ON THE SKEIN POLYNOMIAL FOR LINKS

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Abstract. We give characterizations of the skein polynomial for links (as well as Jones and Alexander-Conway polynomials derivable from it), avoiding the usual “smoothing of a crossing” move. As by-products we have characterizations of these polynomials for knots, and for links with any given number of components.

1. Introduction

The skein polynomial (as called in [7, Chapter 8], also known as HOMFLY or HOMFLY-PT polynomial), \( P_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \), is an invariant for oriented links. Here \( \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \) is the ring of Laurent polynomials in two variables \( a \) and \( z \), with integer coefficients. It is defined to be the invariant of oriented links satisfying the axioms

\[ a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right) = z \cdot P \left( \begin{array}{c|c} & \end{array} \right); \]

\[ P \left( \begin{array}{c|c} & \end{array} \right) = 1. \]

The Alexander-Conway polynomial \( \Delta_L \in \mathbb{Z}[t^{\pm \frac{1}{2}}] \) and the Jones polynomial \( V_L \in \mathbb{Z}[t^{\pm \frac{1}{2}}] \) are related to the skein polynomial:

\[ \Delta_L(t) = P_L(1, t^{\frac{1}{2}} - t^{-\frac{1}{2}}), \quad V_L(t) = P_L(t, t^{\frac{1}{2}} - t^{-\frac{1}{2}}). \]

Our main result is

**Theorem 1.1.** The skein polynomial \( P_L \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \) is the invariant of oriented links determined uniquely by the following four axioms.

\[ a^{-2} \cdot P \left( \begin{array}{c|c} & \end{array} \right) + a^2 \cdot P \left( \begin{array}{c|c} & \end{array} \right) = (2 + z^2) \cdot P \left( \begin{array}{c|c} & \end{array} \right); \]

\[ a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right) = a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right); \]

\[ a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right) = a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right); \]

\[ a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right) = a^{-1} \cdot P \left( \begin{array}{c|c} & \end{array} \right) - a \cdot P \left( \begin{array}{c|c} & \end{array} \right); \]

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A parallel result is for the Jones polynomial. It is not a direct corollary of the above theorem, because the substitutions $a \mapsto t$ and $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ do not send $\mathbb{Z}[a^{\pm1}, z^{\pm1}]$ into $\mathbb{Z}[t^{\pm\frac{1}{2}}]$.

**Theorem 1.2.** The Jones polynomial $V_L \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is the invariant of oriented links determined uniquely by the following four axioms.

\[
\begin{align*}
(\text{II}_V) & \quad t^{-2} \cdot V \left( \begin{array}{c} \circ \end{array} \right) + t^2 \cdot V \left( \begin{array}{c} \boxed{} \end{array} \right) = (t + t^{-1}) \cdot V \left( \begin{array}{c} \boxed{} \end{array} \right); \\
(\text{III}_V) & \quad t^{-1} \cdot V \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) - t \cdot V \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) = t^{-1} \cdot V \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) - t \cdot V \left( \begin{array}{c} \boxed{\backslash} \end{array} \right); \\
(\text{IO}_V) & \quad V \left( \begin{array}{c} \circ \end{array} \right) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot V \left( \begin{array}{c} \boxed{} \end{array} \right); \\
(\text{O}_V) & \quad V \left( \begin{array}{c} \boxed{} \end{array} \right) = 1.
\end{align*}
\]

For the Alexander-Conway polynomial, the result takes a slightly different form. We switch to a $(\Phi)$-type axiom because the $(\text{IO})$-type one degenerates into a consequence of $(\text{II})$ and $(\text{III})$ (see Corollary 4.4).

**Theorem 1.3.** The Alexander-Conway polynomial $\Delta_L \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is the invariant of oriented links determined uniquely by the following four axioms.

\[
\begin{align*}
(\text{II}_\Delta) & \quad \Delta \left( \begin{array}{c} \circ \end{array} \right) + \Delta \left( \begin{array}{c} \boxed{} \end{array} \right) = (t + t^{-1}) \cdot \Delta \left( \begin{array}{c} \boxed{} \end{array} \right); \\
(\text{III}_\Delta) & \quad \Delta \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) - \Delta \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) = \Delta \left( \begin{array}{c} \boxed{\backslash} \end{array} \right) - \Delta \left( \begin{array}{c} \boxed{\backslash} \end{array} \right); \\
(\Phi_\Delta) & \quad \Delta \left( \begin{array}{c} \boxed{} \end{array} \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta \left( \begin{array}{c} \boxed{} \end{array} \right); \\
(\text{O}_\Delta) & \quad \Delta \left( \begin{array}{c} \boxed{} \end{array} \right) = 1.
\end{align*}
\]
If we restrict our attention to oriented links with a fixed number \( \mu > 0 \) of components, the axiom (IO) becomes irrelevant but we must pick a suitable normalization. Let \( U_\mu \) denote the \( \mu \)-component unlink, and let \( C_\mu \) denote the \( \mu \)-component oriented chain where adjacent rings have linking number +1. (In terms of closed braids, \( U_\mu \) is the closure of the trivial braid \( e \in B_\mu \), and \( C_\mu \) is the closure of the braid \( \sigma_1^2 \sigma_2^2 \ldots \sigma_{\mu-1}^2 \in B_\mu \).) We can use either \( U_\mu \) or \( C_\mu \) (but \( U_\mu \) is preferred) to normalize the skein or Jones polynomial, but for Alexander-Conway polynomial we can only use \( C_\mu \).

**Theorem 1.4.** The skein polynomial \( P_L \) is the invariant of oriented \( \mu \)-component links determined uniquely by the following three axioms.

\[
(\text{II}) \quad a^{-2} \cdot P \left( \begin{array}{c} \circ \end{array} \right) + a^2 \cdot P \left( \begin{array}{c} \circ \end{array} \right) = (2 + z^2) \cdot P \left( \begin{array}{c} \circ \end{array} \right);
\]

\[
(\text{III}) \quad a^{-1} \cdot P \left( \begin{array}{c} \circ \circ \end{array} \right) - a \cdot P \left( \begin{array}{c} \circ \circ \end{array} \right) = a^{-1} \cdot P \left( \begin{array}{c} \circ \circ \end{array} \right) - a \cdot P \left( \begin{array}{c} \circ \circ \end{array} \right);
\]

\[
(\text{U}) \quad P(U_\mu) = (z^{-1}(a^{-1} - a))^{\mu-1}.
\]

**Theorem 1.5.** The Jones polynomial \( V_K \) is the invariant of oriented \( \mu \)-component links determined uniquely by the following three axioms.

\[
(\text{II}_V) \quad t^{-2} \cdot V \left( \begin{array}{c} \circ \end{array} \right) + t^2 \cdot V \left( \begin{array}{c} \circ \end{array} \right) = (t + t^{-1}) \cdot V \left( \begin{array}{c} \circ \end{array} \right);
\]

\[
(\text{III}_V) \quad t^{-1} \cdot V \left( \begin{array}{c} \circ \circ \end{array} \right) - t \cdot V \left( \begin{array}{c} \circ \circ \end{array} \right) = t^{-1} \cdot V \left( \begin{array}{c} \circ \circ \end{array} \right) - t \cdot V \left( \begin{array}{c} \circ \circ \end{array} \right);
\]

\[
(\text{U}_V) \quad V(U_\mu) = (-t^\frac{1}{2} + t^{-\frac{1}{2}})^{\mu-1}.
\]

**Theorem 1.6.** The Alexander-Conway polynomial \( \Delta_K \) is the invariant of oriented \( \mu \)-component links determined uniquely by the following three axioms.

\[
(\text{II}_\Delta) \quad \Delta \left( \begin{array}{c} \circ \end{array} \right) + \Delta \left( \begin{array}{c} \circ \end{array} \right) = (t + t^{-1}) \cdot \Delta \left( \begin{array}{c} \circ \end{array} \right);
\]

\[
(\text{III}_\Delta) \quad \Delta \left( \begin{array}{c} \circ \circ \end{array} \right) - \Delta \left( \begin{array}{c} \circ \circ \end{array} \right) = \Delta \left( \begin{array}{c} \circ \circ \end{array} \right) - \Delta \left( \begin{array}{c} \circ \circ \end{array} \right);
\]

\[
(\text{C}_\Delta) \quad \Delta(C_\mu) = (t^\frac{1}{2} - t^{-\frac{1}{2}})^{\mu-1}.
\]
Note that the foundational relation (I) cannot appear in Theorems 1.4–1.6 because it involves links with different number of components.

Our approach is via closed braids. We explain the language of relators in Section 2 and give an algebraic reduction lemma in Section 3. This approach is adapted from the corresponding sections of [5] on Conway’s potential function for colored links. The current context of uncolored links makes the reduction argument more transparent. Section 4 discusses closed braids with different number of strands. The theorems are proved in the last two sections.

2. Braids and skein relators

For braids, we use the following conventions: Braids are drawn from top to bottom. The strands of a braid are numbered at the top of the braid, from left to right. The product \( \beta_1 \cdot \beta_2 \) of two \( n \)-braids is obtained by drawing \( \beta_2 \) below \( \beta_1 \). The set \( B_n \) of all \( n \)-braids forms a group under this multiplication, with standard generators \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \).

It is well known that links can be presented as closed braids. The closure of a braid \( \beta \in B_n \) will be denoted \( \hat{\beta} \). Two braids (possibly with different number of strands) have isotopic closures if and only if they can be related by a finite sequence of two types of moves:

1. Conjugacy move: \( \beta \leftrightarrow \beta' \) where \( \beta, \beta' \) are conjugate in a braid group \( B_n \);
2. Markov move: \( \beta \in B_n \leftrightarrow \beta \sigma_n^{\pm 1} \in B_{n+1} \).

Let \( \Lambda \) be the Laurent polynomial ring \( \mathbb{Z}[a^{\pm 1}, z^{\pm 1}] \). Let \( \Lambda B_n \) be the group-algebra on \( B_n \) with coefficients in \( \Lambda \).

**Definition 2.1.** We say that an element \( \lambda_1 \cdot \beta_1 + \cdots + \lambda_k \cdot \beta_k \) of \( \Lambda B_n \) is a **skein relator**, or equivalently, say that the corresponding formal equation (in which \( P_{L_{\beta_h}} \) stands for the \( P \) of the link \( L_{\beta_h} \))

\[
\lambda_1 \cdot P_{L_{\beta_1}} + \cdots + \lambda_k \cdot P_{L_{\beta_k}} = 0
\]

is a **skein relation**, if the following condition is satisfied: For any links \( L_{\beta_1}, \ldots, L_{\beta_k} \) that are identical except in a cylinder where they are represented by the braids \( \beta_1, \ldots, \beta_k \) respectively, the formal equation becomes an equality in \( \Lambda \).

**Example 2.2.** To every element \( \lambda_1 \cdot \beta_1 + \cdots + \lambda_k \cdot \beta_k \in \Lambda B_n \), by taking braid closures we have a corresponding element

\[
\lambda_1 \cdot \hat{\beta_1} + \cdots + \lambda_k \cdot \hat{\beta_k} \in \Lambda.
\]

The latter vanishes if the former is a skein relator.

**Example 2.3.** The skein relations (I), (II) and (III) in Section 1 correspond to the following relators, respectively: (The symbol \( e \) stands for the trivial braid.)

\[
(I_B) := a^{-1} \cdot \sigma_1^2 - a \cdot \sigma_1^{-2} - z \cdot e;
\]
Proposition 2.4. Assume that
\[(II_B) := a^{-2} \cdot \sigma_i^2 + a^2 \cdot \sigma_i^{-2} - (2 + z^2) \cdot e;\]
\[(III_B) := a^{-1} \cdot \sigma_1 \sigma_2 \sigma_i^{-1} + a \cdot \sigma_i^{-1} \sigma_1 \sigma_i - a^{-1} \cdot \sigma_i^{-1} \sigma_2 \sigma_i - a \cdot \sigma_i^2 \sigma_i^{-1}.\]

**Definition 3.1.** Let \(I\) and \(I\) be represented differently by braids \(\beta\) and \(\beta\), respectively. In the upper half cylinder they are represented by braids \(\beta_1, \ldots, \beta_k\). So the assumption implies the first equality. Similarly for the second equality.

By Definition 2.1, this means skein relations form a two-sided ideal. \(\square\)

3. **An algebraic reduction lemma**

**Definition 3.2.** Modulo \(I\), every braid \(\beta \in B_n\) is equivalent to a \(\Lambda\)-linear combination of braids of the form \(\alpha \sigma_k^{n-1} \gamma\) with \(\alpha, \gamma \in B_{n-1}\) and \(k \in \{0, \pm 1, 2\}\).

A braid \(\beta \in B_n\) can be written as
\[\beta = \beta_0 \sigma_{n-1}^{k_1} \beta_1 \sigma_{n-1}^{k_2} \ldots \sigma_{n-1}^{k_r} \beta_r,\]
where \(\beta_j \in B_{n-1}\) and \(k_j \neq 0\). We allow that \(\beta_0\) and \(\beta_r\) be trivial, but assume other \(\beta_j\)’s are nontrivial. The number \(r\) will be denoted as \(r(\beta)\).

The lemma will be proved by induction on the double index \((n, r)\). Note that the lemma is trivial when \(n = 2\), or \(r(\beta) \leq 1\).

It is enough to consider the case \(r = 2\), because induction on \(r\) works beyond \(2\). Indeed, if \(r(\beta) > 2\), let \(\beta' = \beta_1 \sigma_{n-1}^{k_2} \ldots \sigma_{n-1}^{k_r} \beta_r\), then \(r(\beta') < r(\beta)\). By inductive hypothesis \(\beta'\) is equivalent to a linear combination of elements of the form \(\alpha' \sigma_{n-1}^{k'} \gamma'\), hence \(\beta\) is equivalent to a linear combination of elements of the form \(\beta_0 \sigma_{n-1}^{k_1} \alpha' \sigma_{n-1}^{k'} \gamma'\). This brings the problem back to the \(r = 2\) case. Henceforth we assume \(r = 2\).
Since the initial and terminal part of $\beta$, namely $\beta_0$ and $\beta_r$, do not affect the conclusion of the lemma, we can drop them. So we assume $\beta = \sigma_{n-1}^k \beta_1 \sigma_{n-1}^{k_2}$, where $\beta_1 \in B_{n-1}$.

By the induction hypothesis on $n$, $\beta_1 \in B_{n-1}$ is a linear combination of elements of the form $\alpha_1 \sigma_{n-2} \gamma_1$. Note that $\alpha_1, \gamma_1 \in B_{n-2}$ commute with $\sigma_{n-1}$. So it suffices to focus on braids of the form $\beta = \sigma_{n-1}^k \sigma_n^{\ell} \sigma_{n-1}^m$.

For the sole purpose of controlling the length of displayed formulas, we assume $n = 3$ below. The proof for a general case is similar, replacing $\sigma_1, \sigma_2$ with $\sigma_{n-2}, \sigma_{n-1}$ and replacing $t_1, t_2, t_3$ with $t_{n-2}, t_{n-1}, t_n$, respectively.

Thus, Lemma 3.2 has been reduced to the following

**Lemma 3.3.** Every $\sigma_2^2 \sigma_1^j \sigma_2^m$ is equivalent (modulo $\mathcal{J}_n$) to a linear combination of braids of the form $\sigma_1^k \sigma_2^\ell \sigma_1^m$ where $\ell$ is 0, $\pm 1$ or 2.

**Proof.** Modulo ($\Pi_B$), we may restrict the exponent $k$ to take values 1, 2 and 3 (we are done if $k$ is 0). If $k > 1$ we can decrease $k$ by looking at $\sigma_2^{k-1}(\sigma_2^l \sigma_2^m)$, so it suffices to prove the case $k = 1$. Again modulo ($\Pi_B$), we can restrict the exponents $\ell, m$ to the values 1 and 2. There are altogether 9 cases to verify.

5 trivial cases (braid identities):

$$
\sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1, \quad \sigma_2 \sigma_1 \sigma_2^{-1} = \sigma_1^{-1} \sigma_2 \sigma_1, \quad \sigma_2 \sigma_1^{-1} \sigma_2^{-1} = \sigma_1^{-1} \sigma_2^{-1} \sigma_1,
$$

$$
\sigma_2 \sigma_1 \sigma_2^2 = \sigma_2^2 \sigma_2 \sigma_1, \quad \sigma_2 \sigma_1 \sigma_2^2 = \sigma_1^{-1} \sigma_2 \sigma_1.
$$

The case $\sigma_2 \sigma_1^{-1} \sigma_2$ : Multiplying ($\Pi_B$) by $\sigma_2$ on the right and $\sigma_1^{-1}$ on the left, and taking braid identities into account, we get the relation

$$
a^{-1} \cdot \sigma_2 \sigma_1^{-1} \sigma_2 + a \cdot \sigma_1^{-1} \sigma_2 \sigma_1^{-1} - a^{-1} \cdot \sigma_1^{-1} \sigma_2 \sigma_1 - a \cdot \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sim 0.
$$

Then $\sigma_2 \sigma_1^{-1} \sigma_2$ is equivalent to a linear combination of braids of the form $\sigma_1^{\pm 1} \sigma_2^{\pm 1} \sigma_1^{\pm 1}$. So the case $\sigma_2 \sigma_1^{-1} \sigma_2$ is verified.

The case $\sigma_2 \sigma_1^{-1} \sigma_2^2$ : Multiplying the previous relation by $\sigma_2$ on the right, and taking braid identities into account, we see that

$$
a^{-1} \cdot \sigma_2 \sigma_1^{-1} \sigma_2 + a \cdot \sigma_1^{-1}(\sigma_2 \sigma_1^{-1} \sigma_2) - a^{-1} \cdot \sigma_2 \sigma_1 - a \cdot \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sim 0.
$$

Similar to the above case, this reduces $\sigma_2 \sigma_1^{-1} \sigma_2^2$ to the verified case $\sigma_2 \sigma_1^{-1} \sigma_2$.

The case $\sigma_2 \sigma_2^2 \sigma_2$ : Multiplying ($\Pi_B$) on the right by $\sigma_1 \sigma_2 \sigma_1$, we get

$$
(\Pi_B) \quad a^{-1} \cdot \sigma_1 \sigma_2^2 \sigma_1 + a \cdot \sigma_2^2 - a^{-1} \cdot \sigma_2 \sigma_1 = a \cdot \sigma_2^2 \sim 0.
$$

This verifies the case $\sigma_2 \sigma_2^2 \sigma_2$.

The case $\sigma_2 \sigma_2^2 \sigma_2^2$ : Multiplying ($\Pi_B$) by $\sigma_2$ on the right, we get

$$
(\Pi_B') \quad a^{-1} \cdot \sigma_2 \sigma_2^2 \sigma_1 + a \cdot \sigma_2^2 - a^{-1} \cdot \sigma_2 \sigma_2^2 \sigma_2 - a \cdot \sigma_1 \sigma_2 \sim 0.
$$

The case $\sigma_2 \sigma_2^2 \sigma_2^2$ is also verified.

We have verified all 9 cases. Modulo ($\Pi_B$) we can assume $\ell \in \{0, \pm 1, 2\}$. Thus Lemma 3.2 is proved. The inductive proof of Lemma 3.2 is now complete. \qed
The resulting $\Lambda$-linear combination of braids of the form $\alpha \sigma_n^{k-1} \gamma$ with $\alpha, \gamma \in B_{n-1}$ in the Lemma is not unique, but the inductive proof gives us a recursive algorithm to find one.

To compare the ideal $I_n$ with the relator ideal $R_n$ of Section 2, we have

**Proposition 3.4.** $I_n \subset R_n$ but $I_n \neq R_n$.

**Proof.** The inclusion is easy. Indeed, $(I_B)$ is in the relator ideal $R_n$, and

$$(II_B) = (I_B)^2 + 2z \cdot (I_B),$$

$$(III_B) = \sigma_2^{-1} \cdot (I_B) \cdot \sigma_2 - \sigma_2 \cdot (I_B) \cdot \sigma_2^{-1}.$$  

So both $(II_B)$ and $(III_B)$ are in $I_n$. Therefore $I_n \subset R_n$.

To show they are not equal, we need the notion of homogeneity. Each $n$-braid $\beta$ has an underlying permutation of $\{1, \ldots, n\}$, denoted $i \mapsto i^\beta$, where $i^\beta$ is the position of the $i$-th strand at the bottom of $\beta$. In this way the braid group $B_n$ projects onto the symmetric group $S_n$. An element of $\Lambda B_n$ is called homogeneous if all its terms (with nonzero coefficients) have the same underlying permutation.

As a $\Lambda$-module, $\Lambda B_n$ splits into a direct sum according to underlying permutations of braids. Under this splitting, every element of $\Lambda B_n$ decomposes into a sum of its homogeneous components.

Since $(II_B)$ and $(III_B)$ are homogeneous, the ideal $I_n \subset \Lambda B_n$ is generated by homogeneous elements. Then every homogeneous component of any element of $I_n$ is also in $I_n$. Now the relator $(I_B) \in R_n$ has a homogeneous component $-z \cdot e$ which is not a relator. Hence $(I_B)$ is not in $I_n$. Thus $I_n$ is strictly smaller than $R_n$.  

\[\square\]

4. Stabilizations

Suppose a braid $\beta \in B_n$ is written as a word in the standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. The same word $\beta$ gives a braid in $B_{n+k}$ for any $k \geq 0$. Thus $B_n$ is standardly embedded in $B_{n+k}$. However, when talking about a closed braid $\hat{\beta}$, the number of strands in $\beta$ does matter. We shall use the notation $[\beta]_n$ to emphasize that $\beta$ is regarded as an $n$-braid, and use $[\beta]_n$ for its closure. For example, $[\beta]_{n+1}$ adds a free circle to $[\beta]_n$. The Markov move says $[\beta \sigma_n^{\pm 1}]_{n+1}$ is isotopic to $[\beta]_n$.

For a braid $\beta \in B_n$ and an integer $k \geq 0$, we shall use $\beta^k \in B_{n+k}$ to denote the $k$-th shifted version of $\beta$, i.e., the braid obtained from the word $\beta$ by replacing each generator $\sigma_i$ with $\sigma_{i+k}$. Its closure $[\beta^k]_{n+k}$ is isotopic to $[\beta]_{n+k}$.

Suppose $\beta, \beta' \in B_n$ and $\gamma \in B_p$. Observe from the diagram defining braid closure that the closed braid $[\beta \sigma_n^{\pm 1} \gamma^n \beta']_{n+p}$ is isotopic to $[\beta \gamma^{p(n-1)} \beta']_{n+p-1}$ (which is in fact a connected sum of oriented links $[\beta \beta']$ and $[\gamma]_p$). By an abuse of language, we will call this a Markov move. If $\beta'$ brings the $n$-th position at its top to the same position at its bottom, then $[\beta \gamma^{p(n-1)} \beta']_{n+p-1}$ is isotopic to $[\beta \beta' \gamma^{p(n-1)}]_{n+p-1}$. We will refer to it as a slide move (in the connected sum, sliding $[\gamma]_p$ down the last strand of $\beta'$).
Lemma 4.1. Assume that $P_L \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is an invariant of oriented links that satisfies skein relations (II) and (III). Then for $\beta \in B_n$ and $\gamma \in B_p$ we have

$$(1 + z^2 - a^2) \cdot P\left([\beta \gamma^{p_n}]_{n+p}^{-}\right) = (a^2 - 1) \cdot P\left([\beta \sigma_n^{2 \cdot \gamma^{p_n}}]_{n+p}^{-}\right).$$

Proof. The braid form of axioms (II) and (III) are the relators (II_B) and (III_B), respectively. Multiplying (III_B) by $\sigma_2 \sigma_1^{-1}$ on the right we get another relator

$$a^{-1} \cdot \sigma_2^{-2} \sigma_2 \beta \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \cdot \sigma_2 - a \cdot \sigma_2^{-2} \sigma_1^{-1} \cdot \sigma_1^{-2}.$$

It gives us an equality between the $P$'s of closed $(n + p + 1)$-braids:

$$a^{-1} \cdot P\left([\beta (\sigma_n^{-1} \sigma_n^2 \sigma_n^{-1}) \gamma^{p(n+1)}]_{n+p+1}^{-}\right) + a \cdot P\left([\beta (\sigma_n^{-1} \sigma_n^2) \gamma^{p(n+1)}]_{n+p+1}^{-}\right) + a^{-1} \cdot P\left([\beta \sigma_n^{-1} \gamma^{p(n+1)}]_{n+p+1}^{-}\right) - a \cdot P\left([\beta \sigma_n^{-1} \sigma_n^2 \gamma^{p(n+1)}]_{n+p+1}^{-}\right) = 0.$$

These closed braids can be simplified via isotopy moves (c=conjugacy, M=Markov and s=slide):

$$[\beta \sigma_n^{-2} \sigma_n^2 \sigma_n^{-1} \gamma^{p(n+1)}]_{n+p+1}^{-} \overset{c}{\sim} [\beta \sigma_n^2 \sigma_n^{-1} \gamma^{p(n+1)}]_{n+p+1}^{-} \overset{M}{\sim} [\beta \sigma_n^{-1} \gamma^{p(n+1)}]_{n+p+1}^{-} \overset{s}{\sim} [\beta \gamma^{p(n+1)}]_{n+p}^{-};$$

$$[\beta \sigma_n^{-1} \sigma_n^2 \gamma^{p(n+1)}]_{n+p+1}^{-} \overset{M}{\sim} [\beta \gamma^{p(n+1)}]_{n+p}^{-} \overset{s}{\sim} [\beta \gamma^{p(n+1)}]_{n+p}^{-};$$

Then

$$a^{-1} \cdot P\left([\beta \gamma^{p(n+1)}]_{n+p}^{-}\right) + a \cdot P\left([\beta \gamma^{p(n+1)}]_{n+p}^{-}\right) = (a^{-1} + a) \cdot P\left([\beta \gamma^{p(n+1)}]_{n+p}^{-}\right).$$

Comparing it with the equality (from (II_B))

$$a^{-2} \cdot P\left([\beta \gamma^{p(n+1)}]_{n+p}^{-}\right) + a^2 \cdot P\left([\beta \gamma^{p(n+1)}]_{n+p}^{-}\right) = (2 + z^2) \cdot P\left([\beta \gamma^{p(n)}]_{n+p}^{-}\right),$$

we get the desired conclusion.

Corollary 4.2. Under the assumption of the above lemma, the following two relations are equivalent to each other:

(II) $P\left(\begin{array}{c} x \\ y \\ \hline z \end{array}\right) = z^{-1}(a^{-1} - a) \cdot P\left(\begin{array}{c} x \\ y \\ \hline z \end{array}\right);$

(III) $P\left(\begin{array}{c} x \\ y \\ \hline z \end{array}\right) = a z^{-1}(1 + z^2 - a^2) \cdot P\left(\begin{array}{c} x \\ y \\ \hline z \end{array}\right).$

Proof. The braid form of these two relations are, respectively,

(II_B) $P\left([\beta]_{n+1}^{-}\right) = z^{-1}(a^{-1} - a) \cdot P\left([\beta]_{n}^{-}\right) \quad \text{for any braid } \beta \in B_n;$

(III_B) $P\left([\beta \sigma_n^{-1}]_{n+1}^{-}\right) = a z^{-1}(1 + z^2 - a^2) \cdot P\left([\beta]_{n}^{-}\right) \quad \text{for any braid } \beta \in B_n.$

They are equivalent to each other by the above lemma with $[\gamma]_p := [e]_1.$
There is a parallel statement for Jones polynomial:

**Corollary 4.3.** Assume that $V_L \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$ is an invariant of oriented links that satisfies skein relations $(\Pi_V)$ and $(\Pi_II_V)$. Then the following two relations are equivalent to each other:

$(\text{IO}_V) \quad V \left( \begin{array}{c} \circ \end{array} \right) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot V \left( \begin{array}{c} \circ \end{array} \right)$;

$(\Phi_V) \quad V \left( \begin{array}{c} \circ \end{array} \right) = -t^2(t + t^{-1}) \cdot V \left( \begin{array}{c} \circ \end{array} \right)$.

For the Alexander-Conway polynomial, we have:

**Corollary 4.4.** Assume that $\Delta_L \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$ is an invariant of oriented links that satisfies skein relations $(\Pi_\Delta)$ and $(\Pi_III_\Delta)$. Then $\Delta(L) = 0$ for any split link $L$. In particular, the following relation holds true:

$(\text{IO}_\Delta) \quad \Delta \left( \begin{array}{c} \circ \end{array} \right) = 0.$

**Proof.** For links $L_1 = [\beta]^n_1$ and $L_2 = [\gamma]^n_1$, the split link $L = L_1 \sqcup L_2 = [\beta \gamma^n]^n_{n+p}$. Then apply Lemma 4.1 with substitutions $a \mapsto 1$ and $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. □

5. **Proof of Theorems 1.1–1.3**

We shall focus on Theorem 1.1, then remark on the other two.

**Proof of Theorem 1.1.** Let us forget about the original definition of the skein polynomial, and regard the symbol $P_L$ as a well-defined invariant of oriented links which satisfies the axioms $(\Pi)$, $(\Pi_III)$, $(\Pi_{IO})$ and $(\Pi_O)$. By Corollary 4.2, $P_L$ also satisfies axiom $(\Phi)$. We shall show that such an invariant $P_L$ is computable, hence uniquely determined.

It suffices to prove the following claim by induction on $n$.

**Inductive Claim**. For every $n$-braid $\beta \in B_n$, $P([\beta]^n)$ is computable.

When $n = 1$, Claim(1) is true because there is only one 1-braid $[e]_1$. Its closure is the trivial knot, whose $P$ must be 1 by axiom $(\Pi_O)$.

Now assume inductively that Claim($n-1$) is true, we shall prove that Claim($n$) is also true.

Suppose $\beta$ is an $n$-braid. By Lemma 3.2, the braid $\beta \in B_n$ is equivalent to (in a computable way) a $\Lambda$-linear combination of braids of the form $\alpha \sigma^{k-1}_{n-1} \gamma$ with $\alpha, \gamma \in B_{n-1}$ and $k \in \{0, \pm 1, 2\}$. By Example 2.2, the (mod $I_n$) equivalence preserves the $P$ of closure of braids. So $P([\beta]^n)$ is a $\Lambda$-linear combination (with computable
coefficients) of \( P \left( [\alpha \sigma^k_{n-1}]_n \right) \)'s. For \( k \in \{\pm 1, 0, 2\} \), respectively, we have
\[
\begin{align*}
P \left( [\alpha \sigma^1_{n-1}]_n \right) &= P \left( [\alpha \gamma]_{n-1} \right) & \text{by isotopy}, \\
P \left( [\alpha \sigma^0_{n-1}]_n \right) &= z^{-1} (a^{-1} - a) \cdot P \left( [\alpha \gamma]_{n-1} \right) & \text{by (IO)}, \\
P \left( [\alpha \sigma^2_{n-1}]_n \right) &= az^{-1} (1 + z^2 - a^2) \cdot P \left( [\alpha \gamma]_{n-1} \right) & \text{by (\Phi)}. 
\end{align*}
\]

Since \( P \left( [\alpha \gamma]_{n-1} \right) \) is computable by the inductive hypothesis Claim\((n - 1)\), we see \( P \left( [\alpha \sigma^k_{n-1}]_n \right) \) is also computable. Thus Claim\((n)\) is proved.

The induction on \( n \) is now complete. Hence \( P \) is computable for every closed braid. □

Remark 5.1. The induction above, together with the reduction argument of Section 3, provides a recursive algorithm for computing \( P \left( [\beta]_n \right) \).

Remark 5.2. A remarkable feature of this algorithm is that it never increases the number of components of links. In fact, all the reductions in Section 3 are by axioms (II) and (III) which respect the components, while in this Section, components could get removed but never added, by axioms (IO) and (\Phi). So if we start off with a knot, we shall always get knots along the way, the axioms (IO) and (\Phi) becoming irrelevant. This observation works even for links with any given number of components, once we set up a suitable normalization. Hence the Theorem 1.4.

Remark 5.3. For the Jones polynomial, the proof above works well with the substitutions \( a \mapsto t \) and \( z \mapsto (t^{1/2} - t^{-1/2}) \).

Remark 5.4. The case of Alexander-Conway polynomial is only slightly different. Corollary 4.4 says (\( \text{IO}_\Delta \)) is a consequence of axioms (II\(_\Delta\)) and (III\(_\Delta\)), and (\( \Phi_\Delta \)) is taken as an axiom. So the proof above also works through with the substitutions \( a \mapsto 1 \) and \( z \mapsto (t^{1/2} - t^{-1/2}) \).

Actually, the argument in Section 4 can be adapted to work for Conway potential function of colored links, to the effect that in [Main Theorem], the relation (IO) is a consequence of axioms (II) and (III) hence can be removed from the list of axioms.

6. Proof of Theorems 1.4–1.6

Suppose \( \mu \) is a given positive integer. Regard \( P_L \) as a well-defined invariant of oriented \( \mu \)-component links which satisfies the axioms (II), (III) and (U). We shall temporarily expand the ring \( \Lambda := \mathbb{Z}[a^\pm 1, z^\pm 1] \), where the invariant \( P_L \) takes value, to \( \tilde{\Lambda} := \mathbb{Z}[a^\pm 1, z^\pm 1, (a^{-2} - 1)^{-1}] \), to allow fractions with denominator a power of \( (a^{-2} - 1) \). We shall show that such an invariant \( P_L \) is computable, hence uniquely determined. It is the normalization (U) that brings the value \( P_L \) back into the original \( \Lambda \).

Lemma 6.1. Suppose \( \beta \in B_n \), \( p \geq 0 \), and \( [\beta]_{n+p} \) has \( \mu \) components. If \( n > 1 \), then \( P \left( [\beta]_{n+p} \right) \) is computable as a \( \tilde{\Lambda} \)-linear combination of terms of the form...
Proof. By Lemma 3.2 the braid $\beta \in B_n$ is equivalent to (in a computable way) a $\Lambda$-linear combination of braids of the form $\alpha \sigma^k_{n-1} \gamma$ with $\alpha, \gamma \in B_{n-1}$ and $k \in \{0, \pm 1, 2\}$. So $P([\beta]_{n+p})$ is a $\Lambda$-linear combination (with computable coefficients) of $P([\alpha \sigma^k_{n-1} \gamma]_{n+p})$'s. For $k \in \{\pm 1, 2\}$, respectively, we have

$$P([\alpha \sigma^\pm_1 \gamma]_{n+p}) = P([\gamma \alpha \sigma^\pm_1]_{n+p})$$
$$= P([\gamma \alpha]_{(n-1)+p})$$
$$P([\alpha \sigma^0_{n-1} \gamma]_{n+p}) = P([\alpha \gamma]_{(n-1)+(p+1)})$$
$$P([\alpha \sigma^2_{n-1} \gamma]_{n+p}) = P([\gamma \alpha \sigma^2_{n-1}]_{n+p})$$

$$= 1 + z^2 - a^2 \overline{a^2} - 1 \cdot P([\gamma \alpha]_{(n-1)+(p+1)})$$

by braid conjugation, by Markov move, obvious, by braid conjugation

The $P$'s on the right hand sides satisfy the required conditions. \qed

Proof of Theorem 1.4. Suppose a link $L$ with $\mu$ components is presented as $[\beta]_n = [\beta]_n^{+0}$. Apply Lemma 6.1 repeatedly until no such reduction is possible. Then $P([\beta]_{n+p})$ is computed as a $\Lambda$-linear combination of terms $P([\beta]_{n+p'})$, each with $n' = 1$ hence $\beta' = [e]$, such that

1. every $[\beta']_{n'+p'} = [e]_{1+p'}$ has $\mu$ components, hence $p' = \mu - 1$, and $[\beta']_{n'+p'} = [e]_\mu = U_\mu$; and
2. the $(a^{-2} - 1)^{-1}$-exponent of every coefficient is at most $p' - 0 = \mu - 1$.

Therefore, $P(L)$ is computable and, by axiom (U), every term $P([\beta']_{n'+p'})$ has a factor $(a^{-2} - 1)^{\mu-1}$ that can cancel the $(a^{-2} - 1)^{-1}$-exponent in its coefficient, so $P(L) \in \Lambda$. \qed

Theorem 1.5 can be proved similarly, but Theorem 1.6 needs modifications. Define $\delta_p := \sigma_1^p \sigma_2^p \ldots \sigma_{p-1}^p \in B_p$ whose closure $[\delta_p]_p$ is the oriented $p$-component chain $C_p$.

Lemma 6.2. Suppose $\beta \in B_{n+1}$, $p \geq 1$, and $[\beta \delta^p_n]_{n+p}$ has $\mu$ components. If $n > 0$, then $\Delta ([\beta \delta^p_n]_{n+p})$ is computable as a $\mathbb{Z}[t^{\pm \frac{1}{2}}]$-linear combination of terms of the form $\Delta ([\beta \delta^p_{n'}]_{n'+p'})$, each with $\mu$ components, $\beta' \in B_{n'+1}$, $n' < n$ and $p' \geq p$.

Proof. By Lemma 3.2 (with substitutions $a \mapsto 1$ and $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$), the braid $\beta \in B_{n+1}$ is equivalent to (in a computable way) a $\mathbb{Z}[t^{\pm \frac{1}{2}}]$-linear combination of braids of the form $\alpha \sigma^k_{n'} \gamma$ with $\alpha, \gamma \in B_n$ and $k \in \{0, \pm 1, 2\}$. Multiplication by $\delta^p_n$ makes the braid $\beta \delta^p_n$ equivalent to a linear combination of braids of the form $\alpha \sigma^k_{n'} \delta^p_n$, so $\Delta ([\beta \delta^p_n]_{n+p})$ is computed as a linear combination of the $\Delta ([\alpha \sigma^k_{n'} \delta^p_n]_{n+p})$'s. For
\[ k \in \{\pm 1, 0, 2\}, \text{ respectively, we have} \]

\[
\Delta \left( [\alpha^{\pm 1}_n \gamma \delta^{n}_{p}]_{n+p} \right) = \Delta \left( [\gamma \alpha^{\pm 1}_n \delta^{n}_{p}]_{n+p} \right) \quad \text{by braid conjugacy,}
\]

\[
= \Delta \left( [\gamma \alpha^{(n-1)}_p]_{n+p-1} \right) \quad \text{by a Markov move,}
\]

\[
\Delta \left( [\alpha^{0}_n \gamma \delta^{n}_{p}]_{n+p} \right) = 0 \quad \text{by Corollary 4.4,}
\]

\[
\Delta \left( [\alpha^{2}_n \gamma \delta^{n}_{p}]_{n+p} \right) = \Delta \left( [\gamma \alpha^{2}_n \delta^{n}_{p}]_{n+p} \right) \quad \text{by braid conjugacy}
\]

\[
= \Delta \left( [\gamma \alpha^{(n-1)}_p]_{n+p} \right) \quad \text{by definition.}
\]

The \( \Delta \)'s on the right hand sides are in the desired form with \( n' = n - 1 \) and with \( \mu \) components.

\( \square \)

**Proof of Theorem 1.6.** Suppose a link \( L \) with \( \mu \) components is presented as \( [\beta]_{n+1} = [\beta \delta^{n}_{p}]_{n+1} \). Apply Lemma 6.2 repeatedly until no such reduction is possible. Then \( \Delta ([\beta \delta^{n}_{p}]_{n+1}) \) is computed as a \( \mathbb{Z}[t^{\pm \frac{1}{2}}] \)-linear combination of terms \( \Delta ([\beta' \delta^{n'}_{p'}]_{n'+p'}) \), with \( n' = 0 \). Hence each \( \beta' = [e]_1, p' = \mu \) the number of components, and each \( [\beta' \delta^{n'}_{p'}]_{n'+p'} = [\delta^{\mu}_{\mu}]_{\mu} = C_{\mu} \). Therefore \( \Delta(L) \) is computable and moreover, by axiom (C\( \Delta \)), divisible by \( \Delta(C_{\mu}) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\mu-1} \).

\( \square \)

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