Algebraic Geometry Approach to the Bethe Equation for Hofstadter Type Models

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Abstract

We study the diagonalization problem of certain Hofstadter-type models through the algebraic Bethe ansatz equation by the algebraic geometry method. When the spectral variables lie on a rational curve, we obtain the complete and explicit solutions for models with the rational magnetic flux, and discuss the Bethe equation of their thermodynamic flux limit. The algebraic geometry properties of the Bethe equation on high genus algebraic curves are investigated in cooperation with physical considerations of the Hofstadter model.

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1 Introduction

The Hofstadter Hamiltonian is defined by

\[ H_{Hof} = \mu (\alpha U + \alpha^{-1} U^{-1}) + \nu (\beta V + \beta^{-1} V^{-1}), \]  

(1)

where \( U, V \) are unitary operators satisfying the Weyl commutation relation for an absolute value 1 complex number \( \omega, U V = \omega V U \), and \( \alpha, \beta, \mu, \nu \) are parameters with \( |\alpha| = |\beta| = 1 \), \( \mu, \nu \in \mathbb{R} \). For a primitive \( N \)th root of unity \( \omega = e^{2\pi \sqrt{-1} \Phi} \), i.e., the phase factor \( \Phi = P/N \) with \( P \) relatively prime to \( N \), one may assume the \( N \)th power identity of \( U, V \): \( U^N = V^N = 1 \). There are several physical interpretations for the Hofstadter model, especially in solid state physics. A prominent one is to consider it as a tight-binding approximation for electrons bound to atomic sites in a two-dimensional crystal and in a strong external magnetic field. The history of the Hofstadter model can trace back to the work of Peierls [20] on Bloch electrons in metals with the presence of a constant external magnetic field. By the pioneering works of the 50s and 60s [1, 10, 14, 19, 25], the model can trace back to the work of Peierls [20] on Bloch electrons in metals with the presence of a constant external magnetic field.

With the discovery of quantum Hall effects, a large number of interesting and important theoretical papers on Hamiltonians of the Hofstadter type appeared in the eighties for the quantum mechanical interpretation of the Hall conductivity plateaus (see, e.g., [13] and references therein). In 1976, Hofstadter [16] found the butterfly figure of the spectral band versus the magnetic flux, in which a beautiful fractal picture is exhibited. A detailed study of the model and other Hofstadter-type models began thereafter on the explanation on the fractal structure through various mathematical approaches, such as the semiclassical approximation, WKB-analysis on difference equation, non-commutative geometry and others [2, 7, 8, 11, 15, 24]. A pedagogical account of this important aspect can be found in a vast literature (e.g., [6, 21] and references therein).

On the other hand, motivated by the work of Wiegmann and Zabrodin [23] on the appearance of quantum \( U_q(sl_2) \) symmetry in problems of magnetic translation, Faddeev and Kashaev [12] pursued the diagonalization problem on the following type of Hamiltonian by the quantum transfer matrix method developed by the Leningrad school in the early eighties (cf, for instant, [22]):

\[ H_{FK} = \mu (\alpha U + \alpha^{-1} U^{-1}) + \nu (\beta V + \beta^{-1} V^{-1}) + \rho (\gamma W + \gamma^{-1} W^{-1}) , \]

where \( U, V, W \) are unitary operators with the Weyl commutation relations and the \( N \)th power identity property, \( U V = \omega V U, V W = \omega W V, W U = \omega U W; U^N = V^N = W^N = 1 \). With \( \rho = 0 \) in \( H_{FK} \), one has the Hofstadter Hamiltonian \( \mathfrak{H} \) with the rational flux. Though the physical content of the extended Hamiltonion \( H_{FK} \) with one more added operator \( W \) has not yet been clarified, this general formulation will provide a nice setting in the study of this kind of Hofstadter-type Hamiltonions by the quantum inverse scattering method. With the solution of Bethe ansatz equation under a certain postulated degree condition (see (5.26) of [12]), the energy expression of \( \mathfrak{H} \) obtained in [23] was reproduced in the \( \rho = 0 \) limit computation. Furthermore, a general frame work of calculating spectra of the Hamiltonian through the six-vertex model by the algebraic Bethe ansatz method was presented in [12]. In this approach, the Bethe ansatz equation is formulated through the Baxter vector [3, 6, 8], visualized on a "spectral" curve associated to the corresponding Hofstadter-type model. In general, the spectral curve is a Riemann surface with a very high genus. Thus the Bethe equation here can be viewed as a version of the Baxter’s T-Q relation [3] on a high

\footnote{It is also called as the "Baxter vacuum state" in other literature. Here we follow the terminology used in [12].}
genus spectral curve. The method relies on a special local monodromy solution of the Yang-Baxter equation for the six-vertex $R$-matrix. This solution also appeared in the study of chiral Potts model \cite{4}. For a finite size $L$, the trace of the $L$-monodromy matrix gives rise to the transfer matrix acting on the quantum space $\mathbb{C}^N$, where $N$ is the order of the rational flux. The transfer matrix is composed of a set of commuting operators, one of which is a Hofstadter-like Hamiltonian. For the physical consideration, the size $L$ is kept to be a fixed number. While motivated by the Hofstadter butterfly spectral figure, the "thermodynamic" limit will be treated in a manner of the rational flux’s order $N$ tending to infinity. This process is a different kind of limiting procedure from many in quantum integrable systems, such as the XYZ spin chain or its degenerated forms. For the study of models in this paper, we are mainly concerned with the problem of diagonalizing the transfer matrix of size $L = 3$ (see \cite{12}, or Sects. 6, 7 of the content). The Bethe ansatz equations of the Hofstadter-like Hamiltonians arising in such manner were clearly proposed in \cite{12}, and the principle could be equally applied to a more general, possibly interesting, models. However, the explicit solutions and their qualitative nature have not yet been known, even in the situation when the spectral curve is a rational one. It is the object of this paper to show that the algebraic geometry method could provide an effective tool for a thorough investigation on the mathematical structure of the Bethe equation along the line of the quantum transfer matrix scheme in \cite{12}. In this work, we obtain the complete solutions of the Bethe equation for models with the rational spectral curve for $L \leq 3$, among which a special kind of Hofstadter-type Hamiltonian $H_{FK}$ is treated. We present a detailed and rigorous mathematical derivation of these solutions of Bethe ansatz equations, including the one in \cite{12}, and further expanding it to all the other sectors. Both the qualitative and quantitative properties of the solutions are studied thoroughly. With the understanding of Bethe solutions for these Hofstadter-like models of the rational flux, we apply the thermodynamic process which enables us to propose the Bethe equation for a generic flux with the desirable solutions from both the mathematical and physical considerations. The analysis we make on the solution of the model via the Bethe ansatz method has clearly revealed the algebraic geometry character of the Bethe equation. We adopt such an interpretation in approaching the diagonalization problem of certain quantum integrable models. Accordingly, the Bethe ansatz method for the Hofstadter model \cite{11} is formulated along this scheme, and the qualitative nature of Bethe solutions is obtained through the approach of algebraic curve theory.

This paper is organized as follows: In Sect. 2, we outline the concept of Baxter vector, a key ingredient for the study of the Bethe (ansatz) equation throughout this work. In order to gain the conceptual clarity, we shall present the subject from the algebraic geometry aspect. In Sect. 3, we first recall some results in \cite{12} relevant to our later discussion, fix notations on the transfer matrix and Bethe equation, then derive some general qualitative properties about eigenvalues of the transform matrix. In the next four sections, we shall consider the case when the spectral data lie on a rational curve, and perform the mathematical derivation of the answer, along with the discussion of their physical applications. In Sect. 4, we briefly review the basic procedure of limit reduction to a rational spectral curve, originally appeared in \cite{12}, and present an explicit form of Baxter vector, which will be used in later discussions. We present the complete solutions of the Bethe equations of all sectors for $L \leq 3$ in Sect. 5. In Sect. 6, we explain the "degeneracy" relations between the Bethe solutions and the eigenspaces in the quantum space of the transfer matrix. We also mention the relationship between our Bethe solutions and some known results by using the Bethe ansatz method in literature, and indicate the difference of our approach of the Bethe equation with the usual Bethe-ansatz-type technique through certain non-physical solutions obtained by the latter method. In Sect. 7, we will address the "thermodynamic" limit problem in the sense that $N$ tends to $\infty$, and discuss the Bethe equation for those Hofstadter-like Hamiltonians appeared previously.
in the context of Sect. 5, but with a generic $q$. In Sect. 8, we study the Hofstadter Hamiltonian [1] by starting from the format in [2]. Then we go through an algebraic geometry analysis of the high genus spectral curve in which solutions of the Bethe equation are represented, and the connection of the spectral curves with elliptic curves are then clarified. We obtain a primary understanding of Bethe solutions through the spectral curve, and their relation with the Heisenberg algebra. We end with the concluding remarks in Sect. 9 with a discussion of the directions of our further inquiry.

**Convention.** In this paper, $\mathbf{R}$, $\mathbf{C}$ will denote the field of real, complex numbers respectively, and $\mathbf{Z}$ the ring of integers. For positive integers $N, n$, we will denote $\otimes_n \mathbf{C}^N$ the tensor product of $n$-copies of the complex $N$-dimensional vector space $\mathbf{C}^N$, and $\mathbf{Z}_N$ the quotient ring $\mathbf{Z}/N\mathbf{Z}$.

# 2 Algebraic Geometry Preliminary and Baxter vector

In this paper, we shall denote $\omega$ a primitive $N$th root of unity, and $q = \omega^{\frac{1}{2}}$ with $q^N = (-1)^{N+1}$, in particular for odd $N$, $q = \omega^N$. Let $Z, X$ be two operators satisfying the Weyl commutation relation with the $N$th power identity,

$$ZX = \omega XZ, \quad Z^N = X^N = I.$$  

The algebra generated by $Z, X$ is called the Weyl algebra, in which the element $ZX$ will be denoted by $Y := ZX$. The following relations hold,

$$XY = \omega^{-1} YX, \quad YZ = \omega^{-1} YZ, \quad Y^N = (-1)^{N-1}.$$  

The canonical irreducible representation of the Weyl algebra is given by the following expressions on the $N$-dimensional space $\mathbf{C}^N$,

$$Z : v \mapsto Z(v), \quad Z(v)_k = q^{2k}v_k,$$

$$X : v \mapsto X(v), \quad X(v)_k = v_{k-1},$$

$$Y : v \mapsto Y(v), \quad Y(v)_k = q^{2k}v_{k-1}. \quad (2)$$

Here a vector $v$ of $\mathbf{C}^N$ is represented by a sequence of coordinates, $v = (v_k)_{k \in \mathbf{Z}_N}$, with the $N$-periodic condition, $v_k = v_{k+N}$. Hence one can consider the index $k$ as an element of $\mathbf{Z}_N$ in what follows if no confusion could arise. Denote $|k|$ the standard basis of $\mathbf{C}^N$, $\langle k \rangle$ the dual basis of $\mathbf{C}^{N*}$ for $k \in \mathbf{Z}_N$. The $k$-th component of a vector $v$ of $\mathbf{C}^N$ is given by $v_k = \langle k | v \rangle$. In the Weyl algebra, we shall consider only the vector subspace spanned by $X, Y, Z$ and the identity $I$, in which we denote the non-zero operator

$$\varphi_{\alpha, \beta, \gamma, \delta} := \alpha Y - \beta X - \gamma Z + \delta I : \mathbf{C}^N \longrightarrow \mathbf{C}^N.$$  

The kernel of the above operator, $\text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta})$, depends only on the ratio of the coefficients, i.e., the element $[\alpha, \beta, \gamma, \delta]$ in the projective 3-space $\mathbf{P}^3$. The non-triviality of $\text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta})$ defines a hypersurface of $\mathbf{P}^3$,

$$\mathcal{F} := \{[\alpha, \beta, \gamma, \delta] \in \mathbf{P}^3 \mid \text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta}) \neq 0\}.$$  

In fact, one has the defining equation of $\mathcal{F}$ as follows:

**Lemma 1** The surface $\mathcal{F}$ is defined by the equation,

$$\mathcal{F} : \alpha^N + \beta^N = \beta^N + \gamma^N, \quad [\alpha, \beta, \gamma, \delta] \in \mathbf{P}^3.$$  

Furthermore for $[\alpha, \beta, \gamma, \delta] \in \mathcal{F}$, $\text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta})$ is an one-dimensional subspace of $\mathbf{C}^N$ generated by a vector $v = (v_m)$ with the ratios, $v_m : v_{m-1} = (\alpha \omega^m - \beta) : (\gamma \omega^m - \delta)$. 

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Proof. For \( v \in \mathbb{C}^N \), it is obvious that the criterion of \( v \) in \( \text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta}) \) is described by the above ratio relations of \( v_m \). For a non-zero vector \( v \) of \( \mathbb{C}^N \), the periodic relation \( v_m = v_{m+N} \) for a non-zero component \( v_m \) gives rise to the equation of \( F \). \( \square \)

Let \( v \) be a basis of \( \text{Ker}(\varphi_{\alpha, \beta, \gamma, \delta}) \) for \([\alpha, \beta, \gamma, \delta] \in \mathcal{F} \). Note that there are \( N^2 \) (projective) lines in \( \mathcal{F} \), defined by \( \alpha^N - \beta^N = 0 \), equivalently, \( \gamma^N - \delta^N = 0 \), which are labelled by \( P^1_{j,k} := \{ \alpha \omega^j - \beta = \gamma - \delta \omega^k = 0 \} \) for \( j, k \in \mathbb{Z}_N \). Outside these lines, the component of \( v \) are all non-zero. For elements in \( P^1_{j,k} \), one can set \( v = |k\rangle \) at \([0, 0, \omega^k, 1]\), and \( v = |j-1\rangle \) at \([1, \omega^j, 0, 0]\); while for the rest of elements, the indices in \( \mathbb{Z}_N \) with the non-zero components of \( v \) form a chain from \( k \) increasing to \( j-1 \).

For the later purpose, we introduce a family of non-homogenous representations of the surface \( \mathcal{F} \), depending on the parameter \( h = [a, b, c, d] \in \mathbb{P}^3 \),

\[
\alpha = \xi' a, \quad \beta = xb, \quad \gamma = -\xi' \xi cx, \quad \delta = -\xi d,
\]

where \((x, \xi, \xi') \in \mathbb{C}^3 \) satisfies the equation,

\[
S_h : \quad \xi'^N a^N - xd^N = (-\xi)^N (x^N \xi'^N c^N - d^N).
\]

For a fixed \( h \in \mathbb{P}^3 \), the operator \( \varphi_{\alpha, \beta, \gamma, \delta} \) corresponding to \((x, \xi, \xi')\) becomes

\[
F(x, \xi, \xi') := \xi' aY - xbX + \xi' \xi cxZ - \xi dI,
\]

and we have

\[
F(x, \xi - 1, \xi') = F(x, \xi, \xi') - \xi' xcZ + dI, \quad F(x, \xi, \xi' - 1) = F(x, \xi, \xi') - \xi xcZ - aY.
\]

For an element \( p = (x, \xi, \xi') \) of \( S_h \), we shall denote \(|p\rangle \) the basis of \( \text{Ker}(\varphi_{\xi', a, x, -\xi' \xi c, -\xi d}) \) with \(|0\rangle = 1 \), equivalently, \(|p\rangle \) is the vector of \( \mathbb{C}^N \) defined by

\[
\langle 0|p\rangle = 1, \quad \frac{|m\rangle\langle p|}{|m-1\rangle\langle p|} = \frac{\xi' a \omega^m - xb}{-\xi (\xi' xc \omega^m - d)}.
\]

We shall call \(|p\rangle \) the Baxter vector associated to \( p \in S_h \). Then

\[
F(x, \xi, \xi')|p\rangle = 0, \quad F(x, \xi - 1, \xi' h)|p\rangle = |\tau_{-} p\rangle \Delta_{-}(p), \quad F(x, \xi, \xi' - 1, h)|p\rangle = -|\tau_{+} p\rangle \Delta_{+}(p),
\]

where \( \Delta_{\pm} \) are the following (rational) functions of \( S_h \),

\[
\Delta_{-}(x, \xi, \xi') = d - x \xi' c, \quad \Delta_{+}(x, \xi, \xi') = \frac{\xi (ad - x^2 bc)}{\xi' a - xb},
\]

and \( \tau_{\pm} \) are the automorphisms of \( S_h \) defined by \( \tau_{\pm}(x, \xi, \xi') = (q^{\pm 1} x, q^{-1} \xi, q^{-1} \xi') \).

### 3 Transfer Matrix and the Bethe Equation

As in \([12]\), it is known that the following \( L \)-operator\(^3\) with the operator-valued entries acting on the quantum space \( \mathbb{C}^N \),

\[
L_h(x) = \begin{pmatrix}
aY & xbX \\
x cZ & d
\end{pmatrix}, \quad x \in \mathbb{C},
\]

\(^3\)The conventions in this paper are different from the ones used in \([12]\) where \( X, Y \) correspond to our \( Z, X \) here.
possesses the intertwining property of the Yang-Baxter relation,
\begin{equation}
R(x/x')(L_h(x) \otimes 1)(1 \otimes L_h(x')) = (1 \otimes L_h(x'))(L_h(x) \otimes 1)R(x/x') ,
\end{equation}
where the script letter "aux" will always indicate an operation taking on the auxiliary space \( \mathbb{C}^2 \), \( R(x) \) is the matrix of a 2-tensor of the auxiliary space with the following numerical expression,
\[
R(x) = \begin{pmatrix}
    x\omega - x^{-1} & 0 & 0 & 0 \\
    0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\
    0 & \omega - 1 & x - x^{-1} & 0 \\
    0 & 0 & 0 & x\omega - x^{-1}
\end{pmatrix}.
\]
By performing the matrix product on auxiliary spaces and the tensor product of quantum spaces, one has the \( L \)-operator associated to an element \( \vec{h} = (h_0, \ldots, h_{L-1}) \in (\mathbb{P}^3)^L \),
\[
L_{\vec{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x) := L_{h_0}(x) \otimes L_{h_1}(x) \otimes \cdots \otimes L_{h_{L-1}}(x) ,
\]
which again satisfies the relation (5). The entries of \( L_{\vec{h}}(x) \) are operators of the quantum space \( L \otimes \mathbb{C}^N \), and its trace defines the transfer matrix,
\[
T_{\vec{h}}(x) = \text{tr}_{\text{aux}}(L_{\vec{h}}(x)) .
\]
Then the commutation relation holds,
\[
[T_{\vec{h}}(x), T_{\vec{h}}(x')] = 0 , \quad x, x' \in \mathbb{C} .
\]
The transfer matrix \( T_{\vec{h}}(x) \) can also be computed by changing \( L_{h_j} \) to \( \bar{L}_{h_j} \) via a gauge transformation in the following manner,
\begin{equation}
\bar{L}_{h_j}(x) = A_j L_{h_j}(x) A_j^{-1} , \quad A_L := A_0 , \quad 0 \leq j \leq L - 1 .
\end{equation}
Set
\[
A_j = \begin{pmatrix}
    1 & \xi_j - 1 \\
    1 & \xi_j
\end{pmatrix} ,
\]
and denote the corresponding \( \bar{L}_{h_j}(x) \) by \( \bar{L}_{h_j}(x, \xi_j, \xi_{j+1}) \). With \( F_h(x, \xi, \xi') \) of (6), we have
\[
\bar{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix}
    F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\
    F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1)
\end{pmatrix} ,
\]
and
\[
T_{\vec{h}}(x) = \text{tr}_{\text{aux}}(\bar{L}_{\vec{h}}(x, \vec{\xi})) , \quad \vec{\xi} := (\xi_0, \ldots, \xi_{L-1})
\]
where
\[
\bar{L}_{\vec{h}}(x, \vec{\xi}) := \bigotimes_{j=0}^{L-1} \bar{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix}
    \bar{L}_{\vec{h},1,1}(x, \vec{\xi}) & \bar{L}_{\vec{h},1,2}(x, \vec{\xi}) \\
    \bar{L}_{\vec{h},2,1}(x, \vec{\xi}) & \bar{L}_{\vec{h},2,2}(x, \vec{\xi})
\end{pmatrix} , \quad \xi_L := \xi_0 .
\]
where \( \Delta \). By the definition of \( \Xi \), for all \( j \) with the condition \( \xi_L = \xi_0 \), imposes the constraint of elements \((p_0, \ldots, p_{L-1})\) on the product of surfaces, \( \prod_{j=0}^{L-1} S_{h_j} \), which form a curve \( \mathcal{C}_\xi \) in \( \prod_{j=0}^{L-1} S_{h_j} \) with the coordinates \((x, \xi, \ldots, \xi_{L-1})\) satisfying the relations,

\[
\mathcal{C}_\xi : \quad \xi_j^N = (-1)^N \frac{c_{j+1}}{c_j} a_j^N - x^N b_{j+1}^N, \quad j = 0, \ldots, L - 1. \tag{8}
\]

For \( p = (p_0, \ldots, p_{L-1}) \in \mathcal{C}_\xi \), the Baxter vector \(|p\rangle\) is now defined by

\[
|p\rangle := |p_0\rangle \otimes \ldots \otimes |p_{L-1}\rangle \in L \otimes \mathbb{C}^N.
\]

By the definition of \( \tilde{L}_{\tilde{h}, \tilde{\xi}} \), the Baxter vector of \( \mathcal{C}_\xi \) shares the following relations to entries of \( \tilde{L}_{\tilde{h}} \) similar to those for \( \tilde{L}_{\tilde{h}}(x, \tilde{\xi}) \) in \( \mathbb{F} \),

\[
\tilde{L}_{\tilde{h};1,1}(x, \tilde{\xi})|p\rangle = |\tau_- p\rangle \Delta_-(p), \quad \tilde{L}_{\tilde{h};2,2}(x, \tilde{\xi})|p\rangle = |\tau_+ p\rangle \Delta_+(p), \quad \tilde{L}_{\tilde{h};2,1}(x, \tilde{\xi})|p\rangle = 0,
\]

where \( \Delta_\pm, \tau_\pm \) are the functions and automorphisms of \( \mathcal{C}_\xi \) defined by

\[
\Delta_-(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} (d_j - x\xi_{j+1}a_j),
\]

\[
\Delta_+(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} \frac{\xi_j(a_jd_j - x^2b_{j+1}c_j)}{\xi_j a_j - x b_j}, \tag{9}
\]

\[
\tau_\pm : (x, \xi_0, \ldots, \xi_{L-1}) \mapsto (q^{\pm 1}x, q^{-1}\xi_0, \ldots, q^{-1}\xi_{L-1}).
\]

Then follows the important relation of the transfer matrix on the Baxter vector of the curve \( \mathcal{C}_\xi \),

\[
T_{\xi}(x)|p\rangle = |\tau_- p\rangle \Delta_-(p) + |\tau_+ p\rangle \Delta_+(p), \quad \text{for } p \in \mathcal{C}_\xi. \tag{10}
\]

As \( T_{\xi}(x) \) are commuting operators for \( x \in \mathbb{C} \), a common eigenvector \(| \varphi \rangle \) is a constant vector of \( L \otimes \mathbb{C}^N \) with an eigenvalue \( \Lambda(x) \in \mathbb{C}[x] \). Define the function \( Q(p) = | \varphi \rangle \langle p | \) of \( \mathcal{C}_\xi \), then it satisfies the following Bethe equation,

\[
\Lambda(x)Q(p) = Q(\tau_-(p))\Delta_-(p) + Q(\tau_+(p))\Delta_+(p), \quad \text{for } p \in \mathcal{C}_\xi. \tag{11}
\]

We are going to make a detailed study of the above Bethe equation for the rest of this paper. Before that, we first derive certain functional properties of the eigenvalue \( \Lambda(x) \).

**Lemma 2** With the entries \( L_{\tilde{h};i,j}(x) \) of \( L_{\tilde{h}}(x) \) in \( \mathbb{F} \), the following properties hold.

(1) Under the interchange of operators, \( a_j Y \leftrightarrow a_j \), \( b_j X \leftrightarrow c_j Z \) for all \( j \), we have the symmetries among \( L_{\tilde{h};i,j}(x)s \), \( L_{\tilde{h};1,1}(x) \leftrightarrow L_{\tilde{h};2,2}(x), \ L_{\tilde{h};1,2}(x) \leftrightarrow L_{\tilde{h};2,1}(x) \).

(II) The polynomial \( L_{\tilde{h};i,j}(x) \) is an even or odd function with the parity \((-1)^{i+j} \), and its degree is equal to \( 2\left[\frac{i+1-\delta_{i,j}}{2}\right] - 1 + \delta_{i,j} \).

**Proof.** We apply the gauge transformation \( \mathbb{F} \) with \( A_j = A_{j+1} \) for all \( j \). When

\[
A_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]


one obtains the interchange of entries of \( L_{h_j}(x) \), hence follows (I). For

\[
A_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

the corresponding \( \tilde{L}_{h_j}(x) \) is equal to \( L_{h_j}(-x) \), which implies the parity of \( L_{h_{i,j}}(x) \). The determination of the degree of \( L_{h_{i,j}}(x) \) can be obtained by certain suitable choices of the values of \( h_{i,j} \).

From the definition of \( T_h(x), \Lambda(x) \), one can easily obtain the following result.

**Proposition 1** The transform matrix \( T_h(x) \) is an operator-coefficient even polynomial of \( x \) with degree \( 2\lfloor \frac{h}{2} \rfloor \), which is invariant under the substitutions in Lemma 2 (I). The constant term of \( T_h(x) \) is given by

\[
T_0 := T_h(0) = \prod_{j=0}^{n-1} a_j \otimes Y + \prod_{j=0}^{n-1} d_j.
\]

Subsequently, the polynomial \( \Lambda(x) \) in (11) is an even function of degree \( \leq 2\lfloor \frac{h}{2} \rfloor \) with \( \Lambda(0) = q! \prod_{j=0}^{n-1} a_j + \prod_{j=0}^{n-1} d_j \) for some \( l \).

By the above proposition, we have \( T_h(x) = \sum_{j=0}^{\lfloor h/2 \rfloor} T_{2j}x^{2j} \) with \( T_0 \) given by (12). With a further study of the expressions of \( T_{2j} \)'s, one can show that they form a commuting family of operators, whose proof we won’t present here. Instead, as an illustration of this fact and also for the later use of this paper, we list below the explicit form of \( T_2 \) for \( L = 2, 3 \) in \( T_h(x) = T_0 + x^2 T_2 \), where the commutation relation of \( T_2 \) and \( T_0 \) is easily verified,

\[
\begin{align*}
L = 2, & \quad T_2 = b_0c_1X \otimes Z + c_0b_1Z \otimes X; \\
L = 3, & \quad T_2 = b_0c_1a_2X \otimes Z \otimes Y + a_0b_1c_2Y \otimes X \otimes Z + c_0a_1b_2Z \otimes Y \otimes X \\
& + c_0b_1d_2Z \otimes X \otimes I + d_0c_1b_2I \otimes Z \otimes X + b_0d_1c_2X \otimes I \otimes Z.
\end{align*}
\]

The above \( T_2 \) for \( L = 3 \) can be written as the Hofstadter-type Hamiltonian \( H_{FK} \) of Faddeev and Kashaev described in Sect. 1 (for the exact identification, see (12)). In the equation (11), \( Q(p) \) is a rational function of \( C_h \) with poles. As the functions of \( C_h, \xi_j \), \( x^Nc_j \) \( -d_j \) and \( (\xi_j + x^Nc_j) - N \), are the same, by the description of the Baxter vector, the poles of \( Q(p) \) are contained in the divisor of \( C_h \) defined by

\[
\prod_{j=0}^{L-1} (\xi_j^N x^Nc_j^N - d_j^N) = 0.
\]

Hence the understanding of the Bethe solutions of (11) relies heavily on the function theory of \( C_h \), the algebraic geometry of the curve will play a key role on the complexity of the problem. We shall specify the spectral curves in the later discussion. For our purpose, in the rest of this paper we shall only consider the situation when \( N \) is an odd integer, and denote the integer \( \lfloor \frac{h}{2} \rfloor \) by \( M \),

\[
N = 2M + 1.
\]

The operators \( X, Y, Z, S, T, U \) in (12) are corresponding to \( Z, X, Y, U, V, W \) respectively here in this paper.
4 The Rational Degenerated Bethe Equation

For the next four sections, the spectral curve $\mathcal{C}_h$ will always be the rational curve under the following assumption of degeneration,

$$a_j = q^{-1}d_j, \quad b_j = q^{-1}c_j \quad \text{for } j = 0, \ldots, L - 1 . \quad (14)$$

To make our presentation self-contained, we shall briefly review in this section the general procedure of reducing the Bethe equation on $\mathcal{C}_h$ to a polynomial equation, originally appeared in \[2\], and present an explicit form of the Baxter vector which will be convenient for the later use. By replacing $c_j, d_j$ by $\frac{c_j}{d_j}, 1$, we may assume $d_j = 1$ for all $j$. For the convenience of mathematical discussion, also suitable for physical applications, we shall assume that the parameters $c_j$s are all generic. The solutions for $\xi_j$s in (8) are given by

$$\xi_0^N = \cdots = \xi_{L-1}^N = \pm 1 ,$$

which possess the structure of a finite abelian group. Therefore $\mathcal{C}_h$ is the union of disjoint copies of the $x$-(complex) line indexed by this finite group. Instead working on the curve $\mathcal{C}_h$, the following $\tau_{\pm}$-invariant subset of $\mathcal{C}_h$ will be sufficient for our discussion of the Bethe equation,

$$\mathcal{C} := \{(x, \xi_0, \ldots, \xi_{L-1}) \mid \xi_0 = \cdots = \xi_{L-1} = q^l, \quad l \in \mathbb{Z}_N\} .$$

The curve $\mathcal{C}$ will be identified with $\mathbb{P}^1 \times \mathbb{Z}_N$:

$$\mathcal{C} = \mathbb{P}^1 \times \mathbb{Z}_N, \quad (x, q^1, \ldots, q^l) \longleftrightarrow (x, l) .$$

The Baxter vectors are now labelled by $|x, l\rangle = \otimes_{j=0}^3 |x, l\rangle_j$, where $|x, l\rangle_j \in \mathbb{C}^N$ is defined by the relations,

$$\langle 0 | x, l \rangle_j = 1 , \quad \frac{\langle k | x, l \rangle_j}{\langle k-1 | x, l \rangle_j} = \frac{q^{2k-1}(1 - xc_j q^{-l - 2k})}{(1 - xc_j q^{l + 2k})} , \quad k \in \mathbb{Z}_N .$$

We shall use the bold letter $\mathbf{k}$ to denote a multi-index vector $\mathbf{k} = (k_0, \ldots, k_{L-1})$ with $k_j \in \mathbb{Z}$; the square-length of $\mathbf{k}$ is defined by $|\mathbf{k}|^2 := \sum_{j=0}^{L-1} k_j^2$. The component-expression of the vector $|x, l\rangle$ is given by

$$\langle \mathbf{k} | x, l \rangle = q^{2|k|^2} \prod_{j=0}^{L-1} \prod_{i=1}^{k_j} \frac{1 - xc_j q^{-l - 2i}}{1 - xc_j q^{l + 2i}} , \quad k_j > 0 . \quad (15)$$

The equation (10) takes the form,

$$T(x)|x, l\rangle = |q^{-1}x, l - 1\rangle \Delta_-(x, l) + |qx, l - 1\rangle \Delta_+(x, l) , \quad (16)$$

where $\Delta_{\pm}$ are given by

$$\Delta_-(x, l) = \prod_{j=0}^{L-1} (1 - xc_j q^l) , \quad \Delta_+(x, l) = \prod_{j=0}^{L-1} \frac{1 - x^2 c_j^2}{1 - xc_j q^{-l - 2}} .$$

As in \[2\], we introduce the following two functions on the curve $\mathcal{C}$,

$$f^c(x, 2n) = \prod_{j=0}^{L-1} \prod_{k=0}^{n} \frac{1 - xc_j q^{-2(n-k)}}{1 - xc_j q^{2(n-k)}} , \quad f^c(x, 2n + 1) = \prod_{j=0}^{L-1} \prod_{k=0}^{n} \frac{1 - xc_j q^{-1 - 2(n-k)}}{1 - xc_j q^{1 + 2(n-k)}} , \quad (17)$$
By (15), we have
\[ |x\rangle^e_m = \sum_{n=0}^{N-1} |x, 2n\rangle f^e(x, 2n)\omega^{mn}, \quad |x\rangle^o_m = \sum_{n=0}^{N-1} |x, 2n+1\rangle f^o(x, 2n+1)\omega^{mn}. \] (18)

By computations, the following relations hold for \( n \in \mathbb{Z}_N \),
\[ \frac{f^e(x, 2n)}{f^o(q^{\pm 1}x, 2n - 1)} \Delta_\pm(x, 2n) = \Delta_\pm(x, 0), \quad \frac{f^o(x, 2n - 1)}{f^e(xq^{\pm 1}, 2n - 2)} \Delta_\pm(x, 2n - 1) = \Delta_\pm(x, -1). \]

Then (16) becomes the system of equations,
\[ T(x)|x\rangle}_m = |q^{-1}x\rangle}_m D^-_m(x) + |qx\rangle}_m D^+_m(x), \quad m \in \mathbb{Z}_N, \]

where
\[ |x\rangle}_m = (|x\rangle^e_m, |x\rangle^o_m), \quad D^\pm_m(x) = \left( \begin{array}{cc} 0 & \Delta_\pm(x, -1) \\ \omega^m \Delta_\pm(x, 0) & 0 \end{array} \right). \]

In what follows, it is convenient to use the notation of shifted factorial,
\[ (a; \alpha)_0 = 1, \quad (a; \alpha)_n = (1 - a)(1 - a\alpha) \cdots (1 - a\alpha^{n-1}), \quad n \in \mathbb{Z}_{\geq 0}. \]

By (15), (17), we have
\[ f^e(x, 2n)\langle k|x, 2n\rangle = q^{|k|^2} \prod_{j=0}^{L-1} \frac{(xc_j; \omega^{-1})_{k_j+n+1}}{(xc_j; \omega)_{k_j+n+1}}, \]
\[ f^o(x, 2n+1)\langle k|x, 2n+1\rangle = q^{|k|^2} \prod_{j=0}^{L-1} \frac{(xc_jq^{-1}; \omega^{-1})_{k_j+n+1}}{(xc_jq; \omega)_{k_j+n+1}}. \]

Note that each ratio-term in the above right hand side is defined when the lower index \( k_j + n + 1 \) is positive, however its value depends only the class modular \( N \). So we shall use the same notation for an arbitrary integer index \( n \) by defining the value equal to that of any positive representative of the class involved in \( \mathbb{Z}_N \), and will keep this convention in what follows. By (18), we have
\[ \langle k|x\rangle^e_m = q^{|k|^2} \sum_{n \in \mathbb{Z}_N} \omega^m \prod_{j=0}^{L-1} \frac{(xc_j; \omega^{-1})_{k_j+n+1}}{(xc_j; \omega)_{k_j+n+1}}, \]
\[ \langle k|x\rangle^o_m = q^{|k|^2} \sum_{n \in \mathbb{Z}_N} \omega^m \prod_{j=0}^{L-1} \frac{(xc_jq^{-1}; \omega^{-1})_{k_j+n+1}}{(xc_jq; \omega)_{k_j+n+1}}. \] (19)

We proceed the diagonalizing procedure on the matrix \( D^\pm_m(x) \) by a gauge transformation using an invertible \( 2 \times 2 \) matrix \( U_m(x) \),
\[ |x\rangle}_m \mapsto |x\rangle}_m := |x\rangle}_m U_m(x), \quad D^\pm_m(x) \mapsto U_m(q^{\pm 1}x)^{-1} D^\pm_m(x) U_m(x). \]

We shall choose the matrix \( U_m(x) \) of the form \[^3\]
\[ U_m(x) = \begin{pmatrix} q^{-m}u(qx) & u(qx) \\ u(x) & -q^m u(x) \end{pmatrix}. \]

\[^3\] The form of the gauge transformation matrix here is slightly different from that of (5.15) in [13], by rearrangement of the entries. As we are not able to produce the required formulation through the expression in [13], we wonder that there might be a misprint there.
Then the diagonalizable criterion of $D_m^\pm(x)$ for all $m$ is equivalent to the following equation of $u(x)$,

$$
\frac{u(\omega x)}{u(x)} = \prod_{j=0}^{L-1} \frac{1 - c_j x}{1 - c_j x q}.
$$

(20)

Note that the right hand side of the above form is equal to $\frac{\Delta^+(x, -1)}{\Delta^+(x, 0)}$, which is the same as $\frac{\Delta^-(qx, -1)}{\Delta^-(qx, 0)}$.

The resulting expression of $D_m^\pm(x)$ becomes

$$
D_m^-(x) = q^m \Delta_-(x, -1) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad D_m^+(x) = q^m \Delta_+(x, 0) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
$$

With the notations

$$
|x\rangle_m = (|x\rangle_m^+, |x\rangle_m^-), \quad (Q_m^+(x), Q_m^-(x)) = (\varphi|\rangle_m,
$$

one has

$$
|x\rangle_m^+ = |x\rangle_m^e q^{-m} u(qx) + |x\rangle_m^o u(x), \quad |x\rangle_m^- = |x\rangle_m^e u(qx) - |x\rangle_m^o q^m u(x).
$$

The Bethe equation (11) now takes the form

$$
\pm q^{-m} \Lambda(x) Q_m^\pm(x) = \prod_{j=0}^{L-1} (1 - xc_j q^{-1}) Q_m^\pm(x q^{-1}) + \prod_{j=0}^{L-1} (1 + xc_j) Q_m^\pm(x q).
$$

(21)

5 Solutions of the Rational Bethe Equation

In this section, we advance the discussion of the last section to obtain the explicit solutions of the Bethe equation (21) for $L \leq 3$, among which a special case will be shown in the next section to coincide with the one in [12, 23].

**Lemma 3** The general solutions of the rational function $u(x)$ for (21) are given by

$$
u(x) = R(x^N) \prod_{j=0}^{L-1} (c_j x; \omega)^{-1}_{M+1},
$$

where $R(x^N)$ is a rational function of $x^N$.

**Proof.** Note that the ratio of any two solutions of (20) is a rational function $r(x)$ with the relation $r(\omega x) = r(x)$, which is equivalent to $r(x) = R(x^N)$ for a rational function $R(x^N)$ of $x^N$. Therefore it suffices to show that $\prod_{j=0}^{L-1} (c_j x; \omega)^{-1}_{M+1}$ is a solution of (21), which is easily seen by $q = \omega^{M+1}$.

By the expressions of $\langle k|x\rangle_m^e, \langle k|x\rangle_m^o$ in (19), in order to have the polynomial functions of $Q_m^\pm(x)$ in (21), we choose the following gauge function $u(x)$ by setting $R(x^N) = \prod_{j=0}^{L-1} (1 - xN c_j^N)^2$ in Lemma 3

$$
u(x) = \prod_{j=0}^{L-1} (1 - xN c_j^N) (xc_j q; q^2)_M.
$$

(22)
The polynomial $Q^\pm_m(x)$ has the degree at most equal to that of $u(x)$, which is $(3M + 1)L$. By Proposition 3, one requires the polynomial solutions $Q^\pm_m(x), \Lambda(x)$ of the Bethe equation (21) with the constraints,

\[
\deg Q^\pm_m(x) \leq (3M + 1)L \\
\deg \Lambda(x) \leq 2\left[\frac{L}{2}\right], \quad \Lambda(x) = \Lambda(\xi), \quad \Lambda(0) = q^l + 1 \text{ for some } l.
\]

**Remark.** For another choice of the gauge function $u(x)$, only the function $Q^\pm_m(x)$ differs by a multiple of a certain rational function of $x^N$, which makes no effect as far as the Bethe equation is concerned. \(\square\)

**Lemma 4** Let $q$ be a primitive $N$th root of unity, $k, l$ be integers with $q^k + q^l \in \mathbb{R}$. Then $q^{k+l} = 1$.

**Proof.** By the odd property of $N$, 1 is the only real number among the $N$th roots of unity. We may assume that $1, q^k, q^l$ are three distinct numbers. The following conditions are equivalent,

\[q^k + q^l \in \mathbb{R} \iff q^k - q^{-l} \in \mathbb{R} \iff q^{k+l} - 1 \in q^l\mathbb{R} \,.
\]

By interchanging $k$ and $l$ in the above relations, one concludes that $(q^{k+l} - 1) \in q^k\mathbb{R} \cap q^l\mathbb{R}$. By $q^k \neq \pm q^l$, $q^k\mathbb{R} \cap q^l\mathbb{R}$ consists of only the zero element. Hence $q^{k+l} = 1$. \(\square\)

For the Bethe equation (21), one needs only to consider the plus-part of the equation because of the following result.

**Proposition 2** For $m \in \mathbb{Z}_N$, we have $Q^-_m(x) = 0$, $|x|^-_m = \bar{0}$, and

\[|x|^+_m = 2q^{-m}|x|^+_m u(x) = 2|x|_m^0 u(x) \,.
\]

**Proof.** Let $Q^-_m(x)$ be a non-zero polynomial solution of (21) for some $\Lambda(x)$ with $\Lambda(0) = q^l + 1$, and write $Q^-_m(x) = x^r Q^-_*m(x)$ with $Q^-_*m(0) \neq 0$. By comparing the $x^r$-coefficients of (21), we have $-q^{-m}\Lambda(0)Q^-_*m(0) = (q^{-r} + q^r)Q^-_*m(0)$, hence

\[-q^{-m}(1 + q^l) = q^{-r} + q^r \,.
\]

By Lemma 4, $q^l = q^{2m}$, which implies $q^r = q^{\pm m}$, a contradiction to the odd property of the integer $N$. Therefore the only solution $Q^-_m(x)$ for the negative-part of (21) is the trivial one. Since $Q^-_m(x)$ is of the form $\langle \varphi | x \rangle^-_m$ for an eigenvector $|\varphi \rangle$ of $T(x)$ in $\bigotimes^L \mathbb{C}^N_*$, and all such vectors $|\varphi \rangle$ form a basis of $\bigotimes^L \mathbb{C}^N_*$, hence $|x|^-_m = \bar{0}$ for all $m$. Then follows the relation (23). \(\square\)

We are going to derive some general properties of the solutions $Q^+_m(x)$ of (21).

**Lemma 5** For a polynomial $\Lambda(x)$ with $\Lambda(0) = q^l + 1$, the necessary and sufficient condition of $\Lambda(x)$ for the existence of a non-zero polynomial solution $Q^+_m(x)$ of the equation (21) with the eigenvalue $\Lambda(x)$ is given by the relation $q^l = q^{2m}$. In this situation, with $Q^+_m(x) = x^r Q^+_*m(x)$ and $Q^+_*m(0) \neq 0$, one has $q^r = q^{\pm m}$.

**Proof.** Let $Q^+_m(x)$ be a non-zero solution of (21) and write $Q^+_m(x) = x^r Q^+_*m(x)$ with $Q^+_*m(0) \neq 0$. By the same argument as in Proposition 3, we have

\[q^{-m}(1 + q^l) = q^{-r} + q^r \,.
\]
hence \( q^l = q^{2m} \) and \( q^r = q^{\pm m} \) by Lemma 4. It remains to show the "sufficient" part of the statement. For \( q^l = q^{2m} \), by (19) (23), one concludes \( |x \rangle_m^+ \) cannot be identically zero. We claim that the solution \( Q_m^+(x) \) in (21) has non-trivial solutions. Otherwise, this implies that \( |x \rangle_m^+ \) is always the zero vector for all \( x \) by the same argument as in Proposition 3, hence a contradiction.

**Proposition 3** Let \( m \) be an integer between 0 and \( M \), and \( Q_m^+(x), Q_{N-m}^+(x) \) be solutions of (21) for \( m, N-m \) respectively which arise from the evaluation of eigenvectors \( \langle \varphi \rangle \) of \( T_h^N(x) \) on the Baxter vector. Then \( Q_m^+(x), Q_{N-m}^+(x) \) are elements in \( x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N) \mathbb{C}[x] \).

**Proof.** The divisibility of \( Q_m^+(x), Q_{N-m}^+(x) \) by \( x^m \) follows easily from Lemma 3 so only the factor \( \prod_{j=0}^{L-1} (1 - x^N c_j^N) \) remains to be verified. As \( Q_m^+(x) \) is of the form \( \langle \varphi | x \rangle^+ \) for some vector \( \langle \varphi \rangle \) in \( \otimes \mathbb{C}^{N,*} \), and by (19) (23), it suffices to show that the following divisibility of polynomials,

\[
\prod_{j=0}^{L-1} (xc_j; w)_{M+1} | u(qx) \prod_{j=0}^{L-1} (xc_j; \omega^{-1})_{k_j+n+1},
\]

\[
\prod_{j=0}^{L-1} (xc_jq^{-1}; \omega^{-1})_{k_j+n+1},
\]

\[
\prod_{j=0}^{L-1} (xc_j; w)_{M} | u(x) \prod_{j=0}^{L-1} (xc_jq; \omega)_{k_j+n+1},
\]

By the form of \( u(x) \) in (22) and the relation \( q \omega^{M+1+j} = \omega^{j+1} \) for \( j \in \mathbb{Z} \), the above relations are easily seen. \( \Box \)

For our purpose, the functions \( Q_m^+(x) \) we shall consider are only those arising from eigenvectors of the transfer matrix, hence \( Q_m^+(x) \) is in the form of Proposition 3. For the rest of this paper, the letter \( m \) will always denote an integer between 0 and \( M \),

\[
0 \leq m \leq M.
\]

We shall conduct our discussion of the plus-part of the equation (21) for the sectors \( m, N-m \) simultaneously by introducing the polynomials \( \Lambda_m(x), Q_m(x) \) via the relation,

\[
(\Lambda_m(x), x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N) Q_m(x)) = (q^{-m} \Lambda(x), Q_m^+(x)), \ (q^m \Lambda(x), Q_{N-m}^+(x)) . \tag{24}
\]

Then the relations (21) for both the \( m \) and \( N-m \) sectors are reduced to the following:

\[
\Lambda_m(x) Q_m(x) = q^{-m} \prod_{j=0}^{L-1} (1 - xc_j q^{-1}) Q_m(xq^{-1}) + q^m \prod_{j=0}^{L-1} (1 + xc_j) Q_m(xq) , \tag{25}
\]

where \( Q_m(x), \Lambda_m(x) \) are polynomials with

\[
\deg Q_m(x) \leq ML - m, \quad \deg \Lambda_m(x) \leq 2 \left[ \frac{L}{2} \right], \quad \Lambda_m(x) = \Lambda_m(-x), \quad \Lambda(0) = q^m + q^{-m} .
\]

The general mathematical problem will be the structure of the solution space of the Bethe equation (25) for a given positive integer \( L \). First we are going to derive the detailed answer of the Bethe solutions for the simplest case \( L = 1 \).

\[ \textbf{L = 1}. \] We have \( \Lambda_m(x) = q^m + q^{-m}, \) and \( \deg Q_m(x) \leq M - m. \)
Theorem 1  The solutions $Q_m(x)$ of $(25)_{L=1}$ forms an one-dimensional vector space generated by the following polynomial of degree $M - m$,

$$B_m(x) := 1 + \sum_{j \geq 1} \left( \prod_{i=1}^{j} \frac{q^{m+i-1} - q^{-m-i}}{q^m + q^{-m} - q^{-m-i} - q^{m+i}} \right) (xc_0)^j. \tag{26}$$

(Note that the coefficients in the above expression are zero for $j > M - m$.)

Proof.  Write

$$Q_m(x) = \sum_{j=0}^{M-m} \beta_j (xc_0)^j.$$

Then $(25)_{L=1}$ is equivalent to the following system of equations of $\beta_j$s,

$$(q^m + q^{-m} - q^{-m-j} - q^{m+j}) \beta_j = (q^{m+j-1} - q^{-m-j}) \beta_{j-1}, \quad j \in \mathbb{Z}_0,$$  \hspace{1cm}  (27)

where $\beta_k$ is defined to be zero for $k$ not between 0 and $M - m$. As the values of $q^m + q^{-m} - q^{-m-j} - q^{m+j}, q^{-m-j} - q^{m+j-1}$ are all non-zero, the polynomial $Q_m(x)$ is determined by $\beta_0$, (or equivalently $\beta_{M-m}$), through the recursive relations (27). With $\beta_0 = 1$, this provides the basis element $B_m(x)$.

Corollary 1  The vector space of all polynomial solutions of $(25)_{L=1}$, (without the restriction of the degree of $Q_m(x)$), is $\mathbb{C}[x^N]B_m(x)$.

Proof.  By using the fact

$$q^{m+j-1} - q^{-m-j} = 0 \iff j \equiv M - m + 1 \pmod{N},$$

$$q^m + q^{-m} - q^{m+j} - q^{-m-j} = 0 \iff j \equiv 0, N - 2m \pmod{N},$$

and the relation (27) for those $j$ with $M - m + 1 + 1N < j \leq (l + 1)N$, ($l \in \mathbb{Z}_0$), any solution $\sum_{k \geq 0} \beta_j (c_0 x)^k$ of $(25)_{L=1}$ must have $\beta_k = 0$ except $k \equiv 0, \ldots, M - m \pmod{N}$, hence is an element of $\mathbb{C}[x^N]B_m(x)$ by Theorem 1. \hfill $\square$

For the Bethe equation (25) with $L > 1$, by the scaling of the variables,

$$x \mapsto \lambda^{-1}x, \quad c_j \mapsto \lambda c_j \quad \text{for } \lambda \in \mathbb{C}^*,$$

we may assume that the $x^j$-coefficients of polynomials, $\Lambda_m(x), Q_m(x)$, are always homogenous functions of $c_0, \ldots, c_{L-1}$ with the degree $j$. As (25) is invariant under permutations of $c_j$s, the coefficients of the polynomials of $x$ involved in (25) depend only on the elementary symmetric functions of $c_j$s,

$$s_j = \sum_{i_1 < \cdots < i_j} c_{i_1} \cdots c_{i_j}, \quad \text{for } j = 1, \ldots, L.$$

We shall denote

$$Q_m(x) = \sum_{j=0}^{d} \alpha_j x^j, \quad d := \deg Q_m(x) \leq LM - m,$$

and define $\alpha_j$ to be zero for $j$ not between 0 and $d$. For the rest of this section, we shall only consider the case, $L = 2, 3$.  

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\( L = 2 \). We have only two elementary symmetric functions of \( c_j \):
\[
s_1 = c_0 + c_1, \quad s_2 = c_0 c_1.
\]

**Lemma 6** Let \( n \) be an odd positive integer, \( A \) be a \( n \times n \)-matrix with the complex entries \( a_{i,j} \) satisfying the relations
\[
a_{i,j} = (-1)^{i+j+1}a_{n-j+1,n-i+1}, \quad \text{for } 1 \leq i, j \leq n.
\]
Then \( A \) is a degenerated matrix.

**Proof.** Write \( n = 2h + 1 \). The determinant of \( A \) can be expressed by
\[
det(A) = \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^{n} (-1)^{i+\sigma(i)+1}a_{n-\sigma(i)+1,n-i+1}
\]
\[
= \text{sgn}(\sigma_0)(-1)^{h+1} \sum_{\sigma'} \text{sgn}(\sigma') \prod_{j=1}^{n} a_{\sigma'(j),j} = \text{sgn}(\sigma_0)(-1)^{h+1}\det(A),
\]
where the indices \( \sigma, \sigma' \), run through all permutations of \( \{1, \ldots, n\} \), and \( \sigma_0 \) is the one defined by \( \sigma_0(i) = n - i + 1 \). By \( \text{sgn}(\sigma_0) = (-1)^h \), we have \( \det(A) = 0 \). \( \square \)

In the equation \( (25)_{L=2} \), \( \Lambda_m(x) \) is an even polynomial of degree \( \leq 2 \), and \( \text{deg} Q_m(x) \leq 2M - m \). By comparing the coefficients of the highest degree of \( x \) in \( (25) \), we have
\[
\Lambda_m(x) = (q^{m+d} + q^{-m-d-2})x^2s_2 + q^m + q^{-m}.
\]
For \( k \in \mathbb{Z} \), we define
\[
v_k = q^k + q^{-k} - q^m - q^{-m},
\]
\[
\delta_k = (q^{k-1} - q^{-k})s_1,
\]
\[
u_k = (q^{k-2} + q^{-k} - q^{m+d} - q^{-m-d-2})s_2. \tag{28}
\]
Then \( (25)_{L=2} \) is equivalent to the system of linear equations of \( \alpha_j \)s,
\[
v_{m+j}\alpha_j + \delta_{m+j}\alpha_{j-1} + u_{m+j}\alpha_{j-2} = 0, \quad j \in \mathbb{Z}_{\geq 0}. \tag{29}
\]
In fact, the non-trivial relations in the above system are those \( j \) between 1 and \( d + 1 \), hence the matrix form of \( (29) \) is given by
\[
\begin{bmatrix}
\delta_{m+d+1} & u_{m+d+1} & 0 & \cdots & 0 & 0 \\
u_{m+d} & \delta_{m+d} & u_{m+d} & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
& & & & & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\alpha_d \\
\alpha_{d-1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\alpha_0
\end{bmatrix} = \vec{0}. \tag{30}
\]

**Theorem 2** The equation \( (25)_{L=2} \) has a non-trivial solution \( Q_m(x) \) if and only if \( \text{deg} Q_m(x) = M - m + m' \) for \( 0 \leq m' \leq M \). For each such \( m' \), the eigenvalue \( \Lambda_m(x) \) in \( (25)_{L=2} \) is equal to \( \Lambda_{m,m'}(x) := q^{\frac{1}{2}}(q^{m'-1} + q^{-m'-2})x^2s_2 + q^m + q^{-m} \), and the corresponding solutions of \( Q_m(x) \) form an one-dimensional space generated by a polynomial \( B_{m,m'}(x) \) of degree \( M - m + m' \) with \( B_{m,m'}(0) = 1 \). ( Here \( q^{rac{1}{2}} := q^{M+1} \). )
Proof. Denote $d = \deg Q_m(x)$. Among those $v_j$s of the entries of the square matrix $Q_m(x)$, there is at most one zero term, which is given by $v_{N-2m} = 0$. If $Q_m(0) = 0$, this implies $Q_m(x) = x^{N-2m}Q_m^*(x)$ with $Q_m^*(0) \neq 0$, and each coefficient of the polynomial $Q_m^*(x)$ is expressed by a $Q_m^*(0)$-multiple of a certain polynomial of $c_0, c_1$. By setting $c_1 = 0, x^{N-2m}Q_m^*(x)Q_m^*(0)^{-1}$ gives rise to a solution of $(25)$, which contradicts the conclusion of Corollary 1. Therefore $Q_m(0) \neq 0$. If $d$ is less than $M - m$, again by setting $c_1 = 0$, the function $Q_m(x)$ gives rise to a solution of $(25)$ of degree $< M - m$, a contradiction to Theorem 1. Hence $d = M - m + m'$ for $0 \leq m' \leq M$. It remains to show for each such $m'$, the solutions $Q_m(x)$ form an one-dimensional vector space. As any non-trivial solution $Q_m(x)$ must have $Q_m(0) \neq 0$, this implies the injectivity of the following linear functional of the solution space,

$$Q_m(x) \mapsto Q_m(0) \in \mathbb{C}.$$ 

So one needs only to show the existence of a non-trivial solution $Q_m(x)$, which is equivalent to the degeneracy of the square matrix of size $d + 1$ in the left hand side of $(30)$. Write this square matrix in the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

(31)

where $C$ is a $(d - 2m') \times (2m' + 1)$-matrix, $A, B$ are the tri-diagonal square matrices of the size $2m' + 1, d - 2m'$ respectively. The explicit form of $A$ is given by

\[
A = \begin{pmatrix} 
\delta_{M+1+m'} & u_{M+1+m'} & 0 & \cdots & 0 & 0 \\
0 & \delta_{M+m'} & u_{M+m'} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & v_{M+2-m'} & \delta_{M+2-m'} & u_{M+2-m'} \\
0 & \cdots & 0 & \cdots & \delta_{M+1-m'} & \cdots & \cdots \\
\end{pmatrix}.
\]

By $(29)$, we have $v_j = v_{N-j}, \delta_j = -\delta_{N+1-j}, u_j = u_{N+2-j}$. This implies that the matrix $A$ satisfies the condition of Lemma 2, hence $\det(A) = 0$. Therefore the square matrix $(31)$ has the zero determinant. \[
\square\]

Remark. By the above theorem, the following data are in one-one correspondence with integers $m, m'$ between 0 and $M$,

$$(m, m') \longleftrightarrow \Lambda_{m,m'}(x) \longleftrightarrow B_{m,m'}(x).$$

The characterization of $B_{m,m'}(x)$ is given as the (unique) polynomial with $\deg B_{m,m'}(x) = M - m + m'$ and $B_{m,m'}(0) = 1$, whose coefficients $c_j$s satisfy the equation $(28)$ with

\[
v_k = q^k + q^{-k} - q^m - q^{-m}, \\
\delta_k = (q^{k-1} - q^k)s_1, \\
u_k = (q^{k-2} + q^{-k} - q^{m'-2} - q^{-m'-2})s_2.
\]

(32)

\[
\square\]

$\mathbf{L = 3.}$ There are three elementary symmetric functions of $c_j$s,

$$s_1 = c_0 + c_1 + c_2, \quad s_2 = c_0c_1 + c_1c_2 + c_2c_0, \quad s_3 = c_0c_1c_2.$$
A non-trivial solution $Q_m(x)$ of $(35)_{L=3}$ has the degree $d \leq 3M - m$, and $\Lambda_m(x)$ is of the form
\[
\Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}.
\]
Note that $\lambda_m$ is a homogeneous function of $c_1 s$ of degree 2. For $k \in \mathbb{Z}$, we define
\[
w_k = q^k + q^{-k} - q^m - q^{-m}, \\
v_k = (q^k - q^{-k}) s_1, \\
\delta_k = (q^{k-2} + q^{-k}) s_2, \\
u_k = (q^{k-3} - q^{-k}) s_3.
\]
The equation $(35)_{L=3}$ is equivalent to the system of linear equations of $\alpha_j s$,
\[
w_{m+j} \alpha_j + v_{m+j} \alpha_{j-1} + (\delta_{m+j} - \lambda_m) \alpha_{j-2} + u_{m+j} \alpha_{j-3} = 0, \quad j \in \mathbb{Z}_{\geq 0}.
\]
The non-trivial relations of the above system are those for $j$ between 1 and $d + 3$.

**Lemma 7** Let $Q_m(x)$ be a non-trivial polynomial solution of $(35)_{L=3}$ for some $\Lambda_m(x)$. Then the degree of $Q_m(x)$ is equal to $3M - m$ with $Q_m(0) \neq 0$.

**Proof.** First we note that for $j$ between $m + 1$ and $3M$, the only possible $w_j s$ with zero value are given by
\[w_{N-m} = w_{m+N} = 0.\]
Let $r$ be the zero multiplicity of $Q_m(x)$ at $x = 0$. If $Q_m(0) = 0$, by $(34)$ we have $r = N - 2m$ or $N$. The polynomial $\alpha_r^{-1} Q_m(x)$ with $c_1 = c_2 = 0$ is a non-trivial solution of $(35)_{L=1}$ with the zero multiplicity $r$. By Corollary 1, $r$ must be equal to $N$ and the degree of $Q_m(x)$ is at least $N + M - m$, which contradicts our assumption, $d \leq 3M - m$. Therefore $Q_m(0) \neq 0$. By the relation of $j = d + 3$ in $(34)$, we have $q^{m+d} = q^{-m-d-3}$, hence the only possible values of $d$ are $M - m - 1$, $3M - m$. If $d = M - m - 1$, by $w_j \neq 0$ for $m + 1 \leq j < M$, the function $Q_m(x)$ with $c_1 = c_2 = 0$ gives rise to a non-trivial solution of $(35)_{L=1}$ with the degree $< M - m$, a contradiction to Theorem 1. Therefore $d = 3M - m$. □

By the above lemma, the ”eigenfunction” $Q_m(x)$ for an eigenvalue $\Lambda_m(x)$ is unique up to a non-zero constant, hence there is an one-one correspondence between the eigenvalues and eigenstates of the Bethe equation $(34)_{L=3}$ for a given $m$. By $d = 3M - m$, the $(d + 3)$-th relation in the system $(34)$ is redundant, hence the matrix form of $(34)$ becomes
\[
\begin{pmatrix}
A - \lambda_m & 0 \\
C & B
\end{pmatrix}
\begin{pmatrix}
\alpha_d \\
\vdots \\
\alpha_0
\end{pmatrix} = 0, \quad d = 3M - m,
\]
where $A$ is the $N \times N$ matrix,
\[
A = \begin{pmatrix}
\delta'_{N-1} & u'_{N-1} & 0 & \cdots & 0 & 0 \\
u'_{N-2} & \delta'_{N-2} & u'_{N-2} & 0 & \cdots & \vdots \\
w'_{N-3} & \delta'_{N-3} & u'_{N-3} & 0 & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & w'_{1} & v'_{1} & \delta'_{1} & u'_{1} \\
0 & \cdots & 0 & w'_{0} & v'_{0} & \delta'_{0}
\end{pmatrix}
\]
with the entries defined by

\[ w_k' = q^{k+\frac{3}{2}} + q^{-k-\frac{3}{2}} - q^m - q^{-m}, \quad v_k' = (q^{k+\frac{3}{2}} - q^{-k-\frac{3}{2}})s_1, \]

\[ \delta_k' = (q^{k+\frac{1}{2}} + q^{-k-\frac{1}{2}})s_2, \quad u_k' = (q^{k+\frac{3}{2}} - q^{-k-\frac{3}{2}})s_3 , \]

and \( B, C \) are the following matrices,

\[
B = \begin{pmatrix}
\delta_{M+1} - \lambda_m & u_{M+1} & 0 & \cdots & 0 & 0 \\
v_M & \delta_{M+1} - \lambda_m & u_M & 0 & \cdots & 0 \\
w_{M-1} & v_{M-1} & \delta_{M+1} - \lambda_m & u_{M-1} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & w_{m+3} & v_{m+3} & \delta_{m+3} - \lambda_m & u_{m+3} \\
0 & \cdots & 0 & w_{m+2} & v_{m+2} & \delta_{m+2} - \lambda_m \\
0 & \cdots & 0 & 0 & w_{m+1} & v_{m+1} \\
0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & 0
\end{pmatrix}, \quad w_M = w_{M+1} = q^{\frac{k}{2}} + q^{\frac{-k}{2}} - q^m - q^{-m}.
\]

Note that the coefficient-matrix of the equation (33) is of the size \((d+2) \times (d+1)\), while there are only \(d+1\) variables \(\alpha_j\)s to be solved. The matrix \(B\) is equal to the upper-left \((M-m+1) \times (M-m)\) submatrix of the square matrix \(A - \lambda_m I\), of which the entries \(a_{ij}, 1 \leq i, j \leq N,\) satisfy the relations \(a_{i,j} = (-1)^{i+j}a_{N-j+1,N-i+1}\).

**Theorem 3** For \(0 \leq m \leq M\), the condition of the eigenvalue \(\Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}\) with a non-trivial solution \(Q_m(x)\) of the equation (25) \(_{L=3}\) is determined by the solution of \(\det(A - \lambda_m) = 0\), where \(A\) is the matrix defined by (33). For each such \(\Lambda_m(x)\), there exists a unique (up to constants) non-trivial polynomial solution \(Q_m(x)\) of (25) \(_{L=3}\), and the degree \(Q_m(x)\) is equal to \(3M - m\) with \(Q_m(0) \neq 0\).

**Proof.** By Lemma 1, one needs only to show the existence of a non-trivial solution \(\alpha_j\)s of (33) for a \(\lambda_m\) with \(\det(A - \lambda_m) = 0\). For \(m = M\), it is obvious as there is no matrix \(B\), and \(C\) is zero. For \(m < M\), with a given \(\lambda_m\), there exists a non-trivial vector in the kernel of \(A - \lambda_m\),

\[
(A - \lambda_m) \begin{pmatrix}
\alpha_{3M-m} \\
\vdots \\
\alpha_m \\
\vdots \\
\alpha_{M-m}
\end{pmatrix} = \vec{0}.
\]

As the \(u_j\)s appearing in the matrix \(B\) are all non-zero, by the fact that the rank of \(B\) is at most \(M - m\), one can extend the above \(\alpha_j\ (M - m \leq j \leq 3M - m)\) to a solution \(\alpha_j\)s of (33). The result then follows. \(\square\)

**Remark.** By the above theorem, the eigenstate \(Q_m(x)\) is unique for a given \(\Lambda_m(x)\). It implies that for each \(m\), the eigenvalues \(\Lambda_m(x)\) and eigenstates \(Q_m(x)\) of the Bethe equation (25) \(_{L=3}\) are in one-one correspondence. Note that these \(Q_m(x)\) are obtained under the constraint of \(\Lambda_m(x)\) with
the form (33), a conclusion by the analysis of the transfer matrix in Proposition 1, which we will refer as the "physical" criterion while comparing the usual Bethe-ansatz-technique in the discussion of the next section. □

6 The Degeneracy and Physical Solution Discussion of the Bethe Ansatz Relation

In this section, we first discuss the degeneracy relation of eigenspaces of the transform matrix \( T(x) \) in \( \otimes L C^{N*} \) with respect to the Bethe solutions we obtained in Sect. 5. As before, \( \Lambda(x) \) denotes the eigenvalues of \( T(x) \). By Proposition 1, the constant term of \( T(x) \) is given by

\[
T_0 = D + 1, \quad D := q^{-L} \otimes Y ,
\]

with the eigenvalue \( \Lambda(0) \), which is of the form \( q^l + 1 \). For \( l \in Z_N \), we denote

\[
E_L^l = \text{the eigensubspace of } \otimes L C^{N*} \text{ of the operator } D \text{ with the eigenvalue } q^l ,
\]

which is of the dimension \( N^{L-1} \). By Lemma 3, for \( 0 \leq m \leq M \), the equation (23) describes the relation of \( \Lambda(x) \) and \( Q^+_s(x) \) through (24) when \( \Lambda(0) = q^{2m} + 1 \) or \( q^{2(N-m)} + 1 \). For \( L = 1, T(x) \) is the constant \( q^{-1}Y + I \), and \( E_L^l \) is the eigenspace of \( Y \) for the eigenvalue \( q^l + 1 \). By the evaluation at the Baxter vector \( |x|m, |x|_{N-m} \) respectively, both the spaces, \( E_m^l \) and \( E_{N-m}^l \), give rise to the same functional space generated by \( x^m(1 - x^N c^N)B_m(x) \) with \( B_m(x) \) in Theorem 1. For \( L = 2,3 \), the expression (13) of \( T_2 \) becomes

\[
L = 2, \quad T_2 = q^{-1}c_0c_1(X \otimes Z + Z \otimes X) ,
\]

\[
L = 3, \quad T_2 = q^{-2}(c_0c_1X \otimes Z \otimes Y + c_1c_2Y \otimes X \otimes Z + c_0c_2Z \otimes Y \otimes X)
+ q^{-1}(c_0c_1Z \otimes X \otimes I + c_1c_2I \otimes Z \otimes X + c_0c_2X \otimes I \otimes Z) .
\]

We shall denote \( O_L \) the operator algebra generated by the \( L \)-tensors of \( X,Y,Z,I \) appeared in the corresponding expression of \( T_2 \). Then \( O_L \) commutes with \( D \), hence one obtains a \( O_L \)-representation on \( E_L^l \) for each \( l \).

\( L = 2 \). The \( O_2 \) is a commutative algebra with the generators \( X \otimes Z,Z \otimes X \), and it contains the element \( D(= ZX \otimes XZ) \). The \( O_2 \)-representation on \( \otimes C^{N*} \) has the eigenspace decomposition indexed by the \( (X \otimes Z,Z \otimes X) \)-eigenvalue \( (q^l,q^j) \), or equivalently, the \( (D,Z \otimes X) \)-eigenvalue \( (q^l,q^j) \), where \( i,j,l \) are elements in \( Z_N \) with the relation \( q^l = q^{j+i} \). In fact, the eigenspace is one dimensional with the basis \( \langle \phi_{j_0,j_1} | \) defined by

\[
\langle \phi_{j_0,j_1} | := \frac{1}{N^2} \sum_{k,k' \in Z_N} \omega^{jk+jk'-kk'} \langle k,k' | ,
\]

where \( (j_0,j_1) \) is related to \( (i,j) \) by \( (\omega^{j_0},\omega^{j_1}) = (q^i,q^j) \). The vectors \( \langle \phi_{j_0,j_1} | \), with \( \omega^{j_0+j_1} = q^l \), form a basis of \( E_L^l \). The permutation of tensor factors of \( \otimes C^{N*} \) induces an automorphism of \( E_L^l \), which interchanges the vectors \( \langle \phi_{j_0,j_1} | \) and \( \langle \phi_{j_1,j_0} | \). The eigenvalues of \( T_2 \) are given by \( c_0c_1(q^{l-j} + q^{j-l}) \) for \( j \in Z_N \), and the corresponding eigenspace is generated by \( \langle \phi_{j_0,j_1} | \) and \( \langle \phi_{j_1,j_0} | \), where \( \omega^{j_1} = q^l, \omega^{j_0+j_1} = q^i \), with the dimension equal to two for all \( j \) except \( j = (M+1)l \). By (24), the index \( (l,j) \) which corresponds to \( (m,m') \) of Theorem 2 is given by the relation,

\[
(q^l,q^j) = (q^{2m},q^{m-m'-\frac{1}{2}}) , \quad (q^{-2m},q^{-m-m'-\frac{1}{2}}) .
\]
With the evaluation at the corresponding Baxter vector, \( \langle \phi_{j_0,j_1} \rangle \) and \( \langle \phi_{j_1,j_0} \rangle \) give rise to the same eigenstate \( B_{m,m'}(x) \) in Theorem 3.

For \( \mathbf{L} = 3 \). We have

\[
qD = (Z \otimes X \otimes I)(X \otimes I \otimes Z)(I \otimes Z \otimes X).
\]

With the identification,

\[
U = D^{-1/2}Z \otimes X \otimes I, \quad V = D^{-1/2}X \otimes I \otimes Z,
\]

\( \mathcal{O}_3 \) is generated by \( U, V \) which satisfy the Weyl relation \( UV = q^2VU \) and the \( N \)th power identity. Hence \( \mathcal{O}_3 \) is the Heisenberg algebra and contains \( D \) as a central element. By (37), \( qD^{n/2}T_2 \) is the following Hofstadter-like Hamiltonian,

\[
c_0c_1(U + U^{-1}) + c_0c_2(V + V^{-1}) + c_1c_2(qD^{5/2}UV + q^{-1}D^{-5/2}V^{-1}U^{-1})
\]

Note that the above Hamiltonian is a special case of the Faddeev-Kashaev Hamiltonian \( H_{FK} \) with \( W = q^{-1}D^{-5/2}V^{-1}U^{-1}, \alpha = \beta = \gamma = 1 \). Our conclusion on the sector \( m = M \) is equivalent to that in [12] as it will become clearer later on. It is known that there is a unique (up to equivalence) non-trivial irreducible representation of \( \mathcal{O}_3 \), denoted by \( \mathbf{C}_n^N \), which is of the dimension \( N \). For each \( l, \mathbf{E}_3^l \) is equivalent to \( N \)-copies of \( \mathbf{C}_n^N \) as \( \mathcal{O}_3 \)-modules: \( \mathbf{E}_3^l \simeq N \mathbf{C}_n^N \). For \( 0 \leq m \leq M \), we consider the space \( \mathbf{E}_3^l \) with \( q^l = q^{+2m} \). The evaluation of \( \mathbf{E}_3^l \) on \( |x\rangle_{\pm_m} \) gives rise to a \( N \)-dimensional kernel in \( \mathbf{E}_3 \). By Theorem 3, there are \( N \) polynomial solutions \( Q_m(x) \) of degree \( 3M - m \) of (23) for each of \( N \) distinct eigenvalues \( \Lambda_m(x) \). The \( N \)-dimensional vector space spanned by those \( Q_m(x) \) realizes the irreducible representation \( \mathbf{C}_n^N \) of the Heisenberg algebra \( \mathcal{O}_3 \).

Now we discuss the relation between the Bethe equation (23) and the usual Bethe ansatz formulation in literature. For the physical interest, we are going to focus our attention only on the case \( L = 3 \). For \( 0 \leq m \leq M \), a solution \( Q_m(x) \) in (23) \( L=3 \) must have \( Q_m(0) \neq 0 \) by Theorem 3. We write

\[
\alpha_{3M-m}^{-1}Q_m(x) = \prod_{l=1}^{3M-m} (x - \frac{1}{z_l}), \quad z_l \in \mathbb{C}^*.
\]

By setting \( x = z_l^{-1} \) in (23) \( L=3 \), we obtain the following relation among \( z_l \)'s, called the Bethe ansatz equation in the physical literature as in the case of usual integrable Hamiltonian chains,

\[
q^{m + \frac{2}{p}} \prod_{j=0}^{2} \frac{z_l + c_j}{qz_l - c_j} = \prod_{n=1,n \neq l}^{3M-m} \frac{qz_l - z_n}{z_l - qz_n}, \quad 1 \leq l \leq 3M - m.
\]

The following lemma is obvious.

**Lemma 8** For a polynomial \( Q_m(x) \) of the form (11), the Bethe ansatz relation (12) for the roots of \( Q_m(x) \) is equivalent to the divisibility of \( q^{-m} \prod_{j=0}^{2} (1 - xc_jq^{-1})Q_m(xq^{-1}) + q^m \prod_{j=0}^{2} (1 + xc_j)Q_m(xq) \) by \( Q_m(x) \). In this situation, the quotient polynomial \( \Gamma(x) \) of the latter pair has the degree at most 2 with \( \Gamma(0) = q^{-m} + q^m \).

As mentioned in the remark of Theorem 3, the eigenvalue of the Bethe equation (23) \( L=3 \) is described by the eigenstate \( Q_m(x) \), which is a polynomial with the pre-described zeros satisfying (12). However, the quotient polynomial \( \Gamma(x) \) in Lemma 5 arising from a solution of (12) might not be an eigenvalue \( \Lambda_m(x) \) with the form (33) of the Bethe equation (23) \( L=3 \), i.e., \( \Gamma(x) \) might not be necessarily an even function. A solution of (12) with a non-even quotient polynomial \( \Gamma(x) \) will be called a "non-physical" one. For the sector \( m = M \), there is no non-physical Bethe ansatz solution of (12) by the following lemma.
Lemma 9 For $m = M$, the quotient polynomial $\Gamma(x)$ in Lemma 8 associated to a solution of (42) is always of the form $\Lambda_M(x)$ in (33).

Proof. By Lemma 8, $\Gamma(x) = \sum_{j=0}^{2} \gamma_j x^j$ and $\gamma_0 = q^{-M} + q^M$. It suffices to show the vanishing of $\gamma_1$. We write $\alpha_2^{-1} Q_M(x) = x^{2M} + \sum_{j=0}^{2M-1} \beta_j x^j$ with $\beta_0 \neq 0$. By comparing the $x$-coefficients of (25)$_{L=3}$, one has

$$\gamma_1 \beta_0 + \gamma_0 \beta_1 = (-q^{-M-1} + q^M) s_1 \beta_0 + (q^{-M-1} + q^{M+1}) \beta_1,$$

which implies $\gamma_1 = 0$. □

With a further argument following the proof of the above lemma, one can obtain the relations (5.26), (5.27) of [12], where the conclusion applies only on the $M$-sector. In fact, the comparison of the $x^2$-coefficients of (25)$_{L=3}$ yields the relation,

$$\lambda_M = (q^{M-1} + q^M) s_2 + (q^{M+1} - q^{M-1}) s_1 \beta_0^{-1} + (q^{M-1} + q^{M+2} - q^{M} - q^{M+1}) \beta_2 \beta_0^{-1},$$

an equivalent expression is the following one,

$$\lambda_M = (q^{\frac{M}{2}} + q^{-\frac{M}{2}}) s_2 + (q^{\frac{M}{2}} - q^{-\frac{M}{2}}) s_1 \sum_{n=1}^{2M} z_n + (q^{\frac{3}{2}} + q^{-\frac{3}{2}} - q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{l<n} z_l z_n. \quad (43)$$

With the substitution, $\mu = q^{\frac{1}{2}} c_0^{-1}, \nu = q^{\frac{1}{2}} c_1^{-1}, \rho = q^{\frac{1}{2}} c_2^{-1}$, the expressions (12)$_{m=M}$, (13) coincide with (5.26) (5.27) in [12]. By Lemma 9, the Bethe ansatz relation (12) has been shown to be equivalent to the Bethe equation (25)$_{L=3}$ for the $M$-sector. However, the parallel statement is not true for the sector $m = M - 1$, in fact, there does exist some ”non-physical” Bethe ansatz solutions of (12). An obvious example is given by the following one. By $3M - m = N$, the collection of inverse of roots of $x^N - \beta = 0$, ($\beta \neq 0$), forms a solution of the Bethe ansatz equation (12), and its associated quotient polynomial $\Gamma(x)$ in Lemma 8 is given by

$$\Gamma(x) = (q^{\frac{M}{2}} + q^{-\frac{M}{2}}) + (q^{\frac{M}{2}} - q^{\frac{1}{2}}) s_1 x + (q^{-\frac{3}{2}} + q^{\frac{3}{2}}) s_2 x^2,$$

which is not an even polynomial, required by the solutions of the equation (25). This shows that the inverse of roots of $x^N - \beta = 0$ for $\beta \neq 0$ provides a ”non-physical” solution of Bethe ansatz equation (12). By the above example, the constraint (13) on the eigenvalue $\Lambda_m(x)$ should be taken into account in the discussion of Bethe ansatz solutions of (12) for the spectrum problem of the transfer matrix in an arbitrary sector. Such a consideration will become more crucial when the problem of thermodynamic flux limit procedure is involved in the next section.

7 Rational Bethe Equation for a generic $q$

In this section, we are going to study the rational Bethe equation for a generic $q$. In particular for $|q| = 1$, it is the infinity flux limit discussion, i.e. $N \rightarrow \infty$, of the models appeared in the last two sections. In the discussion of this section, $q$ will always mean a generic one. By using the formula (2), one has the canonical representation on $C^\infty$ of the Weyl algebra generated by $Z, X$ with the relation

$$ZX = q^2 XZ, \quad Y := ZX. \quad (6)$$

$\text{The M in our paper is denoted by P in [12].}$
With \( \vec{h} \in (P^3)^L \), one defines the transfer matrix \( T_L(x) \) as in the finite \( N \) case, and then discusses its spectrum \( \Lambda(x) \). Under the degenerating assumption \((34)\), the method of the previous three sections can equally be applied to reduce the diagonalization problem of the transfer matrix to the Bethe solutions of the equation \((23)\). It is important to notice that the form of \((23)\) is valid for all finite \( N \) while keeping the size \( L \) fixed. This enforces us to use the same form of the Bethe equation for a generic \( q \) as the one in the finite \( N \) case, so that the formulation becomes compatible with the infinity \( N \) limiting process. An eigenvalue of the transfer matrix, the same for the Bethe equation, should be the generic analogy of \( \Lambda(x) \) appeared in the first relation of \((24)\), now denoted by \( \Lambda_m(x) = q^m \Lambda_m(x) \); however, for the eigenstate of the Bethe equation, we will keep the infinity \( N \) version of \( Q_m(x) \) in \((25)\). The Bethe equation for a generic \( q \) for \( m \in \mathbb{Z}_{\geq 0} \) is now described by

\[
\tilde{\Lambda}_m(x) Q_m(x) = \prod_{j=0}^{L-1} (1 - xc_j q^{-1}) Q_m(xq^{-1}) + q^{2m} \prod_{j=0}^{L-1} (1 + xc_j) Q_m(xq) ,
\]

where \( \tilde{\Lambda}_m(x) \) is an even polynomial of degree \( \leq 2[\frac{L}{2}] \) with \( \tilde{\Lambda}_m(0) = q^{2m} + 1 \), and \( Q_m(x) \) is a formal (power) series of \( x \), i.e.,

\[
Q_m(x) = \sum_{j \geq 0} \alpha_j x^j , \in \mathbb{C}[[x]] .
\]

As in the discussion of the last two sections, we are going to make a similar analysis on solutions of \((14)\) for \( L \leq 3 \). For \( L = 1 \), \( \Lambda_m(x) = q^{2m} + 1 \), and \( B_m(x) \) in Theorem \( 3 \) defines a formal series, which becomes the basis of the solution space of \((14)_{L=1} \). Note that the description of \( \Lambda_m(x) \) is consistent with the spectrum of the operator \( Y \). For \( L = 2, 3 \), by the expression \((34)\) of \( T_{2,3} \), we have the operator algebra \( \mathcal{O}_L \) as before, in which \( T_0 \) is a central element.

\( \mathbf{L = 2} \). The equation \((14)_{L=2} \) is equivalent to the linear systems \((24)\) of \( \alpha_j \)'s, where the coefficients of the equations are defined by \((32)\). By Theorem \( 2 \) the eigenvalue \( \Lambda_m(x) \) is given by

\[
\tilde{\Lambda}_{m,m'}(x) = (q^m + q^{-m'}) x^2 + q^{2m} + 1 , \quad \Lambda_{m,m'}(x) \in \mathbb{Z}_{\geq 0}.
\]

For \( m, m' \in \mathbb{Z}_{\geq 0} \), the solutions \( Q_m(x) \) of \((14)_{L=2} \) form an one-dimensional space generated by an element \( B_{m,m'}(x) \in \mathbb{C}[[x]] \) with \( B_{m,m'}(0) = 1 \). Note that the algebra \( \mathcal{O}_2 \) is commutative, and the spectra of the \( \mathcal{O}_2 \)-representation \( \otimes \mathbb{C}^{\infty \times} \) give rise to the eigenvalues of \( T(x) \), which coincide with the above \( \Lambda_{m,m'}(x) \) for \( m, m' \in \mathbb{Z}_{\geq 0} \).

\( \mathbf{L = 3} \). We consider the operator \( qDx^2 T_2 \), which is the Hamiltonian \((10)\). For the sector \( m \), its eigenvalue is given by \( q^{1-m} \tilde{\lambda}_m \), where \( \tilde{\lambda}_m \) is related to the polynomial \( \tilde{\Lambda}_m(x) \) by

\[
\tilde{\Lambda}_m(x) = \tilde{\lambda}_m x^2 + q^{2m} + 1 .
\]

With \( \lambda_m := q^{-m} \tilde{\lambda}_m \), the \( \lambda_m \)'s form the spectra of the quadric-diagonal square matrix \( A \) \((36)\) for a generic \( q \), which is now of the infinity size as \( M \) increases to \( \infty \). For each \( \tilde{\lambda}_m \), the equation \((14)_{L=3} \) of \( Q(x) \) is equivalent to the system \((34)\). It is easy to see that there exists a unique solution \( Q_m(x) \) (up to a constant) by the generic property of \( q \). Note that the same conclusion of \( Q_m(x) \) equally holds for an arbitrary complex number \( \tilde{\lambda}_m \), including those not in the spectra of \( A \). This means that solutions of the equation \((14)_{L=3} \) alone contain, but not sufficiently determine the eigenvalues of the transfer matrix \( T(x) \). Those \( \lambda_m \)'s not from the spectra of \( A \) correspond to the non-physical Bethe ansatz solutions of the finite \( N \) case, as discussed in the last section.
8 High Genus Curves and the Hofstadter Model

We are now going back to the general situation in Sect. 3. Note that the values $\xi_i^N$ s of the curve $C_h$ in (8) are determined by $\xi_0^N$ and $x^N$. Denote

$$ y = x^N, \quad \eta = \xi_0^N. $$

The variables $(y, \eta)$ defines the curve

$$ \mathcal{B}_h : \quad C_h(y)\eta^2 + (A_h(y) - D_h(y))\eta - B_h(y) = 0, \quad (45) $$

where the functions $A_h, B_h, C_h, D_h$ of $y$ are the following matrix elements,

$$ \begin{pmatrix} -A_h(y) & B_h(y) \\ C_h(y) & -D_h(y) \end{pmatrix} = \prod_{j=0}^{L-1} \begin{pmatrix} -a_{j}^N & yb_{j}^N \\ yc_{j}^N & -d_{j}^N \end{pmatrix}. $$

Note that $\mathcal{B}_h$ is a double cover of $y$-line, and $C_h$ is a $(\mathbb{Z}_N)^{L+1}$-branched cover of $\mathcal{B}_h$. The automorphisms $\tau_\pm$ generate a covering transformation group of $C_h$ over $\mathcal{B}_h$. In this section, we shall only consider the case

$$ L = 3, \quad a_0 = d_0 = 0, \quad b_0 = c_0 = 1, $$

and we assume the variables $h_1, h_2$ to be generic. The expression of $T(x)$ is given by

$$ T(x) = x^2(c_1 a_2 X \otimes Z \otimes Y + a_1 b_2 Z \otimes Y \otimes X + b_1 d_2 Z \otimes X \otimes I + d_1 c_2 X \otimes I \otimes Z), $$

equivalently, $x^2 D_{\mathcal{B}_h} T(x)$ is equal to the Hofstadter Hamiltonian (1) with $U, V$ given by (39) and $\mu, \nu, \alpha, \beta$ related to $h_1, h_2$ by

$$ \mu^2 = qb_1 c_1 a_2 d_2, \quad \alpha^2 = q^{-1} b_1 c_1^{-1} a_2^{-1} d_2, \quad \nu^2 = qa_1 d_1 b_2 c_2, \quad \beta^2 = q^{-1} a_1^{-1} d_1 b_2^{-1} c_2. $$

The curve $\mathcal{B}_h$ is defined by (45) with

$$ A_h(y) = -y^2(c_1 a_2 y^N + d_1 c_2 y^N), \quad B_h(y) = y(q^2 c_1 b_2 y^N + d^N c_2 y^N), $$

$$ C_h(y) = y(q^2 b_1 c_2^N + a_1 b_2^N), \quad D_h(y) = -y^2(a_1 b_2^N + b_1 c_2^N). $$

By factoring out the $y$-component, we consider only the main irreducible component of $\mathcal{B}_h$, denoted by

$$ \mathcal{B} : \quad (y^2 b_1 c_2^N + a_1 b_2^N)\eta^2 + (a_1 b_2^N + b_1 d_2^N - c_1 a_2^N - d_1 c_2^N)y\eta - (y^2 c_1 b_2^N + d_1 d_2^N) = 0, \quad (46) $$

which is a double-cover of $y$-line with four branched points, hence it defines an elliptic curve. For the curve $C_h$, the variables $\xi_0$ and $\xi_1$ are related by $\xi_0^N = \xi_1^{-N}$. This implies that $C_h$ consists of $N$ irreducible components, each one is isomorphic to the same curve $W$ defined by the equations in the variable $p = (x, \xi_0, \xi_2)$,

$$ \mathcal{W} : \quad \xi_{0}^{-N} = -\frac{\xi_{0}^{N} a_1^N + x^N b_1^N}{x^N \xi_{0}^N c_1^N - d_1^N}, \quad \xi_{2}^N = \frac{\xi_{0}^{N} a_2^N + x^N b_2^N}{x^N \xi_{0}^N c_2^N - d_2^N}. $$

It is easy to see that $W$ is a $N^3$-fold (branched) cover of the elliptic curve (45), and the genus of $W$ is $6N^3 - 6N^2 + 1$. We shall label the irreducible components of $C_h$ by $s \in \mathbb{Z}_N$ and denote the
elements of $C_n$ by $(p, s)$ with $p \in W$, whose $\xi_1$-coordinate is given by $\xi_0 \xi_1 = q^{2s-1}$. The relation (41) now becomes

$$T(x)|p, s\rangle = |\tau_-(p), s-1\rangle q^{2s-1} \Delta_-(p) + |\tau_+(p), s-1\rangle q^{2s} \Delta_+(p) ,$$

where $\tau_\pm$ are transformations of of $W$ defined by (42), with the coordinates only involving $(x, \xi_0, \xi_2)$, and $\Delta_\pm$ are the following functions on $W$,

$$\Delta_-(p) = -x \xi_0^{-1}(x \xi_2 c_1 - d_1)(x \xi_0 c_2 - d_2) ,$$

$$\Delta_+(p) = x \xi_2 \frac{(a_1 d_1 - x^2 b_1 c_1)(a_2 d_2 - x^2 b_2 c_2)}{(\xi_2 a_1 - x b_1)(\xi_2 a_2 - x b_2)} , \quad p = (x, \xi_0, \xi_2) \in W .$$

By averaging the vectors $|p, s\rangle$ over an element $p$ of $W$, one defines the following Baxter vector on $W$

$$|p\rangle := \frac{1}{N} \sum_{s=0}^{N-1} |p, s\rangle q^{2s} .$$

The action of the transform matrix $T(x)$ on $|p, s\rangle$ can be descended to the Baxter vector of $W$ as follows:

$$x^{-2}T(x)|p, s\rangle = |\tau_-(p)\rangle \tilde{\Delta}_-(p) + |\tau_+(p)\rangle \tilde{\Delta}_+(p) , \quad (47)$$

where $\tilde{\Delta}_\pm$ are the functions of $W$,

$$\tilde{\Delta}_-(x, \xi_0, \xi_2) = \frac{(x \xi_2 c_1 - d_1)(x \xi_0 c_2 - d_2)}{-x \xi_0} ,$$

$$\tilde{\Delta}_+(x, \xi_0, \xi_2) = \frac{\xi_2 (a_1 d_1 - x^2 b_1 c_1)(a_2 d_2 - x^2 b_2 c_2)}{x(\xi_2 a_1 - x b_1)(\xi_2 a_2 - x b_2)} .$$

By the following component expression of the Baxter vector on $C_3$,

$$\langle k_0, k_1, k_2|p, s\rangle q^{2s} = q^{(s-k_0-k_1)q^{-2k_0^2-2k_0k_1-k_1^2+k_1}} \prod_{i=1}^{k_1} \xi_0 (-\xi_2 a_1 \omega^i + x b_1) \prod_{j=1}^{k_2} \xi_2 (-\xi_0 a_2 \omega^j + x b_2) ,$$

the Baxter vector on $W$ is given by

$$\langle k_0, k_1, k_2|p\rangle = \frac{\sum_{n=0}^{N-1} q^{n^2} q^{-2k_0^2-2k_0k_1-k_1^2+k_1} \prod_{i=1}^{k_1} \xi_0 (-\xi_2 a_1 \omega^i + x b_1) \prod_{j=1}^{k_2} \xi_2 (-\xi_0 a_2 \omega^j + x b_2)}{N} . \quad (48)$$

Note that the $k_j$s in the above formula are integers modular $N$. Each product-term of the right hand side means the one for a positive integer $k_j$ representing its class in $\mathbb{Z}_N$. For an eigenvector $\langle \varphi| \otimes \mathbb{C}^N$ of the operator $x^{-2}T(x)$ with the eigenvalue $\lambda$, by (47), the function $Q(p) := \langle \varphi|p\rangle$ of $W$ satisfies the following Bethe equation,

$$\lambda Q(p) = Q(\tau_-(p)) \tilde{\Delta}_-(p) + Q(\tau_+(p)) \tilde{\Delta}_+(p) , \quad \lambda \in \mathbb{C} . \quad (49)$$

The above equation possesses a $\mathbb{Z}_2$-symmetry with respect to the following involution of $W$,

$$\sigma : W \rightarrow W , \quad p = (x, \xi_0, \xi_2) \mapsto \sigma(p) = (-x, -\xi_0, -\xi_2) .$$

In fact, the commutativity of $\sigma$ and $\tau_\pm$, and the $\sigma$-invariant property of $\tilde{\Delta}_\pm(p)$ are easily seen. Then by (48), Baxter vector $|p\rangle$ is invariant under $\sigma$, i.e., $|p\rangle = |\sigma(p)\rangle$ for $p \in W$, which implies
that \( Q(p) \) is a \( \sigma \)-invariant function. Furthermore the rational function \( Q(p) \) has the poles contained in the following divisor,

\[
\xi^{N-1} \prod_{i=1}^{N-1} (\xi_2 x c_1 \omega^i - d_1)(\xi_0 x c_2 \omega^i - d_2) = 0 ,
\]

in particular, it is regular at \( x = 0, \infty \). Note that the finite values of \( Q(p) \) at \( x = 0, \infty \) are consistent with the asymptotic values of \( \Delta_{\pm} \) at \( x = 0, \infty \) in the equation \( (43) \),

\[
\tilde{\Delta}_{\pm}(x, \xi_0, \xi_2) = \pm x^{-1}\xi_0^{-1}d_1 d_2 + O(1) \quad \text{as} \quad x \to 0 ,
\]

\[
\Delta_{\pm}(x, \xi_0, \xi_2) = \pm x \xi_2 c_1 c_2 + O(1) \quad \text{as} \quad x \to \infty .
\]

With the \( x \)-coordinate, \( W \) is a \( 2N^2 \)-cover over the \( x \)-line, unramified at points over \( x = 0, \infty \) whose \((x, \xi_0, \xi_2)\)-coordinates are given by

\[
0^\pm_{i,i'} = \pm (0, q^i \sqrt{\frac{d_1 d_2}{a_1 a_2}}, q^{i'} \sqrt{\frac{d_1 d_2}{a_1 a_2}}) , \quad \infty^\pm_{i,i'} = \pm (\infty, q^i \sqrt{\frac{c_1 c_2}{b_1 b_2}}, q^{i'} \sqrt{\frac{c_1 c_2}{b_1 b_2}}) , \quad i, i' \in \mathbf{Z}_N .
\] (50)

Consider the \( D \)-eigenspace decomposition of \( \otimes_3 \mathbf{C}^{N^2} = \bigoplus_{l \in \mathbf{Z}_N} \mathbf{E}_3^l \) in Sect. 6. The evaluation on the Baxter vector of \( W \) gives rise to the following linear transformation,

\[
\varepsilon_l : \mathbf{E}_3^l \longrightarrow \{ \text{rational functions of } \mathcal{W} \} , \quad v \mapsto \varepsilon_l(v)(p) := \langle v | p \rangle , \quad \text{for } l \in \mathbf{Z}_N .
\]

**Theorem 4** For \( l \in \mathbf{Z}_N \), the linear map \( \varepsilon_l \) is injective, hence it induces an identification of \( \mathbf{E}_3^l \) with a \( N^2 \)-dimensional functional space of \( \mathcal{W} \).

**Proof.** Define the following vectors in \( \otimes_3 \mathbf{C}^{N^2} \),

\[
\langle \psi_{k_0,k_1,k_2} \rangle = (\sum_{n=0}^{N-1} q^{n^2})^{-1} \sum_{k' \in \mathbf{Z}_N} \omega^{k_0 k_1 k'} \langle k_0, k', k_2 \rangle ,
\]

\[
\langle \phi_{j_0,j_1,j_2} \rangle = \sum_{k \in \mathbf{Z}_N} q^{2(k-j_0+j_1+j_2)-k(k-1)} \langle \psi_{j_1-k,k,k_2-k} \rangle ,
\]

where \( k_i, j_i \in \mathbf{Z}_N \). We have

\[
\langle \psi_{k_0,k_1,k_2} \rangle | Z \otimes X \otimes I = \omega^{k_0+k_1} \langle \psi_{k_0,k_1,k_2} \rangle ,
\]

\[
\langle \psi_{k_0,k_1,k_2} \rangle | X \otimes I \otimes Z = \omega^{k_2} \langle \psi_{k_0-1,k_1,k_2} \rangle ,
\]

\[
\langle \psi_{k_0,k_1,k_2} \rangle | I \otimes Z \otimes X = \langle \psi_{k_0,k_1+1,k_2-1} \rangle .
\]

By \( (38) \), one has

\[
\langle \psi_{k_0,k_1,k_2} \rangle | D = q^{-1+2(k_0+k_1+k_2)} \langle \psi_{k_0-1,k_1+1,k_2-1} \rangle , \quad \langle \phi_{j_0,j_1,j_2} \rangle | D = q^{-1+2j_0} \langle \phi_{j_0,j_1,j_2} \rangle .
\]

For a given \( l \in \mathbf{Z}_N \), let \( j_0 \) be the element in \( \mathbf{Z}_N \) defined by \( q^l = q^{-1+2j_0} \). Then the vectors \( \langle \phi_{j_0,j_1,j_2} \rangle \) with \( j_1, j_2 \in \mathbf{Z}_N \) form a basis of \( \mathbf{E}_3^l \). By \( (38) \), we have

\[
\langle \psi_{k_0,k_1,k_2} | p \rangle = N^{-1} q^{-2k_0} \sum_{k' \in \mathbf{Z}_N} q^{-k'2+(2k_1-2k_0+1)k'} \prod_{i=1}^{k'} \frac{\xi_0(\xi_2 a_1 \omega^i + xb_1)}{\xi_2 x c_1 \omega^i - d_1} \prod_{j=1}^{k_2} \frac{-\xi_0 a_2 \omega^j + xb_2}{\xi_0 x c_2 \omega^j - d_2} ,
\]

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of the theory, and hope to have elucidated the mathematical structure of the Bethe-ansatz-style algebraic geometry, even in the degenerated rational spectral curve case in order for gaining Hofstadter Hamiltonian (1), the study of the high genus spectral curve is a necessary step in solving associated model. The spectral curve is in general with a large genus, and the relations among zeros a certain "Strum-Liouville-like" difference equation on the curve, called the Bethe equation of the study of diagonalizing a Hamiltonian with the rational magnetic flux is reduced to the problem of

9 Conclusions and Perspectives

hence

\[ \langle \phi_{j_0,j_1,j_2} | 0_{\mathfrak{l},i}^{\pm} \rangle = N^{-1} q^{-2j_0^2 + j_0^2 + j_2 + j_2 (1-i-i')} \sum_{k,k' \in \mathbb{Z}_N} q^{-2k^2 + k(-2j_0 + 6j_1 + 2j_2 + 1) - k'^2 + 4k'k + (-2j_1 + 1)k'} \prod_{i=1}^{k'} \frac{\xi_0(-\xi_2a_1 + x_{i_0})}{\xi_2z_{i_0} \omega^2 - d_{ij}} \prod_{j=1}^{j_2 - k - \xi_2a_2 + x_{i_0}} \frac{\xi_0(-\xi_2a_2 + x_{i_0})}{\xi_2z_{i_0} \omega^2 - d_{ij}}. \]

Set \( p = 0_{\mathfrak{l},i}^{\pm} \) defined in (50). By the relations of their \((\xi_0, \xi_2)\)-coordinates, \( \xi_0 \xi_2 a_1 d_1^{-1} = q^{i-i'} \), \( \xi_0 \xi_2 a_1 a_2 d_2^{-1} = q^{i-i'} \), we obtain

\[ \langle \phi_{j_0,j_1,j_2} | 0_{\mathfrak{l},i}^{\pm} \rangle = N^{-1} q^{-2j_0^2 + j_0^2 + j_2 + j_2 (1-i-i')} \sum_{k,k' \in \mathbb{Z}_N} q^{-2k^2 + k(-2j_0 + 6j_1 - i + i') - k'^2 + 4k'k + (-2j_1 + 2 + i + i') + 4} \]

hence

\[ q^{j_0(1-j_1) - 2j_0^2 + j_0^2 + j_2 + j_2 (1-i-i')} \frac{i_{i,i}^{(i'+i')+2j_2(1-i') - j_1(3j_2 + 2j_2(1-i') + j_2(1-i') + j_2(1-i') + j_2(1-i'))}}{i_{i,i}^{(i'+i') + (i+i') + 3i' + 2)} = q^{i_{i,i}^{(i+i') + 2j_2(1-i')} + j_1(3j_2 + 2j_2(1-i') - j_1(3j_2 + 2j_2(1-i') + j_2(1-i')) - j_1(3j_2 + 2j_2(1-i'))}. \]

As the correspondence, \((j_1, j_2) \mapsto (-3j_1 + 2j_2, -j_1 - 2j_2)\), defines an automorphism of \( \mathbb{Z}_N^2 \), the relation (51) gives rise to an isomorphism between \( \mathfrak{E}_l \) and the space of Baxter vectors \( 0_{\mathfrak{l},i}^{\pm} \) (or equivalently \( 0_{\mathfrak{l},i}^{\pm} \)’s). This implies the injectivity of \( \varepsilon_l \). □

By the discussion in Sect. 6, \( \mathfrak{E}_l^l \) is equivalent to \( N \) copies of the standard representation as the Heisenberg algebra \( \mathcal{O}_3 \)-modules. Hence by Theorem 4, there exists an \( \mathcal{O}_3 \)-module structure on \( \varepsilon_l(\mathfrak{E}_3^l) \), inherited from the representation space \( \mathfrak{E}_3^l \). The mathematical structure of the functional space \( \varepsilon_l(\mathfrak{E}_3^l) \) by incorporating the divisor theory of Riemann surfaces into the Heisenberg algebra representation remains an algebraic geometry problem for further study.

9 Conclusions and Perspectives

We follow the framework in [12] by the quantum integrable method to study the diagonalization problem of some Hofstadter-like models. Through the Baxter vector of the spectral curve, the study of diagonalizing a Hamiltonian with the rational magnetic flux is reduced to the problem of a certain "Strum-Liouville-like" difference equation on the curve, called the Bethe equation of the associated model. The spectral curve is in general with a large genus, and the relations among zeros and poles for a solution of the Bethe equation yields a system of algebraic equations. Such systems of relations among zeros of the Bethe polynomial solution on a rational spectral curve are usually referred to the Bethe ansatz equation in literature. For certain models of physical interest, e.g., the Hofstadter Hamiltonian [1], the study of the high genus spectral curve is a necessary step in solving the spectrum problem through the algebraic Bethe-ansatz-technique. In this paper, we clarify some finer mathematical manipulations in [12], then go through all the delicate points one must consider in order to obtain the explicit Bethe solutions. A careful analysis of the mathematical nature of the the Bethe equation reveals the vital role of algebraic geometry in a thorough understanding of the Bethe ansatz method, so is the same with the need to obtain the physical answer of the associated model. For this reason, we have examined in this paper the Bethe ansatz equation in the context of algebraic geometry, even in the degenerated rational spectral curve case in order for gaining the mathematical insight of the Bethe equation. We further extend the approach to some more general situations. Above all, we have endeavoured to present a clear and self-contained account of the theory, and hope to have elucidated the mathematical structure of the Bethe-ansatz-style
method in the physical literature. The main content of this paper is in the discussions after Sect. 5, where the detailed mathematical derivation and analysis comparable to physical considerations are presented. The topics between Sect. 4 and Sect. 7 are devoted to the degenerated case, where the Bethe equations are related to models with the rational spectral curve. With an explicit gauge choice, we have conducted the mathematical investigation of the Bethe equation and obtained the complete Bethe solutions for all sectors in Sect. 5. With these mathematical results, we are able to further advance the study of the relevant physical problems, namely, the "degeneracy" of eigenstates of the transfer matrix to the Bethe solutions and the thermodynamic limit discussion in Sects. 6 and 7. Furthermore in Sect. 6, the explicit calculation we have performed for the Bethe solutions, when specializing on one particular sector, provides the results parallel to those using the usual Bethe ansatz method in [12]. Meanwhile, the finding of some non-physical Bethe ansatz solutions in other sectors supports the justification of our approach of the problem. The method we employ here can apply equally well to any number of size $L$. However, to keep things simple, we restrict our attention in this paper only to the case $L \leq 3$ in the discussion of Bethe solutions; the analysis on $L = 1, 2$ are mainly for the mathematical purpose to pave the way of discussing the models with a higher size $L$. The Hofstadter-like model is related to $L = 3$, of which the Bethe ansatz equation has been mathematically discussed in details here. For $L = 4$, it is expected that the problem should be closely related to the discrete quantum pendulum by the works of [9, 18]. The results we have obtained in this paper through the Bethe equation approach strongly indicate a promising direction to the spectra problem of other models, e.g., the discrete quantum pendulum. For the thermodynamic limit discussion, the analysis in Sect. 7 on the special Hofstadter-like Hamiltonian (40) for a generic $q$ shows that the Bethe relation on the spectrum and eigenstates we proposed are in accordance with the diophantine approximation process of an irrational flux. The comparison of our Bethe ansatz method with the $C^*$-algebra approach of semiclassical analysis employed in [6] for the multifractal spectrum structure is a fascinating problem. We plan to address the question of such program elsewhere.

For the original Hofstadter model (1), the Bethe equation is formulated as a "difference" equation of functions on a high genus spectral curve. In Sect. 8, we have made a primary investigation on its solutions. As the spectral curve is a Galois abelian cover of an elliptic curve with the covering group determined by the order $N$ of the rational flux, it would be essential to have a detailed algebraic geometry study of such a high genus curve in cooperation with the Bethe solutions, so that the base elliptic curve and the classical elliptic functions could be engaged in the theory. With our finding in the rational spectral curve case as a guidance for some appropriate direction of calculation, the analysis we have made in this paper will serve as a basis for further study of the Hofstadter model through the elliptic curve techniques in algebraic geometry. This approach would allow us to follow a similar path as in the rational spectral curve case for the study of the spectrum problem of the Hofstadter model. This rich structure requires further study, and a scheme along this line of interpretation is under current investigation. Indeed, we hope that our efforts would eventually shed new light on the role of algebraic geometry in exactly solvable integrable models. In this paper we restrict our attention only to certain Hofstadter-like models, and leave possible generalizations and applications to future work.

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