Star Spectroscopy in the Constant B field Background

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Abstract

In this paper we calculate the spectrum of Neumann matrix with zero modes in the presence of the constant B field in Witten’s cubic string field theory. We find both the continuous spectrum inside $[-\frac{1}{3}, 0)$ and the constraint on the existence of the discrete spectrum. For generic $\theta$, $-1/3$ is not in the discrete spectrum but in the continuous spectrum. For each eigenvalue in the continuous spectrum there are four twist-definite degenerate eigenvectors except for $-1/3$ at which the degeneracy is two. However, for each twist-definite eigenvector the twist parity is opposite among the two spacetime components. Based upon the result at $-1/3$ we prove that the ratio of brane tension to be one as expected. Furthermore, we discuss the factorization of star algebra in the presence of B field under zero-slope limit and comment on the implications of our results to the recent proposed map of Witten’s star to Moyal’s star.
1 Introduction

The non-perturbative physics of Witten’s cubic string field theory [1, 2, 3] has been enhanced a lot since the proposal of vacuum string field theory (VSFT) by Rastelli, Sen and Zwiebach in [4] to [10], see also the related works [11] to [17]. The basic idea is to assume the universality of the BRST operator around the closed string vacuum after tachyon condensation such that the ghost and matter sector decompose and the classical solution of the matter string field is a projector, or the so called sliver state and its tachyonic lump generalization [5].

These projector states are understood as the non-perturbative D-brane soliton of the VSFT, so the ratio of the branes’ tension provide a nontrivial test of the VSFT proposal. The ratio is mainly determined by the spectrum of the Neumann matrix with and without the zero modes as following [3, 17]

\[ R = \frac{T_p}{2 \pi \sqrt{\alpha'} T_{p+1}} = \frac{3(V_{00} + \frac{b}{2})^2 \det(1 - M')^{\frac{1}{4}}(1 + 3M')^{\frac{1}{4}}}{\sqrt{2 \pi b^3} \det(1 - M)^{\frac{1}{4}}(1 + 3M)^{\frac{1}{4}}}, \]  

where \( M \) is the Neumann matrix in the oscillator basis without zero modes, and \( M' \) is the one with zero modes. The ratio has been first tested numerically to be almost one in [4] and then proved to be one analytically by Okuyama in [17]. Since the spectrum of both \( M \) and \( M' \) contain \(-1/3\) so that the ratio blows up and requires properly regularization related to the issue of twist anomaly investigated in [13].

The spectrum of the Neumann matrix \( M \) is constructed in [19] which shows that it has an doubly degenerate continuous spectrum except at \(-1/3\) which is non-degenerate. Later on, the explicit calculation of the spectrum of the Neumann matrix \( M' \) is also carried out in [20]. The result shows that there are additional discrete spectrum besides the continuous one found in [19]. However, it also leads to further puzzle that \( M' \) has doubly degenerate eigenvectors at \(-1/3\) so that naively the ratio R could be zero instead of one as expected. More detailed study of the density of states is required to resolve the issue.

On the other hand, Witten’s cubic SFT is believed to be background independent although there is no general proof for the background independence except some general arguments based upon BCFT in the context of VSFT [6] to [8]. It is then an issue to check the background independence case by case. A simple case is to turn on the constant background B field, some discussions for Witten’s SFT on this background have been discussed in [21, 22, 23], there they argue that the string vertexs can be obtained by an unitary transformation from the one without the B-field. Although this is the case, it is still interesting to understand the background independence issue in VSFT where the non-perturbative D-brane physics can be worked out.

The construction of the sliver state and the tachyonic lump state in the constant B field background based upon [21, 22, 23] is done in [26], and the ratio of the brane tension between \( Dp \) and \( D(p + 2) \) branes is given by

\[ \mathcal{R} = \frac{(\text{Det} G)^{\frac{1}{2}} T_p}{(2 \pi \sqrt{\alpha'})^2 T_{p+2}} = \frac{(\theta^2 + 12(V_{00} + \frac{b}{2})^2)}{4 \sqrt{2 \pi b^3 (\text{Det} G)^{\frac{1}{8}}}} \frac{\det(1 - \mathcal{M}')^{\frac{1}{4}}(1 + 3\mathcal{M}')^{\frac{1}{4}}}{\det(1 - \mathcal{M})^{\frac{1}{4}}(1 + 3\mathcal{M})^{\frac{1}{4}}}, \]  

where \( \mathcal{M} \) and \( \mathcal{M}' \) are the Neumann matrices in the oscillator basis with and without the zero modes as following [22, 23].
where $G$ stands for the open string metric. $\mathcal{M}'$ is the Neumann matrix with the zero modes in the oscillator basis, and due to the B field there is mixing between two coordinate directions so that the dimension of $\mathcal{M}'$ is twice larger than the one of $\mathcal{M}$. In order to determine the ratio one needs to have the spectrum of $\mathcal{M}'$, at least one should make sure if $\mathcal{M}'$ has an eigenvalue at $-1/3$ in order for the ratio to be finite $^1$.

Another interesting issue to which our results will be relevant is the generalization of the recent proposed map of Witten’s star to Moyal’s star $[27]$ to the case with zero modes in the constant B-field. There the Moyal conjugate pairs are the Fourier transform of the twist-even coordinate modes and twist-odd momentum modes, the Fourier basis are the eigenvectors of $\mathcal{M}$ or $\mathcal{M}'$ in our case, and the twist parity is defined with respect to left and right part of the open string in Witten’s cubic string field theory. So we need to construct the complete orthogonal eigenvector basis of $\mathcal{M}'$ in order to find out the proper Moyal conjugate pairs such that the proposed map makes sense. If so, then Witten’s cubic string field theory can be reformulated in the language of the noncommutative field theory, for which the soliton is just a projector.

In section 2, we will summarize the basic of the VSFT by setting up our conventions and notations, and warm up by calculating the $1 + \mathcal{M}'$ part of the ratio $\mathcal{R}$. Section 3 contains the main results of our paper in which we will calculate both the continuous and discrete spectrum of the Neumann matrix. In section 4 we will conclude our paper by discussing the implication of our results to the ratio $\mathcal{R}$ and to the map of Witten’s star to Moyal star. In the appendix we compute the ratio $\mathcal{R}$ explicitly and show that it is exactly one.

Note added: Upon finishing our paper we find that there appears two papers $[28]$ and $[29]$ dealing with the same topic, and the first one has substantial overlap with ours.

### 2 Neumann matrix in the constant B field background

#### 2.1 3-string vertex in the constant B-field background

To construct the sliver state which is D25-brane in VSFT, we need to use the 3-string vertex for the star product in the constant B field background $[24, 25, 26]$, which is given by

$$|V_3 > = \int dp^{(1)}dp^{(2)}dp^{(3)}\delta(p^{(1)} + p^{(2)} + p^{(3)}) \exp \left( -\frac{1}{2} \sum_{r,s \leq 3} \sum_{m,n \geq 1} V_{rs}^{mn}a_m^{(r)\dagger} \cdot a_n^{(s)\dagger} + 2 \sum_{m \geq 1} V_{r0}^{m0}a_m^{(r)\dagger} \cdot p^{(s)} + V_{00}^{rs}p^{(r)} \cdot p^{(s)} \right) - i\theta \sum_{r < s} p^{(r)} \times p^{(s)}$$

$$|p^{(1)} > \otimes |p^{(2)} > \otimes |p^{(3)} >$$

$^1$The finiteness of the ratio is more subtle since some regularization is required, moreover, in order to precisely determine the density states of the continuous spectrum. However, $-1/3$ is at the end of the spectrum so that we believe the match of the degeneracy of $-1/3$ for $\mathcal{M}$ and $\mathcal{M}'$ is essential for the ratio to be well-defined.
where
\[ |p> = \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{1}{2}(p \cdot p) + \sqrt{2}(a_0^\dagger \cdot p) - \frac{1}{2}(a_0 \cdot a_0^\dagger)\right)|0> \quad (4) \]
satisfy \( \hat{p}|k> = k|k> \). The dot products are with respect to the closed string metric along directions without \( B \) field and to the open string metric \( G_{\alpha\beta} \) along directions with \( B \) field. The \( \theta \) is the noncommutative parameter and the cross product means the contraction with the anti-symmetric symbol \( \epsilon^{\alpha\beta} \).

As shown in [21, 22, 23] the \( B \) field will change the Neumann boundary conditions of the free open string to the mixed one and introduce the Moyal-like phase in the string vertex besides replacing the closed string metric by the open string one in the \( B \)-field directions. Obviously one can decompose the above vertex into the transverse part \( |V_3^\perp> \) and the parallel part \( |V_3^\parallel> \) with respect to the \( B \) directions.

To construct the lower dimensional brane to which the transverse directions is no longer translational invariant, we need to use the 3-string vertex in the oscillator basis as the star product, which can be obtained by integrating out the momentum in (3),

\[ |V_3^\parallel> = \left(\frac{N_\theta \pi^{1/2}}{(1 + V_{00})}\right)^{d/2} \exp\left(-\frac{1}{2} \sum_{r,s \leq 3} \sum_{m,n \geq 0} V_{rs}^{\alpha\beta, mn} a_m^{(r)\alpha} a_n^{(s)\beta}\right) (|\Omega> \otimes |\Omega> \otimes |\Omega>), \quad (5) \]

where \( V_{00} \equiv V_{rr} \).

The relations between \( V \) and \( \tilde{V} \) is obtained in [26] and we summarize the results in the following

\[ \gamma_{\alpha\beta, mn}^{rs} = G_{\alpha\beta} V_{mn}^{rs} - \frac{N_\theta}{b} \sum_{t,v=1}^3 V_{m0}^{rv} U_{\alpha\beta,vt}^{st} V_{0n}^{ts} \quad (6) \]
\[ \gamma_{0n}^{\alpha\beta, rs} = \frac{N_\theta}{\sqrt{b}} \sum_{t=1}^3 U_{\alpha\beta,rt} V_{0n}^{ts} \quad (7) \]
\[ \gamma_{00}^{\alpha\beta, rs} = G_{\alpha\beta} \delta^{rs} - N_\theta U_{\alpha\beta, rs} \quad (8) \]

where
\[ U_{\alpha\beta, rs} = G_{\alpha\beta} \phi^{rs} - ia\epsilon_{\alpha\beta} \chi^{rs}, \quad (9) \]

with
\[ N_\theta \equiv \frac{8b(V_{00} + \frac{b}{2})}{\theta^2 det(G) + 12(V_{00} + \frac{b}{2})^2}, \quad a \equiv \frac{\theta}{4(V_{00} + \frac{b}{2})}, \quad (10) \]

and
\[ \chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad \phi^{rs} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \quad (11) \]

Instead of using \( V^{rs} \) one defines the matrices \( M^{rs} \) as following
\[ M^{rs} = CV^{rs}, \quad (12) \]
where $C_{mn} = (-1)^n \delta_{m,n}, m, n \geq 1$ is the twist matrix. The matrices $M^{rs}$ are commuting to each other so that it is useful to solve the algebraic equation for the projector states. Similarly, as shown in [26] the matrices

$$\mathcal{M}^{rs} = C' \mathcal{V}^{rs}$$

are also commuting to each other. The generalized twist matrix $C'_{mn} = (-1)^n \delta_{m,n}, m, n \geq 0$. Although there are nine Neumann matrices, only three of them are independent, in the following we will only focus on one of them, that is, $M \equiv M^{11}$, and the one with zero modes in the constant B-field background $M' \equiv M'^{11}$. These two are the only matrices appear in the ratio of brane tension in (2), and will be our focus to find out their spectrum.

### 2.2 The form of $\mathcal{M}'$

In the following we write down the explicit form of $\mathcal{M}'$ by adopting the basis introduced firstly in [5]. After some calculations, we arrive

$$\mathcal{M}'_{\alpha \beta}^{\alpha \beta} = G^{\alpha \beta} (1 - N_\theta),$$

$$\mathcal{M}'_{\alpha \beta}^{\alpha \beta}_{0n} = - \sqrt{2} \, N_\theta (G^{\alpha \beta} < v_e| - i \epsilon^{\alpha \beta} \frac{2a}{\sqrt{3}} | v_o|),$$

$$\mathcal{M}'_{\alpha \beta}^{\alpha \beta}_{mn} = G^{\alpha \beta} (M_{mn} - \frac{2}{b} N_\theta (| v_e > < v_e| - | v_o > < v_o|))$$

$$+ i \epsilon^{\alpha \beta} \frac{2a}{\sqrt{3}} \frac{2}{b} N_\theta (| v_e > < v_o| + | v_o > < v_e|), \quad m, n \geq 1,$$

where the vectors $| v_e >$ and $| v_o >$ are the same as defined in [20] and [17]. Note that $|(v_{e,o} >)^T = < v_{e,o}|$ and

$$C| v_e >= | v_e >, \quad C| v_o >= - | v_o >.$$  

Explicitly,

$$< v_e| = \left(0, \frac{A_2}{\sqrt{2}}, 0, \frac{A_4}{\sqrt{4}}, 0, \ldots \right), \quad < v_o| = \left(A_1, 0, \frac{A_3}{\sqrt{3}}, 0, \ldots \right),$$

where

$$\left(\frac{1 + iz}{1 - iz}\right)^{1/3} = \sum_{n \in \text{even}} A_n z^n + i \sum_{n \in \text{odd}} A_n z^n.$$  

Unlike the case without B field, the Neumann matrix $\mathcal{M}'$ carries both space-time and level indices, and there are imaginary part associated with nonzero $\theta$. Moreover, the mixing of the space-time indices due to the nonzero $\theta$ will make the diagonalization of the Neumann matrix non-trivial.

Notice that, the open string metric

$$G_{\alpha \beta} = (1 + (2\pi \alpha' B)^2) \eta_{\alpha \beta}$$

4
is diagonal and proportional to $\eta_{\alpha\beta}$. And in the expression of the three-string vertex, the inner product is respect to this open string metric. However, since

$$[a^\alpha_m, a^{\beta\dagger}_n] = G^{\alpha\beta} \delta_{mn}$$  \hspace{1cm} (21)

we can rescale either the oscillators or the metric such that we have normalized metric and commutator. This will give us the freedom to use the well-known results on sliver state construction and spectrum analysis, and the only effect is the overall normalization factor $(DetG)^{\frac{1}{2}}$ appearing in the ratio formula (2) and also in the normalization of the corresponding 3-string vertex and sliver state. From now on, we will work under the convention $G^{\alpha\beta} = \eta^{\alpha\beta}, [a^\alpha, a^{\beta\dagger}] = \eta^{\alpha\beta}$ with $\eta^{\alpha\beta} = \delta^{\alpha\beta}$.

2.3 The spectrum of $M$ and the ratio $R = 1$

The spectrum of $M$ has been calculated in [19], and the result is that it has a continuous spectrum labelled by $k$ such that

$$M[k >] = M(k)[k >] ,$$ \hspace{1cm} (22)

where

$$M(k) = -\frac{1}{2 \cosh \frac{k}{2} + 1} ,$$ \hspace{1cm} (23)

and the eigenvector

$$|k > = (v^k_1, v^k_2, v^k_3, \ldots)^T ,$$ \hspace{1cm} (24)

is generated by

$$f_k(z) = \sum_{n=1}^{\infty} \frac{v^n_n z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k \tan^{-1} z}) , .$$ \hspace{1cm} (25)

The eigenvectors constitute a set of complete orthogonal basis and

$$\int_{-\infty}^{\infty} |k > < k| \mathcal{K}(k) = 1 ,$$ \hspace{1cm} (26)

where

$$\mathcal{K} \equiv \frac{2}{k} \sinh \frac{\pi k}{2} .$$ \hspace{1cm} (27)

However, $k$ is not an twist definite eigenstate but

$$C[k >] = -| - k > .$$ \hspace{1cm} (28)

Since $C$ commutes with $M^\dagger M$ so that each eigenvalue has doubly degenerate twist-definite states except for $|k = 0 >$ which is twist odd [19].

\footnote{However, $C$ does not commute with $M^{12}$ and $M^{21}$.}
Later on it is also useful to have the following facts \[14, 17\] about \(|v_{e,o} >,\)

\[
\begin{align*}
<k|v_e>&=\frac{1}{k}\frac{\cosh\frac{\pi k}{2} - 1}{\cosh\frac{\pi k}{2} + 1}, \\
<k|v_o>&=\frac{\sqrt{3}}{k}\frac{\sinh\frac{\pi k}{2}}{\cosh\frac{\pi k}{2} + 1},
\end{align*}
\]

(29)

\[
\begin{align*}
<v_e|\frac{1}{1+3M}|v_e>&=\frac{1}{4}V_{00}, \\
<v_o|\frac{1}{1-M}|v_o>&=\frac{3}{4}V_{00}.
\end{align*}
\]

(30)

(31)

Since there is an eigenvalue of \(M\) at \(-1/3\) so that the ratio \(R\) is not well-defined unless \(-1/3\) is also an eigenvalue of \(M'\) with multiplicity two. In order to calculate \(R\) we need to study the spectrum of \(M'\), especially at \(-1/3\), this will be done in the next section.

However, it is straightforward to calculate the ratio of the tension. As for \(Det(1-M)\), based upon the above expression of \(M'\) and the facts

\[
\begin{align*}
\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det A \det(D - CA^{-1}B), \\
\det(1 + |u><v|) &= 1 + <v|u>,
\end{align*}
\]

(32)

(33)

one obtains

\[
\begin{align*}
\frac{Det^{1/2}(1-M')}{\det(1-M)} &= N_\theta \det\left(1 - N_\theta(1 + \frac{4a^2}{3})\frac{1}{1-M}|v_o><v_o|\right) \\
&= \frac{4b^2}{\theta^2 + 12(V_{00} + \frac{1}{2})^2}.
\end{align*}
\]

(34)

when \(\theta = 0\), this recovers the result in \[17\].

As for the calculation of \(Det(1+3M')\) part, one has to introduce the regulator to overcome the difficulty brought by the existence of \(-1/3\) eigenvalue. It has been worked out in \[29\] and at last one find that the ratio of the brane tension is exactly 1, as expected. For completeness, the detailed calculation is given in the appendix with a slightly different way of calculating the \(Det(1+3M')\) from the one used in \[29\].

3 Finding the spectrum of \(M'\)

In this section we will find out the spectrum of \(M'\) by using the method in \[20\].

3.1 The eigen-equations

Let’s assume the form of the eigenstate as

\[
v = \begin{pmatrix} g_1 \\ g_2 \\ \int h_1(k)|k> \\ \int h_2(k)|k> \end{pmatrix}
\]

(35)
where \( g_1, g_2 \) are some complex numbers corresponding to the zero modes and \( h_1(k), h_2(k) \) are the coefficients of the expansion on the \( |k \rangle \) basis. Note that due to the complex elements in \( \mathcal{M}' \), we assume that \( h_i(k) \) could be complex. Let \( \lambda \) be the corresponding eigenvalue, then \( \mathcal{M}'v = \lambda v \) encodes the following relations:

\[
\lambda g_1 = (1 - N_\theta)g_1 - \sqrt{2} \frac{N_\theta}{b} \int h_1(k) < v_e |k \rangle + \frac{2ia}{\sqrt{3}} \int h_2(k) < v_o |k \rangle \rangle, \quad (36)
\]

\[
\lambda g_2 = (1 - N_\theta)g_2 - \sqrt{2} \frac{N_\theta}{b} \int h_2(k) < v_e |k \rangle - \frac{2ia}{\sqrt{3}} \int h_1(k) < v_o |k \rangle \rangle, \quad (37)
\]

\[
\lambda \int h_1(k)|k \rangle = - \sqrt{2} \frac{N_\theta}{b} \int h_1(k) < v_e |k \rangle - |v_o \rangle \int h_1(k) < v_o |k \rangle > - \frac{2N_\theta}{b} \int h_2(k) < v_o |k \rangle > + |v_o \rangle \int h_2(k) < v_e |k \rangle > + \frac{i4aN_\theta}{\sqrt{3}b} \int (|v_e \rangle \int h_2(k) < v_o |k \rangle > + |v_o \rangle \int h_2(k) < v_e |k \rangle >) \]

\[
\lambda \int h_2(k)|k \rangle = - \sqrt{2} \frac{N_\theta}{b} \int h_2(k) < v_e |k \rangle - |v_o \rangle \int h_2(k) < v_o |k \rangle > - \frac{2N_\theta}{b} \int h_1(k) < v_o |k \rangle > + |v_o \rangle \int h_1(k) < v_e |k \rangle > + \frac{i4aN_\theta}{\sqrt{3}b} \int (|v_e \rangle \int h_1(k) < v_o |k \rangle > + |v_o \rangle \int h_1(k) < v_e |k \rangle >) . \quad (38)
\]

Define

\[
C_e[h_1(k)] = \int h_1(k) < v_e |k \rangle \quad C_o[h_1(k)] = \int h_1(k) < v_o |k \rangle \]

\[
D_e[h_2(k)] = \int h_2(k) < v_e |k \rangle \quad D_o[h_2(k)] = \int h_2(k) < v_o |k \rangle \quad (40)
\]

And from the first two relations, we can rewrite \( g_1, g_2 \) as

\[
g_1 = \frac{1}{1 - N_\theta - \lambda} \sqrt{2} \frac{N_\theta}{b} (C_e + \frac{2ia}{\sqrt{3}} D_o) \quad (42)
\]

\[
g_2 = \frac{1}{1 - N_\theta - \lambda} \sqrt{2} \frac{N_\theta}{b} (D_e - \frac{2ia}{\sqrt{3}} C_o) \quad (43)
\]

Put the above expressions into the Eqs.(36,37), and using the expansions

\[
|v_e \rangle = \int \frac{< k |v_e \rangle}{\mathcal{K}} |k \rangle \quad (44)
\]

\[
|v_o \rangle = \int \frac{< k |v_o \rangle}{\mathcal{K}} |k \rangle \quad (45)
\]
one have
\[
\int (\lambda - M(k))h_1(k)|k > = \{ c_1C_e + ic_2D_o \} \int \frac{< k|v_e >}{\mathcal{K}}|k > + \{ c_3C_o + ic_2D_e \} \int \frac{< k|v_o >}{\mathcal{K}}|k >
\]
(46)
\[
\int (\lambda - M(k))h_2(k)|k > = \{ c_1D_e - ic_2C_o \} \int \frac{< k|v_e >}{\mathcal{K}}|k > + \{ -ic_2C_e + c_3D_o \} \int \frac{< k|v_o >}{\mathcal{K}}|k >
\]
(47)
where
\[
c_1 = \frac{-2N_\theta}{b} \frac{1 - \lambda}{1 - \lambda - N_\theta}
\]
\[
c_2 = \frac{2N_\theta}{b} \frac{2N_\theta - 1 + \lambda 2a}{1 - N_\theta - \lambda \sqrt{3}}
\]
\[
c_3 = (\frac{2N_\theta}{b})(1 - \frac{4a^2}{3} \frac{N_\theta}{1 - N_\theta - \lambda})
\]

From the argument in [20], one could expect
\[
-\delta(k - k_0)r_1(k) = -(\lambda - M(k))h_1(k) + (c_1C_e + ic_2D_o) < k|v_e > \frac{1}{\mathcal{K}} + (c_3C_o + ic_2D_e) < k|v_o > \frac{1}{\mathcal{K}}
\]
\[
-\delta(k - k'_0)r_2(k) = -(\lambda - M(k))h_2(k) + (c_1D_e - ic_2C_o) < k|v_e > \frac{1}{\mathcal{K}} + (c_3D_o - ic_2C_e) < k|v_o > \frac{1}{\mathcal{K}}
\]
where \( r_i(k) \) undetermined, with \( r_1(k_0) = 0, r_2(k'_0) = 0 \). Now solve \( h_1(k), h_2(k) \) from the above two relations, and applying the operations \( \int dk < v_e|k > \) and \( \int dk < v_o|k > \), we obtain
\[
C_e = A_{ee}(c_1C_e + ic_2D_o) + B_{1e}
\]
(48)
\[
C_o = A_{oo}(c_3C_o + ic_2D_e) + B_{1o}
\]
(49)
\[
D_e = A_{ee}(c_1D_e - ic_2C_o) + B_{2e}
\]
(50)
\[
D_o = A_{oo}(c_3D_o - ic_2C_e) + B_{2o}
\]
(51)
where
\[
A_{ee}(\lambda) \equiv \int \frac{< k|v_e >}{\mathcal{K} \cdot (\lambda - M(k))} < v_e|\frac{1}{\lambda - M(k)}|v_e >
\]
(52)
\[
A_{oo}(\lambda) \equiv \int \frac{< k|v_o >}{\mathcal{K} \cdot (\lambda - M(k))} < v_o|\frac{1}{\lambda - M(k)}|v_o >
\]
(53)
\[
B_{1e} \equiv \int \frac{\delta(k - k_0)r_i(k)}{\lambda - M(k)} < v_e|k >
\]
(54)
\[
B_{1o} \equiv \int \frac{\delta(k - k_0)r_i(k)}{\lambda - M(k)} < v_o|k >
\]
(55)
It’s easy to find that one can group the above equations into two sets:

\[
\begin{align*}
(1 - c_1 A_{ee}) C_e - ic_2 A_{ee} D_o &= B_{1e} \\
 ic_2 A_{oo} C_e + (1 - c_3 A_{oo}) D_o &= B_{2o}
\end{align*}
\]

and

\[
\begin{align*}
(1 - c_3 A_{oo}) C_o - ic_2 A_{oo} D_e &= B_{1o} \\
 ic_2 A_{ee} C_o + (1 - c_1 A_{ee}) D_e &= B_{2e}
\end{align*}
\]

### 3.2 Continuous spectrum

If the determinant factor

\[
Det \equiv (1 - c_1 A_{ee})(1 - c_3 A_{oo}) - c_2^2 A_{oo} A_{ee}
\]

is not zero, then we can invert the above two sets of equation (57) to (60) and obtain the formal solutions

\[
\begin{align*}
C_e &= \frac{B_{1e}(1 - c_3 A_{oo}) + ic_2 A_{ee} B_{2o}}{Det}, \\
C_o &= \frac{B_{1o}(1 - c_1 A_{ee}) + ic_2 A_{oo} B_{2e}}{Det}, \\
D_e &= \frac{B_{2e}(1 - c_3 A_{oo}) - ic_2 A_{ee} B_{1o}}{Det}, \\
D_o &= \frac{B_{2o}(1 - c_1 A_{ee}) - ic_2 A_{oo} B_{1e}}{Det}
\end{align*}
\]

In order to have nontrivial solutions, that is, the above $B$ functions are not all zero, we need to have $\lambda - M(k) = 0$ at $k = k_0 = k' \neq 0$ to cancel the zero of $r_{1,2}(k)$ at $k = k_0$. In this case, we will have a continuous spectrum $\lambda = M(k_0) = -\frac{1}{2 \cosh \frac{\pi k_0}{2} + 1} \in [-1/3, 0)$. Explicitly,

\[
\begin{align*}
\lambda - M(k) &= M(k_0) - M(k) = -M^{(1)}(k_0)(k - k_0) - \frac{1}{2} M^{(2)}(k_0)(k - k_0)^2 + \ldots, \\
&= -\frac{\pi \sinh \frac{\pi k_0}{2}}{(1 + 2 \cosh \frac{\pi k_0}{2})^2} (k - k_0) - \frac{\pi^2 \cosh \frac{\pi k_0}{2} - \pi^2 \sinh \frac{\pi k_0}{2}}{(1 + 2 \cosh \frac{\pi k_0}{2})^3} (k - k_0)^2 + \ldots
\end{align*}
\]

where $M^{(n)}(k_0) \equiv \frac{d^n M}{dk^n}|_{k_0}$. Note that $M^{(1)}(k_0)$ is non-vanishing except at $k_0 = 0$, however, $M^{(2)}(0) \neq 0$. Therefore, if $k_0 = 0$ then $\lambda - M(k) \sim -\frac{1}{2} M^{(2)}(0) k^2$, otherwise $\lambda - M(k) \sim -M^{(1)}(k_0)(k - k_0)$, to the leading order.

In this subsection we focus on the case of $k_0 \neq 0$ and will discuss the special case $\lambda = -1/3(k_0 = 0)$ in the next subsection. Based upon the behavior of $\lambda - M(k)$ near $k = k_0$.
we will have three kind of choices for the behavior of \( r_i(k) \) around \( k = k_0 \). The first one is to let \( r_i(k) \sim d_i(k - k_0) \) for \( i = 1, 2 \), then we have nonzero

\[
B_{ie}(k_0) = \frac{-d_i(\cosh \frac{\pi k_0}{2} - 1)}{k_0(2 \cosh \frac{\pi k_0}{2} + 1)M^{(1)}(k_0)}, \tag{67}
\]

and the corresponding eigenvector is

\[
v(k_0) = \begin{pmatrix}
\frac{\sqrt{2}N_o}{1 - \lambda - N_o} (1 + \frac{2(-3 + 4\omega^2)N_o}{\omega^2}A_{oo}) B_{1e} + ic_2 B_{2e} \\
0 \\
\frac{2}{\sqrt{3}} \frac{\sqrt{2}N_o}{1 - \lambda - N_o} (1 + \frac{2(-3 + 4\omega^2)N_o}{\omega^2}A_{oo}) B_{1o} - ic_2 B_{2o} \\
\end{pmatrix}
\]

Note that \( v(k_0) \) is degenerate with \( v(-k_0) \) which is in general different from \( v(k_0) \) because

\[
B_{ie}(-k_0) = B_{ie}(k_0), \quad B_{io}(-k_0) = -B_{io}(k_0) \tag{70}
\]

and \( A_{ee} \) and \( A_{oo} \) are also even functions of \( k \).

The second choice is \( r_1(k) \sim (k - k_0) \) but \( r_2(k) \sim (k - k_0)^2 \) or higher power of \( (k - k_0) \) so that \( B_{1e} \) and \( B_{1o} \) are the same as the ones in (67) and (68) but

\[
B_{2e} = B_{2o} = 0. \tag{71}
\]

In this case we can form a pair of ”twist-definite” states from \( v(\pm k_0) \) as following,

\[
v_{++}(k_0) = \frac{1}{2} (v(k_0) + v(-k_0))|_{B_{2e} = B_{2o} = 0} = B_{1e} \begin{pmatrix}
\frac{\sqrt{2}N_o}{1 - \lambda - N_o} (1 + \frac{2(-3 + 4\omega^2)N_o}{\omega^2}A_{oo}) \\
0 \\
\frac{2}{\sqrt{3}} \frac{\sqrt{2}N_o}{1 - \lambda - N_o} (1 + \frac{2(-3 + 4\omega^2)N_o}{\omega^2}A_{oo}) \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0 \\
\frac{1}{2M^{(1)}(k_0)}(|k_0 > - | - k_0 >)
\end{pmatrix}
\]

\[
\text{and}
\]

\[
v_{-+}(k_0) = \frac{1}{2} (v(k_0) - v(-k_0))|_{B_{2e} = B_{2o} = 0} = B_{1o} \begin{pmatrix}
0 \\
\frac{i2a}{\sqrt{3}} \frac{\sqrt{2}N_o}{1 - \lambda - N_o} (1 + \frac{4N_o}{\omega^2}A_{ee}) \\
(c_3 - (c_1 - c_2^2)A_{ee}) \frac{1}{\lambda - M} |v_e > \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0 \\
\frac{1}{2M^{(1)}(k_0)}(|k_0 > + | - k_0 >)
\end{pmatrix}
\]

10
Here the subscript indices + or − indicate the twist parity of the third and fourth components with respect to the twist matrix $C_{mn}$. Note that both $v_{+-}$ and $v_{-+}$ are the ”twist-definite” eigenstates although their third and fourth components have different twist parity, this implies that two spacetime directions cannot have the same twist parity for each ”twist-definite” eigenstate.

The third choice is $r_2(k) \sim (k - k_0)$ but $r_1(k) \sim (k - k_0)^2$ or higher power of $(k - k_0)$ so that $B_{2e}$ and $B_{2o}$ are the same as the ones in [71] and [73] but

$$B_{1e} = B_{1o} = 0 .$$  \hspace{1cm} (74)

We will then get the corresponding eigenvectors similar to (72) and (73),

$$u_{-+}(k_0) = \frac{1}{2}(v(k_0) + v(-k_0))|_{B_{1e} = B_{1o} = 0}$$

$$= \frac{B_{2e}}{Det} \begin{pmatrix} 0 \\ -\frac{\sqrt{2}N_0}{1-\lambda-2N_0} (1 + \frac{2(3+4a^2)N_0}{3b} A_{oo}) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2M^{(1)}(k_0)}(|k_0+| - k_0>) \end{pmatrix} \hspace{1cm} (75)$$

and

$$u_{+-}(k_0) = \frac{1}{2}(v(k_0) - v(-k_0))|_{B_{1e} = B_{1o} = 0}$$

$$= \frac{B_{2o}}{Det} \begin{pmatrix} i\frac{2a}{\sqrt{2}} \frac{\sqrt{2}N_0}{1-\lambda-2N_0} (1 + \frac{A_{ee}}{b}) \\ 0 \\ i\frac{c_2}{2\sqrt{2}} |v_e> \\ (c_3 - (c_1c_3 - c_2^2) A_{ee}) \frac{1}{\sqrt{1-M^2}} |v_o> \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2M^{(1)}(k_0)}(|k_0+| - k_0>) \end{pmatrix} \hspace{1cm} (76)$$

and it is easy to see that they are independent set from (72) and (73); moreover, the general state (73) is the linear combinations of the four independent ”twist-definite” eigenstates. We then conclude that there are four ”twist-definite” eigenstates for each eigenvalue except $-1/3$. This is a natural generalization of the results for $M$ [19] and $M'$ [21] since our $M'$ involves two spatial directions, also we have four independent parameters $B_{1e}$ and $B_{1o}$ in the general eigenstate (53).

If we define the inner product of the vectors $u, v$ as $u^tv$, then it is not hard to see that among the four independent eigenvectors, the ones with different twist parities are orthogonal to each other.

3.3 The $\lambda = -1/3$ case

Now, let us take $\lambda = -1/3$ as an eigenvalue in the continuous spectrum. Then we have $k_0 = 0$ such that $M(k_0) = -1/3$. From the requirement that $r_i(k)/(k - M(k_0))$ should have no poles at $k_0$, one can determine the form of the functions $r_i(k)$ up to a scale factor. Taking the freedom of choosing the scale factor and to the leading order, we set

$$r_1(k) = r_1k^2, \quad r_2(k) = r_2k^2$$  \hspace{1cm} (77)
where \( r_{1,2} \) are arbitrary constants\(^3\). And

\[
B_{2e} = B_{1e} = 0
\]

\[
r_2^{-1}B_{2o} = r_1^{-1}B_{1o} = -\frac{6\sqrt{3}}{\pi}
\]

Then we obtain

\[
C_e = \frac{ic_2 A_{ee} B_{2o}}{\text{Det}} \tag{78}
\]

\[
C_o = \frac{(1 - c_1 A_{ee}) B_{1o}}{\text{Det}} \tag{79}
\]

\[
D_e = \frac{-ic_2 A_{ee} B_{1o}}{\text{Det}} \tag{80}
\]

\[
D_o = \frac{(1 - c_1 A_{ee}) B_{2o}}{\text{Det}} \tag{81}
\]

Note that all of \( C, D \)s are dependent on \( A_{ee} \) explicitly but dependent on \( A_{oo} \) only through \( \text{Det} \).

It is straightforward to get the eigenvector for the \( \lambda = -1/3 \) eigenvalue. The explicit form is

\[
v = \begin{pmatrix}
\frac{ic_2 B_{2o}}{\text{Det}} \frac{1}{\lambda - M} | v_e > + \frac{(c_3 - (c_1 - c_2 A_{ee}) B_{2o})}{\text{Det}} \frac{1}{\lambda - M} | v_o > - \frac{36 \pi^2}{\lambda^2} | k = 0 > \ \\
-\frac{ic_2 B_{1o}}{\text{Det}} \frac{1}{\lambda - M} | v_e > + \frac{(c_3 - (c_1 - c_2 A_{ee}) B_{1o})}{\text{Det}} \frac{1}{\lambda - M} | v_o > - \frac{36 \pi^2}{\lambda^2} | k = 0 >
\end{pmatrix}.
\] \[\tag{82}\]

In \[20\], it has been argued that the \(-1/3\) eigenvalue is doubly degenerate, with two independent eigenvectors. At first looking, it seems that we have only one eigenvector. This is not the case. In fact, due to our freedom to choose \( r_i \), we can set either \( r_1(k) \) or \( r_2(k) \) to have higher power expansion in terms of \( (k - k_0) \) such that one of \( B_{io}, i = 1, 2 \) vanishes while the other finite. This will not spoil the whole story. On the contrary, we will find that the choice will lead to two independent eigenvectors and the above one is just the superposition of them.

First let \( B_{1o} = 0 \), which could be achieved by choosing \( r_1(k) \propto k^3 \) or higher power of \( k \). In this case, the above eigenvector turns to be

\[
v_{+ - \frac{1}{\lambda}} = \frac{B_{2o}}{\text{Det}} \begin{pmatrix}
\sqrt{\frac{2}{3b}} \frac{6 N_o}{4 - 3 N_o} (1 + \frac{4 N_o}{b} A_{ee}) \\
0 \\
\frac{ic_2}{\lambda - M} | v_e > \\
(c_3 - (c_1 - c_2 A_{ee}) \frac{1}{\lambda - M} | v_o >
\end{pmatrix} - \frac{36 \pi^2}{\lambda^2} \begin{pmatrix}
0 \\
0 \\
0 \\
| k = 0 >
\end{pmatrix}.
\] \[\tag{83}\]

\(^3\)There is a freedom to choose the value of \( r_{1,2} \). In this paper, if we don’t let the \( r_i \) vanishing, then we just choose \( r_i = 1 \). Choosing one of \( r_i \) vanishing equivalent to choosing higher order of \( k \).
On the other hand, one could choose $B_{2\omega} = 0$ by choosing $r_{1}(k) \propto k^{3}$ or higher power of $k$. Then the eigenvector has the form

$$v_{-+,\frac{1}{3}} = \frac{B_{3\omega}}{\text{Det}} \begin{pmatrix} 0 \\ -\frac{2}{36} \frac{6iN_{b}a}{1+4N_{b}A_{ee}} (1 + \frac{4N_{b}}{b}A_{ee}) (c_{3} - (c_{1}c_{3} - c_{2}^{2})A_{ee} \frac{1}{\lambda-M |v_{o} >}) \\ -i\frac{c_{2}}{\lambda-M |v_{e} >} \end{pmatrix} - \frac{36}{\pi^{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot v_{k = 0 >}.$$  \hspace{1cm} (84)

It is not hard to see that the general eigenvector (82) is just the superposition of $v_{-+,\frac{1}{3}}$ and $v_{-+,\frac{4}{3}}$. Moreover $v_{-+,\frac{1}{3}}$ and $v_{-+,\frac{4}{3}}$ are kind of twist definite states as mentioned in the $k_{0} \neq 0$ case.

One may wonder if (83) (84) are really the eigenvectors of $\mathcal{M}'$ with eigenvalue $-1/3$. This could be checked directly with the explicit form of $\mathcal{M}'$. Unlike the matrix $M$ which has only one twist odd state $|k = 0 >$ at $-1/3$, our $\mathcal{M}'$ is doubly degenerate at $-1/3$ in the continuous spectrum. However taking into account the space dimensions involved in our discussion, we find that the degeneracy at $-1/3$ in the case with $B$ field is only half of the one in the case without $B$ field. Nevertheless, the degeneracy of $\mathcal{M}'$ at $-1/3$ matches the one of $M$.

A subtlety is about the discontinuity of $A_{oo}$ at $-1/3$ discovered in [20]. As shown there, $A_{oo}((\frac{-1}{3})^{+}) = \frac{3}{4}ln27$ but $A_{oo}((\frac{-1}{3})^{-}) = -\infty$. Since in our (83) and (84) $A_{oo}$ only appears in $\text{Det}$ which blows up if $A_{oo} \rightarrow -\infty$, in this case the vectors in (72) and (73) are just reduced to $(0, 0, |k = 0 >, 0)^{T}$ or $(0, 0, 0, |k = 0 >)^{T}$ which obviously cannot satisfy the eigen-equations. So the ambiguity is lifted and we should choose $A_{oo}((\frac{-1}{3})^{+})$ instead of $A_{oo}((\frac{-1}{3})^{-})$; therefore the degeneracy at $-1/3$ is two.

### 3.4 Discrete spectrum

When the determinant factor $\text{Det}$ vanish, then one could have nontrivial solutions only if $B_{v} = B_{3\omega} = 0, i = 1, 2$. The solutions of $\text{Det} = 0$ correspond to the so-called discrete spectrum. Being the function of $\lambda$, the equation has an integral form, which make it very hard to solve analytically. However, what we are really interested in is to see if $\lambda = -1/3$ is an eigenvalue or not. The reasons are twofold: on the one hand, from the form of the ratio of tension, the $-1/3$ eigenvalue of $\mathcal{M}'$ is essential to cancel the zero from $\text{det}(1+3M)$. In order to have the finite ratio, one should expect the degeneracy at $\lambda = -1/3$ of $\mathcal{M}'$ match the one of $M$. On the other hand, from the experience of $\mathcal{M}$ without $B$ field, $-1/3$ exists as an eigenvalue in the discrete spectrum and make the degeneracy higher [20]. Therefore, it is quite important to investigate the possibility of $-1/3$ as an eigenvalue in the discrete spectrum. To this aim, taking $A_{ee} = -\frac{3}{4}V_{00}, \lambda = -1/3$, we have

$$\text{Det} = \frac{\theta^{3}[\theta^{2} + 12(\theta^{2} + 2b/2)^{2}]^{2} - 384A_{oo}V_{00}b(V_{00} + b/2)]}{(\theta^{2} + 12(V_{00} + b/2)^{2})^{2} \cdot (\theta^{2} + 12V_{00}(V_{00} + b/2))} \hspace{1cm} (85)$$

Obviously, $\theta = 0$ is a solution. In fact, this recovers the well-known result that $\lambda = -1/3$ belongs to the discrete spectrum, no matter what $b$ is, in the case of $\theta = 0$. (see [20]).
Certainly it is possible for specified values of $\theta$, $\text{Det}(\lambda = -1/3) = 0$. In this case, one find

$$\theta^2 = \sqrt{384A_\infty V_{00}b(V_{00} + b/2) - 12(V_{00} + b/2)^2}. \quad (86)$$

Since $\theta$ is real, we have to require

$$8bA_\infty V_{00} - 3(V_{00} + b/2)^2 \geq 0 \quad (87)$$

The above relation set the range of $b$, in which it is possible to find a $\theta$ such that $\lambda = -1/3$ is a discrete eigenvalue. However, one could expect that for generic value of $\theta$, $\lambda = -1/3$ does not belong to discrete spectrum, although it exists in the continuous spectrum.

We will not discuss carefully the discrete spectrum here. In [28], the careful discussion on discrete spectrum and eigenvector has been worked out.

### 3.5 The spectrum of $\mathcal{M}'^{12}$ and $\mathcal{M}'^{21}$

Due to the fact that the matrices $\mathcal{M}'^{rs}$ enjoy the same property as the usual matrices $M'$, it is straightforward to obtain the spectrum of $\mathcal{M}'^{12}$ and $\mathcal{M}'^{21}$, once we know the spectrum of $\mathcal{M}'$. More precisely, since the matrices $\mathcal{M}'^{rs}$ are commuting with each other and satisfy the relations

$$\mathcal{M}' + \mathcal{M}'^{12} + \mathcal{M}'^{21} = (\mathcal{M}')^2 + (\mathcal{M}'^{12})^2 + (\mathcal{M}'^{21})^2 = 1, \quad \mathcal{M}'^{12}\mathcal{M}'^{21} = \mathcal{M}'(\mathcal{M}' - 1), \quad (88)$$

they share the same eigenvectors and their eigenvalues satisfy

$$\lambda^{12}(k) - \lambda^{21}(k) = \pm \sqrt{(1 - \lambda(k))(1 + 3\lambda(k))}$$

$$\lambda^{12}(k) + \lambda^{21}(k) = 1 - \lambda(k).$$

From the above two relations, it’s easy to read out the spectrum of $\mathcal{M}'^{12}$ and $\mathcal{M}'^{21}$.

### 4 Discussions and Conclusions

In this paper we have calculated the spectrum of the Neumann matrix with zero modes in the constant B field background. We find both the continuous and discrete spectrum. The existence of the discrete spectrum will set a constraint on $b$ for a given $\theta$; moreover, for the generic $\theta$ there is no $-1/3$ in the discrete spectrum. We will take this as a good point to the finite and nonzero ratio of tension $\mathcal{R}$.

The continuous one is similar to the one of $M$ but with four degenerate twist-definite eigenstates at each eigenvalue except at $-1/3$ where it is only doubly degenerate. The doubling of the degeneracy at each eigenvalue compared to the one of $M$ is expected since the B field mixes two spatial directions and doubles the dimensions of the eigen-space; moreover, the double degeneracy at $-1/3$ is required for the the ratio of the tension $\mathcal{R}$ to be well-defined. However, to determine precisely the ratio from the spectrum requires the
details of the density of states which is beyond the scope of this paper. On the other hand there is an analytic way to determine the ratio $R$ to be one a la the regularization method of Okuyama [17] as shown in [29]. The agreement adds more weights to VSFT and its background independence.

Another issue on which our results are concerned is the new proposed map of Witten star to Moyal star [27] where the Moyal pair is the Fourier transform of a twist-even coordinate and the twist odd momentum. The Fourier basis are the twist-definite eigenstates of $M$. In [18] and [27] it is shown that at the zero slope limit, the star algebra factorizes into two subalgebras which was first proposed in [24]: $A \rightarrow A_0 \otimes A_1$. The $A_0$ corresponds to the subalgebra of zero-momentum sector and the $A_1$ corresponds to the $C^*$-algebra of spacetime functions. The 3-string vertex factorizes accordingly

$$|V_3 > \rightarrow |V_3^{(0)} > \otimes |V_3^{(1)} > ,$$

where $< x_1 | \otimes < x_2 | \otimes < x_3 | |V_3^{(1)} >$ can be identified as the the kernel for the usual Moyal product with zero noncommutativity [27].

In the presence of $B$ field, the directions without $B$ field could be treated along the above way, while the directions with $B$ field are a little different. It is easy to see that the factorization of the star algebra still works. However, one should replace commutative $C^*$-algebra $A_1$ with the noncommutative one. The zero slope limit is the equivalent to (set $b = 2$)

$$V_{\alpha \beta, rs}^{\alpha \beta, rs} \rightarrow G_{\alpha \beta, rs} + O(\epsilon^2)$$

$$V_{00, rs}^{\alpha \beta, rs} \rightarrow 0 + O(\epsilon)$$

$$V_{00, rs}^{\alpha \beta, rs} \rightarrow G_{\alpha \beta, rs} - \frac{16}{12 + \theta^2} \delta^{rs} + i \epsilon^{\alpha \beta} \frac{4 \theta}{12 + \theta^2} \chi^{rs} ,$$

and we can obtain the $A_1$ part of the longitudinal 3-string vertex $|V_3 >$, it is

$$\frac{\sqrt{\pi}}{2} \frac{1}{1 + \frac{\theta^2}{12}} \exp \left( -\frac{1}{2} \left( -\frac{4 + \theta^2}{12 + \theta^2} \sum_r a_0^{(r)} \cdot a_0^{(r)} - \frac{8}{12 + \theta^2} \sum_{r<s} a_0^{(r)} \cdot a_0^{(s)} - \frac{4i \theta}{12 + \theta^2} \sum_{r<s} a_0^{(r)} \times a_0^{(s)} \right) \right) |0 >$$

which is exactly the same as the 3-string vertex for the canonical Moyal product (see (2.26) constructed in [27]). This confirms that the spacetime algebra is a noncommutative one, with Moyal product replacing the usual pointwise product.

Moreover, it deserves to mention that: in the recent paper [29] they show that the zero mode of the tachyonic lump state in the constant B field background is nothing but the usual noncommutative soliton as a projector discovered in [25]. This is complimentary to our above observation that the low energy zero mode of the SFT in constant B field background is governed by the noncommutative $C^*$-algebra.

It is interesting to generalize the proposed map of Moyal star to Witten star in [27] to the 3-string vertex with zero modes and with B field. This requires the eigenvectors of $\mathcal{M}$ to

\[4\text{It is up to an overall constant factor } 3\pi/4.\]
form a complete orthogonal set in order to obtain the proper Moyal conjugate pairs which are the twist-definite eigenvector by the Fourier transform based on these eigenvectors. Future work are required to carry out the program.

In this paper we did find the twist-definite eigenvectors of $\mathcal{M}'$, however, for each twist-definite eigenvector the twist parity is opposite among the two spatial components. This implies that we need to choose opposite twist parity assignments for the Moyal conjugate pairs between the two spatial directions parallel to the B field, that is, if we choose the twist-even coordinate modes and the twist-odd momentum modes as the Moyal conjugate pairs for one parallel direction to the B field, we should choose the twist-odd coordinate modes and the twist-even momentum modes as the conjugate pairs for another parallel direction. This kind of the flip of the twist parity for the Moyal pairs in two spatial directions is unexpected, and it deserves more study to understand its physical implication to the equivalence between Witten’s string field theory and the generalized noncommutative field theory.

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5 Appendix: The calculation of the ratio $R$

In this appendix, we show how to get the ratio of tension $R = 1$, even in the case with $B$ field.

The formula for the ratio $R$ has been given in (2), and in section 2.3, we have found

$$\frac{\text{Det}^{1/2}(1 - \mathcal{M}')}{\text{det}(1 - M)} = \frac{4b^2}{\theta^2 + 12(V_{00} + \frac{b}{2})^2}. \tag{94}$$

As for the part involving $(1 + 3\mathcal{M}')$, we need to use regulator as in [17] since there are double degenerate eigenvectors at $-1/3$ as shown section 3.3. First we can rewrite

$$\text{Det}(1 + 3\mathcal{M}') = (4 - 3N_0^2)\text{Det}(\eta^{\alpha\beta}(1 + 3M + Q) + ie^{\alpha\beta}P) \tag{95}$$

where

$$Q = -\alpha|v_e><v_e| + \beta|v_o><v_o|$$
$$P = \gamma(|v_e><v_e| + |v_o><v_o|)$$

with

$$\alpha = \frac{48(V_{00} + \frac{b}{2})}{\theta^2 + 12V_{00}(V_{00} + \frac{b}{2})}.$$
\[ \beta = \frac{48V_{00}}{\theta^2 + 12V_{00}(V_{00} + \frac{b}{2})} \]
\[ \gamma = \frac{8\sqrt{3}\theta}{\theta^2 + 12V_{00}(V_{00} + \frac{b}{2})} \]

It is easy to see that
\[ \alpha\beta + \gamma^2 = \frac{4}{V_{00}} \beta, \tag{96} \]

which will be useful later.

Further simplifying (95) we get
\[ \frac{\text{Det}(1 + 3M')}{\text{det}(1 + 3M)^2} \]
\[ = (4 - 3\mathcal{N}_\theta)^2 \text{det}(1 + \frac{1}{1 + 3M}Q) \text{det} \left( 1 + \frac{1}{1 + 3M} \left( Q - \frac{1}{1 + 3M + Q} \right) P \right), \tag{97} \]
\[ = (4 - 3\mathcal{N}_\theta)^2 (1 - \alpha H_{ee})(1 + \beta H_{oo})(1 - (\alpha + q\gamma^2)H_{ee})(1 + (\beta - p\gamma^2)H_{oo}), \tag{98} \]

where
\[ p = <v_e|\frac{1}{1 + 3M + Q}|v_e> = \frac{H_{ee}}{1 - \alpha H_{ee}} \]
\[ q = <v_o|\frac{1}{1 + 3M + Q}|v_o> = \frac{H_{oo}}{1 + \beta H_{oo}} \]

and
\[ H_{ee} = <v_e|\frac{1}{1 + 3M}|v_e> = \frac{1}{4}V_{00} \]
\[ H_{oo} = <v_o|\frac{1}{1 + 3M}|v_o>. \tag{100} \]

Note that $H_{oo}$ is not well-defined as shown in \[20\] and it requires proper regularization. Here we just keep it formally and will regularize it later.

We then combine the 2st(3rd) and the 4th(5th) factors in the (99), and use (96) we get
\[ \frac{\text{Det}(1 + 3M')}{\text{det}(1 + 3M)^2} = (4 - 3\mathcal{N}_\theta)^2 \left( 1 - \alpha H_{ee} + \beta \left( 1 - \frac{4}{V_{00}}H_{ee} \right) H_{oo} \right)^2. \tag{102} \]

The factor before $H_{oo}$ is formally zero, however, we know $H_{oo}$ is divergent. Using the proper regularization as done in \[17\], one has the final result as following
\[ \left( 1 - \frac{4}{V_{00}}H_{ee} \right) H_{oo} = \frac{\pi^2}{12V_{00}} \]
\[ \tag{103} \]

In the end we get
\[ \frac{\text{Det}(1 + 3M')}{\text{det}(1 + 3M)^2} = \left( \frac{\theta^2 + 4\pi^2}{\theta^2 + 12(V_{00} + \frac{b}{2})^2} \right)^2 \tag{104} \]

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Combing with (104) and using the fact that $\text{Det}G = (1 + (\frac{\theta}{2\pi})^2)^2$ which appears in the ratio formula (2), we obtain
\[
\mathcal{R} = 1
\] (105)
as expected.

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