Vacuum Structure and $\theta$ States of Adjoint QCD in Two Dimensions

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Abstract

We address the issue of topological angles in the context of two dimensional SU(N) Yang-Mills theory coupled to massive fermions in the adjoint representation. Classification of the resulting multiplicity of vacua is carried out in terms of asymptotic fundamental Wilson loops, or equivalently, charges at the boundary of the world. We explicitly demonstrate that the multiplicity of vacuum states is equal to $N$ for SU($N$) gauge group. Different worlds of the theory are classified by the integer number $k = 0, 1, \ldots, N - 1$ (superselection rules) which plays an analogous role to the $\theta$ parameter in QCD. We study the physical properties of these unconnected worlds as a function of $k$. We achieve this by using two completely independent approaches: First, we apply the well known machinery of the loop calculus in order to calculate the effective string tensions in the theory as function of $k$. The second way of doing the same physics is the standard particle/field theoretic calculation for the binding potential of a pair of infinitely massive fermions. We also calculate the vacuum energy as function of $k$.

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1 Introduction

The existence of topological parameters in certain quantum field theories, such as the $\theta$-angle of quantum chromodynamics (QCD)\cite{1} or Schwinger model (\cite{2}, \cite{3}) are interesting possibilities which can have a profound effect on the physical properties of the models. As is known, many problems such as the $U(1)$ problem, chiral symmetry breaking phenomenon, confinement and multiplicity of vacuum states are related to each other and to $\theta$ dependence. Therefore, in spite of the fact that we live in a certain vacuum state (superselection rule), the variation of physical values with $\theta$ is an extremely important characteristic of the theory. Unfortunately we do not know the nature of such a dependence in a physically interesting theory like four-dimensional QCD which appears too complicated to deal with. The $\theta$ dependence in Schwinger model can be found exactly, however this model is too trivial to share some important QCD-features.

An example of a theory which admits such angles and shares many features of four dimensional gauge theories is 1+1-dimensional QCD with adjoint matter as was pointed out many years ago by Witten\cite{4}. This model demonstrates a complex spectrum \cite{5}, a Hagedorn transition at finite temperature (\cite{3}, \cite{2}, \cite{7}) and is a testing ground for answering questions about the phenomena of screening and confinement \cite{3}, and many other questions, some of which were pointed out above. Studying these features in the context of a 1 + 1-dimensional field theory is desirable since a lack of transverse degrees of freedom in two dimensional space-time, and consequently dynamical gluons, makes explicit calculations possible.

In order to gain a full comprehension of these features it is important to understand the vacuum structure of the theory. In particular there are many outstanding issues pertaining to the link between the existence of topological parameters, multiple vacua, fermion condensates and instantons. For example if one considers the bosonized version of the theory there appear to be fermion condensates for any SU($N$) gauge group and such a condensation is also supported by very general quark-hadron duality arguments \cite{3}. However the instanton calculations only show the existence of condensates when $N = 2$ \cite{9}. The path to resolving this paradox was defined in \cite{10} where the quantum mechanics of the vacua in the finite volume limit for $N = 2, 3$ was investigated (see also \cite{11} for the finite volume calculation with $N = 2$). Here our goal is to proceed with this investigation of the properties of the multiple
vacua in this model for arbitrary $N$ via a different route with a continuum description from the outset. We will consider the limiting case of infinite fermion mass and use the known methods of Wilson loop calculations to gain insight into the interactions of a pair of adjoint particles as a function of topological parameters. In order to make definite contact with the particle interpretation of these loop calculations we also consider the theory through explicit calculation of the effective Hamiltonian in the limit of large fermion mass.

The standard method of classifying the multiplicity of vacua in a particular gauge theory with adjoint matter hinges on identifying the effective gauge group. Here, since gauge transformations operate by adjoint action on all fields, the true gauge group is the quotient of the gauge group and its center. This quotient is multiply connected. For simply connected semi-simple gauge group $G$ with center $Z$,

$$\pi_1(G/Z) = \pi_0(Z) = Z$$

This gives a classification of gauge fields which are constrained to be flat connections at infinity. In that case

$$\lim_{|x|\to\infty} A_\mu(x) = ig^\dagger(x)\nabla_\mu g(x)$$

Where $g(x)$ is a mapping of the circle at infinity to the gauge group $G/Z$. Since $G/Z$ is a symmetry of the Hamiltonian, we expect that all physical states carry a representation of $\pi_1(G/Z)$. In the case where the center of the group is Abelian all of its irreducible representations are one dimensional and further, when $Z \sim \mathbb{Z}_N$, as in the case of $\text{SU}(N)$, we are lead to a classification of all physical states in terms of a single integer parameter $k$. If $Z$ is a generator of $Z$ and $|\psi>$ is a physical state we have

$$Z|\psi> = e^{2\pi i k/N}|\psi>$$

An alternate method of classifying the vacua in such a model follows the example of the Schwinger model ([2], [3]) and involves coupling the gauge fields to external static charges which reside at the boundary of the world. This method was used by Witten [4] to identify $\theta$-vacua in two dimensional non-Abelian field theories. Here we will further develop these ideas and obtain quantitative results using Wilson loop calculations. First though, let us review this classification scheme.
As pointed out by Coleman in the case of the massive Schwinger model \cite{2}, one introduces the parameter $\theta$ as the strength of a fractional charge $e\theta/2\pi$ at the right boundary of the world and an opposite charge at the left boundary of the world. Hence we see that topological parameters here can be considered as a form of generalized boundary conditions for the theory. The most important aspect of this picture is interpreting these charges in terms of a Wilson loop enclosing the world which can be explicitly included in the action of the theory

\[
Z \rightarrow \int D\psi D\bar{\psi} D\mathcal{A} \exp \left( -\int d^2 x \mathcal{L} \right) \exp \left( \frac{i\theta}{2\pi} \oint_{C\rightarrow\infty} dx^\mu A_\mu(x) \right) = \int D\psi D\bar{\psi} D\mathcal{A} \exp \left( -\int d^2 x \mathcal{L}_\theta \right), \quad \mathcal{L}_\theta \equiv \mathcal{L} - \frac{i\theta \epsilon_{\mu\nu} F_{\mu\nu}}{4\pi} \tag{4}
\]

Similarly, in the non-Abelian case we consider static colour charges $T_R$ and $T_{R\bar{}}$ at either end of the world. Here the $T$’s are the generators of the colour group in the representation $R$ and its conjugate, respectively. Unlike the Abelian case though, we do not have any continuous parameter only the discrete choice of representation of the boundary charges. If $\text{tr}_R$ is the trace in the representation $R$ of the gauge group then our theory is modified \cite{4} to

\[
Z \rightarrow \int D\psi D\bar{\psi} D\mathcal{A} \exp \left( -\int d^2 x \mathcal{L} \right) \text{tr}_R \mathcal{P} \exp \left( i \oint_{C\rightarrow\infty} dx^\mu A_\mu(x) \right) \tag{5}
\]

The number of distinct choices of representation in which to take the trace of the boundary loop is directly related to the transformation properties of the loops under $Z$, the center of the gauge group. We note that unlike the Abelian case the formula (5) can not be written in terms of some local lagrangian $\mathcal{L}_\theta$ (4).

As we have seen, the effective gauge group is divided into $N$ different classes by modding out the center of SU(N) and so we expect that the only important feature of any external charges that we may add to the model are their transformation properties under the center. In fact we may think that all we need to do is choose representatives from each class and do our calculations with those. This is indeed the case if we take into account the stability of our choice of external charge. Since all charges are coupled to dynamical fermions there is always the possibility of the vacuum producing adjoint pairs to screen charges whenever this is energetically advantageous. Witten pointed out that the only stable vacua are those for which the boundary charges/loops are taken in one of the $N$ (antisymmetric) fundamental representations (including the trivial one) of SU(N), each of which transform differently under...
the center. An explicit example of unstable representations will arise when we consider the external charges in terms of Wilson loops at the boundary of the world.

Labeling the $N$ different stable vacuum states by the index $k$, we see the analysis in terms of Wilson loops is consistent with the previous homotopy arguments. While it is an open question if this agreement holds for an arbitrary gauge group, we will concentrate here on the case of adjoint QCD with a gauge group $SU(N)$ and center $Z \sim \mathbb{Z}_N$. **Our goal is to answer questions about the physics in each sector of the theory as a function of the discrete topological angle label $k \in \{0, 1, \ldots, N - 1\}$.**

We will now proceed via two routes, one group theoretical and the other particle/field theoretical. First we will introduce the machinery originally due to Rusakov [14] for the calculation of Wilson loop correlators in pure non-Abelian gauge theories. Using these techniques we will calculate the effects of fundamental boundary loops on a world that contains a single adjoint loop which corresponds to a pair of adjoint fermions in the particle picture of the problem. In terms of physical variables these calculations correspond to the evaluation of the string tension for the heavy meson constructed from a pair of adjoint quarks (adjoint internal loop) in $k$-th vacuum state (fundamental boundary loop).

The same system can be analyzed by the standard, pure Hamiltonian, approach. As such we will introduce the model of a single flavour of adjoint fermions in two dimensional QCD and discuss the introduction of the boundary loop in terms of a modified colour electric field due to charges at the edges of the world. Once the model is set we will look at its quantum behaviour via canonical quantization in the limit of infinite fermion mass. Here, as in the massive Schwinger model, the field theory problem is reduced to one dimensional quantum mechanics and we show that the number and qualities of bound and bleached states have a clear interpretation in terms of the previous Wilson loop calculation. Finally we present a discussion and conclusion with ideas for future directions with this model.

## 2 Wilson Correlators in 2D Gauge Theory

The fundamental gauge invariant object in the study of gauge theories is the Wilson loop operator which is defined in terms of a path ordered product of gauge field operators about
some closed path (loop) in space, $C$

$$W(C) = \text{tr}_R P \exp \left( \oint_C dx^\mu A_\mu(x) \right)$$  

(6)

Explicit calculation of the averages of such operators, which form a natural set of observables in two dimensional Yang-Mills theory, is an old topic which arose in the loop formulation of QCD by Makeenko and Migdal [12]. Using Schwinger-Dyson methods it is possible to derive correlators for arbitrary loop configurations on the plane [13] from a system of partial differential equations which relate the correlators of loops with $n$ self-intersections to those with fewer crossings. While in principle these equations are soluble in general, we are required to solve an entire system in order to find an expression for a single Wilson loop average. Another, group theoretic approach by Rusakov [14] developed later allows straightforward calculation of correlators for nested loops on the plane and is more useful for our applications.

Since we have invariance under area preserving diffeomorphisms in two dimensional Yang-Mills theory we expect that for a series of nested loops on the plane $C_1 \ldots C_n$ each taken in representation $R_1 \ldots R_n$, the weight of the entire configuration is a function only of the areas between successive loops $S_1 \ldots S_n$. While the explicit form of the Wilson correlator $< W(C_1 \ldots C_n) >$ is cumbersome to write in a transparent fashion, the general method of calculation follows easily from generalization of the basic examples.

In evaluating Wilson correlators there is a physical picture one can consider due to Minahan and Polychronakos [15] (see also [16]). Consider nested loops $\{C_k\}$ as rings around a cylinder, with periodic space components and time running along the axis of the cylinder, each associated with a character of the appropriate representation $\{\chi_U(R_k)\}$. Information is then carried through time -the area on the cylinder between successive loops $\{S_k\}$- via the propagator $K(R_i, R_j)$ which is the exponential of the quadratic Casimir of the representation under transport

$$K(R_i, R_j) = \delta_{R_i R_j} \exp \left( -g^2/2 \ C(R_i) S_i \right)$$

(7)

The interaction of two loops is due to the property of the group characters

$$\chi_U(R_i) \chi_U(R_j) = \sum_k \chi_U(R_k)$$

(8)

Where $R_i \otimes R_j = \sum_k R_k$ is the decomposition of the product representation into irreducible components. The Wilson correlator is then given as an integral of the sum over branches
of such products of representations with the relevant propagators where each term carries a factor for the dimension \( \{ \chi_{U^\dagger}(R_k)d(R_k) \} \) of each representation that makes it to the end of the cylinder.

For example, we would like to calculate the Wilson correlator of a single adjoint SU(N) loop which encloses an area \( S \) in Euclidean two dimensional space-time. In terms of the previous picture we have a loop on the cylinder an area \( S \) from the future end of the cylinder which is closed to a point. Hence

\[
< W(C) > = \int dU \chi_U(R_A) \exp \left( -g^2/2 \, C(R_A)S \right) \chi_{U^\dagger}(R_k)d(R_k) = d(R_A) \exp \left( -g^2/2 \, C(R_A)S \right) = (N^2 - 1) \, e^{-g^2NS}
\]  

where we have used the orthogonality condition

\[
\int dU \chi_U(R_i)\chi_{U^\dagger}(R_j) = \delta_{R_iR_j}
\]

Similarly we can write down the Wilson correlator for a pair of nested loops each in the \( N \)-dimensional representation \( (R_F) \) of SU(N). If the outer loop \( C_1 \) occupies a total area of \( S_1 + S_2 \) and the inner \( C_2 \) loop \( S_2 \) then

\[
< W(C_1, C_2) > = \int dU \chi_U(R_F) \exp \left( -g^2/2 \, C(R_F)S_1 \right) \chi_{U^\dagger}(R_F) \sum_k \chi_{U^\dagger}(R_k)d(R_k) \exp \left( -g^2/2 \, C(R_k)S_2 \right)
\]

Now we decompose the product of representations \( R_F \otimes R_F = R_1 \oplus R_2 \) and integrate over the group manifold

\[
< W(C_1, C_2) > = \exp \left( -g^2/2 \, C(R_F)S_1 \right) [d(R_1) \exp \left( -g^2/2 \, C(R_1)S_2 \right) + d(R_2) \exp \left( -g^2/2 \, C(R_2)S_2 \right)]
\]

\[
= \frac{N}{2} \, e^{-g^2\frac{N^2-1}{2N}(S_1+S_2)} \left[ (N+1) \, e^{-g^2\frac{(N-1)(N+3)}{2N}S_2} + (N-1) \, e^{-g^2\frac{(N+1)(N-3)}{2N}S_2} \right]
\]

As pointed out in the introduction, we would like to investigate the effect of external fundamental loops on the Wilson correlator of a single adjoint loop in the world. These fundamentals are antisymmetric combinations of single boxes in the Young Tableaux notation
each transforming distinctly under an element of the center of SU(N). In this notation the $k$-fundamental representation is denoted by a single column of $k$ boxes. In tensor notation, which we will also use, these representations are given as contractions of the N-dimensional completely anti-symmetric tensor with $k$ single index fundamental charges $\epsilon^{i_1 \cdots i_{N-k} m_1 \cdots m_k} u^{i_1} \cdots u^{i_k}$.

Clearly from this last definition $k$ defined modulo N only. Later we will discuss the quantum field theory situation with adjoint charges in the limit of infinite mass and make comparisons with the following calculation. Precisely we will use the previously introduced machinery to calculate the Wilson correlator of a $k$-fundamental loop $C_1$ of total area $S_1 + S_2$ within which lies an adjoint loop $C_2$ of area $S_2$. A straightforward generalization of the previous examples leads to the result

$$< W(C_1, C_2) >= \frac{N!}{(N-k)!k!} e^{-g^2(S_1+S_2)k(N-k)/(N+1)} \left[ 1 + \frac{(N-k-1)(N+1)}{k+1} e^{-g^2S_2(N-k)} + \frac{(N+1)(k-1)}{N-k+1} e^{-g^2S_2k} + \frac{kN(N+2)(N-k)}{(k+1)(N-k+1)} e^{-g^2S_2(N+1)} \right]$$

(14)

Where we will take the leading factor which is just the contribution of a k-fundamental loop of total area $S_1 + S_2$ to be normalization and ignore it in any further discussions.

The first interesting observation to be made about (14) is the effect of the addition of a boundary loop on the number of energetically distinct singlet configurations. Without an external loop ($k = 0$) the adjoint loop can form only a single stable configuration (10) while with $k \neq 0$ the system allows up to four distinct configurations. Also, for $k = N$ we expect the external loop to form a singlet itself and not contribute to the effective Wilson correlator for the adjoint loop. This expectation is met in (14) and we explicitly see that the physics of adjoint loops in this model depends on $k \mod N$ in analogy with the continuous $\theta$-angle which is periodic in $2\pi$.

Also of note is a symmetry under changing the representation of the external loop by $k \to N-k$. Mathematically this is the procedure of replacing the gauge fields along the path $C_1$ with their conjugates, which we denote $\bar{C}_1$, and should lead to changes in (14) for loops of arbitrary representation in the world. The situation here is special though since we are dealing with $C_2$ in an adjoint representation which is invariant under charge conjugation, $C_2 = \bar{C}_2$. This invariance is a manifestation of the general invariance of the adjoint representation under transformations in the center of the gauge group which leads to the phenomena of $\theta$-vacua. A consequence of this symmetry we see that the vacua corresponding to $k$ and $N-k$ are
degenerate in energy and hence there are only \((N+2)/2\) and \((N+1)/2\) distinct loop interaction configurations possible for \(N\) even and odd, respectively. However the total number of states is equal to \(N\) as was expected.

The usual method for systematically characterizing these configurations is through the definition of the string tension between the fermions. Disregarding the leading factor in each term identifies a different charge configuration where the prefactor denotes the degeneracy and the exponent, energy. For each configuration we define the string tension \(\sigma\) to be this energy divided by the area of the loop. For the case of \(k = 1\) we find string tensions of

\[
0, \quad g^2(N - 1), \quad g^2(N + 1)
\]

for gauge group SU(N). Later we will compare these string tension calculations with similar ones derived through a different analysis of the same physical problem, but first some comments on having a zero string tension. The vanishing of the string tension means that we have states which are color singlets formed from the two quarks with zero binding energy. Therefore, those quarks can travel in any direction and can be separated for a long time at any distance. At the same time this system is a color singlet state. Such a behavior corresponds to a quark-antiquark correlation at large distances and might be a demonstration of a long-distance order. Therefore, this effect presumably is related somehow to condensation of the fermion fields which will be discussed elsewhere. We would like to recall here that similar behavior was analyzed in a different model: QED2 with \(N\) flavours and nonzero \(\theta\) angle, [17]. Namely, it was found that for the special values \(\theta = \frac{\pi}{N} k, \quad k = 0, 1, \ldots, N\) the system possesses some “exotic” massless states which were called “screened quarks”. We note that those \(\theta = \frac{\pi}{N} k\) are very similar to our vacuum states where such “exotic” states are also exist. Moreover, those values are precisely vacuum states for which the model becomes \(P\) and \(T\) invariant— a property which holds in our model too as we will see in section 4.

As mentioned in the introduction, we only consider the external loops in the \(k\)-fundamental representations since all other representations are unstable due to pair production of dynamical adjoint charges. While this can be argued from basic principles [4], here we can demonstrate this phenomena explicitly in the formalism of loop averages. For simplicity we consider the gauge group to be SU(2) for a moment and label the external representations by their
spin $l/2$. In the same geometry as (14) we find that the correlator of an adjoint loop in this background is

$$< W(C_1, C_2) > = e^{-g^2 l(l+2) (S_1+S_2) + (l+3) e^{-g^2 (l+2) S_2} + (l-1) e^{g^2 l S_2}}$$

(16)

If we consider the prefactor to be the normalization due to a single loop of spin $l/2$ and area $S_1 + S_2$, then we find that for $l \geq 2$ the last term is divergent for large values of $S_2$. This configuration has a negative string tension and is associated with a large amount of energy that could be used to pair creation had this calculation involved dynamical fermions. As it stands, we see that in order to obtain meaningful results with loop calculations it is necessary to consider only stable configurations of the full model.

In passing it is interesting to note that these calculations lead to non-perturbative results for observables pure gluodynamics in the presence of topological angles. For example we consider the world encircled by a single Wilson loop in one of the $k$-fundamental representations. By (5) the expectation value of such a loop in nothing more than the partition function of the problem in the $k - th$ vacuum state and hence we can calculate the vacuum energy as a function of the external parameter $k$. Since our original (Euclidean) action is

$$\frac{1}{4g^2} \int F_{\mu\nu}(x) F^{\mu\nu}(x) d^2x,$$

differentiating the expectation value of the loop with respect to $1/g^2$ we find

$$-\frac{1}{4} \int d^2x \langle k | F^{a}_{\mu\nu}(x) F^{a\mu\nu}(x) | k \rangle = V g^4 \frac{k(N-k)(N+1)}{2N}$$

(17)

And the Euclidean vacuum energy $T_{00}$ in volume $V$ is simply

$$T_{00}/V = -g^2 \frac{k(N-k)(N+1)}{2N}$$

(18)

A few remarks are in order. First, the sign (18) corresponds to the positive vacuum energy for the original Minkowski theory for all $k$ except $k = 0$ and $k = N$ where the vacuum energy equals zero. We note also that our calculation of the vacuum energy corresponds to the nonperturbative part with subtracted perturbative pieces : $E_{\text{vac}} = E_{\text{total}}^{\text{vac}} - E_{\text{pert}}^{\text{vac}}$ (see for example [18]). Therefore, such a vacuum expectation value could have, in principle, any sign, positive or negative. For example, the positiveness of the operator (in Euclidean space) $F^{a}_{\mu\nu}(x) F^{a\mu\nu}(x)$ does not guarantee the positiveness of its vacuum expectation value. In particular, in 4D QCD one expects a negative vacuum energy while in the 2D $CP^{N-1}$ model for large $N$ the vacuum energy is positive [18]. In $QCD_2(N = \infty)$ with massless
fermions in the fundamental representation (t’Hoft model) the vacuum energy has negative
sign again [19] (all these results quoted for the Minkowski space). We also note that our
normalization corresponds to the case when vacuum energy is zero when no external Wilson
loop is inserted. Therefore, we essentially study an energy difference between different $|k\rangle$
vacuum states. Finally we point out the dependence on $N$ in (18) comes in the standard
combination $g^2N$ for large $N$ with an extra factor $\frac{k}{N}$. The appearance of $k$ in this way
is reminiscent of $\theta$ dependence in the large $N$ limit of gauge theories where we expect the
combination $\frac{\theta}{N}$ to occur [20].

3 2D adjoint QCD ($m \to \infty$)

Now we change our perspective on the problem at hand and consider the particle/field the-
etric model of adjoint fermions coupled to an SU(N) gauge theory. We will show that this
model, in the limit of infinite fermion mass, is nothing more than a different formulation
of the Wilson loop problem for the adjoint loop in a $k$-fundamental background which we
have been considering. In the past calculations have been based mainly on bosonization of
the problem and utilized the technology of light-cone coordinates [5], but here we will take
a much simpler approach. We will follow closely Coleman’s analysis of topological angles in
the massive Schwinger model [4] and consider the case of extremely massive fermions. As we
will see, in this limit the field theory problem is reduced to that of one dimensional quantum
mechanics. Using such a simplified version of $QCD_2$ is sufficient to carry out our analysis of
the vacuum structure and general properties of the spectrum, each as a function of vacuum
angle $k$.

We begin with the Lagrangian density of a single flavour of adjoint fermion minimally
coupled to a colour electric field in two dimensions

$$\mathcal{L} = \text{tr}\left[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD - m)\psi\right]$$  \hspace{1cm} (19)

Where $iD^\mu \psi = i\partial^\mu \psi - g[A^\mu, \psi]$ and the fields are expanded on a basis of hermitean matrices
$T^a$

$$\psi = T^a \psi^a , \quad F_{\mu\nu} = F_{\mu\nu}^a T^a$$  \hspace{1cm} (20)
which are normalized $trT^aT^b = \frac{1}{2}\delta^{ab}$. Here we use the $\gamma$ matrices

$$\gamma^0 = \sigma_2 \quad , \quad \gamma^1 = i\sigma_1 \quad , \quad \gamma^5 = \gamma^0\gamma^1$$

(21)

Hence in more explicit notation where $f^{abc}$ are the structure constants of the colour group

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \frac{1}{2}[\bar{\psi}^a(\gamma^0\gamma^\mu\partial_\mu - \gamma^0m)\psi^a] + \frac{g}{2}\bar{\psi}^a\gamma^0\gamma^\mu\psi^b f^{abc}A^c_\mu$$

(22)

From here we can read off the colour-electric current

$$J^a_\mu = \frac{1}{2}\bar{\psi}^b\gamma_\mu\psi^c f^{bac}$$

(23)

There are gauge degrees of freedom in this problem which we would like to eliminate and in order to do so we make the most convenient choice, the (axial, Coulomb) gauge $A_1 = 0$. Under this choice the field strength tensor becomes $F^a_{01} = -\partial_1 A^a_0 = E^a$ and if we now consider the equations of motion for the gauge field we find the two dimensional version of Gauss' Law

$$\partial_1 E^a = -gJ^a_{0T}$$

(24)

Where $J^a_{0T}$ represents the total colour charge distribution of the system. The general solution of this equation is

$$E^a(x) = -\frac{g}{2}\int dy \epsilon(x - y)J^a_{0T}(y)$$

(25)

As described in the introduction, the inclusion of loops of gauge fields in the action explicitly brings topological angles into the model and leads to a background colour-electric field. Here we will generate this background field as was done in [2] and [4] by introducing external charges at the edges ($x = \pm L$) of our world. This involves taking the total colour charge to be the usual charge associated with the fermions $J^a_0$ plus a boundary contribution

$$J^a_{\text{ext}}(x) = \delta(x + L)T^a_F + \delta(x - L)T^a_F$$

(26)

where the operators $T^a_F, T^a_F$ act on independent subspaces and can be physically regarded as being a static charge of representation $F(\bar{F})$ at $x = L(-L)$. We are now interested in the energy of such a configuration

$$\int E^a(x)E^a(x)\, dx = -\frac{g^2}{2}\int |y - z|J^a_{0T}(y)J^a_{0T}(z)\, dydz$$

(27)
In order to have a finite energy associated with a charge configuration it is necessary that we only deal with charge singlets. Here this is defined by the statement that the total charge operator $Q_T^a$ annihilates any physical states where

$$Q_T^a = \int J_0^a(x) \, dx = \int J_0^a(x) \, dx + T_F^a + T_{\bar{F}}^a$$  \hspace{1cm} (28)$$

With this restriction we obtain

$$\int E^a(x)E^a(x) \, dx = -\frac{g^2}{2} \int |y - z|J_0^a(y)J_0^a(z) \, dydz - g^2 \int dy \, yJ_0^a(y)(T_{\bar{F}}^a - T_F^a) \hspace{1cm} (29)$$

$$+ 2Lg^2C_2(F)$$

We note that the last term is simply the contribution that one would expect to the energy of a system consisting of two static, conjugate charges separated by a distance of $2L$. This is consistent with the exponent of the calculation for a single Wilson loop as we have seen, for example, in \([10]\). 

Calculating the conjugate momentum and adding the energy of the colour-electric field \([22]\) we arrive at the Hamiltonian formulation of the problem

$$H = \frac{1}{2} \int dx \, \psi^\dagger(x)(i\gamma_0\gamma_1\partial_1 + \gamma_0m)\psi^a(x) + \int E^a(x)E^a(x) \, dx$$ \hspace{1cm} (30)$$

$$= \frac{1}{2} \int dx \, \psi^\dagger(x)(i\gamma_0\gamma_1\partial_1 + \gamma_0m)\psi^a(x)$$ \hspace{1cm} (31)$$

$$- \frac{g^2}{2} \int |y - z|J_0^a(y)J_0^a(z) \, dydz - g^2 \int dy \, yJ_0^a(y)(T_{\bar{F}}^a - T_F^a)$$

where we have dropped an infinite constant.

### 4 Quantization

We will now proceed to determine the quantum behavior of (31) in the limit of large fermion mass. The first thing to note is the Hamiltonian we have constructed contains particle number changing terms which in general require a full relativistic field theory treatment. Considerable simplification can be realized though if we restrict ourselves to a two-particle subspace and consider number changing interactions as a perturbation. In fact the limit of large fermion mass will be seen to coincide with the weak coupling limit of the original theory and as demonstrated by the analogous calculation of Coleman for the massive Schwinger model \([4]\).
we can systematically find the effective Hamiltonian to zeroth order in \( \hbar \). In fact the only (minor) difference between our calculation and the previous results stems from the fact we use a Majoranna field to represent our adjoint fermions. We use the following mode decomposition of the field

\[
\psi^a(x) = \int \frac{dp}{\sqrt{2\pi}} [b^a(p) u(p) e^{-ip.x} + b^{a\dagger}(p) v(p) e^{ip.x}]
\]  

(32)

where equal time anti-commutators satisfy

\[
\{b^r(p), b^s(q)\} = \{b^{r\dagger}(p), b^{s\dagger}(q)\} = 0 , \quad \{b^r(p), b^{s\dagger}(q)\} = \delta^{rs}\delta(p-q)
\]  

(33)

and we define the spinors in our choice of \( \gamma \) matrices

\[
u = v^* = \frac{i}{\sqrt{2E(p)(E(p) + p)}} \begin{pmatrix} -im \\ E(p) + p \end{pmatrix}
\]  

(34)

where \( E(p) = \sqrt{p^2 + m^2} \).

The construction of states which are annihilated by the total charge operator (28) is straightforward and involves considering the possible contractions of indices in the product state of the fermion, external charge and adjoint external charge state vectors. To be explicit we have in the most general case

\[
(T^\mu)_{ab}(T^{\nu})_{cd}b^{\mu\dagger}(u)b^{\nu\dagger}(v)|0 > \otimes \epsilon^{i_1 \ldots i_N - k m_1 \ldots m_k} |m_1 > \ldots |m_k > \otimes \epsilon^{j_1 \ldots j_N - k n_1 \ldots n_k} |n_1 > \ldots |n_k >
\]  

(35)

where the first factor is the configuration of charges for the fermions with \( T^\mu \) a generator of the colour Lie algebra. The last two factors represent the external charges at the boundary of the world, the second for the the state with external charge \( k \) and the third for its adjoint. All indices run over the dimension of the defining representation of SU(N) and \( \epsilon^{i_1 \ldots i_N} \) is the completely anti-symmetric invariant tensor in \( N \) dimensions. One can easily convince oneself that there are only four independent ways to contract the indices \( \{a, b, c, d, i_1, \ldots, i_{N-k}, j_1, \ldots, j_{N-k}\} \) which are

\[
|1 > \sim b^{\mu\dagger}(u)b^{\nu\dagger}(v)|0 > \otimes \epsilon^{i_1 \ldots i_{N-k} - k m_1 \ldots m_k} |m_1 > \ldots |m_k > \otimes \epsilon^{j_1 \ldots j_{N-k} - k n_1 \ldots n_k} |n_1 > \ldots |n_k >
\]  

(36)

\[
|2 > \sim (T^\mu T^{\nu})_{i_1 j_1}b^{\mu\dagger}(u)b^{\nu\dagger}(v)|0 > \otimes \epsilon^{i_{1 + 2} \ldots i_{N-k} - k m_1 \ldots m_k} |m_1 > \ldots |m_k > \otimes \epsilon^{j_{1 + 2} \ldots j_{N-k} - k n_1 \ldots n_k} |n_1 > \ldots |n_k >
\]  

(37)
\[ |3 > \sim (T^\nu T^\mu)_{i_1 j_1} b^{\mu i}(u) b^{\nu i}(v) |0 > \]
\[ \otimes \varepsilon^{i_1 i_2 \ldots i_{N-k} m_1 \ldots m_k} |m_1 > \ldots |m_k > \otimes \varepsilon^{j_1 j_2 \ldots j_{N-k} n_1 \ldots n_k} |\bar{n}_1 > \ldots |\bar{n}_k > \]  

\[ |4 > \sim (T^\mu)_{i_1 j_1} (T^\nu)_{i_2 j_2} b^{\mu i}(u) b^{\nu i}(v) |0 > \]
\[ \otimes \varepsilon^{i_1 i_2 i_3 \ldots i_{N-k} m_1 \ldots m_k} |m_1 > \ldots |m_k > \otimes \varepsilon^{j_1 j_2 j_3 \ldots j_{N-k} n_1 \ldots n_k} |\bar{n}_1 > \ldots |\bar{n}_k > \]  

In general we are now in a position to calculate the spectrum of the two particle subspace for general \( N \) and \( k \) but for simplicity we will consider only the simplest case of boundary charge \( k = 1 \) where the four state vectors form an overdetermined system. For SU(2) only two of the states are independent and for SU(N), \( N \geq 3 \) there are only three possible singlet configurations. These claims are most clearly verified via the methods of Young tableaux. Essentially these states correspond to diagonal, anti-symmetric and symmetric charge combinations where in the case of SU(2) diagonal and symmetric combinations are identified. We denote the normalized state vectors

\[ |1 > = \frac{1}{\sqrt{2N(N^2 - 1)}} \delta^{\mu \nu} \delta^{mn} \]  
\[ |2 > = \frac{1}{\sqrt{2N(N^2 - 1)}} [T^\mu_F, T^\nu_F]_{mn} \]  
\[ |3 > = \sqrt{\frac{N}{(N^2 - 1)(N^2 - 4)}} \left( \{T^\mu_F, T^\nu_F\}_{mn} - \frac{1}{N} \delta^{\mu \nu} \delta^{mn} \right) \]

where \( m = m_1 \) and \( n = n_1 \) in our previous notation.

Considering only particle number conserving terms in a normal ordering of the Hamiltonian (31), under Wick’s theorem we find a number of different contributions to the effective theory in the two particle subspace. Considerable simplification is achieved though by keeping only terms up to \( O(\hbar^0) \) determined via the dimension counting arguments of [2]. Hence, in operator form, in center of mass coordinates with momentum \( p \) we have, acting on the colour indices, the effective Hamiltonian

\[ H = 4(p^2 + m^2)^{1/2} \delta^{\lambda \mu} \delta^{\rho \sigma} \delta^{rm} \delta^{sn} + 2g^2 |x| (T^a_A)_{\lambda \nu} (T^a_A)_{\mu \rho} \delta^{rm} \delta^{sn} \]
\[ + 2g^2 x(T^a_A)_{\lambda \mu} \delta^{\rho \sigma} [(T^a_F)_{rm} \delta^{sn} + (T^a_F)_{ns} \delta^{mr}] \]

In basis of states (40) we find the Hamiltonian (41) is a matrix on the colour charge subspace. Since we are only interested in the effect of the Hamiltonian up to \( O(\hbar^0) \), we treat
the operators in this matrix as c-numbers and diagonalize. The result is a set of independent one dimensional quantum mechanical problems given by the set of Hamiltonians

\[ H_1 = 2(p^2 + m^2)^{1/2}, \quad H_2 = 2(p^2 + m^2)^{1/2} + (N+1)g^2|x|, \quad H_3 = 2(p^2 + m^2)^{1/2} + (N-1)g^2|x| \quad (42) \]

For large fermion mass, the classical approximation to zeroth order in \( \hbar \) is both justified and calculable. In this limit the solutions of the relevant Schrödinger equations are given by Airy functions and from these solutions we can determine the meson spectrum in this model. Without direct calculation though we can make a few observations. First consider the Schrödinger equation \( H \psi = E \psi \) and define string tension here as the binding energy of the adjoint pair per unit separation

\[ \sigma_{QM} = \frac{E - 2m}{|x|} \quad (43) \]

In the limit of infinite fermion mass we neglect the momentum in (42) and we find string tensions \( 0, g^2(N-1) \) and \( g^2(N+1) \) for \( H_1, H_2 \) and \( H_3 \) respectively. Comparing these results with the string tension as defined in (15) for adjoint loops in pure Yang-Mills theory with \( k = 1 \) we find complete agreement. This establishes a connection between the group theoretic calculation of (14) in the loop picture of Makeenko and Migdal and the more standard approach of particle field theory with a topological term inserted. More importantly, we can make a one to one identification between four terms in the loop formula (14) and four independent contractions we discussed previously (36-39). In the specific case with \( k = 1 \) (12) one can understand a connection between towers of states which can be derived from Hamiltonians (12) and loop formula (14). From this connection it is easy to interpret the terms in (14) which have different string tensions: they simply correspond to the different basic states (40), each of which interact distinctly with the external (topological) sources.

Of note is the state which corresponds to the vanishing string tension which, in the quantum mechanical problem, is an unbound pair of fermions. This pair, through interaction with external charges, still forms a colour charge singlet but might be described as free quarks. We already discussed this state previously with remark that the analogous massless “exotic” states appear in \( N \) flavour Schwinger model for arbitrary mass with specific value of \( \theta = \pi k/N \). In fact these states were noted by Witten long ago [4] and arise for an arbitrary number of adjoint charges in the system. This is due to the nature of the index structure of
the adjoint representation which allows the background electric flux string generated by the boundary charges to be spliced with adjoint charges with the only energetic cost being the bare mass of the charge. If we consider a single flux string in the system \((k = 1)\) with three charges in the adjoint representation then the index structure corresponding to vanishing string tension is \(\bar{T}_j U^j_k V^k W^l W^m \) (Fig. 1). Hence we see that one line of flux enters the charge on the upper index and leaves on the lower so that the string configuration is unchanged. Additionally, as long as the charges do not collide we see that their position on the string is irrelevant and so there is no (long range) force acting between them and they are unbound. All of these observations are made for static charges though and one may expect that for finite fermion mass exchange interactions may modify this behavior. However, the \(N\) flavour Schwinger model shows that these states may persist even for finite mass.

5 Conclusions

By considering the boundary conditions of two dimensional QCD with adjoint fermions we have shown in the limit of large fermion mass that the multiplicity of vacua and their physical properties can be considered in two equivalent ways. The first involves considering the physics in terms of Wilson loops and using the well established methods of calculation to show the properties of the stable configurations, including periodicity in the representation of the external loops and a multiplication of distinct singlet configurations.

Subsequently we examined the same model in terms of a particle/field theoretic picture where the boundary conditions were specified by static colour charges. In the limit of large fermion mass we were able to consider the two particle subspace of the Hamiltonian acting on all possible singlet configurations of a pair of adjoint fermions and the boundary charges. For a particular choice of boundary charge we showed that the resulting string tensions between the fermions matched exactly with the calculations for adjoint Wilson loops.

From the field theory approach we were able to find exact solutions for the meson wave functions in terms of Airy functions. These results may be of use in the future for the calculation of the spectra of these mesons and fermion condensates in each of the different vacuum states. In addition to considering the problem of condensates, one may try to use our
results with heavy quark mass to generalize the string picture originally derived in [21] for pure gauge theory in 2D without dynamical degrees of freedom. Using heavy quarks we have essentially introduced physical degrees of freedom without noticeably changing the internal gluodynamics. Therefore, we expect that the introduction of the heavy quark might be the first step in the direction of the string description of dynamical degrees of freedom.

6 Acknowledgements

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References

[1] For a pedagogical discussion see S. Coleman, “The uses of Instantons” in Aspects of Symmetry, Cambridge University Press, 1985.

[2] S. Coleman, Ann. Phys. 101, 239, (1976)

[3] S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. 93, 267, (1975)

[4] E. Witten, Nuovo Cimento 51 A, 325, (1979)

[5] G. Bhanot, K. Demeterfi and I.R. Klebanov, Phys. Rev. D48, 4980, (1993), hep-th/9307111; D. Kutasov, Nucl. Phys. B144, 33, (1994), hep-th/9306013

[6] I.I. Kogan and A.R. Zhitnitsky, Nucl. Phys. B 465, 99, (1996), hep-ph/9509322

[7] G. Semenoff, O. Tirkkonen and K. Zarembo, hep-th/9605172

[8] D.J. Gross, A.V. Smilga, A. Matytsin and I.R. Klebanov, hep-th/9511104; E. Abdalla, R.Mohayaee and A. Zadra, hep-th/9604063

[9] A.V. Smilga, Phys. Rev. D49, 6836, (1994), hep-th/9402060

[10] F. Lenz, M. Shifman and M. Thies, Phys. Rev. D51, 7060, (1995), hep-th/9412113

[11] S. Pinsky, D. Robertson and K. Harada, hep-th/9512135

[12] Yu. Makeenko and A.A. Migdal, Nucl. Phys. B 188, 269, (1981)

[13] V. Kazakov and I. Kostov, Nucl. Phys. B 176, 199, (1980); V. Kazakov, Nucl. Phys. B 179, 283, (1981)

[14] B. Rusakov, Mod. Phys. Lett. A5, 9, 693, (1990)

[15] J.A. Minahan and A.P. Polychronakos, Phys. Lett. B312, 155, (1993), hep-th/9303153

[16] D. J. Gross and A. Matytsin, Nucl. Phys., B437, 541, (1995), hep-th/9410054

[17] L.V. Belvedere, J.A. Swieca, K.D. Rothe and B. Schroer Nucl. Phys., B153, 112, (1979)

[18] V. Novikov, M. Shifman, A. Vainshtein, V. Zakharov, Phys. Rep. 116, 103, (1984)
[19] A.R. Zhitnitsky, Phys. Lett. B165, 405, (1985); Sov. J. Nucl. Phys. 44, 139, (1986)

[20] E. Witten, Ann. Phys. 128, 363, (1980)

[21] D. J. Gross and W. Taylor, Nucl. Phys. B 400, 181, (1993); Nucl. Phys. B 403, 395, (1993).
Figure Caption

Figure 1 Adjoint particles interacting with a single \((k = 1)\) string of electric flux
