Testing for Gaussianity
Through the Three Point Temperature Correlation Function

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ABSTRACT

One of the crucial aspects of density perturbations that are produced by the standard inflation scenario is that they are Gaussian where seeds produced by topological defects tend to be non-Gaussian. The three point correlation function of the temperature anisotropy of the cosmic microwave background radiation (CBR) provides a sensitive test of this aspect of the primordial density field. In this paper, this function is calculated in the general context of various allowed non-Gaussian models. It is shown that by COBE and the forthcoming South Pole and Balloon CBR anisotropy data may be able to test provide a crucial test of Gaussianity.

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Testing for the Gaussianity of the primordial fluctuation spectrum is of critical importance to many cosmological models. In particular, traditional cosmic inflation [1] specifically predicts a Gaussian density fluctuation spectrum. The scale invariant quantum fluctuations generated during the inflationary epoch are expected to serve as the primordial density perturbations which develop into the large scale structures we observe today [2]. Competing models for structure formation, including topological defects originating from cosmological phase transitions [3] and non-standard inflation models [4], will also generate a scale invariant (or nearly scale invariant) power spectrum for density perturbations similar to that of inflation. However, the statistics of these latter fluctuations are non-Gaussian. Thus, the Gaussianity of the fluctuations provides a unique handle in discriminating different structure formation scenarios. In this letter, we will discuss how to test this aspect of the primordial density field through the temperature anisotropy of the cosmic microwave background radiation (CBR).

As we showed [5], in momentum space, the lowest order deviation from Gaussianity is described by the bispectrum of the gravitational potential $\phi$,

$$P_\phi(k_1, k_2) = \langle \phi_{k_1} \phi_{k_2} \phi_{-k_1-k_2} \rangle.$$  

When the perturbation is adiabatic so that the temperature anisotropy is related to the gravitational potential $\phi$ at the last scattering surface through the Sachs-Wolfe [6] formula:

$$\frac{\delta T}{T} = \frac{\phi}{3},$$  

the three point temperature correlation function is related to the bispectrum through

$$\xi_T(\hat{m}, \hat{n}, \hat{l}) = \frac{1}{27} \int P_\phi(k_1, k_2, k_3) e^{i(k_1 \hat{m} + i k_2 \hat{n} + i k_3 \hat{l})} \eta_0^3 \delta^3 (\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^9},$$  

where $\eta_0 = 2H_0^{-1}$ is the distance to the last scattering surface, and $\hat{m}, \hat{n}, \hat{l}$ are the beam directions. A non-vanishing three point function clearly indicates that the bispectrum is not zero. Note that for Gaussian primordial perturbations, the bispectrum is strictly zero in all cases. Thus, the three
point temperature correlation function is a clean test of the Gaussianity of primordial fluctuations.

In [7], Falk et al. found that in an inflationary model with cubic self-interaction, \( P_\phi(k_1, k_2, k_3) \) is given by:

\[
P_\phi = \beta (k_1 k_2 k_3)^{-3} (k_1^3 + k_2^3 + k_3^3),
\]

(3)

where \( \beta \sim 10^{-6} \). In this paper, we will show that without invoking any new assumptions about inflationary models, the non-linear gravitational evolution of the initial Gaussian perturbations will give rise to a three point correlation function, which has a similar angular dependence to that which certain non-Gaussian inflationary models predict, but a much larger amplitude than the one considered in [7]. Then, we extend the analysis in [7] to include more general cases of inflation, which produce not only the scale invariant but also the “tilted” perturbation spectrum. The extended analysis is helpful in discussing the effect of spectral index on the angular dependence of the three point function. To choose different non-Gaussian inflationary models through three point temperature correlation function will be hard because of the gravitational evolutionary effects. However, the three point correlation function produced by a cosmological phase transition tends to have a distinctive angular dependence, which should enable one to prove or disprove the scenario through observations. Finally, we briefly discuss the effect of noise in the sky signal of CBR measurements on the analysis of three point temperature correlation function.

By taking into account non-linear gravitational evolution, it is found that there are two terms which contribute to the CBR temperature anisotropy:

\[
\frac{\delta T}{T} = \frac{\phi}{3} + 2 \int \frac{\partial \phi}{\partial \eta} d\eta,
\]

(4)

where the first term is the Sachs-Wolfe effect due to the gravitational potential at the last scattering surface, and the second term is the generalized Rees-Sciama effect due the evolution of the gravitational potential along
the photon path. When we adopt a flat cosmological model \((\Omega = 1)\), the quasi-nonlinear analysis gives \([9, 10, 5]\):

\[
\phi(k, \eta) = \phi_i(k) + a(\eta) \int J(\vec{k}, \vec{k}', \vec{k} - \vec{k}') \phi_i(k') \phi_i(k - k'),
\]

(5)

where \(\phi_i\) is the gravitational potential at the last scattering surface, \(a(\eta) = (\frac{\eta}{\eta_i})^2\) is the expansion factor of the universe after decoupling and

\[
J(\vec{k}, \vec{l}, \vec{m}) = 2(\vec{l} \cdot \vec{m}) + \frac{5(\vec{k} \cdot \vec{l})m^2}{k^2} + \frac{5(\vec{k} \cdot \vec{m})l^2}{k^2}.
\]

(6)

We first estimate the amplitude of the second term relative to the first term: since the expansion factor \(a\) after decoupling is \(\sim (1 + z_{\text{dec}}) \sim 1000\), the amplitude of the gravitational potential at the last scattering surface is around \(10^{-5}\) as suggest by COBE, thus the ratio of the Rees-Sciama term to the Sachs-Wolfe term is of order \(0.01 - 0.1\). As the non-linear effects are contained in the Rees-Sciama term, it is this term that contributes significantly to the three point correlation function. For comparison, the non-linear term considered in \([7]\) is \(10^{-6}\) times smaller than the linear term. This is a potential problem on testing inflationary models through three point temperature correlation function. To be observable the amplitude of the non-Gaussianity produced in these models has to be large enough so that the gravitational evolution cannot completely dominate.

The generic form of the bispectrum in inflationary models is given by:

\[
P_{\phi}(k_1, k_2, k_3) = \lambda[P_{\phi}(k_1)P_{\phi}(k_2) + P_{\phi}(k_2)P_{\phi}(k_3) + P_{\phi}(k_3)P_{\phi}(k_1)],
\]

(7)

where \(\lambda\) is a constant and \(P_{\phi} = \langle \phi(k)\phi(-k) \rangle\) is related to the power spectrum \(P(k)\) simply through \(P_{\phi}(k) = P(k)k^{-4}\). The cubic self-interaction model corresponds to the case where \(\lambda \sim 10^{-6}\) with a scale invariant density perturbation spectrum, or equivalently, \(P_{\phi}(k) \sim k^{-3}\); the non-linear gravitational evolution effect corresponds to the case where \(\lambda \sim 2(1 + z_{\text{dec}})/9 \sim 200\) and \(P_{\phi}(k) \approx k^{-2}\).
Note that the two point temperature correlation function is related to $P_\phi$ through:

$$C_2(\hat{m}, \hat{n}) = \frac{1}{9} \int P_\phi(k) e^{ik(\hat{m} - \hat{n})\eta_0} \frac{d^3k}{(2\pi)^3}. \quad (8)$$

The three point correlation function calculated from the bispectrum given by Eq.7 is:

$$\xi_T(\hat{m}, \hat{n}, \hat{l}) = 3\lambda [C_2(\hat{m}, \hat{n})C_2(\hat{n}, \hat{l}) + C_2(\hat{m}, \hat{n})C_2(\hat{m}, \hat{l}) + C_2(\hat{m}, \hat{l})C_2(\hat{n}, \hat{l})]. \quad (9)$$

This is a theoretical relation between three point function and two point function since the finite beam size effects haven’t taken into account yet. The formal treatment of the finite beam effect in CBR experiment can be found in [12, 13]. The beam can be well approximated as a Gaussian:

$$f(|\hat{m} - \hat{n}|, \sigma) = \frac{1}{2\pi\sigma^2} e^{-|\hat{m} - \hat{n}|^2/2\sigma^2}, \quad (10)$$

and the observed temperature correlation function will be the convolution of the theoretical correlation (infinite thin beam) with the beam, which is

$$C_3(|\hat{m}, \hat{n}, \hat{l}|, \sigma) = \int d\Omega_1' d\Omega_2' d\Omega_3' f(|\hat{m} - \hat{m}'|, \sigma) f(|\hat{n} - \hat{n}'|, \sigma) f(|\hat{l} - \hat{l}'|, \sigma) C_3(|\hat{m}', \hat{n}', \hat{l}'|, 0). \quad (11)$$

For a special configuration of three beams where $\hat{m} \cdot \hat{n} = \hat{n} \cdot \hat{l} = \hat{l} \cdot \hat{m} = \cos \alpha$, the beam-smoothed three point function is well approximated as $[C_2(\cos \alpha|\sigma)]^2$ where $C_2(\cos \alpha|\sigma)$ is the two point function with the monopole, dipole and quadruple terms removed. Since the three point function is the products of two 2-point functions, it has a stronger dependence on the power spectra index $n$. Multipole expansion of the 2-point function gives:

$$C_2(\hat{m}, \hat{n}) = \sum_l C_l(2l + 1)P_l(\hat{m} \cdot \hat{n}). \quad (12)$$

For a power law spectrum $P(k) \sim k^n$, $C_l$ is given by

$$C_l = \frac{1}{5} \left( \frac{Q_{rms}}{T_0} \right)^2 \frac{\Gamma(2l + n - 1)}{\Gamma(2l + 5 - n)} \frac{\Gamma(n/2)}{\Gamma(3-n/2)}, \quad (13)$$
where \( Q_{rms} \) is the COBE measured quadruple\[11\] and \( T_0 \) is the black-body temperature of CBR. From the analysis of two point correlation function, COBE can only put a loose bound on power spectra index\[11\]: \( n = 1.1 \pm 0.5 \).

In Fig. 1, we plot the three point function for two different power spectra: a scale invariant \( n = 1 \) spectrum and a “tilted” spectrum where \( n = 0.7 \). Notice how the three point function depends strongly upon the power spectra index. Thus it is anticipated that the analysis of the three point correlation function will put a more stringent bound on \( n \).

In order to test cosmological structure formation scenarios through the three point temperature correlation function, we should have a clear handle on what various models predict. In the following, we will show that the cosmological phase transition can produce distinctive angular dependences other than the form given above.

Cosmological phase transitions are widely discussed in the context of the structure formation \[14\]. In the case of a primordial phase transition, the horizon size at the epoch of phase transition is small and topological defects will form according to the Kibble mechanism \[15\]. The analysis of the three point correlation for defect-induced temperature anisotropy depends crucially on the evolution of the defect-network and the work along this line is still in progress. In this paper, we will show that it is instructive to consider initially the three point function in the late-time phase transition (LTPT) scenario \[16\]. The calculation is considerably simplified in LTPT models for the following reasons: (1) the last scattering surface is assumed smooth in LTPT models, thus temperature anisotropies are solely produced by the generalized Rees-Sciama effect,

\[
\frac{\delta T}{T} = 2 \int \frac{\partial \phi}{\partial \eta} d\eta.
\]

since the fluctuations in density and gravitational potential are generated by the critical fluctuations at the critical point of the phase transition, \( \frac{\partial \phi}{\partial \eta} = \phi \delta(\eta - \eta_p) \), where \( \eta_p \) is the conformal time at the phase transition point.
Thus, in this latetime phase transition model, the temperature anisotropy takes the following simple form:

$$\frac{\delta T}{T} = 2\phi \eta = \eta_p.$$  

(15)

(2) For LTPT, the horizon size is large so that the finite horizon-size effect is negligible. We can calculate the three point correlation function from symmetry considerations. As pointed out by Polyakov [17], the three point correlation function of the fluctuating field $\psi$ is completely determined up to a dimensionless constant by the conformal symmetry of the system at the critical point. The explicit form for the three point function is given by

$$\xi_3 = \langle \psi(x_1)\psi(x_2)\psi(x_3) \rangle = \eta c_2(x_1, x_2)(c_2(x_2, x_3)c_2(x_3, x_1),$$  

(16)

where $c_2(x_1, x_2) = \langle \psi(x_1)\psi(x_2) \rangle$ is the two point function and $\eta$ is a constant. In this letter, we assume that the gravitational potential $\phi$ is directly proportional to the underlying fluctuating field $\psi$. For this case, the three point temperature correlation function has the following simple relation to the two point function:

$$\xi_T(\mathbf{m}, \mathbf{n}, \mathbf{l}) = A \cdot C_2(\mathbf{m}, \mathbf{n})C_2(\mathbf{n}, \mathbf{l})C_2(\mathbf{m}, \mathbf{l}),$$  

(17)

where $A$ is a dimensionless constant. The full beam-smearing effects of the three point correlation function given above is messy and we will report it elsewhere. However, in the special case when $\mathbf{m} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{l} = \mathbf{l} \cdot \mathbf{m} = \cos \alpha$, it can be approximated as $\left[C_2(\cos \alpha | \sigma)\right]^3$, where $C_2(\cos \alpha | \sigma)$ is the 2-pt function with finite beam width $\sigma$, with monopole, dipole and quadruple terms subtracted. We plot the approximated three point function generated by the phase transition in Fig. (2).

The result obtained from Eq. (9) & Eq. (13) strongly suggest that the general form of the three point function, expressed in terms of two point functions, is given by:

$$\xi_T(\mathbf{m}, \mathbf{n}, \mathbf{l}) = Q \cdot [C_2(\mathbf{m}, \mathbf{n})C_2(\mathbf{n}, \mathbf{l}) + C_2(\mathbf{m}, \mathbf{n})C_2(\mathbf{n}, \mathbf{l}) + C_2(\mathbf{m}, \mathbf{l})C_2(\mathbf{n}, \mathbf{l})]$$

$$+ A \cdot C_2(\mathbf{m}, \mathbf{n})C_2(\mathbf{n}, \mathbf{l})C_2(\mathbf{m}, \mathbf{l}),$$  

(18)
where $Q$ and $A$ are constants. This is the archetype form of the three point correlation function that the experimental analysis should be compared with.

The Gaussianity can be tested through the existing COBE and the forthcoming South Pole and Balloon CBR anisotropy data by three point temperature correlation function. In this letter, we have focused on the COBE data although the idea and method discussed can equally apply to South Pole and Balloon experiments. The data set from COBE DMR is especially suitable for carrying out this test. On the one hand, the beam width of COBE is $7^\circ$, which is much larger than the horizon size at decoupling ($\sim 2^\circ$). Most non-linear causal processes which may lead to non-Gaussian signatures on the cosmic microwave background (CMB) sky are smoothed out by the beam. On the other hand, the COBE CMB map covers the whole sky. Thus, the boundary effects will be minimized. However, the detected sky signal contain both the intrinsic CBR temperature fluctuation and the instrumental noises,

$$\frac{\delta T}{T}_{\text{obs}} = \frac{\delta T}{T}_{\text{CBR}} + \frac{\delta T}{T}_{\text{noise}}.$$ (19)

The signal to noise ratio of the COBE data is 1:1 and this is typical in all current CBR temperature anisotropy experiments. Thus, it is important to consider the noise term seriously in the analysis of the three point correlation function. Even if future analysis of the COBE data do find a non-vanishing three-point temperature correlation, it may due to the instrumental noise. However, if one adopts the usual assumption about the noise term: (1) the noise is random Guassian noise which is not correlated temporally or spatially; (2) the noise is not correlated with the CBR signal, then

$$\xi_{\text{obs}} = \xi_{\text{CBR}}.$$ (20)

the three point correlation calculated from the raw observational data will reflect directly the three point temperature correlation of the CBR, even if the noise term is comparable to the signal. This is the another advantage to using the three point function to test Gaussianity of the initial perturbations.
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FIGURE CAPTIONS

Fig. 1: The dependence of the 3-pt temperature correlation on the power spectra index and angular separation. The solid line is for a scale invariant density perturbation; the dash line is for a “tilted” spectrum with spectra index = 0.7.

Fig. 2: The angular dependence for the three point function in different scenarios: the solid line represents the 3-pt function generated by inflation; the dash line represents the one generated by a cosmological phase transition.