Asymptotics of Toeplitz Determinants Generated
by Functions with Fourier Coefficients
in Weighted Orlicz Sequence Classes

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Abstract. We prove asymptotic formulas for Toeplitz determinants generated by functions with sequences of Fourier coefficients belonging to weighted Orlicz sequence classes. We concentrate our attention on the case of nonvanishing generating functions with nonzero Cauchy index.

1. Introduction and main results

1.1. Strong Szegő’s limit theorem for positive generating functions.
Let \( \mathbb{T} \) be the unit circle. For a complex-valued function \( a \in L^1(\mathbb{T}) \), let \( \{a_k\}_{k=-\infty}^{\infty} \) be the sequence of the Fourier coefficients of \( a \),

\[
a_k := \frac{1}{2\pi} \int_{0}^{2\pi} a(e^{i\theta})e^{-ik\theta} d\theta.
\]

Consider the determinants \( D_n(a) \) of the finite Toeplitz matrices \( T_n(a) \),

\[
D_n(a) = \det T_n(a) = \det(a_{j-k})_{j,k=0}^{n} \quad (n \in \mathbb{Z}_+),
\]

where, as usual, \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \) and \( \mathbb{N} := \{1, 2, \ldots\} \). In 1952, Gabor Szegő [S52] proved that if \( a \) is a positive function with Hölder continuous derivative, then

\[
D_n(a) = G(a)^{n+1} E(a) \{1 + o(1)\} \quad \text{as} \quad n \to \infty,
\]

where

\[
G(a) := e^{(\log a)e} = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log a(e^{i\theta})d\theta \right)
\]

and

\[
E(a) := \exp \left( \sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k} \right)
\]

with \( (\log a)_k \) the Fourier coefficients of \( \log a \). Basor [B85] writes: “It is interesting to note that this formula was an important aspect of Lars Onsager’s derivation for the spontaneous magnetization of a two-dimensional Ising lattice. The formula, for

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some special $a$, was proposed to Szegő by S. Kakutani, who heard it from Onsager. For the importance of this asymptotic formula in the Ising model, see [MW73] and also [BS99] Section 5.2. The smoothness conditions needed by Szegő were subsequently relaxed by many authors including Kac, Baxter, Hirschman, Krein, Devinatz, and others. Finally, Ibragimov proved in 1968 that (1.1) is true if the constants $E(a)$ and $G(a)$ are well defined and $a$ is positive. For several different proofs of this beautiful result, see Simon [S05] Chap. 6.

1.2. Nonvanishing functions with zero Cauchy index. Let $C(T)$ be the Banach algebra of all complex-valued continuous functions with the maximum norm. We will denote the Cauchy index of a function $a \in C(T)$ by ind $a$. Baxter (1963) and Hirschman (1966) were the first to replace the positivity of $a$ in (1.1) by the condition that

$$a(t) \neq 0 \quad \text{for all} \quad t \in T \quad \text{and} \quad \text{ind} \ a = 0.$$  

(1.2)

To formulate results of this kind precisely, we will define some smoothness classes in terms of the decay of the Fourier coefficients.

1.3. The Wiener algebra. Let $W$ be the Wiener algebra of all complex-valued functions $a$ on $T$ of the form

$$a(t) = \sum_{k=-\infty}^{\infty} a_k t^k \quad (t \in T) \quad \text{for which} \quad \|a\|_W := \sum_{k=-\infty}^{\infty} |a_k| < \infty.$$  

It is well known that $W$ is a Banach algebra under the norm $\| \cdot \|_W$ and that $W$ is continuously embedded into $C(T)$.

1.4. Functions with Fourier coefficients in weighted Orlicz sequence classes. Let $p : [0, \infty) \to [0, \infty)$ be a right-continuous non-decreasing function such that $p(0) = 0$, $p(t) > 0$ for $t > 0$, and $\lim_{t \to \infty} p(t) = \infty$. Then the function $q(s) = \sup\{t : p(t) \leq s\}$ (defined for $s \geq 0$) has the same properties as the function $p$. The convex functions $\Phi$ and $\Psi$ defined by the equalities

$$\Phi(x) := \int_{0}^{x} p(t)dt, \quad \Psi(x) := \int_{0}^{x} q(s)ds \quad (x \geq 0)$$

are called complementary $N$-functions (see, e.g., [KR61] Section 1.3, [M89] Ch. 8, [M83] Section 13). An $N$-function $\Phi$ is said to satisfy the $\Delta^0_2$-condition if

$$\limsup_{x \to 0} \frac{\Phi(2x)}{\Phi(x)} < \infty.$$  

Any sequence $\{\nu_k\}_{k=0}^{\infty}$ of positive numbers is called a weight sequence. We denote by $W$ the set of all weight sequences $\{\nu_k\}_{k=0}^{\infty}$ such that

(a) $\nu_0 = 1$;
(b) $\nu_{k-1} \leq \nu_k$ for $k \in \mathbb{N}$;
(c) $\{\nu_k\}_{k=0}^{\infty}$ satisfies the $\Delta^0_2$-condition, that is, there is a constant $C_\nu \in (0, \infty)$ such that $\nu_{2k} \leq C_\nu \nu_k$ for $k \in \mathbb{N}$.

It is easy to see that $C_\nu \geq 1$. 
Given $N$-functions $\Phi, \Psi$ and weight sequences $\varphi = \{\varphi_k\}_{k=0}^{\infty}$, $\psi = \{\psi_k\}_{k=0}^{\infty}$, denote by $F_{\varphi,\psi}^{p,q}$ the set of all complex-valued functions $a \in L^1(\mathbb{T})$ satisfying

\begin{equation}
\sum_{k=1}^{\infty} \Phi(|a-k|\varphi_k) + \sum_{k=0}^{\infty} \Psi(|a_k|\psi_k) < \infty.
\end{equation}

If $\Phi(t) = t^p/p, \Psi(t) = t^q/q$ with $p, q \geq 1$ or $\varphi(k) = (k+1)^{\alpha}, \psi(k) = (k+1)^{\beta}$ with $\alpha, \beta \geq 0$, we will simply write $F_{\varphi,\psi}^{p,q}$ or $F_{\alpha,\beta}^{p,q}$ instead of $F_{\varphi,\psi}^{p,q}$.

### 1.5. The strong Szegő limit theorem à la Hirschman

Hirschman [H66] proved that $W \cap F_{1/2,1/2}^{2,2}$ is a Banach algebra under pointwise multiplication and if $a \in W \cap F_{1/2,1/2}^{2,2}$ is a complex-valued function satisfying (1.2), then (1.1) is valid. Böttcher and Silbermann [BS80] (see also [BS06, Corollary 10.45]) proved a non-symmetric version of the latter result. Suppose $\alpha \in [0,1], p > 1$, and, in addition, $W \cap F_{\alpha,1}^{p,p/(p-1)}$ is a Banach algebra under pointwise multiplication. If $a \in W \cap F_{\alpha,1}^{p,p/(p-1)}$ satisfies (1.2), then (1.1) holds. They conjectured also that $W \cap F_{\alpha,1}^{p,p/(p-1)}$ is always a Banach algebra whenever $\alpha \geq 0, \beta \geq 0$ and $p \geq 1, q \geq 1$. This conjecture was proved by Horbach in 1984 (see [BS06, Theorem 6.54]).

Using the ideas of the proof of [BS06, Theorem 6.54], the author [K04] proved that if $\Phi, \Psi$ are $N$-functions both satisfying the $\Delta_2^0$-condition and $\varphi, \psi \in W$, then $W \cap F_{\varphi,\psi}^{p,q}$ is a Banach algebra under pointwise multiplication and an appropriate norm (see also Lemma 2.4). Using this result and ideas of the proof of [BS06, Corollary 10.45], the author and Santos [KS05] obtained the following version of the strong Szegő limit theorem.

**Theorem 1.1.** Suppose $\Phi, \Psi$ are complementary $N$-functions both satisfying the $\Delta_2^0$-condition, $\varphi = \{\varphi_k\}_{k=0}^{\infty}$, $\psi = \{\psi_k\}_{k=0}^{\infty}$ are weight sequences in $W$, and there exists a constant $M \in (0,\infty)$ such that $k \leq M \varphi_k \psi_k$ for all $k \in \mathbb{Z}_+$. If $a \in W \cap F_{\varphi,\psi}^{p,q}$ satisfies (1.2), then (1.1) holds.

### 1.6. Further results on asymptotics of Toeplitz determinants

There exist generalizations of the strong Szegő limit theorem into different directions. For instance, Widom [W76] extended (1.1) to the case of matrix-valued generating functions and Böttcher and Silbermann [BS94] extended further this result to the case of operator-valued generating functions.

On the other hand, if one of the assumptions in (1.2) is not satisfied, then the asymptotic formula (1.1) may fail. The asymptotics of Toeplitz determinants with generating functions which do not satisfy one of the conditions in (1.2) were first considered by Fisher and Hartwig [FH68, FH69]. In particular, they conjectured the asymptotic behavior of Toeplitz determinants with some interesting singular generating functions (the so-called Fisher-Hartwig generating functions). Later their conjecture was proved in many particular cases in papers by Basor, Böttcher, Silbermann, Widom, and others. Basor and Tracy [BT91] provided a series of counterexamples to the Fisher-Hartwig conjecture and stated a generalized conjecture. Ehrhardt [E97] made a significant progress in proving the Basor-Tracy conjecture. We refer for the status of this problem to Ehrhardt’s survey [E01] and also to [BS06, Sections 10.57–10.80].

Many aspects of asymptotic behavior of Toeplitz determinants are considered in the monographs by Grenander and Szegő [GS58], Böttcher and Grudsky [BG05],...
Böttcher and Silbermann [BS83, BS99, BS06], and Simon [S05], where the reader can find further results, historical remarks, and references.

1.7. The case of a nonvanishing Cauchy index: known results. In this paper we embark on the case of nonsingular generating functions with nonzero Cauchy index. Fisher and Hartwig [FH68, FH69] proved asymptotic formulas for Toeplitz determinants generated by functions of the form \( a(t)e^\gamma \), where \( a \) satisfies (1.2) and \( \kappa \in \mathbb{Z} \). Clearly, in that case \( \text{ind}[a(t)e^\gamma] = \kappa \). Their results were extended to the case of matrix-valued generating functions by Böttcher and Silbermann [BS80, Theorem 14]. They considered generating functions from the (matrix version of) Hölder-Zygmund spaces. Let \( \omega \) follow from part (a) and from [BS83, Theorem 10.47]. Part (c)

\[ \mathbb{D} := \{ a \in L^\infty(T) : a_{\pm n} = 0 \text{ for } n \in \mathbb{N} \}. \]

**Theorem 1.2.** Let \( \gamma > 0 \) and \( \mathbb{A} = C^\gamma \). If \( a \in \mathbb{A} \) satisfies (1.1), then

(a) there exist functions \( a_- \) and \( a_+ \) such that \( a = a_-a_+ \) and

\[ a_{\pm 1} \in \mathbb{A} \cap H^\infty, \quad a_{\pm n} \in \mathbb{A} \cap H^\infty; \]

(b) the following refinement of (1.1) is valid:

\[ D_n(a) = G(a)^{n+1} E(a) \{ 1 + \delta_1(n) \} \quad \text{as } n \to \infty \]

with \( \delta_1(n) = O(a^{1-\gamma}) \):

(c) if we put \( b = a_-a_+^{-1} \) and \( c = a_-^{-1}a_+ \), then for every \( \kappa \in \mathbb{N} \),

\[ D_n[a(t)e^\kappa] = G(a)^{n+1} E(a) \{ 1 + \delta_1(n) \} \]

\[ \times (-1)^{(n+\kappa)} G(c)^{\kappa} \left\{ \begin{array}{c} b_{n+1} \ldots \ b_{n-\kappa+2} \\ \vdots \quad \ddots \quad \vdots \\ b_{n+\kappa} \ldots \ b_{n+1} \end{array} \right\} + \delta_2(n) \]

and

\[ D_n[a(t)e^\kappa] = G(a)^{n+1} E(a) \{ 1 + \delta_1(n) \} \]

\[ \times (-1)^{(n+\kappa)} G(b)^{\kappa} \left\{ \begin{array}{c} c_{-n-1} \ldots \ c_{-n+\kappa-2} \\ \vdots \quad \ddots \quad \vdots \\ c_{-n-\kappa} \ldots \ c_{-n-1} \end{array} \right\} + \delta_3(n) \]

as \( n \to \infty \) with \( \delta_1(n) = O(n^{1-2\gamma}) \) and \( \delta_2(n) = \delta_3(n) = O(n^{-3\gamma}) \).

The representation in (a) is called a Wiener-Hopf factorization of \( a \) in the algebra \( \mathbb{A} \).

The proof of part (a) can be found in [PS91, Section 6.25(vi)]. Part (b) follows from [BS83, Sections 6.18–6.20] or [BS06, Theorems 10.35 and 10.37]. Part (c) follows from part (a) and from [BS83, Theorem 6.24] or [BS06, Theorem 10.47].

An alternative approach to the asymptotics of Toeplitz determinants with scalar-valued generating functions of nonvanishing Cauchy index is suggested in
a series of papers by Carey and Pincus [CP99, CP01, CP06]. In [CP06, Theorem A], they found an exact formula for Toeplitz determinants generated by functions of nonvanishing Cauchy index. Their approach is based on a heavy use of results and methods of algebraic $K$-theory of the algebras of operators having trace class commutators and it is by no means elementary. Very recently Böttcher and Widom [BW06] have found an elementary proof of the above mentioned exact formula and, therefore, have obtained a new proof of Theorem 1.2(c).

The author [K07b] (see also [K06, K07a]) has obtained an analog of Theorem 1.2 for matrix-valued generating functions in weighted Wiener algebras. Its scalar version for weight sequences in $\mathbb{W}_0$ class commutators and it is by no means elementary. Very recently Böttcher and results and methods of algebraic $K$-theory of nonvanishing Cauchy index. Their approach is based on a heavy use of a series of papers by Carey and Pincus [1.2(a)–(c).]

1.2. Theorem Suppose $\varphi = \{\varphi_j\}_{j=0}^{\infty}$, $\psi = \{\psi_j\}_{j=0}^{\infty}$ are weight sequences in $\mathbb{W}$ and $\sum_{j=1}^{\infty}|\varphi_j\psi_j|^{-1} < \infty$. If $a \in W \cap F_{1,\alpha,\beta}^{1,1,1}$ satisfies (1.2), then the results of Theorem 1.2 are valid with $A = W \cap F_{1,\alpha,\beta}^{1,1,1}$ and

$$\delta_1(n) = o\left(\sum_{j=n+1}^{\infty}|\varphi_{j+1}\psi_{j+1}|^{-1}\right),$$

$$\delta_2(n) = o(\varphi_{n+1}\psi_{n+1}^{-2}),$$

$$\delta_3(n) = o(\varphi_{n+1}\psi_{n+1}^{-2}).$$

This result follows from [K07b, Theorem 1.2] and [K06, Proposition 22].

In particular, if $A = W \cap F_{1,\alpha,\beta}^{1,1,1}$ with $\alpha, \beta \geq 0$ and $\alpha + \beta > 1$, then the results of Theorem 1.2(a)–(c) are valid with

$$\delta_1(n) = o(n^{1-\alpha-\beta}), \quad \delta_2(n) = o(n^{-2\beta}), \quad \delta_3(n) = o(n^{-2\alpha-\beta}).$$

1.8. The case of a nonvanishing Cauchy index: new results. Our main result is the following version of Theorem 1.2 for generating functions in $W \cap F_{\varphi,\psi}^{1,1,1}$.

1.4. Theorem Suppose $\Phi, \Psi$ are complementary $N$-functions both satisfying the $\Delta^0_2$-condition, $\varphi = \{\varphi_k\}_{k=0}^{\infty}$, $\psi = \{\psi_k\}_{k=0}^{\infty}$ are weight sequences in $\mathbb{W}$, and there exists a constant $M \in (0, \infty)$ such that $k \leq M \varphi_k \psi_k$ for all $k \in \mathbb{Z}_+$. If $a \in W \cap F_{\varphi,\psi}^{1,1,1}$ satisfies (1.2), then the results of Theorem 1.2(a), (c) are valid with $A = W \cap F_{\varphi,\psi}^{1,1,1}$ and

$$\delta_1(n) = o(1), \quad \delta_2(n) = o(\psi_n), \quad \delta_3(n) = o(1/\varphi_n).$$

Part (a) was proved in [K04, Corollary 2.3] (see also Lemma 2.4(b)). Part (c) will be proved in Section 3.4.

If $\Phi$ and $\Psi$ are arbitrary $N$-functions, then from the well known conditions for the embeddings of Orlicz classes (see, e.g., [M89, Theorem 3.4(c)] or [M83, Section 8]) it follows that $F_{1,\varphi,\psi}^{1,1,1} \subset F_{\varphi,\psi}^{1,1,1}$. Thus the class $W \cap F_{\varphi,\psi}^{1,1,1}$ of generating functions in Theorem 1.4 is larger than the class $W \cap F_{\varphi,\psi}^{1,1,1}$ of generating functions in Theorem 1.3. On the other hand, one has better speed of convergence in Theorem 1.3 than in Theorem 1.4.

The paper is organized as follows. In Section 2 we give definitions of weighted Orlicz sequence spaces and classes, define the norm in the class $F_{\varphi,\psi}^{1,1,1}$ and collect necessary results about this class and its intersection with the Wiener algebra $W$. Finally we formulate basic results on Toeplitz and Hankel operators acting on non-weighted Orlicz sequence spaces. In Section 3 we give the proof of Theorem 1.3(c)
following the approach developed by Böttcher and Silbermann in \cite{BS80} Section 8 (see also \cite{BS06} Section 10.47). We apply Jacobi’s theorem on the first step. It allows us to represent $D_n - \kappa [a(t)] t^\kappa$ as the product of $D_n(a)$ (it is treated by Theorem 1.1) and a minor of $T_n^{-1}(a)$. This minor is represented in Lemma 3.5 as a sum of two terms. The first term gives the leading term in the asymptotic formula after some computations. If the generating function is sufficiently smooth (in our case $a \in W \cap F_{\ell, \Phi, \varphi}$), then the norm of the second term is asymptotically small (in our case $o(1/\varphi_n - \kappa)$ or $o(1/\psi_n - \kappa)$). Gathering all these pieces together, we finish the proof in Section 3.4.

2. Auxiliary results

2.1. One result from numerical linear algebra. The following result was stated in \cite{BS06} Section 10.47 without a proof, a sketch of the proof can be found in \cite{K07b}, Proposition 2.3.

Proposition 2.1. Suppose \( \{\gamma_n\}_{n=1}^\infty \) is a sequence of positive numbers and \( \{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty \) are sequences of $m \times m$ matrices. If \( \sup_{n \in \mathbb{N}} \|A_n\| < \infty \) and \( \|A_n - B_n\| = o(\gamma_n) \) as \( n \to \infty \), where \( \| \cdot \| \) is any matrix norm, then

\[ \det A_n = \det B_n + o(\gamma_n) \quad \text{as} \quad n \to \infty. \]

2.2. Weighted Orlicz sequence spaces and classes. Let $I$ be either $\mathbb{N}$ or $\mathbb{Z}_+$. Suppose $\Phi$ is an $N$-function and $\varphi = \{\varphi_k\}_{k=0}^\infty$ is a weight sequence. The set $\ell_\varphi^\Phi(I)$ of all sequences $c = \{c_k\}_{k \in I}$ of complex numbers such that

\[ \sum_{k \in I} \Phi \left( \frac{|c_k| \varphi_k}{\lambda} \right) < \infty \]

for some $\lambda = \lambda(c) > 0$ is a Banach space when equipped with the norm

\[ \|c\|_{\ell_\varphi^\Phi(I)} = \inf \left\{ \lambda > 0 : \sum_{k \in I} \Phi \left( \frac{|c_k| \varphi_k}{\lambda} \right) \leq 1 \right\}. \]

The space $\ell_\varphi^\Phi(I)$ is called a weighted Orlicz sequence space. If $\varphi_k = 1$ for all $k \in \mathbb{Z}_+$, then we will simply write $\ell^\Phi(I)$ instead of $\ell_\varphi^\Phi(I)$ and we will say that $\ell^\Phi(I)$ is an Orlicz sequence space. The set $\ell_\varphi^\Phi(I)$ of all sequences $c = \{c_k\}_{k \in I}$ of complex numbers for which (2.1) holds with $\lambda = 1$ is called a weighted Orlicz sequence class. Weighted Orlicz sequence spaces are the partial case of so-called Musielak-Orlicz sequence spaces (= modular sequence spaces). Good sources for the theory of Musielak-Orlicz sequence spaces are \cite{LT77}, Section 4.d and \cite{M83} for Orlicz sequence spaces see also \cite{M89}; for Orlicz function spaces over finite measure spaces, see \cite{KR61}.

Applying the results of \cite{M83} Theorem 8.14(b) (see also \cite{LT77} Proposition 4.d.3) to the sequence of $N$-functions $\Phi_k(x) = \Phi(x, \varphi_k)$, we get the following.

Theorem 2.2. Suppose $\Phi$ is an $N$-function satisfying the $\Delta^0_2$-condition and $\varphi = \{\varphi_k\}_{k=0}^\infty$ is a weight sequence. Then $\ell_\varphi^\Phi(I) = \ell_\varphi^{\Phi_k}(I)$. 
2.3. The algebra $W \cap F^\varphi,\psi_{\varphi,\psi}$. Let $\Phi, \Psi$ be $N$-functions and let $\varphi = \{\varphi_k\}_{k=0}^\infty$, $\psi = \{\psi_k\}_{k=0}^\infty$ be weight sequences. If $\Phi$ and $\Psi$ both satisfy the $\Delta^0$-condition, then from Theorem [22] it follows that the set $F^\varphi,\psi_{\varphi,\psi}$ of all functions $a \in L^1(\mathbb{T})$ with the Fourier coefficients $\{a_k\}_{k=0}^\infty$ satisfying (1.3) is a Banach space with respect to the norm

$$\|a\|_{F^\varphi,\psi_{\varphi,\psi}} := \|a_k\|_{E^\varphi_k(n)} + \|\{a_k\}_{k \in \mathbb{Z}_+}\|_{F^\varphi_\psi(\mathbb{Z}_+)}.$$ 

For $a \in L^1(\mathbb{T})$ and $n \in \mathbb{N}$, put

$$a^{(n)}(t) = \sum_{k=-n}^n a_k t^k \quad (t \in \mathbb{T}).$$

Lemma 2.3. (see [KS05 Lemma 3, Proposition 2]). Let $\Phi, \Psi$ be $N$-functions both satisfying the $\Delta^0$-condition and let $\varphi = \{\varphi_k\}_{k=0}^\infty$, $\psi = \{\psi_k\}_{k=0}^\infty$ be weight sequences.

(a) There is a constant $C(\varphi, \psi, \Phi, \Psi) > 0$ depending only on $\varphi, \psi$ and $\Phi, \Psi$ such that for all $a \in F^\varphi,\psi_{\varphi,\psi}$,

$$\|a\|_{F^\varphi,\psi_{\varphi,\psi}} \leq C(\varphi, \psi, \Phi, \Psi) \|a\|_{F^\varphi,\psi_{\varphi,\psi}}.$$

(b) If $a \in F^\varphi,\psi_{\varphi,\psi}$, then

$$\lim_{n \to \infty} \|a - a^{(n)}\|_{F^\varphi,\psi_{\varphi,\psi}} = 0.$$

We equip the set $W \cap F^\varphi,\psi_{\varphi,\psi}$ with the norm

$$(2.2) \quad \|a\|_{W \cap F^\varphi,\psi_{\varphi,\psi}} := \|a\|_W + \|a\|_{F^\varphi,\psi_{\varphi,\psi}}.$$ 

The following result generalizes Horbach’s theorem [BS06 Theorem 6.54]. It was recently proved in [K04] in a slightly more general form.

Lemma 2.4. Let $\Phi, \Psi$ be $N$-functions both satisfying the $\Delta^0$-condition and let $\varphi = \{\varphi_k\}_{k=0}^\infty$, $\psi = \{\psi_k\}_{k=0}^\infty$ be weight sequences in $W$.

(a) If $a, b \in W \cap F^\varphi,\psi_{\varphi,\psi}$, then

$$\|ab\|_{W \cap F^\varphi,\psi_{\varphi,\psi}} \leq (1 + 2C_\varphi + 2C_\psi) \|a\|_{W \cap F^\varphi,\psi_{\varphi,\psi}} \|b\|_{W \cap F^\varphi,\psi_{\varphi,\psi}}.$$

(b) If $a \in W \cap F^\varphi,\psi_{\varphi,\psi}$ satisfies (1.2), then $a$ has a logarithm in $W \cap F^\varphi,\psi_{\varphi,\psi}$. If we let for $t \in \mathbb{T}$,

$$a_-(t) := \exp\left(\sum_{k=1}^\infty (\log a)_{-k} t^{-k}\right), \quad a_+(t) := \exp\left(\sum_{k=0}^\infty (\log a)_{k} t^{k}\right),$$

then $a = a_- a_+$ and

$$a_{\pm} \in (W \cap F^\varphi,\psi_{\varphi,\psi}) \cap H^\infty, \quad a_{\pm} \in (W \cap F^\varphi,\psi_{\varphi,\psi}) \cap H^\infty.$$ 

2.4. Toeplitz and Hankel operators on Orlicz sequence spaces. Let $a$ be a function in $L^1(\mathbb{T})$ with the Fourier coefficients $\{a_k\}_{k=\infty}^{\infty}$ and let $\{c_k\}_{k=\infty}^{\infty}$ be a sequence of complex numbers. We formally define the Laurent operator with the symbol $a$ by

$$L(a) : \{c_k\}_{k=\infty}^{\infty} \mapsto \left\{ \sum_{k=-\infty}^\infty a_{j-k} c_k \right\}_{j=\infty}^{\infty}.$$
and the operators $P, Q,$ and $J$ as follows
\[(Pc)_k := \begin{cases} 0 & \text{for } k < 0, \\ c_k & \text{for } k \geq 0, \end{cases} \quad (Qc)_k := \begin{cases} 0 & \text{for } k \geq 0, \\ c_k & \text{for } k < 0, \end{cases} \quad (Jc)_k = c_{-k-1}.
\]

For $t \in T$, put $\tilde{a}(t) := a(1/t)$. Define Toeplitz operators
\[T(a) := PL(a)P| \text{Im } P, \quad T(\tilde{a}) := JQL(a)QJ| \text{Im } P\]
and Hankel operators
\[H(a) := PL(a)QJ| \text{Im } P, \quad H(\tilde{a}) := JQL(a)P| \text{Im } P.\]

For the spaces $\ell^p(Z_+)$, the following result is well known (see [BS83 Chap. 2], [BS99 Chap. 1], [BS06 Chap. 2]), for Orlicz sequence spaces the proofs are actually the same (see [KS05 Section 3]).

**Lemma 2.5.** Suppose $\Psi$ is an $N$-function.
(a) If $a \in W$, then the operators $T(a)$ and $T(\tilde{a})$ are bounded on $\ell^\Psi(Z_+)$ and the operators $H(a)$ and $H(\tilde{a})$ are compact on $\ell^\Psi(Z_+)$.  
(b) If $a, b \in W$, then
\[
\begin{align*}
T(ab) &= T(a)T(b) + H(a)H(b), \\
H(ab) &= T(a)H(b) + H(a)T(b).
\end{align*}
\]
(c) If $a_\kappa \in W \cap H^\infty$, then $H(a_\kappa) = 0$. If $a_\kappa \in W \cap H^\infty_+$, then $H(\tilde{a_\kappa}) = 0$.

### 3. Proof of the main result

#### 3.1. Application of Jacobi’s theorem

Let $\Psi$ be an $N$-function. Denote by $B(\ell^\Psi(Z_+))$ the Banach algebra of all bounded linear operators on the Orlicz sequence space $\ell^\Psi(Z_+)$. For $n \in Z_+$, define the operators $P_n$ and $Q_n$ by
\[P_n : \{c_k\}_{k=0}^\infty \mapsto \{c_0, c_1, \ldots, c_n, 0, 0, \ldots\}, \quad Q_n := I - P_n.
\]

Obviously, $P_n, Q_n \in B(\ell^\Psi(Z_+))$ and $P_n^2 = P_n, Q_n^2 = Q_n$.

We will identify the operator $P_nT(a)P_n : P_n\ell^\Psi(Z_+) \to P_n\ell^\Psi(Z_+)$ with the finite Toeplitz matrix $T_n(a) = [a_{j-k}]_{j,k=0}^n$; if this operator is invertible we will simply write $T_n^{-1}(a)$ instead of $(P_nT(a)P_n)^{-1}P_n$.

Fisher and Hartwig [FH68, FH69] recognized that the following result is the key to treating the asymptotics of Toeplitz determinants generating by functions of nonvanishing Cauchy index.

**Lemma 3.1.** Let $\kappa \in \mathbb{N}$ and $n \geq \kappa$. If $a \in W$ is such that $T_n(a)$ is invertible, then
\[
\begin{align*}
\det [(P_n - P_{n-\kappa})T_n^{-1}(a)P_{\kappa-1}] &= (-1)^{n\kappa} \frac{D_{n-\kappa}[a(t)t^{-\kappa}]}{D_n(a)}, \\
\det [P_{\kappa-1}T_n^{-1}(a)(P_n - P_{n-\kappa})] &= (-1)^{n\kappa} \frac{D_{n-\kappa}[a(t)t^\kappa]}{D_n(a)}.
\end{align*}
\]

**Proof.** This lemma follows from Jacobi’s theorem on the conjugate minors of the adjugate matrix (see, e.g., [G59 Chap. I, Section 4]). This theorem is applied to $T_n(a)$ and the $\kappa \times \kappa$ minor standing at the left lower (resp. right upper) corner of $T_n^{-1}(a)$.

\[\square\]
3.2. The Böttcher-Silbermann asymptotic analysis. Our starting point is the following simple and important fact.

**Proposition 3.2.** (see [KS05 Proposition 11]). If $\Psi$ is an $N$-function satisfying the $\Delta_2^0$-condition, then the sequence $P_n$ converges strongly to the identity operator $I$ on the space $\ell^\Psi(\mathbb{Z}_+)$.

In [KS05 Lemma 6(a)] we proved that the so-called finite section method is applicable to Toeplitz operators on Orlicz sequence spaces. One of the equivalent forms of this property can be stated as follows (see, e.g., [BS06 Proposition 7.3]).

**Lemma 3.3.** Let $\Psi$ be an $N$-function satisfying the $\Delta_2^0$-condition. Suppose $a^{\pm1} \in W \cap H_\infty$, $a^{\pm1}_+ \in W \cap H_\infty^\ell$, and put $a = a_+$. Then the sequence $\{T_n(a)\}_{n=0}^\infty$ is stable in $\ell^\Psi(\mathbb{Z}_+)$, that is, for all sufficiently large $n$, say $n \geq n_0$, the matrices $T_n(a)$ are invertible and

$$
\sup_{n \geq n_0} \|T_n^{-1}(a)\|_{\mathcal{B}(\ell^\Psi(\mathbb{Z}_+))} < \infty.
$$

Böttcher and Silbermann [BS80] developed further Widom’s ideas [W76] and suggested an approach to study of asymptotics of Toeplitz determinants based on the Wiener-Hopf factorization. We formulate their key identities in the setting of Orlicz sequence spaces (originally stated in the setting of $\ell^2(\mathbb{Z}_+)$).

**Lemma 3.4.** Let $\Psi$ be an $N$-function satisfying the $\Delta_2^0$-condition. Suppose $a^{\pm1} \in W \cap H_\infty$, $a^{\pm1}_+ \in W \cap H_\infty^\ell$, and put $a = a_+$. Then for all sufficiently large $n$, say $n \geq n_0$, the matrices $T_n(a)$ are invertible and

$$
T_n^{-1}(a) = P_nT(a^{-1})P_n \left\{ I - \sum_{m=0}^{\infty} F_{n,m} \right\} P_nT(a^{-1})P_n,
$$

where, for $m,n \in \mathbb{N}$,

$$
F_{n,0} := P_nT(c)Q_nT(b)P_n, \quad F_{n,m} := P_nT(c)(Q_nH(b)H(c)Q_n)^mT(b)P_n,
$$

and the convergence in (3.2) is understood in the sense of $\mathcal{B}(\ell^\Psi(\mathbb{Z}_+))$.

**Proof.** This statement can be found in [BS83 Section 6.15] or [BS06 Section 10.34] in the case of $\ell^2(\mathbb{Z}_+)$. We refer also to [KS05 Lemma 8], where a similar formula for $P_0T_n^{-1}(a)P_0$ is proved in detail for $\ell^\Psi(\mathbb{Z}_+)$.

Below we give a sketch of the proof. By Lemma 2.3 the operators $T(a)$ and $T(a^{-1})$ are invertible on $\ell^\Psi(\mathbb{Z}_+)$ and

$$
T^{-1}(a) = T(a_+^{-1})T(a^{-1}), \quad T^{-1}(a^{-1}) = T(a_+)T(a_-).
$$

By Lemma 3.3 there exists a number $n_0$ such that the matrices $T_n(a)$ are invertible for all $n \geq n_0$ and (3.1) is fulfilled.

Applying [BS06 Proposition 7.15], (3.4) and Lemma 2.5(c), we obtain that the operators $Q_nT^{-1}(a)Q_n$ are invertible on $Q_n\ell^\Psi(\mathbb{Z}_+)$ and

$$
T_n^{-1}(a) = P_nT(a_+^{-1})P_n \times \left\{ I - P_nT(a^{-1})Q_n(Q_nT^{-1}(a)Q_n)^{-1}Q_nT(a_+^{-1})P_n \right\} \times P_nT(a^{-1})P_n.
$$
By Lemma 2.5, \( T^{-1}(a) = T(a^{-1}) - K \), where the operator \( K := H(a_+^{-1})H(a_-^{-1}) \) is compact on \( \ell^q(\mathbb{Z}_+) \). From the identities

\[
P_n T(a_+^{\pm 1}) Q_n = Q_n T(a_+^{\pm 1}) P_n = 0
\]

and Lemma 2.6, it follows that the operators \( A_n := Q_n T(a_-^{-1}) Q_n \) are uniformly bounded. Since \( K \) is compact on \( \ell^q(\mathbb{Z}_+) \), taking into account Proposition 3.2, we get \( \|K_n\|_{\mathcal{B}(\ell^q(\mathbb{Z}_+))} \to 0 \) as \( n \to \infty \), where \( K_n := Q_n K Q_n \). Hence \( \|A_n^{-1} K_n\|_{\mathcal{B}(\ell^q(\mathbb{Z}_+))} \to 0 \) as \( n \to \infty \) and

\[
(Q_n T^{-1}(a) Q_n)^{-1} Q_n = (Q_n T(a_-^{-1}) Q_n - Q_n K Q_n)^{-1} Q_n = (A_n - K_n)^{-1} Q_n
\]

\[
= (I - A_n^{-1} K_n)^{-1} A_n^{-1} Q_n = \sum_{m=0}^\infty (A_n^{-1} K_n)^m A_n^{-1} Q_n.
\]

Combining this identity with (3.5), we arrive at (3.3) with

\[
F_n,0 \ := \ P_n T(a_-^{-1}) Q_n A_n^{-1} Q_n T(a_+^{-1}) P_n,
\]

\[
F_n,m \ := \ P_n T(a_-^{-1}) Q_n (A_n^{-1} K_n)^m Q_n T(a_+^{-1}) P_n \quad (m \in \mathbb{N}).
\]

Taking into account \((3.6) - (3.7)\) and Lemma 2.5(b), (c), one can prove that the operators \( F_{n,m} \) can be written in the form (3.3).

**Lemma 3.5.** Under the assumptions of Lemma 3.4

\[
(P_n - P_{n-\kappa}) T^{-1}(a) P_{n-1}
\]

\[
= (P_n - P_{n-\kappa}) T(a_-^{-1})(P_n - P_{n-\kappa}) T(b) P_{n-1} T(a_+^{-1}) P_{n-1} + X_{n,\kappa}
\]

and

\[
P_{n-1} T^{-1}(a) (P_n - P_{n-\kappa})
\]

\[
= P_{n-1} T(a_-^{-1}) P_{n-1} T(c) (P_n - P_{n-\kappa}) T(a_+^{-1}) (P_n - P_{n-\kappa}) + Y_{n,\kappa},
\]

where

\[
X_{n,\kappa} \ := \ (P_n - P_{n-\kappa}) H(a_+^{-1}) H(\overline{c}) Q_n T(b) P_{n-1} T(a_-^{-1}) P_{n-1}
\]

\[
- (P_n - P_{n-\kappa}) T(a_+^{-1}) P_n T(c)
\]

\[
\times \sum_{m=1}^\infty (Q_n H(b) H(\overline{c}) Q_n)^m T(b) P_{n-1} T(a_-^{-1}) P_{n-1},
\]

\[
Y_{n,\kappa} \ := \ P_{n-1} T(a_-^{-1}) P_{n-1} T(c) Q_n H(b) H(\overline{a_-^{-1}}) (P_n - P_{n-\kappa})
\]

\[
- P_{n-1} T(a_+^{-1}) P_{n-1} T(c)
\]

\[
\times \sum_{m=1}^\infty (Q_n H(b) H(\overline{c}) Q_n)^m T(b) P_n T(a_-^{-1}) (P_n - P_{n-\kappa}),
\]

and the convergence in (3.8), (3.9) is understood in the sense of \( \mathcal{B}(\ell^q(\mathbb{Z}_+)) \).

The first identity involving \( X_{n,\kappa} \) is actually proved in [BS80] Section 8 in the setting of \( \ell^2(\mathbb{Z}_+) \) (see also [BS81] Theorem 3 and [BG03] Theorem 2.2). An identity similar to the second one was used in [BS81] Theorem 4. A proof of the second identity is also given in [K07b] Lemma 2.2 in the setting of \( \ell^2(\mathbb{Z}_+) \). Once we have at hands Lemmas 2.5, 3.3 and 3.4 the proof of Lemma 3.5 can...
be developed by analogy with [BS80 Section 8] or [K07b Lemma 2.2] using the factorization technique of the proof of Lemma 3.4.

**3.3. The norms of \(X_{n, \kappa}\) and \(Y_{n, \kappa}\) are asymptotically small.** In this section we will show that the norms of of operators \(X_{n, \kappa}\) and \(Y_{n, \kappa}\) are asymptotically small whenever the generating function \(a\) is sufficiently smooth.

Put \(\Delta_0 := P_0\) and \(\Delta_j := P_j - P_{j-1}\) for \(j \in \{1, \ldots, n\}\). First we will prove the following auxiliary estimate for truncations of Toeplitz operators.

**Lemma 3.6.** Suppose \(\Phi, \Psi\) are complementary \(N\)-functions both satisfying the \(\Delta_0^N\)-condition and \(\varphi = \{\varphi_k\}_{k=0}^\infty, \psi = \{\psi_k\}_{k=0}^\infty\) are weight sequences in \(W\). There exists a constant \(C > 0\) depending only on \(\Phi, \Psi\) and \(\varphi, \psi\) such that if \(a \in W \cap F^\Phi_{\varphi, \psi},\) then for every \(n \in \mathbb{N}\) and every \(j \in \{0, \ldots, n\},\)

\[
\|Q_n T(a) \Delta_j \|_{B(\ell^\Phi(\mathbb{Z}_+))} \leq C \frac{\|a - a^{(n-j)}\|_{F^\Phi_{\varphi, \psi}}}{\psi_{n-j+1}}
\]

\[
\|\Delta_j T(a) Q_n \|_{B(\ell^\Phi(\mathbb{Z}_+))} \leq C \frac{\|a - a^{(n-j)}\|_{F^\Phi_{\varphi, \psi}}}{\varphi_{n-j+1}}
\]

**Proof.** This statement is proved similarly to [KS05 Lemma 9]. Suppose \(c = \{c_k\}_{k=0}^\infty \in \ell^\Phi(\mathbb{Z}_+) \setminus \{0\}\). Clearly,

\[
\Psi\left(\frac{|c_j|}{\|c\|_{\ell^\Phi(\mathbb{Z}_+)}}\right) \leq \sum_{k=0}^\infty \Psi\left(\frac{|c_k|}{\|c\|_{\ell^\Phi(\mathbb{Z}_+)}}\right) \leq 1.
\]

Therefore,

\[
(3.10) \quad |c_j| \leq \Psi^{-1}(1)\|c\|_{\ell^\Phi(\mathbb{Z}_+)}. \tag{3.10}
\]

It is easy to check that

\[
(Q_n T(a) \Delta_j c)_k = \begin{cases} 0, & 0 \leq k \leq n, \\ a_{k-j} c_j, & k > n. \end{cases} \tag{3.11}
\]

Without loss of generality assume that \(\|(a - a^{(n-j)})_k\|_{\ell^\Phi_+(\mathbb{Z}_+)} > 0\). Then taking into account (3.10), (3.11), and that \(a_k = (a - a^{(n-j)})_k\) for \(k > n - j\), we have

\[
\sum_{k=0}^\infty \Psi\left(\frac{|(Q_n T(a) \Delta_j c)_k|\psi_{n-j+1}}{\Psi^{-1}(1)\|a - a^{(n-j)}\|_{F^\Phi_{\varphi, \psi}}\|c\|_{\ell^\Phi(\mathbb{Z}_+)}}\right)
\]

\[
= \sum_{k=n+1}^\infty \Psi\left(\frac{|a_{k-j} \psi_{n-j+1}|}{\|a - a^{(n-j)}\|_{F^\Phi_{\varphi, \psi}}\|c\|_{\ell^\Phi(\mathbb{Z}_+)}}\frac{|c_j|}{\Psi^{-1}(1)\|c\|_{\ell^\Phi(\mathbb{Z}_+)}}\right)
\]

\[
\leq \sum_{k=n+1}^\infty \Psi\left(\frac{|a_{k-j} \psi_{n-j+1}|}{\|(a - a^{(n-j)})_k\|_{\ell^\Phi_+(\mathbb{Z}_+)}}\right)
\]

\[
= \sum_{k=n-j+1}^\infty \Psi\left(\frac{|(a - a^{(n-j)})_k|\psi_k}{\|(a - a^{(n-j)})_k\|_{\ell^\Phi_+(\mathbb{Z}_+)}}\right) \leq 1.
\]

Therefore,

\[
(3.12) \quad \|Q_n T(a) \Delta_j \|_{B(\ell^\Phi(\mathbb{Z}_+))} \leq \frac{\Psi^{-1}(1)}{\psi_{n-j+1}}\|a - a^{(n-j)}\|_{F^\Phi_{\varphi, \psi}}.
\]
The second estimate is proved by using the duality argument. Obviously, $Q^*_n = Q_n$, $Δ^*_n = Δ_n$, and $(T(a))^* = T(\overline{a})$. Since $Φ$ and $Ψ$ are complementary $N$-functions, by \[M83\] Section 13, $[ℓ^Ψ(Z_+)]^* = ℓ^Φ(Z_+)$ and
\[\tag{3.13}
∥Δ_nT(a)Q_n∥_{B(ℓ^Ψ(Z_+))} ≤ 2∥Q_nT(\overline{a})Δ_n∥_{B(ℓ^Ψ(Z_+))}.
\]
By Lemma 2.3(a), $ \overline{a} ∈ Fℓ^Ψ,Φ$ and
\[\tag{3.14}
∥\overline{a} - \overline{a}^{(n-j)}∥_{Fℓ^Ψ,Φ} ≤ C(φ, ψ, Φ, Ψ)∥a - a^{(n-j)}∥_{Fℓ^Ψ,Φ}.
\]
Then applying (3.12) to $ \overline{a} ∈ Fℓ^Ψ,Φ$, we get
\[\tag{3.15}
∥Q_nT(\overline{a})Δ_n∥_{B(ℓ^Ψ(Z_+))} ≤ \frac{2φ^{-1}(1)}{φ_{n-j+1}}C(φ, ψ, Φ, Ψ)∥a - a^{(n-j)}∥_{Fℓ^Ψ,Φ}.
\]
Combining (3.13)–(3.15), we arrive at
\[\tag{3.16}
∥Δ_nT(a)Q_n∥_{B(ℓ^Ψ(Z_+))} ≤ \frac{2φ^{-1}(1)}{φ_{n-j+1}}C(φ, ψ, Φ, Ψ)∥a - a^{(n-j)}∥_{Fℓ^Ψ,Φ}.
\]
The lemma is proved.

The main result of this section is the following.

**Lemma 3.7.** Suppose $Φ, Ψ$ are complementary $N$-functions both satisfying the $Δ^Ψ$-condition and $φ = \{φ_k\}_{k=0}^{∞}, ψ = \{ψ_k\}_{k=0}^{∞}$ are weight sequences in $W$. Let
\[a_{\pm} ∈ (W ∩ ℓ^Φ,Ψ) ∩ H^∞, \quad a_{\pm} ∈ (W ∩ ℓ^Φ,Ψ) ∩ H^∞\]
and put $a = a_{-}a_{+}, b = a_{-}a^{-1}_{+},$ and $c = a^{-1}_{-}a_{+}$. Then the norms of the operators $X_{\kappa, \kappa}, Y_{\kappa, \kappa}$ defined by (3.3), (3.9) satisfy
\[\|X_{\kappa, \kappa}\|_{B(ℓ^Ψ(Z_+))} = o(1/φ_{n-κ}), \quad \|Y_{\kappa, \kappa}\|_{B(ℓ^Ψ(Z_+))} = o(1/φ_{n-κ}) \quad (n → ∞).
\]

**Proof.** By Lemma 2.3(a), $a, b, c ∈ W ∩ ℓ^Φ,Ψ ⊂ W$. Hence, by Lemma 2.5(a), the operator $H(b)H(\overline{c})$ is compact on $ℓ^Ψ(Z_+)$. In view of Proposition 3.2, the sequence of operators $Q_n = I - P_n$ tends strongly to the zero operator on $ℓ^Ψ(Z_+)$, whence $∥Q_nH(b)H(\overline{c})Q_n∥_{B(ℓ^Ψ(Z_+))} → 0$ as $n → ∞$. Therefore, for all sufficiently large $n$,
\[\|\sum_{m=1}^{∞} (Q_nH(b)H(\overline{c})Q_n)^m\|_{B(ℓ^Ψ(Z_+))} ≤ M_1(a_{-}, a_{+}) < ∞ \tag{3.16}\]
and
\[\|Y_{\kappa, \kappa}\|_{B(ℓ^Ψ(Z_+))} ≤ M_2(a_{-}, a_{+})\|P_{\kappa-1}T(c)Q_n\|_{B(ℓ^Ψ(Z_+))}, \tag{3.17}\]
where $M_1(a_{-}, a_{+})$ is a positive constant depending only $a_{\pm}$ and
\[M_2(a_{-}, a_{+}) := \|T(a_{-})Q_n\|_{B(ℓ^Ψ(Z_+))}∥H(b)H(a^{-1}_{+})∥_{B(ℓ^Ψ(Z_+))} - \|T(a^{-1}_{+})\|_{B(ℓ^Ψ(Z_+))}M_1(a_{-}, a_{+})∥T(b)∥_{B(ℓ^Ψ(Z_+))}∥T(a^{-1}_{+})∥_{B(ℓ^Ψ(Z_+))}.
\]
On the other hand, by Lemma 3.6
\[\|P_{\kappa-1}T(c)Q_n\|_{B(ℓ^Ψ(Z_+))} ≤ \sum_{j=0}^{\kappa-1} ∥Δ_jT(c)Q_n∥_{B(ℓ^Ψ(Z_+))} \leq C \sum_{j=0}^{\kappa-1} \frac{∥c - c^{(n-j)}∥_{Fℓ^Ψ,Φ}}{φ_{n-j+1}}.
\]
Since \( \|c - c^{(n-j)}\|_{F_{(\psi,\varphi)}} \) and \( 1/\varphi_{n-j+1} \) are monotonically increasing with respect to \( j \), from \( (3.10) \) and \( (3.11) \) it follows that
\[
\|Y_{n,k}\|_{B(\ell^2(\mathbb{Z}_+))} \leq M_2(a-, a+)C_{K_1} \frac{\|c - c^{(n-k)}\|_{F_{(\psi,\varphi)}}}{\varphi_{n-k}}.
\]

By Lemma \( (2.33) \) b, \( \|c - c^{(n-k)}\|_{F_{(\psi,\varphi)}} = o(1) \) as \( n \to \infty \). Thus \( \|Y_{n,k}\|_{B(\ell^2(\mathbb{Z}_+))} = o(1/\varphi_{n-k}) \) as \( n \to \infty \). The equality \( \|X_{n,k}\|_{B(\ell^2(\mathbb{Z}_+))} = o(1/\varphi_{n-k}) \) as \( n \to \infty \) is proved analogously.

Notice that one can get better estimates for the norms of \( X_{n,k} \) and \( Y_{n,k} \) in the setting of the Hilbert space \( \ell^2(\mathbb{Z}_+) \). In that case effective estimates for the norms of \( Q_n H(b) \) and \( H(\tilde{c}) Q_n \) are easily available (see [BS06, Section 10.35] or [K06, Section 5]). Therefore one can conclude not only that
\[
\left\| \sum_{m=1}^{\infty} (Q_n H(b) H(\tilde{c}) Q_n)^m \right\|_{B(\ell^2(\mathbb{Z}_+))} = O(1) \quad (n \to \infty)
\]
but that this norm tends to zero as \( n \to \infty \) with some determined speed depending on the smoothness of \( a \). For instance, if \( a \in W \cap F_{(\psi,\varphi)}^{1,1} \) and \( \alpha, \beta \geq 0, \alpha + \beta > 1 \), then \( \|Q_n H(b)\|_{B(\ell^2(\mathbb{Z}_+))} = o(n^{-\beta}), \|H(\tilde{c}) Q_n\|_{B(\ell^2(\mathbb{Z}_+))} = o(n^{-\alpha}) \), and
\[
\left\| \sum_{m=1}^{\infty} (Q_n H(b) H(\tilde{c}) Q_n)^m \right\|_{B(\ell^2(\mathbb{Z}_+))} = o(n^{1-\alpha-\beta}) \quad (n \to \infty).
\]
This observation explains why the results of Theorem 1.4 (based on estimates of of truncations of Toeplitz operators on \( \ell^2(\mathbb{Z}_+) \)) are less precise than the results of Theorems 1.2 and Theorems 1.3 (based on estimates of of truncations of Toeplitz and Hankel operators on \( \ell^2(\mathbb{Z}_+) \)).

3.4. Proof of Theorem 1.4 (c). We will prove the asymptotic formula for \( D_n[a(t)t^n] \) and \( \kappa \in \mathbb{N} \) following [BS06, Section 10.47] (see also [K07b, Section 3.3]). In view of Theorem 1.4 (a), there exist functions \( a_- \) and \( a_+ \) such that \( a = a_- a_+ \) and
\[
a_{\pm}^1 \in (W \cap F_{(\psi,\varphi)}^{1,1}) \cap H^\infty \subset W \cap H^\infty, \quad a_{\pm}^1 \in (W \cap F_{(\psi,\varphi)}^{1,1}) \cap H^\infty \subset W \cap H^\infty.
\]
Thus all the conditions of Lemmas 3.3, 3.5 and 3.7 are fulfilled. Therefore the matrices \( T_n(a) \) are invertible for all sufficiently large \( n \). By Lemma 3.1
\[
D_{n-k}[a(t)t^n] = (-1)^{n\kappa} D_n(a) \det[P_{\kappa-1} T_n^{-1}(a)(P_n - P_{n-k})].
\]
By Lemmas 3.5 and 3.7
\[
\|P_{\kappa-1} T^{-1}(a)(P_n - P_{n-k}) \|_{B(\ell^2(\mathbb{Z}_+))} = o(1/\varphi_{n-k}) \quad (n \to \infty). \]
Applying Proposition 2.1 to the above matrices (which are of the size \( m = \kappa \), we get
\[
\det[P_{\kappa-1} T_n^{-1}(a)(P_n - P_{n-k})] = \det[P_{\kappa-1} T(a_{\pm}^{-1}) P_{\kappa-1} T(c)(P_n - P_{n-k}) T(a_{\pm}^{-1})(P_n - P_{n-k})] + o(1/\varphi_{n-k})
\]
\[
= D_{n-k}(a_{\pm}^{-1}) \det[P_{\kappa-1} T(c)(P_n - P_{n-k})] D_{n-k}(a_{\pm}^{-1}) + o(1/\varphi_{n-k})
\]
as $n \to \infty$. From (3.18) and (3.19) we get

$$\det[P_{k-1}T(c)(P_{n+k} - P_n)] = \det \begin{pmatrix} c_{n-1} & \cdots & c_{n-k-2} \\ \vdots & \ddots & \vdots \\ c_{n-k} & \cdots & c_{n-1} \end{pmatrix}. \tag{3.23}$$

Combining (3.20)–(3.23), we arrive at the desired formula

$$D_n[a(t)t^\kappa] = G(a)^{n+1}E(a)\{1 + o(1)\}$$

as $n \to \infty$. The proof of the asymptotic formula for $D_n[a(t)t^{-\kappa}]$ is analogous. \hfill \Box

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