Rotating cylindrical wormholes

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Abstract

We consider stationary, cylindrically symmetric configurations in general relativity and formulate necessary conditions for the existence of rotating cylindrical wormholes. It is shown that in a comoving reference frame the rotational part of the gravitational field is separated from its static part and forms an effective stress-energy tensor with exotic properties, which favors the existence of wormhole throats. Exact vacuum and scalar-vacuum solutions (with a massless scalar) are considered as examples, and it turns out that even vacuum solutions can be of wormhole nature. However, solutions obtainable in this manner cannot have well-behaved asymptotic regions, which excludes the existence of wormhole entrances appearing as local objects in our Universe. To overcome this difficulty, we try to build configurations with flat asymptotic regions by the cut-and-paste procedure: on both sides of the throat, a wormhole solution is matched to a properly chosen region of flat space at some surfaces $\Sigma_{-}$ and $\Sigma_{+}$. It is shown, however, that if we describe the throat region with vacuum or scalar-vacuum solutions, one or both thin shells appearing on $\Sigma_{-}$ and $\Sigma_{+}$ inevitably violate the null energy condition. In other words, although rotating wormhole solutions are easily found without exotic matter, such matter is still necessary for obtaining asymptotic flatness.

1 Introduction

Wormholes, a subject of active discussion in the current literature, are hypothetic objects where two large or infinite regions of space-time are connected by a kind of tunnel. These two regions may lie in the same universe or even in different universes. The existence of sufficiently stable (traversable, Lorentzian) wormholes can lead to physical effects of utmost interest, such as a possibility of realizing time machines or shortcuts between distant regions of space [1–3]. If wormholes exist on astrophysical scales of lengths and times, they can lead to quite a number of unusual observable effects [4, 5].

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It is well known that a wormhole geometry can only appear as a solution to the Einstein equations if the stress-energy tensor (SET) of matter violates the null energy condition (NEC) at least in a neighborhood of the wormhole throat [1–3, 6]. This conclusion, however, has been proved under the assumption that the throat is a compact 2D surface, having a finite (minimum) area (at least in the static case, since for dynamic wormholes it has proved to be necessary to generalize the notion of a throat [7]; see, however [8] for other definitions of a throat). In other words, it was assumed that, as seen from outside, a wormhole entrance is a local object like a star or a black hole.

But, in addition to such objects, the Universe may contain structures which are infinitely extended along a certain direction, like cosmic strings. And, while starlike structures are, in the simplest case, described in the framework of spherical symmetry, the simplest stringlike configurations are cylindrically symmetric.

The opportunity of building wormhole models in the framework of cylindrical symmetry was recently discussed in [9]. It has been shown, for the case of static configurations, that the necessary conditions for the existence of wormhole solutions differs from their spherically symmetric counterparts, but it is still rather hard to obtain more or less realistic cylindrically symmetric wormhole models. Indeed, quite a number of such wormhole solutions have been obtained, but none of them have two flat (or string, i.e., flat up to an angular deficit) asymptotic regions, which exclude the existence of wormhole entrances appearing as local objects in our Universe. Moreover, it has been shown that the existence of a static, cylindrically symmetric, twice asymptotically flat wormhole requires a matter source with negative energy density [9].

In this paper we extend the consideration to stationary configurations containing a vortex gravitational field. Such fields can lead to effective stress-energy tensors with rather exotic properties [10–12], which give us a hope to obtain realistic wormhole models in the framework of general relativity.

A vortex gravitational field is described by the 4-dimensional curl of the tetrad $e^a_\mu$: its kinematic characteristic is the angular velocity of tetrad rotation

$$\omega^\mu = \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} e_{m \nu} e_{n \rho} e_{\sigma}^m,$$

where Greek indices correspond to the four world coordinates $x^\mu$ while the Latin letters $m, n, \ldots$ are used for Lorentz indices. The vector $\omega^\mu$ determines the effective angular momentum density of the gravitational field

$$g S^\mu = \omega^\mu / \kappa,$$

where $\kappa = 8\pi G$ is Einstein’s gravitational constant.

The simplest example of a space-time possessing a stationary vortex gravitational field is that with the cylindrically symmetric metric

$$ds^2 = A du^2 + C dz^2 + B d\phi^2 + 2E dt d\phi - D dt^2,$$

where all metric coefficients depend on the “radial” coordinate $u$ whose range is not a priori specified; $z \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$ are the longitudinal and azimuthal coordinates, respectively. The geometric properties of the spatial sections $t = \text{const}$ of the space-time (3) are determined by the 3D metric

$$dl^2 = A du^2 + C dz^2 + \frac{BD + E^2}{D} d\varphi^2.$$
A vortex gravitational field can both be free (i.e., exist without a matter source) and be created by certain fields with polarized spin, such as a spinor field; it can certainly exist in the presence of matter without polarized spin, such as a perfect fluid. Some examples of cylindrically symmetric wormhole solutions with such sources and the metric (3) have been obtained in [10–12].

In the present paper, we will discuss the general conditions for the existence of cylindrical wormholes with the stationary metric (3) with rotation and try to obtain a wormhole model with two flat asymptotic regions without invoking exotic matter. Section 2 is devoted to working out some general relations for the metric (3). It is shown that in a reference frame comoving to the matter source of gravity (that is, an azimuthal matter flow is absent), then the rotational part of the Ricci and Einstein tensors can be separated from its “static” part, which makes much easier the subsequent analysis. Even more than that, the rotational part of the Einstein tensor can be viewed as an addition to the stress tensor of matter, and the unusual properties of this addition can hopefully provide wormhole construction.

Further on, in Section 3, we discuss the possible definitions of cylindrical wormhole throats by analogy with [9]. With these definitions, we find the necessary conditions for the throat existence, which generalize those formulated in [9] for the static case and are actually easier to satisfy due to the rotational contribution.

As the simplest example, in Section 4 we consider exact rotating wormhole solutions which exist in vacuum (due to a vortex gravitational field) or in the presence of a massless scalar field. It turns out that rotating wormhole solutions obtainable in the present framework cannot be asymptotically flat in principle because the angular velocity does not vanish at large radii. It is therefore suggested to construct asymptotically flat configurations by the cut-and-paste procedure: on each side of the throat, a wormhole solution can be cut and matched with a properly chosen region of flat space, which should be taken in a rotating reference frame to allow matching. The inevitable jump in the extrinsic curvature tensor of the junction surfaces $\Sigma_+$ and $\Sigma_-$ corresponds to the existence of thin shells of matter whose surface SET $S_{ab}$ can be calculated in the well-known Darmois-Israel formalism. It is of utmost interest whether or not $S_{ab}$ can respect the weak and null energy conditions (WEC and NEC, respectively).

This general procedure is discussed in Section 5, while Section 6 describes an attempt to construct such a compound wormhole model using the previously obtained vacuum and scalar-vacuum solutions for the inner region containing the throat. It is shown that one or both SETs of the thin shells on $\Sigma_+$ and $\Sigma_-$ inevitably violate the NEC. This means that a twice asymptotically flat wormhole model cannot be constructed without exotic matter in the framework under consideration, at least with the aid of vacuum or scalar-vacuum solutions. So, even though rotation strongly favors the existence of cylindrical wormhole throats, the problem of building a “realistic” wormhole model remains unsolved.

Section 7 is a brief conclusion, and in the Appendix we outline the ways of obtaining exact rotating cylindrically symmetric solutions for scalar fields with nonzero self-interaction potentials, postponing their detailed analysis for the future.
2 Basic relations for stationary rotating cylindrically symmetric space-times

Let us rewrite the metric (3) in a slightly different notation, singling out the three-dimensional linear element (4):

\[ ds^2 = A \, du^2 + C \, dz^2 + r^2 \, d\varphi^2 - D [dt - (E/D) \, d\varphi]^2 \]

\[ = e^{2\alpha} \, du^2 + e^{2\mu} \, dz^2 + e^{2\beta} \, d\varphi^2 - e^{2\gamma} (dt - E \, e^{-2\gamma} \, d\varphi)^2, \]

where \( A, B, C, D, E \) as well as \( \alpha, \beta, \gamma, \mu \) are functions of the radial coordinate \( x^1 = u, \ x^4 = t \) is the time coordinate, and the different notations (given here to facilitate comparison with different references) are related by

\[ A = e^{2\alpha}, \quad C = e^{2\mu}, \quad D = e^{2\gamma}, \quad e^{\beta} = r(u), \quad r^2 = \Delta/D, \quad \Delta = BD + E^2. \]  

The absolute value of the metric determinant is

\[ g = | \det(g_{\mu\nu}) | = AC \Delta = e^{2\alpha+2\beta+2\gamma+2\mu}. \]  

In the gauge \( A = C \) (equivalently, \( \alpha = \mu \)) the vortex \( \omega = \sqrt{\omega_\alpha \omega^\alpha} \) is [12, 13]

\[ \omega = \frac{E' D - E D'}{2D \sqrt{A \Delta}} = \frac{1}{2} (E e^{-2\gamma})' e^{\gamma-\beta-\mu}. \]  

The Ricci tensor components in the gauge \( \alpha = \mu \) can be expressed as follows (see also [14]; the prime stands for \( d/du \)):

\[ R^1_1 = -e^{-2\mu}[\beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'(\beta' + \gamma')] + 2\omega^2; \]  

\[ R^2_2 = -e^{-2\mu}[\beta'' + \mu'(\beta' + \gamma')]; \]  

\[ \sqrt{g} R^3_3 = -\left[ \frac{DB' + EE'}{2\sqrt{\Delta}} \right]' = -\left[ \beta' \, e^{\beta+\gamma} - E \omega e^\mu \right]'; \]  

\[ \sqrt{g} R^4_4 = -\left[ \frac{D'B + EE'}{2\sqrt{\Delta}} \right]' = -\left[ \gamma' \, e^{\beta+\gamma} + E \omega e^\mu \right]'; \]  

\[ \sqrt{g} R^3_4 = -\left[ \frac{DE' - ED'}{2\sqrt{\Delta}} \right]' = -\left( \omega e^{2\gamma+\mu} \right)'; \]  

\[ \sqrt{g} R^4_3 = -\left[ \frac{B'E - BE'}{2\sqrt{\Delta}} \right]' \].

Assuming that our rotating reference frame is comoving to the matter source of gravity, that is, the azimuthal flow \( T^3_4 = 0 \), we find from \( R^3_4 = 0 \) that

\[ \omega = \omega_0 e^{-\mu-2\gamma}, \quad \omega_0 = \text{const.} \]  

\(^{3}\)It is this component of the stress-energy tensor that describes the flow in the \( x^3 \) direction, or, more precisely, \( T^3_4/\sqrt{|g_{44}|} \), as follows from Zel’manov’s prescription for spatially covariant and chronometrically invariant vector components [15]; this expression can also be verified, though with more effort, using the tetrad formalism, see, e.g., [16].
Substituting (16) into the expression \( (E\omega e^\mu)' \), taking into account (9), we find that

\[
(E\omega e^\mu)' = 2\omega_0^2 e^{\beta - 3\gamma} = 2\omega^2 e^{2\mu + \beta + \gamma} = 2\omega^2 \sqrt{g}. \tag{17}
\]

As a result, in an arbitrary gauge the diagonal components of the Ricci tensor can be written as follows:

\[
R^1_1 = -e^{-2\alpha} [\beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'^2 - \alpha'(\beta' + \gamma' + \mu')] + 2\omega^2; \tag{18}
\]

\[
R^2_2 = \Box_1 \mu; \tag{19}
\]

\[
R^3_3 = \Box_1 \beta + 2\omega^2; \tag{20}
\]

\[
R^4_4 = \Box_1 \gamma - 2\omega^2. \tag{21}
\]

where, for any \( f(u) \), \( \Box_1 f = -g^{-1/2}[\sqrt{g} g^{11} f]' = -e^{-2\alpha} [f'' + f'(\beta' + \gamma' + \mu' - \alpha')] \). We see that the diagonal part of the Ricci tensor splits into the static part \( R^\nu_{\mu} \) and the rotational part \( R^\omega_{\mu} \), where

\[
R^\nu_{\mu} = \omega^2 \text{diag}(2, 0, 2, -2) \tag{22}
\]

(the coordinates are ordered as follows: \( u, z, \varphi, t \)). The corresponding Einstein tensor \( G^\nu_{\mu} = R^\nu_{\mu} - \frac{1}{2} \delta^\nu_{\mu} R \) splits in a similar manner,

\[
G^\nu_{\mu} = G^\nu_{\mu} + G^\omega_{\mu}; \quad G^\omega_{\mu} = \omega^2 \text{diag}(1, -1, 1, -3). \tag{23}
\]

One can check that the tensors \( G^\nu_{\mu} \) and \( G^\omega_{\mu} \) (each separately) satisfy the “conservation law” \( \nabla_\alpha G^\alpha_{\mu} = 0 \) with respect to the static metric (the metric (6) with \( E \equiv 0 \)).

The Einstein equations are written as

\[
G^\nu_{\mu} = -\kappa T^\nu_{\mu}, \quad \kappa = 8\pi G, \tag{24}
\]

where, as usual, \( G \) is Newton’s constant of gravity and \( T^\nu_{\mu} \) is the stress-energy tensor (SET) of all kinds of matter. Equivalently,

\[
R^\nu_{\mu} = -\kappa \tilde{T}^\nu_{\mu}, \quad \tilde{T}^\nu_{\mu} = T^\nu_{\mu} - \frac{1}{2} \delta^\nu_{\mu} T^\alpha. \tag{25}
\]

It is clear that the tensor \( G^\omega_{\mu} \) (divided by \( \kappa \)) behaves in the Einstein equations as an additional SET with very exotic properties (for instance, the effective energy density is \( -3\omega^2/\kappa < 0 \)), acting in an auxiliary static space-time with the metric (6) where \( E \equiv 0 \).

Noteworthy, there remains the off-diagonal component (15) which is, in general, nonzero. If we assume, as before, \( T^3_4 = 0 \) (the comoving reference frame), then

\[
R^3_4 = \frac{E}{D}(R^3_3 - R^4_4). \tag{26}
\]

Thus, if the diagonal components of the Einstein equations have been solved, the \( (4, 3) \) component holds automatically and need not be considered. The same relation as (26) holds for the SET components: \( T^3_4 = (E/D)(T^3_3 - T^4_4) \), it is thus nonzero if \( T^3_3 \neq T^4_4 \).
3 The cylindrical wormhole geometry and its existence conditions

Let us begin with a definition formulated by analogy with Definition 1 in [9].

**Definition 1.** We say that the metric (6) describes a wormhole geometry if the circular radius \( r(u) \equiv e^{\beta(u)} \) has a minimum \( r(u_0) > 0 \) at some \( u = u_0 \), such that on both sides of this minimum \( r(u) \) grows to much larger values than \( r(u_0) \), and, in some range of \( u \) containing \( u_0 \), all metric functions in (6) are smooth and finite (which guarantees regularity and absence of horizons).

The cylinder \( u = u_0 \) is then called a throat (or an \( r \)-throat).

The notion of a wormhole is, as in other similar cases, not rigorous because of the words “much larger”, but the notion of a throat as a minimum of \( r(u) \) is exact.

Now, let us take the diagonal part of the SET \( T_{\mu}^{\nu} \) in the most general form

\[
T_1^1 = -p_r, \quad T_2^2 = -p_z, \quad T_3^3 = -p_\varphi, \quad T_4^4 = \rho, \tag{27}
\]

where \( \rho \) is the density and \( p_i \) are pressures of any physical origin in the respective directions.

It is straightforward to find out how the SET components should behave on a wormhole throat. At a minimum of \( r(u) \), due to \( \beta' = 0 \) and \( \beta'' > 0 \),\(^4\) we have \( R^2_3 - 2\omega^2 < 0 \), and from the corresponding component of (25) it follows

\[
\rho - p_r - p_z + p_\varphi - 2\omega^2/\kappa < 0. \tag{28}
\]

As noted in [9], in the general case of anisotropic pressures, (28) does not necessarily violate any of the standard energy conditions even for non-rotating configurations; it is clear, however, that the inequality (28) cannot hold if \( \rho \) is substantially larger than any of the pressures; for isotropic (Pascal) fluids, with all \( p_i = p \), the condition (28) leads to \( \rho < p \) if \( \omega = 0 \). With nonzero rotation it is much easier to build a configuration with a throat, as is illustrated by numerous examples [10–12], and some new ones will be presented below.

The above definition is, however, not unique: thus, it is rather natural, also by analogy with spherical symmetry, to define a throat and a wormhole in terms of the area function \( a(u) = e^{\beta+\mu} \) instead of \( r(u) \) [9]. We will call it an \( a \)-throat for brevity.

**Definition 2.** In a space-time with the metric (6), an \( a \)-throat is a cylinder \( u = u_a \) where the function \( a(u) = e^{\beta+\mu} \) has a regular minimum.

A configuration where on both sides of \( u_a \) the function \( a(u) \) grows to values \( a \gg a(u_a) \), and, in some range of \( u \) containing \( u_a \), all metric functions in (6) are smooth and finite, is called an \( a \)-wormhole.

Let us look what are the existence conditions of an \( a \)-throat. By Definition 2, at \( u = u_1 \) we have \( \beta' + \mu' = 0 \) and \( \beta'' + \mu'' > 0 \). The minimum occurs in terms of any admissible coordinate \( u \), in particular, in terms of the harmonic coordinate (34). Using it in Eqs. (25) with (19) and (20), we find that the condition \( \beta'' + \mu'' \) implies

\[
R_2^2 + R_3^2 = -e^{-2\alpha} (\beta'' + \mu'') + 2\omega^2 < 2\omega^2 \quad \Rightarrow \quad T_1^1 + T_4^4 = \rho - p_r < 2\omega^2/\kappa. \tag{29}
\]

\(^4\)Here and henceforth we restrict ourselves for convenience to generic minima, at which \( \beta'' > 0 \). If there is a special minimum at which \( \beta'' = 0 \), we still have \( \beta'' > 0 \) in its vicinity, along with all consequences of this inequality. The same concerns minima of \( a(u) \) discussed below.
In addition, substituting $\beta' + \mu' = 0$ into the Einstein equation $G^1_1 = -\kappa T^1_1$, we find
\[
G^1_1 = e^{-2\alpha} \beta' \mu' + \omega^2 = -e^{-2\alpha} \beta'' + \omega^2 \leq \omega^2 \quad \Rightarrow \quad -T^1_1 = p_r \leq \omega^2 / \kappa.
\] (30)

For nonrotating configurations ($\omega = 0$), the requirements (29) and (30) yield [9]
\[
\rho < p_r \leq 0 \quad \text{at} \quad u = u_1,
\]
i.e., there is necessarily a region with negative energy density $\rho$ at and near an $a$-throat. With $\omega \neq 0$, these requirements leave an opportunity of having a cylindrical wormhole geometry without violating the standard energy conditions.

This, however, concerned only the existence of throats, leaving aside the asymptotic behavior of possible solutions. We only note that if we wish to have regular asymptotic behaviors far from the throat, such that, in particular, $\mu \to \text{const}$ while $r \to \infty$, then it will be a wormhole geometry by both definitions.

### 4 Vacuum and massless scalar field solutions

Consider a minimally coupled scalar field $\phi$ with the Lagrangian
\[
L_s = -\frac{1}{2} \epsilon (\partial \phi)^2 - V(\phi)
\] (31)
as a source of the geometry (6). Here, $\epsilon = +1$ corresponds to a normal scalar field and $\epsilon = -1$ to a phantom one.

Let us assume $\phi = \phi(u)$ and the comoving reference frame, so that $T^3_4 = 0$, and the Ricci tensor has the form (18)–(21). The stress-energy tensor of $\phi$ is
\[
T^\nu_\mu(\phi) = \frac{1}{2} e^{-2\alpha} \phi^2 \text{diag}(-1, -1, -1, 1) + \delta^\nu_\mu V(\phi).
\] (32)

It is convenient to use the Einstein equations in the form
\[
R^\nu_\mu = -\kappa (T^\nu_\mu - \frac{1}{2} \delta^\nu_\mu T_\alpha^\alpha) = -\kappa (\epsilon \partial_\mu \phi \partial_\nu \phi - \delta^\nu_\mu V).
\] (33)

For further consideration we focus on vacuum configurations ($T^\nu_\mu \equiv 0$) and those with a massless scalar field ($V(\phi) \equiv 0$). Some solutions can also be obtained with nonzero potentials $V(\phi)$, as will be outlined in the Appendix.

Using the harmonic radial coordinate $u$ such that
\[
\alpha = \beta + \gamma + \mu,
\] (34)
the expressions (18)–(21) are substantially simplified, in particular, $\Box_1 f = -e^{-2\alpha} f''$ for any $f(u)$.

In this case three of the four diagonal components of (33) for a massless scalar field give
\[
R^2_2 = 0 \quad \Rightarrow \quad \mu'' = 0,
\] (35)
\[
R^3_3 = 0 \quad \Rightarrow \quad \beta'' - 2\omega^2 e^{2\alpha} = 0,
\] (36)
\[
R^4_4 = 0 \quad \Rightarrow \quad \gamma'' + 2\omega^2 e^{2\alpha} = 0,
\] (37)
whence it follows

\[ \mu = -mu \quad [\text{with a certain choice of } z \text{ scale}], \]

\[ \beta + \gamma = 2hu \quad [\text{with a certain choice of } t \text{ scale}], \]

\[ \beta'' - \gamma'' = 4\omega_0^2 e^{2\beta-2\gamma}. \]

In obtaining (40), Eq. (16) has been taken into account. Eq. (40) is a Liouville equation whose solution can be written in the form

\[ e^{\gamma-\beta} = 2\omega_0 s(k, u), \quad s(k, u) = \begin{cases} k^{-1} \sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1} \sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases} \]

Here, \( h, k, m = \text{const} \), and the other three integration constants have been suppressed by choosing scales along the \( z \) and \( t \) axes in (38) and (39) and the origin of \( u \) (in (41)). Now it is straightforward to obtain

\[ e^{2\beta} = \frac{e^{2hu}}{2\omega_0 s(k, u)}, \quad e^{2\mu} = e^{-2mu}, \]

\[ e^{2\gamma} = 2\omega_0 s(k, u) e^{2hu}, \quad e^{2\alpha} = e^{(4h-2m)u}, \quad \omega = \frac{e^{mu-2hu}}{2s(k, u)}, \]

\[ E = e^{2hu} s(k, u) \int \frac{du}{s^2(k, u)} = e^{2hu} [E_0 s(k, u) - s'(k, u)], \quad E_0 = \text{const}. \]

The scalar field equation reads \( \phi'' = 0 \), whence \( \phi = Cu \) (fixing the inessential zero point of \( \phi \)); the constant \( C \) has the meaning of scalar charge density. Lastly, the Einstein equation \( G^1_1 = -\kappa T^1_1 \), which is first-order, leads to a relation between the integration constants:

\[ k^2 \text{sign } k = 4(h^2 - 2hm) - 2\kappa\varepsilon C^2. \]

This completes the solution.

A wormhole geometry by Definition 1 corresponds to the cylindrical radius \( r = e^\beta \to \infty \) at both ends of the \( u \) range. It is clear that \( r \to \infty \) and \( e^\gamma \to 0 \) as \( u \to 0 \) in all these solutions. In the same limit, the vortex \( \omega \to \infty \), which probably indicates a singularity, although all components of the scalar field SET (32) are finite (since \( \alpha \) is finite and \( \phi' = C \)), hence, by the Einstein equations, the same is true for the components of the Ricci tensor; however, taken separately, the static and vortex parts of the Ricci tensor diverge.

As to the other end of the \( u \) range, the situation is more diverse:

1. \( k > 0 \). In this case we have \( e^{2\beta} \sim e^{(2h-k)u} \) and \( e^{2\gamma} \sim e^{(2h+k)u} \) at large \( u \), hence a wormhole geometry by Definition 1 takes place for \( 0 < k < 2h \); we also have \( e^\gamma \to \infty \) at large \( u \). Definition 2 requires in a similar way \( 0 < k < 2(h - m) \).

2. \( k = 0 \). In this case we have \( e^{2\beta} \sim u^{-1} e^{2hu} \) and \( e^{2\gamma} \sim u e^{2hu} \), hence we have a wormhole geometry by Definition 1 as long as \( h > 0 \) and by Definition 2 as long as \( h - m > 0 \). We have again \( e^\gamma \to \infty \) at large \( u \).

3. \( k < 0 \). It is clear that a wormhole geometry (by both definitions 1 and 2) is described by all solutions with \( k < 0 \), for which the range of \( u \) is \( 0 < u < \pi/|k| \) without loss of generality. At both ends, \( e^\beta \to \infty \) and \( e^\gamma \to 0 \), while \( e^{\beta+\gamma} \) and \( e^\mu \) remain finite, and \( \omega \sim e^{-2\gamma} \to \infty \) at large \( u \).
From (43) it is evident that \( k < 0 \) is compatible with any \( \varepsilon \), and, curiously, \( \varepsilon = +1 \) even favors negative \( k \), unlike similar solutions for the static case [9].

A vacuum solution corresponds to \( C = 0 \), it has been considered previously in [11, 12]. It is clear that its special case must correspond to flat space in a rotating reference frame. One can verify that this special case is \( m = 0, \ k = 1, \ h = -1/2 \): we then obtain the metric (44) (see below) transformed to the harmonic coordinate \( u \).

The limit \( \omega_0 \to 0 \) leads to the (scalar-)vacuum solution without rotation in the form given in [9] (corresponding to \( \beta'' = \gamma'' = \mu'' = 0 \)); however, the transition is not very simple. It is only possible for \( k > 0 \) and \( k > 2h \); it is carried out by making a shift \( u \mapsto u + u_0 \) and then turning \( u_0 \) to infinity in a special way, requiring a finite limit of the expression \( (\omega_0/k) e^{(k-2h)u_0} \); then, the scales along the \( t \) and \( z \) axes must be adjusted.

## 5 Asymptotic flatness and thin shells

If we wish to describe wormholes in a flat or weakly curved background universe, so that they could be visible to distant observers like ourselves, it is necessary to describe them as systems with flat (or string) asymptotic behaviors. However, when dealing with cylindrically symmetric systems, it is rather hard to obtain such solutions: indeed, even the Levi-Civita vacuum solution has such an asymptotic only in the special case where the space-time is simply flat.

We have proved previously [9] for static cylindrical wormholes that to have such asymptotics on both sides of the throat, it is necessary to invoke matter with negative energy density. This result does not depend on particular assumptions on the nature of matter. If we want to avoid negative densities (and exotic matter in general) in wormhole building, it is reasonable to try rotating configurations. It is, however, still harder to obtain asymptotically flat solutions with rotation than without it. One of the difficulties is connected with the fact that the Minkowski metric, being written in a rigidly rotating reference frame, is stationary only inside the “light cylinder”, so in order to reach an asymptotic region, the rotation should be differential, with vanishing angular velocity at large radii. On the other hand, looking at Eq. (16), we see that at a hypothetic flat infinity, where \( \mu \) and \( \gamma \) should tend to constants, the quantity \( \omega \) does not vanish.

We have seen that even vacuum configurations with rotation can yield cylindrical wormhole throats; however, none of these wormholes are asymptotically flat.

A possible way out is to try to cut a non-asymptotically flat wormhole configuration at some cylinders \( u_+ \) and \( u_- \) at different sides of the throat and to join it there to properly chosen flat space regions. The junction surfaces \( u = u_+ \) and \( u = u_- \) will comprise thin shells with certain surface densities and pressures, which can in principle be physically plausible.

It is clear that, for such a purpose, the flat-space metric should be taken in a rotating reference frame. To do so, in the Minkowski metric \( ds^2 = dx^2 + dz^2 + x^2d\varphi^2 - dt^2 \) we can make the substitution \( \varphi \to \varphi + \Omega t \) to obtain

\[
ds^2_M = dx^2 + dz^2 + x^2(d\varphi + \Omega dt)^2 - dt^2,
\]

where \( \Omega = \text{const} \) is the angular velocity of the reference frame. The relevant quantities defined above are

\[
-g_{44} = e^{2\gamma} = 1 - \Omega^2 x^2, \quad r^2 \equiv e^{2\beta} = \frac{x^2}{1 - \Omega^2 x^2},
\]

\[
E = \Omega x^2, \quad \omega = \frac{\Omega}{1 - \Omega^2 x^2}
\]
[we are using the general notations according to (6)]. This metric is stationary and suitable for matching with the internal metric at $|x| < 1/\Omega$, inside the “light cylinder” $|x| = 1/\Omega$ at which the linear rotational velocity reaches that of light.

To match two cylindrically symmetric regions at a certain surface $\Sigma$: $u = u_0$, it is necessary first of all to identify this surface as it is seen from both sides, hence to provide coincidence of the two metrics at $u = u_0$, or specifically,

$$[\beta] = 0, \quad [\mu] = 0, \quad [\gamma] = 0, \quad [E] = 0,$$

(46)

where the square brackets denote, as usual, a discontinuity of a given quantity across the surface in question,

$$[f] := f(u_0 + 0) - f(u_0 - 0).$$

One should note that in general the two metrics to be matched may be written using different choices of the radial coordinate $u$, but it does not matter since the quantities present in (46) are insensitive to the choice of $u$.

The second step is to determine the material content of the matching surface $\Sigma$ according to the Darmois-Israel formalism [17]: in our case of a timelike $\Sigma$, the surface stress-energy tensor $S_{ab}$ is given by

$$S^b_a = \frac{1}{8\pi}[\tilde{K}^b_a], \quad \tilde{K}^b_a := K^b_a - \delta^b_a K,$$

(47)

where $K = K^a_a$, $K^b_a$ being the extrinsic curvature of the surface $\Sigma$. The latter, in turn, is defined in terms of a covariant derivative of the unit normal vector $n_\mu$ of $\Sigma$ drawn in a chosen direction: if $\Sigma$ is defined by the equation $f(x^\alpha) = 0$ and is parametrized by the coordinates $\xi^a$, then

$$K_{ab} = \partial_{x^\alpha} dx^\beta \nabla_\alpha n_\beta = -n^\gamma \left[ \frac{d^2 x^\gamma}{d\xi^a d\xi^b} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^a} \frac{dx^\beta}{\partial \xi^b} \right].$$

(48)

In our case, we take the surface $x^1 = u = \text{const}$. Choosing the directions of the normal $n^\alpha$ to growing $x^1$ and the natural parametrization $\xi^a = x^a$, $a = 2, 3, 4$, we obtain

$$K_{ab} = -e^{\alpha(u)} \Gamma^1_{ab} = \frac{1}{2} e^{-\alpha(u)} \frac{\partial g_{ab}}{\partial u}.$$  

(49)

The indices of $K_{ab}$ are raised by the surface metric tensor $g^{ab}$, the inverse of $g_{ab}$, $a, b = 2, 3, 4$, namely,

$$(g_{ab}) = \begin{pmatrix} e^{2\mu} & e^{2\beta} - E^2 e^{-2\gamma} & E \\ E & -e^{2\gamma} \end{pmatrix} ,$$

$$(g^{ab}) = \begin{pmatrix} e^{-2\mu} & e^{-2\beta} & E e^{-2\beta-2\gamma} \\ E e^{-2\beta-2\gamma} & -e^{-2\gamma} + E^2 e^{-2\beta-4\gamma} \end{pmatrix}.$$  

(50)

\footnote{Here and henceforth we use the units where $G = 1$, so that $\kappa = 8\pi$.}
As a result, we can write down the following expressions for the trace $K$ and the nonzero components of $\tilde{K}^b_a$ in terms of the metric (6):

\begin{align*}
K &= e^{-\alpha}(\beta' + \gamma' + \mu'), \\
\tilde{K}^2_2 &= -e^{-\alpha}(\beta' + \gamma'), \\
\tilde{K}^3_3 &= -e^{-\alpha}(\mu' + \gamma') - E\omega e^{-\beta-\gamma}, \\
\tilde{K}^4_4 &= -e^{-\alpha}(\beta' + \mu') + E\omega e^{-\beta-\gamma}, \\
\tilde{K}^3_4 &= \omega e^{\gamma-\beta}.
\end{align*}

One can again notice that all quantities (51)–(55) are insensitive to the choice of the radial coordinate.

The expression (55) is of interest because its discontinuity describes the surface matter flow in the $\varphi$ direction. Hence, if we assume that the thin shell at a junction is at rest in the reference frame in which the ambient space-time is considered, we must put $[\omega] = 0$.

### 6. An attempt to build an asymptotically flat wormhole model

Let us try to build a model with the following structure:

\[ M_- \cup \Sigma_- \cup V \cup \Sigma_+ \cup M_+, \]

where $M_-$ and $M_+$ are regions of Minkowski space described by the metric (44), $V$ is a space-time region described by the vacuum (or scalar-vacuum) solution (42)–(43) with a vortex gravitational field, while $\Sigma_-$ and $\Sigma_+$ are junction surfaces endowed with certain surface densities and tensions.

Both $r$- and $a$-throats should be located in $V$, therefore the junction surface $\Sigma_-$ is assumed to be located at $u = u_-$ small enough, such that, in particular, $r'(u_-) < 0$ while the surface $\Sigma_+$ is located at $u = u_+$ such that $r'(u_+) > 0$.

The surfaces $\Sigma_- \in V$ and $\Sigma_+ \in V$ are identified with the surfaces $x = x_-$ and $x = x_+$ belonging to $M_-$ and $M_+$, respectively. For the flat metrics in $M_+$ and $M_-$ we must admit arbitrariness of scales along the $z$ and $t$ axes to provide matching on $\Sigma_{\pm}$, whereas in $V$ with the metric (42)–(43) these scales have already been fixed. The angular frequencies $\Omega = \Omega_{\pm}$ can also be different in $M_+$ and $M_-$. So $M_{\pm}$ will be characterized by the metric coefficients

\begin{align*}
e^{2\beta} &= \frac{x^2}{1 - \Omega_{\pm}^2 x^2}, & e^\mu &= e^{2\gamma_{\pm}} = e^{2\gamma_{\pm}} (1 - \Omega_{\pm}^2 x^2), \\
e^\alpha &= 1, & \omega &= \frac{\Omega_{\pm}}{1 - \Omega_{\pm}^2 x^2}, & E &= \Omega_{\pm} x^2 e^{\gamma_{\pm}},
\end{align*}

where $\gamma_{\pm}$, $\mu_{\pm}$, $\Omega_{\pm}$ are constant parameters characterizing the regions $M_+$ and $M_-$. Moreover, in $M_+$ it is natural to put, as usual, $x > 0$, so that the range of $x$ is there $x_- < x < \infty$, but the junction can only be located in the stationary region, therefore, $x_+ < 1/\Omega_+$. Unlike that, in $M_-$ we put $x < 0$ to adjust the directions of the normal vector $n^\mu$ on both sides of the surface $\Sigma_-$, and we similarly have $|x_-| < 1/\Omega_-$.  

Now the task is to choose the surfaces $\Sigma_{\pm}$, to perform matching and to calculate the surface densities and pressures. It should be stressed again that it is quite unnecessary to adjust the choice
of the radial coordinate in different regions since all relevant quantities are reparametrization-invariant. One should only bear in mind using Eqs. (52)–(54) that the prime means a derivative with respect to the radial coordinate used in the corresponding region.

The matching conditions (46) on $\Sigma_{\pm}$ can be written as follows:

$$-mu = \mu_{\pm},$$  \hspace{1cm} (58)

$$2\omega_0 e^{2hu} s(k,u) = e^{2\gamma_{\pm}} (1 - \Omega^2 x^2),$$  \hspace{1cm} (59)

$$\frac{e^{2hu}}{2\omega_0 s(k,u)} = \frac{x^2}{1 - \Omega^2 x^2},$$  \hspace{1cm} (60)

where we have dropped the index $'\pm'$ with $u, x, \Omega$. We do not write explicitly the condition $[E] = 0$ but assume that it holds. The expression for $E$ in (42) contains the integration constant $E_0$, and its choice makes it easy to provide $[E] = 0$ on one of the surfaces $\Sigma_+$ or $\Sigma_-$. The same condition on the other surface then leads to one more constraint on the system parameters, which, however, does not affect our further reasoning.

If we assume, in addition (though it is not necessary), that the surface matter is at rest in our reference frame, we must put $[\omega] = 0$ which gives

$$\frac{e^{(m-2h)u}}{2s(k,u)} = \frac{\Omega}{1 - \Omega^2 x^2}.$$  \hspace{1cm} (61)

The conditions (58) and (59) fix the constants $\mu_{\pm}$ and $\gamma_{\pm}$ and do not affect other constants. So the only constraint connecting the parameters of the internal and external regions is (60).

Now, the main question is: Can both surface stress-energy tensors be physically plausible and non-exotic under some values of the system parameters? A criterion for that is the validity of the WEC which includes the requirements

$$\frac{S_{44}}{g_{44}} = \sigma \geq 0, \quad S_{ab}\xi^a\xi^b \geq 0,$$  \hspace{1cm} (62)

where $\xi^a$ is an arbitrary null vector ($\xi^a \xi_a = 0$) in $\Sigma = \Sigma_{\pm}$, i.e., the second inequality in (62) comprises the NEC. The conditions (62) are equivalent to

$$[\bar{K}_{44}/g_{44}] \leq 0, \quad [K_{ab}\xi^a\xi^b] \leq 0.$$  \hspace{1cm} (63)

Let us choose the following two null vectors on $\Sigma$ in the $z$ and $\phi$ directions:

$$\xi^a_{(1)} = (e^{-\mu}, 0, e^{-\gamma}),$$

$$\xi^a_{(2)} = (0, e^{-\beta}, e^{-\gamma} + E e^{-\beta - 2\gamma})$$  \hspace{1cm} (64)

(the components are enumerated in the order $a = 2, 3, 4$). Then the conditions (63) read

$$[e^{-\alpha}(\beta' + \mu')] \leq 0,$$

$$[e^{-\alpha}(\mu' - \gamma')] \leq 0,$$

$$[e^{-\alpha}(\beta' - \gamma') + 2\omega] \leq 0,$$  \hspace{1cm} (65)
respectively. Now we can apply these requirements to our configuration at both junctions. On Σ− with \( x_− < 0 \) we obtain
\[
e^{(m-2h)u} \left( -m + h - \frac{s'}{2s} \right) + \frac{1}{|x|(1 - \Omega^2 x^2)} \leq 0, \tag{66}
\]
\[
e^{(m-2h)u} \left( -m - h - \frac{s'}{2s} \right) + \frac{\Omega^2 |x|}{1 - \Omega^2 x^2} \leq 0, \tag{67}
\]
\[
e^{(m-2h)u} \left( 1 - \frac{s'}{s} \right) + \frac{(1 + \Omega x)^2}{|x|(1 - \Omega^2 x^2)} \leq 0 \tag{68}
\]
(where the label “minus” with the symbols \( u, x, \Omega \) is omitted), and on \( \Sigma_+ \) with \( x > 0 \) we have similarly
\[
e^{(m-2h)u} \left( m - h + \frac{s'}{2s} \right) + \frac{1}{x(1 - \Omega^2 x^2)} \leq 0, \tag{69}
\]
\[
e^{(m-2h)u} \left( m + h + \frac{s'}{2s} \right) + \frac{\Omega^2 x}{1 - \Omega^2 x^2} \leq 0 \tag{70}
\]
\[
e^{(m-2h)u} \left( \frac{s' - 1}{s} \right) + \frac{(1 + \Omega x)^2}{x(1 - \Omega^2 x^2)} \leq 0, \tag{71}
\]
(with the “plus” label omitted). Here \( s \) and \( s' = ds/du \) refer to the function \( s = s(k, u) \) introduced in (41).

The inequalities (66), (67), (69), (70) involve all the parameters \( m, h \) and \( k \) and are comparatively hard to explore, whereas (68) and (71) depend in essence on \( k \) only. It turns out that the latter two lead to the conclusion that the matter content of both \( \Sigma_+ \) and \( \Sigma_- \) cannot satisfy the NEC (hence also the WEC).

Indeed, (68) can hold only if \( 1 - s'(k, u) < 0 \) at \( u = u_- \). But \( s'(k, u) = \{ \cosh k, 1, \cos |k|u \} \) for \( k > 0 \), \( k = 0 \), and \( k < 0 \), respectively, and only at \( k > 0 \) we have \( 1 - s' < 0 \). Thus the NEC for \( \Sigma_- \) definitely requires \( k > 0 \) in the solution valid in \( \mathbb{V} \).

In a similar way, (71) can hold only if \( 1 - s'(k, u) > 0 \) at \( u = u_+ \), and this is only possible if \( k < 0 \). All this means that whatever particular solution (with fixed parameters including \( k \)) is taken to describe the space-time region \( \mathbb{V} \), the inequalities (68) and (71) cannot hold simultaneously. Hence the NEC is violated at least on one of the surfaces \( \Sigma_+ \) and \( \Sigma_- \).

### 7 Conclusion

The main purpose of the present study was to make clear whether or not rotating cylindrical wormholes can be obtained without exotic matter and whether or not such wormholes can be asymptotically flat\(^6\) on both sides of the wormhole throat. An answer to the first question is “yes” because the vortex gravitational field creates an effective SET with the exotic structure (23). Meanwhile, the probable answer to the second question is “no”.

It is well known that the nontrivial solution of the Laplace equation in flat space has a logarithmic asymptotic behavior (\( \sim \ln r \)) instead of vanishing at large \( r \). This simple fact actually

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\(^6\)Instead of asymptotic flatness, one could consider a “string” behavior at large radii, with a finite angular deficit or excess, but it is quite clear that our conclusions would be the same.
extends to solutions of the Einstein equations, beginning with the Levi-Civita static vacuum solution, and makes it difficult to inscribe cylindrical sources into a weakly curved environment. As we have seen, the problem is only enhanced if we invoke rotation.

We have tried to overcome this difficulty by considering a configuration that consists of (i) a strong-field region $V$ with the throat, described by the simplest rotating wormhole solution (in vacuum or with a massless scalar field), (ii) rotating thin shells $\Sigma_{\pm}$ placed on both sides of $V$, and (iii) two flat-space regions around the shells. It has turned out that the surface SET on $\Sigma_{\pm}$ inevitably violates the NEC at least on one of the shells (but maybe on both). Let us stress that this result has been obtained without any assumption on the angular velocity of the shells: they can be at rest in the rotating reference frame in which our system is considered or rotate with respect to it at an arbitrary rate.

Thus a twice asymptotically flat rotating cylindrical wormhole cannot be built without exotic matter even though rotation favors the formation of throats. This result has been obtained for vacuum and scalar-vacuum solutions in $V$, but there is a certain hope that it can change with another kind of non-exotic matter filling this internal region.

**Appendix: some solutions with nonzero $V(\phi)$**

We will outline here the way of obtaining exact solutions to the field equations for $V(\phi) \neq 0$, in the following cases:

- An exponential potential, $V(\phi) = V_0 e^{2\lambda \phi}$, $V_0, \lambda = \text{const}$.
- An arbitrary potential $V(\phi)$.

In the latter case, solutions can be obtained by the inverse problem method by specifying one of the metric functions.

A detailed analysis of these solutions and their possible usage for wormhole construction is postponed to future publications.

We are dealing with the metric (6) and Eqs. (33), where $\phi = \phi(u)$, the Ricci tensor components are given by (18)–(21) and $\omega = \omega_0 e^{-\mu - 2\gamma}$ according to (16).

Let us use, as in Sec. 4, the harmonic radial coordinate $u$, such that $\alpha = \beta + \gamma + \mu$. We have, as before, $R_3^3 = R_4^4$, which leads to the Liouville equation (40) and finally to (41), giving us the difference $\eta := \gamma - \beta$. Furthermore, we have $R_3^1 + R_4^1 = 2R_2^2$, whence

$$2\mu'' = \beta'' + \gamma'' \Rightarrow 2\mu = \beta + \gamma + au, \quad a = \text{const},$$

suppressing the second integration constant by choosing the scale along the $z$ axis. We also see that $\alpha'' = 3\mu'' = (3/2)(\beta + \gamma)'$. Thus all metric coefficients are expressed in terms of $\eta$ and $\alpha$, namely,

$$2\beta = \eta + \frac{1}{3}(2\alpha - au),$$
$$2\gamma = -\eta + \frac{1}{3}(2\alpha - au),$$
$$2\mu = \frac{1}{3}(2\alpha + 2au),$$

and the off-diagonal component $E$ is then obtained by combining (9) and (16) [where Eq. (9) should be rewritten in the form $2\omega = (E e^{-2\gamma})' e^{\gamma - \beta - a}$, allowing for an arbitrary $u$ gauge]:

$$(E e^{-2\gamma})' = 2\omega_0 e^{2\beta - 2\gamma}.$$  (A.3)
For the two remaining unknowns $\phi$ and $\alpha$, we can use the scalar field equation and the component $R_2^2 = ...$ of the Einstein equations that yield

\begin{align*}
\varepsilon e^{-2\alpha} \phi'' &= dV/d\phi, \\
e^{-2\alpha} \alpha'' &= -3\kappa V.
\end{align*}

(A.4)

(A.5)

It is also necessary to write the first-order constraint equation $G_1^1 = ...$ which has the form

\begin{align*}
\frac{1}{2} \kappa \varepsilon \phi'^2 &= \kappa V e^{2\alpha} + \omega_0^2 e^{\mu + 2\beta} + \beta' \gamma' + \beta' \mu' + \gamma' \mu'.
\end{align*}

(A.6)

If a solution is found by integrating the second-order Einstein equations, then (A.6) verifies the validity of this solution and yields a relation among the integration constants.

**An exponential potential and a cosmological constant.** For the potential $V(\phi) = V_0 e^{2\lambda \phi}$, Eqs. (A.4) and (A.5) read

\begin{align*}
\varepsilon \phi'' &= 2\lambda V_0 e^{2\lambda \phi + 2\alpha}, \\
\alpha'' &= -3\kappa V_0 e^{2\lambda \phi + 2\alpha}.
\end{align*}

(A.7)

(A.8)

Two combinations of Eqs. (A.7) and (A.8) are easily solved: one of them simply connects $\alpha''$ and $\phi''$ while the other is a Liouville equation for a linear combination of $\alpha$ and $\phi$:

\begin{align*}
3\varepsilon \kappa \phi'' + 2\lambda \alpha'' &= 0, \\
(\alpha + \lambda \phi)' &= (2\varepsilon \lambda^2 - 3\kappa) e^{2(\alpha + \lambda \phi)}. 
\end{align*}

(A.9)

(A.10)

It is then necessary to substitute the solution to (A.6) to find a relation among the integration constants.

The solutions are easily found for any values of $\varepsilon$, $V_0$ and $\lambda$. In particular, if we restrict ourselves to $\varepsilon = +1$ and $V_0 > 0$, we have three branches of solutions depending on the sign of $2\lambda^2 - 3\kappa$.

The case of a nonzero cosmological constant $\Lambda$ is simply the special case where the potential is constant: $\lambda = 0$, $\Lambda = \kappa V_0$.

**An arbitrary potential.** It is in general hard to solve Eqs. (A.4) and (A.5) when $V(\phi)$ is specified; however, specifying $\alpha(u)$, we can solve the problem completely. Indeed, for given $\alpha$, (A.5) readily gives $V(u)$, while $\phi(u)$ is found in terms of the metric coefficients (which are all known by now) from the first-order equation (A.6).

It remains to select solutions with physical properties of interest.

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