HIGGS FIELDS, BUNDLE GERBES AND STRING STRUCTURES

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Abstract. We use bundle gerbes and their connections and curvings to obtain an explicit formula for a de Rham representative of the string class of a loop group bundle. This is related to earlier work on calorons.

1. Introduction

In this paper we bring together calorons (monopoles for the loop group), bundle gerbes and string structures to produce a formula for the string class of a principal bundle with structure group the loop group. If $K$ is a compact Lie group and $L(K)$ is the group of smooth maps $\gamma$ from $[0,2\pi]$ to $K$ such that $\gamma(0) = \gamma(2\pi)$ it is well known [14] that there is a central extension

$$0 \to U(1) \to \hat{L}(K) \to L(K) \to 0$$

where $\hat{L}(K)$ is the Kac-Moody group. If $P \to M$ is a principal bundle with structure group $L(K)$ it has a characteristic class, the string class, in $H^3(M, \mathbb{Z})$ which is the obstruction to lifting $P$ to a $\hat{L}(K)$ bundle.

In [12] Murray introduced the lifting bundle gerbe. This is a bundle gerbe whose Dixmier-Douady class is the obstruction to a principal $G$ bundle lifting to a principal $\hat{G}$ bundle when

$$0 \to \mathbb{C}^\times \to \hat{G} \to G \to 0$$

is a central extension. In the case that $G = L(K)$ the Dixmier-Douady class of the lifting bundle gerbe is the string class. We use the methods of [12] to calculate an explicit formula for the Dixmier-Douady class of the lifting bundle gerbe when $G$ is the loop group and hence provide an explicit differential three form representative for the de Rham image of the string class in real cohomology. This three form is defined in terms of a connection and Higgs field for the loop group bundle.

In [6] it was shown that there is a bijective correspondence between principal bundles over a manifold $M$ with structure group $L(K)$ and $K$ bundles over $S^1 \times M$. This was used to set up a correspondence between periodic instantons, or calorons, and loop group valued monopoles. In particular a connection for the $K$ bundle corresponded to a connection and Higgs field for the $L(K)$ bundle. We apply this correspondence to show that the string class of an $L(K)$ bundle on $M$ is the integral over the circle of the Pontrjagin class of the corresponding $K$ bundle over $S^1 \times M$.

Finally we relate these results to earlier work [9, 4] on string structures. Recall that if $Q$ is a $K$ bundle over a manifold $X$ we can take loops everywhere and form a loop group bundle $P = L(Q)$ over $M = L(X)$. In that case it was known
from work of Killingback [8] that the Pontrjagin class of the bundle $Q$ in $H^4(X, \mathbb{Z})$ transgressed to define the string class in $H^3(L(X), \mathbb{Z})$. The transgression consists of pulling back by the evaluation map

$$ev: S^1 \times L(X) \to X$$

and pushing down to $L(X)$ by integrating over the circle. If we apply the correspondence of [8] to the principal $L(K)$ bundle $L(Q)(L(X), L(K))$ it produces a $K$ bundle over $S^1 \times L(X)$ and we show that this is just the pull-back of $Q$ by the evaluation map. This recovers the result of Killingback [9].

As the present work was being completed a preprint was received from Kiyonori Gomi [6] which also defines connections and curvings on lifting bundle gerbe using the notion of reduced splittings and building on results in [1]. We discuss the relationship between reduced splittings and Higgs fields in 5.3.

2. Some preliminaries

2.1. $\mathbb{C}^\times$ bundles. Let $P \to X$ be a $\mathbb{C}^\times$ bundle over a manifold $X$. We shall denote the fibre of $P$ over $x \in X$ by $P_x$. Recall [6] that if $P$ is a $\mathbb{C}^\times$ bundle over a manifold $X$ we can define the dual bundle $P^*$ as the same space $P$ but with the action $p^*g = (pg^{-1})^*$ and, that if $Q$ is another such bundle, we can define the product bundle $P \otimes Q$ by $(P \otimes Q)_x = (P_x \times Q_x)/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by $(p,q)w = (pw, qw^{-1})$. We denote an element of $P \otimes Q$ by $p \otimes q$ with the understanding that $(pw)\otimes q = p \otimes (qw) = (p \otimes q)w$ for $w \in \mathbb{C}^\times$. It is straightforward to check that $P \otimes P^*$ is canonically trivialised by the section $x \mapsto p \otimes p^*$ where $p$ is any point in $P_x$.

If $P$ and $Q$ are $\mathbb{C}^\times$ bundles on $X$ with connections $\mu_P$ and $\mu_Q$ then $P \otimes Q$ has an induced connection we denote by $\mu_P \otimes \mu_Q$. The curvature of this connection is $R_P + R_Q$ where $R_P$ and $R_Q$ are the curvatures of $\mu_P$ and $\mu_Q$ respectively. The bundle $P^*$ has an induced connection whose curvature is $-R_P$.

2.2. Simplicial spaces. Recall [6] that a simplicial manifold $X$ is a collection of spaces $X_0, X_1, X_2, X_3, \ldots$ with maps $d_i: X_p \to X_{p-1}$ for $i = 1, \ldots, p$, and $s_j: X_p \to X_{p+1}$ for $j = 1, \ldots, p$, satisfying the, so-called, simplicial identities:

\begin{align*}
(1) \quad &d_id_j = d_{j-1}d_i, \quad i < j \\
(2) \quad &s_is_j = s_{j+1}s_i, \quad i \leq j \\
(3) \quad &d_is_j = \begin{cases} s_{j-1}d_i, & i < j \\
\text{id}, & i = j, i = j + 1 \\
sjadi, & i > j + 1. \end{cases}
\end{align*}

Let $\Omega^p(M)$ denote the space of all differentiable $p$ forms on a manifold $M$. Define a homomorphism $\delta: \Omega^p(X_p) \to \Omega^p(X_{p+1})$ by

$$\delta = \sum_{i=1}^{p} (-1)^{i-1}d_i^*.$$

It is straightforward to check that $\delta^2 = 0$ and it clearly commutes with exterior derivative $d$. Hence we have a complex

\begin{align*}
\Omega^p(X_0) &\xrightarrow{\delta} \Omega^p(X_1) \xrightarrow{\delta} \Omega^p(X_2) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^p(X_p) \xrightarrow{\delta} \cdots
\end{align*}
We remark that from a simplicial space we can define a topological space called its realisation\[^1\]. The double complex $\Omega^p(X^q)$ with the differentials $d$ and $\delta$ has total cohomology the real cohomology of the realisation.

If $P \to X_p$ is a $\mathbb{C}^\times$ bundle then we can define a $\mathbb{C}^\times$ bundle over $X_{p+1}$ denoted $\delta(P)$ by

$$
\delta(P) = d_1^{-1}(P) \otimes d_2^{-1}(P) \otimes d_3^{-1}(P) \otimes \ldots.
$$

If $s$ is a section of $P$ then it defines $\delta(s)$ a section of $\delta(P)$ and if $\mu$ is a connection on $P$ with curvature $R$ it defines a connection $\delta(\mu)$ on $\delta(P)$ with curvature $\delta(R)$. If we consider $\delta(\delta(P))$ it is a product of factors and because of the simplicial identities\[^3\] every factor occurs with its dual so $\delta(\delta(P))$ is canonically trivial. If $s$ is a section of $P$ then under this identification $\delta\delta(s) = 1$ and moreover if $\mu$ is a connection on $P$ then $\delta\delta(\mu)$ is the flat connection on $\delta\delta(P)$ with respect to $\delta(\delta(s))$.

If $X$ is a simplicial space then a simplicial line bundle\[^2\] is a $\mathbb{C}^\times$ bundle $P$ over $X_1$ with a section $s$ of $\delta(P)$ over $X_2$ with the property that $\delta(s)$ is the canonical section of $\delta^2(P)$.

2.3. **Locally split maps.** We will be interested in maps $\pi: Y \to M$ which admit local sections. That is, for every $x \in M$ there is an open set $U$ containing $x$ and a local section $s: U \to Y$. For want of a better term we will call maps like this locally split. Note that a locally split map is necessarily surjective and that if we are dealing with the smooth category a locally split map is just a submersion. Locally trivial fibrations are, of course, locally split but the converse is not true.

Let $Y \to M$ be locally split. Then we denote by $Y^{[2]} = Y \times_Y Y$ the fibre product of $Y$ with itself over $\pi$, that is the subset of pairs $(y, y')$ in $Y \times Y$ such that $\pi(y) = \pi(y')$. More generally we denote the $p$th fold fibre product by $Y^{[p]}$.

For $p = 1, 2, \ldots$ we have $p$ projection maps $\pi_i: Y^{[p+1]} \to Y^{[p]}$ for $i = 1, 2, \ldots, p + 1$ given by omitting the $i$-th factor, so

$$
\pi_i(y_1, y_2, \ldots, y_{p+1}) = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{p+1}).
$$

The spaces $X_0 = Y, X_1 = Y^{[2]}, \ldots$ define a simplicial manifold. For this simplicial manifold we can augment the complex\[^4\] by adding at the beginning the space $\Omega^n(M)$ and the map which pulls back forms from $M$ to $Y$ to obtain a complex

$$
\Omega^n(M) \to \Omega^n(Y) \to \Omega^n(Y^{[2]}) \to \ldots
$$

It is a fundamental result of\[^12\] that the complex\[^3\], has no cohomology. So if $\eta \in \Omega^n(Y^{[p]})$ satisfies $\delta(\eta) = 0$ then we can solve the equation $\eta = \delta(\rho)$ for some $\rho \in \Omega^n(Y^{[p-1]})$ (we define $Y^{[0]} = M$). Note that a solution $\rho$ is not unique, any two solutions $\rho$ and $\rho'$ will differ by $\delta(\zeta)$ for some $\zeta \in \Omega^n(Y^{[p-1]})$.

3. **Central extensions**

We give here details of a method of constructing central extensions presented in\[^12\]. Let $\mathcal{G}$ be a Lie group. Recall that from $\mathcal{G}$ we can construct a simplicial manifold $N\mathcal{G} = \{Ng_g \}$ with $Ng_g = \mathcal{G}^p$ with face operators $d_i: \mathcal{G}^{p+1} \to \mathcal{G}^p$ defined by

$$
d_i(g_1, \ldots, g_{p+1}) = \begin{cases} (g_2, \ldots, g_{p+1}), & i = 0, \\ (g_1, \ldots, g_{i-1}g_i, g_{i+1}, \ldots, g_{p+1}), & 1 \leq i \leq p - 1, \\ (g_1, \ldots, g_p), & i = p. \end{cases}
$$

\[^1\]It makes no difference for the discussion in this paper if it is the fat or geometric realisation.
Consider a central extension
\[ \mathbb{C}^\times \to \hat{G} \xrightarrow{\pi} G. \]
Following Brylinski and McLaughlin we think of this as a \( \mathbb{C}^\times \) bundle \( \hat{G} \to G \) with a product \( M : \hat{G} \times \hat{G} \to \hat{G} \) covering the product \( m = d_1 : G \times G \to G \).

Because this is a central extension we must have that \( M(pz, qw) = M(p, q)zw \) for any \( p, q \in P \) and \( z, w \in \mathbb{C}^\times \). This means we have a section \( s \) of \( \delta(P) \) given by
\[ s(p, q) = p \otimes M(p, q) \otimes q \]
for any \( p \in P_g \) and \( q \in P_h \). This is well-defined as \( pw \otimes M(pw, qz) \otimes qz = pw \otimes M(p, q)(wz)^{-1} \otimes qz = p \otimes M(p, q) \otimes q \). Conversely any such section gives rise to an \( M \).

Of course we need an associative product and it can be shown that \( M \) being associative is equivalent to \( \delta(s) = 1 \). To actually make \( \hat{G} \) into a group we need more than multiplication we need an identity \( \hat{e} \in \hat{G} \) and an inverse map. It is straightforward to check that if \( e \in G \) is the identity then, because \( M : \hat{G}_e \times \hat{G}_e \to \hat{G}_e \), there is a unique \( \hat{e} \in \hat{G}_e \) such that \( M(\hat{e}, \hat{e}) = \hat{e} \). It is also straightforward to deduce the existence of a unique inverse.

Hence we have the result from [2] that a central extension of \( G \) is a \( \mathbb{C}^\times \) bundle \( P \to G \) together with a section \( s \) of \( \delta(P) \to G \times G \) such that \( \delta(s) = 1 \). In [2] this is phrased in terms of simplicial line bundles.

For our purposes we need to phrase this result in terms of differential forms.

We call a connection for \( \hat{G} \to \hat{G} \), thought of as a \( \mathbb{C}^\times \) bundle, a connection for the central extension. An isomorphism of central extensions with connection is an isomorphism of bundles with connection which is a group isomorphism on the total space \( \hat{G} \). Denote by \( C'(G) \) the set of all isomorphism classes of central extensions of \( G \) with connection.

Let \( \mu \in \Omega^1(\hat{G}) \) be a connection on the bundle \( \hat{G} \to \hat{G} \) and consider the form
\[ \hat{\delta}(\mu) = d_0^*(\mu) - d_1^*(\mu) + d_2^*(\mu) \]
on \( \hat{G} \times \hat{G} \). There is a projection map
\[ \hat{\pi} : \hat{G} \times \hat{G} \to \hat{G} \otimes \hat{G} \]
whose fibre at \( p \otimes q \) is all \( (pz, qz^{-1}) \) for \( z \in \mathbb{C}^\times \). If \( X \in \mathbb{C} \) denote the vector tangent to \( t \mapsto p \exp(tX) \) by \( \iota_p(X) \). Then the tangent space to the fibres of the projection at \( (p, q) \) is spanned by all vectors of the form \( (\iota_p(X), -\iota_q(X)) \). Apply \( \hat{\delta}(\mu) \) to the vector \( (\iota_p(X), -\iota_q(X)) \). Because \( \mu \) is a connection form we have
\[ d_0^*(\mu)(\iota_p(X), -\iota_q(X)) = X \]
and
\[ d_2^*(\mu)(\iota_p(X), -\iota_q(X)) = -X. \]
Consider the multiplication map \( M : \hat{G} \times \hat{G} \to \hat{G} \). Because \( \mathbb{C}^\times \) is central we must have
\[ T_{(p_1, p_2)} M(\iota_{p_1}(X_1), \iota_{p_2}(X_2)) = \iota_{p_1 p_2}(X_1 + X_2) \]
and hence
\[ d_1^*(\mu)(\iota_p(X), -\iota_q(X)) = 0. \]
It follows that \( \hat{\delta}(\mu) \) is zero on vertical vectors. As it is also clearly invariant under \( \mathbb{C}^\times \) it descends to \( \hat{G} \otimes \hat{G} \). The descended form is the tensor product connection discussed earlier and denoted \( \delta(\mu) \).
Let $\alpha = s^*(\delta(\mu))$. We then have that
\[
\delta(\alpha) = (\delta(s)^*)(\delta\delta(\mu))
\]
\[
= (1)^*(\delta^2(\mu))
\]
\[
= 0
\]
as $\delta^2(\mu)$ is the flat connection on $\delta^2(P)$. Also $d\alpha = s^*(d\delta(\mu)) = \delta(R)$.

In more detail $\alpha$ and $R$ satisfy:
\[
d_0^*R - d_1^*R + d_2^*R = d\alpha
\]
\[
d_0^*\alpha - d_1^*\alpha + d_2^*\alpha - d_3^*\alpha = 0.
\]

Let $\Gamma(\mathcal{G})$ denote the set of all pairs $(\alpha, R)$ where $R$ is a closed, $2\pi i$ integral, two form on $\mathcal{G}$ and $\alpha$ is a one-form on $\mathcal{G} \times \mathcal{G}$ with $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$.

We have constructed a map $C(\mathcal{G}) \to \Gamma(\mathcal{G})$. In the next section we construct an inverse to this map by showing how to define a central extension from a pair $(\alpha, R)$. For now notice that isomorphic central extensions with connection clearly give rise to the same $(\alpha, R)$ and that if we vary the connection, which is only possible by adding on the pull-back of a one-form from $\mathcal{G}$, then we change $(\alpha, R)$ to $(\alpha + \delta(\eta), R + d\eta)$.

3.1. Constructing the central extension. Recall that given $R$ we can find a principal $\mathbb{C}^\times$ bundle $P \to \mathcal{G}$ with connection $\mu$ and curvature $R$ which is unique up to isomorphism. It is a standard result in the theory of bundles that if $P \to X$ is a bundle with connection $\mu$ which is flat and $\pi_1(X) = 0$ then $P$ has a section $s: X \to P$ such that $s^*(\mu) = 0$. Such a section is not unique of course it can be multiplied by a (constant) element of $\mathbb{C}^\times$. Consider now our pair $(R, \alpha)$ and the bundle $P$. As $\delta(R) = d\alpha$ we have that the connection $\delta(w) - \pi^*(\alpha)$ on $\delta(P) \to G \times G$ is flat and hence we can find a section $s$ such that $s^*(\delta(w)) = \alpha$.

The section $s$ defines a multiplication by
\[
s(p, q) = p \otimes M(p, q)^* \otimes q.
\]
Consider now $\delta(s)$ this satisfies $\delta(s)^*(\delta(\delta(w))) = \delta(s^*(\delta(w))) = \delta(\alpha) = 0$. On the other hand the canonical section 1 of $\delta(\delta(P))$ also satisfies this so they differ by a constant element of the group. This means that there is a $w \in \mathbb{C}^\times$ such that for any $p$, $q$ and $r$ we must have
\[
M(M(p, q, r) = wM(p, M(q, r)).
\]
Choose $p \in \hat{\mathcal{G}}_\epsilon$ where $\epsilon$ is the identity in $\mathcal{G}$. Then $M(p, p) \in \hat{\mathcal{G}}_\epsilon$ and hence $M(p, p) = pz$ for some $z \in \mathbb{C}^\times$. Now let $p = q = r$ and it is clear that we must have $w = 1$.

So from $(\alpha, R)$ we have constructed $P$ and a section $s$ of $\delta(P)$ with $\delta(s) = 1$. However $s$ is not unique but this is not a problem. If we change $s$ to $s' = sz$ for some constant $z \in \mathbb{C}^\times$ then we have changed $M$ to $M' = Mz$. As $\mathbb{C}^\times$ is central multiplying by $z$ is an isomorphism of central extensions with connection. So the ambiguity in $s$ does not change the isomorphism class of the central extension with connection. Hence we have constructed a map
\[
\Gamma(\mathcal{G}) \to C(\mathcal{G})
\]
as required. That it is the inverse of the earlier map follows from the definition of $\alpha$ as $s^*(\delta(\mu))$ and the fact that the connection on $P$ is chosen so its curvature is $R$. 
3.2. **An explicit construction.** First we show how the pair \((\alpha, R)\) can be used to recover the original central extension with connection. We will then show this gives a construction of a central extension from any pair \((\alpha, R) \in \Gamma(G)\). Finally we have to show that this gives an inverse to the map defined in the preceding section.

Let \(PG\) denote the space of all paths in \(G\) which begin at the identity and define \(\pi: PG \to G\) to be the map which evaluates the endpoint of the path. We can use this map to pullback the central extension \(\hat{G} \to G\) to a central extension of \(PG\) by \(C^\times\) defined by

\[
p^{-1}(\hat{G}) = \{(f, \hat{g}) \mid f(1) = \pi(\hat{g})\}.
\]

Because \(PG\) is contractible this must be trivial and indeed we can map \((f, z) \in PG \times C^\times\) to \((f, \hat{f}(1)z)\) where \(\hat{f}\) is the (unique) lift of \(f\) to a horizontal path in \(\hat{G}\) starting at the identity. The two projection maps define a commuting diagram:

\[
\begin{array}{ccc}
PG \times C^\times & \to & \hat{G} \\
\downarrow & & \downarrow \\
PG & \to & \hat{G}
\end{array}
\]

The product on \(p^{-1}(\hat{G})\) induces a product on \(PG \times C^\times\) which must take the form \((f, z)(g, w) = (fg, c(f, g)zw)\) for some \(C^\times\) valued cocycle on \(PG\) which we now calculate. Let \(f\) and \(g\) be paths in \(G\) starting at the identity and let \(\hat{f}\) and \(\hat{g}\) be horizontal lifts to \(\hat{G}\) beginning at the identity. Then \(c(f, g)\) satisfies

\[
(\hat{f}\hat{g})(1) = c(f, g)\hat{f}(1)\hat{g}(1).
\]

Notice that \(\hat{f}\hat{g}\) and \(\hat{fg}\) are both lifts of \(fg\). Hence there is a map \(\xi: [0, 1] \to C^\times\) with \(\xi(0) = 1\) and \(\xi(1) = c(f, g)\) such that \(\hat{f}\hat{g} = \hat{fg}\xi\). As \(\hat{fg}\) is horizontal it is straightforward to integrate and prove that

\[
c(f, g) = \exp(-\int_{\hat{g}} \mu).
\]

Let \((f, g)\) denote the path in \(G \times G\). From the definition of \(\alpha\) we see that

\[
\int_{(f, g)} \alpha = \int_{(f, \hat{g})} \pi^*(\alpha)
= \int_f \mu - \int_{\hat{f}} \mu + \int_{\hat{g}} \mu
\]

Using the fact that \(\hat{f}\) and \(\hat{g}\) are horizontal we find that

\[
c(f, g) = \exp(\int_{(f, g)} \alpha).
\]

As in [11] we can now identify the kernel of the homomorphism \(PG \times C^\times \to \hat{G}\). This is all pairs \((h, z)\) such that the holonomy of the connection is equal to \(z^{-1}\). As we are assuming that \(G\) is simply connected we can extend any loop \(h\) to a map \(\tilde{h}\) from a disk \(D\) into \(G\) and define

\[
H(h, R) = \exp(\int_{\tilde{h}(D)} R).
\]

Notice that the \(2\pi i\) integrality of \(R\) implies that \(H(h, R)\) is well-defined. The kernel is therefore the subgroup of all pairs \((h, H(h, R)^{-1})\).
We can now see how to define a central extension given the pair \((\alpha, R)\). First we define \(c(f, g)\) by \(c(f, g) = \exp\left(\int_{(f, g)} \alpha\right)\) and it can be checked, as in [11], that this cocycle makes \(PG \times \mathbb{C}^\times\) into a group. Then define \(H(h, R)\) as above and consider the subset of all pairs \((h, H(h, R)^{-1})\). It can be shown that this is a normal subgroup and the quotient defines the central extension.

We define a connection 1-form \(\mu\) on the principal \(\mathbb{C}^\times\) bundle \(\hat{G}\) using the same technique as in [12]. We use the map \(PG \times \mathbb{C}^\times \to \hat{G}\) to pullback the connection one-form \(\mu\) on \(\hat{G}\) to a connection one-form \(\hat{\mu}\). A straightforward calculation shows that this is given by

\[
\hat{\mu} = \int_{[0, 2\pi]} \iota_{(0, \frac{d}{dt}) \text{ev}}^* R
\]

and \(\text{ev}: PG \times [0, 2\pi] \to G\) is the evaluation map \(\text{ev}(f, t) = f(t)\).

In the case that we start with a pair \((\alpha, R)\) it is straightforward to check that this connection descends to a connection on \(\hat{G}\).

We now have a procedure for constructing from a given central extension a pair \((\alpha, R)\) and from a pair \((\alpha, R)\) a central extension. It is clear from the construction that if we start with a central extension, construct \((\alpha, R)\) and then construct a central extension we get back to where we started from. Consider now what happens if we start with an \((\tilde{\alpha}, \tilde{R})\), construct the central extension and then construct a pair \((\tilde{\alpha}, \tilde{R})\). It is straightforward to show that we have

\[
\pi^*(\tilde{\alpha}) = c^{-1} dc + \delta(\tilde{\mu})
\]

and

\[
\pi^*(\tilde{R}) = d\tilde{\mu}.
\]

Using the definition of \(c\) and \(\tilde{\mu}\) we can show that \(\alpha\) and \(R\) satisfy the same equations and hence deduce that \(\tilde{\alpha} = \alpha\) and \(\tilde{R} = R\) as \(\pi^*\) is injective.

4. Loop groups

There are a number of variants of the loop group that we wish to consider. To define these let \(K\) be a compact group and consider first

\[
L(K) = \{\gamma: [0, 2\pi] \to K \mid \gamma(0) = \gamma(2\pi)\}
\]

this has a subgroup of based loops

\[
L_0(K) = \{\gamma: [0, 2\pi] \to K \mid \gamma(0) = \gamma(2\pi) = 1\}.
\]

We assume here that the maps \(\gamma\) are smooth on \([0, 2\pi]\). Now map \([0, 2\pi]\) to the circle \(S^1\) by \(\theta \to \exp(i\theta)\). We therefore have

\[
\Omega(K) = C^\infty(S^1, K)
\]

the space of smooth maps from the circle to \(K\) with a subgroup of \(LK\) and we let \(\Omega_0(K) = \Omega(K) \cap L_0K\). These groups are all Frechet Lie groups and their Lie algebras are the analogous spaces of maps of \([0, 2\pi]\) to \(\mathfrak{t}\) and denoted by \(L(\mathfrak{t})\), \(L_0(\mathfrak{t})\), \(\Omega(\mathfrak{t})\) and \(\Omega_0(\mathfrak{t})\).
In the case where $G = L(K)$ there is a well known expression for the curvature $R$ of a left invariant connection on $L(K)$ — see [14]. We can also write down a 1-form $\alpha$ on $L(K) \times L(K)$ such that $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$. These are:

\begin{equation}
R = \frac{i}{4\pi} \int_{S^1} \langle \Theta, \partial_\theta \Theta \rangle d\theta
\end{equation}

\begin{equation}
\alpha = \frac{i}{2\pi} \int_{S^1} \langle d^*_\theta \Theta, d^*_\theta Z \rangle d\theta
\end{equation}

Here $\Theta$ denotes the Maurer-Cartan form on the Lie group $K$, that is $\Theta(k)(kX) = X$, $Z$ is the function on $LG$ defined by $Z(g) = (\partial_\theta g)g^{-1}$ and $\langle \ , \rangle$ is an invariant inner product normalised so that the longest root has length squared equal to 2. Note that $F$ is left invariant and that $A$ is left invariant in the first factor of $G \times G$.

The formulae (10) contain implicit wedge products and to avoid confusion let us explain what these are. Let $\omega_i$ be differential forms of degree $d_i$ with values in a vector space $V_i$ for $i = 1, \ldots, k$. If $p: V_1 \times \cdots \times V_k \to \mathbb{C}$ is a $k$ linear map then we define $p(\omega_1, \ldots, \omega_k)$, a differential form of degree $d = d_1 + \cdots + d_k$ by:

\begin{equation}
p(\omega_1, \ldots, \omega_k)(X_1, \ldots, X_k) = \sum_{\pi \in \mathcal{S}_d} \text{sign}(\pi)p(\omega_1(X_{\pi(1)}, \ldots, X_{\pi(d_1)}), \ldots, \omega_k(X_{\pi(d-d_k+2)}, \ldots, X_{\pi(d)}))
\end{equation}

where the sum is over all permutations $\pi$ of $\{1, \ldots, d\}$ with $\text{sign}(\pi)$ the sign of the permutation. Note the potential confusion that if each of the $\omega_i$ is of degree 1 and $p$ is an antisymmetric function then $p(\omega_1(X_1), \ldots, \omega_d(X_d))$ is already an anti-symmetric function of $X_1, \ldots, X_d$. It is, in fact, $1/d!$ times $p(\omega_1, \ldots, \omega_d)$ applied to $X_1, \ldots, X_d$. For later use we record here some identities relating $\Theta$ and $Z$. At a point $g \in LK$ we have

\begin{equation}
\partial_\theta \Theta = \text{ad}(g^{-1}(dZ))
\end{equation}

and if $X$ is in $L\mathfrak{t}$ then

\begin{equation}
\partial_\theta (\text{ad}(g^{-1}(X))) = \text{ad}(g^{-1})([X, Z]) + \text{ad}(g^{-1})\partial_\theta X.
\end{equation}

5. Lifting bundle gerbes and the string class

5.1. The string class. Consider a central extension

\[ \mathbb{C}^\times \to \hat{G} \to G \]

and let $P$ be a principal bundle over a manifold $M$ with structure group $G$. There is a characteristic class in $H^3(M, \mathbb{Z})$ which is the obstruction to lifting $P$ to a $\hat{G}$ bundle. This is easily described in Čech cohomology. Let $\{U_\alpha\}_{\alpha \in I}$ be a good cover of $M$ with respect to which $P$ has transition functions $g_{\alpha\beta}$. As the double intersections are contractible we can lift the transition functions to $\tilde{g}_{\alpha\beta}$ with values in $\hat{G}$ such that $\pi\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$. Because $g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$ it follows that

\begin{equation}
d_{\alpha\beta\gamma} = \tilde{g}_{\beta\gamma}g_{\alpha\gamma}^{-1}\tilde{g}_{\alpha\beta}
\end{equation}

which defines a class in $H^2(M, \mathbb{C}^\times)$ and it is straightforward to check that this class is zero if and only if the bundle lifts to $\hat{G}$ [11]. A standard calculation with the short exact sequence

\[ \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \]
identifies this class with a class in \(H^3(M, \mathbb{Z})\). In that case that \(\mathcal{G}\) is the loop group and \(\hat{\mathcal{G}}\) the central extension of the loop group this class is called the \textit{string class} \([12]\).

### 5.2. Lifting bundle gerbes.

We first briefly review the notion of a \textit{bundle gerbe} from \([13]\) cast in the language of simplicial line bundles. Let \(\pi : Y \to M\) be a local split map. Recall that it defines a simplicial space \(Y^{[n]}\). A bundle gerbe over a manifold \(M\) is a pair \((P, Y)\) where \(P\) is a simplicial line bundle over the simplicial space defined by \(Y\). This means that \(P\) is a \(\mathbb{C}^\times\) bundle over \(Y^{[2]}\).

A bundle gerbe \(P\) has a characteristic class \(\text{DD}(P)\) in \(H^3(M; \mathbb{Z})\) associated to it — the Dixmier-Douady class. We refer to \([12]\) for the definition and properties of \(\text{DD}(P)\). We can realise the image of the Dixmier-Douady class in real cohomology in a manner analogous to the Chern class. A bundle gerbe \((P, Y, M)\) can be equipped with a \textit{bundle gerbe connection}. This is a connection \(\nabla\) on \(P\) which is compatible with the bundle gerbe product in the sense that \(\delta(\nabla)\) is the flat connection for the trivialisation \(s\) of \(\delta(P)\). It is not hard to see that bundle gerbe connections always exist, indeed if \(\nabla\) is any connection then \(\delta(s^*(\nabla)) = 0\) and hence from \([1]\) \(s^*(\nabla) = \delta(\rho)\) for some \(\rho\) on \(Y^{[2]}\) and \(\nabla - \pi^*(\rho)\) is a bundle gerbe connection. If \(F_\nabla\) denotes the curvature of the bundle gerbe connection \(\nabla\) then it is easy to see that we have \(\delta(F_\nabla) = 0\). Hence we can solve the equation \(F_\nabla = \delta(f)\) for some \(f \in \Omega^3(Y)\).

A choice of \(f\) is called a \textit{curving} for the bundle gerbe connection \(\nabla\). Since \(F_\nabla\) is closed, we get \(\delta(df) = 0\) and so we have \(df = 2\pi i \pi^* \omega\) for some necessarily closed 3-form \(\omega\) on \(M\). In \([12]\) it is shown that \(\omega\) is an integral 3-form and represents the image of the Dixmier-Douady class \(\text{DD}(P)\) of \(\omega\) in \(H^3(M, \mathbb{R})\).

Suppose that \(G\) is a Lie group forming part of a central extension
\[
\mathbb{C}^\times \to \hat{G} \to G.
\]

Recall from \([13]\) that we can associate to a principal \(G\) bundle \((P, M, \mathcal{G})\) a bundle gerbe \((\hat{P}, P, M)\) — the so called \textit{lifting bundle gerbe}. \(\hat{P} \to P^{[2]}\) is the pull-back \(\hat{P} = \tau^{-1}\hat{G}\) of \(\hat{G}\) to \(G\) by the natural map \(\tau : P^{[2]} \to \hat{G}\) defined by \(p_2 = p_1 \tau(p_1, p_2)\) for \(p_1\) and \(p_2\) points of \(P\) lying in the same fibre. \(\tau\) satisfies the property \(\tau(p_1, p_2)\tau(p_2, p_3) = \tau(p_1, p_3)\) for points \(p_1, p_2, p_3\) all lying in the same fibre. \(\hat{P}\) inherits a bundle gerbe product from the product in \(\hat{G}\). The Dixmier-Douady class of the bundle gerbe \(\hat{P}\) measures the obstruction to lifting the structure group of the principal \(G\) bundle to \(\hat{G}\). It follows that when \(\mathcal{G}\) is a loop group the Dixmier-Douady class is (the image in real cohomology of) the string class of the bundle \(P\).

From the principal bundle we can construct a simplicial space as above and if
\[
\tau : P^{[2]} \to \hat{G}
\]
is defined by \(p_2 = p_1 \tau(p_1, p_2)\) then we can define
\[
\tau : P^{[k+1]} \to \hat{G}^k
\]
by
\[
\tau(p_1, \ldots, p_{k+1}) = (\tau(p_1, p_2), \ldots, \tau(p_k, p_{k+1}).
\]
It is straightforward to check that this is a \textit{simplicial} map, that is it commutes with the face and degeneracy maps. It follows that pullback of differential forms by \(\tau^*\) commutes with \(\delta\). Suppose that \(\mu\) is a connection on \(\hat{G}\). Then the natural connection \(\hat{\mu} = \tau^* \mu\) on \(\hat{P}\) is not a bundle gerbe connection on \(\hat{P}\). Indeed from the equation \(\delta(\mu) = \pi^*(\alpha)\) we see that \(\delta(\tau^*(\mu)) = \tau^*(\alpha)\). However the form \(\beta = \tau^*(\alpha)\) satisfies \(\delta(\beta) = \tau^*(\delta(\alpha)) = 0\) as \(\delta(\alpha) = 0\). So we can solve the equation \(\delta(\epsilon) = \beta\)
for some 1-form \( \epsilon \) on \( P[2] \). Then the connection \( \tilde{\mu} - \epsilon \) is a bundle gerbe connection on \( \tilde{P} \). Its curvature is given by \( \tau^*(R) - d\epsilon \), where \( R = d\mu \) is the curvature of \( \mu \). The curving is therefore a two-form \( f \) on \( P \) satisfying \( \delta(f) = \tau^*(R) - d\epsilon \) for some 2-form \( f \) on \( P \). From \( f \) the Dixmier-Douady class \( \omega \) is obtained as \( df = \tau^*(\omega) \).

To proceed further we need to concentrate on a specific example so we will let \( G = L(K) \). So we have a principal \( L(K) \) bundle \( P(M, L(K)) \) on \( M \) for \( K \) a compact Lie group, we can form the lifting bundle gerbe \( (\tilde{P}, \tilde{P}, \tilde{M}) \) associated to the central extension

\[
\mathbb{C}^\times \rightarrow \tilde{L}(K) \rightarrow L(K)
\]

where \( \tilde{L}(K) \) is the Kac-Moody group. We can form a bundle gerbe connection \( \nabla \) on \( \tilde{P} \) in the manner described above using the natural connection on \( \tilde{L}(K) \) and the 1-form \( \alpha \) on \( L(K) \times L(K) \). We will show that it is possible to write down an expression for the three curvature \( \omega \) of the bundle gerbe connection \( \nabla \).

Suppose we have chosen a connection 1-form \( A \) on the principal \( G \) bundle \( P \rightarrow M \). Then this is a one-form on \( P \) with values in \( \mathfrak{g} \). It is straightforward to show that

\[
\pi_1^*(A) = \text{ad}(\tau^{-1})\pi_2^*(A) + \tau^*(\Theta) \quad (13)
\]

Let \( \tau_{ij}(p_1, \ldots, p_k) = \tau(p_1, p_i) \). Then using \( \beta = (\tau_{12} \times \tau_{23})^*\alpha \), the definition of \( \alpha \) from \( 9 \) and the identity \( 13 \) we obtain

\[
\beta = \frac{i}{2\pi} \int_{S^1} \langle \pi_{13}^*A - \text{ad}(\tau_{12})^{-1}\pi_{23}^*A, \partial_\theta(\tau_{23})\tau_{23}^{-1}\rangle d\theta \quad (14)
\]

where \( \pi_{12}(p_1, p_2, p_3) = p_3 \), etc.

Define a 1-form \( \epsilon \) on \( P[2] \) by

\[
\epsilon = \frac{i}{2\pi} \int_{S^1} \langle \pi_2^*A, \tau^*(Z) \rangle d\theta.
\]

Using the fact that \( \tau_{12}\tau_{23} = \tau_{13} \) we obtain

\[
\tau_{12}^*(Z) + \text{ad}(\tau_{12})(\tau_{23}^*(Z)) = \tau_{13}^*(Z)
\]

and with this and \( 13 \) we can show that \( \delta(\epsilon) = \beta \).

To solve the equation \( \tau^*R - d\epsilon = \delta(f) \) for some choice of curving \( f \) we first need an explicit expression for \( \tau^*R - d\epsilon \). Using \( 9 \) we have that

\[
\tau^*R = \frac{i}{4\pi} \int_{S^1} \langle \tau^*\Theta, \partial_\theta(\tau^*\Theta) \rangle d\theta
\]

Using the standard fact that \( d\Theta = -(1/2)[d\Theta, d\Theta] \) and the identities \( 11 \) and \( 12 \) we can show that we have

\[
\tau^*R - d\epsilon = \frac{i}{4\pi} \int_{S^1} \langle \pi_1^*(A), \partial_\theta\pi_1^*(A) \rangle - \langle \pi_2^*(A), \partial_\theta\pi_2^*(A) \rangle
\]

and

\[
- \langle [\pi_2^*(A), \pi_2^*(A)], \tau^*(Z) \rangle - 2\langle \pi_2^*(dA), \tau^*(Z) \rangle d\theta.
\]

Recalling that \( F \), the curvature of \( A \) satisfies \( F = dA + 1/2[A, A] \) we have that \( \tau^*R - d\epsilon \) is equal to

\[
\delta \left( \frac{i}{4\pi} \int_{S^1} \langle A, \partial_\theta A \rangle - \frac{i}{2\pi} \int_{S^1} \langle \pi_2^*(F), \tau^*(Z) \rangle d\theta \right)
\]
where $\delta = \pi_1^* - \pi_2^*$. We want to solve $\tau^* R - d\epsilon = \delta(f)$ so we now need to write the two-form
\[
\frac{i}{2\pi} \int_{S^1} \langle \pi_2^*(F), \tau^*(Z) \rangle d\theta
\]
as $\delta$ of a two-form on $P$. To this end, choose a map $\Phi: P \rightarrow C^\infty([0, 2\pi], \mathfrak{g})$ satisfying
\[
\Phi(pg) = \text{ad}(g^{-1})\Phi(p) + g^{-1}\frac{\partial g}{\partial \theta}.
\]
We call any such function a twisted Higgs field for $P$. As a convex combination of twisted Higgs fields is a twisted Higgs field and they clearly exist when $P$ is trivial it is straightforward to use a partition of unity on $M$ to construct a twisted Higgs field for $P$. Notice that a twisted Higgs field will not generally live in $L^\infty$. However if we start with a bundle $P(M, \Omega(K))$ then the twisted Higgs field can be required to take values in $\Omega(k)$. Choose then $\Phi$ a twisted Higgs field for $P$. It satisfies
\[
\text{ad}(\tau)(\pi_1^*(\Phi)) = \pi_2^*(\Phi) + \tau^*(Z)
\]
and hence
\[
\langle \pi_2^*(F), \tau^*(Z) \rangle = \langle \pi_2^*(F), \text{ad}(\tau)(\pi_1^*(\Phi)) - \pi_2^*(\Phi) \rangle
\]
\[
= \langle \text{ad}(\tau^{-1})\pi_2^*(F), \pi_1^*(\Phi) \rangle - \langle \pi_2^*(F), \pi_2^*(\Phi) \rangle
\]
\[
= \pi_1^*([F, \Phi]) - \pi_2^*([F, \Phi]).
\]
Finally we can write $\tau^* R - d\epsilon = \delta(f)$ where $f$ is the two-form on $P$ defined by
\[
f = \frac{i}{2\pi} \int_{S^1} \left( \frac{1}{2} \langle A, \partial_\theta A \rangle - \langle F, \Phi \rangle \right) d\theta
\]
and hence
\[
df = \frac{i}{2\pi} \int_{S^1} \langle dA, \partial_\theta A \rangle - \langle dF, \Phi \rangle - \langle F, d\Phi \rangle d\theta
\]
Using the Bianchi identity for $F$ we obtain
\[
df = -\frac{i}{2\pi} \int_{S^1} \langle F, \nabla \Phi \rangle d\theta
\]
where
\[
\nabla \Phi = d\Phi + [A, \Phi] - \partial_\theta A.
\]
It is straightforward to show that
\[
\pi_1^*(\nabla \Phi) = \text{ad}(\tau^{-1})\pi_2^*(\nabla \Phi)
\]
which shows that $\nabla \Phi$ descends to a one form on $M$ with values in the adjoint bundle of $P$, just as $F$ descends to a two-form on $M$ with values in the adjoint bundle of $P$.

Finally we have

**Theorem 5.1.** Let $P \rightarrow M$ be a principal $LK$ bundle and let $A$ be a connection for $P$ with curvature $F$ and $\Phi$ be a twisted Higgs field for $P$. Then the string class of $P$ is represented in de Rham cohomology by the three-form
\[
\frac{1}{4\pi^2} \int_{S^1} \langle F, \nabla \Phi \rangle d\theta
\]
where
\[
(17) \quad \nabla \Phi = d\Phi + [A, \Phi] - \partial \theta A.
\]

5.3. Reduced splittings. In [7] the concept of reduced splitting is used to calculate a connection and curving for the lifting bundle gerbe. In this section we explain how this fits in with our constructions in the case of the loop group.

First we have

**Definition 5.2 ([7])**. Define the group cocycle \( Z : L(K) \times L(\mathfrak{k}) \to i\mathbb{R} \) by
\[
(0, Z(g, X)) = \text{ad}(\hat{g}^{-1})(g^{-1}(X, 0)) - \text{ad}(g^{-1})(X, 0)
\]
where here we assume a splitting of \( \hat{L}(K) \) as \( L(K) \oplus i\mathbb{R} \) and \( \hat{g} \) is a lift of \( g \) to the central extension. The group cocycle \( Z \) encapsulates the information of the central extension in a similar manner to our \( \alpha \). Indeed in the case of the loop group with the extension defined by the \((R, \alpha)\) in (9) then we have
\[
Z(g^{-1}, X) = -\alpha(1, g)(X, 0) = \frac{i}{2\pi} \int_{S^1} \langle X, \partial \theta(g)g^{-1} \rangle d\theta.
\]

**Definition 5.3 ([7])**. A reduced splitting for a loop group principal bundle \( P(M, L(K)) \) is a map
\[
\ell : P \times L(\mathfrak{k}) \to \mathbb{R}
\]
which is linear in the second factor and satisfies \( \ell(p, X) = \ell(pg, \text{ad}(g^{-1})(X)) + Z(g^{-1}, X) \).

A straightforward calculation shows that if \( \Phi \) is a Higgs field then
\[
\ell(p, X) = \frac{i}{2\pi} \int_{S^1} \langle \Phi, X \rangle d\theta
\]
is a reduced splitting.

5.4. The path fibration. We will illustrate the above discussion for the case of the \( L_0(K) \) bundle \( \mathcal{P} K(K, L_0(K)) \), where \( \mathcal{P} K \to K \) is the path fibration of the group \( K \) — see [1]. In this case there is a canonical closed 3-form on \( K \) representing the image of the generator of \( H^3(K; \mathbb{Z}) = \mathbb{Z} \) inside \( H^3(K; \mathbb{R}) \) given by
\[
\omega_3 = \frac{1}{48\pi^2} \langle [\hat{\Theta}, \hat{\Theta}], \hat{\Theta} \rangle
\]
where \( \hat{\Theta} \) is the right invariant Maurer-Cartan form on \( K \) see for example [1]. We will show that for the path fibration the string class (16) is \( \omega_3 \).

First we need to define a connection. The tangent space at \( p \in \mathcal{P} K \) is the set of all vector fields along the path \( p \) and such a vector field is vertical if it vanishes at \( 2\pi \). A right invariant splitting is given by
\[
H_p = \{ \theta \mapsto (\theta/2\pi)R_{\pi(\theta)}X \mid X \in \mathfrak{k} \}.
\]
Clearly this satisfies \( R_g H_p = H_{pg} \) and the corresponding connection one-form is
\[
A = \Theta - \frac{\theta}{2\pi} \text{ad}(p^{-1})\pi^*(\hat{\Theta}).
\]
A calculation shows that the curvature of \( A \) is
\[
F = \left( \frac{\theta^2}{8\pi^2} - \frac{\theta}{4\pi} \right) \text{ad}(p^{-1})[\pi^*(\hat{\Theta}), \pi^*(\hat{\Theta})].
\]
A suitable Higgs field is given by
\[ \Phi(p) = p^{-1} \frac{\partial p}{\partial \theta} \]
and another calculation shows that
\[ \nabla \Phi = \frac{1}{2\pi} \text{ad}(p^{-1})\pi^* (\Theta). \]
Putting these into the formula for the string class (16) gives the required result.

6. Calorons and the string class

In [6] as part of the study of calorons a correspondence was introduced between $K$ bundles $\tilde{P}$ on a manifold $M \times S^1$ and $\Omega(K)$ bundles $P$ on $M$. This correspondence also related a connection $\tilde{A}$ on $\tilde{P}$ to a connection $A$ on $P$ and a section $\Phi$ of the twisted adjoint bundle. We will describe this correspondence and show that the integral over the circle of the Pontrjagin class of $\tilde{A}$ is the representative for the string class (16).

If $P \to M$ is an $\Omega(K)$ bundle we define
\[ \tilde{P} = (P \times K \times S^1)/\Omega(K) \]
where $\Omega(K)$ acts on $P \times K \times S^1$ by $g(p, k, \theta) = (pg^{-1}, g(\theta)k, \theta)$. Letting $[p, k, \theta]$ denote the equivalence class of $(p, k, \theta)$ we have a right $K$ action on $\tilde{P}$ given by $[p, k, \theta]h = [p, kh, \theta]$ and a projection map $\pi: \tilde{P} \to M \times S^1$ defined by $\pi ([p, k, \theta]) = (\pi(p), \theta)$. The fibres of $\pi$ are the orbits of $K$ and $\tilde{P}$ is a principal $K$ bundle.

Starting instead with a principal $K$ bundle $\tilde{P} \to M \times S^1$ we define a bundle $P \to M$ whose fibre at $m$ is all the sections of $\tilde{P}$ restricted to $\{m\} \times S^1$. This is clearly acted on by $\Omega(K)$.

Given a connection 1-form $A$ on $P(M, \Omega(K))$ we can define a connection 1-form $\tilde{A}$ on $\tilde{P}$ by pushing forward the 1-form on $P \times K \times S^1$ which is also denoted by $\tilde{A}$ and is given by
\[ \tilde{A} = \text{ad}(k^{-1})(A) + \Theta + \text{ad}(k^{-1})d\theta \Phi. \]
To check that we can push forward $\tilde{A}$ to a connection 1-form on $\tilde{P}$ we first have to show that $\tilde{A}$ is invariant under the action of $L(K)$ on $P \times K \times S^1$ and that $\tilde{A}$ kills vectors tangent to the fibering $P \times K \times S^1 \to \tilde{P}$. Let $(p, k, \phi)$ be a point of $P \times K \times S^1$ and suppose that $(X, \eta, \lambda)$ is a tangent vector at $(p, k, \phi)$. Then if $g \in \Omega(K)$ we have
\[ R^*_g \tilde{A}((p, k, \phi); (X, \eta, \lambda)) = \text{ad}(k^{-1})(A(p; X)(\phi)) + L_{k^{-1}}(\eta) + \text{ad}(k^{-1})(\lambda \Phi(\phi)). \]
so $\tilde{A}$ is invariant under the action of $L(K)$. We now check that $\tilde{A}$ kills vectors tangent to the fibering $P \times K \times S^1 \to \tilde{P}$. The vectors tangent to this fibering are of the form $(\iota_p(X), -X(\theta)k, 0)$ where $X$ is in $\Omega(\mathfrak{g})$ the Lie algebra of $\Omega(K)$ and where $\iota_p(X) = \frac{d}{dt} |_{t=0} \exp(tX)$. It is easy to see that these vectors lie in the kernel of $\tilde{A}$. We now need to check that the pushed forward 1-form on $\tilde{P}$ (which we will also denote by $\tilde{A}$) is a connection 1-form on $\tilde{P}$. It is a straightforward matter to check that $\tilde{A}$ is equivariant under the action of $K$ on $\tilde{P}$. Now let $\xi \in \mathfrak{g}$ and let $\tilde{p} = (p, k, \phi)$ be a point of $P \times K \times S^1$. Then we have $\iota_{\tilde{p}}(\xi) = (0, L_k(\xi), 0)$ and $\tilde{A}(0, L_k(\xi), 0) = \xi$. Hence $\tilde{A}$ pushes forward to define a connection 1-form on $\tilde{P}$. 
If we compute the curvature $\tilde{R} = d\tilde{A} + 1/2[\tilde{A}, \tilde{A}]$ for the connection $\tilde{A}$ we obtain:

$$\tilde{R} = \text{ad}(k^{-1})(F + \nabla(\Phi)d\theta)$$

(20)

where $\nabla(\Phi)$ is defined as in (17) and $F$ denotes the curvature of the connection $A$ on the principal $L(K)$ bundle $\tilde{P}$. The Pontrjagin form of the connection $A$ on the bundle over $S^1 \times M$ is

$$-\frac{1}{8\pi^2}(\tilde{R}, \tilde{R}) = -\frac{1}{8\pi^2}(\langle F, F \rangle + 2\langle F, \nabla(\Phi) \rangle)$$

(21)

and integrating over the circle gives the string class of the bundle $P(M, \Omega(K))$. We have hence proved

**Theorem 6.1.** Let $P \to M$ be an $\Omega(K)$ bundle and $\tilde{P} \to M \times S^1$ the corresponding $K$ bundle. Then the string class of $P$ is obtained from the Pontrjagin class of $\tilde{P}$ by integrating over the circle.

This result enables an easy proof of a result of Killingback also proved in [4, 7]. We start with a $K$ bundle $Q \to X$ and let $P$ and $M$ be the loop spaces of $Q$ and $X$ respectively. Then $P$ is an $\Omega(K)$ bundle over $M$ and Killingback shows that the string class of $P$ is obtained from the Pontrjagin class of $Q$ by pulling back by the evaluation map

$$\text{ev}: M \times S^1 \to X$$

(22)

and integrating over the circle. If $P$ is the space of loops in $Q$ there is a natural map $P \times K \times S^1 \to Q$ given by $(p, k, \theta) \mapsto p(\theta)k$. This is constant on the orbits of the $\Omega(K)$ action and hence defines a map $\tilde{P} \to Q$ which is $K$ equivariant and covers the evaluation map (22). This shows that $\tilde{P}$ is the pull-back of $Q$ by the evaluation map. Hence the Pontrjagin class of $\tilde{P}$ is the pull-back by the evaluation map of the Pontrjagin class of $Q$. Thus the string class of $P$ is the integral over the circle of the pull-back by the evaluation map of the Pontrjagin class of $Q$.

### 7. Conclusion

If we try and apply the caloron construction to an $L(K)$ bundle with a connection we obtain a $K$ bundle on $S^1$ which is only smooth on $(0, 2\pi)$. There should be a theory of $K$ bundles on $[0, 2\pi]$ with connection which patch together over $\{0\} \times M$ and $\{2\pi\} \times M$ in such a way as to recover the result of [3] relating $L(K)$ bundles on $M$ to such bundles.

Notice that this approach could be used to show that any $L(K)$ bundle over $M$ has a three class which we could define by transgressing the Pontrjagin class of the induced $K$ bundle over $S^1 \times M$. However it would not be clear that this three class was the string class, the obstruction to lifting to $\tilde{L}(K)$. For this we needed the theory of bundle gerbes and the connections and curvings which provide a bridge between the Cech description of the string class and a de Rham realisation of it.

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