Quantum affine Toda solitons

N. J. MacKay*

contributed talk at the 12th International Congress of Mathematical Physics,
Brisbane, July 1997

Dept of Applied Maths and Theoretical Physics,
Cambridge University,
Cambridge, CB3 9EW, UK

ABSTRACT

We review some of the progress in affine Toda field theories in recent years, explain why known dualities cannot easily be extended, and make some suggestions for what should be sought instead.

1 A brief review

The affine Toda field theories are models of real scalar fields, in one space dimension (here taken to be the real line \(\mathbb{R}\)), with exponential interactions. The Lagrangian is

\[
\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{m^2}{\beta^2} \sum_{j=0}^{n} n_j (e^{\beta\alpha_j \cdot \phi} - 1),
\]

where the field \(\phi(x, t)\) is an \(n\)-dimensional vector, \(n\) being the rank of the finite Lie algebra \(g\). The \(\alpha_j\) \((j = 1, \ldots, n)\) are the simple roots of \(g\); \(\alpha_0\) is the lowest root, so that \(\{\alpha_0, \alpha_j\}\) are described by one of the extended Dynkin diagrams of an affine algebra \(\hat{g}\). The fields can be rescaled so that \(\beta\) only appears in \(\mathcal{L}\) through an overall factor \(1/\beta^2\); expanding in powers of \(\beta^2\) is thus equivalent in the quantum theory to expanding in \(\hbar\).

---

*\text{n.j.mackay@damtp.cam.ac.uk}

†The theory on the half-line is another story.\[\]
A great deal has been discovered about these models in the last ten years, and the resulting spectra of perturbative and solitonic particles are surprisingly rich. It is clear, however, that vastly more remains to be learned about the setting for these results. This paper does not provide a pedagogic review: excellent introductions already exist\cite{1,2}, and the reader is referred to these for full details of masses, S-matrices and so on. Rather I shall provide a brief qualitative summary, and then describe the current frontier. In particular I want to explain why there appears to be no duality among the full spectra of solitons and particles.

1.1 Real $\beta$, simply-laced $g$

The Lagrangian (1), with $\beta$ real, is amenable to simple techniques. The vacuum $\phi = 0$ is unique; the exponentials may be expanded about it, and the masses (of order $m$) and the three-point couplings of the mass eigenstates may then be read off\cite{3,4}. The masses form an eigenvector of the Lie algebra’s Cartan matrix\cite{5}, and so may be put into correspondence with spots on its Dynkin diagram - we shall refer to this as the particle’s ‘species’.

The theory is further known to be classically integrable, with local, commuting conserved charges of spins equal (for $g$ simply-laced) to the exponents of the Lie algebra\cite{6}: the first exponent is always one, corresponding to energy-momentum. Their conservation constrains the admissible three-point couplings very strongly, indeed to such an extent that it might be surprising that solutions exist at all. Nevertheless an admissible set does exist for each algebra, as is made beautifully clear in the construction due to Dorey\cite{7,8}. When two particles fuse to form a third, the conservation of each charge requires the existence of a triangle with sides of lengths equal to the charges and angles determined by the particles’ momenta. In Dorey’s construction, each three-point coupling corresponds to a triangle of roots, each being chosen from an orbit of a root under a Coxeter element of the Weyl group. The projection of this equation in the (higher-dimensional) root space onto various planes, on which the Coxeter element acts as a rotation, gives the triangles for the charges.

These results at tree-level can be quantized perturbatively; and at one-loop order, for simply-laced $g$, the particle masses all renormalize in the same ratio. The expectation is that this will be true to all orders, and exact S-matrices (which, since the particles are scalar, will be scalar factors, in fact products of simple trigonometric functions) may be
hypothesized for the particles\cite{3,4}, with rigid mass ratios and hence pole structure, described by Dorey’s rule. Further, the dependence on $\beta$ is such that there is a strong↔weak coupling duality: the S-matrices are invariant under $\beta \mapsto 4\pi/\beta$.

1.2 Imaginary $\beta$, simply-laced $g$

If in the simplest case, $g = a_1$, we take $\beta$ to be imaginary, we have moved from the sinh-to the sine-Gordon model. The vacuum is now degenerate, and there exist solitons, with masses proportional to $m/\beta^2$, and ‘breathers,’ consisting of an oscillating soliton-antisoliton pair (at imaginary rapidity difference), with a continuous mass spectrum. Exact S-matrices, now with matrix structure, may be proposed for the solitons, and the bound states include a discretized spectrum of breathers. The perturbative particle is still present, and its mass and S-matrix are equal to those of the lowest breather state\cite{10}. It is therefore natural to suspect that in the exact quantum theory the breather and the particle are identical.

For other $g$, however, $\mathcal{L}$ is complex for imaginary $\beta$. Nevertheless there exist solitons\cite{13}, constructed by finding Hirota $\tau$-functions\cite{13} or with vertex operators\cite{14}, of each species and with topological charges in (but not filling) the corresponding fundamental representation\cite{16}, interpolating the (now degenerate) vacua. They have real masses, of order $1/\beta^2$ and proportional to those of the particles, and real higher conserved charges\cite{15}, and they ‘fuse’ - double solitons truncate and appear as single solitons - according to Dorey’s rule. They also obey an exclusion rule: static multisolitons exist, but only if no two of the constituent single solitons are of the same species. There are also breathers in every species, and excited or ‘breathing’ solitons in some\cite{17}; these have the expected continuous mass spectra.

Any quantization of such an apparently non-unitary theory must be very suspect, but it is possible to plough ahead with, for example, semiclassical quantization, and obtain sensible results: that, as with the particles, the soliton mass ratios do not renormalize. The expectation is that, in a sense yet to be discovered, there is an embedded unitary theory, of solitons and their bound states, within the larger non-unitary one\cite{18}. Whether such a unitary theory can be defined for all values of the coupling constant is more equivocal. Finally it is worth pointing out that, in contrast to the sine-Gordon↔massive Thirring

\footnote{Issues of reality and stability are still in doubt\cite{3,11,12}.}
model relationship, there is no known quantum field theory of which the affine Toda solitons are the elementary excitations, though the particle S-matrices have been expressed as exchange relations\[19\].

As with sine-Gordon solitons, we can try to construct exact soliton S-matrices. Traditionally the route used has been the implication from conservation of the local charges that multiparticle scattering must factorize into products of two-particle scattering processes, as described by the Yang-Baxter equation (YBE). However, it is now recognized that underlying the YBE is a set of conserved charges forming a quantum group\[20\]. In contrast to the charges described earlier, these are non-local: that is, if measured on two asymptotically separate physical states, instead of yielding simply the sum of the values of the charges on the individual states, there are further interaction terms. Further, they do not commute - in fact they form a quantum group - and they have indefinite or non-integer spin. The multiplets in the theory will form representations of them (scalar for the particles, non-scalar for the solitons), and their conservation can be used to determine the S-matrices up to a scalar factor.

In the affine Toda theories the charges have non-integral spins of order \(4\pi/\beta^2 - 1\), and form a quantized affine algebra\[21\] in the principal gradation, with deformation parameter \(q = e^{4i\pi^2/\beta^2}\). There is a (topological charge-dependent) similarity transformation to the homogeneous gradation, in which the spectral parameter \(x\) is \(e^{(4\pi/\beta^2 - 1)\theta}\) (where \(\theta\) is the incoming particles’ rapidity difference). It is important to distinguish this realization of the quantum group from that obtained by quantizing the auxiliary algebra of the Lax pair, where the deformation parameter is \(q = e^{i\beta^2}\), and which thus has the conventional classical \((\beta \to 0)\) limit, the undeformed algebra. In contrast, in the classical limit of the S-matrices, whilst the quantum charge algebra itself becomes highly deformed, the ambiguities (multiples of \(2i\pi\) in the exponents) in \(\theta\) at the poles (which occur at \(x\) equal to some power of \(q\)) become relatively small, and the discrete bound state spectrum becomes continuous, as expected.

The two forms of \(q\) are very suggestive of some strong-weak coupling relationship between the auxiliary and quantum spaces. This is not a phenomenon only of affine Toda theories: it also occurs with Yangians\[22\], where there is a similar \(h \leftrightarrow 1/h\) relationship. It is unclear how or whether to attach any significance to this, relating as it does a mathematical artefact (the auxiliary algebra) to physical charges (the quantum algebra).
The physical states are expected to fall into fundamental representations of the quantized affine algebra. These are reducible representations of the \((q\text{-deformed})\) Lie subalgebra, containing its corresponding fundamental representation as a component. If we believe (as all the classical and semiclassical information leads us to) that these multiplets are quantum solitons, an outstanding issue is how they succeed in filling representations which are classically only part-filled. Pressing on, we encounter the outstanding open problem of the general construction of the relevant solutions of the YBE. The only simply-laced case for which it is solved completely is \(a_n\), and the results are precisely what one might hope for\[23, 24\]. If we apply the bootstrap principle, that all poles are to be interpreted in terms of on-shell diagrams of physical particles, and that, when the diagram is tree-level, S-matrices for the intermediate state can be constructed by fusion, the spectrum needed to close the bootstrap consists of the solitons and discretized spectra of breathers and breathing solitons. The analysis of the particles for real \(\beta\) can still be carried out for imaginary \(\beta\), and their masses and S-matrices are the same as for the lowest breathers, leading us to identify them.

Even in the absence of explicit S-matrices for other \(g\), something can still be said about their expected pole structure. Mathematically, the S-matrix fusings are the analogue for the quantum group of the Clebsch-Gordan rule for Lie algebras (to which Dorey’s rule is related\[25\], but not identical): they are its rule for tensoring representations. For the fundamental representations, this rule has recently been shown to be precisely Dorey’s rule\[26\]. The proof is exhaustive, however, so its truth is very suggestive of unseen mathematical structure beyond our current understanding of quantum groups. Mirroring these facts about the algebraic structure is a general observation that the scalar prefactors of the expected S-matrices are related to the ‘quantum dilogarithm’\[27\]. A relationship between two such different pieces of mathematics would come as no surprise: as we have seen, ATFTs have many ways of exhibiting Dorey’s rule.

1.3 \(g\) Nonsimply-Laced

When \(g\) is nonsimply-laced, the classical soliton masses are no longer proportional to those of the particles. In fact, they are proportional to those of the particles in the dual theory\[14\], based on \(g^\vee\), with the root lengths inverted; we shall call this ‘Lie duality’.
In addition to these untwisted algebras, whose affine extensions we denote $g^{(1)}$ and $g^{\vee(1)}$, we must now also consider the twisted algebras $g^{(1)\vee}$, obtained from a nonsimply-laced $g$ by inverting the lengths of both the $\alpha_i$ and $\alpha_0$. (Because of the involvement here of $\alpha_0$, we shall call this ‘affine duality’.) Here, both the particles and the solitons are a subset of those for a simply-laced algebra $h^{(1)}$, of which the twisted algebra is a subalgebra invariant under an automorphism of order $k$: in the usual notation, $g^{(1)\vee} = h^{(k)}$.

For nonsimply-laced algebras, the particle mass ratios\[3, 32\] (computed to one-loop order) are no longer $\beta$-independent. In fact the strong↔weak duality mentioned earlier extends to this case, using affine duality of algebras: the masses\[28\] and S-matrices\[28, 29, 30\] of the $(g^{(1)}, \beta)$ theory are those of the (twisted) $(g^{(1)\vee}, 4\pi/\beta)$ theory.

Dorey’s rule has not yet been fully generalized to nonsimply-laced cases. The rule for $g$ predicts the correct (tree-level) three-point couplings, but careful examination of the particle S-matrices\[29, 30\] shows that not all of these are allowed fusions in the exact quantum theory. In fact Dorey’s rule can be extended to cover twisted algebras\[35, 26\], and it is the intersection (which we shall denote $D(g)$) of the sets of couplings allowed by the rules for $g$ and for $g^{(1)\vee}$ which gives the correct set. However, a full generalization needs also to give the correct flexible mass and charge triangles, and, despite some progress, this has not yet been achieved.

The soliton masses\[31, 32, 33\] (computed semiclassically) no longer renormalize in the same ratio either: in fact the quantum-to-classical soliton mass ratio is, up to a species-independent factor, that of the particle\[9\] Because of the Lie duality mentioned above, however, there is no affine duality of soliton masses. This might have been expected: remember that whereas the particles have masses of order $m$, solitons have masses of order $m/\beta^2$, heavy in the weak limit, which would be expected to become very light for strong coupling. Further, the S-matrices, as can be seen from $x$ and $q$ (whose form is similar for nonsimply-laced algebras), will look very different for strong and weak couplings.

In calculating the soliton S-matrices, we encounter the subtlety that for the $\hat{g}$ theory, the non-local charges form $U_q(\hat{g}^{\vee})$. Their spins now depend on the root length, with the algebra in what has been termed the ‘spin gradation’. It can still be transformed to the homogeneous gradation, however, though with a slightly different form for $x$.

\[\text{8}^{\text{Although almost certainly true}}\] this matter is not settled: the semiclassical calculations have never been completed entirely satisfactorily.
The first complete set of S-matrices for a nonsimply-laced theory was found for the twisted algebra $d_{n+1}^{(2)}$, and contains the expected results: solitons, excited solitons and breathers, with the lowest breather state identifiable with the particle. The soliton fusing rule is $D(c_n)$, as expected.

The most subtle case is that of the untwisted nonsimply-laced algebras, where we must construct YBE solutions corresponding to twisted affine algebras. This was done for the $b_n^{(1)}$ theory, and the classical result mentioned in the first paragraph for the particle was found to apply to the lowest breather state as well: its mass is not always proportional to that of the soliton. Thus we now believe that in all cases the lowest breather is to be identified with the particle; in support of this, the exact masses and S-matrices turn out to be identical. The soliton fusing rule turns out to be $D(c_n)$, suggesting a Lie duality (not yet proven in general): that $g$ solitons have a $D(g^\vee)$ fusing rule.

2 Towards an overview

We can discuss these facts within the following scheme:

\[
\begin{matrix}
g^{\vee(1)}, \epsilon & \text{solitons} & \left( g^{(1)\vee}, \frac{1}{\epsilon} \right. & \text{solitons} \\
\uparrow \text{Lie} & & \downarrow \text{?} \\
g^{(1)}, \epsilon' & \text{particles} & \xleftrightarrow{\text{flying}} & g^{(1)\vee}, \frac{1}{\epsilon} & \text{particles}
\end{matrix}
\]

Here $\epsilon \equiv \beta^2/4\pi$ where $\beta$ is the real coupling: thus in the top-left, classical solitons would be obtained when $\epsilon$ is negative. Everything in this diagram has the same (exact) mass ratios, extracted from the proposed S-matrices, and the same $D(g)$ fusings. Solitons are in the top row, particles in the bottom; the left column is weakly-coupled, the right strongly-coupled.

2.1 Lie duality

The only fact present which has not already been described is the relationship of $\epsilon'$ to $\epsilon$. Recall that Lie duality operated classically (at zeroth order in $\epsilon$) in the left-hand column. At first order (i.e. from semiclassical calculations) it also applied with $\epsilon' = -\epsilon$ (with
real-coupled particles and imaginary-coupled solitons). However, on examination of the exact mass ratios we find that it works to all orders only with

\[
\frac{\epsilon'}{1 + \frac{h}{h^\vee} \epsilon'} = - \frac{\epsilon}{1 + \frac{h^\vee}{h^\vee} \epsilon} \quad \Rightarrow \quad \epsilon' = - \frac{\epsilon}{1 + \frac{h^\vee+\tilde{h}^\vee}{h} \epsilon},
\]

where \(h\) is the Coxeter number (of \(g\) and \(g^\vee\)), \(h^\vee\) is the dual Coxeter number of \(g\), and \(\tilde{h}^\vee\) that of \(g^\vee\).

To see this we require that breather poles in S-matrices for general algebras work analogously to the \(b_n\) case, the only case in which the S-matrices have been calculated and thoroughly examined. Our conjecture (which we do not believe to be too strong, since the form of the poles is highly constrained) is that for any Lie algebra and any species,

\[
m = 2M \sin \left( \frac{\pi|\alpha|^2}{2h\lambda} \right), \quad \text{where} \quad \lambda = -\frac{1}{\epsilon} - \frac{h^\vee}{h};
\]

\(m\) denotes particles, \(M\) solitons, and \(\alpha\) is the relevant simple root, with \(|\text{long root}|^2 = 2\).

A simple illustrative example is given by \(g_2\), which is Lie self-dual. Hence there is only one classical mass ratio, \(\sqrt{3}\). Duality exchanges the long and short roots, and hence also the heavy and light species of solitons compared with particles. Under the strong↔weak affine duality, the particles’ exact quantum mass ratio is

\[
\frac{m_h}{m_l} = 2 \cos \frac{\pi}{H(\epsilon)}
\]

\((h\ \text{means ‘heavy’, } l\ ‘light’)\) with

\[
\frac{1}{H(\epsilon)} = \frac{1}{6} - \frac{1}{18} \frac{1}{1 + \frac{2}{3} \epsilon}
\]

which runs to the affine dual (\(d_4^{(3)}\)) ratio in its strong-coupling limit (checked in perturbation theory to order \(\beta^4\)). Does this expression also give the exact soliton mass ratio? - that is, does the \(g_2^{(1)} \leftrightarrow d_4^{(3)}\) pair contain only one exact quantum mass ratio to describe both solitons and particles\(^\dagger\)? The answer is no. Whilst this ratio’s strong limit gives the correct ratio for the \(d_4^{(3)}\) soliton masses, we can see from \(\epsilon' \neq \epsilon\) in (2) that its weak value does not give the correct \(g_2\) soliton mass ratio.

\(^\dagger\)Thanks to Ed Corrigan for persistently asking me this question.
Unfortunately, whilst some of the soliton S-matrices in the $g_2^{(1)} \leftrightarrow d_4^{(3)}$ pair have been calculated, they are not enough to prove (1) for $g_2$, which, with $h^\vee = 4, h = 6$, yields

$$m_l = 2M_h \sin \left( \frac{\pi}{18\lambda} \right)$$
$$m_h = 2M_l \sin \left( \frac{\pi}{6\lambda} \right).$$

The solitons then have a new mass ratio whose $\epsilon \rightarrow 0$ limit is $\sqrt{3}$,

$$\frac{M_h}{M_l} = 2 \cos \frac{\pi}{H(\epsilon')} ,$$

with $\epsilon'$ related to $\epsilon$ by (2). In the strong limit $\epsilon \rightarrow \infty$ this ratio runs not to the $d_4^{(3)}$ ratio $2 \cos(\pi/12)$ but to $\sqrt{2}$.

Is there some overall duality scheme into which these results among the masses fit? Suppose we seek a duality between the $(\hat{g}, \epsilon)$ and $(\hat{g}', \epsilon')$ theories, in which, for some functional dependence of $\epsilon'$ on $\epsilon$, solitons and particles are exchanged. Such a relationship would of course take no account of excited solitons or higher breathers, and so might seem unlikely. Consideration of the classical masses requires $\hat{g} = g^{(1)}, \hat{g}' = g^{\vee(1)}$: that is, Lie duality. For $g_2$ we seek a relationship

$$\epsilon \leftrightarrow \epsilon'$$
$$M_h/M_l = m_h/m_l$$
$$m_h/m_l = M_h/M_l$$
$$M_h/M_l = m_h/m_l$$
$$m_h/M_l = M_h/m_l$$

The first two are what we considered above: they require

$$2 \cos \frac{\pi}{H(\epsilon')} = 2 \cos \frac{\pi}{H(\epsilon)}. \quad (4)$$

For all to be satisfied we require

$$\sin \left( \frac{\pi}{18\lambda} \right) \sin \left( \frac{\pi}{6\lambda} \right) = \frac{1}{4} \quad (5)$$
$$\sin \left( \frac{\pi}{6\lambda} \right) \sin \left( \frac{\pi}{18\lambda} \right) = \frac{1}{4} \quad (6)$$

(We find that (4) is equivalent to (5)/(6).) But there is no $\epsilon' = f(\epsilon)$ which solves (5) and (6), and hence no exact quantum Lie duality in this form.
This analysis generalizes to the $b_n \leftrightarrow c_n$ case. The $b_n$ spectrum is extracted from the S-matrices, and gives (3) with $h^\vee = 2n - 1, h = 2n$:

$$\frac{M_a}{M_n} = \frac{\sin \frac{a\pi}{H}}{\sin \frac{n\pi}{H}} \quad (a = 1, 2, \ldots, n-1)$$

$$m_a = 2M_a \sin \frac{\pi}{2n\lambda}$$

$$m_n = 2M_n \sin \frac{\pi}{4n\lambda}$$

$$\Rightarrow \frac{m_a}{m_n} = 2 \sin \frac{a\pi}{H},$$

For $c_n$, the known particle mass ratios

$$\frac{m_a}{m_n} = \frac{\sin \frac{a\pi}{H}}{\sin \frac{n\pi}{H}}$$

(again for $a = 1, 2, \ldots, n-1$) will lead via (3), now with $h^\vee = n + 1, h = 2n$, to soliton masses satisfying

$$m_a = 2M_a \sin \frac{\pi}{4n\lambda'}$$

$$m_n = 2M_n \sin \frac{\pi}{2n\lambda'},$$

giving in turn

$$\frac{M_a}{M_n} = 2 \sin \frac{a\pi}{H'},$$

(where our priming of quantities simply denotes their association with $c_n$ rather than $b_n$).

We now seek $\epsilon' = f(\epsilon)$ such that

$$\begin{align*}
(b_n, \epsilon) & \leftrightarrow (c_n, \epsilon') \\
M_i/M_j &= m_i/m_j \\
m_i/m_j &= M_i/M_j \\
M_i/m_j &= m_i/M_j \\
m_i/M_j &= M_i/m_j
\end{align*}$$

for all $i, j = 1, 2, \ldots, n$. The first of these is what we considered earlier, and is satisfied if (3) holds, giving a weak$\leftrightarrow$weak relationship.
The full set is satisfied if
\[
\sin \left( \frac{\pi}{4n\lambda} \right) \sin \left( \frac{\pi}{2n\lambda} \right) = \frac{1}{4} \tag{7}
\]
\[
\sin \left( \frac{\pi}{2n\lambda} \right) \sin \left( \frac{\pi}{4n\lambda} \right) = \frac{1}{4} \tag{8}
\]
holds. In the semiclassical limit (eqns (6.5,6.6) of [32]) it was pointed out that full \( b_n \leftrightarrow c_n \) Lie duality requires a strong\( \leftrightarrow \)weak relationship,
\[
\beta^2 \beta'^2 = 8h^2
\]
(that paper contained a misprint), which is the same as the leading term obtained from either (7) or (8),
\[
\frac{\pi^2}{2h^2\lambda\lambda'} = \frac{1}{4}.
\]
However, at all-orders (7) and (8) again have no solution for arbitrary \( \epsilon \), and we find no Lie duality in this form. There will be solutions for certain fixed \( \epsilon, \epsilon' \), but we have no indication that these would be significant.

2.2 Strongly-coupled solitons

As we pointed out, such a duality would seem too naive given the presence in the spectrum of many other excited states. However, as the coupling becomes strong (\( \epsilon \) large and negative) such states disappear from the spectrum. Strongly-coupled solitons are problematic classically and semiclassically, but there is no problem with examining their proposed exact S-matrices at such values of the coupling. How do solitons behave at strong coupling \( \epsilon \to \infty \)? Their S-matrices are of course non-diagonal, and we can never hope to identify soliton and particle S-matrices. However, in the infinite coupling limit the leading term in the S-matrix is just the identity: might the solitons behave like particles in this limit?

If we were to use the discussion at the beginning of this section as a guide, we might expect to see a relation between \( (g^{(1)}v, \epsilon = -1/\alpha) \) solitons and \( (g^{(1)}V, \alpha - \frac{hV + hV'}{h}) \) particles (where \( \alpha \) is small and positive). In the \( a_2^{(1)} \) case (to pick the simplest example), the relation would be between \( (a_2^{(1)}, -1/\alpha) \) solitons and \( (a_2^{(1)}, \alpha - 2) \) particles, which does not occur: the particle \( S_{11} \), for example, is certainly not unity at \( \epsilon = -2 \), and indeed is so only in the weak- and strong-coupling limits. The unlikelihood of (4) in simply-laced cases (where the rigid mass ratios give us no guidance on how \( \epsilon' \) and \( \epsilon \) are to be related) leads us to suspect
that it is only an artefact. If the solitons are to behave like weakly-coupled particles at all, their masses, of order $m/\epsilon$, seem to make it more relevant to investigate mid-ranges of the coupling, $\epsilon \sim 1$.

2.3 An observation on soliton S-matrices

In our initial scheme, the $g^{(1)}$ quantum solitons carry representations of $U_q(g^{(1)})$ non-local charges (and thus the $g^{(1)}$ solitons, $U_q(g^{(1)})$ charges). Both contain a $U_q(g)$ subalgebra, although the multiplets form different representations of it.

Specializing to $g = c_n$, we find that we are comparing $a^{(2)}_{2n-1}$ with $c^{(1)}_n$ YBE solutions, and these share a very similar algebraic structure, only differing in certain algebraic parameters, and hence in the form of dependence on the quantum group spectral parameter $x$. That is, the change $(n + 1, 2n) \mapsto (2n, 2n - 1)$ in the $c^{(1)}_n$ solutions yields those associated with $a^{(2)}_{2n-1}$. Unfortunately this cannot be achieved by a transformation on $\epsilon$: if we effect the composite Lie-affine transformation on $\epsilon$ implicit in our scheme we also rescale the exponents in $x$ and $q$. This is inevitable: the change in $\epsilon$ exchanges strong and weak couplings, whereas the $(n + 1, 2n) \mapsto (2n, 2n - 1)$ observation relates S-matrices which are both at weak coupling.

2.4 Folding non-local charges

Whereas quantum $b^{(1)}_n$ soliton multiplets carry representations of $U_q(a^{(2)}_{2n-1})$ charges, classically the $b_n$ solitons are ‘folded’ from $d^{(1)}_{n+1}$: that is, they are $d^{(1)}_{n+1}$ solitons invariant under an automorphism of the $d^{(1)}_{n+1}$ Dynkin diagram. The quantum $d^{(1)}_{n+1}$ soliton multiplets represent $U_q(d^{(1)}_{n+1})$ charges, and linear combinations of these, invariant under a twisting automorphism, form a $U_q(d^{(2)}_{n+1})$ subalgebra. To describe the quantum $b^{(1)}_n$ solitons, however, we would need to add roots and so multiply charges, and would expect to find a $U_q(a^{(2)}_{2n-1})$ subalgebra of $U_q(d^{(1)}_{n+1})$. It would be useful to understand how the two are related.
2.5 Further directions

All we have done here is to set up a few easy targets to shoot down. Perhaps the overall point is that we should seek to understand strongly-coupled solitons on their own terms rather than expect straightforward dualities with particles. Certainly, more investigation of the known soliton S-matrices at strong and, especially, at mid-range coupling is needed. However, we should also stand back a little and look for alternative ways forward.

Of primary importance is the generalization of the fusing rule to the full spectrum. In simply-laced cases, and for solitons only or particles only, mass ratios do not renormalize and Dorey’s rule is therefore rigid. Any extension of it, either to nonsimply-laced cases or to a rule for the rich arrangement of couplings between solitons and particles or breathers, requires a flexible, $\beta$-dependent and therefore intrinsically quantum construction. Steps have been made towards both former and latter, but the full construction remains some way off.

At a grander level, the goal is an algebraic scheme in which $g^{(1)}$, $g^{(1)}\wedge$, $g^{(1)}\vee$ and $g^{(1)}\wedge\vee$ all make appearances. Such a scheme would have eventually to incorporate all of our results, of course, but let us point out some of those likely to be more salient. First, the Coxeter number plays a central role in Dorey’s construction, and so the incorporation of the flexible Coxeter number $H$ is a correspondingly crucial step. Second, an emerging fact is that when (the non-commuting, non-additive, non-integer spin) quantum group charges exist, so do Wilson’s local charges (which are commuting, additive and have spins equal to the exponents). Unfortunately, the constructions are always rather model-specific; what we would like is a model-independent construction incorporating both. Finally, in the background remains our lack of understanding of quantum group representations: why Dorey’s rule applies to their tensor products remains a central unexplained fact.

Remarkably, such a construction may be emerging in the theory of deformed W-algebras. Much remains conjectural, but it is becoming clear that different limits of these deformations yield the centres of different quantized affine algebras, dual in various ways, and that the W-algebras can be used to construct representations of these. Discovering the facts of ATFTs within this picture is likely to make an interesting quest.
Acknowledgments

I should like to thank Gérard Watts for helpful comments, and Pembroke College, Cambridge, for a Stokes Fellowship.

References

[1] E. Corrigan, Recent developments in affine Toda quantum field theory, preprint DTP-94/55, hep-th/9412213

[2] G. M. Gandenberger, Exact $S$-matrices for quantum affine Toda solitons and their bound states, Ph.D. thesis, Cambridge University 1996, available at http://www.damtp.cam.ac.uk/user/hep/publications.html

[3] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, Affine Toda field theory and exact $S$-matrices, Nucl. Phys. B338 (1990), 689, and Multiple poles and other features of affine Toda field theory, Nucl. Phys. B356 (1991), 469

[4] P. Christe and G. Mussardo, Elastic $S$-matrices in (1+1) dimensions and Toda field theories, Int. J. Mod. Phys. A5 (1990), 4581

[5] M. Freeman, On the mass spectrum of affine Toda field theory, Phys. Lett. B261 (1991), 57

[6] G. Wilson, The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras, Ergod. Th. Dyn. Sys. 1 (1981), 1

[7] P. E. Dorey, Root systems and purely elastic $S$-matrices (I), Nucl. Phys. B358 (1991), 654; and (II), Nucl. Phys. B374 (1992), 741

[8] A. Fring, H.-C. Liao and D. I. Olive, The mass spectrum and coupling in affine Toda field theories, Phys. Lett. B66 (1991), 82; A. Fring and D. I. Olive, The fusing rule and the scattering matrix of affine Toda theory, Nucl. Phys. B379 (1992), 429

[9] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, Quantum $S$-matrix of the 1+1-dimensional Toda chain, Phys. Lett. B87 (1979), 389

[10] A. B. Zamolodchikov and Al. B. Zamolodchikov, Factorized $S$-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, Ann. Phys. (NY) 120 (1979), 253

[11] J. M. Evans, Complex Toda theories and twisted reality conditions, Nucl. Phys. B390 (1993), 225

[12] S. P. Khastgir and R. Sasaki, Instability of solitons in imaginary-coupling affine Toda theory, Prog. Theor. Phys. 95 (1996), 503

[13] T. J. Hollowood, Solitons in affine Toda field theory, Nucl. Phys. B384 (1992), 523
[14] D. I. Olive, N. Turok and J. W. R. Underwood, *Solitons and the energy-momentum tensor for affine Toda theory*, Nucl. Phys. B401 (1993), 663; and *Affine Toda solitons and vertex operators*, Nucl. Phys. B409 (1993), 509.

[15] M. Freeman, *Conserved charges and soliton solutions in affine Toda theory*, Nucl. Phys. B433 (1995), 657; hep-th/9408092.

[16] W. A. McGhee, *The topological charges of the \( a_n^{(1)} \) affine Toda solitons*, Int. J. Mod. Phys. A9 (1994), 2648, hep-th/9307035; and Durham Ph.D. thesis (1994, unpublished).

[17] U. Harder, A. Iskandar and W. McGhee, *On the breathers of \( a_n^{(1)} \) affine Toda field theory*, Int. J. Mod. Phys. A10 (1995), 1879; hep-th/9409035.

[18] J. Underwood and B. Spence, *Restricted quantum theory of affine Toda solitons*, Phys. Lett. B349 (1995), 83; hep-th/9501078.

[19] E. Corrigan and P. Dorey, *A representation of the exchange relation for affine Toda field theory*, Phys. Lett. B273 (1991), 237; hep-th/9109056.

[20] D. Bernard and A. LeClair, *Quantum group symmetries and non-local currents in 2-dimensional quantum field theories*, Comm. Math. Phys. 142 (1990), 99.

[21] G. Felder and A. LeClair, *Restricted quantum affine symmetry of perturbed minimal conformal models*, Int. J. Mod. Phys. A7 (Proc.Suppl.) 1A (1992), 239.

[22] N. J. MacKay, *Lattice quantization of Yangian charges*, Phys. Lett. B349 (1995), 94; hep-th/9505179.

[23] T. J. Hollowood, *Quantizing sl(N) solitons and the Hecke algebra*, Int. J. Mod. Phys. A8 No.5 (1993), 947.

[24] G. M. Gandenberger, *Exact S-matrices for bound states of \( a_2^{(1)} \) affine Toda solitons*, Nucl. Phys. B449 (1995), 375; hep-th/9501136.

[25] H. W. Braden, *A note on affine Toda couplings*, J. Phys. A25 (1992), L15.

[26] V. Chari and A. Pressley, *Yangians, integrable quantum systems and Dorey’s rule*, Comm. Math. Phys. 181 (1996), 265; hep-th/9505085.

[27] P. R. Johnson, *Exact quantum S-matrices for solitons in simply-laced affine Toda field theories*, Nucl. Phys. B496 (1997), 505; hep-th/9611117.

[28] G. W. Delius, M. T. Grisaru and D. Zanon, *Exact S-matrices for non simply-laced affine Toda theories*, Nucl. Phys. B382 (1992), 365.

[29] P. E. Dorey, *A remark on the coupling-constant dependence in affine Toda field theories*, Phys. Lett. B312 (1993), 291; hep-th/9304149.
[30] E. Corrigan, P. E. Dorey and R. Sasaki, *On a generalised bootstrap principle*, Nucl. Phys. B408 (1993), 579; \texttt{hep-th/9304065}

[31] T. J. Hollowood, *Quantum soliton mass corrections in sl(n) affine Toda field theory*, Phys. Lett. B300 (1993), 73; \texttt{hep-th/9209024}

[32] N. J. MacKay and G. M. T. Watts, *Quantum mass corrections for affine Toda solitons*, Nucl. Phys. B441 (1995), 277; \texttt{hep-th/9411169}

G. M. T. Watts, *Quantum mass corrections for c_2^{(1)} affine Toda solitons*, Phys. Lett. B338 (1994), 40; \texttt{hep-th/9404065}

[33] G. W. Delius and M. T. Grisaru, *Toda soliton mass corrections and the particle-soliton duality conjecture*, Nucl. Phys. B441 (1995), 259; \texttt{hep-th/9411176}

[34] G. Takacs, *Quantum affine symmetry and scattering amplitudes of the imaginary-coupled d_4^{(3)} affine Toda field theory*, Nucl. Phys. B502 (1997), 629; \texttt{hep-th/9701118}; and *The R-matrix of the U_q(d_4^{(3)}) algebra and g_2^{(1)} affine Toda field theory*, Nucl. Phys. B501 (1997), 711; \texttt{hep-th/9702193}

[35] P. E. Dorey, unpublished.

[36] G. M. Gandenberger and N. J. MacKay, *Remarks on excited states of affine Toda solitons*, Phys. Lett. B390 (1997), 18; \texttt{hep-th/9608055}

[37] G. M. Gandenberger and N. J. MacKay, *Exact S-matrices for d_{n+1}^{(2)} affine Toda solitons and their bound states*, Nucl. Phys. B457 (1995), 240; \texttt{hep-th/9506169}

[38] G. W. Delius, M. D. Gould and Y.-Z. Zhang, *Twisted quantum affine algebras and solutions to the Yang-Baxter equation*, Int. J. Mod. Phys. A11 (1996), 3415; \texttt{math.QA/9508012}

[39] G. M. Gandenberger, N. J. MacKay and G. M. T. Watts, *Twisted algebra R-matrices and exact S-matrices for b_n^{(1)} affine Toda solitons and their bound states*, Nucl. Phys. B465 (1996), 329; \texttt{hep-th/9509007}

[40] H. S. Cho, I.-G. Koh and J. D. Kim, *Duality in the d_4^{(3)} affine Toda theory*, Phys. Rev. D47 (1993), 2625

[41] T. Oota, *q-deformed Coxeter element in nonsimply-laced affine Toda field theories*, Yukawa Inst (Kyoto) preprint; \texttt{hep-th/9706054}

[42] A. A. P. Iskandar, *On breathers in affine Toda theories*, Ph.D. thesis, Durham, 1995.

[43] J. M. Evans, N. J. MacKay and M. Ul-Hassan, *Conserved charges and supersymmetry in principal chiral models*, DAMTP-97-120, \texttt{hep-th/9711140}

[44] E. Frenkel and N. Reshetikhin, *Deformations of W-algebras associated to simple Lie algebras*, \texttt{math.QA/9708006};

P. Bouwknegt and K. Pilch, *On deformed W-algebras and quantum affine algebras*, \texttt{math.QA/9801112}

[45] E. Corrigan, *Integrable field theory with boundary conditions*, \texttt{hep-th/9612138}