GROUND STATE SOLUTIONS FOR ASYMPTOTICALLY PERIODIC QUASILINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

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Abstract. In this paper, we are concerned with the existence of ground state solutions for the following quasilinear Schrödinger equation:

\[-\Delta u + V(x)u - \Delta(u^2)u = K(x)|u|^{2^* - 2}u + g(x,u), \quad x \in \mathbb{R}^N (1)\]

where \(N \geq 3\), \(V, g\) are asymptotically periodic functions in \(x\). By combining variational methods and the concentration-compactness principle, we obtain a ground state solution for equation (1) under a new reformative condition which unify the asymptotic processes of \(V, g\) at infinity.

1. Introduction and main result. Quasilinear Schrödinger equations of the form

\[i \frac{\partial \psi}{\partial t} = -\Delta \psi + W(x)\psi - l(|\psi|^2)\psi - \kappa [\Delta \rho(|\psi|^2)] \rho'(|\psi|^2)\psi, \quad (2)\]

have been derived as models of several physical phenomena and have been the subject of extensive study in recent years, where \(W : \mathbb{R}^N \to \mathbb{R}\) is a given potential, \(\kappa\) is a positive constant and \(l, \rho\) are real functions. Here we consider the existence of standing wave solutions for quasilinear Schrödinger equations of form (2) with \(\rho(s) = s, \kappa = 1\). Seeking solutions of the type stationary waves, namely, the solutions of the form \(\psi(t,x) = \exp(-iEt)u(x), \quad E \in \mathbb{R}\), we get an equation of elliptic type which has the formal structure:

\[-\Delta u + V(x)u - \Delta(u^2)u = K(x)|u|^{2^* - 2}u + g(x,u), \quad x \in \mathbb{R}^N, \quad (3)\]

where \(V(x) = W(x) - E\) is the new potential function and \(K(x)|u|^{2^* - 2}u + g(x,u) = l(x, u^2)u\) is the new nonlinearity.

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The main mathematical difficulty with problem (3) is caused by the nonlinearity involving second order derivatives \( \Delta (u^2)u \), the natural functional corresponding to problem (3) is not well defined for all \( u \in H^1(\mathbb{R}^N) \) if \( N \geq 2 \). To overcome this difficulty, various arguments have been developed, such as a constrained minimization argument (see [20, 22, 28, 29]), the perturbation method (see [19, 23, 24]) and a change of variables (see [1], [3]-[12], [21, 25, 27], [30]-[33], [36]).

For the critical case, we would like to mention [5, 6, 8, 11, 25, 27, 31, 33] and the references therein. It seems that Moameni [27] first studied the critical case when the potential \( V \) is radial and satisfies some geometric conditions. In [8], they obtained a positive classical solution by using the concentration compactness principle of Lions [15]. In [5], by using a change of variables and minimization argument, they obtained a sign-changing minimizer with \( k \) nodes of a minimization problem. Reference [25] established the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method. The method is to analyze the behavior of solutions for subcritical problems to pass limit as the exponent approaches to the critical exponent. In [11], they got the existence, concentration and multiplicity of weak solutions by employ the minimax theorems and Ljusternik-Schnirelmann theory. For asymptotically periodic nonlinearities with critical exponent there are considerably fewer results, here we mention [33].

It is worth pointing out that the related semilinear equation with the asymptotically periodic condition has been extensively studied, see [13, 17, 18, 34, 37, 38] and their references. In [13, 34, 37, 38], they discussed the existence of solutions for problem (3) without the second order derivatives \( \Delta (u^2)u \), when the problem is strongly indefinite, that is, 0 lies in a spectral gap of \( -\Delta + V \). We would like to point out that in a recent paper [17] and [18], Liu et al. have given reformative conditions which unify the asymptotic processes of \( V, g \) at infinity. The asymptotic processes is weaker than those in [13, 34, 37, 38]. We borrow an idea from [17] and [18] to obtain the ground state solution for problem (3). The only work with quasilinear asymptotically periodic Schrödinger equations with critical growth is our reference [33], they obtained only the existence of nontrivial solutions for problem (3) by using the mountain pass theorem. Here, we consider the ground state solution, which has great physical interests.

The main purpose of present paper is to establish the existence of a positive ground state solution for problem (3) and the corresponding periodic problem. There are several difficulties in our paper. The mainly one is the reformative condition which unifies the asymptotic processes of \( V, g \) at infinity. So we need rather careful estimates between \( g_0 \) and \( g, V_0 \) and \( V \). Besides, the nonlinear term \( g \) in our paper need not be differentiable, then the constrained manifold need not be of class \( C^1 \) in our case. We should employ a similar argument in [17] to conquer it. At last, the possible lack of compactness due to the criticality of the growth and the unboundedness of the domain, in order to obtain the existence of the solutions we will turn to the concentration compactness lemma due to [15, 16].

We suppose that \( V \) satisfies the following assumption:

(V) \( 0 < V_{\min} \leq V(x) \leq V_0(x) \in L^\infty(\mathbb{R}^N) \) and \( V(x) - V_0(x) \in \mathcal{F}_0 \), where \( V_{\min} \) is a positive constant, \( \mathcal{F}_0 := \{ k(x) : \forall \epsilon > 0, \lim_{|y| \to \infty} \text{meas}(x \in B_1(y) : |k(x)| \geq \epsilon) = 0 \} \), and \( V_0 \) satisfies \( V_0(x + z) = V_0(x) \) for all \( x \in \mathbb{R}^N \) and \( z \in \mathbb{Z}^N \).

We also assume the following conditions on \( K \):

(K) the function \( K(x) \in C(\mathbb{R}^N, \mathbb{R}) \) is 1-periodic in \( x_i, 1 \leq i \leq N \), and there is a point \( x_0 \in \mathbb{R}^N \) such that
Theorem 1.1. Suppose that $V$ possesses a positive ground state solution. (3)

Suppose that $g$ possesses a positive ground state solution. (4)

Remark 1. We compare our results with [33] as follows:

1. In [33], the authors obtained a nontrivial solution by using the mountain pass theorem.
2. We consider a new reformative condition which unify the asymptotic processes of $V,g$ at infinity, which means $\mathcal{F}$ and $\mathcal{F}_0$ contain more elements than those in [33].
3. The aim of $(g_3)$ is to obtain a ground state solution. In [33], the authors had employed other different types of conditions on $g$.

Notation: In this paper, we use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm $\|u\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right)dx$.
- $L^r(\mathbb{R}^N)$ is the usual Banach space endowed with the norm $\|u\|_{L^r}^r = \int_{\mathbb{R}^N} |u|^r dx, \forall s \in [1, +\infty)$.
\(\|u\|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)|\) denotes the usual norm in \(L^\infty(\mathbb{R}^N)\).
\(E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < \infty\}\) is endowed with the norm
\[
\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx.
\]
\(B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}\).
\(u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\}\).
\(|\Omega|\) denote the Lebesgue measure of the set \(\Omega\).
\(C, C_1, C_2, \cdots\) denote various positive (possibly different) constants.

2. Some preliminary results. We observe that formally problem (3) is the Euler-Lagrange equation associated with the energy functional
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x)|u|^{22^*}dx\right)
- \int_{\mathbb{R}^N} G(x,u)dx.
\]
From the variational point of view, the first difficulty we have to deal with problem (3) is to find an appropriate function space where the above functional is well defined. In the spirit of the argument developed in [3]. We make a change of variables \(v := f^{-1}(u)\), where \(f\) is defined by
\[
f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}}, \quad t \in [0, +\infty),
\]
\[
f(t) = -f(-t), \quad t \in (-\infty, 0].
\]
After the change of variables from \(J\), we obtain the following functional:
\[
I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v)dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x)|f(v)|^{22^*}dx
- \int_{\mathbb{R}^N} G(x,f(v))dx.
\]
Then \(I(v) = J(u) = J(f(v))\) and \(I\) is well defined on \(E, I \in C^1(E, \mathbb{R})\) under the hypotheses \((V)\) and \((g_1) - (g_6)\). Moreover, we observe that if \(v\) is a critical point of the functional \(I\), then the function \(u = f(v)\) is a solution of problem (3) (see [3]).

Below we summarize the properties of \(f\), which have been proved in [3, 33] and [7].

Lemma 2.1. The function \(f\) satisfies the following properties:

1. \(f\) is uniquely defined, \(C^\infty\) and invertible;
2. \(|f'(t)| \leq 1\) for all \(t \in \mathbb{R}\);
3. \(|f(t)| \leq |t|\) for all \(t \in \mathbb{R}\);
4. \(\frac{f(t)}{t^2} \to 1\) as \(t \to 0\);
5. \(\frac{f(t)}{t} \to 2^{1/4}\) as \(t \to \infty\);
6. \(\frac{f(t)}{t^4} \leq t^2 f'(t) \leq f(t)\) for all \(t > 0\);
7. \(|f(t)| \leq 2^{1/4}|t|^{1/2}\) for all \(t \in \mathbb{R}\);
8. \(f^2(t) - f(t)f'(t) t \geq 0\) for all \(t \in \mathbb{R}\);
9. there exists a positive constant \(C\) such that \(|f(t)| \geq C|t|\) for \(|t| \leq 1\) and \(|f(t)| \geq C|t|^{1/2}\) for \(|t| \geq 1\);
10. \(|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}\) for all \(t \in \mathbb{R}\);
11. the function \(f(t)f'(t)t^{-1}\) is strictly decreasing for \(t > 0\);
Lemma 2.4. Suppose that \((g_1) - (g_4)\) hold, then for all \((x, s) \in \mathbb{R}^N \times \mathbb{R}\), we have
\[
\frac{1}{2} g(x, f(s)) f'(s) s - G(x, f(s)) \geq \frac{1}{4} g(x, f(s)) f(s) - G(x, f(s)) \geq 0. \tag{5}
\]
For any \(\delta > 0\), there exist \(r_\delta > 0\), \(C_\delta > 0\) and \(\alpha \in (2, 2^*)\) such that
\[
0 \leq g_0(x, s) \leq g(x, s) \leq \delta |s|, \quad \forall (x, s) \in \mathbb{R}^N \times [-r_\delta, r_\delta],
\]
\[
0 \leq g_0(x, s) \leq g(x, s) \leq \delta |s| + C_\delta |s|^{2^*_N - 1}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R},
\]
\[
0 \leq g_0(x, s) \leq g(x, s) \leq C_\delta |s| + \delta |s|^{2^*_N - 1}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R},
\]
\[
0 \leq g_0(x, s) \leq g(x, s) \leq \delta (|s| + |s|^{2^*_N - 1}) + C_\delta |s|^{2\alpha - 1}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R},
\]
\[
0 \leq G_0(x, s) \leq G(x, s) \leq \frac{\delta}{2} |s|^2 + \frac{\delta}{2\alpha} |s|^{2^*_N} + \frac{C_\delta}{2\alpha} |s|^{2\alpha}, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}.
\tag{11}
\]
Proof. We only need to prove the first inequality (5). It follows from \((g_1), (g_4)\) that \(G(x, s) \geq 0\) and for \(s > 0\),
\[
G(x, s) = \int_0^s g(x, t) dt = \int_0^s \frac{g(x, t)}{t^3} t^3 dt \leq \frac{g(x, s)}{s^3} \int_0^s t^3 dt = \frac{1}{4} g(x, s)s.
\]
Combining with Lemma 2.1-(6), one has
\[
\frac{1}{2} g(x, f(s)) f'(s) s - G(x, f(s)) \geq \frac{1}{4} g(x, f(s)) f(s) - G(x, f(s)) \geq 0.
\]
The rest inequalities follow from \((g_1) - (g_4)\) immediately. \(\square\)

Lemma 2.3 \([33]\). Suppose that \((V), (K)\) are satisfied. Let \(\{u_n\} \subset E\) be a \((C)_c\) sequence with \(c > 0\), and \(u_n \to 0\) in \(L^\alpha(\mathbb{R}^N)\) for \(\alpha \in (2, 2^*)\). Then we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) \left( f^2(u_n) - f(u_n) f'(u_n) u_n \right) dx = 0. \tag{12}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left( 2^{2^*_N} |u_n|^{2^*_N - 2} - |f(u_n)|^{2^*_N} \right) dx = 0. \tag{13}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left( \frac{1}{2} |f(u_n)|^{2^*_N - 2} f(u_n) f'(u_n) u_n - \frac{1}{2} 2^{2^*_N - 2} |u_n|^{2^*_N} \right) dx = 0. \tag{14}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, f(u_n)) f'(u_n) u_n dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} G(x, f(u_n)) dx = 0. \tag{15}
\]

Lemma 2.4 \([17]\). Suppose that condition \((V)\) holds. Then there are two positive constants \(C_1\) and \(C_2\) such that \(C_1 \|u\|^2 \leq \|u\|^2 \leq C_2 \|u\|^2\) for all \(u \in E\).
3. Proof of Theorem 1.1. In this section, we present the result by some lemmas. Define
\[
I_0(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V_0(x) f^2(v) dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(v)|^{22^*} dx
- \int_{\mathbb{R}^N} G_0(x, f(v)) dx,
\]
\[\mathcal{A} = \{u \in E : \langle I_0(u), u \rangle = 0, u \neq 0\}, \quad \mathcal{A}_0 = \{u \in E : \langle I_0(u), u \rangle = 0, u \neq 0\},\]
\[c = \inf_{u \in \mathcal{A}} I(u), \quad c_0 = \inf_{u \in \mathcal{A}_0} I_0(u).\]

Lemma 3.1. Suppose that \((V)\) and \((g_1) - (g_5)\) hold, then for each \(u \in E\), \(u \neq 0\), there is a unique \(t_u > 0\) such that \(t_u u \in \mathcal{A}\). Moreover, the maximum of \(I(tu)\) for \(t \geq 0\) is achieved at \(t_u\).

Proof. Define \(h(t) := I(tu)\). It follows from (10) and Lemma 2.1-(3)(7) that
\[
h(t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(tu) dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(tu)|^{22^*} dx
- \int_{\mathbb{R}^N} G(x, f(tu)) dx
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(tu) dx - \frac{1}{2} \int_{\mathbb{R}^N} f^2(tu) dx
- \frac{C_\delta}{22^*} \int_{\mathbb{R}^N} |f(tu)|^{22^*} dx - \frac{22^*/2}{22^*} \int_{\mathbb{R}^N} K(x) |u|^2 dx
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{V_{\text{min}}}{2} - \frac{\delta}{2} \int_{\mathbb{R}^N} f^2(tu) dx - \frac{22^*/2}{22^*} C_\delta \int_{\mathbb{R}^N} |u|^2 dx
- \frac{22^*/2}{22^*} \|\nabla\|_{\infty} \int_{\mathbb{R}^N} |u|^2 dx.
\]
We can get \(h(t) > 0\) whenever \(0 < \delta < V_{\text{min}}\) and \(t > 0\) is small enough.

By Lemma 2.1-(3)(9) and \(G(x, s) \geq 0\), we have
\[
h(t) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \frac{1}{22^*} \int_{\{x : |u(x)| \geq 1\}} K(x) |f(tu)|^{22^*} dx
- \int_{\mathbb{R}^N} G(x, f(tu)) dx
\leq \frac{t^2}{2} \|u\|^2 - \frac{Ct^2}{22^*} \int_{\{x : |u(x)| \geq 1\}} |u|^2 dx.
\]
Hence \(h(t) \to -\infty\) as \(t \to \infty\) and \(h\) has a positive maximum. The condition \(h'(t) = 0\) is equivalent to
\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\{x : u(x) \neq 0\}} \frac{g(x, f(tu)) f'(tu)}{tu} u^2 dx
+ \int_{\{x : u(x) \neq 0\}} \frac{K(x) f'(tu) |f(tu)|^{22^* - 2} f(tu)}{tu} u^2 dx
- \int_{\{x : u(x) \neq 0\}} \frac{V(x) f'(tu) f(tu)}{tu} u^2 dx.
\]
Let
\[
Z(s) := \frac{g(x, f(s))f'(s)}{s} + \frac{K(x)f'(s)|f(s)|^{2^{*}-2}f(s)}{s} - \frac{V(x)f'(s)f(s)}{s}.
\]
By \((g_3)\) and Lemma 2.1-(12), the function
\[
\frac{g(x, f(s))f'(s)}{s} = \frac{g(x, f(s))}{f^3(s)} f'(s) f^3(s)
\]
is strictly increasing for \(s > 0\). Hence also \(s \mapsto Z(s)\) is strictly increasing according to Lemma 2.1-(11)(12). So there is a unique \(t_u > 0\) such that \(h'(t_u) = 0\). The conclusion is an immediate consequence of the fact that \(h'(t) = t^{-1}\langle f'(tu), tu\rangle\).

From Lemma 3.1, we can get the following lemma easily.

**Lemma 3.2.** Suppose that \((V)\) and \((g_1) - (g_3)\) are satisfied. Define
\[
S_{\rho} := \{u \in E : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} V(x)u^2 \, dx = \rho^2\}.
\]
Then the functional \(I\) satisfies the following mountain pass geometry:
(i) there are \(\rho > 0, \alpha > 0\) such that \(|I|_{S_{\rho}} \geq \alpha > 0\);
(ii) there exists \(e \in E, \|e\| > \rho\) such that \(I(e) < 0\).

**Lemma 3.3.** Suppose that \((V), (K)\) and \((g_1) - (g_3)\) hold, then there exists a bounded \((C)_{c}\) sequence \(\{v_n\} \subset E\) associated with the functional \(I\).

**Proof.** From Lemma 3.2, we can get the existence of the \((C)_{c}\) sequence, we only need to prove \(\{v_n\}\) is bounded. First of all, we observe that if a sequence \(\{v_n\} \subset E\) satisfies
\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x)v_n^2 \, dx \leq C_1,
\]
for some constant \(C_1 > 0\), then the sequence \(\{v_n\}\) is bounded in \(E\). For that, we simply need to demonstrate that \(\int_{\mathbb{R}^N} v_n^2 \, dx\) is bounded. In fact, by Lemma 2.1-(9) and \((V)\), we observe that
\[
\int_{\{x : |v_n(x)| \leq 1\}} v_n^2 \, dx \leq \frac{1}{C_2^2} \int_{\{x : |v_n(x)| \leq 1\}} f^2(v_n) \, dx \leq \frac{1}{C_2^2 V_{\min}} \int_{\mathbb{R}^N} V(x)f^2(v_n) \, dx \leq \frac{C_1}{C_2^2 V_{\min}}.
\]
Moreover, by the Sobolev inequality and Lemma 2.1-(9), one deduces
\[
\int_{\{x : |v_n(x)| > 1\}} v_n^2 \, dx \leq \int_{\{x : |v_n(x)| > 1\}} v_n^2 \, dx \leq C_2 \left( \int_{\{x : |v_n(x)| > 1\}} |\nabla v_n|^2 \, dx \right)^{\frac{2^*}{2}} \leq C_2 \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{\frac{2^*}{2}} \leq C_2 C_1^{\frac{2^*}{2}}.
\]
Hence there is a constant \(C_3 > 0\) such that
\[
\int_{\mathbb{R}^N} v_n^2 \, dx = \int_{\{x : |v_n(x)| \leq 1\}} v_n^2 \, dx + \int_{\{x : |v_n(x)| > 1\}} v_n^2 \, dx \leq C_3.
\]
Therefore, it remains to show that \(\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} V(x)f^2(v_n) \, dx\) is bounded.
Let \( \{v_n\} \subset E \) be an arbitrary Cerami sequence for \( I \) at level \( c > 0 \), that is \( I(v_n) \to c \) and \( (1 + \|v_n\|^2)\|I'(v_n)\| \to 0 \), namely
\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n) dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^*} dx \\
- \int_{\mathbb{R}^N} G(x, f(v_n)) dx = c + o_n(1),
\]
and for any \( \varphi \in E \),
\[
\langle I'(v_n), \varphi \rangle = \int_{\mathbb{R}^N} \nabla v_n \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) \varphi dx \\
- \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^* - 2} f(v_n) f'(v_n) \varphi dx - \int_{\mathbb{R}^N} g(x, f(v_n)) f'(v_n) \varphi dx \\
= o_n(1).
\]
Choosing \( \varphi = \varphi_n = \sqrt{1 + 2f^2(v_n)} f(v_n) = \frac{f(v_n)}{f'(v_n)} \), from Lemma 2.1-(6), we get \( \|\varphi_n\|_2 \leq 2\|v_n\|_2 \) and
\[
|\nabla \varphi_n| = \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right) |\nabla v_n| \leq 2|\nabla v_n|.
\]
Thus there exists a constant \( C_4 > 0 \) such that \( \|\varphi_n\| \leq C_4\|v_n\| \). Recalling that \( \{v_n\} \subset E \) is a (C) sequence, we get
\[
\langle I'(v_n), \varphi_n \rangle = \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right) |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x) f^2(v_n) dx \\
- \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^*} dx - \int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) dx \\
= o_n(1).
\]
By computing (16) - \( \frac{1}{4} \) (17), one gets
\[
c + o_n(1) = \frac{1}{4} \int_{\mathbb{R}^N} \frac{1}{1 + 2f^2(v_n)} |\nabla v_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V(x) f^2(v_n) dx \\
+ \int_{\mathbb{R}^N} \left(\frac{1}{4} g(x, f(v_n)) f(v_n) - G(x, f(v_n))\right) dx \\
+ \left(\frac{1}{4} - \frac{1}{22^*}\right) \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^*} dx.
\]
Thanks to (5), we get
\[
\frac{1}{4} \int_{\mathbb{R}^N} \frac{1}{1 + 2f^2(v_n)} |\nabla v_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V(x) f^2(v_n) dx \\
+ \frac{1}{2N} \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^*} dx \leq c + o_n(1).
\]
Denote \( w_n = f(v_n) \), then \( |\nabla w_n|^2 = (1 + 2w_n^2)|\nabla v_n|^2 \). We can rewrite (16), (18) as follows.
\[
\frac{1}{2} \int_{\mathbb{R}^N} (1 + 2w_n^2) |\nabla w_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) w_n^2 dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |w_n|^{22^*} dx \\
- \int_{\mathbb{R}^N} G(x, w_n) dx = c + o_n(1),
\]
Lemma 3.4

This completes the proof.

Lemma 3.5.

Assume that $\{w_n\}$ is bounded in $E$ and there is $C_5 > 0$ such that

$$\int_{\mathbb{R}^N} K(x)|w_n|^{2^*} dx \leq C_5. \quad (21)$$

It follows from (10), (21) and (K) that

$$\int_{\mathbb{R}^N} G(x, w_n) dx \leq \int_{\mathbb{R}^N} \left( \frac{\delta}{2} |w_n|^2 + \frac{C_5}{2^*} |w_n|^{2^*} \right) dx \leq C_6.$$

By the above inequality and (19), one has

$$\frac{1}{2} \int_{\mathbb{R}^N} (1 + 2w_n^2) |\nabla w_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)w_n^2 dx \leq C_6 + \frac{C_5}{2^*} + c + o_n(1),$$

namely

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(w_n) dx \leq C_6 + \frac{C_5}{2^*} + c + o_n(1).$$

This completes the proof.

Lemma 3.4 ([17]). Suppose that $(V)$ and $(g_1) - (g_5)$ are satisfied. If $u \in \mathcal{N}$ and $I(u) = c$, then $u$ is a ground state solution of problem (3).

Lemma 3.5. Assume that $(V)$, $(g_1)$, $(g_2)$ and (1) of $(g_4)$ hold. If $\{u_n\}$ is bounded in $E$ and $u_n \to 0$ in $L_{loc}^\alpha(\mathbb{R}^N)$ for $\alpha \in [2, 2^*)$, one has

$$A_{n1} := \int_{\mathbb{R}^N} \left( V(x) - V_0(x) \right) f^2(u_n) dx = o_n(1). \quad (22)$$

$$A_{n2} := \int_{\mathbb{R}^N} \left[ G(x, f(u_n)) - G_0(x, f(u_n)) \right] dx = o_n(1). \quad (23)$$

Proof. (i) The proof of (22).

Firstly, when $k(x) \in \mathcal{F}_0$, we claim that for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{\{x : |k(x)| \geq \epsilon\}} u^2 dx \leq C_0 \int_{B_{R_{\epsilon}+1}(0)} u^2 dx + C_1 \epsilon^{2/N} ||u||_2^2, \quad \forall u \in E, \quad (24)$$

where $C_0, C_1$ are positive constants and independent on $\epsilon$. (24) has already been proved in [17], we sketch the proof as following. By the definition of $\mathcal{F}_0$, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\text{meas}\{ x \in B_1(y) : |k(x)| \geq \epsilon \} < \epsilon, \quad \forall |y| \geq R_\epsilon.$$

Let $\Omega_i = \{ x \in B_1(y_i) : |k(x)| \geq \epsilon \}$, then $|\Omega_i| < \epsilon$, for all $|y_i| \geq R_\epsilon$. Now, covering $\mathbb{R}^N$ by balls $B_1(y_i)$, $i \in \mathbb{N}$, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N + 1$ balls. Without loss of generality, we suppose that $|y_i| < R_\epsilon$, $i = 1, 2, \ldots, n_\epsilon$ and $|y_i| \geq R_\epsilon$, $i = n_\epsilon + 1, n_\epsilon + 2, \ldots, +\infty$. By the Hölder and Sobolev inequalities, we have

$$\int_{\{x : |k(x)| \geq \epsilon\}} |u|^2 dx \leq \sum_{i=1}^{+\infty} \int_{\Omega_i} |u|^2 dx.$$
\[
\int_{\Omega} |u|^2 dx + \int_{\Omega} |u|^2 dx \\
\leq (N + 1) \int_{\Omega} |u|^2 dx + \sum_{i=n+1}^{+\infty} |\Omega_i|^{\frac{2}{N}} \left( \int_{\Omega} |u|^2 dx \right)^{\frac{N-2}{N}} \\
\leq (N + 1) \int_{B_{R\epsilon}(0)} |u|^2 dx + C_2 \epsilon^{\frac{2}{N}} \sum_{i=1}^{+\infty} \int_{B_{R\epsilon}(y_i)} \left( |\nabla u|^2 + u^2 \right) dx \\
\leq C_0 \int_{B_{R\epsilon}(0)} u^2 dx + C_1 \epsilon^{\frac{2}{N}} \|u\|^2_{L^2},
\]

where \( C_0 = N + 1 \) and \( C_1 = C(N + 1) \).

Let \( k(x) := V(x) - V_0(x) \in \mathcal{F}_0 \), using Lemma 2.1-(3) and (24), one has
\[
|A_{n1}| \leq \int_{\mathbb{R}^N} |k(x)f^2(u_n)| dx \\
= \int_{\{x: |k(x)| \geq \epsilon\}} |k(x)u_n^2| dx + \int_{\{x: |k(x)| < \epsilon\}} |k(x)u_n^2| dx \\
\leq 2\|V_0\|_{\infty} \left[ C_0 \int_{B_{R\epsilon}(0)} u_n^2 dx + C_1 \epsilon^{\frac{2}{N}} \|u_n\|^2_{L^2} \right] + \epsilon \int_{\mathbb{R}^N} |u_n|^2 dx \\
= o_n(1) + C_2 \epsilon^{\frac{2}{N}} + C_3 \epsilon.
\]

Let \( \epsilon \to 0 \), (22) is proved.

(ii) The proof of (23).

Set \( h(x, s) := g(x, s) - g_0(x, s) \in \mathcal{F} \). Similar to the proof of (24), for any \( \epsilon > 0 \), there exists \( R_\epsilon > 0 \) such that
\[
\text{meas}\{x \in B_1(y) : |h(x, s)| \geq \epsilon\} < \epsilon, \quad \forall |y| \geq R_\epsilon, \quad |s| \leq 1/\epsilon.
\]

Covering \( \mathbb{R}^N \) by balls \( B_i(y_i), \ i \in \mathbb{N} \), in such a way that each point of \( \mathbb{R}^N \) is contained in at most \( N + 1 \) balls. Without loss of generality, we suppose that \( |y_i| < R_\epsilon, \ i = 1, 2, \ldots, n_\epsilon \) and \( |y_i| \geq R_\epsilon, \ i = n_\epsilon + 1, n_\epsilon + 2, \ldots, +\infty \). By the mean value theorem, there exists \( t_n \in [0, 1] \) such that
\[
G(x, f(u_n)) - G_0(x, f(u_n)) = [g(x, t_n f(u_n)) - g_0(x, t_n f(u_n))] f(u_n).
\]

Set
\[
\Omega^1 := \{x \in B_1(y_i) : |h(x, t_n f(u_n))| < \epsilon\}, \\
\Omega^2 := \{x \in B_1(y_i) : |t_n f(u_n)| \leq 1/\epsilon, \ |h(x, t_n f(u_n))| \geq \epsilon\}, \\
\Omega^3 := \{x \in B_1(y_i) : |t_n f(u_n)| > 1/\epsilon, \ |h(x, t_n f(u_n))| \geq \epsilon\}.
\]

Then we have
\[
|A_{n2}| \leq \int_{\mathbb{R}^N} |h(x, t_n f(u_n)) f(u_n)| dx \\
\leq \sum_{i=1}^{n_\epsilon} \int_{B_1(y_i)} |h(x, t_n f(u_n)) f(u_n)| dx + \sum_{i=n_\epsilon+1}^{+\infty} \int_{B_1(y_i)} |h(x, t_n f(u_n)) f(u_n)| dx.
\]
By using (6) and Lemma 2.1-(3), we obtain

\[
\sum_{i=1}^{n_i} \int_{B_i(y_i)} |h(x, t_n f(u_n))| f(u_n) dx + \sum_{i=n+1}^{+\infty} \int_{\Omega |} |h(x, t_n f(u_n))| f(u_n) dx + \sum_{i=n+1}^{+\infty} \int_{\Omega^3} |h(x, t_n f(u_n))| f(u_n) dx
\]

\[= I_1 + I_2 + I_3 + I_4.\]

It follows from (8) and Lemma 2.1-(7) that

\[
I_1 \leq (N + 1) \int_{B_{r,\epsilon}(0)} |h(x, t_n f(u_n))| f(u_n) dx
\]

\[\leq (N + 1) \int_{B_{r,\epsilon}(0)} 2[C\delta|t_n f(u_n)| + \delta|t_n f(u_n)|_{22}^{22 - 1}] |f(u_n)| dx
\]

\[\leq 2(N + 1)C\delta \int_{B_{r,\epsilon}(0)} |u_n|^2 dx + 2(N + 1)\delta^{2/2} \int_{B_{r,\epsilon}(0)} |u_n|^2 dx
\]

\[= o_n(1) + C_4 \delta.\]

Let

\[
\Omega^{11} := \{ x \in B_1(y_i) : |h(x, t_n f(u_n))| < \epsilon, \ |t_n f(u_n)| \leq r_\delta \},
\]

\[
\Omega^{12} := \{ x \in B_1(y_i) : |h(x, t_n f(u_n))| < \epsilon, \ |t_n f(u_n)| > r_\delta \}.
\]

By using (6) and Lemma 2.1-(3), we obtain

\[
I_2 = \sum_{i=n+1}^{+\infty} \int_{\Omega^{11}} |h(x, t_n f(u_n))| f(u_n) dx + \sum_{i=n+1}^{+\infty} \int_{\Omega^{12}} |h(x, t_n f(u_n))| f(u_n) dx
\]

\[\leq \sum_{i=n+1}^{+\infty} \int_{\Omega^{11}} 2\delta |t_n f(u_n)| f(u_n) dx + \sum_{i=n+1}^{+\infty} \int_{\Omega^{12}} \frac{\epsilon}{r_\delta} |f(u_n)|^2 dx
\]

\[\leq 2\delta \sum_{i=n+1}^{+\infty} \int_{\Omega^{11}} |u_n|^2 dx + \frac{\epsilon}{r_\delta} \sum_{i=n+1}^{+\infty} \int_{\Omega^{12}} |u_n|^2 dx
\]

\[\leq 2(N + 1)\delta \int_{\Omega^2} |u_n|^2 dx + \frac{(N + 1)\epsilon}{r_\delta} \int_{\Omega^2} |u_n|^2 dx
\]

\[\leq C_5 \delta + C_6 \epsilon.
\]

It follows from (8), Lemma 2.1-(3) and the Hölder and Sobolev inequalities that

\[
I_3 \leq \sum_{i=n+1}^{+\infty} \int_{\Omega^2} 2[C\delta|f(u_n)|^2 + \delta|f(u_n)|_{22}^{22}] dx
\]

\[\leq \sum_{i=n+1}^{+\infty} \left[ 2C\delta \int_{\Omega^2} |u_n|^2 dx + 22^{2/2} \delta \int_{\Omega^2} |u_n|^2 dx \right]
\]

\[\leq 2C_\delta \sum_{i=n+1}^{+\infty} \left[ \Omega^{11} \left( \int_{\Omega^{11}} |u_n|^2 dx \right)^{\frac{n-2}{n-2}} + 2(N + 1)\delta^{2/2} \int_{\Omega^2} |u_n|^2 dx \right]
\]

\[\leq 2C_\delta^{\frac{4}{n}} \sum_{i=n+1}^{+\infty} C \int_{\Omega^2} (|\nabla u_n|^2 + |u_n|^2) dx + C_7 \delta
\]

\[\leq 2C_\delta^{\frac{4}{n}} (N + 1)C \int_{\Omega^2} (|\nabla u_n|^2 + |u_n|^2) dx + C_7 \delta.
\]
Hence we have

\[ |A_{n2}| \leq o_n(1) + C_4 \delta + C_5 \epsilon + C_6 \epsilon^\frac{2}{3} + C_7 \epsilon + C_9 \delta + C_{10} \epsilon^{2^{2^{-\alpha}}}. \]

Let \( \epsilon \to 0 \) and then \( \delta \to 0 \), we complete the proof of (23).  \( \square \)

**Lemma 3.6.** Assume that \((V), (g_1), (g_2)\) and (1) of \((g_4)\) hold. Suppose \( \{u_n\} \subset E \) is bounded, \( |z_n| \to +\infty \). Then for any \( \varphi \in C^\infty_0(\mathbb{R}^N) \), we have

\[ B_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x)) f(u_n) f'(u_n) \varphi(x - z_n) dx = o_n(1). \]  \( (25) \)

\[ B_{n2} := \int_{\mathbb{R}^N} [g(x, f(u_n)) - g_0(x, f(u_n))] f'(u_n) \varphi(x - z_n) dx = o_n(1). \]  \( (26) \)

**Proof.** (i) The proof of (25).

Since \( \varphi \in C^\infty_0(\mathbb{R}^N) \), we get that

\[ \int_{B_{n+1}(0)} |\varphi(x - z_n)|^2 dx = o_n(1). \]  \( (27) \)

Let \( k(x) := V(x) - V_0(x) \in \mathcal{F}_0 \), by using Lemma 2.1-(2)(3), (24), (27) and the Hölder inequality, one has

\[ |B_{n1}| = \int_{\{x: k(x) \geq \epsilon\}} |k(x) f(u_n) f'(u_n) \varphi(x - z_n)| dx \]
\[ + \int_{\{x: k(x) < \epsilon\}} |k(x) f(u_n) f'(u_n) \varphi(x - z_n)| dx \]
\[ \leq 2 \|V_0\|_\infty \int_{\{x: k(x) \geq \epsilon\}} |u_n \varphi(x - z_n)| dx + \epsilon \int_{\{x: k(x) < \epsilon\}} |u_n \varphi(x - z_n)| dx \]
\[ \leq 2 \|V_0\|_\infty \|u_n\|_2 \left( \int_{\{x: k(x) \geq \epsilon\}} |\varphi(x - z_n)|^2 dx \right)^{1/2} + \epsilon \|u_n\|_2 \|\varphi\|_2 \]
\[ \leq C_{11} \left( C_0 \int_{B_{n+1}(0)} |\varphi(x - z_n)|^2 dx + C_1 \epsilon^{2^{2^{-\alpha}} \|\varphi\|_H^2} + C_{12} \epsilon \right) \]
\[ = o_n(1) + C_{13} \epsilon^{1/2} + C_{12} \epsilon. \]

Let \( \epsilon \to 0 \), (25) is proved.

(ii) The proof of (26).
Set $h(x,s) := g(x,s) - g_0(x,s) \in \mathcal{F}$. As the proof of Lemma 3.5, covering $\mathbb{R}^N$ by balls $B_1(y_i)$. Let

$$
\Omega^4 := \{ x \in B_1(y_i) : |h(x,f(u_n))| < \epsilon \},
$$

$$
\Omega^5 := \{ x \in B_1(y_i) : |f(u_n)| \leq 1/\epsilon, \ |h(x,f(u_n))| \geq \epsilon \},
$$

$$
\Omega^6 := \{ x \in B_1(y_i) : |f(u_n)| > 1/\epsilon, \ |h(x,f(u_n))| \geq \epsilon \}.
$$

we have

$$
|B_n| \leq \int_{\mathbb{R}^N} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\leq \sum_{i=1}^{n_1} \int_{B_1(y_i)} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\quad + \sum_{i=n_1+1}^{\infty} \int_{B_1(y_i)} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
= \sum_{i=1}^{n_1} \int_{B_1(y_i)} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\quad + \sum_{i=n_1+1}^{\infty} \int_{\Omega^4} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\quad + \sum_{i=n_1+1}^{\infty} \int_{\Omega^5} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\quad + \sum_{i=n_1+1}^{\infty} \int_{\Omega^6} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
:= I_5 + I_6 + I_7 + I_8.
$$

It follows from (8), Lemma 2.1-(2)(3)(7) and (27) that

$$
I_5 \leq (N+1) \int_{B_{R_{n+1}}(0)} |h(x,f(u_n)) f'(u_n) \varphi(x-z_n)| dx
$$

$$
\leq (N+1) \int_{B_{R_{n+1}}(0)} 2 \left[ C_\delta |f(u_n)| + \delta |f(u_n)|^{2^*_n-1} \right] |f'(u_n) \varphi(x-z_n)| dx
$$

$$
\leq 2(N+1)C_\delta \int_{B_{R_{n+1}}(0)} |u_n \varphi(x-z_n)| dx
$$

$$
+ 2(N+1)\delta \int_{B_{R_{n+1}}(0)} 2^{\frac{2^*_n-1}{2^*_n}} |u_n|^{2^*_n} |\varphi(x-z_n)| dx
$$

$$
\leq 2(N+1)C_\delta \|u_n\|_2 \left( \int_{B_{R_{n+1}}(0)} |\varphi(x-z_n)|^2 dx \right)^{1/2}
$$

$$
+ 2^{\frac{2^*_n+1}{2^*_n}} (N+1)\delta \|u_n\|_{2^*_n}^2 \|\varphi\|_\infty
$$

$$
= o_n(1) + C_{14}\delta.
$$
By using Lemma 2.1-(2), we obtain
\[
I_6 = \sum_{i=n+1}^{+\infty} \int_{\Omega^i} |h(x, f(u_n))f'(u_n)|\varphi(x-z_n) \, dx \\
\leq \epsilon(N+1) \int_{\mathbb{R}^N} |\varphi(x-z_n)| \, dx \\
= C_{15} \epsilon.
\]

It follows from (8), Lemma 2.1-(2)(3)(7) and the H"{o}lder, Young and Sobolev inequalities that
\[
I_7 \leq \sum_{i=n+1}^{+\infty} \int_{\Omega^i} 2C\delta|u_n\varphi(x-z_n)| \, dx + \sum_{i=n+1}^{+\infty} \int_{\Omega^i} 2^{2^*+1} \delta|u_n|^{2^*} |\varphi(x-z_n)| \, dx \\
\leq 2C\delta \sum_{i=n+1}^{+\infty} |\Omega^i|^\frac{2}{2^*} \left( \int_{\Omega^i} |u_n\varphi(x-z_n)| \frac{N-2}{N} \, dx \right) \frac{N-2}{N} + C_{16} \delta \\
\leq 2C\delta \sum_{i=n+1}^{+\infty} \left[ \left( \frac{1}{2} \int_{\Omega^i} |u_n|^2 \, dx \right)^\frac{N-2}{N} + \left( \frac{1}{2} \int_{\Omega^i} |\varphi(x-z_n)|^{2^*} \, dx \right)^\frac{N-2}{N} \right] + C_{16} \delta \\
\leq 2C\delta \sum_{i=n+1}^{+\infty} \left[ \frac{1}{2} \int_{\Omega^i} |\nabla u_n|^2 + |u_n|^2 \, dx \right] \\
+ C_{16} \delta \\
= C_{17} \frac{\delta}{\epsilon} + C_{16} \delta.
\]

By using (9), Lemma 2.1-(2)(3)(7) and the H"{o}lder inequality, one has
\[
I_8 \leq \sum_{i=n+1}^{+\infty} \int_{\Omega^i} 2 \left[ \delta|f(u_n)| + \delta|f(u_n)|^{2^*-1} + C\delta|f(u_n)|^{2\alpha-1} \right] |f'(u_n)|\varphi(x-z_n) \, dx \\
\leq \sum_{i=n+1}^{+\infty} \int_{\Omega^i} \left[ 2\delta|u_n| + 2^{2^*+1} \delta|u_n|^{2^*} + C\delta 2^{\frac{\alpha+1}{\alpha}}|u_n|^{\alpha} \right] |\varphi(x-z_n)| \, dx \\
\leq 2\delta(N+1) \int_{\mathbb{R}^N} |u_n\varphi(x-z_n)| \, dx + 2^{2^*+1} \delta(N+1) \int_{\mathbb{R}^N} |u_n|^{2^*} |\varphi(x-z_n)| \, dx \\
+ 2^{\frac{\alpha+1}{\alpha}} C\delta \epsilon^{2^*-\alpha} \sum_{i=n+1}^{+\infty} \int_{\Omega^i} |u_n|^{2^*} |\varphi(x-z_n)| \, dx \\
\leq 2\delta(N+1) \|u_n\|_2 \|\varphi\|_2 + 2^{2^*+1} \delta(N+1) \|u_n\|_2 \|\varphi\|_\infty \\
+ 2^{\frac{\alpha+1}{\alpha}} C\delta \epsilon^{2^*-\alpha} \|u_n\|_2 \|\varphi\|_\infty.
Then there exist positive constants \( k \), \( \delta \), \( \epsilon \), such that
\[
|B_{n2}| \leq o_1(1) + C_{14} \delta + C_{15} \epsilon + C_{17} \epsilon^2 + C_{16} \delta + C_{18} \delta + C_{19} \epsilon^{2-\alpha}.
\]
Hence we obtain
\[
|B_{n2}| \leq o(1) + C_1 \delta + C_2 \epsilon + C_3 \epsilon^2 + C_4 \delta + C_5 \delta + C_6 \epsilon^{2-\alpha}.
\]
Let \( \epsilon \to 0 \) and then \( \delta \to 0 \), we complete the proof of (26).

As the argument in [35] (p.73, Theorem 4.2), we could obtain the following lemma which is also valid for functional \( I \).

**Lemma 3.7.** Suppose that \( (V_0) \) holds, \( g_0 \) satisfies \((g_1)-(g_4)\), then
\[
c_0 = \inf_{u \in A_0} I_0(u) = \inf_{u \in E} \max_{t>0} I_0(tu) = \inf_{\gamma \in \Gamma \in [0,1]} I_0(\gamma(t)),
\]
where \( \Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, I_0(\gamma(t)) < 0 \} \).

Next, we do some estimates and the method comes from the pioneering work [2] due to Brezis and Nirenberg. Without loss of generality, we assume that \( x_0 \) given by the condition \((K)\) is the origin of \( \mathbb{R}^N \) and that \( B_2(0) \subset \Omega \) given by the condition \((g_3)\).

Given \( \epsilon > 0 \), we consider the function \( w_\epsilon : \mathbb{R}^N \to \mathbb{R} \) defined by
\[
w_\epsilon(x) = C(N) \epsilon^{(N-2)/2} \frac{\epsilon^{(N-2)/2}}{N(N-2)}
\]
where
\[
C(N) = [N(N-2)]^{(N-2)/4}.
\]
We observe that \( \{ w_\epsilon \} \) is a family of functions on which the infimum that defines the best constant \( S \) for the Sobolev embedding \( D^{1,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \), is attained.

Let \( \phi \in C_0^\infty(\mathbb{R}^N, [0,1]) \) be a cut-off function satisfying \( \phi \equiv 1 \) in \( B_1(0) \), \( \phi \equiv 0 \) in \( \mathbb{R}^N \setminus B_2(0) \). Define
\[
u_\epsilon = \phi w_\epsilon, \quad v_\epsilon = \epsilon^{N-2} \frac{u_\epsilon}{\left( \int_{\mathbb{R}^N} K(x) u_\epsilon^{2^*} \, dx \right)^{1/2^*}}.
\]
Then there exist positive constants \( k_1, \, k_2 \) and \( \epsilon_0 \) such that
\[
\int_{\mathbb{R}^N \setminus B_1(0)} |\nabla v_\epsilon|^2 \, dx = O(\epsilon^{N-2}), \text{ as } \epsilon \to 0^+.
\]
(28)
\[
k_1 < \int_{\mathbb{R}^N} K(x) u_\epsilon^{2^*} \, dx < k_2, \text{ for all } 0 < \epsilon < \epsilon_0.
\]
(29)
\[
\int_{\{x \in \mathbb{R}^N : |x| \leq 1\}} |x|^{N-2} u_\epsilon^{2^*} \, dx = O(\epsilon^{N-2}), \text{ as } \epsilon \to 0^+.
\]
(30)
\[
\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 \, dx \leq \| K \|_{2,\mathbb{R}^N}^{2,\mathbb{R}^N} S + O(\epsilon^{N-2}), \text{ as } \epsilon \to 0^+.
\]
(31)
As \( \epsilon \to 0 \), we have
\[
\|v_\epsilon\|_2^2 = \begin{cases} O(\epsilon), & \text{if } N = 3, \\ O(\epsilon^2|\ln \epsilon|), & \text{if } N = 4, \\ O(\epsilon^2), & \text{if } N \geq 5. \end{cases}\]
(32)
\[
\|v_\epsilon\|_{2,\mathbb{R}^N}^{2^* - \frac{4}{N}} = O(\epsilon^{(N-2)/4}), \text{ if } N \geq 3.
\]
(33)
The proofs of (28)-(33) are standard, we omit them.

**Lemma 3.8.** Suppose that \( (V) \), \( (K) \), \( (g_1) \) and \( (g_2) \) hold, consider \( t_\epsilon > 0 \) such that \( I(t_\epsilon v_\epsilon) = \max_{t \geq 0} I(tv_\epsilon) \). Then, there exist \( \epsilon_0 > 0 \) and positive constants \( T_1 \) and \( T_2 \) such that \( T_1 \leq t_\epsilon \leq T_2 \) for every \( 0 < \epsilon < \epsilon_0 \).

The proof of this lemma can be found in [33], we omit it.

**Lemma 3.9.** Suppose that \( (V) \), \( (K) \), \( (g_1) \), \( (g_2) \) and \( (g_3) \) are satisfied. Then there is \( v \in H^1(\mathbb{R}^N) \) \( \setminus \{ 0 \} \) such that

\[
\max_{t \geq 0} I(tv) < \frac{1}{2N} \| K \|_{\infty}^{\frac{2N}{N-2}} S_N^2.
\]

**Proof.** We claim that there is a constant \( C_0 > 0 \) such that

\[
\frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(t_\epsilon v_\epsilon)|^{22^*} dx \geq \frac{t_\epsilon^{22^*}}{2} 2^{2^*} - C_0 \int_{\mathbb{R}^N} (t_\epsilon v_\epsilon)^{2^*} - \frac{1}{2} dx,
\]

where \( t_\epsilon \) as defined by Lemma 3.8. In fact, note that if \( 0 \leq |s| < R \), there exists \( C_1 > 0 \) such that \( |s|^2 \leq C_1 |s|^{2^* - \frac{4}{N}} \). Hence

\[
\int_{\{x : t_\epsilon v_\epsilon < R\}} (t_\epsilon v_\epsilon)^{2^*} dx \leq C_1 \int_{\{x : t_\epsilon v_\epsilon < R\}} (t_\epsilon v_\epsilon)^{2^* - \frac{1}{2}} dx 
\]

Therefore, combining with Lemma 2.1-(3), one has

\[
\frac{1}{22^*} \int_{\{x : t_\epsilon v_\epsilon < R\}} K(x) |f(t_\epsilon v_\epsilon)|^{22^*} dx \
\leq \frac{\| K \|_{\infty}}{22^*} \int_{\{x : t_\epsilon v_\epsilon < R\}} \left( |f(t_\epsilon v_\epsilon)|^{22^*} + 2^{2^*} (t_\epsilon v_\epsilon)^{2^*} \right) dx \
\leq \frac{1}{22^*} 2^{2^*} \| K \|_{\infty} \int_{\{x : t_\epsilon v_\epsilon < R\}} (t_\epsilon v_\epsilon)^{2^*} dx \\
\leq \frac{1}{22^*} 2^{2^*} \| K \|_{\infty} C_1 \int_{\{x : t_\epsilon v_\epsilon < R\}} (t_\epsilon v_\epsilon)^{2^* - \frac{1}{2}} dx.
\]

Invoking (35), Lemma 2.1-(13), and observing that \( \int_{\mathbb{R}^N} K(x)v^{2^*} dx = 1 \), we have

\[
\frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(t_\epsilon v_\epsilon)|^{22^*} dx \\
= \frac{1}{22^*} \int_{\{x : t_\epsilon v_\epsilon < R\}} K(x) |f(t_\epsilon v_\epsilon)|^{22^* - 2^{2^*}} (t_\epsilon v_\epsilon)^{2^*} dx \\
+ \frac{1}{22^*} \int_{\{x : t_\epsilon v_\epsilon \geq R\}} K(x) |f(t_\epsilon v_\epsilon)|^{22^* - 2^{2^*}} (t_\epsilon v_\epsilon)^{2^*} dx \\
+ \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) 2^{2^*} (t_\epsilon v_\epsilon)^{2^*} dx \\
\geq \frac{1}{22^*} 2^{2^*} \| K \|_{\infty} C_1 \int_{\{x : t_\epsilon v_\epsilon < R\}} (t_\epsilon v_\epsilon)^{2^* - \frac{1}{2}} dx \\
- M \| K \|_{\infty} \int_{\mathbb{R}^N} (t_\epsilon v_\epsilon)^{2^* - \frac{1}{2}} dx + \frac{t_\epsilon^{2^*}}{22^*} 2^{2^*} \| K \|_{\infty} \int_{\mathbb{R}^N} (t_\epsilon v_\epsilon)^{2^* - \frac{1}{2}} dx,
\]

where \( C_0 = \frac{1}{22^*} 2^{2^*} \| K \|_{\infty} C_1 + \frac{M \| K \|_{\infty}}{22^*} \). Then we complete the proof of (34).
In order to prove Lemma 3.9, we need to verify that

\[ l(t) := \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx - \frac{t^2}{2^*} 2^* \frac{t^2}{2^*} - 1. \]

It is very standard to get that \( l(t) \) achieves its maximum at

\[ t_0 = \frac{1}{\sqrt{2}} \left[ \int_{\mathbb{R}^N} |\nabla \psi|^2 dx \right]^{\frac{1}{2^*}} \]

and

\[ l(t_0) = \frac{1}{2N} \left[ \int_{\mathbb{R}^N} |\nabla \psi|^2 dx \right]^{\frac{N}{2}}. \quad (36) \]

It follows from Lemma 2.1-(3), (34) and (36) that

\[
I(t, \psi) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(t, \psi) dx \\
- \frac{1}{2^*} \int_{\mathbb{R}^N} K(x)|f(t, \psi)|^{2^*} dx - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx \\
\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x)|\psi|^2 dx - \frac{t^2}{2^*} 2^* \frac{t^2}{2^*} - 1 \\
+ C_0 \int_{\mathbb{R}^N} (t, \psi)^2 - \frac{1}{2} dx - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx \\
= \frac{1}{2N} \left[ \int_{\mathbb{R}^N} |\nabla \psi|^2 dx \right]^{\frac{N}{2}} + C_2 \|\psi\|_2^2 + C_3 \|\psi\|_2^{2^*} \frac{1}{2} - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx. 
\]

Applying (31) and the inequality

\[(b + c)^\zeta \leq b^\zeta + \zeta (b + c)^{\zeta - 1} c, \quad b, c \geq 0, \quad \zeta \geq 1,\]

we have

\[
I(t, \psi) \leq \frac{1}{2N} \left[ \|K\|_{2^{-N}} S^{2^{-N}} + \mathcal{O}(\epsilon^{N-2}) \right]^{\frac{N}{2}} + C_2 \|\psi\|_2^2 + C_3 \|\psi\|_2^{2^*} \frac{1}{2} - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx \\
\leq \frac{1}{2N} \|K\|_{2^{-N}} S^{2^{-N}} + C_2 \|\psi\|_2^2 + C_3 \|\psi\|_2^{2^*} \frac{1}{2} - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx + \mathcal{O}(\epsilon^{N-2}). 
\]

Next, we analyze four cases.

Case (1): \( N = 3 \). By using (32) and (33), we have

\[
I(t, \psi) \leq \frac{1}{2N} \|K\|_{2^{-N}} S^{2^{-N}} + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^{\frac{N-2}{2}}) - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx + \mathcal{O}(\epsilon^{N-2}) \\
= \frac{1}{2N} \|K\|_{2^{-N}} S^{2^{-N}} + \epsilon^{\frac{N-2}{2}} \left( C_4 - \int_{\mathbb{R}^N} G(x, f(t, \psi)) dx \right). 
\]

In order to prove Lemma 3.9, we need to verify that

\[
\lim_{\epsilon \to 0^+} \frac{\int_{\mathbb{R}^N} G(x, f(t_0, \psi)) dx}{\epsilon^{\frac{N-2}{2}}} > C_4. \quad (37) 
\]
Using (10), one has
\[ G(x, s) \geq A_0 s^{22^r - 1}. \] (38)
Consider the function \( \eta_\epsilon : [0, \infty) \to \mathbb{R} \) defined by
\[ \eta_\epsilon(r) = \frac{\epsilon^{(N-2)/2}}{(\epsilon^2 + r^2)^{(N-2)/2}}. \]
Since \( \phi \equiv 1 \) in \( B_1(0) \), in view of (29), we find a constant \( C_5 > 0 \) such that \( v_r(x) \geq C_5 \eta_\epsilon(|x|) \) for \( |x| < 1 \). Notice that \( \eta_\epsilon \) is decreasing and \( f \) is increasing, there is a positive constant \( \alpha \) such that, for \( |x| < \epsilon \),
\[ f(t \epsilon v_r(x)) \geq f(T_1 C_5 \eta_\epsilon(|x|)) \geq f(T_1 C_5 \eta_\epsilon(\epsilon)) \geq f(\alpha \epsilon^{\frac{2 - N}{2}}), \]
where \( T_1 \) is given by Lemma 3.8. Then we can choose \( \epsilon_1 > 0 \) such that
\[ \alpha \epsilon^{\frac{2 - N}{2}} \geq 1, \quad f(t \epsilon v_r(x)) \geq f(\alpha \epsilon^{\frac{2 - N}{2}}) \geq R, \] (39)
for \( |x| < \epsilon, 0 < \epsilon < \epsilon_1 \). It follows from \( B_\epsilon(0) \subset \Omega \), (38), (39) and Lemma 2.1-(9)
that, for any \( x \in B_\epsilon(0) \) and \( 0 < \epsilon < \epsilon_1 \),
\[ G(x, f(t \epsilon v_r)) \geq A_0 |f(t \epsilon v_r)|^{22^r - 1} \geq A_0 f^{22^r - 1}(\alpha \epsilon^{\frac{2 - N}{2}}) \]
\[ \geq A_0 C^{22^r - 1} \alpha^{\frac{22^r - 1}{2}} \epsilon^{\frac{(2 - N)(22^r - 1)}{4}} \]
\[ \geq A_0 C^{22^r - 1} \alpha^{\frac{22^r - 1}{2}} \epsilon^{\frac{(22^r - 1)}{4}}. \] (40)
Using (10), one has
\[ G(x, s) + s^2 \geq 0, \quad \forall x \in \Omega, \ s \geq 0. \] (41)
Since \( B_2(0) \subset \Omega \), by (40), (41) and Lemma 2.1-(3), for \( 0 < \epsilon < \epsilon_1 \), we have
\[ \int_{\mathbb{R}^N} G(x, f(t \epsilon v_r)) dx = \int_{B_2(0)} G(x, f(t \epsilon v_r)) dx + \int_{\Omega \setminus B_2(0)} G(x, f(t \epsilon v_r)) dx \]
\[ \geq A_0 C^{22^r - 1} \alpha^{\frac{22^r - 1}{2}} \epsilon^{\frac{N - 2}{N}} |B_\epsilon(0)| - \int_{\Omega \setminus B_2(0)} f^2(t \epsilon v_r) dx \]
\[ \geq A_0 C^{22^r - 1} \alpha^{\frac{22^r - 1}{2}} \epsilon^{\frac{N - 2}{N}} \omega_N - T_{2}^2 \|v_r\|_2^2, \] (42)
where \( T_2 \) is given by Lemma 3.8. It follows from (32) and (42) that
\[ \frac{\int_{\mathbb{R}^N} G(x, f(t \epsilon v_r)) dx}{\epsilon^{\frac{N - 2}{2}}} \geq A_0 C^{22^r - 1} \alpha^{\frac{22^r - 1}{2}} \omega_N - O(\epsilon^{3/4}). \] (43)
Choosing \( A_0 > 0 \) sufficiently large, (43) establishes (37).

Case (2): \( N = 4 \). It follows from (32) and (33) that
\[ I(t \epsilon v_r) \leq \frac{1}{2N} \|K\|_{\mathbb{R}^N}^{2N} S^N + O(\epsilon^2 |\ln \epsilon|) + O(\epsilon^{N - 2}) \]
\[ - \int_{\mathbb{R}^N} G(x, f(t \epsilon v_r)) dx + O(\epsilon^{N - 2}) \]
\[ = \frac{1}{2N} \|K\|_{\mathbb{R}^N}^{2N} S^N + \epsilon^{\frac{N - 2}{4}} \left(C_5 - \frac{\int_{\mathbb{R}^N} G(x, f(t \epsilon v_r)) dx}{\epsilon^{\frac{N - 2}{2}}} \right). \]
Proof of Theorem 1.1. Firstly, we invoke Lemma 3.2 to find a (C) sequence on level $c$.

Then we can get the conclusion by choosing $A_0$ large enough.

Case (3): $5 \leq N < 10$. Thanks to (32) and (33), we have

$$I(t_{\epsilon}v_{\epsilon}) \leq \frac{1}{2N}\|K\|_{\infty}^{\frac{2-N}{N}}S^\frac{N}{2} + O(\epsilon^2) + O(\epsilon^{N-2})$$

By (ii) of (g5), (39), and Lemma 2.1-(9), one has

$$G(x, f(t_{\epsilon}v_{\epsilon})) \geq A_0 f^4(t_{\epsilon}v_{\epsilon}) \geq A_0 f^4(\alpha \epsilon^{\frac{2-N}{2}}) \geq A_0 C^4 \alpha^2 \epsilon^{2-N},$$

for $x \in B_{\epsilon}(0)$ and $0 < \epsilon < \epsilon_1$. Hence, similar to the proof of (42), we have

$$\int_{\mathbb{R}^N} G(x, f(t_{\epsilon}v_{\epsilon})) dx \geq A_0 C^4 \alpha^2 \epsilon^2 \omega_N - T_2 \|v_{\epsilon}\|^2_2,$$

where $0 < \epsilon < \epsilon_1$. Consequently, one gets

$$\frac{\int_{\mathbb{R}^N} G(x, f(t_{\epsilon}v_{\epsilon})) dx}{\epsilon^2} \geq A_0 C^4 \alpha^2 \omega_N - O(1).$$

Let $A_0$ large enough, we obtain

$$I(t_{\epsilon}v_{\epsilon}) < \frac{1}{2N}\|K\|_{\infty}^{\frac{2-N}{N}}S^\frac{N}{2}.$$

Remark 2. From Lemma 3.9, we can get that $c < \frac{1}{2N}\|K\|_{\infty}^{\frac{2-N}{N}}S^\frac{N}{2}$.

Proof of Theorem 1.1. Firstly, we invoke Lemma 3.2 to find a (C) sequence on level $c$, that is $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)|I'(u_n)| \to 0$ with $c = \inf_{u \in E} I(u)$. Applying Lemma 3.3, we may get, up to a subsequence, $u_n \rightharpoonup u$ in $E$, $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, one has

$$0 = \langle I'(u_n), \phi \rangle + o_n(1) = \langle I'(u), \phi \rangle,$$
i.e. \( u \) is a weak solution of problem (3). The proof of this result is standard, so we omit it here.

(i) The case \( u \neq 0 \).

Firstly, by Lemma 2.1-(6), we have

\[
\frac{1}{2} \int_{\mathbb{R}^N} K(x)|f(u)|^{2^*-2} f(u)f'(u)udx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x)|f(u)|^{22^*} dx
\geq \frac{1}{2N} \int_{\mathbb{R}^N} K(x)|f(u)|^{22^*} \geq 0.
\]

Since \( u \) is a weak solution of problem (3), \( I(u) \geq c \). By Lemma 2.1-(8), (5), (44) and the Fatou lemma, one has

\[
c = \liminf_{n \to \infty} \left( I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle \right)
= \liminf_{n \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(u_n)dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x)f(u_n)f'(u_n)u_n dx \right.
+ \frac{1}{2} \int_{\mathbb{R}^N} g(x, f(u_n))f'(u_n)u_n dx
- \int_{\mathbb{R}^N} G(x, f(u_n))dx
\left. + \frac{1}{2} \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*-2} f(u_n)f'(u_n)u_n dx - \frac{1}{22^*} \int_{\mathbb{R}^N} K(x)|f(u_n)|^{22^*} dx \right]
\geq \frac{1}{2} \int_{\mathbb{R}^N} V(x)|f(u)|^{2} dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x)f(u)f'(u)udx
+ \frac{1}{2} \int_{\mathbb{R}^N} g(x, f(u))f'(u)u dx - \int_{\mathbb{R}^N} G(x, f(u))dx
+ \frac{1}{2} \int_{\mathbb{R}^N} K(x)|f(u)|^{22^*-2} f(u)f'(u)u dx
- \frac{1}{22^*} \int_{\mathbb{R}^N} K(x)|f(u)|^{22^*} dx
= I(u) - \frac{1}{2} \langle I'(u), u \rangle
= I(u)
\]

Hereafter \( I(u) = c \) and \( I'(u) = 0 \), which implies that \( u \) is a ground state solution of problem (3).

(ii) The case \( u = 0 \). Define

\[
\beta := \limsup_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} u_n^2 dx.
\]

If \( \beta = 0 \), by the Lions lemma [35], we get \( u_n \to 0 \) in \( L^\alpha(\mathbb{R}^N) \) for \( \alpha \in (2, 2^*) \). It follows from Lemma 2.3 that

\[
c + o_n(1) = I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle
= \int_{\mathbb{R}^N} V(x) \left( f^2(u_n) - f(u_n)f'(u_n)u_n \right) dx
+ \frac{1}{2} \int_{\mathbb{R}^N} g(x, f(u_n))f'(u_n)u_n dx
- \int_{\mathbb{R}^N} G(x, f(u_n))dx
+ \int_{\mathbb{R}^N} K(x) \left( \frac{1}{2}\left| f(u_n) \right|^{22^*-2} f(u_n)f'(u_n)u_n - \frac{1}{22^*} \left| f(u_n) \right|^{22^*} \left| u_n \right|^{22^*} \right) dx
+ \int_{\mathbb{R}^N} K(x) \frac{1}{2} 2^{2^*-2} \left| u_n \right|^{22^*} dx
- \int_{\mathbb{R}^N} K(x)2^{2\beta} \left| u_n \right|^{2\beta} dx
\]
+ \int_{\mathbb{R}^N} K(x) \left( 2^{\frac{2^*}{2^* - 2}} |u_n|^{2^*} - |f(u_n)|^{2^*} \right) dx
\]
\[= \int_{\mathbb{R}^N} K(x) |u_n|^{2^*} \left[ \frac{1}{2} 2^{\frac{2^*}{2^* - 2}} - \frac{1}{2} 2^{\frac{2^*}{2^* - 2}} \right] dx + o_n(1)
\]
\[= 2^{\frac{2^*}{2^* - 2}} \frac{1}{N} \int_{\mathbb{R}^N} K(x) |u_n|^{2^*} dx + o_n(1).
\]

Hence
\[\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{2^*} dx = 2^{-\frac{2^*}{2^* - 2}} Nc. \tag{45}\]

Thanks to (14), we have
\[\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) |f(u_n)|^{2^* - 2} f(u_n) f'(u_n) u_n dx = Nc. \tag{46}\]

By using Lemma 2.1-(8)(6)(4), one has
\[0 \leq f^2(t) - f(t) f'(t) t \leq f^2(t) - \frac{1}{2} f^2(t) \to 0(t \to 0). \tag{47}\]

It follows from (47) and (15) that
\[o_n(1) = \langle f'(u_n), u_n \rangle = \|u_n\|^2 + \int_{\mathbb{R}^N} V(x) [f(u_n) f'(u_n) u_n - u_n^2] dx
\]
\[- \int_{\mathbb{R}^N} K(x) [f(u_n)]^{2^* - 2} f(u_n) f'(u_n) u_n dx - \int_{\mathbb{R}^N} g(x, f(u_n)) f'(u_n) u_n dx
\]
\[= \|u_n\|^2 - \int_{\mathbb{R}^N} K(x) [f(u_n)]^{2^* - 2} f(u_n) f'(u_n) u_n dx + o_n(1).
\]

Combining with (46), we obtain
\[\lim_{n \to \infty} \|u_n\|^2 = Nc. \tag{48}\]

By (45), (48) and the Sobolev inequality, one has
\[2^{-\frac{2^*}{2^* - 2}} Nc = \lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{2^*} dx \leq \|K\|_{\infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx
\]
\[\leq \|K\|_{\infty} S^{-\frac{2^*}{2^* - 2}} \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{\frac{2^*}{2}} \leq \|K\|_{\infty} S^{-\frac{2^*}{2^* - 2}} \lim_{n \to \infty} \|u_n\|^{2^*}
\]
\[= \|K\|_{\infty} S^{-\frac{2^*}{2^* - 2}} (Nc)^{\frac{2^*}{2}}.
\]

Therefore
\[c \geq \frac{1}{2N} \|K\|_{\infty}^{\frac{2^*}{2^* - 2}} S^{\frac{2^*}{2^* - 2}},
\]

which is a contradiction with Remark 2, thus \(\beta > 0\). By the definition of \(\beta\), there exist \(R > 0\) and \(z_n \in \mathbb{Z}^N\) such that
\[\int_{B_R(0)} u_n^2(x + z_n) dx = \int_{B_R(z_n)} u_n^2(x) dx > \frac{\beta}{2}.
\]

If \(z_n\) is bounded, there exists \(R' > 0\) such that
\[\int_{B_{R'}(0)} u_n^2 dx \geq \int_{B_R(z_n)} u_n^2 dx > \frac{\beta}{2}.
\]
which is a contradiction with \( u_n \to u = 0 \) in \( L^2_{loc}(\mathbb{R}^N) \). Thus \( z_n \) is unbounded, going if necessary to a subsequence, \( |z_n| \to \infty \). Let \( w_n(x) := u_n(x + z_n) \), then there is a function \( w \in E \) such that \( w_n \rightharpoonup w \) in \( E \), \( u_n \to w \) in \( L^2_{loc}(\mathbb{R}^N) \) and \( w_n(x) \to w(x) \) a.e. in \( \mathbb{R}^N \).

For any \( \varphi \in C^0_0(\mathbb{R}^N) \), it follows from (25) and (26) that

\[
0 = (I'(u_n), \varphi(x-z_n)) + o(1)
\]

\[
= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x-z_n)dx + \int_{\mathbb{R}^N} V(x)f(u_n)f'(u_n)\varphi(x-z_n)dx
\]

\[
- \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*}-2f(u_n)f'(u_n)\varphi(x-z_n)dx
\]

\[
- \int_{\mathbb{R}^N} g(x,f(u_n))f'(u_n)\varphi(x-z_n)dx + o_n(1)
\]

\[
= \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \varphi(x-z_n)dx + \int_{\mathbb{R}^N} V_0(x)f(w_n)f'(w_n)\varphi(x-z_n)dx
\]

\[
- \int_{\mathbb{R}^N} K(x)|f(w_n)|^{2^*}-2f(w_n)f'(w_n)\varphi(x-z_n)dx
\]

\[
- \int_{\mathbb{R}^N} g_0(x,f(w_n))f'(w_n)\varphi(x-z_n)dx + o_n(1)
\]

i.e. \( w \) is a weak solution of equation (4).

On the one hand, by Lemma 3.1, for any \( u \in E \), \( u \neq 0 \), there is a unique \( t_u > 0 \) such that \( t_u u \in \mathcal{N} \). Moreover, the maximum of \( I(tu) \) for \( t \geq 0 \) is achieved at \( t_u \). Combining with \( V(x) \leq V_0(x) \) and \( G(x,s) \geq G_0(x,s) \), we obtain

\[
c \leq I(t_u u) \leq I_0(t_u) \leq \max_{t>0} I_0(tu),
\]

hence \( c \leq \inf_{u \in E} \max_{t>0} I_0(tu) \). It follows from Lemma 3.7 that \( c \leq c_0 \).

On the other hand, by (22), (23), \( V(x) \leq V_0(x) \), \( g(x,s) \geq g_0(x,s) \), Lemma 2.1-(8), (5), (44) and the Fatou lemma, we obtain

\[
c = I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle + o_n(1)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(u_n)dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x)f'(u_n)f(u_n)u_n dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^N} g(x,f(u_n))f'(u_n)u_n dx - \int_{\mathbb{R}^N} G(x,f(u_n))dx + o_n(1)
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*}-2f(u_n)f'(u_n)u_n dx - \frac{1}{2^*} \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*} dx
\]

\[
\geq \int_{\mathbb{R}^N} \frac{1}{2} V_0(x)f^2(u_n)dx - \int_{\mathbb{R}^N} \frac{1}{2} V_0(x)f'(u_n)f(u_n)u_n dx
\]
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REFERENCES
[1] S. Adachi and T. Watanabe, Uniqueness of the ground state solutions of quasilinear
Schrödinger equations, Nonlinear Anal., 75 (2012), 819–833.
[2] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical
Sobolev exponents, Comm. Pure Appl. Math., 36 (1983), 437–477.
[3] M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: a dual approach,
Nonlinear Anal., 56 (2004), 213–226.
[4] M. Colin, L. Jeanjean and M. Squassina, Stability and instability results for standing waves
of quasilinear Schrödinger equations, Nonlinearity, 23 (2010), 1353–1385.
[5] Y. B. Deng, S. J. Peng and J. Wang, Nodal soliton solutions for quasilinear Schrödinger
equations with critical exponent, Journal of Mathematical Physics, 54 (2013), 011504.
[6] Y. B. Deng, S. J. Peng and S. S. Yan, Critical exponents and solitary wave solutions for
generalized quasilinear Schrödinger equations, Journal of Differential Equations, 260 (2016),
1228–1262.
[7] J. M. do Ó and U. Severo, Quasilinear Schrödinger equations involving concave and convex
nonlinearities, Commun. Pure Appl. Anal., 8 (2009), 621–644.
[8] J.M. do Ó, O. H. Miyagaki and S. H. M. Soares, Soliton solutions for quasilinear Schrödinger equations with critical growth, Journal of Differential Equations, 248 (2010), 722–744.

[9] X. D. Fang and A. Szulkin, Multiple solutions for a quasilinear Schrödinger equation, J. Differential Equations, 254 (2013), 2015–2032.

[10] F. Gladiali and M. Squassina, Uniqueness of ground states for a class of quasi-linear elliptic equations, Adv. Nonlinear Anal., 1 (2012), 159–179.

[11] Y. He and G. B. Li, Concentrating soliton solutions for quasilinear Schrödinger equations involving critical Sobolev exponents, Discrete and Continuous Dynamical Systems, 36 (2016), 731–762.

[12] L. Jeanjean, T. J. Luo and Z. Q. Wang, Multiple normalized solutions for quasi-linear Schrödinger equations, J. Differential Equations, 259 (2015), 3894–3928.

[13] G. B. Li and A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, Commun. Contemp. Math., 4 (2002), 763–776.

[14] H. F. Lins and E. A. B. Silva, Quasilinear asymptotically periodic elliptic equations with critical growth, Nonlinear Anal., 71 (2009), 2980–2995.

[15] P. L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case, I, Ann. Inst. H. Poincare Anal. Non Lineaire, 1 (1984), 223–283.

[16] P. L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case, II, Ann. Inst. H. Poincare Anal. Non Lineaire, 1 (1984), 109–145.

[17] J. Liu, J. F. Liao and C. L. Tang, A positive ground state solution for a class of asymptotically periodic Schrödinger equations, Comput. Math. Appl., 71 (2016), 965–976.

[18] J. Liu, J. F. Liao and C. L. Tang, A positive ground state solution for a class of asymptotically periodic Schrödinger equations with critical exponent, Comput. Math. Appl., 72 (2016), 1851–1864.

[19] J. Q. Liu, X. Q. Liu and Z. Q. Wang, Multiple sign-changing solutions for quasilinear elliptic equations via perturbation method, Comm. Partial Differential Equations, 39 (2014), 2216–2239.

[20] J. Q. Liu, Y. Q. Wang and Z. Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, Comm. Partial Differential Equations, 29 (2004), 879–901.

[21] J. Q. Liu, Y. Q. Wang and Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations II, J. Differential Equations, 187 (2003), 473–493.

[22] J. Q. Liu and Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations, I, Proc. Amer. Math. Soc., 131 (2003), 441–448.

[23] X. Q. Liu, J. Q. Liu and Z. Q. Wang, Quasilinear elliptic equations via perturbation method, Proc. Amer. Math. Soc., 141 (2013), 253–263.

[24] X. Q. Liu, J. Q. Liu and Z. Q. Wang, Quasilinear elliptic equations with critical growth via perturbation method, Journal of Differential Equations, 254 (2013), 102–124.

[25] X. Q. Liu, J. Q. Liu and Z. Q. Wang, Ground states for quasilinear Schrödinger equations with critical growth, Calculus of Variations and Partial Differential Equations, 46 (2013), 641–669.

[26] R. D. Marchi, Schrödinger equations with asymptotically periodic terms, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 145 (2015), 745–757.

[27] A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$, Journal of Differential Equations, 229 (2006), 570–587.

[28] M. Poppenberg, K. Schmitt and Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, Calc. Var. Partial Differential Equations, 14 (2002), 329–344.

[29] D. Ruiz and G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equation, Nonlinearity, 23 (2010), 1221–1233.

[30] A. Selvitella, Uniqueness and nondegeneracy of the ground state for a quasilinear Schrödinger equation with a small parameter, Nonlinear Anal., 74 (2011), 1731–1737.

[31] H. X. Shi and H. B. Chen, Generalized quasilinear asymptotically periodic Schrödinger equations with critical growth, Comput. Math. Appl., 71 (2016), 849–858.

[32] E. A. B. Silva and G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with subcritical growth, Nonlinear Anal., 72 (2010), 2935–2949.

[33] E. A. B. Silva and G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, Calc. Var. Partial Differential Equations, 39 (2010), 1–33.

[34] X. H. Tang, Non-Nehari manifold method for asymptotically periodic Schrödinger equations, Sci. China Math., 58 (2015), 715–728.

[35] M. Willem, Minimax Theorems, Birkhäuser Boston, 1996.
[36] X. Wu, Multiple solutions for quasilinear Schrödinger equations with a parameter, *J. Differential Equations*, **256** (2014), 2619–2632.

[37] H. Zhang, J. X. Xu and F. B. Zhang, On a class of semilinear Schrödinger equations with indefinite linear part, *J. Math. Anal. Appl.*, **414** (2014), 710–724.

[38] H. Zhang, J. X. Xu and F. B. Zhang, Ground state solutions for asymptotically periodic Schrödinger equations with indefinite linear part, *Math. Methods Appl. Sci.*, **38** (2015), 113–122.

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