Making chiral fermion actions (almost) gauge invariant using Laplacian gauge fixing

J.C. Vink

\textsuperscript{a}University of California at San Diego, Department of Physics, La Jolla, CA 92093, USA

Straight forward lattice descriptions of chiral fermions lead to actions that break gauge invariance. I describe a method to make such actions gauge invariant (up to global gauge transformations) with the aid of gauge fixing. To make this prescription unambiguous, Laplacian gauge fixing is used, which is free from Gribov ambiguities.

1. LATTICE CHIRAL FERMIONS

There are many proposals to describe chiral fermions on the lattice (see refs. \cite{1,2} for recent reviews), which all appear to work when the fermions couple to smooth external gauge fields. When the gauge fields are dynamical, however, these approaches fail: e.g. because opposite handedness mirror fermions are generated dynamically \cite{3}, or because the chiral fermions are shielded from the gauge fields and no longer couple to it \cite{4,5}. The lesson learned from these attempts is that the ‘longitudinal mode’ of the gauge field, which correspond to gauge transformations, is responsible for the breakdown of these models and must be controlled.

To constrain the dangerous longitudinal mode, the Rome group has proposed to include gauge fixing in the definition of the path integral \cite{6}. Now the dynamical, but gauge fixed gauge fields should be sufficiently smooth on the lattice scale such that the chiral fermions remain chiral, as was the case with classical external gauge fields. To recover the gauge invariant target model it should then be sufficient to include counter terms in the action and tune their coefficients such that BRST invariance is recovered. This approach looks promising, if it can be implemented non-perturbatively. This is difficult, however, since gauge fixing requires a Faddeev-Popov determinant which is not positive definite and also Gribov ambiguities may pose problems.

Here I discuss a practical method to implement the Rome idea, without having to put gauge fixing and ghost terms in the action. As an example I consider Wilson fermions with left coupling to a gauge field $U^g$,

$$S = \sum_{xy} \bar{\psi}_x (\not{D}(U^g)_{xy} P_L + \not{\partial}_{xy} P_R) \psi_y$$

$$- \sum_x (\bar{\psi}_x U^g_{\mu x} \psi_{x+\hat{\mu}} + \bar{\psi}_{x+\hat{\mu}} (U^g)_{\mu x} \psi_x - 2\bar{\psi}_x \psi_x),$$

with $\not{D}$ and $\not{\partial}$ the gauge covariant and free lattice Dirac operators respectively, $P_{R,L} = (1 \pm \gamma_5)/2$. The Wilson term breaks the left-gauge invariance, $\psi_x \rightarrow (P_R + g_x P_L) \psi_x$, $g_x \in G$. However, the breaking term is $\propto a$ and should be irrelevant for $a \rightarrow 0$, at least for classical fields.

The link field $U^g$ is a gauge transform of $U$. When integrating over all $U$ in the path integral (without gauge fixing) it is equivalent to integrate with measure $DU^g$, $DU^g$ or $DUDg$. Then the field $g$ represents the longitudinal gauge mode and could be interpreted as a spurious Higgs field, which should not affect the physics. However, this non-gauge fixing approach was found not to work \cite{3–5}. In the Rome approach one adds gauge fixing and ghost terms (and counter terms, of course), such that $U^g$ is constrained on a smooth gauge section.

In the present approach $U^g$ also satisfies a smooth-gauge condition. However, this is achieved not by putting a gauge fixing term in the action, but by actually computing the gauge transformation $g(U)$ which fixes $U$. In the path integral one still integrates over all $U$, but the field to which the fermions couple is not $U$ but $U^g(U)$. Assuming that $g(U)$ is uniquely defined for all $U$, it follows that under a gauge
transformation $U \rightarrow U^h$, $g(U)$ transforms as $g(U^h)_x = h_0 g(U)_x h_0^\dagger_x$, with $h_0$ a global transformation and one can verify that the action (2) is invariant up to a global transformation with $h_0$, $S(\psi, \overline{\psi}, U^h) = S(\psi, \overline{\psi}, U^{h_0})$. Notice that the fermion fields are not transforming. With the unitary (gauge) transformation $\psi \rightarrow (P_R + g P_L) \psi$, $\overline{\psi} \rightarrow \overline{\psi} (g P_R + P_L)$ the $g$ can be removed from $D$ and the fermion fields transform now as usual. For abelian gauge groups $U^{h_0} = U$ and then the action is actually gauge invariant (see ref. [9] for the details).

Several remarks are now in order.
1) It should be stressed once more, that the action (2) with $g \equiv g(U)$, is a function over the full gauge field configurations space. The integration is with the usual Haar measure $D U$, not with the Faddeev-Popov measure that includes gauge fixing and ghost terms. Note that counter terms should still be included in the action.

2) The field $g(U)$ depends on all link variables $U_{\mu x}$, and therefore the action is nonlocal. I assume here that at least for anomaly free fermion representations, the action effectively becomes local in the scaling region.

3) The gauge transformed field must be sufficiently smooth, such that one can hope that the good behavior found with classical $U$ fields also persists for dynamical fields $U^{g(U)}$.

4) It must be possible to compute the gauge transformation $g(U)$ without Gribov ambiguities, in a reasonable amount of computer time.

5) For practical purposes it is essential that the nonlocal action can be simulated efficiently.

In ref. [3] points 4) and 5) are addressed and it is shown there that a hybrid Monte Carlo algorithm can be used to simulate the action. The details depend on the choice of gauge condition. In order to comply with point 4), the Laplacian gauge can be used, which was introduced in ref. [7]. Here $g(U)$ is computed from the eigenfunction of the gauge covariant Laplacian with the smallest eigenvalue (for $G = \text{U}(1)$ or $\text{SU}(2)$). For the HMC algorithm it is then required to compute this lowest eigenfunction numerically and to invert a Laplacian-like matrix, at each HMC time step. This is time consuming, but when the fermions are dynamical anyway, the additional work is relatively small. The Laplacian gauge is unambiguous, provided that this lowest eigenvalue is nondegenerate. A second singularity arises when the eigenfunction would vanish at some lattice site. Even though these ‘Gribov horizons’ are of measure zero in the path integral, one should check point six:

6) Gribov horizons should not pose a problem in practice.

2. SOME TESTS

Points 3) and 6) raised above can be addressed to some extent, using quenched dynamical gauge fields. This is done in ref. [10], which contains the details. A measure for the smoothness of the gauge fixed field is the value of the average link $\langle U \rangle$. The Landau gauge maximizes this quantity. It turns out that the results of the Landau and Laplacian gauges are very similar. Fig. 1 shows the average link in the $\text{U}(1)$ model in two dimensions. It is seen that Gribov copies (local maxima) of the Landau gauge make $\langle U \rangle$ actually slightly smaller than in the Laplacian gauge for $\beta > 3$. Similar measurements with dynamical $\text{SU}(2)$ gauge fields in four dimensions show even less difference between $\langle U \rangle$ in the Landau and Laplacian gauges.

![Figure 1. Average link after standard Landau (solid), Laplacian (dashed) and Laplacian followed by Landau gauge fixing (dotted), for $\text{U}(1)$ gauge fields on a $20^2$ lattice.](image-url)
Even if the Laplacian gauge leads to equally smooth gauge fixed fields as the Landau gauge, one can still fear that this is not smooth enough for the fermions. In particular when there is a gauge singularity, as in the core of an instanton, the $U_g(U)$ field locally has to deviate far for one and this might give problems for the fermions. To test for this, we have computed the eigenvalue spectrum of the fermion matrix defined implicitly in eq. (2), using test configurations with nonzero topological charge. With these backgrounds the fermion matrix should have a zero mode, cf. [2]. It is found that the left-coupled Wilson fermion matrix has an (almost) real eigenvalue, with a small nonzero real part which is found to scale to zero $\propto a$. This shows that in the continuum limit, the fermion matrix indeed has its desired zero mode.

![Flow of the smallest two eigenvalues in the 2d U(1) model at $\beta = 4$ on an 8 x 20 lattice.](image)

Figure 2. Flow of the smallest two eigenvalues in the $2d$ U(1) model at $\beta = 4$ on an $8 \times 20$ lattice.

Also point 6) is addressed in ref. [10]. Using the U(1) model in two dimensions, I have computed the lowest two eigenvalues of the Laplacian after each time step of a HMC simulation of the quenched model. For values of the gauge coupling in the scaling region, $\beta \geq 3$, the separation between the lowest two eigenvalues only becomes comparable with the fluctuations of the individual eigenvalues when the lattice is (unnecessarily) large in at least one direction. It is found that the eigenvalues occasionally approach each other, but never come so close that the computer code runs into trouble computing them. As an example we show part of the level flow found at $\beta = 4$ on an $8 \times 20$ lattice.

Also in SU(2), where I monitored the level flow during Metropolis updating, I only found avoided crossing, with a smallest separation between the eigenvalues which is many orders of magnitude larger than the precision with which these eigenvalues can be computed. Also the other kind of singularity, the vanishing of the eigenfunction at some site, never came close in these test runs. These results indicate that the Laplacian gauge can be computed very efficiently and without numerical problems. Hence it seems worthwhile to further investigate this approach to chiral models on the lattice.

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