Numerical comparisons based on new NCP-functions

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Abstract. We consider studying numerical comparisons based on variants of new NCP-functions denoted by $\phi^p_k$. The nonlinear programming (NLP) can be converted to nonlinear complementarity problem (NCP) by employing the Karush-Kuhn Tucker (KKT) optimality conditions. One of the most popular ways to solve NCP is Lagrangian globalization (LG) method by transforming NCP as a system of nonsmooth (semismooth) equations. The second one is a novel method named the fictitious time integration method (FTIM). We reformulate NCP as a system of nonlinear algebraic equations (NAEs) and then construct an ordinary differential equation (ODE) by utilizing time-like functions. A group preserving scheme (GPS) is a set of ODEs which is a tool for systematically reformulating into the nonlinear dynamical system (NDS). Afterward, the NDS can be manipulated to numerical equations through activating the Lorentz group $SO_0(n,1)$ and its Lie algebra $so(n,1)$. The FTIM will be operated on this numerical equation in numerical simulations for getting approximation solutions. All of the numerical experiments are carried out in performance profile theories. The comparisons of new NCP-functions by utilizing FTIM and LG method will be discussed in numerical experiments. Lastly, an accurate test of both FTIM and LG method is going to be committed by performance profile concepts as well.

1. Motivation and Introduction

We investigate deeply the interdisciplinary field of optimization as the main subject of this study. This part will be an overview of the main idea of this research started by the fundamental parts comprising of motivation and introduction.

1.1. Motivation

One of the major topics to be investigated in this field is the fictitious time integration method (FTIM) as an alternative way to solve the nonlinear complementarity problem (NCP) which has have better approximation results than others. This research constitutes a relatively new area which has emerged from one of their studies [1]; however, they used an FTIM for solving a plentiful problem in engineering outlook. Other than that their research mainly exploited an FTIM to accomplish nonlinear algebraic equations (NAEs), to be appeared in [14, 18, 19]. Many of the researchers clarified an FTIM for modeling in engineering combined with computer science [20, 21, 22, 15]. The NCP has gotten much applications in economics, engineering and a variety of equilibrium models [7, 10, 26]. All of the numerical methods for solving
NCPs have been developed, see [3, 5, 8, 13, 25, 30, 27, 1]. To enrich the methods for resolving the NCP, we started thinking novel numerical methods and one of them is an FTIM. Nothing researcher developed an FTIM in mathematics angle and most of the researchers will be more comfortable to use this method in either engineering or computing. Because of the above reasons, a study of an FTIM in mathematics viewpoints could have been delightful to be deemed.

1.2. Introduction

The nonlinear programming (NLP for concisely) can be defined by

\[
\begin{align*}
\min & \quad f(x) \\
\text{subject to} & \quad -g_i(x) \leq 0, [i = 1, 2, \ldots, l], \\
& \quad h_i(x) = 0, [i = 1, 2, \ldots, k], \\
& \quad x \in X,
\end{align*}
\]

where \( X \) be a nonempty open set in \( \mathbb{R}^n \), and let \( f : \mathbb{R}^n \to \mathbb{R}, g_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, \ldots, l \), and \( h_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, \ldots, k \), are differentiable functions. The Lagrange function is given by

\[
L(x, \lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_k) = f(x) - \sum_{i=1}^{l} \lambda_i g_i(x) + \sum_{i=1}^{k} \mu_i h_i(x)
\]

under some constraint qualifications (e.g. linear independence constraint qualification (LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ)), there exist \( \lambda \) and \( \mu \) such that the Karush-Kuhn Tucker (KKT) optimality conditions of NLP are described as follows

- **optimality condition:** \( \nabla_x L(x, \lambda, \mu) = 0 \),
- **feasible condition:** \( -g_i(x) \leq 0, h_i(x) = 0 \),
- **multiplier condition:** \( \lambda_i \geq 0 \),
- **complementarity condition:** \( \langle \lambda_i, g_i(x) \rangle = 0 \).

Of the above system, these constraints

\[
\lambda_i \geq 0, \quad g_i(x) \geq 0, \quad \langle \lambda_i, g_i(x) \rangle = 0 \iff \phi_k(\lambda_i, g_i(x)) = 0
\]

consist of a NCP. The NCP aims to find a point \( x \in \mathbb{R}^n \) such that

\[
x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product and \( F := (F_1, \ldots, F_n)^T \) is a mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Presently, there exist five primary means for solving these NCPs, which are the merit function approach, nonsmooth approach, smoothing function approach, regularization approach, and neural networks approach. Focusing on this topic, plenty of NCPs have been investigated [28, 2, 12, 11, 29, 5, 17]. Of [2, 11, 24, 23], a variety
of definitions for new NCP-functions $\phi^p_k$, $[k = 1, 2, 3, 4, 5; \ p > 1]$, such as

$$
\begin{align*}
\phi_1(a, b) & := \sqrt{|a|^p + |b|^p} - (a + b), \\
\phi_2(a, b) & := \left(\sqrt{a^2 + b^2}\right)^p - (a + b)^p, \\
\phi_3(a, b) & := \frac{1}{2^p} \left[(a + b)^p - |a - b|^p\right], \\
\phi_4(a, b) & := \begin{cases} 
  a^p - (a - b)^p, & \text{if } a > b, \\
  a^p = b^p, & \text{if } a = b, \\
  b^p - (b - a)^p, & \text{if } a < b,
\end{cases} \\
\phi_5(a, b) & := \begin{cases} 
  a^pb^p - (a - b)^pa^p, & \text{if } a > b, \\
  a^pb^p = a^2p^2, & \text{if } a = b, \\
  a^pb^p - (b - a)^pa^p, & \text{if } a < b.
\end{cases}
\end{align*}
$$

One of the contributions of this paper is a great advantage to easily extend to higher dimensional nonlinear optimization problems (NOPs) with nonlinear equality and inequality constraints. This paper will offer the FTIM as an alternative simulation better in NOPs. The organizations of this paper consist of, in Sect. 1, we review a renowned modification for getting NCP based on NLP terminated by introducing some new NCP-functions. In Sect. 2, the basis of this paper elucidates both the fictitious time function and the Lorentz group $SO_{0}(n, 1)$ including its Lie algebra $\mathfrak{so}(n, 1)$; connected by transforming NAEs until getting hold of the FTIM numerical equations. In Sect. 3, we propose a descent algorithm for solving the unconstrained nonsmooth optimization problem and investigate $P^*_n$ is well-defined and globally convergent. Aside from those, an overview of reformulating NCP as a system of semismooth equations can be seen in this part. In Sect. 4, the performance profiles proposed by [6] will be used in this section for surveys of either numerical methods or new NCP-functions. In the end, all of the primary summaries of this paper will be explained in Sect. 5.

2. FTIM for NCP

The fictitious time function summarized from [16] is

$$
q(\tau) = (1 + \tau)^\gamma, \\
q(0) = 1, \ q(\infty) = \infty, \ [0 \leq \gamma \leq 1],
$$

where $\tau$ is a fictitious time variable. Another one is the Lorentz group $SO_{0}(n, 1)$ together with its Lie algebra $\mathfrak{so}(n, 1)$ adapted from [9]. The Lorentz group $SO_{0}(n, 1)$ is the proper orthochronous Lorentz group and its Lie algebra is $\mathfrak{so}(n, 1)$. Both of them can be defined as below

$$
SO_{0}(n, 1) = \{Y = (y_{ij}) \in SO(n, 1)|y_{n+1,n+1} > 0 \text{ for } y_{i=1,\ldots,n+1 \ j=1,\ldots,n+1} \in \mathbb{R}\}, \\
\mathfrak{so}(n, 1) = \{Y \in M_{(n+1)\times(n+1)}(\mathbb{R})|Y^T \mathbf{J} + \mathbf{J} Y = \mathbf{0}\},
$$

where

$$
SO(n, 1) = \{G \in GL(n + 1, \mathbb{R})|G^T \mathbf{J} G = \mathbf{J} \text{ and } \det G = 1\}, \\
GL(n + 1, \mathbb{R}) = \{G \in M_{(n+1)\times(n+1)}(\mathbb{R})|G \text{ invertible}\}, \\
M_{(n+1)\times(n+1)}(\mathbb{R}) = \left\{ \begin{bmatrix} B & u^T \\ u & 0 \end{bmatrix} | u \in \mathbb{R}^n, \ B^T = -B \right\}.
$$
We consider a system of NAEs below
\[ F_i(x_j) = 0, \ [i, j = 1, ..., n]. \]  
(3)

By employing (2), we have
\[ y_i(\tau) = (1 + \tau)^\gamma x_j, \]  
(4)

where \( \tau \) is a variable which is independent of \( x_j \). The derivative of (4) over \( \tau \) can be obtained by
\[ \dot{y}_i = \frac{dy_i}{d\tau} = \gamma (1 + \tau)^{\gamma - 1} x_j. \]  
(5)

If \( z \neq 0 \), then (3) is equivalent to
\[ 0 = -z F_i(x_j). \]  
(6)

We manipulate (6) as of
\[ 0 = -z F_i\left( \frac{y_i}{(1 + \tau)^\gamma} \right). \]  
Combined with (5) into
\[ \frac{\dot{y}_i}{(1 + \tau)^\gamma} = \gamma \frac{y_i}{(1 + \tau)^\gamma - 1} - z F_i\left( \frac{y_i}{(1 + \tau)^\gamma} \right) \]  
(7)

Multiplying of both sides (7) by the integrating factor \( 1/(1 + \tau)^\gamma \) to
\[ \frac{\dot{y}_i}{(1 + \tau)^\gamma} = \gamma y_i - z F_i\left( \frac{y_i}{(1 + \tau)^\gamma} \right). \]

Lastly,
\[ \frac{d}{d\tau} \left( \frac{y_i}{(1 + \tau)^\gamma} \right) = -\frac{z}{(1 + \tau)^\gamma} F_i\left( \frac{y_i}{(1 + \tau)^\gamma} \right) \]  
(8)

Rewritten (8) as a vector form such as
\[ \dot{x} = f(x, \tau) = -\frac{z}{(1 + \tau)^\gamma} F(x), \]  
(9)

where \( N = n + k + l \) is the dimensional state vector of algebraic equations. We currently will define the NAEs system of (3) as follows
\[ F_i(x_j) := \phi_k(P_i(x_j), Q_i(x_j)) \implies F_i = \phi_k(P_i, Q_i) = 0. \]
Expanded (9) as a system of ODEs i.e.

\[ Q_1: \dot{x}_i = -\frac{z_1}{1 + \tau} F_1(x_i), \quad [x_i \in \mathbb{R}^n], \]
\[ Q_2: \mu_i = -\frac{z_2}{1 + \tau} F_2(\mu_i), \quad [\mu_i \in \mathbb{R}^k], \]
\[ H_k: \lambda_i = -\frac{z_3}{1 + \tau} F_3(\lambda_i), \quad [\lambda_i \in \mathbb{R}]. \]

The different coefficients of \(z_1, z_2, \) and \(z_3\) can be used to enhance the stability of numerical integrations of their equations. More far away,

\[ Q_1 = -\frac{z_1}{1 + \tau} \left[ \frac{\partial f}{\partial x_i} + \sum_{j=1}^{k} \mu_j \frac{\partial h_j}{\partial x_i} - \sum_{j=1}^{t} \lambda_j \frac{\partial g_j}{\partial x_i} \right], \]
\[ Q_2 = -\frac{z_2}{1 + \tau} h_i, \]
\[ H_k = -\frac{z_3}{1 + \tau} \phi_k(\lambda_i, g_i). \]

Is (10) combined, the nonlinear dynamical system (NDS) can be gotten as following

\[ \dot{x} = f(x, \tau) \iff \dot{x}^T = [f(x, \tau)]^T, \quad [x \in \mathbb{R}^N], \]

with \(\tau\) is a time variable, and \(f \in \mathbb{R}^N\) is vector-valued functions of \(x\) and \(\tau\). The augmented vectors are written by

\[ X := (x^T, ||x||)^T = (x, ||x||) \in \mathbb{R}^{N+1}, \]
\[ X_{r+1} = G(r)X_r. \]

Referring to [9], specifically, we are able to redefine (11) as

\[ \omega := ||\dot{x}|| = ||f_r(x, \tau)||, \quad [r \in \mathbb{N} \cup \{0\}], \]

with the result that

\[ Y(r) = \begin{bmatrix} 0 \\ f_r(x, \tau) \\ 0 \end{bmatrix} \in \mathfrak{so}(n, 1). \]

The Lie group \(G(r)\) can be generated from \(Y(r) \in \mathfrak{so}(n, 1)\) by an exponential mapping

\[ \exp: \mathfrak{so}(n, 1) \rightarrow SO_0(n, 1), \]

wherever

\[ e^{Y(r)} = \begin{bmatrix} I_n + \frac{(\cosh \omega - 1)}{\omega} f_r f_r^T & \sinh \omega / \omega f_r \\ \sinh \omega / \omega f_r & \cosh \omega \end{bmatrix}, \]
\[ = \begin{bmatrix} I_n + \frac{(\cosh ||f_r|| - 1)}{||f_r||} f_r f_r^T & \sinh ||f_r|| / ||f_r|| f_r \\ \sinh ||f_r|| / ||f_r|| f_r & \cosh ||f_r|| \end{bmatrix}. \]

We are allowed to modify (13) as

\[ e^{\Delta ||x||} Y(r) = \begin{bmatrix} I_n + \frac{(\cosh (\Delta ||x|| - 1))}{\sinh (\Delta ||x||)|f_r| ||f_r||} f_r f_r^T & \sinh (\Delta ||x||) / ||f_r|| f_r \\ \sinh (\Delta ||x||) / ||f_r|| f_r & \cosh (\Delta ||x||) \end{bmatrix}. \]
Given $\Omega := \frac{\Delta t}{\|x_r\|}$, we have

$$e^{\Omega Y(r)} = \begin{bmatrix} I_n + \frac{(\cosh(\Omega \|f_r\|) - 1) f_r f_r^T}{\|f_r\|^2} & \frac{b_r f_r}{\|f_r\|} \\ \frac{b_r f_r}{\|f_r\|} & a_r \end{bmatrix} \frac{\sinh(\Omega \|f_r\|)}{\|f_r\|} f_r^T \cosh(\Omega \|f_r\|).$$

Therefore,

$$G(r) := e^{\Omega Y(r)} = \begin{bmatrix} I_n + \frac{(a_r - 1) f_r f_r^T}{\|f_r\|^2} & \frac{b_r f_r}{\|f_r\|} \\ \frac{b_r f_r}{\|f_r\|} & a_r \end{bmatrix} \in \text{SO}_0(n, 1), \quad (14)$$

where

$$a_r := \cosh(\Omega \|f_r\|), \quad b_r := \sinh(\Omega \|f_r\|).$$

Substituting for (14) into (12) is obtained

$$\begin{bmatrix} x_{r+1} \\ \|x_{r+1}\| \end{bmatrix} = \begin{bmatrix} I_n + \frac{(a_r - 1) f_r f_r^T}{\|f_r\|^2} & \frac{b_r f_r}{\|f_r\|} \\ \frac{b_r f_r}{\|f_r\|} & a_r \end{bmatrix} \begin{bmatrix} x_r \\ \|x_r\| \end{bmatrix} = \begin{bmatrix} x_r + \frac{x_r (a_r - 1) f_r f_r^T}{\|f_r\|^2} + \frac{b_r f_r}{\|f_r\|} \|x_r\| \\ \|x_r\| \end{bmatrix}.$$  

Since known that $f_r^T = f_r$, accordingly

$$x_{r+1} = x_r + \frac{x_r (a_r - 1) f_r f_r^T}{\|f_r\|^2} + \frac{b_r f_r}{\|f_r\|} \|x_r\| = x_r + \frac{x_r (a_r - 1) f_r f_r^T}{\|f_r\|^2} + \frac{b_r f_r}{\|f_r\|} \|x_r\| = x_r + \left[ \frac{x_r (a_r - 1) f_r f_r^T}{\|f_r\|^2} + \frac{b_r f_r}{\|f_r\|} \|x_r\| \right] f_r = x_r + \left( a_r - 1 \|f_r\|^2 + b_r f_r \|x_r\| \right) f_r \quad (15)$$

whereupon

$$\eta_r := \frac{(a_r - 1) f_r \cdot x_r + b_r \|x_r\| \|f_r\|}{\|f_r\|^2}.$$  

Besides, we have possession of

$$\|x_{r+1}\| = \frac{b_r f_r^T}{\|f_r\|} x_r + a_r \|x_r\| = a_r \|x_r\| + \frac{b_r f_r}{\|f_r\|} f_r \cdot x_r. \quad (16)$$
In order to reveal an adaptive factor of \( \eta_r > 0 \), it must be started by a definition of Schwartz inequality \( f_r \cdot x_r \geq -\|f_r\|\|x_r\| \) so as

\[
(a_r - 1)f_r \cdot x_r \geq -(a_r - 1)\|f_r\|\|x_r\|
\]

\[
\Rightarrow b_r\|x_r\|\|f_r\| + (a_r - 1)f_r \cdot x_r \geq b_r\|x_r\|\|f_r\| - (a_r - 1)\|f_r\|\|x_r\|
\]

\[
\Rightarrow b_r\|x_r\|\|f_r\| + (a_r - 1)f_r \cdot x_r \geq \frac{1 - a_r + b_r\|f_r\|\|x_r\|}{\|f_r\|^2}
\]

\[
\Rightarrow \frac{b_r\|x_r\|\|f_r\| + (a_r - 1)f_r \cdot x_r}{\|f_r\|^2} \geq \frac{1 - a_r + b_r\|f_r\|\|x_r\|}{\|f_r\|^2}
\]

\[
\Rightarrow \eta_r \geq \left[1 - \cosh\left(\frac{\Delta t\|f_r\|}{\|x_r\|}\right) + \sinh\left(\frac{\Delta t\|f_r\|}{\|x_r\|}\right)\right]\|x_r\|
\]

\[
\Rightarrow \eta_r \geq \left[1 - \left(\cosh\left(\frac{\Delta t\|f_r\|}{\|x_r\|}\right) - \sinh\left(\frac{\Delta t\|f_r\|}{\|x_r\|}\right)\right)\right]\|x_r\|
\]

\[
\Rightarrow \eta_r \geq \left[1 - \exp\left(-\frac{\Delta t\|f_r\|}{\|x_r\|}\right)\right]\|x_r\| \|f_r\| > 0, \forall \Delta t > 0.
\]

This scheme preserves the group properties for all \( \Delta t > 0 \), further it is called the GPS.

3. LG method for NCP

The NCP is reformulated as an equivalent system of nonsmooth equations

\[
\Phi(x) = 0,
\]

where \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is locally Lipschitz but not differentiable and defined by

\[
\Phi(x) = \begin{bmatrix}
\phi(x_1, F_1(x)) \\
\vdots \\
\phi(x_n, F_n(x))
\end{bmatrix},
\]

with \( \phi \) is NCP-function. The LG method associates an objective function \( f \) with system (17) to produce an equality constrained optimization problem of the form

\[
\min f(x) \text{ s.t. } \Phi(x) = 0,
\]

wherever \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is assumed to be continuously differentiable. Both Lagrangian and augmented Lagrangian of (18) are respectively

\[
L(x, \lambda) := f(x) + \lambda^T \Phi(x),
\]

\[
P_c(z) := P_c(x, \lambda) := L(x, \lambda) + \frac{1}{2}\|\Phi(x)\|^2,
\]

whereupon \( \lambda \in \mathbb{R}^n \) is the Lagrange multiply vector, \( c \) is a nonnegative real parameter and \( z := (x, \lambda) \). Next, we propose a descent algorithm for solving the unconstrained nonsmooth optimization problem

\[
\min_{z \in \mathbb{R}^n} P_c(z).
\]
We presently should show that the negative of generalized gradients of $P_c$ at $z = (x, \lambda)$ is a descent direction of $P_c$ is $z$ is not a critical point of $P_c$, see [4, 27]. For any $i \in \{1, ..., n\}$, each element of $H_i \in \partial \Phi(x)^T$ and let $V \in \frac{\partial}{\partial x} P_c(x, \lambda)$ be written as follows

\[ V = \nabla f(x) + H[\lambda + c \Phi(x)], \quad [H = (H_1, ..., H_n)]. \]

Of (19), we deduce

\[ P_c(z; q) = \Phi(x)^T h + \nabla f(x)^T d + [\lambda + c \Phi(x)]^T \Phi'; \quad V^T d = \nabla f(x)^T d + [\lambda + c \Phi(x)]^T H^T d. \quad (20) \]

By utilizing (20), we have

\[ P_c(z; q) - V^T d - \Phi(x)^T h = [\lambda + c \Phi(x)]^T [\Phi'(x; d) - H^T d], \quad (21) \]

with $P_c(z)$ at $z = (x, \lambda) \in \mathbb{R}^{2n}$ in a direction $q = \left(\frac{d}{h}\right)$ and $d, h \in \mathbb{R}^n$ called the directional derivative.

**Proposition 3.1** The function $\phi_1 := \phi_{\mathbb{R}^n}$.

(i) Let

\[ \Phi^p_{\mathbb{R}^n}(x) = \begin{bmatrix} \phi^p_{\mathbb{R}^n}(x_1, F_1(x)) \\ \vdots \\ \phi^p_{\mathbb{R}^n}(x_n, F_n(x)) \end{bmatrix} \quad \text{and} \quad \begin{cases} \mathbb{T}(x) = I_1(x) \cup I_2(x) \\ I(x) = I_{0+}(x) \cup I_{+0}(x) \cup I_{00}(x) \end{cases}, \]

where

\[ \begin{cases} I_1(x) = \{i | x_i > 0, \quad F_i(x) > 0\}, \\ I_2(x) = \{i | x_i < 0, \quad F_i(x) < 0\}, \quad \text{and} \quad \begin{cases} I_{0+}(x) = \{i | x_i = 0, \quad F_i(x) > 0\}, \\ I_{+0}(x) = \{i | x_i > 0, \quad F_i(x) = 0\}, \\ I_{00}(x) = \{i | x_i = 0, \quad F_i(x) = 0\}. \end{cases} \]

(ii) For any $i \in \{1, ..., n\}$, each element of $H_i \in \partial \Phi^p_{\mathbb{R}^n}(x)^T$ and $V \in \frac{\partial}{\partial x} P_c(x, \lambda)$ can be written as follows

\[ H_i = a_i(x) E_i + b_i(x) \nabla F_i(x) \quad \text{and} \quad V = \nabla f(x) + H[\lambda + c \Phi_{\mathbb{R}^n}(x)], \]

where $E_i$ is the $i$th column of the $n \times n$ unit matrix, $H = (H_1, ..., H_n)$,

\[ (a_i(x), b_i(x)) = \begin{cases} \nabla \phi^p_{\mathbb{R}^n}(x_i, F_i(x)), \quad i \in \mathbb{T}(x), \\ (\rho_i, 0), \quad i \in I_{0+}(x), \\ (0, \rho_i), \quad i \in I_{+0}(x), \\ (\xi_i, \eta_i), \quad i \in I_{00}(x), \end{cases} \]

with $\rho_i \in [-1, 1]$, $(\xi_i, \eta_i) \in \Xi$ and $\Xi = \text{conv}\{\Xi_1 \cup \Xi_2\}$,

\[ \begin{cases} \Xi_1 = \{(1 - \xi, 1 - \eta) | \xi \geq 0 \text{ and } \eta \geq 0, \quad |\xi|^\frac{p-1}{2} + |\eta|^\frac{p-1}{2} \leq 1\}, \\ \Xi_2 = \{(\xi - 1, \eta - 1) | \xi \leq 0 \text{ or } \eta \leq 0, \quad |\xi|^\frac{p-1}{2} + |\eta|^\frac{p-1}{2} \leq 1\}, \end{cases} \]

where $\xi = \frac{\text{sgn}(a) |a|^{p-1}}{\|a\|_p^{p-1}}$ and $\eta = \frac{\text{sgn}(b) |b|^{p-1}}{\|a\|_p^{p-1}}$. 
(iii) The directional derivative of $\Phi_{|\varphi_n|}^p(x, F_i(x))$ at $x$ in the direction $d$ is given by

$$
\Phi_{|\varphi_n|}^p(x; d) = \begin{cases} 
H_i^T d, & i \in T(x), \\
|d_i|, & i \in I_{0+}(x), \\
|\nabla F_i(x)^T d|, & i \in I_{+0}(x), \\
\phi_{|\varphi_n|}^p(d_i, \nabla F_i(x)^T d), & i \in I_{00}(x).
\end{cases}
$$

(iv) For any $(s, t) \in \Omega'$,

$$
\phi_{|\varphi_n|}^p(a, b) - (as + bt) \geq 0,
$$

where

$$
\Omega' = \{(s, t) | -1 \leq s \leq 0, -1 \leq t \leq 0\} \cup \{((s, t) | \|(s + 1, t + 1)\|_q \leq 1\}, \\
\left[ p, q \in [1, \infty] \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \right].
$$

**Theorem 3.1** Assumed that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and $\lambda$ is non-positive. Let

$$
q = \begin{bmatrix} -V \\ -\phi_{|\varphi_n|}^p(x) \end{bmatrix},
$$

where $V \in \partial P_c(x, \lambda)$, with $\rho_i \in [-1, 1]$ and $(\xi_i, \eta_i) \in \Omega'$. Then, $P_c'(z; q) < 0$ i.e. $q$ is a descent direction of $P_c$ at $z$.

**Proposition 3.2** The function $\phi_2 := \phi_{|\varphi_n|}^p$.

(i) Let

$$
\phi_{|\varphi_n|}^p(x) = \begin{bmatrix} \phi_{|\varphi_n|}^p(x_1, F_1(x)) \\ \vdots \\ \phi_{|\varphi_n|}^p(x_n, F_n(x)) \end{bmatrix}
$$

where

$$
\begin{align*}
I_1(x) &= \{i | x_i > 0, F_i(x) > 0\}, \\
I_2(x) &= \{i | x_i < 0, F_i(x) < 0\}, \\
I_{0+}(x) &= \{i | x_i = 0, F_i(x) > 0\}, \\
I_{+0}(x) &= \{i | x_i > 0, F_i(x) = 0\}, \\
I_{00}(x) &= \{i | x_i = 0, F_i(x) = 0\}.
\end{align*}
$$

(ii) For any $i \in \{1, \ldots, n\}$, each element of $H_i \in \partial \Phi_{|\varphi_n|}^p(x)^T$ and $V \in \partial P_c(x, \lambda)$ can be written as follows

$$
H_i = a_i(x) E_i + b_i(x) \nabla F_i(x) \text{ and } V = \nabla f(x) + H[\lambda + c \Phi_{|\varphi_n|}^p(x)],
$$

where $E_i$ is the $i$th column of the $n \times n$ unit matrix, $H = (H_1, \ldots, H_n)$,

$$
(a_i(x), b_i(x)) = \begin{cases} 
\nabla \phi_{|\varphi_n|}^p(x_i, F_i(x)), & i \in \overline{T}(x), \\
(\rho_i \cdot p(F_i(x))^{p-1}, 0), & i \in I_{0+}(x), \\
(0, \rho_i \cdot px_i^{p-1}), & i \in I_{+0}(x), \\
(0, 0), & i \in I_{00}(x),
\end{cases}
$$

with $\rho_i \in [-1, 1]$. 

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(iii) The directional derivative of $\Phi^p_{[D-FB]}(x_i, F_i(x))$ at $x$ in the direction $d$ is given by

$$
\Phi^p_{[D-FB]}(x_i, F_i(x)) = \begin{cases} 
H_i^T d, & i \in \overline{T}(x), \\
|d_i| \cdot p(F_i(x))^{p-1}, & i \in I_{0+}(x), \\
|\nabla F_i(x)^T d| \cdot p x_i^{p-1}, & i \in I_{+0}(x), \\
0, & i \in I_{00}(x).
\end{cases}
$$

**Theorem 3.2** Assumed that $z = (x, \lambda)$ is not a critical point of $P_c(z)$ and $\lambda$ is non-positive. Let

$$
q = \begin{bmatrix} -V \\
-\Phi^p_{[D-FB]}(x) \end{bmatrix},
$$

where $V \in \frac{\partial}{\partial x} P_c(x, \lambda)$ and $\rho_i \in [-1, 1]$. Then, $P_c'(z; q) < 0$ i.e. $q$ is a descent direction of $P_c$ at $z$.

**Proposition 3.3** The function $\phi_3 := \varphi_{[SN]}^p$.

(i) Let

$$
\tilde{\varphi}_{[SN]}^p(x) = \begin{bmatrix} \phi_{[SN]}^p(x_1, F_1(x)) \\
\vdots \\
\phi_{[SN]}^p(x_n, F_n(x)) \end{bmatrix}
$$

and

$$
\begin{align*}
T(x) &= I_1(x) \cup I_2(x), \\
I(x) &= I_{0+}(x) \cup I_{+0}(x) \cup I_{00}(x), \\
I_{0+}(x) &= \{ i | x_i = 0, F_i(x) > 0 \}, \\
I_{+0}(x) &= \{ i | x_i > 0, F_i(x) = 0 \}, \\
I_{00}(x) &= \{ i | x_i = 0, F_i(x) = 0 \}.
\end{align*}
$$

(ii) For any $i \in \{1, ..., n\}$, each element of $H_i \in \partial \tilde{\varphi}_{[SN]}^p(x)^T$ and $V \in \frac{\partial}{\partial x} P_c(x, \lambda)$ can be written as follows

$$
H_i = a_i(x) E_i + b_i(x) \nabla F_i(x) \text{ and } V = \nabla f(x) + H[\lambda + c \tilde{\varphi}_{[SN]}^p(x)],
$$

where $E_i$ is the $i$th column of the $n \times n$ unit matrix, $H = (H_1, ..., H_n)$,

$$
(a_i(x), b_i(x)) = \begin{cases} 
\nabla \varphi_{[SN]}^p(x_i, F_i(x)), & [i \in \overline{T}(x)], \\
\rho_i \cdot p \left( \frac{F_i(x)}{\lambda} \right)^{p-1}, & [i \in I_{0+}(x)], \\
0, & [i \in I_{+0}(x)], \\
0, & [i \in I_{00}(x)],
\end{cases}
$$

with $\rho_i \in [-1, 1]$.

(iii) The directional derivative of $\tilde{\varphi}_{[SN]}^p(x_i, F_i(x))$ at $x$ in the direction $d$ is given by

$$
\tilde{\varphi}_{[SN]}^p(\overline{T}(x); d) = \begin{cases} 
H_i^T d, & [i \in \overline{T}(x)], \\
|d_i| \cdot p \left( \frac{F_i(x)}{\lambda} \right)^{p-1}, & [i \in I_{0+}(x)], \\
|\nabla F_i(x)^T d| \cdot p \left( \frac{F_i(x)}{\lambda} \right)^{p-1}, & [i \in I_{+0}(x)], \\
0, & [i \in I_{00}(x)].
\end{cases}
$$
Theorem 3.3 Assumed that \( z = (x, \lambda) \) is not a critical point of \( P_c(z) \) and \( \lambda \) is non-negative. Let
\[
q = \begin{bmatrix} -V \\ -\tilde{\Phi}^p_{\text{NR}}(x) \end{bmatrix},
\]
where \( V \in \frac{\partial}{\partial x} P_c(x, \lambda) \) and \( \rho_i \in [-1, 1] \). Then, \( P_c'(z; q) < 0 \) i.e. \( q \) is a descent direction of \( P_c \) at \( z \).

Proposition 3.4 The function \( \phi_4 := \phi^p_{\text{NR}} \).

(i) Let
\[
\Phi^p_{\text{NR}}(x) = \left[ \begin{array}{c} \Phi^p_{\text{NR}}(x_1, F_1(x)) \\ \vdots \\ \Phi^p_{\text{NR}}(x_n, F_n(x)) \end{array} \right] \quad \text{and} \quad \begin{cases} \mathcal{I}(x) = I_1(x) \cup I_2(x) \cup I_3(x) \cup I_4(x), \\ I_0(x) = I_{0+}(x) \cup I_{0+}^0(x) \cup I_{00}(x), \\ I_c(x) = I_{++}(x) \cup I_{--}(x). \end{cases}
\]
where
\[
\begin{cases} I_1(x) = \{i|x_i > F_i(x) > 0\}, \\ I_2(x) = \{i|F_i(x) > x_i > 0\}, \\ I_3(x) = \{i|x_i > 0, F_i(x) \leq 0, \text{ or } x_i > F_i(x)\}, \\ I_4(x) = \{i|x_i \leq 0, F_i(x) > 0, \text{ or } F_i(x) > x_i\}, \end{cases}
\]
and
\[
\begin{cases} I_{0+}(x) = \{i|x_i = 0, F_i(x) > 0\}, \\ I_{00}(x) = \{i|x_i > 0, F_i(x) = 0\}, \\ I_{00}(x) = \{i|x_i = 0, F_i(x) = 0\}, \\ I_{++}(x) = \{i|x_i = F_i(x) > 0\}, \\ I_{--}(x) = \{i|x_i = F_i(x) < 0\}. \end{cases}
\]

(ii) For any \( i \in \{1, ..., n\} \), each element of \( H_i \in \partial \Phi^p_{\text{NR}}(x)^T \) and \( V \in \frac{\partial}{\partial x} P_c(x, \lambda) \) can be written as follows
\[
H_i = a_i(x) E_i + b_i(x) \nabla F_i(x) \quad \text{and} \quad V = \nabla f(x) + H[\lambda + c \Phi^p_{\text{NR}}(x)],
\]
where \( E_i \) is the \( i \)th column of the \( n \times n \) unit matrix, \( H = (H_1, ..., H_n) \),
\[
(a_i(x), b_i(x)) = \begin{cases} \nabla \Phi^p_{\text{NR}}(x_i, F_i(x)), & [i \in \mathcal{I}(x)], \\ (\rho_i \cdot p(F_i(x))^{p-1}, 0), & [i \in I_{0+}(x)], \\ (0, \rho_i \cdot px_i^{p-1}), & [i \in I_{00}(x)], \\ (0, 0), & [i \in I_{00}(x)], \\ (\alpha \cdot px_i^{p-1}, (1 - \alpha) \cdot px_i^{p-1}), & [i \in I_{++}(x)], \\ (-\alpha \cdot px_i^{p-1}, -(1 - \alpha) \cdot px_i^{p-1}), & [i \in I_{--}(x)], \end{cases}
\]
with \( \rho_i \in [-1, 1] \) and \( \alpha \in [0, 1] \).

(iii) The directional derivative of \( \Phi^p_{\text{NR}}(x_i, F_i(x)) \) at \( x \) in the direction \( d \) is given
Assume that

\[ \text{Theorem 3.5} \]

\[ V \]

where \( \phi \) the functions

4. Numerical experiments

The performance profile is a numerical test from [6] to compare the performance of

\[ \text{problem}, \text{there are} n \text{, solver, and also} n_p \text{, is test problem from the test set} P \text{which is generated randomly. For every problem, there are} p \text{and solver} s \text{given by} \]

\[ f_{p,s} = \text{iteration number is required to solve} \]

\[ \text{problem} p \text{by solver} s. \]
The performance ratio is

\[ r_{p,s} := \frac{f_{p,s}}{\min\{f_{p,s} : s \in S\}}, \]

where \( S \) is the five solvers set. Assuming of a parameter \( r_{p,s} \leq r_M, \forall p, s \) which is chosen, and \( r_{p,s} = r_M \) if and only if solver \( s \) does not solve the problem \( p \). In order to obtain an overall approximation for each solver, we now define

\[ \rho_s(\tau) := \frac{1}{n_p} \text{size}\{p \in P : r_{p,s} \leq \tau\} \]

which is called the performance profile of the number of iteration for solver \( s \). Then, \( \rho_s(\tau) \) is the probability for solver \( s \in S \) i.e. a performance ratio \( f_{p,s} \) inside a factor \( \tau \in \mathbb{R} \) of the best possible ratio.

We report numerical experiments carried out in MATLAB running on a PC with Intel i7 of 1.50 GHz CPU processor, 4.00 GB memory, 64-bit operating system windows 10, and random examples. The new NCP-functions (1) and parameters used:

\[ z_1 = 0.5, \ z_2 = 0.3, \ z_3 = 0.8, \ \Delta t = 0.002. \]

**Figure 1.** Performance profile of \( \phi_k^p \) in \( n \) dimensions solved by FTIM where \( p > 1 \) and \( \epsilon = 10^{-7} \).

**Figure 2.** Performance profile of \( \phi_k^p \) in \( n \) dimensions solved by LG method where \( p > 1 \) and \( \epsilon = 10^{-7} \).
We summarize all the comparison results as below:

(i) Figures 1 and 2 are the performance profile of computing times of \( \phi^p_k \). Both figures show that the case of \( \phi^3_k \) is the best in either the FTIM or LG method.
(ii) Figures 3 and 4 are the performance profile of iteration numbers of $\phi_k^p$. The performance of $\phi_k^p$ is the best for the FTIM and LG method respectively.

(iii) Figure 5 compares numerical methods used in this research and the FTIM is better than LG method.

5. Conclusions

The FTIM offers an approximate solution at NOPs better than LG method. The way to reformulate from NAEs into ODEs by means of the fictitious time function until obtained numerical equations is a long way to be conducted; however, the results show up more satisfying such as a higher stability, an approximate accuracy, and also efficient loop in algorithm when used the performance profile to find computing times and number of iterations. The LG method does not depend on some specific NCP-functions and solves a variety of NCPs. We encourage the readers to pursue research at the convergence rate of all NCP-functions; which may need to be considered in future research.

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