PARISIAN RUIN PROBABILITY FOR TWO-DIMENSIONAL BROWNIAN RISK MODEL

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Abstract: Let \((W_1(s), W_2(t)), s, t \geq 0\) be a bivariate Brownian motion with standard Brownian motion marginals and constant correlation \(\rho \in (-1, 1)\). Parisian ruin is defined as a classical ruin that happens over an extended period of time, the so-called time-in-red. We derive exact asymptotics for the non-simultaneous Parisian ruin of the company conditioned on the event of non-simultaneous ruin happening. We are interested in finding asymptotics of such problem as \(u \to \infty\) and with the length of time-in-red being of order \(\frac{1}{u^2}\), where \(u\) represents initial capital for the companies. Approximation of this problem is of interest for the analysis of Parisian ruin probability in bivariate Brownian risk model, which is a standard way of defining prolonged ruin models in the financial markets.

Key Words: multidimensional Brownian motion; Stationary random fields; Extremes;

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1. Introduction

Consider the following Brownian risk model for two portfolios

\[ R_i(t) = u_i + c_i t - W_i(t), \quad i = 1, 2, \]

where the claims \(W_i(t), t \geq 0\) are modeled by two standard dependent Brownian motions, initial capitals \(u_i > 0\) and premium rates \(c_i\). The following representation of the dependence between the claims has been proposed in [1] and [2]

\[ (W_1(s), W_2(t)) = (B_1(s), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)), \quad s, t \geq 0, \]

where \(B_1, B_2\) are two independent standard Brownian motions and \(\rho \in [-1, 1]\). The ruin probability of a single portfolio in the time horizon \([0, T], T > 0\) is given by (see e.g., [3])

\[
\pi_T(c_i, u) := \mathbb{P}\left\{ \inf_{t \in [0, T]} R_i(t) < 0 \right\} = \mathbb{P}\left\{ \sup_{t \in [0, T]} W_i(t) - c_i t > u \right\} = \Phi\left( -\frac{u}{\sqrt{T}} - c_i \sqrt{T} \right) + e^{-2c_i u} \Phi\left( -\frac{u}{\sqrt{T}} + c_i \sqrt{T} \right)
\]
for \( i = 1, 2, u \geq 0 \), with \( \Phi \) the distribution function of an \( N(0, 1) \) random variable. Since from self-similarity of Brownian motion we have the following equalities in distribution for \( c_1 = \frac{c_1}{\sqrt{T}}, u' = \frac{u}{\sqrt{T}} \)

\[
B(tT) - c_1 t > u \Leftrightarrow \sqrt{T}B(t) - c_1 t > u \Leftrightarrow B(t) - c_1' t > u',
\]

then without loss of generality one can assume \( T = 1 \). There are at least two different approaches on how to define the extension of the above to the two-dimensional model. Denote \( W_i^\star(s) = W_i(s) - c_i s, i = 1, 2 \).

Define the simultaneous ruin probability as

\[
\pi_{A,\rho}(c_1, c_2, u, v) = \mathbb{P}\{\exists s \in A : W_1^\star(s) > u, W_2^\star(s) > v\}
\]

which has been recently studied in [4] for \( A = [0, 1] \). Similarly, define non-simultaneous ruin probability as

\[
\pi_{A \times B,\rho}(c_1, c_2, u, v) = \mathbb{P}\{\exists s \in A, t \in B : W_1^\star(s) > u, W_2^\star(t) > v\}
\]

which has been studied for the case \( A = B = [0, 1] \) in [5]. In this contribution we focus on an extensions of the non-simultaneous results of ruin for two-dimensional risk portfolios. In [6] Loeffen, Czarna and Palmowski studied the so-called Parisian ruin of a single portfolio, which is defined as

\[
P_{A,H(u)}^\star(c, u) := \mathbb{P}\{\exists s' \in A \ \forall s \in [s', s'+H(u)] W^\star(s) > u\},
\]

for some \( H(u) \geq 0 \) and \( A = [0, T] \). This model defines the concept of the ruin as crossing the barrier over the extended period of time, the so-called time in red. It seems more natural than the classical ruin approach, since it allows for easier practical investigations whether the ruin has occurred. This model has also been studied for various sets \( A \) and various processes in many other contributions, e.g. [7], [8], [9]. To analyse the model in two-dimensional framework we use the following definition of the ruin probability

\[
P_{A \times B,H(u)}^\star(c_1, c_2, u, v) := \mathbb{P}\{\exists s' \in A, t' \in B \ \forall s \in [s', s'+H_1(u)] \ \forall t \in [t', t'+H_2(u)] W_1^\star(s) > u, W_2^\star(t) > v\},
\]

for some \( H_1(u), H_2(u) \geq 0 \) and intervals \( A, B \). We refer to [10], where one can find an application of Parisian ruin to actuarial risk theory, where \( R_i \) is treated as a surplus process of an insurance company with initial capital \( u_i \). For more general intervals \( A, B \) we have the following comparison between Parisian and classical ruin

\[
\pi_{A \times B,\rho}(c_1, c_2, u, au) \geq P_{A \times B,H(u)}^\star(c_1, c_2, u, au).
\]

Since Parisian ruin probability cannot be determined explicitly for general Gaussian risks, our aim is to investigate the asymptotic behaviour of the Parisian ruin conditioned on the classical ruin occurring, for which the results are known. Hence we calculate

\[
\mathcal{P}_{[0,1]^2,H(u)}^\star(c_1, c_2, u, au) :=
\]
for $u \to \infty$ and also find for which $H(u)$ we have that for some $C > 0$

$$
\lim_{u \to \infty} \mathcal{P}([0,1]^2, H(u))(c_1, c_2, u, au) = C.
$$

We prove that the above is true for $H(u) = \left(\frac{S_1}{u^2}, \frac{S_2}{u^2}\right) := \left(\frac{S_1, S_2}{u^2}\right)$ for some $S_1, S_2 > 0$. To simplify notation we denote

$$
\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) := \mathcal{P}^*([0,1]^2, \frac{(S_1, S_2)}{u^2})(c_1, c_2, u, au)
$$

and similarly

$$
\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) := \mathcal{P}^*([0,1]^2, \frac{(S_1, S_2)}{u^2})(c_1, c_2, u, au).
$$

Arbitrary choice of $H(u) = \left(\frac{S_1}{u^2}, \frac{S_2}{u^2}\right)$ is closely connected to the length of the intervals with comparable variance for the Brownian motion (see [11]). For the choice of $H(u) = o\left(\frac{1}{u^2}\right)$ following the same line of proof we have that

$$
\lim_{u \to \infty} \mathcal{P}^*([0,1]^2, H(u))(c_1, c_2, u, au) = 1.
$$

On the other hand, if we choose $H(u)$ such that $u^2 H(u) \to \infty$, $H(u) < 1$, then the methods employed in this contribution are not sufficient and the asymptotics are of different order, even in the one-dimensional setting.

### 2. Main results

Based on the relation between $a$ and $\rho$, either both of the coordinates impact the asymptotics, or one of the coordinates is negligible (up to a constant). We begin with cases where one of the coordinates dominates the other one and hence the results can be derived from one-dimensional models. Denote by $\Psi$ the survival function of a standard Normal random variable and by $\phi_{1*}$ the probability density function of $(W_1(1), W_2(t^*))$. Let $C_P = \mathbb{E}\left\{\exp\left(\sup_{t \geq 0} \inf_{s \in [0, \frac{2t}{\sqrt{B}}]} \sqrt{2B}(t - s) - 2|t - s|1(t > s)\right)\right\}$, which by [8][Cor 3.5] is positive and finite.

**Theorem 2.1.** If $a \leq \rho$, then

$$
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) = \frac{C_P}{2}.
$$

Our next results are separated into different cases, based on a relative relation between $\rho$ and $A_a = \frac{1}{4a}(1 - \sqrt{4a^2 + 1})$. Function $A_a$ has been found by analytical calculations. Heuristically, when $\rho < 0$ is relatively big compared to $a$ (in terms of absolute value), then it is less likely that the ruin will occur
simultaneously and the asymptotics should be significantly different than the ones that have been discov-

ered for simultaneous ruin in [12].

Denote \( t_* = \frac{\rho}{\rho(2\rho-1)} \) and introduce the following notation for the one-dimensional constants

\[
P(w_1, w_2, f(u)) := \int_{\mathbb{R}} \mathbb{P} \{ \exists s' \in [0, \infty), \forall s \in [s', s' + f(u)] : B(s) - w_1 s > x \} e^{w_2 x} dx,
\]

\[
\mathcal{H}(w_1, w_2, f(u)) := \lim_{\Delta \to \infty} \int_{\mathbb{R}} \mathbb{P} \{ \exists t' \in [0, \Delta), \forall t \in [t', t' + f(u)] : B(t) - w_1 t > x \} e^{w_2 x} dx,
\]

\[
\mathcal{R}_{S_1, S_2} = \int_{\mathbb{R}^2} \mathbb{P} \{ \exists s', t' \in [0, \infty), \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : W_1(s) - s > x \}
\quad \{ W_2(t) - at > y \} e^{\lambda_1 x + \lambda_2 y} dx dy \in (0, \infty).
\]

In each particular case, finitess and positivity of \( P \) and \( \mathcal{H} \) has been proven in Lemma 3.5.

**Theorem 2.2.** Let \( \rho \in (-1, 1) \) and \( a \in (\max(0, \rho), 1] \) be given.

(i) If \( \rho > A_a \), then

\[
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) = \frac{\mathcal{R}_{S_1, S_2}}{\mathcal{R}_{0, 0}}.
\]

(ii) If \( \rho = A_a \) and \( a < 1 \), then

\[
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) = \frac{(1 - ap)\mathcal{P}(\frac{1 - ap}{1 - \rho}, \frac{1 - ap}{1 - \rho^2}, \rho)\mathcal{H}(a, 2a, S_2)}{2a(1 - \rho^2)}.
\]

(iii) If \( \rho = A_a \), \( a = 1 \), then

\[
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) = \frac{C_{4,1}C_{4,1} + C_{4,2}C_{4,2}}{C_4},
\]

where \( C_{4,1} = \mathcal{P}(2, 2, S_1)\mathcal{H}(1, 2, S_2), C_{4,2} = \mathcal{P}(2, 2, S_2)\mathcal{H}(1, 2, S_1) \) and

\[
C_{4,1}' = \begin{cases} 
-2(\frac{1}{3} c_1 + c_2)^2 \Phi(c_2 + \frac{1}{2} c_1), & -\frac{1}{2} c_1 < c_2, \\
1, & \text{otherwise,}
\end{cases}
\]

\[
C_{4,2}' = \begin{cases} 
-2(\frac{1}{3} c_1 + c_2)^2 \Phi(c_1 + \frac{1}{2} c_2), & -\frac{1}{2} c_2 < c_1, \\
1, & \text{otherwise,}
\end{cases}
\]

\[
C_4 = \begin{cases} 
-2(\frac{1}{3} c_1 + c_2)^2 \Phi(c_2 + \frac{1}{2} c_1) + e^{-2(\frac{1}{3} c_1 + c_2)^2} \Phi(c_1 + \frac{1}{2} c_2), & c_2 > \max(-\frac{1}{2} c_1, -2c_1), \\
e^{-2(\frac{1}{3} c_1 + c_2)^2} \Phi(c_2 + \frac{1}{2} c_1) + \frac{1}{2}, & -\frac{1}{2} c_1 < c_2 \leq -2c_1, \\
e^{-2(\frac{1}{3} c_1 + c_2)^2} \Phi(c_1 + \frac{1}{2} c_2), & -2c_1 < c_2 \leq -\frac{1}{2} c_1, \\
1, & c_2 \leq \min(-\frac{1}{2} c_1, -2c_1).
\end{cases}
\]

(iv) If \( a < 1, \rho < A_a \), then

\[
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) = -\frac{\mathcal{P}(\frac{1 - ap}{1 - \rho^2}, \frac{1 - ap}{1 - \rho^2}, \rho)\mathcal{H}(a, 2a, S_2)}{2\rho}.
\]
(v) If \( a = 1, \rho < A_\alpha \), then

\[
\lim_{u \to \infty} \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) = -\frac{C_6}{2 \rho},
\]

where \( t_* = \frac{1}{\rho(2\rho - 1)} \), \( C_6 = \begin{cases} \mathcal{P}(\frac{1-\rho}{1-\rho^2 t_*}, \frac{1-\rho^2 t_*}{1-\rho^2 t_*}, S_1) H(\frac{1}{t_*}, \frac{2}{t_*}, S_2) & c_1 \leq c_2 \\ \mathcal{P}(\frac{1-\rho^2 t_*}{1-\rho^2 t_*}, \frac{1-\rho}{1-\rho^2 t_*}, S_2) H(\frac{1}{t_*}, \frac{2}{t_*}, S_1) & c_1 > c_2 \end{cases} \). 

3. Proofs

We recall that

\[
\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) = \frac{\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au)}{\pi_{[0,1]^2,\rho}(c_1, c_2, u, au)}
\]

Therefore in the proofs we can also focus on investigating the asymptotic behaviour of \( \mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) \), since the asymptotics for \( \pi_{[0,1]^2,\rho}(c_1, c_2, u, au) \) has been calculated in [5].

3.1. Proof of Theorem 2.1. We divide the proof into two parts - \( a < \rho \) and \( a = \rho \), since the methods used are quite different. Further define \( S_{1,2} = \max(S_1, S_2) \), which will be commonly used notation in both parts of the proof.

Case (i): \( a < \rho \). First note that

\[
\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) \leq \mathbb{P}\left\{ \exists \tau' \in [0,1] \forall s \in [\tau', \tau' + \frac{s}{u}] W_1^*(s) > u \right\}.
\]

On the other hand

\[
\begin{align*}
\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) & \geq \mathbb{P}\left\{ \exists \tau' \in [0,1] \forall \tau \in [\tau', \tau' + \frac{s}{u}] B_1^*(\tau) > u, \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t) - c_2 t > au \right\} \\
& \geq \mathbb{P}\left\{ \exists \tau' \in [0,1] \forall \tau \in [\tau', \tau' + \frac{s}{u}] B_1^*(\tau) > u - \frac{1}{\sqrt{u}} \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t) - c_2 t > au \right\} \\
& \geq \mathbb{P}\left\{ \exists \tau' \in [0,1] \forall \tau \in [\tau', \tau' + \frac{s}{u}] B_1^*(\tau) > u - \frac{1}{\sqrt{u}} \rho (u + c_1 t) + \sqrt{1 - \rho^2} B_2(t) - c_2 t > au \right\} \\
& = \mathbb{P}\left\{ \exists \tau' \in [0,1] \forall \tau \in [\tau', \tau' + \frac{s}{u}] B_1^*(\tau) > u - \frac{1}{\sqrt{u}} \rho (u + \frac{u}{\sqrt{u}}) + (c_2 - \rho c_1) t \right\} \\
& \geq \mathbb{P}\left\{ \forall s \in [0,1] B_2(s) > \frac{(a - \rho) u + (c_2 - \rho c_1) s}{\sqrt{1 - \rho^2}}, \exists \tau' \in [0,1] \forall \tau \in [\tau', \tau' + \frac{s}{u}] B_1^*(\tau) > u - \frac{1}{\sqrt{u}} \rho \right\} \\
& \geq \mathbb{P}\left\{ \forall s \in [0,1] B_2(s) > \frac{(a - \rho) u + (c_2 - \rho c_1) s}{\sqrt{1 - \rho^2}} \right\}.
\end{align*}
\]
Since $a < \rho$, we have that
\[ \lim_{u \to \infty} \Pr \left\{ \forall s \in [0,1] \ B_2(s) > \frac{(a - \rho)u + (c_2 - \rho c_1)s}{\sqrt{1 - \rho^2}} \right\} = 1. \]

Further from independence of increments of Brownian motion we have for $B$ a Brownian motion independent of $B_1, B_2$
\[ \Pr \left\{ \exists t' \in [0,1] \forall t \in [t', t' + \frac{s_1}{u^2}] \ W_1^*(t) > u \right\} \]
\[ \geq \left\{ \exists t' \in [0,1] \forall t \in [t', t' + \frac{s_1}{u^2}] \ W_1^*(t) > u - \frac{1}{\sqrt{u}} \right\} \]
\[ = \left\{ \exists t' \in [0,1] \forall t \in [t', t' + \frac{s_1}{u^2}] \ W_1^*(t) > u \right\} \ Pr \left\{ \forall s \in [0, \max(s_2 - s_1, 0)] \ B(s) + \frac{c_1 s}{u} < \frac{1}{\sqrt{u}} \right\}. \]

Finally we have that
\[ \lim_{u \to \infty} \Pr \left\{ \forall s \in [0, \max(s_2 - s_1, 0)] \ B(s) + \frac{c_1 s}{u} < \sqrt{u} \right\} = 1 \]
and from [8][Cor 3.5] we have

\[ (3.1) \quad \Pr_{[0,1], \frac{s_1}{u^2}}(c_1, u) \sim C_P \Psi(u + c_1) \]
with $C_P = \mathbb{E} \left\{ \sup_{t \geq 0} \inf_{s \in [0, \frac{1}{u^2}]} e^{\sqrt{t} B(t - s) - 2|t-s|1(t > s)} \right\} \in (0, \infty)$. Further we recall that from [5][Thm 2.1] we have that
\[ \pi_{[0,1]^2, \rho}(c_1, c_2; u, au) \sim 2 \Psi(u + c_1). \]

This completes the proof of case (i).

Case (ii): $a = \rho$. Notice that for $\Delta > 0$
\[ \Pr_{s_1, s_2}(c_1, c_2; u, au) \leq \Pr_{[1 - \frac{\Delta}{u^2}, 1]^2, \frac{s_1}{u^2}}(c_1, c_2; u, au) + \pi_{[0,1]^2 \setminus [1 - \frac{\Delta}{u^2}, 1]^2, \rho}(c_1, c_2; u, au). \]

Denote $\Delta(u) = [1 - \frac{1}{\sqrt{u}}, 1]$. Then we have that as $u \to \infty$
\[ \Pr_{[1 - \frac{\Delta}{u^2}, 1]^2, \frac{s_1}{u^2}}(c_1, c_2; u, au) \]
\[ \leq \Pr \left\{ \exists t', t' \in \Delta(u) \forall (s, t) \in [s', s' + \frac{s_1}{u^2}] \times [t', t' + \frac{s_1}{u^2}] \ W_1^*(s) > u, W_2^*(t) > au, \forall v \in \Delta(u) W_1^*(v) < u + \frac{1}{\sqrt{u}} \right\} \]
\[ + \Pr \left\{ \exists v \in \Delta(u) W_1^*(v) > u + \frac{1}{\sqrt{u}} \right\} : = \Pr_1 + \Pr_2 \]

Next observe that
\[ \Pr_1 \leq \Pr \left\{ \exists t', t' \in \Delta(u) \forall (s, t) \in [s', s' + \frac{s_1}{u^2}] \times [t', t' + \frac{s_1}{u^2}] \ B_1(s) - c_1 s > u, B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1 - \rho^2}} t > -\frac{\rho}{\sqrt{u}}, \forall v \in \Delta(u) W_1^*(v) < u + \frac{1}{\sqrt{u}} \right\} \]
\[
\leq \mathbb{P}\left\{ \exists s' \in \mathcal{D}(u) \forall s \in [s', s + \frac{s_1}{u^2}] B_1(s) - c_1 s > u \right\} \mathbb{P}\left\{ \exists t' \in \mathcal{D}(u) \forall t \in [t', t + \frac{s_2}{u^2}] B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1 - \rho^2}} t > 0 \right\}
\]
\[
\leq \mathbb{P}\left\{ \exists s' \in \mathcal{D}(u) \forall s \in [s', s + \frac{s_1}{u^2}] B_1(s) - c_1 s > u \right\} \mathbb{P}\left\{ \exists t' \in \mathcal{D}(u) B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1 - \rho^2}} t > 0 \right\}
\]
\[
= \mathcal{P}_{[0,1], \frac{s_1}{u^2}}^*(c_1, u) \Phi \left( \frac{\rho c_1 - c_2}{\sqrt{1 - \rho^2}} \right) (1 + o(1)), \ u \to \infty.
\]
Notice that with [8] we have for some \( C_1 > 0 \)
\[
\lim_{u \to \infty} \frac{\mathcal{P}_{\mathcal{D}(u), \frac{s_1}{u^2}}^*(c_1, u)}{\Psi(u + c_1)} = C_1
\]
and with [3] we have for some \( C_2 > 0 \)
\[
\lim_{u \to \infty} \frac{\pi_{\mathcal{D}(u), \rho}^*(c_1, u)}{\Psi(u + c_1)} = C_2.
\]
Hence
\[
\lim_{u \to \infty} \frac{\mathcal{P}_{\mathcal{D}(u), \frac{s_1}{u^2}}^*(c_1, u)}{\pi_{\mathcal{D}(u), \rho}^*(c_1, u)} = \frac{C_1}{C_2}.
\]
Since from [5] [Thm 2.1] we have
\[
\mathbb{P}_2 = o \left( \pi_{\mathcal{D}(u), \rho}^*(c_1, u) \right), \quad \pi_{[0,1]^2, \mathcal{D}(u), \rho}^*(c_1, c_2; u, au) = o \left( \pi_{\mathcal{D}(u), \rho}^*(c_1, u) \right),
\]
hence as \( u \to \infty \)
\[
\mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) \leq \Phi \left( \frac{\rho c_1 - c_2}{\sqrt{1 - \rho^2}} \right) \mathcal{P}_{[0,1], \frac{s_1}{u^2}}^*(c_1, u).
\]
Finally, following calculations from case (i) we have that for \( B \) a Brownian motion independent of \( B_1, B_2 \)
\[
\mathcal{P}_{\mathcal{D}(u)^2, \frac{s_1}{u^2}}^*(c_1, c_2; u, au)
\]
\[
\geq \mathbb{P}\left\{ \exists t' \in \mathcal{D}(u) \forall t \in [t', t + \frac{s_1}{u^2}] W_1(t) > u, \sqrt{1 - \rho^2} B_2(t) > (c_2 - \rho c_1) t \right\}
\]
\[
\geq \mathbb{P}\left\{ \exists t' \in \mathcal{D}(u) \forall t \in [t', t + \frac{s_1}{u^2}] W_1(t) > u \right\} \mathbb{P}\left\{ \forall s \in [0, \max(S_2 - S_1, 0)] B(s) + \frac{c_1 s}{u} < \sqrt{u} \right\}
\]
\[
\mathbb{P}\left\{ \exists t' \in \mathcal{D}(u) \forall t \in (t', t + \frac{s_1}{u^2}) \sqrt{1 - \rho^2} B_2(t) > (c_2 - \rho c_1) t \right\}.
\]
Further we have
\[ P\left\{ \forall t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\} \leq P\left\{ \exists t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\}. \]

On the other hand with self-similarity and independence of increments of Brownian motion we have that for \( B \), \( B \) Brownian motions independent of \( B_1, B_2 \)
\[ P\left\{ \forall t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\} \geq P\left\{ \forall t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\} \]
\[ \geq P\left\{ \forall t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\} \geq P\left\{ \forall s \in [0, \sqrt{u}] B(s) - (c_2 - \rho c_1) s < \frac{1}{\sqrt{u}} \right\} \]
\[ = P\left\{ \forall s \in [0, 1] \frac{1}{\sqrt{u}} B(s) - \frac{1}{\sqrt{u}} (c_2 - \rho c_1) s < \frac{1}{\sqrt{u}} \right\} \]
\[ = P\left\{ \forall s \in [0, 1] B(s) - \frac{1}{\sqrt{u}} (c_2 - \rho c_1) s < \frac{1}{\sqrt{u}} \right\} \sim 1 \]

Finally
\[
\lim_{u \to \infty} P\left\{ \exists t \in S(u) \sqrt{1 - \rho^2} B(t) > (c_2 - \rho c_1) t \right\} = \lim_{u \to \infty} P\left\{ \sqrt{1 - \rho^2} B(1 - \frac{1}{\sqrt{u}}) > (c_2 - \rho c_1) (1 - \frac{1}{\sqrt{u}}) + \frac{1}{\sqrt{u}} \right\} = \Phi \left( \frac{\rho c_1 - c_2}{\sqrt{1 - \rho^2}} \right).
\]

Hence the claim follows from (3.1) and from [5][Thm 2.1], which gives
\[
\pi_{[0,1]^2, \rho} (c_1, c_2; u, au) \sim 2 \Phi \left( \frac{\rho c_1 - c_2}{\sqrt{1 - \rho^2}} \right) \Psi(u + c_1).
\]

\[ \square \]

3.2. Proof of Theorem 2.2. We again recall that
\[ \mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) = \frac{\mathcal{P}_{S_1, S_2}(c_1, c_2, u, au)}{\pi_{[0,1]^2, \rho}(c_1, c_2, u, au)} \]
and as in the proof of the Theorem 2.1 we focus on investigating the asymptotic behaviour of \( \mathcal{P}_{S_1, S_2}(c_1, c_2, u, au) \). Before we begin the proof we need few technical lemmas. First let
\[ \Sigma_{s,t} = \begin{pmatrix} \rho \min(s,t) & \rho \min(s,t) \\ \rho \min(s,t) & t \end{pmatrix} \]
be the covariance matrix of \((W_1(s), W_2(t))\). In [5] it was noted that the drift has a significant impact on the optimization problem that was used to determine asymptotics for the classical ruin. We denote below for \( a = (1 + \frac{c_1 s}{u}; a + \frac{c_2 t}{u})^T \)
\[ q_a(s, t) := a^T \Sigma_{s,t}^{-1} a, \quad b(s, t) := \Sigma_{s,t}^{-1} a \]
and set

\begin{equation}
q_a^*(s, t) = \min_{x \geq a} q_x(s, t) , \quad q_a^* = \min_{s, t \in [0, 1]} q_a^*(s, t). \tag{3.2}
\end{equation}

Note that for \( a > \rho \) and large enough \( u \) we have \( b(s, t) \sim (\frac{t-a}{st-\rho^2(\min(s, t))^2}) \rightarrow 0 \). From [13] we have that for any \( s, t \) positive the following logarithmic asymptotics occurs

\begin{equation}
\lim_{u \to \infty} \frac{1}{u^2} \log \mathbb{P}\left\{ \exists_{s, t \in [0, 1]} \min(\min(s, t), \bar{t}) > u \right\} = -\frac{q_a^*(s, t)}{2}. \tag{3.3}
\end{equation}

Hence we will use the function \( q_a^*(s, t) \) to reflect the asymptotics of \( \mathbb{P}\{W_1(s) > u, W_2(t) > au\} \). Below we present the main lemma that solves the optimization problem stated above and was first derived in [5].

**Lemma 3.1.** For all large \( u \) we have:

(i) If \( a = 1, \rho < \frac{1}{2} \), then \( q_{a, u(s, t)}^*(s, t) \) attains its unique local minima on \([0, 1]^2\) at

\[
(s_u, t_u) := \left( 1, \frac{1}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}} \right) , \quad (s_u, \bar{t}_u) := \left( 1, \frac{1}{\rho(2\rho - 1) + \frac{c_1 - \rho c_2}{u}} \right).
\]

(ii) If \( a = 1, \rho = \frac{1}{2} \), then \( q_{a, u(s, t)}^*(s, t) \) attains its unique local minima on \([0, 1]^2\) at

\[
(s_u, t_u) := \left( 1, \min\left( 1 + \frac{c_2}{u}, 1 \right) \right) , \quad (s_u, \bar{t}_u) := \left( \min\left( 1 + \frac{c_1 + 2c_2}{u}, 1 \right) \right).
\]

(iii) For any other \( a \in \max(0, \rho), 1, \rho \in (-1, 1) \), \( q_{a, u(s, t)}^*(s, t) \) attains its unique minimum on \([0, 1]^2\) at

\[
(s_u, t_u) := \begin{cases} 
(1, \frac{a}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}}), & \text{if } \frac{a}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}} \in [0, 1] \\
(1, 1), & \text{otherwise}.
\end{cases}
\]

In the rest of the paper we denote

\[
t^* := \lim_{u \to \infty} t_u,
\]

where \( t_u \) is defined as in Lemma 3.1. We moreover recall the comparison between the behaviour of variance in \( t_u \) and \( t^* \) from [5][Rem. 3.3].

**Remark 3.2.** For \( a \in \max(0, \rho), 1 \) and \( t_u < 1 \) as in Lemma 3.1 we have

\[
\varphi_t^*(u + c_1, au + c_2 t_u) \sim e^{-\frac{(c_1 - \rho c_2)^2}{2(1-\rho^2)}}, \quad u \to \infty.
\]

Denote \( k_u = 1 - \frac{(k-1)\Delta}{u^2}, l_u = t_u - \frac{(l-1)\Delta}{u^2}, u > 0, \Delta > 0 \) and set

\[
E_{u, k} = \left[ (k + 1)u, k_u \right], E_{u, k, l} = E_{u, k} \times E_{u, l}, \quad E = [-\Delta, 0] \times [-\Delta, 0].
\]

Define also \( \eta_{u, k, l}(s, t) := (\eta_{1, u, k}(s), \eta_{2, u, l}(t)) := u(W_1(\frac{u}{u^2} + k_u) - W_1(k_u) - c_1 \frac{u}{u^2}, W_2(\frac{u}{u^2} + l_u) - W_2(l_u) - c_2 \frac{u}{u^2}). \)

The following lemma is used to calculate the ruin probability on an interval of size of order \( O(\frac{1}{u^2}) \).
Lemma 3.3. Let \( \rho \in (-1, 1), a \in (\max(0, \rho), 1], l, k = O\left(\frac{u \log(u)}{2} \right) \) and \( \Delta, S_1, S_2 > 0 \) be given constants. Then, as \( u \to \infty \)

\[
P^*_{E, u, k, l, (S_1, S_2)}(c_1, c_2, u, au) \sim u^{-2} \varphi^* (u + c_1, au + c_2 t_u) I_1(\Delta) e^{-\frac{1}{2} u^2 (q_u^*(k_u, l_u) - q_u^*(1, t_u))},
\]

where

\[I_1(\Delta) = \begin{cases} 
\int_{\mathbb{R}^2} P \left\{ \exists s', t' \in [0, \Delta] \forall s \in [s', s'+S_1], t \in [t', t'+S_2] : W_1(s) - s > x \right\} e^{\lambda_1 x + \lambda_2 y} dxdy & l_u = k_u \\
\int_{\mathbb{R}^2} P \left\{ \exists t' \in [0, \Delta] \forall t \in [t', t'+S_2] : W_2(t) - \frac{a-\rho}{t^* - \rho^*} t > y \right\} e^{\lambda_1 x + \lambda_2 y} dxdy & l_u > k_u,
\end{cases}
\]

and \( \lambda_1 = \begin{cases} 
\frac{1}{1-\rho} \quad & l_u = k_u \\
\frac{t^* - \rho^*}{t^* - \rho^*} \quad & l_u > k_u,
\end{cases} \quad \lambda_2 = \begin{cases} 
\frac{a-\rho}{t^* - \rho^*} \quad & l_u = k_u \\
\frac{a-\rho}{t^* - \rho^*} \quad & l_u < k_u.
\end{cases}
\]

Additionally

\[
\lim_{u \to \infty} \sup_{l, k = O(u \log u)} \int_{\mathbb{R}^2} P \left\{ \exists (s', t') \in E \forall s \in [s', s'+S_1], t \in [t', t'+S_2] : \eta_{u, k, l}(s, t) > (x, y) \right\} e^{\lambda_1 x + \lambda_2 y} dxdy < \infty.
\]

Remark 3.4. Limit (3.4) is necessary so that dominated convergence theorem can be used while proving Lemma 3.3.

The following lemmas are used to show that the constants \( I_1 \) in Lemma 3.3 are positive and finite.

Lemma 3.5. i) For any \( b, c > 0, S \geq 0 \) such that \( 2b > c \) we have

\[
\int_{\mathbb{R}} P \{ \exists t' \geq 0 \forall t \in [t', t'+S] W(t) - bt > x \} e^{cx} dx \in (0, \infty).
\]

ii) For any \( b > 0, S \geq 0 \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{R}} P \{ \exists t \in [0, T] \forall t' \in [t', t'+S] W(t) - bt > x \} e^{2bx} dx \in (0, \infty).
\]

Proof of Lemma 3.5

Ad i) We have that

\[
\int_{\mathbb{R}} P \{ \exists t' \geq 0 \forall t \in [t', t'+S] W(t) - bt > x \} e^{cx} dx = \int_{\mathbb{R}} P \{ \exists t' \geq 0 \forall t \in [0, S] W(t + t') - b(t + t') > x \} e^{cx} dx
\]

\[
= \frac{1}{b} \int_{\mathbb{R}} P \{ \exists t' \geq 0 \forall t \in [0, S] W(t + t') - b^2(t + t') > x \} e^{c/x} dx
\]
We split the proof into several cases which depend on the behaviour of the constant from the fact that the function that we integrate is positive on a set of positive mass. Finitness of the constant follows straightforwardly from Lemma 3.6. Take any 

\[
\lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{R}} \mathbb{P}\{\exists \nu \in [0, T] \forall t \in [t', t'+S] W(t) - bt > x\} e^{2bx} dx
\]

This constant is the same as in [8](8) for \( T = \frac{S}{b^2} \), \( \alpha = 1 \), \( \beta = 1 \), \( b_1 = b_2 = \frac{2b - c}{2b} \).

Ad ii) Notice that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{R}} \mathbb{P}\{\exists \nu \in [0, T] \forall t \in [t', t'+S] W(t) - bt > x\} e^{2bx} dx
\]

\[
= \lim_{T \to \infty} \frac{1}{bT} \int_{\mathbb{R}} \mathbb{P}\{\exists \nu \in [0, T] \forall t \in [0, S] W(t) - b(t + t') > x\} e^{2bx} dx
\]

\[
= \lim_{T \to \infty} \frac{1}{bT} \int_{\mathbb{R}} \mathbb{P}\{\exists \nu \in [0, T] \forall t \in [0, S] bW(t + t') - b^2(t + t') > x\} e^{2x} dx
\]

\[
= \lim_{T \to \infty} \frac{1}{bT} \int_{\mathbb{R}} \mathbb{P}\{\exists \nu \in [0, T] \forall t \in [0, S] W(t + t') - (t + t') > x\} e^{2x} dx.
\]

This constant is the same as in [7](2.5) for \( \alpha = 1 \) and \( S = \frac{T}{b^2} \), \( T = \frac{S}{b^2} \). \( \Box \)

**Lemma 3.6.** Take any \( a > \max(0, \rho), S_1, S_2 \geq 0 \). Then

\[
\int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists s', t' \in \mathbb{R}_+ \forall s \in [s', s'+S_1], t \in [t', t'+S_2] : \begin{cases} W_1(s) - s > x \\ W_2(t) - at > y \end{cases} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy \in (0, \infty),
\]

where \( \lambda_1 = \frac{1 - \alpha \rho}{1 - \rho^2} \), \( \lambda_2 = \frac{a - \rho}{1 - \rho^2} \).

**Proof of Lemma 3.6** Positivity comes of the constant from the fact that the function that we integrate is positive on a set of positive mass. Finitness of the constant follows straightforwardly from

\[
\int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists s', t' \in \mathbb{R}_+ \forall s \in [s', s'+S_1], t \in [t', t'+S_2] : \begin{cases} W_1(s) - s > x \\ W_2(t) - at > y \end{cases} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy
\]

\[
\leq \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists s', t' \in \mathbb{R}_+ : \begin{cases} W_1(s') - s' > x \\ W_2(t') - at' > y \end{cases} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy
\]

and [5] [Lemma 3.6]. \( \Box \)

**Proof of Theorem 2.2** We split the proof into several cases which depend on the behaviour of the variance and the optimization point we get from Lemma 3.1. Let next

\[
N_u := \left\lfloor \frac{u \log(u)}{\Delta} \right\rfloor, \quad K_u^{(1)} = \frac{(c_2 - c_1 \rho)u}{\Delta}, \quad K_u^{(2)} = \frac{(c_1 - c_2 \rho)u}{\Delta},
\]

\[
E_{u,m}^1 := [(m + 1)u, m_u], \quad E_{u,j}^2 := [(j + 1)u, j_u],
\]

where \( m_u = 1 - \frac{(m-1)\Delta}{u^2}, j_u = t_u - \frac{(j-1)\Delta}{u^2} \). For \( \Delta > 0 \) we have

\[
\mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) \geq \mathcal{P}_{F_u, (S_1, S_2)}^*(c_1, c_2; u, au).
\]
On the other hand

\[ \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) \leq \mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au) + \pi_{[0,1] \setminus F_u}(c_1, c_2; u, au). \]

We have

\[ \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) \leq \frac{\mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au) + \pi_{[0,1] \setminus F_u}(c_1, c_2; u, au)}{\pi_{[0,1]}(c_1, c_2; u, au)}. \]

Since from [5][Thm 2.2] we have that

\[ \lim_{u \to \infty} \frac{\pi_{[0,1] \setminus F_u}(c_1, c_2; u, au)}{\pi_{[0,1]}(c_1, c_2; u, au)} = 0 \]

therefore

\[ \mathcal{P}_{S_1, S_2}(c_1, c_2; u, au) \sim \frac{\mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au)}{\pi_{[0,1]}(c_1, c_2; u, au)}, \]

where \( F_u \) is case dependant.

**Case (i):** \( \rho > \frac{1}{2a}(1 - \sqrt{8a^2 - 1}) \). According to Lemma 3.1 \( t^* = t_u = 1 \). From [5][Thm 2.2, case (i)] we have

\[ F_u := E_{u,1}^2. \]

Using Lemma 3.3 and Lemma 3.6 and taking \( u \to \infty \) and then \( \Delta \to \infty \), we get that

\[ \mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au) \sim I u^{-2} \varphi_1(u + c_1, au + c_2), \]

where

\[ I = \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists s', t' \in [0, \infty) \forall s \in [s', s + S_1], t \in [t', t' + \frac{S_2}{u}] : \begin{cases} W_1(s) - s > x \\ W_2(t) - at > y \end{cases} \right\} e^{\lambda_1 x + \lambda_2 y} dx dy. \]

With that, the proof of case (i) is complete.

**Case (ii):** \( \rho = \frac{1}{2a}(1 - \sqrt{8a^2 - 1}) \). We split this case into two subcases since the behaviour of the optimizing point is slightly different. First let \( c_2 - \rho c_1 \leq 0 \). According to Lemma 3.1 \( t^* = t_u = 1 \). From [5][Thm 2.2, case (ii)] we have

\[ F_u := [1 - \frac{\Delta}{u^2}, 1] \times [1 - \frac{\log(u)}{u}, 1 - \frac{\Delta}{u^2}]. \]

Using Bonferroni inequality we have that

\[ \mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au) \]

\[ \geq \sum_{l=2}^{N_u} \mathbb{P} \left\{ \exists s' \in E_{u,1}, t' \in E_{u,1}^2 \forall s \in [s', s + \frac{S_2}{u}], t \in [t', t' + \frac{S_2}{u}] : W_1^*(s) > u, W_2^*(t) > au \right\} \]

\[ - \sum_{l=2}^{N_u} \sum_{m=l+1}^{N_u} \mathbb{P} \left\{ \exists s \in E_{u,1}^1, t_1 \in E_{u,1}^2, t_2 \in E_{u,2}^2 : W_1^*(s) > u, W_2^*(t_1) > au, W_2^*(t_2) > au \right\} \]

\[ := P_{u,\Delta} - D_{u,\Delta}. \]

Further we have

\[ \mathcal{P}_{F_u, \frac{(S_1, S_2)}{u^2}}^*(c_1, c_2; u, au) \leq P_{u,\Delta} + D_{u,\Delta}. \]
From Lemma 3.3 we have as \( u \to \infty \)

\[
P_{u,\Delta} \sim C^{(1)}_{2,p}(\Delta)C^{(2)}_{2,p}(\Delta)u^{-2}\varphi_{\psi}(u + c_1, au + c_2) \sum_{l=2}^{N_u} e^{-\frac{3}{2}u^2(q_a(k_u, l_u) - q_a(1, 1))},
\]

where

\[
C^{(1)}_{2,p}(\Delta) = \int_{\mathbb{R}} \mathbb{P} \left\{ \exists s' \in [0, \Delta] \forall s \in [s', s + \delta_1] : W_1(s) - \frac{1 - a\rho}{1 - \rho^2} s > x \right\} e^{\frac{u}{1 - \rho^2} x} \, dx
\]

and

\[
C^{(2)}_{2,p}(\Delta) = \int_{\mathbb{R}} \mathbb{P} \left\{ \exists t' \in [0, \Delta] \forall t \in [t', t + \delta_2] : W_2(t) - at > x \right\} e^{2ax} \, dx.
\]

Using Taylor expansion we have that for \( k < l \)

\[
u_2(q_a(k_u, l_u) - q_a(1, 1)) = \tau_1(k - 1)\Delta + \tau_4 \frac{(l - 1)^2\Delta^2}{u^2} + o\left(\frac{\Delta^2}{u^2}\right),
\]

where \( \tau_1 = \frac{(1 - a\rho)^2}{1 - \rho^2} > 0 \) and \( \tau_4 = \frac{\rho^2 - 2a\rho^2 + a^2\rho^2}{(1 - \rho^2)^2} > 0 \). Therefore with [5][Lem 3.6] we have as \( u \to \infty \)

\[
P_{u,\Delta} \sim C^{(1)}_{2,p}(\Delta)C^{(2)}_{2,p}(\Delta)u^{-2}\varphi_{\psi}(u + c_1, au + c_2) \sum_{l=2}^{N_u} e^{-\frac{\tau_4 (l - 1)^2\Delta^2}{2u^2}}
\]

\[
= \frac{1}{\sqrt{\tau_4}} \frac{C^{(1)}_{2,p}(\Delta)\Delta}{C^{(2)}_{2,p}(\Delta)} u^{-1}\varphi_{\psi}(u + c_1, au + c_2) \sum_{l=2}^{N_u} \sqrt{\tau_4} \Delta u^{-2} e^{-\frac{\tau_4 (l - 1)^2\Delta^2}{2u^2}}
\]

\[
\sim C^{(1)}_{2,p}(\Delta)\Delta \sqrt{\pi} u^{-1}\varphi_{\psi}(u + c_1, au + c_2).
\]

From Lemma 3.5 we have that \( \lim_{\Delta \to \infty} C^{(1)}_{2,p}(\Delta) = C^{(1)}_{2,p} \) and \( \lim_{\Delta \to \infty} \frac{C^{(2)}_{2,p}(\Delta)}{\Delta} = C^{(2)}_{2,p} \). Hence

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{P_{u,\Delta}}{C^{(1)}_{2,p}C^{(2)}_{2,p} \sqrt{\frac{\pi}{2\tau_4}}} = 1.
\]

From the proof of [5][Theorem 2.2, case (ii)] we have that for \( C = C^{(1)}_{2,p}C^{(2)}_{2,p} \frac{\sqrt{\pi}}{\sqrt{2\tau_4}} \)

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u,\Delta}}{P_{u,\Delta}} = \lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u,\Delta}}{Cu^{-1}\psi_{\psi}(u + c_1, au + c_2)} = 0.
\]

i.e. the double sum is negligible compared to the single sum.

Now we consider the case of \( c_2 - \rho c_1 > 0 \). According to Lemma 3.1 there is exactly one minimizer of \( q_{a_u(s,t)}(s,t) \) on \([0, 1]^2\) at \( (s_u, t_u) \) if

\[
(s_u, t_u) \left( \frac{a}{\rho(2a\rho - 1) + \frac{c_2 - \rho c_1}{u}} \right),
\]

where from \( c_1 - \rho c_2 > 0 \) we obtain that

\[
\frac{a}{\rho(2a\rho - 1) + \frac{c_2 - \rho c_1}{u}} \sim 1
\]
as $u \to \infty$. From [5][Thm 2.2, case (iii)] we have $F_u := [1 - \frac{\Delta}{u^2}, 1] \times [t_u - \frac{\log(u)}{u}, 1 - \frac{\Delta}{u^2}]$. The proof follows the path of the proof of case (ii). Again from Lemma 3.3 we have

\[
P_{u,\Delta} \sim C_{3,\bar{p}}^{(1)}(\Delta)C_{3,\bar{p}}^{(2)}(\Delta)u^{-2} \varphi_{1^*} (u + c_1, au + c_2) \sum_{l = -K_u^{(1)}}^{N_u} e^{-\frac{1}{2}u^2(q_u l - q_u(1,1))},
\]

where

\[
C_{3,\bar{p}}^{(1)}(\Delta) = \int \mathbb{P} \left\{ \exists s^* \in [0, \Delta] \forall s \in [s', s' + s_1] : W_1(s) - \frac{1 - a \rho}{1 - \rho^2} s > x \right\} \frac{1 - a \rho}{1 - \rho^2} dx
\]

and

\[
C_{3,\bar{p}}^{(2)}(\Delta) = \int \mathbb{P} \left\{ \exists t^* \in [0, \Delta] \forall t \in [t', t' + s_2] : W_2(t) - at > x \right\} e^{2ax} dx.
\]

Observe that the Taylor expansions mentioned in case (ii) are independent on $c_1, c_2$ and hence remain the same here. Therefore with [5][Lem 3.6] we have

\[
P_{u,\Delta} \sim C_{3,\bar{p}}^{(1)}(\Delta)C_{3,\bar{p}}^{(2)}(\Delta)u^{-2} \varphi_{1^*} (u + c_1, au + c_2) \sum_{l = -K_u^{(1)}}^{N_u} e^{-\frac{1}{2}u^2 (u - l)^2 \frac{1}{u^2}}
\]

\[
\sim C_{3,\bar{p}}^{(1)}(\Delta)C_{3,\bar{p}}^{(2)}(\Delta) \frac{\sqrt{2\pi}}{\Delta} e^{-\frac{(c_1 \rho - c_2)^2}{2\rho^2(1 - \rho)}} \Phi (c_2 - \rho c_1) u^{-1} \varphi_{1^*} (u + c_1, au + c_2).
\]

From Lemma 3.5 we have that $\lim_{\Delta \to \infty} C_{3,\bar{p}}^{(1)}(\Delta) = C_{3,\bar{p}}^{(1)}$ and $\lim_{\Delta \to \infty} \frac{C_{3,\bar{p}}^{(2)}(\Delta)}{\Delta} = C_{3,\bar{p}}^{(2)}$. Hence we have that

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{P_{u,\Delta}}{C_{3,\bar{p}}^{(1)}C_{3,\bar{p}}^{(2)} \sqrt{2\pi} e^{-\frac{(c_1 \rho - c_2)^2}{2\rho^2(1 - \rho)}} \Phi (c_2 - \rho c_1) u^{-1} \varphi_{1^*} (u + c_1, au + c_2)} = 1.
\]

Since $D_{u,\Delta}$ is the same as [5](3.16), we have from [5] that

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u,\Delta}}{C_{3,\bar{p}}^{(1)}C_{3,\bar{p}}^{(2)} \sqrt{2\pi} e^{-\frac{(c_1 \rho - c_2)^2}{2\rho^2(1 - \rho)}} \Phi (c_2 - \rho c_1) u^{-1} \varphi_{1^*} (u + c_1, au + c_2)} = 0.
\]

With that the proof of case (ii) is complete.

Case (iii): $\rho = -\frac{1}{2}, a = 1$. According to Lemma 3.1 $t^* = 1$. The proof is analogous to case (ii). We use (3.6) and (3.5) with

\[
F_u := [1 - \frac{\Delta}{u^2}, 1] \times [1 - \frac{\log(u)}{u}, 1 - \frac{\Delta}{u^2}] \cup [1 - \frac{\log(u)}{u}, 1 - \frac{\Delta}{u^2}] \times [1 - \frac{\Delta}{u^2}, 1],
\]

\[
P_{u,\Delta} = \sum_{l = 2}^{N_u} \mathbb{P} \left\{ \exists s^* \in E_{u,1} \forall s \in (s', s' + H) : W_1^*(s) > u, W_2^*(t) > u \right\}
\]

\[
+ \sum_{l = 2}^{N_u} \mathbb{P} \left\{ \exists s^* \in E_{u,2} \forall s \in (s', s' + H) : W_1^*(s) > u, W_2^*(t) > u \right\}
\]

\[
:= P_{u,\Delta}^{(1)} + P_{u,\Delta}^{(2)},
\]
\[ D_{u, \Delta} = \sum_{l=2}^{N_u} \sum_{m=l+1}^{N_u} \mathbb{P}\left\{ \exists s \in E_{u, l}, t \in E_{u, m} : W_1^*(s) > u, W_2^*(t) > au \right\} \]

\[ + \sum_{k=2}^{N_u} \sum_{m=k+1}^{N_u} \mathbb{P}\left\{ \exists s_1 \in E_{u, k}, s_2 \in E_{u, m} : W_1^*(s_1) > u, W_1^*(s_2) > u, W_2^*(t) > u \right\} \]

Notice that using calculations from case (ii) and (iii) for \( a = 1, \rho = -\frac{1}{2} \) we have that

\[
P_{u, \Delta}^{(1)} = \begin{cases} 
\sum_{l=2}^{N_u} \mathbb{P}\left\{ \exists s' \in E_{u, l}, t' \in E_{u, l} : s' \in [s', \frac{s}{\sqrt{2}}, t' \in [t', \frac{t}{\sqrt{2}}] : W_1^*(s) > u \right\}, & c_2 + 2c_1 \leq 0 \\
\sum_{l=-K_u}^{N_u} \mathbb{P}\left\{ \exists s' \in E_{u, l}, t' \in E_{u, l} : s' \in [s', \frac{s}{\sqrt{2}}, t' \in [t', \frac{t}{\sqrt{2}}] : W_1^*(s) > u \right\}, & c_2 + 2c_1 > 0 
\end{cases}
\]

which leads us to

\[
P_{u, \Delta}^{(1)} \sim \begin{cases} 
C_{4, \rho}^{(1)}(\Delta)C_{4, \rho}^{(2)}(\Delta) \sqrt{\frac{m}{2}} u^{-1} \varphi_{t^*}(u + c_1, u + c_2), & c_2 + 2c_1 \leq 0 \\
C_{4, \rho}^{(1)}(\Delta)C_{4, \rho}^{(2)}(\Delta) \sqrt{\frac{m}{2}} e^{-\frac{2(4c_1 + c_2)^2}{1}} \Phi (c_2 + \frac{1}{2}c_1) u^{-1} \varphi_{t^*}(u + c_1, u + c_2), & c_2 + 2c_1 > 0 
\end{cases}
\]

as \( u \to \infty \) with

\[
C_{4, \rho}^{(1)}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{ \exists s' \in [0, \Delta] : W_1(s) - 2s > x \right\} e^{2x} dx
\]

and

\[
C_{4, \rho}^{(2)}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{ \exists t' \in [0, \Delta] : W_2(t) - t > x \right\} e^{2x} dx.
\]

Similarly, by interchanging the roles of \( c_1 \) and \( c_2 \) and using previous results we have that as \( u \to \infty \)

\[
P_{u, \Delta}^{(2)} \sim \begin{cases} 
C_{4, \rho}^{(1)}(\Delta)C_{4, \rho}^{(2)}(\Delta) \sqrt{\frac{m}{2}} u^{-1} \varphi_{t^*}(u + c_1, u + c_2), & c_1 + 2c_2 \leq 0 \\
C_{4, \rho}^{(1)}(\Delta)C_{4, \rho}^{(2)}(\Delta) \sqrt{\frac{m}{2}} e^{-\frac{2(4c_1 + c_2)^2}{1}} \Phi (c_1 + \frac{1}{2}c_2) u^{-1} \varphi_{t^*}(u + c_1, u + c_2), & c_1 + 2c_2 > 0. 
\end{cases}
\]

Therefore with Lemma 3.5 we can write that

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{P_{u, \Delta}}{CC_{4, \rho}^{(1)}C_{4, \rho}^{(2)}u^{-1} \varphi_{t^*}(u + c_1, u + c_2)} = 1,
\]
where

\[
C = \begin{cases} \\
\left(\frac{1}{3} + \frac{1}{c_1 + c_2}\right)^2 \Phi \left(2 + \frac{1}{c_1}\right) + e^{-2 \left(\frac{1}{3} + \frac{1}{c_1 + c_2}\right)^2} \Phi \left(2 + \frac{1}{c_1 + \frac{1}{3} c_2}\right), & c_2 > \max(-\frac{1}{2} c_1, -2 c_1) \\
\left(\frac{1}{3} + \frac{1}{c_1 + c_2}\right)^2 \Phi \left(2 + \frac{1}{c_1}\right) + 1, & -\frac{1}{2} c_1 < c_2 \leq -2 c_1 \\
1 + e^{-2 \left(\frac{1}{3} + \frac{1}{c_1 + c_2}\right)^2} \Phi \left(2 + \frac{1}{c_1 + \frac{1}{3} c_2}\right), & -2 c_1 < c_2 \leq -\frac{1}{2} c_1 \\
2, & c_2 \leq \min(-\frac{1}{2} c_1, -2 c_1).
\end{cases}
\]

\(D_{u, \Delta}\) is the same as \([5](3.15)\) or \((3.16)\), exactly in the same way as in the proof of \([5][\text{Thm. 2.2, case (iv)}]\) and hence

\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u, \Delta}}{C u^{-1} \varphi_1 (u + c_1, au + c_2)} = 0.
\]

With that the proof of case (iii) is complete.

Case (iv): \(\rho < \frac{1}{\rho} \left(1 - \sqrt{8 a^2 + 1}\right)\). From Lemma 3.1 we have exactly one minimizer of \(q^{*}_{a, u}(s, t)\) on \([0, 1]^2\) which is \((s_u, t_u) = (1, \frac{a}{\rho(2 a \rho - 1) + 2 - \frac{u}{u_1}})\) and for large enough \(u\) we have \(t_u < 1\). The proof is analogous to case (ii) with

\[
F_u := \left[1 - \frac{\Delta}{u^2}, 1\right] \times \left[t_u - \frac{\log(u)}{u}, t_u + \frac{\log(u)}{u}\right],
\]

\[
P_{u, \Delta} := \sum_{l = -N_u}^{N_u} \mathbb{P}\left\{\exists s' \in E_{u, 1} \cup E_{u, \varphi} \forall s \in [s', s + \frac{2}{u^2}] : W_1^{*}(s) > u, W_2^{*}(t) > au\right\},
\]

\[
D_{u, \Delta} := \sum_{l = -N_u}^{N_u} \sum_{m = l}^{N_u} \mathbb{P}\left\{\exists s' \in E_{u, 1} \cup E_{u, \varphi} \forall s \in [s', s + \frac{2}{u^2}] : W_1^{*}(s) > u, W_2^{*}(t_1) > au, W_2^{*}(t_2) > au\right\}.
\]

Using Taylor expansion we have

\[
u^2 (q_0(k_u, l_u) - q_0(1, t^*)) = \tau_1(k - 1) \Delta + \tau_4 \frac{(l - 1)^2 \Delta^2}{u^2} + o(\frac{k^2}{u^2}) + o(\frac{l^3}{u^4}),
\]

where \(\tau_1 = (1 - 2 a \rho)^2 > 0, \tau_4 = -\frac{\rho^4 (1 - 2 a \rho)^4}{a(1 - a \rho)} > 0\). Using Lemma 3.3 and Remark 3.2 we get

\[
P_{u, \Delta} \sim C_{5, 1}^{(1)}(\Delta) C_{5, 2}^{(2)}(\Delta) u^{-2} \varphi_1 (u + c_1, au + c_2 t^*) \sum_{l = -N_u}^{N_u} e^{-\frac{1}{2} u^2 (q_0(k_u, l_u) - q_0(1, 1))}
\]

\[
\sim C_{5, 1}^{(1)}(\Delta) C_{5, 2}^{(2)}(\Delta) u^{-2} e^{-\frac{(s_u - s')^2}{\rho^2}} \varphi_1 (u + c_1, au + c_2 t^*) \sum_{l = -N_u}^{N_u} e^{-\frac{1}{2} u^2 (q_0(k_u, l_u) - q_0(1, 1))},
\]

where

\[
C_{5, 1}^{(1)}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists s' \in [0, \Delta] \forall s \in [s', s + \frac{1}{u}] : W_1(s) - \frac{1 - a \rho}{1 - \rho^2 t^*} s > x\right\} e^{\frac{1 - a \rho}{1 - \rho^2 t^*} x} dx
\]

and

\[
C_{5, 2}^{(2)}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists t' \in [0, \Delta] \forall t \in [t', t + \frac{2}{u}] : W_2(t) - \frac{a}{t^*} t > x\right\} e^{\frac{a}{t^*} x} dx.
\]
With [5][Lemma 3.7] we have
\[
\sum_{l=-N_u}^{N_u} e^{-\frac{1}{2} u^2(q_u(k_u, t_u) - q_u(1, 1))} = \frac{u}{\sqrt{\tau_4}} \sum_{l=1}^{N_u} \sqrt{\tau_4 u} e^{-\frac{u^2}{2} \frac{(l-1)^2 \Delta^2}{u^2}}
\sim \frac{u \sqrt{2\pi}}{\Delta}.
\]

From Lemma 3.5 we have that \(\lim_{\Delta \to \infty} C^{(1)}_{5, p}(\Delta) = C^{(1)}_{5, p}\) and \(\lim_{\Delta \to \infty} \frac{C^{(2)}_{5, p}(\Delta)}{\Delta} = C^{(2)}_{5, p}\). Hence
\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{P_{u, \Delta}}{C^{(1)}_{5, p} C^{(2)}_{5, p} \sqrt{\frac{2\pi}{\Delta}}} = 1.
\]

\(D_{u, \Delta}\) is exactly the same as [5](3.18) and hence
\[
\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u, \Delta}}{Cu^{-1} \varphi_t^* (u + c_1, au + c_2)} = 0,
\]
which means that the double sum is negligible. With that the proof of case (iv) is complete.

**Case (v): \(a = 1, \rho < A_u\).** According to Lemma 3.1, there are two optimal points:
\[
(\bar{s}_u, \bar{t}_u) = \left(1, \frac{1}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}}\right), \quad (\bar{\bar{s}}_u, \bar{\bar{t}}_u) = \left(1, \frac{1}{\rho(2\rho - 1) + \frac{c_1 - \rho c_2}{u}}\right),
\]
where for large enough \(u\) we have
\[
\frac{1}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}} > 1.
\]

We can use (3.5) and (3.6) with
\[
F_u := [1 - \Delta/\sqrt{u^2}, 1] \times [t_u - \log(u)/u, t_u + \log(u)/u] \cup [t_u - \log(u)/u, t_u + \log(u)/u] \times [1 - \Delta/\sqrt{u^2}, 1],
\]
\[
P_{u, \Delta} := \mathbb{P}\left\{ \exists (s', t') \in [1 - \Delta/\sqrt{u^2}, 1] \times [t_u - \log(u)/u, t_u + \log(u)/u] \forall s \in [s', s'] + [s + S_2/\sqrt{u^2}] \right. t \in [t', t'+ S_2/\sqrt{u^2}] W_1^*(s) > u, W_2^*(t) > u \right\}
\]
\[
+ \mathbb{P}\left\{ \exists (s', t') \in [t_u - \log(u)/u, t_u + \log(u)/u] \times [1 - \Delta/\sqrt{u^2}, 1] \forall s \in [s', s'] + [s + S_2/\sqrt{u^2}] t \in [t', t' + S_2/\sqrt{u^2}] W_1^*(s) > u, W_2^*(t) > u \right\}
\]
\[
:= P_{u, \Delta}^{(1)} + P_{u, \Delta}^{(2)}.
\]

\[
D_{u, \Delta} := \mathbb{P}\left\{ \exists (s_1, t_1) \in [1 - \Delta/\sqrt{u^2}, 1] \times [t_u - \log(u)/u, t_u + \log(u)/u] \forall (s_2, t_2) \in [t_u - \log(u)/u, t_u + \log(u)/u] \times [1 - \Delta/\sqrt{u^2}, 1] s_1 > u \right. \right\}
\]
\[
:= \mathbb{P}\left\{ \begin{array}{l}
W_1^*(s_1) > u \\
W_1^*(s_2) > u \\
W_2^*(t_1) > u \\
W_2^*(t_2) > u
\end{array} \right\}.
\]

Using the same calculations as in case (iv) for \(a = 1\) we have that
\[
P_{u, \Delta}^{(1)} \sim C^{(1)}_{6, p} C^{(2)}_{6, p} \sqrt{\frac{2\pi}{\Delta^4/\sqrt{u^2}}} u^{-1} \varphi_t^* (u + c_1, u + c_2 t_u)
\]
\[
= C^{(1)}_{6, p} C^{(2)}_{6, p} \sqrt{\frac{2\pi}{\Delta^4/\sqrt{u^2}}} \left(\frac{c_2^2 - 2c_1 v_2 + v_2^2}{2\rho(1 - \rho)}\right) u^{-1} \varphi_t^* (u + c_1, u + c_2 t_u)\]
where

\[ C^{(1)}_{6, P}(\Delta) = \int_{\mathbb{R}} \left\{ \exists s' \in [0, \Delta] \forall s \in [s', s' + S_1] : W_1(s) - \frac{1 - \rho}{1 - \rho^2 t^*} s > x \right\} e^{\frac{1 - \rho}{1 - \rho^2 t^*} x} dx \]

and

\[ C^{(2)}_{6, P}(\Delta) = \int_{\mathbb{R}} \left\{ \exists t' \in [0, \Delta] \forall t \in [t', t' + S_2] : W_2(t) - \frac{1}{t^*} t > x \right\} e^{\frac{1}{t^*} x} dx. \]

By interchanging the roles of \( c_1 \) and \( c_2 \) we can analogously get

\[ P_{u, \Delta}^{(2)} \sim C^{(1)}_{6, P}(\Delta) C^{(2)}_{6, P}(\Delta) \left( \frac{2\pi e^{-\frac{(\min(c_1, c_2) - \max(c_1, c_2))^2}{2(1 - \rho)}}}{\sqrt{1 - \rho}} \right) \frac{c_1^2 - 2c_1c_2 + c_2^2}{2(1 - \rho)} \varphi_t(u + c_2, u + c_1 t^*). \]

From the proof of [5] [Thm 2.2, case (vi)] we have

1. For \( c_1 > c_2 : \varphi_t(u + c_1, u + c_2 t^*) = o(\varphi_t(u + c_2, u + c_1 t^*)) \),
2. For \( c_1 < c_2 : \varphi_t(u + c_2, u + c_1 t^*) = o(\varphi_t(u + c_1, u + c_2 t^*)) \),
3. For \( c_1 = c_2 : \varphi_t(u + c_1, u + c_2 t^*) = \varphi_t(u + c_2, u + c_1 t^*) \).

With that we obtain that

\[ \lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{P_{u, \Delta}}{C_{6, P} C_{6, P}^{-1} \varphi_t(u + \min(c_1, c_2), u + \max(c_1, c_2) t^*)} = 1 \]

with

\[ C_6 = \begin{cases} e^{-\frac{(\min(c_1, c_2) - \max(c_1, c_2))^2}{2(1 - \rho)}} \frac{\sqrt{2\pi}}{\sqrt{1 - \rho}}, & c_1 \neq c_2 \\ 2e^{-\frac{c_1^2 (1 - \rho)}{2(1 - \rho)}} \frac{\sqrt{2\pi}}{\sqrt{1 - \rho}}, & c_1 = c_2 \end{cases} \]

\( D_{u, \Delta} \) is the same as [5] (3.19) and hence

\[ \lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{D_{u, \Delta}}{C u^{-1} \varphi_t(u + c_1, au + c_2)} = 0. \]

With that the proof of case (v) is complete. \( \square \)

4. Appendix

Proof of Lemma 3.3 Let \( S_1, S_2 > 0 \). For all the cases we can write:

\[ \mathcal{P}_{E_{u, k, t}^{\left( s_1, s_2 \right)}}(c_1, c_2, u, au) \]

\[ = \int_{\mathbb{R}^2} \mathbb{P} \left( \exists (s', t') \in E \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : \begin{array}{c} W_1^*(\frac{u}{x_u} + k_u) > u \\ W_1^*\left(\frac{u}{x_u} + l_u\right) > au \end{array} \right) \times u^{-2} \varphi_{k_u, l_u}(u + c_1 k_u - \frac{x}{u}, au + c_2 l_u - \frac{y}{u}) dx dy \]

\[ = \int_{\mathbb{R}^2} \mathbb{P} \left( \exists (s', t') \in E \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : \begin{array}{c} W_1^*(\frac{u}{x_u} + k_u) - W_1(k_u) + W_1(l_u) > u \\ W_2^*(\frac{u}{x_u} + l_u) - W_2(l_u) + W_2(l_u) > au \end{array} \right) \times u^{-2} \varphi_{k_u, l_u}(u + c_1 k_u - \frac{x}{u}, au + c_2 l_u - \frac{y}{u}) dx dy \]
\[
\int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists (s', t') \in E \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : \eta_{u,k,l}(s, t) > (x, y) \bigg| W_1^*(k_u) = u - \frac{x}{u}, W_2^*(l_u) = au - \frac{y}{u} \right\} \\
\times u^{-2} \varphi_{k_u,l_u}(u + c_1 k_u - \frac{x}{u}, au + c_2 l_u - \frac{y}{u}) dxdy.
\]

From [5] [Lemma 3.3] we have that both for \( k_u > l_u \) and \( k_u < l_u \), as \( u \to \infty \)

\[
(4.1) \quad \varphi_{k_u,l_u}(u + c_1 k_u - \frac{x}{u}, au + c_2 l_u - \frac{y}{u}) \sim \varphi_{l_u}(u + c_1, au + c_2 t_u) \\
\times e^{-\frac{1}{2}u^2(q_{\operatorname{au}}(k_u,l_u)(k_u,l_u) - q_{\operatorname{au}}(1,t_u)(1,t_u))} e^{\lambda_1 x + \lambda_2 y}.
\]

Hence it remains to investigate

\[
\int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists (s', t') \in E \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : \eta_{u,k,l}(s, t) > (x, y) \bigg| W_1^*(k_u) = u - \frac{x}{u}, W_2^*(l_u) = au - \frac{y}{u} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy.
\]

Since the interplay between \( k_u \) and \( l_u \) influences the behaviour of the integrals above, we split the proof into three parts : \( k_u = l_u, k_u < l_u, k_u > l_u \).

(i) If \( k_u = l_u \), then \( \eta_{u,k,l,x,y}^*(s, t) = (\eta_{1,u,k,l,x,y}^*(s), \eta_{2,u,k,l,x,y}^*(t)) \): \( \eta_{u,k,l}(s, t) \to (\eta_{1,u,k,l}(s), \eta_{2,u,k,l}(t)) \).

We have that

\[
\mathbb{E}\{\eta_{u,k,l,x,y}^*(s, t)\} = -\frac{1}{uk_u} \begin{pmatrix} s(u + c_1 k_u - \frac{x}{u}) \\ t(u + c_2 k_u - \frac{y}{u}) \end{pmatrix}
\]

and the covariance matrix is equal to

\[
\Sigma(\eta_{u,k,l,x,y}^*(s, t)) = \begin{pmatrix} s & \rho \min(s, t) \\ \rho \min(s, t) & t \end{pmatrix} - u^{-2} \begin{pmatrix} s & \rho s \\ \rho t & t \end{pmatrix} \begin{pmatrix} k_u & \rho k_u \\ \rho k_u & l_u \end{pmatrix}^{-1} \begin{pmatrix} s & \rho t \\ \rho s & t \end{pmatrix}
\]

\[
= \begin{pmatrix} s & \rho \min(s, t) \\ \rho \min(s, t) & t \end{pmatrix} - O\left(\frac{\log(u)}{u}\right) \begin{pmatrix} s^2 & \rho^2 st \\ \rho^2 st & t^2 \end{pmatrix}, s, t \in [0, \Delta].
\]

Similarly

\[
\Sigma(\eta_{u,k,l,x,y}^*(s_1, t_1) - \eta_{u,k,l,x,y}^*(s_2, t_2)) = \begin{pmatrix} |s_1 - s_2| & \rho \min(s_1 - s_2, t_1 - t_2) \\ \rho \min(s_1 - s_2, t_1 - t_2) & |t_1 - t_2| \end{pmatrix} - O\left(\frac{\log(u)}{u}\right) \begin{pmatrix} |s_1 - s_2|^2 & (\rho \min(s_1 - s_2, t_1 - t_2))^2 \\ (\rho \min(s_1 - s_2, t_1 - t_2))^2 & |t_1 - t_2|^2 \end{pmatrix}.
\]

Together with the continuous mapping theorem we get, as \( u \to \infty \)

\[
\int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists (s', t') \in E \forall s \in [s', s' + S_1], t \in [t', t' + S_2] : \eta_{u,k,l,x,y}^*(s, t) > (x, y) \right\} e^{\lambda_1 x + \lambda_2 y} dxdy
\]
\[ \sim \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists s', t' \in [0, \Delta] \forall s \in [s', s' + S_1], t \in [t', t' + S_2]: \begin{align*}
W_1(s) - s > x \\
W_2(t) - at > y
\end{align*} \right\} e^{\lambda_1x + \lambda_2y} dx dy. \]

It remains to show that (3.4) is finite and to justify the use of the dominated convergence theorem, but using (1.1) we can bound the (3.4) by replacing the Parisian ruin with simple supremum and the finitness then follows from [5] (3.8).

(ii) If \( k_u < l_u \), then observe that the increments \( W_1(s + k_u^2) - W_1(k_u^2), W_2(t + l_u^2) - W_2(l_u^2) \) are independent. Hence

\[
\int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists s', t' \in [0, \Delta] \forall s \in [s', s' + S_1], t \in [t', t' + S_2]: \eta_{u,k,t}(s, t) > (x, y) \ \left| \begin{align*}
W_1^*(k_u) &= u - \frac{x}{u} \\
W_2^*(l_u) &= au - \frac{y}{u}
\end{align*} \right. \right\} e^{\lambda_1x + \lambda_2y} dx dy
\]

where \( v_{1,u,x,y}^*(s) := \left( \eta_{1,u,k}(s) \ \left| \begin{align*}
W_1^*(k_u) &= u - \frac{x}{u} \\
W_2^*(l_u) &= au - \frac{y}{u}
\end{align*} \right. \) is a Gaussian process with

\[
\mathbb{E}\{v_{1,u,x,y}^*(s)\} = \frac{1}{uk_u} s(u + c_1 - \frac{x}{u}) - c_1 s u,
\]

\[
\text{Var} \{v_{1,u,x,y}^*(s)\} = s - \frac{s^2}{u^2 k_u}
\]

and \( v_{2,u,x,y}^*(t) := \left( \eta_{2,u,j}(t) \ \left| \begin{align*}
W_1^*(k_u) &= u - \frac{x}{u} \\
W_2^*(l_u) &= au - \frac{y}{u}
\end{align*} \right. \) is a Gaussian process with

\[
\mathbb{E}\{v_{2,u,x,y}^*(t)\} = \frac{1}{u(au + c_2 - \frac{y}{u})} (k_u t(au + c_2 - \frac{y}{u}) - \rho k_u t(u + c_1 - \frac{x}{u})) - c_2 t u,
\]

\[
\text{Var} \{v_{2,u,x,y}^*(t)\} = t - \frac{t^2}{u^2 (au + \rho^2 k_u^2)}.
\]

Moreover, for each \( 0 \leq s < t \leq -\Delta \), \( (v_{1,u,x,y}^*(s) - v_{1,u,x,y}^*(t)) \) is Normally distributed with

\[
\text{Var} \left( v_{1,u,x,y}^*(s) - v_{1,u,x,y}^*(t) \right) = (s - t) - \frac{(s - t)^2}{u^2 k_u}
\]

while \( (v_{2,u,x,y}^*(s) - v_{2,u,x,y}^*(t)) \) is Normally distributed with

\[
\text{Var} \left( v_{2,u,x,y}^*(s) - v_{2,u,x,y}^*(t) \right) = (s - t) - \frac{(s - t)^2}{u^2 (au + \rho^2 k_u^2)}.
\]

Hence, using that \( \text{Var} \left( v_{i,u,x,y}^*(s) - v_{i,u,x,y}^*(t) \right) \leq 2|s - t| \) for all large enough \( u \),

\[
v_{1,u,x,y}^*(s), s \in [0, \Delta] \text{ weakly converges in } C[0, \Delta] \text{ to } W_1(s) - s, s \in [0, \Delta] \text{ and }\]

\[
v_{2,u,x,y}^*(t), s \in [0, \Delta] \text{ weakly converges in } C[0, \Delta] \text{ to } W_2(s) - \frac{\rho}{\rho^2} s, s \in [0, \Delta].\]

It remains to show that (3.4) is finite and to justify the use of the dominated convergence theorem. As in previous case it falls
directly from (1.1) and [5](3.8). Combining it with the proven weak convergence and with the dominated convergence theorem, we obtain that

\[
\int_{\mathbb{R}^2} P\left\{ \exists (s', t') \in E \forall s \in [s', s' + 1], t \in [t', t + 1/2] \colon \eta_{h, k, l}(s, t) > (x, y) \right\} \left( W_1^*(k_u) = u - \frac{u}{u} \right) \left( W_2^*(l_u) = au - \frac{y}{u} \right) e^{\lambda_1 x + \lambda_2 y} dx dy
\]

\[
\sim \int_{\mathbb{R}^2} P\left\{ \exists s \in [0, \Delta] \forall s \in [s', s' + 1] \colon W_1(s) > s > x \right\} \times P\left\{ \exists t \in [0, \Delta] \forall t \in [t', t + 1/2] : W_2(t) - \frac{a - \rho}{t' - t^2} > y \right\} e^{\lambda_1 x + \lambda_2 y} dx dy
\]

(iii) If \( k_u > l_u \), then again the increments \( W_1(s + k_u u^2) - W_1(k_u u^2) \), \( W_2(t + l_u u^2) - W_2(l_u u^2) \) are independent. Hence

\[
\int_{\mathbb{R}^2} P\left\{ \exists (s', t') \in E \forall s \in [s', s' + 1], t \in [t', t + 1/2] \colon \eta_{h, k, l}(s, t) > (x, y) \right\} \left( W_1^*(k_u) = u - \frac{u}{u} \right) \left( W_2^*(l_u) = au - \frac{y}{u} \right) e^{\lambda_1 x + \lambda_2 y} dx dy
\]

\[
= \int_{\mathbb{R}^2} P\left\{ \exists s \in [0, \Delta] \forall s \in [s', s' + 1] : \nu_1^*(s) > x \right\} \times P\left\{ \exists t \in [0, \Delta] \forall t \in [t', t + 1/2] : \nu_2^*(t) > y \right\} e^{\lambda_1 x + \lambda_2 y} dx dy,
\]

where \( \nu_1^*(s) := \left( \eta_{1, u}(s) \right) \mid \left( W_1^*(k_u) = u - \frac{u}{u} \right) \left( W_2^*(l_u) = au - \frac{y}{u} \right) \) is a Gaussian process with

\[
\mathbb{E}\{\nu_1^*(s)\} = \frac{1}{u(k_u - \rho^2 l_u)} (s l_u (u + c_1 - \frac{x}{u}) - ps l_u (au + c_2 - \frac{y}{u})) - c_1 \frac{s}{u},
\]

\[
\text{Var} (\nu_1^*(s)) = s - \frac{s^2}{u^2(k_u - \rho^2 l_u)}
\]

and \( \nu_2^*(t) := \left( \eta_{2, u}(t) \right) \mid \left( W_1^*(k_u) = u - \frac{u}{u} \right) \left( W_2^*(l_u) = au - \frac{y}{u} \right) \) is a Gaussian process with

\[
\mathbb{E}\{\nu_2^*(t)\} = \frac{1}{u l_u} t (au + c_2 - \frac{y}{u}) - c_2 \frac{t}{u},
\]

\[
\text{Var} (\nu_2^*(t)) = t - \frac{t^2}{u^2 l_u}.
\]

Moreover, for each \( 0 \leq s > t \geq \Delta \), \( \left( \nu_1^*(s) - \nu_1^*(t) \right) \) is normally distributed with

\[
\text{Var} (\nu_1^*(s) - \nu_1^*(t)) = (s - t) - \frac{(s - t)^2}{u^2(k_u - \rho^2 l_u)}
\]

and \( \left( \nu_2^*(s) - \nu_2^*(t) \right) \) is normally distributed with

\[
\text{Var} (\nu_2^*(s) - \nu_2^*(t)) = (s - t) - \frac{(s - t)^2}{u^2 l_u}.
\]
Hence, using that $\text{Var} \left( \nu^*_{i,u,x,y}(s) - \nu^*_{i,u,x,y}(t) \right) \leq 2|s-t|$ for all $u$ large enough,

$\nu^*_{1,u,x,y}(s), s \in [0, \Delta]$ weakly converges in $C[0, \Delta]$ to $W_1(s) - \frac{1-a\rho}{1-\rho^2} s$, $s \in [0, \Delta]$ and

$\nu^*_{2,u,x,y}(t), t \in [0, \Delta]$ weakly converges in $C[0, \Delta]$ to $W_2(t) - \frac{1}{t} t$. This leads to

$$\int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists (s', t') \in E \forall s' \in [s', s'+S_1], t \in [t', t'+S_2] : \eta_{u,k,l}(s, t) > (x, y) \quad \left| \begin{array}{c} W_1^*(k_u) = u - \frac{x}{a} \\ W_2^*(l_u) = au - \frac{y}{a} \end{array} \right. \right\} e^{\lambda_1 x + \lambda_2 y} dxdy$$

$$\sim \int_{\mathbb{R}^2} \mathbb{P} \left\{ \exists s' \in [0, \Delta] \forall s \in [s', s'+S_1] : W_1(s) - \frac{1-a\rho}{1-\rho^2} s > x \right\}$$

$$\times \mathbb{P} \left\{ \exists t' \in [0, \Delta] \forall t \in [t', t'+S_2] : W_2(t) - \frac{1}{t} t > y \right\} e^{\lambda_1 x + \lambda_2 y} dxdy.$$  

The finiteness of $(3.4)$ and the application of the dominated convergence theorem can be proven identically as in the previous case. This completes the proof. \hfill \Box

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