CRITICAL $(\Phi^4)_3, \epsilon$

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ABSTRACT

The Euclidean $(\phi^4)_3, \epsilon$ model in $\mathbb{R}^3$ corresponds to a perturbation by a $\phi^4$ interaction of a Gaussian measure on scalar fields with a covariance depending on a real parameter $\epsilon$ in the range $0 \leq \epsilon \leq 1$. For $\epsilon = 1$ one recovers the covariance of a massless scalar field in $\mathbb{R}^3$. For $\epsilon = 0$ $\phi^4$ is a marginal interaction. For $0 \leq \epsilon < 1$ the covariance continues to be Osterwalder-Schrader and pointwise positive. After introducing cutoffs we prove that for $\epsilon > 0$, sufficiently small, there exists a non-gaussian fixed point (with one unstable direction) of the Renormalization Group iterations. These iterations converge to the fixed point on its stable (critical) manifold which is constructed.

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1. INTRODUCTION, MODEL, RG TRANSFORMATION

1.1 Introduction

Let \( \phi \) be a mean zero Gaussian scalar random field on \( \mathbb{R}^3 \) with covariance

\[
E(\phi(x)\phi(y)) = \text{const} \frac{1}{|x - y|^{(3-\epsilon)/2}}
\]

\[
= \int \frac{d^3 p}{(2\pi)^3} e^{ip \cdot (x-y)} (p^2)^{-\frac{3}{4}(3+\epsilon)/4}
\]

(1.1)

Here \( p^2 = |p|^2 \) is the standard Euclidean norm in \( \mathbb{R}^3 \) and \( p \cdot x \) is the standard scalar product. \( \epsilon \) is a real parameter which we take in the region \( 0 \leq \epsilon < 1 \). Note that for \( \epsilon = 1 \) we have the standard massless free scalar field in \( \mathbb{R}^3 \).

This covariance \( (-\Delta)^{-\frac{3}{4}(3+\epsilon)/4} \) has interesting physical properties. For example, Osterwalder-Schrader positivity plays no role in the present paper, but it is of interest because scaling limits of these theories will be Euclidean quantum field theories. \( (-\Delta)^{-\frac{3}{4}(3+\epsilon)/4} \) is Osterwalder-Schrader positive as well as being pointwise positive not only for \( \epsilon = 1 \) but also in the range that we consider, namely \( 0 \leq \epsilon < 1 \). In the latter range we have the convergent integral representation

\[
(-\Delta)^{-\frac{3}{4}(3+\epsilon)/4}(x-y) = \frac{1}{c_\epsilon} \int_0^\infty ds \ s^{-(3+\epsilon)/4} (-\Delta + s)^{-1}(x-y)
\]

(1.2)

where

\[
c_\epsilon = \int_0^\infty ds \ s^{-(3+\epsilon)/4}(1 + s)^{-1}
\]

(1.3)

The Osterwalder-Schrader and pointwise positivities now follow from those of \( (-\Delta + s)^{-1}(x-y) \).

Furthermore \( (-\Delta)^{-\frac{3}{4}(3+\epsilon)/4} \) is the Green’s function (or potential) for a stable Lévy process in \( \mathbb{R}^3 \) with parameters \( (\alpha, \beta) \) in the Lévy-Khintchine notation [KG], with the characteristic exponent \( \alpha = (3 + \epsilon)/2 \), and \( \beta = 0 \). This process has jumps and diffuses very fast. This property also plays no role in the present paper but we expect that self-avoiding Lévy processes are accessible to the methods of this paper.

These properties make the study of the above gaussian random field and its non-linear perturbations worthwhile. In particular we will study here the critical properties of a model corresponding to the partition function

\[
Z = \int d\mu_C(\phi) \ e^{-V_0(\phi)}
\]

(1.4)

where \( d\mu_C \) is the Gaussian measure with covariance \( C \) and \( C \) is \( (-\Delta)^{-\frac{3}{4}(3+\epsilon)/4} \) with an ultraviolet cutoff described below and

\[
V_0(\phi) = V(\phi, C, g_0, \mu_0) = g_0 \int d^3 x : \phi^4 : C (x) + \mu_0 \int d^3 x : \phi^2 : C (x)
\]

(1.5)

and the coupling constant \( g \) is held strictly positive. Moreover in order to define the model completely we must also introduce a volume cutoff. These cutoffs will be introduced presently when we give a precise definition of the covariance \( C \) in (1.5) as well as that of the model.

From (1.1) we can read off the canonical scaling dimension \( [\phi] \) of \( \phi \)

\[
[\phi] = (3 - \epsilon)/4
\]

(1.6)
This, together with (1.4), implies that we can assign to \( g, \mu \) the dimensions

\[
[g] = \epsilon, \quad [\mu] = (3 + \epsilon)/2
\]  

(1.7)

Note that we have not put in a term

\[
\frac{z}{2} \int d^3x |\nabla \phi(x)|^2
\]

in (1.5). This is because the dimension \([z] = -(1 - \epsilon)/2\). Hence for \( \epsilon < 1 \) this is a candidate for an irrelevant (stable) direction.

We thus expect from Wilson’s theory of critical phenomena [KW] that for \( \epsilon > 0 \) the critical (infra-red) properties of the model are dominated by a non-Gaussian fixed point of Renormalization Group iterations with \( g = g_0 = O(\epsilon) \) provided the unstable parameter \( \mu \) is fine tuned to a critical function \( \mu_c(g) \) which determines the stable (critical) manifold of the fixed point. In the present work we will prove the existence of the non-Gaussian fixed point for \( \epsilon > 0 \) held sufficiently small and, on the way, construct the stable manifold.

The mathematical analysis of Renormalization Group (henceforth denoted RG) transformations has by now a long history [F,BG]. Our particular line of attack is influenced by a series of works which started with [BY], developed further in [DH1,DH2, BDH-est,BDH-eps], with more recent developments in [MS]. We shall be concerned here with these latter developments. In [MS] fluctuation covariances of finite range were exploited, and this simplifies considerably the RG analysis. In particular the analysis of the fluctuation integration becomes a matter of geometry and one no longer needs the cluster expansion and analyticity norms. In the continuum approach of [MS] that is also adopted here the existence of fluctuation covariances with finite range follows easily from a judicious choice of a class of ultraviolet cutoffs. This raises the general question of which gaussian random fields can be decomposed into sums of fluctuation fields with covariances with finite range. A partial answer which includes the standard massless Euclidean field with lattice cutoff will be given in [Gu]. Only the existence of multiscale decompositions with the above finite range property (together with some regularity and positivity properties) is required in what follows.

The present work borrows many technical considerations introduced in [BDH-est]. Although these are independent of the manner of treating the fluctuation step, we repeat some of them because the simpler norms in this paper allow easier proofs. We also borrow some ideas from [BDH-eps] where a related model (which however does not possess the physical properties mentioned earlier) was considered and the existence of a non-Gaussian fixed point was proved. We use this paper as an opportunity to improve previous arguments. In particular, in Section 4, there are much simpler formulas for the remainder after approximating the RG step by second order perturbation theory.

We plan to study in a subsequent work critical properties of Self-Avoiding Lévy Flights in \( \mathbb{Z}^3 \) for the Lévy-Khintchine parameters given above with \( \alpha > \alpha_c \) and \( \alpha - \alpha_c \) very small. Here \( \alpha_c = 3/2 \) at which value we expect (heuristically) mean field behaviour.

### 1.2 Multiscale decomposition

We introduce a special type of ultraviolet cutoff as follows: Let \( g \) be a non-negative, \( C^\infty, O(3) \) invariant function of compact support in \( \mathbb{R}^3 \) such that \( g(x) \) vanishes for \( |x| \geq 1/2 \). Define \( u = g * g \). Thus \( u \) is positive, \( C^\infty \), and of compact support: \( u(x) = 0 \) for \( |x| \geq 1 \). First we note note that

\[
\frac{\text{const}}{|x - y|^{(3-\epsilon)/2}} = \int_0^\infty \frac{dl}{l} \, l^{-(3-\epsilon)/2} \, u\left(\frac{x - y}{l}\right)
\]

by scaling the variable of integration. Define
\[ C(x - y) = \int_{1}^{\infty} \frac{dl}{l} l^{-(3-\epsilon)/2} \frac{u(x - y)}{l} \]  
\[ (1.8) \]

Because the lower limit is 1 and not 0 this \( C \) is pointwise positive and \( C^{\infty} \).

**Remark**

We can exhibit \( C \) in the traditional form with an ultraviolet cutoff function. Let \( \hat{u} \) be the Fourier transform of \( u \). Because of \( O(3) \) invariance we can write \( \hat{u}(p) = v(p^2) \). Then it is easy to see from the above that

\[ C(x - y) = \int \frac{d^3 p}{(2\pi)^3} e^{ip \cdot (x - y)} F(p^2) (p^2)^{-(3+\epsilon)/4} \]  
\[ (1.9) \]

where

\[ F(p^2) = 1/2 \int_{p^2}^{\infty} \frac{ds}{s} s^{(3+\epsilon)/4} v(s) \]  
\[ (1.10) \]

From this we see that \( F \) is positive, continuous and of fast decrease (since \( v(p^2) = |\hat{g}(p)|^2 \) and \( g \) is of compact support) and can be thus identified as the ultraviolet cutoff function.

Let \( L \geq 2 \) be an integer. Let

\[ \Gamma(x - y) = \int_{1}^{L} \frac{dl}{l} l^{-(3-\epsilon)/2} \frac{u(x - y)}{l} \]  
\[ (1.11) \]

Clearly \( \Gamma \) is \( C^{\infty} \), pointwise positive and of finite range:

\[ \Gamma(x - y) = 0 : \ |x - y| \geq L \]  
\[ (1.12) \]

\( \Gamma \) is our *fluctuation covariance* and it satisfies

\[ C(x - y) = \Gamma(x - y) + L^{-(3-\epsilon)/2} C\left(\frac{x - y}{L}\right). \]  
\[ (1.13) \]

Moreover because of our choice of the form of \( u \), namely \( u = g \ast g \), \( \Gamma \) and \( C \) are generalized positive definite. Now it can be shown (see e.g. Section 1.1 of [MS]) that under these conditions for any compact set \( \Lambda_N \) in \( \mathbb{R}^3 \) there exists a Gaussian measure \( d\mu_C \) of mean 0 and covariance \( C \) supported on the Sobolev space \( H_s(\Lambda_N) \) for any \( s > 0 \). Choosing \( s > 3/2 + 2 \) is sufficient for our purpose. Then, by Sobolev embedding, sample paths are at least twice differentiable.

Define the compact set

\[ \Lambda_N = \left[-\frac{1}{2} L^N, \frac{1}{2} L^N\right]^3 \subset \mathbb{R}^3. \]  
\[ (1.14) \]

Our model with an ultraviolet cutoff and volume cutoff is now defined by the partition function

\[ Z = \int d\mu_C(\phi) \ Z_0(\Lambda_N, \phi) \]  
\[ (1.15) \]

where

\[ Z_0(\Lambda_N, \phi) = e^{-V(\Lambda_N, \phi)} \]  
\[ (1.16) \]

and the potential has now been restricted to the volume \( \Lambda_N \). The Wick ordering in the potential is with respect to the infinite volume covariance \( C \) and this is well defined because the Wick constants are finite.
1.3 Renormalization Group transformation

From (1.13), if we now define the rescaled field

$$\mathcal{R}\phi(x) = \phi_{L^{-1}}(x) = L^{-(3-\epsilon)/4}\phi(x/L)$$  \hspace{1cm} (1.17)

we get

$$\int d\mu_C(\phi)Z_0(\Lambda_N, \phi) = \int d\mu_C(\phi)Z_1(\Lambda_{N-1}, \phi)$$  \hspace{1cm} (1.18)

where

$$Z_1(\Lambda_{N-1}, \phi) = \int d\mu(\zeta)Z_0(\Lambda_N, \zeta + \phi_{L^{-1}})$$  \hspace{1cm} (1.19)

The iteration of (1.19) will constitute our RG transformations. After $n$ steps we have

$$\int d\mu_C(\phi)Z_0(\Lambda_N, \phi) = \int d\mu_C(\phi)Z_n(\Lambda_{N-n}, \phi)$$  \hspace{1cm} (1.20)

where

$$Z_n(\Lambda_{N-n}, \phi) = \int d\mu(\zeta)Z_{n-1}(\Lambda_{N-n+1}, \zeta + \phi_{L^{-1}})$$  \hspace{1cm} (1.21)

After $N - 1$ steps we arrive at $Z_{N-1}(\Lambda_1, \phi)$ where $\Lambda_1$ is the $L$-block $[-\frac{1}{2}L, \frac{1}{2}L]^3$.

In order to analyse the RG transformations, it is convenient to write the partition function density in a polymer gas representation. Pave $\mathbb{R}^3$ with closed unit blocks denoted henceforth $\Delta$. Now let $\Lambda \subset \mathbb{R}^3$ be the volume after a certain number of RG steps. We take $\Lambda$ with the induced paving. A connected polymer $X$ is a connected union of a subset of these closed unit blocks and is thus closed. A polymer activity $K(X, \phi)$ is a map $X, \phi \to \mathbb{R}$ where the fields $\phi$ depend only on the points of $X$. We shall only consider polymer activities supported on connected polymers. This notion will be preserved by the RG operations. We then write, suppressing the dependence on $\phi$,

$$Z(\Lambda) = \sum_{N=0}^{\infty} \frac{1}{N!}e^{-V(\Lambda \setminus X)} \sum_{X_1, \ldots, X_N} \prod_{j=1}^{N} K(X_j)$$  \hspace{1cm} (1.22)

where the connected polymers $X_j$ are disjoint, $X = \cup Y X_j$ and $V(Y) = V(Y, \phi, C, g, \mu)$ is given by (1.5) with parameters $g, \mu$ and integration over $Y$. Initially of course $K$ is absent, but they are naturally generated by the RG operations and the above form is stable.

It is possible to rewrite the above in a more compact form if we extend the polymer algebra by cells as done in [BY]. A cell may be the interior of a block, an open face, an open edge or a vertex. A polymer, which is a union of closed blocks, can be uniquely written as a disjoint union of cells, but the point is that all other sets generated by our manipulations, such as complements of polymers, can also be uniquely written as disjoint unions of cells. We define a commutative product, denoted $\circ$, on functions of sets (unions of cells), in the following way $(F_1 \circ F_2)(X) = \sum_{Y, Z, X = Y \cup Z, X = Y \circ Z} F_1(Y) F_2(Z)$ where $X = Y \circ Z$ iff $X = Y \cup Z$ and $Y \cap Z = \emptyset$. The identity $\mathcal{I}$ is defined by $\mathcal{I}(X) = 1$ if $X = \emptyset$ and otherwise vanishes. Finally we can define an exponential operation on functions of unions of cells as the usual power series based on the $\circ$ product and the definition of the $\circ$ identity $\mathcal{I}$. This has the properties of the usual exponential. We also define a space filling function $\mathcal{S}$ by $\mathcal{S}(X) = 1$ if $X$ is a cell and otherwise vanishes. We then have $\mathcal{E}xp\mathcal{S} = 1$ and it easy to see that (1.22) can be rewritten as

$$Z(\Lambda) = [\mathcal{E}xp(\mathcal{S} e^{-V} + K)](\Lambda)$$  \hspace{1cm} (1.23)

The Formal Infinite Volume Limit
By writing the integrands in the form (1.23), the $i$th RG transformation induces a map

$$f_{N-i} : (K_{i-1}, V_{i-1}) \mapsto (K_i, V_i)$$

which will be described in detail in the next section. The subscript $N - i$ is there because this map has dependence on the region $\Lambda_{N-i}$. For any set $X$,

$$V_i(X) = \sum_{\Delta \subset X} V_i(\Delta)$$

so $V_i$ is determined by $\{V_i(\Delta) : \Delta \subset \Lambda_{N-i}\}$. Our formulas for the RG map will show that for any fixed set $X \subset \Lambda_{N-i}$, the polymer activity $K_i(X, \phi)$ is independent of $N$ for all $N$ large enough so that $\Lambda_{N-i}$ contains $X$ and likewise $V_i(\Delta)$ is independent of $N$ for all $N$ large enough so that $\Lambda_{N-i}$ contains $\Delta$ and a neighborhood of $\Delta$. A detailed look at our formulas, particularly the section on extraction, shows that the neighborhood has diameter 8. $\lim_{N \to \infty} K_i(X)$ and $\lim_{N \to \infty} K_i(\Delta)$ exist and, in this pointwise sense, the infinite volume limit $f = \lim_{N \to \infty} f_{N-i}$ exists. In this paper we prove that $f$ has a fixed point in a Banach space of polymer activities $K$.

The finite volume RG could also be studied by these methods by including in $V$ a surface integral over the boundary of $\Lambda_{N-i}$ which fits naturally in this scheme as an object associated to $D-1$ cells on the boundary.

**2. REGULATORS, DERIVATIVES AND NORMS**

**2.1 Regulators**

We first introduce a large field regulator which measures the growth of polymer activities in the fields $\phi$, actually in $\partial \phi$:

$$G_\kappa(X, \phi) = e^{\kappa \|\phi\|_{X,1,\sigma}^2}$$

where

$$\|\phi\|_{X,1,\sigma}^2 = \sum_{1 \leq |\alpha| \leq \sigma} \|\partial^\alpha \phi\|_{X}^2$$

Here $\|\phi\|_X$ is the $L^2$ norm and $\alpha$ is a multi-index. We take $\sigma > 3/2 + 2$ so that this norm can be used in Sobolev inequalities to control $\phi$ and its first two derivatives pointwise. After the function $u$ is fixed, the parameter $\kappa > 0$ is fixed, for the whole paper, by a choice depending only on $u$, so that for all $L \geq 1$ the large field regulator satisfies the stability property

$$\int d\mu(\zeta) G_\kappa(X, \zeta + \phi) \leq 2^{|X|} G_{2\kappa}(X, \phi)$$

where $X$ is a polymer and $|X|$ is the number of unit blocks in $X$. This can be shown in the same way as in the proof of the stability property of the large field regulator in Section 2.4 of [BDH-est].

Now hold $L$ sufficiently large and recall that $\epsilon < 1$. Then we get after rescaling

$$\int d\mu(\zeta) G_\kappa(X, \zeta + \phi_{L^{-1}}) \leq 2^{|X|} G_{\kappa}(L^{-1}X, \phi)$$

This follows easily from the scaling property(1.17) of the fields $\phi$ which gives

$$\|\phi_{L^{-1}}\|_{X,1,\sigma}^2 \leq L^{-(1-\epsilon)/2} \|\phi\|_{L^{-1}X,1,\sigma}^2$$
Next we introduce a large set regulator. Let $X$ be a connected polymer. We define
\[ A_p(X) = 2^p|X|L^{(D+2)|X|} \]  
where for us the dimension of space $D = 3$, and $p \geq 0$ is an integer.

Call a connected polymer small if $|X| \leq 2^D$. A connected polymer which is not small is called large. Let $\bar{X}_L$ be the $L$-closure of $X$. This is the smallest union of $L$-blocks containing $X$. Let $L$ be sufficiently large. Then we have from Lemma 1 of [BDH-est] the following two facts:

For any connected polymer $X$ and for any integer $p \geq 0$
\[ A(L^{-1}\bar{X}_L) \leq c_p A^{-p}(X) \]  
where $c_p = O(1)$ is a constant independent of $L$.

2.2 Derivatives in fields and polymer activity norms

The following definitions are motivated by those in [BDH-est] the main difference being that we will need only a finite number of field derivatives and for them only natural norms. The kernel norms defined below are different.

For a polymer activity $K(X, \phi)$ define the $n$-th derivative in the direction $(f_1, \ldots, f_n)$ as:
\[ (D^nK)(X, \phi; f^{\times n}) = \prod_{j=1}^{n}(\partial/\partial s_j) K(X, \phi + \sum s_j f_j) \mid_{s_j=0 \forall j} \]  
where the shorthand notation $f^{\times n}$ stands for $f_1, f_2, \ldots, f_n$. The functions $f_j$ belong to $C^2(X)$ and we assume that such derivatives exist for $n \leq n_0$ for some $n_0$.

We will measure the size of such derivatives, which are multilinear functionals on $C^2(X)$, by the norm:
\[ \| (D^nK)(X, \phi) \| = \sup_{\|f_j\|_{C^2(X)} \leq 1 \forall j} \| (D^nK)(X, \phi; f^{\times n}) \| \]  
where
\[ \|K(X, \phi)\|_h = \sum_{j=0}^{n_0} \frac{h^j}{j!} \| (D^jK)(X, \phi) \| \]  
and then
\[ \| (K(X))\|_{h, G_n} = \sup_{\phi \in C^2(X)} \| K(X, \phi) \|_h G_n^{-1}(X, \phi) \]  
These norms differ from norms used in [BDH-est] in having the supremum over $\phi$ outside the sum over derivatives, as well as involving only finitely many derivatives. They are easier to use and retain the product property
\[ \| K_1(X_1, \phi)K_2(X_2, \phi) \|_h \leq \| K_1(X_1, \phi) \|_h \| K_2(X_2, \phi) \|_h. \]
which was the basis for proofs in [BDH-est]. We assume as earlier that the activity \( K \) is supported on connected polymers. We then define

\[
\|K\|_{h,G_{\kappa},A} = \sup_{\Delta} \sum_{X \supset \Delta} \|(K(X))\|_{h,G_{\kappa},A}(X)
\]

(2.12)

In addition we define *kernel norms*:

\[
|K(X)|_{h'} = \sum_{j=0}^{n_0} h'_j \| (D^j K)(X,0) \|
\]

(2.13)

where \( h' \) is a real parameter and \( h' \geq 0 \). We define

\[
|K|_{h',A} = \sup_{\Delta} \sum_{X \supset \Delta} |K(X)|_{h'} A(X)
\]

(2.14)

These definitions are also different from the kernel norms in [BDH-est,BDH-eps]), but retain the product property. When using these norms for our model we will choose \( n_0 = 9, h = \varepsilon^{-1/4} \) and either \( h' = h \) or \( h' = h_* \), where

\[
h_* = L^{(3-2[\phi])/2} = L^{(3+\varepsilon)/4}
\]

The kernel norms with \( h' = h_* \) will be useful for controlling flow coefficients.

3. RG STEP

In this section we describe the RG step. This consists of two parts: Fluctuation integration and rescaling, followed by the extraction of relevant parts.

3.1 Fluctuation integration and rescaling

The integration over the fluctuation field exploits the independence of \( \zeta(x) \) and \( \zeta(y) \) when \( |x-y| \geq L \). To do this we pave \( \Lambda \) by blocks of side \( L \) called \( L \)-blocks so that each \( L \)-block is a union of the original \( 1 \)-blocks. Let \( \bar{X}^L \) denote the \( L \)-closure of a set \( X \), namely the smallest union of closed \( L \)-blocks containing \( X \). The polymers will be combined into larger \( L \)-polymers which by definition are closed, connected and unions of \( L \)-blocks. The combination is performed in such a way that the new polymers are associated to independent functionals of \( \zeta \).

We start with the representation (1.25) of Section 1.3.

\[
Z(\Lambda) = [\mathcal{E}xp(ie^{-V} + K)](\Lambda)
\]

(3.1)

with \( K(X) = 0 \) unless \( X \) is closed and connected. By definition we have:

\[
\mathcal{E}xp(ie^{-V} + K)(\Lambda) = \sum_{N} \frac{1}{N!} \sum_{(X_j)} e^{-V(\Lambda \cup X_j)} \prod_{j=1}^{N} K(X_j)
\]

(3.2)

where the sum is over sequences of disjoint polymers \((X_j) = (X_1,\ldots,X_N)\).

Let us define: \( \cup X_j = X \) and \( \Lambda \setminus X = X_c \). \( X_c \) is an open set. We denote by \( \bar{X}_c \) its closure. Obviously, \( V(X_c) = V(\bar{X}_c) \), since \( V(X_c) \) is given by a Lebesgue integral over \( X_c \). Hence we can write:

\[
e^{-V(X_c,\zeta+\phi)} = \prod_{\Delta \subset \bar{X}_c} e^{-V(\Delta,\zeta+\phi)}
\]
Define the polymer activity $P$, supported on closed unit blocks, as:

$$P(\Delta, \zeta, \phi) = e^{-V(\Delta, \zeta + \phi)} - e^{-\tilde{V}(\Delta, \phi)}$$  \hspace{1cm} (3.3)$$

with $\tilde{V}$ to be chosen. In the following $V, K$ has field argument $\zeta + \phi$ whereas $\tilde{V}$ depends only on $\phi$. The dependence of $P$ on $\zeta, \phi$ is as defined above.

Now write:

$$e^{-V(X_e)} = \prod_{\Delta \subset X_e} [e^{-\tilde{V}(\Delta)} + P(\Delta)]$$

then expand the product and insert the expansion into (3.2):

$$\mathcal{E}_{xp(a e^{-V} + K)}(\Lambda) = \sum_{N} \frac{1}{N! M!} \sum_{(X_j), (\Delta_i)} e^{-\tilde{V}(X_0)} \prod_{j=1}^{N} K(X_j) \prod_{i=1}^{M} P(\Delta_i)$$  \hspace{1cm} (3.4)$$

where $X_0 = \Lambda \setminus (\cup X_j) \cup (\cup \Delta_i)$. Let $Y$ be the $L$–closure of $(\cup X_j) \cup (\cup \Delta_i)$ and let $Y_1, \ldots, Y_P$ be the connected components of $Y$. These are $L$–polymers. Let $f$ be the function that maps $\pi := (X_j), (\Delta_i)$ into $\{Y_1, \ldots, Y_P\}$. Now we perform the sum over $(X_j), (\Delta_i)$ in (3.4) by summing over $\pi \in f^{-1}(\{Y_1, \ldots, Y_P\})$ and then $\{Y_1, \ldots, Y_P\}$. The result is:

$$\mathcal{E}_{xp(a e^{-V} + K)}(\Lambda) = \mathcal{E}_{xp_L(a e^{-\tilde{V}} + BK)}(\Lambda)$$  \hspace{1cm} (3.5)$$

where the subscript on $\mathcal{E}_{xp_L}$ indicates that that the domain is functionals of $L$–polymers and

$$(BK)(Y) = \sum_{N+M \geq 1} \frac{1}{N! M!} \sum_{(X_j), (\Delta_i) \rightarrow Y} e^{-\tilde{V}(X_0)} \prod_{j=1}^{N} K(X_j) \prod_{i=1}^{M} P(\Delta_i)$$  \hspace{1cm} (3.6)$$

where the $\rightarrow$ is the map $f$. This representation (3.5) is a sum over products of polymer activities $(BK)(Z_j)$ where the closed disjoint polymers $Z_j$ are necessarily separated by a distance $\geq L$ and the spaces between the polymers are filled with $e^{-\tilde{V}}$ which are independent of the fluctuation field $\zeta$. The covariance $\Gamma(x - y) = 0$ if $|x - y| \geq L$. So the fluctuation integral factorises and we obtain:

$$\int d\mu_{\Gamma}(\zeta)[\mathcal{E}_{xp(a e^{-V} + K)}](\Lambda, \zeta + \phi) = \mathcal{E}_{xp_L(a e^{-\tilde{V}} + (BK^\sharp))(\Lambda, \phi)}$$ \hspace{1cm} (3.7)$$

where the superscript $\sharp$ (“sharp”) denotes integration with the measure $d\mu_{\Gamma}(\zeta)$.

Now we perform the rescaling. This is accomplished by replacing $\phi$ by the rescaled field $\mathcal{R}\phi$ where

$$(\mathcal{R}\phi)(x) = \phi_{L^{-1}}(x) = L^{-[\phi]}\phi(x/L)$$

$$(\mathcal{R}K)(L^{-1}X, \phi) = K(X, \phi_{L^{-1}})$$

If we now define $\mathcal{S} = \mathcal{R} \mathcal{B}$ then we have:

$$\int d\mu_{\Gamma}(\zeta)[\mathcal{E}_{xp(a e^{-V} + K)}](\Lambda, \zeta + \phi_{L^{-1}}) = \mathcal{E}_{xp(a e^{-\tilde{V}_L} + (SK^\sharp))(L^{-1}\Lambda, \phi)}$$ \hspace{1cm} (3.8)$$

where $\tilde{V}_L(\Lambda, \phi) = \tilde{V}(L\Lambda, \phi_{L^{-1}})$ and the superscript $\natural$ (“natural”) denotes integration with the measure $d\mu_{\Gamma_L}(\zeta)$. Here:

$$\Gamma_L(x - y) = L^{2[\phi]}\Gamma(L(x - y))$$
We have returned to a functional of the form $\mathcal{E}xp[e^{-V} + K](\Lambda)$ with $V \to \tilde{V}$ and $K \to (\tilde{S} K)^2$. Thus the operation is an evolution of the interaction described in coordinates $V, K$.

We will refer in the future to (3.8) as the fluctuation step. The RG step will be completed by removing relevant parts from $(\tilde{S} K)^2$ and compensating by a new local potential, this operation being called Extraction.

### 3.2 Extraction

The objective is to cancel parts $F e^{-V}$ of $K$ in $\mathcal{E}xp[e^{-V} + K](\Lambda)$ by a change in $V$, adding terms $V_F$ to $V$. This will be possible for functionals $F$ of a special form which we first describe.

Let $\mathcal{P}$ and $\mathcal{P}_j$ be polynomials. For $Y$, any union of cells, $V$ has the form:

$$V(Y) = \int_Y dx \, \mathcal{P}(\phi(x))$$

We define polymer activities

$$F(X) = \sum_{\Delta \subset X} F(X, \Delta)$$

(3.9)

where $F(X, \Delta)$ has the form

$$F(X, \Delta) = \sum_j \int_{\Delta} dx \, \alpha_j(X, \Delta, x) \mathcal{P}_j(\phi(x))$$

(3.10)

and $F(X, \Delta) = 0$ for $\Delta \not\subset \Lambda$. We also define:

$$V_F(\Delta) = \sum_{X \supset \Delta} F(X, \Delta)$$

(3.11)

Then

$$V_F(\Delta) = \sum_j \int_{\Delta} dx \, \alpha_j(\Delta, x) \mathcal{P}_j(\phi(x)) = \sum_j \int_X dx \, \alpha_j(x) \mathcal{P}_j(\phi(x))$$

where $\alpha(x) = \alpha(\Delta, x)$ with $\Delta \ni x$ and

$$\alpha_j(\Delta, x) = \sum_{X \ni \Delta} \alpha_j(X, \Delta, x)$$

(3.12)

and for any polymer $X$

$$V_F(X) = \sum_{\Delta \subset X} V_F(\Delta) = \sum_j \int_X dx \, \alpha_j(x) \mathcal{P}_j(\phi(x))$$

(3.13)

We define $V_F(X_c)$ by the last member of (3.13), replacing $X$ by $X_c$.

Following [BDH-est] Section 4.2,

**Extraction Formula:**

$$\mathcal{E}xp[\mathfrak{g} e^{-V} + K](\Lambda) = \mathcal{E}xp[\mathfrak{g} e^{-V'} + \mathcal{E}(K, F)](\Lambda)$$

(3.14)

where
\[ V' = V - V_F \]
\[ \mathcal{E}_1(K, F) = K - F e^{-V} \] (3.15)

where \( \mathcal{E}_1(K, F) \) is the linearization of \( \mathcal{E}(K, F) \) in \( K \) and \( F \).

The complete formula for \( \mathcal{E}(K, F) \) is described at the end of this section, but in fact we will only need a crude estimate on the nonlinear part of it which will be quoted from [BDH-est].

We will need a variant of the extraction formula in which vacuum energy is factored out completely as follows. Suppose \( F \) has an additive piece \( F_0 \) which is field independent, i.e. if it is of the form
\[ F = F_1 + F_0 \] (3.16)
we have
\[ V_F' = V - V_F = V_{F_1} - V_{F_0} \]
and then
\[ \exp(\mathbf{1} e^{-V_F'} + \mathcal{E}(K, F)(\Lambda)) = e^{V_{F_0}(\Lambda)} \exp(\mathbf{1} e^{-V_{F_1}} + \mathcal{E}(K, F_0, F_1)(\Lambda)) \] (3.17)
where
\[ \mathcal{E}(K, F_0, F_1) = e^{-V_{F_0}}[\mathcal{E}(K, F)] \] (3.18)

The extraction formula (3.17) is applied to (3.8) so that the combination of integrating out the fluctuation field and extracting returns \( \exp(\mathbf{1} e^{-V} + K)(\Lambda) \) to a functional of the same form with \( V \to V_{F_1} \) and \( K \to \mathcal{E}((SK)^3, F_0, F_1) \). This is a complete RG step.

**Formulas for \( \mathcal{E}(K, F) \):**

\[
\mathcal{E}(K, F)(W) = \sum_{M,N} \frac{1}{M! N!} \sum_{(Z_j)_1(Y_k) \to W} e^{-V'(W \setminus Y)} \\
\prod_{j=1}^M (e^{-F(Z_j, Z_j \cap Y_c)} - 1) \prod_{k=1}^N \tilde{K}(Y_k) 
\] (3.19)

where \( N \geq 1, \ M \geq 0 \) and \( Y = \cup_{k=1}^N Y_k \), the \( Y_k \) are disjoint and:

1) the polymers \((Z_j), (Y_k)\) are connected and such that \( W = (\cup Z_j) \cup (\cup Y_k) \)

2) for every \( j, Z_j \not\subset Y_c, Z_j \not\subset Y \)

and
\[
F(Z, Z \cap \bar{Y}_c) = \sum_{\Delta \subset Z \cap \bar{Y}_c} F(Z, \Delta) 
\] (3.20)

\[
\tilde{K}(X) = K(X) - e^{-V(X)}(e^F - 1)^+(X) 
\] (3.21)

\[
J^+(X) = \sum_{N>1} \frac{1}{N!} \sum_{(X_j) \to X} \prod J(X_j) 
\]

where \((X_j) \to X\) iff \( X = \cup X_j \) and \( X_j \) are distinct (as opposed to disjoint) sets. \( J(X) = e^F(X) - 1 \). \( J^+ \) is a polymer activity (vanishes when \( X \) is not closed and connected).

**Remark:** The proof of (3.19) is step by step the same as Sec. 4.2 of [BDH-est] with appropriate changes to take into account the use of closed polymers instead of open polymers.
Appendix: For convenient later reference we collect here the notations associated with rescaling that will be used in this paper. Some of them refer to objects not yet introduced.

\[ R_\phi(x) = \phi_{L^{-1}}(x) = L^{-[\phi]} \phi(x/L) \] (3.22)

\[ (RK)(L^{-1}X, \phi) = K_L(L^{-1}X, \phi) = K(X, \phi_{L^{-1}}) \]

\[ V_L(\Delta, \phi) = V(L\Delta, \phi_{L^{-1}}) \]

\[ \Gamma_L(x - y) = L^{[\phi]} \Gamma(L(x - y)) \]

For an integral kernel \( u(x - y) \), e.g. a covariance, we define a rescaled version

\[ u_L(x - y) = L^{[\phi]} u(L(x - y)) = L^{\frac{3 - \varepsilon}{2}} u(L(x - y)) \] (3.23)

\[ \tilde{V}_L(\Delta, g, \mu) = V(\Delta, \phi, C, g_L, \mu_L) \]

\[ g_L = L^\varepsilon g, \quad \mu_L = L^{\frac{3}{2} - \varepsilon} \mu \] (3.24)

A set \( X \) is said to be small if \( X \) is connected and \(|X| \leq 2^D\). In the present case the dimension \( D = 3 \). \( \bar{X}^L \)

is the \( L \)-closure of \( X \), which by definition is smallest union of closed \( L \)-blocks containing \( X \).

The superscript \( \natural \) ("natural") denotes integration with the measure \( d\mu_L(\zeta) \) and the superscript \( \sharp \) ("sharp") denotes integration with the measure \( d\mu_V(\zeta) \).

4. APPLICATION OF RENORMALIZATION GROUP STEP

In this section we specify a particular RG map by making choices for \( \tilde{V}, F \) in (3.3) and (3.15). We define the second order approximation to the RG map and derive formula for the error after second order.

Recall:

\[ V(\Delta, \phi) = V(\Delta, \phi, C, g, \mu) = g \int_\Delta d^3 x : \phi^4 :_C (x) + \mu \int_\Delta d^3 x : \phi^2 :_C (x) \] (4.1)

We define \( \tilde{V} \) by

\[ \tilde{V}(\Delta, \phi) = V(\Delta, \phi, C_{L^{-1}}, g, \mu) = g \int_\Delta d^3 x : \phi^4 :_{C_{L^{-1}}} (x) + \mu \int_\Delta d^3 x : \phi^2 :_{C_{L^{-1}}} (x) \] (4.2)

Recall that the RG acts on functionals written in the form \( \text{Exp}(\Phi e^{-V} + K)(\Lambda) \). In this section we refine this description by writing

\[ K = Q e^{-V} + R \] (4.3)

where \( Q \) is an explicit polymer activity which we will call the "second order polymer activity". It will be derived from second order perturbation theory in powers of \( g \) and is defined as follows: \( Q \) is supported on connected polymers \( X, |X| \leq 2 \). We write

\[ Q(X, \phi) = Q(X, \phi; C, \omega, g) = g^2 \sum_{j=1}^3 n_j Q^{(j;j)}(\bar{X}, \phi; C, \omega^{(4-j)}) \] (4.4)
where \((n_1, n_2, n_3) = (48, 36, 8)\) and \(w = (w^{(1)}, w^{(2)}, w^{(3)})\) is a triple of integral kernels to be obtained inductively and

\[
\tilde{X} = \begin{cases} 
\Delta \times \Delta & \text{if } X = \Delta \\
(\Delta_1 \times \Delta_2) \cup (\Delta_2 \times \Delta_1) & \text{if } X = \Delta_1 \cup \Delta_2 \\
0 & \text{otherwise}
\end{cases}
\]

\[(4.5)\]

\[
Q^{(m,m)}(\tilde{X}, \phi; C, w^{(4-m)}) = \frac{1}{2} \int_{\tilde{X}} d^3x d^2y : (\phi^m(x) - \phi^m(y))^2 :_{C, w^{(4-m)}} (x - y) \quad \text{for } m = 1, 2
\]

\[(4.6)\]

Next we define the second order approximation to the RG map. We say that an activity \(p(X)\) is supported on (closed/open) unit blocks if \(p(X) = 0\) for \(X\) not a (closed/open) unit block. A block is closed by default. Let \(p\) be the activity supported on unit blocks defined by

\[
p(\Delta, \zeta, \phi) = V(\Delta, \zeta + \phi) - \tilde{V}(\Delta, \phi) = p_g + p_\mu
\]

\[(4.7)\]

where

\[
p_g = g \int_{\Delta} d^3x : \zeta^4 : \Gamma(x) + 4\phi(x) : \zeta^3 : \Gamma(x) + 6 : \phi^2 :_{C_{L-1}} (x) : \zeta^2 : \Gamma(x) + 4 : \phi^3 :_{C_{L-1}} (x) \zeta(x)
\]

\[
p_\mu = \mu \int_{\Delta} d^3x (2\phi(x)\zeta(x) + : \zeta^2 : \Gamma(x))
\]

\[(4.8)\]

We insert a parameter \(\lambda\) into our previous definitions in such a way that (i) at \(\lambda = 1\) our \(\lambda\) dependent objects correspond with the previous definitions. (ii) The expansion through order \(\lambda^2\) is second order perturbation theory in \(g\) counting \(\mu = O(g^2)\). (iii) Powers of \(\lambda\) are determined so as to correspond with leading powers of \(g\) buried inside polymer activities. (iv) All functions will turn out to be norm analytic in \(\lambda\). Thus

\[
P(\lambda) = e^{-\tilde{V}} (-\lambda p_g - \lambda^2 p_\mu + \frac{1}{2} \lambda^2 p^2_g) + \lambda^3 r_1
\]

\[(4.9)\]

where \(r_1\) is defined by the condition \(P(\lambda = 1) = P = e^{-V} - e^{-\tilde{V}}\). Similarly, we define

\[
K(\lambda) = \lambda^2 e^{-\tilde{V}} Q + \lambda^3 (e^{-V} - e^{-\tilde{V}}) Q + R
\]

which, for \(\lambda = 1\) coincides with \(K = e^{-V} Q + R\), where \(Q\) will be an explicit polymer activity obtained by second order perturbative calculation and \(R\) will be the remainder after second order perturbation theory. Corresponding to (3.6) we define

\[
B(\lambda, K)(Y) = \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(X_j, (\Delta_i) \rightarrow Y)} e^{-\tilde{V}(X_0)} \prod_{j=1}^N K(\lambda, X_j) \prod_{i=1}^M P(\lambda, \Delta_i)
\]

\[(4.11)\]

\[
X_0 = Y \setminus (\cup X_j) \cup (\cup \Delta_i)
\]

\[(4.12)\]

Let \(S(\lambda, K) = R \circ B(\lambda, K)\), where \(R\) is the rescaling defined in the last section. The RG evolution for \(K\) with parameter \(\lambda\) is \(f_K : K \mapsto E(S(\lambda, K)^3, F(\lambda))\), where

\[
F(\lambda) = \lambda^2 F_Q + \lambda^3 F_R
\]

\[(4.13)\]
will be specified later, and $\zeta$ is the rescaled integration over $\zeta$, and, as usual, $F(\lambda) = F$, when $\lambda = 1$. Given a function $f(\lambda)$ let
\[
T_\lambda f = f(0) + f'(0) + \frac{1}{2}f''(0)
\] (4.14)
be the Taylor expansion to second order evaluated at $\lambda = 1$. Then the second order approximation to the RG map is $f^{(\leq 2)} = (f_K^{(\leq 2)}, f_V^{(\leq 2)})$ with
\[
f_K^{(\leq 2)}(K, V) = T_\lambda E(S(\lambda, K)^3, F(\lambda)) = E_1(T_\lambda S(\lambda, K)^3, F_Q)
\] (4.15)
\[
f_V^{(\leq 2)}(K, V) = V'_F
\] (4.16)
Note also that only the linearized $E_1$ intervenes, because it will turn out that the nonlinear part of extraction generates terms only at order $\lambda^3$ or higher.

**Proposition 4.1**

There is a choice of $F_Q$ such that the form of $Q$ remains invariant under the RG evolution at second order. In more detail, $f^{(\leq 2)}(V, e^{-V}) = (V'_F, Q'_V)$ where the parameters in $V'_F$ evolved according to
\[
V'_F(\Delta) = V(\Delta, C, g'_{(\leq 2)}, \mu'_{(\leq 2)})
\]
\[
g'_{(\leq 2)} = L^g(1 - L^a g)
\]
\[
\mu'_{(\leq 2)} = L^{3+2\epsilon} \mu - L^{2\epsilon} g^2
\] (4.17)
The parameters in $Q'_V$ evolved according to
\[
Q'_V(\Delta) = Q(C, w', g_L)
\]
\[
w' = v + w_L
\]
\[
v^{(1)} = \Gamma_L, \quad v^{(p)} = (C_L)^p - (C)^p \quad 2 \leq p \leq 4
\] (4.18)

**Proof:** We define a polymer activity $\hat{Q}$ supported on connected polymers $X$ with $|X| \leq 2$
\[
\hat{Q}(X, \zeta, \phi) = \begin{cases} \frac{1}{2}(p_{A}(\Delta, \zeta, \phi))^2 & |X| = \Delta \\ \frac{1}{2} \sum_{\Delta_1, \Delta_2 \in X} p_{A}(\Delta, \zeta, \phi) p_{A}(\Delta_1, \Delta_2, \zeta, \phi) & |X| = 2, \text{ connected} \end{cases}
\] (4.19)
It is easy to check that
\[
T_\lambda S(\lambda, K, \lambda) = -p_L e^{-\hat{V}_L} + \left\{ e^{-\hat{V}_L} \hat{Q}_L + e^{-\hat{V}_L} Q_L \right\}
\] (4.20)
Let
\[
\hat{Q}(C, \lambda, g_L) = g_L^2 \sum_{j=1}^4 n_j \hat{Q}^{(j)}(\hat{X}, \phi; C, v^{(4-j)})
\] (4.21)
where $(n_1, n_2, n_3, n_4) = (48, 36, 8, 12)$ and
\[
\hat{Q}^{(m,n)}(\hat{X}, \phi; C, u) = \frac{1}{2} \int_{\hat{X}} d^3 x d^3 y : \phi^m(x) \phi^n(y) : C u(x - y)
\] (4.22)
We then have
\[ T_\lambda S(K, \lambda)^2 = e^{-\tilde{v}_L} \left( \tilde{Q}(C, v, g_L) + Q(C, w_L, g_L) \right) \]  \hspace{1cm} (4.23)

Define
\[ F_Q = \tilde{Q}(C, v, g_L) - Q(C, v, g_L) \]  \hspace{1cm} (4.24)

Then we have from (4.23) and (4.24)
\[ \mathcal{E}_1 \left( T_\lambda S(\lambda, K)^2, F \right) = T_\lambda S(\lambda, K)^2 - F_Q e^{-\tilde{v}_L} = e^{-\tilde{v}_L} Q(C, w^{(\leq 2)}, g_L) \]  \hspace{1cm} (4.25)

We write
\[ F_Q(X, \phi) = F_{1,Q}(X, \phi) + F_{0,Q}(X) \]  \hspace{1cm} (4.26)

where \( F_{0,Q} \) is the field independent part of \( F_Q \). Then
\[ F_{1,Q} = g_L^2 \left\{ 36 \tilde{Q}^{(4,0)}(C, v^{(2)}) + 48 \tilde{Q}^{(2,0)}(C, v^{(3)}) \right\} \]
\[ F_{0,Q} = 12g_L^2 \tilde{Q}^{(0,0)}(C, v^{(4)}) \]  \hspace{1cm} (4.27)

\( F_{1,Q}(X) \) can be written as:
\[ F_{1,Q}(X) = \sum_{\Delta \subset X} F_{1,Q}(X, \Delta) \]  \hspace{1cm} (4.28)

where
\[ F_{1,Q}(X, \Delta) = 36g_L^2 F_{1,Q}^{(4)}(X, \Delta) + 48g_L^2 F_{1,Q}^{(2)}(X, \Delta) \]  \hspace{1cm} (4.29)

and
\[ F_{1,Q}^{(m)}(X, \Delta) = \int_{\Delta} d^3x : \phi^m : C(x) f_{1,Q}^{(m)}(x, X, \Delta) \]  \hspace{1cm} (4.30)

with
\[ f_{1,Q}^{(m)}(x, X, \Delta) = \begin{cases} \int_{\Delta} d^3y v^{(m')}(x - y) & X = \Delta \\ \int_{\Delta'} d^3y v^{(m')}(x - y) & X = \Delta \cup \Delta', \text{ connected} \end{cases} \]  \hspace{1cm} (4.31)

and
\[ m' = 4 - m/2 \]

\[ V(F_{1,Q}, \Delta) = \sum_{X \supset \Delta} F_{1,Q}(X, \Delta) = 36g_L^2 \sum_{X \supset \Delta} F_{1,Q}^{(4)}(X, \Delta) + 48g_L^2 \sum_{X \supset \Delta} F_{1,Q}^{(2)}(X, \Delta) \]  \hspace{1cm} (4.32)

In the following we will use the:

Claim:
\( v^{(j)} \), \( 1 \leq j \leq 4 \) are \( C^\infty \), positive, and have support \( v^{(j)}(x - y) = 0 \) for \( |x - y| \geq 1 \).

Proof: That they are \( C^\infty \) follows from their definition (4.25), \( \Gamma \) is \( C^\infty \) and \( C \) is \( C^\infty \) . For the support property, this is obvious for \( v^{(1)} = \Gamma_L \), and for \( p \geq 2 \) : \( v^{(p)} = C_L^p - C^p \) with pointwise multiplication. The latter has \( \Gamma_L \) as a factor because \( C_L = C + \Gamma_L \) and \( \Gamma_L \) has the required support property. The positivity follows from that of \( \Gamma_L \) and \( C \). This proves the claim.

Now return to (4.32).
\[ \sum_{X \supset \Delta} F_{1,Q}^{(m)}(X, \Delta) = \int_{\Delta} d^3x : \phi^m : C(x) \left[ \int_{\Delta} d^3y v^{(m')}(x - y) + \sum_{\Delta' \not\supset \Delta} \int_{\Delta'} d^3y v^{(m')}(x - y) \right] \]
On the r.h.s. use $v^{(m')} (x-y) = 0$ for $|x-y| \geq 1$ to extend the sum on $\Delta'$ to all the $\Delta' \neq \Delta$. We then get

$$
\sum_{X \supset \Delta} F^{(m')}_{1,Q}(X, \Delta) = \int_{\Delta} d^3 x : \phi^m : (x) \left[ \int d^3 y v^{(m')} (x-y) \right]
$$

Hence from (4.32) and above we get

$$
V(F_{1,Q}, \Delta) = a g_L^2 \int_{\Delta} d^3 x : \phi^4 : (x) + b g_L^2 \int_{\Delta} d^3 x : \phi^2 : (x)
$$

(4.33)

where

$$
a = 36 \int d^3 y v^{(2)} (y) > 0 = O(\log L) > 0
$$

(4.34)

$$
b = 48 \int d^3 y v^{(3)} (y) > 0 = O(L^{3/2}) > 0
$$

That $a$ and $b$ are well defined and positive is easy to see using the claim above. That $a = O(\log L), b = O(L^{3/2})$ is proved later in Section 5 (see Lemma 5.12). End of proof.

The exact RG evolution for $K = Qe^{-V} + R$.

The exact map

$$
K \mapsto K' = f_K(\lambda, K, V)|_{\lambda=1} = \mathcal{E}(S(\lambda, K)^2, F(\lambda))|_{\lambda=1}
$$

induces an evolution of the remainder $R$ which is studied by Taylor series around $\lambda = 0$ with remainder written using the Cauchy formula:

$$
f_K(\lambda = 1) = \sum_{j=0}^3 \frac{f_K^{(j)}(0)}{j!} + \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^4 (\lambda - 1)} f_K(\lambda)
$$

The terms $j = 0, 1, 2$ are the second order part $f_K^{(\leq 2)}$. In the $j = 3$ term there are no terms mixing $R$ with $Q, P$ because of the $\lambda^3$ in front of $R$. Therefore it splits

$$
\frac{f_K^{(3)}(0)}{3!} = R_1 + R_2
$$

into the third order derivative at $R = 0$, which we write using the Cauchy formula as

$$
R_1 \equiv R_{\text{main}} = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^2} \mathcal{E}(S(\lambda, Qe^{-V})^2, F_Q(\lambda))
$$

(4.36)

and terms linear in $R$:

$$
R_2 \equiv R_{\text{linear}} = (S_1 R)^2 - F_R e^{-\tilde{V}_L}
$$

$$
S_1 R(Z) = \sum_{X : L^{-1} \hat{X} = Z} e^{-\tilde{V}_L(Z \setminus L^{-1} X)} R_L(L^{-1} X)
$$

(4.37)

The remainder term in the Taylor expansion is

$$
R_3 = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^2 (\lambda - 1)} \mathcal{E}(S(\lambda, K)^2, F(\lambda))
$$

(4.38)

In Proposition 4.1 the coupling constant in $Q$ is not the same as the coupling constant in $V^{(\leq 2)}$. Furthermore, the coupling constant in $V^{(\leq 2)}$ will further change because of the contribution from $F_R$. To take this into account we introduce
\[ V'(\Delta) = V(\Delta, C, g', \mu') \]
\[ g' = L^\varepsilon g(1 - L^\varepsilon ag) + \xi_R \]
\[ \mu' = L^{3+\varepsilon} \mu - L^{2\varepsilon} bg^2 + \rho_R \]

where the remainders \( \xi_R, \rho_R \) anticipate the effects of a yet-to-be-specified \( F_R \). Then we set

\[ R_4 = e^{-\tilde{V}} Q(C, w', g') - e^{-\tilde{V}_L} Q(C, w', g_L) \]  

(4.40)

With these definitions we have written the RG as a map

\[ \mathcal{E} \exp(\xi e^{-V} + Qe^{-V} + R)(\Lambda) \mapsto \mathcal{E} \exp(\xi' e^{-V'} + Q'e^{-V'} + R')(L^{-1} \Lambda) \]

with \( Q' = Q(C, w', g') \) and the RG map \( f_K \) has induced a map \( f_R \)

\[ R' \equiv f_R(V, R) = R_{\text{main}} + R_{\text{linear}} + R_3 + R_4 \]  

(4.41)

Construction of \( F_R \)

To complete the RG step we must specify the relevant part from the remainder \( F_R \). The goal is to choose \( F_R \) so that the map \( R \rightarrow R_{\text{linear}} \) will be contractive. As we will later prove, this will be the case provided the small set part of \( R_{\text{linear}} \) is normalized so that certain derivatives with respect to \( \phi \) vanish at \( \phi = 0 \). We say that a functional \( J \) is normalized if, for all small sets \( X \),

\[ J(X, 0) = 0 \]
\[ D^2 J(X, 0; 1, 1) = 0 \]
\[ D^2 J(X, 0; 1, x_\mu) = 0 \]
\[ D^4 J(X, 0; 1, 1, 1, 1) = 0 \]  

(4.42)

Define

\[ \tilde{F}_R(X, \phi) = \sum_P \int_X d^3 x \; \tilde{\alpha}_P(X) P(\phi(x), \partial \phi(x)) \]  

(4.43)

where \( P \) runs over the relevant monomials, which , in this model are

\[ P = 1, \phi^2, \phi^4, \phi \partial_\mu \phi \] with \( \mu = 1, 2, 3 \).

Choose the coefficients \( \tilde{\alpha}_P \) so that

\[ J = R^4 - \tilde{F}_Re^{-\tilde{V}} \]  

(4.44)

is normalized (details are given below). We define the relevant part, a functional supported on small sets, by

\[ F_R(Z) = \sum_{X: \text{small sets}} \tilde{F}_{R,L}(L^{-1} X) \]  

(4.45)

Then, from the definition of \( R_{\text{linear}} \) in (4.37),

\[ R_{\text{linear}}(Z) = \sum_{X: \text{small sets} \atop L^{-1} X = Z} e^{-\tilde{V}_L(Z) L^{-1} X} J_L(L^{-1} X) + \sum_{X: \text{large sets} \atop L^{-1} X = Z} e^{-\tilde{V}_L(Z) L^{-1} X} J_L(L^{-1} X) \]  

(4.46)
Therefore the first sum in $R_{\text{linear}}$ is also normalized because normalization is preserved under multiplication by smooth functionals of $\phi$ and rescaling.

Having defined $F_R$, we must show that it has the form (3.9), (3.10) required for the extraction operation. Define $F_R(X,Y)$ by replacing the integral over $X$ by integration over $X \cap Y$ in (4.43). Let

$$F_R(Z, \Delta, \phi) = \sum_{X \text{small set} \subset \Delta, L^{-1} X = Z} \bar{F}_R(X, L \Delta \cap X, \phi_{L^{-1}})$$

(4.47)

It is clear that (3.9) holds. For each monomial $\alpha P(X)$ in (4.43) define

$$\alpha_P(Z, \Delta, x) = \sum_{X \text{small set} \subset \Delta, L^{-1} X = Z} \bar{\alpha}_P(X)L^{-[P]+3}1_{\Delta \cap L^{-1} X}(x)$$

(4.48)

where $[P]$ is the dimension of the monomial $P$, $(n[\phi]$ for $\phi^n$ and $2[\phi] + 1$ for $\phi \partial \phi$). Then

$$F_R(Z, \Delta, \phi) = \sum_P \int d^3 x \alpha_P(Z, \Delta, x)P(\phi(x), \partial \phi(x))$$

(4.49)

which shows that (3.10) holds.

In order to compute $V_{F_R}$, note that by translation invariance,

$$\sum_{Z \supset \Delta} \alpha_P(Z, \Delta, x) = \sum_{Z \supset \Delta} \sum_{X \text{small set} \subset \Delta, L^{-1} X = Z} \alpha_P(X)L^{-[P]+3}1_{\Delta \cap L^{-1} X}(x)$$

(4.50)

$$= \sum_{X \text{small set} \subset \Delta} \alpha_P(X)L^{-[P]+3}1_{\Delta \cap L^{-1} X}(x)$$

is independent of $x, \Delta$, (equality a.s. because the equality fails for $x$ in boundaries of blocks). This can be written more simply as

$$\alpha_P = L^{-[P]+3} \sum_{X \text{small set} \supset \Delta(x)} \alpha_P(X)$$

(4.51)

where $\Delta(x)$ is the block that contains $x$, excluding $x$ in a boundary. Therefore, with coefficients derived as in (4.51),

$$V(F_R, \Delta) = \sum_{Z \supset \Delta} F_R(Z, \Delta) = \sum_P \alpha_P \int d^3 x P(\phi)$$

(4.52)

$$=: \int d^3 x \left\{ \alpha_0 + \alpha_{2,0} \phi^2 + \alpha_{4,0} \phi^4 \right\}$$

where the term $\alpha_{2,1} \phi(x) \partial \mu \phi(x)$ is absent because $\alpha_{2,1} = 0$ by reflection invariance. We have to rewrite this in a $C$ Wick ordered basis in order to compute $V'$

$$V(F_R, \Delta) = \int d^3 x \left\{ \beta_0 + \rho_R : \phi^2 :_C (x) + \xi_R : \phi^4 :_C (x) \right\}$$

(4.53)

where
\[
\beta_0 = \alpha_0 + C(0)\alpha_{2,0} + 3C^2(0)\alpha_{4,0} \\
\rho_R = \alpha_{2,0} + 6C(0)\alpha_{4,0} \\
\xi_R = \alpha_{4,0}
\]

which are formulas for the error terms in (4.17).

**Determining coefficients from (4.42)**

Note that the odd derivatives \(D^j J(X, 0; f^x)\), \(j=\text{odd integer}\), vanish identically by \(\phi \mapsto -\phi\) symmetry. Taking derivatives of (4.44) we get

\[
J(X, 0) = R^2(X, 0) - \tilde{F}_R(X, 0)e^{-V(X, 0)} \\
D^2 J(X, 0; f^{x^2}) = D^2 R^2(X, 0; f^{x^2}) - D^2 \tilde{F}_R(X, 0; f^{x^2})e^{-V(X, 0)} + \\
\quad + \tilde{F}_R(X, 0)D^2 \tilde{V}(X, 0; f^{x^2})e^{-V(X, 0)} \\
D^4 J(X, 0; f^{x^4}) = D^4 R^2(X, 0; f^{x^4}) + D^4 \tilde{F}_R(X, 0; f^{x^4})e^{-V(X, 0)} + \\
\quad + 4D^2 \tilde{F}_R(X, 0; f^{x^2})D^2 \tilde{V}(X, 0; f^{x^2})e^{-V(X, 0)} \\
\quad + \tilde{F}_R(X, 0)D^4 \tilde{V}(X, 0; f^{x^4})e^{-V(X, 0)}-3\tilde{F}_R(X, 0)(D^2 \tilde{V}(X, 0; f^{x^2}))^2e^{-V(X, 0)}
\]

Now imposing successively the conditions (4.42) we get

\[
\tilde{F}_R(X, 0) = \tilde{\alpha}_0(X)|X| \\
D^2 \tilde{F}_R(X, 0; 1, 1) = 4|X|\tilde{\alpha}_{2,0}(X) \\
D^2 \tilde{F}_R(X, 0; 1, x_\mu) = |X|\tilde{\alpha}_{2,1}(X, \mu) \\
D^4 \tilde{F}_R(X, 0; 1, 1, 1, 1) = 24|X|\tilde{\alpha}_{4}(X)
\]

Note that from (4.43)

\[
\tilde{\alpha}_0(X) = \frac{1}{|X|}R^2(X, 0)e^{-V(X, 0)} \\
\tilde{\alpha}_{2,0}(X) = \frac{1}{4|X|}e^{-V(X, 0)} \left[ D^2 R^2(X, 0; 1, 1) + R^2(X, 0)D^2 \tilde{V}(X, 0; 1, 1) \right] \\
\tilde{\alpha}_{2,1}(X, \mu) = \frac{1}{|X|}e^{-V(X, 0)} \left[ D^2 R^2(X, 0; 1, x_\mu) + R^2(X, 0)D^2 \tilde{V}(X, 0; 1, x_\mu) \right] \\
\tilde{\alpha}_4(X) = \frac{1}{24|X|}e^{-V(X, 0)} \left[ D^4 R^2(X, 0; 1, 1, 1, 1) \\
\quad + D^2 \tilde{V}(X, 0; 1, 1) \left( [D^2 R^2(X, 0; 1, 1) + R^2(X, 0)D^2 \tilde{V}(X, 0; 1, 1)] \\
\quad + R^2(X, 0) \left( D^4 \tilde{V}(X, 0; 1, 1, 1, 1) - 3(D^2 \tilde{V}(X, 0; 1, 1))^2 \right) \right] \right]
\]

Note that the leading contributions to the \(\tilde{\alpha}_p(X)\) are obtained by setting \(\tilde{V} = 0\) in the above formulae.

**Resume**

We have thus produced at the end of the RG step the promised map:

\[
(V, Q, R) \rightarrow (V', Q', R')
\]
$V'$ is the same as $V$ with evolved coupling $g \to g'$, $\mu \to \mu'$ given by the flow ((4.39)), with $a, b$ given in (4.34) and $\xi R$, $\rho R$ in (4.54). $Q'$ is the same as $Q$ with the change $w \to w'$, (4.18), and $R'$ is given by (4.41) with intervening quantities defined earlier.

5. ESTIMATES.

We will assume $L$ large but fixed and then $\varepsilon$ sufficiently small depending on $L$. $O(1)$ denotes a constant independent of $L$ and $\varepsilon$. Constants C are independent of $\varepsilon$ but may depend on $L$. These constants may change from line to line. It will not be necessary to keep track of these changes.

Throughout we will assume that $w$ at a generic step has been obtained by successive iterations (4.18) with initial $w_0 = 0$.

We make in this section the following hypothesis in terms of the norms introduced in Section 2.2.

Hypothesis

\begin{align}
|g - \bar{g}| &\leq \varepsilon^{3/2}, \quad |\mu| \leq \varepsilon^{2-\delta} \\
\|R\|_{h, G_{\alpha}, A} &\leq \varepsilon^{3/4-\eta} \\
|R|_{h_*, A} &\leq \varepsilon^{11/4-\eta}
\end{align}

where $\delta, \eta = O(1) > 0$ are very small fixed numbers, say $1/64$, and $h = c\varepsilon^{-1/4}$ with $c = O(1)$ a very small number. Further more we take $h_* = L^{(3+\varepsilon)/4}$ and choose $n_0 = 9$. Moreover $\bar{g}$ is the approximate fixed point in the flow of the coupling constant $g$ obtained from the first equation in (4.39) by ignoring the remainder $\xi R$. Namely,

$$\bar{g} = \frac{L^\varepsilon - 1}{L^\varepsilon a} = O(\varepsilon) > 0$$

for $\varepsilon$ sufficiently small depending on $L$. We have used the estimate $a = O(\log L) > 0$ which is proved below in Lemma 5.12 (independent of the Hypothesis). Note that the Hypothesis now implies that $g = O(\varepsilon)$ for $\varepsilon$ sufficiently small.

Recall the definitions of $\rho R$ and $\xi R$ from (4.54). We will prove in this section the following

Theorem 1

Given the Hypothesis above we have for $L$ sufficiently large and then $\varepsilon$ sufficiently small

\begin{align}
|\xi R| &\leq L^\varepsilon \varepsilon^{11/4-\eta} \\
|\rho R| &\leq L^{(3+\varepsilon)/2} \varepsilon^{11/4-\eta} \\
|g' - \bar{g}| &\leq \varepsilon^{3/2}, \quad |\mu'| \leq C\varepsilon^{2-\delta} \\
|g' - g| &\leq \varepsilon^2 \\
\|R'\|_{h, G_{\alpha}, A} &\leq L^{-1/4} \varepsilon^{3/4-\eta} \\
|R'|_{h_*, A} &\leq L^{-1/4} \varepsilon^{11/4-\eta}
\end{align}
The following long series of lemmas, except Lemmas 5.1-5.4 and 5.14, are proved under the Hypothesis above, and will serve to prove Theorem 1. Lemmas 5.21, 5.22, 5.23 and 5.27 are the major parts of the program. \( R_{\text{main}} \) is bounded in Lemma 5.21 and this result determines the qualitative form of the bound on the remainder. \( R_3 \) and \( R_4 \) are seen, in Lemmas 5.22, 5.23 to be negligible in comparison. \( R_{\text{linear}} \) is the crux of the program and it is bounded in Lemma 5.27. The remaining Lemmas are auxiliary results on the way to these Lemmas.

These auxiliary lemmas implement some the following principles: in bounds by \( G, h, \mathcal{A} \) norms, a fluctuation field \( \zeta(x) \) contributes \( C_L \) and a field \( \phi \) contributes \( O(1) g^{-1/4} \). In bounds by the 1, \( \mathcal{A} \) norms, fluctuation fields and \( \phi \) fields contribute \( O(1) \).

**Lemma 5.1**
Let \( Z \) be a 1-polymer, \( Y \) be a \( L^{-1} \)-polymer or \( \emptyset, Y \subset Z \) and \( \text{vol}(Z \setminus Y) \geq \frac{1}{2} \). Choose \( \gamma = O(1) > 0, \kappa = O(1) > 0 \). Let \( \sigma \) sufficiently large. Then for any \( x \in Z \), there exists a constant \( O(1) \) depending on \( \kappa, \gamma, j \) such that

\[
\|\phi\|_{C^2(Z)}^2 \leq O(1) g^{-\frac{j}{4}} e^{\gamma g} \int_{Z \setminus Y} d^3 y |\phi(y)|^4 G_{\kappa}(Z, \phi)
\]  

(5.11)

**Proof**
This is a simple variant of an analogous lemma in [BDH-est]. Write

\[
\phi(x) = \frac{1}{\text{vol}(Z \setminus Y)} \int_{Z \setminus Y} d^3 y (\phi(y) + \phi(x) - \phi(y))
\]

and bound

\[
|\phi(x)| \leq \frac{1}{\text{vol}(Z \setminus Y)} \int_{Z \setminus Y} d^3 y |\phi(y)| + \frac{1}{\text{vol}(Z \setminus Y)} \int_{Z \setminus Y} d^3 y |\phi(x) - \phi(y)|
\]

\[
\leq O(1) \left( \|\phi\|_{L^4(Z \setminus Y)} + \|\phi\|_{Z, 1, \sigma} \right)
\]

where the first term was bounded using the Hölder inequality. The second term was bounded by writing the difference as the integral of \( \nabla \phi \) and using

\[
|\nabla \phi(x)| \leq O(1) \|\phi\|_{\Delta, 1, \sigma}, \quad \text{for } \Delta \ni x
\]

which is the Sobolev embedding theorem, valid for \( \sigma > 3/2 + 2 \). We also have under the same condition on \( \sigma \)

\[
\|\nabla^2 \phi\|_{C(Z)} \leq O(1) \|\phi\|_{Z, 1, \sigma}
\]

Hence

\[
\|\phi\|_{C^2(Z)}^2 \leq O(1) \left( \|\phi\|_{L^4(Z \setminus Y)}^2 + \|\phi\|_{Z, 1, \sigma}^2 \right)
\]

\[
\leq O(1) g^{-j/4} e^{\gamma g} \int_{Z \setminus Y} d^3 y |\phi(y)|^4 G_{\kappa}(Z, \phi)
\]

where \( O(1) \) depends on \( \kappa, \gamma, j \). This proves the lemma.

For fluctuation fields \( \zeta \), we will have occasion to use the following lemmas

Define

\[
\tilde{G}_{\kappa, \alpha}(X, \zeta) = e^{\alpha \|\zeta\|_{L^2(X)}^2} G_{\kappa}(X, \zeta), \quad \alpha, \kappa > 0
\]  

(5.12)

\( \kappa \) is \( O(1) \) and will be held sufficiently small. The choice of \( \alpha \) is dictated by Lemma 5.3 below.

**Lemma 5.2**
For any $x \in X$

$$|\zeta(x)|^2 \leq C_{\alpha,j} \tilde{G}_{\kappa,\alpha}(X,\zeta) \quad (5.13)$$

where

$$C_{\alpha,j} = (\alpha)^{-(j/2)}O(1)$$

and $O(1)$ depends on $j$ and $\kappa$. We have isolated out the $\alpha$ dependence in the bound.

**Lemma 5.3**

$$\int d\mu_\Gamma(\zeta) \tilde{G}_{\kappa,\alpha}(X,\zeta) \leq 2^{|X|} \quad (5.14)$$

for $\alpha = \alpha(L) > 0$ sufficiently small and $\kappa = O(1) > 0$ sufficiently small. Here

$$\alpha(L) = L^{-(3-2[\phi])} \kappa' = L^{-(3+\varepsilon)/2} \kappa'$$

and $\kappa' = O(1) > 0$ is held sufficiently small.

The proof of Lemma 5.2 follows the lines of Lemma 5.1 except that we replace the $L^4$ norm there by the $L^2$ norm in the appropriate places and $Y = \emptyset$ and $Z = \Delta \ni x$. The proof of Lemma 5.3 is the same as the one referenced for (2.3).

It is convenient, for the control of norms of our polymer activities in intermediate steps, to introduce some new regulators and some intermediate norms in the following way.

Define

$$\hat{G}_{\kappa,\alpha}(X,\zeta,\phi) = G_{\kappa}(X,\zeta + \phi)G_{\kappa}(X,\phi)\tilde{G}_{\kappa,\alpha}(X,\zeta) \quad (5.15)$$

$\hat{G}_{\kappa,\alpha}$ is a regulator.

**Lemma 5.4**

$$\int d\mu_\Gamma(\zeta) \hat{G}_{\kappa,\alpha}(X,\zeta,\phi) \leq 2^{|X|} G_{3\kappa}(X,\phi) \quad (5.16)$$

for $\alpha = \alpha(L) > 0$ sufficiently small and $\kappa = O(1) > 0$ sufficiently small.

**Proof:** use Cauchy-Schwartz, stability of $G_{\kappa}$, (2.3) and Lemma 5.3.

For polymer activity $K(X,\zeta,\phi)$ define the norms

$$\|K(X)\|_{h,\hat{G}_{\kappa,\alpha}} = \sup_{\phi,\zeta} \|K(X,\zeta,\phi)\|_h \hat{G}_{\kappa,\alpha}^{-1}(X,\zeta,\phi) \quad (5.17)$$

$$\|K(X)\|_{h,\hat{G}_{\kappa,\alpha}} = \sup_{\zeta} \|K(X,\zeta,0)\|_h \hat{G}_{\kappa,\alpha}^{-1}(X,\zeta) \quad (5.18)$$

where in (5.17) and (5.18) the functional derivatives in $\|K(X,\zeta,\phi)\|_h$ and in $\|K(X,\zeta,0)\|_h$ are computed with respect to the field $\phi$.

The norms above are useful because before fluctuation integration we will encounter activities $K(X,\zeta,\phi)$ which are not just functions of $\zeta + \phi$.

The following lemma is a variant of Theorem 1 [BDH-est] adapted to our purposes.

**Lemma 5.5**

For $V(Y,\phi,\zeta) = V(Y,\phi + \zeta, C, g, \mu)$ or $V(Y,\phi + \zeta, C_{L-1}, g, \mu)$,

$$\|e^{-V(Y,\phi+\zeta)}\|_h \leq 2^{|Y|} e^{-g/4} \int_Y d^3x(\phi+\zeta)^4(x) \quad (5.19)$$
\[ \| e^{-V(\Delta, \phi)} \|_{h_*} \leq 2^{|Y|} \] (5.20)

for \( \varepsilon > 0 \) sufficiently small, and, for the second bound, depending on \( L \).

**Proof:**

\[ \tilde{V}(\Delta, \phi) = V(\Delta, \phi, C_{L-1}, g, \mu) = g \int_{\Delta} d^3 x : \phi^4 :_{C_{L-1}}(x) + \mu \int_{\Delta} d^3 x : \phi^2 :_{C_{L-1}}(x) \]

Undo the Wick ordering. Wick constants are finite and \( O(1) \). Recall from the initial Hypothesis that \( g = O(\varepsilon) \) and \( |\mu| \leq O(\varepsilon^2) \), and \( h = c\varepsilon^{-1/4} \), with \( c \) small enough. Hence

\[ \tilde{V}(\Delta, \phi) = g \int_{\Delta} d^3 x \phi^4(x) - O(1)g \int_{\Delta} d^3 x \phi^2(x) - O(\varepsilon) \]

\[ \tilde{V}(\Delta, \phi) - \frac{g}{2} \int_{\Delta} d^3 x \phi^4(x) = \frac{g}{2} \int_{\Delta} d^3 x \phi^4(x) - O(1)g \int_{\Delta} d^3 x \phi^2(x) - O(\varepsilon) = \]

\[ = \frac{g}{2} \int_{\Delta} d^3 x \left( (\phi^2(x) - O(1))^2 - O(1) \right) \geq -O(\varepsilon) \]

Hence

\[ e^{-\tilde{V}(\Delta, \phi)} \leq (1 + O(\varepsilon))e^{-g/2 \int_{\Delta} d^3 x \phi^4(x)} \]

Compute now the derivatives \( D^k e^{-\tilde{V}} \). We get for \( 1 \leq k \leq n_0 \)

\[ \frac{h^k}{k!} \left\| (D^k e^{-\tilde{V}})(\Delta, \phi) \right\| \leq c^k \sum_{j=1}^k \frac{1}{j!} \sum_{1 \leq l \leq 4} \prod_{i=1}^j \left( \frac{\varepsilon^{-1/4}}{l!} \right) \left\| D^j \tilde{V}(\Delta, \phi) \right\| e^{-\tilde{V}(\Delta, \phi)} \]

\[ \leq c^k \sum_{j=1}^k \frac{1}{j!} \sum_{1 \leq l \leq 4} \frac{(\varepsilon^{-1/4})^j}{l!} \left\| D^j \tilde{V}(\Delta, \phi) \right\| \left\| D^j \tilde{V}(\Delta, \phi) \right\| e^{-\tilde{V}(\Delta, \phi)} \]

\[ \leq O(1)c^k e^{-\frac{\varepsilon}{4} \int_{\Delta} d^3 x \phi^4(x)} e^{\sum_{1 \leq l \leq s} \frac{(\varepsilon^{-1/4})^j}{l!} \left\| D^j \tilde{V}(\Delta, \phi) \right\|} \]

Take the expression for \( \tilde{V} \) where the Wick ordering has been undone. Then it is easy to see that

\[ \sum_{1 \leq l \leq 4} \frac{(\varepsilon^{-1/4})^j}{l!} \left\| D^j \tilde{V}(\Delta, \phi) \right\| \leq O(1) \int_{\Delta} d^3 x \sum_{j=0}^3 \frac{\varepsilon^{j/4}}{j!} \left\| D^j \tilde{V}(\Delta, \phi) \right\| \]

\[ \leq O(1) \int_{\Delta} d^3 x \phi^4(x) + O(1) \]

Hence

\[ \frac{h^k}{k!} \left\| (D^k e^{-\tilde{V}})(\Delta, \phi) \right\| \leq O(1)c^k e^{-\frac{\varepsilon}{4} \int_{\Delta} d^3 x \phi^4(x)} \]

The sum over \( k \) is \( O(c) \) if \( c \) is small enough. The proof of (5.19) follows easily. The proof of (5.20) follows the same lines but we must take \( \varepsilon \) sufficiently small depending on \( L \).

**Lemma 5.6**

Let \( p_g(\Delta, \zeta, \phi), p_\mu(\Delta, \zeta, \phi) \) be as given in (4.8). \( h = c\varepsilon^{-1/4}, g, \mu \) as in the inductive hypothesis, and \( h_* \) as defined earlier. Then for any \( \alpha > 0 \), \( \kappa = O(1) > 0 \), \( \xi = O(1) > 0 \), \( 0 \leq s < 1 \)

\[ \| p_g(\Delta, \zeta, \phi) \|_h \leq C_\alpha e^{1/4} (1 - s)^{-3/4} G_{\kappa, \alpha}(\Delta, \zeta) G_{\kappa}(\Delta, \phi) e^{g(1-s)} \int_{\Delta} d^3 x \phi^4(x) \] (5.21)

\[ \| p_\mu(\Delta, \zeta, \phi) \|_h \leq C_\alpha e^{7/4-\delta} (1 - s)^{-1/2} \tilde{G}_{\kappa, \alpha}(\Delta, \zeta) G_{\kappa}(\Delta, \phi) e^{g(1-s)} \int_{\Delta} d^3 x \phi^4(x) \] (5.22)
\[\|p_g(\Delta, \zeta, 0)\|_{h_*} \leq C_{\alpha, L} \varepsilon \tilde{G}_{\kappa, \alpha}(\Delta, \zeta)\]  
(5.23)

\[\|p_\mu(\Delta, \zeta, 0)\|_{h_*} \leq C_{\alpha, L} \varepsilon^{2-4} \tilde{G}_{\kappa, \alpha}(\Delta, \zeta)\]  
(5.24)

**Proof**

Undoing the Wick ordering we have

\[p_g(\Delta, \zeta, \phi) = g \int_\Delta d^3x \sum_{j=1}^3 a_j \zeta^{4-j}(x) \phi^j(x)\]

where the constants \(a_j\) are \(O(1)\).

\[h^k \|D^k p_g(\Delta, \zeta, \phi; f^{x_k})\| \leq \varepsilon^{-k/4} g \int_\Delta d^3x \sum_{j=1}^3 a_j |\zeta^{4-j}(x)||\phi^{j-k}(x)||f||x_k(\Delta)\]

Note that since \(p_g\) is a third degree polynomial in \(\phi\), derivatives with \(k > 3\) vanish. Moreover on the right hand side \(j \geq k\). Now use Lemma 5.1, 5.2 to get

\[h^k \|D^k p_g(\Delta, \zeta, \phi)\| \leq C_\alpha g \varepsilon^{-k/4} g^{-\frac{k}{2}} (1 - s)^{-3/4} \tilde{G}_{\kappa, \alpha}(\Delta, \zeta) G_\kappa(\Delta, \phi) e^{g(1-s)} \xi \int_\Delta d^3x \phi^k(x)\]

Use \(g = O(\varepsilon)\), multiply by \(\frac{1}{h^k}\) and take sum over \(k\) to obtain (5.21). The remaining parts are obtained along the same lines.

Define \(p(s) = p(s, \Delta, \phi, \zeta)\) by

\[p(s) = sp_g + s^2p_\mu\]  
(5.25)

Then \(r_1 = r_1(\Delta, \phi, \zeta)\) defined by (4.9) is given by

\[r_1 = \frac{1}{2} \int_0^1 ds (1 - s)^2 e^{-p(s)} - (p'(s))^3 + 6p'(s)p_\mu\]  
(5.26)

with \(p'(s) = \frac{dp(s)}{ds} = p_g + 2sp_\mu\) and \(p''(s) = 2p_\mu\).

**Lemma 5.7**

\[\|r_1(\Delta)\|_{h, \tilde{G}_{\kappa, \alpha}} \leq C_\alpha \varepsilon^{3/4}\]  
(5.27)

\[\|r_1(\Delta)\|_{h, \tilde{G}_{\kappa, \alpha}} \leq C_{\alpha, L} \varepsilon^{3-\delta}\]  
(5.28)

**Proof**

The hypotheses for \(g, \mu\) also hold for \(sg, s^2\mu\) and for \([1-s]g, [1-s^2]\mu\). Since \(V + p(s) = V_1(s) + V_2(s)\) with

\[V_1(s) = V(\Delta, \phi + \zeta, C, sg, s^2\mu), \quad V_2(s) = V(\Delta, \phi, C_{L-1}, [1 - s]g, [1 - s^2]\mu)\]

\[\|r_1(\Delta, \zeta, \phi)\|_h \leq \frac{1}{2} \int_0^1 ds (1 - s)^2 e^{-V_1(s)} \|e^{-V_2(s)}\|_h \left( \|p'(s)\|_h^3 + 6\|p'(s)\|_h\|p_\mu\|_h \right)\]
By Lemma 5.5, the exponential terms are bounded by $4 \exp(-(g[1 - s]/4) \int \phi^4)$. Use Lemma 5.6 choosing for the regulators the constants $\frac{1}{4}$ and $\frac{1}{2}$ to obtain

$$\|r_1(\Delta, \zeta, \phi)\|_h \leq C_{\alpha} \varepsilon^{3/4} \hat{G}_{\alpha, \alpha} \int_0^1 ds (1 - s)^2 (1 - s)^{-\frac{g}{4}} e^{-\frac{g}{4} (1 - s)^3} \int_\Delta d^3x \phi^4(x)$$

Choose $0 < \xi < \frac{1}{12}$ to get (5.27). Equation (5.28) is proved similarly.

**Lemma 5.8**

Consider $P(\lambda)$ given in (4.9). Then for $C$ independent of $\varepsilon$, but dependent on $L$,

$$\|P\|_{h, \hat{G}_{\alpha, \alpha}, A} \leq C_L |\lambda\varepsilon|^{1/4} \quad \text{for } |\lambda\varepsilon^{1/4}| \leq 1 \quad (5.29)$$

$$\|P\|_{h, \hat{G}_{\alpha, \alpha}, A} \leq C_L |\lambda\varepsilon|^{1-\delta/2} \quad \text{for } |\lambda\varepsilon^{1-\delta/2}| \leq 1 \quad (5.30)$$

**Proof**

(5.29) and (5.30) are immediate from Lemma 5.7, noting that $|\lambda\varepsilon^{1-\delta/2}|$ in (5.30) is the largest of the several combinations of $\lambda, \varepsilon$ that arise. It comes from the term involving $\mu\lambda^2$.

**Estimates for $Qe^{-V}$**

We now turn to the estimate of $Qe^{-V}$. From (4.4)

$$Q(C, w, g) = g^2 \sum_{m=1}^{3} a_m Q(m, m)(C, w^{(4-m)}) \quad (5.31)$$

where the $a_m$ are numerical coefficients and the $Q(m, m)$ are given in (4.6). Under an iteration, see Proposition 4.1, we have

$$u^{(p)} \to u^{(p)'} = u^{(p)} + w^{(p)}_L$$

where $p = 1, 2, 3$ and the $u^{(p)}$ are given in Proposition 4.1. Starting with $u_0^{(p)} = 0$ we get after $n$ iterations

$$w_n^{(p)} = \sum_{j=0}^{n-1} u_j^{(p)} \quad (5.32)$$

We need to first estimate $w_n^{(p)}$ and the limit $\lim_{n \to \infty} w_n^{(p)}$ under appropriate norms.

We consider Banach spaces $W_p$ of measurable functions with norms $\| \cdot \|_p, p = 1, 2, 3$, defined as follows

$$\|f\|_p = \text{ess. sup}_x \left(\frac{6p+1}{p} |x| \right)$$

(5.33)

We define the Banach space $W_1 \times W_2 \times W_3$ consisting of vectors $w$ with the norm

$$\|w\| = \sup_p \|w_p\|_p$$

Then we have

**Lemma 5.9**

For $L$ sufficiently large and $\varepsilon > 0$ sufficiently small there exists a constant $c = O(1)$ such that
\[ \|w_n\| \leq c/4 \ \forall n \]

and

\[ \|w_{n+1} - w_n\| \leq c/8 \ L^{-n/4} \]

so that \( w_n \to w_* \) in the norm \( \| \cdot \| \), and

\[ \|w_*\| \leq c/4 \]

**Proof**

Let us note first some weak uniform (in \( L \)) bounds. Recall that \( |\phi| = \frac{3}{p} - \varepsilon \).

\[ |\Gamma_L(x)| \leq O(1)|x|^{-2|\phi|} \quad (5.34) \]

\[ |C(x)| \leq O(1) \quad (5.35) \]

To see this observe that from the definition of \( \Gamma \) (see Section 1)

\[ |\Gamma_L(x)| \leq \int_{|x|}^{\infty} \frac{dl}{l} l^{-2|\phi|} \left| u \left( \frac{x}{l} \right) \right| \]

Let \( x \neq 0 \). Then, using support properties of \( u \),

\[ |\Gamma_L(x)| \leq \int_{|x|}^{\infty} \frac{dl}{l} l^{-2|\phi|} \left| u \left( \frac{x}{l} \right) \right| = \frac{2}{3 - \varepsilon} \frac{1}{|x|^{2|\phi|}} \| u \| \infty \]

which proves (5.34). To prove (5.35) recall

\[ |C(x)| \leq \int_{1}^{\infty} \frac{dl}{l} l^{-2|\phi|} \left| u \left( \frac{x}{l} \right) \right| \leq O(1) \| u \| \infty \]

which proves (5.35).

For \( p = 1, 2, 3 \),

\[ |v^{(p)}(x)| = |C_L^p(x) - C^p(x)| \]

\[ \leq p \sup_{1 \leq q \leq p} |\Gamma_L(x)|^q \left| C(x) \right|^{p-q} \leq O(1) \begin{cases} |x|^{-p|\phi|} & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| \leq 1 \end{cases} \]

where we exploited \( C_L = \Gamma_L + C \) and \( \Gamma_L(x) = 0 \) if \( |x| \geq 1 \) and (5.34) and (5.35). Hence

\[ \|v^{(p)}\|_p \leq O(1) \]

and by scaling \( x = L^j x' \)

\[ \|v^{(p)}_L\|_p \leq L^{jp^2|\phi|} L^{-j\left(\frac{2p+1}{2} - j|\phi|\right)} \|v^{(p)}\|_p \leq c_p/8 \ L^{-j/4} \quad (5.36) \]

Define the constant \( c = \sup_p c_p \). Then the above estimates lead immediately to the proof of Lemma 5.9, because of (5.32).

**Lemma 5.10**
\(Q(X, \phi)e^{-V(X, \phi)}\) satisfies the bounds
\[
\|Qe^{-V}\|_{h, G, \lambda, \alpha_p} \leq C_{\rho, \varepsilon}^{1/2}
\] (5.37)
\[
|Qe^{-V}|_{h, \lambda, \alpha_p} \leq C_{\rho, \varepsilon}^2
\] (5.38)

Proof
\[
Qe^{-V} = g^2 \sum_{m=1}^{3} a_m Q^{(m,m)}(C, u^{(4-m)})e^{-V}
\]
\[
\|Q(X, \phi)e^{-V(X, \phi)}\|_h \leq g^2 \sum_{m=1}^{3} |a_m| \left\|Q^{(m,m)}(C, u^{(4-m)}, X, \phi)\right\|_h \left\|e^{-V(X, \phi)}\right\|_h
\] (5.39)

Here \(X\) is a small set, because of the support property of \(Q\). The last factor in the sum will be estimated by Lemma 5.5. From (4.6)
\[
Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)}) = \frac{1}{2} \int_{\tilde{X}} d^3x d^3y : \phi^3(x)\phi^3(y) : C \ w^{(1)}(x - y)
\]

Undo the Wick ordering, which produces lower order terms with finite coefficients.
Apply \(h^k D^k\) with \(h = c\varepsilon^{-1/4}\) and use Lemmas 5.1, 5.9 and \(g = O(\varepsilon)\). We get
\[
\frac{h^k}{k!} \left\|D^k Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)})\right\| \leq O(1)g^{-3/2} w^{(1)}_1 \int_{\tilde{X}} d^3x d^3y \frac{1}{|x - y|^{13/4}} G_{\kappa/4}(X, \phi)e^{g/4} \int_x d^3x \phi^4(x)
\] (5.40)

Next turn to \(Q^{(m,m)}\), \(m = 1, 2\), again in (4.6)
\[
Q^{(m,m)}(\tilde{X}, \phi; C, u^{(4-m)}) = -\frac{m^2}{2} \sum_{\mu, \nu=1}^{3} \int_0^1 ds_1 ds_2 \int_{\tilde{X}} d^3x d^3y
\]
\[
(x - y)_\mu(x - y)_\nu u^{(4-m)}(x - y) : (\phi^{m-1}\nabla_\mu \phi)(y + s_1(x - y))(\phi^{m-1}\nabla_\nu \phi)(y + s_2(x - y)) : C
\]

We consider in turn the cases \(m = 2, 1\). We apply \(h^k D^k\) with \(h = a\varepsilon^{-1/4}\) and use Lemmas 5.1-5.9 and the Sobolev inequality to dominate the \(\nabla \phi\) pointwise by the large field regulators.
\[
\frac{h^k}{k!} \left\|D^k Q^{(2,2)}(\tilde{X}, \phi; C, w^{(2)})\right\| \leq O(1)g^{-1/2} w^{(2)}_2 \int_{\tilde{X}} d^3x d^3y \frac{|(x - y)_\mu|(x - y)_\nu|}{|x - y|^{13/4}} G_{\kappa/4}(X, \phi)e^{g/4} \int_x d^3x \phi^4(x)
\]
\[
\leq c g^{-1/2} G_{\kappa/4}(X, \phi)e^{g/4} \int_x d^3x \phi^4(x)
\] (5.41)

We can estimate in the same way the case \(m = 1\). Note that
\[
\int_{\tilde{X}} d^3x d^3y \frac{|(x - y)_\mu|(x - y)_\nu|}{|x - y|^{19/4}} \leq O(1)
\]

We have
\[
\frac{h^k}{k!} \left\|D^k Q^{(1,1)}(\tilde{X}, \phi; C, w^{(3)})\right\| \leq c G_{\kappa/4}(X, \phi)e^{g/4} \int_x d^3x \phi^4(x)
\] (5.42)
From (5.40), (5.41) and (5.42) we get
\[ g^2 \left\| Q^{(m,m)}(C, w^{(4-m)}) \right\|_{h_\epsilon} \leq c \varepsilon^{1/2} G_{\kappa/4}(X, \phi) e^{g/4} \int_X d^3x \phi^4(x) \tag{5.43} \]

We estimate the r.h.s. of (5.39) using (5.43) and Lemma 5.5.
\[ \left\| Q(X)e^{-V(X)} \right\|_{h, G_{\kappa}} \leq O(1) \varepsilon^{1/2} \tag{5.44} \]

Since the \( Q \) are supported on small sets, we get from (5.44)
\[ \left\| Qe^{-V} \right\|_{h, G_{\kappa}, A_p} \leq C_p \varepsilon^{1/2} \tag{5.45} \]

This proves (5.37). To prove (5.38) we estimate the r.h.s. of (5.39) at \( \phi = 0 \) after undoing the Wick ordering, set \( h = h_\epsilon \), using Lemma 5.1. Lemma 5.10 has been proved.

It is useful to state the following lemma for \( Qe^{-V} \) treated as a function of \( \zeta, \phi \)

**Lemma 5.11**
\[ \left\| Qe^{-V} \right\|_{h, \hat{G}_{\kappa}, A_p} \leq C_p \varepsilon^{1/2} \tag{5.46} \]
\[ \left\| Qe^{-V} \right\|_{h_\epsilon, \hat{G}_{\kappa}, A_p} \leq C_p \varepsilon^2 \tag{5.47} \]

**Proof**
(5.46) follows from (5.37), since \( \hat{G}_{\kappa, \alpha} \geq G_{\kappa} \). For (5.47) we bound derivatives with respect to \( \phi \) of \( (Qe^{-V})(\zeta + \phi, X) \) at \( \phi = 0 \)
\[ \left\| Q(X, \zeta)e^{-V(X, \zeta)} \right\|_{h_\epsilon} \leq g^2 O(1) \sum_{m=1}^3 \left\| Q^{(m,m)}(C, w^{(4-m)}, X, \zeta, X) \right\|_{h_\epsilon} \hat{G}_{\alpha/2}(X, \zeta) \]

since
\[ \left\| e^{-V(X, \zeta)} \right\|_{h_\epsilon} \leq 2 |X| \]

and \( X \) is a small set. Now we proceed as in Lemma 5.10, but dominate the \( \zeta \) using Lemma 5.2, to complete the proof.

We now control the perturbative flow coefficients \( a, b \) given in (4.34).

**Lemma 5.12**
\[ a = O(\log L), \quad b = O(L^{3/2}), \quad \int d^3x \nu^{(4)}(x) = O(L^3) \tag{5.48} \]
for \( L \to \infty \) and \( \epsilon = o(L) \).

**Proof**
From (4.18), for \( p = 2, 3 \), using \( C_L = \Gamma_L + C \)
\[ v^{(p)} = C_L^p - C^p = \Gamma_L^{p-1} + p \Gamma_L^{p-2} C + \delta_{p,3} 3 C^2 \]
with pointwise multiplication. The common factor of \( \Gamma_L \) implies that \( v^{(p)}(x) \) has support in the unit ball. This, together with (5.35), implies that the leading divergence is in the contribution from \( \Gamma_L^p \). Therefore is sufficient to calculate

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\[ I^{(p)} = \int d^3 x (\Gamma_L(x))^p \]

Recall that:
\[ \Gamma_L(x) = \int_{L^{-1}}^1 \frac{dl}{l} (l^{-2|\phi|} u \left( \frac{x}{l} \right)) \quad (5.50) \]

Let \( p = 2 \) so that
\[ I^{(2)} = 2! \int_{l_1 l_2 \leq l \leq \frac{1}{l}} \frac{dl_1 dl_2}{l_1 l_2} (l_1 l_2)^{-2|\phi|} \int d^3 x u \left( \frac{x}{l_1} \right) u \left( \frac{x}{l_2} \right) \]

Suppose that \( L = 2^{N-1} \) for some integer \( N \) and break up the range of integration into disjoint regions
\[ R_n = \{ 2^{-n-1} < l_1 \leq l_2 \leq 2^{-n} \} \quad n = 0, \ldots, N - 1 \]
so that
\[ I^{(p)} = \sum_{n=0}^{N-1} I^{(p)}(R_n) \]

where
\[ I^{(p)}(R) = 2! \int_R \frac{dl_1 dl_2}{l_1 l_2} (l_1 l_2)^{-2|\phi|} \int d^3 x u \left( \frac{x}{l_1} \right) u \left( \frac{x}{l_2} \right) \]

By scaling \( l_1, l_2, x \) by \( 2^n \), \( I^{(p)}(R_n) = 2^{2|\phi|-3} I(R_0) \). Therefore
\[ I^{(p)} = I^{(p)}(R_0) \sum_{n=0}^{N-1} 2^{(2|\phi|-3)n} = O(1) \sum_{n=0}^{N-1} 2^{(2|\phi|-3)n} = O(1) \frac{1 - L^{2|\phi|-3}}{1 - 2^{2|\phi|-3}} \]

This is also true for \( p = 3, 4 \), by the same argument with the appropriate multiple integral expression for \( I^{(p)}(R_0) \). For \( p = 2, 2p|\phi| - 3 = -4\varepsilon \to 0 \) and therefore \( I^{(p)} = O(\log L) \) by L'Hospital's rule. For \( p = 3, 2p|\phi| - 3 \to 3/2 \) and therefore \( I^{(p)} = O(L^{3/2}) \), etc. (5.48) is proved.

**Lemma 5.13**
\[ \left\| Q(e^{-\hat{V}} - e^{-\tilde{V}}) \right\|_{h, G_{x,\alpha}, \mathcal{A}_p} \leq C_p e^{3/4} \quad (5.51) \]
\[ \left\| Q(e^{-\hat{V}} - e^{-\tilde{V}}) \right\|_{h, G_{x,\alpha}, \mathcal{A}_p} \leq C_p e^{3} \quad (5.52) \]

**Proof**
\( Q(X) \) is supported on connected polymers with size \( |X| \leq 2 \). Without loss of generality we do the estimates for \( |X| = 1 \). We can write:
\[ Q(\Delta, \zeta + \phi)(e^{-V(\Delta, \zeta + \phi)} - e^{-\tilde{V}(\Delta, \phi)}) = Q(\Delta, \zeta + \phi)e^{-\frac{1}{2}V(\Delta, \zeta + \phi)} \int_0^{1/2} ds p(\Delta, \zeta, \phi) e^{-\left(\frac{1}{2} - s\right)V(\Delta, \zeta + \phi) - s\tilde{V}(\Delta, \phi) +} \]
\[ + \int_{1/2}^{1} ds Q(\Delta, \zeta + \phi)p(\Delta, \zeta, \phi)e^{-(1-s)V(\Delta, \zeta + \phi) - s\tilde{V}(\Delta, \phi)} \]
whence
\[
\|Q(\Delta, \zeta + \phi)(e^{-V(\Delta, \zeta + \phi)} - e^{-\bar{V}(\Delta, \phi)})\|_h \leq O(1)(A + B)
\] (5.53)
where
\[
A = \|Q(\Delta, \zeta + \phi)e^{-\frac{1}{2}V(\Delta, \zeta + \phi)}\|_h \int_0^{1/2} ds \|p(\Delta, \zeta, \phi)\|_h e^{-s\bar{V}(\Delta, \phi)}_h
\]
and
\[
B = \int_0^{1/2} ds \|Q(\Delta, \zeta + \phi)\|_h \|p(\Delta, \zeta, \phi)\|_h e^{-s\bar{V}(\Delta, \phi)}_h
\]
To estimate \( A \), use Lemma 5.11, still true for \( V \) replaced by \( \frac{1}{2}V \), Lemma 5.6, (5.21) with \( 1 - s \) replaced by \( s \), \( \delta = \frac{1}{8} \), and lemma 5.5 with \( g \) replaced by \( s g \). Observe that \( s^{-3/4} \) is integrable. We get
\[
A \leq C_\alpha \varepsilon^{3/4} \hat{G}_{\kappa, \alpha}(\Delta, \zeta, \phi)
\] (5.54)
To estimate \( B \) we use again Lemma 5.5 with \( g \) replaced by \( s g \). We estimate \( \|p\|_h \) using Lemma 5.6, (5.21) with \( 1 - s \) replaced by \( s \), \( \delta = \frac{1}{8} \). We estimate \( \|Q(\Delta, \zeta + \phi)\|_h \) as in the proof of Lemma 5.10 with the following change. Expand out polynomials in \( \zeta + \phi \), and dominate the \( \zeta \) using Lemma 5.3. We then get a modified estimate (5.43) replacing \( g \) with \( \frac{2}{3}g \), and \( G_{\kappa}(X, \phi) \) with \( \hat{G}_{\kappa, \alpha}(X, \zeta, \phi) \) together with an overall multiplicative factor \( C_\alpha s^{-3/2} \) which is integrable in the range under consideration. Then we use lemma 5.5 with \( g \) replaced by \( s g \). Putting all this together we get
\[
B \leq C_\alpha \varepsilon^{3/4} \hat{G}_{\kappa, \alpha}(\Delta, \zeta, \phi)
\] (5.55)
Using (5.54) and (5.55) we get for (5.53)
\[
\|Q(\Delta, \zeta + \phi)(e^{-V(\Delta, \zeta + \phi)} - e^{-\bar{V}(\Delta, \phi)})\|_h \leq C_\alpha \varepsilon^{3/4} \hat{G}_{\kappa, \alpha}(\Delta, \zeta, \phi)
\] (5.56)
It is easy to show that the same estimate holds if \( |X| = 2 \), connected. Hence
\[
\|Q(e^{-V} - e^{-\bar{V}})\|_{h, \hat{G}_{\kappa, \alpha}, \mathcal{A}} \leq C_{\alpha, \lambda} \varepsilon^{3/4}
\]
This proves (5.51). The proof of (5.52) is similar except that we have only fluctuation fields \( \zeta \) to dominate using Lemma 5.3. This proves Lemma 5.13.

**Lemma 5.14**

\( K(\lambda) \) given by (4.10), satisfies the bounds
\[
\|K(\lambda)\|_{h, \hat{G}_{\kappa, \alpha}, \mathcal{A}} \leq C_{\alpha} |\lambda \varepsilon^{1/4} - \eta/3|^2 \quad \text{for } |\lambda \varepsilon^{1/4} - \eta/3| \leq 1
\] (5.57)
\[
\|K(\lambda)\|_{h, \hat{G}_{\kappa, \alpha}, \mathcal{A}} \leq C_{\alpha, \lambda} |\lambda \varepsilon^{11/12} - \eta/3|^2 \quad \text{for } |\lambda \varepsilon^{11/12} - \eta/3| \leq 1
\] (5.58)

**Proof**

This follows from lemmas 5.11 and 5.13 and the hypotheses (5.2) and (5.3) on \( R \). The \( \lambda \varepsilon^{1/4} - \eta/3 \) and \( \lambda \varepsilon^{11/12} - \eta/3 \) originate from \( \lambda^3 R \) contributions.

**Lemma 5.15**

For any polymer activity \( K \):
\[
\|K(X, \zeta)\|_{h_*} \leq O(1) \tilde{G}_{\alpha, \kappa}(X, \zeta) \left[ \|K(X)\|_{h_*} + h^{-n_0} h_*^{n_0} \|K(X)\|_{h, G_\alpha} \right] \tag{5.59}
\]
\[
\|K(Y, \phi)\|_h \leq O(1) e^{\gamma g} \int_{x \in X} d^p y \|\phi(y)\|_s G_\kappa(Z, \phi) \left[ (K(Y)|_h + L^{-n_0}[\phi] \|K(Y)\|_{L[dG h, G_\alpha]} \right] \tag{5.60}
\]
\[
|K^2(X)|_{h_*} \leq O(1)2^{1|x|} \left[ |K(X)|_{h_*} + h^{-n_0} h_*^{n_0} \|K(X)\|_{h, G_\alpha} \right] \tag{5.61}
\]

where \( \tilde{G}_{\alpha, \kappa} \) is as defined in (5.12), and \( n_0 \) is the maximum number of derivatives appearing in the definition of Kernel and \( h \) norms. In (5.60), \( Y, Z, \gamma \) are as described in Lemma 5.1.

Recall that \( n_0 = 9 \). The superscript \( \sharp \) stands for \( d\mu_r(\zeta) \) integration. \( \alpha \) is chosen as in Lemma 5.3. Note that we have then

\[
\alpha = \alpha(L) = \frac{\kappa'}{h_*^2} \tag{5.62}
\]

**Proof**

First observe:
\[
\frac{1}{n_0!}\|\left(D^{n_0}K\right)(X, \zeta)\| \leq h^{-n_0} \tilde{G}_{\alpha, \kappa}(X, \zeta) \|K(X)\|_{h, G_\alpha} \tag{5.63}
\]

since \( G_\kappa(X, \zeta) \leq \tilde{G}_{\alpha, \kappa}(X, \zeta) \).

Now let \( n < n_0 \). We expand in Taylor series with remainder
\[
\left(D^nK\right)(X, \zeta; f^{\times n}) = \sum_{m=0}^{n_0-n-1} \frac{1}{m!} \left(D^{m+n} K\right)(X, 0; f^{\times n}, \zeta^{\times m}) + \]
\[
+ \frac{1}{(n_0-n-1)!} \int_0^1 ds (1-s)^{n_0-n-1} \left(D^{n_0} K\right)(X, s\zeta; f^{\times n}, \zeta^{\times n_0-n})
\]

whence
\[
\|\left(D^nK\right)(X, \zeta)\| \leq \sum_{m=0}^{n_0-n-1} \frac{1}{m!} \|\zeta\|_{C^m(X)} \|\left(D^{m+n} K\right)(X, 0)\| +
\]
\[
+ \frac{1}{(n_0-n-1)!} \int_0^1 ds (1-s)^{n_0-n-1} \|\zeta\|_{C^m(X)} \|G_\kappa(X, s\zeta)\| \|\left(D^{n_0} K\right)(X)\|_{G_\alpha}
\]

By Lemma 5.2, with \( \zeta \) replaced by \( \sqrt{1-s^2} \zeta \), and (5.62),
\[
h_*^{p} \|\zeta\|_{C^2(X)}^p \leq O(1) \tilde{G}_{\alpha, \kappa}(X, \sqrt{1-s^2} \zeta) \frac{1}{(1-s^2)^{p/2}}
\]

where \( O(1) \) depends on \( \kappa' \) and \( p \) and \( 0 \leq s < 1 \). With \( s = 0 \) this bound is applied to the terms in the sum over \( m \). For the Taylor remainder term we note that \( (1-s)^{(n_0-n)/2-1} \) is integrable since \( n_0 > n \). Hence:
\[
\frac{n_0!}{n!} \|\left(D^nK\right)(X, \zeta)\| \leq O(1) \tilde{G}_{\alpha, \kappa}(X, \zeta) \left[ \sum_{m=0}^{n_0-n-1} \frac{(n+m)!}{m!} \|K(X)\|_{h_*} +
\]
\[
+ \frac{n_0! h^{-n_0} h_*^{n_0}}{n!(n_0-n-1)!} \|K(X)\|_{h, G_\alpha} \right] \leq
\]
\[
\leq O(1) \tilde{G}_{\alpha, \kappa}(X, \zeta) \left[ |K(X)|_{h_*} + h^{-n_0} h_*^{n_0} \|K(X)\|_{h, G_\alpha} \right] \tag{5.64}
\]
Summing (5.64) over $0 \leq n \leq n_0 - 1$ and adding (5.63) after multiplication by $h_n^{\alpha_0}$ proves (5.59). Equation (5.61) follows from (5.59) using Lemma 5.3. Equation (5.60) is proved in the same way as (5.59) with Lemma 5.1 in the place of Lemma 5.2, $h_*, h$ replaced by $h, Lh$. This proves Lemma 5.15.

The next lemma gives bounds on $S(\lambda, K) = R \circ B(\lambda, K)$ given in (4.11).

**Lemma 5.16**
For any $q > 0$, there exists $c_L$ such that, for $L$ large,

$$\|S(\lambda, K)^2\|_{h, G_{\alpha, A_p}} \leq q \quad \text{when } |\lambda\varepsilon^{1/4-\eta/3}| \leq c_L$$

(5.65)

$$|S(\lambda, K)^2|_{h, A_p} \leq q \quad \text{when } |\lambda\varepsilon^{11/12-\eta/3}| \leq c_L$$

(5.66)

When $R = 0$ we may set $\eta = 0$ in (5.65) and replace $\lambda\varepsilon^{11/12-\eta/3}$ by $\lambda\varepsilon^{1-\delta/2}$ in (5.66).

**Proof**
From the definition of reblocking (see (4.11)) and subsequent rescaling

$$\|S(\lambda, K)(Z, \phi, \zeta)\|_h \leq \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_j), (\Delta_i) \rightarrow LZ} \|e^{-V(X_0, \phi_{L-1})}\|_h \prod_{j=1}^N \|K(\lambda, X_j, \phi_{L-1}, \zeta_{L-1})\|_h \prod_{i=1}^M \|P(\lambda, \Delta_i, \phi_{L-1}, \zeta_{L-1})\|_h$$

(5.67)

We rewrite $V(X_0, \phi_{L-1}) = \tilde{V}_L(L^{-1}X_0, \phi)$ and apply Lemma 5.5 (the rewriting gives a better bound by saving factors of 2),

$$\|S(\lambda, K)(Z, \phi, \zeta)\|_h \leq 2|Z| \tilde{G}_{\kappa, \alpha}(LZ, \zeta_{L-1}, \phi_{L-1}) \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_j), (\Delta_i) \rightarrow LZ} \prod_{j=1}^N \|K(\lambda, X_j)\|_{h, \tilde{G}_{\kappa, \alpha}} \prod_{i=1}^M \|P(\lambda, \Delta_i)\|_{h, \tilde{G}_{\kappa, \alpha}}$$

(5.68)

where

$$h_L = L^{-(3-\varepsilon)/4}$$

Using Lemma 5.4 and $G_{3\kappa}(LZ, \phi_{L-1}) \leq G_\kappa(Z, \phi)$ for $L$ large,

$$\|S(\lambda, K)^2(Z)\|_{h, G_{\alpha, A_p}} \leq 2|Z| \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_j), (\Delta_i) \rightarrow LZ} \prod_{j=1}^N \|K(\lambda, X_j)\|_{h, \tilde{G}_{\kappa, \alpha}} \prod_{i=1}^M \|P(\lambda, \Delta_i)\|_{h, \tilde{G}_{\kappa, \alpha}}$$

$$\|S(\lambda, K)^2\|_{h, G_{\alpha, A_p}} \leq \sup_{\Delta} \sum_{X: L^{-1}X \cap \Delta \neq \emptyset} A_{2+p}(L^{-1}X) \|\tilde{K}_k(X)\|_{h, \tilde{G}_{\kappa, \alpha}}$$

Multiply by $A_{2+p}(Z)$ on the left and, using

$$A_{2+p}(L^{-1}X) \leq O(1)A_{-3}(X)$$

(Lemma 1 in [BDH-est]), by

$$O(1) \prod_{j=1}^N A_{-3}(X_j) \prod_{i=1}^M A_{-3}(\Delta_i)$$
on the right. Fix any $\Delta$ and sum both sides over $Z \ni \Delta$. The spanning tree argument of Lemma 7.1 of [BY]
controls the sums over $N, M, Z, (X_j), (\Delta_i) \to LZ$ with the result
\[ \|S(\lambda, K)^2\|_{h, G_{\alpha} A} \leq O(1) \sum_{N \geq 1} O(1)^N L^{2N} \left( \|K(\lambda)\|_{h, \hat{G}_{\alpha} A} + \|P(\lambda)\|_{h, \hat{G}_{\alpha} A} \right)^N \]

The proof of (5.65) is completed by Lemmas 5.8 and 5.14. When $R = 0$ we can replace Lemma 5.14 by
Lemmas 5.13 and 5.8 which gives the result with $\eta = 0$.

For (5.66) we use (5.67) with $\phi = 0$ and replace $h$ by $h_\ast$. We estimate the $\zeta$ dependence by the regulator
$\hat{G}_{\kappa, \alpha}$ introduced in (5.12), to obtain, in the place of (5.68),
\[ \|S(\lambda, K)(Z, 0, \zeta)\|_{h_\ast} \leq 2^{\|Z\|} \hat{G}_{\kappa, \alpha}(LZ, \zeta_L, \zeta_{L-1}) \sum_{N+M \geq 1} \frac{1}{N! M!} \sum_{(X_j), (\Delta_i) \to LZ} \prod_{j=1}^N \|K(\lambda, X_j)\|_{h_\ast, \hat{G}_{\kappa, \alpha}} \prod_{i=1}^M \|P(\lambda, \Delta_i)\|_{h_\ast, \hat{G}_{\kappa, \alpha}} \]

Then Lemma 5.15 is used to estimate the $\zeta$ in $|S(\lambda, K)^2(Z)|_{h_\ast}$, and the rest is as before. End of proof of
Lemma 5.16.

Estimates on relevant parts and flow coefficients from the remainder

Let $(\hat{\alpha}_P)$ be the coefficients $(\hat{\alpha}_0, \hat{\alpha}_{2,0}, \hat{\alpha}_{2,1}, \hat{\alpha}_4)$ defined in (4.43) and (4.55). The flow coefficients $\xi_R, \rho_R$ and
$\beta_0$ are given in (4.54).

\textbf{Lemma 5.17}

\[ \|R^2\|_{h, G_{\alpha} A} \leq \epsilon^{3/4 - \eta} \]
\[ |R^2|_{h_\ast A} \leq O(1) \epsilon^{11/4 - \eta} \]
\[ |\hat{\alpha}_P|_{A} \leq O(1) \epsilon^{11/4 - \eta} \]
\[ |\beta_0| \leq C \epsilon^{3/4 - \eta} \]
\[ |\xi_R| \leq C \epsilon^{3/4 - \eta} \]
\[ |\rho_R| \leq C \epsilon^{3/4 - \eta} \]

\textbf{Proof}

Recall that $\hat{\alpha}_P(X)$, are supported on small sets. (5.70) follows from the hypothesis (5.2) and the stability
of the large field regulator $G_{\kappa}$. (5.71) follow from the hypotheses (5.3) and lemma 5.16 with $n_0 = 9$ and $\epsilon$
sufficiently small depending on $L$. (5.72) follows from (5.71), and (4.55). In fact the dominant contribution
comes by setting $V = 0$ because the difference gives additional powers of $\epsilon$. Then we have
\[ \|\hat{\alpha}_P\|_{A} \leq O(1)n(\beta)|h_\ast^{-n(P)}|1_S R^2|_{h_\ast A} \]
where $n(\beta)$ is the number of fields in the monomial $P$ and $1_S$ is the indicator function on small sets. Now
use (5.71) to get (5.72). (5.73), (5.74), (5.75) follow from (5.72), and the definitions (4.54), (4.48) and Wick
coefficients are $O(1)$. Lemma 5.17 has been proved.

\textbf{Corollary 5.18}

For $\epsilon$ sufficiently small
\[ |g' - g| \leq \epsilon^{3/2}, \quad |\mu'| \leq C \epsilon^{2 - \delta} \]
Proof
It is easy to check from the first of the flow equations (4.17) and the definition of $\bar{g}$ that

$$g' - \bar{g} = (g - \bar{g})(1 - L^{2\varepsilon}ag) + \xi_R$$

and

$$g' - g = (g - \bar{g})(-L^{2\varepsilon}ag) + \xi_R$$

with $a = O(\log L) > 0$ and the initial Hypothesis implies $g = O(\varepsilon)$ so that for $\varepsilon$ sufficiently small $0 < 1 - L^{2\varepsilon}ag < 1 - ag$. The domain of $g$ in the Hypothesis and the bound (5.95) of Lemma 5.17 imply, for $\varepsilon$ sufficiently small, that $\xi_R$ is smaller than the other terms, which gives the two bounds concerning $g'$. The bound on $\mu'$ follows from the second of the flow equations (4.17), the hypothesis on $\mu$, and the bound (5.96) on $\rho_R$. The corollary has been proved.

The following lemma proves the stability of $V$ with respect to perturbations by relevant parts $F$ in our model. We state it in the form enunciated as the stability hypothesis in Section 4.2, (103), [BDH-est]. This lemma will be very useful for the control of the extraction formula, as explained in the reference above.

Recall from (4.13) that $F(\lambda) = \lambda^2 F_Q + \lambda^3 F_R$ and from (3.9) that (each part of) $F$ decomposes: $F(X) = \sum_{\Delta \subset X} F(X, \Delta)$.

Lemma 5.19
For any $R > 0$ and $\xi := R \max(\|\lambda^2|\varepsilon|\lambda^3|\varepsilon^{7/4-\eta})$ sufficiently small,

$$\| e^{-\tilde{V}_L(\Delta)} - \sum_{X \supset \Delta} z(X) F(\lambda, X, \Delta) \|_{h, G, \alpha} \leq 2^2$$

(5.78)

where $z(X)$ are complex parameters with $|z(X)| \leq R$.

Proof
First note that Lemma 5.5 still holds if we replace $\tilde{V}$ by $\tilde{V}_L$ provided $\varepsilon$ is sufficiently small. We then have

$$\| e^{-\tilde{V}_L(\Delta)} - \sum_{X \supset \Delta} z(X) F(\lambda, X, \Delta) \|_h \leq 2 e^{-g_L/4} \int_\Delta d^3x f^\sigma(x) + \sum_{\Delta \supset \Delta} R \| F(\lambda, X, \Delta) \|_h$$

(5.79)

Recall that the $F(X, \Delta)$ are supported on small sets $X$. The proof now follows easily from the following

Claim: For $\varepsilon$ sufficiently small

$$\| F(\lambda, X, \Delta) \|_h \leq C \varepsilon \xi \left( \int_\Delta d^3x \varepsilon \phi^4(x) + \varepsilon^{1/2} \| \phi \|_{\Delta, 1, \sigma}^2 + 1 \right)$$

(5.80)

where $\| \phi \|_{\Delta, 1, \sigma}^2$ is the norm defined in (2.2).

Proof of the Claim: We have

$$\| F(\lambda, X, \Delta) \|_h \leq |\lambda|^2 \| F_Q(X, \Delta) \|_h + |\lambda|^3 \| F_R(X, \Delta) \|_h$$

(5.81)

From (4.26)-(4.31)

$$\| F_Q(X, \Delta) \|_h \leq O(1) \varepsilon^2 \left( \sum_{m=2,4} \| \int_\Delta d^3x : \phi^m : C (x) \|_h \sup_{x \in \Delta} | f^{(m)}_1 (x, X, \Delta) | + | Q^{(0,0)} (X, C, \phi^4) | \right)$$

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Undoing the Wick ordering produces lower order terms with $O(1)$ coefficients. Now it is easy to see that for $m = 2, 4$

$$
\varepsilon \| \int_{\Delta} d^3 x : \phi^m \phi^4 (x) \|_h \leq O(1) \varepsilon \int_{\Delta} d^3 x \phi^4 (x) + O(1)
$$

By Lemma 5.12,

$$\sup_{x \in \Delta} | f_{1,\tilde{Q}}^{(m)} (x, X, \Delta) | \leq C_L, \quad | \tilde{Q}^{(0,0)} (X, C, v^{(4)}) | \leq C_L$$

Therefore

$$|\lambda^2 R \| F_Q (X, \Delta) \|_h \leq C_L R |\lambda|^2 \varepsilon \left( \varepsilon \int_{\Delta} d^3 x \phi^4 (x) + 1 \right) \quad (5.82)$$

Next consider $F_R$, supported on small sets, defined in (4.43), (4.45). Recall (4.49),

$$F_R (X, \Delta, \phi) = \sum_P \int_{\Delta} d^3 x \alpha_P (X, \Delta, x) P(\phi(x), \partial \phi(x))$$

By Lemma 5.17 and (4.48)

$$|\alpha_P (X, \Delta, x) | \leq C_L \varepsilon^{11/4 - \eta}$$

so that

$$|\lambda^3 |z(X)| \| F_R (X, \Delta) \|_h \leq C_L R |\lambda|^3 \varepsilon^{11/4 - \eta} \sum_P \int_{\Delta} d^3 x \| P(\phi(x), \partial \phi(x)) \|_h$$

$$\leq C_L R |\lambda|^3 \varepsilon^{7/4 - \eta} \left( \varepsilon \int_{\Delta} d^3 x \phi^4 (x) + \varepsilon^{1/2} \| \phi \|^2_{\Delta, \sigma} + 1 \right)$$

The claim follows by combining this with (5.82).

Note that in the above inequality the Sobolev norm arises only when estimating the $j$-term corresponding to $\phi \partial \mu \phi$. For this we can bound $|\phi \nabla \phi| \leq 1/2(|\phi|^2 + |\nabla \phi|^2)$ and then use the Sobolev embedding inequality. End of proof of Lemma.

**Lemma 5.20**

For any $R > 0$ and $\xi : = R \max (|\lambda^2 \varepsilon^2, |\lambda|^3 \varepsilon^{11/4 - \eta})$ sufficiently small,

$$|e^{-\tilde{V}_\lambda (\Delta)} - \sum_X \phi^4 (x) F(\lambda, X, \Delta) |_{h_*} \leq 2^2 \quad (5.83)$$

where $z(X)$ are complex parameters with $|z(X)| \leq R$.

**Proof**

This is the same as the last proof except that we can use the estimate

$$|F(\lambda, X, \Delta) |_{h_*} \leq C_L \xi$$

in place of (5.80). (No need to ensure $R |F(\lambda, X, \Delta) |_{h_*}$ is smaller than $\varepsilon \phi^4$ because stability away from $\phi = 0$ is not an issue with the kernel norm). End of proof of Lemma.

Recall (4.36)

$$R_{\text{main}} = \frac{1}{2 \pi i} \oint \frac{d \lambda}{\lambda} \varepsilon \left( S(\lambda, Q e^{-V}), F_Q (\lambda) \right) \quad (5.84)$$
Lemma 5.21

\[ \|R_{\text{main}}\|_{h,G_e,A} \leq C_L \varepsilon^{3/4} \]  \hspace{1cm} (5.85)

\[ |R_{\text{main}}|_{h,A} \leq C_L \varepsilon^{3-3\delta/2} \]  \hspace{1cm} (5.86)

Proof

Let

\[ J(\lambda) = S(\lambda, Qe^{-V})^2 \]

Suppose that \( F(\lambda) \) splits, \( F(\lambda) = F_0(\lambda) + F_1(\lambda) \), into a field independent part \( F_0 \) and \( F_1 \) satisfies stability as in (5.78). According to Theorem 6 on page 780 of [BDH-est],

\[ \|\mathcal{E}(J(\lambda), F_0, F_1)\|_{h,G_e,A} \leq O(1) \left( \|J(\lambda)\|_{h,G_e,A_2} + \|f\|_{A_4} \right) \]  \hspace{1cm} (5.87)

provided the norms on the right hand side are less than a small constant \( R^{-1} = O(1) \). In the above \( |f(X)| \leq 2|z(X)|^{-1} \) where the \( z(X) \) are the complex parameters introduced in Lemma 5.19. The \( f(X) \) are supported on small sets. We choose \( |\lambda| = c_L \varepsilon^{-1/4} \). By Lemma 5.19 we have stability (5.78) if \( \varepsilon \) is small. Therefore (5.87) holds and by combining it with Lemma 5.16,

\[ \|\mathcal{E}(J(\lambda), F_0, F_1)\|_{h,G_e,A} \leq q + O(R^{-1}) \leq O(1) \]

(5.85) follows by choosing the contour \( \gamma \) in (5.84) to be a circle of radius \( c_L \varepsilon^{-1/4} \).

The proof of (5.86) follows the same steps but with contour \( \gamma \) being a circle of radius \( c_L \varepsilon^{-1+\delta/2} \) chosen so that \( \xi \) in Lemma 5.20 is small and the hypothesis of Lemma 5.16 is satisfied. End of proof.

Recall from (4.38) that

\[ R_3 = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^4(\lambda - 1)} \mathcal{E}(S(\lambda, K)^2, F(\lambda)) \]  \hspace{1cm} (5.88)

Lemma 5.22

\[ \|R_3\|_{h,G_e,A} \leq C_L \varepsilon^{1-4\eta/3} \]  \hspace{1cm} (5.89)

\[ |R_3|_{h,A} \leq C_L \varepsilon^{11/3-4\eta/3} \]  \hspace{1cm} (5.90)

Proof

This proof follows the same steps as the proof of Lemma 5.21 with contours \( |\lambda| = c_L \varepsilon^{1-\eta/3} \) and \( |\lambda| = c_L \varepsilon^{11/12-\eta/3} \). End of proof.

Lemma 5.23

\( R_4 \) as defined in (4.40) satisfies

\[ \|R_4\|_{h,G_e,A} \leq C \varepsilon^{3/2} \]

\[ |R_4|_{h,A} \leq C \varepsilon^3 \]
Proof

From (4.40)

\[ R_4 = \left( e^{-V'} - e^{-\tilde{V}_L} \right) Q(C, w', g') + e^{-\tilde{V}_L} \left( Q(C, w', g') - Q(C, w', g_L) \right) \]

First we observe from (4.39), Lemma 5.17 and Lemma 5.9 the following bounds

\[ |g' - g_L| \leq C \varepsilon^2 \]  \hspace{1cm} (5.91)
\[ |\mu' - \mu_L| \leq C \varepsilon^2 \]  \hspace{1cm} (5.92)
\[ ||w'|| \leq c/4 \]  \hspace{1cm} (5.93)

We estimate in turn the two terms in the expression for \( R_4 \) above. Because of \( Q \) each term is supported on small sets.

We write the first term as

\[ \left( e^{-V'} - e^{-\tilde{V}_L} \right) Q(C, w', g') = Q(C, w', g') e^{-\tilde{V}_L} \int_0^1 ds (\tilde{V}_L - V') e^{-(\frac{1}{2} - s)V'} - sV_L \]

Then we bound

\[ \left\| \left( e^{-V'} - e^{-\tilde{V}_L} \right) Q(C, w', g') \right\|_{h, G_\kappa} \leq \left\| Q(C, w', g') \right\|_{h, G_\kappa} \int_0^1 ds ||\tilde{V}_L - V'||_h ||e^{-(\frac{1}{2} - s)V'}||_h ||e^{-sV_L}||_h \]

Using (5.91) and (5.92) we can bound

\[ ||\tilde{V}_L(X) - V'(X)||_h \leq C \varepsilon \frac{O(1)}{\gamma} \int_X d^3 x \phi^4(x) \]  \hspace{1cm} (5.95)

for any \( \gamma = O(1) > 0 \).

By Lemma 5.5 we can bound

\[ ||e^{-(\frac{1}{2} - s)V'(X)}||_h \leq 2 |X| e^{-(\frac{3}{2} - s)} \int_X d^3 x \phi^4(x) \]  \hspace{1cm} (5.96)
\[ ||e^{-sV_L(X)}||_h \leq 2 |X| e^{-s} \int_X d^3 x \phi^4(x) \]  \hspace{1cm} (5.97)

We now plug into (5.94) the bounds (5.96) and (5.97). We then write the \( s \)-integration in (5.94) as the union of the intervals \([0, \frac{1}{4}] \) and \([\frac{1}{4}, 1] \). In the first interval we insert the bound (5.95) with \( \gamma \) replaced by \((\frac{1}{2} - s)\gamma\). In the second interval we insert the same bound with \( \gamma \) replaced by \( s\gamma \) and in both cases take \( \gamma = O(1) \) sufficiently small. Then the \( s \)-integral factor in (5.94) is bounded by \( C \varepsilon \). On the other hand the first factor in (5.94) is bounded by \( O(1) \frac{e^{\frac{3}{2}}}{\varepsilon} \) by virtue of Lemma 5.10. (The factor of \( \frac{1}{2} \) in the exponent does not make a difference). Putting these bounds together and recalling that the \( Q \) are supported on small sets we obtain

\[ \left\| \left( e^{-V'} - e^{-\tilde{V}_L} \right) Q(C, w', g') \right\|_{h, G_\kappa, A} \leq C \varepsilon^{\frac{3}{2}} \]

(5.98)

We can now easily bound the second term in the expression for \( R_4 \) by noting that

\[ |g^2 - g_L^2| \leq C \varepsilon^3 \]
and then using Lemma 5.10. We again get the bound
\[ \| e^{-\hat{V}} \left( Q(C, w', g') - Q(C, w', g_L) \right) \|_{h, G_\kappa, A} \leq C \varepsilon^{\frac{3}{2}} \]
(5.99)

Adding together (5.98) and (5.99) finishes the proof of the first bound in the Lemma. The second bound is easy to prove since all derivatives in the \( h_* \) norm are at \( \phi = 0 \). End of proof of Lemma 5.23.

Lemma 5.24
Let \( X \) be a small set and let \( J \) be normalized as in (4.42). Then we have
\[ |D^{2}J(X, 0; f_{L^{-1}}^{\times 2})| \leq O(1) L^{-(\varepsilon - \varepsilon')/2} \| D^{2}J(X, 0) \| \prod_{j=1}^{2} \| f_{j} \|_{C^{2}(L^{-1}X)} \]
(5.100)
\[ |D^{4}J(X, 0; f_{L^{-1}}^{\times 4})| \leq O(1) L^{-(\varepsilon - \varepsilon')/2} \| D^{4}J(X, 0) \| \prod_{j=1}^{4} \| f_{j} \|_{C^{2}(L^{-1}X)} \]
(5.101)

Proof: See Lemma 15 [BDH-est].

Corollary 5.25
Let \( Y = L^{-1}X \) where \( X \) is a small set, \( Z = L^{-1}X \) and let \( J \) be normalized as in (4.42). Then
\[ |J_{L}(Y)|_{h} \leq O(1) L^{-(\varepsilon - \varepsilon')/2} |J(X)|_{h} \]
(5.102)
\[ \| J_{L}(Y)e^{-\tilde{V}_{L}(Z \setminus Y)} \|_{h, G_\kappa} \leq O(1) L^{-(\varepsilon - \varepsilon')/2} \left[ |J(X)|_{h} + \| J(X) \|_{h, G_{3\kappa}} \right] \]
(5.103)

Proof
(5.102) follows immediately from Lemma 5.24. For (5.103) we write
\[ \| J_{L}(Y, \phi)e^{-\tilde{V}_{L}(Z \setminus Y, \phi)} \|_{h} \leq \| J_{L}(Y, \phi) \|_{h} \| e^{-\tilde{V}_{L}(Z \setminus Y, \phi)} \|_{h} \]
and use Lemmas 5.5 and Lemma 5.15,
\[ \leq O(1) G_{\kappa}(Z, \phi) \left[ |J_{L}(Y)|_{h} + L^{-n_{0}[\phi]} \| J_{L}(Y) \|_{L^{1}(Z, h, G_{\kappa})} \right] \]
followed by (5.102), and rewriting the second term by moving the scaling from \( J \) to the norm,
\[ \leq O(1) G_{\kappa}(Z, \phi) \left[ L^{-(\varepsilon - \varepsilon')/2} |J(X)|_{h} + L^{-n_{0}[\phi]} \| J(X) \|_{h, G_{3\kappa}} \right] \]
(5.103) follows by multiplying both sides by \( G_{\kappa}^{-1}(Z, \phi) \) and taking the supremum over \( \phi \). End of proof.

Lemma 5.26
\[ \| \tilde{F}_{R}e^{-\tilde{V}} \|_{h, G_{\kappa}, A} \leq O(1) \varepsilon^{7/4 - \eta} \]
(5.104)
\[ |\tilde{F}_{R}e^{-\tilde{V}}|_{h, *, A} \leq O(1) \varepsilon^{11/4 - \eta} \]
(5.105)
and \( J = R^2 - \hat{F}_RE^{-\bar{V}} \) satisfies on small sets the bounds

\[
\|J\|_{h,G_{3c},A} \leq O(1)\varepsilon^{\frac{3}{4} - \eta} \\
|J|_{h,,A} \leq O(1)\varepsilon^{\frac{3}{4} - \eta}
\]

**Proof**

First we prove (5.104).

Take the definition of \( \hat{F}_R \) given in (4.43) and (4.44). \( \hat{F}_R \) is supported on small sets. We have

\[
\|
\hat{F}_R(X,\phi)\|_{h} \leq \sum_P |\hat{\alpha}_P(X)| \int_X d^3x \|P(\phi(x), \partial \phi(x))\|_h \\
\leq O(1) \sum_P |\hat{\alpha}_P(X)| \varepsilon^{-1} \left( \varepsilon \int_X d^3x \phi^4(x) + \varepsilon^{1/2} \|\phi\|_{X,1,\sigma}^2 + 1 \right) \\
\leq O(1) \sum_P |\hat{\alpha}_P(X)| \varepsilon^{-1} G_\kappa(X,\phi) e^{\gamma g} \int_X d^3y \phi^4(y)
\]

for any \( \gamma = O(1) > 0 \). Hence, using Lemma 5.5

\[
\|
\hat{F}_R(X\phi)e^{-\bar{V}(X\phi)}\|_{h} \leq \|
\hat{F}_R(X\phi)\|_{h}\|e^{-\bar{V}(X\phi)}\|_{h} \leq O(1) \sum_P 2^{|X|} |\hat{\alpha}_P(X)| \varepsilon^{-1} G_\kappa(X,\phi)
\]

We thus obtain (remembering that \( \hat{\alpha}_P \) are supported on small sets) on using (5.72)

\[
\|
\hat{F}_R(X\phi)e^{-\bar{V}(X\phi)}\|_{h,G_{3c},A} \leq O(1)\varepsilon^{-1} \sum_P |\alpha_P|_A \leq O(1)\varepsilon^{7/4 - \eta}
\]

This proves (5.104).

Now we turn to the proof of (5.105).

As observed in the proof of Lemma 5.17, for \( \varepsilon \) sufficiently small (depending on \( L \)),

\[
|\hat{\alpha}_P|_A \leq n(P)!h_{\kappa}^{-n(P)}|1_{S}\|_{h,,A} \leq O(1)h_{\kappa}^{-n(P)}\varepsilon^{11/4 - \eta}
\]

We have from the definition of \( \hat{F}_R \) given in (4.43)

\[
|\hat{F}_R(X)|_h \leq O(1) \sum_P |\hat{\alpha}_P(X)| h_{\kappa}^{n_P}
\]

whence

\[
|\hat{F}_R|_{h,,A} \leq \sum_P |\hat{\alpha}_P|_A h_{\kappa}^{n_P} \leq O(1)\varepsilon^{11/4 - \eta}
\]

which proves (5.105).

To get these bounds for \( J = R^2 - \hat{F}_RE^{-\bar{V}} \) we apply (5.104) and (5.105) to the \( \hat{F}_RE^{-\bar{V}} \) part. We bound \( R^2 \) by Lemma 5.17. End of proof.

**Corollary 5.27**

\[
\|R_{\text{linear}}\|_{h,G_{3c},A} \leq O(1)L^{-(1-\varepsilon)/2}\varepsilon^{3/4 - \eta} \\
|R_{\text{linear}}|_{h,,A} \leq O(1)L^{-(1-\varepsilon)/2}\varepsilon^{11/4 - \eta}
\]
Proof

We recall from (4.44) that
\[ J = R^4 - \tilde{F}e^{-\tilde{V}} \]
is normalized. Let \(1_{s,s}(X)\) be the indicator function of the event that \(X\) is small. Referring to (4.44), the first term in \(R_{\text{linear}}(Z)\) is
\[ R_{\text{linear},s,s}(Z) := \sum_{X: L^{-1}X^L = Z} e^{-\tilde{V}_L(Z \setminus Y)} 1_{s,s}(X) J_L(Y) \] (5.109)
where \(Y = L^{-1}X\). By Corollary 5.25 this is bounded in \(h, G_\kappa\) norm by
\[ O(1)L^{(7-\epsilon)/2} \sum_{X: L^{-1}X^L = Z} 1_{s,s}(X) \left[ |J(X)|_h + \|J(X)\|_{h,G_3w} \right] \] (5.110)
Multiply both sides by \(A(Z)\), using \(A(Z) \leq A(X)\) on the right hand side. Then sum over \(Z\) to get the \(A\) norm and use the bounds on \(J\) in Lemma 5.17 and Lemma 5.26 to obtain
\[ \|R_{\text{linear},s,s}\|_{h,G_\kappa,A} \leq O(1)L^{-(1-\epsilon)/2} \epsilon^{3/4-\eta} \]
where we used an argument on page 790 of [BDH-est] to control the \(\sum_{X: L^{-1}X^L = Z} L^D\) times the sum over \(X\) in the definition of \(A\) norm. Similarly the bound on the kernel norm by \(O(1)L^{-(1-\epsilon)/2} \epsilon^{11/4-\eta}\) comes from the kernel norm bound in Lemma 5.17, Lemma 5.26 and Lemma 5.15.

Let \(1_{l,s}(X)\) be the large set indicator function. The second term in \(R_{\text{linear}}(Z)\) is
\[ R_{\text{linear},l,s}(Z) := \sum_{X: L^{-1}X^L = Z} e^{-\tilde{V}_L(Z\setminus L^{-1}X)} 1_{l,s}(X) R^\sharp_L(L^{-1}X) \] (5.111)
where we have used \(J(X) = R^\sharp(X)\) because the subtraction is supported on small sets. This is bounded in the same way as above using Lemma 5.17 except that the necessary \(L^{-(1-\epsilon)/2}\) is obtained for a different reason: For large sets, by (2.7), \(A(Z) \leq c_p L^{-4} A_{-p}(X)\), where \(c_p = O(1)\). The Corollary is proved.

Proof of Theorem 1 Concluded

From (4.35)-(4.41), \(R\) is the sum of \(R_i\) where \(i = 1, 2, 3, 4\). By Lemmas 5.21, 5.22, 5.23 and 5.27 with \(L\) large and \(\kappa\) small depending on \(L\), the sum satisfies bounds (5.9) and (5.10).

6. STABLE MANIFOLD AND CONVERGENCE TO NON-GAUSSIAN FIXED POINT

6.1
Let \(\tilde{g}\) be the approximate fixed point of the \(g\) flow given by (5.4). Let us define
\[ \hat{g} = g - \tilde{g} \] (6.1)
and
\[ u = (\hat{g}, \mu, R, w) \] (6.2)
Then the RG iteration given by (4.39), (4.41), (4.18) can be written as
\[ u' = f(u) \] (6.3)
with components
\[ \tilde{g} = f_{\tilde{g}}(u) = (2 - L^\varepsilon) \tilde{g} + \tilde{\xi}(u) \] (6.4)
\[ \mu = f_{\mu}(u) = L^{3/2} \mu + \tilde{\rho}(u) \] (6.5)
\[ R' = r_R(u) = U(u) \] (6.6)
\[ w' = f_{w}(u) = v + w_L \] (6.7)
with initial
\[ u_0 = (\tilde{g}_0, \mu_0, 0, 0) \] (6.8)

Here
\[ \tilde{\xi}(u) = -L^{2\varepsilon} a \tilde{g}^2 + \xi(u) \] (6.9)
\[ \tilde{\rho}(u) = -L^{2\varepsilon} b(g + \tilde{g})^2 + \rho(u) \] (6.10)

Note that the \( w \) flow is autonomous and solved by (5.32). In Lemma 5.9 it was proved that the \( w \) tends to the fixed point \( w_* \) in an appropriate norm.

Let \( E \) be the Banach space consisting of elements \( u \) with the (box) norm
\[ \|u\| = \sup(\varepsilon^{-3/2} g, \varepsilon^{-(2-\delta)} |\mu|, \varepsilon^{-(11/4-\eta)} |||R||, c^{-1} \|w\|) \] (6.11)

Here
\[ |||R||| = \sup(\varepsilon^{2} |||R||h,G,A,|R||h,A) \] (6.12)
and
\[ \|w\| = \sup_p \|w^{(p)}\|_p \]

The \( \| \cdot \|_p \) and \( \|w\| \) norms were defined in (5.33) and the constant \( c \) is that of Lemma 5.9.

Let \( B(r) \subset E \) be the closed ball of radius \( r \), centered at the origin:
\[ B(r) = \{ u \in E : \|u\| \leq r \} \] (6.13)

Let \( D \) be the domain of \( (g, \mu, R) \) specified in (5.1)-(5.3) of the hypothesis stated at the beginning of Section 5. Then we have
\[ u \in B(1) \Rightarrow (g, \mu, R) \in D \] (6.14)
and then Theorem 1 of Section 5 holds.

Theorem 1, together with the autonomous Lemma 5.9, shows that the ball \( B(1) \) would be stable under the RG flow \( f \), except for the unstable direction \( \mu \) (as evident from (6.5)). The initial unstable parameter \( \mu_0 \) will have to be fine tuned to a critical function \( \mu_c(g_0) \) which would determine the critical or stable submanifold in which we expect to get a contraction to a fixed point. As observed in [BDH-eps] this is part of the stable manifold theorem in the theory of hyperbolic dynamical systems [S]. In the following we put ourselves in this framework (see appendix 2, Chapter 5 of [S] and Section 5 of [BDH-eps]).

Suppose \( u \in B(1) \). Then from theorem 1 of Section 5 we have for \( \varepsilon > 0 \) sufficiently small (depending on \( L \)), which implies in particular \( L^\varepsilon = O(1) \),
\[ \|\xi(u)\| \leq O(1) \varepsilon^{11/4-\eta} \]
\[ |\rho(u)| \leq O(1) L^{3/2} \varepsilon^{11/4-\eta} \] (6.15)
\[ |||U(u)||| \leq L^{-1/4} \varepsilon^{11/4-\eta} \]

(From (6.9), (6.10) we have by virtue of (6.15) and the estimates \( a = O(\log L), b = O(L^{3/2}) \), see Lemma 5.12, for \( u \in B(1) \):
\[
|\tilde{\xi}(u)| \leq O(1)\varepsilon^{11/4-\eta} \\
|\tilde{\rho}(u)| \leq \varepsilon^{2-\delta}
\]  
(6.16)

\[\tilde{\xi}, \tilde{\rho}, U, w\] satisfy the following Lipschitz bounds in \(B(1/4) \subset B(1)\)

**Lemma 6.1**

Let \(u, u' \in B(1/4)\). Then we have the Lipschitz bounds:

\[
|\tilde{\xi}(u) - \tilde{\xi}(u')| \leq O(1)\varepsilon^{11/4-\eta}\|u - u'\|
\]

\[
|\tilde{\rho}(u) - \tilde{\rho}(u')| \leq O(1)\varepsilon^{11/4-\eta}\|u - u'\|
\]

\[
|\|\|U(u) - U(u')\|\| \leq O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|u - u'\|
\]

\[
|\|f_w(u) - f_w(u')\| \leq cL^{-1/4}\|u - u'\|
\]

**Proof**

We shall use the fact that \(\xi, \rho, U\) are analytic in \(B(1)\). This follows from the algebraic operations in Section 4 together with the analyticity of the extraction map.

Let \(\Delta u = u - u'\). Then

\[U(u) - U(u') = \int_0^1 dt \frac{\partial}{\partial t} U(u + t\Delta u)\]

By the Cauchy integral formula

\[
\frac{\partial}{\partial t} U(u + t\Delta u) = \frac{1}{2\pi i} \oint_{\gamma} dz \frac{U(u + z\Delta u)}{(z-t)^2}
\]

here \(0 \leq t \leq 1\) and \(\gamma\) is the closed contour

\[\gamma : z - t = re^{i\theta}, \ r = 1/4\|\Delta u\|^{-1}\]

Note that

\[u + z\Delta u = u + t\Delta u + \frac{1}{4}\frac{\Delta u}{\|\Delta u\|}e^{i\theta}\]

Hence

\[\|u + z\Delta u\| \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1\]

So that for \(z \in \gamma\), \(u + z\Delta u \in B(1)\), and hence from (6.15)

\[|\|\|U(u + z\Delta u)\|\| \leq L^{-1/4}\varepsilon^{11/4-\eta}\]

so that

\[\sup_{0 \leq t \leq 1} |\|\|\frac{\partial}{\partial t} U(u + t\Delta u)\|\| \leq O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|\Delta u\|\]

and thus

\[|\|U(u) - U(u')\| \leq O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|u - u'\|\]

This proves the last inequality of (6.17).

On using (6.16) we have in the same way

\[|\tilde{\xi}(u) - \tilde{\xi}(u')| \leq O(1)\varepsilon^{11/4-\eta}\|u - u'\|
\]

To get the Lipschitz bound for \(\tilde{\rho}\) we first use

\[|\rho(u) - \rho(u')| \leq O(1)\varepsilon^{11/4-\eta}\|u - u'\|
\]
Then from (6.10)

\[ |\hat{\rho}(u) - \hat{\rho}(u')| \leq O(L^{3/2})|\bar{g}' + \tilde{g} + 2\tilde{g}' - \tilde{g}| + |\rho(u) - \rho(u')| \]

\[ \leq \left( O(L^{3/2})O(\varepsilon)^{3/2} + O(1)L^{3/2}\varepsilon^{11/4-\eta} \right) \|u - u'\| \]

\[ \leq O(L^{3/2})\varepsilon^{5/2} \|u - u'\| \]

Finally from (6.7) and the definition of the norms in (5.33) we have

\[ \|f_w(u) - f_w(u')\| \leq \sup_{1 \leq p \leq 3} \max \sup_x \left( |x|^{\frac{6}{p}+1} \left| w^{(p)}(x) - w^{(p)}(x') \right| \right) \]

Now \( w^{(p)}(x) = L^2[\phi]w^{(p)}(Lx) \). We then get easily

\[ \|f_w(u) - f_w(u')\| \leq L^{-1/4} \|w - w'\| \leq cL^{-1/4} \|u - u'\| \]

and we are done. Lemma 6.1 has been proved.

Consider now the RG flow (6.3):

\[ u_k = f(u_{k-1}) \]

with initial condition

\[ u_0 = (\tilde{g}_0, \mu_0, 0, 0) \quad \tilde{g}_0 = g_0 - \bar{g} \]

**Theorem 6.2**

There exists \( \mu_0 \) such that for \( u_0 \in B(1/32), u_k = f(u_{k-1}) \in B(1/4) \) for all \( k \geq 1 \).

**Remark**

The following proof of existence of global solutions is a textbook argument in the theory of dynamical systems adapted to the present context.

**Proof**

From the flows (6.4), (6.5) we easily derive after \( n \) steps of the RG

\[ \tilde{g}_k = (2 - L^\varepsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} (2 - L^\varepsilon)^{k-1-j} \tilde{\xi}(u_j), \quad 1 \leq k \leq n \]

\[ \mu_k = L^{-\frac{3+\varepsilon}{2}(n-k)} \mu_n - \sum_{j=k}^{n-1} L^{-\frac{3+\varepsilon}{2}(j+1-k)} \hat{\rho}(u_j), \quad 0 \leq k \leq n - 1 \]

Let us fix \( \mu_n = \mu_f \) and take \( n \to \infty \). We have

\[ \tilde{g}_k = (2 - L^\varepsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} (2 - L^\varepsilon)^{k-1-j} \tilde{\xi}(u_j), \quad k \geq 1 \]

(6.18)

\[ \mu_k = -\sum_{j=k}^{\infty} L^{-\frac{3+\varepsilon}{2}(j+1-k)} \hat{\rho}(u_j), \quad k \geq 0 \]

(6.19)

together with

\[ R_k = U(u_{k-1}), \quad k \geq 1 \]

(6.20)
We can take the autonomous \( w \) flow, given by (6.7), as solved by (5.32) and need no longer consider it as a flow variable.

Note that for \( \varepsilon \) sufficiently small (depending on \( L \))

\[ 0 < 2 - L \varepsilon < 1 \]  

(6.21)

Then for \( u_j \in B(1) \) the infinite sum of (6.19) converges by (6.21) and (6.16). So \( \mu_0 \) has now been determined provided (6.18)-(6.20) has a solution.

It is easy to verify that any solution of (6.18)-(6.19), together with the autonomous \( w \) flow, is a solution of the RG flow

\[ u_k = f(u_{k-1}) \]

Now write (6.18)-(6.20) in the form

\[ u_k = F_k(u) \]  

(6.22)

where \( u = (u_0, u_1, u_2, ...) \) and \( F_k \) has components \( (F_k^{(q)}, F_k^{(\mu)}, F_k^{(R)}) \) given by the r.h.s. of (6.18), (6.19) and (6.20) respectively.

If we write

\[ F(u) = (F_0(u), F_1(u), ...) \]

then (6.22) can be written as a fixed point equation

\[ u = F(u) \]  

(6.23)

Consider the Banach space \( E \) of sequences \( u = (u_0, u_1, u_2, ...) \) with norm

\[ \|u\| = \sup_{k \geq 0} \|u_k\| \]  

(6.24)

and the closed ball \( B(r) \subset E \)

\[ B(r) = \{ u : \|u\| \leq r \} \]  

(6.25)

We shall seek a solution of (6.23) in the closed ball \( B(1/4) \) with initial data \( u_0 = (\tilde{g}_0, \mu_0, 0, 0) \) in \( B(1/32) \) and \( \tilde{g}_0 \) held fixed.

We shall need

**Lemma 6.3**

\[ u \in B(1/32) \Rightarrow F(u) \in B(1/16) \]  

(6.26)

Moreover, for \( u, u' \in B(1/4) \)

\[ \|F(u) - F(u')\| \leq \frac{1}{2}\|u - u'\| \]  

(6.27)

We postpone the proof of this lemma. Given the above lemma 6.3 the proof of theorem 6.2 now follows easily. To this end consider a sequence \( u^{(n)}, n = 1, 2, ..., \) defined by

\[ u^{(n)} = F(u^{(n-1)}) \]  

(6.28)

\[ u^{(0)} \in B(1/32) \]  

(6.29)

**Claim:** \( u^{(n)} \in B(1/4) \) for all \( n \).

To prove the claim first observe that (6.26) and (6.29) imply

\[ \|F(u^{(0)}) - u^{(0)}\| \leq \frac{3}{32} \]
Similarly from (6.19) and (6.16) we have

$$
\|u^{(n)} - u^{(n-1)}\| = \|F(u^{(n-1)}) - F(u^{(n-2)})\| \leq \frac{1}{2^{n-1}}\|u^{(1)} - u^{(0)}\| \leq \frac{1}{2^{n-1}} \frac{3}{32}
$$

Write

$$
u^{(n)} = u^{(0)} + \sum_{j=1}^{n} (u^{(j)} - u^{(j-1)})
$$

Then

$$
\|u^{(n)}\| \leq \frac{1}{32} \left( 1 + 3 \sum_{j=1}^{n} \frac{1}{2^{j-1}} \right) \leq \frac{7}{32} \leq \frac{1}{4}
$$

and the claim has been proved.

By virtue of the claim and \(\|u^{(n)} - u^{(n-1)}\| \to 0\) as \(n \to \infty\) we have \(u^{(n)} \to u\) in \(B(1/4)\) and this \(u\) satisfies

$$
u = F(u)
$$

We are done. Thus it only remains to prove lemma 6.3 to complete the proof of theorem 6.2.

**Proof of lemma 6.3**

First we prove (6.26), and thus take \(u \in B(1/32)\). From (6.18) and the estimates in (6.16) we have

$$
\varepsilon^{-3/2}|F_k^{(g)}(u)| \leq (2 - L^\varepsilon) \frac{1}{32} + O(1)\varepsilon^{5/4-\eta} \sum_{j=0}^{k-1} (2 - L^\varepsilon)^{k-1-j} \leq \frac{1}{32} + O(1)\varepsilon^{1/4-\eta} \leq \frac{1}{16}
$$

since \(\eta < \frac{1}{4}\) and \(\varepsilon\) is sufficiently small.

Similarly from (6.19) and (6.16) we have

$$
\varepsilon^{-(2-\delta)}|F_k^{(u)}(u)| \leq \sum_{j=k}^{\infty} L^{-\frac{3s}{2} (j+1-k)} \leq L^{-\frac{3s}{2}} (1 - L^{-\frac{3s}{2}}) \leq \frac{1}{16}
$$

for \(L\) sufficiently large.

Finally from (6.20) and (6.15)

$$
\varepsilon^{-(11/4-\eta)}|F_k^{(R)}(u)| \leq L^{-1/4} \leq \frac{1}{16}
$$

This proves (6.26). To prove (6.27), take \(u, u' \in B(1/4)\). We can then use the Lipshitz estimates of lemma 6.1. Note that the initial coupling \(g_0\) is held fixed. Then we have

$$
\varepsilon^{-3/2}|F_k^{(g)}(u) - F_k^{(g)}(u')| \leq \sum_{j=0}^{k-1} (2 - L^\varepsilon)^{k-1-j} \varepsilon^{-3/2}|\xi(u_j) - \xi(u'_j)| \leq \frac{1}{2}\|u - u'\|
$$

$$
\leq O(1)\varepsilon^{5/4-\eta}\|u - u'\| \sum_{j=0}^{k-1} (2 - L^\varepsilon)^{k-1-j} \leq O(1)\varepsilon^{1/4-\eta}\|u - u'\| \leq \frac{1}{2}\|u - u'\|
$$
Similarly,
\[ \varepsilon^{-(2-\delta)}|F_k^{(p)}(u) - F_k^{(p)}(u')| \leq L^{-\frac{3\delta}{2}} \sum_{j=k}^{\infty} L^{-\frac{3\delta}{2}(j-k)} \varepsilon^{-(2-\delta)}|\tilde{\rho}(u_j) - \tilde{\rho}(u'_j)| \leq L^{-\frac{3\delta}{2}} O(\log L)\varepsilon^{1/2} \|u - u'\| \leq \frac{1}{2} \|u - u'\| \]

Thus (6.27) has also been proved. This proves lemma 6.3, and the proof of theorem 6.2 has now been completed.

6.2. Stable manifold and convergence to fixed point.
Write \(E = E_1 \times E_2\) with \(u \in E\) represented as \(u = (u_1, u_2)\).
Here \(u_1 = (\hat{g}, R, w)\) and \(u_2 = \mu\). \(E_1\) and \(E_2\) thus represent the contracting and expanding directions for the RG map \(f\).
Let \(p_i, i = 1, 2\), denote the projector onto \(E_i\) and \(f_i = p_i \circ f\).
Note that the norm \(\| \cdot \|\) on \(E\) being a box norm we also have
\[ \|u\| = \sup(\|u_1\|, \|u_2\|) \]
The following Lemma 6.4, the definition 6.5 of the stable manifold \(W^s\) and our final theorem 6.6 are Irwin’s proof of the stable manifold theorem as presented in appendix 2, chapter 5 of [S]. Part of Irwin’s proof is replaced by theorem 6.2.

**Lemma 6.4**
Let \(u, u' \in B(1/4)\). Then we have
\[ \|f_1(u) - f_1(u')\| \leq (1 - \varepsilon)\|u - u'\| \quad (6.30) \]
and, if \(\|u_2 - u'_2\| \geq \|u_1 - u'_1\|\) then
\[ \|f_2(u) - f_2(u')\| \geq (1 + \varepsilon)\|u - u'\| \quad (6.31) \]

**Proof of lemma 6.4**
Because \(u, u' \in B(1/4)\) we can use throughout lemma 6.1. As always \(L\) is sufficiently large and then \(\varepsilon\) sufficiently small. First we prove (6.30). \(f_1\) has components \((f_{\hat{g}}, f_R, f_\omega)\). From (6.4)
\[ f_{\hat{g}}(u) = (2 - L^s)\hat{g} + \tilde{\xi}(u) \]
Thus using lemma 6.1
\[ \varepsilon^{-3/2}|f_{\hat{g}}(u) - f_{\hat{g}}(u')| \leq (2 - L^s)\|u - u'\| + \varepsilon^{-3/2} \|\tilde{\xi}(u) - \tilde{\xi}(u')\| \leq (2 - L^s + O(1)\varepsilon^{5/4-\eta})\|u - u'\| \leq (1 - \varepsilon)\|u - u'\| \]
for \(\varepsilon\) sufficiently small.

Since \(f_R(u) = U(u)\), we have from lemma 6.1
\[ \varepsilon^{-(11/4-\eta)}\|f_R(u) - f_R(u')\| \leq (1 - \varepsilon)\|u - u'\| \]

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for $L$ sufficiently large. Finally from the same lemma

$$c^{-1}\|f_w(u) - f_w(u')\| \leq (1 - \varepsilon)\|u - u'\|$$

These three inequalities prove (6.30).

Next we turn to (6.31). In this case by assumption $\|u_2 - u'_2\| \geq \|u_1 - u'_1\|$ and hence, since our norms are box norms, we have

$$\|u - u'\| = \|u_2 - u'_2\| = \varepsilon^{-(2-\delta)}|\mu - \mu'|$$

From (6.5)

$$f_\mu(u) = L^{\frac{3+\varepsilon}{2}}\mu + \tilde{\rho}(u)$$

Then, using lemma 6.1,

$$\varepsilon^{-(2-\delta)}|f_\mu(u) - f_\mu(u')| \geq L^{\frac{3+\varepsilon}{2}}\|u - u'\| - \varepsilon^{-(2-\delta)}|\tilde{\rho}(u) - \tilde{\rho}(u')|$$

$$\geq (L^{\frac{3+\varepsilon}{2}} - O(\log L)\varepsilon^{5/2})\|u - u'\|$$

$$\geq (1 + \varepsilon)\|u - u'\|$$

which proves (6.31) and thus completes the proof of lemma 6.4.

Let $f^k$ be the $k$-fold composition of the RG map $f$.

**Definition 6.5**

The stable manifold of $f$ is defined by

$$W^s(f) = \{ u \in B(1/32) : f^k(u) \in B(1/4) \ \forall k \geq 0 \} \quad (6.32)$$

Write the initial points $u$ as $u = (u_1, u_2)$ with $u_1 = (\tilde{g}_0, 0, 0)$ and $u_2 = \mu_0$. Observe that theorem 6.2 says that there exists for $u \in B(1/32)$ a $u_2$ such that $f^k(u) \in B(1/4) \ \forall k \geq 0$. We now have

**Theorem 6.6**

$W^s(f)$ is the graph $\{(u_1, h(u_1))\}$ of a function $u_2 = h(u_1)$ with $h$ Lipshitz continuous with $\text{Lip}h \leq 1$. Moreover $f|W^s(f)$ contracts distances and hence has a unique fixed point which attracts all points of $W^s(f)$.

**Proof**

To prove the first statement it is enough to prove that if in $W^s(f)$ we take two points $u = (u_1, u_2)$ and $u' = (u'_1, u'_2)$ then

$$\|u_2 - u'_2\| \leq \|u_1 - u'_1\| \quad (6.33)$$

because then for a given $u_1$ we would have at most one $u_2$, and by theorem 6.2 there exists such a $u_2$. This means that $W^s(f)$ is the graph of a function $h$, $u_2 = h(u_1)$, and moreover

$$\|h(u_1) - h(u'_1)\| \leq \|u_1 - u'_1\|$$

Suppose (6.33) is not true. Then

$$\|u_2 - u'_2\| > \|u_1 - u'_1\| \quad (6.34)$$

Then by (6.31) followed by (6.30) gives

$$\|f_2(u) - f_2(u')\| \geq (1 + \varepsilon)\|u - u'\| > (1 - \varepsilon)\|u - u'\| \geq \|f_1(u) - f_1(u')\|$$

and hence

$$\|f(u) - f(u')\| \geq (1 + \varepsilon)\|u - u'\|$$
Now
\[\|f^2(u) - f^2(u')\| = \|f(f(u)) - f(f(u'))\|\]
and by the above and the second part of Lemma 6.4
\[\|f^2(u) - f^2(u')\| \geq (1 + \varepsilon)\|f(u) - f(u')\| \geq (1 + \varepsilon)^2\|u - u'\|\]
By induction we can prove for all \(k \geq 0\)
\[\|f^k(u) - f^k(u')\| \geq (1 + \varepsilon)^k\|u - u'\|\]
Since \(u, u' \in W^s(f)\) the l.h.s. is bounded above by \(\frac{1}{2}\) and hence for \(k \to \infty\) we have a contradiction because \(u \neq u'\) under (6.34).
Hence (6.33) is true and the first statement of theorem 6.6 has been proved.
Now we prove that \(f|W^s(f)\) is a contraction. Note that if \(u, u' \in W^s(f)\), then
\[\|f_2(u) - f_2(u')\| \leq \|f_1(u) - f_1(u')\|\] (6.35)
We can prove this just as we proved (6.33). Namely assume the contrary and then show in the same way
\[\|f^k(u) - f^k(u')\| \geq (1 + \varepsilon)^{k-1}\|f(u) - f(u')\|\]
The l.h.s. is bounded by \(\frac{1}{2}\) and so as \(k \to \infty\) we get a contradiction because \(f(u) \neq f(u')\) under the negation of (6.35). This proves (6.35) which now implies
\[\|f(u) - f(u')\| = \|f_1(u) - f_1(u')\| \leq (1 - \varepsilon)\|u - u'\|\]
by lemma 6.4, and we are done. Theorem 6.6 has been proved.
Note that theorem 6.6 tells us that \(\mu_0 = h(\tilde{g}_0) = \mu_c(g_0)\) which defines \(\mu_c\). If \(\tilde{g}_* = g_* - \bar{g}\) is one of the coordinates of the fixed point \(u_*\) then \(g_* \neq 0\) since \(u_* \in B(1/4)\). In fact the latter implies
\[|g_* - \bar{g}| \leq \frac{1}{4}\varepsilon^{3/2}\]
and we know \(\bar{g} = O(\varepsilon)\) and this excludes \(g_* = 0\). So our fixed point is non trivial (non-Gaussian).

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