A detailed proof of the fundamental theorem of STF multipole expansion in linearized gravity

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The linearized field equations of general relativity in harmonic coordinates are given by an inhomogeneous wave equation. In the region exterior to the matter field, the retarded solution of this wave equation can be expanded in terms of 10 Cartesian symmetric and tracefree (STF) multipoles in post-Minkowskian approximation. For such a multipole decomposition only three and rather weak assumptions are required:

1. No-incoming radiation condition.
2. The matter source is spatially compact.
3. A spherical expansion for the metric outside the matter source is possible.

During the last decades, the STF multipole expansion has been established as a powerful tool in several fields of gravitational physics: celestial mechanics, theory of gravitational waves and in the theory of light propagation and astrometry. But despite its formidable importance, an explicit proof of the fundamental theorem of STF multipole expansion has not been presented thus far, while only some parts of it are distributed into several publications. In a technical but more didactical form, an explicit and detailed mathematical proof of each individual step of this important theorem of STF multipole expansion is represented.
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I. INTRODUCTION

The field equations of gravity, Einstein (1915,1916) [1, 2], constitute a set of 10 coupled nonlinear partial differential equations which relate the metric tensor $g^{\alpha\beta}$ of curved spacetime to the stress-energy tensor of matter $T^{\alpha\beta}$. Due to the inherited mathematical difficulties of solving these field equations in closed form, exact and physically well interpretable solutions of general theory of relativity are on rare occasions [3]. The most well-known examples for the case of massive isolated sources are the metric of a spherically symmetric massive body derived by Schwarzschild (1916) [4], the solution for a spherically symmetric and electrically charged body found by Reissner (1916) [5] and Nordström (1918) [6], and the metric for rotating bodies obtained by Kerr (1963) [7]. However, for more realistic scenarios, like an accelerated body, an asymmetric body, or a N-body system, exact solutions for the metric field are far out of reach or even do not exist. Therefore, approximative approaches of general relativity are essential for further progress in the theory of gravity. An important approximative approach is the theory of linearized gravity, where in harmonic gauge the coupled field equations of Einstein’s theory are simplified to a set of decoupled inhomogeneous wave equations for each of the 10 components of the metric tensor, Einstein (1916) [8]:

$$\Box_x h^{\alpha\beta} (t, x) = -\frac{16 \pi G}{c^4} T^{\alpha\beta} (t, x) ,$$

which is valid up to order $O(G^2)$, and $G$ is the gravitational constant. In Eq. (1), $\Box_x$ is the d’Alembert operator, $h^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta}$ is the metric perturbation ($g = \text{determinant of } g^{\alpha\beta}$, $\eta^{\alpha\beta} = \text{diag } (-1, +1, +1, +1)$ is the metric of flat space-time), and $c$ is the speed of light; the curved spacetime is assumed to be covered by harmonic coordinates $(t, x)$.

The mathematical structure of linearized field equations (1) resembles the field equations of classical electrodynamics in Lorentz gauge, $\Box_x A^\mu = -\frac{4 \pi}{c} j^\mu$, with $A^\mu$ being the four-potential and $j^\mu$ the four-current, but with the addition that in classical electrodynamics the space-time is Minkowskian, while the space-time in linearized gravity is in fact curved. Especially, the Green functions of both field equations are formally the same, and the harmonic coordinates $(t, x)$ are treated as though they were Cartesian coordinates in flat Minkowski space [19]. Hence, like in classical electrodynamics, a solution of (1) is given by, Einstein (1916) [8]:

$$\overline{h}^{\alpha\beta} (t, x) = \frac{4 G}{c^4} \int_V d^3 x' \frac{T^{\alpha\beta} (t_{\text{ret}}, x')}{|x' - x|} ,$$

where the integral runs over some finite spatial volume $V$ of the extended matter field, $t_{\text{ret}} = t - \frac{|x' - x|}{c}$ is the retarded time from a point inside the matter source with spatial coordinate $x'$ to a field point with spatial coordinate $x$. The so-called advanced solution, where $t_{\text{ret}}$ in (2) is replaced by $t_{\text{adv}} = t + \frac{|x' - x|}{c}$, is usually be regarded unphysical because it violates the causality condition and will not be considered here.

The multipole decomposition of (2) in terms of spherical harmonics is a highly effective approach to further analyze this solution. That tool of multipole expansion has originally been applied a long time ago in classical electrodynamics [10] and later been transformed into the case of linearized gravity. In this respect, a benchmark was the investigation of Campbell, Macek & Morgan (1977) [11] who have worked out a multipole decomposition of
the scalar (gravitational potential), vectorial (electrodynamical four-potential) and tensorial (linearized gravity) field outside the matter source in terms of spherical harmonics.

However, the use of so-called Cartesian symmetric and tracefree (STF) multipole moments [12–18] instead of spherical harmonics simplifies considerably the calculations in gravitational physics [19–22]: the mathematical relations and expressions in gravitational theory become simpler, the numerical algorithms can be performed more efficiently, and the whole approach of gravitational theory becomes more elegant. By now, the STF multipole expansion, in post-Newtonian approximation ("weak-field slow-motion approximation", i.e. $g_{00}, g_{ij}$ exact to order $O(c^{-2})$, $g_{0i}$ exact to order $O(c^{-3})$) and post-Minkowskian approximation ("weak-field approximation", i.e. $g_{\alpha\beta}$ exact to order $O(G)$), has been established as an important tool in linearized gravity and has found a wide range of applications: in celestial mechanics [22–24], in the theory of gravitational waves [25–27], and in the theory of light propagation in curved space-time [28–31] which is a fundamental aspect of relativistic astrometry. Meanwhile, the STF multipole expansion in linearized gravity has a remarkable history and encompasses some decades of period of time. Let us mention some important contributions which are considered as cornerstones in the theory of multipole expansion; further historical facts can be found, for instance, in Box 1 in [19], introductory sections in [20] and [32], and in Section 4.4 in [33].

First, the approach developed in [11] has been established in terms of STF tensors in a pioneering work by Thorne (1980) [19] in post-Newtonian approximation, where some first steps of earlier investigations [12, 13, 34, 35] have considerably been generalized. Especially, Thorne (1980) [19] has shown that the metric outside the matter source can be expanded in terms of 10 STF multipoles as follows (Eqs. (8.4) in [19]):

$$h^{\alpha\beta}(t, \mathbf{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[ \frac{\hat{F}_{L}^{\alpha\beta}(u)}{r} \right],$$

(3)

where $\partial_L = \frac{\partial}{\partial x^1 \ldots \partial x^l}$ are $l$ spatial derivatives, $r = |\mathbf{x}|$ is the spatial distance between the origin of coordinate system and the field point with spatial coordinate $\mathbf{x}$, $\hat{F}_{L}^{\alpha\beta}$ are 10 STF multipoles, and $u = ct - r$. Moreover, Thorne (1980) [19] has shown, using energy-momentum conservation (Eqs. (8.6) and (8.7) in [19]) and a sophisticated gauge transformation (Eqs. (8.9) in [19]) which preserves the harmonic gauge, that outside the matter the metric can finally be expressed in terms of 2 independent multipoles in post-Newtonian approximation: mass multipoles $\hat{M}_L$ and spin multipoles $\hat{S}_L$ (Eqs. (8.13) in [19]). However, the multipoles (Eqs. (5.32) in [19]) were still formally divergent at spatial infinity.

Consequently, Blanchet & Damour (1986) [20] have further developed the approach in [19] and have demonstrated that Thorne’s post-Newtonian multipoles are physically meaningful if one makes a rigorous use of the compact-support source of energy-momentum tensor. This important result has been achieved with the aid of the theory of distributions by means of which Blanchet & Damour (1986) [20] were able to extract the physically relevant and non-divergent part of Thorne’s multipoles in post-Newtonian approximation.

Finally, Blanchet & Damour (1989) [21] have established a powerful theorem in post-Minkowskian approximation which states that outside of an isolated source the metric can be expanded in terms of 10 Cartesian STF multipoles (Eqs. (B.2) and (B.3) in [21]).
defined by

$$\hat{F}^{\alpha\beta}_{L}(u) = \int_{V} d^{3}x' \hat{x}'_{L} \int_{-1}^{+1} dz \delta_{l}(z) T^{\alpha\beta}(\frac{u + z r'}{c}, x'),$$  \hspace{1cm} (4)$$

where $$\hat{x}'_{L} = \text{STF}_{a_{1}, ..., a_{l}}(x'_{a_{1}}, ..., x'_{a_{l}})$$, and $$r' = |x'|$$ is the distance between the origin of coordinate system and a point inside of the source with spatial coordinate $$x'$$; the coefficient functions in (4) are given by

$$\delta_{l}(z) = (2l + 1)!! \frac{(1 - z^{2})^{l}}{2^{l+1} l!},$$ \hspace{1cm} (5)$$

which are normalized: $$\int_{-1}^{+1} dz \delta_{l}(z) = 1$$. As we will see, the expansion (3) is valid in regions $$r > r_{0}$$, where $$r_{0}$$ is the radius of the smallest possible sphere which contains completely the source of matter. The expansion (3) - (4) represents the fundamental theorem of STF multipole expansion in linearized gravity, e.g. Eqs. (B.2) - (B.3) in [21], Eqs. (5.3) - (5.4) in [36], Eqs. (56) - (57) in [26], or Eq. (25) in [27], and stands for a solution of linearized field equations (1) in post-Minkowskian approximation, hence it is even valid in case of ultra-relativistic motion of matter inside the source.

After all, using energy-momentum conservation (Eqs. (5.14) and (5.18) in [36]) and applying a sophisticated gauge choice (Eq. (5.31) in [36]) Damour & Iyer (1991) [36] have demonstrated, footing on the pioneering works of Thorne (1980) [19] and Blanchet & Damour (1986, 1989) [20, 21], that also in post-Minkowskian approximation the family of these 10 multipoles can be reduced to finally only 2 independent multipoles: mass multipoles $$\hat{M}_{L}$$ (Eq. (5.33) in [36]) and spin multipoles $$\hat{S}_{L}$$ (Eq. (5.35) in [36]):

$$T^{\alpha\beta} = \hat{T}^{\alpha\beta}(\hat{M}_{L}, \hat{S}_{L}).$$ \hspace{1cm} (6)$$

The demonstration, that the metric in (3) which depends on 10 multipoles $$\hat{F}^{\alpha\beta}_{L}$$ can be reduced to the form in (6) where the metric depends only on 2 multipoles ($$\hat{M}_{L}, \hat{S}_{L}$$), is a rather ambitious assignment of a task and makes extensive use of irreducible Cartesian tensor techniques originally introduced in [16, 18]. Damour & Iyer (1991) [36] have also demonstrated that to order $$O(c^{-4})$$ their post-Minkowskian multipoles coincide with the post-Newtonian multipoles of Blanchet & Damour (1989) [21] (Eqs. (5.38) and (5.41) in [36]). So the investigation in [36] has been the final touch in the approach of STF multipole expansion to order $$O(G)$$. This elaborated work of Damour & Iyer (1991) [36] will, however, not be on the scope of the present investigation. Instead, we will be focussed on theorem (3) - (4), which is the heart and the core part of STF multipole expansion. An explicit proof of this important theorem is not so straightforward as one might believe and has not been presented in detail thus far; only some parts of it are published but scattered in several publications [11, 19, 21]. Here, in view of its formidable relevance in the theory of linearized gravity, we will outline a more detailed mathematical proof of each individual step of multipole expansion (3) in post-Minkowskian approximation.

The paper is organized as follows: In section II a compendium of the exact field equations of gravity is provided. The linearized approximation of general relativity is given in section
Section IV is devoted to the main part of our investigation, where a detailed proof of
the fundamental theorem (3) is represented and the required assumptions for its validity are
defined. A summary is finally given in section V.

We shall use fairly standard notations of the STF tensor approach [13, 19, 20, 22, 36]:

- Lower case Latin indices $i, j, \ldots$ take values $1, 2, 3$.
- Lower case Greek indices $\mu, \nu, \ldots$ take values $0, 1, 2, 3$.
- $\delta_{ij} = \delta_{ij} = \text{diag}(+1, +1, +1)$ is Kronecker delta.
- $n! = n(n-1)(n-2)\cdots2\cdot1$ is the faculty for positive integer; $0! = 1$.
- $n!! = n(n-2)(n-4)\cdots(2\ or\ 1)$ is the double faculty for positive integer; $0!! = 1$.
- $L = i_1i_2\ldots i_l$ and $Q = i_1i_2\ldots i_q$ are Cartesian multi-indices of a given tensor $T$, that
  means $T_L \equiv T_{i_1i_2\ldots i_l}$ and $T_Q \equiv T_{i_1i_2\ldots i_q}$, respectively.
- two identical multi-indices imply summation: $A_L B_L \equiv \sum_{i_1\ldots i_l} A_{i_1\ldots i_l} B_{i_1\ldots i_l}$.
- The symmetric part of a Cartesian tensor $T_L$ is, cf. Eq. (2.1) in [19]:
  \[ T_{(L)} = T_{(i_1\ldots i_l)} = \frac{1}{l!} \sum_{\sigma} A_{i_{\sigma(1)}\ldots i_{\sigma(l)}} , \]  
  \[ \text{where} \ \sigma \ \text{is running over all permutations of (1, 2, ..., l)}. \]
- The symmetric tracefree part of a Cartesian tensor $T_L$ (notation: $\tilde{T}_L \equiv \text{STF}_L T_L$) is, cf. 
  Eq. (2.2) in [19]:
  \[ \tilde{T}_L = \sum_{k=0}^{[l/2]} a_{lk} \delta_{i_1i_2\ldots i_{2k}} S_{i_{2k+1}\ldots i_l} a_{i_1} a_{i_2} a_{i_{2k}} , \]  
  \[ \text{where} \ [l/2] \ \text{means the largest integer less than or equal to} \ l/2 , \ \text{and} \ S_L \equiv T_{(L)} \ \text{abbreviates the symmetric part of tensor} \ T_L . \ \text{For instance,} \ T_{(L)}^{\alpha \beta} \ \text{means STF with respect to indices} \ L \ \text{but not with respect to indices} \ \alpha, \beta . \ \text{The coefficient in} \ [8] \ \text{is given by} \]
  \[ a_{lk} = (-1)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!! (2k)!!} . \]  
  \[ \text{As instructive examples of} \ [8] \ \text{let us consider the cases} \ l = 2 \ \text{and} \ l = 3 : \]
  \[ \tilde{T}_{ij} = T_{(ij)} - \frac{1}{3} \delta_{ij} T_{ss} , \]
  \[ \tilde{T}_{ijk} = T_{(ijk)} - \frac{1}{3} \left( \delta_{ij} T_{(kss)} + \delta_{jk} T_{(iss)} + \delta_{ki} T_{(jss)} \right) . \]

Further STF relations can be found in [19, 20]. We also will make use of Einstein’s sum 
convention, that mans repeated indices are implicitly summed over.
II. EINSTEIN’S FIELD EQUATIONS

The gravitation is described by 10 coupled nonlinear partial differential equations for the metric tensor, *Einstein* (1915,1916) \[1, 2\], which can be written in the form:

\[
R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8 \pi G}{c^4} T^{\alpha\beta},
\]

(12)

and which discover a fundamental relation between the metric of space-time and the matter field. Essentially, (12) represents a relation among contravariant tensors, of which \( R^{\alpha\beta} \) is the Ricci curvature tensor, \( g^{\alpha\beta} \) is the metric tensor with signature \((- , + , + , +)\), and \( T^{\alpha\beta} \) is the energy-momentum tensor of matter; \( R = R^\alpha_\alpha \) is the Ricci scalar of curvature. The field equations (12) are valid in any coordinate system, that means the coordinates are still arbitrary. For an asymptotically flat space-time, it is useful to decompose the metric tensor as follows:

\[
\sqrt{-g} g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta},
\]

(13)

where \( g \) is the determinant of metric tensor \( g^{\alpha\beta} \), \( h^{\alpha\beta} \) is the metric perturbation which describes the deviation of the metric tensor of curved space-time from the metric tensor of Minkowskian flat space-time given by

\[
\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag} (-1, +1, +1, +1). \tag{14}
\]

In harmonic gauge, also known as de Donder gauge,

\[
\partial_\beta h^{\alpha\beta} = 0, \tag{15}
\]

*Einstein’s* field equations (12) read (Eq. (36.37) in \[9\], Eq. (5.2b) in \[19\]):

\[
\square_x h^{\alpha\beta} = \frac{16 \pi G}{c^4} \left( \tau^{\alpha\beta} + t^{\alpha\beta} \right), \tag{16}
\]

where \( \square_x = \eta^{\mu\nu} \partial_{x^\mu} \partial_{x^\nu} \) is the d’Alembert operator. These both tensors in (16) are given by (Eq. (5.3) in \[19\])

\[
\tau^{\alpha\beta} = (-g) T^{\alpha\beta}, \tag{17}
\]

\[
t^{\alpha\beta} = (-g) t^{\alpha\beta}_{\text{LL}} + \frac{c^4}{16 \pi G} \left( \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} \mathcal{R}^{\alpha\beta}_{\mu\nu} - \mathcal{R}^{\alpha\beta}_{\mu\nu} \mathcal{R}_{\mu\nu} \right), \tag{18}
\]

where \( t^{\alpha\beta}_{\text{LL}} \) is the Landau-Lifschitz pseudotensor of gravitational field, in explicit form given by Eq. (20.22) in \[9\] or by Eqs. (96.8) and (96.9) in \[37\]. The field equations (16) are exact and the gravitational field is not necessarily weak, because the only assumptions made so far are the decomposition (13) and the choice of a harmonic coordinate system (15).
III. LINEARIZED THEORY OF GRAVITY AND STF MULTIPOLAR EXPANSION

The tensors (17) and (18) can be expanded in terms of the coupling constant, cf. Eq. (3.528) and Eq. (3.529) in [33]:

$$\tau^{\alpha\beta} = T^{\alpha\beta} + \frac{8\pi G}{c^4} \tau_1^{\alpha\beta} + \mathcal{O}(G^2),$$  \quad (19)

$$t^{\alpha\beta} = \frac{8\pi G}{c^4} t_1^{\alpha\beta} + \mathcal{O}(G^2).$$  \quad (20)

As it can be deduced from Eq. (16) and expansions (19) - (20), in harmonic gauge and up to terms of the order $\mathcal{O}(G^2)$, the Einstein’s field equations are simplified to d’Alembert’s wave equation for each of the 10 components of the metric tensor, Einstein (1916) [8]:

$$\Box_x h^{\alpha\beta} (t, x) = -\frac{16\pi G}{c^4} T^{\alpha\beta} (t, x),$$  \quad (21)

which is called linearized gravity, a term which refers to the fact that the approximative field equations (21) are linear partial differential equations, to be contrary to the nonlinear exact field equations of gravity (12) or (16); recall the harmonic coordinates are denoted by $(t, x)$.

Actually, there are formally infinitely many solutions of wave equation (21). These solutions of (21) consist of a general solution of the homogeneous wave equation $\Box_x h^{\alpha\beta}_{\text{hom}} = 0$ plus one particular solution of inhomogeneous wave equation (21): $h^{\alpha\beta} = h^{\alpha\beta}_{\text{hom}} + h^{\alpha\beta}_{\text{inhom}}$. For an unique solution of (21) one has to impose initial and boundary conditions. In case of an infinite space-time, there are no boundary conditions, and a well-posed problem (i.e. existence of one and only one unique solution) is given by the initial value problem at initial time $t''$ (Cauchy problem):

$$h^{\alpha\beta}_{\text{hom}} (t'', x'') = \int_{\partial S} d\Omega'' \left[ \frac{\partial}{\partial r''} \left( r'' h^{\alpha\beta}_{\text{hom}} (t'', x'') \right) + \frac{\partial}{\partial c t''} \left( r'' h^{\alpha\beta}_{\text{inhom}} (t'', x'') \right) \right],$$  \quad (22)

These initial conditions are valid in the entire three-dimensional space. According to Kirchhoff’s rigorous integration of the wave equation, Kirchhoff (1883) [38], an unique solution of (21) - (22) is given in terms of these initial conditions by an integral over an arbitrarily shaped but sufficiently smooth surface which contains completely the field point $x$ and the spatially compact matter field described by the energy-momentum tensor $T^{\alpha\beta}$; an explicit expression of Kirchhoff’s solution can be found, for instance, in Eq. (13) on page 420 in [39]. Here, without loss of generality, the surrounding surface $\partial S$ is assumed to be the surface of a sphere $S$. Then, the unique solution of (21) - (22) can be written as follows:

$$\bar{h}^{\alpha\beta} (t, x) = h^{\alpha\beta}_{\text{hom}} (t, x) + h^{\alpha\beta}_{\text{inhom}} (t, x),$$  \quad (23)

$$h^{\alpha\beta}_{\text{hom}} (t, x) = \frac{1}{4\pi} \int_{\partial S} dS'' \left[ \frac{\partial}{\partial r''} \left( r'' h^{\alpha\beta}_{\text{hom}} (t'', x'') \right) + \frac{\partial}{\partial c t''} \left( r'' h^{\alpha\beta}_{\text{inhom}} (t'', x'') \right) \right],$$  \quad (24)

$$h^{\alpha\beta}_{\text{inhom}} (t, x) = \frac{4G}{c^4} \int_{-\infty}^{t} dt' \int_{V} d^3x' T^{\alpha\beta} (t', x') \frac{\delta \left( t' - t + \frac{|x' - x|}{c} \right)}{|x' - x|}.$$

\end{verbatim}
Here, $\delta(x)$ is Dirac’s delta-distribution, normalized by $\int_{-\infty}^{\infty} dx \delta(x) = 1$. For a graphical elucidation of Eqs. (23) - (25) see Fig. 1. According to (24), the solution of homogeneous wave equation is given by a surface integral over a sphere, while (25) is the particular solution of inhomogeneous wave equation which is called retarded solution. In (24) we use $r'' = |x'' - x|$, and for the retarded time between field point $x$ and any point $x''$ on the surface of sphere we use $t'' = t - \frac{r''}{c}$. The homogeneous solution (24) contains the initial conditions (22), that means $h_{\text{hom}}^{\alpha\beta}(t, x)$ in the whole space-time is uniquely determined by its initial values (22) on surface $\partial S$; cf. Eq. (9) in [26]. The surface integral is given in terms of spherical coordinates $(r'', \theta'', \phi'')$ and the origin of the spherical coordinate system is located at the center of the sphere $S$, so that $d\Omega'' = \sin \theta'' d\theta'' d\phi''$. The integration in (24) runs over the surface with radius $r''$.

**FIG. 1: Graphical representation of Kirchhoff’s solution (23) - (25).** The global reference frame with harmonic coordinates $(t, x)$ is denoted by $\Sigma(t, x)$. The vector $x$ points from the origin of global coordinate system to the field point with spatial coordinate $x$. The matter field is described by an energy-momentum tensor $T^{\alpha\beta}$. The matter field and the field point are enclosed by a virtual sphere $S$ with surface $\partial S$. The center of the sphere is located at spatial coordinate $x$ of the field point, so that the spatial distance between the field point and a point inside the matter field is given by $|x' - x|$, and the spatial distance between the field point and the surface $\partial S$ is given by $|x'' - x|$. The matter field is assumed to be isolated, that means: 1. outside the region of some finite spatial volume $V$ (gray colored) the matter field vanishes, and 2. there is no gravitational radiation from outside through the surface $\partial S$ of sphere $S$. The metric field $\overline{h}^{\alpha\beta}$ at field point $(t, x)$ and at surface point $(t'', x'')$ has also been indicated.

Physically, Kirchhoff’s theorem [38] states that the homogeneous solution (24) is uniquely determined by source points which form a sphere with arbitrarily large radius $r''$. We will
assume that the matter source $T^{\alpha\beta}$ in (21) is isolated, that means the source is spatially compact and does not receive any radiation from other sources far away; note, however, that the matter source itself can emit gravitational radiation. Accordingly, since the radius of sphere $S$ can be arbitrarily large, we are allowed to take the limit up to spatial infinity and can replace the initial conditions (22) by the so-called no-incoming-radiation condition, cf. Eq. (10) in [26], and cf. Eqs. (2.5) and (2.6) in [40]:

$$\lim_{r'' \to +\infty} \left[ \frac{\partial}{\partial r''} (r'' \tilde{h}_{\text{hom}}^{\alpha\beta} (x'', t'')) + \frac{\partial}{\partial ct''} (r'' \tilde{h}_{\text{hom}}^{\alpha\beta} (x'', t'')) \right] = 0.$$  \hspace{1cm} (26)

If we impose the no-incoming-radiation condition (26), then the unique solution of (21) is given by the retarded solution (25), Einstein (1916) [8]:

$$\tilde{h}^{\alpha\beta} (t, x) = \frac{4}{c^4} \int_{-\infty}^{t} dt' \int_{V} d^3 x' \ T^{\alpha\beta} (t', x') \ \delta \left( t' - t + \frac{|x' - x|}{c} \right).$$  \hspace{1cm} (27)

According to the fundamental theorem of STF multipole expansion, outside the matter field of an isolated source the retarded solution in (27) can be decomposed in terms of 10 STF multipoles: Eqs. (3) - (4). In what follows, we will present a detailed proof of key formulae of this STF multipole expansion.

**IV. PROOF OF STF MULTIPOLE EXPANSION**

The inhomogeneous wave equation (21) is valid for any component of the tensors $\tilde{h}^{\alpha\beta}$ and $T^{\alpha\beta}$, so we consider the inhomogeneous wave equation just for one of the field components:

$$\Box x \tilde{h} (t, x) = -4 \pi T (t, x),$$  \hspace{1cm} (28)

so that $\tilde{h}$ stands either for $\tilde{h}^{00}$, $\tilde{h}^{0i}$, or $\tilde{h}^{ij}$, while $T$ stands either for $\frac{4}{c^4} T^{00}$, $\frac{4}{c^4} T^{0i}$, or $\frac{4}{c^4} T^{ij}$, respectively. As it has been discussed above, if the source is isolated (i.e. source is spatially compact and no-incoming radiation) then there exists one and only one solution of (28), namely (cf. Eq. (27)):

$$\tilde{h} (t, x) = \int_{-\infty}^{t} dt' \int_{V} d^3 x' \ T (t', x') \ G_R (t' - t, x' - x),$$  \hspace{1cm} (29)

where the spatial integration runs over the volume $V$ of the source, and the retarded Green function is given by

$$G_R (t' - t, x' - x) = \frac{\delta \left( t' - t + \frac{|x' - x|}{c} \right)}{|x' - x|}.$$  \hspace{1cm} (30)
The assumption that the source in (29) is spatially compact is formulated as follows (cf. text above Eq. (B1 a) in [21], cf. text above Eq. (3.1) in [36], or cf. text in section II A in [41])

\[ T (t', x') = 0 \quad \text{for} \quad |x'| > r_0 , \quad (31) \]

where \( r_0 \) is the radius of some sphere which contains completely the source.

Consider the retarded Green function in (30), which can be expanded in a series of Legendre polynomials \( P_q \), cf. Eqs. (D1) and (D2a) in [20]:

\[ G_R (t' - t, x' - x) = \Theta (t - t') \Theta (1 - |\nu|) \sum_{q=0}^{\infty} \frac{(2q+1)!!}{q!} \hat{n}_Q (\phi', \theta') \hat{n}_Q (\phi, \theta) P_q (\nu) , \quad (32) \]

where \( r = |x|, r' = |x'| \), and

\[ \Theta (s) = 0 \quad \text{for} \quad s < 0 , \]
\[ \Theta (s) = 1 \quad \text{for} \quad s \geq 0 , \quad (33) \]

is the Heaviside step function, and

\[ \nu = \frac{r^2 + r'^2 - c^2 (t - t')^2}{2rr'} , \quad (34) \]

is the argument of the Legendre polynomial.

**Proof 1:** We will show the validity of Eq. (32). Some parts of this proof have been presented in [11] in terms of spherical harmonics, while here we present a proof in terms of STF tensors. The Legendre polynomials can be defined by (Rodrigues’ formula, Eq. (12.65) in [42])

\[ P_l (z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} (z^2 - 1)^l , \quad (35) \]

and the normalization is (Eq. (12.48) in [42])

\[ \int_{-1}^{+1} dz P_n (z) P_m (z) = \frac{2}{2n+1} \delta_{nm} . \quad (36) \]

Consider two directions given by two normalized vectors \( n = \frac{x}{r}, n' = \frac{x'}{r'} \), with \( r = |x| \) and \( r' = |x'| \), and \( \gamma \) is the angle between \( n (\theta, \phi) \) and \( n' (\theta', \phi') \), that means \( \cos \gamma = n \cdot n' \); this angle satisfies the trigonometric identity: \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \) (Eq. (12.168) in [42]). Then, let us consider a function \( F (z, r, r') \) which depends on \( z = \cos \gamma \). Further, we assume the function \( F \) to be an element of Hilbert space \( V = L^2 \) given by \( V : = L^2 (z = [-1, +1] ; \mathcal{R}) \), that means the function \( F \) is square-integrable over the surface of the unit sphere. Then, such a function \( F \) can be expanded in terms of Legendre polynomials (Eq. (12.49) in [42]):

\[ F (z, r, r') = \sum_{l=0}^{\infty} P_l (z) F_l (r, r') , \quad (37) \]
where the coefficients are given by (cf. Eq. (12.50) in \[42\]; \(x = \cos \gamma\)),

\[
F_l (r, r') = \frac{2l + 1}{2} \int_{-1}^{+1} dx P_l (x) F (x, r, r') .
\]

(38)

The Legendre polynomial addition theorem states (Eq. (8.189) or Eq. (12.170) in \[42\], for a detailed proof see chapter 12 in \[42\]):

\[
P_l (x) = \frac{4 \pi}{2l + 1} \sum_{m=-l}^{l} Y_{lm} (\theta, \phi) Y_{lm}^* (\theta', \phi') ,
\]

(39)

where \(Y_{lm}\) are the spherical harmonics as defined in \[42\]. By inserting (38) and (39) into (37), we obtain

\[
F (z, r, r') = \frac{4 \pi}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{-1}^{+1} dx P_l (x) F (x, r, r') .
\]

(40)

Now we use a relation between spherical harmonicals and STF-tensors (Eq. (2.11) in \[19\], or Eq. (2.19) in \[22\])

\[
Y_{lm} (\theta, \phi) = \hat{Y}_{lm} L_{\hat{n}} L \hat{n}_{L L} (\theta, \phi) ,
\]

(41)

where

\[
\hat{n}_{L L} (\theta, \phi) = \text{STF} \frac{x_{i_1}}{r} \frac{x_{i_2}}{r} \ldots \frac{x_{i_l}}{r} ,
\]

(42)

and \(n_x + i n_y = e^{i \phi} \sin \theta, n_z = \cos \theta\) (Eq. (2.10) in \[19\]). The coefficients \(\hat{Y}_{lm} L\) (given by Eqs. (A6a) - (A6c) in \[20\], or by Eq. (2.21) in \[22\]) depend on \(l, m\) and on \(L\), but they are independent of \((\theta, \phi)\). Using (41) we verify

\[
F (z, r, r') = \frac{4 \pi}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{Y}_{lm} L \hat{n}_{L L} (\theta, \phi) Y_{lm}^* (\theta', \phi') \int_{-1}^{+1} dx P_l (x) F (x, r, r') .
\]

(43)

By implementing the inversion of Eq. (41) (see Eq. (2.23) in \[22\])

\[
\sum_{m=-l}^{l} \hat{Y}_{lm} L Y_{lm}^* (\theta', \phi') = \frac{(2l + 1)!!}{4 \pi l!} \hat{n}_{L L} (\theta', \phi') ,
\]

(44)

we obtain

\[
F (z, r, r') = \frac{1}{2} \sum_{l=0}^{\infty} \frac{(2l + 1)!!}{l!} \hat{n}_{L L} (\theta, \phi) \hat{n}_{L L} (\theta', \phi') \int_{-1}^{+1} dx F (x, r, r') P_l (x) .
\]

(45)

This expansion of a function of Hilbert space \(V = L^2\) into a series of Legendre polynomials has been given by Eq. (A.26) in \[20\]. According to Eq. (30), the function \(F\) as part of the integrand in Eq. (45) is given by:

\[
F (x, r, r') = \frac{\delta \left( r' - l + \sqrt{r^2 + r'^2 - 2 r r' x} \right)}{\sqrt{r^2 + r'^2 - 2 r r' x}} .
\]

(46)
Note, that the Green function (46) is automatically retarded since $t \geq t'$. Using the formula

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - \nu_i)}{|f'(\nu_i)|},$$  \hspace{1cm} (47)$$

where $\nu_i$ are the roots of $f$, i.e. $f(\nu_i) = 0$, and $f'(\nu_i) = \frac{\partial f(x)}{\partial x}|_{x=\nu_i}$, and taking into account that the only root of $f(x) = t' - t + \sqrt{r^2 + r'^2 - 2rr'x}$ is given by (cf. Eq. (54)):

$$\nu = \frac{r^2 + r'^2 - c^2 (t-t')^2}{2rr'},$$  \hspace{1cm} (48)$$

we can rewrite the function (46) as follows:

$$F(x, r, r') = \Theta(t-t') \Theta(1-|\nu|) \frac{e \delta(x-\nu)}{rr'}.$$  \hspace{1cm} (49)$$

Here, by the Heaviside function $\Theta(t-t')$ we have taken into account the fact that the Green function (46) is retarded, i.e. $t > t'$. Moreover, since the root $\nu$ in (48) can take arbitrarily large numerical values, we have to consider the fact that $-1 \leq x \leq +1$, which is taken into account by the Heaviside function $\Theta(1-|\nu|)$ in (49). By inserting relation (49) into Eq. (50) we can calculate the integral and get

$$F(n \cdot n', r, r') = \frac{c}{2rr'}\Theta(t-t') \Theta(1-|\nu|) \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} \hat{n}_L(\theta, \phi) \hat{n}_L(\theta', \phi') P_l(\nu).$$  \hspace{1cm} (50)$$

This result is in agreement with Eq. (52). q.e.d.

Furthermore, the source $T(t', \mathbf{x}')$ in (29) is expanded in spherical harmonics (cf. Eq. (B.4) in [21]), which means in STF notation:

$$T(t', \mathbf{x}') = \sum_{l=0}^{\infty} \hat{n}_L(\phi', \theta') \hat{T}_L(t', r'),$$  \hspace{1cm} (51)$$

where $\hat{T}_L$ are some STF tensorial functions, but their explicit structure is not relevant here because later the inversion of (51) will be used, see Eq. (126). Inserting the expansions (52) and (51) into (29) yields (in spherical coordinates we have $d^3x' = dr'd^2\theta' d\phi'$ and $r = |\mathbf{x}|, r' = |\mathbf{x}'|)$:

$$T(t, \mathbf{x}) = \frac{c}{2r} \int_{-\infty}^{t} dt' \int_V dr' r' \sum_{l=0}^{\infty} \hat{T}_L(t', r') \Theta(t-t') \Theta(1-|\nu|)$$

$$\times \sum_{q=0}^{\infty} \frac{(2q+1)!!}{q!} P_q(\nu) \hat{n}_Q(\phi, \theta) \int_{0}^{2\pi} d\theta' \sin \theta' \int_{0}^{\pi} d\phi' \hat{n}_L(\phi', \theta') \hat{n}_Q(\phi', \theta').$$  \hspace{1cm} (52)$$

For the integration over the angles $\theta'$ and $\phi'$ we obtain (see Eq. (2.5) in [19]):

$$\int_{0}^{2\pi} d\theta' \sin \theta' \int_{0}^{\pi} d\phi' \hat{n}_L(\phi', \theta') \hat{n}_Q(\phi', \theta') = \frac{4\pi l!}{(2l+1)!!} \delta_{lq}.$$  \hspace{1cm} (53)$$
Thus, we arrive at (cf. Eq. (D 3) in [20])

\[
\overline{h}(t, \mathbf{x}) = \frac{4\pi c}{2\nu} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \int_{-\infty}^{t} dt' \int_0^\infty dr' r' \hat{T}_L(t', r') P_l(\nu) \Theta(t - t') \Theta(1 - |\nu|) .
\]  

(54)

Now we introduce the following four variables (cf. Eqs. (D4 a) and (D 4 b) in [20]) which are independent of each other:

\[
u = ct - r , \quad \nu' = ct' - r' ,
\]  

(55)

and

\[
u = ct + r , \quad \nu' = ct' + r' .
\]  

(56)

After coordinate transformation (55) - (56), the previous integration domain of (54), \(D = \{(t', r'): -\infty \leq t' \leq t \text{ and } 0 \leq r' < r_0\}\), is given by (cf. comments above Eq. (D5) in [20])

\[
D = \{(u', v'): u \leq v' \leq v \text{ and } u' \leq u\} .
\]  

(57)

Proof 2: We will show that the integration domain \(D\) is given by (57). From the definition of the new variables (55) and (56) follows

\[
u = \frac{u + v}{2} , \quad \nu' = \frac{u' + v'}{2} ,
\]  

(58)

and

\[
u = \frac{v - u}{2} , \quad \nu' = \frac{v' - u'}{2} .
\]  

(59)

Let us consider the Heaviside function \(\Theta(1 - |\nu|)\), i.e. the relation:

\[
\left| \frac{r^2 + r'^2 - c^2 (t - t')^2}{2rr'} \right| \leq 1 .
\]  

(60)

This relation can also be written as follows:

\[
(r + r')^2 \geq c^2 (t - t')^2 ,
\]  

(61)

\[
(r - r')^2 \leq c^2 (t - t')^2 .
\]  

(62)

First we consider condition (61). Due to Heaviside function \(\Theta(t - t')\), i.e. \(t \geq t'\), we can rewrite (61) as follows:

\[
r + r' \geq c(t - t') .
\]  

(63)

By inserting (58) and (59) into (63) we find

\[
v' \geq u .
\]  

(64)

Now let us consider condition (62), which can also be written as

\[
c(t' - t) \leq r - r' \leq c(t - t') .
\]  

(65)

By inserting (58) and (59) into (63) we obtain from both conditions in (65):

\[
v' \leq v \quad \text{and} \quad u' \leq u .
\]  

(66)
Finally, the conditions (63) and (66) can be summarized as follows:

\[
D = \{(u', v') : u \leq v' \leq v \text{ and } u' \leq u\},
\]

which is just the integration domain (57). q.e.d.

Then, the integral (54) in these new variables (55) - (56) is given by (cf. Eq. (D5) in [20])

\[
\tilde{h}(t, x) = \frac{4\pi}{4(v - u)} \sum_{l=0}^{\infty} \hat{n}_L (\phi, \theta) \int_{D} du' dv' (v' - u') \hat{T}_L \left(\frac{u' + v'}{2 c}, \frac{v' - u'}{2}\right) \\
\times P_l \left(1 - 2 \frac{(u - u')(v - v')}{(v - u)(v' - u')}\right).
\]

Proof 3: We will show how to arrive at (68). First we note, by means of relations (58) and (59), that

\[
d' = \frac{1}{2} \left(\frac{v' - u'}{2}\right) \quad \text{from (59)}, \quad \text{we obtain from (54) as intermediate step:}
\]

\[
\tilde{h}(t, x) = \frac{4\pi}{4(v - u)} \sum_{l=0}^{\infty} \hat{n}_L (\phi, \theta) \int_{D} du' dv' (v' - u') \hat{T}_L (t', \nu) P_l (\nu),
\]

where we also have implemented the integration domain \(D\) in virtue of (57). Now, for the both arguments of the function \(\hat{T}_L\) we use \(t' = \frac{v' + u'}{2 c}\) according to (58) and \(r' = \frac{v' - u'}{2}\) according to (59) and obtain a further intermediate step:

\[
\tilde{h}(t, x) = \frac{4\pi}{4(v - u)} \sum_{l=0}^{\infty} \hat{n}_L (\phi, \theta) \int_{D} du' dv' (v' - u') \hat{T}_L (t', r') P_l (\nu).
\]

Finally, we have to reexpress the argument \(\nu\) of Legendre polynomial \(P_l\) in terms of the new variables \(u, v, u', v'\). First, from the definition of \(\nu\) given by Eq. (34) and the new variables given by Eqs. (53) - (55)

we get:

\[
\nu = \frac{(v - u)^2 + (v' - u')^2}{2} - \frac{(u + v - u')^2}{2} = 1 - 2 \frac{(u - u')(v - v')}{(v - u)(v' - u')},
\]

which can easily be checked. Thus, inserting (71) into (70) yields

\[
\tilde{h}(t, x) = \frac{4\pi}{4(v - u)} \sum_{l=0}^{\infty} \hat{n}_L (\phi, \theta) \int_{D} du' dv' (v' - u') \hat{T}_L \left(\frac{u' + v'}{2 c}, \frac{v' - u'}{2}\right) \\
\times P_l \left(1 - 2 \frac{(u - u')(v - v')}{(v - u)(v' - u')}\right).
\]
which is just in coincidence with expression (79). q.e.d.

Then we use the following relation for Legendre polynomial (cf. Eq. (D6) in [20]):

\[
P_l \left( 1 - 2 \frac{(u - u') (v - v')}{(v - u) (v' - u')} \right) = \frac{(-1)^l}{l!} \frac{(v - u)^{l+1}}{(v' - u')^l} \frac{\partial^l}{\partial u^l} \left[ \frac{(u - u') (v - v')}{(v - u)^{l+1}} \right].
\] (73)

Proof 4: Blanchet & Damour (1986) [20] have found an elegant way to show the validity of (73) via Euler-Poisson-Darboux differential equation, see text below Eq. (D6) in [20]. Here, we will demonstrate (73) straightaway. According to Eq. (73), the Legendre polynomial under consideration is given by

\[
P_l (\nu) = P_l \left( 1 - 2 \frac{(u - u') (v - v')}{(v - u) (v' - u')} \right).
\] (74)

Using Rodrigues’ formula (35) we verify

\[
P_l (\nu) = \frac{1}{2l!} \frac{d^l}{d\nu^l} (\nu^2 - 1)^l = \frac{2^l}{l!} \frac{d^l}{d\nu^l} \left( \frac{(u - u')^2 (v - v')^2}{(v - u)^2 (v' - u')^2} - \frac{(u - u') (v - v')}{(v - u) (v' - u')} \right)^l
\]

\[
= \frac{2^l}{l!} \frac{d^l}{d\nu^l} \left( \frac{(v - v') (v - u') (u - v')}{(v' - u')^2} \right)^l.
\] (75)

Now we use the relation

\[
\frac{d}{d\nu} = \left( \frac{du}{d\nu} \right) \frac{d}{du},
\] (76)

while for any value of \( l \) we obtain

\[
\frac{d^l}{d\nu^l} = \left[ \left( \frac{du}{d\nu} \right) \frac{\partial}{\partial u} \right] \times \left[ \left( \frac{du}{d\nu} \right) \frac{\partial}{\partial u} \right] \times \ldots \times \left[ \left( \frac{du}{d\nu} \right) \frac{\partial}{\partial u} \right] = \left[ \left( \frac{du}{d\nu} \right) \frac{\partial}{\partial u} \right]^l.
\] (77)

Let us calculate the factor in (78). For that we have to reconvert

\[
\nu = 1 - 2 \frac{(u - u') (v - v')}{(v - u) (v' - u')}
\] (78)

in terms of \( u \), and get

\[
u = \frac{u' + 1 - 2 \frac{u'}{v - v'}}{1 + 2 \frac{u'}{v - v'}}.
\] (79)

With allowance for expression (78), we have

\[
\left( \frac{du}{d\nu} \right) = \frac{1}{2} \frac{(v' - u') (v - u)^2}{(v' - v) (v - u)}.
\] (80)

In Eq. (80), after performing the differentiation, the expression (78) has been reinserted. Inserting operator (77) into (75), using (80), yields

\[
P_l (\nu) = \frac{(-1)^l}{l!} \frac{1}{(v' - u')^l} \left( \frac{v - u)^2}{(v - u')} \frac{\partial}{\partial u} \right)^l \left( \frac{(u - u') (u - v')}{(v - u)^2} \right)^l.
\] (81)
Now we apply the following relation, which is proven in appendix A:

\[
(v-u)^2 \frac{\partial}{\partial u} (u-u')^l (u-v')^l (v-u)^{l+1} = (v-u)^{l+1} \frac{\partial^l}{\partial u^l} (u-u')^l (u-v')^l .
\]  

(82)

By inserting (82) into (81) we obtain

\[
P_l (v) = \left( -1 \right)^l \frac{(v-u)^{l+1}}{(v'-u')^l} \frac{\partial^l}{\partial u^l} (u-u')^l (u-v')^l ,
\]

(83)

which represents the asserted relation (73). q.e.d.

Now, by means of relation (73) and with the aid of (cf. Eq. (A35 b) in [20])

\[
\frac{1}{l!} \hat{n}_L (\theta, \phi) (v-u)^l \frac{\partial^l}{\partial u^l \partial v^l} \left( \frac{F(u)}{v-u} \right) = \frac{1}{2} \hat{\partial}_L \left( \frac{F(ct-r)}{r} \right)
\]

(84)

Proof 5: We will show the validity of relation (81). The function \( F \) on the right-hand side in Eq. (84) does not depend explicitly on three-vector \( x \) but only on its absolute value \( r = |x| \). Therefore, it is meaningful to rewrite the differential operator \( \hat{\partial}_L \) in a form where the vectorial dependence is projected out of the differential process. This can be achieved with virtue of the following relation, see also Eq. (A.30) in [20].

\[
\hat{\partial}_L \left( \frac{F(ct-r)}{r} \right) = \hat{n}_L l! \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^l \left( \frac{F(ct-r)}{r} \right)
\]

(85)

For proofing (85) recall \( \frac{\partial f(r)}{\partial x^l} = \frac{x^l}{r} \frac{\partial f(r)}{\partial r} \) and one verifies \( \hat{\partial}_L f(r) = n_L l! \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^l f(r) \) plus terms containing at least one Kronecker delta which, however, vanish after STF operation; e.g. \( \text{STF} \delta_{ab} = 0, \text{STF} \delta_{ab} x^c = 0 \), etc.

By using the variables \( u = ct - r, v = ct + r \), see Eqs. (55) and (56), we can rewrite (85) as follows:

\[
\hat{\partial}_L \left( \frac{F(ct-r)}{r} \right) = 2\hat{n}_L (v-u)^l \left( \frac{1}{v-u} \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right) \right)^l \left( \frac{F(u)}{v-u} \right),
\]

(86)

where we have used the chain rule: \( \frac{\partial}{\partial r} = \left( \frac{\partial}{\partial v} \right) \frac{\partial v}{\partial r} + \left( \frac{\partial}{\partial u} \right) \frac{\partial u}{\partial r} = \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \). Now we apply the following identity which can easily be proven with the aid of mathematical induction (i.e. show the validity of (87) for \( l = 1 \) and then prove that the validity of (87) for any one natural number \( l \) implies the validity of (87) for the next natural number \( l + 1 \)):

\[
\left( \frac{1}{v-u} \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right) \right)^l \left( \frac{F(u)}{v-u} \right) = \frac{1}{l!} \frac{\partial^l}{\partial u^l \partial v^l} \frac{F(u)}{v-u} .
\]

(87)

By inserting (87) into (86) we obtain

\[
\frac{1}{2} \hat{\partial}_L \left( \frac{F(ct-r)}{r} \right) = \hat{n}_L \left( v-u \right)^l \left( \frac{\partial^l}{\partial u^l \partial v^l} \left( \frac{F(u)}{v-u} \right) \right) ,
\]

(88)

which is just relation (83). q.e.d.
we can rewrite Eq. (88) using $\partial_L$ as follows (cf. Eq. (D7) in [20]):

$$\bar{h}(t, x) = \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{l!} \int_D du' dv' \left( \frac{u' + v' - u}{2c}, \frac{v' - u}{c} \right) \hat{T}_L \left( \frac{(ct - r - u')(ct - r - v')}{r} \right) \partial_L \left[ \frac{(u - u')^l (v - v')^l}{l!} \right].$$

(89)

**Proof 6:** In order to obtain from Eq. (88) the expression in Eq. (89), the relation (73) is used, which yields

$$\bar{h}(t, x) = \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \int_D du' dv' (v' - u') \hat{T}_L \left( \frac{u' + v' - u}{2c}, \frac{v' - u}{2c} \right)$$

$$\times (-1)^l \frac{(v - u)^l}{l!} \partial_L \left[ \frac{(u - u')^l (v - v')^l}{(v - u)^{l+1}} \right].$$

(90)

For being able to apply relation (84), we have to rewrite the term

$$\frac{\partial^l}{\partial u^l} \left[ \frac{(u - u')^l (v - v')^l}{(v - u)^{l+1}} \right].$$

(91)

By means of this relation we find for the term (91) the following expression:

$$\frac{\partial^l}{\partial u^l} \left[ \frac{(u - u')^l (v - v')^l}{(v - u)^{l+1}} \right] = (-1)^l \frac{1}{l!} \partial_u \left[ \frac{(u - u')^l (v - v')^l}{v - u} \right].$$

(93)

Inserting relation (84) into Eq. (89) yields

$$\bar{h}(t, x) = \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \int_D du' dv' (v' - u') \hat{T}_L \left( \frac{u' + v' - u}{2c}, \frac{v' - u}{2c} \right)$$

$$\times \frac{1}{l!} \frac{(v - u)^l}{(v' - u')^l} \frac{1}{l!} \partial_u \partial_v \left[ \frac{(u - u')^l (v - v')^l}{v - u} \right],$$

(94)

where we have taken into account that $(-1)^l (-1)^l = 1$. Now we can apply relation (84) and obtain

$$\bar{h}(t, x) = \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{l!} \int_D du' dv' \left( \frac{u' + v' - u}{2c}, \frac{v' - u}{2c} \right) \hat{T}_L \left( \frac{(ct - r - u')(ct - r - v')}{r} \right) \partial_L \left[ \frac{(ct - r - u')^l (ct - r - v')^l}{r} \right],$$

(95)

which is just relation (89). **q.e.d.**

By means of the transformation

$$u' = s, \quad v' = s + 2y,$$

(96)
a straightforward calculation shows that (89) can be written as follows (cf. Eq. (D9) in [20]):

\[
\bar{h}(t, x) = \sum_{l=0}^{\infty} \frac{4 \pi}{2l+1!} \int_{-\infty}^{ct-r} ds \int_{-\infty}^{\frac{1}{2}(ct-r-s)} dy y^{l-1} \hat{T}_L \left(\frac{s + y}{c}, y\right) \partial_L \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r}\right].
\]

Furthermore, this expression can be written in the following form (cf. Eq. (D8) in [20]):

\[
\bar{h}(t, x) = -\sum_{l=0}^{\infty} \frac{4 \pi}{2l+1!} \int_{-\infty}^{ct-r} ds \hat{\partial}_L \left[\int_{a}^{\frac{1}{2}(ct-r-s)} dy y^{l-1} \hat{T}_L \left(\frac{s + y}{c}, y\right) \left(\frac{ct + r - s}{r}\right)^l \left(\frac{ct + r - s - 2y}{r}\right)^l\right].
\]

Here, we have commuted the operator \( \hat{\partial}_L \) with the integrals, because all differentiations of the upper limits \( \frac{1}{2} (ct - \epsilon r - s) \) with \( \epsilon = \pm 1 \) vanish, due to the factor \( (ct - \epsilon r - s - 2y)^l \) inside the integrals.

**Proof 7:** We will show how to obtain (98) from (97). First, we separate the second integral in (97) into two parts as follows:

\[
\bar{h}(t, x) = -\sum_{l=0}^{\infty} \frac{4 \pi}{2l+1!} \int_{-\infty}^{ct-r} ds \int_{-\infty}^{\frac{1}{2}(ct-r-s)} dy y^{l-1} \hat{T}_L \left(\frac{s + y}{c}, y\right) \partial_L \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r}\right] + \sum_{l=0}^{\infty} \frac{4 \pi}{2l+1!} \int_{-\infty}^{ct-r} ds \int_{a}^{\frac{1}{2}(ct-r-s)} dy y^{l-1} \hat{T}_L \left(\frac{s + y}{c}, y\right) \partial_L \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r}\right],
\]

where \( a \) is an arbitrarily chosen constant which separates the region of integration variable \( y \); in the first line in (99) the minus-sign in front of the integral takes into account that we have interchanged the upper and lower limits of integration. Now let us recall the fundamental theorem of integral calculus:

\[
\frac{d}{dx} \int_{a}^{x} dy f(y) = f(x) \bigg|_{y=x} = f(x).
\]

The differential operator \( \hat{\partial}_L \) in (99) contains terms like \( \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial y}\right) \frac{\partial}{\partial y} \). Accordingly, in the first line of (99) we can take the differential operator \( \hat{\partial}_L \) in front of the integral, because the differentiation of the upper limit \( \frac{1}{2} (ct - r - s) \) would yield a term

\[
\left. (ct - r - s - 2y)^l \right|_{y=\frac{1}{2}(ct-r-s)} = 0,
\]

\[
\left. \frac{\partial}{\partial y} \left(\frac{ct - r - s - 2y}{c}\right)^l \right|_{y=\frac{1}{2}(ct-r-s)} = 0.
\]
due to the term \((ct - r - s - 2y)^i\) in the argument of the integral; the differentiation of the lower limit \(a\) gives zero because \(a\) is a constant. Thus, instead of \((102)\) we can write

\[
\hat{h}(t, x) = -\sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \hat{L} \int_{a}^{ct-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right] + \sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \int_{a}^{ct-r-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \hat{L} \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right].
\]  

(102)

Now let us consider the second line in \((102)\), especially the term:

\[
\hat{L} \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right] = \hat{L} \left[ \frac{(ct-r)^l}{r} \right] \text{ if } i = 0, 1, ..., 2l,
\]

(104)

and obtain for \((103)\) the expression

\[
\hat{L} \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right] = \hat{L} \left[ \frac{(ct+r)^l}{r} \right] \text{ if } i = 0, 1, ..., 2l.
\]

(105)

Inserting \((105)\) into \((102)\) yields

\[
\hat{h}(t, x) = -\sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \hat{L} \int_{a}^{ct-r-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right] + \sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \int_{a}^{ct+r-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \hat{L} \left[ \frac{(ct+r-s)^l (ct+r-s-2y)^l}{r} \right].
\]

(106)

Now, also for the second line of \((106)\) we can take the differential operator \(\hat{L}\) in front of the integral, because the differentiation of the upper limit yields terms like

\[
(ct+r-s-2y)^l \bigg|_{y=\frac{k}{c}} = 0.
\]

(107)

Accordingly, \((106)\) can be written as follows:

\[
\hat{h}(t, x) = -\sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \hat{L} \int_{a}^{ct-r-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \left[ \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} \right] + \sum_{l=0}^{\infty} \frac{4\pi}{2l+1!} \int_{-\infty}^{ct-r} ds \hat{L} \int_{a}^{ct+r-s} dy \frac{dy}{y^{t+1}} \hat{T}_L \left( \frac{s + y}{c}, y \right) \left[ \frac{(ct+r-s)^l (ct+r-s-2y)^l}{r} \right].
\]

(108)
which is nothing else but relation (98). q.e.d.

It is readily seen that the expression (98) can also be written as (cf. Eq. (6.4) in [20])

\[
\mathcal{h}(t, x) = -4 \pi \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} ds \hat{\partial}_L \left[ \frac{R_a \left( \frac{1}{2} (ct - r - s), s \right) - R_a \left( \frac{1}{2} (ct + r - s), s \right)}{r} \right],
\]

(109)

with the function (cf. Eq. (B.6) in [21])

\[
R_a(r, s) = r^l \int_{\alpha}^r dy \left( \frac{r - y}{l!} \right) (\frac{2}{r})^{l-1} \hat{T}_L \left( \frac{s + y}{c}, y \right).
\]

(110)

Furthermore, by commuting the differential operator \( \hat{\partial}_L \) with the integral of the first term in (109) (for the proof one can use the very same arguments as presented in detail in Proof 7), the expression (109) can be written as follows (see Eq. (6.8) in [20] or see Eq. (B.5) in [21]):

\[
\mathcal{h}(t, x) = -\sum_{l=0}^{\infty} \hat{\partial}_L \left[ \frac{4\pi}{r} \int_{-\infty}^{\infty} ds R_a \left( \frac{ct - r - s}{2}, s \right) \right] - \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} ds \hat{\partial}_L \left[ \frac{4\pi}{r} R_a \left( \frac{ct + r - s}{2}, s \right) \right].
\]

(111)

The solution (111) is independent of the choice of \( a \). In case of a field point outside the source \( r > r_0 \), the first argument of the second term in (111) will satisfy \( \frac{1}{2} (ct + r - s) > r_0 \). Hence, if we choose \( a = r_0 \) it becomes evident from Eq. (110) that the second term in (111) will vanish when \( r > r_0 \), because the source is spatially compact (cf. Eq. (31)); note that the sequence of transformations (55), (56) and (96) yields \( y = |x'| \):

\[
\hat{T}_L \left( \frac{s + y}{c}, y \right) = 0 \quad \text{for} \quad y > r_0.
\]

(112)

This argumentation immediately yields (Eq. (B.7) in [21])

\[
\mathcal{h}(t, x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \left[ \frac{\hat{F}_L(u)}{r} \right],
\]

(113)

with (cf. Eq. (B.8) in [21])

\[
\hat{F}_L(u) = -4\pi (-1)^l \int_{-\infty}^{u} ds \left( \frac{u - s}{2} \right)^l \frac{(u-s)^{l/2}}{r_0} \int_{r_0}^{u} dy \left( \frac{u - s}{2} - y \right)^l \frac{2}{r} \left( \frac{2}{y} \right)^{l-1} \hat{T}_L \left( \frac{s + y}{c}, y \right).
\]

(114)
By a change of variables \( s = u + (z - 1)y \), the expression (114) can be transformed into (cf. Eq. (B.9) in [21]):

\[
\hat{F}_L(u) = \frac{4\pi}{2l+1} \int_{-1}^{1} dz (1-z^2)^l \int_0^{r_0} dy y^{l+2} \hat{T}_L \left( \frac{u + z y_c}{c}, y \right). \tag{115}
\]

**Proof 8:** We will show the validity of Eq. (115). Consider the expression given by Eq. (114):

\[
\hat{F}_L(u) = -4\pi (-1)^l \int_{-\infty}^{u} ds \left( \frac{u - s}{2} \right)^{(u-s)/2} \int_{r_0}^{r} dy \left( \frac{u - s}{2} - y \right)^l \hat{T}_L \left( \frac{s + y y_c}{c}, y \right). \tag{116}
\]

The transformation reads

\[
s = u + (z - 1)y, \tag{117}
\]

and the differentials

\[
ds dy = \left| \begin{array}{cc}
\frac{\partial s}{\partial z} & \frac{\partial s}{\partial y} \\
\frac{\partial y}{\partial z} & \frac{\partial y}{\partial y}
\end{array} \right| dz dy = y dz dy.
\]

Thus, one obtains

\[
\hat{F}_L(u) = -4\pi \frac{(-1)^l}{2l+1} \int_{-\infty}^{u} dz (1-z^2)^l \int_{r}^{r_0} dy y^{l+2} \hat{T}_L \left( \frac{u + z y_c}{c}, y \right). \tag{118}
\]

Now we have to transform the integration limits. First, we take into account that (cf. Eqs. 61 and 112)

\[
\hat{T}_L \left( \frac{u + z y_c}{c}, y \right) = 0 \quad \text{for} \quad y > r_0. \tag{119}
\]

Consequently, we conclude

\[
y_{\min} = \frac{u - s}{2}, \quad y_{\max} = r_0, \tag{120}
\]

and write (118) as follows:

\[
\hat{F}_L(u) = \frac{4\pi}{2l+1} \int_{s_{\min}}^{s_{\max}} dz (1-z^2)^l \int_{r_0}^{r} dy y^{l+2} \hat{T}_L \left( \frac{u + z y_c}{c}, y \right). \tag{121}
\]

From (117) we conclude

\[
s_{\min} = \frac{s_{\min} - u}{y_{\max}} + 1, \quad s_{\max} = \frac{s_{\max} - u}{y_{\min}} + 1. \tag{122}
\]

From (120) and taking into account the upper limit in (118), i.e. \( s \leq u \), we immediately get

\[
s_{\min} = u - 2r_0, \quad s_{\max} = u. \tag{123}
\]
Then, by inserting (120) and (123) into (122), we obtain the limits:

\[ z_{\text{min}} = -1, \quad z_{\text{max}} = +1. \]  

Accordingly, the integral (124) reads

\[
\hat{F}_L(u) = \frac{4\pi}{2^{l+1}} \int_{-1}^{+1} dz \left( -1 - z^2 \right) \int_{0}^{\pi} dy \, y^{l+2} \hat{T}_L \left( \frac{u + z y}{c}, y \right),
\]

which is just in coincidence with Eq. (115). q.e.d.

Finally, we use the inversion of Eq. (51) (see Eq. (A9 b) in [20] or Eq. (B.10) in [21])

\[
\hat{T}_L(t, y) = \left( \frac{2l + 1)!}{4 \pi l!} \right) \int_{0}^{2\pi} \sin \theta d\theta \int_{0}^{\pi} d\phi \, \hat{n}_L(\theta, \phi) T(t, y, \theta, \phi),
\]

Inserting (126) into (115), yields for (113) the following expression (cf. Eq. (B.2) in [21])

\[
\overline{h}(t, \mathbf{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[ \hat{F}_L(u) \frac{r}{r} \right],
\]

where \( r = |\mathbf{x}| \) is the spatial distance between the origin of coordinate system and the field point. By a transformation from spherical coordinates \( y = (y, \theta, \phi) \) to Cartesian-like coordinates \( \mathbf{x}' = (x'_1, x'_2, x'_3) \), the symmetric and tracefree multipole moments of the source are given by

\[
\hat{F}_L(u) = \int_{V} d^3\mathbf{x}' \hat{x}^L \int_{-1}^{+1} dz \, \delta_l(z) \left( \frac{u + z r'}{c}, \mathbf{x}' \right),
\]

where the spatial integral runs over the volume \( V \) of the source, \( r' = |\mathbf{x}'| \) is the spatial distance between the origin of coordinate system and a point inside the source with spatial coordinate \( \mathbf{x}' \), and \( u = ct - r \), cf. Eq. (55). In order to derive the form of Eq. (127), we also have used the relation \( \hat{\partial}_l \hat{F}_L(u) = \partial_l \hat{F}_L(u) \) since \( \hat{F}_L \) are STF multipoles, that means the trace over any pair of indices in \( \hat{F}_L \) vanishes: e.g. for \( l = 2 \) we would have \( \hat{\partial}_{i_1 i_2} \hat{F}_{i_1 i_2} = 0 \), we have \( \hat{\partial}_{i_1 i_2} \hat{F}_{i_1 i_2} = \partial_{i_1 i_2} \hat{F}_{i_1 i_2} \), and so on.

The functions in (128) are given by

\[
\delta_l(z) = \frac{(2l + 1)!}{2^{l+1} l!} (1 - z^2)^l.
\]

In view that \( \overline{h} \) in (127) stands either for \( \overline{h}^{\alpha 0}, \overline{h}^{\alpha \beta}, \) or \( \overline{h}^{\beta} \), while \( T \) in (128) stands either for \( 4G \hat{T}^{\alpha 0}, 4G \hat{T}^{\alpha \beta}, \) or \( 4G \hat{T}^{\beta} \), respectively, we can rewrite Eq. (127) and (128) in terms of their explicit tensorial structure:

\[
\overline{h}^{\alpha \beta}(t, \mathbf{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[ \hat{F}_L^{\alpha \beta}(u) \frac{r}{r} \right],
\]
where the STF multipoles are given by

\[ \hat{F}^{\alpha \beta}_{L}(u) = \int_{V} d^{3}x' \hat{x}_{L}^{\alpha} \int_{-1}^{+1} dz \delta_{l}(z) T^{\alpha \beta} \left( \frac{u + z r'_{c}}{c}, x' \right). \]  

(131)

The equations (130) and (131) represent the fundamental theorem of STF multipole expansion in post-Minkowskian approximation, as previously emphasized by Eqs. (3) - (4) in the introductory section. In virtue of equation (31) (compact support source) it is obvious that the multipole expansion (130) and (131) is valid for regions \( r > r_{0} \), where \( r_{0} \) is the radius of the smallest possible sphere which encloses completely the matter source. Finally, it should be noted that a straightforward application of theorem (127) and (128) for the case of electrodynamics leads immediately to the STF expansion given by Eqs. (4.2) and (4.3) in [36].

V. SUMMARY

In linearized gravity the Einsteins field equations are given by an inhomogeneous partial differential equation (1) for each of the 10 components of the metric tensor. In the region exterior to the source the retarded solution (2) can be expanded in terms of 10 Cartesian STF multipoles in post-Minkowskian approximation: Eqs. (3) - (4) (= Eqs. (130) and (131)). These 10 multipoles in (4) are not independent of each other, because using energy-momentum conservation (four relations) and gauge transformation (four relations) they can be reduced to finally 2 independent STF multipoles: mass multipoles and spin multipoles, \( \hat{M}_{L} \) and \( \hat{S}_{L} \), respectively, in post-Newtonian approximation demonstrated by Thorne (1980) [19] and Blanchet & Damour (1986, 1989) [20, 21], while in post-Minkowskian approximation this fact has been established by Damour & Iyer (1991) [36].

Meanwhile, the STF multipole expansion has become an important tool in linearized gravity and has demonstrated its efficiency for a wide spectrum of applications: in celestial mechanics [22–24], in the theory of gravitational waves [25–27], and in high precision astrometry where a particularly important aspect thereof is the theory of light propagation in curved space-time [28–31].

The theorem (3) - (4) is the fundamental theorem and the heart part of STF multipole expansion; see Eqs. (B.2) - (B.3) in [21], Eqs. (5.3) - (5.4) in [36], Eqs. (56) - (57) in [26], or Eq. (25) in [27]. But despite its formidable importance, an explicit proof of Eqs. (3) - (4) has not been presented so far, while some parts of the mathematical proof are distributed into several publications [11, 19–21]. In this investigation, a detailed proof of the STF multipole decomposition in form of a more didactical manuscript has been presented. Only three and rather weak assumptions are required for the validity of the STF multipole expansion:

1. No-incoming radiation condition, Eq. (26).
2. The source is spatially compact, Eq. (31).
3. A spherical expansion of the metric outside the source is possible, Eq. (51).

We hope that our investigation elucidates fundamental aspects of the main theorem of STF multipole expansion (3), where the multipoles in post-Minkowskian approximation are defined by (4).
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Appendix A: Proof of Eq. (82)

Relation (82) contains only derivatives with respect to variable $u$, and since $u$ and $v$ are independent variables, here we can treat $v$ as a constant. Accordingly, we introduce a new variable $x = u - v$ with $\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$, and rewrite relation (82) as follows:

$$\left[ x^2 \frac{\partial}{\partial x} \right]^l \frac{(x + a)^l (x + b)^l}{x^{2l}} = x^{l+1} \frac{\partial^l}{\partial x^l} \frac{(x + a)^l (x + b)^l}{x^{l+1}},$$  \hspace{1cm} (A1)

where $a = v - u'$ and $b = v - v'$, and the independent variable $u'$ and $v'$ are also considered as constant quantities. In order to show the validity of relation (A1) we apply the binomial theorem:

$$(x + a)^l = \sum_{p=0}^{l} \binom{l}{p} x^{l-p} a^p, \quad (x + b)^l = \sum_{q=0}^{l} \binom{l}{q} x^{l-q} b^q,$$  \hspace{1cm} (A2)

where the binomial coefficients are defined by

$$\binom{l}{p} = \frac{l!}{(l-p)!p!}, \quad \binom{l}{q} = \frac{l!}{(l-q)!q!}.$$  \hspace{1cm} (A3)

Inserting (A2) into (A1) yields

$$\sum_{p,q=0}^{l} \binom{l}{p} \binom{l}{q} a^p b^q \left[ x^2 \frac{\partial}{\partial x} \right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} = \sum_{p,q=0}^{l} \binom{l}{p} \binom{l}{q} a^p b^q x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}}.$$  \hspace{1cm} (A4)

Let us consider each individual term in (A4). One can easily show the validity of the following both relations by means of mathematical induction:

$$\left[ x^2 \frac{\partial}{\partial x} \right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} = (-1)^{l} x^{-(p+q-l)} \prod_{k=0}^{l-1} (p + q - k),$$  \hspace{1cm} (A5)

$$x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}} = (-1)^{l} x^{-(p+q-l)} \prod_{k=0}^{l-1} (p + q - k).$$  \hspace{1cm} (A6)

Accordingly, we can conclude the following identity for each individual term in (A4):

$$\left[ x^2 \frac{\partial}{\partial x} \right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} = x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}}.$$  \hspace{1cm} (A7)

That means, each individual term on the left-hand side in (A4) coincides with the corresponding term on the right-hand side in (A4). Thus, we have shown the validity of relation (A1) and, therefore, the validity of relation (A1) and (82), respectively.
Appendix B: Proof of Eq. (104)

Let us consider both expressions in (104), which we write as follows (for a proof of relation (B1) see Eqs. (85) - (88), while the proof of (B2) is very similar, see also relations (A35 b) and (A36 c) in [20]):

\[
\hat{\partial}_L \frac{(ct - r)^n}{r} = \frac{2}{l!} n_L (v - u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l} \frac{u^n}{v - u}, \tag{B1}
\]

\[
\hat{\partial}_L \frac{(ct + r)^n}{r} = \frac{2}{l!} n_L (v - u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l} \frac{v^n}{v - u}, \tag{B2}
\]

where \( u = ct - r \) and \( v = ct + r \). By subtraction of (B1) from (B2) one obtains

\[
\hat{\partial}_L \frac{(ct + r)^n}{r} - \hat{\partial}_L \frac{(ct - r)^n}{r} = \frac{2}{l!} n_L (v - u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l} \frac{v^n - u^n}{v - u}. \tag{B3}
\]

Now we recall the generalized version of third binomial theorem,

\[
v^n - u^n \over v - u = \sum_{j=0}^{n-1} v^{n-j-1} u^j. \tag{B4}
\]

Due to \( 2l \geq n \), the \((2l)\)th derivative of the polynomial in (B4) yields zero:

\[
\frac{\partial^{2l}}{\partial u^l \partial v^l} \frac{v^n - u^n}{v - u} = \frac{\partial^{2l}}{\partial u^l \partial v^l} \sum_{j=0}^{n-1} v^{n-j-1} u^j = 0. \tag{B5}
\]

Thus, inserting (B5) into (B3) yields

\[
\hat{\partial}_L \frac{(ct + r)^n}{r} - \hat{\partial}_L \frac{(ct - r)^n}{r} = 0, \tag{B6}
\]

which is just relation (104).

[1] A. Einstein, *Die Feldgleichungen der Gravitation*, Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin 2 (1915) 844.
[2] A. Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Physik (Ser. 4) 49 (1916) 769.
[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein’s Field Equations*, Cambridge University Press, Second Edition, 2003.
[4] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin 7 (1916) 189.
[5] H. Reissner, *Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie*, Annalen der Physik 50 (1916) 106.
[6] G. Nordström, *On the Energy of the Gravitational Field in Einstein’s Theory*, Verhandl. Koninkl. Ned. Akad. Wetenschap., Afd. Natuurk., Amsterdam 26 (1918) 1201.

[7] R.P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett. 11 (1963) 237.

[8] A. Einstein, *Näherungsweise Integration der Feldgleichungen der Gravitation*, Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin 1 (1916) 688.

[9] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Palgrave Macmillan, 1973.

[10] J.D. Jackson *Classical Electrodynamics*, John Wiley & Sons, Third Edition, New York, 1998.

[11] W. B. Campbell, J. Macek, T. A. Morgan, *Relativistic time-dependent multipole analysis for scalar, electromagnetic, and gravitational fields*, Physical Review D 43 (1977) 2156.

[12] R. Sachs, *Gravitational waves in general relativity. VI. The outgoing radiation condition*, Royal Society of London Proceedings Series A 264 (1961) 309.

[13] F.A.E. Pirani, *Lecture on General relativity*, ed. by A. Trautman, F.A.E. Pirani and H. Bondi, Prentice-Hall, Englewood Cliffs, 1964.

[14] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, 1953.

[15] I.M. Gelfand, R.A. Minlos, Z.Ya. Shapiro, *Representation of the Rotation and Lorentz groups*, Pergamon, Oxford, 1963.

[16] J.A.R. Coope, R.F. Snider, F.R.F. Mc Court, *Irreducible Cartesian Tensors*, Journal of Chemical Physics 43 (1965) 2269.

[17] J.A.R. Coope, R.F. Snider, *Irreducible Cartesian Tensors. II. General Formulation*, Journal of Mathematical Physics 11 (1970) 1003.

[18] J.A.R. Coope, *Irreducible Cartesian Tensors. III. Clebsch-Gordan Reduction*, Journal of Mathematical Physics 11 (1970) 1591.

[19] K.S. Thorne, *Multipole expansions of gravitational radiation*, Rev. Mod. Phys. 52 (1980) 299.

[20] L. Blanchet, T. Damour, *Radiative gravitational fields in general relativity: I. General structure of the field outside the source*, Phil. Trans. R. Soc. London A 320 (1986) 379.

[21] L. Blanchet, T. Damour, *Post-Newtonian generation of gravitational waves*, Annales de l’Institut Henri Poincare I Physique Theorique 50 (1989) 377.

[22] T. Hartmann, M.H. Soffel, T. Kioustephanis, *On the use of STF-tensors in celestial mechanics*, Celestial Mechanics and Dynamical Astronomy 60 (1994) 139.

[23] T. Damour, M. Soffel, C. Xu, *General-relativistic celestial mechanics. I. Method and definition of reference systems*, Physical Review D 43 (1991) 3273.

[24] T. Damour, M. Soffel, C. Xu, *General-relativistic celestial mechanics. II. Translational equations of motion*, Physical Review D 45 (1992) 1017.

[25] S.M. Kopeikin, G. Schäfer, C.R. Quinn, T.M. Eubanks, *Astrometric and timing effects of gravitational waves from localized sources*, Physical Review D 59 (1999) 084023.

[26] L. Blanchet, S.A. Kopeikin, G. Schäfer, *Gravitational radiation theory and light propagation*, Lecture Notes in Physics 562 (2001) 141.

[27] L. Blanchet, *Gravitational radiation from post-Newtonian sources and inspiralling compact binaries*, Living Reviews in Relativity 9 (2006) 4.

[28] S.M. Kopeikin, *Propagation of light in the stationary field of multipole gravitational lens*, Journal of Mathematical Physics 38 (1997) 2587.

[29] S.M. Kopeikin, P. Korobkov, A. Polnarev, *Propagation of light in the field of stationary and radiative multipoles*, Classical Quantum Gravity 23 (2006) 4299.

[30] S.M. Kopeikin, P. Korobkov, *General relativistic theory of light propagation in the field of radiative gravitational multipoles*, arXiv: gr-qc/0510084.
[31] C. Le Poncin-Lafitte, P. Teyssandier, *Influence of mass multipole moments on the deflection of a light ray by an isolated axisymmetric body*, Physical Review D 77 (2008) 044029.

[32] L. Blanchet, *On the multipole expansion of the gravitational field*, Classical Quantum Gravity 15 (1995) 1971.

[33] S.M. Kopeikin, M. Efroimsky and G. Kaplan, *Relativistic Celestial Mechanics of the Solar System*, Wiley-VCH, Weinheim, Germany, 2011.

[34] R. Epstein, R.V. Wagoner, *Post-Newtonian generation of gravitational waves*, Astrophysical Journal 197 (1975) 717.

[35] R.V. Wagoner, *Low-frequency gravitational radiation from collapsing systems*, Physical Review D 19 (1979) 2897.

[36] T. Damour, B.R. Iyer, *Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors*, Physical Review D 43 (1991) 3259.

[37] L.D. Landau, F.M. Lifschitz, *The Classical Theory of Fields*, Addison-Wesley, Fourth Edition, 1975.

[38] G.R. Kirchhoff, *Vorlesungen über mathematische Physik*, Annalen der Physik 18 (1883) 663.

[39] M. Born, E. Wolf, *Principles of Optics*, Cambridge University Press, Cambridge, Seventh Edition, 1999.

[40] S.A. Klioner, S.M. Kopeikin, *Microarcsecond Astrometry in Space: Relativistic Effects and Reduction of Observations*, Astronomical Journal 104 (1992) 897.

[41] L. Blanchet, *Second-post-Newtonian generation of gravitational radiation*, Physical Review D 51 (1995) 2559.

[42] G.B. Arfken, H.J. Weber, *Mathematical methods for physicists*, London, Academic Press, 4th Edition, 1995.