Mellin transform techniques for zeta-function resummations

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Abstract

Making use of inverse Mellin transform techniques for analytical continuation, an elegant proof and an extension of the zeta function regularization theorem is obtained. No series commutations are involved in the procedure; nevertheless the result is naturally split into the same three contributions of very different nature, i.e. the series of Riemann zeta functions and the power and negative exponentially behaved functions, respectively, well known from the original proof. The new theorem deals equally well with elliptic differential operators whose spectrum is not explicitly known. Rigorous results on the asymptoticity of the outcoming series are given, together with some specific examples. Exact analytical formulas, simplifying approximations and numerical estimates for the last of the three contributions (the most difficult to handle) are obtained. As an application of the method, the summation of the series which appear in the analytic computation (for different ranges of temperature) of the partition function of the string —basic in order to ascertain if QCD is some limit of a string theory— is performed.

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1 Introduction

Zeta-function resummation formulas are essential in the zeta-function regularization procedure [1]. In particular, they constitute the key ingredient for the proof (done in several steps by Weldon, Actor, Elizalde and Romeo [2, 3]) of the zeta function regularization theorem. The importance of the final outcome of the theorem —and also, of its extension to multiple, generalized Epstein-Hurwitz series with arbitrary exponents [3]— has been stressed by several authors [4]. Specifically, it is necessary for actual, practical applications to estimate or put a bound to the error one introduces by neglecting the non-polynomial terms which arise in the series commutation process. These terms have proven to be very difficult to handle and, due to the fact that they are (very) small in some cases, usually they have just been dropped off from the final formulas. To put an end to such an unpleasant situation is one of the purposes of this paper.

Another motivation is to present a completely different method for the derivation of these additional contributions, which comes about from a very elegant approach that has its roots in an elaborate admixture of the Mellin transform technique and the heat kernel method. Our new procedure yields a very convenient, closed expression for the zeta function corresponding to very general elliptic operators in terms of complex integrals over movable vertical lines in the complex plane. The final result is the same as the one painstakingly obtained via the original method of the series commutation —with the great advantage that the proof of asymptoticity of the resulting series turns to be now immediate and also, that the calculation of bounds to the additional term is considerably eased. Moreover, cast in this form the theorem admits a quite natural generalization to the case where the spectrum of the operator is not known.

In the next section of the paper, we describe the Mellin transform method as applied to generalized heat kernel operators. The method is extended in section 3 to the case of trace formulas of generalized zeta-function type. In section 4 some mathematical theorems on asymptoticity of series, basic for practical applications of the procedure in rather general situations are presented. In section 5 we obtain some initial (but already non-trivial) results on the additional term by making convenient use of the Poisson summation formula. Specific numerical evaluations and an analytical bound on the elusive additional term which appears as a byproduct of, e.g., the series commutation procedure,
are obtained in section 6, together with some convenient integral expressions for this term. As a particular example of the usefulness of the whole procedure, in section 7 we obtain exact expressions for the analytical continuation of highly non-trivial, zeta-type series which appear in the calculation of the partition functions of strings and membranes. This issue happens to be essential in deciding if QCD can (or cannot) in fact be described as a (super)string (or membrane) theory. Finally, section 8 is devoted to conclusions.

2 Inverse Mellin transform method for generalized heat kernel operators

We shall here rederive the (by now well known) zeta-function resummation formulas —using the Mellin transform techniques that have been employed for obtaining high-temperature expansions, both in field theory as well as in string theory [5]— in a very straightforward way.

Let us consider a non-negative elliptic operator $A$ of order $d$, defined on a compact $n$-dimensional manifold. Let $\{\lambda_i, \phi_i\}$ be the spectral resolution of $A$. (The explicit knowledge of the spectral resolution of the operator will not be basic to our method, and we shall make only a very limited, circumstantial use of it.) In what follows we shall be interested in the quantity

$$F_t(s) = A^{-s}e^{-tA},$$

which can be interpreted as a generalized heat kernel operator. It satisfies the functional differential equation

$$\partial_s F_t(s) + F_t(s - 1) = 0$$

and, from the operator identity

$$e^{-tA} = \frac{1}{2\pi i} \int_{\text{Re } z = c} dz \Gamma(z) (tA)^{-z},$$

one obtains the complex integral representation

$$F_t(s) = \frac{1}{2\pi i} \int_{\text{Re } z = c} dz \Gamma(z) t^{-z} A^{-s-z}.$$
Since $A$ is assumed to be elliptic, the diagonal part of its heat kernel as well as the kernel of the related zeta function exist. A theorem of Seeley yields

$$\zeta_A(s+z)(x) \simeq \frac{1}{\Gamma(s+z)} \left[ \sum_r \frac{K_r(x)}{s+z+r-n/d} + \hat{J}(s+z, x) \right]$$

(5)

(where $\simeq$ is the symbol for asymptotical equivalence), being $K_r(x)$ the heat kernel expansion coefficients and $\hat{J}$ the analytical part of $\zeta_A$, which is not known explicitly, in general. As a result of all the preceding expressions, one arrives at

$$F_t(s,x) \simeq \frac{1}{2\pi i} \int \text{Re } z = c \ dz \ t^{-z} \frac{\Gamma(z)}{\Gamma(s+z)} \left[ \sum_r \frac{K_r(x)}{s+z+r-n/d} + \hat{J}(s+z, x) \right].$$

(6)

The corresponding trace formula can be obtained by integrating over $x$, with the result

$$\text{Tr } F_t(s) \simeq \sum_i \lambda_i e^{-t\lambda_i} = \frac{1}{2\pi i} \int \text{Re } z = c \ dz \ t^{-z} \frac{\Gamma(z)}{\Gamma(s+z)} \left[ \sum_r \frac{K_r(x)}{s+z+r-n/d} + \hat{J}(s+z) \right].$$

(7)

In the rest of the section we will employ this form of complex integral representation, with the understanding that all the conclusion we shall draw using this method can be also validated for the corresponding non-integrated quantities. In order to obtain the so-called resummation formulas (see Actor, Weldon, Elizalde and Romeo [2, 3]) one needs to perform a very accurate application of the residues theorem. For a generic $s$, we encounter simple poles at $z = -k$, $k = 0,1,2,\ldots$, and at $z = \frac{n-r}{d} - s$ in the expression above. Furthermore, in general there will be a non-zero contribution coming from the integral evaluated along the semicircumference at infinity (see Elizalde and Romeo, refs. [2, 3]).

As a result, we can decompose $F_t(s)$ into three parts, very different in nature,

$$F_t(s) \simeq F_t^{(1)}(s) + F_t^{(2)}(s) + F_t^{(3)}(s),$$

(8)

which turn out to be, in our language,

$$F_t^{(1)}(s) = \sum_k \frac{(-t)^k}{k!} \zeta_A(s-k),$$

(9)

$$F_t^{(2)}(s) = \sum_r \frac{\Gamma(n-r)}{\Gamma(n-r/d)} t^{s+r-n/d} K_r$$

(10)

and

$$F_t^{(3)}(s) = \frac{1}{2\pi i} \int_C dz \ \Gamma(z) \ t^{-z} \zeta_A(s+z),$$

(11)
being a convenient contour on the complex plane. Some remarks are in order. First, these results are in complete agreement with the ones originally obtained by one of the authors (see also the recent paper by Actor [6]) for specific situations. However, in our new approach, we explicitly see how the heat kernel coefficients $K_r$ enter the game, in the general case considered from the beginning (eq. (1)). These are, in principle, computable quantities (for a recent reference see Branson and Gilkey [7]), in terms of which the characteristics of an elliptic operator are always given. The analytic part of the final result —which is generally unknown— enters into the first and third contributions. In some cases (see, for instance, Elizalde and Romeo [2]), it can be explicitly evaluated. In particular, it is interesting to consider also the case of hyperbolic compact manifolds, where the analytic part can be related to a zeta function of Selberg type (Bytsenko et al. [5]), even though yet in this case the spectrum is usually not known. Also, we should notice that a similar result has been obtained by Avramidi in a particular case [8].

A very important remark concerns the range of the sums in eqs. (9)-(11). In previous works [2, 3, 4] this range was formally taken to be infinite: the sums were series and the integration contour $C$ in $F^{(3)}$ was at infinite distance from the origin. Then these expressions can only have a formal character: on the one hand the series converge only in very specific cases and, on the other, the contribution of the contour is usually infinite. This will not be any more our philosophy in the present paper. In the following sections we shall prove that (under very general conditions) the series in (9)-(11) are asymptotic; thus, we will always consider them as cut after the most favourable term (yielding always finite sums), and the contour $C$ will be a vertical line at a constant finite abscissa $c$ (i.e., $\text{Re } z = c$), together with the two corresponding horizontal segments at infinity. This is the central part of the full method in this paper, and must be kept always in mind, from now on, when considering expressions like (9)-(11). Rigorous specifications on this point will be given in the next sections.
3 Extension to trace formulas of generalized zeta-function type

The above technique can be extended to trace formulas of the kind

\[ F(s,t) = \sum_i \lambda_i^{-s} f(t\lambda_i), \quad (12) \]

provided \( f(y) \) admits an invertible Mellin transform, namely

\[ f(y) = \frac{1}{2\pi i} \int_{\text{Re } z = c} dz \ y^{-z} M[f](z), \quad (13) \]

with

\[ M[f](z) = \int_0^\infty dy \ y^{z-1} f(y). \quad (14) \]

Being more precise, it is assumed that the function \( f \) is integrable over the open interval \((t_1, t_2)\) provided that \(0 < t_1 < t_2\). Defining

\[ \alpha \equiv \inf \{ \alpha^* \in \mathbb{R} \mid f(0^+) = O(t^{-\alpha^*}) \}, \quad \beta \equiv \sup \{ \beta^* \in \mathbb{R} \mid f(+\infty) = O(t^{-\beta^*}) \}, \quad (15) \]

we shall also consider that \( \alpha < \beta \). This is no restriction since if, on the contrary, \(-\infty < \beta \leq \alpha < \infty\), one just needs to define (for \( R > 0 \)) \( f_1(t) \equiv \theta(R-t)f(t) \), \( f_2(t) \equiv \theta(t-R)f(t) \) and \( F_j(s,t) = \sum_i \lambda_i^{-s} f_j(t\lambda_i), \ j = 1, 2 \), and then consider each \( F_j \) separately (with \( \alpha_1 = \alpha, \ \beta_1 = +\infty, \ \alpha_2 = -\infty, \ \beta_2 = +\beta \)).

In general, there will exist a value \( s_0 \in \mathbb{R} \) such that for \( \text{Re } s > s_0 \) the series \( \sum_i \lambda_i^{-s} \) will be absolutely convergent and for \( \text{Re } s < s_0 \) it will be not. Then, a sufficient condition in order that (12) makes sense is \( \text{Re } s + \beta > s_0 \). In this case,

\[ F(s,t) = \sum_i \lambda_i^{-s} \frac{1}{2\pi i} \int_{\text{Re } z = c} dz \ (\lambda_i t)^{-z} M[f](z), \quad (16) \]

with \( c \in (\alpha, \beta) \) and \( \text{Re } (s) + c > s_0 \). We take \( M[f](c+iy) \in L^1(-\infty < y < \infty) \), so that, finally, making use of all these equations, we can write

\[ F(s,t) = \frac{1}{2\pi i} \int_{\text{Re } z = c} dz \ t^{-z} M[f](z) \zeta_A(s+z). \quad (17) \]

Now, we must suppose that one can solve the problem of the analytic continuation of the functions \( \zeta_A(z) \) and \( M[f](z) \) to the left in \( \text{Re } z \). Under very general conditions, the
function $M[f]$ can be extended as a meromorphic function, but for a general $\zeta_A$ this is not so easy to do in practice. In general one must calculate all the Seeley-De Witt coefficients of $A$ and use them to infer the meromorphic behavior of $\zeta_A$ (in the second example below we show an alternative way to extend $\zeta_A$ using our knowledge of the $\lambda_i$ only). We also need to know the behavior of $\zeta_A(z)$ and $M[f](z)$ for $|\text{Im} z| \to \infty$.

As before, for a generic $s$, we can split

$$F(s, t) = F^{(1)}(s, t) + F^{(2)}(s, t) + F^{(3)}(s, t),$$

where now

$$F^{(1)}(s, t) = \sum_k \text{Res} M[f](z_k) t^{-z_k} \zeta_A(s + z_k),$$

$$F^{(2)}(s, t) = \sum_r M[f]\left(\frac{n-r}{d} - s\right) \frac{\Gamma\left(\frac{n-r}{d}\right)}{r^{s+r-n}} K_r$$

and

$$F^{(3)}(s, t) = \frac{1}{2\pi i} \int_C dz M[f](z) t^{-z} \zeta_A(s + z),$$

being $z_k$ all the (simple) poles of $M[f](z)$. If double poles occur, logarithmic terms in $t$ show up. This may happen for particular values of $s$, once the other free parameters are held fixed.

Let us now consider some illustrative examples.

**Example 1.** For the Riemann zeta function itself, we have

$$\zeta(s) = O\left(|t|^{p(r)} \ln |t| \right), \quad s = r + it,$$

with

$$p(r) = \begin{cases} 
\frac{1}{2} - r, & -K \leq r \leq -\delta, \\
\frac{1}{2} - r, & -\delta \leq r \leq 0, \\
\frac{1}{2} - r, & 0 \leq r \leq \delta, \\
1 - r, & \delta \leq r \leq 1 - \delta, \\
1, & 1 - \delta \leq r \leq 1, \\
0, & 1 \leq r \leq 1 + \delta, \\
0, & r \geq 1 + \delta,
\end{cases}$$

where $K > 0$ and $\delta > 0$ (a standard choice is $\delta < 1$ and $K$ arbitrarily big). Actually, $\ln |t|$ can be supressed in the first, fourth and seventh intervals. It is remarkable that eq. (22) is valid uniformly in each of the intervals of (23).
**Example 2.** Take $\lambda_n = n$ and add a degeneration $g(n)$ (being $g$ an analytical function with good behavior in order to fulfill what follows), i.e.

$$\zeta_g(z) \equiv \sum_{n=1}^{\infty} \frac{g(n)}{n^z} = -\frac{i}{2\pi} \int_C \frac{\ln[\sin(\pi t)]}{t-z} g(t) \, dt,$$

being $C$ the contour of the sector of the complex plane defined by $|\arg z| \leq \theta_0$ and $|z| \geq \epsilon$, $0 < \epsilon < 1$. Decomposing $C$ into its upper and lower parts, $C^\pm$, and assuming $g$ to be analytical on the sector, we obtain

$$\zeta_g(z) = -\int_{\epsilon}^{\infty} d\rho \rho \frac{d}{d\rho} (\rho^{-z} g) - \frac{1}{2} \epsilon^{-z} g(\epsilon) - \frac{i}{2\pi} \sum_{\pm} \int_{C^\pm} dt \ln \left(1 - e^{\pm 2\pi it}\right) \frac{d}{dt} (t^{-z} g). \quad (25)$$

In general, the second and third terms on the r.h.s. are integer functions of $z$, and only the first one needs to be continued to the left in $\text{Re} \, z$. This can be done by applying the Mellin transform techniques.

## 4 A more explicit study of the additional term

Of the three terms which appear in each of the splittings (8) and (18), the third one is the most difficult to tackle. In fact, it was completely overlooked in the first formulations of the zeta function regularization theorem, what led to several erroneous results. In the procedure above it has shown up in a nice and elegant way. However, its apparently very simple form is somewhat deceiving. In fact, in its primitive form [2, 3], it required an evaluation of the integrand — which involves the zeta function of the elliptic operator — all the way along a contour of infinite radius. We shall here show explicitly how to obtain an alternative asymptotic expansion of expressions (8) and (18) in which the residual term consists of a complex integration on a vertical line of constant, finite abscissa. We start by proving the following theorem.

**Theorem 1.** Let $\theta_0 > 0$ and $h$ be an analytic function in the domain defined by $z \neq 0$ and $|\arg z| \leq \theta_0$, and such that

$$h(z) \sim \sum_{n=0}^{\infty} a_n z^{\beta_n}, \quad |z| \to 0,$$

and (only to simplify the discussion, by putting $\beta = \infty$)

$$h(z) \ll O \left(z^{-R}\right), \quad \forall R \in \mathbb{R}, \quad |z| \to \infty,$$
where we understand that $\text{Re} \beta_n \leq \text{Re} \beta_{n+1}$ and $\text{Re} \beta_n \to \infty$. Define

$$Y_r(s, \tau) \equiv \sum_{n=1}^{\infty} \frac{h(n^r \tau)}{n^s},$$

with $r, \tau > 0$ and $s \in \mathbb{C}$. Then, $Y_r(s, \tau)$ is an integer function with respect to the argument $s$ and, denoting as before the Mellin transform of a function $f$ by $M[f]$, one has

$$Y_r(s, \tau) \sim \tau^{-(1-s)/r} \left[ \frac{s}{r} M[h] \left( \frac{1-s}{r} \right) - M[h]' \left( \frac{1-s}{r} + 1 \right) \right] - \frac{1}{2} \tau^{s/r} \left[ \frac{s}{r} M[h] \left( -\frac{s}{r} \right) \right] - M[h]' \left( 1 - \frac{s}{r} \right) + \sum_{n=0}^{\infty} a_n \zeta(s - \beta_n r) \tau^{\beta_n}. \quad (28)$$

**Proof.** Write

$$Y_r(s, \tau) = -\frac{1}{2\pi i} \int_C dz \pi \cot(\pi z) z^{-s} h(z^r \tau) = \frac{1}{2\pi i} \int_C dz \ln(\pi z) \frac{d}{dz} \left[ z^{-s} h(z^r \tau) \right], \quad (30)$$

$C$ being the circuit made up of the two straight lines at angles $\pm \theta_0$. From the first hypothesis it turns out that the two integrals in which (30) splits exist for any $z \neq 0$ with $|\arg z| \leq \theta_0 - \delta$, $\forall \delta > 0$. Thus we deform the circuit $C$ to $C_1$ corresponding to the lines at angles $\pm(\theta_0 - \delta)$ and with a small circular arc of radius $\epsilon$ at the origin.

Notice now the asymptotic behavior of a function of the following type

$$\int_{\epsilon}^{\infty} d\rho \rho^{-s} h(\rho^r \tau) \sim \tau^{-(1-s)/r} \left[ \frac{s}{r} M[h] \left( \frac{1-s}{r} \right) - M[h]' \left( \frac{1-s}{r} + 1 \right) \right] - \frac{1}{2} \tau^{s/r} \left[ \frac{s}{r} M[h] \left( -\frac{s}{r} \right) \right] - M[h]' \left( 1 - \frac{s}{r} \right) + \sum_{n=0}^{\infty} a_n \zeta(s - \beta_n r) \tau^{\beta_n}, \quad (31)$$

provided there are no pole coincidences. Then, defining

$$Y_1(s, \tau) = \frac{is}{2\pi} \int_C dz \ln(\pi z) z^{-s-1} h(z^r \tau) \equiv Y_1^+ + Y_1^-,$$  

(32)

corresponding naturally to the two branches of the circuit, $C_\pm$, the analysis of each of these integrals is quite simple. The second part can in fact be treated as the first, by noticing that

$$Y_2(s, \tau) = \frac{r \tau}{2\pi i} \int_C dz \ln(\pi z) z^{r-s-1} h'(z^r \tau) = \frac{-r \tau}{s-r} Y_1^{[h \rightarrow h']}_1(s-r, \tau). \quad (33)$$

Collecting everything together, one arrives at the desired expression (28), valid in principle for any positive integer $n$ such that $s \neq 1 + \beta_n r$ and $s \neq \beta_n r$. These cases must be considered specially. Either one starts over again or, what is better, one proceeds
According to the following hint: to perform a shift \( s = 1 + \beta_n r + \delta \) or, correspondingly, \( s = \beta_n r + \delta \), and then make use of the well known lemma

**Lemma 1.** Let \( D \) an open domain of \( C \), \( s_0 \in D \), \( D^* = D - \{ s_0 \} \), and \( \beta_n \) a sequence in \( C \) such that \( \text{Re} \beta_n \leq \text{Re} \beta_{n+1} \) and \( \text{Re} \beta_n \to \infty \). Let \( Y(s, \tau) \) analytic as a function of the first argument for \( s \in D \) and \( \tau \in (0, \alpha) \), \( \alpha > 0 \). Suppose that for \( \forall s \in D^* \) one has \( Y(s, \tau) \sim \sum_{n=0}^{\infty} a_n(s) \tau^{\beta_n} \), where the \( a_n(s) \) are analytic on \( D \), and that this expansion is valid uniformly for \( s \in C \), \( C \) being a path contained in \( D \) and enclosing \( s_0 \).

Then it turns out that also \( Y(s_0, \tau) \sim \sum_{n=0}^{\infty} a_n(s_0) \tau^{\beta_n} \). The proof is very simple.

In order to determine the behavior of \( M[f](x + iy) \) for \( |y| \to \infty \) the following theorems are useful

**Theorem 2.** Let \( f \in C^\alpha(0, \infty) \) and suppose there exists \( x_0 \in \mathbb{R} \) such that, for \( x > x_0 \), \( \lim_{t \to \infty} (td/dt)^p(t^xf) = 0 \), for \( p = 0, 1, 2, \ldots, n \), and that \( t^{-1}f_p \) is absolutely integrable, where \( f_p(t, x) \equiv (td/dt)^p(t^xf) \). Then \( M[f](z) = O(|y|^{-n}) \) for \( y \to \pm \infty \).

**Proof.** Follows immediately by wishful application of Riemann-Lebesgue’s lemma.

**Theorem 3.** Let \( f \in C^\alpha(0, \infty) \) with the asymptotic behavior for \( t \to 0^+ \): \( f(t) \sim \sum_{m=0}^{\infty} p_m t^{a_m} \), with \( a_m \) monotonically increasing towards infinity. Suppose also that asymptotic expansions for \( t \to 0^+ \) for the successive derivatives of \( f \) are obtained by taking the derivative, term by term, of the above expansion for \( f \). Suppose also that \( \lim_{t \to \infty} f_p(t, x) = 0 \), for \( p = 0, 1, 2, \ldots, n \), and that \( t^{-1}f_p \) is absolutely integrable for \( x > - \text{Re} a_0 \). Then it follows that \( M[f](z) = O(|y|^{-n}) \) for \( y \to \pm \infty \) and for any \( x \), that is, in the whole domain of the function, after its continuation.

**Proof.** Let \( \rho \in \mathbb{R} \) be large enough and \( \mu \) such that \( \text{Re} a_{\mu-1} < \rho \leq \text{Re} a_\mu \), where for a positive integer \( \delta \) satisfying \( \text{Re} a_0 + \delta > \text{Re} a_\mu \), we define \( \sigma_\rho(t) \equiv \exp(-t^\delta) \sum_{m=0}^{\mu-1} p_m t^{a_m} \), and \( \hat{f} \equiv f - \sigma_\rho \). Applying the Theorem 2 to \( \hat{f} \) and noticing that \( M[\sigma](z) \sim \sum_{m=0}^{\mu-1} p_m / \delta \Gamma((z + a_m)/\delta) \) has exponential behavior for \( |y| \to \infty \), from \( M[f] = M[\hat{f}] + M[\sigma] \) (which gives the continuation to \( -\text{Re} a_\mu < \text{Re} z < \beta \)) and the fact that \( \rho \) is arbitrarily big we prove the theorem.

For \( x_0, \theta_0 \in \mathbb{R} \), let us define that a function \( h \) belongs to \( K \), \( h \in K(x_0, \theta_0) \) iff for any \( x > x_0 \) and \( \epsilon > 0 \) we have \( M[h](x + iy) = O[\exp(-(|\theta_0 - \epsilon||y|))] \), for \( |y| \to \infty \). Also, denote the sector \( S(\theta_0) = \{ t \neq 0 \mid \arg t < \theta_0 \} \).
Theorem 4. Suppose that in the sector $S(\theta_0)$: (i) $h$ is analytic; (ii) $h = O(t^\alpha)$, $t \to 0^+$; and (iii) $h = \exp(-at^\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} (\ln t)^n t^{-r_m}$, $t \to \infty$, with $\Re a \geq 0$, $\nu > 0$ and $\Re r_m$ monotonically increasing towards infinity. Then: (i) if $a = 0$, $h \in K(-\Re \alpha, \theta_0)$; (ii) if $\Re a > 0$, $h \in K(-\Re \alpha, \theta)$, where $\theta = \min(\theta_0, (\pi - 2|\arg a|)/(2\nu))$.

Proof. We only do the first case (the second is analogous). Observe that $M[h](z)$ is analytical for $\Re z > -\Re \alpha$. Define $\theta' = \theta - \epsilon$, $\theta$ as given in the theorem and $\epsilon > 0$; deforming the circuit by an angle $\pm \theta'$ and making the change of variable $t = re^{\pm i\theta'}$ ($r$, real, will be the new variable of integration) in the Mellin transform integral, one easily shows that $M[h](z) = O[\exp(-\theta'|y|)]$, for $|y| \to \infty$, $x > \Re -\alpha$.

With a similar strategy as that used in Theorem 3, with convenient choices of the function $\sigma$, one can prove results analogous to Theorem 4 (so called theorems of exponential decrease) valid for a range of $x$ which moves towards the left. The only proviso is that $h$ have a convenient asymptotic expansion around 0.

Also to be noticed is the fact that in order to apply these theorems, with the aim of displacing the contour of integration in (17), one must make sure that the conclusions of Theorems 2-4 are valid uniformly on any segment of the $x$ variable along which we want to displace the integration contour. If we can perform the translation from $x = c$ to $x = c'$, $c' < c$, and we check that $\zeta_A(s + c' + iy)M[f](c' + iy)$ is absolutely integrable, then

$$F(s, t) = \sum_{c' < \Re z < c} \text{Res} \left[ \zeta_A(s + z) t^{-z} M[f](z) \right] + \frac{t^{-c'}}{2\pi i} \int_{\Re z = c'} \text{dz} \zeta_A(s + z) t^{-iy} M[f](z).$$

To finish this section, let us consider a different example which cannot be resolved by direct application of Theorem 4 above (since it corresponds to $a \neq 0$ and $\Re a = 0$).

Example 3. Let

$$F(s, t) = \sum_{n=1}^{\infty} \frac{J_{\mu}(nt)}{n^s},$$

that is, $\lambda_n = n$, $f = J_{\mu}$, $\alpha = -\mu$, $\beta = 1/2$. We have

$$M[J]_{\mu}(z) = \frac{2^{z-1}\Gamma\left(\frac{z+\mu}{2}\right)}{\Gamma\left(\frac{\mu-z+2}{2}\right)},$$

with $s_0 = 1$, $\Re s > 1/2$ (in this case the domain of $s$ can be enlarged), and $c > 1$– $\Re s < 1/2$. $M[J]_{\mu}$ has poles for $z = -\mu + 2n$, $n \in \mathbb{N}$, with residues $(-1)^n 2^{\mu-2n}/[n!(\mu+n)!]$. 

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while $\zeta(s + z)$ has a pole at $z = -s + 1$. From (36) and from the behavior of the gamma function we know that in any closed interval of $x = \Re z$ we get the behavior

$$M[J]_\mu(z) = O(|y|^{1/2 - \epsilon_2}), \quad |y| \to \infty,$$

(37)

uniformly in $x$. We have $\Re s = 1/2 + \epsilon_1$, $\epsilon_1 > 0$, and choose $c = 1/2 - \epsilon_2$, with $0 < \epsilon_2 < \epsilon_1$, and any $\delta$, $0 < \delta < \min \{1/2, \epsilon_1 - \epsilon_2\}$.

Consider now the following, segmentwise, displacement

$$G(z) = M[J]_\mu(z) \zeta(s + z).$$

Then it turns out that

1. for $1/2 + \delta - \epsilon_1 \leq x \leq 1/2 - \epsilon_2$, $|M[J]_\mu(z)| = O(|y|^{-1/2 - \epsilon_2})$ and $|\zeta(s + z)| = O(1)$, therefore

$$|G(z)| = O(|y|^{-1/2 - \epsilon_2});$$

(39)

2. for $1/2 - \epsilon_1 \leq x \leq 1/2 + \delta - \epsilon_1$, $|M[J]_\mu(z)| = O(|y|^{-1/2 - \epsilon_1})$ and $|\zeta(s + z)| = O(\ln |y|)$, therefore

$$|G(z)| = O(|y|^{-1/2 - \epsilon_1} \ln |y|);$$

(40)

3. for $-1/2 + \delta - \epsilon_1 \leq x \leq 1/2 - \epsilon_1$, $|M[J]_\mu(z)| = O(|y|^{-3/2 + \delta - \epsilon_1})$ and $|\zeta(s + z)| = O(|y|^{1 - \delta} \ln |y|)$, therefore

$$|G(z)| = O(|y|^{-1/2 - \epsilon_1} \ln |y|);$$

(41)

4. for $-1/2 - \epsilon_1 \leq x \leq -1/2 + \delta - \epsilon_1$, $|M[J]_\mu(z)| = O(|y|^{-3/2 + \delta - \epsilon_1})$ and $|\zeta(s + z)| = O(|y|^{1/2})$, therefore

$$|G(z)| = O(|y|^{-1 + \delta - \epsilon_1});$$

(42)

5. finally, for $-K - 1/2 - \epsilon_1 \leq x \leq -1/2 - \epsilon_1$, $|M[J]_\mu(z)| = O(|y|^{-x - 1})$ and $|\zeta(s + z)| = O(|y|^{-x - \epsilon_1})$, therefore

$$|G(z)| = O(|y|^{-1 - \epsilon_1}),$$

(43)

where $K$ is arbitrarily large.
5 The additional term and the Poisson resummation formula

We would like first to examine a particular but important case, as an illustration of the difficulties associated with the determination of the additional term. Such case will be considered also in the next section but under a different point of view.

Let us consider the selfadjoint operator $A = H^\beta$, where $H$ is the Dirichlet Laplacian on $[0, 1]$ with eigenvalues $\lambda_n = n^2$. Putting $2\beta = \alpha$, we can write

$$\zeta_A(z) = \sum_{n=1}^{\infty} n^{-\alpha z} = \zeta(\alpha z).$$  (44)

The related heat kernel trace reads ($t > 0$)

$$\text{Tr} e^{-tA} = \sum_n e^{-n^\alpha t} \equiv S_\alpha(t).$$  (45)

Making use of the Mellin representation discussed in Sec. 2, we get

$$S_\alpha(t) = \frac{1}{2\pi i} \int_C dz t^{-z} \zeta(\alpha z) \Gamma(z) = \frac{\Gamma(\frac{1}{\alpha})}{\alpha t^{\frac{1}{\alpha}}} + \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \zeta(-\alpha k) + \Delta_\alpha(t)$$

$$= \frac{\Gamma(\frac{1}{\alpha})}{\alpha t^{\frac{1}{\alpha}}} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \zeta(-\alpha k) + \Delta_\alpha(t),$$  (46)

where the power series in $t$ will converge for $|t| < b$ (for some $b$, abscissa of convergence).

Then we can analytically continue to the remaining values of $t$. The quantity $S_\alpha(t)$ may be evaluated by making use of the Poisson formula, which states that for $f(x) = f(-x)$ and $f(x) \in L_1$, the following equation holds:

$$\sum_{n=1}^{\infty} f(n) = -\frac{1}{2} f(0) + \int_{0}^{\infty} dx f(x) + 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} dx f(x) \cos 2\pi nx.$$  (47)

Let us consider the function

$$f(x) = e^{-t|x|^\alpha}.$$  (48)

An elementary computation permits to write

$$S_\alpha(t) = -\frac{1}{2} + \frac{\Gamma(\frac{1}{\alpha})}{\alpha t^{\frac{1}{\alpha}}} + 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} dx e^{-|x|^\alpha t} \cos 2\pi nx$$  (49)

and by comparing eqs. (46) and (49), we get

$$\Delta_\alpha(t) = 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} dx e^{-|x|^\alpha t} \cos 2\pi nx - \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \zeta(-\alpha k).$$  (50)
The above expression can be checked immediately. In fact, for $\alpha = 2$, i.e. $\beta = 1$, one has $\zeta(-2k) = 0$ and the additional term is nonvanishing, i.e.

$$\Delta_2(t) = 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} dx e^{-x^2 t} \cos 2\pi nx = \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{t}}.$$

(51)

As a consequence, eq. (46) becomes a well known Jacobi’s theta function identity.

Furthermore, if $\alpha < 2$, it turns out that $\Delta_{\alpha}(t)$ can be made very small, and we can write

$$2 \sum_{n=1}^{\infty} \int_{0}^{\infty} dx e^{-|x|^\alpha t} \cos 2\pi nx \simeq \sum_{k} \frac{(-t)^k}{k!} \zeta(-\alpha k),$$

(52)

which actually looks quite non trivial. This sum must be cut after its minimal term. Again, for $\alpha = 1$, one can perform exactly the elementary integrations involved, and the result is

$$\sum_{n=1}^{\infty} \frac{2t}{t^2 + 4\pi^2 n^2} = -\sum_{r=1}^{\infty} \frac{t^{2r}}{(2r-1)!} \zeta(1-2r).$$

(53)

The sum on the left hand-side can be done with elementary methods and we end up with

$$\frac{1}{2} \cot \left( \frac{t}{2} \right) - \frac{1}{t} = -\sum_{r=1}^{\infty} \frac{t^{2r}}{(2r-1)!} \zeta(1-2r).$$

(54)

which is a well known but certainly non trivial identity.

All this goes through for $\alpha = 2, 4, 6, \ldots$. A connection with the theory of Brownian processes may be established at that point [9]. The instabilities that are known to appear outside the range $0 < \alpha < 1$ and outside these particular values of $\alpha$ can be traced back in our procedure to the fact that both the full series in $\zeta$’s and the additional term on the contour at infinity diverge. Then, both are useless, in practice, and we must come back to our new method as stated before (end of sect. 2) and reflected in (52). This will be illustrated in the following sections in great detail.

6 Numerical estimates of the additional term

In general, the additional term, that is, the contribution of the semicircumference ‘at infinity’ —whose existence was discovered by one of the authors a few years ago [4]— has been very difficult to handle, even when computed at ‘finite distance’ $\text{Re } z = c$ or for finite radius $|z| = R$, either analytically or numerically (see the considerations by Actor
in ref. [6]). Being such an elusive term to any kind of treatment, let us (for the moment) consider it in its most simple form

\[ \Delta_\alpha \equiv \frac{1}{2\pi i} \int_K dz \, \Gamma(z) \, \zeta(\alpha z), \quad (55) \]

where \( K \) is the semicircumference of (finite) radius \( R \), \( z = Re^{i\theta} \), with \( \theta \) going from \( \theta = \pi/2 \) to \( \theta = 3\pi/2 \). To begin with, this expression is not defined at \( \theta = \pi \) and one must start the calculation by using the well known reflection formulas for \( \Gamma \) and \( \zeta \). That yields

\[ \Delta_\alpha = i \int_K \frac{dz}{z(2\pi)^{1-\alpha}} \left( \frac{\sin(\pi \alpha z/2)}{\sin(\pi z)} \right) \frac{\Gamma(1-\alpha z)}{\Gamma(-z)} \zeta(1-\alpha z). \quad (56) \]

Such quantity is, in principle, well defined but, on the other hand, rather bad behaved. In modulus, the integrand clearly diverges for \( |z| \to \infty \) but, due to the highly oscillating sinus factors the final result turns out to be finite. Actually, it has been proven rigorously in [3], that it has the very specific value

\[ \Delta_2 = -\sqrt{\pi} S(\pi^2), \quad S(t) \equiv \sum_{n=1}^{\infty} e^{-\pi^2 t}, \quad (57) \]

for \( \alpha = 2 \), and that it can be made \( < 1 \) for any value of \( \alpha > 2 \) and reasonably high values of \( R \) (the ones actually needed for practical applications). A possible way to handle expression (56) is to employ the integral equation

\[ \Gamma(z) \zeta(\beta z) = \int_0^\infty S_\beta(t) t^{z-1} dt, \quad \text{Re} \, z > 0, \quad S_\beta(t) \equiv \sum_{n=1}^{\infty} e^{-\pi^2 t}, \quad (58) \]

what yields

\[ \Delta_\alpha = -i\alpha \int_K dz \frac{\sin(\pi \alpha z/2)}{\sin(\pi z)} \frac{\Gamma(-\alpha z)}{\Gamma(-z) \Gamma(\frac{1}{\alpha} - z)} \int_0^\infty S_\alpha((2\pi)^\alpha t) t^{\frac{1}{\alpha} - z - 1} dt. \quad (59) \]

Using now Stirling’s formula and a simple approximation for the sinus fraction we get immediately

\[ \Delta_\alpha = -\frac{i}{\sqrt{2\pi} \alpha} \int_K dz \varphi_\alpha(z) \int_0^\infty S_\alpha((2\pi/\alpha)^\alpha t) t^{\frac{1}{\alpha} - z - 1} dt, \quad (60) \]

being

\[ \varphi_\alpha(z) \equiv \exp \left\{ \left[ (2-\alpha)z + \left( \frac{1}{2} - \frac{1}{\alpha} \right) \right] \ln z + (\alpha - 2)z + \left( \frac{\alpha}{2} - 1 \right) \pi |\text{Im} \, z| \right. \]
\[ + \left. i \, \text{sgn} \left( \text{Im} \, z \right) \left( \frac{\alpha}{2} - 1 \right) \pi |\text{Re} \, z| \right\}. \quad (61) \]
We see that, in fact, for $\alpha < 1$, $\varphi_\alpha(z) \to 0$, when $|z| \to \infty$, while for $\alpha = 2$ it is $\varphi_2(z) \equiv 1$ and $\Delta_2 = -\sqrt{\pi}S_2(\pi^2)$, as anticipated before (this is nothing but the famous Jacobi theta function identity). On the other hand, if we substitute in (60) its mean value, $\mu$, for the function $\varphi_\alpha$, we obtain

$$\Delta_\alpha = -\mu \sqrt{\frac{2\pi}{\alpha}} S_\alpha \left( \left( \frac{2\pi}{\alpha} \right)^\alpha \right).$$

An analytical but approximate evaluation carried out for the particular cases $\alpha = 2N$, with $N$ a positive integer (these values are most simple to deal with), suggests that the behavior of the mean value $\mu$ in terms of $\alpha$ can be bounded from above by an expression of the kind

$$|\mu| \leq \frac{e - 2}{2e} \sqrt{\frac{2\pi}{\alpha}}$$

for $\alpha$ large. That gives for the additional term

$$|\Delta_\alpha| \leq \frac{e - 2}{2e} S_\alpha \left( \left( \frac{2\pi}{\alpha} \right)^\alpha \right),$$

which is convergent for $\alpha \to \infty$.

We have checked this analytical bound with a numerical, direct evaluation of the integral (56) written in the following equivalent form, obtained by straightforward manipulations,

$$\Delta_\alpha = -\int_{-\pi/2}^{\pi/2} d\theta \frac{\sin(\pi z/2)}{(2\pi)^{z+1}} \frac{\Gamma(z + 1)}{\sin(\pi z/\alpha) \Gamma(z/\alpha)} \zeta(z + 1), \quad z = Re^{i\theta}, \quad R >> 1.$$  

In particular, the case $\alpha \to \infty$ can be easily handled

$$\Delta_\infty = -\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \frac{\sin(\pi z/2)}{2(2\pi)^{z+1}} \Gamma(z + 1) \zeta(z + 1), \quad z = Re^{i\theta}, \quad R >> 1.$$  

As discussed before, it is quite difficult to evaluate such integrals for $R$ big. Any numerical procedure breaks down for $R$ large enough. This makes the method developed in sects. 2 and 3 of this paper even more valuable. As explained in detail there, by using this technique we never need, in practice, to go to very large $R$'s in the evaluation of the additional term — at the expense of just calculating a few first terms of an asymptotic expansion on the zeta function of the differential operator one is dealing with (the asymptotic series must be optimally cut in the usual way). Put in the abstract (simple) situation which concerns us here, we need only obtain numerical values for the expressions (65) and (66) when $R$ is just of the order of 20 or 30 (what is already involved enough!), since the usual
asymptotic series that appear in practice yield their best result for a number of terms of this (or very often less) order.

There seems to be no definite regular behavior in the numerical values of the expressions above, though all of them, in the wide range $8 \leq R \leq 30$, lie in the quite narrow interval $0.02 \leq |\Delta| \leq 0.07$. This is in very good agreement with the results obtained from the analytical approximation (64), which are $\Delta_4 \simeq 0.04$, $\Delta_6 \simeq 0.07$ and $\Delta_\infty \simeq 0.13$. Only the tendency of this formula for large $\alpha$ deviates from the numerical result by a factor of 10 —namely the numerical results are about ten times lower than the upper bound values given by the formula (64). But this is not strange, since for big $\alpha$’s this expression (64) is rather bad ($S_\alpha$ is then a very slowly convergent series), while, on the contrary, the limiting expression for $\Delta_\infty$ (66) remains in very good shape.

Summing up, eqs. (65), (66) and (64) are the best we could obtain, from an analytical point of view, for the treatment of the additional term which appears both in the series commutation procedure and in the Mellin transform method. However, for the numerical treatment, a clever use of the methods of sect. 3 and 4 drastically reduces the problem, as we have just seen in the present section: the calculation must be performed not at $R = \infty$ but at reasonable values of $R$, and in this case (65) and (66) can be handled through standard numerical integration procedures.

7 Example: exact summation of the string partition function for different ranges of temperature

The problem of deciding if QCD is actually (some limit of) a theory of extended objects (strings, membranes or $p$-branes in general) is almost twenty years old and goes back to ’t Hooft. In a very recent contribution, Polchinski has tried to answer this question both for the Nambu-Goto and for the rigid strings by calculating their partition functions and seeing if they actually match with the one corresponding to QCD for different ranges of temperature [10]. This is no place to go into the details of the procedure [10], which will be given in a forthcoming paper and we shall here concentrate in the mathematical aspects of the problem only.
The first two terms in the loop expansion

\[ S_{\text{eff}} = S_0 + S_1 + \cdots \]  

(67)

of the effective action corresponding to the rigid string

\[ S = \frac{1}{2\alpha_0} \int d^2 \sigma \left[ \rho^{-1} \partial^2 X^\mu \partial^2 X_\mu + \lambda^{ab} (\partial_a X^\mu \partial_b X_\mu - \rho \delta_{ab}) \right] + \mu_0 \int d^2 \sigma \rho, \]  

(68)

being \( \alpha_0 \) the dimensionless, asymptotically free coupling, \( \rho \) the intrinsic metric, \( \mu_0 \) the explicit string tension (important at low energy) and \( \lambda^{ab} \) the Lagrange multipliers, are given, in the world sheet \( 0 \leq \sigma^1 \leq L \) and \( 0 \leq \sigma^2 \leq \beta t \) (an annulus of modulus \( t \)), by

\[ S_0 = \frac{L \beta t}{2\alpha_0} \left[ \lambda^{11} + \lambda^{22} t^{-2} + \rho (2\alpha_0 \mu_0 - \lambda^{aa}) \right] \]  

(69)

(tree level) and

\[ S_1 = -\frac{d-2}{2} \ln \det \left( \partial^4 - \rho \lambda^{ab} \partial_a \partial_b \right) \]

\[ = \frac{d-2}{2} L \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \ln \left( \left( k^2 + \frac{4\pi^2 n^2}{\beta^2 t^2} \right)^2 + \rho \left( \lambda^{11} k^2 + \frac{4\pi^2 n^2}{\beta^2 t^2} \lambda^{22} \right) \right) \]  

(70)

(one loop). This is a highly non-trivial calculation, which has been performed in ref. 10 only in the very strict limits \( T \to 0 \) and \( T \to \infty \) around some extremizing configuration, the parameters being \( \rho, \lambda^{11}, \lambda^{22} \) and \( t \). As a first step of the zeta function method we write

\[ S_1 = - (d-2) \frac{d}{ds} \left. \zeta_A(s/2) \right|_{s=0}, \quad \zeta_A(s/2) = L \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( k^2 + y_+^2 \right)^{-s/2} \left( k^2 + y_-^2 \right)^{-s/2}, \]  

(71)

where

\[ y_\pm = \frac{a}{t} \left[ n^2 + \rho t^2 \lambda^{11} \right] \]  

\[ - \sqrt{\rho} \frac{t}{a} \left( (\lambda^{11} - \lambda^{22}) n^2 \pm \rho t^2 \lambda^{11} \right) \]  

\[ \left( \frac{4a^2}{2a^2} \right)^{1/2} \right]^{1/2}, \quad a \equiv \frac{2\pi}{\beta}. \]  

(72)

One may consider two different approximations of overlapping validity 10, one for low temperature, \( \beta^{-2} \ll \mu_0 \), and the other for high temperature, \( \beta^{-2} \gg \alpha_0 \mu_0 \). Both these approximations (overlapping included) can be obtained from the exact expression above. It can be written in the form

\[ \zeta_A(s/2) = \frac{L}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \frac{y_-}{(y_+ y_-)^s} F(s/2, 1/2; s; 1-\eta), \quad \eta \equiv \frac{y_-^2}{y_+^2}. \]  

(73)
This is an exact formula. For high temperature, the ordinary expansion of the confluent hypergeometric function \( F \) is in order

\[
F(s/2, 1/2; s; 1 - \eta) = \sum_{k=0}^{\infty} \frac{(s/2)_k (1/2)_k}{k! (s)_k} (1 - \eta)^k \rightarrow 1 + \eta^{-1/2}, \ s \rightarrow 0, \quad (74)
\]

since \( y_{\pm} \) can be written as

\[
y_{\pm} \simeq \frac{a}{t} \left[ (n \pm b)^2 + c^2 \right]^{1/2}, \quad b = \frac{\beta t}{4\pi} \sqrt{\rho(\lambda^{11} - \lambda^{22})}, \quad c \simeq \frac{\alpha_0}{4\pi}. \quad (75)
\]

After the appropriate analytic continuation, the derivative of the zeta function yields

\[
\left. \frac{d}{ds} \zeta_A(s/2) \right|_{s=0} = -\frac{L}{4} \sqrt{b^2 + c^2} + \frac{2\pi L}{\beta t} \left[ b^2 + \frac{1}{6} - \left( \frac{1}{2} \ln(-b^2) + \psi(-1/2) + \gamma \right) c^2 \right]. \quad (76)
\]

In order to obtain this result, which comes from elementary Hurwitz zeta functions, \( \zeta(\pm 1, b) \), we have used the binomial expansion in (75) what is completely consistent with the approximation (notice the extra terms coming from the contribution of the pole of (75) to the derivative of \( \zeta_A(s/2) \)).

The low temperature case is more involved and now the methods of sect. 3 prove to be very useful. The term \( n = 0 \) must be treated separately. It gives

\[
\zeta_A^{(n=0)}(s/2) = \frac{L}{2\pi} \frac{\Gamma((1-s)/2)\Gamma(s-1/2)}{\Gamma(s/2)} (\lambda^{11}\rho)^{1/2-s}, \quad \frac{1}{2} < \text{Re}(s) < 1. \quad (77)
\]

This is again an exact result, which yields

\[
\left. \frac{d}{ds} \zeta_A^{n=0}(s/2) \right|_{s=0} = -\frac{L}{2} \sqrt{\lambda \rho} \quad (78)
\]

and

\[
S_1^{n=0} = (d - 2) \frac{L}{2} \sqrt{\lambda \rho}. \quad (79)
\]

Such contribution must be added to the one coming from the remaining terms (eq. (73) above)

\[
\zeta_A'(s/2) = \frac{L}{\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{y_-}{(y_+ - y_-)^s} F(s/2, 1/2; s; 1 - \eta), \quad \text{Re}(s) > 1. \quad (80)
\]

(The prime is no derivative, it just means that the term \( n = 0 \) is absent here.) Within this approximation, and working around the classical, \( T = 0 \) solution: \( \lambda^{11} = \lambda^{22} = \alpha_0 \mu_0 \), \( \rho = t^{-2} = 1 \), we obtain

\[
\zeta_A'(s/2) = \frac{L}{\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \left( \frac{a}{t} \right)^{1-2s} \left[ F_0(s) + F_1(s) + F_2(s) \right], \quad (81)
\]
where
\[ F_0(s) \equiv \sum_{n=1}^{\infty} \frac{n^{1-s}}{(n^2 + \sigma^2)^{s/2}} \left[ F(s/2, 1/2; s; \sigma^2/(n^2 + \sigma^2)) - 1 - \frac{\sigma^2}{4n^2} \right], \quad \sigma^2 \equiv \frac{\lambda \rho t^2 \beta^2}{4\pi^2}, \]
\[ F_1(s) \equiv \sum_{n=1}^{\infty} \frac{n^{1-s}}{(n^2 + \sigma^2)^{s/2}}, \quad F_2(s) \equiv \frac{\sigma^2}{4} \sum_{n=1}^{\infty} \frac{n^{-1-s}}{(n^2 + \sigma^2)^{s/2}}. \quad (82) \]

The study of these functions is done in appendix A. It involves the procedures of sect. 3 and is very related with example 2 there. Substituting the results of the appendix back into (81), we obtain
\[
\zeta_A'(s/2) = \frac{L}{\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \left( \frac{a}{t} \right)^{1-2s} \left[ \frac{\sigma^2}{8s} - \frac{\sigma^2}{8} - \frac{\sigma^2}{4} \ln \left( \frac{\sigma}{2} \right) - \frac{\sigma}{4} \right.
- \frac{1}{24} \left. + \frac{1}{2\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}} + O(s) \right], \quad (83)
\]
therefore
\[
\zeta_A'(0) = -\frac{La\sigma^2}{4t} \quad (84)
\]
and
\[
\frac{d}{ds} \zeta_A'(s/2) \bigg|_{s=0} = \frac{-La}{t} \left[ \psi(-1/2) \frac{\sigma^2}{4} - \ln(a/t) \frac{\sigma^2}{2} - \frac{\sigma^2}{4} - \frac{\sigma^2}{2} \ln(\sigma/2) - \frac{\sigma}{2} \right.
+ \gamma \frac{\sigma^2}{4} - \frac{1}{12} + \left. \frac{1}{2\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}} \right]. \quad (85)
\]
Finally
\[
S_1^{(n \neq 0)} = (d - 2) \frac{La}{t} \left[ \psi(-1/2) \frac{\sigma^2}{4} - \ln(a/t) \frac{\sigma^2}{2} - \frac{\sigma^2}{4} - \frac{\sigma^2}{2} \ln(\sigma/2) - \frac{\sigma}{2} \right.
+ \gamma - \frac{1}{4} + \left. \frac{1}{2\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}} \right]. \quad (86)
\]

Adding (49) and (80) we obtain the desired expansion of the one loop effective action, \( S_1 \), near \( T = 0 \). The physical implications of these results will be discussed elsewhere.

8 Discussion and conclusions

The series commutation techniques that are essential in the proof of the zeta function regularization theorem —which, on its turn, is the basic tool in the general procedure of
zeta function regularization when the spectrum of the operator is explicitly known—have
been promoted in this paper to an extremely elegant mathematical method, by making
use of the Mellin transforms of convenient heat kernel operators, in combination with a
rigorous treatment of the asymptotic series involved. The laborious analysis of the series
to be commuted, the artificial picking up of a convenient function in order to mimick
such series through pole residues on the complex plane, and the process of commutation
itself, with the appearance of additional terms ‘at infinity’, have now (almost completely)
disappeared, in favor of a quite natural Mellin transform (better, inverse Mellin transform)
analysis of the heat kernels (see also [8]). Moreover, the identification of the three different
contributions to the final result [2], namely, the naive commuted series (which results in
a sum of Riemann or Hurwitz zeta functions), the ordinary commutation remnants (a
polynomial function) and the very elusive, additional term of negative power-like behavior,
appears now in a clear, natural, almost misterious way. The last contribution [3], which
was originally very difficult to handle from a numerical point of view, has been given
here a completely new treatment, which allows the calculation of explicit numbers with
reasonable ease.

Our new method has the additional advantage that it is equally well fitted for the treat-
ment of general elliptic differential operators whose spectrum is not known (notice that
the procedures in [2,3,6] depend heavily on the explicit knowledge of the full spectrum).
The sum over eigenvalues is then naturally replaced —within the same procedure— by a
sum over heat-kernel or Seeley-De Witt coefficients. A huge mathematical industry has
been generated for the calculation of these coefficients, and one can now get full profit
from these result in the new context of the zeta function regularization theorem.

The physical applications one can envisage for the techniques here developed keep
growing every day. Aside from the many calculations, carried out in different contexts, of
the vacuum energy and the Casimir effect in quantum field theory, condensed matter and
solid state physics [11], in the last section we have already hinted at the straightforward
use one can make of our expressions in the calculation of the partition functions of strings
and membranes. As pointed out by Polchinski et al. [10], these results are essential in
order to decide, once and forever, if QCD can possibly be a certain limit of some string
(or membrane, or p-brane) theory, through the analysis of the high temperature behavior.
of the corresponding partition functions. These kind of calculations are considered to be very difficult even by the mathematical-physics community (see the specific description in [10]). As we have proven in the last section of this paper, they can be handled in a relatively simple way through our methods. We hope to be able to elaborate further on these and other applications in the near future.

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A Appendix: Auxiliar expressions for the string partition function at low temperature

We make here a careful study of the functions $F_i$ defined in (82). $F_1$ and $F_2$ can be analytically continued without problems:

$$F_1(s) = \zeta(2s - 1) - \frac{s\sigma^2}{2}\zeta(2 + 1) + \sum_{n=1}^{\infty} n^{1-s} \left[ (n^2 + \sigma^2)^{-s/2} - n^{-s} + \frac{s\sigma^2}{2}n^{-s-2} \right],$$

so that, in particular,

$$F_1(0) = -\frac{1}{12} - \frac{\sigma^2}{4},$$

(87)

and

$$F_2(s) = \frac{\sigma^2}{4}\zeta(2 + 1) + \frac{\sigma^2}{4} \sum_{n=1}^{\infty} n^{-1-s} \left[ (n^2 + \sigma^2)^{-s/2} - n^{-s} \right],$$

(89)

which has a pole at $s = 0$,

$$F_2(s) = \frac{\sigma^2}{8s} + \frac{\gamma\sigma^2}{4} + O(s).$$

(90)

Concerning $F_0$, by using

$$\lim_{s \to 0} F(s/2, 1/2; s; \sigma^2/(n^2 + \sigma^2)) = \frac{1}{2} \left[ 1 + \sqrt{1 + \left( \frac{\sigma}{n} \right)^2} \right],$$

(91)
we can write
$$F_0(0) = \frac{1}{2} \sum_{n=1}^{\infty} \left[ (n^2 + \sigma^2)^{1/2} - n - \frac{\sigma^2}{2n^2} \right] = \frac{1}{2} \lim_{\tau \to 0} \sum_{n=1}^{\infty} \left[ \left( (n^2 + \sigma^2)^{1/2} - n - \frac{\sigma^2}{2n^2} \right) e^{-\tau n} \right].$$

This yields
$$\sum_{n=1}^{\infty} n e^{-\tau n} = \frac{1}{\tau^2} - \frac{1}{12} + O(1),$$
for the middle term. For the other two we shall make explicit use of the results obtained in sect. 3 of this paper. For the last term, we get
$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-\tau n} = \text{Res}_{z=0} \left[ \tau^{-z} \Gamma(z) \zeta(z+1) \right] + O(1) = -\ln \tau + O(1).$$

In order to apply this procedure to the first term we need some more specific knowledge of the function
$$\zeta_{-1/2}(z) \equiv \sum_{n=1}^{\infty} n^{-z} (n^2 + \sigma^2)^{1/2},$$
which appears when considering (see sect. 3)
$$\sum_{n=1}^{\infty} (n^2 + \sigma^2)^{1/2} f(\tau n) = \sum_{n=1}^{\infty} (n^2 + \sigma^2)^{1/2} \frac{1}{2\pi i} \int dz (n\tau)^{-z} M[f](z)$$
$$= \frac{1}{2\pi i} \int dz \tau^{-z} M[f](z) \zeta_{-1/2}(z).$$

The study of this function can be reduced to example 2 of sect. 3. For $\text{Re} \ z > -2(p + 1)$ we get an analytical continuation which is a meromorphic function with poles at $z = 2(1 - k)$ of residues $\left( \frac{1}{k} \right) \sigma^{2k}$, $k = 0, 1, 2, 3, \ldots$

$$\zeta_{-1/2}(z) = \int_{\epsilon}^{\infty} dr \ r^{1-z} \left[ \sqrt{1 + \frac{\sigma^2}{r^2} - \theta(p-1) \sum_{k=0}^{p} \left( \frac{1}{k} \right) \left( \frac{\sigma}{r} \right)^{2k} \right]$$
$$+ \sum_{k=0}^{p} \left( \frac{1}{k} \right) \frac{\sigma^{2k}}{z + 2(k-1)} + e^{1-z}(\epsilon^2 + \sigma^2)^{1/2} - \frac{1}{2} e^{-z}(\epsilon^2 + \sigma^2)^{1/2}$$
$$- \frac{i}{2\pi} \sum_{\pm} \int_{C(\pm)} dt \ln \left( 1 - e^{\pm 2\pi i t} \right) \frac{d}{dt} \left[ t^{-z}(t^2 + \sigma^2)^{1/2} \right],$$

where the three last terms are integer functions and the contours $C(\pm)$ are chosen avoiding the points $\pm i\sigma$.

In our case it is sufficient to take $p = 1$. Considering now $-2 < \text{Re} \ z < 0$, the limit $\epsilon \to 0$ can be taken naively, with the result
$$\zeta_{-1/2}(z) = \int_{0}^{\infty} dr \ r^{1-z} \left[ \sqrt{1 + \frac{\sigma^2}{r^2} - \theta(p-1) \sum_{k=0}^{p} \left( \frac{1}{k} \right) \left( \frac{\sigma}{r} \right)^{2k} \right]$$

(98)
\[ + \sum_{k=0}^{1} \left( \frac{1/2}{k} \right) \frac{\sigma^{2k}}{z + 2(k - 1)} - \frac{i}{2\pi} \sum_{\pm} \int_{C(\pm)} dt \ln \left( 1 - e^{\pm 2\pi it} \right) \frac{d}{dt} \left[ t^{-z}(t^2 + \sigma^2)^{1/2} \right]. \]

It is the last term the one which prevents the continuation to the right of \( z = 0 \) due to its divergence at \( t = 0 \). This needs a special treatment (see again sect. 3), and the final result is

\[ \zeta_{-1/2}(z) = \frac{\sigma^2}{4} - \frac{\sigma^2}{2} \ln \left( \frac{\sigma}{2} \right) - \frac{\sigma^2}{2z} + \frac{1}{\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}} + O(z). \tag{99} \]

The integral term is awkward but harmless: it is exponentially suppressed as compared to the rest. Now we can go back to (96)

\[ \sum_{n=1}^{\infty} (n^2 + \sigma^2)^{1/2} e^{-\tau n} = \frac{1}{2\pi i} \int dz \tau^{-z} \Gamma(z) \zeta_{-1/2}(z) = \frac{1}{\tau^2} - \frac{\sigma^2}{2} \ln \tau \tag{100} \]

\[ - \frac{\gamma \sigma^2}{2} + \frac{\sigma^2}{4} - \frac{\sigma^2}{2} \ln \left( \frac{\sigma}{2} \right) - \frac{\sigma}{2} + \frac{1}{\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}} + O(1). \]

And this yields for \( F_0(0) \):

\[ F_0(0) = -\frac{\gamma \sigma^2}{4} + \frac{\sigma^2}{8} - \frac{\sigma^2}{4} \ln \left( \frac{\sigma}{2} \right) - \frac{\sigma}{4} + \frac{1}{24} + \frac{1}{2\pi} \int_{\sigma}^{\infty} dr \ln \left( 1 - e^{2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma^2}}. \tag{101} \]
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