Perturbations of integrable systems and Dyson-Mehta integrals

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Abstract

We show that the existence of algebraic forms of quantum, exactly-solvable, completely-integrable $A-B-C-D$ and $G_2,F_4,E_{6,7,8}$ Olshanetsky-Perelomov Hamiltonians allow to develop the algebraic perturbation theory, where corrections are computed by pure linear algebra means. A Lie-algebraic classification of such perturbations is given. In particular, this scheme admits an explicit study of anharmonic many-body problems. The approach also allows to calculate the ratio of a certain generalized Dyson-Mehta integrals algebraically, which are interested by themselves.

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1 Introduction

This talk is devoted to a study of perturbations of the quantum-mechanical integrable systems. The feature of integrability of these systems attract a lot of attention which in our opinion is not fully justified. The main problem of quantum mechanics is the finding the spectra of the Hamiltonian via solving the Schroedinger equation. However, a knowledge of operators commuting to the Hamiltonian does not help to solve the main problem of quantum mechanics. At least, for the moment such a connection is not known. The simplest illustration of the situation is provided by the one-dimensional quantum dynamics - by definition, any one-dimensional dynamics is completely- and super-integrable - but it does not imply that a general one-dimensional Schroedinger equation with arbitrary potential can be solved exactly in whatever sense. The problem remains to be transcendental and is equivalent to a diagonalization of a generic matrix of infinite size. Existence of the integrals commuting to the Hamiltonian gives a chance to assign definite quantum numbers to the eigenstate, which also can be done constructively for the only situation of knowledge of exact solutions of the original Schroedinger equation. However, among known integrable systems in quantum mechanics there exists a class of the systems with an outstanding property – these integrable systems are exactly-solvable in a sense that their eigenfunctions and eigenvalues can be found exactly. The notion exactly needs to be explained and we introduce a notion of exact-solvability.

Let us take Hamiltonian

\[ \mathcal{H} = -\Delta + V(x), \quad x \in \mathbb{R}^d \]

The system is called integrable, if there exist \( k \) functionally-independent operators commuting with \( \mathcal{H} \),

\[ [\mathcal{H}, I_i] = 0, \quad i = 1, 2 \ldots k \]

If \( k > d - 1 \), the system is called superintegrable. If \( k = d - 1 \) and for all \( i, j \)

\[ [I_i, I_j] = 0 \]

the system is called completely-integrable.

The main problem of quantum mechanics is to solve the Schroedinger equation

\[ \mathcal{H}\Psi(x) = E\Psi(x), \quad \Psi(x) \in L^2(\mathbb{R}^d). \]

Integrability in whatever sense does not help to solve the Schroedinger equation. However, among integrable systems there are several exactly-solvable systems. They can be used as zero approximations to study realistic physical systems.
Now we introduce a definition of \textit{exact-solvability}. Let us assume that a linear operator $h$ possesses infinitely-many finite-dimensional invariant subspaces $\mathcal{V}_n$, \( n = 0, 1 \ldots \), which can ordered
\[
\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n \subset \ldots \mathcal{V},
\]
thus forming an \textit{infinite flag (filtration)} $\mathcal{V}$. Hence the operator $h$ preserves the flag $\mathcal{V}$.

\textbf{General Definition} \cite{4}

An operator $h$ which preserves an infinite flag of finite-dimensional spaces $\mathcal{V}$ is called \textit{exactly-solvable operator with flag $\mathcal{V}$}.

If given $h$ preserves several flags and among them there is a flag for which $\dim \mathcal{V}_n$ is \textit{maximal} for any given $n$, such a flag is called \textit{minimal}.

Below we deal with certain linear spaces of polynomials in several variables. It leads to a notion \textit{‘characteristic vector’} \cite{10}. Let us consider the triangular linear space of polynomials in $k$ variables
\[
\mathcal{P}_n^{(\alpha_1, \ldots, \alpha_k)} = \langle s_1^{p_1} s_2^{p_2} \ldots s_k^{p_k} | 0 \leq \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k \leq n \rangle, \quad (1.1)
\]
where $\alpha$’s are positive numbers and $n = 0, 1, 2, \ldots$. Characteristic vector is a vector with components which are equal to the coefficients (weights) $\alpha_i$ in front of $p_i$:
\[
\vec{f} = (\alpha_1, \alpha_2, \ldots, \alpha_k). \quad (1.2)
\]
Taking the spaces with $n = 0, 1, 2, \ldots$ we arrive at the flag which has $\mathcal{P}_n^{(\alpha_1, \ldots, \alpha_k)}$ as generating linear space. We call this flag $\mathcal{P}^{(\alpha_1, \ldots, \alpha_k)}$. The flag with $\vec{f}_0 = (1, 1, \ldots, 1)$ is called \textit{basic} and denoted $\mathcal{P}^{(k)} \equiv \mathcal{P}^{(1,\ldots,1)}$. It is clear that is minimal among possible flags ever. This flag has a Lie-algebraic interpretation.

Let us take the algebra $gl_{k+1}$ in almost degenerate or totally symmetric representation characterized by spins $(n, 0, 0, \ldots 0)$,
\[
\mathcal{J}_i^- = \frac{\partial}{\partial x_i}, \quad i = 1, 2 \ldots k, \\
\mathcal{J}_{ij}^0 = x_i \frac{\partial}{\partial x_j}, \quad i, j = 1, 2 \ldots k,
\]
\[ J^0 = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} - n, \]  

\[ J^+_i = x_i J^0 = x_i \left( \sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j} - n \right), \quad i = 1, 2 \ldots k. \]

where \( n \) is any real or complex number. It is easy to check that \( J \)'s obey the commutation relations of the algebra \( \mathfrak{gl}_{k+1} \). Furthermore, if \( n = 0, 1, 2 \ldots \), then the finite-dimensional irreps appear with representation spaces

\[ \mathcal{P}^{(k)}_n = \langle x_1^{p_1} x_2^{p_2} \ldots x_d^{p_d} | 0 \leq \Sigma p_i \leq n \rangle \]

**Remark.** The flag \( \mathcal{P}^{(k)} \) is made out of irreducible finite-dimensional representation spaces of the algebra \( \mathfrak{gl}_{k+1} \) taken in realization (1.3).

There exist other flags associated with irreducible finite-dimensional representation spaces of the Lie algebras of differential (difference) operators.

**Definition:**

The operator \( h \) is called algebraic, if it preserves a flag of polynomials. It has the form \( \sum \text{Pol}_n \cdot \partial^n \). This form is called algebraic.

It is evident that the existence of an algebraic form does not guarantee that the operator preserves the flag of polynomials.

**Theorem 1.**

Linear differential operator \( h \) preserves the flag \( \mathcal{P}^{(k)} \) iff \( h = P(J(b \subset \mathfrak{gl}_{k+1})) \), where \( P \) is a polynomial in generators of the maximal affine subalgebra \( b \) of the algebra \( \mathfrak{gl}_{k+1} \), being taken in realization (1.3).

In particular, exactly-solvable second order differential operator \( h \) (preserving the flag \( \mathcal{P}^{(k)} \)) has the form

\[ h = P_2^{(ij)}(x) \partial_i \partial_j + P_1^{(i)}(x) \partial_i \]  

where \( P_2^{(ij)}(x) \) and \( P_1^{(i)}(x) \) are the 2nd and 1st degree polynomials in coordinates \( x \)'s with arbitrary coefficients. It is the standard multidimensional hypergeometrical operator. It was shown in [5, 7] that the Calogero and Sutherland models (equivalently, the \( A_k \)-rational and trigonometric models), and the \( BC_k \)-rational and trigonometric...
models are of the hypergeometrical type (1.3). In order to get these models in the form (1.4) in the Hamiltonian the ground state eigenfunction should be taken as a gauge factor and then Weyl-invariant (trigonometric) polynomials have to be taken as new variables.

Another natural hypergeometrical operator appears in connection to the flag $P^{(1,2)}$. Let us consider the algebra $g^{(2)}$ generated by

\[
L^1 = \partial_1 , \quad L^2 = x_1 \partial_1 - n/3 , \\
L^3 = x_2 \partial_2 - n/6 , \quad L^4 = x_1^2 \partial_1 + 2x_1x_2 \partial_2 - nx_1 , \\
L^5 = \partial_2 , \quad L^6 = x_1 \partial_2 , \\
L^7 = x_2^2 \partial_2 , \quad T = x_2 \partial_{11} ,
\]

(1.5)

(see [8]), where $n$ is any real or complex number. If $n = 0, 1, 2, \ldots$, the algebra $g^{(2)}$ has finite-dimensional irreps with representation spaces given by $P^{(1,2)}_n$. These representation spaces form the flag $P^{(1,2)}$. It is rather obvious that except for the operator $L^4$ the remaining operators in (1.5) has the space $P^{(1,2)}_n$ as common invariant subspace for any $n$. Therefore the second degree differential operator preserving the flag $P^{(1,2)}$ is constructed by taking the second degree polynomial in $L_{1,2,3,5,6}$ plus the generator $T$. Its explicit form is

\[
h = (a_1^{(11)} x_1^2 + a_2^{(11)} x_1 + a_3^{(11)} x_2 + a_4^{(11)}) \partial_1 \partial_1 \\
+(a_1^{(12)} x_1^3 + a_2^{(12)} x_1^2 + a_3^{(12)} x_1 x_2 + a_4^{(12)} x_4 + a_5^{(12)} x_2 + a_6^{(12)}) \partial_1 \partial_2 + \\
(a_1^{(22)} x_1^4 + a_2^{(22)} x_1^3 + a_3^{(22)} x_1^2 x_2 + a_4^{(22)} x_1^2 + a_5^{(22)} x_2^2 + a_6^{(22)} x_1 x_2 + a_7^{(22)} x_1 + a_8^{(22)} x_2 + a_9^{(22)}) \partial_2 \partial_2 + \\
(b_1^{(1)} x_1 + b_2^{(1)}) \partial_1 + (b_1^{(2)} x_2^2 + b_2^{(2)} x_1 + b_3^{(2)} x_2 + b_4^{(2)}) \partial_2 .
\]

(1.6)

It was shown in [8] that the algebraic form of both rational and trigonometric $G_2$ models are a particular type of (1.6). It can be easily checked that the flag $P^{(1,2)}$ is minimal for these models.

There exist other hypergeometrical operators related with flags of polynomials other than $P^{(k)}$ or $P^{(1,2)}$. In Table I we enlist the characteristic vectors of minimal flags of the Olshanetsky-Perelomov Hamiltonians.
2 Perturbation theory

Existence of algebraic forms of exactly-solvable operators allows the construction of an algebraic perturbation theory, where the procedure of finding corrections is linear algebra procedure.

Consider the spectral problem,

\[ (h_0 + \lambda h_1)\phi = E\phi \tag{2.7} \]

where \( \lambda \) is a formal parameter, the solution of that we look for in perturbation theory,

\[ \phi = \sum \lambda^k \phi_k, \quad E = \sum \lambda^k E_k \tag{2.8} \]

Then the following theorem holds:

**Theorem 2 [14]**

*Let \( h_0 \) be an exactly-solvable operator with flag \( \mathcal{V} \). Let the perturbation \( h_1 = v_1 \) is such that (i) \( v_1 \) is an element of a finite-dimensional space \( \mathcal{V}_n \) from the flag and (ii) we look for \( \phi \in \mathcal{V} \). Then the perturbation theory is algebraic: \( \exists p(k) \) such that kth correction \( \phi_k \in \mathcal{V}_{p(k)} \) and hence can be found algebraically.*

Equation to solve to find kth correction is:

\[ (h_0 - E_0)\phi_k = \sum_{i=1}^{k} E_i \phi_{k-i} - v_1 \phi_{k-1} \]

As an immediate consequence of Theorem it can be proven the following. If \( h_0 \) depends on some parameters holomorphically that the coefficients in \( \phi_k \) and \( E_k \) are rational functions parameters. In the case of absence of parameters in \( h_0 \) the coefficients in \( \phi_k \) and \( E_k \) are rational numbers.

This form of the perturbation theory is characterized by a certain self-similarity property – a calculation of kth correction is similar to the calculation of the first correction but with a modified perturbation potential

\[ v_k = - (\sum_{i=1}^{k-1} E_i \phi_{k-i} - v_1 \phi_{k-1})/\phi_0 \].

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3 Algebraic forms of the Olshanetsky-Perelomov Hamiltonians

As was indicated in the previous Section the existence of the algebraic forms of exactly-solvable problems allows for a certain classes of perturbations to construct perturbation theory algebraically. Thus, the exactly-solvable Olshanetsky-Perelomov Hamiltonians for which there exist algebraic forms can be possible unperturbed problems for developing such a perturbation theory. We present those algebraic forms below.

In order to find the algebraic forms of the Olshanetsky-Perelomov Hamiltonians \[2\] the following strategy is used \[5, 7, 8, 11\],

- Gauging away ground state eigenfunction, \((Ψ_0)^{-1} \mathcal{H} Ψ_0\),
- Olshanetsky-Perelomov Hamiltonians possess different symmetries (permutations, translation-invariance, reflections, periodicity etc). These symmetries correspond to the Weyl group acting on the root space. By coding these symmetries to new coordinates we find 'premature' operators to these Hamiltonians,
  
  - Rational case – Weyl-invariant variables:
    \[ t^{(\Omega)}_a(x) = \sum_{\alpha \in \Omega} (\alpha, x)^a \],
    
    where \(a\)'s are the degrees of the Weyl group \(W\) and \(\alpha\) is an orbit. They are defined ambiguously depending on the orbit taken, but they **always lead to algebraic forms**,
  
  - Trigonometric case – trigonometric (periodic) Weyl-invariant variables, 
    \[ τ^{(\Omega)}_a(x) = \sum_{\alpha \in \Omega} \sin^a(\alpha, x) \],
    
    Not always algebraic forms appear in this case, only for very special combination of \(τ^{(\Omega)}_a\) it emerges.
3.1 HISTORY

Rational Case:

- Flat-space metric $g^{\mu\nu}$ being written in $t_\alpha^{(\Omega)}$-coordinates has polynomial matrix elements (V.I. Arnold, ’76 [12])
- Laplacian and $A_n$-extended Laplacian (it means $A_n$-rational model or Calodero Model) in $t_\alpha^{(\Omega)}$ – invariants of $A_n$ root space – have an algebraic form and are written in generators of $gl_n$-algebra of 1st order differential operators (1.3) (W.Ruhl, A.T., ’95 [5])
- Laplacian and $(B,C,BC)_n$-extended Laplacian (in other words, $(B,C,BC)_n$-rational model) in $t_\alpha^{(\Omega)}$ – invariants of $(B,C,BC)_n$ root space – have an algebraic form and are written in generators of $gl_n$-algebra of 1st order differential operators (1.3) (L.Brink, A.T., N.Wyllard, ’98 [7])
- Similar for $G_2$ and $F_4$ rational models but with hidden algebras $g^{(2)}$ and $f^{(4)}$, resp. (M.Rosenbaum et al, ’98 [8], W.Ruhl et al, ’00 [10] and K.G.Boreskov et al, ’01 [11]) as well as $E_{6,7,8}$ [13].

Trigonometric Case:

- Laplacian and $A_n$-extended Laplacian ($A_n$-trigonometric model or Sutherland Model) in $\tau_\alpha^{(\Omega)}$ – trig.invariants of $A_n$ root space – have an algebraic form and are written in generators of $gl_n$-algebra of 1st order differential operators (1.3) (W.Ruhl & A.T., ’95 [5])
- Laplacian and $(B,C,BC)_n$-extended Laplacian ($(B,C,BC)_n$-trigonometric model) in $\tau_\alpha^{(\Omega)}$ – trig.invariants of $(B,C,BC)_n$ root space – have an algebraic form and are written in generators of $gl_n$-algebra of 1st order differential operators (1.3) (L.Brink, A.T., N.Wyllard, ’98 [7])
- Similar for $G_2$ and $F_4$ trigonometric models but with hidden algebras $g^{(2)}$ and $f^{(4)}$, resp. (M.Rosenbaum et al, ’98 [8], W.Ruhl et al, ’00 [10] and K.G.Boreskov et al, ’01 [11])
3.2 Algebraic forms

The algebraic forms of the Olshanetsky-Perelomov Hamiltonians are given by the following differential operator

\[ h = A_{ij}(\tau) \frac{\partial^2}{\partial \tau_i \partial \tau_j} + B_i(\tau) \frac{\partial}{\partial \tau_i} \quad (3.9) \]

with polynomial coefficient functions:

1. **A\(_n\) rational case (Calogero model)**

\[
A_{ij} = \frac{(n - i + 1)j}{n + 1} \tau_i \tau_j + \sum_{l \geq \max(1,j-i)} (j - i - 2l) \tau_{i+l} \tau_{j-l},
\]

\[
B_i = -\frac{1}{n + 1}(1 + \nu + \nu n)(n - i + 2)(n - i + 1) \tau_{i-1} + 2\omega (i + 1) \tau_i, \quad (3.10)
\]

where \(i, j = 1, 2, \ldots n\).

2. **A\(_n\) trigonometric case (Sutherland model)**

\[
A_{ij} = \frac{(n + 1 - i)j}{n + 1} \tau_i \tau_j + \sum_{l \geq \max(1,j-i)} (j - i - 2l) \tau_{i+l} \tau_{j-l},
\]

\[
B_i = \frac{1}{n + 1}(1 + \nu + \nu n) i (n + 1 - i) \tau_i, \quad (3.11)
\]

where \(i, j = 1, 2, \ldots n\).

3. **BC\(_n\) rational model**

\[
A_{ij} = 4 \sum_{l \geq 0} (2l + 1 + j - i) \tau_{i-l-1} \tau_{j+l},
\]

\[
B_i = 2 [1 + \nu_2 + 2\nu(n - i)] (n - i + 1) \tau_{i-1} - 4 \omega i \tau_i, \quad (3.12)
\]

where \(i, j = 1, 2, \ldots n\).

4. **BC\(_n\) trigonometric model**
\[
A_{ij}=\sum_{l \geq 0} \left[ (i-l) \tau_{i-l} \tau_{j+l} + (l+j-1) \tau_{i-l-1} \tau_{j+l-1} - (i-2-l) \tau_{i-2-l} \tau_{j+l} - (l+j+1) \tau_{i-l-1} \tau_{j+l+1} \right],
\]
\[
B_i=\frac{\nu_3}{2} (i-n-1) \tau_{i-1} - \left[ 1 + \nu_2 + \frac{\nu_3}{2} + \nu (2n-i-1) \right] i \tau_i - \nu (n-i+1)(n-i+2) \tau_{i-2},
\]
where \(i, j = 1, 2, \ldots n\).

5. \(G_2\) rational model

\[
A_{11} = 2 \tau_1, \quad A_{12} = 12 \tau_2, \quad A_{22} = -\frac{8}{3} \tau_1^2 \tau_2,
\]
\[
B_1 = \frac{4}{3} \omega \tau_1 + 2(1 + 3\mu + 3\nu), \quad B_2 = 4\omega \tau_2 - \frac{4}{3}(1 + 2\nu)\tau_1^2.
\] (3.14)

6. \(G_2\) trigonometric model

\[
A_{11} = (2\tau_1 + \frac{\alpha^2}{2} \tau_1^2 - \frac{\alpha^4}{24} \tau_2), \quad A_{12} = (12 + \frac{8\alpha^2}{3} \tau_1) \tau_2, \quad A_{22} = -\left( \frac{8}{3} \tau_1^2 \tau_2 - 2\alpha^2 \tau_2^2 \right),
\]
\[
B_1 = 2(1 + 3\mu + 3\nu) + \frac{2}{3}(1 + 3\mu + 4\nu)\alpha^2 \tau_1, \quad B_2 = -\frac{4}{3}(1 + 2\nu)\tau_1^2 + [\frac{7}{3} + 4(\mu + \nu)]\alpha^2 \tau_2.
\] (3.15)

Algebraic forms for the \(F_4\) rational and trigonometric models as well as \(E_6, 7, 8\) rational models can be found in [11] and [13], respectively.

All rational models possess a remarkable property - each of them is characterized by appearance of infinite family of eigenstates depending on single variable, which is the second Weyl invariant \(t_2\) (see definition above) \(^3\). This family always includes the ground state. Finding these states is reduced to solving a simple eigenvalue problem for one-dimensional differential operator, which can be written in the form

\[
h_r = -2t_2 \partial_{t_2}^2 + (4\omega t_2 - 1 - 2a_r) \partial_{t_2},
\]

\(^3\)The invariant \(t_2\) has unique feature – it is the same for all Weyl groups.
where the parameter \( a_r \) depends on the model studied. It can be easily demonstrated that the operator \( h_r \) is reduced to the Hamiltonian of the \( A_1 \) rational model

\[
\mathcal{H}_{A_1} = -\frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2} (x_1 - x_2)^2 + \frac{g}{(x_1 - x_2)^2}
\]

where the coupling constant \( g = a_r(1 - a_r) \).

### 4 Perturbation theory (examples)

In this Section we present several seemingly interesting examples of perturbative calculations for the ground state for different perturbations. Other examples can be found at Ref. [14].

1. \( G_2 \) rational model (see 3.9 with coefficients (3.14)).

   Let us take \( \tau_2 \) as a perturbation which in Cartesian coordinates looks like

   \[
v_1^{(r)} = \tau_2 = (x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2
\]

   The ground state is

   \[
   \phi_0 = 1 , \quad \epsilon_0 = 0
\]

   After simple calculations we get

   \[
   \phi_1 = -\frac{(1 + 2\nu)}{8\omega^2} \tau_2 - \frac{1}{4\omega} \tau_2 + \frac{3}{8\omega^3} (1 + 2\nu)(2 + 3\mu + 3\nu) \tau_1 ,
   \]

   \[
   \epsilon_1 = \frac{3}{4\omega^3} (1 + 2\nu)(1 + 3\mu + 3\nu)(2 + 3\mu + 3\nu) . \quad (4.16)
\]

   It is not very complicated to calculate next several corrections.

2. \( G_2 \) trigonometric model (see 3.9 with coefficients (3.15)).

   (i). Take \( \tau_1 \) as a perturbation, which in Cartesian coordinates is

   \[
v_1^{(t)} = \tau_1 = -\frac{2}{\alpha^2} [\sin^2 \frac{\alpha}{2}(x_1 - x_2) + \sin^2 \frac{\alpha}{2}(x_2 - x_3) + \sin^2 \frac{\alpha}{2}(x_3 - x_1)]
\]

   The first correction to the ground state is given by

   \[
   \phi_1 = -\frac{3}{2(1 + 3\mu + 4\nu)\alpha^2} \tau_1 ,
   \]
and
\[ \epsilon_1 = -\frac{3(1 + 3\mu + 3\nu)}{(1 + 3\mu + 4\nu)\alpha^2}. \tag{4.17} \]

It is worth noting that \( \phi_1 \) depends on the single variable \( \tau_1 \). However, it will not be the case for higher corrections they begin to depend on both \( \tau_1, \tau_2 \). The expression (4.17) has quite amusing property. If \( \nu = 0 \) that corresponds to the three-body Sutherland model, \( \epsilon_1 \) does not depend on \( \mu \). Of course, it does not hold for higher corrections but leads to rather non-trivial property of some correlation function (see below).

(ii). Take \( \tau_2 \) as a perturbation, which in Cartesian coordinates has the form
\[ v_1^{(t)} = \tau_2 = \frac{16}{\alpha^6} \left[ \sin \alpha(x_1 - x_2) + \sin \alpha(x_2 - x_3) + \sin \alpha(x_3 - x_1) \right]^2. \]

In the limit \( \alpha \to 0 \), the perturbation \( v_1^{(t)} \) becomes \( v_1^{(r)} \). The first correction to the ground state is given by
\[ \phi_1 = -\frac{1}{N} \left[ \frac{12(1 + 2\nu)}{\alpha^2} \tau_1^2 + 3(7 + 12\mu + 16\nu)\tau_2 - \frac{72(2 + 3\mu + 3\nu)(1 + 2\nu)}{(1 + 3\mu + 4\nu)\alpha^4} \tau_1 \right] \]
and
\[ \epsilon_1 = \frac{144(1 + 2\nu)(1 + 3\mu + 3\nu)(2 + 3\mu + 3\nu)}{(1 + 3\mu + 4\nu)N\alpha^4}, \tag{4.18} \]

where \( N = [7(7 + 12\mu + 16\nu) - (1 - 84\mu - 82\nu - 144\mu^2 - 336\mu\nu - 192\nu^2)\alpha^2] \). It is not very complicated to calculate next several corrections.

3. A\textsubscript{n} rational model (in other words, the \((n+1)\)-body Calogero problem, (see with coefficients (3.10))). Let us take \( \tau_3 \) as a perturbation. In Cartesian coordinates it looks like
\[ v_1^{(r)} = \tau_3 = \sigma_4(y) = \sum_{i_1,i_2,i_3,i_4} y_{i_1}y_{i_2}y_{i_3}y_{i_4}, \]
where \( \sigma_4(y) \) is the fourth order elementary symmetric polynomial of the Perelomov coordinates as arguments,
\[ Y = \sum x_i, \quad y_i = x_i - \frac{1}{n+1}Y, \quad i = 1, \ldots, (n+1). \]

Easy calculation gives quite simple answer for the first correction
\[ -\phi_1 = \frac{1}{8\omega} \tau_3 + \frac{[1 + \nu(n + 1)](n - 1)(n - 2)}{32(n + 1)\omega^2} \tau_1, \]
\[ \epsilon_1 = \frac{[1 + \nu(n + 1)]^2n(n - 1)(n - 2)}{32(n + 1)\omega^2}. \] (4.19)

It can be shown that for non-physical value \( n = 1 \) (somehow ‘one-body’ problem) the energy correction \( \epsilon_1 \) as well as all other corrections vanish. A meaning of this result is not clear so far.

5 Generalized Dyson-Mehta integrals

Let us take one of crystallographic root spaces \( R(A_n, B_n, C_n, D_n, BC_n, G_2, F_4, E_{6,7,8}) \). Suppose \( P_c(t) \) is a Weyl-invariant polynomial of finite order in \( R \) and \( D_c = \text{Weyl chamber} \)

\[ E_\nu(P_c) = \int_{D_c} P_c(t)(\text{Weyl det})^\nu \exp(-\omega t_2)dt \]

\( t_2 \) is 2nd invariant. Similarly one can introduce \( P_t(\tau) \) as the Weyl-invariant trigonometric polynomial of finite order in \( R \) and \( D_t = \text{Weyl alcove} \)

\[ T_\nu(P_t) = \int_{D_t} P_t(\tau)(\text{Trig. Weyl det})^\nu d\tau \]

\( E_\nu, T_\nu \) are related to the Selberg integrals and we call them \textit{generalized Dyson-Mehta integrals}. If \( P_c(t) \) \( (P_t(\tau)) = 1 \), they become the Dyson-Mehta integrals (for discussion see [2]).

\textbf{Theorem 3.}

(i) \( \frac{E_\nu(P_c)}{E_\nu(P_c = 1)} \) is rational function in \( \nu, \omega \) with rational coefficients,

(ii) \( \frac{T_\nu(P_t)}{T_\nu(P_t = 1)} \) is rational function in \( \nu \) with rational coefficients.

\textbf{Proof.}

- Weyl determinant defines the ground state eigenfunction of \( R \)-inspired rational (trigonometric) model,
• ∃ a set of Weyl invariants which being taken as variables lead to algebraic form of the Hamiltonian, they are generators of algebra of Weyl-invariant polynomials.

• Following the Theorem 2 the perturbation theory for any $P_c(t)$ ($P_1(\tau)$) is algebraic. Hence the first energy correction to the ground state is found by pure algebraic means. Since parameters enter to the algebraic forms of the unperturbed operators linearly or at most quadratically, it is evident that the answer can contain not more than rational function of parameters. Q.E.D.

It is worth to remind that for $A_N$,

\[
(Weyl \ det) = \prod_{i<j}^N (x_i - x_j)
\]

\[
(Trig. \ Weyl \ det) = \prod_{i<j}^N \sin(x_i - x_j)
\]

are anti-symmetric invariants.

A constructive way to realize Theorem 3 is to perturb one of the Olshanetsky-Perelomov Hamiltonians with a perturbation, which meets the conditions of the Theorem. Then make two calculations of the first energy correction: (i) one is based on the algebraic perturbation theory and (ii) another is based on the Rayleigh-Schroedinger Perturbation Theory (RSPT). Now let us consider the examples presented in the previous section.

**EXAMPLES**

Case 2(i).

\[
\int_{D_3(G_2)} \left[ \sin^2 \frac{\alpha}{2} (x_1 - x_2) + \sin^2 \frac{\alpha}{2} (x_2 - x_3) + \sin^2 \frac{\alpha}{2} (x_3 - x_1) \right] W^2(x_1, x_2, x_3) d^3x = \frac{3(1 + 3\mu + 3\nu)}{2(1 + 3\mu + 4\nu)}.
\]
where $D_t(G_2)$ is the $G_2$ Weyl alcove and

$$W(x_1, x_2, x_3) = \prod_{i<j}^3 \sin^\nu \frac{\alpha}{2} (x_i - x_j) \prod_{i<j}^3 \sin^\mu \frac{\alpha}{2} (x_i - x_j - 2x_k).$$

Case 2(ii).

$$\int_{D_t(G_2)} \left[ \sin \alpha (x_1 - x_2) + \sin \alpha (x_2 - x_3) + \sin \alpha (x_3 - x_1) \right]^2 W^2(x_1, x_2, x_3) \, d^3x = \int_{D_t(G_2)} W^2(x_1, x_2, x_3) \, d^3x$$

$$= \frac{9\alpha^2(1 + 2\nu)(1 + 3\mu + 3\nu)(2 + 3\mu + 3\nu)}{(1 + 3\mu + 4\nu)N},$$

Case 3.

$$\int_{D_c(A_{n-1})} \left( \sum_{\{i_1, i_2, i_3, i_4\}} y_{i_1}y_{i_2}y_{i_3}y_{i_4} \prod_{i<j} |y_i - y_j|^{2\nu} e^{-\omega \sum y_i^2} d^{n-1}y \right) = \int_{D_c(A_{n-1})} \prod_{i<j} |y_i - y_j|^{2\nu} e^{-\omega \sum y_i^2} d^{n-1}y$$

$$= \frac{1}{32\omega^2} \frac{(1 + n\nu)^2(n - 1)(n - 2)(n - 3)}{n},$$

where $D_c(A_{n-1})$ is the $A_{n-1}$ Weyl chamber.

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