On general relation between quantum ergodicity and fidelity of quantum dynamics

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General relation is derived which expresses the fidelity of quantum dynamics, measuring the stability of time evolution to small static variation in the Hamiltonian, in terms of ergodicity of an observable generating the perturbation as defined by its time correlation function. Fidelity for ergodic dynamics is predicted to decay exponentially on time-scale \( \propto \delta^{-2} \), \( \delta \sim \) strength of perturbation, whereas faster, typically gaussian decay on shorter time scale \( \propto \delta^{-1} \) is predicted for integrable, or generally non-ergodic dynamics. This surprising result is demonstrated in quantum Ising spin-1/2 chain periodically kicked with a tilted magnetic field where we find finite parameter-space regions of non-ergodic and non-integrable motion in thermodynamic limit.

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The quantum signatures of various types of classical motion, ranging from integrable to ergodic, mixing and chaotic, are still lively debated issues (see e.g. [4]). Most controversial is the absence of exponential sensitivity to variation of initial condition in quantum mechanics which prevents direct definition of quantum chaos [3]. However, there is an alternative concept which can be used in classical as well as in quantum mechanics [5]: One can study stability of motion with respect to small variation in the Hamiltonian. Clearly, in classical mechanics this concept, when applied to individual trajectories, is equivalent to sensitivity to initial conditions. Integrable systems with regular orbits are stable against small variation in the Hamiltonian (the statement of KAM theorem), whereas for chaotic orbits varying the Hamiltonian has similar effect as varying the initial condition: exponential divergence of two orbits for two nearby chaotic Hamiltonians.

The quantity of the central interest here is the fidelity of quantum motion. Consider a unitary operator \( U \) being either (i) a short-time propagator, or (ii) a Floquet map \( U = \mathcal{T} \exp(-i \int_0^\tau d\tau H(\tau)/\hbar) \) of (periodically time-dependent) Hamiltonian \( H \) \((H(\tau + p) = H(\tau))\), or (iii) a quantum Poincaré map. The influence of a small perturbation to the unitary evolution, which is generated by a hermitean operator \( A \), \( U_\delta = U \exp(-i A \delta) \), \( \delta \) being a small parameter, is described by the overlap \( \langle \psi_\delta(t) | \psi(t) \rangle \) measuring the Hilbert space distance between exact and perturbed time evolution from the same initial pure state \( | \psi(t) \rangle = U^\dagger | \psi \rangle \). \( \langle \psi_\delta | \psi \rangle = U_\delta^\dagger | \psi \rangle \), where integer \( t \) is a discrete time (in units of period \( p \)) [6]. This defines the fidelity

\[
F(t) = \langle U_\delta^{-t} U^t \rangle,
\]

where the average is performed either over a fixed pure state \( \langle . \rangle = \langle \psi | . | \psi \rangle \), or, if convenient, as a uniform average over all possible initial states \( \langle . \rangle = (1/N) \text{tr} \langle . \rangle \), \( N \) being the Hilbert space dimension. The quantity \( F(t) \) has already raised considerable interest, though under different names and interpretations: First, it has been proposed by Peres [3] as a measure of stability of quantum motion. Second, it is the Loschmidt echo measuring the dynamical irreversibility of quantum phases, used e.g. in spin-echo experiments [7] where one is interested in the overlap between the initial state \( | \psi \rangle \) and a state \( U_\delta^{-1} U^t | \psi \rangle \) obtained by composing forward time evolution, imperfect time inversion with a residual interaction described by the operator \( A \delta \), and backward time evolution. Third, the fidelity has become a standard measure characterizing the loss of phase coherence in quantum computation [8]. Fourth, it was used to characterize “hypersensitivity to perturbation” in related studies [7], though in different contexts of stochastically time-dependent perturbation.

The main result of this paper is a relation of the fidelity to ergodic properties of quantum dynamics, more precisely to the time autocorrelation function of the generator of the perturbation \( A \). Quantum dynamics of finite and bound systems has always a discrete spectrum since the effective Hilbert space dimension \( N \) is finite, hence it is non-ergodic and non-mixing [9]: time correlation functions have fluctuating tails of order \( \sim 1/N \).

In order to reach genuine complexity of quantum motion with possibly continuous spectrum one has to enforce \( N \to \infty \) by considering one of the following two limits: quasi-classical limit of effective Planck’s constant \( \hbar \to 0 \), or thermodynamic limit (TL) of number of particles, or size \( L \to \infty \). Our result is surprising in the sense that it predicts the average fidelity to exhibit exponential decay on a time scale \( \propto \delta^{-2} \) for ergodic systems (i.e. such that the integrated time auto-correlation of \( A \) is finite), but much faster, typically gaussian decay on a shorter time scale \( \propto \delta^{-1} \) for integrable and general non-ergodic systems (i.e. such that time averaged auto-correlation of \( A \) is non-vanishing). Our theory on fidelity is very general and can be extended to any perturbed unitary evolution, either in quantum, quasi-classical, or even classical (Liouvillean) context. In this paper we apply it to the quantum many-body problem in TL, in particular in the Kicked Ising model (KI), namely the Ising spin 1/2 chain periodically kicked with a tilted homogeneous magnetic field. KI is particularly interesting since it possesses parameter-space regions with positive measure of non-ergodic behavior in TL surrounding the integrable cases [10] of vanishing measure, which is an additional evidence for a conjecture [8] on existence of intermedi-
ate, non-integrable and non-ergodic quantum motion of disorderless interacting many-body systems in TL.

We start by rewriting the fidelity (1) in terms of the Heisenberg evolution of the perturbation $A_t := U^{-t} A U^t$
\[ F(t) = \langle e^{iA_0}\delta e^{iA_1}\delta \ldots e^{iA_{t-1}}\delta \rangle = \tilde{T}(\prod_{t'=0}^{t-1} \exp(iA_t\delta)) \]  
which is achieved by $t$ insertions of the unity $U^{-t'} U^{t'}$ and recognizing $U^{-t'-1}U^{t'} = \exp(i\delta A_{t'-1})$. $\tilde{T}$ is a left-to-right time ordering. Next we make an expansion in $\delta$
expressing the fidelity in terms of correlation functions
\[ F(t) = 1 + \sum_{m=1}^{\infty} \frac{im m^2}{m!} T \sum_{t_1,t_2 \ldots t_m=0}^{t-1} \langle A_{t_1} A_{t_2} \ldots A_{t_m} \rangle. \]  
Being interested mainly in the absolute value $|F(t)|$ we will in the following choose perturbations with vanishing first moment $a := (1/t) \sum_{t'=0}^{t-1} A_{t'} = 0$ so that the series starts at $m = 2$, since a shift by a multiple of unity $A \rightarrow A + a I$ simply rotates the fidelity $F(t) \rightarrow \exp(-i\delta\delta) F(t)$. On the other hand, we can eliminate not only the first, $m = 1$, but all odd orders in the expansion (3) by considering the symmetrized fidelity $F(t) = \langle (U^{t/2} U^{-t/2})^m \rangle$. To second order in $\delta$ we have
\[ F(t) = 1 - \frac{\delta^2}{2} \sum_{t'=-t}^t (t - |t'|) C_A(t') + O(\delta^3), \]  
where it is assumed that 2-point time correlation function is homogenous $C_A(t' - t) := \langle A_{t'} A_t \rangle$, as is the case for uniform average over initial states $\langle \cdot \rangle = \text{tr} \langle \cdot \rangle / N$. Eq. (4) reveals a simple general rule: the stronger correlation decay, the slower is decay in fidelity, and vice versa. Below we discuss two different cases in the limit $N \rightarrow \infty$:

1. **Ergodicity and fast mixing.** Here we assume that $C_A(t) \rightarrow 0$ sufficiently fast that the total sum converges, $S_A := (1/2) \sum_{t=-\infty}^{\infty} C_A(t) = |S_A| < \infty$. For times $t$ much larger than the so-called mixing time scale $t \gg t_{\text{mix}}$ which effectively characterizes the correlation decay, e.g. $t_{\text{mix}} = \sum_{t} |t C_A(t)|^2 \sum_{t'} |C_A(t')|$; it follows that the fidelity drops linearly in time $F_e(t) = 1 - t/\tau_e + O(\delta^3)$ on a scale
\[ \tau_e = S_A^{-1} \delta^{-2}. \]  

In order to show even stronger result we further assume fast mixing with respect to product observables $B_t := A_t A_s$ with $\langle B_{t'} \rangle = C_A(t' - t)$, of order $k \geq 2$, namely $B_{t_1 t_2} B_{t_3 t_4} \ldots B_{t_{2k-1} t_{2k}} \rightarrow \prod_{s=1}^{2k-1} \langle B_{t_{2s-1} t_{2s}} \rangle$ as $t_{1,2,3,4,\ldots}$ are ordered and $t_{2j+1} - t_{2j} \rightarrow \infty$. Therefore, the leading contribution for large $t$ to each $m$-term of (3) comes from sequences $(t_1, t_2, \ldots, t_m)$ where consecutive pairs $(t_{2j-1}, t_{2j})$ are close to each other, $t_{2j} - t_{2j-1} \sim t_{\text{mix}}$. Since for odd $m$ time indices cannot be paired these terms should vanish asymptotically (as $t \rightarrow \infty$) relatively to even $m$ terms. Thus we can evaluate $(2k - 1)!$ equivalent even $m = 2k$ terms in Eq. (3) as $k$-tuple of independent sums over $t_j' = t_{2j} - t_{2j-1}$ giving, for $t \gg t_{\text{mix}}$
\[ F_e(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k - 1)!}{2k!} \delta^{2k} S_A^{2k} \exp(-t/\tau_e). \]  
Note that formulae (3,4) remain valid in a more general case of inhomogeneous time correlations where one should take $S_A := \lim_{t \rightarrow \infty} (1/t) \sum_{t'=0}^{t} \langle B_{t'} \rangle$.

II. **Non-ergodicity.** Here we assume that autocorrelation function of the perturbation does not decay asymptotically but has a non-vanishing time-average, $D_A := \lim_{t \rightarrow \infty} (1/t) \sum_{t'=0}^{t} \langle C_A(t') \rangle$, though the first moment is vanishing $\langle A \rangle = 0$. For times $t$ larger than the averaging time $t_{\text{ave}}$ in which a finite time average effectively relaxes into the stationary value $D_A$, we can write fidelity to second order which decays quadratically in time, $F_{ne}(t) = 1 - (1/2) (t/t_{\text{ave}})^2 + O(\delta^2)$, on a scale
\[ t_{\text{ave}} = D_A^{-1/2} \delta^{-1}. \]  
More general result can be formulated in terms of a time averaged operator $\bar{A} := \lim_{t \rightarrow \infty} (1/t) \sum_{t'=0}^{t} A_{t'}$, namely for $t \gg t_{\text{ave}}$ Eq. (3) can be rewritten as
\[ F_{ne}(t) = 1 + \sum_{m=2}^{\infty} \frac{i m m^2}{m!} \langle \bar{A}^m \rangle = \langle \exp(i\bar{A} t) \rangle. \]  

Global behavior of $F_{ne}(t)$ for non-ergodic systems, where higher $m$-terms of (3) become important, depends generally on the full sequence of moments $\langle \bar{A}^m \rangle$. We argue below, by giving an example of spin 1/2 chains, that there are large classes of perturbing operators where these moments can be shown to possess normal gaussian behavior, yielding Eq. (4). Non-ergodic behavior is certainly present for generic observables in completely integrable systems where a sequence of conservation laws can be used to estimate the time-averaged correlator $D_A$ (1), but we wish to make a stronger statement, namely that there is a generic regime of intermediate dynamics in non-integrable systems displaying non-ergodic behavior.

Let us now apply our theory to quantum spin-1/2 chains described by Pauli operators $\sigma_j$ on a periodic lattice of size $L$, $j = 0, \ldots, L - 1$, acting on a Hilbert space of dimension $N = 2^L$, fix the average $\langle \cdot \rangle = \text{tr} \langle \cdot \rangle / N$, and assume that our Floquet-operator $U$ is translationally invariant (TI) on a lattice. It is useful to introduce a set of local TI observables $Z_{2j} = L^{-1/2} \sum_{k=0}^{L-1} \sigma_{2j+k} \sigma_{2j+k+1} \cdot \cdot \cdot \sigma_{2j+n}$, of order $n \ll L$, where $Z_{2j} = [s_{j}, s_{j+1}, \ldots, s_{j+n}]$, $s_j \in \{0, x, y, z\}$, $1 \leq j \leq n - 1$, and $\sigma_0 := 1$. Using $\langle \sigma_\alpha \sigma_\beta \rangle = \delta_{\alpha \beta} \delta_{x,y,z}$ one may derive a contraction formula
\[ \langle Z_{2j} Z_{2j+2} \ldots Z_{2j+n} \rangle = \sum_{\alpha, \beta} \prod_{k=1}^{n} \delta_{\alpha, \beta} + O(L^{-1}), \]  
while for odd number $\langle Z_{2j} Z_{2j+2} \ldots Z_{2j+n} \rangle = O(L^{-1})$, hence $Z_{2j}$ become independent gaussian field variables.
in TL depending on a multi-index \( s \) of variable but finite length. Therefore, any TI pseudo-local (PL) observable \( A \), having by definition [8] \( l^2 \)-expansion in the basis \( Z_n \) (when \( L = \infty \)), namely \( A = \sum_a a_a Z_a \),

\[
\langle A^2 \rangle = \sum_a a_a^2 < \infty.
\]

possesses normal gaussian moments \( \langle A^2 k \rangle = (2k - 1)!!(A^2)^k(1 + O(L^{-1})) \). Further, for a general TI PL observable \( A \), its time average \( \bar{A} \) is also TI PL, since it can be formally expanded in terms of \( Z_n \) due to construction of \( A \), and such expansion is \( l^2 \) since \( \langle \bar{A}^2 \rangle = \langle AA \rangle = D_A < \langle A^2 \rangle [12] \). However, for a more general non-TI PL observable \( A \), i.e. such that its linear projection to the space of TI observables

\[
(1/L) \sum_{n=0}^{L-1} A|\sigma_j=\sigma_{j+n}
\]

is PL, one cannot generally show that \( \bar{A} \) is TI PL although we believe that this is a typical situation, which we can prove in two cases: (i) If the spectrum of propagator \( U \) is non-degenerate (for any finite \( L \)), then the matrix of \( \bar{A} \) is diagonal in the eigenbasis of \( U \) and \( \bar{A} \) is TI due to Bloch theorem. (ii) If the system is integrable having a complete set of TI PL conservation laws \( Q_n \), \( n = 1, 2 \ldots \) in the sense that \( \{Q_n\} \) is a complete set of eigenvectors of the Heisenberg map \( UA = U^\dagger AU \) for eigenvalue 1 then the time average is a projection \( \bar{A} = \sum_n \langle Q_n|A|Q_n \rangle (\text{assuming that} \langle Q_n|Q_m \rangle = \delta_{sm}) \) which is TI PL. This is the case for KI model studied below. Finally, assuming either (i), (ii), or simply TI PL perturbation \( A \), we find that moments of time-average \( \bar{A} \) are gaussian \( \langle A^{2k} \rangle = (2k - 1)!!D_A^k(1 + O(L^{-1})) \). Summing up the formula [8] produces gaussian decay

\[
F_{\text{ne}}(t) = \exp\left(- (t/t_{\text{ave}})^2/2\right),
\]

for \( t \gg t_{\text{ave}} \), on a time scale [8], which can be computed in a typical integrable situation (ii) as shown below.

Few remarks on the case of finite dimension \( N < \infty \) are in order: (1) \( F(t) \) will then start fluctuating around zero with magnitude \( F_{\text{fluct}} = N^{-1/2} \) for very long times \( t > t^*(N) \) where the time scale \( t^*(N) \) is determined from the condition \( F(t^*)|_{N=\infty} = N^{-1/2} \). (2) \( F(t) \) decays all the way down to \( N^{-1/2} \) only for a typical or random initial state \( \psi \) with \( \sim N \) non-vanishing components when expanded in the eigenbasis of \( U \), or for an average over \( \psi \). If on the other hand one considers the initial state which, when expanded either in the eigenbasis of \( U \) or of \( U_\delta \), contains essentially only few, say \( m \) dominating components, like the regular coherent state of Peres [8], then \( F(t) \) is a quasi-periodic function with \( m \) small frequencies \( \sim \delta \) and amplitudes \( \sim 1/m \). (3) Even in asymptotically ergodic situation the correlation \( C_A(t) \) has a plateau for finite \( N \), which can be estimated using random matrix model for the propagator \( U^t \) as \( D_A \sim D_A^0(N) := c_A/N \) where \( c_A \) is some constant with respect to \( N \). The non-vanishing correlation plateau gives a dominant contribution to Eq. (4) resulting in a quadratic (or gaussian) decay of \( F(t) \) as soon as \( t_{\text{ave}} > S_A|_{N=\infty}/D_A^0 \), i.e. when \( \delta < \delta_0(N) := S_A^{-1} c_A^2 N^{-1/2} \). This perturbative regime of very small perturbation strength, existing for finite \( N \) only, is consistent with the first order perturbation expansion of eigenstates of \( U_\delta \) in terms of the eigenbasis of \( U [13] \).

Consider an example of KI model with the hamiltonian

\[
H_{\text{KI}}(t) = \sum_{j=0}^{L-1} \{J_z \sigma^+_j \sigma^-_{j+1} + \delta_p(t)(h_x \sigma^+_j + h_z \sigma^-_j)\}
\]

where \( \delta_p(t) = \sum_m \delta(t - mp) \), with a Floquet-map \( U = \exp(-iJ_z \sum_{i} \sigma^+_i \sigma^-_{i+1}) \exp(-i \sum_j (h_x \sigma^+_j + h_z \sigma^-_j)) \), where we take units such that \( \hbar = 1 \), depending on a triple of independent parameters \( (J_z, h_x, h_z) \). KI is integrable for longitudinal \( h_x = 0 \) and transverse \( h_z = 0 \) fields [10], and has finite parameter regions of ergodic and non-ergodic behaviors for a tilted field (see Fig. 1). The non-trivial integrability of a transverse kicking field, which somehow inherits the solvable dynamics of its well-known autonomous version [14], is quite remarkable since it was shown [10] that the Heisenberg dynamics can be calculated explicitly for observables which are bilinear in fermi operators \( c_j = (\sigma^+_j - i \sigma^-_j) \Pi_{j+1}^{\infty} \sigma^+_j \), with time correlations decaying to the non-ergodic stationary values as \( |C_A(t) - D_A| \sim t^{-3/2} \) [15]. For \( D_A \) we find explicit expressions, the simplest,
\[ D_{M} = LD_{\sigma z} \]

and \( D_M = LD_{\sigma z} \), for the component of spin \( \sigma_z \), and the component of magnetization \( M = \sum \sigma_j \), respectively.

In a general situation of non-integrable KI we wish to test our theory by a numerical experiment. We consider a line in 3d parameter space with fixed \( J = 1, h_x = 1.4 \) and varying \( h_z \) exhibiting all different types of dynamics: (a) \( h_z = 0 \) integrable, (b) \( h_z = 0.4 \) intermediate (non-integrable and non-ergodic), and (c) \( h_z = 1.4 \) ergodic and mixing. In all cases we fix the operator \( A = M \) which generates the perturbation of KI model with \( h_x \rightarrow h_x + (h_x^2 + h_x^2 \cot h) \delta / h^2 + O(\delta^2), h_z \rightarrow h_z + h_x h_z (1 - h \cot h) \delta / h^2 + O(\delta^2) \), where \( h = h_x^2 + h_z^2 \), and vary \( L \) and \( \delta \). Since we want the perturbation strength to be size \( L \)-independent we scale it by fixing \( \delta' = \delta \sqrt{L/L_0} \) where \( L_0 := 24 \). Time evolution has been computed efficiently by iterating the factored Floquet map (in terms of 1-spin and 2-spin propagators - ‘quantum gates’), requiring \( \propto L^2 \) computer operations per iteration per initial state. In integrable case (a) we confirm saturation of correlations to the theoretical value \( D_M = 0.485126 \times L \) (Fig. 1a), as well as gaussian decay of fidelity \( |F(t)| \) with time-scale \( \tau_{\text{ne}} \) given by \( \tau_{\text{ne}} \approx (\ln N)^{1/2} \) (Fig. 2a) In non-integrable (intermediate) case (b), we find persisting non-ergodic and non-mixing behavior since rescaled correlation functions of typical observables \( C_A(t)/\langle A^2 \rangle \) relax on a short \( L \)-independent time scale to a non-vanishing value \( D_A/\langle A^2 \rangle \) and converge to TL very quickly with increasing size \( L \) (Fig. 1b), but as opposed to integrable case (a) the relaxation appears to be exponential \( |C_M(t) - D_M|/L \approx \exp(-t/t_{\text{mix}}) \) with \( t_{\text{mix}} \approx 7.2 \) (inset 1b). Such behavior has been observed for other two components of the magnetization \( M^y, M^z \) and supports existence of intermediate dynamics observed previously in kicked t-V model \( \delta \). In Fig. 2b we confirm gaussian decay of \( F(t) \) predicted \( \delta \) from numerically observed value of \( D_M = 0.293 \times L \), again up to time \( t_{\text{ne}} \) (2e). In ergodic case (c) we find fast decay of correlation functions fitting well to an exponential \( |C_M(t)|/L \approx \exp(-t/t_{\text{mix}}) \), with \( t_{\text{mix}} \approx 6.0 \). Consequently we find exponential decay of \( F(t) \) of eqs. \( \delta \) using \( S_z = (1/2) \sum C_M(t) \approx 2.54 \times L \), up to the saturation time \( t_{\text{ne}} = (1/2) \tau_{\text{ne}} \ln \mathcal{N} \) (Fig. 2c).

In conclusion, we have presented a simple theory for the stability of quantum motion with respect to a static perturbation of the evolution operator in the limit of Hilbert space dimension \( \mathcal{N} \rightarrow \infty \), characterized by the fidelity measuring the distance between time evolving states. The fidelity was expressed in terms of integrable time-correlation functions of the perturbing operator, showing that faster decay of correlations gives slower decay of fidelity, meaning that ‘chaotic’ dynamics is more stable in Hilbert space than ‘regular’ one (unless the state that one is looking at is simply related to the eigenstates of the system)! In the two limiting cases of mixing and integrable (or more generally, non-ergodic) dynamics we find, respectively, exponential and gaussian decay. For example, our finding has strong implication for the stability of quantum computation with respect to static imperfections (e.g. uncontrollable residual interaction among qubits) \( \delta \). In other words, Eq. \( \delta \) is a version of the fluctuation-dissipation formula for the ‘dissipation coefficient’ \( 1/\tau_{\text{ne}} \) of Eq. \( \delta \) which diverges in non-ergodic regime. If the system has a well defined classical limit then our formula \( \delta \) has a clear and simple classical limit \( h \rightarrow 0 \) too, with an integrated classical autocorrelation function substituting the quantum one \( \delta \). We speculate that our finding is a manifestation of “the structural invariance” \( \delta \) of quantum chaotic dynamics. Although in this paper our theory has been demonstrated in a specific kicked many-body problem, namely the quantum kicked Ising spin 1/2 chain, we should emphasize that it should be generally valid (within the time and perturbation scales depending on the Hilbert space dimension) and thus applicable to any unitary evolution, in particular also to any experimentally interesting quantum dynamics.
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