A NOTE ON LAGRANGIAN SUBMANIFOLDS OF TWISTOR SPACES AND THEIR RELATION TO SUPERMINIMAL SURFACES

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Abstract. In this paper a bijective correspondence between superminimal surfaces of an oriented Riemannian 4-manifold and particular Lagrangian submanifolds of the twistor space over the 4-manifold is proven. More explicitly, for every superminimal surface a submanifold of the twistor space is constructed which is Lagrangian for all the natural almost Hermitian structures on the twistor space. The twistor fibration restricted to the constructed Lagrangian gives a circle bundle over the superminimal surface. Conversely, if a submanifold of the twistor space is Lagrangian for all the natural almost Hermitian structures, then the Lagrangian projects to a superminimal surface and is contained in the Lagrangian constructed from this surface. In particular this produces many Lagrangian submanifolds of the twistor spaces \( \mathbb{C}P^3 \) and \( F_{1,2}(\mathbb{C}^3) \) with respect to both the Kähler structure as well as the nearly Kähler structure. Moreover, it is shown that these Lagrangian submanifolds are minimal submanifolds.

1. Introduction

The twistor space of an oriented Riemannian 4-manifold is inspired by Penrose’s twistor program and introduced in [2], where the authors use it to classify self-dual solutions of the Yang-Mills equations on \( S^4 \). By now many other spaces which also go by the name of twistor spaces have been constructed; see for example [22, 1, 6, 3, 21]. These twistor spaces have, among other things, been used to study all kinds of harmonic maps and relate these to holomorphic maps; see [9, 10, 23]. Particular instances of this are superminimal surfaces in 4-manifolds. These are minimal submanifolds and thus yield a harmonic immersion. These types of surfaces had been extensively studied from different points of view; see [4, 11, 16, 17]. Bryant proved in [5] that every compact Riemann surface admits a superminimal immersion into \( S^4 \) using the twistor fibration \( \mathbb{C}P^3 \to S^4 \). This inspired many studies of minimal surfaces from this twistor bundle point of view; see for example [12, 8, 18, 15]. There are not many results about Lagrangian submanifolds of these twistor spaces. In [7] Lagrangian submanifolds of the Kähler \( \mathbb{C}P^3 \) are constructed, which contain the ones constructed in this paper.

1.1. Results. In [24] all totally geodesic and all homogeneous Lagrangian submanifolds of \( F_{1,2}(\mathbb{C}^3) \), the nearly Kähler manifold of full flags in \( \mathbb{C}^3 \), are classified. The flag manifold \( F_{1,2}(\mathbb{C}^3) \) is the twistor space of \( \mathbb{C}P^2 \). This paper started out as a continuation of the study of Lagrangian submanifolds in \( F_{1,2}(\mathbb{C}^3) \). The relation found

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between superminimal surfaces of $\mathbb{CP}^2$ and Lagrangian submanifold of $\mathbb{F}_{1,2}(\mathbb{C}^3)$ turns out to apply for any twistor space $Z$ of an oriented Riemannian 4-manifold. For this we have to define almost Hermitian structures on the twistor space. The twistor spaces come equipped with two natural almost complex structures, which we denote by $J^\pm$; see (2.2) below. In (2.3) we pick a natural family of metrics $g_\lambda$ for $\lambda > 0$ on $Z$ making $(Z, g_\lambda, J^\pm)$ an almost Hermitian space. We will prove the following theorem in Section 4.

**Theorem A.** For a superminimal surface $\Sigma \subset M^4$ there exists a submanifold $L_\Sigma \subset Z$ which is a Lagrangian submanifold for all the almost Hermitian structures $(g_\lambda, J^\pm)$ simultaneously. This Lagrangian $L_\Sigma$ projects under the twistor fibration to $\Sigma$ and the restriction of the twistor fibration to $L_\Sigma$ determines a circle bundle over $\Sigma$. Conversely, given a submanifold $L \subset Z$ of the twistor space which is Lagrangian with respect to all the almost Hermitian structures $(g_\lambda, J^\pm)$ simultaneously, then the twistor fibration projects $L$ to a surface $\Sigma$ which is superminimal and $L$ is contained in $L_\Sigma$.

From the classification of totally geodesic Lagrangian submanifolds of $\mathbb{F}_{1,2}(\mathbb{C}^3)$ in [24] it follows that each of these is congruent under the symmetry group of the nearly Kähler structure of $\mathbb{F}_{1,2}(\mathbb{C}^3)$ to one which projects to a superminimal surface. With the correspondence of Theorem A we obtain infinitely many new examples of Lagrangian submanifolds of both the Kähler and nearly Kähler structures on $\mathbb{F}_{1,2}(\mathbb{C}^3)$ and $\mathbb{CP}^3$ by relating them to superminimal surfaces. Finally, it is shown that the submanifolds $L_\Sigma \subset (Z, g_\lambda)$ are minimal submanifolds.

2. Twistor space

In this section we give a short introduction to the twistor fibration over a 4-manifold. Let $(M^4, g)$ be an oriented Riemannian 4-manifold. For a given complex structure $J \in \text{End}(T_xM)$ which is compatible with the metric, i.e. $J \in O(T_xM)$, let $\omega_J(\cdot, \cdot) = g(J\cdot, \cdot) \in \Lambda^2 T^*_x M$. The complex structure $J$ is said to be compatible with the orientation if $\omega_J \land \omega_J$ is equal to the orientation of $M^4$ at $x$, which is denoted as $\omega_J \land \omega_J \gg 0$.

**Definition 2.1.** The twistor bundle $\pi : Z \to M^4$ of an oriented Riemannian manifold $(M^4, g)$ is the bundle whose fiber over a point $x \in M^4$ consists of all complex structures on the vector space $T_xM^4$ which are compatible with the Riemannian metric and the orientation, i.e.

$$\pi^{-1}(x) = \{ J \in \text{End}(T_x M^4) : J^2 = -1, J^* g = g \text{ and } \omega_J \land \omega_J \gg 0 \}.$$  

The fiber is isomorphic to $SO(4)/U(2) \cong \mathbb{CP}^1$.

Alternatively, the twistor bundle can be defined as the associated bundle of the principal $SO(4)$-frame bundle $\mathcal{F}$ of $(M^4, g)$ by

$$Z \cong \mathcal{F} \times_{SO(4)} SO(4)/U(2). \quad (2.1)$$

Here $SO(4)/U(2)$ is identified with all complex structures $J$ on $\mathbb{R}^4$ such that $J \in SO(4)$ and $\omega_J \land \omega_J \gg 0$. Elements of $Z$ are equivalence classes $[u, J]$, where $u : \mathbb{R}^4 \to T_xM$ is a frame and $J$ is a complex structure on $\mathbb{R}^4$. The equivalence relation is given by $[u \cdot g, J] \sim [u, g \cdot J]$ for $g \in SO(4)$. The Levi-Civita connection
on $M^4$ induces a horizontal subbundle $T^hZ$ which is complementary to the vertical subbundle $T^vZ$ of the tangent space of $Z$, i.e.

$$TZ = T^vZ \oplus T^hZ.$$ 

The derivative of $\pi$ restricted to $T^hZ$ yields an isomorphism between $T^h_IZ$ and $T_{\pi(I)}M$ for all $I \in Z$. Two natural almost complex structures on $Z$ are defined by

$$J^\pm_I = \pm J_{CP^1} + I \in \text{End}(T^I Z),$$

where $J_{CP^1}$ is the natural complex structure on the fiber $CP^1$ and $I \in Z$ is a complex structure on the vector space $T_{\pi(I)}M \cong T^h_IZ$. In [2] it is shown that $J^+$ is integrable if and only if $(M^4, g)$ is anti-self-dual, which is an essential ingredient for the classification of instantons of $S^4$ in [2]. Moreover, $J^+$ is conformally invariant; see [2]. The almost complex structure $J^-$ is never integrable; see [23].

A third point of view on the twistor space is the taken in [1]. Where the twistor space is naturally identified as a quotient of $F$, namely $Z := F/ U(2)$, where $U(2) \subset SO(4)$ is the stabilizer subgroup of some fixed complex structure $J_0$ on $R^4$. The natural isomorphism $F/ U(2)$ to our previous definition of the twistor space is given by $u \cdot U(2) \mapsto [u,J_0]$. This map does not depend on the representative $u$ of the coset and yields a bundle isomorphism. Let $so(5) = so(4) \oplus p$ be the Cartan decomposition. Let $u(2) \subset so(4)$ be the stabilizer algebra of $J_0$. This yields a decomposition $so(5) = u(2) \oplus n \oplus p$, where $n \subset so(4)$ is the orthogonal complement of $u(2)$ with respect to the Killing form. Throughout this work $B$ will denote the Killing form on $so(5)$. We will denote $m := n \oplus p$ and $h := u(2) \subset so(5)$. The advantage of this point of view is that $Z$ comes equipped with a principal $U(2)$-bundle $F \to Z$ together with a Cartan connection

$$\phi : F \to so(5) = h \oplus m,$$

which is just the Levi-Civita connection on the principal frame bundle combined with the soldering form. The homogeneous space $SO(5)/U(2)$ is an adjoint orbit of a central element $z \in Z(U(2))$. This induces a Kähler-Einstein structure on $SO(5)/U(2)$. The metric at the identity coset is given by

$$g_K = -2B|_n \oplus -B|_p.$$ 

The corresponding symplectic form $\omega_K$ is the Kirillov-Kostant-Souriau symplectic form and is up to a scaling factor given by $\text{ad}(z)$. The associated complex structure is above denoted by $J^+$. The almost complex structure $J^-$ is in this setting given by

$$J^-|_p = J^+|_p \quad \text{and} \quad J^-|_n = -J^+|_n.$$ 

More generally, The model space $SO(5)/U(2)$ comes equipped with a natural family of $U(3)$ structures

$$(g_\lambda, J^\pm),$$

where $g_\lambda = -\frac{1}{4\lambda}B|_n \oplus -B|_p$. The nearly Kähler structure on $SO(5)/U(2)$ is given by $(SO(5)/U(2), g_1, J^-)$. These $U(3)$-structures can be pulled back to $Z$ by the Cartan connection. Moreover, these structures are parallel with respect to this
connection. In the same way it is possible to induce a natural family of $SU(3)$-structures on $Z$. Let $\Upsilon$ denote the natural complex volume form induces from the nearly Kähler structure on $SO(5)/U(2)$. Then

$$(g_\lambda, J^\pm, \lambda \Upsilon) \quad (2.4)$$

defines a family of $SU(3)$-structures on $Z$. The results here do not depend on the particular value of $\lambda$. Some obvious properties of $g_\lambda$ are that $(Z, g_\lambda, J^\pm)$ is an almost Hermitian manifold and the fibers $\pi^{-1}(x) \subset Z$ are totally geodesic submanifolds.

3. SUPERMINIMAL SURFACES

Let $\Sigma$ be an oriented 2-dimensional surface in a 4-dimensional oriented Riemannian manifold $(M^4, g)$. Let $N\Sigma \subset TM^4$ be the normal bundle of $T\Sigma$. Let $J_0$ be the complex structure on $TM^4|\Sigma$ defined by a rotation by $\frac{\pi}{2}$ in $T_x\Sigma$ and a rotation by $\frac{\pi}{2}$ in $N_x\Sigma$. This determines a lift $F_0 : \Sigma \rightarrow Z$ of the inclusion $i : \Sigma \rightarrow M^4$:

$$\begin{array}{ccc}
\Sigma & \xrightarrow{i} & M^4 \\
\downarrow F_0 & & \downarrow \pi \\
Z & \xrightarrow{} & \\
\end{array}$$

with $\pi \circ F_0 = i$.

**Remark 3.1.** For an oriented surface $\Sigma \subset M^4$ there is a natural $U(2)$-reduction of the restricted frame bundle $\mathcal{F}|\Sigma$. Whenever we mention the group $U(2)$ below we are referring to this $U(2)$-reduction given by the complex structure $F_0$ along $\Sigma$.

Fix an oriented orthonormal local frame $u = (e_1, e_2, e_3, e_4)$ of $TM^4$ such that $(e_1, e_2)$ is an oriented basis of $T\Sigma$. The complex structure $J_0$ is determined by $J_0(e_1) = e_2$ and $J_0(e_3) = e_4$. In the identification of $Z$ with the associated bundle as in (2.1) the lift $F_0 : \Sigma \rightarrow Z$ is locally given by

$$F_0(x) = [u(x), J_0], \quad (3.1)$$

where we make some slight abuse of notation by denoting the complex structure on $\mathbb{R}^4$ corresponding to $J_0$ in the frame $u$ also by $J_0$.

**Definition 3.2.** An oriented surface $\Sigma \subset M^4$ is superminimal if the vertical component $(dF_0)^v$ of $dF_0$ vanishes.

**Remark 3.3.** It is easy to see that a superminimal surface is in particular minimal. An oriented minimal surface $\Sigma \subset M^4$ is superminimal if in addition $F_0 : \Sigma \rightarrow (Z, J^+)$ is holomorphic with respect to the complex structure of $\Sigma$ induced from the conformal structure; see [12].

The following lemma is quite trivial, but nevertheless it is a useful equivalent condition for superminimal surfaces.

**Lemma 3.4.** For an oriented surface $\Sigma \subset M^4$ the following are equivalent

(i) the surface $\Sigma$ is superminimal,

(ii) the complex structure $F_0$ on $TM|\Sigma$ is parallel, i.e. $\text{Hol}(\nabla|\Sigma) \subset U(2)$, where $U(2)$ is defined with respect to $F_0$. 

Proof. Let $\gamma : I \to \Sigma$ be a curve. We will use the description of $Z$ as associated bundle $\mathcal{F} \times_{SO(4)} SO(4)/U(2)$. Thus the local formula (3.1) gives

$$
(d_{\gamma'}(t)F_0)^v = \left(\frac{d}{dt}[u(\gamma(t)), J_0]\right)^v = \left(\frac{d}{dt}[u^h(\gamma(t)) \cdot g(t), J_0]\right)^v = \left(\frac{d}{dt}[u^h(\gamma(t)), g(t)J_0]\right)^v = \frac{d}{dt}g(t)J_0,
$$

where $u^h(\gamma(t))$ is a horizontal lift of $\gamma$ and $g(t) \in SO(4)$ is the parallel translation along $\gamma$ with respect to $\nabla|_Z$ in the frame $u$. In the last equality we identified the fiber $\pi^{-1}(\gamma(t))$ with the space of complex structures on $\mathbb{R}^4$ via $u^h(\gamma(t))$. This implies that $g(t) \in \text{stab}(J_0) \cong U(2) \subset SO(4)$ if and only if $(d_{\gamma'}(t)F_0)^v = 0$. Consequently, the holonomy of the $\nabla|_Z$ is contained in $U(2)$ if and only if $\Sigma$ is superminimal. □

There are a couple of interesting other equivalent formulations of superminimal surfaces. For example a surface $\Sigma \subset M^4$ is superminimal if and only if the indicatrix, also known as the curvature ellipse, of $\Sigma$ is a circle centered at zero. Or alternatively, $\Sigma$ is superminimal if and only if it is negatively oriented-isoclinic. This was proven in [17] for $M = \mathbb{R}^4$ and the general case is proven in [12]. These two properties of surfaces in a 4-manifold have been studied in the beginning of the 20th century. The equivalence of these other characterizations to Definition 3.2 is proven in [12].

4. A CORRESPONDENCE BETWEEN LAGRANGIANS AND SUPERMINIMAL SURFACES

Let $\Sigma \subset M^4$ be a superminimal surface. Below we will construct a submanifold $L_\Sigma \subset Z$, which is Lagrangian with respect to all of the almost Hermitian structures $(Z, g_\lambda, J^\pm)$. The restriction of the twistor fibration $\pi|_{L_\Sigma} : L_\Sigma \to \Sigma$ will be a circle bundle and the fibers of $\pi|_{L_\Sigma}$ are geodesics in $(Z, g_\lambda)$. In particular $L_\Sigma$ is a $\lambda$-ruled Lagrangian submanifold in the sense of [19].

Let $\Sigma$ be a superminimal surface in $(M^4, g)$ and let $N\Sigma \subset TM^4$ denote the normal bundle of $T\Sigma$. Let $u = (e_1, e_2, e_3, e_4)$ be an oriented orthonormal local frame of $M$ such that $(e_1, e_2)$ is an oriented orthonormal basis of $T\Sigma$. Define a submanifold $L_\Sigma$ of $Z$ as a bundle over $\Sigma$ by

$$
L_\Sigma \cap \pi^{-1}(x) := \{J \in Z_x(M^4) : J(T_x\Sigma) = N_x\Sigma\}.
$$

Note that $\pi|_{L_\Sigma} : L_\Sigma \to \Sigma$ is a circle bundle and the fibers can in terms of the frame be expressed as the complex structures determined by

$$
J_\theta(e_1) = \cos(\theta)e_3 + \sin(\theta)e_4, \quad J_\theta(e_2) = \sin(\theta)e_3 - \cos(\theta)e_4, \quad (4.1)
$$

for $\theta \in S^1$. Just as we did with $J_0$ we will also denote the complex structure on $\mathbb{R}^4$ corresponding to $J_\theta$ in the frame $u$ by $J_\theta$. These complex structures at a tangent space $T_xM^4$ for $x \in \Sigma$ can be depicted as in Figure 1. The complex structures $J_\theta$ form the equator of the fiber $\pi^{-1}(x)$ with respect to the poles $\{J_0, -J_0\}$.

Remark 4.1. The important property of the equator $S^1_x = \{J_\theta\}_{\theta \in S^1}$ of complex structures is that the subalgebra which preserves this equator is exactly the stabilizer algebra of $J_0$, i.e. $u(2) \subset \mathfrak{so}(4)$.

Proof of Theorem A. Suppose $\Sigma$ is a superminimal surface. Fix an oriented orthonormal local frame $u = (e_1, e_2, e_3, e_4)$ of $TM^4$ such that $(e_1, e_2)$ is an oriented basis of $T\Sigma$. First we show that the tangent space of $L_\Sigma$ is compatible
with the splitting of $Z$ into the vertical and horizontal subbundles. A point of $L_{\Sigma}$ is in the frame $u$ expressed as $J_0$ for some $\theta$. Note that tangent space of $L_{\Sigma}$ which is contained in the vertical subbundle of $TZ$ is equal to $T_{J_0}S^1_x$, where $\pi(J_0) = x \in \Sigma$. Let $F_\theta : \Sigma \to L_{\Sigma} \subset Z$ be the local lift given by $F_\theta(y) = [u(y), J_0]$. By Lemma 3.4 we know that parallel translation in $Z$ along a curve in $\Sigma$ preserves $J_0$. Thus by Remark 4.1 it also preserves the subbundle $L_{\Sigma} \subset Z$. From this we obtain that the vertical component of $dF_\theta(x)$ is contained in $T_{J_0}S^1_x$. Clearly $\text{im}(dF_\theta(x)) \oplus T_{J_0}S^1_x = T_{J_0}L_{\Sigma}$ holds. Consequently, the tangent space of $L_{\Sigma}$ at $J_0$ splits into a vertical and a horizontal subspace, i.e.

$$T_{J_0}L_{\Sigma} = T_{J_0}^vL_{\Sigma} \oplus T_{J_0}^hL_{\Sigma}.$$ 

Let $X \in T_{J_0}L_{\Sigma}$ and write it as $X = X^v + X^h$, with $X^v \in T_{J_0}^vL_{\Sigma}$ and $X^h \in T_{J_0}^hL_{\Sigma}$ and similarly for $Y \in T_{J_0}L_{\Sigma}$. We have

$$g_\lambda(J^\pm(X), Y) = \pm g_\lambda(J_{CP^1}(X^v), Y^v) + g_\lambda(J_\theta(X^h), Y^h) = 0,$$

where $J_{CP^1}(X^v)$ is perpendicular to $Y^v$ because $X^v$ and $Y^v$ are linearly dependent and $J_\theta(X^h)$ is perpendicular to $Y^h$ because $J_\theta$ maps $T_{\Sigma}$ to $N_{\Sigma}$. Thus $J^\pm(TL_{\Sigma})$ is perpendicular to $TL_{\Sigma}$ with respect to the metric $g_\lambda$. We conclude that $L_{\Sigma} \subset (Z, g_\lambda, J^\pm)$ is a Lagrangian submanifold with respect to all the almost Hermitian structures.

Conversely, suppose we are given a submanifold $L \subset (Z, g_\lambda)$ which is Lagrangian for both $J^+$ and $J^-$. Let $\omega_\pm$ be the Kähler forms of these almost Hermitian structures, i.e.

$$\omega_\pm(X, Y) = g_\lambda(J^\pm(X), Y).$$

Let $\omega_\pm = \omega^v_\pm + \omega^h_\pm$ be the decomposition into their vertical and horizontal parts. The vertical and horizontal parts of $\omega_\pm$ satisfy $\omega^v_\pm = - \omega^h_\pm$ and $\omega^h_\pm = \omega^h_\pm$. Thus both 2-forms $\omega^v_\pm$ and $\omega^h_\pm$ vanish on $L$. The projection of $T_I L$ on the horizontal distribution $T^h_I Z$ is a Lagrangian subspace for every $I \in L$ with respect to the restricted almost Hermitian structure, because $\omega^h_\pm$ vanish on $L$. Thus this projection necessarily has a non-trivial kernel. The projection of $T_I L$ onto the vertical distribution $T^v_I Z$ can be at most 1-dimensional, because $\omega^v_\pm$ vanishes on $L$. Consequently, the projection of $TL$ onto the vertical distribution is equal to its intersection with the vertical distribution and the splitting

$$T_I L = T^v_I L \oplus T^h_I L.$$
holds. In particular \( \Sigma := \pi(L) \subset M^4 \) is a 2-dimensional surface. As before we fix some oriented orthonormal local frame \( u = (e_1, e_2, e_3, e_4) \) on \( M^4 \) such that \( (e_1, e_2) \) is an oriented basis of \( T\Sigma \). A complex structure in \( L \) is then of the form \( J_0 \) as in (4.1) for some \( \theta \), because \( L \) is Lagrangian. A lift of \( \pi|_L : L \to \Sigma \) is given by \( F_\theta = [u, J_\theta] : \Sigma \to L \). This implies \( (dF_\theta)^\nu \in T^\nu_{J_\theta}L = T_{J_\theta}S^1 \) and by Remark 4.1 this implies that the holonomy group of \( \nabla|_\Sigma \) is contained in \( U(2) \). Therefore, by Lemma 3.4 we obtain that \( \Sigma \) is superminimal.

4.1. Minimal Submanifolds. For Lagrangian submanifolds in nearly Kähler manifolds it is known that they are minimal submanifolds; see [13]. Apart from these two homogeneous nearly Kähler twistor spaces the almost Hermitian manifolds \((Z, g_\lambda, J^\pm)\) are never nearly Kähler. However, in this section we will show that the Lagrangians constructed in Section 4 are minimal submanifolds for all of these spaces \((Z, g_\lambda)\) simultaneously.

In this section we denote the Levi-Civita connection on \( Z \) by \( \nabla^g \), where the index \( \lambda \) is excluded from the notation. The connection on \( Z \) obtained from the Cartan connection \( \phi : TF \to \mathfrak{so}(5) = \mathfrak{h} \oplus \mathfrak{m} \) on \( F \to Z \) is denoted by \( \nabla; \) see also Section 2 and [1] for more details. Furthermore, let \( A = \nabla^g - \nabla \). We will use \( \nabla \) to compute the mean curvature. Let

\[
B_0 := \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

and \( B_\theta := \exp(\theta J_0)B_0 \) for \( \theta \in S^1 \). These matrices are chosen such that \( B_\theta J_\theta B_\theta^{-1} = J_\theta \). Let \( u \) be an adapted frame along \( \Sigma \) as before. We define a lift \( \hat{u} : L_\Sigma \to F \) of \( \pi : F \to Z \) by

\[
\hat{u}([u(x), J_\theta]) = u(x) \cdot B_\theta.
\]

This is a local lift, because

\[
\pi(\hat{u}([u(x), J_\theta])) = [u(x) \cdot B_\theta, J_0] = [u(x), B_\theta \cdot J_0] = [u(x), J_\theta].
\]

Let us denote \( p_{x, \theta} := \hat{u}([u(x), J_\theta]) \). Let \( R_h : F \to F \) denote the right principal bundle action by an element \( h \in SO(4) \). We define an orthonormal frame of \( L_\Sigma \) as follows. Let

\[
\hat{v}_3(p_{x, \theta}) = \frac{\lambda}{\sqrt{12}} \frac{d}{d\theta} R_{B_\theta} (u(x)) \in T_{p_{x, \theta}} F
\]

and for \( i = 1, 2 \) we let

\[
\hat{v}_i(p_{x, \theta}) = R_{B_\theta} u_\ast e_i \in T_{p_{x, \theta}} F.
\]

Furthermore, if we put \( v_i = \pi_\ast \hat{v}_i \), then \( (v_1, v_2, v_3) \) is an orthonormal frame for \( L_\Sigma \) with respect to the metric \( g_\lambda \). The vector \( v_3 \) is vertical for the bundle \( Z \to M \) and the vectors \( v_1 \) and \( v_2 \) are horizontal.

The mean curvature will by computed by

\[
\sum_{i=1}^3 (\nabla_{v_i} v_i)^\perp = \sum_{i=1}^3 (\nabla_{\hat{v}_i} v_i)^\perp + (A(v_i)v_i)^\perp.
\]

We will show that \( \sum_{i=1}^3 (\nabla_{v_i} v_i)^\perp = \sum_{i=1}^3 (A(v_i)v_i)^\perp = 0 \) and thus that the mean curvature vector of \( L_\Sigma \subset (Z, g_\lambda) \) vanishes.
First we consider \( \sum_{i=1}^{3}(\nabla_{v_{i}}v_{i})^{\perp} \). We identify a vector field \( X \) on \( Z \) with its \( U(2) \)-equivariant function \( \Psi(X) : F \rightarrow m \), where \( m \) is defined in Section 2. The covariant derivative \( \nabla \) can be expressed in terms of the Cartan connection by

\[
\Psi(\nabla_{v_{i}}v_{i}) = d_{h,i}\phi_{m}(v_{i}) + \phi_{b}(v_{i})\cdot \phi_{m}(v_{i}),
\]

where \( \cdot \) denotes the restricted adjoint action of \( h \) on \( m \). First of all we have \( \phi(\hat{v}_{3}) = \frac{1}{\sqrt{2}}\text{Ad}(B_{0})^{-1} \cdot J_{0} \in n \), where \( J_{0} \) is seen as an element of \( h \subset \mathfrak{so}(5) \). This is a constant function and \( \phi_{b}(\hat{v}_{3}) = 0 \), thus we find \( \nabla_{v_{3}}v_{3} = 0 \). Furthermore,

\[
\sum_{i=1}^{2}\Psi(\nabla_{v_{i}}v_{i})^{\perp} = 2\sum_{i=1}^{2}(d_{h,i}\phi_{m}(R_{B_{0}}u_{i}e_{i}) + \phi_{b}(R_{B_{0}}u_{i}e_{i})\cdot \phi_{m}(R_{B_{0}}u_{i}e_{i}))^{\perp}
\]

\[
= 2\sum_{i=1}^{2}(\text{Ad}(B_{0})^{-1}d_{h,i}\phi_{m}(u_{i}e_{i}) + \text{Ad}(B_{0})^{-1}\phi_{b}(u_{i}e_{i})\text{Ad}(B_{0})\cdot \phi_{m}(R_{B_{0}}u_{i}e_{i}))^{\perp}
\]

\[
= (\text{Ad}(B_{0})^{-1}\sum_{i=1}^{2}d_{h,i}\phi_{m}(u_{i}e_{i}) + \phi_{b}(u_{i}e_{i})\cdot \phi_{m}(u_{i}e_{i}))^{\perp}
\]

\[
= 0,
\]

where \( B_{0} \in O(4) \) and in the last equality the sum expresses the mean curvature vector of \( \Sigma \subset M \) which is zero, because \( \Sigma \) is superminimal.

Next we consider \( \sum_{i=1}^{3}A(v_{i})v_{i} \). The connection form \( A \) with respect to the Levi-Civita connection can be expressed in terms of the torsion \( T \) of \( \nabla \) as

\[
g_{\lambda}(A(X)Y, Z) = \frac{1}{2}(T(X,Y,Z) - T(Y, Z, X) + T(Z, X, Y)).
\]

Thus we find

\[
\sum_{i=1}^{3}g_{\lambda}(A(v_{i})v_{i}, Z) = \sum_{i=1}^{3}\frac{1}{2}(T(v_{i}, v_{i}, Z) - T(v_{i}, Z, v_{i}) + T(Z, v_{i}, v_{i}))
\]

\[
= \sum_{i=1}^{3}T(Z, v_{i}, v_{i}).
\]

The torsion tensor \( T \) is identified with an equivariant function \( \hat{T} : F \rightarrow \Lambda^{2}m^{*} \otimes m \), which is given by

\[
\hat{T} = \kappa_{m} - t,
\]

where \( \kappa_{m} \) is the \( m \)-component of the curvature function of \( \phi \) and \( t \) is the constant function which takes the value \( g_{\lambda}([t(x,y), z]) = g_{\lambda}([x, y], z) \). The torsion of the Cartan connection is given by

\[
\kappa_{m} = \kappa_{a} + \kappa_{p} = R_{n} - [\phi_{a}, \phi_{p}],
\]

where \( \kappa_{n} = R_{n} \) is the Riemannian curvature tensor of \((M^{4}, g)\) followed by a projection onto \( n \); see also [1].

**Lemma 4.2.** If either \( X \in n \) or \( X \in p \), then for all \( Z \in m \) the following hold:

(i) \( g_{\lambda}([t(Z,X), X]) = 0 \),

(ii) \( g_{\lambda}(R(Z,X)_{n}, X) = 0 \),

(iii) \( g_{\lambda}([\phi_{a}(Z), \phi_{p}(X)] - [\phi_{a}(X), \phi_{p}(Z)], X) = 0 \).
Proof. For (i) we define \( t_\lambda \in \bigotimes^3 m^* \) by \( t_\lambda(X, Y, Z) = g_\lambda([X, Y], Z) \). Note that \( t_1 \in \Lambda^3 m^* \), because \( g_1 \) is equal to minus the Killing form of \( so(5) \) and thus invariant under the adjoint action. Suppose \( Z \in n \) and \( X \in n \), then \( t_\lambda(Z, X, X) = 0 \). Similarly, if \( Z \in p \) and \( X \in n \), then \( [Z, X]_m \in p \) and thus \( t_\lambda(Z, X, X) = 0 \). Lastly, if \( Z \in n \) and \( X \in p \), then \( t_\lambda(Z, X, X) = t_1(Z, X, X) = 0 \). By a similar argument we obtain (iii) holds.

From this lemma we obtain
\[
\sum_{i=1}^{3} T(Z, v_i, v_i) = 0,
\]
for all \( Z \). We conclude that the Lagrangian submanifold \( L_{\Sigma} \subset (Z, g_\lambda) \) is a minimal submanifold.

Remark 4.3. A \( k \)-form \( \alpha \) which satisfies \( \alpha(\eta) \leq 1 \) for every normalized \( k \)-multivector is called a generalized calibration; see [13]. A \( k \)-dimensional submanifold \( L \) which satisfies \( \alpha(\eta_x) = 1 \) for every \( x \in L \) is called calibrated, where \( \eta_x \) is a normalized top degree multivector on \( T_x L \). If in addition the form \( \alpha \) satisfies \( d\alpha = 0 \), then it is called a calibration form. Calibrated submanifolds of calibrations are automatically volume minimizing within their homology class; see [14]. An interesting phenomenon occurs for calibrated submanifolds of nearly Kähler manifolds. In [13] it is shown that special Lagrangian submanifolds are automatically minimal and in [20] it is shown that pseudo-holomorphic curves are also minimal. Furthermore, there are no pseudo-holomorphic surfaces of (strict) nearly Kähler structures. Thus every calibrated submanifold of a natural generalized calibration form of a nearly Kähler manifold is minimal. We have shown that the Lagrangian submanifolds \( L_{\Sigma} \), which we constructed here are also minimal. It might be interesting to point out that they are also calibrated by the complex volume form \( \lambda_\Upsilon \), which was defined in (2.4) and is a generalized calibration.

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