In this work we implement the Minimal Geometric Deformation method to obtain the isotropic sector and the decoupler matter content of any anisotropic solution of the Einstein field equations with cosmological constant in $2+1$ dimensional space–times. We obtain that the solutions of both sectors can be expressed analytically in terms of the metric functions of the original anisotropic solutions instead of formal integral as in its $3+1$ counterpart. As a particular example we study a regular black hole solution and we show that, depending on the sign of the cosmological constant, the solutions correspond to regular black holes violating the null energy condition or to a non–regular black hole without exotic hair. The exotic/non–exotic and the regular/non–regular black hole dualities are discussed.

I. INTRODUCTION

The interest in the Minimal Geometric Deformation (MGD) as a powerful method to decouple the Einstein field equations to obtain new solutions has considerably increased. Among the main applications of the method we find studies of local anisotropies in spherically symmetric systems, hairy black holes and new anisotropic solutions in $2+1$ dimensional space–times.

The method has been extended to solve the inverse problem, namely, given any anisotropic solution of the Einstein field equations it is possible to recover the isotropic sector and the decoupler matter content which, after gravitational interaction, led to the anisotropic configuration. In that work, it was found that, for an anisotropic solution with exotic matter sector (negative energy density), the free parameters involved in the MGD can be setted such that both the isotropic and the decoupler sectors satisfy all the energy conditions. It was the first time that a kind of exotic/non–exotic matter was found using the method.

As another extension of MGD and the inverse problem, in Ref. the method have been studied considering Einstein’s equations with cosmological constant and implemented in a polytropic black hole which is a solution with a matter content satisfying all the energy conditions. The main finding there was that the isotropic sector is deeply linked with the appearance of exotic matter, although it can be located inside the horizon. In this sense, the work shows again how the apparition of exotic matter seems unavoidable but one could, in principle, control the energy conditions by tuning the isotropy/anisotropy of a black hole solution.

The MGD-decoupling have been implemented in $2+1$ circularly symmetric and static spacetimes obtaining that both the isotropic and the anisotropic sector fulfill Einstein field equations in contrast to the cases studied in $3+1$ dimensions, where the anisotropic sector satisfies quasi-Einstein field equations. In this sense, the isotropic and the decoupler sector leads to a pair of new solution of Einsteins equations, one for each source.

In this work we study MGD in $2+1$ circularly symmetric and static spacetimes with different purposes and interests. First, as a difference to the previous work (see Ref.), we consider $2+1$ Einstein’s equation with cosmological constant. This is because, as the BTZ is a vacuum solution of this configuration, the set of equations coming from MGD method could serve as the starting point to extend interior $2+1$ solutions to anisotropic domains taking into account suitable matching conditions between the compact objects and a BTZ vacuum. Second, we study the inverse MGD problem to explore, among other aspects, the exotic/non–exotic matter content duality previously reported in the $3+1$ dimensional case. As we shall see later, the inverse method leads to more tractable expressions to deal with because they correspond to exact analytic instead to formal equations as previously reported.

This work is organized as follows. In the next section we briefly review the MGD-decoupling method. Then, in section we obtain the isotropic sector and the decoupler matter content considering a regular black hole as anisotropic solution. In section we study the energy conditions to explore the apparition of exotic matter in the solutions and some final comments and conclusion are in the last section.
Let us consider the Einstein field equations
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}^{\text{tot}},
\]  
(1)
and assume that the total energy-momentum tensor is given by
\[
T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^{(m)} + \theta_{\mu\nu},
\]  
(2)
As usual, the energy–momentum tensor for a perfect fluid is given by \(T_{\mu\nu}^{(m)} = \text{diag}(-\rho, p, p, p)\) and the decoupler matter content reads \(\theta_{\mu\nu} = \text{diag}(-\rho^\theta, p^\theta, p^\theta, p^\theta)\). In what follows, we shall assume circularly symmetric space–times with a line element parametrized as
\[
ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\phi^2,
\]  
(3)
where \(\nu\) and \(\lambda\) are functions of the radial coordinate \(r\) only. Considering Eq. (3) as a solution of the Einstein Field Equations, we obtain
\[
\kappa^2 \ddot{\rho} = -\Lambda + \frac{e^{\lambda'} - 1}{2r},
\]  
(4)
\[
\kappa^2 \ddot{p}_r = \Lambda + \frac{e^{\lambda'\nu'}}{2r},
\]  
(5)
\[
\kappa^2 \ddot{p}_\perp = \Lambda + \frac{1}{4} e^{-\lambda} (-\lambda'\nu' + 2\nu'' + \nu'^2),
\]  
(6)
where the prime denotes derivation with respect to the radial coordinate and we have defined
\[
\ddot{\rho} = \rho + \rho^\theta
\]  
(7)
\[
\ddot{p}_r = p + p^\theta
\]  
(8)
\[
\ddot{p}_\perp = p + p^\theta.
\]  
(9)
The next step consists in decoupling the Einstein Field Equations (4), (5) and (6) by implementing the minimal deformation
\[
e^{-\lambda} = \mu + \alpha f.
\]  
(10)
As usual, Eq. (10) leads to two sets of differential equations: one describing an isotropic system sourced by the conserved energy–momentum tensor of a perfect fluid \(T_{\mu\nu}^{(m)}\) and the other set corresponding to Einstein field equations sourced by \(\theta_{\mu\nu}\). Now, as in a previous work [35], we interpret the cosmological constant as an isotropic deformation: if \(\dot{c}_1 \to 0\) there is not deformation at all.

The matter content of the isotropic sector reads
\[
\rho = -\Lambda - \frac{c_1 e^{-\nu} (2\nu'' + \nu' (\nu' - 2))}{2\nu'^3}
\]  
(20)
\[
p = \Lambda - \frac{c_1 r e^{-\nu}}{2\nu'},
\]  
(21)
and the decoupler matter content satisfies
\[
\ddot{\rho}^\theta = \frac{1}{2} \left( \frac{c_1 e^{-\nu} (2\nu'' + \nu' (\nu' - 2))}{\nu'^3} + \frac{e^{-\lambda'\nu'}}{\alpha r} \right)
\]  
(22)
\[
\ddot{p}_r^\theta = \frac{c_1 r e^{-\nu} + e^{-\lambda'\nu'}}{2\nu'}
\]  
(23)
\[
\ddot{p}_\perp^\theta = \frac{1}{4} \left( \frac{2c_1 r e^{-\nu} + e^{-\lambda} (-\lambda'\nu' + 2\nu'' + \nu'^2)}{\nu'} \right)
\]  
(24)

At this point, some comments are in order. First, Eqs. (11), (12) and (13) correspond to Einstein field equations with cosmological constant for a perfect fluid. Second, Eqs. (14), (15) and (16) corresponds to Einstein field equations without cosmological constant and anisotropic decoupler fluid. Note that this set of equations together with those of the isotropic sector allows us to...
decouple Einstein’s equation with cosmological constant in 2 + 1 dimensional space–times for any anisotropic fluid. What is more, the above expressions can be used to extend isotropic solutions embedded in a BTZ vacuum to anisotropic domains after the implementation of suitable matching conditions. Finally, that the inverse problem leads to exact analytical expressions entails the “isotropization” of a broader set of systems than its 3 + 1 dimensional counterpart which is given in terms of formal integrals. In this sense, the inverse problem in 2 + 1 dimensions can be implemented starting from any anisotropic solutions at difference to the 3 + 1 case where depending on the particular form of the anisotropic solution, the inverse problem would yield to formal instead exact analytical solutions. In the next section we shall implement the inverse problem using a well known regular and circularly symmetric black hole solution as anisotropic system.

III. ISOTROPIC SECTOR OF A REGULAR BLACK HOLE IN 2+1 DIMENSIONS

In this section we shall illustrate the inverse MGD problem using as anisotropic solution a regular black hole metric. The reason to consider a regular black hole solution is twofold: to explore the conditions for the apparition of exotic matter and to study if the MGD inverse problem can affect the regularity of the solution. Let us consider the black hole solution \([40]\) with metric functions

\[
e^{\nu} = -M - \Lambda r^2 - q^2 \log (a^2 + r^2) \quad \text{(25)}
\]

\[
e^{-\lambda} = -M - \Lambda r^2 - q^2 \log (a^2 + r^2) \quad \text{(26)}
\]

where \(M, \Lambda, a\) and \(q\) are free parameters. This geometry is sustained by a matter content given by

\[
\rho = \frac{q^2}{8\pi (a^2 + r^2)} \quad \text{(27)}
\]

\[
\rho_r = \frac{-q^2}{8\pi (a^2 + r^2)} \quad \text{(28)}
\]

\[
\rho_\perp = \frac{q^2 (r^2 - a^2)}{8\pi (a^2 + r^2)^2} \quad \text{(29)}
\]

This solution corresponds to a black hole whenever \(-M - \Lambda r^2 - q^2 \log (a^2 + r^2) = 0\) leads to two real roots (or one real root in the extreme case) for some values of the parameters \(\{M, \Lambda, a, q\}\) \([40]\). What is more, the solution is regular everywhere, which can be deduced from the invariants

\[
R = \frac{2q^2 (3a^2 + 2r^2)^2}{(a^2 + r^2)^2} + 6\Lambda \quad \text{(30)}
\]

\[
Ricc = K = \frac{4q^4 (3a^4 + 2a^2r^2 + r^4)}{(a^2 + r^2)^4} + \frac{8\Lambda q^2 (3a^2 + 2r^2)^2}{(a^2 + r^2)^2} + 12\Lambda^2, \quad \text{(31)}
\]

where \(R, Ricc\) and \(K\) correspond to the Ricci, Ricci squared and Kretschmann scalar respectively. From now on we shall apply the inverse MGD problem to obtain the isotropic generator and the decoupler matter content associated with this regular black hole solution. From Eq. \(18\), the decoupler function \(f\) reads

\[
f = -\frac{c_1 \left( a^2 + r^2 \right)^2 \left( q^2 \log (a^2 + r^2) + M + \Lambda r^2 \right)}{4 \left( \Lambda (a^2 + r^2) + q^2 \right)^2} - \frac{q^2 \log (a^2 + r^2) + M + \Lambda r^2}{\alpha} \quad \text{(32)}
\]

Now, from Eq. \(19\), the radial metric function of the isotropic sector is given by

\[
\mu = \frac{\alpha c_1 \left( a^2 + r^2 \right)^2 (q^2 \log (a^2 + r^2) + M + \Lambda r^2)}{4 \left( \Lambda (a^2 + r^2) + q^2 \right)^2} \quad \text{(33)}
\]

Replacing the above result in the set of isotropic Einstein equations, Eqs. \(11\), \(12\) and \(13\), the perfect fluid reads

\[
\rho = -\frac{\alpha c_1 a_r^2 \left( 2Mq^2 (a^2 + r^2) + \Lambda a_r^2 + 2Mq^2 + q^4 \right)}{32\pi \lambda_r^3} - \frac{2\alpha c_1 q^4 a_r^2 \log (a^2 + r^2) + 4\Lambda a_r^3}{32\pi \lambda_r^3} \quad \text{(34)}
\]

\[
p = \frac{\alpha c_1 (a^2 + r^2)}{32\pi \left( \Lambda (a^2 + r^2) + q^2 \right)} + \frac{\Lambda}{8\pi} \quad \text{(35)}
\]

where

\[
a_r^3 := a^2 + r^2 \quad \text{(36)}
\]

\[
\Lambda_r := \Lambda a_r^2 + q^2 \quad \text{(37)}
\]

At this point some comments are in order. First, note that, as in the 3 + 1 case, the inverse problem does not modify the position of the killing horizon. In fact, the horizon appears whenever \(-M - \Lambda r^2 - q^2 \log (a^2 + r^2) = 0\) which, as discussed above, leads to one or two real roots for the black hole solution. Second, the regularity of the solution depends on the positivity of the parameter \(\Lambda\). In fact, the invariants

\[
R = -\alpha c_1 a_r^2 \left( \frac{2Mq^2 + 3q^4}{2\Lambda_r^2} - \frac{2q^4 \log a_r^2}{2\Lambda_r^2} \right)
\]

\[
- \frac{2\Lambda q^2 (3a^2 + 4r^2)}{2\Lambda_r^3} - \frac{3\Lambda^2 a_r^3}{2\Lambda_r^3} \quad \text{(38)}
\]

\[
Ricc = K = \frac{\alpha^2 c_2 a_r^4 (\mathcal{F} + 2\Lambda_r^2)}{4\Lambda_r^6} \quad \text{(39)}
\]

where

\[
\mathcal{F} = \left( 2q^4 \log a_r^2 + 2\Lambda q^2 a_r^2 + \Lambda^2 a_r^4 + 2Mq^2 + q^4 \right)^2, \quad \text{(40)}
\]

reveal that the solution is regular everywhere whenever \(\Lambda_r = \Lambda (a^2 + r^2) + q^2 \neq 0\), which can be satisfied if \(\Lambda > 0\).
In particular, from the horizon condition we can obtain the condition for \( \Lambda > 0 \) in terms of the other parameters as
\[
\Lambda = -\frac{M + q^2 \log(a^2 + r_H^2)}{r_H^2} \tag{41}
\]
from where it must be imposed that \( r_H^2/a^2 < 1 \) and \( q^2 \log(a^2 + r_H^2) > M \), with \( r_H \) the horizon radius. In this case, the solution corresponds to a regular isotropic black hole solution.

Note that, in the case \( \Lambda < 0 \) the solution has a critical radius \( r_c \) as frequently found in the application of MGD. This result would lead to a naked singularity for \( r_c > r_H \) or to a non regular black hole solution for \( r_c < r_H \).

In the last case, we say that the isotropic sector of the regular black hole corresponds to a non-regular black hole solution.

Now we focus our attention into the decoupler sector. In this case, the metric functions are \( \{\nu, f\} \) and the decoupler matter content reads
\[
\rho^0 = \frac{c_1 a^2_r (2\Lambda q^2 (a^2 + r^2) + \Lambda^2 a^4_r + 2M q^2 + q^4)}{4A^3 r} + \frac{q^2}{aa^2 r} + \frac{c_1 q^4 a^2_r \log a^2_r + \Lambda}{2A^3 r} + \frac{\Lambda}{\alpha} \tag{42}
\]
\[
p^r_\nu = \frac{-c_1 a^2_r}{4A^2 r} - \frac{q^2}{aa^2 r} + \frac{\Lambda}{\alpha} \tag{43}
\]
\[
p^r_\lambda = \frac{q^2 (r^2 - a^2)}{aa^2 r} - \frac{c_1 a^2_r}{4A^2 r} - \frac{\Lambda}{\alpha} \tag{44}
\]
The above solution corresponds to an anisotropic regular black hole solution for \( \Lambda > 0 \). In fact, the solution has a killing horizon when \( -M - \Lambda r^2 - q^2 \log (a^2 + r^2) = 0 \).

The invariants are given by
\[
R = \frac{\mathcal{H}_1}{(a^2 + r^2)^2 (\Lambda (a^2 + r^2) + q^2)^3} \tag{45}
\]
\[
Ricc = \frac{\mathcal{H}_2}{(a^2 + r^2)^4 (\Lambda (a^2 + r^2) + q^2)^6} \tag{46}
\]
\[
\mathcal{K} = \frac{\mathcal{H}_3}{(a^2 + r^2)^4 (\Lambda (a^2 + r^2) + q^2)^6} \tag{47}
\]
where \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \) are (too long) regular functions in terms of polynomials of \( r \) and \( \log(a^2 + r^2) \). Note that, as discussed above, the regularity of the solution depends on the sign of \( \Lambda \). More precisely, for \( \Lambda < 0 \) the solution is the one of a non-regular black hole solution.

It is worth mentioning that the solutions obtained here could be considered as “hairy” black holes and that the nature of such a hair fields depends on the energy conditions that we shall discuss in what follows.

### IV. ENERGY CONDITIONS

In this section we study the energy conditions of the obtained solution for the distinct cases outlined before.

#### A. Case I. \( \Lambda > 0 \)

As previously discussed, this case corresponds to regular black hole solutions for the isotropic and decoupler sector with the horizon radius located at \( -M - \Lambda r_H^2 - q^2 \log(a^2 + r_H^2) = 0 \). Given the nature of the solution, a numerical analysis is mandatory. However, setting suitable values of \( M, \Lambda, q, a \) to obtain real horizons and \( \Lambda > 0 \), we infer that the behaviour of the energy density can be written as
\[
\lim_{r \to 0} \rho = A\alpha c_1 - B \tag{48}
\]
\[
\lim_{r \to 0} \rho = -C\alpha c_1 - B \tag{49}
\]
with \( A, B, C \) real and positive numbers. In particular, for \( \Lambda = 2, M = 1, q = 1 \) and \( a = 0.1 \) we obtain
\[
\lim_{r \to 0} \rho = 0.00578336\alpha c_1 - 0.0795775 \tag{50}
\]
\[
\lim_{r \to 0} \rho = -0.00497359\alpha c_1 - 0.0795775 \tag{51}
\]

Note that the apparition of exotic matter is unavoidable. In fact, if \( \alpha c_1 \) is a positive (negative) quantity such that \( \lim \rho > 0 \) (\( \lim \rho < 0 \)), it is obtained that \( \lim \rho > 0 \) (\( \lim \rho < 0 \)) necessarily. In figure 1 we show the profile of the energy density for some values of \( \alpha c_1 \).

![FIG. 1: Energy density for for \( \alpha c_1 = -200 \) (black solid line), \( \alpha c_1 = -140 \) (dashed blue line) \( \alpha c_1 = 400 \) (short dashed red line) and \( \alpha c_1 = 500 \) (dotted green line).](image)

For the decoupler sector we obtain that the negative values for the energy density can be avoided with a suitable choice of the parameter. Without loss of generality, let us set \( \Lambda = 2, M = 1, q = 1 \) and \( a = 0.1 \) to ensure that the horizon radius is real and positive. Now
\[
\lim_{r \to 0} \rho^\theta = \frac{102}{\alpha} - 0.0145352 c_1 \tag{52}
\]
\[
\lim_{r \to 0} \rho^\theta = \frac{2}{\alpha} + \frac{c_1}{8} \tag{53}
\]
Note that for $\alpha$ and $c_1$ positive values, the exotic matter content can be avoided whenever $c_1 \leq \frac{701746}{\alpha}$. In figure 2 we show the profile of $\rho^\theta$ for $\alpha = 1$ and some values for $c_1$.

![FIG. 2: Energy density for for $c_1 = 1$ (black solid line), $c_1 = 10$ (dashed blue line) $c_1 = 50$ (short dashed red line) and $c_1 = 100$ (dotted green line).](image)

We would like to conclude this section by emphasizing that for $\Lambda > 0$ the exotic matter can be partially avoided. In fact, while the perfect fluid solution contain negative energy density for suitable values of $M, q, \Lambda, a$, the apparition of exotic matter in the decoupler sector can be circumvented for certain values of the parameters involved. In the next section we shall study the energy conditions for $\Lambda < 0$.

B. Case II. $\Lambda < 0$

In this case, $\Lambda < 0$ as long as $r_H^2 + a^2 < 1$ and $q^2 |\log(a^2 + r_H^2)| > M$ or $r_H^2 + a^2 > 1$. For example, we can choose the values $M = 2, q = 1, a = 1$ from where $\Lambda = -0.902359$. For these values, the horizon radius is located at $r_H = 2$ and the critical radius is at $r_c = 0.328946$ such that the solution corresponds to a non–regular black hole solution. For this particular choosing of the parameters and for $\alpha c_1 > 0$, the asymptotic behaviour of the energy density is given by

$$\lim_{r \to r_c} \rho = \infty$$

$$\lim_{r \to \infty} \rho = 0.0110235\alpha c_1 + 0.0359037.$$  (55)

In figure in figure 3 we show the profile of $\rho$ as a function of the radial coordinate for different values of $\alpha c_1 > 0$.

![FIG. 3: Energy density for for $c_1 = 0.1$ (black solid line), $c_1 = 0.2$ (dashed blue line) $c_1 = 0.5$ (short dashed red line) and $c_1 = 1$ (dotted green line).](image)

Now, let us turn out our attention in the decoupler sector. For $M = 2, q = 1, a = 1$ and $\Lambda = -0.902359$ we obtain

$$\lim_{r \to \infty} \rho^\theta = -\frac{0.902359}{\alpha} - 0.277051 c_1.$$  (56)

so that for $\alpha < 0, c_1 < 0$ the energy density reach a positive value asymptotically. In particular for $\alpha = -1$ we obtain that

$$\lim_{r \to r_c} \rho^\theta \to \infty$$  (57)

and we obtain that the exotic matter can be avoided. In figure 4 we show the profile of the energy density $\rho^\theta$ for $\alpha = -1$ and different values of $c_1$.

At his point a couple of comments are in order. First, note that in both cases (isotropic and decoupler sector) the exotic content can be avoided. Second, for suitable choices of the parameters, the solution corresponds to a non–regular black hole containing a non–vanishing critical radius. In this sense, we conclude that although the exotic/non–exotic duality can be circumvented for $\Lambda < 0$, the nature of the black hole solution of the isotropic and decoupler sector leads to a kind of regular/non–regular duality.
V. CONCLUSIONS

In this work we have extended the Minimal Geometric Deformation method in 2 + 1 dimensional space–times to decouple the Einstein field equations including cosmological constant. We obtained that the isotropic sector obeys Einstein’s equation with cosmological constant but the decoupler part consists in a system without cosmological term. In this sense, we can combine any 2 + 1 isotropic, static and circularly symmetric interior solution of the Einstein field equations with cosmological constant embedded in a BTZ vacuum with certain decoupler matter solution and suitable matching conditions to obtain new anisotropic interior solutions in the three dimensional realm.

We showed that the inverse problem leads to exact analytical solutions for the decoupling and the isotropic metric in terms of the original anisotropic solution instead to formal integrals obtained in the 3 + 1 counterpart. The scope of this result to obtain analytical solutions is broad. Indeed, it can be implemented taking into account any anisotropic solution as the starting point because it does not involve formal integrals as the 3 + 1 case. As a particular example we implemented the inverse problem to a regular 2 + 1 black hole solution. We obtain that for a positive cosmological constant the isotropic sector corresponds to a regular isotropic black hole in presence of a “exotic” hair (negative energy density), and the decoupler sector is a regular anisotropic black hole which, under certain circumstances, can be surrounded by a matter content with positive energy density so that the apparition of exotic matter can be avoided. For negative cosmological constant both the isotropic and the decoupler sector corresponds to non–regular black hole solution where the existence of exotic hair can be avoided with a suitable choice of the free parameters. It is worth mentioning that on one hand the non–regular black hole solution is singular in a non–vanishing radius as often occur in the implementation of the Minimal Geometric Deformation protocol. On the other hand, the exotic matter can be avoided but the price that it has to be paid is that the the solutions are not regular anymore. In this sense, the kind of exotic/non-exotic matter duality appearing in previous works transmute to a regular/non-regular duality in the cases where the exotic content can be avoided.

VI. ACKNOWLEDGEMENT

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