A possible experimental determination of $m_s/\hat{m}$ from $K_{\mu 4}$ decays

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Abstract

$K\pi$ scattering and $K_{\mu 4}$ decays are studied at leading order of improved chiral perturbation theory. It is shown that high precision $K_{\mu 4}$ experiments at, e.g., DAΦNE should allow for a direct measurement of the quark mass ratio $m_s/\hat{m}$.
The light flavour symmetry breaking sector of QCD involves various parameters whose precise determination is of a fundamental importance. In particular, the products of running quark masses $m_u, m_d, m_s$ with the quark-antiquark condensate of the massless theory,

$$B_0 \equiv -F_0^{-2} \langle \bar{u}u \rangle = -F_0^{-2} \langle \bar{d}d \rangle = -F_0^{-2} \langle \bar{s}s \rangle,$$

are well defined renormalization group invariant quantities which are in principle measurable and which are not determined within the standard model. ($F_0$ denotes the chiral limit of the pion decay constant $F_\pi = 93.1$ MeV.) While quark masses can be chosen freely within QCD, the magnitude of the scale dependent condensate $B_0$ is an intrinsic property of the theory, reflecting the mechanism of spontaneous symmetry breaking. Since the latter is not yet clearly understood in QCD, the order of magnitude of $B_0$ is hard to estimate a priori: $B_0$ could be as large as the scale $\Lambda_H$ of formation of massive bound states, $\Lambda_H \sim 1$ GeV, or it could be as small as the fundamental order parameter $F_0 \sim 90$ MeV.

For sufficiently small quark masses, the expansions of Goldstone boson masses are dominated by the linear terms

$$M_{\pi^+}^2 = (m_u + m_d)B_0 + \ldots, \quad M_{K^+}^2 = (m_u + m_s)B_0 + \ldots, \quad M_{K^0}^2 = (m_d + m_s)B_0 + \ldots. \quad (2)$$

How small the quark masses should actually be in order to ensure this dominance is controlled by the size of $B_0$: For the pseudoscalar meson $\bar{ab}$ ($a, b = u, d, s; a \neq b$) such a dominance requires

$$m_a + m_b \ll B_0/A_0,$$

where $A_0$ is a dimensionless parameter of order one characteristic of contributions to the expansion (2) coming from terms which are quadratic in quark masses. ($A_0$ has been defined in Ref. [3] in terms of a two-point QCD correlator.) For $B_0$ of the order of the bound state scale $\Lambda_H \sim 1$ GeV, the condition (3) is likely to be satisfied for actual values of quark masses. In this case, the standard chiral perturbation theory ($\chi$PT) - which counts each insertion of quark mass as two powers of pion momentum - should describe the low energy data well already within a few lowest orders. If, on the other hand, $B_0$ turned out to be comparable to the fundamental order parameter $F_0 \sim 90$ MeV, the condition (3) would certainly be violated already for $m_a$ or $m_b$ equal to the strange quark mass, but also for non strange quark masses in the range 20 - 30 MeV, where they are still small as compared to $\Lambda_H$. The first term in the expansion (1) would then be considerably lower than the pseudoscalar masses $M_{\pi^+}^2$, and consequently, the standard expansion of the symmetry breaking part of the QCD effective lagrangian should be rearranged in order to improve its convergence. An improved $\chi$PT has been proposed in references [3], [6]: it is an expansion in pion momentum $p/\Lambda_H$, in quark masses $M_q/\Lambda_H$ and in powers of $B_0/\Lambda_H$, with $M_q$ and $B_0$ counting as a single power of $p$. This modified counting rule leads to
a consistent redefinition of individual chiral orders. The leading $O(p^2)$ order now consists of 5
independent terms: in a standard notation,

$$\tilde{\mathcal{L}}_2 = \frac{F_0^2}{4} \{ \langle D^\mu U^+ D_\mu U \rangle + 2B_0 \langle \mathcal{M}_q U + \mathcal{M}_q U^+ \rangle \\
+ A_0 \langle (\mathcal{M}_q U)^2 + (\mathcal{M}_q U^+)^2 \rangle + Z_0^S \langle \mathcal{M}_q U + \mathcal{M}_q U^+ \rangle^2 \\
+ Z_0^P \langle \mathcal{M}_q U - \mathcal{M}_q U^+ \rangle^2 \} \ .$$  (4)

The terms quadratic in the quark mass matrix $\mathcal{M}_q$ that are usually relegated to the $O(p^4)$ order
[4] can now give contributions comparable to the $B_0$-term, reflecting the violation of condition
(3). (Notice that $Z_0^S$ and $Z_0^P$ violate the Zweig rule in the 0$^+$ and 0$^-$ channels, respectively. $Z_0^P$
will play no role in the present work.)

The improved $\chi$PT generalizes the standard expansion since at each order the former
contains additional terms, that the latter relegates to higher orders. Consequently, it is less
predictive, but constitutes a more appropriate theoretical framework for an unbiased experi-
mental determination of symmetry breaking parameters such as the ratios of quark masses,
$m_q B_0$ and other non-perturbative characteristics of the massless QCD vacuum. It is convenient
to use the improved $O(p^2)$ expression for $M_\pi^2$ and $M_K^2 = \frac{1}{2}(M_{K^+}^2 + M_{K^0}^2),$

$$M_\pi^2 = 2\hat{m}\mu_0 + 4\hat{m}^2 A_0 \ ,$$  (5)

$$M_K^2 = (\hat{m} + m_s)\mu_0 + (\hat{m} + m_s)^2 A_0 \ ,$$

where $\hat{m} = \frac{1}{2}(m_u + m_d)$, $\mu_0 = B_0 + 2(m_s + 2\hat{m})Z_0^S$, and to express the low energy constants of
$\tilde{\mathcal{L}}_2$ in terms of

$$r = \frac{m_s}{\hat{m}} \ , \ \zeta = \frac{Z_0^S}{A_0} \ .$$  (6)

(In a similar way, the constant $Z_0^P$ can be expressed in terms of the $\eta$ mass.) The masses $M_\pi^2,$
$M_K^2,$ the quark mass ratio $r$ and the Zweig rule violating constant $\zeta$ are independent parameters,
except for the restriction

$$r_1 \equiv 2 \frac{M_K}{M_\pi} - 1 \leq r \leq r_2 \equiv 2 \frac{M_K^2}{M_\pi^2} - 1 \ ,$$  (7)

arising from the requirement of vacuum stability. The leading order of the standard $\chi$PT [4]
is reproduced for the particular choice $r = r_2 \sim 25.9$, $Z_0^S = 0$, implying $A_0 = 0$. The other
extreme, viz. $r = r_1 \sim 6.3$, $\zeta = 0$, corresponds to the order parameter $B_0 = 0$. The value of
the quark mass ratio $r$ should ultimately be determined from experiment, which may confirm
or invalidate the a priori estimate $r \sim r_2$. Actually, a recent analysis of the deviations from
the Goldberger-Treiman relation suggests that $r$ might be less than 25 by a factor of 2 or 3 [3].
There are not many physical processes directly accessible to experiment that exhibit a strong dependence on the quark mass ratio \( r \) already at the order \( O(p^2) \). One of them is the \( \pi^-\pi^+ \) scattering. At the tree level, the corresponding amplitude can be parametrized as

\[
A(s|t, u) = \frac{\beta_{\pi\pi}}{F_\pi^2}(s - \frac{4}{3}M_\pi^2) + \frac{\alpha_{\pi\pi}}{3F_\pi^2}M_\pi^2, \tag{8}
\]

where, at leading \( O(p^2) \) order \([5], [6]\),

\[
\alpha_{\pi\pi}^\text{lead} = 1 + 6 \frac{r_2 - r}{r_2 - 1} (1 + 2\zeta), \quad \beta_{\pi\pi}^\text{lead} = 1. \tag{9}
\]

As \( r \) decreases from \( r_2 \) to \( r_1 \), \( \alpha_{\pi\pi}^\text{lead} \) increases from the canonical value \( \alpha_{\pi\pi}^\text{lead} = 1 \) to \( \alpha_{\pi\pi}^\text{lead} = 4 \). A method of determining \( r \) from the forthcoming precise low-energy \( \pi^-\pi^+ \) scattering data has been discussed in Ref. [6].

The main purpose of this letter is to point out that there exists another independent case of similar interest: the decay

\[ K^+ \to \pi^+\pi^-\mu^+\nu_\mu, \tag{10} \]

which, although less abundant than the standard \( K_{e4} \) decay, can be easily accessible at future high statistics Kaon factories, e.g. at DAΦNE [10]. To the leading \( O(p^2) \) order, the axial-vector part of the \( K_{l4} \) matrix element receives two contributions, shown in Fig.1. While the direct interaction vertex of Fig.1a is independent of \( r \), the \( K^- \) pole contribution of Fig.1b exhibits an \( r \) dependence through the virtual \( \pi^-K \) scattering amplitude. The latter gives rise to a contribution proportional to the lepton mass, and hence invisible in \( K_{e4} \) decays. In this paper, the question of whether the \( r \) dependence can be observed in the \( K_{\mu4} \) decays is answered positively within the leading order. The loop corrections to this result will be presented elsewhere.

Ignoring isospin breaking effects (from now on \( m_u = m_d = \hat{m} \)), the amplitude for the \( \pi^-K \) scattering process

\[ \pi^a + K^i \to \pi^b + K^j, \tag{11} \]

\( a, b = 1, 2, 3, \ i, j = \pm 1/2 \), is described in terms of two invariant amplitudes \( A^\pm(s, t, u) \),

\[
A^{\pi^a + K^i \to \pi^b + K^j}(s, t, u) = \delta^{ab}\delta^{ij}A^+(s, t, u) - ie^{abc}(\tau^c)^{ij}A^-(s, t, u), \tag{12}
\]

with

\[
A^\pm(s, t, u) = \pm A^\pm(u, t, s). \tag{13}
\]
They are related to the isospin amplitudes $A^I(s,t,u)$, $I = \frac{1}{2}, \frac{3}{2}$, by

$$A^{3/2} = A^+ + A^-$$

$$A^{1/2} = A^+ - 2A^- .$$

Upon neglecting $O(p^4)$ terms, the tree level amplitudes are described by three constants: $\alpha_{\pi K}$, $\beta_{\pi K}$, analogous to the constants $\alpha_{\pi \pi}$ and $\beta_{\pi \pi}$ occurring in the low energy parametrization of the $\pi - \pi$ tree level amplitude in Eq.(8), and $\gamma_{\pi K}$,

$$A^+(s,t,u) = \frac{\beta_{\pi K}}{4F_\pi^2}(t - \frac{2}{3}M_\pi^2 - \frac{2}{3}M_K^2) + \frac{1}{6F_\pi^2}\{(M_K-M_\pi)^2 + 2M_\pi M_K \alpha_{\pi K}\}$$

$$A^-(s,t,u) = \frac{\gamma_{\pi K}}{4F_\pi^2}(s - u) .$$

At leading order, these constants read

$$\alpha_{\pi K}^{lead} - 1 = \frac{r + 1}{r_1 + 1}(\alpha_{\pi \pi}^{lead} - 1) , \quad \beta_{\pi K}^{lead} = 1 = \gamma_{\pi K}^{lead} .$$

At this stage, the $r$ dependence enters the $K - \pi$ amplitude through the constant $\alpha_{\pi K}$ only (cf. the similar situation in the $\pi - \pi$ case). When $r$ differs from $r_2$ this leads to an enhancement of the $K - \pi$ amplitude: as it is defined, $\alpha_{\pi K}^{lead}$, like $\alpha_{\pi \pi}^{lead}$, varies from 1 (the standard case \[4], \[8]) to 4, for $r = r_1$. At the same order, the scattering lengths $a^I_l$ and slope parameters $b^I_l$ ($l = 0, 1, I = \frac{1}{2}, \frac{3}{2}$) read

$$a^{1/2}_0 = \frac{M_\pi M_K}{32\pi F_\pi^2} \frac{5}{3} + \alpha_{\pi K}^{lead} ,$$

$$a^{1/2}_1 = \frac{1}{64\pi F_\pi^2} ,$$

$$a^{3/2}_0 = \frac{M_\pi M_K}{32\pi F_\pi^2} \frac{4}{3} - \alpha_{\pi K}^{lead} ,$$

$$a^{3/2}_1 = 0 ,$$

$$b^{1/2}_0 = \frac{-1}{32\pi F_\pi^2} \left[\frac{3}{2} - \frac{(M_\pi + M_K)^2}{M_\pi M_K}\right] ,$$

$$b^{3/2}_0 = \frac{-1}{32\pi F_\pi^2} \frac{(M_\pi + M_K)^2}{M_\pi M_K} .$$

One notices that the combination $2a^{3/2}_0 + a^{1/2}_0$, which vanishes in the standard case, is the most sensitive one to departures of $r$ from $r_2$. The combination $a^{3/2}_0 - a^{1/2}_0$, which may, in principle, be determined through an accurate measurement of the lifetime of $K - \pi$ atoms \[9],

\[1\]For the definition of the threshold parameters $a^I_l$ and $b^I_l$ we follow the conventions of Ref. \[4\]
does not depend on $r$ at leading order. (The lifetime of $\pi^-\pi$ atoms similarly gives access to the combination $a_0^2 - a_0^0$ of $\pi^-\pi$ scattering lengths, which still depends on $r$ at leading order [3].) On the other hand, model independent informations on the $I = \frac{1}{2}$ phase shifts may be extracted from high precision data on $D_l^4$ decays (e.g. $D^+ \to K^+\pi^-e^+\nu_e$). Quite generally, and as already noticed in the case of $\pi^-\pi$ scattering [6], for $r \sim 10$, the improved leading order modifies the scattering lengths in the same direction and by roughly the same amount as the standard loop corrections [4], [8].

Next, we turn to the $K_{l4}$ decays, $l = e, \mu$; we shall concentrate on the process

$$K^+(k) \to \pi^+(p_+)\pi^-(p_-)l^+(p_l)\nu_l(p_\nu).$$

(19)

The axial current matrix element is described by three form factors, $F, G, R$,

$$<\pi^+(p_+)\pi^-(p_-)|A_{\mu}^{4-i5}|K^+(k)> =$$

$$= \frac{-i}{M_K} \left[ (p_+ + p_-)_{\mu} F + (p_+ - p_-)_{\mu} G + (k - p_+ - p_-)_{\mu} R \right],$$

(20)

while the vector current matrix element requires only one form factor, $H$,

$$<\pi^+(p_+)\pi^-(p_-)|V_{\mu}^{4-i5}|K^+(k)> = \frac{H}{M_K^3} \epsilon_{\mu\nu\rho\sigma} k^\nu (p_+ + p_-)^\rho (p_+ - p_-)^\sigma.$$

(21)

These form factors are functions of the invariants

$$s_\pi = (p_+ + p_-)^2,$$
$$s_\ell = (k - p_+ - p_-)^2,$$
$$\Delta = -2k \cdot (p_+ - p_-).$$

(22)

Contributions to $H$ start only at order $O(p^4)$ in the effective lagrangian with the Wess-Zumino term which gives

$$H = -\frac{\sqrt{2}M_K^3}{8\pi^2 F_0^2}.$$

(23)

At leading order, the form factors $F$ and $G$ are also constant and read

$$F = G = \frac{M_K}{\sqrt{2}F_0}.$$

(24)

The form factor $R$ is the sum

$$R = R_{\text{direct}} + R_{K-\text{pole}}.$$

(25)
$R_{direct}$ arises from diagrams where the axial current $A_{\mu}^{4-i5}$ couples directly to three pseudoscalar mesons (Fig. 1a), while $R_{K-pole}$ is obtained from diagrams where $A_{\mu}^{4-i5}$ couples to a single internal pseudoscalar line (Fig. 1b). At leading order, one obtains

$$R_{direct} = \frac{M_K}{\sqrt{2}F_0} \frac{2}{3}, \quad (26)$$

and

$$R_{K-pole} = \frac{M_K}{\sqrt{2}F_0} \frac{1}{s_l - M_K^2} \left\{ \frac{1}{2}s_\pi + \frac{1}{2}\Delta - \frac{1}{6}(s_l - M_K^2) + \frac{2}{3}M_\pi M_K (\alpha_{lead} - 1) \right\}, \quad (27)$$

At leading order, the dependence on $r$ appears only in $R_{K-pole}$, through the $K-\pi$ scattering parameter $\alpha_{lead}^{\pi K}$. In the differential decay rate, the contributions of $R$ appear always with a multiplicative factor $m^2_l$ (a review of the kinematics of $K_{l4}$ processes and explicit formulae for differential decay rates may be found in Refs. [11], [12]). Hence, $K_{\mu4}$ decays will be quite insensitive to the value of $r$. On the other hand, $K_{l4}$ processes offer the possibility for a direct experimental determination of $m_s/\hat{m}$. In Fig. 2, we have plotted the differential decay rate $d\Gamma/ds_l$ for different values of $r$ using the $O(p^2)$ expressions for the form factors $F$, $G$, $R$, and formula (23) for $H$. As $r$ varies between $r_1$ and $r_2$, one sees an overall effect of 20 - 25 %, and which is not sensitive to the value of the Zweig rule violating parameter $\zeta$ taken in the range 0 - 0.2. (One could, in principle, obtain both the values of $r$ and $\zeta$ from separate measurements of $\alpha_{\pi\pi}$ and of $\alpha_{\pi K}$.) A statistical sample of 30.000 events, which might be obtained at DAΦNE, should be sufficient for an experimental determination of $m_s/\hat{m}$. At order $O(p^2)$, one computes the total decay rate for the process (19) to be $\Gamma = 156$ s$^{-1}$ for $r = r_2$ and $\Gamma = 112$ s$^{-1}$ for $r = r_1$. The loop corrections are expected to modify the above results in a nonnegligible way. There is however no reason to believe that they would destroy the sensitivity with respect to $r$ exhibited at leading order. The experience from the $\pi-\pi$ analysis shows that loops rather tend to amplify the tree level effects. The results of the loop calculations will be presented elsewhere.

**References**

[1] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. 175 (1968) 2195 ; S. Glashow and S. Weinberg, Phys. Rev. Lett. 20 (1968) 224.

[2] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.

[3] N. H. Fuchs, H. Sazdjian and J. Stern, Phys. Lett. B238 (1990) 380.

[4] J. Gasser and H. Leutwyler, Ann. Phys. 158 (1984) 142; Nucl. Phys. B250 (1985) 465.
[5] N. H. Fuchs, H. Sazdjian and J. Stern, Phys. Lett. B269 (1991) 183.

[6] J. Stern, H. Sazdjian and N. H. Fuchs, preprint IPNO/TH 92-106, to appear in Phys. Rev. D.

[7] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.

[8] V. Bernard, N. Kaiser and U.-G. Meissner, Phys. Rev. D43 (1991) R2757; Nucl. Phys. B357 (1991) 129.

[9] L. L. Nemenov, Sov. J. Nucl. Phys. 41 (1985) 629; A. A. Bel’kov, V. N. Pervushin and F. G. Tkebuchava, Sov. J. Nucl. Phys 44 (1986) 300; S. Wycech and A. M. Green, preprint HU-TFT-93-9, Helsinki (1993).

[10] The DAΦNE Handbook (1992), edited by L. Maiani, G. Pancheri and N. Paver.

[11] J. Bijnens, G. Ecker, J. Gasser, contribution to the DAΦNE Handbook (1992), edited by L. Maiani, G. Pancheri and N. Paver.

[12] J. Bijnens, Nucl. Phys. B337 (1990) 635; C. Riggenbach, J. Gasser, J. F. Donoghue and B. R. Holstein, Phys. Rev. D43 (1991) 127.
Figure Captions

Figure 1. The matrix element of the axial current $A_{\mu}^{4-i5}$ (wavy line) between an incoming $K$ (solid line) and two outgoing $\pi$’s (broken lines), showing: a) the direct contribution and, b) the $K$-pole contribution to the form factor $R$.

Figure 2. The differential $K_{\mu 4}$ decay rate $d\Gamma/ds_I$ (in units of $M_{\pi}^{-1}$) as a function of $s_I$ (in units of $M_{\pi}^2$) plotted for different values of $r$ and for $\zeta = 0.1$. Starting from the bottom curve, corresponding to $r = r_1 \sim 6.3$, the subsequent curves correspond to $r = 10$, $r = 15$ and $r = r_2 \sim 25.9$, respectively.