On the almost everywhere and norm convergences of $T$ means with respect to Vilenkin systems

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Preface

The classical Fourier Analysis has been developed in an almost unbelievable way from the first fundamental discoveries by name Fourier. Especially a number of wonderful results have been proved and new directions of such research has been developed e.g. concerning Wavelets Theory, Gabor theory, Time-Frequency Analysis, Fast Fourier Transform, Abstract Harmonic Analysis, etc. One important reason for this is that this development is not only important for improving the "State of the art", but also for its importance in other areas of mathematics and also for several applications (e.g. theory of signal transmission, multiplexing, filtering, image enhancement, coding theory, digital signal processing and pattern recognition).

The classical theory of Fourier series deals with decomposition of a function into sinusoidal waves. Unlike these continuous waves the Vilenkin (Walsh) functions are rectangular waves. The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series. Because of this it is inevitable to compare results of Vilenkin series to those on trigonometric series. There are many similarities between these theories, but there exist differences also. Much of these can be explained by modern abstract harmonic analysis, which studies orthonormal systems from the point of view of the structure of a topological group.

The aim of my master thesis is to discuss, develop and apply the newest developments of this fascinating theory connected to modern harmonic analysis. In particular, we investigate $T$ means but only in the case when their coefficients are monotone and prove convergence in Lebesgue and Vilenkin-Lebesgue points. Since almost everywhere points are Lebesgue and Vilenkin-Lebesgue points for any integrable functions we obtain almost everywhere convergence of such summability methods.
Key words

Key words: Vilenkin systems, Vilenkin groups, $T$ means, Nörlund means, a.e. convergence, Lebesgue points, Vilenkin-Lebesgue points.
Key words
Chapter 1

Introduction

1.1 Vilenkin Groups and Functions

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$, $\mathbb{Z}$ the set of the integers, $\mathbb{R}$ the real numbers, $\mathbb{R}_+$ the positive real numbers, $\mathbb{C}$ the complex numbers. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_k}$ with the product of the discrete topologies of $Z_{m_k}$.

The direct product $\mu$ of the measures $\mu_k(j) := 1/m_k \ (j \in Z_{m_k})$ is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded, then $G_m$ is said to be an unbounded Vilenkin group.

In this book we discuss only bounded Vilenkin groups, i.e. the case when $\sup_{n \in \mathbb{N}} m_n < \infty$.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_j \in Z_{m_j}).$$

It is easy to give a base for the neighborhoods of $G_m$:

$$I_0(x) : = G_m,$$

$$I_n(x) : = \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \ n \in \mathbb{N}).$$
CHAPTER 1. INTRODUCTION

We call subsets \( I_n(x) \subseteq G_m \) Vilenkin intervals. Let

\[ e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m \quad (n \in \mathbb{N}) . \]

If we define \( I_n := I_n(0) \), for \( n \in \mathbb{N} \) and \( \overline{I}_n := G_m \setminus I_n \), then

\[ \overline{I}_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} = \left( \bigcup_{k=0 \atop l=k+1}^{N-2} I_{N}^{k,l} \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_{N}^{k,N} \right) , \quad (1.1.1) \]

where

\[ I_{N}^{k,l} := \begin{cases} I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots), \\ \text{for} \quad k < l < N, \\ I_N(0, \ldots, 0, x_k \neq 0, x_{k+1} = 0, \ldots, x_{N-1} = 0, x_N, \ldots), \\ \text{for} \quad l = N. \end{cases} \]

If we define the so-called generalized number system based on \( m \) in the following way :

\[ M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}), \]

then every \( n \in \mathbb{N} \) can be uniquely expressed as

\[ n = \sum_{j=0}^{\infty} n_j M_j, \]

where \( n_j \in \mathbb{Z}_{m_j} \quad (j \in \mathbb{N}+) \) and only a finite number of \( n_j \)'s differ from zero.

The Vilenkin group can be metrizable with the following metric:

\[ \rho (x, y) := |x - y| := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{M_{k+1}}, \quad (x, y \in G_m). \]

For the natural numbers \( n = \sum_{j=1}^{\infty} n_j M_j \) and \( k = \sum_{j=1}^{\infty} k_j M_j \) we define

\[ n^* = \sum_{j=1}^{\infty} (m_j - n_j) M_j, \quad n_j \in \mathbb{Z}_{m_j} \quad (j \in \mathbb{N}), \]

\[ n + k := \sum_{i=0}^{\infty} (n_i \oplus k_i) M_{i+1} \]

and
\[ n \hat{k} := \sum_{i=0}^{\infty} (n_i \ominus k_i)M_{i+1}, \]

where

\[ a_i \oplus b_i := (a_i + b_i) \mod m_i, \quad a_i, b_i \in \mathbb{Z}_{m_i} \]

and \( \ominus \) is the inverse operation for \( \oplus \).

Next, we introduce on \( G_m \) an orthonormal system, which is called Vilenkin system (see [89, 90, 91]).

At first, we define the complex-valued function \( r_k(x) : G_m \to \mathbb{C} \), the generalized Rademacher functions, by

\[ r_k(x) := \exp \left( 2\pi i x k/m_k \right), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}). \quad (1.1.2) \]

Now, define the Vilenkin systems \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) by:

\[ \psi_n(x) := \prod_{k=0}^{\infty} r_k^n (x), \quad (n \in \mathbb{N}). \quad (1.1.3) \]

Specifically, we call this system the Walsh-Paley system when \( m \equiv 2 \).

**Proposition 1.1.1** (see [1]) Let \( n \in \mathbb{N} \). Then

\[
\begin{align*}
|\psi_n(x)| &= 1, \\
\psi_n(x+y) &= \psi_n(x) \psi_n(y), \\
\psi_n(-x) &= \psi_n^*(x) = \overline{\psi_n(x)}, \\
\psi_{n+k}(x) &= \psi_k \psi_n(x), \quad (k, n \in \mathbb{N}, \ x, y \in G_m).
\end{align*}
\]

A character on a commutative group \( I \) is a continuous complex-valued function which satisfies

\[ f(x + y) = f(x) f(y) \quad \text{and} \quad |f(x)| = 1, \]

for all \( x, y \in I \).

Let us denote by \( \hat{I} \) the set of all character functions of \( I \).

**Proposition 1.1.2** The Vilenkin functions are characters of \( G_m \). Therefore every character has the form \( (1.1.3) \).
The direct product \( \mu \) of the measures
\[
\mu_k (\{j\}) := 1/m_k \quad (j \in \mathbb{Z}_{m_k})
\]
is the Haar measure on \( G_m \) with \( \mu (G_m) = 1 \). Translation of a subset \( I_n(x) \in G_m \) by \( y \) is defined by
\[
\tau_y (I_n(x)) = \{I_n(x) + y\}.
\]
Since \( \mu \) is a product measure we get that
\[
\mu (I_n(x)) = \prod_{k=0}^{n-1} \mu_k (\{x_k\}) \prod_{k=0}^{\infty} \mu_k (\{0, \ldots, m_k - 1\})
\]
\[
= \prod_{k=0}^{n-1} \frac{1}{m_k} \prod_{k=0}^{\infty} 1 = \frac{1}{M_n}.
\]
On the other hand,
\[
\mu (\tau_y (I_n(x))) = \mu (I_n(x + y)) = 1/M_n
\]
for all \( y \in G_m \). Hence,
\[
\mu (\tau_y (I_n(x))) = \mu (I_n(x)).
\]

In particular, \( G_m \) is a locally compact abelian group and it can be equipped with the Haar measure (see Pontryagin [55]), which coincides with the product measure \( \mu \). We shall not construct Haar measure here because for the groups we consider, the construction is trivial.

**Proposition 1.1.3** Let \( n, k \in \mathbb{N} \). Then
\[
\int_{G_m} \psi_n d\mu = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}
\]
Moreover, the Vilenkin systems are orthonormal, that is,
\[
\int_{G_m} \psi_n \psi_k d\mu = \begin{cases} 1 & n = k, \\ 0 & n \neq k. \end{cases}
\]
1.2 \( L_p \) and weak-\( L_p \) Spaces

By a Vilenkin polynomial we mean a finite linear combination of Vilenkin functions. We denote the collection of Vilenkin polynomials by \( \mathcal{P} \).

Let \( L^0(G_m) \) represent the collection of functions which are almost everywhere limits with respect to a measure \( \mu \) of sequences in \( \mathcal{P} \).

For \( 0 < p < \infty \) let \( L^p(G_m) \) represent the collection of \( f \in L^0(G_m) \) such that

\[
\|f\|_p := \left( \int_{G_m} |f|^p \, d\mu \right)^{1/p}
\]

is finite.

Denote by \( L^\infty(G_m) \) the space of all \( f \in L^0(G_m) \) for which

\[
\|f\|_\infty := \inf \{ C > 0 : \mu \{ x \in G_m : |f| > C \} = 0 \} < +\infty.
\]

The space \( C(G_m) \) consist all continuous function for which

\[
\|f\|_C := \sup_{x \in G_m} |f(x)| < c < \infty.
\]

It is evident that if \( f \in C(G_m) \), then

\[
\|f\|_C = \|f\|_\infty \quad \text{and} \quad C(G_m) \subset L^\infty(G_m).
\]

Moreover, if \( 1 < p_1 < p_2 \leq \infty \), then \( L^{p_2}(G_m) \subset L^{p_1}(G_m) \).

**Proposition 1.2.1** (see \([86]\)) If \( f \in L^p(G_m) \), \( \forall p > 0 \), then \( f \in L^\infty(G_m) \) and

\[
\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.
\]

By using Minkowski’s inequality we easily obtain that

\[
\|f\|_p = 0 \iff f = 0 \quad \text{a.e.}
\]

\[
\|cf\|_p = |c| \|f\|_p \quad (c \in \mathbb{C}),
\]

\[
\|f + g\|_p \leq c_p \left( \|f\|_p + \|g\|_p \right),
\]

where \( c_p = 1 \), for \( p \geq 1 \) and \( c_p \leq 2^{1/p} \) for \( p < 1 \). Because of the last property \( \|\cdot\|_p \) is a norm for \( p \geq 1 \) and quasi-norm for \( p < 1 \).
Proposition 1.2.2 (see [86]) Well-known Minkowski’s integral inequality is given by

\[ \left\| \int_{G_m} f(\cdot, t) \, dt \right\|_p \leq \int_{G_m} \| f(\cdot, t) \|_p \, dt, \quad \text{for all } p \geq 1. \]

The convolution of two functions \( f, g \in L^1(G_m) \) is defined by

\[ (f \ast g)(x) := \int_{G_m} f(x - t) g(t) \, dt \quad (x \in G_m). \]

It is easy to see that

\[ (f \ast g)(x) = \int_{G_m} f(t) g(x - t) \, dt \quad (x \in G_m). \]

Proposition 1.2.3 Let \( f \in L^r(G_m), g \in L^1(G_m) \) and \( 1 \leq r < \infty \). Then \( f \ast g \in L^r(G_m) \) and

\[ \| f \ast g \|_r \leq \| f \|_r \| g \|_1. \]

First we present the following very important proposition:

Proposition 1.2.4 Since the Vilenkin function \( \psi_m \) is constant on \( I_n(x) \) for every \( x \in G_m \) and \( 0 \leq m < M_n \), it is clear that each Vilenkin function is a complex-valued step function, that is, it is a finite linear combination of the characteristic functions. On the other hand, notice that, by Lemma 1.3.3 (Paley’s Lemma), it yields that

\[ \chi_{I_n(t)}(x) = \frac{1}{M_n} \sum_{j=0}^{M_n-1} \psi_j(x - t), \quad x \in I_n(t), \]

for each \( x, t \in G_m \) and \( n \in \mathbb{N} \). Thus each step function is a Vilenkin polynomial. Consequently, we obtain that the collection of step functions coincides with a collection of Vilenkin polynomials \( \mathcal{P} \).

Since the Lebesgue measure is regular it follows that given \( f \in L^1 \) there exist Vilenkin polynomials

\[ P_1, P_2, \ldots, \text{ such that } P_n \rightarrow f \quad \text{a.e. as } n \rightarrow \infty. \]

Moreover, any \( f \in L^p(G_m) \) can be written in the form \( f = g - h \) where the functions \( g, h \) are almost everywhere limits of increasing sequences of non-negative Vilenkin polynomials. In particular, \( \mathcal{P} \) is dense in the space \( L^p \), for all \( p \geq 1 \).
The space \( \text{weak} - L^p(G_m) \) consists of all measurable functions \( f \), for which
\[
\|f\|_{\text{weak} - L^p} := \sup_{y > 0} \{\mu(f > y)\}^{1/p} < +\infty.
\]

The following properties of the weak-\( L^p(G_m) \) spaces can be shown easily:
\[
\|f\|_{\text{weak} - L^p} = 0 \iff f = 0 \text{ a.e.}
\]
\[
\|cf\|_{\text{weak} - L^p} = |c| \|f\|_{\text{weak} - L^p} \quad (c \in \mathbb{C}),
\]
\[
\|f + g\|_{\text{weak} - L^p} \leq c_p \left( \|f\|_{\text{weak} - L^p} + \|g\|_{\text{weak} - L^p} \right),
\]
where \( c_p = \max(2, 2^{1/p}) \). Because of the last property \( \|\cdot\|_{\text{weak} - L^p} \) is a quasi-norm.

**Proposition 1.2.5** (see [86]) If \( 0 < p \leq \infty \), then \( L^p(G_m) \subset \text{weak} - L^p(G_m) \) and
\[
\|f\|_{\text{weak} - L^p} \leq \|f\|_p.
\]

**Proof.** It is easy to see that
\[
\int_{G_m} |f(x)|^p \, dx \geq \int_{\{x:|f(x)| > y\}} |f(x)|^p \, dx \geq y^p \mu(|f| > y),
\]
which proves the proposition.

Note that the inclusion \( L^p(G_m) \subset \text{weak} - L^p(G_m) \) is proper for any \( 0 < p < \infty \). Indeed, let
\[
h(x) := |x|^{-1/p}.
\]
Then obviously \( h \notin L^p(G_m) \), but \( h \in \text{weak} - L^p(G_m) \) because
\[
y^p \mu(\{x:|x|^{-1/p} > y\}) = 2y^p y^{-p} = 2.
\]
Recall that the space weak-\( L^p(G_m) \) is also complete for each \( p \geq 1 \).

An operator \( T \) which maps a linear space of measurable functions on \( G_m \) in the collection of measurable functions on \( G_m \) is called sublinear if
\[
|T(f + g)| \leq |T(f)| + |T(g)| \quad \text{a.e. on } G_m \quad \text{and} \quad |T(\alpha f)| = |\alpha||T(f)|
\]
for all scalars \( \alpha \) and all \( f \) in the domain of \( T \).

An operator \( T \) from \( L^p(G_m) \) into \( L^0(G_m) \) is said to be of strong type \( (p,q) \) (or \( (p,q) \) type) for some \( 1 \leq p \leq \infty \) and \( 1 < q < \infty \) if there is a constant \( A > 0 \) such that.
\[ \|Tf\|_q \leq A \|f\|_p \]

for all \( f \in L^p(G_m) \).

An operator \( T \) from \( L^p(G_m) \) into \( L^q(G_m) \) is said to be of weak type \((p, q)\) for some \( 1 \leq p \leq \infty \) and \( 1 < q < \infty \) if there is a constant \( A > 0 \) such that

\[ \|Tf\|_{\text{weak-}L^q} \leq A \|f\|_p \]

for all \( f \in L^p(G_m) \). Hence, any operator of type \((L^p, L^q)\) is necessarily of weak type \((p, q)\).

**Theorem 1.2.6** (see [86]) *(Marcinkiewicz interpolation theorem)* Let \( 1 \leq p_1 < p_2 \) and suppose that the sublinear operator \( T \) satisfies

\[ y^{p_i} \lambda_{Tf}(y) \leq C_i \int_{G_m} |f(y)|^{p_i} \, d\mu(y), \]

for all \( y > 0, f \in L^p(G_m) \) and \( i = 1, 2 \), where the \( C_i \)'s are absolute constants independent of \( y \) and \( f \). In the case when \( p_2 = \infty \) suppose that \( T \) is of strong type, i.e. that

\[ \|Tf\|_{\infty} \leq C_2 \|f\|_{\infty}. \]

Then

\[ \|Tf\|_p \leq p^{2/p} C_1^{p_2 - p} C_2^{p_1 - p_1} \left( \frac{1}{p - p_1} + \frac{1}{p_2 - p} \right) \|f\|_p, \]

for all \( f \in L^p(G_m) \), \( p_1 < p < p_2 \).

In particular, if \( p_1 < p < p_2 \) and operator \( T \) is of weak \((p_1, p_1)\) type and weak \((p_2, p_2)\) type, then it will be of \((p, p)\) type.

**Theorem 1.2.7** (see [99, 100]) Suppose that \((X, \|\|_p)\) are normed linear spaces in \( L^0 \) and \( X_0 \) is dense in \( X \). Let \( T, T_n : X \to L^p(G_m) \) be sub-linear operators, for some \( 1 \leq p < \infty \) with \( T \) bounded and \( T_n f \to Tf \) a.e. on \( G_m \) as \( n \to \infty \), for each \( f \in X_0 \). Set

\[ T^* f := \sup_{n \in \mathbb{N}} |T_n f|, \quad f \in X. \]

If there is a constant \( C > 0 \), independent of \( f \) and \( n \), such that

\[ y^p \mu(\{|Tf| > y\}) \leq C \|f\|_X^p \] (1.2.1)

and

\[ y^p \mu(\{|T^* f| > y\}) \leq C \|f\|_X^p, \] (1.2.2)

for all \( y > 0 \) and \( f \in X \), then

\[ Tf = \lim_{n \to \infty} T_n f \]

a.e. on \( G_m \), for every \( f \in X \).
1.3 Dirichlet and Vilenkin-Fejér Kernels

If $f \in L^1(G_m)$ we can define the Fourier coefficients, the partial sums of Vilenkin-Fourier series, the Dirichlet kernels with respect to Vilenkin systems in the usual manner:

$$
\hat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \quad (n \in \mathbb{N}),
$$

$$
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}^+),
$$

$$
D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}^+),
$$

respectively. It is easy to see that

$$
S_n f (x) = \int_{G_m} f(t) \sum_{k=0}^{n-1} \psi_k (x - t) d\mu(t) = \int_{G_m} f(t) D_n (x - t) d\mu(t) = (f * D_n)(x).
$$

The next well-known identities with respect to Dirichlet kernels (see Lemmas 1.3.1 and 1.3.2, Lemma 1.3.3) will be used many times in the proofs of our main results:

**Lemma 1.3.1 (see [1])** Let $n \in \mathbb{N}$. Then

$$
D_{j+M_n} = D_{M_n} + \psi_{M_n} D_j = D_{M_n} + r_n D_j, \quad j \leq (m_n - 1) M_n \quad (1.3.1)
$$

and

$$
D_{M_n-j}(x) = D_{M_n}(x) - \overline{\psi}_{M_n-1}(-x) D_j(-x) = D_{M_n}(x) - \psi_{M_n-1}(x) D_j(x), \quad j < M_n. \quad (1.3.2)
$$

**Lemma 1.3.2 (see [1])** Let $n \in \mathbb{N}$ and $1 \leq s_n \leq m_n - 1$. Then

$$
D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n-1} \psi_{k M_n} = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k \quad (1.3.3)
$$

and

$$
D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{k=m_j-n_j}^{m_j-1} r_j^k \right), \quad (1.3.4)
$$

for $n = \sum_{i=0}^{\infty} n_i M_i$. 

Lemma 1.3.3  (see [1] and [29])  (Paley’s Lemma)  Let $n \in \mathbb{N}$. Then
\[
D_{M_n}(x) = \begin{cases} 
M_n, & x \in I_n, \\
0, & x \notin I_n.
\end{cases}
\]

We also need the following estimate:

Lemma 1.3.4  (see [1])  Let $n \in \mathbb{N}$. Then
\[
\|D_{M_n}\|_1 = 1.
\]

It is obvious that
\[
\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} (D_k * f)(x)
\]
\[
= (f * K_n)(x) = \int_{G_m} f(t) K_n(x-t) \, d\mu(t),
\]
where $K_n$ are the so called Fejér kernels:
\[
K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k.
\]

Using Abel transformation we get another representation of Fejér means
\[
\sigma_n f(x) = \sum_{k=0}^{n-2} \left(1 - \frac{k}{n}\right) \hat{f}(k) \psi_k(x)
\]

We frequently use the following well-known result:

Lemma 1.3.5  (see [22])  Let $n > t$, $t, n \in \mathbb{N}$. Then
\[
K_{M_n}(x) = \begin{cases} 
\frac{M_n}{1-r(x)}, & x \in I_t \setminus I_{t+1}, \\
\frac{M_n+1}{2}, & x \in I_n, \\
0, & \text{otherwise}.
\end{cases}
\]

The proof of the next lemma can easily be done by using Lemma 1.3.5

We also need the following useful result:

Lemma 1.3.6  (see [1] and [29])  Let $t, s_n, n \in \mathbb{N}$, and $1 \leq s_n \leq m_n - 1$. Then
\[
s_n M_n K_{s_n M_n} = \sum_{l=0}^{s_n-1} \left( \sum_{i=0}^{l-1} r_n^i \right) M_n D_{M_n} + \left( \sum_{l=0}^{s_n-1} q_n^l \right) M_n K_{M_n} \quad (1.3.5)
\]
The next equality for Fejér kernels is very important for our further investigations:

**Lemma 1.3.7** (see [1] and [29]) Let \( n = \sum_{i=1}^{r} s_{n_i} M_{n_i} \), where \( n_1 > n_2 > \cdots > n_r \geq 0 \) and \( 1 \leq s_{n_i} < m_{n_i} \) for all \( 1 \leq i \leq r \) as well as \( n^{(k)} = n - \sum_{i=1}^{k} s_{n_i} M_{n_i} \), where \( 0 < k \leq r \). Then

\[
nK_n = \sum_{k=1}^{r} \left( \prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) s_{n_k} M_{n_k} K_{s_{n_k} M_{n_k}} + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) n^{(k)} D_{s_{n_k} M_{n_k}}. \tag{1.3.6}
\]

We will also frequently use the next estimation of the Fejér kernels:

**Corollary 1.3.8** Let \( n \in \mathbb{N} \). Then

\[
n |K_n| \leq c \sum_{l=(n)}^{[n]} M_l |K_{M_l}| \leq c \sum_{l=0}^{[n]} M_l |K_{M_l}| \tag{1.3.7}
\]

where \( c \) is an absolute constant.

**Lemma 1.3.9** (see [1] and [29]) Let \( n \in \mathbb{N} \). Then, for any \( n, N \in \mathbb{N}_+ \), we have that

\[
\int_{G_m} K_n(x) \, d\mu(x) = 1, \tag{1.3.8}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |K_n(x)| \, d\mu(x) \leq c < \infty, \tag{1.3.9}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |K_n(x)| \, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \tag{1.3.10}
\]

where \( c \) is an absolute constant.

**Lemma 1.3.10** (see [3, 4, 11, 22]) Let \( \{ q_k : k \in \mathbb{N} \} \) be a sequence of non-decreasing numbers. Then, for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_n(x) \, d\mu(x) = 1, \tag{1.3.11}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n(x)| \, d\mu(x) \leq c < \infty, \tag{1.3.12}
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_n(x)| \, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \tag{1.3.13}
\]

where \( c \) is an absolute constant.
Chapter 2

$T$ Means of Vilenkin-Fourier series in Lebesgue Spaces

2.1 Introduction

In the literature, a point $x \in G_m$ is called a Lebesgue point of $f \in L^1(G_m)$, if

$$\lim_{n \to \infty} M_n \int_{I_n(x)} f(t) \, dt = f(x) \quad a.e. \ x \in G_m.$$ 

It is well-known (for details see [86] and [59]) that if $f \in L^1(G_m)$ then almost every point is a Lebesgue point and the following important result holds true:

**Proposition 2.1.1** Let $f \in L^1(G_m)$. Then, for all Lebesgue points $x$,

$$\lim_{n \to \infty} S_{M_n} f(x) = f(x)$$

Fejér’s theorem shows that (see e.g books [15] and [16]) if one replaces ordinary summation by Cesàro summation $\sigma_n$ defined by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

then the Féjer means of Fourier series of any integrable function converges a.e on $G_m$ to the function.
Goginava and Gogoladze [25] introduced the operator $W_A$ defined by

$$W_A f(x) := \sum_{s=0}^{A-1} \sum_{r_s=1}^{M_s-1} \int_{I_A(x-r_se_s)} |f(t) - f(x)| \, d\mu(t).$$

and define a Vilenkin-Lebesgue point of function $f \in L^1(G_m)$, as a point for which

$$\lim_{A \to \infty} W_A f(x) = 0$$

Moreover, they also proved that the following result is true:

**Proposition 2.1.2** Let $f \in L^1(G_m)$. Then

$$\lim_{n \to \infty} \sigma_n f(x) = f(x)$$

for all Vilenkin-Lebesgue points of $f$.

If we consider the maximal operator of Féjer means $\sigma^*$ defined by:

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

then

$$y \mu \{ \sigma^* f > y \} \leq c \| f \|_1, \quad f \in L^1(G_m), \quad y > 0.$$ 

This result can be found in Zygmund [100] for trigonometric series, in Schipp [50] for Walsh series and in Pál, Simon [45] for bounded Vilenkin series.

The boundedness does not hold from Lebesgue space $L^1(G_m)$ to the space $L^1(G_m)$. On the other hand, if we consider restricted maximal operator $\tilde{\sigma}^*_\#$ of Féjer means defined by

$$\tilde{\sigma}^*_\# f := \sup_{n \in \mathbb{N}} |\sigma_{M_n} f|$$

then there exists a function $f \in L^1(G_m)$ such that

$$\| \tilde{\sigma}^*_\# f \|_1 = \infty.$$
Introduction

In the one-dimensional case Yano \cite{95} proved that

$$\|\sigma_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty, \quad (f \in L^p(G_m), \quad 1 \leq p \leq \infty).$$

However (see \cite{30, 59}) the rate of convergence can not be better then $O(n^{-1})$ ($n \to \infty$) for non-constant functions, i.e., if $f \in L^p$, $1 \leq p \leq \infty$ and

$$\|\sigma_n f - f\|_p = o\left(\frac{1}{M_n}\right), \quad \text{as} \quad n \to \infty,$$

then $f$ is a constant function.

It is also known that (see e.g. the books \cite{1} and \cite{59}) for any $1 \leq p \leq \infty$ and $n \in \mathbb{N}$ we have the following estimate

$$\|\sigma_n f - f\|_p \leq c_p \omega_p \left(\frac{1}{M_N}, f\right) + c_p \sum_{s=0}^{N-1} \frac{M_s}{M_N} \omega_p \left(\frac{1}{M_s}, f\right).$$

where $\omega_p \left(\frac{1}{M_n}, f\right)$ is the modulus of continuity of function $f \in L^p$. By applying this estimate, we immediately obtain that if $f \in \text{lip}(\alpha, p)$, i.e.,

$$\omega_p \left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n}\right), \quad n \to \infty,$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\left(\frac{1}{M_N}\right), \quad \text{if} \quad \alpha > 1, \\ O\left(\frac{N}{M_N}\right), \quad \text{if} \quad \alpha = 1, \\ O\left(\frac{1}{M_N}\right), \quad \text{if} \quad \alpha < 1. \end{cases}$$

Convergence and approximation in various norms of Vilenkin-Fejér means can be found in Blahota, Gát and Goginava \cite{5, 6}, Blahota, and Tephnadze \cite{8, 9, 10, 11, 12, 13, 14}, Fine \cite{17, 18}, Fridli \cite{19}, Fujii \cite{20}, Goginava \cite{23, 24}, Gogolashvili Nagy and Tephnadze \cite{26, 27, 28}, Persson and Tephnadze \cite{47, 48} (see also \cite{3, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16}), Pál and Simon \cite{17, 18}, Schipp \cite{56, 57, 58}, Simon \cite{60, 61}, Tephnadze \cite{84, 85, 86, 87, 88} (see also \cite{66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94}), Tutteridze \cite{87, 88}, Weisz \cite{92, 93, 94} and Zhizhiashvili \cite{96, 97, 98}. Similar problems for the two-dimensional case can be found in Nagy \cite{36, 37, 38, 39}, Nagy and Tephnadze \cite{41, 42, 43, 44}. 

The properties established in Lemma 1.3.9 ensure that kernel of the Fejér means \( \{ K_N \}_{N=1}^{\infty} \) form what is called an approximation identity.

**Definition 2.1.3** The family \( \{ \Phi_n \}_{n=1}^{\infty} \subset L^\infty(G_m) \) forms an approximate identity provided that

\[
\begin{align*}
(A1) & \quad \int_{G_m} \Phi_n(x) d(x) = 1 \\
(A2) & \quad \sup_{n \in \mathbb{N}} \int_{G_m} |\Phi_n(x)| d\mu(x) < \infty \\
(A3) & \quad \sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |\Phi_n(x)| d\mu(x) \to 0, \text{ as } n \to \infty, \text{ for any } N \in \mathbb{N}_+.
\end{align*}
\]

The term "approximate identity" is used because of the fact that

\[ \Phi_n * f \to f \quad \text{as } n \to \infty \]

in any reasonable sense. In particular, the following results holds true (for details see the books [21] and [35]):

**Proposition 2.1.4** Let \( f \in L^p(G_m) \), where \( 1 \leq p \leq \infty \) and the family

\( \{ \Phi_n \}_{n=1}^{\infty} \subset L^\infty(G_m) \)

forms an approximate identity. Then

\[ \| \Phi_n * f - f \|_p \to 0 \quad \text{as } n \to \infty. \]

Another well-known example of approximation identity are Riesz logarithmic means \( R_n \) defined by

\[ R_n f := \frac{1}{l_n} \sum_{k=1}^{n} S_k f, \]

where

\[ l_n := \sum_{k=1}^{n} \frac{1}{k}. \]

If we consider the maximal operator \( R^* \) of the Riesz logarithmic means, defined by

\[ R^* f := \sup_{n \in \mathbb{N}} |R_n f|, \]
then
\[ y\mu \{ R^* f > y \} \leq c \| f \|_1, \quad f \in L^1(G_m), \quad y > 0. \]

Moreover, for any \( f \in L^1(G_m) \), we have that
\[ \lim_{n \to \infty} R_n f(x) = f(x) \]
for all Vilenkin-Lebesgue points of \( f \).

The boundedness of the maximal operator of Reisz logarithmic means does not hold from \( L^1(G_m) \) to the space \( L^1(G_m) \). However
\[ \| R_n f - f \|_p \to 0, \quad n \to \infty, \quad (f \in L^p(G_m), \quad 1 \leq p \leq \infty). \]

It is well-known that the \( n \)-th Nörlund mean \( t_n \) and \( T \) means \( T_n \) for the Fourier series of \( f \) are, respectively, defined by
\[ t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f, \quad (2.1.1) \]
and
\[ T_n f := \frac{1}{Q_n} \sum_{k=0}^{n-1} q_k S_k f, \quad (2.1.2) \]
where \( \{ q_k : k \in \mathbb{N} \} \) is a sequence of nonnegative numbers and
\[ Q_n := \sum_{k=0}^{n-1} q_k. \]

Let \( \{ q_k : k \geq 0 \} \) be a sequence of nonnegative numbers where \( q_0 > 0 \). Then the summability method (2.1.2) generated by \( \{ q_k : k \geq 0 \} \) is regular if and only if (see [34])
\[ \lim_{n \to \infty} Q_n = \infty. \]

The representations
\[ t_n f (x) = \int_{G_m} f(t) F_n (x - t) \, d\mu(t) \]
and
\[ T_n f (x) = \int_{G_m} f(t) F_n^{-1} (x - t) \, d\mu(t) \]
Nörlund and $T$ Means of Vilenkin-Fourier series in Lebesgue Spaces

play central roles in the sequel, where

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k \quad (2.1.3)$$

and

$$F_n^{-1} := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k \quad (2.1.4)$$

are called the kernels of Nörlund and $T$ means, respectively.

$T$ means are generalizations of the Fejér and the Reisz logarithmic means. This means that all Nörlund and $T$ means are approximation identity. According to all these facts it is of prior interest to study the behavior of operators related to $T$ means of Fourier series with respect to orthonormal systems.

Let us define maximal operator of $T$ means by

$$T^* f := \sup_{n \in \mathbb{N}} |T_n f| .$$

If $\{q_k : k \in \mathbb{N}\}$ is non-increasing and satisfying the condition

$$\frac{1}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty, \quad (2.1.5)$$

or if $\{q_k : k \in \mathbb{N}\}$ is non-decreasing, satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty, \quad (2.1.6)$$

then

$$y \mu \{T^* f > y\} \leq c \|f\|_1, \quad f \in L^1(G_m), \quad y > 0.$$

The boundedness of the maximal operator of $T$ means does not hold from $L^1(G_m)$ to the space $L^1(G_m)$. However

$$\|T_n f - f\|_p \to 0, \quad \text{as} \quad n \to \infty, \quad (f \in L^p(G_m), \quad 1 \leq p \leq \infty).$$
2.2. WELL-KNOWN AND NEW EXAMPLES OF $T$ MEANS

We define $B_n^{-1}$ means as the class of $T$ means, with monotone and bounded sequence $\{ q_k : k \in \mathbb{N} \}$, such that

$$0 < q < \infty \quad \text{where} \quad q_\infty := \lim_{n \to \infty} q_n.$$  

If the sequence $\{ q_k : k \in \mathbb{N} \}$ is non-decreasing, then we have that

$$nq_0 \leq Q_n \leq nq_\infty.$$  

In the case when the sequence $\{ q_k : k \in \mathbb{N} \}$ is non-increasing, then

$$nq_\infty \leq Q_n \leq nq_0.$$  

In both cases we can conclude that conditions (2.1.6) and (2.1.5) are fulfilled.

Well-known examples of Nörlund and $T$ means with monotone and bounded sequence $\{ q_k : k \in \mathbb{N} \}$ is Fejér means $\sigma_n$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f$$

It is evident that in this case conditions (2.1.6) and (2.1.5) are fulfilled.

The Cesàro means $\sigma_n^\alpha$ (sometimes also denoted $(C, \alpha)$) and its inverse $\sigma_n^{\alpha-1}$ are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f$$

and

$$\sigma_n^{\alpha-1} f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n-1} A_{k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \quad A_n^\alpha := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, ...$$

It is well-known that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \quad (2.2.1)$$

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1} \quad \text{and} \quad A_n^\alpha \sim n^\alpha. \quad (2.2.2)$$
It is obvious that
\[
\frac{|q_n - q_{n+1}|}{n^{\alpha-2}} = O(1), \quad \text{as} \quad n \to \infty, \quad (2.2.3)
\]
\[
\frac{q_0}{Q_n} = O\left(\frac{1}{n^{\alpha}}\right), \quad \text{as} \quad n \to \infty, \quad (2.2.4)
\]
and
\[
\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty. \quad (2.2.5)
\]

Let \(V_{\alpha,-1}^n\) denote the \(T\) mean, where
\[
\{q_k = (k+1)^{\alpha-1} : \quad k \in \mathbb{N}, \quad 0 < \alpha < 1\},
\]
that is
\[
V_{\alpha,-1}^n f := \frac{1}{Q_n} \sum_{k=1}^{n} (k+1)^{\alpha-1} S_k f.
\]

It is obvious that
\[
\frac{|q_n - q_{n+1}|}{n^{\alpha-2}} = O(1), \quad \text{as} \quad n \to \infty, \quad (2.2.6)
\]
\[
\frac{q_0}{Q_n} = O\left(\frac{1}{n^{\alpha}}\right), \quad \text{as} \quad n \to \infty, \quad (2.2.7)
\]
and
\[
\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty. \quad (2.2.8)
\]

We just remind again that \(n\)-th Riesz \(R_n\) logarithmic means are defined by the sequence \(\{q_k = 1/k, \quad k \in \mathbb{N}_+\}\):
\[
R_n f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k},
\]
where
\[
l_n := \sum_{k=1}^{n} \frac{1}{k}.
\]

It is evident that
\[
\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as} \quad n \to \infty. \quad (2.2.9)
\]
and
\[
\frac{q_0}{Q_n} = O \left( \frac{1}{\ln n} \right), \quad \text{as} \quad n \to \infty. \quad (2.2.10)
\]

Let \( U_\alpha^n \) denote the T mean, where
\[
\left\{ q_k = \frac{1}{(k+3)\ln^\alpha(k+3)} : k \in \mathbb{N}, \ 0 < \alpha \leq 1 \right\},
\]
that is
\[
U_\alpha^n f := \frac{1}{Q_n} \sum_{k=1}^{n} \frac{S_k f}{(k+3) \ln^\alpha(k+3)}.
\]

It is obvious that
\[
\frac{q_n^{-1}}{Q_n} = O \left( \frac{1}{n} \right), \quad \text{as} \quad n \to \infty \quad (2.2.11)
\]
and
\[
\frac{q_0}{Q_n} = \begin{cases} 
O \left( \frac{1}{\ln(\ln n)} \right), & \text{if } \alpha = 1, \\
O \left( \frac{1}{\ln^{1-\alpha} n} \right), & \text{if } 0 < \alpha < 1.
\end{cases} \quad (2.2.12)
\]

Let \( \alpha \in \mathbb{R}_+ \). If we define the sequence \( \{q_k : k \in \mathbb{N}\} \) by
\[
\{q_k = \log^\alpha(k+1) : k \in \mathbb{N} : \alpha > 0\},
\]
then we get the class of T means with non-increasing coefficients:
\[
\beta_\alpha^n f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^\alpha(k+1) S_k f.
\]

It is obvious that
\[
\frac{n}{2} \log^\alpha(n/2) \leq Q_n \leq n \log^\alpha n.
\]

It follows that
\[
\frac{1}{Q_n} \leq \frac{c}{n \log^\alpha n} = O \left( \frac{1}{n} \right) \to 0, \quad \text{as} \quad n \to \infty. \quad (2.2.13)
\]
and
\[
\frac{q_n^{-1}}{Q_n} \leq \frac{c \log^\alpha(n-1)}{n \log^\alpha n} = O \left( \frac{1}{n} \right) \to 0, \quad \text{as} \quad n \to \infty. \quad (2.2.14)
\]
2.3 Kernels of T Means

In this Section we investigate kernels of T means with respect to Vilenkin systems. We invoke Abel transformations for $a_j = A_j - A_{j-1}$, $j = 1, \ldots, n$, we find that

\[
\sum_{j=1}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_0 b_1 + \sum_{j=1}^{n-2} A_j (b_j - b_{j+1}) \quad (2.3.1)
\]

and

\[
\sum_{j=M_N}^{n-1} a_j b_j = A_{n-1} b_{n-1} - A_{M_N-1} b_{M_N} + \sum_{j=M_N}^{n-2} A_j (b_j - b_{j+1}). \quad (2.3.2)
\]

If we invoke

\[b_j = q_j, \ a_j = 1 \ \text{and} \ A_j = j \ \text{for} \ j = 0, 1, \ldots, n,
\]

then (2.3.1) and (2.3.2) give the following identities:

\[
Q_n := \sum_{j=0}^{n-1} q_j = q_0 + \sum_{j=1}^{n-1} q_j = q_0 + \sum_{j=1}^{n-2} (q_j - q_{j+1}) j + q_{n-1} (n - 1) \quad (2.3.3)
\]

and

\[
\sum_{j=M_N}^{n-1} q_j := \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) j + q_{n-1} (n - 1) - (M_N - 1) q_{M_N}. \quad (2.3.4)
\]

Moreover, if we use that $D_0 = K_0 = 0$ for any $x \in G_m$ and invoke the the Abel transformations (2.3.1) and (2.3.2) for

\[b_j = q_j, \ a_j = D_j \ \text{and} \ A_j = j K_j \ \text{for any} \ j = 0, 1, \ldots, n - 1
\]

we get identities:

\[
F_n^{-1} = \frac{1}{Q_n} \sum_{j=0}^{n-1} q_j D_j = \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1}) j K_j + q_{n-1} (n - 1) K_{n-1} \right) \quad (2.3.5)
\]

and
Kernels of T Means

\[
\frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j
\]

\[
= \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) jK_j + q_{n-1}(n-1)K_{n-1} - q_{M_N}(M_N - 1)K_{M_N-1} \right).
\]

Analogously, if we use that \( S_0 f = \sigma_0 f = 0 \), for any \( x \in G_m \) and apply the Abel transformations (2.3.1) and (2.3.2) for

\[ b_j = q_j, \ a_j = S_j \] and \( A_j = j\sigma_j \) for any \( j = 0, 1, ..., n - 1 \)

we get the identities:

\[
T_n f = \frac{1}{Q_n} \sum_{j=0}^{n-1} q_j S_j f
\]

\[
= \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_j - q_{j+1}) j\sigma_j f + q_{n-1}(n-1)\sigma_{n-1} f \right)
\]

and

\[
\frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j S_j f
\]

\[
= \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) j\sigma_j f + q_{n-1}(n-1)\sigma_{n-1} f - q_{M_N}(M_N - 1)\sigma_{M_N-1} f \right).
\]

First we consider kernels of T means with respect to Vilenkin systems generated by non-increasing sequences \( \{q_k : k \in \mathbb{N}\} \):

**Lemma 2.3.1** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers, satisfying the condition (2.1.5). Then, for some constant \( C \), we have that

\[
|F_n^{-1}| \leq \frac{C}{n} \left\{ \sum_{j=0}^{n} M_j |K_{M_j}| \right\}.
\]
Proof. Let the sequence \( \{q_k : k \in \mathbb{N}\} \) be non-increasing. Then, by using (2.3.3) we get that

\[
\frac{1}{Q_n} \left( q_0 + \sum_{j=1}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-2} (q_0 + q_j - q_{j+1}) + q_{n-1} \right) \\
\leq \frac{2q_0}{Q_n} \leq \frac{c}{n}.
\]

Hence, if we apply 1.3.7 and use the equality (2.3.5) we can conclude that

\[
|F_n^{-1}| \leq \left( q_0 + \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_{n-1} \right) \right) \sum_{i=0}^{[n]} M_i |K_{M_i}|
\]

\[
\leq \frac{2q_0}{Q_n} \sum_{i=0}^{[n]} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{[n]} M_i |K_{M_i}|.
\]

The proof is complete by just combining the estimates above. ■

Next, we generalize Lemma 2.3.1 but now without any restriction like the condition (2.1.5):

Lemma 2.3.2 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers and \( n > M_N \). Then

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j (x) \right| \leq \frac{c}{M_N} \left\{ \sum_{j=0}^{[n]} M_j |K_{M_j}(x)| \right\},
\]

where \( c \) is an absolute constant.

Proof. Since the sequence \( \{q_k : k \in \mathbb{N}\} \) is non-increasing we can readily conclude that

\[
\frac{1}{Q_n} \left( q_{M_N} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \leq \frac{1}{Q_n} \left( q_{M_N} + \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) + q_{n-1} \right)
\]

\[
= \frac{2q_{M_N}}{Q_n} \leq \frac{2q_{M_N}}{Q_{M_N+1}} \leq \frac{c}{M_N}.
\]

If we apply the Abel transformation (2.3.6) and (2.3.9) combined with (1.3.7) in Lemma 1.3.8 we get that

\[
\left| \frac{1}{Q_n} \sum_{j=M_N}^{n-1} q_j D_j \right| = \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-2} (q_j - q_{j+1}) j K_j + q_{n-1}(n-1) K_{n-1} - q_{M_N} (M_N - 1) K_{M_N-1} \right)
\]

\[
\leq \frac{c}{Q_n} \left( q_{M_N} + \sum_{j=M_N}^{n-2} |q_j - q_{j+1}| + q_{n-1} \right) \sum_{i=0}^{n-1} M_i |K_{M_i}|
\]

\[
\leq \frac{c}{M_N} \sum_{i=0}^{n-1} M_i |K_{M_i}|.
\]

The proof is complete.

The next Lemma is very important for our further investigation in this chapter to prove norm convergence in Lebesgue spaces of T means generated by non-increasing sequences \( \{q_k : k \in \mathbb{N}\} \):

**Lemma 2.3.3** Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then, for any \( n, N \in \mathbb{N}_+ \),

\[
\int_{G_m} F_n^{-1}(x) d\mu(x) = 1, \quad (2.3.10)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m} |F_n^{-1}(x)| d\mu(x) \leq c < \infty, \quad (2.3.11)
\]

\[
\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_n} |F_n^{-1}(x)| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad (2.3.12)
\]

where \( c \) is an absolute constant.

**Proof.** According to Lemma 1.3.4 we easily obtain (2.3.10). By using (1.3.9) in Lemma 1.3.9 combined with (2.3.3) and (2.3.5) we get that

\[
\int_{G_m} |F_n^{-1}(x)| d\mu(x)
\]

\[
\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \int_{G_m} |K_j| d\mu + q_{n-1}(n-1) \int_{G_m} |K_{n-1}| d\mu
\]

\[
\leq \frac{c}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j + \frac{q_{n-1}(n-1)}{Q_n} \leq c < \infty.
\]
so also (2.3.11) is proved. By using (1.3.10) in Lemma 1.3.9 and inequalities (2.3.3) and (2.3.5) we can conclude that

\[
\int_{G_m \setminus I_N} |F_n^{-1}(x)| \, d\mu(x) \leq \frac{1}{Q_n} \sum_{j=0}^{n-1} (q_j - q_{j+1}) j \int_{G_m \setminus I_N} |K_j(x)| \, d\mu(x) \\
+ q_{n-1}(n - 1) \int_{G_m \setminus I_N} |K_{n-1}(x)| \, d\mu(x) \\
\leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j + \frac{q_{n-1}(n - 1) \alpha_{n-1}}{Q_n} \\
:= I + II,
\]

where

\[\alpha_n \to 0, \quad \text{as } n \to \infty.\]

Since the sequence \(\{q_k : k \in \mathbb{N}\}\) is non-increasing, we can conclude that

\[II = \frac{q_{n-1}(n - 1) \alpha_{n-1}}{Q_n} \leq \alpha_{n-1} \to 0, \quad \text{as } n \to \infty.\]

Moreover, for any \(\varepsilon > 0\) there exists \(N_0 \in \mathbb{N}\), such that

\[\alpha_n < \varepsilon \quad \text{when } n > N_0.\]

Furthermore,

\[
\frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_j - q_{j+1}) j \alpha_j \\
+ \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j = I_1 + I_2.
\]

The sequence is non-increasing and therefore

\[|q_j - q_{j+1}| < 2q_0, \quad I_1 \leq \frac{2q_0 N_0}{Q_n} \to 0, \quad \text{as } n \to \infty\]

and

\[I_2 = \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j \\
\leq \frac{\varepsilon}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j < \varepsilon.
\]

and we can conclude that also \(I_2 \to 0\) so the proof is complete. \(\blacksquare\)
Next lemmas we investigate $T$ means with respect to Vilenkin systems generated by non-decreasing sequences $\{q_k : k \in \mathbb{N}\}$:

**Lemma 2.3.4** Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers, satisfying the condition (2.1.6). Then

$$|F^{-1}_n(x)| \leq \frac{c}{n} \left\{ \sum_{j=0}^{\lfloor n \rfloor} M_j |K_{M_j}| \right\},$$

where $c$ is an absolute constant.

Now we prove a Lemma, which is very important for our further investigation in this chapter to prove norm convergence in Lebesgue spaces of $T$ means generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$:

**Corollary 2.3.5** Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers, satisfying the condition (2.1.6). Then for any $n, N \in \mathbb{N}_+$,

$$\int_{G_m} F^{-1}_n(x) d\mu(x) = 1, \quad (2.3.13)$$

$$\sup_{n \in \mathbb{N}} \int_{G_m} |F^{-1}_n(x)| d\mu(x) \leq c < \infty, \quad (2.3.14)$$

$$\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F^{-1}_n(x)| d\mu(x) \to 0, \text{ as } n \to \infty, \quad (2.3.15)$$

where $c$ is an absolute constant.

**Proof.** The proof is analogously to that of Lemma 2.3.3 so we leave out the details. ■

Finally we study some special subsequences of kernels of Nörlund and $T$ means:

**Lemma 2.3.6** Let $n \in \mathbb{N}$. Then

$$F_{M_n}(x) = D_{M_n}(x) - \psi_{M_{n-1}}(x) F^{-1}_{M_n}(x) \quad (2.3.16)$$

and

$$F^{-1}_{M_n}(x) = D_{M_n}(x) - \psi_{M_{n-1}}(x) F_{M_n}(x). \quad (2.3.17)$$
Proof. By using (1.3.2) in Lemma 1.3.1 we get that

\[ F_{M_n}(x) = \frac{1}{Q_{M_n}} \sum_{k=1}^{M_n} q_{M_n-k} D_k(x) = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k D_{M_n-k}(x) \]

\[ = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k (D_{M_n}(x) - \psi_{M_n-1}(x) D_j(x)) \]

\[ = \frac{1}{Q_{M_n}} \sum_{k=0}^{M_n-1} q_k (D_{M_n}(x) - \psi_{M_n-1}(x) D_{M_n-1}(x)) \]

Hence, (2.3.16) is proved. Identity (2.3.17) is proved analogously so the proof is complete. ■

Next four lemmas will be used to prove norm convergence and almost everywhere convergence of subsequences of Nörlund and T means:

Corollary 2.3.7 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then, for any \( n, N \in \mathbb{N}_+ \),

\[ \int_{G_m} F_{M_n}^{-1}(x) d\mu(x) = 1, \quad (2.3.18) \]

\[ \sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}^{-1}(x)| d\mu(x) \leq c < \infty, \quad (2.3.19) \]

\[ \sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}^{-1}(x)| d\mu(x) \to 0, \text{ as } n \to \infty, \quad (2.3.20) \]

Proof. According to (2.3.17) the proof is a direct consequence of Lemmas 1.3.3 and 1.3.10 and Lemma 1.3.4. The proof is complete. ■

Corollary 2.3.8 Let \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then, for any \( N \in \mathbb{N}_+ \),

\[ \int_{G_m} F_{M_n}(x) d\mu(x) = 1, \quad (2.3.21) \]

\[ \sup_{n \in \mathbb{N}} \int_{G_m} |F_{M_n}(x)| d\mu(x) \leq c < \infty, \quad (2.3.22) \]

\[ \sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} |F_{M_n}(x)| d\mu(x) \to 0, \text{ as } n \to \infty, \quad (2.3.23) \]

Proof. According to (2.3.16) the proof is a direct consequence of Lemmas 1.3.3 and 2.3.3 and Lemma 1.3.4. The proof is complete. ■
2.4 Norm Convergence of T Means in Lebesgue Spaces

Now we consider norm convergence of T means with respect to Vilenkin systems:

**Theorem 2.4.1** Let \( f \in L^p(G_m) \) for \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-increasing numbers. Then

\[
\|T_n f - f\|_p \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** According to Corollary 2.3.5 we conclude that conditions (A1), (A2) and (A3) in Proposition 2.1.4 are fulfilled so also this norm convergence follows from this theorem.

The proof is complete. □

**Theorem 2.4.2** Let \( f \in L^p(G_m) \) for \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers, satisfying condition (2.1.5). Then

\[
\|T_n f - f\|_p \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** According to Lemma 1.3.10 we conclude that the conditions (A1), (A2) and (A3) in Proposition 2.1.4 are fulfilled and from this fact follows the stated norm convergence.

The proof is complete. □

According to Theorems 2.4.1 and 2.4.2 we get the following result for T means:

**Corollary 2.4.3** Let \( f \in L^p(G_m) \) for \( p \geq 1 \). Then

\[
\|B_n^{-1} f - f\|_p \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
\|\sigma_{n-1} f - f\|_p \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
\|V_{n-1} f - f\|_p \to 0 \quad \text{as} \quad n \to \infty,
\]
and

\[ \|U_n^{\alpha-1} f - f\|_p \to 0 \text{ as } n \to \infty, \]

**Proof.** In the case when \(B_n^{-1} f\) are \(T\) means generated by non-decreasing sequences \(\{q_k : k \in \mathbb{N}\}\) satisfying the condition (2.1.5) then the statement follows from Theorem 2.4.2.

When \(B_n^{-1} f\) are \(T\) means generated by non-increasing sequence \(\{q_k : k \in \mathbb{N}\}\) the statement is a direct consequence of Theorem 2.4.1.

Analogously, the summability methods

\[ \sigma_n^{\alpha-1} f, \quad V_n^{\alpha-1} f \quad \text{and} \quad U_n^{\alpha-1} f \]

are \(T\) means generated by non-increasing sequences \(\{q_k : k \in \mathbb{N}\}\) and the stated norm convergence follows from Theorem 2.4.4.

The proof is complete. \(\blacksquare\)

Now, we consider subsequences of \(T\) means, generated by non-decreasing sequences, but without any restrictions on the sequence \(\{q_k : k \in \mathbb{N}\}\), In particular, the following is true:

**Theorem 2.4.4** Let \(f \in L^p(G_m)\) for \(p \geq 1\) and \(\{q_k : k \in \mathbb{N}\}\) be a sequence of non-decreasing numbers. Then

\[ \|T_{M_n} f - f\|_p \to 0 \text{ as } n \to \infty. \]

**Proof.** According to Corollary 2.3.7 we conclude that conditions (A1), (A2) and (A3) in Proposition 2.4.4 are fulfilled so also this result is a consequence of this theorem.

The proof is complete. \(\blacksquare\)

Since \(\beta_n^{\alpha-1} f\) are \(T\) means generated by non-decreasing sequence \(\{q_k : k \in \mathbb{N}\}\) we have:

**Corollary 2.4.5** Let \(f \in L^p(G_m)\) for \(p \geq 1\). Then

\[ \|\beta_n^{\alpha-1} f - f\|_p \to 0 \text{ as } n \to \infty. \]
2.5 Convergence of $T$ Means in Vilenkin-Lebesgue points

Our first main result concerning convergence of Nörlund means reads:

**Theorem 2.5.1**  
\textit{a) Let $p \geq 1$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-increasing numbers. If the function $f \in L^1(G_m)$ is continuous at a point $x$, then}

\[ T_n f(x) \to f(x), \quad \text{as } n \to \infty. \]

\textit{Moreover,}

\[ \lim_{n \to \infty} T_n f(x) = f(x) \]

\textit{for all Vilenkin-Lebesgue points of $f \in L^p(G_m)$.}

\textit{b) Let $p \geq 1$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-decreasing numbers satisfying the condition (2.1.6). If the function $f \in L^1(G_m)$ is continuous at a point $x$, then}

\[ T_n f(x) \to f(x), \quad \text{as } n \to \infty. \]

\textit{Furthermore,}

\[ \lim_{n \to \infty} T_n f(x) = f(x) \]

\textit{for all Vilenkin-Lebesgue points of $f \in L^p(G_m)$.}

**Proof.** Let $\{q_k : k \in \mathbb{N}\}$ be a non-increasing sequence. Suppose that $x$ is either point of continuity of function $f \in L^p(G_m)$ or Vilenkin-Lebesgue point of function $f \in L^p(G_m)$. According to Proposition 2.1.2 we can conclude that

\[ \lim_{n \to \infty} |\sigma_n f(x) - f(x)| = 0. \]

By combining (2.3.3) and (2.3.7) we get that

\[ \begin{align*}
|T_n f(x) - f(x)| & \leq \frac{1}{Q_n} \left( \sum_{j=0}^{n-2} (q_j - q_{j+1}) j |\sigma_j f(x) - f(x)| + q_{n-1}(n-1)|\sigma_{n-1} f(x) - f(x)| \right) \\
& \leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j \alpha_j + \frac{q_{n-1}(n-1)}{Q_n} \alpha_{n-1} \\
& := I + II,
\end{align*} \]
where \( \alpha_n \to 0 \), as \( n \to \infty \).

Since the sequence \( \{q_k : k \in \mathbb{N} \} \) is non-increasing we can conclude that
\[
II = \frac{q_{n-1}(n-1)\alpha_{n-1}}{Q_n} \leq \alpha_{n-1} \to 0, \quad \text{as} \quad n \to \infty.
\]

Obviously, \( \alpha_n < A \) where \( n \in \mathbb{N} \) and for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \), such that \( \alpha_n < \varepsilon \) when \( n > N_0 \). Moreover,
\[
I = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_j - q_{j+1}) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j := I_1 + I_2.
\]

Since the sequence \( \{q_k : k \in \mathbb{N} \} \) is non-increasing, we can conclude that
\[
|q_j - q_{j+1}| < 2q_0,
\]
\[
I_1 = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_j - q_{j+1}) j \alpha_j \leq \frac{2q_0N_0}{Q_n} \to 0, \quad \text{as} \quad n \to \infty
\]
and
\[
I_2 = \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j \alpha_j
\]
\[
\leq \frac{\varepsilon}{Q_n} \sum_{j=N_0+1}^{n-2} (q_j - q_{j+1}) j
\]
\[
\leq \frac{\varepsilon}{Q_n} \sum_{j=0}^{n-2} (q_j - q_{j+1}) j < \varepsilon.
\]

Therefore, also \( I \to 0 \) so that a) is proved.

Now we assume that the sequence \( \{q_k : k \in \mathbb{N} \} \) is non-decreasing and satisfying condition (2.1.6). By combining (2.3.3) and (2.3.7) we get that
\[
|T_n f(x) - f(x)| \leq \frac{1}{Q_n} \sum_{j=0}^{n-2} (q_{j+1} - q_j) j \alpha_j + \frac{q_{n-1}(n-1)\alpha_n}{Q_n}
\]
\[
:= III + IV,
\]
where \( \alpha_n \to 0 \), as \( n \to \infty \). It is evident that
\[
IV \leq \frac{q_{n-1}(n-1)\alpha_n}{Q_n} \leq \alpha_n \to 0, \quad \text{as} \quad n \to \infty.
\]
Moreover, for any \( \varepsilon > 0 \) there exists \( N_0 \in \mathbb{N} \), such that \( \alpha_n < \varepsilon/2 \) when \( n > N_0 \). We can write that

\[
III = \frac{1}{Q_n} \sum_{j=0}^{N_0} (q_{j+1} - q_j) j \alpha_j + \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{j+1} - q_j) j \alpha_j
\]

\[
:= III_1 + III_2.
\]

Since the sequence is non-decreasing we can conclude that

\[
|q_{j+1} - q_j| < 2q_{j+1} < 2q_{n-1}.
\]

Hence,

\[
III_1 \leq \frac{2q_0 N_0}{Q_n} \to 0, \quad \text{as} \quad n \to \infty
\]

and

\[
III_2 \leq \frac{1}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{j+1} - q_j) j \alpha_j
\]

\[
\leq \frac{\varepsilon(n-1)}{Q_n} \sum_{j=N_0+1}^{n-2} (q_{j+1} - q_j)
\]

\[
\leq \frac{\varepsilon(n-1)}{Q_n} (q_{n-1} - q_{n-N_0})
\]

\[
\leq \frac{2q_{n-1} \varepsilon(n-1)}{Q_n} < \varepsilon.
\]

Thus, also \( III \to 0 \) so also the proof of b) is complete. ■

**Corollary 2.5.2** Let \( f \in L^p(G_m) \), where \( p \geq 1 \). Then, for all Lebesgue points of \( f \in L^p(G_m) \),

\[
B_{n}^{-1} f \to f, \quad \text{as} \quad n \to \infty, \quad \sigma_{n}^{-1} f \to f, \quad \text{as} \quad n \to \infty,
\]

\[
V_{n}^{-1} f \to f, \quad \text{as} \quad n \to \infty, \quad U_{n}^{-1} f \to f, \quad \text{as} \quad n \to \infty.
\]

**Theorem 2.5.3** Let \( p \geq 1 \) and \( \{q_k : k \in \mathbb{N}\} \) be a sequence of non-decreasing numbers. Then

\[
\lim_{n \to \infty} T_{M_n} f(x) = f(x), \quad \text{for all Lebesgue points of} \quad f \in L^p(G_m).
\]
Proof. If we use first identity in Lemma 2.3.6 to get that

\[ T_{M_n} f(x) = \int_{G_m} f(t) F^{-1}_{M_n} (x - t) \, d\mu(t) \]

\[ = \int_{G_m} f(t) D_{M_n} (x - t) \, d\mu(t) - \int_{G_m} f(t) \psi_{M_n}(x - t) F_{M_n}(x - t) \]

:= I - II.

By applying Proposition 2.1.1 we get that \( I = S_{M_n} f(x) \to f(x) \) for all Lebesgue points of \( f \in L^p(G_m) \), where \( p \geq 1 \).

Moreover, according to Proposition 1.1.1 we find that

\[ \psi_{M_n}(x - t) = \psi_{M_n}(x) \psi_{M_n}(t) \]

so that

\[ II = \psi_{M_n}(x) \int_{G_m} f(t) F_{M_n}(x - t) \psi_{M_n}(t) \, d(t) \]

By combining Proposition 1.2.3 and Corollary 2.3.5 we find that the function

\[ f(t) F_{M_n}(x - t) \in L^p \quad \text{where} \quad p \geq 1 \quad \text{for any} \quad x \in G_m, \]

and \( II \) are the Fourier coefficients of an integrable function. Thus, according to the Riemann-Lebesgue Lemma we get that

\[ II \to 0 \quad \text{for any} \quad x \in G_m. \]

The proof is complete. \( \blacksquare \)

Corollary 2.5.4 Let \( f \in L^p(G_m) \), where \( p \geq 1 \). Then, for all Lebesgue points of \( f \in L^p(G_m) \),

\[ \beta_{M_n} f \to f, \quad \text{as} \quad n \to \infty. \]
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