Analysis of the instantaneous Bethe-Salpeter equation for $q\bar{q}$-bound-states

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We investigate the structure of the instantaneous Bethe-Salpeter equation for $q\bar{q}$-bound states in the general case of unequal quark masses and develop a numerical scheme for the calculation of mass spectra and Bethe-Salpeter amplitudes. In order to appreciate the merits of the various competing models beyond the reproduction of the mass spectra we present explicit formulas to calculate electroweak decays. The results for an explicit quark model will be compared to experimental data in a subsequent paper [1].

I. INTRODUCTION

Despite many efforts the bound state problem of QCD is still far from being well understood. One of the main tasks is to investigate the relevance of quarks as dynamical degrees of freedom in hadronic bound states. Since a relativistic treatment for the quarks in deeply bound states is essential, the Bethe-Salpeter (BS)-equation [2,3] provides a suitable starting point. Unlike in QED one cannot use perturbation theory to obtain useful approximations for the interaction kernel in QCD. Therefore our knowledge of the interaction between quarks is still quite fragmentary and various phenomenological alternatives have to be tested. In the present work we will restrict ourselves to $q\bar{q}$-states. The use of general $q\bar{q}$-interaction kernels depending on the relative time variable leads to serious conceptual and practical problems [4]. Therefore it is very useful at this point to make the simplifying assumption that the BS-kernel can be approximated by an effective interaction that is instantaneous in the rest frame of the bound state. The BS-equation then reduces to the (full) Salpeter equation [5] that has been investigated for $q\bar{q}$-states e.g. by Llewellyn Smith [6], Le Yaouanc and coworkers [7] and recently by Lagaè [8].

In the present paper we analyze the properties of the Salpeter equation for the general case of unequal quark masses in Sec. I. In Sec. II we present a flexible numerical treatment of this equation based on the variational principle of ref. [9]. We use the block structure of the Salpeter amplitude to derive an expansion in terms of a complete set of basis functions which leads to a matrix equation analogous to the RPA (random phase approximation) equation well known in nuclear theory [10]. Our numerical method can be applied to a wide class of phenomenological interaction kernels. In Sec. III we show how to reconstruct the full BS-amplitudes from the Salpeter amplitudes and present the calculation of some decay observables within the Mandelstam formalism including the decay $\pi^0 \rightarrow 2\gamma$. Concluding remarks given are in Sec. IV.

The method obtained in this paper will be applied to an explicit quark model for light mesons in a subsequent paper [1].

II. GENERAL PROPERTIES OF THE SALPETER EQUATION

A. Formulating the Salpeter equation

The BS-amplitude $\chi$ for a fermion-antifermion bound state $|P\rangle$ is defined by

$$[\chi_P(x_1, x_2)]_{\alpha\beta} = \langle 0 | T \Psi_\alpha(x_1) \bar{\Psi}_\beta(x_2) | P \rangle$$  \hspace{1cm} (1)

where $P$ is the four-momentum of the bound state, $T$ denotes the time ordering for the fermion operators $\Psi, \bar{\Psi}$ and $\alpha, \beta$ stand for spinor, flavor and color indices. Due to translational invariance the Fourier transformation can be written as

$$\chi_P(x_1, x_2) = e^{-iP^\mu X_\mu} \int \frac{d^4p}{(2\pi)^4} e^{-ip^\mu x} \chi_P(p)$$  \hspace{1cm} (2)

where $x_1 = X + \eta_1 x$, $x_2 = X - \eta_2 x$ with the conjugate momenta $p_1 = \eta_1 P + p$, $p_2 = \eta_2 P - p$. Here $\eta_1, \eta_2$ are two arbitrary real numbers satisfying $\eta_1 + \eta_2 = 1$. The BS-equation for $\chi_P(p)$ then reads

$$\chi_P(p) = S^0(p_1) \int \frac{d^4p'}{(2\pi)^4} [-i K(P, p, p') \chi_P(p')] S^0(p_2)$$  \hspace{1cm} (3)

also shown in graphical form in Fig. I. The interaction kernel $K(P, p, p')$ generally acts on $\chi_P(p')$ as

$$[K(P, p, p') \chi_P(p')]_{\alpha\beta} = \sum_{\alpha', \beta'} K(P, p, p')_{\alpha\alpha', \beta\beta'} \chi_P(p')_{\alpha'\beta'}$$  \hspace{1cm} (4)

For an interaction that is instantaneous in the rest frame of the bound state with momentum $P = (M, \vec{0})$ the BS-kernel can be written as

$$K(P, p, p') \big|_{P=(M, \vec{0})} = V(\vec{p}, \vec{p'})$$  \hspace{1cm} (5)
which can also be formulated in a covariant way as
\[ K(P, p, p') = V(p_\perp, p'_\perp) \]
where \( p_\perp = p - (Pp/P^2) P \) is perpendicular to \( P \). In practical calculations one has to justify this ansatz a posteriori by investigating its consequences in the framework of explicit models.

Furthermore we will approximate the full quark propagators \( S^F(p) \) by bare propagators \( S^H_i(p_i) \approx i (\gamma p_i - m_i + i\epsilon)^{-1} \) where \( m_1 \) and \( m_2 \) are interpreted as effective masses for the quark and the antiquark. This approximation has been criticized \[14\] because free propagators might be incompatible with a confining kernel. On the other hand one can argue that this choice naturally leads to nonrelativistic potential models that have been applied successfully to heavy quarkonia (a recent model calculation is presented in \[12\], for an extensive review see \[13\]). We thus feel that free propagators should be a reasonable effective parameterization at least for heavy quarks. It is still an open question whether free propagators can also be applied to light quarks, and one has to investigate this problem within explicit models.

With an instantaneous BS-kernel and bare propagators with effective quark masses one can perform the \( p^0 \) integrals in the BS-equation in the rest frame of the bound state with mass \( M \) (see e.g. \[14\]) and thus arrives at the (full) Salpeter equation
\[
\Phi(p) = \int \frac{d^3p'}{(2\pi)^3} \Lambda^-_1(p') \gamma^0 (V(p', p') \Phi(p')) \gamma^0 \Lambda^-_2(-p') \]
\[ \frac{\Lambda^+_1(p') \gamma^0 [V(p', p') \Phi(p')\gamma^0 \Lambda^+_2(-p')]}{M + \omega_1 + \omega_2} \]
\[ \frac{-\int \frac{d^3p'}{(2\pi)^3} \Lambda^+_1(p') \gamma^0 [V(p', p') \Phi(p')\gamma^0 \Lambda^+_2(-p')]}{M - \omega_1 - \omega_2} \]
(7)
with \( \omega_i = \sqrt{\tilde{p}^2 + m_i^2} \) and the projection operators \( \Lambda^\pm_i(p) = (\omega_i \pm H_i(p))/2\omega_i \) on positive and negative energies. Here \( H_i(p) = \gamma^0 (\gamma\tilde{p} + m_i) \) is the standard Dirac Hamiltonian. We also have introduced the Salpeter amplitude \( \Phi \) by
\[
\Phi(p) = \int dp^0 \chi_P(p^0, p) \bigg|_{P=(M,0)} \]
(8)
For weakly bound states with \( |\tilde{p}|/m_i \ll 1 \) and \( M \approx m_1 + m_2 \) one has
\[
\frac{1}{M + \omega_1 + \omega_2} \ll \frac{1}{M - \omega_1 - \omega_2} \]
(9)
so that the first term in eq.(8) can be dropped. This leads to the so called reduced Salpeter equation, which has been used in various studies of relativistic bound states (see e.g. the work of Gara et al. \[15\] and references therein). In the case of light quarks, however, the use of the reduced Salpeter equation is dubious, especially for deeply bound states like the pion. Quark models for light quarks should therefore be based on the full Salpeter equation eq.(8).

Let \( \alpha, \beta \) in eq.(8) refer to Dirac indices in the standard Dirac representation of ref. \[14\]. Then \( \Phi \) is a 4\times4-matrix in spinor space that can be written in block matrix form as
\[
\Phi = \begin{pmatrix} \Phi^++ & \Phi^+_- \\ \Phi^-+ & \Phi^-_- \end{pmatrix} \]
(10)
where each component is a 2\times2-matrix. Applying \( \Lambda^+_1(p) \) from the left hand side and \( \Lambda^-_2(-p) \) from the right hand side to the Salpeter equation leads to
\[
\begin{align*}
\Lambda^+_1(p) \Phi(p) \Lambda^-_2(-p) &= 0 \\
\Lambda^-_1(p) \Phi(p) \Lambda^+_2(-p) &= 0
\end{align*}
\]
(11)
These relations allow us to express \( \Phi^+, \Phi^- \) in terms of \( \Phi^+, \Phi^- \) as
\[
\begin{align*}
\Phi^+ &= +c_1 \Phi^{++} - c_2 s \Phi^{--} \\
\Phi^- &= -c_1 \Phi^{--} + c_2 s \Phi^{++}
\end{align*}
\]
(12)
with the shorthand notation \( s = \delta\tilde{p}^2 \). \( c_1 = \omega_1/(\omega_1 m_2 + \omega_2 m_1) \). We thus find that \( \Phi \) can be written as
\[
\Phi = \hat{\Phi} (\Phi^{++}, \Phi^{--})
\]
(13)
with \( \hat{\Phi} \) being a bilinear function. One can interpret \( \Phi^{++} \) as the upper component and \( \Phi^{--} \) as the lower component of \( \Phi \), as can be seen in the nonrelativistic limit where \( \Phi^{--} \) vanishes for solutions that fulfill \( M \approx m_1 + m_2 \) and where \( \Phi^{++} + i\sigma_2 \) becomes the usual Schrödinger wave function.

For further discussion it is useful to rewrite the Salpeter equation in the form of an eigenvalue problem for the bound state mass \( M \). We follow the treatment of Lagaë \[14\] and define
\[
\psi(p) := \Phi(p) \gamma^0 [W(p, p') \psi(p')]
\]
(14)
\[
\psi(p) := \Phi(p) \gamma^0 [W(p, p') \psi(p')]
\]
(15)
The Salpeter equation can now be written as
\[
\mathcal{H}(\psi)(p) = M \psi(p)
\]
(16)
where
\[
\mathcal{H}(\psi)(p) = H_1(p) \psi(p) - \psi(p) H_2(p)
\]
\[
- \int \frac{d^3p'}{(2\pi)^3} \Lambda^+_1(p) [W(p, p') \psi(p')] \Lambda^-_2(p)
\]
\[
+ \int \frac{d^3p'}{(2\pi)^3} \Lambda^-_1(p) [W(p, p') \psi(p')] \Lambda^+_2(p)
\]
(17)
The equivalence of eq.(14) and eq.(15) can be shown by applying the projectors \( \Lambda^\pm_i \) to both equations from both sides. From eq.(14) one obtains e.g.
\[
\begin{align*}
\Lambda^+_1(p) \psi(p) \Lambda^+_2(p) &= 0 \\
\Lambda^-_1(p) \psi(p) \Lambda^-_2(p) &= 0
\end{align*}
\]
(18)
as in eq.(15), due to the relation \( \Lambda^\pm_i(p) \gamma^0 = \gamma^0 \Lambda^\pm_i(-p) \).
Note that eq.(18) can also be written in the concise form
\[
\frac{H_1}{\omega_1} \psi + \frac{H_2}{\omega_2} \psi = 0
\]
(19)
B. Normalization condition and scalar product

The normalization for general BS-amplitudes has been given for bound states with conserved quantum numbers by Nishijima \(^{[15]}\) and Mandelstam \(^{[16]}\). We follow Cutkosky \(^{[3]}\), who treated the more general case where no current has to be conserved. As this has already been treated within textbooks (e.g. \(^{[19]}\)), we will only give the result for the normalization. Let the bound state be normalized as \((P | P') = (2\pi)^3 2p^0 \delta^3(\vec{P} - \vec{P}')\). Then the normalization condition in graphical representation is given by Fig.2 (compare e.g. \(^{[21]}\)). Contracting with the momentum of the bound state, this reads explicitly:

\[
\int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \text{tr} \left[ \frac{\chi^\dagger(p)P^\mu d}{dP^\mu} \left( I(P, p, p') + iK(P, p, p') \right) \chi(p') \right] = 2iM^2
\]

(20)

The summation over color indices is suppressed here. \(I\) denotes the product of the inverse quark-propagators:

\[
I(P, p, p')_{\alpha\alpha', \beta\beta'} = \delta^{(4)}(p - p') (2\pi)^4 \left( S^e_{\alpha\alpha'}^{-1}(\eta_1 P + p) S^F_{\beta\beta'}^{-1}(-\eta_2 P + p) \right)
\]

(21)

Note that the vectorial condition of Fig.2 and the scalar normalization (20) are in fact equivalent which follows from: (a) the formal covariance of the equation and (b) the fact that in the rest frame the time component of Fig.2 gives eq.(20) and the space components vanish, as the derivative \(d/(dP^0)|I + K\) is proportional to \(p^i, p'^i\) or \(\gamma^i\) so that the integrals or the trace on the rhs. of eq.(21) give zero. For an interaction-kernel, which is instantaneous in the rest frame, i.e. of the type of eq.(22), we have:

\[
P^\mu d\frac{d^3p}{dP^\mu} V(p_\perp, p'_\perp) = 0
\]

(22)

so that the contributions of the interaction kernel to the normalization vanish. At this point we would like to mention that the BS-equation and the normalization condition for an instantaneous interaction may be formulated covariantly, so that the corresponding amplitudes \(\chi\) are correctly normalized in any frame. The explicit normalization for the corresponding Salpeter amplitudes \(\Phi\) \(^{[4]}\) will be performed in the rest frame. First we define the vertex functions:

\[
\Gamma_{\rho}(p) := [S^F(p_1)]^{-1} \chi(p) [S^F(-p_2)]^{-1}
\]

\[
\Gamma_{\rho'}(p) := [S^F(-p_2)]^{-1} \chi^\dagger(p) [S^F(p_1)]^{-1}
\]

(23)

With the BS equation (3) we then obtain the following important result:

\[
\Gamma_{\rho}(p) \big|_{p=(M, \delta)} = \Gamma(\vec{p}) = -i \int \frac{d^3\vec{p}}{(2\pi)^3} \text{tr} [V(\vec{p}, \vec{p}') \Phi(\vec{p}')] (24)
\]

i.e. the vertex-function depends only on the relative three-momentum \(\vec{p}\). This formula allows the reconstruction of the vertex function \(\Gamma\) and therefore of the full BS-amplitude \(\chi\) from the Salpeter amplitude \(\Phi\). Inserting eq.(23) into the normalization condition (21), the dependence on \(p^0\) is completely determined by the quark-propagators, so that the \(p^0\)-integration may be performed analytically.

We use the general relation between the BS-amplitude \(\chi\) and its adjoint \(\chi^\dagger\) for spin-1/2-fermions (see e.g. \(^{[14]}\) for the scalar case):

\[
\chi(p) = \frac{-1}{2\pi i} \int dq^0 \left( \frac{f(q^0, \vec{p})}{p^0 - q^0 + i\epsilon} + \frac{g(q^0, \vec{p})}{p^0 - q^0 - i\epsilon} \right)
\]

(25)

\[
\chi(p') = \frac{-1}{2\pi i} \gamma^0 \int dq^0 \left( \frac{f(q^0, \vec{p})}{p^0 - q^0 + i\epsilon} + \frac{g(q^0, \vec{p})}{p^0 - q^0 - i\epsilon} \right) \gamma^0
\]

with matrix valued functions \(f\) and \(g\). From this we derive the following relations in the special case of an instantaneous interaction:

\[
\Gamma(\vec{p}) = -\gamma_0 \Gamma^\dagger(\vec{p}) \gamma_0 \quad \Phi(\vec{p}) = \gamma_0 \Phi^\dagger(\vec{p}) \gamma_0
\]

(26)

This leads to the normalization condition for the Salpeter-amplitudes in the rest frame:

\[
\int \frac{d^3p}{(2\pi)^3} \text{tr} \left\{ \Phi^\dagger(\vec{p}) \Lambda_1^+ (\vec{p}) \Phi(\vec{p}) \Lambda_2^- (-\vec{p}) \right\} = (2\pi)^2 2M
\]

(27)

It also may be expressed in terms of the \(2 \times 2\) amplitudes \(\Phi^{++}\) and \(\Phi^{--}\) defined in eq.(10) as:

\[
\int \frac{d^3p}{(2\pi)^3} \frac{2\omega_1 \omega_2}{\omega_1 m_2 + \omega_2 m_1} \text{tr} \left\{ (\Phi^{++}(\vec{p}))^\dagger \Phi^{++}(\vec{p}) - (\Phi^{--}(\vec{p}))^\dagger \Phi^{--}(\vec{p}) \right\} = (2\pi)^2 2M
\]

(28)

This form also shows the connection to the nonrelativistic norm: in the NR limit \(\Phi^{++}(\vec{p}) i\sigma_0 \) becomes the usual Schrödinger wave function and \(\Phi^{--}(\vec{p})\) goes to zero as \(\vec{p}^2/m^2 \cdot \Phi^{++}(\vec{p})\). Furthermore the weight function becomes equal to unity so that we obtain the usual Schrödinger normalization. For deeply bound states however we have appreciable deviations from this norm, as the lower amplitude \(\Phi^{--}\) is of the same order as the upper component \(\Phi^{++}\) (see Sec.IIC).

Eq.(27) motivates the definition of a scalar product for amplitudes \(\psi_1 = \Phi^{++} \) and \(\psi_2 = \Phi^{--}\) as:

\[
\langle \psi_1 | \psi_2 \rangle = \int \text{tr} \left( \psi_1 \Lambda_1^+ \psi_2 \Lambda_2^- - \psi_1 \Lambda_1^- \psi_2 \Lambda_2^+ \right) = \frac{1}{2} \int \text{tr} \left[ \psi_1 \left( \frac{H_1}{\omega_1 \psi_2 - \psi_2 \frac{H_2}{\omega_2} \psi_2} \right) \right]
\]

(29)

with all quantities depending on \(\vec{p}\) and the notation \(\int = \int d^3p/(2\pi)^3\). Note that this scalar product is not positive
definite. The normalization condition \( \langle \psi | \psi \rangle = (2\pi)^2 2M \) for solutions of the Salpeter equation is then given as

\[
\langle \psi | \psi \rangle = (2\pi)^2 2M
\]

(30)

The following discussion will be restricted to amplitudes satisfying eq. (19). In that case one has

\[
\langle \psi_1 | H \psi_2 \rangle = \int (\omega_1 + \omega_2) \text{tr} \left( \psi_1^\dagger W \psi_2 \right) - \int \int \text{tr} \left( \psi_1^\dagger W \psi_2^\dagger \right)
\]

(31)

where the prime indicates the dependence of \( \psi_2 \) on \( \bar{p}' \).

The first point can be seen easily from the structure of the BS-equation under charge conjugation. The general case of unequal quark masses and with opposite sign. In the following we will extend this to the equal mass case we first investigate solutions with nonzero norm \( \langle \psi_1 | \psi_2 \rangle \neq 0 \).

C. Structure of the Solutions

The Salpeter equation exhibits some further general structures connecting solutions with positive and negative eigenvalues. For the case of equal quark masses J.F. Lagaé has shown that for kernels satisfying \( \langle W \psi | \psi \rangle = W \psi^\dagger \) the eigenvalues will come in pairs of opposite sign, the corresponding eigen functions having normalizations with opposite sign. In the following we will extend this result to the general case of unequal masses and compare the block structure of the conjugate solutions. We further show that nondegenerate bound states with mass \( M \) have zero norm \( \langle \psi | \psi \rangle = 0 \). The discussion of physical acceptable solutions is postponed to the end of this section.

To deal with the unequal mass case we first investigate the structure of the BS-equation under charge conjugation. The details are shown in the appendix with the result that solutions of

\[
\langle H_{f_1 f_2} \psi_{f_1 f_2} | \bar{p} \rangle = M \psi_{f_1 f_2} (\bar{p}) \\
\langle H_{f_2 f_1} \psi_{f_2 f_1} | \bar{p} \rangle = M \psi_{f_2 f_1} (\bar{p})
\]

(33)

are related through

\[
\psi_{f_1 f_2} (\bar{p}) = -S_C \psi_{f_2 f_1} (-\bar{p}) S_C
\]

(34)

with \( S_C \) given in eq. (34) in the appendix. The indices of \( H \) denote the flavor dependence of \( H_1 \) and \( \Lambda_1^\pm \) in eq. (11). For simplicity only the case without flavor mixing is considered, the generalization being straightforward.

In the following discussion we will assume that the BS-kernel satisfies \( (W_{f_1 f_2} \psi_{f_1 f_2})^\dagger = W_{f_2 f_1} (\psi_{f_2 f_1})^\dagger \) which is fulfilled e.g. for kernels of the form \( W_{f_1 f_2} \psi_{f_1 f_2} = \int ((\bar{p} - \bar{p}'^2)^2 \Gamma_1 \psi_1 \psi_2^\dagger \Gamma_2 \) with hermitian matrix \( \Gamma \).

The hermitian conjugate of eq. (16) thus leads to

\[
- (H_{f_1 f_2} \psi_{f_1 f_2})^\dagger = H_{f_2 f_1} \psi_{f_2 f_1}^\dagger = -M^* \psi_{f_2 f_1}^\dagger
\]

(35)

Renaming \( f_1 \leftrightarrow f_2 \) and comparing this equation to eq. (34) we thus have shown that

- solutions of the Salpeter equation come in pairs \( \langle \psi_{f_1 f_2}, M \rangle \) and \( \langle \psi_{f_2 f_1}, -M^* \rangle \) where eq. (34) connects the two solutions.

Consider the normalization of \( \psi_{f_1 f_2} \) and \( \psi_{f_2 f_1}^\dagger \): With eq. (34) and the relation \( S_C \Lambda_1^\pm (\bar{p}) S_C = -\Lambda_1^\mp (\bar{p}) \) one finds \( \langle \psi_{f_2 f_1} | \psi_{f_2 f_1}^\dagger | \psi_{f_1 f_2} \rangle = \langle \psi_{f_2 f_1}^\dagger | \psi_{f_2 f_1} | \psi_{f_1 f_2} \rangle \) where the indices of the scalar product determine the flavors of \( \Lambda_1^\pm \) and \( \Lambda_2^\pm \) in eq. (23). On the other hand cyclic permutation under the trace shows that

\[
\langle \psi_{f_2 f_1} | \psi_{f_2 f_1}^\dagger | \psi_{f_1 f_2} \rangle = -\langle \psi_{f_2 f_1}^\dagger | \psi_{f_2 f_1} | \psi_{f_1 f_2} \rangle
\]

(35)

and we find that

- the states with eigenvalues \( M \) and \( -M^* \) have opposite norm.

Let us now compare the block matrix structure of the two conjugated solutions. With the angular momentum decomposition eq. (35) (see appendix)

\[
\Phi^{++}(\bar{p}) = \sum_{L S} R_{LS}^{(+)}(p) [Y_L(\Omega_p) \otimes \varphi_S]^J
\]

\[
\Phi^{--}(\bar{p}) = \sum_{L S} R_{LS}^{(-)}(p) [Y_L(\Omega_p) \otimes \varphi_S]^J
\]

(37)

with the 2×2-matrices \( \varphi_{00} = 1/\sqrt{2}, \varphi_{1q} = \sigma_q/\sqrt{2} \) it is straightforward to show that

\[
\left[ \Phi_{f_2 f_1}^{+} \right]_{J M J} (M_J) = (-1)^{-J-M_J} \sum_{L S} (-1)^{L+S} \left[ R_{LS}^{(+)}(p) \right]_{f_2 f_1} [Y_L(\Omega_p) \otimes \varphi_S]^J_{-M_J}
\]

(38)

and similarly for \( \Phi^{--} \). It is shown in the appendix that

\[
\left[ R_{LS}^{(+)}(p) \right]_{f_2 f_1} = (-1)^{L+S} \left[ R_{LS}^{(+)}(p) \right]_{f_1 f_2}
\]

Since \( L \) and \( S \) are integer the phase vanishes and we obtain the result
\[
\left(\Phi^{++}_{f_2 f_1}\right)_{JM_J}^\dagger = (-1)^{J-M_J}\left(\Phi^{++}_{f_1 f_2}\right)_{J-M_J}^\dagger \tag{39}
\]

According to eqs.\((13), (14)\) we write
\[
\psi^\dagger_{f_2 f_1} = \gamma^0 \left[\Phi_{f_2 f_1}(\phi_{f_2 f_1}^{++}, \phi_{f_2 f_1}^{--})\right]^\dagger \tag{40}
\]
where the indices of \(\Phi\) indicate the flavor dependence of \(c_i\) in eq.\((12)\). The hermitian conjugate of eq.\((18)\) with \(f_1\) and \(f_2\) interchanged gives \(\Lambda^+_f \psi^\dagger_{f_2 f_1} \Lambda^+_f = 0\) and \(\Lambda^-_f \psi^\dagger_{f_2 f_1} \Lambda^-_f = 0\) so that we can write
\[
\psi^\dagger_{f_2 f_1} = \Phi_{f_1 f_2} \left(\xi^{++}_{f_1 f_2}, \xi^{--}_{f_1 f_2}\right) \gamma^0 \tag{41}
\]
with some amplitudes \(\xi^{++}, \xi^{--}\) that can be determined by comparing eq.\((11)\) to eq.\((41)\) with the result \(\xi_{f_1 f_2}^{++} J_{M_J} = -\left[\Phi_{f_2 f_1}^{++}\right]_{J-M_J}^\dagger = (-1)^{1-J-M_J}\left[\Phi_{f_2 f_1}^{--}\right]_{J-M_J}^\dagger\) and the same expression with \(++\) and \(--\) interchanged. According to eq.\((11)\) we thus can write
\[
\left(\psi^{J_{M_J}}_{f_2 f_1}\right)^\dagger = (-1)^{1-J-M_J} \Phi_{f_1 f_2} \left[\Phi_{f_2 f_1}^{--} \right]^\dagger J_{M_J} \tag{42}
\]
where
\[
\left[\Phi_{f_2 f_1}^{--} \right]^\dagger J_{M_J} = \left[\Phi_{f_1 f_2}^{++} \right]^\dagger J_{M_J} \tag{43}
\]

We thus have the result that

- exchanging the functions \(\Phi^{++}\) and \(\Phi^{--}\) in \(\Phi = \Phi^{++} \oplus \Phi^{--}\) turns an amplitude with eigenvalue \(M\) into an amplitude with eigenvalue \(-M^*\).

With the relations obtained above it is easy to investigate the eigenvalue \(M = 0\) which is assumed to be not degenerate apart from the trivial degeneracy in the angular momentum projection \(M_J\). From eqs.\((15), (16)\) we have
\[
\mathcal{H}_{f_1 f_2} \psi_{f_1 f_2} = 0 \quad \mathcal{H}_{f_2 f_1} \psi^\dagger_{f_2 f_1} = 0 \tag{43}
\]
which through eq.\((42)\) implies \(\psi^{J_{M_J}}_{f_2 f_1} = \lambda \psi^{J_{-M_J}}_{f_2 f_1}\) with |\(\lambda| = 1\). Then eq.\((30)\) gives
\[
\left<\psi^{J_{M_J}}_{f_2 f_1} \right| \psi^{J_{M_J}}_{f_2 f_1} \right>_{f_2 f_1} = -\left<\left(\psi^{J_{M_J}}_{f_2 f_1}\right)^\dagger \left(\psi^{J_{M_J}}_{f_2 f_1}\right)^\dagger\right>_{f_2 f_1} = -\left<\psi^{J_{-M_J}}_{f_1 f_2} \right| \psi^{J_{-M_J}}_{f_1 f_2} \right>_{f_1 f_2} \tag{44}
\]
Since the scalar product is invariant under rotations one can substitute \(-M_J\) by \(M_J\) and obtains
\[
\left<\psi^{J_{M_J}}_{f_1 f_2} \right| \psi^{J_{M_J}}_{f_1 f_2} \right>_{f_1 f_2} = 0 \tag{45}
\]
So we find that

- nondegenerate eigen functions with eigenvalue \(M = 0\) have zero norm.

From eq.\((42)\) it is clear that \(\psi^{J_{M_J}}_{f_2 f_1} = \lambda \psi^{J_{-M_J}}_{f_1 f_2}\) is equivalent to setting
\[
\Phi^{++} = \pm \Phi^{--} \quad \tag{46}
\]
This equation illustrates a common aspect of the Salpeter equation: in the nonrelativistic limit with \(M \approx m_1 + m_2\) the large component \(\Phi^{++}\) dominates over the small component \(\Phi^{--}\), but if one goes to deeply bound states (e.g. by increasing the coupling constant of an attractive interaction) the two components become more and more equal until finally \(\Phi^{++} = \pm \Phi^{--}\) is achieved for \(M = 0\).

From the discussion above it has become clear that we have to identify the physically acceptable solutions. There are two criteria making sure that a solution is acceptable:

- The norm of the solution has to be nonzero which automatically implies that \(M\) is real.
- The eigenvalue \(M\) and the norm have to be positive in order to fulfill the normalization condition \(\langle \psi | \psi \rangle = (2\pi)^2 2M\).

We would like to mention that the typical doubling of the physical eigenvalues is well known from the RPA equations in nonrelativistic many particle theory \([10]\). For a hamiltonian \(H\) with spectrum \(E_n\) the RPA equations have solutions \(E_n, -E_n\). This doubling can be traced back to the appearance of the time ordering operator \(T\) in the definition of the particle-hole propagator. Therefore neglecting negative mass eigenvalues is consistent with the RPA structure of the Salpeter equation (compare chap.\([11]\)).

The role of the solutions with \(M = 0\) and \(\langle \psi | \psi \rangle = 0\) is not clear. On one hand there is a priori no contradiction with the normalization condition. On the other hand the Salpeter equation has been obtained in the rest frame of the bound state, i.e. one first performs the limit \(\vec{P} \to 0\) and then investigates the case \(M \to 0\). However, the correct procedure for massless bound states is first to perform the limit \(M \to 0\) in the BS-equation. In the resulting equation one then can study the limit \(\vec{P} \to 0\). It cannot generally be expected that exchanging the limits for \(\vec{P}\) and \(M\) leads to equivalent results (compare \([3]\) for a more detailed discussion of this problem). Furthermore the definition of the instantaneous interaction kernel eq.\((3)\) becomes dubious since \(p_\perp\) is not well defined for \(P \to 0\). We therefore prefer to require \(M > 0\) and \(\langle \psi | \psi \rangle > 0\) for physically acceptable solutions.

### III. NUMERICAL TREATMENT

From the definition of \(\Phi\) in eq.\((13)\) it is easy to derive a basis expansion for \(\psi = \Phi \gamma^0\). Let
\[
E_i(\vec{p}) = R_{n, L_i}(\vec{p}) \left[ Y_{L_i}(\Omega_{n_p}) \otimes \phi_{S_i}\right]_{M_J} \tag{47}
\]
be a complete set of $2 \times 2$ basis functions with real radial functions $R_{n,L}(p)$. The basis functions are chosen orthonormal with respect to the usual scalar product given by

$$
(E_i | E_j) = \int \frac{d^3 p}{(2\pi)^3} \mathrm{tr} \left[ E_i^+(\tilde{p}) E_j(\tilde{p}) \right] = \delta_{ij}
$$

(48)

where the trace just gives the usual scalar product for the spin matrices $\mathrm{tr} \varphi_{SM_S'}\varphi_{SM_S} = \delta_{SS'} \delta_{M_S M_S'}$. Note that the angular structure of the basis functions matches the structure of $\Phi^{++}$ and $\Phi^{-}$ as given in eq. (B4). We choose $R_{n,L}(p)$ to be real functions. It is now possible to expand

$$
\Phi^{++}(\tilde{p}) = \sum_{i=1}^{\infty} a_i^{(+)} E_i(\tilde{p})
$$

(49)

$$
\Phi^{-}(\tilde{p}) = \sum_{i=1}^{\infty} a_i^{-} E_i(\tilde{p})
$$

(50)

which implies $\mathcal{R}_{LS}^I(p) = \sum_{i=1}^{\infty} a_i^{(+) R_{n,L}(p)} \delta_{LL}, \delta_{SS}$, for the radial wave function. Since $\mathcal{R}_{LS}^I(p)$ are real functions in most cases of interest, the coefficients $a_i^{(\pm)}$ then also have to be real. Now define the $4 \times 4$-amplitudes

$$
e^{(+) } = \tilde{\Phi} (E_1, 0) \gamma^0
$$

$$
e^{-} = \tilde{\Phi} (0, E_1) \gamma^0
$$

(51)

Note that these functions are not orthogonal with respect to the scalar product given in eq. (29). Since $\Phi$ is bilinear we nevertheless can expand $\psi$ as

$$
\psi = \sum_{i=1}^{\infty} \left( a_i^{(+)} e^{(+)} + a_i^{-} e^{-} \right)
$$

(52)

so that the constraint $\Lambda_1^+ \psi \Lambda_2^+ = \Lambda_1^- \psi \Lambda_2^- = 0$ is automatically fulfilled. The Salpeter equation $\mathcal{H} \psi = M \psi$ can now be written as the matrix equation

$$
\begin{pmatrix}
H^{++} & H^{-+} \\
H^{-+} & H^{--}
\end{pmatrix}
\begin{pmatrix}
(a_i^{(+)} \\
(a_i^{-})
\end{pmatrix} = M
\begin{pmatrix}
N^{++} & N^{+-} \\
N^{+-} & N^{--}
\end{pmatrix}
\begin{pmatrix}
(a_i^{(+)} \\
(a_i^{-})
\end{pmatrix}
$$

(53)

with $H_{ij}^{ss'} = \langle e_i^{(s)} | \mathcal{H} | e_j^{(s')} \rangle$ and $N_{ij}^{ss'} = \langle e_i^{(s)} | e_j^{(s')} \rangle$. From the definition of the scalar product one easily sees that $N^{++} = -N^{--}$ and $N^{+-} = N^{+-} = 0$. Furthermore we find from eqs. (32), (33) that $e^{(+)}$ and $e^{(-)}$ are connected by $| (e_i^{(+)}, f_{M,J}) \rangle = (1)^{1-J-M_S^2} (e_i^{(-)}) J^{-M_S} f_{M,J}$, so that we can use eqs. (32), (33) as well as the invariance of the scalar product under flavor exchange and under the replacement $-M_J \rightarrow M_J$ to obtain $H^{ii}_{ij} = (H_{ij}^{++})^*$ and $H^{ii}_{ij} = (H_{ij}^{--})^*$. The matrix representation of the Salpeter equation thus takes the form

$$
\begin{pmatrix}
H^{++} & H^{-+} \\
(H^{--})^* & (H^{+-})^*
\end{pmatrix}
\begin{pmatrix}
(a_i^{(+)} \\
(a_i^{-})
\end{pmatrix} = M
\begin{pmatrix}
N^{++} & 0 \\
0 & N^{--}
\end{pmatrix}
\begin{pmatrix}
(a_i^{(+)} \\
(a_i^{-})
\end{pmatrix}
$$

(54)

which is of the same form as the well-known RPA equations in nuclear physics [10]. Let $(a^{(+)}, a^{(-)})$ be an eigenvector with eigenvalue $M$. Then eq. (54) shows that $(a^{(-)} *) ^* , (a^{(+)} *) ^*$ is an eigenvector with eigenvalue $-M*$ which is just the result of the previous section. Usually the Salpeter hamiltonian $\mathcal{H}$ has the property that the matrix elements $H_{ij}^{ss'}$ and also the eigenvector coefficients $a_i^{(\pm)}$ are real within the basis given above. In that case and since $N_{ij}^{ss'}$ is real, also $M$ must be real. This result has already been shown before for eigen vectors with nonzero norm.

Furthermore we see that if $M = 0$ is an eigenvalue we expect from eq. (54) that the eigenvector fulfills $a^{(+) } = \pm a^{(-)}$ and we reobtain the result that this solution has zero norm.

In a numerical treatment only a finite basis $i \leq i_{\text{max}} \approx 10$ can be taken into account. Then eq. (54) becomes a finite matrix equation that can be solved with standard numerical methods. One thus obtains an approximate eigenvalue $M_{\beta}$ and an approximate eigen function $\psi_{\beta}$ to the Salpeter equation that exactly fulfill the relation

$$
\langle \psi_{\beta} | \mathcal{H} | \psi_{\beta} \rangle = M_{\beta} \langle \psi_{\beta} | \psi_{\beta} \rangle
$$

(55)

The index $\beta$ indicates that the basis states $E_i$ and thus $\psi$ depend on a variational parameter $\beta$ of dimension $MeV^{-1}$ that sets the absolute scale for the momentum dependence via $E^2(p) = \beta^2/2 E^2_{\text{th}}(p\beta)$. If solutions of nonzero norm are considered the Salpeter equation is equivalent to the variational problem

$$
\delta M[\psi] = \frac{\delta \langle \psi | \mathcal{H} | \psi \rangle}{\langle \psi | \psi \rangle} = 0
$$

(56)

where the variation $\delta$ is taken over all functions $\psi$ with nonzero norm that fulfill $\Lambda_1^+ \psi \Lambda_2^+ = \Lambda_1^- \psi \Lambda_2^- = 0$. According to eq. (52) we make the variational ansatz $\psi = \beta \chi$ implying $M[\psi] = M_\beta$ and look for stationary points of $M_\beta$ as a function of $\beta$ (which also fixes $\beta$ for each meson).

The calculation of the matrix elements within an explicit model will be postponed to our second paper. At this point we would only like to make a few technical comments. The matrix elements of the interaction kernel can be efficiently calculated by inserting two complete sets of basis functions written schematically as

$$
\langle i | f_1(p) V(r) f_2(p') | j \rangle = \sum_{g,h} \langle i | f_1(p) | g \rangle \langle g | V(r) | h \rangle \langle h | f_2(p') | j \rangle
$$

(57)

so that $V(r)$ can be parameterized in coordinate space. A suitable choice for the basis functions is given by the functions $R_{n,L}(y) = N_{n,L} y^2 L_n^{2L+2}(y) e^{-y^2/2}$ with $y = p\beta$ and $L_n^{2L+2}(y)$ being a Laguerre polynomial. We found that about ten basis states are sufficient to solve the Salpeter equation with rather high accuracy. The choice of 3-dimensional harmonic oscillator functions is less favored.
since their asymptotic behavior \( \sim e^{-y^2/2} \) for \( y \to \infty \) turns out to be not appropriate for our quark model (especially for deeply bound states like the Pion).

### IV. DECAY OBSERVABLES

Apart from describing the mass spectrum of mesons, any realistic model must also be able to describe mesonic transitions and decays. The important question arises whether a good description of the extremely deep bound states as pion or kaon can be combined with a reasonable description of confinement. The BS formalism offers a natural framework, as the role of the lower component of the wave function turns out to be crucial for the correct normalization and calculation of the decays. This can be seen most clearly in the following formulas for the leptonic decays and the weak decay constants.

We furthermore show how to reconstruct the full BS amplitude, which may be used as a starting point for the calculation of electromagnetic or hadronic transitions.

#### A. Leptonic decay width and weak decay constant

The transition or decay of bound states are calculated from BS amplitudes using the formalism given by Mandelstam [17]. We will merely sketch it by considering first the leptonic decays of vector mesons. The corresponding Feynman diagram is given by Fig. 3 where \( K_{\ell\bar{r}} \) denotes the internal line for the incoming \( q\bar{q} \)-pair and the outgoing \( l^+l^- \)-pair. If we consider only graphs of leading order in the electromagnetic coupling constant we obtain the approximation on the right hand side.

The exceptional role of these decays (together with the weak decay constants) is that if the BS kernel would be not appropriate for our quark model (especially for deeply bound states like the Pion).

The radial amplitudes \( \Phi^{\pm\pm}_{\ell M}(p) \) are determined completely by the functions \( \Phi^{++} \) and \( \Phi^{--} \). For a \( 1^- \) meson the latter have the form:

\[
\Phi^{\pm\pm}_{1 M}(p) = \mathcal{R}^{(\pm)}_{01}(p) [Y_0(\Omega_\mu) \times \varphi_1]_{1 M} + \mathcal{R}^{(\pm)}_{21}(p) [Y_2(\Omega_\mu) \times \varphi_1]_{1 M}
\]  

(60)

The integration and trace in eq.(58) for the current pick up only the s-wave amplitude \( \Phi^{01}_{01} \). The usual spin summation and averaging leads to the decay width:

\[
\Gamma(1^- \to l^+l^-) = 24 \frac{\alpha^2 e^2}{M^3} \left| \int \frac{p^2 dp}{(2\pi)^3} \left( \mathcal{R}^{(+)}_{01}(p) - \mathcal{R}^{(-)}_{01}(p) \right) \right|^2\]

(61)

where \( \alpha = 1/137 \) is the electromagnetic coupling constant and \( \tilde{e}_q \) is the fraction of the quark charge compared to the electron. Apart from the contribution of the lower component \( \Phi^{--} \) to the normalization eq.(25), its importance comes out here very clearly. For \( M \) we use the experimental meson mass to obtain the correct phase space.

The next observables to be considered are the weak decay constants \( f_\pi \) and \( f_K \). They are defined by the matrix element of the axial current \( [21] \) (with this definition \( f^{(exp)}_\pi = 132 \text{ MeV} \)):

\[
i f_\pi P_\mu = \left< 0 \left| J_\mu^{(0)}(0) \right| P 0^- \right>  
\]

(62)

Again for an instantaneous interaction this can be evaluated from the Salpeter amplitude

\[
f_\pi = \left| \frac{\sqrt{3}}{M} \int \frac{d^3 p}{(2\pi)^3} tr (\Phi(p) \gamma_5 \tilde{e}_q) \right| \quad (63)
\]

and in terms of the \( \Phi^{\pm\pm} = \mathcal{R}^{(\pm)}_{00} Y_{00} \varphi_0 \):

\[
f_\pi = \left| \frac{\sqrt{3}}{M} \int \frac{d^3 p}{(2\pi)^3} tr (\Phi^{++}(p) - \Phi^{--}(p)) \right| \quad (64)
\]

#### B. The decay \( \pi^0, \eta \to 2\gamma \)

These decays provide another test for the description of the low lying pseudoscalar mesons. To our knowledge they have not been calculated in the framework of the full Salpeter equation (for a slightly more restricted ansatz see Mitra et al. [22]). The basic idea here is to reconstruct the vertex function \( \Gamma \) from the Salpeter amplitude \( \Phi \) by means of the BS-eq.(24) itself, which gives the full fourdimensional structure of the BS amplitude \( \chi \). This has to be taken into account for a correct description of decays. The corresponding Feynman diagrams for
the neutral pseudoscalar decay in lowest order of the 
interaction are given in Fig.4. Of course this is not correct 
to any order in the strong interaction like e.g. in the case 
of the pion decay constant, as we obviously neglect the 
strong interaction of the intermediate quark. Therefore 
we expect these calculations to be less accurate.

The T-matrix element in lowest order for a meson with 
mass \( M \) decaying into two photons with momenta \( k_1, k_2 \) 
given by

\[
T = -i \sqrt{3} (ie_q)^2 \int \frac{d^4p}{(2\pi)^3} \text{tr} \left\{ S^F(p/2 + p) \Gamma(p \bar{p}) \right. \\
\left. + \frac{1}{2} S^F(p/2 + p - k_1) \hat{\theta}_2 \\
+ \frac{1}{2} S^F(p/2 + p - k_2) \hat{\theta}_1 \right\}
\]

(65)

Therefore we have to reconstruct the vertex function 
\( \Gamma \) from the Salpeter amplitude \( \Phi \) by means of eq. (24). 
Both of them have an angular decomposition according 
to eq. (22). The four scalar amplitudes depending only 
on \( p_\perp \) are expanded in a basis of radial wave functions. 
The quark propagator can be written as

\[
S^F(p) = i \left( \frac{\Lambda^+(\bar{p} \gamma^0)}{p_0 - \omega + i\epsilon} + \frac{\Lambda^-(\bar{p} \gamma^0)}{p_0 + \omega - i\epsilon} \right) \gamma^0
\]

(66)

which is suitable to perform the \( q^0 \) integration using the 
residue theorem. Then we substitute \( \bar{p} \to -\bar{p} \) in the 
second term of eq. (23) and use \( \Gamma(-\bar{p}) = -\gamma^0 \Gamma(\bar{p}) \gamma^0 \) 
for pseudoscalar mesons which follows from eq. (23). We use \( \{ \gamma^0, \hat{\theta}_i \} = 0 \) for transversal photons and define the quantities

\[
\Theta^\pm(\bar{p} \to k) := \hat{\theta}_2 \Lambda^\pm(\bar{p} \gamma^0) \gamma^0 \hat{\theta}_1 - \hat{\theta}_1 \Lambda^\pm(\bar{p} \gamma^0) \gamma^0 \hat{\theta}_2
\]

(67)

with \( \omega_k := ((\vec{k} - \vec{p})^2 + m^2)^{1/2} \). Furthermore the product of the vertex function with the projection operators \( \Lambda^\pm \) 
is denoted as:

\[
\Gamma^+(\bar{p}) := \Lambda^+(\bar{p}) \gamma^0 \Gamma(\bar{p} \gamma^0) \gamma^0 \Lambda^-(\bar{p})
\]

(68)

and \( \Gamma^{++} \) etc. analogously. In concrete calculations these 
quantities are evaluated using a matrix formalism, which 
represents the multiplication of \( \Gamma \) with a Dirac matrix 
by a transformation amongst the scalar amplitudes (see \( \Xi^3 \)). After some calculation we obtain:

\[
T = -i \sqrt{3} (ie_q)^2 \int \frac{d^3p}{(2\pi)^3} \sum_{\pm} \text{tr} \left\{ \pm \frac{\Gamma^{\pm\pm} \Theta^\pm(\bar{p} \to k)}{(M/2 \pm \omega_k \pm \omega)} \right. \\
\left. \mp \frac{\Gamma^{\pm\pm} \Theta^\pm(\bar{p} \to -k)}{(-M/2 \pm \omega_k \pm \omega)} \right\}
\]

(69)

where the summation \( \pm \) runs over the upper and lower 
sign. For \( \Theta \) we obtain

\[
\Theta^\pm(\bar{p} \to k) = i(\varepsilon_1 \times \varepsilon_2) \cdot \Sigma^\pm(\bar{p} \to k)
\]

(70)

with

\[
\Sigma^\pm(\bar{p} \to k) := \gamma^5 \left( \frac{\gamma_4 \pm 1}{\omega_k} (-\gamma^0(\bar{p} \to k) + m \gamma^0 \gamma_5) \right)
\]

(71)

To avoid the zeros in the denominator for \( M > 2m \) in 
eq (74) we use the BS equation in the form (7) to obtain 
finally (with \( \bar{\Sigma}^0 = \bar{\Sigma}^+ - \bar{\Sigma}^- \)):

\[
T = \sqrt{3} e_q^2 (\varepsilon_1 \times \varepsilon_2) \cdot \bar{S}(\bar{k})
\]

(72)

\[
\bar{S}(\bar{k}) := \int \frac{d^3p}{(2\pi)^3} \frac{1}{M^2/4 - (\omega_k + \omega)^2} \text{tr} \left\{ \Gamma^{++}(\bar{p}) \bar{\Sigma}^-(\bar{p} \to k) - \Gamma^{--}(\bar{p}) \bar{\Sigma}^+(\bar{p} \to k) \right. \\
\left. - \frac{M}{M/2 + \omega_k + \omega} \Gamma^0(\bar{p}) \bar{\Sigma}^0(\bar{p} \to k) \right\}
\]

(73)

As for transverse photons we have \( \varepsilon_1 \perp \vec{k} \perp \varepsilon_2 \), for 
\( \vec{k} = k \varepsilon_z \) the only nonvanishing component of \( \varepsilon_1 \times \varepsilon_2 \) 
is in z-direction and we only need to calculate the z-
component of \( \bar{S} \). The standard formula for the decay 
rate of a particle with mass \( M \) then yields:

\[
\Gamma(\pi^0, \eta \to 2\gamma) = \frac{3 \alpha^2 e_q^4}{16 \pi M^2} |S_z(k\varepsilon_z)|^2
\]

(74)

V. CONCLUSION

We have investigated the structure of the instantaneous 
BS-equation (Salpeter equation) for the general 
case of unequal quark masses. Furthermore we have de-
developed a numerical scheme to solve the Salpeter equa-
tion which enables the calculation of mass spectra and 
Salpeter amplitudes. In order to test various models 
beyond the mere reproduction of the mass spectra, we 
have further given explicit formulas for the computation 
of weak meson decay constants (\( f_\pi, f_K \) etc.), the decay 
widths into two photons and into an electron-positron 
pair.

Because of the relativistic kinematics, the correct relativis-
tic normalization of the amplitudes and the dynamical 
treatment of the lower component \( \Phi^- \) we expect the 
Salpeter equation to provide a framework for quark mod-
els that is superior to other treatments like the reduced 
Salpeter equation or the nonrelativistic quark model. We 
will investigate an explicit quark model based on the 
Salpeter equation in a subsequent paper [1].
APPENDIX A: SPECIAL LORENTZ TRANSFORMATIONS

Let $\Lambda$ be a special Lorentz transformation and $g$ be the corresponding element of the covering group $SL(2,C)$, given by

$$g \sigma(x) g^\dagger = \sigma(\Lambda x) \quad (A1)$$

with $\sigma(x) = x^\nu \sigma_\nu$ and $(\sigma_\nu) = (1, \vec{0})$. The transformation matrix $S_g$ for Dirac spinors in the Weyl representation is then given by

$$S_g = \begin{pmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{pmatrix} \quad (A2)$$

With the transformation matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (A3)$$

we can use $S_g^B = BS_g^W B^{-1}$ to transform $S_g$ into the standard Dirac basis employed in this work. The result reads

$$S_g = \frac{1}{2} \begin{pmatrix} g + (g^\dagger)^{-1} & g - (g^\dagger)^{-1} \\ g - (g^\dagger)^{-1} & g + (g^\dagger)^{-1} \end{pmatrix} \quad (A4)$$

Note that the relation $N_\nu \gamma^\nu = S_g^{-1} \gamma^\nu S_g$ is automatically fulfilled. For a boost $\Lambda$ with $\Lambda(M, \vec{0}) = P$ and $P^2 = M^2$ one has explicitly

$$g = \left[ \sigma \left( \frac{P}{M} \right) \right]^{1/2} = \sqrt{\frac{M}{2(M + P^0)}} \begin{pmatrix} 1 + \sigma \left( \frac{P}{M} \right) \\ 1 - \sigma \left( \frac{P}{M} \right) \end{pmatrix} \quad (A5)$$

and for a 3-dimensional rotation $\Lambda = R$ one has $g = u \in SU(2)$ and therefore

$$S_u = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \quad (A6)$$

in the Weyl basis as well as in the standard Dirac basis.

The transformation of a Dirac field operator $\Psi(x)$ is given by

$$U_g \Psi(x) U_g^{-1} = S_g^{-1} \Psi(\Lambda x) \quad (A7)$$

$$U_g \bar{\Psi}(x) U_g^{-1} = \bar{\Psi}(\Lambda x) \quad (A8)$$

with $U_g$ being a unitary operator. Consider a state with momentum $P$ and $P^2 = M^2$, total angular momentum $J$ and 3-component $M_J$ normalized as $\langle P', J', M'_J | P, J, M_J \rangle = (2\pi)^3 2P^0 \delta^3(\vec{P}' - \vec{P}) \delta_{J,J'} \delta_{M'_J, M_J}$. Then the action of $U_g$ on this state is given by

$$U_g |P, J, M_J \rangle = \sum_{M'_J} |\Lambda P, J, M'_J \rangle D_{M'_J, M_J}^\dagger (u) \quad (A9)$$

where

$$D_{M'_J, M_J}^\dagger (u)$$

is a Wigner D-function with $D_{M'_J, M_J}^\dagger (u) = u_{M'_J, M_J}$ and $u = |\sigma(\Lambda P/M)|^{1/2} g |\sigma(P/M)|^{1/2}$ is the Wigner rotation.

From the definition of the BS-amplitude in Sec. II A one can see by inserting the unity operator $1 = U_g^{-1} U_g$ that the BS-amplitude transforms as

$$\chi_{P'}^P (p) = \sum_{M'_J} S_g^{-1} \chi_{AP}^P (Ap) S_g D_{M'_J, M_J}^\dagger (u) \quad (A10)$$

The BS-equation is compatible with this transformation law for covariant kernels.

APPENDIX B: ANGULAR DECOMPOSITION OF THE $2 \times 2$-AMPLITUDES

Let $\Lambda = R$ be a 3-dimensional rotation and $u = u$ be the corresponding matrix $\in SU(2)$. With $\chi_{AP}$ and the block matrix structure of $\Phi$ given in eq.$\chi_{AP}$ as well as the relation $(-i\sigma_2) u^{-1} i\sigma_2 = u$ we find

$$\begin{aligned}
\Phi^{++}(p) & = \sum_{M'_J} \left\{ [u \times u] \phi^{++} (-i\sigma_2) \right\} D_{M'_J, M_J}^\dagger (u^{-1}) \\
\Phi^{-+}(p) & = \sum_{M'_J} \left\{ [u \times u] \phi^{-+} (-i\sigma_2) \right\} D_{M'_J, M_J}^\dagger (u^{-1})
\end{aligned} \quad (B1)$$

and the same for the other amplitudes where we use the tensor notation $\{ [u \times u] \phi^{++} \}$

$$\Phi^{i\sigma_2}(p) = \sum_{M'_J} \left\{ [u \times u] \phi^{i\sigma_2} \right\} D_{M'_J, M_J}^\dagger (u^{-1})$$

Let $\chi_{SM_S}$ be the spin matrix of the two quarks coupled to the total spin $S$. Define $\varphi_{S_q} = i\sigma_2 = \chi_{S_q}$, i.e.

$$\varphi_{00} = \frac{1}{\sqrt{2}} 1 \quad , \quad \varphi_{1q} = \frac{1}{\sqrt{2}} \sigma_q \quad (B2)$$

or explicitly

$$\begin{aligned}
\varphi_{00} & = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \varphi_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\varphi_{10} & = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \varphi_{1-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned} \quad (B3)$$

Then eq.$\chi_{AP}$ implies that we can decompose $\Phi^{++}$, $\Phi^{--}$ as

$$\Phi^{++}(p) = \sum_{LS} R_{LS}^{++}(p) \chi_{L}(\Omega_p) \otimes \varphi_S$$

$$\Phi^{--}(p) = \sum_{LS} R_{LS}^{--}(p) \chi_{L}(\Omega_p) \otimes \varphi_S$$

with the spin $S$ and the orbital angular momentum $L$ coupled to $J$. We assume that the BS-kernel allows choosing $R_{LS}^{++}(p)$ to be a real function. The sum goes over all values $L, S$ that are compatible with parity and charge parity of the bound state, see below.
APPENDIX C: PARITY TRANSFORMATION OF THE BS-AMPLITUDE

The Dirac field operator $\Psi(x)$ and the bound state with parity number $\pi_P = \pm 1$ transform under parity transformation $P$ as

$$U_P \Psi(x) U_P^{-1} = \gamma^0 \Psi(\hat{P}x) \quad (C1)$$

$$U_P \Psi(x) U_P^{-1} = \Psi(\hat{P}x) \gamma^0 \quad (C2)$$

$$U_P [K, J, M_J, \pi_P] = \pi_P [K, J, M_J, \pi_P] \quad (C3)$$

where $K$ is the momentum of the bound state and $\hat{P}x = (x^0, -\vec{x})$. For the BS-amplitude this implies

$$\chi_K(p) = \pi_P \gamma^0 \chi_{\tilde{K}}(\hat{P}p) \gamma^0 \quad (C4)$$

For the block matrix structure one finds

$$\begin{pmatrix} \Phi^+(-\vec{p}) & \Phi^+(\vec{p}) \\ \Phi^-(-\vec{p}) & \Phi^-(\vec{p}) \end{pmatrix} = \pi_P \begin{pmatrix} \Phi^+(-\vec{p}) & -\Phi^+(\vec{p}) \\ -\Phi^-(-\vec{p}) & \Phi^-(\vec{p}) \end{pmatrix}$$

To be compatible with the angular decomposition eq.(B5) and $\sim$ like $f$ eq.(B5) we find the well-known condition $\pi_P = (-1)^{L+S}$. For the block matrix structure one finds

$$\begin{pmatrix} \Phi^+(-\vec{p}) & \Phi^+(\vec{p}) \\ \Phi^-(-\vec{p}) & \Phi^-(\vec{p}) \end{pmatrix} = \pi_P \begin{pmatrix} \Phi^+(-\vec{p}) & -\Phi^+(\vec{p}) \\ -\Phi^-(-\vec{p}) & \Phi^-(\vec{p}) \end{pmatrix}$$

APPENDIX D: CHARGE CONJUGATION

Let $|f_1, f_2, P\rangle$ be a $q\bar{q}$-bound state with flavors $f_1$ and $f_2$ and momentum $P$. The charge conjugation then acts like

$$U_C \Psi_{\alpha}(x) U_C^{-1} = \sum_{\beta} (S_C)_{\alpha \beta} \Psi_{f_\beta}(x) \quad (D1)$$

$$U_C \Psi^+_{\alpha}(x) U_C^{-1} = \sum_{\beta} (S_C)_{\alpha \beta} \Psi_{f_\beta}(x) \quad (D2)$$

$$U_C |f_1 f_2, P\rangle = |f_2 f_1, P\rangle \quad (D3)$$

and we find for the BS-amplitude

$$\chi^P_{f_1 f_2}(p) = -(S_C \gamma^0) \chi^P_{f_2 f_1}(-p) (S_C \gamma^0) \quad (D4)$$

with the matrix $S_C \gamma^0$ given in the standard basis as

$$S_C \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (D5)$$

Making the choice $\eta_1 = \eta_2 = 1/2$ implies

$$\begin{pmatrix} \Phi^+_{f_1 f_2}(\vec{p}) & \Phi^+_{f_2 f_1}(\vec{p}) \\ \Phi^-_{f_1 f_2}(\vec{p}) & \Phi^-_{f_2 f_1}(\vec{p}) \end{pmatrix} = \begin{pmatrix} \Phi^+_{f_2 f_1}(\vec{p}) & \Phi^+_{f_2 f_1}(\vec{p}) \\ \Phi^-_{f_2 f_1}(\vec{p}) & \Phi^-_{f_2 f_1}(\vec{p}) \end{pmatrix}$$

with $\bar{\Phi}^+ = -i\sigma_2 (\Phi^+^T) i\sigma_2$. With the angular decomposition eq.(B5) and $\bar{\varphi}_{S_q} = (-1)^S \varphi_{S_q}$ one finds

$$\left[R_{LS}^{(\pm)}(p)\right]_{f_1 f_2} = (-1)^{L+S} \left[R_{LS}^{(\pm)}(p)\right]_{f_2 f_1} \quad (D7)$$

For an eigenstate $|P, \pi_C\rangle$ of the charge conjugation we have $f_1 = f_2$ and $U_C |P, \pi_C\rangle = \pi_C |P, \pi_C\rangle$ which implies the well-known condition $\pi_C = (-1)^{L+S}$.

To investigate the compatibility of eq.(D4) with the BS-equation we use the relation $S_{f_i}(p_i) = -(S_C \gamma^0)^T S_{f_i}(p_i) (S_C \gamma^0)$ for the fermion propagator. Let $\chi_{f_1 f_2}(p)$ be a solution of the BS-equation with $f_1$ being the quark flavor and $f_2$ being the antiquark flavor. It is straightforward to show that $\chi_{f_1 f_2}(p)$ as given in eq.(D4) is a solution of the BS-equation with interchanged flavors. For the Salpeter equation this implies that the solution $\psi_{f_1 f_2}$ and the solution of the equation with flavors interchanged $\psi_{f_2 f_1}$ are connected by

$$\psi_{f_1 f_2}(\vec{p}) = -S_C \psi_{f_2 f_1}(-\vec{p}) S_C \quad (D8)$$

APPENDIX E: ANGULAR DECOMPOSITION OF THE 4×4 AMPLITUDE

From the angular decomposition of the $2\times2$ blocks and the property under parity transformation we also obtain the structure of the $4\times4$ amplitudes.

With the spherical components $\sigma_m$ of the Pauli matrices we define the two tensors

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -\sigma_m \\ \sigma_m & 0 \end{pmatrix} \quad (E1)$$

which are again Dirac matrices. We obtain the following representation for the BS amplitudes in the rest frame

$$\chi^{JM,J,\pi_P}(p) = \sum_{i=1}^{8} \chi_i(p^0, |\vec{p}|) \Gamma_i(\Omega_p) = \chi^{JM,J,\pi_P}(p) \quad (E2)$$

The sum over $i$ covers all the values of $s_i, a_i, g_i \in \{0,1\}$ and $l_i \in \{J, J \pm 1\}$ that are compatible with $(-1)^{s_i+a_i+g_i} = \pi_P$, where $\pi_P$ is the parity of the meson. In general we have eight scalar amplitudes $\chi_i(p^0, |\vec{p}|)$ (except for mesons with spin 0, where there are only four).

The same structure holds for the Salpeter amplitudes $\Phi$, the vertex functions $\Gamma$ and any amplitude that is built out of these by multiplication with e.g. $\gamma^0, \gamma^0 \Omega_p$ or $\Lambda^{\pm}$. It is therefore convenient to introduce a matrix formalism for such multiplications which is useful for the calculation of decays or transitions and simplifies the Dirac algebra.

From eq.(E2) we may write the amplitude in the form

$$\Phi(\vec{p}) = \sum_{i=1}^{8} \chi_i(|\vec{p}|) \Gamma_i(\Omega_p) \quad (E3)$$
so that we can express the multiplication from the left or right hand side with a matrix $\tilde{\gamma} \in \{ \gamma^0, \vec{\gamma} \vec{p}, \ldots \}$ by a transformation $D_L(\tilde{\gamma})$ or $D_R(\tilde{\gamma})$ in the 8 dimensional space of the radial amplitudes:

$$\tilde{\gamma} \Phi(\vec{p}) = \sum_{i,k=1}^{8} \Phi_i(|\vec{p}|) D_L(\tilde{\gamma})_{ik} \Gamma_k(\Omega_p)$$  \hspace{1cm} (E4)

and analogously with $D_R$ for multiplication from the right side. We also use the following formula to calculate the trace in Dirac space and the angular integration:

$$\int d\Omega_p \, tr \left\{ \Gamma_i^{\dagger}(\Omega_p) \Phi(\vec{p}) \right\} = 4 \Phi_i(|\vec{p}|)$$  \hspace{1cm} (E5)

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[1] C.R. Münz, J. Resag, B.C. Metsch, H.R. Petry: A Bethe-Salpeter model for light mesons: spectra and decays
[2] E.E. Salpeter, H.A. Bethe: Phys.Rev. 84, 132 (1951)
[3] M. Gell-Mann, F. Low: Phys.Rev. 84, 350 (1951)
[4] N. Nakanishi: Suppl.Theor.Phys., No.43, 1 (1969)
[5] N. Nakanishi: Prog.Theor.Phys., Suppl. No.95, 1 (1988)
[6] E.E. Salpeter, Phys.Rev. 87, 328 (1952)
[7] C.H. Llewellyn Smith, Ann.Phys. 53, 521 (1969)
[8] A. Le Yaouanc, L. Oliver, S. Ono, O. Pene, J.C. Raynal, Phys.Rev. D31, 137 (1985)
[9] J. F. Lagaè: Phys.Rev. D45, 305, 317 (1992)
[10] P. Ring, P. Schuck: The Nuclear Many-Body Problem, Berlin: Springer-Verlag 1980
[11] M. Böhm, Nucl.Phys. B91, 494 (1975)
[12] M. Beyer, U. Bohn, M.G. Huber, B.C. Metsch, J. Resag: Z.Phys.C-Particles and Fields 55, 307 (1992)
[13] W. Lucha, F. F. Schöberl, D. Gromes: Phys.Rep. 200, No.4, 127 (1991)
[14] C. Itzykson, J.-B. Zuber: Quantum Field Theory, New York: McGraw-Hill 1985
[15] A. Gara, B. Durand, L. Durand, L. J. Nickisch: Phys.Rev. D40, 843 (1989)
A. Gara, B. Durand, L. Durand: Phys.Rev. D42, 1651 (1990)
[16] K. Nishijima, Prog.Theor.Phys. 10, 549 (1953); 12, 279 (1954); 13, 305 (1955)
[17] S. Mandelstam, Proc. Roy. Soc. 233, 248 (1955)
[18] R. E. Cutkosky, M. Leon, Phys.Rev. 135 B, 1445 (1964)
[19] D. Lurie: Particles and Fields, New York: Interscience Publishers 1968
[20] C. H. Llewellyn Smith, Nuovo Cim. 60A, 348 (1969)
[21] O. Nachtmann: Elementarteilchenphysik Phänomene und Konzepte, Braunschweig: Vieweg 1986
[22] A. N. Mitra, A. Pagnamenta, N. N. Singh Phys.Rev.Lett. 59, 2408 (1987)