Monochromatic cycles and the monochromatic circumference in 2-coloured graphs

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Abstract

Li, Nikiforov and Schelp [11] conjectured that a 2-edge coloured graph \( G \) with order \( n \) and \( \delta(G) > \frac{3}{4}n \) contains a monochromatic cycle of length \( \ell \), for all \( \ell \in [4, \lceil \frac{n}{2} \rceil] \). We prove this conjecture for sufficiently large \( n \) and also find all 2-edge coloured graphs with \( \delta(G) = \frac{3}{4}n \) that do not contain all such cycles. Finally we show that, for all \( \delta > 0 \) and \( n > n_0(\delta) \), a 2-edge coloured graph \( G \) of order \( n \) with \( \delta(G) \geq \frac{3}{4}n \) either contains a monochromatic cycle of length at least \( (\frac{2}{3} + \frac{\delta}{2})n \), or contains a monochromatic cycle of length \( \ell \), in the same colour, for all \( \ell \in [3, (\frac{2}{3} - \delta)n] \).

1 Introduction

A well-known theorem of Dirac [6] states that a graph with order \( n \geq 3 \) and minimal degree at least \( \frac{1}{2}n \) contains a Hamilton cycle.

Theorem 1 (Dirac [6]). Let \( G \) be a graph of order \( n \geq 3 \). If \( \delta(G) \geq \frac{1}{2}n \), then \( G \) is hamiltonian.

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In fact, as noted by Bondy \[3\], an immediate corollary of the following theorem is that such a graph will contain cycles of all lengths \(\ell \in [3, n]\). We call such a graph \textit{pancyclic}.

**Theorem 2** (Bondy \[3\]). If \(G\) is a hamiltonian graph of order \(n\) such that \(|E(G)| \geq \frac{n^2}{4}\), then either \(G\) is pancyclic or \(n\) is even and \(G \cong K_{n/2,n/2}\).

**Corollary 3.** Let \(G\) be a graph of order \(n \geq 3\). If \(\delta(G) \geq \frac{1}{2}n\), then either \(G\) is pancyclic or \(n\) is even and \(G \cong K_{n/2,n/2}\).

Given a graph \(G\) with edge set \(E(G)\), a 2-edge colouring of \(G\) is a partition \(E(G) = E(R) \cup E(B)\), where \(R\) and \(B\) are spanning subgraphs of \(G\). In a recent paper \[11\], Li, Nikiforov and Schelp made the following conjecture, which will give an analogue of Corollary 3 for 2-edge coloured graphs.

**Conjecture 4.** Let \(n \geq 4\) and let \(G\) be a graph of order \(n\) with \(\delta(G) > \frac{3}{8}n\). If \(E(G) = E(R) \cup E(B)\) is a 2-edge colouring, then \(C_\ell \subseteq R\) or \(C_\ell \subseteq B\) for all \(\ell \in [4, \lceil\frac{1}{2}n\rceil]\).

Note that we may only ask for \(\ell\) in this range. For example, taking the 2-colouring of \(K_5\) as a red and blue \(C_5\) and blowing up, we get a graph with \(\delta(G) = \frac{3}{8}|G|\) but no monochromatic \(C_3\). Similarly letting \(R\) be the complete bipartite graph with vertex classes of order \(\lfloor\frac{n}{2}\rfloor\) and \(\lceil\frac{n}{2}\rceil\), and letting \(B\) be the complement, we obtain a 2-colouring of the complete graph with no monochromatic odd cycle of length \(\ell > \lceil\frac{n}{2}\rceil\). In \[11\], Li, Nikiforov and Schelp proved the following partial result.

**Theorem 5.** Let \(\epsilon > 0\), let \(G\) be a graph of sufficiently large order \(n\), with \(\delta(G) > \frac{3}{4}n\). If \(E(G) = E(R) \cup E(B)\) is a 2-edge colouring, then \(C_\ell \subseteq R\) or \(C_\ell \subseteq B\) for all \(\ell \in [4, \lceil(\frac{1}{2} - \epsilon) n\rceil]\).

We will prove the conjecture for sufficiently large \(n\), but first we will define a set of 2-edge coloured graphs showing that the degree bound \(\frac{3}{4}n\) is tight.

**Definition.** Let \(n = 4p\) and let \(G\) be isomorphic to \(K_{p,p,p,p}\). A 2-bipartite 2-edge colouring of \(G\) is a 2-edge colouring \(E(G) = E(R) \cup E(B)\) such that both \(R\) and \(B\) are bipartite.

If \(G \cong K_{p,p,p,p}\) and \(G\) has a 2-bipartite 2-edge colouring, let \(V_1 \cup V_2\) be the bipartition of \(R\) and \(W_1 \cup W_2\) be the bipartition of \(B\). Let \(U_{i,j} = V_i \cap W_j\) for all \(i, j \in \{1, 2\}\). Then the \(U_{i,j}\) are four independent sets of \(G\) covering the vertices, and so must be the four independent sets of order \(p\). So, a 2-bipartite 2-edge colouring of \(K_{p,p,p,p}\) forces a labelling of the independent sets \(\{U_{i,j} : i, j \in \{1, 2\}\}\) such that:
• all edges between $U_{1,1}$ and $U_{1,2}$ and between $U_{2,1}$ and $U_{2,2}$ are blue;

• all edges between $U_{1,1}$ and $U_{2,1}$ and between $U_{1,2}$ and $U_{2,2}$ are red;

• edges between $U_{1,1}$ and $U_{2,2}$ and between $U_{2,1}$ and $U_{1,2}$ can be either colour.

Hence, if 4 divides $n$, the graph $K_{n/4,n/4,n/4,n/4}$ with a 2-bipartite 2-edge colouring has minimal degree $\frac{3}{4}n$ and no monochromatic odd cycles. Note that for a fixed labelling of the graph, there are $2^p$ 2-bipartite 2-edge colourings of $K_{p,p,p,p}$. However, $K_{p,p,p,p}$ has $24 (p!)^4 = 2^{O(p \log p)}$ automorphisms and so there are $2^p + O(p \log p)$ distinct 2-bipartite 2-edge colourings of $K_{p,p,p,p}$. In fact we will prove that $K_{n/4,n/4,n/4,n/4}$ is the only extremal graph; although any 2-bipartite 2-edge colouring of $K_{n/4,n/4,n/4,n/4}$ is extremal.

**Theorem 6.** Let $n$ be sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring. Then either $C_{\ell} \subseteq R$ or $C_{\ell} \subseteq B$ for all $\ell \in [4, \left[ \frac{1}{2} n \right]]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring.

We define the monochromatic circumference of a $k$-edge coloured graph $G$ to be the length of the longest monochromatic cycle. In [11], the authors also posed the following question.

**Question 7.** Let $0 < c < 1$ and $G$ be a graph of sufficiently large order $n$. If $\delta(G) > cn$ and $E(G)$ is 2-coloured, how long monochromatic cycles are there?

For graphs $G$ with $\delta(G) \geq \frac{3}{4}n$ we show that the monochromatic circumference is at least $(1 + o(1)) \frac{3}{4}n$. In fact, we show the following result.

**Theorem 8.** Let $n$ be sufficiently large and $0 < \delta \leq \frac{1}{180}$. Let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring. Then either $G$ has monochromatic circumference at least $(\frac{3}{4} + \frac{\delta}{2})n$, or one of $R_G$ and $B_G$ contains $C_{\ell}$ for all $\ell \in [3, (\frac{3}{4} - \delta)n]$.

Note that the last statement requires monochromatic cycles of all lengths in some prescribed set of integers, as in Theorem 6. However, here these cycles are required to be of the same colour. Also, the upper bound on $\delta$ is of a technical nature, and we are interested in small $\delta$. There are similar technical bounds throughout.

For integers $t \leq s$, we define the following 2-edge coloured graph, which with $s = 2t$ shows that Theorem 8 is asymptotically sharp.
Definition. Let $F_{s,t}$ be the complete graph on $t+s$ vertices, with $V = V(F_{s,t})$. Let $A \subseteq V$ be a set of order $s$. We 2-edge colour $F_{s,t}$ by letting all edges between $A$ and $V \setminus A$ be blue, and all other edges be red. The blue graph is bipartite and has circumference $2|V \setminus A| = 2t$. The red graph has circumference $s$. Thus the monochromatic circumference of $F_{s,t}$ is $\max\{s, 2t\}$.

Let $n = 3t$. Then $|F_{2t,t}| = n$, $\delta(F_{2t,t}) = n - 1$ and $F_{2t,t}$ has monochromatic circumference $\frac{2}{3}n$. Hence Theorem 8 is asymptotically sharp.

We shall show that a linear dependence between the two occurrences of $\delta$ in Theorem 8 is correct. Fix $\delta > 0$. Let $G \sim F_{n-\lceil(\frac{2}{3}-\delta)n\rceil, \lceil(\frac{2}{3}-\delta)n\rceil}$. Then the monochromatic circumference of $G$ is at most $\frac{2}{3}n + 2\delta n$. However, $G$ contains no monochromatic cycle of length $\ell$ where $\ell$ is whichever of $\lceil(\frac{2}{3}-\delta)n\rceil + 1, \lceil(\frac{2}{3}-\delta)n\rceil + 2$ is odd.

In Section 2 we will introduce some theorems that will be used in our proofs. We will then prove Theorem 6 in two parts. Section 3 will deal with short (up to constant length) cycles and Section 4 will deal with long cycles. This will rely on a number of lemmas, which are proved in Section 5. In Section 6 we will look at the length of the longest monochromatic cycle, and in particular prove Theorem 8. We conclude in Section 7 with some open problems.

2 Results used in the proof

In order to prove Theorem 6, we shall use the common extremal graph theory method of the Regularity Lemma and Blow-up Lemmas to find long cycles. Before introducing these, we make some preliminary definitions.

Definition. Let $G$ be a graph and $X$ and $Y$ be disjoint subsets of $V(G)$. The density of the graph $G[X,Y]$ is the value

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|}.$$ 

We define a regular pair to be one where the density between not-too-small subgraphs of $X$ and $Y$ is close to the density between $X$ and $Y$.

Definition (Regularity). Let $\epsilon > 0$. Let $G$ be a graph and $X$ and $Y$ be disjoint subsets of $V(G)$. We call $(X,Y)$ an $\epsilon$-regular pair for $G$ if, for all $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geq \epsilon |X|$ and $|Y'| \geq \epsilon |Y|$, we have

$$|d(X,Y) - d(X',Y')| < \epsilon.$$
It is often useful to have a bound on the degree of vertices in $X$ and $Y$.

**Definition** (Super-regularity). Let $\epsilon, \delta > 0$. Let $G$ be a graph and $X$ and $Y$ be disjoint subsets of $V(G)$. We call $(X,Y)$ an $(\epsilon,\delta)$-super-regular pair for $G$ if, for all $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geq \epsilon |X|$ and $|Y'| \geq \epsilon |Y|$,  

$$e(X',Y') > \delta |X'||Y'|,$$

and furthermore, $d_Y(v) > \delta |Y|$ for all $v \in X$ and $d_X(v) > \delta |X|$ for all $v \in Y$.

Note that a super-regular pair need not be regular, as the number of edges between subsets is only bounded below.

We will use the following 2-coloured version of the Szemerédi Regularity Lemma [12] (see, for example, the survey paper of Komlós and Simonovits [10] for an edge-coloured version).

**Theorem 9** (Degree form of 2-coloured Regularity Lemma). For every $\epsilon > 0$ there is an $M = M(\epsilon)$ such that if $G = (V,E)$ is any 2-coloured graph and $d \in [0,1]$ is any real number, then there is $k \leq M$, a partition of the vertex set $V$ into $k+1$ clusters $V_0, V_1, \ldots, V_k$, and a subgraph $G' \subseteq G$ with the following properties:

- $|V_0| \leq \epsilon |V|$,  
- all clusters $V_i$, $i \geq 1$, are of the same size $m \leq \lfloor \epsilon |V| \rfloor$,  
- $d_{G'}(v) > d_G(v) - (2d + \epsilon) |V|$ for all $v \in V$,  
- $e(G'(V_i)) = 0$ for all $i \geq 1$,  
- for all $1 \leq i < j \leq k$, the pair $(V_i,V_j)$ is $\epsilon$-regular for $R_{G'}$ with a density either 0 or greater than $d$ and $\epsilon$-regular for $B_{G'}$ with a density either 0 or greater than $d$, where $E(G') = E(R_{G'}) \cup E(B_{G'})$ is the induced 2-edge colouring of $G'$.

Having applied the above form of the Regularity Lemma to a 2-coloured graph $G$, we make the following definition, based on the clusters $\{V_i : 1 \leq i \leq k\}$. Note that this definition depends on the parameters $\epsilon$ and $d$.

**Definition** (Reduced graph). We define a $(\epsilon,d)$-reduced 2-edge coloured graph $H$ on vertex set $\{v_i : 1 \leq i \leq k\}$ as follows:

- let $\{v_i,v_j\}$ be a blue edge of $H$ when $B_{G'}[V_i,V_j]$ has density at least $d$;  
- let $\{v_i,v_j\}$ be a red edge of $H$ when it is not a blue edge and $R_{G'}[V_i,V_j]$ has density at least $d$.
We aim to use subgraphs of the reduced graph $H$ to find subgraphs of $G$. To do so we will use the Embedding Lemma and the Blow-up Lemma of Komlós, Sárközy and Szemerédi [9].

**Theorem 10** (Embedding Lemma). Given $d > \epsilon > 0$, a graph $H$, and a positive integer $m$, let us construct a graph $G$ by replacing each vertex of $H$ with a set of order $m$, and replacing the edges of $H$ with $\epsilon$-regular pairs of density at least $d$. For a fixed integer $t$, let $H(t)$ be the graph defined by replacing each vertex of $H$ with a set of order $t$, and replacing the edges of $H$ with the complete bipartite graph.

Let $F$ be a subgraph of $H(t)$ with $f$ vertices and maximum degree $\Delta > 0$, and let $\eta = d - \epsilon$ and $\epsilon_0 = \eta \Delta / (2 + \Delta)$. If $\epsilon \leq \epsilon_0$ and $t - 1 \leq \epsilon_0 m$, then $F \subseteq G$, and in fact $G$ contains at least $(\epsilon_0 m) f$ vertex disjoint copies of $F$.

Note that in the Embedding Lemma, the graphs $F$ we embed into $G$ have order at most $t |H| = t m |G|$. We will need to embed much larger graphs into $G$; for this we will need the Blow-up Lemma. Note that here we consider super-regular pairs.

**Theorem 11** (Blow-up Lemma). Given a graph $H$ of order $r$ and positive parameters $\delta$, $\Delta$ and $c$, there exist positive numbers $\epsilon = \epsilon(\delta, \Delta, r, c)$ and $\alpha = \alpha(\delta, \Delta, r, c)$ such that the following holds. Let $t$ be an arbitrary positive integer, and replace the vertices $v_1, \ldots, v_r$ of $H$ with pairwise disjoint sets $V_1, \ldots, V_r$ of order $t$. We construct two graphs on the same vertex set $V = \bigcup V_i$. The first graph $G_1$ is obtained by replacing each edge $\{v_i, v_j\}$ of $H$ with the complete bipartite graph between the corresponding vertex sets $V_i$ and $V_j$. A sparser graph $G_2$ is constructed by replacing each edge $\{v_i, v_j\}$ arbitrarily with an $(\epsilon, \delta)$-super-regular pair between $V_i$ and $V_j$. If a graph $F$ with $\Delta(F) \leq \Delta$ is embeddable into $G_1$ then it is also embeddable into $G_2$. This remains true even if for every $i$ there are certain vertices $x$ to be embedded into $V_i$ whose images are a priori restricted to certain sets $C_x \subseteq V_i$, provided that:

(i) each $C_x$ is of order at least $\epsilon t$;

(ii) the number of such restrictions within a set $V_i$ is not more than $\alpha t$.

In the proof of Theorem 6 we shall frequently show that there is a subset $S$ of $V$ on which one of $R_G$ or $B_G$ is hamiltonian, and apply Theorem 2. To prove hamiltonicity, it will normally be sufficient to use Dirac’s Theorem (Theorem 1). However, we will also need the following generalisation.

**Theorem 12** (Chvátal [5]). Let $G$ be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that

$$d_k \leq k < \frac{1}{2} n \Rightarrow d_{n-k} \geq n - k.$$
Then $G$ contains a Hamilton cycle.

In Section 5, we will need to use Tutte’s 1-factor Theorem \[13\], for which we need the following definition.

**Definition.** For any graph $G$, let $q(G)$ be the number of components of $G$ of odd order.

In fact we shall use the following defect version of Tutte’s Theorem, noted by Berge \[1\].

**Theorem 13.** A graph $G$ contains a set of independent edges covering all but at most $d$ of the vertices if, and only if

$$q(G - S) \leq |S| + d$$

for all $S \subseteq V$.

The next result of Bollobás \[2, p.150\] will be used in Section 3, as will the three following results.

**Theorem 14** (Bollobás \[2\]). If $G$ is a graph of order $n$, with $e(G) > \frac{1}{4}n^2$, then $G$ contains $C_k$ for all $k \in \left[3, \left\lceil \frac{n}{2} \right\rceil \right]$.

**Theorem 15** (Bondy and Simonovits \[4\]). Let $G$ be a graph of order $n$ and let $k$ be an integer. If $e(G) > 100k^{1+1/k} n$, then $G$ contains a cycle of length $2k$.

**Theorem 16** (Erdős and Gallai \[7\]). If $G$ is a graph with order $n$ and circumference at most $L$, then $e(G) \leq \frac{1}{2}(n - 1) L$. If $G$ is a graph with order $n$ with no paths of length at least $L + 1$, then $e(G) \leq \frac{1}{2}nL$.

**Theorem 17** (Györi, Nikiforov and Schelp \[8\]). Let $k,m$ be positive integers. There exist $n_0 = n_0(k,m)$ and $c = c(k,m) > 0$ such that for every nonbipartite $G$ on $n > n_0$ vertices with minimum degree

$$\delta > \frac{n}{2(2k + 1)} + c,$$

if $C_{2s+1} \subseteq G$, for some $k \leq s \leq 4k + 1$, then $C_{2s+2j+1} \subseteq G$ for every $j \in [m]$.
3 Existence of short cycles

In this section we shall prove that unless we are in the extremal case, we have monochromatic cycles of all lengths $\ell \in [4, K]$ for a given integer $K$.

Lemma 18. Let $K$ be an integer. Let $n$ be sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. If $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring, then either $C_{\ell} \subseteq R$ or $C_{\ell} \subseteq B$ for all $\ell \in [4, K]$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring.

To prove this we shall use the following claim. The proof of Claim 19 follows exactly the method used in [11] to show the existence of short odd cycles. Note that we can not appeal directly to Theorem 5, as the assumption there is that $\delta(G) > \frac{3}{4}n$, whereas in Theorem 6 we assume only that $\delta(G) \geq \frac{3}{4}n$.

Claim 19. Let $L$ be an integer. Let $n$ be sufficiently large and let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring. If there is a monochromatic $C_3$ or $C_5$, then there is a monochromatic $C_\ell$ for all odd $\ell \in [5, 2L + 1]$.

Proof. Suppose first that $\Delta(B) > \frac{1}{2}n + 4L$. Let $v$ be a vertex with $d_B(v) = \Delta(B)$, and $U = \Gamma_B(v)$. If $B[U]$ contains a path of length $2L$, then using the vertex $v$, there is a blue $C_\ell$ for all $\ell \in [3, 2L + 1]$. Hence $B[U]$ does not contain a path of length $2L$, and hence by Theorem 16 we have $e(B[U]) \leq L|U|$. However, any vertex $v \in U$ has at most $\frac{1}{4}n$ non-neighbours in $U$ and so at least $|U| - \frac{1}{4}n$ neighbours. Hence

$$e(G[U]) = \frac{1}{2} \sum_{u \in U} d_G[u](u) \geq \frac{1}{2}|U|(|U| - \frac{1}{4}n) \geq \frac{1}{2}|U|\left(\frac{1}{2}|U| + 2L\right).$$

Hence $e(R[U]) = e(G[U]) - e(R[U]) > \frac{1}{4}|U|^2$, and so by Theorem 14, $R[U]$ has cycles of all lengths from 3 to $\frac{1}{2}|U|$.

So we may assume that $\Delta(B) \leq \frac{1}{2}n + 4L$, and hence

$$\delta(R) \geq \frac{1}{4}n - 4L > \frac{1}{6}n + c(1, L),$$

where $c(1, L)$ is the constant from Theorem 17. Similarly we may assume that $\delta(B) \geq \frac{1}{6}n + c(1, L)$. Suppose that there is a monochromatic $C_3$ or $C_5$
and assume without loss of generality that it is red. Applying Theorem 17 to $R$ with $L = 1$ and $m = K$, there is a red $C_\ell$ for all odd $\ell \in [5, 2L + 1]$ as required.

\begin{proof}[Proof of Lemma 18] Note that the existence of monochromatic $C_\ell$ for all even $\ell \in [4, K]$ is immediate from Theorem 15. Hence, by Claim 19, it is sufficient to prove that either there is a monochromatic $C_3$ or $C_5$, or $n = 4p$, $G \cong K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge colouring. Suppose that, in fact, none of these occur. Any 2-edge colouring of $K_5$ contains a monochromatic $C_3$ or $C_5$. Hence we may assume that $K_5 \not\subseteq G$. By Turán’s Theorem, we must therefore have that $G \cong T_4(n)$. However, $\delta(G) \geq \frac{3}{4}n$ implies that in fact $n = 4p$ and hence $G \cong K_{p,p,p,p}$. Let $U_i$ $(1 \leq i \leq 4)$ be the independent sets of $G$ of order $p$.

We may assume that $R$ is not bipartite. Let $C = v_1 v_2 \ldots v_r$ be a shortest odd cycle of $R$; we may assume that $r \geq 7$. We may properly 4-vertex colour $C$ by setting $c(v_i) = j$ when $v_i \in U_j$. As $C$ is an odd cycle, there must be three consecutive vertices with different colours under $c$. Without loss of generality, assume that $c(v_3) = 1$, $c(v_4) = 2$ and $c(v_5) = 3$.

We will aim to show that $G[V(C)]$ contains a triangle or 5-cycle which is edge-disjoint from $C$. Then we may assume that an edge of the triangle or 5-cycle is red, else we have a monochromatic $C_3$ or $C_5$. But this red edge, together with $C$, will create a shorter red odd cycle than $C$, contradicting our assumption that $C$ was minimal. We shall find such a triangle or 5-cycle by case analysis.

If $c(v_1)$ is 2 or 4, then $G$ contains the triangle $v_1 v_3 v_5$, as these vertices lie in different $U_j$. Hence $c(v_1) \notin \{1, 3\}$, and similarly $c(v_7) \notin \{1, 3\}$.

If $c(v_5) = 4$, then $G$ contains the triangle $v_1 v_5 v_6$. So we may assume that $c(v_6) \neq 4$ and similarly $c(v_2) \neq 4$. Hence $c(v_2) \in \{2, 3\}$ and $c(v_6) \in \{1, 2\}$. If $c(v_2) = 3$ and $c(v_6) = 1$, then $G$ contains the triangle $v_2 v_4 v_6$. Hence, by symmetry, we may assume that $c(v_2) = 2$ and $c(v_6) \in \{1, 2\}$.

If $c(v_2) = 1$, then $G$ contains the triangle $v_2 v_5 v_7$. Hence $c(v_7) = 3$.

If $|C| = 7$, then as $c$ is a proper colouring, we have $c(v_1) = 1$. But then $v_1 v_5 v_2 v_7 v_4$ is a 5-cycle in $G$, not containing any edges of $C$. So we may assume that $|C| > 7$, and in particular $v_1 v_7 \notin E(C)$.

If $c(v_1) = 1$, then $v_1 v_4 v_7$ is a triangle in $G$. Hence $c(v_1) = 3$. But now, if $c(v_6) = 1$, then $G$ contains the triangle $v_1 v_4 v_6$, while if $c(v_6) = 2$, $G$ contains the triangle $v_1 v_3 v_6$, giving a contradiction.

Hence, in fact, our assumption was false, and one of the cases of the lemma holds.\end{proof}
4 Existence of long cycles

In order to find long monochromatic cycles, we will use the Regularity Lemma. Recall from Section 2 that having applied the Regularity Lemma to \( G \), we define a reduced graph \( H \). Note that the Regularity Lemma implies that the minimal degree of the reduced graph \( H \) is not too much smaller than \( \frac{k}{n} \) times the minimal degree of \( G \).

Suppose that the red edges of our reduced graph \( H \) contain a large set of independent edges. Then we can use the Blow-up Lemma to create lots of long red paths in \( G \), which we can hope to join together into long red cycles. One situation in which we could join together the paths in \( G \) obtained from a monochromatic matching in \( H \) is when the matching is contained in a component of the relevant colour in \( H \). Then we can use the properties of regular pairs, and in particular the Embedding Lemma, to join the paths. The following lemma, proved in Section 5 using extremal arguments, shows that if there is no monochromatic component of \( H \) containing a large matching, then the reduced graph has one of two particular forms.

**Lemma 20.** Let \( 0 < \delta < \frac{1}{36} \) and let \( G \) be a graph of sufficiently large order \( n \) with \( \delta(G) \geq \left( \frac{3}{4} - \delta \right) n \). Suppose that we are given a 2-edge colouring \( E(G) = E(R) \cup E(B) \). Then one of the following holds.

(i) There is a component of \( R \) or \( B \) which contains a matching on at least \( \left( \frac{2}{3} + \delta \right) n \) vertices.

(ii) There is a set \( S \) of order at least \( \left( \frac{3}{4} - \frac{\delta}{2} \right) n \) such that either \( \Delta(R[S]) \leq 10\delta n \) or \( \Delta(B[S]) \leq 10\delta n \).

(iii) There is a partition \( V(G) = U_1 \cup \cdots \cup U_4 \) with \( \min |U_i| \geq \left( \frac{1}{4} - 3\delta \right) n \) such that there are no red edges from \( U_1 \cup U_2 \) to \( U_3 \cup U_4 \) and no blue edges from \( U_1 \cup U_3 \) to \( U_2 \cup U_4 \).

In the first case, we will need the following lemma, which is also proved in Section 5.

**Lemma 21.** Let \( 0 < \delta < \frac{1}{6} \) and let \( G \) be a graph of sufficiently large order \( n \) with \( \delta(G) \geq \left( \frac{3}{4} - \delta \right) n \). Suppose that we are given a 2-edge colouring \( E(G) = E(R) \cup E(B) \). Suppose that there is a monochromatic component containing a matching on at least \( \left( \frac{2}{3} + \delta \right) n \) vertices. Then there is a monochromatic component \( C \) containing a matching on at least \( \left( \frac{1}{2} + \delta \right) n \) vertices such that either \( C \) contains an odd cycle, or \( |C| \geq (1 - 5\delta) n \).
By analysing the original graph, we will use the following two lemmas to show that, in the second and third cases of Lemma 20, we will have the desired monochromatic cycles. In both of the following two lemmas, we assume the following setup. We have constants $0 < \epsilon \ll d \ll \delta < \frac{1}{144}$. Let $n$ be sufficiently large and $G$ a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring. We find a regular partition of $G$ using Theorem 9 and as defined in Section 2 let $H$ be the $(\epsilon, d)$-reduced 2-edge coloured graph obtained from this partition.

The following results will be useful, and will be proved in Section 5.

Lemma 22. If $B_G$ has an independent set $S$ with $|S| > \frac{1}{2}n$, then $\mathcal{C}_\epsilon \subseteq R_G$ for all $\ell \in [3, |S|]$. Further, if $B_G$ is bipartite, then either $\mathcal{C}_\epsilon \subseteq R_G$ for all $\ell \in [4, \lceil \frac{n}{2} \rceil]$, or $n$ is divisible by four, $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring.

Lemma 23. If there is a set $S \subseteq V(H)$ of order at least $(\frac{2}{3} - \frac{1}{2})k$ such that $\Delta(R_H[S]) \leq 10\delta k$, then $G$ contains a blue cycle of length $\ell$ for all $\ell \in [3, (\frac{3}{5} - \delta)n]$.

Lemma 24. Suppose that there is a partition $V(H) = U_1 \cup \cdots \cup U_4$ with $\min_i |U_i| \geq (\frac{1}{4} - 3\delta)k$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$. Then $G$ contains a monochromatic cycle of length at least $\left(1 - 59\delta\right)n$ and monochromatic cycles of length $\ell$ for all $\ell \in [4, \lceil \frac{1}{2}n \rceil]$.

We now prove Theorem 6 by applying the lemmas above to the reduced graph obtained from the Regularity Lemma.

Proof of Theorem 6. Choose $0 < \delta < \frac{1}{144}$ and $d \ll \delta$ (where, as usual, $\ll$ means sufficiently smaller than). Let $\epsilon := \frac{1}{4} \left(\frac{2}{3}, 2, \frac{d}{2}\right)$ be defined from $d$ as in the Blow-up Lemma; we may also assume that $\epsilon \ll d$ by taking a smaller $\epsilon$ if necessary. In particular, we may choose $\epsilon$ and $d$ so that we may apply the Embedding Lemma. Also, by Lemma 18, we have, for any fixed integer $K$, that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring or $G$ contains a monochromatic $C_\ell$ for all $\ell \in [4, K]$. Hence it is sufficient to prove that either $G \cong K_{n/4,n/4,n/4,n/4}$ and the colouring is a 2-bipartite 2-edge colouring, or there is some fixed integer $K$ such that $G$ contains a monochromatic $C_\ell$ for all $\ell \in [K, \lceil \frac{n}{2} \rceil]$.

Let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. We apply the degree form of the 2-colour Regularity Lemma to $G$, with parameters $d$ and $\epsilon$. Let $V_0, V_1, \ldots, V_k$ be the clusters (with $|V_i| = m$ for $i \geq 1$), and $G'$ be the subgraph of $G$ defined by Theorem 9. Let $H$ be the $(\epsilon, d)$-reduced graph defined from $G'$ earlier, with 2-edge colouring $E(H) = E(R_H) \cup E(B_H)$. 

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We have \( \delta\left(G'\right) \geq \left(\frac{3}{4} - d - \epsilon\right)n \). Suppose that \( \delta\left(H\right) < \left(\frac{3}{4} - \delta\right)k \): then there is some \( i \geq 1 \) with \( d_H\left(V_i\right) < \left(\frac{3}{4} - \delta\right)k \). For a vertex \( v \in V_i \), it has neighbours in \( G' \) only in \( V_0 \), or in \( V_j \) for those \( j \) such that \( v_iv_j \) is an edge of \( H \). Hence

\[
d_G'(v) < \left(\frac{3}{4} - \delta\right)km + |V_0| \leq \left(\frac{3}{4} - \delta + \epsilon\right)n
\]

which is a contradiction, as \( \delta \gg d + 2\epsilon \). Hence \( \delta\left(H\right) \geq \left(\frac{3}{4} - \delta\right)k \).

Applying Lemma 20, we have one of the following.

(i) There is a component of \( R_H \) or \( B_H \) which contains a matching on at least \( \left(\frac{3}{4} + \delta\right)k \) vertices.

(ii) There is a set \( S \) of order at least \( \left(\frac{3}{4} - \delta\right)k \) such that either \( \Delta(R_H[S]) \leq 10\delta k \) or \( \Delta(B_H[S]) \leq 10\delta k \).

(iii) There is a partition \( V(H) = U_1 \cup \cdots \cup U_4 \) with \( \min_i |U_i| \geq \left(\frac{1}{4} - 3\delta\right)k \) such that there are no blue edges from \( U_1 \cup U_2 \) to \( U_3 \cup U_4 \) and no red edges from \( U_1 \cup U_3 \) to \( U_2 \cup U_4 \).

If we are in the second or third case, we are done immediately by Lemma 23 and Lemma 24 respectively. Hence we assume that there is a component of \( R_H \) or \( B_H \) which contains a matching on at least \( \left(\frac{3}{4} + \delta\right)k \) vertices. By Lemma 21 we may assume that there is a component \( R'_H \) of \( R_H \) which contains a matching on at least \( \left(\frac{1}{2} + \delta\right)k \) vertices, and that either \( R'_H \) contains an odd cycle or \( |R'_H| \geq (1 - 5\delta)k \).

Take a matching in \( R'_H \) with a maximal number of vertices. Let \( r \) be the number of edges in the matching and \( C_1, C_2, \ldots, C_r \) be the edges of the matching, with \( C_i = v_{i,1}v_{i,2} \). For \( 2 \leq i \leq r \), let \( P_i \) be a shortest path of \( R_H \) from \( C_i \) to \( C_1 \). We may assume that the end-point of \( P_i \) in \( C_i \) is \( v_{i,1} \). For all \( i \geq 2 \), let \( v_{i,j} \) be the endpoint of \( P_i \) in \( C_1 \). Note that the path \( P_i \) may pass through vertices of \( C_j \) for \( j \neq i \).

We wish to apply the Blow-up Lemma to the clusters \( V_{i,j} \) corresponding to vertices \( v_{i,j} \). However, the Blow-up Lemma applies to super-regular pairs and currently we may only assume regularity. We show that by removing a small number of vertices, we may assume that all the edges of our odd-extended matching are super-regular pairs. Let \( W_{1,1} \) be the set of vertices of \( V_{1,1} \) with at least \( \left(d - \epsilon\right)m \) neighbours in \( V_{1,2} \) and \( W_{1,2} \) be the set of vertices of \( V_{1,2} \) with at least \( \left(d - \epsilon\right)m \) neighbours in \( V_{1,1} \). Then it is immediate from regularity that \( |W_{1,j}| \geq (1 - \epsilon)|V_{1,j}| \) for \( j \in \{1, 2\} \). We can check that \( (W_{1,1}, W_{1,2}) \) is a \( \left(\frac{3}{4}\epsilon, d - 2\epsilon\right) \)-super-regular pair. Similarly, we can obtain \( W_{i,j} \subseteq V_{i,j} \) for
all $i \geq 2$ and $j \in \{1, 2\}$ such that $|W_{i,j}| \geq (1 - \epsilon)|V_{i,j}|$ and $(W_{i,1}, W_{i,2})$ is $(\frac{2}{7}\epsilon, d - 2\epsilon)$-super-regular for $R_G$.

Suppose first that there is an odd cycle in the component $R'_H$ of $R_H$. Either this cycle contains $v_{1,1}$, or there is a path from the cycle to $v_{1,1}$. Using Theorem 10, we can find a red path $Q_1 \subseteq R_G$ of odd length at most $2k$ between two vertices $v_1$ and $v_2$ of $W_{1,1}$. We can use the super-regular pair $(W_{1,1}, W_{1,2})$ to find a red path $Q'_1$ of length four from $v_1$ to $v_2$, which does not intersect $Q_1$.

We now construct vertex-disjoint red paths $Q_2, Q'_2, \ldots, Q_r, Q'_r$ in $V \setminus (V(Q_1) \cup V(Q_2))$ such that, for all $2 \leq i \leq r$, both $Q_i$ and $Q'_i$ start in $W_{i,1} \subseteq V_{i,1}$ and pass through the clusters $V_\ell$ corresponding to vertices of $P_\ell$, before terminating in $W_{1,i} \subseteq V_{1,i}$. Indeed suppose that we have constructed such paths $Q_2, Q'_2, \ldots, Q_{s-1}, Q'_{s-1}$ for some $2 \leq s \leq r$. For $i \geq 1$, each path $Q_i$ or $Q'_i$ uses each cluster at most twice and so between them the paths $Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_{s-1}, Q'_{s-1}$ contain at most $4(s - 1) \leq 2k$ vertices in each cluster. Hence, we may remove the vertices of $Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_{s-1}, Q'_{s-1}$ without affecting the regularity of pairs of clusters. Hence, we can find the paths $Q_s$ and $Q'_s$ by Theorem 10.

Each path $Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_r, Q'_r$ has at most two internal vertices contained in each $W_{i,j}$. Letting $W'_{i,j}$ be the vertices of $W_{i,j}$ not used as an internal vertex of some path $Q_2, Q'_2, \ldots, Q_r, Q'_r$, we thus have $|W'_{i,j}| \geq |W_{i,j}| - 2k \geq (1 - 3\epsilon)m$. Deleting vertices where appropriate, we may assume that each $W'_{i,j}$ has order $m' = \lceil (1 - 3\epsilon)m \rceil$. Note that the pairs $(W'_{i,1}, W'_{i,2})$ are $(2\epsilon, \frac{2}{7})$-super-regular.

Consider the subgraph $G''$ of $R_G$ consisting of the super-regular pairs $(W'_{i,1}, W'_{i,2})$ and the paths $Q_1, Q'_1, Q_2, Q'_2, \ldots, Q_r, Q'_r$, as shown in Figure 1. (The super-regular pairs are shown with thick lines and the paths with thin lines.) Note that each of the paths contains at most $k$ vertices, and there are at most $k$ of them. However the union of the $W'_{i,j}$ has order at least $\left(\frac{3}{2} + \frac{2}{7}\right)n$. Let $K = 3 + |Q_1|$. If we replaced all of the super-regular pairs in $G''$ by complete bipartite graphs, it is clear that the resultant graph would contain $C_\ell$ for all $\ell \in [K, \lceil \frac{3}{2}n \rceil]$. By applying the Blow-up Lemma individually to each pair $(W'_{i,1}, W'_{i,2})$, we may thus embed $C_\ell$ into $R_G$ for all $\ell \in [K, \lceil \frac{3}{2}n \rceil]$. Note that we must sometimes restrict vertices of our embedding to lie in the neighbourhood of endvertices of some $Q_\ell$. However, there are only a bounded number $(O(k))$ of such restrictions, and they are all to sets of order at least $\frac{1}{2}dn'$, as we have made our pairs super-regular. Hence we are done in the case that $R'_H$ contains an odd cycle.

Suppose now that the component $R'_H$ of $R_H$ contains no odd cycles and hence $|R'_H| \geq (1 - 5\delta)k$. Then $R'_H$ is bipartite, with classes $H_1$ and $H_2$. 

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Applying the Blow-up Lemma as above, we deduce that $C_\ell \subseteq R_G'$ for all even $\ell \in [4, (\frac{1}{2} + \frac{\delta}{2}) n]$. Hence we are done if we can show that there is a fixed integer $K$ such that $G$ contains a monochromatic $C_\ell$ for all odd $\ell \in [K, \lceil \frac{1}{2} n \rceil]$.

As $R'_H$ is a connected component with order at most $k$ there is a red path of length at most $k - 1$ between any pair of vertices in $R'_H$. Let $X$ be the union of all the clusters in $H_1$, and $Y$ be the union of all the clusters in $H_2$, so that $X$ and $Y$ are subsets of $V$. Using the Blow-up Lemma as above we see that, after removing at most $\epsilon |X|$ vertices from $X$ and $\epsilon |Y|$ vertices from $Y$, the following holds.

- Between any vertices $u, v \in X$ there are, in $R_G'[X \cup Y]$, paths of length $\ell$ for all even $\ell \in [2k, (\frac{1}{2} + \frac{\delta}{2}) n]$ (and similarly for $Y$).

- Between any $u \in X$ and any $v \in Y$ there are, in $R_G'[X \cup Y]$, paths of length $\ell$ for all odd $\ell \in [2k - 1, (\frac{1}{2} + \frac{\delta}{2}) n]$.

If either $X$ or $Y$ contains an internal red edge, then we have cycles of all odd lengths between $2k + 1$ and $(\frac{1}{2} + \delta) n$. Hence we may assume that $R_G[X \cup Y]$ is bipartite. Then

$$|X \cup Y| \geq (1 - 5\delta) km(1 - \epsilon)$$

$$\geq (1 - 6\delta)n.$$
However, if \( \max\{|X|, |Y|\} > \frac{1}{2}n \), we are done by Lemma 22. Hence we may assume that \( \min\{|X|, |Y|\} \geq \left( \frac{1}{2} - 6\delta \right)n \).

If any vertex \( v \) of \( V \setminus (X \cup Y) \) has at least one red neighbour in both \( X \) and \( Y \), then using the paths between a red neighbour of \( v \) in \( X \) and a red neighbour of \( v \) in \( Y \), we have cycles of all odd lengths between \( 2k + 1 \) and \( \left( \frac{1}{2} + \delta \right)n \). Hence all vertices of \( V \setminus (X \cup Y) \) have no red neighbours in at least one of \( X \) or \( Y \).

Define disjoint sets \( X' \) and \( Y' \) by letting \( X' \) be the set of vertices of \( V \setminus (X \cup Y) \) with at least two red neighbours in \( Y \), and \( Y' \) be the set of vertices of \( V \setminus (X \cup Y) \) with at least two red neighbours in \( X \). Then there are no red edges between \( X' \) and \( X \) or between \( Y' \) and \( Y \). If there is a red edge \( uv \) within \( X' \), let \( u' \) and \( v' \) be distinct vertices of \( Y \) with \( uu' \) and \( vv' \) both red edges. Then \( uu'v'v \) is a red path of length three between vertices of \( Y \), with internal vertices in \( V \setminus (X \cup Y) \). Using the \( u'v' \)-paths obtained from the Blow-up Lemma, we have red cycles of all odd lengths between \( 2k + 3 \) and \( \left( \frac{1}{2} + \delta \right)n \).

So we may assume that \( R_G\left[ X \cup X' \cup Y \cup Y' \right] \) is bipartite with classes \( X \cup X' \) and \( Y \cup Y' \). We may assume that both \( X \cup X' \) and \( Y \cup Y' \) have order at most \( \frac{1}{2}n \), else we are done by Lemma 22.

A vertex not in \( X \cup X' \cup Y \cup Y' \) has at least \( \left( \frac{3}{4} - 6\delta \right)n \) neighbours in \( X \cup Y \). Let \( X'' \) be the set of vertices not in \( X \cup X' \cup Y \cup Y' \) with at least \( \left( \frac{3}{8} - 3\delta \right)n \) neighbours in \( X \), and \( Y'' \) be the set of vertices not in \( X \cup X' \cup Y \cup Y' \) with at least \( \left( \frac{3}{8} - 3\delta \right)n \) neighbours in \( Y \). Letting \( X_0 = X \cup X' \cup X'' \) and \( Y_0 = Y \cup Y' \cup Y'' \) we see that \( V \) is the (not necessarily disjoint) union of \( X_0 \) and \( Y_0 \).

Without loss of generality, we may suppose that \( |X_0| \geq \frac{1}{2}n \). By definition, all vertices in \( X'' \) have at least \( \left( \frac{3}{8} - 3\delta \right)n \) neighbours in \( X \). However vertices in \( X'' \) have at most one red neighbour in \( X \), else they would have been in \( Y' \). All vertices in \( X \cup X' \) have at most \( \frac{1}{4}n \) non-neighbours in \( G \) and so at least \( |X_0| - \frac{1}{4}n \) neighbours in \( X_0 \). As there at most \(|X''|\) red edges between \( X'' \) and \( X \), the set \( X'' \) of vertices in \( X \) with a red neighbour in \( X'' \) has order at most \( |X''| \). Vertices in \( X \setminus X'' \) have no red neighbours in \( X_0 \), while vertices in \( X' \setminus X'' \) have no red neighbours in \( X_0 \setminus X'' \). Hence

\[
d_{B_G[X_0]}(v) \geq \begin{cases} |X_0| - \frac{1}{4}n & v \in X \setminus X'' \\ |X_0| - \frac{1}{4}n - |X''| & v \in X'' \cup X' \\ \left( \frac{3}{8} - 3\delta \right)n - 1 & v \in X''. \end{cases} \tag{1}
\]

Since \( |X' \cup X'' \cup X'''| \leq 6\delta n \), the conditions of Theorem 12 are satisfied on the graph \( B_G[X_0] \) and so \( B_G[X_0] \) is Hamiltonian.
However, using (1), we have

\[
e (B_G[X_0]) \geq \frac{1}{2} \left( |X_0| - \frac{1}{4} n \right) |X' \setminus X''| + \left( \left( \frac{3}{16} - \frac{3\delta}{2} \right) n - \frac{1}{2} \right) |X''|
\]

\[
+ \frac{1}{2} \left( |X_0| - \frac{1}{4} n - |X''| \right) \left( |X''| + |X'| \right)
\]

\[
= \frac{1}{2} |X_0| \left( |X_0| - \frac{1}{4} n \right)
\]

\[
+ \left( \left( \frac{3}{16} - \frac{3\delta}{2} \right) n - \frac{1}{2} \right) |X_0| - \frac{1}{2} \left( |X''| + |X'| \right) |X''|
\]

\[
\geq \frac{1}{4} |X_0|^2 + \left( \left( \frac{1}{16} - 6\delta \right) n - \frac{1}{2} \right) |X''|.
\]

Here we have used

\[
\frac{1}{4} |X_0| + \frac{1}{2} \left( |X''| + |X'| \right) \leq \frac{1}{4} (|X| + |X'|) + \frac{1}{2} |X'| + \frac{3}{4} |X''|
\]

\[
\leq \frac{1}{8} n + \frac{3}{4} |V \setminus (X \cup Y)|
\]

\[
\leq \left( \frac{1}{8} + \frac{9\delta}{2} \right) n.
\]

Hence, from Theorem [2] we see that either \(B_G[X_0]\) is pancyclic, in which case \(C_\ell \subseteq B_G\) for all \(\ell \in [3, |X_0|]\), or \(B_G[X_0] \cong K_{|X_0|/2, |X_0|/2}\) and \(e(B_G[X_0]) = \frac{1}{4} |X_0|^2\). Hence, in the latter case, \(X'' = \emptyset\). Similarly, if \(|Y_0| \geq \frac{1}{4} n\), then either \(B_G[Y_0]\) is pancyclic, or \(Y'' = \emptyset\). Hence we may assume that \(X'' = Y'' = \emptyset\) and hence \(B_G\) is bipartite. Thus by Lemma [22] we are done.

If we are in the second or third case of Lemma [20] we are done by Lemma [23] and Lemma [24] respectively.

\[\square\]

5 Proof of Lemmas

In this section we shall prove the lemmas used in the proof of Theorem [6]. Later in the section, we will prove Lemma [23] and Lemma [24]. However, we begin with the proofs of Lemma [20] and Lemma [21]. Throughout both proofs we shall assume that \(R'\) is a largest component of \(R\), and that \(B'\) is a largest component of \(B\). We let \(W_1 = V(B') \cap V(R')\), \(W_2 = V(R') \setminus V(B')\), \(W_3 = V(B') \setminus V(R')\) and \(W_4 = V - (W_1 \cup W_2 \cup W_3)\). We will need the following claim about the component structure.
Claim 25. Let $0 < \delta < \frac{1}{36}$ and let $G$ be a graph of sufficiently large order $n$ with $\delta(G) \geq (\frac{3}{4} - \delta) n$. Suppose that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$. Then one of the following holds.

- One of $R$ or $B$ is connected.
- $V(G) = V(R') \cup V(B')$ and both $R'$ and $B'$ have order at least $(\frac{3}{4} - \delta) n$.
- There is a partition $V(G) = U_1 \cup \cdots \cup U_4$ with $\min|U_i| \geq (\frac{1}{4} - 3\delta) n$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.

Proof. If neither of the above statements holds, then both $R$ and $B$ are disconnected. Suppose first that $|R'| \leq (\frac{5}{12} - \delta) n$. Then $\Delta(R) < (\frac{5}{12} - \delta) n$ and hence $\delta(B) > \frac{1}{4} n$. Since $B$ is disconnected, we see that $B$ has exactly two components $B_1$ and $B_2$ with $\frac{1}{4} n < |B_2| \leq |B_1| < \frac{2}{3} n$. Then $W_i = V(B_i) \cap V(R')$, for $i \in \{1, 2\}$.

Suppose that $W_i \neq \emptyset$, for some $i \in \{1, 2\}$. Let $v \in W_i$. Then $v$ has no neighbours outside $R' \cup B_i$, and so $\Gamma_G(v) \subseteq R' \cup B_i$. Hence, as $W_{3-i} = R' \setminus B_i$,

$$|W_{3-i}| \geq |\Gamma_G(v)| - |B_i| > \left(\frac{1}{12} - \delta\right) n.$$ 

In particular $W_{3-i} \neq \emptyset$, and so $W_1$ is non-empty if and only if $W_2$ is non-empty. As $V(R') = W_1 \cup W_2$, we see that both $W_1$ and $W_2$ are therefore non-empty.

But then

$$|R'| = |W_1| + |W_2| \geq \left(\frac{3}{4} - \delta\right) n - |B_1| + \left(\frac{3}{4} - \delta\right) n - |B_2| = \left(\frac{1}{2} - 2\delta\right) n.$$ 

This contradicts our assumption. We may therefore assume that $|R'| > (\frac{5}{12} - \delta) n$, and similarly $|B'| > (\frac{5}{12} - \delta) n$.

Note that there are no edges (of either colour) between $W_1$ and $W_4$ or between $W_2$ and $W_3$. If $W_4 = \emptyset$, then $V(G) = V(R') \cup V(B')$. As neither $R$ nor $B$ is connected, we must have that $W_2$ is non-empty. Let $v \in W_2$: since $\Gamma_G(v) \cap W_3 = \emptyset$, we see that $|W_3| \leq (\frac{1}{4} + \delta) n$ and so $|R'| \geq (\frac{3}{4} - \delta) n$. We may similarly show that $|B'| \geq (\frac{3}{4} - \delta) n$.

If, however, $W_4 \neq \emptyset$, choose $x \in W_4$. As $\Gamma_G(x) \cap W_1 = \emptyset$, we have $|W_1| \leq (\frac{1}{4} + \delta) n$. However both $R'$ and $B'$ have order at least $(\frac{5}{12} - \delta) n$ and hence both $W_2$ and $W_3$ are non-empty. Thus, arguing as for $W_4$, we see
that both $W_2$ and $W_3$ have order at most $(\frac{1}{4} + \delta) n$ and so $W_1$ is non-empty. This in turn implies that $W_4$ has order at most $(\frac{1}{4} + \delta) n$. Hence each $W_i$ has order at least $(\frac{1}{4} - 3\delta) n$.

\medskip

**Proof of Lemma 20.** We assume throughout that $n$ is sufficiently large. Let $0 < \delta < \frac{1}{36}$. Suppose that $G$ is a graph of order $n$ with $\delta(G) \geq (\frac{3}{4} - \delta) n$ and that we are given a 2-edge colouring $E(G) = E(R) \cup E(B)$.

If $\min_i |W_i| \geq (\frac{1}{4} - 3\delta) n$, then we are in the third case of Lemma 20. Hence, we assume throughout that

$$\min_i |W_i| < \left(\frac{1}{4} - 3\delta\right) n. \quad (2)$$

We will also assume that neither $R'$ nor $B'$ contains a matching on at least $(\frac{2}{3} + \delta) n$ vertices, and refer to this as our main assumption.

We are aiming to show that there is a large set on which one of the colours has a very low density. We show that the orders of $B'$ and $R'$ can not take certain values.

**Claim 26.** Either $|B'| < (\frac{1}{3} + \frac{\delta}{2}) n$ or $|B'| > (\frac{2}{3} - \frac{\delta}{2}) n$.

**Proof.** Suppose that $(\frac{1}{3} + \frac{\delta}{2}) n \leq |B'| \leq (\frac{2}{3} - \frac{\delta}{2}) n$. Then $R$ is connected, by Claim 25.

Let $V_1$ be the smaller of $V(B')$ and $V \setminus V(B')$. Let $V_2 = V \setminus V_1$ and $F$ be the bipartite graph between $V_1$ and $V_2$. There are no blue edges between $V_1$ and $V_2$ and so all edges of $F$ are red. For a subset $S$ of $V_1$ we shall find a lower bound on $|\Gamma_F(S)|$ by splitting into the cases that $|S| > (\frac{1}{4} + \delta) n$ and $|S| \leq (\frac{1}{4} + \delta) n$.

If $S \subseteq V_1$ and $|S| > (\frac{1}{4} + \delta) n$, consider a vertex $v \in V_2$. Then, as $d_G(v) \geq (\frac{3}{4} - \delta) n$, $v$ must have a neighbour in $S$. Hence $\Gamma_F(S) = V_2$, and so $|\Gamma_F(S)| = |V_2| = |V_1| \geq |S|$.

If $S \subseteq V_1$ and $|S| \leq (\frac{1}{4} + \delta) n$, then any vertex in $S$ has at least $|V_2| - (\frac{1}{4} + \delta) n$ neighbours in $V_2$. Hence

$$|\Gamma_F(S)| \geq |V_2| - (\frac{1}{4} + \delta) n$$

$$= |S| - \left(\left(\frac{1}{4} + \delta\right) n + |S| - |V_2|\right)$$

$$\geq |S| - \left(\left(\frac{1}{2} + 2\delta\right) n - |V_2|\right).$$
Thus by the defect form of Hall’s Theorem, $F$ contains a matching with at least
\[ |V_1| - \max \left\{ 0, \left( \frac{1}{2} + 2\delta \right) n - |V_2| \right\} \]
edges. As $|V_1| + |V_2| = n$ and $|V_1| \geq \left( \frac{1}{3} + \frac{\delta}{2} \right) n$, this matching contains at least $\left( \frac{2}{3} + \delta \right) n$ vertices. As $R$ is connected, this contradicts our main assumption.

By Claim 25 and the assumption (2), we may assume that either one of $R$ or $B$ is connected or $V(G) = V(R') \cup V(B')$ and $\min\{|V(R')|, |V(B')|\} \geq \left( \frac{2}{3} - \delta \right) n$. In either case, there will be a monochromatic component of order at least $\left( \frac{2}{3} - \delta \right) n$. Let $X_R = V \setminus V(R')$ and $X_B = V \setminus V(B')$. We make the following definitions when $R'$ or $B'$ is large.

**Definition.** Fix an integer $t \geq \delta^{-1}$. Suppose that $|R'| \geq \left( \frac{2}{3} + \delta \right) n$.

- Let $S_R \subseteq V(R')$ be a set such that
  \[ q(R[V(R') - S_R]) > |S_R| + |V(R')| - \left( \frac{2}{3} + \delta \right) n. \]
  Note that, in view of our main assumption, such a set exists by Theorem T3.

- For $1 \leq i \leq n$, let $T_{R,i}$ be the set of vertices which lie in components of $R[V(R') - S_R]$ of order $i$.

- Let $T_R = \bigcup_{1 \leq i \leq t} T_{R,i}$.

If $|B'| \geq \left( \frac{3}{4} + \delta \right) n$, we define $S_B$, $T_{B,i}$ and $T_B$ similarly.

We shall use the following result throughout. Note that, as with Claim 26, we may exchange the roles of $R$ and $B$ to obtain a symmetrical version of this result.

**Claim 27.** Suppose that $|V(R')| \geq \left( \frac{3}{4} + \delta \right) n$. Then $|S_R| < \left( \frac{1}{3} + \frac{1}{2}\delta \right) n$ and
\[ |X_R \cup T_R| > |S_R| + \left( \frac{1}{3} - 2\delta \right) n. \]

Further, if $C_B$ is a component of $B$ with $|C_B| \leq \left( \frac{5}{12} - 2\delta \right) n$, then $C_B \subseteq S_R$. 

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Thus by the defect form of Hall’s Theorem, $F$ contains a matching with at least  
\[ |V_1| - \max \left\{ 0, \left( \frac{1}{2} + 2\delta \right) n - |V_2| \right\} \]  
edges. As $|V_1| + |V_2| = n$ and $|V_1| \geq \left( \frac{1}{3} + \frac{\delta}{2} \right) n$, this matching contains at least $\left( \frac{2}{3} + \delta \right) n$ vertices. As $R$ is connected, this contradicts our main assumption.

By Claim 25 and the assumption (2), we may assume that either one of $R$ or $B$ is connected or $V(G) = V(R') \cup V(B')$ and $\min\{|V(R')|, |V(B')|\} \geq \left( \frac{2}{3} - \delta \right) n$. In either case, there will be a monochromatic component of order at least $\left( \frac{2}{3} - \delta \right) n$. Let $X_R = V \setminus V(R')$ and $X_B = V \setminus V(B')$. We make the following definitions when $R'$ or $B'$ is large.

**Definition.** Fix an integer $t \geq \delta^{-1}$. Suppose that $|R'| \geq \left( \frac{2}{3} + \delta \right) n$.

- Let $S_R \subseteq V(R')$ be a set such that  
  \[ q(R[V(R') - S_R]) > |S_R| + |V(R')| - \left( \frac{2}{3} + \delta \right) n. \]  
  Note that, in view of our main assumption, such a set exists by Theorem T3.

- For $1 \leq i \leq n$, let $T_{R,i}$ be the set of vertices which lie in components of $R[V(R') - S_R]$ of order $i$.

- Let $T_R = \bigcup_{1 \leq i \leq t} T_{R,i}$.

If $|B'| \geq \left( \frac{3}{4} + \delta \right) n$, we define $S_B$, $T_{B,i}$ and $T_B$ similarly.

We shall use the following result throughout. Note that, as with Claim 26, we may exchange the roles of $R$ and $B$ to obtain a symmetrical version of this result.

**Claim 27.** Suppose that $|V(R')| \geq \left( \frac{3}{4} + \delta \right) n$. Then $|S_R| < \left( \frac{1}{3} + \frac{1}{2}\delta \right) n$ and  
\[ |X_R \cup T_R| > |S_R| + \left( \frac{1}{3} - 2\delta \right) n. \]  

Further, if $C_B$ is a component of $B$ with $|C_B| \leq \left( \frac{5}{12} - 2\delta \right) n$, then $C_B \subseteq S_R$. 

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Proof. All vertices of \( V(R') \) lie in \( S_R \) or some component of \( R[V(R') - S_R] \). Hence

\[
|V(R')| \geq |S_R| + q(R[V(R') - S_R])
\]

\[
> 2|S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n.
\]

This implies that \(|S_R| < \left(\frac{1}{3} + \frac{1}{2}\delta\right)n\).

There are at most \(|T_R| \) components of \( R[V(R') - S_R] \) of order at most \( t \). However, there are at most \( \frac{1}{2}n \leq \delta n \) components of \( R[V(R') - S_R] \) of order at least \( t \). Hence \(|T_R| \geq q(R[V(R') - S_R]) - \delta n \). As \( X_R \) and \( T_R \) are disjoint, we have

\[
|X_R \cup T_R| \geq n - |V(R')| + q(R[V(R') - S_R]) - \delta n
\]

\[
> (1 - \delta)n - |V(R')| + |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n
\]

\[
= |S_R| + \left(\frac{1}{3} - 2\delta\right)n.
\]

Finally, suppose that \( C_B \) is a component of \( B \) with \(|C_B| \leq \left(\frac{5}{12} - 2\delta\right)n\). A vertex in \( C_B \) has blue degree at most \(|C_B| - 1 \). Hence any vertex in \( C_B \) must have red degree at least

\[
\delta(G) - |C_B| + 1 \geq \left(\frac{3}{4} - \delta\right)n - \left(\frac{5}{12} - 2\delta\right)n + 1
\]

\[
= \left(\frac{1}{3} + \delta\right)n + 1.
\]

(3)

A vertex in \( X_R \) has red degree at most

\[
|X_R| - 1 \leq \left(\frac{1}{3} - \delta\right)n - 1.
\]

However, a vertex in \( T_R \) is in a component of \( R[V(R') - S_R] \) of order at most \( t \). Hence, for all \( v \in T_R \),

\[
d_R(v) \leq t + |S_R|
\]

\[
< \left(\frac{1}{3} + \frac{1}{2}\delta\right)n + t.
\]

Hence (3) and \( \delta(G) \geq \left(\frac{3}{4} - \delta\right)n \) imply that \( C_B \cap (X_R \cup T_R) = \emptyset \).
Suppose that there exists \(v \in C_B \setminus S_R\). A blue neighbour of \(v\) lies in \(C_B\), and so \(v\) has no blue neighbours in \(X_R \cup T_R\). However, \(C_B \subseteq V(R')\) and so \(v\) has no red neighbours in \(X_R\). The only vertices with red neighbours in \(T_R\) are those in \(S_R \cup T_R\), and so we see that \(v\) also has no red neighbours in \(T_R\). Hence \(v\) has no neighbours in \(X_R \cup T_R\), and so

\[
d_G(v) \leq n - |X_R \cup T_R| < \left(\frac{2}{3} + 2\delta\right)n.
\]

This contradicts \(\delta(G) \geq (\frac{3}{4} - \delta)n\), and so \(C_B \subseteq S_R\). \(\square\)

We may thus assume that \(S_R\) is not much bigger than \(\frac{1}{3}n\). The following result shows that if \(S_R\) has order approaching \(\frac{1}{3}n\) and \(R'\) is very large, \(R[V \setminus S_R]\) is the required graph with low density.

**Claim 28.** Suppose that \(|V(R')| \geq (1 - \frac{5\delta}{2})n\) and that \(|S_R| \geq (\frac{1}{3} - 2\delta)n\). Then \(R[V \setminus S_R]\) is a graph on at least \((\frac{2}{3} - \frac{\delta}{2})n\) vertices with maximum degree at most \(10\delta n\).

**Proof.** That \(R[V \setminus S_R]\) is a graph of order at least \((\frac{2}{3} - \frac{\delta}{2})n\) follows immediately from Claim 27.

For all \(1 \leq i \leq n\), there are exactly \(\frac{1}{i}|T_{R,i}|\) components of order \(i\) in \(R[V(R') - S_R]\). Hence

\[
\sum_{i \geq 1} \frac{1}{2i - 1}|T_{R,2i-1}| = q(R[V(R') - S_R]) > |S_R| + |V(R')| - \left(\frac{2}{3} + \delta\right)n.
\]

However,

\[
\sum_{i \geq 1} \frac{1}{2i - 1}|T_{R,2i-1}| \leq |T_{R,1}| + \frac{1}{3} \sum_{i \geq 2} |T_{R,2i-1}|
\]

\[
\leq |T_{R,1}| + \frac{1}{3} (|V(R')| - |S_R| - |T_{R,1}|).
\]

Combining these inequalities, and using the bounds on \(|V(R')|\) and \(|S_R|\), we
have
\[
\frac{2}{3} |T_{R,1}| > \frac{4}{3} |S_R| + \frac{2}{3} |V(R')| - \left(\frac{2}{3} + \delta\right)n
\]
\[
\geq \frac{4}{3} \left(\frac{1}{3} - 2\delta\right)n + \frac{2}{3} \left(1 - \frac{5\delta}{2}\right)n - \left(\frac{2}{3} + \delta\right)n
\]
\[
= \left(\frac{4}{9} - \frac{16\delta}{3}\right)n.
\]

Hence \(|T_{R,1}| > \left(\frac{2}{3} - 8\delta\right)n\).

However, \(T_{R,1}\) is a set of isolated vertices in \(R[V \setminus S_R]\). As \(|V \setminus S_R| \leq \left(\frac{2}{3} + 2\delta\right)n\), we see that \(R[V \setminus S_R]\) has maximum degree at most \(10\delta n\). \(\square\)

We may now complete the proof of Lemma 20 using the preceding claims. By Claim 25, we may without loss of generality assume that \(|V(R')| \geq \left(\frac{2}{3} - \delta\right)n\). We divide into cases depending on the order of \(B'\).

If \(|B'| \leq \left(\frac{1}{3} + \frac{2}{3}\right)n\), then any component of \(B\) has order at most \(\left(\frac{1}{3} + \frac{2}{3}\right)n\). By Claim 27, \(|S_R| < \left(\frac{1}{3} + \frac{2}{3}\delta\right)n\) and \(S_R\) contains any small blue components. Thus \(S_R\) contains all components of \(B\), and hence has order \(n\), a contradiction.

We can not have \(\left(\frac{1}{3} + \frac{2}{3}\right)n \leq |B'| \leq \left(\frac{2}{3} - \frac{\delta}{2}\right)n\) by Claim 26. If \(\left(\frac{2}{3} - \frac{\delta}{2}\right)n < |B'| < \left(\frac{2}{3} - \frac{\delta}{2}\right)n\), then, by Claim 25, \(R\) is connected. Also, all components of \(B\) other than \(B'\) have order at most \(\left(\frac{1}{3} + \frac{2}{3}\right)n\). Hence, by Claim 27, \(S_R\) contains \(X_B\) and so \(|S_R| > \left(\frac{2}{3} - \delta\right)n\). Thus we are done by Claim 28.

Finally, suppose that \(|B'| \geq \left(\frac{2}{3} + \delta\right)n\). We may assume that there is a set \(S_B \subseteq V(B')\) such that

\[
q(B | V(B') - S_B|) > |S_B| + |V(B')| - \left(\frac{2}{3} + \delta\right)n.
\]

By Claim 27, we see that \(X_R \subseteq S_B\) and \(X_B \subseteq S_R\).

Suppose that there is a vertex \(v \in T_R \cap T_B\). Then \(v\) has at most \(|S_R| + t\) red neighbours and at most \(|S_B| + t\) blue neighbours. As \(|S_R|\) and \(|S_B|\) both have order at most \(\left(\frac{1}{3} + \frac{2}{3}\right)n\), this contradicts the minimal degree of \(G\). Hence \(T_R \cap T_B = \emptyset\).

Suppose that \(T_B \setminus S_R\) is non-empty and let \(v \in T_B \setminus S_R\). As \(X_R \subseteq S_B\), we have \(T_B \subseteq V(R')\). Hence \(v\) has no red neighbours in \(X_R\). Vertices in \(T_R\) only have red neighbours in \(T_R \cup S_R\). However, \(T_B \cap T_R = \emptyset\) and so \(v \notin T_R \cup S_R\). In particular \(v\) has no red neighbours in \(X_R \cup T_R\).

Hence, \(v\) has at least \(|X_R \cup T_R| - \left(\frac{1}{3} + \delta\right)n\) blue neighbours in \(X_R \cup T_R\), as \(v\) has at most \(\left(\frac{1}{3} + \delta\right)n\) non-neighbours. However \(v \in T_B\) and so all but \(t\)
of its blue neighbours are in \(S_B\). Hence
\[
|S_B| \geq |X_R \cup T_R| - \left(\frac{1}{4} + \delta\right) n - t > |S_R| + \left(\frac{1}{12} - 3\delta\right) n - t,
\]
where the second inequality uses Claim 27.

Similarly, if \(T_R \setminus S_B\) is non-empty, then
\[
|S_R| > |S_B| + \left(\frac{1}{12} - 3\delta\right) n - t.
\]
As these can not both occur, one of \(T_R \setminus S_B\) or \(T_B \setminus S_R\) is empty.

We assume without loss of generality that \(T_B \subseteq S_R\). Then \(S_R\) contains
the disjoint sets \(T_B\) and \(X_B\). Hence, using Claim 27, again
\[
|S_R| \geq |T_B \cup X_B| > |S_B| + \left(\frac{1}{3} - 2\delta\right) n.
\]
Thus \(|S_R| \geq \left(\frac{1}{3} - 2\delta\right) n\). As \(|S_R| < \left(\frac{1}{3} + \frac{1}{2}\delta\right) n\), we must have \(|S_B| \leq \frac{5\delta}{2} n\).
As \(X_R \subseteq S_B\), we see that \(|V(R')| \geq \left(1 - \frac{5\delta}{2}\right) n\). Hence, by Claim 28, we are done.

We now prove Lemma 21, using similar methods to those used in the proof of Lemma 20.

**Proof of Lemma 21** We assume throughout that \(n\) is sufficiently large. Let \(0 < \delta < \frac{1}{6}\). Suppose that \(G\) is a graph of order \(n\) with \(\delta(G) \geq \left(\frac{3}{4} - \delta\right) n\) and
that we are given a 2-edge colouring \(E(G) = E(R) \cup E(B)\).

Suppose that \(R'\) contains a matching on at least \(\left(\frac{2}{3} + \delta\right) n\) vertices. We may assume that \(|V(R')| < (1 - 5\delta) n\) and \(R'\) is bipartite with classes \(Y_R\) and \(Z_R\), otherwise we are done. Without loss of generality, we assume that \(|Z_R| \geq |Y_R|\) and so \(|Y_R| \leq \frac{1}{2}|V(R')| < \left(\frac{1}{2} - \frac{5\delta}{2}\right) n\). As each edge of the matching contains one vertex from \(Z_R\) and one from \(Y_R\), we have
\[
\left(\frac{1}{3} + \frac{\delta}{2}\right) n \leq |Y_R| \leq |Z_R| < \left(\frac{2}{3} - \frac{11\delta}{2}\right) n.
\]

As in the proof of Lemma 20, we let \(X_R = V \setminus V(R')\) and \(X_B = V \setminus V(B')\). Note that \(5\delta n < |X_R| \leq \left(\frac{1}{3} - \delta\right) n\). As \(|V(R')| \geq \left(\frac{3}{4} + \delta\right) n\), we are not in case (iii) of Claim 25. Hence we may assume that \(|X_B| \leq \left(\frac{1}{4} - \delta\right) n\) and \(X_B \cap X_R = \emptyset\).

We will first show that \(B'\) contains a matching on at least \(\left(\frac{1}{2} + \delta\right) n\) vertices. Suppose not; then by Theorem 13 there is a set \(S \subseteq V(B')\) such that
\[
q (B | V(B') - S]) > |S| + |V(B')| - \left(\frac{1}{2} + \delta\right) n.
\]
We will apply the same arguments as used in Claim 27 to the set $S$.

All vertices of $V(B')$ lie in $S$ or some component of $B[V(B') - S]$. Hence

$$|V(B')| \geq |S| + q(B[V(B') - S])$$

$$> 2|S| + |V(B')| - \left(\frac{1}{2} + \delta\right)n.$$ 

This implies that $|S| < \left(\frac{1}{4} + \frac{1}{2}\delta\right)n$.

Let $t$ be an integer with $t \geq \delta^{-1}$. We let $T$ be the set of vertices in components of $B[V(B') - S]$ with order at most $t$. Then $|T| \geq q(B[V(B') - S]) - \delta n$.

Any vertex in $T$ has blue degree at most

$$|S| + t \leq \left(\frac{1}{4} + \delta\right)n + t$$

and any vertex in $X_B$ has blue degree at most

$$|X_B| - 1 \leq \left(\frac{1}{4} + \delta\right)n - 1.$$ 

Also any vertex in $X_R$ has red degree at most

$$|X_R| - 1 \leq \left(\frac{1}{3} - \delta\right)n - 1$$

and any vertex in $Z_R$ has red degree at most

$$|Y_R| < \left(\frac{1}{2} - \frac{5\delta}{2}\right)n.$$ 

Hence any vertex in the intersection of $T \cup X_B$ and $Z_R \cup X_R$ has degree at most $\left(\frac{3}{4} - \frac{3\delta}{2}\right)n - 1$. As $\delta(G) \geq \left(\frac{3}{4} - \delta\right)n$, we deduce that $T \cup X_B$ does not intersect $Z_R \cup X_R$.

Hence $T \cup X_B \subseteq Y_R$. However, $T$ and $X_B$ are disjoint sets, and so

$$|Y_R| \geq |T \cup X_B|$$

$$\geq q(B[V(B') - S]) - \delta n + n - |V(B')|$$

$$> |S| + |V(B')| - \left(\frac{1}{2} + \delta\right)n + (1 - \delta) n - |V(B')|$$

$$\geq \left(\frac{1}{2} - 2\delta\right)n.$$
a contradiction. So \( B' \) contains a matching on at least \( (\frac{1}{2} + \delta) n \) vertices.

We will show that \( B' \) contains all vertices in \( X_R \cup Z_R \). All vertices of \( G \)
have at most \( (\frac{1}{4} + \delta) n \) non-neighbours, and so any two vertices have at least
\( (\frac{1}{2} - 2\delta) n \) common neighbours. As \( |Y_R| \leq (\frac{1}{2} - \frac{5\delta}{2}) n \), any pair of vertices
in \( Z_R \) have a common neighbour in \( V \setminus Y_R \). As all vertices in \( Z_R \) have
no red neighbours in \( V \setminus Y_R \), any two vertices in \( Z_R \) have a common blue
neighbour. Hence all vertices of \( Z_R \) lie in the same blue component. Similarly,
if \( |Z_R| < (\frac{1}{2} - 2\delta) n \) all vertices of \( Y_R \) lie in a single blue component.

Any vertex in \( X_R \) has at most \( (\frac{1}{4} + \delta) n \) non-neighbours in both \( Y_R \) and
\( Z_R \). Thus, by [4], every vertex in \( X_R \) has at least one neighbour in both \( Z_R \)
and \( Y_R \), which is necessarily blue. Hence \( X_R \cup Z_R \) lies within a component
of \( B \) and, if \( |Z_R| < (\frac{1}{2} - 2\delta) n \), then \( B \) is connected. If \( B \) is not connected,
then the component of \( B \) containing \( X_R \cup Z_R \) has order at least \( n - |Y_R| \geq \)
\( (\frac{1}{2} + \frac{3\delta}{2}) n \), and hence this component is \( B' \).

Suppose now that \( |V(B')| < (1 - 5\delta) n \) and \( B' \) is bipartite, with classes
\( Z_B \) and \( Y_B \). Both \( Z_B \cap Z_R \) and \( Y_B \cap Z_R \) are independent sets of \( G \)
and hence have order at most \( (\frac{1}{4} + \delta) n \). If \( |Z_R| < (\frac{1}{2} - 2\delta) n \), then, by the
above argument, \( B \) is connected. So we may assume that \( |Z_R| \geq (\frac{1}{2} - 2\delta) n \).
Hence, as \( Z_R \subseteq B' = Y_B \cup Z_B \), both \( Z_B \cap Z_R \) and \( Y_B \cap Z_R \) have order at
least \( (\frac{1}{4} - 3\delta) n \).

Let \( v \in X_R \). As \( X_R \cap X_B = \emptyset \), we see that \( v \in Z_B \cup Y_B \). We may assume
without loss of generality that \( v \in Z_B \). Then \( v \) has no blue neighbours in \( Z_B \),
and no red neighbours in \( Z_R \). In particular, \( v \) has no neighbours of either
colour in \( Z_B \cap Z_R \), which is a set of order at least \( (\frac{1}{4} - 3\delta) n \). As \( v \) has
at most \( (\frac{1}{4} + \delta) n \) non-neighbours, it thus has at most \( 4\delta n \) non-neighbours
in \( Y_R \subseteq V \setminus (Z_R \cap Z_B) \). However, all edges from \( v \) to \( Y_R \) are blue. Thus
tall but at most \( 4\delta n \) vertices in \( Y_R \) lie in the same blue component as \( v \).
However, \( v \in V(B') \), and \( X_R \cup Z_R \subseteq B' \). Hence \( B' \) contains all but \( 4\delta n \)
vertices, contradicting our assumption that \( |V(B')| < (1 - 5\delta) n \). Hence,
either \( |V(B')| \geq (1 - 5\delta) n \) or \( B' \) contains an odd cycle, and we are done. \( \square \)

We shall now prove Lemma 22, Lemma 23, and Lemma 24, which deal with
particular cases arising from the reduced graph. In both of these lemmas, we
shall be using the graph \( G' \subseteq G \) defined by the Regularity Lemma.

Proof of Lemma 22. Suppose that \( B_G \) has an independent set \( S \) with \( |S| \geq \)
\( \frac{1}{4} n \). All vertices in \( S \) have at most \( \frac{1}{4} n \) non-neighbours in \( G \), and so \( \delta(G[S]) \geq |S| - \frac{1}{4} n \geq \frac{1}{2}|S| \). Hence, by Theorem 1, \( R_G[S] \) is hamiltonian. However, the
minimal degree condition implies that \( e(G[S]) \geq \frac{1}{4}|S|^2 \) and so, by Theorem 2, either \( R_G[S] \) is pancyclic, or \( R_G[S] \cong K_{|S|/2, |S|/2} \). In the latter case,
\( \delta(G[S]) = \frac{1}{2}|S| \), and so \( |S| = \frac{1}{2} n \). Hence, if \( |S| > \frac{1}{2} n \), then \( C_\ell \subseteq R_G \) for all
\( \ell \in [3, |S|] \).
Suppose that $B_G$ is bipartite with classes $S_1$ and $S_2$, chosen so that $|S_1| \geq |S_2|$. If $|S_1| > \frac{1}{2} n$, then $C_\ell \subseteq R_G$ for all $\ell \in [3, |S_1|]$ and we are done. Hence we may assume that $n$ is even and $|S_1| = |S_2| = \frac{1}{2} n$. But by the above, we must have either that $C_\ell \subseteq R_G$ for all $\ell \in [3, \frac{1}{2} n]$, or both $R_G[S_1]$ and $R_G[S_2]$ are isomorphic to $K_{n/4, n/4}$. This implies that $n$ is divisible by four. Also, both $B_G[S_1]$ and $B_G[S_2]$ are isomorphic to the empty graph and so $G \cong K_{n/4, n/4, n/4}$.

For $i \in \{1, 2\}$, let $S_{i,1}$ and $S_{i,2}$ be the independent sets of $G$ partitioning $S_i$. Then if $R_G$ is not bipartite, without loss of generality, there are red edges between $S_{i,1}$ and both $S_{2,1}$ and $S_{2,2}$. Hence there is a red path of length either two or four between a vertex of $S_{2,1}$ and a vertex of $S_{2,2}$, with all internal vertices in $S_1$. As $R_G$ is complete between $S_{2,1}$ and $S_{2,2}$, $R_G$ contains $C_\ell$ for all $\ell \in [4, \lceil \frac{1}{2} n \rceil]$. If, however, $R_G$ is bipartite then the colouring is a 2-bipartite 2-edge colouring. \hfill \square

**Proof of Lemma 2.4** Let $S' \subseteq V(G)$ be the union of the clusters in $S$. Then $|S'| \geq \left(\frac{3}{4} - \delta\right) n$. Let $v \in S'$. The only red neighbours of $v$ in $G'[S']$ lie in clusters adjacent in $R_G$ to the cluster containing $v$. Hence $v$ has at most $10 \delta km \leq 10 \delta n$ red neighbours in $G'[S']$. However, the Regularity Lemma implies that $d_{G'}(v) > d_G(v) - (d + \epsilon) n$. Hence $\Delta(R_G[S']) \leq 11 \delta n$.

Any vertex $v \in S'$ has at least $|S'| - \frac{1}{4} n$ neighbours in $S'$ and so

$$\delta(B_G[S']) \geq |S'| - \left(\frac{1}{4} + 11 \delta\right) n$$

$$\geq \frac{1}{2} |S'|.$$

Thus, by Theorem 1 and Theorem 2, the graph $B_G[S']$ is pancyclic. In particular $C_\ell \subseteq B_G$ for all $\ell \in [3, \left(\frac{3}{4} - \delta\right) n]$. \hfill \square

**Proof of Lemma 2.4** For $1 \leq i \leq 4$, let $W_i \subseteq V$ be the union of the clusters in $U_i$, so that $\min_{i} |W_i| \geq \left(\frac{1}{4} - 4\delta\right) n$. Then $V_0$ is the set of all remaining vertices. Note that in $G'$ there are no blue edges from $W_1 \cup W_3$ to $W_2 \cup W_4$ and no red edges from $W_1 \cup W_2$ to $W_3 \cup W_4$.

Recall that $\delta(G') \geq \left(\frac{3}{4} - \delta\right) n$ and hence vertices in $W_1 \cup \cdots \cup W_4$ have at most $\left(\frac{1}{4} + \delta\right) n$ non-neighbours in $G'$. For a vertex in $W_1$, at least $\left(\frac{1}{4} - 4\delta\right) n$ of these non-neighbours are in $W_4$. Hence vertices in $W_1$ are adjacent in $G'$ (and hence in $G$) to all but at most $5\delta n$ vertices in $W_1 \cup W_2 \cup W_3$. Similar results hold for $W_2$, $W_3$ and $W_4$. Hence $\delta(G[W_i]) \geq |W_i| - 5\delta n$ for all $1 \leq i \leq 4$. Also, the bipartite graphs $B_G[W_1, W_3]$, $B_G[W_2, W_4]$, $R_G[W_1, W_2]$ and $R_G[W_3, W_4]$ have minimal degree at least $\left(\frac{1}{4} - 9\delta\right) n$. 26
Our main tools to prove the lemma will be the following two claims. The first excludes a particular case, while the second gives us long monochromatic paths.

**Claim 29.** There is no set $S$ of order at most three such that $V \setminus S$ can be partitioned into non-empty sets $X_1, \ldots, X_4$ such that, for $i = 1, \ldots, 4$ $G$ has no edges between $X_i$ and $X_{5-i}$.

**Proof.** Suppose that there is such a set $S$. Then $\sum_{i=1}^{4} |X_i| \geq n - 3$, and so, for some $1 \leq i \leq 4$, we have $|X_i| \geq \frac{1}{4}(n - 3)$. As $X_{5-i} \neq \emptyset$, we may consider a vertex $v \in X_{5-i}$. Then $v$ has no neighbours in $X_{5-i}$ and is also not adjacent to itself. Hence $d_G(v) \leq n - (|X_i| + 1) < \frac{1}{4}n$, contradicting the minimal degree of $G$. \hfill $\Box$

**Claim 30.** For any two vertices $u$ and $w$ of $W_1 \cup W_2$, the graph $R_G[W_1 \cup W_2]$ contains a $u$-$w$ path of length $\ell$ for all $\ell \in [2, \left(\frac{1}{2} - 29\delta\right)n]$ of a given parity (odd if $u \in W_1$ and $w \in W_2$ or vice versa and even otherwise). If there is a red edge in $W_1$ or $W_2$, other than $uw$, then $R_G[W_1 \cup W_2]$ contains a red $u$-$w$ path of length $\ell$ for all $\ell \in [6, \left(\frac{1}{2} - 29\delta\right)n]$.

Furthermore $R_G[W_1 \cup W_2]$ contains a cycle of length $\ell$ for all even $\ell \in [4, \left(\frac{1}{2} - 29\delta\right)n]$. If there is a red edge in $W_1$ or $W_2$, then $R_G[W_1 \cup W_2]$ contains a cycle of length $\ell$ for all $\ell \in [4, \left(\frac{1}{2} - 29\delta\right)n]$.

**Proof.** Let $r \leq \min\{|W_1|, |W_2| - 10\delta n\}$. We may assume either that $u$ and $w$ both are in $W_1$, in which case we let $v_1 = u$, or that $u \in W_2$ and $w \in W_1$, in which case we let $v_1$ be some red neighbour of $u$ in $W_1 \setminus \{v\}$.

Let $v_2, \ldots, v_{r-1}$ be a sequence of vertices in $W_1 \setminus \{v_1, w\}$. Recall that, for all $1 \leq i \leq r-1$, the vertex $v_i$ is adjacent in $R_G$ to all but at most $5\delta n$ vertices of $W_2$. Hence, for all $1 \leq i \leq r-2$, the vertices $v_i$ and $v_{i+1}$ have at least $|W_2| - 10\delta n - 1 \geq r - 1$ common red neighbours in $W_2 \setminus \{u\}$. Hence, there are distinct vertices $w_i \in W_2 \setminus \{u\}$ for $1 \leq i \leq r-1$ such that $v_iw_i$ and $w_iv_{i+1}$ are edges of $R_G$. For all $1 \leq i \leq r-1$, the vertices $w_i$ and $v_i$ have at least $r$ common red neighbours in $W_2$, and so at least one in $W_2 \setminus \{w_1, \ldots, w_{r-2}, u\}$. Hence, for all $1 \leq i \leq r-1$, there are (not necessarily distinct) vertices $u_i$ in $W_2 \setminus \{w_1, \ldots, w_{r-2}, u\}$ such that $u_i$ is adjacent to both $w_i$ and $v_i$ in $B_{G'}$. Similarly, for all $1 \leq i \leq r-1$, there are (not necessarily distinct) vertices $u_i'$ in $W_2 \setminus \{w_1, \ldots, w_{r-2}, u\}$ such that $u_i'$ is adjacent to both $v_i$ and $v_i$ in $B_{G'}$.

Then, for all $1 \leq i \leq r-1$, the graph $R_{G'}[W_1 \cup W_2 \setminus \{u\}]$ contains a $v_1$-$w$ path

$v_1w_1v_2w_2 \ldots w_{i-1}v_iu_iw$

and, for $i \geq 2$ a cycle

$v_1w_1v_2w_2 \ldots w_{i-1}v_iu_i'v_1$,
both of length $2i$. Note that we thus have the required even cycles and fixed parity $u$-$w$ paths.

Suppose that there is a red edge $xy \neq uw$ in $W_1$ or $W_2$. If the edge is in $W_1$, we choose the sequence $v_2, \ldots, v_{r-1}$ so that one of $v_1v_2$ and $v_2v_3$ is the edge $xy$ (which depends on whether $w \in \{x, y\}$). Then, for all $3 \leq i \leq r-1$, the graph $R_G[W_1 \cup W_2 \setminus \{u\}]$ contains a $v_1$-$w$ path

$$v_1v_2w_2 \ldots w_{i-1}v_iu_iw \text{ or } v_1w_1v_2v_3 \ldots w_{i-1}v_iu_iw$$

of length $2i - 1$.

Note that we also have a cycle

$$v_1v_2w_2 \ldots w_{i-1}v_iu_iw'v_1 \text{ or } v_1w_1v_2v_3 \ldots w_{i-1}v_iu_iw'v_1$$

of length $2i - 1$. The case when the edge is in $W_2$ is similar. □

Similar results hold for each of $R_G[W_3 \cup W_4]$, $B_G[W_1 \cup W_3]$ and $B_G[W_2 \cup W_4]$. In particular, there is an edge of some colour in $W_1$ and so $G$ contains monochromatic cycles of length $\ell$ for all $\ell \in [4, (\frac{1}{2} - 29\delta)n]$. To complete the proof, we need to show that $G$ contains a monochromatic cycle of length $\ell$ for all $\ell \in [(\frac{1}{2} - 29\delta)n, \frac{1}{2}n]$, and a monochromatic cycle of length at least $(1 - 59\delta)n$.

Suppose now that there are two disjoint paths $P_1$ and $P_2$ from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $R_G$. Let $P_1$ have endpoints $u$ in $W_1 \cup W_2$ and $u'$ in $W_3 \cup W_4$ and let $P_2$ have endpoints $w$ in $W_1 \cup W_2$ and $w'$ in $W_3 \cup W_4$. By restricting to a smaller path if necessary, we may assume that all internal vertices of $P_1$ and $P_2$ are in $V_0$. Then, if there is any red edge in $W_1$, other than $uw$, we may use Claim 30 to find $u$-$w$ paths of length $\ell$ for all $\ell \in [6, (\frac{1}{2} - 29\delta)n]$ in $R[W_1, W_2]$. However, Claim 30 also implies that $R_G[W_3, W_4]$ contains $u'$-$w'$ paths of length $\ell$ for all $\ell \in [6, (\frac{1}{2} - 29\delta)n]$ of a given parity. By concatenating these paths with $P_1$ and $P_2$ we see that in this case we have monochromatic cycles of length $\ell$ for all $\ell \in [3, (1 - 58\delta)n]$.

So we may assume that if there are two disjoint paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $R_G$, then $\sum_{i=1}^4 e(R_G[W_i]) \leq 2$. Similarly, if there are two disjoint paths from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $B_G$, then $\sum_{i=1}^4 e(B_G[W_i]) \leq 2$. However, $e(G[W_i]) \geq (\frac{1}{2} - \frac{9\delta}{2})n|W_i|$ for all $1 \leq i \leq 4$ and so without loss of generality we may assume that there are no two disjoint red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $R_G$.

By a corollary of Menger’s Theorem, there is a vertex $v_R$ such that there are no red paths from $W_1 \cup W_2$ to $W_3 \cup W_4$ in $G - \{v_R\}$. If there is also a vertex $v_B$ such that there are no blue paths from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v_B\}$, then taking $S = \{v_R, v_B\}$, we would have a contradiction to Claim
Hence we may assume that there are two disjoint blue paths between $W_1 \cup W_3$ and $W_2 \cup W_4$. Hence, applying Claim \[30\] to the ends of these paths as above there is a blue cycle of length at least $(1 - 59\delta) n$. Thus $G$ contains a monochromatic cycle of length at least $(1 - 59\delta) n$.

To complete the proof, we need to show that $G$ contains a monochromatic cycle of length $\ell$ for all $\ell \in [(\frac{1}{2} - 29\delta) n, \lceil \frac{3}{2} n \rceil]$. We shall find a lower bound on the red degree of each vertex. Recall that we are assuming that there are two disjoint blue paths between $W_1 \cup W_2$ and $W_3 \cup W_4$. Hence we may assume that $e(B_G[W_1]) \leq 1$, or else we have blue cycles of length $\ell$ for all $\ell \in [3, (1 - 58\delta) n]$ as above. If $v \in W_1$, then in $G'$, $v$ has at most $5\delta n$ non-neighbours in $W_1 \cup W_2$ and no blue neighbours in $W_2$. However $v$ has at most one blue neighbour in $W_1$ in $G$ and hence in $G'$. Thus all vertices in $W_1$ have red degree at least $|W_1| + |W_2| - 5\delta n - 1$ in $G'$ and hence in $G$. Similar bounds hold for all vertices in $\bigcup_{i=1}^3 W_i$.

Suppose that some vertex $v \in V_0$ has at least $(\frac{1}{2} + 8\delta) n + 3$ blue neighbours. Then it must have at least two blue neighbours in at least three of the sets $W_i$. Suppose that there is a blue path $P$ from $W_1 \cup W_3$ to $W_2 \cup W_4$ in $G - \{v\}$. Without loss of generality, we may assume that $P$ has endpoints $u' \in W_1$ and $u \in W_2$ and all internal vertices of $P$ are in $V_0$. Suppose that $v$ has at least two blue neighbours in each of $W_1$, $W_2$ and $W_3$, the other cases being similar. We may find $w' \in W_1$, $w \in W_2$ and $w'' \in W_3$ with \{u, u'\} \∩ \{w, w', w''\} = \emptyset such that each of $w$, $w'$ and $w''$ are blue neighbours of $v$.

By Claim \[30\] we have the following paths:

- for all even $\ell \in [6, (\frac{1}{2} - 29\delta) n]$, $B[W_2, W_4]$ contains a $u-w$ path $P_\ell$ of length $\ell$;
- for all even $\ell' \in [6, (\frac{1}{2} - 29\delta) n]$, $B[W_1, W_3]$ contains a $u'-w'$ path $P'_{\ell'}$ of length $\ell'$;
- for all odd $\ell'' \in [7, (\frac{1}{2} - 29\delta) n]$, $B[W_1, W_3]$ contains a $u-w''$ path $P''_{\ell''}$ of length $\ell''$.

Then, for all even $\ell, \ell' \in [6, (\frac{1}{2} - 29\delta) n]$, the path

$$uP_\ell w w' P'_{\ell'} u'$$

is a blue $u-u'$ path of length $2 + \ell + \ell'$ which is internally disjoint from $P$. Similarly, for all even $\ell \in [6, (\frac{1}{2} - 29\delta) n]$ and odd $\ell'' \in [6, (\frac{1}{2} - 29\delta) n]$, the path

$$uP_\ell w w'' P''_{\ell''} u'$$

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is a blue $u$-$u'$ path of length $2 + \ell + \ell''$ which is internally disjoint from $P$.

Hence, for all $L \in [14, (1 - 58\delta) n]$, there is a blue $u$-$u'$ path of length $L$ which is internally disjoint from $P$. Since $|P| \leq |V_0| + 2 \leq \delta n$, this gives blue cycles of length $L$ for all $L \in [\delta n + 14, (1 - 58\delta) n]$. As we have already shown that $G$ contains monochromatic cycles of length $\ell$ for all $\ell \in [3, (\frac{1}{2} - 29\delta) n]$, we are done. Hence, if there is a vertex $v \in V_0$ with blue degree at least $(\frac{1}{2} + 8\delta) n + 3$, there are no blue paths from $W_1 \cup W_3$ in $W_2 \cup W_4$ in $G - v$. This contradicts Claim 29, with $S = \{v, v_R\}$. Thus each vertex in $V_0$ has blue degree at most $(\frac{1}{2} + 8\delta) n + 3$, and so red degree at least $(\frac{1}{4} - 8\delta) n - 3$.

Let $C_1$ be the red component of $G - \{v_R\}$ containing $W_1 \cup W_2$ and $C_2$ be the red component of $G - \{v_R\}$ containing $W_3 \cup W_4$. We know that $R_G[W_1 \cup W_2]$ and $R_G[W_3 \cup W_4]$ are connected, and the minimal red degree condition in $V_0$ ensures that there are at most two components in $R_G[V - \{v_R\}]$. As $v_R$ has red degree at least $(\frac{1}{4} - 8\delta) n - 3$, it has at least $(\frac{1}{8} - 5\delta) n$ red neighbours in at least one of $C_1$ or $C_2$. Let $C'_i$ be the set $C_i$, with $v_R$ added if it has at least $(\frac{1}{8} - 5\delta) n$ red neighbours in $C_i$.

Then $|C'_1| + |C'_2| \geq n$ and so we may assume without loss of generality that $|C'_1| \geq \lceil \frac{1}{2} n \rceil$. All vertices in $C'_1$ have degree in $R[C'_1]$ at least $(\frac{1}{8} - 5\delta) n$. Further, all vertices in $C'_1 \setminus V_0$ have degree in $R[C'_1]$ at least $|C'_1| - 6\delta n$. As $|C'_1| \leq (\frac{1}{2} + 8\delta) n$ and $|V_0| \leq \epsilon n$, the condition of Theorem 12 holds on $R[C'_1]$ and so $R[C'_1]$ is hamiltonian. But we also have

$$e(R[C'_1]) \geq \frac{1}{2} (|C'_1| - 6\delta n) (|C'_1| - |V_0|) \geq \frac{1}{4}|C'_1|^2.$$ 

Hence, by Theorem 2, $R[C'_1]$ is pancyclic and we are done. \qed

6 \hspace{1cm} Monochromatic circumference

In this section we shall look at the monochromatic circumference of a graph.

We begin by proving Theorem 8.

Proof of Theorem 8. As in the proof of Theorem 6 we consider the reduced graph $H$, which has order $k$ and minimal degree at least $(\frac{3}{4} - \delta) k$. Applying Lemma 20 we have one of the following.

(i) There is a component of $R_H$ or $B_H$ which contains a matching on at least $(\frac{2}{3} + \delta) k$ vertices.

(ii) There is a set $S$ of order at least $(\frac{2}{3} - \delta) k$ such that either $\Delta(R_H[S]) \leq 10\delta k$ or $\Delta(B_H[S]) \leq 10\delta k$. 

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(iii) There is a partition \( V(H) = U_1 \cup \cdots \cup U_4 \) with \( \min|U_i| \geq \left( \frac{1}{3} - 3\delta \right) k \) such that there are no blue edges from \( U_1 \cup U_2 \) to \( U_3 \cup U_4 \) and no red edges from \( U_1 \cup U_3 \) to \( U_2 \cup U_4 \).

In the first case, we use the Blow-Up Lemma as in Theorem 6 to find a monochromatic cycle of length at least \( \left( \frac{2}{3} + \frac{\delta}{2} \right) n \). In the second case, assume without loss of generality that \( \Delta(R_H[S]) \leq 10\delta k \). Then, by Lemma 23, \( G \) contains a blue cycle of length \( \ell \) for all \( \ell \in [3, \left( \frac{2}{3} - \delta \right) n] \). In the third case, \( 2^{\frac{3}{4}} \) implies that \( G \) contains a monochromatic cycle of length at least \( (1 - 59\delta) n \). \( \square \)

We will make the following definition.

**Definition.** For \( 0 < c < 1 \), let \( \Phi = \Phi_c \) be the supremum of values \( \phi \) such that any graph \( G \) of sufficiently large order \( n \) with \( \delta(G) > cn \) and a 2-colouring \( E(G) = E(R) \cup E(B) \) has monochromatic circumference at least \( \phi n \).

For \( c \geq \frac{3}{4} \), Theorem 8 implies that \( \Phi_c \geq \frac{2}{3} \). However, the example given after Theorem 8 shows that \( \Phi_c \leq \frac{3}{5} \) for all \( c \). We can also find upper and lower bounds for \( \Phi_c \) when \( c < \frac{3}{4} \), and we collect them into the following theorem.

**Theorem 31.** For all \( c \geq \frac{3}{4} \), we have \( \Phi_c = \frac{2}{3} \). For all \( c \in (0, 1) \), we have \( \Phi_c \geq \frac{1}{2} c \). Also, there are the following upper bounds on \( \Phi_c \).

\[
\Phi_c \leq \begin{cases} 
\frac{1}{2} & \text{if } c \in \left[ \frac{3}{7}, \frac{3}{4} \right) \\
\frac{2}{5} & \text{if } c \in \left[ \frac{5}{9}, \frac{3}{5} \right) \\
\frac{1}{7} & \text{if } c \in \left( \frac{2r-1}{r^2}, \frac{2r-1}{r^2} \right) \text{ for all } r \geq 3.
\end{cases}
\]

Note that, as \( c \to 0 \), we may use the last upper bound to show that \( \frac{\Phi_c}{4c} \to 1 \). Hence, asymptotically, as \( c \to 0 \), the upper and lower bounds on \( \Phi_c \) agree.

**Proof of Theorem 31.** For \( c \in (0, 1) \), a 2-edge coloured graph with \( \delta(G) > cn \) has at least \( \frac{c}{2} n^2 \) edges. Hence there are at least \( \frac{c}{2} n^2 \) edges of one colour. We may deduce from Theorem 16 that, in that colour, there is a cycle of length at least \( \frac{1}{2} c \). Hence \( \Phi_c \geq \frac{1}{2} c \) for all \( c \in (0, 1) \). We now prove the upper bounds on \( \Phi_c \).

For \( c \in \left[ \frac{3}{7}, \frac{3}{4} \right) \), let \( t \) be an integer such that \( t > \frac{1}{3 - 3c} \). We define a graph \( G_t' \) as follows. Let \( S_1 \) and \( S_2 \) be sets of order \( 2t \) and \( T \) be a set of order \( t \). Let \( R \) be the union of the complete graph on \( S_1 \) and the complete graph on \( S_2 \). Then \( R \) has circumference \( 2t \). Let \( B \) be the union of the complete graph on \( T \) and the complete bipartite graph between \( T \) and \( S_1 \cup S_2 \). Then, any
two consecutive vertices of a cycle in \( B \) must contain a vertex of \( T \) and hence \( B \) has circumference at most \( 2t \). Let \( G'_t \) be the union of \( R \) and \( B \). Then \( \delta(G) = 3t - 1 > c|G'_t| \) and so \( \Phi_c \leq \frac{2}{3} \).

If, for \( r \geq 2 \), we have \( c \in (0, \frac{2r-1}{r^2}) \), let \( t \) be an integer such that \( t > \frac{1}{2r-1-\sqrt{c}} \). Define a family \( \{ A_{i,j} : 1 \leq i \leq r, 1 \leq j \leq r \} \) of sets of order \( t \). We define the following graphs on vertex set \( \bigcup_{i,j} A_{i,j} \):

\[
E(B) = \{ uv : u \in A_{i,j}, v \in A_{i,j'} \text{ for some } 1 \leq i \leq r \text{ and } j \neq j' \};
E(R) = \{ uv : u \in A_{i,j}, v \in A_{i',j} \text{ for some } 1 \leq j \leq r \}.
\]

Let \( G_t^{(r)} \) be the union of the graphs \( R \) and \( B \), as illustrated in Figure 2 for the case \( r = 3 \). Then \( |G_t^{(r)}| = r^2t \) and \( \delta(G_t^{(r)}, (2r-1)t - 1 > c|G_t^{(r)}| \). However as all monochromatic components have order \( rt \), there are no monochromatic cycles of length greater than \( \frac{1}{r} |G_t^{(r)}| \). Hence \( \Phi_c \leq \frac{1}{r} \). Note that this case includes the bound \( \Phi_c \leq \frac{1}{2} \) for \( c < \frac{3}{4} \).

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{figure2.png}
    \caption{The graph \( G_t^{(3)} \)}
\end{figure}

7 Conclusion

Theorem 6 is a 2-colour version of the uncoloured (or 1-coloured) result of Bondy that graphs with order \( n \) and minimal degree at least \( \frac{1}{2}n \) are
pancyclic. We may hope to generalise to \( k \) colours. In this case, we let \( E(G) = \bigcup_{i=1}^{k} E(G_i) \) be an edge colouring, where each \( G_i \) is a spanning subgraph of \( G \), representing the edges coloured \( i \). Our extremal graph was found by letting both \( R \) and \( B \) be subgraphs of the extremal graph in the uncoloured case, and we again use this method to find \( k \)-coloured graphs with high minimum degree but no odd cycles.

**Definition.** Let \( n = 2^k p \) and let \( G \) be isomorphic to the \( 2^k \)-partite graph with classes all of order \( p \). A \( k \)-bipartite \( k \)-edge colouring of \( G \) is a \( k \)-edge colouring \( E(G) = \bigcup_{i=1}^{k} E(G_i) \) such that each \( G_i \) is bipartite.

As in the 2-coloured case, we can deduce that a \( k \)-bipartite \( k \)-edge colouring of the \( 2^k \)-partite graph with classes all of order \( p \) induces a labelling \( U_\alpha \) (\( \alpha \in \{1, 2\}^k \)) of the classes such that, for all \( i \), the graph \( G_i \) is bipartite with classes

\[
\bigcup_{\alpha : \alpha_i = 1} U_\alpha
\]

and

\[
\bigcup_{\alpha : \alpha_i = 2} U_\alpha.
\]

Note that this implies that, if \( \alpha \) and \( \beta \) in \( \{1, 2\}^k \) differ only in the \( i \)th place, then all edges between \( U_\alpha \) and \( U_\beta \) are coloured with \( i \). As this graph has minimum degree \( (1 - \frac{1}{2k}) n \), we make the following conjecture.

**Conjecture 32.** Let \( n \geq 3 \), and \( k \) be an integer. Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq (1 - \frac{1}{2k}) n \). If \( E(G) = \bigcup_{i=1}^{k} E(G_i) \) is a \( k \)-edge colouring, then either:

- for all \( \ell \in \left[ \min\{2^k, 3\}, \left\lceil \frac{1}{2k} n \right\rceil \right] \) there is some \( 1 \leq i \leq k \) such that \( C_\ell \subseteq G_i \), or;

- \( n = 2^k p \), \( G \) is the complete \( 2^k \)-partite graph with classes of order \( p \), and the colouring is a \( k \)-bipartite \( k \)-edge colouring.

Note that the case when \( k = 1 \) is Bondy’s Theorem, and the case \( k = 2 \) is Theorem \ref{Theorem 31}.

We pose the following problem about the monochromatic circumference.

**Problem 33.** What is the value of \( \Phi_c \) for \( c < \frac{\pi}{4} \)?

Note that Theorem \ref{Theorem 31} shows that \( \Phi_c = \frac{\pi}{2} \) for all \( c \geq \frac{\pi}{4} \). In this case, we make the following conjecture with an exact bound on the monochromatic circumference.
Conjecture 34. Let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{3}{4}n$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. If $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge colouring, then $G$ has monochromatic circumference at least $2t + r$.

Note that Theorem 8 is an asymptotic version of this conjecture. By considering the graph $F_{2t+r,t}$ as defined in Section 1, we see that this conjecture is best possible.

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