Universal $R$–matrix for null–plane quantized
Poincaré algebra

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Abstract

The universal $R$–matrix for a quantized Poincaré algebra $\mathcal{P}(3 + 1)$ introduced by Ballesteros et al is evaluated. The solution is obtained as a specific case of a formulated multidimensional generalization to the non–standard (Jordanian) quantization of $sl(2)$.

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1 Introduction

Recently Ballesteros et al built a quantum deformation of the Poincaré algebra \[1\]. The quantization found was generated by a triangular classical \(r\)-matrix and, according to Drinfeld’s theory \[2\], should be a twisting of \(U(P(3 + 1))\). An explicit form of solution \(F\) to the twist equation and universal matrix \(R = \tau(F^{-1})F\) were not given. This problem is solved in the present paper. To know twisting element \(F\) is very important because it deforms not only the symmetry algebra but the geometry of the space–time as well. Twisting a universal enveloping algebra induces coherent transformations in modules and allows to obtain important objects automatically, for example, to construct invariant equations and their solutions. In order to solve the problem we resort to the theory of quantizing Lie algebras with quasi–Abelian dual groups (semidirect product of two Abelian subgroups) \[3, 4, 5\]. In the present communication we consider first a class of algebras along the line of that theory. That class may be regarded as a direct generalization of the triangular or non-standard deformation of the Borel subalgebra in \(sl(2)\) \[6, 7\]. We find general expression for twisting elements and universal \(R\)–matrices and then apply the developed technique to the specific case of \(P(3 + 1)\).

2 General consideration

Let \(L = \mathbf{H} \circledS \mathbf{V}\) be a Lie algebra splitting into a semidirect sum of its two Abelian subalgebras, with the basic elements \(H_i \in \mathbf{H}\) and \(X_\mu \in \mathbf{V}\):

\[
[H_j, X_\mu] = B^i_{\mu j} H_i
\]

Suppose that its dual algebra \(L^*\) has the structure of a semidirect sum \(\mathbf{H}^* \circledS \mathbf{V}^*\) as well, which is defined via a commutative set of matrices \((\alpha^i)^\mu_\nu\). To match the consistency condition upon \(L\) and \(L^*\) it is necessary to require

\[
(\alpha^i)^\mu_\nu B^j_{\mu k} = (\alpha^j)^\mu_\nu B^i_{k \mu}.
\]
There exists a quantization $U_\alpha(L)$ of the universal enveloping algebra $U(L)$ with the relations on the generators [4]

$$[H_j, X_\mu] = \left(\frac{e^{2\alpha \cdot H} - I}{2\alpha \cdot H}\right)_\mu^\nu B^i_j H_i$$ \hfill (1)

and the coproduct

$$\Delta_\alpha(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta_\alpha(X_\mu) = (\epsilon^{2\alpha \cdot H})^\nu_\mu X_\nu + X_\mu \otimes 1.$$ \hfill (2)

Symbol $I$ stands for the unity matrix $I^\mu_\nu = \delta^\mu_\nu$, and $\alpha \cdot H$ means $\sum_i \alpha^i H_i$. The apparent counit is $\varepsilon(X_\mu) = \varepsilon(H_i) = 0$, and the antipode may be readily found from the coproduct with the use of the defining axioms. Its explicit form is irrelevant for our study.

We are interested only in such $U_\alpha(L)$ which are obtained by twisting classical universal enveloping algebras $U(L)$. To find the explicit form of element $F \in U(L) \otimes U(L)$ governing that process and the universal $R$–matrix of algebra $U_\alpha(L)$ is the goal of our investigation. Unexpectedly, it appears easier to start from $U_\alpha(L)$ rather than from the classical algebra, find a solution $\Phi$ to the twist equation, and then return to $U(L)$ (we are going to use the group properties of twisting and $F = \Phi^{-1}$ in particular [8]).

We will seek a solution to the twist equation

$$(\Delta_\alpha \otimes id)(\Phi)\Phi_{12} = (id \otimes \Delta_\alpha)(\Phi)\Phi_{23}$$ \hfill (3)

in the form

$$\Phi = \exp(r^{ij} H_i \otimes X_\mu) \in U_\alpha(L) \otimes U_\alpha(L).$$

The classical skew–symmetric $r$–matrix then will be $r = r^{ij}(X_\mu \otimes H_i - H_i \otimes X_\mu)$, and matrices $\alpha^i$ will be expressed through the structure constants of $L$ by the formula

$$(\alpha^i)^\mu_\nu = \frac{1}{2} r^{j\mu} B^i_j.$$ \hfill (4)

Without any loss of generality we suppose $r$ to be non–degenerate, since otherwise we may restrict ourselves with the image $r(L^*) = r^i(L^*) \subset L$, which is a subalgebra in $L$, and twisting a subalgebra induces that of whole $L$. 2
Calculating both sides of equation (3)

\[(\Delta \otimes id)(e^{r^{\mu}H_i \otimes X_{\mu}})e^{r^{\mu}H_i \otimes X_{\mu} \otimes 1} = e^{r^{\mu}(H_i \otimes 1 \otimes X_{\mu} + 1 \otimes H_i \otimes X_{\mu})}e^{r^{\mu}H_i \otimes X_{\mu} \otimes 1} =
\]

\[= e^{r^{\mu}H_i \otimes 1 \otimes X_{\mu}}e^{r^{\mu}H_i \otimes X_{\mu} \otimes 1 + r^{\mu \nu}r^{\nu}H_i \otimes [H_j, X_{\mu}] \otimes X_{\nu}} \times
\]

\[(id \otimes \Delta_{\alpha})(e^{r^{\mu}H_i \otimes X_{\mu}})e^{r^{\mu}1 \otimes H_i \otimes X_{\mu}} = e^{r^{\mu}(H_i \otimes (e^{2\alpha \cdot H})_{\mu} \otimes X_{\nu} + H_i \otimes X_{\mu} \otimes 1)}e^{r^{\mu}1 \otimes H_i \otimes X_{\mu}},
\]

and then comparing them with each other, taking into account commutation properties of the generators \(H_i\) and \(X_{\mu}\) we come to condition

\[r^{\mu}H_i \otimes 1 \otimes X_{\mu} + r^{\mu}r^{\nu}H_i \otimes [H_j, X_{\mu}] \otimes X_{\nu} = r^{\mu}H_i \otimes (e^{2\alpha \cdot H})_{\mu} \otimes X_{\nu},
\]

which, in its turn, yields \(r^{\mu}(\delta_{\mu} + r^{\nu}B_{\mu}(H)) = r^{\mu}(e^{2\alpha \cdot H})_{\mu}\). The latter is fulfilled provided that \(r^{\mu}B_{\mu}(H) = (e^{2\alpha \cdot H})_{\mu} - \delta_{\mu}\) which holds in view of (3) and (4).

Our next goal is to show that formula \(\Delta(h) = \Phi^{-1}\Delta_{\alpha}(h)\Phi, h \in U_{\alpha}(L)\), defines the classical comultiplication on universal enveloping algebra \(U(L)\). Due to non–degeneracy of \(r\)–matrix, we can lift and drop indices: \(H^{\mu} = r^{\mu}H_i\), \((\alpha_{\mu})_{\nu} = \alpha_{\mu \nu} = r_{\mu}(\alpha^i)_{\nu}\). Matrix \(r_{\mu}\) is the inverse to \(r^{\mu}\): \(r^{\mu}r_{\mu} = \delta_{\mu}^{\nu}\). The relations in \(U_{\alpha}(L)\) take the form

\[[H^{\mu}, X_{\nu}] = (e^{2\alpha \cdot H} - I)_{\mu}^{\nu}.
\]

With a set of numbers \(\xi^{\nu}\) fixed, let us introduce entities \(K^{\mu}\) defining them as

\[K^{\mu} = \xi^{\nu}(I - e^{-2\alpha \cdot H})_{\mu}^{\nu},
\]

and evaluate commutation relations between \(K^{\mu}\) and \(X_{\nu}\):

\[[K^{\mu}, X_{\nu}] = \xi^{\beta}(e^{-2\alpha \cdot H})_{\beta}^{\nu}(2\alpha_{\mu \sigma})_{\nu}(e^{2\alpha \cdot H} - I)_{\nu}^{\sigma}.
\]

From commutativity of matrices \(\alpha_{\mu}\) and condition \(\alpha_{\mu \nu} = \alpha_{\mu \nu}\) following from the classical Yang–Baxter equation we find \(\alpha_{\mu \sigma}(e^{2\alpha \cdot H} - I)\sigma = \alpha_{\mu \sigma}(e^{2\alpha \cdot H} - I)\sigma\). This is verified by a simple induction over powers of matrix \((\alpha \cdot H)\). Finally, we have

\[[K^{\mu}, X_{\nu}] = 2\alpha_{\sigma \nu}K^{\sigma},
\]
i.e. the classical commutation relations. The coproduct on the new generators is
\[ \Delta_{\alpha}(K^\mu) = K^\mu \otimes 1 + (e^{-2\alpha \cdot H})^\mu_\nu K^\nu. \]

With the use of this formula we calculate twisted coproduct which comes out to be
\[ \Delta(K) = e^{-H \otimes X} \Delta_{\alpha}(K) e^{H \otimes X} =
= K \otimes 1 + (e^{2\alpha \cdot H} e^{-2\alpha \cdot H}) \otimes K = K \otimes 1 + 1 \otimes K,
\]
\[ \Delta(X) = e^{-H \otimes X} \Delta_{\alpha}(X) e^{H \otimes X} = e^{2\alpha \cdot H} \otimes X + e^{-H \otimes X} (X \otimes 1) e^{H \otimes X} =
= e^{2\alpha \cdot H} \otimes X + X \otimes 1 - (e^{2\alpha \cdot H} - I) \otimes X = 1 \otimes X + X \otimes 1. \quad (6) \]

We might consider our goal achieved were we sure that the number of independent generators \( K^\mu \) be the same as the dimensionality of space \( \mathbf{H} \). That is not the case in general, and it is determined by a particular choice of \( \xi^\mu \). In the classical limit we have
\[ K^\mu = \xi^\nu (2 \alpha \cdot H)^\nu_\mu = [H^\mu, \xi^\nu X_\nu]. \]

Thus, while \( \xi^\mu \) takes all possible values lineal \( \text{Span}(K^\mu) \) fills up the subspace \( \mathbf{H}' = [\mathbf{H}, \mathbf{V}] \subset \mathbf{H} \). If that subspace coincides with whole \( \mathbf{H} \) we may state that, indeed, twisting \( U_\alpha(\mathbf{L}) \) with element \( \Phi \) results in \( U(\mathbf{L}) \). Let us show that \( \dim(\mathbf{H}') < \dim(\mathbf{H}) \) if and only if subalgebra \( \mathbf{V} \) and the center of \( \mathbf{L} \) have a nontrivial intersection. Indeed, because of non–degeneracy of the classical \( r \)–matrix there exists an isomorphism between linear spaces \( \mathbf{V}^\ast \) and \( \mathbf{H} \) (basic elements \( H^\mu \) and \( X_\mu \) turn out to be mutually dual). Subspace \( \mathbf{H}' \) is less than \( \mathbf{H} \) if and only if there is an element \( X_0 = \xi^\mu_0 X_\mu \in \mathbf{H}^\ast \), orthogonal to entire \( \mathbf{H}' \):
\[ 0 = (X_0, [H^\mu, X_\nu]) = 2\xi^\sigma_0 \alpha^\mu_{\sigma\nu}. \]

Due to the lower indices symmetry of tensor \( \alpha^\mu_{\sigma\nu} \), the latter expression is nothing else than the matrix of the adjoint representation \( \text{ad}(X_0) \) restricted to subspace \( \mathbf{V} \). Let us summarize the results of our study.

**Theorem 1** Element \( \Phi = \exp(r^{i\mu} H_i \otimes X_\mu) \in U_\alpha(\mathbf{L}) \otimes U_\alpha(\mathbf{L}) \) is a solution to the twist equation. Twisting \( U_\alpha(\mathbf{L}) \) with the help of \( \Phi \) gives classical universal enveloping algebra \( U(\mathbf{L}) \) unless \( \mathbf{V} \) contains central elements. The universal \( \mathcal{R} \)–matrix for \( U_\alpha(\mathbf{L}) \) is
\[ \mathcal{R} = \exp(r^{i\mu} X_\mu \otimes H_i) \exp(-r^{i\mu} H_i \otimes X_\mu) \quad (7) \]
Expression (7) is an apparent generalization of the $\mathcal{R}$–matrix for $U_h(sl(2))$ in the form by O. Ogievetsky.

It becomes clear from the above study that the algebras $U_\alpha(L)$ covered by the theorem are completely specified by the set of commutative matrices $\alpha_\mu$, satisfying requirement $(\alpha_\mu)^\sigma_\nu = (\alpha_\nu)^\sigma_\mu$. Such matrices define an associative commutative multiplication $X_\mu \circ X_\mu \equiv \alpha_\mu^\sigma X_\sigma$ on the subspace $V$ and vice versa. For an example of this kind let us take $(\alpha_\mu)^\sigma_\nu = \delta^\sigma_{\mu+\nu}$, where $\mu, \nu, \sigma = 0, \ldots, n$. Thus introduced, $\alpha_\mu$ form an Abelian matrix ring and define a semidirect sum $L = H \triangleleft V$ with $H \sim V^*$. It complies with the condition of the theorem since matrix $\alpha_0$, the unity of the ring, has the maximum rank equal to $n + 1$, and we may set $\xi_\mu = \delta^0_\mu$ in transformation (3).

3 Universal $\mathcal{R}$–matrix for the quantum Poincaré algebra

Let us apply the developed technique to the quantum universal enveloping Poincaré algebra. The deformation is generated by twisting of the subalgebra $L = \text{Span}(E_1, E_2, P_1, P_2, P_+, K_3)$, in terms of [4]. Notations $H^i = -zE_i$, $H^3 = -zP_+$, $Y_i = 2P_i$, $Y_3 = -2K_3$, $i = 1, 2$, having been introduced, the coproduct in $U_\alpha(L)$ reads:

\[
\Delta_\alpha(H_\mu) = H_\mu \otimes 1 + 1 \otimes H_\mu, \\
\Delta_\alpha(Y_1) = e^{H^3} \otimes Y_1 + Y_1 \otimes e^{-H^3}, \\
\Delta_\alpha(Y_2) = e^{H^3} \otimes Y_2 + Y_2 \otimes e^{-H^3}, \\
\Delta_\alpha(Y_3) = e^{H^3} \otimes Y_3 + Y_3 \otimes e^{-H^3} + e^{H^3} H^1 \otimes Y_1 - Y_1 \otimes H^1 e^{-H^3} + e^{H^3} H^2 \otimes Y_2 - Y_2 \otimes H^2 e^{-H^3}.
\]

Non–vanishing commutators are

\[
[H^i, Y_i] = 2 \sinh(H^3), \quad [H^i, Y_3] = 2 \cosh(H^3)H^i, \quad [H^3, Y_3] = 2 \sinh(H^3), \quad i = 1, 2.
\]
Correspondence with the notations of the previous paragraph is achieved through the transformation \( X_\mu = Y_\nu (e^{\alpha H})_\mu^\nu \), the matrices \( \alpha \) being

\[
\alpha_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Explicitly this results in the following change of variables:

\[
X_1 = Y_1 e^{H_3}, \quad X_2 = Y_2 e^{H_3}, \quad X_3 = (Y_1 H^1 + Y_2 H^2 + Y_3) e^{H_3}.
\]

Transition to the classical generators is completed by transformation (5), where we may assume \( \xi^1 = \xi^2 = 0, \xi^3 = 1/2 \), for matrix \( \alpha_3 \) has the maximum rank 3.

\[
K^1 = H^1 e^{-2H_3}, \quad K^2 = H^2 e^{-2H_3}, \quad K^3 = \frac{1}{2} (1 - e^{-2H_3}).
\]

Elements \( K^\mu \) and \( X_\nu \) obey the ordinary, non–quantum, commutation relations of \( U(L) \):

\[
[K^i, X_i] = 2K^3, \quad [K^i, X_3] = 2K^i, \quad [K^3, X_3] = 2K^3, \quad i = 1, 2.
\]

Quantization \( U(L) \to U_\alpha(L) \) is controlled by the twisting element

\[
\mathcal{F} = \exp\left( \frac{K^1}{2K^3 - 1} \otimes X_1 + \frac{K^2}{2K^3 - 1} \otimes X_2 + \frac{1}{2} \ln(1 - 2K^3) \otimes X_3 \right) = \exp(-H^\mu \otimes X_\mu),
\]

and the quantum universal \( \mathcal{R} \)–matrix of the algebra \( U_\alpha(L) \) is given by

\[
\mathcal{R} = \exp(X_\mu \otimes H^\mu) \exp(-H^\mu \otimes X_\mu).
\]

4 Conclusion

The present investigation continues the series of works [3, 4, 5] devoted to a method of constructing quantum Lie algebras with the use of a classical object, the dual group. Based on the duality principle [9] viewing a quantum universal Lie algebra as a set of non–commutative functions on the dual group, that method reduces the quantization problem to finding a deformed Lie biideal consistent with a given coproduct.
Because of complicated structure of a generic Lie group, that program is feasible, however, for simple classes or in separate particular cases. So is the set of quasi-Abelian groups which are decomposed into a semidirect composition of two Abelian subgroups. Quantization theory for such Lie algebras was developed in [4]. Simple as dual groups of that type may seem, they occur rather frequently, especially in low dimensions [1,10,11,12,13], and corresponding quantum algebras possess very diverse and nontrivial properties. So, that class includes two non-isomorphic deformations of $U(sl(2))$. A generalization to the standard quantization was studied in detail in our work [5]. Here we have considered a generalization to the non-standard quantization of $sl(2)$ or, to be more exact, its Borel subalgebra. The universal $R$–matrix for the Jordanian $U_b(sl(2))$ has been found by O. Ogievetsky and A. A. Vladimirov [6, 7, 14]. In the present paper we have obtained explicit expression for $F$ and $R$ for the multi-dimensional generalization of the Borel subalgebra in $sl(2)$. The technique developed made it possible to built the universal $R$–matrix for the null-plane quantization of the Poincaré algebra $\mathcal{P}(3 + 1)$ found in [1].

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References

[1] Ballesteros A, Herranz F J, del Olmo V A and Santander M, 1995 Phys.Lett.B 351 137

[2] Drinfeld V G 1983 Sov.Math.Dokl 28 667

[3] Lyakhovsky V and Mudrov A 1992 J. Phys. A: Math. Gen. 25 L1139.

[4] Mudrov A I 1994 Vest.SPbSU 4 3.

[5] Mudrov A I 1997 J. Math. Phys. 38 476.

[6] Ogievetsky O Max–Plank–Institut preprint VPI–Ph/92–99.

[7] Vladimirov A A 1993 Mod. Phys. Let. A 8 2573

[8] Drinfeld V G 1989 Alg.&Anal. 1 114

[9] Semenov–Tian–Shanski M A 1992 Theor. Math. Phys. 93 302

[10] Ballesteros A, Herranz F J, del Olmo V A and Santander M, 1995 J.Math.Phys. 36 631

[11] Ballesteros A, Herranz F J 1996 J. Phys. A: Math. Gen. 29 4307

[12] Ballesteros A, Herranz F J, del Olmo V A, Santander M and Perena C M 1995 J. Phys. A: Math. Gen. 28 7113

[13] Ballesteros A, Gromov N A, Herranz F J, del Olmo V A and Santander M, 1995 J.Math.Phys. 16 5916

[14] Vladimirov A A 1994 Sym.Meth.Phys., Dubna 2 574