Nonlinear parameter-gauge coupling approach to generalization of generalized Thouless pumps and \(-1\)-form anomaly

Yuan Yao\(^1\),\(^*\)

\(^1\)Institute for Solid State Physics, University of Tokyo, Kashiwa, Chiba 277-8581, Japan

Abstract

We study the physical consequences of nontrivial topology of parameter space on the transport properties of fermionic systems in arbitrary dimensions. By a nonlinear parameter-gauge coupling effective action, we find that electromagnetic responses at each order can be “pumped” by a space-time dependent parameter field and their changes are quantized corresponding to a certain cohomology class of the parameter space. Our work generalizes the Thouless pump and its generalizations. Various dynamical consequences of the interfaces and their quantum phase transitions are manifested and the quantized jumps of related response observables are shown to be robust due to the topological reasons. We also generalize the \(-1\)-form anomaly by the proposed nonlinear coupling action.

\(^*\) smartyao@issp.u-tokyo.ac.jp
I. INTRODUCTION

Transportation properties of quantum matter at zero temperature play essential roles in manifesting nontrivial topology of ground-state wavefunction, e.g. the Nobel-prize winning result Thouless-Kohmoto-Nightingale-Nijs formula [1] relates the integer quantum Hall conductance with the Chern number of filled electronic bands. Later, the topological nature of an adiabatic cycling of gapped one-dimensional $U(1)$-charge symmetric system was investigated by D. Thouless through a quantized charge transport — the Thouless charge pump [2, 3]. It is stated that, in a gapped free-electronic system with a unique ground state, the total charge flowing across a fixed section of a one-dimensional ring within one period of parameter changing is an integer.

The Thouless charge pump can be also seen in a familiar (bosonic though) interacting system, e.g. a spin-1 antiferromagnetic (AFM) Heisenberg chain:

$$H_{AFM} = \sum_i \vec{S}_i \cdot \vec{S}_i + 1,$$

which is defined on a spatial ring with a periodic boundary condition and possesses a nonzero gap above the unique ground state [4, 5]. This system has a full $SO(3)$ spin-rotation symmetry, but let us simply focus on the $U(1)_z$ subgroup which is the rotation symmetry along $z$-axis. Let us impose an external magnetic field in the $z$-direction which is significantly large only in a finite regime of the chain. The external field still respects $U(1)_z$ symmetry and, we can switch on other necessary local $U(1)_z$-symmetric interactions where the magnetic field starts vanishing so that the energy spectrum is eventually still gapped with a unique ground state. Thus the ground state can potentially have a nonzero $S_z$ value (which is the $U(1)_z$-charge) magnetized by the magnetic field and such a $U(1)_z$ charge is quantized trivially. In the low-energy limit, we can assume that the system, which has a single finite-energy state asymptotically, is topological and conformal, e.g. it possesses a Lorentz invariance and an S-invariance — we can exchange the (one-dimensional) space and the time. Let us S-transform the spacetime as in FIG. (1), and then the former magnetism in the presence of the static magnetic field is interpreted as the total spin flowing across a site along the new spatial direction within the new temporal period. Such a net spin flow is quantized by the original space-like viewpoint, by which we can also see this net flow is robust against perturbation. It is because, if we want to smoothly change the magnetism in the former
FIG. 1. The static non-uniform magnetic field in the space-like “pump” or jump (left) is $S$-transformed to a cycling (right).

picture, it is inevitable to close the many-body gap so that the $S_z$-eigenvalue of the ground state can be changed. Such a gap closing formally results in large correlation lengths, respectively, in spatial and temporal directions, in two pictures transformed to each other by $S$-transformation, which contradicts with the correlation length vanishing in the low-energy limit of the later viewpoint.

The argument above implies that a nontrivial Thouless pump reflects a nontrivial topology of the parameter space of the gapped lattice model [6–8] that we will see later. For instance, the gapless point in the full parameter space can obstruct the contractibility of the gapped-system parameter space. Such gapless defects and their stabilities have been investigated in band topological insulators and superconductors [9]. Recently, the generalized Thouless pump has been proposed in higher $d$-dimensional space by a linear gauge-field coupling with a $d$-dimensional lattice parameters with $U(1)$ symmetry, and Wess-Zumino-Witten terms as generalized Berry curvatures [10, 11] are applied to describe various adiabatic phases without symmetries [6–8].

In this work, we will generalize the generalized Thouless pump to higher order of response. In the viewpoint of space-like Thouless pump around Eq. (1), the $U(1)$-charge difference there can be seen as a charge change within the finite regime exerted by the external magnetic field. Such a finite regime is zero-dimensional, which, in a large scale, can be seen as a point-like interface created by the external field. We can generalize this idea to higher dimensions, for example, three-dimensional space, the interface is a two-dimensional space. Let us use the space-like picture where, e.g. the charge pump is $S$-transformed to a spatial charge-jump accumulation, such as the magnetism in the spin-chain example before. Although the charge jump of such an interface can still be discussed, we will later see that it vanishes
once we do not perturb sufficiently many parameters to be spatially dependent. In this case, after S-transformed, the Thouless pump cannot distinguish topologically distinct cycles. Nevertheless, integer quantum Hall conductance of this interface can obtain a nontrivial jump in its normal direction, e.g. we can smoothly insert an integer quantum Hall ($\sigma_H = 1$) interface of a finite thickness in the vacuum as (a) in FIG. (2). Such a smooth insertion can be effectively realized by a massive Dirac fermion with its mass term smoothly winding once by a space-dependent chiral transformation. Alternatively as (b) in FIG. (2), we can insert a trivial phase $\sigma_H = 0$ with the same total charge $Q_0$. These two interfaces cannot be adiabatically deformed to each other without closing the gap (of the whole system including the vacuum) although they accumulate the same number of charges $Q_0$ on the interface. Therefore, after S-transformed to the temporal pump cycle, they represent two topologically distinct cycles and cannot be distinguished by the charge pump. It implies a generalization of the Thouless pump in higher dimensions, where the charge pump is generalized to higher-order $U(1)$ response pump (charge being the 0-th order response), e.g. Hall-conductance pump, to distinguish those two cycles above.

Furthermore, there is a reduction from higher-order response pumps to (0-th order) charge pumps by topologically nontrivial background fields as follows. In the (3 + 1)-dimensional example above, we can artificially insert a unit static flux through the interface. Then, if the Hall conductances of the interfaces are differed by some integer, there will be the same quantized number of charge differences accumulated on the interface as in FIG. (2). S-transformed back to the temporal picture, there will be a nontrivial Thouless pump as a detector to distinguish these two cycles. Therefore, our generalization have observables through the Thouless pump with a nontrivial instanton background [12]. Although motivated by the space-like picture above, we will still discuss the general pattern including the traditional (time-like) pump.

Moreover, general interface settings can be non-compact, where the gapped interface can carry fractional charges on the charge-conjugation explicitly broken edge of (1 + 1)-dimensional massive Dirac fermion [13] or fractional Hall conductances on the time-reversal explicitly broken surface of topological insulators [14, 15]. The interface phase transitions of various settings are discussed in our work. We will generalize this idea to arbitrary dimensions and orders of responses by an effective action nonlinearly coupling parameter fields with $U(1)$ background gauge field.
FIG. 2. In the space-like pump, a unit flux “⊗” through the interface inserted in the vacuum induces nonzero additional charge if the interface supports nontrivial integer quantum Hall phase.

S-transformed to the (temporal) pumps, such a flux can distinguish these two cycles.

The physical observable on the interfaces can also have nontrivial manifestation on the conflicting nature between parameter-space identification and symmetries, called \(-1\)-form anomaly [16, 17] as follows. Gapped systems with a unique ground state are indistinguishable by the partition function in the low-energy limit. When we perform renormalization-group (RG) transformation on them, the asymptotic fixed-point models all have a single state. It is attempting to identify their parameters. However, such an identification can be potentially ill-defined once we couple the system to a background gauge field of a certain symmetry, e.g. \(U(1)\)-charge. A typical example is the integer quantum Hall systems with distinct Hall conductances. To detect such an ambiguity, we can further couple the system to a background parameter-gauge field. Such a parameter-gauge bundle, in our example, will paste two distinct quantum Hall phases together at infinity. Nevertheless, as long as \(U(1)\)-symmetry is respected, the gap must be closed somewhere, such as on an interface, so that the Hall conductance can be changed. Thus, the whole system cannot be gapped with a unique ground state any more. As a special situation, there can be chiral modes at some interface separating two distinct bulk quantum Hall phases. Such an inevitable ingappabilities signals an anomaly in the analog of the anomalous boundary modes on nontrivial symmetry-protected topological phases [18–21]. This anomaly is called \(-1\)-form symmetry anomaly since it is associated with the parameter space and the parameter change.
cannot be generated by quantum operators in real spacetime. We will generalize the $-1$-form anomaly to parameter fields depending on spacetime in a more general way. Alternatively, two parameter fields of distinct adiabatic phases to be identified necessarily depends on more spacetime coordinates than the example above.

This article is organized as follows. The nonlinear parameter-gauge coupling action will be proposed in Sec. (II), which implies a potentially nontrivial topology of parameter space. In Sec. (IV), we will discuss the generalization of the Thouless pump to higher-order responses and their detection through the conventional charge pump by background fluxes. Other consequences of nontrivial topology of parameter space, e.g. symmetry constraints on the interface phase transitions, will be present in Sec. (V). Finally, we will generalize the concept of $-1$-form anomaly in Sec. (VI).

II. NONLINEAR PARAMETER-GAUGE COUPLING

We consider spinless fermionic Hamiltonians respecting $U(1)$-charge symmetry. Let us assume that the gapped system is parameterized by a series of parameters $\{m_\alpha\} \in \mathcal{C}_D$ in the large-scale limit, where $\mathcal{C}_D$ is the set of parameters of $U(1)$-symmetric Hamiltonian with a unique ground state in $D = d + 1$ spacetime dimension(s). In the continuum limit, the Lagrangian density takes the form as $\mathcal{L}(\{x_\mu\}; \{m_\alpha\})$ with $\{x_\mu\}|_{\mu=0,\ldots,d} \equiv (t, x_1, x_2, \cdots, x_d)$.

We can also perturb the parameters to be space-time dependent:

$$\mathcal{L}(\{x_\mu\}; \{m_\alpha\}) \Rightarrow \mathcal{L}(\{x_\mu\}; \{\Phi_\alpha(x_\mu)\}),$$

where $\Phi_\alpha(x_\mu)$ is smooth and adiabatic compared with the energy scale of the system. Then we can classify various types of parameter “field” $\Phi_\alpha(x_\mu)$ by its coordinate dependence. Let us first take the spacetime as $\mathbb{R}^4$ and compactify it depending on $\Phi_\alpha(x_\mu)$. A typical parameter field is the interface:

$$\Phi_\alpha(\{x_\mu\}|_{\mu=0,\ldots,d}) = \Phi_\alpha(x_n) = \begin{cases} m_\alpha^L, & x_n \to -\infty; \\ m_\alpha^R, & x_n \to +\infty, \end{cases}$$

which enables us to compactify the spacetime as $S^1_{(0)} \times S^1_{(1)} \times \cdots S^1_{(n-1)} \times \mathbb{R}^1_{(n)} \times S^1_{(n+1)} \times \cdots S^1_{(d)}$.

We can also further perturb the parameter space away from constant at each slice as:

$$\Phi_\alpha(\{x_\mu\}|_{\mu=0,\ldots,d}) = \Phi_\alpha(\{x_\mu\}|_{\mu=n,\ldots,d}) = \begin{cases} \Phi_\alpha^L(\{x_\mu\}|_{\mu=n+1,\cdots,d}), & x_n \to -\infty; \\ \Phi_\alpha^R(\{x_\mu\}|_{\mu=n+1,\cdots,d}), & x_n \to +\infty, \end{cases}$$
where, without loss of generality, we assume that the parameter field depends on the last \((d - n + 1)\) coordinates of spacetime. In addition, we can still restrict to the functions that enable us to compactify the spacetime as \(S^1_0 \times S^1_1 \times \cdots S^1_{(n-1)} \times \mathbb{R}^1_{(n)} \times S^1_{(n+1)} \times \cdots S^1_{(d)}\). We denote such a spacetime interface depending on \((D - n)\) coordinates as “co-\((D - n)\)” interface, and co-\((D - 0)\) interfaces are of the most general forms up to a spacetime-coordinate reordering. In this notation, the interface in Eq. (3) is co-1, but, without extra specification, we will call co-1 interfaces simply as interfaces throughout this article.

If the parameter perturbation is sufficiently smooth and locally insignificant, the system is still gapped with a unique ground state and the ground state does not merge into higher energy, so all the matter fields can be integrated out. Generalizing the linear “\(A \wedge \cdots\)” coupling [6–8], we can write down the gradient expansion of \(\Phi_\alpha(x_\mu)\) of the topological actions of a general nonlinear coupling (NLC) among parameters and gauge field in \(D\)-dimensional spacetime \(S^1_0 \times M\):

\[
Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A(x)] = Z_0 \exp \left( i \int_{S^1_0 \times M} \sum_{k=0}^{|d/2|} \Phi^*_{\lambda_k} \wedge \mathcal{L}_{\text{C-S}}^{(2k+1)}[A] \right),
\]

where \(Z_0\) is independent on the gauge field and the Chern-Simons (C-S) density for general gauge field \(A\) is only defined in odd dimensions:

\[
\mathcal{L}_{\text{C-S}}^{(1)}[A] = \text{Tr}A;
\]

\[
\mathcal{L}_{\text{C-S}}^{(3)}[A] = \frac{1}{4\pi} \text{Tr} \left( AdA + \frac{2}{3} A^3 \right); \cdots ,
\]

which, for \(U(1)\) gauge fields, take the form as

\[
\mathcal{L}_{\text{C-S}}^{(2k+1)} = \frac{1}{(k + 1)!} A \wedge \left( \frac{dA}{2\pi} \right)^k
\]

The gradient expansion form \(\lambda_k(m_\alpha)\) is:

\[
\lambda_k \equiv \sum_{\{\beta\}} \lambda_{k;\{\beta\}}(m_\alpha) dm_{\beta_1} \wedge dm_{\beta_2} \wedge \cdots \wedge dm_{\beta_{d-2k}},
\]

which is pulled back by \(\Phi^*\) to a differential form on the spacetime \(S^1_0 \times M\) from the parameter space \(\mathcal{C}_D\):

\[
\Phi^*[\lambda_k] = \sum_{\{\beta\},\{\gamma\}} \lambda_{k;\{\beta\}}[\Phi_\alpha(x)] \partial_{\gamma_1} \Phi_{\beta_1} d\gamma_1 \wedge \cdots \wedge \partial_{\gamma_{d-2k}} \Phi_{\beta_{d-2k}} dx^{\gamma_{d-2k}},
\]

by which we can explicitly see the reason why the action (5) is topological — a total antisymmetric tensor pops out so that the Lagrangian density is a Lorentz scalar without a prefactor \(\sqrt{-g}\) where \(g\) is the determinant of the metric.
III. NONTRIVIAL TOPOLOGY OF THE PARAMETER SPACE $C_D$

Let us put the theory whose low-energy response is characterized by Eq. (5) on a compact spin manifold $S^1_{(0)} \times M$. Then we do a gauge transformation $A \rightarrow A + d\theta$ on Eq. (5), where $\theta(x)$ is a $2\pi$-periodic quantity.

$$Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A + d\theta] = Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A] \exp \left\{ i \int_{S^1_{(0)} \times M} \sum_k \Phi^*[\lambda_k] \wedge d\theta \wedge \frac{1}{k!} \left( \frac{dA}{2\pi} \right)^k \right\} : (11)$$

which is restricted by the gauge-invariance:

$$Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A + d\theta] = Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A]. \quad (12)$$

A “small” gauge transformation — $\theta(x)$ is single-valued and arbitrary — gives the closedness condition

$$d\Phi^*[\lambda_k] = \Phi^*[d\lambda_k] = 0, \quad (13)$$

by integration by part and that the differentiation commutes with the pull-back $d\Phi^* = \Phi^*d$. Since $\Phi$ can be perturbed locally from a constant function, we have the closeness for $\lambda_k$:

$$d\lambda_k = 0. \quad (14)$$

For large gauge transformations where $\theta(x)$ is multi-valued, we denote the Poincare dual of $d\theta/(2\pi)$ as $L_\theta$ which is $d$-cycle in $S^1_{(0)} \times M$. The effective action transforms as

$$Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A + d\theta] = Z_{\text{NLC}}[\{\Phi_\alpha(x)\}, A] \exp \left\{ i2\pi \oint_{L_\theta} \sum_k \Phi^*[\lambda_k] \wedge \frac{1}{k!} \left( \frac{dA}{2\pi} \right)^k \right\}. \quad (15)$$

The gauge invariance (12) together with Eq. (14) and that $\theta(x)$ and $A$ are arbitrary implies

$$\oint_{L_{d-2k}} \Phi^*[\lambda_k] = \oint_{\Phi_*(L_{d-2k})} \lambda_k \in \mathbb{Z}, \quad (16)$$

for each $k \in \{0, 1, \cdots, \lfloor d/2 \rfloor \}$ and arbitrary $(d - 2k)$-cycle $L_{d-2k} \in Z_{d-2k}(S^1_{(0)} \times M, \mathbb{Z})$, and here $\Phi_* : Z_{d-2k}(S^1_{(0)} \times M, \mathbb{Z}) \rightarrow Z_{d-2k}(C_D, \mathbb{Z})$ is the push-forward.

If the nontrivial period in Eq. (16) can be realized by some real system characterized by $\lambda_k(m_\alpha)$ with a closed brane $L_{d-2k}$ and $\Phi$, we can conclude that

$$\mathcal{H}_{d-2k}(C_D, \mathbb{Z}) \supset \mathbb{Z}, \quad (17)$$
where $H_n(C_D, \mathbb{Z})$ is the $n$-th homology of the parameter space $C_D$. It is because, otherwise, the form $\lambda_k(m_\alpha)$ is exact, which sufficiently makes the integration in Eq. (16) vanish. Geometrically, a nontrivial integration in Eq. (16) means that there exists a gap-closing parameter in codimension $(D - 2k)$.

IV. GENERALIZATIONS OF GENERALIZED THOULESS PUMPS

Let us take the following compactifiable co-$(D - 0)$ interface:

$$
\Phi_P(x_\mu|_{\mu=0, \cdots, d}) = \begin{cases} 
\phi_{\alpha}(x_\mu|_{\mu=1, \cdots, d}), & t = x_0 \to -\infty; \\
\phi_{\alpha}(x_\mu|_{\mu=1, \cdots, d}), & t = x_0 \to +\infty,
\end{cases}
$$

which means the system is periodic in time and we choose an adiabatic spacetime-dependent $\Phi_P(x_\mu)$ above so that the system remains gapped with a unique ground state at any time. For simplicity, we still assume the space factorizing as $M = T^d$ so that our spacetime is compactified as $T^D$, but we will relax this condition later. We consider the order-$k$ response pump $\Sigma_{P, k; \mu_1, \cdots, \mu_{2k+1}}$ formally as the coefficient in the front of “$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{2k+1}}$” component of $L^{(2k+1)}_{C-S}$ in Eq. (5) across a spacetime section perpendicular to $S^1_{(\nu_1)} \times \cdots \times S^1_{(\nu_{d-2k})}$:

$$
\Sigma_{P, k; \mu_1, \cdots, \mu_{2k+1}} \equiv \frac{1}{(d - 2k)!} \varepsilon^{\nu_1 \cdots \nu_{d-2k} \mu_1 \cdots \mu_{2k+1}} \oint_{\Phi_P[S^1_{(\nu_1)} \times \cdots \times S^1_{(\nu_{d-2k})}]} \lambda_k \in \mathbb{Z},
$$

where $\{\nu\}$ is summed implicitly. Nontrivial contributions to the integration above result from any oriented spacetime section $S^1_{(\nu_1)} \times \cdots \times S^1_{(\nu_{d-2k})}$ perpendicular to the $x_{\mu_1} \cdots x_{\mu_{2k+1}}$-axes, and such an integration is well-defined since any pair of these sections are cobordant and $\Phi_P[S^1_{(\nu_1)} \times \cdots \times S^1_{(\nu_{d-2k})}]$ is closed. Actually, we can have a more general setting where $S^1_{(\nu_1)} \times \cdots \times S^1_{(\nu_{d-2k})}$ is replaced by any other spin manifold. In a similar sense, even the temporal pump can be made space-like by switching $x_0$ from $t$ to any other spatial component. Since $\lambda_k$ is locally a $(d - 2k)$-form, we need at least $(d - 2k)$ parameters to construct $\Phi_P$ so that the period (19) can be nontrivial. In addition, if $d - 2k = 0$, then the period above vanishes. Physically, it implies the fact that the highest response in even-dimensional space is constant as long as the gap does not close.

We calculate the net charge flow through the section in one cycle $t \in (-\infty, +\infty)$ perpen-
icular to the $x_i$-direction as:

$$Q^P_i = \oint_{S^1(0) \times \cdots \times S^1(i-1) \times S^1(i) \times S^1(i+1) \cdots S^1(d)} \frac{\delta}{i\delta A_i} \ln(Z_{NLC})$$

$$= (-1)^{d-i} \sum_k \oint_{S^1(0) \times \cdots \times S^1(i-1) \times S^1(i) \times S^1(i+1) \cdots S^1(d)} \Phi^P[A_k] \wedge i^1 \left[ \frac{1}{k!} \left( \frac{dA}{2\pi} \right)^k \right]$$

$$= (-1)^{d-i} \sum_k \oint_{\Phi^P} [M_k] \lambda_k,$$  

(20)

where $i$ is the inclusion of the oriented $S^1(0) \times \cdots \times S^1(i-1) \times S^1(i) \times S^1(i+1) \cdots S^1(d)$ into the full spacetime, and $M_k^{(i)}$ is the Poincare dual to the pullback $i^1[(dA/2\pi)^k/k!]$. Since the integration in the second line above is done on a closed form, it is independent of the choice $S^1(0) \times \cdots \times S^1(i-1) \times S^1(i) \times S^1(i+1) \cdots S^1(d)$ within the same homology. It is related to the terms in Eq. (20) as follows.

A. Reduction to $D > 0, k = 0$: Thouless pumps in high dimensions

The Thouless pump (T-P) in high dimensions [7] implies that a spacetime-dependent (depending on $d$ coordinates) parameter, will change by an integer number the total charges in the section perpendicular to the hypersurface spanned by those $d$ coordinates. It can be deduced from our result in Eq. (20) as a special case of $k = 0$ with $x_i$ the $i$-th spatial component and $dA = 0$:

$$Q^P_i = \sum_{0;i}^P = (-1)^{d-i} \oint_{\Phi^P} [S^1(0) \times \cdots \times S^1(i) \times S^1(i+1) \cdots S^1(d)] \lambda_0 \in \mathbb{Z},$$  

(21)

where $^\wedge$ means a deletion. Since $\lambda_0$ is locally a $d$-form, it is necessary to have at least $d$ parameters in order that the period above can be nontrivial.

B. Detection related to higher responses: nontrivial background instantons

As mentioned in the Introduction part, we can measure the higher-order response pump by adiabatic flux insertions [22] or static flux configuration, or formally, winding surfaces of spacetime manifold around monopoles — instantons. In the presence of nontrivial background instanton, we obtain that the total charge pump along the $x_i$-direction is determined
by higher-order responses:

\[ Q^P_i = \sum_{k=1}^{\lfloor d/2 \rfloor} \frac{1}{(2k)!} \mathcal{P}_{\mu_1\cdots\mu_{2k}} \sum_{\mu_{2k+1} = i} \cdots \sum_{\mu_{k+1} = i} P_{\mu_1\cdots\mu_k} \]  \hfill (22)

where the spacetime indices \( \{\mu\} \) are summed implicitly in advance to the summation of \( k \), and \( \mathcal{P}_{\mu_1\cdots\mu_{2k}} \) is the instanton number:

\[ \mathcal{P}_{\mu_1\cdots\mu_{2k}} \equiv \oint_{S^1_{(\mu_1)} \times \cdots \times S^1_{(\mu_{2k})}} \frac{1}{k!} \left( \frac{dA}{2\pi} \right)^k \in \mathbb{Z}. \]  \hfill (23)

If we switch \( i \)-th direction above to be temporal, we would have obtained the corresponding spatial-like Thouless pump as the spin example in the Introduction.

1. **Example: adiabatic flux insertion in \( D = 3 + 1 \)**

Let us adiabatically insert a flux threading the \( x_1 \)-loop:

\[ \oint A_1(t, \vec{x}) dx^1 = 2\pi N_{\text{flux}} \frac{t}{T}, \quad t \in [0, T], \]  \hfill (24)

where \( T \) is large enough so that the initial many-body state remains staying at the ground state, and other background gauge-field components vanish \( A_0 = A_2 = A_3 = 0 \). This flux insertion procedure is a cycling of the period \( T \) because the flux at \( t = T \) can be eliminated by a large gauge transformation. Then we can obtain:

\[ Q^P_i = \Sigma^P_{0;i} + N_{\text{flux}} \Sigma^P_{1;01}, \quad i \in \{2, 3\}. \]  \hfill (25)

In addition to the first term which is Thouless-pumped charge, the second term is understood as the transverse charge flow induced by the electric field \( \partial_0 A_1 \) due to the Hall conductance of a spatial slice spanned by \( S^1_{(1)} \times S^1_{(i)} \). Therefore, the second term here is not a consequence resulting from the temporal cycling of parameter fields \( \Phi^P_\alpha \)'s (which do not include \( A_\mu \)).

2. **Example: static flux configuration in \( D = 3 + 1 \)**

We add a static background flux through the plane \( S^1_{(1)} \times S^1_{(2)} \):

\[ \oint \left[ \nabla \times \vec{A} \right]_{x_3} = \oint \partial_1 A_2 - \partial_2 A_1 = 2\pi N_{\text{flux}}, \]  \hfill (26)
with other background components vanishing \( A_0 = A_3 = 0 \). Then

\[
Q^p_3 = \Sigma^p_{0,3} + N_{\text{flux}} \Sigma^p_{1;123}.
\] (27)

The first term \( \Sigma^p_{0,3} \) is the net Thouless-pumped charge flowing along \( x_3 \)-direction in the absence of background fluxes. As a special case, the second term can result from the net Hall conductance of interface (spanned by \( S_{(1)}^1 \times S_{(2)}^1 \)) flowing along \( x_3 \)-direction is \( \Sigma^p_{1;123} \) and such interfaces can carry charge due to the static flux \( 2\pi N_{\text{flux}} \) through \( S_{(1)}^1 \times S_{(2)}^1 \).

V. OTHER CONSEQUENCES: INTERFACE PHASE TRANSITIONS

In this part, we will discuss other physical consequences brought by the potentially non-trivial periods of the integration (16) by various types of interfaces. We will propose that such nontrivial periods will contribute to the change of physical observables during the phase transitions between various interfaces. Here we define that the co-(\( D - n \))-interface phase transition is driven by the interactions that do not change the asymptotical parameter fields, e.g. the \( x_n \to \pm \infty \) behavior in Eq. (4).

We will first discuss the most visualizable \( D = 3 + 1 \) case beyond the linear coupling in (5), followed by the generalizations of both the coupling orders and spacetime dimensions.

A. Phase transition between co-1 interfaces

The linear coupling cases where \( k = 0 \) are interpreted as a field-theoretical Thouless pumps because the integral \( \oint L \Phi^* [\lambda_0] \) is the net charge flow across the brane \( L \). Formally, our NLC generalization in Eq. (5) can be understood as the generalized nonlinear Thouless pumps in the presence of nontrivial background instanton \( \oint (dA)^k \neq 0 \) in various higher dimensions.

First, we will manifest the physical implication of our NLC action with \( D = 3 + 1 \) and \( k = 1 \) in the following interface approach. Let us take the interface to stay between two asymptotic bulk parameters \( \{ m^L_\alpha \} \) and \( \{ m^R_\alpha \} \), respectively:

\[
\Phi_\alpha(t, x_1, x_2, x_3) = \Phi_\alpha(x_3) = \begin{cases} 
m^L_\alpha, & x_3 \to -\infty; 
m^R_\alpha, & x_3 \to +\infty, 
\end{cases}
\] (28)
and we can see the interface as a finite thickness in $x_3 \in [-h, +h]$ where $\Phi_\alpha(x_\mu)$ deviates the deep-bulk values on its two sides. Since the parameters do not converge at the infinity, the spacetime is inevitably non-compact. We can obtain the quantum Hall conductance of this interface from Eq. (5)

$$\sigma_H = \int_{x_3 \in (-\infty, +\infty)} \Phi^* [\lambda_1]$$

$$= \int_{\Phi_{x_3}} \lambda_1,$$

(29)

where $\Phi_{x_3}$ is the path in $C_D$ parameterized by $x_3$ determined by $\Phi_\alpha(x_3)$ in Eq. (28).

The interface undergoes a quantum phase transition and the resultant gapped interface is characterized $\tilde{\Phi}_\alpha(t, x_1, x_2, x_3)$ which also satisfies

$$\tilde{\Phi}_\alpha(t, x_1, x_2, x_3) = \tilde{\Phi}_\alpha(x_3) = \begin{cases} m_L^\alpha, & x_3 \to -\infty; \\ m_R^\alpha, & x_3 \to +\infty, \end{cases}$$

(30)

because, by the definition of the interface quantum phase transition, the interactions are imposed beyond the deep bulks on two sides. Similarly, its quantum Hall conductance can be calculated as

$$\tilde{\sigma}_H = \int_{x_3 \in (-\infty, +\infty)} \tilde{\Phi}^* [\lambda_1]$$

$$= \int_{\tilde{\Phi}_{x_3}} \lambda_1,$$

(31)

where the path $\tilde{\Phi}_{x_3}$ is the path in $C_D$ parametrized by $\tilde{\Phi}(x_3)$ in Eq. (30).

We are interested in the conductance difference brought by such an interface quantum phase transition:

$$\Delta \sigma_H \equiv \tilde{\sigma}_H - \sigma_H$$

$$= \oint_{\Psi} \lambda_1 \in \mathbb{Z},$$

(32)

where $\Psi$ and $S^1_{(3)}$ is determined by the following “compactifiable” setting:

$$\Psi_\alpha(t, x_1, x_2, x_3) = \begin{cases} \tilde{\Phi}_\alpha(t, x_1, x_2, x_3 + H), & x_3 \in (-\infty, 0]; \\ \Phi_\alpha(t, x_1, x_2, -x_3 - H), & x_3 \in (0, +\infty), \end{cases}$$

(33)

where $H \gg h$ and $S^1_{(3)}$ is the one-point compactification of the axis $x_3 \in (-\infty, +\infty)$, which can be done due to $\Psi$ satisfies

$$\lim_{x_3 \to \pm \infty} \Psi_\alpha(t, x_1, x_2, x_3) = m^L_\alpha,$$

(34)
because of Eqs. (28,30). The interface quantum Hall conductance jump $\Delta \sigma_H$ during the interface quantum phase transitions is quantized and constrained by the cohomology class $\lambda_1$. This result is consistent with the gapped boundary of topological insulators, which were understood by stacking of additional integer quantum Hall states.

Let us further consider how the total number of charges change across the interface quantum phase transition. The charge can be read from the coefficient in the front of $L_{C\text{-S}}^{(1)}$ and thus its change is

$$\Delta Q_0 = 0,$$

because $\Phi_{x_3}$ and $\tilde{\Phi}_{x_3}$ are paths in $C_{D=4}$ while $\lambda_0(m_\alpha)$ is locally a $(D - 1)$-dimensional form. Similarly, the net charge $\Delta Q_i$ flowing along $x_i$-direction does not change, either. The geometric interpretation of $\Delta Q_i = 0$ in the current case is obvious because the singularity of $\lambda_0(m_\alpha)$ occurs in at least co-dimension 4 and the two-dimensional area bounded by a loop can generically avoid to intersect it. Physically, the interface parameter fields $\Phi(x_3)$ and $\tilde{\Phi}(x_3)$ induce (massive) modes within the finite thickness $x_3 \in [-h,+h]$, so it can be seen as a $(2+1)$-dimensional spacetime. The phase transitions for such a $(2 + 1)$-dimensional system occur around its gap closing and reopening, during which the charge cannot obtain any finite change. The analysis here implies that charge change cannot characterize the co-1 interface quantum phase transition among gapped phases for $D \geq 3$.

Generally, if $D \in \mathbb{Z}_{\text{even}}$, we have two interfaces as:

$$\Phi_\alpha(t, x_1, \cdots, x_d) = \Phi_\alpha(x_d) = \begin{cases} m_\alpha^L, & x_d \to -\infty; \\ m_\alpha^R, & x_d \to +\infty, \end{cases}$$

$$\tilde{\Phi}_\alpha(t, x_1, \cdots, x_d) = \tilde{\Phi}_\alpha(x_d) = \begin{cases} m_\alpha^L, & x_d \to -\infty; \\ m_\alpha^R, & x_d \to +\infty. \end{cases}$$

Only the highest response corresponding to $L_{C\text{-S}}^{(2[d/2]+1)}$ in Eq. (5) can characterize the quantum phase transition between them:

$$\Delta \Sigma_{[d/2]} = \oint_{\Psi_{\alpha}(S_{[d]}^1)} \lambda_{[d/2]} \in \mathbb{Z},$$

where $S_{[d]}^1$ is defined as the one-point compactification of $d$-th spatial component of the composite system:

$$\Psi_\alpha(t, x_1, \cdots, x_d) = \Psi_\alpha(x_d) = \begin{cases} \tilde{\Phi}_\alpha(x_d + H), & x_d \in (-\infty, 0]; \\ \Phi_\alpha(-x_d - H), & x_d \in (0, +\infty). \end{cases}$$
Since the integrated domains are still one-dimensional paths,

\[ \Delta \Sigma_k \in \begin{cases} 
\mathbb{Z}, & k = \lfloor d/2 \rfloor; \\
\{0\}, & \text{otherwise}, 
\end{cases} \quad (40) \]

where \( \Sigma_0 \equiv Q_\mu \) and \( \Sigma_1 \equiv \sigma_{H,\mu\nu\rho} \) for instance [23].

On the other hand, if \( D \in \mathbb{Z}_{\text{odd}} \), even the highest response also gives a trivial constraint. It is because the gauge invariance requires the pull-back \( \Phi^*[\lambda_{d/2}(m_\alpha)] \) to be a constant sufficiently pinned by the value at \( x_3 \to \pm \infty \), which is unchanged during the interface quantum phase transition. Nevertheless, we will see that a nontrivial obstruction to identifying points in \( C_D \) can have physical observable for any integer \( D \in \mathbb{Z} \), which signals a \(-1\)-form anomaly.

**B. Example: constraints from \( U(1) \) and magnetic translations**

On a two-dimensional lattice, we can insert uniform magnetic fluxes \( 2\pi p/q \) per unit cell and the translation symmetries \( T_{1,2} \in \mathcal{M} \) (here \( \mathcal{M} \) means magnetic translations [24–27]) have the following property:

\[ T_1 T_2 T_1^{-1} T_2^{-1} = \exp \left( i 2\pi \frac{p}{q} \hat{F} \right), \quad (41) \]

where \( \hat{F} \) is the fermion number operator and \( \gcd(p,q) = 1 \). In this part, we will enhance the symmetry to \( U(1) \times \mathcal{M} \) and discuss its effects on the quantum phase transitions among gapped interfaces.

Let us assume that such a two-dimensional lattice is dynamically generated on some interface between two three-dimensional bulks. Therefore, the quantum phase transitions in the original two-dimensional system can be equivalently seen as the interface quantum phase transition. This assumption is physically sensible if we make the energy gap of the bulks on two sides much larger than the characteristic energy scale around interface. The background gauge field for \( U(1) \times \mathcal{M} \) is \( A = A \otimes \mathbb{I}_q \times q \) with a \( \mathcal{M} \)-twisted boundary condition. It means we should replace \( A \) in Eq. (5) with \( A \) and the trace over the indices in \( \mathcal{M} \) introduces an extra \( q \)-factor for the constraint: in addition to \( \Delta Q_{p/q} = 0 \),

\[ \Delta \sigma_{H,p/q} = q \oint_{\Psi_+[\Sigma_1]} \lambda_1 \in q\mathbb{Z}, \quad (42) \]

where \( \Psi \) is defined in Eq. (33). Such a Hall conductance jump which is a multiplier of \( q \) is consistent with the free electron consideration by the \( q \)-fold critical Dirac cones around
the gap closing [28]. Indeed, when the two-dimensional interface system is half-filled and
$q \in \mathbb{Z}_{\text{even}}$ [29], the function $\Phi(x_3)$ and $\tilde{\Phi}(x_3)$ here can be taken as the chiral mass term of
the bulk massive $q$-flavor Dirac fermion:

$$\Phi(x_3) = m_0 \exp \left[ i \gamma^5 \alpha(x_3) \right] = \begin{cases} m_0, & x_3 \to -\infty; \\ -m_0, & x_3 \to +\infty, \end{cases}$$

(43)

$$\tilde{\Phi}(x_3) = m_0 \exp \left[ i \gamma^5 \tilde{\alpha}(x_3) \right] = \begin{cases} m_0, & x_3 \to -\infty; \\ -m_0, & x_3 \to +\infty, \end{cases}$$

(44)

which are diagonal in flavor space due to the $\mathcal{M}$-symmetry, and $\gamma^5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Since $\alpha$ and $\tilde{\alpha}$ are both defined mod $2\pi$, their winding number difference gives the integer factor in
Eq. (42) obtained by calculating a global chiral anomaly [27].

C. Transitions between co-(D – n) interfaces

The transition between co-1 interfaces can be classified by the jump constraint of the highest response quantized corresponding to $\lambda_{[d/2]}$, so it can be used to diagonalize the (co)homology of the parameter space $\mathcal{C}_D$. In this part, we will show that the lower responses can obtain nonzero jump in the case of transition between co-(D – n) interfaces:

$$\Phi_\alpha(\{x_\mu\}_{\mu=0,\ldots,d}) = \Phi_\alpha(\{x_\mu\}_{\mu=n,\ldots,d}) = \begin{cases} \Phi_\alpha(\{x_\mu\}_{\mu=n+1,\ldots,d}), & x_n \to -\infty; \\ \Phi_\alpha(\{x_\mu\}_{\mu=n+1,\ldots,d}), & x_n \to +\infty, \end{cases}$$

(45)

and

$$\tilde{\Phi}_\alpha(\{x_\mu\}_{\mu=0,\ldots,d}) = \tilde{\Phi}_\alpha(\{x_\mu\}_{\mu=n,\ldots,d}) = \begin{cases} \tilde{\Phi}_\alpha(\{x_\mu\}_{\mu=n+1,\ldots,d}), & x_n \to -\infty; \\ \tilde{\Phi}_\alpha(\{x_\mu\}_{\mu=n+1,\ldots,d}), & x_n \to +\infty. \end{cases}$$

(46)

Similarly, we can define the following compactifiable setting:

$$\Psi_\alpha(\{x_\mu\}_{\mu=0,\ldots,d}) = \Psi_\alpha(\{x_\mu\}_{\mu=n,\ldots,d}) = \begin{cases} \tilde{\Phi}_\alpha(\{x_\mu + \delta_{\mu}^n H\}_{\mu=n,\ldots,d}), & x_n \in (-\infty, 0]; \\ \Phi_\alpha(\{(-1)\delta_{\mu}^n x_\mu - \delta_{\mu}^n H\}_{\mu=n,\ldots,d}), & x_n \in (0, +\infty). \end{cases}$$

(47)

Thus we can obtain

$$\Delta \Sigma_{\{(n-1)/2;0,\ldots,n-1\}} = \oint_{\Psi^* (s_{(n)}^1 \times \cdots \times s_{(d)}^1)} \lambda_{(n-1)/2} \in \mathbb{Z}.$$  (48)
In addition, any lower response does not gain finite jump:

$$
\Delta \Sigma_k \in \begin{cases} 
\mathbb{Z}, & k = [(n-1)/2]; \\
0, & k < (n-1)/2,
\end{cases}
$$

(49)

because $\Phi^*\{S^1_{(n)} \times \cdots \times S^1_{(d)}\}$ and $\tilde{\Phi}^*\{S^1_{(n)} \times \cdots \times S^1_{(d)}\}$ are $(d-n+1)$-cycle on which $\lambda_k$, a $(d-2k)$-cocycle, vanishes if $k < (n-1)/2$.

It should be noted that we can replace $S^1_{(n+1)} \times \cdots \times S^1_{(d)}$ by any other spin manifold without changing the arguments above. Here our specification to high-dimensional torus is only for sake of convenience in presence and its relevance with square lattices.

VI. GENERALIZED $-1$-FORM ANOMALIES

Since the $D$-dimensional systems parametrized by $\mathcal{C}_D$ are defined to be gapped, their Hilbert spaces, at the low-energy limit, consisting of one single state cannot be distinguished by the partition function. It is attempting to identify their RG fixed point parameters in $\mathcal{C}_D$. However, they might be distinguishable once coupled with a background gauge field of symmetry. Therefore, the certain symmetry obstructs such an identification. As we will see below, this obstruction has the physical consequence as inevitable gap closing of interfaces.

Actually, we have already seen such phenomena in symmetry-respected interfaces between topological insulators or other invertible phases, e.g. $U(1)$-respected interfaces carrying chiral modes between two distinct integer quantum Hall phases.

Before considering the general $D$ dimensions, let us motivate the idea in $D = 3$ which is relevant to the interfaces sandwiched by distinct quantum Hall phases. Then one of the relevant constraints on the gradient expansion (5) is

$$
d\Phi^*[\lambda_1] = 0,
$$

(50)

which restricts the form of $\Phi$.

However, if we identify two RG fixed parameters $\tau_\alpha(x_\mu) \sim \kappa_\alpha(x_\mu) : S^1_{(0)} \times M \to \mathcal{C}_3$ with

$$
\tau^*[\lambda_1] \neq \kappa^*[\lambda_1] \in Z^0(S^1_{(0)} \times M, \mathbb{Z}),
$$

(51)

which makes the following co-1 boundary condition periodic:

$$
\Phi_\alpha(t, x_1, x_2) = \Phi_\alpha(x_2) = \begin{cases} 
\tau_\alpha, & x_2 \to -\infty; \\
\kappa_\alpha, & x_2 \to +\infty,
\end{cases}
$$

(52)
where, for simplicity, we have assumed \( \tau_\alpha(x_\mu) \) and \( \kappa_\alpha(x_\mu) \) to be constant function that is consistent with Eq. (51) since \( \lambda_1 \) here is locally 0-form. The boundary condition (52) is the background gauge field for the \( -1 \)-form symmetry that the partition functions are the same for the parameters \( \tau_\alpha \) and \( \kappa_\alpha \) at the low-energy limit without background \( U(1) \)-gauge field. However, Eq. (50) has no solution in the presence of Eq. (52). Thus, if we physically impose the boundary condition (52), the whole system cannot be gapped and the bulk must undergo a gap closing at some \( x_2 \). Since \( \lambda_1 \) here exactly corresponds to the Hall conductance of the bulk, we re-derive the fact that the interface between distinct integer quantum Hall phases cannot have a unique ground state, from our general \( D = 3 \) treatment above. Thus we say that the system with the gapped parameter space \( C_3 \) has \( -1[0] \)-form anomaly if there exist two RG fixed points with distinct \( \lambda_1 \) values. Here the level “[0]” of \( -1[0] \)-form is defined as the degree of the form \( \lambda_1 \) in Eq. (51).

Furthermore, we assume that our identification is performed within the restricted image space in \( C_3 \), in which \( \lambda_1 \) is constant. One nontrivial choice is to identify two RG-fixed parameter fields \( \Gamma_\alpha(x_\mu) \sim K_\alpha(x_\mu) : S^1_{(0)} \times M \to C_3 \) with distinct 2-cycles:

\[
\oint_M \Gamma^*[\lambda_0] \neq \oint_M K^*[\lambda_0]. \tag{53}
\]

The system has a \( -1 \)-form symmetry that the low-energy partition function is the same for \( \Gamma_\alpha \) and \( K_\alpha \) parameters, denoted by \( -1[2] \)-form symmetry where the level “[2]” corresponds to the degree of the cycle \( \lambda_0 \) above. Such a symmetry is also inconsistent with being gapped after we introduce a background gauge field. Let us assume that our system is time-dependent and obeys a “periodic” boundary condition along time:

\[
\Phi_\alpha(t, x_1, x_2) = \begin{cases} 
\Gamma(x_1, x_2), & t \to -\infty; \\
K(x_1, x_2), & t \to +\infty,
\end{cases} \tag{54}
\]

where we also simply assume \( \Gamma \) and \( K \) depends on two coordinates, namely co-3 interfaces, consistently with Eq. (53). The boundary condition (54) is a \( -1[2] \)-form symmetry background gauge field. However, this boundary condition globally conflicts with the gauge-invariance condition:

\[
d\Phi^*[\lambda_0] = 0, \tag{55}
\]

so the system must undergo a gap closing at some time point \( t \). Physically, the \( U(1) \)-charge jumps by \( \oint_M K^*[\lambda_0] - \oint_M \Gamma^*[\lambda_0] \neq 0 \) and it sufficiently implies that the gap must be closed
during this procedure although the Hall conductance does not gain a net change. Such an anomalous identification $\Gamma_\alpha(x_\mu) \sim K_\alpha(x_\mu)$ is denoted by $-1_{[2]}$-form anomaly.

It is straightforward to generalize the arguments above to arbitrary dimensions. We can define a $-1_{[n]}$-form anomaly for $n = d \mod 2$, in the following way. Two RG-fixed parameter functions are identified $\mathcal{T}_\alpha(x_\mu) \sim \mathcal{K}_\alpha(x_\mu) : S^1_{(0)} \times M \rightarrow C_D$ with distinct $n$-cycles:

$$\oint \mathcal{T}^* [\lambda_{(d-n)/2}] \neq \oint \mathcal{K}^* [\lambda_{(d-n)/2}],$$

where the integration is performed on the subspace spanned by the last $n$ space-time coordinates. The anomaly of such an identification can be detected by the following “periodic” boundary condition written in spacetime:

$$\Phi_\alpha(\{x_\mu\}) = \begin{cases} 
\mathcal{T}(\{x_\mu\}|_{\mu=d-n+1,\cdots,d}), & x_{\mu=d-n} \rightarrow -\infty; \\
\mathcal{K}(\{x_\mu\}|_{\mu=d-n+1,\cdots,d}), & x_{\mu=d-n} \rightarrow +\infty,
\end{cases}$$

where we also assume that $\mathcal{T}$ and $\mathcal{K}$ are co-$(n+1)$ interface consistently with Eq. (56). The background parameter-gauge field (57) for $-1_{[n]}$-form symmetry is inconsistent with the gauge-invariance condition

$$d\Phi^* [\lambda_{(d-n)/2}] = 0.$$ 

It means that the gap must be closed at some slice along $x_{\mu=d-n}$. We denote the symmetry obstruction to the identification $\mathcal{T}_\alpha(x_\mu) \sim \mathcal{K}_\alpha(x_\mu)$ here as $-1_{[n]}$-form anomaly. Our discussion above generalizes the original $-1$-form anomaly.

ACKNOWLEDGMENTS

The author was supported by JSPS fellowship. This work was supported by JSPS KAKENHI Grant Nos. JP19J13783. A part of the present work was performed at Kavli Institute for Theoretical Physics, University of California at Santa Barbara, supported by US National Science Foundation Grant No. NSF PHY-1748958.

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).

[2] D. J. Thouless, Phys. Rev. B 27, 6083 (1983).
[3] Q. Niu, Phys. Rev. B 34, 5093 (1986).
[4] F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).
[5] I. Affleck, Journal of Physics: Condensed Matter 1, 3047 0953 (1989).
[6] A. Kapustin and L. Spodyneiko, arXiv preprint arXiv:2001.03454 (2020).
[7] A. Kapustin and L. Spodyneiko, arXiv preprint arXiv:2003.09519 (2020).
[8] P.-S. Hsin, A. Kapustin, and R. Thorngren, arXiv preprint arXiv:2004.10758 (2020).
[9] J. C. Teo and C. L. Kane, Phys. Rev. B 82, 115120 (2010).
[10] A. Abanov and P. B. Wiegmann, Nucl. Phys. B 570, 685 (2000).
[11] D. S. Freed, Journal of Differential Geometry 80, 45 (2008).
[12] In general dimensions, the fluxes through intersecting surfaces result in nontrivial instantons.
[13] R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).
[14] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B 78, 195424 (2008).
[15] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[16] C. Cordova, D. S. Freed, H. T. Lam, and N. Seiberg, arXiv preprint arXiv:1905.09315 (2019).
[17] C. Cordova, D. S. Freed, H. T. Lam, and N. Seiberg, SciPost Physics Proceedings 8 (2020).
[18] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010).
[19] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
[20] E. Witten, Rev. Mod. Phys. 88, 035001 (2016).
[21] E. Witten and K. Yonekura, arXiv preprint arXiv:1909.08775 (2019).
[22] M. Oshikawa, Phys. Rev. Lett. 84, 1535 (2000).
[23] Responses $\Sigma_k$ are defined for $(2k + 1)$-dimensional subspace of spacetime.
[24] J. Zak, Phys. Rev. 134, A1602 (1964).
[25] J. Zak, Phys. Rev. 134, A1607 (1964).
[26] Y.-M. Lu, Y. Ran, and M. Oshikawa, Ann. Phys., 168060 0003 (2020).
[27] Y. Yao and M. Oshikawa, arXiv preprint arXiv:1906.11662 (2019).
[28] M. Oshikawa, Phys. Rev. B 50, 17357 (1994).
[29] The half filling here is in the sense of physical unit cell and a higher dimensional Lieb-Schultz-Mattis theorem requires an even $q$ [22, 30].
[30] M. B. Hastings, Phys. Rev. B 69, 104431 (2004).