WEAK $q$-CONCAVITY CONDITIONS FOR $CR$-MANIFOLDS

MAURO NACINOVICH AND EGMONT PORTEN

Abstract. We introduce various notions of $q$-pseudo-concavity for abstract $CR$ manifolds and we apply these notions to the study of hypo-ellipticity, maximum modulus principle and Cauchy problems for CR functions.

Contents

Introduction 1
1. $CR$- and $Z$-manifolds: preliminaries and notation 3
2. Hypo-ellipticity and the maximum modulus principle 6
3. The kernel of the Levi form and the $(H)$ property 9
4. Malgrange’s theorem and some applications 14
5. Hopf lemma and some consequences 17
6. Convex cones of Hermitian forms 24
7. Notions of pseudo-concavity 28
8. Cauchy problem for $CR$ functions - Uniqueness 31
9. Cauchy problem for $CR$ functions - Existence 35
References 36

Introduction

The definition of $q$-pseudo-concavity for abstract $CR$ manifolds of arbitrary $CR$-dimension and $CR$-codimension, given in [19], required that all scalar Levi forms corresponding to non-characteristic codirections have Witt index larger or equal to $q$. Important classes of homogeneous examples (see e.g. [2, 3, 4, 5, 34, 32]) show that these conditions are in fact too restrictive and that weaker notions of $q$-pseudo-concavity are needed. For example, the results on the non validity of the Poincaré lemma for the tangential Cauchy-Riemann complex in [10, 22] only involve scalar Levi forms of maximal rank. In [20] the classical notion of $1$-pseudo-concavity was extended by a trace condition, that was further improved in [1, 17, 21]. These notions are relevant to the behavior of $CR$ functions, being related to hypo-ellipticity, weak and strong unique continuation, hypo-analiticity (see [37]) and the maximum modulus principle.

In this paper, we continue these investigations. A key point of this approach is the simple observation that the Hermitian symmetric vector valued Levi form $L$ of a $CR$ manifold $M$ defines a linear form on $T^{1,1}M = T^{1,0}M \otimes M T^{0,1}M$. Our notion

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\text{$^1$The Witt index of a Hermitian form of signature $(p, q)$ is min$(p, q)$.}
of pseudoconcavity is the request that its kernel contains elements \( \tau \) which are positive semidefinite. To such a \( \tau \) we can associate an invariantly defined degenerate-elliptic real partial differential operators \( P_\tau \), which turns out to be related to the \( dd^c \)-operator of \( [31] \). By consistently keeping this perspective, we prove in this paper some results on \( C^\omega \)-hypoellipticity, the maximum modulus principle, and undertake the study of boundary value problems for \( CR \) functions on open domains of abstract \( CR \) manifolds, testing the effectiveness of a new notion of weak two-pseudo-concavity by its application to the Cauchy problem for \( CR \) functions.

The general plan of the paper is the following. In the first section we define the notion of \( \mathcal{Z} \)-structure, that generalizes \( CR \) structures insofar that all formal integrability and rank conditions can be dropped while our focus are \( CR \) functions, only considered as solutions of a homogeneous overdetermined system of first order p.d.e.’s, and set the basic notation that will be used throughout the paper. In particular, we introduce the kernel \([\ker L] \) of the Levi form as a subsheaf of the sheaf of germs of semipositive tensors of type \((1,1)\).

In \( \S 2 \) we show how the maximum modulus principle relates to \( C^\omega \)-regularity and weak and strong unique continuation of \( CR \) functions. We also make some comments on generic points of non-embeddable \( CR \) manifolds, where, by using the results of \( [37] \), we can prove, in Proposition 2.3, a result of strong unique continuation and partial hypo-analiticity (cf. \( [47] \)).

In \( \S 3 \) we show how that to each semi-positive tensor \( \tau \) in the kernel of the Levi form we can associate a real degenerate elliptic scalar p.d.o. of the second order \( P_\tau \). Real parts of \( CR \) functions are \( P_\tau \)-harmonic and the modulus of a \( CR \) function is \( P_\tau \)-subharmonic at points where it is different from zero. Then, by using some techniques originally developed for the generalized Kolmogorov equation (cf. \( [23, 24, 28] \)) we are able to enlarge, in comparison with \( [11] \), the set of vector fields entralled by \( \mathcal{Z} \). Thus we can improve, by Theorem 3.2, some hypo-ellipticity result of \( [1] \), and, by Theorem 3.7, a propagation result of \( [21] \), for the case in which this hypo-ellipticity fails.

In \( \S 4 \) we prove the \( CR \) analogue of Malgrange’s theorem on the vanishing of the top degree cohomology under some subellipticity condition. Our result slightly generalizes previous results of \( [9, 29, 30] \), also yielding a Hartogs-type theorem on abstract \( CR \) manifolds, to recover a \( CR \) function on a relatively compact domain from boundary values satisfying some momentum condition (Proposition 4.3).

In \( \S 5 \) we use the \( dd^c \)-operator of \( [31] \) to show that the operators \( P_\tau \) are invariantly defined in terms of sections of \([\ker L] \) (Corollary 5.8). The Hopf Lemma for \( P_\tau \) is used to deduce pseudo-convexity properties of the boundary of a domain where a \( CR \) functions has a peak point (Proposition 5.15). This leads to a notion of convexity/concavity for points of the boundary of a domain (Definition 5.4). Most of these notions can be formulated in terms of the scalar Levi forms associated to the covectors of a half-space of the characteristic bundle.

Thus in \( \S 6 \) we have found it convenient to consider properties of convex cones of Hermitian symmetric forms satisfying conditions on their indices of inertia, which are preliminary to the definitions of the next section.

In \( \S 7 \) we propose various notions of weak-\( q \)-pseudoconcavity, give some examples, and show in Proposition 7.7 that on an essentially-2-pseudo-concave manifold strong-1-convexity/concavity at the boundary becomes an open condition, i.e.
stable under small perturbations. This is used in the last two sections to discuss existence and uniqueness for the Cauchy problem for \( CR \) functions, with initial data on a hypersurface.

In §8 after discussing uniqueness in the case of a locally embeddable \( CR \) manifold, we turn to the case of an abstract \( CR \) manifold, proving, via Carleman-type estimates, that the uniqueness results of \([13, 20, 21]\) can be extended by using some convexity condition (see Proposition 8.2). In §9 an existence theorem for the Cauchy problem is proved for locally embeddable \( CR \) manifolds, under some convexity conditions.

1. \( CR \)- and \( Z \)-manifolds: preliminaries and notation

Let \( M \) be a real smooth manifold of dimension \( m \).

**Definition 1.1.** A \( Z \)-structure on \( M \) is the datum of a \( C^\infty_M \)-submodule \( Z \) of the sheaf \( \mathcal{X}^\C_M \) of germs of smooth complex vector fields on \( M \). It is called

- formally integrable if \( [Z, Z] \subset Z \);
- of \( CR \)-type if \( Z \cap \overline{Z} = \emptyset \) (the 0-sheaf);
- almost-\( CR \) if \( Z \) is of \( CR \)-type and locally free of constant rank;
- quasi-\( CR \) if it is of \( CR \)-type and formally integrable;
- \( CR \) if \( Z \) is of \( CR \)-type, formally integrable and locally free of constant rank.

A \( Z \)-manifold is a real smooth manifold \( M \) endowed with a \( Z \)-structure. Since \( C^\infty_M \) is a fine sheaf, \( Z \) can be equivalently described by the datum of the space \( Z(M) \) of its global sections.

When \( M \) is a smooth real submanifold of a complex manifold \( X \), then

\[
Z(M) = \{ Z \in \mathcal{X}^\C(M) \mid Z_p \in T^{0,1}_p X, \forall p \in M \}
\]

is formally integrable. Hence \( Z(M) \) defines a quasi-\( CR \) structure on \( M \), which is \( CR \) if the dimension of \( T^{0,1}_p X \cap \mathbb{C} T_p M \) is constant for \( p \in M \). This is always the case when \( M \) is a real hypersurface in \( X \).

A complex embedding (immersion) \( \phi : M \hookrightarrow X \) of a quasi-\( CR \)-manifold \( M \) into a complex manifold \( X \) is a smooth embedding (immersion) for which the \( Z \)-structure on \( M \) is the pullback of the complex structure of \( X \):

\[
Z(M) = \{ Z \in \mathcal{X}^\C(M) \mid d\phi(Z_p) \in T^{0,1}_{\phi(p)} X, \forall p \in M \}.
\]

**Example 1.1.** Let \( M = \{ w = z_1 \bar{z}_1 + iz_2 \bar{z}_2 \} \subset \mathbb{C}^3_{w,z_1,z_2} = X \). We can take the real and imaginary parts of \( z_1, z_2 \) as coordinates on \( M \), which therefore, as a smooth manifold, is diffeomorphic to \( \mathbb{C}^2_{z_1,z_2} \). The embedding \( M \hookrightarrow \mathbb{C}^3 \) yields the quasi-\( CR \) structure

\[
Z(M) = C^\infty(M) \left[ z_2 \frac{\partial}{\partial z_1} + iz_1 \frac{\partial}{\partial z_2} \right]
\]
on \( M \). Then \( M \setminus \{ 0 \} \) is a \( CR \)-manifold of \( CR \)-dimension 1 and \( CR \)-codimension 2, while all elements of \( Z(M) \) vanish at \( 0 \in M \).

A \( Z \)-manifold \( M \) of \( CR \)-type contains an open dense subset \( M \) whose connected components are almost-\( CR \) for the restriction of \( Z \). Likewise, any quasi-\( CR \) manifold \( M \) contains an open dense subset \( M \) whose connected components are \( CR \) manifolds.
We shall use $\mathcal{O}$ and $\mathcal{A}$ for the sheaves of germs of complex-valued and real-valued alternate forms on $M$ (subscripts indicate degree of homogeneity). Starting with the case of an almost-CR manifold $M$, we introduce the notation:

\[ T^{0,1} = \bigcup_{p \in M} (T^0_p M = \{ Z_p \mid Z \in \mathbb{C}(M) \}) \subset CTM, \quad T^{1,0} = \overline{T^{0,1}}, \]

\[ H^0 M = \bigcup_{p \in M} (H^0_p M = \{ \xi \in T^*_p M \mid \xi(H_p M) = \{ 0 \} \}) \subset T^*M, \]

\[ J_M : H_p M \to H_p M, \quad X_p + iJ_M X_p \in T^{0,1}_p M, \quad \forall X_p \in H_p M, \]

(partial closed bundle structure),

\[ \mathcal{H} = \{ \text{Re } Z \mid Z \in \mathbb{Z} \}, \]

\[ \pi_M : TM \to TM/HM \quad \text{(projection onto the quotient),} \]

\[ \mathcal{A}(M) = \{ \mathcal{A} \in \bigoplus_{k=1}^{\nu} \mathcal{O}^k(M, \mathbb{C}) \mid \mathcal{A}[T^{0,1} M] = 0 \}, \quad (\mathcal{A} \text{ is the ideal sheaf}), \]

\[ H^0 M = \bigcup_{p \in M} (H^0_p M = \{ \xi \in T^*_p M \mid \xi(H_p M) = \{ 0 \} \}) \subset T^*M, \]

\[ H^{1,1} M = \bigcup_{p \in M} (H^{1,1}_p M = \text{Convex hull of } \{ (Z_p \otimes \bar{Z}_p) \mid Z \in \mathcal{Z}(M) \}), \]

\[ H^{1,1,(r)} M = \bigcup_{p \in M} (H^{1,1,(r)}_p M = \{ \tau \in H^{1,1}_p M \mid \text{rank } \tau = r \}). \]

Note that $T^{0,1} M$, $T^{1,0} M$, $H^0 M$, $H^{1,1} M$, $H^{1,1,(r)} M$ define smooth vector bundles because we assumed that the rank $n$ of $Z$ is constant. $n$ is called the CR-dimension and the difference $k = m - 2n$ the CR-codimension of $M$.

For a general $Z$-manifold, we use the same symbols

\[ T^{0,1} M, T^{1,0} M, H^0 M, H^{1,1} M, H^{1,1,(r)} M \]

for the closures of

\[ T^{0,1} M, T^{1,0} M, H^0 M, H^{1,1} M, H^{1,1,(r)} M \]

in $T^C M, T^C M, T^C M/HM, T^{1,1} M, T^{1,1,(r)} M$, respectively.

**Example 1.2.** For the $M$ in Example 1.1, the fiber $T^{0,1}_p M$ has dimension 1 at all points $p$ of $M = M \setminus \{ 0 \}$, while $T^{0,1}_0 M = \mathbb{C}[\partial/\partial z_1, \partial/\partial z_2]$ has dimension 2. By contrast, as we already observed, all elements of $\mathcal{Z}(M)$ vanish at 0.

If $\mathcal{F}$ is a subsheaf of the sheaf of germs of (complex valued) distributions on $M$, an element $f$ of $\mathcal{F}$ is said to be CR if it satisfies the equations $Z f = 0$ for all $Z \in \mathcal{Z}(M)$. The CR germs of $\mathcal{F}$ are the elements of a sheaf that we denote by $\mathcal{F} \mathcal{O}_M$.

We will simply write $\mathcal{O}_M$ for $\mathcal{C}^\infty \mathcal{O}_M$.

We will assume in the rest of this section that $M$ is an almost-CR manifold.

The fibers of $H^{1,1} M$ are closed convex cones, consisting of the positive semidefinite Hermitian symmetric tensors in $T^{0,1} M \otimes M T^{1,0} M$. The characteristic bundle $H^0 M$ is the dual of the quotient $TM/HM$.

Let us describe more carefully the bundle structure of $H^{1,1,(r)} M$. Set $V = T^{0,1} M$ and consider the non-compact Stiefel space $St_r(V)$ of $r$-tuples of linearly independent vectors of $V$. Two different $r$-tuples $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ in $St_r(V)$ define the same $\tau_p$, i.e. satisfy

\[ \tau_p = v_1 \otimes \bar{v}_1 + \cdots + v_r \otimes \bar{v}_r = w_1 \otimes \bar{w}_1 + \cdots + w_r \otimes \bar{w}_r, \]

if and only if there is a matrix $a = (a_j^i) \in U(r)$ (the unitary group of order $r$) such that $w_j = \sum a_j^i v_i$. In fact the span of $v_1, \ldots, v_r$ is determined by the tensor $\tau_p$, so
that \( w_j = \sum_j a'_j v_i \) for some \( a = (a'_j) \in \text{GL}_r(\mathbb{C}) \) and
\[
\sum_{i=1}^r w_i \otimes \bar{w}_i = \sum_{i=1}^r \sum_{j=1}^r a'_j a''_j \bar{v}_i \otimes \bar{v}_h = \sum_{j=1}^r (\sum_{i=1}^r a'_j a''_j) v_i \otimes \bar{v}_h
\]
shows that \( a \in U(r) \). Hence \( H^{1,1(r)} M \) is the quotient bundle of the non-compact complex Stiefel bundle of \( r \times r \)-frames in \( T^{0,1} M \) by the action of the unitary group \( U(r) \). By using the Cartan decomposition
\[
U(r) \times p(r) \ni (x, X) \rightarrow x \cdot \exp(X) \in \text{GL}_r(\mathbb{C}),
\]
where \( p(r) \) is the vector space of Hermitian symmetric \( r \times r \)-matrices, we see that \( H^{1,1(r)} M \) can be viewed as a rank \( r^2 \) real vector bundle on the Grassmanian \( \mathcal{G} r'(M) \) of \( r \)-planes of \( T^{0,1} M \). Thus it is a smooth vector bundle when \( M \) is almost-\( CR \).

**Scalar and vector valued Levi forms.** The map
\[(1.1) \quad Z_p \otimes \bar{Z}_p \rightarrow -\pi_M([Z, \bar{Z}]_p), \quad \forall p \in M, \; \forall Z \in \mathcal{Z}(M), \]
extends to a linear map
\[(1.2) \quad \mathcal{L} : H^{1,1} M \rightarrow TM/\text{HM}, \]
that we call the vector-valued Levi-form.

To each characteristic co-direction \( \xi \in H^0_p M \) we associate the Hermitian quadratic form
\[
\mathcal{L}_\xi(Z_p, \bar{Z}_p) = \mathcal{L}(Z_p \otimes \bar{Z}_p) = -\langle [\xi][Z, \bar{Z}]_p \rangle, \; \forall Z \in \mathcal{Z}(M).
\]
It extends to a convex function on \( H^1_p M \), which is the evaluation by the co-vector \( \xi \) of the vector-valued Levi form. Thus the scalar Levi forms are
\[(1.3) \quad \mathcal{L}_\xi(\tau) = \langle \xi(\mathcal{L}(\tau)) \rangle, \quad \forall p \in M, \; \xi \in H^0_p M, \; \tau \in H^{1,1} M. \]

The range \( \Gamma_p M \) of the vector-valued Levi form is a convex cone of \( T_p M/H_p M \), whose dual cone is
\[
\Gamma^0_p M = \{ \xi \in H^0_p M \mid \mathcal{L}_\xi \geq 0 \}.
\]
Thus we obtain

**Lemma 1.3.** An element \( v \in T_p M/H_p M \) belongs to the closure or the range of the vector-valued Levi-form if and only if
\[(1.4) \quad \langle v|\xi \rangle \geq 0, \quad \forall \xi \in H^0_p M \text{ such that } \mathcal{L}_\xi \geq 0. \]

**Remark 1.4.** Note that \( \Gamma_p M \) need not be closed. An example is provided by the quadric \( M = \{ \Re z_3 = z_1 \bar{z}_2, \; \Re z_4 = \Re(z_1 \bar{z}_2) \} \subset \mathbb{C}^4 \): the cone \( \Gamma^0 M \) is the union of the origin and of an open half-plane.

It is convenient to introduce the notation:
\[
[\ker \mathcal{L}]^{(q)} = H^{1,1(q)} M \cap \ker \mathcal{L}, \quad \overline{[\ker \mathcal{L}]} = \bigoplus_{q \geq 0} [\ker \mathcal{L}]^{(q)} [\ker \mathcal{L}] = \bigoplus_{q < 0} [\ker \mathcal{L}]^{(q)}.
\]

**Definition 1.2.** We call \( [\ker \mathcal{L}] \) the kernel of the Levi form.

We note that this definition is at variance with a notion that appears in the literature (see e.g. [12]), where the kernel of the Levi form consists of the \((1,0)\)-vectors which are isotropic for all scalar Levi forms. These vectors are related to \( [\ker \mathcal{L}]^{(1)} \), which is trivial in several examples of \( CR \) manifolds which are not of hypersurface type and have a non trivial \( [\ker \mathcal{L}] \).
Let \( \gamma \) be a generalized distribution of real vector fields on \( M \) and \( p \in U^{\text{open}} \subset M \). The Sussmann leaf of \( \gamma \) through \( p \) in \( U \) is the set \( \ell(p; \gamma, U) \) of points \( p' \) which are ends of piecewise \( C^\infty \) integral curves of \( \gamma \) starting from \( p \) and lying in \( U \). We know that \( \ell(p; \gamma, U) \) is always a smooth submanifold of \( U \) (see [45]).

Let \( \mathcal{H} = \{ \text{Re} Z \mid Z \in \mathbb{Z} \} \). A \( \mathbb{Z} \)-manifold \( M \) is called \textit{minimal} at \( p \) if \( \ell(p; \mathcal{H}, U) \) is an open neighborhood of \( p \) for all \( U^{\text{open}} \subset M \) and \( p \in U \). (This notion was introduced in [46] for embedded CR manifolds.) In the following, by a \textit{Sussmann leaf of } \mathcal{H} \textit{ we will mean a Sussmann leaf of } \mathcal{H}.

A smooth real submanifold \( N \) of \( M \) (of arbitrary codimension \( \ell \)) is said to be non-characteristic, or generic, at \( p_0 \in N \), when
\[
(1.5) \quad T_{p_0} N + H_{p_0} M = T_{p_0} M.
\]
If this holds for all \( p \in N \), then \( N \) is a generic CR submanifold of \( M \), of type \((n-\ell, k+\ell)\), as \( T^0_{p,1} N = T^0_p N \cap T^0_{p,1} N \) and \( H^0_{p,1} N = H^0_p M \oplus J^\ast_p (T_p N)^0 \) for all \( p \in N \).

To distinguish from the Levi form \( \mathcal{L} \) of \( M \), we write \( \mathcal{L}^N \) for the Levi form of \( N \).

A Sussmann leaf for \( \mathcal{Z} \) which is not open is characteristic at all points.

More generally, when \( \Xi(M) \) is any distribution of complex valued smooth vector fields on \( M \), we say that \( N \) is \( \Xi \)-non-characteristic at \( p_0 \in N \) if
\[
(1.6) \quad T_{p_0} N + \{ \text{Re} Z_{p_0} \mid Z \in \Xi(M) \} = T_{p_0} M.
\]
In this terminology non-characteristic is equivalent to \( \mathcal{Z} \)-non-characteristic.

We note that the \( \Xi \)-non-characteristic points make an open subset of \( N \).

2. \textit{Hypo-ellipticity and the maximum modulus principle}

In [37] we proved that, for locally embedded CR manifolds, the hypo-ellipticity of its tangential Cauchy-Riemann system is equivalent to the holomorphic extendability of its CR functions. Thus hypo-ellipticity may be regarded as a weak form of pseudo-concavity. The regularity of CR distributions implies a strong maximum modulus principle for CR functions (see [20] Theorem 6.2).

**Proposition 2.1.** Let \( M \) be a \( \mathcal{Z} \)-manifold. Assume that all germs of CR distributions on \( M \), that are locally \( L^2 \), are smooth. Then, for every open connected subset \( \Omega \) of \( M \), we have
\[
(2.1) \quad |f(p)| < \sup_{\Omega} |f|, \quad \forall p \in \Omega, \quad \text{for all non constant } f \in \mathcal{O}_M(\Omega).
\]

**Proof.** We prove that an \( f \in \mathcal{O}_M(\Omega) \) for which \( |f| \) attains a maximum value at some inner point \( p_0 \) of \( \Omega \) is constant. Assume that \( p_0 \in \Omega \) and \( |f(p_0)| = \sup_{\Omega} |f| \).

If \( f(p_0) = 0 \), then \( f \) is constant and equal to zero on \( \Omega \). Assume that \( f(p_0) \neq 0 \). After rescaling, we can make \( f(p_0) = |f(p_0)| = 1 \).

Let \( E \) be the space \( \mathcal{O}_M(\Omega) \) endowed with the \( L^2_{\text{loc}} \) topology. By the hypo-ellipticity assumption, \( E \) is Fréchet. Then, by Banach open mapping theorem, the identity map \( E \to \mathcal{O}_M(\Omega) \) is an isomorphism of topological vector spaces. In particular, for all compact neighborhoods \( K \) of \( p_0 \) in \( \Omega \) there is a constant \( C_K > 0 \) such that
\[
|u(p_0)|^2 \leq C_K \int_K |u|^2 d\lambda, \quad \forall u \in \mathcal{O}_M(\Omega).
\]
Applying this inequality to \( f^\ast \), we obtain that
\[
1 \leq \int_K |f|^2 d\lambda \leq \int_K d\lambda.
\]
The sequence \{f^n\} is compact in \(\mathcal{O}_M(\Omega)\), because, by the hypo-ellipticity assumption and the Ascoli-Arzelà theorem, restriction to a relatively compact subset of CR functions is a compact map. Hence we can extract from \{f^n\} a sequence that converges to a CR function \(\phi\), which is non-zero because it has a positive square-integral on every compact neighborhood of \(p_0\). We note now that \(|\phi|\) is continuous, and takes only the values 1, at points where \(|f| = 1\), and 0 at points where \(|f| < 1\). Since \(\phi \neq 0\), we have \(|\phi| \equiv 1\) on \(\Omega\) and hence \(|f| \equiv 1\) on \(\Omega\). By applying the preceding argument to \(p \to \frac{1}{2}(1 + f(p))\), we obtain that \(|1 + f(p)| = 2\) on \(\Omega\). Hence \(\text{Re } f \equiv 1\), which yields \(f \equiv 1\), on \(\Omega\).

Under the assumptions of Proposition 2.1 a CR function \(f \in \mathcal{O}_M(\Omega)\) is constant on a neighborhood of any point where \(|f|\) attains a local maximum.

Then we have

**Proposition 2.2.** Assume that

(i) all germs of CR distribution on \(M\) are smooth;

(ii) the weak unique-continuation principle for CR functions is valid on \(M\).

Then any CR function \(f\), defined on a connected open subset \(\Omega\) of \(M\), for which \(|f|\) attains a local maximum at some point of \(\Omega\), is constant. □

We recall that the weak unique-continuation property (ii) means that a CR function \(f \in \mathcal{O}_M(\Omega)\) which is zero on an open subset \(U\) of \(\Omega\) is zero on the connected component of \(U\) in \(\Omega\).

**Definition 2.1.** We say that \(M\) has property (H) if (i) holds, and property (WUC) if (ii) holds. We say that (H) (or (WUC)) holds at \(p\) if it holds when \(M\) is substituted by a sufficiently small open neighborhood of \(p\) in \(M\).

For a locally CR-embeddable CR manifold \(M\) the implication (H) \(\Rightarrow\) (WUC) is a consequence of [37]. In fact, (H) implies minimality, which implies (WUC) when \(M\) is locally CR-embeddable (see [40] [43]). In fact, in this case (H) implies the strong unique continuation principle for CR functions.

**Proposition 2.3.** Assume that \(M\) is a CR submanifold of a complex manifold \(X\) and that \(M\) has property (H). Then a CR function, defined on a connected open subset \(\Omega\) of \(M\) and vanishing to infinite order at a point \(p_0\) of \(\Omega\) is identically zero in \(\Omega\).

**Proof.** Let \(f \in \mathcal{O}_M(\Omega)\). It is sufficient to prove that the set of points where \(f\) vanishes to infinite order is open in \(\Omega\). This reduces the proof to a local statement, allowing us to assume that the embedding \(M \hookrightarrow X\) is generic. By [37], any CR function \(f\) extends to a holomorphic function \(\tilde{f}\), defined on a connected open neighborhood \(U\) of \(p\) in \(X\). By the assumption that \(M \hookrightarrow X\) is generic, \(\tilde{f}\) is uniquely determined by the Taylor series of \(f\) at \(p\) in any coordinate chart, and thus vanishes to infinite order at a point \(p' \in U \cap \Omega\) if and only if \(f\) does. Hence \(f\) vanishes to infinite order at \(p\) if and only if \(\tilde{f}\) vanishes on \(U\), and this is equivalent to the fact that \(f\) vanishes identically on \(U \cap \Omega\). The proof is complete. □

When \(M\) is not locally embeddable, there should be smaller local rings of CR functions, so that in fact properties of regularity and unique continuation should even be more likely true. Let us shortly discuss this issue. Set

\[ T_{p,1}^0 M = \{ \zeta \in \mathbb{C} T_p^* M \mid \zeta(Z) = 0, \forall Z \in T_{p,1}^0 M \}. \]
Note that, with
\[ T_p^{0,1} M = T_p^{1,0} M = \{ \zeta \in \mathbb{C} T_p^* M \mid \zeta(\bar{Z}) = 0, \forall Z \in T_p^{0,1} M \}, \]
the intersection
\[ T_p^{1,0} M \cap T_p^{0,1} M = \mathbb{C} H^0_p M \]
is the complexification of the fiber of the characteristic bundle, and therefore different from zero, unless \( Z \) is an almost complex structure. Differentials of smooth CR functions are sections of the bundle \( T^{1,0}_p M \). Thus, for a fixed \( p \), we can consider the map
\[ (2.2) \quad \mathcal{O}_{M,p} \ni f \longrightarrow d f(p) \in T_p^{1,0} M. \]
Clearly we have

**Lemma 2.4.** A necessary and sufficient condition for \( M \) to be locally CR-embeddable at \( p \) is that (2.2) is surjective. \( \square \)

We can associate to the map (2.2) a pair \((\nu_p, k_p)\) of nonnegative integers, with \( k_p = \text{dim} C(\{f(p) \mid f \in \mathcal{O}_{M,p}\}) \cap \mathbb{C} H^0_p M \), and \( \nu_p + k_p = \text{dim} C(\{f(p) \mid f \in \mathcal{O}_{M,p}\}) \).

The numbers \( \nu_p \) and \( \nu_p + k_p \) are upper semicontinuous functions of \( p \) and hence locally constant on a dense open subset \( \tilde{M} \) of \( M \). Thus we can introduce

**Definition 2.2.** We call generic the points of the open dense subset \( \tilde{M} \) of \( M \), where \( \nu_p \) and \( \nu_p + k_p \) are locally constant.

**Proposition 2.5.** Assume that \( M \) has property (H). Then the strong unique continuation principle is valid at generic points \( p_0 \) of \( M \). This means that \( f \in \mathcal{O}_{M,p_0} \) is the zero germ if and only if \( f \) vanishes to infinite order at \( p_0 \).

Moreover, there are finitely many germs \( f_1, \ldots, f_{\mu} \in \mathcal{O}_{M,p_0} \), vanishing at \( p_0 \) such that, for every \( f \in \mathcal{O}_{M,p_0} \) we can find \( F \in \mathcal{O}_{C^0,0} \) such that \( f = F(f_1, \ldots, f_{\mu}) \).

**Proof.** By the assumption that \( p_0 \) is generic, we can fix a connected open neighborhood \( U \) of \( p_0 \) in \( M \) and functions \( f_1, \ldots, f_{\mu} \in \mathcal{O}_M(U) \), vanishing at \( p_0 \), such that \( d f_1(p) \wedge \cdots \wedge d f_{\mu}(p) \neq 0 \) for all \( p \in U \) and \( d f_1(p), \ldots, d f_{\mu}(p) \) generate the image of (2.2) for all \( p \in U \). Then, by shrinking \( U \), if needed, we can assume that
\[ \phi : U \ni p \longrightarrow (f_1(p), \ldots, f_{\mu}(p)) \in N \subset C^0 \]
is a smooth real vector bundle on a generic CR submanifold \( N \) of \( C^0 \), of CR-dimension \( \nu_{p_0} \) and CR-codimension \( k_{p_0} \).

In fact, we can assume that \( \text{Re} d f_1, \ldots, \text{Re} d f_{\mu}, \text{Im}(d f_1), \ldots, \text{Im}(d f_{\mu}) \) are linearly independent on \( U \). We can fix local coordinates \( x_1, \ldots, x_{\mu} \) centered at \( p_0 \) with \( x_1, \ldots, x_{\mu+\nu} \) equal to \( \text{Re} f_1, \ldots, \text{Re} f_{\mu}, \text{Im} f_1, \ldots, \text{Im} f_{\mu} \). By the assumption, in these local coordinates \( \text{Im} f_{\mu+1}, \ldots, \text{Im} f_{\mu+\nu} \) are smooth functions of \( x_1, \ldots, x_{\mu+\nu} \) and this yields a parametric representation of \( N \) as a graph of \( C^0 \times \mathbb{R}^{\mu+\nu} \) in \( C^0 \), which is therefore locally a generic CR-submanifold of type \((\nu, \mu - \nu)\) of \( C^0 \). The map \( \phi : U \rightarrow N \) is CR and therefore the pullback of germs of continuous CR function on \( N \) define germs of continuous CR function on \( M \). If \( M \) has property (H), then the \( \mathcal{C}^{\infty} \) regularity of their pullbacks implies the \( \mathcal{C}^{\infty} \) regularity of the germs on \( N \). Thus \( N \) also has property (H), and, since it is embedded in \( C^0 \), by [37], all CR functions on an open neighborhood \( \omega_0 \) of \( 0 \) in \( N \) are the restriction of holomorphic functions on a full open neighborhood \( \bar{\omega}_0 \) of \( 0 \) in \( C^0 \), with \( \omega_0 = \bar{\omega}_0 \cap N \). Since \( f_i = \phi'(z_i) \) for the holomorphic coordinates \( z_1, \ldots, z_{\mu} \) of \( C^0 \), we obtain that all
germs of CR functions at $p_0 \in M$ are germs of holomorphic functions of $f_1, \ldots, f_r$. This clearly implies the validity at $p_0$ of the strong unique continuation principle. The proof is complete. □

3. The kernel of the Levi form and the (H) property

To a finite set $Z_1, \ldots, Z_r$ of vector fields in $\mathcal{Z}(M)$ we associate the real valued vector field

$$Y_0 = \frac{1}{2i} \sum_{j=1}^{r} [Z_j, Z_j].$$

Any CR function $u$ on $M$ satisfies the degenerate-Schrödinger-type equation

$$Su = 0, \quad \text{with}$$

$$S = -iY_0 + \frac{1}{2} \sum_{j=1}^{r} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) = -iY_0 + \sum_{j=1}^{2r} X_j^2,$$

where $X_j = \text{Re} Z_j$, $X_{j+r} = \text{Im} Z_j$ for $1 \leq j \leq r$. In fact, by (3.1), we have

$$S = \frac{1}{2} \sum_{j=1}^{r} \bar{Z}_j Z_j,$$

and thus the operator $S$ belongs to the left ideal, in the ring of scalar linear partial differential operators with complex smooth coefficients, generated by $\mathcal{Z}(M)$.

We note that $S$ is of the second order, with a real principal part which is uniquely determined by $\tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes \bar{Z}_r \in \Gamma(H^{1,1}M)$, while a different choice of the $Z_j$‘s would yield a new $Y_0$, differing form $Y_0$ by the addition of a linear combination of $X_1, \ldots, X_{2r}$.

If we assume that $\tau \in \ker(\mathcal{L})$, then

$$\sum_{i=1}^{r} [Z_i, Z_i] = L_0 - L_0$$

for some $L \in \mathcal{Z}(M)$, which is uniquely determined by $\tau$ modulo a linear combination with $\mathcal{C}^\infty$ coefficients of $Z_1, \ldots, Z_r$. Thus the distributions of real vector fields

$$\begin{cases} Q_1(\tau) = (\text{Re} Z_1, \ldots, \text{Re} Z_r, \text{Im} Z_1, \ldots, \text{Im} Z_r), \\ \mathcal{P}_1(\tau) = \mathcal{U}(Q_1(\tau)), \\ \mathcal{P}_2(\tau) = \mathcal{U}(Q_1(\tau) + \text{Re} L_0), \end{cases}$$

are uniquely determined by $\tau$ and $2$. By $\mathcal{U}(\ldots)$ we indicate the formally integrable distribution of real vector fields, which is generated by the elements of the set inside the parentheses and their iterated commutators. Note that $\mathcal{P}_1(\tau) \subseteq \mathcal{P}_2(\tau)$ and, while $Y_0 = \text{Im} L_0 \in \mathcal{P}_1(\tau)$, the vector field $X_0 = \text{Re} L_0$ may not belong to $\mathcal{P}_1(\tau)$. We also introduce, for further reference, the distributions of complex vector fields

$$\begin{cases} \Theta(\tau) = (Z_1, \ldots, Z_r) \quad \text{and} \quad \Theta = \bigcup_{\tau \in \ker(\mathcal{L})} \Theta(\tau), \\ \bar{\Theta}(\tau) = \Theta(\tau) + (L_0) \quad \text{and} \quad \bar{\Theta} = \bigcup_{\tau \in \ker(\mathcal{L})} \bar{\Theta}(\tau). \end{cases}$$

When there is a $\tau \in \ker(\mathcal{L})(\Omega^{\text{open}})$, we utilize (3.4) to show that the real and imaginary parts of CR functions or distributions on $\Omega \subset M$ are solutions of a real degenerate-elliptic scalar second order differential equation. Indeed, if $f$ is a CR function, or distribution, in $\Omega$, then

$$L_0 f = 0, \quad Z_j f = 0 \implies (L_0 + L_0) f = \sum_{i=1}^{r} (Z_i \bar{Z}_j + \bar{Z}_j Z_j) f.$$
This is a consequence of the algebraic identities
\[(3.7) \quad \frac{1}{2} \left( \sum_{i=1}^{r} (Z_i Z_j + Z_j Z_i) - (L_0 + L_0) \right) = \sum_{i=1}^{r} Z_i Z_j - L_0 = \sum_{i=1}^{r} Z_j Z_i - L_0.\]

It terms of the real vector fields \(X_0 = \text{Re } L_0\) and \(X_j = \text{Re } Z_j, X_{r+j} = \text{Im } Z_j\), for \(1 \leq j \leq r\), the linear partial differential operator of (3.7) is
\[(3.8) \quad P_t = -X_0 + \sum_{i=1}^{2r} X_j^2,\]
which has real valued coefficients and is degenerate-elliptic according to [9]. Thus the real and imaginary parts of a CR function, or distribution, both satisfy the homogeneous equation \(P_t \phi = 0\).

Actually, \(P_t\) is independent of the choice of \(Z_1, \ldots, Z_r\) in the representation of \(\tau\), as we will later show by Proposition 3.6 by representing \(P_t\) in terms of the \(dd^c\) operator on \(M\). We also have (see [21]):

**Lemma 3.1.** If \(u \in \mathcal{D}_\Omega(M)\), then
\[(3.9) \quad P_t |u| \geq 0, \quad \text{on } \Omega \cap \{u \neq 0\}.

**Proof.** On a neighborhood of a point where \(u \neq 0\) we can consistently define a branch of \(\log u\). This still is a CR function, and from the previous observation it follows that \(P_t(\log |u|) = P_t(\text{Re } \log u) = 0\) on \(\Omega \cap \{u \neq 0\}\). Hence
\[
P_t |u| = P_t \exp(\log |u|) = |u| \left( P_t(\log |u|) + \sum_{i=1}^{r} |Z_i(\log |u|)|^2 \right) = |u| \sum_{i=1}^{r} |Z_i(\log |u|)|^2 \geq 0
\]
there. \(\Box\)

We can use the treatment of the generalized Kolmogorov equation in [24] §22.2] to slightly improve the regularity result of [1 Corollary 1.15]. Let us set
\[(3.10) \quad \mathfrak{V}_2 = \mathfrak{V} \left( \bigcup_{\tau \in [\ker L]_1} \mathfrak{V}_2(\tau) \right), \quad \mathfrak{Y} = \mathfrak{V}(\mathfrak{V}_2; \mathcal{H}),\]
where we use \(\mathfrak{V}(\mathfrak{V}_2; \mathcal{H})\) for the \(\mathfrak{V}_2\)-Lie module generated by \(\mathcal{H}\), which consists of the linear combinations, with smooth real coefficients, of the elements of \(\mathcal{H}\) and their iterated commutators with elements of \(\mathfrak{V}_2\):
\[(3.11) \quad \mathfrak{V}(\mathfrak{V}_2; \mathcal{H}) = \mathcal{H} + \mathfrak{V}_2 \mathcal{H} + \mathfrak{V}_2 \mathfrak{V}_2 \mathcal{H} + \mathfrak{V}_2 \mathfrak{V}_2 \mathfrak{V}_2 \mathcal{H} + \cdots\]
Note that \(\mathfrak{V}_2 \subset \mathfrak{V}(\mathfrak{V}_2; \mathcal{H})\) and that both \(\mathfrak{V}_2\) and \(\mathfrak{Y}\) are fine sheaves.

**Theorem 3.2.** \(M\) has property (H) at all points \(p\) where \(\{Y_p| \ Y \in \mathfrak{Y}(M)\} = T_pM\).

Before proving the theorem, let us introduce some notation. For \(\epsilon > 0\) we denote by \(S_\epsilon(M)\) the set of real vector fields \(Y \in \mathfrak{X}(M)\) such that, for every \(p \in M\) there is a neighborhood \(U_{\text{open}} \Subset M\) of \(p\), a constant \(C \geq 0\), \(\tau_1, \ldots, \tau_h \in [\ker L](M)\) and complex vector fields \(Z_1, \ldots, Z_\ell \in \mathcal{Z}(M)\) such that
\[(3.12) \quad \|Yf\|_{l-1} \leq C \left( \sum_{j=1}^{h} \|P_{\tau_j} f\|_0 + \sum_{j=1}^{\ell} \|Z_j f\|_0 + \|f\|_0 \right), \quad \forall f \in \mathfrak{V}_0^\infty(U).
\]
The Sobolev norms of real order (and integrability two) in (3.12) are of course computed after fixing a Riemannian metric on \(M\). Different choices of the metric yield equivalent norms (see e.g. [11] [16] for technical details). Beware that the \(Z_j\)
in the right hand side of (3.12) are not required to be related to those entering the definition of the $P_j$’s. Set
\begin{equation}
S(M) = \bigcup_{\epsilon > 0} S_\epsilon(M).
\end{equation}

Theorem 3.2 will follow from the inclusion $\mathcal{X}(M) \subseteq S(M)$. The following Lemmas 3.3 and 3.4 were proved in [1, 20].

**Lemma 3.3.** If $\tau \in [\ker \mathcal{L}]_1(M)$ and $P_\epsilon = -X_0 + \sum_{i=1}^{2r} X_i^2$, then $X_1, \ldots, X_{2r} \in S_1(M)$ and, for every $U^\text{open} \subseteq M$ there is a constant $C > 0$ and $Z_1, \ldots, Z_{2r} \in \mathcal{X}(M)$ such that
\begin{equation}
(3.14) \quad \sum_{j=1}^{2r} \|X_jf\|_0 \leq C \left(\|f\|_0 + \sum_{j=1}^{2r} \|Z_jf\|_0\right), \quad \forall f \in \mathcal{X}_0^\text{open}(U).
\end{equation}

Set $\mathcal{X}_1 = \left[H \mathcal{X}_1, Y f \mathcal{X}_1, Y f^* \mathcal{X}_1, \mathcal{X}_1, \mathcal{X}_1, \mathcal{X}_1\right] + \cdots$.

**Lemma 3.4.** We have the inclusion $\mathcal{X}(\mathcal{X}_1; \mathcal{X}) \subseteq S_1$.

To prove Theorem 3.2, we add the following lemma.

**Lemma 3.5.** Let $\tau \in [\ker \mathcal{L}]_1(M)$, with $P_\epsilon = -X_0 + \sum_{i=1}^{2r} X_i^2$. Then
\begin{equation}
(3.15) \quad [X_0, S_\epsilon(M)] \subseteq S_{\epsilon/4}(M).
\end{equation}

**Proof.** Let $Q_\epsilon = P_\epsilon + c$, for a suitable nonnegative real constant $c$, to be precised later. We decompose $Q_\epsilon$ into the sum $Q_\epsilon = Q'_\epsilon + iQ''_\epsilon$, where $Q'_\epsilon = \frac{1}{2}(Q_\epsilon + Q_\epsilon^*)$ and $Q''_\epsilon = \frac{1}{2}(Q_\epsilon - Q_\epsilon^*)$ are selfadjoint. In particular, $Q''_\epsilon = Q''_\epsilon - iQ''_\epsilon$. We can rewrite $Q'_\epsilon$ as a sum $Q'_\epsilon = -\sum_{j=1}^{2r} X_j^* X_j + iT + c$, for a p.d.o. $T$ of order $\leq 1$, whose principal part of order $1$ is a linear combination with $C^\omega$ coefficients of $X_1, \ldots, X_{2r}$. Moreover, we note that $P_\epsilon - P_\epsilon^* = Q'_\epsilon - Q''_\epsilon$. The advantage in dealing with $Q_\epsilon$ instead of $P_\epsilon$ is that, for $c$ positive and sufficiently large,
\begin{equation}
(Q_\epsilon f)(0) = (Q'_\epsilon f)(0) \geq 0, \quad \forall f \in \mathcal{X}_0^\text{open}(U).
\end{equation}

This is the single requirement for our choice of $c$.

In [1] it was shown that $[X_j, S_\epsilon] \subseteq S_{\epsilon/4}$ for $i = 1, \ldots, 2r$ and all $\epsilon > 0$. Then (3.15) is equivalent to the inclusion $[Q''_\epsilon, S_\epsilon] \subseteq S_{\epsilon/4}$.

Let $Y \in S_\epsilon(M)$ and $U^\text{open} \subseteq M$. We need to estimate $\|Q''_\epsilon Y f\|_{\epsilon-1}$ for $f \in \mathcal{X}_0^\text{open}(U)$. Let $A$ be any properly supported pseudodifferential operator of order $\frac{1}{2} - 1$. We have
\begin{align}
i\langle [Q''_\epsilon, Y f] A f \rangle & = i\langle [Q''_\epsilon - Q''_\epsilon^*] Y f A f \rangle_0 + i\langle (Q_\epsilon - Q_\epsilon^*) f Y^* A f \rangle_0 \\
& = i\langle Q''_\epsilon Y f A f \rangle_0 - i\langle Y f Q''_\epsilon A f \rangle_0 + i\langle (Q_\epsilon f) Y^* A f \rangle_0 - i\langle (Q'_\epsilon f) Y^* A f \rangle_0.
\end{align}

While estimating the sumsmands in the last expression, we shall indicate by $C_1, C_2, \ldots$ positive constants independent of the choice of $f$ in $\mathcal{X}_0^\text{open}(U)$.

Let us first consider the second and third summands. We have
\begin{align}
\|Y f Q''_\epsilon A f\|_0 & \leq \|Y f\|_{\epsilon-1} \|Q''_\epsilon A f\|_{1-\epsilon} \leq \|Y f\|_{\epsilon-1} (\|A Q''_\epsilon f\|_{1-\epsilon} + \|A, Q''_\epsilon f\|_{1-\epsilon}) \\
& \leq C_1 \|Y f\|_{\epsilon-1} \left(\frac{1}{2}\|Q''_\epsilon f\|_{\epsilon-\frac{1}{2}} + \left\|A, \sum_{j=1}^{2r} X_j^2 \right\|_{1-\epsilon} + \|f\|_{1-\frac{1}{2}}\right).
\end{align}

We have
\begin{align}
\left[A, \sum_{j=1}^{2r} X_j^2 \right] = -\sum_{j=1}^{2r} (2[X_j, A]X_j + [X_j, [X_j, A]]).
\end{align}
Since \([X_j, A]\) and \([X_j, [X_j, A]]\) have order \(\frac{4}{3} - 1\), and \(P_t\) and \(Q_t\) differ by a constant, we obtain
\[
|Y f Q_t A f|_0 \leq C_2 \|Y f\|_{\frac{4}{3} - 1} \left(\|P_t f\|_{\frac{4}{3}} + |f|_{\frac{4}{3}} + \sum_{j=1}^{2r} \|X_j f\|_{\frac{4}{3}}\right).
\]

Analogously, for the third summand we have, since \((Y + Y^*)\) has order zero,
\[
|Q_t f Y^* A f|_0 \leq |Q_t f|_0 (\|A Y^* f\|_0 + \|[Y^*, A] f\|_0)
\leq C_2 (\|P_t f\|_0 + |f|_0) \left(\|Y f\|_{\frac{4}{3} - 1} + |f|_{\frac{4}{3}}\right).
\]

Next we consider
\[
|Q_t Y f A f|_0 = \|Y f \|_{Q_t^* A f} \leq \|Y f \|_{Q_t A f} + \|Y f \|_{[Q_t^*, A] f}.
\]

Let us first estimate the second summand in the last expression.

We have \(Q_t^* = \sum_{r=1}^{2r} X_r^2 + R_0^*\) for a first order p.d.o. \(R_0^*\) whose principal part is a linear combination of \(X_1, \ldots, X_2^r\). Hence
\[
Q_t^* = [R_0^*, A] + \sum (2[X_i, A]X_i + [X_i, [X_i, A]]),
\]
with pseudodifferential operators \([R_0^*, A], [X_i, A], [X_i, [X_i, A]]\) of order \(\leq \left(\frac{4}{3} - 1\right)\). Thus we obtain
\[
|Y f |Q_t^*, A f|_0 \leq C_3 \|Y f\|_{\frac{4}{3} - 1} \left(\|f\|_{\frac{4}{3}} + \sum_{j=1}^{2r} \|X_j f\|_{\frac{4}{3}}\right).
\]

Because of (10), we have the Cauchy inequality
\[
|Q_t f|_0 \leq \sqrt{|Q_t f|_1 (Q_t f|_2)}, \quad \text{for } f_1, f_2 \in C_0^\infty (U).
\]

Hence
\[
|Y f |Q_t^*, A f|_0 \leq \|Q_t f\|_0 \|f\|_0 \leq (\|P_t f\|_0 + \|f\|_0) \|f\|_0.
\]

Let us estimate the first factors. We get
\[
\|Q_t^* A^* Y f A^* Y f\|_0 = \|Q_t A^* Y f A^* Y f\|_0 \leq \|Q_t A^* Y f\|_{\frac{4}{3}} \|A^* Y f\|_{\frac{4}{3}}
\leq \|A^* Y f\|_{\frac{4}{3}} \left(\|A^* Y Q_t f\|_{\frac{4}{3}} + \|Q_t, A^* Y f\|_{\frac{4}{3}}\right)
\leq C_3 \|Y f\|_{\frac{4}{3} - 1} \left(\|Q_t f\|_0 + \|[A^* Y, Q_t] f\|_{\frac{4}{3}}\right).
\]

We need to estimate the second summand inside the parentheses in the last expression. We note that
\[
[A^* Y, Q_t] = [A^* Y, P_t] = -[A^* Y, X_0] + \sum_{j=1}^{2r} \left(2[A^* Y, X_j]X_j + [X_j, [A^* Y, X_j]]\right).
\]

Since the operators \([A^* Y, X_0], [A^* Y, X_j], [X_j, [A^* Y, X_j]]\) have order \(\frac{4}{3}\), we obtain
\[
\|A^* Y, Q_t f\|_{\frac{4}{3}} \leq C_4 \left(\|f\|_0 + \sum_{j=1}^{2r} \|X_j f\|_0\right).
\]

Finally,
\[
\|Q_t^* Y^* A f Y^* A f\|_0 = \|Q_t Y^* A f Y^* A f\|_0 \leq \|Y^* A f\|_{\frac{4}{3}} \left(\|Y^* A Q_t f\|_{\frac{4}{3}} \|Y^* A f\|_{\frac{4}{3}} \|Q_t, Y^* A f\|_{\frac{4}{3}}\right)
\leq C_3 \|Y^* A f\|_{\frac{4}{3}} \left(\|Q_t f\|_0 + \|Q_t, Y^* A f\|_{\frac{4}{3}}\right).
\]
Since
\[ [Y^*A, Q_1] = [Y^*A, P_1] = -[Y^*A, X_0] + \sum_{j=1}^{2r} (2[Y^*A, X_j]X_j + [X_j, [Y^*A, X_j]]) \]
and the operators \([Y^*A, X_0], [Y^*A, X_j], [X_j, [Y^*A, X_j]]\) have order \(\frac{r}{2}\), we obtain that
\[ \| [Q_1, Y^*A]f \|_{-\frac{r}{2}} \leq C \left( \|f\|_0 + \sum_{j=1}^{2r} \|X_j f\|_0 \right). \]
Moreover,
\[ Y^*A = -AY + (Y + Y^*)A + [A, Y], \]
with \(((Y + Y^*)A + [A, Y])\) of order \(\frac{r}{2} - 1\), because \(Y + Y^*\) has order 0. Hence
\[ \|Y^*Af\|_{\frac{r}{2}-1} \leq C \|Yf\|_{r-1} + \|f\|_0. \]
Putting all these inequalities together, we conclude that
\[ \|([X_0, Y^*A])f\|_0 \leq C_8 \left( \|f\|_0^2 + \|Yf\|_{r-1}^2 + \|P_t f\|_0^2 + \sum_{j=1}^{2r} \|X_j f\|_0^2 \right), \quad \forall f \in \mathcal{C}_0^\infty(U). \]
By taking \(A = \Lambda_{\frac{r}{2}-1}[X_0, Y]\) for an elliptically properly supported pseudodifferential operator \(\Lambda_{\frac{r}{2}-1}\) of order \(\frac{r}{2} - 1\), we deduce that
\[ \|([X_0, Y])f\|_{\frac{r}{2}-1} \leq C_9 \left( \|f\|_0 + \|Yf\|_{r-1} + \|P_t f\|_0 + \sum_{j=1}^{2r} \|X_j f\|_0 \right) \]
and therefore, since \(X_1, \ldots, X_{2r} \in S_1(M)\) and \(Y \in S_\infty(M)\), that \([X_0, Y] \in S_{\frac{r}{2}}(M)\). □

**Corollary 3.6.** We have
\[ (3.16) \quad \mathcal{Q}(\mathcal{V}_2; S) \subset S. \]

**Proof of Theorem 3.2** By the assumption, \(\{Y_q \mid Y \in S(M)\} = T_q M\) for all \(q\) in an open neighborhood of \(p\) in \(M\). Thus there are \(p \in U^{\text{open}} \subset M\), \(\tau_1, \ldots, \tau_h \in \{\ker \mathcal{L}(M)\}, \ Z_1, \ldots, Z_\ell \in \mathcal{Z}(M)\) and \(C > 0\) such that
\[ (3.17) \quad \|f\|_{r} \leq C \left( \|f\|_0 + \sum_{j=1}^{h} \|P_{\tau_j} f\|_0 + \sum_{j=1}^{\ell} \|Z_j f\|_0 \right), \quad \forall f \in \mathcal{C}_0^\infty(U). \]
Let \(P_{\tau_j} = -[X_{0, j} + \sum_{s=1}^{2r_j} X_{s, j}^2]\) with \(Z_{s, j} = X_{s, j} + iX_{s+r_j, j} \in \mathcal{Z}(M)\) for \(1 \leq s \leq r_j\), and let \(Z_{0, j}\) be the vector field in \(\mathcal{Z}(M)\) with \(\text{Re} \ Z_{0, j} = X_{0, j}\). If \(A\) is a properly supported pseudodifferential operator, then
\[ [P_{\tau_j}, A] = -[X_{0, j}, A] + \sum_{s=1}^{2r_j} \left( 2X_{s, j} [X_{s, j}, A] + [[X_{s, j}, A], X_{s, j}] \right). \]
If \(A\) has order \(\delta\), and is zero outside a compact subset \(K\) of \(U\), and \(\chi\) is a smooth function with compact support which equals one on a neighborhood of \(K\), then we obtain
\[ \|P_{\tau_j} A(\chi f)\|_0 \leq \|A(\chi P_{\tau_j} f)\|_0 + \|P_{\tau_j} A(\chi f)\|_0 \]
\[ \leq C' \left( \|\chi P_{\tau_j} f\|_0 + \|\chi f\|_0 + \sum_{s=1}^{2r_j} \|X_{s, j} A(\chi f)\|_0 \right) \]
\[ \leq C'' \left( \|\chi P_{\tau_j} f\|_0 + \|\chi f\|_0 + \sum_{j=0}^{r_j} \|Z_{s, j} A(\chi f)\|_0 \right) \]
\[ \leq C''' \left( \|\chi P_{\tau_j} f\|_0 + \|\chi f\|_0 + \sum_{s=0}^{r_j} \|\chi Z_{s, j} f\|_0 \right), \quad \forall f \in \mathcal{C}_0^\infty(U), \]
for suitable positive constants $C', C'', C'''$, uniform with respect to $f$. By using similar argument to estimate $\|Z_A f\|_0$, we obtain that
\[
\|A(\chi f)\|_\epsilon \leq \text{const} \left( \|\chi f\|_0 + \sum_{j=1}^h \|\chi P_{\tau_j} f\|_0 + \sum_{i=1}^\ell \|\eta Z_i f\|_0 \right), \quad \forall f \in \mathcal{C}^\infty_0(U).
\]
This shows that for any pair of functions $\chi_1, \chi_2 \in \mathcal{C}^\infty_0(U)$ with $\text{supp}(\chi_1) \subset \{\chi_2 > 0\}$ we obtain the estimate
\[
\|\chi_1 f\|_{\epsilon+\delta} \leq \text{const} \left( \|\chi_2 f\|_0 + \sum_{j=1}^h \|\chi_2 P_{\tau_j} f\|_0 + \sum_{i=1}^\ell \|\chi_2 Z_i f\|_0 \right), \quad \forall f \in \mathcal{C}^\infty_0(U),
\]
for some constant const = const$(\chi_1, \chi_2) \geq 0$. By $[15]$, this inequality is valid for all $f \in W^\infty_0(U)$ with $P_\tau f, Z_i f \in W^\infty_0(U)$, where $W^\infty_0(U)$ is the space of distributions $\phi$ in $U$ such that, for all $\chi \in \mathcal{C}^\infty_0(U)$, the product $\chi \cdot \phi$ belongs to the Sobolev space of order $\delta$ and integrability two. This implies in particular that any $\text{CR}$ distribution which is in $W^\infty_0(U)$ belongs in fact to $W^\infty_0(U)$, and this implies property $(H)$. □

Let us consider the case where $\mathcal{U}(\nu_2; \mathcal{H})$ does not contain all smooth real vector fields. In this case we have a propagation phenomenon along the leaves of $\nu_2$. Let $\tau \in \ker L(M)$, and $X_0, Y_0, X_1, \ldots, X_r$ the vector fields introduced above for a given representation of $\tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes \bar{Z}_r$. As we already noticed, while $Y_0 = \text{Im} \sum[Z_r, \bar{Z}_r]$ belongs to the Lie subalgebra of $\mathfrak{x}(M)$ generated by $X_1, \ldots, X_r$, the real part $X_0$ of $L_0 = X_0 + iY_0 \in \mathcal{Z}(M)$ may not belong to $\nu_1(\tau)$. Thus the following result improves $[21]$ Theorem 5.2, where only the smaller distribution $\nu_1(\tau)$ was involved.

**Theorem 3.7.** Let $\Omega^\text{open} \subset M$ and assume that $\nu_2$ has constant rank in $\Omega$. If $f \in \mathcal{C}^\infty_M(\Omega)$ and $|f|$ attains a maximum at a point $p_0$ of $\Omega$, then $f$ is constant on the leaf through $p_0$ of $\nu_2$ in $\Omega$.

**Proof.** On the integral manifold $N$ of $\nu_2$ through $p_0$ in $\Omega$ we can consider the $\mathcal{Z}'$-structure defined by the span of the restrictions to $N$ of the elements of $\Theta$. Indeed, the $\text{CR}$ functions on $\Omega$ restrict to $\text{CR}$ functions for $\mathcal{Z}'$ on the leaf $N$. By Corollary 3.6 and Theorem 3.2, the $\mathcal{Z}'$-manifold $N$ has property $(H)$ and therefore the statement is a consequence of Proposition 2.1. □

4. **Malgrange’s theorem and some applications**

In this section we state the obvious generalization of Malgrange’s vanishing theorem and its corollary on the extension of $\text{CR}$ functions under momentum conditions, slightly generalizing results of $[9, 29, 30]$ to the case where $M$ has property $(SH)$. In this section we require that $M$ is a $\text{CR}$ manifold.

We recall that the tangential Cauchy-Riemann complex can be defined as the quotient of the de Rham complex on the powers of the ideal sheaf (for this presentation we refer to $[18]$): since $d \mathcal{I} \subset \mathcal{I}$, we have $d \mathcal{I}^a \subset \mathcal{I}^a$ for all nonnegative integers $a$ and the tangential $\text{CR}$-complex $(\mathcal{D}^a, \partial_M)$ on $a$-forms is defined by the commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{I}^{a+1} & \longrightarrow & \mathcal{I}^a & \longrightarrow & \mathcal{D}^a & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \partial_M & & \\
0 & \longrightarrow & \mathcal{I}^{a+1} & \longrightarrow & \mathcal{I}^a & \longrightarrow & \mathcal{D}^a & \longrightarrow & 0,
\end{array}
\]
where $\mathcal{Q}^a$ is the quotient $\mathcal{F}^a/\mathcal{F}^{a+1}$. In turn $\bar{\partial}_M$ is a degree 1 derivation for a $\mathbb{Z}$-grading $\mathcal{Q}^a = \bigoplus_{q=0}^\infty \mathcal{Q}^{a,q}$, where the elements of $\mathcal{Q}^{a,q}$ are equivalence classes of forms having representatives in $\mathcal{F}^a \cap \mathcal{F}^{a+q}$.

We denote by $\mathcal{E}$ the sheaf of germs of smooth complex valued functions on $M$. The the $\mathcal{Q}^{a,q}$ are all locally free sheaves of $\mathcal{E}$-modules, and therefore we can form the corresponding sheaves and co-sheaves of functions and distributions. We will consider the tangential Cauchy-Riemann complexes $(\mathcal{Q}^{a,*}, \bar{\partial}_M)$ on smooth forms with compact support, $(\mathcal{E}^{a,*}, \bar{\partial}_M) = (\mathcal{Q}^{a,*}, \bar{\partial}_M)$ on smooth forms with closed support, $(\mathcal{Q}^{a,*}, \bar{\partial}_M)$ on form-distributions, $(\mathcal{E}^{a,*}, \bar{\partial}_M)$ on form-distributions with compact support. We use the notation $H^q(\mathcal{F}^{a,*}(\Omega), \bar{\partial}_M)$ for the cohomology group in degree $q$ on $\Omega^{\text{open}} \subset M$, for $\mathcal{F}$ equal to either one of $\mathcal{E}$, $\mathcal{Q}$, $\mathcal{E}'$, $\mathcal{Q}'$.

**Proposition 4.1.** If $M$ has property $(SH)$, and either $M$ is compact or has property $(WUC)$, then $\bar{\partial}_M : \mathcal{E}^{a,0}(M) \rightarrow \mathcal{E}^{a,1}(M)$ and $\bar{\partial}_M : \mathcal{Q}^{a,0}(M) \rightarrow \mathcal{Q}^{a,1}(M)$ have closed range for all integers $a = 0, \ldots, m$.

**Proof.** We can assume that $M$ is connected. It is convenient to fix a Riemannian metric on $M$, and smooth Hermitian products on the complex linear bundles $\mathcal{Q}^{a,q} M$ corresponding to the sheaves $\mathcal{Q}^{a,q}$, to define $L^2$ and Sobolev norms, by using the associated smooth regular Borel measure.

By property $(SH)$, we have a subelliptic estimate: for every $K \Subset M$ we can find constants $C_K \geq 0$, $c_K > 0$, $\epsilon_K > 0$ such that
\begin{equation}
(\bar{\partial}_M u, u) + C_K \|u\|_0^2 \geq c_K \|u\|_{\epsilon_K}^2, \quad \forall u \in \mathcal{Q}^{a,0}(K).
\end{equation}

In a standard way we deduce from (4.2) that
\begin{equation}
u \in \mathcal{Q}^{a,0}(M), \quad \bar{\partial}_M \nu \in [W^{r+\epsilon}_r]^{a,1}(M) \implies \nu \in [W^{r+\epsilon}_r]^{a,1}(M), \quad \forall K \Subset M,
\end{equation}
and that for all $K \Subset M$ and real $r$ there are constants $C_{r,K} \geq 0$, $c_{r,K} > 0$ such that
\begin{equation}
(\bar{\partial}_M \nu, \nu) + C_{r,K} \|\nu\|_r^2 \geq c_{r,K} \|\nu\|_{r+\epsilon_K}^2, \quad \forall \nu \in [\nu \in \mathcal{E}^{a,0}(M) \mid \bar{\partial}_M \nu \in [W^{r+\epsilon}_r]^{a,1}(M), \text{ supp}(\nu) \subset K].
\end{equation}

This suffices to obtain the thesis when $M$ is compact.

Let us consider the case where $M$ is connected and non-compact. Let $\{u_v\}$ be a sequence in $\mathcal{E}^{a,0}(M)$ such that all $\bar{\partial}_M u_v$ have support in a fixed compact subset $K$ of $M$ and there is $r \in \mathbb{R}$ such that $\bar{\partial}_M u_v \subset [W^r(M)]$, $\text{supp}(\bar{\partial}_M u_v) \subset K$ for all $v$ and $\bar{\partial}_M u_v \rightarrow f$ in $[W^{r+\epsilon}_r]^{a,1}(M)$. We can assume that $M \setminus K$ has no compact connected component. Then, since $M$ has property $(WUC)$, it follows that $\text{supp}(u_v) \subset K$ for all $v$, because the $u_v |_{M \setminus K}$ define elements of $\mathcal{O}_M(M \setminus K)$ which vanish on a nonempty open subset of each connected component of $M \setminus K$, and thus on $M \setminus K$. Moreover, this also implies that (4.4) holds with $C_{r,K} = 0$. Then $\{u_v\}$ is uniformly bounded in $[W^{r+\epsilon}_r]^{a,0}(M)$ and hence contains a subsequence which weakly converges to a solution $u \in [W^{r+\epsilon}_r]^{a,0}(M)$ of $\bar{\partial}_M u = f$.

The closedness of the image of $\bar{\partial}_M$ in $\mathcal{Q}^{a,1}(M)$ follows from the already proved result for $\mathcal{E}^{a,1}(M)$ and the hypoellipticity of $\bar{\partial}_M$ on $(a,0)$-forms. \hfill $\square$

We remind that if $M$ is embedded and has property $(H)$, or is (abstract and) essentially pseudoconcave, then has property $(WUC)$.

As in [9] one obtains
Proposition 4.2. Assume that $M$ is a connected non-compact CR manifold of CR dimension $n$ which has properties (SH) and (WUC). Then $H^a(\mathcal{E}^{a,q}(M), \overline{\partial}_M)$ and $H^a(\mathcal{E}^{a,q,n}(M), \overline{\partial}_M)$ are 0 for all $a = 0, \ldots, m$.

Proof. By Proposition 4.1 the sequences

$$
\begin{array}{c}
0 \longrightarrow \mathcal{E}^{a,0}(M) \longrightarrow \mathcal{E}^{a,1}(M), \\
0 \longrightarrow \mathcal{E}^{a,0}(M) \longrightarrow \mathcal{E}^{a,1}(M),
\end{array}
$$

are exact and all maps have closed range.

Assume that $M$ is oriented. Then we can define duality pairings between $\mathcal{E}^{a,q}(M)$ and $\mathcal{E}^{n+k-a,n-q}(M)$ and between $\mathcal{E}^{a,q}(M)$ and $\mathcal{E}^{n+k-a,n-q}(M)$, extending

$$
\langle [\alpha], [\beta] \rangle = \int_M \alpha \wedge \beta,
$$

where $\alpha \in \mathcal{A}^{a,q}(M) \cap \mathcal{I}^{a}(M)$ has compact support and is a representative of $[\alpha] \in \mathcal{E}^{a,q}(M)$ and $\beta \in \mathcal{A}^{m-a,q}(M) \cap \mathcal{I}^{m-k-a,q}(M)$ of $[\beta] \in \mathcal{E}^{n+k-a,n-q}(M)$. Then by duality (see e.g. 41) we obtain exact sequences

$$
\begin{array}{c}
0 \longleftarrow \mathcal{E}^{n+k-a,n}(M) \longleftarrow \mathcal{E}^{n+k-a,n-1}(M), \\
0 \longleftarrow \mathcal{E}^{n+k-a,n}(M) \longleftarrow \mathcal{E}^{n+k-a,n-1}(M),
\end{array}
$$

proving the statement in the case where $M$ is orientable.

If $M$ is not orientable, then we can take its oriented double covering $\pi : \tilde{M} \to M$, which is a CR-bundle with the total space $\tilde{M}$ being a CR manifold of the same CR dimension and codimension. From the exact sequences

$$
\begin{array}{c}
0 \longleftarrow \mathcal{E}^{n+k-a,n}(\tilde{M}) \longleftarrow \mathcal{E}^{n+k-a,n-1}(\tilde{M}), \\
0 \longleftarrow \mathcal{E}^{n+k-a,n}(\tilde{M}) \longleftarrow \mathcal{E}^{n+k-a,n-1}(\tilde{M}),
\end{array}
$$

we deduce that statement for the nonorientable $M$ by averaging on the fibers. \hfill $\square$

We also obtain the analogue of the Hartogs-type theorem in 29.

Proposition 4.3. Let $\Omega^{\text{open}} \subseteq M$ be relatively compact, orientable, and with a piece-wise smooth boundary $\partial \Omega$. If $u_0$ is the restriction to $\partial \Omega$ of an $(a,0)$-form $u_0$ of class $\mathcal{C}^2$ on $M$, with $\overline{\partial}_0 u_0$ vanishing to the second order on $\partial \Omega$, and

$$
\int_{\partial \Omega} u_0 \wedge \phi = 0, \quad \forall \phi \in \ker(\overline{\partial}_M) : \mathcal{E}^{n+k-a,n-1}(M') \to \mathcal{E}^{n+k-a,n}(M'),
$$

then there is $u \in \mathcal{E}^{a,0}(\Omega) \cap \mathcal{E}^{1}(\tilde{\Omega})$ with $\overline{\partial}_M u = 0$ on $\Omega$ and $u = u_0$ on $\partial \Omega$.

Proof. We restrain for simplicity to the case $a = 0$. The general case can be discussed in an analogous way. If $M$ is not orientable, then the inverse image of $\Omega$ in the double covering $\pi : \tilde{M} \to M$ consists of two disjoint open subsets, both CR-diffeomorphic to $\Omega$. Thus we can and will assume that $M$ is orientable.

Let $E$ be a discrete set that intersects each relatively compact connected component of $M \setminus \Omega$ in a single point and $M' = M \setminus E$. Note that $M'$ has been chosen in such a way that no connected component of $M' \setminus \Omega$ is compact.

Extending $\overline{\partial}_M u_0$ by 0 outside of $\Omega$, we define a $\overline{\partial}_M$-closed element $f$ of $\mathcal{E}^{0,1}(M')$, with support contained in $\tilde{\Omega}$. The map $\overline{\partial}_M : \mathcal{E}^{0,0}(M') \to \mathcal{E}^{0,1}(M')$ has a closed image by Proposition 4.1. Hence to get existence of a solution $\nu \in \mathcal{E}^{0,0}(M')$ to
\(\bar{\partial}_M v = f\) it suffices to prove that \(f\) is orthogonal to the kernel of \(\bar{\partial}_M : \mathcal{E}^{n+k,n-1}(M') \to \mathcal{E}^{n+k,n}(M')\). This is the case because
\[
\int_{M'} f \wedge \phi = \int_{\Omega} (\bar{\partial}_M \bar{u}_0) \wedge \phi = \int_{\Omega} (du_0) \wedge \phi = \int_{\partial \Omega} u_0 \phi - \int_{\Omega} u_0 d\phi
\]
for all \(\phi \in \mathcal{E}^{n+k,n-1}(M') = \mathcal{A}^C_{m-1}(M') \cap \mathcal{J}^{n+k}(M')\), and the last summand in the last term vanishes when \(d\phi = \bar{\partial}_M \phi = 0\). A \(v \in \mathcal{E}^{r,0}(M')\) satisfying \(\bar{\partial}_M v = f\) defines a CR function on \(M' \setminus \Omega\) that vanishes on some open subset of each connected component of \(M' \setminus \Omega\). Thus, for (WUC) and the regularity (4.3), which are consequences of (SH), the solution \(v\) is \(\mathcal{C}^1\) and has support in \(\Omega\). In particular it vanishes on \(\partial \Omega\) and therefore \(u = \bar{u}_0 - v\) satisfies the thesis. \(\square\)

**Remark 4.4.** An analogue of this momentum theorem for functions on one complex variable states that a function \(u_0\), defined and continuous on the boundary of a rectifiable Jordan curve \(c\), is the boundary value of a holomorphic function on its enclosed domain if and only if \(\int_c u_0(z) p(z) dz = 0\) for all holomorphic polynomials \(p(z) \in \mathbb{C}[z]\).

### 5. Hopf Lemma and Some Consequences

In complex analysis properties of domains are often expressed in terms of the indices of inertia of the complex Hessian of its exhausting function. Trying to mimic this approach in the case of an (abstract) CR manifold \(M\), we are confronted with the fact that pluri-harmonicity and pluri-sub-harmonicity are well defined only for functions on one complex variable states that a function \(u_0\), defined and continuous on the boundary of a rectifiable Jordan curve \(c\), is the boundary value of a holomorphic function on its enclosed domain if and only if \(\int_c u_0(z) p(z) dz = 0\) for all holomorphic polynomials \(p(z) \in \mathbb{C}[z]\).

#### Hopf Lemma

The classical Hopf Lemma also holds for degenerate-elliptic operators. We have, from [14] Lemma 4.3:

**Proposition 5.1.** Let \(\Omega\) be a domain in \(M\) and \(u \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R})\) satisfy \(P_t u \geq 0\) on \(\Omega\), for the operator \(P_t = -X_0 + \sum_{i=1}^2 X_i^2\) of (3.8). Assume that \(p_0 \in \partial \Omega\) is a \(\mathcal{C}^2\) non-characteristic point of \(\partial \Omega\) for \(P_t\) and that there is an open neighborhood \(U\) of \(p_0\) in \(M\) such that
\[
(5.1) \quad u(p) < u(p_0), \quad \forall p \in \Omega \cap U.
\]
Then
\[
(5.2) \quad du(p_0) \neq 0.
\]

The condition that \(\partial \Omega\) is non-characteristic at \(p_0\) for \(P_t\) means that, if \(\Omega\) is represented by \(p < 0\) near \(p_0\), with \(p \in \mathcal{C}^2\) and \(dp(p_0) \neq 0\), then \(\sum_{i=1}^2 |X_i|^2 = 0\).
Remark 5.2. If $M$ has property (H), then (5.1) is automatically satisfied if $u = |f|$, for $f \in \mathcal{O}_M(\Omega) \cap C^0(\bar{\Omega})$, when $u(p_0)$ is a local maximum and $f$ is not constant on a half-neighborhood of $p_0$ in $\Omega$.

Corollary 5.3. Let $\Omega$ be an open subset of $M$ and $f \in \mathcal{O}_M(\Omega) \cap C^2(\bar{\Omega})$, $p_0 \in \partial \Omega$ with

$$
|f(p)| < |f(p_0)|, \quad \forall p \in \Omega.
$$

If $\partial \Omega$ is smooth and $\Theta$-non-characteristic at $p_0$, then $d|f|(p_0) \neq 0$.

Proof. By the assumption that $\partial \Omega$ is $\Theta$-non-characteristic at $p_0$, the function $u = |f|$ is, for some open neighborhood $U$ of $p_0$ in $M$, a solution of $P_t u \geq 0$ on $\Omega \cap U$, for an operator $P_t$ of the form (5.3), obtained from a section $\tau$ of $[\ker L](U)$, and for which $\partial \Omega$ is non-characteristic at $p_0$. □

The complex Hessian and the operators $dd^c$, $P_t$. Denote by $\mathcal{A}$ the sheaf of germs of smooth real valued 1-forms on $M$, by $\mathcal{I}$ its subsheaf of germs of sections of $H^0M$ and by $\mathcal{I}$ the degree 1-homogeneous elements of the ideal sheaf of $M$. The elements of $\mathcal{I}$ are the germs of smooth complex valued 1-forms vanishing on $T^{0,1}M$.

Let $\Omega$ be an open subset of $M$.

Lemma 5.4. If $\alpha \in \mathcal{A}_1(\Omega)$, then we can find $\xi \in \mathcal{A}_1(\Omega)$ such that $\alpha + i\xi \in \mathcal{I}_1(\Omega)$.

Proof. The sequence

$$
0 \longrightarrow \mathcal{I}_1 \overset{i:}{\longrightarrow} \mathcal{I}_1 \overset{\text{Re}}{\longrightarrow} \mathcal{I}_1 \longrightarrow 0
$$

of fine sheaves is exact, and thus splits on every open subset $\Omega$ of $M$. □

If $\rho$ is a smooth, real valued function on $\Omega^{\text{open}} \subset M$, by Lemma 5.4 we can find $\xi \in \mathcal{A}_1(\Omega)$ such that $d\rho + i\xi \in \mathcal{I}_1(\Omega)$. If $Z \in Z(M)$, then $d\rho(Z) = -i\xi(Z)$, $d\rho(\bar{Z}) = -i\bar{\xi}(Z)$, and we obtain

$$
Z\bar{Z}\rho = Z(d\rho(\bar{Z})) = iZ[\xi(\bar{Z})], \quad \bar{Z}Z\rho = \bar{Z}(d\rho(Z)) = -i\bar{Z}[\bar{\xi}(Z)].
$$

Hence

$$
[Z\bar{Z} + \bar{Z}Z]\rho = i(Z[\xi(\bar{Z})] - \bar{Z}[\bar{\xi}(Z)]) = id\xi(Z, \bar{Z}) + i\bar{\xi}(Z, Z).
$$

We note that $i\xi$ is only defined modulo the addition of a smooth section $\eta \in \mathcal{I}_1(\Omega)$ of the characteristic bundle $H^0M$, for which

$$
\text{id} \eta Z, Z = -\text{Re} \eta(Z, \bar{Z}) = \omega_n(Z, \bar{Z}), \quad \forall Z \in Z(M).
$$

Definition 5.1. The complex Hessian of $\rho$ at $p_0$ is the affine subspace

$$
\text{Hess}^{1,1}_{p_0}(\rho) = \{id\xi_{p_0} \mid \xi \in \mathcal{A}_1(\Omega), \quad d\rho + i\xi \in \mathcal{I}_1(\Omega)\}.
$$

Fix a point $p_0$ where $d\rho(p_0) \notin H^0_{p_0}M$, i.e. $\partial_\rho(p_0) \neq 0$, and consider the level set $N = \{p \in U \mid \rho(p) = \rho(p_0)\}$, in a neighborhood $U$ of $p_0$ in $\Omega$ where $\partial_\rho(p)$ is never 0. Then $N$ is a smooth real hypersurface and a CR-submanifold, of type $(n-1, k+1)$.

Lemma 5.5. For every $p \in N$, we have

$$
\{\xi_N \mid \xi \in T^*_pM \mid d\rho(p) + i\xi \in T^{*1,0}_{p}M\} \subset H^0_{p}N.
$$

The left hand side of (5.5) is an affine hypersurface in $H^0_{p}N$, with associated vector space $H^0_{p}M$. 

Proof. When \( Z \in \mathcal{Z}(U) \) is tangent to \( N \), we obtain \( 0 = dp(\xi_p) = -i\bar{\xi}(\xi_p) \) and hence \( \bar{\xi}(\Re Z_p) = \bar{\xi}(\Im Z_p) = 0 \) because \( \bar{\xi} \) is real. This gives \( \bar{\xi}|_M \in H^0_2 N \). The last statement is a consequence of the previous discussion of the complex Hessian.

**Definition 5.2.** If \( \rho \) is a smooth real-valued function defined on a neighborhood \( \Omega \) of a point \( p_0 \in N \) and \( \xi \in \mathcal{A}_1(\Omega) \) is such that \( dr + i\bar{\xi} \in \mathcal{A}_1(\Omega) \), then we set

\[
(5.6) \quad \ddbar{\rho}_{p_0}(\tau) := \frac{1}{4} d\bar{\xi}(\tau), \quad \forall \tau \in [\ker\mathcal{L}]_{p_0}.
\]

Let \( \tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes \bar{Z}_r \in [\ker\mathcal{L}](\Omega) \), with \( L_0 - L_0 = \sum_{i=1}^r [Z_j, \bar{Z}_j] \) and \( L_0, Z_1, \ldots, Z_r \in \mathcal{Z}(\Omega) \). Let \( \xi \in \mathcal{A}_1(\Omega) \) be such that \( d\rho + i\bar{\xi} = \mathcal{A}_1(\Omega) \). Then

\[
dp(Z_j) + i\bar{\xi}(Z_j) = 0 \implies dp(Z_j) - i\bar{\xi}(Z_j) = 0
\]

\[
\implies d\bar{\xi}(Z_j, \bar{Z}_j) = i \left( Z_j \bar{\xi}(\bar{Z}_j) - \bar{Z}_j \bar{\xi}(Z_j) - \bar{\xi}([Z_j, \bar{Z}_j]) \right)
\]

\[
= Z_j dp(Z_j) + \bar{Z}_j dp(Z_j) - i\bar{\xi}([Z_j, \bar{Z}_j])
\]

\[
= (Z_j \bar{Z}_j + \bar{Z}_j Z_j)p - i\bar{\xi}([Z_j, \bar{Z}_j]).
\]

We recall that \( \sum_{i=1}^r [Z_j, \bar{Z}_j] = \bar{L}_0 - L_0 = 2i \Im L_0 \), with \( L_0 \in \mathcal{Z}(\Omega) \). We have

\[
(dp + i\bar{\xi})(L_0) = 0 \implies dp(\Re L_0) = \bar{\xi}(\Im L_0), \ dp(\Im L_0) = -\bar{\xi}(\Re L_0)
\]

and therefore

\[
2d\ddbar{\rho}(\tau) = \sum_{i=1}^r (Z_j \bar{Z}_j + \bar{Z}_j Z_j)p - i\bar{\xi} \sum_{i=1}^r [Z_j, \bar{Z}_j] = 2P_4 \rho.
\]

As a consequence, we obtain:

**Proposition 5.6.** If \( \rho \) is a real valued smooth function on the open set \( \Omega \) of \( M \) and \( \tau \in [\ker\mathcal{L}](\Omega) \), then

\[
(5.7) \quad d\ddbar{\rho}(\tau) = P_4 \rho \quad \text{on } \Omega.
\]

**Corollary 5.7.** The operator \( P_4 \) only depends on the section \( \tau \) of [\ker\mathcal{L}] and is independent of the choice of the vector fields \( Z_1, \ldots, Z_r \in \mathcal{Z} \) in \( (3.7) \).

**Corollary 5.8.** Let \( \Omega^{\text{open}} \subset M \). If \( \rho_1, \rho_2 \in \mathcal{C}^{\infty}(\Omega) \) are real valued functions which agree to the second order at \( p_0 \in \Omega \), then

\[
(5.8) \quad d\ddbar{\rho}_1(\tau_0) = d\ddbar{\rho}_2(\tau_0), \quad \forall \tau_0 \in [\ker\mathcal{L}]_{p_0}.
\]

In particular, \( d\ddbar{\rho} \) is well defined and continuous on the fibers of [\ker\mathcal{L}] for functions \( \rho \) which are of class \( \mathcal{C}^2 \).

**Remark 5.9.** There is a subtle distinction between \( d\ddbar{\rho} \), which is the \( (1,1) \)-part of an alternate form of degree two, and Hess\(_{1,1}(\rho)\), which is the \( (1,1) \)-part of a symmetric bilinear form. In fact we multiplied by \((i/2)\) the differential in \( (5.6) \), and identified the two concepts, as multiplication by \(i\) interchanges skew-Hermitian and Hermitian-symmetric matrices.

We have:

**Lemma 5.10.** Let \( \rho \) be a smooth real-valued function defined on a neighborhood of \( p_0 \in M \), with \( dp(\rho(p_0)) \neq 0 \) and \( N = \{ p \mid \rho(p) = \rho(p_0) \} \). The following statements:
(i) every $h \in \text{Hess}_{p_0}^{1,1}(p)$ has a non-zero positive index of inertia;
(ii) there exists $\tau \in [\ker L]_{p_0} \cap H^{1,1}_{p_0} N$ such that $d\bar{d}\rho_{p_0}(\tau) > 0$;
(iii) the restriction of every $h \in \text{Hess}_{p_0}^{1,1}(p)$ to $T^{0,1}_{p_0} N$ has a non-zero positive index of inertia;

are related by

$$(ii) \iff (iii) \implies (i).$$

Set $U^- = \{ p \in U \mid \rho(p) < \rho(p_0) \}$.

**Definition 5.3.** We set

$$H^0_{M,p_0}(U^-) = \bigcup_{\lambda > 0} \{ \xi \mid \xi \in T^*_{p_0} \partial U^- \mid \lambda dp(p_0) + i\xi \in T^{1,0}_p M \}. $$

This is an open half-space in $H^0_p N$. Note that $H^0_{M,p_0}(U^-)$ does not depend on the choice of the defining function $\rho$.

**Real parts of CR functions.** In this subsection, we try to better explain the meaning of $d\bar{d}^c$ by defining a differential operator $d^c_\lambda$ which associates to a real smooth function a real one-form. Its definition depends on the choice of a CR gauge $\lambda$ on $M$, but $[d^c_\lambda]$’s corresponding to different choices of $\lambda$ differ by a differential operator with values in $\mathcal{F}_1$, so that all the $d\bar{d}^c_\lambda$ agree with our $d\bar{d}^c$ on $[\ker L]$.

A CR function (or distribution) $\tilde{f}$ is a solution to the equation $du \in \mathcal{F}_1$. In this subsection we study the characterization of the real parts of CR functions.

**Lemma 5.11.** Let $\Omega$ be open in $M$. If $M$ is minimal, then a real valued $f \in \mathcal{O}_M(\Omega)$ is locally constant.

**Proof.** A real valued $f \in \mathcal{O}_M(\Omega)$ satisfies $X f = 0$ for all $X \in \Gamma(M,H M)$ and therefore is constant on the Sussmann leaves of $\Gamma(M,H M)$. □

We have an exact sequence of fine sheaves (the superscript $\mathbb{C}$ means forms with complex valued coefficients)

$$0 \longrightarrow \mathcal{F}_1^\mathbb{C} \xrightarrow{\alpha \rightarrow (\alpha, -\alpha)} \mathcal{F}_1 \oplus \tilde{\mathcal{F}}_1 \xrightarrow{(\alpha, \beta) \rightarrow \alpha + \beta} \mathcal{F}_1^\mathbb{C} \longrightarrow 0.$$  

In [31] §2A] the notion of a balanced real CR-gauge was introduced. It was shown that it is possible to define a smooth morphism

$$\lambda : \mathbb{C} TM \longrightarrow T^{1,0} M$$

of $\mathbb{C}$-linear bundles which defines a special splitting of (5.10); with

$$\tilde{\lambda} : \mathbb{C} TM \ni \alpha \longrightarrow \overline{\lambda(\alpha)} \in T^{0,1} M,$$

we have

$$\alpha = \lambda(\alpha) + \tilde{\lambda}(\alpha), \quad \forall \alpha \in \mathcal{F}_1^\mathbb{C},$$

$$\lambda(\alpha) = \bar{\lambda}(\alpha) = \frac{i}{2} \alpha, \quad \forall \alpha \in \mathcal{F}_1^\mathbb{C}.$$  

Note that

$$\tilde{\lambda}(\mathcal{F}_1) \subset \mathcal{F}_1, \quad \lambda(\tilde{\mathcal{F}}_1) \subset \mathcal{F}_1, \quad \lambda \circ \tilde{\lambda} = \bar{\lambda} \circ \lambda.$$  

Explicitly, the splitting of (5.10) is provided by

$$0 \longrightarrow \mathcal{F}_1^\mathbb{C} \xrightarrow{\alpha \rightarrow (\lambda(\alpha), \bar{\lambda}(\alpha))} \mathcal{F}_1 \oplus \tilde{\mathcal{F}}_1 \xrightarrow{(\alpha, \beta) \rightarrow \lambda(\alpha) - \lambda(\beta)} \mathcal{F}_1^\mathbb{C} \longrightarrow 0.$$  

Furthermore, we get
\[ \mathcal{A}_1^C = \ker \lambda \oplus \mathcal{J}_1^C \oplus \ker \lambda, \quad \mathcal{J}_1 = \ker \bar{\lambda} \oplus \mathcal{J}_1^C, \quad \mathcal{J}_1 = \mathcal{J}_1^C \oplus \ker \lambda. \]
\[ \lambda(a) = \alpha, \quad \forall \alpha \in \ker \lambda, \quad \bar{\lambda}(a) = \alpha, \quad \forall \alpha \in \ker \lambda, \quad \lambda(a) = \bar{\lambda}(a) = \frac{1}{2} \alpha, \quad \forall \alpha \in \mathcal{J}_1^C. \]

Let us introduce the first order linear partial differential operator
\[ (5.15) \quad d^*_\alpha f = \frac{1}{2}(\lambda(df) - \bar{\lambda}(df)), \quad \forall f \in \mathcal{C}^\infty(M). \]

We note that \( d^*_\alpha u \) is real: this means that \( d^*_\alpha u \) is a real valued function. Indeed, for a real valued \( u \in \mathcal{C}^\infty(M) \), we have
\[ d^*_\alpha u = 2 \Im \lambda(du) = -2 \Im(\lambda(du)). \]

**Lemma 5.12.** We have \( dd^*_\alpha u \in \mathcal{J}_2 \) for every \( u \in \mathcal{A}_0 \).

**Proof.** For any germ of real valued smooth function \( u \) the differential \( dd^*_\alpha u \) is real and we have
\[ i dd^*_\alpha u = d(\lambda(du) - \bar{\lambda}(du)) = d(2\lambda(du) - du) = 2d \lambda(du) \in \mathcal{J}_2, \]
\[ = d(du - 2\bar{\lambda}(du)) = -2d\bar{\lambda}(du) \in \mathcal{J}_2, \]
so that \( dd^*_\alpha u \in \mathcal{J}_2 \cap \mathcal{J}_2 \cap \mathcal{J}_2 = \mathcal{J}_2. \]

**Proposition 5.13.** Let \( \Omega \) be a simply connected open set in \( M \). A necessary and sufficient condition for a real valued \( u \in \mathcal{C}^\infty(\Omega) \) to be the real part of an \( f \in \mathcal{O}_M(\Omega) \) is that there exists a section \( \xi \in \mathcal{J}_1(\Omega) \) such that
\[ (5.16) \quad d[dd^*_\alpha u + \xi] = 0 \quad \text{on} \quad \Omega. \]

If \( M \) is minimal, then \( \xi \) is uniquely determined.

**Proof.** Assume that \((5.16)\) is satisfied by some \( \xi \in \mathcal{J}_1(\Omega) \). Then \( d^*_\alpha u + \xi = dv \) for some real valued \( v \in \mathcal{C}^\infty(\Omega) \) and, with \( f = u + iv \) we obtain
\[ \lambda(du) - \bar{\lambda}(du) = i[\lambda(dv) + \bar{\lambda}(dv) - \xi] \quad \implies \quad \bar{\lambda}(df) = \lambda(du - idv) - i\xi \in \mathcal{J}_1^C(\Omega) \]
\[ \implies \quad df \in \mathcal{J}_1(\Omega) \iff f \in \mathcal{O}_M(\Omega). \]

Assume vice versa that \( f = u + iv \in \mathcal{O}_M(\Omega) \), with \( u \) and \( v \) real valued smooth functions. Write \( df = du + idv = \alpha + \zeta \), with \( \alpha \in \mathcal{J}_1(\Omega) \), \( \zeta \in \mathcal{J}_1^C(\Omega) \), and \( \bar{\lambda}(\alpha) = 0 \). From
\[ \bar{\lambda}(du) + i\bar{\lambda}(dv) = \frac{1}{2}\zeta \implies \lambda(du) - \bar{\lambda}(dv) = \frac{1}{2}\zeta, \]
we obtain
\[ id^*_\alpha u = \lambda(du) - \bar{\lambda}(dv) = \bar{\lambda}(dv) + \frac{1}{2}\zeta + i\bar{\lambda}(dv) = i dv - \frac{1}{2} (\zeta - \bar{\zeta}). \]
This is \((5.16)\) with \( \xi = (i/2)(\zeta - \bar{\zeta}). \)

To complete the proof, we note that, if \( \xi \in \mathcal{J}_1(\Omega) \) and \( d\xi = 0 \), then \( \xi = d\phi \) for some real valued function \( \phi \in \mathcal{C}^\infty(\Omega) \). If \( \xi_{p_0} \neq 0 \) for some \( p_0 \in \Omega \), then \( \{\phi(p) = \phi(p_0)\} \) defines a germ of smooth hypersurface through \( p_0 \) which is tangent at each point to the distribution \( HM \), contradicting the minimality assumption. 

**The Aeppli complex for pluri-harmonic functions on the CR manifold \( M \) is**
\[
0 \quad \xrightarrow{(u, \xi) \quad \mapsto \quad d^*_\alpha u + d\xi} \quad \mathcal{A}_0 \oplus \mathcal{J}_1 \quad \xrightarrow{d} \quad \mathcal{J}_2 \quad \xrightarrow{d} \quad \mathcal{J}_3 \quad \xrightarrow{d} \quad \mathcal{J}_m \quad \xrightarrow{d} \quad 0.
\]
We note that \( J_t = 0 \) if \( M \) is a complex manifold (we reduce to the classical case) and \( J_q = \neq 0 \) for \( q > 0 \) if \( M \) is totally real. In general, the terms of degree \( \geq k + 2 \) make a sub-complex of the de Rham complex.

**Peak points of CR functions and pseudo-convexity at the boundary.** A non-characteristic point of the boundary of a domain, where the modulus a CR function attains a local maximum, is pseudo-convex, in a sense that will be explained below.

**Lemma 5.14.** Let \( \partial M \) \( \Omega \in M \) and assume there is \( f \in \mathcal{O}_{M}(\Omega) \cap \mathcal{C}^{2}(\overline{\Omega}) \) such that \( |f| \) attains a local isolated maximum value at \( p_{0} \in \partial \Omega \). If if \( \partial \Omega \) is smooth, non-characteristic at \( p_{0} \) and, moreover, \( df(p_{0}) \neq 0 \), then there is a non-zero \( \xi \in H_{M,0}^{0}(\Omega) \) with \( \mathcal{L}_{\xi}^{\partial \Omega} \geq 0 \).

**Proof.** Let \( U \) be an open neighborhood of \( p_{0} \) in \( M \), and \( \rho \in \mathcal{C}^{\infty}(U, \mathbb{R}) \) a defining function for \( \Omega \) near \( p_{0} \), with \( U^{-} = \Omega \cap U = \{ p \in U \mid \rho(p) < 0 \} \), and \( d\rho(p) \neq 0 \) for all \( p \in U \).

We can assume that \( f(p_{0}) = |f(p_{0})| > 0 \) and exploit the fact that the restriction of \( u = \text{Re}f \) to \( \partial \Omega \) takes a maximum value at \( p_{0} \). Since \( d_{\partial \Omega}u(p_{0}) = 0 \), the real Hessian of \( u \) on \( \partial \Omega \) is well defined at \( p_{0} \), with
\[
\text{hess}(u)(X_{\rho_{0}}, Y_{\rho_{0}}) = (XYu)(p_{0}), \quad \forall X, Y \in \mathfrak{X}(\partial \Omega),
\]
and \( \text{hess}(u)(p_{0}) \leq 0 \) by the assumption that the restriction of \( u \) to \( \partial \Omega \) has a local maximum at \( p_{0} \). In particular, it follows that
\[
(Z\overline{Z}u)(p_{0}) = (\overline{Z}Zu)(p_{0}) \leq 0, \quad \forall Z \in \mathcal{Z}(\partial \Omega).
\]

Let \( v = \text{Im}f \). Then \( df = du + i dv \), and the condition that \( d_{\partial \Omega}u(p_{0}) = 0 \) implies that \( (Zv)(p_{0}) = 0 \) for all \( Z \in \mathcal{Z}(\partial \Omega) \) and thus \( \xi = dv(p_{0}) \in H_{\partial \Omega}^{0} \). Moreover,
\[
(5.17) \quad (Zu)(p) = -i(Zv)(p), \quad (\overline{Z}u)(p) = i(\overline{Z}v)(p), \quad \forall Z \in \mathcal{Z}(\partial \Omega), \quad \forall p \in \partial \Omega.
\]

Hence
\[
2ZZu(p_{0}) = (Z\overline{Z} + \overline{Z}Z)u(p_{0}) = i(Z\overline{Z} - \overline{Z}Z)v(p_{0}) = i(\xi)([Z, \overline{Z}])
\]
and thus the condition on the real Hessian of \( u \) implies that \( \mathcal{L}_{\xi}^{\partial \Omega} \geq 0 \). We note that \( du(p_{0}) \) is different from 0 and proportional to \( d\rho(p_{0}) \). Indeed, near \( p_{0} \) we have
\[
|f| = u \sqrt{1 + (v^{2}/u^{2})} \geq u(1 + \frac{1}{2}(v^{2}/u^{2}) = u + \Theta(p_{0})
\]
since \( v(p_{0}) = 0 \). Thus \( df(p_{0}) = du(p_{0}) \neq 0 \).

By the assumption that \( \partial \Omega \) is non-characteristic at \( p_{0} \), we have that \( du(p_{0}) \) is non-zero and equal to \( \lambda d\rho(p_{0}) \) for some \( \lambda > 0 \): therefore \( \xi = dv(p_{0}) \in H_{\partial \Omega}^{0} \) and this proves our claim.

**Proposition 5.15.** Let \( \Omega \) be an open subset of \( M \), and assume that there is a CR function \( f \in \mathcal{O}_{M}(\Omega) \cap \mathcal{C}^{2}(\overline{\Omega}) \) and a point \( p_{0} \in \partial \Omega \) such that:

(a) \[ |f(p_{0})| > |f(p)|, \quad \forall p \in \Omega, \]

(b) \( \partial \Omega \) is \( \Theta \)-non-characteristic at \( p_{0} \).

Then we can find \( 0 \neq \xi \in H_{M,0}^{0}(\Omega) \) with \( \mathcal{L}_{\xi}^{\partial \Omega} \geq 0 \).

[For the meaning of non-characteristic see (1.6).]
Proof. To apply Lemma 5.14 we need to check that \( d|f|(p_0) \neq 0 \). By the assumption that \( \partial \Omega \) is \( \Theta \)-non-characteristic at \( p_0 \), there is an open neighborhood \( U \) of \( p_0 \) in \( M \) and \( \tau \in [\ker \mathcal{L}](U) \) such that \( \partial \Omega \) is non-characteristic for \( F_\tau \) at \( p_0 \). Since \( P_\tau|f| \geq 0 \), by the Hopf lemma, \( d|f|(p_0) \neq 0 \) and therefore \( du(p_0) \) is a positive multiple of \( dp(p_0) \). Then \( \xi = d \Im f(p_0) \in H^0_{M,p_0}(\Omega) \) and we obtain the statement. \( \square \)

For characteristic peak points in the boundary of \( \Omega \) we have:

**Lemma 5.16.** Let \( \Omega \) be an open subset of \( M \), and assume that there is a CR function \( f = u + iv \in \mathscr{O}_M(\Omega) \cap \mathcal{C}^2(\overline{\Omega}) \), with \( u \) and \( v \) real valued, and \( p_0 \in \partial \Omega \) such that:

\[
\begin{align*}
(a) & \quad v(p_0) = 0, \; du(p_0) = H^0_{p_0,N}, \; u(p_0) > u(p), \; \forall p \in \Omega, \\
(b) & \quad 0 \neq \xi = dv(p_0).
\end{align*}
\]

Then \( \xi \in H^0_{p_0,M} \) and \( \ell_\xi \geq 0 \).

**Proof.** Set \( \eta = du(p_0) \). Then \( \xi = dv(p_0) \in H^0_{p_0,M} \), because \( df(p_0) = \eta + \xi \) is zero on \( Z(M) \), and hence \( \xi \), vanishing on \( Z(M) \) and being real, belongs to \( H^0_{p_0,M} \). The conclusion follows by the argument of Lemma 5.14 taking into account that this time all vectors in \( T^{0,1}_{p_0} M \) are tangent to \( \partial \Omega \) and that \( (5.17) \) is valid for \( Z \in \mathbb{Z}(M) \) at all points where \( f \) is defined and \( \mathcal{C}^1 \).

\( \square \)

Proposition 5.15 suggest to introduce some notions of convexity/concavity for boundary points of a domain in \( M \). Let \( \Omega \) be a domain in \( M \), \( p_0 \in \partial \Omega \) a smooth point of \( \partial \Omega \), and \( \rho \) a defining function for \( \Omega \) near \( p_0 \).

**Definition 5.4.** We say that \( \Omega \) is at \( p_0 \)

- strongly-1-concave if there is \( \tau \in [\ker \mathcal{L}] \cap H^{1,1}_{p_0} \partial \Omega \) such that \( dd^c \rho(p_0)(\tau) < 0 \);
- strongly-1-convex if there is \( \tau \in [\ker \mathcal{L}] \cap H^{1,1}_{p_0} \partial \Omega \) such that \( dd^c \rho(p_0)(\tau) > 0 \).

Points where the boundary is strictly 1-concave cannot be peak points for the modulus of CR functions.

**Proposition 5.17.** Assume that \( M \) has property \((H)\). Let \( \Omega \) be a relatively compact open domain in \( M \) and \( N \subset \partial \Omega \) a smooth part of \( \partial \Omega \) consisting of points where \( \partial \Omega \) is smooth, \( \Theta \)-non-characteristic, and strongly-1-concave. Then

\[
|u(p)| < \sup_{q \in \partial \Omega \cap N} |u(q)|, \quad \forall p \in \Omega \cup N,
\]

for every non constant \( u \in \mathscr{O}_M(\Omega) \cap \mathcal{C}^2(\overline{\Omega}) \).

**Proof.** Since \( M \) has property \((H)\), by Proposition 2.1 we have \( |f(p)| < \max_{\partial \Omega}|f| \), for all \( p \in \Omega \) and all non constant \( f \in \mathscr{O}_M(\Omega) \). The statement then follows from Proposition 5.15 because \( |f| \) cannot have a maximum on \( N \).

\( \square \)

1-concavity/concavity at the boundary and the vector-valued Levi form. Let \( \Omega^\text{open} \subset M \) have piece-wise smooth boundary and denote by \( N \) the CR submanifold of type \((n-1,k+1)\) of \( M \) consisting of the smooth non-characteristic points of \( \partial \Omega \).

The quotient \((TN \cap HM)/HN \subset TN/HN\) is a real line bundle on \( N \).

The partial complex structure \( J_M : HM \rightarrow HM \) restricts to the partial complex structure on \( HN \) and the tangent vectors \( v \) in \((HM \cap TN) \setminus HN\) are characterized by the fact that \( J_M(v) \notin TN \). Fix a point \( p_0 \in N \) and a defining function \( \rho \) of \( \Omega \) on a neighborhood \( U \) of \( p_0 \) in \( N \), so that \( 0 \neq dp(p_0) \) is an outer conormal to \( \Omega \) at \( p_0 \).
The elements $\xi_0 \in H_{M,p_0}^0 \Omega$ are defined, modulo multiplication by a positive scalar, by the condition that $dp(p_0) + i\xi_0 \in T_{p_0}^{0,1}M$. Since $\nu + iJ_M \nu \in T_{p_0}^{0,1}M$, we have

$$0 = \langle dp(p_0) + i\xi_0, (\nu + iJ_M \nu) \rangle = i\langle dp(p_0), J_M \nu \rangle + i\langle \xi_0, \nu \rangle - \langle \xi_0, J_M \nu \rangle$$

$$\implies \langle \xi_0, J_M \nu \rangle = 0, \quad \langle \xi_0, \nu \rangle = -\langle dp(p_0), J_M \nu \rangle.$$

The restriction $\xi_0|_N$ is an element of $H_{p_0}^0 N$, with $\langle \xi_0, \nu \rangle \neq 0$ if $p_0$ is non-characteristic. Therefore we have shown:

**Lemma 5.18.** Let $\nu = J_M \nu_{p_0}$ for an outer normal vector in $p_0 \in N \subset \partial \Omega$ to $\Omega$, with $\nu \in H_{p_0} M$. If $[\nu]$ belongs to the range of the vector-valued Levi form $\mathcal{L}^N$, then $\Omega$ is strongly-1-convex at $p_0$.

Vice versa, if $\Omega$ is strongly-1-convex at $p_0$, then $[\nu]$ belongs to the range of the vector valued Levi form. $\Box$

As usual, we used $[\nu]$ to denote the image of $\nu$ in the quotient $TN/HN$.

A similar statement holds for strong-1-concavity.

## 6. Convex cones of Hermitian forms

In a CR manifold of arbitrary CR-codimension, the scalar Levi forms associate to each point a linear space of Hermitian symmetric quadratic forms. Different notions of pseudo-concavity in \[ \[1, 20, 21\] \] originate from the observation that the polar of a subspace of forms with positive Witt index contains positive definite tensors. As we showed in §5 the analogue on a CR manifold $M$ of the complex Hessian of a smooth real function yields an affine subspace of Hermitian symmetric forms. Therefore it was natural to associate to a non-characteristic point of the boundary of a domain in $M$ an open half-space of Hermitian symmetric forms. In this section we describe some properties of duals of convex cones of Hermitian symmetric forms, to better understand the notions of pseudo-concavity that are relevant to discuss the extensions of some facts of analysis in several complex variables to the case of CR manifolds.

**Convexity in Euclidean spaces.** (cf. \[27, 38\]) Let us recall some notions of convex analysis. Let $V$ be an $n$-dimensional Euclidean real vector space. A nonempty subset $C$ of $V$ is a convex cone (with vertex 0) if

$$v_1, v_2 \in C, \ t_1 > 0, \ t_2 \geq 0 \implies t_1 v_1 + t_2 v_2 \in C.$$

The dual cone of $C$ is

$$C^* = \{ \xi \in V \mid (\nu|\xi) \geq 0, \ \forall \nu \in C \}.$$

By the Hahn-Banach theorem, one easily obtains:

**Lemma 6.1.** For any nonempty convex cone $C$ in $V$ we have $C^{**} = C$.

**Proof.** If $w \notin \mathcal{C}$, then, by the Hahn-Banach separation theorem we can find $\xi \in V$ such that $\inf_{v \in \mathcal{C}} (\nu|\xi) > (\nu|\xi)$. Since $C$ is a cone, this implies that $(\nu|\xi) \geq 0$ for all $\nu \in C$, i.e. $\xi \in C^*$, and then $(\nu|\xi) < 0$ shows that $w \notin C^{**}$. This proves that $C^{**} \subset C$. The opposite inclusion trivially follows from the definition. $\Box$

We call salient a convex cone which does not contain any real line: this means that if $0 \neq \nu \in C$, then $-\nu \notin C$. By Lemma 6.1 we have
Lemma 6.2. A nonempty closed convex cone \( C \) is salient if and only if \( C^* \) has a nonempty interior.

Proof. If \( C \) contains a vector subspace \( W \), then \( C^* \) is contained in the orthogonal \( W^* = W^\perp \), which is a proper linear subspace of \( V \) and therefore \( C^* \) has an empty interior. Vice versa, if \( C^* \) has an empty interior, then its linear span \( U \) is a proper linear subspace of \( V \) and \( W = U^* = U^\perp \) is a linear subspace of \( V \) of positive dimension contained in \( C = \mathbb{C} \).

Lemma 6.3. Let \( C \) be a salient closed convex cone and \( W \) a linear subspace of \( V \) with \( W \cap C = \{0\} \). Then we can find a hyperplane \( W' \) with \( W \subset W' \) and \( W' \cap C = \{0\} \).

Proof. For each \( v \in V \) we write \( v = v' + v'' \) for its decomposition into the sum of its component \( v' \in W \) and its component \( v'' \in W^\perp \). We claim that the orthogonal projection \( C'' \) of \( C \) into \( W^\perp \) is still a closed salient cone. Closedness follows by the fact that \( ||v''|| \leq ||v|| \) for some \( C > 0 \) for all \( v \in C \). To prove that \( C'' \) is salient, we argue by contradiction. Assume that \( C'' \) contains two opposite nonzero vectors \( \pm w'' \). Then there are \( w'_+, w'_- \in W \) such that \( w'_+ + w'', w'_- - w'' \in C \). The sum of these two nonzero vectors is nonzero by the assumption that \( C \) is salient, but

\[
0 \neq (w'_+ + w'') + (w'_- - w'') = (w'_+ + w'_-) \in C \cap W
\]

yields a contradiction.

By Lemma 6.2 the interior of the dual cone of \( C'' \) in \( W^\perp \) is nonempty. This means that there is a \( \xi \in W^\perp \) with \( (v''|\xi) > 0 \) for all \( v'' \in C'' \) and hence \( (\xi|v) > 0 \) for all \( v \in C \), since \( C \subset C'' + W \).

A closed convex cone \( C \) with \( \mathring{C^*} = \emptyset \) contains a linear subspace \( E_C \) of \( V \) and is called a wedge with edge \( E_C \). Lemma 6.3 generalizes to the case of closed wedges.

Lemma 6.4. If \( C \) is a closed wedge with edge \( E_C \) and \( W \) a linear subspace of \( V \) with \( W \cap C \subset E_C \), then there is a hyperplane \( W' \) with \( W \subset W' \) and \( W' \cap C = E_C \).

Proof. \( C \) contains all affine subspaces \( v + E_C \), for \( v \in C \). If \( \pi : V \to V/E_C \) is the projection into the quotient, then \( \pi(C) \) is a pointed cone and \( \pi(W) \cap \pi(C) = \{0\} \). By Lemma 6.3 there is a hyperplane \( H \) in \( V/W \) with \( \pi(W) \subset H \) and \( H \cap \pi(C) = \{0\} \). Then \( W' = \pi^{-1}(H) \) is a hyperplane in \( V \) which contains \( W \) and has \( C \cap W' = E_C \).

Convex cones in the space of Hermitian symmetric forms. Let us denote by \( \mathbb{P}_n \) the \( n^2 \)-dimensional real vector space of \( n \times n \) Hermitian symmetric forms on \( \mathbb{C}^n \). It is a Euclidean space with the scalar product \( (h_1|h_2) = \sum_{i,j=1}^n h_1(e_i, e_j)h_2(e_j, e_i) \), where \( e_1, \ldots, e_n \) is any basis of \( \mathbb{C}^n \). It will be convenient however to avoid fixing any specific scalar product on \( \mathbb{P}_n \), and formulate our statements in a more invariant way, involving the dual \( \mathbb{P}_n^* \) of \( \mathbb{P}_n \). It consists of the Hermitian symmetric covariant tensors, that we write as sums \( \pm v_1 \otimes \bar{v}_1 \pm \cdots \pm v_r \otimes \bar{v}_r \), for \( v_1, \ldots, v_r \in \mathbb{C}^n \). The identification of \( \mathbb{P}_n \) with \( \mathbb{P}_n^* \) provided by the choice of a scalar product on \( \mathbb{P}_n \) allows us to apply the previous results of convex analysis in this slightly different formulation.

A matrix corresponding to a Hermitian symmetric form \( h \) has real eigenvalues. The number of positive (resp. negative) eigenvalues is called its positive (resp. negative) index of inertia, the smallest of the two its Witt index, the sum of the two its rank.
Set $\hat{P}_n^+ = \{ h \geq 0 \}$ and $\hat{P}_n^0 = \hat{P}_n^+ \setminus \{ 0 \}$, $\hat{P}_n^- = \{ h > 0 \}$, and, likewise, $\tilde{P}_n^- = \{ h \leq 0 \}$ and $\tilde{P}_n^+ = \tilde{P}_n^- \setminus \{ 0 \}$. We shall use the simple

Lemma 6.5.

$$[\hat{P}_n^+]^* = [\hat{P}_n^{-}]^* = \bigcup_r \{ v_1 \otimes \tilde{v}_1 + \cdots + v_r \otimes \tilde{v}_r \mid \langle v_1, \ldots, v_r \rangle = \mathbb{C}^r \},$$

$$\{ \psi \in P_n^+ \mid \psi(h) > 0, \forall h \in \hat{P}_n^+ \} = \{ v_1 \otimes \tilde{v}_1 + \cdots + v_n \otimes \tilde{v}_n \mid \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n \},$$

$$\{ \psi \in P_n^+ \mid \psi(h) > 0, \forall h \in \hat{P}_n^+ \} = \{ v_1 \otimes \tilde{v}_1 + \cdots + v_r \otimes \tilde{v}_r \mid r > 0, \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n \}.$$

Proposition 6.6. Let $\mathcal{W}$ be a convex closed cone, with vertex in 0, in $P_n$. Assume that every nonzero element of $\mathcal{W}$ has a non-zero positive index of inertia. Then there is a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that

$$(6.1) \sum_{i=1}^n h(e_i, e_i) \geq 0, \quad \forall h \in \mathcal{W}.$$  

Proof. Both $\mathcal{W}$ and $\mathcal{W}^+ = \{ h_1 + h_2 \mid h_1 \in \mathcal{W}, h_2 \geq 0 \}$ are proper closed convex cones in $P_n$. Since $\mathcal{W}^+$ does not contain any negative semidefinite nonzero form, its edge has empty intersection with $\hat{P}_n^+ = \{ h \geq 0, h \neq 0 \}$. By Lemma 6.4 we can find a $\psi \in P_n^+$ such that

$$\psi(h) > 0, \forall h \in \mathcal{W}^+ \quad \text{and} \quad \mathcal{W}^+ \cap \{ \psi = 0 \} = E_{\mathcal{W}^+}.$$  

In particular, $\psi(h) > 0$ for $h \in P_n^+$ and hence, by Lemma 6.5, $\psi$ is of the form $\psi(h) = \sum_{i=1}^n h(e_i, e_i)$ for a basis $e_1, \ldots, e_n$ of $\mathcal{W}$.  

We obtain, as a corollary, the result of [20] Lemma 2.4, which motivated the definition of essential pseudo-concavity.

Corollary 6.7. If $\mathcal{W}$ is a linear subspace of $P_n$ such that each nonzero element of $\mathcal{W}$ has a positive Witt index, then there exists a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that

$$\sum_{i=1}^n h(e_i, e_i) = 0.$$  

Proposition 6.8. Let $\mathcal{W}$ be a relatively open convex cone with vertex at 0 of $P_n$, and such that every element $h$ of $\mathcal{W}$ has a non-zero positive index of inertia. Then the elements of $\hat{P}_n^-$, which are contained in $\mathcal{W}$, are all degenerate.

All the elements of maximal rank in $\mathcal{W} \cap \hat{P}_n^-$ have the same kernel, which has a positive dimension $r$ and a basis $e_1, \ldots, e_r$ such that

$$(6.2) \sum_{i=1}^r h(e_i, e_i) > 0, \quad \forall h \in \mathcal{W}.$$  

Proof. Let $\hat{P}_n^- = \{ h \in P_n \mid h < 0 \}$. Then $\mathcal{W}$ and $\hat{P}_n^-$ are disjoint relatively open convex cones of $P_n$ with vertex in 0 and therefore (see e.g. [43] Theorem 2.7) are separated by a hyperplane, defined by a linear functional $\psi$, which is positive on $\mathcal{W}$ and negative on $\hat{P}_n^-$. Being negative on $\hat{P}_n^-$, by Lemma 6.5, $\psi$ has the form (6.2).

This implies that all elements of $\mathcal{W} \cap \hat{P}_n^-$ are degenerate. Since $\mathcal{W} \cap \hat{P}_n^-$ is a cone, all its elements of maximal rank belong to its relative interior and have the same kernel, say $U \subset \mathbb{C}^r$, whose positive dimension we denote by $r$. In fact, for a pair of negative semidefinite forms $h_1, h_2$, we have $\ker (h_1 + h_2) = \ker h_1 \cap \ker h_2$. The statement follows by applying Proposition 6.6 to $\mathcal{W}_U = \{ h \mid h \in \mathcal{W} \}$, which is a closed cone in $P_n$, in which all nonzero elements have a nonzero positive index of inertia. In fact, if there is a nonzero $h \in \mathcal{W}_U$ whose restriction to $U$ is seminegative, and $h_0$ is an element of maximal rank in the cone $\mathcal{W} \cap \hat{P}_n^-$, then, for $C > 0$ and large, $h + Ch_0$ would be a negative definite element in $\mathcal{W} \cap \hat{P}_n^-$.  

Proposition 6.9. Let $\mathcal{W}$ be a cone in $\mathbb{P}_n$, with the property that all its elements of maximal rank have a nonzero positive index of inertia. Then all forms in $\mathcal{W} \cap \tilde{\mathcal{P}}_n^+$ are degenerate; those of maximal rank have all the same kernel, of dimension $r > 0$, which contains a basis $e_1, \ldots, e_r$ such that

$$
\sum_{i=1}^{r'} h(e_i, e_i) \geq 0, \quad \forall h \in \mathcal{W}.
$$

Proof. Let $\tilde{\mathcal{P}}_n^+ = \{ h \in \mathbb{P}_n \mid h > 0 \}$. Then $\mathcal{W} + \tilde{\mathcal{P}}_n^+$ is an open cone in $\mathbb{P}_n$ such that all its elements have a nonzero positive index of inertia.

Since $\mathcal{W} + \tilde{\mathcal{P}}_n^+ \cap \tilde{\mathcal{P}}_n^- = (\mathcal{W} + \tilde{\mathcal{P}}_n^+ \cap \tilde{\mathcal{P}}_n^-) \cap \bar{\mathcal{P}}_n^+$, we know from Proposition 6.8 that all elements of maximal rank in $\mathcal{W} \cap \tilde{\mathcal{P}}_n^-$ have the same kernel $U$, which is a subspace of $\mathbb{C}^n$ of positive dimension $r$ and contains a basis $e_1, \ldots, e_r$ for which

$$
\sum_{i=1}^{r'} h(v_i, v_i) > 0, \quad \forall h \in \mathcal{W} + \tilde{\mathcal{P}}_n^+.
$$

This implies (6.3). \hfill \Box

Analogous results can be given to characterize cones of Hermitian forms having some given amount of positive (or negative) eigenvalues. In this case we need to consider the behavior of the restriction of forms to subspaces of $\mathbb{C}^n$. We use the notation $\mathcal{G}_n^r(\mathbb{C}^n)$ for the Grassmannian of complex linear $h$-planes of $\mathbb{C}^n$.

Proposition 6.10. Let $\mathcal{W}$ be a proper closed convex cone in $\mathbb{P}_n$, with vertex in 0 and $q$ an integer with $0 < q \leq n$. Assume that every nonzero form in $\mathcal{W}$ has a positive index of inertia $\geq q$. Then, for every $V \in \mathcal{G}_n^{r+1}(\mathbb{C}^n)$, we can find a basis $v_1, \ldots, v_{n-q+1}$ of $V$ such that

$$
\sum_{i=1}^{n-q+1} h(v_i, v_i) \geq 0.
$$

Proof. It suffices to apply Proposition 6.6 to the restrictions to $V \in \mathcal{G}_n^{r+1}(\mathbb{C}^n)$ of the forms in $\mathcal{W}$. By the assumption, $h|_V$ has a nonzero positive index of inertia for all $h \in \mathcal{W} \setminus \{0\}$. \hfill \Box

An analogous statement to Proposition 6.8 can be formulated for relatively open convex cones of Hermitian forms with positive index of inertia $\geq q$.

Proposition 6.11. Let $\mathcal{W}$ be a relatively open convex cone in $\mathbb{P}_n$ and assume that each $h$ in $\mathcal{W}$ has a positive index of inertia $\geq q$, for an integer $0 < q \leq n$. Then for every $V \in \mathcal{G}_n^{r+1}(\mathbb{C}^n)$ we can find an integer $r_V > 0$ and linearly independent $v_1, \ldots, v_{r_V} \in V$ such that

$$
\sum_{i=1}^{r_V} h(v_i, v_i) > 0, \quad \forall h \in \mathcal{W}.
$$

Proof. For every $V \in \mathcal{G}_n^{r+1}(\mathbb{C}^n)$, the set $\mathcal{W}_V = \{ h|_V \mid h \in \mathcal{W} \}$ is a relatively open convex cone in $\mathbb{P}_{n-q+1}$ such that all of its elements $h|_V$ have a nonzero positive index of inertia. The thesis follows by applying Proposition 6.8 to $\mathcal{W}_V$. \hfill \Box

Proposition 6.12. Let $W$ be a convex cone in $\mathbb{P}_n$ such that the elements of maximal rank of $W$ have a positive index of inertia $\geq q$ ($q$ is an integer with $0 < q \leq n$). Then for every $V \in \mathcal{G}_n^{r+1}(\mathbb{C}^n)$ we can find an integer $r_V > 0$ and linearly independent $v_1, \ldots, v_{r_V} \in V$ such that

$$
\sum_{i=1}^{r_V} h(v_i, v_i) \geq 0, \quad \forall h \in \mathcal{W}.
$$
Proof. It suffices to apply Proposition 6.11 to \( W + \hat{\partial}^r \) and note that (6.5) for all \( h \in W + \hat{\partial}^r \) implies (6.6) for all \( h \in W \). \( \square \)

Remark 6.13. The positive integer \( r \) of Propositions 6.11, 6.12 is the dimension of the kernel of any form of maximal rank in \( W \cap \hat{\partial}^r \).

7. Notions of pseudo-concavity

In [22] it was proved that the Poincaré lemma for the tangential Cauchy-Riemann complex of locally CR-embeddable CR manifolds fails in the degrees corresponding to the indices of inertia of its scalar Levi forms of maximal rank. On the other hand, in [17] it was shown that the Lefschetz hyperplane section theorem for \( q \)-dimensional complex submanifolds generalizes to weakly-\( q \)-pseudo-concave CR submanifolds of complex projective spaces.

This suggests to seek for suitable weakening of the pseudoconcavity conditions to allow degeneracies of the Levi form. A natural condition of weak 1-pseudoconcavity is to require that no semi-definite scalar Levi form has maximal rank. Under some genericity assumption, by using Proposition 6.12, this translates into the fact that \([\ker L]\) is non-trivial. Indeed, this hypothesis implies maximum modulus and unique continuation results analogous to those for holomorphic functions of one complex variable. We expect that properties that are peculiar to holomorphic functions of several complex variables would generalize to CR functions under suitable (weak) 2-pseudo-concavity conditions. This motivates us to give below a tentative list of conditions, motivated partly by the discussion in §6 and partly by the results of the next sections.

Notation 7.1. If \( \mathcal{V} \subset \mathcal{Z} \) is a distribution of complex vector fields on \( \Omega^{\text{open}} \subset M \), we use the notation \([\ker L]_{\mathcal{V}}\) for the semi-positive tensors \( \sum_{i=1}^q Z_i \otimes Z_i \) of \([\ker L]\) with \( Z_i \in \mathcal{V} \).

Definition 7.1. Let \( p_0 \in M \). We say that \( M \) is

\[
(\Psi^{(S)}_{p_0}(q)): \text{strongly-}q\text{-pseudo-concave at } p_0 \text{ if all } \mathcal{L}_\xi, \text{ with } \xi \in H^0_{p_0}M \setminus \{0\}, \text{ are nonzero and have Witt index } \geq q;
\]

\[
(\Psi^{(W)}_{p_0}(q)): \text{weakly-}q\text{-pseudo-concave at } p_0 \text{ if its scalar Levi forms of maximum rank at } p_0 \text{ have Witt index } \geq q;
\]

\[
(\Psi^{(E)}_{p_0}(q)): \text{essentially-}q\text{-pseudo-concave at } p_0 \text{ if, for every distribution of smooth complex vector fields } \mathcal{V} \subset \mathcal{Z}, \text{ of rank } n-q+1, \text{ defined on an open neighborhood } U \text{ of } p_0, \text{ we can find an open neighborhood } U' \text{ of } p_0 \text{ in } U \text{ and a } \tau \in [\ker L]_{\mathcal{V}}^{n-q+1}(U').
\]

\[
(\Psi^{(E^t)}_{p_0}(q)): \text{essentially-}q\text{-pseudo-concave at } p_0 \text{ if, for every distribution of smooth complex vector fields } \mathcal{V} \subset \mathcal{Z}, \text{ of rank } n-q+1, \text{ defined on an open neighborhood } U \text{ of } p_0, \text{ we can find an open neighborhood } U' \text{ of } p_0 \text{ in } U \text{ and a } \tau \in [\ker L]_{\mathcal{V}}(U').
\]

We drop the reference to the point \( p_0 \) when the property is valid at all points of \( M \).

We also consider the (global) condition

\[
(\Psi^{(G)}_{p}(q)) \text{ For all } p \in M \text{ and } \mathcal{V} \subset \mathcal{Z} \text{ of rank } n-q+1 \text{ on a neighborhood } U \text{ of } p, \bigcup_{p' \in U}[\ker L]_{\mathcal{V},p'} \text{ is a bundle with nonempty fibers and such that for every sequence } \{p_i\} \subset M, \text{ converging to } p \in M, \text{ every } \tau \in [\ker L]_{\mathcal{V},p} \text{ is a cluster point of } \bigcup_{n}[\ker L]_{\mathcal{V},p,n}.
\]
Recall that, according to the notation introduced on page 5, the elements of \([\ker L](U')\) are different from zero at each point of \(U'\).

**Remark 7.1.** If \(q > 1\), then \(\Psi^*_p(q) \Rightarrow \Psi^*_p(q-1)\) for \(\star = s, w, e, e^*\), and (cf. Proposition 6.6 and \([20] \S 2\))

\[
\Psi^w(q) \Leftrightarrow \Psi^e(q) \Rightarrow \Psi^e(q), \quad \text{for } q \geq 1.
\]

**Lemma 7.2.** Assume that \(M\) is essentially-\(q\)-pseudo-concave. Then, for every rank \(n-q+1\) distribution \(\mathcal{V}' \subset \mathcal{Z}\) on an \(\Omega^{\text{open}} \subset M\), we can find a global section \(\tau \in \{\ker L\}_{\mathcal{V}'}(\Omega)\).

**Proof.** By the assumption, for each \(p \in \Omega\), there is an \(U^{\text{open}} \subset \Omega\) with \(p \in U_p\) and \(\tau_p = \sum_{i=1}^{n-q+1} Z_i \otimes Z_i \in \{\ker L\}_{\mathcal{V}'} (U_p)\) with \(Z_i \in \mathcal{V}(U_p)\). The global \(\tau\) can be obtained by gluing together the \(\tau_p\)'s by a nonnegative smooth partition of unity on \(\Omega\) subordinate to the covering \([U_p]\).

In the same way we can prove

**Lemma 7.3.** Assume that \(M\) is essentially\(^*\)-\(q\)-pseudo-concave. Then, for every rank \(n-q+1\) distribution \(\mathcal{V}' \subset \mathcal{Z}\) on an \(\Omega^{\text{open}} \subset M\), we can find a global section \(\tau \in \{\ker L\}_{\mathcal{V}'}(\Omega)\).

**Example 7.4.** Let \(F_{\ell_1,\ldots,\ell_h}(\mathbb{C}^m) \subset Gr_{\ell_1}(\mathbb{C}^m) \times \cdots \times Gr_{\ell_h}(\mathbb{C}^m)\) denote the complex flag manifold consisting of the \(r\)-tuples \((\ell_{h_1},\ldots,\ell_{h_k})\) with \(\ell_{h_1} \subset \cdots \subset \ell_{h_k}\), for an increasing sequence \(1 \leq h_1 > \cdots > h_k < m\). Here, as usual, \(\ell_h\) is a generic \(\mathbb{C}\)-linear subspace of dimension \(h\) of \(\mathbb{C}^m\).

For an increasing sequence of integers \(1 \leq i_1 < i_2 < \cdots < i_v < m\), of length \(v \geq 2\), we define the \(CR\)-sub-manifold \(M\) of \(F_{i_1,i_2,\ldots}(\mathbb{C}^m) \times F_{i_2,i_3,\ldots}(\mathbb{C}^m)\) consisting of pairs \(((\ell_{i_1},\ell_{i_2},\ldots), (\ell_{i_2},\ell_{i_3},\ldots))\) with \(\ell_h \subset \ell_{h+1}\) for \(0 < h < v\). Set

\[
d_0 = i_1, \quad d_1 = i_2 - i_1, \quad \cdots, \quad d_h = i_{h+1} - i_h, \quad \cdots, \quad d_{v-1} = i_v - i_{v-1}, \quad d_v = m - i_v.
\]

This \(M\) is a minimal (i.e. \(Z(M) + \overline{Z}(M)\) and their iterated commutators yield all complex vector fields on \(M\)), compact \(CR\) manifold of \(CR\)-dimension \(n\) and \(CR\)-codimension \(k\), with

\[
n = \sum_{i=0}^{v-1} d_id_{i+1}, \quad k = 2 \sum_{1 \leq j < \nu \leq v} d_id_j,
\]

as was explained in \([33] \S 3.1\). Then, with \(q = \min_{1 \leq i \leq 1} d_i\), our \(M\) is essentially, but not strongly, \(q\)-pseudo-concave when \(v \geq 3\), because the non-vanishing scalar Levi forms generate at each point a subspace of dimension \(2\sum_{i=1}^{\nu-1} d_id_{i+2} < k\).

In \([33]\) several classes of homogeneous compact \(CR\) manifolds are discussed, from which more examples of essentially, but not strongly, \(q\)-pseudo-concave manifolds can be extracted.

**Example 7.5.** Let us consider the 11-dimensional real vector space \(\mathcal{W}\) consisting of \(4 \times 4\) Hermitian symmetric matrices of the form

\[
h = \begin{pmatrix} A & B \\ B^* & -A \end{pmatrix}
\]

with \(A, B \in \mathbb{C}^{2 \times 2}, A = A^*, \text{trace}(A) = 0\).

We claim that all nonsingular elements of \(\mathcal{W}\) have Witt index two. In fact, for an element \(h\) of \(\mathcal{W}\), either \(A = 0\), or \(A\) is nondegenerate. If \(A = 0\), the matrix \(A\) is nondegenerate iff \(\det(B) \neq 0\), and in this case the Witt index is two as the two-plane of the first two vectors of the canonical basis of \(\mathbb{C}^4\) is totally isotropic. If
Example 7.6. Let \( M \) be the minimal orbit of \( \text{SU}(p, p) \) in the complex flag manifold \( F_{1,2p-2}(\mathbb{C}^{2p}) \), for \( p \geq 3 \). Its points are the pairs \((\ell_1, \ell_{2p-2})\) consisting of an isotropic line \( \ell_1 \) and a \((2p-2)\)-plane \( \ell_{2p-2} \) with \( \ell_1 \subset \ell_{2p-2} \subset \ell_1^\perp \), where perpendicularity is taken with respect to a fixed Hermitian symmetric form of Witt index \( p \) on \( \mathbb{C}^{2p} \).

Then \( M \) is a compact \( CR \) submanifold of \( F_{1,2p-2}(\mathbb{C}^{2p}) \), of \( CR \) dimension \((2p-3)\) and \( CR \) codimension \((4p-4)\), which is essentially \( 1 \)-pseudo-concave and, when \( p > 3 \), weakly and weakly\(^*\)-(\(p-2\))-pseudo-concave, but not essentially\(^*\)-\(2\)-pseudo-concave.

7.1. Convexity/concavity at the boundary and weak pseudoconcavity. Let us comment on the notion of \( 1 \)-convexity/concavity at a boundary point of a domain \( \Omega \) of \( \mathbb{R}^n \) in the light of the discussion on Hermitian forms of \( \mathbb{R}^n \).

Let \( \rho \) be a real valued smooth function on \( \Omega^{\text{open}} \subset M \) and \( p_0 \) a point of \( \Omega \) with the property that, for each \( i \) \( d \xi_{p_0} \) in \( H^{1,1}_{p_0}(\rho) \), the restriction of \( i \) \( d \xi_{p_0} \) to the space \( \{ Z_{p_0} \in T_{p_0}^1 M \mid Z_{p_0} \rho = 0 \} \) has a nonzero positive index of inertia. The positive multiples of these Hermitian symmetric forms make a relatively open convex cone.
\( W \) in the space \( \mathcal{E}_{n-1} \) of Hermitian symmetric forms on \( T_{p_0}^{0,1} M \cap \ker dp(p_0) \). By Proposition 6.8 we can find an \( r > 0 \) and \( \tau_0 \in H^{1,1,(r)}_{p_0} \) such that
\[
\text{id}\xi(\tau_0) > 0, \quad \forall \xi \in \mathcal{A}_1(\Omega), \quad \text{s.t.} \quad dp(p_0) + i\xi_{p_0} \in T_{p_0}^{s,1,0} M.
\]
Since \( H^{1,1}_p(M) \) is affine with underlying vector space \( \{L_\eta \ | \ \eta \in H^0_\eta M\} \), it follows that actually \( \tau_0 \in [\ker L]^{(r)}_{p_0} \). The same argument applies to the case of a nonzero negative index of inertia.

Thus, by Lemma 5.18 the condition for \( \Omega_{p(p_0)} = \{p \in \Omega \ | \ \rho(p) < \rho(p_0)\} \) to be strongly-(1)-convex, or strongly-(1)-concave at \( p_0 \) is that
\[
\exists \tau_0 \in [\ker L]_{\ker dp, p_0} \text{ such that } \begin{cases}
dd^c\rho(\tau_0) > 0, & \text{(strongly-1-convex)}, \\
\dd^c\rho(\tau_0) < 0, & \text{(strongly-1-concave)}.
\end{cases}
\]
A glitch of the notion of strong-1-convexity (resp. -concavity) is that it is not, in general, stable under small perturbations. This can be ridden out by adding the global assumption of essential-2-pseudo-concavity of \( M \). Set, for simplicity of notation, \( \rho(p_0) = 0 \) and \( dp(p_0) \neq 0 \).

**Proposition 7.7.** Suppose that \( M \) is essentially-2-pseudo-concave and that \( \Omega_0 = \{p \in \Omega \ | \ \rho(p) < 0\} \) is strongly-1-concave at \( p_0 \in \partial \Omega_0 \). Then

1. We can find \( \tau_0 \in [\ker L]^{(n-1)}_{\ker dp, p_0} \) such that \( \dd^c\rho(\tau_0) < 0 \);
2. We can find an open neighborhood \( U \) of \( p_0 \) in \( \Omega \) such that at every \( p' \in U \) the open set \( \Omega_{p(p')} = \{p \in \Omega \ | \ \rho(p) < \rho(p')\} \) is smooth and strongly-1-concave at \( p' \).

\( \square \)

### 8. Cauchy Problem for CR Functions - Uniqueness

In this section we discuss uniqueness for the initial value problem for CR functions, with data on a non-characteristic smooth initial hypersurface \( N \subset M \).

Uniqueness is well understood when \( M \) is a CR submanifold of a complex manifold (see e.g. [40]). Let \( \Omega \subset M \) be an open neighborhood of a non-characteristic point \( p_0 \) of \( N \), such that \( \Omega \setminus N \) is the union of two disjoint connected components \( \Omega^\pm \).

**Proposition 8.1.** Assume that \( M \) is a minimal CR submanifold of a complex manifold \( X \). If \( f \in \mathcal{O}_M(\Omega^+) \cap \mathcal{C}^0(\Omega^+) \) and \( f|_N \) vanishes on an open neighborhood of a non-characteristic point \( p_0 \) of \( N \), then \( f \equiv 0 \) on \( \Omega^+ \).

\( \square \)

We have a similar statement for CR distributions.

**Proposition 8.2.** Assume that \( M \) is either a real-analytic CR manifold, or a CR submanifold of a complex manifold \( X \) that is minimal at every point. Let \( N \) be a \( \mathcal{Z} \)-non-characteristic hypersurface of \( M \), such that \( M \setminus N \) is the union of two disjoint connected open subsets \( M_\pm \). Then there is an open neighborhood \( U \) of \( N \) in \( M \) such that any CR distribution on \( M_+ \) having vanishing boundary values on \( N \), vanishes on \( U \cap M_+ \).

**Proof.** An \( f \in \mathcal{C}'(M_\pm) \) is CR if \( Zf = 0 \) in \( M_\pm \), in the sense of distributions, for all \( Z \in \mathcal{Z}(M) \). We say that \( f \) has zero boundary value on \( N \) if for each \( p \in N \) we can find an open neighborhood \( U_p \) of \( p \) in \( M \) and a CR-distribution \( f \in \mathcal{C}'(U_p) \) which extends \( f|_{M_\pm \cap U_p} \) and is zero on \( U_p \setminus \tilde{M_\pm} \). Note that, since \( N \) is non-characteristic, all CR distributions defined on a neighborhood of \( N \), admit a restriction to \( N \).
The case where $M$ is a real-analytic CR manifold reduces to the classical Holmgren uniqueness theorem.

In the other case, where $M$ is $C^\infty$ smooth, but is assumed to be minimal, we first choose a slight deformation $N_d$ of $N$ such that $N_d$ is contained in $M^+$ and coincides with $N$ near $p$. Moreover we can achieve that the CR orbit $\partial(p, N_d)$ of $p$ in $N_d$ intersects $N_d \cap M^+$. Since $M^+$ is minimal at every point, CR distributions holomorphically extend to open wedges attached to $M^+$. In particular this holds for the boundary value of $f|_{M^+}$ ($M^+$ being the side of $N_d$ containing $M^+$) at any point of $N_d \cap M^+$.

Using that wedge extension propagates along CR orbits we get wedge extension from $N_d$ at $p$. Examining how the wedges are constructed by analytic disc techniques, one more precisely obtains a neighborhood $V$ of $N$ in $M$ such that any given $\theta$ satisfies (8.1), which is continuous on $M$, and vanishes on $U$ (8.1) defining an open wedge in $M$ restricted to $M^+$.

Condition (8.1), with $\Omega(M)$ instead of $\bar{\Omega}(\tau)$, naturally arises when we consider CR sections of a complex CR line bundle (see [20, §7]).

We note that the hypersurface $N$ is non-characteristic at a point $p_0$ with respect to the distribution $\partial N$ if it is non-characteristic at $p_0$ for $\partial N(\tau)$ for some $\tau \in \ker L(M)$.

**Proposition 8.4.** Let $\Omega(\tau)$ be a smooth $\Theta$-non-characteristic hypersurface in $M$. Then there is a neighborhood $U$ of $N$ in $M$ such that any solution $f$ of (8.1), which is continuous on $\Omega$ and vanishes on $N$, is zero on $U \cap \Omega$. 

Remark 8.3. Thanks to the extension result proved in [20, 35], see also [36], it suffices to assume that $M^+$ is globally minimal, i.e. that $M^+$ consists of only one CR orbit.
Proof. We note that the assumption of constancy of rank in unessential and never used in the proof of [21, Theorem 4.1]. We reduce to that situation by considering the case of two closed half-manifolds \( M_+ \) and \( M_- \) with \( \Omega = M_- \) and \( \partial \Omega = N \). A continuous solution \( f \) of (8.1) in \( M_- \) vanishing on \( N \), when extended by 0 on \( M_+ \), defines a continuous solution \( \tilde{f} \) of (8.1) in \( M_- \) with \( \text{supp} \tilde{f} \subset M_- \). In fact, since \( L \in \mathcal{\Theta}(M) \) is first order, \( L \tilde{f} \) equals \( Lf \) on \( M_- \) and 0 on \( M_+ \), as one can easily check by integrating by parts and using the identity of weak and strong extensions of [15]. Hence \( \tilde{f} \) still satisfies (8.1) and vanishes on an open subset of \( M \). By proving Carleman estimates, similar to those in [20, Theorem 5.2], we obtain that \( \tilde{f} \) vanishes along the Sussmann leaves of \( \tilde{\Theta} \) transversal to \( N \) (see [21, 45]). These leaves fill a neighborhood of \( N \) in \( M \), where \( \tilde{f} \) vanishes. This proves our contention.

\[ \square \]

Remark 8.5. Note that \( \mathbb{C}^n \times \mathbb{R}^r \) is weakly pseudo-concave (but not essentially pseudo-concave). Thus we need the genericity assumption (1.5) to get uniqueness in this case. The uniqueness for the non-characteristic Cauchy problem in the case of a single partial differential operator of [11, 44] may be considered a special case of this proposition, when the \( CR \) dimension is one.

Uniqueness in the case where \( N \) can be characteristic for \( \Theta \), but not for \( \mathcal{Z} \), will be obtained by adding a pseudo-convexity hypothesis.

First we prove a Carleman-type estimate.

Lemma 8.6. Let \( \tau \) be a section of \( (\ker L) \) and \( \psi \) a real valued smooth function on \( M \). Then there is a smooth real valued function \( \kappa \) on \( M \) such that

\[ \| \exp(\tau \psi) L_0 f \|_0^2 + \sum_{i=1}^r \| \exp(\tau \psi) Z_i f \|_0^2 \geq \int (2 t \cdot df \psi(\tau) + \kappa) \| f \|^2 e^{2\psi} d\mu, \]

\[ \forall f \in \mathcal{C}_0^\infty(M), \forall t > 0. \]

Here the \( L^2 \)-norms and the integral are defined by utilizing the smooth measure \( d\mu \) associated to a fixed Riemannian metric on \( M \).

Proof. Let \( \tau = \sum_{i=1}^r Z_i \otimes \bar{Z}_i, \sum_{i=1}^r [Z_i, \bar{Z}_i] = \bar{L}_0 - L_0 \), with \( Z_i, L_0 \in \mathcal{Z}(M) \). We will indicate by \( \kappa, \kappa \) smooth functions on \( M \) which only depend on \( Z_1, \ldots, Z_r \). For \( f \in \mathcal{C}_0^\infty(M) \), and a fixed \( t > 0 \), set \( \nu = f \cdot \exp(\tau \psi) \). Integration by parts yields

\[ \sum_{i=1}^r \| Z_i \nu - t \nu Z_i \psi(\tau) \|^2_0 = \sum_{i=1}^r \| Z_i \nu - t \nu \bar{Z}_i \psi(\tau) \|^2_0 + \int \sum_{i=1}^r [Z_i, \bar{Z}_i] \nu \cdot \bar{\nu} d\mu + \text{Re} \int \left( \kappa_0 + \sum_{i=1}^r 2 \tau(Z_i \bar{Z}_i) \right) |\nu|^2 d\mu, \]
where the superscript star stands for formal adjoint with respect to the Hermitian scalar product of $L^2(\mu)$. For the second summand in the right hand side we have

$$\int \sum_{i=1}^r [Z_i, \bar{Z}_i]v \cdot \bar{v} \, d\mu = \int L_0v \cdot \bar{v} \, d\mu - \int L_0v \cdot \bar{v} \, d\mu$$

$$= -\int L_0 v \cdot \bar{v} \, d\mu - \int v \cdot \overline{L_0} v \, d\mu - \int \kappa_1 |v|^2 \, d\mu$$

$$= -2 \Re \int L_0 v \cdot \bar{v} \, d\mu - \int \kappa_1 |v|^2 \, d\mu$$

$$\geq -2\|L_0v - tvL_0\psi\|_0^2 - \int (\kappa_1 + 2\Re L_0\psi)|v|^2 \, d\mu$$

$$\geq -\|L_0v - tvL_0\psi\|_0^2 - \int (1 + \kappa_1 + 2\Re L_0\psi)|v|^2 \, d\mu.$$ 

Therefore we obtain the estimate

$$\|L_0v - tvL_0\psi\|_0^2 + \sum_{i=1}^r \|Z_i v - tvZ_i\psi\|_0^2$$

$$\geq \int (\kappa_1 + 2\Re L_0\psi - \kappa_2)|v|^2 \, d\mu = \int (2tP_v + \kappa)|v|^2 \, d\mu.$$ 

By Proposition 5.6, this yields (8.2). \qed

From the Carleman estimate (8.2) we obtain a uniqueness result under convexity conditions, akin to the one of [25, §28.3] for a scalar p.d.o.

**Proposition 8.7.** Assume there is a section $\tau \in \ker L$ and $\psi \in \mathcal{C}^\infty(M, \mathbb{R})$ such that

(8.3) $d\psi(p_0) \neq 0$, $\partial^\nu \psi(\tau) > 0$.

Then there is an open neighborhood $U$ of $p_0$ in $M$ with the property that any solution $f$ of (8.1) which vanishes a.e. on $U \cap \{p \mid \psi(p) > \psi(p_0)\}$ also vanishes a.e. on $U$.\qed

**Remark 8.8.** In fact, it suffices to require that (8.1) is satisfied by the operators $Z_1, \ldots, Z_r, L_0$.

Let $\Omega$ be an open domain in $M$, and $p_0 \in \partial \Omega$ a smooth point of the boundary.

**Proposition 8.9.** If $\Omega$ is either $\Theta$-non-characteristic or strictly 1-convex at $p_0$ (according to Definition 5.4), then any $f$ satisfying (8.1) in $\Omega$, and having zero boundary values on a neighborhood of $p_0$ in $\partial \Omega$, is $0$ a.e. on the intersection of $\Omega$ with a neighborhood of $p_0$ in $M$. \qed

**Proof.** With $P_t$ defined by (3.7), (3.8), and a real parameter $s$, we have

$$e^{-yw}P_t(e^{yw}) = s \left( \frac{1}{2} \sum_{i=1}^r (Z_i \bar{Z}_i + \bar{Z}_i Z_i)\psi - X_0\psi \right) + s^2 \sum_{i=1}^r |Z_i\psi|^2$$

$$= s \partial^\nu \psi(\tau) + s^2 \sum_{i=1}^r |Z_i\psi|^2.$$ 

Thus the condition of Proposition 8.7 is satisfied for a suitable $\tau \in \ker L$ near $p_0$ when $\partial \Omega$ is $\Theta$-non-characteristic at $p_0$, by taking $s \gg 1$, or, in case $\partial \Omega$ is $\Theta(\tau)$-characteristic at $p_0$, if $dd^c \psi(\tau)(p_0) > 0$. \qed

**Remark 8.10.** We observe that strict 1-convexity at $p_0$ implies that $\partial \Omega$ is $\Theta$-non-characteristic at $p_0$. 

9. Cauchy problem for CR functions - Existence

In this section we will investigate properties of CR functions on CR manifolds satisfying weak 2-pseudo-concavity assumptions.

**Proposition 9.1.** Let $\Omega$ be an open subset of a CR manifold $M$ enjoying property $\Psi^{\text{w}}(2)$. Assume that $p_0$ is a smooth, strongly-1-convex, $\Theta$-non-characteristic point of $\partial \Omega$. Then, for every relatively compact open neighborhood $U$ of $p_0$ in $M$, we can find an open neighborhood $U'$ of $p_0$ in $U$ such that

\begin{equation}
|f(p)| \leq \sup_{U \cap \bar{\Omega}} |f|, \forall p \in U' \cap \Omega, \forall f \in \mathcal{O}_M(\Omega) \cap \mathcal{C}^2(\bar{\Omega}),
\end{equation}

and strict inequality holds if $f$ is not a constant on $U'$ if $U' \cap \Omega$.

**Proof.** We can assume that $\Omega$ is locally defined near $p_0$ by a real valued $\rho \in \mathcal{C}^\infty(U)$:

\[ U \cap \Omega = \{p \in U \mid \rho(p) < 0\}, \quad \text{and} \quad \exists Z \in \Theta(U) \text{ s.t. } (Zp)(p_0) \neq 0. \]

To make local bumps of $\partial \Omega$ near $p_0$, we fix smooth coordinates $x$ centered at $p_0$, that we can take for simplicity defined on $U$, and for a nonnegative real valued smooth function $\chi(t) \in \mathcal{C}^\infty_{0}(\mathbb{R})$, equal to 1 on a neighborhood of 0, set $\phi_\epsilon(p) = e^{-1/\epsilon} \chi(|x|/\epsilon)$. Then we consider the domains

\[ U_\epsilon = \{p \in U \mid -\phi_\epsilon(p) < \rho(p) < 0\}. \]

There is $\epsilon_0 > 0$ such that $U_\epsilon \subset U$ and the points of $N_\epsilon' = \partial U_\epsilon \cap \Omega$ are smooth and $\Theta$-non-characteristic for all $0 < \epsilon \leq \epsilon_0$. In fact $N_\epsilon'$ is a small deformation of $N_\epsilon = \{\phi_\epsilon > 0\} \cap \partial \Omega$, which is smooth and $\Theta$-non-characteristic for $0 < \epsilon \ll 1$.

We claim that, for sufficiently small $\epsilon > 0$, the modulus $|f|$ of any function $f \in \mathcal{O}_M(U_\epsilon') \cap \mathcal{C}^2(U_\epsilon')$ attains its maximum on $N$. We argue by contradiction. If our claim is false, then for all $0 < \epsilon \leq \epsilon_0$ we can find $p_\epsilon \in N_\epsilon'$ and $f_\epsilon \in \mathcal{O}_M(U_\epsilon') \cap \mathcal{C}^2(U_\epsilon')$ with $|f(p_\epsilon)| > |f(p)|$ for all $p \in U_\epsilon'$. In fact $\Psi^{\text{w}}(2)$ implies the maximum modulus principle and therefore the maximum of $|f_\epsilon|$ is attained on the boundary of $U_\epsilon'$. By Proposition 5.15 this implies that there is $\tilde{x}_\epsilon \in H_{\mathcal{M},p_\epsilon}(U_\epsilon')$ such that $\xi_\epsilon^{N_\epsilon'} \geq 0$. By the strong-1-convexity assumption, there is $\tau_0 \in [\ker \mathcal{L}]_{d\rho^{1,p_0}}$ (see Notation 7.1) such that $dd^c \rho(\tau_0) > 0$. For $\epsilon_\nu \to 0$, the sequence $\{p_\epsilon\}$ converges to $p_0$. We can take a function $\tilde{\rho} \in \mathcal{C}^\infty(U)$ such that $\tilde{\rho}$ agrees to the second order with $(\rho + \phi_\epsilon)$ at $p_\epsilon$, for all $\nu$, and with $\rho$ at $p_0$.

We obtain a contradiction, because $\tau_0$ belongs to $[\ker \mathcal{L}]_{d\rho^{1,p_0}} = [\ker \mathcal{L}]_{d\rho^{1,p_0}}$ and therefore, by $\Psi^{\text{w}}(2)$, is a cluster point of a sequence of elements $\tau_\nu, \epsilon_\nu \in [\ker \mathcal{L}]_{d\rho^{1,p_\nu}} = [\ker \mathcal{L}]_{d(\rho + \phi_\nu)^{1,p_\nu}}$ and $dd^c \tilde{\rho}(\tau_\nu) = dd^c (\rho + \phi_\nu)(\tau_\nu) \leq 0$ by Proposition 5.15 and Corollary 5.8. In fact, $dd^c (\rho + \phi_\nu)(\tau_\nu) \to dd^c (\rho)(\tau_0)$ when $\tau_\nu \to \tau_0$.

**Theorem 9.2.** Let $\Omega$ be an open subset of a CR manifold $M$ enjoying property $\Psi^{\text{w}}(2)$ and $N$ a relatively open subset of $\partial \Omega$, consisting of smooth, strongly-1-convex, $\Theta$-non-characteristic points. If $M$ is locally CR-embeddable at all points of $N$, then we can find an open neighborhood $U$ of $N$ in $M$ such that for every $f_0 \in \mathcal{O}_M(N)$ there is a unique $f \in \mathcal{O}_M(U \cap \Omega) \cap \mathcal{C}^\infty(U \cap \Omega)$ with $f = f_0$ on $N$.

**Proof.** The result easily follows from the approximation theorem in [7] and the estimate of Proposition 9.1. \qed
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M. Nacinovich: Dipartimento di Matematica, II Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma (Italy)
E-mail address: nacinovi@mat.uniroma2.it

E. Porten: Department of Mathematics, Mid Sweden University, 85170 Sundsvall, Sweden, and, Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Poland
E-mail address: Egmont.Porten@miun.se