Hydrodynamics and transport coefficients in an infrared-deformed soft-wall AdS/QCD model at finite temperature

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Abstract

We extend an infrared-deformed soft-wall anti de-Sitter/QCD model at zero temperature to a model at finite temperature and perform hydrodynamics. To have the infalling boundary condition to make the hydrodynamic analysis possible, we treat the infrared energy scale factor in our metric as a temperature-depending parameter. Then, by carrying out the hydrodynamic analysis, we compute the transport coefficients, the diffusion constant, and the shear viscosity through the linear response theory.
1 Introduction

To carry out the analysis in the strongly coupled region would be the key of many important unsolved problems in contemporary high-energy theoretical physics. One of these problems is the low-energy dynamics of QCD. Actually, many ideas for the nonperturbative analysis of QCD have been proposed. Among these, there are two interesting classes, the lattice gauge theory [1] and holography [2, 3] (gauge/gravity correspondence, anti-de-Sitter (AdS)/QCD, etc.

In the lattice gauge theory, Minkowski space-time in the original theory is replaced with a finite volume Euclidian discretized space-time with an analytic continuation for the time direction to the imaginary time direction. As a result, the degree of freedom of theories becomes finite, and the nonperturbative numerical analysis for the action itself, which is a Monte Carlo simulation, becomes available. However, Monte Carlo simulation is plagued by a notorious problem named as the sign problem, when fermions are involved in the Monte Carlo. Currently, no fundamental means to overcome the sign problem has been invented yet, and the current analyses in the lattice are always carried out by getting around it [4].

On the other hand, it is well known that the holography is a duality between the strongly coupled field theories and the weakly coupled gravities. It originates in the superstring theory, in which the quantized open and closed strings at low energy can be identified with particles known in field theories and gravities [5]. As a result, behaviors of the low-energy open strings describe the supersymmetric gauge theory. Then, as an important matter in the correspondence, the $U(N)$ supersymmetric gauge theory arises on $N$ overlapped D-branes from the open strings sticking to these $N$ overlapped D-branes at low energy. On the other hand, D-branes can be identified with black branes in the supergravity [6].

Thus, one has two ways to describe a low-energy D-branes. As a result, it is known that one can conjecture the duality between $p$-dimensional large-$N$ supersymmetric $SU(N)$ gauge theory with large ’t Hooft coupling and near-horizon geometries of the black $p + 1$-brane in the condition that the quantum effect of gravity and the length of the string can be neglected. Although there is no exact proof for this correspondence at this moment, it is particularly expected that the duality between the $\mathcal{N} = 4$ four-dimensional large-$N$ $SU(N)$ supersymmetric gauge theory [2, 3] and the five-dimensional anti-de-Sitter space is valid.

One of the great advantages in the gauge/gravity correspondence compared with the lattice gauge theory would be that it is irrelevant to the problem arising when one involves fermions like the sign problem, because the main analyses are carried out in analytic ways in the weakly coupled gravity side.

However, the current gauge/gravity correspondence also has a problem. It is that the dual gauge
theories are always nonrealistic as long as the gravity side is a solution (top-down model). Because of this, many results in the field theory side in the current gauge/gravity correspondence are the ones independent of detail of theories or no more than qualitative ones just in supersymmetric models. On the other hand, once getting away from the study based on a solution in the gravity side, constructing holographic models in bottom-up way is also conducted energetically [7–9].

Anyway, the point that the gauge/gravity correspondence can be irrelevant to the notorious problem in the treatment of fermions in the lattice gauge theory would be one of the great advantages. For this reason, the low-energy dynamics of QCD has recently been studied intensively in the framework of the gauge/gravity correspondence, and this is the motivation of the soft-wall AdS/QCD model, which is a kind of the holographic bottom-up model [8, 9].

Recently, we have done the extension of an IR-deformed AdS/QCD model [10] to the finite temperature system [11,12], in which the deformed bulk vacuum and potential term have been introduced for the scalar field to satisfy the equations of motion. This is because if one straightforwardly extends the model to the finite temperature system, it turns out that the solution of the equation of motion diverges due to the dilaton. Only by deforming the bulk vacuum and potential term, one can obtain the smooth solution for the dilation. With such a treatment, we have examined the critical temperature of chiral symmetry breaking [11] through the analysis of the quark number susceptibility and the meson spectrum [12]. In the analysis of the meson spectrum, we have carried out the numerical analysis of the equation of motion for the fluctuations on the bulk gravity. Such a numerical analysis for the mass spectrum can be considered as a basic method of analyses in holographic QCD as well as the hydrodynamics.

One of the crucial points of the hydrodynamics is that it can be considered to be independent of the detail of theories, and the transport coefficients are also so independent of the detail of theories, for which the transport coefficients are the ones defined in the framework of the hydrodynamic analysis. In particular, the ratio between the shear viscosity ($\eta$) and the entropy density ($s$) $\eta/s$ is the quantity characterizing the actual QCD. For this reason, the ratio $\eta/s$ has been examined very much in the gauge/gravity correspondence [13]. Besides, the holographic hydrodynamics has played an important role in the long-standing problem in the causal hydrodynamics [14], and quark-gluon plasma described by the Bjorken flow [15].

For such circumstances, turning to the hydrodynamics, in this paper, we are going to work out the holographic hydrodynamics in an IR-deformed AdS/QCD model at finite temperature studied [11, 12], to perform the interesting studies mentioned above in the future.

Now, we would like to mention the organization of this paper. In Sec\textsuperscript{2} we review the hydrodynamics
and the transport coefficients obtained from the linear response theory. In Sec.2, we will introduce our holographic model and show how the model is extended to the finite temperature system. It is shown that, if one takes the same way as in Ref. [12], it turns out that the analysis becomes too complicated to be carried out. Therefore, in this paper, we will propose another way, which is simpler than the one in Ref. [12]. Then, it will be seen that there are four options in the numerical calculations for some factors, while only one of them is physically acceptable. In Sec.4, we will sort out the notation used in this paper. In Sec.3, we will carry out the hydrodynamic analysis for the fluctuation of the baryon current in the scalar mode. In the analysis, a necessity to cancel the divergence at the horizon arises as usual. Our model involves the dilaton which makes the analysis complicated. However, we will show how to contain it by exploiting the integral constant. Then, using the Gubser, Klebanov, Polyakov, and Witten relation (using GKP-W) relation [3], we will read out the retarded Green function for the $U(1)$ baryon current in the scalar mode and its diffusion constant. In Sec.4, we will carry out the hydrodynamic analysis for the fluctuation of the baryon current in the scalar mode and its diffusion constant. In Sec.6, we will carry out the hydrodynamic analysis for the fluctuation of the baryon current in the vector mode as well as in Sec.5. Then, using the GKP-W relation [3], we will read out the retarded Green function and evaluate the ratio between the viscosity ($\eta$) and entropy density ($s$) as $\eta/s$. Our summary and conclusions will be presented in Sec.8.

2 Brief review on hydrodynamics and linear response theory

To begin with, we would like to describe the basic matters in hydrodynamics and the linear response theory used in this paper. The description in this section is basically following the review papers [16].

The hydrodynamics is an effective theory to describe the macroscopic dynamics at large distances and time scales. Conserved quantities are considered to survive in such large distances and time scales, and the energy-momentum tensor $T^{\mu\nu}$ is one of conserved quantities. The hydrodynamics is formulated by the hydrodynamic equation for conserved quantities instead of the action principle. The hydrodynamic equation for the energy-momentum tensor is given as

$$\partial_{\mu}T^{\mu\nu} = 0,$$

(2.1)

where

$$T^{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + \tau^{\mu\nu}$$

(2.2)
with energy density \( \varepsilon \), pressure \( P \), local fluid velocity \( u^\mu \) and the \( \tau^{\mu\nu} \) given as

\[
\tau^{\mu\nu} = -\eta \left( \partial^\mu u^\nu + \partial^\nu u^\mu - \frac{2}{3} \eta^{\mu\nu} \partial_\alpha u^\alpha \right) - \zeta \eta^{\mu\nu} \partial_\alpha u^\alpha.
\] (2.3)

Here \( \eta \) and \( \zeta \) mean shear and bulk viscosities, respectively. In a curved space, it is given as

\[
\tau^{\mu\nu} = -P^{\mu\alpha} P^{\nu\beta} \left\{ \eta \left( \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \nabla u \right) - \zeta g_{\alpha\beta} \nabla u \right\},
\] (2.4)

where \( P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \). In the above, \( \eta \) and \( \zeta \) are regarded as the transport coefficients. In what follows we consider the fluid in the rest frame, \( u^\mu = (1, 0, 0, 0) \).

In Eq. (2.2), substituting \( u^\mu = (1, 0, 0, 0) \) and expanding as \( g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \) [\( \eta^{\mu\nu} = \text{diag}(−1, 1, 1, 1) \) and \( h^{\mu\nu} \) means fluctuations], it turns out that \( \delta \langle \tau^{xy} \rangle \) is given as

\[
\delta \langle \tau^{xy} \rangle = i\omega \eta h^{xy},
\] (2.5)

where we have represented the formula in the momentum space. It is known from the linear response theory that the response of an operator \( O \) for the external field \( \phi(0)(k) \) is given as

\[
\delta \langle O(k) \rangle = -G_{R}^{O} O(k),
\] (2.6)

where \( G_{R}^{O}(k) \equiv -i \int_{-\infty}^{\infty} d^4x e^{-ikx} \langle [O(t, x), O(0, 0)] \rangle \theta(t) \) is the retarded Green function. Then, by comparing Eq. (2.3) with Eq. (2.6), we can obtain the Kubo formula with regard to the shear viscosity as

\[
\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G_{R}^{xy xy},
\] (2.7)

where \( G_{R}^{xy xy}(k) \equiv -i \int_{-\infty}^{\infty} d^4x e^{-ikx} \langle [T^{xy}(t, x), T^{xy}(0, 0)] \rangle \theta(t) \).

Next we turn to the diffusion in the conserved current and the energy-momentum tensor. Toward a conserved current \( j^\mu \) with \( 0 = \partial_\mu j^\mu \), the constitutive equation for the conserved current is given as

\[
\dot{j} = \rho u - D \nabla j^t = -D \nabla j^t,
\] (2.8)

where \( j^t = \rho \) and \( D \) mean the charge density and the diffusion constant respectively, and we have taken account of the fluid rest frame. Then, Fick’s law is valid as

\[
\partial_t \rho - D \nabla^2 \rho = 0.
\] (2.9)

This gives the following dissipation relation for the charge density:

\[
\omega = -iDk^2,
\] (2.10)

which will be the pole in the retarded Green function in the charge density, \( G_{R}^{\rho}(k) \equiv -i \int_{-\infty}^{\infty} d^4x e^{-ikx} \langle [\rho(t, x), \rho(0, 0)] \rangle \theta(t) \). Next, we consider the diffusion in the the energy-momentum tensor \( T^{ti} \) with \( i = x, y \).
First, from the constitutive equation in the curved space, one can obtain

\[ T^{zi} = -\frac{\eta}{\varepsilon + P} \partial_z T^{ti} = -\eta \partial_z u^i. \]  \hspace{1cm} (2.11)

Here, we have arranged the \( k \) along with the \( x^3 \) axis as \( k = (0, 0, k) \) for simplicity. Then, from \( \partial_\mu T^{\mu i} = 0 \), one can obtain

\[ \partial_t T^{ti} - \frac{\eta}{\varepsilon + P} \partial^2 T^{ti} = 0. \]  \hspace{1cm} (2.12)

This gives the dissipation relation in the energy-momentum tensor \( T^{ti} \),

\[ \omega = -i \frac{\eta}{\varepsilon + P} k^2, \]  \hspace{1cm} (2.13)

which will be the pole in the retarded Green function \( G^{ti}_{R ti} \).

Finally, we write down the decomposition of the energy-momentum current and the \( U(1) \) current under the little group \( SO(2) \) toward the \( i \) direction (\( i = x, y \)):

- scalar mode: \( T_{00}, T_{03}, T_{33}, T^i_i \) and \( J_0, J_3 \),
- vector mode: \( T_{0i}, T_{3i} \) and \( J_i \),
- tensor mode: \( T_{ij} - \delta_{ij} T_{kk}/2 \).

3 IR-deformed AdS/QCD model at finite temperature

We will start with the following geometry, which is deformed from Schwarzschild \( AdS_5 \) black hole geometry in the IR-region by a factor \( \mu_g \) as

\[ ds^2 = a^2(z) \left( -f(z) dt^2 + \sum_{i=1}^{3} dx_i^2 + \frac{dz^2}{f(z)} \right) \]  \hspace{1cm} (3.1)

with \( a^2(z) = (1/z^2 + \mu_g^2)/l^2 \) and \( f(z) = 1 - (z/z_0)^4 \) (\( z_0 \) means the location of the horizon) and we have put the AdS radius \( l \) as 1 in what follows. The coordinate \( z \) is in the relation with the usual coordinate \( r \) as \( z = 1/r \). This geometry has Hawking temperature \( T = 1/(\pi z_0) \) and is asymptotically \( AdS_5 \) space-time.

We will consider the following \( U(1) \times SU_L(2) \times SU_R(2) \) soft-wall model with the scalar field on the background (3.1) as

\[ S = \int d^5 x \sqrt{-g} e^{-\Phi(z)} (R + 12) + \int d^3 x \sqrt{-g} e^{-\Phi(z)} \left[ -\frac{1}{4g_{U(1)}^2} F_{MN} F^{MN} + \text{Tr} \left\{ -\frac{1}{4g_{SU(2)}^2} (F_{L,MN} F^{MN}_L + F_{R,MN} F^{MN}_R) + |D_M X(z)|^2 - m_X^2 |X(z)|^2 - \frac{\lambda}{4} |X(z)|^4 \right\} \right], \]  \hspace{1cm} (3.2)
where $m_X^2 = -3$ [7,8] and $l = 1$. The trace is performed for the $SU(2)$ algebra mentioned in what follows.

We write $F_{MN}$, $F_{LMN}$ and $F_{RMN}$ as $F_{MN} \equiv \partial_M A_N - \partial_N A_M$, and $F_{LMN} \equiv \partial_M B_{L,N} - \partial_N B_{L,M} - i[B_{L,M}, B_{L,N}]$, where $B_{L,M} = B_{a,L,M} t^a$ with the $SU(2)$ Lie algebra $t^a (a = 1, 2, 3)$, and now we have skipped describing the $R$ part. As for the dual operators for these in the boundary theory, for example see the table in Ref. [7]. Using these, the covariant derivative can be written as $D_M X = \partial_M X + i(B_{L,M} X - X B_{R,M})$.

We write the bulk vacuum of the scalar field as

$$X(z) = \frac{v(z)}{2} 1_2, \tag{3.3}$$

where $1_2$ means a $2 \times 2$ unit matrix, and $v(z)$ is given in Table 1. $v(z)$ behaves around the boundary as

$$v(z) = m_q \zeta z + \frac{\sigma}{\zeta} z^3 + O(z^5). \tag{3.4}$$

From AdS/CFT correspondence, $m_q$ and $\sigma$ can be interpreted as the quark mass and quark condensate, respectively. As for $\zeta$, see Tables 1 and 2.

| Model | $v(z)$ | Parameters |
|-------|--------|------------|
| IIb   | $z(A + Bz^2)(1 + Cz^4)^{-5/8}$ | $A \equiv m_q \zeta$, $B \equiv \sigma / \zeta$, $C \equiv (B^2 / (\mu_d \gamma^2))^{4/5}$ |

Table 1: These are taken from Ref. [10]. The numerical values for the parameters appearing here are given in Table 2 where these are fixed by minimizing the breaking of the Gell-Mann-Oakes-Renner relation, $f_\pi^2 m_\pi^2 = 2 m_q \sigma$, at the 1% level and the experimental values: $m_\pi = 139.6$ MeV and $f_\pi = 92.4$ MeV.

| Model | $\lambda$ | $m_q$ (MeV) | $\sigma^\frac{1}{2}$ (MeV) | $\gamma$ | $\mu_q$ | $\mu$ |
|-------|-----------|-------------|-----------------|---------|--------|------|
| IIb   | 0         | 4.07        | 272             | 0.112   | 257    | 0.205 |
| IIb   | 9         | 6.79        | 229             | 0.20    | 257    | 0.205 |

Table 2: The numerical values for the parameters appearing in Table 1. These are taken from Ref. [10]. $\mu \equiv \mu_q / (2\pi T)$ is evaluated at $T = 0.2$ GeV; $\zeta = \sqrt{3}/(2\pi)$ and $\mu_q = \sqrt{3} \mu_d$. Here, we notice that we will treat $\mu_q$ as a temperature-depending parameter. This way is different from the way in our previous paper [12]. We will discuss this matter in this section.

Here, we would like to mention the field theory dual to our model (3.2) with the geometry (3.1). Although this is not an exact statement because our model is a bottom-up model and does not stand on the configuration of the D-branes, the dual field theory we will assume in this paper would be $D = 1 + 3$ $SU(N_c)$ gauge theory in large ’t Hooft coupling and the large-$N_c$ limit at finite temperature, which has a $U(1)$ baryon symmetry and $SU(N_f)_L \times SU(N_f)_R$ chiral flavor symmetry with $N_f = 2$ as global symmetries.

We have shown [11] that the quark number susceptibility in the model (3.2) on the geometry (3.1) blows up when the temperature is around $160 \sim 190$ MeV, which is considered to have a relation with
where \( u \) and \( v \) from taking the infalling boundary condition, where the infalling boundary condition is determined at the explicit form for \( v \) since \( \Phi \). We have used the relation \( \lambda \) parameter \( v \).

It will be noticed that in our previous paper [12], we fixed \( X(z) \) with considering a regularized term \( v_1(z) \ln f(z) \) as \( X(z) = \frac{1}{4} \left\{ v_1(z) \ln f(z) + v_0(z) \right\} \). (and in accordance with this extra term, the coupling parameter \( \lambda \) has to be modified), where \( v_0(z) \) in this equation corresponds to \( v(z) \) in this paper. The explicit form for \( v_1(z) \) is referred to in Ref. [12]. As it can be seen that without considering the extra term \( v_1(z) \ln f(z) \), the dilaton \( \Phi'(u) \) in the vicinity of the horizon starts with the order \((u - 1)^{-1}\) as

\[
\Phi'(u) = \frac{\Phi_{-1}(T, \mu)}{u - 1} + \mathcal{O}(1)
\]

with the numerators,

\[
\Phi_{-1}(T, \mu) = -\frac{1}{A(6C(\mu)z_0^2 - 4) - 2Bz_0^2(C(\mu)z_0^2 + 6)} \left[ A \left\{ m_X^2 (4\mu^2 + 1) \left( C(\mu)z_0^4 + 1 \right) - 6C(\mu)z_0^4 + 4 \right\} + Bz_0^2 \left\{ m_X^2 (4\mu^2 + 1) \left( C(\mu)z_0^4 + 1 \right) + 2 \left( C(\mu)z_0^4 + 6 \right) \right\} \right],
\]

where \( u \equiv z^2/z_0^2 \) and \( \mu \equiv \mu_0/(2\pi T) \) [later, they are defined at Eqs. (4.11) and (4.12)], and \( A, B \) and \( C(\mu) \) are given in Table II. Then, it turns out that the contribution \( \Phi_{-1}(T, \mu)/(u - 1) \) appears at the order \((u - 1)^{-1}\) in the equation of motion for the fluctuations at the vicinity of the horizon. Therefore, since \( \Phi_{-1}(T, \mu)/(u - 1) \) is multiplied by \((u - 1)^{-1}\) in the equation of motion, finally the contribution \( \Phi(\mu) \) appears at the order \((u - 1)^{-2}\) in the equation of motion. This prevents the solutions of the fluctuations from taking the infalling boundary condition, where the infalling boundary condition is determined at the
order \((1-u)^{-2}\) in the equations of motion. For this reason, it has been found in Ref. [12] introducing the extra term \(v_1(z) \ln f(z)\) is useful in regularizing the divergence and obtaining the infalling boundary condition healthily.

However, it turns out in this paper that the extra term \(v_1(z) \ln f(z)\) makes the analysis very complicated. Despite this, the reason we performed an analysis in our previous paper [12] was because we used numerical analyses using a shooting method. In this paper we will discard the way of using the extra term \(v_1(z) \ln f(z)\), and we will take \(\mu\) as a temperature-depending parameter \(\mu(T)\) so that one can adjust the parameter \(\mu\) to make the numerator \(\Phi_{-1}(T, \mu)\) vanishes at the order \((1-u)^{-1}\). As a result, we obtain the \(\mu\) which can vanish the numerator \(\Phi_{-1}(T, \mu)\) at each temperature as shown in Fig.1 and finally, we can take the infalling boundary condition.

![Figure 1: We plot the positive and real values of \(\mu\) found as the function of temperature. They are obtained from vanishing the numerator \(\Phi_{-1}(T, \mu)\) at the order \((u-1)^{-1}\) in Eq.(3.9). As a consequence, one can healthily take the infalling boundary condition. The left and right plots are different just in the scale of the x axis. The lower curve (red points) and upper curve (blue points) in each figure represent two branches of the solution for a given temperature.](image)

In Fig.1 one can find that there are two branches for the solution represented by the lower curve (red) and upper curve (blue) for a given temperature. Then, the question is which branch is physically meaningful. Before answering the question, we will make a comment on the relation of \(\mu\) in our previous paper [12] which is shown in the rightmost column of Table 2 and \(\mu\) in this paper given in Fig.1.

For the result of model IIb, the branch starts at \(T_c = 0.2\) GeV, and the value of \(\mu\) at the point at which the branch arises is mostly the same value of \(\mu\) in the rightmost column of Table 2 despite that these two are obtained independently. Then, let us consider which branches we should take.

First, we recall that \(\mu\) is the factor appearing in the factor \(a^2(z)\) in Eq.(3.1) as \(a^2(z) = 1/z^2 + \mu_y^2 = 1/z^2 + (2\pi T \mu)^2\). Then, one can see that, if \(\mu\) remains finite at a high temperature, as the temperature increases, our bulk space-time becomes completely different from the Schwarzschild AdS\(_5\) black hole space-time. It means that if \(\mu\) does not vanish as the temperature goes up, the symmetry in the gravity side
corresponding to the conformal symmetry in the dual field theory side vanishes, and the gauge/gravity correspondence in our paper becomes invalid. On the other hand, if $\mu$ vanishes at a high temperature, as the temperature increases, our bulk space-time goes back to the Schwarzschild $AdS_5$ blackhole space-time, and the symmetry in the gravity side can go back to the symmetry of $AdS_5$. At that time, since our model has the extra terms (dilaton, scalar field and gauge fields) other than the Einstein-Hilbert action, the dual field theory is not the $\mathcal{N} = 4$ supersymmetric gauge theory even if the background geometry is the Schwarzschild $AdS_5$ blackhole space-time. However, it would be a consistent condition as a holographic AdS/QCD model that the background geometry goes back to the Schwarzschild $AdS_5$ blackhole space-time in the high-temperature limit. In this sense, eventually, the physically acceptable branch would be the one represented by the lower curve (red points) which goes to zero numerically in the region above around $T = 0.35$ GeV.

An interesting point is that the effect of $\mu$ does not disappear abruptly but gradually disappears as the temperature increases. The effect of $\mu$ is a factor characterizing our AdS/QCD model, and it would be interesting to study its effect more in the phenomenology in the future. Further, it would also be interesting that the branch starts at about $T = 0.2$ GeV, where the temperature $T_c = 0.2$ GeV is roughly consistent with the critical temperature for the hadron/plasma transition.

4 Preliminaries for hydrodynamic analysis

Before making a hydrodynamic analysis, let us first clarify the indices used in this paper

\begin{align*}
M, N &= x_0, x_1, x_2, x_3, z \text{ or } 0, 1, 2, 3, 4 \\
\mu, \nu &= x_0, x_1, x_2, x_3 \text{ or } 0, 1, 2, 3 \\
i, j, k &= x_1, x_2 \text{ or } 1, 2 \\
\alpha, \beta &= x_1, x_2, x_3 \text{ or } 1, 2, 3 \\
a, b, c &= 1, 2, 3 \text{ [indices for the } SU(2) \text{ algebra]}
\end{align*}

where $x_0$ is coordinate of time; $x_1$, $x_2$ and $x_3$ are coordinates of space on the boundary, and $z$ is radial direction of the bulk.

We are going to examine the retarded Green function for the scalar mode of the $U(1)$ baryon current, the vector mode of the $SU(2)$ flavor current and the shear viscosity from the tensor mode of the energy-momentum tensor through AdS/CFT correspondence. For this purpose, we will analyze the hydrodynamics
in the bulk gravity of the fluctuations,

\[ g_{MN} \rightarrow G_{MN} = g_{MN} + h_{MN}, \quad (4.6) \]
\[ g_{MN} \rightarrow G^{MN} = g^{MN} - h^{MN}, \quad (4.7) \]
\[ A_M \rightarrow B_{0M} = 0 + A_M, \quad (4.8) \]
\[ B^a_M \rightarrow B^a_{0M} = 0 + B^a_M, \quad (4.9) \]

where the background of the gauge fields is vanishing. Without confusing, we simply use the same character for the total gauge fields and the fluctuation of gauge fields. These fluctuations are in linear order in the equations of motion. In this paper, we will treat the dilaton as the static background.

We show the classification of fluctuations in the \( SO(2) \) little group for the \( x \) and \( y \) directions as

- **Scalar mode:** \( h_{00}, h_{03}, h_{33}, h_{0z}, h_{3z}, h_{k}^k \) and \( A_0, A_3, A_z \),
- **Vector mode:** \( h_{0i}, h_{3i}, h_{zi} \) and \( A_i \),
- **Tensor mode:** \( h_{ij} - \delta_{ij}h_{k}^k/2 \).

One may see the correspondence of the above classification with the one given in Sec[2].

In AdS/CFT correspondence, the fluctuations of the bulk gauge field \( A_\mu \) act as the source for the \( R \)-charge current \( J_\mu \) on the dual field theory. Since it is known that the \( U(1) \) baryon number can be regarded as an analog of \( R \) charge in AdS/CFT correspondence, we will consider that \( A_\mu \) acts as the source for the global \( U(1) \) baryon charge current \( J_\mu \) (For example, see Ref. [20]). Further, \( B_{L,\mu} \) and \( B_{R,\mu} \) also act as the source for the global \( SU(N_f)_L \times SU(N_f)_R \) chiral flavor currents in which \( N_f = 2 \) in this paper. The gravitational perturbations in the bulk act as the source for the stress-energy tensor \( T_{\mu\nu} \) on the dual theory.

In our following analysis, for convenience, we will define the normalized radial coordinate

\[ u \equiv \frac{z^2}{z_0^2}, \quad (4.11) \]

and the normalized frequency, momentum and the factor

\[ \omega \equiv \frac{\omega}{2\pi T}, \quad q \equiv \frac{q}{2\pi T} \quad \text{and} \quad u \equiv \frac{\mu g}{2\pi T}. \quad (4.12) \]

Finally, let us mention the gauge fixing condition in our present consideration. Like other papers (for example, Ref. [21]), we choose the axial gauge condition and the Landau gauge as follows

\[ A_z = 0, \quad B^a_z = 0 \quad \text{and} \quad h_{Mz} = 0, \quad (4.13) \]
\[ \partial^\mu A_\mu = 0 \quad \text{and} \quad \partial^\mu B_{L,\mu} = \partial^\mu B_{R,\mu} = 0, \quad (4.14) \]
where the Landau gauge is imposed in the four-dimensional space-time on the boundary.

5 Analysis on the $U(1)$ baryon current

In this section, we will first perform a hydrodynamic analysis in the scalar mode shown in Table 4.10 of the $U(1)$ baryon current. To begin with, we can write the equation of motion for the fluctuation of $A_M$ in terms of the field strength as

$$\partial_M \left( \sqrt{-g(u)} e^{-\Phi(u)} F^{MN}(u) \right) = 0,$$

(5.1)

where $g(u) \equiv \det g_{MN}(u)$. We will carry out the analysis in the momentum space by performing the plane wave expansion as

$$A_\mu(u, x_0, x_3) = \int \frac{dw dq}{(2\pi)^2} e^{-i\omega x_0 + iq x_3} A_\mu(u, \omega, q),$$

(5.2)

where we have rotated the direction of momentum to $x_3$ and paid attention to the same notation used in the gauge fields before and after Fourier transformation:

$$\omega g^{00}(u) A'_0(u) - q g^{33}(u) A'_3(u) = 0,$$

(5.3)

$$\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{uu}(u) A'_0(u) \right) = e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{33}(u) \left( \omega A_3(u) + q^2 A_0(u) \right),$$

(5.4)

$$\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{33}(u) g^{uu}(u) A'_3(u) \right) = e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{33}(u) \left( \omega A_0(u) + w^2 A_3(u) \right),$$

(5.5)

$$\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{33}(u) g^{uu}(u) A'_0(u) \right) = e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{33}(u) \left( g^{00} \omega A_0(u) + g^{33} q^2 A_0(u) \right),$$

(5.6)

where '$\equiv \partial_u$.'

From Eqs. 5.3 and 5.4, the equation of motion for $A'_0$ can be obtained as follows:

$$\left( \frac{z_0}{2} \right)^2 \frac{d}{du} \left[ \frac{\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{ uu}(u) A'_0(u) \right)}{e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{33}(u)} \right] - \left( \frac{g^{00}(u)}{g^{33}(u)} \right) m^2 + q^2 A'_0(u) = 0.$$

(5.7)

For a technical reason, we will perform a rescaling as

$$- \frac{2}{z_0^2} e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{uu}(u) A'_0(u) \equiv A'_0(u),$$

(5.8)

where we attach the factor $-2/z_0^2$ in the front to make the leading term of the rescaling as a unit in the expansion around $u = 0$. Note that when taking $\mu = 0$, it recovers the case of the Schwarzschild $AdS_5$ black hole geometry. With the above rescaling, the equation of motion is given by

$$\left( \frac{z_0}{2} \right)^2 \frac{d}{du} \left[ \frac{\partial_u A'_0(u)}{e^{-\Phi(u)} \sqrt{-g(u)} g^{00}(u) g^{33}(u)} \right] = \frac{g^{00}(u)}{g^{33}(u)} m^2 + q^2 e^{-\Phi(u)} A'_0(u).$$

(5.9)

To make the hydrodynamic analysis, let us formally write down the solution to the first-order hydrodynamics as follows:

$$A'_0(u) = C_0^{(0)}(u) \frac{e^{\Phi(u)}}{\sqrt{1 + 4\mu^2 u}} (1 - u)^{-im/2} \left( 1 + m F_1^{(0)}(u) + q^2 G_1^{(0)}(u) + \cdots \right),$$

(5.10)

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where $C_0^{(0)}(u)$, $F_1^{(0)}(u)$ and $G_1^{(0)}(u)$ are going to be fixed below. It is noticed that the infalling boundary condition has been used to yield the factor $(1 - u)^{-i\nu/2}$ in the solution. This is realized by taking the special solution of $\mu$, as shown in Fig.1, with requiring the numerator in the contribution at the order of $(u - 1)^{-1}$ to be vanishing in the dilaton $\Phi'(u)$. Then, we can see readily that $C_0^{(0)}$ can be treated as a constant,

$$C_0^{(0)}(u) \equiv C_0^{(0)},$$  

and the rest coefficients are obtained by the following equations:

$$\frac{z_0^2}{8} \partial_u \left\{ \frac{iC_0^{(0)} + 2(1 - u)\partial_u F_1^{(0)}(u)}{(1 - u) e^{-\Phi(u)} - g(u)g^{00}(u)g^{33}(u)} \right\} = 0,$$  

$$\frac{z_0^2}{4} \partial_u \left\{ \frac{\partial_u G_1^{(0)}(u)}{e^{-\Phi(u)} - g(u)g^{00}(u)g^{33}(u)} \right\} = \frac{C_0^{(0)}}{e^{-\Phi(u)} - g(u)g^{00}(u)g^{33}(u)},$$  

where the former and latter can be obtained from the order at $w$ and $q^2$ in Eq. (5.10). Let us solve the above equations by taking the following three steps:

1. performing the integration on the whole equations to get rid of the derivative $\partial_u \{ \cdots \}$ or $\partial_u (\cdots)$,

2. rewriting the equations in the form “$\partial_u F_1^{(0)} = \cdots$” or “$\partial_u G_1^{(0)} = \cdots$”,

3. carrying out the integration over the whole equations to obtain $F_1^{(0)}$ and $G_1^{(0)}$.

We then obtain the coefficients as follows

$$F_1^{(0)}(u) = C_0^{(0)} \left( F_B^{(0)} - \frac{2F_H^{(0)} e^{-\Phi(u)} \log(u)}{z_0^2} \right) - C_0^{(0)} \left( i \frac{1}{2} - \frac{2F_H^{(0)} e^{-\Phi(u)} (\Phi'(u) |_{u=0} - 2\mu^2)}{z_0^2} \right) u + O(u^2),$$

$$G_1^{(0)}(u) = C_0^{(0)} \left( G_B^{(0)} - \frac{2F_H^{(0)} e^{-\Phi(u)} \log(u)}{z_0^2} \right) + C_0^{(0)} \left( 1 + \frac{2G_H^{(0)} e^{-\Phi(u)} (\Phi'(u) |_{u=0} - 2\mu^2)}{z_0^2} \right) u + O(u^2)$$

with

$$C_0^{(0)} = -\frac{q^2(z_0^2(A_0^{(0)} q + A_3^{(0)} w))}{2(G_H q^2 + F_H w)}. $$

Here $C_0^{(0)}$ has been fixed by using Eq. (5.14) with the Dirichlet boundary condition that the boundary value of $A_0$ and $A_3$ are given by $A_0^{(0)}$ and $A_3^{(0)}$. Note that if the full integration in step 1 or 3 is difficult, instead of it, one can first carry out the Tyler expansion around $u = 0$, and after that, perform integration toward its low-order terms. As we will use eventually the GKP-W relation, the resulting solutions are the ones around the boundary, where $F_H^{(0)}$, $F_B^{(0)}$, $G_H^{(0)}$ and $G_B^{(0)}$ are the integral constants. $F_H^{(0)}$ and $G_H^{(0)}$ appear
from the integration in step 3, and they can be taken arbitrarily. On the other hand, $F_B^{(0)}$ and $G_B^{(0)}$ appear from the integration in step 1, and they are, in general, fixed such that the solutions do not diverge at $u = 1$, where the subscripts “$H$” and “$B$” denote the relevant constants which are fixed from the horizon and the boundary, respectively.

To put it more concretely, after step 2, one can see that $F_1^{(0)'}(u)$ and $G_1^{(0)'}(u)$ behave as $F_1^{(0)'}(u) = G_1^{(0)'}(u) \sim 1/(u - 1)$ around $u = 1$, which generally leads to the solution with logarithmic divergence, such a solution is ill defined. To obtain a physically meaningful solution, a simple way is to vanish these contributions by exploiting the integral constants $F_H^{(0)}$ and $G_H^{(0)}$ so that the numerators become zero.

In the actual calculation, $F_H^{(0)}$ can easily be fixed to be

$$F_H^{(0)} = -\frac{iC_0^{(1)} e^{\Phi(1) z_0^2}}{2\sqrt{1 + \mu^2}}, \quad (5.17)$$

while fixing $G_H$ is, in general, complicated as the integrating becomes difficult due to the dilaton. For this reason, we have to perform integration around $u = 1$ to fix the integral constant $G_H$. It will be shown that, as long as the integral constant $G_H$ is chosen appropriately, we can arrive at the needed solution which has no logarithmic divergence and becomes well defined in the whole bulk. Let us conduct step 1 with an expansion around $u = 1$:

$$\int e^{-\Phi(u)} \sqrt{-g(u)g^{00}(u)} g^{03}(u) \frac{\partial_u G_1^{(0)}(u)}{e^{-\Phi(u)} \sqrt{-g(u)g^{00}(u)}}$$

$$= \int du \frac{C_0^{(0)}}{e^{-\Phi(u)} \sqrt{-g(u)g^{00}(u)}}$$

$$= \int du \left\{ C_0 + C_1 (u - 1) + C_2 (u - 1)^2 + C_3 (u - 1)^3 + \cdots \right\}$$

$$= G_H^{(0)} + C_0 u + C_1 \left( \frac{u^2}{2} - u \right) + C_2 \left( \frac{u^3}{3} - u^2 + u \right) + C_3 \left( \frac{u^4}{4} - u^3 + \frac{3}{2} u^2 - u \right) + \cdots$$

$$= G_H^{(0)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} C_n + C_0 t + C_1 \left( \frac{t^2}{2} - t \right) + C_2 \left( \frac{t^3}{3} - t^2 + t \right) + C_3 \left( \frac{t^4}{4} - t^3 + \frac{t^2}{2} - t \right) + \cdots \quad (5.18)$$

with

$$C_n \equiv \frac{C_0^{(0)}}{n!} \delta^{(n)} \left( \frac{1}{e^{-\Phi(u)} \sqrt{-g(u)g^{00}(u)}} \right) \bigg|_{u=1} \quad \text{and} \quad t \equiv u - 1, \quad (5.19)$$

where “$n$” means the number of the derivative with regard to $u$. In the above equations, we have performed an expansion from the second to the third lines and written its result in a symbolic way. From the third to the fourth lines, we have explicitly written the integral constant $G_H^{(0)}$. From the fourth to the fifth lines, we have changed the variable $u$ to $t \equiv u - 1$. It is seen that, when we rewrite it in the form that “$\partial_u G_1^{(0)} = \cdots$”, there is no term with $1/(u - 1)$ in the expansion beyond the constant term. Therefore, we can readily obtain the expression of $G_H^{(0)}$ to be

$$G_H^{(0)} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} C_n \equiv \frac{C_0^{(0)} z_0^2}{2e^{-\Phi(1)} \sqrt{1 + 4\mu^2}} G_A(\mu, \{ \Phi^{(n)} \}), \quad (5.20)$$

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where we have rewritten symbolically the concise expression for convenience in the following actual calculation. Here, $G_A(\mu, \{\Phi^{(n)}\})$ is a function with $\{\Phi^{(n)}\} \equiv \{\Phi^{(1)}(u)|_{u=1}, \Phi^{(2)}(u)|_{u=1}, \Phi^{(3)}(u)|_{u=1}, \cdots \}$, and when $\mu = 0$ and $\{\Phi^{(n)}\} = \{0\}$, one has $G_A(0, \{0\}) = 1$. In this paper, $G^0_H$ will be treated symbolically.

It can be shown that the solutions near the boundary behave as follows:

$$A'_0(u) = -\frac{2C_0^{(0)}}{z_0} \left( G_B^{(0)} q^2 + F_B^{(0)} w \right) \log(u) + C^{(0)} e^{\Phi^{(0)}} \left( 1 + G_B^{(0)} q^2 + F_B^{(0)} w \right),$$

$$A'_3(u) = \frac{2C_0^{(0)}}{z_0} w \left( G_B^{(0)} q + F_B^{(0)} w \right) \log(u) - C_0^{(0)} e^{-\Phi^{(0)}} \frac{w}{q} \left( 1 + G_B^{(0)} q^2 + F_B^{(0)} w \right).$$

The boundary action $S_0$ is given from the on-shell action in the quadratic order of fluctuation as

$$S_0 = \frac{1}{g^2 z_0^2} \int \frac{dw \ dq}{(2\pi)^2} \left\{ \left( 1 + \frac{4\mu^2 \sqrt{u}}{z_0} \right) A_0(u) A'_0(u) - \left( 1 + \frac{4\mu^2 \sqrt{u}}{z_0} \right) \sum A_\alpha(u) A'_\alpha(u) \right\}. \quad (5.23)$$

By using the prescription for obtaining the retarded Green function in AdS/CFT correspondence [17], we arrive at the two-point retarded Green function of the $U(1)$ baryon charge current in the scalar mode as

$$G^{00}_{(R)} = \frac{N^2 T^2}{8} \frac{e^{\Phi^{(0)-\Phi^{(1)}}} \sqrt{4\mu^2 + i k^2}}{2i \pi T \omega - G_0(\mu, \{\Phi^{(n)}\}) k^2}. \quad (5.24)$$

$$G^{33}_{(R)} = \frac{N^2 T^2}{8} \frac{e^{\Phi^{(0)-\Phi^{(1)}}} \sqrt{4\mu^2 + i \omega^2}}{2i \pi T \omega - G_0(\mu, \{\Phi^{(n)}\}) k^2}. \quad (5.25)$$

From these results, we can read out the diffusion constant \[^{[29]}\] as

$$D = \frac{G_A(\mu, \{\Phi^{(n)}\})}{2\pi T}. \quad (5.26)$$

6 Analysis on the $SU(2)$ vector and axial-vector flavor currents

In this section, we will perform the hydrodynamic analysis for the vector mode shown in Table 4.10. To begin with, let us combine the $SU(2)$ gauge fields into the vector field $V_M$ and the axial-vector field $W_M$,

$$V_M(u) \equiv \frac{1}{2} (B_{L,M}(u) + B_{R,M}(u)) \quad \text{and} \quad W_M(u) \equiv \frac{1}{2} (B_{L,M}(u) - B_{R,M}(u)). \quad (6.1)$$

The equations of motion for the vector field $V^a_M$ and the axial-vector field $W^a_M$ are given as

$$V : D_{V,M} \left( \sqrt{-g(u)} e^{-\Phi(u)} F^{a,MN}_V(u) \right) = 0, \quad (6.2)$$

$$AV : D_{W,M} \left( \sqrt{-g(u)} e^{-\Phi(u)} F^{a,MN}_W(u) \right) + g^{[SU(2)]} e^{-\Phi(u)} \sqrt{-g(u)} v^2(u) g^{MN}(u) W^a_M(u) = 0, \quad (6.3)$$

where $v(u)$ has been defined in Eq. 6.3, and “$V$” and “$AV$” mean the field strength consisting of the vector and the axial-vector fields, respectively. $F^{a,MN}_V(u)$ and $D_{V,M}(u)$ are dictated as $F^{a,MN}_V(u) \equiv \partial_M V^a_M(u) - \partial_N V^a_M(u) + i f^{abc} V^b_M(u) V^c_M(u)$ [the SU(2) structure constant], and $D_{V,M} \equiv \partial_M(u) - i V_M(u)$ [$F_{W,M}(u)$ and $D_{W,M}(u)$ are similar]. It is seen that Eq. 6.2 can be obtained from Eq. 6.3 by dropping
the term proportional to \( g_{SU(2)}^2 \). Thus, our main task here will focus on solving Eq. (6.3).

Now let us write down the equation of motion (6.3) to the linear order, and its result is given by

\[
\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{\alpha\alpha}(u) g^{uu}(u) W_\alpha'(u) \right) = e^{-\Phi(u)} g^{\alpha\alpha}(u) \left( g^{00}(u) \omega^2 + g^{33}(u) k^2 - g_{SU(2)}^2 v^2(u) \right) W_\alpha'(u).
\]

(6.4)

which cannot simply be solved by the double integral dictated above Eq. (5.14) due to the extra term proportional to \( g_{SU(2)}^2 v^2(u) \) in the potential part. Thus, we may factorize it as follows:

\[
W_\alpha(u) \rightarrow \rho(u) W_\alpha^0(u).
\]

(6.5)

Then, Eq. (6.4) becomes the following form

\[
\partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{\alpha\alpha}(u) g^{uu}(u) W_\alpha'(u) \right) + \rho(u) \left\{ \Omega(u) - e^{-\Phi(u)} \sqrt{-g} g^{\alpha\alpha}(u) (g^{00}(u) \omega^2 + g^{33}(u) k^2) \right\} = 0
\]

(6.6)

with

\[
\Omega(u) \equiv \partial_u \left( e^{-\Phi(u)} \sqrt{-g(u)} g^{\alpha\alpha}(u) g^{uu}(u) \rho'(u) \right) - g_{SU(2)}^2 e^{-\Phi(u)} \sqrt{-g} \rho(u) g^{\alpha\alpha}(u) v^2(u).
\]

(6.7)

which indicates that when \( \rho(u) \) satisfies \( \Omega(u) = 0 \), it becomes available to obtain the solution \( W_\alpha^0(u) \) from the double integral. Nevertheless, to fully solve the equation \( \Omega(u) = 0 \) is technically difficult. Practically, considering that the needed solution at the last stage is the one only in the vicinity of the horizon and the boundary, we shall obtain the solutions in the expansion around \( u = 0 \) and \( u = 1 \).

First, let us obtain the solution in the expansion around \( u = 0 \). For this purpose, expanding \( \Omega(u) \) given in Eq. (6.7), we arrive at the following form

\[
(\alpha_0 + \mathcal{O}(u)) \rho''(u) + (\beta_0 + \mathcal{O}(u)) \rho'(u) + \left( \frac{\gamma_0}{u} + \gamma_1 + \mathcal{O}(u) \right) \rho(u) = 0
\]

(6.8)

with

\[
\alpha_0 \equiv \frac{2}{z^0}, \quad \beta_0 \equiv -\frac{2}{z^0} \left( \Phi'(u) \right|_{u=0} - 2 \mu^2), \quad \gamma_0 \equiv \frac{1}{2} g_{SU(2)}^2 A^2, \quad \gamma_1 \equiv g_{SU(2)}^2 A^2 (B z_0^2 + 3 A \mu^2).
\]

(6.9)

As a result, we obtain the general solution \( \rho(u) \) as a linear combination of two confluent hypergeometric functions \( _1F_1 \) and \( U \) with two integral constants. We set two integral constants in such a way that \( _1F_1 \) vanishes, and the leading of \( \rho(u) \) in the expansion of \( u = 0 \) becomes 1. Eventually, we obtain \( \rho(u) \) satisfying Eq. (6.9) to the expanded order as

\[
\rho(u) = \text{Exp} \left( -\frac{\Delta + \beta_0}{2 \alpha_0} u \right) \Gamma \left( 1 - \frac{\gamma_0}{\Delta} \right) U \left( -\frac{\gamma_0}{\Delta}, 0, \frac{\Delta}{\alpha_0} u \right)
\]

(6.10)
with $\Delta \equiv \sqrt{(\beta_0)^2 - 4\alpha_0\gamma_1}$, where the functions $\Gamma(u)$ and $U(u)$ denote the gamma function and the confluent hypergeometric function, respectively.

Now we shall try to solve $\Omega(u) = 0$ in the vicinity of $u = 1$. It turns out that a general solution is given by a linear coupling of the confluent hypergeometric function $U$ and Laguerre polynomials $L_n$. However, it is difficult to fix the integral constants analytically, such that the constants in the solution at $u = 1$ become common with the ones at $u = 0$ of Eq. (6.10), due to the lack of information for the solution of $\rho(u)$ in the interior domain of the bulk. Therefore, we will only treat the solution $\rho(u)$ around $u = 1$ symbolically.

Having obtained $\rho(u)$ satisfying $\Omega(u) = 0$ as in Eq. (6.10), we can solve the equation of motion by using the double integral similar to the previous chapter. With the results of $\mu$ given in Table 1, we can formally write down the solution of $W^\mu$ in the hydrodynamic expansion with regard to $\omega$ and $q^2$ as

$$W^\mu = C^{(a)}_\alpha(u)(1 - u)^{-i\omega/2} \left(1 + \omega F^{(a)}_1(u) + q^2 G^{(a)}_1(u) + \cdots\right).$$  \hspace{1cm} (6.11)

With the same way stated above Eq. (5.14), the coefficients can be written as

$$C^{(a)}_\alpha(u) = \frac{W^\mu_{\alpha}(0)}{1 + G^{(a)}_B q^2 + F^{(a)}_B \omega}$$ \hspace{1cm} (6.12)

and

$$F^{(a)}_1(u) = F^{(a)}_B + \left(\frac{2F^{(a)}_H e^{\Phi(0)}}{\rho(0)^2} - \frac{iC^{(a)}_0}{2}\right) u + \mathcal{O}(u^2),$$ \hspace{1cm} (6.13)

$$G^{(a)}_1(u) = G^{(a)}_B + \left\{\frac{C^{(a)}_0 (\log(u) - 1) + 2G^{(a)}_H e^{\Phi(0)}}{\rho^2(0)}\right\} u + \mathcal{O}(u^2),$$ \hspace{1cm} (6.14)

where $W^\mu_{\alpha}(0)$ denotes the boundary value of $W^\mu_{\alpha}$, and $F^{(a)}_B$, $G^{(a)}_B$ and $F^{(a)}_H$, $G^{(a)}_H$ mean integral constants which are fixed at the boundary and the horizon as the same as the integral constants that appeared in Eqs. (5.13) and (5.14). To be explicit, they are fixed as

$$F^{(a)}_H = \frac{1}{2} iC^{(a)}_0 \sqrt{4\mu^2 + 1} e^{-\Phi(1)},$$ \hspace{1cm} (6.15)

$$G^{(a)}_H = -\frac{1}{2} C^{(a)}_0 \sqrt{4\mu^2 + 1} e^{-\Phi(1)} G_W(\mu, \{\Phi^{(n)}\}),$$ \hspace{1cm} (6.16)

where $G_W(\mu, \{\Phi^{(n)}\})$ is a function as the same as Eq. (5.24) and $G_W(0, \{0\}) = 1$. From the above analysis, we can write down the solutions in the vicinity of $u = 0$ with taking the factor $\rho$

$$W^\mu = W^\mu_{\alpha}(0) \rho(0) + W^\mu_{\alpha}(0) \left\{\frac{4 \rho(0) e^{\Phi(0)} \left(F^{(a)}_H (\omega + G^{(a)}_H q^2) + \rho(0)^2 (2q^2 \log(u) - 2q^2 - i\omega)\right)}{2 \rho(0) \left(F^{(a)}_B \omega + G^{(a)}_B q^2 + 1\right)}
\right\} u + \mathcal{O}(u^2).$$ \hspace{1cm} (6.17)

It is noticed that $V^\mu_{\alpha}$ can simply be read from the solution of $W^\mu_{\alpha}$ by taking $g^2_{SU(2)} = 0$.  

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Finally, we can obtain the two-point retarded Green function in the dual field theory through the GKP-W relation. For this purpose, the boundary action in quadratic order at \( u = 0 \) is needed. One can see that it is given as

\[
S_0 = -\frac{2}{g^2_{SU(2)}} \left( 1 + \frac{4\mu^2\sqrt{u}}{\varepsilon_0} \right) \int \frac{dw \, dq}{(2\pi)^2} \sum_\alpha W^\alpha(u)W^{\alpha'}(u). \tag{6.18}
\]

which enables us to obtain the two-point retarded Green function in the vector mode as

\[
G_{\alpha\alpha}^{AV,R}(u) = \frac{iN^2T^2}{8\rho(0)} \left\{ 4\rho(0)e^{\Phi(0)} \left( F_H^{(\alpha)} \omega + G_H^{(\alpha)} q^2 \right) + \rho(0)^2 \left( 2q^2 \log(u) - 2q^2 - i\omega \right) \right\} + \left( \rho(1) + \frac{i\rho(0)\omega}{2} \right). \tag{6.19}
\]

The two-point retarded Green function \( G_{\alpha\alpha}^{AV,R}(u) \) for \( V_{\alpha}^\alpha(u) \) can simply be resulted from the \( G_{\alpha\alpha}^{AV,R}(u) \) given above with vanishing \( g^2_{SU(2)} \).

### 7 Analysis on the tensor mode

In this section, we will carry out the analysis of the tensor mode of the fluctuation in the dual gravity shown in Table (4.10). Then, from these results, we will calculate the shear viscosity (2.7) and its ratio to the entropy density in the dual field theory. For this purpose, we will start with the equation of motion for the bulk gravity, which can be generally given as

\[
R_{MN} - \frac{1}{2}G_{MN}R + \Lambda G_{MN} = \kappa^2T_{MN}, \tag{7.1}
\]

where \( \kappa^2 = 8\pi G_5 \) [\( G_5 \) is given below Eq. (3.5)], \( \Lambda = -6/l^2 \) (\( l \) denotes the curvature radius of the AdS space and is taken to be 1), and \( T_{MN} \) means the energy-momentum tensor. From Eq. (6.1), it is given as

\[
T_{MN} = g_{MN}\mathcal{L} + g^{PQ} \left\{ \frac{1}{g^2_{SU(1)}} F_{PM}F_{QN} + \frac{1}{g^2_{SU(2)}} \text{Tr} \left( F_{L,PM}F_{L,QN} + F_{R,PM}F_{R,QN} \right) \right\} - 2\text{Tr}D_MXD_NX, \tag{7.2}
\]

where \( \mathcal{L} \) means the Lagrangian (3.2) except for the Einstein-Hilbert part. It is noticed that, when substituting the background (3.1) into the above equation of motion for gravity, there is a deviation proportional to \( \mu_g \), which indicates that the background (3.1) is not an exact solution of the equation of motion for gravity as the backreacted geometry is not considered here. Nevertheless, the effect from the backreacted geometry was found to be small [19], and also it turns out that the deviated part appears only in the diagonal part of the equation of motion, while in our present treatment it involves only the off-diagonal part of the equation of motion, the \((x, y)\) component, so the deviated part will be discarded in this paper.
To evaluate the viscosity, we will consider the $(x,y)@ component in Eq. (7.1). Thus, the energy-momentum tensor at the linear order becomes as $T_{xy} = \text{Tr} \left( g^{\mu\nu} \partial_\mu X |^2 - m^2 X |^2 - \frac{1}{4} |X|^4 \right) h_{xy}$, and the equation of motion for $h_{xy}$ can be obtained as

$$0 = h''_{xy}(u) + g_1'(u) h'_{xy}(u) + g_2(u) h_{xy}(u) \tag{7.3}$$

with

$$g_1(u) = \frac{\sqrt{a f(u)}}{a(u)},$$

$$g_2(u) = g_3(u) + \frac{z_0^2}{4 u a f(u)^2} (\omega^2 - k^2 f(u)),$$

$$g_3(u) = \frac{1}{2 u a f(u)} \left[ 16 u f(u) a''(u) + 8 a'(u) (2 u f'(u) + f(u)) + a(u) (2 u f''(u) + f'(u)) \right] + \frac{\kappa^2 z_0^2 a^4(u)}{4 u a f(u)} T_{xy}(u). \tag{7.4}$$

Here, $g_3(u)$ represents the constant part toward $\omega$ and $k$ in the potential term $g_2(u)$. It is seen that $g_3(u)$ is the obstacle to use the double integral dictated above Eq. (5.14), which is similar to the case of Eq. (6.4). Thus, we may adopt the same analysis as the one for Eq. (6.4), namely exploiting the factorization,

$$h_{xy}(u) = \chi(u) \phi(u), \tag{7.5}$$

and vanishing the extra term $g_3(u)$.

Under the factorization (7.5), we can rewrite Eq. (7.3) as

$$\left( g_1(u) \chi^2(u) \phi'(u) + g_1(u) \chi^2(u) \right) \left\{ \Psi(u) + \frac{z_0^2}{4 u a f(u)^2} (\omega^2 - k^2 f(u)) \right\} \phi(u) = 0 \tag{7.6}$$

with

$$\Psi(u) \equiv \chi''(u) + g_1'(u) \chi'(u) + g_3(u) \chi(u). \tag{7.7}$$

Then, $\chi(u)$ is obtained by requiring $\Psi(u)$ to satisfy $\Psi(u) = 0$. However, it can be shown that it is difficult to solve the equation $\Psi(u) = 0$ fully. Similarly to the previous section, as the needed solution eventually is the one in the vicinity of the horizon and the boundary, we may consider the solution of $\chi(u)$ only in the vicinities at $u = 0$ and $u = 1$. Nevertheless, from the reason mentioned below, one cannot obtain analytically the solution in the expansion around $u = 1$.

Then, we will try to obtain the solution of $\chi(u)$ in the vicinity of $u = 0$. To this purpose, expanding the constant part $\Psi(u)$ around $u = 0$, we will consider the following equation:

$$\chi''(u) + \left( \frac{1}{u} - 2 \mu^2 \right) \chi'(u) + \left[ -\frac{1}{u^2} + \frac{1}{2 u} \left( 40 \mu^2 + \kappa^2 T_{xy}'(u) \right) \right] \chi(u) = 0. \tag{7.8}$$
To obtain the solution $\chi(u)$ in the vicinity of $u = 0$, we may assume the following form as the solution of $\chi(u)$,

$$
\chi(u) = \frac{\chi_0}{u} + \chi_1 u + \chi_L u \log(u) \quad \text{with} \quad \chi_{-1} \equiv 1/z_0^2,
$$

(7.9)

where $\chi_0$, $\chi_1$ and $\chi_L$ are the coefficients to be determined further. While we have fixed $\chi_{-1} = 1/z_0^2$. It can be seen that if we assume the form of solution as in Eq. (7.9), then $\chi_{-1}$ can, in general, be arbitrary [and each coefficients except for $\chi_{-1}$ can be fixed so as to satisfy Eq. (7.8)]. Thus, one should fix $\chi_{-1}$ in the first place as shown in Ref. [18]. By doing so, one can see that $\chi(u)$ plays the same role as the background metric $g_{\alpha\alpha}$ at the vicinity of $u = 0$, where $g_{\alpha\alpha}(u)$ in the vicinity of $u = 0$ behaves as $g_{\alpha\alpha}(u) = \frac{1}{z_0^2} + \frac{4\mu^2}{z_0^2} + O(u)$. As a consequence, we can put the source of $\phi(u)$ as $h^{(0)}(t)\chi(u)$ with $h^{(0)}(t)\chi(u)$ the boundary value.

By substituting $\chi(u)$ with the form in Eq. (7.9) and solving the equations in each order of it, we can fix the coefficients as

$$
\chi_0 = \frac{\chi_{-1}}{2} \left( 44\mu^2 + \kappa_2 T''_{xy}(u) \right)_{u=0},
$$

(7.10)

$$
\chi_1 = \frac{\chi_{-1}}{24 \left( 36\mu^2 + \kappa_2 T'_{xy}(u) \right)_{u=0} \left[ 3 \log(u) \left( 36\mu^2 + \kappa_2 T'_{xy}(u) \right)_{u=0} \right] \left[ 1728\mu^4 + 92\kappa_2 \mu^2 T'_{xy}(u) \right]_{u=0} + \kappa_2 \left( T''_{xy}(u) \right)_{u=0} + 2\kappa_2 \left( T'_{xy}(u) \right)_{u=0} \left( 268\mu^4 + 5\kappa_2 \mu^2 T'_{xy}(u) \right)_{u=0} - 16128\mu^6 \right],
$$

(7.11)

$$
\chi_L = -\frac{\chi_{-1}}{8} \left( 1728\mu^4 + 92\kappa_2 \mu^2 T''_{xy}(0) + \kappa_2 \left( T''_{xy}(u) \right)_{u=0} + \kappa_2 \left( T'_{xy}(u) \right)_{u=0} \right)^2 \right),
$$

(7.12)

where we have obtained the solution of $\chi(u)$ to the order of $u$, which will become necessary in the analysis in what follows.

Now, we come to discuss the solution $\chi(u)$ in the vicinity of $u = 1$. From the expansion of the coefficients of Eq. (7.9) around $u = 1$, one can see that a general solution in the vicinity of $u = 1$ is given as a linear coupling of the confluent hypergeometric function $U$ and Laguerre polynomials $L_n$. However, similar to the case of $\rho(u)$ in the previous section, it is difficult to fix the integral constants analytically, so that the constants are common between the ones at $u = 1$ and $u = 0$ [Eq. (7.10)] with the coefficients (7.11)] , due to the lack of the information of the solution of $\chi(u)$ in the interior domain of the bulk. Thus, $\chi(u)$ around $u = 1$ will be treated symbolically in the present consideration.

In the same way as in the previous sections, we can obtain the hydrodynamic solution for the off-diagonal component of the fluctuation $h_{xy}(u)$ in Eq. (7.10) with $\chi(u)$ determined above. Consequently, it can be written as

$$
h_{xy}(u) = \chi(u)\phi(u) = C_{0}^{(\phi)}(\chi(u) (1-u)^{-i\omega/2} \left( 1 + \omega F_1^{(\phi)}(u) + q^2 G_1^{(\phi)}(u) + \cdots \right)
$$

(7.13)
with

\[ C_0^{(\phi)} = \frac{h^{(0)}_{xy}}{1 + G^{(\phi)}_B + F_B^{(\phi)} q^2}, \]  

\[ F_1^{(\phi)}(u) = F_B^{(\phi)} - \frac{i}{2} u + \frac{i}{2} \left( \frac{1}{2} + \frac{F_H^{(\phi)}}{z_0^2 \chi-1} \right) u^2, \]  

\[ G_1^{(\phi)}(u) = G_B^{(\phi)} - u + \left\{ \frac{2 \chi-1 z_0 \log(u) \left( \chi_0 - \mu^2 \chi-1 \right)}{2(\chi-1)^2 z_0} + \frac{G_B^{(\phi)}}{2(\chi-1)^2 z_0^2} \right\} u^2, \]

where \( h^{(0)}_{xy} \) has been mentioned above Eq. (7.10). \( F_H^{(\phi)}, F_B^{(\phi)} \) and \( G_H^{(\phi)}, G_B^{(\phi)} \) represent the integral constants fixed at boundary and horizon, which are similar to the integral constants that appeared in Eqs. (5.14) and (5.15). More explicitly, they are fixed as

\[ F_H^{(\phi)} = \frac{1}{\sqrt{4 \mu^2 + 1}}, \]  

\[ G_H^{(\phi)} = -\frac{1}{\sqrt{4 \mu^2 + 1}} G_0^{(\phi)}(\mu, \{ \Phi^{(n)} \}). \]

Here we have used \( \chi_{-1} = 1/z_0^2 \). It enables us to write down the solution of \( h_{xy}(u) \) in the vicinity of \( u = 0 \) as

\[ h_{xy}(u) = \frac{h^{(0)}_{xy} \chi-1}{u} + h^{(0)}_{xy}(\chi_0 - q^2 \chi-1) + \frac{h^{(0)}_{xy}}{2 z_0^2 \chi-1} \left\{ i F_H^{(\phi)} \omega + q^2 \left( G_H^{(\phi)} + \chi_{-1} z_0^2 \left( \chi_0 - \mu^2 \chi-1 \right) \right) \right\} + 2 z_0^4 q^2 \chi-1 \left( \chi_0 - \mu^2 \chi-1 \right) \log(u) u + \mathcal{O}(u^2) \]  

which shows that the expression diverges at \( u = 0 \). However, such a divergence arises from the contribution of \( \chi(u) \) given in Eq. (7.11), and the solution of \( \phi(u) \) itself has been obtained healthily, as can be seen from Eq. (7.13). Such a situation is the same as the one discussed in Ref. [18] and also other papers based on Ref. [18].

Now that we have obtained the solution, let us evaluate the viscosity through Kubo formula. The boundary action \( S_0 \) is given by

\[ S_0 = S_{\text{on-shell}} + S_{\text{GH}} + S_{\text{ct}}, \]  

with

\[ S_{\text{GH}} = \frac{1}{8 \pi G_5} \int d^4 x \sqrt{-g^{(4)}} K, \]

\[ S_{\text{ct}} = -\frac{1}{8 \pi G_5} \int d^4 x \sqrt{-g^{(4)}} \left( \frac{3}{4} + \frac{1}{4} R^{(4)} \right), \]

where \( S_{\text{GH}} \) and \( S_{\text{ct}} \) denote the Gibbons-Hawking term and the counter term respectively, and \( g^{(4)}_{\mu \nu} \) is the four-dimensional induced metric, and \( K \) and \( R^{(4)} \) mean the extrinsic curvature and the curvature on the boundary.
boundary toward our geometry \([\ref{a}]\), respectively. They are found to be

\[ S_{\text{on-shell}} = \frac{1}{2\kappa^2 z_0} \int \frac{dw\ dq}{(2\pi)^2} e^{-\Phi(u)} \sqrt{a(u)} \left\{ 4a'(u) f(u) h_{xy}(u)^2 - 3a(u) f(u) h_{xy}(u) h_{xy}'(u) \right\} \bigg|_{u=0}, \]

\[ S_{\text{GH}} = \frac{1}{2\kappa^2 z_0} \lim_{u \to 0} \int \frac{dw\ dq}{(2\pi)^2} e^{-\Phi(u)} \sqrt{u} \left\{ h_{xy}(u)^2 f'(u) + 4f(u) h_{xy}(u) h_{xy}'(u) \right\}, \]

\[ S_{\text{et}} = \frac{1}{8\kappa^2} \lim_{u \to 0} \int \frac{dw\ dq}{(2\pi)^2} e^{-\Phi(u)} \frac{12a(u)^2 f(u) + k^2 f(u) - \omega^2}{a(u)^2 \sqrt{f(u)}} h_{xy}^2(u). \]

The retarded Green function is obtained by using the GKP-W relation

\[ G^{(R)}_{xy}(\omega, k) = -\frac{N^2 T^2}{16} \left\{ G^{(\phi)} H^2 k^2 + 2\pi T \left( iF_H \omega - \chi^2 \right) \right\}, \]

which allows us to evaluate the shear viscosity via the Kubo formula

\[ \eta = -\lim_{\omega \to 0} \frac{\text{Im} \left( G^{(R)}_{xy}(\omega, k = 0) \right)}{\omega} = \frac{\pi}{8} F_H N^2 T^3. \]

With this result, it is then not difficult to obtain the ratio

\[ \frac{\eta}{s} = \frac{1}{4\pi \sqrt{1 + 4\mu^2(1 + 16\pi^2 T^2)^{3/2}}}, \]

where the entropy density is given in Eq. (3.5).

The ratio \(\eta/s\) as the function of temperature is shown in Fig. 2 where the left and the right plots are different in the scale of the \(x\) axis. The red points and the blue points represent the results obtained by using the \(\mu\) in Fig. 1 and each plot in Figs. 2 and 1 corresponds each other. The dashed horizontal line represents the KSS bound, \(1/(4\pi)\) \([22]\). It can be seen that there are two branches with regard to the solutions of \(\mu\) in Fig. 1. Among the red points and the blue points, the red points are the ones that we have chosen as the physically acceptable branch in the last part of Sec. 3.

### 8 Conclusions and Remarks

In this paper, we have extended an IR-deformed AdS/QCD model presented in Ref. [10] to a finite temperature system by a way different from the one proposed in Ref. [12], worked out the hydrodynamics, and computed the transport coefficients. In the actual analysis, we have found several branches, and we have chosen the branch in which our model can be consistent as a holographic model.

The things characterizing our model are the parameters \(\mu_g, g_{SU(2)}, \lambda\), the scalar field \(X\) and the dilaton \(\Phi\). Among these, the ones entering our analysis in this paper have been \(\mu_g\) through \(\mu \equiv \mu_g/(2\pi T)\) and dilaton. We have treated the effects of these in a symbolic way. The remaining things will enter if one considers the background dual to the system at the finite chemical potential. However, the analysis with the finite chemical potential will be too complicated to perform the hydrodynamic analysis.
Figure 2: We plot $\eta/s$ given in Eq. (7.28) using $\mu$ given in Fig. 1. The left and right plots are different just in the scale of the $x$ axis. The red and blue points represent the results obtained by using the $\mu$ in Fig. 1. These plots and the plots in Fig. 1 correspond each other. The dashed horizontal line represents the KSS bound, $1/(4\pi)$ [22]. In these plots, the result represented by the red points corresponds to the results with regard to the $\mu$ we have chosen as the physically acceptable one.

One of the further directions of this study will be the holographic description of the Bjorken flow [23]. The Bjorken Flow is an effective model having been invented to describe evolution of quark-gluon plasma produced in high energy collision experiment. This has the property of the ideal fluid, and diffuses with the form of distribution in the boost-invariant way from a point in Minkowski space-time in which the collision happens. It is considered that the quark-gluon plasma being produced and diffusing in the RHIC or LHC, etc can be described by using the Bjorken Flow. However, it is known that understanding of the Bjorken Flow in the framework of the field theory is hard due to its strong coupling and the real-time evolution. In such a situation, the gauge/gravity correspondence would be useful. For the relevant studies, we would like to refer the reader to the reference in the review papers [15]. In the analyses of these, we can see that hydrodynamics is a necessity. Therefore, developing the hydrodynamic analysis performed in this paper more, we are going to try to the Bjorken flow in our model.

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