Theory of Disordered Itinerant Ferromagnets I: Metallic Phase

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A comprehensive theory for electronic transport in itinerant ferromagnets is developed. We first show that the Q-field theory used previously to describe a disordered Fermi liquid also has a saddle-point solution that describes a ferromagnet in a disordered Stoner approximation. We calculate transport coefficients and thermodynamic susceptibilities by expanding about the saddle point to Gaussian order. At this level, the theory generalizes previous RPA-type theories by including quenched disorder. We then study soft-mode effects in the ferromagnetic state in a one-loop approximation. In three-dimensions, we find that the spin waves induce a square-root frequency dependence of the conductivity, but not of the density of states, that is qualitatively the same as the usual weak-localization effect induced by the diffusive soft modes. In contrast to the weak-localization anomaly, this effect persists also at nonzero temperatures. In two-dimensions, however, the spin waves do not lead to a logarithmic frequency dependence. This explains experimental observations in thin ferromagnetic films, and it provides a basis for the construction of a simple effective field theory for the transition from a ferromagnetic metal to a ferromagnetic insulator.

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1. INTRODUCTION

The theoretical treatment of many-fermions systems is a hard problem, especially in the presence of quenched disorder. Traditional approaches have been Landau Fermi-liquid theory and many-body perturbation theory. Impressive progress has been made within the framework of the latter. The random phase approximation (RPA) has been developed for the Fermi liquid phase, and similar theories have been used to describe magnetic and superconducting phases. In a more recent development, the corrections to Fermi-liquid theory known as “weak-localization effects” have been derived using many-body diagrammatic techniques. Even more recently, field-theoretic methods have been applied to the many-fermion problem, which have certain advantages over the traditional techniques. Most importantly, they allow for a straightforward application of the renormalization group (RG), implementing, inter alia, an old program of describing the various phases of many-body systems in terms of stable RG fixed points. So far this program has been carried out for clean and disordered Fermi liquids. It has also been shown that the weak-localization effects can be understood as the leading corrections to scaling near a disordered Fermi-liquid fixed point. A further advantage of the field theoretic approach is that it allows for a systematic identification and analysis of the soft or massless modes that are responsible for the long-distance, long-time properties of the system. If desirable, it also allows for the derivation of effective theories that keep only the soft modes explicitly, while integrating out all other degrees of freedom in some simple approximation. This ability to focus on soft modes is particularly important for studies of the zero-temperature (T = 0) properties of many-body systems, since it turns out that there are many soft modes at T = 0 that acquire a mass at nonzero temperature. Conservation laws, and analogies to classical fluids, are therefore of limited value in identifying the soft modes of a quantum system at T = 0.

An example of soft modes that have no analogs in classical physics are the diffusive modes that cause the weak-localization effects mentioned above, which are long-wavelength/low-frequency nonanalyticities in the frequency and wavenumber dependence of transport coefficients and thermodynamic quantities. These nonanalyticities, which generically take the form of power laws, are examples of a more general phenomenon known as generic scale invariance. Correlation functions which, as functions of space and time, are power laws and thus homogeneous functions (as opposed to, e.g., exponentials), contain no intrinsic length or time scales, hence the name “scale invariance”. The most widely known example of scale invariance, which is caused by soft modes, occurs at critical points as a result of the critical modes. Critical points are exceptional points in a phase diagram, and reaching them requires a fine tuning of parameters. Generic scale invariance, on the other hand, does not require any fine tuning and is due to soft modes that occur over large regions of parameter space, like the dif-
fusive “diffuson” modes in disordered electron systems that cause the weak-localization effects. However, generic scale invariance is in no way restricted to quantum systems: it has been discussed primarily in classical systems ranging from classical fluids and liquid crystals to sandpiles.

It is the purpose of the present paper to develop a comprehensive field theoretic method for describing disordered itinerant ferromagnets, which so far have not been studied with these techniques, with particular emphasis on the consequences of the soft modes in such systems. Specifically, we generalize the theory of Ref. 8 to the magnetic case. Simple approximations within this theory yield results that correspond to disordered Stoner theory for the equation of state, and to disordered RPA-type approximations for the transport properties. We then study generic scale invariance effects on the background of ferromagnetism. An interesting question in this context is the role of the Goldstone modes or spin waves that arise from the spontaneously broken spin rotational symmetry, which represent additional soft modes compared to a system without magnetic long-range order. Since the soft mode structure of ferromagnets is thus different from that of nonmagnetic systems, one also expects generic scale invariance phenomena that differ from the usual weak-localization nonanalyticities. Indeed, we find that the spin waves contribute to the leading nonanalyticic frequency dependence of the conductivity in three-dimensions, but not to that of the density of states. Also, we find that they do not lead to a logarithmic frequency dependence of the conductivity in two-dimensions. This explains experimental observations that have found the frequency anomaly in thin ferromagnetic films to be the same as those in nonmagnetic metals in an external magnetic field.

This paper is organized as follows. In Sec. II we recall the general Q-matrix field theory for disordered interacting electrons that was developed in Ref. 8, and we show that this theory allows for a saddle-point solution that corresponds to a disordered Stoner theory. In Sec. III we calculate the spin and density susceptibilities, as well as the conductivity, in a Gaussian approximation. This produces generalizations of well-known results to the case of disordered magnets. In Sec. IV we proceed to perform a one-loop calculation of the density of states and the conductivity, and calculate the contribution of the Goldstone modes to the leading nonanalyticities. In Sec. V we conclude with a discussion of our results. The ferromagnetic-metaltopermagnetic-insulator transition will be discussed in a second paper[14] which we will refer to as II.

II. MATRIX FIELD THEORY

A. Q-matrix theory for fermions

We start with a general field theory for electrons. For any fermionic system, the partition function can be written as a functional integral over fermionic (i.e., Grassmann valued) fields \( \bar{\psi} \) and \( \psi \)

\[
Z = \int D[\bar{\psi}, \psi] \exp \left( S[\bar{\psi}, \psi] \right),
\]

where \( S \) is the action. We consider an action that consists of a free-fermion part \( S_0 \), a part \( S_{\text{dis}} \) describing the interaction of the electrons with quenched disorder, and a part \( S_{\text{int}} \) describing the electron-electron interaction,

\[
S = S_0 + S_{\text{dis}} + S_{\text{int}}.
\]

Each field \( \psi \) or \( \bar{\psi} \) carries a Matsubara frequency index \( n \) and a spin index \( \sigma = \uparrow, \downarrow = +, - \). Since we will deal with the quenched disorder by means of the replica trick, each field also carries a replica index \( \alpha \). It is useful to introduce a matrix of bilinear products of the fermion fields,

\[
B_{12} = \frac{i}{2} \begin{pmatrix}
-\psi_{1\uparrow}\bar{\psi}_{2\uparrow} - \psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\
-\psi_{1\downarrow}\bar{\psi}_{2\uparrow} - \psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\
-\psi_{1\downarrow}\bar{\psi}_{2\downarrow} - \psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow}
\end{pmatrix}
\]

\[
\cong Q_{12}.
\]

where all fields are understood to be taken at position \( x \), and \( 1 \equiv (n_1, \alpha_1) \), etc. The matrix elements of \( B \) commute with one another, and are therefore isomorphic to classical or number-valued fields that we denote by \( Q \).

This isomorphism maps the adjoint operation on products of fermion fields, which is denoted above by an overbar, on the complex conjugation of the classical fields. We use the isomorphism to constrain \( B \) to the classical field \( Q \), and exactly rewrite the partition function

\[
Z = \int D[\bar{\psi}, \psi] e^{S[\bar{\psi}, \psi]} \left( \int D[Q] \delta[Q - B] \right)
\]

\[
= \int D[\bar{\psi}, \psi] e^{S[\bar{\psi}, \psi]} \left( \int D[Q] D[\bar{\Lambda}] e^{\text{Tr}(\bar{\Lambda}(Q-B))} \right)
\]

\[
= \int D[Q] D[\bar{\Lambda}] e^{A[Q, \bar{\Lambda}]}.
\]

We have introduced an auxiliary bosonic matrix field \( \bar{\Lambda} \) to enforce the functional delta-constraint in the first line of Eq. (2.3), and the last line defines the action \( A \). The matrix elements of both \( Q \) and \( \bar{\Lambda} \) are spin-quaternions, i.e., elements of \( Q \times Q \) with \( Q \) the quaternion field. From Eq. (2.2) we see that expectation values of the \( Q \) matrix elements yield single-particle Green functions, and \( Q-Q \) correlation functions describe four-fermion correlation functions. The physical meaning of \( \bar{\Lambda} \) is that its expectation value plays the role of a self energy (see Ref. 8 and Sec. III below).
It is convenient to expand the $4 \times 4$ matrix in Eq. (2.2) in a spin-quaternion basis,
\[
Q_{12}(x) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)_{r}^{i} Q_{12}(x) \quad (2.4)
\]
and analogously for $\Lambda$. Here $\tau_0 = s_0 = 1_2$ is the $2 \times 2$ unit matrix, and $\tau_j = -s_j = -\sigma_j$, $(j = 1, 2, 3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i = 0$ and $i = 1, 2, 3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation using Eq. (2.2) reveals that $r = 0, 3$ corresponds to the particle-hole channel (i.e., products $\tilde{\psi} \psi$), while $r = 1, 2$ describes the particle-particle channel (i.e., products $\tilde{\psi} \tilde{\psi}$ or $\psi \psi$). From the structure of Eq. (2.2), one obtains the following formal symmetry properties of the Q matrices:
\[
\begin{align*}
0_r Q_{12} &= (-)^{r} 0_r Q_{21} \quad (r = 0, 3) \quad , \quad (2.5a) \\
1_r Q_{12} &= (-)^{r+1} 1_r Q_{21} \quad (r = 0, 3; \; i = 1, 2, 3) \quad , \quad (2.5b) \\
0_r Q_{12} &= 0_r Q_{21} \quad (r = 1, 2) \quad , \quad (2.5c) \\
1_r Q_{12} &= -1_r Q_{21} \quad (r = 1, 2; \; i = 1, 2, 3) \quad , \quad (2.5d) \\
1_r Q_{12} &= -1_r Q_{21}^{\alpha, \alpha \omega} \quad \text{discrete indices that are not explicitly shown}. \quad (2.5e)
\end{align*}
\]

The star in Eq. (2.5e) denotes complex conjugation.

For the purposes of the present paper, we will be particularly interested in the matrix elements $0_0 Q$ and $3_3 Q$. From Eqs. (2.2) and (2.4) we find
\[
\begin{align*}
\langle 0_{Q_{12}}(x) \rangle &\approx \delta_{12} \frac{i}{\pi} \sum_{\sigma} \langle \bar{\psi}_{1\sigma}(x) \psi_{1\sigma}(x) \rangle \quad , \quad (2.6a) \\
\langle 3_{Q_{12}}(x) \rangle &\approx \delta_{12} \frac{i}{\pi} \sum_{\sigma} \langle \bar{\psi}_{3\sigma}(x) \psi_{3\sigma}(x) \rangle \quad , \quad (2.6b)
\end{align*}
\]
where $(\ldots)$ denotes an average taken with the full action. From these expressions we obtain various observables in terms of expectation values of the Q fields, for instance the particle number density $n$, the frequency or energy dependent density of states $N$, with energy measured from the chemical potential $\mu$ (or, at $T = 0$, from the Fermi energy $\epsilon_F$), and the magnetization $M$,
\[
\begin{align*}
n &= -4i T \sum_{n} \langle 0_{Q_{mn}}(x) \rangle \quad , \quad (2.7a) \\
N(\mu + \omega) &= \frac{4}{\pi} \text{Re} \langle 0_{Q_{mn}} \rangle |_{\omega_n - \omega + i0} \quad , \quad (2.7b) \\
M &= -4i \mu_B T \sum_{n} \langle 3_{Q_{mn}}(x) \rangle \quad , \quad (2.7c)
\end{align*}
\]
with $\mu_B$ the Bohr magneton. Here in what follows we denote fermionic Matsubara frequencies by $\omega_n = 2\pi T (n + 1/2)$, and bosonic ones by $\Omega_n = 2\pi T n$. We use units such that $\hbar = k_B = 1$.

By using the delta constraint in Eq. (2.3) to rewrite all terms that are quartic in the fermion field in terms of $Q$, we can achieve an integrand that is bilinear in $\psi$ and $\bar{\psi}$. The Grassmannian integral can then be performed exactly, and we obtain for the action $A$
\[
A[Q, \Lambda] = A_{\text{dis}} + A_{\text{int}} + \frac{1}{2} \text{Tr} \ln \left( G^{-1}_0 - i\Lambda \right) + \int dx \text{ tr} \left( \Lambda(x) Q(x) \right) . \quad (2.8a)
\]

Here
\[
G_0^{-1} = -\partial_\tau + \partial_x^2 / 2m_e + \mu \quad , \quad (2.8b)
\]
is the inverse free electron Green operator, with $\partial_\tau$ and $\partial_x$ derivatives with respect to imaginary time and position, respectively, and $m_e$ is the electron mass. Tr denotes a trace over all degrees of freedom, including the continuous position variable, while $tr$ is a trace over all those discrete indices that are not explicitly shown. For the disorder part of the action one finds
\[
A_{\text{dis}}[Q] = \frac{1}{\pi N_F T} \int dx \text{ tr} \left( Q(x) \right)^2 . \quad (2.9)
\]
with $N_F$ the density of states at the Fermi level in saddle-point approximation (see Ref. 8 and Sec. III C below), and $T$ the single-particle scattering or relaxation time. The electron-electron interaction $A_{\text{int}}$ is conveniently decomposed into four pieces that describe the interaction in the particle-hole and particle-particle spin-singlet and spin-triplet channels.9 For reasons that will be explained in Sec. III B below, for our purposes we can neglect the particle-particle channel. We thus drop $r = 1, 2$ from our spin-quaternion basis, Eq. (2.3). In particular, we write the interaction as the sum of the particle-hole spin-singlet and triplet terms,
\[
A_{\text{int}}[Q] = A_{\text{int}}^{(s)} + A_{\text{int}}^{(t)} \quad , \quad (2.10a)
\]
\[
A_{\text{int}}^{(s)} = \frac{T \Gamma^{(s)}}{2} \int dx \sum_{r=0,3} (-1)^r \sum_{n_1, n_2, m_\alpha} \sum_{\alpha} \left[ \text{tr} \left( (\tau_r \otimes s_0) Q^{\alpha, \alpha}_{n_1, n_2, m_\alpha}(x) \right) \right] \\
A_{\text{int}}^{(t)} = \frac{T \Gamma^{(t)}}{2} \int dx \sum_{r=0,3} (-1)^r \sum_{n_1, n_2, m_\alpha} \sum_{\alpha} \sum_{i=1}^{3} \left[ \text{tr} \left( (\tau_r \otimes s_1) Q^{\alpha, \alpha}_{n_1, n_2, n_3}(x) \right) \right] \
\]
Here $\Gamma^{(s)} > 0$ and $\Gamma^{(t)} > 0$ are the spin-singlet and spin-triplet interaction amplitudes, respectively. $\Gamma^{(t)}$ is responsible for producing magnetism.
B. The Stoner saddle point

We now look for a saddle-point solution of the field theory derived in the previous subsection. We are interested in an itinerant ferromagnet, i.e., a state where both the density of states at the Fermi level and the magnetization are nonzero. We are further looking for a homogeneous state, so we drop the real space dependence of the fields. Equations (2.7) suggest the following ansatz for the saddle-point fields,

\[
\begin{align*}
\hat{\nu}_{12}^{sp} & = \delta_{12} \left[ \delta_{\tau_0} \delta_{\tau_0} G_{n_1} + \delta_{\tau_3} \delta_{\tau_3} F_{n_1} \right], \\
\hat{\Lambda}_{12}^{sp} & = \delta_{12} \left[ -\delta_{\tau_0} \delta_{\tau_0} i \Sigma_{n_1} + \delta_{\tau_3} \delta_{\tau_3} i \Delta_{n_1} \right].
\end{align*}
\]

Substituting this into Eqs. (2.8) - (2.10), and using the saddle-point condition \( \delta \Lambda / \delta Q = \delta A / \delta \Lambda = 0 \), we obtain the saddle-point equations

\[
\begin{align*}
G_n & = \frac{i}{2V} \sum_k G_n(k) , \\
F_n & = \frac{i}{2V} \sum_k F_n(k) , \\
\Sigma_n & = \frac{-2i}{\pi N_F \tau} G_n - 4i \Gamma(s) T \sum_m e^{i \omega_n m} 0 G_m , \\
\Delta_n & = \frac{2i}{\pi N_F \tau} F_n - 4i \Gamma(t) T \sum_m e^{i \omega_n m} 0 F_m .
\end{align*}
\]

Here

\[
\begin{align*}
G_n(k) & = \frac{1}{2} \left[ G^+(n)(k) + G^-(n)(k) \right] , \\
F_n(k) & = \frac{1}{2} \left[ G^+(n)(k) - G^-(n)(k) \right],
\end{align*}
\]

are Green functions given in terms of

\[
G^\pm_n(k) = \frac{1}{i \omega_n - \xi_k \pm \Delta_n - \Sigma_n} ,
\]

with \( \xi_k = k^2/2m - \mu \). For later reference we also define a matrix saddle-point Green function

\[
G_{sp} = \left( G_0^{-1} - i \hat{\Lambda} \right)^{-1}|_{sp} ,
\]

whose matrix elements are given by

\[
(G_{sp})_{12} = \delta_{12} \left[ G_{n_1} \left( \tau_0 \otimes \sigma_0 \right) + F_{n_1} \left( \tau_3 \otimes \sigma_3 \right) \right] .
\]

Clearly, \( \Sigma \) and \( \Delta \) are self-energies that describe the interaction in Hartree-Fock approximation, and the disorder in self-consistent Born approximation. The Green functions \( G \) and \( F \) obey the equations

\[
\begin{align*}
(i \omega_n - \xi_k - \Sigma_n) G_n(k) + \Delta_n F_n(k) & = 1 , \\
(i \omega_n - \xi_k - \Sigma_n) F_n(k) + \Delta_n G_n(k) & = 0 .
\end{align*}
\]

If we absorb the second, Hartree-Fock, contribution to \( \Sigma \) in Eq. (2.12d), which is purely real and frequency independent, into a redefinition of the chemical potential, we can rewrite Eqs. (2.12d) in the form

\[
\begin{align*}
\Sigma_n & = \frac{1}{\pi N_F \tau} \frac{1}{V} \sum k G_n(k) , \\
\Delta_n & = \frac{1}{\pi N_F \tau} \frac{1}{V} \sum k F_n(k) ,
\end{align*}
\]

where

\[
\Delta = 2 \Gamma(t) T \sum_n \frac{1}{V} \sum k G_n(k) .
\]

From Eqs. (2.7c), (2.11), (2.13), and (2.15) we see that \( \Delta \) is related to the magnetization in saddle-point approximation, \( \Delta = \Gamma(t) M/\mu_B \).

To discuss our saddle-point solution, let us first consider the clean limit, \( \tau = \infty \), \( \Sigma_n = 0 \), \( \Delta_n = \Delta \). From Eqs. (2.14, 2.13) we then obtain an expression for the equation of state that is familiar from Stoner theory:

\[
1 = -2 \Gamma(t) T \sum_n \frac{1}{V} \sum k \frac{1}{(i \omega_n - \xi_k)^2 - \Delta^2} .
\]

In particular, the threshold value of \( \Gamma(t) \) for the onset of ferromagnetism is given by the Stoner criterion

\[
N_F \Gamma(t) = 1 ,
\]

and the usual physical interpretation of Stoner theory applies. For instance, the \( G^\pm_n(k) \) are Hartree-Fock Green functions with the chemical potential shifted by \( \pm \Delta_n \), which is essentially the magnetization.

The discussion of the disordered case proceeds in exact analogy to the BCS-Gorkov theory of disordered superconductors (this is the main reason why we have chosen to write Stoner theory in a form analogous to BCS theory). The equation of state then takes the form

\[
1 = -2 \Gamma(t) T \sum_n \frac{1}{V} \sum k \frac{1}{(i \omega_n - \xi_k + \frac{t}{\pi \sigma \text{sgn} \omega_n})^2 - \Delta^2} .
\]

The integral is independent of the disorder to lowest order in \( 1/\tau \), and so is therefore the Stoner criterion, Eq. (2.17). This is the magnetic analog of Anderson’s theorem in superconductivity.

We conclude that our saddle-point solution of the field theory, Eqs. (2.8) - (2.10), describes a ferromagnetic state in a disordered Stoner approximation that is analogous to BCS-Gorkov theory for disordered superconductors.

III. GAUSSIAN APPROXIMATION
A. Gaussian action

We now write $Q = Q_{sp} + \delta Q$, and $\bar{\Lambda} = \bar{\Lambda}_{sp} + \delta \bar{\Lambda}$, and expand the action, Eq. (2.8a), in powers of $\delta Q$ and $\delta \bar{\Lambda}$. To Gaussian order we obtain $A = A_{sp} + A_G$ with

$$A_G = A_{\text{dis}}[\delta Q] + A_{\text{int}}[\delta Q] + \frac{1}{4} \text{Tr} \left( G_{sp} \delta \bar{\Lambda} G_{sp} \delta \bar{\Lambda} \right)$$

$$+ \text{Tr} \left( \delta \bar{\Lambda} \delta Q \right) . \quad (3.1)$$

Defining Fourier transforms of the fields, $\bar{\Lambda}(\mathbf{k}) = \int d\mathbf{x} \exp(i\mathbf{k}\mathbf{x}) \Lambda(\mathbf{x})$, and analogously for $Q$, the piece of $A_G$ that is quadratic in $\bar{\Lambda}$ can be written

$$\frac{1}{V} \sum \sum \sum_{ij} \sum_{rs} \sum_{ij}^r \delta \bar{\Lambda}_{124}(\mathbf{k}) \delta_{sp} A_{12,34}(\mathbf{k}) \delta_{sp} \delta \bar{\Lambda}_{34}(\mathbf{k}) , \quad (3.2a)$$

with the matrix $A$ given by

$$A_{12,34}(\mathbf{k}) = \delta_{13} \delta_{24} \left[ \varphi_{00}^{00} (\mathbf{k}) m^{00}_{rs,ij} + \varphi_{03}^{n0} (\mathbf{k}) m^{03}_{rs,ij} + \varphi_{33}^{n0} (\mathbf{k}) m^{33}_{rs,ij} \right] \equiv \delta_{13} \delta_{24} A_{12}(\mathbf{k}) . \quad (3.2b)$$

Here we have defined conventions of Green functions

$$\varphi_{ij}^{rm} (\mathbf{k}) = \frac{1}{V} \sum \sum_{ij} G_m^{ij}(\mathbf{p}) G_m^{ij}(\mathbf{p} + \mathbf{k}) , \quad (3.3a)$$

where $i, j = 0, 1, 2, 3$ and $G^0 \equiv \mathcal{G}$, $G^3 \equiv \mathcal{F}$. The matrices $m_{rs,ij}$ etc. are defined as

$$m_{rs,ij}^{00} = \frac{1}{4} \text{tr} \left( \tau_r \tau_s \right) \text{tr} (s_i s_j) , \quad (3.3b)$$

$$m_{rs,ij}^{03} = \frac{1}{4} \text{tr} \left( \tau_r \tau_s \tau_3 \right) \text{tr} (s_i s_j) , \quad (3.3c)$$

$$m_{rs,ij}^{30} = \frac{1}{4} \text{tr} \left( \tau_r \tau_s \tau_3 \right) \text{tr} (s_i s_j) , \quad (3.3d)$$

$$m_{rs,ij}^{33} = \frac{1}{4} \text{tr} \left( \tau_r \tau_s \tau_3 \right) \left( \begin{array}{c} + \\ - \end{array} \right) \text{tr} (s_i s_j) , \quad (3.3e)$$

where $\left( \begin{array}{c} + \\ - \end{array} \right)_i = \delta_{i0} + \delta_{i1} + \delta_{i2} - \delta_{i3}$, etc. $Q$ and $\bar{\Lambda}$ can now be decoupled by shifting and scaling the $\bar{\Lambda}$ field. If we define a new field $\tilde{\Lambda}$ by

$$\delta \bar{\Lambda}(\mathbf{k}) = 2 A^{-1} \left( \delta \tilde{\Lambda}(\mathbf{k}) - \delta Q(\mathbf{k}) \right) , \quad (3.4)$$

then $\Lambda$ and $Q$ decouple. We can thus integrate out $\delta \bar{\Lambda}$ and obtain the Gaussian action entirely in terms of $Q$,

$$A_G[Q] = -\frac{4}{V} \sum \sum \sum \sum_{ij} \delta \tilde{Q}_{124}(\mathbf{k}) \delta_{sp} A_{12,34}(A^{-1})_{12,34}(\mathbf{k})$$

$$\times \delta \tilde{Q}_{34}(\mathbf{k}) + A_{\text{dis}}[\delta Q] + A_{\text{int}}[\delta Q] , \quad (3.5)$$

with $A^{-1}$ the inverse of the matrix $A$ defined in Eq. (2.2b). It is convenient to rewrite this result as

$$A_G[Q] = -\frac{4}{V} \sum \sum \sum \sum_{ij} \delta \tilde{Q}_{124}(\mathbf{k}) \delta_{sp} A_{12,34}(A^{-1})_{12,34}(\mathbf{k})$$

$$\times \delta \tilde{Q}_{34}(\mathbf{k}) + A_{\text{dis}}[\delta Q] + A_{\text{int}}[\delta Q] , \quad (3.5)$$

$$\times \delta \tilde{Q}_{34}(\mathbf{k}) , \quad (3.6a)$$

where

$$M_{12,34}(\mathbf{k}) = \delta_{13} \delta_{24} \left[ \left( A_{12} \right)^{-1}(\mathbf{k}) - \mathbb{1} / \pi N \tau \right]$$

$$- \delta_{1-2,3-4} \delta_{a1,a2} \delta_{a3,a4} B , \quad (3.6b)$$

with $\mathbb{1}$ the unit $8 \times 8$ matrix, and $B$ a matrix whose elements are

$$B_{rs,ij} = -\delta_{rs} \delta_{ij} 2 T \Gamma^{(i)} , \quad (3.6c)$$

where $\Gamma^{(0)} = -\Gamma^{(s)}$ and $\Gamma^{(1,2,3)} = \Gamma^{(t)}$.

B. Gaussian propagators

The Gaussian $Q$ propagators are given in terms of the inverse of the matrix $\mathcal{M}$ defined in Eq. (3.6f). As in Ref. [8] we solve the redundancy problem inherent in the matrix field theory due to the symmetry relations, Eqs. (2.3), by formulating the theory entirely in terms of matrix elements $Q_{12}$ with $n_1 \geq n_2$. Then we have

$$\langle \delta \tilde{Q}_{12}(\mathbf{k}) \delta \tilde{Q}_{34}(\mathbf{k}) \rangle_G = \delta_{k,p} \frac{1}{16} \sum_{ij} \delta_{rs} M_{12,34}(\mathbf{k}) \quad (3.7a)$$

where

$$ij_{rs} = 1 + \delta_{12} \left[ -1 + J^{ij}_{rs} \right] , \quad (3.7b)$$

with

$$J^{ij}_{rs} = \frac{1}{4} \text{tr} \left( \tau_r \tau_s \tau_3 \right) \delta_{r0} \left[ \text{tr} (s_i s_j) + \left( \begin{array}{c} + \\ - \end{array} \right) \text{tr} (s_i s_j) \right]$$

$$+ \frac{1}{4} \text{tr} \left( \tau_r \tau_s \tau_3 \right) \delta_{r3} \left[ \text{tr} (s_i s_j) - \left( \begin{array}{c} + \\ - \end{array} \right) \text{tr} (s_i s_j) \right]$$

$$+ 2 \delta_{rs} \delta_{ij} \left[ \delta_{r0} + \delta_{r3} (1 - \delta_{i0}) \right] . \quad (3.7c)$$

Here and in the following, $\langle \ldots \rangle_G$ denotes an average taken with the Gaussian action. To determine the inverse of $\mathcal{M}$ we notice that the only nonzero matrix elements of $ij_{rs} A$, Eq. (2.2b), are the diagonal and the antidiagonal elements in our $\tau \otimes s$ basis, and that the same is true for the inverse of $A$. We find

$$ij_{rs} M^{-1}_{12,34}(\mathbf{k}) = \delta_{13} \delta_{24} \left[ C_{n1,n2}(\mathbf{k}) + \delta_{1-2,3-4} \delta_{a1,a2} \delta_{a3,a4} \right]$$

$$\times E_{n1,n2,n3,n4}(\mathbf{k}) , \quad (3.8a)$$

where
\[ E_{n_1n_2n_3n_4}(\mathbf{k}) = C_{n_1n_2}(\mathbf{k}) B \]
\[
\times \left[ 1 - \sum_{n_5n_6} \delta_{n_1n_2,n_5n_6} C_{n_5n_6}(\mathbf{k}) B \right]^{-1} C_{n_3n_4}(\mathbf{k}) .
\]
(3.8b)

Here \( B \) is the matrix given by Eq. (3.6a), and \( C \) is defined as
\[
C_{nm}(\mathbf{k}) = (m^{00} + m^{33}) (D^+_nm(\mathbf{k}) + D^-nm(\mathbf{k}))/4 + (m^{03} + m^{30}) \bar{\Lambda}^+(\mathbf{k})/\pi N_F \tau
\]
\[
+ (m^{00} - m^{33}) \bar{\Lambda}^-(\mathbf{k})/\pi N_F \tau
\]
\[
+ (m^{03} - m^{30}) \bar{\Lambda}(\mathbf{k})/\pi N_F \tau
\]
\[
(3.8c)
\]

Here we have defined
\[
D^\pm_{nm}(\mathbf{k}) = \frac{\phi^\pm_{nm}(\mathbf{k})}{[1 - \phi^\pm_{nm}(\mathbf{k})/\pi N_F \tau]} ,
\]
(3.8d)
\[
\mathcal{E}^\pm_{nm}(\mathbf{k}) = \frac{\eta^\pm_{nm}(\mathbf{k})}{[1 - \eta^\pm_{nm}(\mathbf{k})/\pi N_F \tau]} ,
\]
(3.8e)

with
\[
\phi^\pm_{nm}(\mathbf{k}) = \varphi^\pm_{nm}(\mathbf{k}) \pm \psi^\pm_{nm}(\mathbf{k}) ,
\]
(3.8f)
\[
\eta^\pm_{nm}(\mathbf{k}) = \varphi^-_{nm}(\mathbf{k}) \pm \psi^-_{nm}(\mathbf{k}) ,
\]
(3.8g)

in terms of
\[
\varphi^\pm_{nm}(\mathbf{k}) = \varphi^0_{nm}(\mathbf{k}) \pm \varphi^3_{nm}(\mathbf{k}) ,
\]
(3.8h)
\[
\psi^\pm_{nm}(\mathbf{k}) = \varphi^0_{nm}(\mathbf{k}) \pm \varphi^3_{nm}(\mathbf{k}) ,
\]
(3.8i)

Let us discuss these results. First of all, we notice that setting the magnetization equal to zero results in \( D^+ = D^- = \mathcal{E}^+ = \mathcal{E}^- = D \), and we recover the results of Ref. [8]. In particular, \( D_{nm} \) is diffusive for \( nm < 0 \), and hence the spin-singlet and spin-triplet channels of the Gaussian propagator \( C \) are all soft. In the magnetic case, this changes. \( D^+ \) and \( D^- \) are still soft, and given by the nonmagnetic result with the Fermi energy shifted by \( \pm \Delta \), respectively,
\[
D^\pm_{nm}(\mathbf{k}) = \frac{\pi N_F}{D^\pm_{nm} k^2 + \left| \Omega_{n-m} \right|} .
\]
(3.9)

This holds for \( nm < 0 \), and in the limit of small frequencies and wavenumbers. For \( nm > 0 \), \( D^\pm \) is finite in that limit. \( N_F^\pm \) and \( D^\pm \) are the density of states at the Fermi level, and the Boltzmann diffusivity, respectively, of an electron system whose Fermi energy has been shifted by \( \pm \Delta \). The spin-singlet, and the longitudinal component of the spin-triplet, are thus still diffusive, as one would expect. However, the transverse component of the spin-triplet is massive. A calculation yields, for \( nm < 0 \),
\[
\mathcal{E}^\pm_{nm}(\mathbf{k}) = \frac{\pi N_F}{\left( \frac{1}{2} + \frac{1}{2} \right) + \left( \left| \Omega_{n-m} \right| + D^\pm_{nm} k^2 \right)/2} ,
\]
(3.10a)

where we have defined
\[
\tau^\pm = \frac{\tau}{1 \pm 2i \Delta \tau} .
\]
(3.10b)

The mass of these “spin-diffusons” is thus proportional to the magnetization. However, the “interacting” part \( E \) of the propagator, Eqs. (3.3), is still soft even in the transverse spin-triplet channels. To see this, consider the saddle-point equations (2.14). Multiplying the first of these equations by \( F^\pm_n(\mathbf{k}) \) and the second one by \( G_n(\mathbf{k}) \), subtracting the second equation from the first one and integrating over the wavevector, we obtain \( \varphi^\pm_{nm}(k = 0) = 2i F^\pm_n / \Delta_n \) with \( F_n \) from Eq. (2.12). Using this, it is easy to show that
\[
T \sum_n \mathcal{E}^\pm_{nm}(k = 0) = -1/2 \Gamma(\tau) .
\]
(3.11)

Therefore, \( ij_n E \) is still massless for \( i, j = 1,2 \). These are of course the magnetic Goldstone modes, and the above calculation just proves that our Gaussian approximation is conserving in the sense that it correctly reflects the symmetries of the problem, and the resulting soft modes. We note that there is no frequency restriction on the softness of the Goldstone modes, \( E_{12,34} \) is massless both for \( n_1n_2 < 0 \) and for \( n_1n_2 > 0 \). From Eqs. (3.8), (3.10), and (3.11) we see that the structure of the Goldstone modes is
\[
g^\pm(p, \Omega_n) = \frac{1}{1 + 2 \Gamma(\tau) T \sum_n \delta_{n_3-n_4} \mathcal{E}^\pm_{nm}(\mathbf{p})} \frac{1}{\pm i \Omega_n - p^2} ,
\]
(3.12a)

with \( c \) and \( d \) constants. The second equality in Eq. (3.12a) holds in the limit of small wavenumbers and frequencies. The magnetization and disorder dependence of \( c \) and \( d \) is quite complicated, and we write down only the clean limit results to leading order for small values of \( \Delta \), where one finds
\[
c = \Delta/6k_F^2 , \quad d = -\Delta/2N_F \Gamma(\tau) .
\]
(3.12b)

For later reference we also determine the Gaussian \( \Lambda \) propagator. Using Eq. (3.4) in (3.2), we find
\[
\langle \hat{\psi}^i(\delta \Lambda)_{12}(\mathbf{k}) \hat{\psi}^j(\delta \Lambda)_{34}(-\mathbf{p}) \rangle_G = \frac{1}{16 \left( \frac{1}{r_s} \right)^3} \delta_{ij} \delta_{k,p} - \frac{1}{16 \left( \frac{1}{r_s} \right)^3} \delta_{ij} A_{12,34}(\mathbf{k}) .
\]
(3.13)

Notice that the \( \varphi^ij(\mathbf{k}) \), Eq. (3.3a), and therefore the matrix \( A \), Eq. (3.2a), reduce to finite numbers in the limit of low frequencies and small wavenumbers. The \( \Lambda \) propagator is thus massive.

Finally, we mention that if one keeps the particle-particle or Cooper channel, the corresponding propagators have a mass proportional to the magnetization, just like the Cooperons for nonmagnetic electrons in an external magnetic field are massive. Since we are interested in universal phenomena that are due to the soft modes in the system, this justifies our having neglected the Cooper channel.
C. Physical correlation functions

We now use the results of the preceding subsections to calculate some correlation functions of physical interest. Let us start with the single-particle density of states (DOS) in saddle-point approximation. From Eqs. (2.7), (2.11), (2.12), (2.13) we find for the DOS as a function of imaginary frequency

\[ N(i\omega_n) = \frac{1}{2} \left[ N_{HF}(\mu + \Delta_n, i\omega_n) + N_{HF}(\mu - \Delta_n, i\omega_n) \right] . \]  

(3.14)

Here \( N_{HF}(\mu, i\omega_n) \) is the DOS for nonmagnetic electrons with chemical potential \( \mu \) as a fermionic expectation value and translating to a \( Q \)-field correlation by means of Eq. (3.2), we find

\[ \chi_n(q, \Omega_n) = i \frac{D^\pm}{m^2} \frac{\delta Q_{m-n}^\pm(q)}{\delta Q_{m-n}^\pm(-q)} . \]  

(3.15)

By adding an appropriate source term, or by writing the density susceptibility \( \chi_n \) as a fermionic expectation value and translating to a \( Q \)-field correlation by means of Eq. (3.2), we find

\[ \chi_n(q, \Omega_n) = \frac{e^{00}}{00} \chi(q, \Omega_n) + \frac{e^{03}}{33} \chi(q, \Omega_n) - i \frac{e^{03}}{03} \chi(q, \Omega_n) \]  

\[ -i \frac{e^{03}}{30} \chi(q, \Omega_n) . \]  

(3.16a)

Similarly, one obtains the longitudinal (L) and transverse (T) spin susceptibilities as

\[ \chi_L(q, \Omega_n) = \frac{11}{11} \chi(q, \Omega_n) + \frac{13}{33} \chi(q, \Omega_n) - i \frac{13}{30} \chi(q, \Omega_n) \]  

\[ -i \frac{13}{30} \chi(q, \Omega_n) . \]  

(3.16b)

\[ \chi_T(q, \Omega_n) = \frac{33}{33} \chi(q, \Omega_n) + \frac{33}{33} \chi(q, \Omega_n) - i \frac{33}{30} \chi(q, \Omega_n) \]  

\[ -i \frac{33}{30} \chi(q, \Omega_n) . \]  

(3.16c)

We can calculate these susceptibilities explicitly in Gaussian approximation by using the results of Sec. IIIB. For the density susceptibility we find

\[ \chi_n(q, \Omega_n) = T \sum_m D_{m+n,m}^\pm(q) . \]  

(3.17b)

with \( D \) from Eq. (3.8d), are disordered Lindhard functions for nonmagnetic electrons with the chemical potential shifted by \( \pm \Delta_n \). For small frequencies and wavenumbers, and to leading order in the disorder, they read

\[ \chi_n(q, \Omega_n) = \frac{N_F D^\pm q^2}{-i\Omega_n + D^\pm q^2} \]  

(3.17c)

with

\[ D^\pm = \frac{2}{m_e} \frac{\mu \pm \Delta}{\tau} . \]  

(3.17d)

diffusion constants for electrons in the spin-up and spin-down bands, respectively, in \( d \) spatial dimensions. Note that our Gaussian approximation respects particle number conservation, as expressed by the fact that \( \chi_n(q \rightarrow 0, \Omega_n) \rightarrow 0 \). However, the structure of \( \chi_n \) is not diffusive. Rather, in the limit of small \( q \) and \( \Omega_n \), it takes the form

\[ \chi_n(q, \Omega_n) = \frac{N_F}{2} \frac{-i\Omega_n q^2 a_1 + q^4 a_2}{(-i\Omega_n)^2 - i\Omega_n q^2 a_3 + q^4 a_4} . \]  

(3.18)

with \( a_1, a_2, a_3, \) and \( a_4 \) four independent parameters that depend on \( D^+, D^-, N_F \Gamma(s), \) and \( \Gamma(t) \).

From \( \chi_n \) we also obtain the conductivity through the identity

\[ \sigma(i\Omega_n) = \lim_{q \rightarrow 0} i\frac{\Omega_n}{q^2} \chi_n(q, i\Omega_n) . \]  

(3.19)

In the static limit we obtain \( \sigma(0) = \sigma_B \), where

\[ \sigma_B = \frac{N_F}{2} (D^+ + D^-) = N_F D . \]  

(3.20)

with \( D \) the Boltzmann diffusion constant of a nonmagnetic system with unshifted chemical potential. In Gaussian approximation, the conductivity of our itinerant ferromagnet has thus the ordinary Boltzmann value.

The longitudinal spin susceptibility can also be expressed in terms of \( \chi_+ \) and \( \chi_- \). From Eqs. (3.16d), (3.17a), (3.7), (3.8) we obtain

\[ \chi_L = \frac{\chi_+ + \chi_- - 4 \Gamma(s) \chi_+ \chi_-}{1 + (\Gamma(t) - \Gamma(s))(\chi_+ + \chi_-) - 4 \Gamma(s) \Gamma(t) \chi_+ \chi_-} . \]  

(3.21a)

We note again that \( \chi_L \), like \( \chi_n \), is massless, but not diffusive. In the clean limit, \( \tau \rightarrow \infty \), \( \chi_+ \) and \( \chi_- \) become Lindhard functions proper with shifted chemical potentials, but that limit we recover the RPA result of Izuyama et al. if we take into account that their Hubbard model with coupling constant \( v \) corresponds to the special case \( \Gamma(t) = \Gamma(s) = -v/2 \) in our notation. Another special case is the nonmagnetic one, where \( \chi_+ = \chi_- = \chi_0 \), with \( \chi_0 \)
the (disordered) Lindhard function with chemical potential $\mu$. In that case, Eq. (3.21) reduces to the ordinary RPA result, $\chi_L = 2\chi_0/(1 + 2\Gamma^{(t)}\chi_0)$.

Finally, we calculate the transverse spin susceptibility. We find

$$\chi_T(q, \Omega_n) = \frac{T \sum_m \mathcal{E}_{m+n,m}^+(q)}{1 + 2\Gamma^{(t)}T \sum_m \mathcal{E}_{m+n,m}^+(q)}$$

with $\mathcal{E}^\pm$ from Eq. (3.8a).

To discuss this result, we use again Eq. (3.11). Inserting this into Eq. (3.22) yields $1/\chi_T(q = 0, \Omega_n = 0) = 0$, which is indicative of the Goldstone modes. An expansion in powers of $q$ and $\Omega_n$ yields the familiar quadratic dispersion relation for magnons in itinerant ferromagnets, viz.

$$\chi_T(q, i\Omega_n) = -\frac{1}{\Gamma^{(t)}} \left[ g^+(q, i\Omega_n) + g^-(q, i\Omega_n) \right],$$

with $g^\pm$ from Eq. (3.12a). In the limit $|k|/k_F, \Omega_n/\epsilon_F << \Delta/\epsilon_F << 1$, this result agrees with the one obtained with elementary methods.

**IV. EFFECTS OF THE GOLDSTONE MODES IN PERTURBATION THEORY**

In this section we examine the contributions of the ferromagnetic Goldstone modes (FMGM) or spin waves discussed in the previous subsection to the DOS, $N(\omega)$, and the electrical conductivity, $\sigma(\Omega)$. In particular, we focus on the contributions of these soft modes to the nonanalytic frequency dependences of these quantities. Our goal is to determine whether or not these additional, compared to a Fermi liquid state, soft modes that are due to the long-range ferromagnetic order, contribute terms to $N(\omega)$ and $\sigma(\Omega)$ that are as strong as the usual weak localization effects. We will show that they do in the case of the conductivity in three-dimensions, but not in two-dimensions, and not in the case of the DOS in any dimension. This strongly suggests that the FMGM are not important, at least near two-dimensions, in determining the properties of the metal insulator transition from a ferromagnetic metal to a ferromagnetic insulator. This last point will be important in (II).

**A. Single-particle density of states**

The computation of the DOS is straightforward using Eq. (2.7b) as a starting point. We first write the action, Eqs. (2.8d) as the saddle-point contribution, plus the Gaussian part, Eq. (3.1), which is quadratic in $Q$ and $\tilde{\Lambda}$, plus non-Gaussian (cubic and higher order) terms, viz. the diffusive modes or diffusons in the spin-singlet and longitudinal spin-triplet channels, and the Goldstone modes or spin waves in the transverse spin-triplet channels. The former, Eq. (3.3), contribute a term that is very similar to the weak localization contribution to the DOS for nonmagnetic electrons in an external magnetic field. We will discuss this contribution in detail in (II). For now we just consider the functional form of these contributions. They take the form of a frequency-momentum integral over a diffuson propagator. The most divergent part of the Gaussian part, Eq. (3.1), which is quadratic in $Q$ and $\tilde{\Lambda}$, plus non-Gaussian (cubic and higher order) terms, viz. the diffusive modes or diffusons in the spin-singlet and longitudinal spin-triplet channels, and the Goldstone modes or spin waves in the transverse spin-triplet channels.

![Diagram](image)

**Fig. 1.** One-loop (a) and two-loop (b) contributions to the one-point vertex function.

$$A[Q, \tilde{\Lambda}] = A[Q_{sp}, \tilde{\Lambda}_{sp}] + A_G[\delta Q, \delta \tilde{\Lambda}] + \sum_{\ell=3}^\infty A_\ell[\delta \tilde{\Lambda}]$$

(4.1a)

where

$$A_\ell[\delta \tilde{\Lambda}] = -\frac{1}{2\ell} \text{Tr} (iG_{sp} \delta \tilde{\Lambda})^\ell.$$  

(4.1b)

By introducing $\delta \tilde{\Lambda}$ again as defined in Eq. (3.4), we can decouple $\delta Q$ and $\delta \tilde{\Lambda}$ in the Gaussian term, at the expense of having the higher order terms depend on both $Q$ and $\Lambda$.

$$A[Q, \tilde{\Lambda}] = A_{sp} + A_G[\delta Q, \delta \tilde{\Lambda}] + \sum_{\ell=3}^\infty A_\ell[2A^{-1}(\delta \tilde{\Lambda} - \delta Q)]$$

(4.1c)

We proceed by writing the one-point $Q$-correlation function on the right hand side of Eq. (2.7b) as,

$$\langle \delta Q_{nn}^{\alpha\alpha}(x) \rangle = \delta Q_{nn}^{\alpha\alpha}_{sp} + \langle \delta Q_{nn}^{\alpha\alpha}(x) \rangle. \quad (4.2)$$

The diagrammatic loop expansion for the irreducible part of $\langle \delta Q \rangle$, or equivalently for the one-point vertex function, is shown in Fig. 1. The one-loop term, Fig. 1(a), is given analytically by

$$\langle \delta Q_{nn}^{\alpha\alpha}(x) \rangle_{1-loop} = \langle \delta Q_{nn}^{\alpha\alpha}(x) \rangle A_3[2A^{-1}(\delta \tilde{\Lambda} - \delta Q)] \rangle_G. \quad (4.3)$$

Equation (4.3) is evaluated by using Wick’s theorem and the Gaussian propagators, Eqs. (3.7) and (3.13). The leading nonanalyticities are given by contributions where the loop in Fig. 1(a) is a soft mode. That is, the loop must be a $Q$-propagator, since the $\tilde{\Lambda}$-propagator is massive, see Eq. (3.13). As we have seen in Sec. III, there are two different types of soft $Q$-propagators, viz. the diffusive modes or diffusons in the spin-singlet and longitudinal spin-triplet channels, and the Goldstone modes or spin waves in the transverse spin-triplet channels. The former, Eq. (3.3), contribute a term that is very similar to the weak localization contribution to the DOS for nonmagnetic electrons in an external magnetic field. We will discuss this contribution in detail in (II). For now we just consider the functional form of these contributions. They take the form of a frequency-momentum integral over a diffuson propagator. The most divergent part of
the latter is the interacting piece, which is denoted by $E$ in Eq. (3.8). Inspection shows that this piece is essentially a diffusion propagator squared, i.e. it scales like $1/\omega^2 \sim 1/k^4$ (here, and in similar arguments below, $k$ and $\omega$ denote generic internal wavevectors and frequencies.) Dimensional analysis then yields
\[ N(\mu + \omega)_{1\text{-loop, diffusons}} = O(\omega^{(d-2)/2}) \quad (4.4a) \]

The other soft mode contribution is due to the FMGM, or spin waves. They take the form of a frequency-momentum integral over the FMGM propagator, Eq. (3.8), which scales like $1/k^2 \sim 1/\omega$. We thus have
\[ N(\mu + \omega)_{1\text{-loop, FMGM}} = O(\omega^{d/2}) \quad (4.4b) \]

We conclude that the FMGM contributions to the DOS are subleading compared to the diffusion contributions. We will discuss this result further in Sec. V below. Here we just mention that it implies that the FMGMs are irrelevant, as far as the DOS is concerned, for all leading universal effects in the limit of long wavelengths and low frequencies, in particular for the critical behavior at the metal-insulator transition from a FM metal to a FM insulator near $d = 2$.

**B. Electrical conductivity**

We now investigate the same point for the electrical conductivity. In terms of Grassmann variables, the dynamical electrical conductivity as a function of the real frequency $\Omega$ is given by the Kubo formula,
\[ \sigma(\Omega) = \frac{im_e}{\Omega} + \frac{iT}{\Omega V m_e^2} \sum_n \sum_{n1n2} \sum_{\sigma'} \int d\mathbf{x} d\mathbf{x}' \]
\[ \times \left\langle \psi^\sigma_{n1\sigma}(\mathbf{x}) \partial_{\mathbf{x}^i} \psi^\sigma_{n1+m,\sigma}(\mathbf{x}) \psi^{\sigma'}_{n2\sigma'}(\mathbf{x}') \right\rangle \bigg|_{\Omega_n \to \Omega \pm \imath \epsilon} \quad (4.5) \]
Here $\mathbf{x} = (x_1, x_2, x_3)$, etc. The gradient operators in Eq. (4.5) imply that $\sigma(\Omega)$ cannot be written as a simple $Q$-correlation function. At this point we have two choices. We can either generalize the fields we work with, or, we can introduce a suitable source field to generate $\sigma$. We choose the second path, following Refs. 13-20.

1. **Source formalism**

To the fermionic action, Eq. (2.11), we add a source term $S_j$,
\[ S_j = - \sum_{\alpha m} \int d\mathbf{x} \sum_n \sum_{\sigma} \psi^\alpha_{n\sigma}(\mathbf{x}) \partial_{\mathbf{x}^j} \psi^{\alpha}_{n-m,\sigma}(\mathbf{x}) \quad (4.6a) \]
After integrating out the Grassmann fields, the action $A[Q, \tilde{\Lambda}, j]$ is given by Eq. (2.8) with the Tr ln term replaced by,
\[ A_{\text{In}}[\tilde{\Lambda}, j] = \frac{1}{2} \text{Tr} \ln (g^{-1} + JL - i\tilde{\Lambda}) \quad (4.6b) \]
with
\[ (JL)_{12} = \frac{1}{2} \sum_m j_m^{\alpha_1} L_{12,m} \partial_{x_1} \quad (4.6c) \]
where
\[ L_{12,m} = \delta_{\alpha_1\alpha_2} [\tau_- \otimes s_0] \delta_{n_1,n_2-m} - (\tau_+ \otimes s_0) \delta_{n_2,n_1+m} \quad (4.6d) \]
with
\[ \tau_\pm = \tau_0 \pm i\tau_3 \quad (4.6e) \]
In the presence of the source, the partition function, Eq. (2.3), turns into a $j$-dependent generating functional,
\[ Z[j] = \int D[Q] \int D[\tilde{\Lambda}] e^{A[Q, \tilde{\Lambda}, j]} \quad (4.7a) \]
In terms of derivatives of $Z[j]$, $\sigma(\Omega)$ is given by,
\[ \sigma(\Omega) = - \frac{im_e}{\Omega} + \frac{iT}{\Omega V m_e^2} \frac{\partial^2}{\partial j^\alpha_{\mathbf{m}} \partial j^\alpha_{\mathbf{m}}^*} \ln Z[j] \bigg|_{\Omega_{\mathbf{m}} \to \Omega + \imath \epsilon} \quad (4.7b) \]

There are several possible ways to proceed at this point. References 13-20 derived a nonlinear $\sigma$ model. Here, we will use instead a simple perturbation expansion. We first expand $A_{\text{In}}$, Eq. (4.6), in powers of $j$ and $\delta \tilde{\Lambda}$:
\[ A_{\text{In}}[\tilde{\Lambda}] = \sum_{i,j=0}^{\infty} \sum_{i,j=0}^{\infty} A(i,j) \quad (4.8) \]
with $A(i,j) = O(j^i, \delta \tilde{\Lambda}^j)$. According to Eq. (4.7b) we can restrict ourselves to terms with $j \leq 2$. The saddle point contribution is,
\[ A(2,0) = - \frac{1}{4} \text{Tr} \left( G_{12} J L G_{12} J L \right) \quad (4.9) \]
In this approximation, and using Eqs. (2.13) and (1.6) in Eq. (4.9), and the result in Eqs. (4.7), one obtains the Boltzmann equation result for $\sigma$ as given by Eq. (3.20).
For the fluctuation corrections to the Boltzmann conductivity, there are two classes of terms: $A(1,\ell)$ and $A(2,\ell)$, both with $\ell \geq 2$. Since there are two $j$-derivatives in Eq. (4.7b) for $\sigma$, the relevant correlation functions are $\langle A(1,\ell) \rangle_G$ and $\langle A(2,\ell) \rangle_G$. The diagrams that give the one-loop contributions to $\sigma$ are shown in Fig. 4. Analytically one needs,
\[
\sigma = \sigma_B + \begin{array}{c}
\includegraphics[width=0.2\textwidth]{loop_diagram.png}
\end{array} + O(2\text{-loop})
\]

FIG. 2. Diagrammatic contributions to the conductivity. Wavy lines denote sources \( J_L \), and solid lines denote \( \delta \Lambda \) propagators.

\[
A_{(1,2)} = -\frac{1}{2} \text{Tr} \left( G_{sp} J L G_{sp} \delta \Lambda G_{sp} \delta \Lambda \right) , \quad (4.10a)
\]

and

\[
A_{(2,2)} = \frac{3}{8} \left[ \text{Tr} \left( G_{sp} J L G_{sp} \delta \Lambda G_{sp} J L G_{sp} \delta \Lambda \right) + \text{Tr} \left( G_{sp} J L G_{sp} J L G_{sp} \delta \Lambda \right) \right] . \quad (4.10b)
\]

Now we write the loop expansion of the conductivity as

\[
\sigma(\Omega \to 0) = \sigma_B + \sum_{i=1}^{\infty} \sigma_i(\Omega \to 0) , \quad (4.11)
\]

with \( \sigma_i \) the \( i \)th term in the loop expansion. To one-loop order, Eqs. (4.7b) and (4.8) give,

\[
\sigma_1(\Omega) = \frac{i T}{\Omega V m^2} \frac{\partial^2}{\partial j_m^i \partial j_m^\alpha} \left[ \langle A_{2,2} \rangle_G + \frac{1}{2} \langle A_{(1,2)}^2 \rangle_G \right]
\]

\[
\equiv \left[ \sigma_{(1,1)} + \sigma_{(1,2)} \right]_{\Omega \to \Omega+i0} , \quad (4.12)
\]

where \( \langle \ldots \rangle^c \) denotes that only connected diagrams are to be taken into account. \( \sigma_{(1,1)} \) and \( \sigma_{(1,2)} \) are defined, in order, as the two terms in the previous equality. They are graphically represented by the first and second diagram, respectively, in Fig. 2.

2. Analysis of the one-loop terms

The evaluation of the one-loop terms given by Eq. (4.12) is straightforward but tedious. \( \sigma_{(1,1)} \) and \( \sigma_{(1,2)} \) are given by two- and four- \( \delta \Lambda \) correlation functions, respectively. Due to the coupling between \( \delta \Lambda \) and \( \delta Q \) these correlation functions contain both diffusion and FMGM contributions, and hence are soft.

We first consider \( \sigma_{(1,1)} \). Structurally, this term is analogous to the one-loop correction to the DOS that was calculated in Sec. [VA]. Therefore, while the diffusion contribution to this diagram leads to the usual weak-localization correction of order \( \Omega^{(d-2)/2} \) in \( d < 4 \), and \( \ln \Omega \) in \( d = 2 \), the FMGM contribution is weaker, of order \( \Omega^{(d-2)/2} \) in \( d < 4 \).

The \( \sigma_{(1,2)} \) contribution, which is represented by the second diagram in Fig. 2, is more complicated. From the spin structure of this diagram it follows that the two propagators that form the loop must either both be diffusons, or both FMGMs; the contribution that mixes these two modes vanishes. The diagram with two diffusions yields again a weak-localization term of order \( O(\Omega^{(d-2)/2}) \), and will be discussed in (II). The one with two FMGMs has the following structure. Each mode contributes a factor of \( 1/k^2 \sim 1/\omega \) to the integrand. Each gradient operator in \( A_{(1,2)} \) leads to a factor of \( k_x \). Expanding the propagators to order \( \Omega \) and integrating the resulting term over frequency and wavenumber then in general leads to an \( \Omega^{(d-2)/2} \) term in \( d \)-dimensions. Detailed calculations confirm this structural argument, see below. The nonanalytic frequency dependence of the FMGM contributions to the conductivity is thus as strong as that of the diffusion contributions. However, we find that the prefactor of the FMGM nonanayticity is of \( O(1) \) in \( d = 2 + \epsilon \), in contrast to that of the diffusion nonanalyticity, which is of \( O(1/\epsilon) \). While the two sets of soft modes thus give comparable contributions to the nonanalyticity in the metallic phase in \( d = 3 \), the FMGMs do not lead to a \( \ln \Omega \) term in \( d = 2 \). As in the case of the DOS, the conclusion is that the FMGMs are irrelevant for the description of the metal-insulator transition in \( d = 2 + \epsilon \).

For the technical derivation of these results, we start by explicitly writing out \( A_{(1,2)} \) given by Eq. (4.10a). Note that according to the argument given above, the singular terms arise from the wavenumber and frequency dependencies of the soft modes, and all other \( k \) and \( \omega \) dependence can be neglected. Using this we can localize \( A_{(1,2)} \) in space and time and obtain,

\[
A_{(1,2)} \approx \sum_{m,\alpha} A_{(1,2) m}^\alpha \jmath_{m}^\alpha , \quad (4.13)
\]

with,

\[
A_{(1,2) m}^\alpha = -\frac{1}{4} \sum_{n_1 n_2 n_3 j} \frac{1}{V} \sum_{k} i k_x \sum_{\sigma = \pm} \sum_{\beta} \frac{N_{\sigma}}{n_1} \frac{N_{\beta}}{n_2} \frac{N_{\sigma}}{n_3} \frac{N_{\beta}}{m} \times \left[ \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \right]_G
\]

\[
\times \int dx \, dy \, G_n^\alpha(\mathbf{x}) \partial_{x_1} G_n^\alpha(\mathbf{x} - \mathbf{y}) y_1 G_n^\alpha(\mathbf{y}) , \quad (4.14a)
\]

Here \( \mathbf{k} = (k_x, k_y, k_z) \) and \( N \) is the matrix element,

\[
\frac{ij \mathbf{N}_n^\sigma}{n} = \sum_{a, b, c = 0, 3} \text{tr} \left( \tau_a \tau_x \tau_y \tau_z \tau_c \tau_s \right) \text{tr} (s_a s_b s_c s_j)
\]

\[
\times \int dx \, dy \, G_n^\alpha(\mathbf{x}) \partial_{x_1} G_n^\alpha(\mathbf{x} - \mathbf{y}) y_1 G_n^\alpha(\mathbf{y}) , \quad (4.14b)
\]

with \( G_n^\alpha \) and \( \mathbf{N}_n^\sigma \) from Eq. (3.3a), and \( \tau_{\pm} \) from Eq. (4.6b).

From Eqs. (4.11) - (4.14a), and using the factorization property of the Gaussian action, we obtain

\[
\sigma_{(1,2)} = -\frac{i T}{16 \Omega V m^2} \sum_{n_1 n_2 n_3} \sum_{i, j, i', j'} \sum_{r, s, r', s'} \sum_{\sigma, \sigma'} \frac{1}{\sqrt{2}} \frac{1}{k_x} \frac{1}{k_z} \sum_{\beta \beta'}
\]

\[
\times \left[ \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \right]_G
\]

\[
\times \left[ \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \right]_G
\]

\[
\times \left[ \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \langle \delta \Lambda \rangle_{n_1, n_2, n_3} \right]_G
\]
We define

\[ \langle j(\delta \Lambda)_{n_1,n_1+n_2}^{\alpha} (k) \rangle_{G} \]

Note that Eqs. (4.14b) and (4.15) show that the FMGMs and diffusons are not coupled together: If \( i \) equals one or two, then \( j \) must be one or two; similarly, if \( i \) is zero or three, then \( j \) must be zero or three for a nonzero \( ijN \).

Next we separate the \( \delta \Lambda \) fluctuations into soft \( \delta Q \) fluctuations and massive \( \delta \Lambda \) fluctuations using Eq. (4.3). To make our procedure more transparent, we will initially integrate out the massive modes in saddle-point approximation, i.e., we simply drop the \( \delta \Lambda \). We will improve on this in Section \[ \text{IV B}3 \] below. In addition to this approximation, we keep only terms that contribute to nonanalyticities of \( O(\Omega^{-d+2}) \) in \( \sigma(\Omega) \). This procedure amounts to the replacement, in Eq. (4.15), and for \( i,j = 1,2 \),

\[ ij_{rs} N_{n}^\sigma = \frac{ij_{rs} N_{n}^\sigma}{B_n} \]

with

\[ B_n = \varphi_{nn}(k = 0) = \frac{1}{V} \sum_p \left[ g_n^2(p) - F_n^2(p) \right] \]

from Eq. (3.8b). In terms of Q-fluctuations, the leading FMGM contribution to \( \sigma(1,2) \) thus is

\[ \sigma(1,2) = - \frac{iT \Omega m_c}{\Omega m_c} \sum_{n_1,n_2,n_3} \sum_{r,s'} \sum_{i,j,i',j'} \sum_{\sigma,\sigma'} \]

\[ \times \left[ \langle j(\delta Q)_{n_1,n_1+n_2}^{\alpha} (k) \rangle_{G} \right] \]

\[ \times \left[ \langle j(\delta Q)_{n_1+n_2,n_3}^{\alpha} m \rangle_{G} (1-
\]
We next perform the sums over \( \sigma \) and \( \sigma' \) in Eq. (1.21a). It turns out that none of the terms that involve \( C^{21} \) or \( C^{21} \) contribute to the leading nonanalytic frequency dependence of the conductivity. We are thus left with only one term, which depends on \( \Omega_{0} \) only. Keeping again only leading terms, the latter can be expressed in terms of the fundamental Goldstone propagator

\[
g(k, i\Omega_{n}) = g^{+}(k, \Omega_{n}) + g^{-}(k, \Omega_{n})
\]

\[
= \frac{-2d\delta p^{2}}{\Omega_{n}^{2} + (\epsilon p)^{2}} , \tag{4.22}
\]

with \( g^{\pm} \) from Eq. (3.12a). Using the Eqs. (3.7) - (3.9) in Eq. (4.21), all of the resulting terms are proportional to constants times integrals over products of Goldstone propagators. We find

\[
\sigma_{(1,2)}(\Omega) = \frac{-ia^{2}r_{T}^{2}}{4m_{a}^{2}\Omega} I(i\Omega_{m} \to \Omega + i0) , \tag{4.23a}
\]

where

\[
I(i\Omega_{m}) = \frac{1}{V} \sum_{k} k^{2} T \sum_{n_{2}} \Theta(n_{2}) g(k, \omega_{n_{2}})
\]

\[
\times \left[ g(k, \omega_{n_{2}} + \Omega_{m}) + g(k, \omega_{n_{2}} - \Omega_{m}) \right] , \tag{4.23b}
\]

and

\[
a = T \sum_{n} \mathcal{E}^{\pm}_{nn}(k = 0) \mathcal{N}_{n}^{(1)}/B_{n}^{2} . \tag{4.23c}
\]

Notice that \( \mathcal{E}^{\pm}_{nn}(k = 0) = \mathcal{E}^{-}_{nn}(k = 0) \), which follows from Eqs. (3.8).

Simple asymptotic analysis shows that \( I(\Omega) - I(0) \propto \Omega^{d/2} \). However, before we discuss the integral in detail, the prefactor, denoted by \( a \) in Eq. (4.23a), warrants a closer look.

3. The prefactor of the nonanalyticity

We now discuss the prefactor \( a \) defined in Eq. (4.23c). To leading order in a disorder expansion, it suffices to calculate \( a \) in the clean limit. Simple manipulations yield

\[
a = \frac{T}{2} \sum_{n} \frac{1}{V} \sum_{k} \left[ (G^{+}_{n}(k))^{2} k_{x} \frac{\partial}{\partial k_{x}} G^{-}_{n}(k)
\right.
\]

\[
- (G^{-}_{n}(k))^{2} k_{x} \frac{\partial}{\partial k_{x}} G^{+}_{n}(k) \right] . \tag{4.24}
\]

In the saddle-point approximation for the \( G^{\pm}_{n} \), Eq. (2.13d), that we have employed so far, one finds \( a = 0 \). However, this is an artifact of that approximation. To see this, we first point out that a necessary and sufficient condition for \( a \neq 0 \) is a wavenumber dependence of the ‘magnetic’ piece of the self energy, \( \Delta_{a} \) in Eq. (2.13c), which is \( k \)-independent in our saddle-point approximation. Such a \( k \)-dependence indeed arises from our formalism if we keep the massive \( \delta \Lambda \)-fluctuations that we neglected so far for simplicity. Consider Eq. (1.6f) again. By keeping \( \delta \Lambda \) in the decomposition of \( \Lambda \) into \( \Lambda \) and \( Q \), Eq. (4.3), and lumping it into the Green function, we obtain, upon integrating over \( \delta \Lambda \) in Gaussian approximation, a generalization of \( G^{\pm} \) in Eq. (4.24) which has a \( k \)-dependent magnetic self-energy. This procedure still constitutes an approximation, as determining the prefactor exactly would require an exact treatment of the massive modes. It serves to demonstrate, however, that the prefactor is in general nonzero, and our initial null result indeed derives from an oversimplifying approximation. To avoid misunderstandings, we also point out that the leading frequency dependence we determine is exact, and only the prefactor we cannot calculate exactly.

The above procedure produces a wavenumber dependent self-energy, and hence a nonzero \( a \), even for our model with a point-like spin-triplet interaction amplitude \( \Gamma^{(l)} \), Eq. (2.10c). A wavenumber dependent \( \Gamma^{(l)} \) would of course also lead to a wavenumber dependence of the self-energy, but capturing this effect would require a generalization of our matrix fields. We conclude that the prefactor \( a \) in Eq. (4.23a) is in general nonzero and nonuniversal, as it depends on microscopic details like the precise structure of the interaction amplitude.

4. The FMGM-induced nonanalyticity

We finally need to consider the integral \( I(i\Omega_{m}) \) defined in Eq. (4.23b). At \( T = 0 \), the relevant dimensionless integral is

\[
J(\Omega) = \int_{0}^{\infty} dk \int_{0}^{\infty} d\omega \frac{\omega}{\omega^{2} + k^{4}} \left[ \frac{k^{2}}{(\omega - \Omega)^{2} + k^{4}} + \frac{k^{2}}{(\omega + \Omega)^{2} + k^{4}} \right] . \tag{4.25a}
\]

First consider the integral in \( d = 2 \). Subtracting the value at \( \Omega = 0 \), it is easy to see that the leading frequency dependence is linear, i.e. there is no term \( \propto \Omega \ln \Omega \). More generally, standard asymptotic analysis yields

\[
J(\Omega \to 0) = J(0) - \frac{\pi^{2}}{2^{3+d/2}\sin(\pi d/4)} \Omega^{d/2} . \tag{4.25b}
\]

Consistent with the absence of a logarithm in \( d = 2 \), the prefactor of the \( \Omega^{d/2} \) nonanalyticity is finite in the limit \( d \to 2 \). The frequency scale for the nonanalyticity is given by the quantity \( \Delta \), Eq. (2.15), which is proportional to the magnetization.

At nonzero temperature, one finds the same qualitative behavior, since the Goldstone modes, which are the source of the nonanalyticity, have the same functional
form at all values of $T$, as long as one stays in the magnetic phase. This is in sharp contrast to the diffuson-induced weak-localization nonanalyticities, which are cut off by a nonzero temperature since the diffusons acquire a mass at $T > 0$. Collecting everything, we obtain our final result,

$$\text{Re } \sigma(\Omega \to 0) = \sigma_B \left[ 1 + \frac{C_{d}}{\epsilon_{\tau}} \frac{\Omega}{\Delta}^{(d-2)/2} \right], \quad (4.26)$$

which is valid for $2 \leq d < 4$. $C_d > 0$ in Eq. (4.26) is a positive constant (for fixed dimensionality $d$) which is nonuniversal; it depends, for instance, on the wavenumber dependence of the spin-triplet interaction amplitude as was discussed in the preceding subsection. The $d$-dependence of $C_d$ is nonsingular, in particular, $C_2$ is a finite number.

V. DISCUSSION

In this paper we have developed a general theoretical framework for disordered itinerant ferromagnets. While microscopic details like band structure etc. could be built into the theory (they would enter in the form of more complicated Green functions), our method is particularly well suited for studying universal properties that are due to the soft modes in the system and that are independent of the details on microscopic scales. We have therefore restricted ourselves to one parabolic band, and have studied the influence of the soft modes on the transport and thermodynamic properties. Our most important conclusion is that, in disordered systems with ferromagnetic long-range order, there are two distinct families of soft modes that contribute to the leading nonanalyticities that lead to generic scale invariance. One family consists of the diffusive “diffusons” that also exist in the absence of magnetic long-range order, while the other are the magnetic Goldstone modes or spin waves that are characteristic of magnetically ordered systems. Even though, there are crucial differences between the effects of the two types of soft modes, which we now discuss in some detail.

As we have seen in Sec. [V] the one-loop correction to the DOS (Fig. 2(a)) and the simple loop contribution to the one-loop correction to the Boltzmann conductivity (the first diagram in Fig. 2) are simply a frequency-momentum integral over a soft propagator. For interacting electrons, the interacting part of the diffuson propagator, i.e. $E$ in Eq. (1.8a) in the diffuson channels, is essentially a diffusion propagator squared, i.e., it scales like $1/\omega^2 \sim 1/k^4$. The integration then leads to a term proportional to $\Omega^{(d-2)/2}$ in $2 < d < 4$, and $\ln \Omega$ in $d = 2$. This is the usual weak-localization contribution. The FMGM contribution, on the other hand, consists of a frequency-momentum integral over a single propagator that scales like $1/\omega \sim 1/k^2$, and is hence less singular. As a result, there is no FMGM contribution to the leading nonanalyticity in the DOS.

For the “football” contribution to the conductivity (the second diagram in Fig. 2) the situation is different, since both internal propagators can be Goldstone modes. Our explicit calculation has shown that this term indeed contributes to the leading, $O(\Omega^{(d-2)/2})$, nonanalytic frequency dependence of the conductivity. In $d = 3$, the contributions from the diffusons and the FMGMs are thus qualitatively the same. However, in $d = 2$ the diffusons lead to the well-known “weak-localization” logarithmic frequency dependence, while the FMGM contribution does not lead to a logarithm. A related feature is that the prefactor of the nonanalyticity in $d = 2 + \epsilon$ goes like $1/\epsilon$ for the diffuson-induced term, but is of $O(1)$ in the FMGM case.

An interesting technical feature is the fact that in the case of the diffusons, both the simple loop and the “football” contribute an $O(\Omega^{(d-2)/2})$ dependence of the conductivity, while for the FMGM only the latter does. As mentioned above, the different results from the simple loop can be understood by realizing that the interacting part of the diffuson is really a diffusion propagator squared that scales like $1/k^4$, while the FMGM scales like $1/k^2$. A question that arises then is why the diffusons in the “football” do not yield a much stronger singularity, of $O(\Omega^{(d-6)/2})$. The technical reason is that the diffusons come with additional frequency restrictions, as was discussed in Sec. [III]. These frequency restrictions lead to additional frequency factors in the integral that determines the “football” contribution, while no such singularity-protecting effect occurs in either the simple loop for the diffusons or in any of the FMGM contributions.

Another very important difference between diffusons and FMGMs is that the former are soft only at $T = 0$, while the latter remain soft at all $T$ below the Curie temperature. As a result, the nonanalytic frequency dependence coming from the diffusons gets replaced by an analogous nonanalytic temperature dependence at $T > 0$. The one induced by the FMGM on the other hand, is still of $O(\Omega^{(d-2)/2})$ even at $T > 0$.

The sign and strength of the nonanalytic term in Eq. (2.26) is interesting and can be understood on very general grounds. First we remember that this nonanalyticity in frequency also exists at finite temperature. This implies that its origin is classical in nature. The sign of classical “mode coupling” or “generic scale invariance” contributions to the transport coefficients is determined by the structure of the hydrodynamic equations describing the long wavelength dynamics of the soft modes. For the spin density dynamics relevant here, the leading nonlinear coupling in the long wavelength limit contains two gradients, as it does in classical dissipative systems such as Lorentz models. As in the Lorentz models, the soft modes therefore have a localizing effect, i.e. the conductivity decreases with decreasing frequency. In contrast, in classical fluids the leading nonlinear coupling
that are even stronger than in classical fluids. In certain liquid crystals leads to mode coupling effects leads to the same power of the frequency as in the fluid systems are the Goldstone modes, but they do not contribute to the leading nonanalyticities in $d = 2$. Indeed, transport measurements on thin ferromagnetic films have found just the ordinary weak-localization effects due to the diffusons, with no additional effects from the Goldstone modes at all. Our theory explains this null result, which a priori is very surprising. For bulk materials, on the other hand, our theory predicts that the FMGMs do contribute a term that is qualitatively the same as the weak-localization effect, and furthermore that this term will remain a nonanalytic frequency dependence even at $T > 0$.

Another consequence is the suggestion that the FMGMs will not be important for the critical behavior at the metal-insulator transition from a FM metal to a FM insulator, at least close to two-dimensions. The reason for this is the absence of a diverging prefactor of the frequency nonanalyticity as $d \to 2$. This divergence is known to drive the transition in $d = 2 + \epsilon$, and since the FMGMs do not contribute to it one expects the universality class of the transition to be unchanged by the presence of ferromagnetic long-range order. We will investigate this point in (II) and find that the metal-insulator transition on the background of ferromagnetism is indeed closely related to the one of nonmagnetic electrons in an external magnetic field.

contains only one gradient, and the prefactor of the nonanalyticity has a delocalizing sign. We further note that the nonanalyticity in Eq. (4.26) is stronger than the one in classical Lorentz gases by a factor of $\Omega$. It is, in fact, as strong as the generic scale invariance effects in classical fluids, even though the structures of the respective nonlinear couplings suggest just the opposite. The resolution of this apparent paradox lies in the fact that there is long-range order in our system, which is manifested by the Goldstone modes. In a mode-coupling calculation, the transverse spin susceptibility, $\chi_t \propto 1/k^2$, appears as a multiplicative factor in the integrands and effectively leads to the same power of the frequency as in the fluid case. It is interesting to note that a similar mechanism in certain liquid crystals leads to mode coupling effects that are even stronger than in classical fluids. For these systems the standard hydrodynamic description breaks down for all $d < 5$. Here, the analogous critical dimension is $d = 2$, see Eq. (4.26).

There are some important experimental consequences of these results. One is that, as far as the weak-localization properties in $d = 2$ are concerned, one does not expect any differences between metals with long-range ferromagnetic order and nonmagnetic metals in an external magnetic field. The reason is that the only difference in the soft mode structures of the two systems are the Goldstone modes, but they do not contribute to the leading nonanalyticities in $d = 2$. Indeed, transport measurements on thin ferromagnetic films have found just the ordinary weak-localization effects due to the diffusons, with no additional effects from the Goldstone modes at all. Our theory explains this null result, which a priori is very surprising. For bulk materials, on the other hand, our theory predicts that the FMGMs do contribute a term that is qualitatively the same as the weak-localization effect, and furthermore that this term will remain a nonanalytic frequency dependence even at $T > 0$.

Another consequence is the suggestion that the FMGMs will not be important for the critical behavior at the metal-insulator transition from a FM metal to a FM insulator, at least close to two-dimensions. The reason for this is the absence of a diverging prefactor of the frequency nonanalyticity as $d \to 2$. This divergence is known to drive the transition in $d = 2 + \epsilon$, and since the FMGMs do not contribute to it one expects the universality class of the transition to be unchanged by the presence of ferromagnetic long-range order. We will investigate this point in (II) and find that the metal-insulator transition on the background of ferromagnetism is indeed closely related to the one of nonmagnetic electrons in an external magnetic field.

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Alternatively, it is easy to see by means of standard diagrammatic many-body theory that the exact self-energy for our model is wavenumber dependent. This holds for the processes considered here. There may be additional processes at $T > 0$ due to the fact that thermally excited spin waves exist. This will be the subject of a separate investigation.

In nonmagnetic systems, in the absence of an external magnetic field and magnetic impurities, there is another mechanism for producing such terms. In such systems, the particle-particle or Cooper channel ($r = 1, 2$) is also diffusive, and perturbation theory yields a simple momentum integral over a diffusive Cooperon, leading again to a term proportional to $f_{q}^{(d-2)/2}$.

We note, however, that this is true only in theoretical descriptions of the metal-insulator transition near $d = 2$. In high dimensions ($d > 6$), the nature of the transition is not related to the (very weak) nonanalyticities in the metallic phase, see T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. 73, 862 (1994).