CHAOTIC ACCRETION IN A
NON-STATIONARY ELECTROMAGNETIC
FIELD OF A SLOWLY ROTATING
COMPACT STAR

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Abstract

We investigate charge accretion in vicinity of a slowly rotating compact star with a non-stationary electromagnetic field. Exact solutions to the general relativistic Maxwell equations are obtained for a star formed of a highly degenerate plasma with a gravitational field given by the linearized Kerr metric. These solutions are used to formulate and then to study numerically the equations of motion for a charged particle in star’s vicinity using the gravitoelectromagnetic force law. The analysis shows that close to the star charge accretion does not always remain ordered. It is found that the magnetic field plays the dominant role in the onset of chaos near the star’s surface.

Keywords: Plasma accretion; compact stars; chaos; chaotic dynamics; magnetic field; relativistic MHD, gravitoelectromagnetic effects

1 Introduction

Accretion is a fundamental process by which stars accumulate matter from their surroundings, it therefore plays an important role in stellar formation and evolution. For normal stars accretion is principally due to the gravitational field of the star. However for compact objects (classed as neutron stars, pulsars, and white dwarfs) accretion is not only due to gravitational but also electromagnetic effects. In vicinity of these stars matter is mostly in the form of a highly conducting electron plasma, whereas the star itself is formed of a degenerate highly conducting quasi-neutral plasma. The compact state of the plasma inside the star is brought about by the intense gravitational field of the star. Exterior to the star gravitational and electromagnetic field produce combined effects on the dynamics of plasma surrounding the star. Here gravitational field has a two-fold
effect on charge accretion. Firstly it effects the charge dynamics through the interaction of particle mass with the gravitational field of the star. Secondly the background spacetime also influences the electromagnetic field of the star which in turn effects the particle orbit via charge interaction. In general, under such conditions, the electromagnetic field of a compact star is not stationary but gains in intensity as the star evolves. Observationally the magnetic field intensity for a newly formed compact star to the final stages of a compact star can vary from $10^4 \text{G}$ to $10^{12} \text{G}$ and hence varies with time$^{[1-3]}$

In this paper we investigate the combined effects of gravitational and a non-stationary electromagnetic field on a charge particle dynamics in vicinity of a compact star. The star is considered to be formed of a degenerate highly conducting plasma. The magnetohydrodynamics (MHD) approximation is applied for the plasma forming the star. Using a generalized definition$^{[4,5]}$ of the electromagnetic field tensor for an MHD plasma in a curved spacetime the Maxwell field equations are explicitly formulated for a non-stationary axially symmetric electromagnetic field. Here the linearized Kerr metric is used to describe the gravitational field outside the star with ZAMO (zero angular momentum observers) as the stationary observer$^{[6]}$. The generalized Maxwell equation are then solved for the electric and magnetic field components using the separation of variable technique. These solutions are then used to obtain the equations of motion (geodesic equations) in linearized Kerr spacetime for a charge particle in the non-stationary electromagnetic field of the star. The numerical solutions to these equations show that at large distances from the star particle trajectories are well defined and ordered, however as the particle falls towards the star its orbit cannot be precisely predicted hence exhibits chaos. The solution show that star’s magnetic field plays the dominant role in bringing the charge accretion to a chaotic state.

The layout of the paper is as follows. In the next section generalized Maxwell equations are formulated using the generalized electromagnetic field tensor. These equations are solved in section 3 for a non-stationary axially symmetric electromagnetic field using a separation of variable ansatz for the field components. Then in section 4 equations of motion are formulated and numerically solved for a charged particle lying in the field of the star. In the conclusion (section 5) numerical results of section 4 are interpreted and the main results of the study are summarized. Throughout we have used the gravitational units in which $G = 1 = c$ unless mentioned otherwise.

2 Formulation of the Electromagnetic Field Equations

General relativistically the gravitational field of a rotating star can be described by the Kerr metric. In the region exterior to a slowly rotating compact star of mass $M$ a linearization of the Kerr metric is given by
\[ ds^2 = -e^{2\Psi(r)}dt^2 - 2\omega(r)r^2 \sin^2\theta dtd\varphi + r^2 \sin^2\theta d\varphi^2 + e^{-2\Psi(r)}d\nu^2 \]  
(1)

where we have used the usual Boyer-Lindquist coordinates. Also in expression (1) \( e^{2\Psi(r)} \) and \( \omega(r) \) are given by the relations \( e^{2\Psi(r)} = 1 - 2M/r, \omega(r) = d\varphi/dt = -g_{t\varphi}/g_{\varphi\varphi} \), where \( \omega(r) \) is the angular velocity of a free falling frame brought into rotation by the frame dragging of the spacetime\[^{[7]}\]. The four velocity components \( u_r \) and \( u_\theta \) of a stationary observer ZAMO circling the star at a fixed radial distance vanish, therefore using \( u^\alpha u_\alpha = -1 \), the components of the four velocity vector for a ZAMO are:

\[ u^\alpha = e^{-\Psi(r)}(1, 0, 0, \omega(r)), \quad u_\alpha = e^{\Psi(r)}(-1, 0, 0, 0). \]  
(2)

For a star formed of highly degenerate plasma the electromagnetic field \((E^\alpha, B^\alpha)\) of the star can be determined by the Maxwell equations. In a curved spacetime, such as the Kerr spacetime the general relativistic form of the Maxwell equations is given by

\[ F_{\alpha\beta,\gamma} + F_{\gamma\beta,\alpha} + F_{\alpha\gamma,\beta} = 0, \]  
(3)

\[ (\sqrt{-g}F_{\alpha\beta})_{,\beta} = 4\pi\sqrt{-g}J^\alpha, \]  
(4)

where \( g \) represents the determinant of the metric tensor \( g_{\alpha\beta} \). Here \( F_{\alpha\beta} \) is the generalized electromagnetic field tensor for an ideal magnetohydrodynamic fluid given by a unique tensor expression:

\[ F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta} u_\gamma B^\delta, \]  
(5)

and \( J^\alpha \) is current four vector. In general the current four vector is the sum of two terms corresponding to a convection and to a conduction current: \( J^\alpha = \epsilon u^\alpha + \sigma u_\beta F^{\beta\alpha} \), where \( \epsilon \) is the proper charge density and \( \sigma \) is the conductivity of the plasma. The volume element 4-form \( \eta_{\alpha\beta\gamma\delta} \) is equal to \( \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta} \) where \( \epsilon_{\alpha\beta\gamma\delta} \) is the Levi-Civita symbol. Since in a compact star the quasi-neutral degenerate plasma is essentially a perfect conductor, we assume that for the region external to the star \( J^\alpha \) can be approximated to zero (see references\[^{[8]}\] and\[^{[9]}\] for a discussion of this often made assumption for a quasi-neutral degenerate plasma).

To determine the non-stationary electromagnetic field of the star we assume that the electric and magnetic field components are dependent not only on the polar coordinates \( r \) and \( \theta \) but also the time coordinate \( t \) in the frame of ZAMO. We then obtain from the Maxwell equations (3) and (4) the following determining equations for the electromagnetic field in the slow rotation approximation\[^{[10]}\]:

\[ \partial_r (r^2 \sin \theta u^t B^r) + \partial_\theta (r^2 u^t \sin \theta B^\theta) = 0, \]  
(6)

\[ r^2 \sin \theta u^t \partial_t B^\theta - \partial_r (u_t E_\varphi) = 0, \]  
(7)

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\[ r^2 \sin \theta u^t \partial_t E^r + u_t \partial_b E_\varphi = 0, \]
\[ r^2 \sin \theta u^t \partial_t B^r + \partial_r (u_t E_\theta - r^2 \sin \theta u^r B^\theta) - \partial_b (u_t E_r - r^2 \sin \theta u^r B^\theta) = 0, \]
\[ \partial_r (r^2 \sin \theta u^t E^r) + \partial_b (r^2 u^t \sin \theta E^\theta) = 0, \]
\[ r^2 \sin \theta u^t \partial_t E^\theta - \partial_r (u_t E_\varphi) = 0, \]
\[ r^2 \sin \theta u^t \partial_t B^\theta + u_t \partial_b E_\varphi = 0, \]
\[ r^2 \sin \theta u^t \partial_t B^\varphi + \partial_r (u_t E_\theta - r^2 \sin \theta u^r B^\theta) - \partial_b (u_t E_r - r^2 \sin \theta u^r B^\theta) = 0. \]

3 Solution for the Electromagnetic Field Exterior to the Star

The non-stationary electromagnetic field of the star is now determined by the system of equations (6) to (13). To obtain the time dependent electric and magnetic field components \( E^\alpha \) and \( B^\alpha \) we assume the following separation ansatz for the field components:

\[ E^r = A(r) a(t) \frac{\alpha}{\sin^2 \theta}, \quad E^\theta = B(r) b(t) \frac{\beta}{\sin \theta}, \quad E^\varphi = C(r) c(t) \frac{\gamma}{\sin^2 \theta} \]
\[ B^r = D(r) d(t) \frac{\delta}{\sin^2 \theta}, \quad B^\theta = E(r) e(t) \frac{\epsilon}{\sin \theta}, \quad B^\varphi = F(r) f(t) \frac{\lambda}{\sin^2 \theta} \]

Substituting into (6) we obtain

\[ \frac{d}{dr} (r^2 u^t D) = 0 \]

This implies that \( D(r) = \text{const.}/r^2 u^t \). Also from equation (10) we obtain \( A(r) = \text{const.}/r^2 u^t \) hence the radial components of the electric and magnetic field reduce as:

\[ E^r(r, \theta, t) = \delta d(t)/r^2 u^t \sin^2 \theta, \quad B^r(r, \theta, t) = \alpha a(t)/r^2 u^t \sin^2 \theta \]

From (13) it therefore follows that

\[ r^2 u^t C \gamma \frac{dc}{dt} + \frac{d}{dr} (r^2 u^t E\varepsilon - r^2 u^t A\alpha) - u_t \sin \theta \frac{dB_r}{d\theta} = 0 \]

Comparing coefficients of \( \sin \theta \) on both sides the last equation implies that \( B_r \) must be independent of \( \theta \). This is compatible with equation (17) iff \( \delta \equiv 0 \). Therefore it follows that \( B^r = 0 \). Similarly from (17) it follows that \( E^r = 0 \). We therefore have after the separation of variables, from equation (13), (11), (9), and (7) respectively, the following determining equations

\[ \frac{1}{\epsilon} \frac{dc}{dt} = \frac{-\varepsilon}{\gamma r^2 u^t C} \frac{d}{dr} (u_t r^2 E) = k_1 \]
\[1 \frac{db}{dt} = \frac{\lambda}{\beta r^2 u^6 B} \frac{d}{dr} (u_t r^2 B) = k_2\]  \hspace{1cm} (20)

\[1 \frac{df}{dt} = \frac{-\beta}{\lambda r^2 u^6 F} \frac{d}{dr} (u_t r^2 F) = -k_3\]  \hspace{1cm} (21)

\[1 \frac{de}{dt} = \frac{\gamma}{\varepsilon r^2 u^6 E} \frac{d}{dr} (u_t r^2 C) = -k_4\]  \hspace{1cm} (22)

where \(k_1, k_2, k_3,\) and \(k_4\) are the separation constants.

### 3.1 TIME DEPENDENT SOLUTIONS

For the separated equations (19) to (22) we have for the time dependent parts \(b(t), c(t), e(t),\) and \(f(t)\) the following set of equations:

\[\frac{dc}{dt} = k_1 c, \quad \frac{db}{dt} = k_2 f\]  \hspace{1cm} (23)

\[\frac{df}{dt} = -k_3 b, \quad \frac{de}{dt} = -k_4 e\]  \hspace{1cm} (24)

Differentiating the first of equations (23), and substituting in the second equation of the pair (24) we obtain

\[c(t) = A_1 \exp(-i \sqrt{k_1 k_4} t)\]  \hspace{1cm} (25)

Substituting back in equation (23) we obtain for \(e:\)

\[e(t) = -\frac{i \sqrt{k_1 k_4}}{k_1} A_1 \exp(-i \sqrt{k_1 k_4} t)\]  \hspace{1cm} (26)

where \(A_1\) is a constant. In a similar manner we obtain for \(f(t)\) and \(b(t)\):

\[b(t) = A_2 \exp(-i \sqrt{k_2 k_3} t)\]  \hspace{1cm} (27)

\[f(t) = -\frac{i \sqrt{k_2 k_3}}{k_2} A_2 \exp(-i \sqrt{k_2 k_3} t)\]  \hspace{1cm} (28)

### 3.2 RADIAL SOLUTIONS

The radial components \(C(r), E(r), F(r),\) and \(B(r)\) are obtainable explicitly from equations (19) to (22) as follows. First from the radial part of the separated equation (19) we have for the component \(C(r)\):

\[C(r) = \frac{\varepsilon \sqrt{1 - \frac{2M}{r}}}{k_1 \gamma r^2} \frac{d}{dr} \left( \sqrt{1 - \frac{2M}{r}} \varepsilon \right)\]  \hspace{1cm} (29)

Substituting into equation (22) gives an expression for \(E(r)\)
\[
\frac{d}{dr}[(1 - \frac{2M}{r}) \frac{d}{dr}(\sqrt{1 - \frac{2M}{r^2}}\mathcal{E})] = \frac{k_1 k_4 r^2 \mathcal{E}}{\sqrt{1 - \frac{2M}{r}}} \tag{30}
\]

This differential equation can be solved explicitly for the component \(\mathcal{E}(r)\) to give

\[
\mathcal{E}(r) = r^{-3/2}(r - 2M)^{-\frac{1}{2} + 2M\sqrt{k_1 k_4}} C_1 \exp(\sqrt{k_1 k_4} r) - \\
r^{-3/2}(r - 2M)^{-\frac{1}{2} - 2M\sqrt{k_1 k_4}} \frac{C_2}{2\sqrt{k_1 k_4}} \exp(-\sqrt{k_1 k_4} r) \tag{31}
\]

The first part of the above solution is asymptotically unbounded, hence we take only the second part of the solution for \(\mathcal{E}(r)\). Putting then for \(\mathcal{E}(r)\) in equation (29) we obtain after some simplifications the component \(C(r)\):

\[
C(r) = \frac{-\epsilon C_2 \sqrt{1 - \frac{2M}{r}}}{2\gamma r^{9/2} \sqrt{k_1 k_4}} (r - 2M)^{-\frac{1}{2} - 2M\sqrt{k_1 k_4}} \{r(2 + \sqrt{k_1 k_4}) - 3M\} \exp(-\sqrt{k_1 k_4} r) \tag{32}
\]

In a similar manner we obtain for \(\mathcal{F}(r)\) and \(\mathcal{B}(r)\):

\[
\mathcal{F}(r) = r^{-3/2}(r - 2M)^{-\frac{1}{2} + 2M\sqrt{k_2 k_3}} C_3 \exp(\sqrt{k_2 k_3} r) - \\
r^{-3/2}(r - 2M)^{-\frac{1}{2} - 2M\sqrt{k_2 k_3}} \frac{C_4}{2\sqrt{k_2 k_3}} \exp(-\sqrt{k_2 k_3} r) \tag{33}
\]

\[
\mathcal{B}(r) = \frac{-\lambda C_4 \sqrt{1 - \frac{2M}{r}}}{2\beta r^{9/2} \sqrt{k_2 k_3}} (r - 2M)^{-\frac{1}{2} - 2M\sqrt{k_2 k_3}} \{r(2 + \sqrt{k_2 k_3}) - 3M\} \exp(-\sqrt{k_2 k_3} r) \tag{34}
\]

These solutions are single valued and bounded for \(r \to \infty\).

Hence the radial solutions (31) to (34) together with the time dependent solutions (25) to (28) determine the electromagnetic field of a compact star in the region exterior to the star.

4 Charge Accretion in the Slow Rotation Approximation

A linearization of the geodesic equation for a slowly rotating gravitational source is given by the gravitoelectromagnetic force of magnitude \(G + v \times H\) per unit mass where \(G\) and \(H\) are the gravitoelectric (GE) and gravitomagnetic (GM) potentials\[^{11,12}\]. In the slow rotation limit the star can be considered as a
homogeneous sphere of mass $M$ and radius $R$. In this case the GE and GM potential are given by

$$ G = - \frac{M}{r^2}, \quad H = - \frac{12}{5} M R^2 \left( \Omega \cdot \frac{r}{r^5} - \frac{1}{3} \frac{\Omega}{r^3} \right), \quad (35) $$

where $\Omega$ is the angular velocity vector of the star. In the presence of an electromagnetic field the force law for a particle with mass $m$ and charge $q$ is given by

$$ \frac{d^2 r}{dt^2} = (G + v \times H) + \frac{q}{m} (E + v \times B). \quad (36) $$

The equations of motion can be studied numerically in Cartesian coordinates by transforming the polar coordinates into the Cartesian coordinates as $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$. The transformation gives for the electric field components $E^x = -r \sin \varphi E^\varphi, E^y = r \cos \varphi E^\varphi, E^z = -r \sin \theta E^\theta$, and similarly for the magnetic field $B^x = -r \sin \varphi B^\varphi, B^y = r \cos \varphi B^\varphi, B^z = -r \sin \theta B^\theta$. In a two dimensional plane $\theta = \pi/2$ (the equatorial plane of the star), the gravitomagnetic effects are maximum, here we get for the equations of motion for the $(x, y)$ plane:

$$ \begin{align*}
\frac{d^2 x}{dt^2} &= -g - E^x + (H - B^z) \frac{dy}{dt} = -g + y E^\varphi + (H + y B^\theta) \frac{dy}{dt} \quad (37)
\frac{d^2 y}{dt^2} &= -g + E^y - (H - B^z) \frac{dx}{dt} = -g + x E^\varphi - (H + y B^\theta) \frac{dx}{dt}, \quad (38)
\end{align*} $$

where from (14) and (15) we have for the real part of $E^\varphi$ and $B^\theta$

$$ \begin{align*}
\text{Re} \, E^\varphi &= E_0 \sqrt{1 - \frac{2M}{r^2}} \left( r - 2M \right)^{-\frac{1}{2}} \{ r(2 + kr) - 3M \} \exp(-kr) \cos kt \quad (39)
\text{Re} \, B^\theta &= B_0 r^{-3/2} (r - 2M)^{-\frac{1}{2}} \exp(-kr) \sin kt \quad (40)
\end{align*} $$

where $r = \sqrt{x^2 + y^2}$, $k = \sqrt{k_1 k_2}$, and $E_0 \equiv -\varepsilon A_1 C_2 / 2 k_1 k_2 \sin^2 \theta$ and $B_0 \equiv \varepsilon A_1 C_2 / 2 k_1 \sin \theta$ are constants for fixed value of $\theta$, such as in the equatorial plane of the star. Also $g \equiv |G| = 4(2 + k r) / (x^2 + y^2)$ and $H = |H| = \mu / (x^2 + y^2)^{3/2}$, whereas $\mu = (4/5 R) \Omega$, $\Omega$ being the magnitude of the angular velocity vector and $R$ is the radius of the star. Transforming the electromagnetic field components in coordinates $(x, y)$ and using expressions (37), (38) (39), and (40) we plot the trajectories of the particle for various values of the parameters $M, \mu, E_0,$ and $B_0$ in figures (1) to (4); where we have taken $x(0) = 2, y(0) = 2, dx/dt |_{t=0} = 0, dy/dt |_{t=0} = 0$ and $k = 1.
5 Conclusions and Summary

In this paper we have investigated charge accretion around a slowly rotating compact star with a non-stationary electromagnetic field. In the region exterior to the star the electromagnetic field is determined by solving the generalized Maxwell equations (6) to (13). Then to study charge accretion in vicinity of a compact star, the equations of motion for a charged particle in the presence of the non-stationary electromagnetic field are solved numerically. For the external region, where the solutions are stable and convergent, our analysis shows that gravitational as well as electromagnetic field determine the charge particle trajectories. However it is the magnetic force that has the dominant effect on the transition from an ordered state of motion to a chaotic one. Close to the star particle motion is particularly sensitive to a change in the magnetic field of the star. Here an increase in the gravitational field strength (gravitoelectric as well as gravitomagnetic) directly causes a suppression in magnetically induced chaos in particle orbit (Figure 3 and 4). The process however requires a substantial (about 10 to 100 times) increase in the gravitoelectric field. In comparison a change in the gravitomagnetic parameter $\mu$ from 2 to 10 units introduces order in particle’s orbit. On the other hand the effects of electric field (Figure 2) on the charged particle motion are more complicated. As the electric field becomes comparable to the magnetic field, the accretion trajectories become precisely determined. Crossing this value, an increase in the electric field causes the charged particle to reverse its direction of motion after which it escapes from falling into the star. Magnetic field effects on charged particle trajectory displayed in figure 1 also indicate that although an increase in the magnetic field strength makes charge accretion chaotic, the general trend is that of particle in-fall due to the dominant gravitational effects. On the other hand in the regions close to the poles of the star, where magnetic field is particularly high, it is expected that chaotic trend in accretion is more dominant provided that the electric field effects on particle’s motion are comparatively weak.

Summarizing the above conclusions, accretion at relatively large distances from the surface of a compact star is more or less ordered due to the dominant effects of gravitational as well as electric fields. However in the star’s vicinity accretion becomes chaotic depending on the magnetic field strength close to surface of the star as compared to the other forces involved, in particular the gravitational and the electric forces.

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**Figure Captions:**

Figure 1: Effects of the magnetic field on charge particle trajectories. The magnetic field strength varies from $B_0 = 10, 100, 1000$, to $10000$ units, while $M = 1$, $\mu = 1$, and $E_0 = 1$ in gravitational units.

Figure 2: Effects of the electric field on charge particle trajectories. The electric field strength varies from $E_0 = 1, 10, 100$, to $1000$ units, while $M = 1$, $\mu = 1$, and $B_0 = 10$ in gravitational units.

Figure 3: Effects of the gravitoelectric field on charge particle trajectories. The gravitoelectric field strength varies from $M = 1, 10, 100$, to $1000$ units, while $\mu = 1$, $E_0 = 1$ and $B_0 = 10$ in gravitational units.

Figure 4: Effects of the gravitomagnetic field on charge particle trajectories. The gravitomagnetic field strength varies from $\mu = 0.05, 1, 3$, to $10$ units, while $M = 1$, $E_0 = 1$ and $B_0 = 10$ in gravitational units.
