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Gradient estimates for nonlinear elliptic equations with first order terms

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Abstract. We study existence and Lorentz regularity of distributional solutions to elliptic equations with measurable coefficients and either a convection or a drift first order term. The presence of such a term makes the problem not coercive. The main tools are pointwise estimates of the rearrangements of both the solution and its gradient.

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1. Introduction and model problems

This paper is concerned with the study of existence and Lorentz regularity of distributional solutions to a class of non coercive nonlinear elliptic partial differential equations with Dirichlet boundary conditions. The non coercivity is given by the presence in the equation of first order terms of convection or drift type. To avoid technicalities, in the introduction we present the linear version of such equations, for the general case see Sect. 2.

Let us consider at first the following model problem with convection first order term

\[
\begin{cases}
- \text{div} (A(x) \nabla u) = - \text{div} (u E(x)) + f(x) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\] (1.1)

where \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), with \( N > 2 \), \( A(x) \) is a matrix with measurable coefficients that satisfies for \( \alpha, \beta > 0 \)

\[
\alpha |\xi|^2 \leq A(x) \xi \cdot \xi, \quad |A(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \quad (1.2)
\]
the vector field \( E(x) \) belongs either to the Lebesgue or the Marcinkiewicz space of order \( N \) and the function \( f(x) \) belongs to a suitable Lorentz space to be precised (see Sect. 2 for the definition of Marcinkiewicz and Lorentz spaces).

Let us provide the notion of solution that we use in the sequel. If \( f \in L^{(2^*)'}(\Omega) \) we can consider the weak formulation of problem (1.1), namely

\[
\begin{align*}
  u &\in W^{1,2}_0(\Omega) : \int_\Omega A(x) \nabla u \nabla \phi = \int_\Omega u E(x) \nabla \phi + \int_\Omega f(x) \phi \quad \forall \phi \in W^{1,2}_0(\Omega). 
\end{align*}
\]  

(1.3)

The natural assumption

\[
E(x) \in \left( L^N(\Omega) \right)^N
\]  

(1.4)

assures that the convection term of (1.3) is well defined since

\[
v E(x) \nabla \phi \in L^1(\Omega) \quad \forall \, v, \phi \in W^{1,2}_0(\Omega).
\]  

(1.5)

Anyway (1.4) is not the most general condition in order to have (1.5). Indeed if we assume

\[
E(x) \in \left( M^N(\Omega) \right)^N,
\]  

(1.6)

it follows that (1.5) continues to hold, thanks to the sharp Sobolev Embedding in Lorentz spaces, namely

\[
W^{1,q}_0(\Omega) \subset L^{q^*,q}(\Omega) \quad \text{with} \ 1 \leq q < \infty.
\]

If \( f \in L^1(\Omega) \), problem (1.1) has to be meant through the following distributional formulation

\[
\begin{align*}
  u &\in W^{1,1}_0(\Omega) : \int_\Omega A(x) \nabla u \nabla \phi = \int_\Omega u E(x) \nabla \phi + \int_\Omega f(x) \phi \quad \forall \phi \in C^1_0(\Omega). 
\end{align*}
\]  

(1.7)

Notice again that, assuming either (1.4) or (1.6), we have that \( v E(x) \in L^1(\Omega) \) for any \( v \in W^{1,1}_0(\Omega) \).

For the same result in the case \( E \equiv 0 \) we refer to [9] for \( f \) in Marcinkiewicz spaces (see also [24]) and [1] for data in Lorentz spaces.

The main feature of (1.1) is the non coercivity of the convection term, as it can be seen with the following heuristic argument. Assuming for simplicity (1.4), \( f \in L^{(2^*)'}(\Omega) \) and letting \( u \in W^{1,2}_0(\Omega) \) be the solution of (1.3), we obtain that

\[
\alpha \| u \|^2_{W^{1,2}_0(\Omega)} \leq \frac{\| E \|_{L^N(\Omega)}}{S} \| u \|^2_{W^{1,2}_0(\Omega)} + \| f \|_{L^{(2^*)'}(\Omega)} \| u \|_{L^{2^*}(\Omega)},
\]

where \( S \) denotes the Sobolev constant. Thus if the value of \( \| E \|_{L^N(\Omega)} \) is large, the first term in the right hand side above cannot be absorbed in the left hand one.
The classical approach in dealing with (1.1) (see for instance [25,30] and [32]) is to assume a smallness condition on the $L^N(\Omega)$-norm,
\[ \| E \|_{L^N(\Omega)} < S\alpha, \tag{1.8} \]
or a sign condition on the distributional divergence of $E(x)$,
\[ \int_{\Omega} E(x) \nabla \phi \leq 0 \quad \forall \phi \in C_0^1(\Omega), \tag{1.9} \]
so that the problem becomes coercive. Alternatively, to restore the lack of coercivity, one can add an absorption term in the left hand side of (1.1) (see for instance [30] or the more recent [21]).

One naturally wonders if assumptions (1.8) or (1.9) are necessary or rather it is possible to achieve a priori estimates for the solution of (1.1) even if the associated operator is not coercive. The answer is given in [6] and [10] where it has been proven the following result.

**Theorem 1.1.** Let us assume (1.2), $E \in \left(L^N(\Omega)\right)^N$ and that $f \in L^m(\Omega)$ with $1 < m < \frac{N}{2}$. Then
(i) If $(2^*)' \leq m < \frac{N}{2}$ there exists $u \in L^{m^*}(\Omega) \cap W^{1,2}_0(\Omega)$ solution of (1.3) ([6]);
(ii) If $1 < m < (2^*)'$ there exists $u \in W^{1,m^*}_0(\Omega)$ solution of (1.7) ([10]).

Thus, not only problem (1.1) is solvable in $W^{1,2}_0(\Omega)$ for any vector field $E$ satisfying (1.4) (no matter the size of its norm), but also the same sharp regularity result of the case $E \equiv 0$ (see [14]) is recovered, even for distributional solutions with data outside $W^{-1,2}(\Omega)$.

We stress that, even if Theorem 1.1 is stated for a linear problem and with $f(x)$ belonging to Lebesgue spaces, in [6] a more general nonlinear versions of (1.1) is treated with data in Lorentz spaces. Moreover in [20] the authors consider an equation with both convection and drift (see (1.11) below) first order terms, assuming a smallness condition on at least one of them. We do not treat these two lower order terms together and the reason is explained at the end of this section.

Let us briefly describe the approaches used in [10] and [6] in order to deal with problem (1.1). The strategy of the first paper hings on an a priori estimate on the measure of the super level sets of $u$. Such estimate bypasses in some sense the non coercivity of the problem and allows the author to recover the integral estimates for $u$ and $|\nabla u|$.

On the other hand, in [6] the authors approach problem (1.1) by symmetrization technique (see [31]): the main idea is to deduce a differential inequality for the decreasing rearrangement of $u$ (see Sects. 2 and 3 for a brief introduction on this issue) and compare it with the rearrangement of the solution of a suitable symmetrized problem. Since the solution of the symmetrized problem is explicit one recovers the a priori estimate for $u$ and, in turn, the energy estimate for the gradient. Let us stress that such a symmetrization approach does not provide any information about the regularity of $|\nabla u|$.
Our main contribution for problem (1.1) (and its nonlinear counterpart) is to complete the relation between the regularity of \( f \) and \( u, \, |\nabla u| \) in the framework of Lorentz spaces under optimal conditions on the summability of \( E(x) \). More in detail we provide the following result (see Theorems 2.2 and 2.3 in Sect. 2 for the general case).

**Theorem 1.2.** Assume (1.2), \( f \in L^{m,q}(\Omega) \), with \( 1 < m < \frac{N}{2}, \, 0 < q \leq \infty, \) \( E \in \left( M^N(\Omega) \right)^N \) and moreover that there exist

\[
\mathcal{F} \in \left( L^\infty(\Omega) \right)^N, \, \mathcal{E} \in \left( M^N(\Omega) \right)^N \text{ with } \|\mathcal{E}\|_{M^N(\Omega)} < \alpha \frac{N-2m}{m}, \quad (1.10)
\]

such that \( E = \mathcal{F} + \mathcal{E} \). Hence there exists a distributional solution \( u \) of (1.7). Moreover the obtained solution satisfies

- If \( 1 < m < (2^*)' \), then \( u \in L^{\frac{Nm}{N-2m},q}(\Omega) \) and \( |\nabla u| \in L^{\frac{Nm}{N-2m},q}(\Omega) \);
- If \( (2^*)' < m < \frac{N}{2} \), then \( u \in W^{1,2}_0(\Omega) \cap L^{\frac{Nm}{N-2m},q}(\Omega) \).

The most interesting part of this Theorem is the first one \( (1 < m < (2^*)') \), namely the case in which \( f(x) \) does not belong to the dual space of \( W^{1,2}_0(\Omega) \) and the regularity of the gradient increases with the regularity of the datum. To prove it we need pointwise estimates not only for \( u \), the decreasing rearrangement of \( u \), but also for \( |\nabla u| \), the decreasing rearrangement of \( |\nabla u| \). Let us stress that, while estimates of \( u \) are already known in the literature for problems similar to (1.1) (see for instance [6] for the case \( (2^*)' < m < \frac{N}{2} \)), the estimate for \( |\nabla u| \) is new.

As far as the the vector field \( E(x) \) is concerned, the assumption (1.10) is optimal in the sense that, if

\[
\|\mathcal{E}\|_{M^N(\Omega)} \geq \alpha \frac{N-2m}{m},
\]

the standard relation between the regularity of \( f \) and \( u \) is lost and the regularity of the solution depends on the value of the Marcinkiewicz norm of \( \mathcal{E} \) (see [17] and Remark 4.5). Let us also notice that assumption (1.10) is less restrictive than (1.4).

In Sect. 2 we generalize Theorem 1.2 considering a more general nonlinear operator. In this nonlinear setting we also deal with solutions in \( W^{1,1}_0(\Omega) \). This represent an additional difficulty due to the lack of compactness of bounded sequences in such a space.

We have also to mention that unfortunately our approach does not cover the case \( m = (2^*)' \). This borderline case has been solved by [27], if \( E \equiv 0 \), using non standard (nonlinear) potential arguments.

An example of the second type of problems that we consider is

\[
\begin{cases}
-\text{div}(A(x)\nabla w) = E(x)\nabla w + f(x) \text{ in } \Omega, \\
w = 0 \text{ on } \partial\Omega,
\end{cases}
\]

with \( A(x) \) satisfying (1.2), \( E(x) \) as in (1.4) or (1.6) and \( f \) that belongs to a Lorentz space. The first order term in the equation above is also called *drift* term.
In this linear setting (1.11) is (at least formally) the dual problem of (1.1) and one can use a duality approach to recover existence and regularity results (see [12,13,21]). Anyway here we treat problem (1.11) independently from (1.1), following the same spirit and aims of the previous case.

Similarly to the convection one, also the drift term makes the operator of (1.11) not coercive, unless an additional smallness assumption on the $L^N(\Omega)$ norm of $E(x)$ is assumed. Once again it is proved that such assumption is unnecessary for the existence of a weak solution, see [6] and [18]. While in the last two papers problem (1.11) is studied with symmetrization techniques, in [18] the authors obtain energy estimates for (1.11) by means of a slice method that is based on continuity properties of some modified distribution function of $w$ (see [7] and the more recent [19] for related results).

Here we adapt the techniques developed for problem (1.1) to recover Lorentz regularity results also for problem (1.11). Being mainly interested in solutions outside the energy space, let us introduce the distributional formulation of (1.11).

$$w \in W^{1,r}_0(\Omega) : \int_{\Omega} A(x) \nabla w \nabla \phi = \int_{\Omega} E(x) \nabla w \phi + \int_{\Omega} f(x) \phi \ \forall \phi \in C^1_c(\Omega),$$

(1.12)

with $r > \frac{N}{N-1}$. Notice that we have to impose that $w \in W^{1,r}_0(\Omega)$, with $r > \frac{N}{N-1}$, so that the first integral on the right hand side of (1.12) is well defined.

Also in this case the key point is to obtain pointwise estimate for $w$ and $|\nabla w|$, the decreasing rearrangements of $w$ and $|\nabla w|$. Let us state our existence and regularity result for problem (1.12).

**Theorem 1.3.** Assume (1.2), $f \in L^{m,q}(\Omega)$, with $1 < m < \frac{N}{2}$, $0 < q \leq \infty$, $E \in \left(M^N(\Omega)\right)^N$ and moreover that there exist

$$\mathcal{F} \in \left(L^\infty(\Omega)\right)^N, \ \mathcal{E} \in \left(M^N(\Omega)\right)^N \text{ with } \|\mathcal{E}\|_{M^N(\Omega)} < \alpha N^ \frac{m-1}{m},$$

such that $E = \mathcal{F} + \mathcal{E}$. Hence there exists a distributional solution $w$ of (1.12). Moreover the obtained solution satisfies

- if $1 < m < (2^*)'$, then $w \in L^{\frac{Nm}{N-2m-2q}}(\Omega)$ and $|\nabla w| \in L^{\frac{Nm}{N-2m-2q}}(\Omega)$;
- if $(2^*)' < m < \frac{N}{2}$, then $w \in W^{1,2}_0(\Omega) \cap L^{\frac{Nm}{N-2m-2q}}(\Omega)$.

We refer to the next Sect. 2 for the nonlinear version of Theorems (1.1) and (1.11).

After studying problems (1.1) and (1.11) separately, one naturally wonders why do not consider the convection and the drift terms at once. This is what is actually done in [30,32] and [20] but still imposing some additional constraints, as smallness assumptions on the $L^N$ norm of at least one of the vector field or divergence free assumptions, as (1.9). One may wonder if, also in this case, these are just technical assumptions, or rather the presence of the two first order terms represent a genuine obstruction to the solvability of the following problem

$$\begin{cases}
-\text{div}(A(x)\nabla u) = -\text{div}(u E(x)) + B(x) \nabla u + f(x) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases}$$

(1.13)
Let us observe that, in the special case $E, B \in (C^1(\Omega))^N$ and $E \equiv B$, problem (1.13) becomes
\[
\begin{cases}
-\text{div}(A(x)\nabla u) = g(x)u + f(x) \text{ in } \Omega, \\
u = 0 \text{ on } \partial\Omega.
\end{cases}
\]
with $g(x) = -\text{div}(E(x))$, that of course is not solvable for a general $E(x) \in (C^1(\Omega))^N$. Thus the presence of the two lower order terms involves some spectral issues and we do not treat it.

2. Main results

In order to state our main results in their full generality, we need to introduce some basic definitions and properties about rearrangements and Lorentz spaces.

For any measurable function $v : \Omega \to \mathbb{R}$, we define the distribution function of $v$ as
\[
A(t) := |\{x \in \Omega : |v(x)| > t\}| \text{ for } t \geq 0,
\]
and the decreasing rearrangement of $v$ as
\[
\overline{v}(s) := \inf\{t \geq 0 : A(t) < s\} \text{ for } s \in [0, |\Omega|].
\]
By construction it follows that
\[
|\{x \in \Omega : |v(x)| > t\}| = |\{s \in \mathbb{R} : \overline{v}(s) > t\}|,
\]namely the function and its decreasing rearrangement are equimeasurable. We define also the maximal function associated to $v$ as
\[
\tilde{v}(s) = \frac{1}{s} \int_0^s \overline{v}(t)dt.
\]
Notice that, since $\overline{v}(s)$ is non increasing, it follows that $\overline{v}(s) \leq \tilde{v}(s)$ for any $s \in [0, |\Omega|]$.

By definition $A(t)$ is right continuous and non increasing, while $\overline{v}(s)$ is left continuous and non increasing. Thus both functions are almost everywhere differentiable in $(0, |\Omega|)$. For a more detailed treatment of $A(t)$ and $\overline{v}(s)$ we refer to [28] and [23].

Let us give now the definition of Lorentz spaces. For $1 \leq m < \infty$ and $0 < q \leq \infty$ we say that a measurable function $f : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{m,q}(\Omega)$ if the quantity
\[
\|f\|_{L^{m,q}(\Omega)} = \begin{cases}
\left(\int_0^\infty t^{\frac{q}{m}} \overline{f}(t)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty \\
\sup_{t \in (0,\infty)} t^{\frac{1}{m}} \overline{f}(t) & \text{if } q = \infty,
\end{cases}
\]
is finite. We recall that $L^{m,m}(\Omega) = L^m(\Omega)$ and that
\[
L^{m,q}(\Omega) \subset L^{m,r}(\Omega) \text{ for any } 0 < q < r \leq \infty.
\]
The space $L^{m,\infty}(\Omega)$, with $1 \leq m < \infty$ is called Marcinkiewicz space of order $m$ and we denote it by $M^m(\Omega)$.

If we replace $\tilde{f}$ with $\tilde{g}$, we define another space $L^{(m,q)}(\Omega)$ given by all the measurable function $f : \Omega \to \mathbb{R}$ such that the quantity

$$[f]_{L^{(m,q)}(\Omega)} = \begin{cases} \left( \int_0^\infty t^\frac{q}{m} \tilde{f}(t)^q \frac{dt}{t} \right)^\frac{1}{q} & \text{if } q < \infty \\ \sup_{t \in (0,\infty)} t^\frac{1}{m} \tilde{f}(t) & \text{if } q = \infty \end{cases}$$

is finite. Since

$$\|f\|_{L^{m,q}(\Omega)} \leq [f]_{L^{(m,q)}(\Omega)} \leq m' \|f\|_{L^{m,q}(\Omega)},$$

it results that $\|\cdot\|_{L^{m,q}(\Omega)}$ and $[\cdot]_{L^{(m,q)}(\Omega)}$ are equivalent if $m > 1$ and $L^{m,q}(\Omega) \equiv L^{(m,q)}(\Omega)$. Anyway in the borderline case $m = 1$ the space $L^{(1,q)}(\Omega)$ is rather unsatisfactory since, for $q < \infty$, it contains only the zero function. This is because by definition $\tilde{f}(s) \approx \frac{1}{s}$ for $s > |\Omega|$. Hence, following [5], we define $\mathbb{L}^{1,q}(\Omega)$ as the set of measurable function $f$ such that

$$\|f\|_{\mathbb{L}^{1,q}(\Omega)} = \begin{cases} \left( \int_0^{|\Omega|} t^q \tilde{f}(t)^q \frac{dt}{t} \right)^\frac{1}{q} & \text{if } q < \infty \\ \sup_{t \in (0,|\Omega|]} t \tilde{f}(t) & \text{if } q = \infty, \end{cases}$$

is finite. Notice that in [5] is proved that $f$ belongs to $\mathbb{L}^{1,1}(\Omega)$ if and only if

$$\int |f| \log(1 + |f|) < \infty.$$

Hence $\mathbb{L}^{1,1}(\Omega) \equiv L \log L(\Omega)$, while the space $\mathbb{L}^{1,q}(\Omega)$ with $1 < q < \infty$ is a diagonal intermediate space between $L \log L(\Omega)$ and $L^1(\Omega)$ (see [5]).

Let us present now our first problem in its general form. Given $1 < p < N$, consider

$$\begin{cases} -\text{div} \left( a(x, \nabla u) \right) = -\text{div} \left( u |u|^{p-2} E(x) \right) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

(2.3)

where the Carathéodory function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (see Remark 2.5) for $0 < \alpha \leq \beta$

$$\begin{align*}
\alpha |\xi|^p & \leq a(x, \xi)|\xi|, \\
|a(x, \xi)| & \leq \beta |\xi|^{p-1}, \\
[a(x, \xi) - a(x, \xi^*)] & (|\xi| - |\xi^*|) > 0,
\end{align*}$$

(2.4)

for $x$ a.e. in $\Omega$ and for any $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$, the datum $f$ belongs to $L^{m,q}(\Omega)$ with $1 \leq m < \frac{N}{p}$, $0 < q \leq \infty$ and the vector field $E : \Omega \to \mathbb{R}^N$ is such that

$$E = \mathcal{F} + \mathcal{E} \text{ with } \mathcal{F} \in \left( L^\infty(\Omega) \right)^N \text{ and } \mathcal{E}(s)$$
\[ \frac{B}{s^{\frac{p-1}{N}}} \text{ with } B < \alpha \left( \frac{\omega_N}{(p-1)m} \right)^{p-1}. \]  

(2.5)

Where \( \omega_N \) is the volume of the unitary ball in \( \mathbb{R}^N \). Assumption (2.5), up to the addition of a whichever bounded vector filed, prescribes a threshold on the \( M^N(\Omega) \)-norm of \( E \) (see also [6] and [11]). As already said in the introduction, this smallness condition is sharp and cannot be weakened (see [17] and Remark 4.5). Let us recall also that \( \|E\|_{M^N(\Omega)\omega_N} \leq B \) (compare the constant in (1.10) with the one in (2.5)).

**Remark 2.1.** Notice that assumption (2.5) is equivalent to require that there exists \( t^* > 0 \) such that

\[ \frac{B}{s^{\frac{p-1}{N}}} \text{ for any } t \in (0, t^*), \]

with \( B \) as above. The decomposition \( E = F + E \) is introduced to simplify computations in the proof of the rearrangement inequalities. The condition \( F \in (L^\infty(\Omega))^N \)

is not restrictive and can be weakened to \( F \in (L^N(\Omega))^N \).

Let us introduce the distributional formulation of Problem (2.3).

\[ u \in W^{1,1}_0(\Omega) : \frac{|\nabla u|^{p-1}}{E^{1}(\Omega)} \in L^1(\Omega), \quad |u|^{p-1}|E(x)| \in L^1(\Omega) \quad \text{and} \]

\[ \int \Omega a(x, \nabla u)\nabla \phi = \int \Omega |u|^{p-2}uE(x)\nabla \phi + \int \Omega f(x)\phi \quad \forall \phi \in C^1_0(\Omega). \]  

(2.6)

The first result of this section reads as follows.

**Theorem 2.2.** Let us assume \( f \in L^{m,q}(\Omega) \) and that conditions (2.4) and (2.5) hold true. Then there exists a distributional solution \( u \) of (2.6). Moreover the obtained solution satisfies:

(i) If \( \max\{1, \frac{N}{N(p-1)+1}\} < m < (p^*)' \) and \( 0 < q \leq \infty \), then

\[ |u| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega); \]

(ii) If \( (p^*)' < m < \frac{N}{p} \) and \( 0 < q \leq \infty \)

\[ |u| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega) \cap W^{1,p}_0(\Omega). \]

As already said in the introduction the main novelty of this Theorem is the first part (see [6] for similar result in the case \( (p^*)' < m < \frac{N}{p} \) and \( 0 < q \leq \infty \)) and the core of the proof relies on the estimate on the decreasing rearrangement of the gradient provided in Lemma 4.3 of Sect. 4.

In order to present the second result of this section, let us recall that, if \( p \) and \( m \) are close to 1, some subtleties arise (see [4] for the case \( E \equiv 0 \)). Roughly speaking this is because the gradient of the expected solution might not be an integrable function. Indeed, if \( 1 < p < 2 - \frac{1}{N} \) and \( 1 < m < \frac{N}{N(p-1)+1} \), the notion of distributional solution is not any more adequate and entropy solutions have to be introduced (see for instance [11]). We do not treat this case and instead focus on the borderline values \( m = \max\{1, \frac{N}{N(p-1)+1}\} \). In the cases \( f \in L^1(\Omega) \) or \( f \in L^{1,q}(\Omega) \), we assume \( m = 1 \) in (2.5).
Theorem 2.3. Let us assume $m = \max\{1, \frac{N}{N(p-1)+1}\}$ and that conditions (2.4) and (2.5) hold true. Then there exists a distributional solution $u$ of (2.6). Moreover the obtained solution satisfies:

(i) If $p > 2 - \frac{1}{N}$ and $f \in L^1(\Omega)$, then

$$|u| \in L^{(p-1)N \over N-p} (\Omega) \quad \text{and} \quad |\nabla u| \in L^{(p-1)N \over N-1} (\Omega);$$

(ii) If $p > 2 - \frac{1}{N}$ and $f \in L^{1,q}(\Omega)$ with $0 < q \leq \infty$, then

$$|u| \in L^{(p-1)N \over N-p}.(p-1)q(\Omega) \quad \text{and} \quad |\nabla u| \in L^{(p-1)N \over N-1}.(p-1)q(\Omega);$$

(iii) If $p = 2 - \frac{1}{N}$ and $f \in L^{1,q}(\Omega)$ with $0 < q \leq \frac{1}{p-1} = \frac{N}{N-1}$, then

$$|u| \in L^{N \over N-1}.1 \frac{N}{N-q} (\Omega) \quad \text{and} \quad |\nabla u| \in L^{1}.(p-1)q(\Omega);$$

(iv) If $p < 2 - \frac{1}{N}$ and $f \in L^{m,q}(\Omega)$ with $m = \frac{N}{N(p-1)+1} 0 < q \leq \frac{1}{p-1}$, then

$$|u| \in L^{N \over N-1}.(p-1)q (\Omega) \quad \text{and} \quad |\nabla u| \in L^{1}.(p-1)q(\Omega).$$

The main observation on Theorems 2.2 and 2.3 is that, also in this nonlinear Lorentz setting, we recover the same results of the case $E \equiv 0$ (see [1,15,27] and reference therein). Let us further comment Theorem 2.3. In (i) and (ii) the summability of the data assures that $|\nabla u|$ belongs to a Lebesgue space smaller (more regular) than $L^1(\Omega)$. On the contrary, in (iii) and (iv), the gradient belongs to Lorentz spaces with first exponent equal to 1. Such spaces are contained at most in $L^1(\Omega)$ and this makes more difficult the proof since $L^1(\Omega)$ is not reflexive (we refer to [15] for corresponding results restricted to the Lebesgue framework and $E \equiv 0$).

Finally let us focus on nonlinear drift term. Let us consider, for $p > 1$,

$$
\begin{cases}
-\text{div} (a(x, \nabla w)) = E(x)|\nabla w|^{p-2} \nabla w + f(x) \text{ in } \Omega, \\
\quad w = 0 \quad \text{on } \partial \Omega,
\end{cases}
\tag{2.7}
$$

where the Carathéodory function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (2.4), the datum $f$ belongs to $L^{m,q}(\Omega)$ with $1 \leq m < {N \over p}$, $0 < q \leq \infty$ and the vector field $E : \Omega \to \mathbb{R}^N$ is such that

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^{\infty}(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \leq {B \over s^{1 \over N}} \quad \text{with} \quad B < \alpha \omega_m^{1 \over N} {N \over m} - 1. \tag{2.8}
$$

Let us recall again that $\|\mathcal{E}\|_{M^N(\Omega)}^{1 \over N} \omega_N^{1 \over N} \leq B$. It is immediate to note that this assumption becomes more and more restrictive as $m$ approaches 1. This is not just a technical inconvenient and prevent us to treat the case $f$ in $L^1(\Omega)$ or $L^{1,q}(\Omega)$ with $1 < q < \infty$. Indeed, for such type of data and assuming (2.8), the expected regularity of the gradient is too low to have the drift term of (2.7) well defined (we
refer the interested reader to [7]). We consider the following weak formulation of problem (2.7).

\begin{align*}
  w \in W^{1,1}_0(\Omega) : & \quad |\nabla w|^{p-1} \in L^1(\Omega), \quad |E(x)||\nabla w|^{p-1} \in L^1(\Omega) \quad \text{and} \\
  & \quad \int_{\Omega} a(x, \nabla w) \nabla \phi = \int_{\Omega} E(x)|\nabla w|^{p-2} \nabla w \phi + \int_{\Omega} f(x) \phi \quad \forall \phi \in C^1_0(\Omega).
\end{align*}

(2.9)

Let us state the existence and regularity result for problem (2.9).

**Theorem 2.4.** Let us assume \( f \in L^{m,q}(\Omega) \) and that conditions (2.4) and (2.8) hold true. Then there exists a distributional solution \( w \) of (2.9). Moreover the obtained solution satisfies:

(i) If \( \max\{1, \frac{N}{N(p-1)+1}\} < m < (p^*)' \) and \( 0 < q \leq \infty \), then

\[ |w| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega) \quad \text{and} \quad |\nabla w| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega), \]

(ii) If \( (p^*)' < m < \frac{N}{p} \) and \( 0 < q \leq \infty \), then

\[ |w| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega) \cap W^{1,p}_0(\Omega). \]

**Remark 2.5.** We stress that all the previously presented Theorems remain valid for more general second order operators of the Leray-Lions type (see [26]), as \(-\div(a(x, u, \nabla u))\), where the Carathéodory function \( a(x, \eta, \xi) \) satisfies for \( 0 < \alpha \leq \beta \) and \( l(x) \in L^{p'}(\Omega) \)

\begin{align*}
  \alpha|\xi|^p & \leq a(x, \eta, \xi)|\xi|, \\
  |a(x, \eta, \xi)| & \leq \beta(|\eta|^{p-1} + |\xi|^{p-1} + l(x)), \\
  [a(x, \eta, \xi) - a(x, \eta, \xi^*)][\xi - \xi^*] & > 0,
\end{align*}

for \( x \) a.e. in \( \Omega \), \( \eta \in \mathbb{R} \) and \( \xi, \xi^* \in \mathbb{R}^N \) with \( \xi \neq \xi^* \). However, we decided to restrict our attention on simpler operators in the form (2.4), to not overload the already technical proofs.

Schematically the strategy of the proof of Theorems 2.2, 2.3 and 2.4 consists of the following steps:

- Finding suitable sequence of approximating solutions \( \{u_n\} \) and \( \{w_n\} \) for problem (2.6) and (2.9) respectively;
- A priori estimates for the sequences \( \{u_n\} \) and \( \{w_n\} \) in the required Lorentz spaces;
- Existence of a converging subsequences to weak limits \( u \) and \( v \);
- Passage to the limit as \( n \to \infty \) to prove that \( u \) and \( v \) are indeed solutions of the initial problems.
The first step is obtained *truncating* problems (2.6) and (2.9). Indeed thanks to [26], for any $n \in \mathbb{N}$ we infer the existence of $u_n \in W_0^{1,p}(\Omega)$ and $w_n \in W_0^{1,p}(\Omega)$ that solve

$$
\int_{\Omega} a(x, \nabla u_n) \nabla \phi = \int_{\Omega} \frac{|u_n|^{p-2}u_n E_n(x) \nabla \phi}{1 + \frac{1}{n}|u_n|^{p-1}} + \int_{\Omega} f_n(x) \phi \quad \forall \phi \in W_0^{1,p}(\Omega)
$$

(2.10)

and

$$
\int_{\Omega} a(x, \nabla w_n) \nabla \varphi = \int_{\Omega} E_n(x) \frac{|
abla w_n|^{p-2} \nabla w_n \varphi}{1 + \frac{1}{n}|
abla w_n|^{p-1}} + \int_{\Omega} f_n(x) \varphi \quad \forall \varphi \in W_0^{1,p}(\Omega),
$$

(2.11)

respectively, where $E_n(x)$ and $f_n(x)$ are the truncation at level $n \in \mathbb{N}$ of $E(x)$ and $f(x)$.

**Remark 2.6.** It is well known that uniqueness is not expected for distributional solutions to problems like (2.6) and (2.9), and that pathological solutions, that do not satisfy the standard regularity relations that we provided above, may exist.

To rule out this kind of solutions, the literature provide us with different tools as Solutions Obtained as Limit of Approximation (SOLA), Entropy Solution, Renormalized Solutions.

Anyway, since we are not able to prove uniqueness for our problems neither in these smaller classes of solutions, we decided to keep the simpler distributional framework. However, the previous Theorems can be read as regularity results for all the distributional Solutions Obtained as Limit of Approximation.

### 3. Preliminaries

In this section we introduce some preliminary results and tools in order to deal with problems with convection or drift lower order term. In Sect. 3.1 we give the basic background on the symmetrization technique for elliptic problems introduced in the seminal paper [31]. In Sect. 3.2 we prove the *almost everywhere convergence of the gradients* for the approximating sequences $\{u_n\}$ and $\{w_n\}$.

#### 3.1. Background on symmetrization techniques

**Proposition 3.1.** For $n \in \mathbb{N}$, let $v, v_n : \Omega \to \mathbb{R}$ be measurable functions such that

$$
|v(x)| \leq \liminf_{n \to \infty} |v_n(x)| \quad a.e. \ x \in \Omega.
$$

Hence

$$
\overline{v}(s) \leq \liminf_{n \to \infty} \overline{v}_n(s) \quad a.e. \ s \in (0, \Omega).
$$
Proof. For the proof see [23] Proposition 1.4.5. □

Let us state and prove the following Proposition.

**Proposition 3.2.** For almost every \( s \in (0, |\Omega|) \)

\[
\frac{d}{ds} A(\overline{v}(s)) \leq 1 \quad \text{and if } \overline{v}'(s) \neq 0 \quad A'((\overline{v}(s)) = \frac{1}{\overline{v}(s)}. \quad (3.1)
\]

**Proof.** Let us consider all the values \( s_i \) with \( i \in \mathbb{N} \) such that the set

\[
B_i = \{ t \in (0, |\Omega|) : |\overline{v}(t)| = |\overline{v}(s_i)\}
\]

has a strictly positive measure. By constriction every \( B_i \) is an half-open proper interval on which \( \overline{v}(s) \) is constant and, since \( \overline{v}(s) \) is not increasing, \( B_i \cap B_j = \emptyset \) for \( i \neq j \) (this assures us that the \( B_i \) are indeed countable). Moreover \( \bigcup_{i \in \mathbb{N}} B_i \) is closed and

\[
A'((\overline{v}(s)) = 0 \quad \forall \ a.e. \ s \in \bigcup_{i \in \mathbb{N}} B_i.
\]

On the other hand setting \( K = (0, |\Omega|) \setminus \bigcup_{i \in \mathbb{N}} B_i \) we have that

\[
\forall s \in K, \quad |\{ |\overline{v}(t)| = \overline{v}(s)\}| = 0 \quad \text{hence} \quad A((\overline{v}(s)) = s.
\]

Since both \( \overline{v}(s) \) and \( A(s) \) are almost a.e differentiable in \((0, |\Omega|)\) and, since for a.e. \( s \in K \) it holds true that \( \overline{v}'(s) \neq 0 \), we have finished. □

Let us state and prove the following useful Lemma (see Lemma 9 of [28]).

**Lemma 3.3.** For every measurable function \( v : \Omega \rightarrow \mathbb{R} \), there exists a set valued map \( s \rightarrow \Omega(s) \subset \Omega \) such that

\[
\begin{cases}
|\Omega(s)| = s & \text{for any } s \in [0, |\Omega|], \\
\Omega(s_1) \subset \Omega(s_2) & \text{whenever } s_1 < s_2, \\
\Omega(s) = \{|v| > \overline{v}(s)\} & \text{if } \quad |\{ |v| = \overline{v}(s)\}| = 0.
\end{cases}
\]

**(3.2)**

**Remark 3.4.** When we use Lemma 3.3 with \( v \equiv u_n \) or \( w_n \) (see (2.10) and (2.11) below for the definition of \( u_n \) and \( w_n \)) the associated set functions are denoted with \( \Omega_n(s) \). When we use Lemma 3.3 with \( v \equiv |\nabla u_n| \) or \( |\nabla w_n| \) the associated set function is denoted with \( \Omega_n(s) \).

**Proof.** By construction \( v(x) \) and \( \overline{v}(s) \) are equimeasurable thus

\[
|\{|v(x)| > \overline{v}(s)\}| = |\{|\overline{v}(\tau)| > \overline{v}(s)\}| \leq s \leq |\{|v(\tau)| \geq \overline{v}(s)\}| = |\{|v(x)| \geq \overline{v}(s)\}|.
\]

Since the Lebesgue measure is not atomic there exists \( \Omega(s) \) such that

\[
|\{|v(x)| > \overline{v}(s)\} \subset \Omega(s) \subset |\{|v(x)| \geq \overline{v}(s)\}| \quad \text{and} \quad |\Omega(s)| = s. \quad (3.3)
\]

Of course if \( |\{|v| = \overline{v}(s)\}| = 0 \), then \( \Omega(s) = |\{|v(x)| > \overline{v}(s)\}|. \quad \square
\]

In the next Lemma we define the pseudo rearrangement of a function \( g \in L^1(\Omega) \) with respect to a measurable function \( v(x) \) (see [2] and [22]).
Lemma 3.5. Let \( v : \Omega \to \mathbb{R} \) a measurable function, \( 0 \leq g(x) \in L^1(\Omega) \) and \( \Omega(s) \) the set valued function associated to \( v(x) \) defined in (3.2). Then
\[
D(s) := \frac{d}{ds} \int_{\Omega(s)} g(x) \, dx, \quad s \in (0, |\Omega|)
\]
is well defined and moreover
\[
i) \quad \int_0^t D(s) \, ds = \int_{\Omega(t)} g(x) \, dx \leq \int_0^t \overline{g}(s) \, ds, \quad t \in (0, |\Omega|) \tag{3.5}
\]
\[
ii) \quad D(A(k))(-A'(k)) = -\frac{d}{dk} \int_{[|v| > k]} g(x) \, dx, \quad k > 0. \tag{3.6}
\]

Proof. Note now that the function defined for \( s \in (0, |\Omega|) \) as
\[
s \to \int_{\Omega(s)} g(x) \, dx
\]
is absolutely continuous in \((0, |\Omega|)\). Thus it is almost everywhere differentiable and, denoting by \( D(s) \) its derivative, (3.5) holds true. Reading equation (3.5) for every \( s \) such that \( s = A(k) \) it follows
\[
\int_0^{A(k)} D(s) \, ds = \int_{\Omega(A(k))} g(x) \, dx = \int_{[|v| > k]} g(x) \, dx,
\]
where we have used that \( \Omega(A(k)) = [|v| > \overline{v}(A(k))] = [|v| > k] \). Differentiating with respect to \( k \) the previous identity we get (3.6). \( \square \)

The following Lemma assures that the pseudo rearrangement of \( g \) has the same summability of \( g \).

Lemma 3.6. Assume that \( g \in L^r(\Omega) \) with \( 1 \leq r \leq \infty \). Then the function \( D(s) \) defined in (3.4) belongs to \( L^r((0, |\Omega|)) \) and \( \|D\|_{L^r((0, |\Omega|))} \leq \|g\|_{L^r(\Omega)} \).

Moreover if we assume that \( g \in M^q((0, |\Omega|)) \) with \( 1 < q < \infty \), then \( D \) belongs to \( M^q((0, |\Omega|)) \).

Proof. Case \( g \in L^r(\Omega) \). This part of the Lemma has already been proved in [2] (Lemma 2.2). For the convenience of the reader we provide here the proof. Let us focus at first on the case \( 1 \leq r < \infty \) and divide the interval \((0, |\Omega|)\) into \( i \in \mathbb{N} \) disjoint intervals of the type \((s_{j-1}, s_j)\), for \( j = 1, \cdots, i \), of equal measure \(|\Omega|/i\). Let us consider the restriction of \( g(x) \) on the set \( \Omega(s_j) \setminus \Omega(s_{j-1}) \) and take its decreasing rearrangement in the interval \((s_{j-1}, s_j)\). Repeating this for any \( j = 1, \cdots, i \) we define a function (up to a zero measure set) on \((0, |\Omega|)\). Clearly this function depends on \( i \) and so we call it \( D_i(s) \). We stress that by construction the decreasing rearrangement of \( D_i(s) \) coincides with the decreasing rearrangement of \( g(x) \), thus for any measurable \( \omega \subset (0, |\Omega|) \)
\[
\int_\omega D_i^r(s) \, ds \leq \int_0^{[\omega]} \overline{g}^r(s) \, ds. \tag{3.7}
\]
Hence the sequence \( \{D_i'(s)\} \) is equi-integrable and there exists a function \( X \in L^r((0, |\Omega|)) \) such that

\[
D_i \rightharpoonup X \text{ in } L^r((0, |\Omega|)) \quad \text{as} \quad i \to \infty.
\]

The proof is concluded if we show that \( X \equiv D \). Let us define the function

\[
\Phi_i(s) := \int_0^s (D_i(t) - D(t))dt
\]

and notice that \( \Phi_i(0) = \Phi_i(|\Omega|) = 0 \). Thus for any \( \varphi(s) \in C^1([0, |\Omega|]) \) it results

\[
\int_0^{|\Omega|} (D_i(s) - D(s))\varphi(s)ds = -\int_0^{|\Omega|} \left[ \int_0^s (D_i(t) - D(t))dt \right] d\varphi(s) \leq \|\Phi_i\|_{L^\infty((0, |\Omega|))} \|\varphi'\|_{L^\infty((0, |\Omega|))}|\Omega|.
\]

(3.8)

By construction \( \Phi_i(s_j) = 0 \) for any \( j = 1 \cdots i \), since

\[
\int_0^{s_j} D_i(t)dt = \sum_{l=1}^j \int_{s_{l-1}}^{s_l} D_i(t)dt = \sum_{l=1}^j \int_{\Omega(s_l)/\Omega(s_{l-1})} g(x)dx = \int_{\Omega(s_j)} g(x)dx = \int_0^{s_j} D(t)dt.
\]

Hence if \( s_{j-1} \leq s \leq s_j \) we have that

\[
\Phi_i(s) = \int_{s_{j-1}}^s (D_i(t) - D(t))dt.
\]

Recalling (3.5) we deduce

\[
-\int_0^{|\Omega|/i} \bar{g}(t)dt \leq \int_{s_{j-1}}^s D(t)dt \leq \int_{s_{j-1}}^s (D_i(t) - D(t))dt \leq \int_{s_{j-1}}^s D_i(t)dt \leq \int_0^{|\Omega|/i} \bar{g}(t)dt,
\]

that implies the following estimate

\[
|\Phi_i(s)| \leq \int_0^{|\Omega|/i} \bar{g}(t)dt.
\]

Hence the right hand side of (3.8) goes to 0 as \( i \) diverges and

\[
\lim_{i \to \infty} \int_0^{|\Omega|} (D_i(s) - D(s))\varphi(s)ds = 0, \quad \forall \varphi \in C^1([0, |\Omega|]).
\]
Gradient estimates for nonlinear elliptic equations with first order terms

Since we already know that \( D_i(s) \) admits \( X(s) \) as weak limit in \( L^r((0, |\Omega|)) \), it follows that \( X(s) \equiv D(s) \) and we conclude that

\[
\|D\|_{L^r((0, |\Omega|))} \leq \|g\|_{L^r(\Omega)} \tag{3.9}
\]

for any \( g \in L^r(\Omega) \) and \( 1 \leq r < \infty \).

In the case \( g \in L^\infty(\Omega) \), we obtain (3.9) for any \( r \geq 1 \), that in turn implies

\[
\|D\|_{L^\infty((0, |\Omega|))} \leq \|g\|_{L^\infty(\Omega)}.
\]

Case \( g \in M^q(\Omega) \). As in the previous step we can construct a sequence \( \{D_i\} \) such that \( D_i(s) = g(s) \) for \( s \in (0, |\Omega|) \) and

\[
\lim_{i \to \infty} \int_0^{|\Omega|} D_i(t)\phi(t)dt = \int_0^{\infty} D(t)\phi(t)dt \quad \forall \phi \in L^\infty(\Omega).
\]

Take \( \phi_A = \chi_A \) with \( A \subset (0, |\Omega|) \) and \( |A| = s \). Recalling (3.7), we deduce that

\[
\int_0^{s} D(t)\phi_A(t)dt \leq \int_0^{s} \overline{g}(t)dt
\]

and, taking the sup with respect to \( A \),

\[
\int_0^{s} D(t)dt \leq \int_0^{s} \overline{g}(t)dt.
\]

Thus

\[
\overline{D}(s) \leq \frac{1}{s} \int_0^{s} D(t)dt \leq \frac{1}{s} \int_0^{s} \overline{g}(t)dt \leq \|g\|_{M^q(\Omega)} \frac{q}{q-1}s^{\frac{1}{q}}.
\]

A key tool in the symmetrization process introduced in [31] is given by the following Proposition.

**Proposition 3.7.** For any \( v \in W^{1,p}_0(\Omega) \) and for any \( s \in \mathbb{R} \)

\[
\sigma_N \leq A(s)^{\frac{1}{p}}\left(-A'(s)\right)^{\frac{1}{p}}\left(-\frac{d}{ds} \int_{A(s)} |\nabla v|^p\right)^{\frac{1}{p}}, \tag{3.10}
\]

where \( \sigma_N = N\omega_N^{\frac{1}{N}} \) and \( \omega_N \) is the volume of the unitary ball in dimension \( N \).

**Proof.** See pages 711 and 712 of [31]. \( \square \)

The next Lemma is used to establish the membership to Lorentz spaces of some integral quantities.

**Lemma 3.8.** Let \( r : (0, +\infty) \to (0, +\infty) \) be a decreasing function and let us define for \( \beta \geq 0 \) and \( \delta \neq 1 \)

\[
R_\delta(t) := \begin{cases} 
\int_0^t s^\beta r(s)ds & \text{if } \delta < 1 \\
\int_t^{+\infty} s^\beta r(s)ds & \text{if } \delta > 1.
\end{cases} \tag{3.11}
\]

Then for every \( \lambda > 0 \) there exists \( C = C(\beta, \delta, \lambda) \) such that

\[
\int_0^{\infty} \left( \frac{R_\delta(t)}{t} \right)^\lambda \frac{t^{\delta\lambda}}{t}dt \leq C \int_0^{\infty} r(t)^\lambda t^{\lambda(\beta+\delta)}dt.
\]

**Proof.** For the proof see [1] Lemma 2.1. \( \square \)
3.2. Others useful results

Let us recall that for any \( v \in W^{1,p}_0(\Omega) \), with \( 1 < p < \infty \), it follows that
\[
T_k(v) = \max\{-k, \min\{v, k\}\} \quad \text{and} \quad G_k(v) = u - T_k(v)
\]
belong to \( W^{1,p}_0(\Omega) \) for any \( k > 0 \).

To pass to the limit in (2.10) and (2.11) we need the almost everywhere convergence of the gradients of \( \{u_n\} \) and \( \{w_n\} \). Such type of result is very well known in the literature after the nowadays classical paper [16] (see also [8,14] and [29]).

**Lemma 3.9.** Let \( \{u_n\} \subset W^{1,p}_0(\Omega) \) be the sequence of function such that \( u_n \to u \) in measure and that
\[
-\text{div}(a(x, \nabla u_n)) = -\text{div}(|u_n|^{p-2}u_n F_n) + g_n
\]
in the sense of distribution, with \( \{F_n\} \) and \( \{g_n\} \) bounded in \( (L^{p'}(\Omega))^N \) and \( L^1(\Omega) \) respectively. Then, up to a further subsequence,
\[
\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.
\]

**Proof.** (See [14,16] and [29]) We only give a sketch of the proof, putting in evidence the differences, with respect to the existing literature, given by the lower order term in divergence form (if \( F_n \equiv 0 \) see Lemma 1 of [14]). We follow Theorem 2.1 of [29]. Taking \( T_k(u_n) \) as test function in (3.12) and using Young inequality it follows that for any \( \epsilon > 0 \)
\[
\alpha \int_\Omega |\nabla T_k(u_n)|^p \leq C_k \epsilon k^p \int_\Omega |F_n|^p' + \epsilon \int_\Omega |\nabla T_k(u_n)|^p + k \int_\Omega |g_n|,
\]
with \( C_\epsilon = \epsilon^{-\frac{1}{p-1}} \). Thanks to the previous estimate we deduce that for every \( k > 0 \)
\[
|\nabla T_k(u)| \in L^p(\Omega) \quad \text{and} \quad T_k(u_n) \to T_k(u) \quad \text{weakly in } W^{1,p}_0(\Omega).
\]

Let us take \( T_h(u_n - T_k(u)) \), with \( 0 < h < k \), as a test function in (3.12), to obtain
\[
\int_\Omega a(x, \nabla T_k(u_n)) \nabla T_h(T_k(u_n) - T_k(u))
\]
\[
\leq h \int_\Omega |g_n| + \int_\Omega |u_n|^{p-2} u_n F_n(x) \nabla T_h(u_n - T_k(u))
\]
\[
+ \int_\Omega |u_n|^{p-2} u_n F_n(x) \nabla T_h(u_n - T_k(u))
\]
\[
+ \int_\Omega |u_n|^{p-2} u_n F_n(x) \nabla T_h(u_n - T_k(u))
\]
Noticing that \( |u_n - T_k(u)| < h \) \( \subset |u_n| \leq h + k \) \( \subset |u_n| \leq 2k \), that the sequence \( \{|a(x, \nabla T_{2k}(u_n))|\} \) is bounded in \( L^{p'}(\Omega) \) and recalling (3.13), we can pass to the limit with respect to \( n \to \infty \) into the previous inequality and obtain
\[
\limsup_{n \to \infty} \int_{\{|T_k(u_n) - T_k(u)| \leq h\}} I_k(x) dx \leq \int_\Omega |u|^{p-2} u F \nabla T_h(G_k(u)) + h \int_\Omega |g_n|}

\begin{align*}
+ \int_{\{k < |u| < k+h\}} \Psi_k \nabla T_k(u).
\end{align*}

where \( \Psi_k, F \in \left( L^{p'}(\Omega) \right)^N \) are the weak limits of \( a(x, \nabla T_{2k}(u_n)) \) and \( F \). Notice that, letting \( h \to 0 \), the right hand side above is zero. Then we obtained the same information of formula (2.6) of Theorem 2.1 of [29]. The rest of the proof follows, unchanged, [29]. \qed

**Lemma 3.10.** Given the function \( \lambda, \gamma, \varphi, \rho \) defined in \((0, +\infty)\), suppose that \( \lambda, \gamma \geq 0 \) and that \( \lambda \gamma, \lambda \varphi \) and \( \lambda \rho \) belong to \( L^1(0, \infty) \). If for almost every \( t \geq 0 \) we have

\[ \varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \lambda(\tau) \varphi(\tau) d\tau, \]

then for almost every \( t \geq 0 \)

\[ \varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \rho(t) \lambda(\tau) e^{\int_t^\tau \lambda(s) \gamma(s) ds} d\tau. \]

**Proof.** See [3] Lemma 6.1. \qed

**Lemma 3.11.** Given the function \( \lambda, \gamma, \varphi, \rho \) defined in \((0, +\infty)\), suppose that \( \lambda, \gamma \geq 0 \) and that \( \lambda \gamma, \lambda \varphi \) and \( \lambda \rho \) belong to \( L^1(0, \infty) \). If for almost every \( t \geq 0 \) we have

\[ \varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \lambda(\tau) \varphi(\tau) d\tau, \]

then for almost every \( t \geq 0 \)

\[ \varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \rho(t) \lambda(\tau) e^{\int_t^\tau \lambda(s) \gamma(s) ds} d\tau. \]

**Proof.** See [3] Lemma 6.1. \qed

### 4. Proof of the results

#### 4.1. Convection term

We need three preliminary Lemmas. The first one is devoted to the achievement of a point-wise estimate for the decreasing rearrangement of \( u_n \), the solution of (2.10), the second Lemma gives the estimate relative to the decreasing rearrangement of \( |\nabla u_n| \), while the third one provides the required Lorentz bounds for the sequences \( \{u_n\} \) and \( \{|\nabla u_n|\} \).
Lemma 4.1. Let us assume (2.4) and (2.5). For any \( n \in \mathbb{N} \), let \( u_n \) be the solution of (2.10) and denote with \( \overline{u}_n \) its decreasing rearrangement. It follows that
\[
\overline{u}_n(t) \leq \overline{v}(t) := \frac{C}{t^\gamma} \int_t^{\|\Omega\|} s^{\frac{\gamma}{N} - 1} \tilde{f}(s)^{\frac{1}{p-1}} ds,
\]
(4.1)
where \( C = C(N, \alpha, p, E, m) \) and \( \gamma < \frac{N-pm}{(p-1)Nm} \).

Remark 4.2. In order to better understand (4.1) let us set, in the special case \( p = 2 \) and with a slight abuse of notation,
\[
v(x) = \overline{v}(\omega_N |x|^N) = \frac{C(\omega_N |x|^N)}{\omega_N |x|^N} \int_{\omega_N |x|^N} |\nabla \omega_N |x|^N| s^{\frac{\gamma}{N} + \frac{2}{N} - 1} \tilde{f}(s) ds,
\]
and notice that it solves
\[
-\Delta v = \gamma NC \text{div} \left( v \frac{x}{|x|^2} \right) + \omega_N N^2 C \tilde{f}(\omega_N |x|^N) \text{ in } B_{\Omega},
\]
(4.2)
on \( \partial B_{\Omega} \),
where \( B_{\Omega} \) is the ball centered at the origin such that \( |B_{\Omega}| = |\Omega| \) and \( C \) and \( \gamma \) are the constant of Lemma (4.1). Thus inequality (4.1) provides the already mentioned comparison between the rearrangements of the solution of the original problem and the symmetrized one.

Proof. We apply to our contest the approach of [6]. Since \( u_n \in W^{1,p}_0(\Omega) \), we are allowed to take \( T_{h(G_k(u_n))} \) with \( h > 0 \) and \( k \geq 0 \) as test function in (2.10), so that we get
\[
\frac{\alpha}{h} \int_{|u_n| < k+h} |\nabla u_n|^p \leq \int_{|u_n| > k} |f| + \frac{(k+h)^{p-1}}{h} \int_{|u_n| < k+h} |E||\nabla u_n|.
\]
(4.3)
Applying Hölder inequality to the last integral in the right hand side above and letting \( h \) go to zero, we obtain
\[
-\frac{d}{dk} \int_{|u_n| > k} |\nabla u_n|^p \leq \int_{|u_n| > k} |f| \frac{1}{\alpha} + \frac{k^{p-1}}{\alpha} \left( -\frac{d}{dk} \int_{|u_n| > k} |\nabla u_n|^p \right)^{\frac{1}{p}} \left( -\frac{d}{dk} \int_{|u_n| > k} |E||\nabla u_n| \right)^{\frac{1}{p'}}.
\]
(4.4)
Recalling that \( E = F + E \), it follows that
\[
\left( \int_{|u_n| > k} |E|^p \right)^{\frac{1}{p'}} \leq \left( \int_{|u_n| > k} |F|^p \right)^{\frac{1}{p'}} + \left( \int_{|u_n| > k} |E|^p \right)^{\frac{1}{p'}}.
\]
Let us moreover introduce the pseudo rearrangements of \( |F|^p \) and \( |E|^p \) with respect to \( u_n \) (see (3.4) for the definition)
\[
D_{1,n}(s) := \frac{d}{ds} \int_{\Omega_n(s)} |F(x)|^p dx \quad \text{and}
\]
\[ D_{2,n}(s) := \frac{d}{ds} \int_{\Omega_n(s)} |E(x)|^{\nu'} dx, \quad \text{with } s \in (0, |\Omega|). \]

Thanks to 3.6 we have that for \( k > 0 \)

\[ D_{1,n}(A_n(k))(-A_n'(k)) = -\frac{d}{dk} \int_{|u_n| > k} |F|^\nu' \quad \text{and} \]

\[ D_{2,n}(A_n(k))(-A_n'(k)) = -\frac{d}{dk} \int_{|u_n| > k} |E|^\nu'. \]

where \( A_n(k) = |\{ |u_n| > k \} \), namely the distribution function of \( u_n \).

Setting \( D_n(s) = D_{1,n}(s)^{\frac{1}{\nu'}} + D_{2,n}(s)^{\frac{1}{\nu'}} \), to have a more compact notation and using (3.10), inequality (4.4) becomes

\[
\left( -\frac{d}{dk} \int_{|u_n| > k} |\nabla u_n|^\nu \right)^{\frac{1}{\nu'}} \leq \frac{A_n(k)^{\frac{1}{\nu} - 1}}{\alpha \sigma_N^{\nu}} \int_{|u_n| > k} |f| \left( -A_n'(k)^{\frac{1}{\nu'}} \right) + \frac{k^{p-1}}{\alpha} D_n(A_n(k)) \left( -A_n'(k)^{\frac{1}{\nu'}} \right),
\]

that can be rewritten, using once more (3.10), as

\[
1 \leq \left[ \frac{A_n(k)^{\frac{1}{\nu} - 1}}{\alpha \sigma_N^{\nu}} \int_{|u_n| > k} |f| + \frac{k^{p-1}}{\alpha \sigma_N^{p-1}} D_n(A_n(k)) A_n(k)^{\frac{1}{\nu} - 1} (\nu - 1)^{(\nu - 1)(p-1)} \right] (-A_n'(k))^{p-1}.
\]

Thanks to the definition of decreasing rearrangement and using Proposition 3.2 in Sect. 3, it results

\[
-\frac{d}{ds} \overline{u}_n(s) \leq \left[ \frac{s^{p(\frac{1}{\nu} - 1)}}{\alpha \sigma_N^{\nu}} \int_0^s \tilde{f} + \frac{1}{\alpha \sigma_N^{p-1}} D_n(s) s^{\frac{1}{\nu} - 1} (\nu - 1)^{(\nu - 1)(p-1)} \overline{u}_n^{p-1}(s) \right]^{\frac{1}{p-1}}.
\]

For any \( a, b > 0 \), let us recall the standard inequality \((a + b)^{\frac{1}{p-1}} \leq a^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}\) for any if \( 2 \leq p < \infty \). On the other hand, if \( 1 < p < 2 \), it is easy to prove that for any \( \delta > 1 \)

\[
(a + b)^{\frac{1}{p-1}} \leq C_{\delta,p} a^{\frac{1}{p-1}} + \delta b^{\frac{1}{p-1}}, \quad C_{\delta,p} = \delta \left( \frac{2 - p}{\delta (p - 1)} - 1 \right)^{\frac{2 - p}{p - 1}} > 1.
\]

Then (4.6) becomes

\[
-\frac{d}{ds} \overline{u}_n(s) \leq \frac{C_{\delta,p}}{\alpha \sigma_N^{p-1}} s^{p(\frac{1}{\nu} - 1)} \left( \int_0^s \tilde{f}(\tau) d\tau \right)^{\frac{1}{p-1}} + \frac{\delta}{\alpha \sigma_N} D_n(s)^{\frac{1}{p-1}} s^{\frac{1}{\nu} - 1} \overline{u}_n(s),
\]
with $\delta \geq 1$ such that

$$\gamma := \frac{\delta^2 B}{\alpha} \leq \frac{N - pm}{(p - 1) N m},$$

where $B$ is the constant of assumption (2.5). Defining the auxiliary function

$$R_n(s) = e^{s \beta \frac{1}{p - 1}} \int_t^s D_n(\tau) \frac{1}{p - 1} \frac{1}{N - 1} d\tau,$$

we finally deduce that

$$- \frac{d}{ds} (R_n(s) \overline{u}_n(s)) \leq C_s \bar{p}' \left( \int_0^s \tilde{f}(\tau) d\tau \right) \frac{1}{p - 1}.$$ 

In order to estimate $R_n(s)$ we recall the definition of $D_n$ to obtain

$$\int_t^s D_n(\tau) \frac{1}{p - 1} \frac{1}{N - 1} d\tau = \int_t^s (D_{1,n}(\tau) \frac{1}{p} + D_{2,n}(\tau) \frac{1}{p'} \frac{1}{N - 1} d\tau$$

$$\leq C_{\delta, p} \int_t^s D_{1,n}(\tau) \frac{1}{p} \frac{1}{N - 1} d\tau + \delta \int_t^s D_{2,n}(\tau) \frac{1}{p'} \frac{1}{N - 1} d\tau,$$

Moreover thanks to Lemma 3.6 we deduce that

$$\int_t^s D_{1,n}(\tau) \frac{1}{p} \frac{1}{N - 1} d\tau \leq ||\mathcal{F}||_{L^{\frac{1}{p - 1}}(\Omega)} N |\Omega| \frac{1}{N}$$

and, using Young Inequality, integration by parts and assumption (2.5), that

$$\int_t^s D_{2,n}(\tau) \frac{1}{p'} \frac{1}{N - 1} d\tau \leq \frac{1}{pB} \int_t^s D_{2,n}(\tau) \frac{1}{p} \frac{1}{N - 1} d\tau + \frac{B \frac{1}{p - 1}}{p} \int_t^s \frac{1}{\tau} d\tau$$

$$\leq \frac{1}{pB} \left[ s^{\frac{p}{N - 1}} \int_0^s \tilde{\mathcal{E}}^{p'} - t^{\frac{p}{N - 1}} \right]$$

$$+ B \frac{1}{p - 1} \log \left( \frac{s}{l} \right).$$

Thus we have that

$$R_n(s) = e^{s \beta \frac{1}{p - 1}} \int_t^s D_n(\tau) \frac{1}{p - 1} \frac{1}{N - 1} d\tau \leq C \left( \frac{s}{l} \right)^{\gamma}.$$ 

Integrating between $t$ and $|\Omega|$ and recalling that by definition of both $\overline{u}_n(|\Omega|) = 0$ and $R_n(t) = 1$, we get

$$\overline{u}_n(t) = - R_n(|\Omega|) \overline{u}_n(|\Omega|) + R_n(t) \overline{u}_n(t) \leq \frac{C_1}{l^p} \int_t^{|\Omega|} s^{\frac{p}{N - 1} + \gamma} \left( \int_0^s \tilde{f}(\tau) d\tau \right)^{\frac{1}{p - 1}} ds.$$

Thus the proof of the Lemma is concluded.
The next Lemma is the core of our main result and provides the estimate relative to the decreasing rearrangement of $|\nabla u_n|$.

**Lemma 4.3.** Let us assume (2.4) and (2.5). Let $|\nabla u_n|$ be the decreasing rearrangement of $|\nabla u_n|$. There exists $C = C(N, \alpha, p, E, m)$ such that

$$
\frac{1}{s} \int_0^s |\nabla u_n|^{p-1} \leq C \left[ \frac{1}{s} \int_0^s (v(t)^{p-1} D_n(t) \frac{1}{p'} + \tilde{f}(t) \frac{1}{p'}) dt + \left( \frac{1}{s} \int_0^s (v(t)^p D_n(t) + \tilde{f}(t)^p t^{p'}) dt \right)^{\frac{1}{p'}} \right], \quad (4.7)
$$

where $v(t)$ is defined in (4.1).

**Proof.** Taking advantage of Lemma 3.3 (see Remark 3.4), it follows that

$$
\int_0^s |\nabla u_n|^{p-1} d\tau = \int_{\Omega_s} |\nabla u_n|^{p-1} dx
$$

$$
= \int_{\Omega_s \cap \{|u_n| > \bar{u}_n(s)\}} |\nabla u_n|^{p-1} dx + \int_{\Omega_s \cap \{|u_n| \leq \bar{u}_n(s)\}} |\nabla u_n|^{p-1} dx
$$

$$
\leq \int_{\{|u_n| > \bar{u}_n(s)\}} |\nabla u_n|^{p-1} dx
$$

$$
+ \left( \int_{\{|u_n| \leq \bar{u}_n(s)\}} |\nabla u_n|^{p} dx \right)^{\frac{1}{p'}} \left| \Omega_s \right|^{\frac{1}{p'}} \leq I_1(s) + I_2 \left( \frac{1}{p'} \right)(s) s^{\frac{1}{p'}}.
$$

As far as $I_2$ is concerned we infer from (4.5) that

$$
\frac{d}{ds} \int_{\{|u_n| > \bar{u}_n(s)\}} |\nabla u_n|^{p} = \frac{d}{d\kappa} \int_{\{|u_n| > \kappa\}} |\nabla u_n|^{p} \left. \right|_{\kappa = \bar{u}_n(s)} \frac{d}{ds} \bar{u}_n(s)
$$

$$
\leq C \left[ \bar{u}_n(s)^p D_n(s) + s^{\frac{p'}{p}} \tilde{f}(s)^{p'} \right]. \quad (4.8)
$$

Integrating between $s$ and $|\Omega|$, we get

$$
I_2 = \int_{\{|u_n| \leq \bar{u}_n(s)\}} |\nabla u_n|^{p} = - \int_{\{|u_n| > \bar{u}_n(s)\}} |\nabla u_n|^{p} + \int_{\Omega} |\nabla u_n|^{p}
$$

$$
\leq C \left[ \int_s^{|\Omega|} \bar{u}_n(t)^p D_n(t) + t^{p'} \tilde{f}(t)^{p'} dt \right].
$$

In order to estimate $I_1$ notice that

$$
\int_{\{|u_n| \leq |u_n| < \bar{u}_n(s+h)\}} |\nabla u_n|^{p-1}
$$

$$
\leq \left( \int_{\{|u_n| \leq |u_n| < \bar{u}_n(s+h)\}} |\nabla u_n|^{p} \right)^{\frac{1}{p'}} \left[ \{|u_n| \leq |u_n| < \bar{u}_n(s+h)\} \right]^{\frac{1}{p'}}.
$$
Passing to the limit as \( h \to 0 \) and recalling that \( |\{ u_n \} > \bar{u}_n(s) \}|' \leq 1 \) thanks to Proposition 3.2, we obtain that

\[
\frac{d}{ds} \int_{\{ u_n \} > \bar{u}_n(s)} |\nabla u_n|^p \leq \left( \frac{d}{ds} \int_{\{ u_n \} > \bar{u}_n(s)} |\nabla u_n|^p \right)^{\frac{1}{p'}} \\
\leq C \left( \bar{u}_n(s)^{p-1} D_n^{\frac{1}{p'}}(s) + \tilde{f}(s) \frac{1}{N} \right).
\]

Hence we have the following estimate for \( I_1 \)

\[
I_1 \leq C \int_0^s \left( \bar{u}_n(t)^{p-1} D_n^{\frac{1}{p'}}(t) + \tilde{f}(t) t \frac{1}{N} \right) dt.
\]

Putting together the obtained information for \( I_1 \) and \( I_2 \) we recover (4.7).

The previous estimates on the decreasing rearrangements of \( u_n \) and \( |\nabla u_n| \) allow us to obtain the following Lorentz estimates in function of the Lorentz summability of the datum \( f \).

**Lemma 4.4.** Let \( \{ u_n \} \) be the sequence of solutions of (2.10). Then there exists \( C = C(N, \alpha, p, E, m) \) such that:

(i) If \( f \in L^{m,q}(\Omega) \) with \( 1 < m < (p^*)' \) and \( 0 < q \leq \infty \), then

\[
\| u_n \|_{L^{(p-1)Nm/(N-pm), (p-1)q}(\Omega)} \leq C \| f \|_{L^{m,q}(\Omega)} \quad \text{and}
\]

\[
\| \nabla u_n \|_{L^{(p-1)Nm/(N-pm), (p-1)q}(\Omega)} \leq C \| f \|_{L^{m,q}(\Omega)};
\]

(ii) If \( f \in L^{1,q}(\Omega) \) with \( 0 < q \leq \infty \),

\[
\| u_n \|_{L^{(p-1)N/(N-p), (p-1)q}(\Omega)} \leq C \| f \|_{L^{1,q}(\Omega)} \quad \text{and}
\]

\[
\| \nabla u_n \|_{L^{(p-1)N/(N-1), (p-1)q}(\Omega)} \leq C \| f \|_{L^{1,q}(\Omega)};
\]

(iii) If \( f \in L^1(\Omega) \), then

\[
\| u_n \|_{L^{(p-1)N/(N-p), \infty}(\Omega)} \leq C \| f \|_{L^1(\Omega)} \quad \text{and}
\]

\[
\| \nabla u_n \|_{L^{(p-1)N/(N-1), \infty}(\Omega)} \leq C \| f \|_{L^1(\Omega)}.
\]

**Proof.** (i) Let us start with the \( f \in L^{m,q}(\Omega) \) with \( 1 \leq m < (p^*)' \) and \( 0 < q < \infty \). Estimate for \( \{ u_n \} \). Using (4.1) we get

\[
\| u \|_{L^{(p-1)Nm/(N-pm), q}(\Omega)}^{p-1} = \int_0^{+\infty} t^{q(N-pm)Nm} u(t)^{(p-1)q} \frac{dt}{t} \\
\leq C \int_0^{+\infty} t^{q(N-pm)Nm - \gamma(p-1)q} \left( \int_{\Omega} s^{p'} \frac{1}{s N^{p-1}} \tilde{f}(s) \frac{1}{s^{p-1}} ds \right)^{(p-1)q} \frac{dt}{t};
\]
\[
\begin{align*}
\leq C \int_0^\infty t^\frac{q}{m} \frac{\#\left(1 - \frac{p}{N}\right)}{t} \frac{dt}{t},
\end{align*}
\]

where the last inequality comes from Lemma 3.8 with \(\delta = \frac{N - pm}{Nm(p-1)} - \gamma + 1 > 1\), thanks to the choice of \(\gamma\).

In the case \(q = +\infty\), we obtain directly from (4.1) that

\[
\bar{u}(s) \leq \frac{C}{s^{Nm(p-1)}} \|f\|_{L^{m\infty}(\Omega)}.
\]

Estimate for \(||\nabla u_n||\). Thank to Lemma 3.6 estimate (4.7) can be rewritten as

\[
\frac{1}{s} \int_0^s |\nabla u_n|^{p-1} \leq C \left[ \frac{1}{s} \int_0^s \left( (v(t))^{p-1} t^{-\frac{p-1}{N}} + \tilde{f}(t)^{1/p} \right) dt 
\right]
\]

\[
+ \left( \frac{1}{s} \int_0^s \left( (v(t))^{p-1} t^{-\frac{p-1}{N}} + \tilde{f}(t)^{1/p} \right) dt \right)^{\frac{1}{p'}}.
\]

(4.10)

In order to prove the membership of the four terms above to \(L^{m*,q}(\Omega)\) we use Lemma 3.8

\[
\int_0^\infty \frac{q}{m^p} \left( \frac{1}{s} \int_0^s (v(t))^{p-1} t^{-\frac{p-1}{N}} dt \right)^q \frac{ds}{s} \leq C \int_0^\infty \frac{q}{s^{Nm(p-1)}} v(s)^{p(q-1)} \frac{ds}{s} < \infty,
\]

\[
\int_0^\infty \frac{q}{m^p} \left( \frac{1}{s} \int_0^s \tilde{f}(t)^{1/p} dt \right)^q \frac{ds}{s} \leq C \int_0^\infty \frac{q}{s^{m\tilde{f}}(s)^{q-1}} \frac{ds}{s} < \infty,
\]

where we take \(\delta = \frac{1}{m^p} < 1\), and

\[
\int_0^{+\infty} \frac{q}{m^p} \left( \frac{1}{s} \int_s^{\infty} (v(t))^{p-1} t^{-\frac{p-1}{N}} dt \right)^q \frac{ds}{s} \leq \int_0^{+\infty} \frac{q}{s^{m\tilde{f}}(s)^{q-1}} \frac{ds}{s} < \infty,
\]

\[
\int_0^{+\infty} \frac{q}{m^p} \left( \frac{1}{s} \int_s^{\infty} \tilde{f}(t)^{1/p} dt \right)^q \frac{ds}{s} \leq \int_0^{+\infty} \frac{q}{s^{m\tilde{f}}(s)^{q-1}} \frac{ds}{s} < \infty.
\]

where we take \(\delta = \frac{p^*}{m^p} < 1\) (recall that \(m < (p^*)^*\)). Hence we have that

\[
\|\nabla u_n\|^{q}_L^{(p-1)Nm,(p-1)q}(\Omega) \leq C \int_0^\infty \tau^q \left( \frac{1}{s} \int_0^\tau |\nabla w_n|^{p-1}(t) dt \right)^q \frac{ds}{s} \leq C \|f\|_{L^{m,q}(\Omega)}.
\]

In the case \(q = \infty\) we obtain by direct calculation from (4.10) that

\[
\|\nabla u\|_L^{(p-1)Nm,\infty}(\Omega) \leq C \|f\|_{L^{m,\infty}(\Omega)}.
\]
(ii) It follows exactly the same argument of (i).
(iii) Inequality (4.1) becomes
\[ \alpha_n(t) \leq \nu(t) \leq C \| f \|_{L^1(\Omega)} \frac{1}{t^{\gamma'}} \int_t^{|\Omega|} s^\gamma \left( \frac{1}{N} - 1 \right)^{\gamma'} ds \leq C \| f \|_{L^1(\Omega)} t^{-\frac{N-p}{(p-1)N}}, \]
where we have used that \( p' \left( \frac{1}{N} - 1 \right) + \gamma + 1 < 0. \) On the other hand we have that
\[ \frac{1}{t} \int_0^t |\nabla u_n|^{p-1} \leq C \| f \|_{L^1(\Omega)} \left[ \frac{1}{t} \int_0^t s^{\frac{1}{N}-1} ds + \left( \frac{1}{t} \int_0^t s^{\gamma} \left( \frac{1}{N} - 1 \right)^{\gamma'} ds \right)^{\frac{1}{p'}} \right] \]
\[ \leq C \| f \|_{L^1(\Omega)} t^{-\frac{N-1}{N}}, \]
and thus the proof is concluded. \( \square \)

Remark 4.5. In order to show that assumption (2.5) is sharp for Theorems 2.2 and 2.3 to hold, let us consider the solution of the symmetrized problem (4.2) in the simple case \( p = 2 \) and \( f \in M^m(\Omega) \) with \( 1 < m < (2^*)' \). For this value of \( p \) notice that \( \gamma = \gamma(B) = \frac{B}{\alpha N \omega_N} \) (see (4.6)). If we take now \( B > 0 \) so that
\[ \frac{N-2m}{Nm} < \gamma(B) < 1 \) ((2.5) is not satisfied) it follows that
\[ \alpha(t) \leq C \| f \|_{M^m(\Omega)} \frac{1}{t^\gamma} \int_t^{|\Omega|} s^{\frac{2}{N} + \gamma - 1 - \frac{1}{m}} ds \leq \tilde{C} t^{-\gamma}. \]
Thus \( u \in M^{1/(\gamma B)}(\Omega) \). Moreover, if \( \frac{N-2m}{Nm} < \gamma(B) < \frac{N-1}{N} \), we deduce by direct computation of the gradient that
\[ |\nabla u|(t) \leq C t^{-\gamma(B) + \frac{1}{N}}. \]
Hence for any \( 1 < r < \frac{Nm}{N+m} \) we can chose \( B \) in order to have \( |\nabla u| \in M^r(\Omega) \). In the borderline case \( \gamma(B) = \frac{N-2m}{Nm} \) estimates with logarithmic corrections are obtained.

This argument shows that if (2.5) is not satisfied the standard relation between the regularity of the data and the solution is lost (for a more detailed description of this fact see [17]).

Now we are in the position of proving Theorems 2.2 and 2.3. We start from the latter.

Proof of Theorem 2.3. Case (i). Let us start with the case \( p > 2 - \frac{1}{N} \) and \( f \in L^1(\Omega) \). From Lemma 4.4 we deduce that the sequence \( \{ |\nabla u_n| \} \) is bounded in \( L^{(p-1)N, \infty}(\Omega) \) and, in turn, in \( L^r(\Omega) \) for any \( 1 < r < \frac{(p-1)N}{N-1} \). Hence there exists \( u \in W^1_r(\Omega) \) such that \( u_n \rightharpoonup u \) in \( W^1_r(\Omega) \). Thanks to the almost everywhere convergence of the gradients proved in Lemma 3.9, we infer that
\[ \nabla u_n \rightharpoonup \nabla u \text{ in } L^{\frac{r}{r-1}}(\Omega). \]
Observing that it is possible to choose \( r \) such that \( \frac{r}{p-1} > 1 \), it follows that

\[
|\nabla u_n|^p \nabla u_n \to |\nabla u|^{p-2} \nabla u \quad \text{in} \quad L^1(\Omega).
\] (4.11)

Thus we can pass to the limit, as \( n \to \infty \), in the left hand side of (2.10) for every \( \phi \in C^1_0(\Omega) \). In order to handle the lower order term, notice that for every measurable \( \omega \subset \Omega \) it follows that

\[
\int_\omega |u_n|^{p-1} |E_n| \leq \int_0^{\omega} v^{p-1}(t) t^{-\frac{p-1}{N}} dt
\]

\[
\leq C \| f \|_{L^1(\Omega)} \int_0^{\omega} t^{-\frac{N-p-1}{N}} \leq C |\omega| \frac{1}{N},
\] (4.12)

where we used Lemma 4.4. Estimate (4.12) implies that the sequence

\[
\left\{ \frac{|u_n|^{p-2} u_n}{1 + \frac{1}{n} |u_n|^{p-1}} E_n(x) \right\}
\]

is equi-integrable. This, together with the a.e. convergence of \( u_n \), allows us to pass to the limit, as \( n \to \infty \), also in the lower order term of (2.10) and conclude that

\[
\int_\Omega a(x, \nabla u) \nabla \phi = \int_\Omega u |u|^{p-2} E(x) \nabla \phi + \int_\Omega f(x) \phi \quad \forall \phi \in C^1_0(\Omega).
\]

Finally from Lemma 3.9 and Proposition 3.1 we easily infer that

\[
|u|^{p-1} \in L^{\frac{N}{N-p}, \infty}(\Omega) \quad \text{and} \quad |\nabla u|^{p-1} \in L^{\frac{N}{N-1}, \infty}(\Omega).
\]

**Case (ii).** If \( p > 2 - \frac{1}{N} \) and \( f \in L^{1,q}(\Omega) \) with \( 0 < q \leq \infty \), we infer from Lemma 4.4 that \( \{u_n\} \) and \( \{|\nabla u_n|\} \) are bounded in \( L^{\frac{(p-1)N}{N-p}, (p-1)q}(\Omega) \) and \( L^{\frac{(p-1)N}{N-1}, (p-1)q}(\Omega) \) respectively. Since \( \frac{N(p-1)}{N-1} > 1 \) we deduce that there exist \( u \in W^{1,r}_0(\Omega) \) such that \( u_n \rightharpoonup u \) in \( W^{1,r}_0(\Omega) \) for any \( 1 < r < \frac{N(p-1)}{N-1} \). Thus following the same arguments of the previous step, we conclude that there exists \( u \) distributional solution of (2.3) such that

\[
|u| \in L^{\frac{(p-1)N}{N-p}, (p-1)q}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{\frac{(p-1)N}{N-1}, (p-1)q}(\Omega).
\]

**Case (iii).** If \( p = 2 - \frac{1}{N} \) and \( f \in L^{1,q}(\Omega) \) with \( 0 < q \leq \frac{1}{p-1} = \frac{N}{N-1} \), Lemma 4.4 implies that \( \{|\nabla u_n|\} \) is bounded in \( L^1(\Omega) \). Since \( L^1(\Omega) \) is not reflexive, this is not enough to assure the existence of a weakly converging subsequence. In order to recover a compactness property for \( \{|\nabla u_n|\} \), we need to prove its equi-integrability. For it, let \( \omega \) be a measurable subset of \( \Omega \) and notice that

\[
\int_\omega |\nabla u_n(x)| dx \leq \int_0^{\omega} |\nabla u_n(t)| dt \leq \int_0^{\omega} \left( \frac{1}{t} \int_0^s |\nabla u_n|^{p-1} \right)^{\frac{N}{N-1}} dt \frac{1}{t}
\]

\[
\leq \int_0^{\omega} \left( \frac{1}{t} \int_0^t (v(s)^{p-1} s^{-\frac{p-1}{N}} + f s^{\frac{1}{N}}) ds \right)
\]

where the last inequality comes from (4.10). Lemma 4.4 with $f \in L_{1, \frac{N}{N-1}}(\Omega)$ implies that

\[
\int_0^{|\Omega|} \left( \frac{1}{t} \int_0^t (v(s)^{p-1}s^{-\frac{p-1}{N}} + \tilde{f} s) ds + \frac{1}{p} \int_0^{|\Omega|} (v(s)^p s^{-\frac{p}{N}} + \tilde{f} s s^p) ds \right)^{\frac{1}{p}} \frac{dt}{t} \leq C \int_0^{|\Omega|} (t \tilde{f}(t))^{\frac{N}{N-1}} dt = C \|f\|_{L_{1, \frac{N}{N-1}}(\Omega)}.
\]

This means that the function

\[
\left( \frac{1}{t} \int_0^t (v(s)^{p-1}s^{-\frac{p-1}{N}} + \tilde{f} s) ds + \frac{1}{p} \int_0^{|\Omega|} (v(s)^p s^{-\frac{p}{N}} + \tilde{f} s s^p) ds \right)^{\frac{1}{p}} \frac{1}{N-1}
\]

belongs to $L^1(0, |\Omega|)$. This consideration and inequality (4.13) imply that for every $\epsilon$ there exists $\delta > 0$ such that

\[
\int_\omega |\nabla u_n(x)| dx \leq \epsilon \quad \forall \omega \subset \Omega \quad \text{with} \quad |\omega| < \delta.
\]

Hence we take advantage of Dunford-Pettis Theorem to infer the existence of a vector field $L \in (L^1(\Omega))^N$ such that

\[
\nabla u_n \rightharpoonup L \quad \text{in} \quad (L^1(\Omega))^N.
\]

By the very definition of weak gradient of a Sobolev function it results that

\[
\int_\Omega \nabla u_n F = - \int_\Omega u_n \text{div}(F) \quad \forall F \in (C^\infty_0(\Omega))^N.
\]

Thanks to the weak convergence of $\nabla u_n$ in $(L^1(\Omega))^N$ and the strong convergence of $u_n$ in $L^1(\Omega)$ (Lemma 4.4 says that indeed $u_n$ strongly converge to $u$ in $L^r(\Omega)$ with $1 < r < \frac{N}{N-1}$), we can pass to the limit in the equation above and deduce that $F \equiv \nabla u$.

Then, thanks to the almost convergence of $\nabla u_n$ to $\nabla u$ (see Lemma 3.9), we can infer that indeed

\[
\nabla u_n \rightarrow \nabla u \quad \text{in} \quad (L^1(\Omega))^N.
\]

Since $p - 1 = 1 - \frac{1}{N} < 1$, we also have that $|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u$ in $L^1(\Omega)$. We follow the arguments of the previous step to conclude that $u$ is a solution
of (2.3). Moreover, thanks to the almost everywhere of both \( \{ u_n \} \) and \( \{ |\nabla u_n| \} \), we apply again Proposition 3.1 to conclude that

\[
|u| \in L^{\frac{N-1}{N-p}}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{1,(p-1)q}(\Omega).
\]

**Case (iv).** The case \( p < 2 - \frac{1}{N} \) and \( f \in L^{m,q}(\Omega) \) with \( m = \frac{N}{N(p-1)+1} \) and \( 0 < q \leq \frac{1}{p-1} \) is handled similarly to the Case (iii). Indeed, for the considered values of \( m \), it results \( \frac{(p-1)N}{N-pm} = 1 \), thus Lemma 4.4 implies that \( \{ |\nabla u_n| \} \) is bounded in \( L^1(\Omega) \). Reasoning as in (4.13), (4.14) and using the almost everywhere convergence of the gradient (see Lemma 3.9), we conclude that

\[
|\nabla u_n| \to |\nabla u| \quad \text{in} \quad \left( L^1(\Omega) \right)^N.
\]

From now on the proof is close to the one of the previous case.

\[\Box\]

**Proof of Theorem 2.2. Case (i).** Following the same argument of the first step of the proof of Theorem 2.3. We infer that there exists \( u \in W^{1,r}_0(\Omega) \) with \( 1 < r < \frac{Nm(p-1)}{N-m} \) such that up to a subsequence

\[
|\nabla u_n| \to |\nabla u| \quad \text{in} \quad L^r(\Omega).
\]

Since it is possible to chose \( r \) such that \( \frac{r}{p-1} > 1 \), we deduce that

\[
|\nabla u_n|^{p-2} \nabla u_n \to |\nabla u|^{p-2} \nabla u \quad \text{in} \quad L^1(\Omega).
\]

In order to pass to the limit in (2.10), it is enough to notice that Lemma 3.9 and (4.12) are still valid. We also have that

\[
|u|^{p-1} \in L^{\frac{N}{N-p}}(\Omega) \quad \text{and} \quad |\nabla u|^{p-1} \in L^{\frac{N}{N-1}}(\Omega).
\]

**Case (ii).** Choosing \( u_n \) as a test function in (2.10) and using Hölder’s inequality we get

\[
\alpha \int_\Omega |\nabla u_n|^p \leq \left( \int_\Omega |E_n|^p |u_n|^p \right)^\frac{1}{p'} \left( \int_\Omega |\nabla u_n|^p \right)^\frac{1}{p} + \frac{1}{\mathcal{S}} \| f \|_{L^{(p^*)'}}(\Omega) \left( \int_\Omega |\nabla u_n|^p \right)^\frac{1}{p'} \leq C
\]

Moreover thanks to (4.9) it results that \( \{ u_n \} \) is bounded in \( L^q(\Omega) \) for \( p^* < q < [(p-1)m^*)^\ast \). Thus

\[
\int_0^{[\Omega]} t^{-\frac{p}{N}} \tilde{v}^q(t) dt \leq \left( \int_0^{[\Omega]} \tilde{v}^q(t) dt \right)^\frac{p}{q} \left( \int_0^{[\Omega]} t^{-\frac{pq}{N(q-p)}} dt \right)^\frac{q-p}{q} \leq C
\]

since \( 1 - \frac{pq}{N(q-p)} > 0 \). Hence

\[
\| \nabla u_n \|_{L^p(\Omega)} \leq \| E \|_{M^N(\Omega)} \left( \int_0^{[\Omega]} t^{-\frac{p}{N}} \tilde{v}^p(t) dt \right)^\frac{1}{p'} + \frac{1}{\mathcal{S}} \| f \|_{L^{(p^*)'}}(\Omega) \leq C
\]

At this point we conclude that up to a subsequence \( \{ \nabla u_n \} \) weakly converge in \( W^{1,p}_0(\Omega) \) to a function \( u \in W^{1,p}_0(\Omega) \). The rest of the proof is the same of Case (i).

\[\Box\]
4.2. Drift term

In the next Lemmas we recover the pointwise estimate for the rearrangement of $w_n$, the solution of (2.11), and its gradient.

**Lemma 4.6.** Let us assume (2.4) and (2.8). The sequence $\{w_n\}$ of solution of (2.11) satisfies the following estimates:

$$
\bar{w}_n(\tau) \leq z(t) := C \int_{\tau}^{1} \left| \bar{\Omega} \right|^p \left( \int_{0}^{t} f(s) s^{-\frac{B}{\alpha p}} ds \right)^{-\frac{1}{p-1}} dt \quad (4.15)
$$

and

$$
\frac{1}{s} \int_{0}^{s} |\nabla w_n|^{p-1} \leq C_1 \left[ \frac{1}{s} \int_{0}^{s} t^{\frac{N}{p-1} - \frac{B}{\alpha p}} \left( \int_{0}^{t} f(\tau) \tau^{-\frac{B}{\alpha p}} d\tau \right) dt \right. \\
\left. + \left( \frac{1}{s} \int_{0}^{s} t^{\frac{N}{p-1} - \frac{B}{\alpha p}} \left( \int_{0}^{t} f(\tau) \tau^{-\frac{B}{\alpha p}} d\tau \right)^{\frac{1}{p'}} dt \right)^{\frac{1}{p'}} \right],
$$

(4.16)

where $C$ and $C_1$ are two constant depending on $N$, $\alpha$, $p$, $E$, $m$.

**Proof.** Let us divide the proof in two steps.

**Step 1.** Estimate for $w_n$.

**Step 2.** Estimate for $\nabla w_n$.

**Step 1.** Let us set for any $n \in \mathbb{N}$, $k > 0$ and $s \in (0, |\Omega|)$ the distribution function of $w_n$

$$
A_n(k) = \{|w_n| > k\},
$$

and the pseudo rearrangements of the two components of $E(x)$ (see (2.8) and (3.4))

$$
Q_{1,n}(s) := \frac{d}{ds} \int_{\Omega_n(s)} |F(x)|^p dx \quad \text{and} \quad Q_{2,n}(s) := \frac{d}{ds} \int_{\Omega_n(s)} |E(x)|^p dx.
$$

As in Lemma 4.1, let us take $T_h(G_k(u_n))$ with $h > 0$ and $k \geq 0$ as test function in (2.11). We obtain that

$$
\frac{\alpha}{h} \int_{|k| < |w_n| < k+h} |\nabla w_n|^p \leq \int_{|w_n| > k} |f| + \int_{|w_n| > k} |E_n| |\nabla w_n|^{p-1}. \quad (4.17)
$$

Recalling (3.6), let us note that the last integral above can be estimate as

$$
\int_{|w_n| > k} |E_n(x)| |\nabla w_n|^{p-1} = \int_{k}^{+\infty} \left( \frac{d}{ds} \int_{|w_n| > s} |E_n(x)| |\nabla w_n|^{p-1} \right) ds \\
\leq \int_{k}^{+\infty} \left( Q_{1,n}(A_n(s))^{\frac{1}{p}} + Q_{2,n}(A_n(s))^{\frac{1}{p}} \right) \left(-A_n'(s)\right)^{\frac{1}{p}} \\
\left( -\frac{d}{ds} \int_{|w_n| > s} |\nabla w_n|^p \right)^{\frac{1}{p'}} ds,
$$
Passing to the limit as $h \to 0$ in (4.17), we recover that

$$\frac{d}{dk} \int_{|w_n|>k} |\nabla w_n|^p \leq \frac{1}{\alpha} \int_{|w_n|>k} |f|$$

$$+ \frac{1}{\alpha} \int_k^{+\infty} Q_n(A_n(s))(-A'_n(s)) \frac{1}{p} \left( \frac{d}{ds} \int_{|w_n|>s} |\nabla w_n|^p \right)^{\frac{1}{p'}} ds,$$

where we have set $Q_n(s) = Q_{1,n}(s)^{\frac{1}{p}} + Q_{2,n}(s)^{\frac{1}{p}}$ to have a more compact notation.

Using (3.10) we obtain

$$\left( \frac{d}{dk} \int_{|w_n|>k} |\nabla w_n|^p \right)^{\frac{1}{p'}} \leq \frac{1}{\alpha \sigma_N} A_n(k)^{\frac{1}{p}-1}(-A'_n(k))^{\frac{1}{p'}} \int_{|w_n|>k} |f|$$

$$+ \frac{1}{\alpha \sigma_N} A_n(k)^{\frac{1}{p}-1}(-A'_n(k))^{\frac{1}{p'}} \int_k^{+\infty} Q_n(A_n(s))(-A'_n(s))^{\frac{1}{p}}$$

$$\left( -\frac{d}{ds} \int_{|w_n|>s} |\nabla w_n|^p \right)^{\frac{1}{p'}} ds.$$

Let us use Lemma 3.11 and make a change of variable to obtain that

$$\left( \frac{d}{dk} \int_{|w_n|>k} |\nabla w_n|^p \right)^{\frac{1}{p'}} \leq \frac{1}{\alpha \sigma_N} A_n(k)^{\frac{1}{p}-1}(-A'_n(k))^{\frac{1}{p'}} \int_0^{A_n(k)} f$$

$$+ \frac{1}{\alpha^2 \sigma_N^2} A_n(k)^{\frac{1}{p}-1}(-A'_n(k))^{\frac{1}{p'}}$$

$$\int_0^{A_n(k)} Q_n(s)^{\frac{1}{p}-1} \left( \int_0^s f(\tau)d\tau \right) e^{\frac{1}{\alpha \sigma_N} \int_0^{A_n(k)} Q_n(\tau)\frac{1}{p-1}d\tau} ds.$$

We note that the integral in the second line above can be written as

$$-\alpha \sigma_N \int_0^{A_n(k)} \left( \int_0^s f(\tau)d\tau \right) \frac{d}{ds} \left( e^{\frac{1}{\alpha \sigma_N} \int_0^{A_n(k)} Q_n(\tau)\frac{1}{p-1}d\tau} \right) ds.$$

Thus integrating by parts we finally obtain

$$\left( \frac{d}{dk} \int_{|w_n|>k} |\nabla w_n|^p \right)^{\frac{1}{p'}} \leq \frac{1}{\alpha \sigma_N} A_n(k)^{\frac{1}{p}-1}(-A'_n(k))^{\frac{1}{p'}}$$

$$\int_0^{A_n(k)} \frac{1}{\sigma_N} f(s)^{\frac{1}{p}} \left( Q_{1,n}(\tau)^{\frac{1}{p}} + Q_{2,n}(\tau)^{\frac{1}{p}} \right)^{\frac{1}{p-1}} d\tau ds. \quad (4.18)$$
Using once more (3.10), estimate (4.18) becomes

\[ 1 \leq \frac{1}{\alpha \sigma_N^p} A_n(k)^p \left( \frac{1}{N} - 1 \right) (-A_n'(k))^{p-1} \]

\[ \int_0^{A_n(k)} \bar{f}(s) e^{\frac{1}{\alpha \sigma_N} \int_0^{A_n(k)} \left( Q_{1,n}(\tau) \frac{1}{p} + Q_{2,n}(\tau) \frac{1}{p} \right) \tau^{\frac{1}{N} - 1} d\tau} ds, \]

and by a change of variable

\[ -\frac{d}{dt} \bar{w}(t) \leq \frac{1}{\alpha \sigma_N^{p'}} \left( \frac{1}{N} - 1 \right) \left( \int_0^t \bar{f}(s) e^{\frac{1}{\alpha \sigma_N} \int_0^s \left( Q_{1,n}(\tau) \frac{1}{p} + Q_{2,n}(\tau) \frac{1}{p} \right) \tau^{\frac{1}{N} - 1} d\tau} ds \right)^{\frac{1}{p' - 1}}. \]

(4.19)

By construction and by Lemma 3.6 we deduce that \( \|Q_{1,n}\|_{L^\infty(\Omega)} \leq C \|\mathcal{F}\|_{L^\infty(\Omega)} \) and moreover, by means of Young Inequality and integration by parts, we have that

\[ \int_0^t Q_{2,n}(\tau) \frac{1}{p} \tau^{\frac{1}{N} - 1} d\tau \leq \frac{NB}{p(N - p)} + B \log \left( \frac{t}{s} \right). \]

Thus integrating (4.19) we recover (4.15).

**Step 2.** Recalling Lemma 3.3 and Remark 3.4, we obtain that

\[ \int_0^s |\nabla w_n|^{p-1} d\tau = \int_{\Omega_n(s)} |\nabla w_n|^{p-1} dx \]

\[ = \int_{\Omega_n(s) \cap \{|w_n| > \bar{w}_n(s)\}} \nabla w_n|^{p-1} dx + \int_{\Omega_n(s) \cap \{|w_n| \leq \bar{w}_n(s)\}} |\nabla w_n|^{p-1} dx \]

\[ \leq \int_{\{|w_n| > \bar{w}_n(s)\}} |\nabla w_n|^{p-1} dx \]

\[ + \left( \int_{\{|w_n| \leq \bar{w}_n(s)\}} |\nabla w_n|^{p} dx \right)^{\frac{1}{p'}} \left( \Omega_n(s) \right)^{\frac{1}{p}} \]

\[ \leq I_1(s) + I_2^{\frac{1}{p'}} (s)s^{\frac{1}{p}}. \]

**Estimate of** \( I_2 \). From (4.18) we also have

\[ -\frac{d}{dk} \int_{\{|w_n| > k\}} |\nabla w_n|^{p} \leq CA_n(k)^p \left( \frac{1}{N} - 1 + \frac{B}{\alpha \sigma_N} \right) \left( \int_0^{A_n(k)} t^{-\frac{B}{\alpha \sigma_N}} \bar{f}(t) dt \right)^{p'} (-A_n'(k)), \]

from which we infer that (see (4.8))

\[ \frac{d}{ds} \int_{\{|w_n| > \bar{w}_n(s)\}} |\nabla w_n|^{p} \leq C s^{p'} \left( \frac{1}{N} - 1 + \frac{B}{\alpha \sigma_N} \right) \left( \int_0^s t^{-\frac{B}{\alpha \sigma_N}} \bar{f}(t) dt \right)^{p'}. \]

Integrating between \( \tau \) and \( \Omega \) we get

\[ I_2 \leq C \int_{\tau}^{\Omega} s^{p'} \left( \frac{1}{N} - 1 + \frac{B}{\alpha \sigma_N} \right) \left( \int_0^s t^{-\frac{B}{\alpha \sigma_N}} \bar{f}(t) dt \right)^{p'} ds. \]
Estimate of $I_1(s)$. As far as $I_1$ is concerned, recalling (3.1), it follows
\[
\frac{d}{ds} \int_{\{|w_n| > \bar{w}_n(s)\}} |\nabla w_n|^{p-1} \leq \left( \frac{d}{ds} \int_{\{|w_n| > \bar{w}_n(s)\}} |\nabla w_n|^p \right)^{\frac{1}{p'}} \leq C s^{\frac{1}{N} - 1 + \frac{B}{\alpha \sigma N}} \int_0^s t^{-\frac{B}{\alpha \sigma N}} f(t) dt.
\]
Integrating between 0 and $\tau$ we get
\[
I_1 = \int_{\{|w_n| > \bar{w}_n(\tau)\}} |\nabla w_n|^{p-1} \leq C \int_0^\tau s^{\frac{1}{N} - 1 + \gamma} \int_0^s t^{-\gamma} f(t) dt ds.
\]
Putting together these two pieces of information we obtain (4.16).

Let us provide now the a priori bound for $\{w_n\}$ and $|\nabla w_n|$ in the required Lorentz spaces.

**Lemma 4.7.** There exist two constant $C = C(\alpha, p, E, N)$ and $\tilde{C} = \tilde{C}(\alpha, p, E, N)$ such that

\begin{enumerate}
\item if $1 < m < \frac{N}{p}$, $0 < q \leq \infty$ then $\|w_n\|_{L^{(p-1)m^*,(p-1)q}(\Omega)} \leq C \|f\|_{L^{m,q}(\Omega)}$
\item if $1 < m < (p^*)'$, $0 < q \leq \infty$ then $\|\nabla w_n\|_{L^{(p-1)m^*,(p-1)q}(\Omega)} \leq \tilde{C} \|f\|_{L^{m,q}(\Omega)}$.
\end{enumerate}

**Proof.** Estimate for $\{w_n\}$. Assume that $q > \infty$. From (4.15) it follows that
\[
\|w_n\|_{L^{(p-1)m^*,(p-1)q}(\Omega)} = \int_0^{+\infty} t^{\frac{q(N-pm)}{Nm} - \frac{q}{m}} \bar{w}_n(t)^q \frac{dt}{t} \leq C \int_0^{+\infty} t^{\frac{q(N-pm)}{Nm} - (p-1)q} \left( \frac{1}{\tau} \int_0^\tau t^{p\left(\frac{1}{N} - 1\right) + \frac{B}{\alpha \sigma N} (p-1)} \right) \left( \int_0^\tau f(s) s^{-\frac{B}{\alpha \sigma N}} ds \right)^{\frac{1}{p-1}} \frac{d\tau}{\tau} \leq C \left[ \int_0^{+\infty} \frac{q^{a + qB}}{t^m} \left( \tau^{-1} \int_0^\tau t^{-\frac{B}{\alpha \sigma N}} f(t) dt \right)^q \frac{d\tau}{\tau} \right] \leq C \int_0^{+\infty} \frac{q^{aq + qB}}{t^m} \frac{dt}{t},
\]
where we used Lemma 3.8 twice, once with $\delta = \frac{N - pm}{(p-1)Nm} + 1 > 1$ and the second time with $\delta = \frac{1}{m} + \frac{B}{\alpha \sigma N} < 1$. If $q = \infty$ directly from (4.15) we obtain that
\[
\bar{w}_n \leq C \|f\|_{L^{m,\infty}(\Omega)} t^{-\frac{N-pm}{(p-1)Nm}}.
\]
Estimate for \(|\nabla w_n|\). Let us start with
\[
\int_0^\infty \frac{q}{\tau} \left( \frac{1}{\tau} \int_0^\tau \frac{1}{SN^{1+\frac{B}{\alpha SN}}} \int_0^s t^{-\frac{B}{\alpha SN}} f(t) dt \right)^{q} \frac{d\tau}{\tau} \leq C \int_0^\infty \frac{q}{\tau} \left( \frac{1}{s} \int_0^s \frac{1}{SN^{1+\frac{B}{\alpha SN}}} \int_0^t s^{\frac{1}{N} - \frac{1}{1+\frac{B}{\alpha SN}}} + B \alpha s^{\frac{1}{N} - \frac{1}{1+\frac{B}{\alpha SN}}} \int_0^t f(t) dt \right)^{q} \frac{d\tau}{\tau} \leq C \int_0^\infty \frac{q}{\tau} \frac{\tau^{\frac{q}{m} - \frac{q}{2N} - \frac{q}{2}}}{\tau} \frac{q}{\tau}.
\]

Moreover
\[
\int_0^\infty \frac{q}{\tau} \left( \frac{1}{\tau} \int_0^\tau \frac{1}{SN^{1+\frac{B}{\alpha SN}}} \int_0^s t^{-\frac{B}{\alpha SN}} f(t) dt \right)^{q} \frac{d\tau}{\tau} \leq C \int_0^\infty \frac{q}{\tau} \left( \frac{1}{s} \int_0^s \frac{1}{SN^{1+\frac{B}{\alpha SN}}} \int_0^t s^{\frac{1}{N} - \frac{1}{1+\frac{B}{\alpha SN}}} + B \alpha s^{\frac{1}{N} - \frac{1}{1+\frac{B}{\alpha SN}}} \int_0^t f(t) dt \right)^{q} \frac{d\tau}{\tau} \leq C \int_0^\infty \frac{q}{\tau} \frac{\tau^{\frac{q}{m} - \frac{q}{2N} - \frac{q}{2}}}{\tau} \frac{q}{\tau},
\]

where we used Lemma 3.8 twice, once with \(\delta = \frac{p'}{m} > 1\) and the second time with \(\delta = \frac{1}{m} + \frac{B}{\alpha SN} < 1\). Hence we have that
\[
\|\nabla w_n\|_{L^q(L^{\frac{p-1}{N-m},\infty}(\Omega))}^q \leq \int_0^\infty \frac{q}{\tau} \left( \frac{1}{\tau} \int_0^\tau \frac{1}{SN^{1+\frac{B}{\alpha SN}}} \int_0^s t^{-\frac{B}{\alpha SN}} f(t) dt \right)^{q} \frac{d\tau}{\tau} \leq \int_0^\infty \frac{q}{\tau} \frac{\tau^{\frac{q}{m} - \frac{q}{2N} - \frac{q}{2}}}{\tau} \frac{q}{\tau}.
\]

If \(q = \infty\) directly from (4.16) we obtain that
\[
|\nabla w_n| \leq C \|f\|_{L^{m,\infty}(\Omega)} t^{-\frac{N-m}{(p-1)Nm}}.
\]

Proof of Theorem 2.4. Case (i). From Lemma 4.7 we infer the existence of a function \(w \in W^{1,r}_0(\Omega)\) with \(1 < r < \frac{(p-1)Nm}{N-m}\) such that, up to a subsequence
\[
w_n \rightharpoonup w \quad \text{in} \quad W^{1,r}_0(\Omega).
\]

Moreover for any measurable \(A \subset \Omega\) it results (recall that \(m > 1\))
\[
\int_A |\nabla w_n|^{p-1} |E_n(x)| \leq \int_0^{|A|} |\nabla w_n|(s) \bar{E}(s) ds \leq C \int_0^{|A|} \frac{t^{-\frac{1}{m} - \frac{1}{N}}}{t^{-\frac{1}{m} - \frac{1}{N}}} \leq C |A|^{-\frac{1}{m}},
\]

that is the equi-integrability of the sequence
\[
\left\{ \frac{\nabla w_n \cdot E(x)}{1 + \frac{1}{n} |\nabla w_n|} \right\}.
\]

This also implies that the hypotheses of Lemma 3.9 are satisfied (with \(F_n \equiv 0\)). The almost everywhere convergence of \(\nabla w_n\) allows us to conclude (see (4.11)) that
\[
|\nabla w_n|^{p-2} \nabla w_n \rightarrow |\nabla w|^{p-2} \nabla w \quad \text{in} \quad L^1(\Omega),
\]
Gradient estimates for nonlinear elliptic equations with first order terms

and that the function \( w \) satisfies (2.9). Moreover thanks to Proposition 3.1 it follows that

\[
\| w \|_{L^{(p-1)Nm/(p-1)m} (\Omega)} + \| \nabla w \|_{L^{(p-1)Nm/(p-1)m} (\Omega)} \leq C \| f \|_{M^m (\Omega)}.
\]

**Case (ii).** From Lemma 4.7 we know that \( \{ w_n \} \) is bounded in \( L^q (\Omega) \) for \( p^* < q < \left( (p-1)m^* \right)^* \). Thus

\[
\int_{\Omega} |E_n|^p |w_n|^p \leq \| E \|_{M^N(\Omega)} \int_0^{\|\Omega\|} t^{-N/p} \overline{w}_n^p (t) dt \leq \| E \|_{M^N(\Omega)} \left( \int_0^{\|\Omega\|} \overline{w}_n^q (t) dt \right)^{\frac{2}{q}} \left( \int_0^{\|\Omega\|} t^{-N(q/p-1)} dt \right)^{\frac{q-p}{q}} \leq C
\]

since \( 1 - \frac{pq}{N(q-p)} > 0 \). Let us take now \( w_n \) as a test function in (2.11). Using Hölder’s inequality we get

\[
\alpha \int_{\Omega} |\nabla w_n|^p \leq \left( \int_{\Omega} |E_n|^p |w_n|^p \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla w_n|^p \right)^{\frac{1}{p'}} + \frac{1}{S} \| f \|_{L^{(p^*)'}(\Omega)} \left( \int_{\Omega} |\nabla w_n|^p \right)^{\frac{1}{p'}} \leq C \left( \int_{\Omega} |\nabla w_n|^p \right)^{\frac{1}{p'}} + \frac{1}{S} \| f \|_{L^{(p^*)'}(\Omega)} \left( \int_{\Omega} |\nabla w_n|^p \right)^{\frac{1}{p'}}.
\]

Hence up to a subsequence \( \{ \nabla w_n \} \) weakly converge in \( W^{1,p}_0 (\Omega) \) to a function \( w \in W^{1,p}_0 (\Omega) \). The rest of the proof is the same of Case (i).

\[ \square \]

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