Particle dynamics on $AdS_2 \times S^2$ background with two–form flux

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Abstract

Different aspects of particle dynamics on $AdS_2 \times S^2$ background with two–form flux are discussed. These include solution of equations of motion, a canonical transformation to conformal mechanics and an $N = 4$ supersymmetric extension.

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1. Introduction

Over the past few years there has been an upsurge of interest in the model of a relativistic particle propagating near the horizon of the extreme Reissner-Nordström black hole \[1\]–\[13\]. The near horizon geometry in this case corresponds to the $AdS_2 \times S^2$ space–time with two–form flux. A peculiar feature of this system is that it admits two dual descriptions known in the literature as the $AdS$ and conformal bases.

Originally, it was demonstrated in \[1\] that in the limit when the black hole mass $M$ is large, the difference between the particle mass and the absolute value of its charge $(m - |e|)$ tends to zero with $M^2 (m - |e|)$ kept fixed, one recovers the conventional $d = 1$ conformal mechanics of \[14\]. In particular, the absence of a normalizable ground state in the conformal mechanics and the necessity to redefine the Hamiltonian \[14\] were given a new black hole interpretation \[1\]. Notice that the angular variables effectively decouple in the aforementioned limit and show up only in an indirect way via the effective coupling constant characterizing the conformal mechanics.

Later on it was argued in \[11, 15\] that, discarding the angular variables, a particle on $AdS_2$ background and the conformal mechanics of \[14\] can be related by an invertible coordinate transformation. In contrast to \[1\], the connection holds for any finite value of the black hole mass and has its origin in the possibility to choose different cosets of the conformal group $SO(2, 1)$ within the method of nonlinear realizations \[11\]. A proper extension of the $d = 1$ conformal mechanics by angular degrees of freedom which is equivalent to a massive charged particle on $AdS_2 \times S^2$ background with two–form flux was constructed in \[8\] (see also a related work \[9\]). In particular, a simple canonical transformation was found which directly relates symmetry generators (including the Hamiltonian) in both the pictures. As the transformation is invertible, different aspects of dynamics in one model can be studied in terms of the other and vice versa. The two pictures were called the $AdS$ and conformal bases.

The extreme Reissner-Nordström black hole solution of Einstein–Maxwell theory can be embedded into $d = 4, N = 2$ supergravity by adding two gravitini (for a review see e.g. \[16\]). As in the near horizon limit there is an enhancement of symmetry, for the particle on $AdS_2 \times S^2$ background one can construct an $N = 4$ supersymmetric extension. The action of the corresponding super 0–brane was found in \[3, 4\] with the use of the supercoset approach. Notice, however, that a consistent gauge fixed Hamiltonian formulation in terms of physical variables obeying canonical commutation relations is unknown.

Most of the developments mentioned above were focused on the case when a magnetic charge of the extreme Reissner-Nordström black hole vanishes. As the presence of a magnetic charge causes essential change in particle dynamics (see e.g. \[17\]), it is interesting to see which is the conformal model in this case and how the correspondence between the $AdS$ and conformal descriptions is altered.

The purpose of this work is to extend the analysis in \[8\] to the case of a nonvanishing magnetic charge. In the next section we briefly discuss the geometry of background fields. In sect. 3 particle dynamics on $AdS_2 \times S^2$ background with two–form flux is analyzed within
the Hamiltonian formalism. The conserved charges are found which allow us to integrate the equations of motion in an efficient way. A conformal picture is considered in sect. 4. An extension of the conformal mechanics [14] by angular variables is given which is characterized by two independent coupling constants. Making use of the rotation invariance, we construct a simple canonical transformation which relates the AdS and conformal bases. An $N = 4$ supersymmetric extension of the system in the conformal picture is discussed in sect. 5. Making use of the Hamiltonian methods we arrive at an on–shell component formulation for the $D(2, 1; \alpha)$–invariant mechanics of [10] with $\alpha = -1$. It is interesting to note that in order to accommodate $N = 4$ supersymmetry in the original bosonic conformal mechanics one has to identify the two couplings. Sect. 6 is devoted to an $N = 4$ supersymmetric generalization of the model in the AdS basis. In particular, we construct a new Hamiltonian formulation in terms of physical variables which obey canonical commutation relations. We summarize the results in sect. 7. Our conventions for dealing with $SU(2)$–spinors and the commutation relations of $d = 1, N = 4$ superconformal algebra are given in Appendix.

2. Geometry of background fields

Our starting point is the extreme Reissner-Nordström black hole solution of Einstein–Maxwell equations (for a review see e.g. [16])

\[ ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \quad A = -\frac{q}{r} dt + p \cos \theta d\varphi. \]  

(1)

Here $M$, $q$, $p$ are the mass, the electric and magnetic charges, respectively, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the standard metric on a sphere. For the extreme solution one has $M = \sqrt{q^2 + p^2}$. Throughout the paper we use units for which $G = 1$.

The near horizon limit is most easily accessible in isotropic coordinates ($r \to r - M$) which cover the region outside the horizon only

\[ ds^2 = -\left(1 - \frac{M}{r}\right)^{-2} dt^2 + \left(1 + \frac{M}{r}\right)^2 \left(dr^2 + r^2 d\Omega^2\right). \]  

(2)

When $r \to 0$ the metric takes the form

\[ ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 + M^2 d\Omega^2, \]  

(3)

while implementing the limit in the two–form field strength, one finds the background vector field

\[ A = \frac{q}{M^2} r dt + p \cos \theta d\varphi. \]  

(4)

The last two lines give the Bertotti-Robinson solution of Einstein–Maxwell equations.

Notice that in the literature on the subject it is customary to use other coordinates where the horizon is at $r = \infty$. In particular, the use of these coordinates facilitates the analysis in [1]. In this paper we refrain from using such a coordinate system.
From (3) it follows that in the near horizon limit the space–time geometry is the product of a two-dimensional sphere of radius \(M\) and a two-dimensional pseudo Riemannian space–time with the metric
\[
ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2.
\] (5)
The latter proves to be the metric of \(\text{AdS}_2\). In order to see this, consider the hyperboloid in \(\mathbb{R}^{2,1}\)
\[-\eta_{AB} x^A x^B = M^2, \quad \eta_{AB} = \text{diag}(-,+,+),\] (6)
parameterized by the Poincaré coordinates \((t,r)\)
\[x^0 = \frac{1}{2r}(1 + r^2(M^2 - t^2)), \quad x^1 = \frac{1}{2r}(1 - r^2(M^2 + t^2)), \quad x^2 = Mr t .\] (7)
Since \(x^0 - x^1 > 0\), the local coordinates cover only half of the hyperboloid\(^1\). Calculating the metric \(ds^2 = \eta_{AB} dx^A dx^B\) induced on the surface (7) and making the shift \(r \rightarrow M^2 r\), one gets precisely (5). Notice that in this picture the black hole mass \(M\) is equal to the radius of \(S^2(\text{AdS}_2)\). It is worth mentioning also that, by construction, the isometry group of the metric (3) is \(SO(2,1) \times SO(3)\).

To summarize, the background geometry is that of the \(\text{AdS}_2 \times S^2\) space–time with 2–form flux.

3. Particle dynamics on \(\text{AdS}_2 \times S^2\)

Having fixed the background fields, we then consider the action of a relativistic particle on such a background
\[S = -\int dt \left( m \sqrt{(r/M)^2 - (M/r)^2 \dot{r}^2 - M^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + eqr/M^2 + e p \cos \theta \dot{\phi}} \right) .\] (8)
Here \(m\) and \(e\) are the mass and the electric charge of a particle, respectively.

The particle dynamics is most easily analyzed within the Hamiltonian formalism. Introducing the momenta \((p_r, p_\theta, p_\varphi)\) canonically conjugate to the configuration space variables \((r, \theta, \varphi)\), one finds the Hamiltonian
\[H = (r/M) \left( \sqrt{m^2 + (r/M)^2 p_r^2 + (1/M)^2 (p_\theta^2 + \sin^{-2} \theta (p_\varphi + e p \cos \theta)^2)} + eq/M \right) ,\] (9)
which generates time translations. In agreement with the isometries of the background metric one also finds the conserved quantities
\[K = M^3 / r \left( \sqrt{m^2 + (r/M)^2 p_r^2 + (1/M)^2 (p_\theta^2 + \sin^{-2} \theta (p_\varphi + e p \cos \theta)^2)} - M^2 / r \right) + t^2 H + 2 t r p_r , \quad D = t H + r p_r ,\] (10)
\(^1\)In order to avoid closed time–like curves, one considers the universal covering of the hyperboloid with \(-\infty < t < \infty, \ 0 < r\).
which generate special conformal transformations and dilatations, respectively. Together
with the Hamiltonian they form $so(2,1)$ algebra
\begin{equation}
\{H,D\} = H , \quad \{H,K\} = 2D , \quad \{D,K\} = K .
\end{equation}

The generators of rotations
\begin{align*}
J_1 &= -p_\varphi \cot \theta \cos \varphi - p_\theta \sin \varphi - ep \cos \varphi \sin^{-1} \theta , \\
J_2 &= -p_\varphi \cot \theta \sin \varphi + p_\theta \cos \varphi - ep \sin \varphi \sin^{-1} \theta , \\
J_3 &= p_\varphi , \quad \{J_a, J_b\} = \epsilon_{abc} J_c , \quad \epsilon_{123} = 1 .
\end{align*}
enter the Hamiltonian via the Casimir element
\begin{equation}
J^a J_a = p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2 + (ep)^2 ,
\end{equation}
and, hence, are conserved due to $su(2)$ algebra they form.

When analyzing solutions of equations of motion, two distinct cases should be examined. First consider the situation when the magnetic charge of the black hole vanishes
\begin{equation}
p = 0 , \quad M = |q| .
\end{equation}
In this case the particle moves on a plane orthogonal to the angular momentum vector $J_i$. Making use of the rotation invariance one can choose the reference frame where $J_i$ is along $x^3$-axis, i.e.
\begin{equation}
\theta = \pi/2 , \quad p_\theta = 0 \quad \rightarrow \quad J_1 = 0 , \quad J_2 = 0 , \quad J_3 = p_\varphi = L ,
\end{equation}
with $L$ a constant\(^2\). Then from the conservation laws (9) and (10) one can fix the dynamics of the radial coordinate
\begin{equation}
r(t) = \frac{EM^2}{\sqrt{a^2(t) + b^2 + c}} , \quad p_r(t) = \frac{a(t)(\sqrt{a^2(t) + b^2} + c)}{EM^2} ,
\end{equation}
where $E = H$ is the energy and we abbreviated
\begin{equation}
a(t) = D - tE , \quad b^2 = m^2 M^2 + L^2 , \quad c = eq .
\end{equation}
The evolution of the angular variable is found by a straightforward integration
\begin{equation}
\varphi(t) = -\frac{L}{\sqrt{b^2 - c^2}} \left( \arctan \frac{a(t)}{\sqrt{b^2 - c^2}} - \arctan \frac{ca(t)}{\sqrt{a^2(t) + b^2} \sqrt{b^2 - c^2}} \right) + \varphi_0 , \quad p_\varphi(t) = L .
\end{equation}
\(^2\)We assume that $L \neq 0$. When $L = 0$ the particle travels towards the horizon at $r = 0$ along a straight line.
It is important to notice that the conserved charges (9), (10) also specify the value of the Casimir element of $so(2, 1)$ algebra realized in the model in terms of the parameters of the particle and those of the background

$$EK - D^2 = b^2 - c^2 = M^2(m^2 - e^2) + L^2. \quad (19)$$

This should correlate with the bound $b^2 - c^2 > 0$ revealed by the explicit solution given above. The latter also assures that the energy of the particle $E = (r/M^2)(\sqrt{a^2(t)} + b^2 + c)$ is positive even if $c$ is negative. Indeed, if $c < 0$ then from the condition $b^2 - c^2 > 0$ one immediately gets

$$(\sqrt{a^2(t)} + b^2 + c)(\sqrt{a^2(t)} + b^2 - c) > 0, \quad (20)$$

which means that the first factor entering (20) is positive.

As $\dot{r}$ is proportional to $a(t)$ with a positive coefficient, depending on the initial data, the particle either goes directly towards the black hole horizon located at $r = 0$, or it moves away for some time, slows down with the turning point at $t = D/E$, and then travels back towards $r = 0$. The orbit looks particularly simple when the particle is electrically neutral, i.e. $c = 0$

$$r(\varphi) = \frac{EM^2}{b} \left| \cos \left( \frac{b(\varphi - \varphi_0)}{L} \right) \right|. \quad (21)$$

The trajectory is a kind of a loop which starts and ends at $r = 0$ and has a symmetry axis typical for rotation invariant systems.

Now consider the case when the magnetic charge $p$ of the black hole does not vanish. In this case the particle moves on the cone (turning to Cartesian coordinates)

$$\frac{x^i J_i}{\sqrt{x^2}} = -ep. \quad (22)$$

As before, one can use the rotation invariance so as to pass to the reference frame where $J_i$ is along $x^3$-axis. This specifies the canonical pair $(\theta, p_\theta)$

$$J_1 = 0, \quad J_2 = 0, \quad J_3 = p_\varphi = L \quad \rightarrow \quad \cos \theta = -ep/L, \quad p_\theta = 0, \quad (23)$$

and imposes the natural bound $|\frac{p_\varphi}{L}| \leq 1$. The solutions of equations of motion for $(r(t), p_r(t))$ and $(\varphi(t), p_\varphi(t))$ prove to maintain their previous form (16), (18) with $a(t)$ and $c$ unchanged, but $b^2$ modified

$$b^2 = m^2M^2 + L^2 - (ep)^2. \quad (24)$$

For $p \neq 0$ the qualitative behavior of a particle is similar to the previous case but this time it is confined to move on the cone (22).

4. A relation to conformal mechanics

The conventional conformal mechanics in one dimension is governed by the action functional [14]

$$S = \frac{1}{2} \int dt \left( \dot{x}^2 - \frac{g}{x^2} \right), \quad (25)$$
where \( g \) is the coupling constant. Passing to the Hamiltonian formalism one finds the conserved charges

\[
H' = \frac{p^2}{2} + \frac{g}{2x^2}, \quad D' = tH' - \frac{1}{2}xp, \quad K' = t^2H' - t(xp) + \frac{1}{2}x^2,
\]

which altogether form \( so(2,1) \) algebra (11). Guided by this observation, the authors of [1] argued that the quantum mechanics of a test particle moving near the horizon of the extreme Reissner-Nordström black hole\(^3\) matches the old conformal mechanics (25) in the limit

\[
M \to \infty, \quad (m - |e|) \to 0,
\]

with \( M^2(m - |e|) \) fixed.

In [8] the conformal mechanics (25) was extended by a couple of angular variables in such a way that the resulting model is related to a particle moving near the horizon of the extreme Reissner-Nordström black hole by a canonical transformation (for a related work see [9]). The construction in [8] does not appeal to any specific limit and is valid for any finite value of the black hole mass.

In this section we generalize the analysis in [8] to the case when a test particle couples to the magnetic charge of the black hole. As compared to the calculation in [8], the use of the rotation invariance notably facilitates the analysis.

Consider a specific extension of the model (25) by two angular degrees of freedom (\( \Theta, \Phi \))

\[
S = \frac{1}{2} \int dt \left( \dot{x}^2 + \frac{1}{4}x^2(\dot{\Theta}^2 + \sin^2\Theta \dot{\Phi}^2) - \frac{g}{x^2} - 2\nu \cos \Theta \dot{\Phi} \right),
\]

where \( \nu \) is a new coupling constants and \( x \) is now treated as a radial coordinate in the enlarged configuration space. This theory arises, in particular, in the bosonic limit of the superconformal mechanics associated with the supergroup \( D(2,1;\alpha) \) for \( \alpha = -1 \) [10]. Notice that in [10] the \( g \) and \( \nu^2 \) couplings were identified (see also the discussion in sect. 5). In non–supersymmetric case they are independent.

That the new degrees of freedom do not destroy the conformal symmetry of the original model is most easily verified within the Hamiltonian formalism. Indeed, given the Hamiltonian

\[
H' = \frac{p^2}{2} + \frac{g}{2x^2} + \frac{2}{x^2}(p_\Theta^2 + \sin^{-2}\Theta(p_\Phi + \nu \cos \Theta)^2),
\]

where \( (p, p_\Theta, p_\Phi) \) designate momenta canonically conjugate to \( (x, \Theta, \Phi) \), the generators of dilatations \( D' \) and special conformal transformations \( K' \) are constructed following the prescription (26) and the full algebra proves to be \( so(2,1) \).

As might be anticipated from the form of the action (28), the system accommodates rotation invariance. The corresponding generators are derived from (12) by the obvious change of the canonical pairs \( (\varphi, p_\varphi) \to (\Phi, p_\Phi), (\theta, p_\theta) \to (\Theta, p_\Theta) \) and the coupling constants

\(^3\)In [1] only the case of the vanishing magnetic charge was discussed.
\( ep \to \nu \). They are trivially conserved because the angular variables enter the Hamiltonian via the Casimir element of \( so(3) \) algebra realized in the model.

Now let us demonstrate that the system (29) and a particle on \( \text{AdS}_2 \times S^2 \) background with 2–form flux are related by a canonical transformation. In order to simplify the analysis, let us use the rotation invariance intrinsic to both the models and pass for each system to the reference frame where the conserved angular momentum vector is along \( x^3 \)–axis\(^4\). This allows one to disregard the pairs \((\theta, p_\theta)\), and \((\Theta, p_\Theta)\)

\[
\begin{align*}
\cos \theta &= -ep/L , \quad p_\theta = 0 , \\
\cos \Theta &= -\nu/L , \quad p_\Theta = 0 .
\end{align*}
\]

Following [8], we then search for a canonical transformation which brings the symmetry generators characterizing the model (8) precisely to those of the system (28). Comparing the conserved charges (including the Hamiltonian) in both the pictures, one immediately finds the desired transformation

\[
\begin{align*}
x &= \left[ \frac{2M^2}{r} \left( \sqrt{m^2 M^2 + (rp_r)^2} + (Lp_\varphi - (ep)^2) / (L^2 - (ep)^2) - eq \right) \right]^{1/2} , \\
p &= -2rp_r \left[ \frac{2M^2}{r} \left( \sqrt{m^2 M^2 + (rp_r)^2} + (Lp_\varphi - (ep)^2) / (L^2 - (ep)^2) - eq \right) \right]^{-1/2} , \\
p_\Phi &= p_\varphi .
\end{align*}
\]

The corresponding Poisson brackets prove to be canonical. When establishing this correspondence, one has to specify the couplings of the conformal mechanics in terms of the parameters characterizing the particle on \( \text{AdS}_2 \times S^2 \)

\[
\nu = ep , \quad g = 4(m^2 M^2 - (eq)^2) .
\]

A transformation law of the last missing variable \( \Phi \) is then found with the help of (31). Imposing the canonical relations

\[
\{ \Phi, x \} = 0 , \quad \{ \Phi, p \} = 0 , \quad \{ \Phi, p_\Phi \} = 1 ,
\]

which are to be calculated with respect to the variables \((r, p_r)\), \((\varphi, p_\varphi)\), and taking the ansatz

\[
\Phi = \varphi + A(s, p_\varphi) ,
\]

with \( s = (rp_r) \) and \( A(s, p_\varphi) \) an arbitrary function, one reduces (33) to a single ordinary differential equation. This yields the solution

\[
A(s, p_\varphi) = -\frac{\alpha}{\sqrt{k^2 - c^2}} \left( \arctan \frac{s}{\sqrt{k^2 - c^2}} + \arctan \frac{cs}{\sqrt{k^2 - c^2} \sqrt{k^2 + s^2}} \right) ,
\]

\(^4\)Our construction implies \( p_\Phi = p_\varphi = L \) on–shell.
where we denoted

\[ \alpha = \frac{L(Lp_\varphi - (ep)^2)}{(L^2 - (ep)^2)} \, , \quad k^2 = m^2M^2 + \frac{(Lp_\varphi - (ep)^2)^2}{(L^2 - (ep)^2)} \, , \quad c = eq \, . \quad (36) \]

In the consideration above we made explicit use of the rotation invariance. Obviously, rotation is a canonical transformation. So, the transformation relating the models (9) and (29) is a superposition of (31), (34) and two rotations. The latter affect angular variables only. Then it is not hard to guess the radial part of the transformation

\[ x = \left[ \frac{2M^2}{r} \left( \sqrt{m^2M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2} - eq \right) \right]^{\frac{1}{2}} , \]

\[ p = -2rp_r \left[ \frac{2M^2}{r} \left( \sqrt{m^2M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2} - eq \right) \right]^{-\frac{1}{2}} , \quad (37) \]

which indeed brings (29) to (9), provided the identification of the couplings (32) holds.

Notice that the momentum squared is invariant

\[ p_\theta^2 + \sin^{-2} \Theta (p_\varphi + \nu \cos \Theta)^2 = p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2 . \quad (38) \]

Although in this picture the transformation of the angular variables appears to be rather complicated\textsuperscript{5}, for practical uses one does not need to know its explicit form and the relations (37) and (38) prove to be sufficient.

To summarize, the canonical transformation exposed above establishes the equivalence relation between the charged massive particle moving near the horizon of the extreme Reissner-Nordström black hole, which has a non–vanishing magnetic charge, and the conformal mechanics (28). Different aspects of dynamics in one model can be studied in terms of the other and vice versa. It is noteworthy that the equivalence holds for any fixed value of the black hole mass and does not refer to any specific limit. This is to be contrasted with the consideration in [1].

5. \textit{N = 4 supersymmetric extension in the conformal basis}

An \( N = 4 \) supersymmetric generalization of the model (28) was constructed in [10] with the use of the method of nonlinear realizations\textsuperscript{6}. The corresponding action was given in terms of superfields. At the component level it involves non–dynamical auxiliary fields needed for the off-shell closure of the \( d = 1, N = 4 \) superconformal algebra realized in the model. Notice that the \( N = 4 \) supersymmetry makes one to identify the \( g \) and \( \nu^2 \) couplings in (28).

Aiming at the construction of an \( N = 4 \) supersymmetric extension of the particle moving near the horizon of the extreme Reissner-Nordström black hole, in this section we discuss

\textsuperscript{5}For the case of the vanishing magnetic charge it was found in [8] by identifying the \( \text{so}(3) \) generators in both the pictures.

\textsuperscript{6}For an earlier work on \( N = 2 \) model see [18].
an $N = 4$ supersymmetric generalization of the bosonic conformal mechanics (28) using the alternative Hamiltonian formalism. The advantage of this approach is that it automatically yields an on–shell component formulation free from non–dynamical auxiliary fields and offers technical simplifications as compared to the method in [10]. Besides, it can be readily applied to construct important multi–particle superconformal systems, including the $N = 4$ superconformal Calogero model (see e.g. [19]–[22]), while the superfield appears to be more involved [23]. For the particular case of the vanishing coupling constants $g = \nu = 0$ the construction was realized in [8].

Within the framework of the Hamiltonian formalism the construction of an $N = 4$ supersymmetric generalization of the system (29) amounts to extending the bosonic phase space by a pair of canonically conjugate $SU(2)$–spinors $\psi_\alpha, \bar{\psi}^\alpha$ (for our conventions see Appendix) and building in the enlarged phase space a representation of $su(1,1|2)$ superalgebra. Along with the $so(2,1)$–, and $su(2)$–generators which comprise bosonic symmetries of the model (29), the superalgebra involves the supersymmetry generators $Q_\alpha, \bar{Q}^\alpha$ and the superconformal ones $S_\alpha, \bar{S}^\alpha$ (the commutation relations are given in Appendix). The conditions that the Poisson bracket of $Q_\alpha$ and $\bar{Q}^\beta$ yields the Hamiltonian and that $Q_\alpha$ anticommutates with itself prove to be strong enough to fix the form of the supercharges

$$Q_\alpha = p\psi_\alpha + \frac{2i}{x}(\sigma^a\psi)_\alpha J_a + \frac{i}{2x} \bar{\psi}_\alpha \psi^2, \quad \bar{Q}^\alpha = p\bar{\psi}^\alpha - \frac{2i}{x}(\bar{\psi}\sigma^a\psi)^\alpha J_a + \frac{i}{2x} \psi^\alpha \bar{\psi}^2,$$

and the extended Hamiltonian

$$H = \frac{p^2}{2} + \frac{2}{x^2} J^a J_a - \frac{2}{x^2} (\bar{\psi}\sigma^a\psi) J_a + \frac{1}{4x^2} \psi^2 \bar{\psi}^2 .$$

Here $J_a$ are the bosonic $su(2)$–generators which are realized as in (12) with the obvious change $(\varphi, p_\varphi) \to (\Phi, p_\Phi), (\theta, p_\theta) \to (\Theta, p_\Theta)$, $ep \to \nu$. Comparing this Hamiltonian with (29) and taking into account (13), one concludes that the supersymmetric extension is only possible if one identifies the $g$ and $\nu$ couplings as follows

$$g = (2\nu)^2 .$$

This is in full agreement with the superfield considerations in [10].

As is obvious from (40), the extended system maintains the conformal symmetry. Given the Hamiltonian, the generators of dilatations and special conformal transformations are constructed in the standard way

$$D = tH - \frac{1}{2} xp, \quad K = t^2 H - t(xp) + \frac{1}{2} x^2 .$$

Then the Poisson brackets of $K$ with $Q_\alpha$ and $\bar{Q}^\alpha$ yield the superconformal generators

$$S_\alpha = x\psi_\alpha - t Q_\alpha, \quad \bar{S}^\alpha = x\bar{\psi}^\alpha - t \bar{Q}^\alpha .$$

It remains to be discussed the $su(2)$ symmetry realized in the extended model (40). As the fermionic degrees of freedom transform as $SU(2)$–doublets, the bosonic generator $J_a$
must be extended so as to include a piece responsible for the fermions. One can either guess
its form or just calculate the Poisson bracket of $Q_\alpha$ with $\bar{S}^\beta$

$$J_a \rightarrow J_a = J_a + \frac{1}{2}(\bar{\psi}\sigma_a\psi).$$  \hspace{1cm} (44)

Finally, it is straightforward to check that the generators introduced above do form a
representation of $su(1,1|2)$ superalgebra. The explicit verification makes heavy use of the
properties of the Pauli matrices and spinor rearrangement rules given in Appendix.

In order to construct a Lagrangian formulation reproducing the Hamiltonian (40), we
first notice that within the Hamiltonian formalism the canonical bracket

$$\{\psi_\alpha, \bar{\psi}^\beta\} = -i\delta_\alpha^\beta$$

is conventionally understood as the Dirac bracket

$$\{A, B\}_D = \{A, B\} - i\{A, \chi^\alpha\}\{\bar{\chi}_\alpha, B\} - i\{A, \bar{\chi}_\alpha\}\{\chi^\alpha, B\}$$ \hspace{1cm} (45)

associated with the fermionic second class constraints

$$\chi^\alpha = p_\psi^\alpha - i\bar{\psi}\psi = 0, \quad \bar{\chi}_\alpha = p_{\bar{\psi}_\alpha} - i\bar{\psi}_\alpha \psi = 0.$$ \hspace{1cm} (46)

Here $(p_\psi^\alpha, p_{\bar{\psi}_\alpha})$ stand for the momenta canonically conjugate to the variables $(\psi_\alpha, \bar{\psi}_\alpha)$, respectively.

Choosing the right derivative for fermionic degrees of freedom, an action functional leading
to the Hamiltonian formulation (40) is straightforward to build

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{i}{2}\bar{\psi}_{\sigma^a}\psi - \frac{i}{2}\bar{\psi}_\sigma\psi + \frac{1}{8}x^2(\dot{\Theta}^2 + \sin^2\Theta\dot{\Phi}^2) - \frac{2\nu^2}{x^2} - \nu\cos\Theta\dot{\Phi} \right.$$ \hspace{1cm} + $(\bar{\psi}_\sigma\psi)\mathcal{L} - \frac{3}{4x^2}\psi^2\bar{\psi}^2).$$ \hspace{1cm} (47)

Here $\mathcal{L}^a$ is the bosonic part of the angular momentum vector written in configuration space

$$\mathcal{L}^1 = -\frac{1}{2}\dot{\Phi}\cos\Theta\sin\Theta\cos\Phi - \frac{1}{2}\dot{\Theta}\sin\Phi - \frac{2\nu}{x^2}\cos\Phi\sin\Theta,$$

$$\mathcal{L}^2 = -\frac{1}{2}\dot{\Phi}\cos\Theta\sin\Theta\sin\Phi + \frac{1}{2}\dot{\Theta}\cos\Phi - \frac{2\nu}{x^2}\sin\Phi\sin\Theta,$$

$$\mathcal{L}^3 = \frac{1}{2}\dot{\Phi}\sin^2\Theta - \frac{2\nu}{x^2}\cos\Theta.$$ \hspace{1cm} (48)

When relating the action (47) from the Hamiltonian (40) the spinor identity

$$(\bar{\psi}_\sigma\psi)(\bar{\psi}_{\sigma^b}\psi) = -\frac{1}{2}\delta_{ab}\psi^2\bar{\psi}^2$$ \hspace{1cm} (49)

proves to be helpful.
Thus, we have constructed an $N = 4$ supersymmetric extension of the conformal mechanics (28) by applying the Hamiltonian methods. The supersymmetry requires the identification (41) of the coupling constants. The model built in this section can be viewed as an on–shell component formulation for the $D(2, 1; \alpha)$–invariant mechanics of [10] with $\alpha = -1$.

6. $N = 4$ supersymmetric extension in the AdS basis

Having constructed an $N = 4$ supersymmetric extension in the conformal basis, let us discuss its AdS partner. In the preceding section, when evaluating Poisson brackets of the $su(1, 1|2)$–generators, we used only the canonical relations

$$\{x, p\} = 1, \quad \{J_a, J_b\} = \epsilon_{abc} J_c, \quad \{\psi_\alpha, \bar{\psi}^\beta\} = -i\delta_\alpha^\beta,$$

and the fact that all other brackets involving $(x, p, J_a, \psi_\alpha, \bar{\psi}^\alpha)$ vanish. Consider the transformation (37) relating the conformal and AdS bases. It respects (50) provided the $su(2)$–generators in the AdS picture are taken as in (12). This is the instance when one does not need to know the explicit form of the canonical transformation of the angular variables but only their specific combinations, e.g. the $su(2)$–charges. The fermionic degrees of freedom are kept inert under the transformation from one picture to another.

In order to accommodate $N = 4$ supersymmetry in the model regarded in the conformal picture, one has to relate the $g$ and $\nu$ couplings as in (41). On the other hand, the transformation (37) to the AdS picture implies the identification (32). Taking into account that for the extreme Reissner-Nordström black hole $M^2 = q^2 + p^2$, one concludes that an $N = 4$ supersymmetric extension of the model (9) is characterized by the additional physical requirement

$$m = |e|.$$  

Thus, the system can be viewed as a BPS superparticle in a BPS background.

Summarizing the above discussion, we can write down the Hamiltonian

$$H = \frac{r}{M^2} \left( \sqrt{m^2 M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi + ep \cos \theta)^2} + eq \right) - \frac{r}{M^2} ((\bar{\psi} \sigma^4 \psi) J_a - \frac{1}{8} \psi^2 \bar{\psi}^2) \left( \sqrt{m^2 M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi + ep \cos \theta)^2} - eq \right)^{-1}$$

and the conserved charges of an $N = 4$ superparticle propagating on $AdS_2 \times S^2$ background with two–form flux

$$K = t^2 H + 2tp_r + M^2 \frac{r}{M^2} \left( \sqrt{m^2 M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi + ep \cos \theta)^2} - eq \right),$$

$$D = tH + rp_r.$$
\[ Q_\alpha = -\frac{2 \left( (rp_r)\psi_\alpha - i(\sigma_\alpha^a \psi) J_a - \frac{i}{4} \bar{\psi}_\alpha \psi^2 \right)}{\left( \frac{2M^2}{r} \left( \sqrt{m^2 M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2 - e q} \right) \right)^{\frac{1}{2}},} \]

\[ S_\alpha = \psi_\alpha \left( \frac{2M^2}{r} \left( \sqrt{m^2 M^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\varphi + ep \cos \theta)^2 - e q} \right) \right)^{\frac{1}{2}} - tQ_\alpha, \]

\[ J_a = \tilde{J}_a + \frac{1}{2} (\bar{\psi} \sigma_a \psi), \quad (54) \]

where \( J_a \) are defined in (12).

A Lagrangian formulation corresponding to the Hamiltonian (52) is straightforward to construct. It proves sufficient to treat the fermionic degrees of freedom like we did in the preceding section and apply the Legendre transform to the Hamiltonian. However, a resulting formulation does not literally coincide with the gauge fixed version of the super 0–brane on \( AdS_2 \times S^2 \) built within the Green–Schwarz approach [3, 4]. A specific field redefinition is to be implemented in order to relate the two systems. The reason is that the Hamiltonian formulation of the super 0–brane involves fermionic second class constraints which depend on the background fields. Introducing the Dirac bracket one can solve the second class constraints and eliminate the fermionic momenta. However, the brackets for the remaining physical variables are not canonical. In particular, bosonic variables have nonvanishing brackets with physical fermions. In general, one has to implement a nontrivial field redefinition so as to bring the brackets to a canonical form. Notice that the canonical brackets are also needed for constructing a conventional quantum mechanical description. The advantage of the model (52) is that the physical variables do obey the canonical relations. So, the system is likely to describe the Hamiltonian formulation of the gauge fixed super 0–brane on \( AdS_2 \times S^2 \) written in proper coordinates. Finding an explicit form of the field redefinition is an interesting problem which deserves further investigation.

7. Conclusion

To summarize, in the present paper we studied the dynamics of a massive charged particle moving on \( AdS_2 \times S^2 \) background with 2–form flux and constructed its conformal partner. The connection between the two models is provided by a specific canonical transformation which relates symmetry generators in both the pictures. An \( N = 4 \) supersymmetric extension of the model in the conformal bases was constructed and then combined with the canonical transformation so as to produce a new Hamiltonian formulation of an \( N = 4 \) superparticle on \( AdS_2 \times S^2 \).

Turning to possible further developments, the first issue is how to generalize the present analysis to the case of \( D(2,1;\alpha) \) supergroup with \( \alpha \neq -1 \). It is interesting to see which background geometry corresponds to the superconformal particle of [10] written in the \( AdS \) basis and what is the geometrical meaning of the parameter \( \alpha \). Then it remains to explore how the equivalence established within the Hamiltonian formalism is translated into the
Lagrangian language and how it is linked to the off-shell map of [11, 15]. A more technical issue is to find a field redefinition that relates our Hamiltonian formulation of the $N = 4$ superparticle on $AdS_2 \times S^2$ to the Hamiltonian formulation of the gauge fixed super 0–brane of [4].

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Appendix

Throughout the text we use a lower Greek index to designate an $SU(2)$–doublet representation. Complex conjugation yields an equivalent representation to which one assigns an upper index

$$(\psi_\alpha)^* = \bar{\psi}^\alpha , \quad \alpha = 1, 2 .$$

As usual, spinor indices are raised and lowered with the use of the $SU(2)$–invariant antisymmetric matrices

$$\psi^\alpha = \epsilon^{\alpha \beta} \psi_\beta , \quad \bar{\psi}_\alpha = \epsilon_{\alpha \beta} \bar{\psi}^\beta ,$$

where $\epsilon_{12} = 1$, $\epsilon^{12} = -1$. For spinor bilinears we stick to the notation

$$\psi^2 = (\psi^\alpha \psi_\alpha) , \quad \bar{\psi}^2 = (\bar{\psi}_\alpha \bar{\psi}^\alpha) , \quad \bar{\psi}\psi = (\bar{\psi}^\alpha \psi_\alpha) ,$$

such that

$$\psi_\alpha \psi_\beta = \frac{1}{2} \epsilon_{\alpha \beta} \psi^2 , \quad \bar{\psi}^\alpha \bar{\psi}^\beta = \frac{1}{2} \epsilon^{\alpha \beta} \bar{\psi}^2 , \quad \psi_\alpha \bar{\psi}_\beta - \psi_\beta \bar{\psi}_\alpha = \epsilon_{\alpha \beta} (\bar{\psi} \psi) .$$

The Pauli matrices $(\sigma_\alpha)_\beta^\alpha$ are taken in the standard form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

which obey

$$(\sigma_\alpha \sigma_\beta)_\gamma^\alpha + (\sigma_\beta \sigma_\alpha)_\gamma^\beta = 2 \delta_\alpha^\gamma \delta_\beta^\beta , \quad (\sigma_\alpha \sigma_\beta)^\beta_\alpha - (\sigma_\beta \sigma_\alpha)^\beta_\alpha = 2 \epsilon_{\alpha \beta \gamma} (\sigma_\gamma)^\beta_\alpha ,$$

$$(\sigma_\alpha \sigma_\beta)^\beta_\alpha = \delta_\alpha^\beta \delta_\beta^\alpha + i \epsilon_{\alpha \beta \gamma} (\sigma_\gamma)^\beta_\alpha , \quad (\sigma_\alpha)_\gamma^\beta (\sigma_\gamma)^\alpha_\beta = 2 \delta_\alpha^\rho \delta_\beta^\gamma - \delta_\alpha^\beta \delta_\gamma^\rho ,$$

$$(\sigma_\alpha)_\beta^\beta \epsilon^\gamma_\alpha = (\sigma_\gamma)^\beta_\alpha \epsilon^\gamma_\beta , \quad \epsilon^{\alpha \beta} (\sigma_\gamma)_\beta^\gamma = \epsilon^\gamma_\beta (\sigma_\gamma)_\beta^\alpha ,$$

where $\epsilon_{abc}$ is the totally antisymmetric Levi-Civitá tensor, $\epsilon_{123} = 1$. Throughout the text we use the abbreviation $\psi \sigma_\alpha \psi = \bar{\psi}^\alpha (\sigma_\alpha)^\beta_\beta \psi_\beta$. 

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When constructing an $N = 4$ supersymmetric extension of the particle model on $AdS_2 \times S^2$ background within the Hamiltonian formalism, one uses a pair of complex conjugate spinors $(\psi_\alpha, \bar{\psi}^\alpha)$ in order to parameterize the odd sector of the phase space. These obey the Poisson bracket
\begin{equation*}
\{ \psi_\alpha, \bar{\psi}^\beta \} = -i\delta_\alpha^\beta.
\end{equation*}
Conserved charges of an ultimate theory must obey $su(1,1|2)$ superalgebra which is taken in the form
\begin{align*}
\{H, D\} &= H, & \{H, K\} &= 2D, \\
\{D, K\} &= K, & \{J_a, J_b\} &= \epsilon_{abc} J_c, \\
\{Q_\alpha, \bar{Q}^\beta\} &= -2iH\delta_\alpha^\beta, & \{Q_\alpha, \bar{S}^\beta\} &= 2(\sigma_a)_\alpha^\beta J_a + 2iD\delta_\alpha^\beta - C\delta_\alpha^\beta, \\
\{S_\alpha, \bar{S}^\beta\} &= -2iK\delta_\alpha^\beta, & \{\bar{Q}^\alpha, S_\beta\} &= -2(\sigma_a)^\alpha_\beta J_a + 2iD\delta_\beta^\alpha + C\delta_\beta^\alpha, \\
\{D, Q_\alpha\} &= -\frac{1}{2}Q_\alpha, & \{D, S_\alpha\} &= \frac{1}{2}S_\alpha, \\
\{K, Q_\alpha\} &= S_\alpha, & \{H, S_\alpha\} &= -Q_\alpha, \\
\{J_a, Q_\alpha\} &= \frac{i}{2}(\sigma_a)^\alpha_\beta Q_\beta, & \{J_a, S_\alpha\} &= \frac{i}{2}(\sigma_a)^\alpha_\beta S_\beta, \\
\{D, \bar{Q}^\alpha\} &= -\frac{1}{2}\bar{Q}^\alpha, & \{D, \bar{S}^\alpha\} &= \frac{1}{2}\bar{S}^\alpha, \\
\{K, \bar{Q}^\alpha\} &= \bar{S}^\alpha, & \{H, \bar{S}^\alpha\} &= -\bar{Q}^\alpha, \\
\{J_a, \bar{Q}^\alpha\} &= -\frac{i}{2}\bar{Q}^\beta(\sigma_a)^\alpha_\beta, & \{J_a, \bar{S}^\alpha\} &= -\frac{i}{2}\bar{S}^\beta(\sigma_a)^\alpha_\beta.
\end{align*}
Here $C$ is the central charge. The missing Poisson brackets prove to vanish.

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