Zamolodchikov tetrahedral equation
and higher Hamiltonians
of 2d quantum integrable systems

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Abstract. The main aim of this work is to develop a method of constructing higher Hamiltonians of quantum integrable systems associated with the solution of the Zamolodchikov tetrahedral equation. As opposed to the series of papers [1] the approach presented here is effective for generic solutions of the tetrahedral equation without spectral parameter. In a sense, this result is a two-dimensional generalization of the method of [2]. The work is a part of the project relating the tetrahedral equation with the quasi-invariants of 2-knots.

Contents

1 Introduction
  1.1 Yang-Baxter equation and its generalizations .................................................... 1
  1.2 Universal integrability in \( d = 1 \) ................................................................. 2
  1.3 Tetrahedral equation ................................................. 3

2 Commutative family
  2.1 The commutativity demonstration in \( d = 1 \) ............................................. 4
  2.2 Regular 3-d lattices and statistical models ................................................. 7
  2.3 Some consequences of the tetrahedral equation ........................................... 8
  2.4 Two families ................................................................. 9
  2.5 The commutativity proof in \( d = 2 \) ...................................................... 10

3 Conclusion ...................................... 12

1 Introduction

1.1 Yang-Baxter equation and its generalizations

This work is mainly focusing on the \emph{matrix} Yang-Baxter equation

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \text{End}(V^{\otimes 3}), \quad R \in \text{End}(V^{\otimes 2}),
\]

and the \emph{matrix} tetrahedral Zamolodchikov equation [3]

\[
\Phi_{123}\Phi_{145}\Phi_{246}\Phi_{356} = \Phi_{356}\Phi_{246}\Phi_{145}\Phi_{123} \in \text{End}(V^{\otimes 6}), \quad \Phi \in \text{End}(V^{\otimes 3}).
\]

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In both cases $V$ is a finite dimensional vector space, the indices denote the numbers of the space copies in which linear operators act non-trivially. We should also mention the universal description of the $n$-simplex equation (e.g. [4]), generalizing the Yang-Baxter and the Zamolodchikov equations.

The work is aimed to generalize one of the existing applications of the theory of the Yang-Baxter equation to the theory of quantum integrable systems. It should be noted that this equation and subsequently the theory of quantum groups marked a new era in the field of exactly-solvable models of mathematical physics. This concerned both models of statistical physics in dimension $2$, and one-dimensional quantum mechanical models of the theory of magnets. The established connection has been effective in the purely physical issues like the effect of the spontaneous magnetization as like as in many mathematical problems. The language of Hopf algebras not only expanded the machine of modern algebra but also allowed to explore new patterns, for example, in topology.

The mentioning of such diverse areas of modern mathematical physics here is not accidental. The transition from the Yang-Baxter equations to the tetrahedral equation appears natural from many points of view: low-dimensional topology, combinatorics, statistical models, topological field theories, homotopy algebraic structures. Here we present a brief table showing some heredity in subjects related to the Yang-Baxter equation and the tetrahedral one.

|                           | Yang-Baxter equation | Zamolodchikov equation |
|---------------------------|----------------------|-------------------------|
| Statistical models        | $d = 2$              | $d = 3$                 |
| Spin chains               | $d = 1$              | $d = 2$                 |
| Homotopy Lie algebras     | Lie algebras         | 2-Lie algebras (eg. thesis of A. Crans) |
| Topological invariants    | Turaev-Reshetikhin-type knot invariants | 2-knot quasi-invariants in [7] |
| Hopf algebras             | Quasi-triangular Hopf algebras | ? |

1.2 Universal integrability in $d = 1$

Besides the equation [1]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

we consider the structural equations in the combinatorial form:

$$R'_{12}R'_{23}R'_{12} = R'_{23}R'_{12}R'_{23}$$

$$R'L \otimes L = L \otimes LR'$$

where $R' \in \text{End}(V)^{\otimes 2}$ and $L \in \text{End}(V) \otimes \text{End}(V_q)$, here $V_q$ is the quantum vector space (i.e. one another vector space, distinguished from $V$.) The transition from [1] to [3] can be realized as follows: if $R$ satisfies [1] then $R'_{12} = R_{12}P_{12}$ as like as $R''_{12} = P_{12}R_{12}$ satisfy [3], here $P_{12}$ is the transposition linear operator in $V \otimes V$. The equation [1] is traditionally called the RLL-relation, it plays a substantial role in Reshetikhin-Takhtadzhyan algebras. A solution example of [4] is provided by

$$L = R_{11}, R_{112} \ldots R_{11k},$$

where $L$ is an operator in the quantum space $V_q = V_{i_1} \otimes \ldots \otimes V_{i_k}$, here $V_{i_j}$ is just a copy of the space $V$.

The work [2] presents a construction of a commutative family in the algebra $\text{End}(V_q)$ containing the trace $I_1 = \text{Tr}_V L$.

**Lemma 1** ([2]) Let us introduce a notation $L_i$ for the corresponding element in $\text{End}(V_{i_1}) \otimes \text{End}(V_q)$. Then the operators

$$I_k = \text{Tr}_{1 \ldots k} L_1 \ldots L_k R'_{12}R'_{23} \ldots R'_{k-1,k}$$

commute in $\text{End}(V_q)$. The trace is meant with respect to the auxiliary spaces $V_{i_j}$.
This statement has an important role in the technique of constructing quantum-mechanical integrable systems. This is applicable in quantum Gaudin systems, Ruijenaars-Schneider system and others. Moreover, this is directly associated with the theory of exactly-solvable models of statistical physics on 2-dimensional lattices. The study of the spectrum of the transfer-matrix is a key ingredient in the problem of finding the partition function asymptotics of some statistical models (eg. [5]).

1.3 Tetrahedral equation

As like as in the Yang-Baxter case for many purposes it is more convenient to consider the set-theoretic tetrahedral equation (STTE) which may be defined as follows: let X be a finite set, we say that there is a solution for STTE on X if there is a map

$$X \times X \times X \xrightarrow{\Phi} X \times X \times X,$$

satisfying the relation (graphically coinciding with (2))

$$\Phi_{135} \circ \Phi_{145} \circ \Phi_{246} \circ \Phi_{356} = \Phi_{356} \circ \Phi_{246} \circ \Phi_{145} \circ \Phi_{123} : X^{	imes 6} \to X^{	imes 6}.$$ (6)

Here, however, unlike (2), $X^{	imes 6}$ denotes the Cartesian 6-th power of X and the subscripts denote the number of factors to which $\Phi$ is applied, in other factors the map acts identically. For example

$$\Phi_{356}(a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, a_2, \Phi_1(a_3, a_5, a_6), a_4, \Phi_2(a_3, a_5, a_6), \Phi_3(a_3, a_5, a_6)) = (a_1, a_2, a'_3, a_4, a'_5, a'_6),$$ (7)

where

$$\Phi(x, y, z) = (\Phi_1(x, y, z), \Phi_2(x, y, z), \Phi_3(x, y, z)) = (x', y', z').$$

One distinguishes the functional tetrahedral equation (FTE), for example the so-called electric solution (eg. [4]), represented as a transformation acting on the space of functions of three variables:

$$\Phi(x, y, z) = (x_1, y_1, z_1):$$

$$x_1 = \frac{xy}{x + z + xy z},$$

$$y_1 = \frac{x + z + x y z}{y z},$$

$$z_1 = \frac{x + z + x y z}{x z}.$$ (8)

This solution is relevant to the well-known star-triangle relation in the theory of electric circuits.

There is another one interpretation of the tetrahedral equation in the task of coloring of 2-faces of the 4-dimensional cube with elements of the set X. Let $\Phi : X \times X \times X \to X \times X \times X$ be a map. A coloring is called admissible if the colors of the incoming faces of all 3-cubes $x, y, z$ are related with the colors of outgoing faces $x', y', z'$ by the action of the map $\Phi$:

$$(x', y', z') = \Phi(x, y, z)$$

At figure 1 we design a projection of the 4-cube to a 3-space and at figure 2- two alternative sequences of coloring steps. It turns out that the condition of equivalence of the coloring obtained by these two ways is equivalent to the tetrahedral equation on $\Phi$. The problem is described in more details in [7].

The N-cube 2-faces coloring problem allows us to construct a complex analogous to those calculating the Yang-Baxter cohomology for the case of set-theoretic tetrahedral equations in [7]. The 3-cocycles of the complex play a special role in this subject, they are determined by the condition:

$$\varphi(a_1, a_2, a_3)\varphi(a'_1, a_4, a_5)\varphi(a'_2, a'_4, a_6)\varphi(a'_3, a'_5, a'_6) =$$

$$= \varphi(a_3, a_5, a_6)\varphi(a_2, a_4, a_6)\varphi(a_1, a'_4, a'_6)\varphi(a'_1, a'_2, a'_3)$$ (9)

in the notation of the picture. In particular the following lemma fulfills:
Lemma 2 (Th. 4 of [7]) Let $\Phi$ be a solution for the STTE, $\varphi$ be a 3-cocycle of the tetrahedral complex. Let $V$ be the vector space generated by the elements of the set $X$. Then let us define a linear operator $A$ on $V^{\otimes 3}$ specifying its values on tensor products of basis vectors. We say that

$$A(s)(e_x \otimes e_y \otimes e_z) = \varphi(x, y, z)^s(e_x' \otimes e_y' \otimes e_z')$$

if and only if $\Phi(x, y, z) = (x', y', z')$. In this case $A(s)$ provides a solution for the matrix tetrahedral equation.

Remark 1 To construct such matrix solution from the electric solution for the FTE one should realize the finite-dimensional reduction of the latter. In lemma 6 of [7] it is demonstrated that the electric solution reduces correctly to the set $X$ obtained as follows: consider the residue ring $\mathbb{Z}/p^k\mathbb{Z}$, where $p$ is either a prime number of the form $p = 4l + 1$, or $p = 2$, and $k$ is an integer $\geq 2$ (the Legendre symbol $\left( \frac{-1}{p} \right)$ equals 1). We fix one the square roots of $-1$ and call it $\varepsilon \in \mathbb{Z}/p\mathbb{Z}$. Our set $X$ will be the following subset of $\mathbb{Z}/p^k\mathbb{Z}$:

$$X = \{x \in \mathbb{Z}/p^k\mathbb{Z} : x = \varepsilon \mod p\}.$$

2 Commutative family

2.1 The commutativity demonstration in $d = 1$

Let us present here our own proof of the Maillet result in lemma [1]. It demonstrates the main technique and suggests the path for generalization. In this section we denote by $R$ and $L$ solutions for the equations (3) and (4) (do not confuse with equation (1)). We assume that $R$ is invertible. We present here the Maillet generators (5) in a slightly more algebraic way:

$$I_k = Tr_{V_1 \otimes \ldots \otimes V_k} L^{\otimes k} R_{12} \ldots R_{k-1,k} R_{k+1,k+2} \ldots R_{k+l-1,k+l} A^{-1} = L^{\otimes k+l} R_{12} \ldots R_{l-1,l} R_{l+1,l+2} \ldots R_{k+l-1,k+l},$$

where the tensor product is taken with respect to the auxiliary spaces $V_i$, the product in quantum space $V_q$ is implied. To prove the commutativity of $I_k$ we demonstrate that there exists a linear operator $A \in \text{End}(V)^{\otimes k+l}$ such that
This yields that the traces of these expressions with respect to the auxiliary spaces coincide. This in turn produce the identity:

\[
[I_k, I_l] = 0.
\]

In what follows \(Ad_g X\) means \(gXg^{-1}\). Let us introduce some accessory notations: \(R_{1k} = R_{12}R_{23}\ldots R_{k-1,k}\). This expression is subject to some relations

**Lemma 3**

\[
Ad_{R_{1k}} R_{m,m+1} = R_{m+1,m+2}, \quad 1 \leq m \leq k-2.
\]  

**Proof**

\[
R_{12}\ldots R_{m-1,m}R_{m,m+1}R_{m+1,m+2}\ldots R_{k-1,k}R_{m,m+1}R_{k-1,k}^{-1}\ldots R_{m+1,m+2}R_{m,m+1}^{-1}R_{m-1,m}R_{m,m+1}\ldots R_{12}^{-1} = R_{12}\ldots R_{m-1,m}R_{m,m+1}R_{m+1,m+2}R_{m,m+1}^{-1}R_{m+1,m+2}R_{m,m+1}^{-1}R_{m-1,m}R_{m,m+1}\ldots R_{12}^{-1} = R_{12}\ldots R_{m-1,m}R_{m+1,m+2}R_{m-1,m}^{-1}R_{12}^{-1} = R_{m+1,m+2}.
\]
One another relation is expressed by the

**Lemma 4**

\[
\text{Ad}_{R_{-1}} \text{Ad}_{R_{-2}} R_{k-1,k} = R_{12}. \tag{12}
\]

**Proof**

We demonstrate the statement by induction. For \( k = 3 \) we have

\[
R_{23} R_{12} R_{23} R_{12}^{-1} R_{23}^{-1} = R_{23} R_{23}^{-1} R_{12} R_{23} R_{23}^{-1} = R_{12}.
\]

Let the statement be true for \( k - 1 \), then

\[
\text{Ad}_{R_{-1}} \text{Ad}_{R_{-2}} R_{k-1,k} = R_{12}.
\]

We may now fabricate an operator \( A \) by the formula:

\[
A = R_{1,k+1} \cdots R_{-1,k+1} \cdots R_{1,k+1}.
\]

**Lemma 5**

\[
\text{Ad}_A (L \otimes R_{12} \cdots R_{k-1,k} R_{k+1,k+2} \cdots R_{k+l-1,k+l}) = L \otimes R_{12} \cdots R_{l-1,l} R_{l+1,l+2} \cdots R_{k+l-1,k+l}.
\]

**Proof**

Let us note that in virtue of \((4)\) we need to demonstrate only

\[
\text{Ad}_A (R_{12} \cdots R_{k-1,k} R_{k+1,k+2} \cdots R_{k+l-1,k+l}) = R_{12} \cdots R_{l-1,l} R_{l+1,l+2} \cdots R_{k+l-1,k+l}.
\]

By the other hand lemma \((3)\) yields

\[
\text{Ad}_A (R_{12} \cdots R_{k-1,k}) = R_{l+1,l+2} \cdots R_{k+l-1,k+l}. \tag{14}
\]

Lemma \((4)\) provides

\[
\text{Ad}_A (R_{k+1,k+2} \cdots R_{k+l-1,k+l}) = R_{12} R_{23} \cdots R_{l-1,l}. \tag{15}
\]

The commutativity argument of the right hand sides of \((14)\) and \((15)\) finishes the proof.

This reasoning can be generalized for generating functions of integrals. Let us introduce a notation:

\[
S(t) = L \otimes N (1 + t R_{12} + t^2 R_{12} R_{23} + \cdots + t^{N-1} R_{12} \cdots R_{N-1,N})
\]

Then

\[
Q(t) = \text{Tr} S(t) = \sum_{k=0}^{N-1} t^k I_k I_1^{N-k}.
\]

If one now considers the conjugation operator

\[
A = R_{N-2N} R_{N-1,2N} \cdots R_{1,N+1}.
\]

then

\[
\text{Ad}_A (S(t) \otimes S(u)) = S(u) \otimes S(t).
\]

This immediately implies

\[
[Q(t), Q(u)] = 0.
\]
2.2 Regular 3-d lattices and statistical models

Consider a periodic three-dimensional lattice of size $K \times L \times M$, we mark the edges incoming to the node $(i,j,k)$ as $x_{i,j,k}, y_{i,j,k}, z_{i,j,k}$. The periodicity conditions imply $*_N+1,j,k = *_{1,j,k}$, and similar identities for other indexes. Consider a statistical model with the Boltzmann weights in the nodes of the lattice sites determined by the value of the 3-cocycle $\varphi$ of the tetrahedral complex. The states are defined as admissible coloring of the edges, i.e. such that in each node the condition fulfills:

$$\Phi(x_{i,j,k}, y_{i,j,k}, z_{i,j,k}) = (x_{i+1,j,k}, y_{i,j,k+1}, z_{i,j,k+1}).$$  \hspace{1cm} (17)

A partition function is defined as follows:

$$Z(s) = \sum_{\text{Col}} \prod_{i,j,k} \varphi(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^s.$$ \hspace{1cm} (18)

To explore the "integrability" of the subsidiary quantum problem one needs to recognize the layer-to-layer transfer-matrix. In order to determine what it is, we need another interpretation of the partition function.

We associate a copy of the space $V$ to each line of the lattice. For convenience, we denote the vertical spaces by characters $V_{ik}$ and the horizontal ones - by $E_i$ and $N_k$.

We construct an operator $A_{ik}(s)$ with the chosen 3-cocycle according to lemma 2 construction.

Let us define the transfer-matrix by a 1-layer product:

$$T(s) = Tr \prod_i \prod_k A_{ik}(s).$$

The formula implies the product and trace of matrices with respect to horizontal spaces. This operator acts on the tensor product of vertical spaces. Here $A_{ik}(s)$ is an operator on the space $E_i \otimes V_{ik} \otimes N_k$. It turns out that the partition function takes the form

$$Z(s) = Tr_{V_{jk}} T(s)^L.$$

Such issues as the asymptotic behavior of partition functions with respect to increasing the size of the lattice may be solved by the study of the spectrum of the transfer-matrix. The integrability condition,
i.e. the possibility of including the transfer-matrix in a large commutative family, simplifies the problem of finding the spectrum.

2.3 Some consequences of the tetrahedral equation

In this section we give a generalization of the Maillet construction in the case of the three-dimensional lattice and the transfer-matrix associated with the solution of the matrix tetrahedral equation. Consider a lattice of size \( K \times L \times M \) and several forms of 1-layer product:

\[
(\Phi_{(i_1\ldots i_k)}^{s_1}(J_1\ldots J_m)) = \prod_{s \in (1,\ldots,k)} \Phi_{i_1,J_1}(J_1) = \prod_{s \in (1,\ldots,k)} \Phi_{i_1,J_1}(J_1) = \prod_{s \in (1,\ldots,k)} \Phi_{i,J_1}(J_1). \tag{19}
\]

The arrows over indexing sets mean the direction in products: for example

\[
\prod_{s \in (1,\ldots,k)} A_s = A_1 \ldots A_k.
\]

The transfer-matrix can be represented as the trace of the expression (19)

\[
T = I_1 = Tr(\Phi_{(i)}^{s}(J)). \tag{20}
\]

Let us write down several identities which are straightforward consequences of the tetrahedral equations. They can be considered as generalizations of the RLL- relations.

**Lemma 6**

\[
\Phi_{123} \Phi_{1}(i) \Phi_{2}(i) \Phi_{3}(i) = \Phi_{3}(i) \Phi_{2}(i) \Phi_{1}(i) \Phi_{123}, \tag{21}
\]

\[
\Phi_{(i)} \Phi_{(j)} \Phi_{(k)} = \Phi_{123} \Phi_{(j)} \Phi_{123}, \tag{22}
\]

\[
\Phi_{(i)}(i') \Phi_{(i)'}(i') = \Phi_{0}(j) \Phi_{(i)'}(i') \Phi_{(i)'}(i'). \tag{23}
\]

where:

\[
\Phi_{1}(i) = \Phi_{1}(i_1\ldots i_k)_{(j_1\ldots j_k)} = \prod_{s \in (1,\ldots,k)} \Phi_{i_1,J_1},
\]

\[
\Phi_{(i)}(j) = \Phi_{(i_1\ldots i_k)(j_1\ldots j_k)} = \prod_{s \in (1,\ldots,k)} \Phi_{i_1,J_1}.\]

We introduce also some twisted versions of solutions for the tetrahedral equation:

\[
\Phi_{LR}^{L_{123}} = P_{12} \Phi_{R123}, \quad \Phi_{LR}^{L_{123}} = P_{23} \Phi_{R123},
\]

\[
\Phi_{LR}^{R_{123}} = \Phi_{R123} P_{23}, \quad \Phi_{LR}^{R_{123}} = \Phi_{R123} P_{12}.
\]

The identities fulfill:

\[
\Phi_{L}^{L_{123}} \Phi_{145} \Phi_{246} \Phi_{356} = \Phi_{R}^{R_{145}} \Phi_{246} \Phi_{123}, \tag{24}
\]

\[
\Phi_{R}^{R_{\alpha \beta \gamma}} \Phi_{\alpha 12} \Phi_{\beta 23} \Phi_{\gamma 13} = \Phi_{R}^{R_{\alpha \beta \gamma}} \Phi_{\alpha 12} \Phi_{\beta 23} \Phi_{\gamma 13}, \tag{25}
\]

\[
\Phi_{145}^{R_{123}} \Phi_{123}^{L_{356}} \Phi_{246}^{L_{356}} = \Phi_{L}^{L_{356}} \Phi_{246}^{L_{356}} \Phi_{145}^{R_{123}}, \tag{26}
\]

\[
\Phi_{123}^{L_{123}} \Phi_{123}^{R_{246}} \Phi_{246}^{R_{356}} = \Phi_{R}^{R_{356}} \Phi_{246}^{R_{356}} \Phi_{123}^{L_{145}}. \tag{27}
\]

They perform a role similar to the Yang-Baxter equation in combinatorial notation. These equalities are also simple consequences of the tetrahedral equation, we do not provide here the proofs with the only purpose to simplify the exposition.
2.4 Two families

First we pay attention to the connection of the tetrahedral equation and the Yang-Baxter equations of
the following type: for an invertible $\Phi$ the formula \( \Phi_{123} \Phi_{1(i)(j)} \Phi_{2(i)(l)} \Phi_{3(j)(l)} \Phi_{1(i)(j)} \) can be transformed to the kind
\[
\Phi_{123} \Phi_{1(i)(j)} \Phi_{2(i)(l)} \Phi_{3(j)(l)} \Phi_{1(i)(j)}^{-1} = \Phi_{3(j)(l)} \Phi_{2(i)(l)} \Phi_{1(i)(j)}.
\]
Taking the trace of both parts with respect to indices 1, 2, 3, one deduces that the expression \( R_{1(i)(j)} = Tr_1 \Phi_{1(i)(j)} \) satisfies the Yang-Baxter equation (1). Moreover as a consequence of (22) we have
\[
\Phi_{1(i)(j')} \Phi_{(i')*(j')} \Phi_{0(j)(j')} \Phi_{(i')(j')}^{-1} = \Phi_{0(j)(j')} \Phi_{(i')*(j')} \Phi_{(i)(j)}.
\]
If one takes the trace of both parts with respect to the spaces \( i, i', 0 \) it turns out that the expression \( L_{s(j)} = Tr_{(j)} \Phi_{(i)*_{(j)}} \) satisfies the RLL relation:
\[
L_{s(i)} L_{s(j)} R_{1(i)(j)} = R_{1(i)(j)} L_{s(j)} L_{s(i)}.
\]

Thus the data
\[
L_{s(j)} = Tr_{(j)} \Phi_{(i)*_{(j)}}, \quad R'_{1(i)(j)} = Tr_1 \Phi_{(i)*_{(j)}}
\]
meets lemma conditions. The immediate consequence of this is that the following set of operators:
\[
I_{0,k} = Tr_{i_1,j_1,...} L_{s(j_1)} \prod_{i_1,...,k} R'_{(j_m)(j_{m+1})}
\]
is commutative and contains \( I_1 = Tr_{j} L_{s(j)} = Tr_{j} \Phi_{(i)*_{(j)}} \). There is a slightly more general form for \( I_{0,k} \)
\[
I_{0,k} = Tr_{i_1,j_1,...} L_{s(j_1)} \prod_{i_1,...,k} R_{(j_m)(j_{m+1})}.
\]

A similar argument can show that, due to lemma the family
\[
I_{n,0} = Tr_{i_1,j_1,...} L_{s(j_1)} \prod_{i_1,...,n} \Phi_{(i)m}(j_m) \prod_{m=1,...,k} L_{s(j_m)(j_{m+1})}
\]
is also commutative and include \( I_1 \). The main accomplishment of the paper is that these two families commute among themselves. In order to give a precise formulation let us introduce some notation.

\[
B_k = \Phi_{(i_1)*_{(j_1)}} \cdots \Phi_{(i_k)*_{(j_k)}} = \prod_{\alpha \in (1,...,k)} \Phi_{(i_{\alpha})*_{(j_{\alpha})}},
\]
\[
R_k^s = \Phi_{s_{1}(j_1)(j_2)} \cdots \Phi_{s_{k-1}(j_{k-1})(j_k)} = \prod_{\alpha \in (1,...,k-1)} \Phi_{s_{\alpha}(j_{\alpha})(j_{\alpha+1})},
\]
\[
L_l^s = \Phi_{l_{1}(j_1)} s_{1} \cdots \Phi_{l_{i-1}(j_{i-1}) s_{i-1}} = \prod_{\alpha \in (1,...,l-1)} \Phi_{l_{\alpha}(j_{\alpha})(j_{\alpha+1})},
\]
\[
A_L^p = \prod_{\alpha \beta} \Phi_{(i_{\alpha})(j_{\beta})p_{\alpha\beta}},
\]
\[
A_R^p = \prod_{\alpha \beta} \Phi_{p_{\alpha\beta}(j_{\alpha})(j_{\beta})},
\]

\( p_{\alpha\beta} \) is the set of accessory indices in last two formulas. In addition, we need two auxiliary elements:

\[
\Phi_p = \prod_{\alpha \beta} \Phi_{s_{\alpha} p_{\alpha+1,\beta} p_{\alpha,\beta}},
\]
\[
\Phi_{p}^* = \prod_{\alpha \beta} \Phi_{p_{\alpha,\beta+1} p_{\alpha,\beta}^*}.\]
2.5 The commutativity proof in \( d = 2 \)

**Definition 1** Let us call a solution of the tetrahedral equation generic if it is invertible and the operator \( Tr_p (A^p_R \Phi^*_p) \) is invertible too.

**Theorem 1** For any generic solution \( \Phi \) of the tetrahedral equation the following is true

\[
[\mathcal{I}_{0,k}, \mathcal{I}_{n,0}] = 0.
\]

**Proof**

Let us consider an expression:

\[
\begin{pmatrix}
A^p_L & B_k & B_l & R^*_k & L^*_l & A^p_R & \Phi^*_p
\end{pmatrix}_{i_\alpha, i_\beta; p, s, a} \begin{pmatrix}
1_{i_\alpha} & 1_{i_\beta} & 1_a & 1_s & 1_{s, a} & 1_{s, a}
\end{pmatrix}_{j_\alpha, j_\beta, s, a, i_\alpha, i_\beta, j_\alpha, j_\beta, p, s, a}.
\]

(28)

The indices under each multiplier indicate the spaces in which it acts. Next, we shall omit the tensor product sign, considering the indices of the corresponding spaces. This expression can be transmuted to the form:

\[
A^p_L B_k B_l \begin{pmatrix}
\Phi^*_p A^p_r \Phi^*_p L^*_l \Phi^*_p
\end{pmatrix} = A^p_L B_k B_l \begin{pmatrix}
A^p_R R^*_k \Phi^*_p L^*_l \Phi^*_p
\end{pmatrix}.
\]

(29)

Then we deduce:

\[
A^p_L \Phi^*_p B_k B_l R^*_k L^*_l A^p_R \Phi^*_p (A^p_L \Phi^*_p)^{-1} = A^p_L \Phi^*_p B_k B_l R^*_k L^*_l.
\]

In particular this implies

\[
Tr_{i_\alpha, i_\beta; p, s, a} B_{k+i} R^*_k L^*_l A^p_R \Phi^*_p = Tr_{i_\alpha, i_\beta; p, s, a} A^p_L \Phi^*_p B_k B_l R^*_k L^*_l.
\]

Let us note that the trace procedure factorizes

\[
Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l) Tr_p (A^p_R \Phi^*_p) = Tr_p (A^p_R \Phi^*_p) Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l).
\]

One may notice that

\[
(Tr_p (A^p_R \Phi^*_p))^{-1} Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l) Tr_p (A^p_R \Phi^*_p) = Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l).
\]

Taking a trace over the remaining auxiliary spaces completes the proof:

\[
I_{0k} I_{0l} = Tr_{i_\alpha, i_\beta; t} (Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l)) = Tr_{i_\alpha, i_\beta; t} (Tr_{i_\alpha, i_\beta; s, a} (B_{k+i} R^*_k L^*_l)) = I_{0k} I_{0l}.
\]

Proceed now to the proof of lemmas.

**Lemma 7**

\[
\Phi^*_p R^*_k A^p_R = A^p_R R^*_k \Phi^*_p.
\]

(30)

**Proof**

Let us deduce a useful formula:

\[
\tilde{\Phi}^L_{\alpha, \beta, \gamma} \Phi^R_{\alpha(i) (j)} \Phi^R_{\beta(j) (k)} \Phi^R_{\gamma(i) (k)} = \Phi^R_{\beta(j) (k)} \Phi^R_{\gamma(i) (k)} \Phi^R_{\alpha(i) (j)} \tilde{\Phi}^L_{\alpha, \beta, \gamma}.
\]

(31)
It is a direct consequence of equation \((25)\). Let us write more thoroughly \((30)\)

\[
\prod_{\alpha} \tilde{\Phi}^L_{s\alpha P_{\alpha + 1}, \beta_0} \prod_{\alpha} \Phi^R_{s\alpha (j_\alpha)} \prod_{\alpha} \Phi_{p\alpha, \beta}(j_\alpha).
\]

Note that the left and right multipliers commute at different values of \(\beta\) so it is enough to check the relation at fixed \(\beta = \beta_0\) performing further dressing starting with the multipliers closest to the center.

\[
\prod_{\alpha} \tilde{\Phi}^L_{s\alpha P_{\alpha + 1}, \beta_0} \Phi^R_{s\alpha (j_\alpha + 1)} \prod_{\alpha} \Phi_{p\alpha, \beta_0}(j_\alpha).
\]

Now we will focus attention on how the elements of the left product are transferred through the right one. This can be done consequently: fix the index \(\alpha = \alpha_0\) in the left product

\[
\tilde{\Phi}^L_{s\alpha_0 P_{\alpha_0 + 1}, \beta_0} \Phi^R_{s\alpha_0 (j_{\alpha_0})} \prod_{\alpha} \Phi_{p\alpha, \beta_0}(j_\alpha).
\]

The only multipliers of the right product which do not commute with the elements on the left have indices \(\alpha = \alpha_0\) and \(\alpha = \alpha_0 + 1\). When moving through them we use equality \((31)\)

\[
\tilde{\Phi}^L_{s\alpha_0 P_{\alpha_0 + 1}, \beta_0} \Phi^R_{s\alpha_0 (j_{\alpha_0} + 1)} \Phi_{p\alpha_0, \beta_0}(j_\alpha) \Phi_{p\alpha_0 + 1, \beta_0}(j_{\alpha_0}) = \Phi_{p\alpha_0 + 1, \beta_0}(j_{\alpha_0} + 1) \Phi_{p\alpha_0, \beta_0}(j_\alpha).
\]

\(\square\)

**Lemma 8**

\[
A^P_L B_k B_l A^P_R = A^P_R B_k B_l A^P_L.
\]

**Proof**

The statement is an immediate consequence of equation \((22)\). We will illustrate the basic techniques of generalization. We write the left part of the expression:

\[
\prod_{\alpha} \Phi_{(i_\alpha)(i_\beta)} \prod_{\alpha} \Phi_{(i_\alpha)(j_\alpha)} \prod_{\beta} \Phi_{(i_\beta)(j_\beta)} \prod_{\alpha} \Phi_{p\alpha, \beta}(j_\alpha).
\]

We are going to move the multipliers of the third product through the multipliers of the second one. For doing this we use a twisting multipliers of the first and fourth factors. Note that the multipliers of the first and fourth factors commute except those with both coinciding indices. It remains to verify that the multiplier order is suitable. We will move the elements of the third product consequently from the left through the second product. To do this, the order relation on \(\beta\) in the first product should be such that the junior members stand on the right, and in the fourth on the left. Similarly, one can check that the order relation on \(\alpha\) in the first product should be such that the senior members are on the right, and in the fourth on the left.

\(\square\)

**Lemma 9**

\[
\Phi^*_p \Phi_p = \Phi^*_p \Phi_p.
\]

**Proof**

Let us write the left side of the expression

\[
\prod_{\alpha} \tilde{\Phi}^L_{s\alpha P_{\alpha + 1}, \beta_0} \prod_{\alpha} \tilde{\Phi}^R_{p\alpha, \beta_0}(j_\alpha).
\]
We move the right product through the left one component-wise, for fixed $\alpha$ on the left and fixed $\beta$ on the right. Let us consider these multipliers:

$$\prod_{\beta} \hat{\Phi}_L^{\alpha_{\alpha_0}, \beta_{\beta_0+1}, \beta_{\beta_0}} \prod_{\beta} \hat{\Phi}_R^{\alpha_{\alpha_0}, \beta_{\beta_0+1}, \beta_{\beta_0}}.$$ 

The only non-commutative elements of these products have neighboring indices $\beta = \beta_0, \beta_0 + 1$ and $\alpha = \alpha_0, \alpha_0 + 1$. We write only them

$$\hat{\Phi}_L^{\alpha_{\alpha_0}, \beta_{\beta_0+1}, \beta_{\beta_0}} \hat{\Phi}_R^{\alpha_{\alpha_0}, \beta_{\beta_0+1}, \beta_{\beta_0+1}} \hat{\Phi}_L^{\alpha_{\alpha_0+1}, \beta_{\beta_0+1}, \beta_{\beta_0}} \hat{\Phi}_R^{\alpha_{\alpha_0+1}, \beta_{\beta_0+1}, \beta_{\beta_0}}.$$ 

Note that the right pair can be moved through the left pair of multipliers according to (26).

□

Lemma 10

$$A_p^L L^t_1 \Phi_p^* = \Phi_p^* L^t_1 A_p^R.$$ 

Proof

Analogously to lemma 8.

□

3 Conclusion

The choice of the notations $I_{l,0}$ and $I_{0,k}$ used in the main part of the work pursued a definitive goal. We hope that the following hypothesis is correct

Hypothesis 1 For any generic solution of the tetrahedral equation it can be constructed a two-parameter commutative family $I_{l,k}$, which includes two families presented above.

Remark 2 Note that in the work [11] it is constructed a two-parameter family of operators commuting with a layer-by-layer transfer-matrix related to the special solution of the tetrahedral equation. It is curious to compare the result in the $q$-oscillator realization case and the result presented here.

However, the main result of this work reveals some important perspectives:

- Primarily, it provides the opportunity to study the spectra of the corresponding quantum models and models of statistical physics in dimension $d = 3$. We hope that in this case the Bethe ansatz technique also can be applied.

- Another interesting direction is the generalization of the notion of integrability in the case of 2-dimensional surfaces comprising the classical case. It is interesting to juxtapose the approach of this work with the language of paper [8] on the theory of the Hitchin systems on surfaces. We would also be enthusiastic in the study of moduli spaces of 2-bundles on surfaces and an appropriate analogue of the Hitchin theory. In this case one has a significant difficulty in constructing nonabelian gerbs.

- In [7] we present a construction of quasi-invariant of 2-knots in the form of a partition function on the graph of double points of the 2-knot diagram. This structure is resembling to the approach of [9]. The similarity of the partition function expressions is intriguing and gives hope for further development activities in low-dimensional topology and topological quantum field theory. Presumably a certain connection of this method with four-dimensional quantum field theories, like the BF-theory, can be established.

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References

[1] V.V. Bazhanov, S.M. Sergeev, *Zamolodchikov’s Tetrahedron Equation and Hidden Structure of Quantum Groups*. J.Phys.A 39:3295-3310,2006,

[2] J.M Maillet, *Lax equations and quantum groups*, Phys. Lett. B 245, 480 (1990).

[3] A. B. Zamolodchikov, *Tetrahedra equations and integrable systems in three-dimensional space*, Zh. Eksp. Teor. Fiz. 79 (1980) 641664. [English translation: Soviet Phys. JETP 52 (1980) 325-326].

[4] J. Hietarinta, *Permutation-type solutions to the Yang-Baxter and other n-simplex equations*. J. Phys. A: Math. Gen., 1997, Vol. 30, 4757.

[5] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*. Academic, London, 1982.

[6] R.M. Kashaev, I.G. Korepanov, S.M. Sergeev, *Functional tetrahedron equation*. Theor Math Phys (1998) 117: 1402.

[7] I.G. Korepanov, G.I. Sharygin, D.V. Talalaev, *Cohomologies of n-simplex relations*. Mathematical Proceedings of the Cambridge Philosophical Society. 2016. Vol. 161, no. 2. P. 203222.

[8] D.V.Osipov, A.B.Zheglov, *On some questions related to the Krichever correspondence*, Mathematical Notes, 2007, Vol. 81, No. 4, pp. 467-476

[9] Carter, J.S., Jelsovsky, D., Langford, L., Kamada, S., Saito, M., *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947-3989.