RIEMANN-ROCH THEORY FOR WEIGHTED GRAPHS AND TROPICAL CURVES

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Abstract. We define a divisor theory for graphs and tropical curves endowed with a weight function on the vertices; we prove that the Riemann-Roch theorem holds in both cases. We extend Baker’s Specialization Lemma to weighted graphs.

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1. Introduction

The notion of vertex weighted graph, i.e., a graph whose vertices are assigned a non negative integer (the weight), arises naturally in algebraic geometry, as every Deligne-Mumford stable curve has an associated weighted “dual” graph, and the moduli space of stable curves, \( \overline{M}_g \), has a stratification with nice properties given by the loci of curves having a certain weighted graph as dual graph; see [ACQ].

On the other hand, and more recently, vertex weighted graphs have appeared in tropical geometry in the study of degenerations of tropical curves obtained by letting the lengths of some edges go to zero. To describe the limits of such families, with the above algebro-geometric picture in mind, one is led to consider metric graphs with a weight function on the vertices keeping track of the cycles that have vanished in the degeneration. Such metric weighted graphs are called weighted tropical curves; they admit a moduli space, \( M^\text{trop}_g \), whose topological properties have strong similarities with those of \( \overline{M}_g \); see [BMV] and [C2].

The connections between the algebraic and the tropical theory of curves have been the subject of much attention in latest times, and the topic presents a variety...
of interesting open problems. Moreover, the combinatorial skeleton of the theory, its graph-theoretic side, has been studied in the weightless case independently of the tropical structure; also in this setting the analogies with the classical theory of algebraic curves are quite compelling; see [BN1] and [BN2].

In this paper we are interested in divisor theory. For graphs and tropical curves with no weights the theory has been founded so that there are good notions of linear equivalence, canonical divisor, and rank of a divisor. One of the most important facts, as in algebraic geometry, is the Riemann-Roch theorem for the rank, which has been proved in [BN1] for loopless, weightless graphs, and in [GK] and [MZ] for weightless tropical curves.

The combinatorial theory is linked to the algebro-geometric theory not only by the formal analogies. Indeed, a remarkable fact that connects the two theories is Baker’s Specialization Lemma, of [B]. This result has been applied in [CDPR] to obtain a new proof of the famous Brill-Noether theorem for algebraic curves, in [B] to prove the Existence theorem (i.e., the non-emptyness of the Brill-Noether loci when the Brill-Noether number is non-negative) for weightless tropical curves, and in [C3] to prove the Existence theorem for weightless graphs. A Specialization Lemma valid also for weighted graphs could be applied to relate the Brill-Noether loci of \( M_g \) with those of \( M_{g,\tro} \), or to characterize singular stable curves that lie in the Brill-Noether loci (a well known open problem).

The main goal of this paper is to set up the divisor theory for weighted graphs and tropical curves, and to extend the above results. We hope in this way to prompt future developments in tropical Brill-Noether theory; see [La], for example. We begin by giving a geometric interpretation of the weight structure; namely, we associate to every weighted graph a certain weightless graph, and to every weighted tropical curve what we call a “pseudo-metric” graph. In both cases, the weight of a vertex has a geometric interpretation using certain “virtual” cycles attached to that vertex; in the tropical case such cycles have length zero, so that weighted tropical curves bijectively correspond to pseudo-metric graphs; see Proposition 5.3.

With these definitions we prove that the Riemann-Roch theorem holds; see Theorem 5.8 for graphs, and Theorem 5.4 for tropical curves. Furthermore, we prove, in Theorem 4.10, that the Specialization Lemma holds. The proof of this result is not a simple consequence of the weightless case, as the argument requires some non-trivial algebro-geometric steps.

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2. Preliminaries

2.1. Divisor theory on graphs. Graphs are assumed to be connected, unless otherwise specified. We here extend the set-up of [BN1] and [B] to graphs with loops, with a few notational changes.

Let \( G \) be a graph and \( V(G) \) the set of its edges. The group of divisors of \( G \), denoted by \( \text{Div}(G) \), is the free abelian group generated by \( V(G) \):

\[
\text{Div}(G) := \{ \sum_{v \in V(G)} n_v v, \ n_v \in \mathbb{Z} \}.
\]

For \( D \in \text{Div}(G) \) we write \( D = \sum_{v \in V(G)} D(v) v \) where \( D(v) \in \mathbb{Z} \). For example, if \( D = v_0 \) for some \( v_0 \in V(G) \), we have

\[
v_0(v) = \begin{cases} 1 & \text{if } v = v_0 \\ 0 & \text{otherwise.} \end{cases}
\]
The degree of a divisor $D$ is $\deg D := \sum_{v \in V(G)} D(v)$. We say that $D$ is effective, and write $D \geq 0$, if $D(v) \geq 0$ for all $v \in V(G)$. We denote by $\text{Div}^+_{\geq}(G)$ the set of effective divisors, and by $\text{Div}^+(G)$ (respectively $\text{Div}^+_e(G)$) the set of divisors (resp. effective divisors) of degree $d$.

Let $G$ be a graph and $\iota : H \hookrightarrow G$ a subgraph, so that we have $V(H) \subset V(G)$. For any $D \in \text{Div}(G)$ we denote by $D_H \in \text{Div}(H)$ the restriction of $D$ to $H$. We have a natural injective homomorphism

$$\iota_* : \text{Div}(H) \rightarrow \text{Div}(G); \quad D \mapsto \iota_* D$$

such that $\iota_* D(v) = D(v)$ for every $v \in V(H)$ and $\iota_* D(u) = 0$ for every $v \in V(G) \setminus V(H)$.

**Principal divisors.** We shall now define principal divisors and linear equivalence. We set

$$T \mapsto (v \cdot w) = \begin{cases} 
\text{number of edges joining } v \text{ and } w & \text{if } v \neq w \\
-\text{val}(v) + 2 \text{loop}(v) & \text{if } v = w
\end{cases}$$

where $\text{val}(v)$ is the valency of $v$, and $\text{loop}(v)$ is the number of loops based at $v$.

This extends linearly to a symmetric, bilinear “intersection” product

$$V(G) \times V(G) \rightarrow \mathbb{Z}.$$

Clearly, this product does not change if some loops are removed from $G$.

For a vertex $v$ of $G$ we denote by $T_v \in \text{Div}(G)$ the following divisor

$$T_v := \sum_{w \in V(G)} (v \cdot w)w.$$

Observe that $\deg T_v = 0$.

The group $\text{Prin}(G)$ of principal divisors of $G$ is the subgroup of $\text{Div}(G)$ generated by all the $T_v$:

$$\text{Prin}(G) = \langle T_v, \ \forall v \in V(G) \rangle.$$ We refer to the divisors $T_v$ as the generators of $\text{Prin}(G)$.

For any subset $Z \subset V(G)$ we denote by $T_Z \in \text{Prin}(G)$ the divisor

$$T_Z := \sum_{v \in Z} T_v.$$

**Remark 2.1.** For any subset $U \subset V(G)$ such that $|U| = |V(G)| - 1$ the set $\{T_v, \ v \in U\}$ freely generates $\text{Prin}(G)$.

Let us show that the above definition of principal divisors coincides with the one given in [BN1]. Consider the set $k(G) := \{f : V(G) \rightarrow \mathbb{Z}\}$ of integer valued functions on $V(G)$. Then the divisor associated to $f$ is defined in [BN1] as

$$\text{div}(f) := \sum_{v \in V(G)} \sum_{c = v \in E(G)} (f(v) - f(w))v,$$

and these are defined as the principal divisors in [BN1]. Now, we have

$$\text{div}(f) = \sum_{v \in V(G)} \left( \sum_{w \in V(G) \setminus v} (f(v) - f(w))(v \cdot w) \right) v$$

$$= \sum_{v \in V(G)} \left[ \left( \sum_{w \in V(G) \setminus v} -f(w)(v \cdot w) \right) - f(v)(v \cdot v) \right] v$$

$$= -\sum_{v \in V(G)} \left( \sum_{w \in V(G)} f(w)(v \cdot w) \right) v.$$

Fix any $v \in V(G)$ and consider the function $f_v : V(G) \rightarrow \mathbb{Z}$ such that $f_v(v) = 1$ and $f_v(w) = 0$ for all $w \in V(G) \setminus v$. Using the above expression for $\text{div}(f)$ one
checks that $T_v = -\text{div}(f_v)$. As the functions $f_v$ generate $k(G)$, and the divisors $T_v$ generate $\text{Prin}(G)$, the two definitions of principal divisors are equal.

We say that $D, D' \in \text{Div}(G)$ are linearly equivalent, and write $D \sim D'$, if $D-D' \in \text{Prin}(G)$. We denote by $\text{Jac}^d(G) = \text{Div}^d(G)/\sim$ the set of linear equivalence classes of divisors of degree $d$; we set

$$\text{Jac}(G) = \text{Div}(G)/\text{Prin}(G).$$

**Remark 2.2.** If $d = 0$ then $\text{Jac}^0(G)$ is a finite group, usually called the Jacobian group of $G$. This group has several other incarnations, most notably in combinatorics and algebraic geometry. We need to explain the connection with $\text{[C1]}$. If $X_0$ is a nodal curve with dual graph $G$ (see section 4), the elements of $\text{Prin}(G)$ correspond to the multi-degrees of some distinguished divisors on $X_0$, called twistors. This explains why we denote by a decorated “$\tau$” the elements of $\text{Prin}(G)$. See [12] for more details. The Jacobian group $\text{Jac}^0(G)$ is the same as the degree class group $\Delta_X$ of $\text{[C1]}$; similarly, we have $\text{Jac}^0(G) = \Delta^d_X$.

Let $D \in \text{Div}(G)$; in analogy with algebraic geometry, one denotes by

$$|D| := \{E \in \text{Div}_+(G) : E \sim D\}$$

the set of effective divisors equivalent to $D$. Next, the rank, $r_G(D)$, of $D \in \text{Div}(G)$ is defined as follows. If $|D| = \emptyset$ we set $r_G(D) = -1$. Otherwise we define

$$r_G(D) := \max\{k \geq 0 : \forall E \in \text{Div}^k_+(G) \ |D - E| \neq \emptyset\}.$$

**Remark 2.3.** The following facts follow directly from the definition.

- If $D \sim D'$, then $r_G(D) = r_G(D')$.
- If $\deg D < 0$, then $r_G(D) = -1$. Let $\deg D = 0$; then $r_G(D) \leq 0$ with equality if and only if $D \in \text{Prin}(G)$.

**Refinements of graphs.** Let $\tilde{G}$ be a graph obtained by adding a finite set of vertices in the interior of some of the edges of $G$. We say that $\tilde{G}$ is a refinement of $G$. We have a natural inclusion $V(G) \subset V(\tilde{G})$; denote by $U := V(\tilde{G}) \smallsetminus V(G)$ the new vertices of $\tilde{G}$. We have a natural map

$$\sigma^* : \text{Div}(G) \rightarrow \text{Div}(\tilde{G}); \quad D \mapsto \sigma^* D$$

such that $\sigma^* D(v) = D(v)$ for every $v \in V(G)$ and $\sigma^* D(u) = 0$ for every $u \in U$. It is clear that $\sigma^*$ induces an isomorphism of $\text{Div}(G)$ with the subgroup of divisors on $\tilde{G}$ that vanish on $U$. The notation $\sigma^*$ is motivated in remark 2.4.

A particular case that we shall use a few times is that of a refinement of $G$ obtained by adding the same number, $n$, of vertices in the interior of every edge; we denote by $G^{(n)}$ this graph, and refer to it as the $n$-subdivision of $G$.

**Remark 2.4.** Let $G$ be a graph and $e \in E(G)$ a fixed edge. Let $\tilde{G}$ be the refinement obtained by inserting only one vertex, $\tilde{v}$, in the interior $e$. Let $v_1, v_2 \in V(G)$ be the end-points of $e$, so that they are also vertices of $\tilde{G}$. Note that $\tilde{G}$ has a unique edge $\tilde{e}_1$ joining $v_1$ to $\tilde{v}$, and a unique edge $\tilde{e}_2$ joining $v_2$ to $\tilde{v}$. Then the contraction of, say, $\tilde{e}_1$ is a morphism of graphs

$$\sigma : \tilde{G} \rightarrow G.$$

There is a natural pull-back map $\sigma^* : \text{Div}(G) \rightarrow \text{Div}(\tilde{G})$ associated to $\sigma$, which maps $D \in \text{Div}(G)$ to $\sigma^* D \in \text{Div}(\tilde{G})$ such that $\sigma^* D(\tilde{v}) = 0$, and $\sigma^* D$ is equal to $D$ on the remaining vertices of $\tilde{G}$, which are of course identified with the vertices of $G$. By iterating, this construction generalizes to any refinement of $G$. 

From this description, we have that the map $\sigma^*$ coincides with the map we defined in (4), and also that it does not change if we define it by choosing as $\sigma$ the map contracting $e_2$ instead of $e_1$.

In the sequel, we shall sometimes simplify the notation and omit to indicate the map $\sigma^*$, viewing (4) as an inclusion.

2.2. Cut vertices. Let $G$ be a graph with a cut vertex, $v$. Then we can write $G = H_1 \cup H_2$ where $H_1$ and $H_2$ are connected subgraphs of $G$ such that $V(H_1) \cap V(H_2) = \{ v \}$ and $E(H_1) \cap E(H_2) = \emptyset$. We say that $G = H_1 \cup H_2$ is a decomposition associated to $v$. Pick $D_j \in \operatorname{Div}(H_j)$ for $j = 1, 2$, then we define $D_1 + D_2 \in \operatorname{Div} G$ as follows

$$(D_1 + D_2)(u) = \begin{cases} D_1(v) + D_2(v) & \text{if } u = v \\ D_1(u) & \text{if } u \in V(H_1) - \{ v \} \\ D_2(u) & \text{if } u \in V(H_2) - \{ v \}. \end{cases}$$

**Lemma 2.5.** Let $G$ be a graph with a cut vertex and let $G = H_1 \cup H_2$ be a corresponding decomposition (as described above). Let $j = 1, 2$.

1. The map below is a surjective homomorphism with kernel isomorphic to $\mathbb{Z}$

$$\operatorname{Div}(H_1) \oplus \operatorname{Div}(H_2) \longrightarrow \operatorname{Div}(G); \quad (D_1, D_2) \mapsto D_1 + D_2$$

and it induces an isomorphism $\operatorname{Prin}(H_1) \oplus \operatorname{Prin}(H_2) \cong \operatorname{Prin}(G)$ and an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \operatorname{Jac}(H_1) \oplus \operatorname{Jac}(H_2) \longrightarrow \operatorname{Jac}(G) \longrightarrow 0.$$

2. We have a commutative diagram with injective vertical arrows

\begin{align*}
0 & \quad \operatorname{Prin}(G) \quad \operatorname{Div}(G) \quad \operatorname{Jac}(G) \quad 0 \\
0 & \quad \operatorname{Prin}(H_j) \quad \operatorname{Div}(H_j) \quad \operatorname{Jac}(H_j) \quad 0
\end{align*}

3. For every $D_1, D_2$ with $D_j \in \operatorname{Div}(H_j)$, we have

$$r_G(D_1 + D_2) \geq \min\{r_{H_1}(D_1), r_{H_2}(D_2)\}.$$  

4. For every $D_j \in \operatorname{Div}(H_j)$, we have $r_{H_j}(D_j) \geq r_G(D_j)$.

**Proof.** Denote $V(H_j) = \{ u_1, \ldots, u_{n_j}, v \}$ and $V(G) = \{ u_1, \ldots, u_{n_1}, v, u^2_2, \ldots, u^2_{n_2} \}$.

An equivalent way of defining the divisor $D_1 + D_2$ is to use the two maps $\iota^j : \operatorname{Div}(H_j) \rightarrow \operatorname{Div}(G)$ defined in (4). Then we have $D_1 + D_2 = \iota^1 D_1 + \iota^2 D_2$. With this description, it is clear that the map in part (4) is a surjective homomorphism.

In addition, the kernel of this map has generator $(v, -v) \in \operatorname{Div}(H_1) \oplus \operatorname{Div}(H_2)$ and is thus isomorphic to $\mathbb{Z}$.

To distinguish the generators of $\operatorname{Prin}(H_j)$ from those of $\operatorname{Prin}(G)$ we denote by $T_{w} \in \operatorname{Prin}(H_j)$ the generator corresponding to $w \in V(H_j)$. We clearly have

$$\iota^j T_{w^j_{h_{j}}} = T_{w^j_{h_{j}}}$$

for $j = 1, 2$ and $h = 1, \ldots, n_j$. As $\operatorname{Prin}(H_j)$ is freely generated by $T_{w_1^j}, \ldots, T_{w_{n_j}^j}$ and $\operatorname{Prin}(G)$ is freely generated by $T_{w_1}, \ldots, T_{w_{n_1}}, T_{w_2^1}, \ldots, T_{w_{n_2}^2}$, the first part is proved.

Part (4) also follows from the previous argument.

Set $r_j = r_{H_j}(D_j)$ and assume $r_1 \leq r_2$. Set $D = D_1 + D_2$; to prove that $r_G(D) \geq r_1$ we must show that for every $E \in \operatorname{Div}^+_{\alpha}(G)$ there exists $T \in \operatorname{Prin}(G)$ such that $D - E + T \geq 0$. Pick such an $E$; let $E_1 = E_{H_1}$ and $E_2 = E - E_1$, so that $E_2 \in \operatorname{Div} H_2$. Since $\deg E_2 \leq r_1 \leq r_j$, we have that there exists $T_j \in \operatorname{Prin}(H_j)$ such
that \( D_j - E_j + T_j \geq 0 \) in \( H_j \). By the previous part \( T = T_1 + T_2 \in \text{Prin}(G) \); let us conclude by showing that \( D - E + T \geq 0 \). In fact
\[
D - E + T = D_1 + D_2 - E_1 - E_2 + T_1 + T_2 = (D_1 - E_1 + T_1) + (D_2 - E_2 + T_2) \geq 0.
\]

\[ \Box \]

Assume \( j = 1 \) and set \( r = r_G(D_1) \). By \([2]\) we are free to view \( \text{Div}(H_1) \) as a subset of \( \text{Div}(G) \). Pick \( E \in \text{Div}^*_r(H_1) \), then there exists \( T \in \text{Prin}(G) \) such that in \( G \) we have \( D_1 - E + T \geq 0 \). By \([1]\) we know that \( T = T_1 + T_2 \) with \( T_i \in \text{Prin}(G_i) \); since \( D_1(u_k^h) = E(u_k^h) = 0 \) for all \( h = 1, \ldots, n_2 \) we have that \( T_2 = 0 \), hence \( D_1 - E + T_i \geq 0 \) in \( H_1 \) \( \Box \)

Now let \( G = H_1 \lor H_2 \) as above and let \( m, n \) be two nonnegative integers; we denote by \( G^{(m,n)} \) the graph obtained by inserting \( m \) vertices in the interior of every edge of \( H_1 \) and \( n \) vertices in the interior of every edge of \( H_2 \). Hence we can write
\[
G^{(m,n)} := H_1^{(m)} \lor H_2^{(n)}
\]
(recall that \( H^{(m)} \) denotes the \( m \)-subdivision of a graph \( H \)). We denote by \( \sigma_{m,n}^* : \text{Div}(G) \rightarrow \text{Div}(G^{(m,n)}) \) the natural map.

**Proposition 2.6.** Let \( G \) be a graph with a cut vertex and \( G = H_1 \lor H_2 \) a corresponding decomposition. Let \( m, n \) be non-negative integers and \( G^{(m,n)} = H_1^{(m)} \lor H_2^{(n)} \) the corresponding refinement. Then
\[
(1) \quad \sigma_{m,n}^*(\text{Prin}(G)) \subset \text{Prin}(G^{(m,n)}),
\]
\[
(2) \quad \text{Assume that } G \text{ has no loops. Then for every } D \in \text{Div}(G), \text{ we have}
\]
\[
r^*_G(D) = r_{G^{(m,n)}}(\sigma_{m,n}^* D).
\]

**Proof.** It is clear that it suffices to prove part \([1]\) for \((0, n)\) and \((0, m)\) separately, hence it suffices to prove it for \((0, m)\). Consider the map (for simplicity we write \( \sigma^* = \sigma_{0,n}^* \))
\[
\sigma^* : \text{Div}(G) = \text{Div}(H_1 \lor H_2) \rightarrow \text{Div}(H_1 \lor H_2^{(m)}) = \text{Div}(G^{(0,m)}).
\]
The group \( \text{Prin}(G) \) is generated by \( \{T_u, \forall u \in V(G) \setminus \{v\}\} \) (see Remark \([2.1]\)). Hence it is enough to prove that \( \sigma^*(T_u) \) is principal for all \( u \in V(G) \setminus \{v\} \). We denote by \( \widehat{u} \in V(G^{(0,m)}) \) the vertex corresponding to \( u \in V(G) \) via the inclusion \( V(G) \subset V(G^{(0,m)}) \).

If \( u \in V(H_1) \setminus \{v\} \) we clearly have \( \sigma^*(T_u) = T_{\widehat{u}}, \) hence \( \sigma^*(T_u) \in \text{Prin}(G^{(0,m)}) \).

Let \( u \in V(H_2 \setminus \{v\} \). Denote by \( E_u(G) \) the set of edges of \( G \) adjacent to \( u \) and pick \( e \in E_u(G) \); as \( G^{(0,m)} \) is given by adding \( m \) vertices in every edge of \( G \), we will denote the vertices added in the interior of \( e \) by
\[
\{w_{i1}^e, \ldots, w_{im}^e\} \subset V(G^{(0,m)}),
\]
ordering \( w_{i1}^e, \ldots, w_{im}^e \) according to the orientation of \( e \) which has \( u \) as target, so that in \( G^{(0,m)} \) we have \((w_{1i}^e \cdot \widehat{u}) = 1 \) and \((w_{2i}^e \cdot \widehat{u}) = 0 \) if \( i < m \) (and \((w_{mi}^e \cdot w_{mi+1}^e) = 1 \) for all \( i \)). One then easily checks that
\[
\sigma^*(T_u) = (m + 1)T_{\widehat{u}} + \sum_{e \in E_u(G)} \sum_{i=1}^m iT_{w_{ii}^e};
\]
hence \( \sigma^*(T_u) \in \text{Prin}(G^{(0,m)}) \), and part \([1]\) is proved.

Part \([2]\). First we note that the statement holds in the case \( m = n \). Indeed, in this case \( G^{(m,n)} = G^{(n)} \) and hence our statement is \([HKN, \text{Cor. 22}]\); see also \([LM]\) Thm 1.3.

Using this fact, we claim that it will be enough to show only the inequality
\[
r^*_G(D) \leq r^*_{G^{(m,n)}}(\sigma_{m,n}^* D).
\]
Indeed, suppose this inequality holds for every divisor \( D \) on every graph of the form \( G = H_1 \lor H_2 \) and for all pairs of integers \((m, n)\). Pick a divisor \( D \in \text{Div}(G) \), we get,
omitting the maps $\sigma^*$ for simplicity (which creates no ambiguity, as the subscript of $r$ already indicates in which graph we are computing the rank)

$$r_G(D) \leq r_{G(m,n)}(D) \leq r_{G(m,n),(n,m)}(D) = r_{G(l,1)}(D) = r_G(D)$$

where $l = m + n + mn$. (We used the trivial fact that for any graph $H$ and positive integers $h, k$ we have $(H^h)^{(k)} = H^{h+k+hk}$). Hence all the inequalities above must be equalities and the result follows.

Thus, we are left to prove Inequality (6). Let $r = r_G(D)$. We have to show that for any effective divisor $E^*$ on $G^{(m,n)}$ of degree $r$ we have

$$r_{G(m,n)}(\sigma^*_{m,n} D - E^*) \geq 0.$$ 

By [Lu] Thm. 1.5] (or [HKN]), $V(G)$ is a rank-determining set in $G^{(m,n)}$. Therefore it will be enough to show the above claim for divisors of the form $E^* = \sigma^*_{m,n} E$ for any effective divisor $E$ of degree $r$ on $G$. Summarizing, we need to show that for every $E \in \text{Div}_G^*(G)$ there exists $T \in \text{Prin}(G^{(m,n)})$ such that

$$T + \sigma^*_{m,n} D - \sigma^*_{m,n} E \geq 0. \quad (7)$$

Now, since $r = r_G(D)$, there exists a principal divisor $\tilde{T} \in \text{Prin}(G)$ such that

$$\tilde{T} + D - E \geq 0.$$

By the previous part, $\sigma^*_{m,n} \tilde{T}$ is a principal divisor of $G^{(m,n)}$; set $T := \sigma^*_{m,n} \tilde{T}$. Then we have

$$0 \leq \sigma^*_{m,n} (\tilde{T} + D - E) = T + \sigma^*_{m,n} D - \sigma^*_{m,n} E.$$

Therefore (7) holds, and we are done.

\section{Riemann Roch for weighted graphs}

\subsection{Divisor theory for graphs with loops.}

Our goal here is to set up a divisor theory for graphs with loops, so that the Riemann-Roch theorem holds. This theorem has been proved for loopless graphs in [BN1]. To generalize it we shall give a more subtle definition for the rank and for the canonical divisor.

\begin{definition}
Let $G$ be a graph and let $\{e_1, \ldots, e_c\} \subset E(G)$ be the set of its loop-edges. We denote by $\hat{G}$ the graph obtained by inserting one vertex in the interior of the loop-edge $e_j$, for all $j = 1, \ldots, c$. Since $V(G) \subset V(\hat{G})$, we have a canonical injective morphism

$$\sigma^*: \text{Div}(G) \rightarrow \text{Div}(\hat{G}).$$

We set

$$r_G^\#(D) := r_{\hat{G}}(\sigma^* D), \quad (9)$$

and refer to $r_G^\#(D)$ as the rank of $D$.

The superscript “$\#$” is used to avoid confusion with the definition which disregard the loops. We often abuse notation and write just $r_{\hat{G}}(D)$ omitting $\sigma^*$.

Observe that $\hat{G}$ is free from loops and has the same genus as $G$. With the above notation, let $u_j \in V(\hat{G})$ the vertex added in the interior of $e_j$ for all $j = 1, \ldots, c$. It is clear that the map (8) induces an isomorphism of $\text{Div}(G)$ with the subgroup of divisors $\hat{D}$ on $\hat{G}$ such that $\hat{D}(u_j) = 0$ for all $j = 1, \ldots, c$.

\begin{example}
Here is an example in the case $c = 1$.
\end{example}
The last two equalities prove (13), hence the result is proved.

Remark 3.3. We have

\[ r_G(D) \geq r_G^\#(D). \tag{10} \]

Indeed, let \( G_0 \) be the graph obtained from \( G \) by removing all its loop-edges; then, by definition, \( r_G(D) = r_{G_0}(D) \). On the other hand, by Lemma 2.4 (4), writing \( \tilde{G} = G_0 \cup H \) for some graph \( H \), we have \( r_{G_0}(D) \geq r_{\tilde{G}}(D) = r_G^\#(D) \), hence (10) follows.

Definition 3.1 may seem a bit arbitrary, as the choice of the refinement \( \tilde{G} \) may seem arbitrary. In fact, it is natural to ask whether adding some (positive) number of vertices, different from one, in the interior of the loop-edges of \( G \) can result in a different rank. This turns out not to be the case, as we now show.

Proposition 3.4. Let \( G \) be a graph and let \( e_1, \ldots, e_c \) be its loop-edges. For every \( \omega = (n_1, \ldots, n_c) \in \mathbb{N}^c \) let \( G(\omega) \) be the refinement of \( G \) obtained by inserting \( n_i \) vertices in the interior of \( e_i \). Then for every \( D \in \text{Div}(G) \) we have

\[ r_G^\#(D) = r_{G(\omega)}(\sigma^* D) \]

where \( \sigma^* : \text{Div}(G) \rightarrow \text{Div}(G(\omega)) \) is the natural map.

Proof. It will be enough to prove the proposition for \( c = 1 \) since the general statement can be obtained easily by induction on the number of loop-edges of \( G \).

Let \( H_1 \) be the graph obtained from \( G \) by removing its loop-edge, \( e \), and let \( v \) be the vertex of \( G \) adjacent to \( e \). We can thus decompose \( G \) with respect to \( v \):

\[ G = H_1 \cup C_1 \]

where, for \( m \geq 1 \) we denote by \( C_m \) the “\( m \)-cycle”, i.e., the 2-regular graph of genus \( 1 \), having \( m \) vertices and \( m \) edges. Observe that for every \( h \geq 1 \) we have (recall that \( C_m^{(h)} \) denotes the \( h \)-subdivision of \( C_m \))

\[ C_m^{(h)} = C_{m(h+1)}. \tag{11} \]

Therefore, with the notation of Proposition 2.6, we have, for every \( n \geq 0 \),

\[ G^{(0,n)} = H_1^{(0)} \cup C_1^{(n)} = H_1 \cup C_{n+1}. \tag{12} \]

For any divisor \( D \) on \( G \), by definition, we have

\[ r_G^\#(D) = r_{G^{(0,1)}}(\sigma_{0,1}^* D). \]

So we need to prove that for any \( n \geq 1 \),

\[ r_{G^{(0,1)}}(\sigma_{0,1}^* D) = r_{G^{(0,n)}}(\sigma_{0,n}^* D). \tag{13} \]

This is now a simple consequence of Proposition 2.6 (2). Indeed, by applying it to the loopless graph \( G^{(0,1)} = H_1 \cup C_2 \) and the \( n \)-subdivision of \( C_2 \), we get, simplifying the notation by omitting the pull-back maps \( \sigma_{\ast}^* \),

\[ r_{G^{(0,1)}}(D) = r_{(G^{(0,1)})^{(0,n)}}(D) = r_{H_1 \cup C_2^{(n)}}(D) = r_{H_1 \cup C_{n+2}}(D) \]

by (11). On the other hand, applying the proposition a second time to \( G^{(0,n)} = H_1 \cup C_{n+1} \) and the \( 1 \)-subdivision of \( C_{n+1} \), we get

\[ r_{G^{(0,n)}}(D) = r_{(G^{(0,n)})^{(0,1)}}(D) = r_{H_1 \cup C_{n+1}^{(1)}}(D) = r_{H_1 \cup C_{n+2}}(D). \]

The last two equalities prove (13), hence the result is proved.
Hence, since the map (8) is a degree preserving homomorphism, we have 
\[ \deg D = \deg D', \]
where, in the last equality, we applied the the Riemann-Roch formula for loopless graphs.

It is thus obvious that if \( D \sim D' \) for divisors in \( \text{Div}(G) \), then \( r_G^\#(D) = r_G^\#(D') \).

The canonical divisor \( K_G^\# \in \text{Div}(G) \) of \( G \) is defined as follows
\[
K_G^\# := \sum_{v \in V(G)} (\text{val}(v) - 2)v.
\]

**Theorem 3.6.** Let \( G \) be a graph with \( c \) loops, and let \( D \in \text{Div}(G) \).

1. (Riemann-Roch theorem)
\[
r_G^\#(D) - r_G^\#(K_G^\# - D) = \deg D - g + 1.
\]
   In particular, we have \( r_G^\#(K_G^\#) = g - 1 \) and \( \deg K_G^\# = 2g - 2 \).

2. (Riemann theorem) If \( \deg D \geq 2g - 1 \) then
\[
r_G^\#(D) = \deg D - g.
\]

**Proof.** Let \( U = \{u_1, \ldots, u_c\} \subset V(\hat{G}) \) be the set of vertices added to \( G \) to define \( \hat{G} \).
The canonical divisor \( K_{\hat{G}} \) of \( \hat{G} \) is
\[
K_{\hat{G}} = \sum_{\hat{v} \in V(\hat{G})} (\text{val}(\hat{v}) - 2)\hat{v} = \sum_{\hat{v} \in V(\hat{G}) \setminus U} (\text{val}(\hat{v}) - 2)\hat{v}
\]
because the vertices in \( U \) are all 2-valent. On the other hand we have an identification \( V(G) = V(\hat{G}) \setminus U \) and it is clear that this identification preserves the valencies. Therefore, by definition (14) we have
\[
\sigma^* K_G^\# = K_{\hat{G}}.
\]
Hence, since the map (8) is a degree preserving homomorphism,
\[
r_G^\#(D) - r_G^\#(K_G^\# - D) = r_G^\#(\sigma^* D) - r_G^\#(K_{\hat{G}} - \sigma^* D)) = \deg D - g + 1
\]
where, in the last equality, we applied the the Riemann-Roch formula for loopless graphs (proved by Baker-Norine in [BNT]), together with the fact that \( G \) and \( \hat{G} \) have the same genus.

Part (2) follows from the Riemann-Roch formula we just proved, noticing that, if \( \deg D \geq 2g - 1 \), then \( \deg K_G^\# - D < 0 \) and hence \( r_G^\#(K_G^\# - D) = -1 \).}

The next Lemma, which we will use later, computes the rank of a divisor on the so called “rose with \( g \) petals”, or “bouquet of \( g \) loops” \( R_g \).

**Lemma 3.7.** Set \( g \geq 1 \) and \( d \leq 2g \). Let \( R_g \) be the connected graph of genus \( g \) having only one vertex (and hence \( g \) loop-edges). For the unique divisor \( D \in \text{Div}^d(R_g) \) we have
\[
r_{R_g}^\#(D) = \left\lfloor \frac{d}{2} \right\rfloor.
\]

**Proof.** Let \( v \) be the unique vertex of \( G = R_g \), hence \( D = dv \). To compute \( r_{R_g}^\#(D) \) we must use the refinement \( \hat{G} \) of \( R_g \) defined above. In this case \( \hat{G} \) is the 1-subdivision of \( R_g \). So \( V(\hat{G}) = \{\bar{v}, u_1, \ldots, u_g\} \) with each \( u_i \) of valency 2, and \( \bar{v} \) of valency 2\( g \). We have \( u_i \cdot v = 2 \) for all \( i = 1, \ldots, g \), and \( u_i \cdot u_j = 0 \) for all \( i \neq j \).
Let \( \tilde{D} = d\tilde{v} \) be the pull-back of \( D \) to \( \tilde{G} \). Set \( r := \left\lfloor \frac{d}{2} \right\rfloor \). We will first prove that \( r_{\tilde{G}}(\tilde{D}) \geq r \). Let \( E \) be a degree \( r \) effective divisor on \( \tilde{G} \); then for some \( I \subset \{1, \ldots, g\} \) we have
\[
E = e_0 \tilde{v} + \sum_{i \in I} e_i u_i
\]
with \( e_i > 0 \) and \( \sum_{i=0}^r e_i = r \). Notice that \( |I| \leq r \). Now,
\[
\tilde{D} - E \sim d\tilde{v} - e_0 \tilde{v} - \sum_{i \in I} e_i u_i - \sum_{i \in I} \left\lfloor \frac{e_i}{2} \right\rfloor T_{u_i} =: F.
\]
Let us prove that \( F \geq 0 \). Recall that \( T_{u_i}(\tilde{v}) = 2 \), hence
\[
F(\tilde{v}) = d - e_0 - 2 \sum_{i \in I} \left\lfloor \frac{e_i}{2} \right\rfloor \geq d - e_0 - \sum_{i \in I} (e_i + 1) \geq 2r - r - |I| = r - |I| \geq 0
\]
as, of course, \( |I| \leq r \). Next, since \( T_{u_i}(u_i) = -2 \) and \( T_{u_i}(u_j) = 0 \) if \( i \neq j \), we have for all \( i \in I \),
\[
F(u_i) = -e_i + 2 \left\lfloor \frac{e_i}{2} \right\rfloor \geq 0,
\]
and \( F(u_j) = 0 \) for all \( u_j \notin I \). Therefore \( r_{\tilde{G}}(\tilde{D}) \geq r \).

Finally, since \( d \leq 2g \), we can apply Clifford’s theorem \([BN1, \text{Cor. 3.5}]\), and therefore equality must hold.

3.2. Divisors on weighted graphs. Let \((G, \omega)\) be a weighted graph, by which we mean that \( G \) is an ordinary graph and \( \omega : V(G) \to \mathbb{Z}_{\geq 0} \) a weight function on the vertices. The genus, \( g(G, \omega) \), of \((G, \omega)\) is
\[
g(G, \omega) = b_1(G) + \sum_{v \in V(G)} \omega(v).
\]

We associate to \((G, \omega)\) a weightless graph \( G^w \) as follows: \( G^w \) is obtained by attaching at every vertex \( v \) of \( G \), \( \omega(v) \) loops (or “1-cycles”), denoted by \( C_1^v, \ldots, C_{\omega(v)}^v \).

We call \( G^w \) the virtual (weightless) graph of \((G, \omega)\), and we say that the \( C_i^v \) are the virtual loops. The initial graph \( G \) is a subgraph of \( G^w \) and we have an identification
\[
V(G) = V(G^w).
\]
It is easy to check that
\[
g(G, \omega) = g(G^w).
\]
For the group of divisors of the weighted graph \((G, \omega)\), we have
\[
\text{Div}(G, \omega) = \text{Div}(G^w) = \text{Div}(G).
\]
The canonical divisor of \((G, \omega)\) is defined as the canonical divisor of \( G^w \), introduced in the previous section, namely,
\[
K_{(G, \omega)} := K_{G^w} = \sum_{v \in V(G^w)} (\text{val}_{G^w}(v) - 2)v.
\]
Note that \( K_{(G, \omega)} \in \text{Div}(G, \omega) \). By \([17]\) and Theorem 3.6 we have
\[
\deg K_{(G, \omega)} = 2g(G, \omega) = 2.
\]
For any \( D \in \text{Div}(G, \omega) \) we define (cf. Definition 3.1)
\[
r_{(G, \omega)}(D) := r^w_{G^w}(D) = r^w_{G}(D).
\]

**Theorem 3.8.** Let \((G, \omega)\) be a weighted graph.

1. For every \( D \in \text{Div}(G, \omega) \) we have
\[
r_{(G, \omega)}(D) = r_{(G, \omega)}(K_{(G, \omega)} - D) = \deg D - g + 1.
\]
(2) For every $D, D' \in \text{Div}(G)$ such that $D \sim D'$, we have $r_{(G, \omega)}(D) = r_{(G, \omega)}(D')$.

**Proof.** The first part is an immediate consequence of Theorem 3.6.

For $\phi$, recall Remark 3.5; we have that $D \sim D'$ on $G$ if and only if $D$ and $D'$ are equivalent on the graph $G_0$ obtained by removing all loop-edges from $G$. Now, $G_0$ is a subgraph of $\hat{G}^\omega$, moreover $\hat{G}^\omega$ is obtained from $G_0$ by attaching a finite set of 2-cycles at some vertices of $G_0$. Therefore, by iterated applications of Lemma 2.5, we have that $D$ is linearly equivalent to $D'$ on $\hat{G}^\omega$. Hence the statement follows from the fact that $r_{\hat{G}^\omega}$ is constant on linear equivalence classes of $\hat{G}^\omega$. \hfill \blacksquare

4. **Specialization Lemma for weighted graphs**

In this section we fix an algebraically closed field and assume that all schemes are of finite type over it. By “point” we mean closed point.

By **nodal curve** we mean a connected, reduced, projective, one-dimensional scheme, having at most nodes (ordinary double points) as singularities. All curves we shall consider in this section are nodal.

Let $X$ be a nodal curve; its **dual graph**, denoted by $G_X$, is such that $V(G_X)$ is identified with the set of irreducible components of $X$, $E(G_X)$ is identified with the set of nodes of $X$, and there is an edge joining two vertices for every node lying at the intersection of the two corresponding components. In particular, the loop-edges of $G_X$ correspond to the nodes of the irreducible components of $X$.

The **weighted dual graph** of $X$, denoted by $(G_X, \omega_X)$, has $G_X$ as defined above, and the weight function $\omega_X$ is such that $\omega_X(v)$ is the geometric genus of the component of $X$ corresponding to $v$. In particular, let $g_X$ be the (arithmetic) genus of $X$, then

$$g_X = b_1(G_X) + \sum_{v \in V(G_X)} \omega_X(v).$$

4.1. **Specialization of families of line bundles on curves.** Let $\phi : \mathcal{X} \to B$ be a family of curves, and denote by $\pi : \text{Pic}_\phi \to B$ its Picard scheme (often denoted by $\text{Pic}_{\mathcal{X}/B}$). The set of sections of $\pi$ is denoted as follows

$$\text{Pic}_\phi(B) := \{ \mathcal{L} : B \to \text{Pic}_\phi : \pi \circ \mathcal{L} = \text{id}_B \}.$$  

(The notation $\mathcal{L}$ indicates that $\mathcal{L}(b)$ is a line bundle on $X_b = \phi^{-1}(b)$ for every $b \in B$.) Let $b_0 \in B$ be a closed point and set $X_0 = \phi^{-1}(b_0)$; denote by $(G, \omega)$ the weighted dual graph of $X_0$. We identify $\text{Div}(G) = \mathbb{Z}^{V(G)}$, so that we have a map

$$\text{Pic}(X_0) \to \text{Div}(G) = \mathbb{Z}^{V(G)}; \quad L \mapsto \deg L$$

where $\deg$ denotes the multidegree, i.e., for $v \in V(G)$ the $v$-coordinate of $\deg L$ is the degree of $L$ restricted to $v$ (recall that $V(G)$ is identified with the set of irreducible components of $X_0$). Finally, we have a specialization map $\tau$

$$\text{Pic}_\phi(B) \to \text{Div}(G); \quad L \mapsto \deg L(b_0).$$

**Definition 4.1.** Let $X_0$ be a nodal curve. A projective morphism $\phi : \mathcal{X} \to B$ of schemes is a **regular one-parameter smoothing** of $X_0$ if:

1. $B$ is smooth, quasi-projective, $\dim B = 1$;
2. $\mathcal{X}$ is a regular surface;
3. there is a closed point $b_0 \in B$ such that $X_0 \cong \phi^{-1}(b_0)$. (We shall usually identify $X_0 = \phi^{-1}(b_0)$.)

**Remark 4.2.** As we mentioned in Remark 2.2 there is a connection between the divisor theory of $X_0$ and that of its dual graph $G$. We already observed in (21) that to every divisor, or line bundle, on $X_0$ there is an associated divisor on $G$. Now we need to identify $\text{Prin}(G)$. As we already said, the elements of $\text{Prin}(G)$
are the multidegrees of certain divisors on $X_0$, called twisters. More precisely, fix $\phi : X \to B$ a regular one-parameter smoothing of $X_0$; we have the following subgroup of $\text{Pic}(X_0)$:

$$\text{Tw}_\phi(X_0) := \{ L \in \text{Pic}(X_0) : L \cong \mathcal{O}_X(D)|_{X_0} \text{ for some } D \in \text{Div} X : \text{Supp } D \subset X_0 \}.$$ 

The set of twisters, $\text{Tw}(X_0)$, is defined as the union of the $\text{Tw}_\phi(X_0)$ for all one-parameter smoothings $\phi$ of $X_0$.

The group $\text{Tw}_\phi(X_0)$ depends on $\phi$, but its image under the multidegree map $\deg$ does not, so that $\deg(\text{Tw}_\phi(X_0)) = \deg(\text{Tw}(X_0))$. Moreover, the multidegree map induces an identification between the multidegrees of all twisters and $\text{Prin}(G)$:

$$\deg(\text{Tw}(X_0)) = \text{Prin}(G) \subset \mathbb{Z}^{V(G)}.$$ 

See [C1, B, Lemma 2.1] or [C3] for details.

**Definition 4.3.** Let $\phi$ be a regular one-parameter smoothing of $X_0$ and let $\mathcal{L}, \mathcal{L}' \in \text{Pic}_0(B)$. We define $\mathcal{L}$ and $\mathcal{L}'$ to be $\phi$-equivalent, writing $\mathcal{L} \sim_{\phi} \mathcal{L}'$, as follows

$$\mathcal{L} \sim_{\phi} \mathcal{L}' \text{ if } \mathcal{L}(b) \cong \mathcal{L}'(b), \forall b \neq b_0.$$ 

**Example 4.4.** Let $\phi$ be as in the definition and let $C \subset X_0$ be an irreducible component. Denote by $\mathcal{L}' = \mathcal{L}(C) \in \text{Pic}_0(B)$ the section of $\text{Pic}_0 \to B$ defined as follows: $\mathcal{L}'(b) = \mathcal{L}(b)$ if $b \neq b_0$ and $\mathcal{L}'(b_0) = \mathcal{L} \otimes \mathcal{O}_X(C_0) \otimes \mathcal{O}_{X_0}$. Then $\mathcal{L}(C) \sim_{\phi} \mathcal{L}$.

The same holds replacing $C$ with any $\mathbb{Z}$-linear combination of the components of $X_0$.

**Lemma 4.5.** Let $\phi$ be a regular one-parameter smoothing of $X_0$ and let $\mathcal{L}, \mathcal{L}' \in \text{Pic}_0(B)$ such that $\mathcal{L} \sim_{\phi} \mathcal{L}'$. Then the following hold.

1. $\tau(\mathcal{L}) \sim \tau(\mathcal{L}')$.
2. If $h^0(X_b, \mathcal{L}(b)) \geq r + 1$ for every $b \in B \setminus b_0$, then $h^0(X_b, \mathcal{L}'(b)) \geq r + 1$ for every $b \in B$.

**Proof.** To prove both parts we can replace $\phi$ by a finite étale base change (see [C3 Claim 4.6]). Hence we can assume that $\mathcal{L}$ and $\mathcal{L}'$ are given by line bundles on $X$, denoted again by $\mathcal{L}$ and $\mathcal{L}'$.

Since $\mathcal{L}$ and $\mathcal{L}'$ coincide on every fiber but the special one, there exists a divisor $D \in \text{Div } X$ such that $\text{Supp } D \subset X_0$ for which

$$\mathcal{L} \cong \mathcal{L}' \otimes \mathcal{O}_X(D).$$

Using Remark [24] we have $\mathcal{O}_X(D)|_{X_0} \in \text{Tw}(X_0)$ and

$$\tau(\mathcal{O}_X(D)) = \deg \mathcal{O}_X(D)|_{X_0} \in \text{Prin}(G)$$

so we are done.

[2] This is a straightforward consequence of the upper-semicontinuity of $h^0$.

By the Lemma, we have a commutative diagram:

$$\text{Pic}_0(B) \xrightarrow{\tau} \text{Div}(G)$$

$$\text{Pic}_0(B)/\sim_{\phi} \xrightarrow{} \text{Jac}(G)$$

and, by Remark [22], the image of $\tau$ contains $\text{Prin}(G)$.
4.2. Proof of the Specialization Lemma. We shall now prove Theorem 4.10, a Specialization Lemma for weighted graphs. Our set-up is similar to that of [C3] where a variant of the original specialization Lemma, [H] Lemma 2.8, is proved. Before proving Theorem 4.10 we need some preliminaries.

Let \( G \) be a connected graph. For \( v, u \in V(G) \), denote by \( d(v, u) \) the distance between \( v \) and \( u \) in \( G \); note that \( d(v, u) \) is the minimum length of a path joining \( v \) with \( u \), so that \( d(v, u) \in \mathbb{Z}_{\geq 0} \) and \( d(v, u) = 0 \) if and only if \( v = u \).

Fix \( v_0 \in V(G) \); we now define an ordered partition of \( V(G) \) (associated to \( v_0 \)) by looking at the distances to \( v_0 \). For \( i \in \mathbb{Z}_{\geq 0} \) set
\[
Z_i^{(v_0)} := \{ u \in V(G) : d(v_0, u) = i \};
\]
we have \( Z_0^{(v_0)} = \{ v_0 \} \) and, obviously, there exists an \( m \) such that \( Z_m^{(v_0)} \neq \emptyset \) if and only if \( 0 \leq n \leq m \). We have thus an ordered partition of \( V(G) \)
\[
V(G) = Z_0^{(v_0)} \cup \ldots \cup Z_m^{(v_0)}.
\]
We refer to it as the distance-based partition starting at \( v_0 \). We will often omit the superscript \((v_0)\).

Remark 4.6. One checks easily that for every \( u \in V(G) \setminus \{ v_0 \} \) with \( u \in Z_i \), we have
\[
u \cdot Z_j \neq 0 \quad \text{if and only if} \quad j = i \pm 1.
\]
Therefore for any \( 0 \leq i \neq j \leq m \), we have \( Z_i \cdot Z_j \neq 0 \) if and only if \( |i - j| = 1 \).

Whenever \( G \) is the dual graph of a curve \( X \), we identify \( V(G) \) with the components of \( X \) without further mention and with no change in notation. Similarly, a subset of vertices \( Z \subset V(G) \) determines a subcurve of \( X \) (the subcurve whose components are the vertices in \( Z \)) which we denote again by \( Z \).

The following result will be used to prove Theorem 4.10.

Proposition 4.7. Let \( X_0 \) be a nodal curve, \( C_0, C_1, \ldots, C_n \subset X_0 \) its irreducible components of arithmetic genera \( g_0, g_1, \ldots, g_n \), respectively, and \( G \) the dual graph of \( X_0 \). Fix \( g : \mathcal{X} \to B \) a regular one-parameter smoothing of \( X_0 \), and \( \mathcal{L} \in \operatorname{Pic}_g(B) \) such that \( h^0(X_b, \mathcal{L}(b)) \geq r + 1 > 0 \) for every \( b \in B \). Consider a sequence \( r_0, r_1, \ldots, r_n \) of non-negative integers such that \( r_0 + r_1 + \cdots + r_n = r \). Then there exists an effective divisor \( E \in \operatorname{Div}(G) \) such that \( E \sim \tau(\mathcal{L}) \) and for any \( 0 \leq i \leq n \)
\[
E(C_i) \geq \begin{cases} 2r_i & \text{if } r_i \leq g_i - 1 \\ r_i + g_i & \text{if } r_i \geq g_i \end{cases}
\]
(viewing \( C_i \) as a vertex of \( G \), as usual).

Proof. Consider the distance-based partition \( V(G) = Z_0 \cup \ldots \cup Z_m \) starting at \( C_0 \), defined in (25). For every \( i \) the vertex set \( Z_i \) corresponds to a subcurve, also written \( Z_i \), of \( X_0 \). We thus get a decomposition \( X_0 = Z_0 \cup \ldots \cup Z_m \).

To simplify the presentation, denote by \( s_i \) the quantity appearing in the right term of inequalities (27): \( s_i := 2r_i \) if \( r_i \leq g_i - 1 \) and \( s_i = r_i + g_i \) if \( r_i \geq g_i \).

The proof is by induction on the number \( \lambda \) of indices \( i \) such that \( r_i \neq 0 \).

For the base of the induction, i.e. the case \( \lambda = 1 \), we can suppose that \( r_0 = r \) and that all \( r_i \) are zero for \( i \geq 1 \). We have to show the existence of an effective divisor \( E \in \operatorname{Div}(G) \) such that \( E \sim \tau(\mathcal{L}) \) and \( E(C_0) \geq s_0 \).

We choose an effective divisor \( E \sim \tau(\mathcal{L}) \) on \( G \) which maximizes the vector \((E(C_0), E(Z_1), \ldots, E(Z_m))\) in the lexicographic order, i.e. we require that \( E(C_0) \) be maximum among all divisors in \( |\tau(\mathcal{L})| \); next, we require that \( E(Z_1) \) be maximum
Claim 4.8. Every global section of $E$ already done, so let

We denote $L_0 = \mathcal{L}(b_0) \in \text{Pic}(X_0)$, and similarly $L'_0 = \mathcal{L}'(b_0) \in \text{Pic}(X_0)$.

Set $W_0 = \overline{X_0 \smallsetminus C_0}$. To prove the claim, set $E' = \tau(\mathcal{L}') = \deg L'_0$, so that $E' \sim E$. Now, for every component $C \subset X_0$ we have

$$E'(C) = \deg_C L'_0 = E(C) - C \cdot C_0;$$

in particular $E'(C_0) > E(C_0)$. Therefore, by the maximality of $E(C_0)$, the divisor $E'$ is not effective. Hence the subcurve $Y_1 \subset X_0$ defined below is not empty

$$Y_1 := \bigcup_{E'(C) < 0} E(C) + C \cdot W_0 < 0 \subset C.$$

The degree of $L'_0$ on every component of $Y_1$ is negative, hence the claim holds on $Y_1$. If $Y_1 = \emptyset$, we are done. Otherwise, we iterate the procedure. Namely, for $h \geq 2$ we set

$$W_{h-1} := X_0 \smallsetminus C_0 \cup Y_1 \cup \ldots \cup Y_{h-1} \quad \text{and} \quad Y_h := \bigcup_{C \in W_{h-1}, \ E(C) + C \cdot W_{h-1} < 0} C.$$

To prove the claim we shall prove that the sections of $L'_0$ vanish identically on $Y_h$, and that $Y_h$ is empty only if $W_{h-1}$ is empty.

The first assertion will be proved by induction on $h$: the base case $h = 1$ is already done, so let $h \geq 2$. Pick $C \subset Y_h$ (hence $C \subset W_{h-1}$); note that $C \cdot W_{h-1} = -C \cdot (C_0 + \sum_{i=1}^{h-1} Y_i)$. By definition of $Y_h$ we have $E(C) < -C \cdot W_{h-1}$, hence

$$\deg_C \mathcal{L}' = E(C) - C_0 \cdot C < -C \cdot W_{h-1} - C_0 \cdot C = C \cdot \sum_{i=1}^{h-1} Y_i.$$

By induction, the sections of $L'_0$ vanish on $Y_i$ for all $i \leq h - 1$, in particular they have to vanish on $\bigcup_{i=1}^{h-1} Y_i \cap C$. By the above estimate, their number of zeroes is higher than $\deg_C \mathcal{L}'$; hence they must vanish identically on $C$, as wanted.

We prove the remaining assertion by contradiction. So, suppose $Y_h = \emptyset$ and $W_{h-1} \neq \emptyset$. Set

$$E_h := E + T_{W_{h-1}}$$

where $T_{W_{h-1}} \in \text{Prin}(G)$ as defined in (2); hence $E_h \sim E$. We have for any $C \subset X_0$

$$E_h(C) = E(C) + W_{h-1} \cdot C.$$

Since $Y_h$ is empty, one easily checks that $E_h \geq 0$. Hence, by the maximality of $E(C_0)$, we have $E_h(C_0) = E(C_0)$, i.e., $W_{h-1} \cdot C_0 = 0$. Hence $W_{h-1} \subset \bigcup_{j \geq 2} Z_j$ and $W_{h-1} \cdot Z_1 \geq 0$. By the maximality of $E(Z_1)$, we must have $E_h(Z_1) = E(Z_1)$, i.e., $W_{h-1} \cdot Z_1 = 0$. Therefore, using (2), we have $W_{h-1} \subset \bigcup_{j \geq 3} Z_j$. Arguing as before, the maximality of $E(Z_2)$ yields $W_{h-1} \cdot Z_2 = 0$.

Iterating, we arrive at $W_{h-1} \subset Z_m$, such that $W_{h-1} \cdot Z_{m-1} = 0$ by the maximality of $E(Z_{m-1})$; hence $W_{h-1} \subset Z_{m+1}$ which is empty. We reached a contradiction. Claim (3) is proved.

Observe now that the claim implies

$$h^0(X_0, L'_0) = h^0(C_0, L'_0(\mathcal{L}_0 \cap W_0)) = h^0(C_0, L_0(\mathcal{L}_0 \cap W_0 - C_0 \cap W_0)) = h^0(C_0, L_0).$$
Therefore, using Lemma 4.8 (2),
\[ r + 1 \leq h^0(X_0, L_0') = h^0(C_0, L_0). \]
Set \( r_0 := h^0(C_0, L_0) - 1 \) so that \( r_0 \geq r = r_0 \). By Clifford’s inequality and Riemann’s theorem (which do hold on the irreducible curve \( C_0 \), we obtain,
\[ E(C_0) = \deg_{E_0} L_0 \begin{cases} \geq 2r_0 & \text{if } r_0 \leq g_0 - 1 \\ = r_0 + g_0 & \text{if } r_0 \geq g_0. \end{cases} \]
Now, suppose \( r \leq g_0 - 1 \) and let us prove that \( E(C_0) \geq 2r \). If \( r_0 \leq g_0 - 1 \) then the first relation gives \( E(C_0) \geq 2r_0 \geq 2r \). If \( r_0 \geq g_0 \) the second relation gives
\[ E(C_0) = r_0 + g_0 \geq r + r + 1 = 2r \]
as claimed. Next, suppose \( r \geq g_0 \); then \( r_0 \geq r \geq g_0 \) so the second relation gives \( E(C_0) = r_0 + g_0 \geq r + g_0 \). This finishes the proof in the case \( \lambda = 1 \).

Consider now \( \lambda > 1 \) and suppose that the statement holds if the number of indices \( i \) with \( r_i \neq 0 \) is at most \( \lambda - 1 \). The argument follows closely the proof in the case \( \lambda = 1 \), modulo some technicalities.

Without loss of generality, we can assume \( r_0 \neq 0 \). By the induction hypothesis we can choose \( L \) so that for the divisor \( E = \tau(L) \), all the Inequalities (27) are verified for \( i \geq 1 \), and \( E(C_0) \) is a divisor. Furthermore, as before, we will assume that \( E \) maximizes the vector \( (E(C_0), E(Z_1), \ldots, E(Z_m)) \) in the lexicographic order, i.e., \( E(C_0) \) is maximum among all elements in \( |\tau(L)| \) verifying Inequalities (27) for \( i \geq 1 \), next that \( E(Z_i) \) is maximum among all such divisors, etc.

In order to prove the proposition, we need to show that \( E(C_0) \geq s_0 \).

Just as for the induction base we consider \( L' = L(-C_0) \in \text{Pic}_0(B) \), and use the same notation as before. We need the following generalization of Claim 4.8.

**Claim 4.9.** The dimension of the space of global sections of \( L'_i \) which identically vanish on \( X_0 \cap C_0 \) is at least \( r_0 + 1 \).

We will prove it following the same pattern as for Claim 4.8. We have
\[ E'(C) = \deg_C L_0' = E(C) - C \cdot C_0, \]
for any \( C \subset X_0 \); hence \( E'(C_0) > E(C_0) \). Therefore, by the maximality of \( E(C_0) \), the divisor \( E'_0 \) does not verify some of the inequalities in (27) for \( i \geq 1 \), and so the subcurve \( Y_1 \subset X_0 \) defined below is not empty
\[ Y_1 := \bigcup_{E'(C_i) < s_i} C_i = \bigcup_{E'(C_i) + C; W \subset s_i} C_i. \]
Since the degree of \( L'_0 \) on each component \( C_i \) of \( Y_1 \) is strictly smaller than \( s_i \), by Clifford’s inequality and Riemann’s theorem for \( C_i \) we have \( h^0(C_i, L'_0) \leq r_i \). Let \( \Lambda_1 \subset H^0(X_0, L'_0) \) be the space of sections which vanish on \( Y_1 \), so that we have
\[ 0 \longrightarrow \Lambda_1 = \ker \rho \longrightarrow H^0(X_0, L'_0) \overset{\rho}{\longrightarrow} H^0(Y_1, L'_0) \overset{\rho}{\longrightarrow} \bigoplus_{C_i \subset Y_1} H^0(C_i, L'_0) \]
where \( \rho \) denotes the restriction.

From this sequence and the above estimate we get
\[ \dim \Lambda_1 \geq h^0(X_0, L'_0) - \sum_{i; C_i \subset Y_1} r_i \geq r + 1 - \sum_{i \geq 1} r_i = r_0 + 1. \]
Therefore, by maximality assumption, we must have sections of \(W\), i.e., conclude that \(X\) smoothing of a projective nodal curve as been done above. Consider \(C_j \subset Y_h\), so that \(E(C_j) < s_j - C_j \cdot W_{h - 1}\), hence

\[E'(C_j) = E(C_j) - C_0 \cdot C_j < s_j - C_j \cdot W_{h - 1} - C_0 \cdot C_j = s_j + C_j \cdot \left(\sum_{i=1}^{h-1} Y_i\right).\]

as \(C_j \cdot W_{h - 1} = -C_j \cdot (C_0 + \sum_{i=1}^{h-1} Y_i)\). Hence \((L'_0)|_{C_j} (-C_j \cdot \sum_{i=1}^{h-1} Y_i)\) has degree smaller than \(s_j\), therefore by Clifford’s inequality and Riemann’s theorem for \(C_j\),

\[h^0(C_j, L'_0(-C_j \cdot \sum_{i=1}^{h-1} Y_i) \leq r_j.\]

Let us denote by \(\rho_h : \Lambda_{h - 1} \rightarrow H^0(Y_h, L'_0)\) the restriction map. Then we have

\[0 \rightarrow \Lambda_h = \ker \rho_h \rightarrow \Lambda_{h - 1} \xrightarrow{\rho_h} \Im \rho_h \rightarrow \bigoplus_{C_j \subset Y_h} H^0(C_j, L'_0(-C_j \cdot \sum_{i=1}^{h-1} Y_i).\]

Hence the codimension of \(\Lambda_h\) in \(\Lambda_{h - 1}\), written \(\text{codim}_{\Lambda_{h - 1}} \Lambda_h\), is at most the dimension of the space on the right, which, by Claim 13, is at most \(\sum_{j; C_j \subset Y_h} r_j\). Therefore

\[\text{codim } \Lambda_h = \text{codim } \Lambda_{h - 1} + \text{codim } \Lambda_{h - 1} \cdot \Lambda_h \leq \sum_{i; C_i \subset Y_h \cup U_{h - 1}} r_i + \sum_{j; C_j \subset Y_h} r_j\]

where we used the induction hypothesis on \(\Lambda_{h - 1}\). The first claim is proved.

For the proof of the second statement, suppose, by contradiction, \(Y_h = \emptyset\) and \(W_{h - 1} \neq \emptyset\). Set, as in Theorem 4.7, \(E_h = E + Tw_{h - 1}\). We have for any \(C \subset X_0\)

\[E_h(C) = E(C) + W_{h - 1} \cdot C.\]

Since \(Y_h\) is empty, we get \(E_h(C) \geq s_i\) for any \(C \subset W_{h - 1}\). On the other hand, for any \(C \subset X \setminus W_{h - 1}\), we have \(E_h(C) \geq E(C)\). Therefore, by the choice of \(E\), and the maximality assumption, we must have \(E_h(C_0) = E(C_0)\), i.e., \(W_{h - 1} \cdot C_0 = 0\). Therefore \(W_{h - 1} \subset U_{j \geq 2} Z_j\) and hence \(W_{h - 1} \cdot Z_1 \geq 0\). In particular, we have \(E_h(Z_1) \geq E(Z_1)\). But, by the maximality of \(E(Z_1)\), we must have \(E_h(Z_1) = E(Z_1)\), i.e., \(W_{h - 1} \cdot Z_1 = 0\). Therefore \(W_{h - 1} \subset U_{j \geq 3} Z_j\). Repeating this argument, we conclude that \(W_{h - 1} \subset Z_{m + 1} = \emptyset\), which is a contradiction. Claim 13 is proved.

The rest of the proof is almost identical to the case \(\lambda = 1\). Let \(\Lambda\) be the set of sections of \(L'_0\) which identically vanish on \(W_0\); by the claim, \(\dim \Lambda \geq r_0 + 1\). We have a natural injection \(\Lambda \hookrightarrow H^0(C_0, L'_0(-C_0 \cap W_0)) = H^0(C_0, L_0)\), hence \(r_0 + 1 = h^0(C_0, L_0)\). Arguing as before, we obtain \(E(C_0) \geq s_0\). Since \(E(C_i) \geq s_i\) already holds for \(i \geq 1\) (by our choice of \(E\)), the proof of Proposition 4.7 is complete.

**Theorem 4.10** (Specialization Lemma). Let \(\phi : X \rightarrow B\) be a regular one-parameter smoothing of a projective nodal curve \(X_0\). Let \((G, \omega)\) be the weighted dual graph of \(X_0\). Then for every \(L \in \text{Pic}_d(B)\) there exists an open neighborhood \(U \subset B\) of \(b_0\) such that for every \(b \in U\) such that \(b \neq b_0\)

\[r(X_b, L(b)) \leq r((\mathcal{G}, \omega))(r(L)).\]
Proof. To simplify the presentation, we will assume \( G \) free from loops, and indicate, at the end, the (trivial) modifications needed to get the proof in general.

Up to restricting \( B \) to an open neighborhood of \( b_0 \) we can assume that for some \( r \geq -1 \) and for every \( b \in B \) we have

\[
 h^0(X_b, L(b)) \geq r + 1
\]

with equality for \( b \neq b_0 \). Set \( D = \tau(L) \); we must prove that \( r(G_\omega)(D) \geq r \).

As in Proposition 4.7 we write \( C_0, C_1, \ldots, C_n \) for the irreducible components of \( X \), with \( C_i \) of genus \( g_i \). We denote by \( v_i \in V(G) \) the vertex corresponding to \( C_i \).

Recall that we denote by \( \hat{G}^\omega \) the weightless, loopless graph obtained from \( G \) by adding \( g_i = \omega(v_i) \) 2-cycles at \( v_i \) for every \( v_i \in V(G) \). We have a natural injection (viewed as an inclusion) \( \text{Div}(G) \subset \text{Div}(\hat{G}^\omega) \) and, by definition, \( r(G_\omega)(D) = r(\hat{G}^\omega)(D) \).

Summarizing, we must prove that

\[
 r(\hat{G}^\omega)(D) \geq r.
\]

The specialization Lemma for weightless graphs gives that the rank of \( D \), as a divisor on the weightless graph \( G \), satisfies

\[
 r_G(D) \geq r.
\]

Now observe that the graph obtained by removing from \( \hat{G}^\omega \) every edge of \( G \) is a disconnected (unless \( n = 0 \)) graph \( R \) of type

\[
 R = \bigcup_{i=0}^{n} R_i
\]

where \( R_i = \hat{R}_0 \) is the refinement of the “rose” \( R_0 \), introduced in \( 3.4 \) for every \( i = 0, \ldots, n \). Note that if \( g_i = 0 \), the graph \( R_i \) is just the vertex \( v_i \) with no edge.

Now, extending the notation of \( 2.5 \) to the case of multiple cut-vertices, we have the following decomposition of \( \hat{G}^\omega \)

\[
 \hat{G}^\omega = G \lor R
\]

with \( G \cap R = \{ v_0, \ldots, v_n \} \). By Lemma \( 3.4.3 \) for any \( D \in \text{Div}(G) \) such that \( r_G(D) \geq 0 \) we have \( r(\hat{G}^\omega)(D) \geq 0 \).

We are ready to prove \( 33 \) using induction on \( r \). If \( r = -1 \) there is nothing to prove. If \( r = 0 \), by \( 34 \) we have \( r_G(D) \geq 0 \) and hence, by what we just observed, \( r(\hat{G}^\omega)(D) \geq 0 \). So we are done.

Let \( r \geq 1 \) and pick an effective divisor \( E \in \text{Div}^r(\hat{G}^\omega) \). Suppose first that \( E(v) = 0 \) for all \( v \in V(G) \); in particular, \( E \) is entirely supported on \( R \). We write \( r_i \) for the degree of the restriction of \( E \) to \( R_i \), so that for every \( i = 0, \ldots, n \), we have

\[
 r_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{n} r_i = r.
\]

It is clear that it suffices to prove the existence of an effective divisor \( F \sim D \) such that the restrictions \( F_{R_i} \) and \( E_{R_i} \) to \( R_i \) satisfy \( r_{R_i}(F_{R_i} - E_{R_i}) \geq 0 \) for every \( i = 0, \ldots, n \).

By Proposition 3.7, there exists an effective divisor \( F \sim D \) so that \( 27 \) holds for every \( i = 0, \ldots, n \), i.e.

\[
 F(C_i) \geq \begin{cases} 
 2r_i & \text{if } r_i \leq g_i - 1 \\
 r_i + g_i & \text{if } r_i \geq g_i.
 \end{cases}
\]

(Proposition 3.7 applies because of the relations \( 35 \)). Now, \( F(C_i) \) equals the degree of \( F_{R_i} \), hence by the above estimate combined with Theorem 3.6.2 and Lemma 3.7 one easily checks that \( r_{R_i}(F_{R_i}) \geq r_i \), hence, \( r_{R_i}(F_{R_i} - E_{R_i}) \geq 0 \).
We can now assume that \( E(v) \neq 0 \) for some \( v \in V(G) \subset V(\tilde{G}^\omega) \). We write \( E = E' + v \) with \( E' \geq 0 \) and \( \deg E' = r - 1 \).

Arguing as for [23, Claim 4.6], we are free to replace \( \phi : X \to B \) by a finite étale base change. Therefore we can assume that \( \phi \) has a section \( \sigma \) passing through the component of \( X_0 \) corresponding to \( v \). It is clear that for every \( b \in B \) we have
\[
  r(X_b, L_b(-\sigma(b))) \geq r(X_b, L_b) - 1 \geq r - 1.
\]

Now, the specialization of \( \mathcal{L} \otimes \mathcal{O}(-\sigma(B)) \) is \( D - v \), i.e.,
\[
  \tau(\mathcal{L} \otimes \mathcal{O}(-\sigma(B))) = D - v.
\]

By induction we have \( \tau_{\tilde{G}^\omega}(D - v) \geq r - 1 \). Hence, the degree of \( E' \) being \( r - 1 \), there exists \( T \in \Prin(\tilde{G}^\omega) \) such that
\[
  0 \leq D - v - E' + T = D - v - (E - v) + T = D - E + T.
\]

We thus proved that \( 0 \leq \tau_{\tilde{G}^\omega}(D - E) \) for every effective \( E \in \Div'(\tilde{G}^\omega) \). This proves (33) and hence the theorem, in case \( G \) has no loops.

If \( G \) admits some loops, let \( G' \subset G \) be the graph obtained by removing from \( G \) all of its loop edges. Then \( \tilde{G}^\omega \) is obtained from \( G' \) by adding to the vertex \( v_i \), exactly \( g_i \), 2-cycles, where \( g_i \) is the arithmetic genus of \( C_i \) (note that \( g_i \) is now equal to \( \omega(v_i) \) plus the number of loops adjacent to \( v_i \) in \( G \)). Now replace \( G' \) by \( G'' \) and use exactly the same proof. (Alternatively, one could apply the same argument used in [23, Prop. 5.5], where the original Specialization Lemma of [13] was extended to weightless graphs admitting loops.)

5. Riemann-Roch on weighted tropical curves

5.1. Weighted tropical curves as pseudo metric graphs. Let \( \Gamma = (G, \omega, \ell) \) be a weighted tropical curve, that is, \( (G, \omega) \) is a weighted graph (see Section 3.2) and \( \ell : E(G) \to \mathbb{R}_{\geq 0} \) is a (finite) length function on the edges. We also say that \( (G, \ell) \) is a metric graph.

If \( \omega \) is the zero function, we write \( \omega = 0 \) and say that the tropical curve is pure.

Weighted tropical curves were used in [BMV] to bordify the space of pure tropical curves; notice however that we use the slightly different terminology of [23].

For pure tropical curves there exists a good divisor theory for which the Riemann-Roch theorem holds, as proved by Gathmann-Kerber in [2K] and by Mikhalkin-Zharkov in [MZ]. The purpose of this section is to extend this to the weighted setting.

**Divisor theory on pure tropical curves.** Let us quickly recall the set-up for pure tropical curves; we refer to [2K] for details. Let \( \Gamma = (G, \omega, \ell) \) be a pure tropical curve. The group of divisors of \( \Gamma \) is the free abelian group \( \text{Div}(\Gamma) \) generated by the points of \( \Gamma \).

A rational function on \( \Gamma \) is a continuous function \( f : \Gamma \to \mathbb{R} \) such that the restriction of \( f \) to every edge of \( \Gamma \) is a piecewise affine integral function (i.e., piecewise of type \( f(x) = ax + b \), with \( a, b \in \mathbb{Z} \)) having finitely many pieces.

Let \( p \in \Gamma \) and let \( f \) be a rational function as above. The order of \( f \) at \( p \), written \( \text{ord}_p f \), is the sum of all the slopes of \( f \) on the outgoing segments of \( \Gamma \) adjacent to \( p \). The number of such segments is equal to the valency of \( p \) if \( p \) is a vertex of \( \Gamma \), and is equal to 2 otherwise. The divisor of \( f \) is defined as follows
\[
  \text{div}(f) := \sum_{p \in \Gamma} \text{ord}_p(f)p \in \text{Div}(\Gamma).
\]
Recall that div \( f \) has degree 0. The divisors of the form div(\( f \)) are called \textit{principal} and they form a subgroup of Div(\( \Gamma \)), denoted by Prin(\( \Gamma \)). Two divisors \( D, D' \) on \( \Gamma \) are said to be linearly equivalent, written \( D \sim D' \), if \( D - D' \in \text{Prin}(\Gamma) \).

Let \( D \in \text{Div}(\Gamma) \). Then \( R(D) \) denotes the set of rational functions on \( \Gamma \) such that div(\( f \)) + \( D \geq 0 \). The rank of \( D \) is defined as follows

\[
\tau(D) := \{ \max k : \forall E \in \text{Div}^k(\Gamma), \ R(D - E) \neq \emptyset \}
\]

so that \( \tau(D) = -1 \) if and only if \( R(D) = \emptyset \).

The following trivial remark is a useful consequence of the definition.

\textbf{Remark 5.1.} Let \( \Gamma_1 \) and \( \Gamma_2 \) be pure tropical curves and let \( \psi : \text{Div}(\Gamma_1) \to \text{Div}(\Gamma_2) \) be a group isomorphism inducing an isomorphism of effective and principal divisors (i.e., \( \psi(D) \geq 0 \) if and only if \( D \geq 0 \), and \( \psi(D) \in \text{Prin}(\Gamma_2) \) if and only if \( D \in \text{Prin}(\Gamma_1) \)). Then for every \( D \in \text{Div}(\Gamma_1) \) we have \( \tau_{\Gamma_1}(D) = \tau_{\Gamma_2}(\psi(D)) \).

To extend the theory to the weighted setting, our starting point is to give weighted tropical curves a geometric interpretation by what we call pseudo-metric graphs.

\textbf{Definition 5.2.} A \textit{pseudo-metric graph} is a pair \((G, \ell)\) where \( G \) is a graph and \( \ell \) a \textit{pseudo-length} function \( \ell : E(G) \to \mathbb{R}_{\geq 0} \) which is allowed to vanish only on loop-edges of \( G \) (that is, if \( \ell(e) = 0 \) then \( e \) is a loop-edge of \( G \)).

Let \( \Gamma = (G, \omega, \ell) \) be a weighted tropical curve, we associate to it the pseudo-metric graph \((G^\omega, \ell^\omega)\), defined as follows. \( G^\omega \) is the “virtual” weightless graph associated to \((G, \omega)\) described in subsection 5.2. \( G^\omega \) is obtained by attaching to \( G \) exactly \( \omega(v) \) loops based at every vertex \( v \); the function \( \ell^\omega : E(G^\omega) \to \mathbb{R}_{\geq 0} \) is the extension of \( \ell \) vanishing at all the virtual loops.

It is clear that \((G^\omega, \ell^\omega)\) is uniquely determined. Conversely, to any pseudo-metric graph \((G_0, \ell_0)\) we can associate a unique weighted tropical curve \((G, \omega, \ell)\) such that \( G_0 = G^\omega \) and \( \ell_0 = \ell^\omega \) as follows. \( G \) is the subgraph of \( G_0 \) obtained by removing every loop-edge \( e \in E(G) \) such that \( \ell_0(e) = 0 \). Next, \( \ell \) is the restriction of \( \ell_0 \) to \( G \); finally, for any \( v \in V(G) = V(G_0) \) the weight \( \omega(v) \) is defined to be equal to the number of loop-edges of \( G_0 \) adjacent to \( v \) and having length 0.

Summarizing, we have proved the following.

\textbf{Proposition 5.3.} The map associating to the weighted tropical curve \( \Gamma = (G, \omega, \ell) \) the pseudo-metric graph \((G^\omega, \ell^\omega)\) is a bijection between the set of weighted tropical curves and the set of pseudo-metric graphs, extending the bijection between pure tropical curves and metric graphs (see [MZ]).

\textbf{5.2. Divisors on weighted tropical curves.} Let \( \Gamma = (G, \omega, \ell) \) be a weighted tropical curve. There is a unique pure tropical curve having the same metric graph as \( \Gamma \), namely the curve \( \Gamma^p := (G, \ell_0) \). Exactly as for pure tropical curves, we define the group of divisors of \( \Gamma \) as the free abelian group generated by the points of \( \Gamma \):

\[ \text{Div}(\Gamma) = \text{Div}(\Gamma^p) = \{ \sum_{i=1}^m n_i p_i, \ n_i \in \mathbb{Z}, \ p_i \in (G, \ell) \} \]

The canonical divisor of \( \Gamma \) is

\[ K_\Gamma := \sum_{v \in V(G)} (\text{val}(v) + 2\omega(v) - 2)v \]

where \( \text{val}(v) \) is the valency of \( v \) as vertex of the graph \( G \). Observe that there is an obvious identification of \( K_\Gamma \) with \( K_{(G, \omega)} \), in other words, the canonical divisor of \( K_\Gamma \) is the canonical divisor of the virtual graph \( G^\omega \) associated to \((G, \omega)\).
Consider the pseudo-metric graph associated to $K_\Gamma$ by the previous proposition: $(G^\omega, \ell^\omega)$. Note that $(G^\omega, \ell^\omega)$ is not a tropical curve as the length function vanishes at the virtual edges. We then define a pure tropical curve, $\Gamma^\omega_\epsilon$, for every $\epsilon > 0$

$$\Gamma^\omega_\epsilon = (G^\omega, 0, \ell^\omega_\epsilon)$$

where $\ell^\omega_\epsilon(e) = \epsilon$ for every edge lying in some virtual cycle, and $\ell^\omega_\epsilon(e) = \ell(e)$ otherwise. Therefore $(G^\omega, \ell^\omega)$ is the limit of $\Gamma^\omega_\epsilon$ as $\epsilon$ goes to zero. Notice that for every curve $\Gamma^\omega_\epsilon$ we have a natural inclusion

$$\Gamma^\omega_0 \subset \Gamma^\omega_\epsilon$$

(with $\Gamma^\omega_0$ introduced at the beginning of the subsection). We refer to the loops given by $\Gamma^\omega_\epsilon \setminus \Gamma^\omega_0$ as virtual loops.

Now, we have natural injective homomorphism for every $\epsilon$

$$\iota_\epsilon : \text{Div}(\Gamma) \hookrightarrow \text{Div}(\Gamma^\omega_\epsilon)$$

and it is clear that $\iota_\epsilon$ induces an isomorphism of $\text{Div}(\Gamma)$ with the subgroup of divisors on $\Gamma^\omega_\epsilon$ supported on $\Gamma^\omega_0$.

**Theorem 5.4.** Let $\Gamma = (G, \omega, \ell)$ be a weighted tropical curve of genus $g$ and let $D \in \text{Div}(\Gamma)$. Using the above notation, the following hold.

1. The number $rt_\Gamma(\iota_\epsilon(D))$ is independent of $\epsilon$. Hence we define $rt_\Gamma(D) := rt_\Gamma(\iota_\epsilon(D))$.

2. (Riemann-Roch) With the above definition, we have

$$rt_\Gamma(D) - rt_\Gamma(K_\Gamma - D) = \deg D - g + 1.$$

**Proof.** The proof of (1) can be obtained by a direct limit argument to compute $rt_\Gamma(D)$, using Proposition 5.3. A direct proof is as follows.

For two $\epsilon_1, \epsilon_2 > 0$, consider the homothety of ratio $\epsilon_2/\epsilon_1$ on all the virtual loops. This produces a homeomorphism $\psi^{(\epsilon_1, \epsilon_2)} : \Gamma^\omega_{\epsilon_1} \rightarrow \Gamma^\omega_{\epsilon_2}$

(equal to identity on $\Gamma$), and hence a group isomorphism

$$\psi^{(\epsilon_1, \epsilon_2)} : \text{Div}(\Gamma^\omega_{\epsilon_1}) \rightarrow \text{Div}(\Gamma^\omega_{\epsilon_2}); \quad \sum_{p \in \Gamma} n_p p \mapsto \sum_{p \in \Gamma} n_p \psi^{(\epsilon_1, \epsilon_2)}(p).$$

Note that $\psi^{(\epsilon_2, \epsilon_1)}$ is the inverse of $\psi^{(\epsilon_1, \epsilon_2)}$, and that $\psi^{(\epsilon_1, \epsilon_2)} \circ \iota_{\epsilon_1} = \iota_{\epsilon_2}$; see (36).

Note also that $\psi^{(\epsilon_1, \epsilon_2)}$ induces an isomorphism at the level of effective divisors.

We claim that $\psi^{(\epsilon_1, \epsilon_2)}$ induces an isomorphism also at the level of principal divisors. By Remark 5.2, the claim implies part (1).

To prove the claim, let $f$ be a rational function on $\Gamma^\omega_{\epsilon_1}$. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be the homothety of ratio $\epsilon_2/\epsilon_1$ on $\mathbb{R}$, i.e., the automorphism of $\mathbb{R}$ given by $\alpha(x) = x\epsilon_2/\epsilon_1$ for any $x \in \mathbb{R}$. Define the function $\alpha \bullet f$ on $\Gamma^\omega_{\epsilon_1}$ by requiring that for any point of $x \in \Gamma$, $\alpha \bullet f(x) = f(x)$, and for any point $u$ of a virtual loop of $\Gamma^\omega_{\epsilon_1}$ attached at the point $v \in \Gamma$ we set

$$\alpha \bullet f(u) = f(v) + \alpha(f(v) - f(u)).$$

The claim now follows by observing that $(\alpha \bullet f) \circ \psi^{(\epsilon_2, \epsilon_1)}$ is a rational function on $\Gamma^\omega_{\epsilon_2}$, and

$$\text{div}((\alpha \bullet f) \circ \psi^{(\epsilon_2, \epsilon_1)}) = \psi^{(\epsilon_1, \epsilon_2)}(\text{div}(f)).$$

Part (1) is proved.

To prove part (2), recall that, as we said before, for the pure tropical curves $\Gamma^\omega_\epsilon$ the Riemann-Roch theorem holds, and hence this part follows from the previous one.
Remark 5.5. It is clear from the proof of Theorem 5.4 that there is no need to fix the same $\epsilon$ for all the virtual cycles. More precisely, fix an ordering for the virtual cycles of $G^\omega$ and for their edges; recall there are $\sum_{v \in V(G)} \omega(v)$ of them. Then for any $\underline{\epsilon} \in \mathbb{R}^{\sum_{v \in V(G)} \omega(v)}$ we can define the pure tropical curve $\Gamma_{\underline{\epsilon}}$ using $\underline{\epsilon}$ to define the length on the virtual cycles in the obvious way. Then for any $D \in \text{Div}(\Gamma)$ the number $\text{rt}_{\underline{\epsilon}}(\iota_{\underline{\epsilon}}(D))$ is independent of $\underline{\epsilon}$ (where $\iota_{\underline{\epsilon}}$ is the analog of (36)).

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