ON NON-COMMUTING SETS AND CENTRALIZERS IN INFINITE GROUP

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Abstract. A subset $X$ of a group $G$ is a set of pairwise non-commuting elements if $ab \neq ba$ for any two distinct elements $a$ and $b$ in $X$. If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements $Y$ in $G$, then $X$ is said to be a maximal subset of pairwise non-commuting elements and the cardinality of such a subset is denoted by $\omega(G)$. In this paper, among other things, we prove that, for each positive integer $n$, there are only finitely many groups $G$, up to isoclinic, with $\omega(G) = n$ (with exactly $n$ centralizers).

Keywords. Pairwise non-commuting elements of a group; Isoclinic groups; $n$-centralizers.

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1. Introduction and results

Let $G$ be a non-abelian group. We call a subset $X$ of $G$ a set of pairwise non-commuting elements if $ab \neq ba$ for any two distinct elements $a$ and $b$ in $X$. If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements $Y$ in $G$, then $X$ is said to be a maximal subset of pairwise non-commuting elements and the cardinality of such a subset is called the clique number of $G$ and it is denoted by $\omega(G)$. By a famous result of Neumann [10] answering a question of Erdos, we know that the finiteness of $\omega(G)$ in $G$ is equivalent to the finiteness of the factor group $G/Z(G)$, where $Z(G)$ is the center of $G$. Moreover, Pyber [11] showed that $\omega(G)$ is also related to the index of the center of $G$. In fact, he proved that there is some constant $c$ such that $[G : Z(G)] \leq c^{\omega(G)}$. The clique number of groups was investigated by many authors, for instance see [1, 6, 7].

It is easy to see that, if $H$ is an arbitrary abelian group and $G$ is a group with $\omega(G) = n$, then $\omega(G \times H) = n$. Therefore, there can be infinitely many groups $K$ with $\omega(K) = n$. In this paper, by using a notion of isoclinic groups [9], first we show that the cardinality of maximal subset of pairwise non-commuting elements of any two isoclinic groups are that same (see Lemma 2.1 below). Next, by this result, we show that, for each positive integer $n$, there are only finitely many groups $G$, up to isoclinic, with $\omega(G) = n$. Clearly, the relation isoclinic is an equivalence relation on any family of groups and any two abelian groups are isoclinic.

Our main results are.

Theorem 1.1. Let $n$ be a positive integer and $G$ be an arbitrary group such that $\omega(G) = n$. Then

1. There are only finitely many groups $H$, up to isoclinic, with $\omega(H) = n$.\n
There exists a finite group $K$ such that $K$ is isoclinic to $G$ and $\omega(G) = \omega(K)$.

By this results, we give a sufficient condition for solvability by its the cardinality of maximal subset of pairwise non-commuting elements.

**Theorem 1.2.** Every arbitrary group $G$ with $\omega(G) \leq 20$ is solvable and this estimate is sharp.

For any group $G$, let $\mathcal{C}(G)$ denote the set of centralizers of $G$. We say that a group $G$ has $n$ centralizers ($G$ is a $C_n$-group) if $|\mathcal{C}(G)| = n$. Finally, we obtain similar results for groups with a finite number $n$ of centralizers (see Lemma 3.2, Theorem 3.3, Theorem 3.4 and also Theorem 3.5, below).

2. Pairwise non-commuting elements

For prove the main results, we need the following Lemma.

Two groups $G$ and $H$ are said to be isoclinic if there are isomorphisms $\varphi : G/Z(G) \to H/Z(H)$ and $\phi : G' \to H'$ such that

$$\text{if } \varphi(g_1Z(G)) = h_1Z(H)$$

and $\varphi(g_2Z(G)) = h_2Z(H)$,

then $\phi([g_1, g_2]) = [h_1, h_2]$.

This concept is weaker than isomorphism and was introduced by P. Hall [9] as a structurally motivated classification for finite groups. A stem group is defined as a group whose center is contained inside its derived subgroup. It is known that every group is isoclinic to a stem group and if we restrict to finite groups, a stem group has the minimum order among all groups isoclinic to it, see [9] for more details.

**Lemma 2.1.** For every two isoclinic groups $G$ and $H$ we have $\omega(G) = \omega(H)$.

**Proof.** Suppose that $G$ and $H$ are two isoclinic groups.

Therefore, according to P. Hall [9], there exist the commutator maps

$$\alpha : G/Z(G) \times G/Z(G) \to G', \quad (xZ(G), yZ(G)) \mapsto ([x, y])$$

and

$$\alpha' : H/Z(H) \times H/Z(H) \to H', \quad (xZ(H), yZ(H)) \mapsto ([x, y])$$

and also isomorphisms

$$\beta : G/Z(G) \to H/Z(H), \text{ and } \gamma : G' \to H'$$

such that

$$\alpha' (\beta \times \beta) = \gamma (\alpha)$$

where

$$\beta \times \beta : G/Z(G) \times G/Z(G) \to H/Z(H) \times H/Z(H).$$

Now assume that the set $X = \{x_1, x_2, \ldots, x_n\}$ is a a maximal subset of pairwise non-commuting elements of $G$. It follows that $x_iZ(G) \neq x_jZ(G)$ for all $1 \leq i < j \leq n$. Therefore there exist $n$ elements $y_i \in H \setminus Z(H)$ such that $\beta(x_iZ(G)) = y_iZ(H)$.

For completes the proof it is enough to show that the set $Y = \{y_1, y_2, \ldots, y_n\}$ is a a subset of pairwise non-commuting elements of $H$. Suppose, on the contrary,
that there exist $y_i, y_j \in H$ for some $1 \leq i \neq j \leq n$, such that $[y_i, y_j] = 1$. Now, as mentioned above, we obtain that
\[
\alpha'(\beta \times \beta)((x_iZ(G), x_jZ(G))) = \gamma(\alpha)(x_iZ(G), x_jZ(G))
\]
and so $\alpha'(y_iZ(H), y_jZ(H)) = \gamma([x_i, x_j])$ and so $1 = [y_i, y_j] = \gamma([x_i, x_j])$. It follows that $[x_i, x_j] = 1$, a contradiction. Thus $\omega(G) = |X| = |Y| \leq \omega(H)$ and so $\omega(G) \leq \omega(H)$. Similarly, we get $\omega(H) \leq \omega(G)$ and this completes the proof. □

By the above Lemma we prove Theorem 1.1.

Proof of Theorem 1.1. (1) Assume that $G$ is a group with $\omega(G) = n$. According to the Pyber [11], there is some constant $c$ such that $[G : Z(G)] \leq \omega(G) \leq f(n)$. Therefore, by Schur’s Theorem, the derived subgroup $G'$ is finite and also $|G'| \leq f(n)^{2f(n)^3}$. Therefore there are finitely many isomorphism types of $G/Z(G)$ and $G'$ which are bounded above by a function of $n$. Therefore for every choice of $G/Z(G)$ and $G'$, there are only finitely many commutator maps from $G/Z(G) \times G/Z(G)$ to $G'$. It follows, in view of Lemma 2.1 that $G$ is determined by only finitely isoclinism types.

(2) As $\omega(G) = n$, by Pyber [11], $G$ is a center-by-finite group. On the other hand, according to the main Theorem of P. Hall [9], p. 135), there exists a group $K$ such that $G$ is isoclinic to $K$ and $Z(K) \leq [K, K] = K'$. It follows, as $G$ is isoclinic to $K$, that $K$ is center-by-finite and so, according to Schur’s Theorem, $K'$ is finite. Therefore $Z(K)$ and $K/Z(K)$ are finite, so $K$ is finite and so Lemma 2.1 completes the proof.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Assume that $G$ is a group with $\omega(G) \leq 20$. Then according to Theorem 1.1, there exists a finite group $K$ such that $G$ is isoclinic to $K$ and $\omega(G) = \omega(K)$. Thus replacing $G$ by the factor group $G/Z(G)$, it can be assumed without loss of generality that $G$ is a finite group with $\omega(G) \leq 20$. But in this case the result follows from the main result of [8] (note that the alternating group of degree 5, $A_5$ is a group with $\omega(A_5) = 21$ and so the estimate is sharp).

3. Groups with a finite number of centralizers

It is now appropriate to consider groups with a finite number $n$ of centralizers ($C_n$-groups), since there exist the interesting relations between centralizers and pairwise non-commuting elements. For instance, as mentioned in the introduction, the finiteness of $\omega(G)$ in $G$ is equivalent to the finiteness of the factor group $G/Z(G)$. On the other hand, because of centralizers are subgroups containing the center of the group, the finiteness of the factor group $G/Z(G)$ follows that $G$ has finite number of centralizers. Also if $G$ has finite number of centralizers then it is easy to see that $\omega(G)$ is finite. Therefore we can summarize the latter results in the following theorem.

Theorem 3.1. For any group $G$, the following statements are equivalent.

(1) $G$ has finitely many centralizers.
(2) $G$ is a center-by-finite group.
(3) $G$ has finitely many of pairwise non-commuting elements.
It is clear that a group is a $C_1$-group if and only if it is abelian. The class of $C_n$-groups was introduced by Belcastro and Sherman in [5] and investigated by many authors, for instance see [2] [3] [12] [13] [15].

As every group $G$ with a finite number of centralizers is center-by-finite and so, by an argument similar to the one in the proof of Lemma 2.1 we will obtain the following result.

**Lemma 3.2.** For every two isoclinic groups $G$ and $H$ we have $|\mathcal{C}(G)| = |\mathcal{C}(H)|$.

**Proof.** Let $x$ be an element of $G$ and $\beta$ is the isomorphism $\beta : G/Z(G) \rightarrow H/Z(H)$. Therefore there exists a subgroup $K$ of $H$ such that $\beta(C_G(x)/Z(G)) = K/H$. By an argument similar to the one in the proof of Lemma 2.1 we show that there exist an element $y \in K$ such that $K = C_H(y)$ and $yZ(H) = \beta(xZ(G))$. Now as the isomorphism $\beta$ induces a bijection between the subgroups of $G$ containing $Z(G)$ and the subgroups of $H$ containing $Z(H)$ the result follows. $\Box$

Again, by an argument similar to the one in the proof of Theorems 1.1 we obtain the following result.

**Theorem 3.3.** Let $n$ be a positive integer and $G$ be an arbitrary $C_n$-group. Then

1. There are only finitely many groups $H$, up to isoclinic, with $|\mathcal{C}(H)| = n$;
2. There exists a finite group $K$ such that $K$ is isoclinic to $G$ and $|\mathcal{C}(G)| = |\mathcal{C}(K)|$.

For any group $G$, it is easy to see that if $x, y \in G$ and $xy \neq yx$, then $C_G(x) \neq C_G(y)$, from which it follows easily that $1 + \omega(G) \leq |\mathcal{C}(G)|$ (note that $C_G(e) = G$, where $e$ is the trivial element of $G$). Thus, by using Theorem 2.1, we generalize Theorem A of [14].

**Theorem 3.4.** Every arbitrary group $G$ with $|\mathcal{C}(G)| \leq 20$ is solvable and this estimate is sharp.

Finally, by using Theorem 3.3 (Case (2)), we generalize the main results of [2] [3] [4] [5] for infinite groups, as follows:

**Theorem 3.5.** Let $G$ be an arbitrary $C_n$-group. Then

1. $G/Z(G) \cong C_2 \times C_2$ if and only if $n = 4$.
2. $G/Z(G) \cong C_3 \times C_3$ or $S_3$ if and only if $n = 5$.
3. $G/Z(G) \cong D_8$, $A_4$, $C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$ whenever $n = 6$.
4. $G/Z(G) \cong C_5 \times C_5$, $D_{10}$ or $\langle x, y | x^5 = y^4 = 1, xy = x^3 \rangle$ if and only if $n = 7$.
5. $G/Z(G) \cong C_2 \times C_2 \times C_2$, $A_4$ or $D_{12}$ whenever $n = 8$.

**Proof.** For prove it is enough to note that there exists a finite $C_n$-group $K$ such that $K$ is isoclinic to $G$ so $G/Z(G) \cong K/Z(K)$ and so the result follows from the main results in [2] [3] [4] [5]. $\Box$

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