Arithmetic theory of $q$-difference equations

The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures

by

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Introduction

In 1941 Birkhoff and Guenther wrote: “Up to the present time the theory of linear $q$-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. In the opinion of the authors the use of the canonical system [...] is destined to carry the theory of $q$-difference equations to a comparable degree of completeness”. In the same paper they announced a program which they did not develop further, and $q$-difference equations theory remains less advanced today than difference and differential equation theories.

In recent years, mathematicians have reconsidered $q$-difference equations for their links with other branches of mathematics such as quantum algebras and $q$-combinatorics, and Birkhoff and Guenther’s program has been continued. Now there are theories of divergence (J-P. Bézivin, J-P. Ramis) and of $q$-summation (Ch. Zhang). There is also a good analogue of the concept of monodromy and thus a description of the $q$-difference Galois group in the regular case (P.I. Etingof, M. van der Put and M. Singer, J. Sauloy).

Bézivin and Ramis’s results on divergent series have applications to rationality criteria for solutions of systems of $q$-difference equations (cf. [BB]) and for systems of $q$-difference and differential equations (cf. [Ra]), which provides an answer to the old problem of finding criteria to establish whether a formal power series is the Taylor expansion of an algebraic or a rational function.

The question was first raised by Schwarz, who established an exhaustive list of hypergeometric differential equations having a full set of algebraic solutions. Grothendieck’s conjecture on $p$-curvatures tries to give a complete answer to this problem. More precisely, when we consider a differential equation

$$Ly = a_\mu(x) \frac{d^\mu y}{dx^\mu} + a_{\mu-1}(x) \frac{d^{\mu-1} y}{dx^{\mu-1}} + \ldots + a_0(x) y = 0,$$

with coefficients in the field $\mathbb{Q}(x)$, we can reduce the equation $Ly = 0$ modulo $p$ for almost all primes $p \in \mathbb{Z}$. Then Grothendieck’s conjecture predicts:

**Grothendieck’s conjecture on $p$-curvatures.** The equation $Ly = 0$ has a full set of algebraic solutions if and only if for almost all primes $p \in \mathbb{Z}$ the reduction modulo $p$ of $Ly = 0$ has a full set of solutions in $\mathbb{F}_p(x)$.

In spite of numerous papers dedicated to this conjecture in which some particular cases are proved (we recall [Ho], [CC], [K2], [K3], [A2], [Bo]), the conjecture remains open.

In this paper we give a proof of an analogous statement for $q$-difference equations. Following [K3], this allows for an arithmetic description of the generic Galois group of a $q$-difference equation. In fact, in [K3] N. Katz proposes a conjectural arithmetic description of the generic Galois group of a differential equation which is equivalent to Grothendieck’s conjecture:
Katz’s conjectural description of the generic Galois group. The Lie algebra of $\text{Gal}(M)$ is the smallest algebraic Lie sub-algebra of $\text{End}_{\mathbb{Q}(x)}(M)$ whose reduction modulo $p$ contains the $p$-curvature $\psi_p$ for almost all $p$.

Let us briefly explain his statement. Let $\mathcal{M} = (M, \nabla)$ be a $\mathbb{Q}(x)$-vector space with a $\mathbb{Q}(x)/\mathbb{Q}$-connection. We define the generic Galois group $\text{Gal}(M)$ of $\mathcal{M}$ to be the algebraic subgroup of $\text{GL}(M)$ stabilizing all the sub-quotients of the mixed tensor spaces $\oplus_{i,j} (M^*)^\otimes (M^*)^\otimes$. We can consider a lattice $\tilde{M}$ of $M$ over a finite type algebra over $\mathbb{Z}$, stable under the connection, and we can reduce $\tilde{M}$ modulo $p$, for almost all primes $p$. The operator $\psi_p = \nabla(\frac{d}{dx})^p$ acting over $\tilde{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$, is called the $p$-curvature. Moreover it makes sense to consider the reduction modulo $p$ for almost all $p$ of $\text{Gal}(M)$ and its Lie algebra.

Theorem 1. The $q$-difference equation $\mathcal{L}y = 0$ has a full set of solutions in $\mathbb{Q}(x)$ if and only if for almost all rational primes $p$ the set of equations $\mathcal{L}_p y = 0$ has a full set of solutions in $\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{F}_p\ast \mathbb{Z}$.

Theorem 1 is a partial answer to a Bézivin’s conjecture (cf. [Be1]).

Let $M$ be a finite dimensional $\mathbb{Q}(x)$-vector space equipped with a $q$-difference operator $\Phi_q : M \longrightarrow M$, i.e., with a $\mathbb{Q}$-linear invertible morphism such that $\Phi_q(fm) = f(qx)\Phi_q(m)$ for all $f(x) \in \mathbb{Q}(x)$ and all $m \in M$. As in the differential case, it is equivalent to consider a $q$-difference equation or a couple $(M, \Phi_q)$.

One can attach to $\mathcal{M} = (M, \Phi_q)$ an algebraic closed subgroup $\text{Gal}(M)$ of $\text{GL}(M)$, that we call the $q$-difference generic Galois group. It is the stabilizer of all $q$-difference sub-modules of all finite sums of the form $\oplus_{i,j} (M^*)^\otimes (M^*)^\otimes$, equipped with the operator induced by $\Phi_q$. We consider the reduction modulo $p^{\ell}$ of $M$ for almost all $p$, by reducing a lattice $\tilde{M}$ of $M$, defined over a $\mathbb{Z}$-algebra and stable by $\Phi_q$. The algebraic group $\text{Gal}(M)$ can also be reduced modulo $p^{\ell}$ for almost all $p$. Then our description of $\text{Gal}(M)$ is the following:

Theorem 2. The algebraic group $\text{Gal}(M)$ is the smallest algebraic subgroup of $\text{GL}(M)$ whose reduction modulo $p^{\ell}$ contains the reduction of $\Phi^\otimes q$ modulo $p^{\ell}$ for almost all $p$.

Taking into account the fact that $\Phi_q$ is a semi-linear endomorphism (which is easier to handle than the higher derivations occurring in the differential case), sometimes it happens that one can calculate all $\Phi^\otimes q$ at once and therefore determine the generic Galois group.

The techniques employed in the proof of theorem 1 are borrowed from the theory of $G$-functions. There are essentially two properties of arithmetic $q$-difference equations which allow us to obtain stronger results than in the differential case:

1) A formal power series with a nonzero radius of convergence, which is a solution of a $q$-difference equation, has infinite radius of meromorphy whenever $|q| > 1$: we say that solutions of $q$-difference equations have good meromorphic uniformization. If the algebraic number $q$ is not a root of unity, one can always find a place, archimedean or not, such that the associated norm of $q$ is greater than 1. This
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is the key-point of the proof: if we had good meromorphic uniformization of solutions of arithmetic differential equations, Grothendieck’s conjecture would become a corollary of \(G\)-function theory.

2) An arithmetic differential equation whose reduction modulo \(p\) can be written as a product of trivial factors for almost all \(p\) is regular singular and has rational exponents (cf. [K1, 13.0]). A \(q\)-difference equation whose reduction modulo \(p\) can be written as a product of trivial factors for almost every \(p\) is not only regular singular, but its “exponents” are in \(q\mathbb{Z}\): this means that the equation has a complete set of solutions in \(K((x))\).

In one instance the techniques used in \(G\)-function theory give a weaker result in the \(q\)-difference case: the \(q\)-analogue of the Katz estimates for the \(p\)-adic generic radius of convergence is very unsatisfactory (cf. §5 below). This is at the origin of many complications in the text (cf. (8.1)): actually the naive \(q\)-analogue of the notion of nilpotent reduction does not allow us to conclude the proof of theorem 1. A deeper analysis of the definition of \(p\)-curvatures for arithmetic differential equation shows that we can define two \(q\)-analogues of the notion of trivial reduction (cf. §3). Both of them are natural and useful. The first one permits us only to obtain the triviality over \(K((x))\), the second one leads to the triviality over \(K(x)\).

Finally, we want to stress the fact that we have very poor information on the sequence of integers \((\kappa_p)_p\) and no control at all over \((\ell_p)_p\). We are just able to prove (cf. (6.1.2)) that the sequence \((\kappa_p)_p\) completely determines the set \(\{q, q^{-1}\}\). The difficulties linked to these numbers and their distribution are the arithmetical counterpart of the classical (archimedean) problem of small divisors. This becomes clearer if we translate the definition of \(\kappa_p\) and \(\ell_p\) as follows:

\[
\kappa_p = \min\{m \in \mathbb{Z} : m > 0, |1 - q^m|_p < 1\}
\]

and \(p^{-\ell_p} = |1 - q^{\kappa_p}|_p\), where \(|\cdot|_p\) is the \(p\)-adic norm over \(\mathbb{Q}\) such that \(|p|_p = p^{-1}\).

Concerning the archimedean problem of small divisors, it would seem natural to assume that for all embeddings \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}\), the image of \(q\) in \(\mathbb{C}\) does not have complex norm 1. Actually, this assumption is not needed since \(q\) is an algebraic number. In fact in (8.3) we need to show that a formal power series \(y(x) \in \mathbb{C}[x]\) which is a solution of a regular singular \(q\)-difference equation with coefficients in \(\mathbb{C}(x)\) is convergent. In [Be2], the author gives some technical sufficient conditions on the estimate of \(|1 - q^n|\) to assure the convergence of the power series \(y(x)\). It is a consequence of Baker’s theorem on linear forms in logarithms that these conditions are always satisfied when \(q\) is an algebraic number: the idea is already present in [Be1]. It is possible that the techniques of (8.3) can be applied to more general problems of small divisors.

** **

This paper is organized as follows.

In the first part we introduce some basic properties of \(q\)-difference modules, in particular a \(q\)-analogue of the cyclic vector lemma. We then recall some results on the formal classification of \(q\)-difference modules. In §2 we prove a characterization of trivial \(q\)-difference modules and of \(q\)-difference modules which are extensions of trivial modules when \(q\) is a root of unity and \(K\) is a commutative ring. This degree of generality is motivated by theorem 1, where we consider a \(q\)-difference equation over a \(\mathbb{Z}/p^m\mathbb{Z}\)-algebra.

Part II is devoted to the \(p\)-adic situation. Section §3 contains some considerations on arithmetic differential modules, with the purpose of motivating the choice of considering two different \(q\)-analogues of the notion of nilpotent reduction. In §4 we introduce \(p\)-adic \(q\)-difference modules and we establish their primary properties. In particular we prove a \(q\)-analogue of the Dwork-Frobenius-Young theorem. In §5 we introduce the two notions of nilpotent reduction and revisit and translate some classical estimates for differential modules having nilpotent reduction in the \(q\)-difference setting (cf. [DGS, page 96]). The results of this section are crucial for the proof of the main theorem (7.1.1), together with the results of §6.

In Part III we consider the arithmetic situation. In §6 we prove a \(q\)-analogue of [K1, 13.0]: as we have already pointed out we obtain a stronger result than in the differential setting. Section §7 contains the statement of theorem 1 and §8 its proof.

In Part IV we introduce the generic \(q\)-difference Galois group and prove theorem 2. In §11 we show how theorem 2 can sometimes be an effective instrument for calculating Galois groups.
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Finally, the appendix contains an analogue of Schwarz’s list for basic hypergeometric series.

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Part I. Generalities on \( q \)-difference modules

1. \( q \)-difference modules

1.1. Summary of \( q \)-difference algebra

Let \( R \) be a commutative ring.

1.1.1. \( q \)-binomials. For any \( a, q \in R \) and any integer \( n \geq 1 \), we shall use the following standard notation:

\[
\begin{align*}
\left[0\right]_q &= 0, \quad \left[n\right]_q = 1 + q + \ldots + q^{n-1}, \\
\left[\frac{n}{i}\right]_q &= 1 + \frac{q}{1-q}, \\
\left(x - a\right)_n &= (x - a)(x - qa)\ldots(x - q^{n-1}a), \\
\left(a; q\right)_n &= (1 - a)_n.
\end{align*}
\]

If \( q \neq 1 \), we have \( \left[n\right]_q = \frac{1-q^n}{1-q} \).

The \( q \)-binomial coefficients \( \left(\begin{array}{c} n \\ i \end{array}\right)_q \) are the elements of \( R \) defined by the polynomial identity

\[
(1 - x)_n = \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array}\right)_q q^{j(j-1)/2} x^j.
\]

It was already known to Gauss that these are polynomials in \( q \) which have the following properties:

\[
\left(\begin{array}{c} n \\ 0 \end{array}\right)_q = \left(\begin{array}{c} n \\ n \end{array}\right)_q = 1,
\]

\[
\left(\begin{array}{c} n \\ i \end{array}\right)_q = \frac{\left[n\right]_q!}{\left[n-i\right]_q! [i]_q!} = \frac{[n]_q[n-1]_q\ldots[n-i+1]_q}{[i]_q!},
\]

\[
\left(\begin{array}{c} n \\ i \end{array}\right)_q = \left(\begin{array}{c} n-1 \\ i-1 \end{array}\right)_q + \left(\begin{array}{c} n-1 \\ i \end{array}\right)_q q^i = \left(\begin{array}{c} n-1 \\ i-1 \end{array}\right)_q q^{n-i} + \left(\begin{array}{c} n-1 \\ i \end{array}\right)_q, \quad \text{for } n \geq i \geq 1.
\]

1.1.2. \( q \)-dilatation. We fix a unit \( q \) in \( R \). We shall consider several rings of functions of one variable \( x \) and uniformly denote by \( \varphi_q \) the automorphism “of dilatation” induced by \( x \mapsto qx \). We shall denote this automorphism either by \( f(x) \mapsto f(qx) \) or by \( f \mapsto \varphi_q(f) \).

We shall informally refer to an \( R \)-algebra \( F \) of functions endowed with the operator \( \varphi_q \) as a \( q \)-difference algebra over \( R \). A morphism \( F \to F' \) of \( q \)-difference algebras is a morphism of \( R \)-algebras commuting with the action of \( \varphi_q \). Moreover, we shall say that a \( q \)-difference algebra \( F \) over \( R \) is essentially of finite type if there exist \( P_1, \ldots, P_n \in F \) such that \( F = R[P_1(q'x), \ldots, P_n(q'x); i \geq 0] \).

Examples. Typical examples of \( q \)-difference algebras are:
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(i) $R((x))$, with the obvious action of $\varphi_q$.

(ii) When $R$ is a field, the subfield $R(x)$ of $R((x))$ is a $q$-difference algebra over $R$.

(iii) The $R$-algebra

$$R[x-a]_q = \left\{ \sum_{n=0}^{\infty} a_n(x-a)^n : a_n \in R \right\}, \text{ for } a \in R, \ a \neq 0,$$

with $\varphi_q(x-a)_n = q^n(x-a)_n + q^{n-1}(q^n - 1)a(x-a)_{n-1}$.

(iv) Let us consider the $q$-difference algebra $R[x]$ and $P_1(x), \ldots, P_n(x) \in R[x]$. Then the $R$-algebra

$$R\left[x, \frac{1}{P_1(q^ix)}, \ldots, \frac{1}{P_n(q^ix)}, i \geq 0\right]$$

is a $q$-difference algebra essentially of finite type over $R$.

**Definition 1.1.3.** The ring $C = \{ f \in F : \varphi(f) = f \}$ is the subring of constants $F$.

**Example 1.1.4.** Let $F = R((x))$. If $q$ is not a root of unity, then $\varphi_q(f)(x) = 0$ if and only if $f \in R$. If $q$ is a primitive root of unity of order $\kappa$, then $\varphi_q(f)(x) = f$ if and only if $f \in R((x^\kappa))$.

**Definition 1.1.5.** A $q$-difference module $M = (M, \Phi_q)$ over a $q$-difference algebra $F$ is a free $F$-module $M$ of finite rank together with an $R$-linear automorphism:

$$\Phi_q : M \rightarrow M$$

satisfying the rule

$$\Phi_q(f(m)) = f(qx)\Phi_q(m), \text{ for every } f(x) \in F \text{ and every } m \in M.$$

**Remark.** The operator $\Phi_q$ is nothing but a $\varphi_q$-semilinear automorphism of the $F$-module $M$.

**Definition 1.1.6.** A morphism $\psi : (M, \Phi_q) \rightarrow (M', \Phi_q')$ is an $R$-linear morphism $M \rightarrow M'$ which commutes with the semilinear automorphisms $\Phi_q$ and $\Phi_q'$.

**Definition 1.1.7.** The $q$-difference module $M_F$ obtained from $M$ by extension of coefficients from $F$ to $F'$ is the $F'$-module $M \otimes_F F'$ equipped with the operator $\Phi_q \otimes \varphi_q$.

**1.1.8. $q$-derivations.** Let $q \neq 1$. Until the end of this subsection, we assume that $F$ is stable with respect to the operator

$$d_q : F \rightarrow F$$

$$f(x) \mapsto \frac{\varphi_q-id}{(q-1)x}f(x) = \frac{f(qx)-f(x)}{(q-1)x}.$$

**Remark.** The operator $d_q$ satisfies the twisted Leibniz rule:

$$d_q(fg)(x) = d_q(f)(x)g(x) + f(qx)d_q(g)(x).$$

More generally, for any positive integer $n$, we have

$$(1.1.8.1) d^n_q(fg)(x) = \sum_{j=0}^{n} \binom{n}{j}_q d^{n-j}_q(f(q^jx))d^j_q(g)(x).$$
Example 1.1.9.
1) Let us consider the $q$-difference algebra $F = R((x))$. For any positive integer $n$, $d_q x^n = [n]_q x^{n-1}$. More generally
\[
\frac{d^s_q}{[s]q!} x^n = \begin{cases} 
0 & \text{if } n < s \\
\binom{n}{s}_q x^{n-s} & \text{otherwise}
\end{cases}
\]

2) Let $F = R[x - a]; \text{then } d_q(x - a)_n = [n]_q(x - a)_{n-1}$.

We have the following relations between $d_q$ and $\varphi_q$:

Lemma 1.1.10. We set $d^0_q = \varphi^0_q = 1$. For any integer $n \geq 1$ we obtain:
\[
\varphi^n_q = \sum_{i=0}^{n} \binom{n}{i}_q (q-1)^i q^{i(i-1)/2} x^i d^i_q
\]
and
\[
d^n_q = \frac{(\varphi_q - 1)_n}{(q-1)^n q^{n(n-1)/2} x^n} = \frac{(-1)^n}{(q-1)^n x^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j}_q \frac{q^{-j(j-1)/2}}{(q-1)^{-j}} \varphi^j_q.
\]

Proof. We remark that $xd_q \circ x^i d^i_q = q^i x^{i+1} d^{i+1}_q + [i]_q x^i d^i_q$, for all $i \geq 1$. For $n = 2$ one has:
\[
\varphi^2_q = (q-1)^2 q x^2 d^2_q + [2]_q (q-1) x d_q + 1.
\]
It follows by induction that
\[
\varphi^{n+1}_q = ((q-1) x d_q + 1) \varphi^n_q
\]
\[
= \sum_{i=0}^{n} \binom{n}{i}_q (q-1)^i q^{i(i-1)/2} ((q-1)q^i x^{i+1} d^{i+1}_q + (q-1)[i]_q x^i d^i_q + x^i d^i_q)
\]
\[
= (q-1)^n + q^{n+1} x^{n+1} d^{n+1}_q + \sum_{i=1}^{n} \binom{n}{i}_q q^i + \binom{n}{i-1}_q (q-1)^i q^{i(i-1)/2} x^i d^i_q + 1
\]
\[
= \sum_{i=0}^{n+1} \binom{n+1}{i}_q (q-1)^i q^{i(i-1)/2} x^i d^i_q.
\]
The second formula in (1.1.10) holds for $n = 1$ by definition of $d_q$. By induction we obtain
\[
d^{n+1}_q = \frac{\varphi_q - 1}{(q-1)x} \frac{\varphi_q - 1}_n}{(q-1)^n q^{n(n-1)/2} x^n}
\]
\[
= \frac{(\varphi_q - q^n)(\varphi_q - 1)_n}{(q-1)^n q^{n+1} x^{n+1}}
\]
\[
= \frac{(\varphi_q - 1)_{n+1}}{(q-1)^n q^{n+1} x^{n+1}}
\]
\[
= \frac{(-1)^n}{(q-1)^n q^{n+1} x^{n+1}} (1 - \varphi_q) (1 - q^{-1} \varphi_q) \cdots (1 - q^{-n} \varphi_q).
\]
We conclude by using (1.1.11). ■

Remark. Let $M$ be a free $F$-module of finite rank. Let $\Delta_q : M \rightarrow M$ be an $R$-linear endomorphism satisfying the twisted Leibniz rule:
\[
(1.1.10.1) \quad \Delta_q(f(x)m) = f(qx)\Delta_q(m) + d_q(f)(x)m, \text{ for every } f(x) \in F \text{ and every } m \in M.
\]
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Then $\Phi_q = (q - 1)x\Delta_q + 1$ is $\varphi_q$-semilinear. Therefore, if it is invertible, it defines a $q$-difference module. Conversely, if $(q - 1)x$ is a unit in $\mathcal{F}$, any $\Phi_q$ gives rise to a twisted derivation $\Delta_q$ as before. We remark that $\Delta_q$ satisfies the generalized Leibniz formula:

$$
(1.1.10.2) \quad \Delta_q^n(f(x)m) = \sum_{i=0}^{n} \binom{n}{i} q^{-i}(q^i x)\Delta_q^i(m), \text{ for all } f \in \mathcal{F} \text{ and } m \in M.
$$

**Lemma 1.1.11.** The analogue of the formulas in (1.1.10) holds:

$$
(1.1.11.1) \quad \Phi_q^n = \sum_{i=0}^{n} \binom{n}{i} (q - 1)^i q^i(i-1)/2 x^i \Delta_q^i
$$

and

$$
(1.1.11.2) \quad \Delta_q^n = \frac{(\Phi_q - 1)_n}{(q - 1)^n q^{n(n-1)/2}x^n} = \frac{(-1)^n}{(q - 1)^n x^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} q^{-j(j-1)/2} \Phi_q^j.
$$

**Proof.** The proof is similar to the proof of (1.1.10). \qed

### 1.2. The $q$-analogue of the Wronskian lemma

**Lemma 1.2.1.** We assume that $q$ is not a primitive root of unity of order $\leq \mu$ and that the ring of constants $C = \{ f \in \mathcal{F} : \varphi_q(f) = f \}$ is a field. Let $u_0, \ldots, u_{\mu-1} \in \mathcal{F}$, then

$$
\dim C \sum_{i=0}^{\mu-1} Cu_i = \text{rank } \text{Cas}(u_0, \ldots, u_{\mu-1}),
$$

where $\text{Cas}(u_0, \ldots, u_{\mu-1})$ is the so-called Casorati matrix

$$
\text{Cas}(u_0, \ldots, u_{\mu-1}) = \begin{pmatrix}
    u_0 & \cdots & u_{\mu-1} \\
    \varphi_q u_0 & \cdots & \varphi_q u_{\mu-1} \\
    \vdots & \ddots & \vdots \\
    \varphi_q^{\mu-1} u_0 & \cdots & \varphi_q^{\mu-1} u_{\mu-1}
\end{pmatrix}.
$$

**Remark.** Of course, if $(q - 1)x$ is a unit of $\mathcal{F}$, lemma 1.1.10 implies that

$$
\text{rank } \begin{pmatrix}
    u_0 & \cdots & u_{\mu-1} \\
    d_q u_0 & \cdots & d_q u_{\mu-1} \\
    \vdots & \ddots & \vdots \\
    d_q^{\mu-1} u_0 & \cdots & d_q^{\mu-1} u_{\mu-1}
\end{pmatrix} = \text{rank } \begin{pmatrix}
    u_0 & \cdots & u_{\mu-1} \\
    \varphi_q u_0 & \cdots & \varphi_q u_{\mu-1} \\
    \vdots & \ddots & \vdots \\
    \varphi_q^{\mu-1} u_0 & \cdots & \varphi_q^{\mu-1} u_{\mu-1}
\end{pmatrix}.
$$

**Proof.** Obviously we have

$$
\dim C \sum_{i=0}^{\mu-1} Cu_i \geq \text{rank } (\varphi_q^j u_0, \ldots, \varphi_q^j u_{\mu-1})_{j=0, \ldots, \mu-1}.
$$

Let us suppose that the rank of $\text{Cas}(u_0, \ldots, u_{\mu-1})$ is $< \mu$. Changing the order of $u_0, \ldots, u_{\mu-1}$, we may assume that

$$
(1.2.1.1) \quad r = \text{rank } (\varphi_q^j u_0, \ldots, \varphi_q^j u_{\mu-1})_{j=0, \ldots, \mu-1} = \text{rank } (\varphi_q^j u_0, \ldots, \varphi_q^j u_{\mu-1})_{j=0, \ldots, \mu-1},
$$
The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures

with $r < \mu$. It is enough to show that $u_r$ is in $\sum_{i=0}^{r-1} C u_i$. By (1.2.1.1), there exists $(a_0, \ldots, a_{r-1}) \in F^r$, such that:

\[(\varphi_q^ju_0, \ldots, \varphi_q^{j-1}u_r)_{j=0, \ldots, r-1} = (\varphi_q^iu_r)_{j=0, \ldots, r-1} . \]

If we apply $\varphi_q$ to (1.2.1.2) and subtract the expression obtained from (1.2.1.2) we get

\[(\varphi_q^j(x), \ldots, \varphi_q^{j-1}(x))_{j=0, \ldots, r-1} \begin{pmatrix} \varphi_q(a_0) - a_0 \\ \vdots \\ \varphi_q(a_{r-1}) - a_{r-1} \end{pmatrix} = 0 \]

Hence $(\varphi_q a_0, \ldots, \varphi_q a_{r-1}) = (a_0, \ldots, a_{r-1})$ and therefore $a_i \in C$. ■

1.3. The $q$-analogue of the cyclic vector lemma

The following $q$-analogue of the classical cyclic vector lemma for differential modules is a classical result (cf. [S2, annexe B.2] and the very old references cited therein). It can also be deduced from the theory of skew fields (cf. for instance [Ch]). We prefer to give an elementary proof here, following [DGS, III, 4.2].

Lemma 1.3.1. Let us assume that $F$ is a field of characteristic zero and that $q$ is not a root of unity. Let $(M, \Phi_q)$ be a $q$-difference module of rank $\mu$ over $F$. Then there exists a cyclic vector $m \in M$, i.e. an element $m$ such that $(m, \Phi_q(m), \ldots, \Phi_q^{\mu-1}(m))$ is an $F$-basis of $M$.

Remark. By (1.1.11.2), if $m$ is a cyclic vector for $\Phi_q$, then it is also a cyclic vector with respect to the operator $\Delta_q$.

Proof. Let us denote the exterior product by $\wedge$. Let

\[\nu = \max \{ l \in \mathbb{Z} : \exists m \in M \text{ s.t. } m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{l-1}(m) \neq 0 \} ; \]

we suppose that $\nu$ is smaller than $\mu$ and we choose $m \in M$ such that

\[m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{\nu-1}(m) \neq 0 . \]

For all $\lambda \in C$, $s \in \mathbb{Z}$, $s \geq 1$, and $m' \in M$, there exists $m_1 \in \wedge^s M$, for $i = 0, \ldots, \nu$, such that we have

\[0 = (m + \lambda x^s m') \wedge \Phi_q(m + \lambda x^s m') \wedge \ldots \wedge \Phi_q^{\nu}(m + \lambda x^s m') = m_0 + m_1 \lambda + \ldots + m_\nu \lambda^\nu . \]

Since the field of constants $C$ is infinite, we have $m_0 = \ldots = m_\nu = 0$; in particular

\[m_1 = x^s \left( \sum_{i=0}^{\nu} q^{si} m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{i-1}(m) \wedge \Phi_q^i(m') \wedge \Phi_q^{i-1}(m') \wedge \ldots \wedge \Phi_q^{\nu}(m) \right) = 0 \]

for all positive integers $s$. It follows that for all $m' \in M$ and all $i = 0, \ldots, \nu$ we have

\[m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{i-1}(m) \wedge \Phi_q^i(m') \wedge \Phi_q^{i-1}(m') \wedge \ldots \wedge \Phi_q^{\nu}(m') = 0 . \]

In particular, for $i = \nu$ we obtain

\[m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{\nu-1}(m) \wedge \Phi_q^\nu(m') = 0 , \forall m' \in M , \]

which implies that $m \wedge \Phi_q(m) \wedge \ldots \wedge \Phi_q^{\nu-1}(m) = 0$. This contradicts the premises and hence $\nu = \mu$. ■

1.4. Formal classification of $q$-difference modules
The \( q \)-anologue of Grothendieck-Katz’s conjecture on \( p \)-curvatures

We recall the definition of regular singularity in the \( q \)-difference case, when \( K = R \) is a field of characteristic zero and \( q \) is not a root of unity. Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over \( K((x)) \) of finite rank \( \mu \).

**Definition 1.4.1.** One says that \( \mathcal{M} \) is regular singular if there exists a \( K((x)) \)-basis \( \mathcal{e} \) of \( M \) in which the matrix \( A(x) \) of \( \Phi_q \) (a priori an element of \( \text{Gl}_\mu(K((x))) \)) belongs to \( \text{Gl}_\mu(K[[x]]) \).

**Remark 1.4.2.** It is in fact equivalent to require the existence of a basis \( \mathcal{e} \) in which the matrix of \( \Phi_q \) is a constant matrix (cf. [PS, Ch. 12]).

One can say more: if \( \Phi_q(\mathcal{e}) = eA(x) \) with \( A(x) \in \text{Gl}_\mu(K[[x]]) \) and any couple \( \alpha, \beta \) of eigenvalues of \( A(0) \) is such that either \( \alpha = \beta \) or \( \alpha \beta^{-1} \notin q^2 \), then we can find a basis \( f \) of \( M \) over \( K((x)) \) such that \( \Phi_q(f) = fA(0) \). Observe that if \( \mathcal{M} \) is regular singular it is always possible to find a basis \( \mathcal{e} \) satisfying these properties (cf. [S2, 1.1.1] for a detailed proof). This remark will be useful in \( \S6 \).

**Definition 1.4.3.** The exponents of a regular singular \( q \)-difference module \( \mathcal{M} \), with respect to a given basis \( \mathcal{e} \) as in the definition above, are the \( q \)-orbits \( q^x a \) of the eigenvalues \( a \) of \( A(0) \).

Let us consider an extension of \( K((x)) \) of the form \( L((t)) \), where \( x = t^d \) and \( L \) is a finite extension of \( K \) containing a root \( \tilde{q} \) of \( q \) of order \( d \). Then \( \varphi_q \) extends to \( L((t)) \) in the following way:

\[
\varphi_q^- : L((t)) \rightarrow L((t)), \quad t \mapsto \tilde{q}t.
\]

The module \( L((t)) \otimes_{K((x))} M \), equipped with the operator

\[
\Phi_q^- : L((t)) \otimes_{K((x))} M \rightarrow L((t)) \otimes_{K((x))} M
\]

\[
f(t) \otimes m \mapsto \varphi_q^- (f(t)) \otimes \Phi_q(m)
\]

is a \( \tilde{q} \)-difference module over \( L((t)) \). We recall the following result that will be useful in the sequel:

**Theorem 1.4.4.** [P, Cor. 9 and §9, 3] Let \( K \) be a field of characteristic zero, \( q \) not a root of unity, \( \mathcal{M} \) a \( q \)-difference module over \( K((x)) \) of rank \( \mu \). Then there exists a divisor \( d \) of \( \mu \) and a finite extension \( L((t)) \) of \( K((x)) \) as above, such that the \( \tilde{q} \)-difference module \( L((t)) \otimes_{K((x))} M \) has an \( L((t)) \)-basis \( \mathcal{e} \) with the following property: the matrix \( A(t) \) defined by \( \Phi_q(\mathcal{e}) = eA(t) \) is a diagonal block matrix and each block has the form

\[
\begin{pmatrix}
\alpha_i & 0 & \cdots & 0 \\
1 & \alpha_i & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_i
\end{pmatrix} t^{1-\lambda_i},
\]

where \( \lambda_i \in (1/d)\mathbb{Z} \), \( \alpha_i \in \sum_{h=0}^{d} \alpha_{i,h} t^h \), with \( \alpha_{i,h} \in L \) and \( \alpha_{i,d} \neq 0 \).

The matrix \( A(t) \) is unique up to permutation of the blocks.

**Remark 1.4.5.** One can prove that a \( q \)-difference submodule of a regular singular \( q \)-difference module is regular singular (cf. [P]).

## 2. Unipotent \( q \)-difference modules

In this section \( R \) is again an arbitrary commutative ring and \( q \in R \) is a root of unity. Let \( \kappa \) denote its order:

\[
(2.0.5.1) \quad \kappa = \min\{m \in \mathbb{Z} : m > 0, \ q^m = 1 \}.
\]

Let \( \mathcal{F} \) be a \( q \)-difference algebra over \( R \) and \( C = \{ f \in \mathcal{F} : \varphi_q(f) = f \} \) the ring of constants of \( \mathcal{F} \). We notice that \( \varphi_q^\kappa = id_{\mathcal{F}} \).
Remark. Let $\mathcal{M} = (M, \Phi_q)$ be a $q$-difference module over $\mathcal{F}$. Then the operator $\Phi_q^\kappa$ is an $\mathcal{F}$-linear automorphism of $M$.

### 2.1. Trivial $q$-difference modules

**Definition 2.1.1.** The $q$-difference module $\mathcal{M} = (M, \Phi_q)$ over $\mathcal{F}$ is trivial if it is isomorphic to a $q$-difference module of the form $(N \otimes_C \mathcal{F}, \text{id}_N \otimes \varphi_q)$, where $N$ is a free $C$-module.

**Proposition 2.1.2.**

1) If $\mathcal{M}$ is trivial over $\mathcal{F}$ then $\Phi_q^\kappa$ is the identity morphism.

2) Let $R$ be a field and $\mathcal{F} = R(x)$. If $\Phi_q^\kappa$ is the identity, then $\mathcal{M}$ is trivial over $\mathcal{F}$.

**Proof.**

1) Let us suppose that $\mathcal{M}$ is trivial over $\mathcal{F}$. By hypothesis there exists a basis $\underline{e}$ of $M$ over $\mathcal{F}$ such that $\Phi_q(\underline{e}) = \underline{e}$ which implies that $\Phi_q^\kappa = 1$.

2) Since $R$ is a field we can consider the operator $\Delta_q = (\Phi_q - 1)/(q-1)x$ on $M$. When $q$ is a root of unity of order $\kappa$ the formula (1.1.11.1) simplifies to

\begin{align*}
\Phi_q^\kappa &= 1 + (q-1)x\Delta_q^\kappa.
\end{align*}

Therefore, under the assumption $\Phi_q^\kappa = 1$, the operator $\Delta_q$ is a $C$-linear nilpotent morphism of order $\kappa$. Let $\mu = \dim_R M$. There exists a basis $\underline{m} = (m_1, \ldots, m_\mu)$ of $M$ over $C = R(x^\kappa)$ such that the matrix of $\Delta_q$ with respect to $\underline{m}$ is an upper triangular nilpotent matrix in canonical form. In particular, this implies that $\Delta_q(m_1) = 0$. To conclude, it is enough to prove that there exists a basis $\underline{m}'$ of $M$ over $\mathcal{F}$ such that $\Delta_q(\underline{m}') = 0$, which is equivalent to $\Phi_q(\underline{m}') = \underline{m}'$.

If $\mu = 1$, it is enough to choose $\underline{m}' = (m_1)$. Let us suppose $\mu > 1$. If for all $i = 2, \ldots, \mu\kappa$ we have $\Delta_q(m_i) = m_{i-1}$, then $\Delta_q$ would be a nilpotent $C$-linear morphism of order $\mu\kappa > \kappa$, therefore there exists $j \in \{2, \ldots, \mu\kappa\}$ such that $\Delta_q(m_j) = 0$. We can suppose $j = 2$. Repeating the reasoning we find that $\Delta_q(m_1) = \ldots = \Delta_q(m_{\mu}) = 0$. We want to show that $(m_1, \ldots, m_\mu)$ is a basis of $M$ over $\mathcal{F}$. Let us suppose that $\sum_{i=1}^{\mu} a_i(x)m_i = 0$, with $a_i(x) \in R[x]$ and $a_1(x) \neq 0$, and that the degree $\deg_x a_1(x)$ of $a_1(x)$ with respect to $x$ is minimal. Then $\Delta_q(\sum_{i=1}^{\mu} a_i(x)m_i) = \sum_{i=1}^{\mu} d_q(a_i)(x)m_i = 0$, with $\deg_x d_q(a_1)(x) \leq \deg_x a_1(x) - 1$, so we get a contradiction. Finally, we have found a basis $\underline{m}' = (m_1, \ldots, m_\mu)$ of $M$ over $\mathcal{F}$ such that $\Delta_q(\underline{m}') = 0$.

If $(q-1)x$ is a unit of $\mathcal{F}$, the operator $\Delta_q$ is defined over $M$ and (2.1.2.1) shows that:

**Corollary 2.1.3.** The operator $\Phi_q^\kappa$ is unipotent if and only if $\Delta_q^\kappa$ is nilpotent.

### 2.2. Extensions of trivial $q$-difference modules

**Proposition 2.2.1.**

1) If the $q$-difference module $\mathcal{M}$ is an extension of trivial $q$-difference modules then the $\mathcal{F}$-linear morphism $\Phi_q^\kappa$ is unipotent.

2) If $R$ is a field, $\mathcal{F} = R(x)$ and $\Phi_q^\kappa$ is unipotent, then $\mathcal{M}$ is an extension of trivial $q$-difference modules.

**Proof.**

1) We have to prove that $\Phi_q^\kappa - 1$ is a nilpotent endomorphism. If $\mathcal{M}$ is an extension of trivial $q$-difference modules over $\mathcal{F}$, by (2.1.2) we can find a basis $\underline{e}$ of $M$ over $\mathcal{F}$ such that $\Phi_q^\kappa(\underline{e}) = \underline{e}(\mathbb{I} + H_\kappa(x))$, where $\mathbb{I}$ is the identity matrix and $H_\kappa(x)$ is a block matrix of the form

\begin{align*}
H_\kappa(x) = \begin{pmatrix}
0 & * & * \\
0 & \ddots & * \\
0 & 0 & 0
\end{pmatrix}.
\end{align*}

The matrix $H_\kappa(x)$ is nilpotent, hence $\Phi_q^\kappa$ is a unipotent endomorphism.
2) If $R$ is a field we can consider the operator $\Delta_q$ associated to $\Phi_q$. Since $\Phi_q^n$ is unipotent, the $C$-linear morphism $\Delta_q$ is nilpotent (cf. (2.1.3)), therefore there exists $m_1 \in M$ such that $\Delta_q m_1 = 0$. Let $\mu = \dim \mathcal{F} M$. If $\mu = 1$ there is nothing more to prove. Let $\mu$ be greater than 1. The operator $\Delta_q$ induces a structure of a $q$-difference module over the quotient $\mathcal{F}$-vector space $M/\mathcal{F} m_1$ which satisfies the hypothesis. By induction we can find a filtration of $M/\mathcal{F} m_1$

$$\tilde{M}_0 = \{0\} \subset \tilde{M}_1 \subset \ldots \subset \tilde{M}_l = M/\mathcal{F} m_1,$$

such that:

1) for all $i = 0, \ldots, l$ the sub-vector space $\tilde{M}_i$ is stable by the operator induced by $\Delta_q$;
2) for all $i = 1, \ldots, l$ the quotient module $\tilde{M}_i/\tilde{M}_{i-1}$ equipped with its natural structure of a $q$-difference module is trivial over $\mathcal{F}$.
Let $\iota : M \to M/\mathcal{F} m_1$ be the canonical projection. Then

$$M_{-1} = \{0\} \subset M_0 = \iota^{-1}(\tilde{M}_0) = \mathcal{F} m_1 \subset M_1 = \iota^{-1}(\tilde{M}_1) \subset \ldots \subset M_l = \iota^{-1}(\tilde{M}_l) = M$$

satisfies the conditions:
1) for all $i = -1, 0, \ldots, l$ the sub-vector space $M_i$ is stable under the operator induced by $\Delta_q$;
2) for all $i = 0, \ldots, l$ the quotient module $M_i/M_{i-1} \cong \tilde{M}_i/\tilde{M}_{i-1}$ equipped with its natural structure of a $q$-difference module is trivial over $\mathcal{F}$.

\[\blacksquare\]

**Remark 2.2.2.** In [H, Ch. 6] we find a classification of $q$-difference modules over $R(x)$ when $q$ is a root of unity and $R$ is a field of characteristic zero. The author defines the Galois group associated to a linear $q$-difference module and proves that it is the smallest algebraic group over $R(x^q)$ containing $\Phi_q^n$.

### Part II. $p$-adic methods

#### 3. Considerations on the differential case

We would like to recall some properties of arithmetic differential modules that are supposed to motivate the structures we will introduce in the sequel. In particular the considerations below show that the two notions of nilpotent reduction introduced in §5 are both natural $q$-analogues of the notion of nilpotent reduction for differential modules.

Let us consider the field of rational numbers $\mathbb{Q}$. For all prime $p \in \mathbb{Z}$ we consider the $p$-adic norm $| \cdot |_p$ over $\mathbb{Q}$, normalized so that $|p|_p = p^{-1}$. By the Gauss lemma, the norm $| \cdot |_p$ can be extended to the $p$-adic Gauss norm $| \cdot |_{p,Gauss}$ over $\mathbb{Q}(x)$, by setting

$$\left| \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{j=0}^{l} b_j x^j} \right|_{p,Gauss} = \frac{\sup_{i=0,\ldots,n} |a_i|_p}{\sup_{j=0,\ldots,m} |b_j|_p}.$$

Let us consider a differential module $(M, \Delta)$ over $\mathbb{Q}(x)$, i.e. a $\mathbb{Q}(x)$-vector space $M$ of finite dimension $\mu$ equipped with a $\mathbb{Q}$-linear morphism $\Delta : M \to M$ such that $\Delta(f m) = \frac{\partial}{\partial x} m + f \Delta(m)$, for all $f \in \mathbb{Q}(x)$ and $m \in M$. We fix a basis $\mathbf{e}$ of $M$ over $\mathbb{Q}(x)$ and set $\Delta^{\mathbf{e}} = \epsilon G_\mathbf{e}(x)$, for all $\epsilon \geq 1$, where $G_\mathbf{e}(x) = G(x)$. The matrix $G_0(x) = \mathbf{I}_\mu$. The matrix $\sum_{n \geq 0} \frac{G_\mathbf{e}_{ij}(x)}{n!} (x-t)^n \in \mathbb{Q}(x)[[x-t]]$ is a formal solution of $\frac{\partial Y}{\partial x} = Y G_1(x)$ at $t$ in the algebraic closure of $\mathbb{Q}$, such that $G_1(x)$ has no pole at $t$.

For almost all primes $p \in \mathbb{Z}$ we have $|G_1(x)|_{p,Gauss} \leq 1$ and we can consider the image of $G_1(x)$ in $M_{\mu \times \mu}(\mathbb{F}(x))$. One usually says that $(M, \Delta)$ has $p$-adic nilpotent reduction of order $n$ if $|G_1(x)|_{p,Gauss} < 1$ or, equivalently, if $G_n(x) \equiv 0$ modulo $p$. If $n = 1$ we say that $(M, \Delta)$ has $p$-curvature zero.

Another equivalent statement of the Grothendieck conjecture (cf. Introduction) is:

**Grothendieck’s conjecture.** If $(M, \Delta)$ has $p$-curvature zero for almost all primes $p$, then $(M, \Delta)$ becomes trivial over an algebraic extension of $\mathbb{Q}(x)$. 

The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures

Let us consider the $p$-adic generic radius of convergence

$$R_p(M) = \inf \left( 1, \liminf_{n \to \infty} \left| \frac{G_n(x)}{n!} \right|_{p, Gauss}^{-1/n} \right).$$

If $(M, \Delta)$ has $p$-adic nilpotent reduction of order $n$ we have:

$$R_p(M) \geq p^{1/n} p^{-1/(p-1)};$$

in particular if $(M, \Delta)$ has $p$-curvature zero, the previous inequality reduces to $R_p(M) \geq p^{-1/p(p-1)}$.

An important and useful property of arithmetic differential modules is that

$$\left( \mu \right) \text{ has a fundamental matrix of solutions } \mathbf{Y} \text{ for the case of characteristic } p.$$  

More accurate analysis of the case of $p$-curvature zero leads to the following remark. Let \( \overline{G}_1(x) \) be the image of \( G_1(x) \) in \( M_{\mu \times \mu}(\mathbb{F}_p(x)) \). By imposing that \( G_p(x) \equiv 0 \) modulo \( p \) we are actually requiring that the differential system in positive characteristic

$$\frac{d\mathbf{Y}}{dx} = \mathbf{Y} \overline{G}_1(x)$$

has a fundamental matrix of solutions \( \mathbf{Y}(x) \in GL_\mu(\mathbb{F}_p(x)) \). Since the derivation \( \frac{1}{p} \frac{d\mathbf{Y}}{dx} \) makes sense in characteristic \( p \) this implies that \( \frac{1}{p} \frac{d\mathbf{Y}}{dx} \equiv \frac{G_p(x)}{p} \mathbf{Y} \) modulo \( p \), with \( |G_p(x)|_{p, Gauss} \leq p^{-1} = |p|_p \).

The problem is that in the $q$-differential case, one can define some $q$-analogue of factorials, but they generally are $p$-adically smaller than the uniformizer \( p \). It turns out that the $q$-analogue of the condition \( G_p(x) \equiv 0 \) modulo \( p \) is not equivalent to the $q$-analogue of the condition \( \left| \frac{G_p(x)}{p} \right|_{p, Gauss} \leq 1 \), but both of them are linked to the property of a suitable $q$-difference system having a fundamental solution matrix in some polynomial ring over a quotient of \( \mathbb{Z} \). Therefore, for a $q$-difference module, we have two natural notions of nilpotent reduction: from a local point of view the notions are not equivalent (cf. §5), but we conjecture that they are globally equivalent.

4. Introduction to $p$-adic $q$-difference modules

Let \( K_v \) be a field of characteristic zero, complete with respect to a non-archimedean norm \( | \cdot |_v \). Let \( \mathcal{V}_v \) be the ring of integers of \( K_v \), \( \mathcal{O}_v \) the uniformizer of \( \mathcal{V}_v \), \( k_v \) its residue field of characteristic \( p > 0 \).

We fix a nonzero element \( q \in K_v \), such that \( q \) is not a root of unity and \( |q|_v = 1 \). Let \( \overline{q} \) be the image of \( q \) in \( k_v \); we suppose that \( \overline{q} \) is algebraic over the prime field \( \mathbb{F}_p \) and we set

$$k_v = \min \{ m \in \mathbb{Z} : m > 0, \overline{q}^m = 1 \} \geq 1.$$  

We notice that \( \overline{q} \in k_v \), satisfies the assumption of §2.

In addition, we assume that

$$|1 - q^{\kappa_v}|_v < |p|_v^{1/(p-1)}$$

If \( q \) is an element of \( \mathbb{Q}_p \subset K_v \) and \( p > 2 \), this holds automatically in fact \( |1 - q^{\kappa_v}|_v \leq |p|_v < |p|_v^{1/(p-1)} \).

4.1. $p$-adic estimates of $q$-binomials

Lemma 4.1.1. Let \( n \geq i \geq 0 \) be two integers. We have

\[ |n|_q^i |_v = |[\kappa_v]_q |_v \left[ \frac{\overline{q}^i}{[\kappa_v]} \right]_v, \]
The \( q \)-analogue of Grothendieck-Katz’s conjecture on \( p \)-curvatures

where \([x]\) is the integer part of \( x \in \mathbb{R} \), and

\[
(4.1.1.2) \quad \left| \binom{n}{i}_q \right|_v \leq 1 .
\]

**Proof.**

1) By the definition of \( \kappa_v \), if \( \kappa_v \) does not divide \( n \), \( |1 - q^n|_v = 1 \). Since \( |1 - q^{n\kappa_v}|_v \leq |1 - q^{|\kappa_v|}|_v < |p|^{1/(p-1)} \) for all \( n \in \mathbb{Z} \), \( n \geq 1 \), we have (cf. for instance [DGS, II, 1.1])

\[
(4.1.1.3) \quad |1 - q^{n\kappa_v}|_v = \log q^{n\kappa_v} = |n\log q^{\kappa_v}|_v = |n|_v 1 - q^{\kappa_v} ,
\]

so \( |n\kappa_v|_v = \left| \frac{1 - q^n}{1 - q} \right|_v \). We obtain

\[
\left| \frac{n|d|}{|d|} \right|_v = \left| \left( \frac{\kappa_v}{\kappa_v} \right) \prod_{i \leq |\kappa_v|} \left| n_i \right|_v \right| \left( \frac{n}{\kappa_v} \right)_v .
\]

Since \( \kappa_v \) is a divisor of \( p^s - 1 \) for a suitable integer \( s \geq 1 \), we have \( (\kappa_v, p) = 1 \), which implies that \( |\kappa_v|_v = 1 \).

2) Since \( q \) is an invertible element of the ring of integers \( \mathcal{V}_v \), we obtain the inequality \( \left| \binom{n}{i}_q \right|_v \leq 1 \) using the relation

\[
(1 - x)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j}_q q^{j(j-1)/2} x^j \in \mathcal{V}_v[x] .
\]

\[
\blacksquare
\]

**4.2. The Gauss norm and the invariant \( \chi_v(M) \)**

By the Gauss lemma, one can extend the norm \( |x|_v \) to the so-called Gauss norm \( |x|_{v, \text{Gauss}} \) over \( K_v(x) \) by setting

\[
\left| \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{j=0}^{m} b_j x^j} \right|_{v, \text{Gauss}} = \frac{\sup_{i=0, \ldots, n} |a_i|_v}{\sup_{j=0, \ldots, m} |b_j|_v} .
\]

Remark that \( | \cdot |_{v, \text{Gauss}} \) is multiplicative.

**Lemma 4.2.1.** For any \( f(x) \in K_v(x) \) and any positive integer \( n \), we have

\[
\left| \frac{d^n f(x)}{[n]_q^q} \right|_{v, \text{Gauss}} \leq |f(x)|_{v, \text{Gauss}} .
\]

**Proof.** By (1.1.9) and (4.1.1) the inequality holds for all \( f(x) \in K_v[x] \). Furthermore we have

\[
\left| d_q \left( \frac{1}{f(x)} \right) \right|_{v, \text{Gauss}} = \left| \frac{d_q f(x)}{f(x)f(qx)} \right|_{v, \text{Gauss}} \leq \left| \frac{1}{f(x)} \right|_{v, \text{Gauss}} .
\]

By the \( q \)-analogue of the Leibniz formula (1.1.10.2) we have

\[
\frac{d^n}{[n]_q^q} \left( \frac{1}{f(x)} \right) = -\frac{1}{f(q^n x)} \sum_{i=0}^{n-1} \frac{h^i}{[i]_q^i} \left( f(q^{n-i} x) \right) \frac{d_q^{n-i}}{[n-i]_q} \left( \frac{1}{f(x)} \right) ;
\]

and therefore by induction

\[
\left| \frac{d^n}{[n]_q^q} \left( \frac{1}{f(x)} \right) \right|_{v, \text{Gauss}} \leq \left| \frac{1}{f(x)} \right|_{v, \text{Gauss}} .
\]
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Finally, if \( g(x) \in K_v[x] \) we obtain

\[
\left| \frac{d^n_q}{[n]_q!} \left( \frac{g(x)}{f(x)} \right) \right|_{v,Gauss} = \left| \sum_{i=0}^{n} \frac{d^n_q}{[i]_q!}(g(q^{-i}x)) \frac{d^{n-i}}{[n-i]_q!} \left( \frac{1}{f(x)} \right) \right|_{v,Gauss} \leq \left| \frac{g(x)}{f(x)} \right|_{v,Gauss}.
\]

Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over \( K_v(x) \). Since \( K_v \) is a field, we have well defined operators

\[
d_q = \frac{q - 1}{(q - 1)x} \quad \text{and} \quad \Delta_q = \Phi_q - \frac{1}{(q - 1)x}
\]

acting over \( K_v(x) \) and \( M \), respectively (cf. (1.1.8)).

We fix a basis \( \mathcal{e} \) of \( M \) over \( K_v(x) \) and define a sequence of matrices \( G_n(x) \in M_{\mu \times \mu}(K_v(x)) \) for any integer \( n \geq 0 \) by setting

\[
(4.2.1.1) \quad \Delta^n_q(\mathcal{e}) = \mathcal{e}_n G_n(x).
\]

The matrices \( G_n(x) \) satisfy the inductive relation

\[
(4.2.1.2) \quad G_0(x) = \mathbb{I}_\mu, \quad G(x) = G_1(x), \quad G_{n+1}(x) = G_1(x)G_n(qx) + d_q G_n(x).
\]

We call

\[ (\mathcal{S}) \quad d_q Y = Y G(x) \]

the \( q \)-difference system associated to \( \mathcal{M} \) with respect to the basis \( \mathcal{e} \). If zero is not a pole of \( G_1(x) \), we obtain a formal solution of \( (\mathcal{S}) \): \( \sum_{n=0}^{\infty} \frac{G_{\mathcal{S}1}(0)}{[n]_q!} x^n \). More generally, if \( q^n a \) is not a pole of \( G_1(x) \) for all positive integers \( n \), the matrix

\[
(4.2.1.3) \quad Y(x) = \sum_{n=0}^{\infty} \frac{G_n(a)}{[n]_q!} (x - a)_n \in M_{\mu \times \mu}(K_v [x - a]_q)
\]

is a formal solution of \( (\mathcal{S}) \).\(^1\)

One can easily check that if \( Y \) is a fundamental matrix for \( (\mathcal{S}) \) with coefficients in some fixed \( q \)-difference algebra \( \mathcal{F} \) (i.e. an invertible matrix solution of \( (\mathcal{S}) \) with coefficients in \( \mathcal{F} \)), any other fundamental matrix of \( (\mathcal{S}) \) in \( GL_\mu(\mathcal{F}) \) is of the form \( Y F \), where \( F \) is an invertible matrix with coefficients in the subring of constants \( \mathcal{F} \).

In the following definition, the sup-norm of a matrix is the maximum of the norms of its entries:

**Definition 4.2.2.** \( \chi_v(\mathcal{M}) = \inf \left( 1, \lim_{n \to \infty} \left| \frac{G_n(x)}{[n]_q!} \right|_{v,Gauss}^{-1/n} \right) \).

**Lemma 4.2.3.** Let

\[ h(n) = \sup_{s \leq n} \log^+ \left| \frac{G_s(x)}{[s]_q!} \right|_{v,Gauss} = \sup_{s \leq n} \log \left| \frac{G_s(x)}{[s]_q!} \right|_{v,Gauss}, \]

\(^1\) The ring \( K_v [x - a]_q \) is neither integral nor local: in particular the remark that \( Y(a) = \mathbb{I}_\mu \) is not enough to conclude that \( Y(x) \in GL_\mu(K_v [x - a]_q) \) (which is true, anyway). Moreover there exists subalgebra of \( K_v [x - a]_q \) that can be identified to the algebra of analytic function over a \( q \)-invariant analytic domain, eventually non connected, via a \( q \)-Taylor expansion. As a consequence, it is possible to generalize (4.2.6) and (4.3.3) below, obtaining more precise statements. The proofs of these facts, that are not relevant for the sequel, are in a paper in preparation by the author on the \( p \)-adic theory of \( q \)-difference equations.
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with $\log^+ x = \log \sup(x, 1)$, for all $x \in \mathbb{R}$. Then

$$\limsup_{n \to \infty} \frac{h(n)}{n} = \log \frac{1}{\chi_v(M)}.$$ 

Moreover, $\chi_v(M)$ is independent of the choice of the $K_v(x)$-basis $e$ of $M$.

**Proof.** We recall that $G_0(x) = I_q$, and therefore the two definitions of $h(n)$ are equivalent.

Let $f = eF(x)$ be another $K_v(x)$-basis of $M$, with $F(x) \in \text{GL}_q(K_v(x))$. For any integer $n \geq 0$, we set:

\[
\begin{aligned}
\Delta_q^n(a) &= eG_n(x), \quad h_e(n) = \sup_{s \leq n} \frac{|G_s(x)|}{|s|q^1}^v_{\text{Gauss}}; \\
\Delta_q^n(f) &= fH_n(x), \quad h_f(n) = \sup_{s \leq n} \frac{|H_s(x)|}{|s|q^1}^v_{\text{Gauss}}.
\end{aligned}
\]

By (1.1.10.2) we have

\[
\Delta_q^n(f) = \frac{\Delta_q^n(eG_n)}{|n|q!} = \frac{\Delta_q^n(eF(x))}{|n|q!},
\]

and hence it follows that

$$h_f(n) \leq \log |F(x)|^{-1}v,Gauss + \log |F(x)|v,Gauss + h_e(n).$$

By symmetry, we deduce that

$$\limsup_{n \to \infty} \frac{h_e(n)}{n} = \limsup_{n \to \infty} \frac{h_f(n)}{n}.$$

Let $h(n) = h_e(n)$. It is a general fact (cf. for instance the proof of [DGS, VII, Lemma 4.1]) that

$$\limsup_{n \to \infty} \frac{h(n)}{n} = \log \frac{1}{\chi_v(M)}.$$

Let $a$ be an element of $K_v$ such that $q^n a$ is not a pole of $G_1(x)$ for any $n \geq 0$. We want to relate $\chi_v(M)$ and the radius of convergence of the matrix $\sum_{n=0}^{\infty} \frac{G_n(a)}{|n|q^1} (x - a)_n$, which solves the linear $q$-difference system associated to $M$, with respect to the basis $e$. First of all, we notice that if $|a|_v \leq 1$, we have

$$\left| \frac{G_n(x)}{|n|q^1}^v_{\text{Gauss}} \right| \geq \left| \frac{G_n(a)}{|n|q^1}^v \right|,$$

and therefore we obtain

$$\chi_v(M) \leq \liminf_{n \to \infty} \left| \frac{G_n(a)}{|n|q^1}^v \right|^{-1/n}.$$

Hence, if zero is not a pole of $G_1(x)$, the matrix $\sum_{n=0}^{\infty} \frac{G_n(0)}{|n|q^1} x^n$ converges at least for $|x|_v < \chi_v(M)$.

**Example 4.2.4.** Let us consider the analogue of the exponential series

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{|n|q^1}.$$

Obviously $\exp_q(x)$ is the solution at zero of the $q$-difference equation $d_q y = y$, which is the system associated to the $q$-difference module $(K_v(x), \Delta_q)$, with $\Delta_q(f(x)) = d_q f(x) + f(qx)$ for all $f(x) \in K_v(x)$. Then $\chi_v(K_v(x), \Delta_q)$ coincides with the radius of convergence of $\exp_q(x)$, that is $\left| \frac{[\kappa_v]_{q^1}^{1/\kappa_v}}{|p|^{1/\kappa_v}} \right|^{(p-1)}$, by (4.1.1.1).
If $a \neq 0$ the situation is slightly more complicated:

**Lemma 4.2.5.** Let $\sum_{n=0}^{\infty} a_n (x - a)_n \in K_v [x - a]_v$ and let $\varrho \in (0,1)$ be a real number. Then if

$$\sup (\varrho, |a|_v)^{1 - \frac{1}{n}} \sup (\varrho, |a||[\kappa_v]_q|)_v \leq \liminf_{n \to \infty} |a_n|^{-1/n}$$

(in particular, if $\sup (\varrho, |a|_v) < \liminf_{n \to \infty} |a_n|^{-1/n}$)

the series $\sum_{n=0}^{\infty} a_n (x - a)_n$ converges in the disk $\{ x \in K_v : |x - a|_v < \varrho \}.$

**Corollary 4.2.6.** Let $a \in V_v$ be such that $|a|_v \leq \chi_v(M),$ then $\sum_{n=0}^{\infty} \frac{G_n(a)}{[n]_q!} (x - a)_n$ converges in the open disk $\{ x \in K_v : |x - a|_v < \chi_v(M) \}.$

**Proof of lemma 4.2.5.** By the Maximum Modulus Principle [DGS, VI, 1.1] and (4.1.1.3) we have

$$\sup_{|x - a|_v < \varrho} |(x - a)|_v = \sup_{|x - a|_v < \varrho} |(x - a) (x - a + a(1 - q)) \cdots (x - a + a (1 - q^{n-1}))|_v$$

$$\leq \varrho \sup (\varrho, |a|_v)^{(n-1) - \left[ \frac{n-1}{\varrho} \right]} \prod_{i=1}^{n} \sup (\varrho, |a_i|_v|[\kappa_v]_q)_v$$

$$\leq \varrho \sup (\varrho, |a|_v)^{(n-1) - \left[ \frac{n-1}{\varrho} \right]} \sup (\varrho, |a|_v|[\kappa_v]_q)_v^{\left[ \frac{n-1}{\varrho} \right]}$$

Finally, $\sum_{n=0}^{\infty} a_n (x - a)_n$ converges if

$$\sup (\varrho, |a|_v)^{1 - \frac{1}{n}} \sup (\varrho, |a||[\kappa_v]_q|)_v \leq \liminf_{n \to \infty} |a_n|^{-1/n}.$$


The following characterization of $\chi_v(M)$ is the $q$-analogue of a result by André (cf. [A, IV, §5]):

**Proposition 4.2.7.** The sequence $\left( \frac{h(n)}{n} \right)_{n \in \mathbb{N}}$ defined in (4.2.3) is convergent:

$$\lim_{n \to \infty} \frac{h(n)}{n} = \log \frac{1}{\chi_v(M)}.$$

**Proof.** By (4.2.3) it is enough to prove the existence of the limit. Let $s, n$ be two positive integers; we have

$$\frac{\Delta_{s+n}^{[s+n]_q!}(x)}{[s+n]_q!} = \frac{\Delta_{s}^{[s]_q!}}{[s]_q!} (\frac{\Delta_{s}^{[s]_q!}}{[s]_q!} (\varrho G_s(x)))$$

$$= \varrho \sum_{i+j=n} \frac{[n]_q! [s]_q!}{[s+n]_q!} \frac{G_i(x) d^j_q (G_q(x))}{[i]_q! [j]_q! [s]_q!}$$

It follows that

$$\frac{G_{s+n}(x)}{[s+n]_q!} = \sum_{i+j=n} \frac{[n]_q! [s]_q!}{[s+n]_q!} \frac{G_i(x) d^j_q (G_q(x))}{[i]_q! [j]_q! [s]_q!}$$

and hence

$$\log \left( \frac{G_{s+n}(x)}{[s+n]_q!} \right)_{v,Gauss} \leq \log \left( \frac{G_s(x)}{[s]_q!} \right)_{v,Gauss} + h(n) - \log \binom{n+s}{s}.$$
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For all $k \in \mathbb{N}$ and $n \geq s$, by induction we obtain

$$\log \left| G_{s+kn}(x) \right|_{v, \text{Gauss}} \leq \log \left| G_{s+(k-1)n}(x) \right|_{v, \text{Gauss}} + h(n) - \log \left( \frac{s+kn}{s+(k-1)n} \right)_{v, \text{Gauss}}$$

$$\leq \log \left| G_s(x) \right|_{v, \text{Gauss}} + kh(n) - \log \prod_{i=1}^{k} \left( \frac{s+in}{s+(i-1)n} \right)_{v, \text{Gauss}}.$$ 

Let $N \in \mathbb{N}$, $N \geq n$; then $N = \left\lceil \frac{N}{n} \right\rceil n + s$, with $0 \leq s < n$, and the previous inequality becomes

$$\log \left| G_N(x) \right|_{v, \text{Gauss}} \leq \left( \left\lceil \frac{N}{n} \right\rceil + 1 \right) h(n) - \log \prod_{i=1}^{\left\lceil \frac{N}{n} \right\rceil} \left( \frac{s+in}{s+(i-1)n} \right)_{v, \text{Gauss}}.$$ 

Since $\log \left| \prod_{i=1}^{\left\lceil \frac{N}{n} \right\rceil} \left( \frac{s+in}{s+(i-1)n} \right)_{v, \text{Gauss}} \right| \leq 0$ is a decreasing function of $N$ we obtain

$$\frac{h(N)}{N} \leq \left( \frac{1}{n} + \frac{1}{N} \right) h(n) - \log \left| \prod_{i=1}^{\left\lceil \frac{N}{n} \right\rceil} \left( \frac{s+in}{s+(i-1)n} \right)_{v, \text{Gauss}} \right|$$

$$\leq \left( \frac{1}{n} + \frac{1}{N} \right) h(n) - \log \left| \left[ \left[ \frac{N}{n} \right] q^1 \right|_{v, \text{Gauss}} \right| \left| \left[ s q^1 \right|_{v, \text{Gauss}} \right|.$$ 

Finally we deduce by (4.1.1.1) that

$$\limsup_{N \to \infty} \frac{h(N)}{N} \leq \limsup_{N \to \infty} \left( \frac{1}{n} + \frac{1}{N} \right) h(n) - \log \left| \left[ \left[ \frac{N}{n} \right] q^1 \right|_{v, \text{Gauss}} \right| \left| \left[ s q^1 \right|_{v, \text{Gauss}} \right|$$

$$\leq \frac{h(n)}{n} - \log \left( \frac{\left\lceil \lceil \kappa \left| q^1 \right|_{v, \text{Gauss}} \right|_{v}^{1/\kappa (p-1)}}{\left\lceil \left\lceil \kappa \left| q^1 \right|_{v, \text{Gauss}} \right|_{v}^{1/n} \right\lceil} \right).$$ 

Therefore

$$\limsup_{N \to \infty} \frac{h(N)}{N} \leq \liminf_{n \to \infty} \frac{h(n)}{n},$$

from which it follows that the sequence $\left( \frac{h(n)}{n} \right)_{n \in \mathbb{N}}$ is convergent.

Now we prove a first estimate for $\chi_v(M)$. In succeeding sections we will prove a more precise estimate linked to the notion of unipotent reduction.

**Proposition 4.2.8.** We have:

$$\chi_v(M) \geq \frac{\left| \left\lceil \left\lceil \kappa \left| q^1 \right|_{v, \text{Gauss}} \right|_{v}^{1/\kappa (p-1)} \right|_{v}^{1/\kappa (p-1)}}{\sup (\left| G(x) \right|_{v, \text{Gauss}}, 1)}.$$ 

**Proof.** By induction, we get

$$\left| \frac{G_n(x)}{\left| n q^1 \right|_{v, \text{Gauss}}} \right|_{v, \text{Gauss}} \leq \sup (\left| G(x) \right|_{v, \text{Gauss}}, 1)^{n}$$

and the conclusion follows by lemma (4.1.1.1). ■

**Remark 4.2.9.** We have assumed that $|1 - q^{\kappa v}|_v < |p|^{1/(p-1)}$. Let us briefly analyze what happens if we drop this assumption.

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We notice that if $|q|_v > 1$ then $|[n]_q|_v = |q|_v^{n(n - 1)}$ for all $n \geq 0$, and therefore

$$\chi_v(M)^{-1} = \limsup_{n \to \infty} \frac{|G_n(x)|^{1/n}_{v,Gauss}}{|[n]_q|^{1/n}_{v,Gauss}}$$

This limit can be zero as well as $\infty$ or a finite value. On the other hand, if $|q|_v < 1$ then $|[n]_q|_v = 1$ for any $n \geq 0$, and therefore

$$\chi_v(M)^{-1} = \limsup_{n \to \infty} \frac{|G_n(x)|^{1/n}_{v,Gauss}}{|[n]_q|^{1/n}_{v,Gauss}} = \limsup_{n \to \infty} |G_n(x)|^{1/n}_{v,Gauss} \leq \sup (|G(x)|_{v,Gauss}, 1)$$

**Proposition 4.2.10.** If $|q|_v = 1$ and $|1 - q^{eK_v}|_v \geq |p|_v^{1/(p - 1)}$ then

$$\sup (|G(x)|_{v,Gauss}, 1) \geq \chi_v(M)^{-1} \geq \frac{\sup (|G(x)|_{v,Gauss}, 1)}{|[\kappa_v|_v|^{1/\kappa_v}_v|}$$

where

$$e = \inf \{ m \in \mathbb{Z} : m > 0, |1 - q^{eK_v}|_v < |p|_v^{1/(p - 1)} \}$$

**Proof.** Since $|1 - q^{eK_v}|_v \leq |1 - q^{eK_v}|_v$ for any positive integer $n$ we have: $|[n]_q|_v \leq |[\kappa_v|_v|^{1/\kappa_v}_v|$. For any positive integer $n$, there exist two positive integers $r, s, < e$, such that $n = se + r$. We obtain

$$|1 - q^{seK_v}|_v = |1 - q^{(se + r)K_v}|_v = |1 - q^{seK_v} + q^{seK_v}(1 - q^{rK_v})|_v \geq |p|_v^{1/(p - 1)} \text{ if } r \neq 0$$

from which we infer that

$$|[\kappa_v|_v|^{1/\kappa_v}_v| \geq \limsup_{n \to \infty} |[n]_q|_v|^{1/n}_v$$

and therefore

$$\sup (|G(x)|_{v,Gauss}, 1) \geq \chi_v(M)^{-1} \geq \frac{\sup (|G(x)|_{v,Gauss}, 1)}{|[\kappa_v|_v|^{1/\kappa_v}_v|}$$

Finally we have

$$\frac{\sup (|G(x)|_{v,Gauss}, 1)}{|p|_v^{1/(p - 1)}|1 - q^{eK_v}|_v^{1/\kappa_v}_v} \geq \chi_v(M)^{-1} \geq \frac{\sup (|G(x)|_{v,Gauss}, 1)}{|[\kappa_v|_v|^{1/\kappa_v}_v|}$$



### 4.3. $q$-analogue of the Dwork-Frobenius theorem

The next proposition is the $q$-analogue of the Dwork-Frobenius-Young theorem [DGS, VI, 2.1], which establishes a relation between $\chi_v(M)$ and the coefficients of the $q$-difference matrix associated to $M = (M, \Phi_q)$ with respect to a cyclic basis, when $\kappa_v = 1$:

**Proposition 4.3.1.** We suppose that $|1 - q|_v < |p|_v^{1/(p - 1)}$. Let $M$ be a $q$-difference module over $K_v(x)$ of rank $\mu$, $m \in M$ a cyclic vector (cf. (1.3.1)) such that

$$\Delta_q(m, \Delta_q(m), \ldots, \Delta_q^{\mu - 1}(m)) = (m, \Delta_q(m), \ldots, \Delta_q^{\mu - 1}(m))$$

$$\Delta_q(m, \Delta_q(m), \ldots, \Delta_q^{\mu - 1}(m)) = (m, \Delta_q(m), \ldots, \Delta_q^{\mu - 1}(m))$$

$$\begin{pmatrix} 0 & \ldots & 0 & a_0(x) \\ \vdots & \ddots & \ddots & \vdots \\ \ldots & & \ldots & a_{\mu - 1}(x) \end{pmatrix}$$
We deduce that 

\[ \chi_v(\mathcal{M}) = \frac{\sup_{i=0,\ldots,\mu-1} |a_i(x)|_{v,Gauss}^{1/(\mu-i)}}{|p|^{1/(p-1)}}. \]

It follows immediately from (4.2.8) that:

**Corollary 4.3.2.** Let \(|1 - q|_v < |p|_{v}^{1/(p-1)}\). Then \(\chi_v(\mathcal{M}) \geq |p|^{1/(p-1)}\) if and only if

\[ \sup_{i=0,\ldots,\mu-1} |a_i(x)|_{v,Gauss} \leq 1. \]

**Proof of proposition 4.3.1.** We recall that \(\chi_v(\mathcal{M})\) is independent of the choice of the basis of \(\mathcal{M}\) over \(K_v(x)\).

Let \(\gamma \in K_v\) be such that \(|\gamma|_v = \sup_{i=0,\ldots,\mu-1} |a_i(x)|_{v,Gauss}^{1/(\mu-i)}\), \(\xi = (\Delta_\gamma(m), \ldots, \Delta_{\mu-1}(m))\) and

\[ H = \begin{pmatrix} \gamma^{\mu-1} & 0 \\ \gamma^{\mu-2} & \ddots \\ 0 & \cdots & 1 \end{pmatrix}. \]

We set \(f = eH\). By a direct calculation we obtain

\[ \Delta_\gamma(f) = H^{-1} \Delta_\gamma(e)H = f^\gamma W(x) \text{, with } W(x) = \begin{pmatrix} 0 & \cdots & 0 & a_0(x)/\gamma^{\mu} \\ & \ddots & \vdots \\ 0 & \cdots & a_{\mu-1}(x)/\gamma^{\mu-1} \end{pmatrix}. \]

We set \(\Delta_\gamma^n(f) = fH_n(x)\), with \(H_1(x) = \gamma W(x)\). We want to prove by induction on \(n\) that \(H_n(x) \equiv \gamma^n W(x) \cdots W(q^{n-1}x) \mod \gamma^n\). We remark that \(H_n(x) \equiv \gamma^n W(x) \cdots W(q^{n-1}x) \mod \gamma^n\) implies that \(|H_n(x)|_{v,Gauss} \leq |\gamma|_v^n\), and therefore that \(|dh_n(x)|_{v,Gauss} \leq |\gamma|_v^n\) (cf. (4.2.1)). Then we have

\[ H_{n+1}(x) = H_1(x)H_n(qx) + d_q H_n(x) \equiv \gamma^{n+1} W(x) \cdots W(q^n x) \mod \gamma^n. \]

We deduce that \(|H_n(x)|_{v,Gauss} \leq |\gamma|_v^n\), for all \(n \geq 1\), and hence that

\[ \chi_v(\mathcal{M}) \geq \frac{|p|^{1/(p-1)}}{\sup_{i=0,\ldots,\mu-1} |a_i(x)|_{v,Gauss}^{1/(\mu-i)}}. \]

Let us prove the reverse inequality. By induction on \(\mu\) one proves that the characteristic polynomial of \(W(x)\) is

\[ (4.3.2.1) \quad X^\mu - \frac{a_{\mu-1}(x)}{\gamma} X^{\mu-1} - \frac{a_{\mu-2}(x)}{\gamma^2} X^{\mu-2} - \cdots - \frac{a_0(x)}{\gamma^\mu}. \]

By our choice of \(\gamma\), the reduction modulo \(\mathcal{O}_v\) of (4.3.2.1) has a nonzero root, hence \(W(x)\) has an eigenvalue of norm 1. Then there exist
- an extension \(L\) of \(K_v(x)\) equipped with an extension of \(|\cdot|_{v,Gauss}\), still denoted \(|\cdot|_{v,Gauss}\),
- \(\Lambda \in L\), such that \(|\Lambda|_{v,Gauss} = 1\),
- \(\overline{V} \in L^\mu\), such that \(|\overline{V}|_{v,Gauss} = 1\),
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satisfying the relation

$$W(x) \cdots W(q^{n-1}x) \overset{\varphi}{\equiv} W(x)^n \overset{\varphi}{\equiv} \Lambda^n \overset{\varphi}{\equiv} \text{in the residue field of } L \text{ with respect to } \mid v, Gauss.$$

We deduce that

$$\gamma^{-n}H_n(x) \overset{\varphi}{\equiv} \Lambda^n \overset{\varphi}{\equiv} \text{in the residue field of } L \text{ with respect to } \mid v, Gauss.$$

Finally we obtain

$$\left\lfloor \frac{\gamma^{-n}H_n(x)}{[n]_q^i} \right\rfloor_{v,Gauss} \geq \left\lfloor \frac{\gamma^{-n}H_n(x)}{[n]_q^i} \right\rfloor_{v,Gauss} \geq \left\lfloor \frac{\Lambda^n}{n\mid q^i} \right\rfloor_{v,Gauss} = \frac{1}{[n]_q^i}$$

and hence

$$\chi_v(M)^{-1} = \sup \left\{ \limsup_{n \to \infty} \left\lfloor \frac{H_n(x)}{[n]_q^i} \right\rfloor_{v,Gauss}^{1/n} \right\}$$

$$\geq \limsup_{n \to \infty} \left\lfloor \frac{H_n(x)}{[n]_q^i} \right\rfloor_{v,Gauss}^{1/n}$$

$$\geq \sup_{i=0, \ldots, i \leq n} \left\lfloor a_i(x) \right\rfloor_{v,Gauss}^{1/(\mu-1)}.$$

\[ \text{■} \]

In the previous proposition we assumed that $\kappa_v = 1$. If $\kappa_v > 1$ we have:

**Proposition 4.3.3.** The $q$-difference module $(M, \Phi_q)$ equipped with the operator $\Phi_q^{\kappa_v}$ (and consequently with $\Delta_q^{\kappa_v} = (\Phi_q^{\kappa_v} - 1_\mu)/(q^{\kappa_v} - 1)x$) is a $q^{\kappa_v}$-difference module and

$$\chi_v(M, \Phi_q) \leq \chi_v(M, \Phi_q^{\kappa_v})^{1/\kappa_v}.$$

**Proof.** Applying successively (1.1.11.1) and (1.1.11.2), we obtain

$$\Delta_q^{\kappa_v} = \frac{(-1)^n}{(q^{\kappa_v} - 1)^{n\kappa_v}} \sum_{i=0, \ldots, \mu} \left( (-1)^i \binom{n}{i} q^{\kappa_v i (i-1)/2} \binom{i\kappa_v}{j} q^{i-1}\right) \Delta_q^j.$$

Let $f$ be a basis of $M$ over $K_v(x)$ such that $\Delta_q^{\kappa_v} f = fH_n(x)$ and $\Delta_q^{\kappa_v} f = fG_n(x)$. We deduce using (4.3.3.1) that

$$\mid H_n(x)\mid_{v,Gauss} \leq \frac{1}{|q^{\kappa_v} - 1|^v} \left( \sup_{s \leq n\kappa_v} |G_s(x)|_{v,Gauss} \right).$$

Recalling the estimates in (4.1.1) and some general properties of $\limsup$ (cf. [AB, II, 1.8]) we obtain

$$\frac{1}{\chi_v(M, \Phi_q^{\kappa_v})} = \sup_{n \to \infty} \left\lfloor \frac{H_n(x)}{n\mid q^i} \right\rfloor_{v,Gauss}^{1/n}$$

$$\leq \limsup_{n \to \infty} \left( \sup_{s \leq n\kappa_v} |G_s(x)|_{v,Gauss} \right)^{1/n}$$

$$= \frac{1}{\chi_v(M, \Phi_q^{\kappa_v})}.$$

\[ \text{■} \]

5. $p$-adic criteria for unipotent reduction
The \( q \)-analogue of Grothendieck-Katz’s conjecture on \( p \)-curvatures

We recall that \( K_v \) is a complete field with respect to the norm \( |\cdot|_v \) and that \( \mathcal{V}_v \) is its ring of integers, \( \mathcal{V}_v \) its uniformizer and \( k_v \) the residue field.

Let \( q \) be an element of \( K_v \) such that \( |q|_v = 1 \), \( q \) is not a root of unity, and the order \( \kappa_v \) of its image in the multiplicative group \( k_v^\times \) is finite.

Let \( \mathcal{F} \subset K_v(x) \) be a \( q \)-difference algebra essentially of finite type over \( \mathcal{V}_v \) (cf. (1.1.2)). Let \( a \) be an ideal of \( \mathcal{V}_v \) and \( \overline{q} \) be the image of \( q \) in \( \mathcal{V}_v/a \). The algebra \( \mathcal{F} \otimes_{\mathcal{V}_v} \mathcal{V}_v/a \) has the natural structure of a \( \overline{q} \)-difference algebra. Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over \( \mathcal{F} \). We consider the free \( \mathcal{F} \otimes_{\mathcal{V}_v} \mathcal{V}_v/a \)-module \( M \otimes_{\mathcal{V}_v} \mathcal{V}_v/a \) equipped with the morphism \( \Phi_\overline{q} \) induced by \( \Phi_q \); it is a \( \overline{q} \)-difference module over \( \mathcal{F} \otimes_{\mathcal{V}_v} \mathcal{V}_v/a \), which satisfies the assumptions of \( \S \) 2.

We are especially interested in the following two cases:
- \( a \) is the maximal ideal of \( \mathcal{V}_v \) generated by \( \mathcal{w}_v \). We will refer to \( (M \otimes_{\mathcal{V}_v} \mathcal{V}_v/(\mathcal{w}_v \mathcal{V}_v), \Phi_\overline{q}) \) as the reduction of \( \mathcal{M} \) modulo \( \mathcal{w}_v \) or over \( k \).
- \( a \) is the ideal of \( \mathcal{V}_v \) generated by \( 1 - q^{\kappa_v} \). We will refer to \( (M \otimes_{\mathcal{V}_v} \mathcal{V}_v/(1 - q^{\kappa_v})\mathcal{V}_v, \Phi_\overline{q}) \) as the reduction modulo \( 1 - q^{\kappa_v} \).

**Remark 5.0.4.** We notice that \( |p|_v = |p|_v \), therefore both reductions are \( q \)-analogues of the reduction modulo \( p \) in the differential case (cf. \( \S \) 3). In \( \S \) 6 we analyze the reduction modulo \( \mathcal{w}_v \), while in our main theorem (7.1.1) we consider the reduction modulo \( 1 - q^{\kappa_v} \).

Motivated by \( \S \) 2, we are particularly interested in \( q \)-difference modules \( \mathcal{M} \) over \( \mathcal{F} \) such that the reduction modulo \( \mathcal{w}_v \) (resp. \( 1 - q^{\kappa_v} \)) of the operator \( \Phi_q^{\kappa_v} \) is unipotent. We shall say briefly that \( \mathcal{M} \) has unipotent reduction of order \( n \) modulo \( \mathcal{w}_v \) (resp. \( 1 - q^{\kappa_v} \)) if the reduction of \( \Phi_q^{\kappa_v} \) modulo \( \mathcal{w}_v \) (resp. \( 1 - q^{\kappa_v} \)) is a unipotent morphism of order \( n \).

The following example shows that \( \mathcal{M} \) can have unipotent reduction modulo \( \mathcal{w}_v \) without having unipotent reduction modulo \( 1 - q^{\kappa_v} \):

**Example.** Let us consider the \( q \)-difference module over \( \mathbb{Q}_p(x) \) associated to the \( q \)-difference system

\[
\begin{pmatrix}
 y_1(qx) \\
 y_2(qx)
\end{pmatrix} = \begin{pmatrix}
 1 & p \\
 0 & 1
\end{pmatrix}
\begin{pmatrix}
 y_1(x) \\
 y_2(x)
\end{pmatrix}.
\]

Then

\[
\begin{pmatrix}
 y_1(q^{\kappa_p}x) \\
 y_2(q^{\kappa_p}x)
\end{pmatrix} = \begin{pmatrix}
 1 & \kappa_p p \\
 0 & 1
\end{pmatrix}
\begin{pmatrix}
 y_1(x) \\
 y_2(x)
\end{pmatrix},
\]

from which it follows that

\[
\left| \begin{pmatrix}
 1 & \kappa_p p \\
 0 & 1
\end{pmatrix} - I_2 \right|_p = |\kappa_p p|_p = |p|_p.
\]

If we choose \( q = 8 \) and \( p = 3 \) then \( \kappa_p = 2 \) and \( |1 - q^{\kappa_p}|_p = |1 - 8^2|_p = |3^2|_p < |3|_p \), and therefore

\[
\begin{pmatrix}
 1 & \kappa_p p \\
 0 & 1
\end{pmatrix} = \begin{pmatrix}
 1 & 6 \\
 0 & 1
\end{pmatrix} \equiv I_2 \mod 3^2,
\]

but

\[
\begin{pmatrix}
 1 & \kappa_p p \\
 0 & 1
\end{pmatrix} = \begin{pmatrix}
 1 & 6 \\
 0 & 1
\end{pmatrix} \not\equiv I_2 \mod 3^2.
\]

Then by (2.1.2) the \( q \)-difference module associated to (5.0.4.1) has trivial reduction modulo \( p = 3 \), but not modulo \( 1 - q^{\kappa_p} = (-7)3^2 \).

We want to relate the property of having unipotent reduction modulo \( \mathcal{w}_v \) (resp. \( 1 - q^{\kappa_v} \)) to an estimate of the invariant \( \chi_v(\mathcal{M}) := \chi_v(\mathcal{M}_{K_v(x)}) \).

\section{5.1. \( q \)-difference modules having unipotent reduction modulo \( \mathcal{w}_v \)}

First we consider \( q \)-difference modules having unipotent reduction modulo \( \mathcal{w}_v \). The following proposition is a \( q \)-analogue of a classical estimate for \( p \)-adic differential modules [DGS, page 96]:

**Proposition 5.1.1.** If \( \mathcal{M} \) has unipotent reduction modulo \( \mathcal{w}_v \) of order \( n \) then

\[
\chi_v(\mathcal{M}) \geq |\mathcal{w}_v|_v^{-1/\kappa_v n} |[\kappa_v]|_q^{1/\kappa_v} |p|_v^{1/\kappa_v (p-1)}.
\]
The proof of (5.1.1) relies on the following lemma:

**Lemma 5.1.2.** Let us assume that $\mathcal{M}$ has unipotent reduction of order $n$ modulo $\varpi_v$. Let $e$ be a basis of $\mathcal{M}$ over $\mathcal{F}$ and let $\Delta^m e = \epsilon G_m(x)$, for any $m \geq 1$, with $G_m(x) \in M_{\mu \times \mu}(K_v(x))$. Then

$$|G_{n_{\kappa_v}}(x)|_{v,\text{Gauss}} \leq |\varpi_v|^s,$$

for every integer $s \geq 1$.

**Proof.** By (1.1.10.2), for all $s \in \mathbb{N}$, $s > 1$ we have

$$\Delta_q^{(s+1)n_{\kappa_v}}(e) = \frac{\epsilon G^{(s+1)n_{\kappa_v}}(x)}{n_{\kappa_v}}$$

$$= \Delta_q^{(s+1)n_{\kappa_v}}(e) G_{n_{\kappa_v}}(x)$$

$$= \sum_{i=0}^{n_{\kappa_v}} \left( \begin{array}{c} n_{\kappa_v} \\ i \end{array} \right) \Delta_q^{n_{\kappa_v} - i} e d_q^i (G_{n_{\kappa_v}}) (q^{n_{\kappa_v} - i} x)$$

$$= \epsilon \sum_{i=0}^{n_{\kappa_v}} \left( \begin{array}{c} n_{\kappa_v} \\ i \end{array} \right) G_{n_{\kappa_v} - i}(x) d_q^{n_{\kappa_v} - i} (G_{n_{\kappa_v}}) (q^{n_{\kappa_v} - i} x),$$

and hence

$$G^{(s+1)n_{\kappa_v}}(x) = \sum_{i=0}^{n_{\kappa_v}} \left( \begin{array}{c} n_{\kappa_v} \\ i \end{array} \right) G_{n_{\kappa_v} - i}(x) d_q^{n_{\kappa_v} - i} (G_{n_{\kappa_v}}) (q^{n_{\kappa_v} - i} x).$$

By (2.1.3), the definition of unipotent reduction modulo $\varpi_v$ is equivalent to the condition

$$|G_{n_{\kappa_v}}(x)|_{v,\text{Gauss}} \leq |\varpi_v|^s.$$

We shall prove the statement by induction on $s > 1$, using (5.1.2.1). We suppose that

$$|G_{n_{\kappa_v}}(x)|_{v,\text{Gauss}} \leq |\varpi_v|^s.$$

Then all the terms occurring in the sum (5.1.2.1) are bounded by $|\varpi_v|^{s+1}$, in fact:

1) If $(\kappa_v, i) = 1$, then

$$\left| \left( \begin{array}{c} n_{\kappa_v} \\ i \end{array} \right) \right|_{q,v} = \left| \frac{[n_{\kappa_v}]_q}{[i]_q} \left( \begin{array}{c} n_{\kappa_v} - 1 \\ i - 1 \end{array} \right) \right|_{q,v} \leq |[\kappa_v]|_v \leq |\varpi_v|_v$$

and the absolute value of the corresponding term in sum (5.1.2.1) is bounded by $|\varpi_v|^{s+1}$.

2) For all $i = 1, \ldots, n$, we have $|d_q^{n_{\kappa_v}} f(x)|_{v,\text{Gauss}} \leq |[\kappa_v]|_v |f(x)|_{v,\text{Gauss}}$ and therefore

$$\left| \left( \begin{array}{c} n_{\kappa_v} \\ i \end{array} \right) \right|_{q,v} G_{n_{\kappa_v} - i}(x) d_q^{n_{\kappa_v} - i} (G_{n_{\kappa_v}}) (q^{n_{\kappa_v} - i} x) \leq |[\kappa_v]|_v |\varpi_v|^s \leq |\varpi_v|^{s+1},$$

for all $i = 1, \ldots, n$.

3) The term of (5.1.2.1) corresponding to $i = 0$ is $G_{n_{\kappa_v}}(x) G_{n_{\kappa_v}} (q^{n_{\kappa_v}} x)$, and therefore it is bounded by $|\varpi_v|^{s+1}$, by induction.

Thus we have proved that $|G^{(s+1)n_{\kappa_v}}(x)|_{v,\text{Gauss}} \leq |\varpi_v|^{s+1}$.

**Proof of proposition 5.1.1.** By the recursive formula (4.2.1.2) we have

$$|G_m(x)|_{v,\text{Gauss}} \leq |G_{\frac{m}{\mu \times \mu}}(x)|_{v,\text{Gauss}}.$$
The \( q \)-analogue of Grothendieck-Katz’s conjecture on \( p \)-curvatures

The estimate (5.1.1.1) follows from previous lemma, since

\[
\chi_v(\mathcal{M}) \geq \inf \left( 1, \liminf_{m \to \infty} \frac{|G_{\frac{\kappa_v}{m}, \epsilon}|_{n, \kappa_v}^{-1/m} (x, \kappa_v)}{|m|^1/n_{v,Gauss}} \right)
\]

\[
\geq \liminf_{m \to \infty} \frac{\varpi_\sigma |\kappa_v|^{-1/m} v}{|m|^1/n_{v,Gauss}} = |\varpi_\sigma|^{-1/n_{v,Gauss}} |\kappa_v|^{1/n_{v,Gauss}} (p-1) .
\]

Corollary 5.1.3. The \( q \)-difference module \( \mathcal{M} \) has unipotent reduction modulo \( \varpi_v \) if and only if

\[
\chi_v(\mathcal{M}) > |\kappa_v|_{q, v}^{1/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} .
\]

Proof. If \( \mathcal{M} \) has unipotent reduction modulo \( \varpi_v \), we immediately deduce by (5.1.1) that \( \chi_v(\mathcal{M}) > |\kappa_v|_{q, v}^{1/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} .
\)

On the other hand, by hypothesis we have

\[
|\kappa_v|_{q, v}^{1/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} < \chi_v(\mathcal{M}) = \inf \left( 1, \liminf_{n \to \infty} |G_n(x)|_{n, n_{v,Gauss}}^{-1/n} \right)
\]

\[
= \inf \left( 1, |\kappa_v|_{q, v}^{1/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} \liminf_{n \to \infty} |G_n(x)|_{v,Gauss}^{-1/n} \right) .
\]

We deduce that

\[
\limsup_{n \to \infty} |G_n(x)|_{v,Gauss}^{1/n} < 1 .
\]

We conclude that there exists \( N \in \mathbb{N} \) such that \( |G_n(x)|_{v,Gauss} < 1 \) for all \( n > N \), which implies that \( \mathcal{M} \) has unipotent reduction modulo \( \varpi_v \).

5.2. \( q \)-difference modules having unipotent reduction modulo \( 1 - q^{s_v} \)

Under the hypothesis of unipotent reduction modulo \( 1 - q^{s_v} \), we obtain a slight but crucial improvement of (5.1.1) that will be fundamental in the proof of the \( q \)-analogue of Grothendieck’s conjecture (7.1.1) below:

Proposition 5.2.1. Let \( \mathcal{M} \) be a \( q \)-difference module over \( F \), with unipotent reduction modulo \( 1 - q^{s_v} \) of order \( n \). Then

\[
\chi_v(\mathcal{M}) \geq |\kappa_v|_{q, v}^{(n-1)/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} .
\]

Proof. Let \( e \) be the basis of \( M \) such that \( \Delta^m_q e = \varrho G_m(x) \), for all \( m \geq 1 \). Then

\[
|G_{\kappa_v}(x)|_{v,Gauss} \leq |\kappa_v|_{q, v} .
\]

The estimates in (5.1.2) show that

\[
|G_{\kappa_v}(x)|_{v,Gauss} \leq |\kappa_v|_{q, v}^s , \quad \forall \ s \geq 1 ,
\]

therefore we conclude that

\[
\chi_v(\mathcal{M}) \geq |\kappa_v|_{q, v}^{(n-1)/n_{v,Gauss}} |p|_{v, \kappa_v}^{1/(p-1)} .
\]

Corollary 5.2.2. The following assertions are equivalent:

1) \( \chi_v(\mathcal{M}) \geq |p|_{v, \kappa_v}^{1/(p-1)} \).
2) There exists a cyclic basis \( e \) of \( (\mathcal{M}_K(x), \phi_{q^s}^e) \) such that \( \Phi_{q^s}^e e \equiv e \) modulo \( 1 - q^{s_v} \).
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**Proof.** The implication “\( 2)\Rightarrow 1 \)” is a consequence of the previous proposition.

We prove “\( 1)\Rightarrow 2 \)”.

The \( \mathcal{F} \)-module \( M \) equipped with the operators \( \Phi_{q^k} \) is a \( q^k \)-difference module.

It follows by (4.3.3) that \( \chi_v(M, \Phi_{q^k}) \geq |p|^{1/(p-1)} \). We know by (1.3.1) that \( M_{K(x)} \) admits a cyclic basis \( \xi \) over \( K(x) \).

Let \( \Phi_q \xi = \xi A_{\kappa_v}(x) \). We deduce from (4.3.1) that

\[
\left| A_{\kappa_v}(x) - I_\mu \right| \leq 1
\]

and hence that \( |A_{\kappa_v}(x) - I_\mu|_{v, \text{Gauss}} \leq |1 - q^{\kappa_v}|_v \).

\[\blacksquare\]

**Part III. A \( q \)-analogue of Grothendieck’s conjecture on \( p \)-curvatures**

**6. Arithmetic \( q \)-difference modules and regularity**

We now establish some notation that will be maintained until the end of the paper:

\( K \) = a number field.

\( \mathcal{O}_K \) = the ring of integers of \( K \).

\( | \cdot |_v \) = a \( v \)-adic absolute value of \( K \). In the non-archimedean case we normalize \( | \cdot |_v \) as follows:

\[
|p|_v = p^{-[K_v:Q]/[K:Q]},
\]

where \( K_v \) is the \( v \)-adic completion of \( K \) and \( v|p \). Similarly, in the archimedean case we normalize \( | \cdot |_v \) by setting

\[
|x|_v = \begin{cases} 
|x|_R^{1/[K:Q]} & \text{if } K_v = \mathbb{R} \\
|x|_C^{2/[K:Q]} & \text{if } K_v = \mathbb{C}
\end{cases},
\]

where \( | \cdot |_R \) and \( | \cdot |_C \) are the usual absolute values of \( \mathbb{R} \) and and of \( \mathbb{C} \) respectively.

\( \Sigma_f \) = the set of finite places \( v \) of \( K \).

\( \varpi_v \) = uniformizer \( \in \mathcal{O}_K \) associated to the finite place \( v \).

\( k_v \) = residue field of \( K \) with respect to a finite place \( v \).

\( \Sigma_\infty \) = the set of archimedean places of \( K \).

**6.1. On cyclic subgroups of \( \mathbb{Q}_x^\times \)**

and their reduction modulo almost every prime

We fix an element \( q \) of \( K \) which is not zero and not a root of unity. For each \( v \in \Sigma_f \) such that \( |q|_v = 1 \), we denote \( \kappa_v \) to be the multiplicative order of the image of \( q \) in the residue field of \( K \) with respect to \( v \). We refer to [BHIV] for the most recent results on the distribution of \( (\kappa_v)_v \).

We recall that the Dirichlet density \( d(S) \) of a set \( S \) of finite places of a number field \( K \) (cf. for instance [N, VII, §13]) is defined by

\[
d(S) = \limsup_{s \to 1+} \frac{\sum_{v \in S} p^{-sf_v}}{\sum_{v \in \Sigma_f} p^{-sf_v}},
\]

where \( f_v = [k_v : \mathbb{F}_p] \), if \( v|p \).

The proposition below is a particular case of a theorem by Schinzel [Sc, Th. 2]. We prefer to give a direct proof here.

**Proposition 6.1.1.** Let \( S \subset \Sigma_f \) be a set of finite places of \( K \) of Dirichlet density 1 and let \( a, b \) be two elements of \( K^\times = K \setminus \{0\} \) such that for all \( v \in S \), the reduction of \( b \) modulo \( \varpi_v \) belongs to the cyclic group generated by the reduction of \( a \) modulo \( \varpi_v \). Then \( b \in a^{\mathbb{Z}} \).
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**Corollary 6.1.2.** Let \( a, b \) be two elements of \( K \), which are not roots of unity, such that for almost all \( v \in \Sigma_f \) the order of \( a \) modulo \( \mathfrak{p}_v \) and the order of \( b \) modulo \( \mathfrak{p}_v \) coincide. Then either \( a = b \) or \( a = b^{-1} \).

**Remark 6.1.3.** This shows that \( \{ q, q^{-1} \} \) is uniquely determined by the family of integers \( (\kappa_v)_v \).

**Proof of corollary 6.1.2.** We recall that \( k^c \) is a cyclic group and that, therefore, its subgroups are determined by their order. By the previous proposition, we know that \( b = a^n \) and \( a = b^m \) for some integers \( n \) and \( m \). We deduce that \( a^{nm} = a \). Since \( a \) is not a root of unity, we have \( mn = 1 \) and hence either \( m = n = 1 \) or \( m = n = -1 \).

**Proof of proposition 6.1.1 (following an argument of P. Colmez).** We fix a rational prime \( \ell \). Let \( \zeta \) be an \( \ell \)-th root of unity. We consider the following Galois extensions of \( K \): \( K_1 = K(a^{1/\ell}, \zeta) \), \( K_2 = K(b^{1/\ell}, \zeta) \) and \( K_{12} = K(a^{1/\ell}, b^{1/\ell}, \zeta) \). We will prove that \( K_1 = K_{12} \), and hence that \( K_2 \subseteq K_1 \), by applying the following corollary of the Čebotarev Density theorem:

\[ \text{[N, VII, (13.6)] Let } \overline{K} \text{ be a Galois extension of the number field } K \text{ and let } P(\overline{K}/K) \text{ be the set of primes of } K \text{ that split totally in } \overline{K}. \text{ Then the Dirichlet density of } P(\overline{K}/K) \text{ is} \]

\[
d(P(\overline{K}/K)) = \frac{1}{[K : K]}.\]

Let \( v \in \Sigma_f \) be a prime of \( K \) such that \( v|p \), \( p \equiv 1 \) mod \( \ell \), and let \( \{w_1, \ldots, w_r\} \subset \Sigma_f \) be the set of all primes \( w \) of \( K_1 \) such that \( w|v \). Let \( e_i \) be the ramification index of \( w_i|v \) and \( f_i \) be the residue degree. Since \( K_1/K \) is a Galois extension we have: \( e = e_1 = \ldots = e_r \) and \( f = f_1 = \ldots = f_r \) (cf. [N, IV, page 55]).

Therefore, \( e = f = 1 \) if and only if we have \( [K_1 : K] = \sum_{i=1}^r e_i f_i = r \) so \( v \) splits totally in \( K_1 \) if and only if \( e = f = 1 \). Then \( P(K_1/K) \) is the set of all primes \( v \in \Sigma_f \) of \( K \) such that:

- \( v|p \) and \( p \equiv 1 \) mod \( \ell \);
- there exists \( a' \in k_v \) such that \( a' \equiv a \) in \( k_v \);
- \( v \) is not ramified in \( K_1 \).

For the same reason, \( P(K_{12}/K) \) is the set of all primes \( v \in \Sigma_f \) such that:

- \( v|p \) and \( p \equiv 1 \) mod \( \ell \);
- there exists \( a' \in k_v \) such that \( a' \equiv a \) in \( k_v \);
- there exists \( b' \in k_v \) such that \( b' \equiv b \) in \( k_v \);
- \( v \) is not ramified in \( K_{12} \).

Let \( v \in P(K_1/K) \cap S \) and let \( a' \in k_v \) be such that \( a \equiv a'^{\ell} \) in \( k_v \). By hypothesis there exists a positive integer \( n(v) \) such that \( b \equiv a^{n(v)} \) in \( k_v \) and hence \( b \equiv (a'^{n(v)})^{\ell} \). Hence if \( vv \in P(K_1/K) \cap S \) is not ramified in \( K_{12} \) then \( v \in P(K_{12}/K) \). Taking into account that \( S \) has density 1 and that there are only finitely many \( v \in \Sigma_f \) which ramify in \( K_{12} \), we have

\[
d(P(K_{12}/K)) = \frac{1}{[K_{12} : K]} \geq d(P(K_1/K)) = \frac{1}{[K_1 : K]}.\]

We conclude that \( K_{12} = K_1 \) and therefore \( K_2 \subseteq K_1 \).

We recall the following fact from Kummer theory:

\[ \text{[N, VII, (3.6)] Let } n \text{ be a positive integer which is relatively prime with respect to the characteristic of the field } K, \text{ and assume that } K \text{ contains the group of } n \text{-th roots of unity. Then the abelian extensions } \overline{K}/K \text{ of exponents } n \text{ are in one-to-one correspondence with the subgroups } \Gamma \subseteq K^x = K \setminus \{0\}, \text{ which contain } K^{1/n}, \text{ via the rule } \Gamma \mapsto \overline{K} = K(\sqrt[n]{\Gamma}). \]

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This statement, applied to \( K_2 \subset K_1 = K_{12} \) and \( n = \ell \), says that

\[
\beta_\ell K^\times \subset a_\beta K^\times \Rightarrow b \in a_\beta K^\times .
\]

Since \( \ell \) is arbitrary, we conclude that \( b \in a_\beta \).

6.2. Unipotent reduction and regularity

Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over a \( q \)-difference algebra \( F \subset K(x) \) essentially of finite type over \( V_K \) (cf. (1.1.2)).

Definition 6.2.1. The \( q \)-difference module \( \mathcal{M} \) over \( F \) is regular singular if both \( \mathcal{M}_{K((x))} \) and \( \mathcal{M}_{K((1/x))} \) are regular singular \( q \)-difference modules.

Let \( \Sigma_{nilp} \) be the set of finite places \( v \) of \( K \) such that \( \mathcal{M} \) has unipotent reduction modulo \( \varpi_v \). The following result is a \( q \)-analogue of a well known result due to Katz (cf. [K1, 13.0]).

Theorem 6.2.2.
1) If \( \Sigma_{nilp} \) is infinite, then \( \mathcal{M} \) is regular singular.
2) If moreover \( \Sigma_{nilp} \) has Dirichlet density 1, the exponents of \( K((x)) \otimes F M \) with respect to some basis \( \xi \) (and hence to any basis) over \( K((x)) \) coincide with \( q \)-curvatures.

Proof.
1) It is enough to prove the statement at zero. Let \( \xi \) be a basis of \( M \) over \( F \) such that \( \Phi_q \xi = x A(x) \), with \( A(x) \in Gl_\mu(F). \) Then \( A(x) \) can be regarded as an element of \( Gl_\mu(K((x))) \), which means that \( A(x) \) has the following form:

\[
A(x) = \frac{1}{x^l} \sum_{i \geq 0} A_i x^i \in \frac{1}{x^l} Gl_\mu(K[[x]]) ,
\]

for some \( l \in \mathbb{Z} \). If \( l = 0 \), the \( q \)-difference module \( \mathcal{M} \) is regular singular at zero, so let us suppose \( l \neq 0 \). For all positive integers \( m \), we have

\[
\Phi_q^m(\xi) = \xi A(x) A(qx) \cdots A(q^{m-1} x) = \xi \left( \frac{A_0^m}{q^m} \right) + \text{h.o.t.}
\]

By hypothesis, for any \( v \in \Sigma_{nilp} \), there exists a positive integer \( n(v) \geq 1 \) such that we have

\[
(A(x)A(qx) \cdots A(q^{n(v)-1} x) - 1)^{n(v)} \equiv 0 \mod \varpi_v ;
\]

we deduce that \( A_0^{n(v)} \equiv 0 \) modulo \( \varpi_v \), for any \( v \in \Sigma_{nilp} \), and hence that \( A_0 \) is a nilpotent matrix.

We suppose that zero is not a regular singularity. By (1.4.4), there exist an extension \( L((t)) \) of \( K((x)) \) and \( \bar{q} \in L \), with \( t^d = x \) and \( \bar{q}^d = q \), such that we can find a basis \( \xi \) of \( L((t)) \otimes F M \) over \( L((t)) \) with the following properties:

\[
\Phi_q(\xi) = f B(t)
\]

and

\[
B(t) = \frac{B_0}{t^k} + \frac{B_{k-1}}{t^{k-1}} + \cdots + \frac{B_1}{t^1} + \bar{B}_0(t) ,
\]

with \( \bar{B}_0(t) \in M_{\mu \times \mu}(L[[t]]) \), \( k \geq 1 \) and \( B_k \in Gl_\mu(L) \) non nilpotent and in Jordan normal form. Let \( F(t) = \xi + \text{h.o.t} \in Gl_\mu(L((t))) \) be such that \( \Phi_q(\xi) = f B(t) \). This implies that \( A(x) = F(t)^{-1} B(t) F(q \bar{q} t) \).

We get a contradiction since the matrix \( F \in Gl_\mu(L) \) satisfies \( A_0 = F^{-1} B_k F \).

2) We know by 1) that \( \mathcal{M} \) has a regular singularity at zero. Then there exists a \( K((x)) \)-basis \( \xi \) of \( K((x)) \otimes F M \) such that \( \Phi_q(\xi) = \xi A \), with \( A \in Gl_\mu(K) \) in Jordan normal form. By remark (1.4.2), we can choose \( \xi \) such that for almost all \( v \in \Sigma_{nilp} \) there exists \( n(v) \geq 1 \) satisfying the equivalence

\[
(A^{n(v)} - 1)^{n(v)} \equiv 0 \mod \varpi_v .
\]
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Therefore the matrix \( A^{\kappa_v} \) is unipotent modulo \( \varpi_v \) for all \( v \in \Sigma_{nilp} \). We deduce that the reduction modulo \( \varpi_v \) of the eigenvalues of \( A \) are \( \kappa_v \)-th roots of unity for almost all \( v \in \Sigma_{nilp} \). This means that the reduction modulo \( \varpi_v \) of any eigenvalues of \( A \) is an element of the cyclic group generated by the reduction of \( q \), for all \( v \in \Sigma_{nilp} \). The conclusion follows by applying lemma (6.1.1).

**Proposition 6.2.3.** Let us assume that the \( q \)-difference module \( M \) over \( F \) has the property that for almost all finite places \( v \) of \( K \) the morphism \( \Phi^v_q \) induces the identity on the reduction of \( M \) modulo \( \varpi_v \). Then \( M \) becomes trivial over \( K((x)) \).

**Remark.** The \( q \)-difference module \( M \) becomes trivial over \( K((x)) \) if and only if there exists a basis \( e \) of \( M_{K((x))} \) over \( K((x)) \) such that the associated \( q \)-difference system has a fundamental matrix of solutions \( Y(x) \) with coefficients in \( K[[x]] \). In this case the matrix \( G(x) \) defined by \( \Delta_q(e) = eG(x) \) has no poles at zero. This implies that the matrix \( G_n(x) \), defined by \( \Delta_q^n(e) = eG_n(x) \), have no poles at zero, for any positive integer \( n \), and hence that

\[
Y(x) = Y(0) \left( \mathbb{I}_\mu + \sum_{n \geq 0} \frac{G_n(0)}{n!} x^n \right).
\]

**Proof.** By theorem (6.2.2) we know that \( M \) is a regular singular \( q \)-difference module. By the formal classification (1.4.4), there exists a \( K((x)) \)-basis \( f \) of \( K((x)) \otimes_F M \) such that \( \Phi_q(f) = fA \), with \( A \in \text{Gl}_\mu(K) \) in Jordan normal form. By (1.4.2), we can choose \( f \) such that for almost all \( v \in \Sigma_F \) we have

\[
A^{\kappa_v} - 1 \equiv 0 \mod \varpi_v.
\]

We deduce that \( A \) is actually a diagonal matrix and that the eigenvalues of \( A \) are in \( q^{\mathbb{Z}} \). We can assume \( A = \mathbb{I}_\mu \), by applying a “shearing transformation” (cf. [PS, page 154]), i.e. a basis change of the form

\[
\begin{pmatrix}
x^{n_1} \mathbb{I}_{\nu_1} & \\ & \ddots & \\ & & x^{n_r} \mathbb{I}_{\nu_r}
\end{pmatrix},
\]

where \( \nu_1, \ldots, \nu_r \) are positive integers such that \( \sum \nu_i = \mu \) and \( n_1, \ldots, n_r \in \mathbb{Z} \). Let \( e \) be a basis of \( M \) over \( F \). Then there exists \( F(x) \in \text{Gl}_\mu(K((x))) \) such that \( e = fF(x) \). It follows that

\[
\Phi_q(e) = fF(qx) = eF(x)^{-1}F(qx).
\]

Then \( F(x) \) is a fundamental matrix of solutions for the \( q \)-difference system associated to \( M \) with respect to the basis \( e \). After a change of basis of the form \( e' = x^mC e \), where \( m \in \mathbb{Z} \) and \( C \) is a constant invertible matrix, we obtain a \( q \)-difference system having a solution \( Y(x) = \mathbb{I}_\mu + \sum_{m \geq 1} Y_m x^m \in \text{Gl}_\mu(K[[x]]) \).

**7. Statement of the \( q \)-analogue of Grothendieck’s conjecture on \( p \)-curvatures**

**7.1. Statement of the theorem**

We recall that \( K \) is a number field, \( \mathcal{V}_K \) its ring of integers, \( v \) is a finite or an infinite place of \( K \), and \( q \) an element of \( K \), which is not a root of unity. The uniformizer of the finite place \( v \) is denoted by \( \varpi_v \). For almost all finite places \( v \), let \( \kappa_v \) be the the multiplicative order of the image of \( q \) in the residue field of \( M \) modulo \( \varpi_v \). Let \( \varpi_{q,v} \) be the integer power of \( \varpi_v \) such that \( |\varpi_{q,v}|_v = |1 - q^{\kappa_v}|_v \).

We consider a \( q \)-difference algebra \( F \subset K(x) \) essentially of finite type over \( \mathcal{V}_K \) and a \( q \)-difference module \( M = (M, \Phi_q) \) over \( F \).

We want to prove the following theorem, which we consider to be the \( q \)-analogue of the Grothendieck conjecture for differential equations with \( p \)-curvature zero for almost all finite places:
Theorem 7.1.1. Let $M = (\mathcal{M}, \Phi_q)$ be a $q$-difference module over $\mathcal{F}$, such that

\[(*) \quad \text{the operator } \Phi_q^{\ast v} \text{ induces the identity on the reduction of } M \text{ modulo } \varpi_{q,v} \text{ for almost all finite places } v.\]

Then $M$ becomes trivial over $K(x)$.

Remark 7.1.2. We recall that the $q$-difference module $M$ over $K(x)$ is trivial if and only if the following equivalent conditions are satisfied:
1) there exists an isomorphism of $q$-difference modules $M^{\ast v} \otimes_K K(x) \cong M$;
2) there exists a $K(x)$-vector space isomorphism $\psi : M \rightarrow K(x)^{\mu}$ such that for all $m \in M$ we have:
\[\psi(\Phi_q(m)) = \varphi_q(\psi(m)),\]
where $\varphi_q$ is defined component-wise on $K(x)^{\mu}$.
3) there exists a basis $e$ of $M$ over $K(x)$ such that, if $\Delta_x e = \Phi_q G(x)$, we can find $Y(x) \in \text{Gl}(K(x))$ satisfying the $q$-difference linear system $d_q Y(x) = Y(x) G(x)$.

It is clear that if a $q$-difference module $M$ over $\mathcal{F}$ becomes trivial over $K(x)$, the hypothesis $(*)$ of the theorem above is satisfied.

By (2.1.2) we immediately obtain:

Corollary 7.1.3. Let $M = (\mathcal{M}, \Phi_q)$ be a $q$-difference module over $\mathcal{F}$ such that the reduction of $M$ modulo $\varpi_{q,v}$ is trivial for almost all $v$. Then $(\mathcal{M}, \Phi_q)$ is trivial over $K(x)$.

In (7.1.1), we assumed that $q$ is not a root of unity: if $q$ is a root of unity, theorem (7.1.1) is an easy consequence of the results in (§2). We notice that in this particular case, we just need the hypothesis of trivial reduction modulo $\varpi_v$:

Proposition 7.1.4. Let $q$ be a primitive $\kappa$-root of unity, with $\kappa \geq 1$. Then the $q$-difference module $M = (\mathcal{M}, \Phi_q)$ over $\mathcal{F}$ becomes trivial over $K(x)$ if and only if $M$ has trivial reduction modulo $\varpi_v$ for an infinite number of $v \in \Sigma_f$.

Proof. Let $e$ be a basis of $M$ over $\mathcal{F}$ and let $\Phi_q^m(e) = eA_m(x)$ for all $m \geq 1$. By (2.1.2) it is enough to prove that

\[A_{\kappa}(x) = \mathbb{I}_\mu \iff A_{\kappa_v}(x) = \mathbb{I}_\mu \text{ modulo } \varpi_v, \text{ infinitely many } v \in \Sigma_f.\]

To conclude, it is enough to notice that $\kappa_v = \kappa$ for almost all $v \in \Sigma_f$. \(\blacksquare\)

7.2. Idea of the proof

The proof of (7.1.1) is inspired by the theory of $G$-functions, from which we derive the definitions below. In the $q$-difference case, they are not as interesting as in the differential case; in fact, as we will see later, the two invariants that we are going to define are finite only when the $q$-difference module is trivial over $K(x)$. In any case, they will be useful in some intermediate steps of the proof.

Definition 7.2.1. Let $y = \sum_{n=0}^{\infty} a_n x^n \in K[x]$. We set $h(y, n, v) = \sup_{|a_n|_v} \log^{+} |a_n|_v$ and we define the size of $y$ (cf. [A1, I, 1.3]) to be the number

\[\sigma(y) = \lim_{n \to \infty} \sup \frac{1}{n} \sum_{m \in \Sigma_f \cup \Sigma_{\infty}} h(y, n, v).\]

Let $\mathcal{M} = (\mathcal{M}, \Phi_q)$ be a $q$-difference module over a $q$-difference algebra $\mathcal{F} \subset K(x)$. We fix a basis $e$ of $\mathcal{M}_{K(x)}$ over $K(x)$ and we set as usual $\Delta^* e = \Phi_q G_n(x)$ and $h(M, n, v) = \sup_{0 \leq s \leq n} \log |G_n(x)|_{s, \text{Gauss}}^{1/(p-1)}$. We define the size of $\mathcal{M}$ to be

\[\sigma(\mathcal{M}) = \lim_{n \to \infty} \sup \frac{1}{n} \sum_{m \in \Sigma_f} h(M, n, v).\]
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The proof (to be given explicitly in the next section) is organized as follows:

\[ M \text{ satisfies the hypothesis (\( \ast \))} \]

\[(6.2.3) \quad (M, \Phi_q) \text{ becomes trivial over } K((x)) \]
\[ \Rightarrow (8.1) \quad \sigma(M) < \infty \]

\[(8.2) \quad \text{There exists a basis } \xi \text{ of } M_{K((x))} \text{ over } K(x) \text{ such that } \Delta_q \xi = \xi G(x) \text{ and } \exists \text{ an invertible matrix } Y(x) \in \text{Gl}_\mu(K[[x]]) \text{ such that } d_q Y(x) = Y(x)G(x) \text{ and the entries of } Y(x) \text{ have finite size.} \]

\[(8.4) \quad Y(x) \in \text{Gl}_\mu(K(x)) \]

8. Proof of (7.1.1)

8.1. Finiteness of the size of \( M \)

**Proposition 8.1.1.** Let \( M \) be a \( q \)-difference module over \( F \subset K(x) \) satisfying (\( \ast \)). Then

\[ \sigma(M) < +\infty . \]

**Proof.** By assumption (cf. (2.1.2.1) and (2.1.3)) there exists a basis \( \xi \) of \( M \) over \( F \) such that \( \Delta_q \xi = \xi G(x) \) and

\[ |G_{\kappa_v}(x)|_{v, \text{Gauss}} \leq \left| \kappa_v \right|_v \text{ for almost all } v \in \Sigma_f. \]

For such \( v \), by (5.2.1.1) we have

\[ \left| \frac{G_n(x)}{[n]_q^{1/2}} \right|_{v, \text{Gauss}} \leq \left| \frac{\left[ n \right]_q^{1/2}}{[n]_q^{1/2}} \right|_{v, \text{Gauss}} \leq \left| \frac{\left[ \kappa_v \right]_q^{1/2}}{\left[ [n]_q^{1/2} \right]_v} \right| \leq |p|_v^{-n/\kappa_v(p-1)}, \]

from which we obtain

\[ h(M, n, v) \leq n \frac{\log |p|_v^{-1}}{\kappa_v(p-1)}. \]

Let

\[ T_1 = \{ v \in \Sigma_f : |1 - q^{\kappa_v}|_v < |p|_v^{1/(p-1)}, |G_{\kappa_v}(x)|_{v, \text{Gauss}} \leq |\kappa_v|_v \} \]

and

\[ T_2 = \{ v \in \Sigma_f : |1 - q^{\kappa_v}|_v < |p|_v^{1/(p-1)}, |G_{\kappa_v}(x)|_{v, \text{Gauss}} > |\kappa_v|_v \} . \]

The assumption (\( \ast \)) implies that \( T_2 \) is finite.
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By (4.2.7.1) we deduce that

\[
\sigma(M) \leq \sum_{v \in T_1} \frac{\log |p|_v^{-1}}{\kappa_v(p-1)} + \sum_{v \in T_2} \log^+ \frac{1}{\chi_v(M)}.
\]

Let us consider the set \( \tilde{\Sigma} \) of all finite places \( v \) of \( K \) such that \( |q|_v = 1 \). Then \( T_1 \) is cofinite in \( \tilde{\Sigma} \); hence, to conclude that \( \sigma(M) \) is finite, it is enough to prove the lemma:

**Lemma 8.1.2.** Let \( q \neq 0 \) be an element of \( K \) which is not a root of unity, \( \tilde{\Sigma} \) be the set of all finite places \( v \) of \( K \) such that \( |q|_v = 1 \), and \( \kappa_v \) be the multiplicative order of the image of \( q \) in the residue field of \( K \) with respect to \( v \in \tilde{\Sigma} \). Then

\[
\sum_{v \in \tilde{\Sigma}} \frac{\log |p|_v^{-1}}{\kappa_v(p-1)} < \infty.
\]

**Proof.** Let \( 0 < \varepsilon < 1 \). We consider the sets:

\[
S_0 = \{ v \in \tilde{\Sigma}, \kappa_v \geq p - 1 \},
\]

\[
S_1 = \{ v \in \tilde{\Sigma}, \kappa_v < p - 1, \kappa_v^2 \geq p^{1+\varepsilon} \},
\]

\[
S_2 = \{ v \in \tilde{\Sigma}, \kappa_v < p - 1, \kappa_v^2 < p^{1+\varepsilon} \}.
\]

Then, for \( \eta = (1 - \varepsilon)/(1 + \varepsilon) \) and \( v \in S_2 \), we have

\[
p > \kappa_v^{2/(1+\varepsilon)} \Rightarrow p - 1 \geq \kappa_v^{2/(1+\varepsilon)} = \kappa_v^{1+\eta},
\]

and hence we obtain

\[
\sum_{v \in \tilde{\Sigma}} \frac{\log |p|_v^{-1}}{\kappa_v(p-1)} \leq \sum_{v \in S_0} \frac{\log |p|_v^{-1}}{(p-1)^2} + \sum_{v \in S_1} \frac{\log |p|_v^{-1}}{p^{1+\varepsilon}} + \sum_{v \in S_2} \frac{\log |p|_v^{-1}}{\kappa_v^{2+\eta}}.
\]

The sums over \( S_0 \) and \( S_1 \) are clearly convergent. Since for almost all \( v \in \tilde{\Sigma} \) we have \( |1 - q^{\kappa_v}v|^{-1} \geq |p|_v^{-1} \), to conclude that the sum over \( S_2 \) is convergent it is enough to prove that

\[
\sum_{v \in \tilde{\Sigma}} \frac{\log |1 - q^{\kappa_v}v|^{-1}}{\kappa_v^{2+\eta}}
\]

is convergent for all \( \eta > 0 \). We recall that \( \tilde{\Sigma} \) is cofinite in \( \Sigma_f \) and that for all integers \( n \geq 1 \) there exists at worst only a finite number of \( v \in \Sigma_f \) such that \( \kappa_v = n \) (since \( |1 - q^n|_v = 1 \) for almost every \( v \in \Sigma_f \)). Therefore by the Product Formula, we get

\[
\sum_{v \in \tilde{\Sigma}} \frac{\log |1 - q^{\kappa_v}v|^{-1}}{\kappa_v^{2+\eta}} = \sum_{n=1}^{\infty} \sum_{v \in \tilde{\Sigma}, \kappa_v = n} \frac{\log |1 - q^n|_v^{-1}}{n^{2+\eta}}
\]

\[
= \sum_{n=1}^{\infty} \sum_{v \in \tilde{\Sigma}, \kappa_v \neq n} \frac{\log |1 - q^n|_v}{n^{2+\eta}}.
\]

For every \( v \in \Sigma_f \) such that \( |q|_v \leq 1 \) we have \( |1 - q^n|_v \leq 1 \). In particular \( |1 - q^n|_v = 1 \) for almost all \( v \in \Sigma_f \). Therefore we obtain

\[
\sum_{v \in \tilde{\Sigma}} \frac{\log |1 - q^{\kappa_v}v|^{-1}}{\kappa_v^{2+\eta}} \leq \sum_{n=1}^{\infty} \sum_{v \in \Sigma_f, |q|_v > 1} \frac{\log |1 - q^n|_v}{n^{2+\eta}}
\]

\[
\leq \sum_{n=1}^{\infty} \left( \sum_{v \in \Sigma_f, |q|_v \leq 1} \frac{\log 2}{n^{2+\eta}} + \sum_{v \in \Sigma_f \setminus \Sigma_\infty, |q|_v > 1} \frac{\log (1 + |q|_v)}{n^{1+\eta}} \right) < \infty.
\]

\[
(8.1.2.1)
\]

\[
\]
8.2. Finiteness of the size of a fundamental matrix of solutions

We have already proved that a q-difference module \( M \) satisfying (*) becomes trivial over \( K((x)) \) (cf. (6.2.3)) and that it has finite size (cf. (8.1.1)).

**Proposition 8.2.1.** Under the hypothesis (*), there exists a basis \( \mathfrak{e} \) of \( M_K((x)) \) over \( K((x)) \) such that the q-difference system associated to \( \mathfrak{e} \) has an invertible solution matrix \( Y(x) \in \text{Gl}_n(K[[x]]) \), whose entries have finite size.

**Remark 8.2.2.** We recall a property of the size of a formal power series that we will use in the proof below. We know (cf. for example [DGS, VI, 4.1]) that

\[
\limsup_{n \to \infty} \frac{1}{n} h(n, v, y) = \log^+ \frac{1}{r_v(y)},
\]

where \( r_v(y) \) is the \( v \)-adic radius of convergence of \( y \). Then \( \sigma(y) < \infty \) if and only if there exists a finite set \( S \subset \Sigma_f \cup \Sigma_\infty \) such that \( y \) has nonzero radius of convergence for all \( v \in S \) and

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{v \in \Sigma_f \cup \Sigma_\infty \setminus S} h(n, v, y) < \infty.
\]

**Proof.** As in the proof of (6.2.3), we can find a basis \( \mathfrak{e} \) of \( M_K((x)) \) such that the associated q-difference system has a solution \( Y(x) \in \text{Gl}_n(K[[x]]) \) such that \( Y(0) = \mathbb{I}_n \). Let \( \Delta^n_\mu \mathfrak{e} = \mathfrak{e}G_n(x) \). Since \( G_0 = Y(0) = \mathbb{I}_n \), we conclude that \( Y(x) = \sum_{n \geq 0} \frac{G_n(0)}{\nu^n} x^n \).

We want to prove that the entries of \( Y(x) \) have finite size. By (4.2.3) and (4.2.10), we deduce that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{v \in \Sigma_f} \sup_{0 \leq s \leq n} \log^+ |[M]_v| \leq \sigma(M) + \sum_{v \in \Sigma_f, |q_v| < 1} \log^+ \frac{1}{\chi_v(M)} < \infty.
\]

By the previous remark it is enough to prove that the entries of \( Y(x) \) have nonzero radius of convergence for all \( v \in \Sigma_\infty \) and for all \( v \in \Sigma_f \) such that \( |q_v| > 1 \). Since \( M \) is a regular q-difference module over \( K((x)) \), each entry of \( Y(x) \) is the solution of a regular singular q-difference equation, by (1.4.5). It follows from [Be, IV] and [BB, IV] that the entries of \( Y(x) \) have nonzero radius of convergence for all \( v \in \Sigma_\infty \) such that \( |q_v| \neq 1 \) and for all \( v \in \Sigma_f \) such that \( |q_v| > 1 \).

The conclusion of the proof of (8.2.1) is the subject of the next section.

8.3. How to deal with the problem of archimedean small divisors

To conclude that the entries of \( Y(x) \) have finite radius of convergence for all infinite places \( v \) such that \( |q_v| = 1 \), we recall the following result:

**[Be, 6.1]** Let \( \mathcal{L} = \sum_{i=0}^{n} \sum_{j=0}^{n'} a_{i,j} x^j y^i \in \mathbb{C}[x, y^i] \) be a q-difference operator and let \( Q(x) = \sum_{i=0}^{n} a_{i,j} x^j \) be a polynomial such that \( j_0 = \min\{j = 0, \ldots, n : a_{i,j} \neq 0\} \). We suppose that \( |q|_v = 1 \) and that there exist two positive real constants \( c_1 \) and \( c_2 \) such that all the roots \( u \) of the polynomial \( (x - 1)Q(x) \) satisfy the inequality \( |q^n - u|_v \geq c_1 n^{-c_2} \) for \( n > 0 \). Then a formal power series \( y \in \mathbb{C}[[x]] \) which solves \( \mathcal{L} y = 0 \) is convergent.

It is enough to prove that for any finite place \( v \) such that \( |q_v| = 1 \), there exist two positive real constants \( c_{1,v} \) and \( c_{2,v} \) such that

\[
|q^n - u|_v \geq c_{1,v} n^{-c_{2,v}}.
\]

(8.3.0.1)

In [Be] and [BB] the authors assume \( |q| < 1 \). This is only a matter of convention and their results translate to our situation.
The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures for $n \gg 0$. To verify (8.3.0.1) we will use the following theorem by Baker(3):

[Se, 8.2, Corollary] Let $K$ be a number field, $\alpha_1, \ldots, \alpha_l \in K$, $\beta_1, \ldots, \beta_l \in \mathbb{Z}$ and $v$ a place of $K$.

If $\alpha_1^{\beta_1} \cdots \alpha_l^{\beta_l} \neq 1$ then

$$\left| \alpha_1^{\beta_1} \cdots \alpha_l^{\beta_l} - 1 \right|_v \geq \sup(4, \beta_1, \ldots, \beta_l)^{-\text{const}},$$

where the constant depends only on $v$ and on $\alpha_1, \ldots, \alpha_l \in K$.

Let $l = 2$, $\alpha_1 = q$, $\alpha_2 = u$, $\beta_1 = n$ and $\beta_2 = -1$. Since for $n \gg 0$ we have $q^n u^{-1} \neq 1$, we obtain $|q^n - u|_v \geq |u|_v n^{-c(u)}$. Here $c(u)$ is a constant depending on $u$, $q$ and $v$. We set

$$c_{1,v} = \sup \{|u|_v : \text{such that } Q(u) = 0 \} \cup \{1\}$$

and

$$c_{2,v} = \sup \{c(u) : u \text{ such that } Q(u) = 0 \text{ or } u = 1 \};$$

then we obtain the desired inequality. This achieves the proof of (8.2.1).

8.4. Conclusion of the proof: a criterion for rationality

We complete the proof of (7.1.1) by applying the following proposition:

**Proposition 8.4.1. (Y. André)** Let $y(x) \in K[x]$ be a formal power series solution of a $q$-difference equation

$$a_\mu(x)d_\mu^q(y)(x) + a_{\mu-1}(x)d_{\mu-1}^q(y)(x) + \cdots + a_0(x)y(x) = 0,$$

with $a_i(x) \in K(x)$ for all $i = 0, \ldots, \mu$. If $\sigma(y) < \infty$ then $y(x)$ is the Taylor expansion of a rational function $\in K(x)$.

**Proof.** It is a general property of $q$-difference equations that for all $v \in \Sigma_f \cup \Sigma_\infty$ such that $|q|_v \neq 1$, the series $y(x)$ with nonzero radius of convergence has infinite radius of meromorphy (cf. for instance [BB, 7.2]). We remark that we can always find such a $v$ since $q$ is not a root of unity. To conclude that $y(x)$ is a rational function it is enough to apply the more general result [A1, VIII, 1.1, Th.], but we prefer to sketch the proof, since it simplifies under the present assumptions. First we prove that $y(x)$ is an algebraic function (steps from 1 to 5), following the proof of [A2, 2.3.1] adapted to this particular case. Then, in step 6, we prove that $y(x)$ is the expansion of a rational function.

**Step 1.** We fix $\eta \in (0,1]$ and an integer $\nu > 1$. Let

$$\vec{Y} = \vec{t}(1, y(x), \ldots, y(x)^{\nu-1}) = \sum_{m \geq 0} \vec{Y}_m x^m \in K[x]^{\nu}.$$

Using Siegel’s Lemma, one can construct a polynomial vector

$$\vec{P}_N(x) = (P_{N,0}(x), \ldots, P_{N,\nu-1}(x)) = \sum_{m \geq 0} \vec{P}_N^{(m)} x^m \in K[x]^{\nu},$$

for all $N \in \mathbb{N}$, such that

i) $M = \text{ord}_0(\vec{P}_N^{(\nu)} \cdot \vec{Y}) \geq N$;

ii) $\deg_x \vec{P}_N^{(\nu)} = \sup_{i = 0, \ldots, \nu-1} (\deg_x P_{N,i}(x)) \leq \frac{1}{\nu} \left(1 + \frac{1}{\eta}\right) N + o(N)$;

iii) $h(\vec{P}_N) \leq \limsup_{m \to \infty} \frac{1}{m} \sum_{v \in \Sigma_f \cup \Sigma_\infty} \log^+ \left(\sup_{m \geq 0} |\vec{P}_N^{(m)}|_v\right) \leq \eta N \sigma(\vec{Y}) + o(N)$.

(3) There is a proof of the additive version of this theorem in [B], but we prefer to cite the version in [Se], which is more suitable to our situation.
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**Step 2.** Let us suppose that $y(x)$ is not algebraic; then $\overline{P}^2_N \cdot \overline{Y} \neq 0$ for all $N \geq 1$ and hence $M < \infty$. We set
\[
\alpha = \frac{1}{M!} \frac{d^M}{dx^M} (\overline{P}^2_N \overline{Y})(0). 
\]
Of course, $\alpha \neq 0$. For all $v \in \Sigma_f$ we have
\[
\log |\alpha|_v \leq \log^+ |\overline{P}^2_N(x)|_{v,Gauss} + \sup_{m \leq M} \log^+ |\overline{Y}_m|_v.
\]

**Step 3.** Let $V$ be a finite subset of $\Sigma_f \cup \Sigma_\infty$ containing $\Sigma_\infty$ and at least one place $v$ such that the radius of meromorphy $M_v(y)$ is infinite, since $q$ is not a root of unity. For all $v \in V$ the formal power series $y(x)$ is the germ at zero of a meromorphic function, therefore we can write $y(x) = f_v(x)/g_v(x)$, where $f_v(x)$ and $g_v(x)$ are $v$-adic analytic functions converging for $|x|_v < M_v(y)$. We can suppose that $g_v(0) = 1$. We set:
\[
\overline{Z}_v(x) = (g_v(x)^{-1}, g_v(x)^{-2} f_v(x), \ldots, f_v(x)^{-1}),
\]
\[
\psi_v(x) = \overline{P}^2_N(x) \cdot \overline{Z}_v(x);
\]
from which it follows that
\[
\overline{P}^2_N(x) \cdot \overline{Y}(x) = \frac{1}{g_v(x)^{\nu-1}} \psi_v(x).
\]
We deduce that:
\[
\alpha = \frac{1}{M!} \frac{d^M}{dx^M} (\psi_v)(0).
\]

**Step 4.** Let us fix $m_v < M_v(y)$ for all $v \in V$. By Cauchy’s estimates we obtain
\[
\log |\alpha|_v \leq -M \log m_v + \log \left( \sup_{|x|_v = m_v} |\psi_v(x)|_v \right)
\]
\[
\leq -M \log m_v + \log^+ \left( \sup_{m \leq N} |\overline{P}^4(m)|_v \right) + m_v \deg_x \overline{P}^2_N + o(N).
\]

**Step 5.** Summing (8.4.1.1) for $v \in (\Sigma_\infty \cup \Sigma_f) \setminus V$ and (8.4.1.2) for $v \in V$, by the Product Formula we obtain
\[
M \sum_{v \in V} \log m_v \leq \sum_{v \in \Sigma_f \cup \Sigma_\infty} \log^+ \left( \sup_{m \leq N} |\overline{P}^4(m)|_v \right)
\]
\[
+ \sum_{v \in V} \sup_{m \leq M} \log^+ |\overline{Y}_m|_v + \deg_x \overline{P}^2_N \sum_{v \in V} m_v + o(N);
\]
dividing by $M \geq N$ and taking the lim sup for $N \to \infty$ we have
\[
\sum_{v \in V} \log m_v \leq (\eta + 1) \sigma(y) + \frac{1}{\nu} \left( 1 + \frac{1}{\eta} \right) \sum_{v \in V} m_v.
\]
Finally we can take the limit for $\nu \to \infty$ and $\eta \to 0$ and obtain
\[
\sum_{v \in V} \log m_v \leq \sigma(y).
\]
Since $\sigma(y) < \infty$, we get a contradiction by letting $m_v \to M_v(y)$.

Thus we have proved that $y(x)$ is an algebraic function.

**Step 6.** Now we prove that $y(x)$ is the Taylor expansion of a rational function. Since $y(x)$ is algebraic over $K(x)$ there exists $P(x) \in K[x]$ such that $g(x) = P(x)g(x)$ satisfies a relation of the form:
\[
(8.4.1.3) \quad g(x)^s - Q_1(x)g(x)^{s-1} - \cdots - Q_s(x) = 0,
\]
with $Q_1(x), \ldots, Q_s(x) \in K[x]$. Let us fix $v \in \Sigma_\infty \cup \Sigma_f$ such that $|q|_v \neq 1$, hence such that $y(x)$ has infinite $v$-adic radius of meromorphy. The relation (8.4.1.3) implies that $g(x)$ is an entire analytic function.
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function. Proving that $y(x)$ is a rational function is equivalent to prove that $g(x) = P(x)y(x)$ is a polynomial. We deduce by (8.4.1.3) that there exist two real positive constants $c_1$ and $c_2$ such that for $|x|_v >> 1$ we have

$$|g(x)|_v \leq c_1|x|_v^{c_2},$$

which implies that $g(x) \in K[x]$. ■

This completes the proof of theorem (7.1.1).

8.5. A corollary

We point out that the following corollary is a consequence the proof of (7.1.1):

**Corollary 8.5.1.** Let $S$ be a subset of $\Sigma_f$ having Dirichlet density 1 and $M$ be a $q$-difference module over a $q$-difference algebra $F \subset K(x)$ essentially of finite type over $V_v$. We assume that for all $v \in S$ the operator $\Phi_q^{\kappa_v}$ induces the identity over the reduction of $M$ modulo $\mathfrak{q}_{q,v}$. We assume moreover that:

$$\sum_{v \in \Sigma_f \setminus S} | |1 - q^{\kappa_v}|_v < |p|^{1/(p-1)}| \log |\chi_v(M)| < \infty.$$  

Then $M$ becomes trivial over $K(x)$.

**Proof.** We notice that in the proof of (8.1.1) we have actually shown that:

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{v \in \Sigma_f \setminus S} h(M, n, v) < \infty.$$  

Let

$$T = \{ v \in \Sigma_f \setminus S : |G(x)|_{v,Gauss} \leq 1 \text{ and } |1 - q^{\kappa_v}|_v < |p|^{1/(p-1)} \}.$$  

If we prove that

$$(8.5.1.1) \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in T} h(M, n, v) \leq \sum_{v \in T} \log \frac{1}{\chi_v(M)}$$

then we obtain $\sigma(M) < \infty$, since $|1 - q^{\kappa_v}|_v < |p|^{1/(p-1)}$ and $|G(x)|_{v,Gauss} \leq 1$ for almost all $v \in \Sigma_f$. Then we can complete the proof as we did the proof of (7.1.1).

To prove (8.5.1.1) we need only notice that for all $v \in \Sigma_f \setminus S$ such that $|1 - q^{\kappa_v}|_v < |p|^{1/(p-1)}$ and $|G(x)|_{v,Gauss} \leq 1$ and for all $n \geq 1$, we have

$$h(M, n, v) \leq \log \left( \frac{|\kappa_v|_v^{-1} |\mathfrak{q}_{v}^{-1} | \mathfrak{q}_{v}^{-1} |}{|p|_v^{-1} |\mathfrak{q}_{v}^{-1} |} \right) \leq n \log \frac{1}{\chi_v(M)}.$$  

Part IV. A $q$-analogue of Katz’s conjectural description of the generic Galois group

9. Definition of the generic $q$-difference Galois group

9.1. Some algebraic constructions

We consider the following algebraic constructions on the category of $q$-difference modules over a fixed $q$-difference algebra $F$ over a field $K$:
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**Dual q-difference module.** Let $\mathcal{M} = (M, \Phi_q)$ be a $q$-difference module over $\mathcal{F}$. Let us consider the dual $\mathcal{F}$-module $\hat{M} = \text{Hom}_{\mathcal{F}}(M, \mathcal{F})$ of $M$: $\hat{M}$ is naturally a $q$-difference module equipped with the $q$-difference operator $\hat{\Phi}_q = \Phi_q^{-1}$, defined by

$$< \hat{\Phi}_q(m), m > = < \hat{m}, \Phi_q^{-1}(m) > , \text{ for all } \hat{m} \in \hat{M} \text{ and } m \in M.$$  

The $q$-difference module $\hat{M} = (\hat{\Phi}_q)$ over $\mathcal{F}$ is the dual $q$-difference module of $\mathcal{M}$.

**Tensor product of $q$-difference modules.** Let $\mathcal{M}' = (M', \Phi_q')$ and $\mathcal{M}'' = (M'', \Phi_q'')$ be two $q$-difference modules of finite rank over $\mathcal{F}$. The tensor product $M' \otimes_{\mathcal{F}} M''$ has the natural structure of a $q$-difference module defined by:

$$\Phi_q(m' \otimes m'') = \Phi_q'(m') \otimes \Phi_q''(m''), \text{ for all } m' \in M' \text{ and } m'' \in M''.$$  

The $q$-difference module $\mathcal{M}' \otimes_{\mathcal{F}} \mathcal{M}'' = (M' \otimes_{\mathcal{F}} M'', \Phi_q \otimes \Phi_q')$ over $\mathcal{F}$ is the tensor product of $\mathcal{M}'$ and $\mathcal{M}''$.

We denote by $<\mathcal{M}>^\otimes$ the full subcategory of the category of the $q$-difference modules over $\mathcal{F}$ containing all the subquotients of the $q$-difference modules obtained as finite sums of the form $\oplus_i T^{i,j}(\mathcal{M})$, where $T^{i,j}(\mathcal{M}) = \mathcal{M}^{\otimes i} \otimes \mathcal{M}^{\otimes j}$.

### 9.2. Definition of the Galois group of a $q$-difference module

Let $K$ be a field and $q$ be a nonzero element of $K$ which is not a root of unity. Let $\mathcal{M} = (M, \Phi_q)$ be a $q$-difference module over $K(x)$.

Sometimes, the tensor category $<\mathcal{M}>^\otimes$ comes equipped with a $K$-linear fiber functor

$$\omega : <\mathcal{M}>^\otimes \rightarrow \{ \text{finite dimensional } K\text{-vector spaces} \}.$$  

In fact, such a fiber functor always exists after replacing $K$ by some finite extension. Here is an explicit construction. The $q$-difference module $\mathcal{M}$ admits a "model" $\hat{\mathcal{M}}$ over some $q$-difference algebra $\mathcal{F} \subset K(x)$ essentially of finite type over $Y_K$. This provides a corresponding "model" $\hat{N}$ for any $N$ in $<\mathcal{M}>^\otimes$. Consider a $K$-valued point $x$ of $\mathcal{F}$ (which exists after passing to a finite extension of $K$). Then the fiber at $x$ provides a "fiber functor". The corresponding tannakian group may be interpreted as the Picard-Vessiot group of a $q$-difference system, as considered in [PS] (this interpretation is not used in the sequel).

It follows (cf. [DM, 2.11] and [A2, III, 2.1.1]) that there exists an algebraic closed subgroup $Gal(\mathcal{M}, \omega)$ of $GL(\omega(\mathcal{M}))$, such that $\omega$ induces a tensor equivalence of categories between $<\mathcal{M}>^\otimes$ and the category of finite type representations of $Gal(\mathcal{M}, \omega)$ over $K$.

**Definition 9.2.1.** The algebraic group $Gal(\mathcal{M}, \omega)$ is the Galois group of $\mathcal{M}$ pointed at $\omega$.

We recall the following results:

**Lemma 9.2.2.** [A2, III, 2.1.1] The algebraic group $Gal(\mathcal{M}, \omega)$ is the subgroup of $GL(\omega(\mathcal{M}))$ which stabilizes $\omega(N)$ for all sub-objects $N$ of a finite sum $T^{i,j}(\mathcal{M}) = \mathcal{M}^{\otimes i} \otimes \hat{\mathcal{M}}^{\otimes j}$.

**Remark 9.2.3.** We notice that $Gal(\mathcal{M}, \omega)$ is a stabilizer in the sense of algebraic groups.

**Lemma 9.2.4.** [A2, III, 2.1.4] The group $Gal(\mathcal{M}, \omega)$ is trivial if and only if $\mathcal{M}$ is a trivial $q$-difference module over $K(x)$ (cf. (2.1.1)).

### 9.3. Definition of the generic Galois group of a $q$-difference module

Let $\mathcal{M} = (M, \Phi_q)$ be a $q$-difference module over $K(x)$. Let us consider the forgetful fiber functor "underlying vector space"

$$\eta : <\mathcal{M}>^\otimes \rightarrow \{ K(x)\text{-vector spaces} \}.$$  

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The functor \( \text{Aut}^\otimes(\eta) \) defined on the category of commutative \( K(x) \)-algebras is representable by an algebraic group \( \text{Gal}(\mathcal{M}, \eta) \) over \( K(x) \).

**Definition 9.3.1.** The algebraic group \( \text{Gal}(\mathcal{M}, \eta) \) is the generic Galois group of \( \mathcal{M} \).

**Remark 9.3.2.** The generic Galois group \( \text{Gal}(\mathcal{M}, \eta) \) admits the following concrete description: it is the closed subgroup of \( \text{GL}(\mathcal{M}) \) which stabilizes all the \( q \)-difference sub-modules in finite sums \( \oplus_{i,j} T^{i,j}(\mathcal{M}) \) (cf. [A2, §III, 2.2]), in the sense of algebraic groups. Since \( \text{GL}(\mathcal{M}) \) is a noetherian algebraic variety, \( \text{Gal}(\mathcal{M}, \eta) \) is defined as the stabilizer of a finite number of \( q \)-difference sub-modules \( N_1, \ldots, N_r \) of some finite sums \( \oplus_{i,j} T^{i,j}(\mathcal{M}) \): this is equivalent to demanding that \( \text{Gal}(\mathcal{M}, \eta) \) be the stabilizer of the maximal exterior power of \( \oplus_{i=1}^r N_i \) (cf. [W, A.2]). This shows that \( \text{Gal}(\mathcal{M}, \eta) \) can be defined as the stabilizer of a \( q \)-difference sub-module of rank 1 of a finite sum \( \oplus_k T^{i,h,j}(\mathcal{M}) \).

**Warning.** A sub-\( K(x) \)-vector space of a finite sum \( \oplus_{i,j} T^{i,j}(\mathcal{M}) \) stabilized by the generic Galois group \( \text{Gal}(\mathcal{M}, \eta) \) is not necessarily a \( q \)-difference module.

**Remark 9.3.3.** If \( \omega \) is a fiber functor over \( \text{Hom}^\otimes(\omega \otimes_K 1_{K(x)}, \eta) \), values in \( K \)-spaces, the functor \( \text{Isom}^\otimes(\omega \otimes_K K(x)) \) is representable by a \( K(x) \)-group scheme \( \Sigma(\mathcal{M}, \omega) \), which is a torsor over \( \text{Gal}(\mathcal{M}, \omega) \otimes_K K(x) \) (cf. [DM, 3.2] and [A2, III, 2.2]), such that \( \text{Gal}(\mathcal{M}, \eta) = \text{Aut}_{\text{Gal}(\mathcal{M}, \omega) \otimes K(x)} \Sigma(\mathcal{M}, \omega) \).

**Lemma 9.3.4.** A \( q \)-difference module \( \mathcal{M} \) over \( K(x) \) is trivial if and only if \( \text{Gal}(\mathcal{M}, \eta) \) is the trivial group.

**Proof.** If a fiber functor exists, this follows from the previous remark and (9.2.4). In general, \( \omega \) exists after replacing \( K \) by a finite extension, and the result follows by an easy Galois descent. ■

10. An arithmetic description of the generic Galois group

Let \( K \) be a number field and \( \mathcal{V}_K \) the ring of integers of \( K \). We denote by \( \Sigma_f \) the set of all finite places \( v \) of \( K \), by \( \varpi_v \in \mathcal{V}_K \) the uniformizer associated to \( v \), and by \( \mathcal{V}_v \) the discrete valuation ring of \( K \) associated to \( v \).

We choose an element \( q \in K \) which is not a root of unity. For every finite place \( v \) such that \( q \) is a unit of \( \mathcal{V}_v \), we denote by \( \kappa_v \) the order of the cyclic group generated by the image of \( q \) in the residue field of \( \mathcal{V}_v \), i.e.:

\[
\kappa_v = \min\{m \in \mathbb{Z} \mid m > 0 \text{ and } 1 - q^m \in \varpi_v \mathcal{V}_v\}.
\]

Let \( \varpi_{q,v} \) be the power of \( \varpi_v \) satisfying \( 1 - q^{\kappa_v} \in \varpi_{q,v} \mathcal{V}_v \) and \( 1 - q^{\kappa_v} \notin \varpi_v \mathcal{V}_v \mathcal{V}_v \). We set \( k_{q,v} = \mathcal{V}_v / \varpi_{q,v} \mathcal{V}_v \).

10.1. Algebraic groups “containing \( \Phi_q^{\kappa_v} \) for almost all \( v \)”

Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over \( K(x) \). One can always find a \( q \)-difference algebra \( \mathcal{F} \subset K(x) \) essentially of finite type over the ring of integers \( \mathcal{V}_K \) of \( K \) and a \( q \)-difference module \( \mathcal{M} = (M, \Phi_q) \) over \( \mathcal{F} \) such that \( \mathcal{M} \) is isomorphic to \( \mathcal{M}_{K(x)} \).

**Remark.** Since \( q^{\kappa_v} \equiv 1 \) modulo \( \pi_{q,v} \), \( \Phi_q^{\kappa_v} \) induces a \( (\mathcal{F} \otimes_{\mathcal{V}_K} k_{q,v}) \)-linear morphism on \( \mathcal{M} \otimes_{\mathcal{V}_K} k_{q,v} \).

Let \( G \) be a closed algebraic subgroup of \( \text{GL}(\mathcal{M}) \). By Chevalley’s theorem, \( G \) is the stabilizer of a one-dimensional sub-\( K(x) \)-vector space \( L \) in a finite sum \( \oplus_k T^{i,h,j}(\mathcal{M}) \). Up to enlarging \( \mathcal{F} \), there exists an \( \mathcal{F} \)-free module \( \tilde{L} \) such that \( L \cong \tilde{L} \otimes_{\mathcal{F}} K(x) \).

**Definition 10.1.1.** The closed algebraic subgroup \( G \) of \( \text{GL}(\mathcal{M}) \) contains \( \Phi_q^{\kappa_v} \) for almost all \( v \in \Sigma_f \) if, for almost every finite place \( v \) of \( K \), \( \tilde{L} \otimes_{\mathcal{V}_K} k_{q,v} \) is stable by \( \Phi_q^{\kappa_v} \) in \( \oplus_k T^{i,h,j}(\mathcal{M}) \otimes_{\mathcal{V}_K} k_{q,v} \).

**Remark 10.1.2.** We notice that the notion of an algebraic group over \( K(x) \) containing \( \Phi_q^{\kappa_v} \) for almost all \( v \in \Sigma_f \) is well defined:
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**Independence of the choice of $F$ and $L$:** let $L'$ and $F'$ be a different choice for $L$ and $F$ in the previous definition. Then there exists a third $V_K$-algebra $F''$ of the same form with $F, F' \subset F''$. So by extension of scalars, we may suppose that $F = F'$ and that $L$ and $L'$ are two different $F$-lattices of $L$. By enlarging $F$, we may suppose that there exists an $F$-linear isomorphism $\psi : L \to L'$. For almost all $v \in \Sigma_f$ the morphism $\psi$ induces a $(F \otimes V_K k_{q,v})$-linear isomorphism $L \otimes V_K k_{q,v} \to L' \otimes V_K k_{q,v}$ commuting with the action of $\Phi_q^{\kappa_v}$.

**Independence of the choice of $L$:** this follows from the fact that $L$ is a direct factor of an $F$-lattice of $\otimes hT^{\kappa_2,jh}(M)$, hence any $F \otimes V_K k_{q,v}$-linear automorphism $\otimes hT^{\kappa_2,jh}(M) \otimes V_K k_{q,v}$ stabilizing $L \otimes V_K k_{q,v}$ comes from an $F$-linear automorphism of $\otimes hT^{\kappa_2,jh}(M)$ stabilizing $L$.

**Lemma 10.1.3.** The smallest closed algebraic subgroup $G_{\Phi_q}(M)$ of $GL(M)$ which contains $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$ is well defined.

**Proof.** Let $G_1$ and $G_2$ be two closed algebraic subgroups of $GL(M)$ containing $\Phi_q^{\kappa_v}$ for almost all $v$. Let $G_1$ (resp. $G_2$) be defined as the stabilizer of a line $L_1$ (resp. $L_2$) in some $\otimes hT^{\kappa_2,jh}(M)$. Then the intersection of $G_1$ and $G_2$ is the algebraic group stabilizing the lines $L_1$ and $L_2$, or equivalently the line $\wedge^2(L_1 \oplus L_2)$ (cf. [W, A2]). This implies that the intersection of two algebraic subgroups of $GL(M)$ containing the $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$ is still an algebraic subgroup of $GL(M)$ containing the $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$, in the sense of definition (10.1.1).

Moreover $GL(M)$ is an algebraic variety of finite dimension, hence any descending chain of closed algebraic subgroups containing the $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$ is stationary. ■

**Lemma 10.1.4.** Let $N = (N, \Phi_q)$ be an object of $\langle M \rangle^\otimes$. Then the natural morphism

$$\Omega_{\Phi_q} : G_{\Phi_q}(M) \to G_{\Phi_q}(N)$$

is surjective.

**Proof.** Since the action of $\Phi_q$ on $N$ is induced by the action of $\Phi_q$ on $M$, the image of the natural morphism $\Omega_{\Phi_q} : G_{\Phi_q}(M) \to GL(N)$ contains $\Phi_q^{\kappa_v}$ for almost all $v$, and hence contains $G_{\Phi_q}(N)$.

Let us choose a line $L$ in some finite sum $\otimes hT^{\kappa_2,jh}(N)$, such that $G_{\Phi_q}(N)$ is the stabilizer of $L$. Let $L$ be an $F$-lattice of $L$, defined over a $q$-difference algebra $F \subset K(x)$ essentially of finite type over $V_K$. Since $N$ is an object of $\langle M \rangle^\otimes$, $L$ is a line in a suitable subquotient of a finite sum $\otimes hT^{\kappa_2,jh}(M)$. Since $L \otimes V_K k_{q,v}$ is stable by $\Phi_q^{\kappa_v}$ for almost all $v$, $L$ is stabilized by $G_{\Phi_q}(M)$, by construction of $G_{\Phi_q}(M)$. It follows that the image of $\Omega_{\Phi_q}$ is precisely $G_{\Phi_q}(N)$. ■

**10.2. Statement of the main theorem**

**Main theorem 10.2.1.** The algebraic group $Gal(M, \eta)$ is the smallest closed subgroup of $GL(M)$ containing $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$.

**Example.** Let us consider the $q$-difference equation $y(qx) = q^{1/2}y(x)$, associated to the $q$-difference module:

$$\Phi_q : \begin{array}{lcl} K(x) & \to & K(x) \\ f(x) & \mapsto & q^{1/2}f(qx) \end{array}$$

The $q$-difference module $(K(x), \Phi_q)$ is trivial over $K(x^{1/2})$, hence the generic Galois group of $(K(x), \Phi_q)$ is the group $\mu_2 = \{1, -1\}$. For all $v$ such that $[q]_v = 1$ and such that the image of $q^{1/2}$ is an element of the cyclic group generated by the image of $q$ in $k_{q,v}$, the module $(K(x), \Phi_q)$ has $\kappa_v$-curvature zero. For every other $v$ such that $[q]_v = 1$, we have $\phi_q^{\kappa_v} \neq 1$ and $\phi_q^{2\kappa_v} \equiv 1$ over $k_{q,v}$, which means that $\phi_q^{\kappa_v} \equiv -1$. So the Galois group is the smallest algebraic subgroup of the multiplicative group $K(x)^\times \cong GL(K(x))$ containing $\Phi_q^{\kappa_v}$ for almost all $v$.

The proof of the last statement relies on the $q$-analogue of Grothendieck’s conjecture on $p$-curvatures (cf. (7.1.1)).

A part of the statement is very easy to prove:

**Proposition 10.2.2.** The algebraic group $Gal(M, \eta)$ contains $\Phi_q^{\kappa_v}$ for almost all $v \in \Sigma_f$. 

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**Proof.** The algebraic group \( \text{Gal}(\mathcal{M}, \eta) \) can be defined as the stabilizer of a \( q \)-difference module \( L \) of rank one over \( K(x) \). The choice of an \( \mathcal{F} \)-lattice \( \tilde{L} \) of \( M \) determines an \( \mathcal{F} \)-lattice \( L \) of \( L \) of rank one. The reduction over \( k_{q,v} \) of \( \tilde{L} \) is stable by the morphism induced by \( \Phi^v_q \) since \( \tilde{L} \) is a \( q \)-difference module, hence stable under \( \Phi_q \).

\[ \Box \]

**10.3. Proof of the main theorem**

Let \( \mathcal{M} = (M, \Phi_q) \) be a \( q \)-difference module over \( K(x) \) and let \( \text{Gal}(\mathcal{M}, \eta) \) be its generic Galois group. We denote by \( G_{\phi^v}(\mathcal{M}) \) the smallest algebraic subgroup of \( GL(M) \) containing \( \Phi^v_q \) for almost all \( v \). Our purpose is to prove that \( \text{Gal}(\mathcal{M}, \eta) = G_{\phi^v}(\mathcal{M}) \) (cf. (10.2.1)).

We recall that we have already proved that \( G_{\phi^v} \subset \text{Gal}(\mathcal{M}, \eta) \) in (10.2.2).

We choose a \( K(x) \)-vector space \( L \) of dimension 1 in a finite sum of the form \( \oplus_h T^{i_h,j_h}(M) \), such that \( G_{\phi^v}(\mathcal{M}) \) is the stabilizer of \( L \).

We denote by \( \mathcal{W} = (W, \Phi_q) \) the smallest \( q \)-difference sub-module of \( \oplus_h T^{i_h,j_h}(M) \) containing \( L \).

Let \( \mathcal{F} \subset K(x) \) be a \( q \)-difference algebra essentially of finite type over \( \mathcal{V}_K \) and \( \tilde{M} \) an \( \mathcal{F} \)-lattice of \( M \), stable by \( \Phi_q \). Let \( m \) be a basis of the \( \mathcal{F} \)-lattice \( \tilde{L} \) of \( L \) determined by \( M \). Then \( m \) is a cyclic vector for a suitable \( \mathcal{F} \)-lattice \( \tilde{W} \) of \( W \). For almost all \( v \in \Sigma_f \), \( \tilde{L} \otimes_{\mathcal{V}_K} k_{q,v} \) is stable with respect to the morphism induced by \( \Phi^v_q \), which means that:

\[
\Phi^v_q(m) \equiv \alpha_v(x)m \text{ in } \tilde{L} \otimes_{\mathcal{V}_K} k_{q,v}, \text{ with } \alpha_v(x) \in \mathcal{F} \otimes_{\mathcal{V}_K} k_{q,v}.
\]

If \( v \) is the rank of \( W \), we obtain:

\[
\Phi^v_q(m, \Phi_q(m), \ldots, \Phi_q^{\nu-1}(m))
\equiv (m, \Phi_q(m), \ldots, \Phi_q^{\nu-1}(m)) \begin{pmatrix} \alpha_v(x) & & 0 & & \ldots & & \alpha_v(q^{\nu-1}x) \\ 0 & & \ddots & & \ldots & & \alpha_v(q^{\nu-1}x) \end{pmatrix} \text{ in } \tilde{W} \otimes_{\mathcal{V}_K} k_{q,v}.
\]

We deduce that the reduction modulo \( \varpi_{q,v} \) of the sub-\( \mathcal{F} \)-module of \( \tilde{W} \) generated by \( \Phi^v_q(m) \), for any \( i = 0, \ldots, \nu - 1 \), is stable by \( \Phi^v_q \), for almost all \( v \). This implies that the \( K(x) \)-vector space generated by \( \Phi^v_q(m) \), for any \( i = 0, \ldots, \nu - 1 \), is stable by \( G_{\phi^v}(\mathcal{M}) \). Let us call \( U \) the sub-\( K(x) \)-vector space of \( W \) generated by \( (\Phi_q(m), \ldots, \Phi_q^{\nu-1}(m)) \). Then \( W = L \oplus U \) is a decomposition of \( W \) in subspaces stable by \( G_{\phi^v}(\mathcal{M}) \). Let us consider the dual decomposition of \( \tilde{W} \): \( \tilde{W} = \tilde{L} \oplus \tilde{U} \). It follows that \( G_{\phi^v}(\mathcal{M}) \) is the group fixing the line \( L \otimes \tilde{L} \) in \( W \otimes \tilde{W} \) (cf. for instance the proof of theorem [D, 3.1]).

Let us consider the line \( L \otimes \tilde{L} \) instead of the line \( L \) to define \( G_{\phi^v}(\mathcal{M}) \) as a stabilizer. Then we are in the following situation: \( G_{\phi^v}(\mathcal{M}) \) is the group fixing the line \( L \) and \( \mathcal{W} = (W, \Phi_q) \) is the smallest \( q \)-difference module containing \( L \). The \( \mathcal{F} \)-lattice \( \tilde{L} \) of \( L \) is a direct factor in a suitable \( \mathcal{F} \)-lattice of \( \oplus_h T^{i_h,j_h}(M) \), hence \( \tilde{L} \otimes_{\mathcal{V}_K} k_{q,v} \) is fixed by \( \Phi^v_q \), for almost all \( v \in \Sigma_f \):

\[
\Phi^v_q(m) \equiv m \text{ in } \tilde{L} \otimes_{\mathcal{V}_K} k_{q,v}, \text{ for all } m \in \tilde{L}.
\]

Let us fix a cyclic vector \( m \in \tilde{L} \) for \( \tilde{W} \). Then we have:

\[
\Phi^v_q(m, \Phi_q(m), \ldots, \Phi_q^{\nu-1}(m)) \equiv (m, \Phi_q(m), \ldots, \Phi_q^{\nu-1}(m))I_{\nu} \text{ in } \tilde{W} \otimes_{\mathcal{V}_K} k_{q,v}.
\]

By (7.1.1), the \( q \)-difference module \( W \) is trivial and hence \( \text{Gal}(W, \eta) = 1 \). Since \( W \in \text{<}M \text{>}^{\otimes} \), we have a natural morphism

\[
\text{Gal}(\mathcal{M}, \eta) \longrightarrow \text{Gal}(W, \eta) = 1,
\]

which proves that \( \text{Gal}(\mathcal{M}, \eta) \) stabilizes each line of \( W \). In particular \( \text{Gal}(\mathcal{M}, \eta) \) stabilizes \( L \), hence \( \text{Gal}(\mathcal{M}, \eta) = G_{\phi^v}(\mathcal{M}) \). This completes the proof.

**11. Examples of calculation of generic Galois groups**

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We conclude with some examples of arithmetic calculations of the generic Galois group.

**Example 11.1.**

Let us consider the $q$-difference equation

$$
\begin{pmatrix}
    y_1(qx) \\
    y_2(qx)
\end{pmatrix} =
\begin{pmatrix}
    1 & a(x) \\
    0 & b(x)
\end{pmatrix}
\begin{pmatrix}
    y_1(x) \\
    y_2(x)
\end{pmatrix}
,$$

with $a(x) \neq 0$. We suppose moreover that $a(x)$ does not have a zero at $x = 0$. Then for all positive integers $n$ we obtain:

$$
\begin{pmatrix}
    y_1(q^n x) \\
    y_2(q^n x)
\end{pmatrix} =
\begin{pmatrix}
    1 & b(q^{n-1} x) \cdots b(x) \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    y_1(x) \\
    y_2(x)
\end{pmatrix}.
$$

Let us consider the $q$-difference modules $\mathcal{M}$ generated by $(e_1, e_2)$ with

$$
\Phi_q(e_1, e_2) = (e_1, e_2)
\begin{pmatrix}
    1 & 0 \\
    a(x) & b(x)
\end{pmatrix}.
$$

Then the $q$-difference linear system (11.1.1) is associated to $\mathcal{M}$ with respect to the basis $e_2$. We distinguish several cases:

1) $y(qx) = b(x)y(x)$ has a solution in $K(x)$

Then the generic Galois group of $\mathcal{M}$ is:

$$
Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} 1 & 0 \\ c(x) & 1 \end{pmatrix} : c(x) \in K(x) \right\}.
$$

2) $y(qx) = b(x)y(x)$ has a solution in an extension $K(q^{1/d})(x^{1/d})$ of $K(x)$, for a suitable integer $d > 1$. We choose $d$ minimal with respect to this property. We obtain:

$$
Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} 1 & 0 \\ c(x) & \zeta \end{pmatrix} : c(x) \in K(x), \zeta \in \mu_d \right\}.
$$

3) none of the previous conditions is satisfied.

We find the algebraic group:

$$
Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} 1 & 0 \\ c(x) & d(x) \end{pmatrix} : c(x), d(x) \in K(x), d(x) \neq 0 \right\}.
$$

**Example 11.2.**

Let us consider the $q$-difference linear system of order two:

$$
\begin{pmatrix}
    y_1(qx) \\
    y_2(qx)
\end{pmatrix} =
\begin{pmatrix}
    0 & r(x) \\
    1 & 0
\end{pmatrix}
\begin{pmatrix}
    y_1(x) \\
    y_2(x)
\end{pmatrix}
,$$

with $r(x) \in K(x)$ and $r(x) \neq 0$. We can easily calculate by induction that for all positive integers $n$ we have:

$$
\begin{pmatrix}
    y_1(q^{2n} x) \\
    y_2(q^{2n} x)
\end{pmatrix} =
\begin{pmatrix}
    r(q^{2n-1} x) \cdots r(q^2 x) r(qx) & 0 \\
    0 & r(q^{2n-2} x) \cdots r(q^2 x) r(qx)
\end{pmatrix}
\begin{pmatrix}
    y_1(x) \\
    y_2(x)
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
    y_1(q^{2n+1} x) \\
    y_2(q^{2n+1} x)
\end{pmatrix} =
\begin{pmatrix}
    0 & r(q^{2n-1} x) \cdots r(q^2 x) r(qx) \\
    r(q^{2n-1} x) \cdots r(q^2 x) r(qx) & 0
\end{pmatrix}
\begin{pmatrix}
    y_1(x) \\
    y_2(x)
\end{pmatrix}.
$$

It follows that for the generic Galois group of the $q$-difference module $\mathcal{M}$ of rank 2 such that for a fixed basis $(e_1, e_2)$ we have:

$$
\Phi_q(e_1, e_2) = (e_1, e_2)
\begin{pmatrix}
    0 & 1 \\
    r(x) & 0
\end{pmatrix}
$$

there are two possibilities:
The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures

1) $y(q^2x) = r(x)y(x)$ has a solution in $K(x)$

Then $\Phi_2^{(v,v)} \equiv 1$ modulo $\varpi_{q,v}$ for almost all $v$ and the generic Galois group of $\mathcal{M}$ is represented as the algebraic linear subgroup $Gl_2(K(x))$ of the form:

$$Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

2) $y(q^2x) = r(x)y(x)$ has a solution in an extension $K(q^{2/d})(x^{1/d})$ of $K(x)$, for a suitable integer $d > 1$. We choose $d$ minimal with respect to this property. Then:

$$Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} : \zeta_1, \zeta_2 \in \mu_d \right\} \cup \left\{ \begin{pmatrix} 0 & \zeta_3 \\ \zeta_4 & 0 \end{pmatrix} : \zeta_3, \zeta_4 \in \mu_d \right\}.$$

3) none of the previous conditions is satisfied.

then the generic Galois group of $\mathcal{M}$ is represented as the infinite algebraic linear subgroup $Gl_2(K(x))$ of the form:

$$Gal(\mathcal{M}, \eta) = \left\{ \begin{pmatrix} a(x) & b(x) \\ 0 & c(x) \end{pmatrix} : a(x), b(x) \in K(x), a(x)b(x) \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c(x) \\ d(x) & 0 \end{pmatrix} : c(x), d(x) \in K(x), c(x)d(x) \neq 0 \right\}.$$

Appendix. A $q$-analogue of Schwarz’s list

Let $a, b, c, q$ be complex numbers. We suppose that $q$ is not zero and not a root of unity.

We consider the basic hypergeometric function:

$$2\phi_1(a, b, c; q, x) = \sum_{n \geq 0} \frac{(a; q)_n(b; q)_n}{(c; q)_n(q; q)_n} x^n,$$

where $(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. It is defined if $c \notin q\mathbb{Z}$ or if $c \in q\mathbb{Z}$ and either $a \in q\mathbb{Z}$, $ac^{-1} \in q\mathbb{Z}$ or $b \in q\mathbb{Z}$, $bc^{-1} \in q\mathbb{Z}$.

It is a $q$-analogue of the Gauss hypergeometric series

$$2F_1(\alpha, \beta; \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} x^n,$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the Pochhammer symbol. If $c$ is a nonpositive integer $2F_1(\alpha, \beta; \gamma; x)$ is defined if and only if either $a \in \mathbb{Z}$, $c \leq a \leq 0$ or $b \in \mathbb{Z}$, $c \leq b \leq 0$.

The series $2\phi_1(a, b, c; q, x)$ is a solution of the basic hypergeometric $q$-difference equation

$$(\mathcal{H}_{a,b,c})\varphi_q^2 y(x) - \frac{(a+b)x - (1+cq^{-1})}{abx - cq^{-1}} \varphi_q y(x) + \frac{x-1}{abx - cq^{-1}} y(x) = 0,$$

which is defined as soon as neither $a = c = 0$ nor $b = c = 0$.

Our purpose is to make a “list” of all the parameters $(a, b, c)$ such that $(\mathcal{H}_{a,b,c})$ has a basis of algebraic solutions (i.e. in a finite extension of $\mathbb{C}(x)$). It may be thought of as an analogue of Schwarz’s list for hypergeometric differential equations having solution $2F_1(\alpha, \beta; \gamma; x)$:

$$(\mathcal{E}_{\alpha,\beta,\gamma})\quad y''(x) + \gamma - \frac{(\alpha + \beta + 1)x}{x(1-x)} y'(x) - \frac{\alpha\beta}{x(1-x)} y(x) = 0,$$
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where $\alpha$, $\beta$, $\gamma$ are complex parameters. We also establish the list of parameters $a, b, c$ such that $(H_{a,b,c})$ has a basis of solutions in $\mathbb{C}(x)$. The complete solution to this problem is actually closely linked with the solution of the analogous problem for $(E_{a,b,c})$.

We obtain:

$$(H_{a,b,c}) \text{ has a basis of solutions in } \mathbb{C}(x)$$

\[\Leftrightarrow\] either it has a basis of solutions in $\mathbb{C}(x)$

or the following conditions are satisfied:

1) $a, b, c \in q\mathbb{Q};$
2) either $a, bc^{-1} \in q\mathbb{Z}$ or $b, ac^{-1} \in q\mathbb{Z};$
3) $ab^{-1}, c \notin q\mathbb{Z}.$

We can state a more precise result. Let

$$\mathcal{Z} = (\mathbb{Z}_{>0} \times \mathbb{Z}_{\leq 0}) \cup (\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{>0}).$$

We know by [G, Ch. III] that:

**Proposition A.0.1.** Let $\alpha, \beta, \gamma \in \mathbb{Z}$. The following assertions are equivalent:
1) $(E_{\alpha,\beta,\gamma})$ has a basis of solutions in $\mathbb{C}(x);$  
2) $|1 - \gamma|, |\gamma - \alpha - \beta|$ and $|\alpha - \beta|$ are the lengths of the sides of a triangle;  
3) the following conditions are satisfied:
   * either $(\alpha, \alpha + 1 - \gamma) \in \mathbb{Z}$ or $(\beta, \beta + 1 - \gamma) \in \mathbb{Z};$
   * either $(\alpha, \beta) \in \mathbb{Z}$ or $(\alpha + 1 - \gamma, \beta + 1 - \gamma) \in \mathbb{Z}.$

Condition 2) must be interpreted in the most elementary way: if $n_1, n_2, n_3$ are positive integers such that $n_1 \leq n_2 \leq n_3$, they can be the lengths of a triangle if they satisfy the condition $n_1 + n_2 = n_3$.

We are thus led to prove the following proposition:

**Proposition A.0.2.** The $q$-difference equation $(H_{a,b,c})$ has a basis of solutions in $\mathbb{C}(x)$ if and only if the following conditions are satisfied:

* there exists $\alpha, \beta, \gamma \in \mathbb{Z},$ such that $a = q^\alpha, b = q^\beta, c = q^\gamma;$
* either $(\alpha, \alpha + 1 - \gamma) \in \mathbb{Z}$ or $(\beta, \beta + 1 - \gamma) \in \mathbb{Z};$
* either $(\alpha, \beta) \in \mathbb{Z}$ or $(\alpha + 1 - \gamma, \beta + 1 - \gamma) \in \mathbb{Z}.$

**Remark A.0.3.** A basis of solutions of $(H_{a,b,c})$ at zero is given generically by

$$\left\{ \begin{array}{l}
2\phi_1(a, b, c; q, x) \\
e_{\gamma-1}(x)2\phi_1(aqc^{-1}, bqc^{-1}, q^2c^{-1}; q, x)
\end{array} \right.$$

where $e_{\gamma-1}(x)$ is a solution of $y(qx) = (q^{-1})y(x)$. A basis of solutions at $\infty$ is given by

$$\left\{ \begin{array}{l}
e_a \left( \frac{1}{x} \right) 2\phi_1 (a, aqc^{-1}, aqb^{-1}; q, \frac{cq}{abx}) \\
e_b \left( \frac{1}{x} \right) 2\phi_1 (b, bq^{-1}, bqa^{-1}; q, \frac{cq}{abx})
\end{array} \right.$$

We notice that they are not always well defined when $a, b, c \in q\mathbb{Z}$.

Let $\alpha, \beta, \gamma \in \mathbb{Z}$ be such that $a = q^\alpha, b = q^\beta, c = q^\gamma.$ Then $e_{\gamma-1}(x) = x^{1-\gamma}, e_a(1/x) = (1/x)^\alpha$ and $e_b(1/x) = (1/x)^\beta.$ The previous proposition says that $(H_{a,b,c})$ has a basis of rational solutions if and only if one of the two basis of solutions above is well defined: in this case one solution is necessarily a
The \( q \)-anologue of Grothendieck-Katz’s conjecture on \( p \)-curvatures polynomial, the other one is a rational function which can be written explicitly using Heine’s formula [GR, 1.4.6]:

\[
2\phi_1(a, b, c; q, x) = \frac{((abc^{-1})x; q)_\infty}{(x; q)_\infty} 2\phi_1(ca^{-1}, cb^{-1}, c; q, abc^{-1}x),
\]
or Heine’s contiguity relations [GR, Ex. 1.10].

In the next section we will give a proof of (A.0.2). We will discuss the case of algebraic solutions later.

### A.1. Logarithmic singularities

Let us consider a \( q \)-difference equation of order two

\[
(A.1.0.1) \quad y(q^2x) + P(x)y(qx) + Q(x)y(x) = 0,
\]
with \( P(x), Q(x) \in \mathbb{C}(x) \). One says that it is regular singular at zero if \( P(x) \) has no pole at zero and if \( Q(x) \) has neither a pole nor a zero at zero.

Let \( e_c(x) \) be a solution of the \( q \)-difference equation \( y(qx) = cy(x) \), with \( c \in \mathbb{C} \) (cf. [S2, 0.1]). Suppose that \((A.1.0.1)\) has a solution of the form \( e_c(x) \sum_{n \geq 0} a_n x^n \). Then \( c \) satisfies the equation

\[
c^2 + P(0)c + Q(0) = 0.
\]

The two roots \( c_1 \) and \( c_2 \) of this equation are called the exponents of \((A.1.0.1)\) at zero.

When \( c_1, c_2 \) satisfy the condition \( c_1 \not\in c_2q^2 \), we know by [S2, 1.1.4, Th.] that \((A.1.0.1)\) has a basis of solutions of the form \( e_{c_1}(x) \sum_{n \geq 0} a_n x^n \), \( e_{c_2}(x) \sum_{n \geq 0} b_n x^n \). Moreover, \( \sum_{n \geq 0} a_n x^n \) and \( \sum_{n \geq 0} b_n x^n \) are convergent complex power series; if \( |q| \not\in 1 \) they are Taylor expansions of meromorphic functions on \( \mathbb{C} \). If the condition \( c_1 \not\in c_2q^2 \) is not satisfied this is in general not true: the two solutions may involve \( q \)-logarithms, which are solutions of the \( q \)-difference equation \( y(qx) = y(x) + 1 \) (cf. [S2, 0.1]).

**Definition A.1.1.** We say that a \( q \)-difference equation which is regular singular at zero does not have a logarithmic singularity at zero if it has a basis of solutions of the form \( e_{c_1}(x) \sum_{n \geq 0} a_n x^n \), \( e_{c_2}(x) \sum_{n \geq 0} b_n x^n \).

One can give an analogous definition for the point \( \infty \), by using the variable change \( t = \frac{1}{x} \). We have:

**Lemma A.1.2.** The equation \((H_{a,b,c})\) has a basis of solutions in \( \mathbb{C}(x) \) if and only if there exists \( a, b, c, \alpha, \beta, \gamma \in \mathbb{Z} \) such that \( a = q^\alpha \), \( b = q^\beta \), \( c = q^\gamma \), and zero and \( \infty \) are not logarithmic singularities.

**Proof.** We notice that the exponents of \((H_{a,b,c})\) at zero (resp. \( \infty \)) are 1 and \( qe^{-1} \) (resp. \( a \) and \( b \)), and that the equation \((H_{a,b,c})\) is actually defined over the field \( \mathbb{Q}(a, b, c, q) \).

Let us suppose that there exists \( a, b, c, \alpha, \beta, \gamma \in \mathbb{Z} \) such that \( a = q^\alpha \), \( b = q^\beta \), \( c = q^\gamma \), and zero and \( \infty \) are not logarithmic singularities. Then \((H_{a,b,c})\) has a basis of solutions at zero of the form

\[
u_0(x), x^{1-\gamma} v_0(x), \text{ with } u_0(x), v_0(x) \in \mathbb{Q}(q)[[x]],
\]

and a basis of solutions at \( \infty \) of the form

\[\frac{1}{u_\infty} u_\infty \left( \frac{1}{x} \right), \frac{1}{v_\infty} v_\infty \left( \frac{1}{x} \right), \text{ with } u_\infty \left( \frac{1}{x} \right), v_\infty \left( \frac{1}{x} \right) \in \mathbb{Q}(q) \left[ \left[ \frac{1}{x} \right] \right].\]

Let \( \mid \cdot \mid_q \) be the \( q \)-adic norm over \( \mathbb{Q}(q) \). Then \( u_0(x), v_0(x) \) (resp. \( u_\infty (1/x), v_\infty (1/x) \)) have infinite radius of meromorphy at zero (resp. \( \infty \)) with respect to the norm \( \mid \cdot \mid_q \), since \( |q|_q < 1 \).

Let us consider the Birkhoff connection matrix (cf. [S2, §2])

\[
P(x) = \begin{pmatrix}
(1/x)\alpha u_\infty (1/x) & (1/x)\beta v_\infty (1/x) \\
(1/qx)\alpha u_\infty (1/qx) & (1/qx)\beta v_\infty (1/qx)
\end{pmatrix}^{-1}
\begin{pmatrix}
u_0(x) & x^{1-\gamma} v_0(x) \\
u_0(qx) & (qx)^{1-\gamma} v_0(qx)
\end{pmatrix}.
\]

The entries of \( P(x) \) are \( q \)-adically meromorphic elliptic functions over \( \mathbb{P}^1_{\mathbb{Q}(q)} \setminus \{0, \infty\} \). Moreover, since zero and \( \infty \) are not logarithmic singularities and the exponents are integral powers of \( q \), the singularities
The $q$-analogue of Grothendieck-Katz’s conjecture on $p$-curvatures

of $P(x)$ are in the $q$-orbits of the singularities of the coefficients of the equation $(H_{a,b,c})$. It follows that $P(x)$ induces a meromorphic elliptic function over $Q(q)/q^2$, having at worst one pole at $q^2$, hence the matrix $P(x)$ must be constant. Therefore $u_0(x)$, $v_0(x)$ are meromorphic functions over $P^1_{Q(q)}$, i.e., they are rational functions.

The other implication of the equivalence is clear.

The following two lemmas achieve the proof of (A.0.2). Their proof is inspired by the proof of the analogous statements in [G, Ch. III] for the hypergeometric differential equation.

Let

$$q^Z = (q^{Z>0} \times q^{Z=0}) \cup (q^{Z=0} \times q^{Z>0}).$$

Lemma A.1.3. Let $c = q^\gamma$ with $\gamma \in \mathbb{Z}$. Then $(H_{a,b,c})$ has a logarithmic singularity at zero if and only if either $(a, qac^{-1}) \in q^Z$ or $(b, qbc^{-1}) \in q^Z$.

Proof. Let $\gamma = 1$. Then we know by [S2, 1.1.4, Th.] that zero is a logarithmic singularity.

Let us suppose that $\gamma \leq 0$. Under this assumption, the series

$$x^{1-\gamma}2\phi_1(aqc^{-1}, bqc^{-1}, q^2e^{-1}; q, x)$$

is a well defined solution of $(H_{a,b,c})$. Zero is not a logarithmic singularity for $(H_{a,b,c})$ if and only if there exists a second solution of the form $\sum_{n \geq 0} c_n x^n$, with $c_0 = 1$. A direct calculation shows that the coefficients $c_i$ must satisfy the relation:

$$(1 - aq^n)(1 - bq^n)c_n = (1 - q^{\gamma+n})(1 - q^{n+1})c_{n+1}.$$

Let us suppose that neither $a = q^\alpha$, with $0 \geq \alpha \geq \gamma$, nor $b = q^\beta$, with $0 \geq \beta \geq \gamma$. Then $c_1, \ldots, c_{-\gamma}$ can be determined inductively: they are necessarily nonzero complex numbers. For $n = -\gamma$, we get 0 on the right hand side of (A.1.3.1), while the left hand side is not zero. Therefore we cannot find a solution of the form $\sum_{n \geq 0} c_n x^n$, with $c_0 = 1$, and zero is a logarithmic singularity.

On the other hand, if either $a = q^\alpha$, with $0 \geq \alpha \geq \gamma$, or $b = q^\beta$, with $0 \geq \beta \geq \gamma$, the series $2\phi_1(a, b, c; q, x)$ is a well defined solution of $(H_{a,b,c})$, satisfying $2\phi_1(a, b, c; q, 0) = 1$.

If $\gamma \leq 0$, we conclude that zero is not a logarithmic singularity if and only if either $a = q^\alpha$, with $0 \geq \alpha \geq \gamma$, or $b = q^\beta$, with $0 \geq \beta \geq \gamma$. This completes the proof in the case $\gamma \leq 0$.

Let $\gamma \geq 2$. A series of the form $x^{1-\gamma}z(x)$ is a solution of $(H_{a,b,c})$ if and only if $z(x)$ is a solution of $(H_{a',b',c'})$, with $a' = aq^{1-\gamma}$, $b' = bq^{1-\gamma}$ and $c' = q^{2-\gamma}$. So we can deduce this case from the case $\gamma \leq 0$.

Lemma A.1.4. Let $ab^{-1} \in q^Z$. Then $(H_{a,b,c})$ has a logarithmic singularity at $\infty$ if and only if either $(a, b) \in q^Z$ or $(qac^{-1}, qbc^{-1}) \in q^Z$.

Proof. Let $t = cq/abx$. Then $y(t) = e_a(ab^{-1}q^{-1}t)z(t)$ is a solution of $(H_{a,b,c})$ if and only if $z(t)$ is a solution of $(H_{a',b',c'})$, with $a' = a$, $b' = qac^{-1}$ and $c' = qab^{-1} \in q^Z$: this fact can be deduced by the remark that, for generic parameters $a, b, c$, a solution of $(H_{a,b,c})$ at $\infty$ is given by $e_a(ab^{-1}q^{-1}t)2\phi_1(a, aq^{-1}, qab^{-1}; q, t)$. We conclude by (A.1.3).

A.2. The case of algebraic solutions

Proposition A.2.1. The $q$-difference equation $(H_{a,b,c})$ has a basis of algebraic solutions if and only if one of the following conditions is satisfied:
1) $(H_{a,b,c})$ has a basis of rational solutions;
2) $a, b, c \in q^2$, $c, ab^{-1} \notin q^Z$ and either $a, bc^{-1} \in q^Z$ or $b, ac^{-1} \in q^Z$.

Remark A.2.2. If the condition 2) is satisfied, the equation $(H_{a,b,c})$ has a well-defined basis of solutions of the form

$$(A.2.2.1) \quad \left(\frac{1}{x}\right)^\alpha 2\phi_1(a, aq^{-1}, qab^{-1}; q, \frac{cq}{abx}), \quad \left(\frac{1}{x}\right)^\beta 2\phi_1(b, bqc^{-1}, bqac^{-1}; q, \frac{cq}{abx}),$$

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where $\phi_1(a, aq^{-1}, ab^{-1}; q, ca/bx)$ and $\phi_1(b, bq^{-1}, ba^{-1}; q, cb/ax)$ are rational functions and $\alpha, \beta \in \mathbb{Q}$ are such that $a = q^\alpha$ and $b = q^\beta$.

The remark above proves one side of the equivalence in (A.2.1). On the other side we have:

**Lemma A.2.3.** The $q$-difference equation $(H_{a,b,c})$ has only algebraic solutions if and only if the following conditions are satisfied:

- $a, b, c \in q\mathbb{Q}$;
- $(H_{a,b,c})$ has no logarithmic singularities at zero or at $\infty$.

The solutions are then of the form $x^\delta u(x)$, where $\delta \in \mathbb{Q}$ and $u(x) \in \mathbb{C}(x)$.

The proof of this lemma is similar to the proof of lemma A.1.2 and it is based on the remark that the equation $y(qx) = cy(x)$ has no algebraic solutions if and only if $c \in \mathbb{C} \setminus q\mathbb{Q}$ (the Galois group being infinite).

By lemmas (A.1.3) and (A.1.4) we deduce that the $q$-difference equation $(H_{a,b,c})$ has only algebraic solutions if and only if the following conditions are satisfied:

- $a, b, c \in q\mathbb{Q}$;
- either $ab^{-1} \notin q\mathbb{Z}$ or $(a, b) \in q\mathbb{Z}$ or $(aq/b, bq/a) \in q\mathbb{Z}$;
- either $c \notin q\mathbb{Z}$ or $(a, aq^{-1}) \in q\mathbb{Z}$ or $(b, bq^{-1}) \in q\mathbb{Z}$;
- either $a, bc^{-1} \in q\mathbb{Z}$ or $b, ac^{-1} \in q\mathbb{Z}$.

The last condition follows from the fact that the solutions are of the form $x^\delta u(x)$, where $\delta \in \mathbb{Q}$ and $u(x) \in \mathbb{C}(x)$.

If $ab^{-1} \notin q\mathbb{Z}$ and $c \notin q\mathbb{Z}$ then the solutions are in a Kummer extension $\mathbb{C}(x^{1/n})$ of $\mathbb{C}(x)$, otherwise they are rational functions.

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