The Abresch-Rosenberg Shape Operator
and applications

José M. Espinar and Haimer A. Trejos

† Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil; e-mail: jespinar@impa.br
‡ Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil; e-mail: aletrejosserna@gmail.com

Abstract

There exists a holomorphic quadratic differential defined on any $H$–surface immersed in the homogeneous space $\mathbb{E}(\kappa, \tau)$ given by U. Abresch and H. Rosenberg [1, 2], called the Abresch-Rosenberg differential. However, there were no Codazzi pair on such $H$–surface associated to the Abresch-Rosenberg differential when $\tau \neq 0$. The goal of this paper is to find a geometric Codazzi pair defined on any $H$–surface in $\mathbb{E}(\kappa, \tau)$, when $\tau \neq 0$, whose $(2,0)$–part is the Abresch-Rosenberg differential.

In particular, this allows us to compute a Simons’ type formula for $H$–surfaces in $\mathbb{E}(\kappa, \tau)$. We apply such Simons’ type formula, first, to study the behavior of complete $H$–surfaces $\Sigma$ of finite Abresch-Rosenberg total curvature immersed in $\mathbb{E}(\kappa, \tau)$. Second, we estimate the first eigenvalue of any Schrödinger operator $L = \Delta + V$, $V$ continuous, defined on such surfaces. Finally, together with the Omori-Yau’s Maximum Principle, we classify complete $H$–surfaces in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, satisfying a lower bound on $H$ depending on $\kappa$ and $\tau$.

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1 Introduction.

W. P. Thurston proved that the building blocks of the Geometrization Conjecture are given by eight maximal model geometries, in other words, a maximal model geometry is a simply connected smooth manifold $\mathcal{M}$ together with a transitive action of the Lie group $G$ on $\mathcal{M}$, which is maximal among groups acting smoothly and transitively on $\mathcal{M}$, with compact stabilizers. Such maximal model geometries can be classified according to the dimension of its isometry group. If the dimension is 6, they correspond to the space forms $\mathbb{M}^3(\kappa)$. When the dimension is 3, the manifold has the geometry of the Lie group $\text{SO}(3)$, and when the dimension is 4, they correspond to a 2-parameter family $\kappa, \tau \in \mathbb{R}$, $\kappa - 4\tau^2 \neq 0,$
of manifolds denoted by $E(\kappa, \tau)$. These manifolds correspond to the product spaces $M^2(\kappa) \times \mathbb{R}$, when $\kappa \neq 0, \tau = 0$, where $M^2(\kappa)$ is the simply connected surface of constant curvature $\kappa$. The Heisenberg space $\text{Nil}_3$, when $\kappa = 0, \tau \neq 0$. The Berger sphere $S^3_B(\kappa, \tau)$, when $\kappa > 0, \tau \neq 0$. It is known that $E(\kappa, \tau)$ is a Riemannian submersion over $M^2(\kappa)$ with fiber bundle curvature $\tau$ and the fibers are integral curves of a unit Killing field defined in $E(\kappa, \tau)$.

Constant mean curvature surfaces $\Sigma$, in short, $H$–surfaces, immersed in homogeneous 3-spaces $E(\kappa, \tau)$ are receiving an impressive number of contributions since U. Abresch and H. Rosenberg [1, 2] showed the existence of a holomorphic quadratic differential, the Abresch-Rosenberg differential, on such surfaces. Hence, they were able to extend the well-known Hopf’s Theorem, i.e., they classified the topological spheres with constant mean curvature as the rotationally symmetric ones in $E(\kappa, \tau)$.

When $\Sigma$ is isometrically immersed in a space form $M^3(\kappa)$, the second fundamental form defines a bilinear symmetric tensor that satisfies the Codazzi equation. The Codazzi equation is fundamental to ensure that the usual Hopf differential is holomorphic on any $H$–surface in a space form; nevertheless, the above fact is no longer true when the surface is isometrically immersed in a homogeneous 3-manifold $E(\kappa, \tau)$. However, J.A. Aledo, J.M. Espinar and J.A. Gálvez [3] obtained a geometric Codazzi pair $(I, II_S)$ on any $H$–surface in $M^2(\kappa) \times \mathbb{R}$ so that the $(2, 0)$-part of $II_S$ with respect to the conformal structure given by the first fundamental form $I$, is the Abresch-Rosenberg differential. However, up to now, despite the existence of a holomorphic quadratic differential on any $H$–surface in $E(\kappa, \tau)$, $\tau \neq 0$, there were no natural (or geometric) Codazzi pair on such $H$–surfaces.

In [27], J. Simons computed the Laplacian of the squared norm of the second fundamental form of a minimal submanifold in $\mathbb{S}^n$. Nowadays, such formula is known as the Simons’ formula and it is a powerful tool to obtain classification theorems for minimal hypersurfaces. Later, Simons’ formula has been extensively generalized by many authors in different situations (cf. [5, 11, 12, 18]). In [10], S.Y. Cheng and S.T. Yau computed an abstract version of the Simons’ formula, that is, they computed the Laplacian of the squared norm of any bilinear symmetric Codazzi $(2, 0)$–tensor defined on a Riemannian manifold $(\mathcal{M}^n, g)$. Specifically, if $\Phi$ is a bilinear symmetric Codazzi $(2, 0)$–tensor whose local expression in a orthonormal coframe $\{\omega_1, \ldots, \omega_n\}$ is given by $\Phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$, then the Laplacian of $|\Phi|^2$ is given by

$$\frac{1}{2} \Delta |\Phi|^2 = \sum_{i,j,k} (\phi_{ij,k})^2 + \sum_i \lambda_i (\text{tr}(\Phi))_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2, \quad (1)$$

where $R_{ijij}$ is the Riemann curvature tensor associated to the metric $g$, $\text{tr}(\Phi)$ is the trace operator, $\lambda_i$, $i = 1, \ldots, n$, are the eigenvalues of $\Phi$ for an orthonormal frame $\{e_1, \ldots, e_n\}$ and $\Phi$ is a Codazzi tensor, that means, $\phi_{ij,k} = \phi_{ik,j}$.

A key ingredient to compute (1) is that $\Phi$ is Codazzi. So, when the hypersur-
face is isometrically immersed in a space form $\mathbb{M}^n(\kappa)$, the second fundamental form satisfies the Codazzi equation, hence we can recover Simons’ original formula. As said above, when $\tau = 0$, the existence of a holomorphic quadratic differential on a $H$–surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ implies the existence of a pair $(I, II_S)$ defined on this $H$–surface, where $I$ is the induced metric of $\mathbb{M}^2(\kappa) \times \mathbb{R}$ and $II_S$ is a symmetric Codazzi $(2, 0)$–tensor with respect to the connection induced by $I$. These type of pairs $(I, II_S)$ are known as Codazzi pairs (cf. [3, 24]) and this property of $(I, II_S)$ were used by M. Batista in [5] to compute a Simons’ type formula for $H$–surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

One of the main points of this paper is to obtain a geometric Codazzi pair $(I, II_{AR})$, where $II_{AR}$ is a symmetric $(2, 0)$–tensor, that we call the Abresch-Rosenberg fundamental form, on any $H$–surface in $\mathbb{E}(\kappa, \tau)$ whose $(2, 0)$–part with respect to the conformal structure induced by $I$ is the Abresch-Rosenberg differential. In other words, $(I, II_{AR})$ is a Codazzi pair on any $H$–surface. In particular, this allows us to compute a Simons’ type formula for $H$–surfaces in $\mathbb{E}(\kappa, \tau)$, i.e., those $H$–surfaces whose $L^2$–norm of the traceless Abresch-Rosenberg fundamental form is finite. Observe that complete $H$–surfaces $\Sigma \subset \mathbb{R}^3$ of finite total curvature, that is, those whose $L^2$–norm of its traceless second fundamental form is finite, are of capital importance on the comprehension of $H$–surfaces in $\mathbb{R}^3$. In the case $H = 0$, Osserman’s Theorem gives an impressive description of them. If $\Sigma$ has constant nonzero mean curvature and finite total curvature, then it must be compact. In our case, we extend the latter result when $H$ is greater than a constant depending only on $\kappa$ and $\tau$. We also estimate the first eigenvalue of any Schrödinger operator $L = \Delta + V$, $V$ continuous, defined on $H$–surfaces of finite Abresch-Rosenberg total curvature. Finally, together with the Omori-Yau’s Maximum Principle, we classify complete $H$–surfaces (not necessarily of finite Abresch-Rosenberg total curvature) in $\mathbb{E}(\kappa, \tau)$, $\tau \neq 0$.

1.1 Outline of the paper

In Section 2, we set up the notation and we review some of the standard facts on Codazzi pairs.

Section 3 is devoted to the Codazzi pair interpretation of the Abresch-Rosenberg differential and its geometric properties. First, we discuss the known case of $H$–surfaces in a product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Later, we obtain a geometric Codazzi pair associated to the Abresch-Rosenberg differential on any $H$–surface immersed in $\mathbb{E}(\kappa, \tau)$ when $\tau \neq 0$. Specifically, Lemma 7 says

**Key Lemma.** Given a $H$–surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, consider the symmetric $(2, 0)$–tensor given by

$$II_{AR}(X, Y) = II(X, Y) - \alpha \langle T_\theta, X \rangle \langle T_\theta, Y \rangle + \frac{\alpha |T|^2}{2} \langle X, Y \rangle,$$

where
Then, \((I, II_{AR})\) is a Codazzi pair with constant mean curvature \(H\). Moreover, the \((2,0)\)-part of \(II_{AR}\) with respect to the conformal structure given by \(I\) agrees (up to a constant) with the Abresch-Rosenberg differential.

In Section 4, we obtain a Simons’ type formula on any \(H\)-surface in \(\mathbb{E}(\kappa, \tau)\), \(\tau \neq 0\) (cf. [5] when \(\tau = 0\)). To do so, we use the Codazzi pair defined on the Key Lemma and [10],

\textbf{Theorem 3.} Let \(\Sigma\) be a \(H\)-surface in \(\mathbb{E}(\kappa, \tau)\). Then, the traceless Abresch-Rosenberg shape operator satisfies

\[
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + 2K|S|^2,
\]

or, equivalently, away from the zeroes of \(|S|\),

\[
|S| \Delta |S| - 2K|S|^2 = |\nabla |S||^2.
\]

Section 5 is devoted to complete \(H\)-surfaces in \(\mathbb{E}(\kappa, \tau)\) of finite Abresch-Rosenberg total curvature, i.e., those \(H\)-surfaces that satisfy

\[
\int_\Sigma |S|^2 < +\infty.
\]

We must point out here that the family of complete constant mean curvature surfaces of finite Abresch-Rosenberg total curvature is large. We focus on \(H = 1/2\) surfaces in \(\mathbb{H}^2 \times \mathbb{R}\) to show this fact. Recall the following result of Fernández-Mira:

\textbf{Theorem [17, Theorem 16].} Any holomorphic quadratic differential on an open simply connected Riemann surface is the Abresch-Rosenberg differential of some complete surface \(\Sigma\) with \(H = 1/2\) in \(\mathbb{H}^2 \times \mathbb{R}\). Moreover, the space of noncongruent complete mean curvature one half surfaces in \(\mathbb{H}^2 \times \mathbb{R}\) with the same Abresch-Rosenberg differential is generically infinite.

We will see that, if we take the disk \(\mathbb{D}\) as our open Riemann surface and a holomorphic quadratic differential on \(\mathbb{D}\) that extends continuously to the boundary, then the \(H = 1/2\) surface \(\Sigma\) constructed in [17, Theorem 16] has finite Abresch-Rosenberg total curvature.

The above examples also show that, despite what happens in \(\mathbb{R}^3\), \(H\)-surfaces \(\Sigma \subset \mathbb{E}(\kappa, \tau)\) of finite Abresch-Rosenberg total curvature are not necessarily conformally equivalent to a compact surface minus a finite number of points, in particular, \(\Sigma\) is not necessarily parabolic. However, we can obtain (cf. [6] when \(\tau = 0\)):
Theorem 6. Let $\Sigma$ be a complete surface in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, of finite Abresch-Rosenberg total curvature. Suppose one of the following conditions holds

1. $\kappa - 4\tau^2 > 0$ and $H^2 + \tau^2 > \frac{\kappa - 4\tau^2}{4}$.
2. $\kappa - 4\tau^2 < 0$ and $H^2 + \tau^2 > -\frac{\sqrt{3} + 2}{4}(\kappa - 4\tau^2)$.

Then, $\Sigma$ is compact.

Also, we extend Simons’ first stability eigenvalue estimate (cf. [27]) to Schrödinger operators $L = \Delta + V$ defined on a complete $H$–surface of finite Abresch-Rosenberg total curvature, $H^2 + \tau^2 \neq 0$, immersed in $\mathbb{E}(\kappa, \tau)$.

Theorem 7. Let $\Sigma$ be a complete two-sided $H$–surface in $\mathbb{E}(\kappa, \tau)$ of finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 \neq 0$. Denote by $\lambda_1(L)$ the first eigenvalue associated to the Schrödinger operator $L := \Delta + V$, $V \in C^0(\Sigma)$. Then, $\Sigma$ is either an Abresch-Rosenberg surface, a Hopf cylinder or

$$\lambda_1(L) < -\inf_{\Sigma} \{V + 2K\}.$$ 

In particular, when $L$ is the Stability (or Jacobi) operator, i.e.,

$$L = \Delta + (|A|^2 + \text{Ric}(N)),$$

where $\text{Ric}(N)$ is the Ricci curvature of the ambient manifold in the normal direction, we obtain the following (cf. [4] when $\Sigma$ is closed):

Theorem 8. Let $\Sigma$ be a complete two sided $H$–surface of finite Abresch-Rosenberg total curvature in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$.

- If $\kappa - 4\tau^2 > 0$, then $\Sigma$ is either an Abresch-Rosenberg $H$–surface, a Hopf cylinder, or
  $$\lambda_1 < -(4H^2 + \kappa).$$

- If $\kappa - 4\tau^2 < 0$, then $\Sigma$ is either an Abresch-Rosenberg $H$–surface, or
  $$\lambda_1 < -(4H^2 + \kappa) - (\kappa - 4\tau^2).$$

Finally, in Section 6, we apply the Simons’ type formula to classify complete $H$–surfaces in $\mathbb{E}(\kappa, \tau)$ under natural geometric conditions using the Omori-Yau’s Maximum Principle. We can summarize Theorem 10 and Theorem 11 as follows (cf. [5] when $\tau = 0$):

Theorems 10 and 11. Let $\Sigma$ be a complete immersed $H$–surface in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$.

- If $\kappa - 4\tau^2 > 0$, assume that $4(H^2 + \tau^2) > \kappa - 4\tau^2$ and
  $$\sup_{\Sigma} |S| < \sqrt{2}\sqrt{(H^2 + \tau^2) + (\kappa - 4\tau^2)},$$
where $S$ is the traceless Abresch-Rosenberg shape operator. Then, $\Sigma$ is an Abresch-Rosenberg surface in $\mathbb{E}(\kappa, \tau)$. Moreover, if
\[
\sup_{\Sigma} |S| = \sqrt{2}\sqrt{(H^2 + \tau^2) + (\kappa - 4\tau^2)}
\]
and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S|$, then $\Sigma$ is a Hopf cylinder.

- If $\kappa - 4\tau^2 < 0$, assume that $(H^2 + \tau^2) > |\kappa - 4\tau^2|$ and
\[
\sup_{\Sigma} |S| < -|\alpha| + \sqrt{2(H^2 + \tau^2) + \frac{\alpha^2}{2}},
\]
where $S$ is the traceless Abresch-Rosenberg shape operator. Then, $\Sigma$ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa, \tau)$.
Moreover, if
\[
\sup_{\Sigma} |S| = -|\alpha| + \sqrt{2(H^2 + \tau^2) + \frac{\alpha^2}{2}}
\]
and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S|$, then $\Sigma$ is a Hopf cylinder.

2 Preliminaries

Here, we mainly follow [3, 5, 13, 14, 15, 24, 26].

2.1 Homogeneous Riemannian Manifolds $\mathbb{E}(\kappa, \tau)$.

The simply connected homogeneous manifold $\mathbb{E}(\kappa, \tau)$ is a Riemannian submersion $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ over a simply connected surface of constant curvature $\kappa$. The fibers, i.e. the inverse image of a point at $\mathbb{M}^2(\kappa)$ by $\pi$, are the trajectories of a unitary Killing field $\xi$, called the vertical vector field.

Denote by $\nabla$ the Levi-Civita connection of $\mathbb{E}(\kappa, \tau)$, then for all $X, Y, Z, W \in \mathfrak{X}(\mathbb{E}(\kappa, \tau))$, we have:
\[
\nabla_X \xi = \tau X \wedge \xi,
\]
where $\tau$ is the bundle curvature. Note that $\tau = 0$ implies that $\mathbb{E}(\kappa, \tau)$ is a product space. Denote by $\tilde{R}$ the Riemann curvature tensor of $\mathbb{E}(\kappa, \tau)$, then,

**Lema 1** ([13]). Let $\mathbb{E}(\kappa, \tau)$ be a homogeneous space with unit Killing field $\xi$. For all vector fields $X, Y, Z, W \in \mathfrak{X}(\mathbb{E}(\kappa, \tau))$, we have:
\[
\langle \tilde{R}(X, Y)Z, W \rangle = (\kappa - 3\tau^2) \langle (X, Z)Y - (Y, Z)X \rangle
\]
\[
+ (\kappa - 4\tau^2) \left( \langle \xi, Y \rangle \langle \xi, Z \rangle X - \langle \xi, X \rangle \langle \xi, Z \rangle Y \right)
\]
\[
- (\kappa - 4\tau^2) \left( \langle Z, Y \rangle \langle \xi, X \rangle \xi + \langle Z, X \rangle \langle \xi, Y \rangle \xi \right).
\]
2.2 Immersed surfaces in $\mathbb{E}(\kappa, \tau)$.

Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be an oriented immersed connected surface. We endow $\Sigma$ with the induced metric of $\mathbb{E}(\kappa, \tau)$, called the first fundamental form, which we still denote by $\langle \cdot, \cdot \rangle$. Denote by $\nabla$ and $R$ the Levi-Civita connection and the Riemann curvature tensor of $\Sigma$ respectively. Also, denote by $A$ the shape operator,

$$AX = -\nabla_X N$$

for all $X \in \mathfrak{X}(\Sigma)$, where $N$ is the unit normal vector field along the surface $\Sigma$. Then $II(X, Y) = \langle AX, Y \rangle$ is the second fundamental form of $\Sigma$.

Moreover, denote by $J$ the oriented rotation of angle $\pi/2$ on $T\Sigma$, $JX = N \wedge X$ for all $X \in \mathfrak{X}(\Sigma)$.

Set $\nu = \langle N, \xi \rangle$ and $T = \xi - \nu N$, then, $\nu$ is the normal component of the vertical vector field $\xi$, called the angle function, and $T$ is a vector field in $\mathfrak{X}(\Sigma)$ called the tangent component of the vertical vector field $\xi$.

Note that, in a product space $M^2(\kappa) \times \mathbb{R}$, we have a natural projection onto the fiber $\sigma : \Sigma \to \mathbb{R}$, hence we can define the restriction of $\sigma$ to the surface $\Sigma$, that is, $h : \Sigma \to \mathbb{R}$, $h = \sigma|_{\Sigma}$. The function $h$ is called the height function of $\Sigma$. So, in $M^2(\kappa) \times \mathbb{R}$, one can easily observes that $\nabla\sigma = \xi$ and hence, $T$ is the projection of $\nabla\sigma$ onto the tangent plane, $\nabla h = T$.

Lema 2 ([13]). Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be an immersed surface with unit normal vector field $N$ and shape operator $A$. Let $T$ and $\nu$ be the tangent component of the vertical vector field and the angle function respectively. Then, given $X, Y \in \mathfrak{X}(\Sigma)$, the following equations hold:

$$K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2,$$

$$T_S(X, Y) = (\kappa - 4\tau^2)\nu(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

$$\nabla_X T = \nu(AX - \tau JX),$$

$$d\nu(X) = \langle \tau JX - AX, T \rangle,$$

$$\|T\|^2 + \nu^2 = 1,$$

where $K$ denotes the Gaussian curvature of $\Sigma$, $K_e$ denotes the extrinsic curvature and $T_S$ is the tensor given by:

$$T_S(X, Y) = \nabla_X AY - \nabla_Y AX - A([X, Y]), \quad X, Y \in \mathfrak{X}(\Sigma).$$

2.3 $H-$Surfaces in $\mathbb{E}(\kappa, \tau)$ and the Abresch-Rosenberg differential.

Let $\Sigma$ be an orientable complete connected $H-$surface immersed in $\mathbb{E}(\kappa, \tau)$. In terms of a local conformal parameter $z$, the first fundamental form $I = \langle \cdot, \cdot \rangle$ and the second fundamental form are given by

$$I = 2\lambda|dz|^2,$$

$$II = Qdz^2 + 2\lambda H|dz|^2 + \overline{Q}d\overline{z}^2,$$

where $Qdz^2 = -\langle \nabla_{\partial_1} N, \partial_2 \rangle dz^2$ is the usual Hopf differential of $\Sigma$. Hence, in this conformal coordinate, the above Lemma reads as:
Lemma 3 ([14, 15, 17]). Given an immersed $H-$surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, the following equations are satisfied:

\[
K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2 \quad \tag{8}
\]
\[
Q_z = \lambda(\kappa - 4\tau^2)\nu t \quad \tag{9}
\]
\[
t_z = \frac{\lambda}{\lambda} t + Q\nu \quad \tag{10}
\]
\[
t_{\bar{z}} = \lambda(H + i\tau)\nu \quad \tag{11}
\]
\[
\nu_z = -(H - i\tau)t - \frac{Q}{\lambda} \bar{t} \quad \tag{12}
\]
\[
|t|^2 = \frac{1}{2}(1 - \nu^2), \quad \tag{13}
\]

where $t = \langle T, \partial_z \rangle$, $\bar{t} = \langle T, \partial_{\bar{z}} \rangle$, $K_e$ is the extrinsic curvature and $K$ is the Gaussian curvature.

For an immersed $H-$surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, there is a globally defined quadratic differential, called the Abresch-Rosenberg differential.

Definition 1 ([1, 2]). Given a local conformal parameter $z$ for $I$, the Abresch-Rosenberg differential is defined by:

\[
Q^{AR} = Q^{AR}dz^2 = (2(H + i\tau)Q - (\kappa - 4\tau^2)t^2)dz^2,
\]

moreover, associated to the Abresch-Rosenberg differential we define the Abresch-Rosenberg map $q^{AR} : \Sigma \to [0, +\infty)$ by:

\[
q^{AR} = \frac{|Q^{AR}|^2}{4\lambda^2}.
\]

Note that $Q^{AR}$ and $q^{AR}$ do not depend on the conformal parameter $z$, hence $Q^{AR}$ and $q^{AR}$ are globally defined on $\Sigma$.

Then using Lemma 3, we can show

Theorem 1 ([1, 2]). Let $\Sigma$ be a $H-$surface in $\mathbb{E}(\kappa, \tau)$, then the Abresch-Rosenberg differential $Q^{AR}$ is holomorphic for the conformal structure induced by the first fundamental form $I$.

2.4 Codazzi Pairs on Surfaces

We shall denote by $\Sigma$ an orientable (and oriented) smooth surface.

Definition 2. A fundamental pair on $\Sigma$ is a pair of real quadratic forms $(I, II)$ on $\Sigma$, where $I$ is a Riemannian metric.

Associated with a fundamental pair $(I, II)$ we define the shape operator $S$ of the pair as:

\[
II(X, Y) = I(S(X), Y) \text{ for any } X, Y \in \mathfrak{X}(\Sigma). \quad \tag{14}
\]

Conversely, it is clear from (14) that the quadratic form $II$ is totally determined by $I$ and $S$. In other words, a fundamental pair on $\Sigma$ is equivalent to a Riemannian metric on $\Sigma$ together with a self-adjoint endomorphism $S$. 

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We define the **mean curvature**, the **extrinsic curvature** and the **principal curvatures** of the fundamental pair \((I, II)\) as one half of the trace, the determinant and the eigenvalues of the endomorphism \(S\), respectively.

In particular, given local parameters \((x, y)\) on \(\Sigma\) such that

\[
I = E\,dx^2 + 2F\,dxdy + G\,dy^2, \quad II = e\,dx^2 + 2f\,dxdy + g\,dy^2;
\]

the mean curvature and the extrinsic curvature of the pair are given, respectively, by

\[
H(I, II) = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \quad K_e(I, II) = \frac{eg - f^2}{EG - F^2},
\]

moreover, the principal curvatures of the pair are

\[
H(I, II) \pm \sqrt{H(I, II)^2 - K_e(I, II)}.
\]

We shall say that the fundamental pair \((I, II)\) is **umbilical** at \(p \in \Sigma\), if

- if both principal curvatures coincide at \(p\), or
- if \(S\) is proportional to the identity map on the tangent plane at \(p\), or
- if \(H(I, II)^2 - K_e(I, II) = 0\) at \(p\).

We define the **Hopf differential** of the fundamental pair \((I, II)\) as the \((2,0)\)-part of \(II\) for the Riemannian metric \(I\). In other words, if we consider \(\Sigma\) as a Riemann surface with respect to the metric \(I\) and take a local conformal parameter \(z\), then we can write

\[
I = 2\lambda |dz|^2, \quad II = Q\,dz^2 + 2\lambda H |dz|^2 + \overline{Q}\,d\bar{z}^2. \tag{15}
\]

The quadratic form \(Q\,dz^2\), which does not depend on the chosen parameter \(z\), is known as the **Hopf differential** of the pair \((I, II)\). We note that \((I, II)\) is umbilical at \(p \in \Sigma\) if and only if \(Q(p) = 0\).

**Remark 1.** *All the above definitions can be understood as natural extensions of the corresponding ones for isometric immersions of a Riemannian surface in a 3-dimensional ambient space, where \(I\) plays the role of the induced metric and \(II\) the role of its second fundamental form.*

A specially interesting case happens when the fundamental pair satisfies the Codazzi equation, that is,

**Definition 3.** *We say that a fundamental pair \((I, II)\), with shape operator \(S\), is a Codazzi pair if*

\[
\nabla_X SY - \nabla_Y SX - S[X, Y] = 0, \quad X, Y \in \mathfrak{X}(\Sigma), \tag{16}
\]

*where \(\nabla\) stands for the Levi-Civita connection associated to the Riemannian metric \(I\) and \(\mathfrak{X}(\Sigma)\) is the set of smooth vector fields on \(\Sigma\).*

Let us also observe that, from (15) and (16), a fundamental pair \((I, II)\) is a Codazzi pair if and only if

\[
Q_z = \lambda H_z.
\]

Thus, one has:
Lemma 4 ([24]). Let \((I, II)\) be a fundamental pair. Then, any two of the conditions (i), (ii), (iii) imply the third:

(i) \((I, II)\) is a Codazzi pair.

(ii) \(H\) is constant.

(iii) The Hopf differential is holomorphic.

Remark 2. We observe that Hopf's Theorem [23, p. 138], on the uniqueness of round spheres among immersed constant mean curvature spheres in Euclidean 3-space, can be easily obtained from this result.

3 Abresch-Rosenberg Differential and Codazzi Pairs

One of the main points in the work of Abresch-Rosenberg [1, 2] is to prove that the quadratic differential \(Q^{AR}\) given in Definition 1 is holomorphic on any \(H\)-surface \(\Sigma\) immersed in \(E(\kappa, \tau)\). Hence, it is easy to see that such quadratic differential must vanish on a topological sphere by the Poincaré-Hopf Index Theorem. Then, the authors showed that if the Abresch-Rosenberg differential vanishes on a \(H\)-surface then it must be invariant under certain one parameter subgroup of isometries of the ambient manifold \(E(\kappa, \tau)\). In particular, when the surface is a topological sphere, it must be rotationally symmetric. Specifically, they proved:

**Theorem 2** ([1, 2]). The only topological spheres in \(E(\kappa, \tau)\) with constant mean curvature \(H\) are the rotational invariant spheres.

Lemma 4 tells us that the existence of a holomorphic quadratic differential should imply the existence of a Codazzi pair on any \(H\)-surface \(\Sigma\) in \(E(\kappa, \tau)\). When \(E(\kappa, \tau)\) is a product manifold, i.e. \(\tau = 0\), such Codazzi pair was found a long time ago (cf. [3]). Our goal here is to obtain a Codazzi pair on any \(H\)-surface such that the Abresch-Rosenberg differential appears as its Hopf differential. First, we recover the case \(\tau = 0\) since it will enlighten the case \(\tau \neq 0\).

3.1 \(H\)-surfaces in \(E(\kappa, \tau)\) with \(\tau = 0\).

Consider a complete immersed \(H\)-surface \(\Sigma \subset M^2(\kappa) \times \mathbb{R}\). According to the notation introduced above, we define the self-adjoint endomorphism \(S\) on \(\Sigma\) as

\[
SX = 2HAX - \kappa(X, T)T + \frac{\kappa}{2} |T|^2 X - 2H^2 X,
\]

where \(X \in \mathfrak{X}(\Sigma)\), \(A\) is the Weingarten operator associated to the second fundamental form, \(T\) is the tangential component of the vertical vector field \(\partial_t\) defined in \(M^2(\kappa) \times \mathbb{R}\).

Consider the quadratic form \(II_S\) associated to \(S\) given by (17). In [3], it was shown that \((I, II_S)\) is a Codazzi pair on \(\Sigma\) if \(H\) is constant. Moreover, it is traceless, i.e., \(\text{tr}(S) = 0 = H(I, II_S)\), and the Hopf differential associated to \((I, II_S)\) is the Abresch-Rosenberg differential \(Q^{AR}\) in \(M^2(\kappa) \times \mathbb{R}\).
3.2 $H$–surfaces in $\mathbb{E}(\kappa, \tau)$ with $\tau \neq 0$.

The main point in this section is to show that the Abresch-Rosenberg differential has an interpretation in terms of a Codazzi pair defined on any $H$–surface in $\mathbb{E}(\kappa, \tau)$ when $\tau \neq 0$. In this case, we have that $H^2 + \tau^2 > 0$. Define $\theta \in [0, 2\pi)$ by

$$e^{2i\theta} = \frac{H - i\tau}{\sqrt{H^2 + \tau^2}}.$$

Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a $H$–surface and $z$ be a local conformal parameter. Then, up to the complex constant $H + i\tau$, we can re-define the Abresch-Rosenberg differential as:

$$Q^{AR}dz^2 = \left( Q - \frac{\kappa - 4\tau^2}{2(H + i\tau)}t^2 \right)dz^2.$$

One can re-write the above differential as:

$$Q^{AR}dz^2 = \left( Q - \frac{\kappa - 4\tau^2}{2\sqrt{H^2 + \tau^2}}(e^{i\theta}t)^2 \right)dz^2.$$

Given the tangential vector field $T$, define $T_\theta = \cos \theta T + \sin \theta J T$, then

$$Q^{AR}dz^2 = \left( \langle A\partial_z, \partial_z \rangle - \alpha \langle T_\theta, \partial_z \rangle^2 \right)dz^2,$$

where $\alpha = \frac{\kappa - 4\tau^2}{2\sqrt{H^2 + \tau^2}}$ and $A$ is the usual shape operator. This leads us to the following definition:

**Definition 4.** Given a $H$–surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, the Abresch-Rosenberg fundamental form is defined by:

$$II_{AR}(X,Y) = II(X,Y) - \alpha \langle T_\theta, X \rangle \langle T_\theta, Y \rangle + \frac{\alpha |T|^2}{2} \langle X,Y \rangle, \quad (18)$$

or equivalently, the Abresch-Rosenberg shape operator $S_{AR}$ is defined by:

$$S_{AR}X = A(X) - \alpha \langle T_\theta, X \rangle T_\theta + \frac{\alpha |T|^2}{2}X - HX, \quad (19)$$

or, the traceless Abresch-Rosenberg shape operator $S$ is defined by:

$$SX = S_{AR}X - HX = A(X) - \alpha \langle T_\theta, X \rangle T_\theta + \frac{\alpha |T|^2}{2}X - HX, \quad (20)$$

where $X,Y \in \mathfrak{X}(\Sigma)$.

First, we examine the geometric properties of the above quadratic form and its relation to the Abresch-Rosenberg differential:

**Proposition 5.** The following equations hold for the fundamental pair $(I, II_{AR})$:

1. $II_{AR}(\partial_z, \partial_z)dz^2 = Q^{AR}dz^2$, where $z$ is a local conformal parameter for $I$. 

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2. \( H(I, II_{AR}) = H(I, II) \).

3. \( K_e(I, II_{AR}) = K_e(I, II) + \alpha (ST_\theta, T_\theta) + \frac{\alpha^2 |T|^4}{4} \).

Moreover, the squared norm of the shape operator \( |A|^2 \) and the squared norm of the traceless Abresch-Rosenberg shape operator \( |S|^2 \) satisfy
\[
|A|^2 = |S|^2 + 2\alpha (ST_\theta, T_\theta) + \frac{\alpha^2}{2} |T|^4 + 2H^2.
\]

Moreover, it holds
\[
\frac{|T|^4}{4} - \frac{(ST_\theta, T_\theta)^2}{|S|^2} = \frac{(ST_\theta, JT_\theta)^2}{|S|^2}.
\]

Proof. Consider a local conformal parameter \( z \) for \( I \). A straightforward computation shows
\[
II_{AR}(\partial_x, \partial_z)dz^2 = Q_{AR}dz^2
\]
and
\[
H(I, II_{AR}) = H(I, II).
\]

Let us compute \( K_e(I, II_{AR}) \). It is clear that \( |T_\theta| = |JT_\theta| = |T| \), then:
\[
II_{AR}(T_\theta, T_\theta) = II(T_\theta, T_\theta) - \frac{\alpha |T|^4}{2}.
\]
\[
II_{AR}(T_\theta, JT_\theta) = II(T_\theta, JT_\theta).
\]
\[
II_{AR}(JT_\theta, JT_\theta) = II(JT_\theta, JT_\theta) + \frac{\alpha |T|^4}{2}.
\]

From the definition of the Abresch-Rosenberg quadratic form, we have \( K_e(I, II_{AR}) = K_e(I, II) \) on the set \( \mathcal{U} = \{ p \in \Sigma : (T_\theta)_p = 0 \} \). Then, take \( p \in \Sigma \setminus \mathcal{U} \) and consider the orthonormal basis in \( T_p\Sigma \) defined by:
\[
e_1 = \frac{T_\theta}{|T|} \text{ and } e_2 = \frac{JT_\theta}{|T|}.
\]

From (23), we obtain:
\[
II_{AR}(e_1, e_1) - II_{AR}(e_2, e_2) = II(e_1, e_1) - II(e_2, e_2) - \alpha |T|^2,
\]
and, since \( \{e_1, e_2\} \) is orthonormal at \( p \) and (23), we have
\[
K_e(I, II_{AR}) = II_{AR}(e_1, e_1)II_{AR}(e_2, e_2) - II_{AR}(e_1, e_2)^2
\]
\[
= II(e_1, e_1)II(e_2, e_2) - II(e_1, e_2)^2
\]
\[
+ \frac{\alpha}{2} (II(e_1, e_1) - II(e_2, e_2)) |T|^2 - \frac{\alpha^2}{4} |T|^4.
\]

On the one hand, substituting (24) into the above formula of \( K_e(I, II_{AR}) \) and simplifying terms, we get at \( p \):
\[
K_e(I, II_{AR}) = K_e(I, II) + \frac{\alpha}{2} (II_{AR}(T_\theta, T_\theta) - II_{AR}(JT_\theta, JT_\theta)) + \frac{\alpha^2}{4} |T|^4.
\]
On the other hand, recall that $S$ is traceless and hence, at a point $p \in \Sigma$, we can consider an orthonormal basis $\{E_1, E_2\}$ of principal directions for $S$, i.e,

$$SE_1 = \lambda E_1, \ SE_2 = -\lambda E_2 \text{ and } |S|^2 = 2\lambda^2.$$ 

Then, there exists $\beta \in [0, 2\pi)$ such that

$$T_\theta = |T|(\cos \beta E_1 + \sin \beta E_2),$$

and hence, one can easily check

$$\langle ST_\theta, T_\theta \rangle = -\langle SJT_\theta, JT_\theta \rangle.$$ \hspace{1cm} (26)

Hence, from (20) and (26), we have

$$II_{AR}(T_\theta, T_\theta) - II_{AR}(JT_\theta, JT_\theta) = \langle ST_\theta, T_\theta \rangle - \langle SJT_\theta, JT_\theta \rangle$$

$$= 2\langle ST_\theta, T_\theta \rangle,$$

thus, substituting the last equation into (25) yields the expression for $K_e(I, II_{AR})$.

Also, a straightforward computation using (26) gives that

$$\frac{|T|^4}{2} - \frac{\langle ST_\theta, T_\theta \rangle^2}{|S|^2} = \frac{\langle ST_\theta, JT_\theta \rangle^2}{|S|^2},$$

which shows (22).

Finally, (21) can be easily obtained by observing that $|A|^2 = 4H^2 - 2K_e$ and $|S|^2 = 2q^{AR} = 2(H^2 - K_e(I, II_{AR})).$ \hfill \Box

Hence, Lemma 4 implies:

**Lema 6.** Given any $H-$surface in $\mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, it holds:

$Q^{AR}$ is holomorphic if and only if $(I, II_{AR})$ is a Codazzi pair.

So, we can summarize the above discussion in the following

**Lema 7 (Key Lemma).** Given a $H-$surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, $H^2 + \tau^2 \neq 0$, consider the symmetric $(2, 0)-$tensor given by

$$II_{AR}(X, Y) = II(X, Y) - \alpha \langle T_\theta, X \rangle \langle T_\theta, Y \rangle + \frac{\alpha |T|^2}{2} \langle X, Y \rangle,$$

where

- $\alpha = \frac{\kappa - 4\tau^2}{2\sqrt{H^2 + \tau^2}},$
- $e^{2\theta} = \frac{H - i\tau}{\sqrt{H^2 + \tau^2}}$ and
- $T_\theta = \cos \theta T + \sin \theta JT.$

Then, $(I, II_{AR})$ is a Codazzi pair with constant mean curvature $H$. Moreover, the $(2, 0)-$part of $II_{AR}$ with respect to the conformal structure given by $I$ agrees (up to a constant) with the Abresch-Rosenberg differential.
And, as a consequence of Lemma 4 we have:

**Corollary 1.** Let \( \Sigma \) be a \( H \)-surface in \( E(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \). Then, the following conditions are equivalent

- \( |S| \) is bounded,
- \( |A| \) is bounded,
- \( |K| \) is bounded.

**Proof.** On the one hand, from (21) is clear that \( |A| \) is bounded if, and only if, \( |S| \) is bounded. On the other hand, from \( 4H^2 - 2K_e = |A|^2 \) and the Gauss equation, we have that \( |K| \) is bounded if, and only if, \( |A| \) is bounded.

**4 Simons’ type formula in \( E(\kappa, \tau) \).**

In this section, we will obtain a Simons’ type formula for the traceless Abresch-Rosenberg shape operator \( S \) defined on a \( H \)-surface \( \Sigma \subset E(\kappa, \tau) \), \( \tau \neq 0 \). The Simons’ type formula follows form the fact that the traceless Abresch-Rosenberg shape operator is Codazzi and the work of Cheng-Yau [10].

**Theorem 3.** Let \( \Sigma \) be a \( H \)-surface in \( E(\kappa, \tau) \). Then, the traceless Abresch-Rosenberg shape operator satisfies

\[
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + 2K|S|^2,
\]

or, equivalently, away from the zeroes of \( |S| \),

\[
|S| \Delta |S| - 2K|S|^2 = |\nabla |S||^2.
\]

**Proof.** Since \( (I, II_{AR}) \) is a Codazzi pair on \( \Sigma \), from Lemma 4 we get that \( (I, II_{AR} - HI) \) is also a Codazzi pair on \( \Sigma \), observe that \( II_{AR} - HI \) is nothing but the traceless Abresch-Rosenberg fundamental form. Hence, from (1) (cf. [10]), we obtain

\[
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + 2K|S|^2,
\]

as claimed. Here, we have used that \( R_{ijij} = K \) since we are working in dimension two. Also, since \( S \) is traceless, the two eigenvalues, \( \lambda_i \), are opposite signs so \( (\lambda_i - \lambda_j)^2 = 4\lambda_i^2 = 2|S|^2 \).

To obtain (28), using \( \Delta |S|^2 = 2|S| \Delta |S| + 2|\nabla |S||^2 \), we get

\[
|S| \Delta |S| + |\nabla |S||^2 = |\nabla S|^2 + 2K|S|^2
\]

Thus, since \( \Sigma \) has dimension two and \( S \) is traceless and Codazzi, it holds (cf. [8])

\[
|\nabla S|^2 = 2|\nabla |S||^2,
\]

and we finally obtain (28).
4.1 Classification results for $H$–surfaces in $\mathbb{E}(\kappa, \tau)$

First, we shall recall the classification theorem when $q^{AR}$ vanishes identically.

**Lemma 8** ([1, 2, 15]). Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface whose Abresch-Rosenberg differential vanishes. Then $\Sigma$ is invariant by certain one parameter subgroup of isometries of $\mathbb{E}(\kappa, \tau)$.

Hence, the above Lemma motivates the following:

**Definition 5.** Let $\Sigma$ be a complete $H$–surface in $\mathbb{E}(\kappa, \tau)$. We say that $\Sigma$ is an Abresch-Rosenberg surface if its Abresch-Rosenberg differential vanishes identically.

Since $\mathbb{E}(\kappa, \tau)$ is a Riemannian submersion $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$, given $\gamma$ a regular curve in $\mathbb{M}^2(\kappa)$, $\Sigma_\gamma := \pi^{-1}(\gamma)$ is a surface in $\mathbb{E}(\kappa, \tau)$ satisfying that $\xi$ is a tangential vector field along $\Sigma_\gamma$, in this case $\nu = 0$. So, $\xi$ is a parallel vector field along $\Sigma_\gamma$ and hence $\Sigma_\gamma$ is flat and its mean curvature is given by $2H = k_\gamma$, where $k_\gamma$ is the geodesic curvature of $\gamma$ in $\mathbb{M}^2(\kappa)$ (cf. [16, Proposition 2.10]). We will call $\Sigma_\gamma := \pi^{-1}(\gamma)$ a Hopf cylinder in $\mathbb{E}(\kappa, \tau)$ over the curve $\gamma$. If $\gamma$ is a closed curve, $\Sigma_\gamma$ is a flat Hopf cylinder and additionally, if $\pi$ is a circle Riemannian submersion, $\Sigma_\gamma$ is a Hopf torus. The latter case occurs when $\mathbb{E}(\kappa, \tau)$ is a Berger sphere (see, [28, Theorem 1]), i.e., $\kappa = 1$ and $\tau \neq 0$.

Now, with the definition of Hopf cylinders in hand, we classify $H$–surfaces in $\mathbb{E}(\kappa, \tau)$ with constant non-zero Abresch-Rosenberg map $q^{AR}$ (cf. [15] for a different proof).

**Theorem 4.** Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface and suppose $q^{AR}$ is a positive constant map on $\Sigma$, then $\Sigma$ is a Hopf cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.

**Proof.** We can assume, without loss of generality, that $\Sigma$ is simply-connected by passing to the universal cover. Since $q^{AR}$ is a positive constant, (cf. [24, Main Lemma]) we have $0 = \Delta \ln q^{AR} = 4K$, that is, the Gaussian curvature vanishes identically on $\Sigma$. Moreover, since $q^{AR} = H^2 - K_\epsilon(I, I_{AR}) = c^2 > 0$ is constant, we obtain that $K_\epsilon(I, I_{AR}) = H^2 - c^2$ is constant on $\Sigma$.

On the one hand, since $\Sigma$ is simply connected, $q^{AR} = c^2 > 0$, there exists a global conformal parameter $z = x + iy$ (cf. [24, Main Lemma]), so that

$$cI = dx^2 + dy^2 \quad \text{and} \quad cI_{AR} = (H + c)dx^2 + (H - c)dy^2.$$

On the other hand, combining the Gauss equation (8) and the expression of $K_\epsilon(I, I_{AR})$ given by item 3 in Proposition 5, we obtain

$$\tau^2 + (\kappa - 4\tau^2)(1 - |T|^2) = \alpha(ST_\theta, T_\theta) + \frac{\alpha^2|T|^4}{4} - H^2 + c^2,$$

or, in other words,

$$\frac{\alpha^2|T|^4}{4} + (\kappa - 4\tau^2)|T|^2 + \alpha(ST_\theta, T_\theta) + c^2 - H^2 - \tau^2 - (\kappa - 4\tau^2) = 0 \text{ on } \Sigma.$$

Set $u = \langle T_\theta, \partial_\tau \rangle$ and $v = \langle T_\theta, \partial_y \rangle$. Thus, the last equations imply that there exists a polynomial in the variables $u$ and $v$, $P(u, v)$, whose leading term...
is $\frac{\alpha}{2}(u^2 + v^2)^2$. Since $\alpha \neq 0$, we obtain that $|T|$ is constant on $\Sigma$ and hence $\nu$ is constant along $\Sigma$, which implies that $\Sigma$ is a vertical cylinder (cf. [14, Theorem 2.2]).

5 Finite Abresch-Rosenberg Total Curvature

Complete $H$–surfaces $\Sigma \subset \mathbb{R}^3$ of finite total curvature, i.e., those whose $L^2$–norm of its traceless second fundamental form is finite, are of capital importance on the comprehension of $H$–surfaces. If $\Sigma$ has constant nonzero mean curvature and finite total curvature, then it must be compact. If $\Sigma$ is minimal, Osserman’s Theorem gives an impressive description of them.

When we consider complete $H$–surfaces in $\mathbb{E}(\kappa, \tau)$, the traceless part of the second fundamental form encodes less information about the surface. So, it is also convenient to consider the traceless part of the Abresch-Rosenberg fundamental form.

**Definition 6.** Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H$–surface, $H^2 + \tau^2 \neq 0$. We say that $\Sigma$ has finite Abresch-Rosenberg total curvature if the $L^2$–norm of the traceless Abresch-Rosenberg form is finite, i.e,

$$\int_\Sigma |S|^2 \, dv_g < +\infty,$$

where $dv_g$ is the volume element of $\Sigma$.

We must point out here that the family of complete constant mean curvature surfaces of finite Abresch-Rosenberg total curvature is large. Obviously, any Abresch-Rosenberg surface has finite Abresch-Rosenberg total curvature. We focus on $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ to show this fact. Recall the following result of Fernández-Miranda:

**Theorem [17, Theorem 16].** Any holomorphic quadratic differential on an open simply connected Riemann surface is the Abresch-Rosenberg differential of some complete surface $\Sigma$ with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the space of noncongruent complete mean curvature one half surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with the same Abresch-Rosenberg differential is generically infinite.

Then, we take the disk $\mathbb{D}$ as our open Riemann surface and a holomorphic quadratic differential $Q$ on $\mathbb{D}$ that extends continuously to the boundary. Let $\Sigma$ be the $H = 1/2$ surface constructed in [17, Theorem 16]. Now, we will see that $\Sigma$ has finite Abresch-Rosenberg total curvature.

For a conformal parameter $z \in \mathbb{D}$, we have $I = 2\lambda|dz|^2$ and hence $Q = f(z)dz^2$, for a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ that extends continuously to the boundary. Then, the squared norm of the traceless Abresch-Rosenberg shape operator is given by $|S|^2 = \frac{|f|^2}{4\lambda}$ and $dv_g = 4\lambda^2 |dz|^2$. Thus, we have

$$\int_\Sigma |S|^2 \, dv_g = \int_{\mathbb{D}} |f(z)|^2 |dz|^2 < +\infty,$$

as claimed.
The study of constant mean curvature surfaces of finite Abresch-Rosenberg total curvature is complementary to the study of surfaces of finite total curvature (see [20, 21] and references therein). Note that we are assuming \( H^2 + \tau^2 \neq 0 \), otherwise we consider the usual Abresch-Rosenberg operator given by (17). In [6], the authors studied complete \( H \)-surfaces of finite Abresch-Rosenberg total curvature in product spaces \( M^2(\kappa) \times \mathbb{R} \). The fundamental tool used in [6] is the Simons’ type formula for \(|S|\) developed in [5] when \( \tau = 0 \). Hence, using our Simons’ type formula (Theorem 3), we can obtain:

**Proposition 9.** Let \( \Sigma \) be an immersed \( H \)-surface, \( H^2 + \tau^2 \neq 0 \), in \( \mathbb{E}(\kappa, \tau) \) and set \( u = |S| \), where \( S \) is the traceless Abresch-Rosenberg form. Then

\[
-\Delta u \leq au^3 + bu,
\]

where \( a, \, b \) are constants depending on \( \kappa - 4\tau^2 \) and \( H \).

**Proof.** First, from (21) and \( 4H^2 - |A|^2 = 2K_e \), we have

\[
K_e = H^2 - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle - \alpha^2 \frac{|T|^4}{4}.
\]

Substituting (30) into the Gauss equation \( K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2 \), we can rewrite the Gaussian curvature \( K \) as follows

\[
K = (\kappa - 4\tau^2)(1 - |T|^2) + \tau^2 + H^2 - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle - \alpha^2 \frac{|T|^4}{4}.
\]

Next, substituting (31) into (28), we obtain:

\[
\Delta |S| \geq 2|S| \left((\kappa - 4\tau^2)\nu^2 + \tau^2 + H^2 - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle - \alpha^2 \frac{|T|^4}{4}\right)
\]

\[
\quad \geq - |S|^3 - |S| \left(- 2 \min\{0, \kappa - 4\tau^2\} - 2\tau^2 - 2H^2 + \frac{1}{2} \alpha^2 + 2\alpha \langle ST_\theta, T_\theta \rangle\right).
\]

Since \( S \) is traceless, we have that \(|ST_\theta| = \frac{1}{\sqrt{T}} |S| |T_\theta|\), and using the Schwarz inequality \(|\langle ST_\theta, T_\theta \rangle| \leq |T_\theta||ST_\theta|\), we see that

\[
-|S| \left(2\alpha \langle ST_\theta, T_\theta \rangle\right) \geq - \frac{2}{\sqrt{T}} |\alpha| |S|^2 \geq - \frac{|\alpha|}{\sqrt{T}} |S|^3 - \frac{|\alpha|}{\sqrt{T}} |S|.
\]

Finally, combining (33) with (32) yields

\[
\Delta |S| \geq - \left(1 + \frac{|\alpha|}{\sqrt{T}}\right) |S|^3 - \left(- 2 \min\{0, \kappa - 4\tau^2\} - 2\tau^2 - 2H^2 + \frac{1}{2} \alpha^2 + \frac{|\alpha|}{\sqrt{T}}\right) |S|,
\]

which proves (29). \( \square \)

Any \( H \)-surface \( \Sigma \) in \( \mathbb{E}(\kappa, \tau) \) satisfies a Sobolev type inequality of the form (cf. [22])

\[
|f|_2 \leq C_0 |\nabla f|_1 + C_1 |f|_1, \quad f \in C_0^\infty(\Sigma),
\]

where \(|f|_p\) denotes the \( L^p(\Sigma) \)-norm of \( f \) and \( C_0, \, C_1 \) are constants that depend only on the mean curvature \( H \). Here, \( C_0^\infty(\Sigma) \), as always, stands for the linear space of compactly supported piecewise smooth functions on \( \Sigma \).
Now, let \( p \in \Sigma \) be a fixed point. Consider the intrinsic distance function \( d(x, p) \) to \( p \) and define the open sets
\[
B(R) = \{ x \in \Sigma : d(p, x) < R \} \quad \text{and} \quad E(R) = \{ x \in \Sigma : d(x, p) > R \},
\]
then, with the above notations, we can show the following:

**Theorem 5.** Let \( \Sigma \subset E(\kappa, \tau) \) be a complete \( H \)-surface, \( H^2 + \tau^2 \neq 0 \), of finite Abresch-Rosenberg total curvature, that is,
\[
\int_{\Sigma} |S|^2 \, dv_g < +\infty,
\]
then \( |S| \) goes to zero uniformly at infinity. More precisely, there exist positive constants \( A, B \) and a positive radius \( R_\Sigma \), determined by the condition \( B \int_{E(R_\Sigma)} |S|^2 \leq 1 \), such that for \( u = |S| \) and for all \( R \geq R_\Sigma \), we have
\[
\| u \|_{\infty, E(2R)} = \sup_{x \in E(2R)} u(x) \leq A \left( \int_{E(R)} |S|^2 \, dv_g \right)^{\frac{1}{2}} \quad (35)
\]
and, there exist positive constants \( D \) and \( E \) such that the inequality \( \int_{\Sigma} |S|^2 \, dv_g \leq D \) implies
\[
\| u \|_{\infty} = \sup_{x \in \Sigma} u(x) \leq E \int_{\Sigma} |S|^2 \, dv_g.
\]

**Proof.** Since the function \( u = |S| \) satisfies the Sobolev type inequality (34) and the inequality (29), we can now work as in the proof of [7, Theorem 4.1] to show that \( u \) satisfies the inequality (35), letting \( R \) go to infinity we get that \( |S| \) goes to zero uniformly at infinity. \( \square \)

Next, we study \( H \)-surfaces \( \Sigma \) in \( E(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \), of finite Abresch-Rosenberg total curvature. Despite what happens in \( \mathbb{R}^3 \), a \( H \)-surface \( \Sigma \subset E(\kappa, \tau) \) with finite Abresch-Rosenberg total curvature is not necessarily conformally equivalent to a compact surface minus a finite number of points, in particular, \( \Sigma \) is not necessarily parabolic. For example, the complete \( H = 1/2 \) surface constructed above is hyperbolic. However, we obtain:

**Theorem 6.** Let \( \Sigma \) be a complete surface on \( E(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \), of finite Abresch-Rosenberg total curvature. Suppose one of the following conditions holds
\[
\begin{align*}
1. \quad &\kappa - 4\tau^2 > 0 \quad \text{and} \quad H^2 + \tau^2 > \frac{\kappa - 4\tau^2}{4}, \quad (36) \\
2. \quad &\kappa - 4\tau^2 < 0 \quad \text{and} \quad H^2 + \tau^2 > -\frac{\sqrt{5} + 2}{4} (\kappa - 4\tau^2). \quad (37)
\end{align*}
\]
Then, \( \Sigma \) must be compact.

**Proof.** From (31), the Gaussian curvature can be written as
\[
K = (\kappa - 4\tau^2)(1 - |T|^2) + \tau^2 + H^2 - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle - \alpha^2 |T|^4. \quad (38)
\]
Now, \( |T_\theta| \leq 1 \), \( S \) traceless and the Schwarz inequality imply
\[
-\alpha \langle ST_\theta, T_\theta \rangle \geq -\frac{|\alpha||S||T_\theta|}{\sqrt{2}} \geq -\frac{|\alpha||S|}{\sqrt{2}},
\]

Next, we study \( H \)-surfaces \( \Sigma \) in \( E(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \), of finite Abresch-Rosenberg total curvature. Despite what happens in \( \mathbb{R}^3 \), a \( H \)-surface \( \Sigma \subset E(\kappa, \tau) \) with finite Abresch-Rosenberg total curvature is not necessarily conformally equivalent to a compact surface minus a finite number of points, in particular, \( \Sigma \) is not necessarily parabolic. For example, the complete \( H = 1/2 \) surface constructed above is hyperbolic. However, we obtain:

**Theorem 6.** Let \( \Sigma \) be a complete surface on \( E(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \), of finite Abresch-Rosenberg total curvature. Suppose one of the following conditions holds
\[
\begin{align*}
1. \quad &\kappa - 4\tau^2 > 0 \quad \text{and} \quad H^2 + \tau^2 > \frac{\kappa - 4\tau^2}{4}, \quad (36) \\
2. \quad &\kappa - 4\tau^2 < 0 \quad \text{and} \quad H^2 + \tau^2 > -\frac{\sqrt{5} + 2}{4} (\kappa - 4\tau^2). \quad (37)
\end{align*}
\]
Then, \( \Sigma \) must be compact.

**Proof.** From (31), the Gaussian curvature can be written as
\[
K = (\kappa - 4\tau^2)(1 - |T|^2) + \tau^2 + H^2 - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle - \alpha^2 |T|^4. \quad (38)
\]
Now, \( |T_\theta| \leq 1 \), \( S \) traceless and the Schwarz inequality imply
\[
-\alpha \langle ST_\theta, T_\theta \rangle \geq -\frac{|\alpha||S||T_\theta|}{\sqrt{2}} \geq -\frac{|\alpha||S|}{\sqrt{2}},
\]
therefore
\[
K \geq (\kappa - 4\tau^2)\nu^2 + (H^2 + \tau^2) - \frac{1}{2}|S|^2 - |\alpha|\frac{|S|}{\sqrt{2}} - \alpha^2\frac{|T|^4}{4}.
\]

If \(\kappa - 4\tau^2 > 0\), then
\[
K \geq (H^2 + \tau^2) - \frac{\alpha^2}{4} - \frac{1}{2}|S|^2 - \alpha\frac{|S|}{\sqrt{2}}.
\]

If \(\kappa - 4\tau^2 < 0\), then
\[
K \geq (\kappa - 4\tau^2) + (H^2 + \tau^2) - \frac{\alpha^2}{4} - \frac{1}{2}|S|^2 + \alpha\frac{|S|}{\sqrt{2}}.
\]

In both cases, the hypothesis and the fact that \(|S|\) goes to zero uniformly imply that there exists a compact set \(\Omega\) and \(\epsilon > 0\) (depending on the compact set) such that the Gaussian curvature satisfies
\[
K(p) \geq \epsilon > 0 \text{ for all } p \in \Sigma \setminus \Omega.
\]

Therefore, Bonnet Theorem implies that \(d(p, \partial \Sigma \setminus \Omega)\) is uniformly bounded for all \(p \in \Sigma \setminus \Omega\). Thus, \(\Sigma\) must be compact.

\(\Box\)

Remark 3. The above result was obtained in [6] when \(\tau = 0\). We could have extended other results from [6, but we left this to the interested reader.

5.1 First Eigenvalue of a Schrödinger operators
We will use the Simons’ type formula (28) to estimate the first eigenvalue \(\lambda_1(L)\) of a Schrödinger operator \(L\) defined on a complete \(H\)-surface \(\Sigma\) in \(\mathcal{E}(\kappa, \tau)\) of finite Abresch-Rosenberg total curvature.

Set \(V \in C^0(\Sigma)\) and consider the differential linear operator, called Schrödinger operator, given by
\[
L : \quad C^\infty_0(\Sigma) \rightarrow \quad C^\infty_0(\Sigma)
\]
\[
f \rightarrow \quad Lf := \Delta f + V f,
\]
where \(\Delta\) is the Laplacian with respect to the induced Riemannian metric on \(\Sigma\).

Given a relatively compact domain \(\Omega \subset \Sigma\), it is well-known (cf. [9, 19]), that there exists a positive function \(\rho : \Sigma \rightarrow \mathbb{R}\) such that
\[
\begin{cases}
-\Delta \rho = (V + \lambda_1(L, \Omega))\rho & \text{in } \Omega \\
\rho = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where
\[
\lambda_1(L, \Omega) = \inf \left\{ \int_\Omega (|\nabla f|^2 - V f^2)dv_g : \int_\Omega f^2 dv_g \right\}.
\]

that is, \(\lambda_1(L, \Omega)\) and \(\rho\) are the first eigenvalue and first eigenfunction, respectively, associated to the Schrödinger operator \(L\) on \(\Omega \subset \Sigma\).
Now, we can consider the infimum over all the relatively compact domains in $\Sigma$ and we can define the infimum of the spectrum of $L$ as

$$\lambda_1(L) := \inf \{ \lambda_1(L, \Omega) : \Omega \subset \Sigma \text{ relatively compact} \},$$

in particular,

$$\lambda_1(L) := \inf_{i \to +\infty} \lambda_1(L, \Omega_i),$$

for any compact exhaustion $\{\Omega_i\}$ of $\Sigma$.

**Remark 4.** It is standard that the regularity conditions above can be relaxed, but this is not important in our arguments.

First, we will relate the Simons' type formula to the first eigenvalue of any Schrödinger operator.

**Lemma 10.** Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H-$surface, $H^2 + \tau^2 \neq 0$, and $\Omega \subset \Sigma$ a relatively compact domain. Denote by $\lambda_1(L, \Omega)$ and $\rho_\Omega$ the first eigenvalue and first eigenfunction, respectively, associated to the Schrödinger operator $L := \Delta + V$ on $\Omega$, $V \in C^0(\Omega)$. Set

$$C_\Omega = |S| (V + \lambda_1(L, \Omega)) + \Delta |S|,$$

where $S$ is the traceless Abresch-Rosenberg shape operator. Given $\phi \in C^\infty(\Omega')$, $\Omega \subset \Omega'$, then

$$\int_\Omega \phi^2 |S| C_\Omega \leq \int_\Omega |S|^2 |\nabla \phi|^2. \quad (36)$$

**Proof.** Set $\rho_\Omega = \rho$ and $\lambda_1(L, \Omega) = \lambda_1$. By the Maximum Principle $\rho > 0$ in $\Omega$. Set $w := \ln \rho$ in $\Omega'$ and note that it is well defined. Moreover, it holds

$$\Delta w = -(V + \lambda_1) - |\nabla w|^2.$$

Set $\psi = \phi |S|$. On the one hand, by Stokes' Theorem

$$0 = \int_\Omega \text{div} (\psi^2 \nabla w)$$

$$= \int_\Omega \psi^2 \Delta w + \int_\Omega 2\psi \langle \nabla \psi, \nabla w \rangle$$

$$= -\int_\Omega \psi^2 (V + \lambda_1) - \int_\Omega \psi^2 |\nabla w|^2 + \int_\Omega 2\psi \langle \nabla \psi, \nabla w \rangle$$

$$\leq -\int_\Omega \psi^2 (V + \lambda_1) + \int_\Omega |\nabla \psi|^2,$$

where we have used $-\psi^2 |\nabla w|^2 + 2\psi \langle \nabla \psi, \nabla w \rangle \leq |\nabla \psi|^2$. In other words, we have

$$\int_\Omega \psi^2 (V + \lambda_1) \leq \int_\Omega |\nabla \psi|^2. \quad (37)$$

On the other hand, using the definition of $\psi$ we get

$$|\nabla \psi|^2 = \phi^2 |\nabla |S||^2 + 2 \phi |S| \langle \nabla \phi, \nabla |S| \rangle + |S|^2 \phi^2 |\nabla \phi|^2,$$

and, since

$$\frac{1}{2} \text{div} \left( \phi^2 \nabla |S|^2 \right) = \phi^2 |S| \Delta |S| + \phi^2 |\nabla |S||^2 + 2 \phi |S| \langle \nabla \phi, \nabla |S| \rangle,$$

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we have that
\[ |\nabla \psi|^2 = \frac{1}{2} \text{div} \left( \phi^2 \nabla |S|^2 \right) - \phi^2 |S| \Delta |S| + |S|^2 |\nabla \phi|^2, \]
so, taking integrals and using Stokes’ Theorem, we obtain
\[ \int_{\Omega} |\nabla \psi|^2 = -\int_{\Omega} \phi^2 |S| \Delta |S| + \int_{\Omega} |S|^2 |\nabla \phi|^2. \]  
(38)

Thus, combining (38) with (37) we get (36).

Now, we can use Lemma 10 to estimate the first eigenvalue \( \lambda_1(L) \) of the Schrödinger operator \( L = \Delta + V \). Specifically,

**Theorem 7.** Let \( \Sigma \) be a complete two-sided \( H \)-surface in \( \mathbb{E}(\kappa, \tau) \) of finite Abresch-Rosenberg total curvature and \( H^2 + \tau^2 \neq 0 \). Denote by \( \lambda_1(L) \) the first eigenvalue associated to the Schrödinger operator \( L := \Delta + V, V \in C^0(\Sigma) \). Then, \( \Sigma \) is either an Abresch-Rosenberg surface, a Hopf cylinder or

\[ \lambda_1(L) < -\inf_{\Sigma} \{V + 2K\}. \]  
(39)

**Proof.** Assume that \( |S| \) is not identically constant on \( \Sigma \), otherwise \( \Sigma \) is either an Abresch-Rosenberg \( H \)-surface or a Hopf cylinder. Then, from (28), we get

\[ C = |S|(V + \lambda_1(L)) + \Delta |S| \geq |S|(V + \lambda_1(L) + 2K). \]

Note that \( C > |S|(V + \lambda_1(L) + 2K) \) at some point since, otherwise, it would imply that \( |S| \) is constant, which is a contradiction.

Suppose \( \lambda_1(L) \geq -\inf_{\Sigma}\{V + 2K\} \), then \( C \geq |S|(V + \lambda_1(L) + 2K) \geq 0 \).

Now, take \( p \in \Sigma \) a fixed point and \( R > 0 \). Denote by \( r(x) = d(x, p) \) the distance function from \( p \) and \( B(p, R) \) the geodesic ball of radius \( R \). Choose \( R' < R \) and define \( \phi \) as follows

\[ \phi(x) = \begin{cases} 
1 & \text{For } x \text{ such that } 0 \leq r(x) \leq \frac{R'}{2}, \\
2 - \frac{R'}{R}r(x) & \text{For } x \text{ such that } \frac{R'}{2} < r(x) \leq R', \\
0 & \text{For } x \text{ such that } R' < r(x) \leq R.
\end{cases} \]

Observe that \( \phi \in C^1_0(B(p, R')) \), \( \overline{B(p, R')} \subset B(p, R) \) and \( B(p, R) \) is a relatively compact set on \( \Sigma \), then Lemma 10 implies

\[ \int_{B(p, R)} \phi^2 |S| C_{B(p, R)} \leq \int_{B(p, R)} |S|^2 |\nabla \phi|^2, \]
and, since \( \phi = 1 \) on \( B(p, \frac{R'}{2}) \) and \( C \geq 0 \) on \( \Sigma \), we get

\[ \int_{B(p, \frac{R'}{2})} |S| C_{B(p, R)} \leq \int_{B(p, R)} \phi^2 |S| C_{B(p, R)} \leq \frac{4}{(R')^2} \int_{\{x \in \Sigma: \frac{R'}{2} < r(x) \leq R' \}} |S|^2. \]

Hence, from the hypothesis that \( \Sigma \) has finite Abresch-Rosenberg total curvature and letting \( R' \to \infty \) we obtain

\[ \int_{\Sigma} |S| C \leq 0. \]
The above equation implies that \( C \equiv 0 \) on \( \Sigma \), hence \(|S|\) must be constant on \( \Sigma \), which is a contradiction since we are assuming that \(|S|\) is not constant. Therefore \( \lambda_1(L) \) must satisfy

\[
\lambda_1(L) < -\inf_{\Sigma} \{ V + 2K \}.
\]

\[\square\]

Obviously, any closed \( H \)–surface in \( \mathbb{E}(\kappa, \tau) \) has finite Abresch-Rosenberg total curvature. Consequently, from [28, Theorem 1] and Theorem 7 we get an estimate of \( \lambda_1(L) \) of any closed surface of \( \mathbb{E}(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \).

**Corollary 2.** Let \( \Sigma \) be a closed \( H \)–surface in \( \mathbb{E}(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \). Denote by \( \lambda_1(L) \) the first eigenvalue associated to the Schrödinger Operator \( L := \Delta + V \), \( V \in C^0(\Sigma) \). Then, \( \Sigma \) is either a rotationally symmetric \( H \)–sphere, a Hopf \( H \)–tori or

\[
\lambda_1(L) < -\inf_{\Sigma} \{ V + 2K \}. \tag{40}
\]

### 5.2 Stability Operator

Now, we obtain estimates for the most natural Schrödinger operator of a complete \( H \)–surface in \( \mathbb{E}(\kappa, \tau) \), the Stability (or Jacobi) operator

\[
J = \Delta + (|A|^2 + \text{Ric}(N)),
\]

where \( \text{Ric}(N) \) is the Ricci curvature of the ambient manifold in the normal direction. Hence, in this case, \( V \equiv |A|^2 + \text{Ric}(N) \).

Note that, since \( \text{Ric}(N) = (\kappa - 4\tau^2)|T|^2 + 2\tau^2 \) and the Gauss equation (3), we have

\[
V + 2K = 4H^2 + 2(K - K_e) + (\kappa - 4\tau^2)|T|^2 + 2\tau^2 = 4H^2 + \kappa + (\kappa - 4\tau^2)\nu^2.
\]

Hence, Theorem 7 and the above equality gives:

**Theorem 8.** Let \( \Sigma \) be a complete two sided \( H \)–surface of finite Abresch-Rosenberg total curvature in \( \mathbb{E}(\kappa, \tau) \), \( H^2 + \tau^2 \neq 0 \).

- If \( \kappa - 4\tau^2 > 0 \). Then, \( \Sigma \) is either an Abresch-Rosenberg \( H \)–surface, a Hopf cylinder, or

\[
\lambda_1 < -(4H^2 + \kappa).
\]

- If \( \kappa - 4\tau^2 < 0 \). Then, \( \Sigma \) is either an Abresch-Rosenberg \( H \)–surface, or

\[
\lambda_1 < -(4H^2 + \kappa) - (\kappa - 4\tau^2).
\]

**Remark 5.** These estimates were obtained by Alías-Meroño-Ortíz [4] for closed surfaces in \( \mathbb{E}(\kappa, \tau) \).
6 Pinching Theorems for $H$–surfaces in $\mathbb{E}(\kappa, \tau)$

In this section we use the Simons’ type formula (27) together with the Omori-Yau Maximum Principle to classify complete $H$–surfaces in $\mathbb{E}(\kappa, \tau)$ satisfying a pinching condition on its Abresch-Rosenberg fundamental form. First, we recall the Omori-Yau Maximum Principle for the reader convenience.

**Theorem 9** ([29]). Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is bounded from above, then there exists a sequence of points $\{p_j\}_{j \in \mathbb{N}} \in M$ such that:

1. $\lim_{j \to \infty} u(p_j) = \sup_M u$.
2. $|\nabla u(p_j)| < \frac{1}{j}$.
3. $\Delta u(p_j) < \frac{1}{j}$.

Second, we study the Simons’ type formula (27) in the set of non-umbilical points of $S$.

**Proposition 11.** Let $\Sigma$ be a $H$–surface in $\mathbb{E}(\kappa, \tau)$. Then, away from the umbilic points of $S$, it holds

$$\frac{1}{2} \Delta |S|^2 \geq |\nabla S|^2 + |S|^2 F(|S|),$$

where $F(x) = -x^2 + bx + a$ is the second degree polynomial given by

$$a = 2(\kappa - 4\tau^2) + 2(H^2 + \tau^2) - 2(\kappa - 4\tau^2)|T|^2 - \alpha^2 \frac{|T|^4}{2},$$

$$b = -2|\alpha||T|^2,$$

where $\alpha = \frac{\kappa - 4\tau^2}{2\sqrt{H^2 + \tau^2}}$.

**Proof.** From (31), the Gaussian curvature can be written as

$$K = (\kappa - 4\tau^2)(1 - |T|^2) + \tau^2 + H^2 - \alpha^2 \frac{|T|^4}{4} - \frac{1}{2} |S|^2 - \alpha \langle ST_\theta, T_\theta \rangle.$$

Since $|\langle ST_\theta, T_\theta \rangle| \leq |S||T|^2$, substituting the above formulas into (27) yields

$$\frac{1}{2} \Delta |S|^2 \geq |\nabla S|^2 + |S|^2 (2(\kappa - 4\tau^2)(1 - |T|^2) + 2(\tau^2 + H^2) - \alpha^2 \frac{|T|^4}{2} - |S|^2 - 2|\alpha||S||T|^2),$$

as claimed.

So, our next step is to study the first positive root $\bar{x} \in \mathbb{R}^+$ of $F(x)$ so that $F(x) > 0$ for all $x \in (0, \bar{x})$. To do so, we set $t = |T|^2 \in [0, 1]$ and hence, we can rewrite:

$$a(t) = 2(\kappa - 4\tau^2) + 2(H^2 + \tau^2) - 2(\kappa - 4\tau^2)t - \frac{\alpha^2}{2} t^2,$$

$$b(t) = -2|\alpha| t.$$
In order to obtain a positive real root, the coefficients of \( F \) must hold \( h(t) = b(t)^2 + 4a(t) > 0 \) and \( a(t) > 0 \) for all \( t \in [0, 1] \). This means that

\[
(H^2 + \tau^2) + (\kappa - 4\tau^2)(1 - t) + \frac{\alpha^2}{4}t^2 > 0 \quad \text{for all } t \in [0, 1],
\]

and

\[
2(H^2 + \tau^2) + 2(\kappa - 4\tau^2)(1 - t) - \frac{\alpha^2}{2}t^2 > 0 \quad \text{for all } t \in [0, 1].
\]

**Proposition 12.** Define \( G : [0, 1] \to \mathbb{R} \) as

\[
G(t) = \frac{b(t) + \sqrt{b(t)^2 + 4a(t)}}{2},
\]

then, assuming \( h(t) > 0 \) and \( a(t) > 0 \) for all \( t \in [0, 1] \), the function \( G(t) \) satisfies:

\[
\min_{t \in [0, 1]} G(t) = \min \left\{ \sqrt{2\sqrt{(H^2 + \tau^2)} + (\kappa - 4\tau^2)}, -|\alpha| + \sqrt{2(H^2 + \tau^2)} + \frac{\alpha^2}{2} \right\}.
\]

**Proof.** We compute the interior critical points of \( G(t) \). To do so, we compute the derivative of \( G \) at an interior point

\[
G'(t) = \frac{b(t)}{2} \left( 1 + \frac{b(t)}{\sqrt{b(t)^2 + 4a(t)}} \right) + \frac{a'(t)}{\sqrt{(b(t))^2 + 4a(t)}}
\]

\[
= -|\alpha| \left( \frac{b(t) + \sqrt{b(t)^2 + 4a(t)}}{\sqrt{b(t)^2 + 4a(t)}} \right) - \frac{\alpha^2 t + 2(\kappa - 4\tau^2)}{\sqrt{(b(t))^2 + 4a(t)}}
\]

\[
= \frac{1}{\sqrt{b(t)^2 + 4a(t)}} \left( -2|\alpha| G(t) - \alpha^2 t - 2(\kappa - 4\tau^2) \right)
\]

Assume that there exists \( \bar{t} \in (0, 1) \) so that \( G'(\bar{t}) = 0 \), then the above equation implies that the function \( \Psi(t) := -2|\alpha| G(t) - \alpha^2 t - 2(\kappa - 4\tau^2) \) satisfies \( \Psi(\bar{t}) = 0 \) and \( \Psi'(\bar{t}) = -\alpha^2 \bar{t} \). Moreover, observe that

\[
G''(t) = R(t)\Psi(t) + \frac{\Psi'(t)}{\sqrt{b(t)^2 + 4a(t)}},
\]

for some smooth function \( R : (0, 1) \to \mathbb{R}^+ \). Hence

\[
G''(\bar{t}) = \frac{\Psi(\bar{t})}{\sqrt{b(t)^2 + 4a(t)}} = -\frac{\alpha^2 \bar{t}}{\sqrt{b(t)^2 + 4a(t)}} < 0.
\]

Therefore, \( G \) does not have an interior minimum. Therefore, in any case,

\[
\min_{t \in [0, 1]} G(t) = \min \{ G(0), G(1) \}.
\]

Next, we compute the minimum of \( G \). To do so, we will distinguish two cases depending on the sign of \( \kappa - 4\tau^2 \).

**Case 1:** \( \kappa - 4\tau^2 > 0 \). In this case, we assume \( 4(H^2 + \tau^2) > \kappa - 4\tau^2 \).

Then,

\[
h(t) > 0 \quad \text{and} \quad a(t) > 0 \quad \text{for all } t \in [0, 1].
\]

Therefore, Proposition 12 implies

\[
\min_{t \in [0, 1]} G(t) = \sqrt{2\sqrt{(H^2 + \tau^2)} + (\kappa - 4\tau^2)} > 0.
\]
Now, we are ready to show a pinching theorem for complete $H-$ surfaces in $\mathbb{E}(\kappa, \tau)$ when $\kappa - 4\tau^2 > 0$.

**Theorem 10.** Let $\Sigma$ be a complete immersed $H-$surface in $\mathbb{E}(\kappa, \tau)$, $\kappa - 4\tau^2 > 0$. Assume that $4(H^2 + \tau^2) > \kappa - 4\tau^2$ and

$$\sup_{\Sigma} |S| < \sqrt{2}\sqrt{(H^2 + \tau^2) + (\kappa - 4\tau^2)},$$

where $S$ is the traceless Abresch-Rosenberg shape operator. Then, $\Sigma$ is an Abresch-Rosenberg surface in $\mathbb{E}(\kappa, \tau)$.

Moreover, if

$$\sup_{\Sigma} |S| = \sqrt{2}\sqrt{(H^2 + \tau^2) + (\kappa - 4\tau^2)}$$

and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S|$, then $\Sigma$ is a Hopf cylinder.

**Proof.** Assume $|S|$ is not identically zero. Set $d := F(\sup_{p \in \Sigma} |S|)$. From Proposition 12 we have

$$F(|S|(p)) \geq d > 0.$$

Hence, Proposition 11 implies that

$$\Delta |S|^2 \geq d |S|^2$$

away from the umbilic points of $S$.

Thus, since $|S|$ is bounded by hypothesis, Corollary 1 implies that the Ricci curvature of $\Sigma$ (nothing but the Gaussian curvature) is bounded from below. Hence, the Omori-Yau Maximum Principle implies that there exists a sequence $\{p_j\}_{j \in \mathbb{N}} \in \Sigma$ such that:

$$\lim_{j \to \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \quad \text{and} \quad \lim_{j \to \infty} \Delta |S|^2(p_j) \leq 0.,$$

which contradicts (44). Hence, $|S| = 0$ on $\Sigma$, that is, $\Sigma$ is an Abresch-Rosenberg surface.

Now, assume that

$$\sup_{\Sigma} |S| = \sqrt{2}\sqrt{(H^2 + \tau^2) + (\kappa - 4\tau^2)}$$

and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_{\Sigma} |S| > 0$. So, from Proposition 12, there is a neighborhood $\Omega$ of $p$ so that

$$\Delta |S| \geq 0.$$

Then, the Interior Maximum Principle implies that $|S|$ is constant (non zero) on $\Omega$, and hence constant on $\Sigma$. Thus, Lemma 4 implies that $\Sigma$ is a Hopf cylinder.

**Case 2:** $\kappa - 4\tau^2 < 0$. In this case, we assume $H^2 + \tau^2 > |\kappa - 4\tau^2|$. Then,

$$h(t) > 0 \quad \text{and} \quad a(t) > 0 \quad \text{for all} \quad t \in [0, 1].$$

Therefore, Proposition 12 implies

$$\min_{t \in [0, 1]} G(t) = -|\alpha| + \sqrt{2(H^2 + \tau^2) + \frac{a^2}{2}} > 0$$
So, arguing as in Theorem 10, we establish (without proof) a pinching theorem for complete $H$-surfaces in $\mathbb{E}(\kappa, \tau)$ when $\kappa - 4\tau^2 < 0$.

**Theorem 11.** Let $\Sigma$ be a complete immersed $H$–surface in $\mathbb{E}(\kappa, \tau)$, $\kappa - 4\tau^2 < 0$. Assume that $H^2 + \tau^2 > |\kappa - 4\tau^2|$ and

$$\sup_\Sigma |S| < -|\alpha| + \sqrt{2(H^2 + \tau^2) + \frac{\alpha^2}{2}},$$

where $S$ is the traceless Abresch-Rosenberg shape operator. Then, $\Sigma$ is an Abresch-Rosenberg surface of $\mathbb{E}(\kappa, \tau)$.

Moreover, if

$$\sup_\Sigma |S| = -|\alpha| + \sqrt{2(H^2 + \tau^2) + \frac{\alpha^2}{2}}$$

and there exists one point $p \in \Sigma$ such that $|S(p)| = \sup_\Sigma |S|$, then $\Sigma$ is a Hopf cylinder.

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