MODULI SPACES AND TARGET SPACE DUALITY SYMMETRIES IN
\((0, 2)\) \(Z_N\) ORBIFOLD THEORIES WITH CONTINUOUS WILSON LINES

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\textbf{ABSTRACT}

We present the coset structure of the untwisted moduli space of heterotic \((0, 2)\) \(Z_N\) orbifold compactifications with continuous Wilson lines. For the cases where the internal 6-torus \(T_6\) is given by the direct sum \(T_4 \oplus T_2\), we explicitly construct the Kähler potentials associated with the underlying 2-torus \(T_2\). We then discuss the transformation properties of these Kähler potentials under target space modular symmetries. For the case where the \(Z_N\) twist possesses eigenvalues of \(-1\), we find that holomorphic terms occur in the Kähler potential describing the mixing of complex Wilson moduli. As a consequence, the associated \(T\) and \(U\) moduli are also shown to mix under target space modular transformations.

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1. Introduction

Orbifold models [1, 2, 3, 4, 5, 6], describing the compactification of the heterotic string from ten dimensions down to four, have been extensively studied in the past due to the fact that these models are exactly solvable and that they can predict semi-realistic physics. Orbifold compactifications possess various continuous parameters, called moduli, corresponding to marginal deformations of the underlying conformal theory. They enter into the 4D $\mathcal{N} = 1$ supersymmetric low-energy effective Lagrangian as chiral matter fields with flat potentials. Moduli take their values in a manifold called moduli space. For models yielding $\mathcal{N} = 1$ space-time supersymmetry this moduli space is, locally, a Kählerian manifold. Its Kähler potential is one of the three functions [7, 8] which describe the coupling of moduli fields to $\mathcal{N} = 1$ supergravity in the 4D low-energy effective Lagrangian. It has been known for quite some time now that a certain class of moduli, called continuous Wilson lines, can occur whenever the gauge twist in the $E_8 \otimes E_8$ root lattice is realized not by a shift, but rather by a rotation [6, 9, 10]. Continuous Wilson lines are of big interest, not only because they enter into the 4D low-energy effective Lagrangian as described above, but also because turning them on leads to $(0, 2)$ models with observable gauge groups smaller than the generic gauge group $E_6 \otimes H$ (where $H = SU(3), SU(2) \otimes U(1), U(1)^2$) occurring in $(2, 2)$ symmetric $Z_N$ orbifold compactifications. This gauge symmetry breaking is related to the stringy Higgs effect [11].

It was only recently [10], however, that a first step towards a complete classification of the untwisted moduli space of $Z_N$ orbifold theories with continuous Wilson lines was taken. After reviewing some of the relevant facts about toroidal and orbifold compactifications in section 2, we will in section 3 derive the local structure of the untwisted moduli space of asymmetric $Z_N$ orbifolds with continuous Wilson lines. We find that its local structure is given by a direct product of $\frac{SU(n,m)}{SU(n) \otimes SU(m) \otimes U(1)}$ and $\frac{SO(r,p)}{SO(r) \otimes SO(p)}$ cosets and that it is entirely determined by the eigenvalues of the twist $\Theta$ on the underlying Narain lattice. We then proceed with a general discussion of target space modular symmetries in asymmetric $Z_N$ orbifolds with continuous Wilson lines. This is presented in section 4. Target space duality symmetries consist of discrete reparametrisations of the moduli fields which change the geometry of the internal space but leave the underlying conformal field theory invariant. This implies that certain points in the moduli space of orbifold models have to be identified. Thus, the moduli space of the underlying conformal field theory is an orbifold and not a smooth manifold. In section 4, we also introduce two sets of standard coordinates on the $\frac{SO(r,p)}{SO(r) \otimes SO(p)}$ cosets, namely real homogenous and real projective coordinates. This is useful because the group of modular transformations acts
in a simple way on these coordinates. Next, we especialise to the case of \((0, 2)\) \(Z_N\) orbifold compactifications with continuous Wilson lines yielding \(\mathcal{N} = 1\) space-time supersymmetry. We then proceed, in section 5, to explicitly construct the Kähler potentials for some of these moduli spaces. Namely, we will focus on the \(Z_N\) orbifold compactifications for which the internal 6-torus \(T_6\) can be decomposed into a direct sum \(T_4 \oplus T_2\). Then, by using well known techniques \([12, 13, 14]\), we will derive the Kähler potentials for the moduli spaces associated with the underlying 2-torus \(T_2\). For the case when the twist operating on the internal \(T_2\) torus has eigenvalues \(-1\) we find that, in the presence of \(r - 2\) complex Wilson lines, the associated coset \(\frac{SO(r, 2)}{SO(r) \otimes SO(2)}\) doesn’t factorise anymore into two submanifolds, contrary to the \(\frac{SO(2, 2)}{SO(2) \otimes SO(2)}\) case with no Wilson lines turned on. Moreover, we find that the associated Kähler potential contains a holomorphic term describing the mixing of complex Wilson lines. Such a term is precisely of the type which recently has been shown \([15, 16]\) to induce a mass term for Higgs particles of the same order of the gravitino mass once supergravity is spontaneously broken at low energies. In section 6 we proceed to explicitly discuss target space modular symmetries \([17]\) of the Kähler potentials constructed in section 5. We show that, for the Kähler potentials we have explicitly constructed, these discrete reparametrisations induce particular Kähler transformations on the Kähler potentials. Hence, these target space duality transformations are symmetries of the 4D \(\mathcal{N} = 1\) tree level low-energy Lagrangian \([17]\). These target-space duality symmetries also manifest themselves in the string threshold corrections that are of importance for the unification of gauge couplings \([13, 14, 20, 21, 22, 23, 24, 25, 26, 27, 28]\). We point out that, for the case where the twist operating on the internal \(T_2\) has eigenvalues of \(-1\), the associated \(T\) and \(U\) moduli mix under target space duality transformations due to the presence of the mixing terms between complex Wilson lines in the Kähler potential. We present our conclusions in section 7.

2. Toroidal and Orbifold Compactifications

Let us first briefly recall some of the relevant facts about toroidal compactifications with general constant background fields \([29, 30]\). If one compactifies the ten–dimensional heterotic \(E_8 \otimes E_8\) string on a \(d\–dimensional\ torus,

\[ T^d = \frac{\mathbb{R}^d}{\Lambda} \quad (2.1) \]

(where \(\Lambda\) is a \(d\–dimensional\ lattice) then the moduli dependent degrees of freedom can be parametrized by \(16 + 2d\) chargelike integer quantum numbers, namely the winding numbers \(n^i\), the internal momentum numbers \(m_i\) and the charges \(q^A\) of the leftmoving
current algebra which is generated by the extra leftmoving sixteen coordinates \((i = 1, \ldots, d, A = 1, \ldots, 16)\).

The moduli dependence of the untwisted states is encoded in the \(16 + 2d\) dimensional Narain lattice \(\Gamma\). If one expands the Narain vector \(P\) of an untwisted state in terms of a standard lattice basis \(\{k^i, \bar{k}_i, l_A\}\), then the quantum numbers appear as components, whereas the moduli dependence is absorbed into the geometry of the lattice:

\[
P = q^A l_A + n^i \bar{k}_i + m_i k^i \in \Gamma
\]

(2.2)

The moduli are usually grouped into a symmetric matrix \(G_{ij}\), which denotes the lattice metric of the \(d\)-dimensional lattice \(\Lambda\), an antisymmetric matrix \(B_{ij}\) and \(d\) sixteen dimensional vectors \(A_i\), called Wilson lines. The standard basis of \(\Gamma\) can then be constructed in terms of a basis \(\{e_A\}\) of the \(E_8 \otimes E_8\) lattice and of bases \(e_i, e^i\) of \(\Lambda\) and \(\Lambda^*\) (the dual of \(\Lambda\)) as a function of the moduli \(G_{ij}, B_{ij}, A_i\) [31]:

\[
k^i = \left(0, \frac{1}{2} e^i; \frac{1}{2} e^i\right)
\]

(2.3)

\[
\bar{k}_i = \left(A_i, (G_{ij} + B_{ij} - \frac{1}{4} (A_i \cdot A_j)) e^j; (-G_{ij} + B_{ij} - \frac{1}{4} (A_i \cdot A_j)) e^j\right)
\]

(2.4)

\[
l_A = \left(e_A, -\frac{1}{2} (e_A \cdot A_i) e^i; -\frac{1}{2} (e_A \cdot A_i) e^i\right)
\]

(2.5)

Defining the \(16 \times d\) matrix \(A_{Aj}\) as

\[
A_{Ai} = e_A \cdot A_i
\]

(2.6)
yields

\[
A_i \cdot A_j = C_{AB} A_{Ai} A_{Bj}
\]

(2.7)

where the metric \(C_{AB} = e_A \cdot e_B\) for lowering and raising \(A\)-indices denotes the Cartan metric of the \(E_8 \otimes E_8\) lattice. Another parametrization of the moduli, which will turn out to be quite useful later on, is given by a \(d \times d\) matrix \(D_{ij}\) defined by

\[
D_{ij} = 2 (B_{ij} - G_{ij} - \frac{1}{4} (A_i \cdot A_j))
\]

(2.8)

At those points in the moduli space, where one of the matrices (2.6) and (2.8) becomes integer valued, both the symmetry of the Narain lattice and the gauge symmetry of the model are enhanced [32].

Another useful representation of the Narain vector is to specify its components with respect to an orthonormal frame, which allows one to separate the \(16 + d\) leftmoving from the \(d\) rightmoving degrees of freedom

\[
P = (P_L; P_R)
\]

(2.9)
In terms of this decomposition the condition \((L_0 - \tilde{L}_0)|\Phi\rangle = 0\) for physical states reads
\[
\frac{1}{2}(P^2_L - P^2_R) + N + \tilde{N} - 1 = 0
\] (2.10)

Since the (moduli independent) contribution of the number operators \(N\) and \(\tilde{N}\) is an integer\(^3\), it is evident that the Narain lattice must be an even lattice with respect to the indefinite bilinear form of type \((+)^{16+d}(-)^d\). As shown by Narain\(^{[29]}\) modular invariance implies that the lattice \(\Gamma\) must also be selfdual. Since even selfdual lorentzian lattices are unique up to isometries, this then implies that the allowed deformations of such a lattice \(\Gamma\) form a group isomorphic to \(O(16 + d, d)\).

The moduli dependent contribution to the mass \(M\) of an untwisted state is given by
\[
\alpha' M^2 = (P^2_L + P^2_R) + \cdots
\] (2.11)

Since not only the spectrum but also the interactions are invariant under the subgroup \(O(16 + d) \otimes O(d)\) of \(O(16 + d, d)\), the moduli space of toroidal compactifications is locally given by the coset space\(^{[29]}\)
\[
\mathcal{M}_T \simeq \frac{O(16 + d, d)}{O(16 + d) \otimes O(d)}
\] (2.12)

In order to get the global geometry one has to take into account further discrete identifications due to duality (also called modular) symmetries of the target space, which will be discussed later.

Toroidal compactifications are, however, not of big phenomenological interest, because they all yield models with an extended \(\mathcal{N} = 4\) space–time supersymmetry, which doesn’t admit chiral matter multiplets, and with gauge groups of rank \(16 + d\)\(^{[29]}\). They are, nevertheless, the natural starting point for the construction of more realistic models, namely orbifold models.

It is well known from the work of Dixon, Harvey, Vafa and Witten\(^{[1, 2]}\) that by modding out rotations both the number of space–time supersymmetries and the rank of the gauge group can be reduced. If one starts with a toroidal compactification these rotations must be automorphisms of finite order of the corresponding Narain lattice \(\Gamma\)\(^{[3]}\). We will study the case in which the point twist group \(\mathcal{P}\) defining the orbifold is a cyclic group
\[
\mathcal{P} = \langle \Theta \rangle = \{\Theta, \Theta^2, \ldots, \Theta^N = 1\}
\] (2.13)

\(^3\)This is true after applying the GSO condition and after absorbing the normal ordering constant of the NS sector into the definition of the rightmoving number operator.
generated by a single twist $\Theta$ satisfying

$$\Theta \in \text{AUT}(\Gamma), \quad \Theta^N = 1 \quad (2.14)$$

As shown by Narain, Sarmadi and Vafa [4] holomorphic factorization and modular invariance imply that the twist must not mix left- and rightmoving degrees of freedom. It must therefore be a rotation\footnote{The asymmetric orbifold construction given in [3] is slightly more general since it also allows for the modding out of a rotation followed by a translation. Note that modding out by translations is much simpler and better understood as it is equivalent to imposing different toroidal boundary conditions. For the purpose of trying to learn more about the effect of modding out rotations, we will keep the situation as simple as possible and, in the following, only consider pure rotations.}(not just a pseudo–rotation)

$$\Theta = \Theta_L \otimes \Theta_R \in O(16 + d) \otimes O(d) \quad (2.15)$$

We will, in the next section, determine the local structure of the moduli spaces for orbifolds defined by a twist as given in (2.15). Since most of the work on orbifolds has, up to now, focused on more special constructions we will, however, first have to recall some more facts and results.

People have, from the beginning, been especially interested in orbifold models that can be interpreted as compactifications on a six–dimensional orbifold [1, 2]. In these cases the twist $\Theta$ of the Narain lattice $\Gamma$ must act in a left-right symmetric way, to be specified below, so as to have well defined coordinates on the internal $d$–dimensional orbifold. More precisely, the twist $\Theta$ must be given in terms of a $d$–dimensional twist $\theta$ which defines this orbifold and an additional gauge twist $\theta'$ which is an automorphism of the $E_8 \otimes E_8$ root lattice. That is, if one decomposes the Narain vector as

$$P = (p^A, p_i^L; p_i^R) \quad (2.16)$$

then the twist $\Theta$ must be given as

$$\Theta = \theta' \otimes \theta \otimes \theta \in \text{diag}(O(16) \otimes (O(d) \otimes O(d))$$

We will refer to all compactifications, for which the twist $\Theta$ is given by (2.17), as orbifold compactifications. Note that (2.17) is a special case of (2.15).

One further restriction that is often used is to consider only Narain lattices of the special form $\Gamma_{16} \oplus \Gamma_{6,6}$ where $\Gamma_{16}$ denotes the root lattice of $E_8 \otimes E_8$. This means that most of the deformation parameters, namely the $16 \cdot d$ parameters corresponding to the Wilson lines $A_{A_i}$, are set to zero. One can then replace the gauge twist $\theta'$ by an equivalent shift (i.e. by a translation) which is much easier to handle. However, the price of this
simplification is quite high, as the rank of the gauge group is then at least 16. Although it is possible to have nonvanishing Wilson lines when using the shift realization, they are then constrained to a discrete set of values and, hence, are not moduli of the orbifold model. Since discrete Wilson lines act like additional shifts, they also cannot reduce the rank of the gauge group but only break (or extend) the gauge group. On the other hand, it was pointed out in [3, 4] that, if one realizes the gauge twist by a rotation, some of the components of the Wilson lines are still moduli and that they can be used to reduce the rank of the gauge group below 16. Thus, it is important to keep the continuous Wilson lines in the game and we will do so in the following.

Clearly, a deformation of the Narain lattice can only lead to a modulus of an orbifold model if the twist $\Theta$ is still an automorphism of the deformed lattice. This was used in [6] to derive a set of equations for the moduli which, in principle, allow one to decide which of the toroidal moduli are still moduli of the orbifold model and which are frozen to discrete values. In [33, 34] it was shown how these equations can be explicitly solved for the moduli in the case of bosonic or heterotic orbifold compactifications without Wilson lines. For the number of surviving $G_{ij}$ and $B_{ij}$ moduli closed formulas were derived. This was later [10] generalized to heterotic orbifold compactifications with continuous Wilson lines. One drawback of the approach used in [10] is that one can derive the number of moduli, but the expected coset structure of the moduli space remains obscure. In the case of vanishing Wilson lines this coset structure was derived in [35, 36] for all the $\mathbb{Z}_N$ orbifold compactifications with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ space–time supersymmetry. In that approach one uses symmetries of the world sheet action to constrain the Kähler potential appearing in the 4D effective action. The associated coset is then obtained from the explicit expression of the Kähler potential. In the next section we will use a different method for determining the local structure of the moduli space of asymmetric $\mathbb{Z}_N$ orbifolds. We will not make use of the effective action, but rather of the compatibility equation between the Narain twist $\Theta$ and the moduli. Note that we will be dealing with orbifolds defined by a twist $\Theta$ as given in (2.13).

\footnote{The rank of the gauge group cannot be reduced by shifts but only by rotations. More precisely, the rank of the gauge group of an asymmetric orbifold is greater or equal to the number of nontrivial eigenvalues of $\Theta_L$, because for each eigenvalue 1 there is a twist invariant leftmoving oscillator and, therefore, an unbroken $U(1)$.}
3. The coset structure of asymmetric $Z_N$ orbifolds

Consider the coset (2.12) which parametrizes the moduli space of toroidal compactifications locally. The simplest way of arriving at the untwisted moduli space of a general $Z_N$ orbifold (locally) is simply to find the subspace of this coset that is compatible with the action of the twist $\Theta$ on the underlying Narain lattice $\Gamma = \Gamma_{16+d,d}$.

Suppose now that $\Gamma$ is a lattice on which $\Theta$ acts as an automorphism. A deformation $\mathcal{T} \in O(16 + d, d)$ of $\Gamma$ is compatible with $\Theta$ if and only if $\Theta$ is also an automorphism of the deformed lattice $\Gamma' = \mathcal{T}(\Gamma)$. But this is, by inspection, equivalent to $\mathcal{T}^{-1}\Theta\mathcal{T}$ being in the point group $\mathcal{P}$ of $\Gamma$. Since we are taking the point group $\mathcal{P}$ to be the cyclic group generated by $\Theta$, this then means that

\[ \mathcal{T}^{-1}\Theta\mathcal{T} = \Theta^k, \quad 1 \leq k < N \]

(3.1)

for some $k$. That is, $\mathcal{T}$ is in the normalizer $\mathcal{N}$ of the point group $\mathcal{P}$ in $O(16 + d, d)$

\[ \mathcal{T} \in \mathcal{N}(\mathcal{P}, O(16 + d, d)) \]

(3.2)

Statement (3.2), namely that $\mathcal{T}$ has to be in the normalizer $\mathcal{N}$ of the point group, also holds for bigger (abelian or non-abelian) point groups $\mathcal{P}$ which have more than one generator. It is, though, intuitively clear that a twist $\Theta$ and a deformation $\mathcal{T}$ do not have to strictly commute, but that they have to commute on orbits, that is up to point transformations as in (3.1).

Of course, only those deformations $\mathcal{T}$ with $k = 1$ can be continuously connected to the identity, whereas the others will describe nontrivial, discrete deformations. This corresponds to the appearance of discrete background fields in the standard approach [37]. On the other hand, any special solution of equation (3.1) with $k \neq 1$ can be continuously deformed by any solution to (3.1) with $k = 1$. This means that, in order to identify the moduli, one has to find the general solution to (3.1) with $k = 1$.

We will, therefore, in the following only deal with the (most general) case of purely continuous background fields and set $k = 1$. After introducing matrices with respect to an orthonormal basis of $\mathbb{R}^{16+d,d}$ we have to solve the homogeneous matrix equation

\[ [\Theta, \mathcal{T}] = 0 \]

(3.3)

\footnote{There may, of course, exist lattices which are more symmetric than required when modding out by $\Theta$ and, therefore, have bigger point groups. It is possible to define orbifolds by modding out these bigger groups. The number of allowed deformations will then be different. For our purpose these models are just a subset of models with extended symmetry because we want to find all lattices whose point symmetry group contains the cyclic group generated by $\Theta$, which then is chosen to be the point twist group.}
for $\mathcal{T}$. We proceed to show that the moduli space of this equation only depends on the eigenvalues of $\Theta$. The method used in the following is a modification of the method used in [13] for $Z_3$ orbifold compactifications without Wilson lines. First recall that $\Theta$ must be a proper rotation (2.15), that is, it must be an element of $O(16 + d) \otimes O(d)$. The eigenvalues of the twist are $N$–th roots of unity. Those which are real must be equal to $\pm 1$, whereas the complex ones come in pairs of complex conjugated numbers of length 1. Let us denote the number of eigenvalues $\pm 1$ in the left (right) part of $\Theta$ by $d_{\pm 1}^{(L)}$ ($d_{\pm 1}^{(R)}$) and the total number by $d_{\pm 1} = d_{\pm 1}^{(L)} + d_{\pm 1}^{(R)}$. Analogously the number of complex pairs $\exp(\pm i\phi_k)$ of eigenvalues of the left (right) part is denoted by $p_k$ ($q_k$).

By relabeling the orthonormal basis of $\mathbb{R}^{16+d,d}$, the matrix of the twist $\Theta$ can be brought to the form

$$
\begin{pmatrix}
R_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & R_1' & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & R_i & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & -1 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \\
\end{pmatrix}
$$

(3.4)

where

$$
R_i = \begin{pmatrix}
c_i & -s_i & 0 & \cdots & \cdots & \cdots & 0 \\
s_i & c_i & 0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}, \quad R_i' = \begin{pmatrix}
c_i & -s_i & 0 & \cdots & \cdots & \cdots & 0 \\
s_i & c_i & 0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
$$

(3.5)

with

$$
c_i = \cos(\phi_i), \quad s_i = \sin(\phi_i)
$$

(3.6)

Then the matrix of an admissible deformation $\mathcal{T}$ with respect to the same basis has the
blockdiagonal form

\[
\begin{pmatrix}
T_1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ddots & \vdots \\
\vdots & T_i & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & & P & 0 \\
0 & \ldots & \ldots & \ldots & 0 & Q
\end{pmatrix}
\]

Since \( T \in O(16 + d, d) \) we get that

\( T_i \in O(2p_i, 2q_i), \ P \in O(d^{(L)}_{-1}, d^{(R)}_1), \ Q \in O(d^{(L)}_1, d^{(R)}_1) \)  

(3.8)

Whereas the matrices \( P \) and \( Q \) are not further constrained by the commutator equation (3.3), the \( T_i \) must commute with the twist matrix restricted to the \( i \)-th complex eigenspace. Decomposing \( T_i \) into suitable blocks as

\[
T_i = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

yields (3.3) as

\[
\begin{bmatrix} R_i & 0 \\ 0 & R'_i \end{bmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0
\]

(3.10)

The blocks \( A, B, C, D \) depend of course on the index \( i \), but since the different eigenspaces decouple it is possible and convenient to suppress this label.

Equation (3.10) implies that \( R_i A = AR_i \) for the \( 2p_i \times 2p_i \) block \( A \). Similar equations hold for \( B, C \) and \( D \). These equations can now again be analyzed blockwise. In the case of \( A \) (3.10) gives

\[
[A, R_i] = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} c_{1p_i} & -s_{1p_i} \\ s_{1p_i} & c_{1p_i} \end{pmatrix} = 0
\]

(3.11)

implying that only two of the four \( p_i \times p_i \) blocks \( A_i \) are independent, namely

\[
A = \begin{pmatrix} A' & -A'' \\ A'' & A' \end{pmatrix}
\]

(3.12)
where \( A' = A_1 = A_4, \ A'' = -A_2 = A_3 \). The other blocks \( B, C, D \) of \( T_i \) have the same structure. The off–diagonal blocks \( B \) and \( C \) are, however, in general not quadratic.

The matrices \( T_i \) form a group called the centralizer of \( \Theta \) (restricted to the \( i \)–th eigenspace) in \( O(2p_i, \ 2q_i) \). The special structure found for these matrices resembles the one appearing in the standard isomorphism between \( GL(n, \mathbb{C}) \) and a \( 2n^2 \) dimensional subgroup of \( GL(2n, \mathbb{R}) \) given by

\[
GL(n, \mathbb{C}) \ni m = m' + im'' \iff M = \begin{pmatrix} m' & -m'' \\ m'' & m' \end{pmatrix} \in GL(2n, \mathbb{R})
\] (3.13)

The only modification needed is a permutation of certain rows and columns in \( T_i \), in order to reposition some of the blocks. Since such a permutation is an automorphism of \( GL(2p_i + 2q_i) \) the composition with (3.13) yields again an isomorphism

\[
T_i = \begin{pmatrix} A' & -A'' & B' & -B'' \\ A'' & A' & B'' & B' \\ C' & -C'' & D' & -D'' \\ C'' & C' & D'' & D' \end{pmatrix} \iff \begin{pmatrix} A' & B' & -A'' & -B'' \\ C' & D' & -C'' & -D'' \\ A'' & B'' & A' & B' \\ C'' & D'' & C' & D' \end{pmatrix} \iff (3.14)
\]

\[
\iff \begin{pmatrix} A' + iA'' & B' + iB'' \\ C' + iC'' & D' + iD'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = t_i \tag{3.15}
\]

Since the \( T_i \) must not only commute with the twist but also be in \( O(2p_i, \ 2q_i) \), we finally have to translate this into a condition for the \( t_i \). The pseudo–orthogonality\(^8\) of \( T_i \) can be expressed in terms of the blocks \( A, B, C, D \) as

\[
A^T A - C^T C = 1, \quad A^T B - C^T D = 0,
\]

\[
B^T A - D^T C = 0, \quad B^T B - D^T D = -1
\] (3.16)

These relations imply that the blocks of \( t_i \) satisfy

\[
a^+ a - c^+ c = 1, \quad a^+ b - c^+ d = 0, \tag{3.17}
\]

\[
b^+ a - d^+ c = 0, \quad b^+ b - d^+ d = -1
\]

\(^8\)The condition on the determinant does not lead to a relation between the matrix elements of \( T_i \), because the determinant of a pseudo–orthogonal matrix can only take discrete values.
This means that $t_i$ is pseudo-unitary, $t_i \in U(p_i, q_i)$. Therefore the group of those deformations in the $i$-th eigenspace that commutes with the twist is (at least locally) isomorphic to $U(p_i, q_i)$.

Combining the above results for all the blocks in the decomposition of a general $T \in O(16 + d, d)$ we have shown that those deformations $T$ commuting with the twist $\Theta$ form a subgroup isomorphic to

$$\bigotimes_{i=1}^{K} U(p_i, q_i) \otimes O(d_{-1}^{(L)}, d_{-1}^{(R)}) \otimes O(d_1^{(L)}, d_1^{(R)})$$

(3.18)

where $K$ is the total number of distinct pairs of complex eigenvalues.

However, those deformations $T$ which are pure rotations, $T \in O(16 + d) \otimes O(d_1)$, do not change the physical content of a model. To get the (untwisted) moduli space (up to modular transformations) we have to factorize this subgroup. The resulting coset space is given by

$$\mathcal{M}_O(\Theta) \simeq \bigotimes_{i=1}^{K} SU(p_i, q_i) \otimes SU(q_i) \otimes U(1) \otimes \frac{SO(d_{-1}^{(L)}) \otimes SO(d_{-1}^{(R)})}{SO(d_1^{(L)}) \otimes SO(d_1^{(R)})} \otimes \frac{SO(d_1^{(L)}, d_1^{(R)})}{SO(d_1^{(L)}) \otimes SO(d_1^{(R)})}.$$ 

(3.19)

Note that we have made use of the local isomorphisms $U(p, q) \simeq SU(p, q) \otimes U(1)$ and $O(p, q) \simeq SO(p, q)$ to bring our result into the form usually used in the supergravity literature. As claimed above, the local structure of the untwisted moduli space is completely determined by the eigenvalues of the twist $\Theta$. The dimension of the moduli space is

$$\dim(\mathcal{M}_O(\Theta)) = 2 \sum_{i=1}^{K} p_i q_i + d_{-1}^{(L)} d_{-1}^{(R)} + d_1^{(L)} d_1^{(R)}$$

(3.20)

It only depends on the multiplicities of the eigenvalues of $\Theta$. Moduli do only exist if an eigenvalue appears both in the left and in the right part of the twist.

We can now compare our result (3.19) with the coset spaces found in [35, 36] for the $\mathbb{Z}_N$ orbifold compactifications without Wilson lines yielding $N = 1$ and $N = 2$ space-time supersymmetry. In order to do so, we simply have to restrict ourselves to an $O(d, d)$ subsector and to set $d = 6$ as well as $\Theta_L = \Theta_R = \theta$. Then, by plugging into (3.19) the wellknown eigenvalues of the symmetric $\mathbb{Z}_N$ twists leading to $N = 1, 2$ space-time supersymmetry [3, 38], we find that all the results agree as expected. As a straightforward generalization we can now similarly write down the cosets for all these models with continuous Wilson lines turned on. The result will, of course, now also depend on the gauge twist $\theta'$ and its eigenvalues. Since the choice of a gauge twist is

\[\text{In fact, it is easily seen that the form of the world sheet action which is crucial in the approach of [36] depends only on the eigenvalues of } \theta \text{ and their multiplicities.}\]
| Gauge Twist | Coset | Max. Gauge Group | Min. Gauge Group |
|------------|------|-----------------|-----------------|
| $\emptyset \otimes \emptyset$ | $\frac{SU(3,3)}{SU(3) \otimes SU(3) \otimes U(1)}$ | $E_8 \otimes E_8'$ | $E_8 \otimes E_8'$ |
| $A_2^2 \otimes A_2$ | $\frac{SU(6,3)}{SU(6) \otimes SU(3) \otimes U(1)}$ | $(SO(14) \otimes U(1)) \otimes (E_7 \otimes U(1))'$ | $(SU(3) \otimes SU(3)) \otimes E_6'$ |
| $A_2^3 \otimes \emptyset$ | $\frac{SU(6,3)}{SU(6) \otimes SU(3) \otimes U(1)}$ | $(E_6 \otimes SU(3)) \otimes E_8'$ | $SU(3) \otimes E_8'$ |
| $A_2^3 \otimes A_2^2$ | $\frac{SU(9,3)}{SU(9) \otimes SU(3) \otimes U(1)}$ | $(E_6 \otimes SU(3)) \otimes (E_6 \otimes SU(3))'$ | $SU(3) \otimes SU(3)'$ |
| $A_2^4 \otimes A_2^2$ | $\frac{SU(9,3)}{SU(9) \otimes SU(3) \otimes U(1)}$ | $SU(9) \otimes (SO(14) \otimes U(1))'$ | $(SU(3) \otimes SU(3))'$ |

Table 1: Table of all $\mathbb{Z}_3$ orbifold compactifications with $\mathcal{N} = 1$ space-time supersymmetry.

also constrained by world sheet modular invariance one has to proceed as follows. First, one has to find all $E_8 \otimes E_8$ Weyl twists $\theta'$ which have the required order and lead to modular invariant twists $\Theta$. Then one has to calculate their eigenvalues in order to get the coset. To carry out this program will require some work because there are, in general, a lot of gauge twists satisfying the constraints from modular invariance, especially for higher $N$. Based on [39] one can estimate that there will be roughly 500 models. This will, therefore, be the subject of a later publication [40]. However, to give an explicit example, we will list the cosets for all modular invariant $\mathbb{Z}_3$ orbifold compactifications with $\mathcal{N} = 1$ space-time supersymmetry. This is easy to do, since both $\theta$ and $\theta'$ consist of several copies of the $A_2$ coxeter twist. This is a rotation by 120 degrees and therefore has the eigenvalues $\exp(\pm 2\pi i/3)$. More precisely, $\theta$ contains three copies of this twist and the gauge twist $\theta'$ is constrained by modular invariance to contain 0, 3 or 6 further copies. This leads to five inequivalent models [4]. In table (1) we list these twists together with the corresponding moduli spaces and the maximal and minimal gauge group. The maximal gauge group is the one of the model with vanishing Wilson lines and can be found in [5] or in the table of $E_8$ automorphisms given in [41]. The minimal gauge group is the one for generic (purely continuous) Wilson lines and can be calculated using the method introduced in [10].

Let us conclude with two further remarks. The first is that the formula (3.20) coincides with the one derived in [10] for orbifold compactifications with continuous Wilson lines and a special choice of the gauge twist. Conversely the results of this section indicate that it should be possible to generalize the results of [10] to the general asymmetric case.

10 $A_2$ is the complex form of $su(3)$. 

12
The final remark concerns the Kählerian structure of the cosets given in (3.19). Whereas the $SU$ cosets are Kählerian for any value of the parameters, the $SO$ cosets are Kählerian if they are isomorphic to $SU$ cosets by some accidental isomorphism of (low dimensional) Lie groups or if one of the parameters equals 2, that is for cosets $\frac{SO(p, 2)}{SO(p) \otimes SO(2)}$ (3.21)

For $\mathcal{N} = 1$ supersymmetric orbifold compactifications it is well known that the eigenvalue $+1$ does not appear, whereas $-1$ does only appear with multiplicities 0 or 2. Therefore, the moduli spaces of $\mathcal{N} = 1$ $\mathbb{Z}_N$ orbifold compactifications with continuous Wilson lines found here are all Kählerian. For general asymmetric $\mathbb{Z}_N$ orbifolds the situation is less clear, since the necessary and sufficient condition for $\mathcal{N} = 1$ space-time supersymmetry is not known. Our result suggests that the only real eigenvalue $\Theta_L$ may have is $-1$, with multiplicity 0 or 2 as in the compactification case. Note, however, that our investigations have been restricted to the untwisted sector and that it has been pointed out recently [44] that space-time supercharges may also appear from twisted sectors in asymmetric orbifolds.

4. Modular symmetries

The fact that the naive moduli spaces will contain several copies of the same model is clear from the beginning since one can easily imagine that there will be large deformations $T \in O(16 + d, d)$ which are at the same time automorphisms of the Narain lattice and therefore do not lead to a different model. Therefore all transformations of the type $T \in O(16 + d, d) \cap \text{AUT}(\Gamma)$ (4.1)

are symmetries of the toroidal moduli space $\mathcal{M}_T$. Those which also fulfill the normalizer constraint for a twist $\Theta$

$T \in \mathcal{N}(\langle \Theta \rangle, O(16 + d, d) \cap \text{AUT}(\Gamma))$ (4.2)

are then symmetries of the orbifold moduli space $\mathcal{M}_O(\Theta)$.

In the following we will recall and extend the analysis performed by Spalinski [43]. In this approach one first finds the action of modular transformations on the quantum numbers and then derives the induced action on the moduli themselves. To do so, one first writes down the indefinite bilinear form in the lattice basis (in matrix notation) as

$P_L^2 - P_R^2 = v^THv$ (4.3)
Here $v$ is a vector consisting of the $16 + 2d$ integer quantum numbers which label a state,

$$v^T = (q^A, n^i, m_i) \in \mathbb{Z}^{16+2d}, \quad A = 1, \ldots, 16, \ i = 1, \ldots, d,$$  \hspace{1cm} (4.4)

and $H$ is the lorentzian lattice metric of $\Gamma$ given as

$$H = \begin{pmatrix}
(l_A, l_B) & (l_A, k_j) & (l_A, k^n) \\
(k_i, l_B) & (k_i, k_j) & (k_i, k^n) \\
(k^m, l_B) & (k^m, k_j) & (k^m, k^n)
\end{pmatrix} = \begin{pmatrix} C & 0 & 0 \\
0 & 0 & I \\
0 & I & 0
\end{pmatrix}$$  \hspace{1cm} (4.5)

where $C$ is the Cartan matrix of $E_8 \otimes E_8$ and $I$ is the $d \times d$ unit matrix. Modular symmetry transformations $T$ can now also be described in terms of their matrices $\Omega$ with respect to the lattice basis, which act as

$$v \rightarrow v' = \Omega^{-1}v$$  \hspace{1cm} (4.6)

on the quantum numbers. To be a symmetry, the matrix $\Omega$ must be integer valued,

$$\Omega \in GL(16 + 2d, \mathbb{Z}) \iff T \in AUT(\Gamma)$$  \hspace{1cm} (4.7)

and it must leave the indefinite bilinear form invariant

$$\Omega^T H \Omega = H \iff T \in O(16 + d, d)$$  \hspace{1cm} (4.8)

These two conditions combined define a matrix group,

$$G_T = \{ \Omega \in GL(16 + 2d, \mathbb{Z}) | \Omega^T H \Omega = H \}$$  \hspace{1cm} (4.9)

which is called the modular invariance group of toroidal compactifications. It is usually denoted by $O(16 + d, d; \mathbb{Z})$ for obvious reasons, although it is not a group of pseudo-orthogonal matrices. To get the modular invariance group for an orbifold one simply has to add the normalizer constraint. Then

$$G_O = \mathcal{N}(\langle \Theta \rangle, G_T) = \{ \Omega \in GL(16 + 2d, \mathbb{Z}) | \Omega^T H \Omega = H, \quad \Omega^{-1} \Theta \Omega = \Theta^k, \exists k : 1 \leq k < N \}$$  \hspace{1cm} (4.10)

This last formula was given by Spalinski [43] but without mentioning its group theoretical interpretation. The calculation of such normalizers is in general a difficult task which must be done case by case. Nevertheless, several examples have been discussed in the literature [13, 13, 10, 31, 27, 28] for models with no or with discrete Wilson lines. As pointed out in [16] a factorization of the modular invariance group into factors corresponding to different eigenvalues of the twist can only be expected if the underlying
lattice itself decomposes into an orthogonal direct sum, which is not the case generically. Therefore the local decomposition of orbifold moduli spaces into a product of coset spaces does not imply a corresponding decomposition of \( G_O \).

The second step is to deduce the action of modular symmetry transformations on the moduli. This can be done through the mass formula (2.11). Therefore we express the euclidean bilinear form in lattice coordinates as

\[
P_L^2 + P_R^2 = v^T \Sigma v
\]  

(4.11)

The mass matrix \( \Sigma \) which encodes the complete moduli dependence of the whole spectrum is the euclidean lattice metric

\[
\Sigma = \begin{pmatrix}
l_A \cdot l_B & l_A \cdot k_j & l_A \cdot k^n \\
l_i \cdot l_B & l_i \cdot k_j & l_i \cdot k_m \cdot k^n \\
l_m \cdot l_B & l_m \cdot k_j & k_m \cdot k^n
\end{pmatrix}
\]  

(4.12)

Introducing matrix notation and working out the scalar products one gets

\[
\Sigma = \begin{pmatrix}
C + \frac{1}{2} AG^{-1} A^T & -\frac{1}{2} AG^{-1} D^T & -\frac{1}{2} AG^{-1} \\
-\frac{1}{2} DG^{-1} A^T & \frac{1}{2} DG^{-1} D^T & I + \frac{1}{2} DG^{-1} \\
-\frac{1}{2} G^{-1} A^T & I + \frac{1}{2} G^{-1} D^T & \frac{1}{2} G^{-1}
\end{pmatrix}
\]  

(4.13)

Here \( G = (G_{ij}) \) and \( G^{-1} = (G^{ij}) \) are the lattice metrics\(^\text{11}\) of the compactification lattice \( \Lambda \) and its dual \( \Lambda^* \). The matrices \( A = (A_{Ai}) \) and \( D = (D_{ij}) \) were defined in (2.6) and (2.8), respectively.

The action of a modular symmetry transformation on the euclidean bilinear form is given by

\[
v^T \Sigma v \rightarrow v^T \Omega T^{-1} \Sigma' \Omega^{-1} v = \frac{1}{2} v^T \Sigma v
\]  

(4.14)

Note that the moduli dependent matrix \( \Sigma \) will in general also transform, if the deformation \( \Omega \) is not a pure rotation of the Narain lattice. The fact that the deformations we consider are symmetry transformations and therefore must leave the euclidean bilinear form invariant fixes the transformation law of \( \Sigma \) to be

\[
\Sigma \rightarrow \Sigma' = \Omega T \Sigma \Omega
\]  

(4.15)

\(^{11}\)Note that some authors choose the lattice metric of \( \Lambda \) to be \( 2G \). With this convention, some of the matrix elements differ by a factor 2.
with $\Omega \in G_T, G_O$ respectively \[13\]. Since the functional dependence of $\Sigma$ on the moduli is known via (4.13), this allows one, in principle, to calculate the transformation law of the moduli. But as the dependence is quite complicated and nonlinear, this is tedious to do in practice. Since $\Sigma$ is symmetric it is tempting to try to factorize it in the form

$$\Sigma = \phi^T \phi$$

hoping that the moduli dependence of $\phi$ might be simpler. In the case of bosonic strings the construction given by Giveon, Porrati and Rabinovici \[47\] does precisely this. But in order to apply their bosonic result to heterotic strings one has do embed the heterotic string into the bosonic one and calculations are still complicated.

We will, therefore, use a different approach where one directly works with the heterotic string. It is also motivated by the question how the moduli $G_{ij}, B_{ij}, A_i$ are related to standard (homogenous and projective) coordinates on cosets, which are known to have a simple transformation under the group action. To explain why these two questions are closely related let us write down the euclidean bilinear form in matrix notation, but now with respect to an orthonormal frame, and then apply a deformation (not a modular transformation) $\mathcal{T} \in O(16 + d, d; \mathbb{R})$. Then

$$P^2_L + P^2_R = u^T u \rightarrow u^T \mathcal{T}^T \mathcal{T} u$$

Note that we have expressed the deformed bilinear form in terms of the old coordinates $u$. The moduli dependence is now completely given by the symmetric matrix $\mathcal{T}^T \mathcal{T}$. The same deformation can be described with respect to lattice coordinates $v$:

$$P^2_L + P^2_R = v^T \Sigma_0 v \rightarrow v^T \Sigma v$$

We consider $\Sigma_0$ as a fixed reference background and $\Sigma$ as a function of the moduli which describe the continuous deformations of $\Sigma_0$. If $N$ is the basis transformation connecting the $u$ and the $v$ frame by $u = N v$ then, by combining (4.17) and (4.18), we get a decomposition of $\Sigma$ which has the desired form (4.16)

$$\Sigma = N^T \mathcal{T}^T \mathcal{T} N$$

As is well known from the case of the Lorentz group, elements of pseudo orthogonal groups can be decomposed into a rotation $R \in O(16 + d) \otimes O(d)$ and a "boost" $B$. The latter can be used as a coset representative. In the case of (4.19) the rotational part always cancels out

$$\mathcal{T} = RB \Rightarrow \mathcal{T}^T \mathcal{T} = B^T R^T R B = B^T B$$
which again shows that the spectrum only depends on the coset. We can therefore expect that it is possible to factorize $\Sigma$ in terms of a matrix $\phi$ which is a product of a coset representative $B \simeq RB$ and a basis transformation $N$.

While this consideration has told us what $\phi$ should be, it didn’t say how to construct it. There is however one obvious way to factorize $\Sigma$ as in (4.16). Namely, we introduce an orthonormal basis

$$\hat{e}_a = (e_a, 0; 0), \quad \hat{e}_\mu^{(L)} = (0, e_\mu; 0), \quad \hat{e}_\mu^{(R)} = (0, 0; e_\mu)$$

(4.21)

$(a = 1, \ldots, 16, \mu = 1, \ldots, d)$, and expand all the vectors appearing in (4.12) in terms of it. This yields:

$$\Sigma = \begin{pmatrix} l_A \cdot e_a & l_A \cdot e_\mu^{(L)} & l_A \cdot e_\mu^{(R)} \\ \overline{k}_i \cdot e_a & \overline{k}_i \cdot e_\mu^{(L)} & \overline{k}_i \cdot e_\mu^{(R)} \\ k^m \cdot e_a & k^m \cdot e_\mu^{(L)} & k^m \cdot e_\mu^{(R)} \end{pmatrix} \begin{pmatrix} e_a \cdot l_B & e_a \cdot \overline{k}_j & e_a \cdot k^n \\ e_\mu^{(L)} \cdot l_B & e_\mu^{(L)} \cdot \overline{k}_j & e_\mu^{(L)} \cdot k^n \\ e_\nu^{(R)} \cdot l_B & e_\nu^{(R)} \cdot \overline{k}_j & e_\nu^{(R)} \cdot k^n \end{pmatrix} = \phi^T \phi \quad (4.22)$$

Working out the scalar products we get

$$\phi^T = \begin{pmatrix} \mathcal{E} & -\frac{1}{2}AE^* & -\frac{1}{2}AE^* \\ A^T \mathcal{E}^* & (2G + \frac{1}{2}D)E^* & \frac{1}{2}DE^* \\ 0 & \frac{1}{2}E^* & \frac{1}{2}E^* \end{pmatrix} \quad (4.23)$$

The ambiguity in the factorization of $\Sigma$ is reflected by the appearance of the $n$–bein matrices $\mathcal{E} = (e_A \cdot e_a)$, $E = (e_i \cdot e_\mu)$ and their duals $\mathcal{E}^* = \mathcal{E}^{T,-1}$, $E^* = E^{T,-1}$. Since we expect $\phi^T$ to be a coset representative we now compare (4.23) to the well known standard form of such an object [42]. Although (4.23) is not in the standard form of a coset representative, its structure is similar enough to allow for the construction of analogues of standard homogenous and projective coordinates. Let us first introduce

$$X = \begin{pmatrix} -\frac{1}{2}AE^* \\ \frac{1}{2}DE^* \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \\ \frac{1}{2}E^* \end{pmatrix}$$

(4.24)

Note that all the moduli appear in $X$, so that the whole information is encoded in it. But $X$ is not a nice coordinate. However, following the standard procedure described in
the \((16 + 2d) \times d\) matrix

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix}
X_1 \\
X_2 \\
Y
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2}AE^* \\
\frac{1}{2}DE^* \\
\frac{1}{2}E^*
\end{pmatrix}
\]

which consists of the last \(d\) columns of \(\phi^T\), should be a homogenous coordinate. This means that it must transform linearly under the left action of the group modulo rotations acting from the right. In our case the group action is given by

\[
\phi^T \rightarrow \Omega^T \phi^T
\]

where

\[
\Omega \in \{ M \in GL(16 + 2d; \mathbb{R}) | M^T H M = H \} \simeq O(16 + d, d)
\]

Decomposing \(\Omega^T\) into appropriate blocks

\[
\Omega^T = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

we see that our coordinate transforms indeed linearly as

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} \rightarrow \begin{pmatrix}
aX + bY \\
cX + dY
\end{pmatrix}
\]

This linear transformation law is (as usual for homogenous coordinates) achieved by treating \(X\) and \(Y\) as being independent of each other. It can be checked, however, that \(X\) and \(Y\) are really constrained to satisfy

\[
X^T_1 C^{-1} X_1 + X^T_2 Y + Y^T X_2 = -I
\]

where \(C^{-1}\) is the inverse of the Cartan matrix of \(E_8 \otimes E_8\). These are, in fact, the constraint equations for a \(O(16 + d, d) \otimes O(d)\) coset as is well known from both the mathematical \([12]\) and the supergravity literature \([12]\). Such equations are crucial for the construction of supergravity actions and especially of the associated Kähler potential. As an example, we will in section 5 discuss the case of an \(SO(4,2) \otimes SO(2)\) coset in big detail.

From the homogenous coordinate it is easy to construct a projective one, that is a coordinate that transforms under the group action by fractional linear transformations.
Defining

\[ Z = XY^{-1} = \begin{pmatrix} -A \\ D \end{pmatrix} \] (4.31)

we see that the group acts in fact on \( Z \) by fractional linear transformations

\[ Z \to (aZ + b)(cZ + d)^{-1} \] (4.32)

Note also that the dependence on rotations from the right that the homogenous coordinate still had, has completely cancelled out in (4.31). This is manifest since the \( d \)-bein variable \( E^* \) has disappeared. This is the second typical feature of a projective coset coordinate [42].

One useful application of the projective coordinate is that the transformation properties of the moduli can be deduced very simply from it. To pass from the deformation group \( O(16 + d, d) \) to the modular invariance group \( O(16 + d, d; \mathbb{Z}) \) one simply has to restrict to the subgroup of integer valued matrices. As an example let us calculate the transformation of the moduli under the duality transformation which generalizes the well known \( R \to \frac{1}{2R} \) duality known from compactification on the circle. On the level of quantum numbers the generalized duality transformation exchanges winding and momentum numbers while leaving the charges invariant. The corresponding matrix in \( O(16 + d, d; \mathbb{Z}) \) is obviously

\[ \Omega^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \] (4.33)

By acting with \( \Omega^T \) on \( Z \) as in (4.32) one immediately gets that

\[ Z = \begin{pmatrix} -A \\ D \end{pmatrix} \to \begin{pmatrix} -AD^{-1} \\ D^{-1} \end{pmatrix} \] (4.34)

a result which is much harder to derive by other methods.

In the orbifold case the transformation law of the real moduli remains, of course, the same but one has to check that the matrix acting on the quantum numbers fulfills the normalizer condition (3.1). There is, however, another problem (for models with \( \mathcal{N} = 1 \) supersymmetry) since one would like to use a complex parametrization of the moduli. This will be discussed in the next section.
As a second application of the projective coordinate (4.31) let us rederive the global structure of the toroidal moduli space. It is well known that the action of the modular invariance group on the moduli is not faithful [43]. Therefore the true modular symmetry group $G_T$ is a factor group of $G_T$ by those transformations which act trivially on the toroidal moduli. It is known that

$$G_T = PO(16 + d, d; Z) = O(16 + d, d; Z)/\{ \pm I \} \quad (4.35)$$

Whereas it is obvious that $-I$ acts trivially on the moduli, it is less obvious that there are no further trivial transformations. However, by using the projective coordinate $Z$ it becomes clear that, if one requires

$$(aZ + b)(cZ + d)^{-1} = Z \quad (4.36)$$

for all $Z$, then it follows that $b = 0$, $c = 0$ and that either $a = I$, $d = I$ or $a = -I$, $d = -I$.

In the orbifold case the connection between $G_O$ and $G_O$ is an open problem [43]. Clearly the group generated by the twist $\Theta$ acts trivially, by construction, but this is all one knows in general. Of course, one is ultimately interested in $G_O$ because it is this group which acts on the moduli and, therefore, decides both about the global geometry of moduli space and the form of effective supergravity actions. At least in the cases where $G_O$ factorizes into groups acting on the cosets in the decomposition (3.19) it should be possible to make a general statement about the connection between $G_O$ and $G_O$, but we will not try to do so in this paper. We will, in the following, rather focus on (3.19) and deal with the problem of its complexification and the derivation of the associated Kähler potentials.

5. Kähler potentials for $Z_N$ orbifold compactifications with continuous Wilson lines

As we have seen above, the untwisted moduli space of $Z_N$ orbifold compactifications with continuous Wilson lines preserving $\mathcal{N} = 1$ space-time supersymmetry is locally given by a direct product of $SU(n,m)/SU(n) \otimes SU(m) \otimes SU(1)$ and $SO(r,2)/SO(r) \otimes SO(2)$ cosets. Each of these cosets is a Kählerian manifold, that is, its metric $g_{\phi \bar{\phi}}$ is locally expressible as $g_{\phi \bar{\phi}} = \partial_\phi \partial_{\bar{\phi}} K(\phi, \bar{\phi})$ in some complex coordinate system $(\phi, \bar{\phi})$. $K$ denotes its Kähler potential. The Kähler structure of the untwisted moduli space is then determined by the full Kähler potential given by the sum of the Kähler potentials of the individual cosets. The Kähler potential, on the other hand, is also one of the three fundamental functions which describe the
tree-level couplings of generic matter multiplets to $4D, \mathcal{N} = 1$ supergravity, as is well known [4, 8]. Thus, in order to determine the tree-level low-energy Lagrangian describing the coupling of the $Z_N$ orbifold moduli fields to supergravity, knowledge of the associated Kähler potential is crucial. This section is devoted towards explicitly constructing the Kähler potentials for some of the moduli spaces discussed in the previous section. Namely, we will focus on the $Z_N$ orbifold compactifications for which the internal 6-torus $T_6$ can be decomposed into a direct sum $T_4 \oplus T_2$. All such cases are given in table [3] [38]. Then, by using well known techniques [12, 13, 14], we will derive the Kähler potentials for the moduli spaces associated with the underlying 2-torus $T_2$. We will first focus on the $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ cosets. As stated earlier, they arise when the twist operating on $T_2$ has two eigenvalues of $-1$. It is well known [17] that, in the case when no continuous Wilson lines are turned on, the resulting $\frac{SO(2,2)}{SO(2)\otimes SO(2)}$ coset factorises into $\frac{SO(2,2)}{SO(2)\otimes SO(2)} = \left(\frac{SU(1,1)}{U(1)}\right)_T \otimes \left(\frac{SU(1,1)}{U(1)}\right)_U$, each of them being coordinatized by one complex modulus, $T$ and $U$, respectively. The associated Kähler potential is then simply given by the sum of two individual Kähler potentials. When turning on Wilson lines, however, the resulting $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ coset will in general not factorise anymore, and the resulting Kähler potential will be much more complicated. Nevertheless, one still expects to find one modified $T$ and one modified $U$ modulus among the complex coordinates of the $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ coset. This is in fact the case, as we shall see below. We will, for concreteness, discuss the $\frac{SO(4,2)}{SO(4)\otimes SO(2)}$ and the $\frac{SO(3,2)}{SO(3)\otimes SO(2)}$ cosets in great detail. They are the simplest non-trivial ones occuring when turning on the two Wilson lines $A_i$ associated with the two directions of the underlying 2-torus $T_2$. Any other $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ coset, however, can in principle be analysed along very similar lines, although arriving at explicit results for the Kähler potential might be quite tedious. Finally, at the end of this section, we will discuss the Kähler potential for the $\frac{SU(n,1)}{SU(n)\otimes U(1)}$ cosets with Wilson lines turned on. They occur whenever the twist acting on the underlying 2-torus $T_2$ doesn’t have eigenvalues $-1$. We will, for concreteness, discuss the $\frac{SU(2,1)}{SU(2)\otimes U(1)} \otimes \frac{SU(n,1)}{SU(n)\otimes U(1)}$ coset in a straightforward manner.

Let us begin by explicitly constructing a suitable set of complex coordinates for a $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ coset along the lines of [12, 13, 14]. A set of real coordinates $\gamma^I_{\mu}$ for a general coset space $\frac{SO(r,2)}{SO(r)\otimes SO(2)}$ was found in (4.25) and is given by

$$
\begin{pmatrix}
-\mathcal{L}_{\mu a} \\
-(L + \tilde{L})_{\mu}^i \\
\frac{1}{2}(L - \tilde{L})_{\mu}^i
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{2}A_{ak}E_{\mu}^k \\
\frac{1}{2}D_{ij}E_{\mu}^j \\
\frac{1}{2}E_{\mu}^i
\end{pmatrix}
$$

(5.1)
Table 2: $Z_N$ orbifold compactifications with $\mathcal{N} = 1$ space-time supersymmetry and $T_6 = T_2 \oplus T_4$, as given in the classification of [19].

The real moduli matrix $D_{ij}$, associated with the underlying 2-torus $T_2$, is given in (2.8). $E^* = (e^\mu \cdot e^k) = (E^{\mu k})$ denotes the inverse zweibein. Note, too, that we have, for later convenience, introduced component Wilson lines $A_{ai}$ as

$$A_{ai} = e_a \cdot A_i, \quad a = 1, \ldots, 16 \quad (5.2)$$

They are defined relative to the orthonormal basis $e_a$ introduced in (4.21). The $A_{ai}$ are not to be confused with the $A_{Ai}$ introduced in (2.6). The latter ones are defined relative to the lattice basis $e_A$. Note that (2.7) can be equivalently expressed as

$$A_i \cdot A_j = A^a_i A_{aj} \quad (5.3)$$

Also note that $r$ can take any value between $2 \leq r \leq 18$.

It can be readily checked that the $\Upsilon_{\mu '}^I$ satisfy the relation

$$\theta_{IJ} \Upsilon_{\mu '}^I \Upsilon_{\nu '}^J = -\delta_{\mu \nu} \quad (5.4)$$

where the metric tensor $\theta_{IJ}$ for pulling down indices is given by

$$(\theta_{IJ}) = \begin{pmatrix}
\delta^{ab} & 0 & 0 \\
0 & 0 & \delta^i_j \\
0 & \delta_i^j & 0
\end{pmatrix} \quad (5.5)$$
Disentangling the coordinates $L$ and $\tilde{L}$ \[12\] yields a new set of real coordinates $\Upsilon_{\mu}^I$ given by

$$
(\Upsilon_{\mu}^I) = \left(\begin{array}{c} \mathcal{L}_{\mu a} \\ \tilde{L}_{\mu}^i \\ L_{\mu}^i \end{array}\right) = \left(\begin{array}{c} \frac{1}{2}A_{ak}E_k^\mu \\ \frac{1}{2}(\frac{1}{2}E_{\mu}^i - E_{\mu i} + E_{\mu j}B_{ji} + \frac{1}{4}E_{\mu}^j(A^a_j A_a)) \\ \frac{1}{2}(E_{\mu}^i + E_{\mu i} + E_{\mu j}B_{ji} + \frac{1}{4}E_{\mu}^j(A^a_j A_a)) \end{array}\right)
$$

(5.6)

and satisfying the $SO(r) \otimes SO(2)$ coset relation

$$
\theta_{IJ} \Upsilon_{\mu}^I \Upsilon_{\nu}^J = -\delta_{\mu\nu}
$$

(5.7)

where the metric tensor $\theta_{IJ}$ is now diagonal and given by

$$
(\theta_{IJ}) = \left(\begin{array}{ccc} \delta^{ab} & 0 & 0 \\ 0 & \delta_{ij} & 0 \\ 0 & 0 & -\delta_{ij} \end{array}\right)
$$

(5.8)

Inserting the metric tensor (5.8) into (5.7) yields

$$
\delta_{ij}L_{\mu}^i L_{\nu}^j = \delta_{\mu\nu} + \delta_{ij}\tilde{L}_{\mu}^i \tilde{L}_{\nu}^j + \delta^{ab}\mathcal{L}_{\mu a}\mathcal{L}_{\nu b}
$$

(5.9)

Next, let us introduce a set of complex coordinates as follows \[12\]. The real matrix $\Upsilon_{\mu}^I$ has two columns and $r+2$ rows. By combining the two real entries in each row into a single complex variable

$$
\phi_I = \Upsilon_1^I + i\Upsilon_2^I
$$

(5.10)

one arrives at a set of $r+2$ complex coordinates. In terms of these new complex variables the orthogonality relation (5.7) now reads

$$
-\phi^T I_{r,2} \phi = 2 \\
\phi^T I_{r,2} \phi = 0
$$

(5.11)

where $\phi$ is a complex column vector with complex entries $\phi_I$ and where $I_{r,2}$ denotes a diagonal matrix with entries $I_{r,2} = \text{diag}(1, \ldots, 1, -1, -1)$. This set of $r + 2$ complex coordinates is not yet a suitable one for constructing Kähler potentials, since they are not unconstrained but rather satisfy the orthogonality properties (5.11). The next step then consists in identifying a particular set of $r$ unconstrained complex coordinates which
the Kähler potential is going to depend on. It is known [14] how to find such a set of analytic coordinates for a general $SO(\tau, 2) \otimes SO(\tau)$ coset. For concreteness and in order to keep the formulae as simple as possible, we will in the following focus on the $SO(4, 2) \otimes SO(2)$ coset and explicitly construct an analytic set of coordinates for it. To proceed, we first introduce an explicit parametrisation of the metric $G_{ij}$ of the underlying $T_2$ torus as follows

$$G_{ij} = \begin{pmatrix} R_1^2 & R_1R_2 \cos \theta \\ R_1R_2 \cos \theta & R_2^2 \end{pmatrix}$$ \hspace{1cm} (5.12)

The associated zweibein $E_{\mu i}$ satisfying $G_{ij} = \delta^{\mu \nu} E_{\mu i} E_{\nu j}$ and its inverse are then given by

$$E_{\mu i} = \begin{pmatrix} R_1 & R_2 \cos \theta \\ 0 & R_2 \sin \theta \end{pmatrix}, \quad E_{\mu i}^{-1} = \begin{pmatrix} 1/R_1 & -1 \cos \theta/R_1 \sin \theta \\ 0 & 1/R_2 \sin \theta \end{pmatrix}$$ \hspace{1cm} (5.13)

Then, by inserting parametrisation (5.12) and (5.13) into (5.10), the six complex coordinates $\phi_I$ can be readily expressed in terms of the real moduli fields $G_{ij}$, $B_{ij}$ and $A_{ai}$ as

$$\phi_1 = \frac{1}{\sqrt{Y}} \left( A_{11} \frac{\sqrt{G}}{G_{11}} + i(-A_{11} G_{12} G_{11}^{-1} + A_{12}) \right)$$

$$\phi_2 = \frac{1}{\sqrt{Y}} \left( A_{21} \frac{\sqrt{G}}{G_{11}} + i(-A_{21} G_{12} G_{11}^{-1} + A_{22}) \right)$$

$$\phi_3 = \frac{1}{\sqrt{Y}} \left( \sqrt{G} (1 - \frac{1}{G_{11}} + \frac{1}{4} A_{a1} A_{a1}) + i(-B_{12} + \frac{G_{12}}{G_{11}} (1 - \frac{1}{4} A_{a1} A_{a1} + \frac{1}{4} A_{a2} A_{a2})) \right)$$

$$\phi_4 = \frac{1}{\sqrt{Y}} \left( \frac{\sqrt{G}}{G_{11}} (G_{12} + B_{12} + \frac{1}{4} A_{a1} A_{a2}) \right.$$

$$\left. + i(-1 + \frac{G}{G_{11}} - B_{12} \frac{G_{12}}{G_{11}} - \frac{1}{4} A_{a1} A_{a2} G_{12} G_{11}^{-1} + \frac{1}{4} A_{a2} A_{a2}) \right)$$

$$\phi_5 = \frac{1}{\sqrt{Y}} \left( \sqrt{G} (1 + \frac{1}{G_{11}} + \frac{1}{4} A_{a1} A_{a1}) + i(-B_{12} - \frac{G_{12}}{G_{11}} (1 + \frac{1}{4} A_{a1} A_{a1} + \frac{1}{4} A_{a2} A_{a2})) \right)$$

$$\phi_6 = \frac{1}{\sqrt{Y}} \left( \frac{\sqrt{G}}{G_{11}} (G_{12} + B_{12} + \frac{1}{4} A_{a1} A_{a2}) \right.$$

$$\left. + i(1 + \frac{G}{G_{11}} - B_{12} \frac{G_{12}}{G_{11}} - \frac{1}{4} A_{a1} A_{a2} G_{12} G_{11}^{-1} + \frac{1}{4} A_{a2} A_{a2}) \right)$$ \hspace{1cm} (5.14)

where $G = \text{det} G_{ij}$ and

$$Y = 4 \frac{G}{G_{11}}$$ \hspace{1cm} (5.15)
Note that there is an overall factor $\sqrt{Y}$ appearing in front of each of the six $\phi_I$. It is therefore convenient [14] to introduce rescaled coordinates

$$y_I = \sqrt{Y} \phi_I$$  \hspace{1cm} (5.16)$$

satisfying the rescaled constraints

$$-y^\dagger I_{4,2} y = 2Y$$  
$$y^T I_{4,2} y = 0$$ \hspace{1cm} (5.17)$$

The importance of the overall factor lays in that it determines the Kähler potential $K$ as

$$K = -\ln Y$$ \hspace{1cm} (5.18)$$

provided one chooses a solution to the constraint equations (5.17) possessing an $SO(1,3)$ symmetry (more generally an $SO(1, r - 1)$ symmetry in the case of an $SO(r,2)SO(r)\otimes SO(2)$ coset) [14]. That is, by choosing four of the $y_I$ as unconstrained coordinates one then seeks a solution of the constraint equations (5.17) for the two remaining $y$-coordinates as well as for $Y$ which exhibits an $SO(1,3)$ symmetry. We will, for concreteness, choose $y_1, y_2, y_3$ and $y_5$ as unconstrained variables. Then, it can be checked that $Y$, given in (5.15), can be expressed in terms of $y_1, y_2, y_3$ and $y_5$ as

$$Y = \frac{1}{4} ((y_5 + \bar{y}_5)^2 - (y_1 + \bar{y}_1)^2 - (y_2 + \bar{y}_2)^2 - (y_3 + \bar{y}_3)^2)$$ \hspace{1cm} (5.19)$$

Note that (5.19) exhibits an $SO(1,3)$ symmetry. Inserting (5.19) into (5.17) yields

$$y_4^2 - y_6^2 = y_5^2 - y_1^2 - y_2^2 - y_3^2$$  
$$|y_4|^2 - |y_6|^2 = -\frac{1}{2}(y_5^2 - y_1^2 - y_2^2 - y_3^2 + y_2^2 - y_1^2 - y_3^2)$$ \hspace{1cm} (5.20)$$

From (5.14), on the other hand, it follows that

$$y_6 - y_4 = 2i$$ \hspace{1cm} (5.21)$$

Then, it can be checked that the following [14] solves (5.20) subject to (5.21)

$$y_4 = -i \left(1 - \frac{1}{4}(y_5^2 - y_1^2 - y_2^2 - y_3^2)\right)$$  
$$y_6 = i \left(1 + \frac{1}{4}(y_5^2 - y_1^2 - y_2^2 - y_3^2)\right)$$ \hspace{1cm} (5.22)$$

Note that the solution (5.22) also exhibits an $SO(1,3)$ symmetry. Thus, (5.19) and (5.22) are a solution to the orthogonality relations (5.17) with manifest $SO(1,3)$ symmetry.
The analytic structure of the Kähler potential \( K = -\ln Y \) can be made manifest by introducing four complex fields \( M_{ij} \) as

\[
(M_{ij}) = \begin{pmatrix}
y_5 + y_3 & y_1 - iy_2 \\
y_1 + iy_2 & y_5 - y_3
\end{pmatrix}
\] (5.23)

Then, \( Y \) is given by

\[
Y = \frac{1}{4} \det (M_{ij} + \bar{M}_{ij})
\] (5.24)

Finally, introducing the linear combinations

\[
T = y_5 + y_3 \\
2U = y_5 - y_3 \\
B = y_1 - iy_2 \\
C = y_1 + iy_2
\] (5.25)

yields

\[
(M_{ij}) = \begin{pmatrix}
T & B \\
C & 2U
\end{pmatrix}
\] (5.26)

It follows from (5.14) that the complex \( T, U, B \) and \( C \) moduli fields are expressed in terms of the real ones as

\[
T = 2 \left( \sqrt{G}(1 + \frac{1}{4} A_{a1}^a A_{a1}^a) - i (B_{12} + \frac{1}{4} A_{a1}^a A_{a2}^a \frac{G_{12}}{G_{11}} - \frac{1}{4} A_{a1}^a A_{a2}^a) \right)
\]

\[
U = \frac{1}{G_{11}} (\sqrt{G} - i G_{12})
\]

\[
B = \frac{1}{G_{11}} \left( A_{11} \sqrt{G} - A_{21} G_{12} + A_{22} G_{11} + i (- A_{11} G_{12} + A_{12} G_{11} - A_{21} \sqrt{G}) \right)
\]

\[
C = \frac{1}{G_{11}} \left( A_{11} \sqrt{G} + A_{21} G_{12} - A_{22} G_{11} + i (- A_{11} G_{12} + A_{12} G_{11} + A_{21} \sqrt{G}) \right)
\] (5.27)

The \( T \) and the \( U \) modulus are related to the geometrical data of the two-dimensional torus \( T_2 \). The \( T \) modulus is associated with deformations of the Kähler class. It reduces to the well-known expression when turning off the real Wilson lines \( A_{ai} \). The \( U \) modulus is associated with deformations of the complex structure. Note that it doesn’t get admixtures of real Wilson lines \( A_{ai} \), that is, it remains given by the well-known
expression \(\text{[15]}\) when no Wilson lines are turned on. Finally, the complex \(B\) and \(C\) moduli are linear expressions in the real Wilson lines \(A_{ai}\). They vanish when turning off the real Wilson lines \(A_{ai}\). Thus, they qualify to be called complex Wilson lines. The Kähler potential reads

\[
K = -\ln Y = -\ln \left( \frac{1}{4} \det (M_{ij} + \bar{M}_{ij}) \right)
\]

A few remarks are at hand. First note that in the absence of Wilson lines \((B = C = 0)\) the Kähler potential splits into the sum \(K = K(T, \bar{T}) + K(U, \bar{U})\), which is the well-known Kähler potential for the coset \(\text{SO}(2,2)\). On the other hand, turning on Wilson lines leads to the Kähler potential \((5.28)\) which doesn’t split into two pieces anymore. This is so, because the \(\text{SO}(4,2)\) coset doesn’t factorise anymore into two submanifolds. Also note that the complex Wilson lines \(B\) and \(C\) do not just give rise to \(\bar{B}B\) and \(\bar{C}C\) terms in the Kähler potential but also to holomorphic \(BC\) and antiholomorphic \(\bar{B}\bar{C}\) pieces. This will have important consequences when discussing target space duality symmetries of the Kähler potential, as discussed in the next section.

Finally, let us point out that, even in the presence of Wilson lines, the Kähler potential is in terms of the real moduli still given as \(K = -\ln \frac{4G}{G_{11}}\) and thus still proportional to the volume of the internal manifold.

We proceed with a discussion of the \(\text{SO}(3,2)/\text{SO}(3)\otimes\text{SO}(2)\) coset. Inspection of \((5.6)\) shows that a \(\text{SO}(3,2)/\text{SO}(3)\otimes\text{SO}(2)\) coset occurs when only retaining the first components \(A_{11}\) and \(A_{12}\) and setting all the other components of the two Wilson lines \(A_{1}\) and \(A_{2}\) to zero. Then, it follows from \((5.14)\) that \(\phi_{2} = 0\) and, hence, \(\bar{y}_{2} = 0\). The corresponding solutions \((5.19)\) and \((5.22)\) then exhibit an \(\text{SO}(1,2)\) symmetry and the Kähler potential is thus given again by \(K = -\ln Y\). The complex moduli can be read off from \((5.25)\), where now \(B = C = \bar{y}_{1}\). The complex \(T, U\) and \(B\) moduli are expressed in terms of the real ones as in \((5.27)\) with \(A_{21} = A_{22} = 0\). It follows from \((5.28)\) that the Kähler potential for the \(\text{SO}(3,2)/\text{SO}(3)\otimes\text{SO}(2)\) coset reads

\[
K = -\ln \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B})^{2} \right) + \text{const} \tag{5.29}
\]

This concludes the discussion of the \(\text{SO}(4,2)/\text{SO}(4)\otimes\text{SO}(2)\) and \(\text{SO}(3,2)/\text{SO}(3)\otimes\text{SO}(2)\) cosets. Let us point out again that any other \(\text{SO}(r,2)/\text{SO}(r)\otimes\text{SO}(2)\) coset can be analysed along very similar lines.

We now discuss the situation when the twist \(\theta = (\theta_{ij})\) acting on the internal 2-torus \(T_{2}\) has eigenvalues different from \(-1\). Introducing \(\theta^{T_{-1}} = (\theta^{T}_{jk})\) and analysing the
consistency conditions \[10\]

\[ G^j_k \theta^j = \theta^i \cdot G^i_j \quad (A \cdot A) \theta^j = \theta^i \cdot (A \cdot A) \]  

\hspace{1cm} (5.30)

for the internal metric \( G^i_j \) and for the matrix \( A \cdot A \) shows that both \( G^i_j \) and \( A \cdot A \) have only one independent entry each. Denoting these independent entries by \( G_{11} \) and \( A_1 \cdot A_1 \), respectively, yields

\[ (G_{ij}) = \frac{G_{11}}{2} \begin{pmatrix} 2 & \alpha \\ \alpha & \beta \end{pmatrix}, \quad (A \cdot A) = \frac{A_1 \cdot A_1}{2} \begin{pmatrix} 2 & \alpha \\ \alpha & \beta \end{pmatrix} \]  

\hspace{1cm} (5.31)

where \( \alpha \) and \( \beta \) are some twist dependent constants. It then follows from \[9\] that the \( U \)-field takes a constant value given by

\[ U = \frac{\sqrt{2 \beta - \alpha^2}}{2} - i\alpha \]  

\hspace{1cm} (5.32)

The \( T \)-field survives as a modulus and is given by

\[ T = 2 \left( \frac{\sqrt{2 \beta - \alpha^2}}{2} G_{11} + \frac{1}{4} A^a A_a - iB_{12} \right) \]  

\hspace{1cm} (5.33)

Note that only the first real Wilson line enters in this expression. In fact, the second real Wilson line is not independent anymore, but rather fixed in terms of the first real Wilson line \[6\]. This follows directly from the consistency condition \[10\]

\[ \theta' A_i = A^A \theta' A^B e_B = \theta^j A_j \]  

\hspace{1cm} (5.34)

for the continuous Wilson lines. Then, \[5.34\] yields

\[ A_1 \rightarrow A_2 = \theta^j A_j = \theta' A_1 = A^A \theta' A^B e_B \]  

\hspace{1cm} (5.35)

which, indeed, expresses \( A_2 \) in terms of \( A_1 \).

It is useful to introduce a complex Wilson line as

\[ \mathcal{A} = A_{11} + iA_{21} \]  

\hspace{1cm} (5.36)

Then, it follows from \[5.27\] that the combination \( B + \bar{C} \) can be written as

\[ B + \bar{C} = \sqrt{2 \beta - \alpha^2} \bar{A} \]  

\hspace{1cm} (5.37)

and the Kähler potential \[5.28\] as

\[ K = -ln \left( T + \bar{T} - \frac{\sqrt{2 \beta - \alpha^2}}{2} \bar{A} A \right) + const \]  

\hspace{1cm} (5.38)
This is the Kähler potential for the $SU(2,1)/SU(2) \otimes U(1)$ coset. More generally, a $SU(n,1)/SU(n) \otimes U(1)$ coset will be parametrised by one complex $T$ modulus and $n$-1 complex Wilson lines $A_l$ given by

$$T = 2 \left( \frac{\sqrt{2} \beta - \alpha^2}{2} (G_{11} + \frac{1}{4} A_l^a A_{al}) - i B_{12} \right)$$

$$A_l = A_{1l} + i A_{2l} \quad \text{(5.39)}$$

The corresponding Kähler potential reads

$$K = -\ln \left( T + \bar{T} - \frac{\sqrt{2} \beta - \alpha^2}{2} \bar{A}_l A_l \right) + \text{const} \quad \text{(5.40)}$$

Finally, let us point out that additional untwisted matter fields, charged or uncharged under the generic gauge group which survives when turning on Wilson lines, enter into the Kähler potentials (5.28) and (5.40) in precisely the same way as the complex Wilson lines $B, C, A_l$. Hence, the modular symmetry properties of these Kähler potentials are preserved by the inclusion of untwisted matter fields.

### 6. Examples of Target Space Duality Symmetries

We are now poised to discuss the symmetry properties of the Kähler potentials we constructed in the previous section. We will, in particular, be concerned with target space duality symmetries, also referred to as modular symmetries. As stated in section 4, the spectrum of untwisted states of an orbifold theory is invariant under certain discrete transformations of the winding and momentum numbers accompanied by discrete transformations of the moduli fields. These transformations of the moduli fields induce particular Kähler transformations of the Kähler potential and, thus, are symmetries of the tree-level low-energy Lagrangian describing the coupling of moduli fields to supergravity.

As explained in section 4, modular transformations act on the vector $v^T = (q^A, n^i, m_i)$ of quantum numbers as integer valued transformations $\Omega$

$$v' = \Omega^{-1} v \quad \text{(6.1)}$$

As discussed earlier, $\Omega$ must satisfy (4.8). Modular transformations (6.1) act on the real moduli matrix $\Sigma$ given in (4.13) as

$$\Sigma \rightarrow \Omega^T \Sigma \Omega \quad \text{(6.2)}$$

We begin by discussing modular symmetries of the $SO(r,2)/SO(r) \otimes SO(2)$ cosets. For concreteness, we will again focus on the $SO(4,2)/SO(4) \otimes SO(2)$ coset. The associated modular group $G_O$
will be called $O(4, 2; \mathbb{Z})$. It is crucial at this point to notice that some care is required in order to specify this group. If the sixdimensional sublattice $\Gamma_{4,2}$ of the Narain lattice $\Gamma_{22,6}$, on which $G_O$ acts, happens to factorize (that is, if there is an orthogonal direct decomposition of $\Gamma_{22,6}$ into $\Gamma_{4,2}$ and its complement) then $G_O$ will be the group given by

$$\{M \in \text{Gl}(6; \mathbb{Z})| M^T H M = H\} \quad (6.3)$$

where $H$ is the lattice metric of $\Gamma_{4,2}$. But such a decomposition will in general not exist [46] and therefore one has the further constraint that the elements of $G_O$ must also act crystallographically on the full lattice $\Gamma_{22,6}$. The resulting constraints should be similar to those found in the case where the internal six–dimensional torus does not factorize [27].

For definiteness and simplicity, we will only consider the following case which is the simplest one. As already explained in the last section, we demand that the internal torus decomposes as $T_4 \oplus T_2$, where the twist $\theta$ acts on $T_2$ as $-I$. The corresponding directions are labeled by $i = 1, 2$. The modular group $G_O$ then contains the well known group

$$O(2, 2; \mathbb{Z}) = \{M \in \text{Gl}(4, \mathbb{Z})| M^T \eta M = \eta\} \quad (6.4)$$

as a subgroup, where

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

A complete set of generators for $O(2, 2, \mathbb{Z})$ can be found in [47]. $O(2, 2, \mathbb{Z})$ acts non-trivially on the components $n^i$, $m_i$ ($i = 1, 2$) corresponding to the four basis vectors $\overline{k}_i$, $k^i$ of the Narain lattice. In order to find the full group $G_O$ we must identify those quantum numbers $q^A$ which will transform non-trivially under it. Since, by definition, $G_O$ is the group of modular transformations in the $-1$ eigenspace of the twist $\Theta$, this then means that we must identify all basis vectors $l_A$ of the Narain lattice which transform non-trivially under the projection of the twist to the $-1$ eigenspace. In [44] it was shown that, in the absence of discrete background fields, the $l_A$ transform under the twist $\Theta$ with the same integer valued matrix as the $E_8 \otimes E_8$ basis vectors $e_A$. Note furthermore that, due to the explicit form (2.5) of the $l_A$, modular transformations of the $q^A$ among themselves and automorphisms of $E_8 \otimes E_8$ are in a one to one correspondence. Starting from these observations we can find choices for the twist which are quite close to the situation where the corresponding lattice factorizes. Namely, we will choose the gauge twist $\theta'$ such that its two eigenvalues $-1$ come from a coxeter twist in an $A_1 \otimes A_1$ sublattice. (Note, however, that this is not the most general situation. A general Coxeter twist of
an subalgebra of $E_8 \otimes E_8$, which may be used to define an $E_8 \otimes E_8$ automorphism, will have several different eigenvalues.)

In order to construct the modular group $G_O$ we proceed in two steps. First consider the sixdimensional sublattice $\Gamma_{4;2}$ of $\Gamma_{22;6}$, which is spanned by $l_i, \overline{k}_i, k^i$ ($i = 1, 2$), where $l_1$ and $l_2$ correspond to the $A_1 \otimes A_1$ sublattice. The group of pseudo–orthogonal automorphisms of $\Gamma_{4;2}$ is then given by

$$\{M \in \text{Gl}(6; \mathbb{Z})\mid M^T H M = H\}$$

(6.5)

with

$$H = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$$

(6.6)

where $C = 2 \text{diag}(1, 1)$ is the Cartan matrix of $A_1 \otimes A_1$.

In a second step we should then identify all elements of this group which also act cristallographically on the full lattice. There are three classes of elements which automatically fulfill this condition, namely all elements of the subgroup $O(2, 2; \mathbb{Z})$, all Weyl automorphisms of $A_1 \otimes A_1$ and shifts of the $q^i$ by multiples of $n^j, m_j$ with $i, j = 1, 2$. To be able to say something about other elements would, however, require a more detailed analysis. Therefore we will, in the following, only consider various particularly interesting elements belonging to these three classes, and we will work out the corresponding transformation properties of the real and complex moduli as well as of the Kähler potential.

There are, actually, three ways of deriving the action of the modular group $G_O$ on the real moduli fields $G_{ij}, B_{ij}$ and $A_{Ai}$. We will, in the following, make use of all three of them. They are as follows. The transformation law of the real moduli can, in principle, be obtained from (6.2). This, however, can prove to be quite cumbersome. An alternative way of obtaining the transformation law of the real moduli fields is given by looking at the transformation law of the projective coset coordinate $Z$ given in (4.31). Modular transformations (6.1) act on $Z$ by fractional linear transformations (4.32). Yet another way for deriving the transformation law of the real moduli fields is to look at the background field matrix $\varepsilon$ given by

$$\varepsilon = \begin{pmatrix} E & F \\ 0 & X \end{pmatrix} = \begin{pmatrix} 2(G + B + \frac{1}{2}A^a A_a)_{ij} & A_{Ai} \\ 0 & (G + B)_{AB} \end{pmatrix}$$

(6.7)
where the \((G + B)_{AB}\)-data on the \(E_8 \otimes E_8\) root lattice are given by

\[
(G + B)_{AB} = C_{AB}, \quad A > B, \quad (G + B)_{AA} = \frac{1}{2}C_{AA}, \quad (G + B)_{AB} = 0, \quad A < B
\]  

(6.8)

Consider an element \(\hat{g} \in O(4, 4, \mathbb{Z})\) and the bilinear form \(\hat{\eta}\)

\[
\hat{\eta} = \begin{pmatrix}
\hat{a} & \hat{b} \\
\hat{c} & \hat{d}
\end{pmatrix}, \quad \hat{\eta} = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

(6.9)

where \(\hat{a}, \hat{b}, \hat{c}, \hat{d}, I\) are \(4 \times 4\)-dimensional matrices. \(\hat{g}\) satisfies \(\hat{g}^T \hat{\eta} \hat{g} = \hat{\eta}\). The action of \(O(4, 4, \mathbb{Z})\) on \(\varepsilon\) is given as \([49, 47]\)

\[
\varepsilon' = \hat{g}(\varepsilon) = (\hat{a} \varepsilon + \hat{b})(\hat{c} \varepsilon + \hat{d})^{-1}
\]

(6.10)

Then, the modular group \(O(4, 2, \mathbb{Z})\) is the subgroup of \(O(4, 4, \mathbb{Z})\) that preserves the heterotic structure of \(\varepsilon\) in \((6.7)\) while acting on \(\varepsilon\) by fractional linear transformations \((6.10)\).

The modular group \(O(4, 2, \mathbb{Z})\) contains an \(O(2, 2, \mathbb{Z})\) subgroup. There is a natural embedding \([49, 47]\) of \(O(2, 2, \mathbb{Z})\) into \(O(4, 2, \mathbb{Z})\) given as follows. Consider an element \(g \in O(2, 2, \mathbb{Z})\) and the bilinear form \(\eta\)

\[
g = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad \eta = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

(6.11)

where \(a, b, c, d, I\) are \(2 \times 2\)-dimensional matrices. \(g\) satisfies \(g^T \eta g = \eta\). Then, the embedding of \(O(2, 2, \mathbb{Z})\) into \(O(4, 2, \mathbb{Z})\) is given as

\[
\hat{a} = \begin{pmatrix}
\hat{a} & 0 \\
0 & I
\end{pmatrix}, \quad \hat{b} = \begin{pmatrix}
b & 0 \\
0 & 0
\end{pmatrix}, \quad \hat{c} = \begin{pmatrix}
c & 0 \\
0 & 0
\end{pmatrix}, \quad \hat{d} = \begin{pmatrix}
d & 0 \\
0 & I
\end{pmatrix}
\]

(6.12)

The action of \(O(2, 2, \mathbb{Z})\) on \(\varepsilon\) yields

\[
\varepsilon' = \hat{g}(\varepsilon) = \begin{pmatrix}
E' & (aE - E'c)F \\
0 & X
\end{pmatrix}
\]

(6.13)

where

\[
E' = (aE + b)(cE + d)^{-1}
\]

(6.14)
Let us now look at the subgroup of $O(2, 2, \mathbb{Z})$ modular transformations. A set of generators for $O(2, 2, \mathbb{Z})$ can be found in the literature [47]. Here, we will, in the following, look at specific $O(2, 2, \mathbb{Z})$ modular transformations and derive the transformation laws for the real moduli fields. Consider the inverse duality transformation given by

$$
\Omega = \begin{pmatrix} I & 0 \\ 0 & g^T \end{pmatrix}, \quad g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$

(6.15)

It acts on the quantum numbers as in (6.1), thus interchanging the winding numbers $n^i$ with the momentum numbers $m^i$. From (6.2) one finds that

$$
G_{ij} \to \frac{1}{4} \left( (G + B + \frac{1}{4} A^a A_a)^{-1} G (G - B + \frac{1}{4} A^a A_a)^{-1} \right)_{ij}
$$

$$
B_{ij} \to -\frac{1}{4} \left( (G + B + \frac{1}{4} A^a A_a)^{-1} B (G - B + \frac{1}{4} A^a A_a)^{-1} \right)_{ij}
$$

$$
A^a i A_{aj} \to \frac{1}{4} \left( (G + B + \frac{1}{4} A^a A_a)^{-1} A^b A_b (G - B + \frac{1}{4} A^a A_a)^{-1} \right)_{ij}
$$

(6.16)

From (6.16) it then follows that

$$
E \to \frac{1}{4} E^{-1}
$$

(6.17)

which is in agreement with what one obtains from (6.14). The transformation law of the $A_{Ai}$ can, alternatively, be obtained from (4.34) in a straightforward way. It is consistent with the transformation property of the $A^a i A_{aj}$ given in (6.16).

We proceed to show that inverse duality is a symmetry transformation of the Kähler potential. To do so, one has to compute the transformation laws of the complex moduli fields $T, U, B$ and $C$ given in (5.27). We will, in the following, only list a few of the lengthy expressions arising when working out (6.16) and (4.34) explicitly. For instance, it can be checked that

$$
G_{11} \to \frac{1}{4} \left( \frac{G_{22}}{(\det (G + B + \frac{1}{4} A^a A_a))_{ij}} \right) \left( (B_{12} + \frac{1}{4} A^a_1 A_a - A^a_2 A_a \frac{G_{12}}{G_{22}})^2 
+ G (1 + \frac{1}{4} A^a_2 A_a \frac{G_{12}}{G_{22}})^2 \right)
$$

(6.18)

and that

$$
G \to \frac{1}{16} \frac{G}{(\det (G + B + \frac{1}{4} A^a A_a))_{ij}}
$$

(6.19)

It can also be verified that (6.18) can be rewritten as

$$
G_{11} \to \frac{1}{16} G_{11} \frac{1}{(\det (G + B + \frac{1}{4} A^a A_a))_{ij}} \left| U \right|^2 \left| T - \frac{1}{2} \frac{BC}{U} \right|^2
$$

(6.20)
Similarly, one finds from (4.34) that
\[
A_{11} \rightarrow -\frac{1}{2} \frac{1}{(\text{det}(G - B + \frac{1}{3} A^a A_a)_{ij})} \left( A_{11}(G_{22} + \frac{1}{4} A^a_2 A_{a2}) - A_{12}(G_{12} + B_{12} + \frac{1}{4} A^a_1 A_{a2}) \right)
\]
(6.21)
Inserting all these expressions into (5.27) yields
\[
\begin{align*}
U & \rightarrow \frac{T}{-UT + \frac{1}{2} BC} \\
T & \rightarrow \frac{U}{-UT + \frac{1}{2} BC} \\
B & \rightarrow \frac{B}{-UT + \frac{1}{2} BC} \\
C & \rightarrow \frac{C}{-UT + \frac{1}{2} BC}
\end{align*}
\]
(6.22)
Note that, in the presence of the complex Wilson lines \( B \) and \( C \), the \( T \) and \( U \) moduli now mix under inverse duality. When switching off the complex Wilson lines \( B \) and \( C \), no mixing occurs and one obtains the familiar transformation law for the \( T \) and \( U \) moduli \[47\]. Finally, inserting (6.22) into (5.28) yields
\[
K \rightarrow K + F + \bar{F}
\]
(6.23)
with the holomorphic \( F \) given by
\[
F = \ln \left( UT(1 - \frac{1}{2} BC \frac{BC}{UT}) \right)
\]
(6.24)
A useful check on (6.24) is to look at how \( Y = \frac{4G}{G_{11}} \) transforms under (6.16). It follows from (6.19) and (6.20) that
\[
Y \rightarrow Y \frac{1}{|U|^2} \frac{1}{|T - \frac{1}{2} BC|^2}
\]
(6.25)
Thus, \( K = -\ln \frac{4G}{G_{11}} \) transforms indeed as in (6.23) and (6.24).

Next, let us look at the subgroup of \( O(2,2,\mathbb{Z}) \) transformations which acts as \( SL(2,\mathbb{Z})_U \) transformations on the \( U \) modulus. Let
\[
\Omega = \begin{pmatrix} I & 0 \\ 0 & g^T \end{pmatrix}, \quad g^T = \begin{pmatrix} A \\ A^{T,-1} \end{pmatrix}
\]
(6.26)
where \( A \in SL(2, \mathbb{Z}) \). \( SL(2, \mathbb{Z}) \) is generated by two elements
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]  
(6.27)

Consider the case where \( A = T^p \) with \( p \in \mathbb{Z} \). Then, the transformation law of the real moduli fields is readily obtained from (6.14) and reads
\[
(G_{ij}) \rightarrow (G_{ij}) + \begin{pmatrix} 1 & pG_{11} \\ pG_{11} & p^2G_{11} + 2pG_{11} \end{pmatrix}
\]
\[
B_{12} \rightarrow B_{12}
\]
\[
\mathcal{A}_a^1 \rightarrow \mathcal{A}_a^1
\]
\[
\mathcal{A}_a^2 \rightarrow \mathcal{A}_a^2 + p\mathcal{A}_a^1
\]  
(6.28)

Inserting (6.28) into (5.27) yields
\[
U \rightarrow U - ip
\]
\[
T \rightarrow T
\]
\[
B \rightarrow B
\]
\[
C \rightarrow C
\]  
(6.29)

and, hence,
\[
K \rightarrow K
\]  
(6.30)

Now, consider the case where \( A = S \). Then, it follows from (6.14) that
\[
G_{11} \leftrightarrow G_{22}
\]
\[
G_{12} \rightarrow -G_{12}
\]
\[
B_{12} \rightarrow B_{12}
\]
\[
\mathcal{A}_a^1 \rightarrow -\mathcal{A}_a^2
\]
\[
\mathcal{A}_a^2 \rightarrow \mathcal{A}_a^1
\]  
(6.31)

Inserting (6.31) into (5.27) yields
\[
U \rightarrow \frac{1}{\bar{U}}
\]
Note the peculiar Wilson line admixture in the transformation law of the $T$ modulus. Its presence is necessary to make the Kähler potential transform properly. Indeed, inserting (6.32) into (5.28) yields

$$K \rightarrow K + F(U) + \bar{F}(\bar{U})$$

(6.33)

where

$$F(U) = \ln U$$

(6.34)

Under more general $SL(2, Z)$ transformations

$$A = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1$$

(6.35)

it can be checked that the following holds

$$U \rightarrow \frac{\alpha U - i \beta}{i \gamma U + \delta}$$

$$T \rightarrow T - \frac{i \gamma BC}{2i \gamma U + \delta}$$

$$B \rightarrow \frac{B}{i \gamma U + \delta}$$

$$C \rightarrow \frac{C}{i \gamma U + \delta}$$

(6.36)

and that

$$F(U) = \ln(i \gamma U + \delta)$$

(6.37)

As it is well known, there is also a subgroup of $O(2, 2, Z)$ transformations which act as $SL(2, Z)_T$ transformations on the $T$ modulus. Let

$$\Omega = \begin{pmatrix} I \\ g^T \end{pmatrix}, \quad g^T = \begin{pmatrix} \alpha I & \gamma J \\ -\beta J & \delta I \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1$$

(6.38)
Then, similarly, it can be shown that

\[
T \rightarrow \frac{\alpha T - i\beta}{i\gamma T + \delta}
\]
\[
U \rightarrow U - \frac{i\gamma}{2} \frac{BC}{i\gamma T + \delta}
\]
\[
B \rightarrow \frac{B}{i\gamma T + \delta}
\]
\[
C \rightarrow \frac{C}{i\gamma T + \delta}
\]

(6.39)

and

\[
K \rightarrow K + F(T) + \bar{F}(\bar{T})
\]

(6.40)

where

\[
F(T) = \ln(i\gamma T + \delta)
\]

(6.41)

Next, let us look at elements of \(O(4,2,Z)\) which are not in the \(O(2,2,Z)\) subgroup. The biggest additional subgroup which commutes with \(O(2,2,Z)\) is given by the group of automorphisms of the \(A_1 \otimes A_1\) sublattice of the \(E_8 \otimes E_8\) root lattice. Since this subgroup acts trivially on the winding numbers \(n^i\) and on the momentum numbers \(m_i\), we will not be interested in it. Consider, however, the following additional generator of \(O(4,2,Z)\), whose action on the quantum numbers is non-trivial, as follows. Consider the generator

\[
W_L = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(6.42)

It satisfies (6.7). Now, look at the action of the group element

\[
\Omega = (W_L)^p, \quad p \in Z
\]

(6.43)

on the quantum numbers as given in (6.1). It produces a shift in the first component of \(q\) (the momentum vector on the \(E_8 \otimes E_8\) lattice)

\[
q^1 \rightarrow q^1 - p n^1
\]

(6.44)
by \( p \) units of the winding number \( n^1 \). The corresponding transformation of the real moduli fields can be read off from the transformation properties (4.28) and (4.32) of the projective coordinate \( Z \) given in (4.31). One finds that

\[
\begin{align*}
G_{ij} &\rightarrow G_{ij} \\
B_{12} &\rightarrow B_{12} - \frac{1}{4} p A_{12} \\
A_{11} &\rightarrow A_{11} + p \\
A_{21} &\rightarrow A_{21} \\
A_{12} &\rightarrow A_{12} \\
A_{22} &\rightarrow A_{22}
\end{align*}
\tag{6.45}
\]

Thus, the shift given in (6.44) is accompanied by a shift in the first component of the first Wilson line. Inserting (6.45) into (5.27) yields

\[
\begin{align*}
T &\rightarrow T + \frac{p}{2} C_{11} U + \frac{p}{2} \sqrt{C_{11}} (B + C) \\
U &\rightarrow U \\
B &\rightarrow B + p \sqrt{C_{11}} U \\
C &\rightarrow C + p \sqrt{C_{11}} U
\end{align*}
\tag{6.46}
\]

and, from (5.28),

\[
K \rightarrow K
\tag{6.47}
\]

We would like to point out that the associated group element \( \hat{g} \in O(4, 2, Z) \) reproducing (6.45) via (6.10) can be constructed and that it is given by (6.3) with

\[
\hat{a} = \begin{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}
p & \begin{pmatrix}
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\end{pmatrix}, \quad \hat{b} = \begin{pmatrix}
-\frac{1}{2} p^2 C_{11} & \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} \\
\frac{1}{2} p C_{11} & \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\end{pmatrix}, \quad \hat{c} = 0, \quad \hat{d} = \hat{a}^T r^{-1}
\tag{6.48}
\]

Note that, since \( C_{11} \) is even integer valued, \( \hat{b} \) is also integer valued.

We now briefly discuss the target-space duality symmetries of the Kähler potential (5.23) for a \( \frac{SO(3,2)}{SO(3) \times SO(2)} \) coset. The associated modular group \( G_O \) is given by \( O(3, 2, Z) \).
Under $O(3,2,Z)$ the complex moduli $T,U$ and $B$ transform as in (6.22), (6.36), (6.39) and (6.46) with $C = B$. The associated Kähler potential (5.29) then transforms as in (5.24), (5.37), (5.41) and (5.47), where again $C = B$.

Finally, let us turn to the target-space duality symmetries of the Kähler potential (5.40) for an $SU(n,1)$ coset. It possesses an $SL(2,Z)_T$ symmetry as in (6.38) with the $T$ modulus and the complex Wilson lines $A_l$ transforming as

$$
T \rightarrow \frac{\alpha T - i\beta}{i\gamma T + \delta} \\
A_l \rightarrow \frac{A_l}{i\gamma T + \delta}
$$

(6.49)

and the Kähler potential transforming as in (6.40) and (6.41). Let us finally remark that the modular transformation rules (6.36), (6.39) and (6.49) agree with previous results [52, 25, 53] for untwisted matter fields.

7. Conclusion

In this paper we showed that the local structure of the untwisted moduli space of asymmetric $Z_N$ orbifolds is given by a product of $SU(n,m)\otimes SU(m)\otimes U(1)$ and $SO(r,p)\otimes SO(r)\otimes SO(p)$ cosets. We then specialised to the case of $(0,2)$ symmetric orbifold compactifications with continuous Wilson lines. For the case where the underlying 6-torus $T_6$ is given by a direct sum $T_4 \oplus T_2$ we showed that interestingly enough, when the twist on the internal torus lattice has eigenvalues $-1$, there are holomorphic terms in the associated Kähler potential describing the mixing of complex Wilson lines. These terms deserve further study since they were recently shown [15, 16] to be of the type which induce a mass term for Higgs particles of the order of the gravitino mass once supergravity is spontaneously broken. We proceeded to identify the associated target space duality symmetry groups and explicitly checked that they induce particular Kähler transformations of the Kähler potentials. In the case where the twist on the internal torus lattice has eigenvalues of $-1$, the associated $T$ and $U$ moduli were shown to mix under target space duality transformations due to the presence of the holomorphic mixing terms in the Kähler potential. In more general terms, the discussed orbifold examples clearly show that for (0,2) compactifications the moduli spaces of the moduli corresponding to the deformations of the Kähler class and of the complex structure respectively do not in general factorize like in the (2,2) compactifications, and that they get mixed by target space modular transformations.

Having thus checked that these target space symmetries are indeed symmetries of the 4D tree level low-energy Lagrangian, it would be very interesting to know how these
modular symmetries manifest themselves in the string loop threshold corrections \cite{19,20,21,26}. There, one expects to find that Wilson lines break some of the duality symmetries and it would be interesting to find out to what subgroups they are broken down. Also, when turning on continuous Wilson lines, one generically expects to find smaller gauge groups than the ones present in $(2,2)$ symmetric orbifold compactifications \cite{11}. Let us point out that it would be interesting to determine the generic gauge groups occurring at generic points in the moduli space of the $(0,2)$ models discussed in this paper. Work along these lines is in progress \cite{40}. Finally, it would also be of interest to extend the above investigations to the twisted sector. On the one hand, twisted moduli are important, because orbifolds can be smoothen out into Calabi-Yau manifolds by assigning non-zero vevs to twisted moduli. On the other hand, it has recently been pointed out \cite{14} that twisted sectors in asymmetric orbifolds may give rise to additional space-time supercharges.

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