ISOMORPHISM INVARIANTS OF RESTRICTED ENVELOPING ALGEBRAS

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Abstract. Let $L$ and $H$ be finite-dimensional restricted Lie algebras over a perfect field $F$ such that $u(L) \cong u(H)$, where $u(L)$ is the restricted enveloping algebra of $L$. We prove that if $L$ is $p$-nilpotent and abelian, then $L \cong H$. We deduce that if $L$ is abelian and $F$ is algebraically closed, then $L \cong H$. We use these results to prove the main result of this paper stating that if $L$ is $p$-nilpotent, then $L/L' + \gamma_3(L) \cong H/H' + \gamma_3(H)$.

1. Introduction

Let $L$ be a restricted Lie algebra with the restricted enveloping algebra $u(L)$. We shall say that a particular invariant of $L$ is determined by $u(L)$, if every restricted Lie algebra $H$ also possesses this invariant whenever $u(L)$ and $u(H)$ are isomorphic as associative algebras. In particular, the restricted isomorphism problem asks whether the isomorphism type of $L$ is determined by $u(L)$. This problem is motivated by the classical isomorphism problem for group rings: is every finite group $G$ determined by its integral group ring $\mathbb{Z}G$? The survey article [11] contains most of the development in this area. In the late 1980’s, Roggenkamp and Scott [8] and Weiss [9] independently settled down the group ring problem for finite nilpotent groups.

There are close analogies between restricted Lie algebras and finite $p$-groups. In particular, the restricted isomorphism problem is the Lie analogue of the modular isomorphism problem that asks: given finite $p$-groups $G$ and $H$ with the property that $\mathbb{F}_p G \cong \mathbb{F}_p H$ can we deduce that $G \cong H$? Here, $\mathbb{F}_p$ denotes the field of $p$ elements. There has been intensive investigation on the modular isomorphism problem, however the main problem is rather far from being completely answered. Unfortunately not every technique from finite $p$-groups can be used for restricted Lie algebras. For example it is known that the class sums form a basis of the center of $FG$. It then follows that the center of $G$ is determined, see Theorem 6.6 in [12]. Whether or not the center of $L$ is determined by $u(L)$ remains an interesting open question.

In analogy with finite $p$-groups we consider the class $\mathcal{F}_p$ of restricted Lie algebras that are finite-dimensional and $p$-nilpotent. Let $L \in \mathcal{F}_p$. It follows from the Engel’s Theorem that $L$ is nilpotent. We shall examine the nilpotence class of $L$ in Corollary [22]. Note that whether or not the

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nilpotence class of \( G \) is determined by \( \mathbb{F}_p G \) has been considered in the recent years, however no major result is reported up-to-date, see [2].

We start the investigation on the restricted isomorphism problem by first considering the abelian case. In Proposition [2.5] we prove that if \( L \in \mathcal{F}_p \) is an abelian restricted Lie algebra over a perfect field \( \mathbb{F} \), then the isomorphism type of \( L \) is determined by \( u(L) \). Furthermore, if \( \mathbb{F} \) is algebraically closed then every abelian restricted Lie algebra is determined by its enveloping algebra, see Corollary [2.8].

It is not clear what is the next step beyond the abelian case in both the modular isomorphism problem and the restricted isomorphism problem. Nevertheless, we have proved in [13] that if \( L \in \mathcal{F}_p \) is a metacyclic restricted Lie algebra over a perfect field then the isomorphism type of \( L \) is determined by \( u(L) \). The main result of this paper that will be proved in Section 3, is another contribution in this direction; a similar result for finite \( p \)-groups was proved by Sandling [10]. Let us recall that for a Lie subalgebra \( I \subseteq L \), we denote by \( I' \) the restricted Lie subalgebra of \( L \) generated by all \( x' \), \( x \in I \). Also, \( \gamma_i(L) \) denotes the \( i \)-th term of the lower central series of \( L \). Our main result is as follows:

**Theorem.** Let \( L \in \mathcal{F}_p \) be a restricted Lie algebra over a perfect field. Then the restricted Lie algebra \( L/(L' + \gamma_3(L)) \) is determined.

2. Preliminaries

Let \( L \) be a restricted Lie algebra with the restricted enveloping algebra \( u(L) \) over a field \( \mathbb{F} \). By the Poincaré-Birkhoff-Witt (PBW) Theorem, see [8], we can view \( L \) as a restricted Lie subalgebra of \( u(L) \). Let \( \omega(L) \) denote the augmentation ideal of \( u(L) \) which is the kernel of the augmentation map \( \epsilon_L : u(L) \to \mathbb{F} \) induced by \( x \mapsto 0 \), for every \( x \in L \).

Let \( H \) be another restricted Lie algebra such that \( \varphi : u(L) \to u(H) \) is an algebra isomorphism. We observe that the map \( \eta : L \to u(H) \) defined by \( \eta = \varphi - \epsilon_H \varphi \) is a restricted Lie algebra homomorphism. Hence, \( \eta \) extends to an algebra homomorphism \( \overline{\eta} : u(L) \to u(H) \). In fact, \( \overline{\eta} \) is an isomorphism that preserves the augmentation ideals, that is \( \overline{\eta}(\omega(L)) = \omega(H) \), see [7] for the proof of similar fact for Lie algebras. So, without loss of generality, we assume that \( \varphi : u(L) \to u(H) \) is an algebra isomorphism that preserves the augmentation ideals.

Recall that \( L \) is said to be nilpotent if \( \gamma_n(L) = 0 \) for some \( n \); the nilpotence class of \( L \), denoted by \( cl(L) \), is the minimal integer \( c \) such that \( \gamma_{c+1}(L) = 0 \). We denote by \( L'_c \) the restricted subalgebra of \( L \) generated by \( L' = \gamma_2(L) \).

The \( n \)-th dimension subalgebra of \( L \) is

\[
D_n(L) = L \cap \omega^n(L) = \sum_{i p^j \geq n} \gamma_i(L)p^j,
\]

see [5].
Recall that $L$ is said to be in the class $\mathcal{F}_p$ if $L$ is finite-dimensional and $p$-nilpotent. The \textit{exponent} of $x \in L$, denoted by $\exp(x)$, is the least integer $s$ such that $x^{p^s} = 0$. Whether or not $L \in \mathcal{F}_p$ is determined by the following lemma, see [5].

**Lemma 2.1.** Let $L$ be a restricted Lie algebra. Then $L \in \mathcal{F}_p$ if and only if $\omega(L)$ is nilpotent.

Now, consider the graded restricted Lie algebra:

$$\text{gr}(L) := \bigoplus_{i \geq 1} D_i(L)/D_{i+1}(L),$$

where the Lie bracket and the $p$-map are induced from $L$. It is well-known that $u(\text{gr}(L)) \cong \text{gr}(u(L))$ as algebras, see [13]. So we may identify $\text{gr}(L)$ as the graded restricted Lie subalgebra of $\text{gr}(u(L))$ generated by $\omega^1(L)/\omega^2(L)$. Thus, $\text{gr}(L)$ is determined. We can now deduce the following:

**Corollary 2.2.** Let $L$ and $H$ be restricted Lie algebras such that $u(L) \cong u(H)$. If $L \in \mathcal{F}_p$ then $|\text{cl}(L) - \text{cl}(H)| \leq 1$.

**Proof.** Let $c = \text{cl}(L)$. We note that

$$\gamma_n(\text{gr}(L)) = \bigoplus_{i \geq n} \gamma_i(L) + D_{i+1}(L)/D_{i+1}(L),$$

for every $n \geq 1$. Since $\text{gr}(L)$ is determined, it follows that $\gamma_{c+1}(\text{gr}(H)) = 0$. Hence, $\gamma_{c+1}(H) \subseteq D_{c+2}(H)$. So, $\gamma_{c+2}(H) = \gamma_{c+3}(H)$. Since $H$ is nilpotent, it follows that $\gamma_{c+2}(H) = 0$. \hfill $\square$

Note that $D_n(\text{gr}(L)) = \bigoplus_{i \geq n} D_i(L)/D_{i+1}(L)$. Thus, $D_n(L)/D_{n+1}(L)$ is determined, for every $n \geq 1$. We remark that methods of [4] and [7] can be adapted to prove that $D_n(L)/D_{2n+1}(L)$ and $D_n(L)/D_{n+2}(L)$ are also determined, for every $n \geq 1$. In particular, $L/D_3(L)$ is determined. We shall need the following analogue of Lemma 5.1 in [7].

**Lemma 2.3.** If $\varphi : u(L) \to u(H)$ is an isomorphism then $\varphi(D_n(L) + \omega^{n+1}(L)) = D_n(H) + \omega^{n+1}(H)$, for every positive integer $n$.

Now suppose that $L$ is an abelian restricted Lie algebra. Note that the conditions on the $p$-map reduces to

$$(x + y)^p = x^p + y^p, \quad (\alpha x)^p = \alpha^p x^p,$$

for every $x, y \in L$ and $\alpha \in \mathbb{F}$. Thus the $p$-map is a semi-linear transformation. Let $\sigma$ be an automorphism of $\mathbb{F}$. Consider the skew polynomial ring $\mathbb{F}[t; \sigma]$ which consists of polynomials $f(t) \in \mathbb{F}[t]$ with multiplication given by

$$\alpha t^i \beta t^j = \alpha \beta^{\sigma^{-1} i} t^{i+j}.$$
$L$ decomposes as a direct sum of cyclic $\mathbb{F}[t;\sigma]$-modules. In particular, the number of these summands is unique. We summarize this in the following, see also [3] or Section 4.3 in [1]. We denote by $\langle x \rangle_p$ the subalgebra generated by $x$.

**Theorem 2.4.** Let $L$ be a finitely generated abelian restricted Lie algebra over a perfect field $\mathbb{F}$. Then there exist a unique integer $n$ and generators $x_1, \ldots, x_n \in L$ such that

$$L = \langle x_1 \rangle_p \oplus \cdots \oplus \langle x_n \rangle_p.$$ 

**Proposition 2.5.** Let $L \in \mathcal{F}_p$ be an abelian restricted Lie algebra over a perfect field $\mathbb{F}$. If $H$ is a restricted Lie algebra such that $u(L) \cong u(H)$, then $L \cong H$.

**Proof.** We argue by induction on $\dim_{\mathbb{F}} L$. Let $A$ be the subalgebra of $\omega(L)$ generated by all $u^p$, where $u \in \omega(L)$. We observe that $A \cong \omega(L^p)$, as algebras. Thus there is an induced isomorphism:

$$\omega(L^p) \cong \omega(H^p).$$

Since $L \in \mathcal{F}_p$, it follows that $\dim_{\mathbb{F}} L^p < \dim_{\mathbb{F}} L$. Thus, by the induction hypothesis, there exists a restricted Lie algebra isomorphism $\phi : L^p \cong H^p$. We now lift $\phi$ to an isomorphism of $L$ and $H$. By Theorem 2.4, there exist generators $x_1, \ldots, x_n \in L$ such that $L = \langle x_1 \rangle_p \oplus \cdots \oplus \langle x_n \rangle_p$. Without loss of generality we assume

$$L^p = \langle x_1^p \rangle_p \oplus \cdots \oplus \langle x_n^p \rangle_p,$$

for some $m \leq n$. Thus, $x_i^p = 0$, for every $i$ in the range $m < i \leq n$. Note that $\dim L = n + \dim L^p$. So, as it is mentioned in Theorem 2.4, $n$ is determined. Let $y_1, \ldots, y_n \in H$ such that $H = \langle y_1 \rangle_p \oplus \cdots \oplus \langle y_n \rangle_p$. Then

$$H^p = \langle y_1^p \rangle_p \oplus \cdots \oplus \langle y_m^p \rangle_p.$$

So, we can assume that $\phi(x_i^p) = y_i^p$, for every $1 \leq i \leq m$. We can verify that the map induced by $x_i \mapsto y_i$, for every $1 \leq i \leq n$, is a restricted Lie algebra isomorphism between $L$ and $H$.

**Corollary 2.6.** Let $L \in \mathcal{F}_p$ be a restricted Lie algebra over a perfect field. Then $L/L'_p$ is determined.

**Proof.** Note that $[u(L), u(L)]u(L) = L'_p u(L)$. Also, we have $u(L/L'_p) \cong u(L)/L'_p u(L)$. Hence, $u(L/L'_p)$ is determined. Since $L/L'_p \in \mathcal{F}_p$, it follows from Proposition 2.5 that $L/L'_p$ is determined.

It turns out that over an algebraically closed field stronger results hold. Before we state the next result we need to recall a well-known theorem, see [3] or Section 4.3 in [1]. Let $T_L = \langle x \in L \mid x^p = x \rangle_2$ and denote by $\text{Rad}(L)$ the subalgebra of $L$ spanned by all $p$-nilpotent elements.

**Theorem 2.7.** Let $L$ be a finite-dimensional abelian restricted Lie algebra over an algebraically closed field $\mathbb{F}$. Then $L = T_L \oplus \text{Rad}(L)$. 

4

HAMID USEFI
Corollary 2.8. Let $L$ be a finite-dimensional abelian restricted Lie algebra over an algebraically closed field $\mathbb{F}$. Let $H$ be a restricted Lie algebra such that $u(L) \cong u(H)$. Then $L \cong H$.

Proof. Note that for every $k \geq 1$,
\[
\dim_\mathbb{F} L/D_{p^k}(L) = \dim_\mathbb{F} L/D_p(L) + \cdots + \dim_\mathbb{F} D_{p^{k-1}}(L)/D_{p^k}(L),
\]
is determined. So $\dim_\mathbb{F} D_{p^k}(L)$ is determined, for every $k \geq 1$. Let $t$ be the least integer such that $\text{Rad}(L)^{p^t} = 0$. It follows that $D_{p^t}(L) = T_L$. Hence, $\dim_\mathbb{F} \text{Rad}(L) = \dim_\mathbb{F} \text{Rad}(H)$, by Theorem 2.7. Note that $L/T_L \cong \text{Rad}(L)$, as restricted Lie algebras. We claim that $\varphi(u(T_L)) = u(T_H)$. Suppose that the claim holds. Then $\varphi(T_L u(L)) = T_H u(H)$. So,
\[
u(L/T_L) \cong u(L)/T_L u(L) \cong u(H)/T_H u(H) \cong u(H/T_H).
\]
Thus, $u(\text{Rad}(L)) \cong u(\text{Rad}(H))$. Since $\text{Rad}(L), \text{Rad}(H) \in \mathcal{F}_p$, Proposition 2.5 then implies that there exists an isomorphism $\phi : \text{Rad}(L) \rightarrow \text{Rad}(H)$. Clearly, $\phi$ can be extended to an isomorphism of $L$ and $H$ by sending $x_i$ to $y_i$.

Now, we prove the claim. Let $z_1, \ldots, z_n$ be a basis of $\text{Rad}(H)$ and $y_1, \ldots, y_s$ be a basis of $T_H$ and assume that every $y_i$ is less than every $z_j$. Let $x \in T_L$ and express $\varphi(x)$ in terms of PBW monomials in the $y_i$ and $z_j$. So we have,
\[
\varphi(x) = u + \sum \alpha y_1^{a_1} \cdots y_s^{a_s} z_1^{b_1} \cdots z_n^{b_n},
\]
where $u$ is a linear combination of PBW monomials in the $y_i$ only and each term in the sum has the property that $b_1 + \cdots + b_n \neq 0$. Note that for a large $k$ we have $\varphi(x)^{p^k} = u^{p^k} \in u(T_H)$. But $\varphi(x) = \varphi(x)^{p^k}$. So, $\varphi(x) \in u(T_H)$. Since $u(T_L)$ is generated by $L$ and $\varphi$ is an algebra homomorphism, it follows that $\varphi(u(T_L)) \subseteq u(T_H)$. But $u(T_L)$ and $u(T_H)$ are finite-dimensional. So we get $\varphi(u(T_L)) = u(T_H)$. This proves the claim and so the proof is complete. $\square$

3. The Quotient $L/L^p + \gamma_3(L)$

We first record a couple of easy statements.

Lemma 3.1. Let $N$ be a restricted subalgebra of $L$. We have,
\[
\omega(L)N + N\omega(L) = [N, L] + N\omega(L).
\]

Lemma 3.2. For every restricted subalgebra $N$ of $L$ we have,
\[
(1) \quad L \cap ([N, L] + N\omega(L)) = [N, L] + N^p,
(2) \quad Nu(L)/\omega(L)N + N\omega(L) \cong N/([N, L] + N^p).
\]

Now write $J_L = \omega(L)L' + \omega'(L) = \omega(L)L'_p + L'_p\omega(L)$. Since both $\omega(L)L'$ and $L'\omega(L)$ are determined, it follows that $J_L$ is determined.

Corollary 3.3. If $L \in \mathcal{F}_p$ then $\dim_\mathbb{F}(L/L^p + \gamma_3(L))$ is determined.
Lemma 3.6. If $\mathfrak{L}'_p u(L)$ and $J_L$ are determined, it follows from Lemma 3.2 that $\dim_{\mathbb{F}}(\mathfrak{L}'_p/L^{p+\gamma_3(L)})$ is determined. The result then follows, since $L/L'_p$ is determined, by Corollary 2.6.

From now on we assume that $L \in \mathcal{F}_p$ and $\mathbb{F}$ is perfect. By Theorem 2.4 there exists $e_1, \ldots, e_n \in L$ such that

$$L/L'_p = \langle e_1 + L'_p \rangle_p \oplus \cdots \oplus \langle e_n + L'_p \rangle_p.$$ 

Let $\bar{X}$ be a basis of $L/L'_p$ consisting of $\bar{e}_i^p$, where $\bar{e}_i = e_i + L'_p$ and $1 \leq i \leq n$. We fix a set $X$ of representatives of $\bar{X}$. So the elements of $X$ are linearly independent modulo $L'_p$.

We define the height of an element $x \in L$, denoted by $\nu(x)$, to be the largest integer $n$ such that $x \in D_n(L)$, if $n$ exists and infinity otherwise. The weight of a PBW monomial $x_1^{a_1} \cdots x_t^{a_t}$ is defined to be $\sum_{i=1}^t a_i \nu(x_i)$. We observe that $\nu(e_i^p) = p^j$, for every $1 \leq i \leq n$ and every $1 \leq j < \exp(\bar{e}_i)$.

Indeed, if $e_i^p \in D_m(L)$, for some $m > p^j$, then $e_i^p = \sum_{k>j} \alpha_k e_i^k$ modulo $L'_p$. It follows then that $e_i^{\exp(\bar{e}_i)-1} \in L'_p$, which is a contradiction. Let $Y$ be a linearly independent subset of $L'_p$ such that $Z = X \cup Y$ is a basis of $\mathfrak{L}$ and the set $\{z + D_{\nu(z)+1} \mid z \in Z\}$ is a basis of $\mathfrak{L}$. One way to construct such a subset $Y$ is to take coset representatives of a basis for

$$\langle i \geq 1 \rangle \mathbb{D}_1(L) \cap (L'_p + \langle X \rangle_{\mathbb{F}})/D_{i+1}(L).$$

We need the following variant of Theorem 2.1 in [3].

Lemma 3.4. Let $L \in \mathcal{F}_p$. Let $\bar{Z}$ be a homogeneous basis of $\mathfrak{L}$ with a fixed set of representatives $Z$. Then the set of all PBW monomials in $Z$ of weight at least $k$ forms a basis for $\omega^k(L)$, for every $k \geq 1$.

Note that $J_L$ is linearly independent with the set of all PBW monomials in $X$. Let $E$ denote the vector space spanned by $J_L$ and all PBW monomials in $X$ of degree at least two. The following lemma is easy to see and so we omit the proof.

Lemma 3.5. The following statements hold.

1. $\omega(L) = L + E$.
2. $(L + J_L) \cap E = J_L = E \cap L'_p u(L)$.
3. $\omega(L)/J_L = L + J_L/ J_L \oplus E/J_L$.

Lemma 3.6. If $L \in \mathcal{F}_p$ then $E/J_L$ is a central restricted Lie ideal of $\omega(L)/J_L$.

Proof. The fact that $E/J_L$ is a central Lie ideal of $\omega(L)/J_L$ easily follows from the identity $[ab, c] = a[b, c] + [a, c]b$ which holds in any associative algebra. So we have to prove that $E/J_L$ is closed under the $p$-map. Since $J_L$ is an associative ideal of $\omega(L)$, it is enough to prove that $u^p \in E$, for every PBW monomial $u$ in $E$. Let $u = e_1^{a_1} \cdots e_n^{a_n}$, where each $a_i$ is in the range $0 \leq a_i < p^{\exp(\bar{e}_i)}$. It is not hard to see that $u^p = e_1^{pa_1} \cdots e_n^{pa_n}$ modulo $J_L$. 


Since $L \in \mathcal{F}_p$, each $\bar{e}_i$ is $p$-nilpotent. If $pa_i < p^{\exp(\bar{e}_i)}$, for every $1 \leq i \leq n$, then $u^p$ is a PBW monomial of degree at least two. Now suppose that $pa_i \geq p^{\exp(\bar{e}_i)}$, for some $i$. If $pa_i = p^{\exp(\bar{e}_i)}$ then $a_i$ is a power of $p$. Since $u$ has degree at least two, there exists $j \neq i$ such that $a_j \neq 0$. It now follows that $u^p \in J_L$. If $pa_i > p^{\exp(\bar{e}_i)}$ then $e^{pa_i}_i \in J_L$ and so $u^p \in E$. □

Lemma 3.7. We have $H \cap \varphi(E) \subseteq J_H$.

Proof. We suppose $J_H = 0$ and prove that $H \cap \varphi(E) = 0$. Let $v \in H \cap \varphi(E) \subseteq \omega^2(H)$. Let $u \in E$ such that $\varphi(u) = v$. So, $u \in \omega^2(L)$. We prove by induction that $u \in \omega^n(L)$, for every $n$. But $\omega(L)$ is nilpotent, by Lemma 2.1 and so $u = 0$. Suppose now, by induction, that $u \in \omega^n(L)$ and we prove that $u \in \omega^{n+1}(L)$. So, $v \in H \cap \omega^n(H) = D_n(H)$. Thus, by Lemma 2.3 $u \in (D_n(L) + \omega^{n+1}(L)) \cap E$. But

$$(D_n(L) + \omega^{n+1}(L)) \cap E \subseteq \omega^{n+1}(L).$$

Indeed, let $u = \sum \alpha_i z_i + w$, where each $z_i \in Z$ has height $n$ and $w \in \omega^{n+1}(L)$. By Lemma 3.4 $w$ is a linear combination of PBW monomials in $Z$ of weight at least $n + 1$. Since $u \in E$ it follows by the PBW Theorem that $\alpha_i = 0$, for every $i$. So, $u = w \in \omega^{n+1}(L)$, as required. □

Lemma 3.8. We have, $\omega(H)/J_H = H + J_H/\varphi(E)/J_H$.

Proof. By Lemma 3.7 it is enough to prove that

$$\omega(H)/J_H \subseteq H + J_H/\varphi(E)/J_H.$$ 

Note that both $\omega(H)/J_H$ and $\varphi(E)/J_H$ are determined. Since $\dim_p(H + J_H/J_H) = \dim_p(H/(H')^p + \gamma_3(H))$ is determined by Corollary 3.3, the result follows from Lemma 3.5. □

We can now finish the proof of our main result. Note that $L + J_L/J_L \cong L/L^p + \gamma_3(L)$, by Lemma 3.2.

Lemma 3.9. The restriction of the natural isomorphism $\omega(L)/J_L \to \omega(H)/J_H$ to $L + J_L/J_L$ induces an isomorphism of $L + J_L/J_L$ and $H + J_H/J_H$.

Proof. We denote by $\varphi$ the induced isomorphism $\omega(L)/J_L \to \omega(H)/J_H$. Let $\varphi_{L+J_L/J_L} = \varphi_1 + \varphi_2$ denote the restriction of $\varphi$ to $L + J_L/J_L$, where $\varphi_1 : L + J_L/J_L \to H + J_H/J_H$. It is enough to prove that $\varphi_1$ is a restricted Lie algebra isomorphism. Since $E/L$ is a central Lie ideal of $\omega(L)/J_L$, by lemma 3.6 $\varphi(E)/J_H$ is a central Lie ideal of $\omega(H)/J_H$. So, for every $x, z \in L$, we have

$$\varphi([x, z] + J_L) = [\varphi(x) + J_H, \varphi(z) + J_H] = [\varphi_1(x), \varphi_1(z)] + J_H.$$ 

So, $\varphi_1$ preserves the Lie brackets. Also,

$$\varphi(x^p + J_L) = \varphi(x)^p + J_H = (\varphi_1(x))^p + (\varphi_2(x))^p + J_H.$$ 

Since $(\varphi_2(x))^p + J_H \in \varphi(E)/J_H$, it follows that $\varphi_1$ preserves the $p$-powers. Furthermore, $\varphi_1$ is injective, by Lemma 3.5. Since $L + J_L/J_L$ and $H +$
$J_H/J_H$ have the same dimension, by Corollary 3.3 it follows that $\varphi_1$ is an isomorphism, as required.

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