Stochastic aspects of hysteresis

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Abstract. A review of recent advances in the area of hysteretic nonlinearities driven by diffusion processes is presented. The analysis of these systems is based on the Preisach formalism for the description of hysteresis to represent complex nonlinearities as a weighted superposition of rectangular loops. The mathematical theory of diffusion processes on graphs is then applied to solve problems for stochastically driven hysteresis loops. Closed form expressions for the expected value and spectral density of the output are obtained, and sample computations for these quantities are presented. Because of the universality of the Preisach model, this approach can be used to investigate stochastic aspects in hysteretic systems of various physical origins.

1. Introduction

The phenomenon of hysteresis has been with us for ages and has attracted the attention of many researchers for a long time. However, the systematic study of hysteresis was only recently attempted and it led to the appearance of the monographs [1-4]. Since then, interest in this topic has been continuously growing and it has extended far beyond the classical areas of magnetism and plasticity. For example, optical hysteresis [5], superconducting hysteresis [6] and economic hysteresis [7] are well-established scientific domains and many pioneering studies have appeared in biology [8, 9], psychology [10, 11], and computer science [12, 13].

The physical origin of hysteresis is due to the multiplicity of metastable states exhibited by hysteretic systems. A physical system can persist in a metastable state for some time, but thermal perturbations usually drive the system to more stable nearby states. In magnetism, these thermally activated relaxations are commonly called “after-effect” or “viscosity”, while in the area of superconductivity they are known as “creep”. Therefore, the behavior of a hysteretic system could be described as a nonlinear hysteretic transformation of a stochastic input that consists of a random internal noise superimposed on a deterministic external input (Figure 1, case A). In other areas such as economics [7], computer science [12, 13] or wireless communication [14], the external input to a hysteretic system is considered as a stochastic process due to external noise (Figure 1, Case B) or its random nature (Figure 1, Case C). Regardless of the reasons that lead to such models, the mathematical study of hysteretic systems driven by stochastic inputs is of relevance to all the previously mentioned areas. This point is also supported by the universality of the Preisach model that has been used to describe hysteretic nonlinearities of various physical origins. Furthermore, the study of multi-valued nonlinear systems with stochastic inputs is still almost virgin territory, being of considerable mathematical complexity [15]. Therefore, the techniques used to study the properties of these systems are of interest in their own right.
Preisach-type models with stochastic input were introduced by Mayergoyz and Korman to offer a unified and detailed description of hysteresis and “after-effect” in magnetic materials [16-19]. Moreover, it has been shown that this approach can be applied successfully to the study of “creep” phenomena in type II superconductors [20]. Key computations in these viscosity models are based on the relation between randomly induced switchings of rectangular loops and the exit problem for stochastic processes, which is a well studied problem in the theory of diffusion processes. Later, another technique for these computations was discovered which uses the recently developed theory of diffusion processes on graphs [21] developed by Freidlin and Wentzell. This theory was first applied to the study of random perturbations of Hamiltonian dynamical systems [21, 22]. Then it was realized that this mathematical technique is naturally suited to the analysis of noise in hysteretic systems [23-26].

The article is organized as follows. In Section 2, a general discussion of the Preisach model driven by a diffusion process is presented. In Section 3, the technique of diffusion processes on graphs is applied to compute the output characteristics for such systems. Sample results of the computations are presented in Section 4. Finally, conclusions are drawn in Section 5.

Figure 1. Schematic representation of various origins of the stochastic input in hysteretic systems: internal noise found in magnetic (“after-effect” phenomena) and superconductor materials (“creep” phenomena); external noise found in wireless communication systems (“hand-off” problem); or pure stochastic nature of the input found in computer science (“multi-server threshold queues” problem) and economics.

2. Preisach systems driven by stochastic input
The Preisach model has been extensively used for the description of hysteresis of various physical types such as magnetic hysteresis, superconductor hysteresis, mechanical hysteresis of consolidated granular materials, hysteresis of shape-memory alloys, and piezoceramics, etc. This clearly reveals the physical universality of the Preisach model. It has been proved that wiping-out and congruency properties are necessary and sufficient conditions for the representation of actual hysteretic nonlinearities by Preisach’s model [2]. Such complex hysteretic nonlinearities can be modeled through the Preisach formalism as weighted superpositions of rectangular loops.
This can be mathematically described as follows:

\[ h(t) = \int_{\alpha \geq \beta} \mu(\alpha, \beta) \gamma_{\alpha\beta} x(t) d\alpha d\beta. \]

Here, \( \mu(\alpha, \beta) \) is a Preisach distribution function, while \( \gamma_{\alpha\beta} \) are rectangular loop operators shown in Figure 2.

It has also been realized that the Preisach model driven by a stochastic input \( x(t) \) is an effective model for thermally activated relaxations commonly known as “after-effect” in magnetism and “creep” in the area of superconductivity. The universality of this approach has made it naturally suitable for applications in the fields of mechanical and structural engineering where the dynamic loading acting on hysteretic systems is usually random in nature. It is expected that other areas where the input has a stochastic nature (such as communications and economics) will benefit from this research direction.

The input process \( x(t) \) is assumed to be described by the Ito stochastic differential equation:

\[ dx(t) = b(x(t)) dt + \sigma(x(t)) dW(t) \]  

with initial condition \( x(0) = x_0 \). Here \( W(t) \) is the Wiener process, while \( b \) and \( \sigma \) are known functions that satisfy a local Lipshitz and linear growth condition [27, 28]. These standard conditions ensure the existence of a nonexploding, unique solution of the equation (1) that satisfies the initial condition.

The output of the rectangular loop operator can be written mathematically in the following form:

\[ i_{\alpha\beta}(t) = \gamma_{\alpha\beta} x(t) = \begin{cases} 1 & \text{if } x(t) > \alpha \\ -1 & \text{if } x(t) < \beta \\ 1 & \text{if } x(t) \in (\beta, \alpha) \text{ and } x(\tau(t)) = \alpha \\ -1 & \text{if } x(t) \in (\beta, \alpha) \text{ and } x(\tau(t)) = \beta, \end{cases} \]

where \( \tau(t) \) is the value of time at which the last threshold (\( \alpha \) or \( \beta \)) was attained, and \( x(t) \) is given by (1). The stochastic nature of the input leads to random switchings of the rectangular loop operators, and therefore, the output of the Preisach model \( h(t) \) is a stochastic process as well. This output can be constructed as a “weighted superposition” of rectangular loop operators that are individually driven by the same stochastic process.

The expected value of the output process can be written as:

\[ E(h(t)) = \int_{\alpha \geq \beta} \mu(\alpha, \beta) E\{\gamma_{\alpha\beta} x(t)\} d\alpha d\beta, \]  

and the second moment reads:

\[ E(h^2(t)) = \int_{\alpha' \geq \beta'} \int_{\alpha \geq \beta} \mu(\alpha, \beta) \mu(\alpha', \beta') E\{\gamma_{\alpha\beta} x(t) \gamma_{\alpha'\beta'} x(t)\} d\alpha d\beta d\alpha' d\beta'. \]

To derive the formula for the spectral density of the output process we observe that the autocorrelation function of the Preisach model can be expressed as a weighted superposition of cross-correlation functions of two-dimensional processes \( i_{\alpha\beta}(t), i_{\alpha'\beta'}(t) \), representing the outputs of two rectangular loops:

\[ C_h(\tau) = \int_{\alpha' \geq \beta'} \int_{\alpha \geq \beta} \mu(\alpha, \beta) \mu(\alpha', \beta') E\{\gamma_{\alpha\beta} x(0) \gamma_{\alpha'\beta'} x(\tau)\} d\alpha d\beta d\alpha' d\beta'. \]

Although the cross-correlation functions \( E\{\gamma_{\alpha\beta} x(0) \gamma_{\alpha'\beta'} x(\tau)\} \) are not even functions, we have the following relation:

\[ E\{\gamma_{\alpha\beta} x(0) \gamma_{\alpha'\beta'} x(\tau)\} = E\{\gamma_{\alpha\beta} x(-\tau) \gamma_{\alpha'\beta'} x(0)\}. \]

This implies that the
correlation function for the Preisach system \( C_h(\tau) \) is an even function. According to the Wiener-Khinchine theorem the process’s spectral density is the Fourier transform of the autocorrelation function. Since we deal with an even correlation function, the spectral density of the output process can be expressed by using formula (4) in the following form:

\[
S_h(\omega) = \int\int_{\alpha \geq \beta} \mu(\alpha, \beta)\mu(\alpha', \beta')S_{\alpha\beta\alpha'\beta'}(\omega)d\alpha d\beta d\alpha' d\beta',
\]

where \( S_{\alpha\beta\alpha'\beta'} \) is the “cross-spectral density” for the two-dimensional process \( (i_{\alpha\beta}(t), x(\tau)) \) and it is related to the cross-correlation function \( E\{\hat{\gamma}_{\alpha\beta} x(0)\hat{\gamma}_{\alpha'\beta'} x(\tau)\} \) as follows:

\[
S_{\alpha\beta\alpha'\beta'}(\omega) = 2\text{Re} \left\{ \int_0^\infty E\{\hat{\gamma}_{\alpha\beta} x(0)\hat{\gamma}_{\alpha'\beta'} x(\tau)\}e^{-j\omega}\,d\tau \right\}.
\]

The Preisach model describes hysteresis nonlinearities with non-local memories. For this reason, the output process \( h(t) \) can not be embedded as a component of some Markovian process. However, it is apparent from formulas (2)-(6) that the moments and the spectral densities for Preisach systems can be expressed as weighted superpositions of moments and spectral densities, respectively, for much simpler processes. These processes are still non-Markovian, but they can be embedded in higher dimensional Markovian processes. This fundamental property opens the door to the computation of stochastic characteristics for the outputs of hysteretic systems.

3. “The Diffusion processes on graphs” approach to stochastically driven hysteresis

According to the formula (2), the calculation of the expected value for the output processes of a Preisach system entails the calculation of the expected value for the output processes of the rectangular loop systems. The output \( i_{\alpha\beta} \) is a random binary process and it is not Markovian. However, the two component process \( y(t) = (i_{\alpha\beta}(t), x(t)) \) is Markovian. This is due to the local memory property of the rectangular loop operator, which means that joint specifications of current values of input and output uniquely define the states of this hysteresis. The process \( y(t) \) is defined on the four edge graph shown in Figure 3.

The process \( y(t) \) has no delay at the graph vertices, the probability for the process to move from the vertex \( V_1 \) along edge \( E_3 \) is zero, while the random motions along \( E_1 \) and \( E_2 \) are equally probable (an analog our situation happens in \( V_2 \)). As a consequence it can be proved (see [23])
that the probability density $\rho(y,t|y_0,0)$ satisfies, on each graph edge, the forward Kolmogorov equation:

$$\frac{\partial \rho(y,t|y_0,0)}{\partial t} + \hat{L}_x \rho(y,t|y_0,0) = 0,$$

(7)

where $\hat{L}_x$ is the second order operator associated with the input diffusion process defined in (1), and it is specified by the expression:

$$\hat{L}_x \rho = -\frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(x) \rho) - \frac{\partial}{\partial x} (b(x) \rho),$$

(8)

with $\sigma(x)$ and $b(x)$ being the diffusion and drift coefficients, respectively, of the input process $x(t)$. The function $\rho(y,t|y_0,0)$ satisfies the initial conditions:

$$\rho(y,0|y_0,0) = \delta_{\beta_i \alpha_i} \delta_{i,j_0} \delta(x,x_0),$$

(9)

and it has to decay to zero for $x$ going to infinity. In addition, the so-called "vertex" type boundary conditions at graph vertices have to be satisfied [23]-[24]. These "vertex" type boundary conditions express the continuity of the transition probability density when the transition from one graph edge occurs without switching of the rectangular loop, and the zero boundary condition is imposed on the third graph edge connected to this vertex. Moreover, the probability current has to be conserved at each vertex. For example, at vertex $V_1$, these conditions read:

$$\rho((-1,\beta^+),t|y_0) = \rho((-1,\beta^-),t|y_0), \quad \rho((1,\beta^+),t|y_0) = 0,$$

$$\frac{\partial \rho}{\partial x} ((1,\beta^+),t|y_0) + \frac{\partial \rho}{\partial x} ((-1,\beta^-),t|y_0) = \frac{\partial \rho}{\partial x} ((-1,\beta^-),t|y_0).$$

(10)

After solving the initial boundary value problem (7)-(10) for $\rho(y,t|y_0,0)$, the expected value for the output process of a rectangular loop operator $\gamma_{\alpha\beta}$ is simply given by:

$$E\{i_{\alpha\beta}(t)\} = 2 \text{Prob}\{i_{\alpha\beta}(t) = 1|y_0\} - 1 = 2 \int_{-\infty}^{\infty} \rho((1,x),t|y_0) dx - 1$$

The solution of the initial boundary value problem (7)-(10) can be found in terms of parabolic cylinder functions and their Laplace transforms in the case when $x(t)$ is the Ornstein-Uhlenbeck process ($dx(t) = -b(x(t) - x_0) dt + \sigma dW(t)$). The Ornstein-Uhlenbeck process is very appealing as a model of noise due to its stationary and Gaussian nature.

Figure 3. Four edge graph on which two component process $y$ is defined.

Next, the "diffusion processes on graphs" technique is used for the calculation of output spectral densities. For the sake of the clarity we restrict our discussion to the case of symmetrical
hysteretic loops. We denote the cross-spectral density of two symmetrical loops \( \hat{\gamma}_\alpha(-\alpha) \) and \( \hat{\gamma}_\beta(-\beta) \) by \( S_{\beta\alpha}(\omega) \) instead of \( S_{\beta(-\beta)\alpha(-\alpha)}(\omega) \). To compute \( S_{\beta\alpha}(\omega) \), let us consider the three component process \( z(t) = (i_\beta(t), i_\alpha(t), x(t)) \) defined on the graph shown in Figure 4. Because the joint specifications of current values of input \( x(t) \) and outputs \( i_\beta(t) \) and \( i_\alpha(t) \) uniquely define the state of this process, \( z(t) \) has the Markov property. On each edge of this graph, the transition probability density function of \( z(t) \) satisfies (see [26]) the following equation:

\[
\frac{\partial \rho(z(t|z_0,0)}{\partial t} + L_x \rho(z(t|z_0,0) = 0.
\]

The appropriate solution is subject to the obvious initial condition:

\[
\rho(z,0|z_0,0) = \delta_{i_\beta 0} \delta_{i_\alpha 0} \delta(x,x_0),
\]

and it has to decay to zero for \( x \) going to infinity. In addition, so-called “vertex” type boundary conditions are imposed at graph vertices \( x = \pm \alpha \) and \( x = \pm \beta \). These “vertex” type boundary conditions express the continuity of the transition probability function when the transition from one graph edge occurs without switching of the rectangular loop, and the zero boundary condition is imposed on the third graph edge connected to this vertex. Moreover, the probability current has to be conserved at each vertex.

Figure 4. Graph on which the three component process \( z \) is defined.

The stationary probability density function satisfies the boundary value problem:

\[
\left\{ \begin{array}{l}
L_x \rho_s(z) = 0 \text{ on each graph edge}, \\
"vertex" \text{ boundary conditions at each graph vertex}
\end{array} \right.
\]

By introducing the “effective” distribution function:

\[
g(z,\tau) = \int_{-\infty}^{\infty} \sum_{i_\alpha, i_\beta} i_{\alpha 0} \rho(z,\tau|z_0,0) \rho_s(z_0) dx_0
\]

and considering its half-line Fourier transform \( G(z,\omega) \), the spectral density \( S_{\beta\alpha}(\omega) \) can be written as:

\[
S_{\beta\alpha}(\omega) = 2\text{Re} \left\{ \int_{-\infty}^{\infty} \sum_{i_\alpha, i_\beta} i_\beta G(z,\omega) dx \right\}.
\]
Stationary expected values of $i_{\alpha(\cdot)}$ as functions of normalized expected value $\nu(= x_0/\alpha)$ of an Ornstein-Uhlenbeck input process $x(t)$, for various values of $\tilde{\alpha} = \alpha(\sqrt{b}/\sigma)$.

By using formulas (11)-(14) the following boundary-value problem for $G(z, \omega)$ can be derived:

$$\begin{align*}
&j\omega G(z, \omega) + \hat{L}_G(z, \omega) = i_\alpha \rho_s(z), \\
&\text{“vertex” - type boundary conditions}
\end{align*}$$

(16)

From equation (16) we can express $G(z, \omega)$ as $(j/\omega)(\hat{L}_G(z, \omega) - i_\alpha \rho_s(z))$. By substituting this expression into formula (15), the calculation of the integral in (15) can be avoided and $S_{\beta\alpha}(\omega)$ can be found in terms of only the derivatives of $G(z, \omega)$ at graph vertices.

In conclusion, by solving the boundary value problems (13) and (16) the cross-spectral densities $S_{\beta\alpha}(\omega)$ are found, and then the spectral noise density of a symmetric Preisach system is computed by using formula (5), adapted to the symmetric case.

4. Sample computations for the expected value and the spectral density

In Figure 5, sample computations for the stationary expected value $E_{i_{\alpha(\cdot)}}$ of the output process of a symmetric rectangular loop operator $\tilde{\gamma}_{\alpha(\cdot)}$ driven by the Ornstein-Uhlenbeck process:

$$dx(t) = -b(x(t) - x_0)dt + \sigma dW(t)$$

are plotted as functions of normalized values of the input expected value $\nu(= x_0/\alpha)$. The curves correspond to various normalized threshold values of $\tilde{\alpha} = \alpha/\lambda$, where $\lambda^2 = \sigma^2/b$ is the variance of the stationary distribution of the input process. The dependence of $E\{i_{\alpha(\cdot)}\}$ on $x_0$ can be interpreted in magnetics as an anhysteretic magnetization curve. This anhysteretic curve depends on the noise variance. This dependence is especially appreciable when the noise standard deviation $\lambda$ is comparable with the switching threshold value $\alpha$.

The spectral density of the output process for a symmetric rectangular loop operator (with $\alpha = 1$) driven by the Ornstein-Uhlenbeck process (with $b = \sigma = 1$) is presented in Figure 6 for various expected values $x_0$ of the input (the external applied magnetic field, for instance). It is apparent that when $x_0$ is increased, output signals “stabilize” around +1 and consequently, the spectral density is diminished.

Next, consider an Ornstein-Uhlenbeck input process with $b = \sigma = 1$ and $x_0 = 0$ (no external field applied). In Figure 7, the variation of the cross-spectral density $S_{\beta\alpha}(\omega)$ with respect to the half-widths $\beta$ and $\alpha$ of the two loops are presented for the frequency $\omega = 1$. The cross-spectral density has negligible values outside of a finite region around the origin and the calculations suggest that this region becomes smaller when the frequency is increased.

As has been stressed in the previous section, once $S_{\beta\alpha}(\omega)$ is computed, the calculation of the spectral noise density for any Preisach system is reduced to the integration of this function with
some specific weight related to Preisach distribution. In our discussion, only symmetric loops were considered, but the methods are applicable for non-symmetric loops as well.

5. Conclusions
A review of the recent advances in the area of hysteretic nonlinearities driven by diffusion processes was presented. The outputs of such systems are non-Markovian processes that represent nonlinear hysteretic transformations of diffusion processes. For this reason, the calculation of their output characteristics is associated with considerable mathematical difficulties. The methods summarized in this article circumvent these difficulties by using the
Preisach description of hysteretic systems as well as the recently developed theory of diffusion processes on graphs. In addition, special techniques are used to simplify the expression for the moments and spectral densities of the output process. The case of Ornstein-Uhlenbeck input noise was briefly discussed and samples of the computational results were presented. The Ornstein-Uhlenbeck input process is of special physical significance as a model of noise because of its stationary and Gaussian nature. Because of the universality of the Preisach model, this approach can be used to investigate stochastic aspects in hysteretic systems of various physical origins.

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