We show that the cylindrical symmetry of the eigenvectors of the photon position operator with commuting components, \( \hat{x} \), reflects the \( E(2) \) symmetry of the photon little group. The eigenvectors of \( \hat{x} \) form a basis of localized states that have definite angular momentum, \( \hat{J}_z \), parallel to their common axis of symmetry. This basis is well suited to the description of "twisted light" that has been the subject of many recent experiments and calculations. Rotation of the axis of symmetry of this basis results in the observed Berry phase displacement. We prove that \( \{ \hat{x}_1, \hat{x}_2, \hat{x}_3 \} \) is a realization of the two dimensional Euclidean \( e(2) \) algebra that effects genuine infinitesimal displacements in configuration space.

### I. INTRODUCTION

Position operators are key but controversial objects in the study of particle localization in field theory. Newton and Wigner (NW) found position operators with commuting components and spherically symmetric eigenvectors for massive particles and for massless particles with spin \( 0 \) and \( \frac{1}{2} \) [3], but their construction failed for photons. Pryce derived a photon position operator, \( \hat{x}_P \), consistent with the NW axioms but its components do not commute so it does not have localized eigenvectors [2]. A photon position operator with commuting components, \( \hat{x} \), does exist [3], but its eigenvectors are cylindrically symmetric and it does not transform like a vector under rotations [4].

We show here that the components of \( \hat{x} \) perpendicular to the axis of symmetry of its eigenvectors together with the rotation operator parallel to this axis are a realization of the two dimensional Euclidean little group \( E(2) \) [2]. The operator \( \hat{x} \) does not transform like a vector under rotations and boosts because an additional term is required to rotate the axis of symmetry of its eigenvectors. This additional term describes the Berry phase [6] displacement of photon position.

The form of the position eigenvectors used here is flexible enough to accomodate Newton-Wigner and covariant normalization. For photons the Heisenberg picture position eigenvectors are

\[
\epsilon^\mu_{\sigma\chi}(k) = k^\sigma e^{i(k \cdot x - kct - \sigma \chi)} \frac{e^\mu_{\sigma\chi} + i \sigma \epsilon^\mu_{\sigma\chi}}{\sqrt{2}}
\]

in momentum space spherical polar coordinates where \( \sigma \) is helicity, \( \chi \) is the Euler rotation angle about \( k \), and \( x \) is displacement from the origin. Each choice of \( \alpha \) corresponds to a choice of normalisation for the position eigenvectors. In the NW case for which \( \alpha = 1/2 \) these eigenvectors are not covariant and they are nonlocal in configuration space due to the factor \( k^{1/2} \). If \( \alpha = 0 \), \( \epsilon^\mu_{\sigma\chi}(k) \) is a four-vector and transformation to configuration space using the Lorentz invariant measure \( d^3k/k \) is a four-vector proportional to the electromagnetic four-potential. The inverse Fourier transform obtained using the trivial measure is the time derivative of the vector potential, proportional to the electric field describing this instantaneously localized position eigenvector. For the definite helicity transverse modes \( \sigma = \pm \), \( B = -i \sigma E \) so the Riemann-Silberstein vector \( E + i \sigma B = 2 \hat{E} \) is again an electric field. Details of the normalization of these position eigenvectors are discussed in [2] and [3], but these details do not affect the expressions derived in this paper.

The basis of eigenvectors of \( \hat{x} \) is ideally suited to the description of optical beams with definite angular momentum (AM) in a fixed direction. Since orbital AM results in a helical wave front, these beams are referred to as "twisted light" [7]. It has been observed that total angular and linear momentum can be transferred from a photon to a particle trapped in a twisted light beam [7] [8]. Focussing of a beam leads to localization on the axis of symmetry [8] so a basis of localized states is well suited to the theoretical description of focussing. Berry’s topological phase has been observed using light beams and optical fibers [10] [12] as a sideways shift of the beam centroid. Twisted light beams are currently very topical as they are of interest as candidates for manipulation of particles, imaging and optical communications based on violation of local realism [14].

The plan of the paper is as follows: In Section II the Poincaré and position operators and their commutation relations are discussed and the AM and boost operators are separated into intrinsic and extrinsic parts by writing them in terms of \( \hat{x} \). In Section III the Wigner little group algebra is briefly summarized and then extended to include \( \hat{x} \) and we prove that the transverse components of \( \hat{x} \) together with rotation about its axis of cylindrical symmetry are a realization of the photon little group. In
Section IV experimental and theoretical work on optical beams is discussed and in Section V we conclude.

II. POINCARE AND POSITION OPERATORS

In this Section, after a brief review of the Poincaré operators, we introduce \( \mathbf{x} \) and its associated Berry phase and then write the AM and boost operators in terms of position operators.

The Poincaré group describes the fundamental kinematic symmetry of a relativistic particle \([15]\). The generators of translations in space and time, rotations and boosts are the momentum, Hamiltonian, AM and Lorentz boost operators, \( \mathbf{P} \), \( \hat{H} \), \( \mathbf{J} \) and \( \mathbf{K} \) respectively. These Poincaré operators satisfy the commutation relations \( \big[ \hat{J}_i, \hat{J}_j \big] = i\hbar \epsilon_{ijk} \hat{J}_k \), \( \big[ \hat{J}_i, \hat{R}_j \big] = i\hbar \epsilon_{ijk} \hat{R}_k \), \( \big[ \hat{R}_i, \hat{R}_j \big] = -i\hbar \epsilon_{ijk} \hat{R}_k \), \( \big[ \hat{J}_i, \hat{P}_j \big] = i\hbar \epsilon_{ijk} \hat{P}_k \), \( \big[ \hat{R}_i, \hat{P}_j \big] = i\hbar \delta_{ij} \hat{H} \), \( \big[ \hat{R}_i, \hat{H} \big] = -i\hbar \hat{P}_i \), \( \big[ \hat{J}_i, \hat{H} \big] = \hat{P}_i \), \( \big[ \hat{R}_i, \hat{H} \big] = 0 \) for \( i = 1, 2, 3 \) \([3]\). This algebra will next be extended to include position operators.

In \( \mathbf{k} \)-space \( \mathbf{P} = \hbar \mathbf{k} \) and the photon position operator with commuting components, \( \mathbf{x} \), is related to the spinless nonrelativistic momentum space position operator \( i\hbar \delta \) by \( \mathbf{x} = k^\alpha \delta i\hbar \delta k^\alpha \) where \([4]\)

\[
\hat{D} = \exp \left( -i\hat{\sigma} \chi \right) \exp \left( -i\hat{S}_3 \phi \right) \exp \left( -i\hat{S}_2 \theta \right). \tag{2}
\]

Here \( \partial_k \) is the \( \mathbf{k} \)-space gradient, \( \hat{S}_i \) are the Cartesian components of the spin operator \( \hat{S} \), \( \hat{\sigma} = \hat{e}_k \cdot \hat{S} \) is the helicity operator, \( \theta \) and \( \phi \) are the \( \mathbf{k} \)-space spherical polar angles, \( \chi (\theta, \phi) \) is the Euler angle and the \( \mathbf{k} \)-space spherical polar unit vectors are \( \hat{e}_\theta \), \( \hat{e}_\phi \) and \( \hat{e}_k \) as sketched in Fig. 1. The definite helicity transverse unit vectors, equal to \( \hat{D} (\mathbf{e}_1 + i\mathbf{e}_2) \), are

\[
\mathbf{e}_\sigma^{(\chi)} = \frac{1}{\sqrt{2}} (\mathbf{e}_\theta + i\mathbf{e}_\phi) e^{-i\sigma \chi}. \tag{3}
\]

The position operator with commuting components is \([4]\)

\[
\hat{x} = i\partial_k - \frac{i}{k^2} \hat{k} \cdot \hat{S} - \hat{\sigma} a (\theta, \phi) \tag{4}
\]

where \( k = |\mathbf{k}| \), \( \alpha = \frac{\pi}{2} \) for the NW basis and

\[
a = \frac{\cos \theta}{k \sin \theta} \mathbf{e}_\phi + \partial_k \chi. \tag{5}
\]

Inspection of Fig. 1 shows that rotation about \( \mathbf{k} \) does not change \( \theta \) or \( \phi \). The Euler angle \( \chi (\theta, \phi) \) is defined as a general rotation about \( \mathbf{k} \). Any possible transverse basis is the set of eigenvectors of \([4]\) for some \( \chi (\theta, \phi) \). Since experiments are often performed on optical beams with definite angular momentum, the case \( \chi = -m\phi \) for which the position eigenvectors have intrinsic AM \( h m \sigma \) in some arbitrary but fixed direction is of special interest. For this choice of \( \chi [5] \) becomes

\[
a^{(m)} = \frac{\cos \theta - m}{k \sin \theta} \mathbf{e}_\phi \tag{6}
\]

It is known that

\[
\hat{\sigma} a_i^{(m)} = i\mathbf{e}_\sigma \cdot \partial_k \mathbf{e}_\sigma \tag{7}
\]

is a Berry connection with curvature \( \partial_k \times \hat{\sigma} a^{(m)} = -\hat{\sigma} e_{\mathbf{k}} / k^2 \) \([4, 10, 17]\). For parallel transport generated by the rotation \( d\xi \), we have \( d\mathbf{k} = d\xi \times \mathbf{k} \), and hence

\[
\left( a^{(m)} \times k \right) \cdot d\xi = -a^{(m)} \cdot d\mathbf{k}, \tag{8}
\]

so the Berry phase shift is \( \sigma \Omega \) where the \( m \)-independent solid angle subtended by a loop of the photon’s path is \([4, 11]\).

\[
\Omega = -\oint a^{(m)} \cdot d\mathbf{k} = 2\pi (1 - \cos \theta). \tag{9}
\]

The position operator with commuting components, \( \mathbf{x} \), will be emphasized here but the properties of the Pryce operator which does not have commuting components will also be discussed because it is this operator that is commonly used. The Pryce operator is

\[
\hat{x}_p = i\partial_k - \frac{i}{2} \frac{k}{k} + \frac{1}{k} \mathbf{k} \times \hat{S}, \tag{10}
\]

where, from \([4]\),

\[
\hat{x} = \hat{x}_p - \hat{\sigma} \mathbf{a}. \tag{11}
\]

Since \( [\hat{x}_p, \mathbf{x}_p] = i\mathbf{e}_k k / k^3 \), which can be written as \( \mathbf{x}_p \times \mathbf{x}_p = -i\mathbf{e}_k k^3 / k^3 \), it is straightforward to verify that the position operator \([4]\) does indeed have commuting components: \( \mathbf{x} \times \mathbf{x} = \mathbf{x}_p \times \mathbf{x}_p - \hat{\sigma} (i\hbar \partial_k \times \mathbf{x}_p) = 0 \).

In addition to having commuting components, the position operator \([4]\) commutes with the helicity operator and satisfies the usual momentum-position commutation relations. Using \( \mathbf{P} = \hbar \mathbf{k} \) and \( \hat{H} = \hbar c \mathbf{k} \), we write

\[
[\hat{x}_i, \mathbf{x}_j] = 0, \quad [\hat{x}_i, \mathbf{k}] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{\sigma}] = 0. \tag{12}
\]

In the Heisenberg picture where \( d\hat{\mathbf{O}} / dt = [\hat{\mathbf{O}}, \hat{\mathbf{H}}] / \hbar \), the momentum space photon velocity operator is thus

\[
\hat{\mathbf{v}} = \hbar \mathbf{x}_p. \tag{13}
\]

The Foldy representation \([18]\) of the Poincaré AM and boost operators is \( \mathbf{J} = i\hbar \partial_k \times \mathbf{k} + \hat{S} \) and \( \mathbf{K} = \frac{1}{2} \hbar (k \partial_k + \hat{\sigma} \mathbf{e}_k) + \hbar \mathbf{e}_k \times \hat{S} \). In terms of the Pryce position operator \( \mathbf{J} = \hbar \mathbf{x}_p \times \mathbf{k} + \hat{\sigma} \mathbf{e}_k \), \( \mathbf{K} = \frac{1}{2} \hbar (k \mathbf{x}_p + \hat{\mathbf{x}} k) \). In
terms of $\mathbf{x}$ whose eigenvectors, $\mathbf{3}$, are localized these operators are partitioned into intrinsic and extrinsic parts $\mathbf{4}$. For the momentum operator

$$\mathbf{J} = \hbar \mathbf{x} \times k + \mathbf{J}^{(0,a)}$$

$$\mathbf{J}^{(0,a)} = \hat{a} \mathbf{h} (a \times k + e_k)$$

where $\mathbf{J}^{(0,a)}$ and $\hbar \mathbf{x} \times k$ are its intrinsic and extrinsic parts. The superscript $(0,a)$ refers to the position eigenvector at the origin for a particular choice of $a$. The boost operator is

$$\mathbf{K} = \frac{\hbar}{2} (k \mathbf{x} + \mathbf{x} k) + \mathbf{K}^{(0,a)}$$

$$\mathbf{K}^{(0,a)} = \hat{a} \hbar k a.$$ Poincaré transformations are generated by the unitary operator $\mathbf{2}$

$$\hat{U}(\xi, \beta, x, t) = \exp \left[ \frac{i}{\hbar} \left( \mathbf{J} \cdot \xi - \mathbf{K} \cdot \beta \right) + i \left( \hat{H} t - \hat{P} \cdot \mathbf{x} \right) / \hbar \right]$$

in which $\xi$ is the rotation angle, $\beta = v/c$, $x$ a spatial displacement and $t$ is time. The infinitesimal change in an operator $\hat{O}$ due to the unitary transformation $\hat{U}^\dagger \hat{O} \hat{U}$ for $\hat{U} (\mathbf{d}x, \mathbf{d}y, \mathbf{d}z, \mathbf{d}t)$ is then

$$\mathbf{d} \hat{O} = \frac{-i}{\hbar} \mathbf{d} \mathbf{x} \cdot \left[ \mathbf{J} \cdot \mathbf{J} - \mathbf{K} \cdot \mathbf{K} \right]$$

$$+ \frac{i}{\hbar} \left\{ \mathbf{d} t \left[ \mathbf{H} - \hat{P} \cdot \mathbf{x} \right] - \mathbf{d} \mathbf{x} \cdot \left[ \mathbf{P} \sigma \cdot \mathbf{O} \right] \right\}.$$\]

### III. LITTLE GROUP AND WIGNER TRANSLATIONS

In this Section the properties of the Wigner little group of massless particles will first be summarized and then their relationship to the position operators $\hat{x}_p$ and $\mathbf{x}$ will be discussed. We will prove that the operators $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ are a realization of the photon little group. Finally, rotation of the axis of symmetry of the basis will be investigated.

The Wigner little group operators for a specific four-momentum $k^\mu$ are defined by $L^\mu_k k^{\nu} = k^{\nu}$. For a zero mass particle, it is common, for convenience, to take $\mathbf{k}$ parallel to the $3$-axis, $k^\mu = (k, 0, 0, k)$, so that the little group operators read $\hat{L} = \{\hat{L}_1, \hat{L}_2, \hat{L}_3\}$, $\hat{L}_1 = \hat{J}_2 + \hat{K}$ and $\hat{L}_2 = -\hat{J}_1 + \hat{K}$. In $\hat{L}_1$ and $\hat{L}_2$ the component of $\mathbf{J}$ needed to compensate for the boost is rotated about $\mathbf{e}_3$ relative to $\hat{K}$ from $\mathbf{e}_1$ to $-\mathbf{e}_2$ or from $\mathbf{e}_2$ to $\mathbf{e}_1$, that is it lags $\hat{K}$ by $\pi/2$. These operators are an $e(2)$ subalgebra of the Poincaré group that satisfy the commutation relations $[\hat{L}_1, \hat{L}_2] = 0$, $[\hat{J}_3, \hat{L}_1] = i \hbar \hat{L}_2$, $[\hat{J}_3, \hat{L}_2] = -i \hbar \hat{L}_1$. Since $\hat{L}_1$ and $\hat{L}_2$ commute they can be simultaneously diagonalized and their linear combinations have a continuum of eigenvalues that is not observed, therefore, their common eigenvalue must be $0$. The operators $\hat{L}_1$ and $\hat{L}_2$ generate gauge transformations $\mathbf{3} \mathbf{20}$.

Photons in a twisted light beam have definite total AM in some fixed direction that here will be called $e_3$. With $\chi = -m\phi$ the $k$-space unit vectors given by $\mathbf{3}$ become

$$e^{(m)}_\sigma(k) = \frac{1}{\sqrt{2}} (e_\theta + i e_\phi) e^{i m \phi}.$$ Position eigenvectors at arbitrary $x$ and $t$ can be obtained by applying the unitary transformation $U$ given by $\mathbf{3} \mathbf{18}$ to the position eigenvectors at the origin, $k^\alpha e^{(m)}_\sigma(k)$, to obtain

$$c^{(m)}_{k\sigma}(k) = k^\alpha e^{(m)}_\sigma(k) \exp \left[ i (k \cdot x - \omega t) \right]$$

consistent with $c^{(m)}_{k\sigma} = (c^{(0)}_{k\sigma}, c^{(1)}_{k\sigma})$ in $\mathbf{11}$. Experiments are usually performed on optical beams for which focusing leads to photon localization in two dimensions. In these beams photon density is independent of $x_3$ and $t$. Using $\int dz \exp \left[ i (k_z - k_{z_0}) z \right] / 2\pi = \delta (k_z - k_{z_0})$ the integral of $\mathbf{21}$ over $z$ gives

$$e^{(m)}_{\sigma, \perp}(k, \perp) = e^{(m)}_{\sigma, \perp}(k, z_{z_0}) \exp \left[ i (k \cdot x - \omega t) \right]$$

$$e^{(m)}_{\sigma, \perp}(k, \perp) = e^{(m)}_{\sigma, \perp}(k, z_{z_0}) \exp \left[ i (k \cdot x - \omega t) \right]$$

$$e_{\beta} = \cos \theta e_3 + \sin \theta e_\perp, \quad e_\perp = \cos \phi e_1 + \sin \phi e_2.$$
Thus localized states have intrinsic AM $\hbar \tilde{\sigma} m \hat{e}_3$. While the momentum, angular momentum and boost operators transform like three dimensional vectors, the $\tilde{\sigma}$ operator algebra based on (12) and (14) to (17) gives

$$\tilde{J}_i, \tilde{x}_j = i\hbar \epsilon_{ijk} \tilde{x}_k - i\partial x_j, \tilde{J}_i^{(0,a)}, (25)$$

$$\tilde{K}_i, \tilde{x}_j = -\frac{\hbar}{2} \left( \frac{k_j}{k} \tilde{x}_i + \tilde{x}_i k_j \right) - i\partial x_j, \tilde{K}_i^{(0,a)}. (26)$$

Since $\tilde{J}_i^{(0,-m\phi)} = \tilde{\sigma} m \hbar$ given by (24) does not depend on $k$ in (27) and the components of $\tilde{x}$ commute, in the basis $\chi = -m \phi$

$$\tilde{x}_1, \tilde{x}_2 = 0, (27)$$

$$\tilde{J}_3, \tilde{x}_1 = i\hbar \tilde{x}_2, (28)$$

$$\tilde{J}_3, \tilde{x}_2 = -i\hbar \tilde{x}_1. (29)$$

Thus $\{ \tilde{x}_1, \tilde{x}_2, \tilde{J}_3 \}$ is a realization of the two dimensional Euclidean $e(2)$ algebra that effects genuine infinitesimal transformations in configuration space. This is the primary result of this paper.

The Poincaré, little group and Pryce position operators are discussed in (17). The commutators

$$\tilde{J}_i, \tilde{x}_j = i\hbar \epsilon_{ijk} \tilde{x}_k, (30)$$

$$\tilde{K}_i, \tilde{x}_j = -\frac{\hbar}{2} \left( \frac{k_j}{k} \tilde{x}_i + \tilde{x}_i k_j \right) - i\hbar \epsilon_{ijk} \frac{k_k}{k^2} (31)$$

are equivalent to (4) and (10) in (15). The position operator $\tilde{x}$ whose components commute has the additional features that it has an axis of symmetry and localized eigenvectors.

To simplify (24) and (25) and obtain their physical interpretation we will write them in vector form and assume constant $d\xi$, $d\beta$, $dt$ and $dx = 0$. We return to the general case since $\mathbf{a}(\theta, \phi)$ is needed to describe a basis with axis of symmetry not parallel to $e_3$. A nonzero commutator between $\tilde{x}$ and $\tilde{J}$, $\tilde{K}$ or $\tilde{H}$ implies a infinitesimal change in the position operator. For a rotation through the angle $d\xi$ and a velocity change $d\beta$, (19) gives $d\tilde{x}^\xi = -(i/\hbar) d\xi \cdot \tilde{J}, \tilde{x}^\xi$ and $d\tilde{x}^\beta = (i/\hbar) d\beta \cdot \tilde{K}, \tilde{x}^\beta$. The corresponding changes in the position operator are

$$d\tilde{x}^\xi = d\tilde{x} \times \tilde{x} - \partial_k \left( d\xi \cdot \tilde{J}^{(0,a)} \right), (32)$$

$$d\tilde{x}^\beta = \frac{k}{k} d\beta \cdot \tilde{x} - \partial_k \left( -d\beta \cdot \tilde{K}^{(0,a)} \right). (33)$$

The first term on the right hand side of (33) arises because the energy and position operators do not commute. This corresponds to the $dt$ term of (19) and is a feature of the quantum mechanics of both massive and massless particles. The terms in round brackets are $\tilde{\sigma}$ multiplied by a change in the Euler angle $\chi(\theta, \phi)$ where, in (32),

$$\tilde{\sigma} d\chi^\xi = \frac{1}{\hbar} d\xi \cdot \tilde{J}^{(0,a)}(\theta, \phi). (34)$$

Rotation about an axis in the 12-plane will change the axis of symmetry of the basis. For rotation through an angle $d\theta$ about a fixed axis that makes an angle $\phi$ with the $e_1$ axis, $d\xi = -d\theta (\cos \phi e_1 + \sin \phi e_2) = -d\theta e_\phi$. For a boost described by $d\beta = d\beta (-\sin \phi e_1 + \cos \phi e_2) = d\beta e_\phi$ leads $d\xi$ by $/2$ so that $d\chi^\xi = -d\chi^\beta = k a(m)(\theta) \cos (\phi - \phi)$ and the change in Euler angle introduced by the boost cancels that due to the rotation. For a finite rotation about $e_\phi$ this Euler change can be integrated over $\theta$ to give $\Delta \chi^\xi = \int^\theta_{\theta_0} a(m)(\theta) d\theta \cos (\phi - \phi)$.

Since, according to (41) and (44), $\tilde{x}$ includes a term $-\tilde{\sigma} \partial_k \chi(\theta, \phi)$ it follows that

$$d\tilde{x}^\xi - d\tilde{x} \times \tilde{x} = d\tilde{x}^\beta - e_k d\beta \cdot \tilde{x}$$

$$= -\tilde{\sigma} d\theta \left( e_\phi \frac{\partial a(m)}{\partial \theta} \cos (\phi - \phi) \right.$$

$$+ e_\phi a(m)(\theta) \sin (\phi - \phi) \right). (35)$$

The position operator $\tilde{x}$ describes the center of AM, while the Pryce operator $\tilde{x}_P = \tilde{x} + \tilde{\sigma} a^{(1)}$ implies AM relative to this center. When applied to the position eigenvector at the origin, $\tilde{x}_P e^{(m)}_\sigma = (\tilde{x} + \tilde{\sigma} a^{(m)}) e^{(m)}_\sigma = \sigma a^{(m)} e^{(m)}_\sigma$ so the orbital AM is $\sigma a^{(m)} \times k$. The position operator $\tilde{x}$ obeys the commutation relations (26) and (20). While for $\tilde{x}_P$ (36) and (37) are satisfied. These commutation relations are similar except that (26) and (20) contain a term that rotates the axis of symmetry, which (36) contains a term $-i\tilde{\sigma} \epsilon_{ijk} k_k/k^2$ not present in (26) due to noncommutativity of the components of the Pryce position operator. The extra term in (36) is equivalent to the second term on the right hand side of (16) and the right hand side of (13) in (15). Since from (11) and the $\tilde{x}_P$ commutation relation following it $\tilde{x}_P, \tilde{x}_j = [\tilde{x}_P, \tilde{x}_j] = -\tilde{\sigma} \left( \tilde{x}_j a^{(m)} - a^{(m)} \tilde{x}_j \right) = i\tilde{\sigma} \partial_k x^{(m)} - i\tilde{\sigma} k/k^3 = 0$, in (24) this ‘Wigner’ term is absorbed into $\tilde{x}$ that has commuting components.

**IV. OPTICAL BEAMS**

In this section application of the position eigenvectors to optical beams will be discussed in the context of the recent experimental and theoretical literature. We will consider the relationship of the configuration space basis to transfer of linear and angular momentum to a particle, focusing, phase shift in an optical fiber and optical communications.
m serves the component of AM parallel to its axis of symmetry as sketched in Fig. 2. Since refraction by the lens changes the diameter of the fiber and the longitudinal component of AM that is transferred to a particle. It is observed that the optical intensity in a high-order Bessel beam is independent of the radial position vector \( \mathbf{r} \) and its transverse profile is a series of bright rings.

A small particle trapped in a bright ring of such a beam simultaneously spins on its axis and orbits the beam center \( \mathbf{c} \). A photon in this beam has transverse wave vector \( \mathbf{k}_\perp = k_\perp \mathbf{e}_3 \), a radial position vector \( \mathbf{x}_\perp \) pointing outward from the beam axis and extrinsic orbital AM \( \hbar \mathbf{e}_3 \). If it is absorbed, its linear momentum \( \hbar \mathbf{k}_\perp \) and total AM \( (\sigma + l) \hbar \mathbf{e}_3 \) will be transferred to the particle causing it to spin on its axis and orbit the beam axis \( \mathbf{c} \). At a fundamental level it is total angular and linear momentum that is conserved.

The eigenvectors of \( \hat{x} \) are an idealization of an ultrashort pulse focused at \( \mathbf{x} \) for an instant. Experiments are usually performed on optical beams for which focusing leads to photon localization in only two dimensions. Eq. (22) is a good basis for description of focusing of a beam. A CP beam with incident center wave vector \( \mathbf{k}_i = k_i \mathbf{e}_3 \) and final wave vector \( \mathbf{k}_f \) is focused to the point \((0,0,x_3)\) as sketched in Fig. 2. Since refraction by the lens conserves the component of AM parallel to its axis of symmetry, the total AM per photon at the focal point is still \( \sigma m \hbar \mathbf{e}_3 \). This conversion has been observed: focusing of a beam carrying spin AM can induce orbital AM which drives the orbital motion of micron-sized metal particles.

In an optical fiber photon position is limited by the diameter of the fiber and the longitudinal component of momentum is determined by its orientation. A right-handed CP beam cycling around a closed circuit in k-space acquires a Berry phase shift relative to a left-handed CP beam of \( \Omega = 4\pi (1 - \cos \theta) \) per loop as predicted in [11], confirmed experimentally in [13], and given here by (9). It was predicted in [22] that electrons and photons experience a universal geometric phase shift even in a straight waveguide that can be described in perturbation theory by spin-orbit coupling which in the paraxial limit is proportional to the spin-orbit coupling. This effect has recently been observed in dispersion-taylored straight few-mode fibers [23] where \( x_3 \) was varied by cutting the fiber. The linear polarization was found to rotate with \( x_3 \) at a rate proportional to the spin-orbit coupling strength. In these experiments, photons in the input beam have extrinsic orbital AM \( \hbar \mathbf{e}_3 \).

Twisted photons can be used to encode information beyond one bit per single photon [14]. Secure communication requires entangled photons and entanglement is the most mysterious property of quantum particles. Tests of quantum mechanics are often performed on photons so the controversy regarding photon localization is relevant to many experiments that have great potential for the performance of quantum tasks.

V. CONCLUSION

Photons are the most important but also the most problematic neutral bosons. They are important because they are the subject of many experiments, including some intended as tests of quantum mechanics itself. They are controversial because most theorists believe that there is no acceptable photon position operator with commuting components that would lead to a basis of position eigenvectors. But such an operator does in fact exist and we show here that its properties are a consequence of the symmetry of the photon little group. The extra term in its commutation relations with the rotation and boost operators describes rotation of the axis of symmetry of its eigenvectors and the observed Berry phase shift.

We have proved in Section III that \( \{ x_1, x_2, J_3 \} \) is a realization of the two dimensional Euclidean \( e(2) \) algebra that affects genuine infinitesimal transformations in configuration space. This answers the question "What is \( x \)" posed by Stone, Dwivede and Zhou [13]: \( x \) is photon position. While still controversial this conclusion illuminates the debate surrounding photon wave mechanics and the localization of light.

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