Relative Bott–Samelson Varieties

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Abstract

We prove that, defined with respect to versal flags, the product of two relative Bott–Samelson varieties over a flag bundle is a resolution of singularities of a relative Richardson variety. This result generalizes Brion’s resolution of singularities of Richardson varieties to the relative setting. As an application, this gives a resolution of singularities, with a modular interpretation, for the Brill–Noether variety with imposed ramification on twice-marked elliptic curves.

Keywords Bott–Samelson variety · Relative Richardson variety · Brill–Noether variety · Resolution of singularities

1 Introduction

A Bott–Samelson variety of type $A$, is an iterated tower of $\mathbb{P}^1$-bundles, or Grassmann bundles. It can serve as a resolution of singularities for a Schubert variety. For example, fix a line in $\mathbb{P}^3$ and consider the Schubert variety in the Grassmannian $\text{Gr}(2, 4)$ that parametrizes lines intersecting the fixed line. The subvariety of the product of Grassmannians $\text{Gr}(1, 4) \times \text{Gr}(2, 4)$, that parametrizes pairs $(p, L)$, where $L$ intersects the fixed line at $p$, is a Bott–Samelson resolution of singularities for the Schubert variety: It is smooth, with a proper birational map to the Schubert variety, which forgets the point $p$ in each pair $(p, L)$.

Furthermore, the intersection of two flag Schubert varieties defined with respect to transverse flags is a Richardson variety and is well-known to have rational singularities. Brion [3] showed that, given a Richardson variety, the product of the Bott–Samelson resolutions for the two intersecting flag Schubert varieties over the complete flag varieties is a resolution of singularities for the Richardson variety.

We are interested in generalizing Brion’s result to the relative setting. This is motivated by the question of how to resolve the singularities of a Brill–Noether variety.
with imposed ramification, studied in [4, 5, 14]. In recent work [6], Chan and Pflueger showed that any local property of the intersection of two relative Schubert varieties is completely controlled by that of the relative Schubert varieties, under the additional condition that the defining families of flags are \textit{versal}. Versality generalizes transversality: in the case when the base scheme is a point, versality recovers transversality. In the case when the base scheme is a 1-dimensional scheme, versality implies that the flags are transverse in most fibers but become almost transverse (i.e., exactly one pair of complementary dimensions of the two flags has a 1-dimensional intersection and the remaining pairs intersect trivially; see [4]) over finitely many reduced points. Their result generalizes [12].

Our primary result is a Bott–Samelson type resolution of singularities for a relative Richardson variety. Throughout, let $k$ be an algebraically closed field of characteristic 0. We state the main result as follows, postponing relevant definitions.

\textbf{Theorem 1.1} Let $n > 0$. Let $S$ be a smooth finite-type $k$-scheme that carries a rank-$n$ vector bundle $\mathcal{H}$. Let $\sigma, \tau \in S_n$. Let $\rho_\sigma$ and $\rho_\tau$ be reduced decompositions for $\sigma$ and $\tau$ respectively. Let $p, q : S \to \text{Fl}(\mathcal{H})$ be versal sections of the associated flag bundle of $\mathcal{H}$. Let $X_\sigma(p)$ and $X_\tau(q)$ be the relative Schubert varieties defined with respect to $p$ and $q$. Let $Z_{\rho_\sigma}(p)$ and $Z_{\rho_\tau}(q)$ be the relative Bott–Samelson varieties defined with respect to $p$ and $q$. The fiber product

$$Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$$

(i) is smooth;

(ii) and the projection to the relative Richardson variety

$$Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q) \to X_\sigma(p) \cap X_\tau(q)$$

is proper and birational.

Therefore, $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$ is a resolution of singularities of the relative Richardson variety.

The theorem reflects the nature that local geometric properties of relative Richardson varieties factor. It also generalizes Brion’s result without explicitly invoking Kleiman’s transversality theorem, a key tool used in [3].

Our second result is an application to Brill–Noether theory. We refer the readers to [1] for a beautiful treatment on Brill–Noether theory, and to [4, 6] on details of such varieties with imposed ramification. To describe briefly, let $d > r \geq 0$, and let $(E; P, Q)$ be a twice-marked elliptic curve where $P - Q$ is not a torsion point of order $d$ in $\text{Pic}^0(E)$. Denote by $a_\bullet$ and $b_\bullet$ sequences $0 \leq a_1 < \cdots < a_r < a_{r+1} \leq d$ and $0 \leq b_1 < \cdots < b_r < b_{r+1} \leq d$. The moduli space of linear series of projective rank $r$ with imposed ramification and of degree $d$ at $P, Q$ prescribed by $a_\bullet, b_\bullet$ respectively is the subscheme of the classical Brill–Noether variety $G^r_d(E)$

$$G^r_d(E, (P, a_\bullet), (Q, b_\bullet)) \subseteq G^r_d(E),$$
and is the intersection of two relative Grassmann Schubert varieties $G^r_d(E, (P, a_\bullet))$, $G^r_d(E, (Q, b_\bullet))$ defined with respect to versal flags. Let $\lambda = (\lambda_i)$ and $\lambda' = (\lambda'_i)$ be partitions where

$$\lambda_i = a_{r+1-(i-1)} - (r + 1 - i),$$

$$\lambda'_i = b_{r+1-(i'-1)} - (r + 1 - i'),$$

for $i, i' \in [r + 1]$. Let $Z_\lambda(p)$ and $Z_{\lambda'}(q)$ be the relative Bott–Samelson resolutions for the relative Grassmann Schubert varieties $Gr^r_d(E, (P, a_\bullet))$ and $Gr^r_d(E, (Q, b_\bullet))$ respectively; see Sect. 3.2. Then we have the following theorem.

**Theorem 1.2** The product of Bott–Samelson resolutions over the Brill–Noether variety $G^r_d(E)$

$$Z_\lambda(p) \times Gr^r_d(E) Z_{\lambda'}(q)$$

is a resolution of singularities of $G^r_d(E, (P, a_\bullet), (Q, b_\bullet))$.

The resolution $Z_\lambda(p) \times Gr^r_d(E) Z_{\lambda'}(q)$ has a natural modular interpretation: it parametrizes pairs of partial flags of linear series on $E$ of projective rank no greater than $r$, with ranks specified by $a_\bullet$ and $b_\bullet$, and with agreeing top-rank pieces. At the end, we formulate two open conjectures regarding a generalization of such construction to partial defining flags and Bott–Samelson-type resolutions of singularities for Brill–Noether varieties on twice-marked curves of higher genus.

## 2 Background

### 2.1 Flag Varieties, Schubert Varieties and the Bruhat Stratification

We start with an introduction to flag varieties. The complete flag variety of a fixed $n$-dimensional vector space $H$ is

$$Fl(H) = \{ F_\bullet = ([0] \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = H) : \dim F_i = i \text{ for all } i = 1, \ldots, n \}. $$

Given a permutation $\sigma \in S_n$ and a flag $F_\bullet \in Fl(H)$, the Schubert cell $X^\sigma_{\sigma, F_\bullet}$ is a subscheme in $Fl(H)$ defined by incidence relations imposed by $\sigma$ with respect to the flag $F_\bullet$. Set-theoretically, it is

$$\left\{ V_\bullet \in Fl(H) : \dim V_i \cap F_j = \# \{ m : m \leq i, \sigma(m) \leq j \} \text{ for all } i, j \right\}.$$

The Schubert variety $X^\sigma_{\sigma, F_\bullet}$ is the Zariski closure of the Schubert cell $X^\sigma_{\sigma, F_\bullet}$ in $Fl(H)$ and is set-theoretically

$$\left\{ V_\bullet \in Fl(H) : \dim V_i \cap F_j \geq \# \{ m : m \leq i, \sigma(m) \leq j \} \text{ for all } i, j \right\}.$$
These incidence relations are indeed determinantal and thus equip the sets of the above form with scheme structures. Furthermore, if a flag $F_\bullet$ in $\text{Fl}(H)$ is fixed, then the flag variety $\text{Fl}(H)$ is the disjoint union of all Schubert cells defined with respect to $F_\bullet$, indexed by $S_n$. Formally,

$$\text{Fl}(H) = \bigsqcup_{\sigma \in S_n} X^\circ_{\sigma,F_\bullet}.$$ 

The set of Schubert varieties $\{X_{\sigma,F_\bullet}\}_{\sigma \in S_n}$ gives a stratification called the Bruhat stratification on the flag variety $\text{Fl}(H)$, and the containment relations of the strata yield a partial order called the Bruhat order on these strata, as well as on $S_n$. It is always possible to write any permutation as a product of adjacent transpositions. If such a product has length

$$\text{inv}(\sigma) = \#\{(i, j) : \sigma(i) > \sigma(j) \text{ for all } 1 \leq i < j \leq n\},$$

it is called a reduced decomposition of $\sigma$. The Bruhat order on $S_n$ can be obtained combinatorially by taking the transitive and reflexive closure of the following relation: for every $1 \leq i < j \leq n$, $\sigma < \sigma t_{ij}$ if $\sigma(i) < \sigma(j)$ where $t_{ij}$ is the transposition swapping $i$ and $j$. One can check that this relation indeed is a partial order on $S_n$. We record two basic facts about Schubert varieties of complete flag varieties.

**Fact 2.1** Fix a flag $F_\bullet$ in $\text{Fl}(k^n)$. Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if $X^\circ_{\sigma,F_\bullet} \subseteq X^\circ_{\tau,F_\bullet}$.

**Fact 2.2** Fix a flag $F_\bullet$ in $\text{Fl}(k^n)$. The dimension of the Schubert cell $X^\circ_{\sigma,F_\bullet}$ is $\text{inv}(\sigma)$.

The combinatorics of $S_n$ records the geometry of complete flag varieties and Schubert varieties. In [13] Lakshmibai–Sandhya gave a characterization of singular Schubert varieties and their singular loci using pattern avoidance of indexing permutations: a Schubert variety is singular if and only if the indexing permutation contains patterns 3412 or 4231 (in one-line notation). Other local properties of Schubert varieties such as being Gorenstein are also characterized using pattern avoidances [16]. The main objects that we will study – the relative Bott–Samelson varieties – for resolving the singularities of relative Schubert varieties also have their geometric information recorded in the symmetric group.

### 2.2 Degeneracy Loci and Relative Schubert Varieties

In this section, we recall the notion of the relative position of a pair of flags. These incidence relations involved are determinantal, and thus define schemes that are degeneracy loci.

**Proposition 2.3** For $P_\bullet, Q_\bullet \in \text{Fl}(H)$, there exists a unique permutation in $S_n$ such that there exists an ordered basis $b_1, \ldots, b_n$ of $H$ with the property that

$$b_i \in (P_i \setminus P_{i-1}) \cap (Q_{\sigma(i)} \setminus Q_{\sigma(i)-1}).$$
Definition 2.4 For $P_\bullet, Q_\bullet \in \text{Fl}(H)$, we define the relative position of $P_\bullet$ and $Q_\bullet$ as the unique permutation $\sigma$ produced in Proposition 2.3, denoted as $r_H(P_\bullet, Q_\bullet)$, or simply $r(P_\bullet, Q_\bullet)$ if the vector space is clear from context.

Example 2.5 Let $\omega$ be the permutation that exchanges 1 with $n$, 2 with $n-1$ and so on, i.e., the order-reversing permutation. Two flags $P_\bullet, Q_\bullet \in \text{Fl}(H)$ are transverse, meaning that

$$\dim P_i \cap Q_{n-i} = 0 \text{ for all } i \in [n-1],$$

if and only if $r(P_\bullet, Q_\bullet) = \omega$.

Example 2.6 Two flags $P_\bullet, Q_\bullet \in \text{Fl}(H)$ are called almost transverse if there exists $t \in [n-1]$ such that

$$\dim P_i \cap Q_{n-i} = \begin{cases} 0 & \text{if } i \neq t, \\ 1 & \text{if } i = t, \end{cases}$$

Equivalently, $P_\bullet, Q_\bullet$ are almost transverse if and only if $r(P_\bullet, Q_\bullet) = \omega s_i$, where $s_i$ is the simple transposition exchanging $i$ and $i+1$.

From the definition, one can easily check that the relative position of two flags is “anti-symmetric” in the following sense.

Proposition 2.7 Let $P_\bullet, Q_\bullet \in \text{Fl}(H)$. Then $r_H(P_\bullet, Q_\bullet) = r_H(Q_\bullet, P_\bullet)^{-1}$.

We now introduce the degeneracy loci in the interest of this paper.

Definition 2.8 Let $\sigma \in S_n$. Let $\phi^*H$ be the pullback of $H$ over $\text{Fl}(H)$ and let $\text{Fl}(\phi^*H)$ be the flag bundle associated with $\phi^*H$ over $S$, with the structure map $\phi : \text{Fl}(H) \to S$. Let $p, v$ be sections $S \to \text{Fl}(H)$ and denote the fibers of $p, v$ over a point $x \in S$ by $p_x, v_x$. The degeneracy locus $D_\sigma(p; v)$ is a subscheme of $S$

$$D_\sigma(p; v) = \{x \in S : r(p_x, v_x) \leq \sigma\}.$$

Such incidence relations in a family turn out to be determinantal, thereby giving scheme structures on degeneracy loci. We are interested in relative Schubert varieties over some base scheme, which are examples of degeneracy loci, as follows.

Definition 2.9 Let $\sigma \in S_n$. Let $\phi^*H$ be the pullback of $H$ over $\text{Fl}(H)$ and $\text{Fl}(\phi^*H)$ be the flag bundle associated with $\phi^*H$. Let $t : \text{Fl}(H) \to \text{Fl}(\phi^*H)$ be the tautological section. In other words, $\text{Fl}(\phi^*H)$ is isomorphic to $\text{Fl}(H) \times_S \text{Fl}(H)$, and the section $t$ is the diagonal map. Given a section $p : S \to \text{Fl}(H)$, we obtain the section $p' : \text{Fl}(H) \to \text{Fl}(\phi^*H)$ canonically induced by $p$; formally, $p' = \text{id}_{\text{Fl}(H)} \times (p \circ \phi)$. The relative Schubert variety is a subscheme of $\text{Fl}(H)$

$$X_\sigma(p) := D_\sigma(p'; t) \subseteq \text{Fl}(H).$$
Example 2.10  Let \( \sigma \in S_n \). When \( S = \text{Spec } k \), \( \mathcal{H} \) is a fixed vector space \( H \) of dimension \( n \), and a section \( p : S \rightarrow \text{Fl}(H) \) yielding a fixed flag \( P_{\bullet} \in \text{Fl}(H) \), the relative Schubert variety \( X_{\sigma}(p) \) is the Schubert variety \( X_{\sigma, P_{\bullet}} \subseteq \text{Fl}(H) \), defined in Sect. 2.1.

2.3 Versality

The geometry of intersections of relative Schubert varieties behaves nicely under certain conditions. Morally speaking, when the defining sections of relative Schubert varieties interact with each other minimally in “most fibers,” the local geometry of the intersection is completely determined by the local geometry of each intersecting relative Schubert variety. In [6], Chan and Pflueger use the term \textit{versality} for such a condition on the defining flags, which we recall in a slightly different notation as follows.

Suppose we are given a finite-type \( k \)-scheme \( S \) with a vector bundle \( H \) of rank \( n \). Denote by \( \varphi : \text{Fl}(H) \rightarrow S \) the flag bundle associated with \( H \) over \( S \), and denote by \( \psi : \text{Fr}(H) \rightarrow S \) the frame bundle associated with \( H \) over \( S \). Let \( p_1, \ldots, p_m : S \rightarrow \text{Fl}(H) \) be sections. Then we obtain a morphism \( \Phi_{p_1, \ldots, p_m} : \text{Fr}(H) \rightarrow \text{Fl}(k^n)^m \) canonically induced by the sections \( p_1, \ldots, p_m \); it is defined on the level of closed points by sending \((s, f)\), where \( s \in S \) and \( f \) is an isomorphism \( k^n \rightarrow \mathcal{H}_s \), to the \( m \)-tuple \((f^{-1}(p_i)_s)_{i=1}^m\), where \( (p_i)_s \) is the fiber of \( p_i \) over \( s \), and for all \( i \in [m] \), \( f^{-1}(p_i)_s \) denotes the flag \[ \{0\} \subset (f^{-1}(p_i)_s)_1 \subset \cdots \subset (f^{-1}(p_i)_s)_n = k^n. \]

Certain properties of the map \( \Phi_{p_1, \ldots, p_m} \) capture the relative positions of the family of defining flags \( \{p_i\}_{i=1}^m \).

Definition 2.11 [6, Definition 3.1] The family of sections \( \{p_i\}_{i=1}^m \) is \textbf{versal} if the morphism

\[ \Phi_{p_1, \ldots, p_m} : \text{Fr}(H) \rightarrow \text{Fl}(k^n)^m \]

is smooth.

We collect facts and examples of versal families, proven by Chan-Pflueger using linear-algebraic criteria in [6, Proposition 3.2].

Fact 2.12 [6, Lemma 3.4] In the situation of Example 2.9, let \( t : \text{Fl}(\mathcal{H}) \rightarrow \text{Fl}(\varphi^* \mathcal{H}) \) be tautological, let \( \{p_i\}_{i=1}^m \) be a versal family of sections \( S \rightarrow \text{Fl}(\mathcal{H}) \), and let \( \{p'_i\}_{i=1}^m \) be the canonically induced family of sections \( \text{Fl}(\mathcal{H}) \rightarrow \text{Fl}(\varphi^* \mathcal{H}) \) defined by \( p'_i = \text{id}_{\text{Fl}(\mathcal{H})} \times (p_i \circ \varphi) \) for all \( i \in [m] \). Then the family \( \{p'_i\}_{i=1}^m \cup \{t\} \) is versal.

Example 2.13 When \( S = \text{Spec } k \), \( H \) is a fixed vector space, and the sections \( p, q : S \rightarrow \text{Fl}(H) \) give a pair of fixed flags in \( P_{\bullet}, Q_{\bullet} \subseteq \text{Fl}(H) \), \( p, q \) are versal, if and only if \( \Phi_{p, q} \) is smooth, if and only if \( P_{\bullet}, Q_{\bullet} \) are transverse.
However, two sections can be versal, but give non-transverse flags in the fiber over a reduced point.

**Example 2.14** Suppose $S$ is a 1-dimensional smooth scheme, and versal sections $p, q : S \to \text{Fl}(\mathcal{H})$

give two complete flags that are transverse in all but one fiber over a reduced point $x \in S$. Suppose they are almost transverse over $x$, or have relative position $\omega_{si}$. We have that at $x \in D_{\omega_{si}}(p; q)$, $x$ is smooth in $S$ and the codimension of $D_{\omega_{si}}$ at $x$ is $\text{inv}(\omega_{si}) = 1$. Hence by [6, Lemma 3.7], the two flags are versal over $S$. One example to visualize is a family of two distinct points on $\mathbb{P}^1$ moving along a 1-dimensional base $S$ and the two points meet over a reduced point $x \in S$. The two flags giving the two points are versal over $x$.

### 2.4 Relative Richardson Varieties

Our ultimate goal is to construct a Bott–Samelson type resolution of singularities for relative Richardson varieties, which we now define.

Let $\sigma, \tau \in S_n$. Let $p, q$ be sections $S \to \text{Fl}(\mathcal{H})$. By Fact 2.12, we obtain the versal family of sections $t, p', q'$, where $p'$ and $q'$ are the sections $\text{Fl}(\mathcal{H}) \to \text{Fl}(\varphi^*\mathcal{H})$ canonically induced by $p$ and $q$ respectively. We denote by $X_{\sigma}(p)$ and $X_{\tau}(q)$ the relative Schubert varieties, defined in Example 2.9. We now define the **relative Richardson variety** as the intersection of the two relative Schubert varieties defined with respect to these versal sections

$$R_{\sigma, \tau}(p, q) = X_{\sigma}(p) \cap X_{\tau}(q).$$

The local properties and cohomology of relative Richardson varieties are directly related to those of the intersecting relative Schubert varieties, shown by Chan and Pflueger in [6]. They generalized Knutson-Woo-Yong theorem [12, Theorem 1.1] to $\ell > 2$ and to a relative setting. We record Chan-Pflueger’s main results.

**Theorem 2.15** [6, Theorem 4.1]

Suppose $P$ is an étale-local property preserved under taking product with affine spaces, and $f_{P, m}$ is an $m$-input function such that for finite-type $k$-schemes $X_1, \ldots, X_m$ and $x \in \prod_{i=1}^m X_i$,

$$P \left( x, \prod_{i=1}^m X_i \right) = f_{P, m}(P(\pi_1(x), X_1), \ldots, P(\pi_m(x), X_m)),$$

where $\pi_j$ is the projection map $\prod_{i=1}^m X_i \to X_j$ for all $j \in [m]$. Let $t, p_1, \ldots, p_m$ be versal sections $S \to \text{Fl}(\mathcal{H})$, let $D_{\sigma_i}$ denote the degeneracy locus $D_{\sigma_i}(p_i; t)$ and let $D_{\sigma_1, \ldots, \sigma_m}$ denote the intersection of $D_{\sigma_i}$ for all $i$. Suppose $y \in D_{\sigma_1, \ldots, \sigma_m}$. Then

$$P(y, D_{\sigma_1, \ldots, \sigma_m}) = f_{P, m}(P(y, D_{\sigma_1}), \ldots, P(y, D_{\sigma_m})).$$
This theorem can be directly applied to relative Richardson varieties and yields the following.

**Theorem 2.16** [6, Theorem 5.3] A relative Richardson variety $R_{\sigma, \tau} (p, q)$ is normal, Cohen-Macaulay and of pure dimension $\text{inv}(\omega \sigma) + \text{inv}(\omega \tau)$ in $\text{Fl}(\mathcal{H})$. Moreover, the smooth locus of $R_{\sigma, \tau} (p, q)$ is the intersection of the smooth loci of $X_{\sigma} (p; t)$ and $X_{\tau} (q; t)$.

### 2.5 Relative Bott–Samelson Varieties

In this section, we recall the definition of relative Bott–Samelson varieties in [10, Appendix C]. They are resolutions of singularities for relative Schubert varieties. Let $\sigma \in S_n$ with length $\ell$ and let $\rho_\sigma = s_{i_1} \cdots s_{i_\ell}$ be a decomposition into adjacent transpositions. For each $i_j$, let $\text{Fl}(i_j; \mathcal{H})$ be the partial flag bundle associated with $\mathcal{H}$ over $S$ where every point in the fiber over a point $s$ is a partial flag of $\mathcal{H}_s$ lacking exactly the $i_j$-dimensional subspace. There is a canonical projection map $\text{Fl}(\mathcal{H}) \to \text{Fl}(i_j; \mathcal{H})$ and we denote by $Z_{s_{i_j}}$ the product $\text{Fl}(\mathcal{H}) \times_{\text{Fl}(i_j; \mathcal{H})} \text{Fl}(\mathcal{H})$. The scheme $Z_{s_{i_j}}$ can be set-theoretically described as

$$\{(s, E_*, F_*): s \in S, E_*, F_* \in \text{Fl}(\mathcal{H}_s), r(E_*, F_*) \leq s_{i_j}\} \subseteq \text{Fl}(\mathcal{H})^2.$$

Consider the product

$$Z_{\rho_\sigma} := Z_{s_{i_1}} \times_{\text{Fl}(\mathcal{H})} \cdots \times_{\text{Fl}(\mathcal{H})} Z_{s_{i_\ell}} \subseteq \text{Fl}(\mathcal{H})^{\ell+1},$$

where the product of each consecutive pair of $Z_{s_{i_j}}$ and $Z_{s_{i_j+1}}$ is taken using the second projection of $Z_{s_{i_j}}$ and the first projection of $Z_{s_{i_j+1}}$. It can be set-theoretically described as

$$\\{(s, F_*^0, \ldots, F_*^\ell): s \in S, F_*^j \in \text{Fl}(\mathcal{H}_s), r(F_*^{j-1}, F_*^j) \leq s_{i_j} \text{ for all } j \in [\ell]\}.$$

Now let $p$ be a section $S \to \text{Fl}(\mathcal{H})$ and denote by $p'$ the section $\text{Fl}(\mathcal{H}) \to \text{Fl}(\varphi^*\mathcal{H})$ canonically induced by $p$. Recall that the relative Schubert variety $X_{\text{id}}(p) \subseteq \text{Fl}(\mathcal{H})$ consists as closed points of $(s, p_s)$ for all $s \in S$. Denote by $\pi_0$ the first projection of $Z_{\rho_\sigma}$ to $\text{Fl}(\mathcal{H})$.

**Definition 2.17** We define the relative Bott–Samelson variety $Z_{\rho_\sigma} (p)$ as

$$Z_{\rho_\sigma} (p) := \pi_0^{-1} X_{\text{id}}(p) \subseteq \text{Fl}(\mathcal{H})^{\ell+1}.$$

The scheme $Z_{\rho_\sigma} (p)$ can be described set-theoretically as

$$\{(s, F_*^0, \ldots, F_*^\ell): s \in S, F_*^j \in \text{Fl}(\mathcal{H}_s), p_s = F_*^0, r(F_*^{j-1}, F_*^j) \leq s_{i_j} \text{ for all } j \in [\ell]\}.$$

**Proposition 2.18** If $S$ is a smooth scheme in the above situation, then the relative Bott–Samelson variety $Z_{\rho_\sigma} (p)$ is smooth.
Every point of the variety can be described as a sequence of choices of a red flag, a green flag, a blue flag, an orange flag and a purple flag. Adjacent flags differ by at most one subspace $V_j$ of dimension $i_j$ given by $\rho_\sigma$

**Proof** Since $Z_{\rho_\sigma}(p)$ is a family of iterated $\mathbb{P}^1$-bundles over $X_{\text{id}}(p)$, $\pi_0$ is a smooth morphism. In addition, $X_{\text{id}}(p)$ is a smooth scheme; therefore, $Z_{\rho_\sigma}(p)$ is smooth. □

**Example 2.19** When $S = \text{Spec } \mathbb{k}$ and $H$ is a fixed $n$-dimensional vector space over $\mathbb{k}$, we recover the absolute Bott–Samelson variety. As an example, let $\sigma = 4231$ with the reduced decomposition $\rho_\sigma = (34)(12)(23)(34)(12)$. A fixed section $p : S \to \text{Fl}(H)$ gives a fixed flag $P_\bullet$, and the Bott–Samelson variety $Z_{\rho_\sigma}(p)$ is the subscheme of $\text{Fl}(H)^6$ whose construction is schematically depicted in Fig. 1. It is a sequence of iterated $\mathbb{P}^1$-fibrations over a point in the flag variety.

In the special case above, $Z_{\rho_\sigma}(p)$ admits a canonical proper morphism to $\text{Fl}(H)$ and is birational to the image – the Schubert variety $X_{\sigma,P_\bullet}$. Therefore, a Bott–Samelson variety is a resolution of singularities for its associated Schubert variety. Bott–Samelson varieties were first introduced by Bott and Samelson [2] and later explicitly constructed by Demazure [8] and Hansen [11] as resolutions of singularities for Schubert varieties in complete flag varieties. Zelevinsky [17] generalized the construction to resolve singularities for Grassmann Schubert varieties. Despite having fibers of large dimensions and hence not a small resolution, the Bott–Samelson resolutions play a central role in the studies of the geometry of Schubert varieties.

When two Schubert varieties are defined with respect to transverse flags, the product of their Bott–Samelson resolutions over the complete flag variety resolves the singularities of their intersection. Formally, suppose $S = \text{Spec } \mathbb{k}$, $H$ is a fixed $n$-dimensional vector space, two versal sections $p, q : S \to \text{Fl}(H)$ give transverse flags $P_\bullet, Q_\bullet \in \text{Fl}(H)$, and $\sigma, \tau \in S_n$ with reduced decompositions $\rho_\sigma, \rho_\tau$. The variety $Z_{\rho_\sigma}(p) \times_{\text{Fl}(H)} Z_{\rho_\tau}(q)$ is smooth and the canonical morphism

$$Z_{\rho_\sigma}(p) \times_{\text{Fl}(H)} Z_{\rho_\tau}(q) \to X_{\sigma,P_\bullet} \cap X_{\tau,Q_\bullet} \subseteq \text{Fl}(H)$$

is proper and birational. Thus $Z_{\rho_\sigma}(p) \times_{\text{Fl}(H)} Z_{\rho_\tau}(q)$ is a resolution of singularities for the Richardson variety $X_{\sigma,P_\bullet} \cap X_{\tau,Q_\bullet}$ [3, Proof of Theorem 4.2.1]. We will generalize this result to products of relative Bott–Samelson varieties defined with respect to versal sections.
3 Products of Relative Bott–Samelson Varieties

3.1 Resolutions for Relative Complete Flag Richardson Varieties

In this section we prove our main theorem, restated as follows.

Theorem 3.1 Let $S$ be a smooth finite-type $k$-scheme that carries a rank-$n$ vector bundle $\mathcal{H}$. Let $\sigma, \tau \in S_n$. Let $\rho_\sigma$ and $\rho_\tau$ be reduced decompositions for $\sigma$ and $\tau$ respectively. Let $p, q : S \to \text{Fl}(\mathcal{H})$ be versal sections. Let $X_\sigma(p)$ and $X_\tau(q)$ be the relative Schubert varieties defined with respect to $p$ and $q$. Let $Z_{\rho_\sigma}(p)$ and $Z_{\rho_\tau}(q)$ be the relative Bott–Samelson varieties defined with respect to $p$ and $q$. The fiber product

$$Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$$

(i) is smooth;

(ii) and the projection to the relative Richardson variety

$$Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q) \to X_\sigma(p) \cap X_\tau(q)$$

is proper and birational.

Therefore, $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$ is a resolution of singularities of the relative Richardson variety.

The crux of the proof is to show that $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$ is smooth, which we accomplish in two steps. First, we relate the local geometry of $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$ with that of a principal $\text{GL}_n(k)$-bundle over $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$. Second, we relate the local geometry of the principal bundle with that of relative Bott–Samelson varieties over the complete flag variety, of whose local geometry we have a good grasp, thanks to the versal defining flags.

We fix notation. Let $\Phi_1 : \text{Fl}(\mathcal{H}) \to \text{Fl}^2$ be the map canonically induced by $p$ and $q$ in the way described in Sect. 2.3, and denote $Z_{\rho_\sigma}(p) \times_{\text{Fl}(\mathcal{H})} Z_{\rho_\tau}(q)$ by $Z_{\sigma, \tau}(p, q)$.

For step one, we construct a principal $\text{GL}_n(k)$-bundle over $Z_{\sigma, \tau}(p, q)$ as follows. Let $\rho_\tau^{-1}$ be the reverse of $\rho_\tau$ and $\rho_\sigma \rho_\tau^{-1}$ be the concatenation of $\rho_\sigma$ and $\rho_\tau^{-1}$. Let $t$ be the tautological section of the trivial bundle $\text{Fl}(\text{Fl} \times k^n) \to \text{Fl}$ over $\text{Fl}$. (In other words, $t$ is the diagonal map $\text{Fl} \to \text{Fl} \times \text{Fl} \text{Fl}$. Consider the scheme $Z_{\rho_\sigma \rho_\tau^{-1}}(t) \subseteq \text{Fl}^{l+e'+2}$. This scheme projects to $\text{Fl}^2$ via the two outer projections, denoted by $\pi_0$ and $\pi_{l+e'}$, and hence we obtain the fiber product $Z_{\rho_\sigma \rho_\tau^{-1}}(t) \times_{\text{Fl}^2} \Phi_1(\mathcal{H})$. There exists a canonical morphism

$$\alpha : Z_{\rho_\sigma \rho_\tau^{-1}}(t) \times_{\text{Fl}^2} \Phi_1(\mathcal{H}) \to Z_{\sigma, \tau}(p, q)$$

that sends a closed point

$$(F_0, \ldots, F_\ell, F_{\sigma, 0}^{\tau^{-1}, 0}, \ldots, F_{\sigma, e'}^{\tau^{-1}, e'}, s, f),$$
where \( s \in S, f : k^n \to \mathcal{H}_S \) is a vector-space isomorphism and \( F_{\bullet}^{-, -} \in \text{Fl}(\mathcal{H}_S) \), to
\[
(p_s = f(F_{\bullet}^{\sigma, 0}), \ldots, f(F_{\bullet}^{\tau^{-1}, 0})) = q_s. 
\]

Denote by \( \beta \) the canonical proper projection \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \to \text{Fr}(\mathcal{H}) \).
Then we obtain the following diagram with Cartesian squares.
\[
\begin{array}{ccc}
Z_{\alpha, \tau}(p, q) & \xleftarrow{\alpha} & Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \\
\downarrow & & \downarrow \beta \\
S & \xleftarrow{} & \text{Fr}(\mathcal{H}) \\
\end{array}
\]

The intuition behind considering the fiber product \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) is that it is the family of sequences of flags where the \((\ell + 1)\)-th flag has relative position at most \( \sigma \) with \( p \) and at most \( \tau \) with \( q \), upon some change of basis, and \( Z_{\rho_{\sigma, 0}^{-1}}(t) \) is smooth, by Proposition 2.18. Furthermore, by construction, we have the following lemma.

**Lemma 3.2** The scheme \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) equipped with the morphisms \( \alpha, \beta \), is the fiber product of \( Z_{\alpha, \tau}(p, q) \) with \( \text{Fr}(\mathcal{H}) \). Hence \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) is isomorphic to \( Z_{\alpha, \tau}(p, q) \times_S \text{Fr}(\mathcal{H}) \), a \( \text{GL}_n(k) \)-principal bundle over \( Z_{\alpha, \tau}(p, q) \), via a canonical isomorphism.

Now we are ready to prove the theorem.

**Proof of Theorem 3.1** To show smoothness of \( Z_{\alpha, \tau}(p, q) \) in (i), let \( x \in Z_{\alpha, \tau}(p, q) \), and we show that \( x \) is a smooth point. By Lemma 3.2, \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) is locally isomorphic to \( Z_{\alpha, \tau}(p, q) \times \text{GL}_n(k) \). Therefore, if \( y \in Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) such that \( \alpha(y) = x \), then \( x \) is smooth in \( Z_{\alpha, \tau}(p, q) \) if and only if \( y \) is smooth in \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \). Since \( \Phi_{p,q} \) is a smooth morphism, and smoothness of morphisms is stable under base change, the morphism \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \to Z_{\rho_{\sigma, 0}^{-1}}(t) \) is smooth. Since \( Z_{\rho_{\sigma, 0}^{-1}}(t) \) is a smooth scheme over \( k \), and composition of smooth morphisms is smooth, \( Z_{\rho_{\sigma, 0}^{-1}}(t) \times_{\text{Fl}^2} \text{Fr}(\mathcal{H}) \) is smooth. Therefore, \( x \) is a smooth point in \( Z_{\alpha, \tau}(p, q) \) and \( Z_{\alpha, \tau}(p, q) \) is smooth.

For (ii), the scheme \( Z_{\alpha, \tau}(p, q) \) admits a canonical morphism \( \pi \) to \( \text{Fl}(\mathcal{H}) \), whose scheme-theoretic image is the relative Richardson variety \( X_\sigma(p) \cap X_\tau(q) \). Since \( Z_{\alpha, \tau}(p, q) \) is a proper scheme, and \( X_\sigma(p) \cap X_\tau(q) \) is separated, the morphism \( \pi \) is proper. Furthermore, let \( R_\sigma^\circ \) be the open subscheme of \( X_\sigma(p) \cap X_\tau(q) \) where the inequalities of relative positions of flags in all fibers are equalities. It is similar to the absolute setting to see that \( \pi \) is an isomorphism on \( R_\sigma^\circ \). Therefore, \( \pi \) is birational.

We conclude that \( Z_{\alpha, \tau}(p, q) \) is a resolution of singularities for the relative Richardson variety \( X_\sigma(p) \cap X_\tau(q) \). \( \square \)

One immediate corollary is the special case when \( S = \text{Spec} \ k \) and sections \( p, q \) give transverse flags \( P_\bullet, Q_\bullet \) of a vector space \( H \), previously proven by Brion using Kleiman’s transversality theorem.

\( \square \) Springer
Corollary 3.3 Given permutations \( \sigma, \tau \in S_n \) with respective reduced decomposition \( \rho_{\sigma}, \rho_{\tau} \) and transverse flags \( P_*, Q_* \) of a vector space \( H \), the product \( Z_{\rho_{\sigma}}(P_*) \times_{Fl} Z_{\rho_{\tau}}(Q_*) \) is the resolution of singularities for the Richardson variety \( X_{\sigma, P_*, \tau, Q_*} \).

3.2 Resolutions for Relative Grassmann Richardsonson Varieties

We have a similar theorem in the case of the versal intersection of two relative Grassmann Schubert varieties. This version will be relevant for an application of resolving the singularities of Brill–Noether varieties with imposed ramification on twice-marked elliptic curves; see Sect. 4.1. Given integers \( n \geq r \geq 1 \), recall that \( Gr(r, n) \) is the Grassmannian parametrizing \( r \)-dimensional subspaces in an \( n \)-dimensional vector space. A partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of \( \sum \lambda_i \) is \( Gr(r, n) \)-admissible if

\[
    n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0.
\]

Alternatively, we can describe \( \lambda \) by collecting parts of the same size; that is, we write

\[
    \lambda = (\mu_1^1, \ldots, \mu_j^j)
\]

where

\[
    n - k \geq \mu_1 > \cdots > \mu_j > 0
\]

and \( \sum i_j = r \). If a partition \( \lambda = (\mu_1^1, \ldots, \mu_j^j) \), we say \( \lambda \) is of type \( j \). For example, the partition \( \lambda = (4^1, 3^2, 2, 1) \) with \( (\mu_1, i_1) = (4, 1), (\mu_2, i_2) = (3, 2), (\mu_3, i_3) = (2, 1), (\mu_4, i_4) = (1, 1) \), and is thus of type 4.

Definition 3.4 Given integers \( n \geq r \geq 1 \), a smooth \( k \)-scheme \( S \) with a vector bundle \( \mathcal{H} \) of rank \( n \), \( \varphi : Gr(r, \mathcal{H}) \to S \) the Grassmann bundle over \( S \), a section \( p : S \to Fl(\mathcal{H}) \) giving a complete flag of subbundles of \( \mathcal{H} \) and a \( Gr(r, n) \)-admissible partition \( \lambda \), the relative Grassmann Schubert variety

\[
    X_{\lambda}(p) := \{ (x, V \subseteq \mathcal{H}_x) : \dim V \cap (p_x)_{n-r+i-i\lambda_i} \geq i \text{ for all } 1 \leq i \leq r \} \subseteq Gr(r, \mathcal{H}).
\]

The incidence conditions are determinantal, thus yielding subschemes of \( Gr(r, \mathcal{H}) \). The codimension of the relative Grassmann Schubert variety given by \( \lambda \) in the Grassmann bundle is precisely \( \sum \lambda_i \).

Now we define relative Zelevinsky resolutions for a relative Grassmann Schubert varieties, generalizing Zelevinsky resolutions in the absolute case [7, 17].

Definition 3.5 Given \( n \geq r \geq 1 \), a smooth \( k \)-scheme \( S \) with a vector bundle \( \mathcal{H} \) of rank \( n \), a section \( p : S \to Fl(\mathcal{H}) \) giving a complete flag of subbundles of \( \mathcal{H} \), a \( Gr(r, n) \)-admissible partition \( \lambda \) written as \( (\mu_1^1, \ldots, \mu_j^j) \), and \( a_s = \sum_{s=1}^{r} i_\ell \), the
relative Zelevinsky resolution for relative Grassmann Schubert varieties

\[ Z_{\lambda}(p) = \left\{ (x, V_1 \subset \cdots \subset V_j) : \dim V_s = a_s, V_s \subseteq (p_\lambda)_{n-r+a_s-\lambda a_s} \text{ for all } 1 \leq s \leq j \right\}, \]

as a subvariety of \( \text{Fl}(a_1, \ldots, a_j; \mathcal{H}) \).

The incidence relations are again determinantal, thus yielding a variety.

**Theorem 3.6** Given integers \( n \geq r \geq 1 \), a smooth k-scheme \( S \) with a vector bundle \( \mathcal{H} \) of rank \( n \), the flag bundle \( \text{Fl}(\mathcal{H}) \to S \) associated with \( \mathcal{H} \), the Grassmann bundle \( \text{Gr}(r, \mathcal{H}) \to S \), two versal sections \( p, q : S \to \text{Fl}(\mathcal{H}) \) and two \( \text{Gr}(r, n) \)-admissible partitions \( \lambda, \lambda' \), the product of relative Zelevinsky resolutions for relative Grassmann Schubert varieties over the Grassmann bundle

\[ Z_{\lambda}(p) \times \text{Gr}(r, \mathcal{H}) Z_{\lambda'}(q) \]

(i) is smooth; and

(ii) the projection to the relative Grassmann Richardson variety

\[ Z_{\lambda}(p) \times \text{Gr}(r, \mathcal{H}) Z_{\lambda'}(q) \to X_{\lambda}(p) \cap X_{\lambda'}(q) \subseteq \text{Gr}(r, \mathcal{H}) \]

is proper and birational.

Therefore, \( Z_{\lambda}(p) \times \text{Gr}(r, \mathcal{H}) Z_{\lambda'}(q) \) is a resolution of singularities for the relative Grassmann Richardson variety.

**Proof** We exploit the idea in the proof of Theorem 3.1 and denote \( Z_{\lambda}(p) \times \text{Gr}(r, \mathcal{H}) Z_{\lambda'}(q) \) by \( Z_{\lambda, \lambda'}(p, q) \). To show smoothness of \( Z_{\lambda, \lambda'}(p, q) \), we construct a principal GL-bundle over \( Z_{\lambda, \lambda'}(p, q) \). Suppose \( \lambda \) and \( \lambda' \) are of type \( j \) and \( j' \) respectively, and define \( Z_{\lambda, \lambda'} \) as follows:

\[ Z_{\lambda, \lambda'} = \left\{ F_1, \ldots, F_n, V_1, \ldots, V_j, U_1, \ldots, U_{j'}, G_1, \ldots, G_n : (*) \right\} \]

where the condition \((*)\) consists of the following:

(i) \( F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n \) and \( G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n \) are complete flags in \( \text{Fl} \);

(ii) \( V_1 \subset V_2 \subset \cdots \subset V_{j-1} \subset V_j \) and \( U_1 \subset U_2 \subset \cdots \subset U_{j'-1} \subset U_{j'} \);

(iii) for all \( 1 \leq \ell \leq j \), \( \dim V_\ell = a_\ell \) and \( V_\ell \subseteq F_{n-r+a_\ell-\lambda a_\ell} \);

(iv) for all \( 1 \leq \ell' \leq j' \), \( \dim U_{\ell'} = b_{\ell'} \) and \( U_{\ell'} \subseteq G_{n-r+b_{\ell'}-\lambda' b_{\ell'}} \); and

(v) \( V_j = U_{j'} \).

The incidence relations are determinantal, thus defining a variety. Furthermore, \( Z_{\lambda, \lambda'} \) is smooth because it is an iterated tower of Grassmann bundles. It admits projections \( \pi, \pi' \) to \( \text{Fl}^2 \) by projecting the complete flags \( \{0\} \subset F_1 \subset \cdots \subset F_n = k^n \) and \( \{0\} \subset G_1 \subset \cdots \subset G_n = k^n \). Again by construction, the product \( Z_{\lambda, \lambda'} \times \text{Fl}^2 \text{Fr}(\mathcal{H}) \) is
Hence isomorphic via a unique isomorphism to the product $\mathbb{Z}_{\lambda,\lambda'}(p, q) \times S \text{Fr}(\mathcal{H})$, and the following diagram commutes:

$$
\begin{array}{cccc}
Z_{\lambda,\lambda'}(p, q) & \xleftarrow{\alpha} & Z_{\lambda,\lambda'} \times_{\text{Fr}^2(\mathcal{H})} \text{Fr}(\mathcal{H}) & \longrightarrow & Z_{\lambda,\lambda'} \\
\downarrow & & \downarrow\beta & & \downarrow\pi,\pi' \\
S & \xleftarrow{\Phi_{p,q}} & \text{Fr}(\mathcal{H}) & \longrightarrow & \text{Fr}^2.
\end{array}
$$

Hence $Z_{\lambda,\lambda'} \times_{\text{Fr}^2} \text{Fr}(\mathcal{H})$ is a principal GL-bundle over $Z_{\lambda,\lambda'}(p, q)$. As similarly argued in the proof of Theorem 3.1, $Z_{\lambda,\lambda'}(p, q)$ is smooth if and only if $Z_{\lambda,\lambda'}$ is smooth, which holds true as previously discussed.

The variety $Z_{\lambda,\lambda'}(p, q)$ admits a canonical projection $\pi_r$ to $\text{Gr}(r, \mathcal{H})$ whose scheme-theoretic image is the relative Grassmann Richardson variety $X_{\lambda}(p) \cap X_{\lambda'}(q)$. Since $Z_{\lambda,\lambda'}(p, q)$ is proper and $X_{\lambda}(p) \cap X_{\lambda'}(q)$ is separated, the morphism $\pi_r$ is proper. Let $R^0_{\lambda,\lambda'}$ be the open subscheme of $X_{\lambda}(p) \cap X_{\lambda'}(q)$ where the inequalities of relative positions are equalities. Since $\pi_r$ is an isomorphism on $R^0_{\lambda,\lambda'}$, $\pi_r$ is birational. \(\square\)

We give the essentially smallest interesting example in the case of relative Grassmann–Schubert intersections.

**Example 3.7** Let $r = 2, n = 4$. Let $S$ be a 1-dimensional smooth $k$-scheme with a rank-$4$ vector bundle $\mathcal{H}$. Further suppose we are given versal sections $p, q$ such that over a unique reduced point $x \in S$,

$$
dim(p_s)_{4-j} \cap (q_s)_j = \begin{cases} 
0, & \text{if } j = 1, 3, 4, \\
1, & \text{if } j = 2,
\end{cases}
$$

and that $p_s$ and $q_s$ are transverse everywhere else; see Fig. 2. Let $\lambda = \lambda'$ be the $\text{Gr}(2, 4)$-admissible partition $(1, 0)$ and for any $s \in S$, let $P_s$ and $Q_s$ denote the projectivization of $(p_s)_2$ and $(q_s)_2$. Then the relative Grassmann Richardson variety $X_{\lambda}(p) \cap X_{\lambda'}(q)$ parametrizes $(s, L)$ where $s \in S$, $L$ is a line in $\mathbb{P}(\mathcal{H}_s)$ and $L$ intersects the two lines $P_s$ and $Q_s$ each at a point.

By [6, Theorem 5.3], the fiber of $X_{\lambda}(p) \cap X_{\lambda'}(q)$ over $x$ is the union of two transverse $\mathbb{P}^2$ intersecting at $\mathbb{P}^1$, and the singular locus of $X_{\lambda}(p) \cap X_{\lambda'}(q)$ is precisely the points $(x, L = P_x)$ and $(x, L = Q_x)$ contained in the $\mathbb{P}^1$ intersection. The smooth points of $X_{\lambda}(p) \cap X_{\lambda'}(q)$ are $(s, L)$ in one of the following situations:

(i) $s = x, L \cap P_x = L \cap Q_x = P_x \cap Q_x$;
(ii) $s = x, L \cap P_x \neq P_x \cap Q_x$ and $L \cap Q_x \neq P_x \cap Q_x$;
Fig. 3 A picture illustrating the incidence relations of \( L \) and \( P_x, Q_x \) in \( \mathbf{P}(\mathcal{H}_x) \) when \((x, L)\) is a singular point in \( X_\lambda(p) \cap X_{\lambda'}(q) \); that is, precisely when \( L = P_x \) or \( L = Q_x \).

Fig. 4 A picture illustrating the incidence relations of \( L \) and \( P_s, Q_s \) in \( \mathbf{P}(\mathcal{H}_s) \) when \((s, L)\) is a smooth point in \( X_\lambda(p) \cap X_{\lambda'}(q) \).

(iii) \( s \neq x \).

See Figs. 3 and 4 for pictures picturing the incidence relations of \( L \) and \( P_s, Q_s \) depending on the smoothness at \((s, L)\) in \( X_\sigma(p) \cap X_\tau(q) \).

The smooth variety \( Z_\lambda(p) \times_{\mathbf{G}_r} Z_{\lambda'}(q) \) parametrizes \((s, L, p_1, p_2)\) where the points \( p_1, p_2 \) are contained in \( L \cap P_x, L \cap Q_x \) respectively. Over \((s \neq x, L)\), or \((x, L \notin \{P_x, Q_x\})\), the line \( L \) uniquely determines \( p_1, p_2 \) because \( p_1 = L \cap P_x, p_2 = L \cap Q_x \) respectively; therefore, the projection \( \pi \) is an isomorphism on the open set of \( X_\lambda(p) \cap X_{\lambda'}(q) \) where equalities of relative positions hold. Over \((x, P_x)\) or \((x, Q_x)\), the fiber of \( \pi \) is isomorphic to \( \mathbf{P}^1 \times \mathbf{P}^1 \).

4 Application to Brill–Noether Theory

In this section, we return to Brill–Noether varieties with imposed ramification, which motivated our consideration of resolution of singularities for relative Richardson varieties. We refer to [1, Chapter V] for a beautiful treatment on Brill–Noether theory.

4.1 Resolution of Singularities for Twice-Pointed Brill–Noether Varieties

Let \( g \geq 0, \ d \geq r \geq 0 \), and let \((X; P, Q)\) be twice-marked smooth genus \( g \) curve. Denote by \( a_\bullet \) and \( b_\bullet \) sequences \( 0 \leq a_1 < \cdots < a_r < a_{r+1} \leq d \) and \( 0 \leq b_1 < \cdots < b_r < b_{r+1} \leq d \). Let \( G'_d(X, (P, a_\bullet), (Q, b_\bullet)) \) denote the moduli space of linear series of projective rank \( r \) of degree \( d \) with imposed ramification at \( P, Q \) prescribed by \( a_\bullet, b_\bullet \) respectively (see [4, 6] for the precise ramification conditions). First considered in [9], it is a subscheme of the classical Brill–Noether variety \( G'_d(X) \), which is known to be smooth. In [4], Chan, Osserman and Pflueger showed that if \( g = 1 \) and \( P - Q \) is not a torsion point of order less than or equal to \( d \) in \( \text{Pic}^0(X) \), then the variety \( G'_d(X, (P, a_\bullet), (Q, b_\bullet)) \) is the intersection of two relative Grassmann Schubert varieties \( G'_d(X, (P, a_\bullet)) \) and \( G'_d(X, (Q, b_\bullet)) \) over the base scheme \( \text{Pic}^d(X) \); see [6, Corollary 6.2]. In other words, it is a relative Grassmann Richardson variety, studied by [6]. Furthermore, [4, Proposition 3.1] and also later [6, Theorem 5.3] state...
that the variety has singular locus precisely the union
\[
\left( G^\text{r}_d(X, (P, a\bullet)) \cap G^\text{r}_d(X, (Q, b\bullet)) \right) \cup \left( G^\text{r}_d(X, (P, a\bullet)) \cap G^\text{r}_d(X, (Q, b\bullet)) \cap G^\text{r}_d(X, (Q, b\bullet)) \right)_{\text{sing}}.
\]

Our Theorem 3.6 can be applied to give a resolution of singularities as follows. Let \( E \) be an elliptic curve marked by \( P, Q \) where \( P - Q \) is not a torsion point of order weakly less than \( d \) in \( \text{Pic}^0(E) \). Let \( \mathcal{L} \) be the Poincaré line bundle on \( E \times \text{Pic}^d(E) \) and let \( \pi \) be the projection \( E \times \text{Pic}^d(E) \to \text{Pic}^d(E) \). Then the base scheme \( \text{Pic}^d(E) \) carries a rank-\( d \) vector bundle \( \mathcal{H} = \pi_* \mathcal{L} \). Over each \( [L] \in \text{Pic}^d(E) \), set \( p_{[L]}, q_{[L]} \) to be the flags of global sections \( H^0(E, L) \)
\[
p_{[L]} = \{0\} \subset H^0(E, L((1 - d)P)) \subset \cdots \subset H^0(E, L(-P)) \subset H^0(E, L),
q_{[L]} = \{0\} \subset H^0(E, L((1 - d)Q)) \subset \cdots \subset H^0(E, L(-Q)) \subset H^0(E, L).
\]

furnishing two sections \( p, q : \text{Pic}^d(E) \to \text{Fl}(\mathcal{H}) \). Let \( a\bullet \) and \( b\bullet \) be increasing sequences of \( r + 1 \) nonnegative numbers no greater than \( d \). The sections \( p, q \) are proven to be versal in [6, Lemma 6.1]. Define \( G'(r + 1, d) \)-admissible partitions \( \lambda = (\lambda_i) \) and \( \lambda' = (\lambda'_i) \) where
\[
\lambda_i = a_{r+1-(i-1)} - (r + 1 - i),
\lambda'_i = b_{r+1-(i'-1)} - (r + 1 - i')
\]
for \( i, i' \in [r + 1] \).

**Theorem 4.1** In the situation above, the product of Bott–Samelson resolutions over the Brill–Noether variety \( G^\text{r}_d(E) \)
\[
Z_\lambda(p) \times G^\text{r}_d(E) Z_{\lambda'}(q)
\]
is a resolution of singularities of \( G^\text{r}_d(E, (P, a\bullet), (Q, b\bullet)) \).

**Proof** This is a direct application of Theorem 3.6. \( \square \)

**Example 4.2** In the situation of Theorem 4.1, \( g = 1, d = 4, r = 1, a\bullet = b\bullet = (0, 2) \) and \( \lambda = \lambda' = (1, 0) \), it is routine to check that the variety \( Z_\lambda(p) \times G^\text{r}_d(E) Z_{\lambda'}(q) \) described as in Example 3.7 is indeed a resolution of singularities for \( G^\text{r}_4(X, (P, (0, 2)), (Q, (0, 2))) \).

### 4.2 Open Problems

We now state two conjectures which generalize Theorem 1.2 to Brill–Noether varieties with twice-marked higher genus curves. First we generalize versality to partial flags. Let \( n, m \geq 1 \), and let \( S \) be a finite-type \( k \)-scheme with a vector bundle \( \mathcal{H} \) of rank \( n \). Let \( \{d^i_{\bullet}\}_{i=1}^m \) be \( m \) sequences such that \( 0 < d^1_1 \leq d^2_2 \leq \cdots \leq d^i_i < n \) and let \( \text{Fl}(d^i_{\bullet}; \mathcal{H}) \to S \) be the partial flag bundle associated with \( \mathcal{H} \) with dimensions specified by \( d^i_i \) for all
i ∈ [m]. Let Fr(\mathcal{H}) → S be the frame bundle over S associated with \mathcal{H}. For each i ∈ [m], suppose \( p_i : S \rightarrow \text{Fl}(d_i^*; \mathcal{H}) \) is a section. The family of partial flag sections \( p_1, \ldots, p_m \) is **versal** if the induced morphism

\[
\Phi_{p_1, \ldots, p_m} : \text{Fr}(\mathcal{H}) \rightarrow \prod_{i=1}^{m} \text{Fl}(d_i^*; \mathcal{H})
\]

is smooth.

Let \( n, r ≥ 0 \), let \( \lambda \) be a \( \text{Gr}(r, n) \)-admissible partition that can be written as \((\mu_1^i, \ldots, \mu_j^i)\) and set \( a_s = \sum_{\ell=1}^{s} i_\ell \). A section \( p : S \rightarrow \text{Fl}(d_\bullet; \mathcal{H}) \) for some increasing sequence of positive integers \( d_\bullet \) less than \( n \) is \( \lambda \)-**compatible** if

\[
\dim(p_x)_{n-r+a_s-\lambda_{a_s}} = d_j ∈ d_\bullet
\]

for some \( j \) for all \( x ∈ S \). Then define relative Grassmann Schubert varieties \( X_\lambda(p) \) with respect to partial flags and Bott–Samelson resolutions \( Z_\lambda(p) \) for relative Grassmann Schubert varieties with respect to partial flags similarly to how those are defined in Definitions 3.4 and 3.5. Given \( \text{Gr}(r, n) \)-admissible partitions \( \lambda, \lambda' \) and versal sections \( p, q \) that are \( \lambda \)-compatible and \( \lambda' \)-compatible respectively, we have the following conjecture.

**Conjecture 4.3** The product of Bott–Samelson resolutions for relative Grassmann Schubert varieties with respect to partial flags \( Z_\lambda(p) ×_{\text{Gr}(r, \mathcal{H})} Z_{\lambda'}(q) \) is smooth and admits a proper birational morphism to the relative Richardson variety with respect to partial flags \( X_\lambda(p) ∩ X_{\lambda'}(q) \).

This conjecture may be specialized to a resolution of singularities for the Brill–Noether varieties on twice-marked curves in higher genus. Let \( g > 1, d > r ≥ 0 \) and let \((X, P, Q)\) be a smooth projective curve of genus \( g \). For nonnegative integers \( N, a, b \) such that \( N, N-a, N-b ≥ 2g-1 \) and \( d = N-a-b \), the variety \( \text{Pic}^N(X) \) carries a vector bundle \( \mathcal{H} \) such that \( \mathcal{H}|_{[L]} = H^0(L(aP+bQ)) \). Then the Brill–Noether variety \( G'_d(X) \) and \( G'_d(X, (P, a_\bullet), (Q, b_\bullet)) \) are isomorphic to relative Richardson varieties with respect to partial flags in \( \mathcal{H} \); first conjectured as [6, Conjecture 6.3] and then proven by [15]. Suppose \( \lambda, \lambda' \) are \( \text{Gr}(r + 1, d - g + 1) \)-admissible partitions given by vanishing sequences \( a_\bullet \) and \( b_\bullet \) as in Sect. 4.1. Then we have a generalization of Theorem 4.1.

**Conjecture 4.4** In the situation above where \( g > 1 \), the product of Bott–Samelson resolutions over the Brill–Noether variety \( G'_d(X) \)

\[
Z_\lambda(p) ×_{G'_d(X)} Z_{\lambda'}(q)
\]

is a resolution of singularities of \( G'_d(X, (P, a_\bullet), (Q, b_\bullet)) \).

It is not clear to the author if the proposed extension of the definition of versality to partial flags is sufficient to obtain similar desingularizations of Bott–Samelson type.
This is especially because when $d_k$ are very different, we do not know whether the technique in Theorems 3.1 and 3.6 can directly apply. We invite the reader to future investigations.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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