ABSTRACT. In this work, we consider the two dimensional tidal dynamics equations in a bounded domain and address some optimal control problems like total energy minimization, minimization of dissipation of energy of the flow, etc. We also examine an another control problem which is similar to that of data assimilation problems in meteorology of obtaining unknown initial data, when the system under consideration is tidal dynamics, using optimal control techniques. For all these cases, different distributed optimal control problems are formulated as the minimization of suitable cost functionals subject to the controlled two dimensional tidal dynamics system. The existence of an optimal control as well as the Pontryagin maximum principle for such systems is established and the optimal control is characterized via adjoint variable. The Pontryagin’s maximum principle gives the first-order necessary conditions of optimality. We also establish the uniqueness of optimal control in small time interval. Finally, we derive a second order necessary and sufficient conditions of optimality for such problems.

1. Introduction

Many mathematical developments in infinite dimensional nonlinear system theory and partial differential equations awarded a new dimension to the control theory of fluid dynamics models (cf. [1, 9, 20, 12, 15, 29, 34] etc). Optimal control theory of fluid dynamic equations has been one of the major research areas of applied mathematics with a good number of applications in Oceanography, Geophysics, Engineering and Technology (see for example [12, 15, 34, 37, 3], etc). Controlling fluid flow and turbulence inside a flow in a given physical domain by means of body forces, boundary data, temperature, initial data, etc, is an interesting control problem in fluid mechanics. Ocean tides have been investigated by many mathematicians and physicists, starting from Galileo Galilei, Isaac Newton etc (see [13, 27]). The ocean tide informations are heavily used in the geophysical areas such as Earth tides, the elastic properties of the Earth’s crust, tidal variations of gravity, and in calculating the orbits of artificial satellites used for space exploration etc (see [22, 23]). Laplace rearranged the rotating shallow water equations into the system that underlies the tides and is known as the Laplace tidal equations. By taking the shallow water model on a rotating sphere, which is a slight generalization of the Laplace model, the tidal dynamics model considered in [18, 22, 23] is obtained. We refer the readers to [22, 23, 28] etc, for extensive study on the

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recent progress in this field. In this article, we consider the controlled two dimensional tidal dynamics equations in bounded domains (see (1.1) and (1.3) below) and study the optimal control problems including total energy minimization problem, minimization of dissipation of energy of the flow, an optimization problem similar to the data assimilation problem in meteorology, etc. We establish the first order necessary conditions of optimality via Pontryagin’s maximum principle. The second order necessary and sufficient conditions of optimality for general cost functionals is also obtained.

Let us now describe the two dimensional tidal dynamics equations with a distributed control. Let \( \Omega \) be a bounded subset of \( \mathbb{R}^2 \) with smooth boundary conditions. That is, \( \Omega \) is a horizontal ocean basin, where tides are induced over the time interval \([0, T]\). The boundary contour \( \partial \Omega \) is composed of two disconnected parts: a solid part \( \Gamma_1 \), coinciding with the edge of the continental and island shelves, an open boundary \( \Gamma_2 \). Let us assume that sea water is incompressible and the vertical velocities are small compared with the horizontal velocities, and hence we are able to exclude acoustic waves. Also long waves, including tidal waves, are stood out from the family of gravitational oscillations. Moreover, in order to reduce computational difficulties, we assume that the Earth is absolutely rigid, and the gravitational field of the Earth is not affected by movements of ocean tides. Also, we ignore the effect of the atmospheric tides on the ocean tides and the effect of curvature of the surface of the Earth on horizontal turbulent friction. Under these commonly used assumptions, we consider the following controlled tidal dynamics model (see [18, 14, 22, 23, 24, 40, 25, 26] etc):

\[
\begin{align*}
\frac{\partial \mathbf{w}}{\partial t} + l \mathbf{k} \times \mathbf{w} + g \nabla \zeta + \frac{r}{h} \mathbf{w} \cdot \mathbf{w} - \kappa_h \Delta \mathbf{w} &= \mathbf{g} + \mathbf{U}, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \zeta}{\partial t} + \text{div}(h\mathbf{w}) &= 0, \quad \text{in } \Omega \times (0, T), \\
\mathbf{w} &= \mathbf{w}^0, \quad \text{on } \partial \Omega \times [0, T], \\
\mathbf{w}(0) &= \mathbf{w}_0, \quad \zeta(0) = \zeta_0, \quad \text{in } \Omega, \\
w^0 &= 0, \quad \text{on } \Gamma_1, \quad \text{and } \int_0^T \int_{\Gamma_2} h \mathbf{w}^0 \cdot \mathbf{n} d\Gamma_2 dt = 0,
\end{align*}
\]

where \( \mathbf{w}(x, t) = (w_1(x, t), w_2(x, t)) \in \mathbb{R}^2 \), the horizontal transport vector, is the averaged integral of the velocity vector over the vertical axis, \( l = 2\rho \cos \theta \) is the Coriolis parameter, where \( \rho \) is the angular velocity of the Earth rotation and \( \theta \) is the colatitude, \( \mathbf{k} \) is a unit vector oriented vertically upward, \( \mathbf{k} \times \mathbf{w} = (-w_2, w_1) \), \( g \) is the free fall acceleration, \( r \) is the bottom friction factor, \( \kappa_h \) is the horizontal turbulent viscosity coefficient, \( \mathbf{n} \) is the unit outward normal to the boundary \( \Gamma_2 \). The scalar \( \zeta(x, t) \in \mathbb{R} \) is the deviations of free surface with respect to the ocean bottom (i.e., \( \zeta = \zeta_s - \zeta_b \), where \( \zeta_s \) is the surface and \( \zeta_b \) is the shift of the ocean bottom). \( \mathbf{g} = \gamma_L g \nabla \zeta^+ \) is the known tide-generating force with \( \gamma_L \) is the Love factor approximately equal to 0.7 and \( \zeta^+ \) is the height of the static tide. Also \( \mathbf{U} \) is the distributed control acting on the system.

Let us now discuss about the boundary conditions satisfied by the averaged velocity field. Remember that the contour \( \partial \Omega \) consists of two parts, a solid part \( \Gamma_1 \) coinciding with the shelf edge and the open boundary \( \Gamma_2 \). The function \( \mathbf{w}^0(x, t) \) is a known function on the boundary. This impermeability condition is given on the solid part of the boundary, i.e., the restriction \( \mathbf{w}^0|_{\Gamma_1} = 0 \) is the no-slip boundary condition on the shoreline, and \( \int_0^T \int_{\Gamma_2} h \mathbf{w}^0 \cdot \mathbf{n} d\Gamma_2 dt = 0, \)
follows from the mass conservation law. In (1.1), \( h(x) \) is the vertical scale of motion, i.e., the depth of the calm sea at \( x \) in the region \( \Omega \) and we assume that it is a continuously differentiable function nowhere becoming zero, so that

\[
0 < \lambda = \min_{x \in \Omega} h(x), \quad \mu = \max_{x \in \Omega} h(x), \quad \max_{x \in \Omega} |\nabla h(x)| \leq M,
\]

(1.2)

where \( M \) is a positive constant which equals to zero at a constant ocean depth \( h \). The initial datum \( w_0(x) \) and \( \zeta_0(x) \) are given. In particular, \( w_0 \) and \( \zeta_0 \) can be set equal to zero, which indicates the fact that initially the ocean is at rest.

The unique global solvability results of the system (1.1) is obtained by simplifying the non-homogeneous boundary value problem to a homogeneous Dirichlet boundary value problem (see [17, 22, 23] for more details).

In order to simplify the non-homogeneous boundary value problem to a homogeneous Dirichlet boundary value problem, we set

\[
u(x,t) = w(x,t) - w_0(x,t),
\]

and

\[
\xi(x,t) = \zeta(x,t) + \int_0^t \text{div}(h(x)w^0(x,s))ds,
\]

which are referred to as the tidal flow and the elevation. The full flow \( w^0 \), which is given a priori on the boundary \( \partial \Omega \), has been extended to the whole domain \( \Omega \times (0,T] \) as a smooth function and still denoted by \( w^0 \). Then the controlled tidal dynamics system (1.1) can be written in the abstract form as:

\[
\begin{cases}
\frac{\partial u(t)}{\partial t} + A u(t) + B(u(t)) + \nabla \xi(t) = f(t) + U(t), \quad \text{in } \Omega \times (0,T), \\
\frac{\partial \xi(t)}{\partial t} + \text{div}(h u(t)) = 0, \quad \text{in } \Omega \times (0,T), \\
u(t) = 0, \quad \text{on } \partial \Omega \times (0,T), \\
u(0) = u_0, \quad \xi(0) = \xi_0, \quad \text{in } \Omega,
\end{cases}
\]

(1.3)

where we scaled \( g \) to unity. For \( U = 0 \), we call the system (1.3) as an uncontrolled tidal dynamics system. In (1.3), \( A \) denotes the matrix operator:

\[
A := \begin{pmatrix}
-\alpha \Delta & -\beta \\
\beta & -\alpha \Delta
\end{pmatrix},
\]

(1.4)

where \( \Delta \) is the Laplacian operator, \( \alpha := \kappa h \), \( \beta := 2\rho \cos \theta \) are positive constants, \( B \) denotes the nonlinear vector operator,

\[
B(u) := \gamma(x)|u + w^0|(u + w^0),
\]

(1.5)

where \( w^0(x,t) \in \mathbb{R}^2 \) is a known deterministic function on the boundary \( \partial \Omega \). The function \( \gamma(x) := \frac{r}{h(x)} \) is a strictly positive smooth function. The function \( f \) and initial datum \( u_0 \) and \( \xi_0 \) are given by

\[
\begin{cases}
f = g - \frac{\partial w^0}{\partial t} + \nabla \int_0^t \text{div}(h w^0)ds + \kappa h \Delta w^0 - l k \times w^0, \\
u_0(x) = w_0(x) - w^0(x,0), \\
\xi_0(x) = \zeta_0(x).
\end{cases}
\]

(1.6)
Let us now discuss some of the solvability results available in the literature for the system (1.1). The existence and uniqueness of a weak solution for the tidal dynamic equations (see systems (1.1) or (1.3)) in bounded domains has been obtained in [17, 22, 23], using compactness arguments. The authors in [24] obtained similar results for the deterministic tidal dynamics system using global monotonicity property of the linear and nonlinear operators. The existence of a periodic solution for the tidal dynamics problem in two dimensional finite domains is obtained in [14]. The existence and uniqueness of weak and strong solutions of the stationary tidal dynamic equations in bounded and unbounded domains is obtained in [25]. The authors in [25] also established a uniform Lyapunov stability of the steady state solution. The dynamic programming method and feedback analysis for an optimal control of 2D tidal dynamics system is carried out in [26]. The global solvability results for stochastic perturbations in bounded and unbounded domains, and different characteristics have been established in [24, 40, 35, 2, 16] etc. Some optimal control problems in tidal power generation and related problems are considered in [4, 30, 31, 39], etc. The authors in [2] formulated a martingale problem of Stroock and Varadhan associated to an initial value control problem and established the existence of optimal controls.

The rest of the paper is organized as follows. In the next section, we give the necessary functional setting needed to obtain the global solvability results of the system (1.3). We also consider the corresponding linearized system in the same section and establish the existence and uniqueness of a global weak solution (Theorem 2.6). A distributed optimal control problem (total energy minimization problem) as the minimization of a suitable cost functional subject to the controlled tidal dynamics system (1.3) is formulated in section 3. The existence of an optimal triplet (Theorem 3.3) as well as the first order necessary conditions of optimality, via Pontryagin’s maximum principle is also established in this section (Theorem 3.4). The optimal control is characterized using the adjoint variable and the solvability of the adjoint system is also discussed (see Theorem 3.2). Similar results for different optimizations problems like dissipation of energy of the flow and initial data optimization problem (a problem similar to the data assimilation problems of meteorology) are also obtained in the same section (Theorems 3.11 and 3.8). The final section is devoted for deriving the second order necessary and sufficient optimality conditions for a general optimal control problem (Theorems 4.4 and 4.5).

2. Mathematical Formulation

In this section, we explain the necessary function spaces needed to obtain the global solvability results and provide the global existence and uniqueness of weak solution of the uncontrolled system (1.3). We also give an insight into the global solvability of the corresponding linearized problem.

2.1. Functional Setting. For \( p \geq 1 \), we denote by \( L^p(\Omega) := L^p(\Omega; \mathbb{R}) \), the space consisting of equivalence classes of measurable real valued functions for which the \( p^{th} \) power of the absolute value is Lebesgue integrable, where functions which agree almost everywhere are identified. We know that \( L^2(\Omega) \) is a Hilbert space, and the norm and inner product in \( L^2(\Omega) \) are denoted by \( \| \cdot \|_{L^2} \) and \( \langle \cdot , \cdot \rangle_{L^2} \). We define \( L^p(\Omega) := L^p(\Omega; \mathbb{R}^2) \) as the Banach space of Lebesgue measurable \( \mathbb{R}^2 \)-valued, \( p \)-integrable functions on \( \Omega \) with the norm:

\[
\| u \|_{L^p} := \left( \int_\Omega |u(x)|^p \, dx \right)^{1/p}.
\]
For \( p = 2 \), \( L^2(\Omega) := L^2(\Omega; \mathbb{R}^2) \) is a Hilbert space equipped with the inner product given by
\[
(u, v)_{L^2} := \int_{\Omega} u(x) \cdot v(x) dx, \quad u, v \in L^2(\Omega).
\]

Then the norm on \( L^2(\Omega) \) is defined by
\[
\|u\|_{L^2} := (u, u)_{L^2}^{1/2} = \left( \int_{\Omega} |u(x)|^2 dx \right)^{1/2}.
\]

Let \( H^1(\Omega) := H^1(\Omega; \mathbb{R}^2) \) denotes the Sobolev space \( W^{1,2}(\Omega) := W^{1,2}(\Omega; \mathbb{R}^2) \) with the norm defined by
\[
\|u\|_{H^1}^2 := \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2, \quad \text{for all } u \in H^1(\Omega).
\]

We also let \( H^1_0(\Omega) := H^1_0(\Omega; \mathbb{R}^2) \) to be the closure of \( C^\infty_0(\Omega; \mathbb{R}^2) \) in \( H^1(\Omega) \) norm, where \( C^\infty_0(\Omega; \mathbb{R}^2) \) is the space of all infinitely differentiable functions with compact support in \( \Omega \). Then \( H^1_0(\Omega) \) is also a Sobolev space under the induced norm. Since \( \Omega \) is a bounded domain, in view of the Poincaré inequality, i.e.,
\[
\|u\|_{L^2} \leq C_\Omega \|\nabla u\|_{L^2},
\]
the norms \( \|\nabla u\|_{L^2(\Omega)} \) and \( \|u\|_{H^1} \) are equivalent in \( H^1_0(\Omega) \). Thus \( H^1_0(\Omega) \) is also a Hilbert space with inner product:
\[
(u, v)_{H^1_0} := (\nabla u, \nabla v)_{L^2} = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx,
\]
and the norm:
\[
\|u\|_{H^1_0} = \|\nabla u\|_{L^2} = \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{1/2}.
\]

We denote the dual of \( H^1_0(\Omega) \) by \((H^1_0(\Omega))' = H^{-1}(\Omega)\). The induced duality between the spaces \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \). Then we have the following continuous and dense embedding:
\[
H^1_0(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset H^{-1}(\Omega).
\]

The above embedding is also compact, since \( \Omega \) is bounded. Using the Gelfand triple \((H^1_0(\Omega), L^2(\Omega), H^{-1}(\Omega))\), we may consider \( \nabla \) or \( \Delta \) as a linear map from \( L^2(\Omega) \) or \( H^1_0(\Omega) \) into \( H^{-1}(\Omega) \) respectively.

We next give the well known inequality due to Ladyzhenskaya (see Lemma 1 and 2, Chapter 1, [19]), which is used in the paper quite frequently.

**Lemma 2.1** (Ladyzhenskaya inequality). For \( u \in C_0^\infty(\Omega; \mathbb{R}^n) \), \( n = 2, 3 \), there exists a constant \( C > 0 \) such that
\[
\|u\|_{L^4} \leq C^{1/4} \|u\|_{L^2}^{1-\frac{n}{4}} \|\nabla u\|_{L^2}^{\frac{n}{4}}, \quad \text{for } n = 2, 3,
\]
where \( C = 2, 4 \), for \( n = 2, 3 \) respectively.

Thus, for \( n = 2 \), we have
\[
\|u\|_{L^4} \leq 2^{1/4} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \leq 2^{1/4} C \|\nabla u\| = \tilde{C}_\Omega \|u\|_{H^1_0},
\]
where \( \tilde{C}_\Omega = 2^{1/4} C_\Omega \) and we also used the Poincaré inequality. Thus, we obtain a continuous and compact embedding \( H^1_0(\Omega) \hookrightarrow L^4(\Omega) \).
2.2. Linear operator. Let us define the non-symmetric bilinear form:
\[ a(u, v) := \alpha [(\nabla u_1, \nabla v_1) + (\nabla u_2, \nabla v_2)] + \beta [(u_1, v_2) - (u_2, v_1)], \]
where \( u = (u_1, u_2), v = (v_1, v_2). \) If \( u \) has a smooth second order derivatives, then
\[ a(u, v) = (Au, v)_{L^2}, \quad \text{for all } v \in H^1_0(\Omega). \]
We consider \( A = -\alpha \Delta u + \beta k \times u \) and an integration by parts twice yields (using the fact that \( u|_{\partial \Omega} = 0 \))
\[ (-\alpha \Delta u + \beta k \times u)_{L^2} = \alpha (\nabla u, \nabla v)_{L^2} + \beta (k \times u, v)_{L^2} \]
\[ = (u, -\alpha \Delta v - \beta k \times v)_{L^2}, \]
and hence \((Au, v)_{L^2} \neq (u, Av)_{L^2}, \) so that \( A \) is not symmetric. The bilinear form \( a(\cdot, \cdot) \) is continuous and coercive in \( H^1_0(\Omega) \), i.e.,
\[ |a(u, v)| \leq C_0 \|u\|_{H^1_0} \|v\|_{H^1_0}, \quad \text{for all } u, v \in H^1_0(\Omega), \quad (2.3) \]
\[ (Au, u)_{L^2} = a(u, u) = \alpha \|\nabla u\|_{L^2}^2 = \alpha \|u\|_{H^1_0}^2, \quad (2.4) \]
for some positive constant \( C_0. \) By means of the Gelfand triple we may consider \( A \), given by \( (\cdot, \cdot) \), as a mapping from \( H^1_0(\Omega) \) into its dual \( H^{-1}(\Omega) \), so that \((Au, u) = \alpha \|\nabla u\|_{L^2}^2 = \alpha \|u\|_{H^1_0}^2. \)

2.3. Nonlinear operator. Let us denote the nonlinear operator \( B(\cdot) \) by
\[ v \mapsto B(v) := \gamma(x)|v + w^0|(v + w^0). \]

Lemma 2.2 \((24, 35, 25)\). The operator \( B \) has the following properties: For all \( u, v, w^0 \in L^4(\Omega), \) we have
(i) \( \|B(u)\|_{L^2} \leq \frac{\gamma}{\lambda}(\|u\|_{L^4} + \|w^0\|_{L^4})^2, \)
(ii) \( B(\cdot) \) is a nonlinear continuous operator from \( H^1_0(\Omega) \) into \( H^{-1}(\Omega), \)
(iii) \( (B(u) - B(v), u - v)_{L^2} \geq 0, \)
(iv) \( (B(u), u)_{L^2} \geq -\frac{\gamma}{\lambda}(\|w^0\|_{L^4}^4 + \|u\|_{L^4}^2), \)
(v) \( \|B(u) - B(v)\|_{L^2} \leq \frac{\gamma}{\lambda}(\|u\|_{L^4} + \|v\|_{L^4} + \|w^0\|_{L^4})\|u - v\|_{L^4}, \)
(vi) The operator \( B(\cdot) \) is Fréchet differentiable with the Fréchet derivative \( B'(u) = 2\gamma|u + w^0| \in L(L^4, L^2) \) and \( (B'(u)v, v)_{L^2} \geq 0. \)
(vii) \( \|B'(u)v\|_{H^{-1}} \leq \frac{2\gamma}{\lambda}(\|u\|_{L^4} + \|w^0\|_{L^4})\|v\|_{L^4}. \)

Proof. We prove only (viii). For \( w \in H^1_0(\Omega), \) we consider
\[ \langle B'(u)v, w \rangle \leq \|B'(u)v\|_{L^2} \|w\|_{L^2} \leq \frac{2\gamma}{\lambda} \|u + w^0\|_{L^4} \|v\|_{L^4} \|w\|_{H^1_0} \]
\[ \leq \frac{2\gamma}{\lambda}(\|u\|_{L^4} + \|w^0\|_{L^4}) \|v\|_{L^4} \|w\|_{H^1_0}, \quad (2.5) \]
so that we have \( \|B'(u)v\|_{H^{-1}} \leq \frac{2\gamma}{\lambda}(\|u\|_{L^4} + \|w^0\|_{L^4})\|v\|_{L^4}. \) \qed

The estimates \((2.4)\) and (iii) easily imply the following:

Lemma 2.3. The operator \( F(u) := Au + B(u) - f \) is a globally monotone operator, i.e.,
\[ \langle F(u) - F(v), u - v \rangle \geq 0, \quad \text{for all } u, v \in H^1_0(\Omega). \]
2.4. Global existence and uniqueness. Let us now give the definition of weak solution and discuss the solvability results available in the literature for the uncontrolled system (1.3) (see [22, 23, 24, 25] for details).

Definition 2.4. The pair

\[(u, \xi) \in (C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))) \times C([0, T]; L^2(\Omega)),\]

with

\[\left(\partial_t u, \partial_t \xi\right) \in L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)),\]

is called a weak solution to the system (1.3), if for \(f \in L^2(0, T; H^{-1}(\Omega)), (u_0, \xi_0) \in L^2(\Omega) \times L^2(\Omega)\) and \((v, \eta) \in H^1_0(\Omega) \times H^1(\Omega)\), \((u, \xi)\) satisfies:

\[\lim_{t \to 0} \int_\Omega u(t)v dx = \int_\Omega u_0v dx, \quad \lim_{t \to 0} \int_\Omega \xi(t)\eta dx = \int_\Omega \xi_0\eta dx,\]

(2.6)

and the energy equality

\[\frac{d}{dt} \left(\|\sqrt{h}u(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2\right) + 2\langle F(u(t)), h u(t)\rangle = 0.\]

(2.7)

Theorem 2.5 (Existence and uniqueness of weak solution, Chapter 2, [22], Propositions 3.6, 3.7 [24]). Let \((u_0, \xi_0) \in L^2(\Omega) \times L^2(\Omega)\) be given. For \(f \in L^2(0, T; H^{-1}(\Omega))\), there exists a unique weak solution \((u, \xi)\) to the uncontrolled system (1.3) satisfying

\[\|u(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2 + \alpha \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \left(\|u_0\|_{L^2}^2 + \|\xi_0\|_{L^2}^2 + \frac{r}{\lambda} \int_0^t \|w^0(s)\|_{L^4}^4 ds + \int_0^t \|f(s)\|_{H^{-1}}^2 ds\right)e^{\kappa t},\]

(2.8)

for all \(t \in [0, T]\), where \(K = \max\{1 + M + \frac{\alpha}{2}(1 + \mu^2) + M\}\).

2.5. The linearized system. Let us linearize the equations (1.3) around \((\hat{u}, \hat{\xi})\) which is the unique weak solution of system (1.3) with control term \(U = 0\) (uncontrolled system), external forcing \(\tilde{g}\), and initial datum \(\hat{u}_0\) and \(\hat{\xi}_0\) are such that \(\tilde{g} \in L^2(0, T; H^{-1}(\Omega))\) and \((u_0, \xi_0) \in L^2(\Omega) \times L^2(\Omega)\). We consider the following linearized system:

\[\frac{\partial w(t)}{\partial t} + Aw(t) + B'(\hat{u}(t))w(t) + \nabla \eta(t) = \tilde{g}(t) + U(t), \quad \text{in } \Omega \times (0, T),\]

\[\frac{\partial \eta(t)}{\partial t} + \text{div}(hw(t)) = 0, \quad \text{in } \Omega \times (0, T),\]

\[w(t) = 0, \quad \text{on } \partial \Omega \times (0, T),\]

\[w(0) = w_0, \quad \eta(0) = \eta_0, \quad \text{in } \Omega,\]

(2.9)

where \(\hat{g} = g - \tilde{g}\), \(B'(\hat{u}) = 2\gamma |\hat{u} + w^0|\) and \((w_0, \eta_0) \in L^2(\Omega) \times L^2(\Omega)\). We now prove an a-priori energy estimate satisfied by the system (2.9). Let us take an inner product with \(w(\cdot)\) to the first equation in (2.9) to obtain

\[\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \alpha \|\nabla w(t)\|_{L^2}^2\]
\begin{align*}
&= -(B'(\hat{\mathbf{u}}(t))\mathbf{w}(t), \mathbf{w}(t))_{L^2} - \langle \nabla \eta(t), \mathbf{w}(t) \rangle + \langle \mathbf{g}(t), \mathbf{w}(t) \rangle + \langle U(t), \mathbf{w}(t) \rangle \\
&\leq \langle \eta(t), \text{div} \mathbf{w}(t) \rangle_{L^2} + \langle \mathbf{g}(t), \mathbf{w}(t) \rangle + \langle U(t), \mathbf{w}(t) \rangle \\
&\leq \|\eta(t)\|_{L^2} \|\text{div} \mathbf{w}(t)\|_{L^2} + \|\mathbf{g}(t)\|_{H^{-1}} \|\nabla \mathbf{w}(t)\|_{L^2} + \|U(t)\|_{H^{-1}} \|\nabla \mathbf{w}(t)\|_{L^2} \\
&\leq \sqrt{2}\|\eta(t)\|_{L^2} \|\nabla \mathbf{w}(t)\|_{L^2} + \frac{\alpha}{4} \|\nabla \mathbf{w}(t)\|^2_{L^2} + \frac{2}{\alpha} \|\mathbf{g}(t)\|_{H^{-1}}^2 + \frac{2}{\alpha} \|U(t)\|_{H^{-1}}^2, \\
&\leq 3\alpha \|\nabla \mathbf{w}(t)\|^2_{L^2} + \frac{4}{\alpha} \|\eta(t)\|_{L^2}^2 + \frac{2}{\alpha} \|\mathbf{g}(t)\|_{H^{-1}}^2 + \frac{2}{\alpha} \|U(t)\|_{H^{-1}}^2, \quad (2.10)
\end{align*}

where we used \((B'(\hat{\mathbf{u}})\mathbf{w}, \mathbf{w})_{L^2} \geq 0\), \(\|\text{div} \mathbf{u}\|_{L^2} \leq \sqrt{2}\|\nabla \mathbf{u}\|_{L^2}\), Cauchy-Schwarz and Young’s inequalities. Let us now take inner product with \(\eta(t)\) to the second equation in \((2.9)\) to get

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\eta(t)\|_{L^2}^2 &= -(\text{div}(h\mathbf{w}(t)), \eta(t))_{L^2} \leq \|\text{div}(h\mathbf{w}(t))\|_{L^2} \|\eta(t)\|_{L^2} \\
&= \|h\text{div} \mathbf{w}(t) + \nabla h \cdot \mathbf{w}(t)\|_{L^2} \|\eta(t)\|_{L^2} \\
&\leq (\|h\text{div} \mathbf{w}(t)\|_{L^2} + \|\nabla h \cdot \mathbf{w}(t)\|_{L^2}) \|\eta(t)\|_{L^2} \\
&\leq \sqrt{2}\|\nabla \mathbf{w}(t)\|_{L^2} \|\eta(t)\|_{L^2} + M \|\mathbf{w}(t)\|_{L^2} \|\eta(t)\|_{L^2} \\
&\leq \frac{\alpha}{8} \|\nabla \mathbf{w}(t)\|^2_{L^2} + \frac{M}{2} \|\mathbf{w}(t)\|^2_{L^2} + \left(\frac{4\mu^2}{\alpha} + \frac{M}{2}\right) \|\eta(t)\|^2_{L^2}, \quad (2.11)
\end{align*}

where we used Hölder’s and Young’s inequalities. Combining \((2.10)\) and \((2.11)\), we find

\begin{align*}
\frac{d}{dt}(\|\mathbf{w}(t)\|^2_{L^2} + \|\eta(t)\|^2_{H^{-1}}) + \alpha \|\nabla \mathbf{w}(t)\|^2_{L^2} &\leq M \|\mathbf{w}(t)\|^2_{L^2} + \left(\frac{8(\mu^2 + 1)}{\alpha} + M\right) \|\eta(t)\|^2_{L^2} + \frac{4}{\alpha} \|\mathbf{g}(t)\|^2_{H^{-1}} + \frac{4}{\alpha} \|U(t)\|^2_{H^{-1}}.
\end{align*}

Integrating the above inequality from 0 to \(t\), we obtain

\begin{align*}
\|\mathbf{w}(t)\|^2_{L^2} + \|\eta(t)\|^2_{H^{-1}} &\leq \|\mathbf{w}_0\|^2_{L^2} + \|\eta_0\|^2_{H^{-1}} + \left(\frac{8(\mu^2 + 1)}{\alpha} + M\right) \int_0^t \|\mathbf{w}(s)\|^2_{L^2} + \|\eta(s)\|^2_{H^{-1}} ds \\
&\quad + \frac{4}{\alpha} \int_0^t \|\mathbf{g}(s)\|^2_{H^{-1}} ds + \frac{4}{\alpha} \int_0^t \|U(s)\|^2_{H^{-1}} ds.
\end{align*}

(2.12)

An application of Gornwall’s inequality in \((2.12)\) yields

\begin{align*}
\|\mathbf{w}(t)\|^2_{L^2} + \|\eta(t)\|^2_{H^{-1}} &\leq \left(\|\mathbf{w}_0\|^2_{L^2} + \|\eta_0\|^2_{H^{-1}} + \frac{4}{\alpha} \int_0^t \|\mathbf{g}(s)\|^2_{H^{-1}} ds + \frac{4}{\alpha} \int_0^t \|U(s)\|^2_{H^{-1}} ds\right) e^{\frac{(8\mu^2 + 1)}{\alpha} + M} t, \quad (2.13)
\end{align*}

for all \(t \in [0, T]\).

In order to obtain the time derivative estimates, from the first equation in \((2.9)\), we find

\begin{align*}
\int_0^T \|\partial_t \mathbf{w}(t)\|^2_{H^{-1}} dt &\leq 5 \int_0^T \|\Lambda \mathbf{w}(t)\|^2_{H^{-1}} dt + \int_0^T \|B'(\hat{\mathbf{u}}(t)) \mathbf{w}(t)\|^2_{H^{-1}} dt + \int_0^T \|\nabla \eta(t)\|^2_{H^{-1}} dt
\end{align*}
is defined as the set of states \( U \) we take the set of all admissible control class \( A \).

\[
\begin{align*}
\text{Definition 3.1} & \quad \text{definition of class of admissible class of solutions.} \\
U & \quad \text{where we used Lemma 2.2 (vii), Hölder’s and Young’s inequalities. Similarly we have}
\end{align*}
\]

\[
\int_0^T \| \partial_t \eta(t) \|^2_{L^2} dt = \int_0^T \| \text{div}(h \omega(t)) \|^2_{L^2} dt \\
\leq 2 \left[ M^2 \int_0^T \| \omega(t) \|^2_{L^2} dt + 2 \mu^2 \int_0^T \| \nabla \omega(t) \|^2_{L^2} dt \right] < +\infty,
\]

and both the estimates are uniformly bounded. Moreover, using a standard Faedo-Galerkin approximation technique, we have the following Theorem.

**Theorem 2.6.** Let \( (m, \eta) \in L^2(\Omega) \times L^2(\Omega) \) be given. For \( \tilde{g} \in L^2(0, T; H^{-1}(\Omega)) \) and \( U \in L^2(0, T; \mathbb{H}^{-1}(\Omega)) \), there exists a unique weak solution to the system (2.9) satisfying

\[
(\omega, \eta) \in (C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}^1_0(\Omega))) \times (C([0, T]; L^2(\Omega)),
\]

with

\[
(\partial_t \omega, \partial_t \eta) \in L^2(0, T; \mathbb{H}^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)).
\]

**3. Optimal Control Problem**

In this section, we formulate a distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled tidal dynamics system \( (\text{L3}) \). The main objective is to prove the existence of an optimal control that minimizes the cost functional given below, subject to the constraint \( (\text{L3}) \) and establish the first order necessary condition via, Pontryagin’s maximum principle. The optimal control is characterized via adjoint variable. The cost functional under our consideration is given by

\[
\mathcal{J}(u, \xi, U) := \frac{1}{2} \int_0^T \| u(t) - u_d(t) \|^2_{L^2} dt + \frac{1}{2} \int_0^T \| \xi(t) - \xi_d(t) \|^2_{H^{-1}} dt + \frac{1}{2} \int_0^T \| U(t) \|^2_{H^{-1}} dt,
\]

where \( u_d(\cdot) \in L^2(0, T; L^2(\Omega)) \) and \( \varphi_d(\cdot) \in L^2(0, T; L^2(\Omega)) \) are the desired states. Note that the cost functional is the sum of the total energy and total effort by controls. In this work, we take the set of all admissible control class \( \mathcal{U}_{ad} = L^2(0, T; \mathbb{H}^{-1}(\Omega)) \). Next, we give the definition of class of admissible class of solutions.

**Definition 3.1** (Admissible class). The admissible class \( \mathcal{A}_{ad} \) of triples

\[
(u, \xi, U) \in (C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}^1_0(\Omega))) \times (C([0, T]; L^2(\Omega)) \times L^2(0, T; \mathbb{H}^{-1}(\Omega))
\]

is defined as the set of states \( (u, \xi) \) solving the system \( (\text{L3}) \) with control \( U \in \mathcal{U}_{ad} \). That is,

\[
\mathcal{A}_{ad} := \left\{ (u, \xi, U) : (u, \xi) \in (C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}^1_0(\Omega))) \times (C([0, T]; L^2(\Omega)) \right\}
\]
is a unique weak solution of \((1.3)\) with control \(U \in L^2(0, T; H^{-1}(\Omega))\).

Clearly \(\mathcal{A}_{ad}\) is a nonempty set as for any \(U \in \mathcal{W}_{ad}\), there exists a unique weak solution of the system \((1.3)\). In view of the above definition, the optimal control problem we are considering can be formulated as:

\[
\min_{(u, \xi, U) \in \mathcal{A}_{ad}} J(u, \xi, U). \tag{3.2}
\]

A solution to the problem \((3.2)\) is called an optimal solution. The optimal triplet is denoted by \((u^*, \xi^*, U^*)\) and the control \(U^*\) is called an optimal control.

### 3.1. The adjoint system.

In order to establish Pontryagin’s maximum principle, we need to find the adjoint system corresponding to \((1.3)\). Remember that optimal control is characterized via adjoint variable. In this subsection, we formally derive the adjoint system corresponding to the problem \((1.3)\). Let us first define

\[
\begin{align*}
N_1(u, \xi, U) &:= -Au - B(u) - g\nabla \xi + U, \\
N_2(u, \xi) &:= -\text{div}(hu).
\end{align*} \tag{3.3}
\]

Then the tidal dynamics system \((1.3)\) can be written as

\[
\{\partial_t u, \partial_t \xi\} = \{N_1(u, \xi, U), N_2(u, \xi)\}.
\]

We define the augmented cost functional \(\tilde{J}\) by

\[
\tilde{J}(u, \xi, U, p, \varphi) := \int_0^T \langle p, \partial_t u - N_1(u, \xi, U) \rangle dt + \int_0^T \langle \varphi, \partial_t \xi - N_2(u, \xi) \rangle dt - J(u, \xi, U),
\]

where \(p\) and \(\varphi\) denote the adjoint variables corresponding to \(u\) and \(\xi\) respectively. Next, we derive the adjoint equations formally by differentiating the augmented cost functional \(\tilde{J}\) in the Gâteaux sense with respect to each of its variables. The adjoint variables \(p, \xi\) and \(U\) satisfy the following system:

\[
\begin{align*}
\frac{\partial p}{\partial t} - [\partial_u N_1]^* p - [\partial_u N_2]^* \varphi &= J_u, \\
\frac{\partial \varphi}{\partial t} - [\partial_\xi N_1]^* p - [\partial_\xi N_2]^* \varphi &= J_\xi, \\
- [\partial_U N_1]^* p - [\partial_U N_2]^* \varphi &= J_U, \\
p|_{\partial \Omega} &= 0, \\
p(T, \cdot) &= 0, \quad \varphi(T, \cdot) = 0.
\end{align*} \tag{3.4}
\]

Note that differentiating \(\tilde{J}\) with respect to the adjoint variables recovers the original non-linear system. Further, we compute \([\partial_u N_1]^* p, [\partial_u N_2]^* \varphi, [\partial_\xi N_1]^* p, [\partial_\xi N_2]^* \varphi\) as

\[
\begin{align*}
[\partial_u N_1]^* p &= -\tilde{A}p - B'(u)p, \\
[\partial_u N_2]^* \varphi &= h\nabla \varphi, \\
[\partial_\xi N_1]^* p &= \text{div} p, \\
[\partial_\xi N_2]^* \varphi &= 0,
\end{align*} \tag{3.5}
\]

where \(\tilde{A}p = -\alpha \Delta p - \beta k \times p\). Since \(A\) is non-symmetric, we have \(A v \neq \tilde{A} v\). But one can easily see that \(\langle Av, v \rangle = \alpha \|\nabla v\|_{L^2}^2 = \langle \tilde{A} v, v \rangle\), for all \(v \in H^1_0(\Omega)\). Since \(H^{-1}(\Omega)\) is a separable,
reflexive Banach space, it should be noted that $f(\cdot) = \frac{1}{2}\|\cdot\|_{\mathbb{H}^{-1}}^2$ is Gâteaux differentiable and we calculate the Gâteaux derivative as
\[
\langle f'(u), v \rangle_{\mathbb{H}_0^1 \times \mathbb{H}^{-1}} = \left. \frac{d}{dt} f(u + \tau v) \right|_{\tau=0} = \frac{1}{2} \left. \frac{d}{dt} \|u + \tau v\|_{\mathbb{H}^{-1}}^2 \right|_{\tau=0} \\
= \frac{1}{2} \left. \frac{d}{dt} \|(-\Delta)^{-1/2}(u + \tau v)\|_2^2 \right|_{\tau=0} = \langle (-\Delta)^{-1/2}u, (-\Delta)^{-1/2}v \rangle_{L^2} \\
= \langle (-\Delta)^{-1}u, v \rangle_{\mathbb{H}_0^1 \times \mathbb{H}^{-1}},
\]
for all $v \in \mathbb{H}^{-1}(\Omega)$. Thus, we have $f'(u) = (-\Delta)^{-1}u \in \mathbb{H}_0^1(\Omega)$. Note that the third condition in (3.4) gives $(-\Delta)^{-1}U = -p$. Thus from (3.4), it follows that the adjoint variables $(p, \varphi)$ satisfy the following adjoint system:
\[
\begin{cases}
-\frac{\partial p(t)}{\partial t} + \tilde{A}(p(t)) + B'(u(t))p(t) - h\nabla \varphi(t) = (u(t) - u_d(t)), \quad \text{in } \Omega \times (0, T), \\
-\frac{\partial \varphi(t)}{\partial t} - \div p(t) = (\xi(t) - \xi_d(t)), \quad \text{in } \Omega \times (0, T), \\
p(t) = 0, \quad \text{on } \partial \Omega \times (0, T), \\
\varphi(T, \cdot) = 0, \quad \varphi(T, \cdot) = 0, \quad \text{in } \Omega.
\end{cases} \tag{3.6}
\]
Let us take $p(T) = p_T \in L^2(\Omega)$ and $\varphi(T) = \varphi_T \in L^2(\Omega)$ and obtain the a-priori energy estimates. We take inner product with $p(\cdot)$ to the first equation in (3.6) to obtain
\[
-\frac{1}{2} \frac{d}{dt} \|p(t)\|_{L^2}^2 + \alpha \|\nabla p(t)\|_{L^2}^2 \\
= -(B'(u(t))p(t), p(t))_{L^2} - \langle h \nabla \varphi(t), p(t) \rangle + ((u(t) - u_d(t)), p(t))_{L^2} \\
\leq (\varphi(t), \div (h p(t)))_{L^2} + ((u(t) - u_d(t)), p(t))_{L^2} \\
\leq \|\varphi(t)\|_{L^2} \|\nabla h \cdot p(t) + h \div p(t)\|_{L^2} + \|u(t) - u_d(t)\|_{L^2} \|p(t)\|_{L^2} \\
\leq (\|\nabla h\|_{L^\infty} \|p(t)\|_{L^2} + \|h\|_{L^\infty} \|\nabla p(t)\|_{L^2}) \|\varphi(t)\|_{L^2} + \frac{1}{2} \|u(t) - u_d(t)\|_{L^2}^2 + \frac{1}{2} \|p(t)\|_{L^2}^2 \\
\leq \frac{M+1}{2} \|p(t)\|_{L^2}^2 + \left(\frac{M}{2} + \mu^2\right) \|\varphi(t)\|_{L^2}^2 + \frac{\alpha}{4} \|\nabla p(t)\|_{L^2}^2 + \frac{1}{2} \|u(t) - u_d(t)\|_{L^2}^2, \tag{3.7}
\]
where we used $(B'(u)p, p)_{L^2} \geq 0$, Cauchy-Schwarz, Hölder and Young’s inequalities. Next, we take inner product with $\varphi(\cdot)$ to the second equation in (3.6) to get
\[
-\frac{1}{2} \frac{d}{dt} \|\varphi(t)\|_{L^2}^2 = \langle \div p(t), \varphi(t) \rangle_{L^2} + ((\xi(t) - \xi_d(t)), \varphi(t))_{L^2} \\
\leq \|\div p(t)\|_{L^2} \|\varphi(t)\|_{L^2} + \|\xi(t) - \xi_d(t)\|_{L^2} \|\varphi(t)\|_{L^2} \\
\leq \frac{\alpha}{4} \|\nabla p(t)\|_{L^2}^2 + 2 \left(\frac{1}{\alpha} + 1\right) \|\varphi(t)\|_{L^2}^2 + \frac{1}{2} \|\xi(t) - \xi_d(t)\|_{L^2}^2. \tag{3.8}
\]
Combining (3.7) and (3.8), we also find
\[
-\frac{d}{dt} \left(\|p(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2\right) + \alpha \|\nabla p(t)\|_{L^2}^2 \\
\leq (M+1) \|p(t)\|_{L^2}^2 + \left[M + 2\mu^2 + 4 \left(\frac{1}{\alpha} + 1\right)\right] \|\varphi(t)\|_{L^2}^2 \\
+ \|u(t) - u_d(t)\|_{L^2}^2 + \|\xi(t) - \xi_d(t)\|_{L^2}^2. 
\]
Calculations similar to (2.14) and (2.15) yield
\[ \|L\|_{L^2} \text{of weak solution to the system } (3.6) \text{ with } p \text{ solvability results of } (3.6). \]
The following theorem gives the global existence and uniqueness
uniformly bounded.

Integrating the above inequality from \( t \) to \( T \), we obtain
\[
\|p(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2 + \alpha \int_t^T \|\nabla p(s)\|_{L^2}^2 ds
\leq \|p(T)\|_{L^2}^2 + \|\varphi(T)\|_{L^2}^2 + \left[M + 2\mu_2 + 4\left(\frac{1}{\alpha} + 1\right)\right] \int_t^T (\|p(t)\|_{L^2}^2 + \|\varphi(s)\|_{L^2}^2) ds
+ \int_t^T \|u(s) - u_d(s)\|_{L^2}^2 ds + \int_t^T \|\xi(s) - \xi_d(s)\|_{L^2}^2 ds. \tag{3.9}
\]
An application of the Gronwall’s inequality in (3.9) yields
\[
\|p(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2 + \alpha \int_t^T \|\nabla p(s)\|_{L^2}^2 ds
\leq \left(\|p_T\|_{L^2}^2 + \|\varphi_T\|_{L^2}^2 + \int_t^T \|u(s) - u_d(s)\|_{L^2}^2 ds + \int_t^T \|\xi(s) - \xi_d(s)\|_{L^2}^2 ds\right)e^{[M + 2\mu_2 + 4(\frac{1}{\alpha} + 1)]T}, \tag{3.10}
\]
for all \( t \in [0, T] \). Since \( p_T \in \mathbb{L}^2(\Omega), \varphi_T \in \mathbb{L}^2(\Omega), \ u, u_d \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \) and \( \varphi, \varphi_d \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \), the right hand side of the estimate in (3.10) is uniformly bounded. Calculations similar to (2.1.11) and (2.1.15) yield \( \|\partial_t p\|_{L^2(0, T; \mathbb{L}^2(\Omega))} \) and \( \|\partial_t \varphi\|_{L^2(0, T; \mathbb{L}^2(\Omega))} \) are also uniformly bounded.

Once again using a Faedo-Galerkin approximation technique, one can obtain the global solvability resuls of (3.6). The following theorem gives the global existence and uniqueness of weak solution to the system (3.6) with \( p(T) = p_T \in \mathbb{L}^2(\Omega) \) and \( \varphi(T) = \varphi_T \in \mathbb{L}^2(\Omega) \).

**Theorem 3.2.** Let \( (p_T, \varphi_T) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \) be given. For \( u_d \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \) and \( \varphi_d \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \), there exists a unique weak solution to the system (3.6) satisfying
\[(p, \varphi) \in (C([0, T]; \mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(0, T; \mathbb{H}^1(\Omega))) \times C([0, T]; \mathbb{L}^2(\Omega)), \]
with
\[(\partial_t p, \partial_t \varphi) \in \mathbb{L}^2(0, T; \mathbb{H}^{-1}(\Omega)) \times \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)). \]

**3.2. Existence of an optimal control.** Our next aim is to show that an optimal triplet \((u^*, \xi^*, U^*)\) exists for the problem (3.2).

**Theorem 3.3** (Existence of an optimal triplet). Let \((u_0, \xi_0) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)\) and \( f \in \mathbb{L}^2(0, T; \mathbb{H}^{-1}(\Omega)) \) be given. Then there exists at least one triplet \((u^*, \xi^*, U^*)\) \( \in \mathcal{A}_{ad} \) such that the functional \( \mathcal{J}(u, \xi, U) \) attains its minimum at \((u^*, \xi^*, U^*)\), where \((u^*, \xi^*)\) is the unique weak solution of the system (1.3) with the control \( U^* \).

**Proof.** Let us first define
\[ \mathcal{J} := \inf_{U \in \mathcal{A}_{ad}} \mathcal{J}(u, \xi, U). \]
Since, \( 0 \leq \mathcal{J} < +\infty \), there exists a minimizing sequence \( \{U_n\} \in \mathcal{A}_{ad} \) such that
\[ \lim_{n \to \infty} \mathcal{J}(u_n, \xi_n, U_n) = \mathcal{J}, \]
where \((u_n, \xi_n)\) is the unique weak solution of the system (1.3) with the control \( U_n \) and also the initial data
\[ u_n(0) = u_0 \in \mathbb{L}^2(\Omega) \] and \( \xi_n(0) = \xi_0 \in \mathbb{L}^2(\Omega). \] \( \tag{3.11} \)
Since $0 \in \mathcal{W}_{ad}$, without loss of generality, we assume that $\mathcal{J}(u_n, \xi_n, U_n) \leq \mathcal{J}(u, \xi, 0)$, where $(u, \xi, 0) \in \mathcal{A}_{ad}$. Using the definition of $\mathcal{J}(\cdot, \cdot, \cdot)$, this immediately gives

$$
\frac{1}{2} \int_0^T \|u_n(t) - u_d(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^T \|\xi_n(t) - \xi_d(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^T \|U_n(t)\|_{H^{-1}}^2 dt \\
\leq \frac{1}{2} \int_0^T \|u(t) - u_d(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^T \|\xi(t) - \xi_d(t)\|_{L^2}^2 dt.
$$

(3.12)

Since $u, u_d \in L^2(0, T; L^2(\Omega))$ and $\xi, \xi_d \in L^2(0, T; L^2(\Omega))$, from the above relation, it is clear that, there exist a $R > 0$, large enough such that

$$
0 \leq \mathcal{J}(u_n, \xi_n, U_n) \leq R < +\infty.
$$

In particular, there exists a large $\bar{C} > 0$, such that

$$
\int_0^T \|U_n(t)\|_{H^{-1}}^2 dt \leq \bar{C} < +\infty.
$$

That is, the sequence $\{U_n\}$ is uniformly bounded in the space $L^2(0, T; H^{-1}(\Omega))$. Since $(u_n, \xi_n)$ is a unique weak solution of the system (1.3) with control $U_n$, from the energy estimates (see (2.5)), we have

$$
\|u_n(t)\|_{L^2}^2 + \|\xi_n(t)\|_{L^2}^2 + \alpha \int_0^t \|\nabla u_n(s)\|_{L^2}^2 ds \\
\leq \left( \|u_0\|_{L^2}^2 + \|\xi_0\|_{L^2}^2 + \frac{\rho}{\lambda} \int_0^t \|w^0(s)\|_{L^2}^4 ds + \int_0^t \|f(s)\|_{H^{-1}}^2 ds + \int_0^t \|U_n(s)\|_{H^{-1}}^2 ds \right) e^{Kt} \\
\leq \left( \|u_0\|_{L^2}^2 + \|\xi_0\|_{L^2}^2 + \frac{\rho}{\lambda} \int_0^t \|w^0(s)\|_{L^2}^4 ds + \int_0^t \|f(s)\|_{H^{-1}}^2 ds + \bar{C} \right) e^{Kt},
$$

for all $t \in [0, T]$. It can be easily see that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega)) \cap T^2(0, T; H_0^1(\Omega))$ and $\{\xi_n\}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$. Hence, by using the Banach-Alglou theorem, we can extract a subsequence $\{(u_{nk}, \xi_{nk}, U_{nk})\}$ such that

$$
\left\{ \begin{array}{l}
\quad u_{nk} \rightharpoonup u^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\
\quad u_{nk} \to u^* \text{ in } L^2(0, T; H^1(\Omega)), \\
\quad \partial_t u_{nk} \to \partial_t u^* \text{ in } L^2(0, T; H^{-1}(\Omega)), \\
\quad \xi_{nk} \rightharpoonup \xi^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\
\quad \partial_t \xi_{nk} \to \partial_t \xi^* \text{ in } L^2(0, T; L^2(\Omega)), \\
\quad U_{nk} \to U^* \text{ in } L^2(0, T; H^{-1}(\Omega)),
\end{array} \right.
$$

(3.13)

as $k \to \infty$. Using Aubin-Lion’s compactness theorem (see Theorem 1, [32]) and the convergence in (3.13), we infer that

$$
\left\{ \begin{array}{l}
\quad u_{nk} \to u^* \text{ in } L^2(0, T; L^2(\Omega)), \text{ a. e., in } \Omega \times (0, T), \\
\quad \xi_{nk} \to \xi^* \text{ in } L^2(0, T; L^2(\Omega)), \text{ a. e., in } \Omega \times (0, T),
\end{array} \right.
$$

(3.14)
as } k \to \infty \text{. It is also a consequence of Aubin-Lion’s compactness theorem that } (u^*, \xi^*) \in C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Omega)) \text{. Note that the initial condition } (3.11) \text{ and the right continuity in time at } 0 \text{ gives }

\begin{align*}
    u^*(0) &= u_0 \in L^2(\Omega) \text{ and } \xi^*(0) = \xi_0 \in L^2(\Omega). \quad (3.15)
\end{align*}

Hence } (u^*, \xi^*) \text{ is a unique weak solution of } (1.3) \text{ with control } U^* \in L^2(0, T; H^{-1}(\Omega)) \text{, the whole sequence } (u^n, \xi^n) \text{ converges to } (u^*, \xi^*) \text{. This easily gives } (u^*, \xi^*, U^*) \in \mathcal{A}_{\text{ad}} \text{.}

Now we show that } (u^*, \xi^*, U^*) \text{ is a minimizer, that is } J = \mathcal{J}(u^*, \xi^*, U^*) \text{. Since the cost functional } \mathcal{J}(\cdot, \cdot, \cdot) \text{ is continuous and convex (see Proposition III.1.6 and III.1.10, [11]) on } L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)) \times \mathcal{U}_{\text{ad}} \text{, it follows that } \mathcal{J}(\cdot, \cdot, \cdot) \text{ is weakly lower semi-continuous (Proposition II.4.5, [11])} \text{. That is, for a sequence}

\begin{align*}
    (u_n, \xi_n, U_n) \overset{w}{\to} (u^*, \xi^*, U^*) \text{ in } L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)) \times L^2(0, T; H^{-1}(\Omega)) ,
\end{align*}

we have

\begin{align*}
    \mathcal{J}(u^*, \xi^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, \xi_n, U_n). \quad \text{(3.12)}
\end{align*}

Therefore, we get

\begin{align*}
    \mathcal{J} \leq \mathcal{J}(u^*, \xi^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, \xi_n, U_n) = \lim_{n \to \infty} \mathcal{J}(u_n, \xi_n, U_n) = \mathcal{J},
\end{align*}

and hence } (u^*, \xi^*, U^*) \text{ is a optimizer of the problem } (3.2) \text{.} \quad \Box

### 3.3. Pontryagin’s maximum principle

In this subsection, we prove the first order necessary condition for the optimal control problem } (3.2) \text{ via Pontryagin’s maximum principle. We also characterize optimal control in terms of the adjoint variable. Remember that our optimal control problem is a minimization of the cost functional given in } (3.1) \text{ and hence we obtain a minimum principle.}

We first give the minimum principle in terms of the Hamiltonian formulation. Let us define the Lagrangian by

\begin{align*}
    \mathcal{L}(u, \xi, U) := \frac{1}{2} (\|u - u_d\|_{L^2}^2 + \|\xi - \xi_d\|_{L^2}^2 + \|U\|_{H^{-1}}^2).
\end{align*}

We define the corresponding Hamiltonian by

\begin{align*}
    \mathcal{H}(u, \xi, U, p, \varphi) := \mathcal{L}(u, \xi, U) + \langle p, N_1(u, \xi, U) \rangle + \langle \varphi, N_2(u, \xi) \rangle,
\end{align*}

where } N_1 \text{ and } N_2 \text{ are defined in } (3.3) \text{. Hence, we obtain Pontryagin’s minimum principle as }

\begin{align*}
    \mathcal{H}(u^*(t), \xi^*(t), U^*(t), p(t), \varphi(t)) \leq \mathcal{H}(u^*(t), \xi^*(t), W, p(t), \varphi(t)), \quad (3.16)
\end{align*}

for all } W \in H^{-1}(\Omega) \text{ and a.e. } t \in [0, T] \text{. That is, the following minimum principle is satisfied by an optimal triplet } (u^*, \xi^*, U^*) \in \mathcal{A}_{\text{ad}} \text{ obtained in Theorem 3.3:}

\begin{align*}
    \frac{1}{2} \|U^*(t)\|_{H^{-1}}^2 + \langle p(t), U^*(t) \rangle \leq \frac{1}{2} \|W\|_{H^{-1}}^2 + \langle p(t), W \rangle, \quad (3.17)
\end{align*}

for all } W \in H^{-1}(\Omega) \text{, and a.e. } t \in [0, T] \text{. For } \mathcal{A}_{\text{ad}} = L^2(0, T; H^{-1}(\Omega)) \text{, from } (3.17) \text{, we see that } -p \in \partial_{\frac{1}{2}} \|U^*(t)\|_{H^{-1}}^2 \text{, where } \partial \text{ denotes the subdifferential. Since } \frac{1}{2} \|\cdot\|_{H^{-1}}^2 \text{ is Gâteaux differentiable, the subdifferential consists of a single point and it follows that }

\begin{align*}
    -p(t) = (-\Delta)^{-1}U^*(t) \in H^1_0(\Omega), \quad \text{a.e. } t \in [0, T].
\end{align*}

The optimal control is given by } U^*(t) = \Delta p(t) \in H^{-1}(\Omega), \text{ a.e. } t \in [0, T].
Let us take inner product with $\tilde{\alpha}$ (Theorem 3.4) corresponding to the controls $U^\ast$

Proof. Since $(\alpha, \varphi) \in (C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))) \times C([0,T]; L^2(\Omega))$ of the adjoint system (3.6) such that

$$\frac{1}{2}\|U^\ast(t)\|_{H^{-1}}^2 + \langle p(t), U^\ast(t) \rangle \leq \frac{1}{2}\|W\|_{H^{-1}}^2 + \langle p(t), W \rangle,$$

for all $W \in H^{-1}(\Omega)$ and a.e. $t \in [0,T]$. Before proceeding further, we prove two important Lemmas, which is used to establish our main Theorem.

Lemma 3.5. Let $(u^*, \xi^*, U^\ast) \in S_{ad}$ be an optimal triplet of the control problem (3.2) obtained in Theorem 3.3. Let $(u_{U^\ast+\tau U}, \xi_{U^\ast+\tau U})$ be the unique weak solution of the problem (1.3) with control $U^\ast + \tau U$, for $\tau \in [0,1]$ and $U \in H^{-1}(\Omega)$.

$$\sup_{t \in [0,T]} \left[ \|u_{U^\ast+\tau U}(t) - u_U(t)\|_{L^2}^2 + \|\xi_{U^\ast+\tau U}(t) - \xi_U(t)\|_{L^2}^2 \right] + \alpha \int_0^T \|\nabla (u_{U^\ast+\tau U}(t) - u_U(t))\|_{L^2}^2 dt$$

$$\leq \left\{ \frac{4\tau^2}{\alpha} \int_0^T \|U(t)\|_{H^{-1}}^2 dt \right\} e^{\left[ \frac{\alpha}{2} (2+\mu^2) + M \right] T}.$$

Proof. Since $(u_{U^\ast+\tau U}, \xi_{U^\ast+\tau U})$ and $(u_U, \xi_U)$ are the unique weak solutions of the system (1.3) corresponding to the controls $U^\ast + \tau U$ and $U^\ast$, respectively, we know that $(\tilde{u}, \tilde{\xi}) = (u_{U^\ast+\tau U} - u_U, \xi_{U^\ast+\tau U} - \xi_U)$ satisfy the following system:

$$\left\{ \begin{array}{l}
\frac{\partial \tilde{u}(t)}{\partial t} + A\tilde{u}(t) + B(u_{U^\ast+\tau U}(t)) - B(u_U(t)) + \nabla \tilde{\xi}(t) = \tau U(t), \text{ in } \Omega \times (0,T), \\
\frac{\partial \tilde{\xi}(t)}{\partial t} + \text{div}(h\tilde{u}(t)) = 0, \text{ in } \Omega \times (0,T), \\
\tilde{u}(t) = 0, \text{ on } \partial \Omega \times (0,T), \\
\tilde{u}(0) = 0, \tilde{\xi}(0) = 0, \text{ in } \Omega.
\end{array} \right.$$

Let us take inner product with $\tilde{u}(\cdot)$ to the first equation in (3.20) to obtain

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 + \alpha \|\nabla \tilde{u}(t)\|_{L^2}^2$$

$$= -(B(u_{U^\ast+\tau U}(t)) - B(u_U(t))(\tilde{u}(t))_{L^2} \langle \nabla \tilde{\xi}(t), \tilde{u}(t) \rangle + \tau \langle U(t), \tilde{u}(t) \rangle$$

$$= -(B'(\theta u_U + (1-\theta)u_{U^\ast+\tau U}(t))\tilde{u}(t), \tilde{u}(t))_{L^2} + \langle \tilde{\xi}(t), \text{div} \tilde{u}(t) \rangle_{L^2} + \tau \langle U(t), \tilde{u}(t) \rangle$$

$$\leq \|\tilde{\xi}(t)\|_{L^2} \|\nabla \tilde{u}(t)\|_{L^2} + \tau \|U(t)\|_{H^{-1}} \|\nabla \tilde{u}(t)\|_{L^2}$$

$$\leq \frac{\alpha}{4} \|\nabla \tilde{u}(t)\|_{L^2}^2 + \frac{4}{\alpha} \|\tilde{\xi}(t)\|_{L^2}^2 + \frac{2\tau^2}{\alpha} \|U(t)\|_{H^{-1}}^2,$$

for $0 < \theta < 1$, where we used Taylor’s formula, Lemma 2.2 (vi), Cauchy-Schwarz and Young’s inequalities. We now take inner product with $\tilde{\xi}(\cdot)$ to the second equation in (3.20) to find

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\xi}(t)\|_{L^2}^2 = -h\text{div} \tilde{u}(t), (\tilde{\xi}(t))_{L^2} - (\nabla h \cdot \tilde{u}(t), (\tilde{\xi}(t))_{L^2}$$
\[ \| \nabla \tilde{u}(t) \|_{L^2}^2 + \| \tilde{\xi}(t) \|_{L^2}^2 + \alpha \int_0^t \| \nabla \tilde{u}(s) \|_{L^2}^2 ds \leq \left[ \frac{4}{\alpha}(2 + \mu^2) + M \right] \int_0^t \| \tilde{u}(s) \|_{L^2}^2 + \| \tilde{\xi}(s) \|_{L^2}^2 ds + \frac{4\tau^2}{\alpha} \int_0^t \| U(s) \|_{H^{-1}}^2 ds. \] (3.23)

An application of Gronwall's inequality in (3.23) yields

\[ \| \tilde{u}(t) \|_{L^2}^2 + \| \tilde{\xi}(t) \|_{L^2}^2 + \alpha \int_0^t \| \nabla \tilde{u}(s) \|_{L^2}^2 ds \leq \left\{ \frac{4\tau^2}{\alpha} \int_0^t \| U(s) \|_{H^{-1}}^2 ds \right\} e^{\left[ \frac{4}{\alpha}(2 + \mu^2) + M \right] t}, \] (3.24)

for all \( t \in [0, T] \), which completes the proof. \( \square \)

The following lemma gives the differentiability of the mapping \( U \mapsto (u_U, \xi_U) \) from \( \mathcal{U}_d \) into \( (C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))) \times C([0, T]; L^2(\Omega)) \).

**Lemma 3.6.** Let \( (u_0, \xi_0) \in (L^2(\Omega) \times L^2(\Omega)) \) and \( f \in L^2(0, T; H^{-1}(\Omega)) \) be given. Then the mapping \( U \mapsto (u_U, \xi_U) \) from \( \mathcal{U}_d \) into \( (C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))) \times C([0, T]; L^2(\Omega)) \) is Gâteaux differentiable. Moreover, we have

\[ \left\{ \lim_{\tau \downarrow 0} \frac{u_{U^* + \tau U} - u_{U^*}}{\tau}, \lim_{\tau \downarrow 0} \frac{\xi_{U^* + \tau U} - \xi_{U^*}}{\tau} \right\} = \{ \mathbf{w}, \eta \}, \] (3.25)

where \( (\mathbf{w}, \eta) \) is the unique weak solution of the linearized system:

\[
\begin{align*}
\frac{\partial \mathbf{w}(t)}{\partial t} + A\mathbf{w}(t) + B'(u_U^*(t))\mathbf{w}(t) + \nabla \eta(t) &= U(t), \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial \eta(t)}{\partial t} + \text{div}(h\mathbf{w}(t)) &= 0, \quad \text{in} \quad \Omega \times (0, T), \quad (3.26)
\end{align*}
\]

\[ \mathbf{w}(t) = 0, \quad \text{on} \quad \partial\Omega \times (0, T), \]

\[ \mathbf{w}(0) = \mathbf{w}_0, \quad \eta(0) = \eta_0, \quad \text{in} \quad \Omega, \]
and the pairs \((\mathbf{u}^*_U, \xi^*_U)\) and \((\mathbf{u}^{*+\tau}_U, \xi^{*+\tau}_U)\) are the unique weak solutions of the controlled system \((3.3)\) with controls \(\mathbf{u}^*_U\) and \(\mathbf{u}^*_U + \tau\mathbf{U}\), respectively. That is, we have

\[
\begin{align*}
\lim_{\tau \to 0} \frac{\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U - \tau \mathbf{w}\|_{L^\infty(0,T;L^2(\Omega))}}{\tau} &= 0, \\
\lim_{\tau \to 0} \frac{\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U - \tau \mathbf{w}\|_{L^2(0,T;\mathbb{H}^1(\Omega))}}{\tau} &= 0, \\
\lim_{\tau \to 0} \frac{\|\xi^{*+\tau}_U - \xi^*_U - \tau \eta\|_{L^\infty(0,T;L^2(\Omega))}}{\tau} &= 0.
\end{align*}
\]

(3.27)

**Proof.** Let us define

\[
(y, \varrho) := (\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U - \tau \mathbf{w}, \xi^{*+\tau}_U - \xi^*_U - \tau \eta).
\]

Then \((y, \varrho)\) satisfies the following system:

\[
\begin{align*}
\frac{\partial y(t)}{\partial t} + A y(t) + B'((\mathbf{u}^*_U(t))) y(t) + \nabla \varrho(t) &= B'(\mathbf{u}^*_U(t))(\mathbf{u}^{*+\tau}_U(t) - \mathbf{u}^*_U(t)) \\
&\quad - B(\mathbf{u}^{*+\tau}_U(t)) + B(\mathbf{u}^*_U(t)), \text{ in } \Omega \times (0,T), \\
\frac{\partial \varrho(t)}{\partial t} + \text{div}(h(y(t))) &= 0, \text{ in } \Omega \times (0,T), \\
y(t) &= \mathbf{0}, \text{ on } \partial \Omega \times (0,T), \\
y(0) &= \mathbf{0}, \varrho(0) = \eta(0) = 0, \text{ in } \Omega.
\end{align*}
\]

(3.28)

Remember that the term \(B'(\mathbf{u}^*_U)((\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U) - B(\mathbf{u}^{*+\tau}_U) + B(\mathbf{u}^*_U) \in L^2(0,T;L^2(\Omega))\) and hence the system \((3.28)\) has a unique weak solution with

\[
(y, \varrho) \in (C([0,T];L^2(\Omega))) \cap L^2(0,T;\mathbb{H}^1_0(\Omega))) \times C([0,T];L^2(\Omega)).
\]

Using Taylor’s series, the right hand side of the first equation in \((3.28)\) can be written as

\[
N(\mathbf{u}^*_U, \mathbf{u}^{*+\tau}_U) := B'(\mathbf{u}^*_U)((\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U) - B(\mathbf{u}^{*+\tau}_U) + B(\mathbf{u}^*_U)
\]

\[
= [B'(\mathbf{u}^*_U) - B'(\theta \mathbf{u}^{*+\tau}_U + (1-\theta)\mathbf{u}^*_U)](\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U)
\]

\[
= 2\gamma[\|\mathbf{u}^*_U + \mathbf{w}^0\| - \|\theta \mathbf{u}^{*+\tau}_U + (1-\theta)\mathbf{u}^*_U + \mathbf{w}^0\|](\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U)
\]

\[
\leq 2\gamma\|\mathbf{u}^*_U + \mathbf{w}^0 - (\theta \mathbf{u}^{*+\tau}_U + (1-\theta)\mathbf{u}^*_U + \mathbf{w}^0)\|\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U\| \\
\leq 2\gamma\|\theta\|\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U\| + 2\gamma||\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U\|^2,
\]

for \(0 < \theta < 1\). Then, using Hölder’s and Ladyzhenskaya inequalities, we have

\[
\|B'(u^*_U)((\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U) - B(\mathbf{u}^{*+\tau}_U) + B(\mathbf{u}^*_U)\|_{L^2}
\]

\[
\leq 2\|\gamma\|_{L^\infty}\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U\|^2_{L^4}
\]

\[
\leq \frac{2\sqrt{2}r}{\lambda}\|\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U\|_{L^2}\|\nabla((\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U))\|_{L^2},
\]

and

\[
\|N(u^*_U, \mathbf{u}^{*+\tau}_U)\|_{L^2(0,T;L^2(\Omega))}
\]

\[
= \|B'(u^*_U)((\mathbf{u}^{*+\tau}_U - \mathbf{u}^*_U) - B(\mathbf{u}^{*+\tau}_U) + B(\mathbf{u}^*_U))\|_{L^2(0,T;L^2(\Omega))}.
\]
\[
\leq \frac{2\sqrt{2r}}{\lambda} \|u_{U^*} - u_{U^*}\|_{L^\infty(0,T;L^2(\Omega))} \|\nabla(u_{U^*} - u_{U^*})\|_{L^2(0,T;L^2(\Omega))}
\]

\[
\leq \left\{ \frac{8\sqrt{2r^2r}}{\alpha\lambda} \int_0^T \|U^*(t)\|_{H^{-1}}^2 dt \right\} e^{\frac{1}{\alpha}(2+\mu^2)M} T,
\]

(3.29)

using (3.19). A calculation similar to (2.12) gives

\[
\|y(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2 + \alpha \int_0^t \|\nabla y(s)\|_{L^2}^2 ds
\]

\[
\leq \int_0^t \|N(u_{U^*}(s), u_{U^*+\tau U}(s))\|_{L^2}^2 ds + K_1 \int_0^t (\|y(s)\|_{L^2}^2 + \|\varphi(s)\|_{H^1}^2) ds,
\]

(3.30)

where \(K_1 = \max\{M + 1, M + 4\mu^2\}\). An application of the Gronwall’s inequality in (3.30) yields

\[
\|y(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2 + \alpha \int_0^t \|\nabla y(s)\|_{L^2}^2 ds \leq \int_0^t \|N(u_{U^*}(s), u_{U^*+\tau U}(s))\|_{L^2}^2 ds \right\} e^{K_1 t},
\]

(3.31)

for all \(t \in [0,T]\). Using (3.29) in (3.30), it can be easily seen that

\[
\|u_{U^*+\tau U} - u_{U^*} - \tau w\|_{L^\infty(0,T;L^2(\Omega))} \leq \|N(u_{U^*}, u_{U^*+\tau U})\|_{L^2(0,T;L^2(\Omega))} e^{K_1 T}
\]

\[
\leq \left\{ \frac{8\sqrt{2r^2r}}{\alpha\lambda} \int_0^T \|U^*(t)\|_{H^{-1}}^2 dt \right\} e^{\frac{(K_1+K_2)T}{2}},
\]

(3.32)

where \(K_2 = \left[\frac{1}{\alpha}(2+\mu^2) + M\right]\). Then, we have

\[
\lim_{\tau \downarrow 0} \frac{\|u_{U^*+\tau U} - u_{U^*} - \tau w\|_{L^\infty(0,T;L^2(\Omega))}}{\tau} \leq \lim_{\tau \downarrow 0} \left\{ \frac{8\sqrt{2r^2r}}{\alpha\lambda} \int_0^T \|U^*(t)\|_{H^{-1}}^2 dt \right\} e^{\frac{(K_1+K_2)T}{2}} = 0,
\]

(3.33)

and

\[
\lim_{\tau \downarrow 0} \frac{\|u_{U^*+\tau U} - u_{U^*} - \tau w\|_{L^2(0,T;H_0^1(\Omega))}}{\tau} = 0.
\]

(3.34)

Using (3.31), a calculation similar to (3.32) also shows that

\[
\lim_{\tau \downarrow 0} \frac{\|\xi_{U^*+\tau U} - \xi_{U^*} - \tau \eta\|_{L^\infty(0,T;L^2(\Omega))}}{\tau} = 0,
\]

which completes the proof. \(\square\)

Next, we prove our main Theorem 3.4 using Lemmas 3.5 and 3.6.

**Proof of Theorem 3.4.** Let \((u^*, \xi^*, U^*) \in \mathcal{A}_{ad}\) be an optimal triplet of the control problem (3.2) obtained in Theorem 3.3. Let \(G(U) = J(u_U, \xi_U, U)\), where \((u_U, \xi_U, U)\) is the solution of the controlled system (1.3) with control \(U \in H^{-1}(\Omega)\), for a.e. \(t \in [0,T]\). Let \(\tau\) be sufficiently small such that \(U^* + \tau U \in \mathcal{Z}_{ad}\) and \((u_{U^*+\tau U}, \xi_{U^*+\tau U}, U^* + \tau U) \in \mathcal{A}_{ad}\). Then, we have

\[
G(U^* + \tau U) - G(U^*) = J(u_{U^*+\tau U}, \xi_{U^*+\tau U}, U^* + \tau U) - J(u_{U^*}, \xi_{U^*}, U^*)
\]
From the above calculations, it is immediate that
\[
\|w\|_{L^2}^2 = 1 - 2 \tau \int_0^T \frac{\|\xi U^* U(t) - \xi_d(t)\|_{L^2}^2}{\|U^* + \tau U(t)\|_{L^2}^2} dt
\]
Using the estimate (3.19) (see Lemma 3.5), we know that
\[
\|w\|_{L^2} \to 0, \quad \text{that is,} \quad \langle G, \xi \rangle = 1
\]
\[
= \frac{1}{2} \int_0^T (u_{U^* + \tau U}(t) - u_{U^*}(t), u_{U^* + \tau U}(t) - u_{U^*}(t))/\|2 dt
\]
\[
+ \frac{1}{2} \int_0^T \|\xi U^* U(t) - \xi_d(t)\|_{L^2}^2 dt - \frac{1}{2} \int_0^T \|\xi U^* U(t) - \xi_d(t)\|_{L^2}^2 dt - \frac{1}{2} \int_0^T \|U^* + \tau U(t)\|_{L^2}^2 dt
\]
From the above calculations, it is immediate that
\[
G(U^* + \tau U) - G(U^*)
\]
\[
= \frac{1}{2} \int_0^T \|u_{U^* + \tau U}(t) - u_{U^*}(t)\|_{L^2}^2 dt + \int_0^T (u_{U^* + \tau U}(t) - u_{U^*}(t), u_{U^* + \tau U}(t) - u_{U^*}(t))/\|2 dt
\]
Using the estimate (3.19) (see Lemma 3.5), we know that \(\|u_{U^* + \tau U} - u_{U^*}\|_{L^2(0,T;L^2(\Omega))}^2\) and
\(\|\xi U^* U - \xi_d\|_{L^2(0,T;L^2(\Omega))}^2\) can be estimated by \(\tau^2 \|U\|_{L^2(0,T;H^{-1}(\Omega))}^2\). Thus dividing by \(\tau\), and then sending \(\tau \downarrow 0\), we easily have \(\|u_{U^* + \tau U} - u_{U^*}\|_{L^2(0,T;L^2(\Omega))} \to 0\) and \(\|\xi U^* U - \xi_d\|_{L^2(0,T;L^2(\Omega))} \to 0\) as \(\tau \downarrow 0\).

Let us denote the Gâteaux derivative of \(G\) at \(U^*\) in the direction of \(U \in H^{-1}(\Omega)\) by \(\langle G', U^* \rangle, U\). Let \((w, \eta)\) satisfy the linearized system (3.26) with control \(U\), and initial data to be equal to zero, that is, \(w(0) = 0\) and \(\eta(0) = 0\). From Lemma 3.6, we also have the convergences (see (3.27)):
\[
\lim_{\tau \downarrow 0} \|u_{U^* + \tau U} - u_{U^*} - \tau w\|_{L^2(0,T;L^2(\Omega))} = 0 \quad \text{and} \quad \lim_{\tau \downarrow 0} \|\xi U^* U - \xi_d - \tau \eta\|_{L^2(0,T;L^2(\Omega))} = 0.
\]
Dividing by $\tau$ and then taking $\tau \downarrow 0$ in (3.35), we obtain
\[
0 \leq \langle \mathcal{G}'(U^*), U \rangle = \lim_{\tau \downarrow 0} \frac{\mathcal{G}(U^* + \tau U) - \mathcal{G}(U^*)}{\tau}
\]
\[
= \int_0^T (w(t), u_{U^*}(t) - u_d(t))_L^2 dt + \int_0^T (x(t), \xi_{U^*}(t) - \xi_d(t))_L^2 dt
\]
\[
+ \int_0^T ((-\Delta)^{-1/2}U(t), (-\Delta)^{-1/2}U^*(t))_L^2 dt
\]
\[
= \int_0^T \langle w(t), -p_t(t) + \tilde{A}p(t) + B'(u(t))p(t) - h\nabla \varphi(t) \rangle dt
\]
\[
+ \int_0^T (\eta(t), -\varphi_t(t)) - \text{div} \ p(t))_L^2 dt + \int_0^T ((-\Delta)^{-1/2}U(t), (-\Delta)^{-1/2}U^*(t))_L^2 dt
\]
\[
= \int_0^T \langle w(t) + A w(t) + B'(\tilde{u}(t))w(t) + \nabla \eta(t), p(t) \rangle dt + \int_0^T (\eta_t(t) + \text{div}(h w(t)), \varphi(t))_L^2
\]
\[
+ \int_0^T ((-\Delta)^{-1/2}U(t), (-\Delta)^{-1/2}U^*(t))_L^2 dt
\]
\[
= \int_0^T \langle U(t), p(t) \rangle dt + \int_0^T ((-\Delta)^{-1/2}U(t), (-\Delta)^{-1/2}U^*(t))_L^2 dt,
\] (3.36)
where we used an integration by parts and the fact that $(u, v)_L^2(\Omega) = \langle u, v \rangle$ for all $u \in H^1_0(\Omega)$, $v \in L^2(\Omega) \subset H^{-1}(\Omega)$. We also used the equation satisfied by $(w, \eta)$ with control $U$ (see (3.26)). Thus, from (3.36), we infer that
\[
0 \leq \langle \mathcal{G}'(U^*(t)), U(t) \rangle = \int_0^T \langle U(t), p(t) \rangle dt + \int_0^T \langle U(t), (-\Delta)^{-1}U^*(t) \rangle dt.
\]
Similarly if we take the directional derivative of $\mathcal{G}$ in the direction of $-U \in H^{-1}(\Omega)$, we obtain $\langle \mathcal{G}'(U^*(t)), U(t) \rangle \leq 0$. Hence, we obtain $\langle \mathcal{G}'(U^*(t)), U(t) \rangle = 0$, and we have
\[
\int_0^T \langle U(t), p(t) + (-\Delta)^{-1}U^*(t) \rangle dt = 0,
\] (3.37)
for all $U \in H^{-1}(\Omega)$. Thus, it is immediate that
\[
\langle U(t), p(t) + (-\Delta)^{-1}U^*(t) \rangle = 0, \text{ for all } U \in H^{-1}(\Omega), \text{ and a.e. } t \in [0, T].
\]
Since the above equality is true for all $U \in H^{-1}(\Omega)$, we get $p(t) + (-\Delta)^{-1}U^*(t) = 0$, a.e. $t \in [0, T]$ and hence we obtain $U^*(t) = \Delta p(t) \in H^{-1}(\Omega)$, a.e. $t \in [0, T]$. Hence, there exists a unique weak solution $(p, \varphi)$ of the adjoint system (3.6) such that for almost every $t \in [0, T]$ and all $W \in H^{-1}(\Omega)$, (3.18) is satisfied. \hfill \Box

3.4. Uniqueness of optimal control in small time interval. In this subsection, we show the uniqueness of optimal control in small time interval for the optimal control problem (3.2). Remember that the control to state mapping is nonlinear and getting global in time unique optimal control is difficult. Thus, we are looking for a time $T$ such that this time ensures uniqueness of optimal control. If we choose the final time $T$ to be sufficiently small, then the state equation for $(u, \xi)$ differ from the corresponding linearized problem $(w, \eta)$ slightly only. In this case, the linearized state equation corresponding to $(w, \eta)$ produces a strictly convex cost functional and the corresponding optimal control is unique.
Theorem 3.7. Let \((u^*, \xi^*, U^*) \in \mathcal{A}_{ad}\) be an optimal triplet for the problem (3.2). Then if the final time \(T\) is sufficiently small such that
\[
e^{2T[(\mu^2+2)(\frac{1}{\lambda}+\delta)+M]} \exp\left[\int_0^T \|u^*(s) + w^0(s)\|_{L^4}^4 \, ds\right] < \frac{\alpha^2}{8},
\]
is satisfied, then an optimal triplet \((u^*, \xi^*, U^*) \in \mathcal{A}_{ad}\) obtained in Theorem 3.3 is unique.

Proof. Let us assume that there exists an another optimal triplet \((\tilde{u}, \tilde{\xi}, \tilde{U}) \in \mathcal{A}_{ad}\). We know that \(U^*(t) = \Delta p^*(t) \in H^{-1}(\Omega), \) a.e. \(t \in [0, T],\) and \(\tilde{U}(t) = \Delta \tilde{u}(t) \in H^{-1}(\Omega),\) a.e. \(t \in [0, T].\) Note also that \((p^*, \xi^*)\) and \((\tilde{p}, \tilde{\xi})\) satisfy the adjoint system (3.6) with the forcing \((u^* - u_d, \xi^* - \xi_d)\) and \((\tilde{u} - u_d, \tilde{\xi} - \xi_d),\) respectively. Thus, for a.e. \(t \in [0, T],\) we have
\[
\|U^*(t) - \tilde{U}(t)\|_{H^{-1}} = \|\Delta p^*(t) - \Delta \tilde{u}(t)\|_{H^{-1}} = \|p^*(t) - \tilde{p}(t)\|_{H^0}.
\]
Thus, we obtain
\[
\int_0^T \|U^*(t) - \tilde{U}(t)\|_{H^{-1}}^2 \, dt = \int_0^T \|p^*(t) - \tilde{p}(t)\|_{H^0}^2 \, dt.
\]
The quantity on the right hand side of the inequality (3.40) can be estimated similarly as in (3.10), as the system satisfied by \((\tilde{p} - p^*, \tilde{\xi} - \xi^*)\) is given by
\[
\left\{
\begin{array}{l}
- \frac{\partial p(t)}{\partial t} + \tilde{A}(p(t)) + (B'(\tilde{u}(t)) - B'(u^*(t)))p(t) - h\nabla \varphi(t) \\
\quad = (\tilde{u}(t) - u^*(t)), \quad \text{in} \quad \Omega \times (0, T),
\end{array}
\right.
\]
\[
- \frac{\partial \varphi(t)}{\partial t} - \text{div} p(t) = (\tilde{\xi}(t) - \xi^*(t)), \quad \text{in} \quad \Omega \times (0, T),
\]
\[
p(t) = 0, \quad \text{on} \quad \partial \Omega \times (0, T),
\]
\[
p(T, \cdot) = 0, \quad \varphi(T, \cdot) = 0 \quad \text{in} \quad \Omega,
\]
where \(p = \tilde{p} - p^*\) and \(\varphi = \tilde{\xi} - \varphi^*.\) Using Lemma 2.2(vi), we know that \((B'(\tilde{u})p, p)_{L^2} \geq 0\) and we estimate \((B'(u^*)p, p)_{L^4}\) as
\[
\|(B'(u^*)p, p)_{L^4}\| = \|2(\gamma^0|u^* + w^0|p, p)_{L^2} \| \leq 2\|\gamma||_{L^\infty} \|u^* + w^0\|_{L^4} \|p\|_{L^4} \|p\|_{L^2}
\]
\[
\leq \frac{2\gamma}{\lambda} \|u^* + w^0\|_{L^4} \|p\|_{L^4} \|p\|_{L^2} \|\nabla p\|_{L^4}^{3/2}
\]
\[
\leq \frac{\alpha}{4} \|\nabla p\|_{L^2}^2 + \frac{27}{4\alpha^3} \|u^* + w^0\|_{L^4}^4 \|p\|_{L^2}^2,
\]
where we used Hölder’s, Ladyzhenskaya and Young’s inequalities. A calculation similar to (3.33) yields
\[
\|p(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2 + \frac{\alpha}{2} \int_t^T \|\nabla p(s)\|_{L^2}^2 \, ds
\]
\[
\leq \left[ M + 2\mu^2 + 4\left(\frac{1}{\alpha} + 1\right) \right] \int_t^T \left( \|p(t)\|_{L^2}^2 + \|\varphi(s)\|_{L^2}^2 \right) \, ds
\]
\[
+ \frac{27}{2\alpha^3} \int_t^T \|u^*(s) + w^0(s)\|_{L^4}^4 \|p(s)\|_{L^2}^2 \, ds + \int_t^T \|\tilde{u}(s) - u^*(s)\|_{L^2}^2 \, ds
\]
\[
+ \int_t^T \|\tilde{\xi}(s) - \xi^*(s)\|_{L^2}^2 \, ds.
\]
An application of Gronwall’s inequality in (3.43) yields
\[
\|p(t)\|^2_{L^2} + \|\varphi(t)\|^2_{L^2} + \frac{\alpha}{2} \int_0^T \|\nabla p(s)\|^2_{L^2} ds
\]
\[
\leq \left\{ \int_0^T \left\| \bar{u}(s) - u^*(s) \right\|^2_{L^2} ds + \int_0^T \left\| \bar{\xi}(s) - \xi^*(s) \right\|^2_{L^2} ds \right\} e^{[M + 2M^2 + 4(\frac{d}{a} + 1)](T-t)}
\]
\[
\times \exp \left[ \int_0^T \|u^*(s) + w^0(s)\|^4_{L^4} ds \right],
\]
for all \( t \in [0, T] \). In particular, for \( t = 0 \), we have
\[
\int_0^T \|\nabla p(s)\|^2_{L^2} ds \leq \left\{ \frac{8}{\alpha^2} \int_0^T \|U^*(t) - \bar{U}(t)\|^2_{H^{-1}} dt \right\} e^{2T[(\mu^2 + 2)(\frac{d}{a} + 1) + \frac{4}{d} + M]}
\]
\[
\times \exp \left[ \int_0^T \|u^*(s) + w^0(s)\|^4_{L^4} ds \right],
\]
where we used (3.49). Combining (3.40) and (3.45), it can be easily seen that
\[
\int_0^T \|U^*(t) - \bar{U}(t)\|^2_{H^{-1}} dt \leq \left\{ \frac{8}{\alpha^2} \int_0^T \|U^*(t) - \bar{U}(t)\|^2_{H^{-1}} dt \right\} e^{2T[(\mu^2 + 2)(\frac{d}{a} + 1) + \frac{4}{d} + M]}
\]
\[
\times \exp \left[ \int_0^T \|u^*(s) + w^0(s)\|^4_{L^4} ds \right].
\]
Note that \( \int_0^T \|u^*(s) + w^0(s)\|^4_{L^4} ds \) is bounded and the bound is given in (2.8). Now, if we choose time \( T \) sufficiently small such that (3.38) is satisfied, then we get that \( \|U^* - \bar{U}\|_{L^2(0,T;H^{-1}(\Omega))} = 0 \). Thus, we obtain \( U^*(t) = \bar{U}(t) \), a.e., \( t \in [0, T] \), for sufficiently small \( T \). This gives the uniqueness of the optimal control up to the time \( T \) satisfying (3.38).

3.5. Minimization of dissipation of energy of the flow. One can consider the following cost functional also:
\[
J(u, \xi, U) := \frac{1}{2} \int_0^T \|\nabla (u(t) - u_d(t))\|^2_{L^2} dt + \frac{1}{2} \int_0^T \|\xi(t) - \xi_d(t)\|^2_{L^2} dt
\]
\[
+ \frac{1}{2} \int_0^T \|U(t)\|^2_{H^{-1}} dt + \frac{1}{2} \|u(T) - u'_d\|^2_{L^2} + \frac{1}{2} \|\xi(T) - \xi'_d\|^2_{L^2},
\]
where \( u_d(\cdot) \in L^2(0, T; H^1(\Omega)) \) and \( \xi_d(\cdot) \in L^2(0, T; L^2(\Omega)) \) are the desired states, and \( u'_d \in L^2(\Omega) \) and \( \xi'_d \in L^2(\Omega) \) are desired velocity and elevation at a final time \( T \). Note that the cost functional given in (3.47) is the sum of total dissipation of energy of the flow, \( L^2 \)-energy of the elevation and the total effort by the distributed controls. In this case one can get the following Pontryagin minimum principle:
Theorem 3.8. Let \((u^*, \xi^*, U^*) \in \mathcal{A}_{ad}\) be an optimal solution of the problem \([3.2]\) obtained in Theorem 3.3. Then there exists a unique weak solution \((p, \varphi) \in (C([0,T];L^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1_0(\Omega))) \times C([0,T];L^2(\Omega))\) of the adjoint system:

\[
\begin{cases}
-\frac{\partial p(t)}{\partial t} + \tilde{A} p(t) + B'(u(t))p(t) - h \nabla \varphi(t) = -\Delta (u(t) - u_d(t)), & \text{in } \Omega \times (0,T), \\
-\frac{\partial \varphi(t)}{\partial t} - \text{div} p(t) = \xi(t) - \xi_d(t), & \text{in } \Omega \times (0,T), \\
p(t) = 0, & \text{on } \partial \Omega \times (0,T), \\
p(T, \cdot) = u(T) - u^f_d, & \varphi(T, \cdot) = \xi(T) - \xi^f_d & \text{in } \Omega,
\end{cases}
\]

such that

\[
\frac{1}{2} \|U^*(t)\|_{\mathbb{H}^{-1}}^2 + \langle p(t), U^*(t) \rangle \leq \frac{1}{2} \|W\|_{\mathbb{H}^{-1}}^2 + \langle p(t), W \rangle,
\]

for all \(W \in \mathbb{H}^{-1}(\Omega)\) and a.e. \(t \in [0,T]\).

Remark 3.9. Note that the term appearing in the right hand side of the first equation in \((3.48)\) can be handled in the following way:

\[
\langle -\Delta (u - u_d), p \rangle = \langle \nabla (u - u_d), \nabla p \rangle_{L^2} \leq \frac{2}{\alpha} \|\nabla (u - u_d)\|_{L^2}^2 + \frac{\alpha}{2} \|\nabla p\|_{L^2}^2.
\]

3.6. Data assimilation problem. Let us now consider a problem similar to the data assimilation problems of meteorology. In the data assimilation problems arising from meteorology, determining the accurate initial data for the future predictions is an important step. This motivates us to consider a similar problem in the tidal dynamics. We pose a optimal data initialization problem, where we find the unknown optimal initial data by minimizing suitable cost functional subject to the tidal dynamics equations (see \([34]\) for the case of incompressible Navier-Stokes equations and \([4]\) for Cahn-Hillard-Navier-Stokes equations).

Let us formulate the initial data optimization problem as finding an optimal initial velocity \(U \in L^2(\Omega)\) such that \((u, \xi, U)\) satisfies the following system:

\[
\begin{cases}
\frac{\partial u(t)}{\partial t} + Au(t) + B(u(t)) + g \nabla \xi(t) = f(t) + U(t), & \text{in } \Omega \times (0,T), \\
\frac{\partial \xi(t)}{\partial t} + \text{div}(h u(t)) = 0, & \text{in } \Omega \times (0,T), \\
u(t) = 0, & \text{on } \partial \Omega \times (0,T), \\
u(0) = U, & \xi(0) = \xi_0, & \text{in } \Omega,
\end{cases}
\]

and minimizes the cost functional

\[
\mathcal{J}(u, \xi, U) := \frac{1}{2} \|U\|_{L^2}^2 + \frac{1}{2} \int_0^T \|u(t) - u_M(t)\|_{L^2}^2 dt + \frac{1}{2} \int_0^T \|\xi(t) - \xi_M(t)\|_{L^2}^2 dt + \frac{1}{2} \|u(T) - u^f_M\|_{L^2}^2 + \frac{1}{2} \|\xi(T) - \xi^f_M\|_{L^2}^2,
\]

where \(u_M\) is the measured average velocity of the fluid and \(\xi_M\) is the measured elevation, \(u^f_M\) and \(\xi^f_M\) are measured velocity and elevation at time \(T\), respectively. In order to make the cost functional given in \((3.52)\) meaningful, we assume that

\[
u_M \in L^2(0,T;L^2(\Omega)), \xi_M \in L^2(0,T;L^2(\Omega)), u^f_M \in L^2(\Omega) \quad \text{and} \quad \xi^f_M \in L^2(\Omega).
\]
In this context, we take the set of admissible control class, $\mathcal{U}_{ad} = L^2(\Omega)$. Also, the admissible class $\mathcal{A}_{ad}$ consists of all triples $(u, \xi, U)$ such that the set of states $(u, \xi)$ is a unique weak solution of the tidal dynamics system $(3.51)$ with control $U \in \mathcal{U}_{ad}$. We define the optimal control problem as:

$$\min_{(u, \xi, U) \in \mathcal{A}_{ad}} J(u, \xi, U).$$

(3.54)

The next theorem provides the existence of an optimal triplet $(u^*, \xi^*, U^*)$ for the problem $(3.54)$. A proof of the Theorem can be established similar to that of Theorem 3.3.

**Theorem 3.10** (Existence of an optimal triplet). Let the initial data $\xi_0 \in L^2(\Omega)$ be given and let $U \in \mathcal{U}_{ad}$. Then there exists at least one triplet $(u^*, \xi^*, U^*) \in \mathcal{A}_{ad}$ such that the functional $J(u, \xi, U)$ given in $(3.52)$ attains its minimum at $(u^*, \xi^*, U^*)$, where $(u^*, \xi^*)$ is the unique weak solution of the system (3.51) with the initial data control $U^* \in \mathcal{U}_{ad}$.

As in the proof of Theorem 3.4 Pontryagin’s minimum principle follows (with some obvious modifications) for this control problem also. Thus, we have the following minimum principle.

$$\frac{1}{2} \|U^*\|^2 + \langle p(0), U^* \rangle_{L^2} \leq \frac{1}{2} \|W\|^2 + \langle p(0), W \rangle_{L^2},$$

for all $W \in \mathcal{U}_{ad}$. Since $\mathcal{U}_{ad} = L^2(\Omega)$, then the optimal control is given by $U^* = -p(0)$, where $(p, \varphi)$ is the unique weak solution of the following adjoint system:

$$\begin{cases}
-\frac{\partial p(t)}{\partial t} + \tilde{A}p(t) + B'(u(t))p(t) - h\nabla \varphi(t) = u(t) - u_M(t), & \text{in } \Omega \times (0, T), \\
-\frac{\partial \varphi(t)}{\partial t} - \text{div } p(t) = \xi(t) - \xi_M(t), & \text{in } \Omega \times (0, T), \\
p(t) = 0, & \text{on } \partial \Omega \times (0, T), \\
p(T, \cdot) = u(T) - u_M, \quad \varphi(T, \cdot) = \varphi(T) - u_M^t & \text{in } \Omega.
\end{cases}
$$

(3.55)

A similar calculation as in Theorem 3.2 yields the existence of a unique weak solution to the system (3.55) such that

$$(p, \varphi) \in (C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{H}_0^1(\Omega))) \times (C([0, T]; L^2(\Omega))).$$

(3.56)

Using the continuity of $p(\cdot)$ in time at $t = 0$ in $L^2(\Omega)$, we know that $p(0) \in L^2(\Omega)$ and hence we get

$$U^* = -p(0) \in \mathcal{U}_{ad} = L^2(\Omega).$$

(3.57)

Thus, we have the following Theorem.

**Theorem 3.11** (Optimal initial control). Let $(u^*, \xi^*, U^*) \in \mathcal{A}_{ad}$ be an optimal triplet. Then there exists a unique weak solution $(p, \varphi)$ of the adjoint system (3.55) satisfying (3.56) such that the optimal control is obtained via (3.57).

4. Second Order Necessary and Sufficient Optimality Condition

In this section, we consider the following cost functional:

$$J(u, \xi, U) := \int_0^T [g(t, u(t)) + h(t, \xi(t)) + \ell(U(t))] dt,$$

(4.1)

and derive the second order necessary and sufficient optimality condition for the optimal control problem (3.2). In (4.1), the Gâteaux differentiable functions $g(\cdot, \cdot), h(\cdot, \cdot)$ and $\ell(\cdot)$
satisfy the following assumptions. Motivated from [8], similar problems for Navier-Stokes equations have been considered in [21, 38] and Cahn-Hillard-Navier-Stokes equations have been addressed in [7].

**Assumption 4.1.** Let us assume that

1. \( g : [0, T] \times \mathbb{L}^2(\Omega) \to \mathbb{R}^+ \) is measurable in the first variable, \( g(t, 0) \in \mathbb{L}^\infty(0, T) \), and for \( \delta_1 > 0 \), there exists \( C_{\delta_1} > 0 \) independent of \( t \) such that

\[
|g(t, u_1) - g(t, u_2)| \leq C_{\delta_1} \|u_1 - u_2\|_{L^2}, \text{ for all } t \in [0, T], \quad \|u_1\|_{L^2} + \|u_2\|_{L^2} \leq \delta_1.
\]

Moreover, the Gâteaux derivatives \( g_u(t, \cdot) \) and \( g_uu(t, \cdot) \) are continuous in \( \mathbb{L}^2(\Omega) \) for all \( t \in [0, T] \).

2. \( h : [0, T] \times \mathbb{H} \to \mathbb{R}^+ \) is measurable in the first variable, \( h(t, 0) \in \mathbb{L}^\infty(0, T) \), and for \( \delta_2 > 0 \), there exists \( C_{\delta_2} > 0 \) independent of \( t \) such that

\[
|h(t, \xi_1) - h(t, \xi_2)| \leq C_{\delta_2} \|\xi_1 - \xi_2\|_{L^2}, \text{ for all } t \in [0, T], \quad \|\xi_1\|_{L^2} + \|\xi_2\|_{L^2} \leq \delta_2.
\]

Further, the Gâteaux derivatives \( h_\xi(t, \cdot) \) and \( h_{\xi\xi}(t, \cdot) \) are continuous in \( \mathbb{L}^2(\Omega) \) for all \( t \in [0, T] \).

3. \( l : \mathbb{L}^2(\Omega) \to (-\infty, +\infty) \) is convex and lower semicontinuous, and the Gâteaux derivatives \( l_U(\cdot) \) and \( l_{UU}(\cdot) \) are continuous in \( \mathbb{L}^2(\Omega) \). Moreover, there exist \( C_1 > 0 \) and \( C_2 \in \mathbb{R} \) such that

\[
l(U) \geq C_1\|U\|_{L^2}^2 - C_2, \text{ for all } U \in \mathbb{L}^2(\Omega).
\]

We take \( \mathcal{U}_{ad} = \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)) \) consisting of admissible controls \( U \). Also, the admissible class \( \mathcal{A}_{ad} \) of triples \( (u, \xi, U) \) is defined as the set of states \( (u, \xi) \in (C([0, T]; \mathbb{H}) \cap \mathbb{L}^2(0, T; \mathbb{H}^1_0(\Omega))) \times C([0, T]; \mathbb{L}^2(\Omega)) \) solving the system (1.3) with the control \( U \in \mathcal{U}_{ad} \). Under the above assumptions, one can prove the following theorems of existence of an optimal control and first order necessary conditions as in Theorems 3.3 and 3.4.

**Theorem 4.2** (Existence of an optimal triplet). Let the initial data \((u_0, \xi_0) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)\) and \( f \in \mathbb{L}^2(0, T; \mathbb{H}^{-1}(\Omega)) \) be given. Then under Assumption 4.1, there exists at least one triplet \( (u^*, \xi^*, U^*) \in \mathcal{A}_{ad} \) such that the functional \( J(u, \xi, U) \) attains its minimum at \((u^*, \xi^*, U^*)\), where \((u^*, \xi^*)\) is the unique weak solution of (1.3) with the control \( U^* \).

**Theorem 4.3** (Pontryagin’s minimum principle). Let \((u^*, \xi^*, U^*) \in \mathcal{A}_{ad} \) be an optimal solution of the problem (3.2) obtained in Theorem 4.2. Then there exists a unique weak solution \((p, \varphi)\) of the adjoint system:

\[
\begin{aligned}
-\frac{\partial p(t)}{\partial t} + \tilde{A}p(t) + B'(u^*(t))p(t) - h\nabla \varphi(t) &= g_u(t, u), \quad \text{in } \Omega \times (0, T), \\
-\frac{\partial \varphi(t)}{\partial t} - \text{div } p(t) &= h_\xi(t, \xi), \quad \text{in } \Omega \times (0, T), \\
p(t) &= 0, \quad \text{on } \partial \Omega, \\
p(T, \cdot) &= 0, \quad \varphi(T, \cdot) = 0 \quad \text{in } \Omega,
\end{aligned}
\]

(4.2)

and for a.e. \( t \in [0, T] \) and \( W \in \mathbb{L}^2(\Omega) \), we have

\[
\ell(U^*(t)) + \|p(t), U^*(t)\|_{L^2} \leq \ell(W) + \|p(t), W\|_{L^2}.
\]

(4.3)
Moreover, we can write (3.18) as
\[ (-p(t), W - U^*(t))_{L^2} \leq \ell(W) - \ell(U^*(t)), \] (4.4)
a.e. \( t \in [0, T] \) and \( W \in L^2(\Omega) \). Since \( \mathcal{U}_a = L^2(0, T; L^2(\Omega)) \), then form the above relation, we see that \(-p \in \partial \ell(U^*(t))\), where \( \partial \) denotes the subdifferential. Whenever \( l \) is Gâteaux differentiable, it follows that
\[ -p(t) = \ell_U(U^*(t)), \quad \text{a.e. } t \in [0, T]. \] (4.5)
If \( l \) is not Gâteaux differentiable then, we have \(-p \in \partial \ell(U^*(t))\).

Let us now establish the second order necessary and sufficient conditions of optimality for the problem (3.2). Let \((\hat{u}, \hat{\xi}, \hat{U}) \in \mathcal{A}_a\) be an arbitrary feasible triplet for the optimal control problem (3.2). Let us set
\[ \mathcal{Q}_{u,\xi,\hat{U}} := \{(u, \xi, U) \in \mathcal{A}_a \} - \{ (\hat{u}, \hat{\xi}, \hat{U}) \}, \] (4.6)
which denotes the differences of all feasible triplets of the problem (3.2) with \((\hat{u}, \hat{\xi}, \hat{U})\). The next two Theorems provide the second order necessary and sufficient optimality condition for the optimal control problem (3.2).

**Theorem 4.4 (Necessary condition).** Let \((u^*, \xi^*, U^*)\) be an optimal triplet for the problem (3.2). Let the Assumption (4.7) holds true and the adjoint variables \((p, \varphi)\) satisfies the adjoint system (1.2). Then for any \((u, \xi, U) \in \mathcal{Q}_{u^*, \xi^*, U^*}\), there exist \(0 \leq \theta_1, \theta_2, \theta_3, \theta_4 \leq 1\) such that
\[
\int_0^T (g_{uu}(t, u^* + \theta_1 u)u, u)dt + \int_0^T (h_{\xi\xi}(t, \xi^* + \theta_2 \xi)\xi, \xi)dt + \int_0^T (\ell_{UU}(U^* + \theta_3 U)U, U)dt \\
- 2\int_0^T ([B'(u^* + \theta_4 u) - B'(u^*)]u, p)_{L^2}dt \geq 0.
\] (4.7)

**Proof.** For any \((u, \xi, U) \in \mathcal{Q}_{u^*, \xi^*, U^*}\), by the definition of (4.6), there exists \((z, \psi, W) \in \mathcal{A}_a\) such that \((u, \xi, U) = (z - u^*, \psi - \xi^*, W - U^*)\). Thus, from (1.3), we can derive that \((u, \xi, U)\) satisfies the following system:
\[
\begin{cases}
\frac{\partial u(t)}{\partial t} + Au(t) + B(u(t) + u^*(t)) - B(u^*(t)) + \nabla \xi(t) = U(t), & \text{in } \Omega \times (0, T), \\
\frac{\partial \xi(t)}{\partial t} + \text{div}(hu(t)) = 0, & \text{in } \Omega \times (0, T), \\
u(t) = 0, & \text{on } \partial \Omega \times (0, T), \\
\xi(0) = 0, & \text{in } \Omega.
\end{cases}
\] (4.8)

Taking inner product of the first two equations in (4.8) with \((p, \varphi)\), integrating over \([0, T]\), and then adding, we obtain
\[
\int_0^T \langle u(t) + Au(t) + B'(u^*(t))u(t) + \nabla \xi(t) - U(t), p(t) \rangle dt \\
+ \int_0^T \langle B(u(t) + u^*(t)) - B(u^*(t)) - B'(u^*(t))u(t), p(t) \rangle dt \\
+ \int_0^T \langle \xi(t) + \text{div}(hu(t)), \varphi(t) \rangle_{L^2} dt = 0.
\] (4.9)
Using an integration by parts, we infer that

\[
\int_0^T \langle \mathbf{u}(t), -\mathbf{p}_t(t) + \tilde{\mathbf{A}} \mathbf{p}(t) + B'(\mathbf{u}^*(t)) \mathbf{p}(t) - h \nabla \varphi(t) \rangle dt \\
+ \int_0^T (B(\mathbf{u}(t) + \mathbf{u}^*(t)) - B(\mathbf{u}^*(t)) - B'(\mathbf{u}^*(t)) \mathbf{u}(t) - U(t), \mathbf{p}(t))_{L^2} dt \\
+ \int_0^T (\xi(t), -\varphi_t(t) - \operatorname{div} \mathbf{p}(t))_{L^2} dt = 0.
\] (4.10)

Since \((\mathbf{u}^*, \xi^*, U^*)\) is an optimal triplet, we know that \((\mathbf{u}^*, \xi^*, U^*)\) satisfies the first order necessary condition

\[
\int_0^T (U(t), \mathbf{p}(t) + \ell_U(U^*(t)))_{L^2} dt = 0, \quad \text{for all } U \in \mathcal{Z}_{ad}.
\] (4.11)

Using (4.10), (4.11) and (4.2), we find

\[
\int_0^T (\ell_U(U^*(t)), U(t))_{L^2} dt + \int_0^T (g_u(t, \mathbf{u}^*(t)), \mathbf{u}(t))_{L^2} dt + \int_0^T (h_\xi(t, \xi^*(t)), \xi(t))_{L^2} dt \\
+ \int_0^T (B(\mathbf{u}(t) + \mathbf{u}^*(t)) - B(\mathbf{u}^*(t)) - B'(\mathbf{u}^*(t)) \mathbf{u}(t), \mathbf{p}(t))_{L^2} dt = 0.
\] (4.12)

Since \((\mathbf{u}, \xi, U) \in \mathcal{Q}_{\mathbf{u}^*, \xi^*, U^*}\), by (4.6), we know that \((\mathbf{u} + \mathbf{u}^*, \xi + \xi^*, U + U^*)\) is a feasible triplet for the problem (3.2). Remembering that \((\mathbf{u}^*, \xi^*, U^*)\) is an optimal triplet, we also obtain

\[
\mathcal{J}(\mathbf{u} + \mathbf{u}^*, \xi + \xi^*, U + U^*) - \mathcal{J}(\mathbf{u}^*, \xi^*, U^*) \\
= \int_0^T [g(t, (\mathbf{u} + \mathbf{u}^*)(t) + h(t, (\xi + \xi^*)(t)) + \ell((U + U^*)(t))] dt \\
- \int_0^T [g(t, \mathbf{u}^*(t)) + h(t, \xi^*(t)) + \ell(U^*(t))] dt \geq 0.
\] (4.13)

Using (4.13), Taylor’s series expansion and the Assumption 4.1 on the cost functional, there exist constants \(0 \leq \theta_1, \theta_2, \theta_3 \leq 1\) such that

\[
\int_0^T \left[ (g_u(t, \mathbf{u}^*(t)), \mathbf{u}(t))_{L^2} + \frac{1}{2} (g_uu(t, \mathbf{u}^*(t) + \theta_1 \mathbf{u}(t)) \mathbf{u}(t), \mathbf{u}(t))_{L^2} \right] dt \\
+ \int_0^T \left[ (h_\xi(t, \xi^*(t)), \xi(t))_{L^2} + \frac{1}{2} (h_\xi\xi(t, \xi^*(t) + \theta_2 \xi(t)) \xi(t), \xi(t))_{L^2} \right] dt \\
+ \int_0^T \left[ (\ell_U(U^*(t)), U(t))_{L^2} + \frac{1}{2} (\ell_{UU}(U^*(t) + \theta_3 U(t)) U(t), U(t))_{L^2} \right] dt \geq 0.
\] (4.14)

From (4.12), it follows that

\[
\frac{1}{2} \int_0^T (g_uu(t, \mathbf{u}^* + \theta_1 \mathbf{u}) \mathbf{u}, \mathbf{u}) dt + \int_0^T (h_\xi\xi(t, \xi^* + \theta_2 \xi) \xi, \xi) dt + \int_0^T (\ell_{UU}(U^* + \theta_3 U) U, U) dt \\
- \int_0^T (B(\mathbf{u} + \mathbf{u}^*) - B(\mathbf{u}^*) - B'(\mathbf{u}^*) \mathbf{u} + \mathbf{p})_{L^2} dt \geq 0.
\] (4.15)

Once again using Taylor’s series expansion, there exists a constant \(0 \leq \theta_4 \leq 1\) such that

\[
B(\mathbf{u} + \mathbf{u}^*) - B(\mathbf{u}^*) - B'(\mathbf{u}^*) \mathbf{u} = B'(\mathbf{u}^* + \theta_4 \mathbf{u}) \mathbf{u} - B'(\mathbf{u}^*) \mathbf{u}.
\] (4.16)
Using (4.16) in (4.15), one finally gets (4.7).

**Theorem 4.5 (Sufficient condition).** Let the Assumption [4.7] holds true and \((u^*, \xi^*, U^*)\) be a feasible triplet for the problem (3.2). Let us also assume that the first order necessary condition holds (see (4.11)), and for any \(0 \leq \theta_1, \theta_2, \theta_3, \theta_4 \leq 1\) and \((u, \xi, U) \in Q_{u^*, \xi^*, U^*}\), the following inequality holds:

\[
\int_0^T (g_{uu}(t, u^*) + \theta_1 u)u, u)dt + \int_0^T (h_{\xi\xi}(t, \xi^* + \theta_2 \xi)\xi)dt + \int_0^T (\ell_U(U^* + \theta_3 U)U, U)dt
\]

\[
- 2 \int_0^T ([B'(u^* + \theta_4 u) - B'(u^*)]u, p)_{L^2}dt \geq 0.
\]

(4.17)

Then \((u^*, \xi^*, U^*)\) is an optimal triplet for the problem (3.2).

**Proof.** For any \((z, \psi, W) \in A_{ad}\), by the definition of (4.6), we know that \((z - u^*, \psi - \xi^*, W - U^*) \in Q_{u^*, \xi^*, U^*}\) and it satisfies:

\[
\left\{
\begin{array}{ll}
\partial(z - u^*)(t) = (W - U^*)(t), & \text{in } \Omega \times (0, T), \\
\partial(\psi - \xi^*)(t) + \text{div}(h(z - u^*)(t)) = 0, & \text{in } \Omega \times (0, T), \\
\partial(z - u^*)(t) = 0, & \text{on } \partial \Omega \times (0, T), \\
(z - u^*)(0) = 0, & \text{in } \Omega.
\end{array}
\right.
\]

(4.18)

Taking an inner product with \((p, \varphi)\) to the first two equations in (4.18), integrating over \([0, T]\) and then adding, we get

\[
\int_0^T \langle z(t) - u^*(t), -p_t(t) + \tilde{A}p(t) + B'(u^*)(t)p(t) - h\nabla \varphi(t)dt\rangle dt
\]

\[
+ \int_0^T (\psi(t) - \xi^*(t), \varphi(t) + \text{div}(hu(t)))_{L^2}dt
\]

\[
+ \int_0^T (B(z(t)) - B(u^*)(t) - B'(u^*)(t)z(t) - u^*(t)) - (W - U^*)(t), p(t) = 0
\]

(4.19)

where we also performed an integration by parts. An application of Taylor’s series expansion yields the existence of a constant \(0 \leq \theta_4 \leq 1\) such that

\[
B(z) - B(u^*) = B'(u^* + \tilde{\theta}_4(z - u^*))(z - u^*).
\]

(4.20)

Note that \((u^*, \xi^*, U^*)\) satisfies the first order necessary condition given in (4.11). It is true for any \(U \in Z_{ad}\), and in particular for \(W - U^* \in Z_{ad}\), that is, we have

\[
\int_0^T (W(t) - U^*)(t), p(t))_{L^2}dt = - \int_0^T (W(t) - U^*)(t), \ell_U(U^*)(t))_{L^2}dt.
\]

(4.21)

Using the adjoint system (4.2), (4.20) and (4.21) in (4.19), we further find

\[
\int_0^T (z - u^*, g_u(t, u^*))_{L^2}dt + \int_0^T (\psi - \xi^*, h_{\xi}(t, \xi^*))_{L^2}dt + \int_0^T (\ell_U(U^*), W - U^*)_{L^2}dt
\]

\[
+ \int_0^T (B'(u^* + \tilde{\theta}_4(z - u^*))(z - u^*) - B'(u^*)(z - u^*), p)_{L^2}dt = 0.
\]

(4.22)
We use the Assumption 4.1 and Taylor’s formula to obtain
\[
\int_0^T [g(t, z(t)) + h(t, \psi(t)) + \ell(W(t))]dt - \int_0^T [g(t, u^*(t)) + h(t, \xi^*(t)) + \ell(U^*(t))]dt
= \int_0^T \left[ (g_u(t, u^*), z - u^*) + \frac{1}{2} (g_{uu}(t, u^* + \tilde{\theta}_1(z - u^*))(z - u^*), (z - u^*)) \right]dt
+ \int_0^T \left[ (h_\xi(t, \xi^*), \psi - \xi^*) + \frac{1}{2} (h_{\xi\xi}(t, \xi^* + \tilde{\theta}_2(\psi - \xi^*))(\psi - \xi^*), (\psi - \xi^*)) \right]dt
+ \int_0^T \left[ (\ell_U(t, U^*), W - U^*) + \frac{1}{2} (\ell_{UU}(t, U^* + \tilde{\theta}_3(W - U^*)) (W - U^*), W - U^*) \right]dt,
\]
for some 0 \leq \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3 \leq 1. Using (1.22) in (1.23), we get
\[
\int_0^T [g(t, z(t)) + h(t, \psi(t)) + \ell(W(t))]dt - \int_0^T [g(t, u^*(t)) + h(t, \xi^*(t)) + \ell(U^*(t))]dt
\geq 0,
\]
where we used (4.17) also. Therefore for any (z, \psi, W) \in A_{ad}, we find that the following inequality holds:
\[
\int_0^T [g(t, z(t)) + h(t, \psi(t)) + \ell(W(t))]dt \geq \int_0^T [g(t, u^*(t)) + h(t, \xi^*(t)) + \ell(U^*(t))]dt,
\]
which implies that the triplet (u^*, \xi^*, U^*) is an optimal triplet for the problem (3.2). \qed

**Remark 4.6.** Since B'(u) = 2\gamma |u + w^0|, and is not Gâteaux differentiable, the second order necessary and sufficient condition obtained in Theorems 4.4 and 4.5 (see (4.17)) can be written as
\[
\int_0^T (g_{uu}(t, u^* + \theta_1 u)u, u)dt + \int_0^T (h_{\xi\xi}(t, \xi^* + \theta_2 \xi)\xi, \xi)dt + \int_0^T (\ell_{UU}(U^* + \theta_3 U)U, U)dt
- 4 \int_0^T (\gamma [u^* + \theta_4 u + w^0] - |u^* + w^0|)u, p)_{L^2}dt \geq 0,
\]
for any 0 \leq \theta_1, \theta_2, \theta_3, \theta_4 \leq 1. But we know that
\[
|u^* + \theta_4 u + w^0| - |u^* + w^0| \leq |u^* + \theta_4 u + w^0 - (u^* + w^0)| \leq |u|.
\]
Using the above estimate and Hölder’s inequality, we infer that
\[
|\langle \gamma [u^* + \theta_4 u + w^0] - |u^* + w^0|, u, p \rangle_{L^2}| \leq ||\gamma||_{L^\infty}||u||_{L^2}^2 ||p||_{L^2}.
\]
Thus, we have
\[-4 \int_0^T (\gamma [u^* + \theta \xi + w^0] - [u^* + w^0]) u, p)_{L^2} \geq -\frac{4r}{\lambda} \int_0^T \|u\|^2_{L^2} \|p\|_{L^2} dt.\]

Hence, if the condition given below is true
\[
\int_0^T (g_{uu}(t, u^* + \theta_1 u, u)dt + \int_0^T (h_{\xi\xi}(t, \xi^* + \theta_2 \xi)\xi)dt + \int_0^T (\ell_{UU}(U^* + \theta_3 U, U)dt
- \frac{4r}{\lambda} \int_0^T \|u\|^2_{L^2} \|p\|_{L^2} dt \geq 0,
\]
then (4.25) is automatically satisfied.

Let us now provide an example to illustrate the results obtained in Theorems 4.4 and 4.5.

**Example 4.7.** Let us consider the following cost functional:
\[\mathcal{J}(u, \xi, U) := \frac{1}{2} \int_0^T \|u(t)\|^2_{L^2} dt + \frac{1}{2} \int_0^T \|\xi(t)\|^2_{L^2} dt + \frac{1}{2} \int_0^T \|U(t)\|^2_{L^2} dt.\]

Let \((u^*, \xi^*, U^*)\) be an optimal solution for the problem (3.2). Then, Pontryagin’s maximum principle gives
\[\int_0^T (U(t), p(t) + U^*(t))_{L^2} dt = 0, \] (4.26)
for all \(U \in H^{-1}(\Omega)\) or \(U^*(t) = -p(t)\), for a.e. \(t \in [0, T]\). It is clear that the Assumption 4.7 holds true and the adjoint variable \((p, \varphi)\) satisfies the adjoint system (4.12) with \(g_u(t, u) = u\) and \(h_\xi(t, \xi) = \xi\). Also, \((u^*, \xi^*, U^*)\) is an optimal solution for the problem (3.2) if and only if
\[
\int_0^T \|u(t)\|^2_{L^2} dt + \int_0^T \|\xi(t)\|^2_{L^2} dt + \int_0^T \|U(t)\|^2_{L^2} dt
- 2 \int_0^T (\gamma [u^*(t) + \theta u(t) + w^0(t)] - [u^*(t) + w^0(t)]) u(t), p(t))_{L^2} dt \geq 0, \] (4.27)
for any \(0 < \theta < 1\). But if the stronger condition
\[
\int_0^T \|u(t)\|^2_{L^2} dt + \int_0^T \|\xi(t)\|^2_{L^2} dt + \int_0^T \|U(t)\|^2_{L^2} dt - \frac{4r}{\lambda} \int_0^T \|u(t)\|^2_{L^2} \|p(t)\|_{L^2} dt \geq 0, \] (4.28)
or if
\[
\int_0^T \|u(t)\|^2_{L^2} \left(1 - \frac{4r}{\lambda} \|p(t)\|_{L^2}\right) dt \geq 0
\]
holds true, then (4.27) is satisfied (see Remark 4.6). It can also be seen that if
\[
\|p(t)\|_{L^2} \leq \frac{1}{4} \|\varphi\|_{L^\infty} \leq \frac{\lambda}{4r}, \text{ a.e. } t \in [0, T],
\]
then also the condition (4.27) is satisfied.

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