FAMILIES OF K3 SURFACES OVER CURVES SATISFYING
THE EQUALITY OF ARAKELOV-YAU’S TYPE AND
MODULARITY

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Let $C$ denote a smooth projective curve of genus $g$ over $\mathbb{C}$, and $S' \subset C$ a finite set of points, and $f : X^0 \rightarrow C \setminus S'$ a smooth family of algebraic K3 surfaces, which extends to a family $f : X \rightarrow C$ with semi-stable singular fibres over $S'$. Let $S \subset S'$ denote the subset where the local monodromies of $R^2f_*\mathcal{O}_{X^0}$ have infinite orders. Let $\omega_{X/C}$ denote the dualizing sheaf. It is known that $f_*\omega_{X/C}$ is ample on $C$ by Fujita if $f$ is not isotrivial [6] (See [9] [29] when the base of higher dimension). Jost and Zuo showed the following inequality [8]:

\[(0.0.1) \quad \deg f_*\omega_{X/C} \leq \deg \Omega^1_C(\log S).\]

If the iterated Kodaira-Spencer map of this family is zero, one shows then a stronger inequality

\[(0.0.2) \quad \deg f_*\omega_{X/C} \leq \frac{1}{2} \deg \Omega^1_C(\log S).\]

These inequalities generalize the original Arakelov inequality

\[\deg f_*\omega_{X/C} \leq \frac{g}{2} \deg \Omega^1_C(\log S')\]

for a family of semi-stable curves of genus $g$. The strict Arakelov inequality was proved in [28] when $g \geq 2$ and $S'$ non-empty. If $S'$ is empty, Miyaoka-Yau inequality implies a stronger inequality

\[\deg f_*\omega_{X/C} \leq \frac{g-1}{6} \deg \Omega^1_C(\log S').\]

Thus the original Arakelov inequality is always strict for $g \geq 2$. But there are families of Jacobians reaching the equality. In general, Yau [31] proved the so-called Yau’s Schwarz type inequality, which can be formulated as follows. Let $(M, ds)$ be a Hermitian manifold with holomorphic sectional curvature bounded above by a negative constant $K$, and let $(C \setminus S, ds_\mu)$ be a Poincaré type metric. Then there exists a positive constant $c$, such that for any holomorphic map $\phi : C \setminus S \rightarrow M$, one has $\phi^*ds \leq cd s_\mu$. It is the reason why we call the inequalities (0.0.1) and (0.0.2) are of Arakelov-Yau’s type. One can choose $c = 1$ in (0.0.1) and $c = 1/2$ in (0.0.2). We will show in this note that both of them are optimal, and have a modularity meaning.

For a family $f : E \rightarrow \mathbb{P}^1$ of non-constant semi-stable elliptic curves, Beauville proved that $f$ has at least 4 singular fibres, which is equivalent to Arakelov

\[\frac{1}{2} \deg \Omega^1_C(\log S').\]

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inequality in this case. He obtained a complete classification for the families of semistable elliptic curves when Arakelov inequality becomes equality. There are exactly 6 such family and \( \mathbb{P}^1 \setminus S' \) is a modular curve \([4]\). In this note we shall study non-isotrivial algebraic families of semi-stable K3 surfaces over curves when the inequality \((0.0.1)\), or \((0.0.2)\) becomes an equality. The corresponding question has been considered in \([30]\) for families of abelian varieties. The final presentation of this note has been influenced by \([30]\). It has also been motivated by Mok’s work on rigidity theorems of locally Hermitian symmetric spaces \([13]\) and \([14]\), where he use the Gaussian curvature of the induced metric on a holomorphic curves in a locally Hermitian symmetric space to characterize when this curve will be a totally geodesic embedding.

To state the main result, we recall some notation. Let \( A^0 \rightarrow C^0 \) be a family of abelian surfaces with a section, then the desingularization \( Z^0 \rightarrow C^0 \) of the quotient \( A^0 / \{ \pm 1 \} \rightarrow C^0 \) is a family of Kummer surfaces (the so called Kummer construction). The rational map \( A^0 \rightarrow Z^0 \) is called a rational quotient of \( A^0 \). The family \( a : A^0 \rightarrow C^0 \) is called the associated family of abelian surfaces of \( Z^0 \rightarrow C^0 \). In general, it is not true that every family of Kummer surfaces has an associated family of abelian surfaces. An involution \( \iota \) on a K3 surface \( X \) is called a Nikulin involution if \( \iota^* \omega = \omega \) for every \( \omega \in H^0(X, \Omega_X^2) \). It is known (Nikulin \([14]\)) that every Nikulin involution \( \iota \) has eight isolated fixed points, and the rational quotient \( X \rightarrow Z \) by \( \iota \) is a K3 surface.

**Theorem 0.1.** Let \( f : X \rightarrow C \) be a family of semi-stable K3 surfaces over \( C \), and suppose that \( R^2 f_* (\mathbb{Z}_{X^0}) \) has infinite local monodromies at a non-empty set \( S \subset C \). If this family reaches the Arakelov bound in \((0.0.1)\). Then we have

a) The general fibres of \( f : X \rightarrow C \) have Picard number at least 19.

b) After passing through a finite étale cover \( \sigma : C' \rightarrow C \), there exist an open set \( C^0 \subset C' \) and a global Nikulin involution \( \iota \) on \( f : X^0 = f^{-1}(C^0) \rightarrow C^0 \) such that the rational quotient \( X^0 \rightarrow Z^0 \) by \( \iota \) is a family of Kummer surfaces over \( C^0 \), which has an associated family of abelian surfaces that is isogenous to the square product of a family of elliptic curves \( g : E \rightarrow C' \). The projective monodromy representation of the local system \( R^1 g_* (\mathbb{Z}_{E^0}) \) extends to

\[
\tau : \pi_1 (C' \setminus \sigma^{-1} S, *) \rightarrow \text{PSL}_2(\mathbb{Z})
\]

such that

\[
C' \setminus \sigma^{-1} S \cong \mathcal{H} / \tau \pi_1 (C' \setminus \tau^{-1} S, *).
\]

A family of K3 surfaces satisfying Property b) will be called coming from Nikulin-Kummer construction of the square product of a family of elliptic curves.

**Theorem 0.2.** If this family has the zero iterated Kodaira-Spencer map, and reaches the Arakelov bound in \((0.0.2)\). Then the general fibres of \( f : X \rightarrow C \) have the Picard number at least 18, after passing through a finite étale cover \( \sigma : C' \rightarrow C \), the monodromy representation \( \rho \) of \( R^2 f_* (\mathbb{Z}_{X^0}) \) is of the form

\[
\rho = \text{rank-2 trivial representation} \otimes (\tau : \pi_1 (C' \setminus \sigma^{-1} S, *) \rightarrow \text{SL}_2(\mathbb{Z})),
\]

and

\[
C' \setminus \sigma^{-1} S \cong \mathcal{H} / \tau \pi_1 (C' \setminus \tau^{-1} S, *).
\]
Remark 0.3. i) Theorem 0.1 can be used to explain the observation of B. Lian and S.-T. Yau ([10], [11]) that the weight-2 VHS attached to \( f \) of certain one dimensional families of \( K3 \) surfaces coming from the Mirror of \( K3 \) surfaces of Picard number \( \geq 1 \) can be expressed as the square products of the weight 1 VHS attached to some modular families of elliptic curves (also see [5]). Note that such a family must reach the Arakelov bound in (0.0.1). We thank A. Todorov for pointing out that to us. Note that, if \( S = \emptyset \) then there is another type families of \( K3 \) surfaces reaching the Arakelov bound (0.0.1). Namely, let \( a : A \to C \) be a modular family of false elliptic curves, i.e. abelian surface whose endomorphism ring is isomorphic to an order of an indefinite quaternion algebra over \( \mathbb{Q} \) ([26]). Then the Kummer construction gives rise to a family \( f : X \to C \) of smooth \( K3 \) surfaces reaching the Arakelov bound (0.0.1), and \( C \) is a Shimura curve. One likes to know what is the mirror pair of this family.

ii) For a family \( f : X \to C \) in Theorem 0.2 one can find a family \( f' : X' \to C \), which comes from the Nikulin-Kummer construction of a product of a modular family of elliptic curves \( g : E_1 \to C \) with an elliptic curve \( E_2 \) over \( C \), and such that sub VHSs of transcendental lattices of \( f \) and \( f' \) are Hodge isometric to each other. Are there more closer geometric relations among these families?

Let \( f : X \to \mathbb{P}^1 \) be a Calabi-Yau 3-fold fibred by non-constant semi-stable \( K3 \) surfaces. The triviality of \( \omega_X \) implies that \( \deg f_* \omega_{X/\mathbb{P}^1} = 2 \).

Corollary 0.4. Let \( f : X \to \mathbb{P}^1 \) be a Calabi-Yau 3-fold fibred by non-constant semi-stable \( K3 \) surfaces. Then the followings hold true:

i) If the iterated Kodaira-Spencer map of \( f \) is non-zero, then \( f \) has at least 4 singular fibres. If \( f \) has 4 singular fibres, then \( X \) is rigid and birational to the Nikulin-Kummer construction of a square product of a family of elliptic curves \( g : E_1 \to \mathbb{P}^1 \). After passing through (if necessary) a double cover \( E' \to E \), the family \( g' : E' \to \mathbb{P}^1 \) is a modular family of elliptic curves from the Beauville's 6 examples.

ii) If the iterated Kodaira-Spencer map of \( f \) is zero, then \( f \) has at least 6 singular fibres. If \( f \) has 6 singular fibres over \( S \subset \mathbb{P}^1 \), then \( X \) is non-rigid, the general fibres have Picard number at least 18, and \( \mathbb{P}^1 \setminus S \simeq \mathcal{H}/\Gamma \), where \( \Gamma \) is a subgroup of \( SL_2(\mathbb{Z}) \) of index 24.

Remark 0.5. i) Any \( K3 \)-fibred Calabi-Yau 3-fold \( f : X \to \mathbb{P}^1 \) in i) is rigid because of the modular construction for \( X \). Since all 6 examples of Beauville are defined over \( \mathbb{Z} \), we may assume that \( X \) has a suitable integral model. The \( L \)-series of \( X \) is defined to be the \( L \)-series of the Galois representation on \( H^3_{et}(\bar{X}, \mathbb{Q}) \). One should be able verify the so-called modularity conjecture for \( X \) as M-H Saito and N. Yui checked one example in [20]. That is, up to a finite Euler factor, \( L(X, s) = L(f, s) \) for \( f \in S_4(\Gamma_0(N)) \).

ii) Does any rigid Calabi-Yau 3-fold fibred by semi-stable \( K3 \) surfaces come from the modular construction in i)?

iii) One can construct an example for the case ii) of Corollary 0.4. Let \( g : E(4) \to X(4) \)
be the modular family of elliptic curves corresponding to the congruence group \( \Gamma(4) \). Then \( X(4) \simeq \mathbb{P}^1 \) with six cusps, and \( \deg g_* \omega_{E(4)/X(4)} = 2 \). The Nikulin-Kummer construction to the product of \( g : E(4) \to X(4) \) with a constant family of elliptic curves gives an K3 fibred Calabi-Yau 3-fold reaching the upper bound in (0.0.2), which is non rigid.

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## 1. Weight-2 VHS and \( \mathbb{R} \)-Splitting

Let \( f : X \to C \) be a family of semi-stable K3 surfaces. Consider its weight-2 variation of Hodge structure (VHS for simplicity)

\[ \mathcal{V} = R^2 f_* (\mathcal{O}_X) \]

Let \( S \subset C \) denote the subset, where the local monodromies of \( \mathcal{V} \) have infinite order. One has the canonical extension of Hodge bundles

\[ E^{p,q} = R^q f_* (\mathcal{O}_X^{p,q}(\log \Delta)), \quad p + q = 2, \]

together with the cup product of Kodaira-Spencer map

\[ \theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_C^1(\log S). \]

\( \theta = \theta^{2,0} + \theta^{1,1} \) is called the Higgs field of \( \mathcal{V} \).

**Lemma 1.1.** We have \( \deg E^{2,0} \leq \deg \Omega_C^1(\log S) \), and if the equality

\[ \deg E^{2,0} = \deg \Omega_C^1(\log S) \]

holds, then there is a real splitting \( \mathcal{V} \otimes \mathbb{R} = \mathcal{W} \oplus \mathcal{U} \), which is orthogonal w.r.t. the polarization, and \( \mathcal{U} \) is unitary. The corresponding Higgs bundle splitting is

\[ (E^{2,0} \oplus E_1^{1,1} \oplus E_2^{0,2}, \theta) \oplus (E_2^{1,1}, 0) \]

where \( E_1^{1,1} = E_1^{1,1} \oplus E_2^{1,1} \) and \( E_1^{1,1} \) is a line bundle of degree zero such that

\[ \theta : E^{2,0} \to E_1^{1,1} \otimes \Omega_C^1(\log S), \quad \theta : E_1^{1,1} \to E_2^{0,2} \otimes \Omega_C^1(\log S) \]

are isomorphisms.

**Proof.** Consider the map \( \theta^{1,1} : E^{1,1} \to E_1^{0,2} \otimes \Omega_C^1(\log S) \), let \( E_2^{1,1} \subset E^{1,1} \) denote the kernel of \( \theta^{1,1} \), then \( (E_2^{1,1}, 0) \) is a Higgs sub-bundle.

**Claim:** \( \deg E_2^{1,1} \leq 0 \), and if the equality holds then the Higgs subbundle

\[ (E_2^{1,1}, 0) \subset (E, \theta) \]

induces a splitting \( (E, \theta) = (E^{2,0} \oplus E_1^{1,1} \oplus E_2^{0,2}, \theta) \oplus (E_2^{1,1}, 0) \), which corresponds to a splitting of the local system over \( C \mathcal{V} \otimes C = \mathcal{W} \oplus \mathcal{U} \).

**Proof of the claim:** Let \( h \) denote the Hodge metric on \( E|_{C \setminus S} \), and let \( \Theta(E|_{C \setminus S}, h) \) be its curvature form. Then we have ([7], Chapter II)
\[ \Theta(E|_{C\setminus S}) + \theta \wedge \bar{\theta} + \bar{\theta} \wedge \theta = 0, \]

where \( \bar{\theta} \) is the complex conjugation of \( \theta \) with respect to \( h \). Consider the \( C^\infty \)-orthogonal (for \( h \)) decomposition \( E|_{C\setminus S} = E^{1,1}_{2}|_{C\setminus S} \oplus E^{1,1}_{2}|_{C\setminus S} \), one has

\[ \Theta(E^{1,1}_{2}|_{C\setminus S}, h) = \Theta(E|_{C\setminus S}, h)|_{E^{1,1}_{2}} + \bar{A} \wedge A = -(\theta \wedge \bar{\theta})|_{E^{1,1}_{2}} - (\bar{\theta} \wedge \theta)|_{E^{1,1}_{2}} + \bar{A} \wedge A, \]

where \( A \in A^{1,0}(\text{Hom}(E^{1,1}_{2}, E^{1,1}_{2})) \) is the second fundamental form of the sub-bundle \( E^{1,1}_{2} \subset E \), and \( \bar{A} \) is the complex conjugation with respect to \( h \). Since \( \theta(E^{1,1}_{2}) = 0 \), we have \( (\theta \wedge \bar{\theta})|_{E^{1,1}_{2}} = 0 \). Hence

\[ \Theta(E^{1,1}_{2}|_{C\setminus S'}, h) = -(\theta \wedge \bar{\theta})|_{E^{1,1}_{2}} + \bar{A} \wedge A. \]

\( \Theta(E^{1,1}_{2}|_{C\setminus S'}, h) \) is negative semidefinite since \( \theta \wedge \bar{\theta} \) is positive semidefinite and \( \bar{A} \wedge A \) is negative semidefinite. Since the local monodromies around points in \( S \) are unipotent, \( Tr \Theta(E^{1,1}_{2}|_{C\setminus S'}, h) \) represents \( c_1(E^{1,1}_{2}) \) as a current. Thus

\[ \text{deg } E^{1,1}_{2} = \int_{C\setminus S} Tr \Theta(E^{1,1}_{2}|_{C\setminus S}, h) \leq 0, \]

and \( \Theta(E^{1,1}_{2}|_{C\setminus S}, h) = 0 \) if \( \text{deg } E^{1,1}_{2} = 0 \). This implies that \( \bar{\theta}(E^{1,1}_{2}) = 0 \) and \( A = 0 \). Altogether show that the sub-Higgs bundle \( (E^{1,1}_{2}, 0) \) of \( (E, \theta) \) induces a splitting of the Higgs bundle

\[ (E, \theta) = (E^{2,0} \oplus E^{1,1}_{1} \oplus E^{0,2}, \theta) \oplus (E^{1,1}_{2}, 0) \]

and the corresponding splitting \( V \otimes \mathbb{C} = W \oplus U \) of the complex local system. Thus the claim is proved.

Let \( I \subset E^{0,2} \otimes \Omega^{1}_C(\log S) \) be the image of \( \theta^{1,1} \), then, by the exact sequence

\[ 0 \to E^{1,1}_{2} \to E^{1,1} \to I \to 0, \]

and note that \( \text{deg } E^{1,1}_{2} = 0 \), one gets

\[ -\text{deg } E^{2,0} + \text{deg } \Omega^{1}_C(\log S) = \text{deg}(E^{0,2} \otimes \Omega^{1}_C(\log S)) \geq \text{deg } I = -\text{deg } E^{1,1}_{2} \geq 0. \]

Hence \( \text{deg } E^{2,0} \leq \text{deg } \Omega^{1}_C(\log S) \) and the equality holds if and only if \( \text{deg } E^{1,1}_{2} = 0 \) and \( I = E^{0,2} \otimes \Omega^{1}_C(\log S) \), which is our \( E^{1,1}_{1} \). It is easy to see that the Higgs field of \( W \) is an isomorphism, thus \( W \) is irreducible over \( \mathbb{C} \). Now we only need to show that the decomposition \( V \otimes \mathbb{C} = W \oplus U \) can be, in fact, defined over \( \mathbb{R} \). Taking the complex conjugation on \( W \) one has

\[ \overline{W} \subset V \otimes \mathbb{C} = V \otimes \mathbb{C}. \]

\( \overline{W} \) is again of the Hodge type \((2,0) + (1,1) + (0,2)\), irreducible and with the non-zero Higgs field. The projection \( p : \overline{W} \subset V \otimes \mathbb{C} \to U \) can not be injective since \( U \) is unitary. Moreover, since \( \overline{W} \) can not have a proper sub local system, this projection must be zero. Thus \( \overline{W} = W \) and we obtain a real sub local system \( W \subset V \otimes \mathbb{R} \). The intersection form restricted to \( W \) is non-degenerated. Thus the orthogonal complement of \( W \) with respect to the intersection form gives the desired real decomposition \( V \otimes \mathbb{R} = W \oplus U \). \( \square \)
Lemma 1.2. If the iterated Kodaira-Spencer map $\theta^{1,1}\theta^{2,0} = 0$, then
\[
\deg E^{2,0} \leq \frac{1}{2} \deg \Omega^1_C(\log S).
\]

When the equality $\deg E^{2,0} = \frac{1}{2} \deg \Omega^1_C(\log S)$ holds, then there is a real splitting
\[
\mathbb{V} \otimes \mathbb{R} = \mathbb{W} \oplus \mathbb{U},
\]
which is orthogonal w.r.t. the polarization, and $\mathbb{U}$ is unitary. The corresponding Higgs bundle splitting is
\[
(E^{2,0} \oplus (E^{1,1}_1 \oplus E^{1,1*}_1) \oplus E^{0,2} \oplus \theta) \oplus (E^{2,1}_2, 0)
\]
where $E^{1,1}_1$ and $E^{1,1*}_1$ are sub line bundles of $E^{1,1}$ with
\[
\deg E^{1,1}_1 = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S),
\]
and $E^{1,1} = E^{1,1}_1 \oplus E^{1,1*}_1 \oplus E^{2,1}_2$. The Higgs field
\[
\theta : (E^{2,0} \oplus (E^{1,1}_1 \oplus E^{1,1*}_1) \oplus E^{0,2} \oplus (E^{1,1}_1 \oplus E^{1,1*}_1) \oplus E^{0,2}) \otimes \Omega^1_C(\log S)
\]
is defined by $\theta = \tau + -\tau^*$, where $\tau : E^{2,0} \simeq E^{1,1}_1 \otimes \Omega^1_C(\log S)$, $E^{1,1}_1 \to 0$.

Proof. Since $\theta^{1,1}\theta^{2,0} = 0$, the map $\theta^{2,0}$ factors through
\[
\theta^{2,0} : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S),
\]
where $E^{1,1}_1 \subset E^{1,1}$ is a sub-line bundle such that $\theta^{1,1}(E^{1,1}_1) = 0$. Thus
\[
(E^{2,0} \oplus E^{1,1}_1, \theta^{2,0}) \subset (E, \theta)
\]
is a rank-2 Higgs sub bundle. By the same arguments in the proof of Lemma 1.1, one has $\deg E^{2,0} \oplus E^{1,1}_1 \leq 0$, thus
\[
\deg E^{2,0} \leq \frac{1}{2} \Omega^1_C(\log S).
\]

If the equality holds, then $\theta^{2,0} =: \tau : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S)$ is an isomorphism with $\deg E^{1,1}_1 = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S)$, and the Higgs sub bundle $(E^{2,0} \oplus E^{1,1}_1, \tau)$ gives rise to a complex sub local system $\mathbb{W}_1 \subset V \otimes \mathbb{C}$. The dual $\mathbb{W}_1 \subset V \otimes \mathbb{C}$ corresponds to Higgs subbundle
\[
(E^{2,0} \oplus E^{1,1}_1)^* = E^{1,1*}_1 \oplus E^{0,2}
\]
together with the Higgs field $-\tau^* : E^{1,1*}_1 \to E^{0,2} \otimes \Omega^1_C(\log S)$. The sub-local system $\mathbb{W} := \mathbb{W}_1 \oplus \mathbb{W}_1$ is real, and the intersection form restricted to $\mathbb{W}$ is non-degenerated. Hence, the orthogonal complement defines the desired decomposition. \hfill \Box
2. Splitting over \( \overline{\mathbb{Q}} \)

We start with a very simple observation. Suppose that \( V \) is a local system defined over \( \overline{\mathbb{Q}} \). Fixing a positive integer \( r \), let \( \mathcal{G}(r, V) \) denote the set of all rank-\( r \) sub-local systems of \( V \). Then \( \mathcal{G}(r, V) \) is a projective variety defined over \( \overline{\mathbb{Q}} \). The following property is well known.

**Lemma 2.1.** If \([W] \in \mathcal{G}(r, V)\) is an isolated point, then \( W \) is defined over \( \overline{\mathbb{Q}} \).

**Lemma 2.2.** The \( \mathbb{R} \)-splittings \( V \otimes \mathbb{R} = W \oplus U \) in Lemma 1.1 and Lemma 1.2 can be defined over \( \overline{\mathbb{Q}} \).

**Proof.** By Lemma 2.1, one only needs to show that \( W \) is a rigid sub-local system of \( V \otimes \mathbb{C} \). Suppose that there is a family of sub-local systems \( \{W_t\} \), \( W_0 = W \). By semi-continuity, the Higgs fields \( \theta_{t=0} \) of \( W_t \) are again isomorphisms for \( t \) being sufficiently closed to 0. Then the projection \( W_t \to V \otimes \mathbb{C} \to U \) must be zero, otherwise, \( W_t \) would contain a non-trivial unitary component, which contradicts that \( \theta_{t=0} \) are isomorphisms. Hence \( W_t = W \).

Similarly, we show that the sub-local system \( W = W_1 \oplus \overline{W}_1 \subset V = W \oplus U \) is rigid. Suppose that there is a family of sub local systems \( \{W_t\} \) with \( W_0 = W \), we decompose \( W_t \) into the direct sum of irreducible components over \( \mathbb{C} \), which has only following possible types up to isomorphism

\[
W_1 \oplus \overline{W}_1; \quad W_1 \oplus U'; \quad \overline{W}_1 \oplus U''; \quad U''',
\]

where \( U', U'', U''' \) are unitary. By semicontinuity, the last three cases are impossible if \( t \) is sufficiently closed to 0 (otherwise \( \theta_{t=0} \) would be zero). Thus

\[
W_t \simeq W_1 \oplus \overline{W}_1,
\]

which implies that the projection \( W_t \to V \otimes \mathbb{C} \to U \) must be zero. Otherwise, \( W_1 \) would contain a non-trivial unitary component, which contradicts that the Higgs fields of \( W \) are isomorphisms. \( \square \)

3. Splitting over \( \mathbb{Q} \) and \( \mathbb{Z} \)-structures

We call the splitting in Lemma 1.1 of type \( (0, 0, 1) \) and the splitting in Lemma 1.2 of type \( (1, 0, 2) \).

**Lemma 3.1.** If \( S \neq \emptyset \), the splittings in Lemma 2.2 can be defined over \( \mathbb{Q} \).

**Proof.** Let \( V \otimes K = W \oplus U \) be the splitting of type \( (1, 0, 1) \) in Lemma 2.2, where \( K \) is a Galois extension of \( \mathbb{Q} \). For any \( \sigma \in Gal(K/\mathbb{Q}) \), we claim that \( \sigma W = W \). Otherwise, the projection \( p : \sigma W \to V \otimes K \to U \) must be non-zero and \( \sigma W \) is isomorphic to a unitary sub local system \( U' \subset U \) under \( p \) since \( W \) is irreducible (thus \( \sigma W \) is also irreducible). Let \( \gamma \) be a short loop around \( s \in S \). Then the monodromy matrix \( \rho_W(\gamma) \) has infinite order, hence \( \rho_{\sigma W}(\gamma) \) has also infinite order, which contradicts that \( \rho_{U'}(\gamma) \) is identity. We proved that \( W \) is invariant under \( Gal(K/\mathbb{Q}) \). Hence \( W \) is defined over \( \mathbb{Q} \) and the orthogonal complement of \( W \subset V \otimes \mathbb{Q} \) w.r.t. the intersection form defines an \( \mathbb{Q} \)-splitting

\[
V \otimes \mathbb{Q} = W \oplus U.
\]
By the same argument, we show that the splitting of type \((0,0,2)\) in lemma 2.2 is also defined over \(\mathbb{Q}\).

**Lemma 3.2.** After passing through a finite etale cover of \(C\) the splittings of type \((0,0,1)\) and \((0,0,2)\) in Lemma 3.1 induce \(\mathbb{Z}\)–sub lattices

\[ V \supset W_{\mathbb{Z}} \oplus Z^{19}, \quad V \supset W_{\mathbb{Z}} \oplus Z^{18} \]

such that \(V \otimes \mathbb{Q} = (W_{\mathbb{Z}} \oplus Z^{19}) \otimes \mathbb{Q}\) and \(V \otimes \mathbb{Q} = (W_{\mathbb{Z}} \oplus Z^{18}) \otimes \mathbb{Q}\), where \(Z^{19}, Z^{18}\) is respectively a rank-19 constant \(\mathbb{Z}\)-lattice of type-(1,1) and a rank-18 constant \(\mathbb{Z}\)-lattice of type-(1,1).

**Proof.** Let \(W_{\mathbb{Z}} = V \cap W, \quad U_{\mathbb{Z}} = V \cap U\). It is easy to check that

\[ W_{\mathbb{Z}} \otimes \mathbb{Q} = W, \quad U_{\mathbb{Z}} \otimes \mathbb{Q} = U, \]

thus \(W_{\mathbb{Z}}\) and \(U_{\mathbb{Z}}\) are lattices in \(W\) and \(U\). Since \(U\) is unitary and carries an \(\mathbb{Z}\)–structure, the monodromy group of \(U\) is finite. Since the local monodromies of \(U\) around \(S\) are trivial, \(U\) extends to a local system on \(C\). Therefore, after passing through the cover corresponding to this monodromy group, \(U\) becomes a constant local system \(Z^{19}, Z^{18}\) respectively.

**Corollary 3.3.** Let \(f : X \to C\) be a family of semi-stable \(K3\) surfaces over a curve \(C\). When it reaches the upper bound \(\deg f_* \omega_{X/C} = \deg \Omega^1_C(\log S)\), then the Picard number of the general fibres is at least 19. If \(\theta_{1,1} \theta_{2,0} = 0\) and \(f\) reaches the upper bound \(\deg f_* \omega_{X/C} = \frac{1}{2} \deg \Omega^1_C(\log S)\), then the Picard number of the general fibres is at least 18.

4. Nikulin and Kummer construction

Let \(f : X \to C\) be a family of semi-stable \(K3\) surfaces, which reaches the upper bound \(\deg f_* \omega_{X/C} = \deg \Omega^1_C(\log S)\). By Lemma 3.2, after passing through a finite étale cover of \(C\), one has

\[ R^2 f_* (Z_{X^0}) \otimes \mathbb{Q} = W \oplus Q^{19}, \]

where \(W\) is an \(\mathbb{C}\)-irreducible representation of \(\pi_1(C \setminus S, *)\) and \(Q^{19}\) is a constant local system of rank 19 such that \(Q^{19}_t \subset NS(X_t) \otimes \mathbb{Q}\) for any \(t \in C \setminus S\). We obtain therefore,

**Lemma 4.1.** For any \(t \in C \setminus S\), the Picard number \(\rho(X_t) \geq 19\) and for any class \(s_t \in Q^{19}_t \subset Pic(X_t) \otimes \mathbb{Q}\) there is a \(\mathbb{Q}\)-divisor \(D \in Div(X) \otimes \mathbb{Q}\) such that \(D|_{X_t} = s_t\).

Let \(Y\) be an algebraic \(K3\) surface and \(H^2(Y, Z) = T_Y \oplus NS(Y)\) be the orthogonal decomposition. \(T_Y\) is the so called transcendental lattice of \(Y\), which is even and has signature \((2, 20 - \rho(Y))\). It is well-known that as lattices

\[ H^2(Y, Z) \cong U^3 \oplus E_8(-1)^2. \]

We recall some results about embeddings of lattices (see [18] and references there)
Lemma 4.2. (Theorem 2.4 of [12], or see Corollary 2.6 of [Mo]) Let $T$ be a non-degenerate even lattice of rank $r$. Then there is a primitive embedding

$$T \hookrightarrow U^r$$

In particular, if $\rho(X) \geq 19$, then there is a primitive embedding

$$T_X \hookrightarrow U^3.$$  

Lemma 4.3. If $12 < \rho \leq 20$, then every even lattice $T$ of signature $(2, 20 - \rho)$ occurs as the transcendental lattice of some algebraic K3 surface and the primitive embedding $T \hookrightarrow U^3 \oplus E_8(-1)^2$ is unique.

Theorem 4.4. ([13]) If $\rho(Y) \geq 19$, then there exists a primitive embedding

$$\varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y, \mathbb{Z})$$

and a Nikulin involution $\tau : Y \to Y$ such that $\tau^* : H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is identity on $(\varphi(E_8(-1)^2))^\perp$.

Proof. By Lemma 4.2, there is a primitive embedding $\phi : T_Y \hookrightarrow U^3$, thus a primitive embedding $\phi \oplus 0 : T_Y \hookrightarrow U^3 \oplus E_8(-1)^2$. By Lemma 4.3 (uniqueness), the above embedding is isomorphic to

$$T_Y = NS(X)^\perp \subset H^2(Y, \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2.$$  

Thus, there is a primitive embedding

$$\psi : E_8(-1)^2 \hookrightarrow T_Y^\perp = NS(Y) \subset H^2(Y, \mathbb{Z}).$$

Let $\{c^1_j\}_{1 \leq j \leq 8}$ and $\{c^2_j\}_{1 \leq j \leq 8}$ be the bases of $E_8(-1) \oplus 0$ and $0 \oplus E_8(-1)$ and

$$g : H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

be defined as: $g(\psi(c^1_j)) = \psi(c^2_j), \ g(\psi(c^2_j)) = \psi(c^1_j)$ and $g(e) = e$ for any $e \in (\psi(E_8(-1)^2))^\perp$. Then, by theorems of Nikulin (see Theorem 5.6 of [Mo]), there is a Nikulin involution $\tau : Y \to Y$ and $w \in W(Y)$ (the group of Picard-Lefschetz reflections) such that $\tau^* = w \cdot g \cdot w^{-1}$. Let

$$\varphi : E_8(-1)^2 \overset{\psi}{\to} H^2(Y, \mathbb{Z}) \overset{w}{\to} H^2(Y, \mathbb{Z}),$$

then $\varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y, \mathbb{Z})$ is another primitive embedding, and

$$\tau^*(\varphi(c^1_j)) = \varphi(c^2_j), \ \tau^*(\varphi(c^2_j)) = \varphi(c^1_j), \ \tau^*(e) = e, \ \forall e \in (\varphi(E_8(-1)^2))^\perp.$$  

Let $t_0 \in C \setminus S$ be a point such that the fibre $X_{t_0}$ satisfying $\rho(X_{t_0}) = 19$. Thus,

$$Q_{t_0}^{19} = NS(X_{t_0}) \otimes \mathbb{Q}.$$  

Since the monodromy action of $\pi_1(C \setminus S, t_0)$ on $Q_{t_0}^{19}$ is trivial, $\varphi(c^1_j)$ and $\varphi(c^2_j)$, $1 \leq j \leq 8$ can be lifted to divisors $D^1_j$ and $D^2_j$, $1 \leq j \leq 8$ on $X$. Then we have

Lemma 4.5. For any $t \in C \setminus S$, let $d^1_{ij} = D^1_j|_{X_t} \in H^2(X_t, \mathbb{Z})$. Then $\{d^1_{ij}\}_{1 \leq j \leq 8}$, $(i = 1, 2)$ generate a sublattice of $H^2(X_t, \mathbb{Z})$, which is isomorphic to $E_8(-1)^2$ such that $E_8(-1)^2 \hookrightarrow H^2(X_t, \mathbb{Z})$ is a primitive embedding, $E_8(-1) \oplus 0$ and $0 \oplus E_8(-1)$ are isomorphic to $\mathbb{Z}\{d^1_{ij}, j = 1, \ldots, 8\}$ and $\mathbb{Z}\{d^2_{ij}, j = 1, \ldots, 8\}$.
Proof. The proof is straightforward. For example, to prove that \( \{d_{ij}^1\}_{1 \leq j \leq 8} \) are \( \mathbb{Z} \)-linearly independent: if \( \sum n_j d_{ij}^1 = 0 \) in \( H^2(X_t, \mathbb{Z}) \), we claim that \( \sum n_j \varphi(c_j^1) = 0 \), which will imply the \( \mathbb{Z} \)-linearly independence of \( \{d_{ij}^1\}_{1 \leq j \leq 8} \). The claim is clear, otherwise there is a \( A \in NS(X_{t_0}) \) such that \( (\sum n_j \varphi(c_j^1), A) \neq 0 \). Let \( \tilde{A} \) be a lifting of \( A \), then

\[
(\sum n_j d_{ij}^1, \tilde{A}|_{X_t}) = (\sum n_j D_j^1|_{X_t}, \tilde{A}|_{X_t}) = (\sum n_j D_j^1|_{X_{t_0}}, \tilde{A}|_{X_{t_0}})
\]

\[
= (\sum n_j \varphi(c_j^1), A) \neq 0.
\]

To see that the embedding \( E_8(-1)^2 \hookrightarrow H^2(X_t, \mathbb{Z}) \) is primitive, let \( B \in H^2(X_t, \mathbb{Z}) \) be a class with \( mB \in \mathbb{Z}\{d_{ij}^1, i = 1, 2, j = 1, \ldots, 8\} \). Then \( B \) is invariant under the monodromy, and thus there is a lifting \( \tilde{B} \) of \( B \). Since \( \varphi : E_8(-1)^2 \hookrightarrow H^2(X_{t_0}, \mathbb{Z}) \) is primitive and \( m\tilde{B}|_{X_{t_0}} \in \varphi(E_8(-1)^2) \), \( \tilde{B}|_{X_{t_0}} = \sum n_j^i \varphi(c_j^1) \). Then

\[
(m(\tilde{B} - \sum n_j^i D_j^1)|_{X_t}, m(\tilde{B} - \sum n_j^i D_j^1)|_{X_t}) = 0
\]

and \( (m(\tilde{B} - \sum n_j^i D_j^1)|_{X_t}, H|_{X_t}) = 0 \), which implies that \( m(\tilde{B} - \sum n_j^i D_j^1)|_{X_t} = 0 \) by Hodge index theorem since \( m(\tilde{B} - \sum n_j^i D_j^1)|_{X_t} = mB - m \sum n_j^i d_{ij}^1 \) is an algebraic class. Thus \( (\tilde{B} - \sum n_j^i D_j^1)|_{X_t} = 0 \) since \( H^2(X_t, \mathbb{Z}) \) is torsion free.

Let \( E = \bigoplus_{p+q=2} E^{p,q} \) be the canonical extension of the Hodge bundle associated to the local system \( R^2 f_*(\mathcal{Z}_{X_0}) \), and \( \mathcal{E}nd(E) \to C \) be the endomorphism bundle over \( C \), which represents the functor

\[
\mathcal{E}nd(E)^\sharp : \{\text{schemes over } C\} \to \{\text{sets}\}
\]

where \( \mathcal{E}nd(E)^\sharp(T) = \{\text{bundle morphism } E_T \to E_T \text{ over } T\} \). For \( t \in C \setminus S \), by Lemma 4.5, we can define an isometric involution

\[
g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})
\]

by \( g_t(d_{ij}^1) = d_{ij}^2 \), \( g_t(d_{ij}^2) = d_{ij}^1 \), \( g_t(e) = e \) for all \( e \in \mathbb{Z}\{d_{ij}^i\}^\perp \) and \( 1 \leq j \leq 8 \). It is easy to see that \( g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z}) \) is a morphism of \( \pi_1(C \setminus S) \)-modules. Thus, they give rise an involution

\[
g : R^2 f_*(\mathcal{Z}_{X_0}) \to R^2 f_*(\mathcal{Z}_{X_0})
\]

of local system, which corresponds to a section \( g \in H^0(C \setminus S, \mathcal{E}nd(E)) \).

**Lemma 4.6.** The section \( g \in H^0(C \setminus S, \mathcal{E}nd(E)) \) defined above can be extended to a section in \( H^0(C, \mathcal{E}nd(E)) \), and thus \( g \) is an algebraic section.

**Proof.** Recall that \( R^2 f_*(\mathcal{Z}_{X_0}) \otimes \mathbb{Q} = \mathcal{W} \oplus \mathcal{Q}^{19} \) and the canonical extension of the Hodge bundle corresponding to \( R^2 f_*(\mathcal{Z}_{X_0}) \) can be written into

\[
(E, \theta) = (E_{\mathcal{W}}, \theta) \oplus (\mathcal{O}_C^{19}, 0),
\]

where \( (E_{\mathcal{W}}, \theta) \) and \( (\mathcal{O}_C^{19}, 0) \) are the canonical extension of the Hodge bundles corresponding to \( \mathcal{W} \) and \( \mathcal{Q}^{19} \) respectively. By the construction of \( g \), it is identity on \( \mathcal{W} \) (thus extended to \( E_{\mathcal{W}} \)), and is well-defined on the constant lattice \( \mathcal{Z}^{19} \). Thus it is clear that \( g \) can be extended on \( C \).
**Lemma 4.7.** Let $H$ be an ample divisor on $X$ and $g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ be the Hodge isometry involutions defined above. Then there exists a non-empty Zariski open set $C^0 \subset C \setminus S$ such that $g_t(H|_{X_t})$ is an ample divisor for any $t \in C^0$. In particular, $g_t$ is an effective Hodge isometry for any $t \in C^0$.

**Proof.** We may write $H|_{X_t} = \sum n_j^1 \varphi(e_j^1) + \sum n_j^2 \varphi(e_j^2) + e$, where $e \in \varphi(E_8(-1)^2)$. Let $E$ be a lifting of $e$ and

$$D = \sum_{j=1}^8 n_j^1 D_j^1 + \sum_{j=1}^8 n_j^2 D_j^2 + E, \quad \tilde{D} = \sum_{j=1}^8 n_j^1 D_j^2 + \sum_{j=1}^8 n_j^2 D_j^1 + E.$$  

Then, for any $t \in C \setminus S$, $H|_{X_t} = D|_{X_t}$ and $g_t(D|_{X_t}) = \tilde{D}|_{X_t}$. Thus $D$ is a relative ample divisor on $f^{-1}(C \setminus S)$ and $\tilde{D}|_{X_{t_0}}$ is ample (here we have chosen $t_0$ such that $g_{t_0}$ is effective). Thus there exists a Zariski open set $C^0 \subset C \setminus S$ such that $\tilde{D}$ is relative ample on $f^{-1}(C^0)$.

**Lemma 4.8.** The $g$ induces an involution $\tau : f^{-1}(C^0) \to f^{-1}(C^0)$ over $C^0$ such that $\tau_t : X_t \to X_t$ (for $t \in C^0$) are Nikulin involutions with $\tau_t^* = g_t$.

**Proof.** Let $\mathcal{L} = D + \tilde{D}$, where $D$ and $\tilde{D}$ are the divisors defined in the proof of Lemma 4.7. Then we know that $\mathcal{L}$ is relative ample on $f^{-1}(C^0)$ and $\mathcal{L}_t = \mathcal{L}|_{X_t}$ is invariant under the involution $g_t$. Let $\pi : Aut^\mathcal{L}(f^{-1}(C^0)/C^0) \to C^0$ denote the automorphism group scheme, which represents the functor

$$Aut^\mathcal{L}_{f^{-1}(C^0)/C^0}(T) = \left\{ \text{Isomorphisms } h : f^{-1}(C^0) \times_{C^0} T \to f^{-1}(C^0) \times_{C^0} T \right\}.$$  

Thus there exists a universal automorphism

$$f^{-1}(C^0) \times_{C^0} Aut^\mathcal{L}(f^{-1}(C^0)/C^0) \xrightarrow{h} f^{-1}(C^0) \times_{C^0} Aut^\mathcal{L}(f^{-1}(C^0)/C^0)$$

and $h^*$ induces an endomorphism $\pi^* E \to \pi^* E$, which gives a homomorphism

$$Aut^\mathcal{L}(f^{-1}(C^0)/C^0) \xrightarrow{\alpha} \text{End}(E) \xrightarrow{\pi} C^0.$$  

By Torelli theorem of K3 surfaces, $\alpha$ is injective. On the other hand, the fibres of $\alpha$ are isomorphic to group schemes, which are smooth. Thus $\alpha$ is an embedding. By Lemma 4.6 and Lemma 4.7, $g(C^0)$ is algebraic and contained in the image of $\alpha$, which gives a section of $\pi : Aut^\mathcal{L}(f^{-1}(C^0)/C^0) \to C^0$. That is an automorphism

$$f^{-1}(C^0) \xrightarrow{\tau} f^{-1}(C^0)$$

such that $\tau_t^* = g_t$ for any $t \in C^0$. Thus $\tau_t$ are Nikulin involutions, i.e. $\tau_t^* \omega = \omega$ for any $\omega \in H^{2,0}(X_t)$.

\[\square\]
Since all fibres $X_t$ are algebraic K3 surfaces, the $\tau_i$ gives rise a Shioda-Inose structure on $X_t$ by theorems of Morrison (see Theorem 6.3 of [13]). Let $g : Z^0 \to C^0$ be the desingularization of $f^{-1}(C^0)/\tau \to C^0$. Then $g : Z^0 \to C^0$ is a family of Kummer surfaces and there exist divisors $N_1, ..., N_8$ on $Z^0$ such that their restrictions $(N_1)_t, ..., (N_8)_t$ on $Z^0_t$ are the exceptional $(-2)$-curves of the double points of $X_t/\tau_t$ (produced by the eight isolated fixed points of $\tau_t$). By Lemma 3.2, we write $R^2 f_*(Z_{f^{-1}(C^0)}^0) = \mathbb{W} \oplus \mathbb{Z}^{19}$. Then we have (see Lemma 3.1 of [15])
\[ R^2 g_*(Z_{Z^0}^\tau) \simeq (\mathbb{W} \oplus \mathbb{Z}^{19\tau})(2) \oplus \mathbb{Z}[N_1, ..., N_8], \]
where $\mathbb{Z}^{19\tau}$ is the invariant sub local system of $\mathbb{Z}^{19}$ under $\tau$, $(\mathbb{W} \oplus \mathbb{Z}^{19\tau})(2)$ has the same underlying local system as $(\mathbb{W} \oplus \mathbb{Z}^{19\tau})$, and with an intersection by multiplication by 2 of the the intersection form on $(\mathbb{W} \oplus \mathbb{Z}^{19\tau})$.

**Lemma 4.9.** By making $C^0$ smaller, there exists a family of abelian surfaces $a : A^0 \to C^0$

with $\rho(A^0) \geq 3$ such that $g : Z^0 \to C^0$ is its Kummer construction.

**Proof.** It is easy to see that, for any $t \in C^0$, $NS(Z^0_t)$ contains a sub-lattice, which is isomorphic to $\mathbb{Z}^{19\tau}(2) \oplus \mathbb{Z}[N_1, ..., N_8]$ as a trivial $\pi_1(C \setminus S)$-modules. Thus $g : Z^0 \to C^0$ is a family of Kummer surfaces with $\rho(Z_t) \geq 19$. Let $t_0 \in C^0$ with $\rho(Z^0_{t_0}) = 19$. Then $NS(Z^0_{t_0}) \supset \mathbb{Z}^{19\tau} \oplus \mathbb{Z}[N_1, ..., N_8]$ and
\[ NS(Z^0_{t_0}) \otimes \mathbb{Q} = (\mathbb{Z}^{19\tau} \oplus \mathbb{Z}[N_1, ..., N_8]) \otimes \mathbb{Q}. \]

Let $E_1, ..., E_{16}$ be the liftings of the sixteen pairwise-disjoint $(-2)$-curves on $Z^0_{t_0}$ to $Z^0$. It is not difficult to see that we can choose $E_i$ ($i = 1, ..., 16$) to be effective divisors on $Z^0$. In fact, since $g_* \mathcal{O}_{Z^0}(E_i) \neq 0$ (because $H^0(E_i|_{Z^0_t}) \neq 0$ for any $t \in C^0$ by Riemann-Roch theorem), we have, for $m$ large enough and a point $p \in C^0$, $H^0(\mathcal{O}_{Z^0}(E_i + mg^{-1}(p))) = H^0(\mathcal{O}_{C^0}(mp) \otimes g_* \mathcal{O}_{Z^0}(E_i)) \neq 0$. Thus there is an effective divisor $D$ on $Z^0$ such that $D|_{Z^0_{t_0}}$ is numerical equivalent to $E_i|_{Z^0_{t_0}}$, which implies that $D|_{Z^0_{t_0}} = E_i|_{Z^0_{t_0}}$ since a nodal class is represented by only one effective divisor. We can choose $E_i$ ($i = 1, ..., 16$) to be irreducible further. In fact, we will show that $E_i|_{Z^0_{t_0}}$ is irreducible if $\rho(Z^0_{t_0}) = 19$. Otherwise, let $E_i|_{Z^0_{t_0}} = D_1 + D_2$, where $D_1$ is irreducible with $D_1^2 = -2$ and $D_2$ is effective. Note that for any lifting of an irreducible curve, whose restriction to any other fibre is equivalent to an effective divisor. Thus if $\tilde{D}_1$ and $\tilde{D}_2$ are the liftings of $D_1$ and $D_2$ ($\tilde{D}_2$ obtained by lifting the irreducible components of $D_2$), we see that $\tilde{D}_1|_{Z^0_{t_0}}$ and $\tilde{D}_2|_{Z^0_{t_0}}$ are equivalent to effective divisors. On the other hand, $E_i|_{Z^0_{t_0}} - \tilde{D}_1|_{Z^0_{t_0}}$ is numerically equivalent to $\tilde{D}_2|_{Z^0_{t_0}}$ since it is so on $Z^0_{t_0}$. But this is impossible since $E_i|_{Z^0_{t_0}}$ is a nodal class. Let $g : Z \to C$ be a compactification of $g : Z^0 \to C^0$ with $Z$ smooth and $E_1, ..., E_{16}$ be extend to $Z$. It is known that $E_1|_{Z^0_{t_0}} + \cdots + E_{16}|_{Z^0_{t_0}} \equiv 2\delta$. Let $\Delta$ be a divisor on $Z$ such that $\Delta|_{Z^0_{t_0}} = \delta$. Then $E_1 + \cdots + E_{16} - 2\Delta$ is numerically equivalent to zero on the general fibres, thus
\[ E_1 + \cdots + E_{16} - 2\Delta \equiv g^* D_a, \quad D_a \in \text{Div}(C). \]

Choose $C^0$ smaller so that $E_i|_{Z_t}$ ($i = 1, ..., 16$) are irreducible for $t \in C^0$ and
\[ E_1 + \cdots + E_{16} \equiv 2\Delta \quad \text{on } Z^0. \]
Let $A^0 \rightarrow Z^0$ be the double covering with branch locus $E_1 + \cdots + E_{16}$, and let $\varpi : A^0 \rightarrow A^0$ be the uniform blow down of the sixteen $(-1)$-curves on the fibres $A^0_{0\ell}$. Then $a : A^0 \rightarrow C^0$ is the family of abelian surfaces with $\rho(A^0) \geq 3$.

5. SPLITTING ON FAMILIES OF ABELIAN SURFACES

Let $a : A^0 \rightarrow C^0$ be the family of abelian surfaces constructed in Lemma 4.9. We take a compactification $a : A \rightarrow C$, (which may not be semi-stable). It is known that the sub VHS $T_a \subset R^2a_*(Z_{A^0})$ of the transcendental part of the weight-2 VHS attached to $a : A^0 \rightarrow C^0$ is Hodge isometric to $T_g(2)$, where

$$T_g \subset R^2g_*(Z_{Z^0})$$

is the sub VHS of the transcendental part of the weight-2 VHS attached to $g : Z^0 \rightarrow C^0$. Furthermore, $T_g$ is Hodge isometric to $T_f(2)$, where

$$T_f = \mathbb{W} \subset R^2f_*(Z_{f^{-1}(C^0)})$$

is the sub VHS of the transcendental part of the weight-2 VHS attached to $f : f^{-1}(C^0) \rightarrow C^0$. Since $\mathbb{W}$ is, in fact, defined on $C \setminus S$, $T_a$ can be extended to $C \setminus S$ as an VHS.

**Lemma 5.1.** The $\mathbb{Q}$-vector space of endomorphisms of

$$R^1a_*(Z_{A^0}) \otimes \mathbb{Q}$$

has dimension 4, and is of $(0,0)$-type.

**Proof.** By the construction of $a : A^0 \rightarrow C^0$, we see $R^2a_*(Z_{A^0}) \otimes \mathbb{Q}$ contains a constant local system of dimension 3 of $(1,1)$-type (this corresponds to a sub-lattice of Picard lattice of $A^0$). Hence, it corresponds to a 3-dimensional subspace of $End(R^1a_*(Z_{A^0})$ of $(0,0)$-type. Using a non-scalar endomorphism of this space, we can split $R^1a_*(Z_{A^0}) \otimes \mathbb{C}$ into the following type

$$R^1a_*(Z_{A^0}) \otimes \mathbb{C} \simeq \mathbb{W}_1 \oplus \mathbb{W}_2,$$

where both $\mathbb{W}_i$ are of rank-2 and irreducible over $\mathbb{C}$. Otherwise $R^1a_*(Z_{A^0}) \otimes \mathbb{C}$ would contain a rank-1 sub-local system with zero Higgs field. This implies that the Higgs field of $(p, g)$-type on $\wedge^2R^1a_*(Z_{A^0}) \otimes \mathbb{C}$ can not be isomorphism, a contradiction. We claim that $\mathbb{W}_1 \simeq \mathbb{W}_2$. Otherwise, the space $End(R^1a_*(Z_{A^0})) \otimes \mathbb{C}$ has at most dimension 2, a contradiction. Let $\mathbb{W}_1 \simeq \mathbb{W}_2 \simeq \mathbb{W}$. We have then

$$\mathcal{E}nd(R^1a_*(Z_{A^0})) \otimes \mathbb{C} \simeq \mathcal{E}nd(\mathbb{W}) \oplus \mathbb{C}^4 = \mathcal{E}nd_0(\mathbb{W}) \oplus \mathbb{C}^4 = \mathbb{W}' \oplus \mathbb{C}^4.$$

Since $\mathbb{W}$ is irreducible, one shows that $\mathbb{W}'$ does not contain any constant sub-local system and the last splitting can be defined over $\mathbb{Q}$. Hence,

$$\dim \mathcal{E}nd(R^1a_*(Z_{A^0})) \otimes \mathbb{Q} = 4.$$

**Lemma 5.2.** The family $a : A^0 \rightarrow C^0$ is isogenous to the square product of a family of elliptic curves $e : E^0 \rightarrow C^0$. 

\[\square\]
Proof. Case 1). Suppose that there is a subset \( T \subset C^0 \) of non-countable many points such that \( A_t \) is isogenous to \( E_t \times E_t, \ t \in T \). Since there are only countable many isomorphic classes of elliptic curves having complex multiplication, we find an \( t_0 \in T \) such that \( \text{End}(E_{t_0}) \otimes \mathbb{Q} = \mathbb{Q} \). Hence, the endomorphism algebra

\[
\text{End}(A_{t_0}) \otimes \mathbb{Q} \simeq M_2(\mathbb{Q}).
\]

In the other words, we have \( \text{End}(R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q})|_{t_0} \simeq M_2(\mathbb{Q}) \). Since

\[
\text{End}(R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q})
\]

is constant local system, we have \( \text{End}(R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q}) \simeq M_2(\mathbb{Q}) \). The element

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \in \text{End}(R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q})
\]

gives a \( \mathbb{Q} \)-splitting \( R^1a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q} = \mathbb{W}_{\mathbb{Q}} \oplus \mathbb{W}_{\mathbb{Q}} \), thus isogeny splitting of \( f : A^0 \to C^0 \) into the square product of a family of elliptic curves \( e : E^0 \to C^0 \).

Case 2). Suppose that there are non-countable many points \( \{t\} \subset C^0 \) such that \( A_t \) is simple. Since the Picard number \( \rho(A_t) \geq 3 \), one checks easily that \( \rho(A_t) = 3 \) and \( \text{End}(A_t) \otimes \mathbb{Q} \) is the totally indefinite quaternion algebra over \( \mathbb{Q} \). An abelian surface with this type endomorphism algebra is called a false elliptic curve. There are countable many projective curves \( \{C_i\}_{i \in \mathbb{N}} \) in the moduli space of polarized abelian surfaces, which are Shimura curves of certain type and parametrize all false elliptic curves. So, the family \( a : A^0 \to C^0 \) induces a morphism \( \phi : C^0 \to C_i \) for some \( i \in \mathbb{N} \), which extends to a morphism \( \phi : C \to C_i \). This implies that the local monodromies of \( R^2a_*(\mathbb{Z}_{A^0}) \) around the singularity has finite order. It contradicts to \( S \neq \emptyset \).

\[ \square \]

6. Proof of theorems and corollary

Proof of Theorem 0.1: Only the modularity of \( C' \setminus \sigma^{-1}S \) needs to be checked. The isogeny \( a : A^0 \to C^0 \sim e^2 : E^0 \times_{C^0} E^0 \to C^0 \) induces an isomorphism \( S^2(R^1e_*(\mathbb{Z}_{E^0})) \simeq \mathbb{W}|_{C^0} \). There are natural group homomorphisms

\[
1 \to \{\pm 1\} \to \text{SL}_2(\mathbb{R}) \to SO(1,2),
\]

which induce an isomorphism between \( \mathcal{H} \) and a connected component of the symmetric space \( SO(1,2)/SO(2) \times O(1), \) say

\[
i : \mathcal{H} \simeq SO^+(1,2)/SO(2) \times O(1).
\]

Since \( \mathbb{W}|_{C^0} \) is the restriction of \( \mathbb{W} \) on \( C \setminus S \) to \( C^0 \), the local monodromies of \( R^1e_*(\mathbb{Z}_{E^0}) \) around \( (C \setminus S) \setminus C^0 \) are either +1, or -1. Thus the projective monodromy representation of \( R^1e_*(\mathbb{Z}_{E^0}) \) is actually defined on \( C \setminus S \), say

\[
\rho_{R^1e_*(\mathbb{Z}_{E^0})} : \pi_1(C \setminus S, *) \to \text{PSL}_2(\mathbb{Z}).
\]

Let \( \tilde{\phi}_{R^1e_*(\mathbb{Z}_{E^0})} : \widetilde{C \setminus S} \to \mathcal{H} \) be the period map corresponding to \( R^1e_*, \mathbb{Z}_{E^0} \) and

\[
\tilde{\phi}_{\mathbb{W}} : \widetilde{C \setminus S} \to SO^+(1,2)/SO(2) \times O(1),
\]

denote the period map corresponding to \( \mathbb{W} \). Then \( \tilde{\phi}_{\mathbb{W}} = i \cdot \tilde{\phi}_{R^1e_*(\mathbb{Z}_{E^0})} \) is an isomorphism. In fact, the tangent map of \( \tilde{\phi}_{\mathbb{W}} \) is precisely the Kodaira-Spencer
map of $\mathcal{W}$: $\theta^{2,0} : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S)$, which is isomorphic at each point by Lemma 1.1. Thus $\tilde{\phi}_W$ is a local diffeomorphism. Since the Hodge metric on the Higgs bundle corresponding to $\mathcal{W}$ has logarithmic growth at $S$ and bounded curvature by Schmid [23], together with the remarks after Proposition 9.1 and Proposition 9.8 in [25], $\tilde{\phi}_W$ is a covering map, hence an isomorphism. This implies that $\tilde{\phi}_{R^1 e_*(Z_{E^0})}$ is an isomorphism. Thus

$$\phi_{R^1 e_*(Z_{E^0})} : C \setminus S \simeq \mathcal{H}/\rho_{R^1 e_*(Z_{E^0})}$$

is an isomorphism.

In order to prove Theorem 0.2, we need the following lemma.

**Lemma 6.1.** Let $f : X \to C$ be a family of semi-stable K3 surfaces, which has zero iterated Kodaira-Spencer map and reaches the Arakelov bound (II)

$$\deg f_* \omega_{X/C} = \frac{1}{2} \deg \Omega^1_C(\log S).$$

Then, after passing through a finite étale covering $C' \to C$, the VHS $\mathcal{W}$ is non-rigid.

**Proof.** One needs to show that, after passing through a finite étale covering of $C$, the local system $R^2 f_* (Z_{X^0}) \otimes \mathbb{C}$ admits a non-zero endomorphism of type $(-1,1)$. By Lemma 3.2, one has splitting

$$R^2 f_* (Z_{X^0}) \supset W \oplus \mathbb{Z}^{18}, \quad R^2 f_* (Z_{X^0}) \otimes \mathbb{Q} = (W \oplus \mathbb{Z}^{18}) \otimes \mathbb{Q}.$$  

By Lemma 1.2, the Higgs bundle corresponds to $\mathcal{W}$ has the form

$$(E^{2,0} \oplus E^{1,1}_1) \oplus (E^{1,1*}_1 \oplus E^{0,2})$$

such that the Higgs fields

$$\tau : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S), \quad \tau^* : E^{1,1*} \to E^{0,2} \otimes \Omega^1_C(\log S)$$

are isomorphisms. These two Higgs subbundles correspond to two sub-local systems $\mathcal{W}_1$ and $\bar{\mathcal{W}}_1$. We claim that, after passing through a finite étale covering of $C$, one has $\mathcal{W}_1 \simeq \bar{\mathcal{W}}_1$. To prove the claim, consider the sub-local system

$$\mathcal{W}_1 \to \mathcal{W}.$$  

If $\mathcal{W}_1$ is not rigid, then there is a small deformation $W_{1,t} \subset W \otimes \mathbb{C}$ such that both projections $W_{1,t} \subset W \otimes \mathbb{C} \to \mathcal{W}_1$ and $W_{1,t} \subset W \otimes \mathbb{C} \to \bar{\mathcal{W}}_1$ are non-zero. Since $\mathcal{W}_1$ is irreducible, one obtains

$$\mathcal{W}_1 \simeq W_{1,t} \simeq \mathcal{W}_1.$$  

If $\mathcal{W}_1$ is rigid, then by Lemma 2.1 $\mathcal{W}_1$ is defined over a number field $K$. Let $\mathcal{O}_K$ denote the ring of algebraic integers in $K$, and let

$$\mathcal{W}_1 \otimes \mathcal{O}_K = \mathcal{W} \otimes \mathcal{O}_K \cap \mathcal{W}_1.$$  

Then $\mathcal{W}_1 \otimes \mathcal{O}_K \cong \mathcal{W}_1$, which means that the corresponding monodromy representation of $\mathcal{W}_1$ can be defined over $\mathcal{O}_K$. The determinant $\det \mathcal{W}_1 = E^{2,0} \otimes E^{1,1}_1$ is a rank-1 unitary local system $\eta \in \text{Pic}^0(C)$ and takes values in
\[ O_K. \text{ By a theorem of Kronecker, } \eta \text{ is a torsion. So, after passing through the finite étale covering corresponding to } \eta, \text{ one obtains } E^{2,0} \simeq E^{1,1}, \text{ and } (E^{2,0} \oplus E^{1,1}, \tau) \simeq (E^{1,1,1} \oplus E^{0,2}, \tau^*). \]

Thus, in any case, we obtain a non-zero endomorphism

\[ (E^{2,0} \oplus E^{1,1}) \oplus (E^{1,1,1} \oplus E^{0,2}) \rightarrow (E^{2,0} \oplus E^{1,1,1}) \oplus (E^{1,1,1} \oplus E^{0,2}) \]

of type \((-1,1)\), which corresponds to an endomorphism of \( R^2 f_*(\mathbb{Z}_{X_0}) \otimes \mathbb{C} \) of type \((-1,1)\).

\[ \square \]

**Proof of Theorem 0.2:** By Lemma 6.1, after passing through a finite étale covering \( C' \rightarrow C \), the VHS \( \mathcal{W} \) is non-rigid. By Corollary 5.6.3 of [21], one has

\[ \text{End}(\mathcal{W}) \otimes \mathbb{Q} \simeq M_2(\mathbb{Q}). \]

Taking an element in \( M_2(\mathbb{Q}) \) with two distinct rational eigenvalues, we get a \( \mathbb{Q} \)-splitting \( \mathcal{W} \otimes \mathbb{Q} = \mathcal{W}_1 \oplus \mathcal{W}_2 \) such that \( \mathcal{W}_1 \) is isomorphic to \( \mathcal{W}_2 \) and the Higgs bundle corresponding to \( \mathcal{W}_1 \) has the form

\[ (L \oplus L^{-1}, \theta), \quad \theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S). \]

\( \mathcal{W}_1 \) has an \( \mathbb{Z} \)-structure defined by \( \mathcal{W}_{1Z} = \mathcal{W}_Z \cap \mathcal{W}_1 \). Again by Proposition 9.1 of [25], the Higgs bundle \( \theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S) \) gives rise to the uniformization

\[ C \setminus S \simeq \mathcal{H}/\rho_{\mathcal{W}_1} \pi_1(C \setminus S, *), \]

where \( \rho_{\mathcal{W}_1} \pi_1(C \setminus S, *) \subset SL_2(\mathbb{Z}) \) of finite index.

**Proof of Corollary 0.4** i) By Theorem 0.1 there exists a family of elliptic curves \( g : E^0 \rightarrow \mathbb{P}^{10} \subset \mathbb{P}^1 \setminus S \) such that the projective representation

\[ p \rho_{R^1 g_* \mathbb{Z}_{E^0}} : \pi_1(\mathbb{P}^1 \setminus S, *) \rightarrow \Gamma' \subset PSL_2(\mathbb{Z}) \]

extends to \( \mathbb{P}^1 \setminus S \) and \( \mathbb{P}^1 \setminus S \simeq \mathcal{H}/\Gamma' \). By [2], \( \Gamma' \subset PSL_2(\mathbb{Z}) \) is of index 12 and conjugates to one of the following 6 subgroups of \( PSL_2(\mathbb{Z}) \), which are images of \( \Gamma(3), \Gamma(5), \Gamma(6), \Gamma(8), \Gamma(10) \) and \( \Gamma(12) \) in \( SL_2(\mathbb{Z}) \) of index 24, where

\[ \Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \big| b \equiv c \equiv 0, a \equiv 1(\text{mod. } n) \right\}, \]

\[ \Gamma_0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \big| c \equiv 0, a \equiv 1(\text{mod. } n) \right\}, \]

\[ \Gamma_0(0) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \big| c \equiv 0(\text{mod. } n) \right\}. \]

In the proof of Theorem 0.1, we have seen already that the monodromy of \( R^1 g_* \mathbb{Z}_{E^0} \) of a short loop around a point of \( (\mathbb{P}^1 \setminus S) \setminus \mathbb{P}^{10} \) is either +1, or -1. If all of them equal to +1, then the representation \( \rho_{R^1 g_* \mathbb{Z}_{E^0}} \) extends to \( \mathbb{P}^1 \setminus S \), and the image of \( \pi_1(\mathbb{P}^1 \setminus S, *) \) under this representation conjugates to one of the above 6 subgroups. Hence \( g : E^0 \rightarrow \mathbb{P}^{10} \) extends to a modular family of elliptic curves \( g : E \rightarrow \mathbb{P}^1 \setminus S \) from one of 6 examples in [2]. Suppose that the monodromies of \( R^1 g_* \mathbb{Z}_{E^0} \) of short loops around some points of \( (\mathbb{P}^1 \setminus S) \setminus \mathbb{P}^{10} \)
equal to $-1$. Then the image of $\pi_1(\mathbb{P}^1 \setminus S, \ast)$ conjugates to the preimage $p^{-1} p \Gamma$, where $\Gamma$ is one of $\Gamma(3), \Gamma_0^0(4) \cap \Gamma(2), \Gamma_0^0(5), \Gamma_0^0(6), \Gamma_0(8) \cap \Gamma_0^0(4)$ and $\Gamma_0(9) \cap \Gamma_0^0(3)$.

The inclusion $\Gamma \subset p^{-1} p \Gamma$ of index 2 defines an étale covering $E_0' \rightarrow E_0$, which is étale along the fibres and the family $g': E_0' \rightarrow \mathbb{P}^1 \setminus S$ corresponding to $\Gamma$.

ii) is straightforward.

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