Additional Fibonacci–Bernoulli relations

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Abstract

We continue our study on relationships between Fibonacci (Lucas) numbers and Bernoulli numbers and polynomials. The derivations of our results are based on functional equations for the respective generating functions, which in our case are combinations of hyperbolic functions. Special cases and some corollaries will highlight interesting aspects of our findings.

1 Introduction

Fibonacci and Lucas numbers satisfy the linear second-order recurrence

\[ u_n = u_{n-1} + u_{n-2}, \quad n \geq 2, \]

with initial conditions \( F_0 = 0, \) \( F_1 = 1 \) and \( L_0 = 2, \) \( L_1 = 1, \) respectively. Both sequences have a long history and are very popular among mathematicians as they appear in important mathematical branches such as number theory, combinatorics and graph theory. They have entries A000045 and A000032 in the On-Line Encyclopedia of Integer Sequences \cite{OEIS}. Excellent references on these sequences are the books \cite{12,14}.
As usual, Bernoulli polynomials are defined by the exponential generating function [11, Chapter 1, Section 1.3]

\[ H(x, z) := \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}, \quad |z| < 2\pi. \]

For \( n \geq 0 \), they satisfy the following functional relations [11, Chapter 1, Section 1.3]:

1. \( B_n(1 + x) - B_n(x) = nx^{n-1}, \)
2. \( B_n(1 - x) = (-1)^n B_n(x), \)
3. \( B_n(-x) = (-1)^n (B_n(x) + nx^{n-1}), \)
4. \( B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n \left( x + \frac{k}{m} \right), \quad m \geq 1. \)

Also, Bernoulli polynomials have the property

\[ B_n(x + z) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x) z^k, \]

from which we get

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \]

where \( B_n = B_n(0) \) are the Bernoulli numbers. The Bernoulli numbers are rational numbers starting with

\[ B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \]

and \( B_{2n+1} = 0 \) for \( n \geq 1 \).

The following identities connect Fibonacci numbers to Bernoulli polynomials and are proved in [7]: for each integers \( n \geq 0, \ j \geq 1, \) and \( x \in \mathbb{C} \),

\[ \sum_{k=0}^{n} \binom{n}{k} F_{jk} (\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j ((\sqrt{5}x + \beta)F_j + F_{j-1})^{n-1}, \]

\[ \sum_{k=0}^{n} \binom{n}{k} F_{jk} (-\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j ((\alpha - \sqrt{5}x)F_j + F_{j-1})^{n-1}, \]

where \( \alpha = \frac{1 + \sqrt{5}}{2} \) is the golden ratio and \( \beta = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\alpha} \). Other results in this direction are contained in [3, 4, 5, 9, 10, 15, 16, 17, 18], among others.

In this paper, we state new relations involving Fibonacci and Lucas numbers and Bernoulli numbers and polynomials. We will work with many exponential generating functions. The results stated are complements of the recent discoveries from [5, 7, 8]. Some of our results were announced without proofs in [2].

2
2 Identities from hidden threefold convolutions

We begin with a known lemma [12, Vol. 1, p. 251].

Lemma 1. Let $n$ and $j$ be positive integers. Then

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk} F_{j(n-k)} = \frac{2^n L_{jn} - 2^n L_j}{5},$$ (6)

$$\sum_{k=0}^{n} \binom{n}{k} L_{jk} L_{j(n-k)} = 2^n L_{jn} + 2^n L_j,$$ (7)

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk} L_{j(n-k)} = 2^n F_{jn}.$$ (8)

Our first main result is the following theorem.

Theorem 2. Let $n$ and $j$ be positive integers. Then

$$\sum_{k=0}^{n} \binom{n}{k} \left(2^k L_{jk} - 2^k L_j\right)(\sqrt{5} F_j)^{n-k} \frac{B_{n-k+2}}{n-k+2} = \frac{2^{n+2} L_{j(n+2)} - 2^n L_j^{n+2}}{5(n+1)(n+2)F_j^2} - L_j,$$ (9)

$$\sum_{k=0}^{n} \binom{n}{k} \left(2^k L_{jk} + 2^k L_j\right)(\sqrt{5} F_j)^{n-k} \frac{-2^{n-k+2} - 1}{n-k+2} B_{n-k+2} = L_j,$$ (10)

and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^k F_{jk}(\sqrt{5} F_j)^{n-k} B_{n-k+2}}{n-k+2} + \frac{2}{\sqrt{5}} \sum_{k=0}^{n-1} \binom{n}{k} L_j^k (\sqrt{5} F_j)^{n-k} \frac{B_{n-k+1}}{n-k+1}$$

$$= \frac{2^{n+3} F_{j(n+2)}}{5(n+1)(n+2)F_j^2} - \frac{2L_j^{n+1}}{5(n+1)F_j}.$$ (11)

Proof. Let $F(z)$ and $L(z)$ denote the exponential generating functions of sequences $(F_{jn})_{n \geq 0}$ and $(L_{jn})_{n \geq 0}$, respectively, with $j \geq 1$. Then, it is easy to derive

$$F^2(z) = \frac{4}{5} e^{L_j z} \sinh^2 \left(\frac{\sqrt{5} F_j}{2} z\right),$$ (12)

$$L^2(z) = 4 e^{L_j z} \cosh^2 \left(\frac{\sqrt{5} F_j}{2} z\right).$$ (13)

3
From the power series for the cotangent [11, Chapter 1.3]
\[
\coth z = \sum_{n=0}^{\infty} 2^{2n} B_{2n} \frac{z^{2n-1}}{(2n)!}
\]
we get
\[
-\frac{d}{dz} \coth z = \frac{1}{\sinh^2 z} = \frac{1}{z^2} - \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n} B_{2n} z^{2n-2}}{(2n)!}.
\]

Hence, we see that the functional equation (12), using (6) and (14), can be written equivalently as
\[
e^{L_j z} = S_1(z) - S_2(z)
\]
with
\[
S_1(z) = \frac{1}{5F_j^2} \sum_{n=0}^{\infty} \frac{(2^n L_jn - 2L_j^n) z^{n-2}}{n!},
\]
\[
S_2(z) = \sum_{n=0}^{\infty} \frac{(2^n L_jn - 2L_j^n) z^n}{n!} \cdot \sum_{n=1}^{\infty} \frac{(2n-1)5^n 2^{2n-2} B_{2n} z^{2n-2}}{(2n)!}.
\]

In the series \(S_1(z)\) the first two terms are zero and therefore
\[
S_1(z) = \frac{1}{5F_j^2} \sum_{n=0}^{\infty} \frac{2^{n+2}L_{j(n+2)} - 2L_j^{n+2} z^n}{(n+2)(n+1) n!}.
\]

The second series \(S_2(z)\) equals
\[
S_2(z) = \sum_{n=0}^{\infty} \frac{(2^n L_jn - 2L_j^n) z^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{(2n+1)5^n F_j 2^{2n} B_{2n+2} z^{2n}}{(2n + 2)!}
\]
and is a simple Cauchy product. Expanding and comparing the coefficients of \(z^n\) proves (9).

Identity (10) follows from the functional equation (10) combined with (7) and
\[
\frac{1}{\cosh^2 z} = \frac{d}{dz} \tanh z = \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n} - 1)B_{2n} z^{2n-2}}{(2n)!}.
\]

The underlying functional equation for identity (11) is
\[
\frac{F(z) L(z)}{\sinh^2 \left( \frac{\sqrt{5}F_j z}{2} \right)} = \frac{4}{\sqrt{5}} e^{L_j z} \coth \left( \frac{\sqrt{5}F_j z}{2} \right).
\]

(15)
By (8), (12) and (13), the LHS of (15) is
\[
\frac{F(z)L(z)}{\sinh^2\left(\frac{\sqrt{5}F_j}{2}z\right)} = \frac{4}{5F_j^2} \sum_{n=0}^{\infty} 2^n F_j z^{n-2} n! - 4 \sum_{n=0}^{\infty} 2^n F_j z^n n! \cdot \sum_{n=0}^{\infty} B_{2n+2} 2^n F_j z^{2n} (2n)! = \frac{8}{5F_j z} + 4 \left( \frac{1}{5F_j^2} \sum_{n=0}^{\infty} 2^{n+2} F_j (n+2) z^n (n+1)! n! \cdot \sum_{n=0}^{\infty} B_{2n+2} 2^n F_j z^{2n} (2n)! \right),
\]
whereas the RHS of (15) equals
\[
\frac{4}{\sqrt{5}} e^{L_j z} \coth\left(\frac{\sqrt{5}F_j}{2}z\right) = \frac{4}{\sqrt{5}} \left( \frac{2}{\sqrt{5}F_j} \sum_{n=0}^{\infty} L_j^n z^n n! + \sum_{n=0}^{\infty} L_j^n z^n \cdot 2 \sum_{n=0}^{\infty} B_{2n} 5^{2n+1} F_j^{2n+1} z^{2n-1} (2n)! \right) = \frac{8}{5F_j z} + \frac{8}{\sqrt{5}} \left( \sum_{n=0}^{\infty} L_j^{n+1} z^n (n+1)! n! + z \sum_{n=0}^{\infty} L_j^n z^n n! \cdot \sum_{n=0}^{\infty} (2n+2) B_{2n+2} 2^{n+1} F_j z^{2n+1} (2n)! \right).
\]

Now, we can apply the Cauchy multiplication theorem on both sides. When simplifying the RHS, use \( \binom{n-1}{k} n^{-k} = \binom{n}{k} \).

Formula (9) has been derived recently in [10].

When \( j = 1 \), then the special cases of Theorem 2 reduce to
\[
\sum_{n-k\equiv 0(\mod 2)}^{n} \binom{n}{k} (2^k L_k - 2)(\sqrt{5})^{n-k} \frac{B_{n-k+2}}{n-k+2} = \frac{2^{n+2} L_{n+2} - 2}{5(n+1)(n+2)} - 1, \tag{16}
\]
and
\[
\sum_{n-k\equiv 0(\mod 2)}^{n} \binom{n}{k} (2^k L_k + 2)(\sqrt{5})^{n-k} \frac{2^{n-k+2} - 1}{n-k+2} B_{n-k+2} = 1
\]

and
\[
\sum_{n-k\equiv 0(\mod 2)}^{n} \binom{n}{k} 2^k F_k (\sqrt{5})^{n-k} \frac{B_{n-k+2}}{n-k+2} + \frac{2}{\sqrt{5}} \sum_{n-k\equiv 0(\mod 2)}^{n-1} \binom{n}{k} (\sqrt{5})^{n-k} \frac{B_{n-k+1}}{n-k+1} = \frac{2}{5(n+1)} \left( \frac{2^{n+1} F_{n+2}}{n+2} - 1 \right).
\]

The identity (16) appeared as a problem proposal in [6].

**Remark 3.** Using the fact that, for any sequence \( (a_n)_{n\geq 0} \),
\[
\sum_{n-k\equiv 0(\mod 2)}^{n} a_k = \sum_{k=0}^{n} \frac{1 + (-1)^{n-k}}{2} a_k, \quad \sum_{n-k\equiv 1(\mod 2)}^{n} a_k = \sum_{k=0}^{n} \frac{1 - (-1)^{n-k}}{2} a_k,
\]

identities (9)–(11) possess the following equivalent forms without the mod-notation (with \( n \) even):
\[
\begin{aligned}
\sum_{k=0}^{n/2} \binom{n}{2k} \frac{n-2k-1}{(k+1)(2k+1)} \cdot \frac{L_j^{2k+2} - 2^{2k+1} L_{2j(k+1)}}{(5F_j^2)^{k+1}} B_{n-2k} &= \left( \frac{L_j}{\sqrt{5} F_j} \right)^n, \\
\sum_{k=0}^{n/2} \binom{n}{2k} \frac{2^{n-2k+2} - 1}{n-2k+2} \cdot \frac{2 L_j^{2k} + 2^k L_{2j}^k}{(5F_j^2)^k} B_{n-2k+2} &= \left( \frac{L_j}{\sqrt{5} F_j} \right)^n, \\
\sum_{k=0}^{n/2} \binom{n}{2k} \left( \frac{4}{5F_j^2} \right)^k \frac{2^{n-2k-1}}{2k+1} \cdot \frac{F_j^{(2k+1)} + F_j \frac{L_j^{2k}}{4k}}{B_{n-2k}} &= 0. 
\end{aligned}
\]

**Theorem 4.** For all positive integers \( n \) and \( j \),
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{k+1} \left( -1 \right)^k \frac{F_j^{(k+1)}}{L_j^k} \left( \frac{L_j}{\sqrt{5} F_j} \right)^n - (1 + (-1)^n) \frac{2^{k+3} - 2}{k+2} F_j B_{k+2} = 0.  \tag{17}
\]
In particular,
\[
\sum_{k=1}^{2n} (-1)^k \binom{2n-1}{k-1} \frac{2^k F_j}{k L_j} = 0.  \nonumber
\]
Similarly,
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k \left( -1 \right)^k \frac{L_{jk}}{L_j^k} \left( \frac{L_j}{\sqrt{5} F_j} \right)^n - \frac{1 + (-1)^n}{n-k+1} B_k = 1 + (-1)^n,  \tag{18}
\]
\[
\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k} \frac{2^k L_{jk}}{L_j^k} = 0.  \nonumber
\]

**Proof.** From the basic identity \( \sinh z = \tanh z \cdot \cosh z \) we get
\[
\frac{\sqrt{5}}{2} F(z)e^{-\frac{L_j}{\sqrt{5} F_j} z} = \tanh \left( \frac{\sqrt{5} F_j}{2} z \right) \cosh \left( \frac{\sqrt{5} F_j}{2} z \right).  \tag{19}
\]
Then
\[
\text{LHS of (19)} = \frac{\sqrt{5}}{2} \sum_{n=1}^{\infty} F_j^n \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} \left( -\frac{L_j}{2} \right)^n \frac{z^n}{n!} = \frac{\sqrt{5}}{2} \sum_{n=0}^{\infty} F_j^{(n+1)} \frac{z^{n+1}}{(n+1)!} \cdot \sum_{n=0}^{\infty} \left( -\frac{L_j}{2} \right)^n \frac{z^n}{n!} = \frac{\sqrt{5}}{2} z \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{F_j^{(k+1)}}{k+1} \left( -\frac{L_j}{2} \right)^{n-k} \frac{z^n}{n!}.  
\]

whereas

\[
\text{RHS of (19)} = \sum_{n=1}^{\infty} 2^{2n}(2^{2n} - 1) \left(\frac{\sqrt{5} F_j}{2}\right)^{2n-1} B_{2n} \frac{z^{2n-1}}{(2n)!} \cdot \sum_{n=0}^{\infty} \left(\frac{\sqrt{5} F_j}{2}\right)^{2n} \frac{z^{2n}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} 2^{2n+2}(2^{2n+2} - 1) \left(\frac{\sqrt{5} F_j}{2}\right)^{2n+1} B_{2n+2} \frac{z^{2n+1}}{(2n+2)!} \cdot \sum_{n=0}^{\infty} \left(\frac{\sqrt{5} F_j}{2}\right)^{2n} \frac{z^{2n}}{(2n)!}
\]

\[
= z \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1 + (-1)^n}{2} \cdot \frac{1 + (-1)^k}{2} 2^{k+2}(2^{k+2} - 1)
\times \left(\frac{\sqrt{5} F_j}{2}\right)^{k+1} \frac{B_{k+2}}{(k+2)!} \left(\frac{\sqrt{5} F_j}{2}\right)^{n-k} \frac{z^n}{(n-k)!}
\]

\[
= z \left(\frac{\sqrt{5} F_j}{2}\right)^{n+1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1 + (-1)^n}{2} \frac{1 + (-1)^k}{2}
\times \left(\frac{n}{k}\right) 2^{k+2}(2^{k+2} - 1) \frac{B_{k+2}}{(k+2)(k+1) n!} \frac{z^n}{(k+2)(k+1) n!}
\]

Note that above we used

\[
\sum_{n=0}^{\infty} a_{2n} z^{2n} \cdot \sum_{n=0}^{\infty} b_{2n} z^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1 + (-1)^n}{2} \frac{1 + (-1)^k}{2} a_k b_{n-k} z^n.
\]

Comparing the coefficients of \(z^n\) after some simple manipulations we have (17).
Identity (18) follows from \(\cosh z = \coth z \cdot \sinh z\) which gives

\[
\frac{1}{2} L(z) e^{-\frac{\sqrt{5} F_j}{2} z} = \coth \left(\frac{\sqrt{5} F_j}{2} z\right) \sinh \left(\frac{\sqrt{5} F_j}{2} z\right).
\]

(20)

Proceeding as before,

\[
\text{LHS of (20)} = \frac{1}{2} \sum_{n=0}^{\infty} L_{jn} \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} \left(-\frac{L_j}{2}\right)^n \frac{z^n}{n!}
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} L_{jk} \left(-\frac{L_j}{2}\right)^{n-k} \frac{z^n}{n!},
\]
and

\[
\text{RHS of (20)} = \sum_{n=0}^{\infty} \sum_{k=0}^{2n-1} \frac{(\sqrt{5}F_j)^{2n-1}}{(2n)!} B_{2n} \frac{z^{2n-1}}{(2n)!} \cdot \sum_{n=0}^{\infty} \frac{(\sqrt{5}F_j)^{2n+1}}{(2n+1)!} z^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n)}{k} \frac{1 + (-1)^n}{2} \frac{1}{2} \frac{1 + (-1)^k}{2} \frac{\sqrt{5}F_j}{2} \frac{(\sqrt{5}F_j)^{k-1}}{n-k+1} B_k \frac{\sqrt{5}F_j}{2} \frac{(\sqrt{5}F_j)^{n-k+1} z^n}{n!}
\]

Comparing the coefficients of \(z^n\) after some simple manipulations we have (18).

In view of the binomial theorem and the Binet formula, (18) is equivalent to

\[
(1 + (-1)^n)^n \sum_{k=0}^{n} \frac{(n)}{k} \frac{2^kB_k}{n-k+1} = 0,
\]

so that we have

\[
\sum_{k=0}^{n} \frac{(n)}{k} \frac{2kB_k}{n-k+1} = 0, \quad n \text{ even}.
\]

**Theorem 5.** For all positive integers \(n\) and \(j\),

\[
\sum_{k=0}^{[n/2]} \frac{(n)}{2k} \frac{(5F_j^2)^k}{2k+1} \left( \frac{F_j(n-2k+1)}{n-2k+1} B_{2k} - \frac{F_j L_{n-2k}}{2n} \right) = 0, \quad (21)
\]

\[
\sum_{k=0}^{[n/2]} \frac{(n)}{2k} \frac{(5F_j^2)^k}{2k+1} \left( \frac{4^k+1}{k+1} L_j(n-2k)B_{2k+2} - \frac{L_j^{n-2k}}{2n} \right) = 0. \quad (22)
\]

**Proof.** Use the exponential generating functions from (12) and (13) in conjunction with

\[
\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_{2n} z^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_{n-2k} b_{2k} z^n
\]

and \(\sinh z = \frac{\cosh z}{\coth z}, \quad \cosh z = \frac{\sinh z}{\tanh z} \).

On account of the identity

\[
2 \sum_{k=0}^{[n/2]} \frac{(n)}{2k} x^{n-2k} z^{2k} = (x + z)^n + (x - z)^n,
\]

formula (21) can also be written as

\[
\sum_{k=0}^{[n/2]} \frac{(n)}{2k} \frac{(5F_j^2)^k F_j(n-2k+1)}{n-2k+1} B_{2k} = \frac{F_j L_{n+1}}{2}.
\]
3 Special Bernoulli polynomial identities

The properties of the Bernoulli polynomials stated in Lemmas 6 and 7 below are direct consequences of the functional relations (1) and (2).

Lemma 6. If \( n \) is any non-negative integer, then

\[
B_n(1 + x) \pm B_n(1 + y) = B_n(x) \pm B_n(y) + n(x^{n-1} \pm y^{n-1}),
\]

\[
B_n(1 + x) \pm B_n(1 + y) = (-1)^n \left( B_n(1 - x) \pm B_n(1 - y) \right) + n(x^{n-1} \pm y^{n-1}),
\]

\[
B_n(-x) \pm B_n(-y) = (-1)^n \left( B_n(x) \pm B_n(y) + n(x^{n-1} \pm y^{n-1}) \right).
\]

(23)

Lemma 7. Let \( n \) be any non-negative integer. If \( x - y = 1 \), then

\[
B_n(x) - B_n(y) = ny^{n-1},
\]

while if \( x + y = 1 \), then

\[
B_n(x) - (-1)^n B_n(y) = 0,
\]

\[
B_n(1 + x) - B_n(1 + y) = \begin{cases} 
  n(x^{n-1} - y^{n-1}), & \text{if } n \text{ is even;} \\
  -2B_n(y) + n(x^{n-1} - y^{n-1}), & \text{otherwise.}
\end{cases}
\]

(25)

Lemma 8. For real or complex \( z \), let a given well-behaved function \( h(z) \) have in its domain the representation

\[
h(z) = \sum_{k=c_1}^{c_2} v_k z^w_k,
\]

where \( v_k \) and \( w_k \) are given real sequences and \(-\infty \leq c_1 < c_2 \leq +\infty\). Let \( i \) and \( m \) be integers. Then

\[
\sum_{k=c_1}^{c_2} F_{i w_k + m} v_k z^{w_k} = \frac{1}{\sqrt{5}} \left( \alpha^m h(\alpha^i z) + \beta^m h(\beta^i z) \right),
\]

(26)

\[
\sum_{k=c_1}^{c_2} L_{i w_k + m} v_k z^{w_k} = \alpha^m h(\alpha^i z) - \beta^m h(\beta^i z).
\]

(27)

Lemma 8 written in a slightly different form we can find in [1, Theorem 1].

Theorem 9. Let \( j \) and \( m \) be integers and \( n \) a non-negative integer. Then

\[
\sum_{k=0}^{n} \binom{n}{k} F_{j k + m} B_{n-k}(x) z^k = \frac{1}{\sqrt{5}} \left( \alpha^m B_n(x + \alpha^j z) - \beta^m B_n(x + \beta^j z) \right),
\]

(28)

\[
\sum_{k=0}^{n} \binom{n}{k} L_{j k + m} B_{n-k}(x) z^k = \alpha^m B_n(x + \alpha^j z) + \beta^m B_n(x + \beta^j z).
\]

(29)
Proof. Use (26) and (27) with
\[ h(z) = B_n(x + z) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x)z^k, \]
so that \( w_k = k, v_k = \binom{n}{k} B_{n-k}(x), c_1 = 0 \) and \( c_2 = n. \)

Setting \( x = 0 \) in (28) and (29) yield the following Fibonacci–Bernoulli and Lucas–Bernoulli relations.

**Corollary 10.** Let \( j \) and \( m \) be integers and \( n \) non-negative integer. Then
\[ \sum_{k=0}^{n} \binom{n}{k} F_{jk+m} B_{n-k} z^k = \frac{1}{\sqrt{5}} \left( \alpha^m B_n(\alpha^j z) - \beta^m B_n(\beta^j z) \right), \quad (30) \]
\[ \sum_{k=0}^{n} \binom{n}{k} L_{jk+m} B_{n-k} z^k = \alpha^m B_n(\alpha^j z) + \beta^m B_n(\beta^j z). \quad (31) \]

**Theorem 11.** Let \( j \) and \( m \) be integers and \( n \) non-negative integer. Then
\[ \sum_{k=0}^{n} \binom{n}{k} F_{jk+m} B_{n-k} \frac{L^k}{L_j} = \begin{cases} F_m B_n \left( \frac{\alpha^j}{L_j} \right), & \text{if } n \text{ is even;} \\ \frac{L_m}{\sqrt{5}} B_n \left( \frac{\alpha^j}{L_j} \right), & \text{if } n \text{ is odd}, \end{cases} \quad (32) \]
\[ \sum_{k=0}^{n} \binom{n}{k} L_{jk+m} B_{n-k} \frac{L^k}{L_j} = \begin{cases} L_m B_n \left( \frac{\alpha^j}{L_j} \right), & \text{if } n \text{ is even;} \\ \sqrt{5} F_m B_n \left( \frac{\alpha^j}{L_j} \right), & \text{if } n \text{ is odd}. \end{cases} \]

Proof. Choose \( x = \frac{\alpha^j}{L_j} \) in (2) and use the Binet formula \( L_j = \alpha^j + \beta^j \) to obtain
\[ B_n \left( \frac{\beta^j}{L_j} \right) = (-1)^n B_n \left( \frac{\alpha^j}{L_j} \right). \quad (33) \]

Now use this information in Corollary 10 with \( z = \frac{1}{L_j}. \)

**Lemma 12.** Let \( a, b, c \) and \( d \) be rational numbers and \( \lambda \) an irrational number. Then \( a + \lambda b = c + \lambda d \) if and only if \( a = c \) and \( b = d. \)

**Corollary 13.** Let \( j \) be an integer and \( n \) a non-negative integer. Then
\[ \sum_{k=0}^{n} \binom{n}{k} F_{jk-1} B_{n-k} = B_n \left( \frac{\alpha^j}{L_j} \right), \quad n \text{ even}, \quad (34) \]
Proof. Since the expression on the left side of (32) is rational, being the finite sum of rational numbers, it follows that \( B_n \left( \frac{\alpha^j}{L_j} \right) \) is a rational number for even \( n \). Now, using (5) and relation \( \alpha^s = \alpha F_s + F_{s-1} \), we have

\[
B_n \left( \frac{\alpha^j}{L_j} \right) = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k \alpha^{j(n-k)}}{L_j^{n-k}}
\]

\[
= \alpha \sum_{k=0}^{n} \binom{n}{k} \frac{B_k F_j (n-k)}{L_j^{n-k}} + \sum_{k=0}^{n} \binom{n}{k} \frac{B_k F_j (n-k-1)}{L_j^{n-k}},
\]

from which identities (34) and (35) follow when we invoke Lemma 12. The proof of (36) and (37) is similar. 

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{F_{jk} B_{n-k}}{L_j^k} = 0, \quad \text{n even, (35)}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{L_{jk-1} B_{n-k}}{L_j^k} = \sqrt{5} B_n \left( \frac{\alpha^j}{L_j} \right), \quad \text{n odd, (36)}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{L_{jk} B_{n-k}}{L_j^k} = 0, \quad \text{n odd. (37)}
\]

Remark 14. We observe from identity (32) that \( \sqrt{5} B_n \left( \frac{\alpha^j}{L_j} \right) \) is rational for \( n \) odd.

Theorem 15. Let \( j \) and \( m \) be integers and \( n \) non-negative integer. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{F_{jk+m}}{L_j^k} B_{n-k} = \begin{cases} 
F_m B_n \left( \frac{\alpha^j}{L_j} \right) + \frac{n F_j (n-1)+m}{L_j^{n-1}}, & \text{if } n \text{ is even;}

-\frac{L_m}{\sqrt{5}} B_n \left( \frac{\alpha^j}{L_j} \right) - \frac{n F_j (n-1)+m}{L_j^{n-1}}, & \text{otherwise.}
\end{cases}
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{L_{jk+m}}{L_j^k} B_{n-k} = \begin{cases} 
L_m B_n \left( \frac{\alpha^j}{L_j} \right) + \frac{n F_j (n-1)+m}{L_j^{n-1}}, & \text{if } n \text{ is even;}

-\sqrt{5} F_m B_n \left( \frac{\alpha^j}{L_j} \right) - \frac{n L_j (n-1)+m}{L_j^{n-1}}, & \text{otherwise,}
\end{cases}
\]

Proof. Identities (3) and (33) give

\[
B_n \left( -\frac{\alpha^j}{L_j} \right) + B_n \left( -\frac{\beta^j}{L_j} \right) = \begin{cases} 
2 B_n \left( \frac{\alpha^j}{L_j} \right) + \frac{n F_j (n-1)+m}{L_j^{n-1}}, & \text{if } n \text{ is even;}

-\frac{n L_j (n-1)+m}{L_j^{n-1}}, & \text{otherwise,}
\end{cases}
\]

\[
B_n \left( -\frac{\alpha^j}{L_j} \right) - B_n \left( -\frac{\beta^j}{L_j} \right) = \begin{cases} 
\frac{\sqrt{5} n F_j (n-1)}{L_j^{n-1}}, & \text{if } n \text{ is even;}

-2 B_n \left( \frac{\alpha^j}{L_j} \right) - \frac{\sqrt{5} n F_j (n-1)}{L_j^{n-1}}, & \text{otherwise.}
\end{cases}
\]
Use these in Corollary 10 with \( z = -\frac{1}{L_j} \). Note the use of the known identities [12, Vol. 1, p. 111]

\[ F_r L_s + F_s L_r = 2F_{r+s}, \quad L_r L_s + 5F_r F_s = 2L_{r+s}. \]

\[ \square \]

**Theorem 16.** Let \( j \) be integer and \( n \) non-negative integer. Then

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{L_j^k} B_{n-k} = \frac{n}{\sqrt{5}} \left( \frac{\sqrt{5}F_j}{L_j} \right)^{n-1}, \quad n \text{ even}, \quad \text{(38)} \]

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{L_j^k} B_{n-k} = n \left( \frac{\sqrt{5}F_j}{L_j} \right)^{n-1}, \quad n \text{ odd}. \quad \text{(39)} \]

**Proof.** Setting \( x = \frac{2\alpha^j}{L_j} - 1 \) in (1) yields

\[ B_n \left( \frac{2\alpha^j}{L_j} \right) - B_n \left( \frac{F_j \sqrt{5}}{L_j} \right) = n \left( \frac{F_j \sqrt{5}}{L_j} \right)^{n-1}. \quad \text{(40)} \]

Setting \( x = \frac{2\beta^j}{L_j} \) in (2) gives

\[ B_n \left( \frac{2\beta^j}{L_j} \right) - B_n \left( \frac{F_j \sqrt{5}}{L_j} \right) = 0, \quad n \text{ even}. \quad \text{(41)} \]

From (40) and (41) we find

\[ B_n \left( \frac{2\alpha^j}{L_j} \right) - B_n \left( \frac{2\beta^j}{L_j} \right) = n \left( \frac{F_j \sqrt{5}}{L_j} \right)^{n-1}, \quad n \text{ even}, \]

from which, upon use in (30), with \( m = 0 \) and \( z = \frac{2}{L_j} \), identity (38) follows.

Using \( x = \frac{2\beta^j}{L_j} \) in (2) gives

\[ B_n \left( \frac{2\beta^j}{L_j} \right) + B_n \left( \frac{F_j \sqrt{5}}{L_j} \right) = 0, \quad n \text{ odd}. \quad \text{(42)} \]

Addition of (40) and (42) produces

\[ B_n \left( \frac{2\alpha^j}{L_j} \right) + B_n \left( \frac{2\beta^j}{L_j} \right) = n \left( \frac{F_j \sqrt{5}}{L_j} \right)^{n-1}, \quad n \text{ odd}, \]

from which, upon use in (31), with \( m = 0 \) and \( z = \frac{2}{L_j} \), identity (39) follows. \[ \square \]

The result of the next theorem exhibits strong similarity to the polynomial identities from Introduction.
Theorem 17. The following identity is valid for all $n \geq 0$, $j \geq 1$, and complex $x$:

$$
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk}(\pm \sqrt{5} F_j)^{n-k} B_{n-k}(x) = n F_j \left( (\pm \sqrt{5} F_j x + L_j)^{n-1} + (\pm \sqrt{5} F_j (x - 1) + L_j)^{n-1} \right).
$$

(43)

Proof. Since

$$
H(x, \sqrt{5} F_j z) = \sqrt{5} F_j e^{\frac{\sqrt{5} F_j (2x-1) z}{2}} 2 \sinh \left( \frac{\sqrt{5} F_j z}{2} \right),
$$

we get the relation

$$
F(z) L(z) H(x, \sqrt{5} F_j z) = 2 F_j z e^{L_j z} e^{\frac{\sqrt{5} F_j (2x-1) z}{2}} \cosh \left( \frac{\sqrt{5} F_j z}{2} \right).
$$

Hence,

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk}(\sqrt{5} F_j)^{n-k} B_{n-k}(x) \frac{z^n}{n!} = F_j z e^{\frac{\sqrt{5} F_j (2x-1) + L_j}{2} z} \left( e^{\frac{\sqrt{5} F_j z}{2}} + e^{-\frac{\sqrt{5} F_j z}{2}} \right)
= F_j z \left( e^{(\sqrt{5} F_j x + L_j) z} + e^{(\sqrt{5} F_j (x-1) + L_j) z} \right).
$$

This proves (43) with the positive root. The second follows upon replacing $x$ by $1 - x$ and using (2).

Setting $x = 0$ in Theorem 4, we have the following.

Corollary 18. For $n \geq 0$ and $j \geq 1$, 

$$
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk}(\sqrt{5} F_j)^{n-k} B_{n-k}(x) = n F_j \left( L_j^{n-1} + (\pm \sqrt{5} F_j + L_j)^{n-1} \right).
$$

Corollary 19. For $n \geq 0$ and $j \geq 1$, 

$$
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk}(-\sqrt{5} F_j)^{n-k} B_{n-k}(\alpha) n F_j 2^{1-n} \left( (\sqrt{5} F_j + L_{j+3})^{n-1} + (\sqrt{5} F_j + L_{j+3})^{n-1} \right),
$$

where $\alpha$ is the golden ratio. Also, for $j \geq 3$, we have the analog identity

$$
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk}(-\sqrt{5} F_j)^{n-k} B_{n-k}(\alpha)
= n F_j 2^{1-n} \left( (\sqrt{5} F_j - L_{j-3})^{n-1} + (-\sqrt{5} F_j - L_{j-3})^{n-1} \right).
$$

(44)
Proof. Set \( x = \alpha \) in Theorem 4 and simplify using \( 5F_n = L_{n+1} + L_{n-1} \).

We mention the special case of (44) for \( j = 3 \):

\[
\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^{n-k} F_{3k} B_{n-k}(\alpha) = (-1)^{n-1} n L_{n-1}.
\]

Also, inserting \( \beta \) in (43) and setting \( j = 1 \) we can state the identity

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_k (\sqrt{5})^{n-k} B_{n-k}(\beta) = (-1)^{n-1} n L_{2n-2}.
\]

Corollary 20. Let \( n, j \) and \( q \) be integers with \( n, j \geq 1 \) and \( q \geq 2 \). Then

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\pm \sqrt{5} F_j)^{n-k} (q^{1-(n-k)} - 1) B_{n-k} = n F_j q^{1-n} \sum_{r=1}^{q-1} \left( (\pm \sqrt{5} F_j r + q L_j)^{n-1} + (\pm \sqrt{5} F_j (r - q) + q L_j)^{n-1} \right).
\]

Proof. Formula (4) gives

\[
(q^{1-n} - 1) B_n = \sum_{r=1}^{q-1} B_n \left( \frac{n}{q} \right).
\]

Therefore, we can write

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\pm \sqrt{5} F_j)^{n-k} (q^{1-(n-k)} - 1) B_{n-k} = n F_j \sum_{r=1}^{q-1} \left( (\pm \sqrt{5} F_j r + q L_j)^{n-1} + (\pm \sqrt{5} F_j (\frac{r}{q} - 1) + L_j)^{n-1} \right) = n F_j q^{1-n} \sum_{r=1}^{q-1} \left( (\pm \sqrt{5} F_j r + q L_j)^{n-1} + (\pm \sqrt{5} F_j (r - q) + q L_j)^{n-1} \right).
\]

We proceed with some examples. The special case \( q = 2 \) takes the form

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{jk} (\sqrt{5} F_j)^{n-k} (2^{1-(n-k)} - 1) B_{n-k} = n F_j 2^{1-n} \left( (L_j + 2\alpha^j)^{n-1} + (L_j + 2\beta^j)^{n-1} \right) = n F_j 2^{1-n} \sum_{m=0}^{n-1} \binom{n-1}{m} 2^m L_{jm} L_{j-1}^{n-1-m}.
\]
For $j = 1$ the left-hand side can be expressed in closed-form and we obtain after some manipulations

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\sqrt{5}}{4}\right)^{k} (2 - 2^{k}) F_{n-k} B_{k} = \frac{nL_{3(n-1)}}{2^{n-1}}.$$  

Similarly the case $q = 3$ is treated. The calculations are lengthy and omitted. The result is

$$\sum_{k=0}^{n} \binom{n}{k} 6^{k} (1 - 3^{n-k-1}) F_{jk} (\sqrt{5} F_{j})^{n-k} B_{n-k} = n F_{j} \sum_{m=0}^{n-1} x \binom{n-1}{m} (2^{n-1} + 4^{m}) L_{j}^{n-1-m} L_{jm}.$$  

For $j = 1$ we get

$$\sum_{k=0}^{n} \binom{n}{k} 6^{k} (\sqrt{5})^{n-k} (1 - 3^{n-k-1}) F_{k} B_{n-k} = n 2^{n-1} L_{2n-2} + \sum_{m=1}^{n} \binom{n}{m} m 4^{m-1} L_{m-1}.$$  

4 Conclusion

In this paper, we have discovered new identities relating Bernoulli polynomials (numbers) to Fibonacci and Lucas numbers. In our future papers, we will discuss the analogue results for Euler polynomials (numbers) and Fibonacci and Lucas numbers as well as identities connecting Bernoulli polynomials (numbers) with Jacobsthal, Pell and balancing numbers.

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