ON THE EXISTENCE OF $C^{1,1}$ ISOMETRIC IMMERSIONS OF SEVERAL CLASSES OF NEGATIVELY CURVED SURFACES INTO $\mathbb{R}^3$

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ABSTRACT. We prove the existence of $C^{1,1}$ isometric immersions of several classes of metrics on surfaces $(\mathcal{M}, g)$ into the three-dimensional Euclidean space $\mathbb{R}^3$, where the metrics $g$ have strictly negative curvature. These include the standard hyperbolic plane, generalised helicoid-type metrics and generalised Enneper metrics. Our proof is based on the method of compensated compactness and invariant regions in hyperbolic conservation laws, together with several observations on the geometric quantities (Gauss curvature, metric components etc.) of negatively curved surfaces.

1. INTRODUCTION

The existence of isometric immersions of a given $n$-dimensional Riemannian manifold $(\mathcal{M}, g)$ into the Euclidean space $\mathbb{R}^N$ is a well-known classical problem in differential geometry and non-linear analysis; cf. [5, 25, 28, 36, 39, 43] and the references cited therein. Even in the case $(n, N) = (2, 3)$, namely the isometric immersions of surfaces into the Euclidean $3$-space, the question of existence remains open in the large.

From the perspective of Partial Differential Equations (PDEs), several distinct approaches have been developed toward the study of isometric immersions. First, in [36], Nash directly tackled the defining equations for isometric immersions: let $g$ be the given Riemannian metric on $\mathcal{M}$; an isometric immersion $f : (\mathcal{M}, g) \to (\mathbb{R}^N, g_{\text{Eucl}})$ satisfies

$$df \cdot df = g,$$  \hspace{1cm} (1.1)

where $g_{\text{Eucl}}$ denotes the Euclidean metric on $\mathbb{R}^N$. Eq. (1.1) forms a first-order non-linear PDE system, which is of no definite type (elliptic, parabolic or hyperbolic). Nash proved the existence of $C^\infty$ isometric immersions for $g \in C^\infty$, provided that the co-dimension $(N - n)$ is sufficiently large. In particular, in the case $n = 2$, we need co-dimension up to 15, even if $\mathcal{M}$ is a compact surface.

The second approach is via the Darboux equation. Fix any unit vector $e \in \mathbb{R}^3$ and consider $\phi := f \cdot e$; the local existence of an isometric immersion $f : (\mathcal{M}, g) \to (\mathbb{R}^3, g_{\text{Eucl}})$ is equivalent to the following PDE of Monge–Ampère type, named after Darboux:

$$\det(\nabla^2 \phi) = \kappa \det(g)(1 - |\nabla \phi|^2),$$  \hspace{1cm} (1.2)

provided that $|\nabla \phi|^2 = g^{ij}\partial_i \phi \partial_j \phi < 1$. Throughout the Einstein summation convention is assumed, and $\kappa$ denotes the Gauss curvature of the surface $(\mathcal{M}, g)$. By establishing the elliptic theory for Monge–Ampère equations, Nirenberg [37] proved the existence of isometric immersions of $(\mathcal{M}, g)$ for $\kappa > 0$, $g \in C^4$. Notice that whenever $\kappa > 0$, $\mathcal{M}$ is a topological sphere, in view of the Gauss–Bonnet theorem. Guan–Li [23] extended Nirenberg’s result to the case $\kappa \geq 0$.

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$g \in C^4$, by passing to the limits of the degenerate-elliptic equation (1.2). The sign of $\kappa$ is crucial to the study of the existence of classical solution to Eq. (1.2), hence, of the existence of isometric immersions. For example, it is well-known that the pseudo-2-sphere (space form of constant Gauss curvature $-1$) cannot be isometrically embedded into $\mathbb{R}^3$ via $C^2$ maps; see Hilbert–Cohn Vossen [27].

The third approach to the existence of isometric immersions is via the analysis of the Gauss–Codazzi equations. Temporarily let us only present the key ideas here; the detailed derivations can be found in §2. Let $(L, M, N)$ be the components of the normalised second fundamental form of $f : (\mathcal{M}, g) \to (\mathbb{R}^3, g_{\text{Eucl}})$. The Codazzi equations are the following first-order PDE system:

$$
\begin{align*}
M_x - L_y &= \Gamma_{22}^1 L - 2\Gamma_{12}^2 M + \Gamma_{11}^1 N, \\
N_x - M_y &= -\Gamma_{22}^1 L + 2\Gamma_{12}^1 M - \Gamma_{11}^1 N,
\end{align*}
$$

where $\{\Gamma_{jk}^i\}_{1 \leq i,j,k \leq 2}$ denote the Christoffel symbols of the Levi–Civita connection on $(\mathcal{M}, g)$. In addition, the Gauss equation is a zeroth-order quadratic relation on $(L, M, N)$:

$$
LN - M^2 = \kappa.
$$

By the fundamental theorem of surfaces (see Eisenhart [20]), for smooth metrics the solubility of the Gauss–Codazzi equations (1.3) and (1.4) are equivalent to the existence of isometric immersions. This has been extended in [30, 31] by Mardare to the case of $W^{1,p}_{\text{loc}}$ isometric immersions; also see Chen–Li [7] for a geometric proof in the arbitrary dimension/co-dimension case.

In the recent years, there have been rapid developments in the study the isometric immersions of surfaces with negative $\kappa$ via the aforementioned third approach, i.e., by analysing the Gauss–Codazzi equations (1.3) (1.4). Chen–Slemrod–Wang [9] (also see the exposition [10]) first observed that the Codazzi equations (1.3) can be viewed as a system of hyperbolic balance laws for $(L, M)$, once we substitute $N$ by $(\kappa + M^2)/L$ using the Gauss equation (1.4). In this formulation, if one suitably interprets the geometric quantities in terms of fluid mechanics, the cases of $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$ correspond to the subsonic, sonic and supersonic fluid flows. In particular, in the supersonic case ($\kappa < 0$) the resulting system of balance laws is strictly hyperbolic, which enables us to prove the existence of weak solutions using tools from hyperbolic PDEs. Indeed, by applying the method of vanishing viscosity and the theory of invariant regions, which had been exploited in the works of Morawetz on transonic flows ([32 33 34]), Chen–Slemrod–Wang established the global existence of $C^{1,1}$ isometric immersions of a one-parameter family $\{g_\beta\}$ of negatively curved metrics, where $g_{\sqrt{2}}$ corresponds to the metric of the classical catenoid. For this reason, the family $\{g_\beta\}$ may be termed as “generalised catenoids”. Subsequently, using another closely related system of balance laws, Cao–Huang–Wang [3] further proved the existence of $C^{1,1}$ isometric immersions of a family of “generalised helicoids”, which includes the usual helicoid as a special case. Moreover, employing another tool from the hyperbolic balance laws, the Lax–Friedrich scheme, Cao–Huang–Wang [3] proved the global existence of $C^{1,1}$ isometric immersions of the metrics of the form $g = E(y)dx^2 + dy^2$, under the conditions $\log(E(y)^2\sqrt{-\kappa})$ being a non-increasing $C^{1,1}$ function, $E'''(y) = -\kappa E(y)$, plus some mild conditions on the initial data ($y = 0$). In particular, [4] extended the results in [3] on the “generalised helicoids”. Furthermore, exploiting the theory of BV solutions of balance laws (cf. Dafermos
Christoforou \cite{12} established the existence of \(C^{1,1}\) isometric immersions of the conformal metrics \(g_{c,q^*} = (\cosh(cx))^2/(q^* - 1)(dx^2 + dy^2)\), where \(c > 0, q > 1\) are constants, with conditions on \(q^*\) and/or the initial data \((y = 0)\). The collection \(\{g_{c,q^*}\}\) neither contains or is contained in the family of “generalised catenoids” as in Chen–Slemrod–Wang \cite{9} or Cao–Huang–Wang \cite{3}. The existence result in \cite{12} requires further \textit{a priori} stability assumptions, entailed by the “BV framework” formulated in the paper.

To summarise, in the recent works \cite{3, 4, 9, 10, 12}, various negatively curved metrics of the “generalised catenoid type” or the “generalised helicoid type” have been proved to possess global \(C^{1,1}\) isometric immersions into \(\mathbb{R}^3\).

The main contribution of the current paper is to establish the global or local existence of \(C^{1,1}\) isometric immersions for several more families of such metrics. This is done by further exploiting the Gauss–Codazzi equations \cite{14, 13} via the theory of hyperbolic balance laws, as well as exploring the detailed structures of relevant geometric quantities, \textit{e.g.}, the Gauss curvature, the metric components and the Christoffel symbols. Before subsequent developments, let us note that the metrics considered in this paper are still far from being general: special structures of the metrics are essential to control the \textit{source terms} of the associated hyperbolic balance laws, which play a crucial role in the theory of these PDEs; see Dafermos \cite{15}.

The remaining parts of the paper is organised as follows. In §2 we derive Gauss–Codazzi equations and recast them into suitable hyperbolic balance laws. Next, in §3 we briefly discuss the theory of compensated compactness and invariant regions, as well as the application to the Gauss–Codazzi system. Then, in §§4–6 respectively, we establish the existence of \(C^{1,1}\) isometric immersions for three families of metrics: the standard hyperbolic (Lobachevsky) plane, “helicoid-type metrics” and “generalised Enneper metrics”. In §7 we discuss the non-existence of invariant regions of “reciprocal-type metrics”. Finally, in §8 we conclude with a few remarks.

2. The PDEs

2.1. Gauss–Codazzi equations. In this paper we let \((\mathcal{M}, g)\) be a 2-dimensional Riemannian manifold, \textit{i.e.}, a surface, with Riemannian metric \(g\). The regularity of \(g\) is assumed to be at least Lipschitz, but is not required to be smooth or analytic in general. We consider the isometric immersion \(f : (\mathcal{M}, g) \rightarrow (\mathbb{R}^3, g_{\text{Eucl}})\): this means that the Euclidean metric pulled back via \(f\) coincides with the given metric \(g\) on \(\mathcal{M}\), namely \(f^*g_{\text{Eucl}} = g\). It is equivalent to Eq. (1.1).

If an isometric immersion \(f : (\mathcal{M}, g) \rightarrow (\mathbb{R}^3, g_{\text{Eucl}})\) exists, for each point \(x \in \mathcal{M}\), the tangent space \(T_{f(x)}\mathbb{R}^3\) splits into orthogonal vector spaces:

\[
T_{f(x)}\mathbb{R}^3 \cong T_x\mathcal{M} \bigoplus [T_x\mathcal{M}]^\perp.
\]

We write \(T\mathcal{M}^\perp\) for the normal bundle, \textit{i.e.}, the vector bundle over \(\mathcal{M}\) with fibres \(T_x\mathcal{M}\). Denoting by \(\nabla\) the trivial Levi–Civita connection on \(T\mathbb{R}^3\) and \(\nabla\) the Levi–Civita connection on \(T\mathcal{M}\), we can define \(\Pi : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \rightarrow \Gamma(T\mathcal{M}^\perp)\) via

\[
\Pi(X,Y) := \nabla_X Y - \nabla_Y X,
\]

where \(\Gamma(\mathcal{E})\) for a vector bundle \(\mathcal{E}\) denotes the space of sections of \(\mathcal{E}\). This tensor field \(\Pi\) takes values in the normal bundle, and it describes the manner in which \(\mathcal{M}\) is immersed in the ambient Euclidean space. This extrinsic quantity is known as the \textit{second fundamental form}. For the immersion \(f\) of surfaces into \(\mathbb{R}^3\), \(T\mathcal{M}^\perp\) is 1-dimensional, so \(\Pi\) can be viewed as a \(\mathbb{R}\)-valued
Let \( 2 \times 2 \) matrix field (namely, a section of \( T^* \mathcal{M} \otimes T^* \mathcal{M} \)), or equivalently, a field of quadratic forms (namely, a section of \( T \mathcal{M} \otimes T \mathcal{M} \)). Therefore, by a slightly abusive notation we may write

\[
\Pi = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \quad \text{or} \quad \Pi = h_{11} dx^2 + 2h_{12} dxdy + h_{22} dy^2, \tag{2.3}
\]

where \( h_{12} = h_{21} \) in view of the definition of \( \Pi \) in Eq.\((2.2)\) and the basic properties of \( \nabla \) and \( \nabla \). In addition, we also write

\[
\Pi(X, Y, \eta) := g_{\text{Eucl}}(\Pi(X, Y), \eta) \quad \text{for} \quad X, Y \in \Gamma(T \mathcal{M}) ; \eta \in \Gamma(T \mathcal{M}^\perp). \tag{2.4}
\]

The Gauss and Codazzi equations express the orthogonal splitting of the zero Riemann curvature of \((\mathbb{R}^3, g_{\text{Eucl}})\) along Eq.\((2.1)\). Let \( X, Y, Z, W \in \Gamma(T \mathcal{M}) \) be tangential vector fields and \( \eta \in \Gamma(T \mathcal{M}^\perp) \) be a normal vector field. Also, denote the inner products induced by the metrics \( g \) or \( g_{\text{Eucl}} \) both by \( \langle \cdot, \cdot \rangle \), and write \( R \) for the Riemann curvature tensor on \( \mathcal{M} \). With these preparations, the Gauss equation reads:

\[
R(X, Y, Z, W) = \langle \Pi(X, Z), \Pi(Y, W) \rangle - \langle \Pi(X, W), \Pi(Y, Z) \rangle, \tag{2.5}
\]

and the Codazzi equations are as follows:

\[
\nabla_Y \Pi(X, Z, \eta) - \nabla_X \Pi(Y, Z, \eta) = 0. \tag{2.6}
\]

Due to our regularity assumptions on \( g \), Eqs.\((2.5)(2.6)\) should be understood in the sense of distributions. Here \( g \) (hence \( R \), due to Gauss’s Theorema Egregium) are given functions and \( \Pi \) consist of the unknowns. For the geometric formulations above, we refer to §6 in do Carmo\[19\] or §2 in Chen–Li\[7\] for detailed discussions.

To proceed, let us derive the form of Gauss–Codazzi equations in local coordinates, namely Eqs.\((1.4)(1.3)\). These equations are regarded as well-known in Han–Hong\[25\] and the aforementioned works\[3, 4, 9, 10, 12\]; nevertheless, it is not easy to find in the literature a careful derivation from the first principles, i.e., the global Eqs.\((2.5)(2.6)\). For the convenience of the readers we shall present the derivation below.

Let \( \{ \partial_1, \partial_2 \} \) be a local coordinate frame on \( T \mathcal{M} \), and \( \partial_3 \) be the unit normal vector field in \( T \mathcal{M}^\perp \). Taking \( X = \partial_1, Y = \partial_2, Z = \partial_1 \) and \( W = \partial_2 \), the Gauss equation \((2.5)\) gives us

\[
R_{1212} = h_{11} h_{22} - (h_{12})^2, \tag{2.7}
\]

where \( R_{ijkl} := R(\partial_i, \partial_j, \partial_k, \partial_l) \). For the Codazzi equation, there are two independent choices of coordinates: \( (i, j, k) = (1, 2, 1) \) and \( (i, j, k) = (1, 2, 2) \). They lead respectively to

\[
\begin{align*}
\nabla_{\partial_2} \Pi(\partial_1, \partial_1, \partial_3) - \nabla_{\partial_1} \Pi(\partial_2, \partial_1, \partial_3) &= 0, \\
\nabla_{\partial_2} \Pi(\partial_1, \partial_2, \partial_3) - \nabla_{\partial_1} \Pi(\partial_2, \partial_2, \partial_3) &= 0. \tag{2.8}
\end{align*}
\]

By the Leibniz rule, we have the identity

\[
\nabla_{\partial_i} \Pi(\partial_j, \partial_k, \partial_\alpha) = \partial_i \Pi(\partial_j, \partial_k, \partial_\alpha) - \Pi(\nabla_{\partial_i} \partial_j, \partial_k, \partial_\alpha)
\]

\[
- \Pi(\partial_j, \nabla_{\partial_i} \partial_k, \partial_\alpha) - \Pi(\partial_j, \partial_k, \nabla_{\partial_i} \partial_\alpha), \tag{2.9}
\]

where \( \nabla^\perp \) is the projection of \( \nabla \) onto \( T \mathcal{M}^\perp \). In our case, the co-dimension of the immersion is 1; hence \( \partial_i \langle \partial_\alpha, \partial_\alpha \rangle = 2 \langle \nabla_{\partial_i} \partial_\alpha, \partial_\alpha \rangle = 0 \), which forces

\[
\Pi(\partial_j, \partial_k, \nabla^\perp_{\partial_i} \partial_\alpha) = 0. \tag{2.10}
\]
Thus, together with the definition for the Christoffel symbols
\[ \nabla_{\partial_i} \partial_j = \Gamma^l_{ij} \partial_l, \]
Eq. (2.9) becomes
\[ \nabla_{\partial_i} \Pi(\partial_j, \partial_k, \partial_\alpha) = \partial_i h_{jk} - \left\{ \Gamma^p_{ij} h_{pk} + \Gamma^p_{ik} h_{pj} \right\}. \] (2.12)
Substituting Eq. (2.12) into Eq. (2.8), after some cancellations we arrive at the following:
\[ \begin{align*}
\frac{\partial}{\partial_i} (h_{11}, h_{12}, h_{22})^\top &= \frac{\partial}{\partial_i} \left( \frac{\partial |g|}{\partial_i} \right) (L, M, N)^\top,
\end{align*} \] (2.13)
Up to now, we have derived Eq. (2.7) and (2.13), which are equivalent to the Gauss and Codazzi equations, respectively. Let us introduce the following normalisation. For brevity, write
\[ |g| := \det(g), \]
which is strictly positive. Then, by definition, the Gauss curvature is
\[ \kappa := \frac{R_{1221}}{|g|}. \] (2.14)
Moreover, we can then define the functions \( L, M, N \) on \( \mathcal{M} \) by the following:
\[ \frac{1}{|g|} \Pi = \frac{1}{|g|} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}. \] (2.15)
So
\[ \partial_i (h_{11}, h_{12}, h_{22})^\top = \sqrt{|g|} \left\{ \partial_i + \frac{\partial_i |g|}{2|g|} \right\} (L, M, N)^\top. \] (2.16)
On the other hand, using the formula for Christoffel symbols in the local coordinates:
\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{ \partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk} \right\} \]
as well as identities for the derivative of the logarithmic of matrices, we can deduce that
\[ \begin{align*}
\Gamma^1_{12} + \Gamma^2_{22} &= \frac{1}{2} \left\{ g^{11} \partial_2 g_{11} + g^{22} \partial_2 g_{22} + 2g^{12} \partial_2 g_{12} \right\} \\
&= \frac{1}{2} \text{trace} \left( \partial_i \log g \right) \\
&= \partial_i \left\{ \log (\det g) \right\} = \frac{\partial_i |g|}{|g|},
\end{align*} \] (2.17)
thanks to the positive definiteness of \( g \). Putting everything together, Eqs. (2.14) and (2.7) become the local form of the Gauss equation:
\[ \kappa = LN - M^2, \] (2.18)
and Eqs. (2.16), (2.17) applied to Eq. (2.13) give us the Codazzi equations:
\[ \begin{align*}
\partial_1 M - \partial_2 L &= \Gamma^2_{22} L - 2 \Gamma^2_{12} M + \Gamma^2_{11} N, \\
\partial_1 N - \partial_2 M &= -\Gamma^2_{22} L + 2 \Gamma^1_{12} M - \Gamma^1_{11} N,
\end{align*} \] (2.19) (2.20)
which reproduce Eqs. (1.4), (1.3).
In the sequel, we shall refer to Eqs. (2.18), (2.19), (2.20) as the Gauss and Codazzi equations. We also write \( \{ \partial_1, \partial_2 \} \) and \( \{ \partial_x, \partial_y \} \) interchangeably.
2.2. **Hyperbolic balance laws.** In this subsection, let us transform the Gauss and Codazzi equations to the associated system of hyperbolic balance laws following Cao–Huang–Wang [3, 4]. To make the paper self-contained, we shall describe the main steps of the transform, with a remark on the necessity of the negative curvature condition $\kappa < 0$.

**Step 1: Fluid formulation.** First of all, let us introduce

$$\gamma := \sqrt{-\kappa},$$

(2.21)

which is a positive $C^{1,1}$ function. We further normalise

$$\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} L & M \\ M & N \end{bmatrix},$$

(2.22)

so that the *normalised Gauss equation* becomes

$$\tilde{L} \tilde{N} - \tilde{M}^2 = -1.$$  

(2.23)

The *normalised Codazzi equations* are computed in Eq. (2.7) of Cao–Huang–Wang [4]:

$$\begin{align*}
\tilde{M}_x - \tilde{L}_y &= \tilde{\Gamma}_{22}^{\gamma} \tilde{L} + 2 \tilde{\Gamma}_{12}^{\gamma} \tilde{M} + \tilde{\Gamma}_{22}^{\gamma} \tilde{N}, \\
\tilde{N}_x - \tilde{M}_y &= -\tilde{\Gamma}_{22}^{\gamma} \tilde{L} + 2 \tilde{\Gamma}_{12}^{\gamma} \tilde{M} - \tilde{\Gamma}_{22}^{\gamma} \tilde{N}.
\end{align*}$$

(2.24)

(2.25)

with $\tilde{\Gamma}_{jk}^i$, the normalised Christoffel symbols, given by

$$\begin{align*}
\tilde{\Gamma}_{22}^{\gamma} &= \gamma_2^2 + \gamma_2 \gamma, & \tilde{\Gamma}_{12}^{\gamma} &= \gamma_2 + \frac{\gamma_x}{2\gamma}, & \tilde{\Gamma}_{22}^{\gamma} &= \gamma_2^2, \\
\tilde{\Gamma}_{22}^{\gamma} &= \gamma_{22}, & \tilde{\Gamma}_{12}^{\gamma} &= \gamma_1 + \frac{\gamma_x}{\gamma}. 
\end{align*}$$

(2.26)

Now, let us introduce the “fluid variables”:

$$U := \begin{bmatrix} \tilde{L} \\ -\tilde{M} \end{bmatrix} \equiv \begin{bmatrix} \rho \\ m \end{bmatrix},$$

(2.27)

where $\rho$ is the density and $m$ the momentum of a 1-dimensional compressible fluid. Thus, the normalised Gauss equation (2.23) gives us

$$\tilde{N} = \frac{m^2 - 1}{\rho},$$

(2.28)

which recasts the normalised Codazzi equations (2.24) (2.25) to the following balance law:

$$U_y + [\mathcal{F}(U)]_x = \mathcal{G}(U),$$

(2.29)

where

$$[\mathcal{F}(U)]_x = \nabla \mathcal{F}(U) \cdot U_x = \begin{bmatrix} 2 \rho \rho_x - \frac{m_x}{\rho^2 - \rho x} - \rho \end{bmatrix},$$

(2.30)

and the right-hand side

$$\mathcal{G}(U) = \begin{bmatrix} -\tilde{\Gamma}_{22}^{\gamma} \rho - 2 \tilde{\Gamma}_{12}^{\gamma} m - \tilde{\Gamma}_{22}^{\gamma} \left( \frac{m^2 - 1}{\rho} \right) \\ -\tilde{\Gamma}_{22}^{\gamma} \rho - 2 \tilde{\Gamma}_{12}^{\gamma} m - \tilde{\Gamma}_{22}^{\gamma} \left( \frac{m^2 - 1}{\rho} \right) \end{bmatrix}.$$  

(2.31)

The matrix $\nabla \mathcal{F}(U)$, where $\nabla$ is the gradient of $\mathcal{F}$ with respect to $U$, is

$$\nabla \mathcal{F}(U) = \begin{bmatrix} 0 & 1 \\ -\frac{m^2 - 1}{\rho^2} & \frac{2m}{\rho} \end{bmatrix},$$

(2.32)
whose eigenvalues are
\[ \lambda_{\pm} = \frac{m \pm 1}{\rho}. \tag{2.33} \]

The above calculation for eigenvalues requires
\[ \rho \neq 0 \iff \tilde{L} \neq 0 \iff L \neq 0. \]

As \( g \) is symmetric and positive definite, this is automatically satisfied.

**Step 2: Parabolic regularisation.** Now let us introduce the parabolic regularisation for the normalised Gauss–Codazzi equations, i.e., the viscous approximation equations in terms of the fluid variables. For each \( \epsilon > 0 \) sufficiently small, we choose the regularisation as follows:
\[
(\rho')_y + (m')_x = \epsilon(\rho')_{xx} - \tilde{\Gamma}_{22}^2(\rho') - 2\tilde{\Gamma}_{12}^2m' - \tilde{\Gamma}_{11}^2\left(\frac{(m')^2 - 1}{\rho}\right), \tag{2.34}
\]
\[
(m')_y + \left(\frac{(m')^2 - 1}{\rho}\right)_x = \epsilon(m'_{xx} - \tilde{\Gamma}_{22}^1(\rho') - 2\tilde{\Gamma}_{12}^1m' - \tilde{\Gamma}_{11}^1\left(\frac{(m')^2 - 1}{\rho}\right). \tag{2.35}
\]

Then, introducing
\[
u := \frac{m}{\rho}, \quad v := \frac{1}{\rho}, \tag{2.36}
\]
(the superscript \( \epsilon \) will be dropped for simplicity), one can write
\[
\rho = \frac{1}{v}, \quad m = \frac{u}{v}, \quad \tilde{N} = \frac{u^2 - v^2}{v}.
\]

The regularised equations \((2.34)\) \((2.35)\) become
\[
v_y - vu_x + w_x = \epsilon v_{xx} - \frac{2\epsilon(v_x)^2}{v} + \epsilon\tilde{\Gamma}_{22}^2 + 2\tilde{\Gamma}_{12}^2uv + \epsilon\tilde{\Gamma}_{11}^2(u^2 - v^2)v, \tag{2.37}
\]
as well as
\[
u_y - vu_x + w_x
= \epsilon u_{xx} - \frac{2\epsilon u_xv_x}{v} + (\tilde{\Gamma}_{22}^2 - 2\tilde{\Gamma}_{12}^1)u + 2\tilde{\Gamma}_{12}^2u^2 + \tilde{\Gamma}_{11}^1(v^2 - u^2) + \tilde{\Gamma}_{11}^1u(u^2 - v^2) - \tilde{\Gamma}_{22}^1. \tag{2.38}
\]

**Step 3: Decoupling.** Taking the sum and the difference of Eqs. \((2.37)\) and \((2.38)\), one can simplify the first- and second-order terms of \( u \) and \( v \). This amounts to introducing the **Riemann invariants** (cf. Part III of Smoller [40]):
\[
w := u + v, \quad z = u - v. \tag{2.39}
\]

Thus, let us rewrite the regularised system in terms of the rotated coordinates \(\{w, z\}\):
\[
w_y + zw_x = \epsilon w_{xx} - \frac{2\epsilon v_xw_x}{v} + S^{(1)}, \tag{2.40}
\]
\[
z_y + wz_x = \epsilon z_{xx} - \frac{2\epsilon v_xz_x}{v} + S^{(2)}, \tag{2.41}
\]
where the source terms \( S^{(1)} \) and \( S^{(2)} \), viewed as functions of \( (u, v) \), are inhomogeneous cubic polynomials in \( (u, v) \) with coefficients involving linear combinations of \( \tilde{\Gamma}_{ij} \):
\[
S^{(1)} = -\tilde{\Gamma}_{22}^1
+ (\tilde{\Gamma}_{22}^2 - 2\tilde{\Gamma}_{12}^1)u + \tilde{\Gamma}_{22}^1v
+ (2\tilde{\Gamma}_{12}^2 - \tilde{\Gamma}_{11}^1)u^2 + \tilde{\Gamma}_{11}^1v^2 + 2\tilde{\Gamma}_{12}^2uv
\]
In this case, symmetric matrix): is assumed to be diagonal (it is natural because this paper, the coefficients of $S$ balance law (Eqs.(2.40)-(2.41)), let us remark that, under the following working assumption of generalities, no further simplifications are available for $w$ and $z$, with the source terms given by $S^{(1)}$ and $S^{(2)}$ in Eqs. (2.42)-(2.43). In full generalities, no further simplifications are available for $S^{(1)}$ and $S^{(2)}$.

Step 4: Formulae of $\tilde{\Gamma}^i_{jk}$ for diagonal metrics. After the derivation of the hyperbolic balance law (Eqs. (2.40)-(2.41)), let us remark that, under the following working assumption of this paper, the coefficients of $S^{(1)}$ and $S^{(2)}$ can be evaluated easily. Throughout this paper $g$ is assumed to be diagonal (it is natural because $g$ is always diagonalisable as a positive definite symmetric matrix):

$$g = \begin{bmatrix} E(x, y) & 0 \\ 0 & G(x, y) \end{bmatrix}.$$  

(2.44)

In this case, $\Gamma^i_{jk}$ and thus $\tilde{\Gamma}^i_{jk}$ are given as follows:

$$\begin{align*}
\Gamma^1_{11} &= \frac{E_x}{2E}, & \Gamma^1_{12} &= \frac{E_y}{2E}, & \Gamma^1_{22} &= -\frac{G_x}{2E}, \\
\Gamma^2_{11} &= -\frac{E_y}{2G}, & \Gamma^2_{12} &= \frac{G_x}{2G}, & \Gamma^2_{22} &= \frac{G_y}{2G},
\end{align*}$$

(2.45)

$$\begin{align*}
\tilde{\Gamma}^1_{11} &= \frac{E_x}{2E} + \frac{\gamma x}{\gamma}, & \tilde{\Gamma}^1_{12} &= \frac{E_y}{2E} + \frac{\gamma y}{2\gamma}, & \tilde{\Gamma}^1_{22} &= -\frac{G_x}{2E}, \\
\tilde{\Gamma}^2_{11} &= -\frac{E_y}{2G} + \frac{\gamma x}{2\gamma}, & \tilde{\Gamma}^2_{12} &= \frac{G_x}{2G} + \frac{\gamma y}{\gamma}.
\end{align*}$$

(2.46)

2.3. $\kappa < 0$ is essential. We now discuss the necessity of the assumption of negative curvature to our approach. Suppose $\kappa \geq 0$ in some region $\mathcal{M}' \subset \mathcal{M}$ and that the normalisation factor, again denoted by $\gamma$, where $\gamma = \sqrt{\kappa} \in C^{1,1}(\mathcal{M}')$. Also, define the normalised second fundamental form $\{\hat{L}, \hat{M}, \hat{N}\}$ as in Eq. (2.22), and introduce the fluid variables $\{\rho, m\}$ as in Eq. (2.27). (Here the sign of $m$ is not essential; what is important is that we do not introduce further derivatives of $\{\hat{L}, \hat{M}, \hat{N}\}$ in the fluid formulation.) Then, notice that the formulae for the normalised Codazzi equations (2.24)-(2.25) remain unchanged; thus, in the balance law (2.29), we have

$$[\mathcal{F}(U)]_x = \begin{bmatrix} m_x \\ \frac{m_x}{\rho} \end{bmatrix},$$

hence

$$\nabla \mathcal{F}(U) = \begin{bmatrix} 0 & 1 \\ -\frac{m^2 + 1}{\rho^2} & \frac{2m}{\rho} \end{bmatrix}.$$  

(2.47)

However, the eigenvalues of the matrix $\nabla \mathcal{F}(U)$ in the region $\mathcal{M}'$ can be computed as follows:

$$\lambda_\pm = \frac{1}{2} \left\{ \frac{2m}{\rho} \pm \sqrt{\left( \frac{2m}{\rho} \right)^2 - 4 \left( \frac{m^2 + 1}{\rho^2} \right)} \right\} = \frac{1}{2} \left\{ \sqrt{\frac{4}{\rho^2} \pm \frac{2m}{\rho}} \right\},$$

(2.48)
which is not a real value. In view of the definition of strict hyperbolicity (cf. Dafermos [15]), we have proved:

**Proposition 2.1.** The balance law (2.29) associated to the fluid formulation of the normalised Gauss and Codazzi equations (2.23) (2.24) (2.25) in §2.2 is strictly hyperbolic if and only if the Gauss curvature $\kappa$ of the manifold $(M, g)$ is negative.

The above proposition explains why we need to restrict to the case $\kappa < 0$ in [9, 10, 3, 4] and this paper. It echoes that in the Darboux equation formulation for the isometric immersion problem, $\kappa > 0$ ensures the ellipticity of Eq. (1.2).

In passing, we comment that the isometric immersion problems have also been transformed to hyperbolic PDEs by taking sufficiently many derivatives of the immersion maps. In the work [24] by Han, the local existence of $C^{r-6}$ isometric immersions near the origin $(x, y) = (0, 0)$ for $g \in C^r$ is proved by analysing the Darboux equation (1.2), for $r \geq 9$, $\kappa(0) = 0$ and $\nabla \kappa(0) \neq 0$. Han [24] simplified the earlier arguments due to Lin [29]. For this purpose, one considers the ansatz $u = u_0 + \epsilon^6 \tilde{u}$ of the classical solution to the Darboux equation (1.2), where $u_0$ is an approximate background solution. The crucial observation in [24] is that certain higher order derivatives of $\tilde{u}$ satisfy a first-order positive symmetric hyperbolic system, which can be solved by carrying out usual energy estimates. Let us also mention that in [6], Chen–Clelland–Slemrod–Wang–Yang obtained a simplified proof of the local existence of a smooth isometric embedding of a smooth 3-dimensional Riemannian manifold with non-zero Riemannian curvature tensor into 6-dimensional Euclidean space, also by reducing to first-order positive symmetric hyperbolic systems. We also refer to Efimov [21], Tunitskii [42], Poznyak [38], Rozendorn [39] and the references therein for the pioneering works on the isometric immersions of surfaces of negative curvatures by the Russian school.

### 3. Compensated Compactness

In this section we discuss our central analytic tool — compensated compactness.

Introduced by Murat [35] and Tartar [41], the theory of compensated compactness has been extensively employed in non-linear hyperbolic PDEs and applications to gas dynamics and elasticity; see Ball [2], Chen–Slemrod–Wang [8], Dafermos [15], Ding–Chen–Luo [16], DiPerna [17, 18], Evans [22], Morawetz [32, 33, 34], Tartar [41] and the references cited therein. Recently, its applications to the isometric immersions problem have been discovered by Chen–Slemrod–Wang [10, 11] and studied by Cao–Huang–Wang [3, 4] and Christofourou [12], among others.

In this paper, as in [10, 3, 4, 12], we prove the existence of weak solutions to the Gauss–Codazzi equations by the method of vanishing artificial viscosity, for which the vanishing viscosity limit is obtained in virtue of the compensated compactness method. The uniform $L^\infty$ estimate that guarantees the existence of the vanishing viscosity limit is established by the method of invariant regions. Finally, the existence of weak solutions to the Gauss–Codazzi equations imply the existence of $C^{1,1}$ isometric immersions of the corresponding surfaces into $(\mathbb{R}^3, g_{\text{Eucl}})$, due to Mardare [30, 31] and Chen–Li [7].

#### 3.1. Uniform $L^\infty$ estimates via invariant regions

To begin with, we shall analyse the system of parabolic equations (2.40) (2.41) for the Riemann invariants $w$ and $z$ (see Step 3 in §2.2). Recall that these PDEs are obtained as the parabolic regularisations of the original hyperbolic...
system of Codazzi equations, via adding to the Codazzi equations the terms of “artificial viscosity”, namely \( \varepsilon w_{xx} \equiv \varepsilon (u')_{xx} \) and \( \varepsilon u_{xx} \equiv \varepsilon (u')_{xx} \). Our goal is to show that, as \( \varepsilon \to 0^+ \), \( \{u',v'\} \) converges to the weak solution of Eqs. (2.40) (2.41).

Introduce the state (column) vector field

\[
U \equiv U' := (w',z')^\top \equiv (w,z)^\top;
\]

we view \( U \) as the map \((u,v)^\top \mapsto (w,z)^\top\), a transform from \( \mathbb{R}^2 \) to itself. For simplicity let us systematically drop the superscripts \( \varepsilon \). Then, with

\[
S = S(u,v) := (S^{(1)},S^{(2)})^\top
\]
the system (2.40) (2.41) can be recast into the following balance law:

\[
U_y = \varepsilon U_{xx} + \begin{bmatrix}
-2\varepsilon v_x v^{-1} - z & 0 \\
0 & -2\varepsilon v_x v^{-1} - w
\end{bmatrix} U_x + S.
\]

Now let us explain the method of invariant regions: our presentation follows closely the classical paper [14] by Chueh–Conley–Smoller. Consider the PDE system for \( U = U(x,t) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \)

\[
U_t = \varepsilon D \cdot \Delta U + \sum_{j=1}^m M^j U_{x_j} + S,
\]
where \( D \) is a positive definite \( n \times n \) matrix, \( \{M^1, \ldots, M^m\} \) are \( n \times n \) matrices and \( S : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \). We say that \( Q \subset \text{Image}(U) \subset \mathbb{R}^n \) is an invariant region of Eq. (3.4) if \( U_0(x) := U(x,t_0) \in Q \) implies that \( U(x,t) \in Q \), whenever \( t_0 \leq t \leq T_* \) (\( T_* \) = the lifespan of the solution). Then, the following characterisation of invariant regions of Eq. (3.4) is at hand:

**Proposition 3.1** (Theorem 4.4 in [14]). In the above setting, consider

\[
Q := \bigcap_{i=1}^d \left\{ z \in \mathbb{R}^n : G_i(z) \leq 0, \ G_i : \text{Image}(U) \to \mathbb{R} \text{ are smooth functions} \right\}.
\]

Then \( Q \) is an invariant region for Eq. (3.4) for all \( \varepsilon > 0 \) if and only if the following three conditions hold:

1. \( \nabla U_0 G_i \) is a left eigenvector for \( D \) and each \( M^j ; j = 1,2, \ldots, m \);
2. For each vector \( \eta \in \mathbb{R}^n \), we have \( \nabla^2_{U_0} G_i(\eta,\eta) \geq 0 \) whenever \( \nabla U_0 G_i(\eta) = 0 \);
3. \( \nabla U_0 G_i \cdot S(U_0) \leq 0 \).

In the above, \( U_0 \) is an arbitrary point on \( \partial Q \).

Specialising to our system (3.3), let us take \( m = 1 \), \( D = \text{Id} \) and \( d = 4 \). The following choice for \( \{G_i = G_i(w,z)\}_{i=1}^4 \) clearly satisfies Conditions (1)(2) in Proposition 3.1

\[
G_1 := w + c_1, \quad G_2 := z + c_2, \quad G_3 := -w + c_3, \quad G_4 := -z + c_4,
\]
where the constants \( c_1, \ldots, c_4 \) are chosen such that \( Q := \bigcap_{i=1}^4 \{G_i \leq 0\} \) is a closed square whose sides intersecting the \( u \) and \( v \)-axes at an angle of \( \pi/4 \). Moreover, in the \((u,v)\)-plane let us denote the edges of \( Q \) by \( e_i := \{G_i = 0\} \) (hence \( \partial Q = \bigcup_{i=1}^4 e_i \)). To fix the notations, we require that the edge \( e_1 \) connects the top and the right vertices of \( Q \) in the \((u,v)\)-plane, \( e_2 \) connects the right and the bottom vertices, \( e_3 \) the bottom and the left vertices, and \( e_4 \) the left and the top vertices. In addition, we use \((\text{top, right, bottom, left vertices}) := (\bullet_t, \bullet_r, \bullet_b, \bullet_l)\).
Thus, Proposition 3.1 immediately leads to the following criterion for invariant regions to the Gauss–Codazzi equations:

**Proposition 3.2.** The square $Q$ specified by Eq. (3.6) is an invariant region for the balance law (3.3) if and only if $S^{(1)} \leq 0$ on $e_1$, $S^{(2)} \leq 0$ on $e_2$, $S^{(1)} \geq 0$ on $e_3$ and $S^{(2)} \geq 0$ on $e_4$.

That is, the uniform $L^\infty$ estimates on the Riemann invariants $w$ and $z$ (which is ensured by the definition of invariant regions) can be deduced if one can draw a square $Q$ in the $(u, v)$-plane such that, on the four sides of it, the source terms $S^{(1)}, S^{(2)}$ have correct signs. On the other hand, $S^{(1)}, S^{(2)}$ are given explicitly by Eqs. (2.42) and (2.43); in various nice cases, their zero loci are determined by cubic algebraic curves (denoted by $\varphi$ in the sequel) in the $(u, v)$-plane. In view of Proposition 3.2, in these cases the analytic problem of proving uniform $L^\infty$ estimates can be translated to the geometric problem of finding suitable squares $Q$ satisfying the sign conditions prescribed by the relative positions of the zero loci of $S^{(1)}, S^{(2)}$.

Before further development, we remark that the method of invariant regions has been employed by Morawetz [32, 33, 34] in the study of transonic flows (also see Chen–Slemrod–Wang [8]), as well as in various reaction-diffusion models (see [13] and the references therein).

### 3.2. $C^{1,1}$ Weak rigidity of Gauss–Codazzi equations and isometric immersions

With the uniform $L^\infty$ estimates on the components of second fundamental forms at hand, we can send $\epsilon \to 0^+$ (i.e., evaluate the vanishing viscosity limit) and show that the weak limit is still a weak solution to the Gauss–Codazzi equations. Moreover, back to the geometric problem, we can find an isometric immersion whose second fundamental form is prescribed by the above weak limit.

The above result is known as the weak continuity of the nonlinear PDEs (Gauss–Codazzi), or the weak rigidity of isometric immersions. In fact, we have the following Proposition 3.3, which is a special case of Theorem 4.1 and Remark 4.1 in Chen–Li [7]. In [7] the isometric immersion of arbitrary dimensions and co-dimensions are studied. We also mention that the last part of the following proposition had been obtained by Mardare in [30, 31].

**Proposition 3.3.** Let $\{L^\epsilon, M^\epsilon, N^\epsilon\}$ be a family of functions on a surface $(\mathcal{M}, g)$ where $g \in W^{1,p}$ for $p \in [2, \infty]$. Suppose that their $L^p$ norms are uniformly bounded on compact subsets of $\mathcal{M}$, and that they approximately satisfy the Gauss–Codazzi equations (1.4) (1.3) in the following sense:

\begin{align}
(L^\epsilon N^\epsilon) - (M^\epsilon)^2 &= O_1(\epsilon), \\
(M^\epsilon)_x - (L^\epsilon)_y &= \Gamma_{22}^2 L^\epsilon - 2 \Gamma_{12}^2 M^\epsilon + \Gamma_{11}^2 N^\epsilon + O_2(\epsilon), \\
(N^\epsilon)_x - (M^\epsilon)_y &= -\Gamma_{22}^1 L^\epsilon + 2 \Gamma_{12}^1 M^\epsilon - \Gamma_{11}^1 N^\epsilon + O_3(\epsilon),
\end{align}

where $O_i(\epsilon) \to 0$ in $W^{1,-r}$ for $r > 1$ as $\epsilon \to 0^+$, $i \in \{1, 2, 3\}$, and the above equalities are understood in the sense of distributions. Then, as $\epsilon \to 0^+$, after passing to subsequences, $\{L^\epsilon, M^\epsilon, N^\epsilon\}$ converges weakly in $L^p_{\text{loc}}$ to $\{L, M, N\}$, which is a weak solution to the Gauss–Codazzi equations. Moreover, there exists an $C^{1,1}$ (or $W^{2,\infty}$) isometric immersion of $(\mathcal{M}, g)$ into $\mathbb{R}^3$, whose second fundamental form is

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix}.
\]

In summary, in order to establish $C^{1,1}$ isometric immersions for negatively curved surfaces, it suffices to find the aforementioned invariant regions, i.e., the squares $Q$. This is what we shall do in the subsequent sections for several special classes of metrics. Before carrying out this project, let us single out a simple identity central to the later developments:
3.3. A lemma on the derivatives of \( \kappa \) and \( \gamma \). The following result enables us to simplify the expressions of the normalised Christoffel symbols \( \tilde{\Gamma}^i_{jk} \) in Eq. (2.40) for special classes of metrics:

**Lemma 3.4.** Let \( \kappa < 0 \) be the Gauss curvature of \((M, g)\) and \( \gamma := \sqrt{-\kappa} \) as in Eq. (2.21). Then
\[
\frac{\nabla \kappa}{2\kappa} = \frac{\nabla \gamma}{\gamma}.
\]

**Proof.** This is immediate from
\[
\frac{\nabla \gamma}{\gamma} = \nabla \log \gamma = \nabla \log \sqrt{-\kappa} = \frac{\nabla \log(-\kappa)}{2} = \frac{\nabla \kappa}{2\kappa}.
\]
\[
\Box
\]

4. The Standard Hyperbolic Plane

In this section we establish the \( C^{1,1} \) isometric immersions of the standard hyperbolic plane \( \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\} \), also known as the Lobachevsky plane.

Recall that \( \mathbb{H}^2 \) is a Lie group when one identifies \((x, y) \in \mathbb{H}^2\) with the proper affine functions \( g_{(x,y)} : \mathbb{R} \to \mathbb{R}, t \mapsto yt + x \), where the group action is the composition of functions (cf. Chapter 1 in do Carmo [19]). The unique left-invariant metric \( g \) on \( \mathbb{H}^2 \) satisfying \( g(0,1) = g_{\text{Eucl}} \) is the standard Lobachevsky metric:
\[
g = \begin{bmatrix}
y^{-2} & 0 \\
0 & y^{-2}
\end{bmatrix}, \quad y > 0.
\]

Its Christoffel symbols are
\[
\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{22} = 0, \quad \Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^2_{22} = \frac{1}{y},
\]

It is well-known that \( \mathbb{H}^2 \) has constant Gauss curvature \( \kappa \equiv -1 \); hence \( \gamma = -\kappa^2 = 1 \) and
\[
\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} \quad \text{for all } i, j, k \in \{1, 2\}.
\]

Therefore, substituting into Eqs. (2.42) (2.43), we get
\[
S^{(1)} = \frac{1}{y}(u - v)
\left( 1 + (u + v)^2 \right),
\]
\[
S^{(2)} = \frac{1}{y}(u + v)
\left( 1 + (u - v)^2 \right).
\]

Notice that \( y > 0 \) and the zero loci of \( S^{(1)} \), \( S^{(2)} \) are \( \{u = v\} \) and \( \{u = -v\} \), respectively. By Proposition 3.2 the square \( Q \) with the vertices
\[
\bullet_t = (0, 2a), \quad \bullet_r = (a, a), \quad \bullet_b = (0, 0), \quad \bullet_l = (-a, a)
\]
is an invariant region for Eq. (3.3), where \( a > 0 \) is an arbitrary positive constant. \( Q \) can be equivalently characterised by
\[
Q = \{(u, v) \in \mathbb{R}^2 : 0 \leq u + v = w \leq 2a, -2a \leq u - v = z \leq 0\}.
\]

Therefore, we can deduce:

**Theorem 4.1.** Let \( y_0 > 0 \) be arbitrary and denote the initial data by
\[
(u, v)|_{y=y_0} =: (u_0, v_0) : \mathbb{R} \to \mathbb{R}^2.
\]
Assume that \( u_0 \pm v_0 \) is bounded on \( \mathbb{R} \) and
\[
\inf_{\mathbb{R}} (u_0 + v_0) > 0, \quad \sup_{\mathbb{R}} (u_0 - v_0) < 0. \tag{4.7}
\]
Then there exists an \( L^\infty \) weak solution to the Gauss–Codazzi equations \((1.4)\) and \((1.3)\) in the half space \( \Omega = \mathbb{R} \times (y_0, \infty) \) for the standard metric \((4.1)\) on \( \mathbb{H}^2 \). Moreover, restricted to the domain \( \Omega, \mathbb{H}^2 \) admits a \( C^{1,1} \) isometric immersion into \( \mathbb{R}^3 \).

**Proof.** We first note that the assumptions on \( u_0, v_0 \) ensure the existence of some point \( x_\star \in \mathbb{R} \) such that
\[
\partial u_0(x_\star), \partial v_0(x_\star) \in Q \text{ for some } a > 0, \text{ where } Q \text{ is the invariant region described above.}
\]
Thus we get the uniform \( L^\infty \) estimate for \((u,v)\) with the initial data \((u_0, v_0)\) on \( \Omega \). On the other hand, we have \((L,M,N) = (1/v, -u/v, (a^2 - v^2)/v)\); so the uniform \( L^\infty \) estimate for \((L,M,N)\) follows, where the superscripts \( \epsilon \) has been dropped. Moreover, they satisfy the Gauss–Codazzi equations approximately in the sense of Proposition 3.3: this can be seen by adopting verbatim the estimates in Chapter 4 of Cao–Huang–Wang [3]. Therefore, the proof is complete by taking the vanishing viscosity limit, in view of Proposition 3.3. \( \Box \)

In this above we have established the \( C^{1,1} \) isometric immersion of \( \mathbb{H}^2 \) “nearly globally”, i.e., away from an arbitrarily thin slot \( \{(x, y) \in \mathbb{R}^2 : 0 < y < y_0\} \). Thus, various isometric models of \( \mathbb{H}^2 \), including the Poincaré disk and the pseudo-sphere, all admit \( C^{1,1} \) “nearly global” isometric immersions into \( \mathbb{R}^3 \). In contrast, these models are long known to be not globally \( C^2 \) embeddable (cf. Hilbert [27]).

5. Generalised Helicoid-type metrics

The goal of this section is two-fold: first, let us prove the existence of isometric immersions of another family of “generalised helicoid-type” metrics, which has not been covered by Cao–Huang–Wang [3, 4] and Christoforou [12]; second, we give a further characterisation of the generalised helicoids considered in [3, 4].

5.1. Global existence of isometric immersions of some helicoid-type metrics. As in [3, 4], let us call the metrics
\[
g = \begin{bmatrix} E(y) & 0 \\ 0 & 1 \end{bmatrix} \tag{5.1}
\]
of the “helicoid type”. When \( E \) is a suitable hyperbolic cosine function of \( y \) (see §5.2 below), \( g \) is the metric for the classical helicoid, or the generalised helicoids considered in [3, 4]. In this case, Eq. (2.45) yields that
\[
\Gamma_{11}^2 = -\frac{E_y}{2}, \quad \Gamma_{12}^1 = \frac{E_y}{2E}, \quad \text{other } \Gamma_{jk}^i = 0. \tag{5.2}
\]
Then, in view of Eqs. (2.46) (2.42) and (2.43), the source terms can be simplified as follows:
\[
S^{(1)} = -\frac{E_y}{E}u + \frac{\gamma_y}{\gamma}v - \frac{E_y}{2E}(u - v)(u + v)^2,

S^{(2)} = -\frac{E_y}{E}u - \frac{\gamma_y}{\gamma}v - \frac{E_y}{2E}(u - v)^2(u + v). \tag{5.3}
\]
Moreover, in this case we can compute the Gauss curvature, e.g., via Brioschi’s formula (see [1]) as follows:
\[
\kappa = -\frac{1}{2\sqrt{E} \partial y \sqrt{E}} \frac{\partial y E}{\partial y} = -\frac{E_{yy}}{2E} + \frac{(E_y)^2}{4E^2}. \tag{5.4}
\]
Thus, by Lemma 3.4, we have
\[
\frac{\gamma_y}{y} = \frac{-E^2 E_{yy} + 2 EE_y E_{yy} - (E_y)^3}{-2E^2 E_{yy} + E (E_y)^2} = \frac{EE_{yy} - (E_y)^2 - E_y}{E}. \tag{5.5}
\]

From now on let us specialise to the following particular family of metrics, which has not established by the generalised helicoid-type metrics in [3, 4, 9, 10, 12]:
\[
g = \begin{bmatrix} Ay^2 + B y + C & 0 \\ 0 & 1 \end{bmatrix}, \quad A > 0 \text{ and } B^2 - 4AC < 0. \tag{5.6}
\]
The conditions on $A, B, C$ ensure that $g$ is indeed a Riemannian metric. In this case, Eq.\((5.4)\) immediately leads to
\[
\kappa = \frac{B^2 - 4AC}{4(Ay^2 + By + C)} < 0, \tag{5.7}
\]
and Eq.\((5.5)\) becomes
\[
\frac{\gamma_y}{\gamma} = \frac{E_y}{E}. \tag{5.8}
\]
Thus, further simplifications of the source terms in Eq.\((5.3)\) are available:
\[
S^{(1)} = -\frac{E_y}{E} \left\{ (u + v) + \frac{(u - v)(u + v)^2}{2} \right\}, \tag{5.9}
\]
\[
S^{(2)} = -\frac{E_y}{E} \left\{ (u - v) + \frac{(u - v)^2(u + v)}{2} \right\}. \tag{5.10}
\]

In the case $E_y/E \geq 0$, i.e., if we further require $2Ay + B \geq 0$, the signs of $S^{(1)}, S^{(2)}$ are determined by the cubic polynomials $\varphi^{(1)}, \varphi^{(2)} \in \mathbb{R}[u, v]$ (cf. Fig.\(5.1)\):
\[
\varphi^{(1)} = -(u + v) - \frac{(u - v)(u + v)^2}{2} = -(u + v) \left[ 1 + \frac{u^2 - v^2}{2} \right], \tag{5.11}
\]
\[
\varphi^{(2)} = (v - u) - \frac{(u - v)^2(u + v)}{2} = (v - u) \left[ 1 + \frac{u^2 - v^2}{2} \right]. \tag{5.12}
\]
For $(u, v) \in [0, 1] \times [0, 1]$ there holds $1 + (u^2 - v^2)/2 > 0$; by a similar computation as in §4, we find that the square $Q$ with vertices
\[
\bullet_l = (0, 2a), \quad \bullet_r = (a, a), \quad \bullet_b = (0, 0), \quad \bullet_t = (-a, a)
\]
is an invariant region whenever $a$ is an arbitrary constant in $(0, 1]$. Using analogous arguments as in the proof of Theorem 4.1, we conclude with the following sufficient conditions for the existence of isometric immersions for the helicoid-type metrics:

**Theorem 5.1.** Let $y_0 > 0$ be arbitrary and denote the initial data by
\[
(u, v)|_{y=y_0} =: (u_0, v_0) : \mathbb{R} \to \mathbb{R}^2.
\]
Assume that
\[
0 < \inf_{\mathbb{R}} (u_0 + v_0) \leq \sup_{\mathbb{R}} (u_0 + v_0) \leq 2, \quad -2 \leq \inf_{\mathbb{R}} (u_0 - v_0) \leq \sup_{\mathbb{R}} (u_0 - v_0) < 0. \tag{5.13}
\]
Then there exists an $L^\infty$ weak solution to the Gauss–Codazzi equations \((1.4)\) and \((1.3)\) in the half space $\Omega = \mathbb{R} \times (y_0, \infty)$ for the helicoid-type metric
\[
g = \begin{bmatrix} Ay^2 + B y + C & 0 \\ 0 & 1 \end{bmatrix} \quad \text{where } A > 0, B \geq 0 \text{ and } B^2 - 4AC < 0.
\]
Moreover, restricted to the domain $\Omega$, $g$ admits a $C^{1,1}$ isometric immersion into $\mathbb{R}^3$.\[14\]
On the other hand, in the case $E_y/E \leq 0$, i.e., $2Ay + B \leq 0$, the square $Q$ with vertices $\bullet_t = (0, a')$ for $a' > \sqrt{2}$, $\bullet_b = (0, a'')$ for $0 \leq a'' < \sqrt{2}$, and the other two vertices $\bullet_l, \bullet_r$ lying on the upper branch of the hyperbola $\{(u, v) : u^2 - v^2 + 2 = 0\}$ is an invariant region. Surprisingly, the computation here coincides with that for the “catenoid-type surfaces” in §3.1 of Cao–Huang–Wang [3]; see Eq. (3.7) on p1439 therein, with the choice $c = \sqrt{2}$. Therefore,

**Theorem 5.2.** Let $y_0 > 0$ be arbitrary and denote the initial data by $(u, v)|_{y=y_0} =: (u_0, v_0): \mathbb{R} \to \mathbb{R}^2$.

Assume that $u_0 \pm v_0$ is bounded on $\mathbb{R}$, and

$$\sup_{\mathbb{R}}(u_0 + v_0) > 0, \quad \inf_{\mathbb{R}}(u_0 - v_0) < 0. \tag{5.14}$$

Then there exists an $L^\infty$ weak solution to the Gauss–Codazzi equations (1.4) and (1.3) in the strip $\Omega = \mathbb{R} \times [y_0, -B/2A]$ for the helicoid-type metric

$$g = \begin{bmatrix} Ay^2 + By + C & 0 \\ 0 & 1 \end{bmatrix}$$

where $A > 0$, $B \leq 0$ and $B^2 - 4AC < 0$.

Moreover, restricted to the domain $\Omega$, $g$ admits a $C^{1,1}$ isometric immersion into $\mathbb{R}^3$.

![Figure 5.1. The zero loci for $\varphi^{(1)}$ (blue) and $\varphi^{(2)}$ (orange)](image)

### 5.2. Absence of isometric deformation in $\mathbb{R}^3$ from generalised catenoids to generalised helicoids.

In this subsection we provide further descriptions of the $C^{1,1}$ isometrically immersed generalised helicoids found by Cao–Huang–Wang in [3, 4]. As is well-known in classical differential geometry (e.g., Abbena–Salamon–Gray [1]), the standard helicoids and standard catenoids, both immersed in $\mathbb{R}^3$, can be isometrically deformed into each other in $\mathbb{R}^3$—there exists a one-parameter family of differentiable immersions $\{\Phi_\tau : \mathbb{R}^2 \to \mathbb{R}^3 : 0 \leq \tau \leq \pi/2\}$ that varies smoothly in $\tau$, in which $\Phi_0$ is an isometric immersion for the standard helicoid, $\Phi_{\pi/2}$ is that for the standard catenoid, and the induced metrics by $\Phi_\tau$ are constant in $\tau$. Thus, if we know that $\Phi_{\pi/2}$ is an isometric immersion for the standard catenoid, and the family $\{\Phi_\tau\}$ is an isometric deformation, then we can immediately deduce that the standard helicoid is isometrically immersible in $\mathbb{R}^3$. 

15
Now, recall that in Chen–Slemrod–Wang [9], Cao–Huang–Wang [3] and Christoforou [12] (the ranges of parameters in these works differ slightly) it has been proved that the metrics of the “generalised catenoid-type” can be $C^{1,1}$ isometrically immersed in $\mathbb{R}^3$. As a subset of such “generalised catenoid-type” metrics, the following metrics
\[
g_\beta^{\text{cat}} = \begin{bmatrix} E(y) = \left( c \cosh \left( \frac{y}{c} \right) \right)^{\frac{1}{\beta^2 - 1}} & 0 \\ 0 & c^\frac{1}{\beta^2 - 1} s \end{bmatrix}, \quad c \neq 0, \quad \beta \geq \sqrt{2} \tag{5.15}
\]
admit isometric immersions $f_\beta^{\text{cat}}$ on $\mathbb{R} \times [-y_0, 0]$ for some $y_0 > 0$:
\[
f_\beta^{\text{cat}}(x, y) = \left( \left( c \cosh \left( \frac{y}{c} \right) \right)^{\frac{1}{\beta^2 - 1}} \sin x, \left( c \cosh \left( \frac{y}{c} \right) \right)^{\frac{1}{\beta^2 - 1}} \cos x, \int_y^0 \frac{1}{\beta^2 - 1} \left( c \cosh \left( \frac{s}{c} \right) \right)^{\frac{1}{\beta^2 - 1}} ds \right)^\top. \tag{5.16}
\]
Notice that $\beta = \sqrt{2}$ corresponds to the standard catenoid; in the sequel, the metric and the standard parametrisation are denoted as $g^{\text{std cat}}$ and $f^{\text{std cat}}$, respectively:
\[
g^{\text{std cat}} = g_\beta^{\text{cat}} = \begin{bmatrix} c^2 \cosh \left( \frac{y}{c} \right)^2 & 0 \\ 0 & \frac{1}{c} \cosh \left( \frac{y}{c} \right)^2 \end{bmatrix}, \tag{5.17}
\]
\[
f^{\text{std cat}}(x, y) = f_\beta^{\text{cat}}(x, y) = \left( c \cosh \left( \frac{y}{c} \right) \sin x, c \cosh \left( \frac{y}{c} \right) \cos x, y \right)^\top. \tag{5.18}
\]
On the other hand, as motivated by the case of classical helicoid and catenoid, a natural parametrisation of the “generalised helicoids” is as follows:
\[
f^{\alpha,\psi}_{\text{hel}}(x, y) = \left( c \sinh \left( \frac{y}{c} \right)^{\alpha} \sin x, -c \sinh \left( \frac{y}{c} \right)^{\alpha} \cos x, \psi(x) \right)^\top, \tag{5.19}
\]
where $\psi$ is a function in $x$ only, and $\alpha > 0$ is a constant. Let us call $g^{\alpha,\text{id}}_{\text{hel}}$ the metrics of the “generalised helicoid-type” induced by $f^{\alpha,\psi}_{\text{hel}}$, which form a subset of the metrics considered in [3]:
\[
g^{\alpha,\psi}_{\text{hel}} := \left\{ f^{\alpha,\psi}_{\text{hel}} \right\} \ast g^{\text{Eucl}}. \tag{5.20}
\]
Here $g^{\text{Eucl}}$ is the Euclidean metric on $\mathbb{R}^3$, and the superscript $\ast$ denotes the pullback operator. The special case $\alpha = 1, \psi = c \text{id}$ gives the parametrisation of the standard helicoid (modulo the transform $y \mapsto c \sinh(y c^{-1})$):
\[
f^{1,\text{id}}_{\text{hel}}(x, y) = f^{1,\text{id}}_{\text{hel}}(x, y) = \left( c \sinh \left( \frac{y}{c} \right) \sin x, -c \sinh \left( \frac{y}{c} \right) \cos x, cx \right)^\top. \tag{5.21}
\]
Let us also denote by $g^{\text{std hel}}$ the metric of the standard helicoid, namely
\[
g^{\text{std hel}} := \left\{ f^{\text{std hel}} \right\} \ast g^{\text{Eucl}}. \tag{5.22}
\]
Moreover, the family $\{ \Phi_\tau : 0 \leq \tau \leq \frac{\pi}{2} \}$ given by
\[
\Phi_\tau(x, y) := \sin \tau \cdot f^{\text{std hel}}(x, y) + \cos \tau \cdot f^{\text{std cat}}(x, y) \tag{5.23}
\]
is the desired family of isometric deformations.

In what follows, we show that the generalised catenoids $g_\beta^{\text{cat}}$ cannot be naturally isometrically deformed into $g^{\alpha,\psi}_{\text{hel}}$, unless both $g_\beta^{\text{cat}} = g^{\text{std cat}}$ and $g^{\alpha,\psi}_{\text{hel}} = g^{\text{std hel}}$:
Proposition 5.3. Let \( \{ \Phi_\tau : \mathbb{R}^2 \to \mathbb{R}^3 : 0 \leq \tau \leq \pi/2 \} \) be the natural one-parameter family of parametrisations
\[
\Phi_\tau := \sin \tau \cdot f^\alpha_{\text{hel}} + \cos \tau \cdot f^\beta_{\text{cat}},
\]
where \( f^\alpha_{\text{hel}} \) and \( f^\beta_{\text{cat}} \) are defined as in Eqs. (5.19) and (5.16). Then \( \Phi_\tau \) is an isometric deformation from the generalised catenoid to the generalised helicoid if and only if
\[
\alpha = 1, \quad \psi = c \text{Id} \quad \text{and} \quad \beta = \sqrt{2}.
\]

Proof. Let \( g^{(\tau)} \) be the metric induced by \( \Phi_\tau \), namely \( g^{(\tau)} := (\Phi_\tau)^* g_{\text{Eucl}} \). Then
\[
g^{(\tau)}_{12} = \frac{\partial \Phi_\tau}{\partial x} \cdot \frac{\partial \Phi_\tau}{\partial y} = \left[ \begin{array}{c}
\cos(\tau) \left[ \frac{c}{\tau} \sinh \left( \frac{\tau}{c} \right) \right]^{\alpha} \cos x - \sin(\tau) \left[ \frac{c}{\tau} \cosh \left( \frac{\tau}{c} \right) \right]^{\frac{1}{\beta^2-1}} \sin x \\
\cos(\tau) \left[ \frac{c}{\tau} \sinh \left( \frac{\tau}{c} \right) \right]^{\alpha} \sin x + \sin(\tau) \left[ \frac{c}{\tau} \cosh \left( \frac{\tau}{c} \right) \right]^{\frac{1}{\beta^2-1}} \cos x \\
\cos(\tau) \psi'(x)
\end{array} \right].
\]

\[
= \cos(\tau) \sin(\tau) c^{\alpha-1} \left[ \sinh \left( \frac{\tau}{c} \right) \right]^{\frac{1}{\beta^2-1}} \left[ \sinh \left( \frac{\tau}{c} \right) \right]^{\alpha-1} \left\{ -\alpha \left[ \cosh \left( \frac{\tau}{c} \right) \right]^2 + \frac{1}{\beta^2-1} \left[ \sinh \left( \frac{\tau}{c} \right) \right]^2 \right\} + \cos(\tau) \sin(\tau) \frac{1}{\beta^2-1} \psi'(x) \left[ \cosh \left( \frac{\tau}{c} \right) \right]^{\frac{1}{\beta^2-1}}.
\]

Thus, for \( \Phi_\tau \) to be independent of \( \tau \), we need
\[
0 = \frac{\psi'(x)}{\beta^2 - 1} + c^{\alpha-1} \left[ \sinh \left( \frac{\tau}{c} \right) \right]^{\alpha-1} \left\{ -\alpha \left[ \cosh \left( \frac{\tau}{c} \right) \right]^2 + \frac{1}{\beta^2 - 1} \left[ \sinh \left( \frac{\tau}{c} \right) \right]^2 \right\}. \tag{5.24}
\]

The first term on the right-hand side of Eq. (5.24) depends only on \( x \), and the second term only on \( y \). So both terms must be constants, and, in particular, \( \psi'(x) = 0 \). Now let us introduce \( z := \sinh(y/c) \); in this new variable, the second term (call it \( S \)) can be expressed as
\[
S(z) = c^{\alpha-1} z^{\alpha-1} \left\{ -\alpha + \left( \frac{1}{\beta^2 - 1} - \alpha \right) z^2 \right\}.
\]

This forces \( \alpha = 1 \) and \( \frac{1}{\beta^2 - 1} - \alpha = 0 \), namely \( \beta = \sqrt{2} \). Therefore, \( S(z) = -c \), and by Eq. (5.24) we can deduce \( \psi(x) = cx \). The necessity has thus been established.

For sufficiency, we compute
\[
g^{(\tau)}_{11} = \frac{\partial \Phi_\tau}{\partial x} \cdot \frac{\partial \Phi_\tau}{\partial x} = \left\{ \cos \tau \left( \frac{c}{\tau} \sinh \left( \frac{\tau}{c} \right) \right) \right\}^{\alpha} \cos x + \sin \tau \left( \frac{c}{\tau} \cosh \left( \frac{\tau}{c} \right) \right) \left[ \frac{1}{\beta^2-1} \sin x \right]^2
\]
\[
+ \left\{ -\cos \tau \left( \frac{c}{\tau} \sinh \left( \frac{\tau}{c} \right) \right) \right\} \sin x + \sin \tau \left( \frac{c}{\tau} \cosh \left( \frac{\tau}{c} \right) \right) \left[ \frac{1}{\beta^2-1} \cos x \right]^2 + \cos^2 \tau [\psi'(x)]^2
\]
\[
= \cos^2 \tau \left( \frac{c}{\tau} \sinh \left( \frac{\tau}{c} \right) \right)^{2\alpha} + \sin^2 \tau \left( \frac{c}{\tau} \cosh \left( \frac{\tau}{c} \right) \right) \left[ \frac{2}{\beta^2-1} \right] + \cos^2 \tau [\psi'(x)]^2,
\]

17
as well as

\[ g^{(\tau)}_{22} = \frac{\partial \Phi_{\tau}}{\partial y} \cdot \frac{\partial \Phi_{\tau}}{\partial y} \]

\[ = \left\{ \cos \tau \cdot \alpha \left[ c \sinh \left( \frac{y}{c} \right) \right]^{\alpha-1} \cosh \left( \frac{y}{c} \right) \sin x + \sin \tau \frac{1}{\beta^2 - 1} \left[ c \cosh \left( \frac{y}{c} \right) \right]^{\frac{1}{\beta^2-1}} \sinh \left( \frac{y}{c} \right) \cos x \right\}^2 \]

\[ + \left\{ - \cos \tau \cdot \alpha \left[ c \sinh \left( \frac{y}{c} \right) \right]^{\alpha-1} \cosh \left( \frac{y}{c} \right) \cos x + \sin \tau \frac{1}{\beta^2 - 1} \left[ c \cosh \left( \frac{y}{c} \right) \right]^{\frac{1}{\beta^2-1}} \sinh \left( \frac{y}{c} \right) \sin x \right\}^2 \]

\[ + \sin^2 \tau \left( \frac{\beta^2 - 1}{\beta^2 - 1} \right)^2 \left[ c^2 \cosh^2 \left( \frac{y}{c} \right) \right]^{\frac{1}{\beta^2-1}} \]

\[ = \cos^2 \tau \alpha^2 \left[ c \sinh \left( \frac{y}{c} \right) \right]^{2\alpha-2} \cosh^2 \left( \frac{y}{c} \right) + \sin^2 \tau \frac{1}{(\beta^2 - 1)^2} \left[ c \cosh \left( \frac{y}{c} \right) \right]^{\frac{2\alpha-2}{\beta^2-1}}. \]

In the case \( \alpha = 1, \beta = \sqrt{2} \) and \( \psi(x) = cx \), the above identities together with \( 1 + \sinh^2 \theta = \cosh^2 \theta \) immediately gives us

\[ g^{(\tau)}_{11} = g^{(\tau)}_{22} = c^2 \cosh^2 \left( \frac{y}{c} \right), \quad g^{(\tau)}_{12} = 0 \quad \text{for all } 0 \leq \tau \leq \frac{\pi}{2}. \]

Indeed, \( \Phi_{\tau} \) is independent of \( \tau \). The proof is now complete.

\[ \square \]

Proposition 5.3 suggests that the “generalised helicoids” considered in Cao–Huang–Wang [3, 4] and the “generalised catenoids” considered in Chen–Slemrod–Wang [9], Cao–Huang–Wang [3] and Christoforou [12] are distinct geometric objects, unless they are precisely the classical helicoids and catenoids. In particular, one cannot prove the existence of isometric immersions of the latter via a natural isometric deformation in the ambient space \( \mathbb{R}^3 \) from the former. This provides further characterisation of the families of isometrically immersible metrics found in [3, 4, 9, 10, 12].

6. Generalised Enneper metrics

In this section, we consider the following one-parameter family of metrics:

\[ g^{(\alpha)} = \begin{bmatrix} (1 + x^2 + y^2)^\alpha & 0 \\ 0 & (1 + x^2 + y^2)^\alpha \end{bmatrix}. \]

(6.1)

When \( \alpha = 2 \), the metric \( g^{(2)} \) gives the well-known classical minimal surface, known as the “Enneper surface” (see [1]). It has the following global isometric immersion:

\[ f_{enn}(x, y) := \left( x - \frac{1}{3} x^3 + xy^2, -y - x^2y + \frac{1}{3} y^3, x^2 - y^2 \right)^\top, \]

(6.2)

with the Gauss curvature

\[ \kappa_{enn} = \frac{-4}{(1 + x^2 + y^2)^4}. \]

(6.3)

We call the metrics \( \{g^{(\alpha)}\} \) in Eq. (6.1) “generalised Enneper metrics”. For simplicity in notations, introduce

\[ \Theta(x, y) := 1 + x^2 + y^2, \]

(6.4)

so that \( E = G = \Theta^\alpha, \) \( F = 0 \) in the metric \( g^{(\alpha)} \). Thus, we have

\[ \begin{cases} E_x = 2\alpha \Theta^{\alpha-1} x, \quad E_y = 2\alpha \Theta^{\alpha-1} y, \\ E_{xx} = 2\alpha \Theta^{\alpha-1} + 4\alpha (\alpha - 1) \Theta^{\alpha-2} x^2, \quad E_{yy} = 2\alpha \Theta^{\alpha-1} + 4\alpha (\alpha - 1) \Theta^{\alpha-2} y^2. \end{cases} \]
The Christoffel symbols can be computed from Eq. (2.45):

\[
\Gamma^1_{11} = \frac{\alpha x}{\Theta}, \quad \Gamma^1_{12} = \frac{\alpha y}{\Theta}, \quad \Gamma^1_{22} = -\frac{\alpha x}{\Theta},
\]

\[
\Gamma^2_{11} = -\frac{\alpha y}{\Theta}, \quad \Gamma^2_{12} = \frac{\alpha x}{\Theta}, \quad \Gamma^2_{22} = \frac{\alpha x}{\Theta}.
\]

Next, the non-trivial component of Riemann curvature becomes

\[
R^1_{212} = \tilde{\partial}_1 \Gamma^1_{22} - \tilde{\partial}_2 \Gamma^1_{12} + \Gamma^1_{22} \Gamma^1_{11} + \Gamma^2_{22} \Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^1_{12} \Gamma^2_{22} = \frac{2\alpha}{\Theta^2};
\]

thus

\[
\kappa = \frac{g_{11} R^1_{212}}{|g|} = -2\alpha \Theta^{-2-\alpha} < 0 \quad \text{whenever} \ \alpha > 0. \quad (6.5)
\]

In addition, we have

\[
\gamma = \sqrt{-\kappa} = \frac{\sqrt{2\alpha}}{\Theta^{2+\alpha}} \quad \text{and} \quad \nabla \gamma = -(2+\alpha) \sqrt{2\alpha} \Theta^{-2-\frac{\alpha}{2}} \begin{bmatrix} x \\ y \end{bmatrix},
\]

so

\[
\frac{\nabla \gamma}{\gamma} = -\frac{2+\alpha}{\Theta} \begin{bmatrix} x \\ y \end{bmatrix};
\]

on the other hand,

\[
\nabla E = \frac{2\alpha \Theta^{\alpha-1}}{\Theta^\alpha} = \frac{2\alpha}{\Theta} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (6.6)
\]

Therefore, we also have

\[
\frac{\nabla \gamma}{\gamma} = \beta \frac{\nabla E}{E} \quad \text{where} \quad \beta = -\frac{2\alpha}{2+\alpha}. \quad (6.7)
\]

The source terms \( S^{(1)} \) and \( S^{(2)} \) are given as follows:

\[
S^{(1)} = -\frac{1}{2} \frac{2\alpha y}{\Theta} (u+v)^2 (u-v) + \frac{\alpha x}{\Theta} (u+v)^2 + \beta v(u+v) \frac{2\alpha}{\Theta} x
\]

\[
+ \frac{1}{2} \frac{2\alpha y}{\Theta} u + \left\{ -\frac{1}{2} \frac{2\alpha y}{\Theta} + \frac{\beta}{2} \frac{2\alpha y}{\Theta} \right\} v + \frac{1}{2} \frac{2\alpha}{\Theta} x
\]

\[
= \frac{\alpha y}{\Theta} \left\{ -(u+v)^2 (u-v) + \mu(u+v)^2 - \frac{2\alpha}{2+\alpha} \mu v(u+v) + u - \frac{5\alpha + 2}{2+\alpha} v + \mu \right\}. \quad (6.8)
\]

and

\[
S^{(2)} = -\frac{1}{2} \frac{2\alpha y}{\Theta} (u+v)(u-v)^2 + \frac{\alpha x}{\Theta} u^2 + \left\{ \frac{\alpha x}{\Theta} + \beta \frac{2\alpha x}{\Theta} \right\} v^2
\]

\[
- \left\{ \frac{2\alpha x}{\Theta} + \beta \frac{2\alpha x}{\Theta} \right\} uv + \left\{ \frac{3}{2} \frac{2\alpha y}{\Theta} + \beta \frac{2\alpha y}{\Theta} \right\} u - \left\{ \frac{\alpha y}{\Theta} + \beta \frac{2\alpha y}{\Theta} \right\} v + \frac{1}{2} \frac{2\alpha}{\Theta}
\]

\[
= \frac{\alpha y}{\Theta} \left\{ -(u+v)(u-v)^2 + \mu u^2 + \frac{2-3\alpha}{2+\alpha} \mu v^2 - \frac{4-2\alpha}{2+\alpha} \mu uv - \frac{2-3\alpha}{2+\alpha} (u+v) + \mu \right\}. \quad (6.9)
\]

Throughout the current section we write \( \mu \) for

\[
x \equiv \mu y. \quad (6.10)
\]

Thus, for \( y \geq 0 \), the signs of \( S^{(1)}, S^{(2)} \) are equal to those of the polynomials \( \varphi^{(1)}, \varphi^{(2)} \):

\[
\varphi^{(1)} := -(u+v)(u-v) + \mu(u+v)^2 - \frac{2\alpha}{2+\alpha} \mu v(u+v) + u - \frac{5\alpha + 2}{2+\alpha} v + \mu, \quad (6.11)
\]

\[
\varphi^{(2)} := -(u+v)(u-v)^2 + \mu u^2 + \frac{2-3\alpha}{2+\alpha} \mu v^2 - \frac{4-2\alpha}{2+\alpha} \mu uv - \frac{2-3\alpha}{2+\alpha} (u+v) + \mu. \quad (6.12)
\]
The zero loci of the above polynomials in the \((u, v)\)-plane are fairly complicated. Fig. 6.1 shows \(\{\psi^{(1)} = 0\}\) and \(\{\psi^{(2)} = 0\}\) for a typical choice of \(\alpha, \mu\), and in the complimentary material in §8 we include the animation of the two-parameter \((\alpha, \mu)\) family of the zero loci. Nevertheless, one distinctive common feature about these zero loci can be observed: the manner in which the rightmost branches of \(\{\psi^{(1)} = 0\}\) and \(\{\psi^{(2)} = 0\}\) intersect guarantees the existence of an invariant region \(Q\) with vertices

\[
\bullet_1 = (a + b, b), \quad \bullet_r = (a + 2b, 0), \quad \bullet_d = (a + b, -b), \quad \bullet_l = (a, 0)
\]

with some \(a\) and \(b > 0\), due to Proposition 3.2. Furthermore, by experimenting with different ranges of the parameters \(\alpha, \mu\), we find that for each \(\alpha \in [1, 10]\) and \(\mu \in [0, 1]\), the above constants \(a, b\) can be chosen independent of \(\mu\). By arguments similar to those in the previous sections, we can deduce:

**Theorem 6.1.** Let \(y_0 > 0\) be arbitrary and denote the initial data by

\[
(u, v)|_{y = y_0} =: (u_0, v_0) : \mathbb{R} \to \mathbb{R}^2.
\]

For each \(\alpha \in [1, 10]\), there exist positive numbers \(a, b > 0\) such that the following holds: Assume that

\[
a \leq \inf_{\mathbb{R}} (u_0 + v_0) \leq \sup_{\mathbb{R}} (u_0 + v_0) \leq a + 2b, \quad -a \geq \sup_{\mathbb{R}} (u_0 - v_0) \leq \inf_{\mathbb{R}} (u_0 - v_0) \leq -a - 2b. \quad (6.13)
\]

Then there exists an \(L^\infty\) weak solution to the Gauss–Codazzi equations (1.4) and (1.3) in the interior of the wedge \(\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, y \leq x\}\) for the generalised Enneper metric

\[
g^{(\alpha)} = \begin{bmatrix}
(1 + x^2 + y^2)^\alpha & 0 \\
0 & (1 + x^2 + y^2)^\alpha
\end{bmatrix}.
\quad (6.14)
\]

By the symmetry of the \(x, y\) variables in \(g^{(\alpha)}\), Theorem 6.1 implies the existence of \(C^{1,1}\) isometric immersions in the whole \(\mathbb{R}^2\), with possible exception on the diagonals \(\{y = \pm x\}\).

![Figure 6.1. The zero loci for \(\psi^{(1)}\) (blue) and \(\psi^{(2)}\) (orange), with \(c = 1, d = 2\)](image)
7. “Reciprocal-type” metrics: Non-existence of invariant regions

In this section we discuss a special class of negatively curved metrics, for which the source terms $S^{(1)}, S^{(2)}$ are easy to compute, but the invariant regions $Q$ of the form in Proposition 3.2 fail to exist. It manifests the potential limitations of the compensated compactness/invariant region method.

We consider the metrics of the “reciprocal-type”:

$$g = \begin{bmatrix} E(y) & 0 \\ 0 & E(y)^{-1} \end{bmatrix}, \quad E(y) > 0.$$  \hfill (7.1)

The Brioschi’s formula (see [1]) gives us

$$\kappa = -\frac{1}{2\sqrt{EG}} \left[ \frac{\partial}{\partial x} \left( \frac{Gx}{\sqrt{EG}} \right) + \frac{\partial}{\partial y} \left( \frac{Ey}{\sqrt{EG}} \right) \right] = -\frac{Eyy}{2}.$$  \hfill (7.2)

Hence, we require $E$ to be strictly convex and positive. Thanks to Eqs. (2.45)-(2.46) we have

$$\tilde{\Gamma}_{22} = -\frac{Ey}{2E} + \frac{\gamma_y}{\gamma}, \quad \tilde{\Gamma}_{11} = -\frac{EE_y}{2}, \quad \tilde{\Gamma}_{12} = \frac{Ey}{2E} + \frac{\gamma_y}{2\gamma}, \quad \text{other } \tilde{\Gamma}_{ij} = 0.$$

The source terms in Eqs. (2.42)-(2.43) thus become:

$$S^{(1)} = -\frac{3Ey}{2E}u + \left( -\frac{Ey}{2E} + \frac{\gamma_y}{\gamma} \right)v - \frac{Ey}{2E}(u-v)(u+v)^2,$$

$$S^{(2)} = -\frac{3Ey}{2E}u + \left( \frac{Ey}{2E} - \frac{\gamma_y}{\gamma} \right)v - \frac{Ey}{2E}(u-v)^2(u+v).$$  \hfill (7.3)

To proceed, let us find conditions on $E$ such that $\gamma_y/\gamma$ is proportional to $Ey/E$: in this case we can factor out $Ey/E$ in $S^{(1)}, S^{(2)}$. Indeed, suppose there exists a constant $\alpha$ (to be determined) such that

$$\frac{\gamma_y}{\gamma} = \frac{E_y}{E}. \hfill (7.4)$$

By Lemma 3.4 and Eq. (7.2), we obtain the ODE:

$$\frac{E_{yyy}}{E_{yy}} = 2\alpha \frac{E_y}{E}.$$  \hfill (7.5)

Taking logarithmic on both sides, there is a constant $C > 0$ such that

$$E_{yy} = CE^\alpha.$$  \hfill (7.6)

From the above ODE, we find that exponential and hyperbolic trigonometric functions are good choices for $E = E(y)$.

7.1. Case 1: exponential. In this case let us consider

$$E(y) = Ae^{\omega y}. \hfill (7.7)$$

Thus, $C = \omega > 0$ and $\alpha = 1$ in Eq. (7.5). We can easily compute that

$$\kappa = -\frac{\omega^2 e^{\omega y}}{2} < 0,$$  \hfill (7.8)

as well as

$$\frac{E_y}{E} = \frac{2\gamma_y}{\gamma} = \omega.$$
Moreover, the source terms are given by
\[ S^{(1)} = -\frac{3\omega}{2} u - \frac{\omega e^{2\omega y}}{2} (u - v)(u + v), \]
\[ S^{(2)} = -\frac{3\omega}{2} u - \frac{\omega e^{2\omega y}}{2} (u - v)^2 (u + v), \] (7.9)
whose signs are determined by \( \wp^{(1)}, \wp^{(2)} \in \mathbb{R}[u, v] \) respectively:
\[ \wp^{(1)} = 3u + k(u - v)(u + v), \] (7.10)
\[ \wp^{(2)} = 3u + k(u + v)(u - v), \] (7.11)
with the factor
\[ k \equiv k(\omega, y) = e^{2\omega y} > 0. \] (7.12)

7.2. **Case 2: hyperbolic cosine.** In this case we consider
\[ E(y) = A \cosh(\omega y), \quad A, \omega > 0. \] (7.13)
Then by Eq. (7.5) we have \( \alpha = 1/2, \) and
\[ \kappa = -\omega^2 \cosh(\omega y)/2 < 0. \] (7.14)
Also, we have
\[ \frac{E_y}{E} = \frac{2\gamma_y}{\gamma} = \omega \tanh(\omega y), \]
and the source terms are given by
\[ S^{(1)} = -\frac{\omega \tanh(\omega y)}{2} \left\{ 3u + [\cosh(\omega y)]^2 (u - v)(u + v)^2 \right\}, \]
\[ S^{(2)} = -\frac{\omega \tanh(\omega y)}{2} \left\{ 3u + [\cosh(\omega y)]^2 (u - v)^2 (u + v) \right\}. \] (7.15)
Thus, whenever \( y > 0, \) the signs of \( S^{(1)}, S^{(2)} \) are opposite to those of
\[ \wp^{(1)} = 3u + \tilde{k}(u - v)(u + v), \] (7.16)
\[ \wp^{(2)} = 3u + \tilde{k}(u + v)(u - v), \] (7.17)
with the factor
\[ \tilde{k} \equiv \tilde{k}(\omega, y) = \cosh^2(\omega y) \geq 1. \] (7.18)
Here \( \wp^{(1)}, \wp^{(2)} \) take the same form as those in Case 1, §6.1; the only difference is in the factor \( \tilde{k}. \)

As seen from Fig. 7.1 (cf. the supplementary materials in §9 for the animation of the one-parameter family \( k \in [0, 3] \)), the zero locus of \( \wp^{(1)} \) (\( \wp^{(2)}, \) resp.) in the \( (u, v)-plane \) is a curve locally look like \( \{ v = u^3 \} \) (\( \{ v = u^3 \}, \) resp.) near the origin. Moreover, in the right half the \( (u, v)-plane \) to the zero locus of \( \wp^{(1)} \) (\( \wp^{(2)}, \) resp.) one has \( \wp^{(1)} > 0 \) (\( \wp^{(2)} > 0, \) resp.) It is clear that the invariant region \( Q \) of the form in Proposition 3.2 cannot exist. Therefore, our method in §§4–6 does not apply to the reciprocal-type metrics (7.1).

8. **Conclusions**

In this paper we study isometric immersions of negatively curved surfaces by the method of compensated compactness. The existence of \( C^{1,1} \) isometric immersions is guaranteed by the existence of invariant regions \( Q; \) the latter is deduced from the relative positions of the zero loci...
of source terms $S^{(1)}$, $S^{(2)}$ of the hyperbolic balance laws associated to the Gauss–Codazzi system, namely Eqs. (2.40) and (2.41).

In Theorem 4.1 we show the existence of isometric immersions of the standard Lobachevsky plane away from the $x$-axis into $\mathbb{R}^3$. In Theorems 5.1 and 5.2 we prove the existence of isometric immersions of an infinite family of helicoid-type metrics into $\mathbb{R}^3$. Finally, in Theorem 6.1 we prove the existence of isometric immersions, with computer assistance, for a one-parameter family of metrics containing the Enneper surface. The regularity for the above isometric immersions is $C^{1,1}$, in view of the method of compensated compactness we adopt in this paper.

The supplementary materials to this work can be downloaded from Wolfram Cloud, Siran Li_isometric immersions of negatively curved surfaces. Mathematica code.nb.

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