Copulas based seemingly unrelated quantile regression

Roengchai Tansuchat, Paravee Maneejuk, Woraphon Yamaka\(^1\), and Songsak Sriboonchitta

Center of Excellent in Econometrics, Faculty of Economics, Chiang Mai University, Chiang Mai, Thailand

\(^1\)Email: woraphon.econ@gmail.com

Abstract. We propose a multivariate copulas based seemingly unrelated quantile regression. We add the multivariate copula density function into the likelihood to relax the strong assumption of multivariate normal distribution of the conventional model. The simulation study is conducted to evaluate the performance of our proposed model. Moreover, we apply our proposed model to the Fama-French equation in order to investigate the systematic risk in the three major stocks in NASDAQ market. The results of this study suggest that our proposed model provides a particularly good description of these stock prices at every quantile level.

1. Introduction

After [1] introduced multivariate regressions, seemingly unrelated regression (SUR) model has become popular in both statistics and econometrics. Particularly, it has many applications in economics such as in the works by [2], [3], and [4]. The seemingly unrelated regression (SUR) has been extended to seemingly unrelated quantile regression (SUQR) as introduced in [5] and [6]. The model has become more robust against outliers in the response measurements and can explain the entire conditional distribution of the outcome variable. Consider the structure of SUQR model with \( n \) equations, the typical setup of SUQR

\[
y_{i,t} = x'_{i,t} \beta_{i,t} + \varepsilon_{i,t}, \quad i = 1, \ldots, n; \quad t = 1, \ldots, T,
\]

where \( x'_{i,t} \beta_{i,t} \) is conditional mean of \( y_{i,t} \) given the \( T \times k \) matrix of \( k \) regressors \( x'_{i,t} \) and \( \varepsilon_{i,t} \) is the error term and is i.i.d. unobservable \( n \)-dimensional random vectors where the distribution depends on the \( \alpha \)th \((0 < \alpha < 1)\) quantile. Thus, \( \alpha \)th conditional quantile of \( y_{i,t} \) given \( x'_{i,t} \) is simply

\[
Q_{\alpha}(y_{i,t} | x'_{i,t}) = x'_{i,t} \beta_{i,t}.
\]

The model takes an advantage of conventional SUR model where the errors are allow to be correlated across equations to gain more efficiency. Thus the vector of error term \( \varepsilon_{i,t} \) is i.i.d over time and.

\[
E[\varepsilon_{i,t} \varepsilon'_{i,t} | x_{i,t}] = \Sigma.
\]

These errors are assumed to join by a multivariate distribution, in particular, the multivariate normal distribution. However, this relationship of this multivariate distribution is linear which might fail to capture the non-linear relationship in the error structure [4]. To solve this drawback, the copula approach is considered as it can capture the nonlinear dependence structure in the multivariate SUQR model.
Similar to our model, the study of [7] extended the copula approach to bivariate seemingly unrelated Tobit model by modeling its dependence structure through copulas. However, this study aims to extend the conditional mean model into conditional quantile model and propose a multivariate copula-based SUQR model. Thus, the model becomes more flexible and can explain the entire conditional distribution of the predictor variable and becomes more robust to outliers.

In this paper, we also propose a Bayesian estimation to implement the multivariate copula-based SUQR model. This estimation provides the entire posterior distribution of a large number of parameters which have a complex function. In addition, it takes into account the uncertainty parameter when making a prediction. To the best of our knowledge, there are no studies on SUQR from a copula-based perspective. Thus, the main contribution of this paper is to propose an alternative method for drawing inferences about conditional quantiles in multiple regression problems via Bayesian estimation. The simulation study is also proposed to evaluate the performance and accuracy of our model. We, then, apply our proposed model real data study.

The remainder of the paper is organized as follows. Section 2 describes the econometric models considered in this study, Section 3 presents posterior estimation, and Section 4 presents the simulation study. The application of our proposed model is reported in Section 5. Finally, Section 6 summarizes and presents the conclusions of this paper.

2. Methodology

2.1. Copulas

The general Sklar’s theorem [8] is this. Let $H$ be an n-dimensional distribution with marginals $F_i$, $i=1,2,...,n$. Then there exists an $n$-copula $C$ such that for all $x_1,...,x_n$ in $\overline{R}$,

$$H(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n)),$$

where $C$ is copula function of a $n$-dimensional random variables. If the marginals are continuous, then the copula $C$ is unique. Otherwise, $C$ is uniquely determined on $\prod_{i=1}^{n} R(F_i)$ where $R(F_i)$ is denoted as the range of the function $F_i$. Conversely, if $C$ is an $n$-copula $C$ and $F_i$, $i=1,2,...,n$ are univariate distributions, then the function $H : \overline{R}^n \rightarrow [0,1]$ is also defined in equation (4). We can model the marginal distribution and joint dependence separately. Then, the copula can be given by

$$C(u_1,...,u_n) = C(F_1^{-1}(u_1),...,F_n^{-1}(u_n))$$

where $u$ is uniform $[0,1]$.

2.2. Posterior density of multivariate copula families

In this study, we consider an Elliptical copula consisting a symmetric Gaussian and t- copulas to join the $\mathcal{E}_{ij}$ of our proposed model. According to Bayes’ rule, we can construct the posterior density of the copula function by multiplying the prior distribution ($p(C)$) with the likelihood function of the copula function.

Following [9] and [10], we can impose the prior distribution on copula density to obtain the posterior densities of the Elliptical copulas as in the following:

- Posterior Gaussian copulas density

  In the general case of $n$-dimensional the posterior Gaussian or Normal copula is

  $$f(R_{u,i}, \Phi_i^{(u_{ai}))}) = p(R_{u,i}) \cdot (\text{det} R_{u,i})^{-1} \cdot \frac{1}{2} \left( \Phi_i^{(u_{ai}))} \right) \cdot \left( R_{u,i}^{-1} - I \right) \cdot \left( \Phi_i^{(u_{ai}))} \right)$$

(6)
where $\Phi$ is $n$-dimensional standard normal cumulative distribution, $R_{\alpha,\sigma}$ is matrix dependecne of Gaussian copula at quantile $\alpha$, and $p(R_{\alpha,\sigma})$ is the prior density of dependencecne parameter, which is assumed to be uniform distribution $[-1, 1]$. To draw the updated dependencecne parameters, we random these parameters from the proposal trancate normal distribution $[-1, 1]$ interval.

- Posterior Student-t copulas density
  
  In the case of $n$-dimensional the posterior student-t copula density is evaluated by
  
  $$f(R_{\alpha,T}, t_v, (u_{a,i})) = p(R_{\alpha,T}) \cdot p(v) \cdot f_{v,\alpha}(t_v^{-1}(u_{a,i})) \cdot \frac{\prod_{i=1}^{n} f_{v,\alpha}(t_v^{-1}(u_{a,i}))}{\prod_{i=1}^{n} f_{v,\alpha}(t_v^{-1}(u_{a,i}))},$$  

  where $f_{v,\alpha}$ is the joint density of a $t_v(v, 0, R)$-distributed random vector where $R_{\alpha,T}$ is the correlation matrix. The prior density for $v$ and $R_{\alpha,T}$ are exponential distribution and uniform $[-1, 1]$, respectively. To draw the updated dependence parameters and degree of freedom ($v$), we random $v$ and $R_{\alpha,T}$ from the proposal trancate normal distribution $[-1, 1]$ interval and distribution $[0, \infty]$ interval.

3. Posterior estimation of SUQR

In this paper, we adopt a Bayesian approach to estimate the parameter sets in equation (1). Thus, the full posterior density of Multivariate copula based SUQR can be formed by multiplying the Asymmetric Laplace density with copula density and prior density. First of all, according to [11], the multivariate density of Asymmetric Laplace distribution is

$$L_{\alpha} = \alpha^T (1-\alpha)^T \frac{1}{\sigma^2} \exp \left( -\frac{\sum_{i=1}^{T} (1-\alpha)(y_{i,t} - x_{i,t} \beta_{a,i})}{\sigma^2} \right) \quad \text{if } y_{i,t} < x_{i,t} \beta_{a,i}$$

$$L_{\alpha} = \alpha^T (1-\alpha)^T \frac{1}{\sigma^2} \exp \left( -\frac{\sum_{i=1}^{T} (-\alpha)(y_{i,t} - x_{i,t} \beta_{a,i})}{\sigma^2} \right) \quad \text{if } y_{i,t} \geq x_{i,t} \beta_{a,i},$$

then the improper uniform prior proposed by [11] are also employed in this study, thus the posterior density for $\beta_{a,i}$ become

$$P_{\alpha}(\beta_{a,i}, \sigma_{a,i}^2) \bigg| y_{i,t}, x_{i,t} = \frac{L_{\alpha} \cdot P_{\alpha}(\beta_{a,i}, \sigma_{a,i}^2)}{\int L_{\alpha} \cdot P_{\alpha}(\beta_{a,i}, \sigma_{a,i}^2) \, d\beta_{a,i}}.$$  

This study takes an advantage of the copula function for inclusion in SUQR model, thus the full posterior density of this model can be written as:

$$P(\Gamma_{\alpha}, y_{i,t}, x_{i,t}) = L_{\alpha} \cdot P_{\alpha}(\beta_{a,i}, \sigma_{a,i}^2) \cdot f_{v,\alpha}(R_{\alpha,T}, \Phi_{a,i}(u_{a,i})) \sim \text{Gaussian Copula}$$

$$P(\Gamma_{\alpha}, y_{i,t}, x_{i,t}) = L_{\alpha} \cdot P_{\alpha}(\beta_{a,i}, \sigma_{a,i}^2) \cdot f_{v,\alpha}(R_{\alpha,T}, t_v, (u_{a,i}), v) \sim \text{Student-t Copula}.$$  

To estimate these parameters, the Metropolis-Hasting algorithm is employed to generate samples from a probability distribution, using joint density function equation (10). In the first step, we establish a starting value for each of the posterior parameters obtained from maximum likelihood method. In second step, we sample candidate parameters $\Gamma_{\alpha} = \Gamma_{\alpha}^{j-1} + N(0, c)$ from the proposal function where $c$ is variance of each parameter. Then, the deference between posterior distribution-based candidate parameters and posterior distribution based previously updated parameters ($\Gamma_{\alpha}^{j-1}$) is computed and compared with the random value of log(uniform[0,1]). Therefore, if

$$P(\Gamma_{\alpha} \big| y_{i,t}, x_{i,t}) - P(\Gamma_{\alpha}^{j-1} | y_{i,t}, x_{i,t}) < \log(\text{uniform}[0,1]) \ ,$$

$$\Gamma_{\alpha}^{j} = \Gamma_{\alpha}^{j-1},$$

$$\text{else} \quad \Gamma_{\alpha}^{j} = \Gamma_{\alpha}^{j}.$$
We run the chain for 10,000 iterations. We discard the first 2,000 iterations as burn-in. The remaining 8,000 parameter set $\Gamma_{\alpha}^{2001}, \ldots, \Gamma_{\alpha}^{10000}$ are divided by 8,000 to obtain the posterior mean parameters.

In the context of Bayesian inference in model selection problem where we wish to employ the copula approach, we face the practical problem choosing an appropriate copula among a finite collection of different families of copula. Since choosing copula is for the purpose of modeling the dependence structure of the SUQR model, we then consider the Deviance Information Criterion (DIC) for comparing various model specifications and the one with the lowest DIC is preferred. For existing literature on the Metropolis-Hasting algorithm and SUQR model, readers can refer to the work by [6, 12].

4. Experiment study

In this section, we employ a simulation study to investigate the accuracy of the estimation in SUQR model, we considered only the Gaussian to model the dependence structure of the SUQR. The simulation is based on SUQR with three equations:

$$
y_{1,t} = \beta_{11,\alpha} + x_{1,t}' \beta_{12,\alpha} + \epsilon_{1,t} 
$$

$$
y_{2,t} = \beta_{21,\alpha} + x_{2,t}' \beta_{22,\alpha} + \epsilon_{2,t} 
$$

$$
y_{3,t} = \beta_{31,\alpha} + x_{3,t}' \beta_{32,\alpha} + \epsilon_{3,t} 
$$

(13)

| Parameter   | True $\alpha = 0.25$ | True $\alpha = 0.50$ | True $\alpha = 0.75$ |
|-------------|-----------------------|-----------------------|-----------------------|
| $\beta_{11,\alpha}$ | 1 (0.114) | 1 (0.058) | 1 (0.052) |
| $\beta_{12,\alpha}$ | 1 (0.105) | 5 (0.036) | 2 (0.045) |
| $\sigma_{1,\alpha}^2$ | 1 (0.100) | 1 (0.053) | 1 (0.060) |
| $\beta_{12,\alpha}$ | 4 (0.092) | 8 (0.395) | 4 (0.210) |
| $\beta_{22,\alpha}$ | -3 (0.101) | -2 (0.152) | -0.2 (0.070) |
| $\sigma_{2,\alpha}^2$ | 2 (0.204) | 2 (0.088) | 2 (0.051) |
| $\beta_{13,\alpha}$ | 2 (0.101) | 5 (0.088) | 1 (0.089) |
| $\beta_{32,\alpha}$ | 3 (0.101) | 2 (0.196) | 3 (0.162) |
| $\sigma_{3,\alpha}^2$ | 3 (0.310) | 3 (0.302) | 3 (0.313) |
| $R_{\alpha 12}$ | 0.5 (0.076) | 0.5 (0.072) | 0.5 (0.085) |
| $R_{\alpha 13}$ | 0.5 (0.067) | 0.5 (0.078) | 0.5 (0.081) |
| $R_{\alpha 23}$ | 0.5 (0.079) | 0.5 (0.058) | 0.5 (0.079) |

Note that the error terms, $\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t}$, are assumed to be asymmetric Laplace distribution with skew parameter $\alpha = (0.25, 0.5, 0.75)$. We simulated 100 samples for the Monte Carlo experiment using
the specified parameters of each case. For each simulation case, we assess the performance of our proposed SQUJR model and compare our results with the true values.

The estimation results from Table 1 shows the estimated posterior mean and we observed that our proposed model and method produce the unbiased parameter estimates. The estimated posterior parameter means are close to the true values and the standard deviation from the mean of each parameter is reasonable. The acceptance rates, which are not reported in the table, are in the acceptable range from 20% to 40%. Thus, it represents an optimal mixing between the proposal distribution and posterior distribution.

5. Empirical result
This study employs the Fama-French model to quantify the systematic risk of stock in the market. We extend the multivariate regression Fama and French Three Factor Model (TFM) of [13] to the multivariate Quantile regression or SUQR. Therefore, our model presents in the simple form as

\[ R_{ij} - R_{fi} = \beta_{a_i} (R_{Mi} - R_{fi}) + \phi_{a_i} SMB_i + \varphi_{a_i} HML_i + \varepsilon_{ij}, \quad i = 1, \ldots, n; \quad t = 1, \ldots, T \]  (14)

where \( R_{ij} \) is return of asset \( i \), \( R_{fi} \) is risk free rate, \( R_{Mt} \) is return of the market, \( SMB_i \) is Small Minus Big (proxy for company size), \( HML_i \) is High Minus Low (proxy for book-to-market values), \( \beta_{a_i} \) is \( \text{cov}(R_i, R_M) / \sigma_M^2 \) and \( \phi_{a_i}, \varphi_{a_i} \) are the level of exposure to size risk and value risk, respectively.

5.1. Data
The data for this study contains 177 monthly returns of 3 stocks in NASDAQ stock market, namely MICROSOFT, ADOBE, and APPLE, for January 2001-October 2015 period, along with the Fama-French factors for the same period. The stock data were obtained fromDataStream and Fama-French factors were obtained from French’s website to calculate the Fama-French coefficients. In this paper, we use Treasury bills as a proxy of the risk-free rate.

5.2. Model comparison
In this section, we compare the proposed model with the Bayesian multivariate Gaussian models on multivariate quantile regression of [6] in the real data analysis. Three Fama-French equations model in different quantile level (\( \alpha = 0.25, 0.50, 0.75 \)) are considered in this application study. The comparison of models is based on Deviance information criterion (DIC) which is the measure of model fit. It is useful in Bayesian model selection problems where the posterior distributions of the models have been obtained by posterior samples. Among the trial runs of several models, the results in Table 2 provides an evidence that Gaussian and Student-\( t \) present a lowest DIC for quantile level at \( \alpha = 0.25, 0.50, 0.75 \) as they show the lowest DIC. This result confirms that the copula-based model outperforms the conventional model.

Table 2. Model selection.

| \( \alpha \) | DIC | Gaussian | Student-\( t \) | Conventional model |
|------------|-----|----------|-----------------|-------------------|
| \( 0.25 \) | -4728.48 | -3602.621 | -5215.135         |
| \( 0.50 \) | -2501.287 | -8515.28 | -9458.117         |
| \( 0.75 \) | -3074.949 | -2982.581 | -4258.548         |

5.3. Empirical evidence
An evaluation is to be made on the factors affecting the returns of MICROSOFT, APPLE, and ADOBE stocks which are traded in the NASDAQ stock market. Table 3 presents the estimated posterior mean parameters with different quantiles from 0.25 to 0.75 for the stock prices which are obtained from Bayesian estimation. Let consider the first stock, the relationship between excess
returns and expected risk or expected volatility of APPLE evolves from negative to positive as the quantile increases. At the lower quantile ($\alpha = 0.25$), the coefficient $R_{\text{NASDAQ,}\alpha=0.25}$ is 1.0604, and this indicates that low quantile of APPLE return has the high expected risk and APPLE price will be more volatile than the NASDAQ market return. For $\alpha = 0.5$, the estimated mean coefficient for $R_{\text{NASDAQ,\alpha=0.5}}$ is 0.4232. This indicates that median quantile of APPLE return has the low expected risk and APPLE price will be less volatile than the NASDAQ market return. For $\alpha = 0.75$, the estimated mean coefficient for $R_{\text{NASDAQ,\alpha=0.75}}$ is 0.6865. This indicates that median quantile of APPLE return has the low expected risk and APPLE price will be less volatile than the NASDAQ market return but it has high expected risk when compared with $\alpha = 0.5$. For the other two factors, SML and HML, we observed that the level of exposure to size risk ($SMB_{\text{APPLE,\alpha}}$) and the level of exposure to value risk ($HML_{\text{APPLE,\alpha}}$) present a small negative effect to APPLE return at every quantile level. By the same token, similar interpretation can be made for the results regarding MICROSOFT and ADOBE returns.

| Case          | $\alpha =0.25$ | $\alpha =0.5$ | $\alpha =0.75$ |
|---------------|----------------|----------------|----------------|
| $\beta_{\text{APPLE,}\alpha}$ | -0.0058 (0.0048) | -0.0005 (0.000001) | 0.1848 (0.0003) |
| $R_{\text{NASDAQ,}\alpha}$ | 1.0604 (0.0151) | 0.4232 (0.007) | 0.6865 (0.0006) |
| $SMB_{\text{APPLE,\alpha}}$ | -0.0012 (0.0016) | -0.0001 (0.0001) | -0.0059 (0.000001) |
| $HML_{\text{APPLE,\alpha}}$ | -0.0006 (0.0025) | -0.0001 (0.0001) | -0.0007 (0.000001) |
| $\beta_{\text{MICRO,\alpha}}$ | -0.0134 (0.0040) | -0.0005 (0.000001) | 0.0330 (0.00078) |
| $R_{\text{NASDAQ,\alpha}}$ | 0.9678 (0.0051) | 0.8000 (0.002) | 0.5544 (0.0051) |
| $SMB_{\text{MICRO,\alpha}}$ | -0.0024 (0.0016) | -0.0018 (0.0001) | -0.0153 (0.0001) |
| $HML_{\text{MICRO,\alpha}}$ | -0.0046 (0.0024) | -0.000001 (0.000001) | -0.0002 (0.000001) |
| $\beta_{\text{ADOBE,\alpha}}$ | -0.0145 (0.0044) | 0.0012 (0.0001) | 0.0328 (0.0071) |
| $R_{\text{NASDAQ,\alpha}}$ | 1.4306 (1.0251) | 1.1533 (0.0053) | 0.9383 (0.1442) |
| $SMB_{\text{ADOBE,\alpha}}$ | -0.0010 (0.0018) | -0.0032 (0.000001) | -0.0465 (0.0049) |
| $HML_{\text{ADOBE,\alpha}}$ | -0.0065 (0.0023) | -0.00038 (0.000001) | -0.0007 (0.000004) |
| Copula Parameter | 0.9754 (0.003) | 0.0001 (0.0732) | 12.625 (0.6812) |
| Copula Parameter | 0.9710 (0.0037) | 0.0029 (0.0001) | |
| Copula Parameter | 0.9791 (0.0013) | 0.0325 (0.000001) | |

On copula parameter, different quantile level presents a different copula function. It is noticeable that the degree of dependence among these stocks is high at the extreme quantile. This indicates a
strong positive relationship among these stocks in the light of extreme event such as bull or bear market. Based on this result, we would suggest that it is not a good strategy to diverse the investment into these three stocks in the bear market.

6. Conclusions
This paper proposes a copula based seemingly unrelated quantile regression (SUQR). We apply our proposed model to quantify and measure the risk of the stock price through the Fama-French model with three-factor analysis. The proposed model is flexible and has a potential to capture the extreme market condition. The study adopts a Bayesian method under Asymmetric Laplace distribution (ALD), together with a Metropolis Hasting algorithm in the parameter sampler. The experiment study is conducted to investigate the performance of our proposed model. In the simulations, we show the accuracy of our proposed model against the simulation data. The results confirm that our proposed model is well estimated unknown parameters for every quantile levels and the estimated mean parameters are close to the true parameter values.

We investigate three major US stocks, namely MICROSOFT, ADOBE, and APPLE and the empirical evidence shows that the expected risk behaves differently across various quantile levels. The expected risk of APPLE evolves from negative to positive as the quantile increases. We also observe that SMB and HML coefficients are negative for all stock prices.

References
[1] Zellner A 1962 An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias Journal of the American statistical Association 57(298) 348-368
[2] White E N and Hewings G J 1982 Space time employment modeling; some results using seemingly unrelated regression estimators Journal of Regional Science 22(3) 283-302.
[3] Adelegan J O 2000 Foreign direct investment and economic growth in Nigeria: A seemingly unrelated model African Review of Money Finance and Banking 5-25
[4] Pastpipatkul P, Maneejuk P, Wiboopongse A and Sriboonchitta S 2016 Seemingly unrelated regression based copula: an application on Thai rice market In Causal Inference in Econometrics Springer Cham. 437-450
[5] Jun S J and Pinkse J 2009 Efficient semiparametric seemingly unrelated quantile regression estimation Econometric Theory 25(5) 1392-1414
[6] Waldmann E and Kneib T 2015 Bayesian bivariate quantile regression Statistical Modelling 15(4) 326-344.
[7] Wichitakorn N and Choya S B 2012 Modeling Dependence of Seemingly Unrelated Tobit Model through Copula: A Bayesian Analysis
[8] Sklar A 1959 Fonctions de répartition à n dimensions et leurs marges Publ. Inst. Statist. Univ. Paris 8 229-231
[9] Smith M S 2011 Bayesian approaches to copula modelling Available at SSRN 1974297
[10] Smith M S, Gan Q and Kohn R J 2012 Modelling dependence using skew t copulas: Bayesian inference and applications Journal of Applied Econometrics 27(3) 500-522
[11] Yu K and Moyeed R A 2001 Bayesian quantile regression. Statistics & Probability Letters 54(4) 437-447
[12] Fama E F and K R French 1996 Multifactor explanations of asset pricing anomalies Journal of Finance 51 55–84
[13] Min A and Czado C 2010 Bayesian inference for multivariate copulas using pair-copula constructions Journal of Financial Econometrics 8(4) 511-546