A QUANTUM ANALOGUE OF GENERIC BASES FOR AFFINE CLUSTER ALGEBRAS

MING DING AND FAN XU

Abstract. We construct quantized versions of generic bases in quantum cluster algebras of finite and affine types. Under the specialization of $q$ and coefficients to 1, these bases are generic bases of finite and affine cluster algebras.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in order to give a combinatorial framework for understanding total positivity in algebraic groups and dual canonical bases in quantum groups [21]. Cluster algebras relate closely to algebraic representation theory via two links. One is by cluster categories ([1]) and the Caldero-Chapoton map ([3]). The other is by stable module categories over preprojective algebras ([22]) and evaluation forms ([23]).

Let $Q = (Q_0, Q_1)$ be a finite quiver without oriented cycles where $Q_0 = \{1, 2, \ldots, n\}$ is the set of vertices and $Q_1$ is the set of arrows. The (coefficient-free) cluster algebra $A(Q)$ is the subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ with a set of generators (called cluster variables).

The cluster category $C_Q$ is the orbit category of $\mathcal{D}b(Q)$ by the functor $\tau \circ [-1]$ (see [1]). In [3], the authors introduced a map $X_M$ from the set of objects in $C_Q$ to $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$, called the Caldero-Chapoton map. There is a bijection between the set of isoclasses of indecomposable rigid objects (i.e., without nontrivial self-extensions) in $C_Q$ and the set of cluster variables in $A(Q)$ (see [5, Theorem 4]). Hence, the cluster algebra $A(Q)$ can be viewed as the subalgebra of $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$ generated by $X_M$ of indecomposable rigid objects $M$.

Quite differently, Geiss, Leclerc and Schröer ([22]) considered the preprojective algebra $\Lambda = \Lambda_Q$ of $Q$ and the category $\text{nil}(\Lambda)$ of finite dimensional nilpotent $\Lambda$-modules. They ([23]) attached to certain preinjective representations $M$ of $Q$ a Frobenius subcategory $C_M$ of $\text{nil}(\Lambda)$. If $M = I \oplus \tau(I)$ for the sum of indecomposable injective representations of $Q$, then the stable category $\hat{C}_M$ is triangle equivalent to the cluster category $C_Q$. Let $A(C_M)$ and $A(C_M)$ be the cluster algebra associated to $C_M$ and $C_M$, respectively.

Let $\Delta_d$ be the affine variety of nilpotent $\Lambda$-modules with dimension vector $d = (d_i)_{i \in Q_0}$. In [23, Section 1.2], the authors defined the evaluation form $\delta_M$ for $M \in \Delta_d$. Define $\langle M \rangle = \{N \in \Delta_d \mid \delta_M = \delta_N\}$. Let $Z$ be an irreducible component of $\Delta_d$. Then $M$ is a generic point in $Z$ if $\langle M \rangle \cap Z$ contains a dense open subset of $Z$. The involving dual semicanonical basis is just the set of $\delta_M$ by taking a generic point $M$ for each irreducible...
component \( Z \). Geiss, Leclerc and Schröer \([23], [24]\) proved that \( A(C_M) \) is spanned by a subset of the dual semicanonical basis and then specializes a basis of \( \hat{A}(C_M) \).

Analogously, Dupont \([13]\) introduced generic variables for \( A(Q) \). Let \( E_d \) be the affine variety of representations of \( Q \) with dimension vector \( d \). For \( M \in E_d \), define \( \langle M \rangle = \{ N \in E_d \mid X_M = X_N \} \). We say \( M \) is generic if \( \langle M \rangle \) contains a (dense) open subset of \( E_d \) and then \( X_d := X_M \) is called the generic variable for \( d \in \mathbb{N}^n \). The definition can be naturally extended to \( d \in \mathbb{Z}^n \). Set \( G(Q) = \{ X_d \mid d \in \mathbb{Z}^n \} \). Dupont conjectured the set \( G(Q) \) is the \( \mathbb{Z} \)-basis of \( A(Q) \) and confirmed it for type \( \tilde{A} \). Recently, the conjecture has been completely proved in \([25]\) (see also \([4]\) for finite type, \([11]\) for affine type). Therefore, the set \( G(Q) \) is called the generic basis of \( A(Q) \). In particular, for an affine quiver \( Q \), one obtain

\[
G(Q) = \{ X_M \mid M \text{ is rigid or } M \cong nE_δ \oplus R \text{ with regular rigid } R \text{ and regular simple } E_δ \}.
\]

Compared with (dual) canonical basis, one would like to define the quantized version of (dual) semicanonical basis for quantum groups. The aim of the present note is to construct the quantized version of generic bases for quantum cluster algebras of finite and affine types. Quantum cluster algebras were introduced by A. Berenstein and A. Zelevinsky \([2]\) as a noncommutative analogue of cluster algebras \([14],[15]\) to study canonical bases. A quantum cluster algebra is generated by a set of generators called the quantum cluster variables inside an ambient skew-field \( F \). Under the specialization, the quantum cluster algebras are exactly cluster algebras which were introduced by S. Fomin and A. Zelevinsky \([14],[15]\).

Let \( q \) be a formal variable and \( A_q(Q) \) be the quantum cluster algebra for \( Q \) (see Section 2 for more details). Recently, a quantum analog \( X_? \) (we use the same notation as above without causing confusion) of the Caldero-Chapoton map has been defined in \([28]\) and refined in \([20]\). In \([20]\), the author further showed that quantum cluster variables could be expressed as images of indecomposable rigid objects under the quantum Caldero-Chapoton map for acyclic equally valued quivers.

The note is organized as follows. In Section 2, we recall the definition of quantum cluster algebras and the quantum Caldero-Chapoton map. Section 3 is contributed to prove two multiplication formulas for acyclic quantum cluster algebras. The first formula (Theorem 3.3) is a quantum version of the formula \( X_{M \oplus N} = X_M X_N \) in cluster algebras. It is also a quantization of the degenerated form of Green’s formula (See \([10],[5]\)). We will apply this formula to show that the image of regular simple modules (over homogeneous tubes with minimal imaginary root \( δ \) as dimension vector) under the quantum Caldero-Chapoton map belong to quantum cluster algebras over \( Q \) of affine type in Section 5. The second formula (Theorem 3.5 and 3.8) generalizes the quantum cluster multiplication formula in \([20]\) to non-rigid objects. The generalization is essential for this note. We apply this formula to characterizing the regular modules over non-homogeneous tubes with dimension vector \( δ \) (Lemma 5.2). In Section 4, we construct the quantum generic basis of a quantum cluster algebra of finite type. By the specialization of \( q \) and coefficients to 1, the basis is no more than the good basis of a finite-type cluster algebra in \([1]\). The main results of this note are shown in Section 5 and 6. We give a basis of an affine-type quantum cluster algebras over \( Q \) in Section 5. By the specialization of \( q \) and coefficients to 1, the basis is just the generic basis of an affine-type cluster algebra in \([11]\) and \([13]\). In Section 6, we prove a formula for quantum cluster algebras of type \( \tilde{A} \) and \( \tilde{D} \). The formula characterizes the difference of regular modules of dimension vector \( δ \) in a homogeneous tube and a non-homogeneous
tube. It is the quantum version of the difference property in [11] and [13]. It helps us to refine the basis in Section 5 to integral bases.

2. The quantum Caldero-Chapoton map

2.1. Quantum cluster algebras. The main reference for quantum cluster algebras is [2]. Here, we also recommend [20] Section 2 as a nice reference. Let \( L \) be a lattice of rank \( m \) and \( \Lambda : L \times L \rightarrow \mathbb{Z} \) a skew-symmetric bilinear form. Let \( q \) be a formal variable and consider the ring of integer Laurent polynomials \( \mathbb{Z}[q^{\pm 1/2}] \). Define the \textit{based quantum torus} associated to the pair \((L, \Lambda)\) to be the \( \mathbb{Z}[q^{\pm 1/2}] \)-algebra \( \mathcal{T} \) with a distinguished \( \mathbb{Z}[q^{\pm 1/2}] \)-basis \( \{X^e : e \in L\} \) and the multiplication given by

\[
X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.
\]

It is easy to see that \( \mathcal{T} \) is associative and the basis elements satisfy the following relations:

\[
X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.
\]

It is known that \( \mathcal{T} \) is an Ore domain, i.e., is contained in its skew-field of fractions \( \mathcal{F} \). The quantum cluster algebra will be defined as a \( \mathbb{Z}[q^{\pm 1/2}] \)-subalgebra of \( \mathcal{F} \).

A \textit{toric frame} in \( \mathcal{F} \) is a map \( M : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\} \) of the form

\[
M(c) = \varphi(X^{\eta(e)})
\]

where \( \varphi \) is an automorphism of \( \mathcal{F} \) and \( \eta : \mathbb{Z}^m \rightarrow L \) is an isomorphism of lattices. By the definition, the elements \( M(c) \) form a \( \mathbb{Z}[q^{\pm 1/2}] \)-basis of the based quantum torus \( \mathcal{T}_M := \varphi(\mathcal{T}) \) and satisfy the following relations:

\[
M(c)M(d) = q^{\Lambda_M(c,d)/2} M(c + d), \quad M(c)M(d) = q^{\Lambda_M(c,d)} M(d)M(c), \quad M(0) = 1, \quad M(-c)^{-1} = M(c),
\]

where \( \Lambda_M \) is the skew-symmetric bilinear form on \( \mathbb{Z}^m \) obtained from the lattice isomorphism \( \eta \). Let \( \Lambda_M \) also denote the skew-symmetric \( m \times m \) matrix defined by \( \lambda_{ij} = \Lambda_M(e_i, e_j) \) where \( \{e_1, \ldots, e_m\} \) is the standard basis of \( \mathbb{Z}^m \). Given a toric frame \( M \), let \( X_i = M(e_i) \). Then we have

\[
\mathcal{T}_M = \mathbb{Z}[q^{\pm 1/2}](X_1^{\pm 1}, \ldots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i).
\]

An easy computation shows that

\[
M(c) = q^{\frac{1}{2} \sum_{i<j} c_i c_j \lambda_{ij}} X_1^{c_1} X_2^{c_2} \cdots X_m^{c_m} =: X^c \quad (c \in \mathbb{Z}^m).
\]

Let \( \Lambda \) be an \( m \times m \) skew-symmetric matrix and let \( \widetilde{B} \) be an \( m \times n \) matrix for some positive integer \( n \leq m \). We call the pair \((\Lambda, \widetilde{B})\) \textit{compatible} if \( \widetilde{B}^T \Lambda = (D|0) \) is an \( n \times m \) matrix with \( D = \text{diag}(d_1, \ldots, d_n) \) where \( d_i \in \mathbb{N} \) for \( 1 \leq i \leq n \). The pair \((M, \widetilde{B})\) is called a \textit{quantum seed} if the pair \((\Lambda_M, \widetilde{B})\) is compatible. Define the \( m \times m \) matrix \( E = (e_{ij}) \) by

\[
e_{ij} = \begin{cases} 
\delta_{ij} & \text{if } j \neq k; \\
-1 & \text{if } i = j = k; \\
\max(0, -b_{ik}) & \text{if } i \neq j = k.
\end{cases}
\]
For $n, k \in \mathbb{Z}, k \geq 0$, denote $[n]_q = \frac{(q^n - q^{-n}) \cdots (q^{n-1} - q^{-n+1})}{(q^k - q^{-k}) \cdots (q^{-k+1} - q^k)}$. Let $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M': \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ as follows:

$$M'(c) = \sum_{p=0}^{c_k} \left[ \begin{array}{c} c_k \\ p \end{array} \right] q^{pk} M(\mathcal{E}c + pb^k), \quad M'(-c) = M'(c)^{-1}. \tag{2.1}$$

where the vector $b^k \in \mathbb{Z}^m$ is the $k$–th column of $\tilde{B}$. Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in direction $k$. In general, two quantum seeds $(M, \tilde{B})$ and $(M', \tilde{B}')$ are mutation-equivalent if they can be obtained from each other by a sequence of mutations, denoted by $(M, \tilde{B}) \sim (M', \tilde{B}')$. Let $\mathcal{C} = \{ M'(e_i) \mid (M, \tilde{B}) \sim (M', \tilde{B}'), i = 1, \ldots, n \}$. The elements of $\mathcal{C}$ are called quantum cluster variables. Let $\mathcal{P} = \{ M(e_{i_j}) : i_j = n + 1, \ldots, m \}$ and the elements of $\mathcal{P}$ are called coefficients. Given $(M', \tilde{B}') \sim (M, \tilde{B})$ and $c = (c_i) \in \mathbb{Z}^m$, a element $M'(c)$ is called a quantum cluster monomial if $c_i \geq 0$ for $i = 1, \ldots, n$ and 0 for $i = n+1, \ldots, m$. We denote by $\mathbb{P}$ the multiplicative group by $q^{\mathbb{Z}}$ and $\mathcal{P}$. Write $\mathbb{Z}\mathbb{P}$ as the ring of Laurent polynomials in the elements of $\mathcal{P}$ with coefficients in $\mathbb{Z}[q^{\pm 1/2}]$. Write $\mathbb{Q}\mathbb{P}$ as the ring of Laurent polynomials in the elements of $\mathcal{P}$ with coefficients in $\mathbb{Q}[q^{\pm 1/2}]$. The quantum cluster algebra $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{C}$. We associate $(M, \tilde{B})$ a $\mathbb{Z}$-linear bar-involution on $\mathcal{T}_M$ defined by

$$q^{r/2} M(c) = q^{-r/2} M(c), \quad (r \in \mathbb{Z}, \ c \in \mathbb{Z}^n).$$

It is easy to show that $\bar{XY} = \bar{X} \bar{Y}$ for all $X, Y \in \mathcal{A}_q(\Lambda_M, \tilde{B})$ and that each element of $\mathcal{C} \cup \mathcal{P}$ is bar-invariant.

Now assume that there exists a finite field $k$ satisfying $|k| = q$. In the same way, we can define based quantum torus $\mathcal{T}_{[k]}$ and specialized quantum cluster algebras $\mathcal{A}_{[k]}(\Lambda_M, \tilde{B})$ by substituting $\mathbb{Z}[[k]^{\pm 1/2}]$ for $\mathbb{Z}[q^{\pm 1/2}]$ in the above definition. By [2] Corollary 5.2, $\mathcal{A}_q(\Lambda_M, \tilde{B})$ and $\mathcal{A}_{[k]}(\Lambda_M, \tilde{B})$ are subalgebras of $\mathcal{T}$ and $\mathcal{T}_{[k]}$, respectively. There is a specialization map $ev : \mathcal{T} \to \mathcal{T}_{[k]}$ by mapping $q^{1/2}$ to $|k|^{1/2}$, which induces a bijection between quantum monomials of $\mathcal{A}_q(\Lambda_M, \tilde{B})$ and $\mathcal{A}_{[k]}(\Lambda_M, \tilde{B})$ ([20 Section 2.2]).

### 2.2. The quantum Caldero-Chapoton map

Let $k$ be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and $\tilde{Q}$ an acyclic quiver with vertex set $\{1, \ldots, m\}$. Denote the subset $\{n+1, \ldots, m\}$ by $C$. The elements in $C$ are called the frozen vertices , and $\tilde{Q}$ is called an ice quiver. The full subquiver $Q$ on the vertices $1, \ldots, n$ is called the principal part of $\tilde{Q}$.

Let $\tilde{B}$ be the $m \times n$ matrix associated to the ice quiver $\tilde{Q}$, i.e., its entry in position $(i, j)$ is

$$b_{ij} = |\{ \text{arrows } i \to j \}| - |\{ \text{arrows } j \to i \}|$$

for $1 \leq i \leq m, 1 \leq j \leq n$. And let $\bar{I}$ be the left $m \times n$ matrix of the identity matrix of size $m \times m$. Further assume that there exists some antisymmetric $m \times m$ integer matrix $\Lambda$ such that

$$\Lambda(-\tilde{B}) = \bar{I} := \begin{bmatrix} I_{m} \\ 0 \end{bmatrix},$$

where $I_n$ is the identity matrix of size $n \times n$. Thus, the matrix $\tilde{B}$ is of full rank.
Let $\tilde{R}$ and $\tilde{R}^{\text{tr}}$ be the $m \times n$ matrix with its entry in position $(i, j)$ is
\[ \tilde{r}_{ij} = \dim_k \Ext^1_{k\tilde{Q}}(S_i, S_j) \]
and
\[ \tilde{r}^*_{ij} = \dim_k \Ext^1_{k\tilde{Q}}(S_j, S_i) \]
for $1 \leq i \leq m$, $1 \leq j \leq n$, respectively. Note that
\[ \dim_k \Ext^1_{k\tilde{Q}}(S_i, S_j) = |\{\text{arrows } j \to i\}|. \]

Denote the principal parts of the matrices $\tilde{B}$ and $\tilde{R}$ by $B$ and $R$ respectively. Note that
\[ \tilde{B} = \tilde{R}^{\text{tr}} - \tilde{R} \quad \text{and} \quad B = R^{\text{tr}} - R \]
where $R^{\text{tr}}$ represents the transposition of the matrix $R$. In general, the matrix $B$ is not of full rank so that there exists no matrix $\Lambda$ compatible with $B$. Hence, one need add some frozen vertices to $Q$ and then obtain an acyclic quiver $\tilde{Q}$ with a compatible pair $(\tilde{B}, \Lambda)$.

Let $\tilde{Q}$ be the cluster category of $k\tilde{Q}$, i.e., the orbit category of the derived category $D^b(\tilde{Q})$ by the functor $F = \tau \circ [-1]$ where $\tau = \tau_{\tilde{Q}}$ is the Auslander-Reiten translation and $[1]$ is the translation functor. We note that the indecomposable objects of the cluster category $C_{\tilde{Q}}$ are either the indecomposable $k\tilde{Q}$-modules or $P_i[1]$ for indecomposable projective modules $P_i (1 \leq i \leq m)$. Each object $M$ in $C_{\tilde{Q}}$ can be uniquely decomposed in the following way:
\[ M \cong M_0 \oplus P_M[1] \]
where $M_0$ is a $k\tilde{Q}$-module and $P_M$ is a projective $k\tilde{Q}$-module. Let $P_M = \bigoplus_{1 \leq i \leq m} m_i P_i$.

We extend the definition of the dimension vector $\dim$ on modules in $\text{mod} k\tilde{Q}$ to objects in $C_{\tilde{Q}}$ by setting
\[ \dim M = \dim M_0 - (m_i)_{1 \leq i \leq m}. \]

The Euler form on $k\tilde{Q}$-modules $M$ and $N$ is given by
\[ \langle M, N \rangle = \dim_k \Hom_{k\tilde{Q}}(M, N) - \dim_k \Ext^1_{k\tilde{Q}}(M, N). \]

Note that the Euler form only depends on the dimension vectors of $M$ and $N$. As in [18], we define
\[ [M, N] = \dim_k \Hom_{k\tilde{Q}}(M, N) \quad \text{and} \quad [M, N]^1 = \dim_k \Ext^1_{k\tilde{Q}}(M, N). \]

The quantum Caldero-Chapoton map of an acyclic quiver $\tilde{Q}$ has been studied in [28] and [20]. Here, we reformulate their definitions to the following map
\[ X_{\tilde{Q}} : \text{obj} C_{\tilde{Q}} \longrightarrow \mathcal{T} \]
defined by the following rule: If $M$ is a $k\tilde{Q}$-module and $P$ is a projective $k\tilde{Q}$-module, then
\[ X_{\tilde{Q}}^M[P] = \sum_{\underline{e}} |\text{Gr}_e M[q^{-\frac{1}{2} \langle m, e \rangle - \langle e, m \rangle}] X^{\tilde{B} e^{-1}}(\tilde{i} - \tilde{R}) \dim P / \text{rad} P, \]
where $\dim M = \underline{m}$ and $\text{Gr}_e M$ denotes the set of all submodules $V$ of $M$ with $\dim V = e$. Usually, we omit the upper index $\tilde{Q}$ in the notation $X_{\tilde{Q}}$ (except Section 4 and Section 5).
It should not cause any confusion. We note that
\[ X_{P[1]} = X_{rP} = X_{\dim P/rad P} = X_{\dim soc I} = X_{I[-1]} = X_{r^{-1}I}. \]
for any projective \( k\overline{Q}\)-module \( P \) and injective \( k\overline{Q}\)-module \( I \) with \( \text{soc} I = P/rad P \). Hereinafter, we denote by the corresponding underlined small letter \( \underline{x} \) the dimension vector of a \( kQ \)-module \( X \) and view \( \underline{x} \) as a column vector in \( \mathbb{Z}^n \).

3. Multiplication theorems for acyclic quantum cluster algebras

Throughout this section, assume that \( \overline{Q} \) is an acyclic quiver and \( Q \) is its full subquiver. In this section, we will prove a multiplication theorem for any acyclic quantum cluster algebra. First, we improve Lemma 5.2.1 and Corollary 5.2.2 in [20], i.e., here we handle the dimension vector of any \( kQ \)-module while in [20] the author only deals with dimension vectors of rigid modules.

Lemma 3.1. For any dimension vector \( m, e, f \in \mathbb{Z}^n_{\geq 0} \), we have

\begin{align*}
(1) \quad & \Lambda((\overline{I} - \overline{R})m, \overline{B}e) = -\langle e, m \rangle; \\
(2) \quad & \Lambda(\overline{B}e, \overline{B}f) = \langle e, f \rangle - \langle f, e \rangle.
\end{align*}

Proof. By definition, we have
\[
\Lambda((\overline{I} - \overline{R})m, \overline{B}e) = m^\text{tr} (\overline{I} - \overline{R})^\text{tr} \overline{B}e = -m^\text{tr} (I_n - R)^\text{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} e = -m^\text{tr} (I_n - R)^\text{tr} e = -e^\text{tr} (I_n - R)m = -\langle e, m \rangle.
\]

As for (2), the left side of the desired equation is equal to
\[
e^\text{tr} \overline{B}^\text{tr} \overline{B}f = -e^\text{tr} \overline{B}^\text{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} f = -e^\text{tr} \overline{B}^\text{tr} f.
\]

The right side is
\[
\langle e, f \rangle - \langle f, e \rangle = e^\text{tr} (I_n - R)f - f^\text{tr} (I_n - R)e
\]
\[
= e^\text{tr} (I_n - R)f - e^\text{tr} (I_n - R)^\text{tr} f
\]
\[
= e^\text{tr} (R^\text{tr} - R)f = -e^\text{tr} (R - R^\text{tr})f = -e^\text{tr} \overline{B}^\text{tr} f.
\]

Thus we prove the lemma. \( \square \)

Corollary 3.2. For any dimension vector \( m, l, e, f \in \mathbb{Z}^n_{\geq 0} \), we have
\[
\Lambda(\overline{B}e - (\overline{I} - \overline{R})m, \overline{B}f - (\overline{I} - \overline{R})l) = \Lambda((\overline{I} - \overline{R})m, (\overline{I} - \overline{R})l) + \langle e, f \rangle - \langle f, e \rangle - \langle e, l \rangle + \langle f, m \rangle.
\]
Theorem 3.3. Let $M$ and $N$ be $kQ$-modules. Then
\[ q^{[M,N]} X_N X_M = q^{-\frac{1}{2} \Lambda((I-R)\mu,(I-R)\mu)} \sum_E \varepsilon_{MN} X_E. \]

Proof. We apply Green’s formula in [16]
\[ \sum_E \varepsilon_{MN} F_{XY}^E = \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-(A,D)} F_{AB}^M F_{CD}^N \varepsilon_{AC} \varepsilon_{BD}. \]

Then
\[ \sum_E \varepsilon_{MN} X_E \]
\[ = \sum_{E,X,Y} \varepsilon_{MN} q^{-\frac{1}{2}(Y,X)} F_{XY}^E X_B(I-R)\mu \]
\[ = \sum_{A,B,C,D,X,Y} q^{[M,N]-[A,C]-[B,D]-(A,D)-\frac{1}{2}(B+D,A+C)} F_{AB}^M F_{CD}^N \varepsilon_{AC} \varepsilon_{BD} X_B(I-R)\mu. \]

Since
\[ X_B(I-R)\mu = X^{B_{(\mu+\lambda)-(I-R)\mu}} \]
\[ = q^{-\frac{1}{2} \Lambda((B_{(\mu+\lambda)-(I-R)\mu})^\mu, B_{(I-R)\mu})} X^{B_{(I-R)\mu}} X^{B_{(I-R)\mu}} \]
\[ = q^{-\frac{1}{2} \Lambda((I-R)\mu, (I-R)\mu)} q^{\frac{1}{2}(D,A)-\frac{1}{2}(B,C)} X^{B_{(I-R)\mu}} X^{B_{(I-R)\mu}} \]
\[ = q^{-\frac{1}{2} \Lambda((I-R)\mu, (I-R)\mu)} q^{\frac{1}{2}(D,A)-\frac{1}{2}(B,C)} X^{B_{(I-R)\mu}} X^{B_{(I-R)\mu}}. \]

Thus
\[ \sum_E \varepsilon_{MN} X_E \]
\[ = q^{\frac{1}{2} \Lambda((I-R)\mu,(I-R)\mu)} \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-(A,D)-\frac{1}{2}(B+D,A+C)+[A,C]^1+[B,D]^1}. \]
\[ q^{\frac{1}{2}(D,A)-\frac{1}{2}(B,C)} F_{AB}^M F_{CD}^N X^{B_{(I-R)\mu}} X^{B_{(I-R)\mu}}. \]
Here we use the following fact
\[
\sum_X \varepsilon_X Y_C = q^{[A,C]^1}, \quad \sum_Y \varepsilon_Y X_D = q^{[B,D]^1}
\]

Note that
\[
[M, N] - [A, C] - [B, D] - (A, D) + [A, C]^1 + [B, D]^1 = [M, N]^1 + (B, C).
\]

Hence
\[
\sum_E \varepsilon_{E E} X_E = q^{|A|} \Lambda((\tilde{I} - \tilde{R})_{M, N}((\tilde{I} - \tilde{R})_{M, N}) M_{M, N}) q^{[M, N]}
\]

\[
= q^{|A|} X_E \Lambda((\tilde{I} - \tilde{R})_{M, N}((\tilde{I} - \tilde{R})_{M, N}) M_{M, N}) q^{[M, N]^1}
\]

This completes the proof.

\[\square\]

**Remark 3.4.** Theorem 3.3 is similar to the multiplication formula in dual Hall algebras. It is reasonable to conjecture that it provides some PBW-type basis ([17]) in the corresponding quantum cluster algebra.

Let \(M, N\) be \(kQ\)-modules and assume that
\[
\dim_k \text{Ext}_1^{kQ}(M, N) = \dim_k \text{Hom}_{kQ}(N, \tau M) = 1.
\]

Then there are two “canonical” exact sequences
\[
\varepsilon : \quad 0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0
\]
\[
\varepsilon' : \quad 0 \rightarrow D_0 \rightarrow N \rightarrow \tau M \rightarrow \tau A \oplus I \rightarrow 0
\]

which induces the \(k\)-bases of \(\text{Ext}_1^{kQ}(M, N)\) and \(\text{Hom}_{kQ}(N, \tau M)\), respectively. We fix them.

Set \(M = M' \oplus P_0, A_0 = A \oplus P_0\) where \(P_0\) is a projective \(kQ\)-module, \(A\) and \(M'\) have no projective summands. The exact sequences also provide the two non-split triangles in \(\mathcal{C}_Q\):
\[
N \rightarrow E \rightarrow M \rightarrow N[1] = \tau N
\]
and
\[
M \rightarrow D_0 \oplus A_0 \oplus I[-1] \rightarrow N \rightarrow \tau M.
\]

Now we state the first part of our multiplication theorem for acyclic quantum cluster algebras, which can be viewed as a quantum analogue of the one-dimensional Caldero-Keller multiplication theorem in [5]. The main idea in the proof comes from [18].

**Theorem 3.5.** With the above notation, assume that \(\text{Hom}_{k\tilde{Q}}(D_0, \tau A_0 \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0\). Then the following formula holds
\[
X_N X_M = \sum \sum \varepsilon_X Y_C X_D
\]

\[
= \left( \frac{q^{[M, N]^1} - 1}{q - 1} \right) X_N X_M.
\]

Here, we note that
\[
q^{[M, N]^1} - 1
\]

\[
= q^{[M, N]^1} X_N X_M.
\]
Proof. By definition, we have
\[
X_NX_M = \sum_{C,D} q^{-\frac{1}{2}(D,C)} F_{CD}^N X^{\tilde{B}}_D - (\tilde{I} - \tilde{R})_M \sum_{A,B} q^{-\frac{1}{2}(B,A)} F_{AB}^M X^{\tilde{B}}_A - (\tilde{I} - \tilde{R})_M
\]
\[
= \sum_{A,B,C,D} F_{AB}^M F_{CD}^N q^{-\frac{1}{2}(D,C) - \frac{1}{2}(B,A) + \frac{1}{2}N(\tilde{B}_D - (\tilde{I} - \tilde{R})_M \tilde{B}_A - (\tilde{I} - \tilde{R})_M)} X^{\tilde{B}}_{D} - (\tilde{I} - \tilde{R})(m+n)
\]
\[
= \sum_{A,B,C,D} F_{AB}^M F_{CD}^N q^{-\frac{1}{2}(B+D,A+C)} q^{B,C} q^{\frac{1}{2}N(\tilde{I} - \tilde{R})_M \tilde{B}}_{D} - (\tilde{I} - \tilde{R})(m+n)
\]
We set
\[
s_1 := \sum_{E \not\equiv M \oplus N} \frac{\varepsilon_{EM}}{q-1} X_E = \sum_{X,Y \not\equiv E \oplus M \oplus N} \frac{\varepsilon_{EM}}{q-1} F_{XY}^E q^{-\frac{1}{2}(Y,X)} X^{\tilde{B}}_Y - (\tilde{I} - \tilde{R})_E
\]
As in the proof of Theorem 3.3 we have
\[
\sum_{X,Y \not\equiv E \oplus M \oplus N} \frac{\varepsilon_{EM}}{q-1} X_E
\]
\[
= \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-(A,D)-\frac{1}{2}(B+D,A+C)} F_{AB}^M F_{CD}^N X^{Y} \tilde{B}_D - (\tilde{I} - \tilde{R})_E
\]
\[
= \sum_{A,B,C,D} q^{[M,N]+(B,C)-\frac{1}{2}(B+D,A+C)} F_{AB}^M F_{CD}^N X^{\tilde{B}}_{D} - (\tilde{I} - \tilde{R})_E
\]
On the other hand
\[
X_{M \oplus N} = \sum_{A,B,C,D} q^{[B,C]-\frac{1}{2}(B+D,A+C)} F_{AB}^M F_{CD}^N X^{\tilde{B}}_{D} - (\tilde{I} - \tilde{R})(m+n)
\]
Thus
\[
s_1 = \sum_{A,B,C,D} q^{[M,N]} - q^{[B,C]} \frac{1}{q-1} q^{B,C} q^{\frac{1}{2}(B+D,A+C)} F_{AB}^M F_{CD}^N X^{\tilde{B}}_{D} - (\tilde{I} - \tilde{R})(m+n).
\]
Thirdly we compute the term
\[
s_2 := \sum_{A,D_0,I,D_0 \not\equiv N} \frac{|\text{Hom}_{kQ}(N,\tau M)_{D_0 A I}|}{q-1} X_{A_0 \oplus D_0 \oplus I \oplus [-1]}.
\]
Here, we use the following notation as in [18]
\[
\text{Hom}_{kQ}(N,\tau M)_{D_0 A I} := \{ f \not= N \rightarrow \tau M | \ker f \cong D_0, \coker f \cong \tau A \oplus I \}.
\]
Note that \( \dim_k \text{Hom}_{kQ}(N,\tau M) = 1 \), we have the following exact sequences
\[
0 \rightarrow B_0 \rightarrow M \rightarrow A_0 \rightarrow 0
\]
\[
0 \rightarrow C \rightarrow \tau B_0 \rightarrow I \rightarrow 0
\]
where \( C = \text{im} f, \ker f = D_0 \).
Here, \( Y = D, K = A, B = B_0 + L \) in the above expression and the equality can be illustrated by the following diagram:

\[
\begin{array}{ccccccccc}
Y & \downarrow & Y & \downarrow & \tau A & \downarrow & \tau A & \downarrow & 0 \\
0 & \rightarrow & D_0 & \rightarrow & N & \rightarrow & \tau M & \rightarrow & \tau A_0 \oplus I & \rightarrow & 0 \\
0 & \rightarrow & X & \rightarrow & C & \rightarrow & \tau B & \rightarrow & \tau L \oplus I & \rightarrow & 0 \\
\end{array}
\]

We must to check the relation between
\[
-\frac{1}{2} \langle Y + L, K + X \rangle + [L, X]
\]
and
\[
-\frac{1}{2} \langle B + D, A + C \rangle + \langle B, C \rangle.
\]

In this case, note that \( D = Y, L = A_0 - A, K = A, [L, X] = [X, \tau L] = 0 \). We have
\[
-\frac{1}{2} \langle Y + L, K + X \rangle + [L, X] = -\frac{1}{2} \langle Y + L, K + X \rangle + \langle L, X \rangle
\]
\[
= -\frac{1}{2} \langle D + A_0 - A, A + D_0 - D \rangle + \langle A - A_0, D_0 - D \rangle
\]
\[
= -\frac{1}{2} \langle D, A \rangle - \frac{1}{2} \langle D, D_0 \rangle + \frac{1}{2} \langle D, D \rangle - \frac{1}{2} \langle A_0, A \rangle + \frac{1}{2} \langle A_0, D_0 \rangle
\]
\[
-\frac{1}{2} \langle A_0, D \rangle + \frac{1}{2} \langle A, A \rangle - \frac{1}{2} \langle A, D_0 \rangle + \frac{1}{2} \langle A, D \rangle.
\]

And
\[
-\frac{1}{2} \langle B + D, A + C \rangle + \langle B, C \rangle
\]
\[
= -\frac{1}{2} \langle M - A + D, A + N - D \rangle + \langle M - A, N - D \rangle
\]
\[
= -\frac{1}{2} \langle M, A \rangle - \frac{1}{2} \langle M, D \rangle + \frac{1}{2} \langle A, A \rangle - \frac{1}{2} \langle A, N \rangle + \frac{1}{2} \langle A, D \rangle
\]
\[
-\frac{1}{2} \langle D, A \rangle - \frac{1}{2} \langle D, N \rangle + \frac{1}{2} \langle D, D \rangle + \frac{1}{2} \langle M, N \rangle.
\]
Hence it is equivalent to compare
\[-\frac{1}{2}\langle D, D_0 \rangle - \frac{1}{2}\langle A_0, A \rangle + \frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle A, D \rangle - \frac{1}{2}\langle A, D_0 \rangle\]
and
\[-\frac{1}{2}\langle D, N \rangle - \frac{1}{2}\langle M, A \rangle + \frac{1}{2}\langle M, N \rangle - \frac{1}{2}\langle M, D \rangle - \frac{1}{2}\langle A, N \rangle.\]
We claim that
\[\langle D, N \rangle + \langle M, D \rangle = \langle D, D_0 \rangle + \langle A_0, D \rangle\]
and
\[\langle A_0, A \rangle + \langle A, D_0 \rangle = \langle M, A \rangle + \langle A, N \rangle.\]
Indeed, we have
\[\langle D, N - D_0 \rangle = \langle D, \tau M - \tau A_0 - I \rangle = \langle D, \tau M - \tau A_0 \rangle = \langle A_0 - M, D \rangle.\]
In the same way, we also have
\[\langle A, N - D_0 \rangle = \langle A_0 - M, A \rangle.\]
Thus
\[s_2 = q^{\frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle M, N \rangle} \sum_{A, B, C, D} q^{[B,C]_1} q^{\frac{1}{2}\langle B, C \rangle - \frac{1}{2}\langle B + D, A + C \rangle} F_A F_B F_C F_D X_{\tilde{L}((\tilde{I} - \tilde{R})(m + n))}.\]
Therefore we have the following multiplication formula
\[X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m} X_E + q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m+\frac{1}{2}(M,N)-\frac{1}{2}\langle A_0, D_0 \rangle} X_{D_0 \oplus A_0 \oplus I[-1]}.\]

There are three canonical special cases satisfying the assumption $\text{Hom}_{k\tilde{Q}}(D_0, \tau A \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0$ in Theorem 3.5.

**Special case I.** Assume that $A_0 = 0 = I$. Then $L = K = 0 = A$. If $B \neq M$, i.e., $B \nsubseteq M$, then there exists $f_1 : N \longrightarrow \tau M$ induced by the above diagram which is not surjective. It is a contradiction to the assumption $\dim_k \text{Hom}_{k\tilde{Q}}(N, \tau M) = 1$. In this case, the multiplication formula is
\[X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m} X_E + q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m+\frac{1}{2}(M,N)} X_{D_0}.\]

**Special case II.** Assume that $D_0 = 0$ and $\text{Hom}_{k\tilde{Q}}(A_0, I) = 0$. Then $Y = X = 0, C = N$. In this case, the multiplication formula is
\[X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m} X_E + q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})m)(\tilde{I} - \tilde{R})m+\frac{1}{2}(M,N)} X_{A_0 \oplus I[-1]}.\]

**Special case III.** Assume that $M, N$ are indecomposable rigid $kQ$-mod and
\[\dim_k \text{Ext}_{C\tilde{Q}}^1(M, N) = 1.\]
Since $D_0 \oplus A_0 \oplus I[-1]$ is rigid, then the assumption $\text{Hom}_{k\tilde{Q}}(D_0, \tau A \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0$ in Theorem 3.5 holds.

**Lemma 3.6.** With the assumption in Special case III, we have $\frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle M, N \rangle = \frac{1}{2}$. 
Proof. Note that we have
\[ \frac{1}{2} \langle A_0, D_0 \rangle - \frac{1}{2} \langle M, N \rangle = \frac{1}{2} \langle A_0, N - N/D_0 \rangle - \frac{1}{2} \langle M, N \rangle. \]

We need to confirm the two equations
(1) \( \langle M, N \rangle = \langle A_0, N \rangle - 1 \) and
(2) \( \langle A_0, N/D_0 \rangle = 0. \)

Note that \( A_0 \oplus N \) is rigid, thus \( [A_0, N]^1 = 0. \) We have the following exact sequences
\[ 0 \to N/D_0 \to \tau M \to \tau A_0 \oplus I \to 0 \]
\[ 0 \to D_0 \to N \to N/D_0 \to 0 \]

Applying the functor \( \text{Hom}_{k\tilde{Q}}(N, -) \), we have the following exact sequences
\[ [N, N/D_0]^1 \to [N, \tau M]^1 \to [N, \tau A_0 \oplus I]^1 \to 0 \]
\[ [N, N]^1 \to [N, N/D_0]^1 \to 0. \]

Hence
\[ \langle M, N \rangle = [M, N] - 1 = [A_0, N] - 1 = \langle A_0, N \rangle - 1. \]

As for the second equation, apply the functor \( \text{Hom}_{k\tilde{Q}}(A_0, -) \) to the exact sequence
\[ 0 \to D_0 \to N \to N/D_0 \to 0 \]
We have the following exact sequence
\[ [A_0, N]^1 \to [A_0, N/D_0]^1 \to 0 \]
Thus \( [A_0, N/D_0]^1 = 0. \) Applying the functor \( \text{Hom}_{k\tilde{Q}}(\tau M, -) \) to the exact sequence
\[ 0 \to N/D_0 \to \tau M \to \tau A_0 \oplus I \to 0 \]
We have the following exact sequence
\[ [\tau M, \tau M]^1 \to [\tau M, \tau A_0 \oplus I]^1 \to 0 \]
Thus we have
\[ [M, A_0]^1 = [A_0, \tau M] = 0. \]
Again applying the functor \( \text{Hom}_{k\tilde{Q}}(A_0, -) \), we have the exact sequence
\[ 0 \to [A_0, N/D_0] \to [A_0, \tau M] = 0 \]
Hence \( [A_0, N/D_0] = 0. \)

By Lemma 3.6, we obtain the following multiplication theorem between quantum cluster variables in \( [20] \).

Corollary 3.7. Let \( M \) and \( N \) be indecomposable rigid \( kQ \)-modules and \( \dim_k \text{Ext}^1_{\tilde{Q}}(M, N) = 1. \) Let
\[ N \to E \to M \to N[1] = \tau N \]
and
\[ M \to D_0 \oplus A_0 \oplus I[-1] \to N \to \tau M \]
be two non-split triangles in \( \tilde{Q} \) as above. Then we have
\[ X_N X_M = q^{-\frac{1}{2}} \Lambda(\tilde{I}-\tilde{R}) \mu(\tilde{I}-\tilde{R}) \nu X_E + q^{-\frac{1}{2}} \Lambda(\tilde{I}-\tilde{R}) \Lambda(\tilde{I}-\tilde{R}) \mu(\tilde{I}-\tilde{R}) \nu \mu X_{D_0 \oplus A_0 \oplus I[-1]}. \]
Now let $M$ be a $kQ$-module and $P$ be a projective $k\tilde{Q}$-module with $[P, M] = [M, I] = 1$, where $I = \nu(P)$, here $\nu = \text{DHom}_{k\tilde{Q}}(-, k\tilde{Q})$ is the Nakayama functor. It is well-known that $I$ is an injective module with $\text{soc} I = P/\text{rad} P$. Fix two nonzero morphisms $f \in \text{Hom}_{k\tilde{Q}}(P, M)$ and $g \in \text{Hom}_{k\tilde{Q}}(M, I)$. The two morphisms induce the following exact sequences

\[ 0 \longrightarrow P' \longrightarrow P \overset{f}{\longrightarrow} M \longrightarrow A \longrightarrow 0 \]

and

\[ 0 \longrightarrow B \longrightarrow M \overset{g}{\longrightarrow} I \longrightarrow I' \longrightarrow 0 . \]

These correspond to two non-split triangles in $C_{\tilde{Q}}$

$M \rightarrow E' \rightarrow P[1] \rightarrow M[1]$ and

$I[-1] \rightarrow E \rightarrow M \rightarrow I$,

respectively, where $E \simeq B \oplus I'[-1]$ and $E' \simeq A \oplus P'[1]$.

Now we state the second part of our multiplication theorem for acyclic quantum cluster algebras.

**Theorem 3.8.** With the above notations, assume that $[B, I'] = [P', A] = 0$. Then we have

\[ X_{\tau P} X_M = q^{\frac{1}{2} \Lambda(\dim P/\text{rad} P, - (\tilde{I} - \tilde{R})_m)} X_E + q^{\frac{1}{2} \Lambda(\dim P/\text{rad} P, - (\tilde{I} - \tilde{R})_m) - \frac{1}{2}} X_{E'} . \]

**Proof.**

\[ X_{\tau P} X_M = X^{\dim P/\text{rad} P} \sum_{G, H} q^{-\frac{1}{2} \langle H, G \rangle} F_{GH}^M \tilde{B}h - (\tilde{I} - \tilde{R})_m \]

\[ = \sum_{G, H} q^{-\frac{1}{2} \langle H, G \rangle} F_{GH}^M q^{\frac{1}{2} \Lambda(\dim P/\text{rad} P, \tilde{B}h - (\tilde{I} - \tilde{R})_m)} X^{\tilde{B}h - (\tilde{I} - \tilde{R})_m + \dim P/\text{rad} P} \]

\[ = q^{\frac{1}{2} \Lambda(\dim P/\text{rad} P, - (\tilde{I} - \tilde{R})_m)} \sum_{G, H} q^{-\frac{1}{2} \langle H, G \rangle} q^{-\frac{1}{2} \langle P, H \rangle} F_{GH}^M \tilde{B}h - (\tilde{I} - \tilde{R})_m + \dim P/\text{rad} P . \]

Here we use the following fact

\[ \Lambda(\dim P/\text{rad} P, \tilde{B}h) = (\dim P/\text{rad} P)^{tr} \Lambda \tilde{B}h = - (\dim P/\text{rad} P)^{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} h = - [P, H] . \]

Note that by assumption $[P, M] = 1$, we have that $[P, H] = 0$ or $1$.

We firstly compute the term

\[ X_E = X_{B \oplus I'[-1]} = \sum_{X, Y} q^{-\frac{1}{2} \langle X, Y \rangle} F_{XY}^B \tilde{B}h - (\tilde{I} - \tilde{R})_h + \dim \text{soc} I' . \]
We have the following diagram

```
  0    0
 / \   / \  
 Y   Y  0  0
 /   \  /   \  
 M   I  B   0
 /   \  /   \  
 G   A  X   0
 0    0 0  0
```

and a short exact sequence

```
0 \rightarrow \text{im}\theta \rightarrow I \rightarrow I' \rightarrow 0.
```

As we assume that \([B, I'] = 0\), thus \([H, I'] = 0\). Then

\[
\langle Y, X \rangle - \langle H, G \rangle = \langle H, X \rangle - \langle H, G \rangle = \langle H, X - G \rangle = \langle H, B - M \rangle = -\langle H, \text{im}\theta \rangle.
\]

Applying the functor \([H, -]\) to the above short exact sequence, we have

```
0 \rightarrow [H, \text{im}\theta] \rightarrow [H, I] \rightarrow [H, I'] \rightarrow [H, \text{im}\theta]^{1} \rightarrow 0.
```

Note that \([H, I] = [H, I'] = 0\), thus \langle H, \text{im}\theta \rangle = 0\). Hence

\[
X_E = \sum_{G,H,[P,H]=0} q^{-\frac{1}{2}(H,G)} F_{GHX}^{M} B_{H-(\tilde{I}-\tilde{R})M+\dim P/\text{rad}P}.
\]

Now compute the term

\[
X_{E'} = X_{A\oplus P'[1]} = \sum_{X,Y} q^{-\frac{1}{2}(Y,X)} F_{XY}^{A} X_{\tilde{Y}_{y-(\tilde{I}-\tilde{R})A+\dim P'/\text{rad}P'}}.
\]

We have the following diagram

```
  0    0
 / \   / \  
 P   H  Y  0
 /   \  /   \  
 M   A  X   0
 /   \  /   \  
 G   X  0  0
 0    0 0  0
```
Applying the functor \([P', -]\) to the following short exact sequence
\[
0 \rightarrow Y \rightarrow A \rightarrow G \rightarrow 0.
\]
we have
\[
0 \rightarrow [P', Y] \rightarrow [P', A] \rightarrow [P', G] \rightarrow 0
\]
As we assume that \([P', A] = 0\), thus \([P', G] = 0\). Then \(\langle P', G \rangle = 0\). Hence we have
\[
\langle Y, X \rangle - \langle H, G \rangle = \langle Y, G \rangle - \langle H, G \rangle = \langle Y - H, G \rangle = \langle A - M, G \rangle
\]
Therefore
\[
X_E' = \sum_{G, H, [P, H] = 1} q^{-\frac{1}{2} \langle H, G \rangle} P_{\bar{G}H} X_{\bar{E}H} - \langle \bar{E}I \rangle + \dim P/\text{rad} P.
\]
This completes the proof. \(\square\)

Note that if \(M\) is indecomposable rigid object in \(C_{\bar{Q}}\) and \([P, M] = [M, I] = 1\), then both \(E = B \oplus I'[1] - 1\) and \(E' = A \oplus P'[1]\) are rigid. Thus the assumptions \(\text{Hom}_{k\bar{Q}}(B, I') = \text{Hom}_{k\bar{Q}}(P', A) = 0\) in Theorem 3.8 naturally hold. The quantum cluster multiplication theorem in [20] deals with this special case.

4. Generic bases in specialized quantum cluster algebras of finite type

Let \(k\) be a finite field with cardinality \(|k| = q\) and \(m \geq n\) be two positive integers and \(\bar{Q}\) an acyclic quiver with vertex set \(\{1, \ldots, m\}\). The full subquiver \(Q\) on the vertices \(\{1, \ldots, n\}\) is the principal part of \(\bar{Q}\). Let \(A_{|k|}(\bar{Q})\) be the corresponding specialized quantum cluster algebra of \(Q\) with coefficients. Then the main theorem in [20] shows that \(A_{|k|}(\bar{Q})\) is the \(\mathbb{Z}P\)-subalgebra of \(\mathcal{F}\) generated by
\[
\{X_M | M \text{ is indecomposable rigid } kQ\text{-mod}\} \cup \{X_{\tau P_i}, 1 \leq i \leq n | P_i \text{ is indecomposable projective } k\bar{Q}\text{-mod}\}.
\]
Let \(i\) be a sink or a source in \(\bar{Q}\). We define the reflected quiver \(\sigma_i(\bar{Q})\) by reversing all the arrows ending at \(i\). An admissible sequence of sinks (resp. sources) is a sequence \((i_1, \ldots, i_l)\) such that \(i_1\) is a sink (resp. source) in \(\bar{Q}\) and \(i_k\) is a sink (resp source) in \(\sigma_{i_k-1} \cdots \sigma_{i_1}(\bar{Q})\) for any \(k = 2, \ldots, l\). A quiver \(\bar{Q}'\) is called reflection-equivalent to \(\bar{Q}\) if there exists an admissible sequence of sinks or sources \((i_1, \ldots, i_l)\) such that \(\bar{Q}' = \sigma_{i_l} \cdots \sigma_{i_1}(\bar{Q})\). A quiver \(\bar{Q}'\) is called reachable from \(\bar{Q}\) if \(\bar{Q}' = \sigma_{i_l} \cdots \sigma_{i_1}(\bar{Q})\) where \(1 \leq i_1, \ldots, i_l \leq n\). Note that mutations can be viewed as generalizations of reflections, i.e, if \(i\) is a sink or a source in a quiver \(\bar{Q}\), then \(\mu_i(\bar{Q}) = \sigma_i(\bar{Q})\) where \(\mu_i\) denotes the mutation in the direction \(i\). Thus if \(\bar{Q}'\) is reachable from \(\bar{Q}\), there is a natural canonical isomorphism between \(A_{|k|}(\bar{Q})\) and \(A_{|k|}(\bar{Q}')\), denoted by
\[
\Phi_i : A_{|k|}(\bar{Q}) \rightarrow A_{|k|}(\bar{Q}').
\]
Let $\Sigma^+_i : \text{mod}(\tilde{Q}) \to \text{mod}(\tilde{Q}')$ be the standard BGP-reflection functor and $R^+_i : C_\tilde{Q} \to C_{\tilde{Q}'}$ be the extended BGP-reflection functor defined by [30]:

$$R^+_i : \begin{cases} X &\mapsto \Sigma^+_i(X), \quad \text{if } X \not\preceq S_i \text{ is a } kQ-\text{module,} \\ S_i &\mapsto P_i[1], \\ P_j[1] &\mapsto P_j[1], \quad \text{if } j \neq i, \\ P_i[1] &\mapsto S_i. \end{cases}$$

By Rupel [28], the following holds:

**Theorem 4.1.** [28] Theorem 2.4, Lemma 5.6] For any $X^\tilde{Q}_M \in A_{|k|}(\tilde{Q})$, we have $\Phi_i(X^\tilde{Q}_M) = X^\tilde{Q}'_{R^+_i M}$.

**Definition 4.2.** (4.) Let $Q$ be an acyclic quiver with associated matrix $B$. $Q$ will be called graded if there exists a linear form $\epsilon$ on $\mathbb{Z}^n$ such that $\epsilon(B\alpha_i) < 0$ for any $1 \leq i \leq n$ where $\alpha_i$ still denotes the $i$-th vector of the canonical basis of $\mathbb{Z}^n$.

If $Q$ is a graded quiver, then it is proved in [4] that we can endow the cluster algebra of $Q$ with a grading. Namely, the results are the following:

For any Laurent polynomial $P$ in the variables $X_i$, the $\text{supp}(P)$ of $P$ is defined as the set of points $\lambda = (\lambda_i, 1 \leq i \leq n)$ of $\mathbb{Z}^n$ such that the $\lambda$-component, that is, the coefficient of $\prod_{1 \leq i \leq n} X_i^{\lambda_i}$ in $P$ is nonzero. For any $\lambda$ in $\mathbb{Z}^n$, let $C_\lambda$ be the convex cone with vertex $\lambda$ and edge vectors generated by the $B\alpha_i$ for any $1 \leq i \leq n$. Then we have the following two propositions as the quantum versions of Proposition 5 and Proposition 7 in [4] respectively.

**Proposition 4.3.** Let $Q$ be a graded acyclic quiver with no multiple arrows and $M = M_0 \oplus P_M[1]$ with $M_0$ is $kQ$-module and $P_M$ projective $kQ$-module. Then, $\text{supp}(X_{M_0 \oplus P_M[1]})$ is in $C_{\lambda_M}$ with $\lambda_M := (-\langle \alpha_i, \dim M_0 \rangle + \langle \dim P_M, \alpha_i \rangle)_{1 \leq i \leq n}$. Moreover, the $\lambda_M$-component of $X_{M_0 \oplus P_M[1]}$ is some nonzero monomials in $\{ |k|^{\pm 1}, X^\pm_{n+1}, \ldots, X^\pm_{m} \}$.

**Proposition 4.4.** Let $Q$ be a graded acyclic quiver with no multiple arrows. For any $m \in \mathbb{Z}$, set

$$F_m = \left( \bigoplus_{\epsilon(v) \leq m} \mathbb{Z}^P \prod_{1 \leq i \leq n} u_i^{e_i} \right) \cap A_{|k|}(\tilde{Q}),$$

then the set $(F_m)_{m \in \mathbb{Z}}$ defines a $\mathbb{Z}$-filtration of $A_{|k|}(\tilde{Q})$.

For any $d \in \mathbb{Z}^n$, define $d^+ = (d^+_i)_{1 \leq i \leq n}$ such that $d^+_i = d_i$ if $d_i > 0$ and $d^+_i = 0$ if $d_i \leq 0$ for any $1 \leq i \leq n$. Dually, we set $d^- = d^+ - d$. The following proposition [4, 5] can be viewed as the categorification of [2] Theorem 7.3.

**Proposition 4.5.** Let $\tilde{Q}$ be an acyclic quiver. Then the set $\{ \prod_{i=1}^n X^d_{S_i} X^d_{P_i[1]} \mid (d_1, \ldots, d_n) \in \mathbb{Z}^n \}$ is a $\mathbb{Z}P$-basis of $A_{|k|}(\tilde{Q})$.

**Proof.** For any $1 \leq i \leq n$, it is easy to check that the following set is a cluster

$$\{ X_{\tau P_1}, \ldots, X_{\tau P_{i-1}}, X_{S_i}, X_{\tau P_{i+1}}, \ldots, X_{\tau P_n} \}$$

obtained by the mutation in direction $i$ of the cluster

$$\{ X_{\tau P_1}, \ldots, X_{\tau P_{i-1}}, X_{\tau P_i}, X_{\tau P_{i+1}}, \ldots, X_{\tau P_n} \}.$$
Then the proposition immediately follows from [2] Theorem 7.3 and [20] Theorem 5.4.3. □

The main result is the following theorem showing the \( \mathbb{Z}P \)-basis in a specialized quantum cluster algebra of finite type. The basis is the good basis in a cluster algebra of finite type in \([4]\) by specializing \( q \) and coefficients to 1 and the existence of Hall polynomials for representation-finite algebras \([27]\).

**Theorem 4.6.** Let \( Q \) be a simple-laced Dynkin quiver with \( Q_0 = \{1, 2, \cdots, n\} \) and \( \widetilde{Q} \) reachable from \( \widetilde{Q}' \) for any graded quiver \( Q' \). Then the set \( \mathcal{B}(Q) := \{X_M | M = M_0 \oplus P_M[1] \text{ with } M_0 \text{ is } kQ\text{-module}, \text{ } P_M \text{ a direct sum of projective } kQ\text{-modules } P_i(1 \leq i \leq n), \text{ } M \text{ rigid object in } C_{\widetilde{Q}}\} \) is a \( \mathbb{Z}P \)-basis of \( A_{|k|}(\widetilde{Q}) \).

**Proof.** Assume that \( \sigma_{i_1} \cdots \sigma_{i_t}(\widetilde{Q}') = \widetilde{Q}(1 \leq i_1, \cdots, i_t \leq n) \). For any \( X_{\widetilde{Q}'}_M \in \mathcal{B}(Q') \) with dimension vector \( \dim M = m = (m_1, \cdots, m_n) \in \mathbb{Z}^n \), we know that \( X_{\widetilde{Q}'}_M \in A_{|k|}(\widetilde{Q}) \). Then by Proposition \( 4.5 \) we have

\[
X_{\widetilde{Q}'}_M = b_m \prod_{i=1}^{n} (X_{\widetilde{Q}'}_{S_{i}})^{m_i} + \sum_{e \in \mathcal{E}(m)} b_e \prod_{i=1}^{n} (X_{\widetilde{Q}'}_{S_{i}})^{e_i} (X_{\widetilde{Q}'}_{P_i[1]})^{e_i}.
\]

where \( \mathcal{E}(m) = (l_i^+ - l_i^-)_{i \in Q_0}, b_m \text{ and } b_e \in \mathbb{Z}P \). As \( Q' \) is a graded quiver, then by Proposition \( 4.4 \) it follows that \( b_m \) must be some nonzero monomial in \( \{q^{\pm \frac{1}{2}}, X_{n+1}^{\pm 1}, \cdots, X_m^{\pm 1}\} \).

Therefore, \( \mathcal{B}(Q') \) is a \( \mathbb{Z}P \)-basis of \( A_{|k|}(\widetilde{Q}') \). There is a natural isomorphism: \( \Phi_{i_1} \cdots \Phi_{i_t} : A_{|k|}(\widetilde{Q}') \to A_{|k|}(\widetilde{Q}) \). By Theorem \( 4.1 \) we obtain that

\[
\Phi_{i_1} \cdots \Phi_{i_t}(X_{\widetilde{Q}'}_M) = X_{\widetilde{Q}'}_{R_{i_t} \cdots R_{i_1}(M)}.
\]

Hence, \( \mathcal{B}(Q) \) is a \( \mathbb{Z}P \)-basis of \( A_{|k|}(\widetilde{Q}) \). □

By the existence of Hall polynomials for representation-finite algebras \([27]\), we have the following corollary.

**Corollary 4.7.** With the above notation, the set \( \mathcal{B}(Q) \) is a \( \mathbb{Z}P \)-basis of \( A_q(\widetilde{Q}) \).

5. **Generic bases in specialized quantum cluster algebras of affine type**

Throughout this section, we assume that \( Q \) is a tame quiver with trivial valuation. A **tame quiver** is an acyclic quiver whose underlying diagram in an extended Dynkin diagram. We recall some facts about representation theory of tame quivers (for example, refer to \([12]\) [8] [26] for more details). Let \( k \) be a finite field with \(|k| = q\). The category \( \text{rep}(kQ) \) of finite-dimensional representations can be identified with the category of mod-\( kQ \) of finite-dimensional modules over the path algebra \( kQ \). For any \( kQ \)-representation \( M \) and \( i \in Q_0 \), we denote by \( (M)_i \) the \( k \)-space at \( i \). It is well-known that indecomposable \( kQ \)-module contains (up to isomorphism) three families (by the Auslander-Reiten quiver): the component of indecomposable regular modules \( R(Q) \), the component of the preprojective
modules $\mathcal{P}(Q)$ and the component of the preinjective modules $\mathcal{I}(Q)$. If $P \in \mathcal{P}(Q)$, $I \in \mathcal{I}(Q)$ and $R \in \mathcal{R}(Q)$, then
\[
\text{Hom}_{kQ}(R, P) \simeq \text{Hom}_{kQ}(I, R) \simeq \text{Hom}_{kQ}(I, P) = 0,
\]
and
\[
\text{Ext}^1_{kQ}(P, R) \simeq \text{Ext}^1_{kQ}(R, I) \simeq \text{Ext}^1_{kQ}(P, I) = 0.
\]
If $M$ and $N$ are two regular indecomposable modules in different tubes, then
\[
\text{Hom}_{kQ}(M, N) = 0 \quad \text{and} \quad \text{Ext}^1_{kQ}(M, N) = 0.
\]

The Auslander-Reiten quiver of $\mathcal{R}(Q)$ consists of tubes. An indecomposable regular module $R$ is regular simple if it contains no nontrivial regular submodule, and call it homogeneous if $\tau_Q R \cong R$. Any regular module at the bottom of a tube is regular simple module. If it is homogeneous, then the tube is a homogeneous tube, otherwise, called a non-homogeneous tube. For a regular module $R$, its degree is the index $[\text{End}_{kQ}(R) : k]$. There are at most $t \leq 3$ non-homogeneous tubes for $Q$. We denote these non-homogeneous tubes by $T_1, \cdots, T_t$. Let $\tau_i$ be the rank of $T_i$ and the regular simple modules in $T_i$ be $E^{(i)}_1, \cdots, E^{(i)}_{\tau_i}$ such that $\tau_Q E^{(i)}_2 = E^{(i)}_1, \cdots, \tau_Q E^{(i)}_1 = E^{(i)}_{\tau_i}$ for $i = 1, \cdots, t$. If we restrict the discussion to one tube, we will omit the index $i$ for convenience. There are $q + 1 - t$ homogeneous tubes which regular simples at the bottoms are of degree 1. Given a regular simple module $E$ in a tube, $E[i]$ is the indecomposable regular module with quasi-socle $E$ and quasi-length $i$ for any $i \in \mathbb{N}$. The minimal imaginary root of $Q$ is denoted by $\delta = (\delta_i)_{i \in Q_0}$. Note that the regular simple module of degree 1 at the bottom in a homogeneous tube is of dimension vector $\delta$. We now prove that the quantum Caldero-Chapoton map does not depend on the modules in the homogeneous tube with dimension vector $\delta$.

**Lemma 5.1.** Let $\lambda$ and $\lambda'$ be in $k$ such that $E(\lambda)$ and $E(\lambda')$ are two regular simple modules of dimension vector $\delta$. Then $X_{E(\lambda)} = X_{E(\lambda')}$.

**Proof.** We only need to prove $|\text{Gr}_e(E(\lambda))| = |\text{Gr}_e(E(\lambda'))|$. Let $e$ be a vertex in $Q_0$ such that $\delta_e = 1$. If $e$ is a sink and $Q$ is a quiver with the underlying graph not of type $A_n$, then there is unique edge $\alpha \in Q_1$ such that $t(\alpha) = e$. It is easy to check $\delta_{s(\alpha)} = 2$. Let $P_e$ be the simple projective module corresponding to $e$ and $I$ be the indecomposable preinjective module of dimension vector $\delta - \text{dim} P_e$. Then $\text{dim} \text{Ext}^1_{kQ}(I, P_e) = 2$. Given any $\varepsilon \in \text{Ext}^1_{kQ}(I, P_e)$, we have a short exact sequence whose equivalence class is $\varepsilon$

\[
\varepsilon : \quad 0 \rightarrow P_e \xrightarrow{(1)} E_\varepsilon \xrightarrow{(0, 1)} I \rightarrow 0
\]

where $(E_\varepsilon)_i = (P_e)_i \oplus (I)_i$ for any $i \in Q_0$, $(E_\varepsilon)_\beta = (I)_\beta$ for $\beta \neq \alpha$ and $(M_\varepsilon)_\alpha$ is

\[
(E_\varepsilon)_\alpha = \begin{pmatrix} 0 & m(\varepsilon, \alpha) \\ 0 & 0 \end{pmatrix}
\]

where $m(\varepsilon, \alpha) \in \text{Hom}_{kQ}((I)_{s(\alpha)}, (P_e)_e)$. Any regular simple $kQ$-module $E$ of dimension vector $\delta$ satisfies that $E_\alpha$ is as follows

\[
(E)_{s(\alpha)} = k^2 \begin{pmatrix} 1 & \lambda \\ \varepsilon & 0 \end{pmatrix}
\]

where $\lambda = (E)_{s(\alpha)}$. (P.e)
where $\lambda \in k^* \setminus \{1\}$. Let $P$ be an indecomposable projective module such that $P \subseteq M(\lambda)$ and $d_{P(e)} = 0$. Then $P$ is also a submodule of $E(\lambda')$ and $d_{\alpha P(\alpha)} = 0$. Let $P$ be an indecomposable projective module such that $P \subseteq E(\lambda)$ and $d_{P(e)} = 1$. Assume that $P_\alpha$ is $k \xrightarrow{a+b\lambda} k$, then there exists $P' \in \text{Gr}_k(E(\lambda'))$ such that $P' \cong P$ and $P'_\alpha$ is $k \xrightarrow{a+b\lambda'} k$. Since $\tau E(\lambda) = E(\lambda)$, we know $\tau^{-i} P$ and $\tau^{-1} P'$ are the submodules of $E(\lambda)$ and $E(\lambda')$ for any $i \in \mathbb{N}$, respectively. Hence, any preprojective submodule $X$ of $E(\lambda)$ corresponds to a preprojective submodule $X'$ of $E(\lambda')$ such that $X \cong X'$. Let $Q$ be of type $\tilde{A}_n$. Then there are two adjacent edge $\alpha$ and $\beta$. Any regular simple module $E$ of dimension vector $\delta$ satisfies that $E_\alpha$ is as follows:

$$
\begin{array}{ccc}
  & 1 & \\
  k & \longrightarrow & k \\
  \lambda & \longrightarrow & k
\end{array}
$$

for some $\lambda \in k^*$. The discussion is similar as above. If $e$ is a source, the discussion is also similar. Therefore, there is a homeomorphism between $\text{Gr}_k(E(\lambda))$ and $\text{Gr}_k(E(\lambda'))$ for any dimension vector $e$. \hfill \square

Now let $Q$ be a graded tame quiver and $E_1, \cdots, E_r$ be regular simple modules in a nonhomogeneous tube with rank $r$ such that $\tau_Q E_2 = E_1, \cdots, \tau_Q E_1 = E_r$. Let $\tilde{Q}$ be a quiver obtained from $Q$ by adding frozen vertices $\{n+1, \cdots, 2n\}$ and arrows $n + i \rightarrow i$ for any $1 \leq i \leq n$. Then we have the following result.

**Lemma 5.2.**

$$
X_{E_i[r-1]} X_{E_{i-1}} = q^{\frac{1}{2} \Lambda((\tilde{I}-\tilde{R})e_i[r-1], (\tilde{I}-\tilde{R})e_{i-1})} X_{E_i[r]} + q^{\frac{1}{2} \Lambda((\tilde{I}-\tilde{R})e_i[r-1], (\tilde{I}-\tilde{R})e_{i-1})}^{-1} X_{E_i[r-2]} \oplus I_{[-1]}
$$

where $I$ is an injective $k\tilde{Q}$-module associated to frozen vertices.

**Proof.** We have the following exact sequences

$$
0 \longrightarrow E_i[r-1] \longrightarrow E_i[r] \longrightarrow E_{i-1} \longrightarrow 0
$$

$$
0 \longrightarrow E_i[r-2] \longrightarrow E_i[r-1] \longrightarrow \tau_Q E_{i-1} \longrightarrow I \longrightarrow 0.
$$

Hence the proof follows from Theorem [3.5]. \hfill \square

Define the set

$$
\mathbf{D}(Q) = \{ \underline{d} \in \mathbb{N}^{Q_0} | \exists \text{ a regular rigid module } R \text{ and a regular simple module } E \text{ with dimension vector } \delta \text{ such that } \dim((E^\oplus n) \oplus R) = \underline{d} \}.
$$

Set $\mathbf{E}(Q) = \{ \underline{d} \in \mathbb{Z}^{Q_0} | \exists M = M_0 \oplus P_M[1] \text{ with } M_0 \text{ is } kQ\text{-module, } P_M \text{ projective } k\tilde{Q}\text{-module, } M \text{ rigid object in } \mathcal{C}_{\tilde{Q}} \text{ with } \dim M = \underline{d} \}$. By the main theorem in [11], we have that $\mathbb{Z}^{Q_0}$ is the disjoint union of $\mathbf{D}(Q)$ and $\mathbf{E}(Q)$. We make an assignment, i.e., a map

$$
\phi : \mathbb{Z}^{Q_0} \rightarrow \text{obj}(\mathcal{C}_{\tilde{Q}})
$$

and set

$$
X_{\phi(\underline{d})} := (X_E)^n X_R
$$

if $\underline{d} \in \mathbf{D}(Q)$ and $|Q_0| > 2$;

$$
X_{\phi(\underline{d})} := X_\delta
$$

for some $\delta$ in a homogeneous tube of degree 1 if $\underline{d} \in \mathbf{D}(Q)$ and $Q$ is the Kronecker quiver;

$$
X_{\phi(\underline{d})} := X_M
$$

for some $\delta$ in a homogeneous tube of degree 1 if $\underline{d} \in \mathbf{D}(Q)$ and $Q$ is the Kronecker quiver;
if $d \in E(Q)$. It is clear that the above assignment is not unique. For simplicity and no confusions, we omit $\phi$ in the notation $X_{\phi(q)}$.

**Theorem 5.3.** Let $k$ be a finite field with $|k| = q \neq 2$. Let $Q$ be a tame quiver with $Q_0 = \{1, 2, \cdots, n\}$ and $Q$ reachable from $Q'$ for any graded quiver $Q'$. Then the set

$$B(Q) := \{X_d|d \in \mathbb{Z}Q_0\}$$

is a $\mathbb{Q}P$-basis of $A|Q|Q' \otimes_{\mathbb{Z}P} \mathbb{Q}P$.

**Proof.** Assume that $e$ is a sink in $Q'$ and $P_e$ projective $kQ$-module associated to $e$. Let $I$ be an indecomposable preinjective $kQ$-module with dimension vector $\delta - \dim P_e$. By Lemma 5.1 and Theorem 5.3, we have

$$q^2 X_{P_e} X_I = q^{-1} \Lambda((\overline{Q} - \overline{Q}_P)) X_{P_e \oplus I} + (q + 1 - t) \varepsilon_{IP_e} X_E + \sum_{i=1}^t \varepsilon_{I_{P_e}} X_{E_1[r]} X_{E_1[r_i]}.$$ 

It follows that

$$q^{\frac{1}{2}} \Lambda((\overline{Q} - \overline{Q}_P)) X_{P_e \oplus I} X_I X_{P_e} = q^{-1} \Lambda((\overline{Q} - \overline{Q}_P)) X_{P_e \oplus I} X_I X_{P_e} = (q + 1 - t) \varepsilon_{IP_e} X_E + \sum_{i=1}^t \varepsilon_{I_{P_e}} X_{E_1[r]}$$

where, by definition, $\varepsilon_{IP_e} = \varepsilon_{I_{P_e}} = q - 1$. By Lemma 5.2, $X_{E_1[r]}$ is in $A|Q|Q' \otimes_{\mathbb{Z}P} \mathbb{Q}P$ for $1 \leq i \leq t$, thus $X_E$ is in $A|Q|Q' \otimes_{\mathbb{Z}P} \mathbb{Q}P$. It follows that for any $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$, $X_m \in B(Q')$. Then by Proposition 4.3 we have

$$X_m^{Q'} = b_m \prod_{i=1}^n (X_{S_i}^{Q})^{m_i} (X_{P_{i}[1]}^{Q})^{m_i} + \sum_{\varepsilon(<c(m)} b_{\varepsilon} \prod_{i=1}^n (X_{S_i}^{Q})^{c_i} (X_{P_{i}[1]}^{Q})^{c_i}$$

where $b_m, b_{\varepsilon} \in \mathbb{Z}P$. As $Q'$ is a graded quiver, then by Proposition 4.3, Proposition 4.4, it follows that $b_m$ must be some nonzero monomial in $\{q^{\pm \frac{1}{2}}, X_{n+1}, \cdots, X_m\}$. Therefore, $B(Q')$ is a $\mathbb{Q}P$-basis of $A|Q|Q' \otimes_{\mathbb{Z}P} \mathbb{Q}P$. By Theorem 5.4, we obtain that $B(Q)$ is a $\mathbb{Q}P$-basis of $A|Q|Q' \otimes_{\mathbb{Z}P} \mathbb{Q}P$. 

By [6] Proposition 5], the quiver Grassmannians $Gr_{\mathbb{P}}(M)$ of a $kQ$-module $M$ are some polynomials in $\mathbb{Z}[q]$. Then by specializing $q$ and coefficients to 1, the bases in Theorem 5.3 induces the integral bases in affine cluster algebras ([11][13]). In the same way, we have the following corollary.

**Corollary 5.4.** With the above notation, the set $B(Q)$ is a $\mathbb{Q}P$-basis of $A_q(Q) \otimes_{\mathbb{Z}P} \mathbb{Q}P$.

**5.1. An example.** Let $Q$ be the tame quiver of type $\overline{A}_1(1)$ as follows

$$\begin{array}{c}
1 \\
\overline{2}
\end{array}$$

It is well known that the regular indecomposable modules decomposes into a direct sum of homogeneous tubes indexed by the projective line $\mathbb{P}^1(k)$. We denote the regular indecomposable modules in the homogeneous tube for $p \in \mathbb{P}^1(k)$ of degree 1 by $\overline{R}_p(n)$ where $n \in \mathbb{N}$ and $\dim \overline{R}_p(n) = (n, n)$.
We consider the following ice quiver $\tilde{Q}$ with frozen vertices 3 and 4:

```
1    2
\downarrow \quad \uparrow
3    4
```

Thus we have

\[
\tilde{R} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \end{pmatrix}.
\]

An easy calculation shows that the following antisymmetric $4 \times 4$ integer matrix

\[
\Lambda = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{pmatrix}
\]

satisfying

\[
\Lambda(-\tilde{B}) = \tilde{I} := \begin{bmatrix} I_2 \\ 0 \end{bmatrix},
\]

where $I_2$ is the identity matrix of size $2 \times 2$. Then we have the following result.

**Lemma 5.5.** Let $R_p(1)$ be the indecomposable regular module of degree 1 as above. Then

\[
X_{R_p(1)} = X_{S_1}X_{S_2} - q^{-\frac{1}{2}}X_1X_2X_4.
\]

**Proof.** By definition, we have

\[
X_{S_1} = X^{(-1,0,1,0)} + X^{(-1,2,0,0)};
\]

\[
X_{S_2} = X^{(0,-1,0,0)} + X^{(2,-1,0,1)};
\]

\[
X_{R_p(1)} = X^{(1,-1,1,1)} + X^{(-1,1,0,0)} + X^{(-1,-1,1,0)}.
\]

Hence the lemma follows from a direct calculation. \(\square\)

By Lemma 3.1, the expression of $X_{R_p(1)}$ is independent of the choice of $p \in \mathbb{P}_k^1$ of degree 1. Hence, we set

\[
X_\delta := X_{R_p(1)}.
\]

**Remark 5.6.**

1. By Lemma 5.5, we know that $X_\delta$ belongs to $A_{|k|}(\tilde{Q})$.

2. By the following Theorem 5.3 $B(Q)$ is a $\mathbb{Z}P$--basis in the quantum cluster algebra $A_{|k|}(\tilde{Q})$. Moreover, if specializing $q$ and coefficients to 1, $B(Q)$ is exactly the generic basis in the sense of [13).

Note that there is an alternative choice of $(\Lambda, \tilde{B})$, i.e., $\tilde{Q} = Q$ and set $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Then we have $\Lambda(-\tilde{B}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Hence, one should consider the category of $KQ$-representations for a field $K$ with $|K| = q^2$. In this way, we obtain a quantum cluster algebra of Kronecker type without coefficients. The multiplication and the
bar-invariant bases in this algebra have been thoroughly studied in [9]. Moreover, under the specialization $q = 1$, the bases in [9] induce the canonical basis, semicanonical basis and generic basis of the cluster algebra of the Kronecker quiver in the sense of [29], [7] and [13], respectively.

6. Remark on the difference property of affine quantum cluster algebras

In this section, we prove that bases in Theorem 5.3 are $\mathbb{Z}P$-bases for quantum cluster algebras of type $\tilde{A}$ and $\tilde{D}$. This refines Theorem 5.3.

6.1. The case in type $\tilde{A}_{r,s}$. Let $Q$ be a quiver of type $\tilde{A}_{r,s}$ as follows.

There are two non-homogeneous tubes (denoted by $\mathcal{T}_0, \mathcal{T}_\infty$) in the set of indecomposable regular modules. The minimal imaginary root of $Q$ is $\delta = (1, 1, \cdots, 1)$. Let $E(\lambda)$ be the regular simple module in the homogeneous tube for $\lambda \in \mathbb{P}^1$ of degree 1. The regular simple modules in $\mathcal{T}_0$ are

$$E^{(0)}_t = S_{r+t+1} \quad 1 \leq t \leq s - 1, \quad E^{(0)}_s : \quad k \quad \cdots \quad k \quad 1 \quad 0 \quad k$$

The regular module $E^{(0)}_1[s]$ has the form as follows

$$k \quad \cdots \quad k \quad 1 \quad 0 \quad k$$

By [11], we have

$$|Gr\lambda(E^{(0)}_1[s])| = |Gr\lambda(E(\lambda))| + |Gr_{\dim S_{r+2}}(E^{(0)}_2[s-2])| \quad (\star)$$

The follow lemma can be viewed as the difference property of quantum cluster algebra of $\tilde{A}_{r,s}$.

Proposition 6.1.

$$X_{E^{(0)}_1[s]} = X_{E(\lambda)} + q^\frac{1}{2} X_{E^{(0)}_2[s-2]}$$
Proof. By the above equation (\(*\)), we have

\[
X_{E_1^{(0)}}[s] = \sum_{\nu} |\text{Gr}_\nu E_1^{(0)}[s]| q^{-\frac{1}{2}(\nu, \delta - \nu)} X^\tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta}
\]

\[
= \sum_{\nu} |\text{Gr}_\nu E(\lambda)| q^{-\frac{1}{2}(\nu, \delta - \nu)} X^\tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta} + \sum_{\nu \neq \delta} |\text{Gr}_{\nu - s_{r+2}} E_2^{(0)}[s - 2]| q^{-\frac{1}{2}(\nu, \delta - \nu)} X^\tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta}
\]

\[
= X_{E(\lambda)} + q^{\frac{1}{2}} \sum_{\nu} |\text{Gr}_\nu E_2^{(0)}[s - 2]| q^{-\frac{1}{2}(\nu', \tilde{e}_2^{(0)}[s - 2] - \nu')} X^{\tilde{B}_{\nu'} - (\tilde{I} - \tilde{R})e_2^{(0)}[s - 2]}
\]

\[
= X_{E(\lambda)} + q^{\frac{1}{2}} X_{E_2^{(0)}[s - 2]}.
\]

Here we use the facts that

\[
\langle \nu', e_2^{(0)}[s - 2] - \nu' \rangle = \langle \nu - s_{r+2}, \delta - \nu - \tau_Q s_{r+2} \rangle
\]

\[
= \langle \nu, \delta - \nu \rangle - \langle \nu, \tau_Q s_{r+2} \rangle - \langle s_{r+2}, \delta \rangle + \langle s_{r+2}, \tau_Q s_{r+2} \rangle + \langle s_{r+2}, \nu \rangle
\]

\[
= \langle \nu, \delta - \nu \rangle - \langle s_{r+2}, \delta \rangle + \langle s_{r+2}, \tau_Q s_{r+2} \rangle + 2\langle s_{r+2}, \nu \rangle
\]

\[
= \langle \nu, \delta - \nu \rangle - 0 - 1 + 2
\]

\[
= \langle \nu, \delta - \nu \rangle + 1.
\]

And

\[
\tilde{B}_{\nu'} - (\tilde{I} - \tilde{R})e_2^{(0)}[s - 2]
\]

\[
= \tilde{B}(\nu - s_{r+2}) - (\tilde{I} - \tilde{R})(\delta - s_{r+2} - \tau_Q s_{r+2})
\]

\[
= \tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta} - \tilde{s}_{r+2} + (\tilde{I} - \tilde{R})(s_{r+2} + \tau_Q s_{r+2})
\]

\[
= \tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta} - (\tilde{R}^{br} - \tilde{R})s_{r+2} + (\tilde{I} - \tilde{R})(s_{r+2} + \tau_Q s_{r+2})
\]

\[
= \tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta} + (\tilde{I} - \tilde{R}^{br})s_{r+2} + (\tilde{I} - \tilde{R})\tau_Q s_{r+2}
\]

\[
= \tilde{B}_\nu - (\tilde{I} - \tilde{R})\tilde{\delta}.
\]

6.2. The case in type $\tilde{D}_r$. Let $Q$ be a tame quiver of type $\tilde{D}_r$ for $r \geq 4$ as follows

\[
\begin{array}{cccccc}
& r & \rightarrow & r - 1 & \rightarrow & \cdots & \rightarrow 4 & \rightarrow & 3 & \rightarrow & 1 & \rightarrow & 2 \\
& 1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & r & \rightarrow & r + 1
\end{array}
\]

There are three non-homogeneous tubes (denoted by $T_0, T_1, T_\infty$). The minimal imaginary root of $Q$ is $\delta = (1, 1, 2, \cdots, 2, 1, 1)$. Let $E(\lambda)$ be the regular simple module in the homogeneous tube for $\lambda \in \mathbb{P}^1$ of degree 1. The regular simple modules in $T_1$ are

\[
E_t^{(1)} = S_{t+2} \text{ for } 1 \leq t \leq r - 3, \quad E_{r-2}^{(1)} =:
\]

\[
\begin{array}{cccccc}
k & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & 1 & \rightarrow & 1
\end{array}
\]

\[
\begin{array}{cccccc}
k & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & 1 & \rightarrow & k
\end{array}
\]

\[
\begin{array}{cccccc}
k & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & 1 & \rightarrow & k
\end{array}
\]
By [11], we have

\[ |Gr_{E_1}(E_1^{(1)}[r-2])| = |Gr_{E}(E(\lambda))| + |Gr_{E_2^{\dim S_1}}(E_2^{(1)}[r-4])|. \]

Similar to Lemma [6.1] we have

**Proposition 6.2.**

\[ X_{E_1^{(1)}[r-2]} = X_{E(\lambda)} + q^2 X_{E_2^{(1)}[r-4]}. \]

As a direct corollary of Proposition [6.1] and [6.2] we obtain the following refinement of Theorem [5.3].

**Corollary 6.3.** Let \( \tilde{Q} \) be reachable from \( \tilde{Q}' \) for any graded quiver \( Q' \) of type \( \tilde{A} \) and \( \tilde{D} \). Then the set

\[ \mathcal{B}(Q) := \{ X_{d|d} \in \mathbb{Z}^{Q_0} \} \]

is a \( \mathbb{Z}P \)-basis of \( \mathcal{A}_{\mathbb{Z}}(\tilde{Q}) \).

Now assume that \( Q \) is a tame quiver. Let \( E \) be a regular simple module in a nonhomogeneous tube with rank \( n \) and \( E(\lambda) \) the regular simple module in the homogeneous tube for \( \lambda \in \mathbb{P}^1 \) of degree 1. We conjecture that:

**Conjecture 6.4.** With the above notations, we have

\[ X_{E[n]} = X_{E(\lambda)} + q^{1/2} X_{(\tau_Q^{-1} E)[n-2]}. \]

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