Subgroup properties of pro-$p$ extensions of centralizers

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Abstract We prove that a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$ with procyclic edge stabilizers is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups with vertex groups being stabilizers of certain vertices of $T$ and edge groups (when non-trivial) being stabilizers of certain edges of $T$, in the following two situations: (1) the action is $n$-acylindrical, i.e., any non-identity element fixes not more than $n$ edges; (2) the group $G$ is generated by its vertex stabilizers. This theorem is applied to obtain several results about pro-$p$ groups from the class $L$ defined and studied in Kochloukova and Zalesskii (Math Z 267:109–128, 2011) as pro-$p$ analogues of limit groups. We prove that every pro-$p$ group $G$ from the class $L$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups with infinite procyclic or trivial edge groups and finitely generated vertex groups; moreover, all non-abelian vertex groups are from the class $L$ of lower level than $G$ with respect to the natural hierarchy. This allows us to give an affirmative answer to questions 9.1 and 9.3 in Kochloukova and Zalesskii (Math Z 267:109–128, 2011). Namely, we prove that a group $G$ from the class $L$ has Euler–Poincaré characteristic zero if and only if it is abelian, and if every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, then $\text{def}(G) \geq 2$. Moreover, we prove that $G$ satisfies the Greenberg–Stallings property and any finitely generated non-abelian subgroup of $G$ has finite index in its commensurator.

This research was partially supported by CNPq.

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1 Introduction

The main structure theorem of the Bass–Serre theory states that a group $G$ acting on a tree $T$ is the fundamental group of a graph of groups whose vertex and edge groups are the stabilizers of certain vertices and edges of $T$. This means that $G$ can be described by taking iterated amalgamated free products and HNN extensions. The analogue of the structure theorem in the pro-$p$ case does not hold in general [5]. Nevertheless, it was proved in [8] (see also [7]) that every finitely generated infinite pro-$p$ group that acts virtually freely on some pro-$p$ tree $D$ is isomorphic to the fundamental pro-$p$ group of a finite graph of finite $p$ groups whose edge and vertex groups are isomorphic to the stabilizers of some edges and vertices of $D$. The first objective of our paper is to prove that such a pro-$p$ version of the Bass–Serre theory structure theorem holds for finitely generated pro-$p$ groups acting on a pro-$p$ tree with procyclic edge stabilizers in any of the following two situations:

1) the action is $n$-acylindrical, i.e., any non-identity element fixes not more than $n$ consecutive edges;
2) the group $G$ is generated by its vertex stabilizers.

Theorem A Let $G$ be a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$ with procyclic edge stabilizers. Suppose that either the action is $n$-acylindrical or $G$ is generated by its vertex stabilizers. Then, $G$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups $(G, \Gamma)$ with procyclic edge groups and finitely generated vertex groups. Moreover, the vertex and non-trivial edge groups of $(G, \Gamma)$ are stabilizers of certain vertices and edges of $T$, respectively, and stabilizers of vertices of $T$ in $G$ are conjugate to subgroups of vertex groups of $(G, \Gamma)$.

The original motivation for this study was an attempt to investigate further the pro-$p$ analogues of abstract limit groups, defined and studied by Kochloukova and the second author in [14]. Limit groups have been studied extensively over the last ten years, and they played a crucial role in the solution of the Tarski problem [10–12, 23–28]. The name limit group was introduced by Sela. There are different equivalent definitions for these groups. The class of limit groups coincides with the class of fully residually free groups; under this name, they were studied by Remeslennikov, Kharlampovich and Myasnikov. One can also define limit groups as finitely generated subgroups of groups obtained from free groups of finite rank by finitely many extensions of centralizers. Starting from this definition, a special class $\mathcal{L}$ of pro-$p$ groups (pro-$p$ analogues of limit groups) was introduced in [14]. The class $\mathcal{L}$ consists of all finitely generated subgroups of pro-$p$ groups obtained from free pro-$p$ groups of finite rank by finitely many extensions of centralizers. In [14], it was shown that many properties that hold for limit groups are also satisfied by the pro-$p$ groups from the class $\mathcal{L}$. In the present paper, we study further the group theoretic structure properties of the pro-$p$ groups from the class $\mathcal{L}$ and prove some other results that are known to hold in the abstract case.
It is well known that a freely indecomposable limit group of height \( h \geq 1 \) is the fundamental group of a finite graph of groups that has infinite cyclic edge groups and has a vertex group that is a non-abelian limit group of height \( \leq h - 1 \); for example, see Proposition 2.1 in [2]. This fact allows one to prove many interesting properties for limit groups using induction arguments. The second main result of this paper is an analogue of this result for pro-\( p \) groups from the class \( \mathcal{L} \). In the pro-\( p \) case instead of height we use a similar hierarchy, termed weight, which was defined in [14].

**Theorem B** Let \( G \) be a pro-\( p \) group from the class \( \mathcal{L} \). If \( G \) has weight \( n \geq 1 \), then it is the fundamental pro-\( p \) group of a finite graph of pro-\( p \) groups that has infinite procyclic or trivial edge groups and finitely generated vertex groups. Moreover, if \( G \) is non-abelian, then it has at least one vertex group that is a non-abelian pro-\( p \) group and all the non-abelian vertex groups of \( G \) are pro-\( p \) groups from the class \( \mathcal{L} \) of weight \( \leq n - 1 \).

Case (1) of Theorem A is the key ingredient in the proof of Theorem B.

Theorem B has some interesting consequences. In [13], Kochloukova proved that any limit group \( G \) has non-positive Euler–Poincaré characteristic \( \chi(G) \) and that \( \chi(G) = 0 \) if and only if \( G \) is abelian. Inspired from this result, in [14], Kochloukova and the second author proved that any pro-\( p \) group \( G \) from the class \( \mathcal{L} \) has a non-positive Euler–Poincaré characteristic and raised the question whether it is true that \( \chi(G) = 0 \) if and only if \( G \) is abelian (see Question 9.3 in [14]). We use Theorem B to give an affirmative answer to this question. In the same paper, Kochloukova and the second author noted that if \( G \) is a limit group such that every abelian subgroup of \( G \) is cyclic and \( G \) itself is not cyclic then the deficiency \( \text{def}(G) \geq 2 \), and they raised the question whether the analogue of this result is also true for pro-\( p \) groups from the class \( \mathcal{L} \) (see Question 9.1 in [14]). We use Theorem B once more to give a positive answer to this question.

In [32], based on results of Greenberg [4], Stallings proved that if \( G \) is a free group and \( H \) and \( K \) are finitely generated subgroups of \( G \) with the property that \( H \cap K \) has finite index in both \( H \) and \( K \), then \( H \cap K \) has finite index in \( \langle H, K \rangle \), where \( \langle H, K \rangle \) denotes the subgroup of \( G \) generated by \( H \) and \( K \). Nowadays this property is known as Greenberg–Stallings property. Kapovich [9] proved that finitely generated word-hyperbolic fully residually free groups satisfy the Greenberg–Stallings property. Nikolaev and Serbin extended it to all limit groups [18]. In this paper, we prove that all pro-\( p \) groups from the class \( \mathcal{L} \) satisfy this property.

In [22], Rosset proved that every finitely generated subgroup \( H \) of a free group \( F \) has a “root”: a subgroup \( K \) of \( F \) that contains \( H \) with \( |K : H| \) finite and which contains every subgroup \( U \) of \( F \) that contains \( H \) with \( |U : H| \) finite. We extend the result of Rosset to the class of all limit groups. We also prove the existence of the root for finitely generated closed subgroups of pro-\( p \) groups from the class \( \mathcal{L} \). This allows us to show that every non-abelian finitely generated closed subgroup \( H \) of a pro-\( p \) group \( G \) from the class \( \mathcal{L} \) has finite index in its commensurator \( \text{Comm}_G(H) \). This property is also satisfied by abstract limit groups [18].

We list our results for the pro-\( p \) analogues of limit groups in the following.
Theorem C Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. Then

1. The Euler–Poincaré characteristic $\chi(G) = 0$ if and only if $G$ is abelian;
2. If every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, then $\text{def}(G) \geq 2$;
3. If every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, then $G$ has exponential subgroup growth;
4. There are only finitely many conjugacy classes of non-procyclic maximal abelian subgroups of $G$;
5. (Greenberg–Stallings property) If $H$ and $K$ are finitely generated subgroups of $G$ with the property that $H \cap K$ has finite index in both $H$ and $K$, then $H \cap K$ has finite index in $\langle H, K \rangle$;
6. If $H$ is a finitely generated subgroup of $G$, then $H$ has a root in $G$;
7. If $H$ is a finitely generated non-abelian subgroup of $G$, then $|\text{Comm}_G(H) : H| < \infty$.

We note that we can not use in our proofs standard combinatorial methods as in the abstract case because not all elements of pro-$p$ groups can be expressed as finite words of generators.

Organization. In Sect. 2 we collect some results from the theory of pro-$p$ groups acting on pro-$p$ trees that we will need in the proof of Theorem A. We prove Theorem A in Sect. 3. In Sect. 4, we prove Theorem B and parts (1), (2), (3) and (4) of Theorem C. Parts (5), (6) and (7) of Theorem C are proved in Sect. 5. In Sect. 6, we note that every finitely generated subgroup of an abstract limit group has a root.

Notation. Throughout the paper, $p$ denotes a prime. The $p$-adic integers are denoted by $\mathbb{Z}_p$. When $G$ is a topological group, then subgroups of $G$ are tacitly taken to be closed, unless otherwise stated; also $d(G)$ tacitly refers to the minimal number of topological generators of $G$. Moreover, homomorphisms between topological groups are tacitly taken to be continuous. For a pro-$p$ group $G$ acting continuously on a pro-$p$ tree $T$ we define $\tilde{G} := \langle G_x \mid x \in T \rangle$, where $G_x$ is the stabilizer of the point $x$.

2 Preliminaries on pro-$p$ groups acting on pro-$p$ trees

In this section, we collect results from the theory of pro-$p$ groups acting on pro-$p$ trees that will be used in the proof of Theorem A.

We start with some definitions, following [20]. A profinite graph is a triple $(\Gamma, d_0, d_1)$, where $\Gamma$ is a boolean space and $d_0, d_1 : \Gamma \to \Gamma$ are continuous maps such that $d_i d_j = d_j$ for $i, j \in \{0, 1\}$. The elements of $V(\Gamma) := d_0(\Gamma) \cup d_1(\Gamma)$ are called the vertices of $\Gamma$ and the elements of $E(\Gamma) := \Gamma - V(\Gamma)$ are called the edges of $\Gamma$. If $e \in E(\Gamma)$, then $d_0(e)$ and $d_1(e)$ are called the initial and terminal vertices of $e$. If there is no confusion, one can just write $\Gamma$ instead of $(\Gamma, d_0, d_1)$.

A profinite graph $\Gamma$ is said to be connected if all its finite quotient graphs are connected. Every profinite graph is an abstract graph, but in general a connected profinite graph is not necessarily connected as an abstract graph. This is true, however,
for graphs of finite diameter. More precisely, if for an abstract graph $\Gamma$ we define $\delta(\Gamma)$ to be the supremum of the diameters of its connected components (see [6]), then we have the following.

**Lemma 2.1** [6, Corollary 4]. A connected profinite graph $\Gamma$ with $\delta(\Gamma) < \infty$ consists of a single path component.

**Lemma 2.2** [6, Lemma 3] Let $\Gamma$ be a profinite graph and $\delta(\Gamma) < \infty$. Then, the path components of $\Gamma$ are exactly the connected components.

Let $(E^*(\Gamma), \ast) = (\Gamma/\text{V}(\Gamma), \ast)$ be a pointed profinite quotient space with $\text{V}(\Gamma)$ as a distinguished point, and let $\mathbb{F}_p[[E^*(\Gamma), \ast]]$ and $\mathbb{F}_p[[\text{V}(\Gamma)]]$ be respectively the free profinite $\mathbb{F}_p$ modules over the pointed profinite space $(E^*(\Gamma), \ast)$ and over the profinite space $\text{V}(\Gamma)$ (cf. [19]). Let the maps $\delta : \mathbb{F}_p[[E^*(\Gamma), \ast]] \to \mathbb{F}_p[[\text{V}(\Gamma)]]$ and $\epsilon : \mathbb{F}_p[[\text{V}(\Gamma)]] \to \mathbb{F}_p$ be defined respectively by $\delta(e) = d_1(e) - d_0(e)$ for all $e \in E^*(\Gamma)$ and $\epsilon(v) = 1$ for all $v \in \text{V}(\Gamma)$. Then, we have the following complex of free profinite $\mathbb{F}_p$ modules

$$0 \rightarrow \mathbb{F}_p[[E^*(\Gamma), \ast]] \xrightarrow{\delta} \mathbb{F}_p[[\text{V}(\Gamma)]] \xrightarrow{\epsilon} \mathbb{F}_p \rightarrow 0.$$

We say that the profinite graph $\Gamma$ is a *pro-$p$ tree* if the above sequence is exact. If $T$ is a pro-$p$ tree, then we say that a pro-$p$ group $G$ acts on $T$ if it acts continuously on $T$ and the action commutes with $d_0$ and $d_1$. For $t \in \text{V}(T) \cup E(T)$, we denote by $G_t$ the stabilizer of $t$ in $G$.

For a pro-$p$ group $G$ acting on a pro-$p$ tree $T$, let $\tilde{G}$ denote the subgroup generated by all vertex stabilizers. Moreover, for any two vertices $v$ and $w$ of $T$, let $[v, w]$ denote the geodesic connecting $v$ to $w$ in $T$, i.e., the (unique) smallest pro-$p$ subtree of $T$ that contains $v$ and $w$.

**Theorem 2.3** Let $G$ be a pro-$p$ group acting on a pro-$p$ tree $T$. Then

(a) [20, Proposition 3.5] The graph $T/\tilde{G}$ is a pro-$p$ tree.

(b) [20, Corollary 3.6] The group $G/\tilde{G}$ is a free pro-$p$ group.

(c) [20, Theorem 3.7] The set

$$T^G = \{ m \in T \mid gm = m \quad \text{for all } g \in G \}$$

of fixed points of $T$ is a pro-$p$ subtree of $T$ (possibly empty).

(d) [20, Corollary 3.8] If $v$ and $w$ are two different vertices of $T$, then $E([v, w]) \neq \emptyset$ and $(G_v \cap G_w) \leq G_e$ for every $e \in E([v, w])$.

**Lemma 2.4** [3, Lemma 2.14] Let $G$ be a profinite group acting on a profinite graph $S$ and let $m_1, m_2$ be elements of a connected component $C$ of $S$. If $g \in G$ with $gm_1 = m_2$, then $g$ leaves $C$ invariant. In other words

$$C/\text{Stab}_G(C) \subseteq S/G,$$

where $\text{Stab}_G(C)$ is the maximal subgroup of $G$ leaving $C$ invariant.
Lemma 2.5 [8, Lemma 4.7 and Lemma 4.8] Let $G := \varinjlim G_i$ be the inverse limit of an inverse system $\{G_i, \varphi_{ij}, I\}$ of pro-$p$ groups and $H_i \leq G_i$ so that $\varphi_{ij}(H_i) \leq H_j$ holds whenever $i \leq j$. Suppose that there is a constant $d$ with $d(G_i) = d$ for all $i \in I$. The following statements hold:

(a) If $d(G) = d$, then there exists $j \in I$ such that the projection $G \to G_j$ is surjective.
(b) For the induced inverse limit $H := \varprojlim H_i \leq G$, we have $H^G = \varprojlim H_i^G$.

Proposition 2.6 [8, Proposition 2.4] A pro-$p$ group $G$ acting on a pro-$p$ tree $T$ with trivial edge stabilizers such that there exists a continuous section $\sigma : V(T)/G \to V(T)$ is isomorphic to a free pro-$p$ product

$$
\left( \coprod_{v \in V(T)/G} G_{\sigma(v)} \right) \sqcup (G/(G_w | w \in V(T))).
$$

If $\Gamma$ is a connected profinite graph, then $\Gamma = \varprojlim \Gamma_i$ is the inverse limit of its finite connected quotient graphs $\Gamma_i$. Fixing a vertex $v$ and considering its images $v_i$ in $\Gamma_i$, we can define the fundamental pro-$p$ group $\pi_1(\Gamma_i)$ as the pro-$p$ completion $\pi_1^{\text{abs}}(\Gamma_i)$ of the abstract fundamental group $\pi_1^{\text{abs}}(\Gamma_i)$ with respect to the base point $v_i$ for each $i$. Then, the inverse system $\{\Gamma_i\}$ induces an inverse system of the fundamental pro-$p$ groups $\pi_1(\Gamma_i)$. The pro-$p$ fundamental group $\pi_1(\Gamma)$ is defined as the inverse limit $\pi_1(\Gamma) = \varprojlim \pi_1(\Gamma_i)$. Since $\pi_1(\Gamma_i)$ is a free pro-$p$ group, so is $\pi_1(\Gamma)$.

As a connected profinite graph $\Delta$ of finite diameter is abstractly connected, we can consider simultaneously $\pi_1(\Delta)$ and $\pi_1^{\text{abs}}(\Delta)$, and then the following holds.

Proposition 2.7 [34, Proposition 2.1] Let $\Delta$ be a connected profinite graph of finite diameter. Suppose that $\pi_1(\Delta)$ is finitely generated. Then, $\pi_1(\Delta)$ is the pro-$p$ completion of $\pi_1^{\text{abs}}(\Delta)$.

Proof By Lemma 2.4 in [34] combined with the definition on page 230 in the same paper, the fundamental pro-$p$ group of $\Delta$ is the projective limit of the Galois groups of all Galois (regular) $p$ coverings of $\Delta$. By Lemma 2.1, we have that $\Delta$ is a connected abstract graph and therefore so is every finite covering of it. Hence, the Galois groups of all Galois $p$ coverings of $\Delta$ are exactly all finite $p$ quotients of the abstract fundamental group $\pi_1^{\text{abs}}(\Delta)$. This exactly means the statement of the lemma.

Let $\Gamma$ be a connected profinite graph and suppose that $\Delta$ is a profinite subgraph of $\Gamma$. Then (see [33, page 486]), it can be defined a quotient graph $\Gamma_\Delta$ as a result of collapsing all connected components of $\Delta$; in fact, if $\Gamma = \varprojlim \Gamma_i$ is an inverse limit of finite graphs $\Gamma_i$, then $\Gamma_\Delta$ is defined as the inverse limit of finite graphs $\Gamma_i \Delta$ obtained by collapsing all connected components of the images $\Delta_i$ of $\Delta$ in $\Gamma_i$ to vertices. Then, we have the following.

Proposition 2.8 (Proposition on page 486 of [33]). The natural map $\Gamma \to \Gamma_\Delta$ induces an epimorphism $\pi_1(\Gamma) \to \pi_1(\Gamma_\Delta)$ of the fundamental groups.
The definition of the fundamental pro-$p$ group of a connected profinite graph of pro-$p$ groups is quite involved (see [36]). However, the fundamental pro-$p$ group $\Pi_1(\G, \Gamma)$ of a finite graph of finitely generated pro-$p$ groups $(\G, \Gamma)$ can be defined as the pro-$p$ completion of the abstract (usual) fundamental group $\Pi_1^{ab}(\G, \Gamma)$ (we use here the fact that every subgroup of finite index in a finitely generated pro-$p$ group is open). We shall need only this case throughout the paper. Thus, $G = \Pi_1(\G, \Gamma)$ has the following presentation

$$\Pi_1(\G, \Gamma) = \langle \G(v), \tau_e | rel(\G(v)), \partial_1(g) = \partial_0(g)^{\tau_e}, g \in \G(e), \tau_e = 1 \text{ for } e \in T;$$

here $T$ is a maximal subtree of $\Gamma$ and $\partial_0 : \G(e) \to \G(d_0(e)), \partial_1 : \G(e) \to \G(d_1(e))$ are monomorphisms.

In contrast to the abstract case, the vertex groups of $(\G, \Gamma)$ do not always embed in $\Pi_1(\G, \Gamma)$, i.e., $\Pi_1(\G, \Gamma)$ is not always proper. However, the edge and vertex groups can be replaced by their images in $\Pi_1(\G, \Gamma)$ and after this replacement $\Pi_1(\G, \Gamma)$ becomes proper. Thus, from now on we shall assume that $\Pi_1(\G, \Gamma)$ is always proper.

The fundamental pro-$p$ group $\Pi_1(\G, \Gamma)$ acts on the standard pro-$p$ tree $S$ (defined analogously to the abstract one) associated to it, with vertex and edge stabilizers being conjugates of vertex and edge groups, and such that $S / \Pi_1(\G, \Gamma) = \Gamma$ (cf. (3.7) in [35] and Section 5 in [36]).

**Theorem 2.9** [8, Theorem 3.8] or [7, Theorem 1.1] A finitely generated pro-$p$ group $G$ acting virtually freely on a pro-$p$ tree $T$ is isomorphic to the fundamental pro-$p$ group $\Pi_1(\G, \Gamma)$ of a finite graph of finite $p$ groups whose edge and vertex groups are isomorphic to the stabilizers of some edges and vertices of $T$.

**Theorem 2.10** [35, Theorem 3.10] or [36, Theorem 5.6] If $K$ is a finite subgroup of $\Pi_1(\G, \Gamma)$, then $K \subseteq g\G(v)g^{-1}$ for some $v \in V(\Gamma)$ and $g \in \Pi_1(\G, \Gamma)$.

Now we state a pro-$p$ version of Proposition 4.4 in [36]. As observed in paragraph 5.4 in [36], it is valid for any extensions closed variety of profinite groups, so in particular for pro-$p$ groups. We use here the fact that the notion of simply connectivity and acyclicity coincide in the pro-$p$ case (see [34]).

**Proposition 2.11** [36, Proposition 4.4] Let $H$ be a pro-$p$ group acting on a pro-$p$ tree $\Sigma$ in such a way that the quotient graph $\Gamma = \Sigma / H$ is finite. Then, $H$ is the fundamental group of the graph of groups $(\G, \Gamma)$, where, for each $m$ in $\Gamma$, $\G(m)$ is isomorphic to the stabilizer in $H$ of some preimage of $m$ in $\Sigma$.

**Lemma 2.12** [19, Lemma 9.1.5] Let $\{G_{1i}, \mu_{1ij}, I_1\}$ and $\{G_{2i}, \mu_{2ij}, I_2\}$ be surjective inverse systems of pro-$p$ groups over posets $I_1$ and $I_2$, respectively. Then

(a) $I_1 \times I_2$ is a poset in a natural way and $\{G_{1i} \amalg G_{2k}, \mu_{1ij} \amalg \mu_{2kr}, I_1 \times I_2\}$ is an inverse system over $I_1 \times I_2$.

(b) $$(\lim_{I_1} G_{1i}) \amalg (\lim_{I_2} G_{2i}) \cong \lim_{I_1 \times I_2} (G_{1i} \amalg G_{2k}).$$
Proposition 2.13 [19, Exercise 9.1.22]. Let $G = G_1 \sqcup \cdots \sqcup G_n$ be a free pro-$p$ product of pro-$p$ groups and let $g_1, \ldots, g_n$ be elements of $G$. Then

$$G = g_1G_1g_1^{-1} \sqcup \cdots \sqcup g_nG_ng_n^{-1}$$

For more details about pro-$p$ groups acting on pro-$p$ trees, see [20] and [35].

3 The decomposition theorem for pro-$p$ groups acting on a pro-$p$ tree $T$ with procyclic edge stabilizers

In this section, we prove Theorem A, stated in the introduction.

We will need the following technical lemma, whose proof is extracted from the proof of Lemma 4.1 in [8]. Recall that given a pro-$p$ group $G$, we denote by $d(G)$ the minimal number of topological generators of $G$.

Lemma 3.1 Let $G$ be a finitely generated pro-$p$ group with $d(G) \geq 2$.

(a) If $G = A \sqcup_C B$ is a free amalgamated pro-$p$ product with $C$ procyclic, then $d(G) \geq d(A) + d(B) - 1$.

(b) If $G = \text{HNN}(H, A, t)$ is a pro-$p$ HNN extension with $A$ procyclic, then $d(G) \geq d(H)$.

Proof For a pro-$p$ group $H$ denote by $\tilde{H}$ the Frattini quotient $H/\Phi(H)$.

(a) Let $N$ be the kernel of the canonical homomorphism $\tilde{A} \sqcup \tilde{B} \to \tilde{G}$. Since $C$ is procyclic, the image $M$ of $N$ via the cartesian map $\tilde{A} \sqcup \tilde{B} \to \tilde{A} \times \tilde{B}$ is also procyclic. The latter map induces an epimorphism from $\tilde{G}$ to the elementary abelian pro-$p$ group $(\tilde{A} \times \tilde{B})/M$. Hence, $d(G) = d(\tilde{G}) \geq d(\tilde{A}) + d(\tilde{B}) - 1 = d(A) + d(B) - 1$.

(b) Suppose that $G = \text{HNN}(H, A, t) = \langle H, t \mid tat^{-1} = f(a) \rangle$, where $\langle a \rangle = A$. Then, there is an obvious epimorphism $\tilde{G} \to \langle \tilde{H} \times \langle \tilde{t} \rangle \rangle/\langle \tilde{t} \tilde{a}(\tilde{t})^{-1}(\tilde{f}(\tilde{a})\tilde{t})^{-1} \rangle$. Thus, $d(G) \geq d(H)$. \qed

Next we prove a preliminary result on the fundamental pro-$p$ group of a finite graph of finite $p$ groups. If $\Pi_1^{\text{abs}}(G, \Gamma)$ is residually $p$, then the vertex groups of $(G, \Gamma)$ embed in $\Pi_1(G, \Gamma)$. Thus in the next result, we assume that $\Pi_1^{\text{abs}}(G, \Gamma)$ is residually $p$ just to ensure this.

Lemma 3.2 Let $(G, \Gamma)$ be a finite graph of finite $p$ groups with cyclic edge groups $\bar{G}(e)$ such that $\bar{G}(e) \neq \bar{G}(v)$ for every edge $e$ in some maximal subtree $T_\Gamma$ of $\Gamma$ and every vertex $v$ incident to $e$. Suppose $\Pi_1^{\text{abs}}(G, \Gamma)$ is residually $p$ and let $G = \Pi_1(G, \Gamma)$ be the fundamental pro-$p$ group of $(G, \Gamma)$. Then, $d(G)$ tends to infinity whenever $|\Gamma|$ tends to infinity.

Proof Since the fundamental group $\Pi_1(\Gamma)$ is a free quotient group of $G$ of rank $|E(\Gamma)| - |V(\Gamma)| + 1$, if $|E(\Gamma)| - |V(\Gamma)| \to \infty$, then $d(G) \to \infty$ and we are done. Therefore, we may assume that $|E(\Gamma)| - |V(\Gamma)|$ is bounded by some constant $k$. Since $G = \text{HNN}(\Pi_1(G, T_\Gamma), \bar{G}(e), t_e, e \in \Gamma \setminus T_\Gamma)$ and $\bar{G}(e)$’s are cyclic, by
Lemma 3.1 (b), it suffices to show that $d(\Pi_1(\mathcal{G}, T_\Gamma))$ grows. Thus, we may assume that $\Gamma$ is a tree, i.e., $\Gamma = T_\Gamma$. Let $P$ be the set of pending vertices. Since $\mathcal{G}(e) \neq \mathcal{G}(v)$, we have that the free pro-$p$ product $\Omega_{e \in P} \mathcal{G}_{e}$ of cyclic groups of order $p$ is a quotient of $\Pi_1(\mathcal{G}, T_\Gamma)$ (one can see this by factoring out the normal subgroup generated by $\mathcal{G}(e)$’s). Thus, $|P|$ is bounded by $d(\mathcal{G})$ and so it suffices to prove the result for $T_\Gamma$ being a segment. Enumerating its edges consequently, we note that the vertex groups of every odd edge generate non-abelian and so non-cyclic group $G_i$, $i = 1, 3, 5\ldots$. Thus we have $\Pi_1(\mathcal{G}, T_\Gamma) = G_1 \cup \mathcal{G}(e_2) G_3 \cup \mathcal{G}(e_4) G_5 \cdots$. Now the result follows by Lemma 3.1 (a). □

**Proposition 3.3** Let $G$ be a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$ with pro-cyclic edge stabilizers. Then $G$ is a surjective inverse limit $G = \lim_{\leftarrow U} \Pi_1(\mathcal{G}_U, \Gamma)$ of fundamental groups of finite graphs of pro-$p$ groups $(\mathcal{G}_U, \Gamma)$ (over the same finite graph $\Gamma$), where the connecting maps $\psi_{U,W}$ map each vertex group $\mathcal{G}_U(v)$ and each edge group $\mathcal{G}_U(e)$ onto a conjugate of the vertex group $\mathcal{G}_W(v)$ and a conjugate of the edge group $\mathcal{G}_W(e)$, respectively. Moreover, the maximal (by inclusion) vertex stabilizers in $G$ are finitely generated and there are only finitely many of them in $G$ up to conjugation. There are also finitely many maximal edge stabilizers $G_e$, up to conjugation, whose images in $\Pi_1(\mathcal{G}_U, \Gamma)$ are conjugates of edge groups and any other edge stabilizer is conjugate to a subgroup of one of these $G_e$.

**Proof** For every open subgroup $U$ of $G$ consider $\tilde{U}$, a subgroup generated by all intersections with vertex stabilizers. Then by Theorem 2.3 (a) and Theorem 2.3 (b), the quotient group $U/\tilde{U}$ acts freely on the pro-$p$ tree $T/\tilde{U}$, and therefore, it is free pro-$p$. Thus $G_U := G/\tilde{U}$ is virtually free pro-$p$. By Theorem 2.9, it follows that $G_U$ is the fundamental pro-$p$ group $\Pi_1(\mathcal{G}_U, \Gamma_U)$ of a finite graph of finite $p$ groups with cyclic edge stabilizers. For a maximal subtree $T_{\Gamma_U}$ of $\Gamma_U$, we may assume that $\mathcal{G}_U(e) \neq \mathcal{G}_U(v)$ for every edge $e$ in $T_{\Gamma_U}$ and every vertex $v$ incident to $e$ (if there is an edge $e \in T_\Gamma$ and a vertex $v$ incident to $e$ such that $\mathcal{G}_U(e) = \mathcal{G}_U(v)$, then we just collapse $e$). Clearly we have $G = \lim_{\leftarrow U} G_U$. Since $d(G_U) \leq d(G)$, by Lemma 3.2 it follows that the number of vertices and edges of $\Gamma_U$ is bounded for each $U$. Since there are only finitely many finite graphs with bounded number of vertices and edges, by passing to a cofinal system of $\{\Gamma_U\}$ if necessary, we can assume that $\Gamma_U = \Gamma$ for each $U$. Fix a maximal subtree $T_\Gamma$ of $\Gamma$ and recall that $G_U = \Pi_1(\mathcal{G}_U, \Gamma)$ has the following presentation:

$$\Pi_1(\mathcal{G}_U, \Gamma) = \langle \mathcal{G}_U(v), t_U(e) \mid rel(\mathcal{G}_U(v)), \partial_1(g) = \partial_0(g)^{t_U(e)}, \rangle,$$

$$g \in \mathcal{G}_U(e), t_U(e) = 1 \quad \text{for} \quad e \in T_\Gamma.$$

Now let $U$ and $W$ be open subgroups of $G$ such that $U \leq W$, let $v \in V(\Gamma)$ and let $\psi_{U,W} : G_U \rightarrow G_W$ be the natural epimorphism. Since $\mathcal{G}_U(v)$ is a finite $p$ group, we have that $\psi_{U,W}(\mathcal{G}_U(v))$ also is a finite $p$ group, and so, by Theorem 2.10, it stabilizes a vertex (under the action of $G_W = \Pi_1(\mathcal{G}_W, \Gamma)$ on its associated pro-$p$ tree). Hence, it is contained in a conjugate of some vertex group of $(\mathcal{G}_W, \Gamma)$. Since $\Gamma$ has only finitely many vertices, by passing to a cofinal system if necessary, for $U \leq W$ we have a homomorphism $\mathcal{G}_U(v) \rightarrow \mathcal{G}_W(v)^{g_{U,W,v}}$, where $g_{U,W,v}$ is some element of $\Pi_1(\mathcal{G}_W, \Gamma)$. 


Let \( e \in E(\Gamma) \) and suppose that \( d_0(e) = u \) and \( d_1(e) = v \). Then, since \( \mathcal{G}_U(e) = \mathcal{G}_U(u) \cap \mathcal{G}_U(v) \), for \( U \leq W \) we have

\[
\psi_{U,W}(\mathcal{G}_U(e)) \leq \mathcal{G}_W(u)^{GU,W,u} \cap \mathcal{G}_W(v)^{GU,W,v} \tag{1}
\]

and the intersection on the right-hand side of (1) is contained in a conjugate of an edge group [(cf. Theorem 2.3 (d)].

Thus, as in the case with vertex groups, for \( U \leq W \) (if necessary we pass to a cofinal subsystem), the group \( \mathcal{G}_U(e) \) maps to the group \( \mathcal{G}_W(e) \), up to conjugation. Thus for every \( e \), we have an inverse system \( \{\mathcal{G}_U(e)^{GU} \mid g_U \in G_U\} \) of conjugates of \( \mathcal{G}_U(e) \). The inverse limits of these families, for every \( e \in E(\Gamma) \), give the family \( \{\mathcal{G}_e\} \) of groups closed under the conjugation by elements of \( G \). Moreover, there are at most \(|E(\Gamma)|\) of them up to conjugation, since this is true for \( \{\mathcal{G}_U(e)^{GU} \mid g_U \in G_U\} \) for each \( U \). Let us choose an element \( G(e) \) of \( \{\mathcal{G}_e\} \). Its images on \( \Pi_1(\mathcal{G}_U, \Gamma) \) under the projection maps form the inverse system \( \{\mathcal{G}_U(e)\} \) (for each \( e \in E(\Gamma) \)); this inverse system can be assumed to be surjective by Lemma 2.5 (a), if \( G(e) \neq 1 \). For each \( U \), the group \( \mathcal{G}_U(e) \) is the stabilizer of an edge of the pro-\( p \) tree \( T/\bar{U} \) by Theorem 2.9 and therefore so is \( \mathcal{G}_U(e) \). Hence, \( G(e) \) stabilizes an edge of the pro-\( p \) tree \( T = \lim \leftarrow U T/\bar{U} \). If \( G(e) = 1 \), then we can factor out the normal closure of \( \mathcal{G}_U(e) \), since by Lemma 2.5 we have \( G = \lim \leftarrow U \mathcal{G}_U/(\mathcal{G}_U(e))^{GU} \) for such \( e \). Thus, we may assume that \( \{\mathcal{G}_U(e)\} \) is surjective for every \( e \). It follows that \( G(e) \) is the stabilizer in \( G \) of an edge of \( T \). Conversely, let \( G_t \) be the stabilizer of an edge \( t \in T \). Then, \( G_t = G_u \cap G_v \) for some vertices \( u, v \) of \( T \) and, with the same argument as at the beginning of the paragraph, \( G_t \bar{U}/\bar{U} \) is conjugate to a subgroup of an edge group \( \mathcal{G}_U(e) \) for some \( e \) (see Theorem 2.10). Since \( \Gamma \) is finite, we may assume that \( e \) is the same for all \( U \). Then \( G_t \) is conjugate to a subgroup of \( G(e) \). This shows that the number of maximal edge stabilizers of \( G \) acting on \( T \) is finite (not exceeding \(|E(\Gamma)|\)), up to conjugation.

Note that the homomorphism \( \mathcal{G}_U(v) \to \mathcal{G}_W(v)^{GU,W,v} \) is an epimorphism. Indeed, suppose that this homomorphism is not surjective. Then, since \( \mathcal{G}_W(v)^{GU,W,v} \) is a finite \( p \) groups, \( \psi_{U,W}(\mathcal{G}_U(v)) \) is contained in a maximal subgroup of \( \mathcal{G}_W(v)^{GU,W,v} \), which is normal and of index \( p \). Using the fact that the homomorphism \( \mathcal{G}_U(e) \to \psi_{U,W}(\mathcal{G}_U(e)) \) is an epimorphism and recalling that \( \psi_{U,W}(\mathcal{G}_U(e)) = \mathcal{G}_W(e)^{GU,W,e} \) for some \( h_{U,W,e} \in \Pi_1(\mathcal{G}_W, \Gamma) \), by factoring out the normal closure of all vertex groups of \( \Pi_1(\mathcal{G}_W, \Gamma) \) except \( \mathcal{G}_W(v) \), it is easy to see that we have a contradiction, since \( \psi_{U,W} \) is an epimorphism.

For every vertex \( v \), we have an inverse system \( \{\mathcal{G}_U(v)^{GU} \mid g_U \in G_U\} \) of conjugates of \( \mathcal{G}_U(v) \). The inverse limits of these families give the family \( \{\mathcal{G}_e\} \) of groups closed under the conjugation by elements of \( G \). Moreover, there are at most \(|V(\Gamma)|\) of them up to conjugation, since this is true for \( \{\mathcal{G}_U(v)^{GU} \mid g_U \in G_U\} \) for each \( U \). Let us choose an element \( G(v) \) of \( \{\mathcal{G}_e\} \). Its images on \( \Pi_1(\mathcal{G}_U, \Gamma) \) under the projection maps form the surjective inverse system \( \{\mathcal{G}_U(v)\} \). For each \( U \), the group \( \mathcal{G}_U(v) \) is the stabilizer of a vertex of the pro-\( p \) tree \( T/\bar{U} \) by Theorem 2.9, and therefore, \( \mathcal{G}_U(v) \) as a conjugate of \( \mathcal{G}_U(v) \) is the stabilizer of a vertex of \( T/\bar{U} \). Hence \( G(v) \) is the stabilizer in \( G \) of a vertex of \( T = \lim \leftarrow U T/\bar{U} \). Conversely, let \( G_t \) be the stabilizer of a vertex \( t \in T \). Then, its image \( G_t \bar{U}/\bar{U} \) is conjugate to a subgroup of a vertex group \( \mathcal{G}_U(v) \) for some \( v \) (see
Theorem 2.10), and since $\Gamma$ is finite, we may assume that $v$ is the same for all $U$. Then, $G_t$ is conjugate to a subgroup of $G(v)$. This shows that the number of maximal vertex stabilizers of $G$ acting on $T$ is finite (not exceeding $|V(\Gamma)|$), up to conjugation.

Finally, note that from the fact that $d(G_U) \leq d(G)$ for each $U$ and Lemma 3.1, it follows easily that $G(v)$ is finitely generated for each $v \in V(\Gamma)$. This finishes the proof of the proposition.

Now we introduce two separate subsections to be treated separately: the case of acylindrical action (that will be used in the rest of the paper) and the case when $G$ is generated by its vertex stabilizers.

### 3.1 Acylindrical action

**Definition 1** Let $G$ be a pro-$p$ group acting on a pro-$p$ tree $T$. We say that this action is $n$-acylindrical if for every non-trivial edge stabilizer $G_e$, the subtree of fixed points $T^G_e$ [cf. Theorem 2.3 (c)] has diameter $n$. Note that, by Lemma 2.1, this means that any element $1 \neq g \in G$ can fix at most $n$ edges in any (profinite) geodesic $[v, w]$ of $T$.

**Lemma 3.4** Let $G$ be a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$ with procyclic edge stabilizers such that $T/G$ has finite diameter. Then, $G$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups $(\mathcal{G}, \Delta)$ with procyclic edge groups and finitely generated vertex groups. Moreover, the vertex and edge groups of $(\mathcal{G}, \Delta)$ are stabilizers of certain vertices and edges of $T$ respectively.

**Proof** By Theorem 2.3 (b), the quotient group $G/\tilde{G}$ acts freely on the pro-$p$ tree $T/\tilde{G}$. Since $\pi_1(T/G) \cong G/\tilde{G}$ (see Theorem 4.8 in [36]), we have that $\pi_1(T/G)$ is finitely generated. Since $T/G$ has finite diameter, by Proposition 2.7 it follows that $\pi_1(T/G)$ is just the pro-$p$ completion of the ordinary fundamental group $\pi_1^{abs}(T/G)$. By Proposition 3.3, we have that there are finitely many maximal stabilizers of vertices $G_{w_1}, G_{w_2}, \ldots, G_{w_m}$ up to conjugation. Let $C_1, C_2, \ldots, C_n$ be simple circuits that are free generators of $\pi_1^{abs}(T/G)$ and let $v_1, v_2, \ldots, v_m$ be the images of $w_1, \ldots, w_m$ in $T/G$. Put $\Delta$ to be a minimal connected subgraph of $T/G$ containing $C_1, C_2, \ldots, C_n$ and $v_1, v_2, \ldots, v_m$; clearly $\Delta$ is finite. By Lemma 2.4, for any connected component $\Omega$ of the preimage of $\Delta$ in $T$ and its setwise stabilizer $\text{Stab}_G(\Omega)$ we have $\Omega/\text{Stab}_G(\Omega) = \Delta$. By Proposition 2.11, a pro-$p$ group acting on a pro-$p$ tree cofinitely is the fundamental group of a finite graph of groups in a standard manner, i.e., in our case $\text{Stab}_G(\Omega) = \Pi_1(\mathcal{G}, \Delta)$. More precisely, $\Delta$ admits a connected transversal $D$ in $\Omega$ with $d_0(e) \in D$ for every $e \in D$. This gives the standard structure of a graph of pro-$p$ groups $(\mathcal{G}, \Delta)$ on $\Delta$, where the vertex and edge groups are stabilizers of vertices and edges of $D$ and we have

$$\Pi_1(\mathcal{G}, \Delta) = \langle G_v, x_e \in \text{Stab}_G(\Omega) \mid v \in V(D), x_e d_1(e) \in D, \text{ for } e \in E(D) \text{ with } d_1(e) \notin D \rangle.$$  

Let $u_1, \ldots, u_m$ be the preimages of $v_1 \ldots v_m$ in $D$. Then, $G_{u_1}, G_{u_2}, \ldots, G_{u_m}$ are conjugates of $G_{w_1}, G_{w_2}, \ldots, G_{w_m}$, so that every vertex stabilizer of $G$ up to con-
jugation is contained in one of them. Therefore, $G$ is generated by $\pi_{1}^{\text{abs}}(T / G)$ and $G_{u_1}, G_{u_2}, \ldots, G_{u_m}$ (see it modulo Frattini). Thus, we have

$$G = \langle G_{u_1}, x_e \in \text{Stab}_G(\Omega) \mid i = 1, \ldots, m, x_e d_1(e) \in D, \text{ for } e \in E(D) \text{ with } d_1(e) \notin D \rangle$$

and so $G = \Pi_1(\mathcal{G}, \Delta)$.

**Theorem 3.5** Let $n$ be a natural number and $G$ be a finitely generated pro-$p$ group acting $n$-acylindrically on a pro-$p$ tree $T$ with procyclic edge stabilizers. Then $G$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ with procyclic edge groups and finitely generated vertex groups. Moreover, the vertex and non-trivial edge groups of $(\mathcal{G}, \Gamma)$ are stabilizers of certain vertices and edges of $T$, respectively, and stabilizers of vertices of $T$ in $G$ are conjugate to subgroups of vertex groups of $(\mathcal{G}, \Gamma)$.

**Proof** By Proposition 3.3, there are only finitely many maximal by inclusion edge and vertex stabilizers in $G$ up to conjugation. Then, since the action is $n$-acylindrical, $T_{G_e}$ has diameter at most $n$ for every non-trivial edge stabilizer $G_e$. It follows that $\bigcup_{G_e \neq 1} T_{G_e} / G$ has finite diameter. Indeed, since there are only finitely many maximal edge stabilizers up to conjugation, it suffices to show that for a maximal edge stabilizer $G_{e'}$ stabilizing an edge $e'$, the tree $\bigcup_{G_{e} \leq G_{e'}} T_{G_e}$ has finite diameter. But for $G_e \leq G_{e'}$ the geodesic $[e, e']$ is stabilized by $G_e$ [cf. Theorem 2.3 (d)] and so has length not more than $n$.

Thus $\bigcup_{G_{e} \neq 1} T_{G_e} / G$ has finite diameter and finitely many connected components. It follows that the closure $\Delta$ of it has also finite diameter (see Lemma 2.2) and finitely many connected components.

Let $\Delta_\alpha$ be a connected component of $\Delta$. By Lemma 2.4, for any connected component $\Omega_\alpha$ of the preimage of $\Delta_\alpha$ in $T$ and its setwise stabilizer $\text{Stab}_G(\Omega_\alpha)$ we have $\Omega_\alpha / \text{Stab}_G(\Omega_\alpha) = \Delta_\alpha$. By Lemma 3.4, we have that $\text{Stab}_G(\Omega_\alpha) = \Pi_1(\mathcal{G}, \Delta_\alpha)$ is the fundamental group of a finite graph of groups in a standard manner, where the vertex and edge groups are stabilizers of vertices and edges of $D_\alpha$ and so

$$\Pi_1(\mathcal{G}, \Delta_\alpha) = \langle G_v, x_e \in \text{Stab}_G(\Omega_\alpha) \mid v \in V(D_\alpha), x_e d_1(e) \in D_\alpha, \text{ for } e \in E(D_\alpha) \text{ with } d_1(e) \notin D_\alpha \rangle.$$}

Collapsing all connected components of the preimage of $\Delta$ in $T$, by Proposition 2.8 we get a pro-$p$ tree $\tilde{T}$ on which $G$ acts with trivial edge stabilizers [since $\tilde{T}_{G_e}$ is connected for every $e \in E(\tilde{T})$ by Theorem 2.3 (c)], so by Proposition 2.6 we have that $G$ is a free pro-$p$ product

$$G = \left( \coprod_{\alpha} \text{Stab}_G(\Omega_\alpha) \right) \wr \left( \coprod_{v \notin \bigcup_{\alpha} D_\alpha} G(v) \right) \wr \pi_1(\tilde{T} / G).$$

Therefore, $\text{Stab}_G(\Omega_\alpha)$, $\pi_1(\tilde{T} / G)$ and $G(v)$ for $v \notin \bigcup_{\alpha} D_\alpha$ are finitely generated.
Since the free pro-$p$ product of the fundamental pro-$p$ groups of finitely many finite graphs of pro-$p$ groups is again the fundamental pro-$p$ group of a finite graph of pro-$p$ groups, we have the needed structure of the fundamental pro-$p$ group of a finite graph of pro-$p$ groups on $G$ in this case.

The last part of the theorem follows from Proposition 3.3.

3.2 Generation by stabilizers

If $G$ is generated by vertex stabilizers, we can prove the structure theorem without $n$-acylindricity. To accomplish this, we need first the following.

**Lemma 3.6** Let $(\G, \Gamma)$ be a finite tree of finite $p$ groups and let $G = \Pi_1(\G, \Gamma)$ be the fundamental pro-$p$ group of $(\G, \Gamma)$. Let $G(\Gamma) = \prod_{v \in V(\Gamma)} G(v)$ be a free pro-$p$ product and let $\psi : G(\Gamma) \longrightarrow G$ be the epimorphism sending $G(v)$ to their copies in $G$. Suppose there is a collection \{ $G(v) = G(\psi(v))$, $v \in V(\Gamma)$, $g_v \in G(\Gamma)$ \} of conjugates of free factors of $G(\Gamma)$ and a collection \{ $G(e) = G(\psi(e))$, $e \in E(\Gamma)$, $g_e \in G$ \} of conjugates of edge groups of $G$ such that $\psi(G(d_1(e))) \cap \psi(G(d_0(e))) = G(e)$. Then, the kernel of $\psi$ is generated as normal subgroup by the set of elements $\psi_{1,e}^{-1}(g_1^{-1})\psi_{0,e}^{-1}(g)$, where $g \in G(e)$ and $\psi_{i,e} = \psi_{i,G(d_i(e))}$, $i = 0, 1$.

**Proof** Note that $\psi(\psi_{1,e}^{-1}(g_1^{-1})\psi_{0,e}^{-1}(g)) = g_1^{-1}g = 1$ and so the elements $\psi_{1,e}^{-1}(g_1^{-1})\psi_{0,e}^{-1}(g)$ belong to the kernel of $\psi$. This means that $\psi$ factors via the natural quotient homomorphism $\pi : G(\Gamma) \longrightarrow \Pi$ modulo the normal closure of the elements $\psi_{1,e}^{-1}(g_1^{-1})\psi_{0,e}^{-1}(g)$, i.e., there exists a natural epimorphism $\pi : \Pi \longrightarrow G$ such that $\pi \circ \psi = \psi$.

Define now a tree of pro-$p$ groups $(\G', \Gamma)$ as follows. Put $G'(v) = \pi(G(\psi(v)))$, $G'(e) = \pi(G_{0,e}(G(e)))$ and define $d_0$, $d_1$ to be the natural embeddings of $G'(e)$ into $G'(d_0(e))$ and into $G'(d_1(e))$. Then, the relations

$$
\psi_{1,e}^{-1}(g_1^{-1})\psi_{0,e}^{-1}(g),
$$

where $g \in G(e)$, define on $\Pi$ the structure of the fundamental group $\Pi_1(\G', \Gamma)$ of the graph $(\G', \Gamma)$ of groups.

Let $F$ be an open free pro-$p$ subgroup of $G$. Then $f^{-1}(F)$ is an open free pro-$p$ subgroup of $\Pi$ of the same index as the index of $F$ in $G$. Then by the Euler characteristic formula (cf. Exercise 3 on page 123 in [31]), that holds here since our groups are the pro-$p$ completions of the corresponding abstract groups, we have

$$
\text{rank}(F) - 1 = |G : F| \left( \sum_{e \in E(\Gamma)} 1/|G(e)| - \sum_{v \in V(\Gamma)} 1/|G(v)| \right)
= |\Pi : f^{-1}(F)| \left( \sum_{e \in E(\Gamma)} 1/|G(e)| - \sum_{v \in V(\Gamma)} 1/|G(v)| \right)
= \text{rank}(f^{-1}(F)) - 1.
$$
Thus the free pro-$p$ groups $F$ and $f^{-1}(F)$ have the same rank, and therefore, they are isomorphic. Since the kernel of $f$ is torsion free, $f$ is an isomorphism, as desired. □

**Theorem 3.7** Let $G$ be a finitely generated pro-$p$ group acting on a pro-$p$ tree $T$ with procyclic edge stabilizers. Suppose $G$ is generated by its vertex stabilizers. Then, $G$ is the fundamental pro-$p$ group of a finite tree of pro-$p$ groups $(\mathcal{G}, \Gamma)$ with procyclic edge groups and finitely generated vertex groups. Moreover, the vertex and edge groups of $(\mathcal{G}, \Gamma)$ are stabilizers of certain vertices and edges of $T$, respectively, and stabilizers of vertices and edges of $T$ in $G$ are conjugate to subgroups of vertex and edge groups of $(\mathcal{G}, \Gamma)$ respectively.

**Proof** By Proposition 3.3, the group $G$ is a surjective inverse limit $G = \lim_{\longrightarrow} G_U$, where $G_U = \Pi_1(\mathcal{G}_U, \Gamma)$ is the fundamental group of a finite graph of pro-$p$ groups $(\mathcal{G}_U, \Gamma)$, where the connecting maps $\psi_{U,W}$ map each vertex group $\mathcal{G}_U(v)$ and each edge group $\mathcal{G}_U(e)$ onto a conjugate of the vertex group $\mathcal{G}_W(v)$ and a conjugate of the edge group $\mathcal{G}_W(e)$, respectively. Moreover, there are only finitely many maximal by inclusion vertex stabilizers in $G$ up to conjugation and also finitely many up to conjugation edge stabilizers $G_e$ whose images in $\Pi_1(\mathcal{G}_U, \Gamma)$ are conjugates of edge groups and any other edge stabilizer is conjugate to a subgroup of one of these $G_e$. Keeping the notation of the proof of Proposition 3.3, we denote by $G(e), G(v)$ some representatives of them. Note that in this case, by Theorem 2.3 (a), it follows that $\Gamma$ is a tree.

**Claim** We can choose the representatives $G(e)$ and $G(v)$ such that for $e \in E(\Gamma)$ one has $G(e) = G(d_0(e)) \cap G(d_1(e))$.

**Proof of the claim** Let $D$ be a maximal subtree of $\Gamma$ such that this holds for all $e \in E(D)$. We show that $D = \Gamma$. Suppose not. Then, there exists $e \in E(\Gamma) \setminus E(D)$ such that a vertex $v$ of $e$ is in $D$. Let $G'(e) = G(e)^{h_v}$ be the image of $G(e)$ in $G_U$. Then, clearly $G_U(v)^{h_v}$ contains $G_U(e)^{h_v}$. Since the image $G'_U(v)$ of $G(v)$ in $G_U$ is a conjugate of $G_U(v)$, it follows that the set $X_U$ of elements $x_U \in G_U$ such that $G'_U(e)^{x_U} \leq G'_U(v)$ is non-empty and clearly these sets form an inverse system $\{X_U\}$. It follows that the inverse limit $X := \lim_{\longleftarrow} X_U$ is non-empty and $G(e)^{x_U} \leq G(v)$ for any $x \in X$. So we replace $G(e)$ by $G(e)^{x}$ (in this way $G'_U(e)$ is replaced by $G'_U(e)^{x_U}$, where $x_U$ is the image of $x$ in $G_U$). Let $w$ be the other vertex of $e$. Similarly, there is an inverse system $\{Y_U\}$ of non-empty subsets of $G_U$ such that $G'_U(e) \leq G'_U(w)^{y_U}$ for each $y_U \in Y_U$. Then the inverse limit $Y := \lim_{\longleftarrow} Y_U$ is non-empty and for each $y \in Y$ we have $G(v) \cap G(w)^y = G(e)$ (since $G'_U(v) \cap G'_U(w)^{y_U} = G_U(e)$ for every $U$). Then, $D \cup \{e\} \cup \{w\}$ satisfies the statement, contradicting the maximality of $D$.

Now consider the projection $\psi : G \to G_U$, and let $G'_U(v) = \psi(U(G(v))), G'_U(e) = \psi(U(G(e)))$ for $G(v), G(e)$ being as in Claim. Let

$$G_U(\Gamma) := \prod_{v \in V(\Gamma)} \mathcal{G}_U(v)$$

and let $f_U : G_U(\Gamma) \to \Pi_1(\mathcal{G}_U, \Gamma)$ be the homomorphism defined by sending $\mathcal{G}_U(v)$ to their copies in $\Pi_1(\mathcal{G}_U, \Gamma)$. We choose an element $g_{U,v} \in G_U(\Gamma)$ such
that \( f_U(G_U(v)^{GU,v}) = G^*_U(v) \). Put \( G_U(v) = G_U(v)^{GU,v} \). Since free products in the pro-\( p \) case do not depend on the conjugation of the factors (see Proposition 2.13), we have \( G_U(\Gamma) = \prod_{v \in V(\Gamma)} G_U(v) \).

Now let \( U \) and \( W \) be open subgroups of \( G \) such that \( U \leq W \). Then, the maps \( G_U(v) \to G_W(v) \) induce an epimorphism \( \varphi_{U,W} : G_U(\Gamma) \to G_W(\Gamma) \), which gives the following commutative diagram

\[
\begin{array}{ccc}
G_U(\Gamma) & \xrightarrow{\varphi_{U,W}} & G_W(\Gamma) \\
\downarrow f_U & & \downarrow f_W \\
\Pi_1(G_U, \Gamma) & \xrightarrow{\psi_{U,W}} & \Pi_1(G_W, \Gamma)
\end{array}
\]

Let \( G(\Gamma) := \prod_{v \in V(\Gamma)} G(v) \). Then the maps \( G(v) \to G_U(v)^{GU,v} \) induce an epimorphism \( \varphi_U : G(\Gamma) \to G_U(\Gamma) \) such that \( \varphi_{U,W} \varphi_U = \varphi_W \). Thus we have a surjective inverse system \( \{G_U(\Gamma)\} \), which by Lemma 2.12 has inverse limit

\[
G(\Gamma) = \lim_{\to \Gamma} G_U(\Gamma) = \prod_{v \in V(\Gamma)} G(v).
\]

Note that \( G^*_U(e) \) and \( G^*_U(v) \) are conjugates in \( G_U \) of \( G_U(e) \) and \( G^*_U(v) \) respectively and, by Claim, the relations of Lemma 3.6 hold for \( G^*_U(e) \) and \( G^*_U(v) \). It follows that \( f_U \) is the epimorphism defined by just imposing on \( G_U(\Gamma) \) the amalgamation relations \( f_{U,1,e}^{-1}(g) = f_{U,0,e}^{-1}(g) \) for \( g \in G^*_U(e), e \in E(\Gamma), \) where \( f_{U,i,e} = (f_U)_{G_U(d_i(e))}, i = 0, 1 \); this means that the kernel of \( f_U \) is generated by the relations \( f_{U,1,e}(g^{-1})f_{U,0,e}(g) \) for \( g \in G^*_U(e), e \in E(\Gamma) \). Let \( f : G(\Gamma) \to G \) be the epimorphism given as the projective limit of the epimorphisms \( f_U \). Put \( f_i,e = f_i(G(d_i(e))), i = 0, 1 \). It follows that imposing on \( G(\Gamma) \) the relations \( f_{i,e}^{-1}(g^{-1})f_{0,e}(g) = 1 \), where \( g \in G(e) \) defines exactly \( f \). This gives the desired structure (i.e., presentation) of the fundamental group of a graph of groups on \( G = \Pi_1(G, \Gamma) \), with vertex end edge groups \( G(v) \) and \( G(e) \) and with the corresponding natural embeddings.

The rest of the proof follows directly from Proposition 3.3. \( \square \)

4 The decomposition theorem for pro-\( p \) groups from the class \( \mathcal{L} \)

In this section, we prove Theorem B and parts (1), (2), (3) and (4) of Theorem C, stated in the introduction.

We say that the amalgamated free pro-\( p \) product \( A \sqcup_C B \) is proper if \( A \) and \( B \) embed in \( A \sqcup C B \). Ribes proved that an amalgamated free pro-\( p \) product with procyclic amalgamation is proper (see Theorem 3.2 in [21]).

It is worth to recall the definition of the class \( \mathcal{L} \) of pro-\( p \) groups [14]. Denote by \( \mathcal{G}_0 \) the class of all free pro-\( p \) groups of finite rank. We define inductively the class \( \mathcal{G}_n \) of pro-\( p \) groups \( G_n \) in the following way: \( G_n \) is a free pro-\( p \) amalgamated product \( G_{n-1} \sqcup_C A \), where \( G_{n-1} \) is any group from the class \( \mathcal{G}_{n-1} \), \( C \) is any self-centralized procyclic pro-\( p \) subgroup of \( G_{n-1} \) and \( A \) is any finite rank free abelian pro-\( p \) group

\[
G(\Gamma) = \lim_{\to \Gamma} G_U(\Gamma) = \prod_{v \in V(\Gamma)} G(v).
\]
such that $C$ is a direct summand of $A$. The class of pro-$p$ groups $\mathcal{L}$ consists of all finitely generated pro-$p$ subgroups $H$ of some $G_n \in \mathcal{G}_n$, where $n \geq 0$. If $n$ is minimal with the property that $H \leq G_n$ for some $G_n \in \mathcal{G}_n$, we say that $H$ has weight $n$. Then, $H$ is a subgroup of a free amalgamated pro-$p$ product $G_n = G_{n-1} \amalg \mathbb{Z} P$ for some $P \leq G_{n-1}$. As was mentioned above, by Theorem 3.2 in [21], this amalgamated pro-$p$ product is proper. Thus, $H$ acts naturally on the pro-$p$ tree $T$ associated with $G_n$ (see [20]), and its edge stabilizers are procyclic.

**Lemma 4.1** Let $H$ and $T$ be as above. Then, the action of $H$ on $T$ is 2-acylindrical.

**Proof** It suffices to prove that the action of $G_n$ on $T$ is 2-acylindrical. Let $G_e$ be a non-trivial edge stabilizer. If the diameter of $T^G_e$ is bigger than 2, then it contains a non-pending vertex $v$ whose stabilizer is conjugate to $G_{n-1}$, and so we may assume without loss of generality that it is $G_{n-1}$. Let $e'$ be another edge incident to $v$ stabilized by $G_e$, i.e., $G_e \leq G_{e'}$. Then, $g = g'$ for some $g \in G_{n-1}$ and so $G_e$ is conjugate to $G_{e'}$, but in a profinite group this is possible only if $G_e = G_{e'}$ (one can see it by looking at finite quotients). Thus, $g \in N_{G_{n-1}}(G_e)$. By Theorem 5.1 in [14], it follows that $N_{G_{n-1}}(G_e) = C_{G_{n-1}}(G_e) = G_e$. Thus $e = e'$, a contradiction.

**Theorem 4.2** Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. If $G$ has weight $n \geq 1$, then it is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups that has infinite procyclic or trivial edge groups and finitely generated vertex groups. Moreover, if $G$ is non-abelian, then it has at least one vertex group that is a non-abelian pro-$p$ group and all the non-abelian vertex groups of $G$ are pro-$p$ groups from the class $\mathcal{L}$ of weight $\leq n - 1$.

**Proof** By Lemma 4.1, the action of $G$ on the standard pro-$p$ tree $T$ associated with $G_n$ is 2-acylindrical; so by Theorem 3.5, it follows that $G = \amalg_{1} (\mathcal{G}, \Gamma)$ is the fundamental pro-$p$ group of a finite graph of pro-$p$ groups with procyclic edge groups and finitely generated vertex groups. Moreover, each vertex group of $G$ is a vertex stabilizer of $G$ in $T$; thus, it is a pro-$p$ group from the class $\mathcal{L}$ contained in a subgroup of $G_n = G_{n-1} \amalg \mathbb{Z} P$ conjugate to $G_{n-1}$ or $A$. If it is non-abelian, then it must be contained in a subgroup of $G_n$ conjugate to $G_{n-1}$ and so it has weight $\leq n - 1$. Thus, in order to finish the proof, it remains to show that at least one of the vertex groups of $G$ is non-abelian.

Let $T_\Gamma$ be a maximal subtree of $\Gamma$. By collapsing the fictitious edges of $T_\Gamma$ (i.e., edges whose edge group is equal to the vertex group of a vertex of this edge), we may assume that all vertex groups contain properly edge groups for incident edges. Then if all vertex groups are abelian, we can have at most one vertex in $\Gamma$ because otherwise the centralizer of the edge group $G(e)$ is not abelian for any edge $e \in T_\Gamma$, contradicting Theorem 5.1 in [14]. Thus, we may assume that $T_\Gamma$ has only one vertex, i.e., $\Gamma$ is a bouquet). Let $H$ be the vertex group of this unique vertex and $A_i$, for $i = 1, \ldots, |E(\Gamma)|$, the edge groups (which are procyclic). Then $G = \amalg_{i} (H, A_i, t_i; i = 1, \ldots, |E(\Gamma)|)$. If $H$ is not procyclic, then since $(H, H^{H_i}) = H \amalg A_i H^{H_i}$ (cf. Proposition 2.11), we get once more a contradiction by Theorem 5.1 in [14]. Now suppose that $H$ is procyclic. Then, the groups $A_i$ are linearly ordered by inclusion, so there exists $j$ such that $A_j \leq A_i$ for every $i = 1, \ldots, |E(\Gamma)|$. Thus, $A_j = A_j^{H_i}$ for every $i$, i.e., $A_j$...
is normalized by all \( t_i \). By Theorem 5.1 in [14], it follows that \( A_j \) is central in \( G \) and \( G \) is abelian, which is a contradiction. Thus, at least one of the vertex groups of \( G \) is non-abelian.

Using the theorem and induction, we can deduce the following.

**Corollary 4.3** Let \( G \) be a pro-\( p \) group from the class \( \mathcal{L} \). Then, there are only finitely many conjugacy classes of non-procyclic maximal abelian subgroups of \( G \).

**Proof** Suppose that \( G \) has weight \( n \) and that the statement in the corollary holds for all pro-\( p \) groups from the class \( \mathcal{L} \) of weight \( \leq n - 1 \). Let \( H \) be a non-procyclic maximal abelian subgroup of \( G \) and consider the action of \( G \) on \( T \) according to the first paragraph in the proof of Theorem 4.2. Then, \( H \) is a subgroup of \( G_{n-1} \mathop{\sqcup} C_A \) and so, by Corollary 5.5 in [14], the group \( H \) is conjugate in \( G_n \) to a subgroup of \( G_{n-1} \) or to a subgroup of \( A \). Thus, it stabilizes a vertex of \( T \), and therefore, by Theorem 3.5, it is contained in a conjugate of a vertex group \( G(v) \) of \( (G, \Gamma) \). The non-abelian vertex groups of \( G \) have weight \( \leq n - 1 \), and therefore, by the induction hypothesis, they have only finitely many conjugacy classes of non-procyclic maximal abelian subgroups. Since \( G \) has only finitely many vertex groups, the result follows. \( \square \)

Recall that if \( cd(G) < \infty \) and if \( \dim_{\mathbb{F}_p} H^k(G, \mathbb{F}_p) < \infty \) for all \( k \geq 0 \), then the Euler–Poincaré characteristic of \( G \) is defined by

\[
\chi(G) := \sum_{k=0}^{\infty} (-1)^k \dim_{\mathbb{F}_p} H^k(G, \mathbb{F}_p).
\]

Moreover, if \( G \) is the fundamental pro-\( p \) group of a finite graph of pro-\( p \) groups \( (G, \Gamma) \) such that the Euler–Poincaré characteristic is well defined for the vertex and edge groups, then the action of \( G \) on the standard tree \( S(G) \), which is defined similarly as the standard tree in the abstract Bass–Serre theory, implies the formula

\[
\chi(G) = \left( \sum_{v \in V(\Gamma)} \chi(G(v)) \right) - \left( \sum_{e \in E(\Gamma)} \chi(G(e)) \right).
\]

The first part of the following theorem coincides with Theorem 8.1 in [14], while the second part generalizes Theorem 8.2 and gives an affirmative answer to Question 9.3 in the same paper.

**Theorem 4.4** Let \( G \) be a pro-\( p \) group from the class \( \mathcal{L} \). Then, \( G \) has a non-positive Euler–Poincaré characteristic. Moreover, \( \chi(G) = 0 \) if and only if \( G \) is abelian.

**Proof** Clearly, \( \chi(G) = 0 \) if \( G \) is abelian. Thus, it suffices to show that \( \chi(G) < 0 \) whenever \( G \) is non-abelian. We will prove this using induction on the weight \( n \) of the group \( G \). Suppose that \( G \) is non-abelian. If \( n = 0 \), then \( G \) is a non-abelian free pro-\( p \) group and so we have \( \chi(G) = 1 - d(G) < 0 \). Now let \( n \geq 1 \) and suppose that every non-abelian pro-\( p \) group from the class \( \mathcal{L} \) which has weight less than \( n \) has a negative Euler–Poincaré characteristic. By Theorem 4.2, the group \( G \) is the fundamental pro-\( p \)
group of a finite graph of pro-$p$ groups $(G, \Gamma)$ with infinite procyclic or trivial edge groups and whose vertex groups are either finitely generated free abelian pro-$p$ groups or non-abelian pro-$p$ groups from the class $L$ of weight $\leq n - 1$. Moreover, there is at least one non-abelian vertex group, say $G(v)$. Thus, by the induction hypothesis, we have $\chi(G(v)) < 0$. Now by the Euler–Poincaré characteristic formula, we have

$$\chi(G) = \left( \sum_{x \in V(\Gamma)} \chi(G(x)) \right) - \left( \sum_{e \in E(\Gamma)} \chi(G(e)) \right) \leq \left( \sum_{x \in V(\Gamma)} \chi(G(x)) \right)$$

$$- \left( \sum_{e \in E(\Gamma)} 0 \right) \leq \chi(G(v)) < 0.$$ 

\[\Box\]

Let $r(G)$ denote the minimal number of relations of $G$, i.e,

$$r(G) := \inf \{|R| \mid G \text{ has a presentation } \langle X \mid R \rangle \text{ with } |X| = d(G)\}.$$ 

It is a well known fact that $d(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$, and if $G$ is finitely generated, then $r(G) = \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$ (see [30]). Recall that if $G$ is a finitely presented pro-$p$ group, then the deficiency of $G$ is defined by

$$\text{def}(G) := d(G) - r(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p).$$

**Lemma 4.5** Let $G$ be a finitely generated pro-$p$ group with $d(G) \geq 2$.

(a) If $G = A \sqcup C \sqcup B$ where $C$ is procyclic, then $\text{def}(G) \geq \text{def}(A) + \text{def}(B) - 2$.

(b) If $G = \text{HNN}(H, A, t)$ where $A$ is procyclic, then $\text{def}(G) \geq \text{def}(H)$.

**Proof** Part (a) follows from Lemma 3.1 (a) and the obvious fact that $r(A \sqcup C \sqcup B) \leq r(A) + r(B) + 1$. For part (b) first suppose that $H = \langle X \mid R \rangle$, where $|X| = d(H)$ and $|R| = r(H)$. From the definition of HNN extensions, we have

$$G = \text{HNN}(H, A, t) = \langle H, t \mid tat^{-1} = f(a), \langle a \rangle = A \rangle = \langle X, t \mid R, tat^{-1} = f(a) \rangle,$$

where $f : A \rightarrow G$ is a monomorphism. By Lemma 1.1 in [16], there exists a presentation $\langle Y \mid S \rangle$ of $G$ such that $|Y| = d(G)$ and $|S| = |R| + 1 - (|X| + 1 - |Y|)$. Hence

$$\text{def}(G) = d(G) - r(G) \geq |Y| - |S| = |X| - |R| = \text{def}(H).$$

\[\Box\]

Now, we are ready to answer positively Question 9.1 in [14].

**Theorem 4.6** Let $G$ be a pro-$p$ group from the class $L$. If every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, then $\text{def}(G) \geq 2$. 

Proof Suppose that every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic. Again, as in the proof of Theorem 4.4, we will use induction on the weight $n$ of the group $G$. If $n = 0$, then it is clear that $\text{def}(G) \geq 2$. Let $n \geq 1$ and suppose that any non-procyclic pro-$p$ group from the class $\mathcal{L}$ which has weight $\leq n - 1$ and in which every abelian pro-$p$ subgroup is procyclic has deficiency $\geq 2$. By Theorem 4.2, the group $G$ is the fundamental pro-$p$ group $\Pi_1(G, \Gamma)$ of a finite graph of pro-$p$ groups with infinite procyclic or trivial edge groups and finitely generated vertex groups. Moreover, each non-abelian vertex group is a pro-$p$ group from the class $\mathcal{L}$ of weight $\leq n - 1$. Let $T_\Gamma$ be a maximal subtree of $\Gamma$, $k := |E(\Gamma)|$ and $l := |E(T_\Gamma)|$. We can obtain $G$ by successively forming amalgamated free products and HNN extensions. Indeed

$$G = A_k \text{ where } A_l := \mathcal{G}(u_1) \amalg \mathcal{G}(u_2) \amalg \cdots \amalg \mathcal{G}(u_l),$$

$$A_{l+1} := \text{HNN}(A_l, \mathcal{G}(e_{l+1}), t_{l+1}) \text{ and } A_j := \text{HNN}(A_{j-1}, \mathcal{G}(e_j), t_j)$$

for $j = l + 2, \ldots, k$.

We want to show that $\text{def}(A_i) \geq 2$ for each $i$. Clearly, we can assume that none of the $\mathcal{G}(u_i)$’s is procyclic. Indeed, if $\mathcal{G}(u_j) \cong \mathbb{Z}_p$, then since $\mathcal{G}(u_j) \amalg \mathcal{G}(u_{j+1})$ is a pro-$p$ group from the class $\mathcal{L}$, we must have $\mathcal{G}(e_j) = \mathcal{G}(u_j)$ and thus $\mathcal{G}(u_j) \amalg \mathcal{G}(u_{j+1}) = \mathcal{G}(u_{j+1})$. Hence, we can assume that the vertex groups $\mathcal{G}(u_i)$ satisfy the hypothesis of the theorem. Thus $\text{def}(\mathcal{G}(u_i)) \geq 2$ for each $i$. Therefore, by Lemma 4.5 (a), we have $\text{def}(A_l) \geq 2$. Moreover, Lemma 4.5 (b) gives

$$2 \leq \text{def}(A_l) \leq \text{def}(A_{l+1}) \leq \cdots \leq \text{def}(A_k) = \text{def}(G).$$

For a finitely generated pro-$p$ group $G$, denote by $s_n(G)$ the number of open subgroups of $G$ of index at most $n$. A pro-$p$ group $G$ is said to have exponential subgroup growth if

$$\limsup_n \frac{\log s_n(G)}{n} > 0.$$ 

Lackenby proved that a finitely generated pro-$p$ group $G$ has exponential subgroup growth if and only if there is a strictly descending chain $\{G_n\}$ of open normal subgroups of $G$ such that $\inf_n \frac{d(G_n) - 1}{|G : G_n|} > 0$ (see [15], Theorem 8.1).

Theorem 4.7 Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. If every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, then $G$ has exponential subgroup growth.

Proof Suppose that every abelian pro-$p$ subgroup of $G$ is procyclic and $G$ itself is not procyclic, and let $\{G_n\}$ be a strictly descending chain of open normal subgroups of $G$. Since $G$ is finitely presented, we have that $\chi_2(G)$ and $\chi_2(G_n)$ are well defined, where $\chi_2(G) := \sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(G, \mathbb{F}_p)$ is the second partial Euler–Poincaré characteristic of $G$. By Lemma 3.3.15 in [17] we have $\chi_2(G_n) \leq |G : G_n| \chi_2(G)$, which implies that $\text{def}(G_n) - 1 \geq |G : G_n|(\text{def}(G) - 1)$. Now from Theorem 4.6 and
the result of Lackenby mentioned above, it follows that $G$ has exponential subgroup growth. 

\[ \square \]

5 Subgroup properties of pro-$p$ groups from the class $\mathcal{L}$

In this section, we prove parts (5), (6) and (7) of Theorem C, stated in the introduction. We will need the following simple lemma.

**Lemma 5.1** Let $G$ be a pro-$p$ group, and let $H$ and $K$ be finitely generated subgroups of $G$. Let $A$ be a subgroup of $G$ that is contained in both $H$ and $K$. If $A$ has finite index in both $H$ and $K$, then $A$ has a finite index subgroup that is normal in $\langle H, K \rangle$.

**Proof** Since the restrictions of the natural epimorphism $\psi : H \amalg_A K \to \langle H, K \rangle$ to $H$ and $K$ are injections, the amalgamated free pro-$p$ product $N = H \amalg_A K$ is proper, i.e., $H$, $K$ and $A$ are subgroups of $N$. If $A$ is one of $H$ or $K$, then the result is clear. Therefore, we can assume that $A$ is different from $H$ and $K$. Note that if $U$ is an open subgroup of $A$ normal in $N$, then $\psi(U)$ is an open subgroup of $A$ normal in $\langle H, K \rangle$. Hence, in order to prove the lemma, it suffices to show that $A$ has an open subgroup which is normal in $N$.

Since $N = H \amalg_A K$ is proper, by Theorem 9.2.4 in [19], there is an indexing set $I$ and families 

$$\{ U_i \mid U_i \trianglelefteq_o H \}_{i \in I} \quad \text{and} \quad \{ V_i \mid V_i \trianglelefteq_o K \}_{i \in I}$$

with the property 

$$\bigcap_{i \in I} U_i = 1 = \bigcap_{i \in I} V_i \quad \text{and} \quad U_i \cap A = V_i \cap A \quad \text{for each} \quad i \in I.$$ 

We say that a family $S$ of subsets of a group $G$ is **filtered from below** if for every pair of subsets $S_1, S_2 \in S$ there exists some $S_3 \in S$ with $S_3 \leq S_1 \cap S_2$. Clearly, we can assume that the families $\{ U_i \}_{i \in I}$ and $\{ V_i \}_{i \in I}$ are filtered from below. Since $A$ is of finite index in both $H$ and $K$, it follows that there is some $k \in I$ such that $U_k \leq A$ and $V_k \leq A$. Thus 

$$U_k = U_k \cap A = V_k \cap A = V_k$$

and consequently $U_k$ is an open normal subgroup of both $H$ and $K$. Hence, $U_k$ is an open subgroup of $A$ which is normal in $N$. This finishes the proof.

Let $G$ be a (profinite) group and let $H$ be a (closed) subgroup of $G$. The **commensurator** of $H$ in $G$, denoted by $\Comm_G(H)$, is the set 

$$\{ g \in G \mid H \cap gHg^{-1} \text{ has finite index in both } H \text{ and } gHg^{-1} \}.$$ 

It is not hard to check that $\Comm_G(H)$ is a subgroup of $G$ (possibly not closed if $G$ is profinite) that contains $N_G(H)$.

The following result is well known; for completeness, we give its proof.
Subgroup properties of pro-\(p\) extensions of centralizers

**Proposition 5.2** Let \(G\) be a group, and let \(H\) and \(K\) be subgroups of \(G\) such that \(K \leq H\). If \(K\) has finite index in \(H\), then \(\text{Comm}_G(K) = \text{Comm}_G(H)\).

**Proof** Let \(g \in \text{Comm}_G(K)\). Then, \(K \cap gKg^{-1}\) has finite index in \(K\), and hence in \(H\). Since \(K \cap gKg^{-1} \subseteq H \cap gHg^{-1}\), we have that \(H \cap gHg^{-1}\) has finite index in \(H\). Similarly \(K \cap gKg^{-1}\) has finite index in \(gKg^{-1}\), and hence \(H \cap gHg^{-1}\) has finite index in \(gHg^{-1}\). Thus \(\text{Comm}_G(K) \subseteq \text{Comm}_G(H)\).

Conversely, let \(g \in \text{Comm}_G(H)\). Then \(H \cap gHg^{-1}\) has finite index in \(H\). Thus, \(K \cap H \cap gHg^{-1} = K \cap gHg^{-1}\) has finite index in \(K \cap H = K\). Similarly, \(K \cap gKg^{-1}\) has finite index in \(K \cap gHg^{-1}\). Hence, \(K \cap gKg^{-1}\) has finite index in \(K\). In a similar way, we can show that \(K \cap gKg^{-1}\) has finite index in \(gKg^{-1}\). Thus, \(\text{Comm}_G(H) \subseteq \text{Comm}_G(K)\).

**Definition 2** Let \(G\) be a (pro-\(p\)) group and let \(H\) be a finitely generated subgroup of \(G\). A root of \(H\) in \(G\), denoted by \(\text{root}_G(H)\), is a subgroup \(N\) of \(G\) that contains \(H\) with \(|N : H|\) finite and which contains every subgroup \(K\) of \(G\) that contains \(H\) with \(|K : H|\) finite.

Note that if \(H\) is a finitely generated subgroup of finite index in \(G\), then it is obvious that \(\text{root}_G(H) = G\).

**Theorem 5.3** Let \(G\) be a pro-\(p\) group from the class \(\mathcal{L}\). Then

1. (Greenberg–Stallings property) If \(H\) and \(K\) are finitely generated subgroups of \(G\) with the property that \(H \cap K\) has finite index in both \(H\) and \(K\), then \(H \cap K\) has finite index in \(\langle H, K \rangle\);
2. If \(H\) is a finitely generated subgroup of \(G\), then \(H\) has a root in \(G\);
3. If \(H\) is a finitely generated non-abelian subgroup of \(G\), then \(|\text{Comm}_G(H) : H| < \infty\).

**Proof** (1) Let \(H\) and \(K\) be finitely generated subgroups of \(G\) with the property that \(H \cap K\) has finite index in both \(H\) and \(K\). Note that if \(\langle H, K \rangle\) is abelian, then the result follows from the structure theorem of the torsion free finitely generated abelian pro-\(p\) groups [see the proof of part (2)]. Thus, we can assume that \(\langle H, K \rangle\) is not abelian. By Lemma 5.1, there exists a finitely generated open subgroup \(U\) of \(H \cap K\) that is normal in \(\langle H, K \rangle\). Hence by Theorem 6.5 in [14], we have \(|\langle H, K \rangle : U| < \infty\). This implies that \(|\langle H, K \rangle : H \cap K| < \infty\).

(2) Let \(H\) be an abelian finitely generated subgroup of \(G\). Note that if \(H \leq A \leq G\) and \(|A : H| < \infty\), then by Corollary 5.4 in [14] it follows that \(A\) is abelian. Consider the set

\[
S(H) = \{A \mid H \leq A \leq G, A \text{ is finitely generated and abelian}\}.
\]

Let \(A_1 \leq A_2 \leq \cdots\) be an ascending chain of elements in \(S(H)\). Then, \(A = \langle \bigcup_{i \geq 1} A_i \rangle\) is abelian. Let \(H \leq G_n = G_{n-1} \sqcup \langle A_{n-1} \rangle\) where \(n\) is the weight of \(H\). Then by Corollary 5.5 in [14], if \(A\) is non-cyclic then it is conjugate to a factor of \(G_n\). By induction on \(n\), it is not hard to see that \(A\) is finitely generated. Thus, every ascending chain in \(S(H)\) has an upper bound. By Zorn’s lemma, it follows that \(S(H)\) has a
maximal element; denote this element by $S$. From the structure theorem of finitely generated free modules over principal ideal domains it follows that there exists a basis $y_1, y_2, \ldots, y_n$ of $S$ such that $p^{a_1}y_1, p^{a_2}y_2, \ldots, p^{a_m}y_m$ is a basis of $H$ where $m \leq n$ and $a_1, a_2, \ldots, a_m$ are nonzero integers with the relation $a_1 \leq a_2 \leq \cdots \leq a_m$. Set $N = \langle y_1, \ldots, y_m \rangle$; it is easy to see that $N = \text{root}_G(H)$.

Now let $H$ be a non-abelian finitely generated subgroup of $G$. By Theorem 4.4, we have that $\chi(H) < 0$. If $H \leq K$ and $|K : H| < \infty$, then from the multiplicativity of the Euler–Poincaré characteristic it follows that $\chi(H) \leq \chi(K) = \frac{\chi(H)}{|K:H|} < 0$. Choose $K$ such that $H \leq K$, the index $|K : H| < \infty$ and $\chi(K)$ is as large as possible. We claim that $K$ is a root of $H$ in $G$. Indeed, suppose that there is some $M \leq G$ such that $H \leq M$, the index $|M : H| < \infty$ and $K$ does not contain $M$. Then, by Greenberg–Stallings property, we have that $H$ is also of finite index in $A = \langle K, M \rangle$. But then $\chi(A) = \frac{\chi(K)}{|A:K|} > \chi(K)$, which is a contradiction. Thus, we must have $K = \text{root}_G(H)$.

(3) Let $H$ be a finitely generated non-abelian subgroup of $G$. By (2), $H$ has a root in $G$. By Proposition 5.2, we have

$$\text{Comm}_G(H) = \text{Comm}_G(\text{root}_G(H)).$$

Since $\text{root}_G(\text{root}_G(H)) = \text{root}_G(H)$, it suffices to prove that if $H = \text{root}_G(H)$, then $H = N_G(H) = \text{Comm}_G(H)$.

Suppose that $H = \text{root}_G(H)$. By Theorem 6.7 in [14], $H$ has finite index in $N_G(H)$. Hence we have

$$H \leq N_G(H) \leq \text{root}_G(H) = H.$$ 

Thus $H = N_G(H)$. Also, it is clear that $N_G(H) \leq \text{Comm}_G(H)$. It remains to show that $\text{Comm}_G(H) \leq N_G(H)$. Let $g \in \text{Comm}_G(H)$. This means that $H \cap gHg^{-1}$ has finite index in both $H$ and $gHg^{-1}$, and as a consequence, we have

$$\text{root}_G(H \cap gHg^{-1}) = \text{root}_G(gHg^{-1}) = \text{root}_G(H) = H.$$ 

It follows that

$$\langle gHg^{-1}, H \rangle = H.$$ 

Suppose that $g \in \text{Comm}_G(H) \setminus N_G(H)$. Then $gHg^{-1} \neq H$, and hence, $H$ is properly contained in $\langle gHg^{-1}, H \rangle = H$, a contradiction. Thus, we must have $\text{Comm}_G(H) \setminus N_G(H) = \emptyset$, i.e., $N_G(H) = \text{Comm}_G(H)$, as desired. This finishes the proof.

**Definition 3** For a given subgroup $H$ of $G$, the *normalizer tower* of $H$ in $G$ is defined as

$$N_G^0(H) = H, \quad N_G^{\alpha+1}(H) = N_G(N_G^\alpha(H))$$

and if $\alpha$ is a limit ordinal, then

$$N_G^\alpha(H) = \bigcup_{\beta < \alpha} N_G^\beta(H).$$
By part (2) of the above theorem and Theorem 6.7 in [14], we have the following.

**Corollary 5.4** Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. If $H$ is a finitely generated non-abelian subgroup of $G$, then the normalizer tower of $H$ in $G$ stabilizes after finitely many steps, i.e., it has finite length.

**Lemma 5.5** Let $G$ be a pro-$p$ group from the class $\mathcal{L}$ and let $H$ be a non-abelian finitely generated subgroup of $G$. Then, $\text{Comm}_G(H) = \text{root}_G(H)$. In particular, the group $H$ has finite index in $\text{Comm}_G(H)$.

**Proof** Was performed in the proof of part (3) of Theorem 5.3. \hfill \Box

The following result generalizes Corollary 6.6 in [14].

**Corollary 5.6** Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. If $F$ is a finitely generated free pro-$p$ subgroup of $G$ with $d(F)$ not congruent to 1 modulo $p$, then

$$F = N_G(F) = \text{root}_G(F) = \text{Comm}_G(F).$$

**Proof** Let $F$ be a finitely generated subgroup of $G$ with $d(F)$ not congruent to 1 modulo $p$. By part (2) of Theorem 5.3, we know that $F$ has a root. Suppose that $\text{root}_G(F) \neq F$. Then, there is a subgroup $H$ of $G$ that contains $F$ and such that $|H : F| = p$. Since $H$ is torsion free, by Serre’s result [29] we have that $H$ is a free pro-$p$ group. From Nielsen–Schreier formula, we have $d(F) = p(d(H) - 1)$. Thus, $d(F) \equiv 1 \pmod{p}$, which is a contradiction. Thus we must have $F = \text{root}_G(F)$. By Theorem 6.7 in [14], $F$ has finite index in $N_G(F)$. Hence, $F \leq N_G(F) \leq \text{root}_G(F) = F$. By the previous lemma, we have $F = N_G(F) = \text{root}_G(F) = \text{Comm}_G(F)$.

A finitely generated subgroup $H$ of a group $G$ is said to be **self-rooted** if it has a root in $G$ and $\text{root}_G(H) = H$. From the above corollary, it follows that if $G$ is a non-abelian pro-$p$ group from the class $\mathcal{L}$, then for any $n \in \mathbb{N}$ there is a self-rooted finitely generated subgroup $F$ of $G$ with $d(F) > n$.

Let $G$ be a pro-$p$ group from the class $\mathcal{L}$. To every finitely generated self-rooted subgroup $H$ of $G$, we associate the set

$$H^* = \{U \mid U \leq H \quad \text{and} \quad |H : U| < \infty\}.$$

Consider the sets

$$\mathcal{M}(G) = \{H \mid H \text{ is a finitely generated subgroup of } G\},$$
$$\overline{G} = \{H \mid H \leq G\},$$
$$\mathcal{L}(G) = \{H^* \mid H \text{ is a finitely generated self-rooted subgroup of } G\}$$

and recall that we can consider $\overline{G}$ as a lattice with the standard meet and joint operations for groups. One can easily prove the following result.
Proposition 5.7 Let $G$ be a pro-$p$ group from the class $\mathcal{L}$.

(a) If $H$ is a finitely generated self-rooted subgroup of $G$, then $H^*$ is a convex sub-lattice of $\overline{G}$ with greatest element $H$ and without a least element.

(b) The set $\mathcal{L}(G)$ forms a partition of $\mathcal{M}(G)$, i.e., any two distinct elements in $\mathcal{L}(G)$ are disjoint and $\mathcal{M}(G)$ is equal to the union of all the elements in $\mathcal{L}(G)$.

6 Abstract limit groups

In [22], as we mentioned in the introduction, Rosset proved that every finitely generated subgroup of a free group has a root. The following theorem generalizes this result to the class of abstract limit groups.

Theorem 6.1 Let $G$ be an abstract limit group. If $H$ is a finitely generated subgroup of $G$, then $H$ has a root in $G$.

Proof We only need to mention that by Theorem 6 in [18], abstract limit groups satisfy the Greenberg–Stallings property and by Lemma 5 in [13], non-abelian abstract limit groups have negative Euler characteristic. The rest of the proof is similar to the proof of part (2) of Theorem 5.3.

By the above theorem and Theorem 1 in [1], we have the following.

Corollary 6.2 Let $G$ be an abstract limit group. If $H$ is a finitely generated non-abelian subgroup of $G$, then the normalizer tower of $H$ in $G$ stabilizes after finitely many steps, i.e., it has finite length.

Finally, let us note that the result of Proposition 5.7 also holds for abstract limit groups.

Acknowledgments This work was carried out while the first author was holding a CNPq Postdoctoral Fellowship at the University of Brasília. He would like to thank CNPq for the financial support and the Department of Mathematics at the University of Brasília for its warm hospitality and the excellent research environment. The authors thank the anonymous referee for carefully reading the manuscript.

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