HILBERT SCHEMES, INTEGRABLE HIERARCHIES, 
AND GROMOV-WITTEN THEORY

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Abstract. Various equivariant intersection numbers on Hilbert schemes of points on the affine plane are computed, some of which are organized into $\tau$-functions of 2-Toda hierarchies. A correspondence between the equivariant intersection on Hilbert schemes and stationary Gromov-Witten theory is established.

1. Introduction

The relations between the cohomology rings of Hilbert schemes of points on algebraic surfaces and $\mathcal{W}$ algebras \cite{LQW} through vertex operators have led to the following question (i.e. Question 5 in Sect. 6 of \cite{QW}): what is the precise connection between Hilbert schemes and integrable hierarchies? The connection between the geometry of Hilbert schemes and vertex operators is made through the so-called Chern character operators \cite{LQW} (also cf. \cite{Lehn}). However the Chern character operators are usually nilpotent for (cohomology) degree reasons. This presents a serious difficulty for a sensible answer to the above question in the framework of ordinary cohomology theory.

The main goal here is to initiate a direct link between equivariant cohomology rings of Hilbert schemes of $n$ points $X^{[n]}$ on a quasi-projective surface $X$ and integrable hierarchies, and to establish a correspondence with (stationary) Gromov-Witten theory. In the present paper, we will treat the case when $X$ is the affine plane, where some main idea is made clear.

A distinguished $T = \mathbb{C}^*$-action on $X = \mathbb{C}^2$ induces an action on the Hilbert schemes $X^{[n]}$ with finitely many fixed points (cf. \cite{ES}). As explained by Vasserot \cite{Vas}, the Heisenberg algebra construction in \cite{Na1} extends to the $T$-equivariant cohomology of Hilbert schemes $X^{[n]}$ (also cf. Nakajima \cite{Na2}). In particular, all information of the equivariant cohomology ring $H^*_T(X^{[n]})$ is encoded in a ring $\mathbb{H}_n = H^{2n}_T(X^{[n]})$. The direct sum $\mathbb{H}_X = \bigoplus_{n \geq 0} \mathbb{H}_n$ becomes the bosonic Fock space of a Heisenberg algebra. The ring $\mathbb{H}_n$ is further identified in \cite{Vas} with the class algebra of the symmetric group $S_n$ (cf. \cite{LT, Wa} for the very relevant study of these class algebras). Under such an identification, we observe that the $k$-th equivariant Chern
characters of the tautological rank $n$ vector bundle over $X^{[n]}$ correspond precisely to the $k$-th power-sum of Jucys-Murphy elements, extending the earlier observation for ordinary Chern characters in [QW]. In particular, from the corresponding result on the class algebra of symmetric group [Wa], the Chern characters give rise to a set of ring generators for $H_n$. Therefore, one way to present the equivariant intersection theory on $X^{[n]}$ is to study the intersection numbers of these Chern characters and their variants.

We further introduce the moduli spaces $\mathcal{M}(m, n)$, where $m \in \mathbb{Z}$, $n \geq 0$. As varieties, $\mathcal{M}(m, n)$ is isomorphic to $X^{[n]}$. The equivariant cohomology ring of $\mathcal{M}(m, n)$ also naturally corresponds to a ring $H^{(m)}_n$ which is isomorphic to $H_n$. We identify $\mathcal{F} = \oplus_{m,n} H^{(m)}_n$ with the fermionic Fock space via the celebrated boson-fermion correspondence. The pioneering work of Kyoto school (cf. [MJD] and the references therein) on connections among Fock spaces, vertex operators, and soliton equations provides much algebraic background for the geometric picture developed here.

The intersection numbers of the equivariant Chern characters in $\mathcal{M}(m, n)$ can be organized into three types of generating functions. The so-called $N$-point function which we can compute has a simple relation with the $N$-point disconnected series of stationary Gromov-Witten invariants of $\mathbb{P}^1$. Note that $N$-point disconnected series is somewhat more complicated than the $N$-point connected series which were computed in [OP]. We relate the second generating function (called the multi-point trace function) in a simple and precise form to the characters on the fermionic Fock space. The latter has been computed by Bloch and Okounkov [BO], and it also computes the stationary Gromov-Witten invariants of an elliptic curve according to Okounkov and Pandharipande [OP]. This trace function is also intimately related to the trace functions in the theory of vertex algebras first studied by Zhu [Zhu]. Yet another generating function which involves the equivariant intersection numbers of the moduli spaces $\mathcal{M}(m, n)$ is shown to be the $\tau$-functions for the 2-Toda hierarchies [UT]. It is interesting to compare with [OP] where $\tau$ functions of 2-Toda hierarchies were constructed from Gromov-Witten invariants and Hurwitz numbers.

The notion of the equivariant Chern character operator, which gives rise to a master operator $\mathcal{H}(z)$ acting on $\mathcal{F}$, underlies the calculations of all the above generating functions. In the study of the class algebras of symmetric groups, a counterpart of this operator has also played a distinguished role [LT, Wa]. The operator $\mathcal{H}(z)$ turns out to be related to another operator $\varepsilon_0(z)$, which has played a key role in the study of stationary Gromov-Witten theory [OP], by the following formula:

\[ \mathcal{H}(z) = \frac{1}{e^{z/2} - e^{-z/2}} \left( \varepsilon_0(z) - \frac{1}{e^{z/2} - e^{-z/2}} I \right). \] (1.1)

The formula (1.1) defines the Gromov-Witten/Hilbert correspondence. Combining with the Gromov-Witten/Hurwitz correspondence in [OP], we also have a Hurwitz/Hilbert correspondence. In fact, the same combinatorics of the symmetric
groups underlies these distinct geometric studies. In particular, the use of Jucys-Murphy elements could also be used to clarify the notion of completed cycles considered in [OP].

An important open question is to establish a direct geometric connection behind the correspondence between (equivariant) intersection theory of Hilbert schemes and (equivariant) Gromov-Witten theory. It is possible that mirror symmetry and the correspondence between (equivariant) intersection theory of Hilbert schemes considered in [OP].

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2. The equivariant setup for Hilbert schemes

2.1. The torus action on the Hilbert schemes. Let \( T = \mathbb{C}^* \), and let \( \theta \) be the 1-dimensional standard module of \( T \) on which \( s \in T \) acts as multiplication by \( s \), and let \( t \) be the associated character. Then the representation ring \( R(T) \) is isomorphic to \( \mathbb{Z}[t, t^{-1}] \).

Let \( X \) be an algebraic variety acted by \( T \). Let \( H^*_T(X) \) and \( H^*_T(X) \) be the equivariant cohomology and the equivariant homology with \( \mathbb{C} \)-coefficient respectively. Note that \( H^*_T(pt) = H^*(BT) = \mathbb{C}[t] \). Then there exist linear maps

\[
\cup : H^i_T(X) \otimes H^j_T(X) \to H^{i+j}_{T}(X), \quad \cap : H^i_T(X) \otimes H^j_T(X) \to H^{i-j}_{T}(X).
\]

If \( X \) is of pure dimension, then there exists a linear map \( D : H^*_T(X) \to H^*_T(X) \). If \( X \) is smooth of pure dimension, \( D \) is an isomorphism. When \( f : Y \to X \) is a \( T \)-equivariant and proper morphism of varieties, we have a Gysin homomorphism \( f_* : H^*_T(Y) \to H^*_T(X) \) of equivariant homology. Moreover, when both \( Y \) and \( X \) are smooth of pure dimension, we have the Gysin homomorphism \( D^{-1} f_* : H^*_T(Y) \to H^*_T(X) \) of equivariant cohomology, which will still be denoted by \( f_* \).

Given an algebraic surface \( X \), the Hilbert schemes of \( n \) points on \( X \) is a nonsingular complex variety of dimension \( 2n \), cf. [Na1] for a general reference.

¿From now on, let \( X = \mathbb{C}^2 \). The torus \( T = \mathbb{C}^* \) acts on the affine coordinate functions \( w \) and \( z \) of \( X \) by \( s(w, z) = (sw, s^{-1}z) \). It induces an action on the Hilbert scheme \( X^{[n]} \) of \( n \) points on \( X \) with finitely many fixed points parametrized by partitions of \( n \) [ES]. Let \( \lambda \) be a partition of \( n \) and \( \xi_\lambda \) be the fixed point on \( X^{[n]} \) corresponding to the partition \( \lambda \). Let \( i_\lambda \) be the inclusion map \( \xi_\lambda \to X^{[n]} \). Let \( 1_{\xi_\lambda} \in H^0_T(\xi_\lambda) \) be the unit, and thus \( [\xi_\lambda] = i_\lambda^*(1_{\xi_\lambda}) \) lies in \( H^0_T(X^{[n]}) \). Here and
below, $[-]$ denotes the equivariant fundamental cycle or its associated equivariant cohomology class.

Denote by $\mathbb{C}[t]'$ the localization of the ring $\mathbb{C}[t]$ at the ideal $(t - 1)$, and denote

$$\iota_n = \bigoplus \lambda i_\lambda : (X^{[n]})^T \to X^{[n]}.$$ 

We define $H^*_T((X^{[n]})^T)' = H^*_T((X^{[n]})^T) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]'$ and define $H^*_T(X^{[n]})'$ similarly. We define

$$\iota_n! : H^*_T((X^{[n]})^T)' \longrightarrow H^*_T(X^{[n]})'$$

to be the induced Gysin map. By the localization theorem, $\iota_n!$ is an isomorphism. The inverse $(\iota_n!)^{-1}$ is given by

$$\alpha \mapsto \left( \frac{(i_\lambda)^*(\alpha)}{e_T(T_{\xi_\lambda})} \right)_\lambda$$

where $e_T(T_{\xi_\lambda})$ denotes the $T$-equivariant Euler class of the tangent bundle of $X^{[n]}$ over $\xi_\lambda$.

We define a bilinear pairing as in Vasserot $\text{[Vas]}$

$$\langle - , - \rangle_n : H^*_T(X^{[n]})' \otimes_{\mathbb{C}[t]'} H^*_T(X^{[n]})' \to \mathbb{C}[t]'$$

by

$$\langle \alpha, \beta \rangle_n = (-1)^n p_n!(\iota_n!)^{-1}(\alpha \cup \beta)$$

where $p_n$ is the projection of the set $(X^{[n]})^T$ of $T$-fixed points to a point.

### 2.2. The ring $\mathbb{H}_n$. For $k \geq n$, from the spectral sequence computation and $H^{2k}(X^{[n]}) = 0$, we see that $H^{2k}_T(X^{[n]}) = t^{k-n} \cup H^{2n}_T(X^{[n]})$. Following Vasserot $\text{[Vas]}$, we have an induced product $\star$ on $\mathbb{H}_n \overset{\text{def}}{=} H^{2n}_T(X^{[n]})$ such that

$$t^n \cup (x \star y) = x \cup y.$$ 

There is also an induced non-degenerate bilinear form $\langle - , - \rangle_n : \mathbb{H}_n \otimes \mathbb{H}_n \to \mathbb{C}$. We define

$$\mathbb{H}_X = \bigoplus_{n=0}^{\infty} \mathbb{H}_n.$$ 

The bilinear forms $\langle - , - \rangle_n$ on $\mathbb{H}_n$ for all $n$ induce a bilinear form on $\mathbb{H}_X$, which is denoted by $\langle - , - \rangle$. Given a linear operator $f \in \text{End}(\mathbb{H}_X)$, we denote by $f^*$ the adjoint operator with respect to this bilinear form. The unit in $H^*_T(X^{[0]})$ will be denoted by $|0\rangle$.

For a fixed point $\xi_\lambda$ of $X^{[n]}$, define $[\lambda] \in \mathbb{H}_n$ as in $\text{[Vas]}$ by

$$t^n \cup [\lambda] = (-1)^n h(\lambda)^{-1} [\xi_\lambda] \quad (2.1)$$
where \( h(\lambda) = \prod_{\square \in \lambda} h(\square) \) is the product of the hook numbers associated to \( \lambda \).

**Remark 2.1.** Passing from \( H^*_T(X^{[n]})' \) to \( \mathbb{H}_n \) does not lose information since we can recover the cup product and bilinear form on \( H^*_T(X^{[n]})' \) from those on \( \mathbb{H}_n \). Thus, understanding the ring \( \mathbb{H}_n \) is the same as understanding the equivariant intersection theory on \( X^{[n]} \).

2.3. **The Heisenberg algebra.** Let \( i > 0 \). Denote by \( Y \) the \( T \)-invariant subspace \( 0 \times \mathbb{C} \subset X \). We define

\[
Y_{n,i} = \{ (\xi, \eta) \in X^{[n+i]} \times X^{[n]} \mid \eta \subset \xi, \text{Supp}(I_\eta/I_\xi) = \{ y \} \in Y \}.
\]

Let \( p_1 \) and \( p_2 \) be the projections of \( X^{[n+i]} \times X^{[n]} \) to the two factors. We define the linear operator \( p_{-i} \in \text{End}(\mathbb{H}_X) \) by

\[
p_{-i}(\alpha) = D^{-1} p_1! (p_2^* \alpha \cap [Y_{n,i}]) \in \mathbb{H}_{n+i}
\]

for \( \alpha \in H^2_T(X^{[n]}) \). Note that the restriction of \( p_1 \) to \( Y_{n,i} \) is proper. We define \( p_i \in \text{End}(\mathbb{H}_X) \) to be the adjoint operator of \( p_{-i} \). Alternatively, letting \( p'_2 \) be the projection of \((X^{[n]})^T \times X^{[n-i]} \) to \( X^{[n-i]} \), we see that

\[
p_i(\alpha) = (-1)^i \cdot D^{-1} p'_2 (I_{n} \times \text{Id})^{-1}(p_2^* \alpha \cap [Y_{n-i,i}]) \in \mathbb{H}_{n-i}
\]

for \( \alpha \in H^2_T(X^{[n]}) \). Finally, we put \( p_0 = 0 \). An argument parallel to the one in [Na1] leads to the following, cf. [Vas].

**Proposition 2.2.** The operators \( p_n, n \in \mathbb{Z}, \) acting on \( \mathbb{H}_X = \bigoplus_{n=0}^{\infty} \mathbb{H}_n \) satisfy the following Heisenberg commutation relation:

\[
[p_m, p_n] = m\delta_{m,-n} I.
\]

Furthermore, the space \( \mathbb{H}_X \) becomes the Fock space (i.e. irreducible module) over the Heisenberg algebra with highest weight vector \( |0\rangle \).

Given a partition \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) of \( n \), we denote \( \lambda = \prod_{r\geq 1} r^{m_r} m_r! \), which is the order of the centralizer of an element in \( S_n \) of cycle type \( \lambda \). We then define

\[
p_{-\lambda} = \frac{1}{\delta_{\lambda}} \prod_{r\geq 1} r^{m_r} p_{-r} \langle 0 \rangle.
\]

The \( p_{-\lambda} \)'s, as \( \lambda \) runs over all partitions of \( n \), form a linear basis of \( \mathbb{H}_n \). One has

\[
\langle p_{-\lambda}, p_{-\mu} \rangle = \frac{1}{\delta_{\lambda}} \delta_{\lambda,\mu}.
\]

\(^1\)There are sign typos in \([Vas]\) on the formula for the annihilation operators and Heisenberg commutation relations.
3. The equivariant Chern character operator

3.1. The equivariant Chern characters. Let \( Z_n = \{ (\xi, x) \in X[n] \times X \mid x \in \text{supp}(\xi) \} \) be the codimension 2 universal subscheme. Denote by \( O[n] \) the tautological rank \( n \) vector bundle \( \pi_1^*(O_{Z_n} \otimes \pi_2^*O) = \pi_1^*(O_{Z_n}) \) on \( X[n] \) induced from the trivial line bundle \( O \) on \( X \), where \( \pi_1, \pi_2 \) denote the projections of \( X[n] \times X \) to the factors. Clearly \( O[n] \) is \( T \)-equivariant over \( X[n] \).

The computation of the torus action on \( X[n] \) (cf. e.g. \[Na2\]) implies that

\[
O[n]|_{\xi_\lambda} = \bigoplus_{\square \in \lambda} c_\square \tag{3.1}
\]

as a \( T \)-module, where \( c_\square \) is the content of the box \( \square \) in the Young diagram associated to \( \lambda \). Denote by \( \text{ch}^{[n]}_{k,T} \) the \( k \)-th \( T \)-equivariant Chern character of \( O[n] \). In particular the zero-th Chern character \( \text{ch}^{[n]}_{0,T} \) equals the rank of the vector bundle \( O[n] \), which is \( n \). Then

\[
\text{ch}^{[n]}_{k,T}|_{\xi_\lambda} = \frac{1}{k!} \sum_{\square \in \lambda} (c_\square t)^k.
\]

By the projection formula, we have in \( H^*_T(X[n])' \) that

\[
\text{ch}^{[n]}_{k,T} \cup [\xi_\lambda] = \frac{1}{k!} \sum_{\square \in \lambda} c_\square k t^k = \frac{1}{k!} \sum_{\square \in \lambda} c_\square k[t]^k [\xi_\lambda]. \tag{3.2}
\]

Let \( k \) be a nonnegative integer. Denote

\[
\widetilde{\text{ch}}^{[n]}_k = t^{n-k} \text{ch}^{[n]}_{k,T} \in \mathbb{H}_n.
\]

We define an operator \( \mathcal{G}_0 \), resp. \( \mathcal{G}_z \), in \( \text{End}(\mathbb{H}_X) \) by sending \( a \in \mathbb{H}_n \) to \( a \ast \sum_{k \geq 0} \widetilde{\text{ch}}^{[n]}_k \), resp. to \( a \ast \sum_{k \geq 0} \widetilde{\text{ch}}^{[n]}_k \), in \( \mathbb{H}_n \) for each \( n \). Similarly, we define an operator \( \mathcal{G}_z \) by sending \( a \in \mathbb{H}_n \) to \( a \ast \sum_{k \geq 0} z^k \widetilde{\text{ch}}^{[n]}_k \) for each \( n \), where \( z \) is a variable. By definition, we have

\[
\mathcal{G}_z = \sum_{k \geq 0} \mathcal{G}_k z^k.
\]

Formula (3.2) is equivalent to

\[
\mathcal{G}_z(\lambda) = \sum_{\square \in \lambda} e^{z c_\square} \cdot [\lambda]. \tag{3.3}
\]

We introduce

\[
\varsigma(z) = e^{z/2} - e^{-z/2}.
\]
Lemma 3.1. Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we have
\[
\mathcal{G}_z([\lambda]) = \frac{1}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i-i+1/2)} - \frac{1}{\varsigma(z)} \right) \cdot [\lambda].
\]

Proof. The contents \( c_{\Box} \)'s of \( \lambda \) are:
\[-i+1, -i+2, \ldots, -i+\lambda_i, \quad i = 1, \ldots, \ell(\lambda)\].
Noting that
\[
e^{z(-i+1)} + \ldots + e^{z(-i+\lambda_i)} = \frac{e^{z(\lambda_i-i+1)} - e^{z(-i+1)}}{e^{z} - 1},
\]
the formula (3.3) can be written as
\[
\mathcal{G}_z([\lambda]) = \frac{1}{\varsigma(z)} \sum_{i=1}^{\infty} \left( e^{z(\lambda_i-i+1/2)} - e^{z(-i+1/2)} \right) \cdot [\lambda]
\]
where we have used \( \lambda_i = 0 \) for \( i > \ell(\lambda) \). Now the lemma follows from the identity \( \sum_{i=1}^{\infty} e^{z(-i+1/2)} = 1/\varsigma(z) \). \qed

Remark 3.2. Comparing the above with the computation in Lascoux-Thibon [LT], Lemma 3.1, we observe that the Jucys-Murphy (JM) elements of the symmetric group \( S_n \) corresponds exactly to the equivariant Chern roots of the rank \( n \) bundle \( \mathcal{O}^{[n]} \). A similar observation in the non-equivariant setup has played an important role in the study of the Chen-Ruan orbifold cohomology ring of the symmetric product [QW]. In particular, the first equivariant Chern character of \( \mathcal{O}^{[n]} \) corresponds to the conjugacy classes of transpositions (compare [FW]). It was showed in [Vas] that there is a ring isomorphism between \( \mathbb{H}_n \) and the class algebra \( R(S_n) \) of the symmetric group \( S_n \), which sends \( [\lambda] \) to the Schur function \( s_{\lambda} \) etc. Our results above and observation in Remark 3.2 make the dictionary between the two rings more explicit. For the convenience of the reader, we compile the following dictionary table.

| Hilbert Scheme Setup | Symmetric Group Setup |
|---------------------|-----------------------|
| the ring \( \mathbb{H}_n \) | the class algebra \( R(S_n) \) |
| fixed-point class \( [\lambda] \) | Schur function \( s_{\lambda} \) |
| Heisenberg monomial \( p_{-\lambda} \) | conjugacy class (c.c.) \( K_{\lambda} \) |
| Chern character \( \widetilde{\text{ch}}^{[n]}_1 \) | c.c. \( K_{21n-2} \) of transpositions |
| Chern character \( \widetilde{\text{ch}}^{[n]}_k \) | \( k \)-th power-sum of JM elements |

Remark 3.3. By the dictionary, Theorem 5.10 in [Wa] on the ring generators of \( R(S_n) \) implies that \( \widetilde{\text{ch}}^{[n]}_k \), \( 0 \leq k < n \), form a set of ring generators of \( \mathbb{H}_n \).
3.2. The moduli space \( M(m, n) \). Denote by \( O_m \) the \( T \)-equivariant line bundle over \( X = \mathbb{C}^2 \) associated to the \( T \)-character \( t^m \), where \( m \in \mathbb{Z} \). Let \( M(m, n) \) be the moduli space which parameterizes all rank-1 subsheaves of \( O_m \) such that the quotients are supported at finitely many points of \( X \) and have length \( n \). Given \( I \in X^{[n]} \), then \( O_m \otimes I \) is an element in \( M(m, n) \).

As before, the study of the equivariant cohomology ring \( H^*_T(M(m, n)) \) leads to a ring \( \mathbb{H}_n^{(m)} \) whose product is denoted by \( \ast \). The natural identification \( M(m, n) \cong X^{[n]} \) leads to the natural identification of the rings \( \mathbb{H}_n^{(m)} \cong \mathbb{H}_n \), which induces a bilinear form \( \langle - , - \rangle_n^{(m)} \) on \( \mathbb{H}_n^{(m)} \) from \( \mathbb{H}_n \). We introduce

\[
\mathcal{F}^{(m)} = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^{(m)}, \quad m \in \mathbb{Z}
\]

\[
\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}^{(m)}.
\]

In particular we identify \( M(0, n) = X^{[n]} \) and \( \mathbb{H}_n^{(0)} = \mathbb{H}_n \).

We denote by \( S \) the isomorphism \( S : \mathbb{H}_n^{(m)} \to \mathbb{H}_n^{(m+1)} \). This induces isomorphisms (which will be denoted by \( S \) again) \( S : \mathcal{F} \to \mathcal{F} \) and \( S : \mathcal{F}^{(m)} \to \mathcal{F}^{(m+1)} \) for all \( m \in \mathbb{Z} \). The bilinear form on \( \mathcal{F} \) induced from \( \langle - , - \rangle_n^{(m)} \) on \( \mathbb{H}_n^{(m)} \) will be again denoted by \( \langle - , - \rangle \).

3.3. Operators on the fermionic Fock space. By the standard boson-fermion correspondence \([MJD]\), \( \mathcal{F} \) can be identified with the fermionic Fock space (or equivalently, the infinite wedge space). The operator \( S \) is exactly the shift operator on the fermionic Fock space. Given an operator \( f \in \text{End}(\mathcal{F}) \), the number \( \langle f \rangle := \langle |0\rangle , f |0\rangle \) is called the vacuum expectation of \( f \).

It has been well known (cf. e.g. \([MJD, Wa]\)) that the completed infinite-rank general linear Lie algebra \( \hat{gl}_\infty \) (whose standard basis is denoted by \( E_{i,j} \), \( i, j \in \mathbb{Z} + 1/2 \)) acts on the fermionic Fock space. As will become clear below (cf. Lemma 3.6 and Lemma 5.1), the study of the equivariant intersection theory on Hilbert schemes naturally leads to the following operator in \( \text{End}(\mathcal{F}) \):

\[
\mathfrak{H}(z) = \frac{1}{s(z)} \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{kz} E_{k,k}.
\]

The operator is further expanded as

\[
\mathfrak{H}(z) = \sum_{k=0}^{\infty} \mathfrak{H}_k z^k.
\]

Remark 3.4. The operator \( \mathfrak{H}(z) \) in a somewhat different form has appeared in the study of the class algebras of the symmetric groups and wreath products \([LT, Wa]\), and it affords a compact expression in terms of vertex operators.
On the other hand, the following operators

\[ \varepsilon_r(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{(k-r/2)} \frac{E_{k-r,k}}{\varsigma(z)}, \quad r \in \mathbb{Z} \quad (3.4) \]

have been introduced by Okounkov and Pandharipande, and they have played a fundamental role in the study of the stationary Gromov-Witten invariants \([OP]\). The operators \(\varepsilon_r(z), \quad r \in \mathbb{Z}\), satisfy the following identities:

\[ [p_k, \varepsilon_r(z)] = \varsigma(kz) \varepsilon_{k+r}(z) \quad (3.5) \]

\[ S^{-1} \varepsilon_0(z) S = e^z \varepsilon_0(z) \quad (3.6) \]

\[ \varepsilon_0(z)([\lambda]) = \sum_{i=1}^{\infty} e^{z(\lambda_i-i+1/2)} \cdot [\lambda] \quad (3.7) \]

for all partitions \(\lambda = (\lambda_1, \lambda_2, \ldots)\) (here we recall \([\lambda] \in \mathbb{H}_X = \mathcal{F}^{(0)}\)).

**Lemma 3.5.** We have the following identities for operators on \(\mathcal{F}\):

\[ S^{-1} \mathcal{H}(z) S = e^z \mathcal{H}(z) \quad (3.8) \]

\[ S^{-m} \mathcal{H}(z) S^m = e^{mz} \mathcal{H}(z) + \frac{e^{mz} - 1}{\varsigma(z)^2} I, \quad m \in \mathbb{Z} \quad (3.9) \]

**Proof.** The first identity follows from the definitions. The second one follows from the first one and (3.6). \(\square\)

**Lemma 3.6.** As an operator on \(\mathcal{F}^{(0)} = \mathbb{H}_X\), we have the identification

\[ \mathcal{G}_z = \mathcal{H}(z) \]

**Proof.** Follows from Lemma 3.1, formula (3.7) and Lemma 3.5. \(\square\)

In light of the interpretation of \(\mathcal{H}(z)\) in Lemma 3.6 and the role of \(\varepsilon_0(z)\) in \([OP]\), the identity (3.8) defines the Gromov-Witten/Hilbert correspondence.

4. The \(n\)-point functions of equivariant intersection numbers

4.1. Equivariant intersection numbers on Hilbert schemes. Keeping in mind Remarks 2.1 and 3.3, one way to understand the equivariant intersection theory on \(X^{[n]}\) is to study the products

\[ \tilde{\text{ch}}_{k_1}^{[n]} \ast \tilde{\text{ch}}_{k_2}^{[n]} \ast \ldots \ast \tilde{\text{ch}}_{k_s}^{[n]} \]

for arbitrary nonnegative integers \(k_1, \ldots, k_s, \quad s \geq 1\).

For partitions \(\lambda\) and \(\mu\) of \(n\), we may consider the intersection numbers

\[ \langle \lambda, \tilde{\text{ch}}_{k_1}^{[n]} \ast \tilde{\text{ch}}_{k_2}^{[n]} \ast \ldots \ast \tilde{\text{ch}}_{k_s}^{[n]} \ast \mathfrak{p}_\mu \rangle_n \quad (4.1) \]
The main idea below and in later sections is to form some suitable generating functions of these intersection numbers and then reformulate them in terms of the operator formalism.

4.2. The $n$-point function. Given partitions $\lambda$ and $\mu$ of $n$, we organize the intersection numbers $\langle \lambda, \widetilde{c}_k^1 \ldots \widetilde{c}_k^N, \mu \rangle$ defined in (4.1) into a generating function by introducing the $N$-point function

$$G_{\lambda,\mu}(z_1, \ldots, z_N) = \sum_{k_1, \ldots, k_N} z_1^{k_1} \ldots z_N^{k_N} \langle \lambda, \widetilde{c}_k^1 \ldots \widetilde{c}_k^N, \mu \rangle_n.$$ 

This can be reformulated in an operator form:

$$G_{\lambda,\mu}(z_1, \ldots, z_N) = \langle p_{-\lambda}, \mathcal{G}_{z_1} \ldots \mathcal{G}_{z_N} p_{-\mu} \rangle_n
= \langle p_{-\lambda}, \mathcal{H}(z_1) \ldots \mathcal{H}(z_N) p_{-\mu} \rangle_n.$$ 

We also define similarly

$$F^*_{\lambda,\mu}(z_1, \ldots, z_N) = \langle p_{-\lambda}, \varepsilon_0(z_1) \ldots \varepsilon_0(z_N) p_{-\mu} \rangle_n.$$ 

According to [OP], Section 3, $F^*_{\lambda,\mu}(z_1, \ldots, z_N)$ has the interpretation as the $N$-point disconnected series of stationary Gromov-Witten invariants of $\mathbb{P}^1$ relative to $0, \infty \in \mathbb{P}^1$.

Note that, in particular for $N = 0$, we have defined $G_{\lambda,\mu}()$ and $F^*_{\lambda,\mu}()$. It is easy to see that

$$G_{\lambda,\mu}() = F^*_{\lambda,\mu}() = \frac{\delta_{\lambda,\mu}}{\delta \lambda}.$$ 

It follows from (3.8) and Lemma 3.6 that the 1-point function is given by

$$G_{\lambda,\mu}(z) = \frac{1}{\varsigma(z)} \left( F^*_{\lambda,\mu}(z) - \frac{1}{\varsigma(z)} F^*_{\lambda,\mu}() \right).$$ 

Therefore, it remains to compute $F^*_{\lambda,\mu}(z)$. Note that the somewhat simpler connected series $F^0_{\lambda,\mu}$ rather than the disconnected series has been computed in [OP], Section 3.2.

We introduce some notations on partitions. Given two partitions $\lambda$ and $\mu$, we denote by $\lambda + \mu$ the partition obtained from combining the parts of $\lambda$ and $\mu$ and rearranging them in a descending order. For two partitions $\lambda = (1^{m_1}2^{m_2} \ldots)$ and $\mu = (1^{n_1}2^{n_2} \ldots)$, we say $\lambda \subset \mu$ if $m_i \leq n_i$ for all $i$, and denote $\mu - \lambda$ the partition $(1^{n_1-m_1}2^{n_2-m_2} \ldots)$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $r = \ell(\lambda)$, we denote

$$\varsigma(\lambda, z) = \varsigma(\lambda_1 z) \ldots \varsigma(\lambda_r z).$$
Given a subset \( U \subset \underline{r} = \{1, \ldots, r\} \), we denote by \( \lambda_U \) the subpartition of \( \lambda \) which consists of the parts \( \lambda_i, i \in U \).

**Proposition 4.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) and \( \mu = (\mu_1, \ldots, \mu_s) \) be partitions of \( n \), where \( r = \ell(\lambda) \) and \( s = \ell(\mu) \). We have

\[
F_{\lambda,\mu}(z) = \sum_{U} \frac{\varsigma(\lambda_U, z) \varsigma(\lambda_U + \mu - \lambda, z)}{3\lambda \delta_{\lambda_U+\mu-\lambda}}
\]

summed over subsets \( U \subset \underline{r} \) such that \( \lambda \subset \lambda_U + \mu \).

**Proof.** We denote by \([p_{\lambda_U}, \varepsilon_0(z)]\) the multi-commutator \([\cdots [p_{\lambda_0}, [p_{\lambda_1}, \varepsilon_0(z)]]] \cdots\), if we write \( U = \{a, b, \ldots\} \). Note that the multi-commutator is independent of the ordering of elements in \( U \) since the \( p_k \)'s \( (k > 0) \) commute with each other.

By moving \( \varepsilon_0(z) \) to the left whenever possible, we have

\[
p_{\lambda_r} \cdots p_{\lambda_1} \varepsilon_0(z) = \sum_{U \subset \underline{r}} [p_{\lambda_U}, \varepsilon_0(z)] \prod_{i \in \underline{r} \setminus U} p_{\lambda_i}
\]

\[
= \sum_{U \subset \underline{r}} \varsigma(\lambda_U, z) \varsigma(\lambda_U, z) \prod_{i \in \underline{r} \setminus U} p_{\lambda_i}
\]

where we have repeatedly used (3.5). It follows that

\[
F_{\lambda,\mu}^*(z) = \frac{1}{3\lambda \delta_{\mu}} \langle p_{\lambda_r} \cdots p_{\lambda_1} \varepsilon_0(z) p_{-\mu_1} \cdots p_{-\mu_s} \rangle
\]

\[
= \frac{1}{3\lambda \delta_{\mu}} \sum_{U \subset \underline{r}} \varsigma(\lambda_U, z) \langle \varepsilon_{|\lambda_U|}(z) \prod_{i \in \underline{r} \setminus U} p_{\lambda_i} \cdot p_{-\mu_1} \cdots p_{-\mu_s} \rangle
\]  

(4.4)

Denote by \( \lambda_U^- \) to be the partition which consists of the parts \( \lambda_i, i \in \underline{r} \setminus U \). Apparently, the vacuum expectation

\[
\langle \varepsilon_{|\lambda_U|}(z) \prod_{i \in \underline{r} \setminus U} p_{\lambda_i} \cdot p_{-\mu_1} \cdots p_{-\mu_s} \rangle = 0 \quad \text{unless } \lambda_U^- \subset \mu.
\]

If \( \lambda_U^- \subset \mu \), or equivalently if \( \lambda \subset \lambda_U + \mu \), then \( \mu - \lambda_U^- = (\lambda_U + \mu) - \lambda \), and we can show by induction that

\[
\prod_{i \in \underline{r} \setminus U} p_{\lambda_i} \cdot p_{-\mu_1} \cdots p_{-\mu_s} |0\rangle = \frac{\delta_{\mu}}{3\lambda_U + \mu - \lambda} p_{-\mu_1} \cdots p_{-\mu_1} |0\rangle
\]

(4.5)

where we have denoted \( \mu - \lambda_U^- = (\mu_1, \ldots, \mu_s) \).

By (4.4) and (4.5), we have

\[
F_{\lambda,\mu}^*(z) = \frac{1}{3\lambda \delta_{\mu}} \sum_{U \subset \underline{r}} \frac{\delta_{\mu}}{3\lambda_U + \mu - \lambda} \varsigma(\lambda_U, z) \langle \varepsilon_{|\lambda_U|}(z) p_{-\mu_1} \cdots p_{-\mu_s} \rangle
\]

(4.6)
Similar to (4.3), we now move \( \varepsilon_{|\lambda_U|}(z) \) to the right in \( \langle \varepsilon_{|\lambda_U|}(z) p_{-\mu_a} \cdots p_{-\mu_t} \rangle \) whenever possible. Note that if \( \varepsilon_K(z) (K > 0) \) results from such a move, then the corresponding vacuum expectation is zero. Therefore

\[
\langle \varepsilon_{|\lambda_U|}(z) p_{-\mu_a} \cdots p_{-\mu_t} \rangle = \langle [\cdots [\varepsilon_{|\lambda_U|}(z), p_{-\mu_a}], \cdots p_{-\mu_t}] \rangle \\
= \varsigma(\lambda_U + \mu - \lambda, z) \langle \varepsilon_0(z) \rangle \\
= \varsigma(\lambda_U + \mu - \lambda, z)/\varsigma(z).
\]  
(4.7)

Now the proposition follows from (4.6) and (4.7).

**Theorem 4.2.** The 1-point function \( G_{\lambda, \mu}(z) \) is given by

\[
G_{\lambda, \mu}(z) = \frac{1}{3\lambda \varsigma(z)^2} \left( \sum_U \frac{\varsigma(\lambda_U, z) \varsigma(\lambda_U + \mu - \lambda, z)}{3\lambda_U + \mu - \lambda} - \delta_{\lambda, \mu} \right)
\]

where \( U \) runs over the subsets of \( \ell(\lambda) \) such that \( \lambda \subset \lambda_U + \mu \).

**Proof.** Follows from (4.2) and Proposition 4.1.

In general, by Lemma 3.6, we have the following.

**Proposition 4.3.**

\[
G_{\lambda, \mu}(z_1, \ldots, z_N) = \frac{1}{\prod_{i=1}^N \varsigma(z_i)} \sum_{U \subseteq N} \frac{(-1)^{|N| - |U|}}{\prod_{i \in N \setminus U} \varsigma(z_i)} F_{\lambda, \mu}^U(z_U).
\]

where we have denoted \( F_{\lambda, \mu}^U(z_U) = \langle p_{-\mu}, \prod_{i \in U} \varepsilon_0(z_i) p_{-\mu} \rangle_n \).

**Remark 4.4.** Note that the \( N \)-point connected series \( F_{\lambda, \mu}^n \) of Gromov-Witten invariants rather than the disconnected series \( F_{\lambda, \mu}^U \) has been computed in [OP], Section 3.3. The strategy used in Proposition 4.1 could be generalized to compute the \( N \)-point disconnected series \( F_{\lambda, \mu}^U(z_1, \ldots, z_N) \), although the notations would be a bit involved. Then the computation of \( G_{\lambda, \mu}(z_1, \ldots, z_N) \) follows by Proposition 4.3.

4.3. **The multi-point trace function.** The \( q \)-trace of an operator \( f \in \text{End}(\mathbb{H}_X) \) is defined to be

\[
\text{Tr}_q f \overset{\text{def}}{=} \sum_{\lambda} \mathfrak{d}_\lambda \langle p_{-\lambda}, f(p_{-\lambda}) \rangle q^{[\lambda]}.
\]

In particular, for the identity operator \( I \) on \( \mathbb{H}_X \), we have \( \text{Tr}_q I = 1/(q; q)_\infty \), where we have used the notation \( (a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots \).

Our main object here is the multi-point trace function \( \text{Tr}_q(\mathfrak{G}_{z_1} \cdots \mathfrak{G}_{z_N}) \) of a product of the operators \( \mathfrak{G}_{z_j} \), which encodes information about the intersection numbers \( (1.1) \). Here \( z_1, \ldots, z_N \) are independent variables.

We denote

\[
\Theta(z) = \Theta(z; q) \overset{\text{def}}{=} \eta(q)^{-3} \sum_{m \in \mathbb{Z}} (-1)^m q^{(m+1/2)^2} e^{(m+1/2)z} \\
= (e^{z/2} - e^{-z/2})(q z; q)_\infty(q z^{-1}; q)_\infty/(q; q)_\infty^2
\]
Here $\eta(q) = q^{1/24}(q; q)_\infty$ is the Dedekind eta function, and the last identity above uses the Jacobi triple product identity. We further define

$$\Theta^{(k)}(z) = \frac{d^k}{dz^k}\Theta(z), \quad k \geq 0.$$ 

We agree that $\Theta^{(k)}(z) = 0$ for $k < 0$.

Given a positive integer $N$, we denote $\mathcal{N} = \{1, 2, \ldots, N\}$. Given a finite set $U$, we denote by $S_U$ the symmetric group on $U$. In particular, $S_{\mathcal{N}} = S_N$. Given $U = \{u_1, \ldots, u_k\} \subset \mathcal{N}$ with $u_1 < \ldots < u_k$ and a permutation $\sigma \in S_U$, we denote by $M_{U, \sigma}$ the $k \times k$ matrix whose $(i, j)$-th entry is $\Theta^{(j-i+1)}(z_{\sigma u_1} \cdots z_{\sigma u_{k-j}})/(j-i+1)!$.

We further denote by $\Theta_{U, \sigma}$ the product $\Theta(z_{\sigma u_1})\Theta(z_{\sigma u_2})\cdots\Theta(z_{\sigma u_k})$.

A main result of Bloch-Okounkov ([BO], Theorem 0.5) is the following formula for $\text{Tr}_q(\varepsilon_0(z_1)\cdots\varepsilon_0(z_N))$ in our notation:

$$\text{Tr}_q(\varepsilon_0(z_1)\cdots\varepsilon_0(z_N)) = \frac{1}{(q; q)_\infty} \sum_{\sigma \in S_{\mathcal{N}}} \frac{\det M_{\mathcal{N}, \sigma}^U}{\Theta_{\mathcal{N}, \sigma}^U}. \quad (4.8)$$

The formula (4.8) also computes the stationary Gromov-Witten invariants of an elliptic curve, according to Okounkov and Pandharipande [OP].

**Theorem 4.5.** We have

$$\text{Tr}_q(\mathfrak{G}_{z_1} \cdots \mathfrak{G}_{z_N}) = \frac{1}{(q; q)_\infty} \prod_{i=1}^N \zeta(z_i) \sum_{U \subset \mathcal{N}} (-1)^{N-|U|} \sum_{\sigma \in S_U} \frac{\det M_{U, \sigma}^U}{\Theta_{U, \sigma}^U}. \quad (4.8)$$

**Proof.** By Lemma 3.3 and Lemma 3.6 we have

$$\text{Tr}_q(\mathfrak{G}_{z_1} \cdots \mathfrak{G}_{z_N}) = \frac{1}{\prod_{i=1}^N \zeta(z_i)} \sum_{U \subset \mathcal{N}} (-1)^{N-|U|} \text{Tr}_q(\prod_{i \in U} \varepsilon_0(z_i)).$$

Now the theorem follows by applying (4.8) by replacing the set $\mathcal{N}$ by $U$. \hfill $\square$

5. **Equivariant intersection and $\tau$-functions of 2-Toda hierarchies**

5.1. **Hilbert schemes and $\tau$-functions.** Let $t = (t_1, t_2, \ldots)$ and $s = (s_1, s_2, \ldots)$ be two sequences of indeterminates. Define the following half vertex operators:

$$\Gamma_{\pm}(t) = \exp \left( \sum_{k>0} t_k p_{\pm k}/k \right).$$

Given a partition $\mu = (\mu_1, \mu_2, \ldots)$ we write $t_\mu = t_{\mu_1}t_{\mu_2}\cdots$. Let $x = (x_1, x_2, \ldots)$ be another sequence of indeterminates. We introduce the following generating function for the intersection numbers $\langle \lambda, \tilde{\chi}_{k_1}, \ldots \tilde{\chi}_{k_n}, \mu \rangle$ defined in [11]:

$$\tau(x, t, s) = \sum_n \sum_{|\lambda|=|\mu|=n} t_\lambda s_\mu \langle \lambda, \exp \left( \sum_{k=0}^{\infty} x_k \tilde{\chi}_k^{[n]} \right), \mu \rangle_n.$$
Note that $\Gamma^-(s) = \sum_{n \geq 0} \sum_{|\lambda|=n} t^{\lambda} p_{-\lambda}$, and $\Gamma^+(t) = \Gamma^-(t)^*$. From the definition of $\tilde{\mathcal{F}}_k$ and Lemma 3.6 we see that the $\tau$-function affords an operator formulation:

$$
\tau(x, t, s) = \left( \Gamma^+(t) \exp \left( \sum_{k=0}^{\infty} x_k \tilde{\mathcal{F}}_k \right) \right) \Gamma^-(s).
$$

5.2. **The Chern character operators from $\mathcal{M}(m, n)$**. We have a universal exact sequence:

$$
0 \to J_m \to \pi_2^* \mathcal{O}_m \to Q_m \to 0
$$

where $\pi_1, \pi_2$ are the projections of $\mathcal{M}(m, n) \times X$ to the two factors. Denote by $\mathcal{O}^{[n]}_m$ the $T$-equivariant vector bundle (i.e. torsion-free sheaf) over $\mathcal{M}(m, n)$ of rank $n$ given by the push-forward $\pi_1^*(Q_m)$, whose fiber over a point $\xi_\lambda \in \mathcal{M}(m, n)$ is given by $\mathcal{O}^{[n]}_m|_{\xi_\lambda}$.

In the same way as defining the operators $\mathcal{G}_z$ and $\mathcal{G}_k(\geq 0)$ acting on $\mathbb{H}_X = \mathcal{F}^{(0)}$, we can define the operators $\mathcal{G}^{(m)}_z$ and $\mathcal{G}^{(m)}_k(\geq 0)$ acting on $\mathcal{F}^{(m)}$ using the cup products with $\sum_{k \geq 0} t^{n-k} \text{ch}_k(\mathcal{O}^{[n]}_m) z^k$ and $t^{n-k} \text{ch}_k(\mathcal{O}^{[n]}_m)$ respectively on $\mathbb{H}^{(m)}_n$. For technical reasons below, we introduce the following modification

$$
\tilde{\text{ch}}_k(\mathcal{O}^{[n]}_m) = t^{n-k} \text{ch}_k(\mathcal{O}^{[n]}_m) + c^{(m)}_k
$$

where the constant $c^{(m)}_k$ is defined by

$$
\frac{e^{mz} - 1}{\zeta(z)^2} = mz^{-1} + \sum_{k \geq 0} c^{(m)}_k z^k / k!.
$$

Equivalently, if we define

$$
\tilde{\mathcal{G}}^{(m)}_z = \mathcal{G}^{(m)}_z + \frac{e^{mz} - 1}{\zeta(z)^2} I. \quad (5.2)
$$

and further write $\tilde{\mathcal{G}}^{(m)}_z = mz^{-1}I + \sum_{k \geq 0} \tilde{\mathcal{G}}^{(m)}_k z^k$, then

$$
\tilde{\mathcal{G}}^{(m)}_k = \mathcal{G}^{(m)}_k + c^{(m)}_k I, \quad k \geq 0
$$

and $\tilde{\mathcal{G}}^{(m)}_k$ acts on $\mathbb{H}^{(m)}_n$ by the product with $\tilde{\text{ch}}_k(\mathcal{O}^{[n]}_m)$. When $m = 0$, we have $c^{(0)}_k = 0$ and $\tilde{\mathcal{G}}^{(0)}_k = \mathcal{G}^{(0)}_k = \mathcal{G}_k$ for all $k$.

Similarly we define the equivariant intersection numbers for $\mathcal{M}(m, n)$, denoted by $\langle - \rangle^{(m)}_n$. When $m = 0$ it reduces to the ones defined earlier.
5.3. **The τ-functions and \( \mathcal{M}(m,n) \).** We form the following generating function of the equivariant intersection numbers on \( \mathcal{M}(m,n) \):

\[
\tau(x, t, s, m) = \sum_n \sum_{|\lambda|=|\mu|=n} t^\lambda s^\mu \left( \lambda, \exp \left( \sum_{k=0}^{\infty} x_k \tilde{c}h_k(\mathcal{O}_m^{[n]}) \right), \mu \right)_n^{(m)}.
\]

In particular we have \( \tau(x, t, s, 0) = \tau(x, t, s) \).

**Lemma 5.1.** As an operator on \( \mathcal{F}^{(m)} \), we have the identification

\[ \tilde{G}^{(m)}_z = \mathcal{H}(z). \]

**Proof.** Under the identification \( S^m : \mathbb{H}_n \rightarrow \mathbb{H}_n^{(0)} \rightarrow \mathbb{H}_n^{(m)} \), we denote by \( [\lambda]^{(m)} \) the image of \( [\lambda] \). By the identification of toric action \( (5.1) \), the same proof as in Lemma 3.1 implies that

\[ \tilde{G}^{(m)}_z([\lambda]^{(m)}) = \left( \frac{e^{mz}}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} \right) - \frac{1}{\varsigma(z)} \right) \cdot [\lambda]^{(m)}. \]

By \( S^m([\lambda]) = [\lambda]^{(m)} \), we have

\[ \mathcal{H}(z)([\lambda]^{(m)}) = \mathcal{H}(z)S^m([\lambda]) = e^{mz}S^m\mathcal{H}(z)([\lambda]) + \frac{e^{mz} - 1}{\varsigma(z)^2}S^m([\lambda]). \]

This can be rewritten as

\[ \mathcal{H}(z)([\lambda]^{(m)}) = \left( \frac{e^{mz}}{\varsigma(z)} \left( \sum_{i=1}^{\infty} e^{z(\lambda_i - i + 1/2)} \right) - \frac{1}{\varsigma(z)} \right) \cdot [\lambda]^{(m)} \]

by lemma 3.1 and Lemma 3.6. This finishes the proof. \( \square \)

**Theorem 5.2.**

1. The function \( \tau(x, t, s, m) \) can be reformulated as:

\[ \tau(x, t, s, m) = \left( S^{-m} \Gamma_+ (t) \exp \left( \sum_{k=0}^{\infty} x_k \mathcal{H}_k \right) \Gamma_-(s) S^m \right). \]

2. The functions \( \tau(x, t, s, m), m \in \mathbb{Z} \), satisfies the 2-Toda hierarchy of Ueno-Takasaki [UT]. The lowest equation among the hierarchy reads:

\[
\frac{\partial^2}{\partial t_1 \partial s_1} \ln \tau(t, s, x, m) = \frac{\tau(t, s, x, m + 1) \tau(t, s, x, m - 1)}{\tau(t, s, x, m)^2}.
\]

**Proof.** Part (1) follows from the definition of \( \tau(x, t, s, m) \) and Lemma 5.1. The second part is standard since the operator \( \mathcal{H}_k \) lies in \( \hat{\mathfrak{gl}}_\infty \). \( \square \)
By setting $t_2 = t_3 = \cdots = s_2 = s_3 = \cdots = 0$ and $x_2 = x_3 = \cdots = 0$ in $\tau(x, t, s)$, we obtain the following generating function of the intersection numbers on Hilbert schemes:

$$\sum_n \left( t_1 s_1 e^{x_0} \right)^n \left\{ \frac{1}{n!} p_{-1}^n \exp \left( x_1 \mathcal{G}_1 \right) \cdot \frac{1}{n!} p_{-1}^n \right\}_n$$

thanks to the fact that $\mathcal{G}_0(\lambda) = |\lambda| \cdot [\lambda]$. Setting $u = x_0 + \ln(t_1 s_1)$, we denote the above generating function by $\tau(u, x_1)$. A simple computation reduces the Toda equation (5.3) to the following:

$$e^{-u} \frac{\partial^2}{\partial u^2} \ln \tau(u, x_1) = \frac{\tau(u + x_1, x_1) \tau(u - x_1, x_1)}{\tau(u, x_1)^2}.$$ 

It is interesting to observe that this $\tau$ function can also be interpreted as generating functions of certain Hurwitz numbers.

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