MODULAR GROUP REPRESENTATIONS AND FUSION IN LOGARITHMIC
CONFORMAL FIELD THEORIES AND IN THE QUANTUM GROUP CENTER

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ABSTRACT. The $SL(2, \mathbb{Z})$-representation $\pi$ on the center of the restricted quantum group $\mathbb{U}_q\mathfrak{sl}(2)$ at the primitive $2p$th root of unity is shown to be equivalent to the $SL(2, \mathbb{Z})$-representation on the extended characters of the logarithmic $(1, p)$ conformal field theory model. The multiplicative Jordan decomposition of the $\mathbb{U}_q\mathfrak{sl}(2)$ ribbon element determines the decomposition of $\pi$ into a “pointwise” product of two commuting $SL(2, \mathbb{Z})$-representations, one of which restricts to the Grothendieck ring; this restriction is equivalent to the $SL(2, \mathbb{Z})$-representation on the $(1, p)$-characters, related to the fusion algebra via a nonsemisimple Verlinde formula. The Grothendieck ring of $\mathbb{U}_q\mathfrak{sl}(2)$ at the primitive $2p$th root of unity is shown to coincide with the fusion algebra of the $(1, p)$ logarithmic conformal field theory model. As a by-product, we derive $q$-binomial identities implied by the fusion algebra realized in the center of $\mathbb{U}_q\mathfrak{sl}(2)$.

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1. INTRODUCTION

We study a Kazhdan–Lusztig-like correspondence between a vertex-operator algebra and a quantum group in the case where the conformal field theory associated with the vertex-operator algebra is logarithmic. In its full extent, the Kazhdan–Lusztig correspondence comprises the following claims:

1. A suitable representation category of the vertex-operator algebra is equivalent to the category of finite-dimensional quantum group representations.
2. The fusion algebra associated with the conformal field theory coincides with the quantum-group Grothendieck ring.
3. The modular group representation associated with conformal blocks on a torus is equivalent to the modular group representation on the center of the quantum group.

Such full-fledged claims of the Kazhdan–Lusztig correspondence [1] have been established for affine Lie algebras at a negative integer level and for some other algebras “in the negative zone.” But in the positive zone, the correspondence holds for rational conformal field models [2] (such as \((p', p)\)-minimal Virasoro models and \(\widehat{\iota}(2)_k\) models with \(k \in \mathbb{Z}_+\)) with certain “corrections.” Notably, the semisimple fusion in rational models corresponds to a semisimple quasitensor category obtained as the quotient of the representation category of a quantum group by the tensor ideal of indecomposable tilting modules. Taking the quotient (“neglecting the negligible” in [3], cf. [4]) makes the correspondence
somewhat indirect; in principle, a given semisimple category can thus correspond to different quantum groups. Remarkably, the situation is greatly improved for the class of logarithmic (nonsemisimple) models considered in this paper, where the quantum group itself (not only a quasitensor category) can be reconstructed from the conformal field theory data.

In this paper, we are mostly interested in Claims 3 and 2. Claim 3 of the Kazhdan–Lusztig correspondence involves the statement that the counterpart of the quantum group center on the vertex-operator algebra side is given by the endomorphisms of the identity functor in the category of vertex-operator algebra representations. This object — morally, the “center” of the associated conformal field theory — can be identified with the finite-dimensional space $\mathcal{Z}_{cft}$ of conformal blocks on a torus. In the semisimple case, $\mathcal{Z}_{cft}$ coincides with the space of conformal field theory characters, but in the nonsemisimple case, it is not exhausted by the characters, although we conveniently call it the (space of) extended characters (all these are functions on the upper complex half-plane). The space $\mathcal{Z}_{cft}$ carries a modular group representation, and the Kazhdan–Lusztig correspondence suggests looking for its relation to the modular group representation on the quantum group center.

We recall that an $SL(2, \mathbb{Z})$-representation can be defined for a class of quantum groups (in fact, for ribbon quasitriangular categories) [5, 6]. Remarkably, the two $SL(2, \mathbb{Z})$-representations (on $\mathcal{Z}_{cft}$ and on the quantum group center $\mathfrak{z}$) are indeed equivalent for the logarithmic conformal field theory models studied here.

The details of our study and the main results are as follows. On the vertex-operator algebra side, we consider the “triplet” W-algebra $\mathcal{W}(p)$ that was studied in [7, 8] in relation to the logarithmic $(1, p)$ models of conformal field theory with $p = 2, 3, \ldots$. The algebra $\mathcal{W}(p)$ has $2p$ irreducible highest-weight representations $\mathcal{X}^{\pm}(s)$, $s = 1, \ldots, p$, which (in contrast to the case of rational conformal field models) admit nontrivial extensions among themselves ($L_0$ is nondiagonalizable on some of extensions, which makes the theory logarithmic). The space $\mathcal{Z}_{cft}$ in the $(1, p)$-model is $(3p - 1)$-dimensional (cf. [9, 10]).

On the quantum-group side, we consider the restricted (“baby” in a different nomenclature) quantum group $\overline{U}_q\mathfrak{sl}(2)$ at the primitive $2p$th root of unity $q$. We define it in 3.1 below, and here only note the key relations $E_p = F_p = 0$, $K^{2p} = 1$ (with $K^p$ then being central). It has $2p$ irreducible representations and a $(3p - 1)$-dimensional center (Prop. 4.4.4 below). The center $\mathfrak{z}$ of $\overline{U}_q\mathfrak{sl}(2)$ is endowed with an $SL(2, \mathbb{Z})$-representation constructed as in [5, 6, 11], even though $\overline{U}_q\mathfrak{sl}(2)$ is not quasitriangular [12] (the last fact may partly explain why $\overline{U}_q\mathfrak{sl}(2)$ is not as popular as the small quantum group).

1.1. Theorem. The $SL(2, \mathbb{Z})$-representations on $\mathcal{Z}_{cft}$ and on $\mathfrak{z}$ are equivalent.

Thus, Claim 3 of the Kazhdan–Lusztig correspondence is fully valid for $\mathcal{W}(p)$ and $\overline{U}_q\mathfrak{sl}(2)$ at $q = e^{\pi i}$. We let $\pi$ denote the $SL(2, \mathbb{Z})$-representation in the theorem.
Regarding Claim 1.2, we first note that, strictly speaking, the fusion for $\mathcal{W}(p)$, understood in its “primary” sense of calculation of the coinvariants, has been derived only for $p = 2$ [15]. In rational conformal field theories, the Verlinde formula [12] allows recovering fusion from the modular group action on characters. In the $(1, p)$ logarithmic models, the procedure proposed in [15] as a non-semisimple generalization of the Verlinde formula allows constructing a commutative associative algebra from the $SL(2, \mathbb{Z})$-action on the $\mathcal{W}(p)$-characters. This algebra $\mathfrak{G}_{2p}$ on $2p$ elements $\chi^\alpha(s)$ ($\alpha = \pm 1$, $s = 1, \ldots, p$) is given by

\begin{equation}
\chi^\alpha(s)\chi^\alpha'(s') = \sum_{s'' = |s - s'| + 1}^{s + s' - 1} \tilde{\chi}^\alpha(s'')
\end{equation}

where

\[ \tilde{\chi}^\alpha(s) = \begin{cases} 
\chi^\alpha(s), & 1 \leq s \leq p, \\
\chi^\alpha(2p - s) + 2\chi^{-\alpha}(s - p), & p + 1 \leq s \leq 2p - 1.
\end{cases} \]

For $p = 2$, this algebra coincides with the fusion in [15], and we believe that it is indeed the fusion for all $p$. Our next result in this paper strongly supports this claim, setting it in the framework of the Kazhdan–Lusztig correspondence between $\mathcal{W}(p)$ and $\mathcal{U}_q\mathfrak{sl}(2)$ at $q = e^{i\pi}$.

1.2. Theorem. Let $q = e^{i\pi}$. Under the identification of $\chi^\alpha(s)$, $\alpha = \pm 1$, $s = 1, \ldots, p$, with the $2p$ irreducible $\mathcal{U}_q\mathfrak{sl}(2)$-representations, the algebra $\mathfrak{G}_{2p}$ in (1.1) is the Grothendieck ring of $\mathcal{U}_q\mathfrak{sl}(2)$.

We emphasize that the algebras are isomorphic as fusion algebras, i.e., including the identification of the respective preferred bases given by the irreducible representations.

The procedure in [15] leading to fusion (1.1) is based on the following structure of the $SL(2, \mathbb{Z})$-representation $\pi$ on $\mathfrak{z}_{\text{ch}}$ in the $(1, p)$ model:

\begin{equation}
\mathfrak{z}_{\text{cft}} = \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1}.
\end{equation}

Here, $\mathcal{R}_{p+1}$ is a $(p + 1)$-dimensional $SL(2, \mathbb{Z})$-representation (actually, on characters of a lattice vertex-operator algebra), $\mathcal{R}_{p-1}$ is a $(p - 1)$-dimensional $SL(2, \mathbb{Z})$-representation (actually, the representation on the unitary $\mathfrak{sl}(2)$-characters at the level $k = p - 2$), and $\mathbb{C}^2$ is the standard two-dimensional $SL(2, \mathbb{Z})$-representation. Equivalently, (1.2) is reformulated as follows. We have two $SL(2, \mathbb{Z})$-representations $\tilde{\pi}$ and $\pi^*$ on $\mathfrak{z}_{\text{cft}}$ in terms of which $\pi$ factors as $\pi(\gamma) = \pi^*(\gamma)\tilde{\pi}(\gamma) \forall \gamma \in SL(2, \mathbb{Z})$ and which commute with each other, $\pi^*(\gamma)\tilde{\pi}(\gamma') = \tilde{\pi}(\gamma')\pi^*(\gamma)$; moreover, $\tilde{\pi}$ restricts to the $2p$-dimensional space of the $\mathcal{W}(p)$-characters.

In view of Theorem 1.1, this structure of the $SL(2, \mathbb{Z})$-representation is reproduced on the quantum-group side: there exist $SL(2, \mathbb{Z})$-representations $\tilde{\pi}$ and $\pi^*$ on the cen-
ter $\mathfrak{z}$ of $\mathcal{U}_q\mathfrak{s}\ell(2)$ in terms of which the representation in $[5, 6]$ factors. Remarkably, these representations $\bar{\pi}$ and $\pi^*$ on $\mathfrak{z}$ can be constructed in intrinsic quantum-group terms, by modifying the construction in $[5, 6]$. We recall that the $T$ generator of $SL(2, \mathbb{Z})$ is essentially given by the ribbon element $v$, and the $S$ generator is constructed as the composition of the Radford and Drinfeld mappings. That $\bar{\pi}$ and $\pi^*$ exist is related to the multiplicative Jordan decomposition of the ribbon element $v = \bar{v}v^*$, where $\bar{v}$ is the semisimple part and $v^*$ is the unipotent (one-plus-nilpotent) part. Then $\bar{v}$ and $v^*$ yield the respective “$T$”-generators $\bar{T}$ and $T^*$. The corresponding “$S$”-generators $\bar{S}$ and $S^*$ are constructed by deforming the Radford and Drinfeld mappings respectively, as we describe in Sec. 5.3 below. We temporarily call the $SL(2, \mathbb{Z})$-representations $\bar{\pi}$ and $\pi^*$ the representations associated with $\bar{v}$ and $v^*$.

1.3. Theorem. Let $v = \bar{v}v^*$ be the multiplicative Jordan decomposition of the $\mathcal{U}_q\mathfrak{s}\ell(2)$ ribbon element (with $\bar{v}$ being the semisimple part) and let $\bar{\pi}$ and $\pi^*$ be the respective $SL(2, \mathbb{Z})$-representations on $\mathfrak{z}$ associated with $\bar{v}$ and $v^*$. Then

1. $\bar{\pi}(\gamma)\pi^*(\gamma') = \pi^*(\gamma')\bar{\pi}(\gamma)$ for all $\gamma, \gamma' \in SL(2, \mathbb{Z})$,

2. $\pi(\gamma) = \bar{\pi}(\gamma)\pi^*(\gamma)$ for all $\gamma \in SL(2, \mathbb{Z})$, and

3. the representation $\bar{\pi}$ restricts to the image of the Grothendieck ring in the center.

The image of the Grothendieck ring in this theorem is under the Drinfeld mapping. The construction showing how the representations $\bar{\pi}$ and $\pi^*$ on the center are derived from the Jordan decomposition of the ribbon element is developed in Sec. 5.3 only for $\mathcal{U}_q\mathfrak{s}\ell(2)$, but we expect it to be valid in general.

1.4. Conjecture. The multiplicative Jordan decomposition of the ribbon element gives rise to $SL(2, \mathbb{Z})$-representations $\bar{\pi}$ and $\pi^*$ with the properties as in Theorem 1.3 for any factorizable ribbon quantum group.

Regarding Claim 1 of the Kazhdan–Lusztig correspondence associated with the $(1, p)$ logarithmic models, we only formulate a conjecture; we expect to address this issue in the future, beginning with [17], where, in particular, the representation category is studied in great detail. In a sense, the expected result is more natural than in the semisimple/rational case because (as in Theorem 1.2) it requires no “semisimplification” on the quantum-group side.

1.5. Conjecture. The category of $\mathcal{W}(p)$-representations is equivalent to the category of finite-dimensional $\mathcal{U}_q\mathfrak{s}\ell(2)$-representations with $q = e^{\frac{2\pi i}{p}}$.

From the reformulation of fusion (1.1) in quantum-group terms (explicit evaluation of the product in the image of the Grothendieck ring in the center under the Drinfeld mapping), we obtain a combinatorial corollary of Theorem 1.2 (see 1.4 for the notation regarding $q$-binomial coefficients):
1.6. Corollary. For \( s + s' \geq n \geq m \geq 0 \), there is the q-binomial identity

\[
\sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} q^{2mi+j(2n-s-s') + ms} \left[ \begin{array}{c} n - i \\ j \\ \end{array} \right] \left[ \begin{array}{c} i + j + s - n \\ m - j \\ \end{array} \right] \left[ \begin{array}{c} m - i - j + s' \\ m - j \\ \end{array} \right] = \\
= q^{2mn} \sum_{\ell=0} \left[ \begin{array}{c} n - \ell \\ m \\ \end{array} \right] \left[ \begin{array}{c} m + s + s' - \ell - n \\ m \\ \end{array} \right].
\]

The multiplication in algebra (1.1), which underlies this identity, is alternatively characterized in terms of Chebyshev polynomials, see 3.3.7 below.

There are numerous relations to the previous work. The fundamental results in [5, 6] regarding the modular group action on the center of a Drinfeld double can be “pushed forward” to \( \mathcal{U}_q\mathfrak{sl}(2) \), which is a ribbon quantum group. We note that in the standard setting [18], a ribbon Hopf algebra is assumed to be quasitriangular. This is not the case with \( \mathcal{U}_q\mathfrak{sl}(2) \), but we keep the term “ribbon” with the understanding that \( \mathcal{U}_q\mathfrak{sl}(2) \) is a subalgebra in a quasitriangular Hopf algebra from which it inherits the ribbon structure, as is detailed in what follows. The structure (1.2), already implicit in [15], is parallel to the property conjectured in [11] for the \( SL(2, \mathbb{Z}) \)-representation on the center of the small quantum group \( \mathcal{U}_q\mathfrak{sl}(2)_{\text{small}} \). Albeit for a different quantum group, we extend the argument in [11] by choosing the bases in the center that lead to a simple proof and by giving the underlying Jordan decomposition of the ribbon element and the corresponding deformations of the Radford and Drinfeld mappings. The \((3p-1)\)-dimensional center of \( \mathcal{U}_q\mathfrak{sl}(2) \) at \( q \) the primitive \( 2p \)th root of unity is twice as big as the center of \( \mathcal{U}_q\mathfrak{sl}(2)_{\text{small}} \) for \( q \) the primitive \( p \)th root of unity (for odd \( p \)) [11, 19]. We actually find the center of \( \mathcal{U}_q\mathfrak{sl}(2) \) by studying the bimodule decomposition of the regular representation (the decomposition of \( \mathcal{U}_q\mathfrak{sl}(2)_{\text{small}} \) under the adjoint action has been the subject of some interest; see [20] and the references therein). There naturally occur indecomposable \( 2p \)-dimensional \( \mathcal{U}_q\mathfrak{sl}(2) \)-representations (projective modules), which have also appeared in [18, 21, 22]. On the conformal field theory side, the \( \mathcal{W}(p) \) algebra was originally studied in [7, 8], also see [23, 24].

This paper can be considered a continuation (or a quantum-group counterpart) of [15] and is partly motivated by remarks already made there. That the quantum dimensions of the irreducible \( \mathcal{W}(p) \)-representations are dimensions of quantum-group representations was noted in [15] as an indication of a quantum group underlying the fusion algebra derived there. For the convenience of the reader, we give most of the necessary reference to [15] in Sec. 2 and recall the crucial conformal field theory formulas there.\(^1\) In Sec. 3 we define the restricted quantum group \( \mathcal{U}_q\mathfrak{sl}(2) \), describe some classes of its representations (most importantly, irreducible), and find its Grothendieck ring. In Sec. 4 we collect

\(\text{\footnotesize \footnote{\text{\footnotesize We note a minor terminological discrepancy: in [15], the “fusion” basis (the one with nonnegative integer structure coefficients) was called canonical, while in this paper we call it the preferred basis, reserving “canonical” for the basis of primitive idempotents and elements in the radical.}}}}\)
the facts pertaining to the ribbon structure and the structure of a factorizable Hopf algebra on $\mathfrak{U}_q\mathfrak{sl}(2)$. There, we also find the center of $\mathfrak{U}_q\mathfrak{sl}(2)$ in rather explicit terms. In Sec. 5 we study $SL(2,\mathbb{Z})$-representations on the center of $\mathfrak{U}_q\mathfrak{sl}(2)$ and establish the equivalence to the representation in Sec. 2 and the factorization associated with the Jordan decomposition of the ribbon element.

The Appendices contain auxiliary or bulky material. In Appendix A we collect a number of standard facts about Hopf algebras that we use in the paper. In Appendix B we construct a Drinfeld double that we use to derive the $M$-matrix and the ribbon element for $\mathfrak{U}_q\mathfrak{sl}(2)$. In Appendix C we give the necessary details about indecomposable $\mathfrak{U}_q\mathfrak{sl}(2)$-modules. The “canonical” basis in the center of $\mathfrak{U}_q\mathfrak{sl}(2)$ is explicitly constructed in Appendix D. As an elegant corollary of the description of the Grothendieck ring in terms of Chebyshev polynomials, we reproduce the formulas for the eigenmatrix in [15]. Appendix E is just a calculation leading to identity (1.3).

**Notation.** We use the standard notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{Z}, \quad [n]! = [1][2] \cdots [n], \quad n \in \mathbb{N}, \quad [0]! = 1$$

(without indicating the “base” $q$ explicitly) and set

$$
\begin{align*}
\begin{bmatrix} m \\ n \end{bmatrix} &= \begin{cases} 0, & n < 0 \, \text{or} \, m - n < 0, \\
\frac{m!}{n!} \frac{[m]!}{[m - n]!} & \text{otherwise.}
\end{cases}
\end{align*}
$$

(1.4)

In referring to the root-of-unity case, we set

$$q = e^{\frac{i\pi}{p}}$$

for an integer $p \geq 2$. The $p$ parameter is as in Sec. 2.

For Hopf algebras in general (in the Appendices) and for $\mathfrak{U}_q\mathfrak{sl}(2)$ specifically, we write $\Delta$, $\epsilon$, and $S$ for the comultiplication, counit, and antipode respectively. Some other conventions are as follows:

- $\mathcal{Z}$ — the quantum group center,
- $\mathcal{Ch}$ — the space of $q$-characters (see A.1),
- $\mu$ — the integral (see A.2),
- $c$ — the cointegral (see A.2),
- $g$ — the balancing element (see A.2),
- $v$ — the ribbon element (see A.6),
- $\bar{M}$ — the $M$-matrix (see A.4.2; $M$ is used for $\mathfrak{U}_q\mathfrak{sl}(2)$ and $M$ in general),
- $\chi$ — the Drinfeld mapping $A^* \to A$ (see A.5),
- $\chi^\pm(s)$ — the image of the irreducible $\mathfrak{U}_q\mathfrak{sl}(2)$-representation $\chi^\pm(s)$ in the center under the Drinfeld mapping (see 4.3),
- $\hat{\phi}$ — the Radford mapping $A^* \to A$ (see A.3).
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\( \hat{\phi}^\pm(s) \) — the image of the irreducible \( \hat{U}_q \mathfrak{sl}(2) \)-representation \( X^\pm(s) \) in the center under the Radford mapping (see [4.5]),

\( X^\pm(s) \) — irreducible \( \hat{U}_q \mathfrak{sl}(2) \)-representations (see [5.2.1]; in [2.1] irreducible \( \mathcal{W}(p) \)-representations.

\( Y^\pm(s) \) — Verma modules (see [3.2.2] and [C.1]),

\( \bar{Y}^\pm(s) \) — contragredient Verma modules (see [C.1]),

\( \mathfrak{p}^\pm(s) \) — projective \( \hat{U}_q \mathfrak{sl}(2) \)-modules (see [3.2.3] and [C.2]),

\( \text{qCh}_X \) — the \( q \)-character of a \( \hat{U}_q \mathfrak{sl}(2) \)-representation \( X \) (see [A.6.1]),

\( G_{2p} \) — the \( \hat{U}_q \mathfrak{sl}(2) \) Grothendieck ring; \( \mathcal{G}(A) \) is the Grothendieck ring of a Hopf algebra \( A \),

\( D_{2p} \) — the Grothendieck ring image in the center under the Drinfeld mapping,

\( R_{2p} \) — the Grothendieck ring image in the center under the Radford mapping.

We write \( x', x'', x''' \), etc. (Sweedler’s notation) in constructions like

\[
\Delta(x) = \sum_{(x)} x' \otimes x'', \quad (\Delta \otimes \text{id})\Delta(x) = \sum_{(x)} x' \otimes x'' \otimes x''', \quad \ldots .
\]

For a linear function \( \beta \), we use the notation \( \beta(?) \), where ? indicates the position of its argument in more complicated constructions.

We choose two elements generating \( SL(2, \mathbb{Z}) \) as \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and use the notation of the type \( S, S^*, \bar{S}, \ldots \) and \( \mathcal{T}, \mathcal{T}^*, \bar{\mathcal{T}}, \ldots \) for these elements in various representations.

2. VERTEX-OPERATOR ALGEBRA FOR THE \((1, p)\)-CONFIRMED FIELD THEORY, ITS CHARACTERS, AND \( SL(2, \mathbb{Z}) \)-REPRESENTATIONS

Logarithmic models of conformal field theory, of which the \((1, p)\)-models are an example, were introduced in [25] and were considered, in particular, in [13, 8, 26, 9, 23, 24, 27, 15, 16] (also see the references therein). Such models are typically defined as kernels of certain screening operators. The actual symmetry of the theory is the maximal local algebra in this kernel. In the \((1, p)\)-model, which is the kernel of the “short” screening operator, see [15], this is the W-algebra \( \mathcal{W}(p) \) studied in [7, 8]. We briefly recall it in [2.1]. In [2.2] we give the modular transformation properties of the \( \mathcal{W}(p) \)-characters and identify the \((3p-1)\)-dimensional \( SL(2, \mathbb{Z}) \)-representation on \( \mathfrak{z}_{cR} \) (the space of extended characters). In [2.3] we describe the structure of this representation.

2.1. VOA. Following [15], we consider the vertex-operator algebra \( \mathcal{W}(p) \) — the W-algebra studied in [7, 8], which can be described in terms of a single free field \( \varphi(z) \) with the operator product expansion \( \varphi(z) \varphi(w) = \log(z - w) \). For this, we introduce the energy-momentum tensor

\[
(2.1) \quad T = \frac{1}{2} \partial \varphi \partial \varphi + \frac{\alpha_0}{2} \partial^2 \varphi, \quad \alpha_+ = \sqrt{2p}, \quad \alpha_- = -\sqrt{\frac{2}{p}}, \quad \alpha_0 = \alpha_+ + \alpha_-,
\]
with central charge $c = 13 - 6(p + \frac{1}{p})$, and the set of vertex operators $V_{r,s}(z) = e^{j(r,s)\varphi(z)}$ with $j(r, s) = \frac{1}{2}\alpha_+ + \frac{1}{2s}\alpha_-$. Let $\mathcal{F}$ be the sum of Fock spaces corresponding to $V_{r,s}(z)$ for $r \in \mathbb{Z}$ and $1 \leq s \leq p$ (see the details in [15]). There exist two screening operators

$$S_+ = \oint e^{\alpha_+\varphi}, \quad S_- = \oint e^{\alpha_-\varphi},$$

satisfying $[S_{\pm}, T(z)] = 0$. We define $\mathcal{W}(p)$ as a maximal local subalgebra in the kernel of the “short” screening $S_-$. The algebra $\mathcal{W}(p)$ is generated by the currents

$$W^-(z) = e^{-\alpha_+\varphi}(z), \quad W^0(z) = [S_+, W^-(z)], \quad W^+(z) = [S_+, W^0(z)]$$

(which are primary fields of dimension $2p - 1$ with respect to energy-momentum tensor (2.1)). The algebra $\mathcal{W}(p)$ has $2p$ irreducible highest-weight representations, denoted as $\mathcal{X}^+(s)$ and $\mathcal{X}^-(s)$, $1 \leq s \leq p$ (the respective representations $\Lambda(s)$ and $\Pi(s)$ in [15]). The highest-weight vectors in $\mathcal{X}^+(s)$ and $\mathcal{X}^-(s)$ can be chosen as $V_{0,s}$ and $V_{1,s}$ respectively.

It turns out that $\ker S_-|_{\mathcal{F}} = \bigoplus_{s=1}^{p} \mathcal{X}^+(s) \oplus \mathcal{X}^-(s)$.

### 2.2. $\mathcal{W}(p)$-algebra characters and the $\text{SL}(2, \mathbb{Z})$-representation on $\mathcal{Y}$

We now recall [15] the modular transformation properties of the $\mathcal{W}(p)$-characters

$$\chi^+_s(\tau) = \text{Tr}_{\mathcal{X}^+(s)} e^{2\pi i \tau (L_0 - \frac{s^2}{24})}, \quad \chi^-_s(\tau) = \text{Tr}_{\mathcal{X}^-(s)} e^{2\pi i \tau (L_0 - \frac{s^2}{24})}, \quad 1 \leq s \leq p$$

(the respective characters $\chi^\Lambda_{s,p}(\tau)$ and $\chi^\Pi_{s,p}(\tau)$ in [15]), where $L_0$ is a Virasoro generator, the zero mode of energy-momentum tensor (2.1). Under the $S$-transformation of $\tau$, these characters transform as

$$\chi^+_s(-\frac{1}{\tau}) = \frac{1}{\sqrt{2p}} \left( \frac{s}{p} \chi^+_p(\tau) + (-1)^{p-s} \chi^-_p(\tau) \right) + \sum_{s'=1}^{p-1} q_{s,s'}^{p-s+s'} \left( \chi^+_p(\tau) + \chi^-_p(\tau) \right) - \sum_{s'=1}^{p-1} (-1)^{p+s+s'} q_{s,s'}^{p-s+s'} \varphi_{s'}(\tau)$$

and

$$\chi^-_s(-\frac{1}{\tau}) = \frac{1}{\sqrt{2p}} \left( \frac{s}{p} \chi^+_p(\tau) + (-1)^s \chi^-_p(\tau) \right) + \sum_{s'=1}^{p-1} q_{s,s'}^{s+s'} \left( \chi^+_p(\tau) + \chi^-_p(\tau) \right) + \sum_{s'=1}^{p-1} (-1)^{s+s'} q_{s,s'}^{s+s'} \varphi_{s'}(\tau),$$

where $q_{s,s'}^{s+s'} = q_{s,s'}^{s+s'} + q_{s,s'}^{-s-s'}$, $q = e^{i\pi/p}$, and we introduce the notation

$$\varphi_s(\tau) = \tau \left( \frac{p-s}{p} \chi^+_s(\tau) - \frac{s}{p} \chi^-_{p-s}(\tau) \right), \quad 1 \leq s \leq p - 1.$$
above, $\tau$ enters only linearly, but much more complicated functions of $\tau$ (and other arguments of the characters) can be involved in nonrational theories, cf. [28]. In the present case, because of the explicit occurrences of $\tau$, the $SL(2, \mathbb{Z})$-representation space turns out to be $(3p - 1)$-dimensional, spanned by $\chi_s^+(\tau), 1 \leq s \leq p$, and $\varphi_s(\tau), 1 \leq s \leq p - 1$. Indeed, we have

$$\varphi_s(-1/\tau) = \frac{1}{\sqrt{2p}} \sum_{s' = 1}^{p-1} (-1)^{p+s+s'}q^{s'}\rho_{s'}(\tau),$$

where for the future convenience we introduce a special notation for certain linear combinations of the characters:

$$\rho_s(\tau) = \frac{p-s}{p}\chi_s^+(\tau) - \frac{s}{p}\chi_{p-s}^-(\tau), \quad 1 \leq s \leq p - 1.$$

Under the $T$-transformation of $\tau$, the $W(p)$-characters transform as

$$\chi_s^+(\tau + 1) = \lambda_{p,s}\chi_s^+(\tau), \quad \chi_{p-s}^-(\tau + 1) = \lambda_{p,s}\chi_{p-s}^-(\tau),$$

where

$$\lambda_{p,s} = e^{i\pi(p^2/2p - 1/4)},$$

and hence

$$\varphi_s(\tau + 1) = \lambda_{p,s}(\varphi_s(\tau) + \rho_s(\tau)).$$

We let $3_{cft}$ denote this $(3p - 1)$-dimensional space spanned by $\chi_s^+(\tau), 1 \leq s \leq p$, and $\varphi_s(\tau), 1 \leq s \leq p - 1$. As noted in the introduction, $3_{cft}$ is the space of conformal blocks on the torus, which is in turn isomorphic to the endomorphisms of the identity functor. Let $\pi$ be the $SL(2, \mathbb{Z})$-representation on $3_{cft}$ defined by the above formulas.

2.3. Theorem. The $SL(2, \mathbb{Z})$-representation on $3_{cft}$ has the structure

$$3_{cft} = \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1},$$

where $\mathcal{R}_{p+1}$ and $\mathcal{R}_{p-1}$ are $SL(2, \mathbb{Z})$-representations of the respective dimensions $p + 1$ and $p - 1$, and $\mathbb{C}^2$ is the two-dimensional representation. This implies that there exist $SL(2, \mathbb{Z})$-representations $\overline{\pi}$ and $\pi^*$ on $3_{cft}$ such that

$$\pi(\gamma) = \pi^*(\gamma)\overline{\pi}(\gamma), \quad \overline{\pi}(\gamma)\pi^*(\gamma') = \pi^*(\gamma')\overline{\pi}(\gamma), \quad \gamma, \gamma' \in SL(2, \mathbb{Z}).$$

Proof. Let $\mathcal{R}_{p+1}$ be spanned by

$$\kappa_0(\tau) = \chi_0^-(\tau),$$

$$\kappa_s(\tau) = \chi_s^+(\tau) + \chi_{p-s}^-(\tau), \quad 1 \leq s \leq p - 1,$$

$$\kappa_p(\tau) = \chi_p^+(\tau)$$

(2.10)
Equations (2.2)–(2.5) then imply that
\[ \mathcal{T}\kappa_s(\tau) = \lambda_{p,s}\kappa_s(\tau) \]
and
\[ S\kappa_s(\tau) = \widehat{\kappa}_s(\tau), \quad S\widehat{\kappa}_s(\tau) = \kappa_s(\tau), \]
where
\[ \widehat{\kappa}_s(\tau) = \frac{1}{\sqrt{2p}}(-1)^{s-p}\kappa_0(\tau) + \sum_{s'=1}^{p-1}(-1)^{s's'}q_1^{s's'}\kappa_{p-s'}(\tau) + \kappa_p(\tau), \quad 0 \leq s \leq p, \]
is another basis in \( \mathcal{R}_{p+1} \).

Next, let \( \mathcal{R}'_{p-1} \) be the space spanned by \( \varphi_s(\tau) \) in (2.4); another basis in \( \mathcal{R}'_{p-1} \) is
\[ \widehat{\varphi}_s(\tau) = -\frac{1}{\sqrt{2p}}\sum_{s'=1}^{p-1}(-1)^{p+s+s'}q_1^{s's'}\varphi_s(\tau), \quad 1 \leq s \leq p - 1. \]

Finally, let another \((p - 1)\)-dimensional space \( \mathcal{R}''_{p-1} \) be spanned by \( \rho_s(\tau) \) in (2.6); another basis in \( \mathcal{R}''_{p-1} \) is given by
\[ \widehat{\rho}_s(\tau) = \frac{1}{\sqrt{2p}}\sum_{s'=1}^{p-1}(-1)^{p+s+s'}q_1^{s's'}\rho_s(\tau), \quad 1 \leq s \leq p - 1. \]

Equations (2.2)–(2.5) then imply that
\[ S\varphi_s(\tau) = \widehat{\rho}_s(\tau), \quad S\widehat{\varphi}_s(\tau) = \rho_s(\tau), \]
\[ S\rho_s(\tau) = \varphi_s(\tau), \quad S\widehat{\rho}_s(\tau) = \varphi_s(\tau), \]
and the \( \mathcal{T} \)-transformations in Eqs. (2.7)–(2.9) are expressed as
\[ \mathcal{T}\begin{pmatrix} \rho_s(\tau) \\ \varphi_s(\tau) \end{pmatrix} = \lambda_{p,s}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \rho_s(\tau) \\ \varphi_s(\tau) \end{pmatrix}, \quad 1 \leq s \leq p-1. \]

Therefore, the representation \( \pi \) has the structure \( \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1} \), where \( \mathbb{C}^2 \otimes \mathcal{R}_{p-1} \) is spanned by \((\varphi_s(\tau), \rho_s(\tau))\), \( 1 \leq s \leq p - 1 \).

We now let \( \widetilde{\pi} \equiv \pi((\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \) and \( \pi^* \equiv \pi^*((\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \) act on \( \mathfrak{z}_{\text{cft}} \) as
\[ \widetilde{\kappa}_s(\tau) = \widehat{\kappa}_s(\tau), \quad \widetilde{\varphi}_s(\tau) = \widehat{\varphi}_s(\tau), \quad \widetilde{\rho}_s(\tau) = -\widehat{\rho}_s(\tau), \]
\[ S^*\kappa_s(\tau) = \kappa_s(\tau), \quad S^*\varphi_s(\tau) = -\varphi_s(\tau), \quad S^*\rho_s(\tau) = \rho_s(\tau), \]
and let \( \widetilde{\mathcal{T}} \equiv \pi((\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \) act as
\[ \widetilde{\mathcal{T}}\kappa_s(\tau) = \lambda_{p,s}\kappa_s(\tau), \quad 0 \leq s \leq p, \]
\[
\tilde{T} \left( \begin{array}{c} \rho_s(\tau) \\ \varphi_s(\tau) \end{array} \right) = \lambda_{p,s} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \rho_s(\tau) \\ \varphi_s(\tau) \end{array} \right), \quad 1 \leq s \leq p - 1,
\]
and
\[
\bar{T}^* \kappa_s(\tau) = \kappa_s(\tau), \quad 0 \leq s \leq p,
\]
\[
\bar{T}^* \left( \begin{array}{c} \rho_s(\tau) \\ \varphi_s(\tau) \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \left( \begin{array}{c} \rho_s(\tau) \\ \varphi_s(\tau) \end{array} \right), \quad 1 \leq s \leq p - 1.
\]

It follows that under \( \pi^* \), we have the decomposition
\[
Z_{\text{cft}} = C \oplus \cdots \oplus C \oplus C^2 \oplus \cdots \oplus C^2
\]
(where \( C \) is the trivial representation) and under \( \bar{\pi} \), the decomposition
\[
Z_{\text{cft}} = \mathcal{R}_{p+1} \oplus \mathcal{R}'_{p-1} \oplus \mathcal{R}''_{p-1}.
\]

It is now straightforward to verify that \( \bar{\pi} \) and \( \pi^* \) satisfy the required relations. \( \square \)

2.3.1. Remarks.

(1) Up to some simple multipliers, \( \pi^* \) is just the inverse matrix automorphy factor in [15] and the restriction of \( \bar{\pi} \) to \( \mathcal{R}_{p+1} \oplus \mathcal{R}''_{p-1} \) is the \( SL(2, \mathbb{Z}) \)-representation in [15] that leads to the fusion algebra (1.1) via a nonsemisimple generalization of the Verlinde formula.

(2) \( \mathcal{R}_{p-1} \) is the \( SL(2, \mathbb{Z}) \)-representation realized in the \( \hat{\mathfrak{s}\ell}(2)_{p-2} \) minimal model [29, 30].

In Sec. 5, the structure described in 2.3 is established for the \( SL(2, \mathbb{Z}) \)-representation on the quantum group center.

3. \( \overline{U}_q\mathfrak{s}\ell(2) \): REPRESENTATIONS AND THE GROTHENDIECK RING

The version of the quantum \( s\ell(2) \) that is Kazhdan–Lusztig-dual to the \( (1, p) \) conformal field theory model is the restricted quantum group \( \overline{U}_q\mathfrak{s}\ell(2) \) at \( q \) the primitive \( 2p \)th root of unity. We introduce it in 3.1, consider its representations in 3.2, and find its Grothendieck ring in 3.3.

3.1. The restricted quantum group \( \overline{U}_q\mathfrak{s}\ell(2) \). The Hopf algebra \( \overline{U}_q\mathfrak{s}\ell(2) \) (henceforth, at \( q = e^{i\pi} \)) is generated by \( E, F, \) and \( K \) with the relations
\[
E^p = F^p = 0, \quad K^{2p} = 1
\]
and the Hopf-algebra structure given by
\[
KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F,
\]

\( \overline{U}_q\mathfrak{s}\ell(2) \): REPRESENTATIONS AND THE GROTHENDIECK RING

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\[
E^p = F^p = 0, \quad K^{2p} = 1
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and the Hopf-algebra structure given by
\[
KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F,
\]
\[ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \]
\[ \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K, \]
\[ \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1, \]
\[ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \]

The elements of the PBW-basis of \( \overline{U}_q\mathfrak{sl}(2) \) are enumerated as \( E^i K^j F^\ell \) with \( 0 \leq i \leq p - 1, 0 \leq j \leq 2p - 1, 0 \leq \ell \leq p - 1 \), and its dimension is therefore \( 2p^3 \).

3.1.1. It follows (e.g., by induction) that

\[
\Delta(F^m E^n K^j) = \sum_{r=0}^{m} \sum_{s=0}^{n} q^{2(n-s)(r-m)+r(m-r)+s(n-s)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \times F^r E^{n-s} K^{r-m+j} \otimes F^{m-r} E^s K^{n-s+j}.
\]

3.1.2. The (co)integral and the comodulus. For \( \overline{U}_q\mathfrak{sl}(2) \), the right integral and the left–right cointegral (see the definitions in \([A.2]\)) are given by

\[
\mu(F^i E^m K^n) = \frac{1}{\zeta} \delta_{i,p-1} \delta_{m,p-1} \delta_{n,p+1}
\]

and

\[
\mathbf{c} = \zeta F^{p-1} E^{p-1} \sum_{j=0}^{2p-1} K^j,
\]

where we choose the normalization as

\[
\zeta = \sqrt{\frac{p}{2}} \frac{1}{(p-1)!^2}
\]

for future convenience.

Next, simple calculation shows that the comodulus for \( \overline{U}_q\mathfrak{sl}(2) \) (see \([A.2]\)) is \( a = K^2 \). This allows us to find the balancing element using \([A.4]\). There are four possibilities for the square root of \( a \), two of which are group-like, and we choose

\[
g = K^{p+1}.
\]

This choice determines a ribbon element for \( \overline{U}_q\mathfrak{sl}(2) \), and hence a particular version of the \( SL(2, \mathbb{Z}) \)-action on the quantum group studied below.

The balancing element \([3.3]\) allows constructing the “canonical” \( q \)-characters of \( \overline{U}_q\mathfrak{sl}(2) \)-representations (see \([A.6.1]\)).
3.1.3. The Casimir element. Let $\mathfrak{Z}$ denote the center of $\mathcal{U}_q\mathfrak{sl}(2)$. It contains the element

$$ C = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}, $$

called the Casimir element. It satisfies the minimal polynomial relation

$$ \Psi_2(C') = 0, $$

where

$$ \Psi_2(x) = (x - \beta_0)(x - \beta_p)\prod_{j=1}^{p-1}(x - \beta_j)^2, \quad \beta_j = \frac{q^j + q^{-j}}{(q - q^{-1})^2}. $$

A proof of (3.5) is given in 4.3 below as a spin-off of the technology developed for the Grothendieck ring (we do not need (3.5) before that).

It follows from the definition of $\mathcal{U}_q\mathfrak{sl}(2)$ that $K^p \in \mathfrak{Z}$. In fact, $K^p$ is in the $2^p$-dimensional subalgebra in $\mathfrak{Z}$ generated by $C$ because of the identity

$$ K^p = \frac{1}{2} \sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p}{p-r} \left( \begin{array}{c} p \vspace{1mm} \\ r \end{array} \right) (-1)^{1-r} \tilde{C}^{p-2r}, $$

where we set

$$ \tilde{C} = (q - q^{-1})^2C. $$

3.2. $\mathcal{U}_q\mathfrak{sl}(2)$-representations. The $\mathcal{U}_q\mathfrak{sl}(2)$-representation theory at $q = e^{\frac{i\pi}{p}}$ is not difficult to describe (also see [18, 21, 22]). There turn out to be just $2p$ irreducible representations. In what follows, we also need Verma modules (all of which except two are extensions of a pair of irreducible representations) and projective modules (which are further extensions). The category of all finite-dimensional $\mathcal{U}_q\mathfrak{sl}(2)$-representations at the primitive $2p$th root of unity is fully described in [17].

3.2.1. Irreducible representations. The irreducible $\mathcal{U}_q\mathfrak{sl}(2)$-representations $X^\alpha(s)$ are labeled by $\alpha = \pm 1$ and $1 \leq s \leq p$. The module $X^\pm(s)$ is linearly spanned by elements $|s,n\rangle^\pm$, $0 \leq n \leq s - 1$, where $|s,0\rangle^\pm$ is the highest-weight vector and the $\mathcal{U}_q\mathfrak{sl}(2)$-action is given by

$$ K|s,n\rangle^\pm = \pm q^{s-1-2n}|s,n\rangle^\pm, $$

$$ E|s,n\rangle^\pm = \pm [n]|s-n\rangle|s,n-1\rangle^\pm, $$

$$ F|s,n\rangle^\pm = |s,n+1\rangle^\pm, $$

where we set $|s,s\rangle^\pm = |s,-1\rangle^\pm = 0$. $X^+(1)$ is the trivial module.

For later use, we list the weights occurring in the module $X^+(s)$, i.e., the eigenvalues that $K$ has on vectors in $X^+(s)$,

$$ q^{-s+1}, q^{-s+3}, \ldots, q^{s-1}, $$

where
and in the module $X^-(p-s)$,
\begin{equation}
q^{s+1}, q^{s+3}, \ldots, q^{2p-s-1}.
\end{equation}

We also note the dimensions and quantum dimensions (see \[A.6.1\])
\[\dim X^\alpha(s) = s \quad \text{and} \quad q\dim X^\alpha(s) = \alpha^{p-1}(-1)^{s-1}[s].\]
It follows that $q\dim X^\alpha(s) = -q\dim X^{-\alpha}(p-s)$ and $q\dim X^\alpha(p) = 0$.

3.2.2. Verma modules. There are $2p$ Verma modules $V^{\pm}(s)$, $1 \leq s \leq p$. First, these are the two Steinberg modules
\[V^{\pm}(p) = X^{\pm}(p).\]
Next, for each $s = 1, \ldots, p-1$ and $\alpha = \pm 1$, the Verma module $V^\alpha(s)$ is explicitly described in \[C.1\] as an extension $0 \to X^{-\alpha}(p-s) \to V^\alpha(s) \to X^\alpha(s) \to 0$; for consistency with more complicated extensions considered below, we represent it as
\[X^\alpha(s) \bullet \to X^{-\alpha}(p-s) \bullet,
\]
with the convention that the arrow is directed to a submodule. We note that $\dim V^\alpha(s) = p$ and $q\dim V^\alpha(s) = 0$ (negligible modules \[3\]).

3.2.3. Projective modules. For $s = 1, \ldots, p-1$, there are nontrivial extensions yielding the projective modules $P^{\pm}(s)$, $0 \to V^-(p-s) \to P^+(s) \to V^+(s) \to 0$, $0 \to V^+(p-s) \to P^-(s) \to V^-(s) \to 0$.

Their structure can be schematically depicted as
\begin{equation}
\begin{array}{c}
\cdots
\end{array}
\end{equation}

It follows that $\dim P^+(s) = \dim P^-(s) = 2p$ and $q\dim P^+(s) = q\dim P^-(s) = 0$. The bases and the action of $U_q\mathfrak{sl}(2)$ in $P^+(s)$ and $P^-(s)$ are described in \[C.2.1\] and \[C.2.2\].

3.3. The Grothendieck ring. We next find the Grothendieck ring of $\bar{U}_q\mathfrak{sl}(2)$.

3.3.1. Theorem. Multiplication in the $\bar{U}_q\mathfrak{sl}(2)$ Grothendieck ring is given by
\[X^\alpha(s) X^{\alpha'}(s') = \sum_{s'' = |s-s'|+1}^{s+s'-1} \bar{X}^{\alpha\alpha'}(s''),\]
where

\[ \tilde{X}_\alpha(s) = \begin{cases} 
  X_\alpha(s), & 1 \leq s \leq p, \\
  X_\alpha(2p - s) + 2X_{-\alpha}(s - p), & p + 1 \leq s \leq 2p - 1.
\end{cases} \]

To prove this, we use (i) a property of the tensor products of any representation with a Verma module, (ii) an explicit evaluation of the tensor product of any irreducible representation with a two-dimensional one, and (iii) the observation that the information gained in (i) and (ii) suffices for finding the entire Grothendieck ring.

We first of all note that the trivial representation \( X^+(1) \) is the unit in the Grothendieck ring and, obviously,

\[ X_\alpha(s) X^-(1) = X^{-\alpha}(s) \]

for all \( s = 1, \ldots, p \) and \( \alpha = \pm 1 \). Moreover,

\[ X_\alpha(s) X^-(s') = X_\alpha(s) X^+(s') X^-(1) = X^{-\alpha}(s) X^+(s'), \]

and it therefore suffices to find all the products \( X_\alpha(s) X^+(s') \) and, furthermore, just the products \( X^+(s) X^+(s') \).

### 3.3.2. Products with Verma modules.

In the Grothendieck ring, the Verma module \( V_\alpha(s) \) (with \( 1 \leq s \leq p - 1 \)) is indistinguishable from \( V^{-\alpha}(p-s) \), and we choose to consider only the \( p + 1 \) Verma modules \( V_a, a = 0, 1, \ldots, p \), given by

\[
V_0 = V^-(p), \quad V_a = V^+(a), \quad 1 \leq a \leq p - 1, \quad V_p = V^+(p).
\]

Their highest weights \( q^{a-1} \) coincide with the respective highest weights of \( X^-(p), X^+(a), X^+(p) \).

Taking the tensor product of a Verma module \( V_a \) and an irreducible representation gives a module that is filtered by Verma modules. In the Grothendieck ring, this tensor product therefore evaluates as a sum of Verma modules; moreover, the Verma modules that occur in this sum are known, their highest weights being given by \( q^{\epsilon_a + \epsilon_i} \), where \( q^{\epsilon_a} \) is the highest weight of \( V_a \) and \( q^{\epsilon_i} \) are the weights of vectors in the irreducible representation. With (3.7), this readily gives the Grothendieck-ring multiplication

\[
V_a X^+(s') = \sum_{s''=a-s'+1}^{a+s'-1} V_{s''},
\]

where we set \( V_{s''} = V_{-s''} \) for \( s'' < 0 \) and \( V_{p+s''} = V_{p-s''} \) for \( 0 < s'' < p \).

### 3.3.3. Lemma.

For \( 2 \leq s \leq p - 1 \), we have

\[ X_\alpha(s) X^+(2) = X_\alpha(s - 1) + X_\alpha(s + 1). \]
Proof. Let $e_k = |s, k\rangle^\alpha$ for $0 \leq k \leq s - 1$ and \{ $f_0 = |2, 0\rangle^+$, $f_1 = |2, 1\rangle^+$ \} be the respective bases in $X^\alpha(s)$ and in $X^+(2)$. Under the action of $F$, the highest-weight vector $e_0 \otimes f_0$ with the weight $\alpha q^{(s+1)-1}$ generates the module $X^\alpha(s + 1)$. The vector $e'_0 = e_1 \otimes f_0 - \alpha q [s - 1] e_0 \otimes f_1$ satisfies the relations

$$Ee'_0 = 0, \quad Ke'_0 = \alpha q^{(s-1)-1} e'_0.$$ 

Under the action of $F$, it generates the module $X^\alpha(s - 1)$. □

As regards the product $X^\alpha(p) X^+(2)$, we already know it from (3.11) because $X^\alpha(p)$ is a Verma module: with the two relevant Verma modules replaced by the sum of the corresponding irreducible representations, the resulting four terms can be written as

$$X^\alpha(p) X^+(2) = 2X^\alpha(p - 1) + 2X^{-\alpha}(1).$$

As we have noted, the products $X^\alpha(s) X^-(2)$ are given by the above formulas with the reversed "\(\alpha\)" signs in the right-hand sides.

3.3.4. We next evaluate the products $X^\alpha(s) X^+(3)$ as

$$X^\alpha(s) X^+(3) = X^\alpha(s) \left( (X^+(2) X^+(2) - X^+(1)) \right),$$

where the products with $X^+(2)$ are already known. By induction on $s'$, this allows finding all the products $X^\alpha(s) X^+(s')$ as

$$X^\alpha(s) X^+(s') = \sum_{s'' = |s - s'| + 1}^{p-1 - |p - s - s'|} X^\alpha(s'') + \delta_{p, s, s'} X^\alpha(p)$$

$$+ \sum_{s'' = 2p - s - s' + 1}^{p-1} \left( 2X^\alpha(s'') + 2X^{-\alpha}(p - s'') \right),$$

where $\delta_{p, s, s'}$ is equal to 1 if $p - s - s' + 1 \leq 0$ and $p - s - s' + 1 \equiv 0 \mod 2$, and is 0 otherwise.

The statement in 3.3.1 is a mere rewriting of (3.12), taken together with the relations $X^\alpha(s) X^-(s') = X^{-\alpha}(s) X^+(s')$. It shows that the $U_q\mathfrak{sl}(2)$ Grothendieck ring is the $(1, p)$-model fusion algebra derived in [15]. This concludes the proof of 3.3.1

3.3.5. Corollary. The $U_q\mathfrak{sl}(2)$ Grothendieck ring contains the ideal $\mathfrak{H}_{p+1}$ of Verma modules generated by

$$X^+(p - s) + X^-(s), \quad 1 \leq s \leq p - 1,$$

$$X^+(p), \quad X^-(p).$$


The quotient $\mathcal{G}_{2p}/\mathfrak{V}_{p+1}$ is a fusion algebra with the basis $\mathcal{X}(s), 1 \leq s \leq p - 1$ (the canonical images of the corresponding $\mathcal{X}^+(s)$) and multiplication

$$\mathcal{X}(s) \mathcal{X}(s') = \sum_{s'' = |s - s'| + 1}^{p-1-|p-s-s'|} \mathcal{X}(s''), \quad s, s' = 1, \ldots, p - 1.$$  

This is a semisimple fusion algebra, which coincides with the fusion of the unitary $\hat{\mathcal{L}}(2)$ representations of level $p - 2$.

3.3.6. Corollary. The $\mathfrak{U}_q\mathcal{L}(2)$ Grothendieck ring $\mathcal{G}_{2p}$ is generated by $\mathcal{X}^+(2)$.

This easily follows from Theorem 3.3.1; therefore, $\mathcal{G}_{2p}$ can be identified with a quotient of the polynomial ring $\mathbb{C}[x]$. Let $U_s(x)$ denote the Chebyshev polynomials of the second kind

$$U_s(2 \cos t) = \frac{\sin st}{\sin t}. \quad (3.13)$$

The lower such polynomials are $U_0(x) = 0$, $U_1(x) = 1$, $U_2(x) = x$, and $U_3(x) = x^2 - 1$.

3.3.7. Proposition. The $\mathfrak{U}_q\mathcal{L}(2)$ Grothendieck ring is the quotient of the polynomial ring $\mathbb{C}[x]$ over the ideal generated by the polynomial

$$\Psi_{2p}(x) = U_{2p+1}(x) - U_{2p-1}(x) - 2. \quad (3.14)$$

Moreover, let

$$P_s(x) = \begin{cases} U_s(x), & 1 \leq s \leq p, \\ \frac{1}{2}U_s(x) - \frac{1}{2}U_{2p-s}(x), & p + 1 \leq s \leq 2p. \end{cases} \quad (3.15)$$

Under the quotient mapping, the image of each polynomial $P_s$ coincides with $\mathcal{X}^+(s)$ for $1 \leq s \leq p$ and with $\mathcal{X}^-(s-p)$ for $p + 1 \leq s \leq 2p$.

Proof. It follows from 3.3.1 that

$$\mathcal{X}^+(2) \mathcal{X}^+(1) = \mathcal{X}^\pm(2), \quad (3.16)$$

$$\mathcal{X}^+(2) \mathcal{X}^\pm(s) = \mathcal{X}^\pm(s-1) + \mathcal{X}^\pm(s+1), \quad 2 \leq s \leq p - 1, \quad (3.17)$$

$$\mathcal{X}^+(2) \mathcal{X}^+(p) = 2\mathcal{X}^+(p-1) + 2\mathcal{X}^-(1), \quad (3.18)$$

$$\mathcal{X}^+(2) \mathcal{X}^-(p) = 2\mathcal{X}^-(p-1) + 2\mathcal{X}^+(1). \quad (3.19)$$

We recall that the Chebyshev polynomials of the second kind satisfy (and are determined by) the recursive relation

$$xU_s(x) = U_{s-1}(x) + U_{s+1}(x), \quad s \geq 2, \quad (3.20)$$

with the initial data $U_1(x) = 1$, $U_2(x) = x$. From (3.20), we then obtain that polynomials (3.15) satisfy relations (3.16)–(3.18) after the identifications $P_s \to \mathcal{X}^+(s)$ for $1 \leq s \leq p$ and $P_s \to \mathcal{X}^-(s-p)$ for $p + 1 \leq s \leq 2p$. Then, for Eq. (3.19) to be satisfied, we must
impose the relation \( xP_{2p}(x) \equiv 2P_{2p-1}(x) + 2P_1(x) \); this shows that the Grothendieck ring is the quotient of \( \mathbb{C}[x] \) over the ideal generated by polynomial (3.14). \( \square \)

3.3.8. Proposition. The polynomial \( \hat{\Psi}_{2p}(x) \) can be factored as

\[
\hat{\Psi}_{2p}(x) = (x - \hat{\beta}_0)(x - \hat{\beta}_p) \prod_{j=1}^{p-1} (x - \hat{\beta}_j)^2,
\]

\( \hat{\beta}_j = q^j + q^{-j} = 2 \cos \frac{\pi j}{p} \).

This is verified by direct calculation using the representation

\[
\hat{\Psi}_{2p}(2 \cos t) = 2(\cos(2pt) - 1),
\]

which follows from (3.13). We note that \( \hat{\beta}_j \neq \hat{\beta}_{j'} \) for \( 0 \leq j \neq j' \leq p \).

4. \( \mathbb{U}_q\mathfrak{sl}(2) \): Factorizable and Ribbon Hopf Algebra Structures and the Center

The restricted quantum group \( \mathbb{U}_q\mathfrak{sl}(2) \) is not quasitriangular [12]; however, it admits a Drinfeld mapping, and hence there exists a homomorphic image \( \mathbb{D}_{2p} \) of the Grothendieck ring in the center. In 4.1, we first identify \( \mathbb{U}_q\mathfrak{sl}(2) \) as a subalgebra in a quotient of a Drinfeld double. We then obtain the \( M \)-matrix in 4.2, characterize the subalgebra \( \mathbb{D}_{2p} \subset Z \) in 4.3, and find the center \( Z \) of \( \mathbb{U}_q\mathfrak{sl}(2) \) at \( q = e^{i\pi/p} \) in 4.4. Furthermore, we give some explicit results for the Radford mapping for \( \mathbb{U}_q\mathfrak{sl}(2) \) in 4.5, and we find a ribbon element for \( \mathbb{U}_q\mathfrak{sl}(2) \) in 4.6.

4.1. \( \mathbb{U}_q\mathfrak{sl}(2) \) from the double. The Hopf algebra \( \mathbb{U}_q\mathfrak{sl}(2) \) is not quasitriangular, but it can be realized as a Hopf subalgebra of a quasitriangular Hopf algebra \( \hat{D} \) (which is in turn a quotient of a Drinfeld double). The \( M \)-matrix (see A.4.2) for \( \hat{D} \) is in fact an element of \( \mathbb{U}_q\mathfrak{sl}(2) \otimes \mathbb{U}_q\mathfrak{sl}(2) \), and hence \( \mathbb{U}_q\mathfrak{sl}(2) \) can be thought of as a factorizable Hopf algebra, even though relation (A.9) required of an \( M \)-matrix is satisfied not in \( \mathbb{U}_q\mathfrak{sl}(2) \) but in \( \hat{D} \) (but on the other hand, (A.11) holds only with \( m_I \) and \( n_I \) being bases in \( \mathbb{U}_q\mathfrak{sl}(2) \)).

The Hopf algebra \( \hat{D} \) is generated by \( e, \phi, \) and \( k \) with the relations

\[
kek^{-1} = qe, \quad k\phi k^{-1} = q^{-1} \phi, \quad [e, \phi] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \]

\[
e^p = 0, \quad \phi^p = 0, \quad k^{4p} = 1, \]

\[
e(e) = 0, \quad e(\phi) = 0, \quad e(k) = 1, \]

\[
\Delta(e) = 1 \otimes e + e \otimes k^2, \quad \Delta(\phi) = k^{-2} \otimes \phi + \phi \otimes 1, \quad \Delta(k) = k \otimes k, \]

\[
S(e) = -ek^{-2}, \quad S(\phi) = -k^2 \phi, \quad S(k) = k^{-1}. \]

A Hopf algebra embedding \( \mathbb{U}_q\mathfrak{sl}(2) \to \hat{D} \) is given by

\[
E \mapsto e, \quad F \mapsto \phi, \quad K \mapsto k^2. \]
In what follows, we often do not distinguish between $E$ and $e$, $F$ and $\phi$, and $K$ and $k^2$.

**4.1.1. Theorem.** $\bar{D}$ is a ribbon quasitriangular Hopf algebra, with the universal $R$-matrix

\[(4.1) \quad \bar{R} = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{n,j=0}^{4p-1} \frac{(q - q^{-1})^m}{[m]!} q^{m(m-1)/2 + m(n-j) - nj/2} e^m k^n \otimes \phi^m k^j\]

and the ribbon element

\[(4.2) \quad \nu = \frac{1 - i}{2\sqrt{p}} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m + mj + \frac{1}{2}(j+p+1)^2} \phi^m e^m k^{2j}.\]

**Proof.** Equation (4.1) follows from the realization of $\bar{D}$ as a quotient of the Drinfeld double $D(B)$ in [B.11]. The quotient is over the Hopf ideal generated by the central element $k\kappa - 1 \in D(B)$. It follows that $\bar{D}$ inherits a quasitriangular Hopf algebra structure from $D(B)$ and $R$-matrix (4.1) is the image of (B.17) under the quotient mapping.

Using $R$-matrix (4.1), we calculate the canonical element $u$ (see (A.12)) as

\[(4.3) \quad u = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{n,r=0}^{4p-1} (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m(m+3) - 2rn/2} \phi^m e^m k^n.\]

We note that actually $u \in \overline{U}_q sl(2)$. Indeed,

\[
u = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{n,r=0}^{4p-1} (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m(m+3) - 2rn/2} \phi^m e^m k^n = \]

\[
= \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \left( \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r(r+2m+2j)} \right) (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m(m+3) \phi^m e^m k^{2j}} \]

\[
+ \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \left( \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r(r+2m+2j+1)} \right) (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m(m+3) \phi^m e^m k^{2j+1}}.
\]

The second Gaussian sum vanishes,

\[
\sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r(r+2m+2j+1)} = 0.
\]

To evaluate the first Gaussian sum, we make the substitution $r \rightarrow r - j - m$:

\[
u = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r^2} \left( \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r^2} \right) (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m(m+3) + \frac{1}{2}(j+m)^2} \phi^m e^m k^{2j}
\]

\[
= \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r^2} \left( \sum_{r=0}^{4p-1} e^{-i\pi \frac{1}{2p} r^2} \right) (-1)^m \frac{(q - q^{-1})^m}{[m]!} q^{-\frac{1}{2}m + m(j-p-1) + \frac{1}{2}j^2} \phi^m e^m k^{2j}.
\]
Then evaluating
\[ \sum_{r=0}^{4p-1} e^{-i \frac{r^2}{p}} = (1 - i)2\sqrt{p}, \]
we obtain
\[ u = \frac{1 - i}{2\sqrt{p}} \sum_{m=0}^{p-1} \sum_{n=0}^{2p-1} \sum_{j=0}^{2p-1} (q - q^{-1})^{m+n} \frac{1}{[m!]^2} q^{-\frac{1}{2}m+m+\frac{1}{2}(j+p+1)^2} e^m \phi^m \bar{M}^{2j+2p+2}. \]

We then find the ribbon element from relation (A.16) using the balancing element \( g = k^{2p+2} \) from (3.3), which gives (4.2).

4.2. The \( M \)-matrix for \( \overline{U}_q\mathfrak{sl}(2) \). We next obtain the \( M \)-matrix (see [A.4.2]) for \( \overline{U}_q\mathfrak{sl}(2) \)
from the universal \( R \)-matrix for \( \bar{D} \) in (4.1). Because \( u \in \overline{U}_q\mathfrak{sl}(2) \), it follows from (A.13) that the \( M \)-matrix for \( \bar{D} \), \( M = \bar{R}_{21}\bar{R}_{12} \), actually lies in \( \overline{U}_q\mathfrak{sl}(2) \otimes \overline{U}_q\mathfrak{sl}(2) \), and does not therefore satisfy condition (A.11) in \( \bar{D} \) (and hence \( \bar{D} \) is not factorizable). But this \( is \) an \( M \)-matrix for \( \overline{U}_q\mathfrak{sl}(2) \subset \bar{D} \). A simple calculation shows that \( \bar{R}_{21}\bar{R}_{12} \) is explicitly rewritten in terms of the \( \overline{U}_q\mathfrak{sl}(2) \)-generators as

\[
\bar{M} = \frac{1}{2p} \sum_{m=0}^{p-1} \sum_{n=0}^{p-2} \sum_{j=0}^{p-1} \frac{1}{[m!]^2} q^m \sum_{m=0}^{p-1} \sum_{n=0}^{2p-1} \sum_{j=0}^{2p-1} (q - q^{-1})^{m+n} \frac{1}{[m!]^2} q^{-\frac{1}{2}m+m+\frac{1}{2}(j+p+1)^2} e^m \phi^m \bar{M}^{2j+2p+2}. \]

4.3. Drinfeld mapping and the \( (1, p) \) fusion in \( \mathcal{Z}(\overline{U}_q\mathfrak{sl}(2)) \). Given the \( M \)-matrix, we can identify the \( \overline{U}_q\mathfrak{sl}(2) \) Grothendieck ring with its image in the center using the homomorphism in [A.6.2]. We evaluate this homomorphism on the preferred basis elements in the Grothendieck ring, i.e., on the irreducible representations. With the balancing element for \( \overline{U}_q\mathfrak{sl}(2) \) in (3.3) and the \( M \)-matrix in (4.4), the mapping in [A.6.2] is

\[ \mathcal{G}_{2p} \to \mathcal{Z} \]
\[ \chi^\pm(s) \mapsto \chi^\pm(s) \equiv \chi(q \text{Ch}_\chi^\pm(s)) = (\text{Tr}_{\chi^\pm(s)} \otimes \text{id})(K^{p-1} \otimes 1) \bar{M}, \quad 1 \leq s \leq p. \]

Clearly, \( \chi^+(1) = 1 \). We let \( \mathcal{D}_{2p} \subset \mathcal{Z} \) denote the image of the Grothendieck ring under this mapping.

4.3.1. Proposition. For \( s = 1, \ldots, p \) and \( \alpha = \pm 1 \),

\[ \chi^\alpha(s) = \alpha^{p+1}(-1)^{s+1} \sum_{n=0}^{s-1} \sum_{m=0}^{n} (q - q^{-1})^{m+n} q^{\frac{s-n+m-1}{2} \left( \begin{array}{c} n \\ m \end{array} \right) E^m \bar{K}^{s-1+\beta \alpha-2n+m}, \]

where we set \( \beta = 0 \) if \( \alpha = +1 \) and \( \beta = 1 \) if \( \alpha = -1 \). In particular, it follows that

\[ \chi^+(2) = -\hat{C} \]
(with $\tilde{C}$ defined in 3.1.3) and

\[ \chi^{-\alpha}(s) = -(-1)^p \chi^{\alpha}(s) K^p. \]

**Proof.** The proof of (4.6) is a straightforward calculation based on the well-known identity (see, e.g., [12])

\[ \prod_{s=0}^{r-1} \left( C - \frac{q^{2s+1} K + q^{-2s-1} K^{-1}}{(q - q^{-1})^2} \right) = F^r E^r, \quad r < p, \]

which readily implies that

\[ \text{Tr} \chi^\alpha(s) E^m F^m K^\alpha = \alpha^{m+a} ([m]!)^2 \sum_{n=0}^{s-1} q^{a(s-1-2n)} \left[ \begin{array}{c} s - n + m - 1 \\ m \end{array} \right] \left[ \begin{array}{c} n \\ m \end{array} \right]. \]

Using this in (4.5) gives (4.6). For $\chi^+(2)$, we then have

\[ \chi^+(2) = -\sum_{n=0}^{1} \sum_{m=0}^{n} (q - q^{-1})^{2m} q^{-(m+1)(m+1-2n)} \left[ \begin{array}{c} 1 - n + m \\ m \end{array} \right] \left[ \begin{array}{c} n \\ m \end{array} \right] E^m F^m K^{1-2n+m} = -q^{-1} K - q K^{-1} - (q - q^{-1})^2 EF. \]

Combining 4.3.1 and 3.3.6 we obtain

4.3.2. **Proposition.** $\mathcal{D}_{2p}$ coincides with the algebra generated by the Casimir element.

The following corollary is now immediate in view of 3.3.7 and 3.3.8.

4.3.3. **Corollary.** Relation (3.5) holds for the Casimir element.

4.3.4. **Corollary.** Identity (1.3) holds.

The derivation of (1.3) from the algebra of the $\chi^{\alpha}(s)$ is given in Appendix E in some detail. We note that although the left-hand side of (1.3) is not manifestly symmetric in $s$ and $s'$, the identity shows that it is.

4.3.5. In what follows, we keep the notation $\mathfrak{M}_{p+1}$ for the Verma-module ideal (more precisely, for its image in the center) generated by

\[ \mathfrak{x}(0) = \chi^{-}(p), \]

\[ \mathfrak{x}(s) = \chi^{+}(s) + \chi^{-}(p-s), \quad 1 \leq s \leq p - 1, \]

\[ \mathfrak{x}(p) = \chi^{+}(p). \]

This ideal is the socle (annihilator of the radical) of $\mathcal{D}_{2p}$. 
4.4. The center of $U_q\mathfrak{sl}(2)$. We now find the center of $U_q\mathfrak{sl}(2)$ at the primitive $2p$th root of unity. For this, we use the isomorphism between the center and the algebra of bimodule endomorphisms of the regular representation. The results are in 4.4.4 and D.1.1.

4.4.1. Decomposition of the regular representation. The $2p^3$-dimensional regular representation of $U_q\mathfrak{sl}(2)$, viewed as a free left module, decomposes into indecomposable projective modules, each of which enters with the multiplicity given by the dimension of its simple quotient:

$$\text{Reg} = \bigoplus_{s=1}^{p-1} s\mathcal{P}^+(s) \oplus \bigoplus_{s=1}^{p-1} s\mathcal{P}^-(s) \oplus p\mathcal{X}^+(p) \oplus p\mathcal{X}^-(p).$$

We now study the regular representation as a $U_q\mathfrak{sl}(2)$-bimodule. In what follows, $\boxtimes$ denotes the external tensor product.

4.4.2. Proposition. As a $U_q\mathfrak{sl}(2)$-bimodule, the regular representation decomposes as

$$\text{Reg} = \bigoplus_{s=0}^{p} \mathcal{Q}(s),$$

where

1. the bimodules

$$\mathcal{Q}(0) = \mathcal{X}^-(p) \boxtimes \mathcal{X}^-(p), \quad \mathcal{Q}(p) = \mathcal{X}^+(p) \boxtimes \mathcal{X}^+(p)$$

are simple,

2. the bimodules $\mathcal{Q}(s)$, $1 \leq s \leq p - 1$, are indecomposable and admit the filtration

$$(4.12) \quad 0 \subset \mathcal{R}_2(s) \subset \mathcal{R}(s) \subset \mathcal{Q}(s),$$

where the structure of subquotients is given by

$$(4.13) \quad \mathcal{Q}(s)/\mathcal{R}(s) = \mathcal{X}^+(s) \boxtimes \mathcal{X}^-(s) \oplus \mathcal{X}^-(p-s) \boxtimes \mathcal{X}^-(p-s)$$

and

$$\mathcal{R}(s)/\mathcal{R}_2(s) = \mathcal{X}^-(p-s) \boxtimes \mathcal{X}^+(s) \oplus \mathcal{X}^-(p-s) \boxtimes \mathcal{X}^+(s) \oplus \mathcal{X}^+(s) \boxtimes \mathcal{X}^-(p-s),$$

and where $\mathcal{R}_2(s)$ is isomorphic to the quotient $\mathcal{Q}(s)/\mathcal{R}(s)$.

The proof given below shows that $\mathcal{R}(s)$ is in fact the Jacobson radical of $\mathcal{Q}(s)$ and $\mathcal{R}_2(s) = \mathcal{R}(s)^2$, with $\mathcal{R}(s)\mathcal{R}_2(s) = 0$, and hence $\mathcal{R}_2(s)$ is the socle of $\mathcal{Q}(s)$. For $s =$
1, \ldots, p - 1, the left $U_q sl(2)$-action on $Q(s)$ and the structure of subquotients can be visualized with the aid of the diagram

\[
\begin{array}{cccc}
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s) \\
X^-(p-s) \otimes X^+(s) & X^-(p-s) \otimes X^+(s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) \\
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s) \\
X^-(p-s) \otimes X^+(s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) \\
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s)
\end{array}
\]

and the right action with

\[
\begin{array}{cccc}
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s) \\
X^-(p-s) \otimes X^+(s) & X^-(p-s) \otimes X^+(s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) \\
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s) \\
X^-(p-s) \otimes X^+(s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) & X^+(s) \otimes X^-(p-s) \\
X^+(s) \otimes X^+(s) & X^-(p-s) \otimes X^-(p-s)
\end{array}
\]

The reader may find it convenient to look at these diagrams in reading the proof below.

**Proof.** First, the category $\mathcal{C}$ of finite-dimensional left $U_q sl(2)$-modules has the decomposition \cite{17}

\[
\mathcal{C} = \bigoplus_{s=0}^{p-1} \mathcal{C}(s),
\]

(4.14)

where each $\mathcal{C}(s)$ is a full subcategory. The full subcategories $\mathcal{C}(0)$ and $\mathcal{C}(p)$ are semisimple and contain precisely one irreducible module each, $X^+(p)$ and $X^-(p)$ respectively. Each $\mathcal{C}(s)$, $1 \leq s \leq p-1$, contains precisely two irreducible modules $X^+(s)$ and $X^-(p-s)$, and we have the vector-space isomorphisms \cite{17}

\[
\text{Ext}^1_{U_q} (X^+(s), X^-(p-s)) \cong \mathbb{C}^2,
\]

(4.15)

where a basis in each $\mathbb{C}^2$ can be chosen as the extensions corresponding to the Verma module $\mathcal{V}^\pm(s)$ and to the contragredient Verma module $\mathcal{V}^\pm(s)$ (see \cite{C.I}).

In view of (4.14), the regular representation viewed as a $U_q sl(2)$-bimodule has the decomposition

\[
\text{Reg} = \bigoplus_{s=0}^{p-1} Q(s)
\]

into a direct sum of indecomposable two-sided ideals $Q(s)$. We now study the structure of subquotients of $Q(s)$. Let $R(s)$ denote the Jacobson radical of $Q(s)$. By the Wedderburn–Artin theorem, the quotient $Q(s)/R(s)$ is a semisimple matrix algebra over $\mathbb{C}$,

\[
Q(s)/R(s) = \text{End}(X^+(s)) \oplus \text{End}(X^-(p-s)), \quad 1 \leq s \leq p - 1,
\]

\[
Q(0) = \text{End}(X^-(p)), \quad Q(p) = \text{End}(X^+(p))
\]
For $1 \leq s \leq p - 1$, we now consider the quotient $\mathcal{R}(s)/\mathcal{R}_2(s)$, where we set $\mathcal{R}_2(s) = \mathcal{R}(s)^2$. For brevity, we write $\mathcal{R} \equiv \mathcal{R}(s)$, $Q \equiv Q(s)$, $X^+ \equiv X^+(s)$ and $X^- \equiv X^-(p-s)$, $V^+ \equiv V^+(s)$, $V^- \equiv V^-(p-s)$, and similarly for the contragredient Verma modules $\overline{V}^\pm$. In view of (4.15), there are the natural bimodule homomorphisms

$$
\pi^\pm : \text{End}(\mathcal{V}^\pm), \quad \pi^\pm : \text{End}(\overline{\mathcal{V}}^\pm).
$$

The image of $\pi^+$ has the structure of the lower-triangular matrix

$$
\text{im}(\pi^+) = \begin{pmatrix}
X^+ \otimes X^+ & 0 \\
X^+ \otimes X^- & X^- \otimes X^-
\end{pmatrix}
$$

Clearly, the radical of $\text{im}(\pi^+)$ is the bimodule $X^+ \otimes X^-$. It follows that $\pi^+(\mathcal{R}) = X^+ \otimes X^-$ and the bimodule $X^+ \otimes X^-$ is a subquotient of $\mathcal{R}$. In a similar way, we obtain that $\pi^-(\mathcal{R}) = X^- \otimes X^+$ and $\pi^\pm(\mathcal{R}) = X^\pm \otimes X^\mp$. Therefore, we have the inclusion

$$
\mathcal{R}/\mathcal{R}^2 \supset X^- \otimes X^+ \oplus X^- \otimes X^+ \oplus X^+ \otimes X^- \oplus X^+ \otimes X^-
$$

Next, the Radford mapping $\Phi : \text{Reg}^* \rightarrow \text{Reg}$ (see A.3) establishes a bimodule isomorphism between $\text{Reg}^*$ and $\text{Reg}$, and therefore the socle of $\mathcal{Q}$ is isomorphic to $\mathcal{Q}/\mathcal{R}$. This suffices for finishing the proof: by counting the dimensions of the subquotients given in (4.16) and (4.17), and the dimension of the socle of $\mathcal{Q}$, we obtain the statement of the proposition.

### 4.4.3. Bimodule homomorphisms and the center

To find the center of $\mathcal{P}_{q}\mathfrak{sl}(2)$, we consider bimodule endomorphisms of the regular representation; such endomorphisms are in a 1:1 correspondence with elements in the center. Clearly,

$$
\text{End}(\text{Reg}) = \bigoplus_{s=0}^{p} \text{End}(\mathcal{Q}(s)).
$$

For each $\mathcal{Q}(s)$, $0 \leq s \leq p$, there is a bimodule endomorphism $e_s : \text{Reg} \rightarrow \text{Reg}$ that acts as identity on $\mathcal{Q}(s)$ and is zero on $\mathcal{Q}(s')$ with $s' \neq s$. These endomorphisms give rise to $p + 1$ primitive idempotents in the center of $\mathcal{P}_{q}\mathfrak{sl}(2)$.

Next, for each $\mathcal{Q}(s)$ with $1 \leq s \leq p - 1$, there is a homomorphism $w^+_s : \mathcal{Q}(s) \rightarrow \mathcal{Q}(s)$ (defined up to a nonzero factor) whose kernel, as a linear space, is given by $\mathcal{R}(s) \oplus X^- (p-s) \otimes X^-(p-s)$ (see (4.12)); in other words, $w^+_s$ sends the quotient $X^+(s) \otimes X^+(s)$ into the subbimodule $X^+(s) \otimes X^+(s)$ at the bottom of $\mathcal{Q}(s)$ and is zero on $\mathcal{Q}(s')$ with $s' \neq s$. Similarly, for each $s = 1, \ldots, p - 1$, there is a central element associated with the homomorphism $w^-_s : \mathcal{Q}(s) \rightarrow \mathcal{Q}(s)$ with the kernel $\mathcal{R}(s) \oplus X^+(s) \otimes X^+(s)$, i.e.,
the homomorphism sending the quotient $X^-(p-s) \boxtimes X^-(p-s)$ into the subbimodule $X^-(p-s) \boxtimes X^-(p-s)$ (and acting by zero on $Q(s')$ with $s' \neq s$). In total, there are $2(p-1)$ elements $w_s^\pm$, $1 \leq s \leq p-1$, which are obviously in the radical of the center.

By construction, the $e_s$ and $w_s^\pm$ have the properties summarized in the following proposition.

**4.4.4. Proposition.** The center $\mathfrak{Z}$ of $\mathfrak{U}_q sl(2)$ at $q = e^{\frac{i\pi}{2}}$ is $(3p-1)$-dimensional. Its associative commutative algebra structure is described as follows: there are two “special” primitive idempotents $e_0$ and $e_p$, $p-1$ other primitive idempotents $e_s$, $1 \leq s \leq p-1$, and $2(p-1)$ elements $w_s^\pm$ ($1 \leq s \leq p-1$) in the radical such that

$$
e_s e_{s'} = \delta_{s,s'} e_s, \quad s, s' = 0, \ldots, p,$$

$$
e_s w_{s'}^\pm = \delta_{s,s'} w_{s'}^\pm, \quad 0 \leq s \leq p, \ 1 \leq s' \leq p-1,$$

$$w_s^+ w_{s'}^- = w_s^- w_{s'}^+ = 0, \quad 1 \leq s, s' \leq p-1.$$

We call $e_s$, $w_s^\pm$ the canonical basis elements in the center, or simply the **canonical central elements**. They are constructed somewhat more explicitly in [D.1.1].

We note that the choice of a bimodule isomorphism $\text{Reg}^* \to \text{Reg}$ fixes the normalization of the $w_s^\pm$.

**4.4.5.** For any central element $A$ and its decomposition

$$A = \sum_{s=0}^p a_s e_s + \sum_{s=1}^{p-1} (c_s^+ w_s^+ + c_s^- w_s^-)$$

with respect to the canonical central elements, the coefficient $a_s$ is the eigenvalue of $A$ in the irreducible representation $X^+(s)$. To determine the $c_s^+$ and $c_s^-$ coefficients similarly, we fix the normalization of the basis vectors as in [C.2] i.e., such that $w_s^+$ and $w_s^-$ act as

$$w_s^+ b_n^{(+,s)} = a_n^{(+,s)}, \quad w_s^- y_k^{(-,s)} = x_k^{(-,s)}$$

in terms of the respective bases in the projective modules $P^+(s)$ and $P^-(p-s)$ defined in [C.2.1] and [C.2.2]. Then the coefficient $c_s^+$ is read off from the relation $A b_n^{(+,s)} = c_s^+ a_n^{(+,s)}$ in $P^+(s)$, and $c_s^-$, similarly, from the relation $A y_k^{(-,s)} = c_s^- x_k^{(-,s)}$ in $P^-(p-s)$.

**4.5. The Radford mapping for $\mathfrak{U}_q sl(2)$.** For a Hopf algebra $A$ with a given cointegral, we recall the Radford mapping $\hat{\phi} : A^* \to A$, see [A.3] (we use the hat for notational consistency in what follows). For $A = \mathfrak{U}_q sl(2)$, with the cointegral $c$ in [3.2], we are interested in the restriction of the Radford mapping to the space of $q$-characters $\mathfrak{ch}$ and, more specifically, to the image of the Grothendieck ring in $\mathfrak{ch}$ via the mapping $X \mapsto q \text{Ch}_X$ (see [A.1.7]). We thus consider the mapping

$$\mathfrak{C}_{2p} \rightarrow \mathfrak{Z},$$
which acts on the irreducible representations as
\[ X^\pm(s) \mapsto \hat{\phi}^\pm(s) = \sum_{(c)} \text{Tr}_{X^\pm(s)}(K^{p-1}c') c'', \quad 1 \leq s \leq p. \]

Let \( \mathfrak{H}_{2p} \) be the linear span of the \( \hat{\phi}^\pm(s) \) (the image of the Grothendieck ring in the center under the Radford mapping). As we see momentarily, \( \mathfrak{H}_{2p} \) is \( 2p \)-dimensional and coincides with the algebra generated by the \( \hat{\phi}^\alpha(s) \).

It follows that
\[ \hat{\phi}^+(1) = c, \]
in accordance with the fact that \( c \) furnishes an embedding of the trivial representation \( X^+(1) \) into \( \overline{U}_q \mathfrak{sl}(2) \). A general argument based on the properties of the Radford mapping (cf. [19]) and on the definition of the canonical nilpotents \( w^\pm_s \) above implies that for \( s = 1, \ldots, p - 1 \), \( \hat{\phi}^+(s) \) coincides with \( w^+_s \) up to a factor and \( \hat{\phi}^-(s) \) coincides with \( w^-_{p-s} \) up to a factor.

We now give a purely computational proof of this fact, which at the same time fixes the factors; we describe this in some detail because similar calculations are used in what follows.

4.5.1. Lemma. For \( 1 \leq s \leq p - 1 \),
\[ \hat{\phi}^+(s) = \omega_s w^+_s; \quad \hat{\phi}^-(s) = \omega_s w^-_{p-s}, \quad \omega_s = \frac{p \sqrt{2p}}{|s|^2}. \]

Also,
\[ \hat{\phi}^+(p) = p \sqrt{2p} e_p; \quad \hat{\phi}^-(p) = (-1)^{p+1} p \sqrt{2p} e_0. \]

Therefore, the image of the Grothendieck ring under the Radford mapping is the socle (annihilator of the radical) of \( \mathfrak{F} \).

Proof. First, we recall (3.2) and use (4.10) and (3.1) to evaluate
\[ (4.19) \quad \hat{\phi}^\alpha(s) = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{2p-1} \sum_{j=0}^{2p-1} \alpha^{i+j} ([i]!)^2 q^{i(s-1-2n)} \binom{s-n+i-1}{i} \binom{n}{i} F^{p-1-i} E^{p-1-i} K^j \]
(the calculation is very similar to the one in 4.3.1).

Next, we decompose \( \hat{\phi}^\alpha(s) \) with respect to the canonical basis following the strategy in 4.4.5. That is, we use (4.19) to calculate the action of \( \hat{\phi}^+(s) \) on the module \( \mathcal{P}^+(s') \) (\( 1 \leq s' \leq p - 1 \)). This action is nonzero only on the vectors \( b_0^{(+,s')} \) (see 4.2.11); because \( \hat{\phi}^+(s) \) is central, it suffices to evaluate it on any single vector, which we choose as \( b_0^{(+,s')} \).

For \( 1 \leq s \leq p - 1 \), using (4.9) and (D.6), we then have
\[ (4.20) \quad \hat{\phi}^+(s) b_0^{(+,s')} = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{n} \sum_{j=0}^{2p-1} ([i]!)^2 q^j (s+s'-2-2n) \left[ \frac{s-n+i-1}{i} \right] \left[ \frac{n}{i} \right] \times \prod_{r=0}^{p-2-i} \left( C - \frac{q^{2r+1} K + q^{-2r-1} K^{-1}}{(q - q^{-1})^2} \right) b_0^{(+,s')} \]

\[ = \zeta \sum_{n=0}^{s-1} \sum_{i=0}^{n} \sum_{j=0}^{2p-1} (-1)^{p+i} ([i]!)^2 q^j (s+s'-2-2n) \left[ \frac{s-n+i-1}{i} \right] \left[ \frac{n}{i} \right] \prod_{r=1}^{p-2-i} \left[ s' + r \right] [r] a_0^{(+,s')}, \]

with the convention that whenever \( p - 2 - i = 0 \), the product over \( r \) evaluates as 1. We simultaneously see that the diagonal part of the action of \( \hat{\phi}^+(s) \) on \( D^+(s') \) vanishes.

Analyzing the cases where the product over \( r \) in (4.20) involves \([p] = 0\), it is immediate to see that a necessary condition for the right-hand side to be nonzero is \( s' \leq s \). Let therefore \( s = s' + \ell \), where \( \ell \geq 0 \). It is then readily seen that (4.20) vanishes for odd \( \ell \); we thus set \( \ell = 2m \), which allows us to evaluate

\[ \hat{\phi}^+(s' + 2m) b_0^{(+,s')} = \]

\[ = 2p \zeta \sum_{i=s'-1}^{m+s'-1} (-1)^{p+i} ([i]!)^2 \left[ \frac{m+i}{i} \right] \left[ \frac{m+s'-1}{i} \right] \left[ \frac{p-2-i+s'}{s'} \right]! \left[ \frac{p-2-i}{s'} \right]! a_0^{(+,s')} . \]

But this vanishes for all \( m > 0 \) in view of the identity

\[ \sum_{j=0}^{m} (-1)^j \left[ \frac{m+s'-1}{j} \right]! \frac{m+s'+m-1}{[j]! [m-j]!} = \frac{1}{m} \sum_{j \in \mathbb{Z}} (-1)^j \left[ \frac{m+s'-1+j}{m-1+j} \right] = 0, \quad m \geq 1. \]

Thus, \( \hat{\phi}^+(s) \) acts by zero on \( D^+(s') \) for all \( s' \neq s \); it follows similarly that \( \hat{\phi}^+(s) \) acts by zero on \( D^-(s') \) for all \( s' \) and on both Steinberg modules \( \chi^\pm(p) \). Therefore, \( \hat{\phi}^+(s) \) is necessarily proportional to \( u^+_s \), with the proportionality coefficient to be found from the action on \( D^+(s) \). But for \( s' = s \), the sum over \( j \) in the right-hand side of (4.20) is zero unless \( n = s - 1 \), and we have

\[ \hat{\phi}^+(s) b_0^{(+,s)} = \frac{2p \zeta}{[s]} \sum_{i=0}^{s-1} (-1)^{p+i} [i]! \left[ \frac{p-2-i}{s-1-i} \right]! a_0^{(+,s)} , \]

where the terms in the sum are readily seen to vanish unless \( i = s - 1 \), and therefore

\[ = 2p \zeta (-1)^{p+s+1} [p-1]! [s-1]! [p-1-s]! [s] a_0^{(+,s)} , \]

which gives \( \omega_s \) as claimed. The results for \( \hat{\phi}^-(s) \) (\( 1 \leq s \leq p - 1 \)) and \( \hat{\phi}^\pm(p) \) are established similarly.

It follows (from the expression in terms of the canonical central elements; cf. [19] for the small quantum group) that the two images of the Grothendieck ring in the center, \( D_{2p} \).
and $\mathfrak{N}_{2p}$, span the entire center:

$$\mathcal{D}_{2p} \cup \mathfrak{N}_{2p} = \mathfrak{Z}.$$ 

We next describe the intersection of the two Grothendieck ring images in the center (cf. [19] for the small quantum group). This turns out to be the Verma-module ideal (see 4.3.5).

### 4.5.2. Proposition

$\mathcal{D}_{2p} \cap \mathfrak{N}_{2p} = \mathfrak{V}_{p+1}$.

**Proof.** Proceeding similarly to the proof of 4.5.1 we establish the formulas

\[(4.21) \quad \hat{\phi}^+(s) + \hat{\phi}^-(p-s) = \frac{\zeta ([p-1]!)^2}{p} \times \left( (-1)^{p-s} \mathfrak{x}(0) + \sum_{s'=1}^{p-1} (-1)^{p+s+s'} (q^{s' s'} + q^{-s' s'}) \mathfrak{x}(s') + \mathfrak{x}(p) \right)\]

for $s = 1, \ldots, p - 1$, and

\[(4.22) \quad \hat{\phi}^+(p) = \frac{1}{\sqrt{2p}} \left( \mathfrak{x}(0) + 2 \sum_{s'=1}^{p-1} \mathfrak{x}(p-s') + \mathfrak{x}(p) \right),\]

\[(4.23) \quad \hat{\phi}^-(p) = \frac{1}{\sqrt{2p}} \left( (-1)^p \mathfrak{x}(0) + 2 \sum_{s'=1}^{p-1} (-1)^{s'} \mathfrak{x}(p-s') + \mathfrak{x}(p) \right),\]

which imply the proposition. The derivation may in fact be simplified by noting that as a consequence of (D.2) and (D.2), $\hat{\phi}^+(s) + \hat{\phi}^-(p-s)$ belongs to the subalgebra generated by the Casimir element, which allows using (D.7). \(\square\)

### 4.6. The $\overline{U}_q\mathfrak{sl}(2)$ ribbon element

We finally recall (see [A.6] and [31]) that a ribbon element $v \in A$ in a Hopf algebra $A$ is an invertible central element satisfying (A.15). For $\overline{U}_q\mathfrak{sl}(2)$, the ribbon element is actually given in (4.2), rewritten as

$$v = \frac{1 - i}{2\sqrt{p}} \sum_{m=0}^{p-1} \sum_{j=0}^{2p-1} \left( \frac{q - q^{-1}}{m!} \right)^{\frac{m}{2} + mj + \frac{1}{2}(j+p+1)^2} F^m E^m K^j$$

in terms of the $\overline{U}_q\mathfrak{sl}(2)$ generators. A calculation similar to the one in the proof of 4.5.1 shows the following proposition.

### 4.6.1. Proposition

The $\overline{U}_q\mathfrak{sl}(2)$ ribbon element is decomposed in terms of the canonical central elements as

$$v = \sum_{s=0}^{p} (-1)^{s+1} q^{-\frac{1}{2}(s^2-1)} e_s + \sum_{s=1}^{p-1} (-1)^{p} q^{-\frac{1}{2}(s^2-1)} [s] \frac{q - q^{-1}}{2p} \hat{\varphi}(s),$$

where

$$\hat{\varphi}(s) = \frac{p-s}{p} \hat{\phi}^+(s) - \frac{s}{p} \hat{\phi}^-(p-s), \quad 1 \leq s \leq p - 1.$$
Strictly speaking, expressing \( v \) through the canonical central elements requires using \[ \ref{4.5.1} \] but below we need \( v \) expressed just through \( \hat{\phi}^\pm(s) \).

5. \( \text{SL}(2, \mathbb{Z}) \)-REPRESENTATIONS ON THE CENTER OF \( \mathcal{U}_q \mathfrak{sl}(2) \)

In this section, we first recall the standard \( \text{SL}(2, \mathbb{Z}) \)-action \[ \ref{5, 6, 11} \] reformulated for the center \( \mathfrak{Z} \) of \( \mathcal{U}_q \mathfrak{sl}(2) \). Its definition involves the ribbon element and the Drinfeld and Radford mappings. From the multiplicative Jordan decomposition for the ribbon element, we derive a factorization of the standard \( \text{SL}(2, \mathbb{Z}) \)-representation \( \pi \), \( \pi(\gamma) = \bar{\pi}(\gamma)\pi^*(\gamma) \), where \( \bar{\pi} \) and \( \pi^* \) are also \( \text{SL}(2, \mathbb{Z}) \)-representations on \( \mathfrak{Z} \). We then establish the equivalence to the \( \text{SL}(2, \mathbb{Z}) \)-representation on \( \mathfrak{Z}_{cft} \) in \[ \ref{2.2} \].

5.1. The standard \( \text{SL}(2, \mathbb{Z}) \)-representation on \( \mathfrak{Z} \). Let \( \pi \) denote the \( \text{SL}(2, \mathbb{Z}) \)-representation on the center \( \mathfrak{Z} \) of \( \mathcal{U}_q \mathfrak{sl}(2) \) constructed, as a slight modification of the representation in \[ \ref{5, 6, 11} \], as follows. We let \( S \equiv \pi(S) : \mathfrak{Z} \to \mathfrak{Z} \) and \( T \equiv \pi(T) : \mathfrak{Z} \to \mathfrak{Z} \) be defined as

\[
S(a) = \hat{\phi}(\chi^{-1}(a)), \quad T(a) = b S^{-1}(v^{-1}(S(a))), \quad a \in \mathfrak{Z},
\]

where \( v \) is the ribbon element, \( \chi \) is the Drinfeld mapping, \( \hat{\phi} \) is the Radford mapping, and \( b \) is the normalization factor

\[
b = e^{i\pi/(2p - 12)}.\]

We call it the standard \( \text{SL}(2, \mathbb{Z}) \)-representation, to distinguish it from other representations introduced in what follows.

We recall that \( S^2 \) acts via the antipode on the center of the quantum group, and hence acts identically on the center of \( \mathcal{U}_q \mathfrak{sl}(2) \),

\[
S^2 = \text{id}_{\mathfrak{Z}}.
\]

5.2. Theorem. The standard \( \text{SL}(2, \mathbb{Z}) \)-representation on the center \( \mathfrak{Z} \) of \( \mathcal{U}_q \mathfrak{sl}(2) \) at \( q = e^{i\pi/p} \) is equivalent to the \((3p - 1)\)-dimensional \( \text{SL}(2, \mathbb{Z}) \)-representation on \( \mathfrak{Z}_{cft} \) (the extended characters of the \((1, p)\) conformal field theory model in \[ \ref{2.2} \]).

We therefore abuse the notation by letting \( \pi \) denote both representations.

**Proof.** We introduce a basis in \( \mathfrak{Z} \) as

\[
\rho(s), \quad 1 \leq s \leq p - 1, \\
\kappa(s), \quad 0 \leq s \leq p, \\
\varphi(s), \quad 1 \leq s \leq p - 1,
\]

where

\[
\rho(s) = \frac{p - s}{p} \chi^+(s) - \frac{s}{p} \chi^-(p - s),
\]
κκκ(s) are defined in (4.11), and

\[ \varphi(s) = \frac{1}{\sqrt{2p}} \sum_{r=1}^{p-1} (-1)^{r+s+p} (q^{rs} - q^{-rs}) \hat{\varphi}(r) \]

(with \( \hat{\varphi}(s) \) defined in (4.23)). That this is a basis in the center follows, e.g., from the decomposition into the canonical central elements.

The mapping

\[ \rho_s \mapsto \rho(s), \quad 1 \leq s \leq p-1, \]
\[ \kappa_s \mapsto \kappa(s), \quad 0 \leq s \leq p, \]
\[ \varphi_s \mapsto \varphi(s), \quad 1 \leq s \leq p-1 \]

between the bases in \( \mathfrak{z}_{\text{cft}} \) and in \( \mathfrak{z} \) establishes the equivalence. Showing this amounts to the following checks.

First, we evaluate \( S(\rho(s)) \) as

\[ S(\rho(s)) = \hat{\varphi} \circ \chi^{-1}(\frac{p-s}{p} \chi^+(s) - \frac{s}{p} \chi^-(p-s)) \]

\[ = \frac{p-s}{p} \hat{\varphi}^+(s) - \frac{s}{p} \hat{\varphi}^-(p-s) = \hat{\varphi}(s), \]

and hence, in view of (5.2),

\[ (5.3) \quad S(\hat{\varphi}(s)) = \rho(s), \quad 1 \leq s \leq p-1. \]

We also need this formula rewritten in terms of \( \hat{\rho}(s) \), that is,

\[ (5.4) \quad S(\varphi(s)) = \hat{\rho}(s), \quad 1 \leq s \leq p-1. \]

Further, we use (4.21) and (4.22) to evaluate \( S(\kappa(s)) \) as

\[ S(\kappa(s)) = \hat{\phi} \circ \chi^{-1}(\chi^+(s) + \chi^-(p-s)) = \hat{\phi}^+(s) + \hat{\phi}^-(p-s) = \]

\[ = \frac{1}{\sqrt{2p}} \left( (-1)^{p-s} \chi(0) + \sum_{s'=1}^{p-1} (-1)^{s'} (q^{s'} + q^{-s'}) \kappa(p-s') + \kappa(p) \right), \quad 0 \leq s \leq p, \]

where we set \( \chi^\pm(0) = \hat{\phi}^\pm(0) = 0 \). This shows that \( S \) acts on \( \rho(s), \kappa(s), \) and \( \varphi(s) \) as on the respective basis elements \( \rho_s, \kappa_s, \) and \( \varphi_s \) in \( \mathfrak{z}_{\text{cft}} \).

Next, it follows from (4.6.1) that \( v \) acts on \( \hat{\phi}^\pm(s) \) as

\[ v \hat{\phi}^+(s) = (-1)^{s+1} q^{-\frac{1}{2}(s^2-1)} \hat{\phi}^+(s), \]
\[ v \hat{\phi}^-(s) = (-1)^{p+1} q^{-\frac{1}{2}(p^2+s^2-1)} \hat{\phi}^-(s), \quad 1 \leq s \leq p. \]
As an immediate consequence, in view of \( T \chi^\pm(s) = bS^{-1}(v^{-1})^{\pm}(s) \), we have
\[
\begin{align*}
T \chi^+(s) &= \lambda_{p,s} \chi^+(s), \\
T \chi^-(s) &= \lambda_{p,p-s} \chi^-(s), \\
\end{align*}
\]
where \( \lambda_{p,s} \) is defined in (2.8). It follows that \( T \) acts on \( \rho(s) \) and \( \kappa(s) \) as on the respective basis elements \( \rho_s \) and \( \kappa_s \) in \( \mathfrak{z}_{\text{eff}} \).

Finally, we evaluate \( T \varphi(s) \). Recalling (4.6.1) to rewrite \( v \) as
\[
v = \sum_{t=0}^{p} (-1)^{t+1} q^{-\frac{1}{2}(t^2-1)} e_t (1 + \varphi(1)),
\]
we use (5.2) and (5.4), with the result
\[
T \varphi(s) = bS v^{-1} \tilde{\rho}(s) = bS \sum_{t=0}^{p} (-1)^{t+1} q^{\frac{1}{2}(t^2-1)} e_t (1 - \varphi(1)) \tilde{\rho}(s).
\]
But (a simple rewriting of the formulas in (D.3))
\[
\tilde{\rho}(s) = (-1)^{p+s} \frac{\sqrt{2p}}{q^s - q^{-s}} \left( e_s e_s - \frac{q^s + q^{-s}}{|s|^2} w_s \right),
\]
and therefore (also recalling the projector properties to see that only one term survives in the sum over \( t \))
\[
T \varphi(s) = -b \frac{\sqrt{2p}}{q^s - q^{-s}} S \sum_{t=0}^{p} (-1)^{t+s+p} q^\frac{1}{2}(t^2-1) e_t (1 - \varphi(1)) \left( e_s e_s - \frac{q^s + q^{-s}}{|s|^2} w_s \right) =
\]
\[
= -\frac{b \sqrt{2p}}{q^s - q^{-s}} S (-1)^p q^\frac{1}{2}(s^2-1) e_s \left( e_s e_s - \frac{q^s + q^{-s}}{|s|^2} w_s - \varphi(1)e_s \right)
\]
\[
= b (-1)^{s+1} q^\frac{1}{2}(s^2-1) S \tilde{\rho}(r) + b (-1)^p \frac{\sqrt{2p}}{q^s - q^{-s}} S \varphi(1)e_s.
\]
Here, \( S \tilde{\rho}(r) = \varphi(r) \) and \( \varphi(1)e_s = (-1)^{s+p+1} \frac{q^s - q^{-s}}{\sqrt{2p}} \varphi(s) \), and hence
\[
T \varphi(s) = \lambda_{p,s} (\varphi(s) + \rho(s)).
\]
This completes the proof. \( \square \)

5.3. Factorization of the standard \( SL(2, \mathbb{Z}) \)-representation on the center. In view of the equivalence of representations, the \( SL(2, \mathbb{Z}) \)-representation \( \pi \) on the center admits the factorization established in (2.3). Remarkably, this factorization can be described in “intrinsic” quantum-group terms, as we now show. That is, we construct two more \( SL(2, \mathbb{Z}) \)-representations on \( \mathfrak{z} \) with the properties described in (1.3).
5.3.1. For the ribbon element $v$, we consider its multiplicative Jordan decomposition

$$v = v^* \bar{v}$$

into the semisimple part

$$\bar{v} = \sum_{s=0}^{\infty} (-1)^{s+1} q^{-\frac{1}{2}(s^2-1)} e_s$$

and the unipotent part

$$v^* = 1 + \varphi(1).$$

With (5.6), we now let $T^*: \mathfrak{z} \to \mathfrak{z}$ and $\bar{T}: \mathfrak{z} \to \mathfrak{z}$ be defined by the corresponding parts of the ribbon element, similarly to (5.1):

$$T^*(a) = S^{-1}(v^*-1S(a)), \quad \bar{T}(a) = bS^{-1}(\bar{v}-1S(a)), \quad a \in \mathfrak{z}.$$

Then, evidently,

$$\mathcal{J} = T^* \bar{T}.$$

5.3.2. We next define a mapping $\xi: \bar{U}_q sl(2)^* \to \bar{U}_q sl(2)$ as

$$\xi(\beta) = (\beta \otimes \text{id})(N),$$

where

$$N = (v^* \otimes v^*) \Delta(S(v^*)).$$

It intertwines the coadjoint and adjoint actions of $\bar{U}_q sl(2)$, and we therefore have the mapping $\xi: Ch(\bar{U}_q sl(2)) \to \mathfrak{z}$, which is moreover an isomorphism of vector spaces. We set

$$S^* = \hat{\phi} \circ \xi^{-1}, \quad \bar{S} = \xi \circ \chi^{-1}.$$

This gives the decomposition

$$S = S^* \bar{S}.$$

5.3.3. Theorem. The action of $S^*$ and $T^*$ on the center generates the $SL(2, \mathbb{Z})$-representation $\pi^*$, and the action of $\bar{S}$ and $\bar{T}$ on the center generates the $SL(2, \mathbb{Z})$-representation $\bar{\pi}$, such that

1. $\bar{\pi}(\gamma) \pi^*(\gamma') = \pi^*(\gamma') \bar{\pi}(\gamma)$ for all $\gamma, \gamma' \in SL(2, \mathbb{Z}),$
2. the representation $\bar{\pi}$ restricts to the Grothendieck ring (i.e., to its isomorphic image in the center), and
3. $\pi(\gamma) = \bar{\pi}(\gamma) \pi^*(\gamma)$ for all $\gamma \in SL(2, \mathbb{Z}),$

and $\pi$ and $\bar{\pi}$ are isomorphic to the respective $SL(2, \mathbb{Z})$-representations on $\mathfrak{z}_{\text{cft}}$ in 2.3.
The verification is similar to the proof of (5.2) with
\[ S^*^{-1}(\hat{\phi}(s)) = \xi(q\text{Ch}_c(s)) = (\text{Tr}_{X^c(s)} \otimes \text{id})( (K^{p-1} \otimes 1)N ) \]
and
\[ S^*^{-1}(\chi(s)) = (\mu \otimes \text{id})(S(\chi(s)) \otimes 1)N \]
(and similarly for \( \bar{S} \)), based on the formula
\[ S(\nu^*) = S(1 + \varphi(1)) = \hat{\phi}^+(1) + \hat{\rho}(1) = c + \hat{\rho}(1). \]

5.3.4. The three mappings involved in (5.8) — \( \hat{\phi} \) defined in (A.5), \( \chi \) defined in (A.14), and \( \xi \) in (5.7) — can be described in a unified way as follows. Let \( A \) be a ribbon Hopf algebra endowed with the standard \( SL(2, \mathbb{Z}) \)-representation. For \( x \in A \), we define
\[ \lambda_x : A^* \to A \]
as
\[ \lambda_x(\beta) = (\beta \otimes \text{id})( (x \otimes x)\Delta(S(x))) , \]
where \( S \) is the standard action of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Taking \( x \) to be the three elements 1, \( \nu \), and \( \nu^* \), we have
\[ \lambda_1 = \hat{\phi}, \quad \lambda_\nu = \chi, \quad \lambda_{\nu^*} = \xi. \]

6. Conclusions

We have shown that the Kazhdan–Lusztig correspondence, understood in a broad sense as a correspondence between conformal field theories and quantum groups, extends into the nonsemisimple realm such that a number of structures on the conformal field theory side and on the quantum group side are actually isomorphic, which signifies an “improvement” over the case of rational/semisimple conformal field theories.

Although much of the argument in this paper is somewhat too “calculational,” and hence apparently “accidental,” we hope that a more systematic derivation can be given. In fact, the task to place the structures encountered in the study of nonsemisimple Verlinde algebras into the categorical context [32, 33, 34, 35] was already formulated in [15]. With the quantum-group counterpart of nonsemisimple Verlinde algebras and of the \( SL(2, \mathbb{Z}) \)-representations on the conformal blocks studied in this paper in the \((1, p)\) example, this task becomes even more compelling.

We plan to address Claim 1 of the Kazhdan–Lusztig correspondence (see page 2) between the representation categories of the \( \mathcal{V}(p) \) algebra and of \( U_{q, \ell}(2) \) [17]. This requires constructing vertex-operator analogues of extensions among the irreducible representations (generalizing the \((1, 2)\) case studied in [27]).

Another direction where development is welcome is to go over from \((1, p)\) to \((p', p)\) models of logarithmic conformal field theories, starting with the simplest such model, \((2, 3)\), whose content as a minimal theory is trivial, but whose logarithmic version may be quite interesting.
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**APPENDIX A. HOPF ALGEBRA DEFINITIONS AND STANDARD FACTS**

We let $A$ denote a Hopf algebra with comultiplication $\Delta$, counit $\epsilon$, and antipode $S$. The general facts summarized here can be found in [36, 37, 38, 39,12].

**A.1. Adjoint and coadjoint actions, center, and $q$-characters.** For a Hopf algebra $A$, the adjoint and coadjoint actions $\text{Ad}_a : A \to A$ and $\text{Ad}_a^* : A^* \to A^*$ ($a \in A$) are defined as

$$\text{Ad}_a(x) = \sum_{(a)} a'S(a'') \quad \text{and} \quad \text{Ad}_a^*(\beta) = \beta(\sum_{(a)} S(a')a'').$$

The center $Z(A)$ of $A$ can be characterized as the set

$$Z(A) = \{ y \in A \mid \text{Ad}_x(y) = \epsilon(x)y \quad \forall x \in A \}.$$

By definition, the space $\mathcal{Ch}(A)$ of $q$-characters is

(A.1) \quad $\mathcal{Ch}(A) = \{ \beta \in A^* \mid \text{Ad}_x^*(\beta) = \epsilon(x)\beta \quad \forall x \in A \}$

$$= \{ \beta \in A^* \mid \beta(xy) = \beta(S^2(y)x) \quad \forall x,y \in A \}.$$

Given an invertible element $t \in A$ satisfying $S^2(x) = txt^{-1}$ for all $x \in A$, we define the linear mapping $q_{ch}^t : A \to \mathbb{C}$ for any $A$-module $\mathcal{X}$ as

(A.2) \quad $q_{ch}^t : A \to \mathbb{C}$

$$q_{ch}^t_A = \text{Tr}_{\mathcal{X}}(t^{-1}?) .$$

**A.1.1. Lemma ([12,39]).** For any $A$-module $\mathcal{X}$ and an element $t$ such that $S^2(x) = txt^{-1}$, we have

1. $q_{ch}^t \in \mathcal{Ch}(A)$
2. if in addition $t$ is group-like, i.e., $\Delta(t) = t \otimes t$, then

$$q_{ch}^t : \mathcal{X} \mapsto q_{ch}^t\mathcal{X}$$

is a homomorphism of the Grothendieck ring to the ring of $q$-characters.
A.2. (Co)integrals, comoduli, and balancing. For a Hopf algebra $A$, a right integral $\mu$ is a linear functional on $A$ satisfying

$$(\mu \otimes \text{id})\Delta(x) = \mu(x)1$$

for all $x \in A$. Whenever such a functional exists, it is unique up to multiplication with a nonzero constant.

A comodulus $a$ is an element in $A$ such that

$$(\text{id} \otimes \mu)\Delta(x) = \mu(x)a.$$ 

The left–right cointegral $c$ is an element in $A$ such that

$$xc = cx = \epsilon(x)c, \quad \forall x \in A.$$ 

If it exists, this element is unique up to multiplication with a nonzero constant. We also note that the cointegral gives an embedding of the trivial representation of $A$ in the bimodule $A$. We use the normalization $\mu(c) = 1$.

Whenever a square root of the comodulus $a$ can be calculated in a Hopf algebra $A$, the algebra admits the balancing element $g$ that satisfies

$$(A.3) \quad S^2(x) = gxg^{-1}, \quad \Delta(g) = g \otimes g,$$

In fact, we have the following lemma.

A.2.1. Lemma ([38]).

(A.4) \quad g^2 = a.

A.3. The Radford mapping. Let $A$ be a Hopf algebra with the right integral $\mu$ and the left–right cointegral $c$. The Radford mapping $\hat{\phi} : A^* \to A$ and its inverse $\hat{\phi}^{-1} : A \to A^*$ are given by

$$(A.5) \quad \hat{\phi}(\beta) = \sum_{(c)} \beta(c')c'', \quad \hat{\phi}^{-1}(x) = \mu(S(x)).$$

A.3.1. Lemma ([40] [41]). $\hat{\phi}$ and $\hat{\phi}^{-1}$ are inverse to each other, $\hat{\phi}\hat{\phi}^{-1} = \text{id}_A$, $\hat{\phi}^{-1}\hat{\phi} = \text{id}_{A^*}$, and intertwine the left actions of $A$ on $A$ and $A^*$, and similarly for the right actions.

Here, the left-$A$-module structure on $A^*$ is given by $a \cdot \beta = \beta(S(a))$ (and on $A$, by the regular action).

A.4. Quasitriangular Hopf algebras and the $R$ and $M$ matrices.
A.4.1. $R$-matrix. A quasitriangular Hopf algebra $A$ has an invertible element $R \in A \otimes A$ satisfying

\begin{align}
\Delta^{\text{op}}(x) &= R\Delta(x)R^{-1}, \\
(\Delta \otimes \text{id})(R) &= R_{13}R_{23}, \\
(id \otimes \Delta)(R) &= R_{13}R_{12},
\end{align}

\[ R_{12}R_{13}R_{23} = R_{23}R_{12}, \]

\[ (\epsilon \otimes \text{id})(R) = 1 = (\text{id} \otimes \epsilon)(R), \]

\[ (S \otimes S)(R) = R. \]

A.4.2. $M$-matrix. For a quasitriangular Hopf algebra $A$, the $M$-matrix is defined as

\[ M = R_{21}R_{12} \in A \otimes A. \]

It satisfies the relations

\begin{align}
(\Delta \otimes \text{id})(M) &= R_{32}M_{13}R_{23}, \\
M\Delta(x) &= \Delta(x)M \quad \forall x \in A.
\end{align}

Indeed, using (A.8), we find \((\Delta \otimes \text{id})(R_{21}) = R_{32}R_{31}\) and then using (A.7), we obtain (A.9). Next, from (A.6), which we write as \(R_{12}\Delta(x) = \Delta^{\text{op}}(x)R_{12}\), it follows that

\[ R_{21}R_{12}\Delta(x) = (R_{12}\Delta(x))^{\text{op}}R_{12} = (\Delta^{\text{op}}(x)R_{12})^{\text{op}}R_{12} = \Delta(x)R_{21}R_{12}, \]

that is, (A.10).

If in addition $M$ can be represented as

\[ M = \sum m_i \otimes n_i, \]

where $m_i$ and $n_i$ are two bases in $A$, the Hopf algebra $A$ is called factorizable.

A.4.3. The square of the antipode \cite{38,5}. In any quasitriangular Hopf algebra, the square of the antipode is represented by a similarity transformation

\[ S^2(x) = uu^{-1} \]

where the canonical element $u$ is given by

\begin{align}
u &= \cdot((S \otimes \text{id})R_{21}), \\
u^{-1} &= \cdot((S^{-1} \otimes S)R_{21})
\end{align}

(where $\cdot(a \otimes b) = ab$) and satisfies the property

\[ \Delta(u) = M^{-1}(u \otimes u) = (u \otimes u)M^{-1}. \]

Any invertible element $t$ such that $S^2(x) = txt^{-1}$ for all $x \in A$ can be expressed as $t = \theta u$, where $\theta$ is an invertible central element.
A.5. The Drinfeld mapping. Given an $M$-matrix (see A.4.2), we define the Drinfeld mapping $\chi : A^* \to A$ as

\begin{equation}
\chi(\beta) = (\beta \otimes \text{id})M = \sum_I \beta(m_I)n_I.
\end{equation}

A.5.1. Lemma (38). In a factorizable Hopf algebra $A$, the Drinfeld mapping $\chi : A^* \to A$ intertwines the adjoint and coadjoint actions of $A$ and its restriction to the space $\mathfrak{Ch}$ of $q$-characters gives an isomorphism of associative algebras

$$\mathfrak{Ch}(A) \xrightarrow{\sim} \mathfrak{Z}(A).$$

A.6. Ribbon algebras. A ribbon Hopf algebra [31] is a quasitriangular Hopf algebra equipped with an invertible central element $v$, called the ribbon element, such that

\begin{equation}
v^2 = uS(u), \quad S(v) = v, \quad \epsilon(v) = 1, \quad \Delta(v) = M^{-1}(v \otimes v).
\end{equation}

In a ribbon Hopf algebra,

\begin{equation}
g = v^{-1}u,
\end{equation}

where $g$ is the balancing element (see A.2).

A.6.1. Let $A$ be a ribbon Hopf algebra and $X$ an $A$-module. The balancing element $g$ allows constructing the “canonical” $q$-character of $X$:

\begin{equation}
q\text{Ch}_X \equiv q\text{ch}_X^g = \text{Tr}_X(g^{-1}?) \in \mathfrak{Ch}(A).
\end{equation}

We also define the quantum dimension of a module $X$ as

$$q\text{dim} X = \text{Tr}_X g^{-1}.$$ 

It satisfies the relation

$$q\text{dim} X_1 \otimes X_2 = q\text{dim} X_1 \cdot q\text{dim} X_2.$$ 

for any two modules $X_1$ and $X_2$.

Let now $A$ be a factorizable ribbon Hopf algebra and let $\mathfrak{G}(A)$ be its Grothendieck ring. We combine the mapping $\mathfrak{G}(A) \to A^*$ given by $X \mapsto q\text{Ch}_X$ and the Drinfeld mapping $\chi : A^* \to A$.

A.6.2. Lemma. In a factorizable ribbon Hopf algebra $A$, the mapping

$$\chi \circ q\text{Ch} : \mathfrak{G}(A) \to \mathfrak{Z}(A)$$

is a homomorphism of associative commutative algebras.
APPENDIX B. THE QUANTUM DOUBLE

In this Appendix, we construct a double of the Hopf algebra $B$ associated with the short screening in the logarithmic conformal field theory outlined in [2.1]. The main structure resulting from the double is the $R$-matrix, which is then used to construct the $M$-matrix $\overline{\mathcal{M}}$ for $\overline{U}_q \mathfrak{sl}(2)$.

B.1. Constructing a double of the “short-screening” quantum group. For $q = e^{\frac{i\pi}{p}}$, we let $B$ denote the Hopf algebra generated by $e$ and $k$ with the relations

\begin{equation}
\begin{aligned}
e^p &= 0, \quad k^{4p} = 1, \quad kek^{-1} = qe, \\
\Delta(e) &= 1 \otimes e + e \otimes k^2, \quad \Delta(k) = k \otimes k, \\
\epsilon(e) &= 0, \quad \epsilon(k) = 1, \\
S(e) &= -ek^{-2}, \quad S(k) = k^{-1}.
\end{aligned}
\end{equation}

The PBW-basis in $B$ is

$$e_{mn} = e^m k^n, \quad 0 \leq m \leq p - 1, \quad 0 \leq n \leq 4p - 1.$$  

The space $B^*$ of linear functions on $B$ is a Hopf algebra with the multiplication, comultiplication, unit, counit, and antipode given by

\begin{equation}
\begin{aligned}
\langle \beta, \gamma, x \rangle &= \sum_{(x)} \langle \beta, x' \rangle \langle \gamma, x'' \rangle, \\
\langle \Delta(\beta), x \otimes y \rangle &= \langle \beta, xy \rangle, \\
\langle 1, x \rangle &= \epsilon(x), \quad \langle \epsilon(\beta), 1 \rangle = \langle \beta, 1 \rangle, \\
\langle S(\beta), x \rangle &= \langle \beta, S^{-1}(x) \rangle
\end{aligned}
\end{equation}

for any $\beta, \gamma \in B^*$ and $x, y \in B$.

The quantum double $D(B)$ is a Hopf algebra with the underlying vector space $B^* \otimes B$ and with the multiplication, comultiplication, unit, counit, and antipode given by Eqs. (B.1) and (B.2) and by

\begin{equation}
x(\beta) = \sum_{(x)} \beta(S^{-1}(x'')x')x'', \quad x \in B, \quad \beta \in B^*.
\end{equation}

B.1.1. Theorem. $D(B)$ is the Hopf algebra generated by $e$, $\phi$, $k$, and $\kappa$ with the relations

\begin{align}
kek^{-1} &= qe, \quad e^p = 0, \quad k^{4p} = 1, \\
k\phi k^{-1} &= q\phi, \quad \phi^p = 0, \quad k^{4p} = 1, \\
k\kappa &= k\kappa, \quad k\phi k^{-1} = q^{-1}\phi, \quad \kappa\phi k^{-1} = q^{-1}e, \\
[k, \phi] &= \frac{k^2 - \kappa^2}{q - q^{-1}}, \\
\Delta(e) &= 1 \otimes e + e \otimes k^2, \quad \Delta(k) = k \otimes k, \quad \epsilon(e) = 0, \quad \epsilon(k) = 1, \\
\Delta(\phi) &= k^2 \otimes \phi + \phi \otimes 1, \quad \Delta(\kappa) = \kappa \otimes \kappa, \quad \epsilon(\phi) = 0, \quad \epsilon(\kappa) = 1, \\
S(e) &= -ek^{-2}, \quad S(k) = k^{-1}, \\
S(\phi) &= -\kappa^{-2}\phi, \quad S(\kappa) = \kappa^{-1}.
\end{align}
Proof. Equations (B.4), (B.7), and (B.9) are relations in \( B \). The unit in \( B^* \) is given by the function 1 such that

\[
\langle 1, e_{mn} \rangle = \delta_{m,0}.
\]

The elements \( \kappa, \phi \in B^* \) are uniquely defined by

\[
\langle \kappa, e_{mn} \rangle = \delta_{m,0} q^{-n/2}, \quad \langle \phi, e_{mn} \rangle = \delta_{m,1} q^{-n} q^{-1}.
\]

For elements of the PBW-basis of \( B \), the first relation in (B.2) becomes

\[
\langle \beta \gamma, e_{mn} \rangle = \sum_{r=0}^{m} \binom{m}{r} \langle \beta, e^{m-r}k^n \rangle \langle \gamma, k^{2m-2r+n} \rangle,
\]

where we use the notation

\[
\langle n \rangle = \frac{q^n - 1}{q^2 - 1} = q^{n-1}[n], \quad \langle n \rangle! = \langle 1 \rangle \langle 2 \rangle \ldots \langle n \rangle, \quad \binom{m}{n} = \frac{(m)!}{\langle n \rangle! (m-n)!}.
\]

We then check that the elements \( \phi^i \kappa^j \) with \( 0 \leq i \leq p-1 \) and \( 0 \leq j \leq 4p-1 \) constitute a basis in \( B^* \) and evaluate on the basis elements of \( B \) as

\[
\langle \phi^i \kappa^j, e_{mn} \rangle = \delta_{m,1} \frac{(i)!}{(q-q^{-1})i} q^{-(i+2)} q^{-n/2 - ij - i(i-1)}
\]

The easiest way to see that (B.12) holds is to use (B.11) to calculate \( \langle \phi^i, e^{m}k^n \rangle \) and \( \langle \kappa^j, e^{m}k^n \rangle \) by induction on \( j \) and then calculate \( \langle \phi^i \kappa^j, e^{m}k^n \rangle \) using (B.11) again, with \( \beta = \phi^i \) and \( \gamma = \kappa^j \).

Next, we must show that \( \phi^i \kappa^j \) are linearly independent for \( 0 \leq i \leq p-1 \) and \( 0 \leq j \leq 4p-1 \). Possible linear dependences are \( \sum_{i=0}^{p-1} \sum_{j=0}^{4p-1} \lambda_{ij} \phi^i \kappa^j = 0 \) with some \( \lambda_{ij} \in \mathbb{C} \), that is,

\[
\sum_{i=0}^{p-1} \sum_{j=0}^{4p-1} \lambda_{ij} \langle \phi^i \kappa^j, e^{m}k^n \rangle = 0
\]

for all \( 0 \leq m \leq p-1 \) and \( 0 \leq n \leq 4p-1 \). Using (B.12), we obtain the system of \( 4p^2 \) linear equations

\[
\sum_{i=0}^{p-1} \sum_{j=0}^{4p-1} \delta_{mi} \frac{(i)!}{(q-q^{-1})i} q^{-(i+2)n/2 - ij - i(i-1)} \lambda_{ij} = \\
= \frac{(m)!}{(q-q^{-1})m} q^{-m(m-1)} \sum_{j=0}^{4p-1} q^{-4j(n+2m)} \sum_{j=0}^{4p-1} \lambda_{mj} = 0
\]

for the \( 4p^2 \) variables \( \lambda_{ij} \). The system decomposes into \( p \) independent systems of \( 4p \) linear equations

\[
\sum_{j=0}^{4p-1} A_{jn} \lambda_{mj} = 0
\]
for $4p$ variables $\lambda_{mj}$, $0 \leq j \leq 4p - 1$ (with $m$ fixed), where $A_{jn} = q^{\frac{j}{4}} j(n+2m)$. The determinant of the matrix $A_{jn}$ is the Vandermonde determinant, which is nonzero because no two numbers among $(q^{\frac{j}{2}}(n+2m))_{0 \leq n \leq 4p-1}$ coincide.

With (B.12) established, we verify (B.5), (B.8), and (B.10).

Next, to verify (B.6), we write (B.3) for $x = k$ and $x = e$ as the respective relations

$$(B.13) \quad k\beta = \beta(k^{-1}k), \quad e\beta = -\beta(k^{-2}e) + \beta(k^{-2}e)k^2$$

valid for all $\beta \in B^*$. The following formulas are obtained by direct calculation using (B.12):

$$\kappa(k^{-1}k) = \kappa, \quad \kappa(k^{-2}e) = 0,$$

$$\phi(k^{-1}k) = q^{-1}\phi, \quad \phi(k^{-2}e) = \frac{\kappa^2}{q - q^{-1}},$$

$$\phi(k^{-2}e) = \phi, \quad \phi(k^{-2}e) = \frac{1}{q - q^{-1}}.$$

These relations and (B.13) imply (B.6), which finishes the proof. \hfill \Box

**B.2. The $R$-matrix.** As any Drinfeld double, $D(B)$ is a quasitriangular Hopf algebra, with the universal $R$-matrix given by

$$(B.14) \quad R = \sum_{m=0}^{p-1} \sum_{i=0}^{4p-1} e_{mi} \otimes f_{mi},$$

where $e_{mi}$ are elements of a basis in $B$ and $f_{ij} \in B^*$ are elements of the dual basis,

$$(B.15) \quad \langle f_{ij}, e_{mn} \rangle = \delta_{im}\delta_{jn}.$$ 

**B.2.1. Lemma.** For $D(B)$ constructed in [B.7] the dual basis is expressed in terms of the generators $\phi$ and $\kappa$ as

$$(B.16) \quad f_{ij} = \frac{(q - q^{-1})^i}{[i]!} q^{i(i-1)/2} \frac{1}{4p} \sum_{r=0}^{4p-1} q^{i(j+r)/2} \phi^i \kappa^r,$$

and therefore the $R$-matrix is given by

$$(B.17) \quad R = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{i,j=0}^{4p-1} (q - q^{-1})^m \frac{1}{m!} q^{m(m-1)/2 + m(i-j) - ij/2} e^m k^i \otimes \phi^m \kappa^{-j}.$$ 

**Proof.** By a direct calculation using (B.12), we verify that Eqs. (B.15) are satisfied with $f_{ij}$ given by (B.16). \hfill \Box
C.1. Verma and contragredient Verma modules. Let $s$ be an integer $1 \leq s \leq p - 1$ and $\alpha = \pm 1$. The Verma module $V^\alpha(s)$ has the basis

\[ \{x_k\}_{0 \leq k \leq s-1} \cup \{a_n\}_{0 \leq n \leq p-s-1}, \]

where $\{a_n\}_{0 \leq n \leq p-s-1}$ correspond to the submodule $X^{-\alpha}(p-s)$ and $\{x_k\}_{0 \leq k \leq s-1}$ correspond to the quotient module $X^\alpha(s)$ in

\[ 0 \to X^{-\alpha}(p-s) \to V^\alpha(s) \to X^\alpha(s) \to 0, \]

with the $\mathbb{U}_{q\mathfrak{sl}(2)}$-action given by

\[ Kx_k = \alpha q^{s-1-2k}x_k, \quad 0 \leq k \leq s-1, \]
\[ Ka_n = -\alpha q^{p-s-1-2n}a_n, \quad 0 \leq n \leq p-s-1, \]
\[ Ex_k = \alpha [k][s-k]x_{k-1}, \quad 0 \leq k \leq s-1 \quad (\text{with } x_{-1} \equiv 0), \]
\[ Ea_n = -\alpha [n][p-s-n]a_{n-1}, \quad 0 \leq n \leq p-s-1 \quad (\text{with } a_{-1} \equiv 0) \]

and

\[ Fx_k = \begin{cases} x_{k+1}, & 0 \leq k \leq s-2, \\ a_0, & k = s-1, \end{cases} \]
\[ Fa_n = a_{n+1}, \quad 0 \leq n \leq p-s-1 \quad (\text{with } a_{p-s} \equiv 0). \]

In addition, there are Verma modules $V^\pm(p) = X^\pm(p)$.

The contragredient Verma module $\tilde{V}^\alpha(s)$ is defined in the basis (C.1) by the same formulas except (C.3) and (C.4), replaced by the respective formulas

\[ Ex_k = \begin{cases} a_{p-s-1}, & k = 0, \\ \alpha [k][s-k]x_{k-1}, & 1 \leq k \leq s-1, \end{cases} \]
\[ Fx_k = x_{k+1}, \quad 0 \leq k \leq s-1 \quad (\text{with } x_s \equiv 0). \]

C.2. Projective modules. The module $P^\pm(s)$, $1 \leq s \leq p - 1$, is the projective module whose irreducible quotient is given by $X^\pm(s)$. The modules $P^\pm(s)$ appeared in the literature several times, see [18, 21, 22]. In explicitly describing their structure, we follow [22] most closely.
C.2.1. $\mathcal{P}^+(s)$. Let $s$ be an integer $1 \leq s \leq p - 1$. The projective module $\mathcal{P}^+(s)$ has the basis
\[
\{ (x_k^{(+,s)}, y_k^{(+,s)}) \}_{0 \leq k \leq p-s-1} \cup \{ a_n^{(+,s)}, b_n^{(+,s)} \}_{0 \leq n \leq s-1},
\]
where $\{ b_n^{(+,s)} \}_{0 \leq n \leq s-1}$ is the basis corresponding to the top module in (3.9), $\{ a_n^{(+,s)} \}_{0 \leq n \leq s-1}$ to the bottom, $\{ x_k^{(+,s)} \}_{0 \leq k \leq p-s-1}$ to the left, and $\{ y_k^{(+,s)} \}_{0 \leq k \leq p-s-1}$ to the right module, with the $\overline{U}_q\mathfrak{sl}(2)$-action given by
\[
K x_k^{(+,s)} = -q^{p-s-1-2k} x_k^{(+,s)}, \quad K y_k^{(+,s)} = -q^{p-s-1-2k} y_k^{(+,s)}, \quad 0 \leq k \leq p - s - 1,
\]
\[
K a_n^{(+,s)} = q^{s-1-2n} a_n^{(+,s)}, \quad K b_n^{(+,s)} = q^{s-1-2n} b_n^{(+,s)}, \quad 0 \leq n \leq s - 1,
\]
\[
E x_k^{(+,s)} = -[k][p - s - k] x_{k-1}^{(+,s)}, \quad 0 \leq k \leq p - s - 1 \quad (\text{with } x_{-1}^{(+,s)} \equiv 0),
\]
\[
E y_k^{(+)s} = \begin{cases} \end{cases} \end{align*}
\[
E a_n^{(+,s)} = [n][s-n] a_{n-1}^{(+,s)}, \quad 0 \leq n \leq s - 1 \quad (\text{with } a_{-1}^{(+,s)} \equiv 0),
\]
\[
E b_n^{(+,s)} = \begin{cases} \end{cases} \end{align*}
and
\[
F x_k^{(+,s)} = \begin{cases} \end{cases} \end{align*}
\[
F y_k^{(+,s)} = y_{k+1}^{(+,s)}, \quad 0 \leq k \leq p - s - 1 \quad (\text{with } y_{p-s}^{(+,s)} \equiv 0),
\]
\[
F a_n^{(+,s)} = a_{n+1}^{(+,s)}, \quad 0 \leq n \leq s - 1 \quad (\text{with } a_{s}^{(+,s)} \equiv 0),
\]
\[
F b_n^{(+,s)} = \begin{cases} \end{cases} \end{align*}
C.2.2. $\mathcal{P}^-(p - s)$. Let $s$ be an integer $1 \leq s \leq p - 1$. The projective module $\mathcal{P}^-(p - s)$ has the basis
\[
\{ (x_k^{(-,s)}, y_k^{(-,s)}) \}_{0 \leq k \leq p-s-1} \cup \{ a_n^{(-,s)}, b_n^{(-,s)} \}_{0 \leq n \leq s-1},
\]
where $\{ y_k^{(-,s)} \}_{0 \leq k \leq p-s-1}$ is the basis corresponding to the top module in (3.9), $\{ x_k^{(-,s)} \}_{0 \leq k \leq p-s-1}$ to the bottom, $\{ a_n^{(-,s)} \}_{0 \leq n \leq s-1}$ to the left, and $\{ b_n^{(-,s)} \}_{0 \leq n \leq s-1}$ to the right module, with the $\overline{U}_q\mathfrak{sl}(2)$-action given by
\[
K x_k^{(-,s)} = -q^{p-s-1-2k} x_k^{(-,s)}, \quad K y_k^{(-,s)} = -q^{p-s-1-2k} y_k^{(-,s)}, \quad 0 \leq k \leq p - s - 1,
\]
\[
K a_n^{(-,s)} = q^{s-1-2n} a_n^{(-,s)}, \quad K b_n^{(-,s)} = q^{s-1-2n} b_n^{(-,s)}, \quad 0 \leq n \leq s - 1,
\]
\[
E x_k^{(-,s)} = -[k][p - s - k] x_{k-1}^{(-,s)}, \quad 0 \leq k \leq p - s - 1 \quad (\text{with } x_{-1}^{(-,s)} \equiv 0),
\]
\[
E y_k^{(-,s)} = \begin{cases} \end{cases} \end{align*}
\[
E a_n^{(-,s)} = [n][s-n] a_{n-1}^{(-,s)}, \quad 0 \leq n \leq s - 1 \quad (\text{with } a_{-1}^{(-,s)} \equiv 0),
\]
\[
E b_n^{(-,s)} = \begin{cases} \end{cases} \end{align*}
It follows that (also cf. [11]; we are somewhat more explicit about the representation-theory side, based on the analysis in polynomials eigenspaces of $\pi$ in $\pi$). Canonical central elements.

Second, we recall polynomial relation (3.5) for the Casimir element and define the projectors

$$
E\psi_k^{(-s)} = \begin{cases} 
-\lfloor k \rfloor [p - s - k] \psi_{k-1}^{(-s)} + \chi_{k-1}^{(-s)}, & 1 \leq k \leq p - s - 1, \\
\alpha_{s-1}^{(-s)}, & k = 0,
\end{cases}
$$

$$
E\alpha_n^{(-s)} = [n] [s - n] \alpha_{n-1}^{(-s)}, \quad 0 \leq n \leq s - 1 \quad \text{(with } \alpha_{-1}^{(-s)} \equiv 0),
$$

$$
E\beta_n^{(-s)} = \begin{cases} 
[n] [s - n] \beta_{n-1}^{(-s)}, & 1 \leq n \leq s - 1,
\chi_{p-s-1}^{(-s)}, & n = 0,
\end{cases}
$$

and

$$
F\psi_k^{(-s)} = \psi_{k+1}^{(-s)}, \quad 0 \leq k \leq p - s - 1 \quad \text{(with } \psi_{p-s}^{(-s)} \equiv 0),
$$

$$
F\alpha_n^{(-s)} = \begin{cases} 
\alpha_{n+1}^{(-s)}, & 0 \leq n \leq s - 2,
\chi_0^{(-s)}, & n = s - 1,
\end{cases}
$$

$$
F\beta_n^{(-s)} = \beta_{n+1}^{(-s)}, \quad 0 \leq n \leq s - 1 \quad \text{(with } \beta_{s}^{(-s)} \equiv 0).
$$

APPENDIX D. CONSTRUCTION OF THE CANONICAL CENTRAL ELEMENTS

D.1. Canonical central elements. To explicitly construct the canonical central elements in $\mathfrak{u}_q sl(2)$ in terms of the $\mathfrak{u}_q sl(2)$ generators, we use the standard formulas in [42, Ch. V.2] (also cf. [11]; we are somewhat more explicit about the representation-theory side, based on the analysis in $\pi$). We first introduce projectors $\pi^+_s$ and $\pi^-_s$ on the direct sums of the eigenspaces of $K$ appearing in the respective representations $X^+(s)$ and $X^-(p - s)$ for $1 \leq s \leq p - 1$, Eqs. (3.7) and (3.8). These projectors are

$$
\pi^+_s = \frac{1}{2p} \sum_{n=0}^{s-1} \sum_{j=0}^{2p-1} q^{(2n-s+1)j} K^j, \quad \pi^-_s = \frac{1}{2p} \sum_{n=s}^{p-1} \sum_{j=0}^{2p-1} q^{(2n-s+1)j} K^j.
$$

It follows that

$$
\pi^+_s + \pi^-_s = \frac{1}{2} (1 - (-1)^s K^p).
$$

Second, we recall polynomial relation (3.5) for the Casimir element and define the polynomials

$$
\psi_0(x) = (x - \beta_p) \prod_{r=1}^{p-1} (x - \beta_r)^2,
$$

$$
\psi_s(x) = (x - \beta_0) (x - \beta_p) \prod_{r=1}^{p-1} (x - \beta_r)^2, \quad 1 \leq s \leq p - 1,
$$

$$
\psi_p(x) = (x - \beta_0) \prod_{r=1}^{p-1} (x - \beta_r)^2,
$$

$$
\psi_p(x) = (x - \beta_0) \prod_{r=1}^{p-1} (x - \beta_r)^2,
$$

$$
\psi_p(x) = (x - \beta_0) \prod_{r=1}^{p-1} (x - \beta_r)^2,
$$

$$
\psi_p(x) = (x - \beta_0) \prod_{r=1}^{p-1} (x - \beta_r)^2.
$$
where we recall that $\beta_j = \frac{q^j + q^{-j}}{(q - q^{-1})^2}$, with $\beta_j \neq \beta_{j'}$ for $0 \leq j \neq j' \leq p$.

**D.1.1. Proposition.** The canonical central elements $e_s$, $0 \leq s \leq p$, and $w_s$, $1 \leq s \leq p - 1$, are explicitly given as follows. The elements in the radical of $\mathfrak{z}$ are

(D.3)  
\[ w_s^\pm = \pi_s^\pm w_s, \quad 1 \leq s \leq p - 1, \]

where

(D.4)  
\[ w_s = \frac{1}{\psi_s(\beta_s)} (C - \beta_s) \psi_s(C). \]

The canonical central idempotents are given by

(D.5)  
\[ e_s = \frac{1}{\psi_s(\beta_s)} (\psi_s(C) - \psi_s'(\beta_s) w_s), \quad 0 \leq s \leq p, \]

where we formally set $w_0 = w_p = 0$.

**Proof.** First, $(C - \beta_s)\psi_s(C)$ acts by zero on $Q(0) = \mathfrak{X}^-(p) \boxtimes \mathfrak{X}^-(p)$ and $Q(p) = \mathfrak{X}^+(p) \boxtimes \mathfrak{X}^+(p)$. We next consider its action on $Q(s)$ for $1 \leq s \leq p - 1$. It follows from (4.12) that the Casimir element acts on the basis of $\mathcal{P}^+(s)$ as

(D.6)  
\[ \begin{align*}
    C b_n^{(+,s)} &= \beta_s b_n^{(+,s)} + a_n^{(+,s)}, \\
    C x_n^{(+,s)} &= \beta_s x_n^{(+,s)}, \\
    C y_n^{(+,s)} &= \beta_s y_n^{(+,s)}, \\
    C a_n^{(+,s)} &= \beta_s a_n^{(+,s)}
\end{align*} \]

for all $0 \leq n \leq s - 1$. Clearly, $(C - \beta_s)^2$ annihilates the entire $\mathcal{P}^+(s)$, and therefore $(C - \beta_r)\psi_r(C)$ acts by zero on each $Q(s)$ with $s \neq r$. On the other hand, for $s = r$, we have

\[ (C - \beta_r)\psi_r(C)b_n^{(+,r)} = \psi_r(C)a_n^{(+,r)} = \psi_r(\beta_r)a_n^{(+,r)}. \]

Similar formulas describe the action of the Casimir element on the module $\mathcal{P}^-(p - s)$. It thus follows that $w_r$ sends the quotient of the bimodule $Q(r)$ in (4.13), i.e., $\mathfrak{X}^+(r) \boxtimes \mathfrak{X}^+(r) \oplus \mathfrak{X}^-(p-r) \boxtimes \mathfrak{X}^-(p-r)$, into the submodule $\mathfrak{X}^+(r) \boxtimes \mathfrak{X}^+(r) \oplus \mathfrak{X}^-(p-r) \boxtimes \mathfrak{X}^-(p-r)$ at the bottom of $Q(r)$. Therefore, $w_r = \text{const} \cdot (w_r^+ + w_r^-)$.

To obtain $w_r^+$ and $w_r^-$, we multiply $w_r$ with the respective operators projecting on the direct sums of the eigenspaces of $K$ occurring in $\mathfrak{X}^+(s)$ and $\mathfrak{X}^-(p-r)$. This gives (D.3) (the reader may verify independently that although the projectors $\pi^\pm_r$ are not central, their products with $w_r$ are). The normalization in (D.4) is chosen such that we have $w_r b_n^{(+,r)} = a_n^{(+,r)}$.

To obtain the idempotents $e_r$, we note that $\psi_r(C)$ annihilates all $Q(s)$ for $s \neq r$, while on $Q(r)$, we have $\psi_r(C)x_n^{(+,r)} = \psi_r(\beta_r)x_n^{(+,r)}$, $\psi_r(C)y_n^{(+,r)} = \psi_r(\beta_r)y_n^{(+,r)}$, $\psi_r(C)a_n^{(+,r)} = \psi_r(\beta_r)a_n^{(+,r)}$, and furthermore, by Taylor expanding the polynomial,

\[ \psi_r(C)b_n^{(+,r)} = \psi_r(\beta_r)b_n^{(+,r)} + (C - \beta_r)\psi_r(\beta_r)b_n^{(+,r)}, \]
with higher-order terms in $(C - \beta_r)$ annihilating $b_n^{(r,s)}$. Similar formulas hold for the action on $P^-(p-s)$. Therefore, $Q(r)$ is the root space of $\frac{1}{\psi_n(\beta_r)} \psi_n(C)$ with eigenvalue 1, and the second term in (D.5) is precisely the subtraction of the nondiagonal part.

D.2. Remarks.

(1) We note that $w_s^+ + w_s^- = w_s$. This follows because $(1 + (-1)^s K^p) w_s = 0$.

(2) For any polynomial $R(C)$, decomposition (4.18) takes the form

\[
R(C) = \sum_{s=0}^{p} R(\beta_s) e_s + \sum_{s=1}^{p-1} R'(\beta_s) w_s.
\]

For example, (D.7) implies that for $\hat{C}$ defined in 3.1.3 we have

\[
\hat{C} = \sum_{s=0}^{p} (q^s + q^{-s}) e_s + (q - q^{-1})^2 \sum_{s=1}^{p-1} w_s.
\]

D.3. Eigenmatrix of the $(1, p)$ fusion algebra. Using (D.7) and expressions through the Chebyshev polynomials in 3.3.7, we recover the eigenmatrix $P$ of the fusion algebra (1.1). This eigenmatrix was obtained in [15] by different means, from the matrix of the modular $S$-transformation on $W(p)$-characters. The eigenmatrix relates the preferred basis (the basis of irreducible representations) and the basis of idempotents and nilpotents in the fusion algebra. Specifically, if we order the irreducible representations as

\[
X^t \equiv (X^+(p), X^-(p), X^+(1), X^-(p-1), \ldots, X^+(p-1), X^-(1))
\]

and the idempotents and nilpotents that form a basis of $D_{2} \equiv \mathfrak{g}_{2}$ as

\[
Y^t \equiv (e_p, e_0, e_1, w_1, \ldots, e_{p-1}, w_{p-1}),
\]

then the eigenmatrix $P(p)$ is defined as

\[
X = P(p) Y.
\]

The calculation of the entries of $P(p)$ via (D.7) is remarkably simple: for example, with $R(\hat{C})$ taken as $U_s(\hat{C})$ (see 3.3.7), we have

\[
R(\hat{\beta}_j) = R(2 \cos \frac{\pi j}{p}) = \sin \frac{\pi j s}{p} \sin \frac{\pi j}{p}
\]

in accordance with 3.13. Evaluating the other case in (3.15) similarly and taking the derivatives, we obtain the eigenmatrix

\[
P(p) = \begin{pmatrix}
P_{0,0} & P_{0,1} & \ldots & P_{0,p-1} \\
P_{1,0} & P_{1,1} & \ldots & P_{1,p-1} \\
\vdots & \vdots & \ddots & \vdots \\
P_{p-1,0} & P_{p-1,1} & \ldots & P_{p-1,p-1}
\end{pmatrix}
\]
with the $2 \times 2$ blocks $^{[15]}^2$

$$P_{0,0} = \begin{pmatrix} p & (-1)^{p+1} p \\ p & -p \end{pmatrix}, \quad P_{0,j} = \begin{pmatrix} 0 & -(-1)^{j+p} \frac{2\lambda_j}{p} \sin \frac{j\pi}{p} \\ 0 & -\frac{2\lambda_j}{p} \sin \frac{j\pi}{p} \end{pmatrix},$$

$$P_{s,0} = \begin{pmatrix} s & (-1)^{s+1} s \\ p-s & (-1)^{s+1}(p-s) \end{pmatrix},$$

$$P_{s,j} = (-1)^s \begin{pmatrix} -\frac{\sin \frac{sj\pi}{p}}{\sin \frac{j\pi}{p}} & \frac{2\lambda_j}{p^2} \left(-s \cos \frac{sj\pi}{p} \sin \frac{j\pi}{p} + \sin \frac{sj\pi}{p} \cos \frac{j\pi}{p}\right) \\ \frac{\sin \frac{sj\pi}{p}}{\sin \frac{j\pi}{p}} & \frac{2\lambda_j}{p^2} \left(-(p-s) \cos \frac{sj\pi}{p} \sin \frac{j\pi}{p} - \sin \frac{sj\pi}{p} \cos \frac{j\pi}{p}\right) \end{pmatrix},$$

for $s, j = 1, \ldots, p-1$, where, for the sake of comparison, we isolated the factor

$$\lambda_j = \frac{p^2}{|j|^3 \sin \frac{\pi}{p}} = \frac{p^2 \left(\sin \frac{\pi}{p}\right)^2}{\left(\sin \frac{\pi}{p}\right)^3},$$

whereby the normalization of each nilpotent element, and hence of each even column of $P$ starting with the fourth, differs from the normalization chosen in $^{[15]}$ (both are arbitrary because the nilpotents cannot be canonically normalized).

### Appendix E. Derivation of the $q$-binomial identity

We derive identity (1.3) from the fusion algebra realized on the central elements $\chi^\pm(s)$. In view of $^{[A.6.2]}$, the central elements $\chi^\alpha(s)$ in (4.6) (with $\alpha = \pm 1$, $s = 1, \ldots, p$) satisfy the algebra

$$(E.1) \quad \chi^\alpha(s) \chi^\alpha'(s') = \sum_{s''=|s-s'|+1}^{s+s'-1} \tilde{\chi}^{\alpha\alpha'}(s''),$$

where

$$\tilde{\chi}^\alpha(s) = \begin{cases} \chi^\alpha(s), & 1 \leq s \leq p, \\ \chi^\alpha(2p - s) + 2\chi^{-\alpha}(s - p), & p + 1 \leq s \leq 2p - 1. \end{cases}$$

We now equate the coefficients at the respective PBW-basis elements in both sides of (E.1). Because of (4.8), it suffices to do this for the algebra relation for $\chi^+(s) \chi^+(s')$. Writing it as in (3.12), we have

\[2\text{The formula for } P_{0,j} \text{ corrects a misprint in } ^{[15]}, \text{ where } (-1)^{j+p} \text{ occurred in a wrong matrix entry.} \]
We first calculate the right-hand side. Simple manipulations with $q$-binomial coefficients show that

$$\chi^+(s) + \chi^-(p-s) = (-1)^{s+1} \sum_{n=0}^{p-1} \sum_{m=0}^{p-1} (q - q^{-1})^{2m} q^{-(m+1)(m+s-1-2n)}$$

$$\times \left[ s + m - n - 1 \right] \left[ \binom{m}{n} \right] E^m F^m K^{s-1-2n+m},$$

where

$$\binom{m}{n} = \begin{cases} 0, & n < 0, \\ \frac{[m - n + 1] \ldots [m]}{[n]!}, & \text{otherwise}, \end{cases}$$

which leads to

$$\text{r.-h. s. of (E.2)} = (-1)^{s+s'} \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \sum_{\ell=0}^{\min(s,s')-1} (q - q^{-1})^{2m} q^{-(m+1)(s+s'-2-2\ell)}$$

$$\times \left[ s + s' - 2 - \ell - n + m \right] \left[ \binom{m}{n} \right] E^m F^m K^{s+s'-2-2n+m}.\]$$

Changing the order of summations, using that the $q$-binomial coefficients vanish in the cases specified in (1.4), and summing over even and odd $m$ separately, we have

$$\text{r.-h. s. of (E.2)} = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \sum_{\ell=0}^{\min(s+s' - 1, s'-1)} (q - q^{-1})^{2m} q^{-(m+1)(s+s'-2-2\ell)}$$

$$\times \left[ s + s' - 2 - \ell - n + m \right] \left[ \binom{m}{n} \right] E^m F^m K^{s+s'-2-2n+} +$$

$$+ \sum_{m=1}^{p-1} \sum_{n=0}^{p-1} \sum_{\ell=0}^{\min(s+s'-1, s'-1)} (q - q^{-1})^{2m} q^{-(m+1)(s+s'-2n-1)}$$

$$\times \left[ s + s' - 2 - \ell - n + m \right] \left[ \binom{m}{n} \right] E^m F^m K^{s+s'-2n-1}.\]$$

Next, in the left-hand side of (E.2), we use that $\chi^+(s)$ are central and readily calculate

$$\text{l.-h. s. of (E.2)} = (-1)^{s+1} \sum_{n=0}^{s-1} \sum_{m=0}^{n} (q - q^{-1})^{2m} q^{-(m+1)(m+s-1-2n)}$$

$$\times \left[ s - n + m - 1 \right] \left[ \binom{m}{n} \right] E^m \chi^+(s') F^m K^{s-1-2n+m} =$$
obtain the coefficients (see (1.4)), the summations over the identity and because
\[ \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} q^{2mi+j(2n+2-s-s')} \left[ -n - i \right] \left[ i + j + s - 1 - n \right] \left[ m - i - j - 1 + s' \right] = \]
\[ = q^{m(2n+1-s)} \sum_{\ell=0}^{\min(s-1,s')-1} \left[ n - \ell \right] \left[ m + s + s' - 2 - \ell - n \right], \]
where 1 \leq m \leq p - 1, n \in \mathbb{Z}_{2p}, 1 \leq s, s' \leq p. Because of the vanishing of q-binomial coefficients (see (1.4)), the summations over \( j \) and \( i \) in the left-hand side can be extended to \( \mathbb{Z} \times \mathbb{Z} \), which gives (1.3) after the shifts \( s \rightarrow s + 1, s' \rightarrow s' + 1 \). In the above derivation, \( q \) was the 2p-th primitive root of unity, but because \( p \) does not explicitly enter the resultant identity and because q-binomial coefficients are (Laurent) polynomials in \( q \), we conclude that (1.3) is valid for all \( q \).

REFERENCES

[1] D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras*, I, J. Amer. Math. Soc. 6 (1993) 905–947; II, J. Amer. Math. Soc. 6 (1993) 949–1011; III, J. Amer. Math. Soc. 7 (1994) 335–381; IV, J. Amer. Math. Soc. 7 (1994) 383–453.

[2] G. Moore and N. Seiberg, *Lectures on RCFT*, in: Physics, Geometry, and Topology (Trieste spring school 1989), p. 263; Plenum (1990).

[3] M. Finkelberg, *An equivalence of fusion categories*, Geometric and Functional Analysis (GAFA) 6 (1996) 249–267.

[4] V.G. Turaev, Quantum Invariants of Knots and 3-Manifolds, Walter de Gruyter, Berlin–New York (1994).
[5] V. Lyubashenko, *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. 172 (1995) 467–516 [hep-th/9405167]; *Modular properties of ribbon abelian categories*, Symposia Gaussiana, Proc. of the 2nd Gauss Symposium, Munich, 1993. Conf. A (Berlin, New York), Walter de Gruyter, (1995) 529–579 [hep-th/9405168]; *Modular Transformations for Tensor Categories*, J. Pure Applied Algebra 98 (1995) 279–327.

[6] V. Lyubashenko and S. Majid, *Braided groups and quantum Fourier transform*, J. Algebra 166 (1994) 506–528.

[7] H.G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. B 259 (1991) 448.

[8] M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. B 386 (1996) 131 [hep-th/9606030].

[9] M.A.I. Flohr, *On modular invariant partition functions of conformal field theories with logarithmic operators*, Int. J. Mod. Phys. A11 (1996) 4147 [hep-th/9509166].

[10] M. Flohr, *On Fusion Rules in Logarithmic Conformal Field Theories*, Int. J. Mod. Phys. A12 (1997) 1943–1958 [hep-th/9605151].

[11] T. Kerler, *Mapping class group action on quantum doubles*, Commun. Math. Phys. 168 (1995) 353–388 [hep-th/9402017].

[12] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press (1994).

[13] M.R. Gaberdiel and H.G. Kausch, *Indecomposable fusion products*, Nucl. Phys. B 477 (1996) 293 [hep-th/9604026].

[14] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B 300 (1988) 360.

[15] J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. 247 (2004) 713–742 [hep-th/0306273].

[16] V. Gurarie and A.W.W. Ludwig, *Conformal field theory at central charge c = 0 and two-dimensional critical systems with quenched disorder*, [hep-th/0409105]

[17] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT*, math.QA/0512621

[18] N.Yu. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys., 127 (1990) 1–26.

[19] A. Lachowska, *On the center of the small quantum group*, math.QA/0107098

[20] V. Ostrik, *Decomposition of the adjoint representation of the small quantum sl2*, Commun. Math. Phys. 186 (1997) 253–264 [q-alg/9512026].

[21] D.V. Gluschenkov and A.V. Lyakhovskaya, *Regular representation of the quantum Heisenberg double \( \{ U_q(\mathfrak{sl}(2)), Fun_q(\mathfrak{sl}(2)) \} (q \text{ is a root of unity})*, hep-th/9311075

[22] M. Jimbo, T. Miwa, and Y. Takeyama, *Counting minimal form factors of the restricted sine-Gordon model*, math-ph/0303059

[23] M.R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory* [hep-th/0111260].

[24] M. Flohr, *Bits and Pieces in Logarithmic Conformal Field Theory*, Int.J.Mod.Phys. A18 (2003) 4497–4592 [hep-th/0111228]

[25] V. Gurarie, *Logarithmic operators in conformal field theory*, Nucl. Phys. B 410 (1993) 535 [hep-th/9303160].

[26] F. Rohsiepe, *Nichtrütige Darstellungen der Virasoro-Algebra mit nichttrivialen Jordanblöcken*, Diploma Thesis, Bonn (1996) [BONN-IB-96-19].

[27] J. Fjelstad, J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic conformal field theories via logarithmic deformations*, Nucl. Phys. B633 (2002) 379 [hep-th/0201091].
[28] A.M. Semikhatov, A. Taormina, and I.Yu. Tipunin, Higher-level Appell functions, modular transformations, and characters, math.QA/0311314.
[29] V.G. Kač, Infinite Dimensional Lie Algebras, Cambridge University Press, 1990.
[30] J. Fuchs, Affine Lie algebras and quantum groups, Cambridge University Press, Cambridge, 1992.
[31] N.Yu. Reshetikhin and M.A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, J. Geom. Phys. 5 (1988) 533–550.
[32] B. Bakalov and A.A. Kirillov, Lectures on Tensor Categories and Modular Functors, AMS (2001).
[33] J. Fuchs, I. Runkel, and C. Schweigert, TFT construction of RCFT correlators I: Partition functions, Nucl. Phys. B 646 (2002) 353 [hep-th/0204148].
[34] J. Fuchs, I. Runkel, and C. Schweigert, TFT construction of RCFT correlators II: Unoriented world sheets, Nucl. Phys. B 678 (2004) 511–637 [hep-th/0306164].
[35] T. Kerler and V.V. Lyubashenko, Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners, Springer Lecture Notes in Mathematics 1765, Springer Verlag (2001).
[36] R.G. Larson and M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, Amer. J. Math., 91 (1969) 75–94.
[37] D.E. Radford, The order of antipode of a finite-dimensional Hopf algebra is finite, Amer. J. Math 98 (1976) 333–335.
[38] V.G. Drinfeld, On Almost Cocommutative Hopf Algebras, Leningrad Math. J. 1 (1990) No.2, 321–342.
[39] C. Kassel, Quantum Groups, Springer-Verlag, New York, 1995.
[40] M.E. Sweedler, Hopf Algebras, Benjamin, New York, NY (1969).
[41] D.E. Radford, The trace function and Hopf algebras, J. Alg. 163 (1994) 583–622.
[42] F.R. Gantmakher, Teoriya Matrits [in Russian], Nauka, Moscow (1988).

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