Time evolution and the Schrödinger equation on time dependent quantum graphs

Uzy Smilansky\textsuperscript{1} and Gilad Sofer\textsuperscript{2,\,*\,*\,*}\textsuperscript{\textcopyright}

\textsuperscript{1} Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 7610001, Israel
\textsuperscript{2} Faculty of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel

E-mail: gilad.sofer@campus.technion.ac.il

Received 30 October 2023; revised 29 December 2023
Accepted for publication 17 January 2024
Published 31 January 2024

Abstract
The purpose of the present paper is to discuss the time dependent Schrödinger equation on a metric graph with time-dependent edge lengths, and the proper way to pose the problem so that the corresponding time evolution is unitary. We show that the well posedness of the Schrödinger equation can be guaranteed by replacing the standard Kirchhoff Laplacian with a magnetic Schrödinger operator with a harmonic potential. We then generalize the result to time dependent families of vertex conditions. We also apply the theory to show the existence of a geometric phase associated with a slowly changing quantum graph.

Keywords: quantum graphs, Schrödinger equation, time evolution

1. Introduction

In the present work, we study the time evolution on quantum graphs whose edge lengths vary in time. A stationary quantum graph consists of a set of vertices connected by edges of prescribed lengths. On each edge, the Schrödinger operator is the one-dimensional Laplacian. The graph is supplemented with appropriate boundary conditions at the vertices, which ensure that the
operator is self-adjoint and so the resulting time evolution is unitary. Our goal is to generalize this model by allowing the edge lengths to be time dependent.

To illustrate the possible difficulties this time dependence causes, we start by considering the simplest example. Namely, a graph that consists of two vertices connected by an edge of length $L(t/T)$. Here, $T^{-1}$ measures the rate of change of the edge lengths.

The evolution for this system is dictated by the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}, \quad x \in \left[ -\frac{L(t/T)}{2}, \frac{L(t/T)}{2} \right]. \quad (1.1)$$

At the two vertices we impose the standard Neumann boundary condition:

$$\frac{\partial \psi(x,t)}{\partial x} = 0 \quad \text{at} \quad |x| = \frac{L(t/T)}{2}, \forall t \geq 0. \quad (1.2)$$

The operator on the right hand side of (1.1) involves only derivatives with respect to $x$. The time dependence is due to the boundary conditions, which are imposed at the time dependent boundary. If one considers the time $t$ as a parameter, the operator on the right is referred to as the instantaneous Hamiltonian.

Given some initial condition $\psi(x,0)$, we let the system evolve according to (1.1). A straightforward computation then gives the following expression for the time derivative of the squared $L^2$ norm of $\psi(x,t)$:

$$\frac{d}{dt} \|\psi\|^2 = \frac{d}{dt} \int_{L(t/T)}^{L(t/T)} |\psi(x,t)|^2 dx \quad = \frac{1}{2T} \frac{d}{dt} \frac{L(t/T)}{2} \left[ \psi \left( \frac{L(t/T)}{2}, t \right) \right]^2 + \psi \left( \frac{L(t/T)}{2}, t \right)^2. \quad (1.3)$$

This rather unexpected change of the $L^2$ norm in time is due to the choice of Neumann boundary condition, which causes the additional term $\int_{L(t/T)}^{L(t/T)} \frac{d}{dt} |\psi(x,t)|^2 dx$ to vanish after integrating by parts.

Recalling the standard interpretation of $|\psi(x,t)|^2 dx$ as the probability for finding the particle in the interval $(x,x+dx)$ at time $t$, the above result implies that there exists a probability flux which equals to the product of the density at the boundary and its speed. Hence, the probability to find the particle within the boundaries changes in time. This implies that the evolution dictated by (1.1) is not unitary. In this context, it is worth noting:

1. The instantaneous $L^2$ norm is constant in the adiabatic limit $T \to \infty$.
2. For the Dirichlet boundary condition ($\psi(x,0) = 0$), the instantaneous $L^2$ norm is constant.

Equation (1.3) shows that the change in norm arises also for the Robin boundary condition:

$$\frac{\partial \psi}{\partial x}(x,t) = C \psi(x,t) \quad \text{at} \quad |x| = \frac{L(t/T)}{2}, C \in \mathbb{R}. \quad (1.4)$$

In section 2 we return to this problem, and derive the boundary conditions which render the time evolution unitary, ensuring probability conservation. We follow previous studies which treat time dependent domains in different contexts. The time dependent Schrödinger equation on domains in $\mathbb{R}^d$ with time dependent boundary was studied in [12, 13, 19]. The propagation of electromagnetic waves in vibrating cavities is a closely related problem studied in [18].
The present work can be considered as an extension of these studies to quantum graphs. We shall note that the time dependent Schrödinger equation on a star graph was studied in [17]. However, the vertex boundary conditions considered there correspond to the Dirichlet condition, and thus do not address the problem discussed above.

Once the problem is properly reformulated in section 2, the effect of the moving boundary is expressed in terms of a gauge field which, when included, renders the Schrödinger operator self-adjoint (theorem 2.1). This is analogous to the derivation for slowly varying time dependent domains presented in [13].

In section 3, we generalize the result for graphs where the vertex conditions depend on time as well (theorem 3.4). Moreover, we apply these results by studying the geometric phase associated with the time evolution in the adiabatic limit. Finally, in appendix we derive the secular equation whose zeros give the instantaneous spectrum at each time.

2. Quantum graphs with time dependent edge lengths and standard boundary conditions

This section consists of three parts, in which the time dependent Schrödinger equation for quantum graphs with time dependent edge lengths is formulated and discussed. The first subsection addresses the problem in the simple case of a single interval. It serves as a primer for the treatment of a general graph which is presented in the second subsection. In the third subsection we demonstrate the theory by studying the time evolution on a time dependent equilateral quantum graph.

2.1. A time dependent interval

Before proceeding further, it is worthwhile to rewrite the Schrödinger equation (1.1) in dimensionless form. Observing that $\hbar^2 / m$ has the dimension of $\text{length}^2 \text{time}$, then scaling $x$ by $L$ and $t$ by $T$ results in the dimensionless quantity $L^2 / T \cdot \hbar^2 / m$. Using the standard convention in which $\hbar^2 / m$ takes the numerical value 1, we remain with a dimensionless $L^2 / T$. We then get the following Schrödinger equation:

$$i \frac{\partial \psi(x,t)}{\partial t} = -\frac{\partial^2 \psi(x,t)}{\partial x^2}, \quad x \in \left[-\frac{1}{2}L(t/T), \frac{1}{2}L(t/T)\right].$$

(2.1)

Following [12, 13, 19], we wish to write the Schrödinger equation in terms of the scaled variables

$$\tau = \frac{t}{T}, \quad \xi = \frac{x}{L(t/T)}.$$  

(2.2)

To do this, we introduce the unitary transformation:

$$\omega(\xi, \tau) := L(\tau)^{1/2} \psi(L(\tau)\xi, T\tau),$$  

(2.3)
where now $\tau \in [0, 1]$ and $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. The Schrödinger equation (2.1) can be rewritten in the new coordinates as

$$i\frac{T}{\tau} \frac{\partial}{\partial \tau} \omega(\xi, \tau) = \frac{1}{L(\tau)^2} \left[ -\left( \frac{\partial}{\partial \xi} - i\xi \frac{\dot{L}(\tau) L(\tau)}{2T} \right)^2 - \xi^2 \left( \frac{\dot{L}(\tau) L(\tau)}{2T} \right)^2 \right] \omega(\xi, \tau).$$  \hspace{1cm} (2.4)

In the new Schrödinger equation, the Schrödinger operator includes a repelling harmonic potential, and the Laplacian appears as a magnetic Laplacian with $A(\xi, \tau) = \xi \frac{\dot{L}(\tau) L(\tau)}{2T}$ playing the role of a vector potential. This suggests that one can render the magnetic operator above self-adjoint by replacing the original Neumann condition with the magnetic Neumann condition:

$$\left( \frac{\partial}{\partial \xi} - i\xi \frac{\dot{L}(\tau) L(\tau)}{2T} \right) \omega(\xi, \tau) = 0 \quad \text{at} \quad |\xi| = \frac{1}{2}. \hspace{1cm} (2.5)$$

This indeed gives a self-adjoint operator on $L^2 \left(-\frac{1}{2}, \frac{1}{2}\right)$. Using (2.3) and (2.5), we obtain the corresponding boundary condition for $\psi(x, t)$:

$$\frac{\partial \psi(x, t)}{\partial x} = \pm \frac{i\dot{L}}{4T} \psi(x, t) \quad \text{at} \quad x = \pm \frac{1}{2} \frac{T(l)}{T}. \hspace{1cm} (2.6)$$

Repeating the computation in (1.3) with the new boundary condition immediately shows that the resulting time evolution is unitary.

The following alternative explanation for the magnetic boundary condition was suggested by Berry [10]. The movement of the edges induces a nontrivial probability current at the boundary. One may account for this probability current by introducing a modified Robin boundary condition:

$$\frac{\partial \psi(x, t)}{\partial x} = C \psi(x, t) \quad \text{at} \quad |x| = \frac{1}{2} \frac{T(l)}{T}, \ C \in \mathbb{C}. \hspace{1cm} (2.7)$$

While the usual Robin condition corresponds to $C \in \mathbb{R}$, a straightforward computation shows that the requirement $\frac{1}{4}||\dot{\psi}||^2 = 0$ gives the condition $\text{Im}(C) = \frac{\dot{L}}{2T}$. We may thus solve the problem by suggesting the boundary condition (2.6). The associated Robin parameter may be considered as an effective magnetic field.

The Schrödinger equation and boundary conditions in (2.4) and (2.5) can be further simplified by introducing the gauge transformation

$$\omega(\xi, \tau) = e^{i\frac{\dot{\theta}(\tau)}{4T}(\xi^2 - \frac{1}{4})} g(\xi, \tau). \hspace{1cm} (2.8)$$

One may then eliminate the magnetic Laplacian in equation (2.4) if the phase is chosen to be $\theta(\tau) = \frac{\dot{L}}{4T}$. Note that in the units used in this work, $\theta$ is dimensionless.

The Schrödinger equation is now given by

$$i\frac{T}{\tau} \frac{\partial}{\partial \tau} g(\xi, \tau) = -\frac{1}{L(\tau)^2} \frac{\partial^2 g(\xi, \tau)}{\partial \xi^2} + \frac{\dot{L} L}{4T^2} \xi^2 g(\xi, \tau) - \frac{\ddot{L} L L}{16T^2} g(\xi, \tau), \ |\xi| \leq \frac{1}{2},$$  \hspace{1cm} (2.9)

and subject to standard Neumann boundary conditions.
A similar equation was presented in [17, 19]. We derived it here as a prelude to the treatment of the time dependent Schrödinger operator on a graph. Note that the factor $\frac{1}{T}$ multiplies the $\tau$ derivative, implying that the adiabatic limit $T \to \infty$ is the analogue of the semi-classical limit.

### 2.2. Compact metric graphs with time dependent edge lengths

Consider now the metric graph $\Gamma = (V, E, \mathcal{L})$, where $V$ is the vertex set and $E$ is the edge set (both assumed finite). Moreover, $\mathcal{L} : [0, T] \to \mathbb{R}^{|E|}$ is a family of edge lengths parameterized as $\{L_e(t/T)\}_{e \in E}$ with $L_e(t/T)$ positive and twice continuously differentiable. A coordinate $x_e$ with $x_e \in \left[ -\frac{1}{2}L_e(t/T), \frac{1}{2}L_e(t/T) \right]$ is assigned to every edge in $E$ (see figure 1). We denote the set of edges connected to the vertex $v$ by $S_v$ and the degree of the vertex by $d_v := |S_v|$.

The time dependent Schrödinger operator for this model consists of the direct sum of $|E|$ one-dimensional Laplacians attached to the edges $\Delta = \bigoplus_{e \in E} \frac{\partial^2}{\partial x_e^2}$:

$$i \frac{\partial}{\partial t} \Psi[x, t] = -\Delta \Psi[x, t], \quad x = \{x_e\}_{e \in E},$$

with

$$\Psi[x, t] = \{\psi_e(x_e, t)\}_{e \in E}, \quad 0 \leq t \leq T. \quad (2.11)$$

We have shown in the preceding section that for the graph which consists of a single interval, the standard Neumann boundary conditions that are used in the time independent case should be modified when the length is time dependent. A similar computation shows that the same is true for arbitrary graphs. Namely, one should replace the commonly used Neumann–Kirchhoff vertex condition:

$$\psi_e(x_e = v, t) = \psi_{e'}(x_{e'} = v, t), \quad \forall e, e' \in S_v,$$

$$\sum_{e \in S_v} \frac{\partial \psi_e(v, t)}{\partial x_e} = 0, \quad \text{(derivatives directed away from the vertex).} \quad (2.12)$$

The alternative formulation should provide a self-adjoint instantaneous Schrödinger operator, which, in the limit of constant edge lengths, should converge to the standard Neumann–Kirchhoff Laplacian in an appropriate sense. We focus on the standard vertex conditions (2.12) in this section; in section 3 we provide a treatment of more general vertex conditions.
Following the example of the interval, it is suggested that the proper way to represent the problem is by introducing a new set of coordinates,

\[ \xi_e = \frac{x_e}{L_e(t/T)}, \tau = \frac{t}{T}, \]

(2.13)

in which the edge lengths are identically equal to one. Denoting the associated (stationary) metric graph by \( \Gamma_0 \), we define the family of unitaries

\[ U_t : L^2(\Gamma) \to L^2(\Gamma_0), \]

(2.14)

\[ U_t \psi_e(x_e, t) = \omega_e(\xi_e, \tau) := L_e(\tau)^{1/2} \psi(L_e(\tau) \xi_e, T \tau). \]

(2.15)

As before, the family \( U_t \) gives us the new Schrödinger equation on each edge:

\[ i \frac{\partial}{\partial \tau} \omega_e(\xi_e, \tau) = \frac{1}{L_e(\tau)^2} \left[ - \left( \frac{\partial}{\partial \xi_e} - i \xi_e \frac{\hat{L}_e(L_e(\tau))}{2T} \right)^2 - \xi_e^2 \left( \frac{\hat{L}_e(L_e(\tau))}{2T} \right)^2 \right] \omega_e(\xi_e, \tau). \]

(2.16)

The vertex boundary conditions which correspond to the magnetic Schrödinger operator above are obtained by precomposing the vertex condition (2.12) with \( U_t^{-1} \), and replacing \( \frac{\partial}{\partial \xi_e} \) with the magnetic derivative associated with (2.16). This gives

\[ L_e^{-1/2}(\tau) \omega_e(v, \tau) = L_e^{-1/2}(\tau) \omega_{e'}(v, \tau), \quad \forall e, e' \in S_v, \]

(2.17)

\[ \sum_{e \in \mathcal{S}_v} L_e^{-1/2}(\tau) \left( \frac{\partial}{\partial \xi_e} - i \xi_e \frac{\hat{L}_e(L_e(\tau))}{2T} \right) \omega_e(\xi_e, \tau) = 0 \quad \text{at} \ |\xi_e| = \frac{1}{2}. \]

(2.18)

This operator is clearly self-adjoint on \( L^2(\Gamma_0) \) for all \( t \). If we switch back to our original time dependent graph, the condition above translates into the following magnetic vertex condition:

\[ \psi_e(v, t) = \psi_{e'}(v, t), \quad \forall e, e' \in S_v, \]

(2.19)

\[ \sum_{e \in \mathcal{S}_v} \left( \frac{\partial}{\partial x_e} - i \text{sgn}(x_e) \frac{\hat{L}_e(t/T)}{4T} \right) \psi_e(x_e, t) = 0 \quad \text{at} \ |x_e| = \frac{L_e(t/T)}{2}. \]

(2.20)

For every \( t \), our (time dependent) magnetic Laplacian is unitarily equivalent to a self-adjoint operator on \( L^2(\Gamma_0) \), and is thus self-adjoint for all \( t \). One can also verify that the corresponding time evolution is now unitary.

We thus conclude that for the problem of the time dependent graph to be well posed, the usual Neumann–Kirchhoff vertex condition should be replaced with the magnetic vertex condition (2.19) and (2.20). Moreover, note that in the limit \( T \to \infty \), one recovers the standard Neumann–Kirchhoff vertex condition of a stationary quantum graph. We state this as the main result for this section:

**Theorem 2.1.** The Schrödinger equation (2.16) on the stationary metric graph \( \Gamma_0 \) equipped with the magnetic vertex conditions (2.17) and (2.18) induces a unitary flow on \( L^2(\Gamma_0) \). This allows to define a solution to the Schrödinger equation (2.10) on the time dependent graph \( \Gamma \) with unitary time evolution, by composing the solutions of (2.16) with the unitary family \( U_t \), (2.14) and (2.15).
This theorem is a particular case of theorem 3.4 in section 3, where the proof can be found. We now go back to our representation of the problem using the stationary graph (2.17) and (2.18). It turns out that using this representation, the problem can be solved more easily.

Note that the magnetic potential of the instantaneous operator is given on each edge by $A_e(\xi_e) = \xi_e L_e / 2T$. Since each edge is parameterized by $[-1/2, 1/2]$ and $A$ is an antisymmetric function, then the integral of $A$ along every edge is equal to zero. In particular, this shows that the total magnetic flux induced by the magnetic potential through every cycle of the graph vanishes. Thus, following the method presented in [9, section 2.6], one can eliminate the magnetic term by applying the gauge transformation

$$
\omega_e(\xi_e, \tau) = e^{i\theta_e(\tau) (\xi_e^2 - 1/4)} g_e(\xi_e, \tau), \quad \theta_e = \frac{L_e L_e}{2T}.
$$

(2.21)

The Schrödinger equation then reads

$$
i\frac{\partial g_e(\xi_e, \tau)}{\partial \tau} = -\frac{1}{L_e^2(\tau)} \frac{\partial^2 g_e(\xi_e, \tau)}{\partial \xi_e^2} + \left(\frac{L_e L_e \xi_e^2}{4T^2} - \frac{L_e L_e + \frac{L_e^2}{16T^2}}{g_e(\xi_e, \tau)}\right), \quad |\xi_e| \leq \frac{1}{2}, \forall e \in \mathcal{E},
$$

(2.22)

where the vertex conditions are now

$$
L_e^{-1/2}(\tau) g_e(v, \tau) = L_e^{-1/2}(\tau) g_{e'}(v, \tau), \quad \forall e, e' \in S_v,
$$

(2.23)

$$
\sum_{e \in \mathcal{E}_v} \frac{1}{L_e^{3/2} (\tau)} \frac{\partial g_e(v, \tau)}{\partial \xi_e} = 0.
$$

(2.24)

In general, the time dependent potential and vertex conditions make the Schrödinger equation (2.22) non-separable, and thus difficult to solve. One way to simplify the equation is by assuming the adiabatic limit, $T \to \infty$, corresponding to the edge lengths changing very slowly compared to the characteristic time scale of the system. In this limit, the solution to the Schrödinger equation is given by

$$
g_e(\xi, \tau) = e^{-i\int_0^\tau H(\tau') d\tau'} g_e(\xi, 0),
$$

(2.25)

where $H(\tau)$ is the instantaneous operator at time $\tau$. Meaning—the time evolution at the adiabatic limit can be obtained by computing the instantaneous spectrum.

Note that as $T \to \infty$, the shifted harmonic potential in (2.22) is negligible. Namely, the difference between our instantaneous Schrödinger operator and the weighted Laplacian without the potential converges in norm to zero. The instantaneous equation for the eigenfunctions $\{g_e(\xi_e, \tau)\}_{e \in \mathcal{E}}$ can thus be solved via the standard scattering approach presented in [16], as described in appendix.

### 2.3. Example—equilateral graph with linearly growing edges

We demonstrate the above theory by computing the time evolution for a very simple system—an equilateral graph whose edge lengths grow linearly in time:

$$
L_e(t/T) := L(t/T) = L_0 + \frac{Vt}{T}.
$$

(2.26)
Figure 2. An equilateral star graph with three edges of varying lengths $\Gamma_t$, along with the associated stationary graph $\Gamma_0$ parametrized so that the central vertex corresponds to $\xi = 1/2$.

To solve this problem, we consider the associated stationary metric graph, $\Gamma_0$. The main feature of this system which simplifies the computation is the fact that $L_e = 0$, and so the harmonic term in the Schrödinger equation (2.22) vanishes, leaving us with the Schrödinger equation on $\Gamma_0$

$$i \frac{\partial g_e(\xi_e, \tau)}{\partial \tau} = \frac{1}{L^2(\tau)} \frac{\partial^2 g_e(\xi_e, \tau)}{\partial \xi_e^2} - \frac{V^2}{16T^2} g_e(\xi_e, \tau), \quad (2.27)$$

where the instantaneous operator is simply the Laplacian shifted by a constant. Moreover, since all edges are of equal length, the boundary conditions (2.23) and (2.24) are simply the standard Neumann–Kirchhoff condition.

Since the vertex conditions for this system are time independent, the given problem is separable. A straightforward computation using separation of variables gives that the solution of (2.27) admits the following series expansion on each edge:

$$g_e(\xi_e, \tau) = \sum_{n \in \mathbb{N}} a_n g_e^{(n)}(\xi_e) e^{i T \lambda_n \frac{\tau}{T}} e^{i V^2 \frac{\tau}{16T^2}}, \quad (2.28)$$

where $\lambda_n$ are the eigenvalues of the stationary graph $\Gamma_0$ imposed with the Neumann–Kirchhoff condition and $g_e^{(n)}$ are the associated eigenfunctions. It is thus enough to consider the time evolution for initial conditions of the form $g_e^{(n)}(\xi_e)$.

As a more concrete example, we consider the equilateral star graph with $E$ edges (see figure 2), parameterized so that central vertex corresponds to $\xi_e = 1/2$. The eigenvalues for this graph can be written explicitly as $\{\lambda_n\}_{n=0}^\infty \cup \{\alpha_n\}_{n=0}^\infty$, where

$$\lambda_n = \left(\frac{\pi}{2} + \pi n\right)^2 - \frac{V^2}{16T^2}, \quad \text{(Multiplicity } E - 1\text{)},$$

and

$$\alpha_n = \left(\frac{\pi}{2} + \pi n\right)^2 - \frac{V^2}{16T^2}, \quad \text{(Multiplicity 1)}.$$
The associated normalized eigenfunctions are given by

\[ g^{(\lambda_n)}_e (\xi) = \frac{1}{\sqrt{E}} \],

\[ g^{(\lambda_n)}_e (\xi) = \sqrt{\frac{2}{E}} \cos \left( n\pi \left( \frac{\xi}{2} + \frac{1}{2} \right) \right) \text{ for } n > 0, \]

\[ g^{(\alpha_e)}_e (\xi) = \sqrt{\frac{2}{E}} \sigma_e^{(n)} \cos \left( n\pi \left( \frac{\pi}{2} + \frac{\pi n}{2} \right) \left( \frac{\xi}{2} + \frac{1}{2} \right) \right), \]

where \( \sigma_e^{(n)} \in \{-1, 0, 1\} \) are chosen on each edge to give \( E - 1 \) linearly independent functions satisfying the vertex condition at the central vertex.

For convenience, we focus on the time evolution for the eigenstates \( g^{(\lambda_n)}_e \) with \( n > 0 \). We can compute the time evolution for these states on the time dependent star graph \( \Gamma \) by applying the inverse gauge transformation to (2.21) and unitary family \( U_t^{-1} \):

\[ \psi^{(\lambda_n)}_e (x, t) = \sqrt{\frac{2}{EL(t/T)}} \cos \left( \frac{x \pi}{L(t/T)} + \frac{1}{2} \right) e^{\frac{2}{EL(t/T)}} e^{\frac{x^2 \pi^2 v^2 t}{2EL(t/T)}}. \tag{2.29} \]

One can easily verify that \( \psi^{(\lambda_n)}_e \) indeed solves the Schrödinger equation (2.10), satisfies the magnetic boundary conditions (2.19) and (2.20), and that \( \| \psi^{(\lambda_n)}_e (x, t) \|_{L^2} = 1 \) for all \( t \). By writing the initial condition as a generalized Fourier series in the functions \( g^{(\lambda_n)}_e, g^{(\alpha_n)}_e \) on each edge, one may compute the time evolution of (2.10) for any initial condition.

Using (2.29), one can for instance compute the average kinetic energy of the states \( \psi^{(\lambda_n)}_e \) over time, as done in [17]:

\[ \langle K(t) \rangle_{\psi^{(\lambda_n)}} := \left\langle \psi^{(\lambda_n)}_e | H | \psi^{(\lambda_n)}_e \right\rangle (t) = \sum_{e \in E} \int_{-L_e/2}^{L_e/2} \left| \frac{\partial \psi^{(\lambda_n)}_e (x, t)}{\partial x} \right|^2 dx \]

\[ = \frac{2}{E L^3} \sum_{e \in E} \int_{-L_e/2}^{L_e/2} \frac{x^2 L^2}{4T^2} \cos^2 \left( n\pi \left( \frac{x}{L} + \frac{1}{2} \right) \right) + \pi^2 n^2 \sin^2 \left( n\pi \left( \frac{x}{L} + \frac{1}{2} \right) \right) dx \]

\[ = \frac{V^2}{2T^2} \left( \frac{1}{4\pi^2 n^2} + \frac{1}{24} \right) + \frac{\pi^2 n^2}{(L_0 + Vt/T)^2}, \quad (n > 0). \tag{2.30} \]

Namely, the average kinetic energy of the particle decreases over time, and approaches a constant which depends entirely on the growth rate of the edges, \( V \). One can show that the same result holds for all eigenstates, except for \( g^{(0)}_e \), whose average kinetic energy remains constant in time.

3. Generalizations and applications

3.1. Generalization of the results to time dependent vertex conditions

The analysis above focused on the standard Neumann–Kirchhoff vertex conditions. Nevertheless, one would generally like to consider various vertex conditions, such as the Dirichlet condition (considered in [17]), or the related \( \delta \) and \( \delta' \) vertex conditions (see [9,
We have seen in section 1 that in the case of the Dirichlet condition, there is no need to reformulate the problem so that it is well posed. Yet, for an arbitrary self-adjoint vertex condition which is valid for a stationary metric graph, one would need to make the appropriate adjustments when considering the moving quantum graph (as in the case of the Neumann–Kirchhoff condition).

In order to present these adjustments, we first give a brief reminder about how general self-adjoint vertex conditions are written for a stationary quantum graph. A more in-depth survey of the topic can be found in [9, section 1.4.1].

Given a Sobolev function \( f \in H^2(\Gamma) \) and a vertex \( v \in V \), we denote by \( F(v) \) the vector of boundary values of \( f \) at the vertex \( v \) along the edges adjacent to \( v \):

\[
F(v) := (f_1(v), \ldots, f_d(v))^T.
\]

Similarly, we denote the vector of (outwards pointing) derivatives of \( f \) at \( v \) by \( F'(v) \).

**Theorem 3.1** ([9, 15]). Let \( \Gamma \) be a compact metric graph. Consider the operator \( \Delta = -\frac{d^2}{dx^2} \) with domain consisting of all functions \( f \in H^2(\Gamma) \) satisfying certain local conditions at each vertex. Then \( \Delta \) is self-adjoint if and only if for every vertex \( v \) of degree \( d_v \) there exist \( d_v \times d_v \) matrices \( A_v \) and \( B_v \) such that

1. The \( d_v \times 2d_v \) matrix \( (A_v | B_v) \) is of maximal rank,
2. The matrix \( A_vB_v^* \) is self-adjoint,
3. Each \( f \in \text{Dom}(\Delta) \) satisfies

\[
A_vF(v) + B_vF'(v) = 0.
\]

In other words, the possible vertex conditions of a stationary quantum graph are parameterized by the pair \( (A_v, B_v) \) at each vertex \( v \).

In general, it would be natural to associate with the moving graph \( \Gamma \) and each \( v \in V \) a continuous family \( (A_v(t), B_v(t))_{t \in [0, T]} \) which satisfies conditions 1–3 above. This would give a family of time varying vertex conditions associated with the time dependent metric graph. Yet, this does not in general provide unitary time evolution (as in the case of the Neumann–Kirchhoff condition). This problem arises once again due to the fact that the changing edges of the graph induce a magnetic potential as seen in equation (2.22), and the associated pair \( (A_v(t), B_v(t)) \) might not render the given magnetic operator self-adjoint. To solve this, we employ the following standard result:

**Theorem 3.2** ([9, theorem 2.6.1]). Let \( \Gamma \) be a quantum graph with a magnetic potential \( \mathcal{A} \). Assume that the magnetic flux of \( \mathcal{A} \) through \( \Gamma \) vanishes. Then for any pair \( (A_v, B_v) \) as in theorem 3.1, the magnetic Laplacian \( -\frac{d^2}{dx^2} - i\mathcal{A}(x) \) is self-adjoint when the graph is endowed with the vertex conditions

\[
A_vF(v) + B_vDF(v) = 0,
\]

where \( DF(v) \) is the vector of magnetic derivatives of \( f \) at \( v \):

\[
Df_e(x) = \left( \frac{\partial}{\partial x_e} - i\mathcal{A}_e(x_e) \right)f_e(x_e).
\]
We can combine the two results above with the derivation in section 2. Motivated by our family of unitaries \( U_t \), we define the modified boundary vectors \( W(v) \), \( W'(v) \) \( \in \mathbb{C}^d \):

\[
W(v) := \left( L_1^{1/2} f_1(v), \ldots, L_d^{1/2} f_d(v) \right)',
\]
\[
W'(v) := \left( L_1^{3/2} f_1(v), \ldots, L_d^{3/2} f_d(v) \right)'.
\]

In this new representation, the condition \( A_v F + B_v F' \) translates into

\[
A_v L_v^{-1/2} W(v) + B_v L_v^{-3/2} W'(v) = 0,
\]
\[
L_v := \text{diag} (L_1, \ldots, L_d).
\]

Then combining the results above immediately provides the following:

**Proposition 3.3.** Let \( \Gamma \) be a time-dependent metric graph. Let \( (A_v(t/T), B_v(t/T))_{t \in [0,T]} \) be a given family of pairs of matrices which satisfy that for all \( t, v \):

1. The \( d_v \times 2d_v \) matrix \( (A_v(t/T)|B_v(t/T)) \) has maximal rank,
2. The matrix \( A_v B_v \) is self-adjoint.

Then for all \( \tau := \frac{t}{T} \in [0,1] \), the following operator on \( L^2(\Gamma_0) \) is self-adjoint:

\[
H_\tau = \frac{1}{L_\tau(\tau)^2} \left\{ \left( \frac{\partial}{\partial \xi_\tau} - i \xi_\tau \frac{\hat{L}_\tau(\tau)}{2T} \right)^2 - \xi_\tau^2 \left( \frac{L_\tau(\tau)}{2T} \right)^2 \right\} \omega_\tau,
\]

\[
\text{Dom}(H_\tau) = \{ \omega \in H^2(\Gamma_0) : A_v(\tau) L_\tau^{-1/2} W(v) + B_v(\tau) L_\tau^{-3/2} D_\tau W(v) = 0 \},
\]

where

\[
W(v) := (\omega_1(v), \ldots, \omega_d(v))',
\]
\[
D_\tau W(v) := (D_\tau \omega_1(v), \ldots, D_\tau \omega_d(v))',
\]

and \( D_\tau \) is the magnetic derivative:

\[
D_\tau \omega_\tau(\xi_\tau) := \left( \frac{\partial}{\partial \xi_\tau} - i \xi_\tau \frac{\hat{L}_\tau(\tau)}{2T} \right) \omega_\tau(\xi_\tau).
\]

We are now in position to prove the following generalization of theorem 2.1:

**Theorem 3.4.** Let \( \Gamma \) be a time-dependent quantum graph, along with a family of matrices \( (A_v(t/T), B_v(t/T))_{t \in [0,T]} \) as in proposition 3.3. Assume that

1. The map \( t \mapsto (A_v(t/T), B_v(t/T)) \) is \( \mathbb{C}^2 \) for all \( t, v \),
2. For each \( v \in V \), the kernel of \( B_v(t/T) \) is independent of \( t \).

Then with the operator \( H_\tau \) from proposition 3.3, the Schrödinger equation on \( \Gamma_0 \)

\[
i \frac{\partial g}{\partial \tau} = H_\tau g
\]

(3.14)
induces a unitary flow on \( L^2(\Gamma_0) \), which allows to define via the unitary family \( U_t \) in (2.14) and (2.15) a solution to the Schrödinger equation on \( \Gamma \):

\[
\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2},
\]

\[
A_v(t/\Gamma) \psi(\tau) + B_v(t/\Gamma) D\psi(\tau) = 0, \quad \forall \psi \in \mathcal{V},
\]

where

\[
D_v \psi_c(x_c) := \left( \frac{\partial}{\partial x_c} - i\xi_c \frac{L_c}{2T} \right) \psi_c(x_c).
\]

**Proof.** The argument is similar to the one presented in [13], where additional details may be found. We outline the idea of the proof, focusing on the relevant changes that should be made for our case. We apply theorem 8.1 in [14]. By [9, theorem 1.4.11], the quadratic form associated with \( H_\tau \) is given by

\[
Q_\tau [\omega] = \sum_{c \in \mathcal{E}} \int_{-1/2}^{1/2} \frac{1}{L_c(\tau)} \left| \frac{\partial \omega_c}{\partial \xi_c} - i\xi_c \frac{L_c(\tau) L_c(\tau)}{2T} \omega_c \right|^2 d\xi_c \\
+ \sum_{c \in \mathcal{E}} \left( \frac{L_c(\tau)}{2T} \right)^2 \xi_c^2 |\omega_c|^2 d\xi_c \\
+ \sum_{v \in \mathcal{V}} \left\langle \Lambda_v P_{R,v} L_v^{1/2} W(v), P_{R,v} L_v^{1/2} W(v) \right\rangle,
\]

with domain

\[
\text{Dom}(Q_\tau) = \left\{ \omega \in H^1(\Gamma_0) | P_{D,v} L_v^{1/2} W(v) = 0 \right\},
\]

where \( P_{D,v} \) is the orthogonal projection onto \( \ker(B_v(\tau)) \), and \( P_{R,v}, \Lambda_v \) are matrices which depend smoothly on \( A(\tau), B(\tau) \).

Due to proposition 3.3, the operator \( H_\tau \) is self-adjoint for all \( \tau \). Combining condition (1) with equation (3.18) yields that the map \( \tau \mapsto Q_\tau[f] \) is \( C^2 \) for all \( f \). Furthermore, since \( P_{D,v} \) is the orthogonal projection onto \( \ker(B_v) \), assumption (2) ensures that the domain of \( Q_\tau \) is independent of \( \tau \). Lastly, a straightforward computation shows that

\[
Q_\tau [\omega] \geq b \| \omega \|_{W^1(\Gamma_0)}^2 - c \| \omega \|_{L^2(\Gamma_0)}^2,
\]

for some \( b > 0, c \in \mathbb{R} \). Under these conditions, [14] shows that the Schrödinger equation (3.14) induces a unitary flow on \( L^2(\Gamma_0) \). Composing this flow with the unitary family \( U_t \) induces a unitary flow on \( L^2(\Gamma) \), which due to the computations above, provides a solution to the Schrödinger equation (3.15).

\[
\square
\]

Note that one may employ the same gauge transformation as in section 2 to bring the Schrödinger equation to the simpler form (2.22), along with the non-magnetic boundary conditions

\[
A_v(\tau) L_v^{-1/2} W(v) + B_v(\tau) L_v^{-3/2} W'(v) = 0, \quad \forall \psi \in \mathcal{V}.
\]

As before, in the adiabatic limit, this equation may be solved spectrally, by applying the scattering formalism with the appropriate scattering matrix.
Figure 3. The first four eigenvalues of the Robin interval as a function of $t/T$. As $t/T \to 1$, two of the eigenvalue curves approach $-\infty$.

Remark 3.5. We would like to illustrate the possible importance of the condition regarding $\ker(B)$ in theorem 3.4. To do so, we consider the stationary quantum graph given by the interval $[0, 1]$, equipped with the time dependent Robin condition at both endpoints $v \in \{0, 1\}$:

$$\sin \left( \frac{\pi}{2T} f(v) \right) = -\cos \left( \frac{\pi}{2T} f'(v) \right), \quad t \in [0, T].$$  \hspace{1cm} (3.22)

We focus on the adiabatic case, where the boundary conditions are slowly varied from the Neumann condition at $t = 0$ to the Dirichlet condition at $t = T$. While the associated family $(A(t/T), B(t/T))$ is analytic, it is evident that $\ker(B(t/T))$ changes at $t = T$. Consequently, one can show that the first two eigenvalues $\lambda_1(t), \lambda_2(t)$ approach the value $-\infty$ as $t \to T$ (see figure 3). While we cannot explicitly compute the time evolution for this system in the non-adiabatic case, this unusual spectral behavior hints that the time evolution for the associated states may display an abrupt change of behavior for $t > T$.

The phenomenon of eigenvalues approaching $-\infty$ as $\ker(B(t/T))$ changes has appeared from different perspectives in several recent and ongoing works. This phenomenon can be identified not only through $\ker(B(t/T))$, but also using quantities such as the spectral flow (see [4, 21]) and the Duistermaat index, as shown by Berkolaiko et al [7].

3.2. Application—the geometric phase

We apply the theory above to show the existence of a geometric phase for slowly varying quantum graphs, similar to the one proposed by Berry [11]. There, Berry considered time evolution of the Schrödinger equation in systems with a time dependent Hamiltonian $H(R(t))$, where the time dependent parameter $R(t) \in \mathbb{R}^n$ is assumed to change adiabatically. He then showed that if the system is initially prepared at an eigenstate $\psi_n(x, 0)$ of $H(R(0))$, then at the adiabatic limit, the time evolution is given by

$$\psi(x, t) = \exp \left( -\frac{i}{\hbar} \int_0^t E_n(R(t')) \, dt' \right) \exp (i\gamma_n(t)) \psi_n(x, 0),$$  \hspace{1cm} (3.23)
where $E_n(R(t))$ is the $n$th eigenvalue of $H(R(t))$. The first exponential term is known as the dynamical phase, and it is a direct analog of the phase acquired by solutions of the Schrödinger equation in time independent systems. The second exponential term, known as the geometric phase (or Berry phase), is given by

$$
\gamma_n(t) = i \int_0^t \langle \psi_n(R(t')) | \nabla_R \psi_n(R(t')) \rangle \cdot \dot{R}(t') \, dt',
$$

where $\psi_n(R(t))$ is the eigenfunction associated with $E_n(R(t))$.

Since its discovery in 1982, the geometric phase has been the subject of many works, and has been experimentally measured in various settings. In [20], Simon showed that if the adiabatic time evolution of the eigenstate is considered as parallel transport in the associated space of parameters, then the geometric phase can be thought of as the corresponding holonomy.

The original derivation by Berry was given for systems with time dependent Hamiltonians on a fixed Hilbert space. This does not exactly fit the time dependence in our setting, where the geometry itself of the system changes in time, causing the Hilbert space to change. Nevertheless, theorem 3.4 shows that the time evolution for our system is equivalent to that of a stationary metric graph with a time dependent Hamiltonian. Utilizing this equivalence, one can repeat the original derivation by Berry to conclude that (3.23) holds for time dependent quantum graphs as well. Namely, for the Schrödinger equation (3.15) on a graph with a time dependent vector of edge lengths $L$, we have the adiabatic result as $T \to \infty$:

$$
\gamma_n(t) = i \int_0^t \langle \psi_n(L(t')) | \nabla_L \psi_n(L(t')) \rangle \cdot \dot{L}(t') \, dt'.
$$

Note that differentiation of the eigenvalues and eigenfunctions with respect to the edge lengths is possible due to [8], under the assumption that the spectrum of $H(L(t))$ is simple along the entire path. In general, the time dependence of the system can also be manifested by parametrically changing the boundary conditions in time (as in theorem 3.4).

Since the potential appearing in (2.22) is real valued, our Hamiltonian possesses time reversal symmetry, and the eigenfunctions of $H(L(t))$ all remain real when parallel transported along the path $L(t)$. Thus, by [20], the geometric phase must be an integer multiple of $\pi$ when the trajectory in parameter space $L(t)$ is taken to be closed. For instance, this happens if the graph $\Gamma$ changes periodically, i.e. $L(t + T) = L(t)$ for some $T > 0$.

Let us consider the simplest nontrivial example, of a star graph with three edges of varying length, where the system is initially prepared at the first non-constant eigenstate, $\psi_2$. The parameter space in this case is $\mathbb{R}^3_+ \setminus \ell$, with the diagonal line $\ell := \text{span} \{(1,1,1)\}$ removed (since the eigenstate $\psi_2$ is degenerate along this line). Due to the invariance of the geometric phase under homotopy, it is enough to compute the geometric phase along a single cycle which winds once around $\ell$. For instance, we can choose some circle around the line $\ell$ which is contained in the plane $L_1 + L_2 + L_3 = 1$. Conveniently, due to the constraint $L_1 + L_2 + L_3 = 1$, the Hamiltonian $H$ can be thought of as depending on only two parameters along the path (say, $L_1$ and $L_2$). Our cycle then encloses the isolated conical point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ on our plane. We may then apply the result proven in [6]:

**Theorem 3.6.** Let $H(\vec{L})$ be a self-adjoint operator which depends analytically on two parameters $\vec{L} = (L_1, L_2)$, with a non-degenerate conical point at some point $\vec{L}_0$. Assume that $H(\vec{L})$ commutes with some anti-unitary involution $V$, i.e.

$$
V(\alpha f) = \pi V(f), \quad V^2 = 1, \quad \langle Vf, Vg \rangle = \langle f, g \rangle.
$$

(3.26)
Then the geometric phase along a cycle enclosing the singularity $\tilde{L}_0$ is $\pi$ modulo $2\pi$.

By applying the theorem above on $H_\tau$ along with the anti-unitary involution $V\psi(x) = \overline{\psi(x)}$, we obtain the following:

**Corollary 3.7.** The geometric phase acquired by $\psi_2$ along a closed cycle $\mathcal{L}(t)$ is equal to $\pi$ times the winding number of $\mathcal{L}(t)$ around the degenerate line $\ell$.

More generally, for any eigenstate $\psi_i$ of the star graph there exists a finite collection $\{\ell_j\}_{j=1}^N$ of lines through the origin along which $\psi_i$ is degenerate. In this case, the same argument shows that the geometric phase will be equal to $\pi$ times the sum of winding numbers of the path around the given lines. For an arbitrary graph, the computation of the geometric phase is more complicated, since the subset $D$ of the parameter space $\mathbb{R}_+^{|E|}$ such that $\psi_i$ is degenerate might be substantially harder to compute (by Wigner and Von-Neumann, it is generically a subset of codimension two). As before, the possible values for the geometric phase will be determined by the topology of the space $\mathbb{R}_+^{|E|} - D$.

We shall mention that an equivalent computation of the geometric phase can be obtained by replacing the parameter space $\mathbb{R}_+^{|E|}$ with a different one, known as the secular manifold. The secular manifold can be thought of as a quotient of our original parameter space, which takes into account the invariance of our system under scaling of the edges (for a thorough introduction to the secular manifold, see [2, 3]). Such a computation for a star graph was recently done by Alon [1] independently of the authors. The advantage of this approach is that it reduces the computation of the geometric phase to that of a single monodromy for all eigenstates $\psi_i$. The drawback of this approach is that the given path that the edge lengths complete in parameter space is harder to express.

**Data availability statement**

No new data were created or analysed in this study.

**Acknowledgments**

We are grateful to Professor Michael Berry for explaining the physical origin of the magnetic phase. Thanks are due to Professor Kirill Cherednichenko for his continuous interest, and for pointing out the similarity between the boundary conditions in this work and the ones corresponding to graphs consisting of composite conductors. We also thank Lior Alon for very helpful discussions which helped understanding the geometric phase, and to Gregory Berkolaiko for extremely helpful remarks which corrected and improved this work. We also appreciate the important and useful comments made by the referees. US thanks the University of Bath for the hospitality while holding a David Parkin Professorship (September 2021–March 2022), and GS thanks the Weizmann Institute for the hospitality while visiting during 2022–2023.

**Appendix. Secular equation for the instantaneous graph**

We now derive the secular equation which provides the eigenvalues and eigenfunctions of the instantaneous operator, and in turn allows to solve for the time evolution at the adiabatic limit.

To solve for the instantaneous spectrum, we write our eigenfunctions on each edge as

$$g_e(\xi, \tau) = a_e(\tau) e^{ikL_e(\tau)(1/2 + \xi)} + \hat{a}_e(\tau) e^{ikL_e(\tau)(1/2 - \xi)}.$$  \hspace{1cm} (A.1)
Using the modified vertex conditions (2.23) and (2.24), straightforward algebraic manipulation (see [5] for further details) then shows that the scattering coefficients satisfy

\[ a_e(\tau) = \frac{2L_e^{1/2}(\tau)}{d_e} \sum_{j \in S_e} \frac{1}{2} e^{i k L_j(\tau)} a_j(\tau) - e^{i k L_e(\tau)} a_e(\tau). \]  

(A.2)

We can equivalently write this in terms of the bond scattering matrix. We say that the directed edge \( j' \) follows the directed edge \( j \) if the starting vertex of \( j' \) is the end vertex of \( j \). With this, we define the instantaneous bond scattering matrix as

\[
S_{j'j}(\tau) := \begin{cases} 
\frac{2}{\pi} \left( \frac{L_j(\tau)}{L_j'(\tau)} \right)^{1/2} & j' = j, \\
\frac{2}{\pi} \left( \frac{L_j(\tau)}{L'_j(\tau)} \right)^{1/2} & j' \text{ follows } j \text{ and } j' \neq j, \\
0, & \text{otherwise}
\end{cases}
\]

The relation (A.2) then translates into

\[
\left( I - S(\tau) e^{ikL(\tau)} \right) \vec{a} = 0,
\]

(A.3)

where \( L(\tau) := \text{diag} \left( L_1(\tau), L_1(\tau), L_2(\tau), L_2(\tau), \ldots, L_{|E|}(\tau), L_{|E|}(\tau) \right) \) is the diagonal matrix of edge lengths and \( \vec{a} := (a_1, a_1, \ldots, a_{|E|}, a_{|E|}) \) is the vector of scattering coefficients.

We now see that \( k^2 > 0 \) is an eigenvalue of the instantaneous problem if and only if it is a root of the secular equation:

\[
\det \left( I - S(\tau) e^{ikL(\tau)} \right) = 0.
\]

(A.4)

Thus, one may obtain the instantaneous spectrum in the adiabatic limit by solving the instantaneous secular equation for each \( \tau \). The coefficients for the associated eigenfunctions may be then solved from equation (A.3).

Note that unlike many standard cases, here the bond scattering matrix \( S(\tau) \) is not unitary, since the edges of the graph were scaled, giving a Laplacian with different (length dependent) weights on each edge. By defining the matrix \( P_{j'j}(\tau) = \frac{1}{\sqrt{L_j(\tau)}} \delta_{j'j} \), one can see that \( S(\tau) = P^{-1}SP \), where \( S \) is the unitary bond scattering matrix of the time dependent metric graph \( \Gamma \) at time \( \tau \). Thus, although \( S \) is not unitary, it is similar to the unitary bond scattering matrix of the non-weighted Neumann–Kirchhoff Laplacian, which keeps the secular equation invariant.

**ORCID iD**

Gilad Sofer © https://orcid.org/0000-0003-0864-7557

**References**

[1] Alon L 2022 private communication
[2] Alon L 2020 Quantum graphs—generic eigenfunctions and their nodal count and Neumann count statistics *PhD Thesis* Mathematics Department, Technion—Israel Institute of Technology
[3] Alon L, Band R and Berkolaiko G 2018 Nodal statistics on quantum graphs *Commun. Math. Phys.* **362** 909–48
[4] Band R, Prokhorova M and Sofer G Spectral flow and the generalized nodal deficiency (in preparation)

[5] Berkolaiko G 2017 An elementary introduction to quantum graphs Geometric and Computational Spectral Theory (Contemporary Mathematics vol 700) (American Mathematical Society) pp 41–72

[6] Berkolaiko G and Comech A 2018 Symmetry and Dirac points in graphene spectrum J. Spectr. Theory 8 1099–147

[7] Berkolaiko G, Cox G, Latushkin Y and Sukhtaiev S 2023 The duistermaat index and eigenvalue interlacing for self-adjoint extensions of a symmetric operator (arXiv:2311.06701)

[8] Berkolaiko G and Kuchment P 2012 Dependence of the spectrum of a quantum graph on vertex conditions and edge lengths Spectral Geometry (Proc. Symp. in Pure Mathematics vol 84) (American Mathematical Society) (arXiv:1008.0369)

[9] Berkolaiko G and Kuchment P 2013 Introduction to Quantum Graphs (AMS Mathematical Surveys and Monographs vol 186) (American Mathematical Society)

[10] Berry M V 2022 private communication

[11] Berry M V 1984 Quantal phase factors accompanying adiabatic changes Proc. R. Soc. A 392 45–57

[12] Doescher S W and Rice M H 1969 Infinite square-well potential with a moving wall Am. J. Phys. 37 1246–9

[13] Duca A and Joly R 2021 Schrödinger equation in moving domains Ann. Henri Poincare 22 2029–63

[14] Kisynski J 1964 Sur les operateurs de Green des problems de Cauchy abstraits Stud. Math. 23 285–328

[15] Kostrykin V and Schrader R 1999 Kirchhoff’s rule for quantum wires J. Phys. A: Math. Gen. 32 595–630

[16] Kottos T and Smilansky U 2000 Chaotic scattering on graphs Phys. Rev. Lett. 85 968–71

[17] Matrasulov D U, Yusupov J R, Sabirov K K and Sobirov Z A 2015 Time-dependent quantum graph Nanosyst. Phys. Chem. Math. 6 173–81

[18] Moore G T 1970 Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity J. Math. Phys. 11 2679

[19] Munier A, Burgan J R, Feix M and Fijalkow E 1981 Schrödinger equation with time-dependent boundary conditions J. Math. Phys. 22 1219–23

[20] Simon B 1983 Holonomy, the quantum adiabatic theorem and Berry’s phase Phys. Rev. Lett. 51 2167–70

[21] Sofer G 2022 Spectral curves of quantum graphs with δs type vertex conditions Master’s Thesis Technion—Israel Institute of Technology