Nonexistence of triples of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs *

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Abstract

In the paper "Broersma and Hoede, Path graphs, J. Graph Theory 13 (1989) 427-444", the authors proposed a problem whether there is a triple of mutually nonisomorphic connected graphs which have an isomorphic connected $P_3$-graph. For a long time, this problem remains unanswered. In this paper, we give it a negative answer that there is no such triple, and thus completely solve this problem.

Keywords: path graph, connected, isomorphism

1 Introduction

Broersma and Hoede [3] generalized the concept of line graphs to that of path graphs by defining adjacency as follows. Let $k$ be a positive integer, and $P_k$ and $C_k$ denote a path and a cycle with $k$ vertices, respectively. Let $\pi_k(G)$ be the set of all $P_k$’s in $G$. The path graph $P_k(G)$ of $G$ is a graph with vertex set $\pi_k(G)$ in which two $P_k$’s are adjacent whenever their union is a path $P_{k+1}$ or a cycle $C_k$. Broersma and Hoede got many results on $P_3$-graphs, especially, described two infinite classes of pairs of nonisomorphic connected graphs which have isomorphic connected $P_3$-graphs. They also raised a number of unsolved problems or questions, all of which have been solved during

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these year, but only the following one remains unanswered.

**Problem.** Whether there exists a triple of mutually nonisomorphic connected graphs which have an isomorphic connected $P_3$-graph?

For $k = 2$, i.e., line graphs, from Whitney’s result (see [4]) it is not difficult to see that the problem has a negative answer. In [5] the authors showed that for $k \geq 4$ there are not only triples of but also arbitrarily many mutually nonisomorphic connected graphs with isomorphic connected $P_k$-graphs. However, interestingly we will show in this paper that for $k = 3$ there does not exist any triple of mutually nonisomorphic connected graphs with an isomorphic connected $P_3$-graph, just like the case for $k = 2$ but very different from the case for $k \geq 4$. Note that If one drops the connectedness of the original graph or its $P_3$-graph, then it is easy to find arbitrarily many mutually nonisomorphic graphs with an isomorphic $P_3$-graph.

## 2 Preliminaries

All graphs in this paper are undirected, finite and simple. We follow the terminology and notations used in [1, 2]. If $\sigma$ is an isomorphism from $G$ to $H$, then $\sigma$ induces a $P_k$-isomorphism $\sigma^*$ from $G$ to $H$, where $\sigma^*(a_1a_2\cdots a_k) = \sigma(a_1)\sigma(a_2)\cdots \sigma(a_k)$ for all $a_1a_2\cdots a_k \in \pi_k(G)$. A $P_k$-isomorphism $\tau$ is induced if $\tau = \sigma^*$ for some isomorphism $\sigma$. If $\tau_1$ is a $P_k$-isomorphism from $G_i$ to $H_i$ for $i = 1$ and $2$, then we say that $\tau_1$ and $\tau_2$ are equivalent if there are isomorphisms $\sigma$ and $\rho$ from $G_1$ to $G_2$ and $H_1$ to $H_2$, respectively, such that $\tau_1 = (\rho^*)^{-1} \circ \tau_2 \circ \sigma^*$.

Define an $i$-thorn to be a $P_3$ with exactly $i$ ($i = 1$ or $2$) terminal ends in $G$. Let $T_i(G)$ be the set of $i$-thorns in $G$. We say that two $P_3$-isomorphisms $\tau_i$ from $G_i$ to $H_i$ for $i = 1$ and $2$, are $T$-related if (i) $G_1$ and $G_2$ differ only in their star components, so do $H_1$ and $H_2$; (ii) $|T_2(G_1)| = |T_2(G_2)|$; and (iii) $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \pi_3(G_1) - T_2(G_1) = \pi_3(G_2) - T_2(G_2)$.

Consider two 1-thorns $abc$ and $abd$ where $\deg(a) \geq 2$ and $\deg(c) = \deg(d) = 1$, then swapping $abc$ and $abd$ gives a $P_3$-isomorphism, which we call a $B$-swap.

Suppose $abcde$ is a $P_3$ in $G$ such that both $abc$ and $cde$ are terminal 1-thorns, i.e., $\deg(a) = \deg(e) = 1$ and $\deg(c) = 2$, then swapping $abc$ and $cde$ gives a $P_3$-isomorphism, which we call an $S$-swap.

For distinct $a, b \in V(G)$, let $D_{a,b}$ denote the subgraph of $G$ consisting of the union of all $P_3$’s with ends $a$ and $b$ and with middle vertex of degree 2 in $G$. If $D_{a,b}$ is nonempty we call it a diamond with ends $a$ and $b$. We usually
write $V(D_{a,b}) - \{a, b\}$ as $\{c_1, c_2, \cdots, c_k\}$ and call $k$ the width of $D_{a,b}$, and refer to $D_{a,b}$ as a $k$-diamond. Note that if $a \sim b$, the edge $ab$ is not included in $D_{a,b}$. To distinguish the two possibilities, we say that the diamond $D_{a,b}$ is braced if $a \sim b$ and unbraced otherwise. For $1 \leq i < j \leq k$, the $P_3$'s $ac_i b$ are called diamond paths while the pair of $P_3$'s $c_i ac_j$ and $c_i bc_j$ is called a diamond pair. Then swapping $c_i ac_j$ and $c_i bc_j$ gives a $P_3$-isomorphism, which we call a $D$-swap.

Suppose $\tau_1$ and $\tau_2$ are $P_3$-isomorphisms from $G$ to $H$. We say that $\tau_1$ and $\tau_2$ are $B$-related if $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of $B$-swaps. The $S$-related and $D$-related are defined similarly. We use joins of these four equivalence relations: for example, two $P_3$-isomorphisms are $TBSD$-related if we can get from one to the other by a chain of zero or more $T$-, $B$-, $S$- and/or $D$-relations.

The following is the main result of [1], based on which we shall solve our problem by case analysis.

**Theorem 2.1** Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$ such that at least one of $G$ or $H$ is connected. Then $\tau$ is one of the following:

(i) $T$-related to a $P_3$-isomorphism of generalized $K_{3,3}$ type;
(ii) of special Whitney type;
(iii) $D$-related to a $P_3$-isomorphism of Whitney type $3, 4, 5$ or $6$;
(iv) $D$-related to a $P_3$-isomorphism of bipartite type; or
(v) $TBSD$-related to an induced $P_3$-isomorphism.

The definition for each of the above types will be given in the successive subsections.

For solving our problem, in Theorem 2.1 we only need to consider that the original graphs $G$ and $H$ are nonisomorphic connected graphs with $T_2(G) = T_2(H) = \emptyset$. Below, we will analyze the types in Theorem 2.1 case by case in details.

### 2.1 Generalized $K_{3,3}$ type

First, we introduce the following notation which is used in the definition of generalized $K_{3,3}$ type. We write $(c, d)ab(e, f) \mapsto uwvxu$ if $G$ contains the edges $ab, ac, ad, be, bf$, $H$ contains the $C_4$ $uwvx$, and $\tau$ maps $cab \mapsto xuv$, $dab \mapsto vwx$, $abe \mapsto uvw$ and $abf \mapsto wxu$. We also write $abc(d, e) \mapsto uvwx$ if $G$ contains the edges $ab, bc, cd, ce$, $H$ contains the $P_5$ $uvwx$, and $\tau$ maps $abc \mapsto vwx$, $bcd \mapsto uvw$ and $bee \mapsto xwy$. This notation will be reversed (e.g.,
abcd \mapsto (w, x)uv(y, z) as needed. Then, define the generalized $K_{3,3}$ type as follows:

Either $\tau$ or $\tau^{-1}$ as in the following cases (i) through (vii), or any equivalent $P_3$-isomorphism, is said to be of generalized $K_{3,3}$ type.

(i) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, and $cad$ and $ebf$ map to $P_3$ components of $H$.
(ii) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, $kebfh \mapsto yv_3u_1(v_1, v_2)$, and $cad$ maps to a $P_3$ component.
(iii) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, $(k, l)eb(a, f) \mapsto u_1v_1u_2v_3u_1$, $(h, i)fb(a, e) \mapsto u_1v_2u_2v_3u_1$, and $cad$, $kel$ and $hfi$ map to $P_3$ components.
(iv) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, $ecadg \mapsto xu_3v_1(u_1, u_2)$, and $cebfh \mapsto yv_3u_1(v_1, v_2)$. Note that $G$ and $H$ are connected and isomorphic.
(v) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, $ebfhe \mapsto (v_1, v_2)u_1v_3(y, z)$, and $cad$ maps to $yv_3z$. Again $G$ and $H$ are connected and isomorphic.
(vi) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, $(c, d)eb(a, f) \mapsto u_1v_1u_2v_3u_1$, $(h, i)fb(a, e) \mapsto u_1v_2u_2v_3u_1$, $acda \mapsto (w, x)u_3v_1(u_1, u_2)$, and $hfi$ maps to $wu_3x$. Again $G$ and $H$ are connected and isomorphic.
(vii) The construction on $K_{3,3}$; $G \cong H \cong K_{3,3}$.

**Remark 1.** For generalized $K_{3,3}$ type, it is easy to get the following results:

1. For cases (i), (ii) and (iii), $G$ and $H$ are nonisomorphic, but $H$ is not connected and there are isolated vertices in $P_3(G)$ and $P_3(H)$.
2. For cases (iv) and (vii), $G$ and $H$ are connected with $T_2(G) = T_2(H) = \emptyset$, but $G$ and $H$ are isomorphic.
3. For cases (v) and (vi), $G$ and $H$ are connected, but are isomorphic and there are isolated vertices in $P_3(G)$ and $P_3(H)$.

Thus there is no pair of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs in generalized $K_{3,3}$ type.

### 2.2 Special Whitney type

Let $SW$ be the graph obtained by subdividing each edge of $K_{1,3}$ exactly once, then $P_3(SW) \cong C_6$. $\tau$ is a $P_3$-isomorphism from $SW$ to $C_6$, then we say $\tau$, $\tau^{-1}$ or any equivalent $P_3$-isomorphism is of *special Whitney type*.

It is clear that $SW$ and $C_6$ are two nonisomorphic connected graphs with isomorphic connected $P_3$-graphs.
2.3 Whitney type 3, 4, 5 or 6

In this subsection, we begin with a general idea which will be used here and in the next subsection. Suppose \( F \) is a graph. A diamond inflation of \( F \) is a graph obtained by replacing each edge \( ab \in E(F) \) by an unbraced \( s_{ab} \)-diamond \( D_{a,b} \) \((s_{a,b} \geq 1)\), and adding \( t_a \) terminal edges incident with each \( a \in V(F) \) \((t_a \geq 0)\). Suppose \( \varphi \) is an edge-isomorphism between graphs \( F \) and \( F' \), and suppose \( I \) and \( I' \) are diamond inflations of \( F \) and \( F' \), respectively, with the following property: for every \( ab \in E(F) \), if \( \varphi(ab) = uv \) then (i) \( s_{uv} = s_{ab} \) and (ii) \( t_a + t_u = t_a + t_b \). Obtain \( G \) and \( H \) from \( I \) and \( I' \), respectively, by adding star components to one of them (if necessary) to make the numbers of 2-thorns equal. Then we can define a \( P_3 \)-isomorphism \( \tau \) from \( G \) to \( H \) and say that \( \tau \) is a diamond inflation of \( \varphi \).

**Remark 2.** If \( D_{a,b} \) is a nontrivial diamond (i.e., \( s_{a,b} > 1 \)) in \( G \), then there exists a unique and nontrivial diamond \( D_{a,v} \) in \( H \) (see the proof in [1]).

The type in this subsection is related to Whitney’s exceptional edge-isomorphisms which is stated as follows:

**Theorem 2.2 (Whitney [6])** Suppose that \( \varphi \) is an edge-isomorphism from \( G \) to \( H \) where \( G \) and \( H \) are both connected. If \( \varphi \) is not induced, then \( i = |E(G)| = |E(H)| \in \{3, 4, 5, 6\} \), \( G \) and \( H \) are isomorphic to \( W_i \) and \( W'_i \) in some order, and \( \varphi \) is equivalent to \( \varphi_i \) or \( \varphi_i^{-1} \), where

(i) \( W_6 \cong W'_6 \cong K_4 \), with \( V(W_6) = \{a, b, c, d\} \), \( V(W'_6) = \{u, v, w, x\} \), and \( \varphi_6 \) maps \( ab \mapsto uv \), \( ac \mapsto uw \), \( ad \mapsto vx \), \( bc \mapsto wx \), and \( cd \mapsto wx \);

(ii) \( W_5 = W_6 - cd \), \( W'_5 = W'_6 - wx \) and \( \varphi_5 = \varphi_6|E(W_5) \);

(iii) \( W_4 = W_6 - \{bd, cd\} \), \( W'_4 = W'_6 - \{vx, wx\} \) and \( \varphi_4 = \varphi_6|E(W_4) \); and

(iv) \( W_3 = W_6 - \{bc, bd, cd\} \cong K_{1,3} \), \( W'_3 = W'_6 - x \cong K_3 \), and \( \varphi_3 = \varphi_6|E(W_3) \).

Then a \( P_3 \)-isomorphism \( \tau \) is said to be of Whitney type \( i \) if \( \tau \) or \( \tau^{-1} \) is equivalent to a diamond inflation of \( \varphi_i \) as above for \( i = 3, 4, 5, 6 \).

Denote by \( t_z \) the number of terminal edges incident with \( z \) for \( z \) in \( \{a, b, c, d\} \) or \( \{u, v, w, x\} \). For Whitney type \( P_3 \)-isomorphisms, according to condition (ii) of Diamond Inflation, gives one equation from each pair of corresponding edges of the original Whitney graphs. Then there is a same solution for all four types:

\[
\begin{align*}
t_u &= \frac{1}{2}(t_a + t_b + t_c - t_d) \\
t_v &= \frac{1}{2}(t_a + t_b - t_c + t_d) \\
t_w &= \frac{1}{2}(t_a - t_b + t_c + t_d) \\
t_x &= \frac{1}{2}(-t_a + t_b + t_c + t_d) \quad \text{(except for type 3)}
\end{align*}
\]
Because we require connected $P_3$-graphs, in the above four equations we must have $t_z = 0$ or $1$ for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$. We write $(t_a, t_b, t_c, t_d) \mapsto (t_u, t_v, t_w, t_x)$. If $t_a, t_b, t_c, t_d = 0$ or $1$, then we get the corresponding solutions for $t_u, t_v, t_w, t_x$ by (1). For example: $(1, 0, 0, 1) \mapsto (0, 1, 1, 0)$ denotes that $t_a = 1, t_b = t_c = 0$ and $t_d = 1$ correspond to solutions $t_u = 0, t_v = t_w = 1$ and $t_x = 0$ by (1). So it is easy to check that there are only the following eight cases satisfying $t_z = 0$ or $1$ for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$:

(i) $(0, 0, 0, 0) \mapsto (0, 0, 0, 0)$.
(ii) $(1, 1, 1, 1) \mapsto (1, 1, 1, 1)$ (except for type 3).
(iii) $(1, 1, 0, 0) \mapsto (1, 1, 0, 1) (ab \mapsto uv)$.
(iv) $(1, 0, 1, 0) \mapsto (1, 0, 1, 0) (ac \mapsto uw)$.
(v) $(1, 0, 0, 1) \mapsto (0, 1, 1, 0) (ad \mapsto vw)$.
(vi) $(0, 1, 1, 0) \mapsto (1, 0, 0, 1) (bc \mapsto ux)$ (except for type 3).
(vii) $(0, 1, 0, 1) \mapsto (0, 1, 0, 1) (bd \mapsto vx)$ (except for type 3 or 4).
(viii) $(0, 0, 1, 1) \mapsto (0, 0, 1, 1) (cd \mapsto wx)$ (except for type 3, 4 or 5).

If a $P_3$-isomorphism $\tau$ or $\tau^{-1}$ is equivalent to a diamond inflation of $\varphi_i$ as above, and falls into one of the above cases (i) through (viii), then $\tau$ is said to be of special Whitney type $i$ for $i = 3, 4, 5$ or $6$. Thus only in special Whitney type $i$ for $i = 3, 4, 5$ or $6$, we can find pairs of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs if we choose suitable diamond widths.

2.4 Bipartite type

First, we also introduce the definition of bipartite type. Start with a positive integer $k$ and an arbitrary bipartite graph $F$ with at least one edge and with a bipartition $(A, B)$. Let $I$ and $I'$ be different diamond inflations of $F$, where each edge $e$ is inflated to a diamond of the same width $s_e$ both times, but in producing $I$ each vertex $v$ has $t_v$ terminal edges added, while in producing $I'$ it has $t'_v$ terminal edges added. where

$$
 t'_v = \begin{cases} 
 t_v - k & \text{if } v \in A \\
 t_v + k & \text{if } v \in B 
\end{cases} \tag{2}
$$

Thus, we need $t_v \geq k$ for all $v \in A$. Let $\varphi$ be the identity edge-isomorphism from $F$ to itself. Clearly $\varphi$, $I$ and $I'$ satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of $F$ has the form $ab$ with $a \in A$ and $b \in B$, so that $t'_a + t'_b = (t_a - k) + (t_b + k) = t_a + t_b$. We can
therefore obtain a $P_3$-isomorphism $\tau$ by diamond inflation; $\tau$ is in general not induced. We say $\tau$ and $\tau^{-1}$, or any equivalent $P_3$-isomorphisms, are of bipartite type.

This case is similar to the above Whitney type. Because we require that the $P_3$-graphs of $I$ and $I'$ are connected, we must have $t_v, t'_v = 0$ or 1 for every $v \in A \cup B$. Since $k \leq t_v(v \in A)$, we have $k = 0$ or 1. If $k = 0$, then $I \cong I'$. If $k = 1$, then $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$. Otherwise, if there is a vertex $u_0 \in A$ with $t_{u_0} = 0$ or a vertex $v_0 \in B$ with $t_{v_0} = 1$, then $t'_{u_0} = -1$ or $t'_{v_0} = 2$ by (2). Therefore we have a $P_3$-isomorphism $\tau_0$ from $I$ to $I'$, where $t_u = 1$ and $t'_u = 0$ for all $u \in A$, $t_v = 0$ and $t'_v = 1$ for all $v \in B$, respectively. Then we say that $\tau_0$ and $\tau_0^{-1}$, or any equivalent $P_3$-isomorphism, are of special bipartite type. Therefore, this is the only case to find pairs of nonisomorphic connected graphs which have isomorphic connected $P_3$-graphs in the bipartite type.

2.5 $TBSD$-related to an induced $P_3$-isomorphism

In this subsection, we require that there is no isolated vertices in $P_3$-graphs. Then all $P_3$-isomorphisms are BSD-related to an induced one. It is clear that if two original graphs $G$ and $H$ are connected with an isomorphic $P_3$-graph, then $G \cong H$ by the definition of BSD-related. Thus in this type, if we require connected $P_3$-graphs, then the original graph and its $P_3$-graph are one to one.

Then from the arguments in above five subsections, we can get the following corollary which is essential to the solution of our problem.

**Corollary 2.3** Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$, where $G$ and $H$ are nonisomorphic connected graphs with an isomorphic connected $P_3$-graph. Then $\tau$ is one of the following:

(i) of special Whitney type;
(ii) $D$-related to a $P_3$-isomorphism of special Whitney type 3, 4, 5 or 6; or
(iii) $D$-related to a $P_3$-isomorphism of special bipartite type.

3 Main result

Now we can state and show the main result of this paper.

**Theorem 3.1** There is no triple of mutually nonisomorphic connected graphs with an isomorphic connected $P_3$-graph.
**Proof.** Assume, to the contrary, that there exists a triple of mutually non-isomorphic connected graphs $G_1$, $G_2$ and $G_3$ which have an isomorphic connected $P_3$-graph. Let $\tau_i$ be a $P_3$-isomorphism from $G_i$ to $G_{i+1}$, then $\tau_i$ will be one of three types in Corollary 2.3 for $i = 1, 2$.

**Case 1.** $\tau_1$ and $\tau_2$ are of the same type.

**Subcase 1.1** $\tau_1$ and $\tau_2$ are both of special Whitney type.

Without loss of generality, let $G_1 \cong SW$ and $G_2 \cong C_6$. Since $\tau_2$ is also of special Whitney type, it is clear that $G_3 \cong SW$. Thus $G_1 \cong G_3$, a contradiction.

**Subcase 1.2** $\tau_1$ and $\tau_2$ are both of $D$-related to a $P_3$-isomorphism of special Whitney type $i$ for $i = 3, 4, 5$ or 6.

Without loss of generality, we assume that $i = 4$. Then $\tau_1$ and $\tau_2$ are $D$-related to a $P_3$-isomorphism of special Whitney type 4, and let $G_1$ and $G_2$ be diamond inflations of $W_4$ and $W_4'$, respectively, where $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$, $t_u = 0$, $t_v = t_w = 1$ and $t_x = 0$ (i.e., $(1, 0, 0, 1) \mapsto (0, 1, 1, 0)$). Since $\tau_1$ and $\tau_2$ are of the same type, $G_3$ is also a diamond inflation of $W_4$, and $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$ by (1). Hence $G_1 \cong G_3$, also a contradiction.

**Subcase 1.3** $\tau_1$ and $\tau_2$ are both of $D$-related to a $P_3$-isomorphism of special bipartite type.

This subcase is similar to Subcase 1.2. Denote by $F$ an arbitrary bipartite graph with a bipartition $(A, B)$. Then assume that $G_1$ and $G_2$ are different diamond inflations of $F$, respectively, where $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in $G_1$; $t_u = 0$ for all $u \in A$ and $t_v = 1$ for all $v \in B$ in $G_2$. Thus we can easily obtain that $G_3$ is also a diamond inflation of $F$ with $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in $G_3$ by the definition of $\tau_2$. Then $G_1 \cong G_3$, contrary to the assumption.

**Case 2.** $\tau_1$ and $\tau_2$ are of different types.

By the definition of special Whitney type, we know that it is also a particular case of special Whitney type 3, with the following restrictions: (i) each edge $e$ with diamond width $s_e = 1$ in $K_{1,3}$ and $K_3$, and (ii) $t_u = 0$ for each vertex $u$ in $K_{1,3}$ and $K_3$. In fact, special Whitney type is the same as special Whitney type 3 in essence. Then in order to solve Case 2, we only need to distinguish the following two subcases:

**Subcase 2.1** $\tau_1$ is of special Whitney type, and $\tau_2$ is $D$-related to a $P_3$-isomorphism of special bipartite type.

By the definition of $\tau_1$, $G_1$ or $G_2$ is a diamond inflation of $W_3 = K_{1,3}$ or $W_3' = K_3$; and also by $\tau_2$, $G_2$ and $G_3$ are different diamond inflations of some bipartite graph, respectively. Thus there is only one possibility: $G_2$ is a diamond inflation of $K_{1,3}$, where $K_{1,3}$ has a bipartition $A = \{a\}$, $B = \{b, c, d\}$. Then $G_1 \cong C_6$ and $G_2 \cong SW$. It is easy to see that $t_a = t_b = t_c = t_d = 0$ in $K_{1,3}$, a contradiction to the definition of special bipartite type,
where \( t_a = 1 \) or \( t_b = t_c = t_d = 1 \).

**Subcase 2.2** \( \tau_1 \) is \( D \)-related to a \( P_3 \)-isomorphism of special Whitney type \( i \) for \( i = 3, 4, 5 \) or 6, and \( \tau_2 \) is \( D \)-related to a \( P_3 \)-isomorphism of special bipartite type.

For \( i = 4, 5 \) or 6, if \( \tau_1 \) is \( D \)-related to a \( P_3 \)-isomorphism of special Whitney type \( i \), then \( G_1 \) and \( G_2 \) are diamond inflations of \( W_i \) and \( W_i' \) which have odd cycles. \( G_2 \) and \( G_3 \) are different diamond inflations of some bipartite graph by the definition of \( \tau_2 \). Then \( \tau_1 \) must be \( D \)-related to a \( P_3 \)-isomorphism of special Whitney type 3. By the same argument as in Subcase 2.1, we obtain that \( G_2 \) is a diamond inflation of \( K_{1,3} \), where \( K_{1,3} \) has a bipartition \( A = \{ a \}, B = \{ b, c, d \} \). By the definition of special Whitney type 3, \( \tau_1 \) falls into one of the following four cases: \((0, 0, 0, 0) \mapsto (0, 0, 0, 0), (1, 1, 0, 0) \mapsto (1, 1, 0, 0), (ab \mapsto uv), (ac \mapsto uv), (1, 0, 1, 0) \mapsto (1, 0, 1, 0), (ad \mapsto uv) \). However, by the definition of special bipartite type, there are only two choices: either \( t_a = 0, t_b = t_c = t_d = 1 \), or \( t_a = 1, t_b = t_c = t_d = 0 \).

Finally, there does not exist any graph \( G_2 \) that has common property of two different types at the same time. So \( \tau_1 \) and \( \tau_2 \) must be of the same type, a contradiction. The proof is thus complete.

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