On the differential invariants of a family of diffusion equations

M. Torrisi † R. Tracinà

Department of Mathematics and Computer Science
University of Catania
Viale A. Doria, 6, 95125 Catania, Italy
E-mail: torrisi@dmi.unict.it and tracina@dmi.unict.it

March 2, 2022

Abstract

The equivalence transformation algebra $L_\xi$ for the class of equations $u_t - u_{xx} = f(u, u_x)$ is obtained. After getting the differential invariants with respect to $L_\xi$, some results which allow to linearize a subclass of equations are showed. Equations like the standard deterministic KPZ equation fall in this subclass.

PACS: 02.30.Jr; 02.20.Tw

Keywords: Nonlinear diffusion equations, equivalence transformation, differential invariants.
1. Introduction

In some previous papers [1], [2], [3] the differential invariants for the family of equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$ have been obtained and applied in order to characterize some linearizable subclasses of that equations.

Here, we consider the following diffusion equations:

$$u_t - u_{xx} = f(u, u_x),$$

which arise in several problems of mathematical physics.

By using the invariance Lie infinitesimal criterion [4], we construct the algebra $L_E$ of the equivalence transformations. These transformations have the property to change any element of a family of PDEs to a PDE which belongs to the same family. An equivalence transformation maps solutions of an equation of the family to solutions of the transformed equation.

Following the method proposed by N.H. Ibragimov in [5], [6] and successively applied in [7], we calculate the differential invariants with respect to the equivalence transformations of the family (1.1).

Starting from these results, we characterize a subclass of equations (1.1) which can be linearized through an equivalence transformation. In this subclass falls the standard deterministic Kardar-Parisi-Zhang (KPZ) equation [8], [9] which models the relaxation of an initially rough surface to a flat surface.

The outline of the paper is the following. In Section 2, we obtain the infinitesimal equivalence generator of equations (1.1). In Section 3, we look for differential invariants and, by following the infinitesimal method [5], [6], we show that the family of equations (1.1) does not admit differential invariants of order zero and one, while second order differential invariants are found. Finally, in Section 4, these last ones are used in order to characterize a subclass of the family (1.1) which can be mapped, by an equivalence transformation, in the Fourier’s equation. The conclusions are given in Section 5.

2. Algebra $L_E$

We recall that a transformation of the type

$$t = t(\tau, \sigma, v), \quad x = x(\tau, \sigma, v), \quad u = u(\tau, \sigma, v),$$

(2.1)
which is locally a $C^\infty$-diffeomorphism and changes the original equation into a new equation having the same differential structure but a different form of the function $f$, is an equivalence transformation [4] (hereafter ET) for the equations (1.1). An invariance transformation can be regarded as particular ET such that the transformed function $f$ has the same form. In the following we consider only continuous groups of equivalence transformations.

The direct search for the equivalence transformations through the finite form of the transformation is connected with considerable computational difficulties. The use of the Lie infinitesimal criterion, suggested by Ovsiiannikov [4], gives an algorithm to find the infinitesimal generators of the ETs that overcame these problems.

In order to obtain a continuous group of ETs of equations (1.1), we consider, by following [4], the arbitrary function $f$ as a dependent variable and apply the Lie infinitesimal invariance criterion to the following system:

$$
\begin{align*}
  u_t - u_{xx} &= f, \\
  f_t &= f_x = f_{uu} = 0,
\end{align*}
$$

where the last three equations of (2.2) are usually called auxiliary equations and give the independence of $f$ on $t$, $x$ and $u_t$.

The infinitesimal equivalence generator $Y$ has the following form:

$$
Y = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \mu \frac{\partial}{\partial f},
$$

where $\xi, \xi$ and $\eta$ are sought depending on $t, x, u, u_t, u_x$ and $f$, while $\mu$ depends on $t, x, u, u_t, u_x$ and $f$, the components $\zeta_1$ and $\zeta_2$, as known, are given by

$$
\begin{align*}
  \zeta_1 &= D_t(\eta) - u_tD_t(\xi^1) - u_xD_t(\xi^2), \\
  \zeta_2 &= D_x(\eta) - u_tD_x(\xi^1) - u_xD_x(\xi^2).
\end{align*}
$$

The operators $D_t$ and $D_x$ denote the total derivatives with respect to $t$ and $x$:

$$
\begin{align*}
  D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + ..., \\
  D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + ....
\end{align*}
$$
The prolongation of operator (2.3), which we need in order to require the invariance of (2.2), is

\[ \tilde{Y} = Y + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \omega_t \frac{\partial}{\partial f_t} + \omega_x \frac{\partial}{\partial f_x} + \omega_{ut} \frac{\partial}{\partial f_{ut}}, \]  

(2.7)

where (see e.g. [7], [10])

\[ \zeta_{22} = D_x(\zeta_2) - u_{tt}D_x(\xi^1) - u_{xx}D_x(\xi^2), \]  

(2.8)

\[ \omega_t = \tilde{D}_t(\mu) - f_u \tilde{D}_t(\eta) - f_{ut} \tilde{D}_t(\zeta_2), \]  

(2.9)

\[ \omega_x = \tilde{D}_x(\mu) - f_u \tilde{D}_x(\eta) - f_{ux} \tilde{D}_x(\zeta_2), \]  

(2.10)

\[ \omega_{ut} = \tilde{D}_{ut}(\mu) - f_u \tilde{D}_{ut}(\eta) - f_{utx} \tilde{D}_{ut}(\zeta_2), \]  

(2.11)

while \( \tilde{D}_t, \tilde{D}_x \) and \( \tilde{D}_{ut} \) are defined by:

\[ \tilde{D}_t = \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} + f_{ut} \frac{\partial}{\partial f_{ut}} + ..., \]  

(2.12)

\[ \tilde{D}_x = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} + f_{tx} \frac{\partial}{\partial f_t} + f_{xx} \frac{\partial}{\partial f_x} + ...., \]  

(2.13)

\[ \tilde{D}_{ut} = \frac{\partial}{\partial u_t} + f_{ut} \frac{\partial}{\partial f} + f_{utx} \frac{\partial}{\partial f_t} + f_{xut} \frac{\partial}{\partial f_x} + .... \]  

(2.14)

Applying the operator (2.7) to the system (2.2) and following the well known algorithm (see e.g. [10], [11]) we obtain

\[ Y = (c_0 + c_1 t) \frac{\partial}{\partial t} + \left( \frac{1}{2} c_1 x + c_2 t + c_3 \right) \frac{\partial}{\partial x} + \varphi(u) \frac{\partial}{\partial u} + \]  

\[ + (-c_1 u_t - c_2 u_x + \varphi' u_t) \frac{\partial}{\partial u_t} + \left( -\frac{1}{2} c_1 u_x + \varphi' u_x \right) \frac{\partial}{\partial u_x} + \]  

\[ + \left( -c_1 f - c_2 u_x + \varphi' f - \varphi'' u_x^2 \right) \frac{\partial}{\partial f}, \]  

(2.15)

where \( c_0, c_1, c_2 \) and \( c_3 \) are arbitrary constants, \( \varphi \) is an arbitrary function of \( u \) and the prime denotes the differentiation with respect to \( u \). So, we have found that the Lie algebra \( L_\mathcal{E} \) for the class of equations (1.1) is infinite-dimensional and generates an infinite continuous group \( G_\mathcal{E} \) of equivalence transformations spanned by the following operators:
\[ Y_0 = \frac{\partial}{\partial t}, \quad Y_1 = t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x} - f \frac{\partial}{\partial f} - u_t \frac{\partial}{\partial u_t} - \frac{1}{2} u_x \frac{\partial}{\partial u_x}, \]

\[ Y_2 = t \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial f} - u_x \frac{\partial}{\partial u_t}, \quad Y_3 = \frac{\partial}{\partial x}, \]

\[ Y_\varphi = \varphi \frac{\partial}{\partial u} + \left( \varphi' f - \varphi'' u_x^2 \right) \frac{\partial}{\partial f} + \varphi' u_t \frac{\partial}{\partial u_t} + \varphi' u_x \frac{\partial}{\partial u_x}. \]

3. Search for differential invariants

Following [5]-[7], we recall that, for the family of equations (1.1), a differential invariant of order s is a function \( J \), of the independent variables \( t, x \), the dependent variable \( u \) and its derivatives \( u_t, u_x \), as well as of the function \( f \) and its derivatives of maximal order \( s \), invariant with respect to the equivalence group \( G_E \).

3.1 Differential invariants of order zero.

Here we search for functions

\[ J = J(t, x, u, u_t, u_x, f) \]

satisfying the invariant condition \( Y(J) = 0 \).

From the invariant tests \( Y_0(J) = 0 \) and \( Y_3(J) = 0 \), easily follows that \( J \) must depend only on \( u, u_t, u_x \) and \( f \).

From invariance test \( Y_\varphi(J) = 0 \), after observing that, being \( \varphi \) an arbitrary function, \( Y_\varphi \) can be splitted in the following three operators

\[ \hat{Y}_\varphi = \frac{\partial}{\partial u}, \quad \hat{Y}_\varphi' = f \frac{\partial}{\partial f} + u_t \frac{\partial}{\partial u_t} + u_x \frac{\partial}{\partial u_x}, \quad \hat{Y}_\varphi'' = -u_x^2 \frac{\partial}{\partial f}, \]

it is a simple matter to get

\[ J = J(q), \]

with

\[ q = \frac{u_t}{u_x}. \]

From \( Y_2(J) = 0 \) we get \( J_q = 0 \), hence the equations (1.1) do not possess differential invariants of zero order.
3. 2 Differential invariants of first order.

The differential invariants of first order involve \( f_u \) and \( f_{ux} \) also, so we need the following first prolongation of operator \( Y \):

\[
Y^{(1)} = Y + \omega_u \frac{\partial}{\partial f_u} + \omega_{ux} \frac{\partial}{\partial f_{ux}},
\]

(3.4)

which, after computing \( \omega_u \) and \( \omega_{ux} \) likewise (2.9)- (2.14), can be written as:

\[
Y^{(1)} = Y + \left( -c_1 f_u + \varphi'' f - \varphi'' u_x f_{ux} - \varphi''' u_x^2 \right) \frac{\partial}{\partial f_u} + \\
- \left( \frac{1}{2} c_1 f_{ux} + c_2 + 2 \varphi'' u_x \right) \frac{\partial}{\partial f_{ux}}.
\]

(3.5)

By observing that:

\[
Y_0^{(1)} = Y_0, \quad Y_3^{(1)} = Y_3, \quad \hat{Y}_\varphi^{(1)} = \hat{Y}_\varphi
\]

it is a simple matter to ascertain that at this step the search is reduced to look for invariant functions of the form

\[
J = J(u_t, u_x, f, f_u, f_{ux})
\]

(3.6)

with respect to the following operators

\[
Y_1^{(1)} = t \frac{\partial}{\partial t} + \frac{1}{2} f_u \frac{\partial}{\partial f_u} - f \frac{\partial}{\partial f} - u_t \frac{\partial}{\partial u_t} - \frac{1}{2} u_x \frac{\partial}{\partial u_x} - f_u \frac{\partial}{\partial f_u} + \\
- \frac{1}{2} f_{ux} \frac{\partial}{\partial f_{ux}},
\]

(3.7)

\[
Y_2^{(1)} = t \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial f} - u_x \frac{\partial}{\partial u_t} - \frac{\partial}{\partial f_{ux}},
\]

(3.8)

\[
\hat{Y}_\varphi^{(1)} = f \frac{\partial}{\partial f} + u_t \frac{\partial}{\partial u_t} + u_x \frac{\partial}{\partial u_x},
\]

(3.9)

\[
\hat{Y}_{\varphi''}^{(1)} = -u_x^2 \frac{\partial}{\partial f} + (f - u_x f_{ux}) \frac{\partial}{\partial f_u} - 2 u_x \frac{\partial}{\partial f_{ux}},
\]

(3.10)

\[
\hat{Y}_{\varphi'''}^{(1)} = -u_x^2 \frac{\partial}{\partial f_u}.
\]

(3.11)
By requiring the invariance of \( J \) with respect to the operator \( \hat{Y}^{(1)} \) it follows
\[
J = J(u_t, u_x, f, f_{ux}),
\]
while the invariant test, applied to (3.12)
\[
\hat{Y}^{(1)}(J) \equiv u_x \frac{\partial J}{\partial f} + 2 \frac{\partial J}{\partial f_{ux}} = 0,
\]
yields that
\[
J = J(u_t, u_x, p_1),
\]
where
\[
p_1 = \frac{f}{u_x} - \frac{f_{ux}}{2}.
\]
Acting by operator \( \hat{Y}^{(1)} \) on the invariant (3.14), one obtains that
\[
\hat{Y}^{(1)}(J) \equiv u_t \frac{\partial J}{\partial u_t} + u_x \frac{\partial J}{\partial u_x} = 0
\]
and hence
\[
J = J(p_1, p_2),
\]
with
\[
p_2 = \frac{u_t}{u_x}.
\]
Finally, from the invariant test \( Y^{(1)}_2(J) = 0 \) we get
\[
J = J(p),
\]
where
\[
p = p_1 - 2p_2 = \frac{f - u_x f_{ux}}{2u_x} - 2 \frac{u_t}{u_x},
\]
and from \( Y^{(1)}_1(J) = 0 \) it follows
\[
Y^{(1)}_1(J) \equiv \left( -\frac{1}{2} \frac{f}{u_x} + \frac{u_t}{u_x} + \frac{1}{4} f_{ux} \right) \frac{\partial J}{\partial p} = 0.
\]
Provided that \(-\frac{1}{2} \frac{f}{u_x} + \frac{u_t}{u_x} + \frac{1}{4} f_{ux} \neq 0\), we get \( \frac{\partial J}{\partial p} = 0 \). Hence the equations (1.1) do not admit differential invariants of first order.
3. 3 Differential invariants of second order.

In the search for second order differential invariants, because of the function $J$ is sought as depending from $f_{uu}$, $f_{ux}$, and $f_{ux^2}$ also, we need the following second prolongation of operator $Y$:

$$Y^{(2)} = Y^{(1)} + \omega_{uu} \frac{\partial}{\partial f_{uu}} + \omega_{ux} \frac{\partial}{\partial f_{ux}} + \omega_{ux^2} \frac{\partial}{\partial f_{ux^2}}$$  (3.22)

where [7], [10]:

$$\omega_{uu} = \tilde{D}_u(\omega_u) - f_{uu} \tilde{D}_u(\eta) - f_{ux} \tilde{D}_u(\zeta_2),$$  (3.23)

$$\omega_{ux} = \tilde{D}_u(\omega_u) - f_{uu} \tilde{D}_u(\eta) - f_{ux} \tilde{D}_u(\zeta_2),$$  (3.24)

$$\omega_{ux^2} = \tilde{D}_u(\omega_{ux}) - f_{uu} \tilde{D}_u(\eta) - f_{ux^2} \tilde{D}_u(\zeta_2).$$  (3.25)

After some calculations we get

$$Y^{(2)} = Y^{(1)} - \left( \varphi' f_{ux^2} + 2 \varphi'' \right) \frac{\partial}{\partial f_{ux^2}} +$$

$$- \left( \frac{1}{2} c_1 f_{ux^2} + \varphi' f_{ux} + \varphi'' u_x f_{ux} - 2 \varphi''' u_x \right) \frac{\partial}{\partial f_{ux}} +$$

$$+ \left[ -c_1 f_{uu} - \varphi' f_{uu} + \varphi'' (f - u_x f_{ux}) + \varphi''' (f - u_x f_{ux}) - \varphi'''' \frac{u_x^3 f_{uu}^2}{f_{uu}} \right] \frac{\partial}{\partial f_{uu}}.$$  (3.26)

By observing that:

$$Y_0^{(2)} = Y_0, \quad Y_3^{(2)} = Y_3, \quad \hat{Y}_\varphi^{(2)} = \hat{Y}_\varphi$$

we ascertain that we must look for invariant functions of the form

$$J = J(u_t, u_x, f, f_u, f_{ux}, f_{uu}, f_{ux^2}, f_{ux^2})$$  (3.27)

which are invariant with respect to the following operators:

$$Y_1^{(2)} = Y_1^{(1)} - f_{uu} \frac{\partial}{\partial f_{uu}} - \frac{1}{2} f_{ux^2} \frac{\partial}{\partial f_{ux^2}},$$  (3.28)

$$Y_2^{(2)} = Y_2^{(1)},$$  (3.29)

$$\hat{Y}_\varphi^{(2)} = Y_\varphi^{(1)} - f_{uu} \frac{\partial}{\partial f_{uu}} - f_{ux} \frac{\partial}{\partial f_{ux}} - f_{ux^2} \frac{\partial}{\partial f_{ux^2}},$$  (3.30)
\[ \hat{Y}^{(2)}_{\varphi''} \varphi'' = Y^{(1)}_{\varphi'} + (f_u - 2u_x f_{uu}x) \frac{\partial}{\partial f_{uu}} - u_x f_{uu}x \frac{\partial}{\partial f_{uu}x} - 2 \frac{\partial}{\partial f_{uu}x} \]  \hspace{1cm} (3.31)

\[ \hat{Y}^{(2)}_{\varphi'''} \varphi''' = Y^{(1)}_{\varphi''} + (f - u_x f_{ux}) \frac{\partial}{\partial f_{uu}} - 2u_x \frac{\partial}{\partial f_{uu}x} \]  \hspace{1cm} (3.32)

\[ \hat{Y}^{(2)}_{\varphi IV} \varphi IV = -u_x^2 \frac{\partial}{\partial f_{uu}}. \]  \hspace{1cm} (3.33)

The invariance condition \( \hat{Y}^{(2)}_{\varphi'''} (J) = 0 \) implies
\[ J = J(u_t, u_x, f, f_u, f_{ux}, f_{uu}x, f_{ux}u_x), \]  \hspace{1cm} (3.34)
while \( \hat{Y}^{(2)}_{\varphi''} (J) = 0 \) yields
\[ J = J(u_t, u_x, f, f_{ux}, f_{u}x_{u}x; p_{1}), \]  \hspace{1cm} (3.35)
where
\[ p_{1} = \frac{f_{u}}{u_{x}} - \frac{f_{uu}x}{2}. \]  \hspace{1cm} (3.36)

Likewise, from \( Y^{(2)}_{2} (J) = 0 \) we obtain
\[ J = J(u_x, f_{ux}u_x, p_{1}, p_{2}, p_{3}), \]  \hspace{1cm} (3.37)
where
\[ p_{2} = f - u_t, \quad p_{3} = \frac{f}{u_{x}} - f_{ux}, \]  \hspace{1cm} (3.38)
and from \( \hat{Y}^{(2)}_{\varphi'} (J) = 0 \),
\[ J = J(q_{1}, q_{2}, q_{3}, q_{4}), \]  \hspace{1cm} (3.39)
where
\[ q_{1} = f_{ux}u_x (f - u_t), \quad q_{2} = \left( \frac{f_{u}}{u_{x}} - \frac{f_{uu}x}{2} \right) (f - u_t), \]  \hspace{1cm} (3.40)
\[ q_{3} = p_{3} = \frac{f}{u_{x}} - f_{ux}, \quad q_{4} = u_x f_{ux}u_x. \]  \hspace{1cm} (3.41)

By applying the operator \( \hat{Y}^{(2)}_{\varphi''} \) to the differential invariant given by (3.39), taking into account (3.40-3.41), we get
\[ \left( -\frac{2q_{1}}{q_{4}} \frac{\partial J}{\partial q_{1}} + \left( \frac{1}{2} q_{1} - \frac{q_{2}q_{4}}{q_{1}} + \frac{q_{1}q_{3}}{q_{4}} \right) \frac{\partial J}{\partial q_{2}} + \frac{\partial J}{\partial q_{3}} - 2 \frac{\partial J}{\partial q_{4}} \right) = 0. \]  \hspace{1cm} (3.42)
The corresponding characteristic equations give

\[ J = J(r_1, r_2, r_3) , \] 

(3.43)

where

\[ r_1 = \frac{f}{u_x} - f u_x + \frac{1}{2} u_x f u_x u_x , \] 

(3.44)

\[ r_2 = f u - \frac{1}{2} u_x f u_x + \frac{1}{2} f f u_x u_x - \frac{1}{2} u_x f u_x f u_x u_x + \frac{1}{4} u_x^2 f u_x^2 u_x , \] 

(3.45)

\[ r_3 = \frac{f - u_t}{u_x} - u_x f u_x . \] 

(3.46)

Finally, the invariant test

\[ Y_1^{(2)}(J) = 0 , \] 

(3.47)

after some calculations, yields

\[ r_1 \frac{\partial J}{\partial r_1} + 2 r_2 \frac{\partial J}{\partial r_2} + r_3 \frac{\partial J}{\partial r_3} = 0 . \] 

(3.48)

From the corresponding characteristic equations, provided that

\[ 2 f - 2 u_t - u_x^2 f u_x u_x \neq 0 , \] 

(3.49)

we get that the general form of second order differential invariants of equation (1.1) is

\[ J = J(\lambda_1, \lambda_2) , \] 

(3.50)

with \( \lambda_1 \) and \( \lambda_2 \) given by

\[ \lambda_1 = \frac{2 f - 2 u_x f u_x + u_x^2 f u_x u_x}{2 f - 2 u_t - u_x^2 f u_x u_x} , \] 

(3.51)

\[ \lambda_2 = \frac{(4 f u - 2 u_x f u_x + 2 f f u_x u_x - 2 u_x f u_x f u_x u_x + u_x^2 f u_x^2 u_x) u_x^2}{(2 f - 2 u_t - u_x^2 f u_x u_x)^2} . \] 

(3.52)

4. Some Applications

Here we wish use the second order invariants \( \lambda_1 \) and \( \lambda_2 \) in order to bring nonlinear equations of the class (1.1) in linear form by using the equivalence transformations of the admitted group \( G_\varepsilon \).
The search for transformations mapping a non-linear differential equation in a linear differential equation has interested several authors. In particular S. Kumei and G. W. Bluman in their pioneering paper [12] gave some necessary and sufficient conditions that, by examining the invariance algebra, allow to affirm whether a non-linear equation is transformable in linear form.

It is worthwhile noticing that the Kumei-Bluman algorithm (see also [13]) constructing the linearizing map, based on the existence of an admitted infinite parameter Lie group transformations, does not require the knowledge, a priori, of a specific linear target equation. The target come out in a natural way during the developments of the algorithm. Here, instead, we search the non-linear equations of the class (1.1) that can be mapped by an equivalence transformation in a linear equation of the subclass

\[ v_\tau - v_{\sigma\sigma} = k_0 v_\sigma, \quad (4.1) \]

with \( k_0 = \text{const.} \)

That is, once fixed a priori the target (4.1) we characterize the whole set of equations (1.1) which can be mapped in (4.1).

For the subclass (4.1) the differential invariants \( \lambda_1 \) and \( \lambda_2 \) are zero. So, taking into account (3.51 - 3.52), we search the functional forms of \( f(u, u_x) \) for which

\[ \begin{cases} 
\lambda_1 = 0 \\
\lambda_2 = 0.
\end{cases} \quad (4.2) \]

Then, solving

\[ 2f - 2u_x f_{u_x} + u_x^2 f_{u_x u_x} = 0, \quad (4.3) \]

we get

\[ f = u_x^2 h(u) + h_1(u) u_x \quad (4.4) \]

where \( h \) and \( h_1 \) are arbitrary functions of \( u \).

By requiring that

\[ \lambda_2 \big|_{f=u_x^2 h(u)+h_1(u)u_x} = 0, \quad (4.5) \]

we get

\[ h_1(u) = h_0 \quad (4.6) \]

where \( h_0 \) is a constant.

We are able, now, to affirm:
Theorem 1 An equation belonging to the class (1.1) can be transformed in a linear equation of the form (4.1) by an ET generated by (2.15) if and only if the function $f$ is given by

$$f = u_x^2 h(u) + h_0 u_x.$$  

(4.7)

Proof. From the eqns. (4.3) and (4.5) it follows that the condition (4.7) is necessary.

In order to demonstrate that it is sufficient, we must show that it exists at least a ET transforming the equations

$$u_t - u_{xx} = u_x^2 h(u) + h_0 u_x$$  

(4.8)

in (4.1).

The finite form of the ETs generated by (2.15) is:

$$t = \tau e^{-\varepsilon_1} - \varepsilon_0, \quad x = (\sigma - \tau \varepsilon_2 - \varepsilon_3)e^{-\frac{1}{2}\varepsilon_1}, \quad u = \psi(v),$$  

(4.9)

where $\psi$ is an arbitrary function, with $\psi'(v) \neq 0$, and $\varepsilon_i$ are arbitrary parameters.

By applying the transformation (4.9) to the equations (4.8), we get

$$v_\tau - v_{\sigma \sigma} = v_\sigma^2 \frac{\psi'^2 h(\psi(v)) + \psi''}{\psi'} + \left(h_0 e^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2\right) v_\sigma.$$  

(4.10)

By choosing as $\psi(v)$ a solution of ODE

$$\frac{\psi'^2 h(\psi(v)) + \psi''}{\psi'} = 0,$$  

(4.11)

the transformed equation (4.10) takes the linear form (4.1) where $k_0 = h_0 e^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2$. □

It is a simple matter to show that it is possible to choose the arbitrary parameters $\varepsilon_i$ in order to make $k_0 = 0$. So we can affirm, by assuming, for sake of simplicity, $\varepsilon_1 = 0$ and $\varepsilon_2 = h_0$:
Corollary 1 The group of ETs
\[ t = \tau - \varepsilon_0, \quad x = \sigma - \tau h_0 - \varepsilon_3, \quad u = H^{-1}(c_0 v + c_1), \quad (4.12) \]
with \( H^{-1} \) denoting the inverse function of \( H(\psi) = \int e^{\int_0^\psi h(z) dz} dw \) and with \( c_0 \neq 0, c_1 \) arbitrary constants, maps the equations of the form (4.8) in the equation
\[ v_\tau - v_{\sigma\sigma} = 0. \]

Example 1 We consider the equation
\[ u_t - u_{xx} = -u_x^2 t g u + u_x. \quad (4.13) \]
In this case is \( h(u) = -tg u \) and \( h_0 = 1 \), so \( H(\psi) = \sin \psi \) and the transformations (4.12) become
\[ t = \tau - \varepsilon_0, \quad x = \sigma - \tau - \varepsilon_3, \quad u = \arcsin(c_0 v + c_1). \quad (4.14) \]
It is simple matter to verify that the transformation (4.14) maps equation (4.13) in
\[ v_\tau - v_{\sigma\sigma} = 0. \]

Remark The standard deterministic KPZ equation has the form
\[ h_t - Dh_{zz} = \lambda h_z^2 \quad (4.15) \]
where \( h(t, z) \) is the height of the surface at time \( t \) above the point \( z \) in the reference plane. \( Dh_{zz} \) describes diffusional relaxation within the surface. \( D \) is the diffusion coefficient. The strength of the nonlinearity \( \lambda \) is proportional to the growth speed.

The above equation, after a trivial change of independent variables \( t = t \) and \( z = \sqrt{D}x \), reads
\[ h_t - h_{xx} = \frac{\lambda}{D} h_x^2. \quad (4.16) \]
A special case of this equation is the Burger’s equation in potential form
\[ u_t - u_{xx} = u_x^2. \quad (4.17) \]
One can ascertain that the transformation (4.12.III) for (4.16) and (4.17) becomes the well known transformation which maps the considered equations in the well studied linear Fourier’s equation
\[ w_t - w_{xx} = 0. \quad (4.18) \]
5. Conclusions

In this paper we considered a family of semilinear diffusion equations and following [5], [6] we have obtained the differential invariants of second order under the equivalence transformations for this family by the infinitesimal method.

As an application, we have proved that a family of generalized diffusion equations can be reduced to the heat equation

\[ v_\tau - v_{\sigma\sigma} = 0 \]  \hspace{1cm} (5.1)

via appropriate equivalence transformations.

Finally, for special equations as standard deterministic KPZ (Burger’s equation in potential form), from the transformation (4.12) we recovery the well known transformation which brings them to the heat equation.

Acknowledgements

This work was supported by INdAM through G.N.F.M., by the M.I.U.R. project: Nonlinear mathematical problems of wave propagation and stability in models of continuous media and by University of Catania (ex fondi 60%).

References

[1] R. Tracinà, 2004, Invariants of a family of nonlinear wave equations, Communications in Nonlinear Science and Numerical Simulation, 9, pp. 127-133.

[2] R. Tracinà, 2002, On some applications of differential invariants of a family of nonlinear wave equations, ISNA-16 Proceedings, 16th International Symposium on Nonlinear Acoustic - Moscow 19-23/8/02, pp 583-586.

[3] M. Torrisi, R. Tracinà, A. Valenti, 2003, On the linearization of semilinear wave equations, Nonlinear Dynamics, to appear.

[4] L. V. Ovsianikov, 1982, Group Analysis of Differential Equations (New York: Academic Press).
[5] Ibragimov NH, 1997, Infinitesimal method in the theory of invariants of algebraic and differential equations, Notices of the South African Mathematical Society, 29, pp 61-70.

[6] Ibragimov NH, 1999, *Elementary Lie Group Analysis and Ordinary Differential Equations* (Chichester: John Wiley & Sons).

[7] N. H. Ibragimov, M. Torrisi, A. Valenti, 2004, Differential invariants of nonlinear equations \( v_{tt} = f(x, v_x) v_{xx} + g(x, v_x) \), Communications in Nonlinear Science and Numerical Simulation, 9, pp 69-81.

[8] M. Kardar, G Parisi, Y.C. Zhang, 1986, Dynamic Scaling of Growing Interfaces, Phys. Rev. Lett., 56, 9, pp 889-892.

[9] J. Krug, H. Spohn, 1988, Universality classes for deterministic surface growth, Phys. Rev. A, 38, 8, pp 4271-4283.

[10] N. H. Ibragimov, M. Torrisi, A. Valenti, 1991, Preliminary Group Classification of equations \( v_{tt} = f(x, v_x) v_{xx} + g(x, v_x) \), J. Math. Phys., 32, 11, pp 2988-2995.

[11] M. Torrisi, R. Tracinà, A. Valenti, 1993, On equivalence transformations applied to a non-linear wave equation, Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, pp 367-375 (Edited by N.H. Ibragimov et al.) Kluwer Academic Publishers.

[12] S. Kumei, G. W. Bluman, 1982, When non linear differential equation are equivalent to linear differential equations SIAM J. Appl. Math., 42, pp 1157-1173.

[13] G. Bluman, S. Kumei, 1989, *Symmetries and Differential Equations* (New York: Springer-Verlag).