CONVEX AND STARSHAPED SETS IN MANIFOLDS
WITHOUT CONJUGATE POINTS

SAMEH SHENAWY

Abstract. Let $W^n$ be the class of $C^\infty$ complete simply connected $n$-dimensional manifolds without conjugate points. The hyperbolic space as well as Euclidean space are good examples of such manifolds. Let $W \in W^n$ and let $A$ be a subset of $W$. This article aims at characterization and building convex and starshaped sets in this class from inside. For example, it is proven that, for a compact starshaped set, the convex kernel is the intersection of stars of extreme points only. Also, if a closed unbounded convex set $A$ does not contain a totally geodesic hypersurface and its boundary has no geodesic ray, then $A$ is the convex hull of its extreme points. This result is a refinement of the well-known Karein-Millman theorem.

1. An Introduction

Let $M$ be a $C^\infty$ complete Riemannian manifold. A vector field $J$ along a geodesic $\alpha$ is called a Jacobi vector field if

$$D_\alpha^2 J + \Re(\alpha', J) \alpha' = 0$$

$D$ is the covariant derivative and $\Re$ is the curvature tensor. Two points on a geodesic $\alpha$ are said to be conjugate to each other if there is a non-trivial Jacobi vector field along $\alpha$ that vanishes at both of them. A geodesic $\alpha$ has no conjugate points if every Jacobi field along $\alpha$ vanishes at most once. A $C^\infty$ complete Riemannian manifold $M$ is called a manifold without conjugate points if every geodesic of $M$ has no conjugate points. In this case, the exponential map is a covering map at every point of $M$. Moreover, if $M$ is simply connected, then $\exp_p$ is a diffeomorphism and $M$ has the property that for every two distinct points $p$ and $q$ in $M$, there is a unique geodesic joining them. Let $W^n$ be the class of $C^\infty$ complete simply connected $n$-dimensional Riemannian manifolds without conjugate points. The hyperbolic space $H^n$, the $n$-dimensional Euclidean space $E^n$ and all manifolds with non-positive curvature are good examples of such manifolds. We refer to [6–12,17,18,20,21] and references therein for more details and examples of these manifolds.

2010 Mathematics Subject Classification. Primary 52A10, 52A30; Secondary 53B20.

Key words and phrases. Manifolds without conjugate points, hyperbolic space, convex sets, stars, starshaped sets, extreme points, kernel.
It is very nice to study the boundary of a closed set \( A \) in \( W \in \mathcal{W}^n \) and get a global properties of \( A \). For instance, the Krein-Milman theorem \(^{2,14–16,19,22}\) in the \( n \)–dimensional Euclidean space \( E^n \) asserts that every compact convex set is the convex hull of its extreme points i.e. given a compact convex set \( A \subset E^n \), one only need its extreme points \( E(A) \) to recover the set shape.

The aim of this paper is to characterize convex and starshaped sets in manifolds without conjugate points using their extreme points. Sufficient conditions for a set \( A \in \mathcal{W}^n \) to be convex, totally geodesic, and starshaped are considered. A generalization of Krein-Milman theorem to the setting of closed unbounded convex sets is given. It is clear that the convex kernel of a starshaped set \( A \subset W, W \in \mathcal{W}^2 \), is the intersection of stars of all points of \( A \). In this work, it is proven that, for a compact starshaped set, the convex kernel is the intersection of stars of extreme points only. Moreover, the original starshaped condition is replaced by a more general condition where the intersection of the stars of certain extreme points is not empty. Thus we get a characterization of starshaped sets in \( \mathcal{W}^2 \).

2. Results

Let \( W \in \mathcal{W}^n \) and let \( A \) be a non-empty subset of \( W \). The geodesic segment joining two points \( p \) and \( q \) is denoted by \([pq]\). If \( p \) is removed we write \((pq)\). The geodesic ray with vertex at \( p \) and passing through \( q \) is denoted by \( R(pq) \) while the geodesic passing through \( p \) and \( q \) is denoted by \( G(pq) \). We say that \( p \) sees \( q \) via \( A \) if \([pq] \subset A \). The set of all points of \( A \) that \( p \) sees via \( A \) is called the star of \( A \) at \( p \) and is denoted by \( A_p \). \( A \) is a starshaped set if there is a point \( p \in A \) that sees every point in \( A \) i.e. \( A_p = A \). The set of all such points \( p \) is called the kernel of \( A \) and is denoted by \( \ker A \). \( A \) is convex if \( \ker A = A \). A point \( p \in A \) is called an extreme point of \( A \) if \( p \) is not a relative interior point of any segment in \( A \). The set of all extreme points of \( A \) is called the profile of \( A \) and is denoted by \( E(A) \). Note that the definition of extreme points is introduced here to a non-convex set so it is somewhat different form the classical one. The convex hull, \( C(A) \), of \( A \) is the intersection of all convex subsets of \( E^n \) that contain \( A \). Three concepts of convex sets were introduced to complete Riemannian manifolds in [1]. The three concepts coincide in complete simply connected Riemannian manifolds without conjugate points since geodesics of these manifolds are global minimizers. \(^{3,5,18,21}\).

We begin with the following important lemmas.

Lemma 1. Let \( W \in \mathcal{W}^n \) and let \( A \) be a closed subset of \( W \). If \( a \) and \( b \) are points of \( A \) and \([ab] \notin A\), then there are two points \( x, y \in \partial A \cap [ab] \) such that \((xy) \cap A = \emptyset\).
Lemma 2. Let $W \in \mathcal{W}^m$ and let $A$ be a compact subset of $W$. Then $A$ has at least one extreme point.

Proof. Let $p$ be in $W \setminus A$. Define the real-valued continuous function $f$ on $A$ by $f(x) = d(p, x), x \in A$. Since $A$ is compact, $f$ attains its maximum value at a point $y \in A$. Thus $A$ is a subset of the closed disc $\overline{B}(p, r)$ with centre at $p$ and radius $r = d(p, y)$ defined by

$$\overline{B}(p, r) = \{x \in W : d(p, x) \leq r\}$$

The point $y$ is an extreme point of $A$ since any geodesic segment containing $p$ in its relative interior cuts the exterior of $A$. □

Theorem 1. Let $W \in \mathcal{W}^2$ and let $A$ be a compact starshaped subset of $W$. Then

$$\ker A = \bigcap_{x \in E(A)} A_x$$

Proof. Let $B = \bigcap_{x \in E(A)} A_x$. By the definition of the kernel of a star-shaped set we have

$$\ker A = \bigcap_{x \in A} A_x \subset \bigcap_{x \in E(A)} A_x = B$$

So, we need only to show that $B \subset A$. Let $x \in B \setminus \ker A$. Then there is a point $y \in A$ such that $[xy] \not\subset A$. By Lemma 1, we find two points $\overline{x}, \overline{y}$ in $\partial A \cap [xy]$ such that $(\overline{x}, \overline{y}) \cap A = \emptyset$. Let $z \in \ker A$, then $z$ sees $\overline{y}$ via $A$ and hence $R(z\overline{y}) \cap A$ is a closed geodesic segment. Let $q \in \partial A$ such that $R(z\overline{y}) \cap A = [zq]$. Suppose that $q \neq \overline{y}$. Since $x \in E(A), q$ sees $x$ via $A$. Then $z$ sees the geodesic segment $[xq]$ via $A$ and consequently $z$ sees $[x\overline{y}]$ via $A$ which is a contradiction and $q$ is not an extreme point i.e. there is a geodesic segment $[ab] \subset A$ such that $p \in (ab)$. It is clear from the choice of $q$ that $(ab) \not\subset R(z\overline{y})$. Since $z \in \ker A$, $z$ sees $(ab)$ via $A$. Thus we get two points $\overline{a} = [za] \cap [xy]$ and $\overline{b} = [zb] \cap [xy]$ such that $\overline{y} \in [\overline{a}\overline{b}]$ which contradicts the choice of $\overline{y}$. So, $q = \overline{y}$. $\overline{y}$ is not an extreme point otherwise $\overline{y}$ sees $x$. Therefore, we get a geodesic segment $[rs]$ such that $\overline{y} \in (rs) \subset A$. The geodesic $G(rs)$ separates the points $x$ and $z$ otherwise, as we do above, $z$ sees $(rs)$ via $A$ and we get a point $\overline{r} = [zr] \cap [xy] \in A$ that contradicts the choice of $\overline{y}$. Let $H_1$ be the closed half space generated by $G(x\overline{y})$ that does not contain $z$ and let $H_2$ be the half space generated by $G(z\overline{y})$ that does not contain $x$. Let $D = A \cap H_1 \cap H_2$. $D$ has a non-empty intersection with the geodesic segment $(rs)$ i.e. $D$ has points close to $\overline{y}$. Since $D$ is compact, $D$ has an extreme point $p \in \partial D$ by Lemma 2. The boundary points of $D$ are either boundary points of $A$ or points of $G(x\overline{y})$. Thus $p$ is an extreme point of $A$ i.e. $p$ sees $x$ via $A$. Since $z \in \ker A$, $z$ sees the geodesic segment $[px]$ via $A$ and consequently $[x\overline{y}] \subset A$ which is a contradiction and the point $x$ does not exist. □
Theorem 2. Let \( W \in W^2 \) and let \( A \) be a compact subset of \( W \). Suppose that \( B = \bigcap_{x \in E(A)} A_x \neq \varnothing \). Then \( \ker A = B \) if and only if for every \( x \notin A \), there is a geodesic ray with vertex at \( x \) having a non-empty intersection with \( A \).

Proof. Suppose that \( \ker A \neq B \) i.e. \( B \notin \ker A \). Thus there is a point \( z \in A \) such that \([yz] \notin A\). Then by Lemma 1, there are two points \( y \) and \( z \) in \( \partial A \cap [yz] \) such that \((yz) \cap A = \varnothing \). Let \( \rho \in [yz] \), then we get a point \( \rho \notin A \) such that the geodesic ray \( R(\rho\overline{\rho}) \) has a non-empty intersection with \( A \). Rotate the ray \( R(\rho\overline{\rho}) \) to touch \( \partial A \) such that \( \rho \) is fixed and the angle between \([\rho\overline{\rho}] \) and \([pz]\) decreases. The intersection of the new geodesic ray and \( A \) has an extreme point \( x \) of \( A \). Thus \( y \) sees \( x \) via \( A \) and \([xy]\) cuts the geodesic ray \( R(\rho\overline{\rho}) \) in a point \( a \) which is a contradiction otherwise \( a \in [yx] \) which is also a contradiction. Thus \( \ker A = B \).

To prove the second implication, let \( p \notin A \) and \( q \in \ker A \). Consider the geodesic ray \( R(qp) \) passing through \( p \). The geodesic ray \( R(qp) \setminus [qp] \) has a non-empty intersection with \( A \) otherwise \( q \notin \ker A \).

Corollary 1. Let \( W \in W^2 \) and let \( A \) be a compact subset of \( W \). Then \( A \) is starshaped if and only if \( \bigcap_{x \in E(A)} A_x \neq \varnothing \) and for every \( x \notin A \), there is a geodesic ray with vertex at \( x \) having a non-empty intersection with \( A \). Moreover, \( \ker A = \bigcap_{x \in E(A)} A_x \).

Theorem 3. Let \( W \in W^n \) and let \( A \) be a non-empty closed subset of \( W \). If \( \partial A \) is convex, then \( A \) is a convex set. Moreover, if \( A \) has a non-empty interior, then \( A \) is unbounded, \( \partial A \) is totally geodesic and \( A^c \) is also convex.

Proof. Suppose that \( A \) is not convex i.e. we get two points \( p \) and \( q \) in \( A \) such that \([pq] \) is not contained in \( A \). Since \( A \) is closed, we find two points \( r, s \) in \( \partial A \) such that \((rs) \cap A = \varnothing \) which is a contradiction and so \( A \) is convex.

To show that \( A \) is unbounded, let \( p \in \text{int}(A) \). Suppose that \( A \) is bounded and so we find a real number \( \varepsilon \) such that \( A \) is contained in the closed ball \( \overline{B}(p, \varepsilon) \) of radius \( \varepsilon \) and center at \( p \). Let \([ab]\) be any chord of \( \overline{B}(p, \varepsilon) \) that runs through \( p \). Since \( A \) and \([ab]\) are both closed and convex sets, we find \( a' \) and \( b' \) in \( \partial A \) such that \( A \cap [ab] = [a'b'] \) which is a contradiction since \([a'b']\) cuts the interior of \( A \). Therefore \( A \) is unbounded.

Assume that \( \partial A \) is not totally geodesic i.e. there are two points \( a \) and \( b \) in \( \partial A \) such that the line \( G(ab) \) passing through \( a \) and \( b \) is not contained in \( \partial A \). Since \( \partial A \) and \( G(ab) \) are closed convex sets, there are
a' and b' in ∂A such that

\[ [ab] \subset \partial A \cap G(ab) = [a'b'] \]

Let \( p \in G(ab) \setminus [a'b'] \) (i.e. \( p \in \text{int}(A) \cap G(ab) \) or \( p \in A^c \cap G(ab) \)) and assume that \( p \in R(b'a') \). If \( p \in \text{int}(A) \cap G(ab) \), then the geodesic convex cone \( C(b, \overline{B}(p, \varepsilon)) \) with vertex \( b \) and base \( \overline{B}(p, \varepsilon) \) for a sufficiently small \( \varepsilon \) shows that \( a \) is an interior point which is a contradiction see Figure 1.

\[ \text{Figure 1. Two cases for the point } p \]

Now we take \( p \in A^c \cap G(ab) \). Let \( q \) be a point of \( \text{int}(A) \). The sets \([pq]\) and \( A \) are closed convex sets and so there is a point \( q' \in \partial A \) such that \([pq] \cap A = [q'q]\). This implies that the intersection \( B = \partial A \cap C(p, \overline{B}(q, \varepsilon)) \), for a small \( \varepsilon \), is a non-empty closed convex set since \( \partial A \) is convex. Therefore, \( B \) is a convex cross section of \( C(p, \overline{B}(q, \varepsilon)) \) that determines a hypersurface \( H \) whose intersection with \( C(p, \overline{B}(q, \varepsilon)) \) is \( B \). At least one of the points \( a \) and \( b \) (say \( a \)) does not lie in \( H \) otherwise the line \( G(ab) \) lies in \( H \) which contradicts the fact that \( p \) is the vertex of the convex cone \( C(p, \overline{B}(q, \varepsilon)) \). Now, the convex cone \( C(a, B) \) has interior points which is a contradiction since both \( a \) and \( B \) are in \( \partial A \) see Figure 1. This contradiction completes the proof.

**Corollary 2.** Let \( W \in \mathcal{W}^n \) and let \( A \) be a non-empty open subset of \( W \) and \( \text{int}(\overline{A}) = A \). If \( \partial A \) is convex, then \( A \) is unbounded convex set and \( \partial A \) is totally geodesic.

**Proof.** It is clear that \( \overline{A} \) satisfies the hypothesis of Theorem 1. Therefore \( \partial \overline{A} = \partial A \) is convex and \( \overline{A} \) is convex and unbounded. Note that if \( A \) is bounded, then \( \overline{A} \) is also bounded and equivalently, \( \overline{A} \) is unbounded implies that \( A \) is unbounded. Since the interior of a closed convex set is also convex, the convexity of \( \overline{A} \) implies that \( A \) is convex. \( \square \)
Theorem 4. Let $W \in \mathcal{W}^n$ and let $A$ be a non-empty closed subset of $W$. If $(pq) \subset \text{int}(A)$ for every pair of boundary points $p, q$ of $A$, then $A$ is strictly convex.

Proof. It is enough to prove that $A$ is convex since the strict convexity of $A$ is direct. Now, we assume that $A$ is not convex i.e. there are $p, q$ in $A$ such that $[pq]$ is not contained in $A$. Since $A$ is closed, there are $p', q'$ in $\partial A$ such that $(p'q') \cap A = \emptyset$ which is a contradiction and $A$ is convex. □

Corollary 3. Let $W \in \mathcal{W}^n$ and let $A$ be a non-empty closed subset of $W$. $A$ is convex if and only if $(pq) \subset \partial A$ or $(pq) \subset \text{int}(A)$ for each pair of boundary points $p, q$.

Since the interior of a closed convex set is again convex, this result is still true for open sets such that $\text{int}(A) = A$. The following example shows that the closeness is important. Let $A$ be a subset of $E^2 \in \mathcal{W}^2$ defined by $A = \{(x, y) : 0 < x < 1, 0 < y < 1\} \cup \{(0, 0), (1, 1), (1, 0), (0, 1)\}$. $A$ is neither closed nor open and $(pq) \subset \partial A$ or $(pq) \subset \text{int}(A)$ for each pair of boundary points $p, q$ but $A$ is not convex.

Proposition 1. Let $W \in \mathcal{W}^n$ and let $A$ be a non-empty closed subset of $W$. If there is a point $p \in A$ that sees $\partial A$ via $A$, then $A$ is starshaped.

Proof. We claim that $p \in \ker A$. Suppose that $p$ is not in $\ker A$ i.e. there is a point $q \in A$ such that $[pq]$ is not contained in $A$. Since $A$ is closed, there are two points $p'$ and $q'$ in $\partial A \cap [pq]$ such that $(p'q') \cap A = \emptyset$. Thus $p$ does not see neither $p'$ nor $q'$. This contradicts the fact that $p$ sees $\partial A$ via $A$ and the proof is complete. □

It is clear that the converse of this result is also true. Thus we can say that this proposition is a characterization of the kernel of the closed starshaped sets. This means that the kernel of a closed starshaped set $A$ is only the points of $A$ that see $\partial A$. The following corollary is direct.

Corollary 4. Let $W \in \mathcal{W}^n$ and let $A$ be a non-empty closed convex subset of $W$. If $\partial A$ is starshaped, then $\ker(\partial A) \subset \ker A$.

In the light of the above results, one can test the convexity and starshapedness of a closed set $A$ using its boundary points. In the next part a minimal subset of these boundary points will build $A$ up from inside.

Theorem 5. Let $W \in \mathcal{W}^n$ and let $A$ be a non-empty closed convex subset of $W$. If $A$ has no hyperplane, then $A = C(\partial A)$.

Proof. Since $A$ is convex, $A$ is connected. We will prove that $C(\partial A)$ is open and closed in the relative topology on $A$ and hence $A = C(\partial A)$.

First, we prove that $C(\partial A)$ is open in $A$. Let $p \in C(\partial A) \subset A$. We have the following cases:
(1) \( p \in C(\partial A) \cap \text{int}(A) \): Let \( B_\delta = B(p, \delta) \cap A \). In this case there exists a real number \( \delta \) such that \( B(p, \delta) \subset A \) and so \( B_\delta = B(p, \delta) \). Suppose that \( p \) is not an interior point of \( C(\partial A) \) i.e. for any \( \delta \), \( B_\delta \) is not contained in \( C(\partial A) \) and so \( p \) is a boundary point of \( C(\partial A) \). Therefore, there is a supporting hyperplane \( H_1 \) of \( C(\partial A) \) (the closure of \( C(\partial A) \) is a closed convex subset of \( A \)) at \( p \) and \( C(\partial A) \) is contained in a closed half-space with boundary \( H_1 \). Let \( x \) be any point of \( B(p, \delta) \) that lies on the other side of \( H_1 \) and let \( H_2 \) be a parallel hyperplane to \( H_1 \) at \( x \). Since \( A \) does not contain a hyperplane, we find a point \( y \in H_2 \setminus A \). The line segment \([xy]\) cuts \( \partial A \) at a point \( z \in H_2 \) which contradicts the fact that \( H_1 \) supports \( C(\partial A) \). This contradiction implies that \( p \) is an interior point of \( C(\partial A) \) in the relative topology of \( A \).

(2) \( p \in C(\partial A) \cap \partial A \): in this case, \( B(p, \delta) \) has a non-empty intersection with \( A \) for any real number \( \delta \). Let \( B_\delta = B(p, \delta) \cap A \). Suppose that \( p \) is not an interior point of \( C(\partial A) \). Then, for any \( \delta \), the set \( B_\delta \) has a point \( x \) which is not in \( C(\partial A) \). But \( C(\partial A) \) is closed convex set and \( x \notin C(\partial A) \), and so we get a hyperplane \( H \) passing through \( x \) that separates \( x \) and \( C(\partial A) \). Since \( A \) does not have a hyperplane, there is a point \( y \) in \( H \setminus A \) where \([xy]\) cuts \( \partial A \). Thus \( H \) cuts \( \partial A \) and so \( H \) cuts \( C(\partial A) \) which is a contradiction and so \( p \) is an interior point of \( C(\partial A) \) in the relative topology on \( A \).

This discussion above implies that \( C(\partial A) \) is an open set in \( A \). Now, we want to prove that \( C(\partial A) \) is closed in \( A \). Let \( p \) be a boundary point of \( C(\partial A) \). If \( p \in \partial A \), then \( p \in C(\partial A) \). Let \( p \in \text{int}A \), then there is a small positive real number \( \delta \) such that \( B(p, \delta) \subset A \). Since \( p \) is a boundary point of \( C(\partial A) \), \( B(p, \delta) \neq B(p, \delta) \cap C(\partial A) \neq \phi \). Therefore, we find a point \( x \) in \( B(p, \delta) \) which is not in \( C(\partial A) \). Since \( C(\partial A) \) is a closed convex set, we get a hyperplane \( H \) passing through \( x \) and does not intersect \( C(\partial A) \). But \( A \) does not have a hyperplane and so \( H \) cuts \( \partial A \) which is a contradiction and \( p \in C(\partial A) \) i.e. \( C(\partial A) \) is closed in the relative topology on \( A \) and the proof is complete. \( \square \)

In general, sets need not have extreme points. The following proposition gives a sufficient condition for the existence of extreme points.

**Proposition 2.** Let \( W \in W^n \) and let \( A \) be a non-empty closed convex subset of \( W \). \( A \) contains at least one extreme point if and only if \( A \) has no geodesic.

**Proof.** Let us assume that \( A \) has a geodesic \( l \). Suppose that \( A \) has an extreme point \( p \). It is clear that \( p \notin l \). Let \( B \) be the closed convex hull of \( p \) and \( l \). \( B \) is a subset of \( A \) since \( A \) is a closed convex set containing both \( p \) and \( l \). It is clear that \( B \) contains a line passing through \( p \) and
parallel to \( l \) i.e. either \( p \) is not an extreme point or the line \( l \) does not exist. \( \square \)

**Lemma 3.** Let \( W \in \mathcal{W}^n \) and let \( A \) be a non-empty closed convex subset of \( W \). If \( H \) is a supporting totally geodesic hypersurface of \( A \), then \( E(H \cap A) \subset E(A) \)

**Proof.** Let \( p \) be an extreme point of \( H \cap A \). Suppose that \( p \notin E(A) \) i.e. there are \( x, y \) in \( \partial A \) such that \( p \in (xy) \). The hypersurface \( H \) supports \( A \) at \( p \) and so \([xy] \subset H \). This implies that \( p \in [xy] \subset H \cap A \) which contradicts the fact that \( p \) is an extreme point of \( H \cap A \). This contradiction completes the proof. \( \square \)

The minimal subset of a compact convex set \( A \) which generates \( A \) is its extreme points. Our next main theorem shows that this property is more general.

**Theorem 6.** Let \( W \in \mathcal{W}^n \) and let \( A \) be a non-empty closed convex subset of \( W \). If \( A \) has no hyperplane and its boundary has no ray, then \( A = C(E(A)) \).

**Proof.** To prove that \( A = C(E(A)) \), it suffices to prove that \( \partial A \subset C(E(A)) \) and by Theorem 3 we get that \( A = C(\partial A) \subset C(E(A)) \subset A \) and hence \( A = C(E(A)) \). Let \( p \in \partial A \). If \( p \) is an extreme point, then \( p \in E(A) \subset C(E(A)) \). Now suppose that \( p \) is not an extreme point i.e. there are \( x, y \) in \( \partial A \) such that \( p \in (xy) \). Since \( A \) is a closed convex set, there is a supporting totally geodesic hypersurface \( H \) of \( A \) at \( p \). It is clear that the set \( H \cap A \) is a non-empty closed convex subset of \( \partial A \). Since \( \partial A \) has no ray, the set \( H \cap A \) is bounded i.e. \( H \cap A \) is a compact convex set. Therefore, \( H \cap A = C(E(H \cap A)) \). But \( E(H \cap A) \subset E(A) \) and so \( p \in C(E(H \cap A)) \subset C(E(A)) \) and the proof is complete. \( \square \)

**References**

[1] Alexander, Stephanie. "Local and global convexity in complete Riemannian manifolds." Pacific Journal of Mathematics 76, no. 2 (1978): 283-289.

[2] Balashov, Maxim Viktorovich. "An Analog of the Krein–Mil’man Theorem for Strongly Convex Hulls in Hilbert Space." Mathematical Notes 71, no. 1-2 (2002): 34-38.

[3] Beltagy, M. "Sufficient conditions for convexity in manifolds without focal points." Comment. Math. Univ. Carolinae 34 (1993): 435-439.

[4] Beltagy, M. "Local and global exposed points." Acta Mathematica Scientia 15, no. 3 (1995): 335-341.

[5] Beltagy, M. *On starshaped sets*, Bull. Malays. Math. Soc., II. Ser. 11, no. 2 (1988): 49–57.

[6] Burns, Keith. "The flat strip theorem fails for surfaces with no conjugate points." Proceedings of the American Mathematical Society (1992): 199-206.

[7] Eberlein, Patrick. "Geodesic flow in certain manifolds without conjugate points." Transactions of the American Mathematical Society 167 (1972): 151-170.
CONVEX AND STARSHAPED SETS IN $W^n$

[8] Emmerich, Patrick. Rigidity of complete Riemannian manifolds without conjugate points. Shaker Verlag Gmbh, Germany, 2013.

[9] Green, Leon William. "Surfaces without conjugate points." Transactions of the American Mathematical Society 76, no. 3 (1954): 529-546.

[10] Goto, Midori S. "Manifolds without focal points." Journal of Differential Geometry 13, no. 3 (1978): 341-359.

[11] Gulliver, Robert. "On the variety of manifolds without conjugate points." Transactions of the American Mathematical Society 210 (1975): 185-201.

[12] Ivanov, Sergei, and Vitali Kapovitch. "Manifolds without conjugate points and their fundamental groups." Journal of Differential Geometry 96, no. 2 (2014): 223-240.

[13] Jaume, Daniel A., and Rubén Puente. "Conjugacy for closed convex sets." Contributions to Algebra and Geometry 46, no. 1 (2005): 131-149.

[14] Lay, Steven R. Convex sets and their applications. Courier Corporation, 2007.

[15] Li, Shougui, and Yicheng Gong. "On the relations of a convex set and its profile." In Integral Geometry And Convexity, pp. 199-211. 2006.

[16] Martínez-Maurica, J., and C. Perez Garcia. "A new approach to the Kreĭn-Milman theorem." Pacific Journal of Mathematics 120, no. 2 (1985): 417-422.

[17] Morse, Marston, and Gustav A. Hedlund. "Manifolds without conjugate points." Transactions of the American Mathematical Society 51 (1942): 362-386.

[18] O'Sullivan, John J. "Manifolds without conjugate points." Mathematische Annalen 210, no. 4 (1974): 295-311.

[19] Oates, D. K. "A non-compact Krein-Milman theorem." Pacific Journal of Mathematics 36, no. 3 (1971): 781-785.

[20] Ruggiero, Rafael Oswaldo. "A note on the divergence of geodesic rays in manifolds without conjugate points." Geometriae Dedicata 134, no. 1 (2008): 131-138.

[21] Shenawy, S. "Tietze-type theorem in 2-dimensional Riemannian manifolds without conjugate points." Balkan Journal of Geometry and Its Applications 16, no. 2 (2011): 133-137.

[22] Valentine, Frederick A. "Convex sets." McGraw-Hill, Inc., New York 1964 (1964).

Basic Science Department, Modern Academy for Engineering and Technology, Maadi, Egypt.

E-mail address: drssshenawy@eng.modern-academy.edu.eg, drshenawy@mail.com