THE DIRICHLET PROBLEM FOR A FAMILY OF TOTALLY DEGENERATE DIFFERENTIAL OPERATORS

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ABSTRACT. In the framework of the Potential Theory we prove existence and uniqueness for the Perron-Wiener-Brelot solution to the Dirichlet problem related to a family of totally degenerate, in the sense of Bony, differential operators. We also state and prove a Wiener-type criterion and an exterior cone condition for boundary regularity. Our results apply to a wide family of strongly degenerate operators that includes the following example \( \mathcal{L} = i^2 \Delta_x + (x, \nabla_y) - \partial_t \), for \((x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}\).

Keywords: Degenerate Kolmogorov equations; Hypoelliptic equations; Boundary value problem, Perron-Wiener-Brelot solution, Boundary regularity.

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1. Introduction

We consider second-order partial differential operators of the form

\[ \mathcal{L} := i^{2\theta} \sum_{i=1}^{m} \partial_{x_i}^2 + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t, \]

where \( z = (x, t) \in \mathbb{R}^{N+1}, \) \( m, N \) and \( \theta \) are non-negative integers with \( 1 \leq m \leq N \). Moreover, \( b_{ij} \) is a real constant for every \( i, j = 1, \ldots, N \). The standing assumption of this article is:

[H.1] The matrix \( B := (b_{ij})_{i,j=1,\ldots,N} \) has the form

\[
B = \begin{pmatrix}
\circ & \circ & \ldots & \circ & \circ \\
B_1 & \circ & \ldots & \circ & \circ \\
\circ & B_2 & \ldots & \circ & \circ \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\circ & \circ & \ldots & \circ & B_k
\end{pmatrix}
\]
where every block $B_j$ is a $m_j \times m_{j-1}$ matrix of rank $m_j$ with $j = 1, 2, \ldots, k$. Moreover, every $m_j$ is a positive integer such that
\[
1.3 \quad m_0 \geq m_1 \geq \ldots \geq m_k \geq 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_k = N.
\]
We agree to let $m_0 := m$ to have a consistent notation, moreover every $\mathcal{O}$ denotes a block matrix whose entries are zeros.

As we will see in the following Proposition 2.1, the condition $[H.1]$ implies that the operator $\mathcal{L}$ is hypoelliptic. This means that, for every open set $U \subseteq \mathbb{R}^{N+1}$, every function $u \in L^1_{\text{loc}}(U)$, which solves the equation $\mathcal{L}u = f$ in the distributional sense, belongs to $C^\infty(U)$ whenever $f \in C^\infty(U)$. In particular, $u$ is a classical solution to $\mathcal{L}u = f$.

In the framework of the Potential Theory, we address the boundary value problem
\[
1.4 \quad \begin{cases} 
\mathcal{L}u = 0 & \text{in } U, \\
u |_{\partial U} = \varphi, 
\end{cases}
\]
where $U$ is any open subset of $\mathbb{R}^{N+1}$ and $\varphi \in C_c(\partial U)$. The Perron-Weiner-Brelot method provides us with a function $H^U_\varphi$ which is a classical solution to $\mathcal{L}H^U_\varphi = 0$ in $U$.

**Theorem 1.1.** Every open set $U \subseteq \mathbb{R}^{N+1}$ is resolutive, i.e. for every $\varphi \in C_c(\partial U)$ it is defined the Perron-Weiner-Brelot solution $H^U_\varphi$ to the problem (1.4). Moreover, $H^U_\varphi \in C^\infty(U)$ is a classical solution to $\mathcal{L}H^U_\varphi = 0$.

Concerning the boundary value datum, it is well known that solution $H^U_\varphi$ to (1.4) does not attain the prescribed boundary datum at every point of $\partial U$. We say that a point $z_0 \in \partial U$ is $\mathcal{L}$-regular if $H^U_\varphi(z) \rightarrow \varphi(z_0)$ as $z \rightarrow z_0$, for every $\varphi \in C_c(\partial U)$. The second main result of this article is a Wiener criterion for the regularity of a boundary point $z_0 \in \partial U$. Its statement requires some notation. For any fixed $\lambda \in (0, 1)$ and for every $n \in \mathbb{N}$ we consider the following set
\[
1.5 \quad U^\lambda_n(z_0) := \left\{ z \in \mathbb{R}^{N+1} \setminus U : \left( \frac{1}{\lambda} \right)^n \log n \leq \Gamma(z; z_0) \leq \left( \frac{1}{\lambda} \right)^{(n+1) \log(n+1)} \right\} \cup \{z_0\}.
\]
Here $\Gamma$ is the fundamental solution of $\mathcal{L}$, whose explicit expression is given in (2.19). Moreover, $\tilde{R}^1_{U^\lambda_n(z_0)}$ denotes the balayage of the constant function 1 on the set $U^\lambda_n(z_0)$ (see Definition 3.21). With this notation we have

**Theorem 2.1.** Let $U \subset \mathbb{R}^{N+1}$ be an open set and let $z_0 \in \partial U$. Then $z_0$ is a $\mathcal{L}$-regular point, if and only if
\[
1.6 \quad \sum_{n=1}^{\infty} \tilde{R}^1_{U^\lambda_n(z_0)}(z_0) = +\infty.
\]

The proof of Theorem 2.1 is based on the explicit expression of the fundamental solution $\Gamma$ of $\mathcal{L}$ and follows the lines of the work [12] of Kogoj, Lanconelli and Tralli, where the regularity of Kolmogorov operator $(1.1)$ with $\theta = 0$ is studied. In particular the article [12] extends to degenerate Kolmogorov equations the Evans Gariepy’s Wiener test for the heat equation [9], and a regularity criterion proved by Landis in [15], which again holds for the heat equation.

Finally we give a Zaremba type criterion for the regularity of boundary points $z_0 = (x_0, 0)$. It is a sufficient geometric condition for the regularity of a boundary point which relies on the definition of cone $\mathcal{C}_{z_0}$ with vertex at $z_0$ (see Definition 3.5).

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1We indicate with $C_c(\partial U)$ the family of continuous functions on $\partial U$ with compact support
**Proposition 1.3.** Let \( U \) be an open set of \( \mathbb{R}^{N+1} \) and let \( z_0 = (x_0, 0) \in \partial U \). If there exists an exterior cone \( C_{z_0} \) with vertex at \( z_0 \), then \( z_0 \) is \( \mathcal{L} \)-regular.

Note that Proposition 1.3 extends the analogous result proved by Manfredini \([18]\) for the case \( \vartheta = 0 \). The requirement that the time coordinate of \( z_0 \) is \( t_0 = 0 \) is needed because the definition of the cone \( C_{z_0} \) requires a dilation-invariance property of \( \mathcal{L} \) which, in the case \( \vartheta > 0 \), is granted only for \( t_0 = 0 \) (see (2.24) below). On the other hand, the exterior cone criterion proved by Manfredini \([18, \text{Theorem 6.3}]\) applies to every point \( z_0 = (x_0, t_0) \) with \( t_0 \neq 0 \).

Let us briefly discuss Proposition 1.3 in the simplest case of \( m_0 = N, \vartheta = 1 \) and \( B = \emptyset \), that is

\[
\mathcal{L} = t^2 \Delta_x - \partial_t.
\]

In this setting the group of dilations is defined as

\[
\delta(r)(x, t) := (r^3 x, r^2 t), \quad \text{for any } (x, t) \in \mathbb{R}^{N+1} \text{ and any } r > 0,
\]

and following Definition 5.5 the cone of vertex \( z_0 := (x_0, 0) \), height \( T > 0 \) and base \( K \subset \mathbb{R}^N \) is given by

\[
C_{z_0} := \left\{ (x_0 + r^3 x, -r^2 T) : x \in K, 0 \leq r \leq 1 \right\}.
\]

Note that (1.7) can be reduced to the heat equation by the change of the time-scale \( u(x, t) := v(x, t^2/3) \). The classical parabolic cone

\[
\tilde{C}_{z_0} := \left\{ (x_0 + rx, -r^2 T) : x \in \tilde{K}, 0 \leq r \leq 1 \right\},
\]

introduced in \([8]\) by Effros and Kazdan, guarantees the regularity of boundary point \( z_0 \) for the solution \( v \) to problem (1.4) relevant to the heat operator. Inverting the time-scale change of variables defined above \( \tilde{C}_{z_0} \) does coincide with \( C_{z_0} \). However, this simple argument does not apply to ultraparabolic operators of the type (1.1). Hence, the result stated in Proposition 1.3 can not be proved trivially with a time-scale change of variables in the general setting we are dealing with.

We next give some comments about our main results. The first one concerns the uniqueness of the solution to the Dirichlet problem (1.4). A first simple answer to the uniqueness problem plainly follows from the maximum principle (see Corollary 3.16 below). In particular, it implies that if \( u \) and \( v \) belong to \( C(\overline{U}) \), for some bounded open set \( U \), \( u \) and \( v \) are both classical solutions to (1.4), and attain the same values on \( \partial U \), then necessarily agree. This result is however unsatisfactory. Indeed, it is well known that, if we consider the Cauchy-Dirichlet problem for the heat equation in a cylinder, then the solution is uniquely defined by the boundary condition on the parabolic boundary of the cylinder. For this reason, we would expect that only the regular boundary points need to be considered in order to have the uniqueness of the solution to (1.4). Unfortunately, this fact is not true even in the case of the heat equation. Indeed, Lukeš proves in Example 3.2 (D) of \([16]\) that there exist bounded open sets that admit different solutions that agree at every regular boundary point.

The classical Perron method for the Laplace equation relies on the Poisson kernel, which provides us with the solution to the Dirichlet problem on any ball of the Euclidean space. In the more general setting of the abstract Potential Theory the Euclidean balls are replaced by the resolutive sets, that are sets such that the Perron-Wiener-Brelot solution is defined. Specifically, it is assumed that there exists a family of resolutive open sets \( \{U_i\}_{i \in I} \), such that \( \{U_i\}_{i \in I} \) is a basis for the topology of the space. Note that, unlike in the case of the Laplace equation, it is not required that all the boundary points of a resolutive set are regular.
The development of the Potential Theory is simpler in the case of the existence of a basis of regular sets, which are resolutive sets whose all the boundary points are regular. For this reason, even in the case of the heat operator, an effort has been done in order to build a basis of regular sets for the space $\mathbb{R}^{N+1}$. In particular, Bauer first pointed out in [3] that the cones defined for $(x_0, t_0) \in \mathbb{R}^{N+1}$, and $r > 0$ as

$$K_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{N+1} \mid |x - x_0| < t_0 - t < r\},$$

have this property. Later Effros and Kazdan introduce in [8] regular sets that are build as follows. Every set is the union of the cylinder

$$\tilde{Q}_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^{N+1} \mid |x - x_0| < r, t_0 - 2r < t \leq t_0 - r\},$$

and the cone $K_r(x_0, t_0)$. The regularity of the boundary points for the above families of sets is proved by a simple barrier argument, which relies on the fact that, for every point of the lateral boundary of the cone, the spatial component $v_x$ of the outer normal $v = (v_x, v_t) \in \mathbb{R}^{N+1}$ is non zero.

Bony considers in [4] the boundary value problem (1.4) for degenerate operators in the form

$$L = \sum_{j=1}^{m} X_j^2 + Y,$$

where $X_1, \ldots, X_m$ and $Y$ are vector fields defined in $\Omega$, with smooth coefficients, satisfying the Hörmander’s condition [10]

$$\text{Lie}(X_1, \ldots, X_m, Y)(z) = \mathbb{R}^{N+1}, \quad \text{for every } z \in \Omega. \quad (1.9)$$

We recall that $\text{Lie}(X_1, \ldots, X_m, Y)$ is the Lie algebra generated by the vector fields $X_1, \ldots, X_m, Y$, that is the vector space generated by $X_1, \ldots, X_m, Y$ and their commutators. The commutator of two given vector fields $W$ and $Z$ is the vector field defined as:

$$[W, Z] := WZ - ZW.$$

In his work, Bony restricts his study to non totally degenerate operators. This means that, for every $z \in \mathbb{R}^{N+1}$, at least one of the vector fields $X_1(z), \ldots, X_m(z)$ is non zero. The non total degeneracy of the operator $L$ allows Bony to build a family of bounded open regular sets by a general method that relies on a barrier argument. Note that for $\vartheta = 0$ the Bony’s theory applies to the operator $L$. We refer the reader to [6], [11], [13] and [18] for the study of the relevant Dirichlet problem.

We remark that the non total degeneracy of the operator $L$ is a mild requirement. Indeed, from the very definition of commutator it follows that

$$W(z) = 0 \text{ and } Z(z) = 0 \quad \Rightarrow \quad [W, Z](z) = 0,$$

thus

$$X_1(z) = 0, \ldots, X_m(z) = 0, Y(z) = 0 \quad \Rightarrow \quad \text{Lie}(X_1, \ldots, X_m, Y)(z) = \{0\}. \quad (1.10)$$

In particular, if $L$ satisfies the Hörmander’s condition, then at least one of the vector fields $X_1, \ldots, X_m, Y$ is different from zero. Concerning the operator $L$, it can be written in the form (1.8) with

$$X_j := t^\vartheta \partial_{x_j}, \quad j = 1, \ldots, m, \quad Y := \langle Bx, D \rangle - \partial_t,$$

and, as we say in Proposition 2.1, the assumption [H.1] is equivalent to the Hörmander’s condition, even though $L$ is totally degenerate at $t = 0$, for $\vartheta \geq 1$.

In this work we rely on the construction of the Perron-Weiner-Brelot solution to the Dirichlet problem (1.4) based on the existence of a family of resolutive sets, as explained in
the monograph [7] by Constantinescu and Cornea. We recall that a family of resolutive sets for operators in the form (1.1) satisfying the assumption [H.1] has been built by Montanari in [19]. We point out that in the particular case of the heat operator, these resolutive sets agree with the standard cylinders.

Further developments. We aim to extend the theory developed in the present article to more general totally degenerate, according to Bony, differential operators for which neither an explicitly expression of the fundamental solution nor an Harnack-type inequality is available.

This article is organized as follows. In Section 2 the notation adopted throughout the rest of the article and some known result about the operator $\mathcal{L}$ are recalled. Moreover, we also give a detailed proof of the hypoellipticity of $\mathcal{L}$. In section 3 we recall all the notions and results from Potential Theory that we need. We also prove a characterization of boundary regularity in the abstract setting of Potential Theory (Theorem 3.31). In Section 4 we construct the Perron-Wiener-Brelot solution of the problem (1.4) and prove Theorem 1.1. In Section 5 we prove Theorem 1.2 and Proposition 1.3.

2. Notation and preliminaries

In this section we specify the notation adopted throughout the rest of the article and provide some known result about the family of operators we are going to study. We also give a detailed proof of the hypoellipticity of $\mathcal{L}$ and of the existence of its fundamental solution. Here and in the following of this note we write the operator $\mathcal{L}$ in the Hörmander’s form

$$\mathcal{L} = \sum_{j=1}^{m} (t^\theta \partial_{x_j})^2 + \langle Bx, D \rangle - \partial_t = \sum_{j=1}^{m} X_j^2 + Y,$$

with

$$X_j := t^\theta \partial_{x_j}, \quad Y := \langle Bx, D \rangle - \partial_t,$$

for $j = 1, \ldots, m$. As usual in the theory of the Hörmander’s operators, we identify any vector field $X$ with the vector valued function whose entries are the coefficients of $X$, specifically

$$X = \sum_{j=1}^{N} c_j(x,t)\partial_{x_j} + c_0(x,t)\partial_t = (c_1(x,t), \ldots, c_N(x,t), c_0(x,t)).$$

We denote by $A$ and $E(s)$ the $N \times N$ matrices defined as

$$A := \begin{pmatrix} I_m & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \quad E(s) := \exp(-sB).$$

where $I_m$ is the $m \times m$ identity matrix and $s$ is any real number. For every $t, \tau \in \mathbb{R}$ we eventually define the matrix

$$C(\tau, t) := \int_{\tau}^{t} s^{2\theta} E(s)AE^T(s)ds.$$

We recall some known facts about the operator $\mathcal{L}$ in the case $\theta = 0$, which will be useful for the study of the case $\theta \geq 1$.

If $\theta = 0$, specifically when

$$\mathcal{L} := \sum_{j=1}^{m} \partial_{x_j}^2 + \langle Bx, D \rangle - \partial_t,$$
The following statements are equivalent to the condition \([H.1]\):

- **(Hörmander’s condition)**: \(\text{rank} \, \text{Lie} (X_1, \ldots, X_m, Y) (x, t) = N + 1\) for every \((x, t) \in \mathbb{R}^{N+1}\);
- **Ker**\((A)\) does not contain non-trivial subspaces which are invariant for \(B^T\);
- **(C)\)** \(C(t, t) > 0\) for every \(t > \tau\);
- **(Kalman’s rank condition)**: \(\text{rank} \left( \begin{array}{c} A, B, \ldots, B^{N-1} A \end{array} \right) = N\).

The equivalence between i) and ii) was first proved by Hörmander in \([10]\). A detailed proof of the equivalence between i), ii), iii) and \([H.1]\) can be found in \([13]\) (see Proposition A.1, and Proposition 2.1). The equivalence between iii) and iv) was pointed out by Lunardi in \([17]\). We next prove that the above result also holds in the case \(\theta \geq 1\).

**Proposition 2.1.** The following statements are equivalent to the condition \([H.1]\):

- **(Hörmander’s condition)**: \(\text{rank} \, \text{Lie} (X_1, \ldots, X_m, Y) (x, t) = N + 1\) for every \((x, t) \in \mathbb{R}^{N+1}\);
- **Ker**\((A)\) does not contain non-trivial subspaces which are invariant for \(B^T\);
- **(C)\)** \(C(t, t) > 0\) for every \(t > \tau\);
- **(Kalman’s rank condition)**: \(\text{rank} \left( \begin{array}{c} A, B, \ldots, B^{N-1} A \end{array} \right) = N\).

**Proof.** As said above, the assertion is known to be true in the case \(\theta = 0\). Moreover, the constant \(\theta\) doesn’t appear in \([H.1]\), ii) and iv), hence the equivalence between \([H.1]\), ii) and iv) trivially follows from the case \(\theta = 0\).

We next prove that \([H.1]\) is equivalent to ii) for every \(\theta \geq 1\). With this aim, we compare condition ii) with \(\theta = 0\) and condition ii) with \(\theta \geq 1\). In order to distinguish the two cases we set, for \(j = 1, \ldots, m,\)

\[
\begin{align*}
\tilde{X}_0^j &= \partial_x x_j, \\
\tilde{X}_k^j &= \left[ \tilde{X}_j^{k-1}, Y \right], \quad k = 1, \ldots, \kappa.
\end{align*}
\]

Moreover, we let

\[
V_k := \text{span} \left\{ \tilde{X}_1^k, \ldots, \tilde{X}_m^k \right\}, \quad k = 0, \ldots, \kappa.
\]

and we set, for \(j = 1, \ldots, m,\)

\[
\begin{align*}
X_j^0 := t^\theta \partial_x x_j, \\
X_j^k &= \left[ X_j^{k-1}, Y \right], \quad k = 1, \ldots, \kappa.
\end{align*}
\]

A direct computation shows that

\[
\begin{align*}
t^\theta \partial_x Y_j &= t^\theta \left[ \partial_x Y_j \right] + \partial t^\theta - \partial \partial_x Y_j, \quad j = 1, \ldots, m,
\end{align*}
\]

that can be written as follows

\[
X_j^0 = t^\theta \tilde{X}_j^0 + t \partial t^\theta - 1 \tilde{X}_j^0, \quad j = 1, \ldots, m.
\]

By iterating the same argument, we find

\[
X_j^2 = t^\theta \tilde{X}_j^2 + 2 \partial t^\theta - 1 \tilde{X}_j^1 + \theta (\theta - 1) t^\theta - 2 \tilde{X}_j^0, \quad j = 1, \ldots, m.
\]

and, for \(k = 3, \ldots, \kappa,\) and \(j = 1, \ldots, m,\)

\[
X_j^k = t^\theta \tilde{X}_j^k + k \partial t^\theta - 1 \tilde{X}_j^{k-1} + \cdots + \theta (\theta - 1) \cdots (\theta - k + 1) t^\theta - k \tilde{X}_j^0.
\]

Note that the last coefficient vanishes whenever \(k > \theta\).

We are now ready to show that the Hörmander’s condition ii) is satisfied by the system of vector fields \(\{X_1^k, \ldots, X_m^k, Y\}\) in the set \(\{t \neq 0\}\). Indeed, we easily see that, in this case,

\[
\text{span} \left\{ \tilde{X}_1^0, \ldots, \tilde{X}_m^0 \right\} = \text{span} \left\{ X_1^0, \ldots, X_m^0 \right\}.
\]
Moreover (2.11) implies that
\begin{equation}
\text{span} \left\{ \tilde{X}^0_i, \ldots, \tilde{X}^0_m, \tilde{X}^1_i, \ldots, \tilde{X}^1_m \right\} = \text{span} \left\{ X^1_i, \ldots, X^1_m \right\},
\end{equation}
for every \( t \neq 0 \). By the same reason, using the above assertions and (2.12), we conclude that \( \text{Lie} \left\{ X^k_1, \ldots, X^k_m, Y \right\} \) agrees with \( \text{Lie} \left\{ \tilde{X}^k_i, \ldots, \tilde{X}^k_m, Y \right\} \) whenever \( t \neq 0 \).

We are left with the set \( \{ t = 0 \} \). In this case we use (2.12) with \( k = \theta \) and we find
\begin{equation}
X^\theta_j(x, 0) = \theta! \tilde{X}_j^0, \quad j = 1, \ldots, m,
\end{equation}
for every \( x \in \mathbb{R}^N \). This means that \( \tilde{X}_j^0 \) belongs to \( \text{Lie} \left\{ X^k_1, \ldots, X^k_m, Y \right\} \) computed at \( t = 0 \). Hence, \( \text{Lie} \left\{ X^k_1, \ldots, X^k_m, Y \right\} \) contains \( \text{Lie} \left\{ \tilde{X}^k_i, \ldots, \tilde{X}^k_m, Y \right\} \) and the Hörmander’s condition (1.9) is satisfied also in the set \( \{ t = 0 \} \). This concludes the proof of the equivalence between [H.1] and ii).

We next prove that ii is equivalent to iii). We follow the Hörmander’s argument. We first note that the matrix \( E(s)AE^T(s) \) is non negative, for every \( s \in \mathbb{R} \). Then \( C(\tau, t) \geq 0 \) whenever \( r \geq \tau \). Moreover, the function \( t \mapsto \langle C(\tau, t)\xi, \xi \rangle = 0 \) is non-decreasing. We claim that the following assertions are equivalent:

1. there exists a \( t_0 > \tau \) such that \( \langle C(\tau, t_0)\xi, \xi \rangle = 0 \);
2. \( \langle C(\tau, t)\xi, \xi \rangle = 0 \) for every \( t > \tau \);
3. \( A(B^T)^k\xi = 0 \), for every non-negative integer \( k \).

We first prove that 1. implies 2. Assume that there exists a \( t_0 > \tau \) and a vector \( \xi \in \mathbb{R}^N \) such that \( \langle C(\tau, t_0)\xi, \xi \rangle = 0 \). Then \( \langle C(\tau, t)\xi, \xi \rangle = 0 \) for every \( t \in [\tau, t_0] \). From the definition of \( C(\tau, t) \) (2.4), it follows that
\begin{equation}
s^{2\theta}(AE^T(s)\xi, E^T(s)\xi) = 0, \quad \text{for every} \quad s \in [\tau, t_0],
\end{equation}
then
\begin{equation}
\left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^{k+2\theta} A(B^T)^k \right) \xi = 0, \quad \text{for every} \quad s \in [\tau, t_0],
\end{equation}
which implies the assertion 2. The implications 3. \( \Rightarrow \) 2. \( \Rightarrow \) 1. are trivial and are omitted.

The proof of the equivalence between ii) and iii) is a direct consequence of the fact that
\begin{equation}
V := \left\{ \tilde{\xi} \in \mathbb{R}^N \mid A(B^T)^k\xi = 0 \text{ for every non-negative integer } k \right\}
\end{equation}
is the greatest subspace of \( \text{Ker}(A) \) which is \( B^T \)-invariant. This completes the proof of Proposition 2.1. \( \square \)

We emphasize that the condition iii) of Proposition 2.1 is very important in our setting. Indeed, by using the Fourier transform we find the explicit expression of the fundamental solution of \( \mathcal{L} \). Indeed, for every \( z = (x, t), \xi = (\xi, \tau) \in \mathbb{R}^{N+1} \) with \( t > \tau \) we have
\begin{equation}
\Gamma(z; \xi) := \frac{(4\pi)^{-N/2}}{\sqrt{\det C(\tau, t)}} \exp \left( -\frac{1}{4} \langle C^{-1}(\tau, t) (x - E(t - \tau)\xi), x - E(t - \tau)\xi \rangle \right).
\end{equation}
Here and in the following we agree to set \( \Gamma(x, t; \xi, \tau) = 0 \) whenever \( t \leq \tau \). The expression (2.19) was first obtained by Kuptsov under a condition equivalent to iv), and used by Montanari in [19].

We spend few words about some geometric aspects related to the operator \( \mathcal{L} \). In the article [13] the composition law
\begin{equation}
(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau) \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1},
\end{equation}
was introduced and it was proved that $\mathbb{K} := (\mathbb{R}^{N+1}, \circ)$ is a non-commutative Lie group with zero element $(0, 0)$ and inverse element given by
\begin{equation}
(x, t)^{-1} = (-E(-t)x, -t) \quad \forall (x, t) \in \mathbb{R}^{N+1}.
\end{equation}
Moreover, the operator $\mathcal{L}$ with $\theta = 0$ is invariant with respect to the left translation (2.20). We refer the reader to the monograph [2] for a general presentation of the theory of Lie groups and to [1] for a survey of results on the operator $\mathcal{L}$. However, the operator $\mathcal{L}$ is not translation invariant as $\theta \geq 1$ (see Proposition 1.2.13 in [2]). Nevertheless, the matrix (2.3) will be used also for $\theta \geq 1$ in order to define a basis of resolutive sets and to state a Harnack inequality.
For every $r > 0$, we denote with $\delta(r)$ the automorphism on $\mathbb{R}^{N+1}$ given by
\begin{equation}
\delta(r) := (\delta_0(r), r^2), \quad \forall r > 0,
\end{equation}
where
\begin{equation}
\delta_0(r) := r^{2\theta} \text{diag}(r^{m_0}, r^{3m_1}, \ldots, r^{2\kappa m_\kappa}),
\end{equation}
and $I_{mj}$ denotes the identity matrix in $\mathbb{R}^{m_j}$, for $j \in \{0, \ldots, \kappa\}$. The operator $\mathcal{L}$ is homogeneous of degree two with respect to the dilation $\delta$, i.e.
\begin{equation}
\mathcal{L} \circ \delta(r) = r^2 \delta(r) \circ \mathcal{L}, \quad \forall r > 0.
\end{equation}
Throughout the sequel we indicate with $Q$ the homogeneous dimension of $\mathbb{R}^N$ with respect to $(\delta_0(r), r > 0)$, i.e.
\begin{equation}
r^Q = \text{det} \delta_0(r), \quad \text{for every} \quad r > 0.
\end{equation}
Furthermore, $Q + 2$ will be said homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(\delta(r))_{r > 0}$.
With this notation, we are now in position to introduce the cylindrical sets with basis at $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ previously used in [19]. For every $T > t_0$ and $r > 0$, we let
\begin{equation}
Q_{r,T}(z_0) := \left\{ z \in \mathbb{R}^{N+1} : t_0 < t < T, \left| \delta_0 \left( \frac{1}{\sqrt{T}} \right) (E(-t)x, E(-t_0)x_0) \right| < 1 \right\},
\end{equation}
where with $|\cdot|$ we indicate the standard Euclidean norm on $\mathbb{R}^N$, and we denote by $\partial_P Q_{r,T}(z_0)$ its parabolic boundary
\begin{equation}
\partial_P Q_{r,T}(z_0) = \partial Q_{r,T}(z_0) \setminus \left\{ z = (x, t) \in \mathbb{R}^{N+1} : \left| \delta_0 \left( \frac{1}{\sqrt{T}} \right) (E(-T)x, E(-t_0)x_0) \right| < 1 \right\}.
\end{equation}
Here $\partial Q_{r,T}(z_0)$ is the topological boundary of $Q_{r,T}(z_0)$. It has been shown by Montanari, in [19], that, for every $Q_{r,T}(z_0)$ and for every $\varphi \in C(\partial Q_{r,T}(z_0), \mathbb{R})$, there exists a unique solution $u \in C^\infty(Q_{r,T}(z_0), \mathbb{R})$ to the problem
\begin{equation}
\left\{ \begin{array}{l}
\mathcal{L} u = 0, \quad \text{in} \ Q_{r,T}(z_0), \\
u |_{\partial P Q_{r,T}(z_0)} = \varphi.
\end{array} \right.
\end{equation}
Moreover, again in [19, Theorem 3.1] a Harnack inequality for positive solution to $\mathcal{L} u = 0$ has been proved.
We introduce some further notation. For every $\beta \in \mathbb{R}$, $0 < \alpha < \gamma < 1$ and $\nu \in (0, \nu_0)$, with $\nu_0 > 0$ depending on $\alpha$ and of the coefficients of the matrix $B$, and for every $\xi \in \mathbb{R}^N$ let us define the following sets
\begin{align*}
Q^+ & := Q_{r,T}((\beta + 1)r)(\xi, (\beta - 1)r) \cap \{ \beta - \gamma \leq t/r \leq \beta - \alpha \}, \\
Q^- & := Q_{r,T}((\beta + 1)r)(\xi, (\beta - 1)r) \cap \{ \beta + \alpha \leq t/r \leq \beta + \gamma \}.
\end{align*}
We state the following Harnack inequality.
Theorem 2.2. There exists a non-negative constant $M(\alpha, \gamma, \beta, \nu) < \infty$ such that for all $r > 0$

\begin{equation}
\max_{\mathcal{Q}} u \leq M \min_{\mathcal{Q}} u,
\end{equation}

for all non-negative $u \in C^\infty(\mathcal{Q}_{vr,(\beta+1)r}, (\xi, (\beta-1)r), \mathbb{R})$ satisfying $\mathcal{L}u = 0$ in $Q_{vr,(\beta+1)r} (\xi, (\beta-1)r)$.

3. Recalls of potential theory

We begin recalling some definitions and results from Potential Theory. We adopt the notation of the monograph [7] by Constantinescu and Cornea. Let us indicate with $(E, d_E)$ a metric space, locally connected and locally compact. Moreover, denoting with $\mathcal{I}$ the topology generated by the metric $d_E$ on $E$, we assume that $(E, \mathcal{I})$ has a countable basis of open sets.

Definition 3.1 (Sheaf of functions). Suppose we are given, for every open set $U \in \mathcal{I}$, a family $\mathcal{H}(U)$ of extended real valued functions $u : U \rightarrow [-\infty, \infty]$. We say that the map

$$
\mathcal{H} : U \mapsto \mathcal{H}(U),
$$

is a sheaf of functions on $E$ if the following properties hold:

(i) if $U_1, U_2 \in \mathcal{I}$ with $U_1 \subseteq U_2$ and $u \in \mathcal{H}(U_2)$ then $u_{|U_1} \in \mathcal{H}(U_1)$;

(ii) if $(U_i)_{i \in I} \in \mathcal{I}$ and $u : \bigcup_{i \in I} U_i \rightarrow [-\infty, \infty]$ is such that $u_{|U_i} \in \mathcal{H}(U_i) \forall i \in I$, then $u \in \mathcal{H}(\bigcup_{i \in I} U_i)$.

A sheaf of functions $\mathcal{H}$ on $E$ will be called harmonic if, for every $U \in \mathcal{I}$, $\mathcal{H}(U)$ is a linear subspace of $C(U, \mathbb{R})$. A sheaf of functions $\mathcal{U}$ on $E$ will be said hyperharmonic if, for any $U \in \mathcal{I}$, the family $\mathcal{U}(U)$ is a convex cone of lower semi-continuous, lower finite functions.

Note that if $\mathcal{U}$ is a hyperharmonic sheaf on $E$, then the map

$$
\mathcal{H}_U : U \mapsto \mathcal{U}(U) \cap (-\mathcal{U}(U)), \quad \forall U \in \mathcal{I},
$$

is a harmonic sheaf on $E$.

Throughout the sequel we indicate with $\mathcal{H}$ (resp. $\mathcal{U}$) a harmonic (resp. hyperharmonic) sheaf on $E$ and $\mathcal{H}_U$-functions (resp. $\mathcal{U}$-functions) will be called harmonic (resp. hyperharmonic). Moreover, a function $u \in (-\mathcal{U})$ will be called hypoharmonic.

Let $U \subseteq E$ be an open subset of $E$ and let $\varphi : U \rightarrow (-\infty, +\infty]$ be a lower semi-continuous function. Then, for any open set $V \subset U$, with compact closure and non-empty boundary, and for any non-negative Radon measure $\mu$ on $\partial V$ we define

\begin{equation}
\int_{\partial V} \varphi \, d\mu := \sup \left\{ \int_{\partial V} g \, d\mu : g \in C(\partial V, \mathbb{R}), g \leq \varphi \text{ on } \partial V \right\}.
\end{equation}

Since $\varphi$ is lower finite and $\partial V$ is compact, $\varphi$ is bounded from below on $\partial V$. Hence the set on the righthand side in (3.1) is not empty. We can give the following definition.

Definition 3.2 ($\mathcal{H}$-sweeping). Let $V$ be an open set of $E$, with compact closure and non-empty boundary. Let us consider a family $\mu^V = \{\mu^V_x\}_{x \in V}$ of non-negative Radon measures on $\partial V$. The family $\mu^V$ will be called a sweeping on $V$.

For any lower semi-continuous function $\varphi : \partial V \rightarrow (-\infty, +\infty]$ we will denote with $\mu_x^V$ the function

$$
\mu_x^V : V \rightarrow (-\infty, +\infty],
$$

$$
x \mapsto \mu_x^V(x) := \int_{\partial V} \varphi \, d\mu^V_x.
$$
If $H$ is a harmonic sheaf on $E$, then the sweeping $\mu^V$ will be called $H$-sweeping if:

(i) $\forall \varphi \in C(\partial V, \mathbb{R})$ the function $\mu^V_\varphi$ is a $H$-function;

(ii) for any $H$-function $h$ defined on an open neighbourhood of $V$ we have $\mu^V_h = h$ on $V$.

We will say that the family

$$\Omega := \{ \mu^V_i = \{ \mu^V_{x_i} : i \in I \} \},$$

(3.2)

is a sweeping system on $E$ if $\{ V_i : i \in I \}$ is a basis for $E$ of relatively compact sets with non-empty boundary and for any $i \in I$ $\mu^V_i$ is a sweeping on $V_i$.

If $H$ is a harmonic sheaf on $E$, then a sweeping system $\Omega$ is called $H$-sweeping system on $E$ if $\mu^V_i$ is a $H$-sweeping on $V_i$ for every $i \in I$.

A hyperharmonic sheaf can be defined starting from a sweeping systems. Indeed, let us consider on $E$ a sweeping system $\Omega$ as defined in (3.2) and give the following definition.

**Definition 3.3.** Let $U \subseteq E$. A l.s.c. function $u : U \to (-\infty, +\infty]$ will be said $\Omega$-hyperharmonic if for any $i \in I$ such that $V_i \Subset U$ we have that $\mu^V_u \leq u$ on $V_i$, i.e.

$$u(x) \geq \int_{\partial V_i} u \, d\mu^V_{x_i}, \quad \forall x \in V_i.$$  

(3.3)

The function $u$ will be said locally $\Omega$-hyperharmonic if there exists an open covering $\mathcal{M}$ of $U$ such that, $\forall W \in \mathcal{M}$, $U \cap W$ is $\Omega$-hyperharmonic on $W$.

**Definition 3.4 (Hyperharmonic sheaf generated by $\Omega$).** Let $\Omega$ be a sweeping system on the space $E$. We call hyperharmonic sheaf generated by $\Omega$ the map $\mathcal{U}$ defined as follows

$$\mathcal{U} : T \ni U \mapsto \mathcal{U}(U) := \{ u : u \text{ is locally } \Omega\text{-hyperharmonic on } U \}.$$ 

Moreover, given the hyperharmonic sheaf $\mathcal{U}$ defined above, we call harmonic sheaf generated by $\Omega$ the harmonic sheaf given by

$$\mathcal{H}_\mathcal{U} : U \mapsto \mathcal{U}(U) \cap (-\mathcal{U}(U)), \quad \forall U \in \mathcal{T}.$$ 

3.1. **Resolutive sets.** Throughout the rest of this section $\mathcal{U}$ will denote a given hyperharmonic sheaf on the space $E$. Let us give the following definition.

**Definition 3.5 (MP-set).** An open set $U \subseteq E$ will be called a minimum principle set, in short a $MP$-set, if every $\mathcal{U}$-function $u$ which is non-negative outside the intersection with $U$ of a compact set $K \subseteq E$ and

$$\liminf_{x \to y} u(x) \geq 0, \quad \forall y \in \partial U,$$

is non-negative on $U$.

**Remark 3.6.** We point out that, if in the previous definition we are considering an open set $U$ with compact closure, we drop the condition that a $\mathcal{U}$-function $u$ is non-negative outside the intersection with $U$ of a compact set $K \subseteq E$.

Let $\mathcal{U}$ be the hyperharmonic sheaf on $E$, $U \subseteq E$ be a $MP$-set and let $\varphi : \partial U \to (-\infty, +\infty]$. Let us consider the set

$$\overline{\mathcal{U}}_{\varphi} := \left\{ u \in \mathcal{U}(U) : \{ u < 0 \} \text{ is a compact, possibly empty, subset of } U \right\}.$$

$$\liminf_{U \ni x \to y} u(x) \geq \varphi(y) \forall y \in \partial U.$$
The sets $\mathcal{U}_f^U$ and $\mathcal{U}_f^{-U} = -\mathcal{U}_f^{-U}$ will be called respectively the set of upper-functions and the set of lower-function. We will call upper-solution and lower-solution the functions:

$$H_f^U := \inf \mathcal{U}_f^U, \quad H_f^{-U} := \sup \mathcal{U}_f^{-U}.$$

The next proposition is a straightforward consequence of the definition of upper and lower solution.

**Proposition 3.7.** Let $\varphi_1, \varphi_2 : \partial U \to \mathbb{R}, \alpha \in \mathbb{R}, \alpha > 0$. Then:

(i) $\varphi_1 \leq \varphi_2 \Rightarrow H_f^{\varphi_1} \leq H_f^{\varphi_2}, H_f^{-\varphi_1} \leq H_f^{-\varphi_2}$

(ii) $H_f^{\varphi_1 + \varphi_2} = H_f^{\varphi_1} + H_f^{\varphi_2}$, $H_f^{\varphi_{1 \varphi_2}} \geq H_f^{\varphi_1} + H_f^{\varphi_2},$ whenever the sums are defined.

(iii) $H_f^{\alpha \varphi_1} = \alpha H_f^{\varphi_1}, \overline{H_f^{-\varphi_1}} = \alpha H_f^{-\varphi_1}, \overline{H_f^{-\alpha \varphi_1}} = -\alpha H_f^{-\varphi_2}.$

(iv) $\varphi_1 \geq 0 \Rightarrow H_f^{\varphi_1}, H_f^{-\varphi_1} \geq 0.$

Let us given now a crucial definition.

**Definition 3.8 (Resolutive set and PWB solution).** A function $\varphi : \partial U \to [-\infty, \infty]$ is called resolutive if the functions $H_f^{\varphi}, H_f^{-\varphi}$ are $\mathcal{H}_f^U$-functions and coincide. In this case we set $H_f^{U} := H_f^{\varphi}$ and we say that $H_f^{U}$ is the generalized solution in the sense of Perron-Weiner-Brelot (in short PWB solution).

An open set $U$ of $E$, with non-empty boundary, is said to be a resolutive set (with respect to $\mathcal{U}$) if every $\varphi \in C_c(\partial U, \mathbb{R})$ is resolutive.

If $U$ is a resolutive set, for any $x \in U$, the map

$$C_c(\partial U, \mathbb{R}) \ni \varphi \mapsto H_f^{U}(x) \in \mathbb{R},$$

is a linear and non-negative functional, hence by the Riesz Theorem, for every $x \in U$, there exists a suitable Radon measure $\mu_x^U$ on $\partial U$ such that

$$H_f^{U}(x) = \int_{\partial U} \varphi(y) \, d\mu_x^U(y).$$

The measure $\mu_x^U$ is called the $\mathcal{H}_f^U$-harmonic measure related to $U$ and $x$. Clearly the family $\mu^U := \{\mu_x^U\}_{x \in U}$ is a sweeping on $U$ and, if $\overline{U}$ is compact, the family $\mu^U$ is a $\mathcal{H}_f^U$-sweeping on $U$.

### 3.2. Harmonic spaces and $\mathcal{B}$-harmonic spaces.

Let us begin defining a harmonic space.

**Definition 3.9 (Harmonic space).** The couple $(E, \mathcal{U})$, where $\mathcal{U}$ is a hyperharmonic sheaf on $E$, is called a harmonic space if the following axioms are satisfied:

(A1)(Positivity): For every $x \in E$ there exists a $\mathcal{H}_f^U$-function, defined in a neighbourhood of $x$, that does not vanish at $x$.

(A2)(Bauer convergence property): Let $(u_n)_{n \in \mathbb{N}}$ be a monotone increasing sequence of $\mathcal{H}_f^U$-functions on an open set $U$ of $E$. Then

$$u := \lim_{n \to +\infty} u_n,$$

is a $\mathcal{H}_f^U$-function whenever it is locally bounded.

(A3)(Resolutivity): The resolutive sets (with respect to $\mathcal{U}$) form a basis for the topology $\mathcal{T}$ on $E$. 
(A4)(Completeness): A lower semi-continuous, lower finite function \( u \) on an open set \( U \) of \( E \) belongs to \( \mathcal{U}(U) \) if, for any relatively compact with non-empty boundary resolutive set \( V \) (with respect to \( \mathcal{U} \)) such that \( \overline{V} \subset U \), we have \( \mu_u^V \leq u \) on \( V \), that is
\[
 u(x) \geq \int_{\partial V} u \, d\mu_x^V, \quad \forall x \in V,
\]
where \( \mu_u^V \) is given by the sweeping constructed with the basis of resolutive sets.

Remark 3.10. In the particular case the hyperharmonic sheaf \( \mathcal{U} \) is generated by a sweeping system \( \Omega \) (see Definition 3.4), the axiom (A4) of Completeness, is trivially satisfied.

In our setting, by using the Harnack inequality for the non-negative solutions to \( Lu = 0 \) given in Theorem 2.2, we will prove the following property which, in turn, implies the Bauer convergence property (A2).

(A2’)(Doob convergence property): If \( \{u_n\}_{n \in \mathbb{N}} \) is a monotonically increasing sequence of \( \mathcal{H}_\mathcal{U} \)-functions on an open set \( U \subset E \) such that the set
\[
 \left\{ x \in U \mid \sup_{n \in \mathbb{N}} u_n(x) < \infty \right\},
\]
is dense in \( U \), then
\[
 u := \lim_{n \to \infty} u_n,
\]
is a \( \mathcal{H}_\mathcal{U} \)-function on \( U \).

Throughout the sequel we indicate with \( (E, \mathcal{U}) \) a harmonic space.

Definition 3.11 (Superharmonic function). A hyperharmonic function \( u \) on a harmonic space \( (E, \mathcal{U}) \) is called superharmonic if, for any relatively compact resolutive set \( V \), the function \( \mu_u^V \) is harmonic. A hypoharmonic function \( u \) will be said subharmonic if \( -u \) is superharmonic.

Remark 3.12. Every superharmonic function \( u \) is finite on a dense subset of its domain. Moreover, if the harmonic sheaf \( \mathcal{H}_\mathcal{U} \) has the Doob convergence property (A2’), then hyperharmonic functions, which are finite on a dense set, are superharmonic.

Definition 3.13 (Potential). A non-negative superharmonic function \( p \) for which any non-negative harmonic minorant vanishes identically is called a potential.

We refer to [7] for some properties of superharmonic functions and potentials. Let us give the following definition.

Definition 3.14 (\( \mathcal{B} \)-harmonic space). A harmonic space \( (E, \mathcal{U}) \) will be called \( \mathcal{B} \)-harmonic space if for any \( x \in E \) there exists a potential \( p \) on \( E \) such that \( p(x) > 0 \).

The following result holds (see [7, Proposition 2.3.2]).

Proposition 3.15. Let \( (E, \mathcal{U}) \) be a harmonic space. The following conditions are equivalent:
(i) \( E \) is a \( \mathcal{B} \)-harmonic space;
(ii) the set \( \mathcal{P}_c \) of finite continuous potentials on \( E \) such that any \( p \in \mathcal{P}_c \) is harmonic outside a compact set, separates the points of \( E \);
(iii) the set of non-negative superharmonic functions on \( E \) separates the points of \( E \);
(iv) for any relatively compact, resolutive set \( V \) and for any \( x \in V \), there exists a non-negative, finite, continuous superharmonic function \( u \) on \( E \) such that
\[
 \int_{\partial V} u \, d\mu_x^V < u(x).
\]
As a consequence of the Proposition above we have the following corollary [7, Corollary 2.3.3].

**Corollary 3.16.** Every open set of a $\mathcal{B}$-harmonic space is an MP-set, according to Definition 3.5.

$\mathcal{B}$-harmonic spaces are really important since the following result holds true [7, Theorem 2.4.2].

**Theorem 3.17.** Any open set of a $\mathcal{B}$-harmonic space with non-empty boundary is resolutive.

The consequence of Theorem 3.17 is that given an open set $U$ of a $\mathcal{B}$-harmonic space the $\mathcal{H}_U$-Dirichlet problem

\[
\begin{cases}
u \in \mathcal{H}_U(U), \\
u|_{\partial U} = \varphi, \quad \forall \varphi \in C_c(\partial U, \mathbb{R}),
\end{cases}
\]

admits a solution $H^U_\varphi$ in the sense of Perron-Wiener-Brelot.

In general, we cannot expect a good behaviour of $H^U_\varphi$ at the boundary points of $U$. In the following section we describe the conditions under which the boundary datum $\varphi$ in (3.4) is attained by the generalized solution $H^U_\varphi$.

### 3.3. Boundary regularity.

Let us give the following definitions.

**Definition 3.18 ($\mathcal{H}_U$-regular point).** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space and let $U$ be an open subset of $E$ with non-empty boundary. A point $x_0 \in \partial U$ is said $\mathcal{H}_U$-regular if

\[
\lim_{x \to x_0} H^U_\varphi(x) = \varphi(x_0), \quad \forall \varphi \in C_c(\partial U, \mathbb{R}).
\]

A point $x_0 \in \partial U$ which is not regular is called $\mathcal{H}_U$-irregular.

**Definition 3.19 (Barrier function).** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space and let $U$ be an open set of $E$ with non-empty boundary, $x_0 \in \partial U$ and let $V$ be an open neighbourhood of $x_0$.

We say that a function $\omega \in \mathcal{U}(V \cap U)$ is a barrier at $x_0$ if:

(i) $\omega > 0$ on $U \cap V$;

(ii) $\lim_{x \to x_0} \omega(x) = 0$.

The first condition for a boundary point to be regular is having a barrier function [7, Proposition 2.4.7].

**Proposition 3.20.** Let $U$ be a resolutive subset of a $\mathcal{B}$-harmonic space $(E, \mathcal{U})$. Then, any boundary point $x_0$ which possesses a barrier is regular.

In order to state some geometrical characterization of regular point we need some further notation. The following notion was introduced by Brelot [5].

**Definition 3.21 (Reduit and balayage function).** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space. For any non-negative function $u$ on $E$ and any subset $A$ of $E$ we call the reduit of $u$ on $A$ the following function

\[
R^u_A := \inf \left\{ v \in \mathcal{U}(E) : v \geq 0 \text{ on } E \text{ and } v \geq u \text{ on } A \right\}.
\]

We call balayage of $u$ on $A$ the lower semi-continuous regularization of the reduit function of $u$ on $A$, that is

\[
\hat{R}^u_A(x) := \lim_{y \to x} R^u_A(y), \quad \forall x \in E.
\]
We list also some further properties of balayage and reduit function which will turn useful in the following parts of the article.

**Proposition 3.22.** For any subsets $A$ and $B$ of $E$ and for every non-negative function $u$ and $v$ on $E$ the following properties hold

(i) $R^u_A = u$ on $A$;
(ii) if $A \subseteq B$ and $u \leq v$ we have $R^u_A \leq R^v_B$;
(iii) $\tilde{R}^u_A = R^u_A$ if $A$ is open;
(iv) $R^u_{A \cup B} + \tilde{R}^u_{A \cap B} \leq \tilde{R}^u_A + \tilde{R}^u_B$, and $R^u_{A \cup B} + R^u_{A \cap B} \leq R^u_A + R^u_B$.

For a proof of the last property in the proposition above we refer to [7, Theorem 4.2.2].

**Proposition 3.23.** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space and let $u$ be a non-negative superharmonic function on $E$ and $A \subset E$. Then, $R^u_A$ is harmonic on $E \setminus \overline{A}$ and $\tilde{R}^u_A$ and $\tilde{R}^u_A$ coincide on $E \setminus \overline{A}$.

**Proposition 3.24.** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space. The balayage of any non-negative superharmonic function on $E$, on any compact subset of $E$, is a potential.

For a proof of the previous propositions we refer to [7, Proposition 5.3.1 and Proposition 5.3.5].

**Definition 3.25 (Polar set).** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space and let $U$ be an open set of $E$. A set $P$ is said polar set in $U$ if there exists a non-negative superharmonic function $p$ on $U$ which is equal $+\infty$ at least on $U \cap P$. In this case, we say that the function $p$ is associated to $P$.

Even though most of the results stated below are proved in [5], we give here their proofs, since our axiomatic setting is slightly different than the one adopted by the author of [5].

**Proposition 3.26.** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space, $P$ a polar set in $E$ and $u$ a non-negative function on $E$. Then, the reduit $R^u_P$ is zero on a dense subset of $E$. Moreover, $\tilde{R}^u_P \equiv 0$.

**Proof:** Suppose that $P$ is a polar set and consider its associated function $p$. Then, $p = +\infty$ on $P$. Choose any point $x_0$ where $p(x_0) < +\infty$. We have that $\lambda p \geq u$ on $P$, for every $\lambda > 0$, moreover $p \geq 0$ outside $P$. Then,

$$\lambda p \geq R^u_P, \quad \forall \lambda > 0.$$  

(3.7)

Since (3.7) holds also in $x_0$ and $p(x_0)$ is finite, taking the infimum on $\lambda > 0$ we get that $R^u_P(x_0) = 0$. By the previous argument we have that $R^u_P$ is zero on every point in which $p$ is finite. Since $p$ is a superharmonic function, it is finite on a dense subset of $E$ (see Remark 3.12). Then $R^u_P = 0$ on a dense subset of $E$. Form this fact it follows that $\tilde{R}^u_P \equiv 0$.  

As an immediate consequence of Proposition 3.26 and 3.22 we have

**Corollary 3.27.** Let $(E, \mathcal{U})$ be a $\mathcal{B}$-harmonic space. If $P$ is a polar set of $E$, then $\tilde{R}^u_{A \cup P} = \tilde{R}^u_{A}$, for any subset $A$ of $E$ and for any non-negative function $u$ on $E$.

The following definition will be used to give a further characterization of regular points.

**Definition 3.28 (Thinnes).** Let $A$ be a subset of a $\mathcal{B}$-harmonic space $(E, \mathcal{U})$ and let us consider a point $x_0 \notin A$. We say that $A$ is thin at $x_0$ if either $x_0 \notin \overline{A}$ or $x_0 \in \overline{A}$ and there exists a non-negative superharmonic function $u$ on $E$ such that

$$u(x_0) < \liminf_{A \ni x \to x_0} u(x).$$  

(3.8)

Let us consider a point $x_0 \in A$. We say that $A$ is thin at its point $x_0$ if $\{x_0\}$ is a polar set in $E$ (according to Definition 3.25) and $A \setminus \{x_0\}$ is thin at $x_0$. 

The following Proposition holds.

**Proposition 3.29.** Let \((E, \mathcal{H})\) be a \(\mathcal{H}\)-harmonic space, let \(A\) be any subset of \(E\), \(x_0 \notin A\) and let \(w\) be a positive function on \(E\), finite and continuous at \(x_0\). Then, \(A\) is thin in \(x_0\) if and only if there exists an open neighbourhood \(V\) of \(x_0\) such that

\[
R^w_{A \cap V}(x_0) < w(x_0) \quad \text{and} \quad \tilde{R}^w_{A \cap V}(x_0) < w(x_0).
\]

**Proof.** Let us begin noticing that the second inequality in (3.9) is a straightforward consequence of the first one. Hence, it is enough to prove the first condition in (3.9).

If \(A\) is thin in \(x_0\) and \(x_0 \notin \overline{T}\) the inequalities trivially follow. Indeed, chosen an open neighbourhood such that \(A \cap V = \emptyset\) we have that \(R^w_{A \cap V} \equiv 0\). Since \(w(x_0) > 0\) inequalities (3.9) follow.

Let us suppose that \(x_0 \in \overline{A}\). By Definition 3.28 there exists a non-negative superharmonic function \(u\) such that (3.8) holds, that is

\[
u(x_0) < \liminf_{A \ni x \to x_0} u(x).
\]

Since \(w(x_0) > 0\) let us choose \(\lambda > 0\) such that,

\[
u(x_0) < \lambda w(x_0) < \liminf_{A \ni x \to x_0} u(x).
\]

Since \(w\) is continuous in \(x_0\) we have that

\[
\liminf_{A \ni x \to x_0} (u(x) - \lambda w(x)) > 0.
\]

Then, there exists a neighbourhood \(V\) of \(x_0\) such that \(u(x) > \lambda w(x)\) for every \(x \in A \cap V\). Hence, \(u \geq R^w_{A \cap V}\). In particular for \(x = x_0\), we have

\[
R^w_{A \cap V}(x_0) \leq \frac{u(x_0)}{\lambda} < w(x_0),
\]

which gives the first inequality in (3.9). Let us prove the vice versa. Since \(R^w_{A \cap V} \geq \tilde{R}^w_{A \cap V}\), it is enough to show that the second condition in (3.9) implies that \(A\) is thin in \(x_0\). Since the balayage function \(\tilde{R}^w_{A \cap V}\) is the lower semi-continuous regularization of \(R^w_{A \cap V}\), we have that

\[
\liminf_{x \to x_0} R^w_{A \cap V}(x) < w(x_0).
\]

Hence, choosing \(\varepsilon > 0\) such that \(\liminf_{x \to x_0} R^w_{A \cap V}(x) + \varepsilon < w(x_0)\), we have, by definition of \(\liminf\) and the particular choice of \(\varepsilon\), we have that

\[
\forall W \in \mathcal{U}_{x_0}, \exists \tilde{x} \in W \setminus \{x_0\} : R^w_{A \cap V}(\tilde{x}) < \liminf_{x \to x_0} R^w_{A \cap V}(x) + \varepsilon < w(x_0).
\]

Therefore, by definition of reduit function, there exists a non-negative superharmonic function \(u\) such that \(u \geq w\) on \(A \cap V\) and

\[
\forall W \in \mathcal{U}_{x_0}, \exists \tilde{x} \in W \setminus \{x_0\} : u(\tilde{x}) < w(x_0).
\]

Thus, for any \(W \in \mathcal{U}_{x_0}\), we have that

\[
\inf_{x \in W \setminus \{x_0\}} u(x) < w(x_0),
\]

which yields

\[
\liminf_{x \to x_0} u(x) := \sup_{W \in \mathcal{U}_{x_0}} \left\{ \inf_{x \in W \setminus \{x_0\}} u(x) \right\} < w(x_0).
\]

By the inequality above and the lower semi-continuity of \(u\) we have that \(u(x_0) < w(x_0)\). Then, by the continuity of \(w\) in \(x_0\) and the fact that \(u \geq w\) on \(A \cap V\) we get that

\[
u(x_0) < \lim_{A \cap V \ni x \to x_0} w(x) \leq \liminf_{A \cap V \ni x \to x_0} u(x).
\]
The thesis follows. \[\square\]

It is important noticing that thanks to Proposition 3.29 our definition of thinnes (Definition 3.28) turns out to be equivalent to the one presented in [7, pp. 149]. Hence, the following Theorem holds true [7, Theorem 6.3.3].

**Theorem 3.30.** Let \( U \) be an open subset of a \( B \)-harmonic space \((E, \mathcal{V})\) and \( x_0 \in \partial U \). Hence, \( x_0 \) is \( \mathcal{H}_E \)-regular if and only if \( U^c := E \setminus U \) is not thin at \( x_0 \).

We conclude this chapter proving the following Theorem, which gives a characterization of boundary regularity in the abstract setting of Potential Theory similar to the one due by Negrini and Scornazzani [20, Theorem 14] for non-totally degenerate Kolmogorov operators.

**Theorem 3.31 (Characterization of regular points).** Let \((E, \mathcal{V})\) be a \( B \)-harmonic space, \( U \) be an open subset of \( E \) and \( x_0 \in \partial U \) such that \( \{x_0\} \) is a polar set in \( E \), according to Definition 3.25. Then, \( x_0 \) is a \( \mathcal{H}_E \)-irregular point if and only if

\[
\inf_{K \in \mathcal{K}(x_0)} \tilde{R}_{U^c \cap K}^1(x_0) = 0,
\]

where the infimum is taken on the family of compact neighbourhoods \( K \) of \( x_0 \) ordered by inclusion and \( \tilde{R}_{U^c \cap K}^1 \) is the balayage of the constant function 1 on \( U^c \cap K \) according to Definition 3.21.

**Proof.** Suppose (3.11) holds. Then, for any \( \varepsilon \in (0, 1) \), there exists a compact neighbourhood \( K_{\varepsilon} \) of \( x_0 \) such that

\[
\tilde{R}_{U^c \cap K_{\varepsilon}}^1(x_0) \leq \varepsilon.
\]

Since \( \{x_0\} \) is a polar set in \( E \) it follows from Corollary 3.27 that

\[
\tilde{R}_{U^c \cap K_{\varepsilon} \setminus \{x_0\}}^1(x_0) = \tilde{R}_{U^c \cap K_{\varepsilon}}^1(x_0) \leq \varepsilon < 1.
\]

Thus, by Proposition 3.29, it follows that \( U^c \setminus \{x_0\} \) is thin in \( \{x_0\} \). Then, since \( \{x_0\} \) is a polar set in \( E \) and \( U^c \setminus \{x_0\} \) is thin in \( \{x_0\} \), it follows that \( U^c \) is thin in \( x_0 \in U^c \) according to Definition 3.28. By Theorem 3.30 it follows that \( x_0 \) is a \( \mathcal{H}_E \)-irregular point. Conversely assume \( x_0 \) is \( \mathcal{H}_E \)-irregular. As a consequence of Theorem 3.30 we get that \( U^c \) is thin in \( x_0 \) according to Definition 3.28, that is \( U^c \setminus \{x_0\} \) is thin in \( \{x_0\} \) and \( \{x_0\} \) is a polar set in \( E \). Then, for any compact neighbourhood \( K \) of \( x_0 \), by Corollary 3.27, we get that

\[
0 \leq \tilde{R}_{U^c \cap K}^1(x_0) = \tilde{R}_{U^c \cap K \setminus \{x_0\}}^1(x_0).
\]

Taking the infimum for \( K \in \mathcal{K}(x_0) \), we obtain the desired equality (3.11). \(\square\)

4. **The Perron-Wiener-Brelot solution for \( \mathcal{L} \)**

We consider the Dirichlet problem

\[
\begin{cases}
\mathcal{L}u = 0 \text{ in } U, \\
u_{|\partial U} = \varphi,
\end{cases}
\]

where \( U \) is an open subset of \( \mathbb{R}^{N+1} \), \( \varphi \in C_c(\partial U, \mathbb{R}) \) and \( \mathcal{L} \) is the operator defined in (1.1) satisfying hypothesis [H.1]. As we are interested in classical solution to \( \mathcal{L}u = 0 \), throughout the sequel of this article we denote with \( \mathcal{H} \) the harmonic sheaf defined as

\[
U \mapsto \mathcal{H}(U) := \{ u \in C^\infty(U, \mathbb{R}) : \mathcal{L}u = 0 \text{ in } U \},
\]

for every open set \( U \) of \( \mathbb{R}^{N+1} \) and we say that a function \( u \) is harmonic in an open set \( U \) if \( u \in \mathcal{H}(U) \).

We next discuss the main steps of the procedure that provides us with the unique solution \( u \) to the boundary value problem (4.1).
4.1. Definition of the sweeping system. We construct the Perron-Wiener-Brelot solution to problem (4.1). With this aim we consider, for any \( z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}, T > t_0 \) and \( r > 0 \), the cylinder \( Q_{r,T}(z_0) \) defined in (2.26), and the relevant Dirichlet problem (2.27). Note that, in the simplest case of the heat operator \( \mathcal{L} = \Delta - \partial_t \), we are considering the usual Cauchy-Dirichlet problem on the parabolic cylinder

\[
Q_r(z_0) := B_r(x_0) \times (t_0, T).
\]

As already recalled in Chapter 2, there exists a unique classical solution \( u \in C^\omega(Q_{r,T}(z_0)) \) to the Dirichlet problem (2.27), which attains the boundary data on \( \partial \mathcal{P} Q_r(z_0) \). By Riesz’s Theorem we have that there exists a Radon measure \( \mu_{z,T}(z_0) \), supported on \( \partial \mathcal{P} Q_r(z_0) \), such that

\[
u(z) := \int_{\partial \mathcal{P} Q_r(z_0)} \varphi(\xi) d\mu_{z,T}(z_0)(\xi), \quad \forall z \in Q_{r,T}(z_0).
\]

Thus, the family

\[
\Omega := \left\{ \mu_{z,T}(z_0) : z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}, r \in \mathbb{R}^+, T > t_0 \right\},
\]

is a sweeping system on \( \mathbb{R}^{N+1} \).

4.2. The hyperharmonic sheaf \( \mathcal{W} \) and the \( \mathcal{B} \)-harmonic space \( (\mathbb{R}^{N+1}, \mathcal{W}) \). We now consider the hyperharmonic sheaf \( \mathcal{W} \) generated by \( \Omega \) in accordance with the Definition 3.4, and we prove that \( (\mathbb{R}^{N+1}, \mathcal{W}) \) is a \( \mathcal{B} \)-harmonic space, according to Definition 3.14.

We first prove that \( (\mathbb{R}^{N+1}, \mathcal{W}) \) is a harmonic space in the sense of Definition 2.7. For every \( Q_{r,T}(\xi, \tau) \subseteq \mathcal{B} \), we choose a point \( (\xi, \tau) \in \mathbb{R}^{N+1} \), and two positive constants \( T \) and \( r \) such that

\[
Q_{r,T}(\xi, \tau) \subseteq U \quad \text{and} \quad \z \in Q^-.
\]

Note that, for every \( p \in \mathbb{N} \), \( \{u_{n+p} - u_n\}_{n \in \mathbb{N}} \) is a sequence of non-negative \( \mathcal{H}_\mathcal{W} \)-functions. Then, by the Harnack inequality stated in Theorem 2.2, we obtain

\[
0 \leq (u_{n+p}(z) - u_n(z)) \leq \max_{Q} (u_{n+p} - u_n) \leq M \min_{Q} (u_{n+p} - u_n)
\]
In the last inequality we have used the fact that there exists a point \( \zeta \in Q^+ \cap V \) such that
\[
\min_{\overline{Q}}(u_{n+p} - u_n) \leq M(u_{n+p}(\zeta) - u_n(\zeta)),
\]
since \( V \) is dense in \( U \). From the compactness of \( K \) it follows that there exists a finite family of cylinders \( \{Q^i\}_{i=1}^m \), contained in \( U \), such that \( K \subset \bigcup_{i=1}^m Q^i \). This proves that \( \{u_n\}_{n \in \mathbb{N}} \) converges uniformly on \( K \).

We next show that \( u \) in (4.6) is a \( \mathcal{H}_\mathcal{U} \)-function. Indeed, fixed \( Q_{r,T}(z_0) \Subset U \), we have that
\[
(4.7) \quad u_n(z) = \int_{\partial P_{Q_{r,T}}(z_0)} u_n(\zeta) d\mu^Q_{Q_{r,T}}(\zeta), \quad \forall z \in Q_{r,T}(z_0), \forall n \in \mathbb{N}.
\]
From the uniform convergence, it follows that the limit function \( u \) satisfies the same identity. Then, \( u \in \mathcal{H}_\mathcal{U}(Q_{r,T}(z_0)) \) and the Doob convergence property follows. This completes the proof of (A2), and thus, that \( (\mathbb{R}^{N+1}, \mathcal{U}) \) is a harmonic space.

In order to show that \( (\mathbb{R}^{N+1}, \mathcal{U}) \) is a \( \mathcal{H} \)-harmonic space we show that \( \mathcal{U} \) separates the points, so that we can rely on Proposition 3.15. Let us consider two points \( z_1 \neq z_2 \). There exists \( T > 0 \) such that \( z_1, z_2 \in U = \mathbb{R}^N \times [-T,T] \). If \( t_1 \neq t_2 \), then we set \( u_{t_1}(z) = e^{t_1} \). If otherwise \( t_1 = t_2 = t \) we choose \( \gamma \in \mathbb{R}^N \) so that \( \langle x_1 - x_2, E(-t)\gamma \rangle \neq 0 \) and \( M > 0 \) so that
\[
u_z(z) = M - \langle x, E(-t)\gamma \rangle > 0, \quad \forall z \in U.
\]
By definition \( u_{t_2} \) satisfies
\[
\mathcal{L} u_{t_2}(z) = \langle Bx, Du_{t_2}(z) \rangle - \partial_t u_{t_2}(z) = -\langle Bx, E(-t)\gamma \rangle + \langle Bx, E(-t)\gamma \rangle = 0.
\]
Hence, \( u_{t_1}(z) \neq u_{t_2}(z) \) and \( u_{t_1}, u_{t_2} \) are both non-negative superharmonic functions. Then, by Proposition 3.15 it follows that \( (\mathbb{R}^{N+1}, \mathcal{U}) \) is a \( \mathcal{H} \)-harmonic space.

4.3. Conclusions. Thanks to Theorem 3.17, we conclude that there exists a generalized solution \( H^U_\varphi \in \mathcal{H}_\mathcal{U}(U) \) to the problem
\[
(4.8) \quad \begin{cases}
u \in \mathcal{H}_\mathcal{U}(U), \\
u|_{\partial U} = \varphi, \quad \forall \varphi \in C_c(\partial U, \mathbb{R}).
\end{cases}
\]
We next show that the generalized solution \( H^U_\varphi \) to (4.8) is also a classical solution to the equation \( \mathcal{L} H^U_\varphi = 0 \), then it is a solution to problem (4.1). This fact is the main consequence of the following

Proposition 4.1. The sweeping system \( \Omega \), defined above, is a \( \mathcal{H} \)-sweeping system, with respect to the sheaf \( \mathcal{H} \) defined in (4.2).

Proof. We show that \( \mu^{Q_{r,T}(z_0)}_{\varphi} \) is a \( \mathcal{H} \)-sweeping, according to Definition 3.2. Clearly fixed \( \varphi \in C(\partial P_{Q_{r,T}(z_0), \mathbb{R}}) \) the function
\[
\mu^{Q_{r,T}(z_0)}_{\varphi} : Q_{r,T}(z_0) \to (-\infty, +\infty],
\]
\[
z \mapsto \mu^{Q_{r,T}(z_0)}_{\varphi}(z) := \int_{\partial P_{Q_{r,T}(z_0)}} \varphi(\zeta) d\mu^Q_{Q_{r,T}}(\zeta),
\]
is a \( \mathcal{H} \)-function, because it is the solution \( H^U_\varphi \) of the Dirichlet problem (2.27) with boundary data \( \varphi \). Consider \( u \in \mathcal{H}(U) \), and let \( Q_{r,T}(z_0) \Subset U \) be any cylinder. Since \( u \) is the solution to (2.27) with boundary data \( \varphi = u \), we have that
\[
\mu^{Q_{r,T}(z_0)}_u(z) := \int_{\partial P_{Q_{r,T}}(z_0)} u(\zeta) d\mu^Q_{Q_{r,T}(z_0)}(\zeta) = u(z), \quad \forall z \in Q_{r,T}(z_0).
\]
Hence, \( \mu_{Q_r, T}^{\varphi} (z_0) = u \) on \( Q_r, T \) \( (z_0) \) and the thesis follows. \(\square\)

The main consequence of the above Proposition is that \( U \subseteq \mathbb{R}^{N+1} \mathcal{H}_\varphi (U) \equiv \mathcal{H} (U) \) for every open set. We are now ready to give the

Proof of Theorem 1.1. Consider the sweeping system \( \Omega \) defined in (4.5) and the hyperharmonic sheaf \( \mathcal{H} \) generated by \( \Omega \). We have that \( (\mathbb{R}^{N+1}, \mathcal{H}) \) is a \( \mathcal{B} \)-harmonic space, according to Definition 3.14, then Theorem 3.17 provides us with the existence of the Perron-Wiener-Brelot solution \( H_\varphi^U \) to (4.8). Moreover, Proposition 4.1 implies that \( H_\varphi^U \) to (4.8) is also a classical solution to the equation \( \mathcal{L} H_\varphi^U = 0 \), then it is a solution to problem (4.1). The uniqueness of the solution in bounded open sets is a consequence of the maximum principle. \( \square \)

5. Boundary regularity

In Section 4 we have shown that there exists the generalized solution \( H_\varphi^U \) to the Dirichlet problem for the operator \( \mathcal{L} \) defined in (1.1) in an arbitrary open set \( U \)

\[
\begin{cases}
\mathcal{L} u = 0, & \text{in } U, \\
u_{\partial U} = \varphi, & \forall \varphi \in C_c (\partial U, \mathbb{R}).
\end{cases}
\]

Moreover, \( H_\varphi^U \) is unique if the open set \( U \) is bounded. In this chapter we describe the conditions under which the boundary datum \( \varphi \) is attained by the generalized solution \( H_\varphi^U \).

In particular, we prove Theorem 1.2 and Proposition 1.3.

In order to use the abstract Theorem 3.31 we begin showing that every singleton \( \{z_0\} \) is a polar set in \( \mathbb{R}^{N+1} \). Our proof follows the same line as [14, Lemma 4.5].

Let us consider the fundamental solution \( \Gamma \) of the operator \( \mathcal{L} \), defined in (2.19). In order to make \( \Gamma \) a lower semi-continuous function on \( \mathbb{R}^{N+1} \) we agree to let \( \Gamma (\zeta; \zeta) = 0 \), so that

\[
(5.1) \quad \Gamma (\zeta; \zeta) = \liminf_{z \to \zeta} \Gamma (z; \zeta), \quad \forall \zeta \in \mathbb{R}^{N+1}.
\]

The following Lemma holds.

Lemma 5.1. Let \( \zeta_0 := (\xi_0, \tau_0) \in \mathbb{R}^{N+1} \) be fixed and let \( u \) be the function defined as follows

\[
u(\zeta) := \Gamma(z; \zeta_0), \quad \forall \zeta \in \mathbb{R}^{N+1}.
\]

Then, \( u \) is a non-negative \( \mathcal{B} \)-function on \( \mathbb{R}^{N+1} \).

Proof. The non-negativity and the lower semi-continuity of \( u \) follow form the properties of the fundamental solution \( \Gamma \) and from (5.1). Let us prove that, fixed a cylinder \( Q_r, T \equiv Q_r, T(\zeta_0) \), \( u \) satisfies the inequality

\[
(5.3) \quad u(z) \geq \int_{\partial Q_r, T} u(\zeta) \, d\mu_{\varphi}^{Q_r, T}(\zeta), \quad \forall z \in Q_r, T.
\]

Let us consider a function \( \varphi \in C(\partial Q_r, T, \mathbb{R}) \) such that \( \varphi \leq u \) on \( \partial Q_r, T \).

Assume that \( \zeta_0 \notin Q_r, T \) and indicate with \( H_\varphi^{Q_r, T} \) the generalized solution to the Dirichlet problem in \( Q_r, T \) with boundary datum \( \varphi \). Form the harmonicity of \( u \) in \( Q_r, T \) and the lower-semicontinuity of \( u \) we get

\[
\liminf_{z \to \zeta} (u(z) - H_\varphi^{Q_r, T}(z)) \geq u(\zeta) - \varphi(\zeta) \geq 0, \quad \forall \zeta \in \partial Q_r, T.
\]

Hence, since by Corollary 3.16 \( Q_r, T \) is a \( MP \)-set, according to Definition 3.5, and \( u - H_\varphi^{Q_r, T} \in \mathcal{H}_\varphi (Q_r, T) \), we have that

\[
u(z) \geq H_\varphi^{Q_r, T}(z) := \int_{\partial Q_r, T} \varphi \, d\mu_{\varphi}^{Q_r, T}, \quad \forall z \in Q_r, T.
\]
Passing to the supremum of every \( \varphi \leq u \) on \( \partial Q_{r,T} \) we obtain the desired inequality (5.2).

On the other hand, let us suppose that \( \zeta_0 \in Q_{r,T} \). Since \( u \equiv 0 \) on the set \( \{ t \leq \tau_0 \} \) we have that \( \varphi \leq 0 \) on \( \partial Q_{r,T} \cap \{ t \leq \tau_0 \} \). If we denote by \( v \) the classical solution to the Dirichlet problem in \( Q_{r,T} \) with boundary datum \( \varphi \), we obtain that \( v \leq 0 \) on \( Q_{r,T} \cap \{ t \leq \tau_0 \} \). Hence, \( v(\zeta_0) \leq 0 \). Moreover, by the uniqueness of the solution to the Dirichlet problem in \( Q_{r,T} \), we obtain that \( v \equiv H_\varphi^{Q_{r,T}} \) in \( Q_{r,T} \). Thus, \( H_\varphi^{Q_{r,T}}(\zeta_0) = v(\zeta_0) \leq 0 \). Let us consider \( \widetilde{Q}_{r,T} := Q_{r,T} \setminus \{ \zeta_0 \} \). By definition of \( \varphi \) and the lower semi-continuity of \( u \) we get

\[
\liminf_{z \to \zeta} (u(z) - H_\varphi^{Q_{r,T}}(z)) \geq u(\zeta) - \varphi(\zeta) \geq 0, \quad \forall \zeta \in \partial \widetilde{Q}_{r,T}.
\]

Since by Corollary 3.16 \( \widetilde{Q}_{r,T} \) is a \( MP \)-set, according to Definition 3.5, and \( u - H_\varphi^{Q_{r,T}} \in \mathcal{H}_\varphi(\widetilde{Q}_{r,T}) \), we have that \( u \geq H_\varphi^{Q_{r,T}} \) on \( \widetilde{Q}_{r,T} \). Moreover in \( \zeta_0 \) we have that \( u(\zeta_0) = 0 \geq H_\varphi^{Q_{r,T}}(\zeta_0) \). Hence, \( u \geq H_\varphi^{Q_{r,T}} \) on \( Q_{r,T}(z) \) and (5.2) follows exactly as for the case \( \zeta_0 \notin \widetilde{Q}_{r,T} \).

**Proposition 5.2.** Every singleton \( \{ \zeta_0 \}, \zeta_0 \in \mathbb{R}^{N+1}, \) is a polar set in \( \mathbb{R}^{N+1} \).

**Proof.** Let \( \zeta_0 := (x_0, t_0) \in \mathbb{R}^{N+1} \) and we use the fundamental solution \( \Gamma \) to built a function \( p \) which satisfies the condition of Definition 3.25.

For any \( \varepsilon > 0 \) let us consider the family of points

\[
\xi_{\varepsilon} := (\xi_\varepsilon, \tau_\varepsilon) = (E(-\varepsilon)x_0, t_0 - \varepsilon),
\]

where \( E(\cdot) \) is the \( N \times N \) matrix defined in (2.3). By the definition (2.19) of fundamental solution we obtain that

\[
\Gamma(\zeta_0; \xi_{\varepsilon}) := \frac{(4\pi)^{N/2}}{\sqrt{\det C(t_0 - \varepsilon, t_0)}} \xrightarrow{\varepsilon \to 0^+} +\infty.
\]

Then, there exists a decreasing sequence \( \{ \varepsilon_n \}_{n \in \mathbb{N}} \) such that

\[
(5.4) \quad \Gamma(\zeta_0; \xi_{\varepsilon_n}) \geq 4^n, \quad \forall n \in \mathbb{N}.
\]

Let us consider the function \( p \) defined as follows

\[
(5.5) \quad p(z) := \sum_{n=1}^{\infty} \frac{\Gamma(z; \xi_{\varepsilon_n})}{2^n}
\]

and show that \( p \) satisfies the condition of Definition 3.25. By assumption (5.4) we have

\[
p(\zeta_0) = \sum_{n=1}^{\infty} \frac{\Gamma(\zeta_0; \xi_{\varepsilon_n})}{2^n} \geq \sum_{n=1}^{\infty} 2^n = +\infty.
\]

Let \( z \neq \zeta_0 \). Then, there exists a positive \( r \) such that

\[
\overline{B}_r(z) \times \overline{B}_r(z) \cap \{(w, \zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} : w = \zeta \} = \emptyset.
\]

As \( \Gamma \) is continuous in \( \{(w, \zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} : w \neq \zeta \} \), we have

\[
M := \max_{(y, \zeta) \in \partial \Gamma(\zeta_0)} \Gamma(y, s; \eta, \sigma) < +\infty.
\]

Moreover, since \( \xi_{\varepsilon_n} \to \zeta_0 \) as \( n \to +\infty \), there exists an index \( \bar{n} > 0 \) such that, for any \( n > \bar{n} \), \( \xi_{\varepsilon_n} \in \overline{B}_r(z_0) \). We show that \( p \) converges uniformly on \( \overline{B}_r(z) \). Indeed,

\[
\sum_{n=\bar{n}}^{\infty} \sup_{w \in \overline{B}_r(z)} \left| \frac{\Gamma(w; \xi_{\varepsilon_n})}{2^n} \right| \leq M \sum_{n=\bar{n}}^{\infty} \frac{1}{2^n} = M.
\]
Hence, it follows that $p$ converges uniformly on $\overline{B}_r(z)$. From Lemma 5.1 it then follows that $p$ is a $\mathcal{H}$-function on $\mathbb{R}^{N+1}$, finite for any $z \neq z_0$. Moreover, since the harmonic space $(\mathbb{R}^{N+1}, \mathcal{H})$ has the Doob convergence property, by Remark 3.12, $p$ is superharmonic. Hence, $\{z_0\}$ is a polar set in $\mathbb{R}^{N+1}$, according to Definition 3.25, with associated function $p$. \hfill $\Box$

Now we apply the results presented in the last part of Chapter 3 to discuss the regularity of boundary points. Indicating with $U$ an open subset of $\mathbb{R}^{N+1}$ with non-empty boundary let us consider, for any $r > 0$ and $z_0 \in \partial U$, the set

$$B_r := \overline{B}_r(z_0) \setminus U,$$  \hfill (5.6)

As a consequence of Theorem 3.31 and Proposition 5.2 we characterize the regularity of boundary points as follows.

**Corollary 5.3.** Let $U \subseteq \mathbb{R}^{N+1}$ be an open set and let $z_0 \in \partial U$. Then, $z_0$ is a $\mathcal{H}_{\mathcal{U}}$-regular point if and only if

$$\lim_{r \to 0^+} \widehat{R}^1_{B_r}(z_0) > 0.$$  \hfill (5.7)

Since the sweeping system $\Omega$, defined in (4.5), is a $\mathcal{H}$-sweeping (see Proposition 4.1), the definition of $\mathcal{L}$-regular point and $\mathcal{H}_{\mathcal{U}}$-regular point coincide.

### 5.1 Wiener-type criterium

We begin proving Theorem 1.2. We extend to our contest the same approach used in [12]. We will use the lemma below.

**Lemma 5.4.** For every $p \in \mathbb{N}$, let us split the set $B_r$ in (5.6) as follows

$$B_r = B_r^p \cup B_r^{*p},$$

where, for any $\lambda \in (0, 1)$, we write

$$B_r^p = \left\{ z \in B_r : \Gamma(z; z_0) \geq \left( \frac{1}{\lambda} \right)^{p \log p} \right\} \cup \{z_0\}$$

and

$$B_r^{*p} = \left\{ z \in B_r : \Gamma(z; z_0) \leq \left( \frac{1}{\lambda} \right)^{p \log p} \right\}.$$  

Then,

$$\lim_{r \to 0^+} \widehat{R}^1_{B_r}(z_0) = \lim_{r \to 0^+} \widehat{R}^1_{B_r^p}(z_0).$$

**Proof.** By the monotonicity property of the balayage and Proposition 3.22, we have

$$\widehat{R}^1_{B_r^p}(z_0) \leq \widehat{R}^1_{B_r}(z_0) \leq \widehat{R}^1_{B_r^p}(z_0) + \widehat{R}^1_{B_r^{*p}}(z_0).$$  \hfill (5.8)

Moreover, note that when $r \to 0^+$, $B_r \to \{z_0\}$. Then, since $z_0 \notin B_r^{*p}$, we have that $B_r^{*p}$ converges to the empty set when $r \to 0^+$, which yields

$$\widehat{R}^1_{B_r^{*p}}(z_0) \to 0.$$  

Passing to the limit in (5.8) we obtain the thesis. \hfill $\Box$

**Proof of Theorem 1.2.** We begin showing the necessary condition, e.g. we prove the implication

$$z_0 \text{ is } \mathcal{H}_{\mathcal{U}}\text{-regular} \Rightarrow \sum_{n=1}^{\infty} \widehat{R}^1_{U_{n \mathcal{U}}}(z_0) = +\infty,$$  \hfill (5.9)
By the hypothesis it follows from Corollary 5.3 that
\[
\lim_{r \to 0^+} \overline{R}_{B_r}^i(z_0) > 0.
\]
Let us assume by contradiction that
\[
\sum_{n=1}^{\infty} \overline{\tilde{R}}_{U_{\mathbb{R}}^c(z_0)}^i(z_0) < +\infty,
\]
where \(U_{\mathbb{R}}^c(z_0)\) is the set defined in (1.5). We are going to prove that the assumption (5.11) is in contradiction with (5.10).

By hypothesis (5.11), for every \(\varepsilon > 0\), there exists \(p_\varepsilon := p(\varepsilon) \in \mathbb{N}\) such that
\[
\sum_{n=p_\varepsilon}^{\infty} \overline{\tilde{R}}_{U_{\mathbb{R}}^c(z_0)}^i(z_0) < \varepsilon.
\]
On the other hand, following the notation of Lemma 5.4, for any positive radius \(r > 0\), we have
\[
B_r^{p_\varepsilon} \subseteq \bigcup_{n=p_\varepsilon}^{\infty} U_{\mathbb{R}}^c(z_0).
\]
Then, by Proposition 3.22, we get
\[
\overline{\tilde{R}}_{B_r^{p_\varepsilon}}^i(z_0) \leq \sum_{n=p_\varepsilon}^{\infty} \overline{\tilde{R}}_{U_{\mathbb{R}}^c(z_0)}^i(z_0) < \varepsilon.
\]
By Lemma 5.4, we get
\[
\lim_{r \to 0^+} \overline{\tilde{R}}_{B_r}^i(z_0) = 0,
\]
which is in contradiction with (5.10). This prove the necessary condition (5.9).

We prove now the sufficient condition of Theorem 1.2, e.g.
\[
\sum_{n=1}^{\infty} \overline{\tilde{R}}_{U_{\mathbb{R}}^c(z_0)}^i(z_0) = +\infty \Rightarrow z_0 \text{ is } \mathcal{R}_\mathbb{R}\text{-regular.}
\]
In particular we prove that, for any \(r > 0\), it holds
\[
\overline{\tilde{R}}_{B_r}^i(z_0) \geq \frac{1}{2}.
\]
Form Corollary 5.3 passing to the limit in the inequality above when \(r\) goes to zero will ensure the regularity of the boundary point \(z_0\).

Let us fix \(r > 0\) and consider the set \(B_r\) defined in (5.6). There exists \(\bar{n} \in \mathbb{N}\) and \(q = q(r), p = p(r) \in \mathbb{N}\), with \(\bar{n} \leq q < p\), such that once defined
\[
G_{q,p} := \bigcup_{n=q}^{p} U_n^c(z_0), \quad \text{and} \quad W_{q,p}(z) := \sum_{n=q}^{p} \overline{\tilde{R}}_{U_n^c(z_0)}^i(z_0),
\]
we have that
\[
G_{q,p} \subset B_r, \quad \text{and} \quad W_{q,p}(z_0) \geq \sum_{n=q}^{p} \overline{\tilde{R}}_{U_n^c(z_0)}^i(z_0) > 2.
\]
We estimate the function \(W_{q,p}(z)\) on \(G_{p,q}\). For any \(n \in \{q, \ldots, p\}\), by Definition 3.21 of reduit function, we have that, for any \(h \in \{q, \ldots, p\}\), there exists a neighbourhood \(O_h \subset B_r\) of \(U_h^c(z_0)\) such that
\[
R_{U_h^c(z_0)}^i(z) \leq \overline{\tilde{R}}_{U_h^c(z_0)}^i(z_0) + \frac{1}{2n}, \quad \forall z \in O_h.
\]
Hence, since (5.14) holds for every $h \in \{q, \ldots, p\}$ we have that it is satisfied on $O = \bigcup_{h=q}^{p} O_h$. Summing for $q \leq n \leq p$, we get that

$$W_{q,p}(z) \leq W_{q,p}(z_0) + \sum_{n=q}^{p} \frac{1}{2^n} < 2 + W_{q,p}(z_0), \quad \text{on } O.$$  
Define the non-negative function

$$v_{q,p}(z) := \frac{W_{q,p}(z)}{2 + W_{q,p}(z_0)}, \quad \forall z \in \mathbb{R}^{N+1}.$$  
Since $v_{q,p} \leq 1$ on $O$ we have that from Proposition 3.22 it follows that $R^1_O \geq R^v_{O,p}$. In particular, by definition of $v_{q,p}$, Proposition 3.22 and the fact that $O \subset B_r$, we eventually arrive at

$$\hat{R}_{B_r}^1(z_0) \geq \hat{R}_O^1(z_0) = \hat{R}_O^1(z_0) \geq R^v_{O,p}(z_0) \geq v_{q,p}(z_0) = \frac{W_{q,p}(z_0)}{2 + W_{q,p}(z_0)}.$$  
Note that the function $s \mapsto \frac{1}{1 + s}$ is increasing. Thus, since $W_{q,p}(z_0) > 2$, we obtain that

$$\hat{R}_{B_r}^1(z_0) \geq \frac{1}{2},$$  
which gives the desired inequality (5.13). Hence, the sufficient condition (5.12) follows. \hfill \Box

5.2. **Exterior cone condition.** Throughout this chapter we indicate with $z_0$ a point $(x_0,0)$, $x_0 \in \mathbb{R}^N$. Let us give the definition of $\mathcal{L}$-cone.

**Definition 5.5** ($\mathcal{L}$-cone of vertex $z_0$). For any $T > 0$ and any compact subset $K$ of $\mathbb{R}^N$ with positive Lebesgue measure we call $\mathcal{L}$-cone of vertex $z_0$, base $K$ and height $T$, the set

$$C_{z_0} := \{ (\delta_0(r)x + x_0, -r^2T) : x \in K, 0 \leq r \leq 1 \},$$

where $\delta_0(r)$ is defined in (2.23).

Given an open subset $U$ of $\mathbb{R}^{N+1}$ and $z_0 \in \partial U$ we say that there exists an exterior cone with vertex in $z_0$ if there exists an $\mathcal{L}$-cone $C_{z_0}$ such that $C_{z_0} \subseteq U^c$.

We prove that if an open set $U$ admits an exterior cone $C_{z_0}$ for a given point $z_0 \in \partial U$ then $z_0$ is regular.

**Proof of Proposition 1.3.** We prove that $U^c := \mathbb{R}^{N+1} \setminus U$ is not thin in $z_0$, according to Definition 3.28. Then, by Theorem 3.30, $z_0$ is a $\mathcal{L}$-regular point.

Since $z_0 \in U^c$ we show that $U^c \setminus \{z_0\}$ is not thin in $\{z_0\}$. Clearly if this holds true then $U^c$ is not thin in $z_0$.

Thanks to Proposition 3.29 it is enough to prove that, for any open neighbourhood $V$ of $z_0$

$$R^1_{U^c \cap (V \setminus \{z_0\})}(z_0) = 1,$$

For any $r > 0$ let us consider the neighbourhoods $B_r$ defined in (5.6), where we have chosen in their definition the open $(N+1)$-dimensional balls $B_r(z_0)$. By Proposition 3.22 we have that

$$R^1_{B_r}(z_0) \leq R^1_{B_r \setminus \{z_0\}}(z_0) + R^1_{\{z_0\}}(z_0).$$

Form (5.15) it follows that it is enough to show that

$$R^1_{B_r \setminus \{z_0\}}(z_0) \geq R^1_{B_r}(z_0) - R^1_{\{z_0\}}(z_0) \geq 1.$$  
With no loss of generality we assume that

$$C_{z_0} \subset B_r,$$

and

$$R^1_{\{z_0\}}(z_0) > 0.$$
Let us fix a point $\xi \in K$. Then, calling $\eta := (\xi, -T)$, there exists an index $\bar{n} = \bar{n}(r) > 0$ such that
\begin{equation}
U_n^c(\eta) \subseteq B_r, \quad \forall n \geq \bar{n},
\end{equation}
where $U_n^c(\eta)$ is the set defined in (1.5) for any
\begin{equation}
0 < \lambda < \Lambda_{K,z_0} := \frac{L^N(K)}{1 + R^1_{\{z_0\}}(z_0)},
\end{equation}
being $L^N$ the Lebesgue $N$-dimensional measure. Note that if $\Lambda_{K,z_0}$ in (5.18) satisfies $\Lambda_{K,z_0} \geq 1$, then any $\lambda \in (0, 1)$ is acceptable. By definition of exterior cone and by (5.17), we have that there exists $\bar{m} = \bar{m}(\varrho) \geq \bar{n} > 0$ such that, for $\varrho \in (0, 1)$ it holds
\begin{equation}
B_r \supseteq U_{\bar{m}}^c(\eta) \supseteq (\delta_0(\varrho)K + x_0) \times \{ -\varrho^2T \} \equiv K_{\varrho}(x_0) \times \{ -\varrho^2T \}.
\end{equation}
Thus, by Definition 3.21 of reduct function $R^1_{B_r}(z_0) = 1$ and that $\int_{\mathbb{R}^N} \Gamma(x, -\varrho^2T; \xi, -T) \, dx = 1$, keeping in mind (5.18), we obtain that
\begin{align*}
R^1_{B_r \setminus \{z_0\}}(z_0) & \geq R^1_{B_r}(z_0) - R^1_{\{z_0\}}(z_0) \\
& \geq \int_{K_{\varrho}(x_0)} \Gamma(x, -\varrho^2T; \xi, -T) \, dx - R^1_{\{z_0\}}(z_0) \\
& = \frac{1}{\varrho^Q} \int_K \Gamma \left( \delta_0(1/\varrho)x', -\varrho^2T; \xi, -T \right) \, dx' - R^1_{\{z_0\}}(z_0) \\
& \geq \frac{L^N(K)}{\varrho^Q \Lambda_{K,z_0}^{-\bar{m}\log \bar{m}}} - R^1_{\{z_0\}}(z_0), \quad \text{(by (5.19) and the definition of $U_{\bar{m}}^c(\eta)$,)} \\
& \geq \frac{L^N(K)}{\Lambda_{K,z_0}^{-\bar{m}\log \bar{m}}} - R^1_{\{z_0\}}(z_0) = 1, \quad \text{(since $\varrho \leq 1$ and $\Lambda_{K,z_0} > \lambda > \Lambda_{K,z_0}^{-\bar{m}\log \bar{m}}$).}
\end{align*}
Then, condition (5.16) is satisfied and the thesis follows. \qed

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References

[1] F. Anceschi, S. Polidoro: A survey on the classical theory for Kolmogorov equation. Le Matematiche Vol LXXV- Issue 1 (2020) 221–258.
[2] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni: Stratified Lie Groups and their sub-Laplacians. Springer Monographs in Mathematics, 2007.
[3] H. Bauer: Harmonische Räume und ihre Potentialtheorie. Lecture Notes in Mathematics, 22, Springer-Verlag, 1966.
[4] J.-M. Bony: Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. Ann. Inst. Fourier (Grenoble) 19 (1969) 277–304.
[5] M. Brelot: Lectures on Potential Theory. Tata Institute of Fundamental Research, Bombay, 1960.
[6] C. Cinti, E. Lanconelli: Riesz and Poisson-Jensen representation formulas for a class of ultraparabolic operators on Lie groups. Potential Anal. 30 (2009) 179–200.

[7] C. Constantinescu, C. Cornea: Potential theory on harmonic spaces Springer-Verlag, Berlin, 1972.

[8] E. G. Effros, J. L. Kazdan: On the Dirichlet problem for the heat equation Indiana Univ. Math. J. 20 (1970/1971) 683–693.

[9] L. C. Evans, R. F. Gariepy: Wiener’s criterion for the heat equation. Arch. Rational Mech. Anal. 78 (1982) 293–314.

[10] L. Hörmander: Hypoelliptic second order differential equations. Acta Math. 119 (1967) 147–171.

[11] A.E. Kogoj: On the Dirichlet Problem for hypoelliptic evolution equations: Perron-Wiener solution and a cone-type criterion. J. Differential Equations 262 (2017) 1524–1539.

[12] A.E. Kogoj, E. Lanconelli, G. Tralli: Wiener-Landis criterion for Kolmogorov-type operators. Discrete Contin. Dyn. Syst. Ser. A 38 (2018) 2467–2485.

[13] E. Lanconelli, S. Polidoro: On a class of hypoelliptic evolution operators. Rend. Sem. Mat. Univ. Pol. Torino 52 (1994) 29–63.

[14] E. Lanconelli, F. Uguzzoni: Potential analysis for a class of diffusion equations: a Gaussian bounds approach. J. Differential Equations 248 (9) (2010) 2329–2367.

[15] E.M. Landis: Necessary and sufficient conditions for the regularity of a boundary point for the Dirichlet problem for the heat equation. Dokl. Akad. Nauk SSSR 185 (1969) 517–520.

[16] J. Lukeš: Théorème de Keldych dans la théorie axiomatique de Bauer des fonctions harmoniques. Czechoslovak Mathematical Journal 24 (1) (1974) 114–125.

[17] A. Lunardi: Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24 (4) (1997) 133–164.

[18] M. Manfredini: The Dirichlet problem for a class of ultraparabolic equations. Adv. Differential Equations 2 (1997) 831–866.

[19] A. Montanari: Harnack Inequality for Totally Degenerate Kolmogorov-Fokker-Planck Operators. Bollettino U.M.I. 10-B (7) (1996) 903–926.

[20] P. Negrini, V. Scornazzani: Superharmonic functions and regularity of boundary points for a class of elliptic-parabolic partial differential operators. Bollettino U.M.I. Analisi Funzionale e Applicazioni Serie VI 3 (1984) 85–107.