Abstract

We present a privacy system that leverages differential privacy to protect LinkedIn members’ data while also providing audience engagement insights to enable marketing analytics related applications. We detail the differentially private algorithms and other privacy safeguards used to provide results that can be used with existing real-time data analytics platforms, specifically with the open sourced Pinot system. Our privacy system provides user-level privacy guarantees. As part of our privacy system, we include a budget management service that enforces a strict differential privacy budget on the returned results to the analyst. This budget management service brings together the latest research in differential privacy into a product to maintain utility given a fixed differential privacy budget.
Contents

1 Introduction 3
   1.1 Contributions .............................................................. 3
   1.2 Related Work ............................................................... 3

2 Preliminaries 4

3 Private Data Analytics 5

4 Privacy System Architecture 7
   4.1 Pinot: a distributed OLAP datastore .................................. 9

5 Differentially Private Algorithms 10
   5.1 Known Domain Algorithms ............................................. 10
   5.2 Unknown Domain with Δ-Restricted Sensitivity ....................... 11
   5.3 Unknown Domain with Unrestricted Sensitivity ...................... 11

6 Privacy Budget Management Service 12
   6.1 Budget Management Implementation .................................. 13
   6.2 Differential Privacy Composition ...................................... 13

7 Results 15

8 Conclusion 17

A Omitted Analysis for Section 5.2 20
1 Introduction

LinkedIn’s Audience Engagement API is a platform that enables marketers (analysts) aggregated insights about members’ content engagements while ensuring member (user) data is protected. Consider an advertiser that is selling a cloud solution and wants to create a sponsored post on LinkedIn. The advertiser might use the Audience Engagement API to do research and find that the target audience engages with GDPR articles. Hence, the advertiser should write about how their cloud solution adheres to GDPR standards, thus increasing engagement. By design, the Audience Engagement API is secure, aggregated, and uses state of the art differentially private algorithms to provide rigorous privacy guarantees. In order to incorporate differential privacy, we carefully balance various resources, including data storage distributed across several servers, real-time query computation, privacy loss quantified by the differential privacy parameters $(\varepsilon, \delta)$, and accuracy. We describe here the overall privacy system deployed at LinkedIn that balances these resources to provide a product that surfaces audience engagement insights while putting members first by safeguarding their data.

Providing scalable, real-time analytics with low latency without differential privacy is challenging enough. Luckily, we have the open source real-time distributed OLAP datastore, called Apache Pinot (incubating) [11]. Pinot enables use cases like Job and Publisher Analytics and Who Viewed My Profile. In order to develop a differentially private system, we need to think how it can be used in conjunction with a (distributed) OLAP system such as Pinot. This would enable us to have scalable privacy systems. Furthermore, we need to implement a budgeting tool into the API so that analysts cannot repeatedly query the dataset thus making noise addition pointless. Our goal is twofold: implement differentially private algorithms that can be used with real-time distributed OLAP systems and incorporate a privacy budget management service to restrict the amount of information an analyst can retrieve. For the privacy budget management service, we incorporate the latest composition bounds for our particular algorithms [4] to extract more utility subject to a given differential privacy budget.

1.1 Contributions

We make several contributions toward making practical privacy systems that leverage differential privacy.

- We provide a suite of differentially private algorithms that cover the data analytics tasks for LinkedIn’s Audience Engagement API, which provide user-level privacy guarantees

- We detail our privacy budget management service that is able to track each analyst’s privacy budget over multiple data centers. Hence, we can ensure the budget is enforced across large scale systems in real-time.

- We showcase empirical results of our algorithms on LinkedIn’s data for various privacy parameters on our deployed system.

1.2 Related Work

Differential privacy has become the standard privacy benchmark for data analytics on sensitive datasets. Despite its popularity in the academic literature, the number of actually implemented
differential privacy systems is limited, but growing. Several of the currently implemented systems with differential privacy are in the local model, where data is individually privatized prior to being aggregated on a central server. The main local differentially private systems include Google’s RAPPOR on their Chrome browser [8], Apple’s iOS and MacOS diagnostics [1], and Microsoft’s telemetry data in Windows 10 Fall Creators Update [3].

The privacy model we are interested in for this work is the global privacy setting, where data is already stored centrally, but we want to ensure each result computed on the data is privatized. In this less restrictive privacy setting, the main industrial differential privacy systems include Microsoft’s PINQ [17], Uber’s FLEX for its internal analytics [12], LinkedIn’s PriPeARL for its ad analytics [16], and the 2020 U.S. Census [2]. In this work, we present a privacy system that incorporates a privacy budget management service to ensure user-level privacy, whereas LinkedIn’s PriPeARL system provides event-level privacy and was focused on providing consistent results, which we also incorporate. The FLEX system points out that a privacy budget management service can be implemented but does not provide a strategy for how to do it. Our service takes into account the various privacy algorithms to take advantage of the latest privacy composition bounds, such as pay-what-you-get composition [5, 4]. Further, our system is part of an API that allows for adaptively chosen queries computed in real-time, which is, to our knowledge, a different model from the future U.S. Census Bureau’s system. There are also open source libraries for differentially private algorithms, such as [10] and the upcoming collaboration project between Harvard and Microsoft [14]. Another related privacy system is PSI (Ψ) from the Harvard Privacy Tools Project [9]. PSI is a private data sharing interface to “enable researchers in the social sciences and other fields to share and explore privacy-sensitive datasets with the strong privacy protections of differential privacy.” Although they support several commonly used statistics, our system covers the necessary algorithms to privatize queries in the Audience Engagement API. Further, our system allows for handling highly distributed datasets via Pinot and still enforce a strict privacy budget that is eventually consistent across data centers.

2 Preliminaries

We now present some notation and fundamental definitions that will be used to describe our privacy system. We will denote the data histogram as \( h \in \mathbb{N}^d \) where \( d \) is the dimension of the data universe, which might be unknown or known. We say that \( h \) and \( h' \) are neighbors, sometimes denoted as \( h \sim h' \), if they differ in the presence or absence of at most one member’s data. We now define differential privacy [7, 6].

**Definition 2.1** (Differential Privacy). An algorithm \( M \) that takes a histogram in \( \mathbb{N}^d \) to some arbitrary outcome set \( Y \) is \((\varepsilon, \delta)-\)differentially private (DP) if for all neighbors \( h, h' \) and for all outcome sets \( S \subseteq Y \), we have \( \Pr[M(h) \in S] \leq e^\varepsilon \Pr[M(h') \in S] + \delta \). If \( \delta = 0 \), then we simply write \( \varepsilon \)-DP.

In our algorithms, we will add noise to the histogram counts. The noise distributions we consider are from a Gumbel distribution where \( \text{Gumbel}(b) \) has PDF \( p_{\text{Gumbel}}(z; b) \) or a Laplace distribution where \( \text{Lap}(b) \) has PDF \( p_{\text{Lap}}(z; b) \), and

\[
p_{\text{Gumbel}}(z; b) = \frac{1}{b} \cdot e^{-(z/b + e^{-z/b})} \quad \text{and} \quad p_{\text{Lap}}(z; b) = \frac{1}{2b} \cdot e^{-|z|/b}.
\]
As an analyst interacts with private algorithms, the resulting privacy parameters increase with each returned result. Hence, we need to account for the overall privacy budget that an analyst can exhaust before the privacy loss is deemed to be too large. We then use the composition property of DP to bound the resulting privacy parameters. We will use bounded range in our composition analysis, which was introduced by Durfee and Rogers [5]. Note that \( \varepsilon \)-BR implies \( \varepsilon \)-DP and \( \varepsilon \)-DP implies \( 2\varepsilon \)-BR.

**Definition 2.2 (Bounded Range).** Given a mechanism \( M \) that takes a histogram in \( \mathbb{N}^d \) to outcome set \( Y \), we say that \( M \) is \( \varepsilon \)-bounded range (BR) if for any \( y_1, y_2 \in Y \) and any neighboring databases \( h, h' \) we have

\[
\frac{\Pr[M(h) = y_1]}{\Pr[M(h') = y_1]} \leq e^\varepsilon \frac{\Pr[M(h) = y_2]}{\Pr[M(h') = y_2]}
\]

where we use the density function instead for continuous outcomes.

We now state the result from Dong et al. [4] that tightens the composition bound from Durfee and Rogers [5] which itself improved on the more general optimal DP composition bounds [15, 19].

**Lemma 2.1.** Let \( M_1, M_2, \cdots, M_t \) each be \( \varepsilon \)-BR where the choice of mechanism \( M_i \) at round \( i \) may depend on the previous outcomes of \( M_1, \cdots, M_{i-1} \), then the resulting composed algorithm is \( (\varepsilon'(\delta), \delta) \)-DP for any \( \delta \geq 0 \) where \( \varepsilon'(\delta) \) is the minimum of \( t\varepsilon \) and

\[
t \left( \frac{\varepsilon}{1 - e^{-\varepsilon}} - 1 - \ln \left( \frac{\varepsilon}{1 - e^{-\varepsilon}} \right) \right) + \varepsilon \sqrt{\frac{t}{2} \ln(1/\delta)}.
\]

We also can use the more complicated composition bound for BR mechanisms [4]. However we cannot use the optimal composition bound from [4] because it only applies to the non-adaptive setting. Here we are interested in the API setting which allows the user to ask adaptive queries.

### 3 Private Data Analytics

To incorporate differential privacy, we needed to consider the various tasks we want the application to handle. We will be focusing on data analytics based on histograms or counts over different domain elements. We will discuss each query type our privacy system handles, but first we need to set up some notation.

In order to provide a **user-level** privacy guarantee where all data records of a user are protected, as opposed to **event-level** where only an individual data record is protected, we consider two types of queries. The first consists of **distinct count** queries where a member can contribute a count of at most 1 to any number of elements, i.e. \( ||h - h'||_\infty \leq 1 \) for any neighbors \( h, h' \) (\( \ell_\infty \)-sensitivity). An example of such a query would be “what are the top-\( k \) articles that are shared among distinct members with a certain skill set?” The second type is **non-distinct count** queries where a member can increase the count of any element by amount \( \tau \geq 1 \), i.e. \( ||h - h'||_\infty \leq \tau \) for any neighbors \( h, h' \). Note that \( \tau = 1 \) gives us the distinct count setting and \( \tau \) can be a parameter for each non-distinct count query.

In either case, distinct count or non-distinct count queries, a member can either affect the count of an arbitrary number of elements \( ||h - h'||_0 \leq d \) for any neighbors \( h, h' \in \mathbb{N}^d \) or a bounded number \( ||h - h'||_0 \leq \Delta \) for \( \Delta < d \) (\( \ell_0 \)-sensitivity). We separate these two cases as the **unrestricted**
sensitivity and \(\Delta\)-restricted sensitivity settings, respectively. In the case of unrestricted sensitivity, we will return only a fixed number of counts, say the top-\(k\), in order to bound the privacy loss.

Scaling our privacy system across several analysts with queries that require data from multiple servers requires algorithms that can run efficiently with runtime that does not scale with the entire data domain size \(d\). For example, for the top-10 articles engaged with by staff software engineers, we do not want to query over all articles, since there could potentially be billions of articles and would be computationally expensive and slow. For this reason, we distinguish the case where the data domain is reasonably sized and known, i.e. \textit{known domain}, from when the data domain is very large or unknown, i.e. \textit{unknown domain}.

We then summarize in Table 1 the set of queries that we want our privacy system to handle into \textit{unrestricted sensitivity} or \(\Delta\)-\textit{restricted sensitivity} as well as \textit{known domain} or \textit{unknown domain} with the corresponding algorithms we will use for each setting. Recall that we can interpolate between distinct count queries and non-distinct count queries with the \(\tau \geq 1\) parameter, so we include \(\tau\) as a parameter to each of our algorithms. Furthermore, each algorithm takes a privacy parameter \(\varepsilon_{\text{per}}\).

|                  | \(\Delta\)-restricted sensitivity | unrestricted sensitivity |
|------------------|-----------------------------------|---------------------------|
| \textbf{Known Domain} | \textit{KnownLap}^{\Delta,\tau} [7] | \textit{KnownGumb}^{k,\tau} [18] |
| \textbf{Unknown Domain} | \textit{UnkLap}^{\Delta,d,\tau} | \textit{UnkGumb}^{k,d,\tau} |

Table 1: DP algorithms for various data analytics tasks

Restricting the \(\ell_{\infty}\)-sensitivity is not part of the algorithm, rather it is done by using distinct count queries (\(\tau = 1\)), knowing a bound a priori, or done via a preprocessing step on the data. For the unknown domain setting, we require a parameter \(\bar{d}\) which tells us how many elements from the original dataset that we can access in our algorithms. We think of \(\bar{d}\) as the maximum number of elements our OLAP system can return without causing significant latency. For the unrestricted sensitivity setting, we require our algorithms to return at most \(k\) elements, such as the top-\(k\). This is due to the fact that a user can change the counts of an arbitrary number of elements. In such cases, to have any hope to bound the privacy loss, we bound the number of elements that can be returned. In Table 1, we refer to the following mechanisms: the standard Laplace mechanism from [7] is denoted as \textit{KnownLap}^{\Delta,\tau}, which adds Laplace noise with scale proportional to \(\tau/\varepsilon_{\text{per}}\) to each count; the \(k\)-peeling exponential mechanism [18], which adds Gumbel noise with scale proportional to \(\tau/\varepsilon_{\text{per}}\) to each count and returns the elements with the top-\(k\) noisy counts, which we denote as \textit{KnownGumb}^{k,\tau}; the generalized restricted sensitivity algorithm from [5] denoted as \textit{UnkLap}^{\Delta,d,\tau}, which is presented in Algorithm 3; the generalized unrestricted sensitivity algorithm from [5] denoted as \textit{UnkGumb}^{k,d,\tau}, which is presented in Algorithm 4.

The primary difference between querying the OLAP datastore for results with privacy as opposed to without privacy is that when querying for the top-\(k\) in the unknown domain setting, we instead fetch the top-\(\bar{d}\) and then use \textit{UnkGumb}^{k,d,\tau} or \textit{UnkLap}^{\Delta,d,\tau}. Ideally, we would want to set \(\bar{d} = d\) to get the full dataset, but that is not practical when the number of elements is large and the existing architecture potentially trims the results for efficiency. The choice of algorithm for each query can be a simple look up of the \textit{group by} clause where if no additional information is given,
then we default to the unknown domain and unrestricted sensitivity.

4 Privacy System Architecture

Existing OLAP datastores are designed to provide real-time data analytics over distributed datasets, with differential privacy not necessarily being incorporated from the beginning. Pinot is the analytics platform of choice at LinkedIn for site-facing use cases. In this section, we detail how we incorporated differential privacy with Pinot and the application. Figure 1 presents the overall system.

![Diagram](image-url)

Figure 1: The overall privacy system with additional components for DP being the DP Library as well as the Budget Management Service and Data Store. The arrows between Analysts and Data Centers show that an analyst may be initially assigned one data center (bold) but can migrate to a different one (dashed).

The application entity, based on the request received from the analyst, generates queries to the underlying database. The queries typically ask for a histogram grouped by some column. In order to apply the right algorithm for the query, the application needs to know the sensitivity and domain setting of the column as shown in Table 1. Also, the query is to be modified to fetch a potentially larger number of rows from the database.

We designed generic interfaces that are implemented by a suite of algorithms. The interfaces allow the application to:

- Retrieve modified query parameters (e.g. changing $k$ to $\bar{d}$).
- Estimate the privacy cost of the query (e.g. return $\Delta$ or $k$).
• Add noise to results based on privacy parameters configured.
• Compute the actual cost of the query, e.g. the number of items returned in the unrestricted sensitivity setting.

The application can independently invoke budget management functions, such as the following:
• Getting the available budget for an analyst to verify whether a query can even start to execute.
• Depleting the available budget once the query has executed.

The cost of a query could be multi-dimensional, including the cost of a making the call and of information retrieved (see Section 6).

Given the query from the analyst and the selected DP algorithm, the application will then interact with the DP library. It will first determine the expected cost of the resulting query to show the application, which is a function of the query that is asked and the selected algorithm. The application then calls the DP library to translate the query to a DP version that will be used to query Pinot. For example, if the query is for top-$k$ and the algorithm is in the unknown domain setting, then the translation could simply be to modify $k$ to $2k$, in which case $\bar{d} = 2k$ in UnkLap$^{\Delta,d,\tau}$ and UnkGumb$^{k,d,\tau}$. On the other hand, if the query is over the known domain setting, then we will want to translate $k$ to $d$ in order to get counts over the full domain, including elements with zero counts.

Now that when the application has the modified query from the DP library, we need to check whether there is enough budget remaining from the budget management service for the query to be evaluated, which are updated parameters ($k^\star, \ell^\star$) that decrease from some fixed values. We typically call the $k^\star$ parameter the information budget and can be thought of as the amount of the $\varepsilon$ parameter in DP we are consuming. Additionally, we refer to $\ell^\star$ as the call budget and is associated with the $\delta$ parameter in DP. We assume that each analyst will have its own budget and each analyst starts with the same budget. If the budget is exhausted for an analyst then the budget management service does not allow the query to be executed and tells the application that the analyst has exhausted their entire budget. If the budget is not depleted, yet what remains is less than the expected cost of the query, then we still do not evaluate the query.

Once the budget management service allows for the query to be evaluated, the application queries Pinot as it would have without the privacy system only now with the translated query. The Pinot result is then returned to the application and then the application makes another call to the DP library with the Pinot result. The DP library will then run the corresponding DP algorithm on the Pinot result and return the DP result. Based on the DP result, the budget management service updates the parameters $k^\star, \ell^\star$, as described in Algorithm 5, and returns the result to the application.

We built the algorithms module and the budget management module to be independent of each other for the following reasons:

• While DP algorithms are running on the application layer, budget management operations require a remote call to a distributed system because the budget management service needs to provide a consistent view to all application instances. Therefore, keeping the budget management independent of algorithms will allow us to scale them independently. The algorithms will need to scale to minimize memory and CPU usage, whereas the budget management service will need to scale in terms of handling higher query-per-second (QPS), while minimizing latency.
• We require that newer (as yet unknown) algorithms still be able to use and manage budgets.
• Multiple implementations of the budget manager are possible depending on system requirements. We need to be able to iterate on these independently and quickly.
• The algorithms need not (and do not) know about the analyst that is querying, and the budget manager does not behave differently depending on the type of query or algorithms used. As long as they both have a common notion of the units \((k^*, \ell^*)\) and dimensions of cost, it makes sense to keep these independent of each other.

4.1 Pinot: a distributed OLAP datastore

Pinot [11] is a distributed, real-time, columnar OLAP data store, currently incubating in Apache. At LinkedIn, we have two main categories of analytics applications: internal applications (such as dashboards, anomaly detection platform, A/B testing, etc.) and site-facing applications (such as Who viewed my profile, Talent Insights, etc.). Internal dashboards need to process a large volume of data (trillions of records), but can tolerate latencies in hundreds of milliseconds. They also have a relatively low query volume. The site-facing applications, on the other hand, serve hundreds of millions of LinkedIn members, and therefore have a very high query volume with a latency budget of a few to perhaps tens of milliseconds.

Pinot has a flexible architecture and supports a wide variety of applications in the spectrum. Pinot production clusters at LinkedIn are serving tens of thousands queries per second, supporting more than 50 analytical use cases, and ingesting over millions of records per second. Other companies such as Uber, Microsoft, and Weibo are also operating production Pinot clusters.

As shown in Figure 2, Pinot has three different components: controller, broker, and server. Controllers handle cluster wide coordination, run periodic tasks for cluster state validation and retention management, and provide a REST API for managing cluster metadata. Brokers receive
queries and federate them to servers so as to cover all the segments (shards) of a table. Servers
execute the query on the segments. Offline servers host segments that are batch ingested while
real-time servers host the segments that are ingested from streaming sources, such as Kafka [13].

For the privacy system at LinkedIn, we naturally decided to use Pinot as an OLAP data
store because Pinot already supported a lot of customer-facing analytics applications like Audience
Engagement. However, it is noteworthy that our architecture keeps the the budget management
service and Pinot as separate components so that we can easily provide DP features to other
analytical query engines such as Presto and Spark SQL.

5 Differentially Private Algorithms

We detail the algorithms for the various tasks in Table 1. These algorithms consist of previous work
from [7], [18], and [5], or slightly modified forms. Each algorithm takes a $\varepsilon_{\text{per}}$ privacy parameter,
which determines the amount of noise to add, while each algorithm in the unknown domain setting
has an additional $\delta > 0$ privacy parameter. We point out that $\text{UnkGumb}^{k,d,\tau}$ is the default algorithm
to use when no other information is known. However, the benefit of knowing the domain is that
when $k$ results are requested, $k$ results will be returned each time, whereas the unknown domain
setting may return fewer than $k$. The benefit of the $\Delta$-restricted sensitivity setting is that the budget
depletes by only $\Delta$, rather than by the number of elements returned, from as in the unrestricted
setting.

5.1 Known Domain Algorithms

We will now state the well known Laplace [7] and Exponential [18] mechanisms. We present the
Laplace mechanism [7] in the context of histogram data with the assumption that the $\ell_\infty$-sensitivity
between any neighbors is bounded by $\tau$. Note that we will use a slightly different scale of noise
in procedure $\text{KnownLap}^{\Delta,\tau}$ in Algorithm 1 than is traditionally used. This is because we want to
compose bounded range algorithms in our privacy budget manager, where each algorithm has the
same parameter $\varepsilon_{\text{per}}$. We go into more detail of the privacy budget service in Section 6.

Algorithm 1 KnownLap$^{\Delta,\tau}$; Laplace mechanism over known domain with $\ell_\infty$-sensitivity $\tau$, and
$\Delta$-restricted sensitivity

**Input:** Histogram $h$, $\Delta$ sensitivity, along with parameter $\varepsilon_{\text{per}}$.

**Output:** Noisy histogram.

for $i \in [d]$ do
    $v_i = h_i + \text{Lap}(2\tau/\varepsilon_{\text{per}})$

Return $\{v_1, \cdots , v_d\}$

We now discuss the Exponential Mechanism [18] in full generality and use the range of a quality
score rather than the global sensitivity of the score, as was presented in [4].

**Definition 5.1 (Exponential Mechanism).** The Exponential Mechanism $M_q : \mathcal{X} \to \mathcal{Y}$ with quality score $q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is written as $M_q(x)$, which samples $y$ with probability proportional to
$\exp(\varepsilon_{\text{per}}q(x,y)/S_q)$ where,

$$
S_q := \sup_{x \sim x'} \{ \max_{y \in \mathcal{Y}} \{ q(x,y) - q(x',y) \} - \min_{y' \in \mathcal{Y}} \{ q(x,y') - q(x',y') \} \}.
$$

10
Note that the Exponential Mechanism is equivalent to adding Gumbel noise $\text{Gumbel}(\Delta q / \varepsilon_{\text{per}})$ to $q(x, y)$ for each $y \in \mathcal{Y}$ and reporting the largest noisy counts [5]. We then have the following result from [18, 4]

**Lemma 5.1.** The Exponential Mechanism is $\varepsilon_{\text{per}}$-BR qnd $\varepsilon_{\text{per}}$-DP.

In our case, the quality score will simply be the heights of the histogram. Note that we have only discussed the Exponential Mechanism to return a single element. In the case where we want to return $k$-elements, we can iteratively apply the Exponential Mechanism by removing the element that is returned in each round and then run the Exponential Mechanism again without the previously returned elements, also known as peeling. However, we can implement this more efficiently by adding Gumbel noise to all the counts and then releasing the top-$k$ elements in a single shot [5]. However, we need to also include counts, so we add independent Laplace noise to the counts of the elements in the noisy top-$k$. We then formally present the KnownGumb $^{k, \tau}$ procedure in Algorithm 2.

**Algorithm 2 KnownGumb $^{k, \tau}$; Exponential Mechanism over known domain with $\ell_\infty$-sensitivity $\tau$ and unrestricted sensitivity**

```
Input: Histogram $h$, number of outcomes $k$, and parameter $\varepsilon_{\text{per}}$.
Output: Ordered set of indices and counts.
for $i \in [d]$ do
    $v_i = h_i + \text{Gumbel}(\tau / \varepsilon_{\text{per}})$
Sort $\{v_i\}$ where $v_i \geq \cdots \geq v_d$
Return $\{(i_1, h_{i_1} + \text{Lap}(2\tau / \varepsilon_{\text{per}})), \ldots, (i_k, h_{i_k} + \text{Lap}(2\tau / \varepsilon_{\text{per}}))\}$
```

We then have the following result which follows from Dwork et al. [7], as well as from McSherry and Talwar [18], Dong et al. [4].

**Lemma 5.2.** Assume that $||h - h'||_\infty \leq \tau$ and $||h - h'||_0 \leq \Delta$ for any neighbors $h, h'$. The procedure KnownLap $^{\Delta, \tau}$ is $\Delta\varepsilon_{\text{per}}/2$-DP and $\Delta\varepsilon_{\text{per}}$-BR. Further, if $\Delta$ is large or unknown then KnownGumb $^{k, \tau}$ is $3k\varepsilon_{\text{per}}/2$-DP and $2k\varepsilon_{\text{per}}$-BR.

### 5.2 Unknown Domain with $\Delta$-Restricted Sensitivity

We present the UnkLap $^{\Delta, d, \tau}$ procedure in Algorithm 3 in a more general form than in [5], which only considered the distinct count case, i.e. $\tau = 1$. Further, the proof of privacy remains true if we release the counts as well as the indices. The proof of the following result follows a similar analysis as presented in [5], which we present in Appendix A for completeness.

**Lemma 5.3** (Durfee and Rogers [5]). Assume that $||h - h'||_\infty \leq \tau$ and $||h - h'||_0 \leq \Delta$ for any neighbors $h, h'$, then the procedure UnkLap $^{\Delta, d, \tau}$ is ($\varepsilon_{\text{per}}/2, \delta$)-DP.

### 5.3 Unknown Domain with Unrestricted Sensitivity

We present the UnkGumb $^{k, d, \tau}$ procedure in Algorithm 4 in a more general form than in [5], which only considered the distinct count case, i.e. $\tau = 1$. The proof of the following theorem follows the same analysis as in [5]. Note that we use the optimal threshold index procedure from Algorithm 6 in Durfee and Rogers [5] by default and return counts by adding Laplace noise to the discovered elements in the top-$k$. 

11
Algorithm 3 UnkLap$^{\Delta, \bar{d}, \tau}$; Laplace mechanism over unknown domain with access to $\bar{d} \geq \Delta$ elements, $\ell_\infty$-sensitivity $\tau$, and $\Delta$-restricted sensitivity.

**Input:** Histogram $h$, $\Delta$ sensitivity, cut off at $\bar{d} + 1$, and $\varepsilon_{\text{per}}, \delta$.

**Output:** Ordered set of indices and counts.

Solve for $\hat{\delta}$: $\delta = \frac{\hat{\delta}}{4} \cdot (e^{\varepsilon_{\text{per}}/2} + 1)(3 + \ln(\Delta/\hat{\delta}))$

Sort $h_{(1)} \geq h_{(2)} \geq \cdots \geq h_{(\bar{d}+1)}$.

Set $v_{\perp} = h_{(\bar{d}+1)} + \tau \cdot (1 + 2\Delta \ln(\Delta/\hat{\delta})/\varepsilon_{\text{per}}) + \text{Lap}(2\tau\Delta/\varepsilon_{\text{per}})$

for $i \leq \bar{d}$ do

Set $v_i = h_{(i)} + \text{Lap}(2\tau\Delta/\varepsilon_{\text{per}})$

Sort $\{v_i\} \cup v_{\perp}$

Let $v_{i_1}, \ldots, v_{i_j}$ be the sorted list until $v_{\perp}$

Return $\{(i_1, v_{i_1}), \ldots, (i_j, v_{i_j}), (\perp, v_{\perp})\}$.

Theorem 1 (Durfee and Rogers [5]). Assume $||h - h'||_\infty \leq \tau$ for any neighbors $h, h'$. Then UnkGumb$^{k, \bar{d}, \tau}$ is $((2k + 1)\varepsilon_{\text{per}}, \delta)$-DP.

Algorithm 4 UnkGumb$^{k, \bar{d}, \tau}$; Exponential mechanism over unknown domain with access to $\bar{d} \geq k$ elements, $\ell_\infty$-sensitivity $\tau$, and unrestricted sensitivity

**Input:** Histogram $h$; outcomes $k$, cut off at $\bar{d} + 1$, and $\varepsilon_{\text{per}}, \delta$.

**Output:** Ordered set of indices and counts.

Sort $h_{(1)} \geq h_{(2)} \geq \cdots \geq h_{(\bar{d}+1)}$.

for $i \in \{k, \cdots, \bar{d}\}$ do

Set $v_i = h_{(i+1)} + \tau + \tau \ln(i/\delta)/\varepsilon_{\text{per}} + \text{Gumbel}(\tau/\varepsilon_{\text{per}})$

Set $\tilde{k} = \text{argmin}\{v_i\}$.

Set $h_{\perp} = h_{(k+1)} + \tau \cdot (1 + \ln(\min\{\tilde{k}, \bar{d} - \tilde{k}\}/\delta)/\varepsilon_{\text{per}})$.

Set $v_{\perp} = h_{\perp} + \text{Gumbel}(\tau/\varepsilon_{\text{per}})$.

for $j \leq \tilde{k}$ do

if $h_{(j)} > h_{(k+1)}$ then

Set $v_{(j)} = h_{(j)} + \text{Gumbel}(\tau/\varepsilon_{\text{per}})$.

Sort $\{v_{(j)}\} \cup v_{\perp}$.

Let $v_{i_1}, \ldots, v_{i_j}, v_{\perp}$ be the sorted list up until $v_{\perp}$.

if $j < k$ then

Return $\{(i_1, h_{i_1} + \text{Lap}(2\tau/\varepsilon_{\text{per}})), \ldots, (i_j, h_{i_j} + \text{Lap}(2\tau/\varepsilon_{\text{per}})), (\perp, v_{\perp})\}$

else

Return $\{(i_1, h_{i_1} + \text{Lap}(2\tau/\varepsilon_{\text{per}})), \ldots, (i_k, h_{i_k} + \text{Lap}(2\tau/\varepsilon_{\text{per}}))\}$.

6 Privacy Budget Management Service

We ultimately want to ensure that no analyst can identify any individual’s data with high confidence. We then impose a strict overall $(\varepsilon^*, \delta^*)$-DP guarantee. In order to compute the parameters
(ε_{per}, δ) that we use in each call to our algorithms\(^1\) over an entire sequence of interactions with the API, we also want to know how many queries the API will allow, denoted as ℓ\(^\star\) that we term the call budget, which will effectively impact δ\(^\star\). Further, we want to track the number of elements we want to return, denoted as k\(^\star\) that we term the information budget, which will effectively impact ε\(^\star\). Note that k\(^\star\) does not necessarily equal the number of elements returned, because we might be in the restricted sensitivity setting, and ℓ\(^\star\) does not precisely equal the number of calls to the API, since we might be in the known domain setting for some queries. Once we have (ε_{per}, δ), we will only use these parameters in each algorithm, hence not allowing for adaptively changing privacy parameters.

6.1 Budget Management Implementation

As mentioned in Section 4, the budget manager needs to be a distributed system so that it can be accessed/updated from different application execution platforms. Each analyst may access data from multiple data centers and each access must deduct from the same budget. Hence, the budget manager maintains eventual consistency across data centers.

The budget can be thought of as an associative array with keys from [ℓ, k] and values as the corresponding units used. Given a particular outcome o from the API, the budget service will update k\(^\star\) ← k\(^\star\) − Δ for Δ-restricted sensitivity queries or k\(^\star\) ← k\(^\star\) − 2|o| for unrestricted sensitivity queries where |o| denotes the number of elements returned in outcome o. Furthermore, the privacy budget management system will update ℓ\(^\star\) ← ℓ\(^\star\) − 1 for each query the analyst makes that is in the unknown domain setting. Once k\(^\star\) or ℓ\(^\star\) are depleted, we prevent the analyst from making any other queries. We address the challenge of computing the individual privacy parameters (ε_{per}, δ) given (ε\(^\star\), δ\(^\star\), k\(^\star\), ℓ\(^\star\)) in Theorem 2.

We adopt a privacy budget management service that assumes any user does not collude with other analysts. Hence each analyst is given her own privacy budget to interact with the Audience Engagement API and her queries do not impact the budget of another analyst. One can imagine variants of this assumption, such as all analysts that belong to the same company must share a budget. Further, the API adheres to the privacy budget up to some time frame. Thus, if an analyst has asked more than ℓ\(^\star\) unknown domain queries, then she will not be allowed any further queries. After this prescribed time frame, the parameters effectively get refreshed and the analyst can continue asking queries. Refresh is acceptable at regular intervals if the underlying data is flushed and replaced at similar intervals, whether through complete snapshot replacement, or rolling windows, such that the user’s data does not remain constant.

The application links with a budget manager client library so as to hide the implementation details of the budget management service because application writers do not need to know the details about the budget database, or the budget refresh mechanisms. The parameters (k\(^\star\), ℓ\(^\star\)) may be configured by the application.

6.2 Differential Privacy Composition

We present pseudocode for the privacy budget management service in Algorithm 5. We then present a way to compute the privacy guarantee of our overall system, which largely follows the analysis from Durfee and Rogers [5]. Essentially, the analysis follows from the fact that each algorithm can be represented as an iterative sequence of ε_{per}-BR algorithms. Note that the algorithms in the

\(^1\)Note that for the Laplace and Exponential mechanisms in the known domain, δ = 0.
unknown domain setting have a probability $\delta$ of larger privacy loss, which we account for in the overall $\delta^*$ in the privacy guarantee.

Algorithm 5 BudgetSystem$^{k^*,\ell^*}$; Budget Management Service

**Input:** An adaptive stream of histograms $h_1, h_2, \ldots$, fixed integers $k^*$ and $\ell^*$, along with per iterate privacy parameters $\varepsilon_{\text{per}}, \delta$.

**Output:** Sequence of outputs $(o_1, o_2, \cdots)$.

**while** $k^* > 0$ and $\ell^* > 0$ **do**

- From previous outcomes, select $h_i \in \mathbb{N}^{d_i}$ with $\ell_i^\infty$-bound $\tau_i$.
- Select $k_i$ and number of elements allowed to access $d_i$.

  **if** histogram has $\Delta$-restricted sensitivity **then**
  
  - **if** $\Delta > k^*$ **then**
    
    Break
  
  - **if** $d_i > d_i$ **then**
    
    $o_i = \text{KnownLap}^\Delta,\tau_i(h_i)$.
    
    Update $k^* \leftarrow k^* - \Delta$.
  
  - **else**
    
    $o_i = \text{UnkLap}^\Delta,d_i,\tau_i(h_i)$
    
    Update $\ell^* \leftarrow \ell^* - 1$.
    
    Update $k^* \leftarrow k^* - 1$.

  **if** histogram has unrestricted sensitivity **then**

  - **if** $2k_i > k^*$ **then**
    
    Break
  
  - **if** $d_i > d_i$ **then**
    
    $o_i = \text{KnownGumb}^{k_i,\tau_i}(h_i)$.
    
    Update $k^* \leftarrow k^* - 2k_i$.
  
  - **else**
    
    $o_i = \text{UnkGumb}^{k_i,d_i,\tau_i}(h_i)$
    
    Update $\ell^* \leftarrow \ell^* - 1$.
    
    Update $k^* \leftarrow k^* - (2|o_i| - 1 \{o_i[-1] = \bot\})$.

**Return** $o = (o_1, o_2, \cdots)$

In order to allow for the budget management service to return counts in the unrestricted sensitivity setting, we need to account for that in our overall budget. Further, in the unknown domain setting, if the last element of $o_i$ is $\bot$ at round $i$, denoted as $o_i[-1]$, then adding Laplace noise with parameter $2\tau_i/\varepsilon_{\text{per}}$ to the counts of each of the discovered $|o_i| - 1$ elements. will ensure $\varepsilon_{\text{per}}$-BR for each count. We can then apply our privacy loss bounds to get an overall DP guarantee by updating $k^* \leftarrow k^* - (2|o_i| - 1)$ and when the last element in $o_i$ is not $\bot$, then we instead update $k^* \leftarrow k^* - 2|o_i|$. Note that if we did not require counts in the results and need only return an ordered list of elements in the top-$k$, then we need only update $k^* \leftarrow k^* - |o_i|$.

**Theorem 2.** For $\delta' \geq 0$ and $\varepsilon_{\text{per}}, \delta > 0$, the BudgetSystem$^{k^*,\ell^*}$ is $(\varepsilon^*, \delta^*)$-DP where $\delta^* = 2\ell^*\delta + \delta'$ and $\varepsilon^*$ is defined as the minimum between $\varepsilon_{\text{per}}$ and the following,

$$k^* \left( \frac{\varepsilon_{\text{per}}}{1 - e^{\varepsilon_{\text{per}}}} - 1 - \ln \left( \frac{\varepsilon_{\text{per}}}{1 - e^{\varepsilon_{\text{per}}}} \right) \right) + \varepsilon_{\text{per}} \sqrt{k^* \ln(1/\delta')}.$$
Proof. For the \( \Delta \)-restricted sensitivity setting, we are deducting the information budget by \( \Delta \) in the known domain setting or we scale the privacy parameter by \( \Delta \) and deduct one from the information budget in the unknown domain setting. For a given histogram in the known domain setting, adding \( \text{Lap}(2\tau/\epsilon_{\text{per}}) \) to each count will ensure \( \Delta \epsilon_{\text{per}} \)-BR. We can also analyze this mechanism as if we iteratively add \( \text{Lap}(2\tau/\epsilon_{\text{per}}) \) to each count and then apply any DP or BR composition bound to obtain a DP guarantee. We need only apply composition for the number of elements that can actually change between neighboring datasets, i.e. \( \Delta \), and not the full dimension of the histogram. For all settings, the application of each Laplace mechanism is \( \epsilon_{\text{per}}/2 \)-DP, hence \( \epsilon_{\text{per}} \)-BR, while each application of Gumbel noise (Exponential Mechanism) is \( \epsilon_{\text{per}} \)-BR. Thus, we apply the composition bounds for BR mechanisms from Dong et al. [4]. The resulting bound applies BR composition over \( k^* \) many \( \epsilon_{\text{per}} \)-BR mechanisms and deducting 1 from the call cost budget \( \ell^* \) if an unknown domain algorithm is used for a query.

Given the total budget for the number of outcomes and queries \( (k^*, \ell^*) \) along with privacy budget \( (\epsilon^*, \delta^*) \) we can solve for the parameter \( \epsilon_{\text{per}} \) that satisfies the budget, which is then used in each algorithm. One approach we can use is the following (somewhat arbitrary) choice for \( \delta = \delta^*/4 \) and \( \delta' = \delta^*/2 \).

7 Results

We now present some preliminary results of our privacy system for the Audience Engagement API. In Figure 3 we present curves for the number of discovered elements in a top-50 query with varying \( \epsilon_{\text{per}} \) and \( \bar{d} \), i.e. the number of elements to collect, in procedure \text{UnkGumb}^{50, \bar{d}, 1} \) from Algorithm 4 with a fixed \( \delta = 10^{-10} \). The query is to find the top articles that distinct members from the San Francisco area are engaging with. We provide intervals that contain the 25th and 75th percentiles over 1000 independent trials. Note that the randomness in each trial is solely from the noise generation and we are using the same dataset each time. We see that with the same level of privacy, increasing the number of elements to fetch allows us to discover more elements. Hence, we see a natural tradeoff not just between privacy \( (\epsilon_{\text{per}}) \) and utility (number of elements returned), but also between run time (fetching more results) and utility. For example, we can return twice as many elements if we fetch four times more elements with Pinot and setting \( \epsilon_{\text{per}} = 0.8 \).

We also empirically evaluate procedure \text{UnkLap}^{\Delta, \bar{d}, 1} \) from Algorithm 3 in the unknown domain, \( \Delta \)-restricted sensitivity setting. In Figure 4 we show both the proportion of times in 1000 trials that each element was returned (right vertical axis) as well as the comparison between the noisy counts (in green) and the true counts (in red) that are returned for the discovered elements for a single trial (left vertical axis). In each plot there is a privacy parameter \( \epsilon_{\text{per}} \in \{0.1, 0.2, 0.3\} \), with fixed \( \delta = 10^{-10} \). The query is to find the top primary job titles of members that engaged with articles about privacy or California. We assume that any one member cannot have more than one primary job title, hence \( \Delta = 1 \), and fetch \( \bar{d} = 1000 \) results from Pinot.

For the budget manager, we have a fixed budget for each marketing partner. Once the privacy budget is depleted, a marketing partner would recycle old queries to get the same results or wait some fixed amount of time for the privacy budget to be refreshed. This policy decision for the rate in which to refresh the budget is dependent on how often the underlying dataset gets renewed and the characteristics of the underlying dataset. In order to maintain consistency across the same queries on the same dataset, we use the same seed in the pseudorandom noise, as in [16].
Figure 3: The number of discovered elements returned in UnkGum^{50,d,1} for a top-50 query with various \( \bar{d} \). We give the empirical average in 1000 trials and the (25%,75%) percentiles.

Figure 4: The noisy counts (left \( y \)-axis) of the discovered elements returned in UnkLap^{1,100,1} for a top-100 query as well as the proportion (right \( y \)-axis) in which various elements in 1000 independent trials were discovered.
8 Conclusion

We have presented a privacy system that incorporates state of the art algorithms for releasing histograms and top-\(k\) results in a differentially private way. Also, we have shown how we track the privacy budget for multiple analysts that can query our API. Combining the budget management service with DP algorithms allows us to make strong privacy guarantees of the overall system for any external partner that is allowed to make multiple, adaptively selected queries. This privacy system allows us to track the amount of information that is being released to external partners via the API in a precise way so that we can make informed decisions in how we can balance privacy safeguards with the usefulness of the product. We hope that this work demonstrates the feasibility of providing rigorous DP guarantees in systems that can scale.

Acknowledgements  We would like to thank Sofus Macskassy, Mark Cesar, Stephen Lynch, Koray Mancuhan, as well as the entire LinkedIn Data Science Applied Research team for their helpful feedback on this work. Further, we thank Ya Xu and Igor Perisic for their support throughout this project.
References

[1] Apple Differential Privacy Team. 2017. Learning with Privacy at Scale. Available at https://machinelearning.apple.com/2017/12/06/learning-with-privacy-at-scale.html.

[2] Aref N. Dajani, Amy D. Lauger, Phyllis E. Singer, Daniel Kifer, Jerome P. Reiter, Ashwin Machanavajjhala, Simson L. Garfinkel1, Scot A. Dahl, Matthew Graham, Vishesh Karwa, Hang Kim, Philip Leclerc, Ian M. Schmutte, William N. Sexton, Lars Vilhuber, and John M. Abowd. 2017. The modernization of statistical disclosure limitation at the U.S. Census Bureau. (2017). Available online at https://www2.census.gov/cac/sac/meetings/2017-09/statistical-disclosure-limitation.pdf.

[3] Bolin Ding, Jana Kulkarni, and Sergey Yekhanin. 2017. Collecting Telemetry Data Privately. https://www.microsoft.com/en-us/research/publication/collection-telemetry-data-privately/

[4] Jinshuo Dong, David Durfee, and Ryan Rogers. 2019. Optimal Differential Privacy Composition for Exponential Mechanisms and the Cost of Adaptivity. CoRR abs/1909.13830 (2019). arXiv:1909.13830 http://arxiv.org/abs/1909.13830

[5] David Durfee and Ryan Rogers. 2019. Practical Differentially Private Top-k Selection with Pay-what-you-get Composition. CoRR abs/1905.04273 (2019). arXiv:1905.04273 http://arxiv.org/abs/1905.04273

[6] Cynthia Dwork, Krishnaram Kenthapadi, Frank McSherry, Ilya Mironov, and Moni Naor. 2006. Our Data, Ourselves: Privacy Via Distributed Noise Generation. In Advances in Cryptology (EUROCRYPT 2006).

[7] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. 2006. Calibrating noise to sensitivity in private data analysis. In Proceedings of the Third Theory of Cryptography Conference. 265–284.

[8] Úlfar Erlingsson, Vasyl Pihur, and Aleksandra Korolova. 2014. RAPPOR: Randomized Aggregatable Privacy-Preserving Ordinal Response. In Proceedings of the 2014 ACM SIGSAC Conference on Computer and Communications Security (CCS ’14). ACM, New York, NY, USA, 1054–1067. https://doi.org/10.1145/2660267.2660348

[9] Marco Gaboardi, James Honaker, Gary King, Kobbi Nissim, Jonathan Ullman, Salil Vadhan, and Jack Murtagh. 2016. PSI (Ψ): a Private data Sharing Interface. In Theory and Practice of Differential Privacy. New York, NY. https://arxiv.org/abs/1609.04340

[10] Miguel Guevara. 2019. Google Developers. https://developers.googleblog.com/2019/09/enabling-developers-and-organizations.html

[11] Jean-François Im, Kishore Gopalakrishna, Subbu Subramaniam, Mayank Shrivastava, Adwait Tumbde, Xiaotian Jiang, Jennifer Dai, Seunghyun Lee, Neha Pawar, Jialiang Li, and Ravi Aringunaram. 2018. Pinot: Realtime OLAP for 530 Million Users. In Proceedings of the 2018 International Conference on Management of Data (SIGMOD ’18). ACM, New York, NY, USA, 583–594.
[12] Noah Johnson, Joseph P. Near, and Dawn Song. 2018. Towards Practical Differential Privacy for SQL Queries. *Proc. VLDB Endow.* 11, 5 (Jan. 2018), 526–539.

[13] Apache Kafka. [n. d.]. A Distributed Streaming Platform. [kafka.apache.org/](http://kafka.apache.org/)

[14] John Kahan. 2019. LinkedIn. [https://www.linkedin.com/pulse/microsoft-harvards-institute-quantitative-social-science-john-kahan/](https://www.linkedin.com/pulse/microsoft-harvards-institute-quantitative-social-science-john-kahan/)

[15] P. Kairouz, S. Oh, and P. Viswanath. 2017. The Composition Theorem for Differential Privacy. *IEEE Transactions on Information Theory* 63, 6 (June 2017), 4037–4049. [https://doi.org/10.1109/TIT.2017.2685505](https://doi.org/10.1109/TIT.2017.2685505)

[16] Krishnaram Kenthapadi and Thanh T. L. Tran. 2018. PriPeARL: A Framework for Privacy-Preserving Analytics and Reporting at LinkedIn. In *Proceedings of the 27th ACM International Conference on Information and Knowledge Management (CIKM ’18).* ACM, New York, NY, USA, 2183–2191.

[17] Frank McSherry. 2010. Privacy Integrated Queries. *Commun. ACM* 53 (September 2010), 89–97.

[18] Frank McSherry and Kunal Talwar. 2007. Mechanism design via differential privacy. In *48th Annual Symposium on Foundations of Computer Science.*

[19] Jack Murtagh and Salil Vadhan. 2016. The Complexity of Computing the Optimal Composition of Differential Privacy. In *Proceedings, Part I, of the 13th International Conference on Theory of Cryptography - Volume 9562 (TCC 2016-A).* Springer-Verlag, Berlin, Heidelberg, 157–175. [https://doi.org/10.1007/978-3-662-49096-9_7](https://doi.org/10.1007/978-3-662-49096-9_7)
A Omitted Analysis for Section 5.2

We now go through the analysis for Algorithm 3. The differences between Algorithm 3 and the version that appeared as Algorithm 4 in [5] is that we are returning counts as well as indices, we do not limit the number of outcomes to be at most $k$ (since it is not a parameter), and we allow for counts to increase or decrease by $\tau \geq 1$ in neighboring datasets. As we will mainly be borrowing the analysis in [5] we will change $\bar{\text{UnkLap}}$ to better match the statements in that work. We then introduce the following algorithm, which we will show has the same distribution as $\bar{\text{UnkLap}}^{\Delta,k,\tau}(h)$.

**Definition A.1** (Limited Histogram Report Noisy Counts). We assume that the $\ell_\infty$ sensitivity between any neighboring histograms is $\tau$. We define the limited histogram report noisy counts to be $\text{LapMax}^{k,\tau}$ that takes as input a histogram along with a domain set of indices and returns an ordered list of counts with the corresponding index, where $\text{LapMax}^{k,\tau}(h, d) = (\{v(1), i(1)\}, \ldots, \{v(\perp), \perp\})$ and $(v(1), \ldots, v(\perp))$ is the sorted list of $v_i = h_i + \text{Lap}(2\tau \Delta / \epsilon_{\text{per}})$ for each $i \in d$ and $v_\perp = h_{(k+1)} + \tau \left(1 + 2 \Delta \ln(\Delta / \delta) / \epsilon_{\text{per}}\right) + \text{Lap}(2\tau \Delta / \epsilon_{\text{per}})$ with $\delta$ given in Algorithm 3 as a function of $\delta > 0$.

We have the following that connects $\text{LapMax}^{k,\tau}$ with $\text{UnkLap}^{\Delta,k,\tau}$.

**Corollary A.1.** For any histogram $h$, we have that both mechanisms $\text{LapMax}^{k,\tau}(h, d^k(h))$ and $\text{UnkLap}^{\Delta,k,\tau}(h)$ produce outcomes that are equal in distribution.

If we fix a domain $d$ beforehand, then we have the following privacy statement. Note that the privacy of $\text{LapMax}^{k,\tau}(h, d)$ follows from the Laplace mechanism [7] being $\epsilon_{\text{per}}$-DP. This is what allows us to output the counts as well as the indices. We just need to ensure that $i_{(k+1)} \notin d$ because then if it was, then changing one index would change the count of both $h_{(k+1)}$ and $h_\perp = h_{(k+1)} + \tau \left(1 + \Delta \ln(\Delta / \delta) / \epsilon_{\text{per}}\right)$.

**Lemma A.1.** For any fixed $d \subseteq \{d\}$ and neighbors $h, h'$ such that $i_{(k+1)}, i'_{(k+1)} \notin d$, then for any set of outcomes $T$,

$$\Pr[\text{LapMax}^{k,\tau}(h, d) \in T] \leq e^{\epsilon_{\text{per}} / 2} \Pr[\text{LapMax}^{k,\tau}(h', d) \in T].$$

As was done in Durfee and Rogers [5], we can carefully account for the good (can bound the privacy loss) and bad (can bound these events with small probability) sets. Note that the outcome set of $\text{UnkLap}^{\Delta,k,\tau}$ is a superset of Algorithm 4 in [5] when $k = k$, and it is straightforward to see that these algorithms have the same distribution with respect to index output (ignoring the counts output from $\text{UnkLap}^{\Delta,k,\tau}$). Therefore, all the bounds on the bad outcomes will still hold for our setting, and the analysis then follows from results in Section 6 of [5], where we state each result here.

**Definition A.2.** Given two neighboring histograms $h, h'$, we define $S_{\text{Lap}}$ as the outcome set of $\text{UnkLap}^{\Delta,k,\tau}(h, d^k(h))$ (both indices and counts) and the outcome set of $\text{UnkLap}^{\Delta,k,\tau}(h', d^k(h'))$ as $S'_{\text{Lap}}$.

We then define the bad outcomes as $S^\delta_{\text{Lap}} := S_{\text{Lap}} \setminus S'_{\text{Lap}}$ and $S^\delta_{\text{Lap}} := S'_{\text{Lap}} \setminus S_{\text{Lap}}$.

**Lemma A.2.** For $\Delta$-restricted sensitivity neighbors $h, h'$, we have

$$\Pr[\text{LapMax}^{k,\tau}(h, d^k(h)) \in \bar{S}_{\text{Lap}}] \leq \delta / 4 \cdot (3 + \ln(\Delta / \delta)) =: \tilde{\delta}$$

(4)
Lemma A.3. For any neighboring histograms $h, h'$ and for any $S \subseteq S_{\text{Lap}} \cap S'_{\text{Lap}}$, we let $d^{\text{per}} = d^k(h) \cap d^k(h')$ and we must have the following for $\bar{\delta}$ given in (4)

$$
\Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S] \leq \Pr[\text{LapMax}^{\bar{k},\tau}(h, d^{\text{per}}) \in S] \\
\leq \Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S] + \bar{\delta}
$$

Lemma A.4. For any neighboring histograms $h, h'$ and any $S \subseteq S_{\text{Lap}}$, then for $\bar{\delta}$ given in (4),

$$
\Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S] \\
\leq \epsilon^{\text{per}/2} \Pr[\text{LapMax}^{\bar{k},\tau}(h', d^k(h')) \in S] + (\epsilon^{\text{per}/2} + 1)\bar{\delta}.
$$

Proof. We use the above results to get the following inequalities.

$$
\Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S] \\
= \Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S \cap \{S_{\text{Lap}} \cap S'_{\text{Lap}}\}] \\
\quad + \Pr[\text{LapMax}^{\bar{k},\tau}(h, d^k(h)) \in S \cap \{S_{\text{Lap}}\}] \\
\leq \Pr[\text{LapMax}^{\bar{k},\tau}(h, d^{\text{per}}) \in S \cap \{S_{\text{Lap}} \cap S'_{\text{Lap}}\}] + \bar{\delta} \\
\leq \epsilon^{\text{per}/2} \Pr[\text{LapMax}^{\bar{k},\tau}(h', d^{\text{per}}) \in S \cap \{S_{\text{Lap}} \cap S'_{\text{Lap}}\}] + \bar{\delta} \\
\leq \epsilon^{\text{per}/2} \left( \Pr[\text{LapMax}^{\bar{k},\tau}(h', d^k(h')) \in S \cap \{S_{\text{Lap}} \cap S'_{\text{Lap}}\}] + \bar{\delta} \right) + \bar{\delta} \\
\leq \epsilon^{\text{per}/2} \Pr[\text{LapMax}^{\bar{k},\tau}(h', d^k(h')) \in S] + (\epsilon^{\text{per}/2} + 1)\bar{\delta}.
$$

$\square$