Some result’s in the theory of fractional integral equations of Volterra-Fridlhom types

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ABSTRACT

The aim of this work is finding some result’s in the existence, uniqueness and stability solutions of new fractional integral equations of Volterra Fridlhom types by using Picard approximation method. Theorems on existence and uniqueness of a solution are established under some necessary and sufficient conditions on compact spaces. Furthermore, the study leads us to improve and extend the above method and the study become more general and detailed than those introduced by Butris results.

Keyords: Existence, uniqueness, and stability solution, fractional integral equations of Volterra-Fridlhom types, Picard approximation method.

1. INTRODUCTION

Over the years, many mathematicians, using their own notation and approach, have found various definitions that fit the idea of a non-integer order integral or derivative. One version that has been popularized in the world of fractional calculus is the Riemann Liouville definition. It is interesting to note that the Riemann-Liouville definition of a fractional derivative gives the same result. Since most of the other definitions of fractional calculus are largely variations of the Riemann-Liouville version.

Many results about the existence, uniqueness and stability solutions of fractional integral equations have been obtained by Picard approximation method that were proposed by many studies [2,3,5,7,9,18,19,20]. They are many subjects in physics and technology using mathematical methods that depends on the studies[1,2,3,8,13,14,15,16,21,22].

Definition1[17]. Let \( \{ f_m(t) \}_{m=0}^\infty \) be a sequence of functions defined on a set, \( E \subseteq \mathbb{R}^1 \). We say that \( \{ f_m(t) \}_{m=0}^\infty \) converges uniformly to the limit function \( f \) on \( E \) if, given \( \varepsilon > 0 \) there exists a positive integer \( N \) such that:

\[
|f_m(t) - f(t)| < \varepsilon, \quad (m \geq N, t \in E)
\]

Definition2[17]: Let \( f \) be a continuous function defined on a domain

\[
D = \{(t,u): a \leq t \leq b, c \leq u \leq d\}.
\]

Then \( f \) is said to satisfy a Lipschitz condition in the variable \( u \) on \( D \), provided that a constant \( L > 0 \) exists with property that

\[
|f(t,u_0) - f(t,u_2)| \leq L|u_1 - u_2|, \quad \text{for all } (t,u_1),(t,u_2) \in D.
\]

The constant \( L \) is called a Lipschitz constant for \( f \).

Definition3[17]: A solution \( u(t) \) is said to be stable if for each \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that any solution \( \tilde{u}(t) \) which satisfies

\[
\| \tilde{u}(t_0) - u(t_0) \| < \delta,
\]

for some \( t_0 \), also satisfies

\[
\| \tilde{u}(t) - u(t) \| < \varepsilon
\]

for all \( t \geq t_0 \).

Definition 4[9]. If \( g(t)e[a,b] \), and, then the operator \( I_0^a \) defined by

\[
I_0^a g(x) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t - s)^{1-\alpha}} \, ds.
\]

for \( a \leq t \leq b \), is called the Riemann-Liouville fractional integral operator of order \( \alpha \). Here \( \Gamma(\alpha) \) is the Gamma function defined by:

\[
\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t} \, dt.
\]

Definition 5[9]. The Caputo fractional derivative of order \( \alpha \geq 0 \) of a continuous function
less than one, i.e.,
\[ \varphi_{\text{max}}(a) = \frac{h^a}{\alpha^a} (K_a + I_a) < 1 \]  \hspace{1cm} (5)

Define a sequence of functions \( \{u_m(t)\}_{m=0}^\infty \) and \( \{w_m(t)\}_{m=0}^\infty \) by the following:
\[ u_m(t) = f(t) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} F(t,s,u_{m-1}(s),w_{m-1}(s)) \, ds \]  \hspace{1cm} (6)

and
\[ w_m(t) = g(t) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} G(t,s,u_{m-1}(s),w_{m-1}(s)) \, ds \]  \hspace{1cm} (7)

2. **Existence Solutions of Fractional Integral Equations of (V) and (F).**

The investigation of the existence solution of (V) and (F) will be introduced by the following theorem:

**Theorem1.** Let the vector functions \( f(t), g(t) \) and \( F(t,u,w), G(t,u,w) \) defined and continuous on the domain (1) suppose these functions are satisfying the inequalities (2), (3) and the conditions (4), (5). Then there exists a sequence of functions (6) and (7) converges uniformly on the domain
\[ D^* = \{ (t,s), f(a), g(a) \} \in [0,h] \times D_1 \times D_2 \]  \hspace{1cm} (8)

to the limit vector function \( \left( \frac{u(t)}{w(t)} \right) \) which is continuous on the domain (1) and satisfies the following integral equations:
\[ \left( \begin{array}{c} u(t) \\ w(t) \end{array} \right) = \left( \begin{array}{c} f(t) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} F(t,s,u_{m-1}(s),w_{m-1}(s)) \, ds \\ g(t) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} G(t,s,u_{m-1}(s),w_{m-1}(s)) \, ds \end{array} \right) \]  \hspace{1cm} (9)

and it is existing solution of (V) and (F).
Similarly, from the sequence of functions (7) and we obtain that
\[ \|w_m(t) - g(a)\| \leq \frac{h^m N_a}{\alpha t^\alpha} + q_2 + \|g(a)\| \]  
(13)
that is \( w_m(t) \in D_G \), for all \( t \in [0, h] \) and \( g(a) \in D_G \).

Next, we shall prove that the sequences of functions (6) and (7) converge uniformly on the domain (8). Then by mathematics induction, we have
\[ \|u_{m+1}(t) - u_m(t)\| \leq K_a \frac{h^a}{\alpha t^\alpha}(\|u_m(t) - u_{m-1}(t)\| + \|w_m(t) - w_{m-1}(t)\|) \]  
(14)
and
\[ \|w_{m+1}(t) - w_m(t)\| \leq L_a \frac{h^a}{\alpha t^\alpha}(\|u_m(t) - u_{m-1}(t)\| + \|w_m(t) - w_{m-1}(t)\|) \]  
(15)
for all \( m=1,2,3,\ldots \). Rewrite (14) and (15) in a vector form we get:
\[ \varphi_{m+1} \leq \omega(t) \varphi_m \]  
(16)
where
\[ \varphi_{m+1} = \left( \|u_{m+1}(t) - u_m(t)\|, \|w_{m+1}(t) - w_m(t)\| \right), \varphi_m = \left( \|u_m(t) - u_{m-1}(t)\|, \|w_m(t) - w_{m-1}(t)\| \right) \]
and
\[ \omega(t) = \left( K_a \frac{h^a}{\alpha t^\alpha}, L_a \frac{h^a}{\alpha t^\alpha} \right) \]
Now we, take the maximum value for the both sides of the inequality (16), we have
\[ \varphi_{m+1} \leq \omega \varphi_m \]  
(17)
where \( \omega = \max_{t \in [0, h]} \omega(t) \).

By repeating (17), we find that \( \varphi_{m+1} \leq \omega^m \varphi_0 \) and also, we get
\[ \sum_{i=1}^{\infty} \varphi_i \leq \sum_{i=1}^{\infty} \omega^{i-1} \varphi_0 \]  
(18)
Using the condition (15), thus the sequence of functions (6) and (7) are uniformly convergent, that is
\[ \lim_{m \to \infty} \sum_{i=1}^{m} \omega^{i-1} \varphi_0 = \sum_{i=1}^{\infty} \omega^{i-1} \varphi_0 = (E - \omega)^{-1} \varphi_0 \]  
(19)
Let
\[ \lim_{m \to \infty} \left( \frac{u_m(t)}{w_m(t)} \right) = \left( \frac{u(t)}{w(t)} \right) \]  
(20)
Since the sequence of function (6) and (7) are defined and continuous in the domain (1), then the limit function \( \left( \frac{u(t)}{w(t)} \right) \) is also defined and continuous in the domain (1).

By using the conditions and inequalities of a theorem, we can prove that the inequalities (10) and (11) will be satisfied for all \( t \in [0, h], f(t) \in D_F, g(a) \in D_G \), \( m = 0,1,2,\ldots \).

3. UNIQUENESS SOLUTIONS OF FRACTIONAL INTEGRAL EQUATIONS OF \( V \) AND \( F \).

The investigation of the uniqueness solutions of \( V \) and \( F \) will be introduced by:

**Theorem 2.** Let all assumptions and conditions of Theorem 3 be satisfied. Then the solution \( \left( \frac{u(t)}{\omega(t)} \right) \) is a unique of \( V \) and \( F \).

**Proof.** Let \( \left( \frac{u(t)}{\omega(t)} \right) \) be another solution of \( V \) and \( F \), that is
\[ u(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(t, s, u(s), \omega(s)) ds \]  
and
\[ \omega(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_a^b (t-s)^{\alpha-1} G(t, s, u(s), \omega(s)) ds \]
Assuming
\[ \|u(t) - \tilde{u}(t)\| \leq K_a \frac{h^a}{\alpha t^\alpha}(\|u(s) - \tilde{u}(s)\| + \|w(s) - \tilde{w}(s)\|) \]  
(21)
and
\[ \|w(t) - \tilde{w}(t)\| \leq L_a \frac{h^a}{\alpha t^\alpha}(\|u(t) - \tilde{u}(t)\| + \|w(t) - \tilde{w}(t)\|) \]  
(22)
By iterating the inequality (21) we have:
\[ \left( \|u(t) - \tilde{u}(t)\| \right) \leq \omega \left( \|u(t) - \tilde{u}(t)\| + \|w(t) - \tilde{w}(t)\| \right) \]  
(23)
Then by the condition (15), we find that:
\[ \left( \|u(t) - \tilde{u}(t)\| \right) \leq \omega^m \left( \|u(t) - \tilde{u}(t)\| + \|w(t) - \tilde{w}(t)\| \right) \]  
(24)
Thus \( \left( \frac{u(t)}{\omega(t)} \right) = \left( \tilde{u}(t) \frac{\omega(t)}{\tilde{w}(t)} \right) \). Hence the solutions \( \left( \frac{u(t)}{\omega(t)} \right) \) of \( V_1 \) and \( V_2 \) is a unique on the domain (1).

4. STABILITY SOLUTIONS OF FRACTIONAL INTEGRAL EQUATIONS OF \( V \) AND \( F \).

In this section, we can study the stability solutions of Volterra integral equations \( V \) and \( F \) respectively.

**Theorem 3.** Suppose that the functions \( F(t, u, w) \) and \( f(t, u, w) \) be continuous in the domain (1) and satisfy the inequalities (2) and (3). Then the solution (9) is stable for all \( t \geq 0 \).
Proof. Taking
\[ \| u(t) - \bar{u}(t) \| \leq \| f(a) - \bar{f}(a) \| + K_\alpha \frac{h^\alpha}{\Gamma(\alpha)} \| u(t) - \bar{u}(t) \| + \| w(t) - \bar{w}(t) \| \] (24)
and
\[ \| w(t) - \bar{w}(t) \| \leq \| g(a) - \bar{g}(a) \| + L_\beta \frac{h^\beta}{\Gamma(\beta)} \| u(t) - \bar{u}(t) \| + \| w(t) - \bar{w}(t) \| , \] (25)
where
\[ \bar{u} = \bar{f}(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(t,s,u(s),\bar{w}(s)) ds \]
and
\[ \bar{w} = \bar{g}(a) + \frac{1}{\Gamma(\alpha)} \int_a^b (t-s)^{\alpha-1} H(t,s,u(s),w(s)) ds \]
Rewrite (24) and (25) in a vector form that is
\[ \left( \begin{array}{c} \| u(t) - \bar{u}(t) \| \\ \| w(t) - \bar{w}(t) \| \end{array} \right) \leq \left( \begin{array}{c} \| f(a) - \bar{f}(a) \| \\ \| g(a) - \bar{g}(a) \| \end{array} \right) + \omega \left( \begin{array}{c} \| u(t) - \bar{u}(t) \| \\ \| w(t) - \bar{w}(t) \| \end{array} \right) \]
For
\[ \| f(a) - \bar{f}(a) \| \leq \delta_1, \| g(a) - \bar{g}(a) \| \leq \delta_2 \]
then
\[ \left( \begin{array}{c} \| u(t) - \bar{u}(t) \| \\ \| w(t) - \bar{w}(t) \| \end{array} \right) \leq \left( \begin{array}{c} \delta_1 \\ \delta_2 \end{array} \right) + \omega \left( \begin{array}{c} \| u(t) - \bar{u}(t) \| \\ \| w(t) - \bar{w}(t) \| \end{array} \right) \]
By using the condition (5) we have:
\[ \left( \begin{array}{c} \| u(t) - \bar{u}(t) \| \\ \| w(t) - \bar{w}(t) \| \end{array} \right) \leq \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right), e_1, e_2 \geq 0. \]
Also, by using the definition of the stability, we find that \( \bar{u}(t), \bar{w}(t) \)
is a stable solution for \( t \geq 0 \) of (V) and (F). Similar results can be obtained for other class of fractional integral equations of Volterra-Friedholm types. In particular, the fractional integral equations which has the form:
\[ u(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(t,s,u(s),w(s)) ds \] (V)
and
\[ w(t) = g(t) + \frac{1}{\Gamma(\beta)} \int_a^b (t-s)^{\beta-1} G(t,s,u(s),w(s)) ds \] (F)
The vector functions \( f(t), g(t) \) and \( F(t,u,w), G(t,u,w) \) are define and continuous on the domain:
\[ D = \{ (t,s,u,w); t,s \in \mathbb{R}^1, |t-s| \leq h, u \in D_1, w \in D_2, 0 < \alpha < \infty, \beta < 1 \} \] (26)
Suppose that the vector functions \( F(t,u,w) \) and \( G(t,u,w) \) satisfy the following inequalities:
\[ \| F(t,s,u,w) \| \leq M_\alpha, \| G(t,s,u,w) \| \leq N_\beta \]
\[ \| F(t,s,u_1,w_1) - F(t,s,u_2,w_2) \| \leq K_\alpha (\| u_1 - u_2 \| + \| w_1 - w_2 \| ) \] (27)
\[ \| G(t,s,u_1,w_1) - G(t,s,u_2,w_2) \| \leq L_\beta (\| u_1 - u_2 \| + \| w_1 - w_2 \| ) \]
For all \( t,s \in \mathbb{R}^1, u_1, u_2 \in D_1, w_1, w_2 \in D_2, \) where \( M_\alpha, N_\beta, K_\alpha \) and \( L_\beta \) are positive constant
We define non-empty sets as follows:
\[ D_F = D_1 - \frac{h^\alpha M_\alpha}{\Gamma(\alpha)} + q_1 + \| f(a) \| \]
\[ D_G = D_2 - \frac{h^\beta N_\beta}{\beta(\beta)} + q_2 + \| g(a) \| \]
Where, \( q_1 = \max_{t \in [0,h]} | f(t) | \) and \( q_2 = \max_{t \in [0,h]} | g(t) | \)
Furthermore, we assume that the largest Eigen-value of the matrix
\[ \omega = \begin{pmatrix} K_\alpha \frac{h^\alpha}{\Gamma(\alpha)} & K_\beta \frac{h^\beta}{\beta(\beta)} \\ L_\beta \frac{h^\beta}{\beta(\beta)} & L_\beta \frac{h^\beta}{\beta(\beta)} \end{pmatrix} \]
less than one, i.e.
\[ \varphi_{\max}(\omega) = K_\alpha \frac{h^\alpha}{\Gamma(\alpha)} + L_\beta \frac{h^\beta}{\beta(\beta)} < 1 \] (29)
Define a sequence of functions \( \{ u_m(t) \}_{m=0}^\infty, \{ w_m(t) \}_{m=0}^\infty \) by the following:
\[ u_m(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(t,s,u_{m-1}(s),w_{m-1}(s)) ds \]
With \( u_0(a) = f(a) \) (30)
and
\[ w_m(t) = g(t) + \frac{1}{\Gamma(\beta)} \int_a^b (t-s)^{\beta-1} G(t,s,u_{m-1}(s),w_{m-1}(s)) ds \]
With \( w_0(a) = g(a) \) (31)
We can state and prove a similar theorem of theorem1 by using the condition (29).

5. EXAMPLES

Example (1): consider the following system of fractional integro-differential equations, for \( t \in [1,5], \alpha=0.1,0.9, \ u(1) = 1, w(1) = 3 \) Picard approximation solutions using MATLAB are show in figure (1):
\[ u(t) = \frac{t^3 + 1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} (-0.8u(s) + \ln (w(s))) ds \]
\[ w(t) = \frac{t^2}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)} \int_1^5 (t-s)^{\alpha-1}(\ln(u(s) + w(s)))ds \]

\[ w(t) = \frac{-1.33e^{t+1}}{\Gamma(\alpha)} + \frac{1}{3 \Gamma(\alpha)} \int_1^5 (t-s)^{\alpha-1} \sin \left(5u(s) + w(s)\right)ds \]

| \( \alpha = 0.1 \) | \( \alpha = 0.9 \) |
|---|---|
| \( t_i \) | \( u(t_i) \) | \( w(t_i) \) | \( u(t_i) \) | \( w(t_i) \) |
| 1 | 3.1022 | 4.0511 | 3.0795 | 4.0397 |
| 1.1 | 3.6995 | 6.776 | 3.4627 | 4.6507 |
| 1.2 | 4.475 | 9.475 | 3.9213 | 5.2864 |
| 1.3 | 5.0723 | 12.2748 | 4.4612 | 5.9755 |
| 1.4 | 5.8366 | 15.0739 | 5.0878 | 6.7267 |
| 1.5 | 6.5849 | 17.9617 | 5.8068 | 7.5465 |

| \( \alpha = 0.8 \) | \( \alpha = 0.01 \) |
|---|---|
| \( t_i \) | \( u(t_i) \) | \( w(t_i) \) | \( u(t_i) \) | \( w(t_i) \) |
| 1 | 0.8503 | -7.4411 | 0.0099 | 0.9011 |
| 1.1 | 0.9987 | -8.2811 | 0.3516 | 1.1664 |
| 1.2 | 1.1476 | -9.2194 | 0.5363 | 0.9343 |
| 1.3 | 1.3060 | -10.2649 | 0.4944 | 0.7012 |
| 1.4 | 1.4762 | -11.4308 | 0.3975 | 0.8203 |
| 1.5 | 1.6593 | -12.7312 | 0.4364 | 0.9213 |

**Example (2):** consider the following system of fractional integro-differential equations, for \( t \in I = [1,5], \alpha = 0.01, 0.8, u_0(1) = 0, w_0(5) = 1 \): numerical solution using MATLAB approximation method are show in figure (2):

\[ u(t) = \frac{0.99e^{t-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_1^t (t-s)^{\alpha-1} \ln(u(s) + 2w(s))ds \]

**6. CONCLUSION**

In this work the existence uniqueness and stability solutions of new fractional integral equations of Volterra-Fridlhom types is studied by using Picard approximation method. Theorems on existence and uniqueness of a solutions are established under some necessary and sufficient conditions on compact spaces.

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