Quantum Interaction $\phi_4^4$:  
the Construction of Quantum Field  
defined as a Bilinear Form  
Edward P. Osipov  
Department of Theoretical Physics  
Institute for Mathematics  
630090 Novosibirsk 90  
RUSSIA  
E-mail address (Relcom): osipov@math.nsk.su

Abstract

We construct the solution $\phi(t, x)$ of the quantum wave equation $\Box \phi + m^2 \phi + \lambda : \phi^3 := 0$ as a bilinear form which can be expanded over Wick polynomials of the free $in$-field, and where $:\phi^3(t, x):$ is defined as the normal ordered product with respect to the free $in$-field. The constructed solution is correctly defined as a bilinear form on $D_\theta \times D_\theta$, where $D_\theta$ is a dense linear subspace in the Fock space of the free $in$-field. On $D_\theta \times D_\theta$ the diagonal Wick symbol of this bilinear form satisfies the nonlinear classical wave equation.

1 Introduction

The construction of an interacting quantum field, which satisfies a system of Wightman axioms (or a physical and/or mathematical analogue) is a central problem in quantum field theory. It seems that the best starting point would be a relativistic dynamical equation of motion with some interpretation of nonlinear terms. The description of a physical vacuum as a measure on the configuration space or on the space of trajectories is closely connected with dynamical equations of motion and quantum mechanics. However here we consider a possible description of dynamics and leave a possible description of the vacuum for the future.

In the present paper we consider a self-interacting scalar quantum field in four-dimensional Minkowski space-time satisfying the following relativistic wave equation

$$\Box \phi(t, x) + m^2 \phi(t, x) + \lambda : \phi^3(t, x): = 0,$$

or in the form of integral equation

$$\phi(t, x) = \phi_{in}(t, x) - \lambda \int_{-\infty}^{t} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y.$$  

1
A principal barrier of this way appears as difficulties associated with the definition of a product of fields given at the same point. This difficulty of the definition of a local product is connected with a singular dependence of the field on space-time coordinates. Usually one tries to solve this problem by renormalization.

We consider Equations (1.1) and (1.2) and the definition of the product of fields at a point in the following way. First of all, we construct the solution of the quantum Yang-Feldman equation (1.2) in the class of bilinear form acting in the Fock Hilbert space corresponding to the free quantum \textit{in}-field. In the other words, we seek the solution of the Yang-Feldman equation (1.2), in fact, in the form of the expansion of the solution in terms of the creation and annihilation operators of the \textit{in}-field. Second, we define the product of the free \textit{in}-field and its normal-ordered product ( = the Wick product ).

In the Fock Hilbert space any operator or any bilinear form (belonging to a wide class of operators or bilinear forms) can be approximated by Wick polynomials. In the Fock space one can define a product of bilinear forms for bilinear forms expanded in terms of normal-ordered Wick polynomials. The normal-ordered functionals of the creation and annihilation operators have a natural dense domain of definition, \( \sigma \subset H_{\text{in}} \), which is some analog to the Schwartz space. The are linear continuous maps from \( \sigma \) into \( \sigma' \), where \( \sigma' \) is the dual of \( \sigma \), i.e. they are bilinear forms on \( \sigma \times \sigma \), see \([1]-[3]\). In our paper we use the construction of Wick polynomials of the free field given in \([4]-[11]\).

We solve the Yang-Feldman equation (1.2) by constructing the expansion of the quantum field in terms of Wick polynomials of the free \textit{in}-field.

Since this expansion converges not on all vectors, so to construct the quantum field as a bilinear form we choose the special subspace \( D_{\theta} \) of vectors in the Fock Hilbert space. Namely, we take coherent vectors near to the vacuum of the \textit{in}-field and their finite linear combinations. Here a coherent vectors near to the vacuum means the vector of the form \( |z\rangle = \exp(za^+_m)\Omega \) with small complex-valued test function \( z \) ( \( \Omega \) is the vacuum vector, \( z \in \mathcal{S} \) and has a small \( F \)-norm, the definition of the \( F \)-norm see in Sect. 2 ). We note that \( D_{\theta} \) is dense in the Fock space.

The considered expansion of the quantum field in terms of the Wick polynomials of the free \textit{in}-field converges on the coherent vectors near to the vacuum and defines the solution of the quantum wave equation (1.1), (1.2) as a bilinear form on \( D_{\theta} \times D_{\theta} \).

The considered construction uses, in fact, the idea that the creation operator is conjugate to the annihilation one, moreover every creation vector is an eigenvector of all annihilation operators. In other words, matrix elements of quantum field on coherent vectors, i.e. Wick symbols, are solutions of classical Yang-Feldman wave equation with complex \textit{in}-data.

However, a complication arises here. This complication is connected with the existence and the construction of complex solutions of classical (real) wave equation. We overcome this complication by using coherent vectors near to the vacuum. To coherent vectors near to the vacuum correspond small matrix elements of the quantum field and small complex \textit{in}-data of classical wave equation. The convergence of solutions of the classical wave equation for small complex initial \textit{in}-data gives us the convergence of the considered expansion and allows us to construct the quantum field as a bilinear form defined on the
subspace, generated by linear combinations of coherent vectors corresponding to small complex \textit{in}-data. Therefore, we construct the bilinear form by using its Wick symbols for coherent vectors near to the vacuum only.

The same consideration allows to construct the bilinear form $\phi_{out}$ corresponding to the quantum \textit{out-field}.

The constructed quantum field is a scalar with respect to the Poincaré transformation

$$U(a, \Lambda)\phi(t, x)U(a, \Lambda)^{-1} = \phi((a, \Lambda)(t, x)),$$

where $U(a, \Lambda) = U_{in}(a, \Lambda)$.

The generator of the translation subgroup (that is, the Hamiltonian and the momentum operator) satisfy the spectrum condition. The constructed field is non-local with respect to the free $\phi_{in}(t, x)$ field. The question about locality of the constructed field is open. This question is closely connected with the question about a structure of the bilinear form and with the question about the existence of a measure corresponding to the vacuum and about its support. It would be interesting to represent the constructed bilinear form $\phi(t, x)$ as an operator-valued generalized function or as an operator-valued hyperfunction.

In conclusion, we remark that the considered construction has been suggested by Heifets [12]. He also constructed the quantum field with the help of small complex initial data, however Heifets [12] used instead of $F$-norm more complicated variant of $R$-norm.

Rączka [13] also tried to construct the quantum field as a bilinear form. He used for the construction a unproved assumption about the Wick symbols of approximations and their convergence to the solution of (real) wave equation for any (not necessary small) complex \textit{in-data}. This assumption is incorrect in general. We go around this difficulty considering small complex initial \textit{in-data} and extending the results of Morawetz and Strauss [14, 15], for this case, see also [16]-[18].

Our consideration is the following. In Sect. 2 we formulate and prove the assertions (Theorems 2.3 and 2.4) that we need for solutions of non-linear (real) wave equation with small complex \textit{in-data}. In Sect. 3 we prove some estimates for the non-linear part of the classical wave equation (Lemmas 2.1 and 2.2). In Sect. 4 we describe field as a bilinear form (Theorems 4.1-4.3) and in Sect. 5 we discuss the obtained results and its connection with other approaches.

2 Solution of the wave equation for small complex initial \textit{in-data}

In this section we consider global complex solutions of classical (real) nonlinear wave equation

$$u_{tt} - \Delta u + m^2 u + \lambda u^3 = 0, \quad m > 0, \quad \lambda > 0.$$  \hspace{1cm} (2.1)

To construct the hermitian (scalar) quantum field we need the solutions of (2.1) for small complex \textit{in-data}. First we rewrite the equation (2.1) in the integral form

$$u(t, x) = u_T(t, x) - \lambda \int_T^t \int R(t - \tau, x - y)u^3(\tau, y)d\tau d^3y.$$ \hspace{1cm} (2.2)
Here $u_T(t, x)$ is the complex solution of the free equation that corresponds to the complex Cauchy data at the time $T$, $R(t, x)$ is the Riemann function of the linear equation, i.e. the free solution with Cauchy data $R(0, x) = 0$, $R_t(0, x) = \delta(x)$. We remark that

$$R(t, x) = -R(-t, x), \quad R(t, x) = \frac{\sin(t(-\Delta + m^2)^{\frac{1}{2}})}{(-\Delta + m^2)^{\frac{1}{2}}} (x)$$

and for $t > 0$

$$R(t, x) = \frac{\delta(t - |x|)}{4\pi t} + \frac{1}{4\pi} \theta(t - |x|) m J_1(m(t^2 - |x|^2)^{1/2}) (t^2 - |x|^2)^{-1/2}.$$

To construct the quantum field we need the solutions of the equation

$$u(t, x) = \lambda \int_{-\infty}^{t} \int R(t - \tau, x - y) u^3(\tau, y) d\tau d^3y \quad (2.3)$$

for small complex $in$-data. Here $u_{in}$ is a solution of the free equation, corresponding to complex $in$-data.

To construct the solutions of the Yang-Feldman equation (2.3) we extend Morawetz’s and Strauss’ results [14] on the case of small complex-valued $in$-data.

In order to describe our results we define the energy norm

$$\|u(t)\|^2_e = \int (|u_t(t, x)|^2 + |\nabla u(t, x)|^2 + m^2|u(t, x)|^2) d^3x$$

and the $F$-norm [14]

$$\|u\|^2_F = \sup_t \|u(t)\|^2_e + \sup_x |u(t, x)|^2 + \int_{-\infty}^{\infty} \sup_x |u(t, x)|^2 dt \quad (2.4)$$

for the Banach space $F^C$ of continuous complex-valued functions with finite $F$-norm.

Define the Banach space $\mathcal{F}^C$ of complex-valued solutions of the free equation

$$u_{tt} - \Delta u + m^2 u = 0,$$

which is a subspace of the Banach space $F^C$. For this purpose we define the subspace $\mathcal{F}_{1}^C$ as the space of the free solutions whose Cauchy data $(\varphi(x), \pi(x))$ at the zero time are such that $\varphi(x)$ with its first and second derivatives belong to $L^1 \cap L^2$, and the third derivatives are in $L^1$, $\pi(x)$ with its first derivatives are in $L^1 \cap L^2$ and second derivatives are in $L^1$. A solution of the free equation with initial data from $\mathcal{F}_{1}^C$ has the uniform decay like $(1 + |t|)^{-3/2}$ for $t \to \infty$ and has a finite $F$-norm (a complex free solution can be considered similar to the real one, see Appendix B [14]). Define $\mathcal{F}^C$ as the completion of $\mathcal{F}_{1}^C$ in the $F$-norm. It is clear that $\mathcal{F}^C$ is a closed subspace of $F^C$.

Let $\mathcal{F}$ and $F$ denote the corresponding real subspaces of the spaces $\mathcal{F}^C$ and $F^C$.

To construct a solution for small complex $in$-data we formulate two lemmas, which we prove in Sect. 3.
To formulate these lemmas we introduce the notations that we need.
Let \( M = [a, b], -\infty \leq a < b \leq +\infty \), and in addition \( M = M_1 \cup M_2 \), where \( M_1 = [a, b] \cap \{ \tau \mid |t - \tau| \geq \delta \} \), \( M_2 = [a, b] \cap \{ \tau \mid |t - \tau| < \delta \} \), \( \delta > 0 \).

We define
\[
[u]_M = \sup_{t \in M} \sup_{x} |u(t, x)| + \left\{ \int_{M} \sup_{x} |u(t, x)|^2 dt \right\}^{1/2}
\]
and
\[
[u] = [u]_{(-\infty, \infty)}.
\]

We introduce the norm
\[
\langle u \rangle_M = \left\{ \sup_{K} \int_{K \cap M} |u|^2 d\bar{S} \right\}^{1/2}, \quad \langle u \rangle = \langle u \rangle_{(-\infty, \infty)},
\]
where \( d\bar{S} \) denotes the measure on the surface of the cone \( K \) and \( K \) runs over all forward and backward light cones in space-time. On the surface of the forward or backward light cone \( K = \{ (t, x) \in K \mid t^2 - |x|^2 \geq 0, \pm t \geq 0 \} \) we use the measure \( d\bar{S} \), where \( d\bar{S} = \theta(\pm t)\delta(|t| - |x|)dt \) and \( dS = t^2d\omega, |\omega| = 1 \) is the sphere measure on the sphere of radius \( t \) in \( \mathbb{R}^3 \).

Lemma 2.1.
Let
\[
I(t, x) = \int_{M_1} \int R(t - \tau, x - y)u(\tau, y)v(\tau, y)w(\tau, y)d\tau dy,
\]
where \( u, v, \) and \( w \) are arbitrary smooth complex-valued functions.

Then
\[
|I(t, x)|^2 \leq c(\delta, \alpha)\langle u \rangle_M \int_{M_1} [||u(\tau)||_\infty^2 ||v(\tau)||_\infty + ||u(\tau)||_\infty ||v(\tau)||_\infty ||u(\tau)||_e]
\]
\[
||v(\tau)||_2^{1 - 2\alpha} ||u(\tau)||_\infty^{2\alpha} |t - \tau|^{-3/2 + 3\alpha} d\tau.
\]
Here \( 0 \leq \alpha \leq 1/2 \), \( c(\delta, \alpha) \) is a constant, depending on \( \delta \) and \( \alpha \) only (and, maybe, on the mass which enters into the Riemann function).

Lemma 2.2.
Let \( u(t, x) \) and \( v(t, x) \) be a pair of arbitrary smooth complex-valued functions. Let
\[
R u(t, x) = \int_{M} \int R(t - \tau, x - y)u^3(\tau, y)d\tau dy.
\]

(a) For any \( 0 \leq \alpha < 1/6 \), we have
\[
||Ru||_F \leq c([u])_M^{1 + \alpha} ||u||_{F, M}^{1 - \alpha} \left[ ||u||_{F, M} + \langle u \rangle_M \right].
\]

(b)
\[
||Ru - Rv||_F \leq c\left( [u]_M + [v]_M \right)^{1/2} \left( ||u||_{F, M} + \langle u \rangle_M + ||v||_{F, M} + \langle v \rangle_M \right)^{3/2}
\]
\[\|u - v\|_{F,M}^{1/2} \sup_{t \in M} \|u(t) - v(t)\|_{e}^{1/2}.\]

Remark. The constants entering into Lemma 2.2 do not depend of \(M\).

For a solution \(u = u(t, x)\) of Equation (2.1) we denote by \(u_T = u_T(t, x)\) the free solution whose Cauchy data at \(t = T\) agree with that of \(u\),
\[u_T(T, x) = u(T, x), \quad \frac{\partial u_T}{\partial t}(T, x) = \frac{\partial u}{\partial t}(T, x),\]
and formulate now the theorems about solutions of Equations (2.2)-(2.3) for small complex initial data and in-data.

**Theorem 2.3 (Cauchy problem).**

There exists a strictly positive \(\theta\) such that for any \(S, -\infty < S < \infty\), and \(u_S \in \mathcal{F}^C, \|u_S\|_F < \theta\), there exists a unique global solution \(u\) of Equation (2.2) with finite \(F\)-norm and whose Cauchy data at time \(S\) equal that of \(u_S\). In addition \(\|u\|_F < 2\theta\). The free solution \(u_T\), whose Cauchy data at time \(T\) equal that of \(u\), also belongs to \(\mathcal{F}^C\). Furthermore, \(u_T\) depends continuously on \(u_S\) in \(\mathcal{F}^C\) and \(\|u_T\|_F < 2\theta\).

There exists a unique free solution \(u_{in}\) and a unique free solution \(u_{out}\) such that
\[\|u_{in}(t) - u(t)\|_e \to 0 \text{ for } t \to -\infty \text{ and } \|u_{out}(t) - u(t)\|_e \to 0 \text{ for } t \to +\infty,\]
in this case \(u_{in}, u_{out} \in \mathcal{F}^C\) and the \(F\)-norm of \(u_{in}\) and of \(u_{out}\) is less than \(2\theta\).

**Theorem 2.4 (Cauchy problem at \(t = -\infty\)).**

There exists a strictly positive \(\theta\) such that for \(u_{in} \in \mathcal{F}^C\) and \(\|u_{in}\|_F < \theta\) there exists a unique global solution \(u\) of Equation (2.3) with finite \(F\)-norm and which converges in the energy norm to \(u_{in}\) for \(t \to -\infty\). In addition \(\|u\|_F < 2\theta\). For any \(T, -\infty < T < +\infty\), the free solution \(u_T\), whose Cauchy data at time \(t = T\) equal that of \(u\), also belongs to \(\mathcal{F}^C\) and in this case \(\|u_T\|_F < 2\theta\). There exists a unique free solution \(u_{out}\) such that
\[\|u(t) - u_{out}(t)\|_e \to 0 \text{ for } t \to +\infty,\]
in addition \(\|u_{out}\|_F < 2\theta\). Furthermore, \(u_T, u_{out}\) depend continuously on \(u_{in}\) in \(\mathcal{F}^C\). \(u(t, x)\) also depends continuously on \(u_{in}\) in \(\mathcal{F}^C\).

Remarks.

1) \(\theta\) depends on the mass and the coupling constant in the non-linear equation (2.1) and its “smallness” depends only on the value of constants in the bounds of Lemmas 2.1 and 2.2.

2) If \(u^3\) is replaced by \(F(u)\) in Equation (2.3), this theorem depends only on the property \(|F''(u)| = O(|u|)\) as \(u \to 0\).

**Proof of Theorems 2.3 and 2.4.**

The proof of Theorem 2.3 is completely the same as the proof of Theorem 2.4. Therefore, we restrict ourselves to the proof of Theorem 2.4.

First consider the uniqueness of a solution of the Cauchy problem at \(t = -\infty\). For the \(u\) with the initial data \(u_{in}\) from \(\mathcal{F}^C_\theta = \mathcal{F}^C \cap \{u \in \mathcal{F}^C| \|u\|_F \leq \theta\}\) and with finite \(F\)-norm, \(\|u\|_F \leq \theta\), there is the representation
\[u_{in}(t) = u(t) + \lambda \int_{-\infty}^{t} R(t - \tau) * u^3(\tau) d\tau.\]
If \( u \) and \( v \) are two solutions as in the statement of Theorem 2.4, then

\[
u(t) - v(t) = -\lambda \int_{-\infty}^{t} R(t - \tau) * [u^3(\tau) - v^3(\tau)]d\tau.
\]

Taking the energy norm, we get

\[
sup_{t \leq S} \|u(t) - v(t)\|_e \leq \frac{\lambda}{m} \sup_{\tau \leq S} \|u(\tau) - v(\tau)\|_e \int_{-\infty}^{S} (\|u(\tau)\|_\infty + \|v(\tau)\|_\infty)^2 d\tau,
\]

or

\[
sup_{t} \|u(t) - v(t)\|_e \leq \frac{\lambda}{m} \sup_{t} \|u(t) - v(t)\|_e \int_{-\infty}^{\infty} (\|u(\tau)\|_\infty + \|v(\tau)\|_\infty)^2 d\tau.
\]

Since

\[
\frac{\lambda}{m} \int_{-\infty}^{\infty} (\|u(\tau)\|_\infty + \|v(\tau)\|_\infty)^2 d\tau \leq \frac{\lambda}{m} (\|u\|_F + \|v\|_F)^2 \leq \frac{4\lambda \theta^2}{m},
\]

choosing \( 4\lambda \theta^2/m^2 < 1 \) we get

\[
sup_{t} \|u(t) - v(t)\|_e = 0
\]

and, since \( u(t) \) and \( v(t) \) belong to \( F^C \) and, thus, are continuous, \( u(t) = v(t) \). This proves the uniqueness.

To construct the solution \( u \), we shall solve the integral equation. Since we shall require \( u_T \in F^C \), the construction must exhibit \( u \) as the limit of smooth functions. For this reason, we first solve the ordinary Cauchy problem with initial data at a time \( S \). Let \( u_S \in F^C \) be a free solution with complex \( C^2 \) data of compact support.

Define

\[
Ru(t, x) = -\int_{S}^{t} R(t - \tau) * u^3(\tau)d\tau.
\]

We solve the equation

\[
u = u_T + \lambda Ru
\]

by successive approximations:

\[
u^{(0)} = u_S, \quad u^{(n)} = u_S + \lambda Ru^{(n-1)}, \quad n = 1, 2, \ldots.
\]

Each \( u^{(n)} \) is of class \( C^2 \) because \( u_S \) is.

We claim, that \( \theta \) so small that if \( \|u_S\|_F < \theta \), then

\[
(i) \quad \|u^{(n)}\|_F \leq \theta,
(ii) \quad \langle u^{(n)} \rangle \leq 2\theta/m
\]

for all \( n = 0, 1, 2, \ldots \).

We prove the claim by induction on \( n \). If \( n = 0 \), \( (i) \) is true by definition. The estimate \( (ii) \) follows from the inequality \( (i) \) and the simple energy inequality \( \langle u_S \rangle \leq \frac{2}{m} \|u_S\|_e \). This energy inequality can be proved with the help of relation (2.7) for the energy-momentum
density (2.8)-(2.9) in the same way as the energy inequality (2.10). We note that \( u^{(0)} = u_S \) is the solution of the free equation and for \( u_S \) the right side of (2.10) is equal to zero.

Next, we have

\[
\|u^{(n)} - u_S\|_F \leq \text{(by Lemma 2.2(a)) } c\|u^{(n-1)}\|_F^2 \left(\|u^{(n-1)}\|_F + \langle u^{(n-1)} \rangle \right)
\]

\[
\leq \text{(by the induction assumption) } \lambda c(1 + \frac{2}{m})\theta^3. \tag{2.5}
\]

Choose \( \theta \) so small that

\[
\lambda c(1 + \frac{2}{m})\theta^3 \leq \frac{1}{4} \theta.
\]

With this choice of \( \theta \) we then have (i).

To prove (ii), note that \( u^{(n)} \) is a solution of

\[
u_{tt} - \Delta u^{(n)} + m^2 u^{(n)} = -\lambda (u^{(n-1)})^3
\]

and therefore enjoys the energy inequality. To prove the energy inequality we use the following identity

\[
(u_{tt} - \Delta u + m^2 u)u^*_t + (u^*_t - \Delta u^* + m^2 u^*)u_t = \frac{\partial E}{\partial t} + \text{div } P, \tag{2.7}
\]

where

\[
E(t, x) = |u_t(t, x)|^2 + |\nabla u(t, x)|^2 + m^2 |u(t, x)|^2, \tag{2.8}
\]

\[
P(t, x) = -\nabla u(t, x)u^*_t(t, x) - \nabla u^*(t, x)u_t(t, x). \tag{2.9}
\]

This identity is fulfilled for any (smooth) function \( u \). Note that \((E(t, x), P(t, x))\) is given by the components \((T_{00}(t, x), T_{0i}(t, x))\) of the energy-momentum tensor, see [19, §XI.14, Addition], [20] ch.1, §2.2, p.23, see also [21] Theorem 2.1, and is the energy-momentum density of the complex field and not of a real one) To obtain the energy inequality we multiply the equation (2.6) by \( u_{t}^{(n)} \), add the conjugate term and integrate over the part \( K_1 \) of the forward or backward light cone. The equality (2.6) implies that the integral over the 4-dimensional divergence (the right side of (2.7) ) is not greater than the right side of (2.10). On the other hand, by the Gauss theorem the integral over the 4-dimensional divergence is equal to the energy-momentum flow (2.8)-(2.9) through the chosen part of the forward or backward light cone and is estimated from below by the left side of (2.10). This gives

\[
2^{-1/2} m^2 \int_{K_1} |u^{(n)}|^2 d\tilde{S} - \sup_t \|u^{(n)}(t)\|_e^2 \leq 2\lambda \int \int |(u^{(n-1)})^3 u_{t}^{(n)}| dtd^3x, \tag{2.10}
\]

and finally we receive

\[
2^{-1/2} m^2 \langle u^{(n)} \rangle^2 - \sup_t \|u^{(n)}(t)\|_e^2 \leq 2\lambda \int \int |(u^{(n-1)})^3 u_{t}^{(n)}| dtd^3x. \tag{2.11}
\]
Next, the right side of (2.11) is less than

\[ 2\lambda \sup \|u_t^{(n)}\|_2 \|u_t^{(n-1)}\|_2 \int_{-\infty}^{\infty} \|u_t^{(n-1)}(t)\|_\infty^2 dt. \quad (2.12) \]

(2.5), the choice of \( \theta \), and the induction assumption imply that (2.12) is not greater than

\[ 2\lambda (\|u_S\|_F + \frac{1}{4}\theta) \theta^3 \leq \frac{5}{2} \lambda \theta^4, \]

that is

\[ (u^{(n)})^2 \leq 2^{1/2} m^{-2} (\theta^2 + \frac{5}{2} \lambda \theta^4) \leq 4\theta^2 / m^2 \]

for \( 2^{1/2}(1 + \frac{5}{2} \lambda \theta^2) \leq 4 \). This proves (ii).

Next, we apply Lemma 2.2(b) to the difference

\[ u^{(n)} - u^{(n-1)} = \lambda R u^{(n-1)} - \lambda R u^{(n-2)}. \]

Using (i) – (ii), we obtain

\[ \|u^{(n)} - u^{(n-1)}\|_F \leq c \lambda \theta^2 (2 + \frac{4}{m})^2 \|u^{(n-1)} - u^{(n-2)}\|_F. \]

Choosing the coefficient on the right to be less than 1/2, through choice of \( \theta \), \( \{u^{(n)}\} \) becomes a Cauchy sequence in the \( F \)-norm. Its limit is the solution \((I - R)^{-1} u_\tau\). Furthermore, this solution is a \( C^2 \) function (if \( u_S \) is ) as a consequence of the estimate

\[ \|Du^{(n)}(t)\|_\infty \leq \|Du_S(t)\|_\infty + c \int_S^t \|Du^{(n-1)}(\tau)\|_\infty d\tau, \]

where \( D \) is a first or second derivative.

If \( u_S \) has compact support in space, so does the solution. This follows from the explicit form of the approximation \( u^{(n)} \), from the fact, that

\[ \text{supp } u^{(n)}(t, \cdot) \subset \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \text{supp}(u_S(S, \cdot), \partial u_S / \partial t(S, \cdot))) \leq |t - S| \} \]

and from the convergence of the series \( u = \sum_n (u^{(n+1)} - u^{(n)}) \). Of course, these statements are valid for every \( t, -\infty < t < \infty \).

The convergence of the series \( u = \sum_n (u^{(n+1)} - u^{(n)}) \) and the restrictions on \( \theta \) imply that \( \|u\|_F < 2\theta \). Moreover, the convergence of \( u^{(n)} \) to \( u \) in the \( F \)-norm implies that \( \langle \cdot \rangle \)-norm of \( u \) is also restricted by \( 2\theta / m \). This follows from the continuity of \( u^{(n)} \) and \( u \) and the convergence and the uniform boundedness of the integral of \( u^{(n)}^2 \) taken over the bounded part of the cone.

Now let \( u_{in,k} \) be a sequence of \( C^2 \) smooth free solutions of compact support which tends to \( u_{in} \) in \( \mathcal{F}^C \), \( \|u_{in}\|_F < \theta \). It is clear that there exists a sequence of such \( u_{in,k} \). Let \( u_k \) be the constructed solution of (2.2) whose Cauchy data at time \( t = -k \) equals that of \( u_{in,k} \) at time \( t = -k \). The limit of \( u_k \) as \( k \to \infty \) will be the required solution. To prove the convergence of \( u_k \) we consider the difference \( u_k(t) - u_l(t) \). For \( k > l \)

\[ u_k(t) - u_l(t) = (u_{in,k}(t) - u_{in,l}(t)) - \lambda \int_{-l}^{t} R(t - \tau) * (u_k^3(\tau) - u_l^3(\tau)) d\tau \]
Consider the $F$-norm. The $F$-norm of the first term goes to zero. Lemma 2.2(b) and the uniform estimate $\langle u_k \rangle \leq 2\theta/m$ imply that the $F$-norm of the second term is less than $c\lambda\theta^2(2 + \frac{4}{m})^2\|u_k - u_l\|_F < \frac{1}{2}\|u_k - u_l\|_F$. The estimate of this term we carry over on the left side.

The final term on the right is estimated as follows. Let $\varepsilon > 0$. Since $u_{in,k}$ converges in $\mathcal{F}C$, there exists an $L = L(\varepsilon)$ such that,

$$\left[u_{in,k}\right]_{(-\infty,-L]} = \sup_{t \leq -L} \|u_{in,k}(t)\|_{\infty} + \left\{ \int_{-\infty}^{-L} \|u_{in,k}(t)\|_{\infty}^2 dt \right\}^{1/2} \leq \sup_{t \leq -L} \|u_{in}(t)\|_{\infty} + \left\{ \int_{-\infty}^{-L} \|u_{in}(t)\|_{\infty}^2 dt \right\}^{1/2} + 2\|u_{in} - u_{in,k}\|_F < \varepsilon$$

for all $k \geq L$.

Lemma 2.2(a) and the equality $u_k = u_{in,k} + \lambda \mathcal{R}u_k$ imply that

$$\left[u_k\right]_{(-\infty,-L]} \leq \left[u_{in,k}\right]_{(-\infty,-L]} + c\lambda\theta^2(1 + \frac{2}{m}) \left[u_k\right]_{(-\infty,-L]}.$$

For sufficiently small $\theta$ and $c\lambda\theta^2(1 + \frac{2}{m}) < \frac{1}{2}$. So $\left[u_k\right]_{(-\infty,-L]} < 2\varepsilon$. Therefore, these arguments, the uniform estimate $\langle u_k \rangle \leq 2\theta/m$ and Lemma 2.2(a) imply that the $F$-norms of the last term of (2.13) are not greater than

$$c\lambda\theta^2(1 + \frac{2}{m}) \left[u_k\right]_{(-\infty,-L]} \leq \varepsilon.$$

It follows from these estimate that $\{u_k\}$ is a Cauchy sequence in the $F$-norm.

Call the limit $u$. By passage to the limit we obtain

$$u(t) = u_{in}(t) - \lambda \int_{-\infty}^{t} R(t - \tau) \ast u^3(\tau)d\tau,$$

whence

$$\|u(t) - u_{in}(t)\|_e \to 0 \quad \text{as} \quad t \to -\infty,$$

This means that we have constructed the solution $u(t)$ for the initial $in$-data $u_{in}(t)$.

Now let $u_T$ be the free solution with the Cauchy data at $t = T$ equal to the Cauchy data of $u$. Let $u_k$ be defined as stated above, that is, it is a solution of (2.2), with Cauchy data at time $t = -k$ equal to the Cauchy data of $u_{in,k}$ at time $t = -k$. Let $u_{k,T}$ be the free solution with the Cauchy data at $t = T$ equal to the Cauchy data of $u_k$. Then $u_{k,T}$ is a smooth free solution given by

$$u_{k,T} = u_k(t) - \lambda \int_{-m}^{T} R(t - \tau) \ast u^3_k(\tau)d\tau.$$
Just as in the above argument, the right side converges in $\mathcal{F}^C$. If the limit of $u_{k,T}$ in $\mathcal{F}^C$ is called $v$, then $v \in \mathcal{F}^C$ and

$$v(t) = u_{in}(t) - \lambda \int_{-\infty}^{T} (R(t - \tau) \ast u^3(\tau))d\tau.$$ 

Since the Cauchy data of the free solution $v$ at $t = T$ agree with those of $u$, $v = u_T$, that is,

$$u_T(t) = u_{in}(t) - \lambda \int_{-\infty}^{T} R(t - \tau) \ast u^3(\tau)d\tau.$$ 

The continuous dependence of $u_T$ in the $F$-norm is a consequence of the construction: $u$ depends continuously on $u_{in}$, and $u_T$ on $u_{in}$.

Now we construct the solution $u_{out}$. $u_{out}$ has been defined as a unique free solution such that $\|u_{out}(t) - u(t)\|_e \to 0$ for $t \to +\infty$. We claim that $u_{out}$ is given by the following formula

$$u_{out}(t) = u_{in}(t) - \lambda \int_{-\infty}^{+\infty} R(t - \tau) \ast u^3(\tau)d\tau,$$

that means that the formula

$$u_{out}(t) = u(t) - \lambda \int_{t}^{+\infty} R(t - \tau) \ast u^3(\tau)d\tau.$$ 

is valid also. Indeed, the right sides are defined correctly, have finite energy and $u(t)$ converges to $u_{out}$ in the energy norm as $t \to +\infty$. Direct differentiation, in the weak sense, shows that $u_{out}$ is a free solution, so that it must coincide with $u_{out}$. We need to show that not only the $F$-norm of $u_{out}$ is finite, but that $u_{out} \in \mathcal{F}^C$.

To prove the statement that $u_{out} \in \mathcal{F}^C$ we approximate $u_{in}$ by smooth solutions $u_{in,k}$ with compact support and as $u_k$ we take the solution of (2.2), whose Cauchy data at time $t = -k$ agree with that of $u_{in,k}$ at time $t = -k$. Then let $u_{out,k}$ be the free solution whose Cauchy data at time $t = k$ agree with the Cauchy data of $u_k$ at time $t = k$.

Of course the Cauchy data of $u_k$ at any time are smooth and of compact support.

We have the integral representation

$$u_{out,k}(t) = u_{in,k}(t) - \lambda \int_{-k}^{k} R(t - \tau) \ast u^3_k(\tau)d\tau,$$

whence, for $k > l$ we have

$$u_{out,k}(t) - u_{out,l}(t) = (u_{in,k}(t) - u_{in,l}(t)) - \lambda \int_{-l}^{l} R(t - \tau) \ast (u^3_k(\tau) - u^3_l(\tau))d\tau$$

$$-\lambda \int_{-k}^{-l} R(t - \tau) \ast u^3_k(\tau)d\tau - \lambda \int_{-l}^{k} R(t - \tau) \ast u^3_l(\tau)d\tau.$$ 

(2.14)

Consider the $F$-norm of the four terms on the right as $k, l \to \infty$. The $F$-norm of the first term goes to zero by assumption, the second term is less than

$$c\lambda \theta^2(2 + \frac{4}{m})^2 \|u_k - u_l\|_F \leq \frac{1}{2} \|u_k - u_l\|_F.$$ 

11
The latter two terms can be estimated analogously to the similar term of (2.13). Let \( \varepsilon > 0 \). There exists \( L = L(\varepsilon) \) such that

\[
[u_{\text{in},k}]_{(-\infty,-L] \cup [L,\infty)} \leq \sup_{|t| \geq L} \|u_{\text{in}}(t)\|_{\infty} + \left\{ \int_{|t| \geq L} \|u_{\text{in}}(t)\|_{2}^{2} dt \right\}^{1/2} + 2\|u_{\text{in}} - u_{\text{in},k}\|_{F} < \varepsilon
\]

for all \( k \geq L \).

By Lemma 2.2(a)

\[
[u_{k}]_{(-\infty,-L]} + [u_{k}]_{[L,\infty)} \leq [u_{\text{in},k}]_{(-\infty,-L]} + [u_{\text{in},k}]_{[L,\infty)} + c\lambda \theta^{2} (1 + \frac{2}{m}) [u_{k}]_{(-\infty,-L]} + c\lambda \theta^{2} (1 + \frac{2}{m}) [u_{k}]_{[L,\infty)}.
\]

For sufficiently small \( \theta \) \( c\lambda \theta^{2} (1 + \frac{2}{m}) < 1/2 \). So

\[
[u_{k}]_{(-\infty,-L]} + [u_{k}]_{[L,\infty)} < 2\varepsilon.
\]

Therefore, the sum of the \( F \)-norm of the last two terms of (2.14) is not greater than

\[
c\lambda \theta^{2} (1 + \frac{2}{m}) \left( [u_{k}]_{(-\infty,-L]} + [u_{k}]_{[L,\infty)} \right).
\]

The obtained bounds imply that \( u_{\text{out},k} \) is a Cauchy sequence in the \( F \)-norm. Call the limit \( v \). By passage to the limit we obtain

\[
v(t) = u_{\text{in}}(t) - \lambda \int_{-\infty}^{t} R(t - \tau) \ast u^{3}(\tau) d\tau,
\]

whence \( v(t) = u_{\text{out}}(t) \), as required.

Theorems 2.3 and 2.4 are proved.

3 Riemann function estimates.

Proof of Lemmas 2.1 and 2.2

Proof of Lemma 2.1.

\( I(t,x) \) consists of an integral \( I_{S} \) over the surface of a light cone and an integral \( I_{C} \) over the interior of the cone. Since \( R(t,x) = -R(-t,x) \), the integrals over forward and backward light cones can be considered similarly. The integral over the surface of a light cone is the following

\[
I_{S} = \pm \frac{1}{4\pi} \int_{M_{1,\pm}} \int_{|x-y|=|t-\tau|} u(\tau,y) v(\tau,y) w(\tau,y) dS \frac{d\tau}{|t-\tau|}
\]

and over the interior

\[
I_{C} = \pm \int_{M_{1,\pm}} \int_{|x-y|>|t-\tau|} k(\mu) u(\tau,y) v(\tau,y) w(\tau,y) d^{3}y d\tau,
\]

12
where \( k(\mu) = c \mu^{-1} J_1(m \mu) \), \( J_1 \) is the Bessel function, \( \mu^2 = (t - \tau)^2 - |x - y|^2 \) and \( M_{1,\pm} = M_1 \cap \{ \tau | \pm (t - \tau) \geq 0 \} \). The measure \( dS \) is defined before the formulation of Lemma 2.1.

To the surface integral we apply Schwarz’ inequality:

\[
I_S^2 \leq \left( \int \int |w|^2 dS d\tau \right) \left( \int \int |u|^2 |v|^2 (t - \tau)^{-2} dS d\tau \right),
\]

where the integrals are taken over the range

\[ |x - y| = |t - \tau|, \quad \tau \in M_1 \]

and \( dS = \rho^2 d\omega, \rho \equiv |x - y|, |\omega| = 1 \). The first factor on the right side is bounded by

\[
\langle w \rangle_M^2 = \sup_K \int_{K \cap (M \times \mathbb{R}^3)} |w|^2 d\tilde{S}.
\]

As for the second factor, we first note that the integration by part gives

\[
\int_{|x - y| = |t - \tau|} \Phi(y) dS = \int_{|x - y| = |t - \tau|} \Phi(y)|x - y||t - \tau|^{-1} dS
\]

\[
= \int_{|x - y| < |t - \tau|} \frac{\partial}{\partial \rho} (\Phi \rho^3) |t - \tau|^{-1} d\rho d\omega
\]

\[
= \int_{|x - y| < |t - \tau|} (\rho \Phi_\rho + 3\Phi) |t - \tau|^{-1} d^3y. \tag{3.1}
\]

Applying this identity to \( \Phi = |v|^2 = vv^* \) and using \( \rho \leq |t - \tau| \), we obtain

\[
\int_{|x - y| = |t - \tau|} |v|^2 dS = \int_{|x - y| < |t - \tau|} (\rho vv_\rho^* + \rho v_\rho v^* + 3vv^*) |t - \tau|^{-1} d^3y
\]

\[
\leq \int_{|x - y| < |t - \tau|} \left[ 2 |vv_\rho| + 3 |v|^2 |t - \tau|^{-1} \right] d^3y
\]

\[
\leq 2 \|v(\tau)\|_{L^2,\rho} \|v_\rho(\tau)\|_2 + 3 \delta^{-1} \|v(\tau)\|_{2,\rho^*} \tag{3.2}
\]

Therefore,

\[
I_S^2 \leq c(\delta) \left( \int \int |u|^2 dS d\tau \right) \left( \int_{M_1} \|u(\tau)\|_{L^\infty}^2 \|v(\tau)\|_{L^\infty} \|v(\tau)\|_{2,\rho^*} (t - \tau)^{-2} d\tau \right).
\]

Here and in the following the notation \( \| \cdot \|_{p,\rho} \) means the \( L_p \)-norm of a complex-valued function over the sphere \( |x - y| < |t - \tau| \).

Now consider the integral \( I_C \) over the interior. The contribution of the forward and backward cones is estimated in the same way. Considering the light cone we use the notation

\[
\rho = |x - y|, \quad \mu^2 = (t - \tau)^2 - \rho^2, \quad y - x = \rho \omega.
\]
and introduce the light cone variables for the forward and backward light cone
\[ \xi = \pm(t - \tau) + \rho, \quad \eta = \pm(t - \tau) - \rho, \quad \tau \in M_{1,\pm}. \]

Thus \( \mu^2 = \xi \eta \). We also introduce a weight factor:
\[
l(\eta) = \begin{cases} \\
\eta^{3/2} & \text{for } \eta \geq \frac{1}{2} \delta, \\
\eta^{3/4} & \text{for } 0 < \eta < \frac{1}{2} \delta.
\end{cases}
\]

Estimating the contribution of the forward and backward cones by Schwarz' inequality we obtain for each such contribution \( |I_C|^2 \leq AB \), where
\[
A = \int \int l(\eta)^{-1} |w|^2 d\tau d^3 y, \\
B = \int_{M_{1,\pm}} \int l(\eta) k(\mu)^2 |u|^2 |v|^2 d\tau d^3 y.
\]

Changing variables, \((\rho, \theta) \to (\xi, \eta)\), we have
\[
A = \frac{1}{2} \int \int l(\eta)^{-1} \int_{|\omega|=1} |w|^2 \rho^2 d\omega d\xi d\eta \\
\leq \frac{1}{2} \int_0^\infty \left[ \int_{K(t, x, M_{1,\pm}, \eta)} \int_{|\omega|=1} |w|^2 \rho^2 d\omega d\xi \right] l(\eta)^{-1} d\eta.
\]

The integral in square brackets is precisely the integral of \( |w|^2 \) over a part \( K(t, x, M_{\pm}, \eta) \) of the surface of the forward or backward light cone \( K_{\pm}(t, x) \) with the top at \((t, x)\). This part of the surface of the cone \( K(t, x, M_{\pm, \eta}) \) is given by the condition \( \tau = \mp \frac{\xi + \eta}{2} + t \in M_{\pm}, \eta \in [0, +\infty) \). Since \( l(\eta)^{-1} \) is integrable, the expression for \( A \) is bounded by
\[
A \leq c(\delta) \sup_K \int_{K \cap (M \times \mathbb{R}^3)} |w|^2 d\tilde{S},
\]
where \( K \) runs over all forward or backward light cones and \( d\tilde{S} \) denotes the usual surface measure on the surface of \( K \). The factor \( c(\delta) \) appears due to the integral of \( l(\eta)^{-1} \), with depends on \( \delta \). In the second factor \( B \), we use the gross asymptotic behavior of \( k(\mu) = c \mu^{-1} J_1(m \mu) \)
\[
k(\mu)^2 \leq c \mu^{-3} \leq c(\eta |t - \tau|)^{-3/2},
\]
see [22, Section 8.45]. Therefore,
\[
B \leq c \int_{M_{1,\pm}} D(\tau) |t - \tau|^{-3/2} d\tau,
\]
where
\[
D(\tau) = \int l(\eta) \eta^{-3/2} |u|^2 |v|^2 d^3 y.
\]

We estimate \( D(\tau) \), dividing the domain for \( \eta \) in two parts: \( \eta > \frac{1}{2} \delta \) and \( \eta < \frac{1}{2} \delta \).
The part of $D(\tau)$ over $\eta > \frac{1}{2}$ is less than
$$\text{const} \|u(\tau)\|_{2*}^2 \|v(\tau)\|_{2*,*}^2.$$ 

For $\eta < \frac{1}{2}\delta$, we have \( I(\eta) \eta^{-3/2} = \eta^{-3/4} \). At each point we have the identity
$$\eta^{-3/4} |u|^2 |v|^2 = \text{div}_y \left( \frac{\mathbf{x} - \mathbf{y}}{\rho} \cdot 4\eta^{1/4} uu^* vv^* \right) + 4\eta^{1/4} (uu^*_\rho + uu^* |v|^2 + |u|^2 (v^*_\rho vv^*) + 8 \rho^{-1} \eta^{1/4} |u|^2 |v|^2).$$ 

Integration of this identity over the range $0 < \eta < \frac{1}{2}\delta$ (that is, over the spherical shell $|t - \tau| - \frac{1}{2}\delta < \rho < |t - \tau|$) gives
$$\int_{\eta < \frac{1}{2}\delta} \eta^{-3/4} |u|^2 |v|^2 d^3 y = 4 \int_{\eta = \frac{1}{2}\delta} \delta^{1/4} |u|^2 |v|^2 dS + 4 \int_{\eta < \frac{1}{2}\delta} (\eta^{1/4} (uu^*_\rho + uu^*) |v|^2 + \eta^{1/4} |u|^2 (v^*_\rho vv^*) + 2 \eta^{1/4} \rho^{-1} |u|^2 |v|^2) d^3 y,$$ 

the contribution of the point $\eta = 0$, that is, the contribution of the spherical shell $\rho = |t - \tau|$, is equal to zero. To estimate the surface integral in (3.3) we use the identity (3.1) and, analogously to (3.2), we obtain the estimate
$$\int_{\eta = \frac{1}{2}\delta} |u|^2 |v|^2 dS \leq 2 \|u(\tau)\|_{2*}^2 \|v(\tau)\|_c \|v(\tau)\|_{2*,*} + (2 + 6\delta^{-1}) \|u(\tau)\|_c \|v(\tau)\|_{2*,*} \|u(\tau)\|_\infty \|v(\tau)\|_\infty.$$

In the volume integral in (3.3) we use $\eta \leq \frac{1}{2}\delta$ and $\rho \geq \delta - \eta \geq \frac{1}{2}\delta$. Therefore, we have
$$D(\tau) \leq c(\delta) \left\{ \|u(\tau)\|_{2*} \|v(\tau)\|_c \|v(\tau)\|_{2*,*} + \|u(\tau)\|_\infty \|v(\tau)\|_\infty \|u(\tau)\|_c \|v(\tau)\|_{2*,*} \right\}.$$ 

Finally, we use the trivial estimate
$$\|v(\tau)\|_{2*,*} \leq c \|v(\tau)\|_\infty |t - \tau|^{3/2},$$
which is raised to an arbitrary power $\alpha$. This is used to estimate the terms with $\| \cdot \|_{2*,*}$ both in the bound for $D(\tau)$ and in the one for $I_5$. Taking into account these estimates, we obtain the estimate of Lemma 2.1.

Lemma 2.1 is proved.

Proof of Lemma 2.2.

Denote by $W(t, \mathbf{x})$ the same integral as $R u(t, \mathbf{x})$ except that $u^3$ is to be replaced by $uwv$ and we shall obtain the estimates for this term. These estimates yields the estimates of Lemma.

According to the definition of the $F$-norm, $\|W\|_F$ consists of three terms (2.4). To estimate the energy norm, we apply the energy relation
$$\|R(t - \tau) * f\|_c = \|f\|_2.$$
the function \( f =uvw \). We obtain
\[
\|W(t)\|_e \leq \int_M \|uvw\|_2 d\tau \leq m^{-1} \sup_{t \in M} \|w(t)\|_e \int_M \|u(\tau)\|_\infty \|v(\tau)\|_\infty d\tau
\]
\[
\leq m^{-1} \sup_{\tau \in M} \|w(\tau)\|_e \left( \int_M \|u(\tau)\|_\infty^2 d\tau \right)^{1/2} \left( \int_M \|v(\tau)\|_\infty^2 d\tau \right)^{1/2}.
\]
We shall obtain the required estimates for Lemma 2.2(a) by setting \( u = v = w \). Using the relation \( u^3 - v^3 = u^2(u - v) + uv(u - v) + v^2(u - v) \) and taking instead of \( u, v, w \), respectively, \( u, u, v \), or \( u, v, u - v \), or \( v, v, u - v \), we obtain the desired estimates of Lemma 2.2(b) for this term.

To estimate the rest of \( F \)-norm we write
\[
W = W_1 + W_2,
\]
where
\[
W_1 = \int_{M_1} \int R u v w d\tau d^3y, \quad W_2 = \int_{M_2} \int R u v w d\tau d^3y
\]
and
\[
M_1 = [a, b] \cap \{ \tau \mid |t - \tau| \geq 1 \}, \quad M_2 = [a, b] \cap \{ \tau \mid |t - \tau| < 1 \}.
\]

To \( W_1 \) we apply Lemma 2.1 with \( \delta = 1 \). Then
\[
|W_1(t, x)| \leq c \langle w \rangle_M \{ \int_{M_1} ... d\tau \}^{1/2}
\]
with the same integrand as in Lemma 2.1. Since \( \alpha < 1/6 \), \( |t - \tau|^{-3/2+3\alpha} \) is integrable. Therefore
\[
\|W_1(t)\|_\infty + \left( \int_{-\infty}^{+\infty} \|W_1(t)\|_\infty^2 dt \right)^{1/2}
\]
\[
\leq c \langle w \rangle_M \sup_{\tau \in M} \|v(\tau)\|_e^{1/2-\alpha} \left[ \|u\|_M \sup_{\tau \in M} \|v(\tau)\|_e + \|u\|_M \|v\|_e \right]^{1/2}.
\]
This implies Lemma 2.2(a) for \( W_1 \) when we set \( u = v = w \). Using the relation \( u^3 - v^3 = u^2(u - v) + uv(u - v) + v^2(u - v) \) and taking instead of \( u, v, w \), respectively, \( u, u - v, u \), or \( u, u - v, v \), \( v, v, u - v \) and setting \( \alpha = 0 \) we obtain the desired estimates of Lemma 2.2(b).

Finally, let us estimate \( W_2(t, x) \). We write it as \( I_S + I_C \), where \( I_S \) is the integral over the surface of the cone and \( I_C \) is the integral over the interior of the cone. For \( I_S \) as in the proof of Lemma 2.1, we use the integration by parts
\[
\int_{\rho = |t-\tau|} \Phi dS = \int_{0 \leq \rho \leq |t-\tau|} \frac{\partial}{\partial \rho} (\Phi \rho^2) d\rho d\omega = \int_{0 \leq \rho \leq |t-\tau|} (\Phi_{\rho} + 2 \frac{\Phi}{\rho}) d^3y
\]
and for \( \Phi = uvw \) we have
\[
I_S = \int_{M_2} \int_{\rho \leq |t-\tau|} (uvw_{\rho} + uv_{\rho}w + u_{\rho}vw + 2 \rho^{-1} uvw) d^3y d\tau \frac{d\tau}{|t-\tau|}. \quad (3.4)
\]
Applying Hölder’s inequality with exponents 3, 6, 2 or 3/2, 6, 6 to the inner integral in (3.4) and using the estimates
\[
\|u(\tau)\|_{3,*,|t-\tau|^{-1}} \leq c\|u(\tau)\|_\infty
\]
and
\[
\|\rho^{-1}u(\tau)\|_{\frac{3}{2},*,|t-\tau|^{-1}} \leq c\|u(\tau)\|_\infty
\]
for \(|t-\tau| \leq 1\), we obtain
\[
|I_S| \leq c \int_{M_2} (\|u(\tau)\|_\infty \|v(\tau)\|_{6,*} \|w_\rho(\tau)\|_{2,*} + \|u(\tau)\|_\infty \|v_\rho(\tau)\|_{2,*} \|w(\tau)\|_{6,*})
\]
\[
+ \|v(\tau)\|_\infty \|u_\rho(\tau)\|_{2,*} \|w(\tau)\|_{6,*} + \|u(\tau)\|_\infty \|v(\tau)\|_{6,*} \|w(\tau)\|_{6,*}) d\tau.
\]
Taking into account that \(J_1(m\mu)\mu^{-1} = O(\mu^{-3/2})\) (see [22, Section 8.45] ) we have for the integral over the interior of the cone
\[
|I_C| \leq c \int_{M_2} \|u(\tau)\|_\infty \|v(\tau)\|_{2,*} \|w(\tau)\|_{2,*} d\tau.
\]
As in the proof of Lemma 2.1 the asterisks indicate the integral in the norm over \(\rho \leq |t-\tau|\) only. We take into account that for \(|t-\tau| \leq 1\) \(\|u(\tau)\|_{p,*} \leq c\|u(\tau)\|_\infty\), and that the integration is taken over \(M_2 = [a,b] \cap \{0 < |t-\tau| < 1\}\) only. Then, we set \(u = v = w\) and obtain
\[
|W_2(t, \mathbf{x})| \leq c_1 \left( \int_{M_2} \|u(\tau)\|_{2,\infty}^2 d\tau \right) \sup_{\tau \in [0,b]} \|u(\tau)\|_e
\]
\[
\leq c_2 \left( \int_{M_2} \|u(\tau)\|_{2,\infty}^2 d\tau \right)^{1/2} \sup_{\tau \in M} \|u(\tau)\|_\infty \sup_{\tau \in M} \|u(\tau)\|_e.
\]
Making in the integral over \(t, \tau\) the change of variables on \(t - \tau, \tau\), we obtain
\[
\left( \int_{-\infty}^{\infty} \sup_{\mathbf{x}} |W_2(t, \mathbf{x})|^2 dt \right)^{1/2} \leq c \left( \int_{M} \|u(\tau)\|_{2,\infty}^2 d\tau \right)^{1/2} \sup_{\tau \in M} \|u(\tau)\|_\infty \sup_{\tau \in M} \|u(\tau)\|_e.
\]
\[
\leq c \|u\|_{F,M}^2 \|u\|_{F,M}.
\]
This yields the part (a) of Lemma.

On the other hand, for the part (b) we use Sobolev’s inequality, \(\|u\|_{6,*} \leq c\|u\|_e\) and the relation
\[
\int_{M_2} \|u(\tau)\|_\infty d\tau \leq \left( \int_{M_2} \|u(\tau)\|_{2,\infty}^2 d\tau \right)^{1/2} \left( \int_{M_2} d\tau \right)^{1/2}.
\]
Thus we get
\[
|W_2(t, \mathbf{x})| \leq c_1 \left( \int_{M_2} \|u(\tau)\|_{2,\infty} d\tau \right) \sup_{\tau \in M} \|v(\tau)\|_e \sup_{\tau \in M} \|w(\tau)\|_e
\]
\[
+ \left( \int_{M_2} \|v(\tau)\|_{2,\infty} d\tau \right) \sup_{\tau \in M} \|u(\tau)\|_e \sup_{\tau \in M} \|w(\tau)\|_e.
\]
\[ \leq c_2 \left( \left( \int_{M_2} \| u(\tau) \|_{\infty} d\tau \right)^{1/2} \sup_{\tau \in M} \| v(\tau) \|_e \sup_{\tau \in M} \| w(\tau) \|_e \right) \]
\[ + \left( \int_{M_2} \| v(\tau) \|_{\infty} d\tau \right)^{1/2} \sup_{\tau \in M} \| u(\tau) \|_e \sup_{\tau \in M} \| w(\tau) \|_e \right), \]
whence
\[ \int_{-\infty}^{+\infty} |W_2(t,x)|^2 dt \leq c \left( [u]_M \sup_{\tau \in M} \| v(\tau) \|_e \sup_{\tau \in M} \| w(\tau) \|_e \right) \]
\[ + [v]_M \sup_{\tau \in M} \| u(\tau) \|_e \sup_{\tau \in M} \| w(\tau) \|_e \right). \]

Again using the relation \( u^3 - v^3 = u^2(u - v) + uv(u - v) + v^2(u - v) \) and taking instead of \( u, v, w, \) respectively, \( u, u, u - v, \) or \( u, v, u - v, \) or \( v, v, u - v, \) we obtain the estimate of Lemma 2.2(b).

Lemma 2.2 is proved.

4 Construction of the quantum field
as a bilinear form

To construct the quantum field as a bilinear form we shall start from the quantum non-linear wave equation written in the form of integral equation (1.2).

We begin with a brief sketch and an outline of the construction of solution of Equation (1.2) and then we turn to the description of the technical details.

We shall construct the solution \( \phi(t,x) \) of Equation (1.2) as a bilinear form. This bilinear form is defined in the Fock space \( \mathcal{H}_{in} \) of the free field \( \phi_{in} \) and can be expanded in terms of creation and annihilation operators. By : : in (1.2) we denote the normal ordering with respect to the free field \( \phi_{in} \), and correspondingly, by product we mean the normal ordered product of the bilinear forms. However, an operator-valued structure of the interacting field is unknown in advance.

Since, in fact, we come from the notion of wave operator, so the natural initial quantum field should be the free quantum \( \textit{in} \)-field that enters into Equation (1.2).

Thus, we need to construct the bilinear form that corresponded to the interacting quantum field and is defined on the whole space-time, that is, to construct the unique solution in the large, that corresponds to the unique initial \( \textit{in} \)-field.

A representation of the solution in the form of a limit of some iterative series is a natural way of the construction of this solution. We construct the iterative series as series expanded in terms of Wick polynomials on the free \( \textit{in} \)-field. Therefore, to obtain the solution in the large it is sufficient to construct a bilinear form corresponding to the interacting field at any time. It can be continued in the large by translation with the Hamiltonian. Nevertheless, we would like to obtain the solution in the large defined as a bilinear form. It is convenient to approximate the solution by bilinear forms defined for all times.

It turns out that it is possible. This is connected with the fact that coherent vectors are the eigenvectors of annihilation operators. Moreover, the Wick polynomial of the
free field is a bilinear form, the free field is the sum of the creation and annihilation operators, and the creation operator is conjugate to the annihilation operator. Taking this into account, we obtain that the matrix elements between coherent vectors are equal to the corresponding polynomial depending on the sum of corresponding eigenvalue of one coherent vector and the complex conjugate of the eigenvalue of another coherent vector.

Therefore, if we consider iterations of the right side of Equation (1.2), that is, the expressions

$$\phi^{(l)}(t, x) = \{\phi_{in} + \lambda N_R\{\phi_{in} + \lambda N_R\{\ldots \{\phi_{in} + \lambda N_R(\phi_{in})\ldots\}\}\}(t, x)$$

(4.1)

where

$$N_R(\phi) = -\int_{-\infty}^{t} \int R(t - \tau, x - y) :\phi^3(\tau, y) : d\tau d^3y,$$

we approximate the quantum field by Wick polynomials, i.e. by bilinear forms. These bilinear forms $\phi^{(l)}(t, x)$ are defined on some sufficiently wide dense subspaces, in particular, on the subspace generated by linear combination of coherent vectors. Consider matrix elements of the constructing bilinear form on the vectors that are equal to a finite linear combination of coherent vectors near to the vacuum we reduce, in fact, these matrix elements to a bilinear combination of iterations. These iterations are the iterations of corresponding solution of classical wave equation with small complex in-data. This allows us to use the theorems proved in Sect. 2.

To prove the convergence of approximations we choose as convenient subspace the set of linear combinations of coherent vectors near to the vacuum (we denote it by $D_\theta$). This subspace is dense in the Fock space. We define explicitly the quantum field on this subspace and with the help of weak estimates of Sect. 2 and 3 we prove the convergence of the approximations (4.1). It is convenient to introduce in addition approximations with space-time and an ultraviolet cut-offs.

Bilinear forms generated by creation and annihilation operators was considered by Kristensen, Mejilbo and Poulsen [1]-[3]. Baez in [10] stated and proved the results that we need about Wick polynomials as bilinear forms. These results can be applied to the approximations that we consider. Note that we use slightly other notations as Baez [10].

Therefore, we construct the quantum field $\phi(t, x)$ as a bilinear form on $D_\theta \times D_\bar{\theta}$ and approximate it by bilinear form corresponding to the iterations (4.1) (with an ultraviolet and space-time cut-off). The limit of these iterations and cut-offs converges and gives the bilinear form, that is, the solution of (1.1) and (1.2).

Let pass to the detailed presentation. Introduce the notations that we need. Let $\mathcal{H}_{in}$ be the Fock Hilbert space of the free $in$-field. The field $\phi_{in}(t, x)$ in terms of the annihilation $a$ and the creation $a^+$ operator has the following form (we shall use the notation of [19, §7] and shall not write in the following the index “$in$” for the creation and annihilation operators):

$$\phi_{in}(t, x) = \frac{1}{(2\pi)^{3/2}} \int (e^{-i\mu(p)t + ipx}a(p) + e^{+i\mu(p)t - ipx}a^+(p)) \frac{d^3p}{\sqrt{2\mu(p)}},$$
where \( \mu(p) = (p^2 + m^2)^{1/2} \),

\[
[a(p), a^+(p')] = \delta(p - p')
\]

\[
[a(p), a(p')] = [a^+(p), a^+(p')] = 0.
\]

In \((t, x)\)-space it is convenient to introduce also the notation for positive- and negative parts

\[
\phi^+_m(t, x) = A^+(t, x) = \frac{1}{(2\pi)^{3/2}} \int e^{i\mu(p)t - ipx} a^+(p) \frac{d^3p}{\sqrt{2\mu(p)}},
\]

\[
\phi^-_m(t, x) = A(t, x) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mu(p)t + ipx} a(p) \frac{d^3p}{\sqrt{2\mu(p)}}.
\]

Let a pair of complex \( z_1, z_2 \in \mathcal{S}(\mathbb{R}^3) \) is given, denote by

\[
u_m(t, x, z_1, z_2) = (2\pi)^{-3/2} \int (e^{-i\mu(p)t + ipx} z_1^-(p) + e^{+i\mu(p)t - ipx} z_2^-(p)) \frac{d^3p}{\sqrt{2\mu(p)}}
\]

the complex solution corresponding to a pair \((z_1, z_2)\). This solution, or its initial data, defines uniquely a pair \((z_1, z_2)\), which corresponds to the the positive- and negative parts of \( \nu_m(t, x, z_1, z_2) \),

\[
z_1(\cdot) = 2^{-1/2}(2\pi)^{3/2}(-\Delta + m^2)^{1/4} u_m(0, -\cdot) + i2^{-1/2}(2\pi)^{3/2}(-\Delta + m^2)^{-1/4} \dot{u}_m(0, -\cdot)
\]

\[
z_2(\cdot) = 2^{-1/2}(2\pi)^{3/2}(-\Delta + m^2)^{1/4} u_m(0, \cdot) - i2^{-1/2}(2\pi)^{3/2}(-\Delta + m^2)^{-1/4} \dot{u}_m(0, \cdot).
\]

Let \( u(t, x, z_1, z_2) \) denotes the solution of (2.3) and \( u_{\text{out}}(t, x, z_1, z_2) \) denotes the \( \text{out-data} \) corresponding to the initial \( \text{in-data} \) \( u_m(t, x, z_1, z_2) \), or, that is equivalent, corresponding to the pair \((z_1, z_2)\).

To define the bilinear forms we introduce the convenient dense subspaces in the Fock Hilbert space \( \mathcal{H}_m \) of the free \( \text{in-field} \). Define first of all the coherent vectors. Let

\[
|z\rangle = \exp(za^+)\Omega, \quad \Omega = |0\rangle = \text{vacuum}.
\]

Here \( za^+ = \int \tilde{z}(k)a^+(k)d^3k, \quad z \in \mathcal{S} \), where \( \mathcal{S} \) is the Schwartz space of rapidly decreasing smooth complex-valued functions on \( \mathbb{R}^3 \), and a tilde denotes the Fourier transform. Equation (4.2) implies that the scalar product in \( \mathcal{H}_m \) of two coherent vectors is equal to

\[
(z_1|z_2) = \exp(\overline{\varphi}_1(z_1, z_2) = \exp(\int \overline{\varphi}_1(x)z_2(x)d^3x).
\]

Let \( D \) be the subspace of all finite linear combinations of coherent vectors

\[
D = \{ \chi \in \mathcal{H}_m \mid \chi = \sum \alpha_j|z_j\rangle, \quad z_j \in \mathcal{S}(\mathbb{R}^3) \}.
\]
We introduce also the subspace $D_\theta$, of linear combinations of coherent vectors near to the vacuum,

$$D_\theta = \{ \chi \in D \mid \chi = \sum \alpha_j |z_j \rangle, z_j \in \mathcal{S}(\mathbb{R}^3),$$

$$\|u_{in}(\cdot, z_j, 0)\|_F < \theta/4, \quad \|u_{in}(\cdot, 0, z_j)\|_F < \theta/4 \}.$$ 

It is clear that the subspace $D_\theta$ is dense in $\mathcal{H}_{in}$.

It is implied, for instance, by the fact that derivatives of coherent vectors are finite-particle vectors,

$$\frac{d^n}{d\alpha^n} |\alpha\rangle \bigg|_{\alpha=0} = (z a^+)^n \Omega,$$

and any vector $\mathcal{H}_{in}$ can be approximated by finite-particle vectors (from the Schwartz space). Derivatives themselves can be approximated by linear combination of coherent vectors near to vacuum. Moreover, for all positive integer $n$ $D_\theta \subset D(H_{in}^n)$, where $H_{in}$ is the free Hamiltonian of the $in$-field. In addition, for all positive $n$ $H_{in}^n$ is essentially self-adjoint on $D_\theta$. The inclusion $D_\theta \subset D(H_{in}^n)$ is the consequence of the simple bound

$$\|H_{in}^n |\zeta\rangle\|^2 = \sum_k \| \left( \sum_{i} \mu(p_i) \right) ^n \frac{\tilde{z}(p_1) \ldots \tilde{z}(p_k)}{k!^{1/2}} \|^2$$

$$\leq \sum_k \frac{1}{k!} \left( \sum_{n_1 \ldots n_k} \frac{n!}{n_1! \ldots n_k!} \left( \sup_{0 \leq j \leq n} \| \mu(p) \|^2 \right) \right)^2$$

$$\leq \sum_k \frac{k^{2n}}{k!} \sup_{0 \leq j \leq n} \| (-\Delta + m^2)^{j/2} \zeta \|^2 < \infty.$$ 

The self-adjointness of $H_{in}^n$ on $D_\theta$ follows from the Nelson’s theorem. A dense subspace of analytical vectors for $H_{in}^n$ is the space of linear combinations of coherent vectors near to the vacuum with test functions with compact support in the momentum space.

Thus, we formulate the main theorem.

Theorem 4.1.

Let

$$\chi_1 = \sum_{j=1}^{n_1} \alpha_j e^{(v_j a^+) \Omega}, \quad \chi_2 = \sum_{k=1}^{n_2} \beta_k e^{(w_k a^+) \Omega},$$

where complex-valued functions $v_j, w_k \in \mathcal{S}(\mathbb{R}^3)$ and for the constant $\theta$ from Theorems 2.3 and 2.4

$$\|u_{in}(\cdot, v_j, 0)\|_F < \frac{\theta}{4}, \quad \|u_{in}(\cdot, 0, v_j)\|_F < \frac{\theta}{4},$$

$$\|u_{in}(\cdot, w_k, 0)\|_F < \frac{\theta}{4}, \quad \|u_{in}(\cdot, 0, w_k)\|_F < \frac{\theta}{4},$$

for all $j, k$.

Then the following expressions give bilinear forms

$$\phi(t, x)(\chi_1, \chi_2) = \sum_{j,k} \bar{\alpha}_j \beta_k e^{(v_j, w_k)} u(t, x, v_j, w_k),$$

(4.5)
These bilinear form are symmetrical and are defined on $D \times D$. Moreover, these bilinear forms satisfies the following equalities

$$\phi(t, x) = \phi_{in}(t, x) - \lambda \int_{-\infty}^{t} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y,$$  \hspace{1cm} (4.8)

$$\phi_{out}(t, x) = \phi(t, x) - \lambda \int_{t}^{\infty} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y$$

$$= \phi_{in}(t, x) - \lambda \int_{-\infty}^{\infty} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y$$

$$= \phi_{in}(t, x) + \int_{-\infty}^{+\infty} R(t - \tau, x - y)(\Box + m^2)\phi(\tau, y)d\tau d^3y,$$ \hspace{1cm} (4.9)

and $\phi_{out}(t, x)$ satisfies the free equation

$$(\Box + m^2)\phi_{out}(t, x) = 0.$$  

In addition, on $D \times D$ the bilinear form $\phi(t + T, x)$ converges to the bilinear form $\phi_{out}(t, x)$ as $T \to +\infty$,

$$\phi(t + T, x)(\chi_1, \chi_2) \xrightarrow{\| \cdot \|_C} \phi_{out}(t, x)(\chi_1, \chi_2) \hspace{1cm} T \to +\infty.$$ \hspace{1cm} (4.10)

Proof of Theorem 4.1.

To prove that (4.5)-(4.7) define a bilinear form we use the approximations (4.1), spatial, time, and ultraviolet cut-offs. Then we obtain an approximation of bilinear form, that is given by the solutions corresponding to the smooth initial in-data with compact support, i.e. by solutions that are analogous to the solutions that appear in the proof of Theorem 2.3, 2.4 with initial in-data that belong to $\mathcal{F}^C$.

This approximation may be obtained in the following.

We change the integral over $(-\infty, t]$ on the integral over $[S, t]$. This change corresponds to the time cut-off. Let $\phi_{in, \sigma, \Lambda}(t, x) = \phi_{in, \sigma}(t, x)\Lambda(x)$, where $\phi_{in, \sigma}(t, x) = \int \phi_{in}(t, x - y)\sigma(y)d^3y$, here reals functions $\sigma, \Lambda \in \mathcal{S}(R^3)$ and $\Lambda(x)$ has a compact support.

The change of $\phi_{in}(t, x)$ on $\phi_{in, \sigma, \Lambda}(t, x)$ corresponds to an ultraviolet and volume cut-offs. These changes correspond to the approximation by bilinear forms given by solutions with smooth initial in-data with compact support. Finally, we approximate the field solution by bilinear forms $\phi_{S_{\sigma, \Lambda}}^{(l)}(t, x)$, where

$$\phi_{S_{\sigma, \Lambda}}^{(l)}(t, x) = \{ \phi_{in, \sigma, \Lambda} + \lambda N_{R,S}\{\ldots\{\phi_{in, \sigma, \Lambda} + \lambda N_{R,S}(\phi_{in, \sigma, \Lambda})\} \ldots\}\}(t, x)$$  \hspace{1cm} (4.11)

and

$$N_{R,S}(\phi) = -\int_{S}^{t} \int R(t - \tau, x - y) : \phi^3(\tau, y) : d\tau d^3y.$$
(4.11) contains Wick polynomial of $\phi_{m,\sigma,\Lambda}$. It is clear that these Wick polynomials are correctly defined bilinear forms, see, for instance, [4], [10, Theorem 3], [19, ch. , §7], [23].

Let us write how these bilinear forms generated by Wick polynomials act in the Fock space. These bilinear forms have the following Wick symbols. Let $\chi_1, \chi_2 \in D^\infty(H_{in})$, then for the bilinear form $\phi_{in}(t_1, x_1)\ldots\phi_{in}(t_n, x_n)$: we have

$$\phi_{in}(t_1, x_1)\ldots\phi_{in}(t_n, x_n) : \langle \chi_1, \chi_2 \rangle$$

$$= \sum_{I \subset \{1, \ldots, n\}} \left( \prod_{i \in I} A(t_i, x_i) \chi_1, \prod_{i \in \{1, \ldots, n\} \setminus I} A(t_i, x_i) \chi_2 \right), \quad (4.15)$$

(see [10, Theorem 3], [24, ch. X, §7]).

It is easy to see that

$$u_{in}(t, x, z_1 + z_2) = u_{in}(t, x, z_1, 0) + u_{in}(t, x, 0, z_2),$$

$$u_{in}(t, x, z_1, z_2) = u_{in}(t, x, z, x_1)$$

$$A(t, x)e^{(za^+)}\Omega = u_{in}(t, x, 0, z)e^{(za^+)}\Omega.$$ The last relation is implied by the following simple calculation

$$a(k)e^{(za^+)}\Omega = a(k)\sum_{n=0}^{\infty} \frac{(za^+)^n}{n!}\Omega = \sum_{n=0}^{\infty} \left[ a(k), \frac{(za^+)^n}{n!} \right]\Omega$$

$$= \sum_{n=1}^{\infty} z(k)\frac{(za^+)^{n-1}}{(n-1)!}\Omega = z(k)e^{(za^+)}\Omega,$$

see, for instance, [23, ch. 9.1].

Therefore, if $\chi_1 = \sum_j \alpha_j |v_j\rangle$, $\chi_2 = \sum_k \beta_k |w_k\rangle$, then (4.12) gives

$$\phi_{in}(t_1, x_1)\ldots\phi_{in}(t_n, x_n) : \langle \chi_1, \chi_2 \rangle$$

$$= \sum_{j,k} \alpha_j \beta_k \sum_{I \subset \{1, \ldots, n\}} \left( \prod_{i \in I} u_{in}(t_i, x_i, 0, v_j) |v_j\rangle, \prod_{i \in \{1, \ldots, n\} \setminus I} u_{in}(t_i, x_i, 0, w_j) |w_j\rangle \right)$$

$$= \sum_{j,k} \alpha_j \beta_k \langle v_j | w_k \rangle \prod_{i \in \{1, \ldots, n\}} \left( u_{in}(t_i, x_i, 0, v_j) + u_{in}(t_i, x_i, 0, w_k) \right)$$

$$= \sum_{j,k} \alpha_j \beta_k \langle v_j | w_k \rangle \prod_{i \in \{1, \ldots, n\}} u_{in}(t_i, x_i, v_j, w_k). \quad (4.14)$$

In particular, it follows from (4.14) that

$$\phi_{in,\sigma,\Lambda}(t_1, x_1)\ldots\phi_{in,\sigma,\Lambda}(t_n, x_n) : \langle \chi_1, \chi_2 \rangle$$

$$= \sum_{j,k} \alpha_j \beta_k \langle v_j | w_k \rangle \prod_{i \in \{1, \ldots, n\}} u_{in,\sigma,\Lambda}(t_i, x_i, v_j, w_k). \quad (4.15)$$
where $u_{in,\sigma \Lambda}(t, \mathbf{x})$ is a free solution with the Cauchy data at time zero given by

$$u_{in,\sigma \Lambda}(0, \mathbf{x}, \overline{\mathbf{v}}, w) = \Lambda(\mathbf{x}) \int u_{in}(0, \mathbf{x} - \mathbf{y}, \overline{\mathbf{v}}, w) \sigma(\mathbf{y}) \, d^3y,$$

$$\dot{u}_{in,\sigma \Lambda}(0, \mathbf{x}, \overline{\mathbf{v}}, w) = \Lambda(\mathbf{x}) \int \dot{u}_{in}(0, \mathbf{x} - \mathbf{y}, \overline{\mathbf{v}}, w) \sigma(\mathbf{y}) \, d^3y,$$

The relations (4.14), (4.15) imply that on $D \times D$ the bilinear forms-iterations (4.11) satisfy the equality

$$\phi^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x})(\chi_1, \chi_2) = \sum_{j,k} \overline{\alpha}_j \beta_k \langle v_j|w_k \rangle u^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k),$$

where $u^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ is the $l$th-iteration of Equation (2.1) - (2.2) with the Cauchy data at time $S$ equal to the Cauchy data of the free solution $u_{in,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$.

Now we show that for $l \to \infty$ the bilinear forms $\phi^{(l)}_{S,\sigma \Lambda}$ and $\phi^{(3)}_{S,\sigma \Lambda}$ converge on $D_\theta \times D_\theta$ to the bilinear forms $\phi_{S,\sigma \Lambda}(t, \mathbf{x})$ and $\phi^{3}_{S,\sigma \Lambda}(t, \mathbf{x})$: and on vectors of the form (4.3) these bilinear forms are equal to

$$\phi_{S,\sigma \Lambda}(t, \mathbf{x})(\chi_1, \chi_2) = \sum_{j,k} \overline{\alpha}_j \beta_k e^{(\overline{\mathbf{v}}_j, w_k)} u_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k),$$

(4.16)

$$\phi^{3}_{S,\sigma \Lambda}(t, \mathbf{x})(\chi_1, \chi_2) = \sum_{j,k} \overline{\alpha}_j \beta_k e^{(\overline{\mathbf{v}}_j, w_k)} u^{3}_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k),$$

(4.17)

where $u_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ is the solution of (2.1) - (2.2) with the Cauchy data at time $S$ equal to the Cauchy data of the free solution $u_{in,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$. In other words, these bilinear forms satisfy the following equation

$$\phi_{S,\sigma \Lambda}(t, \mathbf{x}) = \phi_{in,\sigma \Lambda}(t, \mathbf{x}) - \lambda \int_{S}^{t} \int R(t - \tau, \mathbf{x} - \mathbf{y}) : \phi^{3}_{S,\sigma \Lambda}(\tau, \mathbf{y}) : d\tau d^3y.$$

Really, first we note that $D_\theta \subset D \subset D^\infty(H_{in})$ and, thus, the approximations $\phi^{(l)}_{S,\sigma \Lambda}$ are correctly defined on $D_\theta \times D_\theta$. Choose, then, an ultraviolet and space cut-offs such, that for the states from $D_\theta \times D_\theta$ the $F$-norm of $u_{in,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ would be less than $\theta/2$. For this purpose, we shall take $\Lambda$ with compact support and $\sigma$ such that $\|\sigma\|_1 \leq 1$, $\|\Lambda\|_\infty \leq 1$. It is obvious that

$$\|u_{in,\sigma \Lambda}(\cdot, \cdot, \overline{\mathbf{v}}_j, w_k) - u_{in}(\cdot, \cdot, \overline{\mathbf{v}}_j, w_k)\|_F \to 0$$

(4.18)

for such $\Lambda$ and $\sigma$ and for $\Lambda \to 1$, $\sigma \to \delta$-function in $S'$ (our choice is $v_j, w_k \in S(\mathbb{R}^3)$).

Therefore, the inequalities (4.4) for $u_{in,\sigma \Lambda}(\cdot, \cdot, \overline{\mathbf{v}}_j, w_k)$ are fulfilled for the states from $D_\theta \times D_\theta$ and with the change $\theta$ on $2\theta$, and, thus, the conditions of Theorem (2.4) are fulfilled and the approximations $u^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ correspond to the initial data at time $S$ equal to $u_{in,\sigma \Lambda}(\cdot, \cdot, \overline{\mathbf{v}}_j, w_k)$. These initial data are smooth and have compact support. As in the proof of Theorem 2.4 we obtain, that the approximations $u^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ converge to $u_{S,\sigma \Lambda}(t, \mathbf{x}, \overline{\mathbf{v}}_j, w_k)$ for $l \to \infty$. This means that the bilinear forms $\phi^{(l)}_{S,\sigma \Lambda}(t, \mathbf{x})$ and $\phi^{3}_{S,\sigma \Lambda}(t, \mathbf{x})$ converge to the bilinear forms (4.16)-(4.17).
Now let $S \to -\infty$, we obtain, as in the proof of Theorem 2.4, that as $S \to -\infty$ $u_{\Lambda}(t, x, \bar{\nu}_j, w_k)$ converges to the solution $u_{\Lambda}(t, x, \bar{\nu}_j, w_k)$ with in-data $u_{in,\Lambda}(\cdot, \cdot, \bar{\nu}_j, w_k)$. On $D_\theta \times D_\theta$ this gives the convergence of the bilinear forms $\phi_{\Lambda}(t, x)$ to the bilinear forms $\phi_{\Lambda}(t, x)$.

On vectors (4.3) from $D_\theta$ these bilinear forms are equal to

$$\phi_{\Lambda}(t, x)(\chi_1, \chi_2) = \sum_{j,k} \bar{\alpha}_j \beta_k e^{(\pi_jw_k)} u_{\Lambda}(t, x, \bar{\nu}_j, w_k),$$

$$\phi_{\Lambda}(t, x): (\chi_1, \chi_2) = \sum_{j,k} \bar{\alpha}_j \beta_k e^{(\pi_jw_k)} u_{\Lambda}(t, x, \bar{\nu}_j, w_k),$$

where $u_{\Lambda}(t, x, \bar{\nu}_j, w_k)$ is the solution of (2.1) - (2.2) with in-data $u_{in,\Lambda}(t, x, \bar{\nu}_j, w_k)$, and satisfy the equation

$$\phi_{\Lambda}(t, x) = \phi_{in,\Lambda}(t, x) - \lambda \int_{-\infty}^{t} \int R(t - \tau, x - y) : \phi_{\Lambda}(\tau, y) : d\tau d^3 y.$$}

Finally let $\Lambda$ tend to 1 and $\sigma$ to $\delta$-function. Since (4.18) fulfills, so $u_{\Lambda}(\cdot, \cdot, \bar{\nu}_j, w_k) \to u(\cdot, \cdot, \bar{\nu}_j, w_k)$ in $F$-norm as $\Lambda \to 1$, $\sigma \to \delta$-function. But this means that $\phi_{\Lambda}(t, x)$ and $\phi_{\Lambda}(t, x)$: converge on $D_\theta \times D_\theta$ to $\phi(t, x)$ and $\phi(t, x)$:

Now we construct the bilinear form for $\phi_{\Lambda}(t, x)$. Equation (4.9) defines $\phi_{\Lambda}(t, x)$ as a bilinear form and (4.5)-(4.6) and Theorem 2.4 imply that the bilinear form is given by the equality (4.7) The same theorem 2.4 implies the convergence (4.10) of the bilinear form $\phi(t + T, x)$ to $\phi_{\Lambda}(t, x)$ as $T \to +\infty$.

The constructed bilinear forms (4.5)-(4.7) are sesquilinear. Really, taking into account (4.13), we have that

$$u(t, x, z_1, z_2) = \sum (u^{(l+1)}(t, x, z_1, z_2) - u^{(l)}(t, x, z_1, z_2) = u(t, x, z_2, z_1),$$

and so

$$u_{out}(t, x, z_1, z_2) = u_{out}(t, x, z_2, z_1).$$

Hence for $\chi_1 = \sum \alpha_j |v_j|$, $\chi_2 = \sum \beta_k |w_k|,$

$$\phi(t, x)(\chi_1, \chi_2) = \phi(t, x)(\chi_2, \chi_1),$$

$$\phi_{\Lambda}(t, x): (\chi_1, \chi_2) = \phi_{\Lambda}(t, x): (\chi_2, \chi_1),$$

$$\phi_{\Lambda}(t, x)(\chi_1, \chi_2) = \phi_{\Lambda}(t, x)(\chi_2, \chi_1),$$

i.e. bilinear forms (4.5)-(4.7) are sesquilinear.

Theorem 4.1 is proved.

The constructed bilinear form as a solution of the quantum equation satisfies the uniqueness condition of the following type. Let

$$\phi_1 = \phi_{in} + N_R(\phi_1),$$

25
\[ \phi_2 = \phi_{in} + N_R(\phi_2). \]

Let \( \phi_1 \) and \( \phi_2 \) be bilinear forms defined on \( D_\theta \times D_\theta \) and such, that for \( \chi_1, \chi_2 \in D_\theta \), \( \phi_1(t, x)(\chi_1, \chi_2) \in F^C \) and \( \phi_2(t, x)(\chi_1, \chi_2) \in F^C \). Let \( \phi_1^\phi(t, x) : \) and \( \phi_2^\phi(t, x) : \) be the bilinear forms and let the relation

\[ \phi^3(t, x) : (|v\rangle, |w\rangle) = e^{-2(\theta, w)} \left( \phi(t, x)(|v\rangle, |w\rangle) \right)^3 \]  

be fulfilled for any pair of coherent vectors \(|v\rangle, |w\rangle\) (The definition of normal ordering!). Then on \( D_\theta \times D_\theta \), \( \phi_1(t, x) \) coincides with \( \phi_2(t, x) \) and with our form \( \phi(t, x) \).

Here we do not consider the proof of the uniqueness and this definition of a normal ordering. We use the relation of the form (4.19) for a Wick polynomial only.

We formulate yet two useful assertions.

Theorem 4.2.

The bilinear forms \( \phi(t, x) \), \( \phi_{out}(t, x) \) transform as scalar under the Poincaré transformation generated by the in-field

\[ U_{in}(a, \Lambda)\phi(t, x)U_{in}(a, \Lambda)^{-1} = \phi((a, \Lambda)(t, x)), \]

\[ U_{in}(a, \Lambda)\phi_{out}(t, x)U_{in}(a, \Lambda)^{-1} = \phi_{out}((a, \Lambda)(t, x)) \]  

Remarks.

1. The Poincaré transformation of the interpolating field and of the in- and out-fields generates the same representation of the Poincaré group, \( U(a, \Lambda) = U_{in}(a, \Lambda) = U_{out}(a, \Lambda) \).

2. The expressions (4.20)-(4.21) are defined as bilinear forms for all \( a \in \mathbb{R}^4 \) and \( \Lambda \) such that \( U(0, \Lambda)D_\theta \in D_2 \), i.e. for \( \Lambda \) sufficiently near to 1. This is connected with the fact that our bilinear form is defined only on \( D_\theta \times D_\theta \) for sufficiently small \( \theta \), and the norm used for the space of initial in-data is Lorentz-noninvariant.

3. It is possible to obtain analyticity of classical solutions for small complex initial data for the space of the initial data \( F^C \). To derive this analyticity one can use the estimates of Lemmas 2.1 and 2.2 and the method analogous to [17] or given by Baez and Zhou [18] for the case of initial data, corresponding to the finite energy. This analyticity allows to extend by continuity the equalities (4.20) and (4.21) on the Poincaré-invariant subspace.

Expressions (4.5), (4.7) of Theorem 4.1 for the bilinear form \( \phi \) and \( \phi_{out} \) imply obviously that the coupling constant \( \lambda \) can be reconstructed uniquely by matrix elements of the interpolating field \( \phi \) or the out-field \( \phi_{out} \). Moreover, the following assertion is valid.

Theorem 4.3.

The coupling constant \( \lambda \) is determined uniquely by matrix elements (4.5) of the interpolating field

\[ \lambda = \lim_{\varepsilon \to 0} \varepsilon^{-4} \lim_{T \to \infty} \left( \int (\phi(t, x)(|v\rangle, |v\rangle))^4 dt d^3x \right)^{-1} \langle v|v\rangle^4 \]

\[ \int (\phi(t + T, x)(|\varepsilon v\rangle, |\varepsilon v\rangle) \phi(t + T, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle) \]
\[-\phi(t + T, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle)\phi(t + T, x)(|\varepsilon v\rangle, |\varepsilon v\rangle) d^3x \tag{4.22}\]

\[
= \lim_{\varepsilon \to 0} \left( \int (\phi(t, x)(|v\rangle, |v\rangle)) dtd^3x \right)^{-1} \langle v|v\rangle^4 \\
- \int (\phi_{\text{out}}(t, x)(|\varepsilon v\rangle, |\varepsilon v\rangle) \phi_{\text{out}}(t, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle) \\
- \phi_{\text{out}}(t, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle) \phi_{\text{out}}(t, x)(|\varepsilon v\rangle, |\varepsilon v\rangle)) d^3x \tag{4.23}\]

for any \(|v\rangle \in D_\theta, v \in S(\mathbb{R}^3, C), v \neq 0.

Proof.

For \(v \in S(\mathbb{R}^3, C)\) the initial in-data \(u_{\text{in}}(t, x, \bar{v}, v)\) are real and belong to \(\mathcal{F}\), and thus, \(u(t, x, \bar{v}, v)\) belongs to \(\mathcal{F}\) also, \(u(t, x, \bar{v}, v) \neq 0\).

Since

\[
0 < \int u(t, x, \bar{v}, v)^4 dtd^3x \\
\leq \left( \sup_{x} \int \sup_{t} u(t, x, \bar{v}, v)^2 dt \right) \sup_{t} \int u(t, x, \bar{v}, v)^2 d^3x \\
\leq \|u\|^4_F,
\]

so \(\int u(t, x, \bar{v}, v)^4 dtd^3x\)^{-1} is correctly defined and, by Theorem 4.1, is equal to

\[
(\int (\phi(t, x)(|v\rangle, |v\rangle)) dtd^3x)^{-1} \langle v|v\rangle^4.
\]

The same theorem implies the existence of the limit as \(T \to +\infty\) and the right side \((4.22)=(4.23)\). Taking into account that \(\langle \varepsilon v|\varepsilon v\rangle \to \langle 0|0 \rangle = 1\) for \(\varepsilon \to 0\), the expressions \((4.22)\) and \((4.23)\) are equal to \(\lambda\). This is the consequence of the equality

\[
\lambda = \lim_{\varepsilon \to 0} \varepsilon^{-4} \left( \int u(t, x, \bar{v}, v)^4 dtd^3x \right)^{-1} \\
\left( \int (\phi_{\text{out}}(t, x, \varepsilon \bar{v}, \varepsilon v) \hat{u}_{\text{out}}(t, x, 2\varepsilon \bar{v}, 2\varepsilon v) \\
- \hat{u}_{\text{out}}(t, x, 2\varepsilon \bar{v}, 2\varepsilon v) \phi_{\text{out}}(t, x, \varepsilon \bar{v}, \varepsilon v)) d^3x, \right),
\]

which is proved in [F]. Theorem 4.3 is proved.

Remarks.

1. It follows that the coupling constant is uniquely defined by matrix elements of the \(\text{out}\)-field only.

2. Expression \((4.23)\) can be rewritten also in the form

\[
\lambda = \lim_{\varepsilon \to 0} \varepsilon^{-4} \left( \int :\phi^4: (t, x)(|v\rangle, |v\rangle) dtd^3x \right)^{-1} \langle v|v\rangle \\
\left( \int (\phi_{\text{out}}(t, x)(|\varepsilon v\rangle, |\varepsilon v\rangle) \phi_{\text{out}}(t, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle) \\
- \phi_{\text{out}}(t, x)(|2\varepsilon v\rangle, |2\varepsilon v\rangle) \phi_{\text{out}}(t, x)(|\varepsilon v\rangle, |\varepsilon v\rangle)) d^3x, \right.
\]

27
where \( \phi^A(t, x) \) is the bilinear form with matrix elements (4.27).

The proved assertion about bilinear forms implies easily that on \( D_\theta \times D_\theta \) the bilinear forms

\[
\phi_{S,\tau,\Lambda}^{(l)}(t, x), \: \phi_{S,\tau,\Lambda}^{(l)}(t, x), \: (\phi_{S,\tau,\Lambda}^{(l)}(t, x))^2, \: \phi_{S,\tau,\Lambda}^{(l)}(t, x), \: \phi_{S,\tau,\Lambda}^{(l)}(t, x), \: \phi_{S,\tau,\Lambda}^{(l)}(t, x), \: \phi_{S,\tau,\Lambda}^{(l)}(t, x),
\]

\[
H_{S,\tau,\Lambda}^{(l)} = \frac{1}{2} \int (\phi_{S,\tau,\Lambda}^{(l)}(t, x) + (\nabla \phi_{S,\tau,\Lambda}^{(l)}(t, x))^2 + m^2 \phi_{S,\tau,\Lambda}^{(l)}(t, x) + \frac{\lambda}{2} \phi_{S,\tau,\Lambda}^{(l)}(t, x)) d^3 x.
\]

are correctly defined. Clearly that as \( l \to \infty, S \to -\infty, \Lambda \to 1, \text{and} \sigma \to \delta\)-function these bilinear form on \( D_\theta \times D_\theta \) converge to the bilinear forms

\[
\phi^2(t, x), \: \phi^2(t, x), \: (\nabla \phi)^2(t, x), \: \phi^4(t, x), \: \phi^4(t, x), \: \phi^4(t, x), \: H = \frac{1}{2} \int (\phi^2(t, x) + (\nabla \phi)^2(t, x) + m^2 \phi^2(t, x) + \frac{\lambda}{2} \phi^4(t, x)) d^3 x.
\]

On \( D_\theta \times D_\theta \) these bilinear forms satisfy the relations

\[
\phi^2(t, x) : (\chi_1, \chi_2) = \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle \dot{u}(t, x, \tau_j, w_k)^2,
\]

\[(\nabla \phi)^2(t, x) : (\chi_1, \chi_2) = \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle (\nabla u(t, x, \tau_j, w_k))^2,
\]

\[
\phi^4(t, x) : (\chi_1, \chi_2) = \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle u(t, x, \tau_j, w_k)^4,
\]

\[
H(\chi_1, \chi_2) = \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle
\]

\[
= \frac{1}{2} \int \left( \dot{u}(t, x, \tau_j, w_k)^2 + (\nabla u(t, x, \tau_j, w_k))^2 + m^2 u(t, x, \tau_j, w_k)^2 + \frac{\lambda}{2} u(t, x, \tau_j, w_k)^4 \right) d^3 x.
\]

Moreover, on \( D_\theta \times D_\theta \) the bilinear form \( H \) is equal to the bilinear form \( H_{in} \),

\[
H(\chi_1, \chi_2) = H_{in}(\chi_1, \chi_2) = \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle
\]

\[
= \frac{1}{2} \int \left( \dot{u}_{in}(t, x, \tau_j, w_k)^2 + (\nabla u_{in}(t, x, \tau_j, w_k))^2 + m^2 u_{in}(t, x, \tau_j, w_k)^2 \right) d^3 x
\]

\[
= \sum \alpha_j \beta_k \langle v_j \vert w_k \rangle \int \mu(k) \overline{v_j(k)} w_k(k) d^3 k.
\]

Therefore, on \( D_\theta \times D_\theta \) the bilinear form \( H(\chi_1, \chi_2) \) is positively definite and \( H_{in} \) is a unique positive self-adjoint operator, which bilinear form on \( D_\theta \times D_\theta \) coincides with the bilinear form of \( H \).

We remark, that the expressions analogous to (4.24)-(4.28) can be written for the momentum and angular momentum operators.
We mention first the review of Callaway [24]. This review contains, in particular, arguments of Fröhlich [25] and Aizenman et al. [26, 27] concerning the triviality of $\phi^4_4$. Fröhlich [25] and Aizenman et al. [26, 27] assert that any construction of $\phi^4_4$, obtained as a limit of ferromagnetic lattice approximation, is trivial. Recently Pedersen, Segal, Zhou [28] gave arguments for nontriviality of (massless) $\phi^4_4$, and, more generally, for nontriviality of $\phi^q_d$ [28], see also [29, 30]. The usual perturbation theory claims that $\phi^4_4$ is a nontrivial and a renormalizable theory.

We interpret the results of Fröhlich [25] and Aizenman et al. [26, 27] as an approximation of a measure. This measure corresponds to the approximation, but this approximation do not catch the nonlinearity (and singularity) of the interaction. Its convergence corresponds to the convergence on the subspace of zero measure (for the true measure), see the analogous interpretation for more singular case [31, 32, 33].

Our approach obtains undoubtedly a nontrivial theory, in particular, the coupling constant is uniquely determined by the matrix elements of the interpolating or the out-field (see Theorem 4.3), but the straightforward proof is up to now unknown and absent. It would be very interesting to understand to what structure corresponds our construction: to the whole quantum field or only to the non-linear “tree” approximation.

This construction is connected with the idea to construct the vacuum with the help of a generalized density and/or its logarithmical derivative, see [34]–[41]. In our case the generalized density is equal to $\rho_{in}(wu)$, i.e. to the vacuum in terms of the interacting field, it is defined on the (whole) space $\mathcal{F}$. Here $\rho_{in}(\cdot)$ is the generalized density of the free vacuum and $w$ is a (quasi) canonical transformation. $\rho_{in}(wu)$ has to consider on finite-dimensional subspaces and then has to be extended on $\mathcal{S}'$ (as a measure generated by this generalized density). To $\rho_{in}(\cdot)$ corresponds a unique state. This state can be obtained by extension from finite-dimensional subspaces of entire holomorphic functions [11, 34, 35]. In our case the variable $\varphi(x) - i((-\Delta_3 + m^2)^{-1/2}\pi)(x)$ corresponds to the complex variable. Some exceptional properties of the state in terms of complex variables are described in [11]. They correspond to a holomorphic representation of Weyl group. In this case the Weyl group is a nuclear infinite-dimensional Lie group.

The further progress will be connected with the possibility to extend a domain of definition of the bilinear form and/or with possibility to obtain the bounds connected with these bilinear forms in some suitable rigging of Fock Hilbert space of the $in$-field. It is very important that the Hamiltonian is correctly defined as an operator, essentially self-adjoint on $D_\theta$, and in the same time it can be expressed as a bilinear form connected with $\phi$. We emphasize that we need such bounds in terms of bilinear form corresponding to the field $\phi$.

Acknowledgments

The author is grateful to Ludwig Faddeev for the interesting question.
References

[1] Kristensen P., Mejlbo L., Thue Poulsen E.: Tempered distributions in infinitely many dimensions. I. Canonical field operators. Commun. Math. Phys. 1 (1965) 175-214.

[2] Kristensen P., Mejlbo L., Thue Poulsen E.: Tempered distributions in infinitely many dimensions. II. Displacement operators. Math. Scand. 14 (1964) 129-150.

[3] Kristensen P., Mejlbo L., Thue Poulsen E.: Tempered distributions in infinitely many dimensions. III. Linear transformations of field operators. Commun. Math. Phys. 6 (1967) 29-48.

[4] Baez J., Segal I., Zhou Z.: Introduction to algebraic and constructive quantum field theory. Princeton, Princeton University Press, 1992.

[5] Segal I.: Nonlinear functions of weak processes, I. J. Funct. Anal. 4 (1969) 404-456.

[6] Segal I.: Nonlinear functions of weak process, II. J. Funct. Anal. 6 (1970) 29-75.

[7] Segal I.: Local noncommutative analysis, in “Problems in Analysis” (R.Gunning, Ed.), pp. 111-130, Princeton Univ. Press, Princeton, NJ, 1970.

[8] Klein A.: Renormalized products of the generalized free field and its derivatives. Pacific J. Math. 45 (1973) 275-292.

[9] Pesenti D.: Produit de Wick des formes sesquilineaires. In: "Séminaire de Théorie du Potentiel", Paris 1972-1974 (M.Brent, G.Choquet, and J.Deny, Eds.), pp. 120-143, Lecture Notes in Mathematics, Vol. 518, Springer-Verlag, New York.

[10] Baez J.C.: Wick products of the free Bose field. J. Funct. Anal. 86 (1989) 211-225.

[11] Paneitz S.M., Pedersen J., Segal E., Zhou Z.: Singular operators on boson fields as forms on spaces of entire functions on Hilbert space. J. Funct. Anal. 100 (1991) 36-58.

[12] Heifets E.P.: The $\phi_4^4$ classical wave equation and the construction of the quantum field as a bilinear form in the Fock space. (In Russian). Novosibirsk, Institute of Mathematics, 1974.

[13] Rączka R.: The construction of solution of nonlinear relativistic wave equation in $\lambda:\phi^4$: theory. Jour. Math. Phys. 16 (1975) 173-176.

[14] Morawetz C.S., Strauss W.A.: Decay and scattering of solutions of a nonlinear relativistic wave equation. Comm. Pure Appl. Math. 25 (1972) 1-31.

[15] Morawetz C.S., Strauss W.A.: On a nonlinear scattering operator. Comm. Pure Appl. Math. 26 (1972) 47-54.
[16] Strauss W.A.: Non-linear scattering theory. Proceedings of Conference on Scattering Theory in Mathematical Physics (Eds. J.A.La Vita and J.-P. Marchand). Denver 1973, p. 53.

[17] Rączka R., Strauss W.: Analyticity properties of the scattering operator in non-linear relativistic classical and prequantized field theories. Rept. Math. Phys. 16 No.3 (1979) 317-327.

[18] Baez J., Zhou Z.-F.: Analyticity of scattering for the $\phi^4$ theory. Commun. Math. Phys. 124 (1989) 9-21.

[19] Reed M., Simon B.: Methods of modern mathematical physics III. Scattering theory Academic Press, New York, San Francisco, London, 1979.

[20] Bogoliubov N.N., Shirkov D.V.: Introduction to quantum field theory. Moscow, Nauka, 1973.

[21] Strauss W.A.: Decay and asymptotics for $\Box u = F(u)$. J. Funct. Anal. 2 (1968) 409-457.

[22] Gradstein I.S., Rizhik I.M.: Tables of integrals, sums, series, and products. (in Russian). Moscow, Fizmatgiz, 1963.

[23] Glimm J., Jaffe A.: Quantum physics. A functional integral point of view. New York, Heidelberg, Berlin, Springer-Verlag, 1981.

[24] Callaway D.: Triviality pursuit: can elementary scalar particles exist? Phys. Reports 167 No 5 (1988) 241-320.

[25] Fröhlich J.: On the triviality of $\lambda \phi^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. Nucl. Phys. B200 [FS4] (1982) 281-296.

[26] Aizenman M.: Geometric analysis of $\phi^4$ fields and Ising models (Part I & II). Commun. Math. Phys. 86 (1982) 1-48.

[27] Aizenman M., Graham R.: On the renormalized coupling constant and the susceptibility in $\phi^4$ field theory and the Ising model in four dimensions. Nucl. Phys. B225 [FS9] (1983) 261-288.

[28] Pedersen J., Segal E., Zhou Z.: Massless $\phi^4$ quantum field theories and the nontriviality of $\phi^4$. Nucl. Phys. B376 (1992) 129-142.

[29] Baumann K.: On relativistic quantum fields fulfilling CCR. J. Math. Phys. 28 (1987) 697-704.

[30] Baumann K.: On canonical irreducible quantum theories describing boson and fermion. J. Math. Phys. 29 (1988) 1225-1230.
[31] Osipov, E.P.: On triviality of the $\exp \lambda \varphi^4$ quantum field theory in a finite volume. Th-103, Novosibirsk, Institute for Mathematics, 1979 & Rept. Math. Phys. 20 (1984) 111-116.

[32] Albeverio S., Gallavotti G., Høegh Krog R.: The exponential interaction in $R^n$. Phys. Lett. 83B (1979) 177-178.

[33] Albeverio S., Gallavotti G., Høegh Krog R.: Some results for the exponential interaction in two or more dimensions. Commun. Math. Phys. 70 (1979) 187-192.

[34] Kirillov A. I. On two mathematical problems of canonical quantization. I. (In Russian). Theor. Math. Phys. 87 (1991) 22-33.

[35] Kirillov A. I. On two mathematical problems of canonical quantization. II. (In Russian). Theor. Math. Phys. 87 (1991) 163-172.

[36] Kirillov A. I. On two mathematical problems of canonical quantization. III. Stochastic mechanics of vacuum. (In Russian). Theor. Math. Phys. 91 (1992) 377-395.

[37] Kirillov A. I. On two mathematical problems of canonical quantization. IV. Theor. Math. Phys. 93 (1992) 249-263.

[38] Albeverio S., Röckner M.: Classical Dirichlet forms on topological vector spaces - the construction of the associated diffusion process. Prob. Th. Rel. Fields. 83 (1989) 405-434.

[39] Albeverio S., Röckner M.: Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula. J. Funct. Anal. 88 (1990) 395-436.

[40] Albeverio S., Kusuoka S., Röckner M.: On partial integration in infinite-dimensional space and applications to Dirichlet forms. J. London Math. Society 42 (1990) 122-136.

[41] Albeverio S., Röckner M.: Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Prob. Th. Rel. Fields 89 (1991) 347-386.