Polynomial functors and opetopes

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Abstract

We give an elementary and direct combinatorial definition of opetopes in terms of trees, well-suited for graphical manipulation (e.g. drawings of opetopes of any dimension and basic operations like sources, target, and composition); a substantial part of the paper is constituted by drawings and example computations. To relate our definition to the classical definition, we recast the Baez-Dolan slice construction for operads in terms of polynomial monads: our opetopes appear naturally as types for polynomial monads obtained by iterating the Baez-Dolan construction, starting with the trivial monad. Finally we observe a suspension operation for opetopes, and define a notion of stable opetopes. Stable opetopes form a least fixpoint for the Baez-Dolan construction. The calculus of opetopes is also well-suited for machine implementation: in an appendix we show how to represent opetopes in XML, and manipulate them with simple Tcl scripts.

Introduction

Among a dozen or so existing definitions of weak higher categories, the opetopic approach is one of the most intriguing, since it is based on a collection of `shapes' that had not previously been studied: the opetopes. Opetopes are combinatorial structures parametrising higher-dimensional many-in/one-out operations, and can be seen as higher-dimensional generalisations of trees. They are important combinatorial structures on their own, `as pervasive in higher-dimensional algebra as simplices are in geometry', according to Leinster [9] p.216. Opetopes and opetopic higher categories were introduced by Baez and Dolan in the seminal paper [11], and the theory has been developed further by Hermida-Makkai-Power [6], Leinster [2] and Cheng [2], [3], [4]. It is in a sense a theory from scratch, compared to several other theories of higher categories which build on large bodies of preexisting machinery and experience, e.g. simplicial methods. The full potential of the opetopic approach may depend on a deeper understanding of the combinatorics of opetopes.

At the conference on n-categories: Foundations and applications at the IMA in Minneapolis, June 2004, much time was dedicated to opetopes, but it became clear that a concise and direct definition of opetopes was lacking, and that there was no practical
way to represent higher-dimensional opetopes on the blackboard. In fact, there did not seem to exist a general method to represent concrete opetopes in any way, algebraic, graphical, or by machine\footnote{In fact a method does exist for algebraic/mechanical representation: Hermida-Makkai-Power \cite{6} final section] explain how in principle any opetope (there called multitope) can be serialised into a string of hash signs and stars, with two sorts of brackets. We admit we do not have any experience with that representation.} The best definitions are very abstract and not very hands-on: e.g. Leinster’s definition in terms of iterated free cartesian monads \cite{9}, or the Hermida-Makkai-Power \cite{6} definition of opetopic sets (there called multitopic sets), followed by a theorem that this category is a presheaf category, hence characterising a category of opetopes (there called multitopes).

As to graphical representations of opetopes in low dimensions, the current method is based on a polytope interpretation of opetopes (which is at the origin of the terminology: the word ‘opetope’ comes from ‘operation’ and ‘polytope’). Leinster \cite[§ 7.4]{9} has constructed a geometric realisation functor which provides support for this interpretation, although the polytopes in general can not be piece-wise linear objects in Euclidean space. Moreover, geometrical objects in dimension higher than 3 are inherently difficult to represent graphically, and currently one resorts to Lego-like drawings in which the individual faces of the polytopes are drawn separately, with small arrows as a recipe to indicate how they are supposed to fit together.

The goal of this paper is to come closer to the combinatorics. Our initial idea was to represent an opetope as a tree with some circles, which we now call constellations. This works in dimension 4 (cf. \ref{1.1} below), but it does not seem to be sufficient to capture the possible opetopes in dimension 5 and higher. Pursuing the idea, what we eventually found was a representation in terms of a sequence of trees with circles, and in fact it is basically the notion of metatree originally proposed by Baez and Dolan. That notion was never really developed, though: in the original paper \cite{1} the claim that metatrees could express opetopes was not really substantiated, and in the subsequent literature there seems to be no mention of the metatree notion. The presence of circles makes a conceptual difference, and it also reveals a certain shortcoming in the original notion of metatree, related to units.

We hasten to point out that our notion of opetope coincides with the notion due to Leinster \cite{9} (cf. the explicit comparison given in \ref{3.1}), not with the original Baez-Dolan definition: we work consistently with non-planar trees, which means our opetopes are ‘un-ordered’ like abstract geometric objects, whereas the original Baez-Dolan opetopes come equipped with an ordering of their faces. In our version, the planar aspect is only a particular feature of low dimensional opetopes.

While our opetopes agree with Leinster’s, the description we provide is completely elementary and does not even make reference to category theory. We think that our description can serve as the famous ‘5-minute definition’ that was previously missing,
and that it can provide a convenient tool for communicating opetopical ideas. We also indicate how our approach is well-suited for machine manipulation.

It should be mentioned that another purely combinatorial description exists: Palm [10] has introduced a notion of dendrotope: dendrotopes are certain decorated Hasse diagrams. He shows that dendrotopic sets correspond to many-to-one computads, and hence by an unpublished result of Harnik, Makkai and Zawadowski, and by the Hermida-Makkai-Power theorem, dendrotopes should correspond to opetopes. However, a direct combinatorial comparison has not been given at this time.

Let us briefly outline the organisation of the exposition. In the first section we give the definition of opetopes in a direct combinatorial way, without reference to category theory. The crucial ingredient is the correspondence between non-planar trees and nestings of circles: an opetope is merely a sequence of such correspondences, with an initial condition. We give the definition in two steps: first the elementary ‘5-minute definition’ with examples, then we develop the involved notions of trees and constellations more formally and compare with Baez-Dolan metatrees. It is possible to jump directly from the ‘5-minute definition’ to Section 4 where the same elementary and hands-on approach is pursued to describe in detail how to compute sources and targets of opetopes, and how to compose them. However, such a reading would ignore the theoretical justification for all the definitions and constructions.

Sections 2 and 3 form the theoretical heart of this work: we use the notions of polynomial functors and polynomial monads to relate the combinatorial definition of Section 1 to the original definition by Baez and Dolan [1], and more specifically to the elegant redefinition due to Leinster [9]. After reviewing some basic facts about polynomial functors in Section 2 we give an easy account of the Baez-Dolan slice construction in the setting of polynomial monads. From the graphical description of polynomial functors we see that the Baez-Dolan construction is about certain decorated trees. The double Baez-Dolan construction gives trees decorated with trees, subject to complicated compatibility conditions. We show that these compatibility conditions are completely encoded by drawing circles in trees. Iterating the Baez-Dolan construction involves the correspondence between trees and nestings, and it readily follows that the opetopes defined in Section 1 arise precisely as types for the monads produced by iterating the Baez-Dolan construction, starting from the trivial monad.

In Section 4 we show by way of examples how the calculus of opetopes works in practice: we are concerned with computing sources and target of opetopes, and with composing them.

In the short Section 5 we observe a suspension operation for opetopes, and define a notion of stable opetopes. The stable opetopes also form a polynomial monad, and we show this is the least fixpoint for the (pointed) Baez-Dolan construction.

In the Appendix we briefly describe a machine implementation of the ‘calculus of opetopes’ based on XML, including a mechanism for automated graphical output.
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1 Opetopes

We first give the quick definition of opetope, through the notions of tree, constellation, and zoom. Afterwards we develop these notions more carefully.

The ‘5-minute definition’ of opetope

1.1 Trees. The fundamental concept is that of a tree. Our trees are non-planar finite rooted trees with boundary: they have any number of input edges (called leaves), and have precisely one output edge (called the root edge) always drawn at the bottom. There is a partial order in which the root is the maximal element and the leaves are minimal elements. The following drawings should suffice to exemplify trees, but beware that the planar aspect inherent in a drawing should be disregarded:

A formal definition of tree is given in 1.14.

1.2 Nestings. Another graphical representation of the same structure is given in terms of nested circles in the plane. We prefer to talk about nested spheres in space to avoid
any idea of planarity when in a moment we combine the notion with trees. A **nesting** is a finite collection of non-intersecting spheres and dots, which either consists of a single dot (and no spheres) or has one outer sphere, containing all the other spheres and dots.

The dots of a nesting correspond to the leaves of the tree. The outer sphere corresponds to the root edge of the tree, and the special case of a nesting which consists solely of one dot corresponds to the dotless tree. The partial order is simply inclusion.

The following drawings of nestings correspond exactly to the five trees drawn above.

![Diagram of nestings corresponding to trees](image)

**1.3 Correspondences.** A **correspondence** between a nesting $S$ and a tree $T$ consists of specified bijections

$$
dots(S) \leftrightarrow \text{leaves}(T)$$

$$
spheres(S) \leftrightarrow \text{dots}(T)$$

respecting the partial orders. Here is a typical picture:

![Diagram of correspondence](image)

The bijections are indicated by the labels $a, b, c, d, e, f, g$.

**1.4 Constellations.** A **constellation** is a superposition of a tree with a nesting with common set of dots, and such that each sphere cuts a subtree. Here is an example:

![Diagram of constellations](image)

More precisely, it is a configuration $C$ of edges, dots, and spheres, such that
(i) edges and dots form a tree (called the underlying tree of $C$),
(ii) dots and spheres form a nesting (the underlying nesting of $C$),
(iii) for each sphere, the edges and dots contained in it form a tree again.

A purely combinatorial definition of constellation is given in 1.16.

Let us briefly take a look at some degenerate examples. In a constellation without a sphere, the underlying nesting is necessarily a single dot. Hence the possibilities in this case are exhausted by the set of trees with only one dot:

![Tree examples](image1)

In a constellation without dots, the underlying tree must be a single edge. There must be an outer sphere, so such constellations may look like these examples:

![Tree examples](image2)

Note that every sphere must contain a segment of the line, since there is no such thing as the empty tree.

Finally, we draw a few examples of constellations without leaves:

![Tree examples](image3)

In 3.7 it is shown that constellations represent, in a precise sense, trees of trees, which is the reason for their importance. We want to iterate the idea of trees of trees by repeating the step of drawing spheres. To do this, we shift the nesting to a tree and iterate. In our terminology, we zoom:

**1.5 Zooms.** A zoom from constellation $A$ to constellation $B$, written

$$A \circlearrowleft B,$$

is a correspondence between the underlying nesting of $A$ and the underlying tree of $B$. In other words, there are specified two bijections:

$$\text{dots}(A) \leftrightarrow \text{leaves}(B)$$

$$\text{spheres}(A) \leftrightarrow \text{dots}(B)$$

respecting the partial orders.

Here is an example:
The bijections are indicated with numbers.
We also wish to exhibit the two most degenerate zooms:

\[
\begin{array}{c}
X_0 \circlearrowleft & X_1 \circlearrowleft & X_2 \circlearrowleft & X_3 \circlearrowleft & \ldots \circlearrowleft & X_n.
\end{array}
\]

1.6 Zoom complexes. A zoom complex of degree \( n \) is a sequence of zooms

1.7 Opetopes. An opetope of dimension \( n \) is defined to be a zoom complex \( X \) of degree \( n \) starting like this:

\[
\begin{array}{c}
X_0 \circlearrowleft & X_1 \circlearrowleft & X_2 \circlearrowleft & X_3 \circlearrowleft & \ldots \circlearrowleft & X_n.
\end{array}
\]

(1)

Here, \( X_0 \) and \( X_1 \) are exactly as drawn, while \( X_2 \) is described verbally as having one dot and one leaf (necessary in order to be in zoom relation with \( X_1 \)), and having any finite number of linearly nested spheres.

1.8 Remark. This definition of opetope should be attributed to Baez and Dolan [1] who introduced the notion of opetope in terms of a slice construction for symmetric operads (a polynomial analogue of which we shall call the Baez-Dolan construction (Section 3)), and offered an alternative description in terms of sequences of trees called metatrees. Definition 1.7 features important adjustments to the Baez-Dolan notion of metatree, as we shall explain in 1.21.
1.9 Examples. A 0-opetope is the zoom complex \( \circ \) (there is only one such), and a 1-opetope is the zoom complex \( \bullet \rightarrow \circ \) (again there is only one such). The 2-opetopes are in bijection with the natural numbers, counting the linearly nested spheres in \( X_2 \).

For \( n \geq 3 \), there are no restrictions on the constellations \( X_n \), except to be in zoom relation with \( X_{n-1} \). For example, if there are \( n \) spheres in \( X_2 \), then the zoom condition forces \( X_3 \) to be a straight line with \( n \) dots on (and the bijection between spheres and dots is uniquely determined since the linear nesting of the spheres in \( X_2 \) must correspond to the linear arrangement of the dots in \( X_3 \)), and any nesting can be drawn on top of that. Here is an example:

\[
\begin{align*}
X_0 & \quad X_1 & \quad X_2 & \quad X_3 \\
\bullet & \quad \circ & \quad \bullet & \quad \circ \\
\end{align*}
\]

Clearly the information encoded in \( X_0 \), \( X_1 \) and \( X_2 \) is redundant, and a 3-opetope is completely specified by a \( X_3 \) of this form: a line with dots and 'spheres'. This is equivalent to specifying a planar tree. The planarity comes about because there is a line organising the dots in \( X_3 \), which in turn is a consequence of the linear nesting of the spheres in \( X_2 \). Here is the planar tree corresponding to the 3-opetope above:

\[
\begin{align*}
\vdots \\
\end{align*}
\]

and here is how this 3-opetope would be represented in the polytope style, as in Leinster's book [9] and in the work of Cheng:

\[
\begin{align*}
\Rightarrow \\
\end{align*}
\]

1.10 Remark. The two-step initial condition in the definition of opetope may look strange, and in any case the first two constellations are redundant in terms of information. (As we just saw, for \( n \geq 3 \) also \( X_2 \) is redundant, since the configuration of dots in \( X_3 \) completely determines \( X_2 \).) The justifications for including \( X_0 \) and \( X_1 \) are first of all to cover also dimension 0 and 1 in an uniform way, and make the opetope dimension
match the degree of the complex. Second, those leading \( \circ \) will play a key role in the notion of stable opetopes in \[5.1\]. From the theoretical viewpoint, which we take up in the next section, the point is that \( X_0 \) and \( X_1 \) represent the trivial polynomial functor (the identity functor on \( \textbf{Set} \)), from which iterated application of the Baez-Dolan construction \[3.2\] will generate all the opetopes in higher dimension. The extra condition imposed on \( X_2 \) (the linear nesting of the spheres) is also explained by that construction. The fact that there are no extra conditions on \( X_n \) for \( n \geq 3 \) expresses a remarkable feature of the double Baez-Dolan construction \[3.7\] at the heart of this paper, namely that the double Baez-Dolan construction generates constellations.

1.11 Example. A 4-opetope is a zoom complex of degree 4 like this example:

\[
Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4
\]

As we discussed, it would be enough to indicate \( Y_3 \circ \overrightarrow{Y_4} \), and if we furthermore take advantage of the linear order in \( Y_3 \) and make the convention that \( Y_4 \) should be a planar tree, where the clockwise planar order expresses the (downwards) linear order in \( Y_3 \), then also \( Y_3 \) is redundant, and we can represent the 4-opetope by the single constellation:

(While such economy can sometimes be practical, conceptually it is rather an obfuscation.)

1.12 Example. We finish with an example of a 5-opetope, just to point out that there is no longer any natural planar structure on the underlying trees in degree \( d \geq 5 \). Arguing as above, to specify a 5-opetope it is enough to specify a single zoom \( Z_4 \circ \overrightarrow{Z_5} \).
provided we understand that the tree in $Z_4$ is planar (and hence allows us to reconstruct the previous constellation). Here is an example of a 5-.opetope represented in this economical manner:

![Diagram of a 5-opetope in $Z_4$ and $Z_5$]

**Formal definitions: trees and constellations**

While the presented definition of opetopes is appealing in its simplicity, scrutiny of the definition raises some questions: what exactly is meant by tree? Is it a combinatorial notion? In that case, what does it mean to draw circles on a tree? In this subsection we give the definitions a more formal treatment. We show in particular that the notion of constellation is purely combinatorial and does not depend on geometric realisation. Secondly, the analysis will clarify the relation to Baez-Dolan metatrees (and uncover the shortcoming with these). Thirdly, the insight provided by the formal viewpoint will be helpful for understanding the constructions in Section 3 and the calculations in Section 4.

**1.13 Graphs.** By a graph we understand a pair $(T_0, T_1)$, where $T_0$ is a set, and $T_1$ is a set of subsets of $T_0$ of cardinality 2. The elements in $T_0$ are called vertices, and the elements in $T_1$ edges. An edge $\{x, y\}$ is said to be incident to a vertex $v$ if $v \in \{x, y\}$. We say a vertex is of valence $n$ if the set of incident edges is of cardinality $n$.

The geometric realisation of a graph is the CW-complex with a 0-cell for each vertex, and for each edge a 1-cell attached at the points corresponding to its two incident vertices.

**1.14 Trees.** By a finite rooted tree with boundary we mean a finite graph $T = (T_0, T_1)$, connected and simply connected, equipped with a pointed subset $\partial T$ of vertices of valence 1, called the boundary. We will not need other kinds of tree than finite rooted trees with boundary, and we will simply call them trees.

The basepoint $t_0 \in \partial T$ is called the output vertex, and the remaining vertices in $\partial T$ are called input vertices. Most of the time we shall not refer to the boundary vertices at all, and graphically a boundary vertex is just represented as a loose end of the incident edge. Edges incident to input vertices are called leaves or input edges of the tree, while
the unique edge incident to the output vertex is called the root edge or the output edge of the tree.

The vertices in $T_0 \setminus \partial T$ are called nodes or dots; we draw them as dots. A tree may have zero dots, in which case it is just a single edge (together with two boundary vertices, which we suppress); we call such a tree a unit tree. Not every vertex of valence 1 needs to be a boundary vertex: those which are not are called null-dots.

The standard graphical representation of trees is justified by geometric realisation. Note that leaves and root are realised by half-open intervals, and we keep track of which are which by always drawing the root at the bottom. By labelling the cells we can recover the abstract tree when needed, and we shall allow ourselves to mix the two viewpoints, although we shall frequently omit the labels.

If $T = (T_0, T_1, \partial T, t_0)$ is a tree, the set $T_0$ has a natural poset structure $a \leq b$, in which the input vertices and null-dots are minimal elements and the output vertex is the maximal element. We say $a$ is a child of $b$ if $a \leq b$ and $\{a, b\}$ is an edge. Each dot has one output edge, and the remaining incident edges are called input edges of the dot.

1.15 Nestings and correspondences. Nestings (as in 1.2) are just another graphical representation of an abstract tree $(T_0, T_1, \partial T, t_0)$. Graphically, a nesting is a collection of non-intersecting spheres and dots, which either consists of a single dot (and no spheres) or has one outer sphere, containing all the other spheres and dots. We identify two nestings if there is an isotopy between them. We shall need some more terminology about nestings, expanding the dictionary between trees and nestings. A sphere that does not contain any other spheres or dots is called a null-sphere. These correspond exactly to the null-dots of a tree. The region bounded on the outside by a sphere $S$ and on the inside by the dots and spheres contained in $S$ is called a layer. The layers of a nesting correspond to the nodes of the tree. An inner sphere mediates between two layers just like an inner edge in a tree sits between two nodes. We will often confuse a layer with its outside bounding sphere.

1.16 Towards a combinatorial definition of constellations. In 1.4 we defined a constellation as a tree with a sphere nesting on top, more precisely as a configuration $C$ of edges, dots, and spheres (in 3-space), such that: (i) edges and dots form a tree, (ii) dots and spheres form a nesting, and (iii) for each sphere, the edges and dots contained in it form a tree again.

This definition has a clear intuitive content, and plays a important role as convenient tool for manipulating constellations and opetopes, just like we usually manipulate trees in terms of their geometrical aspect, not in terms of abstract graphs. However,
The definition depends on geometric realisation, and it is not clear at this point of our exposition that it is a rigorous notion at all. It is likely that the definition can be formalised geometrically by talking about isotopy classes of such configurations of (progressive) line segments, dots, and spheres in Euclidean space. We shall not go further into this. We wish instead to stress that the notion can be given in purely combinatorial terms.

The idea is to draw the tree corresponding to the underlying nesting, and specify some bijections. For this to work it is necessary to mark the position of the null-spheres by temporarily turning them into dots. This is formalised through the notion of subdivision of trees:

**1.17 Subdivision and kernels.** A *linear tree* is a tree in which every dot has exactly one input edge. The unit tree is an example of a linear tree. There is a linear tree for each natural number. A *subdivision* of a tree $T$ is a tree $T'$ obtained by replacing each edge by a linear tree. We draw the new dots as white dots. Here is a picture of a tree and a subdivision:

$$T \quad T'$$

When we speak about dots of a subdivided tree we mean the union of old and new dots:

$$\text{dots}(T') = \text{blackdots}(T') + \text{whitedots}(T')$$

(note that $\text{blackdots}(T') = \text{dots}(T)$).

If $T$ is a tree, we call a subset $K \subset \text{dots}(T)$ a *kernel* if $K$ is non-empty and connected as a subset of the graph $T$. In other words, a sphere containing exactly the dots of a kernel cuts a tree, as in condition (iii) of 1.4. In the following picture, $K = \{r, u, v\}$ is an example of a kernel:

$$\text{(When we speak of kernels of a subdivided tree we refer to all dots, black and white.)}$$
1.18 Combinatorial definition of constellation. A constellation $C : T \to N$ between two trees $T$ and $N$ is a triple $(T', \sigma_\bullet, \sigma_\circ)$, where $T'$ is a subdivision of $T$, and $\sigma_\bullet$ and $\sigma_\circ$ are bijections

$$
\sigma_\bullet : \text{blackdots}(T') \xrightarrow{\sim} \text{leaves}(N)
$$
$$
\sigma_\circ : \text{whitedots}(T') \xrightarrow{\sim} \text{nulldots}(N)
$$

such that the sum map $\sigma := \sigma_\bullet + \sigma_\circ$ satisfies the kernel rule:

for each $x \in \text{dots}(N)$, the set $\{ t \in \text{dots}(T') \mid \sigma(t) \leq x \}$ is a kernel in $T'$.

Here is an example of a constellation in this sense:

![Constellation Diagram](image)

(The white dots are not a part of $T$; they represent the subdivision of $T$ which is a part of the data constituting $C$.)

Let us compare the definitions of constellation given in 1.16 and 1.4. Given a constellation according to definition 1.16, as in Figure (3), for each dot $x$ in $N$ that is not a null-dot, draw a sphere in $T'$ around the dots in $T'$ corresponding to the descendant leaves and null-dots of $x$ in $N$. (One case deserves a comment, perhaps, and is also illustrated in the figure: if dot $p$ has only one child which is a dot $q$ (not a null-dot), then $p$ and $q$ have the same set of descendant leaves and null-dots, and hence the two corresponding spheres in $T'$ should contain the same dots. In this case the sphere corresponding to $p$ should be drawn so as to contain the sphere corresponding to $q$, since $q$ is a child of $p$.) To finish the construction, replace the white dots in $T'$ by null-spheres.

![Constellation Diagram](image)

It is now clear that the left-hand side of the picture is a constellation in the sense of 1.4. Conversely, given an constellation $C$ in the sense of 1.4 with underlying tree $T$,
the preceding arguments can be reversed to construct a constellation according to the combinatorial definition 1.16: first draw the tree $N$ corresponding to the underlying nesting of $C$ (with a specified bijection) (this gives Figure (4)), then erase all the spheres in $C$ except the null-spheres, and draw the null-spheres so small that they look like (white) dots — they constitute now a subdivision of $T$. At this point we have a constellation in the sense of 1.16: the bijections $\sigma_*$ and $\sigma_\circ$ are already part of the correspondence between the underlying nesting of $C$ and the tree $N$, and each dot $x \in \text{dots}(N)$ corresponds to a sphere in $C$, so the kernel rule is just a reformulation of the condition that each sphere cuts a tree.

It is clear that the two constructions are inverse to each other.

1.19 **Zooms and zoom complexes, revisited.** Let us plug the combinatorial definition of constellation 1.16 into the definition of zoom 1.5 and zoom complex 1.6. Given a zoom

we can interpret each constellation as a relationship between trees, like this:
The defining property of zoom means the two trees in the middle coincide (modulo the subdivision, which is rather a part of the structure of $C_2$), so we can overlay the two constellations:

In conclusion, a zoom is a sequence of three trees connected by constellations:

$$T_0 \xrightarrow{C_1} T_1 \xrightarrow{C_2} T_2.$$

Similarly, a zoom complex is a sequence of trees and constellations

$$T_0 \xrightarrow{C_1} T_1 \xrightarrow{C_2} T_2 \cdots \xrightarrow{C_{n-1}} T_n.$$

1.20 Opetopes, revisited. We saw in 1.19 that an opetope of dimension 2 can be represented by a linear tree, and an opetope of dimension 3 by a planar tree, which is the same thing as a nesting on a linear tree. In other words, an opetope of dimension 3 can be represented as a constellation $T_2 \rightarrow T_3$, where $T_2$ is a linear tree. In general, an opetope of dimension $n \geq 2$ can be represented by a sequence of trees and constellations $T_2 \rightarrow T_3 \rightarrow \cdots \rightarrow T_n$, with $T_2$ a linear tree. For $n \geq 3$, it is also represented by the zoom complex $C_3 \xrightarrow{\cdots} C_n$, where $C_3$ is the constellation associated to a planar tree as in 1.19. In this zoom complex, $C_k$ is the constellation $T_{k-1} \rightarrow T_k$. The sequence $C_3 \xrightarrow{\cdots} C_n$ is redundant compared to the sequence $T_2 \rightarrow T_3 \rightarrow \cdots \rightarrow T_n$, but drawing the redundant spheres is very practical as they explicitly witness the validity of the kernel rule.

1.21 Relation with Baez-Dolan metatrees. The viewpoint on zoom complexes given in 1.19 provides an explicit comparison with the notion of metatree introduced by Baez and Dolan [1]. There are two important differences.

A metatree (cf. [1], pp. 176–177) is essentially a sequence of trees $T_0, \ldots, T_n$ not allowed to have null-dots, with specified bijections $\sigma_i : \text{dots}(T_{i-1}) \sim \text{leaves}(T_i)$ satisfying the kernel rule. In other words, it is the special case of a zoom complex where the trees have no null-dots, and hence there is no subdivision involved in the constellations. Null-dots represent nullary operations of the operads or polynomial functors of
the Baez-Dolan construction \[3.2\] and nullary operations do arise. Therefore the Baez-
Dolan metatrees seem to be insufficient to reflect the Baez-Dolan construction and to
describe opetopes. Our zoom complexes may be what Baez and Dolan really envis-
aged with the notion of metatree.

The second difference is of another nature: Baez and Dolan worked with planar
trees, but introduced a notion of combed tree, in which the leaves are allowed to cross
each other in any permutation. The trees in Baez-Dolan metatrees are in fact combed.
These artefacts come from working with symmetric operads. The effect on the defi-
nition of opetope is that each opetope comes equipped with an ordering of its faces.
We work instead with non-planar trees and polynomial monads, and the resulting
opetopes (which agree with Leinster’s, cf. \[3.1\]) are ‘un-ordered’ like abstract geometric
objects. Planarity is revealed to be a special feature of dimension 3, cf. \[1.9\].

Let us remark that we think the spheres are an important conceptual device for
understanding opetopes in terms of sequences of trees. Baez and Dolan stressed that
a key feature of the slice construction is that operations are promoted to types, and
 reduction laws are promoted to operations. This two-level correspondence comes to
the fore with the notion of zoom: the types are represented by the leaves, the operations
are the dots, and the reduction laws are expressed by the spheres. The zoom relation
shifts dots to leaves and spheres to dots.

2 Polynomial functors and polynomial monads

2.1 Polynomial functors. We recall some facts about polynomial functors (see \[8\] for
details). A diagram of sets and set maps like this

\[
\begin{array}{c}
E \\
\downarrow \scriptstyle s \\
I \\
\downarrow \\
\end{array}
\begin{array}{c}
p \\
\downarrow \\
B \\
\downarrow \scriptstyle t \\
J
\end{array}
\]

\[\text{(5)}\]

gives rise to a polynomial functor \( P : \text{Set} / I \rightarrow \text{Set} / J \) defined by

\[
\text{Set} / I \overset{s^*}{\longrightarrow} \text{Set} / E \overset{p^*}{\longrightarrow} \text{Set} / B \overset{h}{\longrightarrow} \text{Set} / J.
\]

Here lowerstar and lowershriek denote, respectively, the right adjoint and the left ad-
joint of the pullback functor upperstar. In explicit terms, the functor is given by

\[
\text{Set} / I \quad \overrightarrow{\sum_{b \in B}} \quad \overrightarrow{\prod_{e \in E_b}} X_{s(e)} \rightarrow \text{Set} / J.
\]
where \( E_b := p^{-1}(b) \) and \( X_i := f^{-1}(i) \), and where the last set is considered to be over \( J \) via \( t \).

We will always assume that \( p : E \to B \) has finite fibres. No finiteness conditions are imposed on the individual sets \( I, J, E, B \), nor on the fibres of \( s \) and \( t \).

### 2.2 Graphical interpretation.

The following graphical interpretation provides the link between polynomial functors and the tree structures of Section II. The important aspects of an element \( b \in B \) are: the fibre \( E_b = p^{-1}(b) \) and the element \( j := t(b) \in J \). We capture these data by picturing \( b \) as a (non-planar) bouquet (also called a corolla)

\[
\begin{array}{c}
  e \\
  \downarrow \\
  b \\
  \downarrow \\
  j
\end{array}
\]

Hence each leaf is labelled by an element \( e \in E_b \), and each element of \( E_b \) occurs exactly once. In virtue of the map \( s : E \to I \), each leaf \( e \in E_b \) acquires furthermore an implicit decoration by an element in \( I \), namely \( s(e) \).

An element in \( E \) can be pictured as a bouquet of the same type, but with one of the leaves marked (this mark chooses the element \( e \in E_b \), so this description is merely an expression of the natural identification \( E = \bigsqcup_{b \in B} E_b \)). Then the map \( p : E \to B \) consists in forgetting this mark, and \( s \) returns the \( I \)-decoration of the marked leaf.

Consider now a set over \( I \), say \( f : X \to I \). Then the elements of \( P(X) \) are bouquets as above, but where each leaf is furthermore decorated by elements in \( X \) in a compatible way:

\[
\begin{array}{c}
  x \\
  \downarrow \\
  e \\
  \downarrow \\
  b \\
  \downarrow \\
  j
\end{array}
\]

The compatibility condition for the decorations is that leaf \( e \) may have decoration \( x \) only if \( f(x) = s(e) \). The set of such \( X \)-decorated bouquets is naturally a set over \( J \) via \( t \) (return the label of the root edge). More formally, \( P(X) \) is the set over \( B \) (and hence over \( J \) via \( t \)) whose fibre over \( b \in B \) is the set of commutative triangles

\[
\begin{array}{c}
  X \\
  \downarrow f \\
  E_b \\
  \downarrow s \\
  I
\end{array}
\]
2.3 Composition of polynomial functors. The composition of two polynomial functors is again polynomial; this is a consequence of distributivity and the Beck-Chevalley conditions [8]. We are mostly interested in the case $J = I$ so that we can compose $P$ with itself. The composite polynomial functor $P \circ P$ can be described in terms of grafting of bouquets: the base set for $P \circ P$, formally described as $p_*(B \times_I E)$, is the set of bouquets of bouquets

The conditions on the individual bouquets are still in force: each dot is decorated by an element in $B$, and for a dot with decoration $b$ the set of incoming edges is in specified bijection with the fibre $E_b$. The compatibility condition for grafting is this:

Compatibility Condition: for an edge $e$ coming out of a dot decorated $c$, we have

$$s(e) = t(c).$$

2.4 Morphisms. A cartesian natural transformation $u : P' \Rightarrow P$ between polynomial functors corresponds to a commutative diagram

whose middle square is cartesian, cf. [8]. In other words, giving $u$ amounts to giving a $J$-map $u : B' \rightarrow B$ together with an $I$-bijection $E'_{b'} \cong E_{u(b')}$ for each $b' \in B'$.

Let $\text{Poly}(I)$ denote the category whose objects are the polynomial endofunctors on $\text{Set} / I$ as in [5] and whose arrows are the cartesian natural transformations as in (6). This is a strict monoidal category under composition, and with the identity functor $\text{Id}$ as unit object. Note that a polynomial functor always preserves cartesian squares, and (under the assumption $E \rightarrow B$ finite) sequential colimits [8].

2.5 Polynomial monads. A polynomial monad is a polynomial endofunctor $P : \text{Set} / I \rightarrow \text{Set} / I$ with monoid structure in $\text{Poly}(I)$. (This should really be called a cartesian polynomial monad, since there is another more general notion of morphism of polynomial
functors \[\text{for which there is also a notion of monad.}\) In other words, there is specified a composition law \(\mu : P \circ P \to P\) with unit \(\eta : \text{Id} \to P\), satisfying the usual associativity and unit conditions, and \(\mu\) and \(\eta\) are cartesian natural transformations. Throughout we indicate monads by their functor part, confident that in each case it is clear what the natural-transformation part is, or explicating it otherwise.

The composition law is described graphically as an operation of contracting trees (formal compositions of bouquets) to bouquets.

We shall refer to \(I\) as the set of \(\text{types}\) of \(P\), and \(B\) as the set of \(\text{operations}\). Since we have a unit, we can furthermore think of \(E\) as the set of \(\text{partial operations}\), i.e. operations all of whose inputs except one are fed with a unit. The composition law can be described in terms of partial operations as a map

\[B \times_I E \to B,\]

consisting in substituting one operation into one input of another operation, provided the types match: \(i(b) = s(e)\).

2.6 The free monad on a polynomial endofunctor. (See also Gambino-Hyland [5].) Given a polynomial endofunctor \(P : \text{Set}/I \to \text{Set}/I\), a \(P\)-set is a pair \((X, a)\) where \(X\) is an object of \(\text{Set}/I\) and \(a : P(X) \to X\) is an arrow in \(\text{Set}/I\) (not subject to any further conditions). A \(P\)-map from \((X, a)\) to \((Y, b)\) is an arrow \(f : X \to Y\) giving a commutative diagram

\[
\begin{array}{ccc}
P(X) & \xrightarrow{P(f)} & P(Y) \\
\downarrow{a} & & \downarrow{b} \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Let \(P\)-\textit{Set}/I denote the category of \(P\)-sets and \(P\)-maps. The forgetful functor \(U : P\text{-Set}/I \to \text{Set}/I\) has a left adjoint \(F\), the free \(P\)-set functor. The monad \(T := U \circ F : \text{Set}/I \to \text{Set}/I\) is the free monad on \(P\). This is a polynomial monad, and its set of operations is the set of \(P\)-trees, as we now explain.

2.7 \(P\)-trees. Let \(P\) denote a polynomial endofunctor given by \(I \leftarrow E \to B \to I\). We define a \(P\)-tree to be a tree whose edges are decorated in \(I\), whose nodes are decorated in \(B\), and with the additional structure of a bijection for each node \(n\) (with decoration \(b\)) between the set of input edges of \(n\) and the fibre \(E_b\), subject to the compatibility condition that such an edge \(e \in E_b\) has decoration \(s(e)\), and the output edge of \(n\) has decoration \(t(b)\). Note that the \(I\)-decoration of the edges is completely specified by the node decoration together with the compatibility requirement, except for the case of a unit tree.
Another description is useful: a $P$-tree is a tree with edge set $A$, node set $N$, and node-with-marked-input-edge set $N'$, together with a diagram

$$
\begin{array}{c}
A \quad N' \quad N \quad A \\
\downarrow \quad \downarrow \quad \downarrow \\
I \quad E \quad B \quad I
\end{array}
$$

Then the vertical maps $\alpha$ and $\beta$ express the decorations, and the commutativity and the cartesian condition on the middle square express the bijections and the compatibility condition. The top row is a polynomial functor associated to a tree, and in short, a $P$-tree can be seen as a cartesian morphism from a tree to $P$ in a certain category of polynomial endofunctors.

The $P$-trees are obtained by freely grafting elements of $B$ onto the leaves of elements of $B$, provided the decorations match (and formally adding a unit tree for each $i \in I$). More formally, the set of $P$-trees, which we denote by $B^*$, is a least fixpoint for the polynomial endofunctor

$$
\mathbb{Set}/I \rightarrow \mathbb{Set}/I
$$

$$X \mapsto I + P(X);$$

it is given explicitly as the colimit

$$B^* = \bigcup_{n \in \mathbb{N}} (I + P)^n(\emptyset).$$

2.8 Explicit description of the free monad on $P$. A slightly more general fixpoint construction characterises the free $P$-set monad $T$: if $A$ is an object of $\mathbb{Set}/I$, then $T(A)$ is a least fixpoint for the endofunctor $X \mapsto A + P(X)$. In explicit terms,

$$T(A) = \bigcup_{n \in \mathbb{N}} (A + P)^n(\emptyset).$$

It is the set of $P$-trees with input edges decorated in $A$. But this is exactly the characterisation of a polynomial functor with operation set $B^*$: let $E^*$ denote the set of $P$-trees with a marked input leaf, then $T : \mathbb{Set}/I \rightarrow \mathbb{Set}/I$ is the polynomial functor given by

$$
\begin{array}{c}
E^* \quad B^* \\
I \quad I
\end{array}
$$

The maps are the obvious ones: return the marked leaf, forget the mark, and return the root edge, respectively. The monad structure of $T$ is described explicitly in terms of grafting of trees. In a partial-composition description, the composition law is

$$B^* \times I E^* \rightarrow B^*$$
consisting in grafting a tree onto the specified input leaf of another tree. The unit is given by $I \to B^*$ associating to $i \in I$ the unit tree with edge decorated by $i$. (One can readily check that this monad is cartesian.)

3 The Baez-Dolan construction for polynomial monads

Throughout this section, we fix a polynomial monad $P : \text{Set}/I \to \text{Set}/I$, represented by

![Diagram](https://via.placeholder.com/150)

We shall associate to the polynomial monad $P : \text{Set}/I \to \text{Set}/I$ another polynomial monad $P^+ : \text{Set}/B \to \text{Set}/B$. The idea of this construction is due to Baez and Dolan [1], who realised it in the settings of symmetric operads. We present two versions of it: first a very formal version, following Leinster [9], but in a polynomial setting, then an explicit graphical version, which also serves to establish that the output of the first version is polynomial.

3.1 The Baez-Dolan construction for polynomial monads, formal version. Denote by $\text{PolyMon}(I)$ the category of polynomial monads on $\text{Set}/I$, i.e. the category of monoids in $\text{Poly}(I)$. Since $P$ is a monad, the slice category $\text{Poly}(I)/P$ has a natural monoidal structure: the composite of $Q \to P$ with $R \to P$ is $R \circ Q \to P \circ P \to P$ and the unit is $\text{Id} \to P$. Let $\text{PolyMon}(I)/P$ denote the category of polynomial monads over $P$, i.e. monoids in $\text{Poly}(I)/P$. The forgetful functor $\text{PolyMon}(I)/P \to \text{Poly}(I)/P$ has a left adjoint, the free $P$-monad functor, hence generating a monad $T : \text{Poly}(I)/P \to \text{Poly}(I)/P$. To complete the Baez-Dolan construction it only remains to reinterpret this monad $T$ as a polynomial monad. The key point is that there is a natural equivalence of categories

$$\text{Poly}(I)/P \simeq \text{Set}/B,$$

given by evaluation at the terminal object $I \to I$, which we denote by 1. In detail, if $Q \to P$ is an object in $\text{Poly}(I)/P$, the associated object in $\text{Set}/B$ is simply $Q(1) \to P(1) = B$. The inverse equivalence basically takes an object $C \to B$ in $\text{Set}/B$ to the object in $\text{Poly}(I)/P$ given by the fibre square

![Diagram](https://via.placeholder.com/150)
The promised monad $P^+ : \text{Set}/B \to \text{Set}/B$ is simply the monad corresponding to $T : \text{Poly}(I)/P \to \text{Poly}(I)/P$ under this equivalence. We shall describe this monad explicitly in a moment and see that it is polynomial.

To compare with Leinster’s version of the Baez-Dolan construction [9], note that the above equivalence induces a monoidal structure on $\text{Set}/B$ which is the tensor product of $P$-collections, for which the monoids are the $P$-operads, in the sense of [9]. Further details can be found in [8]. Hence we also get an equivalence of categories between $\text{PolyMon}(I)/P$ and the category of $P$-operads, and the free $P$-monad functor on $\text{Poly}(I)/P$ corresponds to the free $P$-operad functor used in Leinster’s version of the Baez-Dolan construction. The equivalence (7) seems to be due to Kelly [7], in the more general setting of clubs.

3.2 The Baez-Dolan construction for a polynomial monad, explicit graphical version. Starting from our polynomial monad $P$, we describe explicitly a new polynomial monad $P^+$, shown afterwards to coincide with the one constructed above. The idea is to substitute into dots of trees instead of grafting at the leaves (so notice that this shift is like in a zoom relation). Specifically, with $B^*$ the set of $P$-trees, define $U^*$ to be the set of $P$-trees with one marked dot. There is now a polynomial functor

$$
\begin{array}{c}
U^* \\
\downarrow \quad \downarrow \quad \downarrow \\
B^* \\
P^+ \\
B
\end{array}
$$

where $U^* \to B^*$ is the forgetful map, $U^* \to B$ returns the bouquet around the marked dot, and $t : B^* \to B$ comes from the monad structure on $P$ — it amounts to contracting all inner edges (or setting a new dot in a unit tree). Graphically:

$$
\begin{align*}
\{ \quad \} & \quad \to \quad \{ \quad \} \\
\{ \quad \} & \quad \to \quad \{ \quad \}
\end{align*}
$$

(In this diagram as well as in the following diagrams of the same type, a symbol $\{ \}$ is meant to designate the set of all bouquets like this (with the appropriate decoration), but at the same time the specific figures representing each set are chosen in such a way
that they match under the structure maps.) Note that since the forgetful map forgets a marked dot, the nullary operations in $P^+$ are precisely the unit trees $\mathcal{I}$, one for each $i \in I$.

This polynomial endofunctor $P^+$ is naturally a monad: the substitution law can be described in terms of a partial composition law

$$B^* \times_B U^* \to B^*$$

defined by substituting a $P$-tree into the marked dot of an element in $U^*$, as indicated in this figure:

![Diagram showing the substitution process]

Of course the substitution makes sense only if the decorations match. This means that $t(F)$, the ‘total bouquet’ of the tree $F$, is the same as the local bouquet of the node $f$. (The letters in the figure do not represent the decorations — they are rather unique labels to express the involved bijections, and to facilitate comparison with Figure (10) below.) The unit for the monad is given by the map $B \to B^*$ interpreting a bouquet as a tree with a single dot. (One can check again that this monad is cartesian.)

### 3.3 Comparison between the two versions of the construction.

We wish to compare the two monads $T$ and $P^+$ under the equivalence of categories $\text{Poly}(I)/P \cong \text{Set}/B$. Recall that an object $C \to B$ of $\text{Set}/B$ corresponds to the polynomial functor $Q$ (over $P$) given by

$$E \times_B C \longrightarrow C$$

$$I \quad E \longrightarrow B \quad I.$$ 

The explicit description of $P^+$ allows us to compute its value on an object $C \to B$ of $\text{Set}/B$: the result is the set of $P$-trees with each node decorated by an element of $C$, compatibly with the arity map $C \to B$ (being a $P$-tree means in particular that each node already has a $B$-decoration; these decorations must match). We claim that this is the same thing as a $Q$-tree. Indeed, since the tree is already a $P$-tree, we already have
$I$-decorations on edges, as well as bijections for each node between the input edges and the fibre $E_b$ over the decorating element $b \in B$. But if $c \in C$ decorates this same node, then the cartesian square specifies a bijection between the fibre over $c$ and the fibre $E_b$ and hence also with the set of input edges. So in conclusion, $P^+$ sends $C$ to the set of $Q$-trees.

On the other hand, $T$ sends the corresponding polynomial functor $Q$ to the free monad on $Q$, with structure map to $P$ given by the monad structure on $P$. Specifically, $T$ produces from $Q$ the polynomial monad

$$
\begin{array}{ccc}
C^* & \rightarrow & C^* \\
\downarrow & & \downarrow \\
E^* & \rightarrow & B^* \\
\downarrow & & \downarrow \\
E & \rightarrow & B
\end{array}
$$

where $C^*$ denotes the set of $Q$-trees, so the two endofunctors agree on objects. The same argument works for arrows, so the two endofunctors agree.

To see that the monad structures agree, note that the set of operations for $P^+ \circ P^+$ is the set of $P$-trees with nodes decorated by $P$-trees in such a way that the total bouquet of the decorating tree matches the local bouquet of the node it decorates. The composition law $P^+ \circ P^+ \Rightarrow P^+$ consists in substituting each tree into the node it decorates. On the other hand, to describe the monad $T$ it is enough to look at the base sets, since each top set is determined as fibre product with $E$ over $B$. In this optic, $T$ sends $B$ to $B^*$, and $T \circ T$ sends $B$ to $B^{**}$ which is the set of $P^*$-trees, which is the same as $P$-trees with nodes decorated by $P$-trees, and edges decorated in $I$, subject to the usual compatibility conditions. Clearly the composition law $T \circ T \Rightarrow T$ corresponds precisely to the one we described for $P^+$. For both monads, the unit is described as associating to a bouquet the corresponding one-dot tree.

In conclusion, the two constructions agree, and, in particular, produce a polynomial functor.

### 3.4 In terms of nestings

We have described the free monad construction and the Baez-Dolan construction in terms of trees, but of course they can equally well be described in terms of nested spheres, as we shall now explain. The interplay between these two descriptions will lead directly to opetopes as defined in Section 1. Let us stress again that trees and nestings are just different graphical expressions of the same combinatorial structure. However, some features of trees can be a little bit subtler to see in terms of nestings.
The basic operations, the elements in \( B \), are configurations of a sphere with dots inside:

We call such a thing a \textit{layer}. The set of dots inside the sphere is in bijection with the set \( E_b \), and via \( s : E \to I \) these dots also carry an implicit decoration by elements in \( I \), the input types. The label \( j \) on the outside of the sphere represents \( t(b) \), the output. We put the label \( b \) on the inside of the sphere it decorates, since it mediates between the input devices (the dots) and the output device (the sphere), just as the dot of a bouquet mediates between the inputs (the leaves) and the output.

Next, \( B^* \) is the set of arbitrary \( P \)-nestings, with layers decorated in \( B \) and spheres and dots decorated in \( I \) (subject to compatibility conditions), and \( E^* \) is the set of arbitrary \( P \)-nestings (compatibly decorated) with a marked dot. The substitution law for the free monad on \( P \) is now described by substituting one \( P \)-nesting into a dot of another, provided the decorations match. (This corresponds to grafting of trees.)

For the Baez-Dolan construction (where we now suppose \( P \) is a monad), \( U^* \) is the set of \( P \)-nestings with a marked sphere, so here is the nesting version of Figure (8):

Note that the map \( t \) consists in erasing all inner spheres, which is just the nesting equivalent of the tree operation of contracting all inner edges – this is always possible for undecorated nestings, but for this to make sense in the \( P \)-decorated case we need the monad structure on \( P \). The map \( s \) consists in returning the \textit{layer} determined by the marked sphere: this means the region delimited on the outside by the marked sphere itself and on the inside by its children, so the operation can also be described as taking the marked sphere and contracting each sphere inside it to a dot. (This is the nesting equivalent of the tree operation of returning the ‘local bouquet’ of a dot.)

The substitution law is perhaps less obvious in this nesting interpretation. Looking at Figure (9) we see that for trees the substitution takes place at a specified dot, and
consists in replacing its 'local bouquet' by a more complicated tree, so the operation is
about refining the tree. Correspondingly for nestings, the operation is about refining
the nesting by drawing some more spheres in the specified layer. Here is the nesting
version of Figure (9):

Again, the $B$-decorations have not been drawn; the letters serve only to specify the
bijections, and to facilitate comparison with Figure (9).

3.5 The double Baez-Dolan construction (slice-twice construction). After applying
the Baez-Dolan construction once (in its tree interpretation), we have a polynomial
functor $B \leftarrow U^* \rightarrow B^* \rightarrow B$ which is a monad for the operation of substituting one
tree into a dot of another tree (subject to some bookkeeping). Now it turns out we
can make abstraction from the underlying trees: these are just a way of specifying the
type of each dot and the type of the whole tree — the types are (decorated) bouquets.
Since the operation is substitution into dots, we are precisely in the situation where
we can apply the Baez-Dolan construction in its nesting manifestation, and this double
Baez-Dolan construction

\[
\begin{array}{c}
U^{**} \\
\downarrow s \\
B^* \\
\downarrow t \\
P^{++} \\
\downarrow t \\
B^*
\end{array}
\]

has the following direct interpretation, as we shall show: $B^{**}$ is the set of $P$-constellations,
and $U^{**}$ consists of $P$-constellations with a marked sphere:
By \emph{P-constellation} we mean a constellation whose underlying tree is a \emph{P-tree}. The structure maps in \emph{P}++ are: \( t \) returns the underlying tree of a constellation, and \( s \) returns the tree contained in the marked layer. The monad structure consists in substituting one constellation into the marked layer of another, provided of course their decorations match.

This heuristic description is justified by the following theorem. First we make explicit what we mean by tree of trees:

\textbf{3.6 Trees of trees.} The set \( B^{**} \) resulting from two consecutive Baez-Dolan constructions is the set of trees of \emph{P}-trees, or \emph{P}+-trees. By definition, a \emph{tree of \emph{P}-trees} is a tree \( M \) whose dots are decorated by \( B \)-trees, and whose edges are decorated by elements in \( B \), and with a specified bijection, for each node \( n \) with decorating tree \( T \), between the set of input edges of \( n \) and the set of dots in \( T \), which in turn can be described as the fibre \( U_T^{**} \) under the forgetful map \( U^* \to B^* \) of \emph{P}+. And then there are the compatibility conditions: an edge \( u \in U_T^* \) must have decoration \( s(u) \), and the output edge of \( n \) must have decoration \( t(T) \) (this is the bouquet obtained from the monad law of \emph{P}, i.e. contracting all inner edges of \( T \)). It is clear that this is precisely a \emph{P}+-tree.

\textbf{3.7 Theorem.} A tree of \emph{P}-trees is the same thing as a \emph{P}-constellation.

\textit{Proof. From \emph{P}-constellation to \emph{P}+-tree.} Given a constellation \( C \), we first get an abstract tree \( M \) by taking the tree corresponding to the underlying nesting of \( C \), cf. 1.3. Let \( L \) denote the set of layers, and \( S \) the set of spheres and dots. To each layer we associate its outside sphere (the output sphere), hence a map \( L \to S \). Let \( \overline{L} \) denote the set of layers with a marked child, and consider the forgetful map to \( L \); finally there is the obvious map \( \overline{L} \to S \) returning the marked child. These maps,

\[ S \leftarrow \overline{L} \to L \to S \]
is the polynomial functor associated to the tree $M$ as in \[2.7\]. We must now decorate this tree, i.e., provide a diagram

\[
\begin{array}{cccccc}
S & \xrightarrow{\alpha} & L & \xleftarrow{\gamma} & L & \xrightarrow{\beta} & S \\
\downarrow & & \downarrow & (3) & \downarrow & (2) & \downarrow \\
B & \xleftarrow{\alpha} & U^* & \xrightarrow{\beta} & B^* & \xrightarrow{\beta} & B.
\end{array}
\]

To define $\alpha$: to each dot in $C$ we associate its local bouquet in the underlying $P$-tree of $C$. To each sphere in $C$, intuitively we can just look which edges come into it and which edge goes out, and this defines the local bouquet of a sphere. Note however that here we are implicitly using the monad structure of $P$, since in reality we are taking the $P$-tree $T$ contained in the sphere and then contracting this tree to a single bouquet $t(T)$. The map $\beta$ is defined similarly: to each layer, return the $P$-tree seen in that layer. This is the $P$-tree contained in the output sphere of the layer but with the subtrees in the children contracted (here again we use the monad structure of $P$). With $\alpha$ and $\beta$ described this way, it is clear that square (1) commutes: both ways around the square amount to taking the bouquet around the output sphere of a given layer.

To define $\gamma : \mathcal{L} \rightarrow U^*$, notice that the $P$-tree seen in a given layer has a node for each child sphere of the layer. So given a layer with a marked child, return the $P$-tree seen in this layer (as in the definition of $\beta$), with the node marked that corresponds to the child. Now (2) is commutative and cartesian by construction.

Finally, both ways around the square (3) amount to returning the bouquet of the marked child, which is the same as the local bouquet of the node in the tree-with-marked-node corresponding to the layer-with-marked-child.

*From $P^+$-tree to $P$-constellation.* Intuitively, the Baez-Dolan construction is a free gluing construction: $B^{**}$ is obtained by freely substituting trees into nodes of a tree, cf. \[9\]. If for each such substitution we keep track of the surgery via the scar it left — that’s a sphere in the tree — then the resulting configurations are precisely the $P$-constellations, i.e. $P$-trees with spheres.

To be more specific, a $P^+$-tree $M$ is viewed as a recipe for how to glue small trees together to a big tree, the small trees being those that label the nodes of $M$. We refer to $M$ as the composition tree. In the end the gluing loci will sit as spheres in the resulting big tree.

We start with the special case where the $P^+$-tree $M$ is the unit tree $\mathcal{L}$, i.e., a single edge decorated by some bouquet $b \in B$. We need a $P$-constellation whose nesting corresponds to a unit tree. Hence this constellation has no spheres, and thus has just a single dot, so it amounts to giving a one-dot $P$-tree. Obviously we just take $b$ itself, considered as a $P$-tree via the unit map for the monad.

If the composition tree $M$ has just one dot $n$, this dot is decorated by a $P$-tree $T$ (of a certain type). We need to provide a sphere nesting with just one sphere, and we just take $T$ with a sphere around it.
If the composition tree $M$ has more than one dot, then it has inner edges, and each inner edge $a$, say from node $c$ down to node $r$ represents a substitution: the tree $T_r$ decorating $r$ has a node for each input edge of $r$; by the compatibility condition, the node corresponding to edge $a$ is decorated $A = t(T_c)$, the output type of $T_c$. Hence it makes sense to substitute $T_c$ into that node of $T_r$, cf. (9). We should perform the substitutions corresponding to all the inner edges of $M$. By associativity of the substitution law, it doesn’t matter at which level we start, and in fact since we have a unit operation at our disposal, we don’t even have to care about levels: we can really make the substitutions edge by edge in any order.

Hence it is enough to explain what happens for a composition tree with a single inner edge, i.e., a two-dot tree. Suppose the composition tree looks like this:

\[
\begin{array}{c}
\text{node } c \\
\text{decorated by the } P\text{-tree } T_c \\
of \text{output type } A \in B
\end{array}
\]

\[
\begin{array}{c}
\text{node } r \\
\text{decorated by the } P\text{-tree } T_r \\
one of whose nodes } f \text{ is decorated by } A \in B
\end{array}
\]

Now the substitution goes like this (cf. (9)):

\[
T_c
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

\[
T_r
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

resulting in

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

This $P$-tree is the underlying $P$-tree of the constellation we are constructing. There should be two spheres: one outer sphere (corresponding to the root edge of $M$) for which there is no choice, and one inner sphere corresponding to the inner edge in $M$. This inner sphere has to be precisely the scar of the surgery.

If the composition tree has more inner edges, each corresponding substitution will produce a sphere in the final tree, and clearly the nesting resulting from all the substitutions will correspond to the composition tree as required.

(A short remark concerning two degenerate cases: If $T_c$ is the unit tree \( \downarrow \), then $A$ is the bouquet \( \uparrow \). The effect of the substitution in this case is simply to erase the dot,
leaving a null-sphere as scar. If \( T_c \) is the a one-dot tree, then we are substituting a single dot into a another dot of the same type, and the resulting tree is unchanged, but a sphere is placed around this dot, as scar of the operation. The fact that the underlying tree stays the same just says that one-dot trees are the units for the substitution law.) □

To appreciate this result, note that a \( P^+ \)-tree is a complicated structure: it is a whole collection of \( P \)-trees (the decorations) satisfying a complicated set of compatibility conditions. The theorem shows that all these data can be encoded in a single \( P \)-constellation, where there are no compatibility conditions to check!

Theorem has the following interesting corollary:

\section*{3.8 Corollary.} For any polynomial monad \( P \), any abstract tree admits a \( P^+ \)-decoration.

In contrast, it is not true that any tree admits a decoration by a monad not of the form \( P^+ \). For example, as we shall see in a moment (3.10), only linear trees can be decorated by the trivial monad.

\begin{proof}

By the theorem, a \( P^+ \)-decoration of a tree is the same thing as a \( P \)-constellation. But every abstract nesting can appear as underlying nesting of a constellation. In fact for any \( P \)-tree, you can draw arbitrary nestings. □

\end{proof}

\section*{The polynomial monads of opetopes}

We shall generate all the opetopes iteratively, starting from the trivial polynomial monad \( P^0 : \text{Set} \to \text{Set} \) given by \( 1 \leftarrow 1 \to 1 \to 1 \), the identity functor on \( \text{Set} \).

\section*{3.9 Basis for the construction.} We define the set of 0-dimensional opetopes to be the singleton set \( \mathcal{Z}^0 := \{ \} \) of types of \( P^0 \) and define the set of 1-dimensional opetopes to be the singleton set \( \mathcal{Z}^1 := \{ \} \) of operations of \( P^0 \), to conform with the standard graphical interpretation (cf. 2.2):

\begin{align*}
\{ \ast \} & \quad \longrightarrow \quad \{ \downarrow \} \\
\{ \mid \} & \quad = \quad P^0 = \text{Id} \quad \{ \mid \}
\end{align*}

\section*{3.10 First Baez-Dolan construction.} We apply the Baez-Dolan construction to \( P^0 \) to get a polynomial monad \( P^1 \) whose set of types is \( \mathcal{Z}^1 \), the singleton again, and whose operations are obtained by freely grafting \( \downarrow \)'s onto the leaf of \( \downarrow \) (and formally adding an operation \( \mid \)). We get:
We denote by $Z^2$ the set of operations of $P^1$. It is naturally in bijection with $\mathbb{N}$. In fact $P^1$ is the free-monoid monad $\mathbf{Set} \to \mathbf{Set}$, $X \mapsto \sum_{n \in \mathbb{N}} X^n$.

3.11 Second Baez-Dolan construction. Performing the Baez-Dolan construction a second time defines $P^2$. By Theorem 3.7, this is about setting spheres in the trees we have got, which are the linear trees. So $P^2$ looks like this:

Let $Z^3$ denote the set of operations. Clearly this set is naturally identified with the set of 3-dimensional opetopes as defined in 1.7.

For the next iteration — trees of trees of trees — a new meta-device is needed. The solution is to take the tree expression of the nesting and set spheres in it like in the previous step. More precisely, by Theorem 3.7 the set $Z^3$ (of constellations whose underlying tree is linear) is also the set of $P^1$-trees, i.e. trees with a certain compatible decoration by linear trees, and we know that to specify such a tree is just to draw the tree corresponding to the nesting, with a specified bijection: all the decorations can then be read off this bijection.

Applying the Baez-Dolan construction a third time just amounts to freely drawing spheres in these composition trees, and it is clear that the resulting new set of operations $Z^4$ is precisely the set of 4-opetopes defined in 1.7 — well, in fact it is precisely the economical version of 4-opetopes where the redundant $X_0$, $X_1$ and $X_2$ are suppressed, as explained in 1.11.
The preceding argument is also the general inductive argument showing that if we know that the \( n \)-opetopes are the types of \( P^n \), then the \( (n + 1) \)-opetopes are the types of \( P^{n+1} \). Hence we can give the following reformulation of our definition of opetopes.

3.12 Opetopes. Let \( P^0 \) denote the trivial polynomial monad \( 1 \leftarrow 1 \rightarrow 1 \rightarrow 1 \) and let \( Z^0 \) denote its set of types (the singleton set). Let \( P^k \) denote the \( k \)th iterated Baez-Dolan construction on \( P^0 \). By definition, the set of \( k \)-dimensional opetopes \( Z^k \) is the set of types for \( P^k \), or equivalently, for \( k > 0 \), the set of operations for \( P^{k-1} \).

For emphasis, we summarise the above discussion:

3.13 Theorem. The two notions of opetopes, 1.7 and 3.12, agree.

3.14 Remark. There exist in the literature four variations of the notion of opetope, not only in formulation but also in content: the original definition of Baez-Dolan \[1\], the multitopes of Hermida-Makkai-Power \[6\], the opetopes in terms of cartesian monads due to Leinster \[9\], and a modification of the Baez-Dolan notion due to Cheng \[2\]. The four notions have been compared by Cheng \[2\], \[3\]. It is clear from 3.1 that our notion coincides with Leinster’s. Our description of Leinster’s sequence of cartesian monads stresses that all these monads are polynomial, and exploits the graphical calculus for polynomial functors to provide the explicit combinatorial description that was missing in Leinster’s work.

4 Calculus of opetopes

In the previous section we have shown how opetopes arise as types and operations for polynomial monads. From that description we immediately get the notions of sources and target of an opetope, and a notion of composition or gluing of opetopes, which we now make explicit and illustrate by examples. A reader who has skipped Sections 2 and 3 can take the following descriptions as definitions.

In this section, by root dot we mean the dot adjacent to the root edge (if there are any dots).

Faces

We follow the polytope-inspired terminology for opetopes, and call their input and output devices facets (i.e. codimension-1 faces):
4.1 Target. The target facet of an \( n \)-opetope \( X \) is the \((n - 1)\)-opetope obtained by omitting the top constellation \( X_n \) and the last zoom in the zoom complex. The target is also called the output facet.

4.2 Sources. Let \( X \) be an \( n \)-opetope. For each sphere \( s \) in \( X_n \), there is a source facet (or input facet), which is an \((n - 1)\)-opetope. You can think of it as the part of the zoom complex you can see by looking only through the layer determined by \( s \), i.e., the region in \( X_n \) delimited on the outside by \( s \) itself and from the inside by the children of \( s \).

So there are three steps in the computation of the source facet corresponding to \( s \):

(i) up in \( X_n \), consider only the layer determined by \( s \). In other words, restrict to the sphere \( s \) and contract all spheres contained in \( s \);

(ii) perform certain corresponding operations on the spheres in \( X_{n-1} \) and in all lower constellations, in order to maintain the constellations in zoom relation;

(iii) omit \( X_n \).

In a moment we shall describe this in detail, but first it is convenient to introduce the notions of globs and drops:

4.3 Globs. An \( n \)-opetope whose top constellation \( X_n \) has precisely one sphere is called a glob. In this case, there is precisely one source facet, and this facet is isomorphic to the target facet. For each \((n - 1)\)-opetope \( F \) there is a unique \( n \)-glob whose target facet is \( F \), obtained by drawing the tree corresponding to the nesting underlying \( F_{n-1} \), and drawing a sphere around it all. This is called the glob over \( F \). In abstract terms, it is nothing but the unit operation of type \( F \), cf. \[3.2\]. Hence the globs in dimension \( n \) are in natural bijection with the \((n - 1)\)-opetopes, via the target map. The term ‘glob’ comes from the classical way of drawing opetopes: in dimension 2 there is only one glob, which is pictured like this:

\[ \bigcirc \]

4.4 Drops. An opetope whose top constellation \( X_n \) has no spheres is called a drop. So a drop has no sources. Since a constellation without spheres necessarily has a unique dot, \( X_{n-1} \) has a unique sphere. Hence the target of a drop is always a glob. In particular the set of all \( n \)-drops is in bijection with the set of all \((n - 2)\)-opetopes, via the target map applied twice. Again the terminology comes from the polytope-style drawing of opetopes, where in dimension 2 one can draw the unique drop as

\[ \bigcirc \]
Notice that also in dimension 3 there is only one drop (since there is only one 1-opetope): it is the 3-opetope whose sole facet is \([13]\).

### 4.5 Sphere operations

The operations involved in computing sources can be described in terms of the following sphere operations on a constellation \(X_i\):

- Erase a sphere which is not the outer sphere.
- Draw a new sphere around a dot or a sphere.
- Contract a sphere to a dot.
- Restrict to a sphere.

Each operation on \(X_i\) implies certain other operations on \(X_{i-1}\), ensuring that the resulting constellations are in zoom relation, and these operations in turn imply other operations on \(X_{i-2}\), and so on. (It is understood that the sequence of operations starts at the top constellation and propagates downwards, so we will not have to worry about consequences on \(X_{i+1}\) of an operation on \(X_i\).)

### 4.6 Erasing a sphere (not the outer sphere), or drawing a new sphere around a dot or a sphere

These operations do not have any consequences in the constellation below.

### 4.7 Contracting a sphere to a dot

Let \(s\) be a sphere in \(X_i\), and let \(T\) denote the tree it cuts. If there is at least one dot in \(T\), then let \(r\) denote the root dot of \(T\). Then we are contracting \(s\) down to \(r\). In \(X_{i-1}\) we must erase the spheres corresponding to each non-root dot in \(T\), and that’s all. If there are no dots in \(T\) (\(T\) consists of just an edge), then we are contracting \(s\) down to a new dot which we denote \(s^*\). Since \(T\) is just a single edge, the dot \(s^*\) will have a unique child \(c\) (either a dot or a leaf). In \(X_{i-1}\) we have to draw a new sphere around the sphere or dot corresponding to \(c\).

### 4.8 Restricting to a sphere

Let \(s\) be a sphere in \(X_i\). Restricting to \(s\) means erasing everything outside it. The new root edge will be the root edge of the tree \(T\) cut by \(s\), and each leaf of \(T\) will be labelled by the dot (or leaf) the edge was connecting to outside \(s\). For each dot \(x\) that is descendant of \(T\) but not in \(T\) itself, contract the corresponding sphere \(x^\circ\) in \(X_{i-1}\). Finally, restrict to the sphere \(r^\circ\) in \(X_{i-1}\) corresponding to the root dot \(r\) of \(T\). (If \(T\) contains no dot, i.e. is just an edge, then instead of a root dot it has a unique leaf \(r\); in that case we are restricting to the corresponding dot \(r^\circ\) in \(X_{i-1}\).)

### 4.9 Example

We will compute the sources of the following 5-opetope:
There are sources corresponding to the spheres 13, 14, 15, and 16; we will denote these source facets by $S_{13}$, $S_{14}$, $S_{15}$, and $S_{16}$.

**4.10 Computation of source $S_{13}$.** Step (i): contract 14, 15, and 16 in $X_5$:

Step (ii): perform the corresponding operations in the lower constellations, according to the sphere operations rules. This means deleting spheres 10 and 11, and drawing a new sphere around sphere 12 (corresponding to the contracted ‘empty’ sphere 16). Finally (iii), omit the top constellation. The end result is:

**4.11 Computation of source $S_{14}$.** Step (i): restrict to sphere 14:
Step (ii) amounts to contracting sphere 9 in $X_4$, and hence erasing sphere 6 and 7 down in $X_3$. End result:

4.12 Computation of source $S_{15}$. Step (i): restrict to sphere 15:

This implies (step (ii)) that in $X_4$ we have to restrict to sphere 9 and contract sphere 12. Down in $X_3$ this means erase sphere 7. End result:

4.13 Computation of source $S_{16}$. Step (i): restrict to sphere 16:

Step (ii): the root of this subtree is the leaf 12, so down in $X_4$ we have contract sphere 12 and then restrict to the resulting dot 12. The contraction has the consequences in $X_3$ of erasing sphere 7 (and we rename sphere 6 to 12). Restricting to dot 12 in $X_4$ means restricting to sphere 12 in $X_3$. Since dot 4 is a descendant which is not inside sphere 12, we have to contract sphere 4 in $X_2$. End result:
Composition tree and gluing

4.14 Composition tree. The composition tree of an opetope is simply the tree corresponding to the nesting of the top constellation (with a specified correspondence). It concisely expresses the incidence relations among the codimension-1 faces, and how these faces are attached to each other along codimension-2 faces. We denote the composition tree of $X$ by $\text{ct}(X)$.

In the composition tree $\text{ct}(X)$, each dot $s$ corresponds to an input facet $S$ (codimension-1 face). The last codimension-1 face of $X$, its target facet, is represented in the composition tree as the ‘total bouquet’, i.e. the bouquet obtained by contracting all inner edges (or setting a dot in the unit tree, if $X$ is a drop).

The edges in $\text{ct}(X)$ correspond to the codimension-2 faces of $X$: There is an incoming edge of dot $s$ for each input facet of $S$, and the output edge of $s$ represents the output facet of $S$. In other words, an edge linking a dot $s$ to its parent dot $p$ represents the codimension-2 face along which $S$ is attached to $P$ (the face corresponding to $p$): this codimension-2 face is the target facet of $S$ and one specific source facet of $P$. This source is easily determined: $p$ is a sphere in $X_n$ and $s$ is another sphere immediately contained in $p$. When computing $P$ we contract the sphere $s$ to a dot, hence it becomes a sphere in $P_{n-1}$, and so represents a source facet of $P$.

The leaves of $\text{ct}(X)$ correspond to the dots in the top constellation, which in turn correspond to the spheres in $X_{n-1}$. These are precisely the input facets of the target of $X$. By the preceding discussion, each of these codimension-2 faces is also the source facet of exactly one source facet of $X$, namely the facet $S$ corresponding to the parent dot $s$ of the leaf.

If there is a dot in $\text{ct}(X)$ (i.e. $X$ is not a drop), then the root dot determines a bottom source, characterised also as the source facet having the same target as the target of $X$ (corresponding to the output edge of $\text{ct}(X)$).

In summary we see that, except if $X$ is a drop, every codimension-2 face of $X$ occurs exactly twice as a facet of a facet. In fact, more generally, if $V$ is a codimension-$(k+2)$ face of an opetope $X$, and $F$ is a codimension-$k$ face of $X$ containing $V$, then the number of codimension-$(k+1)$ faces $E$ such that $V \subset E \subset F$ is either 1, or 2. It is 1 if and only if $F$ is a drop (in which case it is the drop on $E$).

4.15 Example (continued from [4.9]). For the opetope $X$ of the example above, the composition tree is
We see that $S_{13}$ (corresponding to dot 13) has four input facets (corresponding to the four input edges of dot 13): the first one (leaf 12) is left vacant, its three other input facets serve as gluing locus for the output facets of $S_{14}$, $S_{15}$, and $S_{16}$. In turn, $S_{14}$ and $S_{15}$ each has two input facets (which are not in use for gluing), while $S_{16}$ has no input facets (i.e., $S_{16}$ is a drop). Note that the root edge represents the output facet of $S_{13}$.

4.16 Gluing and filling. As explained in the proof of Theorem 3.7, a decorated composition tree serves as a recipe for gluing together $n$-dimensional opetopes $S_i$, producing one big $n$-dimensional opetope $T$, and finally filling the whole thing with an $n$-dimensional opetope $X$ in such a way that the original opetopes $S_i$ become the input facets of $X$, and $T$ becomes the output facet.

The first part consists in producing the 'composite' opetope $T$ from the $S_i$ according to the recipe specified by the composition tree. This can be done in steps: it is enough to explain what happens when the composition tree has a single inner edge, i.e., a simple gluing. The second part (4.19) consists in constructing the filling $(n + 1)$-opetope $X$.

4.17 Gluing. Given an $n$-opetope $R$ with a specified source $F$, and another $n$-opetope $S$ with target $F$, then their composite $T$ is again an $n$-opetope, whose target is the target of $R$, and whose set of sources is

$$\text{sources}(S) \cup \text{sources}(R) \setminus \{F\}.$$

The recipe composition tree looks something like this:

Every such situation arises as follows. Write down an arbitrary $n$-opetope $R$ (but not a drop), pick one of its source facets, and write down this $(n - 1)$-opetope $F$. Next we need to provide an $n$-opetope $S$ having $F$ as its target. By definition of the target map, $S$ is obtained from $F$ by drawing its composition tree and then drawing some arbitrary spheres in it.
4.18 Example. Let us illustrate the situation with an example. Here is $S$:

\[ S_{n-1} \]

And here comes $R$:

\[ R_{n-1} \]

Now $F$ is the target of $S$ and at the same time the source of $R$ corresponding to sphere $f$:

\[ F_{n-1} \]

We need to construct a new $n$-opetope $T$ whose target is the same as the target of $R$. This means that it differs from $R$ only in the top constellation, where the configuration of spheres is different. The difference in sphere layout is expressed nicely in terms of the composition trees of $S$ and $R$. The recipe prescribes that we should glue $S$ onto the $F$-facet of $R$. In terms of the composition trees of $S$ and $R$ this means that we must substitute the whole tree $ct(S)$ into the node $f$ of $ct(R)$. Since the target of $S$ is $F$, this will again produce a valid decorated composition tree which will be $ct(T)$. In the current example, the situation is this:
The new dots that appear in the composition tree of $T$ specify that new spheres should be drawn in $R_n$ in order to obtain $T_n$. These spheres are drawn in the layer between the sphere $f$ and the spheres contained in $f$. The dot substitution performed on the composition trees is not enough information though: there is an ambiguity for the spheres corresponding to the childless dots in $\text{ct}(T)$: where should those null-spheres be drawn? But the missing bit is clearly encoded in $S_n$ itself. In fact, substituting $\text{ct}(S)$ into the $f$ node of $\text{ct}(R)$ is just the composition-tree expression of copying over the non-outer spheres from $S_n$ to $R_n$: copy those four spheres, and paste them into the layer between the sphere $f$ and its children. The children of $f$ (dots and spheres immediately contained in $f$) are in 1–1 correspondence with the dots in $S_n$ (since $F$ is the target of $S$ and the $f$ source of $R$). Here is the result, with the four new spheres highlighted in fat black:

4.19 The filler. The filling $(n + 1)$-opetope $X$ should have $T$ as target, so $X_k = T_k$ for $k \leq n$. The underlying tree of $X_{n+1}$ must be the composition tree of $T$; it remains to draw some spheres in this tree. These spheres are determined by the original recipe composition tree (Figure (11)): there are precisely two spheres to be drawn, corresponding to the two dots $S$ and $R$ in the composition tree: one sphere is the outer sphere (corresponding to the root dot $R$), the other sphere is the ‘scar’ of the gluing operation (corresponding to $S$) — this sphere was already drawn dashed in Figure (12).
So here is the final $X$ of our running example:

\[
X_n = T_n
\]

It is clear from the construction that it has $S$ and $R$ as sources and $T$ as target.

5 Suspension and stable opetopes

5.1 Suspension. The suspension $S(X)$ of an $n$-opetope $X$ is the $(n + 1)$-opetope defined by setting

\[
S(X)_0 := \bigcirc \\
S(X)_{k+1} := X_k \quad \text{for } 0 \leq k \leq n.
\]

In other words, just prepend a new $\bigcirc$ to the zoom complex, raising the indices.

The operations ‘source’, ‘target’, and ‘composition of opetopes’ all commute with suspension. Indeed, these operations are defined on the top constellations, and the repercussions down through the zoom complex can never reach the degree-1 term in the complex.

5.2 Stable opetopes. The suspension defines a map $S : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ for each $n \geq 0$. Let $\mathbb{Z}^\infty$ denote the colimit of this sequence of maps,

\[
\mathbb{Z}^\infty = \bigcup_{n \geq 0} \mathbb{Z}^n.
\]

This is the set of all opetopes in all dimensions, where we identify two opetopes if one is the suspension of the other. The elements in $\mathbb{Z}^\infty$ are called stable opetopes. Note that a stable opetope has a well-defined top constellation, and that therefore the notions of source, target, and composition make sense for stable opetopes.
Define $\mathbb{Z}^\infty := \bigcup_{n \geq 0} \mathbb{Z}^n$, the set of stable opetopes with a marked input facet. Now consider the polynomial monad of stable opetopes

$$P^\infty : \textbf{Set}/\mathbb{Z}^\infty \rightarrow \textbf{Set}/\mathbb{Z}^\infty$$

defined by the diagram

$\begin{array}{ccc}
\mathbb{Z}^\infty & \xrightarrow{t} & \mathbb{Z}^\infty \\
\downarrow{s} & & \downarrow{t} \\
\mathbb{Z}^\infty & \xrightarrow{t} & \mathbb{Z}^\infty
\end{array}$

As usual, $t$ returns the target, $s$ returns the source, and $\mathbb{Z}^\infty \rightarrow \mathbb{Z}^\infty$ is the forgetful map. This polynomial functor is a least fixpoint for the pointed Baez-Dolan construction, as we shall now explain.

5.3 The category of polynomial monads. Let $\textbf{PM}$ denote the category of all polynomial monads [8]. The arrows in this category are diagrams

$\begin{array}{ccc}
E & \xrightarrow{t} & B \\
\downarrow{a} & & \downarrow{I} \\
E' & \xrightarrow{t} & B'
\end{array}$

which respect the monad structure. This is most easily expressed in the partial-composition viewpoint where it amounts to requiring that these two squares commute:

$\begin{array}{ccc}
B \times_1 E & \xrightarrow{t} & B \\
\downarrow{a} & & \downarrow{I} \\
B' \times_1 E' & \xrightarrow{t} & B'
\end{array}$

The suspension map $S : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n+1}$ induces an arrow in $\textbf{PM}$:

$$S : P^n \rightarrow P^{n+1}$$
In other words, there is a natural diagram

\[
\begin{array}{ccc}
\mathbb{Z}^{n+1} & \rightarrow & \mathbb{Z}^{n+1} \\
\downarrow & & \downarrow \\
\mathbb{Z}^n & \rightarrow & \mathbb{Z}^n \\
\downarrow & & \downarrow \\
\mathbb{Z}^{n+2} & \rightarrow & \mathbb{Z}^{n+2} \\
\downarrow & & \downarrow \\
\mathbb{Z}^{n+1} & \rightarrow & \mathbb{Z}^{n+1}
\end{array}
\]

The middle square is cartesian because marking a sphere in the top constellation is independent of suspension. It is a monad map since suspension commutes with partial composition.

5.4 The Baez-Dolan construction is functorial. We claim that the Baez-Dolan construction extends to a functor \(BD : PM \rightarrow PM\). To show this we have to explain what it does on arrows (and then it will be clear that composition of arrows and identity arrows are respected). The Baez-Dolan construction on \(\alpha\) given in (17) is:

\[
\begin{array}{ccc}
U^* & \rightarrow & B^* \\
\downarrow & & \downarrow \\
B & \rightarrow & B \\
\downarrow & & \downarrow \\
U'^* & \rightarrow & B'^* \\
\downarrow & & \downarrow \\
B' & \rightarrow & B'
\end{array}
\]

Here \(\alpha^* : B^* \rightarrow B'^*\) is defined already on the level of the free monad construction. The right-hand square commutes because \(\alpha\) is a monad morphism. The rest is pure combinatorics, about setting marks in trees. Since \(\alpha^*\) is defined ‘node-wise’, there is also an evident map \(U^* \rightarrow U'^*\) which makes the two other squares commute, and for which the middle square is cartesian. Finally one can check that \(\alpha^*\) is a monad morphism:

\[
\begin{array}{ccc}
B^* \times_B U^* & \rightarrow & B^* \\
\downarrow & & \downarrow \\
B & \rightarrow & B \\
\downarrow & & \downarrow \\
B'^* \times_B U'^* & \rightarrow & B'^* \\
\downarrow & & \downarrow \\
B' & \rightarrow & B'
\end{array}
\]

Again this is a purely combinatorial matter: the horizontal maps are defined in terms of substituting trees into nodes of trees. Since the two rows are just two instances of this, but with different decorations, the diagram commutes.
5.5 Pointed polynomial monads. The Baez-Dolan functor has a rather boring least fixpoint: it is simply the initial polynomial monad $\emptyset \leftarrow \emptyset \rightarrow \emptyset \rightarrow \emptyset$. We are more interested in the notion of pointed polynomial monads and the pointed analogue of the Baez-Dolan functor.

By a pointed polynomial monad we understand a polynomial monad equipped with a monad map from the trivial monad

$$
\begin{array}{ccc}
1 & \rightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
$$

A morphism of pointed polynomial monads is one that respects the map from $\text{Id}$. This defines a category $\mathcal{PM}_\ast$. If $i : \text{Id} \rightarrow M$ is a pointed polynomial monad, then $\text{BD}(M)$ is naturally pointed again, so the Baez-Dolan construction defines also a functor $\mathcal{PM}_\ast \rightarrow \mathcal{PM}_\ast$. To see this, note that by functoriality we get a map $\text{BD}(\text{Id}) \xrightarrow{\text{BD}(i)} \text{BD}(M)$. On the other hand we have $\text{Id} = P^0$, the polynomial monad of 0-opetopes, and $\text{BD}(\text{Id}) = P^1$, and the suspension map provides $\text{Id} \rightarrow \text{BD}(\text{Id})$. (Note that $P^1 : \text{Set} \rightarrow \text{Set}$ is the free-monoid monad.)

Now it follows readily from the standard Lambek iteration argument that

5.6 Proposition. The polynomial monad $P^\infty$ of stable opetopes is a least fixpoint for the pointed Baez-Dolan construction $\mathcal{PM}_\ast \rightarrow \mathcal{PM}_\ast$.

Indeed, $P^\infty$ can be characterised as the colimit of

$$
\begin{array}{ccc}
\text{Id} & \rightarrow & \text{BD}(\text{Id}) \\
& \rightarrow & \text{BD}^2(\text{Id}) \\
& & \vdots
\end{array}
$$

Appendix: Machine implementation

Our description of opetopes naturally lends itself towards machine implementation. The involved data grow only linearly with the dimension of the opetopes, and being fundamentally a tree structure, it is straightforward to encode in XML, as we shall now explain.

A.1 Trees-only representation. For the sake of machine implementation, we have adopted a variation of the trees-only representation of opetopes given in [1.20] instead of having the white dots (i.e. the null-spheres) explicitly, we let each null-dot refer to the unique child of the corresponding null-sphere in the previous constellation (be it a dot or a leaf). Now, more than one null-sphere may sit on the same edge, in which case it is not enough for the corresponding null-dots to refer to that edge. But the fact that
these spheres sit on the same edge means there is induced an ordering among them, and this ordering can be expressed on the level of null-dots by letting them refer to each other in a chain, with only the last null-dot referring to something in the previous constellation (corresponding to the null-sphere farthest away from the root). This system in turn requires some careful bookkeeping in connection with sphere operations, since the reference of null-dot $x$ to a null-dot $y$ becomes invalid if $y$ is contracted. Keeping track of these references is not difficult, but tedious and unenlightening.

A.2 File format. XML (Extensible Mark-up Language, cf. http://www.w3.org/XML/) is a lot like HTML, except that you define your own tags to express a grammar. This is done in a Document Type Definition (DTD). The opetope DTD looks like this:

```xml
<!ELEMENT opetope (constellation+)>
<!ELEMENT constellation (dot|leaf)>  
<!ELEMENT dot (dot|leaf)+>
<!ELEMENT leaf EMPTY>
<!ATTLIST opetope name CDATA #REQUIRED>
<!ATTLIST constellation name CDATA #REQUIRED>
<!ATTLIST dot name CDATA #REQUIRED ref CDATA #IMPLIED>
<!ATTLIST leaf name CDATA #REQUIRED>

The first block declares the tags for opetope, constellation, dot, and leaf, specifying which sort of children they can have. In the second block it is specified that each tag must have a name attribute, and that the dot tag is also allowed an optional ref attribute, used only for null-dots.

Here is an XML representation of the zoom complex in Example 1.12 interpreted as a 5-opetope:

```xml
<?xml version="1.0" encoding="UTF-8"?>
<!DOCTYPE opetope SYSTEM "opetope.dtd">
<opetope name="Z">
  <constellation name="Z4">
    <dot name="b">
      <dot name="a">
        <leaf name="1"/>
      </dot>
      <dot name="c">
        <leaf name="2"/>
        <leaf name="3"/>
      </dot>
    </dot>
  </constellation>
  <constellation name="Z5">
    <dot name="p">
      <dot name="x" ref="b"/>
      <dot name="y">
        <leaf name="a"/>
        <leaf name="b"/>
        <leaf name="c"/>
      </dot>
    </dot>
  </constellation>
  <constellation name="ct(Z)">
```

(The indentation is only for the benefit of the human reader; the XML parser ignores whitespace between the tags.) Notice how the null-dots \( x \) and \( w \) are provided with a reference to dots in the previous constellations, indicating where the corresponding spheres belong.

### A.3 Scripts.

The algorithms for sphere operations have been implemented in the scripting language Tcl, using the tDOM extension (cf. http://www.tdom.org/) for parsing and manipulating XML. There are among other things procedures for computing sources, targets, and compositions, and writing the results back to new XML files. These scripts can be run from the unix prompt, provided Tcl and the tDOM extension are available on the system. The script `computeAllFacets` takes as argument the name of an opetope XML file, and computes all its codimension-1 faces, writing the resulting opetopes to separate XML files. The script `glueOnto` takes three arguments: the bottom opetope (name of XML file), the name of the gluing locus, and the top opetope (as XML file). The result is written to a new XML file.

Precise instruction for installation and usage can be found in the readme file and manual pages accompanying the scripts. XML files for all the examples of this paper are also included, together with the XML representation of a 10-opetope with 15 input facets.

### A.4 Automatic generation of graphical representation.

DOT\(^2\) is a language for specifying abstract graphs in terms of node-edge incidences, and generate a graphical representation of the graph, for example in PDF format. We provide a short Tcl script `opetope2pdf` which produces a DOT file from an opetope XML file, and, if the dot interpreter is present on the system, also generates a pdf file. This can be helpful to get an overview of a complicated opetope and its faces, but unfortunately the output is not quite as nice as the drawings in this paper (hand-coded \LaTeX\); specifically, there is no support for drawing the spheres.

Here is what the output looks like when the script is run on the XML file listed above:

\begin{verbatim}
<opetope>
  <constellation>
    <leaf name="p"/>
    <leaf name="x"/>
    <leaf name="y"/>
  </constellation>
</opetope>
\end{verbatim}

\(^2\)See E. GANSNER, E. KOUTSOFIOS, and S. NORTH, Drawing graphs with DOT, http://www.research.att.com/sw/tools/graphviz/dotguide.pdf.
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