Product of Boundary Distributions

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Abstract

1) We identify new parameter branches for the ultra-local boundary Poisson bracket in $d$ spatial dimension with a $(d-1)$-dimensional spatial boundary. There exist $2^{r(r-1)/2}$ $r$-dimensional parameter branches for each $d$-box, $r$-row Young tableau. The already known branch (hep-th/9912017) corresponds to a vertical 1-column, $d$-box Young tableau. 2) We consider a local distribution product among the so-called boundary distributions. The product is required to respect the associativity and the Leibnitz rule. We show that the consistency requirements on this product correspond to the Jacobi identity conditions for the boundary Poisson bracket. In other words, the restrictions on forming a boundary Poisson bracket can be related to the more fundamental distribution product construction. 3) The definition of the higher functional derivatives is made independent of the choice of integral kernel representative for a functional.

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1 Introduction

It is fair to say that up to now, the Literature \[1, 2, 3, 4, 5\] on the boundary Poisson bracket has revolved around the boundary Poisson bracket in “integrated” form. That is: as a composition of two local functionals $F$ and $G$ into a new local functional $\{F, G\}$, \textit{i.e.} where all objects are integrated over the full physical space.

We shall in this paper elaborate on a singular, distributional formalism for the boundary Poisson bracket.

When addressing this problem, one immediately runs into the well-known problem \[8\] of rendering proper sense of multiplying very singular distributions together. Preferably, the distribution product should respect standard rules of calculus, such as: commutativity, associativity and the Leibnitz rule. Perhaps surprisingly, it turns out, as we shall show in this paper, that a few descriptive rules like the above are enough to reproduce, in a one-to-one fashion, the family of boundary Poisson bracket solutions, of which a $d$-parameter branch was already identified in \[5\]. From the boundary Poisson bracket point of view, the restricting principle is the Jacobi identity.

Conceptionally, an important question is raised by the infinitely many versions of the distribution product that the above mentioned analysis produces. How do we account for this infinity? The paradigm will be that a priori they are all equally-good. A specific choice relies on external data. That is, a specific choice of the “boundary background”, a specific choice of renormalization scheme for the “boundary theory”, etc. We shall simply argue that is no unique, canonical, intrinsic distribution product.

The main feature of the boundary calculus is the fact that we can not freely discard boundary terms, when integrating by part. Hence, the adjoint differentiation is not just the ordinary differentiation (with a minus in front). Curiously, the adjoint differentiation turns out to be more fundamental than the differentiation itself. The logic behind the distributional construction can roughly be sketched by the following chain:

\[
\begin{align*}
\text{Distributions} & \downarrow \\
\text{Adjoint Differentiation} & \downarrow \\
\text{Boundary Distributions} & \downarrow \\
\text{Product} & \downarrow \\
\text{Differentiation} &
\end{align*}
\] (1.1)

The organization of the paper is as follows: We start by reviewing some basic notions in the Schwartzian \[7\] formulation of distribution theory, adapted to the situation with a boundary. The distribution product itself is postponed to Section \[3\]. We consider the ultra-local case in Section \[4\], where the main results are listed. Further developments of the general theory are done in Section \[3\]. The boundary Poisson bracket is discussed in Section \[5\]. Finally, we consider an alternative basis for the product in Section \[7\]. Conclusions are in Section \[8\]. The reader might want to focus on Section \[3-4\] and Section \[6\] for a first-time reading.
2 Elementary Distribution Theory

Let the physical space $\Sigma$ be a closed subset of $\mathbb{R}^d$. Consider the ring $C^\infty(\mathbb{R}^d, \mathcal{A})$ of complex-valued functions equipped with the topology corresponding to the uniform convergence on compact subsets. We are not interested in the specific behavior of the functions outside the physical space $\Sigma$, so we mode out by the two-sided ideal

$$I \equiv \left\{ f \in C^\infty(\mathbb{R}^d, \mathcal{A}) \mid f(\Sigma) \subseteq \{0\} \right\} \subseteq C^\infty(\mathbb{R}^d, \mathcal{A}) .$$

(2.1)

The relevant quotients are two rings and one multiplicative group:

$$\mathcal{E} \equiv C^\infty \left( \mathbb{R}^d, \mathcal{A} \right) / I ,$$

$$\mathcal{D} \equiv C^\infty_c \left( \mathbb{R}^d, \mathcal{A} \right) / I \subseteq \mathcal{E} ,$$

$$\mathcal{E}^\times \equiv C^\infty(\mathbb{R}^d, \mathcal{A}\{0\}) / I \subseteq \mathcal{E} .$$

(2.2)

Elements of the second subquotient ring $\mathcal{D}$ are required to have a representative with compact support. We shall loosely refer to this subring as the test functions with compact support. Elements of the multiplicative group $C^\infty(\mathbb{R}^d, \mathcal{A}\{0\})$ are required to have a non-vanishing representative.

The vector space $\mathcal{D}'$ of distributions is here defined as the space of functionals $u : \mathcal{E}^\times \times \mathcal{D} \to \mathcal{A}$, which are continuous and linear wrt. the second entry. The second entry carries a test function $f \in \mathcal{D}$ while the first entry carries a non-vanishing volume density $\rho \in \mathcal{E}^\times$. The support supp$(u)$ of a distribution $u \in \mathcal{D}$ is defined as

$$\text{supp}(u) = \bigcap_A \left\{ A \text{ closed } \subseteq \mathbb{R}^d \mid \forall f \in C^\infty \left( \mathbb{R}^d, \mathcal{A} \right) : f(A) \subseteq \{0\} \Rightarrow u[\mathcal{E}^\times, f] \subseteq \{0\} \right\} .$$

(2.3)

The ring $\mathcal{E}$ of test functions with not necessarily compact support is embedded into the distribution space $\mathcal{D}'$ via the embedding $i : \mathcal{E} \to \mathcal{D}'$:

$$i(f)[\rho, g] \equiv \int_\Sigma \rho \, d^d x \, f \, g , \quad \rho \in \mathcal{E}^\times , \quad f \in \mathcal{E} , \quad g \in \mathcal{D} .$$

(2.4)

We have that supp$(i(f)) \subseteq \Sigma \cap \text{supp}(f)$. Notation: The $i$-map will often not be written explicitly in formulas. Hopefully, this will not cause any confusion.

Generally, we have a product $\mathcal{E} \times \mathcal{D}' \to \mathcal{D}'$ of a test function $f$ and a distribution $u$ via

$$(f \cdot u)[\rho, g] = u[\rho, fg] , \quad u \in \mathcal{D}' , \quad \rho \in \mathcal{E}^\times , \quad f \in \mathcal{E} , \quad g \in \mathcal{D} .$$

(2.5)

Note that supp$(fu) \subseteq \text{supp}(f) \cap \text{supp}(u)$.

2.1 Adjoint Differentiation

Initially, the differential operator $\partial_i \equiv \frac{\partial}{\partial x_i} : \mathcal{E} \to \mathcal{E}$, $i = 1, \ldots, d$, is only well-defined on smooth functions. Not every distribution will be differentiable when a boundary $\partial \Sigma$ is present. Rather, the

1 The reader may substitute the word space with his favorite term: Cauchy surface, space-time, world volume, etc, for most of the paper. When addressing the boundary Poisson bracket, $\Sigma$ will an equal-time-surface. The time variable will be suppressed from the notation.
distributions are adjointly differentiable. In detail, the adjoint differential operator $\partial^\dagger_i : \mathcal{D}' \to \mathcal{D}'$ acts on a distribution $u \in \mathcal{D}'$ via

$$\partial^\dagger_i u[\rho, f] \equiv u[\rho, \partial_i f] , \quad \rho \in \mathcal{E}^\times , \quad f \in \mathcal{D} . \quad (2.6)$$

This definition of course mimics $\int_\Sigma \rho \, d^dx \, (\partial^\dagger_i f) g = \int_\Sigma \rho \, d^dx \, f \, \partial_i g$ for test functions $f, g \in \mathcal{D}$. We have that $\text{supp}(\partial^\dagger_i u) \subseteq \text{supp}(u)$.

We immediately get that $[\partial^\dagger_i, \partial^\dagger_j] u = 0$ as a consequence of $[\partial_i, \partial_j] f = 0$. The commutator

$$[\partial^\dagger_i, f] = - (\partial_i f) \quad (2.7)$$

of the adjoint derivative $\partial^\dagger_i$ with a test function $f \in \mathcal{E}$ is again a test function, as the following short calculation shows:

$$[\partial^\dagger_i, f] u[\rho, g] = \partial^\dagger_i f u[\rho, g] - u[\rho, \partial_i (fg)] = - u[\rho, (\partial_i f)g] = - (\partial_i f) u[\rho, g] . \quad (2.8)$$

We emphasis that the adjoint derivative $\partial^\dagger_i$ does not satisfy the Leibnitz rule, not even among test functions. Nevertheless, equation (2.7) can still be viewed as a Leibnitz-type rule for the adjoint differential operator $\partial^\dagger_i$:

$$\partial^\dagger_i (fu) = - (\partial_i f) u + f \partial^\dagger_i u . \quad (2.9)$$

2.2 Boundary Distributions

We can now define the main object of this paper: The vector space $\mathcal{B}$ of boundary distributions

$$\mathcal{B} \equiv \left\{ \sum_k f_k (\partial^\dagger)^k g_k \left| \text{All but finitely many of } f_k, g_k \in \mathcal{E} \text{ are zero} \right. \right\} \subseteq \mathcal{D}' . \quad (2.10)$$

We have employed a multi-index notation: For instance, the index

$$k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d , \quad \mathbb{N}_0 \equiv \{0, 1, 2, \ldots \} , \quad (2.11)$$

runs over the $d$-dimensional non-negative integers, and

$$(\partial^\dagger)^k = (\partial^\dagger_1)^{k_1} \cdots (\partial^\dagger_d)^{k_d} . \quad (2.12)$$

By using property (2.7), we may write

$$\mathcal{B} \subseteq \bigoplus_k \mathcal{B}_k , \quad \mathcal{B}_k = \mathcal{E} \cdot (\partial^\dagger)^k 1 . \quad (2.13)$$

The adjoint derivatives $\partial^\dagger_i$, $i = 1, \ldots, d$, give rise to a $d$-dimensional grading of $\mathcal{B}$. The elements $(\partial^\dagger)^k 1, k \in \mathbb{N}_0^d$, constitute a $\mathcal{E}$-linear basis for $\mathcal{B}$, called the adjoint basis. Note that the support of the basis elements is the whole set $\Sigma$:

$$\text{supp}((\partial^\dagger)^k 1) = \Sigma . \quad (2.14)$$

Here it is crucial that the volume density $\rho$ is an argument of the distributions, so that we can vary $\rho$. As a consequence, the $\mathcal{E}$-coefficient function inside $\mathcal{B}_k$ is unique. (Of course, it is sufficient to vary $\rho$ infinitesimally, for instance around the constant function $\rho = 1$.)
2.3 Functional and Functional Derivative

A functional \( F \in \mathcal{F}(\mathcal{E}^\times, \mathcal{C}) \) is a map \( F : \mathcal{E}^\times \to \mathcal{C} \) from the group \( \mathcal{E}^\times \) of volume densities to the complex numbers \( \mathcal{C} \), which in general depends on a number of fields \( \phi^A \). We shall not adapt the latter dependence explicitly in the notation because it would appear too heavy. The functional is called differentiable wrt. the fields \( \phi^A \) if there exist a distributions \( u_A : \mathcal{E}^\times \times \mathcal{D} \to \mathcal{C} \) that satisfies

\[
\forall \rho, \delta \phi^B : \sum_A u_A[\rho, \delta \phi^A] = \delta F[\rho]
\]

for all smooth infinitesimal variations \( \delta \phi^A \). For a differentiable functional \( F \), we shall write \( u_A = \delta F/\delta \phi^A \) and call it the functional derivative of \( F \) wrt. \( \phi^A \).

The Lebesque (or the Riemann) integral provides an embedding \( \int_\Sigma : \mathcal{D} \to \mathcal{F}(\mathcal{E}^\times, \mathcal{C}) \), of functions \( f \in \mathcal{D} \) with compact support into the functional space \( \mathcal{F}(\mathcal{E}^\times, \mathcal{C}) \):

\[
\int_\Sigma f[\rho] \equiv \int_\Sigma \rho \, d^d x \, f , \quad \rho \in \mathcal{E}^\times , \quad f \in \mathcal{D} .
\]

The integration kernel function \( f \) is unique – i.e. the embedding \( \int_\Sigma \) is injective – because the volume density \( \rho \) can be varied. The local functionals are of this form \( \hat{F} = \int_\Sigma f \). They are differentiable, and their functional derivatives are boundary distributions:

\[
\frac{\delta F}{\delta \phi^A} = \sum_k (\partial^\dagger)^k P_{A(k)} f \quad \text{and} \quad \sum_k E_{A(k)} f \cdot (\partial^\dagger)^k 1 \in \mathcal{B} .
\]

This formula is similar to [1, Statement 5.2] of Soloviev. Here

\[
P_{A(k)} f(x) \equiv \frac{\partial f(x)}{\partial \phi^{A(k)}(x)}
\]

and

\[
E_{A(k)} f(x) \equiv \sum_{m \geq k} \left( \begin{array}{c} m \\ k \end{array} \right) (-\partial)^{m-k} P_{A(m)} f(x)
\]

denote the higher partial derivatives and the higher Euler-Lagrange derivatives, respectively, cf. [3, 6].

2.4 Geometric Consideration

One may consider coordinate transformations \( x^i \to x'^{i'} \). To make this construction geometric, one should introduce covariant derivatives \( \partial_i \sim D_i \) by specifying a choice of connection. More generally, the pertinent multi-index structure becomes non-commutative words of a \( d \)-letter alphabet \( \mathcal{A} \), if the derivatives \( [D_i, D_j] \neq 0 \) cease to commute.

3 Distribution Product

3.1 Introduction

As is very well known, even without a boundary, i.e. \( \Sigma = \mathbb{R}^d \), there is no truly natural product for two generic distributions \( u, v \in \mathcal{D}' \). A severe blow against such hopes is dealt by the Schwartz’ Impossibility Result [5]: It states that one cannot in a satisfactory manner embed the whole algebra
of continuous functions \( C(\mathbb{R}^d) \) inside a differential algebra. In this paper, the term differential algebra denotes an associative and commutative algebra equipped with \( d \) mutually commutative, linear derivations \( \partial_i, i = 1, \ldots, d \), that are defined for all algebra elements in the algebra and that satisfy the Leibnitz rule. In the case with no boundary, Colombeau [4] has embedded \( \mathcal{D}' \) into a differential algebra. As an illustrative manifestation of the Schwartz’ Impossibility Result at work, the product of Colombeau does not coincide with the classical product in \( C(\mathbb{R}^d) \). However, it preserves the classical product for smooth functions \( C^\infty(\mathbb{R}^d) \). It would be interesting to know if and/or how Colombeau’s construction can be adapted to incorporate boundaries.

An old idea uses the convolution in the Fourier transformed space \([4, 11]\) to define a product. However, we are concerned, that the important features of the boundary and the notion of locality would be “washed away” by the Fourier transform if one is not careful.

### 3.2 Boundary Poisson Bracket

Here we shall take a more modest approach, and only consider a distribution product within the much smaller class \( \mathcal{B} \) of boundary distributions. Our main motivation and main application is to reformulate the boundary Poisson bracket \([1, 3, 4, 5]\) using the boundary distributions.

First step is to make sense of the following product:

\[
\{ f, g \} = \frac{\delta[f \Sigma f]}{\delta \phi^A} \omega^{AB} \frac{\delta[f \Sigma g]}{\delta \phi^B} = \left( \sum_k E_{A(k)} f \cdot (\partial^\dagger)^k 1 \right) \omega^{AB} \left( \sum_\ell E_{B(\ell)} g \cdot (\partial^\dagger)^\ell 1 \right) .
\] (3.1)

In 1993, Soloviev \([1, \text{Rule 5.4}]\) showed that the boundary Poisson brackets would have an alternative description in terms of distribution products. He inquired, without pursuing it further, whether a distribution product could be given a rigorous meaning.

### 3.3 Distribution Product

A priori, no canonical choice for such a product is preferred. A certain choice for the product represents an additional input of informations into the system. This should be specified in order to have a well-posed problem. In a fundamental and geometrically formulated theory, like for instance a complete, non-perturbative formulation of string theory, the appearance of boundaries themselves, like branes, is presumably only a good low energy description. By analyzing the product of the boundary distributions, we are merely collecting some left-over low energy data of the full dynamical boundary theory. As a physicist, one may view the product as a result of a renormalization without a regularization. As a mathematician, it is simply going to be a definition that depends on external parameters.

Let us start by specifying the product on the basis of \( \mathcal{B} \). There should exist smooth structure functions \( c_n^{k,\ell}(x) \in \mathcal{E} \), such that

\[
(\partial^\dagger)^k 1 \cdot (\partial^\dagger)^\ell 1 = \sum_n c_n^{k,\ell} (\partial^\dagger)^n 1 ,
\] (3.2)

i.e. the algebra \( \mathcal{B} \) closes. On a technical note, to avoid cluttering up the paper with arguments of convergence, we assume that

\[
\forall k, \ell : c_n^{k,\ell} = 0 \text{ for all but finitely many } n .
\] (3.3)
Next we extend the product to the whole $B \times B$ by demanding that the coefficient functions themselves carry no surprises:

\[
(f_k \cdot (\partial^1)^k 1) \cdot (g\ell \cdot (\partial^1)^\ell 1) = (f_k g_\ell) \cdot ((\partial^1)^k 1 \cdot (\partial^1)^\ell 1). \tag{3.4}
\]

We shall furthermore assume:

- **Naturalness**: $u \cdot 1 = u = 1 \cdot u$, or $c_n^{k,0} = c_0^k = \delta_n^k$.
- **Commutativity**: $u \cdot v = v \cdot u$, or $c_n^{k,\ell} = c_n^{\ell,k}$. \tag{3.5}
- **Associativity**: $(u \cdot v) \cdot w = u \cdot (v \cdot w)$, or $\sum_n c_n^{k,\ell} c_p^{\ell,m} = \sum_n c_p^{k,n} c_n^{\ell,m}$.

To ensure that the boundary Poisson bracket (See Section 3) does not change the bulk properties, one can impose the decoupling requirement:

- **Decoupling**: $u \perp 1 \lor v \perp 1 \Rightarrow u \cdot v \perp 1$, or $c_0^{k,\ell} = \delta_0^k \delta_0^\ell$. \tag{3.6}

Strictly speaking, we cannot exclude the possibility that going to a bigger algebra than the boundary distribution algebra $B$ might kill some of the solutions by additional requirements. However, this does not jeopardize the $B$ product construction as an isolated construction. It rests in itself. In fact, rather we predict that one can construct deformed versions of the Colombeau algebra.

### 3.4 Differentiation

Given the multiplication in $B$, there now is a canonical choice for a differential structure $\partial_i$ in $B$. This consists in requiring that the Leibnitz-type relation (2.7) which previously only applied to smooth functions should continue to hold for all boundary distributions $u \in B$:

\[
\partial_i u \equiv u \cdot \partial^1_i 1 - \partial^\dagger_i u. \tag{3.7}
\]

Note how the existence of the product is vitale for this definition. One may easily check that $\partial_j i(f) = i(\partial_j f)$ for smooth functions $f \in \mathcal{E}$, cf. (2.4), and that this definition (3.7) respects the Leibnitz rule wrt. the standard product $\cdot \colon \mathcal{E} \times B \to B$, cf. (2.5). We shall also impose the Leibnitz rule for the full product $\cdot \colon B \times B \to B$:

\[
\partial_i (u \cdot v) = (\partial_i u) \cdot v + u \cdot \partial_i v. \tag{3.8}
\]

The Leibnitz rule (3.3) leads to a tower of quadratic consistency relations among the structure functions:

\[
c_n^{k+e_i,\ell} + c_n^{k,\ell+e_i} - \sum_p c_p^{k,\ell} c_n^{p,e_i} = c_n^{k,\ell}_{n-e_i} - \partial_i c_n^{k,\ell}. \tag{3.9}
\]

Here $e_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$’th unit vector of the index lattice. We shall extensively investigate this quadratic Leibnitz condition in the following two Sections 4-5. With the implementation of the Leibnitz rule, we may refer to $\partial_i$ as a (first order) linear derivation, while $\partial^\dagger_i$ is a (first order) affine differential operator. Hence their commutator becomes a (zeroth order) left multiplication operator:

\[
[\partial^\dagger_i, \partial^\dagger_j] = 0,
\]

\[
[\partial_i, \partial_j] \overset{(3.8)}{=} 0,
\]

\[
[\partial_i, \partial^\dagger_j] \overset{(3.8)}{=} \partial_i \partial^\dagger_j 1 \equiv \partial^\dagger_i 1 \cdot \partial^\dagger_j 1 - \partial^\dagger_i \partial^\dagger_j 1. \tag{3.10}
\]
4 Ultra-Local Product

4.1 Definition

The product is called \textit{ultra-local}, if the $c_n^{k,\ell}$ coefficient functions are of the form

\begin{equation}
    c_n^{k,\ell} = c_n^{k+\ell} \delta_n^{k+\ell}.
\end{equation}

In this distinguished case the structure constants $c_n^{k,\ell}$ are dimensionless, and the finiteness assumption (3.3) is automatically satisfied. The decoupling requirement follows from the naturalness assumption. Moreover, the other assumptions reduces to:

- Naturalness : $c_n^{k,0} = 1 = c_n^{0,k}$.
- Commutativity : $c_n^{k,\ell} = c_n^{\ell,k}$.
- Associativity : $c_n^{k,\ell} c_n^{k+\ell,m} = c_n^{k,\ell+m} c_n^{\ell,m}$.
- Leibnitz rule : $c_n^{k+p,\ell} c_n^{k,\ell} - c_n^{k,\ell+p} c_n^{k,\ell} = c_n^{k,\ell} \land \partial_i c_n^{k,\ell} = 0$.

Note that the Leibnitz rule has split into two conditions. The last condition says that there will only be locally constant solutions, i.e. they are constant on each connected component of $\Sigma$. Let us restrict ourselves to one of the connected components, so we can speak of a constant solution. Then the commutativity and the associativity conditions define a 2-cocycle in the Abelian multiplicative monoid $(\mathbb{C}, \cdot)$ of complex numbers. (A monoid is a semi-group with a neutral element. Contrary to the notion of a group, elements are not assumed to have an inverse element.)

4.2 Ultra-Local Analysis

The generic solutions are 2-coboundaries:

\begin{equation}
    \exists \{b_k\}_k \subseteq \mathbb{C} \cup \{\infty\} : c_n^{k,\ell} = \frac{b_k b_\ell}{b^{k+\ell}}, \quad b^{k=0} = 1,
\end{equation}

that satisfies the first of the Leibnitz conditions. By use of the exactness, it can be cast into:

\begin{equation}
    \frac{1}{c_n^{k,e_i}} + \frac{1}{c_n^{\ell,e_i}} = \frac{1}{c_n^{k+\ell,e_i}} + 1.
\end{equation}

It follows that there exists constants $s_{ij} \in \mathbb{C} \cup \{\infty\}$, $i,j = 1, \ldots, d$, such that

\begin{equation}
    \frac{b^{k+e_i}}{b^k b^{e_i}} = \frac{1}{c_n^{k,e_i}} = 1 + \sum_{j=1}^d \frac{k_j}{s_{ij}}.
\end{equation}

Let us calculate the first few lattice vectors in terms of the basis elements:

\begin{align*}
    b^{e_i+e_j} &= b^{e_i} b^{e_j} (1 + \frac{1}{s_{ij}}), \\
    b^{e_i+e_j+e_k} &= b^{e_i} b^{e_j} b^{e_k} (1 + \frac{1}{s_{ij}})(1 + \frac{1}{s_{ik}} + \frac{1}{s_{jk}}).
\end{align*}

Enforcing the symmetry in the first formula and the associativity in the last formula we are lead to consider the following algebraic variety in $(\mathbb{C} \cup \{\infty\})^{d^2}$:

\begin{equation}
    s_{ij} = s_{ji} \land s_{ij} (s_{ik} - s_{jk}) = 0.
\end{equation}
(no sums.) It turns out that there are no further requirements and that we can associate a solution to each parameter value \( \{s_{ij}\}_{ij} \) on this algebraic variety. The latter requirement in (4.7) has the solutions

\[
s_{ij} = 0 \lor s_{ij} = \infty \lor \forall k = 1, \ldots, d : s_{ik} = s_{jk}.
\]  

(4.8)

In words, if \( s_{ij} \in \mathcal{C}\setminus\{0\} \), then the \( i \)'th and the \( j \)'th row (and column) are pairwise equal. This inspires an equivalence relation \( \sim \) in \( \{1, \ldots, d\} \):

\[
i \sim j \overset{\text{def}}{\iff} i = j \lor s_{ij} \in \mathcal{C}\setminus\{0\}.
\]  

(4.9)

(It is reflexive by construction. The symmetry and the transitivity follow from (4.7).) Let us call the number of equivalence classes the rank, and denote it \( r \). The cardinality \( n_i \) of the equivalence classes has the sum \( \sum_{i=1}^d n_i = d \). Clearly, the equivalence classes can be mapped into a \( d \)-box, \( r \)-row Young tableau of row lengths \( (n_1, \ldots, n_r) \), where \( n_1 \geq \ldots \geq n_r \geq 1 \). (We remind the reader that a Young tableau is a decorated Young diagram, where the decoration consists in a numbering \( 1, \ldots, d \) of the boxes. The boxes within the same row and rows of equal lengths can still freely be permuted.) We note that after the row (and column) reduction \( \sim \), the \( s_{ij} \) matrix is an \( r \times r \) symmetric matrix with \( \frac{r(r-1)}{2} \) free off-diagonal elements \( s_{ij} = 0 \) or \( s_{ij} = \infty \), that can take \( 2^{\frac{r(r-1)}{2}} \) different combinations.

When all diagonal elements \( s_i \equiv s_{ii} \in \mathcal{C}\setminus\{0\} \), we are on a \( r \)-dimensional branch of the algebraic variety. Branch points have more than one of the diagonal elements \( s_i \) equal to \( s_i = 0 \) or \( s_i = \infty \). Branch points only exist in higher dimensions \( d > 1 \).

### 4.3 Ultra-Local Results

Our main result is that we can write the generic solution as a product of two factors \((I)\) and \((II)\):

\[
c^{k,\ell}_{(I)} = c^{k,\ell}_{(I)} c^{k,\ell}_{(II)},
\]

\[
c^{k,\ell}_{(I)} = \frac{(s)_{i(k)} (s)_{i(\ell)}}{(s)_{i(k+\ell)}} = \frac{\Gamma(i(k)+s) \Gamma(i(\ell)+s)}{\Gamma(i(k+\ell)+s) \Gamma(s)} = \frac{B(i(k)+s,i(\ell)+s)}{B(i(k+\ell)+s,s)} = \prod_{i=0}^{i(k)-1} \prod_{j=0}^{i(\ell)-1} \frac{i+j+s}{i+j+1+s},
\]

\[
c^{k,\ell}_{(II)} = \prod_{i,j} \delta_0^{\min(k_i+\ell_i,k_j+\ell_j)},
\]  

(4.10)

for arbitrary diagonal parameters \( s \in (\mathcal{C} \cup \{\infty\})^{r} \) on \( r \) copies of the Riemann sphere, \( r = 1, \ldots, d \). Here \( (s)_n = \Gamma(s+n)/\Gamma(s) \) is the Pochhammer symbol in \( r \) dimensions and \( \Gamma(s) \equiv \prod_{i=1}^{r} \Gamma(s_i) \).

The linear map \( i : \mathcal{A}^d \rightarrow \mathcal{A}^r \) maps the basis vectors \( e_i \) to the basis vectors \( i(e_j) = e_{j/\sim} \). That is:

\[
i(e_j) = i(e_k) \iff j \sim k.
\]  

(4.11)

The solutions (4.10) are precisely the allowed coefficients for the ultra-local boundary Poisson bracket. As we shall see, this is by no means a coincidence. It merely reflects a deep correspondence of notions and properties between the two kinds of boundary calculi that will be further illuminated in Section 3. In [5] we found one \( d \)-dimensional branch.
4.4 Examples

As an example, consider the vertical, full-rank Young diagram \( (n_1=1, \ldots, n_d=1) \). If all the off-diagonal parameters \( s_{ij} = \infty \) blow up, the solution reads:

\[
(c_{k,\ell}) = \frac{(s)_k(s)_\ell}{(s)_{k+\ell}} = \frac{\Gamma(k+s)\Gamma(\ell+s)}{\Gamma(k+\ell+s)\Gamma(s)}.
\] (4.12)

This is the \( d \)-dimensional branch found in [3].

If all the off-diagonal parameters \( s_{ij} = 0 \) vanish instead, the solution reads:

\[
(c_{k,\ell}) = \begin{cases} 
\frac{(s)_k(s)_\ell}{(s)_{k+\ell}} = \frac{\Gamma(|k|+s)\Gamma(|\ell|+s)}{\Gamma(|k|+|\ell|+s)\Gamma(s)} & \text{if } \exists i = 1, \ldots, d : k \parallel e_i \parallel \ell, \\
0 & \text{otherwise}.
\end{cases}
\] (4.13)

As a third example, consider the horizontal, maximally symmetric, rank-1 Young diagram \( (n_1=d) \), where all the parameters \( s_{ij} = s \) are equal. In this case, the solution is simply:

\[
(c_{k,\ell}) = \frac{(s)|k|(s)|\ell|}{(s)|k+\ell|} = \frac{\Gamma(|k|+s)\Gamma(|\ell|+s)}{\Gamma(|k|+|\ell|+s)\Gamma(s)}.
\] (4.14)

where \( |k| \equiv \sum_{i=1}^d k_i \). This distinguished 1-dim branch has contrary to all other branches an alternative \( \Delta^{(n)} \) basis, as we shall see in Section 7.3.

5 Supplementary Formalism

5.1 Integration of Distributions

Note that when a distribution has compact support, its test functions do not also have to have compact support, understood in the appropriate sense. Hence, we may extend the definition of integration \( \int_{\Sigma} : D \rightarrow \mathcal{F}(\mathcal{E}^\mathcal{X},\mathcal{U}) \), cf. (2.16), to an arbitrary boundary distribution \( u \in \mathcal{B}_{\mathcal{C}} \) with compact support, simply by:

\[
\int_{\Sigma} \rho \, d^d x \, u \equiv \int_{\Sigma} u[\rho] = u[\rho, 1].
\] (5.1)

The integral (5.1) satisfies

\[
\int_{\Sigma} \rho \, d^d x \left( u \cdot \partial^1_i 1 - \partial^i u \right) = \int_{\Sigma} \rho \, d^d x \, \partial^1_i u = \partial^i u[\rho, 1] = u[\rho, \partial_i 1] = u[\rho, 0] = 0.
\] (5.2)

This together with the Leibnitz rule yields \( \int_{\Sigma} \rho \, d^d x \left( \partial^1_i u \right) \cdot v = \int_{\Sigma} \rho \, d^d x \, u \cdot \partial_i v \) for boundary distributions \( u, v \in \mathcal{B}_{\mathcal{C}} \). We see from the second expression in (5.2) that even with a boundary \( \partial \Sigma \) present, the integral of a total adjoint derivative vanishes. The opposite is also true, as will become clear below:

\[
\int_{\Sigma} u = 0 \quad \Leftrightarrow \quad \exists u^1, \ldots, u^d \in \mathcal{B}_{\mathcal{C}} : u = \sum_{i=1}^d \partial^1_i u^i.
\] (5.3)

(Here it is important that \( \rho \) is free.)
More explicitly, given an arbitrary boundary distribution \( u = \sum_k f_k \cdot (\partial^\dagger)^k 1 \in \mathcal{B}_c \), with finitely many smooth coefficient functions \( f_k \in \mathcal{D} \), there exists a unique smooth function \( f = \sum_k \partial^k f_k \in \mathcal{D} \),

\[
(5.4)
\]

which has the same integral:

\[
\int_{\Sigma} u = \int_{\Sigma} \sum_k f_k \cdot (\partial^\dagger)^k 1 = \sum_k (\partial^\dagger)^k 1[f_k] = \int_{\Sigma} \sum_k \partial^k f_k
= \int_{\Sigma} f = i(f)[\cdot, 1] = \int_{\Sigma} i(f), \quad f_k \in \mathcal{D}.
(5.5)
\]

See Soloviev \[2, p.2\] for a related formula. The non-injective linear map \( u \mapsto i_{\Sigma} u \) has the following kernel:

\[
\int_{\Sigma} u = 0 \iff f = 0.
(5.6)
\]

We can see this in a more elaborated manner by rewriting \( u \) as

\[
u = \sum_k f_k \cdot (\partial^\dagger)^k 1 \overset{(2.7)}{=} \sum_k (\partial^\dagger)^k \sum_{m \geq k} \binom{m}{k} \partial^{m-k} f_m
= f + \text{(total adjoint derivative terms)},
(5.7)
\]

This also establishes a proof of (5.3). We shall sometimes refer to the unique smooth function \( f \) as \( f \equiv C^{\infty}(u) \).

The functional \( i_{\Sigma} u \) is differentiable as a consequence of (5.5) and our previous discussion in Section 2.3 of the \( C^{\infty} \) case. It follows from straightforward manipulations, that (2.17) is valid for distributions as well:

\[
\frac{\delta[i_{\Sigma} u]}{\delta \phi^A} = \frac{\delta[i_{\Sigma} f]}{\delta \phi^A} = \sum_k E_{A(k)} f \cdot (\partial^\dagger)^k 1 = \sum_k (\partial^\dagger)^k P_{A(k)} f
= \sum_k (\partial^\dagger)^k P_{A(k)} u = \sum_k E_{A(k)} u \cdot (\partial^\dagger)^k 1,
(5.8)
\]

where we have independently extended the definition \( P_{A(k)} u \equiv \sum_n P_{A(k)} f_n \cdot (\partial^\dagger)^n 1 \) and

\[
E_{A(k)} u \equiv \sum_{m \geq k} \binom{m}{k} (-\partial)^{m-k} P_{A(m)} u
(5.9)
\]

to an arbitrary boundary distribution \( u \in \mathcal{B} \).

The functionals can in principle also depend on the mixed derivatives \( P(\partial, \partial^\dagger) \phi^A \) of the field variables \( \phi^A \), where \( P(\partial, \partial^\dagger) \) is a polynomial in the non-commutative \( \partial_i \) and \( \partial^\dagger_i \). However, we shall for simplicity assume throughout this paper that adjoint derivatives \( \partial^\dagger_i \) acting on the fundamental field variables \( \phi^A \) have been reexpressed in terms of the ordinary derivatives and multiplication operators by repeated use of (3.7). Let us also note for later that

\[
[\partial^\dagger_i, P_{A(k)}] = [P_{A(k)}, \partial_i] = P_{A(k-e_i)}.
(5.10)
\]
5.2 Generating Functions and Fourier Transform

In order to carry out the analysis most efficiently, it is convenient to resum the distributions in a generating series

\[
e^{\partial^iy} u \equiv \sum_k \frac{y^k}{k!} (\partial^i)^k u ,
\]

\[
e^{\partial y} u \equiv \sum_k \frac{y^k}{k!} \partial^k u = \frac{e^{-\partial^iy} u}{e^{-\partial^iy_1}}.
\]  

(5.11)

The last expression should be understood by a formal Taylor expansion around \(y = 0\). Then it merely reproduces the previous definition (3.7) of differentiation \(\partial_i\) and its iterated consequences (assuming the Leibnitz rule). The Leibnitz-type rule (2.9) for the adjoint differential operator \(\partial^i\) is

\[
e^{\partial^iy}(f \cdot u) = e^{-\partial y f} \cdot e^{\partial^iy} u , \quad f \cdot e^{\partial^iy} u = e^{\partial^iy}(e^{\partial y} f \cdot u) ,
\]  

(5.12)

while the Leibnitz rule (3.8) for the differential operator \(\partial_i\) becomes

\[
e^{\partial y}(u \cdot v) = e^{\partial y} u \cdot e^{\partial y} v ,
\]  

(5.13)

or equivalently,

\[
e^{\partial^iy}(u \cdot v) = \frac{e^{\partial^iy} u \cdot e^{\partial^iy} v}{e^{\partial^iy} 1} .
\]  

(5.14)

We have the following “affine Leibnitz rule”:

\[
e^{u_A + u_B + \partial y} (u \cdot v) = e^{u_A + \partial y} u \cdot e^{u_B + \partial y} v , \quad u_A, u_B, u, v \in \mathcal{B} .
\]  

(5.15)

The interplay (3.10) between \(\partial^i\) and \(\partial_i\) reads:

\[
e^{-\partial y} e^{\partial^iy} u = \frac{e^{-\partial y} e^{\partial^iy} 1}{e^{\partial^iy} 1} e^{\partial^iy} e^{-\partial y} u = \frac{e^{\partial^i(y+\bar{y})} 1}{e^{\partial^i y} 1} e^{\partial^i\bar{y}} e^{-\partial y} u .
\]  

(5.16)

The product (3.2) is summed up as

\[
T(y_A, y_B, q) \equiv \sum_{k,\ell,n} \frac{y_A^k y_B^\ell}{k! \ell!} c_{k,\ell}^n q^n .
\]  

(5.17)

It has the Fourier transform\(^2\)

\[
T(y_A, y_B, y) \equiv \int dq \, e^{-qy} T(y_A, y_B, q) .
\]  

(5.18)

---

\(^2\) Notation: A \(q\)-integration \(\int dq \sim \int_{-\infty}^{+\infty} dq\) is always taken to be along the real axis \(\mathbb{R}\), while a \(y\)-integration \(\int dy \sim \int_{-\infty}^{+\infty} \frac{dy}{i\pi}\) is along the imaginary axis \(i\mathbb{R}\), and normalized with a factor \(2\pi i\).

\(^3\) We have shuffled the order of the arguments \(T(y, y_A, y_B) \sim T(y_A, y_B, y)\) compared with a similar notation used in \(\mathcal{B}\).
The boundary algebra then reads

- **Closure**:
  \[ e^{\partial_j y_A} e^{\partial_i y_B} = \int d^d y \ T(y_A, y_B, y) \ e^{\partial_i y_1} . \]

- **Naturalness**:
  \[ T(y_A, 0, y) = \delta^d(y_A - y) = T(0, y_A, y) . \]

- **Commutativity**:
  \[ T(y_A, y_B, y) = T(y_B, y_A, y) . \]

- **Associativity**:
  \[ \int d^d y \ T(y_A, y_B, y) \ T(y_C, y_D, y) = \int d^d y \ T(y_A + y_C, y_B + y_D, y) . \]

- **Leibnitz Rule**:
  \[ \int d^d y \ e^{\partial_j y_B} T(y_A, y_B + y_B, y) e^{-\partial_j y_A} T(y_C, y_D, y) = (A \leftrightarrow D, B \leftrightarrow C) . \]

The last equation corresponds to the quadratic Leibnitz condition (3.9). An important version of the Leibnitz rule reads:

\[ \int d^d y \ e^{\partial_j y_B} T(y_A, y_B + y_B, y + y_B) e^{-\partial_j y_A} T(y_C, y_D, y + y_C) = (A \leftrightarrow D, B \leftrightarrow C) . \]  

(5.20)

This identity was (in the constant case) written down in [5, Equation (6.1)] in the context of the boundary Poisson bracket as a sufficient condition for the Jacobi identity to hold. A proof is given in Appendix A.

### 5.3 Change of Product

To deal with the non-ultra-local product, we invoke the following Ansatz: There exists an invertible matrix-valued smooth function \( \Lambda^k_a \) and a fiducial solution \( c^{a,b}_{(0)c} \) such that

\[ c^{k,f}_{n} = \Lambda^k_a \Lambda^f_b \ c^{a,b}_{(0)c} \Lambda^c_n . \]  

(5.21)

It would be obvious to use a ultra-local solution \( c^{a,b}_{(0)c} = \delta^a_{c+b} \) as the fiducial solution, but any solution will do. We may view \( \Lambda^a_k \) as a change of the renormalization scheme; also known as finite renormalization.

We sum up the \( \Lambda^k_a \) as

\[ \Lambda(y, q^{(0)}) = \sum_{k,a} \frac{y^k}{k!} \Lambda^k_a q^{(0)a} . \]  

(5.22)

The Fourier transform reads

\[ \Lambda(y, y^{(0)}) = \int d^d q^{(0)} e^{-q^{(0)} y^{(0)}} \Lambda(y, q^{(0)}) . \]  

(5.23)

The Ansatz (5.21) then reads:

\[ T(y_A, y_B, y) = \int d^d y^{(0)} A_{y_B}^{d} \int d^d y^{(0)} A_{y_A}^{d} \Lambda(y_A, y^{(0)}_A) \Lambda(y_B, y^{(0)}_B) \ T^{(0)}_{y_B} \ y^{(0)}_A \ y^{(0)}_B \ y^{(0)}_C \Lambda^{-1}(y^{(0)}, y) . \]  

(5.24)

The Ansatz (5.21) automatically satisfies the commutativity and the associativity requirements. The quadratic Leibnitz condition (3.9) is met by the following generic condition:

\[ \partial_i \Lambda^k_a = \Lambda^k_{a-e_i} - \Lambda^{k+e_i} a + \Lambda^k_b \ c^{b,d}_{(0)a} \Lambda^{e_i d} - \Lambda^k_b \ c^{b,e_i}_{(0)a} , \]  

(5.25)
or equivalently, in terms of the inverse matrix:
\[
\partial_i \Lambda^a \equiv -\Lambda^a\ell \Lambda^\ell_k \partial_i \Lambda^\ell_b = \Lambda^a_{k-e_i} - \Lambda^{a+e_i}_k - \Lambda^{e_i}_d \epsilon^{d}_{(0)b} \Lambda^b_k + c^{e_i a}_{(0)} \Lambda^b_k .
\] (5.26)

This system is integrable. For an infinitesimal variation, \( \Lambda^a_k = \delta^a_k + \delta \Lambda^a_k \), the condition (5.25) becomes a linearized, homogeneous, first order partial differential equation:
\[
\partial_i \delta \Lambda^a_k = (A_i \delta \Lambda)^k_a \equiv \delta \Lambda^k_{a-e_i} + \delta \Lambda^{k+e_i}_a + \delta \Lambda^{e_i}_d \epsilon^{d}_{(0)a} , \quad [A_i, A_j] = 0 ,
\] (5.27)

with solution
\[
\delta \Lambda(x) = \exp \left( \sum_{i=1}^d \int_{x^{(0)}}^{x} dx^i A_i \right) \delta \Lambda(x^{(0)}) ,
\] (5.28)
independent of the integration path. Other requirements are:

- **Naturalness** : \( \Lambda^a_{k=0} = \delta^a_0 \).

- **Decoupling** : \( \Lambda^{k=0}_a = \delta^a_0 \).

It is easy to see that the solution (5.28) for \( \Lambda(x) = 1 + \delta \Lambda(x) \) respects naturalness (and decoupling) requirements for all points \( x \), if it does so for one point \( x^{(0)} \).

## 6 Boundary Poisson Bracket

A compelling smooth-to-smooth version of the \( x \)-pointwise boundary Poisson bracket (3.1) is, cf. Section 5.1:
\[
\{ f, g \}_\infty = C^\infty \left[ \frac{\delta [f; g]}{\delta \phi_A} \omega^{AB} \frac{\delta [f; g]}{\delta \phi_B} \right] = C^\infty \left[ \sum_{k} E_A(k) f \cdot (\partial^k)^1 \omega^{AB} \left( \sum_{\ell} E_B(\ell) g \cdot (\partial^\ell)^1 \right) \right]
\]
\[
= C^\infty \sum_{k, \ell, n} E_A(k) f \omega^{AB} E_B(\ell) g \ c^{k,\ell}_{n,1}
\]
\[
= \sum_{k, \ell, n} \partial^n \left( E_A(k) f \ c^{k,\ell}_{n,1} \omega^{AB} E_B(\ell) g \right) , \quad f, g \in \mathcal{E} .
\] (6.1)

One of the main points is that the boundary Poisson bracket can be defined via the last equality as a composition between smooth functions into smooth functions without ever introducing the boundary distributions. Historically [1, 2, 3, 4, 5], this is what have been done. But as the first three lines clearly indicate, the distribution product is “lurking just beneath the surface”.

### 6.1 Mono-Local Tensors

The notion of a tensor field acquires some new features when a boundary \( \partial \Sigma \) is present. These are worthwhile pointing out. First of all, the Poisson bi-vector \( \omega^{AB} = \omega^{AB}(x, \phi(x)) \) is a function of the field variables \( \phi^A(x) \) and it may have explicit \( x \)-dependence, but it does not depend on the derivatives \( \partial^k \phi^A(x), k \neq 0 \), of the field variables. It is antisymmetric \( \omega^{BA} = -\omega^{AB} \). Furthermore, it satisfies the Jacobi identity:
\[
0 = \sum_{\text{cycl.}A,B,C} \omega^{AD} P_{D(0)} \omega^{BC} .
\] (6.2)
This implies the Jacobi identity
\[ 0 = \sum_{\text{cycl.}\ f,g,h} \{ f, \{ g, h \} \}_{\infty}, \quad f, g, h \in \mathcal{E}, \] (6.3)
for the boundary Poisson bracket (6.1) itself, as we shall see in Appendix B.

The Poisson bi-vector \( \omega^{AB}(x) \) transforms covariantly as a tensor, i.e.
\[ \omega^{AB} \rightarrow \omega'^{A'B'} = P_{A(0)} \phi'^{A'} \omega^{AB} P_{B(0)} \phi'^{B'} \] (6.4)
under a change of coordinates
\[ \phi^A(x) \rightarrow \phi'^A(x) = \phi'^A(\phi(x), x), \] (6.5)
which does not involve the derivatives \( \partial^k \phi^A(x), k \neq 0 \), of the field variables. The presence of the boundary \( \partial \Sigma \) implies that the 2-tensor \( \omega^{AB}(x) \) cannot maintain its form under a more general field transformation. Hence, it is natural to consider only a Poisson bi-vector \( \omega^{AB}(x) \) which does not depend on the derivatives \( \partial^k \phi^A(x), k \neq 0 \), in the first place since the allowed class of field redefinitions cannot induce them later-on. We should perhaps stress that we continue to allow for derivatives \( \partial^k \phi^A(x) \) inside \( f \) and \( g \) in (3.1). After all, this is the raison d’etre of a boundary Poisson bracket, while on the other hand, the applications usually deal with a Poisson structure on Darboux form, i.e. where \( \omega^{AB} \) is a constant.

6.2 Multi-Local Formalism

Let us mention, without going into lengthy details, that to allow for arbitrary field redefinitions, one should consider a bi-local Poisson bracket
\[ \{ F, G \} = \int_{\Sigma \times \Sigma} \rho \, d^d x_A \rho \, d^d x_B \frac{\delta F}{\delta \phi^A(x_A)} \omega^{AB}(x_A, x_B) \frac{\delta G}{\delta \phi^B(x_B)} \]
\[ 0 = \sum_{\text{cycl.} A, B, C} \int_{\Sigma} \rho \, d^d x_D \omega^{AD}(x_A, x_D) \frac{\delta \omega^{BC}(x_B, x_C)}{\delta \phi^D(x_D)}. \] (6.6)

By the word bi-local we are merely referring to that \( \omega^{AB}(x_A, x_B) \) (in the distributional sense) may depend on \( x_A \) and \( x_B \). It may very well be that the support is along the diagonal \( x_A = x_B \), but this does not necessarily imply that the Poisson bi-vector \( \omega^{AB}(x_A, x_B) \) can be written with one entry only, due to the very nature of distributions. Whenever one can bring \( \omega^{AB}(x_A, x_B) \) on the diagonal form
\[ \omega^{AB}(x_A, x_B) = \omega^{AB}(x_A) \delta_{\Sigma}(x_A, x_B), \] (6.7)
one may reduce the boundary Poisson bracket (6.6) to the mono-local case. The point is twofold:

- The diagonal Ansatz (6.7) is not stable under general field redefinitions, when a boundary \( \partial \Sigma \) is present.
- Even if \( \omega^{AB}(x_A, x_B) \) is of the diagonal form (6.7), the pertinent Jacobi condition (6.6) may not acquire a mono-local form, if \( \omega^{AB}(x_A) \) depends on the field derivatives \( \partial^k \phi^A(x), k \neq 0 \).
Here the Dirac delta distribution is defined as
\[
\forall f : \int_{\Sigma \times \Sigma} \rho \, d^d x_A \rho \, d^d x_B \, f(x_A, x_B) \, \delta_{\Sigma}(x_A, x_B) = \int_{\Sigma} \rho \, d^d x \, f(x, x). \tag{6.8}
\]
It has the property that
\[
\left( \partial^i_{x_A} - \partial^i_{x_B} \right) \delta_{\Sigma}(x_A, x_B) = 0. \tag{6.9}
\]
The functional derivative may formally be written as
\[
\frac{\delta}{\delta \phi^{A}(x)} = \sum_i \sum_k (\partial^i_k \delta_{\Sigma}(x, z^{(i)})) P^{(z^{(i)})}_{A(k)}, \tag{6.10}
\]
where the \( i \)-sum extends over the finitely many \( d \)-tuple variables \( z^{(i)} = (z_1^{(i)}, \ldots, z_d^{(i)}) \) of the implicitly attacked multi-local functional. That is, the functional derivative operation “peers” beyond the integral symbols of the unwritten multi-local functional (if any integrals).

In the multi-local framework, the proof of the Jacobi identity for the boundary Poisson bracket will follow from the traditional argument: Namely, as a consequence of 1) the Jacobi identity (6.6) for \( \omega^{AB}(x_A, x_B) \), 2) the Leibnitz rule and 3) the fact that two functional derivatives commute:
\[
\left[ \frac{\delta}{\delta \phi^{A}(x_A)}, \frac{\delta}{\delta \phi^{B}(x_B)} \right] = 0. \tag{6.11}
\]
While multi-local objects are a straightforward and natural generalization, we shall devote our attention to the important mono-local case. First of all, because it is interesting in its own right. Secondly, since we have only sketched the multi-local formalism, we would like to provide a proof of the Jacobi identity that rest within the mono-local formalism itself. (See Appendix B). Unfortunately, the mono-local proof does not have the attractive simplicity of the multi-local proof.

### 6.3 Fourier Transform

It is convenient to resum the higher derivatives in a series,
\[
P^{A}(q)f \equiv \sum_{k=0}^{\infty} q^k P^{(k)}_{A} f, \]
\[
E^{A}(q)f \equiv \sum_{k=0}^{\infty} q^k E^{(k)}_{A} f = \exp \left[ -\partial \frac{\partial}{\partial q} \right] P^{A}(q)f, \tag{6.12}
\]
and to introduce the Fourier transform
\[
P^{A}(y)f \equiv \int d^d q \, e^{-qy} P^{A}(q)f, \]
\[
E^{A}(y)f \equiv \int d^d q \, e^{-qy} E^{A}(q)f = e^{-\partial y} P^{A}(y)f. \tag{6.13}
\]
Then we may write a functional derivative as
\[
\frac{\delta[f_{\Sigma}f]}{\delta \phi^{A}} = \int d^d y \, e^{\partial^i y} P^{A}(y)f = \int d^d y \, E^{A}(y)f \cdot e^{\partial^i y}. \tag{6.14}
\]
As a consequence of (5.10), we have the following Heisenberg group structure [3]:

\[
e^{\partial y} P_A(y_A) = P_A(y_A + y) e^{\partial y},
\]

\[
E_A(y_A) = E_A(y_A + y) e^{\partial y},
\]

\[
P_A(y_A)e^{\partial_1 y} = e^{\partial_1 y} P_A(y_A + y),
\]

\[
E_A(y_A)e^{\partial_1 y} = E_A(y_A + y).
\]

The \(x\)-pointwise boundary Poisson bracket (6.1) can be stated as

\[
\{ f, g \}_\infty = C^\infty \left[ \delta[f]_\Sigma \frac{\delta[j]}{\delta \phi^A} \omega^{AB} \frac{\delta[j]}{\delta \phi^B} \right]
\]

\[
= C^\infty \int d^d y_A d^d y_B d^d y E_A(y_A) f \omega^{AB} E_B(y_B) g T(y_A, y_B, y) e^{\partial_1 y_1}
\]

\[
= \int d^d y_A d^d y_B d^d y e^{\partial y} \left( E_A(y_A) f \omega^{AB} T(y_A, y_B, y) E_B(y_B) g \right). \tag{6.15}
\]

The Jacobi identity for this un-integrated, \(x\)-pointwise boundary Poisson bracket \(\{ f, g \}_\infty\) is satisfied. Here (and everywhere else in this paper), we are referring to the Jacobi identity in its strongest form, i.e. without any residual total derivative terms nor any total adjoint derivative terms leftover. It is proven in Appendix B. Interestingly, it continues to hold if we replace the smooth functions \(f, g\) and \(\omega^{AB}\) with general boundary distributions \(u, v\) and \(\omega^{AB}\) and take the last equation in (6.1) or in (6.16) as a definition:

\[
0 = \sum_{\text{cycl., } u, v, w} \{ u, \{ v, w \}_\infty \}_\infty, \quad u, v, w \in \mathcal{B}. \tag{6.17}
\]

### 7 Alternative \(\Delta^{(n)}1\) Basis

#### 7.1 Heuristic Remarks

Heuristically, the adjoint basis \((\partial^1)^k 1\) may be viewed as a renormalization/definition of what should be meant by the singular expression \((\rho_{1\Sigma})^{-1} (-\partial)^k (\rho_{1\Sigma})\), where

\[
1_\Sigma(x) \equiv \begin{cases} 
1 & \text{if } x \in \Sigma \\
0 & \text{otherwise}
\end{cases} \tag{7.1}
\]

is the characteristic function for the set \(\Sigma\) inside \(\mathbb{R}^d\).

In the same way, we shall now consider an alternative basis \(\Delta^{(n)}1\), which provides a rigorous meaning to \(\partial^n \ln(\rho_{1\Sigma})\). It will be convenient in the rigorous treatment to exclude \(n = 0\).

For the ultra-local product, we shall use this basis to argue that the parameter \(s\) itself has an interpretation as a renormalization of \(\ln(\rho_{1\Sigma})/\Gamma(0)\).
7.2 \( \Delta^{(n)} \) Operators

After the above heuristic introduction, we turn to a rigorous definition of the operators \( \Delta^{(n)} : \mathcal{B} \to \mathcal{B} \) for \( n \in \mathbb{N}_0 \setminus \{0\} \):

\[
\Delta(y)u = \sum_{n \neq 0} \frac{y^n}{n!} \Delta^{(n)}u = \int_0^1 \frac{d\alpha}{e^{-\mathbf{\partial}_1^\dagger \alpha y}} \frac{d}{d\alpha} e^{-\mathbf{\partial}_1^\dagger \alpha y} u \quad , \quad u \in \mathcal{B} ,
\]

\[
\Delta(y=0)u = 0 . \tag{7.2}
\]

The integral expressions should be understood by first Taylor expanding around \( y = 0 \) and then performing the \( \alpha \)-integration. Up to second order in \( y \), we get:

\[
\Delta^{(e_i)}u = -\mathbf{\partial}_i^\dagger u ,
\]

\[
\Delta^{(e_i+e_j)}u = \frac{1}{2} \left( \mathbf{\partial}_i^\dagger \mathbf{\partial}_j^\dagger u - \mathbf{\partial}_i^\dagger \mathbf{\partial}_j^\dagger u \right) + (i \leftrightarrow j) . \tag{7.3}
\]

Note that this definition relies heavily on the existence of the distribution product (3.2). For \( u \equiv 1 \) the definition (7.2) simplifies to

\[
\Delta(y)1 = \ln \left( e^{-\mathbf{\partial}_1^\dagger y} \right) . \tag{7.4}
\]

An equivalent definition is given by

\[
y \frac{\partial}{\partial y} \Delta(y)u = \sum_{n \neq 0} \frac{|n|}{n!} \Delta^{(n)}u = \frac{1}{e^{-\mathbf{\partial}_1^\dagger y}} y \frac{\partial}{\partial y} e^{-\mathbf{\partial}_1^\dagger u} , \quad u \in \mathcal{B} . \tag{7.5}
\]

7.3 Inverse Relations

Of particular interest is the case where the \( \Delta^{(n)} 1, n \neq 0 \), become linearly independent. Because of the triangular form of the \( \Delta^{(n)} 1, n \neq 0 \), this happens precisely when

\[
\forall n \neq 0 : \Delta^{(n)} 1 \neq 0 . \tag{7.6}
\]

Granted that this is the case, the inverse relations read:

\[
e^{\mathbf{\partial}_1^\dagger y} u = \sum_{k} \frac{y^k}{k!} \left( \mathbf{\partial}_1^\dagger \right)^k u = \int_0^1 d\alpha \ \exp \left( \Delta(-\alpha y) 1 \right) \cdot \frac{d}{d\alpha} \Delta(-\alpha y) u \\
= \int_0^1 d\alpha \ \exp \left( \Delta(-\alpha y) 1 \right) \cdot \frac{\partial}{\partial y} \Delta(-\alpha y) u , \quad u \in \mathcal{B} ,
\]

\[
\frac{\partial}{\partial y} e^{\mathbf{\partial}_1^\dagger y} u = \sum_{k} \frac{|k|}{k!} \frac{y^k}{\mathbf{\partial}_1^\dagger} \left( \mathbf{\partial}_1^\dagger \right)^k u = \exp \left( \Delta(-y) 1 \right) \cdot \frac{\partial}{\partial y} \Delta(-y) u ,
\]

\[
e^{\mathbf{\partial}_1^\dagger y} 1 = \exp \left( \Delta(-y) 1 \right) \equiv \sum_{k} y^k \ S^{(k)} \left( \{ \left( -\frac{1}{n!} \Delta^{(n)} 1 \}_{n \neq 0} \} \right) . \tag{7.7}
\]

In the last line we have indicated that the adjoint basis elements

\[
\frac{\left( \mathbf{\partial}_1^\dagger \right)^k 1}{k!} = S^{(k)} \left( \{ \left( -\frac{1}{n!} \Delta^{(n)} 1 \}_{n \neq 0} \} \right)
\]
\[
(-1)^k \sum_{\{m_n\}_{n\neq 0}} \prod_{n\neq 0} \frac{1}{m_n!} \left( \frac{1}{n!} \Delta^{(n)} 1 \right)^{m_n}, \quad k \in \mathbb{N}_0^d, \tag{7.8}
\]

up to a normalization, become the \(d\)-dimensional Schur polynomials of the alternative basis elements \((-1)^{n} \Delta^{(n)} 1, n \in \mathbb{N}_0^d \setminus \{0\}\).

### 7.4 Derivative of \(\Delta^{(n)}\) operators

The derivative \(\partial_i\) acts very simple on the \(\Delta^{(n)}\) operators, as the reader might have suspected from the heuristic arguments given in Section 7.1:

\[
\partial_i \Delta^{(n)} u = \Delta^{(n+e_i)} u ,
\]

\[
e^{\partial_i \bar{y}} \Delta(y)u = \Delta(y+\bar{y})u - \Delta(y)u . \tag{7.9}
\]

This follows straightforwardly from the definitions (3.7) and (7.2). Note that we can describe the derivative \(\partial_i\) in terms of the \(\Delta^{(n)}\) operators without using the distribution product (3.2). This is the key advantage of the \(\Delta^{(n)}\) operators. As one might expect, the action of the adjoint derivatives \(\partial^\dagger_i\) now becomes complicated:

\[
e^{\partial^\dagger_i \bar{y}} \Delta(y)u = \frac{\Delta(y-\bar{y})u - \Delta(-\bar{y})u}{\exp (\Delta(-\bar{y})1)} . \tag{7.10}
\]

### 7.5 Product in \(\Delta^{(n)} 1\) Basis

Starting from the \(\Delta^{(n)} 1\) basis, we can describe the distribution product as

\[
\Delta^{(k)} 1 \cdot \Delta^{(\ell)} 1 = \sum_{n \neq 0} \tilde{c}^{k,\ell}_n \Delta^{(n)} 1 , \tag{7.11}
\]

for some smooth structure functions \(\tilde{c}^{k,\ell}_n \in \mathcal{E}\). From this point of view we would have the same basic requirements, such as the commutativity and the associativity assumption and the Leibnitz rule. (The naturalness assumption is automatically fulfilled, because \(n \neq 0\).)

The product (3.2) is summed up in

\[
\tilde{T}(y_A, y_B, q) \equiv \sum_{k,\ell, n \neq 0} \frac{y_A^k}{k!} \frac{y_B^\ell}{\ell!} \tilde{c}^{k,\ell}_n q^n . \tag{7.12}
\]

It has Fourier transform

\[
\tilde{T}(y_A, y_B, y) \equiv \int d^d q \ e^{-qy} \tilde{T}(y_A, y_B, q) . \tag{7.13}
\]

We note for later that:

\[
\int d^d y \tilde{T}(y_A, y_B, y) = \tilde{T}(y_A, y_B, q=0) = 0 . \tag{7.14}
\]

The closeness (7.11) of the boundary algebra then reads

\[
\Delta(y_A) 1 \cdot \Delta(y_B) 1 = \int d^d y \tilde{T}(y_A, y_B, y) \Delta(y) 1 . \tag{7.15}
\]
Contrary to the adjoint basis, the Leibnitz condition is linear in the structure functions
\[
\partial_i \tilde{c}_n^{k,\ell} = \tilde{c}_n^{k+e_i,\ell} + \tilde{c}_n^{k,\ell+e_i} - \tilde{c}_n^{k,\ell},
\]
\[
e^{\partial y} \tilde{T}(y_A, y_B, y - \bar{y}) = \tilde{T}(y_A + \bar{y}, y_B + \bar{y}, y) - \tilde{T}(y_A + \bar{y}, \bar{y}, y) - \tilde{T}(\bar{y}, y_B + \bar{y}, y) + \tilde{T}(\bar{y}, \bar{y}, y).
\]

(7.16)

7.6 Ultra-Local Product

We have that \( \tilde{c}_n^{k,\ell} = \tilde{c}_n^{k,\ell} \delta^k \) is ultra-local in the \( \Delta(n)1 \) basis if and only if \( c_n^{k,\ell} = c_n^{k,\ell} \delta^k \) is ultra-local in the adjoint basis \( (\partial^\dagger)^k1 \). This is due to the homogeneity of the \( \Delta(n) \) operators of homogeneity weights \( n \in \mathbb{N}_0^d \). Here the homogeneity weights are wrt. a scaling of the adjoint differential operator \( \partial_i^\dagger \rightarrow \lambda_i \partial_i^\dagger, i = 1, \ldots, d \). Or vice-verse. Only the Leibnitz rule is slightly changed compared with the adjoint case in Section 4.1. It still splits into two conditions:
\[
\partial_i \tilde{c}_n^{k,\ell} = 0.
\]

(7.17)

The generic solutions are 2-coboundaries of locally constant invertible elements:
\[
\exists \tilde{b}^k \in \mathcal{C}^\times : \tilde{c}_n^{k,\ell} = \frac{\tilde{b}^k \tilde{b}^\ell}{\tilde{b}^{k+\ell}}.
\]

(7.18)

that satisfies the first of the Leibnitz conditions – below written slightly simplified by use of the exactness property:
\[
\frac{1}{\tilde{c}_n^{k,\ell}} + \frac{1}{\tilde{c}_n^{k,\ell+e_i}} = \frac{1}{\tilde{c}_n^{k+e_i,\ell}}.
\]

(7.19)

It follows that there exists constants \( s_{ij} = s_{ji}, i, j = 1, \ldots, d \), such that
\[
\frac{1}{\tilde{c}_n^{k,\ell+e_i}} = \sum_{j=1}^d k_j s_{ij}.
\]

(7.20)

A check of the product confirms that these constants \( s_{ij} \) coincide with the \( s_{ij} \) introduced in the adjoint case in Section 4.2. In fact, the ultra local solution \( (4.10) \) becomes
\[
\tilde{c}_n^{k,\ell} = \tilde{c}_n^{k,\ell} \delta_n^{k,\ell},
\]
\[
\tilde{c}_n^{k,\ell} = s B(i(k), i(\ell)),
\]
\[
\tilde{c}_n^{k,\ell} = \prod_{i, j \atop i \neq j} \delta_0^\min(k_i + e_i, k_j + e_j),
\]

(7.21)

i.e. all the diagonal parameter \( s \) enter multiplicatively. The arguments \( i(k) \) in the Euler Beta function are non-negative integers. Recall that the Euler Beta function has a pole located in zero. All other branches besides the 1-dimensional branch with the horizontal Young diagram \( (n_1 = d) \) are “infected” with this pole. This is an indirect manifestation of the fact that the generators \( \Delta(n)1, n \neq 0 \), on the other branches are not linearly independent.
7.7 Non-Ultra-Local Product

To deal with the non-ultra-local case, we invoke the following Ansatz: There exists an invertible matrix-valued smooth function $\Lambda^{k}_{a}$ and a fiducial solution $\tilde{c}^{a,b}_{(0)c}$ such that

$$
\tilde{c}^{k,\ell}_{n} = \sum_{a,b,c\neq 0} \Lambda^{k}_{a} \Lambda^\ell_{b} \tilde{c}^{a,b}_{(0)c} \Lambda^{c}_{n} .
$$

(7.22)

It would be obvious to use the ultra-local solution $\tilde{c}^{a,b}_{(0)c} = B(|a|, |b|) \delta^{a+b}_c$ as the fiducial solution, but in fact any solution will do.

Again we sum up the $\Lambda^{k}_{a}$ as

$$
\Lambda(y, q^{(0)}) = \sum_{k,a\neq 0} \frac{q^{(0)}_b}{k!} \Lambda^{k}_{a} q^{(0)a} ,
$$

(7.23)

and the Fourier transform

$$
\Lambda(y, y^{(0)}) = \int d^d q^{(0)} e^{-q^{(0)} y^{(0)}} \Lambda(y, q^{(0)}) .
$$

(7.24)

We note for later that:

$$
\int d^d y^{(0)} \Lambda(y, y^{(0)}) = \Lambda(y, q^{(0)} = 0) = 0 .
$$

(7.25)

The Ansatz (7.22) then reads:

$$
\tilde{T}(y_A, y_B, y) = \int d^d y^{(0)} A d^d y^{(0)} B d^d y^{(0)} \Lambda(y^{(0)} A, y^{(0)} B) \Lambda(y^{(0)} B, y^{(0)} A) \Lambda^{-1}(y^{(0)}, y) .
$$

(7.26)

The Ansatz (7.22) automatically satisfies the commutativity and the associativity requirements. The linear Leibnitz condition (7.16) is met by the following generic condition:

$$
\partial_i \Lambda^{k}_{a} = \Lambda^{k+\epsilon_i}_{a} - \Lambda^{k-\epsilon_i}_{a} ,
$$

$$
e^{(q^{(0)} + \partial^\ell) y^{(0)}} \Lambda(y, q^{(0)}) = \Lambda(y + \bar{y}, q^{(0)}) - \Lambda(y, q^{(0)}) ,
$$

$$
e^{\partial^{\bar{y}} y^{(0)} - \bar{y}} \Lambda(y, y^{(0)} - \bar{y}) = \Lambda(y + \bar{y}, y^{(0)}) - \Lambda(y, y^{(0)}) .
$$

(7.27)

or equivalently, in terms of the inverse matrix:

$$
\partial_i \Lambda^{a}_{k} = - \Lambda^{a}_{k} \partial_i A^{\ell}_{b} + \Lambda^{a+\epsilon_i}_{k} - \Lambda^{a-\epsilon_i}_{k} .
$$

(7.28)

The shift operators $(E^n)^k_{\ell} = \delta^{k+n}_\ell$ span a commutative and associative algebra $\bar{E}^n \bar{E}^m = \bar{E}^{n+m}$, with a trivial differential structure $[\partial_i, \bar{E}^n] = 0$. Hence, we can write the condition (7.27) as

$$
[\partial_i, \bar{E} \epsilon^i, \Lambda] = 0 .
$$

(7.29)

This system is clearly integrable with solution

$$
\Lambda(x + \bar{y}) = \exp \left( \sum_{i=1}^d \bar{y}^i \bar{E}^{\epsilon_i} \right) \Lambda(x) \exp \left( - \sum_{i=1}^d \bar{y}^i \bar{E}^{\epsilon_i} \right)
$$
\[
\sum_{k,a} \frac{\bar{y}^k}{k!} \bar{E}^k x \Lambda(x) \frac{(-\bar{y})^a}{a!} \bar{E}^a,
\]
\[
\Lambda(x+\bar{y})^k a = \sum_{\ell,b} \frac{\bar{y}^\ell}{\ell!} \Lambda(x)^{k+\ell}_{a-b} \frac{(-\bar{y})^b}{b!},
\]  
(7.30)

where \( x \) is a reference point. This can of course also be derived by expanding the second expression in (7.27).

8 Conclusions

When imposing the Jacobi identity for a boundary Poisson bracket, an infinite tower of conditions emerges. (Plus an expected standard Jacobi condition on the Poisson bi-vector \( \omega^{AB} \).) In the ultra-local case, this tower was already known [3]. We have demonstrated that there is a well-defined distribution product behind the boundary Poisson bracket construction. We have shown that the tower of conditions on the boundary Poisson bracket can really be viewed as conditions on this distribution product instead. As a result, we have simplified the boundary calculus of Hamiltonian field theory considerably. We hope that these developments will stimulate further interests in the physical systems with a boundary.

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A Proof of the Leibnitz Identity (5.20)

Consider the following \((A \leftrightarrow D, B \leftrightarrow C)\) symmetric object:

\[
ed^y_B e^{\partial_1 y_A} e^{\partial_1 y_C} e^{\partial_1 y_D}
\]

\[
= e^{-\partial_1 y_C} \left[ e^{\partial(y_B-y_C)} e^{\partial_1 y_A} e^{\partial_1 y_C} e^{\partial_1 y_D} \right]
\]

\[
= e^{-\partial_1 y_C} \left[ e^{\partial(y_B-y_C)} e^{\partial_1 y_A} \int d^d \tilde{y} T(y_C, y_D, \tilde{y} + y_C) e^{\partial_1 (\tilde{y} + y_C)} \right]
\]

\[
= \int d^d \tilde{y} e^{-\partial_1 y_B} \left[ e^{\partial_1 y_A} \int d^d \tilde{y} T(y_C, y_D, \tilde{y} + y_C) e^{\partial_1 (\tilde{y} + y_C)} \right]
\]

\[
= \int d^d \bar{y} e^{-\partial_1 y_B} \left[ \int d^d y T(y_A, \bar{y} + y_B, y + y_B) e^{\partial_1 (y + y_B)} e^{\partial(y_C - y_B)} T(y_C, y_D, \bar{y} + y_C) \right]
\]

\[
= \int d^d \bar{y} d^d \bar{y} e^{\partial y_B} T(y_A, \bar{y} + y_B, y + y_B) e^{\partial y_C} T(y_C, y_D, \bar{y} + y_C) e^{\partial_1 y_1}. \quad (A.1)
\]

Here we have applied the Leibnitz rule three times plus various definitions. The desired Leibnitz identity (5.20) now follows from the unique decomposition of a boundary distribution into its \( C^\infty \) coefficient functions, cf. (2.13).
B Mono-Local Proof of the Jacobi Identity (6.17)

Consider three boundary distributions \( u, v \) and \( w \). When calculating a “double bracket”, using the last formula in (6.10), we get three terms:

\[
\{ u, \{ v, w \}_\infty \}_\infty = T_1(u, v, w) + T_2(u, v, w) - T_1(u, w, v).
\]

(B.1)

Here the first term \( T_1 \) is

\[
T_1(u, v, w) \equiv \int d^6y \, e^{\partial y} \left[ E_A(y_A) u \, T(y_A, y_B, y) \, \omega^{AB} e^{-\partial y_B} P_B(y_B) e^{\partial (\tilde{y} - y_C)} P_C(y_C) v \right. \\
\times e^{\partial (\tilde{y} - y_B)} \left( \omega^{CD} T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right) \\
= \int d^6y \, e^{\partial y} \left[ \omega^{AB} e^{\partial (\tilde{y} - y_B)} \left( \omega^{CD} T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right) \right]
\]

(B.2)

We have used the following shorthand notation for the integration measure

\[
d^6y \equiv d^d y_A \, d^d y_B \, d^d y_C \, d^d y_D \, d^d \tilde{y},
\]

(B.3)

and we have performed the following change of integration variables

\[
\begin{pmatrix}
  y'_A \\
y'_B \\
y' \\
y'_C \\
y'_D \\
\tilde{y}'
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 1 & 0 & -1 \\
  0 & -1 & 1 & -1 & 0 & 1 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  y_A \\
y_B \\
y \\
y_C \\
y_D \\
\tilde{y}
\end{pmatrix},
\]

(B.4)

which has the Jacobian equal to 1. The full Jacobi identity contains six \( T_1 \) terms. It is easy to see that they cancel by symmetry.

The second term \( T_2 \) in (B.3) is

\[
T_2(u, v, w) \equiv \int d^6y \, e^{\partial y} \left[ E_A(y_A) u \, T(y_A, y_B, y) \, \omega^{AB} e^{-\partial y_B} P_B(y_B) e^{\partial (\tilde{y} - y_C)} P_C(y_C) v \right. \\
\times e^{\partial (\tilde{y} - y_B)} \left( \omega^{CD} T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right) \\
= \int d^6y \, e^{\partial y} \left[ \omega^{AB} e^{\partial (\tilde{y} - y_B)} \left( \omega^{CD} T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right) \right]
\]
\[
= \int d^5 y \, e^{\partial y} \left[ E_A(y_A) u \, T(y_A, y_B + \tilde{y}, y) \, \omega^{AB} \right.
\times e^{-\partial y_B} \left( P_B(y_B) \omega^{CD} E_C(y_C) v \, T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right)
\]
\[
= \int d^5 y \, e^{\partial y} \left[ E_A(y_A) u \, T(y_A, \tilde{y} + y_B, y) \, \omega^{AB} \right.
\times e^{-\partial y_B} \left( \delta^d (y_B) P_B(0) \omega^{CD} E_C(y_C) v \, T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right)
\]
\[
= \int d^5 y \, e^{\partial y} \left[ E_A(y_A) u \, T(y_A, \tilde{y}, y) \, \omega^{AB} \right.
\times P_B(0) \omega^{CD} E_C(y_C) v \, T(y_C, y_D, \tilde{y}) \, E_D(y_D) w \right].
\]

(B.5)

In the next-to-last step we used the assumption that \( \omega^{CD} \) does not depend on the derivatives \( \partial^k \phi^A(x), \ k \neq 0 \). It is now easy to see that in the full Jacobi identity, the three \( T_2 \) terms cancel by use of the Jacobi identity (5.2) for \( \omega^{AB} \) and the associativity (5.19) of the product \( T(y_A, y_B, y) \).

\[ \square \]

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