A superspace formulation of
Abelian antisymmetric tensor gauge theory

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Abstract

We apply a superspace formulation to the four-dimensional gauge theory of a massless Abelian antisymmetric tensor field of rank 2. The theory is formulated in a six-dimensional superspace using rank-2 tensor, vector and scalar superfields and their associated supersources. It is shown that BRS transformation rules of fields are realized as Euler-Lagrange equations without assuming the so-called horizontality condition and that a generating functional $\bar{W}$ constructed in the superspace reduces to that for the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field. The WT identity for this theory is derived by making use of the superspace formulation and is expressed in a neat and compact form $\partial \bar{W} / \partial \theta = 0$.

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I. INTRODUCTION

Gauge theories of Abelian rank-2 antisymmetric tensor fields have become of interest for various reasons. Kalb and Ramond first realized that Abelian rank-2 antisymmetric tensor fields could interact with classical strings [1]. This interaction has been applied to the Lorentz-covariant description of vortex motion in an irrotational, incompressible fluid [2], and to the dual formulation of the Abelian Higgs model [3]. Abelian rank-2 antisymmetric tensor fields are also involved in supergravity multiplets [4] and in excited states of quantized (super)strings [5]. They are crucial for superstring theories to realize anomaly-cancellation mechanism and to estimate dualities in extended objects. In addition, it has been shown that an Abelian rank-2 antisymmetric tensor field generates effective mass for an Abelian vector gauge field through a topological coupling between these two fields [6]. A geometric aspect of Abelian rank-2 antisymmetric tensor fields has been discussed in a U(1) gauge theory in loop space [7].

Covariant quantization of an Abelian rank-2 antisymmetric tensor field was first attempted by Townsend [8] and has been studied by many authors [9,10] in systematic manners based on the Becchi-Rouet-Stora (BRS) formalism. It was found in the covariant quantization that a naive gauge-fixing term containing the antisymmetric tensor field is itself invariant under a secondary gauge transformation and that commuting ghost fields are required for complete gauge fixing.

In the present paper, we consider a superspace formulation of the four-dimensional gauge theory of a massless Abelian rank-2 antisymmetric tensor field. Until recently, the BRS formalism for Abelian rank-2 antisymmetric tensor fields has been discussed by several authors using superfields on a six-dimensional superspace [10]. However, in these superspace formulations, BRS transformation rules of fields are put in by hand and superfields are not free but are constrained to satisfy the so-called horizontality condition. As a consequence, covariance of the superfields is spoiled under superspace rotations that mix spacetime and anticommuting coordinates.
To avoid these limitations, we apply the superspace formulation proposed by Joglekar [11] to the gauge theory of Abelian rank-2 antisymmetric tensor field. This formulation also uses a six-dimensional superspace but has remarkable features: (i) Unlike earlier superspace formulations [10,12], superfields are not a priori restricted by the horizontality conditions and superspace rotations can be carried out; BRS transformation rules are realized as Euler-Lagrange equations. (ii) The theory in superspace is constructed by taking into account generalized gauge invariance and the Lagrangian density (without gauge-fixing and source terms) is a scalar under OSp(3,1|2) transformations in the superspace. (iii) The whole action including all source terms is accommodated in a single action written in terms of superfields. For details see Ref. [13]. After the application, we show that the generating functional in the superspace formulation contains in itself all the necessary information of the generating functional for the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field. We further show that BRS transformation rules can be obtained without assuming the horizontality condition and that the WT identity for the theory can be expressed in a neat, compact, and mathematically convenient form $\partial \bar{W} / \partial \theta = 0$. This will lead to a simplified treatment of the renormalization problem in the gauge theory of Abelian rank-2 antisymmetric tensor field.

The present paper is organized as follows. We briefly review the gauge theory of Abelian rank-2 antisymmetric tensor field in Sec.IIA and provide for the superspace formulation in Sec.IIB. In Sec.III, we apply the superspace formulation proposed in Ref. [11] to the gauge theory of Abelian rank-2 antisymmetric tensor field and construct a generating functional from superfields. We show that this generating functional reduces to one considered in the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field. In Sec.IV, we see that BRS transformation rules are realized as Euler-Lagrange equations. We also discuss a relation between the BRS transformation and a six-dimensional gauge transformation. In Sec.V, we derive a simple form of the WT identity by making use of the superfield formulation. Section VI is devoted to a summary and discussion.
II. PRELIMINARY

A. Gauge theory of Abelian rank-2 antisymmetric tensor field

In this section, we briefly review the gauge theory of Abelian rank-2 antisymmetric tensor field. Let $B_{\mu\nu}(x)$ be an Abelian antisymmetric tensor field on four-dimensional Minkowski space $M^4$ with a space-time coordinate system $(x^\mu)$ ($\mu = 0, 1, 2, 3$). We consider the Abelian gauge theory defined by the action

$$S_0 = \frac{1}{12} \int d^4 x F_{\mu\nu\rho} F^{\mu\nu\rho}, \quad (2.1)$$

where $d^4 x \equiv dx^0 dx^1 dx^2 dx^3$ and $F_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. This action is invariant under the gauge transformation $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ with a vector gauge parameter $\Lambda_\mu(x)$.

To covariantly quantize $B_{\mu\nu}$ using the BRS formalism [9,10], it is necessary to introduce the following ghost and auxiliary fields: anticommuting vector fields $\rho_\mu(x)$ and $\tilde{\rho}_\mu(x)$, a commuting vector field $\beta_\mu(x)$, anticommuting scalar fields $\chi(x)$ and $\tilde{\chi}(x)$, and commuting scalar fields $\sigma(x)$, $\varphi(x)$ and $\tilde{\sigma}(x)$. The BRS transformation $\delta$ is defined for $B_{\mu\nu}$ by replacing $\Lambda_\mu$ in the gauge transformation by the ghost field $\rho_\mu$, and for other fields it is defined so as to satisfy the nilpotency condition $\delta^2 = 0$:

$$\delta B_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$$

$$\delta \rho_\mu = -i \partial_\mu \sigma, \quad \delta \sigma = 0,$$

$$\delta \tilde{\rho}_\mu = i \beta_\mu, \quad \delta \beta_\mu = 0,$$

$$\delta \tilde{\sigma} = \tilde{\chi}, \quad \delta \tilde{\chi} = 0,$$

$$\delta \varphi = \chi, \quad \delta \chi = 0. \quad (2.2)$$

Covariant quantization of $B_{\mu\nu}$ can be performed with the action

$$S = S_0 + S_1 + S_2, \quad (2.3)$$

with the gauge-fixing terms
\[ S_1 = -i \int d^4 x \delta [\tilde{\rho}_\nu (\partial_\mu B^{\mu\nu} + k_1 \beta^\nu)] , \quad (2.4) \]
\[ S_2 = -i \int d^4 x \delta [\tilde{\sigma} \partial_\mu \rho^\mu + \varphi (\partial_\mu \tilde{\rho}^\mu - k_2 \tilde{\chi})] , \quad (2.5) \]

where \( k_1 \) and \( k_2 \) are gauge parameters. Owing to the nilpotency property of \( \delta \), these gauge-fixing terms are invariant under the BRS transformation. The first term \( S_1 \) breaks the gauge invariance of \( S_0 \) explicitly. The second term \( S_2 \) is necessary to break the invariance of \( S_1 \) under the secondary gauge transformation \( \delta \rho_\mu = \partial_\mu \varepsilon, \delta \tilde{\rho}_\mu = \partial_\mu \bar{\varepsilon} \). The gauge-fixing procedure for quantization of \( B_{\mu\nu} \) is complete with \( S_1 + S_2 \). Carrying out the BRS transformation in Eqs. (2.4) and (2.5), we obtain

\[ S_1 + S_2 = \int d^4 x [ -i \partial_\mu \tilde{\rho}_\nu (\partial^\rho \rho^\nu - \partial^\nu \rho^\rho) + \partial_\mu \tilde{\sigma} \partial^\rho \sigma + \beta_\nu (\partial_\mu B^{\mu\nu} + k_1 \beta^\nu - \partial^\nu \varphi) - i \tilde{\chi} \partial_\mu \rho^\mu - i \chi (\partial_\mu \tilde{\rho}^\mu - k_2 \tilde{\chi})] . \quad (2.6) \]

The action \( S \) describes a massless system. We can read from \( S \) how many physical degrees of freedom \( B_{\mu\nu} \) has: The total degrees of freedom of the commuting fields \( B_{\mu\nu}, \beta_\mu, \sigma, \varphi \) and \( \tilde{\sigma} \) are naively 13, but some of them are not independent because of the four constraints \( \partial_\mu B^{\mu\nu} + k_1 \beta^\nu - \partial^\nu \varphi = 0 \) derived from \( S \). Their genuine degrees of freedom are thus 9. The total degrees of freedom of the anticommuting fields \( \rho_\mu, \tilde{\rho}_\mu, \chi \) and \( \tilde{\chi} \) are naively 10, but some of them are also not independent because of the two constraints \( \partial_\mu \rho^\mu = 0 \) and \( \partial_\mu \tilde{\rho}^\mu - k_2 \tilde{\chi} = 0 \) derived from \( S \). Their genuine degrees of freedom are thus 8. Subtracting the genuine degrees of freedom of the anticommuting fields from those of the commuting fields, we conclude that \( B_{\mu\nu} \) has one physical degree of freedom, describing a spin less particle.

**B. Superspace and superfields**

We shall work in a superspace of six dimensions. The superspace used in this paper possesses a local coordinate system \( (\tilde{x}^i) \equiv (x^\mu, \lambda, \theta) \) \((i = 0, 1, 2, 3, 4, 5)\) with anticommuting real coordinates \( x^4 \equiv \lambda \) and \( x^5 \equiv \theta \), and will be denoted by \( M^{4/2} \). We introduce to \( M^{4/2} \) a metric tensor \( g_{ij} \) whose non-vanishing components are
\[ g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{45} = g_{54} = 1. \] (2.7)

The set of linear homogeneous transformations that leave \( g_{ij} \bar{x}^i \bar{x}^j \) invariant forms the pseudo-orthosymplectic supergroup \( \text{OSp}(3,1|2) \). This is nothing but a supersymmetric generalization of the Lorentz group.

Let \( \bar{X}(\bar{x}) = \bar{X}(x, \lambda, \theta) \) be an arbitrary superfield on \( M^{4/2} \). Since \( \lambda \) and \( \theta \) are nilpotent, \( \bar{X} \) can be expanded as

\[ \bar{X}(\bar{x}) = X(x) + \lambda X_\lambda(x) + \theta X_\theta(x) + \lambda \theta X_{\lambda\theta}(x), \] (2.8)

where \( X, X_\lambda, X_\theta \) and \( X_{\lambda\theta} \) are component fields on \( M^4 \). In terms of \( \bar{X}, X, X_\lambda, X_\theta \) and \( X_{\lambda\theta} \), Eq. (2.8) is written as

\[ \bar{X}(\bar{x}) = X(x) + \lambda \bar{X}_\lambda(\bar{x}) + \theta \bar{X}_\theta(\bar{x}) - \lambda \theta X_{\lambda\theta}(x). \] (2.9)

Equation (2.10) can be regarded as a constraint in the five fields \( \bar{X}, X, \bar{X}_\lambda, \bar{X}_\theta \) and \( X_{\lambda\theta} \); we now choose \( \mathcal{X} \equiv (\bar{X}, \bar{X}_\lambda, \bar{X}_\theta, X_{\lambda\theta}) \) as a set of independent fields.

Let us define

\[ \bar{f} \equiv \int d^4x f(\mathcal{X}(\bar{x}), \partial_\mu \mathcal{X}(\bar{x})) \] (2.11)

from a polynomial \( f \) in \( \mathcal{X} \) and \( \partial_\mu \mathcal{X} \). For an arbitrary function \( \mathcal{F} \) of \( \bar{f} \), we can readily show that

\[ \int \prod_{\bar{x}} d\bar{X}(\bar{x}) d\bar{X}_\lambda(\bar{x}) d\bar{X}_\theta(\bar{x}) dX_{\lambda\theta}(x) \mathcal{F} = \int \prod_{x} dX(x) dX_\lambda(x) dX_\theta(x) dX_{\lambda\theta}(x) \mathcal{F}. \] (2.12)

\[ ^1 \text{In this paper we shall attach "overbar" to all the superfields on } M^{4/2}. \]
Equation (2.12) holds whether $\bar{X}$ is a commuting superfield or an anticommuting superfield. It should be noted that the integration in Eq. (2.11) and the multiplication in Eq. (2.12) are carried out over $M^4$, not over $M^4/2$. Consequently, Eq. (2.12) is still a function of $\lambda$ and $\theta$. If $f$ does not depend on $X_{\lambda\theta}$ and $\partial_{\mu}X_{\lambda\theta}$, we can formally divide the both sides of Eq. (2.12) by $\int \prod_x dX_{\lambda\theta}(x)$, arriving at

$$\int \prod_x d\bar{X}(\bar{x}) d\bar{X}_\lambda(\bar{x}) d\bar{X}_\rho(\bar{x}) F = \int \prod_x dX(x) dX_\lambda(x) dX_\theta(x) F. \quad (2.13)$$

Using this formula twice, we have

$$\frac{\partial}{\partial\theta} \int \prod_x d\bar{X}(\bar{x}) d\bar{X}_\lambda(\bar{x}) d\bar{X}_\rho(\bar{x}) F = \int \prod_x d\bar{X}(\bar{x}) d\bar{X}_\lambda(\bar{x}) d\bar{X}_\rho(\bar{x}) \frac{\partial F}{\partial\theta}. \quad (2.14)$$

This is an important formula used in the superspace formulation of gauge theories.

**III. ACTION AND GENERATING FUNCTIONAL IN SUPERSPACE**

In this section, we apply the superfield formulation proposed by Joglekar [11] to the gauge theory of Abelian rank-2 antisymmetric tensor field. To this end, we now generalize the antisymmetric tensor field $B_{\mu\nu}(x)$ to a superfield $\bar{B}_{ij}(\bar{x})$ on $M^4/2$ satisfying the antisymmetric property in superspace, $\bar{B}_{ij} = (-1)^{|i||j|} \bar{B}_{ji}$, and the commuting property $\bar{x}^k \bar{B}_{ij} = (-1)^{|i|(|j|+|i|)} B_{ij} \bar{x}^k$. Here $|i|$ is a function of $i$, defined as $|i| = 0$ for $i = 0, 1, 2, 3$, and $|i| = 1$ for $i = 4, 5$. The superfield $\bar{B}_{ij}$ is assumed to transform as a rank-2 covariant tensor under coordinate transformations characterized by OSp$(3,1|2)$. The field strength of $\bar{B}_{ij}$ is defined by

$$\bar{F}_{ijk} \equiv \partial_i \bar{B}_{jk} + (-1)^{|i|(|j|+|k|)} \partial_j \bar{B}_{ki} + (-1)^{|k|(|i|+|j|)} \partial_k \bar{B}_{ij}, \quad (3.1)$$

which is invariant under the generalized gauge transformation

$$\delta \bar{B}_{ij} = \partial_i \bar{\Lambda}_j - (-1)^{|i||j|} \partial_j \bar{\Lambda}_i \quad (3.2)$$
with a vector gauge parameter \( \bar{\Lambda}_i(x) \) satisfying the commuting property \( \bar{x}^k \bar{\Lambda}_i = (-1)^{|i|+1} \bar{\Lambda}_i \bar{x}^k \). We consider the following generalization of the action (2.1):

\[
\bar{S}_0 = -\frac{1}{12} \int d^4x F^{ijk}(\bar{x}) F_{kji}(\bar{x}).
\]  

Note that the integration above is carried out over \( M^4 \). Obviously, \( \bar{S}_0 \) is invariant under the gauge transformation (3.2).

In addition to \( \bar{B}^{ij} \), we introduce a vector superfield \( \bar{\zeta}_i(x) \) satisfying the anticommuting property \( \bar{x}^k \bar{\zeta}_i = (-1)^{|i|+1} \bar{\zeta}_i \bar{x}^k \) and a commuting scalar superfield \( \bar{\eta}(x) \). Furthermore, we introduce supersources (source superfields) \( \bar{K}^{ij}(x), \bar{t}^i(x) \) and \( \bar{u}(x) \) which are associated with the superfields \( \bar{B}^{ij}, \bar{\zeta}_i \) and \( \bar{\eta} \), respectively. It is assumed that the (inner) products \( \bar{K}^{ij} \bar{B}_{ji}, \bar{\zeta}_i \bar{t}^i \) and \( \bar{\eta} \bar{u} \) are anticommuting scalars under OSp(3,1|2) transformations; that is, \( \bar{K}^{ij} \) is an anticommuting tensor supersource, \( \bar{t}^i \) a commuting vector supersource, and \( \bar{u} \) an anticommuting scalar supersource.

The superspace formulation of the gauge theory of Abelian rank-2 antisymmetric tensor field is begun with the action

\[
\bar{S} = \bar{S}_0 + \bar{S}_{GS}
\]

with the gauge-fixing and source terms

\[
\bar{S}_{GS} = \int d^4x \frac{\partial}{\partial \theta} \left[ \bar{K}^{ij}(x) \bar{B}_{ji}(x) \\
+ \bar{\zeta}_i(x) \left\{ \delta^i_{\alpha} \partial_{\mu} \bar{B}^{\mu\alpha}(x) + k_1 \delta^i_{\nu} \bar{\zeta}_\nu,\theta(x) + \bar{t}^i(x) \right\} \\
+ \bar{\eta}(x) \left\{ \partial_{\mu} \bar{\zeta}^{\mu}(x) + k_2 \bar{\zeta}_4,\theta(x) + \bar{u}(x) \right\} \right],
\]

where \( \alpha = 0, 1, 2, 3, 4 \) and \( k_1 \) and \( k_2 \) are gauge parameters.

Let us collectively denote the superfields by \( \bar{\Phi}(x) \) and the supersources by \( \bar{\Sigma}(x) \): \( \bar{\Phi} = (\bar{B}^{ij}, \bar{\zeta}_i, \bar{\eta}) \) and \( \bar{\Sigma} = (\bar{K}^{ij}, \bar{t}^i, \bar{u}) \). In accordance with the discussion in Sec.2.2, we treat \( \bar{\Phi}(x) \), \( \bar{\Phi}_{\lambda}(x) \) and \( \bar{\Phi}_{\vartheta}(x) \) as a set of independent fields, and \( \bar{\Sigma}(x) \) and \( \bar{\Sigma}_{\lambda}(x) \) as a set of independent sources. (The field \( \Phi_{\lambda\vartheta}(x) \) and the sources \( \bar{\Sigma}_{\lambda}(x) \) and \( \Sigma_{\lambda\vartheta}(x) \) do not occur at this stage.) Now, defining the path-integral measure
\[ \{ \mathcal{D}\Phi \} \equiv \mathcal{D}\Phi \mathcal{D}\Phi, \lambda \mathcal{D}\Phi, \theta , \]  
(3.6) 

with 
\[ \mathcal{D}\Phi \equiv \prod_x d\Phi(\bar{x}) , \]  
\[ \mathcal{D}\Phi, \lambda \equiv \prod_x d\Phi, \lambda (\bar{x}) , \quad \mathcal{D}\Phi, \theta \equiv \prod_x d\Phi, \theta (\bar{x}) , \]  
(3.7) 

we consider the generating functional 
\[ \bar{W}[\bar{\Sigma}, \bar{\Sigma}, \theta; \lambda, \theta] = \int \{ \mathcal{D}\bar{B}_{ij} \} \{ \mathcal{D}\bar{\zeta}_i \} \{ \mathcal{D}\bar{\eta} \} \exp(i\bar{S}) . \]  
(3.8) 

Since the integrations in Eqs. (3.3) and (3.5) and the multiplications in Eq. (3.7) are carried out over \( M^4 \), the generating functional \( \bar{W} \) should be understood to be a function of \( \theta \) and \( \lambda \) as well as a functional of \( \Sigma \) and \( \Sigma, \theta \).

The integrations over \( \bar{B}_{ij, \lambda} \) and \( \bar{B}_{ij, \theta} \) in Eq. (3.8) lead to a form of \( \bar{W} \) that is proportional to \( \prod_{i,x} \delta(\bar{K}^{4i}(\bar{x})) \). Then, carrying out the integrations over \( \bar{B}_{4i}, \bar{\zeta}_5, \bar{\zeta}_{i, \lambda}, \bar{\zeta}_{5, \theta}, \) and \( \bar{\eta}_{\lambda} \), we finally arrive at
\[ \bar{W} = N \prod_{i,x} \delta(\bar{K}^{4i}(\bar{x})) \delta(\bar{K}^{4i, \theta}(\bar{x})) \prod_x \delta(\bar{\ell}_4(\bar{x})) \delta(\bar{\ell}_{4, \theta}(\bar{x})) \]  
\[ \times \int \mathcal{D}\bar{\mathcal{M}} \exp(i\bar{S}') , \]  
(3.9) 

where \( N \) is a constant,
\[ \mathcal{D}\bar{\mathcal{M}} \equiv \mathcal{D}\bar{B}_{\mu\nu} \mathcal{D}\bar{B}_{\mu5} \mathcal{D}\bar{B}_{55} \mathcal{D}\bar{\zeta}_4 \mathcal{D}\bar{\zeta}_{4, \theta} \mathcal{D}\bar{\eta}_{\lambda} \mathcal{D}\bar{\eta}_{\lambda} , \]  
(3.10) 

and
\[ \bar{S}' = \bar{S}'_0 + \bar{S}'_{1,2} + \bar{S}'_\Sigma , \]  
(3.11) 

with 
\[ \bar{S}'_0 = \frac{1}{12} \int d^4x \mathcal{F}_{\mu\nu\rho} \mathcal{F}^{\mu\nu\rho} , \]  
(3.12) 
\[ \bar{S}'_{1,2} = \int d^4x \left[ -\partial_\mu \bar{\zeta}_\nu (\partial^\mu \bar{B}_5^\nu - \partial^\nu \bar{B}_5^\mu) - \frac{1}{2} \partial_\mu \bar{\zeta}_4 \partial^\mu \bar{B}_{55} \right] \]
We also consider the generating functional up to the gauge-fixing terms from the superspace formulation characterized by the action

\[ S_\Sigma = \int d^4x \left[ -\bar{K}^{\mu\nu}(\partial_\mu \bar{B}_{\nu5} - \partial_\nu \bar{B}_{\mu5}) - \bar{K}^{55} \partial_\mu \bar{B}_{55} - \bar{K}^{\mu\nu,\theta} \bar{B}_{\mu\nu} - 2\bar{K}^{\mu5,\theta} \bar{B}_{\mu5} + \bar{K}^{55,\theta} \bar{B}_{55} + \bar{\bar{u}} \bar{\bar{\bar{u}}} \right]. \tag{3.13} \]

We also consider the generating functional \( W \) defined by

\[ W[\bar{K}^{\mu\nu}, \bar{K}^{\mu5}, \bar{K}^{\mu\nu,\theta}, \bar{K}^{55,\theta}, \bar{\bar{u}}, \bar{\bar{\bar{u}}}, \bar{\bar{\bar{t}}}, \bar{\bar{t}} \mu, \bar{\bar{t}} 5, \bar{\bar{t}} \mu, \bar{\bar{t}} 5, \bar{\bar{u}}, \bar{\bar{\bar{u}}}; \lambda, \theta] = \int \mathcal{D}\bar{M} \exp(i\bar{S}'), \tag{3.15} \]

with which \( \bar{W} \) is written as

\[ \bar{W} = N \prod_{i,x} \delta(\bar{K}^{4i}(\bar{x})) \delta(\bar{K}^{4i,\theta}(\bar{x})) \prod_{x} \delta(\bar{t}_4(\bar{x})) \delta(\bar{t}_4,\theta(\bar{x})) W. \tag{3.16} \]

Integrating Eq. (3.13) over \( \bar{K}^{4i}, \bar{K}^{4i,\theta}, \bar{t}_4 \) and \( \bar{t}_4,\theta \), we have

\[ W = \int \mathcal{D}\bar{K}^{4i} \mathcal{D}\bar{K}^{4i,\theta} \mathcal{D}\bar{t}_4 \mathcal{D}\bar{t}_4,\theta \bar{W}. \tag{3.17} \]

With the identification

\[
\begin{align*}
\bar{B}_{\mu\nu} &= B_{\mu\nu}, \\
\bar{B}_{\mu5} &= i\rho_\mu, \\
\bar{B}_{55} &= -2i\sigma, \\
\bar{\bar{c}}_\mu &= \bar{\rho}_\mu, \\
\bar{\bar{c}}_{\mu,\theta} &= \bar{\beta}_\mu, \\
\bar{\bar{c}}_4 &= -i\bar{\sigma}, \\
\bar{\bar{c}}_{4,\theta} &= -i\bar{\chi}, \\
\bar{\eta} &= \varphi, \\
\bar{\bar{\eta}}_{\theta} &= -i\bar{\chi},
\end{align*}
\tag{3.18}
\]

at \( \lambda = \theta = 0 \), Eqs. (3.12) and (3.13) agree with Eqs. (2.1) and (2.6), respectively. Hence, the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field is correctly reproduced up to the gauge-fixing terms from the superspace formulation characterized by the action.
In addition, the source term $S_\Sigma$ and the generating functional $W$ have the same forms as those in the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field. The generating functional $\bar{W}$ has a neat form, while $W$ is directly related to the ordinary gauge theory, although $W$ is still a function of $\lambda$ and $\theta$. These functional are related to each other by Eqs. (3.16) and (3.17).

We now mention a difference between earlier superspace formulations [10] and our superspace formulation: in the earlier formulations, all the components of $\bar{B}_{ij}$ except $\bar{B}_{\mu \nu}$ are identified with the ghost fields $\rho_\mu$, $\bar{\rho}_\mu$, $\sigma$, $\varphi$ and $\bar{\sigma}$. On the other hand, in our formulation, only two components $\bar{B}_{\mu 5}$ and $\bar{B}_{55}$ are identified with the ghost fields $\rho_\mu$ and $\sigma$, while the other components except $\bar{B}_{\mu \nu}$ are treated as auxiliary fields. Instead, $\bar{\zeta}_\mu$, $\bar{\zeta}_4$ and $\bar{\eta}$ are identified with ghost fields $\bar{\rho}_\mu$, $\bar{\sigma}$ and $\varphi$, respectively. As will be seen in the next section, the treatment of $\bar{B}_{ij}$ in our formulation makes possible to determine BRS transformation rules without assuming the so-called horizontality condition.

IV. BRS TRANSFORMATION AND SIX-DIMENSIONAL GAUGE TRANSFORMATION

We shall see in this section that some of the BRS transformation rules can be realized as the Euler-Lagrange equations and that a six-dimensional gauge transformation is related to the BRS transformation. Taking into account Eq. (3.18), we define the BRS transformation rules of the superfields so that they can reduce to Eq. (2.2):

$$
\delta \bar{B}_{\mu \nu} = -i \partial_\mu \bar{B}_{\nu 5} + i \partial_\nu \bar{B}_{\mu 5} \nonumber
$$

$$
\delta \bar{B}_{\mu 5} = \frac{i}{2} \partial_\mu \bar{B}_{55} , \quad \delta \bar{B}_{55} = 0 , \quad (4.1)
$$

and

$$
\delta \bar{\zeta}_\mu = i \bar{\zeta}_\mu, \quad \delta \bar{\zeta}_\mu = 0 , \nonumber
$$

$$
\delta \bar{\zeta}_4 = i \bar{\zeta}_4, \quad \delta \bar{\zeta}_4 = 0 , \nonumber
$$

$$
\delta \bar{\eta} = i \bar{\eta}, \quad \delta \bar{\eta} = 0 . \quad (4.2)
$$
The transformation rules (4.2) indicate that the BRS transformation \( \delta \) may be represented as the derivative with respect to \( \theta \). If the equations \( \partial_\mu \bar{B}_{\nu 5} - \partial_\nu \bar{B}_{\mu 5} = -\bar{B}_{\mu \nu , \theta} , \partial_\mu \bar{B}_{55} = 2\bar{B}_{\mu 5 , \theta} , \) and \( \bar{B}_{55 , \theta} = 0 \) are satisfied, the BRS transformation defined by Eqs. (4.1) and (4.2) is represented as

\[
\delta = i \frac{\partial}{\partial \theta} . \tag{4.3}
\]

Remarkably, these desirable equations are derived from the action (3.4) as the Euler-Lagrange equations for \( \bar{B}_{\mu \nu , \lambda} , \bar{B}_{\mu 4 , \lambda} \) and \( \bar{B}_{44 , \lambda} \):

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{\mu \nu , \lambda}} = -\frac{1}{2} (\bar{B}_{\mu \nu , \theta} + \partial_\mu \bar{B}_{\nu 5} - \partial_\nu \bar{B}_{\mu 5} ) = 0 , \tag{4.4}
\]

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{\mu 4 , \lambda}} = -2\bar{B}_{\mu 5 , \theta} + \partial_\mu \bar{B}_{55} = 0 , \tag{4.5}
\]

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{44 , \lambda}} = \frac{3}{2} \bar{B}_{55 , \theta} = 0 . \tag{4.6}
\]

Note here that the superfields are functions of \( x^\mu , \lambda \) and \( \theta \), while \( \bar{S} \) is a function of \( \lambda \) and \( \theta \). It should be emphasized that with Eq. (4.3), the BRS transformation rules (4.1) can be obtained as the Euler-Lagrange equations. This situation is quite different from that in the earlier superspace formulations, in which the BRS transformation rules are determined by putting in the horizontality condition by hand.

We now extend the BRS transformation to the superfields \( \bar{B}_{\mu 4} , \bar{B}_{44} \) and \( \bar{B}_{45} \), utilizing Eq. (4.3) and the Euler-Lagrange equations

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{\mu 5 , \lambda}} = \bar{B}_{\mu 4 , \theta} - \partial_\mu \bar{B}_{45} + \bar{B}_{\mu 5 , \lambda} = 0 , \tag{4.7}
\]

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{55 , \lambda}} = \frac{1}{2} \bar{B}_{44 , \theta} + \bar{B}_{45 , \lambda} = 0 , \tag{4.8}
\]

\[
\frac{\partial \bar{S}}{\partial \bar{B}_{45 , \lambda}} = -2\bar{B}_{45 , \theta} - \bar{B}_{55 , \lambda} = 0 . \tag{4.9}
\]

Since \( \bar{S}_{SG} \) in Eq. (3.4) does not contain the superfields \( \bar{B}_{ij , \lambda} \), Eqs (4.4)-(4.9) can collectively be written as

\[
\frac{\partial \bar{S}_{0}}{\partial \bar{B}_{ij , \lambda}} = 0 . \tag{4.10}
\]
We can understand Eq. (4.10) as the BRS transformation rule of $\bar{B}_{ij}$.

Now, choosing the gauge parameter $\bar{\Lambda}_i$ in Eq. (3.2) to be a particular form

$$\bar{\Lambda}_i(\bar{x}) = \bar{B}_{i5}(\bar{x})\Lambda,$$

we define the six-dimensional gauge transformation

$$\hat{\delta}\bar{B}_{ij} = (\partial_i \bar{B}_{j5} - (-1)^{|i||j|}\partial_j \bar{B}_{i5})\Lambda,$$

where $\Lambda$ is an anticommuting infinitesimal constant. Using Eqs. (4.4)-(4.9), we readily show that

$$\hat{\delta}\bar{B}_{ij} = \Lambda \bar{B}_{ij,\theta} = \Lambda \frac{\partial \bar{B}_{ij}}{\partial \theta}.$$  (4.13)

Differentiations of Eq. (4.13) with respect to $\lambda$ and $\theta$ lead to

$$\hat{\delta}\bar{B}_{ij,\lambda} = \Lambda \frac{\partial \bar{B}_{ij,\lambda}}{\partial \theta},$$

$$\hat{\delta}\bar{B}_{ij,\theta} = \Lambda \frac{\partial \bar{B}_{ij,\theta}}{\partial \theta}. $$  (4.15)

As seen from Eqs. (4.3) and (4.13)-(4.15), the six-dimensional gauge transformation $\hat{\delta}$ for $\bar{B}_{ij}$, $\bar{B}_{ij,\lambda}$ and $\bar{B}_{ij,\theta}$ is nothing other than the BRS transformation for them with the parameter $-i\Lambda$. Note that Eq.(4.15) vanishes because of $\partial^2/\partial\theta^2 = 0$ or the nilpotency property of the BRS transformation.

V. WT IDENTITY

In this section we derive the WT identity for the gauge theory of Abelian rank-2 antisymmetric tensor field by making use of the superspace formulation discussed above. Since $\tilde{S}_0$ is a functional of the superfields $\bar{B}_{ij}$, $\bar{B}_{ij,\lambda}$ and $\bar{B}_{ij,\theta}$, the use of Eqs.(4.13)-(4.15) gives

$$\hat{\delta}\tilde{S}_0 = \Lambda \frac{\partial \tilde{S}_0}{\partial \theta}.$$  (5.1)

Then, noting the invariance of $\tilde{S}_0$ under the gauge transformation (4.12), we have
\[ \frac{\partial \tilde{S}_0}{\partial \theta} = 0, \]  
(5.2)

which shows the BRS invariance of \( \tilde{S}_0 \). Differentiating Eq. (3.8) with respect to \( \theta \), we obtain

\[ \frac{\partial \bar{W}}{\partial \theta} = \int \{ \mathcal{D} \bar{B}_{ij} \} \{ \mathcal{D} \tilde{c}_i \} \{ \mathcal{D} \bar{\eta} \} \frac{\partial}{\partial \theta} \exp(i\tilde{S}) \]
\[ = \int \{ \mathcal{D} \bar{B}_{ij} \} \{ \mathcal{D} \tilde{c}_i \} \{ \mathcal{D} \bar{\eta} \} i \frac{\partial \tilde{S}_0}{\partial \theta} \exp(i\tilde{S}), \]
(5.3)

where the formula (2.14) has been applied. From Eq. (5.2), it follows that

\[ \frac{\partial \bar{W}}{\partial \theta} = 0. \]  
(5.4)

Instead of using the six-dimensional gauge invariance of \( \tilde{S}_0 \), we can show Eq. (5.4) by directly calculating the last term of Eq. (5.3). In this alternative method, field theoretical analogies of the formula \((x - y)\delta(x - y) = 0\) are repeatedly used.

Differentiation of Eq. (3.17) with respect to \( \theta \) is simply written as

\[ \frac{\partial W}{\partial \theta} = \int \mathcal{D} \bar{K}^4i \mathcal{D} \bar{K}^4i,\theta \mathcal{D} \bar{t}^4 \mathcal{D} \bar{t}^4,\theta \frac{\partial \bar{W}}{\partial \theta} \]  
(5.5)

by taking into account \( \partial \bar{K}^4i,\theta / \partial \theta = \partial \bar{t}^4i,\theta / \partial \theta = 0 \), and \( \int d\bar{K}^4i,\theta d\bar{K}^4i,\theta f_1(\bar{K}^4i,\theta) = \int d\bar{t}^4i,\theta d\bar{t}^4i,\theta f_2(\bar{t}^4i,\theta) = 0 \) satisfied for arbitrary functions \( f_1 \) and \( f_2 \). As a result, Eq. (5.4) gives

\[ \frac{\partial W}{\partial \theta} = 0. \]  
(5.6)

On the other hand, calculating \( \partial W / \partial \theta \) from the expression (3.13), we obtain

\[ \frac{\partial W}{\partial \theta} = \int d^4x \left[ \frac{1}{2} \bar{K}^{\mu\nu},\theta \frac{\partial W}{\partial K^{\mu\nu}} + \bar{K}^{\mu 5},\theta \frac{\partial W}{\partial K^{\mu 5}} \right. \]
\[ + \bar{t}^{\mu},\theta \frac{\partial W}{\partial t^{\mu}} + \bar{u},\theta \frac{\partial W}{\partial \bar{u}} \right] \]
\[ = -i \int d^4x \int \mathcal{D} \bar{M} \left[ \bar{K}^{\mu\nu},\theta (\partial_{\mu} \bar{B}_{\nu 5} - \partial_{\nu} \bar{B}_{\mu 5}) \right. \]
\[ + \bar{K}^{\mu 5},\theta \partial_{\mu} \bar{B}_{55} - \bar{t}^{\mu},\theta \bar{c}_{\mu,\theta} + \bar{t}_{5,\theta} \bar{c}_{4,\theta} + \bar{u},\theta \bar{\eta},\theta \right] \]
\[ \times \exp(i\tilde{S}'). \]  
(5.7)
We now introduce sources $J^{\mu\nu}$, $J^{\mu}$, $j^{\mu}$, $j$ and $\tilde{j}$ on $M^4$ that are defined, at $\lambda = \theta = 0$, by $J^{\mu\nu} = -i\bar{K}^{\mu\nu,\theta}$, $J^{\mu} = 2i\bar{K}\nu^{5,\theta}$, $j^{\mu} = \bar{t}^{\mu,\theta}$, $j = \bar{t}_{5,\theta}$ and $\tilde{j} = i\bar{u},\theta$. With these sources and Eq. (3.18), it follows from Eqs. (5.6) and (5.7) that

$$\int d^4x \int \mathcal{D}M \left[J^{\mu\nu}(\partial_{\mu}\rho_{\nu} - \partial_{\nu}\rho_{\mu}) + J^{\mu}\partial_{\mu}\sigma + j^{\mu}\beta_{\mu} + j\bar{\chi} + \tilde{j}\chi\right] = 0,$$

(5.8)

where

$$\mathcal{D}M \equiv \mathcal{D}B_{\mu\nu}\mathcal{D}\rho_{\mu}\mathcal{D}\sigma\mathcal{D}\bar{\rho}_{\mu}\mathcal{D}\sigma\mathcal{D}\beta_{\mu}\mathcal{D}\bar{\chi}\mathcal{D}\varphi\mathcal{D}\chi.$$

(5.9)

This is nothing other than the WT identity in the ordinary gauge theory of Abelian rank-2 antisymmetric tensor field. Hence, Eq. (5.4) is understood as the WT identity.

Differentiating Eq. (3.16) with respect to $\theta$, we have

$$\frac{\partial W}{\partial \theta} = N \prod_{i,x} \delta(\bar{K}^{4i}(\bar{x}))\delta(\bar{K}^{4i,\theta}(\bar{x})) \prod_{x} \delta(\bar{t}_{4}(\bar{x}))\delta(\bar{t}_{4,\theta}(\bar{x})) \frac{\partial W}{\partial \theta},$$

(5.10)

where $\partial\{\delta(\bar{K}^{4i})\delta(\bar{K}^{4i,\theta})\}/\partial \theta = \delta'(\bar{K}^{4i})\bar{K}^{4i,\theta}\delta(\bar{K}^{4i,\theta}) = 0$ and $\partial\{\delta(\bar{t}_{4})\delta(\bar{t}_{4,\theta})\}/\partial \theta = \delta'(\bar{t}_{4})\bar{t}_{4,\theta}\delta(\bar{t}_{4,\theta}) = 0$ have been used. From Eqs. (5.5) and (5.10), we see that Eq. (5.4) is equivalent to Eq. (5.6). Therefore Eq. (5.4) is considered the WT identity written in terms of the superspace formulation.

VI. SUMMARY AND DISCUSSION

In this paper, we have found that the superspace formulation proposed by Joglekar works well not only in the Yang-Mills theory but also in the gauge theory of Abelian rank-2 antisymmetric tensor field. As we have seen, all information on the quantization of an Abelian rank-2 antisymmetric tensor field is contained in the simple generating functional $W$ in Eq. (3.8), with which the WT identity is expressed in a compact and elegant form

$$\frac{\partial W}{\partial \theta} = 0.$$

(6.1)
In the superspace formulation of Yang-Mills theory, the WT identities are also cast in the same form as Eq. (6.1) [14]. It was shown in Ref. [15] that this simple form can directly be derived from partial OSp(3,1|2) invariance of the generating functional $\overline{W}$ defined in the superspace formulation of Yang-Mills theory. Furthermore, several subjects concerning the renormalization of Yang-Mills theory [16] and generalizations of the BRS transformation [17] have been studied in the context of the superspace formulation of Yang-Mills theory. Based on the superspace formulation considered in the present paper, we will be able to extend the discussions in Ref. [15-17] to the gauge theory of Abelian rank-2 antisymmetric tensor field.

The superspace formulation can readily be applied to the gauge theories of Abelian antisymmetric tensor fields of higher rank. It is also possible to generalize the superspace formulation in the present paper to several of the gauge theories containing both the Yang-Mills and rank-2 antisymmetric tensor fields. One of such theories is the theory involving the Chapline-Manton coupling [18] that is defined by the action

$$ S_{\text{CM}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right], \quad (6.2) $$

with $H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} + k\omega_{\mu\nu\rho}$. Here $F_{\mu\nu}^a$ is the field strength of Yang-Mills fields $A_\mu^a$, $k$ a constant with dimensions of length, and $\omega_{\mu\nu\rho}$ the Chern-Simons three-form consisting of $A_\mu^a$. Another theory that one thinks is a massive gauge theory of non-Abelian rank-2 antisymmetric tensor field [19] defined by the action

$$ S_{\text{NA}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{12} \hat{H}_{\mu\nu\rho}^a \hat{H}^{\mu\nu\rho a} - \frac{1}{4} m^2 \hat{B}_{\mu\nu}^a \hat{B}^{\mu\nu a} \right], \quad (6.3) $$

with $\hat{B}_{\mu\nu}^a \equiv B_{\mu\nu}^a - m^{-1} (D_\mu \phi_\nu^a - D_\nu \phi_\mu^a)$ and $\hat{H}_{\mu\nu\rho}^a \equiv D_\mu \hat{B}_{\nu\rho}^a + D_\nu \hat{B}_{\rho\mu}^a + D_\rho \hat{B}_{\mu\nu}^a$. Here $B_{\mu\nu}^a$ is a non-Abelian antisymmetric tensor field, $\phi_\mu^a$ a non-Abelian vector field, $m$ a constant with dimensions of mass, and $D_\mu$ denotes the covariant derivative defined from $A_\mu^a$. In addition to the theories defined by the Lagrangians (6.2) and (6.3), there are gauge theories with topological terms consisting of the Yang-Mills and antisymmetric tensor fields [20,21]. To apply the superspace formulation proposed by Joglekar to those theories, it is necessary to consider how topological terms are defined in the superspace.

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