Supermodular $f$-divergences and bounds on lossy compression and generalization error with mutual $f$-information

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Abstract

In this paper, we introduce super-modular $f$-divergences and provide three applications for them: (i) we introduce Sanov’s upper bound on the tail probability of sum of independent random variables based on super-modular $f$-divergence and show that our generalized Sanov’s bound strictly improves over ordinary one, (ii) we consider the lossy compression problem which studies the set of achievable rates for a given distortion and code length. We extend the rate-distortion function using mutual $f$-information and provide new and strictly better bounds on achievable rates in the finite blocklength regime using super-modular $f$-divergences, and (iii) we provide a connection between the generalization error of algorithms with bounded input/output mutual $f$-information and a generalized rate-distortion problem. This connection allows us to bound the generalization error of learning algorithms using lower bounds on the rate-distortion function. Our bound is based on a new lower bound on the rate distortion function that (for some examples) strictly improves over previously best-known bounds. Moreover, super-modular $f$-divergences are utilized to reduce the dimension of the problem and obtain single-letter bounds.

1 Introduction

The generalized relative entropy $D_f(\cdot \| \cdot)$, also known as the $f$-divergence, was introduced by Ali-Silvey [1] and Csiszar [2, 3] as a measure of dissimilarity between the two distributions defined on the same sample space. $f$-divergences have found various applications in information theory, statistics and machine learning among other fields. Some applications and properties of $f$-divergence are given in [4–8].

In this paper we introduce a class of $f$-divergences that satisfy a supermodularity property. More specifically, given an arbitrary distribution $p_{X_1, \ldots, X_n}$ and an arbitrary product distribution $q_{X_1, \ldots, X_n} = \prod_{i=1}^n q_{X_i}$, we consider the function $g : 2^{[n]} \to \mathbb{R}$ defined as

$$g(A) = D_f(p_{X_A} \| q_{X_A}), \quad \forall A \subset [n],$$

where $X_A = (X_i : i \in A)$. We say that $f$-divergences satisfies a supermodularity property if $g(A)$ is a super-modular set function. We provide an explicit class of convex functions $f$ that lead to supermodular $f$-divergences.

Next, we give some applications of supermodular $f$-divergences as follows:

- We generalize the upper bound part of Sanov’s theorem [9] on the tail probability of the sum of random variables using supermodular $f$-divergences. We show that our extension
of Sanov’s bound based on some choice of supermodular $f$-divergence strictly improves over the ordinary Sanov’s upper bound in the non-asymptotic regime.

- $f$-divergences can be used to define a mutual $f$-information $I_f(X;Y)$ between two random variables $X$ and $Y$. Mutual $f$-information is a measure of dependence between two random variables and generalizes Shannon’s mutual information. We show that supermodular $f$-divergences imply that the resulting mutual $f$-information satisfies the following property: for any random variables $A, B, C$ we have
\[ I_f(A, B; C) \geq I_f(A; C) + I_f(B; C) \tag{1} \]
as long as $A$ and $B$ are independent. We call the above property the $AB$-property. The $AB$-property is stronger than the data processing inequality $I_f(AB; C) \geq I_f(A; C)$.

We continue by giving an explicit application of the $AB$-property. We note that in the converse proof of lossy source coding problem (also known as the rate-distortion problem), one can rewrite the proof in such a way that instead of using the chain-rule property of mutual information, we use the $AB$-property to complete the proof. This insight shows that we can mimic the standard proof and replace Shannon’s mutual information with $f$-information in all places in the proof. We define a notion of $f$-rate-distortion function and show that it can provide new and strictly better bounds on the achievable rates for a given distortion in the finite blocklength regime. We also extend a previous work by Ziv and Zakai in [5].

- It is known that under certain assumptions, the generalization error of a learning algorithm can be bounded from above in terms of the mutual information between the input and output of the algorithm [10, 11]. Similar bounds are obtained in [12–20] for various generalizations and extensions using other measures of dependence. In this paper we are interested in the generalization error for the class of algorithms with bounded input/output mutual $f$-information. For this class of algorithms, we give a novel connection between the $f$-rate-distortion function and the generalization error. Moreover, this leads to a new upper bound on the generalization error using the $f$-rate-distortion function that strictly improves over the previous bounds in [11, 12]. Finally, $f$-rate-distortion function defined using super-modular $f$-divergences enjoy certain properties that facilities its evaluation when the number of data samples is large.

As stated above, we provide a novel connection between generalization error and the rate-distortion theory. This connection allows us to strictly improves over the bound in [11]. In order to show that our rate-distortion bound strictly improves over the bound by Xu and Raginsky [11], we provide a new lower bound on the rate-distortion function (and on $f$-rate distortion in general). Not only this bound allows us to relate our bound to the bound by Xu and Raginsky, but it also strictly improves over the previously known bounds on the rate-distortion function in some cases [11,12].

We also give some variants of the rate-distortion bound in Section 6.1.1 and Appendix A that can tighten the ordinary rate-distortion bounds on the generalization error. In particular, in Section 6.1.1 we define the notion of the rate of consistency of algorithms and use it to add constraints to the rate-distortion bound. This is shown to yield a tight bound on the generalization error for the Gaussian mean estimation problem using the empirical risk minimization (ERM) algorithm.

This paper is organized as follows: in Section 2, we define supermodular $f$-divergence and discuss its application in Sanov’s theorem. In Section 3, we explore various definitions on
mutual f-information and their properties. Lossy compression with mutual f-information is discussed in Section 4. Lower bounds on (f-rate-distortion function and its connection with generalization error of learning with bounded input/output mutual f-information are discussed in Sections 5 and 6 respectively.²

1.1 Notation and preliminaries

Random variables are shown in capital letters, whereas their realizations are shown in lowercase letters. We show sets with calligraphic font. For a random variable \(X\) generated from a distribution \(\mu\), we use \(E_X\) or \(E_\mu\) to denote the expectation taken over \(X\) with distribution \(\mu\). \(P_Z\) means the distribution over \(Z\). We use \(\log_2(x)\) and \(\ln(x)\) to denote the logarithm in base two and in base \(e\) respectively.

2 f-divergence

The generalized relative entropy of Ali-Silvey [1] and Csiszar [2] [3] (also called the “f-divergence”) is defined as follows:

**Definition 1.** Let \(f : \mathbb{R}_+ \rightarrow \mathbb{R}\) be a convex function with \(f(1) = 0\). Let \(P\) and \(Q\) be two probability distributions on a measurable space \((\mathcal{X}, \sigma(\mathcal{X}))\). If \(P \ll Q\) then the f-divergence is defined as

\[
D_f(P \parallel Q) = E_Q \left( \frac{dP}{dQ}(X) f \left( \frac{dP}{dQ} \right) \right)
\]

where \(\frac{dP}{dQ}\) is a Radon-Nikodym derivative and \(f(0) \triangleq f(0^+)\).

Define the conditional f-divergence as follows:

\[
D_f(P_Y \mid X \parallel Q_Y \mid X \mid P_X) = E_{X \sim P_X} \left[ D_f(P_Y \mid X \mid Q_Y \mid X \mid P_X) \right],
\]

**Theorem 1.** (Properties of f-divergences) [22, Chap. 6]

- **Non-negativity:** \(D_f(P \parallel Q) \geq 0\) and equality holds if and only if \(P = Q\).
- **Joint convexity:** \((P, Q) \rightarrow D_f(P \parallel Q)\) is a jointly convex function. In particular, this property implies that conditioning does not decrease f-divergence: Let \(P_X \xrightarrow{P_Y \mid X} P_Y\) and \(P_X \xrightarrow{Q_Y \mid X} Q_Y\). Then

\[
D_f(P_Y \parallel Q_Y) \leq D_f(P_Y \mid X \parallel Q_Y \mid X \mid P_X).
\]

- **Data processing inequality:** Let \(P_X \xrightarrow{P_Y \mid X} P_Y\) and \(Q_X \xrightarrow{Q_Y \mid X} Q_Y\). Then

\[
D_f(P_Y \parallel Q_Y) \leq D_f(P_X \parallel Q_X).
\]

For the special case of \(f(t) = t \ln(t)\), \(D_f(\cdot \mid \cdot)\) reduces to the KL divergence. For the special case of \(f_\alpha(t) = t^\alpha - 1\) for \(\alpha \geq 1\), the \(f_\alpha\)-divergence can be written as

\[
D_{f_\alpha}(\mu \parallel \nu) := \int \left( \frac{d\mu}{d\nu} \right)^\alpha d\nu(x) - 1
\]

²Some parts of Section 6 were presented at 2021 IEEE International Symposium on Information Theory (ISIT) [21].

³We say \(\mu \ll \nu\), i.e., \(\mu\) is absolutely continuous with respect to \(\nu\) if \(\nu(A) = 0\) for some \(A \in \mathcal{X}\), then \(\mu(A) = 0\).
Renyi’s divergence of order \( \alpha \) can be derived by \( D_\alpha(\mu||\nu) := \frac{1}{\alpha-1} \ln(1 + D_{\alpha}(\mu||\nu)) \). In particular, we have \( D_2(\mu||\nu) = \ln (1 + \chi^2(\mu||\nu)) \) where \( \chi^2 - \text{divergence} \) is defined as

\[
\chi^2(\mu||\nu) = E_\nu \left( \frac{d\mu}{d\nu} - 1 \right)^2. \tag{2}
\]

2.1 Supermodular f-divergences

**Definition 2.** Given a convex function \( f \) with \( f(1) = 0 \), we say that the \( D_t \) is super-modular if for any joint distribution \( p_{X_1,X_2,X_3} \) and any product distribution \( q_{X_1,q_{X_2},q_{X_3}} \) on arbitrary alphabets \( X_1, X_2, X_3 \) we have

\[
D_t(p_{X_1,X_2,X_3}||q_{X_1,q_{X_2},q_{X_3}}) + D_t(p_{X_3}\|q_{X_3}) \geq D_t(p_{X_1,X_3}||q_{X_1,q_{X_3}}) + D_t(p_{X_2,X_3}||q_{X_2,q_{X_3}}). \tag{3}
\]

**Remark 1.** The reason for using the term super-modularity is that (3) implies that the function \( g : \ell^n \rightarrow \mathbb{R} \) defined as

\[
g(A) = D_t(p_{X_A}||q_{X_A}), \quad \forall A \subset [n]
\]

is a super-modular set function for any product distribution \( q_{X^n} = \prod_{i=1}^n q_x \), and any arbitrary joint distribution \( p_{X^n} \). Here, \( X_A = (X_i : i \in A) \)

**Corollary 1.** Let \( P_{X_1,\ldots,X_n,W} \) and \( Q_{X_1,\ldots,X_n,W} \) be two distributions on \( n + 1 \) random variables. Assume that \( Q_{X_1,\ldots,X_n,W} = Q_{X_1,Q_{X_2},\ldots,Q_{X_n},Q_W} \). Then, by induction on \( n \), (3) implies that

\[
D_t(P_{X^n,W}||Q_{X^n,W}) - D_t(P_W||Q_W) \geq \sum_{i=1}^n [D_t(P_{X_i,W}||Q_{X_i,W}) - D_t(P_W||Q_W)]. \tag{4}
\]

To proceed, we need the following definition:

**Definition 3.** Define \( \mathcal{F} \) be the class of convex functions \( f(t) \) on \([0, \infty)\) that are not affine (not of the form \( t \mapsto at + b \) for some constants \( a \) and \( b \)), \( f(1) = 0 \), \( f'' \) is strictly positive, and \( 1/f'' \) is concave.

The above class of convex functions is important because it makes \( \Phi \)-entropy subadditive [23, Theorem 14.1]. Alternative equivalent definitions for the class \( \mathcal{F} \) are given in [23].

**Remark 2.** This class of convex functions appears in other contexts [24] [25, Appendix B]. For instance, it makes the definition of \( \Phi \)-entropic measures of correlation possible [24].

**Proposition 1.** \( D_t \) is super-modular for any function \( f \in \mathcal{F} \).

The proof is given in Appendix 8.1.

The functions \( f(t) = t \ln t \) and \( f(t) = \frac{1}{\alpha-1}(t^\alpha - 1) \) for \( \alpha \in (1,2] \) are two examples in class \( \mathcal{F} \). The following lemma (proven in Appendix 8.2) shows that \( t \ln t \) and \( t^2 \) are “extreme” members of \( \mathcal{F} \) in the sense that the growth rate of any function \( f(t) \in \mathcal{F} \) (as \( t \) converges to infinity) is no smaller than \( t \ln t \) and no larger than \( t^2 \).

**Lemma 1.** Take an arbitrary \( f(t) \in \mathcal{F} \). Then, \( t \mapsto tf''(t) \) is a non-decreasing function and

\[
\lim_{t \to \infty} \frac{f(t)}{t \ln t} = \lim_{t \to \infty} tf''(t) > 0. \tag{5}
\]

Next, \( t \mapsto f''(t) \) is a non-negative and non-increasing function and

\[
\lim_{t \to \infty} \frac{t^2}{f(t)} = \lim_{t \to \infty} \frac{1}{f''(t)} > 0. \tag{6}
\]
2.1.1 Application

As an application for super-modular f-divergence, we extend Sanov’s upper bound [9]. Upper bound part of Sanov’s theorem states that for n i.i.d. random variables distributed according to $P_X$, we have the following upper bound:

$$
P \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \geq \delta \right] \leq \exp \left( -n \inf_{Q_X : \text{E}_{Q_X}[X] \geq \delta} D(Q_X \parallel P_X) \right).$$

Our extension of Sanov’s upper bound is two-fold: firstly, we generalize it to sum of dependent random variables and secondly we replace the KL-divergence by the f-divergence.

**Theorem 2.** Let $S = (X_1, \cdots, X_n)$ be a sequences of n i.i.d. random variables according to $P_X$ and $W \sim P_W$ be independent of $S$. Let $\ell(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. Let

$$\alpha \triangleq P \left[ \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W) \geq \delta \right].$$

Then for any $f \in \mathcal{F}$,

$$\alpha f \left( \frac{1}{\alpha} \right) + (1 - \alpha) f(0) \geq n \cdot \inf_{Q \lesssim X, W} \left\{ D_{f}(Q_{X,W} \parallel P_{XQW}) \right\} - \frac{n-1}{n} D_{f}(Q_{W} \parallel P_{W}). \tag{7}$$

**Remark 3.** We can think of $\ell(\cdot, \cdot)$ as the loss function of a learning algorithm, $W$ as the chosen hypothesis, and $S$ as the test sample drawn independently of $W$.

**Remark 4.** If we set $f(x) = x \ln(x)$ and $W$ to be constant, the above theorem reduces to Sanov’s theorem. If $W$ is not a constant and $f(x) = x \ln(x)$, the above bound reduces to

$$\ln \left( \frac{1}{\alpha} \right) \geq n \cdot \inf_{Q \lesssim X, W} \left\{ D(Q_{X,W} \parallel P_{XQW}) \right\} - \frac{n-1}{n} D(Q_{W} \parallel P_{W}). \tag{8}$$

On the other hand, the Chernoff’s bound is

$$P \left[ \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W) \geq \delta \right] = \mathbb{E}_{W} \left[ P \left[ \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W) \geq \delta \mid W \right] \right] \leq e^{-n\alpha \delta} \mathbb{E}_{W} \left[ \left( \mathbb{E}[\alpha^{\ell(X,C)}|W]\right)^n \right]. \tag{9}$$

Hence, the Chernoff bound yields that $P\left[\frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W) \geq \delta \right] \leq e^{-nE_{CH}}$ where

$$E_{CH} = \sup_{\alpha > 0} \left\{ \alpha \delta - \frac{1}{n} \ln \mathbb{E}_{W} \left[ \left( \mathbb{E}[\alpha^{\ell(X,C)}|W]\right)^n \right] \right\}. \tag{10}$$

We show in Appendix 8.3 that the bound in (8) matches the Chernoff’s bound.

**Proof of Theorem 2.** First, note that given any random variable $Z \in \mathcal{Z}$ and any $A \subseteq \mathcal{Z}$ where $P_{Z}(A) > 0$, we have

$$P_{Z}(A) f \left( \frac{1}{P_{Z}(A)} \right) + (1 - P_{Z}(A)) f(0) = D_{f}(P_{Z_{\{Z\in A\}}} \parallel P_{Z}).$$
Let
\[ A \triangleq \{(x_1, \ldots, x_n, w) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \times \mathcal{W} : \frac{1}{n} \sum_{i=1}^{n} \ell(x_i, w) \geq \delta \}. \]

Then,
\[
\alpha f\left(\frac{1}{\alpha}\right) + (1 - \alpha)f(0) = D_f(P_{(S,W)}|(S,W)\in A)\|P_{S,W}).
\] (11)

Note that under \( \tilde{P}_{S,W} = P_{(S,W)|(S,W)\in A} \), we have
\[ \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W)\right] \geq \delta. \]

Thus,
\[ D_f(P_{(S,W)}|(S,W)\in A)\|P_{S,W}) \geq \inf_{R_{S,W}} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W)\right] \geq \delta \] (12)

From Corollary 1 for any arbitrary \( R_{S,W} \) we have
\[ D_f(R_{S,W}\|P_{S,W}) \geq -(n-1)D_f(R_W\|P_W) + \sum_i D_f(R_{X,i}\|P_{X,i}) \]
\[ \geq -(n-1)D_f(R_W\|P_W) + nD_f(R_{X,W}\|P_{X,W}) \]

where \( \bar{R}_{X,W}(x, w) = \frac{1}{n} \sum_i R_{X,i,W}(x, w) \) and the last step follows convexity of \( D_f \) (See Theorem 1). Also, note that \( \mathbb{E}_R\left[\frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W)\right] = \mathbb{E}_R\left[\ell(X, W)\right] \). Therefore,
\[ D_f(R_{S,W}\|P_{S,W}) \geq \inf_{R_{S,W}} \mathbb{E}_R\left[\frac{1}{n} \sum_{i=1}^{n} \ell(X_i, W)\right] \geq \delta \]

From (11)-(13), we get the desired result. \( \square \)

**Example 1.** Let \( f(x) = x^2 - x \). Then,
\[ D_f(Q\|P) = \chi^2(Q\|P). \]

Let \( W \) to be a constant random variable. Then, the function \( \ell \) depends only on \( x \) and we can denote it by \( \ell(x) \). Then, we get
\[
\frac{1}{\alpha} - 1 \geq n \cdot \inf_{Q} \chi^2(Q_X\|P_X) = n \left(-1 + \exp\left(\inf_{Q} D_2(Q_X\|P_X)\right)\right) \geq 0.
\] (14)

As a concrete example, let \( X_i \sim \text{Bern}(1/2 - \delta) \) and let \( \alpha \) be the probability that their average exceeds 1/2. Then, we get
\[
\frac{1}{\alpha} - 1 \geq n \cdot \inf_{Q} \chi^2(Q_X\|P_X) = n \left(\frac{1}{4(1/2 - \delta)} + \frac{1}{4(1/2 + \delta)} - 1\right) = \frac{4n\delta^2}{1 - 4\delta^2}
\] (15)

\[ \frac{1}{\alpha} - 1 \geq n \cdot \inf_{Q} \chi^2(Q_X\|P_X) = n \left(\frac{1}{4(1/2 - \delta)} + \frac{1}{4(1/2 + \delta)} - 1\right) = \frac{4n\delta^2}{1 - 4\delta^2}
\] (16)
or
\[ \alpha \leq U_{x_{2-x}} := \frac{1}{1 + \frac{4n^2}{1 - 4\delta^2}}. \]

From Sanov’s theorem we get
\[ \alpha \leq U_{x_{lnx}} := \exp \left( -n \left( \frac{1}{2} \ln \left( \frac{1}{1 - 4\delta^2} \right) \right) \right) = (1 - 4\delta^2)^{n/2}. \]

We will show that for a finite number of samples \( n \geq 1 \) and \( \delta \in (0, \frac{1}{2\sqrt{n}}) \), we have \( U_{x_{2-x}} < U_{x_{lnx}} \).

Assume that \( \delta = \frac{c}{2\sqrt{n}} \) for some \( c < 1 \). Then, we get \( U_{x_{lnx}} = (1 - \frac{c^2}{n})^{n/2} \geq 1 - \frac{c^2}{2} \) for \( n \geq 2 \) since \( (1 - x)^n \geq 1 - nx \) for \( x \in [0,1] \) and \( n \geq 1 \). On the other hand, we have
\[
U_{x_{2-x}} = \frac{1}{1 + \frac{c^2}{1 - c^2}} = \frac{n - c^2}{(1 + c^2)n - c^2} = \frac{1}{1 + \frac{c^2}{1 + c^2}} \leq \frac{1}{1 + c^2}.
\]

As \( c \in (0,1) \), we get \( \frac{1}{1 + c^2} < 1 - c^2/2 \) and the claim is established.

### 3 f-information

The following two proposals for defining a mutual f-information in terms of the f-divergence are known: The first is (see [5] [26, Eq. 3.10.1]):
\[ I^{CKZ}_t(A; B) := D_t(P_{AB}||P_A \times P_B), \quad (17) \]
and has been studied in the literature (e.g. see [6,28–30]). Another definition is given in [31, Eq. 79]:
\[ I^{PV}_t(A; B) := \min_{Q_B} D_t(P_{AB}||P_A \times Q_B), \quad (18) \]
where the minimum is over all \( Q_B \) such that \( P_B \ll Q_B \). Note that \( I^{CKZ}_t(A; B) = I^{CKZ}_t(B; A) \) is symmetric but \( I^{PV}_t(A; B) \) is not symmetric in general. Moreover, when \( D_t(\cdot||\cdot) \) is the KL divergence, both of these f-informations reduce to Shannon’s mutual information.

**Example 2.** Let \( f(x) = x^\alpha - 1 \) for \( \alpha \in [1,2] \). Then, for random variables \( A \) and \( B \), we have
\[ I^{PV}_t(A; B) = \min_{Q_B} e^{(\alpha - 1)D_t(P_{AB}||P_AQ_B)} - 1 = e^{(\alpha - 1)\min_{Q_B} D_t(P_{AB}||P_AQ_B)} - 1 \quad (19) \]
\[ = e^{(\alpha - 1)I_\alpha(A; B)} - 1 = \left( \mathbb{E}_{P_B} \left[ \mathbb{E}_{P_A} \left( \frac{dP_B}{dP_A} \right)^{\alpha} \right]^{1/\alpha} \right)^\alpha - 1 \quad (20) \]
where \( I_\alpha \) is the \( \alpha \)-mutual information according to Sibson’s proposal [32].

There is yet another definition for mutual f-information in [25, Appendix B] as follows:
\[ I^{MBGYA}_t(A; B) = \min_{Q_B} \{ D_t(P_{AB}||P_A \times Q_B) - D_t(P_B||Q_B) \} \quad (21) \]
where the minimum is over all \( Q_B \) such that \( P_B \ll Q_B \). It is clear from the definitions above that
\[ I^{CKZ}_t(A; B) \geq I^{PV}_t(A; B) \geq I^{MBGYA}_t(A; B). \]

**Example 3.** Let \( f_\alpha(t) = t^\alpha - 1 \) for \( 1 \leq \alpha \leq 2 \). Then it is shown in [25, Theorem 33] that
\[ I^{MBGYA}_{f_\alpha}(A; B) = \frac{1}{\alpha - 1} \left( \mathbb{E}_{P_B} \left[ \mathbb{E}_{P_A} \left( \left( \frac{dP_B}{dP_A} \right)^{\alpha} - 1 \right) \right]^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha}}. \]
When \( \alpha = 2 \), interestingly this definition coincides with [33, Definition 1] (derived independently) in the context of exploration bias.

\(^4\)A further generalization is given in [27].
3.1 Properties of mutual f-information

Definition 4. Let \( \rho(X; Y) \) be a mapping that assigns a non-negative real number to arbitrary random variables \( X \) and \( Y \). We say that \( \rho \) is a measure of dependence if it satisfies the faithfulness and data processing properties defined as follows: we say that \( \rho(\cdot; \cdot) \) satisfies the faithfulness property if \( \rho(X; Y) = 0 \) if and only if \( X \) and \( Y \) are independent. We say that \( \rho(\cdot, \cdot) \) satisfies the data processing property if \( \rho(X; Y) \geq \rho(W; Z) \) whenever \( W - X - Y - Z \) forms a Markov chain.

Theorem 3. [25, 31] For any convex function \( f \), \( I^PV_t \), \( I^{CKZ}_t \) are measures of dependence (see Definition 4). For any function \( f \in \mathcal{F} \) as defined in Definition 3, the mutual f-information \( I^{MGBYA}_t \) is a measure of dependence.

Theorem 4. Assume that \( dP_{Y|X}(y, x) := \frac{dP_{XY}(y, x)}{dP_X(x)} \) exists and we call it \( p(y|x) \).

(i). For every convex function \( f \), \( I^PV_t(X; Y) \) and \( I^{MBGYA}_t(X; Y) \) are concave in \( P_X \) when \( p(y|x) \) is fixed. Equivalently, \( I^PV_t(X; Y) \geq I^PV_t(X; Y|Q) \) and \( I^{MBGYA}_t(X; Y) \geq I^{MBGYA}_t(X; Y|Q) \) for any \( p_X, q_Y \).

(ii). For every convex function \( f \), \( I^{CKZ}_t(X; Y) \) and \( I^{PV}_t(X; Y) \) are convex functions of \( p(y|x) \) when \( P_X \) is fixed. Equivalently, \( I^{CKZ}_t(X; Y) \leq I^{CKZ}_t(X; Y|Q) \) and \( I^{PV}_t(X; Y) \leq I^{PV}_t(X; Y|Q) \) for any \( q_X, p_Y \).

(iii). Assume that \( f \in \mathcal{F} \). Let \( X^n = (X_1, \cdots, X_n) \) be a sequence of \( n \) independent random variables. Assume that \( U \) and \( X^n \) are arbitrary distributed. Then,

\[
I^{CKZ}_t(X^n; U) \geq \sum_i I^{CKZ}_t(X_i; U), \tag{22}
\]

\[
I^{MBGYA}_t(X^n; U) \geq \sum_i I^{MBGYA}_t(X_i; U). \tag{23}
\]

(iv). For \( f \in \mathcal{F} \) and every \( p(x, y) \), we have

\[
I(X; Y) \cdot \left[ \lim_{t \to \infty} tf''(t) \right] \geq I^{CKZ}_t(X; Y) \geq I^{PV}_t(X; Y) \geq I^{MBGYA}_t(X; Y). \tag{26}
\]

The proof of Theorem 4 is given in Appendix 8.4.

Definition 5. Assume that random variable \( X \) takes value in a discrete set \( \mathcal{X} \). The f-entropy of \( X \) is defined as follows:

\[
H^{CKZ}_t(X) \triangleq I^{CKZ}_t(X; X) = f(0) \left( 1 - \sum_{x \in \mathcal{X}} P^2_X(x) \right) + \sum_{x \in \mathcal{X}} P^2_X(x) f \left( \frac{1}{P_X(x)} \right) \tag{24}
\]

\[
H^{PV}_t(X) \triangleq I^{PV}_t(X; X) = \min_{Q_X} \left\{ f(0) \left( 1 - \sum_{x \in \mathcal{X}} P_X(x) Q_X(x) \right) + \sum_{x \in \mathcal{X}} P_X(x) Q_X(x) f \left( \frac{1}{Q_X(x)} \right) \right\} \tag{25}
\]

\[
H^{MBGYA}_t(X) \triangleq I^{MBGYA}_t(X; X) = \min_{Q_X} \left\{ f(0) \left( 1 - \sum_{x \in \mathcal{X}} P_X(x) Q_X(x) \right) + \sum_{x \in \mathcal{X}} P_X(x) Q_X(x) f \left( \frac{1}{Q_X(x)} \right) - \sum_{x \in \mathcal{X}} Q_X(x) f \left( \frac{P_X(x)}{Q_X(x)} \right) \right\} \tag{26}
\]
We will prove some properties of $f$-entropy. We use $H_f$ if the statement is true for both $H^\text{PV}_f(X)$ and $H^\text{CKZ}_f(X)$.

**Theorem 5.** [Properties of $f$-entropy]

(i). For every convex function $f$ defined on $I = [0, +\infty)$, $H^\text{MBGYA}_f(X)$ is a concave function of $p(x)$. With the extra condition $f''(x) \leq 0$ for $x > 0$, $H^\text{CKZ}_f(X)$ is a concave function of $p(x)$. Moreover, $f \in \mathcal{F}$ implies that $f''(x) \leq 0$ for $x > 0$.

(ii). The function $P(x) \mapsto H^\text{MBGYA}_f(X)$ is maximized at the uniform distribution for every convex function $f(x)$. A similar statement holds $H^\text{CKZ}_f(X)$ if $f''(x) \leq 0$ for $x > 0$.

(iii). Let $Y$ take value in a finite set $\mathcal{Y}$, then for convex $f : \mathbb{R}_+ \to \mathbb{R}$,

$$I^\text{CKZ}_f(X;Y) \leq H^\text{CKZ}_f(Y),$$

$$I^\text{PV}_f(X;Y) \leq H^\text{PV}_f(Y),$$

$$I^\text{MBGYA}_f(X;Y) \leq H^\text{MBGYA}_f(Y).$$

The proof of Theorem 5 is given in Appendix 8.5.

4 Lossy compression with mutual $f$-information

Consider a memoryless source $X \sim \mu_X$ and a distortion function $d(x, \hat{x})$ for $x \in \mathcal{X}$ and $\hat{x} \in \hat{\mathcal{X}}$ where $\mathcal{X}$ and $\hat{\mathcal{X}}$ are the source and reconstruction alphabet sets. While the literature commonly assumes that the reconstruction alphabet set is identical with the source alphabet set $\hat{\mathcal{X}} = \mathcal{X}$, we do not make this assumption here. Moreover, the distortion function $d(x, \hat{x})$ is generally assumed to be non-negative. Here, unlike the source-coding literature, we allow $d(x, \hat{x})$ to take negative values. The same standard proofs (of the rate-distortion theory) go through when $\hat{\mathcal{X}} \neq \mathcal{X}$ or when $d(x, \hat{x})$ becomes negative.

An $(n, R, D)$ lossy source code consists of an encoder $\mathcal{E} : \mathcal{X}^n \mapsto \{1, 2, \cdots, 2^nR\}$ and a decoder $\mathcal{D} : \{1, 2, \cdots, 2^nR\} \mapsto \hat{\mathcal{X}}^n$ such that the reconstruction sequence

$$\hat{X}^n = \mathcal{D}(\mathcal{E}(X^n))$$

satisfies the expected requirement

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \leq D.$$

A rate-distortion pair $(R, D)$ is said to be achievable if for every $\epsilon > 0$, one can find an $(n, R, D+\epsilon)$ lossy source code for some blocklength $n$. Given a rate $R$, let $\mathcal{R}(R)$ be the minimum $D$ such that the rate-distortion pair $(R, D)$ is achievable. Similarly, given a distortion $D$, let $\mathcal{R}(D)$ be the minimum $R$ such that the rate-distortion pair $(R, D)$ is achievable. The following characterization of the rate-distortion function $\mathcal{R}(R)$ is known [34, Theorem 3.5].

$$\mathcal{R}(D) = \inf_{P_{X;X} : \mathbb{E}[d(X, \hat{X})] \leq D} I(X; \hat{X}).$$

\(^5\)Some technical conditions are needed for (30) and (31) when $d(x, \hat{x})$ is unbounded. For instance, a sufficient condition is existence of $\hat{x}_0 \in \hat{\mathcal{X}}$ such that $\mathbb{E}[d(X, \hat{x}_0)] < \infty$ [35, Theorems 7.2.4 & 7.2.5].
\[ \mathcal{D}(R) = \inf_{P_{\hat{X}|X} : I(\hat{X}; X) \leq R} \mathbb{E}[d(X, \hat{X})], \]  

(31)

One can also formally define a variant of the rate-distortion function by replacing Shannon’s mutual information with other measures of correlation. For instance, using mutual \( f \)-information defined in (17) or (18), we define:

\[ \mathcal{R}_t^{CKZ}(D) = \inf_{P_{\hat{X}|X} : \mathbb{E}[d(X, \hat{X})] \leq D} I_t^{CKZ}(X; \hat{X}), \]  

(32)

\[ \mathcal{R}_t^{CKZ}(R) = \inf_{P_{\hat{X}|X} : I_t^{CKZ}(X; \hat{X}) \leq R} \mathbb{E}[d(X, \hat{X})], \]  

(33)

and similarly,

\[ \mathcal{R}_t^{MBGYA}(D) = \inf_{P_{\hat{X}|X} : \mathbb{E}[d(X, \hat{X})] \leq D} I_t^{MBGYA}(X; \hat{X}), \]  

(34)

\[ \mathcal{R}_t^{MBGYA}(R) = \inf_{P_{\hat{X}|X} : I_t^{MBGYA}(X; \hat{X}) \leq R} \mathbb{E}[d(X, \hat{X})]. \]  

(35)

We call (32) and (34) \( f \)-rate-distortion functions. Note that for the special case of \( f(t) = t \ln(t) \), the \( f \)-rate-distortion functions in (32) and (34) reduce to the ordinary rate-distortion functions in (30). The ordinary rate-distortion function in (30) with Shannon’s mutual information is a meaningful quantity with an operational interpretation as the solution to the lossy compression problem. What about \( \mathcal{R}_t^{CKZ}(D) \) or \( \mathcal{R}_t^{PV}(D) \)? Ziv and Zakai in [5] consider the classical proof for the lossy source coding and attempt to mimic the proof by replacing Shannon’s mutual information with the mutual \( f \)-information in the proof steps. They show that the function \( \mathcal{R}_t^{CKZ}(D) \) is useful in obtaining new infeasibility results for the lossy source coding problem when blocklength \( n = 1 \). In fact, the bound obtained using \( f \)-rate-distortion functions can strictly improve over the ordinary rate-distortion function (30) when we restrict to codes with blocklength \( n = 1 \). Authors in [5] simply use the fact that mutual \( f \)-information (as defined in (17) or (18)) satisfies the data-processing property. However, we would like to highlight that the argument in [5] does not yield a computationally tractable bound for arbitrary \( n > 1 \); this is due to the fact that [5] only yields a bound in the multi-letter form for \( n > 1 \) (not a single-letter bound). This limitation stems from the fact that \( f \)-mutual-information does not enjoy the chain-rule property of Shannon’s mutual information.

Even though mutual \( f \)-information does not have the chain rule property of Shannon’s mutual information, in this paper we state new properties for mutual \( f \)-information which could be utilized in lieu of the chain rule to obtain a single-letter bound. Using these new properties, we relate \( \mathcal{R}_t^{CKZ}(D) \) and \( \mathcal{R}_t^{MBGYA}(D) \) to the lossy source coding problem when blocklength \( n \) is arbitrary. More specifically, our result allows us to generalizes the result in [5] to arbitrary blocklength \( n \).

**Theorem 6.** For any \((n, R, D)\) lossy source code, there is some \( p(\hat{x}|x) \) such that

- \( \mathbb{E}[d(X, \hat{X})] \leq D \)
- For any arbitrary function \( f \in \mathcal{F} \) as defined in Definition 3, we have

\[ \frac{1}{n} H_t^{CKZ}(K) \geq I_t^{CKZ}(X; \hat{X}), \]  

(36)

\[ \]  

\[ \text{We are only interested in the } f \text{-rate-distortion function in the context of the lossy source coding problem. The } f \text{-rate-distortion function is also of interest in other applications such as privacy or security; interested readers can refer to [36, Section IV.A, Section IV.C] for such applications of } f \text{-rate-distortion functions.} \]
where $K$ is a uniform random variable over $\{1, 2, \cdots, 2^{nR}\}$. In particular, equation (36) can be equivalently written as

$$\frac{1}{n} \left\{ f(0) \left( 1 - 2^{-nR} \right) + 2^{-nR} f(2^{nR}) \right\} \geq \mathcal{R}_f^{CKZ}(X; \hat{X}).$$

Proof of Theorem 6. Take an encoder $E : X^n \mapsto \{1, 2, \cdots, 2^{nR}\}$ and a decoder $D : \{1, 2, \cdots, 2^{nR}\} \mapsto \hat{X}^n$ such that the reconstruction sequence \(\hat{X}^n = D(E(X^n))\) satisfies

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \leq D.$$

Let $M = E(X^n) \in \{1, 2, \cdots, 2^{nR}\}$ be the compression of $X^n$. That is, $M$ and $K$ are defined on the same alphabet set, and $K$ is uniformly distributed. Take a time-sharing random variable $Q$ uniform over $\{1, 2, \cdots, n\}$ and independent of previously defined variables. Then,

$$\mathbb{E}[d(X_Q, \hat{X}_Q)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \leq D.$$

For either the CKZ or MBGYA notions of the mutual $f$-information, we have

$$H_f(K) \geq H_f(M) \quad \text{(38)}$$

$$= I_f(M; M) \quad \text{(39)}$$

$$\geq I_f(X^n; \hat{X}^n) \quad \text{(40)}$$

$$\geq \sum_{i=1}^{n} I_f(X_i; \hat{X}^n) \quad \text{(41)}$$

$$\geq \sum_{i=1}^{n} I_f(X_i; \hat{X}_i) \quad \text{(42)}$$

$$\geq n \cdot I_f(X_Q; \hat{X}_Q), \quad \text{(43)}$$

where (38) follows from property (ii) of Theorem 5, (39) follows from the definition of $f$-entropy, (40), (42) from data processing property of $f$-information (Theorem 3), and finally (41) follows from property (iii) of Theorem 4.

Corollary 2. We obtain

$$\frac{1}{n} H_f^{CKZ}(K) \geq \mathcal{R}_f^{CKZ}(D), \quad \text{(44)}$$

$$\frac{1}{n} H_f^{MBGYA}(K) \geq \mathcal{R}_f^{MBGYA}(D), \quad \text{(45)}$$

where $K$ is a uniform random variable over $\{1, 2, \cdots, 2^{nR}\}$. Furthermore, for the CKZ notion to compute the minimum in $\mathcal{R}_f^{CKZ}(D)$, it suffices to compute the minimum over all conditional distributions $P_{X|\hat{X}}$ for $\hat{X} \in \hat{X}$ such that the support of $\hat{X}$ has size at most $|\mathcal{X}| + 1$. 

\(\square\)
Proof. Inequalities (44) and (45) follow immediately from Theorem 6. It remains to show the bound on the size of the support of $\hat{X}$ when computing $R_{t}^{CKZ}(D)$. The cardinality bounds on the auxiliary random variable $W$ comes from the standard Caratheodory-Bunt [37] arguments and is omitted. We just point out that similar to the ordinary mutual information, f-information $I_{t}^{CKZ}(X;\hat{X})$ has the following property: fix a conditional distribution $P_{X|\hat{X}}$ and consider the set of marginal distributions $P_{\hat{X}}$ that induce a given marginal distribution on $P_{X}$. Then,

$$I_{t}^{CKZ}(X;\hat{X}) = \mathbb{E}_{P_{X}\hat{P}_{X}} f\left(\frac{dP_{X|\hat{X}}}{dP_{\hat{X}}}(X,\hat{X})\right)$$

is linear in the marginal distribution $P_{\hat{X}}$.

\[\square\]

Remark 5. In the asymptotic regime when the blocklength $n$ tends to infinity, a full characterization of the achievable rate-distortion pairs $(R,D)$ is known and given in (30). As a sanity check, let us restrict the above theorem to the case of $n$ tending to infinity. Assume that a rate-distortion pair $(R,D)$ is achievable as $n$ tends to infinity. Letting $n$ converge to infinity we obtain

$$\lim_{n \to \infty} \frac{1}{n} \left\{ f(0)\left(1-2^{-nR}\right) + 2^{-nR}f\left(2^nR\right) \right\} \geq R_{t}^{CKZ}(D)$$

or equivalently,

$$R \lim_{t \to \infty} \frac{f(0)(t-1) + f(t)}{t \ln(t)} \leq R_{t}^{CKZ}(D).$$

If $R = R(D)$ lies on the rate-distortion curve, from Lemma 1 in Section 2.1 we obtain

$$R(D) \left[ \lim_{t \to \infty} tf''(t) \right] \geq R_{t}^{CKZ}(D).$$

The correctness of the above inequality can be verified using part (iv) of Theorem 4.

Example 4. Let $X \sim B(1/2)$ be a uniform binary source (i.e., $X \in \{0,1\}$ is uniform). Let $\mathcal{X} = X = \{0,1\}$ and $d(x,\hat{x}) = 1(x \neq \hat{x})$ be the Hamming distance. So, the average distortion is in fact the probability of mismatch: $\mathbb{E}[d(X,\hat{X})] = \mathbb{P}[X \neq \hat{X}]$. The classical rate-distortion function in this case is given by [34, Example 11.1]:

$$R(D) = 1 - H_{2}(D), \quad \forall \ D \in [0,0.5].$$

where $H_{2}(x) = -x \log_{2}(x) - (1-x) \log_{2}(1-x)$ is the binary entropy function.

Let $f(x) = x^2 - 1$ which belongs to $\mathcal{F}$. Then, $R_{t}^{CKZ}(D)$ is achieved by a binary symmetric channel $P_{X|\hat{X}}(1|0) = P_{X|\hat{X}}(0|1) = D$ and

$$R_{t}^{CKZ}(D) = (2D - 1)^2, \quad \forall \ D \in [0,0.5].$$

The statement of the theorem then implies that for any $(n,R,D)$ lossy source code, we have

$$R \geq LB_{t}(D,n) := \frac{1}{n} \log_{2}(nR_{t}^{CKZ}(D) + 1).$$

From the rate-distortion theory, we know that $R \geq R(D)$ (which holds for arbitrary values of $n$). Figure 2 plots the lower bounds $LB_{t}(D,n)$ and $R(D)$ for different values of $n$. As the figure indicates, the lower bound $LB_{t}(D,n)$ is a non-trivial lower bound for a finite blocklength lossy source code.

Claim 1. We have that $LB_{t}(D,n) \geq R(D)$ for $\frac{1}{2} - \frac{1}{2\sqrt{n}} \leq D \leq \frac{1}{2}$ and arbitrary block length $n \geq 1$.  

12
Figure 2: Lower bounds on achievable rate $R$ versus $D$ for a code of length $n$ for a binary source $X \sim \text{Bernoulli}(1/2)$ and Hamming distortion. The new lower bound $LB_t(D, n) = \frac{1}{n} \log_2(nR_{CKZ}(D) + 1)$ for $f(x) = x^2 - x$ is plotted along with $\mathcal{R}(D)$ for $n = 2, 8, 32$.

The proof is given in Appendix 8.6.

The following theorem shows single-letterization of the $f$-rate distortion function when $f \in \mathcal{F}$.

**Theorem 7.** Take some arbitrary $f \in \mathcal{F}$. Let $X^n$ be i.i.d. according to some $P_X$ for some arbitrary natural number $n$. Then, we have

$$\inf_{P_{X^n|X^n}: I_{f}^{CKZ}(X^n; \hat{X}^n) \leq nR} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \geq R_{f}^{CKZ}(R). \quad (46)$$

Similarly,

$$\inf_{P_{X^n|X^n}: I_{f}^{MBGYA}(X^n; \hat{X}^n) \geq nR} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \geq R_{f}^{MBGYA}(R). \quad (47)$$

The proof is given in Appendix 8.7.

# 5 Lower bounds on the (f-)rate-distortion function

The (f-)rate-distortion function is expressed in terms of an optimization problem. It does not have an explicit expression. However, there are some explicit lower bounds on the rate-distortion function such as Shannon’s lower bound. Such lower bounds are of independent interest and have been previously studied in the literature. A new motivation for studying such lower bounds stems from a connection that we provide between lower bounds on the rate-distortion function and upper bounds on the generalization error of learning algorithms. This connection is discussed later in Section 6.1.

In this section we discuss explicit lower bounds on the (f-)rate-distortion function. We provide new bounds that outperform the existing bounds in some cases. We begin our discussion with the ordinary rate-distortion function and extend the result to (f-)rate-distortion function afterwards.
5.1 Review of the existing bounds and ideas in the literature

One known idea to obtain a lower bound on the rate-distortion function is as follows: consider a random variable $X$. Then, we have

\[ \mathcal{R}(R) = \inf_{P_{X|X}: I(X;X) \leq R} \mathbb{E} \left[ d(X, \hat{X}) \right] \]

(48)

where

\[ = \inf_{P_{X|X}} \sup_{\lambda \geq 0} \mathbb{E} \left[ d(X, \hat{X}) \right] - \lambda R + \lambda D(P_{X|X} \| P_{X} P_{X}) \]

(49)

\[ \geq \sup_{\lambda \geq 0} \inf_{P_{X|X}} \mathbb{E} \left[ d(X, \hat{X}) \right] - \lambda R + \lambda D(P_{X|X} \| P_{X} P_{X}) \]

(50)

\[ = \sup_{\lambda \geq 0} \inf_{Q_{X}} \mathbb{E}_{P_{XX}} \left[ d(X, \hat{X}) \right] - \lambda R + \lambda D(P_{X|X} \| Q_{X} P_{X}) \]

(51)

\[ = \sup_{\lambda \geq 0} \inf_{Q_{X}} -\lambda R - \lambda \mathbb{E}_{X \sim P_{X}} \left[ \ln \left\{ \mathbb{E}_{\hat{X} \sim Q_{X}} \left[ e^{-\frac{d(X, \hat{X})}{\lambda}} \right] \right\} \right] \]

(52)

where (50) follows from minimax theorem and (51) follows from the following equality:

\[ D(P_{X|X} \| Q_{X} P_{X}) = D(P_{X|X} \| P_{X} P_{X}) + D(P_{X} \| Q_{X}). \]

To obtain (52), note that for every fixed $Q_{X}$, the minimizing $P_{X|X}$ in (51) is the Gibbs measure:

\[ dP_{X|X}^{*}(\hat{x}|x) := \frac{dQ_{X}(\hat{x}) e^{-\frac{d(x, \hat{x})}{\lambda}}}{\mathbb{E}_{Q_{X}} \left[ e^{-\frac{d(x, \hat{x})}{\lambda}} \right]}, \]

Then (52) follows from substituting $P_{X|X}^{*}(\hat{x}|x)$ in (51). The above idea can be used to obtain the following explicit lower bound:

**Theorem 8.** \cite[Theorem 55]{38}, \cite[Lemma 2]{39} Consider a random variable $X \sim P_{X}$ distributed on the measure space $(\mathcal{X}, \mathcal{A}, \mu)$, a measurable space $(\mathcal{Y}, \mathcal{B})$, and a distortion function $d : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ satisfying 1) $\inf_{y \in \mathcal{Y}} d(x, y) = 0$ for all $x \in \mathcal{X}$. 2) there exists a finite set $\mathcal{B} \subseteq \mathcal{Y}$ such that $\mathbb{E}[\min_{y \in \mathcal{B}} d(X, y)] = 0$. Suppose that $\mu$ is a reference measure for random variable $X$ (i.e. $P_{X} \ll \mu$), then

\[ \mathcal{R}(D) \geq h_{\mu}(X) - \inf_{\lambda \geq 0} \{ \lambda D + \ln \nu(\lambda) \} \]

(53)

where

\[ h_{\mu}(X) = -\mathbb{E} \left[ \ln \left( \frac{dP_{X}}{d\mu}(X) \right) \right], \quad \nu(\lambda) = \sup_{y \in \mathcal{Y}} \int e^{-\lambda d(x, y)} d\mu(x), \quad \forall \lambda \in [0, \infty). \]

**Remark 6.** Theorem 8 holds for all values of distortion $D \geq 0$. In \cite[40], Marton studied the asymptotic of $\mathcal{R}(D)$ when $D \to 0$, showing that for discrete sources

\[ \mathcal{R}(D) \geq H_{2} (P_{X}) + c \cdot D \log_{2} D + O(D) \]

where $c$ is a constant determined by the source.

Another known idea (originally developed by Shannon) for finding a lower bound on the rate-distortion function when $\mathcal{X} = \hat{\mathcal{X}} = \mathbb{R}^{k}$ is as follows:

\[ I(X; \hat{X}) = h(X) - h(X|\hat{X}) \]
\[ = h(X) - h(X - \hat{X} | \hat{X}) \]
\[ \geq h(X) - h(X - \hat{X}) \]
\[ \geq h(X) - \sup_{p_{\hat{X}|X} : \mathbb{E}[d(X, \hat{X})] \leq D} h(X - \hat{X}). \]

Therefore, for any arbitrary \( d(x, \hat{x}) \), we deduce that
\[
\mathcal{R}(D) \geq h(X) - \sup_{p_{\hat{X}|X} : \mathbb{E}[d(X, \hat{X})] \leq D} h(X - \hat{X}).
\]

Assume that we have a distortion measure \( d(\cdot, \cdot) \) of the form
\[ d(x, \hat{x}) = L(x - \hat{x}) \quad \forall x, \hat{x} \in \mathbb{R}^k \]
for some function \( L : \mathbb{R}^k \to \mathbb{R} \). Then, we obtain Shannon’s lower bound on the rate-distortion function: [41, Lemma 4.6.1]
\[
\mathcal{R}(D) \geq h(X) - \sup_{p_X : \mathbb{E}[L(X)] \leq D} h(X).
\]

Moreover, the solution to the maximization problem
\[
\max_{p_X : \mathbb{E}[L(X)] \leq D} h(X)
\]

is of the form
\[
p^*(x) = \frac{e^{-bL(x)}}{\int e^{-bL(x)} dx}
\]
where the constant \( b \) is chosen such that
\[
\int L(x)p^*(x)dx = D
\]

Note that Theorem 8 recovers this lower bound with the choice of \( \mu \) being the Lebesgue measure.

**Corollary 3.** [41, Section 4.8] Let \( X \sim P_X \) on \( \mathbb{R}^k \). Assume that \( X = \hat{X} = \mathbb{R}^k \), and take a distortion function \( d(x, \hat{x}) = \|x - \hat{x}\|_r^r \) for all \( x, \hat{x} \). Then we have
\[
\mathcal{R}(D) \geq h(X) - \ln \left( \left( \frac{Dre}{k} \right)^{\frac{k}{r}} \Gamma(1 + k/r)V_k \right)
\]
where \( V_k \) is the volume of the unit radius ball in \( \mathbb{R}^k \), i.e.,
\[
V_k \triangleq \text{Vol} \left[ \{x \in \mathbb{R}^k : \|x\|_r \leq 1 \} \right].
\]

### 5.2 A new lower bound on the rate-distortion function

Using Jensen’s inequality on Equation (52), we get the following theorem:
**Theorem 9.** Take an arbitrary source \( X \sim P_{X} \) defined on a set \( \mathcal{X} \) and a distortion function \( d(x, \hat{x}) \in \mathbb{R} \) (which may be negative). Then, the following two statements hold: We have

\[
\mathcal{D}(R) \geq \sup_{\lambda \geq 0} \inf_{Q_{X}} \left\{ -\frac{R}{\lambda} - \frac{1}{\lambda} \ln \mathbb{E}_{P_{X}Q_{X}} \left[ e^{-\lambda d(X, \hat{X})} \right] \right\} \geq \sup_{\lambda \geq 0} \left\{ -\frac{R}{\lambda} - \frac{1}{\lambda} \phi(-\lambda) \right\}
\]

where \( \phi(\cdot) \) is a function defined as follows:

\[
\phi(\lambda) = \sup_{\hat{x}} \ln \mathbb{E}_{P_{X}} \left[ e^{\lambda d(X, \hat{x})} \right].
\]

Note that if for some \( \lambda \), \( \mathbb{E}_{P_{X}} \left[ e^{\lambda d(X, \hat{x})} \right] = \infty \) for some \( \hat{x} \), we set \( \phi(\lambda) = \infty \).

**Remark 7.** The bound in part (i) of Theorem 9 can be used to find bounds on the generalization error of learning algorithms, while the extra assumptions of Theorem 8 preclude their application in the context of generalization error.

**Corollary 4.** Assume that \( d(X, \hat{x}) \) is unbiased (i.e., \( \mathbb{E}[d(X, \hat{x})] = 0 \)) and \( \sigma^{2} \)-subgaussian for every \( \hat{x} \in \mathcal{X} \). Then \( \phi(\lambda) = \lambda^{2} \sigma^{2} / 2 \) and

\[
\mathcal{D}(R) \geq \sup_{\lambda \geq 0} \left\{ -\frac{R}{\lambda} - \frac{1}{2} \lambda \sigma^{2} \right\} = -\sqrt{2\sigma^{2}R}
\]

**Example 5.** The lower bound given in part (i) of Theorem 9 can be stronger than the lower bound of Theorem 8 (taken from \([39, \text{Lemma 2}]\)). For instance, assume that \( \mathcal{X} = \{1, 2, 3\} \) and \( P_{X}(1) = 0.5, P_{X}(2) = 0.01, P_{X}(3) = 0.49 \). Let \( d(x, \hat{x}) = \mathbf{1}\{x \neq \hat{x}\} \). Theorem 8 gives a lower bound on \( \mathcal{D}(D) \). It implies the following lower bound on \( \mathcal{D}(R) \):

\[
\text{LB}_{1}(R) = \sup_{\lambda \geq 0} \left\{ -\frac{R}{\lambda} - \frac{1}{\lambda} \left( \ln \sup_{\hat{x}} \left( \sum_{x} e^{-\lambda d(x, \hat{x})} \right) - H(P_{X}) \right) \right\}.
\]

On the other hand, the lower bound of Theorem 9 evaluates to

\[
\text{LB}_{2}(R) = \sup_{\lambda \geq 0} \left\{ -\frac{R}{\lambda} - \frac{1}{\lambda} \ln \sup_{\hat{x}} \mathbb{E}_{P_{X}} \left[ e^{-\lambda d(X, \hat{x})} \right] \right\}.
\]

These two lower bounds are plotted in Figure 3. This plot indicates that the lower bound given in part (i) of Theorem 9 can be stronger than the lower bound of Theorem 8.

### 5.3 Lower bound on the f-rate-distortion function

The lower bounds of the previous section can be generalized to the f-rate-distortion functions defined in (33) and (35) (note that for the special case of \( f(t) = t \ln(t) \), the f-rate-distortion functions in (32) and (34) reduce to the ordinary rate-distortion functions in (30)). Note that the function \( f : [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\} \) is not necessarily in class function \( \mathcal{F} \) in this section. We also use \( f^{*} : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\} \) to denote the convex conjugate function of \( f \) defined as follows:

\[
f^{*}(x) = \sup_{t \geq 0} \{tx - f(t)\}, \quad \forall x \in \mathbb{R}.
\]

For instance, if \( f(t) = t \ln(t) \), we have \( f^{*}(t) = e^{t} - t + 1 \).

Since \( I_{t}^{CKZ}(\hat{X}; X) \geq \tilde{I}_{t}^{PV} (\hat{X}; X) \), we have \( \mathcal{D}_{t}^{CKZ}(R) \geq \mathcal{D}_{t}^{PV}(R) \) for every rate \( R \). Therefore, any lower bound on \( \mathcal{D}_{t}^{PV}(R) \) is also a lower bound on \( \mathcal{D}_{t}^{CKZ}(R) \). In what follows, we only consider \( \mathcal{D}_{t}^{PV}(R) \):

\[
-\mathcal{D}_{t}^{PV}(R) = \sup_{P_{X|X}} \mathbb{E}_{P_{XX}}[-d(X, \hat{X})]
\]

16
Figure 3: With the setting of Remark 5, LB$_1$(R) is the lower bound of distortion-rate function in Theorem 8 (taken from [39, Lemma 2]) and LB$_2$(R) is the lower bound proposed in part (i) of Theorem 9.

\[
\begin{align*}
= & \sup_{Q_{\hat{X}}} \sup_{P \mid \hat{X}: D_{\text{f}}(P \hat{X}, Q_{\hat{X}}) \leq R} \mathbb{E}_{P \hat{X}}[-d(X, \hat{X})] \\
\leq & \sup_{Q_{\hat{X}}} \sup_{\nu \mid \hat{X}, Q_{\hat{X}}: D_{\text{f}}(\nu \hat{X}, Q_{\hat{X}}) \leq R} \mathbb{E}_{\nu \hat{X}}[-d(X, \hat{X})] \\
= & \sup_{Q_{\hat{X}}} \mathbb{F}_{D_{\text{f}}, Q_{\hat{X}}, R}[-d(X, \hat{X})],
\end{align*}
\] (61)

where for any arbitrary random variable $Z \sim \mu$ and any function $g(\cdot)$ we define

\[
\mathbb{F}_{\mu, R}^{D_{\text{f}}}[g(Z)] \triangleq \sup_{\nu \sim \mu: D_{\text{f}}(\nu, \mu) \leq R} \mathbb{E}_{Z \sim \nu}[g(Z)].
\] (63)

The definition of smoothed expectation (63) also appears in the context of distributionally robust optimization problems [42–45]. In [42], the authors give an equivalent characterization of smoothed expectation. Theorem 15 in Appendix 8.9 gives a similar characterization. We state and prove the following theorem which is derived by Theorem 15.

**Theorem 10.** Let $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ be a convex function. Take an arbitrary source $X \sim P_X$ defined on a set $X$ and a distortion function $d(x, \hat{x}) \in \mathbb{R}$ (which may be negative). Then, the following two statements hold:

We have

\[
D_{\text{f}}^{PV}(R) \geq \sup_{\lambda \geq 0, a \in \mathbb{R}} \left\{ -\frac{1}{\lambda} [a + R] - \frac{1}{\lambda} \phi_{\ell}(-\lambda, a) \right\},
\] (64)

where $\phi(\cdot, \cdot)$ is a function defined on $[0, \infty) \times \mathbb{R}$ as follows:

\[
\phi_{\ell}(\lambda, a) = \sup_{\hat{x}} \mathbb{E}_{P_X} f^*(\lambda d(X, \hat{x}) - a).
\]
Remark 8. The lower bound in (64) yields the lower bound in (56) when specializing to \( f(x) = x \ln x - x + 1 \) for \( x \geq 0 \) \( (f(0) := 0) \). For this function \( f^*(y) = e^y - 1 \) for \( y \in \mathbb{R} \). The lower bound in Part (a) of Theorem 10 yields:

\[
\mathcal{Q}_t^{PV}(R) \geq \sup_{\lambda \geq 0, \ a \in \mathbb{R}} \left\{ -\frac{1}{\lambda} \mathbb{E}_P \left[ f^*(x^*) - f^*(x^0) \right] \right\}
\]

\[
= \sup_{\lambda \geq 0} \left\{ -\frac{1}{\lambda} R \sup_x \left\{ e^{-\lambda a} - e^{-a} \right\} \right\}
\]

\[
= \sup_{\lambda \geq 0} \left\{ -\frac{1}{\lambda} R \ln \sup_x \left\{ e^{-\lambda d(x, x^0)} \right\} \right\}.
\]

6 Generalization error of learning algorithms with bounded input/output mutual \( f \)-information

Consider a learning problem with an instance space \( Z \), a hypothesis space \( \mathcal{W} \) and a loss function \( \ell : \mathcal{W} \times Z \to \mathbb{R} \). Assume that the test and training samples are produced (in an i.i.d. fashion) from an unknown distributions \( \mu \) on \( Z \) respectively. A training dataset of size \( n \) is shown by the \( n \)-tuple, \( S = (Z_1, Z_2, \ldots, Z_n) \) \( \in \mathbb{Z}^n \) of i.i.d. random elements according to an unknown distribution \( \mu \). A learning algorithm is characterized by a probabilistic mapping \( A(\cdot) \) (a Markov Kernel) that maps training data \( S \) to the random variable \( W = A(S) \in \mathcal{W} \) as the output hypothesis. The population risk of a hypothesis \( w \in \mathcal{W} \) is computed on the test distribution \( \mu \) as follows:

\[
L_\mu(w) \triangleq \mathbb{E}_\mu[\ell(w, Z)] = \int_Z \ell(w, z) \mu(dz), \quad \forall w \in \mathcal{W}.
\]

The goal of learning is to ensure that under any data generating distribution \( \mu \), the population risk of the output hypothesis \( W \) is small, either in expectation or with high probability. Since \( \mu \) is unknown, the learning algorithm cannot directly compute \( L_\mu(w) \) for any \( w \in \mathcal{W} \), but can compute the empirical risk of \( w \) on the training dataset \( S \) as an approximation, which is defined as

\[
L_S(w) \triangleq \frac{1}{n} \sum^n_{i=1} \ell(w, Z_i).
\]

The true objective of the learning algorithm, \( L_\mu(W) \), is unknown to the learning algorithm while the empirical risk \( L_S(W) \) is known. The generalization gap is defined as the difference between these two quantities as \([46, 47]\)

\[
\text{gen}_\mu(W, S) = L_\mu(W) - L_S(W),
\]

where \( W = A(S) \) is the output of the algorithm \( A \) on the input \( S \sim (\mu)^{\otimes n} \). In common algorithms such as empirical risk minimization (ERM) and gradient descent, \( L_S(W) \) is minimized \([48, 49]\). Therefore, to control \( L_\mu(W) \) we need to bound \( \text{gen}_\mu(W, S) \) from above (in expectation or with high probability). Observe that \( \text{gen}_\mu(W, S) \), as defined in (69), is a random variable and a function of \((S, W)\). The generalization error is the expected value of \( \text{gen}_\mu(W, S) \):

\[
\text{gen}(\mu, A) = \mathbb{E}[L_\mu(W) - L_S(W)].
\]
Designing algorithms with low generalization error is a key challenge in machine learning. The following upper bound on the generalization error is given in [11] (see also [10]):

**Theorem 11.** [11] Suppose \( \ell(w,Z) \) is \( \sigma^2 \)-subgaussian under \( Z \sim \mu \) for all \( w \in \mathcal{W} \). Take an arbitrary algorithm \( A \) that runs on a training dataset \( S \). Then the generalization error is bounded as
\[
\text{gen}(\mu,A) \leq \sqrt{\frac{2\sigma^2}{n} I(S;A(S))}.
\]

In [12], a strengthened version of Theorem 11 is given as follows:

**Theorem 12.** [12] Suppose that the loss function \( \ell(w,Z) \) is \( \sigma^2 \)-subgaussian under the distribution \( \mu \) on \( Z \) for any \( w \in \mathcal{W} \). We have:
\[
\text{gen}(\mu,A) \leq \frac{1}{n} \sum_{i=1}^{n} \sqrt{2\sigma^2 I(Z_i;A(S))}.
\] (71)

### 6.1 From lossy compression to generalization error

Theorem 11 provides an upper bound on the generalization error in terms of \( I(S;A(S)) \). Let us now write the sharpest possible bound on the generalization error given an upper bound \( R \) on \( I(S;A(S)) \):
\[
\mathcal{U}_1(R) \triangleq \sup_{P_{W|S}: I(S;W) \leq R} \mathbb{E} \left[ L_\mu(W) - L_S(W) \right]
\] (72)

where the supremum in (72) is over all Markov kernels \( P_{W|S} \) with a bounded input/output mutual information and \( S \sim (\mu)^\otimes n \).

We claim that \( \mathcal{U}_1(R) \) is related to the rate-distortion function. To see this, consider a rate-distortion problem where the input symbol space is \( S \), the reproduction space is \( W \) and the following distortion function between a symbol \( w \) and an input symbol \( s \) is used:
\[
\Delta(w,s) = L_s(w) - L_\mu(w).
\]

With this definition, from (72), we obtain
\[
-\mathcal{U}_1(R) = \inf_{P_{W|S}: I(S;W) \leq R} \mathbb{E} [\Delta(W,S)]
\] (73)

which is in the rate-distortion form. With \( \mathcal{U}_1(R) \) defined as in (72), it follows that for any arbitrary algorithm \( A \) with \( I(S;A(S)) \leq R \) we have
\[
\text{gen}(\mu,A) \leq \mathcal{U}_1(R).
\]

This upper bound does not require any subgaussianity assumption on the loss function. From this viewpoint, Theorem 11 is just a convenient and explicit lower bound on a rate-distortion function under an extra assumption on the loss function. In fact, Corollary 4 shows that
\[
\mathcal{U}_1(R) \leq \sqrt{\frac{2\sigma^2}{n} R},
\] (74)

yielding Theorem 11.

\[7\]While the literature commonly takes the reproduction space to be the same as the input symbol space, the rate-distortion theory does not formally require that.
Note that by a similar argument, if instead of an upper bound on $I(S; A(S)) \leq R$, we know $I(Z_i; A(S)) \leq R_i$ for $1 \leq i \leq n$, the sharpest bound on the generalization error would be

$$\text{gen}(\mu, A) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[L_\mu(W) - \ell(W, Z_i)] \leq \frac{1}{n} \sum_{i=1}^{n} U_2(R_i).$$

Here,

$$U_2(R) \triangleq \sup_{P_{W|Z}: I(W; Z) \leq R} \mathbb{E} \left[ L_\mu(W) - \ell(W, Z) \right]$$

where $Z \in \mathcal{Z}$ is distributed according to $\mu$. As before, $-U_2(R)$ is in the rate-distortion form. Then, Corollary 4 yields Theorem 12.

As discussed above, one of our contributions is the novel connection between the generalization error and the rate-distortion theory. This connection is at a formal level and follows from the similarity of the expression of the generalization error and that of the rate-distortion function. In a subsequent work [50], Sefidgaran et al. give an operational interpretation of generalization error in terms of the rate-distortion function.

### 6.1.1 Improving rate-distortion type bound via rate of consistency

As we saw above the sharpest possible bound on the generalization error given an upper bound $R_i$ on $I(Z_i; A(S)) \leq R_i$ was

$$\text{gen}(\mu, A) \leq \frac{1}{n} \sum_{i=1}^{n} U_2(R_i).$$

As an example, consider the problem of learning the mean of a Gaussian random vector $Z \sim \mathcal{N}(\beta, \sigma^2 I_d)$ with loss function $\ell(w, z) = \|w - z\|^2$ using the ERM algorithm. The generalization error of the ERM algorithm can be computed exactly as

$$\text{gen}(P_Z, P_{W|S}) = \frac{2\sigma^2 d}{n}.$$  \hspace{1cm} (77)

To compare (77) with (76), note that the output of ERM algorithm $W_{ERM} = \frac{1}{n} \sum_{i=1}^{n} Z_i$ and $W_{ERM} \sim \mathcal{N}(\beta, \frac{\sigma^2}{n} I_d)$, and therefore

$$I(W_{ERM}; Z_i) = \frac{d}{2} \ln \frac{n}{n-1}$$

where it is obtained in [12]. However, we claim that $U_2(R) = \infty$ for this example. To see this, note that

$$L_\mu(\hat{w}) - \ell(\hat{w}, z) = \sigma^2 d + \|\beta - \hat{w}\|^2 - \|\hat{w} - z\|^2 = \sigma^2 d + \|\beta\|^2 - \|z\|^2 + 2\hat{w}^t(z - \beta)$$

Take some constant $c$ and let $\hat{W} = c(Z - \beta) + T$ where $T \sim \mathcal{N}(0, \tilde{\sigma}^2 I_d)$ is independent of $Z$. For every $c$, by letting $\tilde{\sigma}^2 \to \infty$ we can ensure that $I(\hat{W}; Z) \leq R$. Next, by letting $c \to \infty$ we get that $\mathbb{E}[L_\mu(\hat{w}) - \ell(\hat{w}, z)] \to \infty$.

Next, we show that one can use the notion of “rate of consistency” in conjunction with rate-distortion bounds to get useful bounds on the generalization error (in fact an exact bound for our example).

**Definition 6.** Let

$$W^* = \arg \min_{w \in \mathcal{W}} \mathbb{E}[\ell(w, Z)]$$

As discussed above, one of our contributions is the novel connection between the generalization error and the rate-distortion theory. This connection is at a formal level and follows from the similarity of the expression of the generalization error and that of the rate-distortion function. In a subsequent work [50], Sefidgaran et al. give an operational interpretation of generalization error in terms of the rate-distortion function.
be the optimal hypothesis set. We cannot compute \( \mathcal{W}^* \) since we do not know the sample distribution. Let \( \kappa(n) \triangleq \mathbb{E} \left[ \min_{w \in \mathcal{W}^*} \| A(S) - w \|^2 \right] \) be the average distance between the output of the algorithm and the set \( \mathcal{W}^* \). We say that an algorithm \( A \) is consistent if \( \kappa(n) \) tends to zero as the number of samples \( n \) tends to infinity. We call \( \kappa(n) \) the consistency rate of the algorithm \( A \).

The sharpest possible bound on the generalization error of an algorithm \( A \) with consistency rate \( \kappa(n) \) and an upper bound \( R_i \) on \( I(Z_i; A(S)) \leq R_i \) is

\[
\text{gen} \left( \mu, A \right) \leq \frac{1}{n} \sum_{i=1}^{n} \hat{U}_2(R_i, \kappa(n)).
\]

where

\[
\hat{U}_2(R, \kappa(n)) = \sup_{P_{W|Z}: I(W; Z) \leq R, E_W \left[ \| W - \beta \|^2 \right] \leq \kappa(n)} \mathbb{E} \left[ L_\mu(W) - \ell(\hat{W}, Z) \right].
\]

We will now compute the bound in (79) for our example of learning the mean of a Gaussian random vector \( Z \sim \mathcal{N}(\beta, \sigma^2 I_d) \) with loss function \( \ell(w, z) = \| w - z \|^2 \). Note that \( \mathcal{W}^* = \{ \beta \} \) for this example. Then,

\[
\hat{U}_2(R, \kappa(n)) = \sup_{P_{W|Z}: I(W; Z) \leq R, E_W \left[ \| W - \beta \|^2 \right] \leq \kappa(n)} \mathbb{E} \left[ L_\mu(W) - \ell(\hat{W}, Z) \right]
\]

\[
\leq \sup_{P_{W|Z}: I(W; Z) \leq R, E_W \left[ \| W - \beta \|^2 \right] \leq \kappa(n)} \mathbb{E}_{P_W P_Z} \| W - Z \|^2 - \inf_{P_{W|Z}: I(W; Z) \leq R} \mathbb{E}[\ell(\hat{W}, Z)]
\]

\[
\leq \sup_{P_{W|Z}: I(W; Z) \leq R, E_W \left[ \| W - \beta \|^2 \right] \leq \kappa(n)} \mathbb{E} \| \hat{W} - \beta \|^2 + \mathbb{E} \| Z - \beta \|^2 - \inf_{P_{W|Z}: I(W; Z) \leq R} \mathbb{E}[\ell(\hat{W}, Z)]
\]

\[
\leq (\kappa(n) + \sigma^2 d - \frac{d}{2e (V_d \Gamma(1 + d/2))^2/d}) \exp \left( \frac{2h(Z) - 2R}{d} \right)
\]

\[
\overset{(a)}{=} \kappa(n) + \sigma^2 d(1 - e^{-2R/d}).
\]

where (a) comes from the Shannon’s lower bound in Corollary 3 and (b) is derived from the fact that \( h(Z) = \frac{d}{2} \ln(2\pi e \sigma^2) \) and \( V_d \Gamma(1 + d/2) = \pi^{d/2} \).

Now, for the ERM algorithm, we have

\[
\kappa(n) = \mathbb{E}_{W_{\text{ERM}}} \left[ \| W_{\text{ERM}} - \beta \|^2 \right] = \frac{\sigma^2 d}{n}.
\]

From (78), we have \( R_i = \frac{d}{2} \ln \frac{n}{n-1} \) and we obtain \( \hat{U}_2(R_i, \kappa(n)) = \frac{2\sigma^2 d}{n} \). Therefore, from (79) we obtain \( \text{gen} \left( \mu, A \right) \leq \frac{2\sigma^2 d}{n} \) which matches (77) and is exactly tight!

**Remark 9.** It is shown [12] that for ERM algorithm on gaussian mean estimation model, Theorem 12 with sub-gaussianity assumption of loss function \( \ell(W, Z) \) under \( P_{W_{\text{ERM}}} P_Z \) yields an upper bound on the generalization error of order \( \frac{1}{\sqrt{n}} \). However, the above rate distortion upper bound is order of \( \frac{1}{n} \).
6.1.2 Computational aspects of the rate-distortion bound

Assuming that an upper bound $R$ on the $I(S; A(S))$ is known, computing the upper bound $\mathcal{U}_1(R)$ is still a concern if the sample size $n$ is large despite the fact that computing $\mathcal{U}_1(R)$ is a convex optimization problem and there are efficient algorithms for solving it for discrete sample spaces [51]. The following theorem addresses this computational concern by providing an upper bound in a “single-letter” form:

**Theorem 13.** For any arbitrary loss function $\ell(w, z)$, and algorithm $A$ that runs on a training dataset $S$ of size $n$, we have

$$\mathcal{U}_1(R) \leq \mathcal{U}_2 \left( \frac{R}{n} \right)$$

where $\mathcal{U}_2(R)$ is defined in (75). Consequently,

$$\text{gen}(\mu, A) \leq \mathcal{U}_2 \left( \frac{I(S; A(S))}{n} \right)$$

Furthermore, to compute the maximum in (75), it suffices to compute the maximum over all conditional distributions $P_{\hat{W}|Z}$ for $\hat{W} \in \mathcal{W}$ such that the support of $\hat{W}$ has size at most $|Z| + 1$.

**Remark 10.** In Appendix A, we show that “auxiliary loss functions” can be utilized to tighten the gap between $\mathcal{U}_1(R)$ and $\mathcal{U}_2(R/n)$.

A generalized version of Theorem 13 is given later in Theorem 14. Therefore, a separate proof is not given for the above theorem here.

Note that $\mathcal{U}_2(R)$ is still in terms of a rate-distortion function and does not admit an explicit closed-form expression in general. However, the Blahut-Arimoto algorithm can be used to compute it [51] even when the cardinality of instance space $Z$ is infinite (see also [52]).

It is shown in [56] that Blahut-Arimoto algorithm in discrete case has only two kinds of convergence speeds depending on the source distribution. One is the exponential convergence, which is a fast convergence, and the other is the convergence of order $O(1/N)$ where $N$ is number of the iterations.

Note that the bound in Theorem 11 is in a very explicit form. Moreover, the bound in Theorem 11 depends only on mutual information $I(S; A(S))$ while the bounds in Theorem 13 depend on $\mu$, and $I(S; A(S))$ (as $\mathcal{U}_1(\cdot)$ and $\mathcal{U}_2(\cdot)$ depend on $\mu$). However, one can obtain a bound from Theorem 13 that does not depend on $\mu$ by maximizing the bound in Theorem 13 over all distributions $\mu$. We show that even after this maximization, the bound in Theorem 13 is still an improvement over Theorem 11. To see this, first take some arbitrary $\mu$ such that $\ell(w, Z)$ is $\sigma^2$-subgaussian for every $w \in \mathcal{W}$ under the distribution $\mu$ on $Z$. Then, observe that the bound in Theorem 13 is always less than or equal to the bound in Theorem 11 since $\mathcal{U}_2(R) \leq \sqrt{2\sigma^2R}$. On the other hand, the following examples show that the maximum of $\mathcal{U}_2(R)$ over all distributions $\mu$ with the $\sigma^2$-subgaussian property can be strictly less than the bound in Theorem 11.

**Example 6.** Figure 4 depicts the bound in Theorem 11 versus the maximum of the bound in Theorem 13 over all distributions $\mu$ on $\{0, 1\}$ for two particular loss functions. Note that the distortion function itself depends on the choice of $\mu$ and this makes it difficult to find a closed form expression for the maximum of the bound in Theorem 13 over all distributions $\mu$. In the left image, we consider $\mathcal{W} = Z = \{0, 1\}$ and a learning problem on a data set $S$ with the size $n = 1$ with loss function $\ell(w, z) = w \cdot z$. In the right picture, we consider $\mathcal{W} = [0, 1]$, $Z = \{0, 1\}$ and a learning problem on a data set $S$ with the size $n = 1$ with loss function $\ell(w, z) = |w - z|$.

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8Rate-distortion theory for continuous or abstract alphabets is discussed at length in the literature, e.g. see [53, 54]. See also [55] for a survey.
Figure 4: Left picture: The bound in Theorem 11 versus the maximum of the upper bound in Theorem 13 over all distributions \( \mu \) when \( W = Z = \{0, 1\} \) and \( \ell(w, z) = w \cdot z \). Right picture: The bound in Theorem 11 versus the maximum of the upper bound in Theorem 13 over all distributions \( \mu \) when \( W = [0, 1] \), \( Z = \{0, 1\} \) and \( \ell(w, z) = |w - z| \).

6.2 Generalization error for algorithms with bounded input/output mutual \( f \)-information

Theorem 11 shows that the generalization error of a learning algorithm can be bounded from above in terms of the mutual information between the input and output of the algorithm [10, 11]. Similar bounds are obtained in [12–20] for various generalizations and extensions using other measures of dependence.

In this paper we are interested in the generalization error using the mutual \( f \)-information instead of Shannon’s mutual information. As before, the sharpest possible upper bound on the generalization error given an upper bound \( R \) on \( I_f(S; A(S)) \) is

\[
\mathcal{U}^1_f(R) \triangleq \sup_{P_{W|S}: I_f(S; W) \leq R} \mathbb{E}[L_{\mu}(W) - L_{S}(W)],
\]

where the training data \( S = (Z_1, Z_2, \ldots, Z_n) \in \mathcal{Z}^n \) is a sequence of \( n \) i.i.d. repetitions of samples generated according to a distribution \( \mu \). It follows that for any algorithm \( A \) satisfying \( I_f(S; A(S)) \leq R \), we have

\[
\text{gen}(\mu, A) \leq \mathcal{U}^1_f(R).
\]

Similarly, assuming \( I_f(Z_i; W) \leq R_i \) for \( 1 \leq i \leq n \), the sharpest bound is

\[
\text{gen}(\mu, A) \leq \frac{1}{n} \sum_{i=1}^{n} \mathcal{U}^2_f(R_i),
\]

where

\[
\mathcal{U}^2_f(R) \triangleq \sup_{P_{W|Z}: I_f(Z; W) \leq R} \mathbb{E}[L_{\mu}(W) - \ell(W, Z)].
\]

As before, the bound \( \mathcal{U}^2_f(R) \) is easier to compute than \( \mathcal{U}^1_f(R) \) because the optimization problem in (86) is for a single symbol \( Z \) whereas the optimization problem in (84) is for a sequence \( S \) of \( n \) symbols.

The following bound follows from the characterization in (73) and Theorem 10 for \( R = I_f(W; S) \).

23
Corollary 5. We have

\[ U^*_{I_t}(R) \leq \Psi^*_{I_{S_t}}(R) \]  

(87)

where \( f^* \) is the convex conjugate function of \( f \) defined in (59) and

\[ \Psi^*_{I_{S_t}}(x) \equiv \inf_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \frac{a + x}{\lambda} + \frac{1}{\lambda} \sup_{w \in \mathcal{W}} \mathbb{E}_{P_Z} [f^* (\lambda L_{S_t}(w) - \lambda \mu(w) - a)] \right\}. \]

In this corollary \( I_t \) can be either the PV or CKZ notions defined in (18) and (17).

Example 7. Let \( f(x) = x \ln x - x + 1 \) and \( \ell(w, Z) \) be \( \sigma^2 \)-subgaussian under \( P_Z \) for all \( w \in \mathcal{W} \). Then \( f^*(y) = e^y - 1 \) and

\[ \Psi^*_{I_{S_t}}(I(S; W)) = \inf_{\lambda \geq 0} \left\{ \frac{I(S; W)}{\lambda} + \frac{1}{\lambda} \sup_{w \in \mathcal{W}} \mathbb{E}_{P_Z} [\exp (\lambda L_{S_t}(w) - \lambda \mu(w))] \right\} \]

\[ \leq \inf_{\lambda \geq 0} \left\{ \frac{I(S; W)}{\lambda} + \frac{\sigma^2}{2n} \right\} = \sqrt{\frac{2\sigma^2 I(S; W)}{n}}, \]  

(88)

recovering Theorem 11.

Thus, from Corollary 5 that

\[ \text{gen}(\mu, P_{W|S}) \leq \Psi^*_{I_{S_t}}(I_t(S; W)) \]  

(89)

Similarly, from (85) and Theorem 10 we obtain

Corollary 6. Let \( f : [0, \infty) \to \mathbb{R} \cup \{\infty\} \) be a convex function. Then

\[ \text{gen}(\mu, P_{W|S}) \leq \frac{1}{n} \sum_{i=1}^{n} U^*_{I_{S_t}}(I_t(Z_i; W)) \leq \frac{1}{n} \sum_{i=1}^{n} \Psi^*_{I_{t(Z_i)}}(I_t(Z_i; W)) \]  

(90)

where

\[ \Psi^*_{I_{t(Z)}}(x) \equiv \inf_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \frac{a + x}{\lambda} + \frac{1}{\lambda} \sup_{w \in \mathcal{W}} \mathbb{E}_{P_Z} [f^* (\lambda \ell(w, Z) - \lambda \mathbb{E}_{P_Z} \ell(w, Z) - a)] \right\}. \]

(91)

In this corollary \( I_t \) can be either the PV or CKZ notions defined in (18) and (17).

Corollary 6 can be used to derive explicit bounds on the generalization error in the following example.

Example 8. (i). Assume \( f(x) = (x - 1)^2 \) for \( x \in [0, \infty) \). Assume \( \ell(w, Z) \) is a random variable with finite variance \( \text{Var}(\ell(w, Z)) \leq \sigma^2 \) for all \( w \in \mathcal{W} \).

\[ \text{gen}(\mu, P_{W|S}) \leq \frac{1}{n} \sum_{i=1}^{n} \sqrt{\sigma^2 \chi^2(P_{WZ_i} || P_{WPZ_i})}. \]

where \( \chi^2(\cdot || \cdot) \) is defined in (2).

(ii). Assume \( \ell(w, Z) \) is a \( \sigma^2 \)-subgaussian random variable under \( P_Z \) for all \( w \in \mathcal{W} \) and \( f(x) = (\alpha + \bar{\alpha} x) \ln (\alpha + \bar{\alpha} x) \) defined on the domain \( \mathbb{R}_+ \) for \( \alpha \in [0, 1] \) and \( \alpha + \bar{\alpha} = 1 \). Then

\[ \text{gen}(\mu, P_{W|S}) \leq \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{2\sigma^2}{\bar{\alpha}^2} D(\bar{\alpha} P_{WZ_i} + \alpha P_{WPZ_i} || P_{WPZ_i}), \quad \forall \alpha \in [0, 1].} \]  

(92)

Proof of Example 8 is given in Section 8.11.
6.3 Upper bound on the generalization error using $f$ in the function class $\mathcal{F}$

The following theorem generalizes Theorem 13 and is helpful from a computational perspective.

**Theorem 14.** Let $f \in \mathcal{F}$. Then for the CKZ and MBGYA notions of mutual $f$-information, we have

$$U_1^f(R) \leq U_2^f(R/n), \quad \forall R \geq 0.$$  \hspace{1cm} (93)

Consequently, for any algorithm $A$ we have

$$\text{gen} (\mu, A) \leq U_2^f \left( \frac{I_f(S; A(S))}{n} \right).$$

Furthermore, to compute the maximum in $U_2^f$ for the CKZ notion of the $f$-information, it suffices to compute the maximum over all conditional distributions $P_{\hat{W}|Z}$ for $\hat{W} \in W$ such that the support of $\hat{W}$ has size at most $|Z| + 1$.

The proof is given in Section 8.10.

For a finite hypothesis space $W$, one can find an explicit upper bound on the generalization error that holds for any arbitrary learning algorithm and depends only on the sample size $n$ as follows:

**Corollary 7.** Assume that $f \in \mathcal{F}$. Then

$$\text{gen} (\mu, A) \leq \Psi_{\ell(Z)}^f \left( \frac{H_f(W_{\text{unif}})}{n} \right)$$

where $H_f$ is defined in Definition 5 and $\Psi_{\ell(Z)}^f(x)$ is defined in (91), and $W_{\text{unif}}$ is a random variable with uniform distribution on $W$.

**Proof.** From Theorem 14, we get

$$\text{gen} (\mu, A) \leq U_1^f (I_f(S; A(S))) \leq U_2^f (I_f(S; A(S))/n) \leq U_2^f (H_f(A(S))/n) \leq U_2^f (H_f(W_{\text{unif}})/n)$$  \hspace{1cm} (94)

(a) comes from Part (iii) of Theorem 5 and (b) comes from Part (ii) of Theorem 5. $\square$

**Example 9.** Let $f(x) = (\alpha + (1 - \alpha)x) \ln(\alpha + (1 - \alpha)x)$ defined on the domain $\mathbb{R}_+$ for $\alpha \in [0, 1]$. We know that $f \in \mathcal{F}$. Let $\ell(w, Z)$ be $\sigma^2$-subgaussian under $P_Z$ for $w \in W$. So from Proposition 7

$$\text{gen} (\mu, A) \leq \Psi_{\ell(Z)}^f \left( \frac{H_f(W_{\text{unif}})}{n} \right) = \sqrt{\frac{2\sigma^2 H_f(W_{\text{unif}})}{(1 - \alpha)^2}}$$

where in this example

$$H_f(W_{\text{unif}}) = \left( 1 - \frac{1}{|W|} \right) \alpha \ln \alpha + \left( \frac{\alpha}{|W|} + 1 - \alpha \right) \ln(\alpha + (1 - \alpha)|W|).$$

In particular, when $\alpha = 0$, we get $\text{gen} (\mu, A) \leq \sqrt{\frac{2\sigma^2 \ln |W|}{n}}$. However, one can minimize

$$\frac{H_f(W_{\text{unif}})}{(1 - \alpha)^2}$$
over $\alpha$ and get strictly better bounds. The minimum of the above expression occurs at $\alpha^* = 1/(|W| - 1)$ and substituting $\alpha^*$, we get

$$\text{gen} (\mu, A) \leq \sqrt{2\sigma^2 \frac{(|W| - 1)^2 \ln(|W| - 1)}{|W| \cdot (|W| - 2)}} \frac{\ln(|W| - 1)}{n},$$

slightly improving over $\sqrt{2\sigma^2 \ln |W| \frac{\ln(|W| - 1)}{n}}$.

7 Future work

The following questions are left for future work:

- Proposition 1 shows that $D_f$ is super-modular for any function $f \in \mathcal{F}$. Is the converse true? One can show that $D_f$ is super-modular if and only if for any product distribution $q_{X_1} q_{X_2}$ and any function $Z(x_1, x_2)$ satisfying $E_q[Z] = 1$ we have

$$E_{X_1, X_2} [f(Z)] \geq E_{X_1} [f(E_{X_2} Z)] + E_{X_2} [f(E_{X_1} Z)].$$

(95)

- It is known that $I_{PV}^f$ has applications in hypothesis testing and channel coding [31, Theorem 8]. Can we find similar applications for $I_{MBGYA}^f$, perhaps to bound the exponent of certain conditional events?

- In Theorem 4, it is shown that $I_{CKZ}$ and $I_{MBGYA}$ satisfy $AB$-property mentioned in the introduction. Does $I_{PV}^f (X; Y)$ satisfy the $AB$-property? The proof does not seem to go through.

- To improve the rate-distortion bound on the generalization error, we used two ideas to add other convex constraints to the set of randomized learning algorithms $P_{W|S}$. Firstly, in Section 6.1.1 we showed that the upper bound on the rate of consistency of learning algorithm can be a useful convex constraint for rate-distortion problems, and in some specific examples of learning model, it yields a tight bound on generalization error. Secondly, in Appendix A we assumed that we know the performance of ERM on some auxiliary loss function (or can estimate the ERM performance from the data itself). We used this knowledge to add a convex constraint to the rate-distortion upper bound. We showed that the modified rate-distortion upper bound improves upon the ordinary rate-distortion upper bound. Exploring other convex constraints which yield tighter upper bounds on generalization error is left as future work.

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8 Proofs of the results

In the following sections we present the proofs of the results stated in the previous sections in their order of appearance.

8.1 Proof of Proposition 1

It is known that f-divergence is related to Φ-entropy: given two distributions p(x) and q(x), let 
\( g(x) = p(x)/q(x) \). Then,

\[
D_f(p||q) = \sum_x q(x)f\left(\frac{p(x)}{q(x)}\right) = \sum_x q(x)f(g(x)) - f(1) = \sum_x q(x)f(g(x)) - f\left(\sum_x q(x)g(x)\right)
\]

Let \( \Phi(x) = f(x) \). Then, \( D_f(p||q) \) is the Φ-entropy of the function \( g(x) \). If the function \( \Phi \in F \), the Φ-entropy becomes subadditive.

We show that for any \( f \in F \), we have (3). Equivalently,

\[
\sum_{x_1,x_2,x_3} q_{X_1}(x_1)q_{X_2}(x_2)q_{X_3}(x_3) \left[ f\left(\frac{p_{X_1,X_2,X_3}(x_1,x_2,x_3)}{q_{X_1}(x_1)q_{X_2}(x_2)q_{X_3}(x_3)}\right)\right] + \sum_{x_3} q_{X_3}(x_3) \left[ f\left(\frac{p_{X_3}(x_3)}{q_{X_3}(x_3)}\right)\right] \\
\geq \sum_{x_1,x_3} q_{X_1}(x_1)q_{X_3}(x_3) \left[ f\left(\frac{p_{X_1,X_3}(x_1,x_3)}{q_{X_1}(x_1)q_{X_3}(x_3)}\right)\right] + \sum_{x_2,x_3} q_{X_2}(x_2)q_{X_3}(x_3) \left[ f\left(\frac{p_{X_2,X_3}(x_2,x_3)}{q_{X_2}(x_2)q_{X_3}(x_3)}\right)\right].
\]

It suffices to show that for any \( x_3 \)

\[
\sum_{x_1,x_2} q_{X_1}(x_1)q_{X_2}(x_2) \left[ f\left(\frac{p_{X_1,X_2,X_3}(x_1,x_2,x_3)}{q_{X_1}(x_1)q_{X_2}(x_2)q_{X_3}(x_3)}\right)\right] - \sum_{x_2} q_{X_2}(x_2) \left[ f\left(\frac{p_{X_2,X_3}(x_2,x_3)}{q_{X_2}(x_2)q_{X_3}(x_3)}\right)\right] \\
\geq \sum_{x_1} q_{X_1}(x_1) \left[ f\left(\frac{p_{X_1,X_3}(x_1,x_3)}{q_{X_1}(x_1)q_{X_3}(x_3)}\right)\right] - f\left(\frac{p_{X_3}(x_3)}{q_{X_3}(x_3)}\right).
\]
Equivalently, we need to show
\[
\sum_{x_1,x_2} q_{x_1}(x_1)q_{x_2}(x_2) \left[ f \left( \frac{p_{x_1,x_2,x_3}(x_1,x_2,x_3)}{q_{x_1}(x_1)q_{x_2}(x_2)q_{x_3}(x_3)} \right) \right] - \sum_{x_2} q_{x_2}(x_2) \left[ f \left( \frac{p_{x_2,x_3}(x_2,x_3)}{q_{x_2}(x_2)q_{x_3}(x_3)} \right) \right] \\
\geq \sum_{x_1} q_{x_1}(x_1) \left[ f \left( \sum_{x_2} q_{x_2}(x_2) \cdot \frac{p_{x_2,x_3}(x_1,x_2,x_3)}{q_{x_2}(x_2)q_{x_3}(x_3)} \right) \right] - \sum_{x_2} q_{x_2}(x_2) \cdot \frac{p_{x_2,x_3}(x_2,x_3)}{q_{x_2}(x_2)q_{x_3}(x_3)}.
\]

The above inequality follows from [23, Exercise 14.2] that for \( f \in \mathcal{F} \) and any non-negative weights \( \lambda_i \)'s adding up to one,
\[
\sum_{i=1}^n \lambda_i f(Z_i) - f \left( \sum_{i=1}^n \lambda_i Z_i \right)
\]
is a jointly convex function of \( Z_i \)'s.

### 8.2 Proof of Lemma 1

For every \( f \in \mathcal{F} \) defined on \([0, \infty)\) we have \( f''(t) + tf'''(t) \geq 0 \) for any \( t \geq 0 \) [57, Exercise 2.3.29]. Hence \( t \to tf'(t) \) is a non-decreasing function. From \( f \in \mathcal{F} \), \( f'(t) > 0 \). Moreover, since \( f \) is not a linear function, \( tf''(t) \) is not zero for all \( t \). Thus, \( \lim_{t \to \infty} tf'(t) > 0 \). Also, from [57, Exercise 2.3.29], we obtain that \( f''(x) \leq 0 \). Hence \( f' \) is a non-increasing function.

Since \( t \to tf'(t) \) is a non-decreasing function, we obtain \( tf'(t) \geq f'(1) \). Therefore,
\[
f''(t) \geq \frac{f''(1)}{t} \quad \forall t \geq 1.
\]

Hence,
\[
f'(t) \geq f'(1) + \int_1^t \frac{f''(1)}{\tau} d\tau.
\]

This shows that \( \lim_{t \to \infty} f'(t) = \infty \). This, in turn, shows that \( \lim_{t \to \infty} f(t) = \infty \).

To show (5) using L’Hospital’s rule we have
\[
\lim_{t \to \infty} \frac{f(t)}{\ln t} = \lim_{t \to \infty} \frac{t f'(t)}{\ln t} = \lim_{t \to \infty} \frac{f(t)}{\ln t} = \lim_{t \to \infty} tf''(t).
\]

To show (6), using L’Hospital’s rule
\[
\lim_{t \to \infty} \frac{t^2}{f(t)} = \lim_{t \to \infty} \frac{2t}{f'(t)} = \lim_{t \to \infty} \frac{2}{f''(t)}.
\]

### 8.3 Proof of the claim in Remark 4

The exponent of Sanov’s bound can be simplified as follows:
\[
E_{SA} = \frac{1}{n} \inf_{Q: \ell(X,W) \geq \delta} \{ nD(Q_{X,W}||P_XQ_W) - (n-1)D(Q_W||P_W) \}
\]
\[
= \frac{1}{n} \inf_{Q} \sup_{\alpha \geq 0} \alpha (\delta - E_{Q_{X,W}}[\ell(X,W)]) + nD(Q_{X,W}||P_XQ_W) - (n-1)D(Q_W||P_W) \quad (99)
\]
\[
= \frac{1}{n} \sup_{\alpha \geq 0} \inf_{Q} \alpha (\delta - E_{Q_{X,W}}[\ell(X,W)]) + nD(Q_{X,W}||P_XQ_W) - (n-1)D(Q_W||P_W)
\]

31
\[
\begin{align*}
&= \sup_{\alpha \geq 0} \alpha \delta + \inf_{Q} \mathbb{E}_{Q} \left[ \ln \frac{Q_{W} \otimes Q_{X|W}^n}{P_{W} \otimes P_{X}^n \exp(\alpha \ell(X, W))} \right] \\
&= \frac{1}{n} \sup_{\alpha \geq 0} \alpha \delta + \inf_{Q_{W}, Q_{X|W}} \mathbb{E}_{Q_{W}} \left[ \mathbb{E}_{Q_{X|W}} \left[ \ln \frac{Q_{W}}{P_{W}} + n \ln \frac{Q_{X|W}}{P_{X} \exp(\frac{\alpha}{n} \ell(X, W))} \right] \right] \\
&= \frac{1}{n} \sup_{\alpha \geq 0} \alpha \delta + \inf_{Q_{W}} \mathbb{E}_{Q_{W}} \left[ \ln \frac{Q_{W}}{P_{W}} + n \ln \frac{1}{\mathbb{E}[\exp(\frac{\alpha}{n} \ell(X, W))|W]} \right] \\
&= \sup_{\alpha \geq 0} \alpha \delta - \frac{1}{n} \ln \mathbb{E}_{W} \left[ \left( \mathbb{E}_{X} [e^{\alpha \ell(X, W)}|W] \right)^{n} \right].
\end{align*}
\]

where (99) follows from Sion’s minimax theorem, (100) and (101) come from solving the optimization problems over \(Q_{X|W}\) and \(Q_{W}\) respectively and substituting the optimizers \(Q_{X|W} = P_{X} e^{\alpha \ell(X, W)} \mathbb{E}_{P_{X}} [e^{\alpha \ell(X, W)}|W]\) and \(Q_{W} = P_{W} \mathbb{E}_{P_{W}} [\mathbb{E}[\exp(\frac{\alpha}{n} \ell(X, W))|W]]\).

### 8.4 Proof of Theorem 4

Proof of (i):

\[
I_{t}^{PV}(X; Y) = \min_{Q_{Y}} \mathbb{E}_{P_{X}Q_{Y}} f \left( \frac{dP_{Y|X}}{dQ_{Y}} \right)
\]

is a minimum over some linear functions of \(P_{X}\) and hence is a concave function.

Next, consider \(I_{t}^{MBGYA}(X; Y)\). Note that

\[
\mathbb{E}_{P_{X}Q_{Y}} \left[ f \left( \frac{dP_{Y|X}}{dQ_{Y}}(Y, X) \right) \right]
\]

is a linear function of \(P_{X}\). The expression

\[
\mathbb{E}_{Q_{Y}} \left[ f \left( \frac{dP_{Y}}{dQ_{Y}}(Y) \right) \right]
\]

is convex in \(P_{Y}\). It is also convex in \(P_{X}\) since \(P_{Y}\) is a linear function of \(P_{X}\) (as \(P_{Y|X}\) is fixed). So

\[
I_{t}^{MBGYA}(X; Y) = \min_{Q_{Y}} \mathbb{E}_{P_{X}Q_{Y}} \left[ f \left( \frac{dP_{Y|X}}{dQ_{Y}}(Y, X) \right) \right] - \mathbb{E}_{Q_{Y}} \left[ f \left( \frac{dP_{Y}}{dQ_{Y}}(Y) \right) \right].
\]

is a minimization over a sum of a linear function of \(P_{X}\) and a concave function of \(P_{X}\). Therefore, it is concave in \(P_{X}\).

Proof of (ii): for simplicity of exposition, we prove the statement for discrete variables. Take channels \(p_{i}(y|x)\) and non-negative weights \(\lambda_{i}\) adding up to one. Let \(\bar{p}(y|x) = \sum_{i} \lambda_{i} p_{i}(y|x)\) be the average channel.

We begin with the CKZ definition of mutual f-information. Define the perspective function of \(f\) as

\[
g(t, z) = tf(z/t), \quad t > 0, z \geq 0.
\]

(104)
It is known that $f$ is a convex function, $g$ is a jointly convex function of $(t, z)$; the Hessian equals
\[
\begin{pmatrix}
\frac{1}{t}f''(\frac{z}{t}) & -\frac{z}{t}f''(\frac{z}{t}) \\
-\frac{z}{t}f''(\frac{z}{t}) & \frac{1}{t}f''(\frac{z}{t})
\end{pmatrix} \succeq 0.
\]

Applying Jensen’s inequality, we obtain $g(t, z) = tf(z/t)$
\[
\sum_i \lambda_i \sum_{x,y} p(x)p_i(y)f\left(\frac{p_i(y|x)}{p_i(y)}\right) \geq \sum_{x,y} p(x) \left\{ \sum_i \lambda_i p_i(y) \right\} f\left(\frac{\sum_i \lambda_i p_i(y|x)}{\sum_i \lambda_i p_i(y)}\right)
= \sum_{x,y} p(x)\bar{p}(y)f\left(\frac{\bar{p}(y|x)}{\bar{p}(y)}\right).
\]
This yields the desired result.
Next, consider the PV definition of mutual $f$-information:
\[
I_{q_Y} F_{PV}(X; Y) = \min_{q_Y} D_{t}(P_{XY} \| P_X \times Q_Y).
\]
Note that $D_t(P_{XY} \| P_X \times Q_Y)$ is a jointly convex function of $(q(y), p(y|x))$ for a fixed $p(x)$. This follows from the joint convexity property of $D_t$. From elementary convex analysis, $\min_{a \in A} \phi(a, b)$ is convex in $b$ when $\phi(a, b)$ is jointly convex function of $(a, b)$ and $A$ is a convex set. Therefore,
\[
I_{q_Y} F_{PV}(X; Y) = \min_{q_Y} D_{t}(P_{XY} \| P_X \times Q_Y)
\]
is jointly convex in $p(y|x)$.
Finally consider the MBGYA notion of mutual $f$-information. Let
\[
J(p_{Y|X}(y)) := \sum_{x \in X} p_X(x)f\left(p_{Y|X}(y, x)\right) - f\left(\sum_{x \in X} p_{Y|X}(y, x)p_X(x)\right).
\]
This function is jointly convex in $p_{Y|X}$. This follows from the following fact about functions in $F$: for non-negative weights $\lambda_i$ adding up to one, from [23, Exercise 14.2] we have that the function
\[
(z_1, \cdots, z_n) \mapsto \sum_{i=1}^n \lambda_i f(z_i) - f\left(\sum_{i=1}^n \lambda_i z_i\right)
\]
is jointly convex. Using the convexity property of the perspective function (defined above in (104)), we obtain that
\[
\sum_{y \in Y'} q_Y(y) J\left(p_{Y|X}(y)\right) q_Y(y)
\]
is a jointly convex function of $q_Y$ and $p_{Y|X}$. Then
\[
I^{MBGYA}_{q_Y}(X; Y) = \min_{q_Y} \sum_{y \in Y'} q_Y(y) J\left(p_{Y|X}(y)\right) q_Y(y)
\]
is convex function in $p_{Y|X}$. Thus, the claim is established.

Proof of (iii): by induction on $n$, it suffices to show that for any independent $X_1$ and $X_2$, we have
\[
I_t(X_1, X_2; U) \geq I_t(X_1; U) + I_t(X_2; U).
\]
To show the above inequality when $f$ belongs to $\mathcal{F}$ we proceed as follows: for the MBGYA notion of mutual $f$-information, it suffices to show that for any arbitrary $Q_U$, we have

$$ D_t(P_{X_1,X_2,U} \| P_{X_1,X_2}Q_U) - D_t(P_U \| Q_U) \geq \sum_{i=1}^{2} \{ D_t(P_{X_i,U} \| P_{X_i}Q_U) - D_t(P_U \| Q_U) \}. $$

(106)

This follows from independence of $X_1$ and $X_2$ and the super-modularity of $D_t$ as defined in (3).

For the CKZ definition of $f$-information, it suffices to consider (106) for $Q_U = P_U$.

Proof of (iv): the inequality $I_t^{CKZ}(X;Y) \geq I_t^{PV}(X;Y) \geq I_t^{MBGYA}(X;Y)$ follows from the definitions of mutual $f$-information. It remains to show that

$$ I(X;Y) \cdot \left[ \lim_{t \to \infty} t f''(t) \right] \geq I_t^{CKZ}(X;Y). $$

If $\lim_{t \to \infty} t f''(t) = \infty$, there is nothing to prove. Assume that $k \triangleq \lim_{t \to \infty} t f''(t) < \infty$. From Lemma 1, $t f''(t)$ is non-decreasing. Thus,

$$ \forall t \in [0, \infty), \ f''(t) \leq \frac{k}{t}. $$

(107)

We claim that

$$ f(z) - f'(1)(z - 1) \leq k(z \ln(z) - z + 1), \quad \forall z. $$

(108)

Take some $1 \leq z < \infty$. Integrating both hands sides of (107) twice, each time from 1 to $z$ yields (108) for all $z \geq 1$. The proof of (108) for $0 < z < 1$ is similar. Integrating both hands sides of (107) from $z$ to 1 yields

$$ f'(1) - f'(z) \leq -k \ln(z). $$

Integrating again from $z$ to 1 yields (108) for all $z \leq 1$. Finally, observe that the upper bound on $f(\cdot)$ in (108) immediately implies that

$$ I_t^{CKZ}(X;Y) \leq k \cdot I(X;Y). $$

8.5 Proof of Theorem 5

Proof of (i):

$$ H_t^{MBGYA}(X) = \min_{Q_X} \left\{ f(0) \left( 1 - \sum_{x \in \mathcal{X}} P_X(x)Q_X(x) \right) + \sum_{x \in \mathcal{X}} P_X(x)Q_X(x) f \left( \frac{1}{Q_X(x)} \right) - \sum_{x \in \mathcal{X}} Q_X(x) f \left( \frac{P_X(x)}{Q_X(x)} \right) \right\} $$

is a minimum over a concave function of $P_X$. Therefore, it is concave in $P_X$.

Next, consider $H_t^{CKZ}(X)$ under the assumption that $f'''(x) \leq 0$. Assume that $\mathcal{X} = \{ x_1, x_2, \cdots, x_n \}$ and $p_i = \mathbb{P}[X = x_i]$ for $1 \leq i \leq n$. We have

$$ H_t^{CKZ}(X) = \sum_{i=1}^{n} p_i^2 f \left( \frac{1}{p_i} \right) + \left( 1 - \sum_{i=1}^{n} p_i^2 \right) f(0) $$

(109)

$$ = \sum_{i=1}^{n} L_t(p_i) $$

(110)
where

\[ L_t(x) \triangleq x^2 f\left( \frac{1}{x} \right) + \left( \frac{1}{x} - x^2 \right) f(0). \]

To show concavity of \( H^{CKZ}_t(X) \), it suffices to show concavity of \( L_t(x) \) for \( x \in [0, 1] \). The second derivative of \( L_t \) equals:

\[ 2f\left( \frac{1}{x} \right) - \frac{2}{x} f\left( \frac{1}{x} \right) + \frac{1}{x^2} f''\left( \frac{1}{x} \right) - 2f(0) \]

(111)

Note that for all \( b \leq a \)

\[ f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2} f''(a) \leq f(b). \]

This holds because the residual term in the second order Taylor expansion of \( f(b) \) at the point \( a \) for \( 0 \leq b \leq a \) is

\[ R_2(b) = \int_a^b \frac{f^{(3)}(t)}{2!} (b - t)^2 dt \]

which is non-negative due to \( f^{(3)}(a) \leq 0 \) for all \( a > 0 \). Letting \( a = 1/x \) and \( b = 0 \), we obtain

\[ 2f\left( \frac{1}{x} \right) - \frac{2}{x} f\left( \frac{1}{x} \right) + \frac{1}{x^2} f''\left( \frac{1}{x} \right) - 2f(0) \leq 0 \]

as desired.

Finally, one of the equivalent forms of \( f \in \mathcal{F} \) presented in [24, Proposition 11] is that \( B_t : (x, y) \rightarrow f(x) - f(y) - f'(x)(x - y) \) is a jointly convex function. Therefore, it is also a separately convex function. If \( B_t \) is seperately convex on \((0, \infty) \times (0, \infty)\), then \( f''(x) \leq 0 \) for \( x \in (0, \infty) \) [57, Exercise 2.3.29]. This completes the proof of part (i).

Proof of (ii): Since the \( f \)-entropy is concave in \( p(x) \) and symmetric, it must be maximized at the uniform distribution.

Proof of (iii): The inequality \( I^{CKZ}_t(X; Y) \leq H^{CKZ}_t(Y) \) is proved in [19, Lemma 5]. The proof for \( I^{PV}_t(X; Y) \leq H^{PV}_t(Y) \) and \( I^{MBGYA}_t(X; Y) \leq H^{MBGYA}_t(Y) \) are similar. Note that

\[ 0 \leq \frac{P_{XY}(x, y)}{P_X(x)Q_Y(y)} \leq \frac{1}{Q_Y(y)} \]

(112)

Therefore, Jensen’s inequality yields

\[ f\left( \frac{P_{XY}(x, y)}{P_X(x)Q_Y(y)} \right) \leq \left( 1 - \frac{P_{XY}(x, y)}{P_X(x)} \right) f(0) + \frac{P_{XY}(x, y)}{P_X(x)} f\left( \frac{1}{Q_Y(y)} \right). \]

Now averaging over \( x, y \) under \( P_X \times Q_Y \), we get

\[ D_t(P_{XY} \parallel P_X \times Q_Y) \leq f(0) \left( 1 - \sum_{y \in Y} P_Y(y)Q_Y(y) \right) + \sum_{y \in Y} P_Y(y)Q_Y(y)f\left( \frac{1}{Q_Y(y)} \right), \]

and

\[ D_t(P_{XY} \parallel P_X \times Q_Y) - D_t(P_Y \parallel Q_Y) \]

\[ \leq f(0) \left( 1 - \sum_{y \in Y} P_Y(y)Q_Y(y) \right) + \sum_{y \in Y} P_Y(y)Q_Y(y)f\left( \frac{1}{Q_Y(y)} \right) - \sum_{y \in Y} Q_Y(y)f\left( \frac{P_Y(y)}{Q_Y(y)} \right). \]

Taking the minimum of both sides over \( Q_Y \), we obtain \( I^{PV}_t(X; Y) \leq H^{PV}_t(Y) \) and \( I^{MBGYA}_t(X; Y) \leq H^{MBGYA}_t(Y) \). Inequality \( I^{CKZ}_t(X; Y) \leq H^{CKZ}_t(Y) \) follows by setting \( Q_Y \equiv P_Y \). It suffices to consider \( X \) being a discrete random variable because of the generalization of the Gel’fand-Yaglom-Peres theorem for \( f \)-divergence in [58, Proposition 1].
8.6 Proof of Claim 1

To show this, we first prove the claim for \( n = 1 \). Note that for \( 0 \leq D \leq \frac{1}{2} \) we have

\[
\mathcal{R}(D) = 1 - H_2(D) = 1 + D \log_2(D) + (1 - D) \log_2(1 - D)
\]

\[
\leq 1 + \log_2 \left( D^2 + (1 - D)^2 \right) = \log_2 \left( 1 + (2D - 1)^2 \right) \tag{113}
\]

\[
= LB_t(D, 1) \tag{114}
\]

where (a) comes from Jensen’s inequality for the logarithm function. For \( n \geq 2 \), we use the Taylor expansion of \( 1 - H_2(D) \) around \( D = 1/2 \):

\[
1 - H_2(D) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{(2D - 1)^{2k}}{2k(2k - 1)}
\]

\[
= \frac{1}{\ln 2} \left[ \frac{(2D - 1)^2}{2} + \frac{(2D - 1)^4}{12} + \frac{(2D - 1)^6}{30} + \frac{(2D - 1)^8}{56} \right] + \frac{1}{\ln 2} \sum_{k=5}^{\infty} \frac{(2D - 1)^{2k}}{2k(2k - 1)}.
\]

Next,

\[
LB_t(D, n) = \frac{1}{n} \log_2(n H_f^{C\tilde{K}Z}(D) + 1)
\]

\[
= \frac{1}{n} \log_2(n(2D - 1)^2 + 1)
\]

\[
= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{n^k(2D - 1)^{2k}(-1)^{k+1}}{nk}
\]

\[
= \frac{1}{\ln 2} \left[ \frac{n(2D - 1)^2}{n} - \frac{n^2(2D - 1)^4}{2n} + \frac{n^3(2D - 1)^6}{3n} - \frac{n^4(2D - 1)^8}{4n} \right]
\]

\[+ \frac{1}{\ln 2} \sum_{k=5}^{\infty} \frac{n^k(2D - 1)^{2k}(-1)^{k+1}}{nk}.
\]

To show \( LB_t(D, n) \geq \mathcal{R}(D) \), it suffices to show the following two claims:

\[
\frac{n(2D - 1)^2}{2} - \frac{n^2(2D - 1)^4}{12} + \frac{n^3(2D - 1)^6}{30} - \frac{n^4(2D - 1)^8}{56} \geq 0 \tag{115}
\]

and for any odd \( k \geq 5 \)

\[
\frac{n^k(2D - 1)^{2k}}{nk} - \frac{n^{k+1}(2D - 1)^{2k+2}}{n(k+1)} - \frac{(2D - 1)^{2k}}{2k(2k - 1)} - \frac{(2D - 1)^{2k+2}}{(2k + 2)(2k + 1)} \geq 0. \tag{116}
\]

Inequality (116) is equivalent with

\[
\frac{n^{k-1}}{k} - \frac{n^k(2D - 1)^2}{k + 1} - \frac{1}{2k(2k - 1)} - \frac{(2D - 1)^2}{(2k + 2)(2k + 1)} \geq 0. \tag{117}
\]

Since \( k \geq 5 \), we have \( \frac{1}{2k(k+1)} \geq \frac{1}{2k(2k-1)} \) and \( \frac{1}{2k(k+1)} \geq \frac{1}{(2k+2)(2k+1)} \) and it suffices to show that

\[
(k + 1)n^{k-1} - kn^k(2D - 1)^2 - \frac{1}{2} - \frac{1}{2}(2D - 1)^2 \geq 0.
\]

which holds as long as \( |D - \frac{1}{2}| \leq \frac{1}{2\sqrt{n}} \).
Next, consider (115):

\[
1 - \frac{n(2D - 1)^2}{2} + \frac{n^2(2D - 1)^4}{3} - \frac{n^3(2D - 1)^6}{4} - \frac{1}{2} \left(\frac{2D - 1}{12} - \frac{3}{30} - \frac{4}{56}\right) \geq 0.
\]

Since \(|D - \frac{1}{2}| \leq \frac{c}{2\sqrt{n}}\), for some constant \(c \leq 1\), we assume that \(D = \frac{1}{2} - \frac{c}{2\sqrt{n}}\). Then we get

\[
1 - \frac{c^2}{2} + \frac{c^4}{3} - \frac{c^6}{4} - \frac{1}{2} - \frac{c^2}{12n} - \frac{c^4}{30n^2} - \frac{c^6}{56n^3} \geq 0.
\]

For \(n \geq 2\), it suffices to show that

\[
1 - \frac{c^2}{2} + \frac{c^4}{3} - \frac{c^6}{4} - \frac{1}{2} - \frac{c^2}{12n} - \frac{c^4}{30n^2} - \frac{c^6}{56n^3} \geq 0.
\]

The last inequality holds for \(c \leq 1\). The claim is established.

8.7 Proof of Theorem 7

We prove the statement for the CKZ notion of mutual f-information. The proof for the MBGYA notion of mutual f-information is similar.

Take some arbitrary \(P_{X^n|X^n}\) satisfying \(I_{f}^{CKZ}(X^n; \hat{X}^n) \leq nR\). Using Part (iii) of Theorem 4 and Theorem 3, we have

\[
nR \geq I_{f}^{CKZ}(X^n; \hat{X}^n) \geq \sum_i I_{f}^{CKZ}(X_i; \hat{X}_i) \geq \sum_i I_{f}^{CKZ}(X_i; \hat{X}_i).
\] (118)

We also have

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d(X_i, \hat{X}_i)] \geq \frac{1}{n} \sum_{i=1}^{n} \mathcal{G}_{f}^{CKZ}(I_{f}^{CKZ}(X_i; \hat{X}_i)) \geq \mathcal{G}_{f}^{CKZ}\left(\frac{1}{n} \sum_{i=1}^{n} I_{f}^{CKZ}(X_i; \hat{X}_i)\right) \geq \mathcal{G}_{f}^{CKZ}(R)
\] (119) \hspace{1cm} (120) \hspace{1cm} (121)

where (119) follows from the definition of \(\mathcal{G}_{f}^{CKZ}\), (120) follows from convexity of \(\mathcal{G}_{f}^{CKZ}(\cdot)\) justified below, and (121) follows from (118) and the fact that \(\mathcal{G}_{f}^{CKZ}(\cdot)\) is a decreasing function. Convexity of \(\mathcal{G}_{f}^{CKZ}(\cdot)\) follows from the fact that Mutual f-information \(I_{f}^{CKZ}(A; B)\) is convex in \(P_{B|A}\) for a fixed distribution on \(P_A\) (Theorem 4, Part (ii)).

Since \(P_{X^n|X^n}\) was an arbitrary conditional distribution satisfying \(I_{f}^{CKZ}(X^n; \hat{X}^n) \leq nR\), we deduce from (119)-(121) the desired inequality.

8.8 Proof of Theorem 10

It follows from (62) that

\[
\mathcal{G}_{f}^{PV}(R) \geq -\sup_{Q_X} \mathbb{P}_{P_X Q_X, R}[-d(X, \hat{X})]
\] (122)
Lemma 2. Lower bound in (64) can be lower bounded by

\[ \sup_{\lambda \geq 0, a \in \mathbb{R}} \left\{ - \frac{1}{\lambda} [a + R] - \frac{1}{\lambda} \phi(\lambda, a) \right\} \geq -2 \max\{1, R\} \cdot \sup_{\hat{x}} \|d(X, \hat{x})\|_{L_t^*}, \]  

where \( \| \cdot \|_{L_t^*} \) is the Orlicz norm as defined in Definition 7.

Proof. The lower bound in the part (ii) for \( f \)-rate-distortion function follows from

\[ \mathcal{G}_{f}^{PV}(R) \geq - \sup_{Q_X} \mathbb{E}_{P_X Q_X} \left[ -d(X, \hat{X}) \right] - \sup_{Q_X} \mathbb{E}_{P_X Q_X} \left[ d(X, \hat{X}) \right] \]

\[ \geq -2 \sup_{Q_X} \left\{ \max\{1, R\} \cdot \|d(X, \hat{X})\|_{L_t^*} \right\}, \]

where in (129) we have \( (X, \hat{X}) \sim P_X Q_X \) and use Part (b) of Theorem 15. Next, observe that for any \( Q_X \) we have

\[ \|d(X, \hat{X})\|_{L_t^*} \leq \sup_{\hat{x}} \|d(X, \hat{x})\|_{L_t^*}, \]

which holds because \( \mathbb{E}_{P_X Q_X} \left[ f^\ast \left( \frac{d(X, \hat{x})^2}{\lambda} \right) \right] \leq \sup_{\hat{x}} \mathbb{E}_{P_X} \left[ f^\ast \left( \frac{d(X, \hat{x})^2}{\lambda} \right) \right] \) for each \( t \geq 0 \) due to the fact that \( f^\ast \) is a non-decreasing function. \( \square \)

Theorem 15. Let \( f : [0, \infty) \to \mathbb{R} \) be a convex function satisfying \( f(1) = 0 \). Take a random variable \( Z \) with a given distribution \( \mu \) on the alphabet set \( Z \). Let

\[ \mathcal{G}_t : = \{ g : Z \to \mathbb{R} : \|g(Z)\|_{L_t^*} < \infty \} \]

where \( f^\ast \) is defined in (59) and \( \|g(Z)\|_{L_t^*} \) is the Orlicz \( L_t^* \) norm on random variable \( g(Z) \) which is defined in Appendix B. Then,

(i). For \( g \in \mathcal{G}_t \), a characterization of \( \mathbb{E}_{\mu, R}^{D_t}[g(Z)] \) using the moment generating function of \( g(Z) \) is as follows:

\[ \mathbb{E}_{\mu, R}^{D_t}[g(Z)] \leq \inf_{\lambda \geq 0, a \in \mathbb{R}} \lambda(R + a) + \lambda \mathbb{E}_{Z \sim \mu} f^\ast \left( \frac{g(Z)}{\lambda} - a \right). \]  

(ii). The set \( \mathcal{G}_t \) is a linear space (under ordinary addition and multiplication of functions) and \( \mathbb{E}_{Q_X R}^{D_t}[\|g(Z)\|] \) is a norm on the space \( \mathcal{G} \). Moreover, this norm relates to the Orlicz \( L_t^* \) norm on random variable \( g(Z) \) as follows:

\[ \mathbb{E}_{\mu, R}^{D_t}[\|g(Z)\|] \leq 2 \max\{1, R\} \|g(Z)\|_{L_t^*}. \]
8.9 Proof of Theorem 15

Proof of the part (a): Note that
\[
\mathcal{F}_{\mu,R}^D[g(Z)] = \sup_{\nu \in \mathcal{F}_R} \mathbb{E}_{Z \sim \nu}[g(Z)]
\] (131)
is a constrained supremum over all probability distributions \(\nu\) satisfying \(D_t(\nu \| \mu) \leq R\). A probability distribution \(\nu\) can be thought of as a non-negative measure satisfying \(\mathbb{E}_{Z \sim \nu} \left[ \frac{d\nu}{d\mu}(Z) \right] = 1\). Observe that
\[
F_{\mu,R}^D[g(Z)] = \sup_{\nu \in \mathcal{F}_R} \mathbb{E}_{Z \sim \nu}[g(Z)]
\] (132)
\[
= \sup_{\nu} \min_{\lambda \geq 0, a \in \mathbb{R}} \mathbb{E}_{Z \sim \nu}[g(Z)] + \lambda(R - D_t(\nu \| \mu)) + a \left(1 - \mathbb{E}_{\mu} \left[ \frac{d\nu}{d\mu}(Z) \right] \right)
\] (133)
where in (133) we take maximum over all non-negative measures \(\nu\) that are not necessarily normalized. Observe that the definition for \(D_t(\nu \| \mu)\) can be extended to unnormalized distributions \(\nu\). Next, one can switch maximum and minimum in (133) because the maximization is a concave optimization problem with a feasible choice of \(\nu = \mu\) meeting the constraint \(D_t(\nu \| \mu) \leq R\). So, the Slater’s condition, a sufficient condition for strong duality is satisfied. Therefore,
\[
\mathcal{F}_{\mu,R}^D[g(Z)] = \min_{\lambda \geq 0, a \in \mathbb{R}} \sup_{\nu} \left\{ \mathbb{E}_{Z \sim \nu}[g(Z)] + \lambda R - \lambda D_t(\nu \| \mu) + a \left(1 - \mathbb{E}_{\mu} \left[ \frac{d\nu}{d\mu}(Z) \right] \right) \right\}
\] (134)
\[
= \min_{\lambda \geq 0, a \in \mathbb{R}} \sup_{\nu} \left\{ \lambda R + a + \lambda \int d\mu(z) \left[ \frac{d\nu}{d\mu}(z) \cdot \frac{g(z) - a}{\lambda} - f \left( \frac{d\nu}{d\mu}(z) \right) \right] \right\}
\] (135)
\[
\leq \min_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \lambda R + a + \lambda \int d\mu(z) \sup_{t(z) \geq 0} \left[ t(z) \cdot \frac{g(z) - a}{\lambda} - f(t(z)) \right] \right\}
\] (136)
\[
= \min_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \lambda(R + a) + \lambda \mathbb{E}_{Z \sim \mu} f^* \left( \frac{g(Z)}{\lambda} - a \right) \right\}
\] (137)

Proof of the part (b): First we show that \(\mathcal{F}_{\mu,R}^D[g(Z)] : \mathcal{G}_t \rightarrow \mathbb{R}_+\) is a norm. Clearly, \(\mathcal{F}_{\mu,R}^D[g(Z)] \geq \mathbb{E}_{Z \sim \mu}[g(Z)] \geq 0\). Moreover, \(\mathcal{F}_{\mu,R}^D[g(Z)] = 0\) implies \(\mathbb{E}_{Z \sim \mu}[g(Z)] = 0\) and hence \(g(Z) = 0\) almost surely with respect to measure \(\mu\). We also have \(\mathcal{F}_{\mu,R}^D[|c| \cdot g(Z)] = |c| \cdot \mathcal{F}_{\mu,R}^D[g(Z)]\). The triangle inequality property is immediate from the sub-additivity of the function \(|\cdot|\).

It remains to show the following inequality for a convex function \(f\) with \(f(1) = 0\):
\[
\mathcal{F}_{\mu,R}^D[g(Z)] \leq 2 \max\{1, R\} \|g(Z)\|_{L^*(\mu)}.
\] (138)

We use the equivalence between the Amemiya norm and the Orlicz norm (see [59])
\[
\|g(Z)\|_{L^*(\mu)} \leq \|g(Z)\|_{L^*(\mu)}^A \leq 2\|g(Z)\|_{L^*(\mu)}
\] (139)
where the Amemiya norm defined as follows:
\[
\|g(Z)\|_{L^*(\mu)}^A \triangleq \inf_{t > 0} \left[ \frac{1 + \mathbb{E}_{Z \sim \mu} f^*(|tg(Z)|)}{t} \right].
\] (140)

From part (a) of Theorem 15 we have
\[
\mathcal{F}_{\mu,R}^D[g(Z)] = \inf_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \lambda(R + a) + \lambda \mathbb{E}_{Z \sim \mu} f^* \left( \frac{|g(Z)|}{\lambda} - a \right) \right\}
\] (141)
\[
\leq R \inf_{\lambda \geq 0} \left\{ \lambda + \lambda \mathbb{E}_{Z \sim \mu} \left( \frac{f^*}{R} \left( \frac{|g(Z)|}{\lambda} \right) \right) \right\} \tag{142}
\]
\[
= R \|g(Z)\|_{L^*_\mu(\mu)} \tag{143}
\]

Consider two cases: if \( R > 1 \), then from (143) and (139) we obtain \( \mathbb{F}_{\mu,R}^D \|g(Z)\| \leq 2R \|g(Z)\|_{L^*_\mu(\mu)} \). On the other hand, for \( R > 1 \) we have \( \|g(Z)\|_{L^*_\mu(\mu)} \leq \|g(Z)\|_{L^*_\mu(\mu)} \), so we get \( \mathbb{F}_{\mu,R}^D \|g(Z)\| \leq 2R \|g(Z)\|_{L^*_\mu(\mu)} \) as desired. Next, if \( R \leq 1 \), then from the definition of \( \mathbb{F}_{\mu,R}^D \), (143) for the choice of \( R = 1 \), and (139) we have

\[
\mathbb{F}_{\mu,R}^D \|g(Z)\| \leq \mathbb{F}_{\mu,R}^D \|g(Z)\| \leq \|g(Z)\|_{L^*_\mu(\mu)} \leq 2 \|g(Z)\|_{L^*_\mu(\mu)}.
\]

This yields the desired inequality in both cases.

### 8.10 Proof of Theorem 14

Let \( \bar{w} = (\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_n) \in \mathcal{W}^n \) be a sequence of length \( n \). Let

\[
\bar{U}_1^S(R) \triangleq \sup_{P_{W|S}} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ L_{\mu}(\bar{W}_i) - \ell(\bar{W}_i, Z_i) \right] \tag{144}
\]

where \( S = (Z_1, Z_2, \cdots, Z_n) \) and \( \bar{W} = (\bar{W}_1, \bar{W}_2, \cdots, \bar{W}_n) \). Observe that if the entries of the vector \( \bar{W} \) are all equal, the expression in (144) reduces to the one in (84). Therefore, in (144) we are taking the supremum over a larger set. Thus, \( \bar{U}_1^S(R) \geq \bar{U}_2^S(R) \). We claim that \( \bar{U}_1^S(R) \leq \bar{U}_2^S(R/n) \). This follows from Theorem 7 since \( -\bar{U}_1^S(R) \) and \( -\bar{U}_2^S(R/n) \) can be expressed as rate-distortion functions. The cardinality bound in the statement of the theorem follows from Corollary 2.

### 8.11 Proof of Example 8

Proof of (i): The convex conjugate of \( f(x) = (1-x)^2 \) for \( x \in \mathbb{R} \) is \( f^*(y) = y + \frac{1}{4} \). Then,

\[
D_1(P\|Q) = \chi^2(P\|Q) = \mathbb{E}_Q \left[ \left( \frac{dP}{dQ} \right)^2 \right] - 1
\]

\[
\mathbb{E}_{P_Z} \left[ \lambda (\ell(w, Z) - \mathbb{E}_Z (\ell(w, Z))) - a + \frac{(\lambda (\ell(w, Z) - \lambda \mathbb{E}_Z (\ell(w, Z)) - a)^2}{4} \right] \leq \frac{\lambda^2 \sigma^2}{4} + \frac{a^2}{4} - a
\]

So

\[
\Psi_\ell^{f^*}(\chi^2(P_{W,Z}||P_{W}P_{Z})) = \inf_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \frac{1}{\lambda} \chi^2(P_{W,Z}||P_{W}P_{Z}) + \frac{a}{\lambda} + \frac{1}{\lambda} \left[ \frac{\lambda^2 \sigma^2}{4} + \frac{a^2}{4} - a \right] \right\}
\]

\[
= \inf_{\lambda \geq 0} \left\{ \frac{1}{\lambda} \chi^2(P_{W,Z}||P_{W}P_{Z}) + \frac{\lambda \sigma^2}{4} \right\} = \sqrt{\sigma^2 \chi^2(P_{W,Z}||P_{W}P_{Z})}.
\]

Then using Theorem 6,

\[
\text{gen}(\mu, P_{W|S}) \leq \frac{1}{n} \sum_{i=1}^n \sqrt{\sigma^2 \chi^2(P_{W,Z}||P_{W}P_{Z})}.
\]

Proof of (ii):

\[
D_1(P\|Q) = D(\alpha P + \alpha Q\|Q)
\]
Its convex conjugate function is \( f^*(y) = e^{\frac{y^2}{2\alpha}} - \frac{y}{\alpha} \) for \( y \in \mathbb{R} \). We use the theorem 6 to characterize an upper bound on generalization error:

\[
\mathbb{E}_Z[\frac{1}{n}[\lambda\ell(w, Z) - \lambda\mathbb{E}_Z\ell(w, Z) - \frac{\lambda}{2} - 1] - \frac{\alpha}{2}\mathbb{E}_Z(\lambda\ell(w, Z) - \lambda\mathbb{E}_Z\ell(w, Z) - a) \leq e^{-\frac{a}{2\alpha}} e^{\frac{\lambda^2\sigma^2}{2}} + \frac{\alpha}{\alpha} \forall \lambda, a \in \mathbb{R}.
\]

Then

\[
\Psi^*_{\ell, (\cdot, Z)}(I_{f, C}^Z(W; Z_i)) \triangleq \inf_{\lambda \geq 0, a \in \mathbb{R}} \left\{ \frac{1}{\lambda} I_{f, C}^Z(W; Z_i) + \frac{a}{\lambda} + \frac{1}{2}\phi(\lambda, a) \right\}
\]

\[
= \inf_{\lambda \geq 0} \left\{ \frac{1}{\lambda} I_{f, C}^Z(W; Z_i) + \frac{1}{2}\lambda^2 \sigma^2 \right\} = \sqrt{\frac{2\sigma^2}{\lambda}} D(\tilde{a}P_{WZ_i} + \alpha P_W P_{Z_i} \| P_W P_{Z_i}).
\]

Obviously \( \alpha = 0 \) is deduced to Example 1. We will also look at regime \( \alpha \to 1 \). Define \( Q_{WZ_i}^\alpha \triangleq \tilde{a}P_{WZ_i} + \alpha P_W P_{Z_i} \) and then \( Q_{WZ_i}^1 = P_W P_{Z_i} \), we get

\[
\text{gen}(\mu, P_{W|S}) \leq \lim_{\alpha \to 1} \frac{1}{n} \sum_{i=1}^n \sqrt{2\sigma^2 \lambda_i} D(Q_{WZ_i}^\alpha \| Q_{WZ_i}^\alpha)} = \frac{1}{n} \sum_{i=1}^n \sqrt{2\sigma^2 \lambda_i} D(Q_{WZ_i} \| P_{WZ_i})
\]

\[
= \frac{1}{n} \sum_{i=1}^n \sqrt{2\sigma^2 \lambda_i} (P_{WZ_i} \| P_{WZ_i})
\]

(a) comes from the fact that the fisher information \( J_\alpha \) defined on \( Q_{WZ_i}^\alpha \) in \( \alpha = 1 \) is equal to \( \lim_{\alpha \to 1} \frac{D(Q_{WZ_i}^\alpha \| Q_{WZ_i}^\alpha)}{(1-\alpha)^2} \).

\[
J_1 = \mathbb{E}_{Q_{WZ_i}} \left[ \left( \frac{\partial Q_{WZ_i}}{\partial \mu} \right) \left( \frac{\partial Q_{WZ_i}}{\partial \nu} \right) \right] = \mathbb{E}_{P_{WZ_i}} \left[ \left( 1 - \frac{dP_{WZ_i}}{dP_{WZ_i}} \right) \right] = \chi^2(P_{WZ_i}, P_{WZ_i}).
\]

### A Improving the rate-distortion upper bound via the auxiliary loss function approach

While \( \mathcal{U}_1(R) \) (as defined in (72)) is the sharpest possible bound on the generalization error given an upper bound \( R \) on \( I(S; A(S)) \), the single-letter bound \( \mathcal{U}_2(R/n) \) in Theorem 13 is not. In fact, the following relaxation is used in the proof of Theorem 13: instead of producing one output hypothesis \( W \) for the entire sequence \( S = (Z_1, Z_2, \cdots, Z_n) \), we produce \( n \) output hypothesis \( W_1, W_2, \cdots, W_n \). To tighten the gap between \( \mathcal{U}_1(R) \) and \( \mathcal{U}_2(R/n) \), one needs to answer the following question: given a joint distribution \( (W, Z_1, Z_2, \cdots, Z_n) \), what are the set of marginal distributions on \( (W, Z_i) \)? For instance if \( W \) is a binary random variable and \( (Z_1, Z_2, \cdots, Z_n) \) are i.i.d., \( W \) cannot have high dependence with all of the \( Z_i \)'s.

Motivated by the above question, in the rest of this section we present a general idea which may be used on its own, or in conjunction with the ideas in the previous section to improve the upper bound given in Theorem 13. Let \( \tilde{\ell}(w, z) \) be an “auxiliary” loss function; an arbitrary loss function of our choice which can be different from the original loss function \( \ell(w, z) \). We show that the average risk of the ERM algorithm on the auxiliary loss function \( \tilde{\ell} \) can be used

\[\text{In particular, using mutual information as the measure of dependence we have the following: for a binary } W \text{ and a sequence } (Z_1, \cdots, Z_n) \text{ of independent random variables, we have } 1 \geq I(W; Z_1, Z_2, \cdots, Z_n) = \sum_i I(W; Z_i). \]
to bound the generalization error of a different algorithm $A$, which runs on the same training data as the ERM algorithm, but with the original loss function $\ell(w, z)$. Let

$$\text{ERM}(z_1, \ldots, z_n) = \min_w \sum_{i=1}^n \frac{1}{n} \tilde{\ell}(w, z_i)$$

be the risk of the ERM algorithm given a training sequence $s = (z_1, z_2, \ldots, z_n)$ according to $\tilde{\ell}$. Let

$$v_n = \mathbb{E}_{S \sim (\mu) \otimes n} \text{ERM}(Z_1, \ldots, Z_n)$$

(146)

be the average risk of the ERM algorithm. Let us, for now, assume that $v_n$ is known to us (estimating $v_n$ is discussed in Section A.1).

Take an arbitrary algorithm $A$. Let $W = A(S)$ Then, the risk of $A$ with respect to $\tilde{\ell}$ is greater than or equal the risk of the ERM algorithm, i.e.,

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \frac{1}{n} \tilde{\ell}(W, Z_i) \right] \geq v_n.$$  

(147)

Let $Q$ be a random variable, independent of all previously defined variables, and uniform on the set $\{1, 2, \ldots, n\}$. Set $\tilde{Z} = Z_Q$. Observe that $\tilde{Z} \sim \mu$ because $Z_i \sim \mu$ for all $i$ and $Q$ is independent of $(Z_1, \ldots, Z_n)$. Using this definition for $\tilde{Z}$, the risk of $A$ with respect to the loss $\tilde{\ell}$ equals

$$\mathbb{E}[\tilde{\ell}(W, \tilde{Z})] = \mathbb{E} \left[ \sum_{i=1}^n \frac{1}{n} \tilde{\ell}(W, Z_i) \right]$$

(148)

and the generalization error with respect to the loss $\ell$ can be characterized as

$$\text{gen} (\mu, A) = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ L_\mu(W) - \ell(W, Z_i) \right] = \mathbb{E} \left[ L_\mu(W) - \ell(W, \tilde{Z}) \right].$$

(149)

From (147), (148) and (149) we obtain the following upper bound on the generalization error of $\ell$:

$$\text{gen} (\mu, A) \leq \max_{P_{W|\tilde{Z}}: \mathbb{E}[\hat{\ell}(W, \tilde{Z})] \geq v_n} \mathbb{E} \left[ L_\mu(W) - \ell(W, \tilde{Z}) \right].$$

(150)

where $\tilde{Z} \in \mathcal{Z}$ is distributed according to $\mu$. The above bound has a similar form as the one given in Theorem 13. Observe that (150) provides a generalization bound on the algorithm $A$ based on the sole assumption that it uses a training data of size $n$. If more is known about the algorithm, e.g. an upper bound on the input and output mutual information, we can write better bounds as follows:

**Theorem 16.** Let

$$\hat{U}_2(R) \triangleq \sup_{P_{W|\tilde{Z}}: \mathbb{E}[\hat{\ell}(W, \tilde{Z})] \leq R, \mathbb{E}[\hat{\ell}(W, \tilde{Z})] \geq v_n} \mathbb{E} \left[ L_\mu(W) - \ell(W, \tilde{Z}) \right]$$

(151)

where $v_n$ is defined in (146). Then,

$$U_1(R) \leq \hat{U}_2(R/n) \leq U_2(R/n).$$
Proof of Theorem 16 can be found in Section A.2.

Example 10. Consider the setting in Example 6. Left picture of Figure 5 illustrates this improvement in $\mathcal{U}_2(R/n)$ when $\ell(w, z) = -1[w \neq z]$ and $n = 10$. Right picture of Figure 5 illustrates this improvement in $\mathcal{U}_2(R/n)$ when $\ell(w, z) = (w - z)^2$ and $n = 10$.

A dual form of $\tilde{U}_2(R)$ is given in the following theorem:

**Theorem 17.** We have the following upper bound on $\tilde{U}_2(R)$:

$$
\tilde{U}_2(R) \leq \min_{\lambda \geq 0, \eta \geq 0} \lambda R - \eta v_n + \lambda \sup_{\hat{w}} \left( \mathbb{E}_{P_z} \left[ e^{\lambda \left( \phi^\mu - \ell(\hat{w}, Z) + \eta \hat{\ell}(\hat{w}, Z) \right)} \right] \right)
$$

(152)

**Proof.**

$$
\tilde{U}_2(R) = \sup_{\lambda \geq 0, \eta \geq 0} \mathbb{E} \left[ L_\mu(\hat{W}) - \ell(\hat{W}, Z) \right]
$$

$$
= \sup_{\lambda \geq 0, \eta \geq 0} \mathbb{E} \left[ L_\mu(\hat{W}) - \ell(\hat{W}, Z) \right] + \lambda \left[ R - D(P_W || P_Z) + \eta \mathbb{E}[\hat{\ell}(\hat{W}, Z)] - v_n \right]
$$

$$
\leq \min_{\lambda \geq 0, \eta \geq 0} \mathbb{E} \left[ L_\mu(\hat{W}) - \ell(\hat{W}, Z) \right] + \lambda \left[ R - D(P_W || P_Z) + \eta \mathbb{E}[\hat{\ell}(\hat{W}, Z)] - v_n \right]
$$

(153)

$$
= \min_{\lambda \geq 0, \eta \geq 0} \sup_{\hat{w}} \left[ \lambda R - \eta v_n + \mathbb{E}_{P_Z} \left[ L_\mu(\hat{W}) - \ell(\hat{W}, Z) + \eta \hat{\ell}(\hat{W}, Z) \right] - \lambda D(P_W || P_Z) \right]
$$

(154)

where (a) comes from the fact that

$$
D(P_W || P_Z) = D(P_W || P_Z) + D(P_W || P_Z);
$$

to obtain (b), note that for every fixed $Q_{\hat{W}}$, the minimizing $P_{W|Z}$ in (153) is the Gibbs measure:

$$
dP^*_{W|Z}(\hat{w} | z) := \frac{dQ_W(\hat{w}) e^{\lambda \left( \phi^\mu - \ell(\hat{w}, Z) + \eta \hat{\ell}(\hat{w}, Z) \right)}}{\mathbb{E}_{Q_W} \left[ e^{\lambda \left( \phi^\mu - \ell(\hat{w}, Z) + \eta \hat{\ell}(\hat{w}, Z) \right)} \right]};
$$

Finally, (c) follows from Jensen’s inequality for the concave function $\ln(\cdot)$. This completes the proof. \qed
Figure 5: Left picture: The bound in Theorem 13 and its improved version via the auxiliary loss function $\ell(w, z) = (w - z)^2$ for the learning setting $W = [0, 1], Z = \{0, 1\}$ and $n = 10$ and the original loss function $\ell(w, z) = |w - z|$. Right picture: The bound in Theorem 13 and its improved version via the auxiliary loss function $\ell(w, z) = -1[w \neq z]$ for $W = Z = \{0, 1\}$ and $n = 10$ and the original loss function $\ell(w, z) = w \cdot z$.

A.1 Estimating $v_n$

In order to use the bound in Theorem 16, one must know the value of $v_n$. However, this is not known in practice. For instance, consider the special case of loss function $\ell(w, z) = (w - z)^2$. Given a training data $(z_1, z_2, \cdots, z_n)$, the output of the ERM algorithm with the quadratic loss is just the average of the training data samples and $v_n$ equals

$$v_n = \frac{n - 1}{n} \text{Var}_n(Z).$$

The variance of the test data is not known, but can be estimated from the training dataset itself. Below we show how to estimate $v_n$ by running the ERM algorithm on the available training data. Assume that the auxiliary loss satisfies $|\ell(w, z) - \ell(w, z')| \leq c$ for all $w, z, z'$. Then, we have

$$\text{ERM}(z_1, z_2, \cdots, z_n) = \min_w \frac{1}{n} \sum_{i=1}^{n} \tilde{\ell}(w, z_i) \leq \min_w \left[ \frac{c}{n} + \frac{1}{n} \tilde{\ell}(w, z_1') + \frac{1}{n} \sum_{i=2}^{n} \tilde{\ell}(w, z_i) \right] = \frac{c}{n} + \text{ERM}(z_1', z_2, \cdots, z_n).$$

Then McDiarmid’s inequality implies high concentration around expected value for the ERM algorithm:

$$\Pr \left[ |\text{ERM} - \mathbb{E}[\text{ERM}]| \geq \ell \right] \leq 2e^{-\frac{2n\ell^2}{c^2}}.$$

Thus, one can find an estimate for $v_n$ with high probability based on the available training data sequence.

At the end, we remark that it is also possible to write bounds based on multiple auxiliary loss functions rather than just one.
A.2 Proof of Theorem 16

It is clear that $\tilde{U}_2(R/n) \leq U_2(R/n)$ from their definitions. By the definition of $v_n$ for any arbitrary $p_{W|S}$ where $S = (Z_1, Z_2, \cdots, Z_n)$ we have

$$\mathbb{E} \left[ \sum_{i=1}^{n} \frac{1}{n} \hat{\ell}(W, Z_i) \right] \geq v_n.$$ 

It follows that

$$U_1(R) = \sup_{P_{W|S}: I(W; S) \leq R} \mathbb{E} [L_\mu(W) - L_S(W)] \leq \sup_{P_{W|S}: I(W; S) \leq R} \mathbb{E} [L_\mu(W) - \ell(W, Z_i)] \geq v_n.$$ 

For any arbitrary $p_{W|S}$ we have

$$I(W; S) \geq \sum_{i=1}^{n} I(W; Z_i).$$

Thus,

$$U_1(R) \leq \sup_{P_{W|S}: \frac{1}{n} \sum_{i=1}^{n} I(W; Z_i) \leq R,} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [L_\mu(W) - \ell(W, Z_i)].$$

Take some arbitrary $p_{W|S}$ and a time-sharing random variable $Q$ uniform on $\{1, 2, \cdots, n\}$, independent of previously defined variables. Note that

$$I(W; Z_Q) \leq I(Q, W; Z_Q)$$
$$= I(W; Z_Q|Q)$$
$$= \frac{1}{n} \sum_{i=1}^{n} I(W; Z_i)$$
$$\leq \frac{R}{n}$$

where (155) follows from the fact that $Z_i$’s are iid. We also have

$$\mathbb{E} \left[ \hat{\ell}(W, Z_Q) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \hat{\ell}(W, Z_i) \right] \geq v_n.$$ 

Thus, the joint distribution $p_{W,Z_Q}$ satisfies the constraints of $\tilde{U}_2(R/n)$. Moreover, $Z_Q \sim \mu$ and

$$\mathbb{E} [L_\mu(W) - \ell(W, Z_Q)] = \frac{1}{n} \sum_{i=1}^{n} [L_\mu(W) - \ell(W, Z_i)]$$

Thus, we deduce that $U_1(R) \leq \tilde{U}_2(R/n)$ as desired.
B Orlicz norm and sub-Gaussian random variables

Definition 7. We define the Orlicz space \( L_\psi \) to be the set of all random variables defined on \( X \) such that
\[
\mathbb{E}[\psi(a|X|)] < \infty
\]
for some \( a > 0 \). The \( \psi \) space is a Banach space with respect to the norm
\[
\|X\|_{L_\psi} \triangleq \inf \left\{ t > 0 : \mathbb{E}[\psi(|X|/t)] \leq 1 \right\}.
\]

Definition 8. Let \( \psi_2(x) = e^{x^2} - 1 \). We define the subgaussian random variables to be the set of all random variables defined on \( X \) such that
\[
\mathbb{E}[e^{a^2|X|^2}] < \infty
\]
for some \( a > 0 \). We also call this set \( L_{\psi_2} \). The \( L_{\psi_2} \) space is a Banach space with respect to the norm
\[
\|X\|_{L_{\psi_2}} \triangleq \inf \left\{ t > 0 : \mathbb{E}[e^{\frac{|X|^2}{t^2}}] \leq 2 \right\}.
\]

Remark 11. There is a useful characterization of subgaussian random variables via moment generating function which is equivalent with finiteness of \( L_{\psi_2} \) norm. The random variable \( X \) is said to be sub-Gaussian with parameter \( \sigma^2 \) if
\[
\mathbb{E} \left[ e^{s(X - \mathbb{E}[X])} \right] \leq e^{\frac{s^2\sigma^2}{2}}, \quad \forall s \in \mathbb{R}. \tag{156}
\]

Using the Chernoff’s bound, we obtain,
\[
\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq e^{\frac{-t^2}{2\sigma^2}}, \quad \forall t \geq 0. \tag{157}
\]