{(\log t')^\delta}} \left( \log \log(e^t + t) \phi(t) \right)^{1/r} + \sup_{e^{(\log t')^\delta} \leq t \leq e^{r'}} \left( \log \log(e^t + t) \phi(t) \right)^{1/r} \leq c \left( \log r' \right)^{\delta/r} \sup_{t \geq e^{(\log t')^\delta}} \phi(t)^{1/r}.
\]

Setting $\beta_k = \sup_{t \geq e^{(\log r_k)^\delta}} \phi(t)^{1/r_k}$ and combining both cases, we obtain
\[
\sup_{t \geq 1} \frac{\Phi(t)^{1/r_k}}{t} \leq c \left( \log r' \right)^{\delta/r_k} + \beta_k \left( \log r' \right)^{1/r_k} \leq c \left( \log r' \right)^{\delta/r_k} + \beta_k \left( \log r' \right)^{1/r_k} \leq c(k^\delta + \beta_k).
\]

Since $\beta_k \to 0$ as $k \to \infty$, we arrive to a contradiction with (4.3), and therefore the theorem is proved. \(\square\)

**Remark 4.2.** The following inequality is contained implicitly in [7]:
\[
\lambda w \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} \leq c \log(r') \| f \|_{L^1(M,w)} \quad (r > 1).
\]
The proof of Theorem 1.1 shows that $\log(r')$ here is optimal, namely, it cannot be replaced by $\varphi(r')$ for any increasing $\varphi$ such that $\lim_{t \to \infty} \frac{\varphi(t)}{\log t} = 0$.

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