Topological currents in the bulk in the absence of gapless states

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We provide evidence that, alongside symmetry protected edge states, two-dimensional topological phases also support bulk currents. These currents are activated by local potential gradients in the bulk, while all parts of the system are adiabatically connected to the same phase. To understand their origin one can view the bulk of a homogeneous topological insulator as a perfectly entangled state of pairs of edge-like currents, adding up to a zero net flow. Potential gradients strain those states, progressively disentangling the hidden currents through a transfer of population. This produces a localised bulk current that is transverse to the strain, even when the potential is always below the energy gap, where one expects only edge currents to appear. Bulk currents are topologically protected and behave like edge currents under external influence, such as temperature or local disorder. The resilience and the tuning of bulk currents with local potentials makes them an appealing medium for technological applications.

INTRODUCTION

When a two-dimensional topological insulator is embedded in a surface with sharp boundaries, it develops gapless symmetry protected edge states [1–5]. Edge states support currents that are robust against imperfections, flowing without losses along a direction given by the Chern number, a topological invariant. Topologically insulating materials have been identified experimentally [6–8] and theoretically [9–11]. About 12% of recorded inorganic crystals are suspected a topological nature [9]. This abundance and the resilience of edge states make topological insulators attractive for applications: from frictionless, directed transport of currents, to transistors, amplifiers and detectors [12–16]. Our work challenges the dualistic vision of robust edge currents surrounding an inert bulk. Studying the influence of potentials on a topological insulator, we find and characterise new symmetry protected bulk currents. These currents arise from a continuous distortion of the valence band of the bulk, share the robustness and applications of edge states, with enhanced tuneability and no geometric constraints.

RESULTS

Edge and bulk currents:– The currents on the edges of a topological insulator are proportional to the potentials acting at the material’s boundary. According to our Methods, a uniform chemical potential $V(\mathbf{r}) = -\mu$ acting on a material with Chern number $\nu$, induces a current

$$I_{\text{edge}} = \frac{\mu}{2\pi^2} \mu,$$

which is robust against temperatures and local disorder. Most surprisingly, we find that a spatially varying potential creates currents in the bulk of the material, even when the potential is so small that it cannot close the insulating gap, cause a phase transition or locally destroy topological order. In particular, any small potential gradient $\nabla V(\mathbf{r})$ induces perpendicular bulk currents

$$I_{\text{bulk}} = \frac{\nu}{2\pi} a_0 |\nabla V(\mathbf{r})|,$$

proportional to the Chern number $\nu$ and the lattice constant $a_0$. These bulk currents share the topological protection of edge currents, but their strength, direction and geometry are tuneable with the potential $V(\mathbf{r})$, unconstrained by the form of the sample.

Later, we provide a clear explanation of bulk currents founded in the tensor-network description of topological systems [17–19]. In this view, the topological bulk insulator is a many-body state of hidden entangled “edge states” that bind together, producing a zero net current. Any inhomogeneous potential strains and progressively disentangles these states, which reveal as localised currents in the bulk, perpendicular to the gradient of the strain field. In extreme gradients, or when the material is cut, edge states unbind and become free currents flowing along the new boundary. We support this interpretation with a study of 1D topological Su-Schrieffer-Heeger (SSH) model [20] under local potentials and a connection to topological insulator tensor-network states [21]. Note that, despite evidence of bulk currents in the special case of Dirac materials [22] and the quantum Hall effect [23], our work provides the first general description for lattice systems that is applicable to any dimension.

Edge currents in lattice models:– For a realistic discussion, we use Haldane’s topological insulator model [24] of fermions $\{c_i, c_i^\dagger\}$ on a honeycomb lattice

$$H = \sum_{\langle ij \rangle} t_1 c_i^\dagger c_j + \sum_{\langle i \rangle} t_2 e^{i\phi_{ij}} c_i^\dagger c_i + \sum_{i} V(r_i) c_i^\dagger c_i.$$

The model’s plaquette in Fig. 1(a) illustrates the real nearest neighbour hopping $t_1$, and the complex next-
nearest neighbour hopping $t_2$ with direction $\nu_{ij} = \pm 1$ and phase $\phi$. The Haldane model has two gapped topological phases with Chern numbers $\nu = \pm 1$, and trivial phase where $\nu = 0$. It is possible to cross between these phases by tuning $\phi, t_1, t_2$. Without loss of generality, we show numerical results for the topological insulator phase with $\nu = +1$, using $t_1 = 1$, $t_2 = 0.1$, $\phi = \pi/2$, on a cylinder with height $L_y$ and circumference $L_x$.

Our study considers a potential $V(\mathbf{r})$, small enough to preserve the topological phases. For a radially symmetric potential $V(\mathbf{r}) = V(y)$, Hamiltonian (3) is a collection of $L_x$ independent 1D problems, labeled by the momenta $p_x$ around the cylinder. In the topological phase $\nu \neq 0$, the cylinder supports two sets of edge states, with an approximately gapless and linear dispersion relation, $\omega \propto p_x$. As shown in Fig. 1(b), there are two modes for each $p_x$, each of them localised at a different boundary of the tube and flowing along opposite directions.

We detect the edge currents in position space, studying the flow of particles $n_i = c_i^\dagger c_i$ across a line $l$ that crosses the boundary (see Methods). Figs. 1(c–d) show the left-to-right flow of particles $I_{\text{edge}}$ as a function of chemical potential and temperature. At low temperatures $k_BT \ll t_1$, this flux is proportional to the chemical potential $V(\mathbf{r}) = \mu$ and satisfies $I = \nu \mu/(2\pi)$. We also observe that the current is invariant over a broad range of temperatures $T$, as expected [25]. Finally, the edge current vanishes when the temperature or the chemical potential approach the insulator gap, due to particles and holes in the conductance and valence bands, respectively.

**Bulk currents in lattice models:**—Let us now explore the physics of topological insulators under an inhomogeneous potentials. Our first example is a cylinder where a central stripe is subject to a different potential

$$V(\mathbf{r}) = \begin{cases} +w, & L_y/4 \leq y \leq 3L_y/4, \\ 0, & \text{else}. \end{cases}$$

(4)

Since the potential is strictly zero at the edges of the cylinder, the edge currents vanish. Instead, we find two bulk currents localised at the boundaries of the stripe, $y = L_y/4$ and $y = 3L_y/4$, flowing with opposite directions and equal strength $I_{\text{bulk}} = |w|/(2\pi)$ [cf. Fig. 2(b)]. The enhanced strength and the direction of these bulk currents are determined by the local potential gradient, $\nabla V(\mathbf{r}) = |w|/a_0$, not the absolute value of the potential. We verify this claim using a potential that interpolates linearly between opposite values 0 and $+w$. As
shown in Fig. 2(f), now the two edge currents add up to the same total value, but they spread across the linear ramp, adopting a locally smaller value \( I_{\text{bulk}}(y) \approx w/L_y \), in agreement with (2).

Despite having a different type of localisation, bulk currents are resilient to temperature changes and to disorder, just like edge currents. Fig. 2(c) shows the same temperature dependency for bulk currents as in 1(c). Moreover, Fig. 2(d) shows that bulk currents are largely insensitive to local random potentials \( V(r) = w \text{sign}(L_y/2 - y) + \epsilon \), with \( \epsilon \) drawn with uniform probability from the interval \( \epsilon \in [-w_{\text{dis}}, w_{\text{dis}}] \). In all cases, temperature and disorder acting on edge or bulk currents, the resilience of the bulk currents is limited to potentials and temperatures below the energy gap.

We use the inverse participation ratio (IPR) — the fourth moment of the wave function —, to identify which eigenstates are localised and contribute to edge and bulk currents [26]. In order to select the localised states that are actually occupied, we define the “filling” \( f_{n,p_x} \) for the \( n \)-th eigenstate \( \psi_{n,p_x} \) with quasimomentum \( p_x \) as

\[
f_{n,p_x} = \langle a_{n,p_x}^\dagger a_{n,p_x} \rangle \sum_y |\psi_{n,p_x}(y)|^4.
\]

As shown in Fig. 3, the conduction band of the topological insulator is empty. Nevertheless, we find two sets of localised states. There are highly localised gapless edge states on the boundary of the cylinder: these states are symmetrically filled and give a zero net edge current. We also find a more diffused family of localised states in the bulk: these states result from a distortion of the original bulk bands around \( p_x \approx \pi/a_0 \), which appears to accom-

modate the bulk currents at potential gradients of our stripe.

**Origin of bulk currents:** Localised bulk states can be also generated in the 1D SSH topological model, which provides a more intuitive picture for their emergence. The SSH Hamiltonian with hopping \( t \) and potential \( V_i \)
reads

\[ H = -\frac{t}{2} \sum_{i}^{2L-1} \left[1 + \epsilon(-1)^i\right] c_{i+1}^\dagger c_i + \sum_{i}^{2L} V_i c_i^\dagger c_i. \] (6)

In the perfectly dimerised regime \( \epsilon = 1 \) the system decouples into pairs of bound states \( \frac{1}{\sqrt{2}}(c_{2i} + c_{2i+1}) \) [cf. Fig. 4(a)], leaving two free edge states \( c_1 \) and \( c_{2L} \). A potential step \( V_i = w \text{sign}(i - L) \) is invisible to most of the strongly coupled fermionic modes, except for the pair \( \{c_L, c_{L+1}\} \) positioned at the step. As shown in Fig. 4(a), the potential step strains and progressively disentangles the strongly coupled fermionic modes, except for the pair \( \{c_L, c_{L+1}\} \), causing the appearance of localised edge particle and hole states, the 1D equivalent of our bulk-currents. As in the cylinder, these new states sit in the bulk of the spectrum, with an energy \(-\sqrt{w^2 + t^2}\) in between the bands \(-t \pm w\) of the left and right half chains. This continuous disentangling is confirmed by the monotonic decrease in the entanglement entropy of half the chain, as a function of \( w \) [cf. Fig. 4(b)]. This plot also reveals that we create the bulk pairs without really causing a phase transition, far from pure dimerisation.

The SSH model is one of many topological phases analysed with Matrix-Product-States (MPS) theory [17–19]. MPS express the wave function as a contraction of tensors, one per site in a one-dimensional structure that mimics the underlying lattice. The contracted “legs” of the tensors represent hidden entangled states of the many-body system, correlated by the tensors through a “projective measurement”. In the SSH model with \( \epsilon = 1 \) this representation is trivial: the “hidden” degrees of freedom map to the bulk superposition states \( c_{2i} + c_{2i-1} \) and the edge states \( \{c_1, c_{2L}\} \) [cf. Fig. 4(a)]. However, if we abandon the fully dimerised limit \( \epsilon \neq 1 \) or move to two dimensions—e.g. the Haldane model—, the fermionic topological phases require a more complex tensor network description [21].

As sketched in Fig. 4(c), general tensor network states have multiple entangled pairs correlated in an extended many-body state. The boundary of such a topological phase leaves unentangled degrees of freedom that group into an edge state with symmetry protected features. As in the SSH, any potential difference between neighbouring sites strains the underlying entanglement, progressively decoupling the tensors. This interpretation reveals a scenario where bulk currents can have tunable strength and direction, through a local control of the potential gradients along the lattice edges. As illustration, Fig. 4(d) shows the currents (arrows) on a 2D slab of the Haldane model, where we have \( V(r) = 0 \) everywhere, except for a rectangular patch with \( V(r) = +w \). Note the emergence of a localised bulk current around the patch. The strength and orientation of the current are controlled according to Eq. (2), while its path and direction adapts to the shape of the potential gradient.

**DISCUSSION**

Summing up, we have found that the bulk of a topological insulator can exhibit robust, symmetry protected bulk currents. They share with edge currents an intrinsic robustness against temperature and local disorder. However, while edge currents are confined at the boundaries of the material, bulk currents are defined by gradients of local potentials and can adopt any shape, extension and strength. This means bulk currents can be used in the same applications as edge states, with greater versatility and without the need for complex sample fabrication and shaping. As a spin-off of this work, we have established a qualitative connection between the appearance of bulk currents and the theory of topological tensor network states. To our knowledge, the appearance of bulk currents is the first practical application of this theory and opens the door to further studies in more complex models, such as fermions with spin [1], topological superconductors or topological phases with interactions.

**MATERIALS AND METHODS**

**Edge currents with a global potential**

We model edge physics with an isolated, one-dimensional, fermionic Hamiltonian

\[ H = \nu \int_{-\infty}^{\infty} dp \varepsilon(p) a_p^\dagger a_p. \]

This integral results in Eq. (1), a current proportional to \( \nu \) and \( \mu \), independent of the temperature \( T \).

**Bulk currents with local gradient**

Consider the action of the integer quantum Hall effect in the absence of a boundary, \( S[A] = \frac{e^2}{h} \int dx dt C_{\mu \lambda} A_\mu \partial_\lambda A_\nu \), where \( A \) is an abelian vector potential. The electric current density [27, 28] is given by

\[ J_{\text{bulk}}^\mu = \frac{\delta S[A]}{\delta A_\mu} = \frac{\nu}{2\pi} e^{\mu \lambda} \partial_\lambda A_\nu. \] (8)

When \( \mu = x, \kappa = y \) and \( \lambda = t \) we obtain the continues version of formula (2).

**Deriving the lattice current formula**

The continuity equation for the single site occupancy operator \( n_i = c_i^\dagger c_i \) reads \( dn_i/dt = \sum_j J_{ij} \). Using the Heisenberg equation for \( n_i \) we can obtain another expression for \( dn_i/dt = -i[n_i, H] \). Assuming a quadratic Hamiltonian \( H = \sum_{m,n} A_{mn} c_m^\dagger c_n \) with Hermitian matrix \( A_{jk} \), and equating the two expressions we find

\[ \sum_j J_{ij} = -i \sum_{m,n} A_{mn} \left[ c_i^\dagger c_j, c_m^\dagger c_n \right]. \]
We can show that the commutator on the right hand side can be rewritten as \([c_i^\dagger c_j, c_m^\dagger c_n] = \delta_{im} c_j^\dagger c_n - \delta_{jn} c_m^\dagger c_i\), then we identify the current operator as
\[
\hat{J}_{ij} = -i \left( A_{ij} c_i^\dagger c_j - A_{ji} c_j^\dagger c_i \right).
\]

For a quadratic model \(H = \sum_{ij} A_{ij} c_i^\dagger c_j\), the continuity equation reveals a set of currents \(J_{ij}\) running from site \(i\) to its neighbours
\[
\frac{d}{dt} \langle n_i \rangle = \sum_j J_{ij} \quad \text{with} \quad J_{ij} = 2 \text{Im} \left\{ A_{ij} \left( c_i^\dagger c_j \right) \right\}. \quad (9)
\]

This equation can be used to estimate the average current around any region. For a line “\(l\)” crossing the edge states from Fig. 1(b), we compute the net particle flux from the ‘left’ (\(L\)) to the ‘right’ (\(R\)) as \(I_{\text{edge}} = \sum_{i \in L,j \in R} J_{ij}\).

**Solutions for the cylinder**

As illustrated in Fig. 1 and Fig. 2, we consider cylindrical systems. The periodic coordinate is labelled \(x\) and the height coordinate \(y\). The length of the periodic direction is \(L_x\) and the total height of the cylinder is \(L_y\). If we consider external potentials \(V(r)\) that only depends on \(y\), then translational invariance is preserved along the \(x\) direction. Therefore, we can introduce a momentum coordinate \(p_x\) in the \(x\) direction and solve for each \(p_x\) independently. The lattice fermions will Fourier transform as,
\[
c_j^\dagger(p_x) = \frac{1}{\sqrt{L_x}} \sum_x e^{-ip_x x} c_j^\dagger(x), \quad (10)
\]

where the momentum can take values \(p_x = 2\pi n/L_x\) for \(n = 1, 2, \ldots, L_x\) and index \(j\) labels the vertical sites of the unit cell. The Hamiltonian for fixed \(p_x\), \(H(p_x)\), can then be obtained, and has the form
\[
H(p_x) = \sum_{j,l=1}^{2L_y} A_{jl}(p_x) c_j^\dagger(p_x) c_l(p_x).
\]

Since the one-dimensional unit cell contains \(2L_y\) sites, \(H(p_x)\) has \(2L_y\) different eigenvalues \(\varepsilon_m(p_x)\) for \(m = 1, 2, \ldots, 2L_y\) that we refer to as the ‘bands’ of the model. The fermionic eigenmodes of \(H(p_x)\) are
\[
a_{m,p_x}^\dagger = \sum_j U_{j,m}(p_x) c_j^\dagger(p_x), \quad (11)
\]

where \(U_{j,m}(p_x)\) is the matrix that diagonalises \(H_{jl}(p_x)\).

The two point correlator between any two sites in real space, \((x,y)\) and \((r,s)\), can be written in terms of the eigenmodes modes as
\[
\langle c_{x,y}^\dagger c_{r,s} \rangle = \frac{1}{L_x} \sum_{p_x} e^{-ip_x(x-r)} \langle c_{y,s}^\dagger(p_x)c_{x,y}(p_x) \rangle = \frac{1}{L_x} \sum_{mp_x} e^{-ip_x(x-r)} U_{y,m}(p_x) U^*_{s,m}(p_x) n(\varepsilon_m(p_x)),
\]

where the mode occupations \(n(\varepsilon_m(p_x)) = \langle a_{m,p_x}^\dagger a_{m,p_x} \rangle\) are given by Fermi Dirac statistics. Temperature enters our calculations through the Fermi functions \(n\).

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