ABSTRACT. We prove that for mappings in $W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$, continuous up to the boundary, with modulus of continuity satisfying a certain divergence condition, the image of the boundary of the unit ball has zero $n$-Hausdorff measure. For Hölder continuous mappings we also prove an essentially sharp generalized Hausdorff dimension estimate.

1. Introduction

Throughout the paper $\mathbb{B}^n$ denotes the unit ball in $\mathbb{R}^n$ and $W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ is the Sobolev space of $L^n(\mathbb{B}^n, \mathbb{R}^m)$-functions $f : \mathbb{B}^n \to \mathbb{R}^m$ with weak first order derivatives in $L^n(\mathbb{B}^n)$.

If $f : \mathbb{B}^2 \to \Omega \subset \mathbb{R}^2$ is a conformal mapping, then the boundary of $\Omega$ can have positive Lebesgue measure even if $f$ extends continuously up to the boundary of the disk. If one requires more, for example uniform Hölder continuity, then $\partial \Omega$ is necessarily of Lebesgue measure zero. In fact, Jones and Makarov proved in $[6]$ that $\partial \Omega$ has measure zero if $f$ satisfies $|f(z) - f(w)| \leq \psi(|z - w|)$ in $\mathbb{B}^2$ for $\psi : (0, \infty) \to (0, \infty)$ with

\[
\int_0^\infty \frac{|\log \psi(t)|^2}{\log t} \, dt = \infty.
\]

This condition is very sharp: if the integral in (1) converges then $[6]$ provides us with a simply connected domain $\Omega$ and a conformal mapping $f : \mathbb{B}^2 \to \Omega$ so that the boundary of $\Omega$ has positive Lebesgue measure and $f$ has the modulus of continuity $\psi$.

Our first result gives a surprisingly general extension of the conformal setting; notice that each uniformly continuous conformal mapping $f : \mathbb{B}^2 \to \Omega$ belongs to $W^{1,2}(\mathbb{B}^2, \mathbb{R}^2)$.

Theorem 1.1. Let $f \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ be a continuous mapping so that

\[
|f(z) - f(w)| \leq \psi(|z - w|)
\]

for all $z, w \in \mathbb{B}^n$, where $\psi : (0, \infty) \to (0, \infty)$ is an allowable modulus of continuity with

\[
\int_0^\infty \left( \frac{|\log \psi(t)|^n}{\log t} \right) \frac{dt}{t} = \infty.
\]

Then $\mathcal{H}^n(f(\partial \mathbb{B}^n)) = 0$.

Above, $\mathcal{H}^n(A)$ denotes the $n$-dimensional Hausdorff measure of a set $A$.

For the definition of an allowable modulus of continuity see Section 2 below. For example, $\psi(t) = Ct^\gamma$, $0 < \gamma < 1$, and

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are allowable for all integers \( l \geq 2 \) and all \( s > 0 \). Notice that \( \psi_{t,s} \) satisfies (3) if and only if \( s \leq 1 \). Here \( C > 0 \), \( \log^{(k)} t \) is the \( k \)-times iterated logarithm and \( C_l \) is any constant with \( \log^{(l)} C_l \geq 1 \).

Let us look at the special case \( n = m = 2 \) of Theorem 1.1 in the Hölder continuous setting: \( \psi(t) = C t^{-\gamma} \), where \( 0 < \gamma \leq 1 \). Consider a space filling (Peano) curve, i.e. a continuous mapping \( g \) from the unit circle onto a square. In the standard construction, \( g \) is Hölder continuous with exponent \( \gamma = 1/2 \). If one takes, say, the Poisson extension \( f \) of \( g \) to the unit disk, then \( f \) is also Hölder continuous. It is easy to check by hand that the partial derivatives of \( f \) do not belong to \( L^2(B^2) \). By Theorem 1.1 no Hölder continuous (or even continuous with control function satisfying (3)) extension \( f \) of a space filling curve can satisfy \( |Df| \in L^2(B^2) \).

In the Hölder continuous case, Jones and Makarov actually proved that the Hausdorff dimension of \( f(\partial B^2) \) is strictly less than two for conformal \( f \). Contrary to the area zero results, this dimension estimate is truly conformal in the following sense.

**Example 1.** Let \( p > 1 \). There exists a locally Hölder continuous homeomorphism \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with \( f \in W^{1,2}_{loc}(B^2, \mathbb{R}^2) \), which maps \( \partial B^2 \) onto a set of positive \( \mathcal{H}^s \)-measure, for the gauge function \( g(t) = t^2 (\log \frac{1}{t})^p \).

Here \( \mathcal{H}^s \) denotes the generalized Hausdorff measure with the function \( g(t) \) as the dimension gauge. The precise definitions are given in Section 2. Our second result gives a rather optimal positive result.

**Theorem 1.2.** Let \( f \in W^{1,n}(B^n, \mathbb{R}^m) \) and fix \( 0 < \gamma \leq 1 \) and \( C_0 > 0 \). If \( f \) satisfies
\[
|f(z) - f(w)| \leq C_0 |z - w|^{\gamma}
\]
for all \( z, w \in B^n \), then \( \mathcal{H}^s(f(\partial B^n)) = 0 \), for the gauge function \( g(t) = t^n \log \frac{1}{t} \).

Jones and Makarov proved their result via harmonic measure and hence this technique does not work in the setting of Theorem 1.1. An alternate approach, relying on the conformal invariance of (quasi)hyperbolic metric, was given in Koskela-Rohde [7], see [11]. Furthermore, Malý and Martio [10] established Theorem 1.1 in the Hölder continuous case via a technique that we have not been able to push further.

Let us briefly describe the idea of the proof of Theorem 1.1. We consider a Whitney decomposition of \( B^n \) and assign each \( Q \in \mathcal{W} \) a vector \( f_Q \in \mathbb{R}^m \) and a radius \( r_Q \). The vector \( f_Q \) will simply be the “average” of \( f \) over \( Q \) and \( r_Q \) the maximum of \( |f_Q - f_Q| \) over all neighbors of \( Q \). Then the \( n \)-integrability of the weak derivatives of \( f \) guarantees, via the Poincaré inequality, that the sequence \( \{ r_Q \}_{Q \in \mathcal{W}} \) belongs to \( l^n \). We realize \( f(\partial B^n) \) as (a part of) the closure of \( \{ f_Q \}_{Q \in \mathcal{W}} \) in \( \mathbb{R}^m \). Those \( f(\omega), \omega \in \partial B^n \), for which one can find a sequence of \( Q \in \mathcal{W} \) with \( |f_Q - f(\omega)| \leq r_Q \) are easily handled. For the remaining \( \omega \in \partial B^n \), we modify our centers \( f_Q \) and radii \( r_Q \), still retaining the \( l^n \)-condition, so that suitably blown up balls cover these points sufficiently many times. This is where the non-integrability condition (3) kicks in. One cannot fully follow the above idea, and our proof below is more complicated.
Our approach is flexible and applies to many related problems. In order to avoid extra technicalities, we do not record such applications here. Let us simply mention that the dimension gap phenomenon from [3] can be shown to extend from conformal mappings to general Sobolev mappings [8].

2. Preliminaries

Let us first agree on some basic notation. Given a number \( a > 0 \), we write \([a]\) for the largest integer less or equal to \( a \). Similarly, \([a]\) is the smallest integer greater or equal to \( a \). If \( A \) is a finite set, \( \#A \) is the number of elements in \( A \). If \( A \subset \mathbb{R}^n \) has finite and strictly positive Lebesgue measure and \( f: \mathbb{R}^n \to \mathbb{R} \) is a Lebesgue integrable function, we denote the average \( \frac{1}{|A|} \sum_A f \) of \( f \) over the set \( A \) by \( \int_A f \) or \( f_A \), where \( |A| \) is the \( n \)-dimensional Lebesgue measure of the set \( A \). For \( f: \mathbb{R}^n \to \mathbb{R}^m \), \( f_A \) is then defined via the component functions of \( f \). Given a point \( x \in \mathbb{R}^n \) and a non-negative number \( r \), \( B(x, r) \) denotes the open ball with centre \( x \) and radius \( r \) and \( Q(x, r) \) denotes the cube \( \{ y \in \mathbb{R}^n : \max(|x_i - y_i|)_{i=1,\ldots,n} \leq r \} \). If \( B = B(x, r) \) is a ball and \( a \) is a positive number, the notation \( aB \) stands for the ball \( B(x, ar) \). We denote the radius of a ball \( B \) by \( r(B) \). If we write \( L = L(t) \), we mean that the number \( L > 0 \) depends on the parameters listed in the parentheses. Finally, \( C \) denotes a positive constant, which may depend only on \( n \) and \( m \), the dimensions of the domain space and the image space, and may differ from occurrence to occurrence.

We write \( \mathcal{H}^h(A) \) for the generalized Hausdorff measure of a set \( A \subset \mathbb{R}^n \), given by

\[
\mathcal{H}^h(A) = \lim_{\delta \to 0} \mathcal{H}_0^h(A),
\]

where

\[
\mathcal{H}_0^h(A) = \inf \left\{ \sum_{i=1}^\infty h(\text{diam } U_i) : A \subset \bigcup_{i=1}^\infty U_i, \text{diam } U_i \leq \delta \right\}
\]

and \( h \) is a dimension gauge (a non-decreasing function with \( \lim_{t \to 0} h(t) = h(0) = 0 \) and with \( h(t) > 0 \) for all \( t > 0 \)). If \( h(t) = t^a \) for some \( a \geq 0 \), we simply write \( \mathcal{H}^a \) for \( \mathcal{H}^h \) and call it the \( a \)-dimensional Hausdorff measure.

We need also a generalized weighted Hausdorff content of a set \( A \subset \mathbb{R}^n \), given by

\[
\lambda^h(A) = \inf \left\{ \sum_{i=1}^\infty c_i h(\text{diam } U_i) : \chi_A(x) \leq \sum_{i=1}^\infty c_i \chi_{U_i}(x), \forall x \in \mathbb{R}^n \right\}.
\]

Also here \( h \) is a gauge function. A sequence of pairs \( (c_i, U_i)_{i=1}^\infty \), where \( c_i \geq 0 \) and \( U_i \subset \mathbb{R}^n \), that satisfies \( \chi_A(x) \leq \sum_{i=1}^\infty c_i \chi_{U_i}(x) \), is called a weighted cover of the set \( A \). Again, we write \( \lambda^h_a = \lambda^h_a \), if \( h(t) = t^a \).

**Lemma 2.1.** Let \( E \subset \mathbb{R}^n \) be bounded. Let \( h \) be a continuous gauge function with \( h(2t) \leq c h(t) \) for some \( c > 0 \). Then \( \mathcal{H}^h(E) \leq c \lambda^h_a(E) \).

**Proof.** The lemma follows from Corollary 8.2 and the proof of Theorem 9.7 of [5] (see also [11, 21.0.24]). \( \Box \)

Recall that for each open subset \( U \) of \( \mathbb{R}^n \) there exist a Whitney decomposition \( U = \bigcup_{i=1}^\infty Q_i \), where \( Q_i \) are cubes with mutually parallel sides, pairwise disjoint interiors and each of edge length \( 2^k \) for some integer \( k \), such that the relation

\[
\frac{1}{4} \leq \frac{\text{diam } Q_i}{\text{dist}(Q_i, \partial \Omega)} \leq 1
\]
holds for all \( i = 1, 2, \ldots \). We write \( Q_1 \sim Q_2 \), if the Whitney cubes \( Q_1 \neq Q_2 \) share at least one point (the so-called neighbor cubes). We have
\[
\frac{1}{4} \leq \frac{\text{diam } Q}{\text{diam } \tilde{Q}} \leq 4,
\]
once \( Q \sim \tilde{Q} \). Therefore, the total number \#(\tilde{Q} : \tilde{Q} \sim Q)\) of all neighbors of a fixed cube \( Q \) does not exceed \( C \). See [12] for details.

Let \( \omega \in \partial B^n \). By \((Q, (\omega))_{m=1}^n\), we mean the sequence of all Whitney cubes in a fixed Whitney decomposition of \( B^n \), intersecting the radius \([0, \omega]\). This sequence starts with a central cube and tends to \( \omega \). For a point \( x \in [0, \omega] \), we denote the number of Whitney cubes intersecting the segment \([0, x]\) by \#\(q(0, x)\). It is easy to see that
\[
c_1 \leq \frac{\#q(0, x)}{\log \frac{1}{|x|}} \leq c_2,
\]
whenever \#\(q(0, x) > c_3\), where \( c_i > 0 \), \( i = 1, 2, 3 \) are constants that may depend on \( n \).

Finally we define the allowable moduli of continuity.

**Definition 2.2.** A continuously differentiable increasing bijection \( \psi : (0, \infty) \to (0, \infty) \) is an *allowable modulus of continuity* if there exists \( t_0 < 1 \) and \( \beta > 0 \) such that for every \( t \leq t_0 \) the following conditions hold:

\[
(\log \frac{1}{\psi^{-1}(t)}) \text{ is differentiable and } \frac{(\psi^{-1})'(t)}{\psi^{-1}(t)} t \text{ is a decreasing function;}
\]

\[
\log \frac{1}{\psi^{-1}(t)} \leq \beta \log \frac{1}{\psi^{-1}(\sqrt{t})};
\]

\[
\frac{(\log \psi(t))' t \log t}{\log \psi(t)} \text{ is a monotone function.}
\]

**Remark 1.**

i) One could replace the monotonicity conditions in (6) and (8) with a *pseudomonotonicity* condition (e.g. there exists a constant \( C > 0 \) such that \( u(t) \leq Cu(s) \) if \( t \leq s \)). This would only affect the constants in the proofs.

ii) The conditions (5) and (7) mean that the function \( \log \frac{1}{\psi^{-1}(t)} \) is a function of logarithmic type in the sense of [11] Definition 4.2.]

3. Proofs

**Proof.** We may assume that \( m, n \geq 2 \). Let \( f \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^m) \) and \( \psi \) be as in the statement of Theorem [11]. Denote \( \psi^{-1}(t) \) by \( u(t) \). It follows from our assumptions (3), (6), (7), (8) and [11] Remark 5.3.\] that

\[
\int_0^t \left( \frac{u(t)}{u'(t)} \right)^{n-1} \frac{dt}{t^n} = \infty.
\]

We define \( a(t) = \frac{u(t)}{u'(t)} \) and \( \lambda(k) = \frac{2^k}{mk2^n} \) for \( k \in \mathbb{N} \). By (6), \( \lambda \) is increasing for large \( k \). For simplicity we assume \( \lambda \) to be increasing.

Let \( \mathcal{W} \) be a fixed Whitney decomposition of \( \mathbb{B}^n \). For each cube \( Q \in \mathcal{W} \), we define a corresponding center \( f_Q \) and a corresponding radius \( r_Q = \max\{|f_Q - f_Q| : Q \sim \tilde{Q}\} \), which determine a family of balls on the image side: \( \mathcal{B} = \{B(f_Q, r_Q) : Q \in \mathcal{W}, r_Q > 0\} \). Note that some balls in \( \mathcal{B} \) may coincide, the simplest way to act in such a situation is to treat them as different balls for certainty (we may identify each ball in \( \mathcal{B} \), with \( (Q, B(f_Q, r_Q)) \),
then different Whitney cubes on the pre-image side generate different pairs), however, identifying such balls would cause no problem either.

We assign two new weighted collections of balls to each ball in $\mathcal{B}$. Given $B = B(x, r) \in \mathcal{B}$, we define concentric subballs $S_i(B) = B(x, r/2^i)$ for all $i \in \mathbb{N}$ and assign the weight $w_{S_i(B)} = 2^i$ to each $S_i(B)$. We set $S_B = \{S_i(B) : i \in \mathbb{N}\}$. Then

$$\sum_{B' \in S_B} w_{B'} r(B')^n = \sum_{i=1}^{\infty} w_{S_i(B)} r(S_i(B))^n = \sum_{i=1}^{\infty} 2^i r(B)^n \leq r(B)^n.$$

The second collection is defined in a similar way. If $B = B(x, r)$ is a ball in $\mathcal{B}$, we choose the smallest number $k_0(B) \in \mathbb{N}$, such that $2^{-k_0(B)} \leq r$. Next, for each $k = k_0(B), k_0(B) + 1, \ldots$, we choose $R_k(B) = B(x, \alpha(2^{-k}))$ and set $R_B = \{R_k(B) : k = k_0(B), k_0(B) + 1, \ldots\}$. The weights we assign this time are $w_{R_k(B)} = \lambda(k)$ for all $k = k_0(B), k_0(B) + 1, \ldots$. Similarly to above:

$$\sum_{B' \in R_B} w_{B'} r(B')^n = \sum_{k=k_0(B)}^{\infty} w_{R_k(B)} r(R_k(B))^n = \sum_{k=k_0(B)}^{\infty} \alpha(2^{-k})^n \lambda(k) \leq \sum_{k=k_0(B)}^{\infty} \alpha(2^{-k})^n \frac{\lambda(k)}{(\lambda(0))^{n-1}}$$

$$= \frac{1}{\lambda(0)^{n-1}} \sum_{k=k_0(B)}^{\infty} 2^{-nk} \leq \frac{2}{\lambda(0)^{n-1}} \cdot 2^{-nk_0(B)} \leq \frac{2}{\lambda(0)^{n-1}} r(B)^n.$$

Finally, we define our weighted collection of balls by setting $\mathcal{F} = \bigcup_{B \in \mathcal{B}} (S_B \cup R_B)$. Again, some of the balls in the united families may coincide; however, we treat them as "different" balls. Distinguishing them is, again, not difficult.

Let us now estimate the weighted sum of the $r$th powers of the radii of the balls in $\mathcal{F}$. Let $N(Q) = Q \cup \bigcup_{Q' \in \partial Q} Q'$ be the union of all neighbors of a cube $Q \in \mathcal{W}$. For neighboring cubes $Q$ and $Q'$, we obtain, via the Hölder and Poincaré inequalities, that

$$|f_Q - f_{Q'}| \leq \int_Q |f - f_{N(Q)}| + \int_{Q'} |f - f_{N(Q)}| \leq C \int_{N(Q)} |f - f_{N(Q)}| \leq C \left( \int_{N(Q)} |f - f_{N(Q)}|^n \right)^{1/n}$$

$$\leq C \left( \int_{N(Q)} |Df|^n \right)^{1/n}.$$

Hence, we have the estimate

$$r_Q^n = \max \{ |f_Q - f_{Q'}|^n : Q \sim Q' \} \leq C \int_{N(Q)} |Df|^n$$

for each $Q \in \mathcal{W}$ and some constant $C > 0$. Next, using the fact that the inequality $\sum_{Q \in W} \chi_{N(Q)}(y) \leq C$ holds for every $y \in \mathbb{R}^n$, we estimate

$$\sum_{B \in \mathcal{F}} w_B r(B)^n \leq C(\lambda(0)) \sum_{B \in \mathcal{B}} r(B)^n = C(\lambda(0)) \sum_{Q \in \partial \mathcal{W}} r_Q^n \leq C(\lambda(0)) \sum_{Q \in \partial \mathcal{W}} \int_{N(Q)} |Df|^n$$

$$\leq C_1 \int_{\bigcup_{Q \in \mathcal{W}} N(Q)} |Df|^n \leq C_1 \int_{\mathcal{B}^*} |Df|^n < \infty,$$

where $C_1 > 0$ is some constant depending on $n, m$ and $\lambda(0)$ only.

We may assume that there is at least one $Q \in \mathcal{W}$ with $r_Q > 0$; otherwise $f(\partial \mathcal{B}^n)$ is a singleton. Let $\omega \in \partial \mathcal{B}^n$. We consider the radius $[0, \omega]$ and the sequence $(Q_j(\omega))_{j=1}^{\infty}$. We fix a large integer $l_0 = l_0(\omega, f) \in \mathbb{N}$ so that there are elements of the sequence $(f_{Q_j(\omega)})_{j=1}^{\infty}$...
outside $B(f(\omega), 2^{-l_0+1})$, if $(f_{Q,\omega})^\infty_{j=1}$ contains at least one element different from $f(\omega)$. If such an integer does not exist, there necessarily is some $Q = Q_{\omega} \in \mathcal{W}$ with $f_{0} = f(\omega)$ and $r_{Q_{\omega}} > 0$. In this case, we choose $l_{0} = l_{0}(\omega, f) \in \mathbb{N}$ so that $2^{-l_{0}} < r_{Q_{\omega}}$. In both cases we also require that $2^{-l_{0}+1} < t_{0}$. This allows us to use the properties (5) and (7).

For the purposes of our "porosity argument", we would like to make the number $l_{0}$ independent of the point $\omega$. This is done by considering the decomposition

$$\partial B^n = \bigcup_{i \in \mathbb{N}} E_i$$ where $E_i = \{\omega \in \partial B^n : l_{0}(\omega, f) \leq i\}$.

Setting $F_{l} = f(E_{l})$, we then have $f(\partial B^n) = \bigcup_{i \in \mathbb{N}} F_{l}$.

Let us fix $l_{0} \in \mathbb{N}$. Our aim is to prove that $H_{\omega}^{\infty}(F_{l_{0}}) = 0$.

Fix $x \in F_{l_{0}}$. Take any $\omega \in E_{l_{0}}$, such that $x = f(\omega)$, and define the sequence of concentric annuli $A_{l}(x) = B(x, 2^{-l+1}) \setminus B(x, 2^{-l})$ with $l = l_{0}, l_{0} + 1, \ldots$. Next, we assign a suitable set $P_{l}(x)$ of cubes from $\mathcal{W}$ to each annulus $A_{l}(x)$, $l = l_{0}, l_{0} + 1, \ldots$. If $f_{Q,\omega}(x) = x$ for all $j \in \mathbb{N}$, we put $P_{l}(x) = \{Q_{\omega}\}$ for each $l \geq l_{0}$, where $Q_{\omega}$ is the cube defined earlier. Otherwise, all the sets $P_{l}(x)$ with $l \geq l_{0}$ consist of elements from $(f_{Q,\omega})^\infty_{j=1}$: if an annulus $A_{l}(x)$ with some $l \geq l_{0}$, contains no centres from $(f_{Q,\omega})^\infty_{j=1}$, we define $P_{l}(x) = \{Q_{m}(\omega)\}$, where an integer $m \in \mathbb{N}$ is chosen so that $f_{Q_{m-1}(\omega)} \notin B(x, 2^{-l+1})$, but $f_{Q_{m}(\omega)} \in B(x, 2^{-l})$; if, in contrast, there is at least one centre $f_{Q_{m}(\omega)}$ in $A_{l}(x)$, we take $P_{l}(x) = \{Q_{k}(\omega) : k = m_{1}, m_{2}, \ldots\}$, where $m_{1}, m_{2} \in \mathbb{N}$ are such that $f_{Q_{m_{1}}(\omega)} \notin B(x, 2^{-l})$, $f_{Q_{m_{2}+1}(\omega)} \in B(x, 2^{-l})$ and $f_{Q_{m}(\omega)} \in A_{l}(x)$ for all $k = m_{1}, \ldots, m_{2}$. Moreover, it is possible to choose the sets $P_{l}(x)$ above so that the inequality $k_{1} \leq k_{2}$ is valid, whenever $Q_{k_{1}}(\omega) \in P_{l_{1}}(x), Q_{k_{2}}(\omega) \in P_{l_{2}}(x)$ and $l_{1} \leq l_{2}$.

Denoting

$$\theta_{l}(x) = \begin{cases} 1, & \text{if } \#P_{l}(x) \leq \tilde{c}_{0} \lambda(l), \\ 0, & \text{otherwise,} \end{cases}$$

for $l \geq l_{0}$ and a constant $\tilde{c}_{0} > \lambda^{-1}(0)$, which we will specify later, we would like to prove that there exists an integer $l_{1} \geq 2l_{0}$, such that

$$\sum_{k=l_{0}}^{l_{1}} \theta_{k}(x) \geq \frac{l}{2}$$

for each $l \geq l_{1}$. In other words, at least half of the annuli do not contain too many centres from $(f_{Q,\omega})^\infty_{j=1}$. There is nothing to prove, if $f_{Q,\omega}(x) = x$ for all $j \in \mathbb{N}$; otherwise, the proof is by contradiction.

Let us assume that (11) does not hold for some $l \geq 2l_{0}$. Take the smallest number $J \in \mathbb{N}$ such that $f_{Q,\omega}(x) \in B(x, 2^{-l})$ for all $j > J$ and let $\omega_{*} \in [0, \omega]$ be the point of $Q_{J}(\omega) \cap [0, \omega]$, which is the closest to $\omega$. Now, the assumption on the continuity of $f$ and the properties of our Whitney decomposition imply

$$2^{-l} \leq |f_{Q,\omega}(x) - x| = |f_{Q,\omega}(x) - f(\omega)| \leq \int_{Q_{J}} |f(y) - f(\omega)| \, dy \leq \psi(2(1 - |\omega|)).$$

That is,

$$\frac{u(2^{-l})}{2} \leq 1 - |\omega|.$$
Using (5), we observe that
\[
\log \frac{2}{u(2^{-i})} \geq \log \frac{1}{1 - |\omega'|} \geq \frac{1}{c_2} \#q(0, \omega').
\]

In the calculation above, we may have to adjust the choice of \(l_0\) to ensure \(\#q(0, \omega') > c_3\) (see (5)). Finally, we obtain a lower bound for \(\#q(0, \omega')\), using the assumption that we have at least \([l/2] - l_0 + 2\) annuli \(A_k(x)\) with \(\theta_k(x) = 0\). We notice that the sets \(P_k(x)\) with \(\theta_k(x) = 0\) contain different cubes for different \(k\)'s, and, if \(k \leq l\), then the cubes in \(P_k(x)\) precede \(Q_j(\omega)\) in \((Q_j(\omega))_{j=1}^\infty\). We have
\[
c_2 \log \frac{2}{u(2^{-i})} \geq \#q(0, \omega') \geq \sum_{k=l_0}^{\lfloor l/2 \rfloor} \#P_k(x) \geq \sum_{k=l_0}^{\lfloor l/2 \rfloor} \tilde{c}_0 \lambda(k) \geq \tilde{c}_0 \sum_{k=l_0}^{\lfloor l/2 \rfloor} \frac{2^{-k}u'(2^{-k})}{u(2^{-k})}.
\]

Choosing \(\tilde{c}_0 > c_2 \beta\), this cannot hold when \(l\) is large enough. Thus, there is a number \(l_1 = l_1(\tilde{c}_0, l_0, u)\), such that (11) holds for all \(l \geq l_1\).

Our next step is to prove that if \(\theta_k(x) = 1\) for some \(k\) and \(P_k(x) = \{Q_1, \ldots, Q_m\}\), then it is possible to find a collection of balls \(\{B_1, \ldots, B_m\}\) from the families \(S_{B(f_0, r_Q)}\) or \(R_{B(f_0, r_Q)}\), having radii at least \(const \cdot \alpha(2^{-k})\) and satisfying \(\sum_{i=1}^m w_{B_i} \geq const \cdot \lambda(k)\). Moreover, we choose different balls (in the sense mentioned above) for different \(k\)'s.

Let us fix \(k \geq l_0\) such that \(\theta_k(x) = 1\). Suppose first that the annulus \(A_k(x)\) contains no centres from \((f_Q(\omega))_{j=1}^\infty\). Then the set \(P_k(x)\) consists of a single cube \(Q \in \mathcal{W}\) with \(f_0 \in B(x, 2^{-k})\). The definitions of \(r_Q\) and \(l_0\) imply \(r_Q > 2^{-k}\), and hence \(k \geq k_0(B(f_Q, r_Q))\). Thus, we may choose the ball \(R_k(B(f_Q, r_Q))\), which, by definition, has radius \(\alpha(2^{-k})\) and weight \(\lambda(k)\). In addition, the centre of this ball lies in \(B(x, 2^{-k})\).

Assume now that the annulus \(A_k(x)\) contains at least one of the centres from \((f_Q(\omega))_{j=1}^\infty\). Then, we have by the definitions of \(P_k(x)\) and \(r_Q\) that
\[
\sum_{Q \in P_k(x)} 2r_Q \geq 2^{-k}.
\]

Since \(\#P_k(x) \leq \tilde{c}_0 \lambda(k)\), we observe that
\[
\sum_{Q \in P_k(x)} 2r_Q \geq \frac{2^{-k}}{2 \tilde{c}_0}.
\]

For each \(Q \in P_k(x)\) with \(2r_Q \geq \frac{\alpha(2^{-k})}{2 \tilde{c}_0}\), we choose a number \(n_Q \in \mathbb{N}\) so that
\[
2^{n_Q-1} \frac{\alpha(2^{-k})}{2 \tilde{c}_0} \leq 2r_Q < 2^n \frac{\alpha(2^{-k})}{2 \tilde{c}_0}
\]
and pick a ball \(\bar{B} = S_{n_Q}(B(f_Q, r_Q)) = B(f_Q, r_Q/2^{n_Q}) \in S_{B(f_0, r_Q)}\). By the definition of \(S_i(B)\), we have \(w_{\bar{B}} = 2^{n_Q}\) and
\[
r(\bar{B}) = \frac{r_Q}{2^{n_Q}} \geq \frac{\alpha(2^{-k})}{8 \tilde{c}_0}.
\]
For the sum of the weights $\sum_Q 2^{n_Q}$ of all the balls obtained in such a manner, we observe that

$$\frac{\alpha(2^{-k})}{2c_0} \sum_{Q \in F(x)} 2^{n_Q} \geq \sum_{Q \in F(x)} 2r_Q \geq \frac{2^{-k}}{2}.$$ 

Hence we have a collection of balls $\{B_1, \ldots, B_m\} \subset F$ with weights sum $\sum_{i=1}^{m} w_{B_i} > c_0 \lambda(k)$ and of radii at least $\alpha(2^{-k})/8c_0$. Moreover, all these balls have their centres in the annulus $A_k(x)$, and hence in the ball $B(x, 2^{-k+1})$.

We have proved that there exists a number $l_1 = l_1(l_0, c_0)$, such that for each $\omega \in E_{l_0}$ and $l \geq l_1$, among the numbers $l_0, \ldots, l$, there are at least $[l/2]$ integers $k \in \{l_0, \ldots, l\}$, such that we are able to find a finite collection of balls $\{B_i\}_{i \in I} \subset F$ with weights sum $\sum_{i \in I} w_{B_i}$ at least $\lambda(k)$ and of radii at least $\alpha(2^{-k})/8c_0$, so that the centres of the balls $B_i$, $i \in I$, lie in the ball $B(x, 2^{-k+1})$. Here, $c_0$ is a positive constant depending only on $\beta$, $n$ and $\lambda(0)$, and the balls are different for a fixed $\omega$ and different $k$'s.

Fix $l \geq l_1$. We modify our family $F$ according to $l$. If $B \in F$ and there is $k \in \{l_0 + 1, \ldots, l\}$ such that $\alpha(2^{-k})/8c_0 \leq r(B) < \alpha(2^{-k+1})/8c_0$, we replace $B$ with the ball $\tilde{B} = \frac{\lambda(k)}{\alpha(l)}B$, and set $w_B = (\lambda(l)/\lambda(k))^n w_B$. The radius of $\tilde{B}$ satisfies $r(\tilde{B}) \geq \frac{\lambda(l)}{\lambda(k)} \alpha(2^{-k})/8c_0 = 2^{-k}/8c_0 \lambda(l)$ and the equality $w_B r(\tilde{B})^n = w_B r(B)^n$ holds. Similarly, we replace a ball $B$ with $r(B) \geq \alpha(2^{-k})/8c_0$ with the ball $\tilde{B} = \frac{\lambda(0)}{\lambda(l)}B$ and set $w_B = (\lambda(l)/\lambda(0))^n w_B$. Again, we have $r(\tilde{B}) \geq 2^{-l}/8c_0 \alpha(l)$ and $w_B r(\tilde{B})^n = w_B r(B)^n$. Finally, $F_l$ is the collection of balls obtained in this manner from the balls in $F$. For this family of balls, we notice (see (10)) that

$$\sum_{B \in F_l} w_B r(B)^n \leq \sum_{B \in F} w_B r(B)^n < \infty. \tag{12}$$

If $\omega \in E_{l_0}$, $x = f(\omega)$ and $k \in \{l_0, \ldots, l\}$ is such that $\theta_k(x) = 1$, then there is a collection $\{B_i\}_{i \in I} \subset F$ with the properties mentioned above. If a ball $B_i$ with some $i \in I$ is replaced by a ball $\tilde{B}_i = \frac{\lambda(k)}{\lambda(0)}B_i$, while creating $F_l$, we necessarily have $k_i \leq k$. Therefore, the inequalities

$$\sum_{i \in I} w_{B_i} = \sum_{i \in I} \left(\frac{\lambda(l)}{\lambda(k_i)}\right)^n w_{B_i} \geq \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \sum_{i \in I} w_{B_i} \geq \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \lambda(k) = \lambda(l)^n \frac{1}{\lambda(k)^{n-1}}$$

and $r(\tilde{B}_i) \geq 2^{-k}/8c_0 \lambda(l) \geq 2^{-k}/8c_0 \lambda(l)$ hold (by (5), $\lambda$ is increasing). Since, for each $i \in I$, the centre of a ball $\tilde{B}_i$ is contained in $B(x, 2^{-k+1})$, we have the inclusion $x \in 16c_0 \lambda(l)B_i$. Hence we observe that

$$\sum_{B \in F_l} w_B \chi_{16c_0 \lambda(l)B}(y) \geq \sum_{\substack{k = l_0, \ldots, l \\theta_k(y) = 1}} \lambda(l)^n \frac{1}{\lambda(k)^{n-1}} \geq \lambda(l)^n \frac{1}{4} \sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}} \geq \lambda(l)^n \frac{1}{4} G_l$$

for each $y \in F_{l_0}$, where $G_l = \sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}}$. That is, $(\frac{4\alpha_n}{\lambda(0) G_l}, 16c_0 \lambda(l)B)_{B \in F_l}$ is a weighted cover of the set $F_{l_0}$. We observe also that diameters of all balls in this cover are at least $2^{-l}$. This information will be used in the proof of Theorem 1.2 below.
Finally, using the weighted cover obtained above and (12), we estimate the weighted Hausdorff $n$-content $\lambda^w_\infty(F_0)$:

$$\lambda^w_\infty(F_0) \leq \frac{4}{\lambda(l)^n} \sum_{B \in F} w_B (\text{diam } 16\tilde{c} \lambda(l) B)^n \leq \frac{4^{2n+1} \tilde{c}^n}{G_l} \sum_{B \in F} w_B (\text{diam } B)^n$$

$$\leq \frac{2^{5n+2} \tilde{c}^n}{G_l} \sum_{B \in F} w_B r(B)^n \leq \frac{A}{G_l},$$

where the constant $A$ depends on $\beta$, $n$, $m$, $\|f\|_{W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)}$ and $\lambda(0)$ but does not depend on $l_0$ or $l$.

Now, Lemma 2.1 implies $\mathcal{H}^w_\infty(F_0) \leq \frac{C \tilde{c}}{G_l}$. Here $C$ depends only on the dimension $n$. Now, we are done as soon as we can show that $G_l \to \infty$ as $l \to \infty$. We have

$$G_l = \sum_{k=1}^l \frac{1}{\lambda(k)^{n-1}} = \sum_{k=1}^l \frac{u(2^{-k})^{n-1}}{2^{-k(2-n-1)u'(2^{-k})^{n-1}}} \geq \int_{2^{-l}}^{2^{-1}} \left( \frac{u(t)}{u'(t)} \right)^{n-1} \frac{dt}{t^n}$$

and the right hand side diverges as $l \to \infty$ by the assumptions on the modulus of continuity.

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1. We only point out the required changes.

Proof of Theorem 1.2 Let $f$ be as in statement of the theorem. Our notation will be the same as in previous proof. That is, $a(t) = \gamma t$ and $\lambda(k) = \frac{1}{k}$.

Fix a small $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$(13) \quad \int_{\mathbb{B}^n \setminus B(0, 1-\delta)} |Df|^n \leq \varepsilon.$$ 

Let $\mathcal{W}^\delta$ be the set of the cubes in $\mathcal{W}$ which are contained in $\mathbb{B}^n \setminus B(0, 1-\delta)$ and whose all neighbour cubes are also contained in $\mathbb{B}^n \setminus B(0, 1-\delta)$. We define our collection of balls to be $\mathcal{B}^\delta = \{B(f_Q, r_0): Q \in \mathcal{W}^\delta\}$. Then, proceeding as in the previous proof, we define $\mathcal{F}^\delta$ analogously to $\mathcal{F}$ and obtain the estimate (see (10))

$$(14) \quad \sum_{B \in \mathcal{F}^\delta} w_B r(B)^n \leq C_1 \varepsilon.$$ 

Let $\omega \in \partial \mathbb{B}^n$. We define the number $l_0 = l_0(\omega, f, \delta)$ as in previous proof, but instead of all cubes in $(Q_f(\omega))^\infty_{j=1}$ we consider only those which are contained in $\mathcal{W}^\delta$. Again, we split $\partial \mathbb{B}^n$ to sets $E_i = \{\omega \in \partial \mathbb{B}^n: l_0(\omega) \leq l\}$ and consider a fixed $f(E_i)$. With the same method as earlier we find, for big $l$ a collection of balls $\mathcal{F}_i^\delta$ with weights such that $\langle \frac{\tilde{w}_B}{1-\lambda(l)} \frac{16\tilde{c}}{\gamma} B \rangle_{B \in \mathcal{F}_i^\delta}$ is a weighted cover of the set $f(E_i)$, the radius of the balls $\frac{16\tilde{c}}{\gamma} B$ is at least $2^{-l}$ and

$$\sum_{B \in \mathcal{F}_i^\delta} w_B r(B)^n \leq C_1 \varepsilon.$$ 

We may assume that our $\varepsilon > 0$ is so small that all balls in our weighted cover have radius smaller than $\frac{1}{2}$. With this weighted cover we obtain
We define the square $\overline{C}_l$ to a set of positive $\mathcal{H}^s$-measure, with $g(t) = t^2 \left( \log \frac{1}{t} \right)^{2p}$.

The mapping is a composition of two locally Hölder continuous mappings. The second mapping is defined in [4, Prop. 5.1]. It is a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, identity outside $[0, 1]^2$, and maps a small Cantor set $C \subset [0, 1]^2$ to a large Cantor set $C' \subset [0, 1]^2$ with positive $\mathcal{H}^s$-measure. It was checked in [9] that this mapping belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ if $p > 1/2$.

Next, we elaborate on the construction of $h$ and prove that it is Hölder continuous in $[0, 1]^2$. Let $\sigma < 1/2$. We use the notation $||x|| = \max(||x_1||, ||x_2||)$. Let $p > 1/2$. We will construct a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, with $f \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, which is locally Hölder continuous and maps $\partial B^2$ to a set of positive $\mathcal{H}^s$-measure, with $g(t) = t^2 \left( \log \frac{1}{t} \right)^{2p}$.

The mapping is a composition of two locally Hölder continuous mappings. The second mapping is defined in [4, Prop. 5.1]. It is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, identity outside $[0, 1]^2$, and maps a small Cantor set $C \subset [0, 1]^2$ to a large Cantor set $C' \subset [0, 1]^2$ with positive $\mathcal{H}^s$-measure. It was checked in [9] that this mapping belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ if $p > 1/2$.

The set $C'$, and sets $Q_{ki}$, $P_{ki}$ and $A_{ki}$, with $k \in \mathbb{N}$ and $i = 1, \ldots, 2^k$ are defined in the same way, using $2r' = (\log 2)^{-p} \cdot 2R' = r' \cdot 2R_k = (\log 2)^{-p} \cdot 2^{k-1}R_k$ and $2R_k = (\log 2)^{-p} \cdot 2^{k-1}(k-1)^{-p}$ for other $k \in \mathbb{N}$.

The mapping $h$ is defined so that it maps the frame $A_{ki}$ to the frame $A_{ki}'$ via a “radial” stretching and is continuous in $[0, 1]^2$. The radial stretching which maps $A = \{x : r_k \leq ||x|| \leq R_k\}$ to $A' = \{x : r_k' \leq ||x|| \leq R_k'\}$ is

$$\rho(x) = (a||x|| + b) \frac{x}{||x||},$$

where $a = \frac{R_k - r_k'}{R_k - r_k}$ and $b = \frac{R_k r_k' - R_k r_k}{R_k - r_k}$.

If $x, y \in A$, then $||x - y|| \leq 2R_k$ if $2^{k-1}(k-1)^{-p}$ and $a \leq \frac{4\sigma}{1 - 2\sigma} C(\sigma)^{-1 - \beta k} \leq \frac{C(\sigma)||x - y||^{\beta - 1}}{1 - 2\sigma}$. Similarly,

$$\frac{|b|}{|r_k|} \leq \frac{4}{1 - 2\sigma} C(\sigma)^{-1 - \beta k} \leq \frac{C(\sigma)||x - y||^{\beta - 1}}{1 - 2\sigma}.$$
The mapping $\rho$ is Hölder continuous with exponent $\beta$,
\[ \|\rho(x) - \rho(y)\| \leq C\|x - y\| + 2\frac{|b|}{|r_k|} \|x - y\| \leq C(\sigma)\|x - y\|^{\beta}. \]

If $x \in A_{k,i}$ and $y \in Q_{k+1,j} \subset P_{k,i}$, then $\|x - y\| \geq R_{k+1} - r_{k+1} = C(\sigma)\sigma^k$ and $\|h(x) - h(y)\| \leq 2R_k \leq 2^{-k}$. These imply
\[ \frac{\|h(x) - h(y)\|}{\|x - y\|^{\beta}} \leq C(\sigma). \]

The $\beta$-Hölder continuity of $h$ easily follows from the continuity estimates obtained above.

The first mapping $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a (locally Hölder continuous) quasiconformal mapping for which $C \subset g(\partial \mathbb{B}^2)$. Such a mapping was constructed in [2].

Finally, the composition $h \circ g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism with $h \circ g(\partial \mathbb{B}^2) \supset C'$. Moreover, it is locally Hölder continuous and $h \circ g \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ by quasiconformality of $g$ and the change of variable formula.

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