Type IIA orientifolds and orbifolds on non-factorizable tori

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Abstract

We investigate Type II orientifolds on non-factorizable torus with and without its orbifolding. We explicitly calculate the Ramond-Ramond tadpole from string one-loop amplitudes, and confirm that the consistent number of orientifold planes is directly derived from the Lefschetz fixed point theorem. We furthermore classify orientifolds on non-factorizable $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, and construct new supersymmetric Type IIA orientifold models on them.
1 Introduction

Many attempts have been made for constructing string vacua using D-branes in order to realize the Standard Model. In Type IIA orientifolds, intersecting D-brane models provide chiral spectra [1–13], which feature some of the properties of the supersymmetric or non-supersymmetric Standard Model (see [14–16] and references therein). We know now vast number of perturbative vacua in the landscape of string theory, and it is of great importance to further investigate possible vacua in string theory construction [18–21].

Most of Type IIA models compactified on six-dimensional spaces have been constructed by orbifold tori given by $\mathbb{Z}_N$ [7, 8], $\mathbb{Z}_2 \times \mathbb{Z}_2$ [9, 10, 17] and $\mathbb{Z}_4 \times \mathbb{Z}_2$ [12, 13], whose point group is defined by the Coxeter elements. In the case of $\mathbb{Z}_N$ orbifold, some models compactified on non-factorizable tori are constructed by Coxeter elements [22], while, in the case of $\mathbb{Z}_N \times \mathbb{Z}_M$ Coxeter orbifolds [23], the compact spaces are factorized to $T^2 \times T^2 \times T^2$. Recently, however, non-factorizable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds were constructed in heterotic string [24–26], and in Type IIA string [21]. One of the authors in this paper recently classified $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models on non-factorizable tori [27, 28]. Non-factorizable orbifolds possess different geometries from factorizable ones because the number of fixed tori, and the Euler numbers, in six-dimensional spaces can be less than those of the factorizable ones. Such non-factorizable orbifolds can be applied to Type IIA string models, which give rise to rather richer structure by inclusion of D-branes.

For the consistency of theory the tadpole cancellation is required (see [3, 30–34], and for review, [35–37]). We explicitly calculate string one-loop amplitudes on the Klein bottle, the annulus and the Möbius strip on non-factorizable tori and orbifolds, and confirm that the consistent number of orientifold planes (O-planes) is directly derived from the Lefschetz fixed point theorem via the cancellations of Ramond-Ramond (RR) tadpole. We give a systematic way to construct various models on non-factorizable orbifolds. Interestingly, we further find new feature of non-factorizable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, in which the numbers of O-planes depend on three-cycles.

This paper is organized as follows: In section 2 we describe the tadpole cancellation condition on generic non-factorizable tori. In this analysis the Lefschetz fixed point theorem makes the cancellation condition simplified and provides an intuitive picture. We apply this formula to orientifold models which have been already well-investigated. In section 3 we explicitly construct Type IIA orientifolds on $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the $D_6$ Lie root lattice. Because the contributions of untwisted sector in orbifolds are given by the same forms of those in tori, the formula, which is derived from the Lefschetz fixed point theorem, provides a necessary condition on non-factorizable orbifolds. We describe general features of orientifold
constructions on non-factorizable orbifolds. Section 4 is devoted to the conclusion. In appendix A we explain details of the classification of orientifolds and orbifolds on the Lie root lattices. In appendix B we summarize a set of useful conventions to describe non-factorizable tori in terms of the lattice space and its dual. In appendix C we briefly review the string one-loop amplitudes which are given by the Klein bottle, the annulus and the Möbius strip as the worldsheet topologies.

2 Orientifold on non-factorizable torus

In this section we will evaluate RR-tadpole cancellation conditions of torus compactification in Type IIA string theory in the presence of D6-branes and orientifold planes (O6-planes). We will show a method to analyze the orientifold models on non-factorizable tori, which can be applied to any kind of torus compactifications. We introduce a set of general formula for the tadpole amplitudes in RR-sector on non-factorizable tori, which are defined by the Lie root lattices. Utilizing the Lefschetz fixed point theorem, we can check the tadpole cancellation condition not only on the usual factorizable tori but also on the non-factorizable ones in a quite simple way. We will further apply this method to orbifold models in section 3.

2.1 RR-tadpole and the Lefschetz fixed point theorem

We consider the Type IIA models compactified on a six-torus $T^6$. A six-torus could be regarded as a six-dimensional Euclidean space $\mathbb{R}^6$ divided by a lattice $\Lambda$, i.e., $T^6 = \mathbb{R}^6/\Lambda$. As we will see, the structure of the lattice $\Lambda$ plays a central role in the analysis of this paper. Here let us consider orientifolds in Type IIA given in the following way:

$$\Omega R$$

(2.1)

where $\Omega$ is the worldsheet parity operator, and $R$ is the orientifold involution which indicates the reflection of three directions in $T^6$. Usually the action $R$ can be given as

$$R : z_i \rightarrow \bar{z}_i.$$  

(2.2)

In order to construct consistent effective theories in four-dimensional spacetime, we study the tadpole cancellation condition in the presence of orientifolds. The tadpole amplitude is derived from the string one-loop graphs whose topologies are the Klein bottle, the annulus, and the Möbius strip. These amplitudes are represented as $K$, $A$ and $M$, respectively. Here let us explicitly describe their amplitudes in terms of a modulus $t$ in the loop channel as follows:

$$K = 4c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{closed}} \left( \frac{\Omega R}{2} \left(1 + \frac{(-1)^F}{2}\right)(-1)^{s}e^{-2\pi t(L_0 + \bar{L}_0)} \right),$$

(2.3a)
\[ \mathcal{A} = c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{open}} \left( \frac{1}{2} \left( \frac{1 + (-1)^F}{2} \right) (-1)^s e^{-2\pi t L_0} \right), \]  
(2.3b)

\[ \mathcal{M} = c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{open}} \left( \frac{\Omega R}{2} \left( \frac{1 + (-1)^F}{2} \right) (-1)^s e^{-2\pi t L_0} \right), \]  
(2.3c)

where \( F \) and \( s \) denote the fermion numbers in the worldsheet and in the spacetime, respectively; the overall coefficient \( c \) is given by \( c \equiv V_4/(8\pi^2\alpha'^2) \), where \( V_4 \) is from the integration over momenta in non-compact directions. Since the divergence from the RR-tadpole should be evaluated in the tree channel, which is described by the \( t \)-modulus, we should rewrite them via the modular transformation, even though the computations of the amplitudes are easier in the loop channel given by \( t \)-modulus. The RR-sectors in the tree channel which we should evaluate in order to see the tadpole cancellation in the presence of orientifold planes and D-branes, correspond to the states with the following insertions in the loop channel [29]:

- Klein bottle: closed string, NS-NS sector, \((-1)^F\)
- annulus: open string, R sector
- Möbius strip: open string, NS sector, \((-1)^F\)

In this paper we calculate these amplitudes for the case cases that D-branes are parallel on O-planes. Then the amplitudes can be written in the form as follows:

\[ \mathcal{K} = c (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \left[ \frac{0}{1/2} \right]^4}{\eta^{12}} \mathcal{L}_K, \]  
(2.5a)

\[ \mathcal{A} = \frac{c}{4} (1_{\text{RR}} - 1_{\text{NSNS}}) \{ (\text{tr}(\gamma_1))^2 \} \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \left[ \frac{0}{1/2} \right]^4}{\eta^{12}} \mathcal{L}_A, \]  
(2.5b)

\[ \mathcal{M} = -\frac{c}{4} (1_{\text{RR}} - 1_{\text{NSNS}}) \{ \text{tr}(\gamma_1^{-1} \gamma_0^T \Omega R) \} \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \left[ \frac{1/2}{0} \right]^4}{\eta^{12}} \mathcal{L}_M, \]  
(2.5c)

where the string oscillation modes are represented with respect to the \( \vartheta \)-function and the Dedekind \( \eta \)-function, while the zero modes are given by \( \mathcal{L}_K, \mathcal{L}_A \) and \( \mathcal{L}_M \). The \( \gamma \) matrices are orientifold actions on the Chan-Paton factors in the notation of [34]. Due to the spacetime supersymmetry, the total amplitudes from RR- and NSNS-sectors should be cancelled to each other, as seen the factor \( (1_{\text{RR}} - 1_{\text{NSNS}}) \) on each amplitude in (2.5). The mapping between the two different moduli \( t \) and \( l \) in these channels is also given as

\[ \text{Klein bottle} : \ t = \frac{l}{4}, \quad \text{annulus} : \ t = \frac{l}{2}, \quad \text{Möbius strip} : \ t = \frac{l}{8}. \]  
(2.6)

To evaluate the RR-tadpole generated by the orientifold, we extract only the contributions from RR-sector in the tree channel. In the IR limit \( l \to \infty \) the divergence from the RR-tadpole
where $\tilde{K}_{RR}$, $\tilde{A}_{RR}$ and $\tilde{M}_{RR}$ are RR-tadpole contributions in the tree channel mapped from $K$, $A$ and $M$ in the loop channel under the modular transformation, respectively.

Now let us evaluate the zero mode contributions $L_{K,A,M}$ in (2.5) given by the momentum modes and the winding modes. $p$ and winding modes $w$ can be written in terms of a set of certain basis vectors $\{p_i\}$ and $\{w_i\}$, respectively:

$$p = \sum_i n_i p_i, \quad w = \sum_i m_i w_i, \quad m_i, n_i \in \mathbb{Z}. \quad (2.8)$$

The zero mode contribution to the loop channel amplitudes is

$$L \equiv \sum_{n_i} \exp \left( -\frac{\pi}{\delta} t n_i M_{ij} n_j \right) \cdot \sum_{m_i} \exp \left( -\frac{\pi}{\delta} t m_i W_{ij} m_j \right), \quad (2.9)$$

where $n_i, m_i \in \mathbb{Z}$, $M_{ij} = p_i \cdot p_j$, $W_{ij} = w_i \cdot w_j$ and $\delta = 1$ for Klein bottle, $\delta = 2$ for annulus and Möbius strip. Using the generalized Poisson resummation formula, we can rewrite

$$\sum_{n_i} \exp \left( -\frac{\pi}{\delta} t n_i A_{ij} n_j \right) = \frac{1}{t^{\dim(A)/2} (\det A)^{\frac{1}{2}}} \sum_{n_i} \exp \left( -\frac{\pi}{\delta t} n_i A^{-1}_{ij} n_j \right). \quad (2.10)$$

When we move to the tree channel by using (2.6), the zero mode contribution $L$ is

$$L = \sum_{n_i} \left( \frac{\alpha l}{\bar{r}} \right)^3 \sqrt{\det M \det W} \exp \left( -\frac{\pi}{\delta} t n_i A_{ij}^{-1} n_j \right) \cdot \sum_{m_i} \exp \left( -\frac{\pi}{\delta} \frac{\alpha l}{t} m_i W_{ij}^{-1} m_j \right), \quad (2.11)$$

which goes to $\left( \frac{\alpha l}{\bar{r}} \right)^3 \sqrt{\det M \det W}$ in the IR limit $l \to \infty$.

We consider a six-torus $T^6$ on a lattice $\Lambda$. Then different two points in $T^6$ are identified in terms of the lattice shift vector $r\alpha_i \in \Lambda$ as

$$T_{\alpha_i} : \mathbf{x} \to \mathbf{x} + r\alpha_i, \quad (2.12)$$

where $r$ is a radius of $T^6$. For simplicity we set $r = 1$ in the following in this paper. Translation operator acting on the momentum states $|p\rangle$ is given by

$$T_{\alpha_i} |p\rangle = \exp(2\pi i p \cdot \alpha_i) |p\rangle. \quad (2.13)$$

Then the momentum modes are expressed by dual vector $\alpha^*_i \in \Lambda^*$,

$$\alpha_i \cdot \alpha^*_j = \delta_{ij}. \quad (2.14)$$
In the Klein bottle amplitude, the momentum modes should be invariant under the action of $\Omega R$. Thus the vector $\alpha_i^*$ consists of the $R$ invariant sublattice in the dual lattice $\Lambda^*$, and we have \[ \sqrt{\det M_K} = \text{Vol}(\Lambda_{R,\text{inv}}^*). \] (2.15)

In the same way, the winding modes $w_i$ are given by the lattice vector $\alpha_i$ invariant under the action $-R$ on the lattice space $\Lambda$ (with the constant $\alpha' = 1$). Then we obtain
\[
\sqrt{\det W_K} = \text{Vol}(\Lambda_{-R,\text{inv}}).
\] (2.16)

One of the simplest way to cancel the RR-tadpole of the O6-plane is to add D6-branes parallel to the O6-planes. Since the O6-planes lie on the $R$ fixed locus, the basis vectors which describe three-cycles of the O6-plane are generated from $R$-invariant sublattice $\Lambda_{R,\text{inv}}$. Then, in the case of the annulus amplitude, the momentum modes are described by the vector in the dual lattice $(\Lambda_{R,\text{inv}})^*$. The winding modes are related to the distances between these D6-branes, and they are the sublattice projected by $-R$, i.e., $\Lambda_{-R,\perp} \equiv \frac{1}{2} R \Lambda$. In the Möbius strip amplitude the momentum modes are same as the ones of the annulus amplitude. On the other hand, the winding modes should be in the invariant sublattice under $-\Omega R$, and it is given by $\Lambda_{-R,\text{inv}}$.

Summarizing the above, we obtain the following descriptions:
\[
\sqrt{\det M_K} = \text{Vol}(\Lambda_{R,\text{inv}}^*), \quad \sqrt{\det M_A} = \sqrt{\det M_M} = \text{Vol}(\Lambda_{R,\perp}^*), \quad \sqrt{\det W_K} = \sqrt{\det W_M} = \text{Vol}(\Lambda_{-R,\text{inv}}), \quad \sqrt{\det W_A} = \text{Vol}(\Lambda_{-R,\perp}),
\] (2.17a-c)

where we used the following relations:
\[
\Lambda_{R,\perp}^* = (\Lambda_{R,\text{inv}})^*, \quad \text{Vol}(\Lambda) = \text{Vol}(\Lambda_{R,\text{inv}}) \cdot \text{Vol}(\Lambda_{-R,\perp}).
\] (2.18)

For the contributions to Chan-Paton factors, we have $\gamma_1 = \mathbb{1}$ so that $\text{tr}(\gamma_1) = N$ is the number of D6-branes. Furthermore we require $\gamma_{\Omega R}^{-1} \gamma_{\Omega R}^T = \mathbb{1}$ in order to cancel the RR-tadpole.

Now we are ready to obtain the RR-tadpole cancellation condition. The sum of RR-tadpole contributions for large $l$ is asymptotically
\[
\tilde{K}_{RR} + \tilde{A}_{RR} + \tilde{M}_{RR}
= c \int d\ell \left( \frac{64}{\sqrt{\det M_K \det W_K}} + \frac{N^2}{16 \sqrt{\det M_A \det W_A}} - \frac{4N}{\sqrt{\det M_M \det W_M}} \right)
= c \int d\ell \frac{1}{16 \text{Vol}(\Lambda_{R,\perp}^*) \text{Vol}(\Lambda_{-R,\perp}) (N - 4N_{O6})^2},
\] (2.19-2.20)

\footnote{See appendix \footnote{[a]} for the definition of $\Lambda_{R,\text{inv}}$ and $\Lambda_{R,\perp}$.}
where \( N_{O6} \) is the number of the O6-planes according to the Lefschetz fixed point theorem:

\[
N_{O6} \equiv \frac{\text{Vol}((1-R)\Lambda)}{\text{Vol}(\Lambda_{-R,\perp}, \Lambda_{-R,\text{inv}})} = 2^3 \cdot \frac{\text{Vol}(\Lambda_{-R,\perp})}{\text{Vol}(\Lambda_{-R,\text{inv}})}. \tag{2.21}
\]

The equation (2.20) indicates that the RR-tadpole is cancelled by D6-branes whose number is four times as many as that of O6-planes. Therefore we find that it is enough to count the number of O6-planes in (2.21) instead of calculating individual amplitudes. For factorizable models, we have \( \text{Vol}(\Lambda_{-R,\perp})/\text{Vol}(\Lambda_{-R,\text{inv}}) = 1 \). The condition (2.20) is also expressed as

\[
N\Pi - 4\Pi_{O6} = 0, \tag{2.22}
\]

where \( \Pi \) and \( \Pi_{O6} \) denote three-cycles in D6-branes and O6-planes, respectively.

This is the case for O6-planes in Type IIA theory. We can generalize this tadpole cancellation condition to an \( O_q \)-plane in type IIA/IIB theory in such a way as

\[
(N - 2^{q-4}N_{Oq})^2 = 0, \tag{2.23}
\]

where the number of \( O_q \)-planes is given by

\[
N_{Oq} \equiv \frac{\text{Vol}((1-R)\Lambda)}{\text{Vol}(\Lambda_{-R,\perp}, \Lambda_{-R,\text{inv}})} = 2^{9-q} \cdot \frac{\text{Vol}(\Lambda_{-R,\perp})}{\text{Vol}(\Lambda_{-R,\text{inv}})}. \tag{2.24}
\]

In the case of an \( O_9 \)-plane, the orientifold action is given by \( \Omega \), i.e., \( R = \mathbb{1} \), and the above equation is ill-defined, however we can calculate it in a same way. Then it is appropriate to set \( \text{Vol}(\Lambda_{-R,\perp})/\text{Vol}(\Lambda_{-R,\text{inv}}) = 1 \) for \( O_9 \)-plane.

## 2.2 Orientifold models on the Lie root lattices

Here let us first review the Type IIA orientifold on a factorizable torus \( T^2 \times T^2 \times T^2 \) to fix our notation. There are two ways to implement \( \Omega R \) of (2.2) in each \( T^2 \). The lattice \( \Lambda_i \) which defines the boundary condition of \( i \)-th \( T^2 \) is given by

\[
\Lambda_i = \left\{ n_{2i-1} \alpha_{2i-1} + n_{2i} \alpha_{2i} \bigg| n_{2i-1}, n_{2i} \in \mathbb{Z} \right\}, \quad i = 1, 2, 3, \tag{2.25}
\]

where, for simplicity, we set \( r = 1 \) in (2.12); \( \alpha_j \) is a simple root of the lattice. Without loss of generality we can define \( \alpha_{2i} \) along the \( x^{2i} \)-direction for the orientifold action \( \Omega R \) in (2.2), which acts crystallographically on the lattice \( \Lambda_i \). Therefore the complex structure \( U_i \) on the \( i \)-th torus \( T^2 \) should satisfy \( RU_i = U_i \) modulo the shift given by \( \Lambda_i \). Then there are only two solutions

\[
U_i = ia \quad \text{or} \quad \frac{1}{2} + ia, \quad a \in \mathbb{R}, \tag{2.26}
\]
which indicates that there are two distinct lattices for the $\mathcal{R}$ action. The one is called $A$-type lattice [38], whose lattice vector is given by

$$\alpha_1^A = \sqrt{2}e_1, \quad \alpha_2^A = \sqrt{2}e_2.$$  \hspace{1cm} (2.27)

Notice that in this case the complex structure of the torus is given by $U = ia$. The other is called $B$-type lattice, which is given by

$$\alpha_1^B = e_1 - e_2, \quad \alpha_2^B = e_1 + e_2.$$  \hspace{1cm} (2.28)

This corresponds to the case $U = \frac{1}{2} + ia$. We can see it by the re-definition of the vector $\alpha_2^B \rightarrow -\alpha_1^B + \alpha_2^B$. Then we have two distinct theories which depend on the choice of $A$-type or $B$-type lattices in Figure 1. For example, the number of fixed loci given by the action of $\mathcal{R}$ is two (for the $A$-type) and one (for the $B$-type), which associate the total O6-plane charges.

Instead of using the $B$-type lattice, we define an equivalent orientifold by an alternative definition for $\mathcal{R}$ on the lattice (2.27),

$$\mathcal{R} : z_j \rightarrow i\bar{z}_j.$$  \hspace{1cm} (2.29)

In order to distinguish the actions on non-factorizable tori from the ones on factorizable torus, let us attach a label to the action (2.29) as $D$, and to the one (2.2) in the previous subsection as $C$ [21]. For example we call the models by following $\mathcal{R}$ action $CCD$ model,

$$\mathcal{R} : z_1 \rightarrow \bar{z}_1, \quad z_2 \rightarrow \bar{z}_2, \quad z_3 \rightarrow i\bar{z}_3.$$  \hspace{1cm} (2.30)

In appendix A, we can see that these actions provide convenient tools for the classifications of orientifold orbifolds on the Lie root lattices.

First let us consider the RR-tadpole cancellation conditions in the factorizable models. Instead of the direct calculations of the zero mode contribution on each $T^2$ and of the oscillator  

\footnote{By T-dualizing this torus this corresponds to $B$-field which is frozen NS-NS closed moduli [15,33].}
modes in the Klein bottle, the annulus and the Möbius strip amplitudes, it is enough to count
the number of O6-planes from [27,20]: The numbers of O6-planes are $N_{O6} = 8$ (for AAA), 4
(for AAB), 2 (for ABB) and 1 (for BBB). The types of the actions in the $T^2 \times T^2 \times T^2$ are
illustrated in Figure 2. Here we obtain the RR-tadpole cancellation conditions:\footnote{Because
these are the models on factorizable tori, and the C- and D-actions lead to the A- and B-models,
respectively.
\footnote{In this work a compactified space which cannot be represented as the direct products of two-torus $T^2$ is
called non-factorizable. For example, six-tori on $D_6$, $A_3 \times A_3$ and $A_3 \times A_2 \times A_1$, while six-tori on $A_2 \times A_2 \times A_2$, $A_2 \times D_2 \times (A_1)^2$ and $(A_1)^6$ are factorizable.}}

\[\begin{array}{ccc}
\text{AAA} & \text{AAB} & \text{BBB} \\
X_5 \rightarrow & \otimes & \rightarrow X_8 \\
X_6 \rightarrow & \otimes & \rightarrow X_8 \\
X_7 \rightarrow \otimes & \rightarrow X_8 \\
X_9 \rightarrow \otimes & \rightarrow X_8 \\
\end{array}\]

\begin{center}
Figure 2: AAA and AAB models on a factorizable torus. The orientifold planes lie on the
dashed blue lines. In this Figure we used a label B as the D-action on the A-lattice.
\end{center}

\begin{align}
\text{AAA} : & \quad (N - 32)^2 = 0, \\
\text{AAB} : & \quad (N - 16)^2 = 0, \\
\text{ABB} : & \quad (N - 8)^2 = 0, \\
\text{BBB} : & \quad (N - 4)^2 = 0. \\
\end{align}

These are trivial results which have already been known. We emphasize that for the classi-
fication of orientifold models on non-factorizable tori and orbifolds it is convenient to fix the
lattices and distinguish the models with respect to the definitions of $\mathcal{R}$.

Next we analyze some typical models on a non-factorizable\footnote{In this work a compactified space which cannot be represented as the direct products of two-torus $T^2$ is
called non-factorizable. For example, six-tori on $D_6$, $A_3 \times A_3$ and $A_3 \times A_2 \times A_1$, while six-tori on $A_2 \times A_2 \times A_2$, $A_2 \times D_2 \times (A_1)^2$ and $(A_1)^6$ are factorizable.} tori $T^6$, which cannot be
expressed as the direct product $T^2 \times T^2 \times T^2$. As an example we consider an orientifold model
on a non-factorizable torus given by the Lie root lattice $D_6$. In this model the lattice $D_6$ can
be given by the simple roots

\[\begin{align}
\alpha_i &= e_i - e_{i+1}, \\
\alpha_6 &= e_5 + e_6, \\
i &= 1, \ldots, 5. \\
\end{align}\]
where \( \mathbf{e}_i \)'s are basis of Cartesian coordinates whose normalization is given as \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \). The orientifold action \( R \) of the CCC-model is

\[
R : \mathbf{e}_{2i-1} \rightarrow \mathbf{e}_{2i-1}, \quad \mathbf{e}_{2i} \rightarrow -\mathbf{e}_{2i}, \quad i = 1, 2, 3.
\]

(2.33)

The number of O6-planes is obtained by means of (2.21). In order to evaluate the Lefschetz fixed point theorem, we should fix the sublattice spaces \( \Lambda_{-R,\perp} \) and \( \Lambda_{-R,\text{inv}} \). \( \Lambda_{-R,\perp} \) is a lattice space projected out by \( -R \), and given by

\[
\Lambda_{-R,\perp} = \left\{ \sum_{i=1}^{3} n_{\perp,i,\alpha_{\perp,i}} \bigg| n_{\perp,i} \in \mathbb{Z} \right\},
\]

(2.34)

whose basis vectors are given by

\[
\alpha_{\perp,1} = \mathbf{e}_2, \quad \alpha_{\perp,2} = \mathbf{e}_4, \quad \alpha_{\perp,3} = \mathbf{e}_6.
\]

(2.35)

On the other hand, the sublattice \( \Lambda_{-R,\text{inv}} \), which is invariant under \( -R \), is given by

\[
\Lambda_{-R,\text{inv}} = \left\{ \sum_{i=1}^{3} n_{\text{inv},i,\alpha_{\text{inv},i}} \bigg| n_{\text{inv},i} \in \mathbb{Z} \right\},
\]

(2.36)

\[
\alpha_{\text{inv},1} = \mathbf{e}_2 - \mathbf{e}_4, \quad \alpha_{\text{inv},2} = \mathbf{e}_4 - \mathbf{e}_6, \quad \alpha_{\text{inv},3} = \mathbf{e}_5 + \mathbf{e}_6.
\]

Then we can easily evaluate the number of the O6-planes for the CCC model as

\[
N_{06} = 2^3 \cdot \frac{\text{Vol}(\Lambda_{-R,\perp})}{\text{Vol}(\Lambda_{-R,\text{inv}})} = 4.
\]

(2.37)

In the same way, we consider the CCD model. The lattices \( \Lambda_{-R,\perp} \) is given by

\[
\Lambda_{-R,\perp} = \left\{ \sum_{i=1}^{3} n_{\perp,i,\alpha_{\perp,i}} \bigg| n_{\perp,i} \in \mathbb{Z} \right\},
\]

(2.38)

\[
\alpha_{\perp,1} = \mathbf{e}_2, \quad \alpha_{\perp,2} = \mathbf{e}_4, \quad \alpha_{\perp,3} = \frac{1}{2}(\mathbf{e}_5 - \mathbf{e}_6),
\]

and \( \Lambda_{-R,\text{inv}} \) is given by

\[
\Lambda_{-R,\text{inv}} = \left\{ \sum_{i=1}^{3} n_{\text{inv},i,\alpha_{\text{inv},i}} \bigg| n_{\text{inv},i} \in \mathbb{Z} \right\},
\]

(2.39)

\[
\alpha_{\text{inv},1} = \mathbf{e}_2 - \mathbf{e}_4, \quad \alpha_{\text{inv},2} = \mathbf{e}_2 + \mathbf{e}_4, \quad \alpha_{\text{inv},3} = \mathbf{e}_5 - \mathbf{e}_6.
\]

Then we obtain \( N_{06} = 2 \). Substituting these numbers into the RR-tadpole cancellation condition (2.20), we easily obtain the number of D-branes. Here we summarize the data of the orientifolds on the non-factorizable \( D_6 \) lattice:

\[
\begin{align*}
\text{CCC} & : (N - 16)^2 = 0, \\
\text{CCD} & : (N - 8)^2 = 0, \\
\text{CDD} & : (N - 4)^2 = 0, \\
\text{DDD} & : (N - 8)^2 = 0.
\end{align*}
\]

(2.40)
These results completely agree with the ones in [21]. The gauge group of these models are $SO(16)$, $SO(8)$, $SO(4)$ and $SO(8)$, respectively. For models on non-factorizable tori, the closed string spectra are the same as that of factorizable models.

We evaluated the the number of O6-planes $N_{O6}$ according to the Lefschetz fixed point theorem, and from (2.21) this give the necessary and sufficient condition for the RR-tadpole condition. This analysis is generic and provides quite a simple rule to calculate the number of O-planes and D-branes in orientifold models on non-factorizable tori in Type II string theory.

3 Supersymmetric $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold models

In this section let us consider Type IIA supersymmetric orientifold models on orbifolds and describe the way to deal with orientifolds on non-factorizable lattices. Since the contributions of the RR-tadpole from untwisted states are calculated in the same way as the ones of the orientifolds on tori, we can easily count the numbers of D-branes via the Lefschetz fixed point theorem (2.21). We also provide detail calculations of the RR-tadpole cancellation condition on $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

3.1 Orbifolds and orientifolds

In the previous section we showed general expressions for orientifolds on non-factorizable tori (2.1). Here let us consider orientifold models on orbifolds given by

$$\frac{\text{Type IIA on } T^6}{\Omega \mathbb{R} \times \mathbb{Z}_N \times \mathbb{Z}_M}.$$ (3.1)

An orbifold is defined as a quotient of torus over a discrete set of isometries of the torus [39], called the point group $P$, i.e.,

$$\mathcal{O} = T^6 / P = \mathbb{R}^6 / S.$$ (3.2)

Here $S$ is called the space group, and is the semi-direct product of the point group $P$ and the translation group $T$. $\mathbb{Z}_N$ orbifolds on the Lie root lattices have been classified in terms of the Coxeter elements or the generalized Coxeter elements. In the case of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, the (generalized) Coxeter elements yield only orbifolds on factorizable lattices. Recently, however, $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on non-factorizable lattices were investigated in heterotic strings [27]. We apply their analyses to Type IIA orientifold models.

Since the point group $P$ of orbifold must act crystallographically on the lattice, we choose these elements from the group generated by the Weyl reflection (A.4) and the outer automorphisms $G_{\text{out}}$. In the case of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold on a Lie root lattice, the point group
elements of the orbifold can be defined by two commutative elements in the group generated from Weyl group and the outer automorphisms, i.e,

\[ [\theta, \phi] = 0, \quad \theta, \phi \in \{W, G_{\text{out}}\}. \] (3.3)

On the complex coordinates of the torus \( T^6 \), the point group elements of the orbifold act in such a way as

\[
\begin{align*}
\theta : (z_1, z_2, z_3) &\Rightarrow (e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3), \\
\phi : (z_1, z_2, z_3) &\Rightarrow (e^{2\pi i w_1} z_1, e^{2\pi i w_2} z_2, e^{2\pi i w_3} z_3),
\end{align*}
\] (3.4)

where \((v_1, v_2, v_3)\) and \((w_1, w_2, w_3)\) are twists of an orbifold. We consider orientifold models with \( N = 1 \) supersymmetry as follows: The requirement of \( SU(3) \) holonomy can be phrased as invariance of the \((3,0)\)-form \( \Omega = dz_1 \wedge dz_2 \wedge dz_3 \), and leads to

\[ v_1 + v_2 + v_3 = w_1 + w_2 + w_3 = 0. \] (3.5)

The twists of the \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifolds which are compatible with \( N = 1 \) supersymmetric orientifolds are listed in Table 1.

| \( (v_1, v_2, v_3) \) | \( (w_1, w_2, w_3) \) | \( (v_1, v_2, v_3) \) | \( (w_1, w_2, w_3) \) |
|------------------|------------------|------------------|------------------|
| \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | \( (\frac{1}{2}, -\frac{1}{2}, 0) \) | \( (0, \frac{1}{2}, -\frac{1}{2}) \) | \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) | \( (\frac{1}{2}, -\frac{1}{2}, 0) \) | \( (0, \frac{1}{4}, -\frac{1}{4}) \) |
| \( \mathbb{Z}_2 \times \mathbb{Z}_6 \) | \( (\frac{1}{2}, -\frac{1}{6}, 0) \) | \( (0, \frac{1}{6}, -\frac{1}{6}) \) | \( \mathbb{Z}_2 \times \mathbb{Z}_6' \) | \( (\frac{1}{2}, -\frac{1}{2}, 0) \) | \( (\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}) \) |
| \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) | \( (\frac{1}{3}, -\frac{1}{3}, 0) \) | \( (0, \frac{1}{3}, -\frac{1}{3}) \) | \( \mathbb{Z}_3 \times \mathbb{Z}_6 \) | \( (\frac{1}{3}, -\frac{1}{3}, 0) \) | \( (0, \frac{1}{6}, -\frac{1}{6}) \) |
| \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) | \( (\frac{1}{4}, -\frac{1}{4}, 0) \) | \( (0, \frac{1}{4}, -\frac{1}{4}) \) | \( \mathbb{Z}_6 \times \mathbb{Z}_6 \) | \( (\frac{1}{6}, -\frac{1}{6}, 0) \) | \( (0, \frac{1}{6}, -\frac{1}{6}) \) |

Table 1: Twists of \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifolds.

As explained in Appendix A there are twelve distinct classes of non-factorizable lattices, see Table 8. The \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) orbifolds are allowed on these non-factorizable lattices (see Table 10 in appendix A). The series of generators \( \theta \) and \( \phi \) of the \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifold as well as the action \( \Omega R \) consist of the orientifold group:

\[
\left\{ \theta^{k_1} \phi^{k_2} \mid \Omega R \theta^{k_1} \phi^{k_2} \mid k_1 = 0, \ldots, N; \ k_2 = 0, \ldots, M \right\},
\] (3.6)

These elements appear in the following string one-loop amplitudes as insertions [23],

\[
\mathcal{K} = 4c \int_0^\infty \frac{dt}{t^3} \text{Tr}_{\text{closed}} \left\{ \frac{\Omega R}{2} P \left( \frac{1 + (-1)^F}{2} \right) (-1)^S e^{-2\pi t (L_0 + \bar{L}_0)} \right\},
\] (3.7a)
\[ A = c \int_{0}^{\infty} \frac{dt}{t^3} \text{Tr}_{\text{open}} \left( \frac{1}{2} P \left( \frac{1 + (-1)^F}{2} \right) (-1)^{s} e^{-2\pi t L_0} \right), \quad (3.7b) \]

\[ M = c \int_{0}^{\infty} \frac{dt}{t^3} \text{Tr}_{\text{open}} \left( \frac{\Omega R \omega^{k_1} \phi^{k_2}}{2} P \left( \frac{1 + (-1)^F}{2} \right) (-1)^{s} e^{-2\pi t L_0} \right). \quad (3.7c) \]

Here
\[ P = \left( \frac{1 + \theta + \cdots + \theta^{N-1}}{N} \right) \left( \frac{1 + \phi + \cdots + \phi^{M-1}}{M} \right). \quad (3.8) \]

After extracting the RR-tadpoles, the insertion of \( \Omega R \omega^{k_1} \phi^{k_2} \) in the Klein bottle amplitude corresponds to the contribution from O-planes fixed by \( R \omega^{k_1} \phi^{k_2} \). Since in the \( \Omega R \omega^{k_1} \phi^{k_2} \) insertion the contributions from untwisted sectors are calculated in the same way as the cases of tori in section 2, we obtain the necessary condition (2.20) for the RR-tadpole cancellation by D-branes parallel to the O-planes. From this necessary condition, we obtain all the numbers of O-planes and D-branes on the orbifold. In the next subsection we will demonstrate a few examples of \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) orientifold models, and evaluate the RR-tadpole cancellation condition.

### 3.2 \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) model

Here we discuss the \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) orientifold model on the Lie root lattice \( D_6 \) \( (2.32) \) in detail because in this case all possible subtleties show up.

There exists only one distinct \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) orbifold on \( D_6 \), whose point group elements \( \theta \) and \( \phi \) are given by

\[
\theta : \begin{cases} 
\textbf{e}_1 \rightarrow \textbf{e}_2 \rightarrow -\textbf{e}_1 \\
\textbf{e}_3 \rightarrow -\textbf{e}_4 \rightarrow -\textbf{e}_3 \\
\textbf{e}_5 \rightarrow \textbf{e}_5 \\
\textbf{e}_6 \rightarrow \textbf{e}_6 
\end{cases} \quad \phi : \begin{cases} 
\textbf{e}_1 \rightarrow \textbf{e}_1 \\
\textbf{e}_2 \rightarrow \textbf{e}_2 \\
\textbf{e}_i \rightarrow -\textbf{e}_i \quad i = 3, 4, 5, 6 
\end{cases} \quad (3.9a)
\]

or, in matrix representation, by

\[
\theta : \begin{pmatrix} 
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}, \quad \phi : \begin{pmatrix} 
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 
\end{pmatrix}. \quad (3.9b)
\]

By using the above elements we can show all the orientifold actions which preserve \( \mathcal{N} = 1 \) supersymmetry by means of \( C \) and \( D \) actions. For example, the reflection \( \mathcal{R} \) on the DDC
model is given by

\[
\mathcal{R} : \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \equiv (b, b, a),
\]

(3.10)

where we used an abbreviation defined by

\[
(m_1, m_2, m_3) \equiv \begin{pmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix}
\]

with \(m_i \in \{\pm a, \pm b, \pm 1\}\)

(3.11)

and

\[
a \equiv \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\quad b \equiv \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\quad 1 \equiv \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\quad 0 \equiv \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

(3.12)

From the Lefschetz fixed point theorem (2.21), the number of O6-plane fixed by \(\mathcal{R}\) is given as \(N_{06} = 1\). If we put four D-branes parallel to this \(\mathcal{R}\)-fixed O6-plane, the RR-tadpole of this model will be cancelled. Similarly, the element \(\mathcal{R}\theta = (a, -a, a)\) gives \(N_{06} = 4\), whose tadpole is cancelled by sixteen D-branes parallel to this four \(\mathcal{R}\theta\)-fixed O6-planes. We similarly evaluate the cases for the other elements of the orientifold group. The relations between the orientifold group elements and the numbers of O-planes are summarized in Table 2.

| Orientifold elements of \(\mathcal{R}\) | \# of O6-planes |
|----------------------------------------|-----------------|
| (\(\pm a, \pm a, \pm a\), (1, -1, \(\pm a\)) | 4               |
| (\(\pm a, \pm a, \pm b\), (1, -1, \(\pm b\)) | 2               |
| (\(\pm b, \pm b, \pm b\))               | 1               |

Table 2: Orientifold group elements and the numbers of O6-planes on the \(D_6\) lattice. The underline indicates a symmetry under the cyclic permutation.

Since the \(\mathbb{Z}_4\) action changes the directions of the O-planes by angle of \(\theta^{1/2}\) in the following way:

\[
\mathcal{R}\theta = \theta^{-1/2}\mathcal{R}\theta^{1/2}.
\]

(3.13)
This action generates the exchange between the action C and D each other. Then we can see that CCC and DDC, CCD and DDD, CDD and DCD models are equivalent with each other, respectively. In the case of the CCC model, for example, two different numbers of O6-planes appear since the orientifold group elements in $R$, $R\theta^2$, $R\phi$ and $R\theta^2\phi$ are given by $(\pm a, \pm a, \pm a)$, whereas the elements in $R\theta$, $R\theta^3$, $R\theta\phi$ and $R\theta^3\phi$ are given by $(\pm b, \pm b, \pm a)$. Analyzing such actions, we obtain all the models for $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifolds on $D_6$ lattice, listed in Table 3.

| Lattice | Label | reps. of $\mathcal{R}$ | $\mathcal{R}$, $\mathcal{R}\theta^2$, $\mathcal{R}\phi$, $\mathcal{R}\theta^2\phi$ | $\mathcal{R}\theta$, $\mathcal{R}\theta^3$, $\mathcal{R}\theta\phi$, $\mathcal{R}\theta^3\phi$ |
|---------|-------|------------------------|---------------------------------|---------------------------------|
| $D_6$   | CCC   | $(a, a, a)$             | 4                               | 1                               |
|         | CCD   | $(a, a, b)$             | 2                               | 2                               |
|         | CDD   | $(a, b, b)$             | 2                               | 2                               |
|         | DCC   | $(b, a, a)$             | 2                               | 2                               |

Table 3: All the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold models on the $D_6$ lattice.

We estimated the RR-tadpole cancellation by counting the O-planes from the equation \eqref{2.20}, which is the necessary condition in the case of the orbifold model. However it is expected that the RR-tadpoles are cancelled even in the orbifold model. These countings also give correct results for well-investigated non-factorizable models on $\mathbb{Z}_N$ orbifolds in \cite{22} and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in \cite{21}. We give the explicit results of the RR-tadpole cancellation for a few models in the following.

### 3.2.1 Klein bottle amplitude

First let us evaluate the Klein bottle amplitude of $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold model on the $D_6$ lattice \eqref{2.32} with the orientifold action

$$\mathcal{R} = (b, a, a), \quad (3.14)$$

which gives the DCC model. The contribution of the oscillator modes are equal in any insertions of the orientifold group because they act as the unit operator in \eqref{3.7a}. In the $\theta^{n_1}\phi^{n_2}$-twisted sector, the oscillator contribution is given by $K_{(n_1,n_2)}^{(m_1,m_2)} \equiv K_{(n_1,k_1)}^{(m_1,k_1)}$ (see, for the notation, \cite{23}). We also need the multiplicities $\chi_{K_{(n_1,k_1)}}^{(n_1,k_1)}$ of the $\theta^{n_1}\phi^{n_2}$-twisted fixed sectors, which are invariant under the insertion $\Omega \mathcal{R}\theta^{n_1}\phi^{n_2}$, which can be seen in Table 4.
Table 4: Multiplicities of the fixed points for the DCC and CCC models.

When an action $\theta^{n_1} \phi^{n_2}$ does not fix certain directions in the compact space, the Kaluza-Klein momentum modes and the winding modes appear as the zero modes in the $\theta^{n_1} \phi^{n_2}$-fixed sector. Let us evaluate such zero modes in the $\theta$-twisted sector. The $\theta$ invariant sublattice $\Lambda^\theta$ is expanded in terms of the basis

$$\{e_5 + e_6, \ e_5 - e_6\}.$$  \hspace{1cm} (3.15)

We can see that the $\cal{R}$ invariant dual sublattice $(\Lambda^\theta)^*_{\cal{R}, \text{inv}}$, whose basis is given by $\{2e_5\}$, and the $-\cal{R}$ invariant sublattice $(\Lambda^\theta)_{-\cal{R}, \text{inv}}$, with its basis $\{2e_6\}$, yield the momentum modes and the winding modes in this sector, respectively. However there are two subtleties in this evaluation, one of which is caused by the momentum doubling, and the other from the appearance of the half winding states [22].

The former subtlety is caused by the shifts associated to the $\Omega \cal{R} \theta^{k_1} \phi^{k_2}$ insertions. In the $\theta$-twisted sector we have two fixed tori given by

$$xe_5, \ \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + xe_5,$$

where $x \in \mathbb{R}$ is a coordinate on the fixed tori. Note that the invariance of fixed points or fixed tori under $\Omega \cal{R} \theta^{k_1} \phi^{k_2}$ is defined modulo the translation generated by the lattice $\Lambda$. The $\cal{R}$ insertion acts on the two fixed tori in such a way as

$$\cal{R} : \begin{cases} xe_5 & \rightarrow x e_5, \\ \frac{1}{2}(e_1 + e_2 + e_3 + e_4) + xe_5 & \rightarrow \frac{1}{2}(e_1 + e_2 + e_3 - e_4) + xe_5 \\ & = \frac{1}{2}(e_1 + e_2 + e_4) + (-1 + x)e_5. \end{cases} \hspace{1cm} (3.17)$$
In the latter case, the translation of a lattice shift \( \alpha_4 = e_1 - e_5 \) is accompanied. Because a momentum mode \(| p \rangle\) picks up a phase factor \( e^{2 \pi p \cdot l} \) under the translation by \( l \), we generally need phase factors in the amplitudes. In the case of (3.17), the phase factor is \( p \cdot l = 2e_5 \cdot e_5 = 2 \), and does not affect the amplitudes. If the phase factor is given as \(-1\), the momentum modes are effectively doubled by interference between modes with and without shifts:

\[
\sum_n (-1)^n \exp(-\pi n^2 p^2) + \sum_n \exp(-\pi n^2 p^2) = 2 \sum_n \exp(-4\pi n^2 p^2).
\]  (3.18)

The latter subtlety occurs in the winding modes. There are special points with the following property:

\[
\theta : \frac{1}{2}(e_1 + e_2) \to \frac{1}{2}(-e_1 + e_2) = \frac{1}{2}(e_1 + e_2) + e_6,
\]  (3.19)

where we used a lattice shift given by \( e_1 + e_6 \). The point does not lie on the \( \theta \)-fixed tori, whereas this shift does generate the winding modes:

\[
X(\sigma, \tau) = \frac{1}{2}(e_1 + e_2) + \frac{\sigma}{2\pi} e_6 + (\tau \text{ dependence}),
\]  (3.20)

There are two points \( \frac{1}{2}(e_1 \pm e_2) \) which are invariant under the action \( R \), and the multiplicity is equal to that of the \( \theta \)-fixed tori which are also invariant under \( R \).

Therefore we conclude that the zero modes in the \( \theta \)-twisted sector with \( R \) insertion are given by the following vectors:

\[
p = 2ne_5, \quad w = me_6,
\]  (3.21)

where \( n, m \in \mathbb{Z} \). In the notation of (C.4), the zero mode contributions in the Klein bottle amplitude is \( L_{2,\frac{1}{2}} \). In a similar way we can evaluate the other twisted sectors in the orbifold model. Note that for non-factorizable orbifolds the zero mode contributions depend on the insertion \( \Omega R \theta^{k_1} \phi^{k_2} \). In the \( \phi \)-twisted sector we have \( L_{2,\frac{1}{2}} \) for the \( \Omega R \theta^{2k_1} \phi^{k_2} \) insertions, and \( L_{4,1} \) for the \( \Omega R \theta^{2k_1+1} \phi^{k_2} \) insertions.

Next we evaluate the zero mode contribution from the untwisted sector given in (2.17a). The basis of dual lattice \( \alpha^* \in \Lambda^* \), which is defined by \( \alpha_i^* \cdot \alpha_j = \delta_{ij} \), is given as

\[
\begin{align*}
\alpha_1^* &= e_1, \\
\alpha_2^* &= e_1 + e_2, \\
\alpha_3^* &= e_1 + e_2 + e_3, \\
\alpha_4^* &= e_1 + e_2 + e_3 + e_4, \\
\alpha_5^* &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6), \\
\alpha_6^* &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6).
\end{align*}
\]  (3.22)
Then the $\mathcal{R}$ invariant dual sublattice $\Lambda_{\mathcal{R},\text{inv}}^*$ in (2.11a), which yields the momentum modes in the Kaluza-Klein states, is expanded by the basis

$$\{e_1 + e_2, e_3, e_5\}.$$  \hfill (3.23)

In the same way, the $-\mathcal{R}$ invariant lattice $\Lambda_{-\mathcal{R},\text{inv}}$ in (2.17c) yielding the winding states is expanded by

$$\{e_1 - e_2, 2e_4, 2e_6\}.$$  \hfill (3.24)

Substituting these elements into (2.9), we obtain the zero mode contribution $L_{\mathcal{K}}^{(0,0)}$. Its modular transformation is given by the factors

$$\sqrt{\det M^\mathcal{K}} = \text{Vol}(\Lambda_{\mathcal{R},\text{inv}}^*) = \sqrt{2},$$

$$\sqrt{\det W^\mathcal{K}} = \text{Vol}(\Lambda_{-\mathcal{R},\text{inv}}) = 4\sqrt{2}.$$  \hfill (3.25)

We also need the zero mode contributions with the other insertions $\Omega R\theta^k_1 \phi^k_2$. Since these elements are given by $R\theta^k_1 \phi^k_2 = (\pm a, \pm b, \pm b)$ for the DCC model, we have the same results as that of the $\mathcal{R}$ insertion.

We obtained all the ingredients to write down the Klein bottle amplitude for the DCC model, which are summarized as

$$\mathcal{K} = c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \left( L_{\mathcal{K}}^{(0,0)} K^{(0,0)} + 2 L_{2,1}^\mathcal{K} K^{(1,0)} + 4 L_{2,1}^\mathcal{K} K^{(2,0)} + 2 L_{2,1}^\mathcal{K} K^{(3,0)} \right. $$

$$\left. + \frac{1}{2} (8 L_{4,1} + 4 L_{1,2}) K^{(0,1)} + 8 K^{(1,1)} \right)$$

$$\left. + \frac{1}{2} (8 L_{4,1} + 4 L_{1,2}) K^{(2,1)} + 8 K^{(3,1)} \right).$$  \hfill (3.26)

Its modular transformation to the tree channel is

$$\tilde{\mathcal{K}} = 16c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \left( \tilde{L}_{\mathcal{K}}^{(0,0)} \tilde{K}^{(0,0)} - 2 \tilde{L}_{2,8} \tilde{K}^{(1,0)} - 4 \tilde{L}_{2,8} \tilde{K}^{(2,0)} - 2 \tilde{L}_{2,8} \tilde{K}^{(3,0)} \right. $$

$$\left. - 2(\tilde{L}_{1,4} + \tilde{L}_{2,8}) \tilde{K}^{(0,1)} + 4 \tilde{K}^{(1,1)} \right)$$

$$\left. - 2(\tilde{L}_{1,4} + \tilde{L}_{2,8}) \tilde{K}^{(2,1)} + 4 \tilde{K}^{(3,1)} \right).$$  \hfill (3.27)

Note that in the IR limit $l \to \infty$, the zero mode contributions $\tilde{L}_{\mathcal{K}}^{(0,0)}$ and $\tilde{L}_{\alpha,\beta}$ in the tree channel (3.27) go to unity, then we obtain $2(\tilde{L}_{1,4} + \tilde{L}_{2,8}) \to 4$. Then we observe that the prefactors are given by the complete projector [31]

$$\prod_{i=1, n_1 v_i + n_2 w_i \neq 0}^3 \left( - 2 \sin(\pi n_1 v_i + \pi n_2 w_i) \right).$$  \hfill (3.28)

This relation implies that only the untwisted sector contributes to the RR-tadpole.
### 3.2.2 Annulus amplitude

In order to cancel the RR-tadpole we introduce D-branes parallel to O-planes. We attach a label \((i_1, i_2)\) to a stock of D-branes which is invariant under the orientifold action \(R\theta^{i_1}\phi^{i_2}\), and define that \((0, 0)\) denotes D-branes invariant under the action \(R\). The three-cycle wrapped by the brane \((0, 0)\) is given by the \(R\) invariant lattice \(\Lambda_{R, \text{inv}}\) whose basis is given by

\[
\{e_1 + e_2, e_3 - e_5, e_3 + e_5\}. \tag{3.29}
\]

From (3.13) the brane \((1, 0)\) is rotated by half the angle of \(\theta\) with respect to the brane \((0, 0)\). The three-cycle wrapped by the brane \((1, 0)\) is given by the \(R\theta\) invariant lattice \(\Lambda_{R\theta, \text{inv}}\) whose basis is given by

\[
\{e_1 - e_5, e_3 + e_4, e_1 + e_5\}. \tag{3.30}
\]

An open string stretching from brane \((i_1, i_2)\) to brane \((i_1 - n_1, i_2 - n_2)\) is localized at intersection of D-branes. It is convenient to call such a state the \(\theta^{n_1}\phi^{n_2}\)-twisted sector.

The three-cycles of brane \((0, 0)\) and brane \((1, 0)\) share a common direction, and the lattice vector in this direction is given by \(2e_5\). The momentum modes are obtained from the dual of the vector \(2e_5\) in such a way as

\[
p = \frac{n}{\sqrt{2}}e_5. \tag{3.31}
\]

where \(n \in \mathbb{Z}\). The basis of the winding modes is related to the distances of the parallel D-branes. Because we put D-branes parallel to the O-planes, the shortest distance corresponds to the lattice vector projected by the actions \(-R\) and \(-R\theta\), i.e., \(\Lambda_{R, \perp} \cap \Lambda_{R\theta, \perp}\). Then the winding modes are

\[
w = \frac{n}{\sqrt{2}}e_6. \tag{3.32}
\]

Then zero mode contribution of the \(\theta\)-twisted sector is expressed as \(L_{1, 1}\).

Let us explain one more case of the \(\phi\)-twisted sector. The winding modes are given as “half winding-like” modes, and are also given by the projected lattice \(\Lambda_{R, \perp} \cap \Lambda_{R\phi, \perp}\) whose basis is

\[
\left\{\frac{1}{2\sqrt{2}}(e_1 + e_2)\right\}, \tag{3.33}
\]

Then the zero modes of open string stretching between the brane \((0, 0)\) and the brane \((0, 1)\) are given by

\[
p = \frac{n}{\sqrt{2}}(e_1 + e_2), \quad w = \frac{n}{2\sqrt{2}}(e_1 + e_2). \tag{3.34}
\]

The zero mode contribution in the annulus amplitude is expressed as \(L_{2, \frac{1}{2}}\). The other zero modes are calculated in a similar way.
Since in the $\theta^{n_1}\phi^{n_2}$-twisted sector the contributions from the oscillator modes do not depend on branes $(i_1, i_2)$, they are given as $\mathcal{A}^{(n_1, k_1)(n_2, k_2)}$. The insertions of $1$, $\theta$, $\phi$ and $\theta^2\phi$ leave D-branes invariant, and perform non-trivial actions on the Chan-Paton factors described as $\gamma_{k_1, k_2}^{(i_1, i_2)}$, which appear in the amplitude as

$$\text{tr} \left( \gamma_{k_1, k_2}^{(i_1-n_1, i_2-n_2)} \right) \text{tr} \left( \gamma_{k_1, k_2}^{(i_1, i_2)} \right)^{-1}$$

in the $\theta^{n_1}\phi^{n_2}$-twisted sector. Sectors of $k_1 \neq 0$ or $k_2 \neq 0$ cannot be cancelled by the other diagrams. Therefore the $\mathbb{Z}_2$ twisted tadpole cancellation condition is required [23, 34]:

$$\text{tr} \left( \gamma_{2,0}^{(i_1, i_2)} \right) = \text{tr} \left( \gamma_{0,1}^{(i_1, i_2)} \right) = \text{tr} \left( \gamma_{2,1}^{(i_1, i_2)} \right) = 0. \quad (3.36)$$

We should also evaluate the multiplicities $\chi_M$ of the open string states, which are given by the intersection number of D-branes. The intersection numbers of two branes can be obtained by the determinant of vectors $v_i$ and $v'_i$ giving the three-cycles in respective D-branes [22]. These vectors can be expanded in terms of the lattice basis as $v_i = \sum v_{ij} \alpha_j$. Then the intersection number is

$$I = \det \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{16} \\ v_{21} & v_{22} & \cdots & v_{26} \\ \vdots & \vdots & \ddots & \vdots \\ v'_{31} & v'_{32} & \cdots & v'_{36} \end{pmatrix}. \quad (3.37)$$

Owing to the above $\mathbb{Z}_2$ twisted tadpole condition, it is sufficient to consider the intersection number $\chi_M$ for $k_1 = k_2 = 0$, which are given in Table 5.

The contribution from the zero modes of the untwisted sector is obtained from (2.17b) and (2.17d). For the brane $(0, 0)$, which is parallel to the $\mathcal{R}$-fixed O6-plane, it is

$$\sqrt{\text{det} M^A} = \text{Vol}(\Lambda_{\mathcal{R}, \perp}^*) = 4,$$

$$\sqrt{\text{det} W^A} = \text{Vol}(\Lambda_{-\mathcal{R}, \perp}) = 4.$$
| $\chi_A$ | CCC | CCD | CDD | DCC |
|---------|-----|-----|-----|-----|
| $(i_1, i_2) - (i_1, i_2)$ | 1 | 1 | 1 | 1 |
| $(i_1, i_2) - (i_1 + 1, i_2)$ | 1 | 1 | 2 | 1 |
| $(2i_1 + 1, i_2) - (2i_1 + 3, i_2)$ | 4 | 2 | 4 | 2 |
| $(2i_1, i_2) - (2i_1 + 2, i_2)$ | 1 | 2 | 4 | 2 |
| $(2i_1 + 1, i_2) - (2i_1 + 1, i_2 + 1)$ | 4 | 2 | 4 | 2 |
| $(2i_1, i_2) - (2i_1, i_2 + 1)$ | 1 | 2 | 4 | 2 |
| $(i_1, i_2) - (i_1 + 1, i_2 + 1)$ | 2 | 2 | 4 | 2 |
| $(2i_1 + 1, i_2) - (2i_1 + 3, i_2 + 1)$ | 4 | 2 | 4 | 2 |
| $(2i_1, i_2) - (2i_1 + 2, i_2 + 1)$ | 1 | 2 | 4 | 2 |

Table 5: Intersection numbers in the annulus amplitudes in the DCC and CCC models.

where $\mathcal{A}^{(n_1, n_2)} = \mathcal{A}^{(n_1, 0)(n_2, 0)}$. The modular transformation to the amplitude in the tree channel yields

$$\tilde{A} = \frac{N_c^2}{4} (1_{RR} - 1_{NSNS}) \int_{0}^{\infty} dl \left( \tilde{\mathcal{L}}_{A}^{(0,0)} \tilde{\mathcal{A}}^{(0,0)} - 2 \tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(1,0)} - 4 \tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(2,0)} - 2 \tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(3,0)} - 2(\tilde{\mathcal{L}}_{1,4} + \tilde{\mathcal{L}}_{2,2}) \tilde{\mathcal{A}}^{(0,1)} + 4 \tilde{\mathcal{A}}^{(1,1)} - 2(\tilde{\mathcal{L}}_{2,2} + \tilde{\mathcal{L}}_{1,4}) \tilde{\mathcal{L}}^{(2,1)} - 4 \tilde{\mathcal{A}}^{(3,1)} \right).$$

(3.40)

We again observe the complete projector in the IR limit.

3.2.3 M"obius strip amplitude

The amplitude of the M"obius strip (3.7c) includes the insertion of $\Omega \mathcal{R}$, and string states should be invariant under these orientifold actions. In $\theta^{n_1} \phi^{n_2}$-twisted sector, the insertion $\Omega \mathcal{R} \theta^{k_1} \phi^{k_2}$ acts on open strings stretching from brane $(i_1, i_2)$ to brane $(i_1 - n_1, i_2 - n_2)$ as

$$\Omega \mathcal{R} \theta^{k_1} \phi^{k_2} : [(i_1, i_2)(i_1 - n_1, i_2 - n_2)] \rightarrow [(-i_1 + n_1 - 2k_1, -i_2 + n_2 - 2k_2)(-i_1 - 2k_1, -i_2 - 2k_2)],$$

(3.41)

Therefore in the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold case the following conditions are required:

$$2(i_1 + k_1) - n_1 = 0 \pmod{4}, \quad (3.42a)$$
$$2(i_2 + k_2) - n_2 = 0 \pmod{2}. \quad (3.42b)$$
Then the sectors with \( n_1 = 0, 2 \) and \( n_2 = 0 \) contribute to the amplitude. The intersection number is obtained in the same way as in the case of annulus. In Table 5 we can see that \( \chi_M = 1 \) for untwisted sectors and \( \chi_M = 2 \) for \( \theta^2 \)-twisted sectors.

The momentum modes are evaluated in a similar way of subsection 3.2.1, however the winding modes are changed due to the insertions. In the untwisted sector with the \( \Omega \mathcal{R} \theta \) insertion, from the condition (3.42) the open string states \([(1, 0)(3, 0)], [(1, 1)(3, 1)], [(1, 1)(3, 1)], [(1, 1)(3, 1)], [(3, 0)(1, 0)], [(1, 0)(3, 0)] \) and \([(1, 0)(3, 0)] \) contribute to the amplitude. For instance, in the open string state \([(1, 0)(3, 0)] \), the momentum modes, which are generated by the dual lattice \( \Lambda_{\mathcal{R} \theta, \text{inv}} \cap \Lambda_{\mathcal{R} \theta^* \text{inv}} \), are given as

\[
P = \frac{n}{\sqrt{2}} \mathbf{e}_5. \tag{3.43}
\]

The winding modes invariant under \(- \Omega \mathcal{R} \theta\) are given by

\[
w = \frac{2n}{\sqrt{2}} \mathbf{e}_6. \tag{3.44}
\]

This can also read from \( \Lambda_{- \mathcal{R} \theta, \text{inv}} \cap \Lambda_{- \mathcal{R} \theta^* \text{inv}} \). The zero mode contribution for this state is represented as \( \mathcal{L}_{1,4} \).

We should take into account of the orientifold actions to the Chan-Paton factors. For the open strings \([(i_1, i_2)(i_1 - n_1, i_2 - n_2)] \), the \( \Omega \mathcal{R} \theta^k \phi^{k^2} \)-insertion contributes in the amplitude as

\[
\text{tr} \left[ (\gamma_{\Omega \mathcal{R} k_1 k_2})^{-1} (\gamma_{\Omega \mathcal{R} k_1 k_2})^T \right]. \tag{3.45}
\]

Since only the sectors with \( n_1 = 0, 2 \) and \( n_2 = 0 \) contribute to the amplitude, we abbreviate

\[
a_{k_1 k_2}^{(n_1)} \equiv \text{tr} \left[ (\gamma_{\Omega \mathcal{R} k_1 k_2})^{-1} (\gamma_{\Omega \mathcal{R} k_1 k_2})^T \right], \tag{3.46}
\]

\[
b_{k_1 k_2}^{(n_1)} \equiv \text{tr} \left[ (\gamma_{\Omega \mathcal{R} k_1 k_2})^{-1} (\gamma_{\Omega \mathcal{R} k_1 k_2})^T \right]. \tag{3.47}
\]

These assignments correspond to two different classes of the D-brane configurations in this model, and are sufficient to evaluate the tadpole cancellation conditions for \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) models. However we will need more independent variables for \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) models.

For the contributions from untwisted sector, we can use the results from the (3.26) and (3.39) owing to the relations (2.17a)-(2.17d).

To summarize, we obtain the Möbius strip amplitude in the loop channel as

\[
\mathcal{M} = -\frac{N_c}{4} (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \left( \frac{a_{0,0}^{(0)} + b_{0,0}^{(0)}}{2} \mathcal{L}_M^{(0,0)(0,0)} \mathcal{M}^{(0,0)(0,0)} + \frac{a_{3,0}^{(2)} + b_{3,0}^{(2)}}{2} \mathcal{L}_{1,4} \mathcal{M}^{(2,3)(0,0)} \right)
\]

\[
+ \frac{a_{2,0}^{(0)} + b_{2,0}^{(0)}}{2} \mathcal{L}_{1,4} \mathcal{M}^{(0,2)(0,0)} + \frac{a_{1,0}^{(2)} + b_{1,0}^{(2)}}{2} \mathcal{L}_{1,4} \mathcal{M}^{(2,1)(0,0)}
\]

21
of (3.36) and (3.51), the gauge groups are determined as: $\mathcal{M}^{(0,0)(0,1)} + 2\frac{a_{3,1}^{(2)} + b_{3,1}^{(2)}}{2} \mathcal{M}^{(2,3)(0,1)}$

+ $\frac{a_{0,1}^{(0)} \mathcal{L}_{2,2} + b_{0,1}^{(0)} \mathcal{L}_{1,4}}{2} \mathcal{M}^{(0,0)(0,1)} + 2\frac{a_{3,1}^{(2)} + b_{3,1}^{(2)}}{2} \mathcal{M}^{(2,3)(0,1)}$

+ $\frac{a_{2,1}^{(0)} \mathcal{L}_{1,4} + b_{2,1}^{(0)} \mathcal{L}_{2,2}}{2} \mathcal{M}^{(0,2)(0,1)} + 2\frac{a_{1,1}^{(2)} + b_{1,1}^{(2)}}{2} \mathcal{M}^{(2,1)(0,1)}$.

The modular transformation to the tree channel yields

$$\tilde{\mathcal{M}} = -4c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_{0}^{\infty} dl \left( a_{0,0}^{(0)} + b_{0,0}^{(0)} \tilde{L}^{(0,0)} \tilde{\mathcal{M}}^{(0,0)} - (a_{1,0}^{(2)} + b_{1,0}^{(2)}) \tilde{L}_{8,2} \tilde{\mathcal{M}}^{(1,0)}
\right.$$

$$+ 2(a_{2,0}^{(2)} + b_{2,0}^{(2)}) \tilde{L}_{8,2} \tilde{\mathcal{M}}^{(2,0)} - (a_{1,0}^{(2)} + b_{1,0}^{(2)}) \tilde{L}_{8,2} \tilde{\mathcal{M}}^{(3,0)}
$$

$$+ 2(a_{0,1}^{(0)} \tilde{L}_{4,4} + b_{0,1}^{(0)} \tilde{L}_{8,2}) \mathcal{M}^{(0,1)} + 2(a_{3,1}^{(2)} + b_{3,1}^{(2)}) \tilde{\mathcal{M}}^{(1,1)}
$$

$$+ 2(a_{2,1}^{(0)} \tilde{L}_{8,2} + b_{2,1}^{(0)} \tilde{L}_{4,4}) \mathcal{M}^{(2,1)} + 2(a_{1,1}^{(2)} + b_{1,1}^{(2)}) \tilde{\mathcal{M}}^{(3,1)} \right).$$

(3.48)

To obtain the complete projector and to cancel the tadpole [23], we set

$$a_{0,0}^{(0)} = a_{1,0}^{(2)} = -a_{2,0}^{(0)} = a_{3,0}^{(2)} = -a_{0,1}^{(0)} = -a_{1,1}^{(2)} = -a_{2,1}^{(0)} = a_{3,1}^{(2)} = N,$$

(3.50)

$$b_{0,0}^{(0)} = b_{1,0}^{(2)} = -b_{2,0}^{(0)} = b_{3,0}^{(2)} = -b_{0,1}^{(0)} = -b_{1,1}^{(2)} = -b_{2,1}^{(0)} = b_{3,1}^{(2)} = N.$$  

(3.51)

Let us focus on the coefficients on the zero mode contributions in the Klein bottle amplitude (3.37), the annulus amplitude (3.40) and the Möbius strip amplitude (3.49). The RR-tadpole cancellation condition (2.7) leads to

$$0 = 16 + \frac{N^2}{4} - 4N = \frac{1}{4}(N - 8)^2.$$  

(3.52)

The number of one stack of the D-branes is $N = 8$ to cancel the RR-tadpole. Taking account of (3.36) and (3.51), the gauge groups are determined as $(Sp(2))^4$ for the DCC model.

For the CCC model, one of whose orientifold actions is given by

$$\mathcal{R} = (a, a, a).$$

(3.53)

On the other hand, the element $\mathcal{R}\theta$ in the orientifold group is given by

$$\mathcal{R}\theta = (-b, b, a).$$

(3.54)

As seen in Table 2, these two elements yield different numbers of O-planes. To show this, we evaluate the RR-tadpole amplitude in the following way: In the tree channel the Klein bottle amplitude is

$$\tilde{\mathcal{K}} = c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_{0}^{\infty} dl \left( 20\tilde{L}^{(0,0)} \tilde{\mathcal{K}}^{(0,0)} - 32\tilde{L}_{2,8}\tilde{\mathcal{K}}^{(1,0)} \right).$$
The prefactors do not correspond to that from the complete projector. The annulus and the Möbius strip amplitudes are also described as

\[
\tilde{A} = \frac{c}{16} (1_{RR} - 1_{NSNS}) \int_0^\infty d\ell \left( (M^2 + 4N^2) \tilde{\mathcal{L}}^{(0,0)}_A \tilde{A}^{(0,0)} - 8 MN \tilde{\mathcal{L}}^{(1,0)}_{2,2} \tilde{A}^{(1,0)} - 4(M^2 + 4N^2) \tilde{\mathcal{L}}^{(2,0)}_{2,2} \tilde{A}^{(2,0)} - 8 MN \tilde{\mathcal{L}}^{(3,0)}_{2,2} \tilde{A}^{(3,0)} - 4(M^2 \tilde{\mathcal{L}}^{(0,1)}_{2,2} + 4N^2 \tilde{\mathcal{L}}^{(1,1)}_{1,4}) \tilde{A}^{(0,1)} + 16 MN \tilde{A}^{(1,1)} \right),
\]

\[
\tilde{M} = -c(1_{RR} - 1_{NSNS}) \int_0^\infty d\ell \left( 2(M + N) \tilde{\mathcal{L}}^{(0,0)}_M \tilde{M}^{(0,0)} - 2(M + 4N) \tilde{\mathcal{L}}^{(1,0)}_{8,2} \tilde{M}^{(1,0)} - 8(M + N) \tilde{\mathcal{L}}^{(2,0)}_{8,2} \tilde{M}^{(2,0)} - 2(M + 4N) \tilde{\mathcal{L}}^{(3,0)}_{8,2} \tilde{M}^{(3,0)} - 8(M \tilde{\mathcal{L}}^{(0,1)}_{8,2} + N \tilde{\mathcal{L}}^{(1,1)}_{4,4}) \tilde{M}^{(0,1)} + 4(M + 4N) \tilde{M}^{(1,1)} \right),
\]

where \( M \) and \( N \) are the numbers of D-branes which are invariant under the set of orientifold actions \( \{ \mathcal{R}, \mathcal{R}\theta^2, \mathcal{R}\phi, \mathcal{R}\theta^2\phi \} \), and under the other set of actions \( \{ \mathcal{R}\phi, \mathcal{R}\phi^3, \mathcal{R}\theta^3, \mathcal{R}\theta^3\phi \} \), respectively. In the Möbius strip amplitude we have set

\[
\begin{align*}
\hat{a}^{(0)}_{0,0} &= a^{(2)}_{2,0} = -a^{(2)}_{0,1} = -a^{(2)}_{1,1} = a^{(2)}_{3,1} = M, \\
\hat{b}^{(0)}_{0,0} &= b^{(2)}_{2,0} = -b^{(2)}_{0,1} = -b^{(2)}_{1,1} = b^{(2)}_{3,1} = N.
\end{align*}
\]

Focus on the coefficient in (3.55), (3.56a) and (3.56b), we obtain the RR-tadpole cancellation conditions (2.7),

\[
\begin{align*}
0 &= 20 + \frac{1}{16}(M^2 + 4N^2) - 2(M + N) = \frac{1}{16}((M - 16)^2 + (N - 4)^2), \\
0 &= -32 - \frac{MN}{2} + 2(M + 4N) = -\frac{1}{2}(M - 16)(N - 4),
\end{align*}
\]

and find \( M = 16 \) and \( N = 4 \). This indicates that we should insert sets of different numbers of D-branes in an appropriate way in several kinds of non-factorizable tori.

The open string massless spectrum is given in Table 5. The multiplicities of twisted states spectra depend on the intersection numbers [23] (see Table 5). We see that the CCD and DCC models are distinct from the CDD model despite the same numbers of O-planes, and actually these four models have different spectra. For the closed string the numbers of massless states are considerably reduced due to their Hodge numbers in [26, 27].
Table 6: Open string massless spectra of $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold on the $D_6$ lattice. The symbols “V” and “C” denote the vector and chiral multiplets, respectively.

### 3.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$, the calculation is similar to the examples in the previous subsection. The new feature in $\mathbb{Z}_2 \times \mathbb{Z}_2$ is that we have more freedom to choose orbifold actions in comparison with the case of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

For $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the $D_6$ lattice (2.32), all the point group elements can be given by the use of $a$ and $b$ in (3.12), see Appendix A. In the case of the CCC orientifold with $R = (a, a, a)$, the point group elements $\theta$ and $\phi$ are

$$
\theta : (-1, -1, 1), \quad \phi : (1, -1, -1).
$$

(3.60)

The orientifold group elements including $\Omega R$ are

$$
\{\Omega R, \Omega R\theta, \Omega R\phi, \Omega R\theta\phi\},
$$

(3.61)

and these elements generate O6-planes respectively. From Table 6 the numbers of O6-planes are read two for each elements.

In the CCD orientifold with $R = (a, a, b)$, we have two distinct pairs of the point group
elements:

\[
\begin{align*}
\theta : & (1, -1, -1) \\
\phi : & (-1, -1, 1)
\end{align*}
\]

\[
\begin{align*}
\theta : & (1, -1, -1) \\
\phi : & (-1, -1, b)
\end{align*}
\] (3.62)

The numbers of O-planes generated by the former orbifold actions are also two. In the latter case, the \(\Omega R\) and \(\Omega R\theta\) (\(\Omega R\phi\) and \(\Omega R\theta\phi\)) generate two (four) O6-planes, respectively. We can classify the distinct orientifold models on the Lie root lattices, and the other possible elements on the \(D_6\) lattice are listed in Table 7. We should notice that even though the numbers of O6-planes are the same in any three-cycles in \(Z_2 \times Z_2\) orientifold models, those of non-factorizable models can be different.

| Lattice | Label | reps. of \(\mathcal{R}\) | Orbifold | \# of O6-planes |
|---------|-------|------------------|---------|----------------|
|         |       | rep. of \(\theta\) | rep. of \(\phi\) | \(\mathcal{R}\) | \(\mathcal{R}\theta\) | \(\mathcal{R}\phi\) | \(\mathcal{R}\theta\phi\) |
| \(D_6\) | CCC   | \((a, a, a)\)    | \((1, -1, -1)\) | \((-1, -1, 1)\) | 4 | 4 | 4 | 4 |
|         | CCD   | \((a, a, b)\)    | \((1, -1, -1)\) | \((-1, -1, 1)\) | 2 | 2 | 2 | 2 |
|         |       |                  | \((1, -1, -1)\) | \((-1, -a, b)\) | 2 | 2 | 4 | 4 |
|         | CDD   | \((a, b, b)\)    | \((1, -1, -1)\) | \((-1, -1, 1)\) | 1 | 1 | 1 | 1 |
|         |       |                  | \((1, -1, -1)\) | \((-1, b, -b)\) | 1 | 1 | 4 | 4 |
|         |       |                  | \((-1, 1, -1)\) | \((a, -1, -b)\) | 1 | 1 | 2 | 2 |
|         |       |                  | \((a, -1, -b)\) | \((-a, b, -1)\) | 1 | 2 | 2 | 4 |
|         | DDD   | \((b, b, b)\)    | \((-1, -1, 1)\) | \((1, -1, -1)\) | 2 | 2 | 2 | 2 |
|         |       |                  | \((1, -1, -1)\) | \((-1, -b, b)\) | 2 | 2 | 2 | 2 |
|         |       |                  | \((-1, -b, b)\) | \((b, -1, -b)\) | 2 | 2 | 2 | 2 |

Table 7: \(Z_2 \times Z_2\) orbifold models on the \(D_6\) Lie root lattice.

Finally we check the RR-tadpole cancellation in the \(Z_2 \times Z_2\) CCC model on the \(D_6\) lattice. The contribution from \(\phi\)- and \(\theta\phi\)-twisted sectors are the same as \(\theta\)-sector for the CCC model on the \(D_6\) lattice. The RR-tadpole cancellation is satisfied with \(N = 4\) as we can see the following amplitudes in the tree channel. The Klein bottle amplitude is given as

\[
\tilde{K} = 32c(1_{RR} - 1_{NSNS}) \int_0^\infty dl \left( \tilde{L}_K^{(0,0)} \tilde{K}^{(0,0)} - 4 \tilde{L}_{2,8} \tilde{K}^{(1,0)} - 4 \tilde{L}_{2,8} \tilde{K}^{(0,1)} - 4 \tilde{L}_{2,8} \tilde{K}^{(1,1)} \right). 
\] (3.63)
The annulus and the Möbius amplitudes are also given as

\[
\tilde{A} = \frac{N^2 c}{8} (1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \left( \tilde{\mathcal{L}}^{(0,0)}_A \tilde{A}^{(0,0)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{A}^{(1,0)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{A}^{(0,1)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{A}^{(1,1)} \right),
\]

\[
\tilde{M} = -4Nc(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty dl \left( \tilde{\mathcal{L}}^{(0,0)}_{\tilde{M}} \tilde{M}^{(0,0)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{M}^{(1,0)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{M}^{(0,1)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{M}^{(1,1)} \right).
\]

(3.64a) (3.64b)

We observe that in any amplitudes the prefactors are given by the complete projector (3.28).

4 Conclusion

In this paper we studied the RR-tadpole cancellation condition in Type II string models compactified on six-tori given by general Lie root lattices. We obtained a simple derivation to count the orientifold planes lying on the lattice by the use of the Lefschetz fixed point theorem. As expected the RR-tadpole contributions are cancelled by adding an appropriate number of D-branes parallel to the O-planes. The Lefschetz fixed point theorem provides an intuitive picture to non-factorizable models, and we easily showed a way to construct orientifold models on tori and orbifolds.

In \( D = 4 \), \( \mathcal{N} = 1 \) \( \mathbb{Z}_N \times \mathbb{Z}_M \) orientifolds, mainly the factorizable models on \( T^2 \times T^2 \times T^2 \) have been constructed and investigated. We gave the classifications in Type IIA orientifold models with O6-planes, and obtained many new models. As explained in detail, the Lefschetz fixed point theorem provide intuitive and convenient tools in model construction. Since the condition derived in (2.20) is the necessary condition for orbifolds, we performed explicit calculations for \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold models, and confirmed the RR-tadpole calculations. It is expected that even in other non-factorizable orbifold models the RR-tadpole cancellation should be checked in the same calculation. We further found many non-factorizable \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifolds in which the numbers of O-planes depend on the three-cycles left invariant under the orbifold projections in Table 7 and in Table 11. These features are not seen in factorizable models, and will provide new possibilities for model constructions. On the other hand, since the metric of non-factorizable tori is changed to B-field via T-duality, our consideration should be related to compactification with such backgrounds. Actually in heterotic orbifolds there are some coincidences between non-factorizable models and factorizable models with generalized discrete torsion [41]. Our results indicate that there would be a possibility to construct various class of \( D = 4 \), \( \mathcal{N} = 1 \) models with different set of chiral spectra from other well-known (non-)factorizable models.
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Appendix

A Six-dimensional Lie root lattice

In this appendix we study basic aspects of the Lie root lattice given by the simple Lie algebra and its application to non-factorizable six-tori. First we review the simple root on the Lie algebra. In terms of Lie root lattices, we find that there are only twelve distinct non-factorizable six-tori and four factorizable ones. By the use of the Weyl reflection and the outer automorphisms, we can classify all the point groups of orbifolds and orientifold actions \( R \) on the tori, which crystallographically act on the Lie root lattices. We give explicit representations of point group elements generated by the Weyl reflections and the outer automorphisms of the Lie root lattices. Some of the point groups can be given by the Coxeter elements from the Cater diagrams or the generalized Coxeter elements as explained later. Beside these elements, we see that point groups which are not included in the (generalized) Coxeter elements are also obtained by the classification.

We utilize these elements for the point groups of our orientifold models, and these elements lead to many new orientifold models as explained in section 3 and in this appendix. Here we give the systematic way to construct orbifolds and orientifolds on the Lie root lattices.

A.1 Lie root lattices

We use the words of the Lie algebra in order to define the shape of tori defined in (2.12). The Lie algebras whose orders are within six are \( A_N \), \( B_N \), \( C_N \), \( D_N \), \( E_6 \), \( F_4 \) and \( G_2 \). The simple
roots $\alpha_i$ of these Lie algebras can be given as follows:

$A_N : \alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, N$

$B_N : \alpha_i = e_i - e_{i+1}, \quad \alpha_N = e_N, \quad i = 1, \ldots, N - 1$

$C_N : \alpha_i = e_i - e_{i+1}, \quad \alpha_N = 2e_N, \quad i = 1, \ldots, N - 1$

$D_N : \alpha_i = e_i - e_{i+1}, \quad \alpha_N = e_{N-1} + e_N, \quad i = 1, \ldots, N - 1$

$E_6 : \alpha_i = e_i - e_{i+1}, \quad \alpha_5 = e_4 + e_5, \quad \alpha_6 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + \sqrt{3}e_6)$

$F_4 : \quad \alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = 2e_3, \quad \alpha_4 = -e_1 - e_2 - e_3 - e_4$

$G_2 : \quad \alpha_1 = e_1 - e_2, \quad \alpha_2 = -e_1 + 2e_2 - e_3,$

where $e_i$'s are unit vectors whose scalar product is defined as $e_i \cdot e_j = \delta_{ij}$. The Dynkin diagrams are drawn in Figure 3. From these diagrams one can easily find a set of equivalence relations (isomorphism) among the simple Lie algebra,

$$A_1 \sim B_1 \sim C_1, \quad B_2 \sim C_2, \quad A_3 \sim D_3. \quad (A.2)$$

We further find the equivalence relations from the Lie root lattice point of view:

$$A_2 \sim G_2, \quad B_2 \sim D_2 \sim (A_1)^2, \quad D_4 \sim F_4, \quad (A.3a)$$

$$B_N \sim (A_1)^N, \quad C_N \sim D_N, \quad (A.3b)$$

where $(A_1)^2 = A_1 \times A_1$. Here we assumed the most symmetric cases, where the lengths of the shortest roots are equal between the lattices given as direct products. Since we are interested in symmetries of the lattices, the assumption is rational. We often use these equivalence relations in the classification of the six-tori.

Taking the direct products of tori generated from these lattices, we obtain six-tori in terms
of the Lie root lattices. We conclude that there are only twelve inequivalent non-factorizable six-tori and four factorizable ones in such a way as in Table 8.

| [non-factorizable tori] |
|------------------------|
| $A_6$ | $D_6$ | $E_6$ |
| $A_5 \times A_1$ | $A_4 \times A_2$ | $A_4 \times (A_1)^2$ |
| $D_5 \times A_1$ | $D_4 \times A_2$ | $D_4 \times (A_1)^2$ |
| $A_3 \times A_3$ | $A_3 \times A_2 \times A_1$ | $A_3 \times (A_1)^3$ |

| [factorizable tori] |
|---------------------|
| $(A_2)^3$ | $(A_2)^2 \times (A_1)^2$ | $A_2 \times (A_1)^4$ | $(A_1)^6$ |

Table 8: All the Lie root lattices in six dimensions.

### A.2 Weyl reflection and graph automorphism

Next we investigate the automorphisms of the above lattices. Both orbifold and orientifold groups should crystallographically act on the lattices. These groups can be classified in terms of the Weyl reflection and the graph automorphism acting on the simple roots of the Lie root lattice. The Weyl group $W$ is generated by the following Weyl reflections $r_{\alpha_k}$ which associate the simple root $\alpha_k$:

$$r_{\alpha_k} : \lambda \rightarrow \lambda - 2\frac{\alpha_k \cdot \lambda}{|\alpha_k|^2} \alpha_k.$$  \hspace{1cm} (A.4)

In the case of the $D_N$ Lie root lattice, for instance, the Weyl reflection $r_{\alpha_k}$ for $k = 1, \ldots, N-1$ is given as

$$r_{\alpha_k} : \begin{cases} 
\alpha_{k-1} &\rightarrow \alpha_{k-1} + \alpha_k \\
\alpha_k &\rightarrow -\alpha_k \\
\alpha_{k+1} &\rightarrow \alpha_{k+1} + \alpha_k
\end{cases}$$  \hspace{1cm} (A.5a)

and $\alpha_m$ for $m \neq k - 1, k, k + 1$ are unchanged. For $k = N$ it is

$$r_{\alpha_N} : \begin{cases} 
\alpha_{N-2} &\rightarrow \alpha_{N-2} + \alpha_N \\
\alpha_N &\rightarrow -\alpha_N
\end{cases}$$  \hspace{1cm} (A.5b)

and the other $\alpha_m$’s are unchanged. For the classification of the automorphisms, it would be convenient to rewrite them in the basis of orthogonal unit vectors $e_i$ as

$$r_{\alpha_k} : e_k \leftrightarrow e_{k+1} \quad k = 1, \ldots, N-1,$$  \hspace{1cm} (A.6a)

Most of other six-tori would be obtained by the continuous deformation of moduli of these tori [26].


\[ r_{\alpha_N} : \mathbf{e}_N \leftrightarrow -\mathbf{e}_{N-1}. \]  
\quad \text{(A.6b)}

On the other hand, the outer automorphism \( g \) of the Cartan diagram is represented as
\[ g : \alpha_{N-1} \leftrightarrow \alpha_N, \quad \text{(A.7)} \]
and the other simple roots are left unchanged. In the unit vector basis, it is
\[ g : \mathbf{e}_N \rightarrow -\mathbf{e}_N. \]  
\quad \text{(A.8)}

In terms of \( \mathbf{e}_i \), we can easily construct any elements generated from \( r_{\alpha_k} \) and \( g \). For example a product of two Weyl reflections which do not commute with each other makes up \( \mathbb{Z}_3 \) element as
\[ r_{\alpha_k}r_{\alpha_{k+1}} : \mathbf{e}_k \rightarrow \mathbf{e}_{k+1} \rightarrow \mathbf{e}_{k+2} \rightarrow \mathbf{e}_k, \quad (k < N - 1). \]  
\quad \text{(A.9)}

This is the permutation group \( S_3 \). Similarly the Weyl reflections \( r_{\alpha_k} \) for \( k = 1, \ldots, N-1 \) generate a permutation group \( S_N \). Adding the other elements \( r_{\alpha_N} \) and \( g \) to \( S_N \), the representation of the group is given by permutations with signs
\[ \mathbf{e}_i \rightarrow \pm \mathbf{e}_j \rightarrow \pm \mathbf{e}_k \rightarrow \cdots \rightarrow \pm \mathbf{e}_i. \]  
\quad \text{(A.10)}

Then the order of the Weyl group \( \mathcal{W} \) and \( \{\mathcal{W}, g\} \) are summarized in Table 9:

|       | \( \mathcal{W} \) | \( \{\mathcal{W}, g\} \) |
|-------|-----------------|----------------|
| \( D_N \) | \( 2^{N-1}N! \) | \( 2^N N! \) |
| \( A_{N-1} \) | \( N! \) | \( 2N! \) |

Table 9: The order of the Weyl group and the graph automorphism.

In the case of the \( A_{N-1} \) Lie root lattice, the Weyl reflections generate permutation group \( S_N \) in terms of \( \mathbf{e}_i \). Its outer automorphism of the Dynkin diagram is given by the following permutation
\[ g : \mathbf{e}_i \leftrightarrow -\mathbf{e}_{N+1-i}, \quad i = 1, \ldots, N \]  
\quad \text{(A.11)}

We can always permute \( g \) to \( g' \) by the elements of \( \mathcal{W} \) such that \( g' \) change the sign of all \( \mathbf{e}_i \)'s, This element is expressed as an identity matrix with negative sign \(-\mathbb{I}_N\), which means \( \{\mathcal{W}, g\} = \{\mathcal{W}, -\mathbb{I}_N\} \). Therefore the order of \( \{\mathcal{W}, g\} \) is twice as many as that of \( \mathcal{W} \), as in Table 9.
Then it is straightforward to obtain all \( \mathbb{Z}_2 \) elements of the \( D_N \) lattice, and they are given by the following sub-elements
\[
\begin{align*}
  e_i & \leftrightarrow e_j, \\
  e_k & \leftrightarrow -e_l, \\
  e_m & \rightarrow -e_m,
\end{align*}
\]
(A.12a)

except for \( D_4 \). The \( \mathbb{Z}_N \) elements are constructed similarly. For example \( \mathbb{Z}_3 \) elements are constructed by the following sub-elements,
\[
\begin{align*}
  e_i & \rightarrow \pm e_j \rightarrow \pm e_k \rightarrow \pm e_i, \quad \text{(\# of terms with} - \text{ sign is even)} \quad (A.13)
\end{align*}
\]
and their permutations. \( \mathbb{Z}_4 \) elements includes the following sub-elements,
\[
\begin{align*}
  e_i & \rightarrow -e_j \rightarrow -e_i, \quad i \neq j, \quad (A.14a) \\
  e_i & \rightarrow \pm e_j \rightarrow \pm e_k \rightarrow \pm e_l \rightarrow \pm e_i, \quad \text{(\# of terms with} - \text{ sign is even).} \quad (A.14b)
\end{align*}
\]

We can similarly deal with the \( A_N \) lattices. Note that the roots of the \( A_N \) and \( D_N \) can be given by
\[
\begin{align*}
  A_N & : \quad e_i - e_j, \quad (A.15a) \\
  D_N & : \quad \pm e_i \pm e_j, \quad i, j = 1, \ldots, N. \quad (A.15b)
\end{align*}
\]

They are symmetric under the permutations of \( i \) and \( j \). Now it is apparent that on the \( D_6 \) lattice, \( (3.9b) \) is the only inequivalent \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) elements, and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) elements can be given by \( a, b \) and \( 1 \) in \( (3.12) \). From Table \( 8 \) we have all the point group elements which can be expressed by the Weyl reflections and the outer automorphism (except for the \( E_6 \) lattice). However there are a few exceptions owing to additional outer automorphisms as follows.

We shortly explain the Coxeter elements and the generalized Coxeter elements\(^6\). The Coxeter element of the Lie root lattice is defined by product of all the Weyl reflections which associate with simple roots,
\[
D_N \equiv r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_N}. \quad (A.16)
\]

The other Coxeter elements, which are generated by different ordering of product, are conjugate to one another, and lead to the same class of orbifolds. There are other elements generated by the Weyl reflections. These orbifolds can be classified by the Carter diagrams \([40]\). The Coxeter elements of \( D_4 \) from the Carter diagrams are, for example,
\[
D_4 = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_4}, \quad (A.17a)
\]

\(^6\)From the definition of the (generalized) Coxeter elements, we can see that the elements do not left any directions invariant for corresponding sub-space. Then it is apparent that for \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifold they lead to factorizable models on \( T^2 \times T^2 \times T^2 \).
\[ D_4(a1) = r_{\alpha_1}r_{\alpha_2}r_{\alpha_3}r_{\alpha_2+\alpha_3+\alpha_4}, \]  
(A.17b)

where \( r_{\alpha_2+\alpha_3+\alpha_4} \) is a Weyl reflection associated with the sum of simple roots \( \alpha_2 + \alpha_3 + \alpha_4 \). Then the order of \( D_4 \) is six, and that of \( D_4(a1) \) is four. However these elements do not include the outer automorphisms. The generalized Coxeter elements are defined by adding outer automorphisms to the Coxeter elements. For example the \( D_N \) Lie root lattice has a graph automorphism \( g \) which exchanges the simple root \( \alpha_{N-1} \) and \( \alpha_N \). The generalized Coxeter element is defined by

\[ C^{[2]} \equiv r_{\alpha_1}r_{\alpha_2}\cdots r_{\alpha_{N-2}}g. \]  
(A.18)

For instance the generalized Coxeter element of \( D_4 \) is

\[ C^{[2]} = r_{\alpha_1}r_{\alpha_2}r_{\alpha_3}g, \]  
(A.19)

and the order of this element is eight.

Actually these (generalized) Coxeter elements and elements from the Cater diagrams are included in the above classification by the use of \( e_i \). An exception occurs in the \( D_4 \) lattice, which has another outer automorphism \( g' \),

\[ g' : \alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1. \]  
(A.20)

The generalized Coxeter element of this outer automorphism is defined by

\[ C^{[3]} \equiv r_{\alpha_1}r_{\alpha_2}g'. \]  
(A.21)

This action corresponds to a rotation of \( (e^{\pi i/6}, e^{5\pi i/6}) \). For this element the classification in the \( e_i \) basis is inconvenient (since for example it acts as \( g' : e_1 \to (e_1 + e_2 + e_3 + e_4)/2 \)). We comment that among \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifolds this element generates new orbifold only for \( \mathbb{Z}_3 \times \mathbb{Z}_3 \), e.g. \( (C^{[3]})^4 \) is rotation of \( (e^{2\pi i/3}, e^{2\pi i/3}) \) and that of \( r_{\alpha_3}g' \) is \( (1, e^{2\pi i/3}) \). Then a torus on the \( D_4 \times A_2 \) lattice allows a \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) orbifold.

In the case that two independent radii of a torus on the \( A_3 \times A_3 \) lattice are equal to each other, there is an additional outer automorphism \( g_{33} \),

\[ g_{33} : \alpha_i \leftrightarrow \pm \alpha_{i+3}, \]  
(A.22)

where \( \alpha_i \) is a simple root of the first (second) \( A_3 \) for \( i = 1, 2, 3 \) \((i = 4, 5, 6)\). From the observation of its eigenvalues, these elements do not generate another \( \mathbb{Z}_N \times \mathbb{Z}_M \) elements. However the orientifold action \( \mathcal{R} \) can be generated from \( g_{33} \), which will be explained in the

---

7There would be complete classifications including the outer automorphisms by mathematicians. However the authors do not know it. Alternatively our approach provides a complete classification and useful formula for the six-dimensional Lie root lattices, except for \( E_6 \).
next subsection. Such outer automorphisms also arise in factorizable tori including sublattices $(A_2)^n$ and $(A_1)^m$. For example, $(A_2)^2$ has an outer automorphism as

$$g_{22} : \alpha_i \rightarrow -\alpha'_i \rightarrow -\alpha_i, \ i = 1, 2.$$  \hspace{1cm} (A.23)

where $\alpha_i$ is a simple root of the first $A_2$ and $\alpha'_i$ is one of the second $A_2$. The eigenvalues of this element are $(e^{\pi i/2}, e^{\pi i/2})$, and generate $\mathbb{Z}_4$ elements. In this case the factorizable tori are actually non-factorizable as orbifolds.

We investigate the other cases similarly, and obtain the allowed $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds in Table 10. In the next subsection we will explain the orientifold actions which are compatible with these non-factorizable orbifolds.

| Lie root lattice | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | $\mathbb{Z}_4 \times \mathbb{Z}_4$ |
|-----------------|----------------------|----------------------|----------------------|----------------------|
| $A_6$           |          |          |          |          |
| $D_6$           | $\checkmark$ | $\checkmark$ |          | $\checkmark$ |
| $E_6$           | $\checkmark$ | $\checkmark$ | $\checkmark$ |          |
| $A_5 \times A_1$ | $\checkmark$ |          |          |          |
| $D_5 \times A_1$ | $\checkmark$ | $\checkmark$ |          |          |
| $A_4 \times A_2$ | $\checkmark$ |          |          |          |
| $A_4 \times (A_1)^2$ | $\checkmark$ |          |          |          |
| $D_4 \times A_2$ | $\checkmark$ | $\checkmark$ |          |          |
| $D_4 \times (A_1)^2$ | $\checkmark$ | $\checkmark$ |          | $\checkmark$ |
| $A_3 \times A_3$ | $\checkmark$ | $\checkmark$ |          |          |
| $A_3 \times A_2 \times A_1$ | $\checkmark$ |          |          |          |
| $A_3 \times (A_1)^3$ | $\checkmark$ | $\checkmark$ |          |          |
| $(A_2)^3$       | $\checkmark$ | $\checkmark$ |          |          |
| $(A_2)^2 \times (A_1)^2$ | $\checkmark$ | $\checkmark$ |          |          |
| $A_2 \times (A_1)^4$ | $\checkmark$ | $\checkmark$ |          |          |
| $(A_1)^6$       | $\checkmark$ | $\checkmark$ |          | $\checkmark$ |

Table 10: Classification of six-dimensional (non-)factorizable tori and possible $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models on them.
A.3 Orientifolds on non-factorizable orbifolds

In the previous subsection we gave a way to obtain orbifolds on non-factorizable tori. In order to preserve supersymmetry, the orbifold action $\theta : (z_1, z_2, z_3) \rightarrow (e^{2\pi i z_1}, e^{2\pi i z_2}, e^{2\pi i z_3})$ should satisfy the equation $v_1 + v_2 + v_3 = 0$. Then only a holomorphic $(3,0)$-form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ and a anti-holomorphic $(0,3)$-form $\bar{\Omega}$ are left invariant, and the other three forms on a six-tori are generally projected out. The orientifold action $\mathcal{R}$ of O6-plane, which preserve $\mathcal{N} = 1$ supersymmetry, should act as

$$\mathcal{R} : (z_1, z_2, z_3) \rightarrow (a\bar{z}_1, b\bar{z}_2, c\bar{z}_3),$$

(A.24)

where $a$, $b$ and $c$ are phase factors. Then the every orientifold group element including $\mathcal{R}$ generates fixed loci of O6-planes.

For their classification we again use the abbreviations $a$, $b$ and $1$ in (3.12). For the $D_6$ lattice we have $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements as $\theta = (-1, -1, 1)$ and $\phi = (1, -1, -1)$. The orientifold actions which are compatible with this orbifold are

$$(\pm a, \pm a, \pm a), (\pm a, \pm a, \pm b), (\pm a, \pm b, \pm b), (\pm b, \pm b, \pm b),$$

(A.25)

where the underlined entries are permuted. For the orbifold elements $\theta = (-1, a, b), \phi = (1, -1, -1)$, the compatible orientifold actions are

$$\pm (a, a, -b), \pm (a, -a, b), \pm (-b, a, -b), \pm (b, -a, b).$$

(A.26)

In other words, the restriction is that the eigenvalues of each orientifold group element $\mathcal{R}$, $\mathcal{R}\theta$, $\mathcal{R}\phi$ and $\mathcal{R}\theta\phi$ should be $(-1, -1, -1, 1, 1, 1)$. Note that there are some equivalent actions due to the symmetry of the lattice. These considerations lead to Table 7 for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models on the $D_6$ lattice.

There exists an exception in this classification for the $A_3 \times A_3$ lattice as mentioned before. We define the lattice $A_3 \times A_3$ by using the simple roots

$$\alpha_1 = e_1 - e_2, \quad \alpha_4 = e_4 - e_5,$$
$$\alpha_2 = e_2 - e_3, \quad \alpha_5 = e_5 - e_6,$$
$$\alpha_3 = e_2 + e_3, \quad \alpha_6 = e_5 + e_6.$$  

(A.27)

In this base $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds are obtained in a similar manner of the $D_6$ lattice. Note that the action $\mathcal{R} = (*, b, *)$, where $*$ is $b$, $a$ or $1$, is forbidden due to the lattice structure.

---

8Note that for this orbifold elements the basis is different from (A.24).

9It may seem that the classification with $b,a$ and $1$ elements is missing the action $\mathcal{R} : \alpha_i \rightarrow -\alpha_i$ with $i = 1, 2, 3$, however this action is included in orientifold groups, e.g. the $\mathcal{R}\theta\phi$ action of DCD model on Table 1.4
outer automorphism between two $A_3$'s generates an exceptional action

$$\mathcal{R} : \alpha_i \leftrightarrow \alpha_{i+3}, \ i = 1, 2, 3.$$  \hspace{1cm} (A.28)

If we redefine the base of $A_3 \times A_3$ as

$$\begin{align*}
\alpha_1 &= e_1 - e_3, & \alpha_4 &= e_2 - e_4, \\
\alpha_2 &= e_3 - e_5, & \alpha_5 &= e_4 - e_6, \\
\alpha_3 &= e_3 + e_5, & \alpha_6 &= e_4 + e_6,
\end{align*}$$ \hspace{1cm} (A.29)

the exceptional action is expressed by $(b, b, b)$ in the orthogonal $e_i$ basis:

$$\mathcal{R} : e_1 \leftrightarrow e_2, \ e_3 \leftrightarrow e_4, \ e_5 \leftrightarrow e_6.$$ \hspace{1cm} (A.30)

Actually this element gives only one inequivalent element including the outer automorphism, and we label it as $(DDD)'$.

Including this orientifold action we obtain all the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the $A_3 \times A_3$ lattice in Table 11.

| Lattice | Label | rep. of $\mathcal{R}$ | Orbifold | # of O6-planes |
|---------|-------|----------------------|----------|----------------|
|         |       | rep. of $\theta$ | rep. of $\phi$ | $\mathcal{R}$ | $\mathcal{R}\theta$ | $\mathcal{R}\phi$ | $\mathcal{R}\theta\phi$ |
| $A_3 \times A_3$ | CCC | $(a, a, a)$ | $(1, -1, -1)$ | $(1, -1, 1)$ | 2 | 2 | 2 | 2 |
|         | | | $(-1, 1, -1)$ | $(a, -1, -a)$ | 2 | 2 | 2 | 8 |
|         | CCD | $(a, a, b)$ | $(1, -1, -1)$ | $(-1, 1, 1)$ | 2 | 2 | 2 | 2 |
|         | | | $(1, -1, -1)$ | $(-1, a, -b)$ | 2 | 2 | 2 | 2 |
|         | | | $(1, 1, -1)$ | $(a, -1, -b)$ | 2 | 2 | 2 | 8 |
|         | DCD | $(b, a, b)$ | $(1, -1, -1)$ | $(-1, 1, 1)$ | 2 | 2 | 2 | 2 |
|         | | | $(1, -1, -1)$ | $(-1, a, -b)$ | 2 | 2 | 2 | 8 |
|         | | | $(1, 1, -1)$ | $(b, -1, -b)$ | 2 | 2 | 2 | 8 |
|         | | | $(1, a, -b)$ | $(b, -a, -1)$ | 2 | 2 | 2 | 8 |
|         | | | $(1, -a, b)$ | $(-b, a, -1)$ | 2 | 2 | 2 | 2 |
|         | $(DDD)'$ | $(b, b, b)$ | $(1, -1, -1)$ | $(1, -1, 1)$ | 1 | 1 | 1 | 1 |

Table 11: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models on the $A_3 \times A_3$ Lie root lattice.
### B Comments on lattices

In this appendix we briefly summarize conventions of the (sub-)lattice and its dual lattice space for a $\mathbb{Z}_2$ action $\mathcal{R}$ in the following way:

- $\Lambda_{\mathcal{R}, \perp}$: lattice projected out by the action $\mathcal{R}$, $\Lambda_{\mathcal{R}, \perp} \equiv \frac{1 + \mathcal{R}}{2} \Lambda$
- $\Lambda_{\mathcal{R}, \text{inv}}$: $\mathcal{R}$ invariant sublattice
- $\Lambda^*$: dual lattice of $\Lambda$, for its base $\alpha_j \cdot \alpha_i^* = \delta_{ji}$, $\alpha_j \in \Lambda$, $\alpha_i^* \in \Lambda^*$

These three lattice spaces are closely related to one another. Introducing a lattice $\Lambda_{-\mathcal{R}, \perp}$ which is projected out by the $-\mathcal{R}$ action on it, then we find the following non-trivial equations:

\begin{align}
\Lambda_{\mathcal{R}, \perp}^* &= (\Lambda_{\mathcal{R}, \text{inv}})^*, \\
\text{Vol}(\Lambda) &= \text{Vol}(\Lambda_{\mathcal{R}, \text{inv}}) \cdot \text{Vol}(\Lambda_{-\mathcal{R}, \perp}), \\
\text{Vol}(\Lambda^*) &= \text{Vol}(\Lambda)^{-1}.
\end{align}

Let us analyze in a more concrete way. For example, we consider the four-dimensional $D_4$ Lie root lattice $\Lambda$ and its dual lattice $\Lambda^*$ based on

\begin{align}
\Lambda &= \begin{cases}
(1, -1, 0, 0) \\
(0, 1, -1, 0) \\
(0, 0, 1, -1) \\
(0, 0, 1, 1)
\end{cases},
\Lambda^* &= \begin{cases}
(1, 0, 0, 0) \\
(1, 1, 0, 0) \\
(1, 0, 1, 0) \\
(1, 2, 1, 0)
\end{cases}
\end{align}

and we give a $\mathbb{Z}_2$ action $\mathcal{R}$ on the $D_4$ lattice as

\[ \mathcal{R} = \text{diag}(1, 1, -1, -1). \]

Then, we can obtain the basis vectors in the lattices $\Lambda_{\mathcal{R}, \perp}$, $\Lambda_{\mathcal{R}, \text{inv}}$, $\Lambda_{\mathcal{R}, \perp}^*$ and $\Lambda_{\mathcal{R}, \text{inv}}^*$ in the following forms:

\begin{align}
\Lambda_{\mathcal{R}, \perp} &= \begin{cases}
(1, -1, 0, 0) \\
(0, 1, 0, 0)
\end{cases}, \\
\Lambda_{\mathcal{R}, \text{inv}} &= \begin{cases}
(1, -1, 0, 0) \\
(0, 2, 0, 0)
\end{cases}, \\
\Lambda_{\mathcal{R}, \perp}^* &= \begin{cases}
(1, 0, 0, 0) \\
(1, 1, 0, 0)
\end{cases}, \\
\Lambda_{\mathcal{R}, \text{inv}}^* &= \begin{cases}
(1, 0, 0, 0) \\
(1, 1, 0, 0)
\end{cases}
\end{align}

Thus we easily see the relation among various lattice spaces:

\[ \Lambda_{\mathcal{R}, \perp} \quad \perp \quad \Lambda \quad \xrightarrow{\text{inv}} \quad \Lambda_{\mathcal{R}, \text{inv}} \]

\[ \Lambda_{\mathcal{R}, \perp}^* \quad \xleftarrow{\text{inv}} \quad \Lambda^* \quad \perp \quad \Lambda_{\mathcal{R}, \perp}^* \]

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C String one-loop amplitudes

In this appendix we summarize descriptions of the string one-loop amplitudes whose topologies are given by the Klein bottle, the annulus and the Möbius strip in the loop channel [23, 38]. These are applied to discuss the RR-tadpole amplitudes in the main part of this paper. Here we start from the forms in which the zero mode and the oscillator modes are factorized:

\[ \mathcal{K} = 4c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \left( \frac{1}{4NM} \sum_{n_1,k_1=0}^N \sum_{n_2,k_2=0}^M \mathcal{K}^{(n_1,k_1)(n_2,k_2)} \mathcal{L}^{(n_1,k_1)(n_2,k_2)}_K \right) , \quad \tag{C.1a} \]

\[ \mathcal{A} = c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \left( \frac{1}{4NM} \sum_{n_1,k_1=0}^N \sum_{n_2,k_2=0}^M \sum_{(i_1,i_2)=(0,0)}^{(N-1,M-1)} \text{tr} \left( \gamma_{k_1k_2}^{(i_1,i_2)} \right) \right) \times \mathcal{A}^{(n_1,k_1)(n_2,k_2)}_{\text{A}} \mathcal{L}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}_A , \quad \tag{C.1b} \]

\[ \mathcal{M} = -c(1_{\text{RR}} - 1_{\text{NSNS}}) \int_0^\infty \frac{dt}{t^3} \left( \frac{1}{4NM} \sum_{n_1,k_1=0}^N \sum_{n_2,k_2=0}^M \sum_{(i_1,i_2)=(0,0)}^{(N-1,M-1)} \text{tr} \left( \gamma_{\Omega Rk_1k_2}^{(i_1,i_2)} \right) \right) \times \mathcal{M}^{(n_1,k_1)(n_2,k_2)}_{\text{M}} \mathcal{L}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}_M , \quad \tag{C.1c} \]

where the values \( \mathcal{K}^{(n_1,k_1)(n_2,k_2)}, \mathcal{A}^{(n_1,k_1)(n_2,k_2)} \) and \( \mathcal{M}^{(n_1,k_1)(n_2,k_2)} \) denote oscillator contributions, and \( \mathcal{L} \) indicates the zero mode contributions in the amplitudes. They belong to the \( \theta^{n_1} \phi^{n_2} \)-twisted sector with \( \theta^{k_1} \phi^{k_2} \)-insertion in the amplitudes. The \( \gamma^{(i_1,i_2)} \)'s are the matrix representations of the orientifold action on the Chan-Paton factors [34], whose superscript \((i_1, i_2)\) labels the different types of D6-branes on which the open string attaches. The location of the brane \((i_1, i_2)\) is defined by rotating brane \((0, 0)\) by the action \( \theta^{-i_1/2} \phi^{-i_2/2} \).

C.1 Contributions from zero modes

The above one-loop amplitudes \([C.1]\) contain the zero mode contributions \( \mathcal{L}_{K,A,M} \) from the sum of the Kaluza-Klein momentum modes and the winding modes, which are expressed in such a way as

\[ \mathcal{L}_{K}^{(n_1,k_1)(n_2,k_2)} = \chi_{K}^{(n_1,k_1)(n_2,k_2)} \text{Tr}_{KK+W}^{(n_1,n_2)} \left( \Omega R \theta^{k_1} \phi^{k_2} e^{-2\pi\ell (L_0+L_0)} \right) , \quad \tag{C.2a} \]

\[ \mathcal{L}_{A}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} = \chi_{A}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} \text{Tr}_{KK+W}^{(i_1,i_2),(1-i_1-i_2-n_2)} \left( \theta^{k_1} \phi^{k_2} e^{-2\pi\ell L_0} \right) , \quad \tag{C.2b} \]

\[ \mathcal{L}_{M}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} = \chi_{M}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} \text{Tr}_{KK+W}^{(i_1,i_2),(1-i_1-i_2-n_2)} \left( \Omega R \theta^{k_1} \phi^{k_2} e^{-2\pi\ell L_0} \right) . \tag{C.2c} \]

\[ ^{10}\text{In this appendix we borrow quite useful conventions and equations in appendix A of [23].} \]

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Note that in the Klein bottle amplitude $\chi_K$ denotes the number of the corresponding fixed points which are left invariant under orientifold group actions $\mathcal{R} \theta^{k_1} \phi^{k_2}$. In the open string amplitudes $\chi_A$ gives the intersection number of the D-branes involved.

When we consider string propagating in the torus $T^6 = \mathbb{R}^6/\Lambda$, the zero modes contributions $\mathcal{L}$ from the momentum modes $p = \sum_i n_i p_i$ and the winding modes $w = m_i w_i$ are given by

$$\mathcal{L} \equiv \sum_{n_i} \exp \left( -\delta \pi t n_i M_{ij} n_j \right) \cdot \sum_{m_i} \exp \left( -\delta \pi t m_i W_{ij} m_j \right),$$

where $t$ is the modulus in the loop channel and $n_i, m_i \in \mathbb{Z}$ are the quanta in the momentum modes and the winding modes [22]. Note that the matrices $M_{ij}$ and $W_{ij}$ are given by the products of $p_i$ and of $w_i$ in such a way as $M_{ij} = p_i \cdot p_j$, $W_{ij} = w_i \cdot w_j$; we set $\delta = 1$ (the Klein bottle), $\delta = 2$ (the annulus and the Möbius strip). Due to this, in two-dimensional torus $T^2 \subset T^6$, we can rewrite the above equations (C.2) in the following form:

$$\mathcal{L}_{\alpha,\beta} \equiv \sum_{m \in \mathbb{Z}} \exp \left( -\alpha \pi t m^2 \rho \right) \cdot \sum_{n \in \mathbb{Z}} \exp \left( -\beta \pi t n^2 \rho \right),$$

where $\rho = r^2/\alpha'$. It is worth rewriting this to the one in the tree channel. According to the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} = \sqrt{t} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t},$$

we find that the zero mode contribution in the tree channel is given as

$$\tilde{\mathcal{L}}_{\alpha,\beta} \equiv \sum_{m \in \mathbb{Z}} \exp \left( -\alpha \pi t m^2 \rho \right) \cdot \sum_{n \in \mathbb{Z}} \exp \left( -\beta \pi t n^2 \rho \right).$$

This formulation is quite useful not only for factorizable torus $T^2 \times T^2 \times T^2$ but also for non-factorizable tori in the main text via a suitable arrangement.

### C.2 Contributions from oscillator modes

Here we move to the discussion on the oscillator modes. These contributions into the one-loop amplitudes (C.1) are given by

$${\mathcal{K}}^{(n_1,k_1)(n_2,k_2)} = \text{Tr}_{\text{NSNS}}^{(n_1,n_2)} \left( \Omega \mathcal{R} \theta^{k_1} \phi^{k_2} (-1)^F e^{-2\pi t (L_0 + \bar{L}_0)} \right),$$

$${\mathcal{A}}^{(n_1,k_1)(n_2,k_2)} = \text{Tr}_{\text{NS}}^{(0,0)(-n_1,-n_2)} \left( \theta^{k_1} \phi^{k_2} (-1)^F e^{-2\pi t L_0} \right),$$

$${\mathcal{M}}^{(n_1,k_1)(n_2,k_2)} = \text{Tr}_{\text{R}}^{(0,0)(-n_1,-n_2)} \left( \Omega \mathcal{R} \theta^{k_1} \phi^{k_2} e^{-2\pi t L_0} \right).$$
The superscript \((0,0)(-n_1, -n_2)\) on the trace \(\text{Tr}^{(0,0)(-n_1, -n_2)}_{\text{NS}}\) in (C.7b) indicates open string states stretching between two distinct branes \((0,0)\) and \((-n_1, -n_2)\), or equivalently, between the brane \((i_1, i_2)\) and the brane \((i_1 - n_1, i_2 - n_2)\). The oscillator contributions (C.7) can be expressed by the use of Jacobi theta functions \(\vartheta\left[\begin{array}{c} \alpha \\ \beta \end{array}\right](t)\) and the Dedekind eta function \(\eta(t)\):

\[
\vartheta\left[\begin{array}{c} \alpha \\ \beta \end{array}\right](t) = \sum_{n \in \mathbb{Z}} q^{(\alpha + \alpha)^2} e^{2\pi i (\alpha + \beta)} , \quad \eta(t) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) , \quad \text{(C.8)}
\]

with \(q = e^{-2\pi t}\). Then the amplitudes are expressed as

\[
K^{(n_1, n_2)} = \frac{\vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right]}{\eta^3} \prod_{n_1v_1 + n_2w_1 \in \mathbb{Z}} \left( \vartheta\left[\begin{array}{c} n_1v_1 + n_2w_1 \\ 1/2 + n_1v_1 + n_2w_1 \end{array}\right] e^{\pi i (n_1v_1 + n_2w_1)} \right)
\]

\[
\times \prod_{n_1v_1 + n_2w_1 \in \mathbb{Z}} \left( \vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right] \right) , \quad \text{(C.9a)}
\]

\[
A^{(n_1, k_1)(n_2, k_2)} = \frac{\vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right]}{\eta^3} \prod_{(n_1v_1 + n_2w_1, k_1v_1 + k_2w_1) \in \mathbb{Z}^2} \left( \vartheta\left[\begin{array}{c} n_1v_1 + n_2w_1 \\ 1/2 + n_1v_1 + n_2w_1 \end{array}\right] e^{\pi i (n_1v_1 + n_2w_1)} \right)
\]

\[
\times \prod_{(n_1v_1 + n_2w_1, k_1v_1 + k_2w_1) \in \mathbb{Z}^2} \left( \vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right] \right) , \quad \text{(C.9b)}
\]

\[
M^{(n_1, k_1)(n_2, k_2)} = \frac{\vartheta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right]}{\eta^3} \prod_{(n_1v_1 + n_2w_1, k_1v_1 + k_2w_1) \in \mathbb{Z}^2} \left( \vartheta\left[\begin{array}{c} n_1v_1 + n_2w_1 \\ 1/2 + n_1v_1 + n_2w_1 \end{array}\right] e^{\pi i (n_1v_1 + n_2w_1)} \right)
\]

\[
\times \prod_{(n_1v_1 + n_2w_1, k_1v_1 + k_2w_1) \in \mathbb{Z}^2} \left( \vartheta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right] \right) . \quad \text{(C.9c)}
\]

Notice that except for the \(\mathbb{Z}_0\) orbifold the values \(K^{(n_1, k_1)(n_2, k_2)}\) are equal for any insertion of \(\theta^{k_1}\theta^{k_2}\), even though the lattice contributions differ [38]. Then we omit the label \(k_i\) in (C.9a). The arguments in the theta and eta functions are \(2t\) in the Klein bottle, \(t + \frac{i}{2}\) in the Möbius strip, and \(t\) in the annulus. Further, we used the notation [38], \(\langle x \rangle \equiv x - \lfloor x \rfloor - \frac{1}{2}\), where the brackets on the rhs denote the integer part and

\[
\delta = \begin{cases} 
1 & \text{if } (n_1v_i + n_2w_i, k_1v_i + k_2w_i) \in \mathbb{Z} \times \mathbb{Z} + \frac{1}{2} \\
0 & \text{otherwise}
\end{cases} \quad \text{(C.10)}
\]

The tree channel expressions \(\hat{K}, \hat{A}\) and \(\hat{M}\) can be evaluated with the help of the modular transformation of (C.8).
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