A NOTE ON RETRACTS OF POLYNOMIAL RINGS IN THREE VARIABLES

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Abstract. For retracts of the polynomial ring, in [1], Costa asks us whether every retract of $k^{[n]}$ is also the polynomial ring or not, where $k$ is a field. In this paper, we give an affirmative answer in the case where $k$ is a field of characteristic zero and $n = 3$.

1. Introduction

Let $A$ and $B$ be commutative rings. We say $A$ is a retract of $B$ if $A$ is a subring of $B$ and there exists an ideal $I$ of $B$ such that $B \cong A \oplus I$ as $A$-modules. The followings are basic properties of retracts.

Proposition 1.1. (cf. [1] Section 1) Let $B$ be an integral domain and let $A$ be a retract of $B$. Then the following assertions hold true.

1. $A$ is algebraically closed in $B$.
2. If $B$ is a UFD, then $A$ is also a UFD.
3. If $B$ is regular, then $A$ is also regular.

Lemma 1.2. Let $k$ be a field. Let $A$ and $B$ be $k$-algebras. If $A$ is a retract of $B$, then $A \otimes_k K$ is a retract of $B \otimes_k K$ for any field $K$ containing $k$.

Proof. Since $A$ is a retract of $B$, there exists an ideal $I$ of $B$ such that $B \cong A \oplus I$ as $A$-modules. Let $K$ be a field containing $k$. Taking a tensor product by $K$ over $k$, we have $B \otimes_k K \cong (A \otimes_k K) \oplus I'$ as $A \otimes_k K$-modules, where $I' := I \otimes_k K$ is an ideal of $B \otimes_k K$. Thus, $A \otimes_k K$ is a retract of $B \otimes_k K$. □

Let $k$ be a field. We denote $k^{[n]}$ by the polynomial ring in $n$ variables over $k$. In [1], Costa asks us the following question.

Question 1.3. Let $k$ be a field and let $B := k^{[n]}$. Then, is every retract of $B$ containing $k$ the polynomial ring?
If \( n \leq 2 \), then the above question is affirmative and proved by Costa ([11, Theorem 3.5]). On the other hand, it is well known that Question 1.3 is related to Zariski’s cancellation problem as below.

**Proposition 1.4.** If Question 1.3 holds for \( n + 1 \), then Zariski’s cancellation problem has an affirmative answer for \( \mathbb{A}^n \), that is, \( X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1} \) implies \( X \cong \mathbb{A}^n \).

*Proof.* Let \( k \) be a field. Suppose that Question 1.3 holds for \( n + 1 \). Let \( X \) be an affine variety over \( k \) such that \( X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1} \) and let \( A \) be the coordinate ring of \( X \). Then \( A[1] \cong k^{[n+1]} \) and \( \text{tr.deg}_k A = n \). It is clear that \( A \) is a retract of \( k^{[n+1]} \). Therefore \( A \cong k^{[n]} \), which implies that \( X \cong \mathbb{A}^n \). \( \Box \)

When \( k \) is a field of positive characteristic, Gupta [4] proved that Zariski’s cancellation problem does not hold for \( \mathbb{A}^n \) if \( n \geq 3 \). Therefore Question 1.3 does not hold in the case where \( k \) is a field of positive characteristic and \( n \geq 4 \). So, the remaining cases are

- the characteristic of \( k \) is positive and \( n = 3 \),
- the characteristic of \( k \) is zero and \( n \geq 3 \).

In this paper, we consider the case where \( k \) is a field of characteristic zero and \( n = 3 \). The main result in this paper is to give an affirmative answer for Question 1.3 in this case.

2. **Main results**

Let \( k \) be an algebraically closed field and let \( X \) be a (not necessarily complete) nonsingular algebraic variety over \( k \). By virtue of Nagata’s Completion Theorem ([9]), there exists a complete algebraic variety \( \overline{X} \) over \( k \) such that \( X \) is open and dense in \( \overline{X} \). We say that \( \overline{X} \) is a resolvable variety if Hironaka’s Main Theorems about resolution of singularities hold for \( \overline{X} \) and \( \overline{X} - X \). For a resolvable variety \( X \), we denote \( \kappa(X) \) by the logarithmic Kodaira dimension.

**Lemma 2.1.** (cf. [6, Theorem 1.1 (a)]) Let \( f : X \to Y \) be a morphism of nonsingular, resolvable algebraic varieties over an algebraically closed field. If \( f \) is dominant and generically separable, then \( \kappa(X) \geq \kappa(Y) \).

The following is a characterization for affine planes (see Miyanishi [7], Fujita [3], Miyanishi–Sugie [8] and Russell [10]).

**Theorem 2.2.** Let \( k \) be an algebraically closed field and \( X \) be a nonsingular affine surface over \( k \). Let \( A \) be the coordinate ring of \( X \), namely \( X = \text{Spec} \ A \). Then \( X \cong \mathbb{A}_k^2 \) if and only if \( A^* = k^* \), \( A \) is a UFD and \( \kappa(X) = -\infty \).
First of all, we shall show some properties of retracts of the polynomial ring over a field.

Lemma 2.3. Let \( k \) be an algebraically closed field and let \( B := k^{[n]} \) be the polynomial ring in \( n \) variables over \( k \). Let \( A \) be a retract of \( B \) containing \( k \) and set \( X = \text{Spec} \, A \). If \( X \) is resolvable and \( Q(B) \) is separably generated over \( Q(A) \), then the following assertions hold true.

1. \( A \) is a finitely generated UFD over \( k \) with \( A^* = k^* \),
2. \( X \) is a nonsingular variety over \( k \),
3. \( \overline{\kappa}(X) = -\infty \),

where we denote \( Q(R) \) by the quotient field of an integral domain \( R \).

Proof. Since \( A \) is a retract of \( B \), \( A \) is a \( k \)-subalgebra of \( B \). Hence it is clear that \( A \) is an integral domain with \( A^* = k^* \). Furthermore, there exists a surjective homomorphism as \( k \)-algebras \( \varphi : B \rightarrow A \) such that the following sequence of \( A \)-modules is exact and split:

\[
0 \rightarrow I \rightarrow B \xrightarrow{\varphi} A \rightarrow 0,
\]

where \( I := \ker f \). Hence \( A \) is finitely generated as a \( k \)-algebra. Also by Proposition 1.1 (2), we see that \( A \) is a UFD.

We consider a morphism \( f : A^n_k \cong \text{Spec} \, B \rightarrow X \) defined by a natural inclusion map \( \iota : A \rightarrow B \). It follows from Proposition 1.1 (3) that \( X \) is nonsingular over \( k \).

Suppose that \( X \) is resolvable and \( Q(B) \) is separably generated over \( Q(A) \). Then \( f \) is dominant and generically separable. Hence by Lemma 2.1, we have \( \overline{\kappa}(X) \leq \overline{\kappa}(A^n_k) = -\infty \), which implies that \( \overline{\kappa}(X) = -\infty \). □

When we consider the polynomial ring in two variables whose ground field is not necessarily algebraically closed, the following result is useful.

Theorem 2.4. (cf. [5] or [2, Theorem 5.2]) Let \( K \) and \( k \) be fields such that \( K \) is separably generated over \( k \). Suppose that \( A \) is a commutative \( k \)-algebra for which \( K \otimes_k A \cong K^{[2]} \). Then \( A \cong k^{[2]} \).

The following is the main result in this paper.

Theorem 2.5. Let \( k \) be a field of characteristic zero and let \( B := k^{[3]} \) be the polynomial ring in three variables over \( k \). Then every retract of \( B \) is isomorphic to the polynomial ring.

Proof. Let \( A \) be a retract of \( B \) and let \( d := \text{tr.deg}_k(A) \). Clearly, if \( d = 0 \), then \( A = k \). By Proposition 1.1, \( A \) is algebraically closed in \( B \). Hence, if \( d = 3 \), then \( A = B = k^{[3]} \). If \( d = 1 \), then we already know that \( A \cong k^{[1]} \) by [1, Theorem 3.5].
Suppose that \( d = 2 \). Let \( K \) be an algebraic closure of \( k \). Set \( A_K := A \otimes_k K \) and \( B_K := B \otimes_k K \). It follows from Lemma 1.2 that \( A_K \) is also retract of \( B_K = K^{[3]} \). Set \( X = \text{Spec } A_K \). By using Lemma 2.3 we have \( X \) is a nonsingular, factorial surface over \( K \) and \((A_K)^* = K^* \). Therefore it follows from Theorem 2.2 that \( A_K \cong K^{[2]} \). Applying Theorem 2.4 for \( A_K \), we have \( A \cong k^{[2]} \). □

Remark 2.6. In Lemma 2.3, we don’t know whether \( Q(B) \) is separably generated over \( Q(A) \) or not in general. Of course, if it is true in general, then Theorem 2.5 holds true for any characteristic.

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REFERENCES

[1] D. Costa, Retracts of polynomial rings, J. Algebra 44 (1977), 492–502.
[2] G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations (second edition), Encyclopedia of Mathematical Sciences vol. 136, Invariant Theory and Algebraic Transformation Groups VII, Springer-Verlag, 2017.
[3] T. Fujita, On Zariski problem, Proc. Japan Acad., Ser. A, 55 (1979), 106–110.
[4] N. Gupta, On Zariski’s cancellation problem in positive characteristic, Adv. Math., 264 (2014), 296–307.
[5] T. Kambayashi, On the absence of non-trivial separable forms of the affine plane, J. Algebra 35 (1975), 449–456.
[6] T. Kambayashi, On Fujita’s strong cancellation theorem for the affine plane, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 27 (1980), no. 3, 535–548.
[7] M. Miyanishi, An algebraic characterization of the affine plane, J. Math. Kyoto Univ., 15 (1975), 169–184.
[8] M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, J. Math. Kyoto Univ., 20 (1980), 11–42.
[9] M. Nagata, Imbedding of an abstract variety in a complete variety, J. Math. Kyoto Univ., Ser. A 2 (1962), 1–10.
[10] P. Russell, On affine ruled rational surfaces, Math. Ann., 255 (1981), 287–302.