DYNAMICS OF CONTINUOUS TIME MARKOV CHAINS WITH APPLICATIONS TO STOCHASTIC REACTION NETWORKS

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Abstract. This paper contributes an in-depth study of properties of continuous time Markov chains (CTMCs) on non-negative integer lattices, with particular interest in one-dimensional CTMCs with polynomial transitions rates. Such stochastic processes are abundant in applications, in particular within biology. We study the classification of states for general CTMCs on the non-negative integer lattices, by characterizing the set of absorbing states (similarly, trapping, escaping, positive irreducible components and quasi-irreducible components). For CTMCs on non-negative integers with polynomial transition rates, we provide threshold criteria in terms of easily computable parameters for various dynamical properties such as explosivity, recurrence, transience, positive/null recurrence, implosivity, and existence and non-existence of passage times. In particular, simple sufficient conditions for exponential ergodicity of stationary distributions and quasi-stationary distributions are obtained. Moreover, an identity for stationary measures is established and tail asymptotics for stationary distributions is given. A similar identity as well as asymptotics is derived for quasi-stationary distributions. Finally, we apply our results to stochastic reaction networks.

1. Introduction

Continuous time Markov chains (CTMCs) on a countable state space are widely used in applications, for example, in genetics [17], epidemiology [44], ecology [19], biochemistry and systems biology [52], sociophysics [50], and queueing theory [23]. For a CTMC on a discrete state space, there are several fundamental topics: classification of the state space, dynamical properties (recurrence, transience, explosivity, etc.), and patterns of potential stationary distributions (SDs).

In applications it is often difficult to classify the state space into irreducible classes and to determine whether a class is absorbing, transient, etc. This has for example been revealed in recent work on extinction events in stochastic reaction networks [2]. Moreover, there are few handy necessary and sufficient conditions for dynamical properties in the literature. Reuter provided necessary and sufficient conditions for explosivity for CTMCs (Reuter’s criterion) [45], but these conditions are difficult to check, except in the case of birth-death processes (BDPs) [35], and competition processes [46], due to the infinitely many algebraic equations. Meyn and Tweedie developed Lyapunov-Foster criteria for dynamical properties (e.g., non-explosivity, and positive recurrence), but it seems there are no criteria for the converse properties, for instance, Lyapunov-Foster conditions for explosivity or null recurrence [39].

Examples are abundant in stochastic reaction network theory. Here, a key issue is to understand how and whether the graphical structure of a reaction network determines...
the dynamics of the corresponding CTMC. For instance, consider the following two reaction networks with one species (S):

\[
\begin{aligned}
S & \xrightarrow{\frac{1}{2}} 2S & & \xrightarrow{\frac{3}{1}} 3S & & \xrightarrow{\frac{6}{1}} 4S & & \xrightarrow{\frac{1}{1}} 5S, \\
S & \xrightarrow{\frac{1}{2}} 2S & & \xrightarrow{\frac{3}{1}} 3S & & \xrightarrow{\frac{1}{1}} 4S.
\end{aligned}
\]

The corresponding Markov chains are on \( \mathbb{N}_0 \). A reaction, \( nS \xrightarrow{\kappa} mS \), encodes jumps from \( x \) to \( x + m - n \) with propensity \( \lambda(x) = \kappa x (x - 1) \ldots (x - n + 1), \kappa > 0 \). The first is explosive while the second is positive recurrent [3], which might be inferred from known BDP criteria [6]. However, these criteria are not easy to compute and blind to the structure of the networks.

In addition, given ergodicity of a CTMC, the form of the SD is desirable but not always achievable. For reaction networks, the form of the SD is known only in special cases, for example, for complex balanced networks, where the SD has Poisson product-form [4, 10, 34, 51] (similar to results in queueing theory [28]), and networks that are also BDPs. Using large deviation theory, the dynamics of a stochastic system might be inferred from a deterministic counterpart (mean field, ODE system), since the latter might be easier to analyse. Such theory is usually valid only over finite-time and for large system size [30, 31]. When the CTMC is absorbing, there is even less work on characterising the patterns of quasi-stationary distributions (QSDs) [35].

Motivated by the above concerns, we study the three topics (classification of states, dynamical properties, and patterns of SDs and QSDs) restricted to CTMCs on \( \mathbb{N}_0 \), particularly with polynomial transition rates. These CTMCs serve as a framework for a large class of stochastic polynomial systems emerging in epidemiology, ecology, chemistry, and molecular biology, and approximate ‘almost’ all CTMCs due to the denseness of polynomials in the spaces of continuous functions [8].

More precisely, given a set of possible jump vectors, we first characterise the sets of different types of states (absorbing, recurrent, etc.) in easily computable terms. When applied to reaction networks, these sets can be directly determined from the reaction graph, and in particular provide a neater representation of the extinction set, rather than using the algorithmic approach in [2]. Based on the classification, we study the dynamical behaviour of CTMCs with polynomial transition rates in terms of four easily computable parameters. We provide threshold criteria for existence and non-existence of passage times, positive recurrence and null recurrence, exponential ergodicity of SDs and QSDs. Especially, we also provide necessary and sufficient conditions for explosivity, recurrence (vs transience) and implosivity. Some of these criteria (positive/null recurrence) even turn out to be necessary and sufficient in the application to reaction networks (paper in preparation). In particular, we provide an explanation for the different dynamics of the two models in (1.1).

We discover simple identities for both stationary measures and QSDs of CTMCs. Built on these identities, we obtain asymptotic estimates of the tail distributions of SDs and QSDs. Despite there are known tail estimates of SDs for discrete time Markov chains (DTMCs) [6, 15, 29, 37], they are not directly applicable to CTMCs since the existence of SDs of CTMCs and that of DTMCs do not imply each other. In contrast, it seems there is almost no work on the asymptotics of tail distributions of QSDs. Interestingly, it turns out that the tails of SDs and QSDs can only be one of three types: Conley-Maxwell-Poisson (CMP), geometric, and Zeta distributions.

The outline of the paper is as follows: In Section 2, the notation is introduced and background on CTMCs is reviewed. Classification of states is provided in Section 3.
Section 4 further contributes to threshold criteria for various dynamical properties of CTMCs with polynomial transition rates on a countable state space. In Section 5, identities for stationary measures as well as QSDs are established, and estimates of the tail distributions of SDs as well as QSDs are established based on the identities. Examples of stochastic reaction networks are provided in Section 6 (a full application to reaction networks will be pursued in a subsequent paper). Finally, proofs of the main results are appended. Additional tools used in the proofs as well as proofs of some elementary propositions are compiled in a supplementary material.

2. Preliminaries

2.1. Notation. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ be the set of real, non-negative, and positive numbers, respectively. Let $\mathbb{Z}$ be the set of integers, $\mathbb{N} = \mathbb{Z} \cap \mathbb{R}_{\geq 0}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the absolute value of $x$, $\lceil x \rceil$ the ceiling function (i.e., the minimal integer $\geq x$), and $\lfloor x \rfloor$ the floor function of $x$ (i.e., the maximal integer $\leq x$). Let $d \in \mathbb{N}$. For $x \in \mathbb{R} \cup \{+\infty\}$, let $x = (x(1), \cdots, x(d)) \in \mathbb{R}^d \cup \{\infty\}$ ($\infty$ is used if at least one coordinate is $\infty$). For $x = (x_1, \cdots, x_d)$, $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, let $(x, y) = \sum_{j=1}^d x_j y_j$ be the inner product of $x$ and $y$, $\|x\| = \max_{1 \leq j \leq d} |x_j|$, and $x \geq y$ (similarly $y \leq x$, $x > y$, $y < x$) if it holds coordinate-wise. For $x, y \in \mathbb{N}_0$, let $x^y = \prod_{j=0}^{y-1} (x-j)^{1_{\mathbb{N}_0}}(x-y)$ be the descending factorial. For $x, y \in \mathbb{N}_0$, let $x^y = \prod_{j=1}^y x_j$.

For $x \in \mathbb{R}^d$, $B \subseteq \mathbb{R}^d$ and $j = 1, \cdots, d$, define

$$\hat{x}_j = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_d), \quad \hat{B}_j = \{\hat{x}_j : x \in B\}, \quad \hat{B}_j = \{x_j : x \in B\}.$$

For $b \in \mathbb{R}$ and $A \subseteq \mathbb{R}^d$, let $A + b = \{a + b : a \in A\}$ and $bA = \{ab : a \in A\}$. For $a \in \mathbb{R}^d$ and $B \subseteq \mathbb{R}$, $ab = Ba = \{ba : b \in B\}$. For nonempty subsets $A, B \subseteq \mathbb{R}^d$, denote

$$A \geq B = \{x \in A : x \geq y, \text{ for some } y \in B\}.$$ 

In particular, if $B = \{b\}$ is a singleton, simply denote $A \geq B$ by $A \geq b$. If $A = \mathbb{N}_0^d$, we write $B_>$ for $A \geq B$ whenever $d$ is clear from the context, for instance, $\mathbb{O}_> = \mathbb{N}_0^d$.

Recall that the relation $\leq$ induces a partial order on $\mathbb{R}^d$. The set $A$ is the minimal set of a non-empty set $B \subseteq \mathbb{R}^d$ if $A$ consists of all elements $x \in B$ such that $x \not\geq y$ for all $y \in B \setminus \{x\}$. Note that $B$ has a non-empty minimal set if and only if $B$ is bounded from below with respect to (w.r.t.) $\leq$. In particular, if $B \subseteq \mathbb{N}_0^d$, then its minimal set is finite, see Proposition S3.1 in the supplement.

Given $u \in \mathbb{Z}^d \setminus \{0\}$, for $x, y \in \mathbb{N}_0^d$, write $x \leq u y$ (or $y \geq u x$) if $y - x \in u \mathbb{R}_{\geq 0}$. Analogously, write $x < u y$ (or $y > u x$) if $y - x \in u \mathbb{R}_{>0}$. We use $|x|_u$ to denote the positive integer $c$ such that $y = nc$. Moreover, for all $x \leq u y$, denote the lattice interval $[x, y]_u = \{z \in \mathbb{Z}^d : x \leq u z \text{ and } z \leq u y\}$, and analogously for $[x, y]_u$, etc. For instance, when $d = 1$, for $x, y \in \mathbb{Z}$ with $x < y$, $[x, y]_1 = \{x, x+1, \cdots, y-1\}$.

Moreover, for $x, y \in \mathbb{Z}^d$, if $x \neq 0$, $x$ divides $y$, denoted $x \mid y$, if there exists $m \in \mathbb{Z}$ such that $y = mx$ and we write $m = \frac{y}{x}$. For $A \subseteq \mathbb{Z}^d$, $x$ is a common divisor of $A$ (or $x$ divides $A$), denoted $x \mid A$ if $x \mid y$, $\forall y \in A$. In particular, $x$ is called a greatest common divisor (gcd) of $A$, denoted gcd($A$) if $\hat{x} \mid A$ for every common divisor $\hat{x}$ of $A$. Obviously, if $x$ and $\hat{x}$ are both gcds of $A$, then $\hat{x} = \pm x$. In other words, a gcd of $A$ is unique up to $\pm$ sign, and we simply denote either of these two gcds by gcd($A$).
For a subset \( A \subseteq \mathbb{Z}^d \), let \( \#A \) be the cardinality of \( A \),

\[
\text{span } A = \left\{ \sum_{i \in I} c_i x^{(i)} \in \mathbb{Z}^d : c_i \in \mathbb{R}, \ x^{(i)} \in A, \ \forall i \in I, \ \text{and } I \subseteq \mathbb{N} \text{ is finite} \right\}
\]

the span of \( A \) and \( \dim \text{span } A \) the dimension of the span. Not all subsets of \( \mathbb{Z}^d \) have a \( \gcd \) (or even a common divisor), e.g., \( A = \{(1,2),(2,1)\} \). Indeed, for \( A \subseteq \mathbb{Z}^d \setminus \{0\} \) non-empty, one can show that \( \dim \text{span } (A) = 1 \) if and only if \( A \) has a common divisor if and only if \( \gcd(A) \) exists (Proposition S2.1). For any vector \( c \in \mathbb{Z}^d \), let \( \gcd(c) = \gcd(\{c_j : j = 1, \ldots, d\}) \in \mathbb{Z} \).

For probability measures \( \mu, \nu \) on \( \mathbb{N}_0^d \), let \( \| \mu - \nu \|_{\text{TV}} \) denote their total variation distance [13, 21]. For every finite measure \( \mu \) on \( \mathbb{N}_0 \), let

\[
T_\mu : \mathbb{N}_0 \rightarrow [0,1], \ x \mapsto \sum_{y \geq x} \mu(x),
\]

be the tail distribution of \( \mu \). The tail distribution defined here is slightly different from the standard definition as it includes the probability of \( x \).

For a function \( f : \mathbb{N}_0^d \rightarrow \mathbb{R}_\geq \), its support is denoted \( \text{supp } f = \{ x \in \mathbb{N}_0^d : f(x) > 0 \} \).

2.2. Markov chains. We first review some basic theory of Markov chains [42, 47]. Let \( Y_t \) \((t \geq 0)\), or \( Y_t \) for short, be a CTMC on a countable state space \( \mathcal{Y} \). For two states \( x, y \in \mathcal{Y} \), \( x \) is accessible from \( y \) (or \( y \) leads to \( x \)) denoted by \( y \rightarrow x \) if \( P_y(Y_t = x \text{ for some } t \geq 0) > 0 \). Moreover, \( x \) communicates with \( y \), denoted \( x \leftrightarrow y \), if both \( x \rightarrow y \) and \( y \rightarrow x \). A non-empty subset \( E \subseteq \mathcal{Y} \) is communicable if \( x, y \in E \) implies \( x \leftrightarrow y \). Hence \( \leftrightarrow \) defines an equivalence relation on \( \mathcal{Y} \), and partitions \( \mathcal{Y} \) into communicating classes (or classes for short).

A non-empty subset \( E \subseteq \mathcal{Y} \) is closed if \( x \in E \) and \( x \rightarrow y \) implies \( y \in E \). A set is open if it is not closed. A state \( x \) is absorbing (escaping) if \( \{x\} \) is a closed (open) class. An absorbing state \( x \) is neutral if \( x \) is not accessible from any other state \( y \in \mathcal{Y} \setminus \{x\} \), and otherwise, it is trapping. A non-singleton closed class is a positive irreducible component (PIC), while a non-singleton open class is a quasi-irreducible component (QIC). Any singleton class is either an absorbing state (neutral or trapping), or an escaping state, whereas any non-singleton class must be either a PIC or a QIC.

A communicating class \( E \) is recurrent (positive or null recurrent), or transient according to the standard meanings of the terms. Moreover, every recurrent class is closed and every finite closed class is recurrent [42]. Hence any open class (QIC or escaping state) is transient.

In the following, we define a class of CTMCs on \( \mathbb{N}_0^d \) in terms of a finite set of jump vectors and a set of non-negative transition functions. Let \( \Omega \subseteq \mathbb{Z}^d \setminus \{0\} \) be finite, let \( \lambda_\omega : \mathbb{N}_0^d \rightarrow \mathbb{R}_\geq, \ \omega \in \Omega \), be non-negative functions on \( \mathbb{N}_0^d \), and let \( \mathcal{F} = \{ \lambda_\omega : \omega \in \Omega \} \). We say, a state \( y \in \mathbb{N}_0^d \) is reachable from \( x \in \mathbb{N}_0^d \) (denoted \( x \rightarrow y \)) if there exists a sequence \( x^{(0)}, \ldots, x^{(m)} \), such that \( x = x^{(0)}, \ y = x^{(m)} \) and \( \lambda_{\omega(i)}(x^{(i)}) > 0 \) with \( \omega^{(i)} = x^{(i+1)} - x^{(i)} \in \Omega, \ i = 0, \ldots, m - 1 \). Now, define a class of CTMCs on \( \mathbb{N}_0^d \) in terms of \( (\Omega, \mathcal{F}) \):

\[
\mathcal{C} = \{ Y_t : Y_0 \in \mathbb{N}_0^d, \ q_{x,y} = \lambda_{y-x}(x) 1_{\Omega}(y-x), \ \forall x, y \in \mathcal{Y}_0 \},
\]

where the state space \( \mathcal{Y}_0 \) consists of \( Y_0 \) and the states reachable from \( Y_0 \), and \((q_{x,y})_{x,y \in \mathcal{Y}_0}\) is the transition operator on \( \mathcal{Y}_0 \) (index 0 is used only when the initial state is given). The notion of accessibility and reachability coincide if \( \mathcal{Y}_0 = \mathbb{N}_0^d \).
A set $A \subseteq \mathbb{N}_0^d$ is a communicating class for $\mathcal{C}$ if it is a communicating class for one (and hence for all) $Y_t \in \mathcal{C}$ with $A \subseteq \mathcal{Y}_0$. A state $x \in \mathbb{N}_0^d$ is a neutral state for $\mathcal{C}$ if it is a neutral state for all $Y_t \in \mathcal{C}$ with $x \in \mathcal{Y}_0$, a trapping state for $\mathcal{C}$ if it is so for one $Y_t \in \mathcal{C}$ with $x \in \mathcal{Y}_0$ (and hence for all with $x \in \mathcal{Y}_0$ and $\#\mathcal{Y}_0 > 1$), and an escaping state for $\mathcal{C}$ if it is so for one $Y_t \in \mathcal{C}$ with $x \in \mathcal{Y}_0$ (and hence for all with $x \in \mathcal{Y}_0$). A set $A \subseteq \mathbb{N}_0^d$ is a PIC (QIC) for $\mathcal{C}$ if it is so for one $Y_t \in \mathcal{C}$ with $A \subseteq \mathcal{Y}_0$.

Let $N$, $T$, $E$, $P$, and $Q$ be the (possibly empty) set of all neutral states, trapping states, escaping states, PICs and QICs for $\mathcal{C}$, respectively. Every $x \in \mathbb{N}_0^d$ belongs to precisely one of the these sets (see Proposition S1.1). By definition, any $Y_t \in \mathcal{C}$ with $\mathcal{Y}_0 \in \mathcal{P}$ is irreducible.

A positive (probability) measure $\pi$ on $\mathbb{N}_0^d$ is a stationary measure (SD) for $\mathcal{C}$ if $\pi$ is a non-negative equilibrium of the so-called master equation [22]:

$$0 = \sum_{\omega \in \Omega} \lambda_\omega (x - \omega) \pi(x - \omega) - \sum_{\omega \in \Omega} \lambda_\omega (x) \pi(x), \quad x \in \mathbb{N}_0^d.$$  

### 3. Classification of states

We impose regularity on the set of transition functions and do not allow a CTMC to jump sporadically. Specifically, we assume

(A1) $\text{supp} \lambda_\omega = \{y \in \mathbb{N}_0^d: \lambda_\omega(x) > 0 \forall x \geq y\}$, for every $\omega \in \Omega$.

For every $\omega \in \Omega$, let $I_\omega$ be the minimal set of $\text{supp} \lambda_\omega$. Denote $O_\omega = I_\omega + \omega$, and let $I = \bigcup_{\omega \in \Omega} I_\omega$ and $O = \bigcup_{\omega \in \Omega} O_\omega$. By Proposition S3.1, $I_\omega$ is a non-empty finite set, and by (A1) we have $(I_\omega)_{\geq} = \text{supp} \lambda_\omega$ and $(O_\omega)_{\geq} = \text{supp} \lambda_\omega + \omega$, and thus

$I_{\geq} = \bigcup_{\omega \in \Omega} \text{supp} \lambda_\omega = \text{supp} \min_{\omega \in \Omega} \lambda_\omega$, \hspace{1cm} $O_{\geq} = \bigcup_{\omega \in \Omega} (\text{supp} \lambda_\omega + \omega)$.

Let $S = \text{span} \Omega$ and define the invariant subspace,

$L_c = (S + c) \cap \mathbb{N}_0^d$, \hspace{1cm} for $c \in \mathbb{Z}^d$,

that is, $Y_t \in L_c$ if $\mathcal{Y}_0 \in L_c$. Furthermore, it is translational invariant: $L_c = L_{c'}$ whenever $c - c' \in S$ and $L_c \cap L_{c'} = \emptyset$ if $c - c' \notin S$.

#### Example 3.1.

Consider $\mathcal{C}$ with $\Omega = \{(2, 1), (2, -1)\}$, $I_{(2,1)} = \{(2, 2), (3, 1)\}$, $I_{(2,-1)} = \{(3, 3)\}$. Hence $O_{(2,1)} = \{(4, 3), (5, 2)\}$, $O_{(2,-1)} = \{(5, 2)\}$. See Figure 1.

![Figure 1. Illustration of Example 3.1. $I_{\geq} = I_{} + \Pi$, $O_{\geq} = \Pi$.](image-url)
3.1. Characterization of different sets of states. Given \((\Omega, \mathcal{F})\), for \(\omega \in \Omega\), let \(I^*_{\omega} = \{x \in (I_{\omega})_{\geq} : x + \omega \rightarrow x\}\), and define \(I_* = \{\omega \in \Omega : I^*_{\omega} = (I_{\omega})_{\geq}\}\).

**Theorem 3.2.** Given \((\Omega, \mathcal{F})\), assume (A1). Then \(N = N_0^d \setminus (O_{\geq} \cup I_{\geq})\), \(T = (O \setminus I_{\geq}) \setminus I\), \(E = I_{\geq} \setminus \bigcup_{\omega \in \Omega} I^*_{\omega}\), \(P \cup Q = \bigcup_{\omega \in \Omega} I^*_{\omega}\), and \(\bigcup_{\omega \in \Omega} ((I_{\omega})_{\geq} \setminus I^*_{\omega}) \subseteq E \cup Q\). In particular,

1. \(N = N_0^d \setminus I\geq\), \(P = I_{\geq}\), \(T = E = Q = \emptyset\) if and only if \(\Omega_* = \Omega\).
2. \(P \cup Q = \emptyset\) and all states are singleton classes (either absorbing or escaping), if and only if \(\sum_{\omega \in \Omega} c_{\omega} \neq 0\) whenever \(c_{\omega} \in \mathbb{N}_0\) for all \(\omega \in \Omega\) such that \(\sum_{\omega \in \Omega} c_{\omega} \neq 0\).
3. \(I_{\geq} \setminus O_{\geq} \subseteq E\), \(P \cup Q \subseteq I_{\geq} \cap O_{\geq}\).

Let \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) be associated with \((\Omega, \mathcal{F})\) and \((\bar{\Omega}, \bar{\mathcal{F}})\), respectively. We say \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) are structurally equivalent if for any \(A \subseteq N_0^d\), \(A\) is a communicating class for \(\mathcal{C}\) if and only if \(A\) is a communicating class for \(\bar{\mathcal{C}}\). Similarly, we say the classes are structurally identical if they are structurally equivalent and \(\Omega = \bar{\Omega}\).

**Example 3.3.** Consider the two CTMC classes \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) associated with the two reaction networks in (1.1). Both \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) are associated with \(\Omega = \{-1, +1\}\), and \(I_1 = \{1\}, I_{-1} = \{2\}\), hence they are structurally identical. Nonetheless, they have quite different dynamics as mentioned in the Introduction.

The next result is a direct but nontrivial consequence of Theorem 3.2, and provides a criterion to show structural equivalence. We use \(\bar{I}_{\omega}\) to denote a minimal set corresponding to \((\bar{\Omega}, \bar{\mathcal{F}})\).

**Theorem 3.4.** Let \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) be associated with \((\Omega, \mathcal{F})\) and \((\bar{\Omega}, \bar{\mathcal{F}})\), respectively. If

1. \(\bar{I}_{\geq} \subseteq \Omega\) and \(\bar{I}_{\omega} = I_{\omega}\) for all \(\omega \in \bar{\Omega}\),
2. \(\Omega \setminus \bar{\Omega} \subseteq \bigcup_{\omega \in \bar{\Omega}} \omega \mathbb{N}\),
3. for every \(\omega \in \bar{\Omega}\), \(I_{\omega}\) is the minimal set of \(\bigcup_{\omega \in \omega \mathbb{N}} I_{\omega}\),

Then \(\mathcal{C}\) and \(\bar{\mathcal{C}}\) are structurally equivalent.

Next we derive a necessary and sufficient condition for \(T = \emptyset\).

**Theorem 3.5.** Given \((\Omega, \mathcal{F})\), assume (A1). Then \(T = \emptyset\) if and only if

\[\forall x \in O, \exists y \in I \text{ such that } x \geq y.\]

Finally, we provide necessary and sufficient conditions for \(#T < \infty\).

**Theorem 3.6.** Given \((\Omega, \mathcal{F})\), assume (A1). The following three statements are equivalent:

1. \(T\) is finite,
2. \((\bar{O}_j)_{\geq} \subseteq (\bar{T}_j)_{\geq}\) for all \(j = 1, \ldots, d\),
3. for all \(j = 1, \ldots, d\) and \(x \in O\), there exists \(y \in \bar{I}\) such that \(\bar{x}_j \geq \bar{y}_j\).

A necessary condition for \(#T < \infty\) is given below, which is also sufficient for \(d = 2\).

**Theorem 3.7.** Given \((\Omega, \mathcal{F})\), assume (A1). If \(T\) is finite, then

\[\min \hat{O}_j \geq \min \bar{T}_j, \forall j = 1, \ldots, d.\]

**Theorem 3.8.** Given \((\Omega, \mathcal{F})\), assume (A1). If \(d = 2\), then \(T\) is finite if and only if (3.1) holds.
Example 3.9. Let \( d = 3 \). Consider \( \mathcal{C} \) with \( \Omega = \{1, -2 \cdot 1\} \), and \( \mathcal{I}_1 = \{(1,1,3), (1,3,1)\} \), \( \mathcal{I}_2 = \{(4, 4, 4)\} \). Hence \( \mathcal{I} = \{(1,1,3), (1,3,1), (4, 4, 4)\} \) and \( \mathcal{O} = \{(2,2,4), (2,2,2), (2,4,2)\} \). For \( j = 1 \) and \( x = (2,2,2) \in \mathcal{O} \), there exists no \( y \in \mathcal{I} \) such that \( x_k \geq y_k \) for \( k = 2, 3 \). Hence, by Theorem 3.6, \( \#T = \infty \). Furthermore,
\[
\min \tilde{J}_j = 2 > \min J_j = 1, \quad \text{for } j = 1, 2, 3,
\]
contradicting (3.1). Hence, extension to higher dimension than two is not possible.

3.2. Classification of states in dimension one. In the following, we focus on classifying states for a class of one-dimensional CTMCs (i.e., \( \dim \mathcal{S} = 1 \)). Assume (A2) \( \omega_s = \gcd(\Omega) \) exists.

(A3) \( (\omega_s)_1 > 0 \).

By definition, \( \omega_s \) is not necessarily in \( \Omega \). Indeed, (A2) is equivalent to \( \dim \mathcal{S} = 1 \) (for a proof, see Proposition S2.1), and \( \mathcal{S} = \omega_s \mathbb{Z} \). Since all jumps occur along a line, we can define a positive direction by \( \omega_s \) and assume (A3) w.o.l.g. Furthermore, \( \leq \omega_s \) defines a complete order on \( \mathcal{L}_c \), for \( c \in \mathbb{N}_0^d \). It is a half lattice line in \( \mathbb{N}_0^d \) if \( \omega_s \in \mathbb{N}_0^d \), while a finite lattice interval in \( \mathbb{N}_0^d \) if \( \omega_s \notin \mathbb{N}_0^d \).

Let \( \Omega_\pm = \{ \omega \in \Omega : \sgn(\omega_k) = \pm \} \) be the sets of forward and backward jumps, respectively. We make the further assumption:

(A4) \( \Omega_+ \neq \emptyset, \Omega_- \neq \emptyset \).

If either \( \Omega_+ = \emptyset \) or \( \Omega_- = \emptyset \), then any CTMC in \( \mathcal{C} \) is a pure birth or death process on a finite or infinite state space. The classification of states as well as the dynamics of such processes are simpler than under (A4). Indeed, one can derive parallel results from the corresponding results under (A4). We leave this to the interested reader.

Denote the union of the positive and negative minimal sets by \( \mathcal{I}_\pm = \cup_{\omega \in \Omega_\pm} \mathcal{I}_\omega \) and \( \mathcal{O}_\pm = \cup_{\omega \in \Omega_\pm} \mathcal{O}_\omega \), and define
\[
\mathcal{L} = \text{min}^{\omega_s} \mathcal{L}_c \cap \left( (\mathcal{I}_+ \geq) \cap (\mathcal{O}_- \geq) \cup (\mathcal{I}_- \geq) \cap (\mathcal{O}_+ \geq) \right),
\]
\[
\mathcal{E} = \text{max}^{\omega_s} \mathcal{E}_c \cap \left( (\mathcal{I}_+ \geq) \cap (\mathcal{O}_- \geq) \cup (\mathcal{I}_- \geq) \cap (\mathcal{O}_+ \geq) \right),
\]
where \( \text{min}^{\omega_s} \) and \( \text{max}^{\omega_s} \) are minimum and maximum (potentially infinite) w.r.t. \( \omega_s \). These are well-defined regardless of the chosen representative \( c \) due to (A3) and translational invariance.

Moreover, we define \( \omega_{ss} = \gcd((\omega_s)_1, \ldots, (\omega_s)_d) \) and \( \Gamma_c = \mathcal{T} \cap \mathcal{L}_c \) (similarly, \( \mathcal{E}_c \), etc.), and for \( k = 1, \ldots, \omega_{ss} \),
\[
\Gamma^{(k)}_c = \left( \omega_s \mathbb{Z} + \frac{k - 1}{\omega_{ss}} \omega_s + c \right) \cap \mathcal{L}_c, \quad \mathcal{T}^{(k)}_c = \Gamma^{(k)}_c \cap \mathcal{T}_c,
\]
(similarly, \( \mathcal{E}^{(k)}_c \), etc.). Let \( \Sigma^+_c = \{ k \in \{1, \ldots, \omega_{ss}\} : \mathcal{T}^{(k)}_c \neq \emptyset \} \), \( \Sigma^-_c = \{1, \ldots, \omega_{ss}\} \setminus \Sigma^+_c \).

Before presenting the classification results, we consider a “messy” example.

Example 3.10. Let \( d = 2 \). Consider \( \mathcal{C} \) with \( \omega_s = (1, -1) \), \( \Omega_+ = \{\omega_s, 2\omega_s\} \), \( \Omega_- = \{-3\omega_s, -4\omega_s\} \), \( \mathcal{I}_+ = \{(1, 2)\} \), \( \mathcal{I}_2\omega_s = \{(0, 4)\} \), \( \mathcal{I}_- = \{(7, 0)\} \), and \( \mathcal{I}_{-4\omega_s} = \{(6, 2)\} \).

In this case, \( \mathcal{E}_c \) is located in the middle of an invariant set \( \mathcal{L}_c \). See Figure 2.

Theorem 3.11. Given \( (\Omega, \mathcal{F}) \), assume (A1)-(A4). There exists \( b \in \mathbb{N}_0^d \) such that for all \( c \in \mathbb{N}_0^d \), \( c \in \mathcal{F} \). Moreover, \( \mathcal{F}_{\omega_s} = \mathcal{E}_c \cup \mathcal{Q}_c \) consists of all non-singleton communicating classes on \( \mathcal{L}_c \), while \( \mathcal{L}_c \setminus \mathcal{F}_{\omega_s} \) is the union of singleton communicating classes, composed of \( \mathcal{N}_c = \mathcal{L}_c \setminus (\mathcal{O}_c \cup \mathcal{I}_c) \), \( \mathcal{T}_c = \mathcal{L}_c \cap \mathcal{O}_c \setminus \mathcal{I}_c \), \( \mathcal{E}_c = \mathcal{I}_c \cap \mathcal{L}_c \setminus (\mathcal{F}_{\omega_s} \cup \mathcal{Q}_c) \).
Furthermore, for \( k \in \Sigma_c^+ \), \( Q_c^{(k)} = \Gamma_c^{(k)} \cap [\overline{\mathcal{G}}, \overline{\mathcal{T}}]_{\omega_*} \) is a QIC trapped into \( T_c^{(k)} \), and for \( k \in \Sigma_c^- \), \( P_c^{(k)} = \Gamma_c^{(k)} \cap [\mathcal{G}, \mathcal{T}]_{\omega_*} \) is a PIC.

Define \( i(c) = \min \omega^* L_c \cap I \geq, i_+(c) = \min \omega^* L_c \cap (I_+) \geq, o(c) = \min \omega^* L_c \cap O \geq, \) and \( o_-(c) = \min \omega^* L_c \cap (O-) \geq, \) for \( c \in N_0^d \). By definition, \( c \geq i(c), o(c) \). The result below provides more detailed characterization of the relevant sets for \( \omega_* \in N_0^d \). The proof is tedious but straightforward and left to the interested reader.

**Corollary 3.12.** Given \( (\Omega, \mathcal{F}) \), assume (A1)-(A4) and \( \omega_* \in N_0^d \). Then for \( c \in N_0^d \), we have \( c = \max \{ i(c), o_-(c) \} \), \( \tau = +\infty \),

\[
N_c = \{ \min \omega^* L_c, \min \omega^* \{ i(c), o(c) \} \} \leq \omega_*, \quad E_c = [ i(c), c \omega_*, \quad T_c = [ o(c), i(c) \omega_* , \quad E_c < \infty .
\]

hence \#\( N_c \), \#\( T_c \), \#\( E_c \) are finite. Moreover, \#\( \Sigma_c^+ \) and 

(i) \( o(c) \geq \omega_* \), \( i(c) \) if and only if \( T_c = \emptyset \).

(ii) \( o(c) = o^-(c) \leq \omega_* \), \( i(c) = i^+(c) \) if and only if \( E_c = \emptyset \).

(iii) \( i(c) - o(c) \geq \omega_* \), \( \omega_* \) if and only if \( P_c = \emptyset \).

(iv) \( i(c) - o(c) < \omega_* \), \( \omega_* \) implies \( i(c) = i^+(c) \).

(v) \( \cup_{k \in \Sigma_c^+} E_c^{(k)} \neq \emptyset \) implies \( T_c \neq \emptyset \), which itself implies \( E_c = \cup_{k \in \Sigma_c^+} E_c^{(k)} \), and every escaping state either leads to another escaping state or a trapping state.

The following example illustrates that a QIC may lead to multiple trapping states, even in the case \( \omega_* \in N_0^d \).

**Example 3.13.** Let \( d = 1 \). Consider \( \mathcal{E} \) with \( \Omega = \{ 1, -2 \} \), \( I_1 = \{ 3 \} \) and \( I_2 = \{ 2 \} \). The flow chart for the state space is as follows:

![Flow chart](image-url)
Here $o(c) = 0$ and $i(c) = 2$. The unique QIC, $N_{\geq 3}$, is trapped into $T = \{0, 1\}$.

**Example 3.14.** PICs and QICs can coexist on the same invariant subspace. Consider the two classes of CTMCs on $N_0^2$, illustrated in Figure 3. (i) Let $\omega_* = (2, -2)$, $\Omega = \{\omega_*, -\omega_*\}$, $I_{\omega_*} = \{(1, 2)\}$ and $I_{-\omega_*} = \{(2, 1)\}$. (ii) Let $\omega_* = (2, 2)$, $\Omega = \{\omega_*, -\omega_*\}$, $I_{\omega_*} = \{(1, 2)\}$ and $I_{-\omega_*} = \{(2, 3)\}$.

4. **Criteria for dynamical properties**

Throughout Section 4 and 5 we assume $d = 1$. Hence (A2) automatically is fulfilled. By (A1), all chains in $C$ have an infinite countable state space, unless they start in an absorbing state or an escaping state leading to a trapping state. To avoid triviality, we only discuss dynamics of infinite chains.

In this situation, $\geq_{\omega_*}$ is the same as $\geq$ (and thus $[a, b]_{\omega_*} = [a, b]_1$ for all $a, b \in N_0$), $\omega_{**} = \omega_*$, and both $I_\omega$ and $O_\omega$ are *singletons* for all $\omega \in \Omega$. For simplicity, we denote the unique element in the sets also by $I_\omega$ and $O_\omega$, respectively. For all $c \in N_0$, we have $L_c = N_0$, and thus the parameters and sets defined in the previous section, e.g., $i(c)$, $\Sigma_{\Sigma^+}$, are independent of $c$, and hence for convenience we put $c = 0$. From Corollary 3.12 it follows that $T = [0, i]_1$, and thus $T = \emptyset$ if and only if $i \leq o$. When $T \neq \emptyset$, we have $T \cup E = [0, i+1]$. Moreover, $\#T, \#E, \#N < \infty$.

For regularity of the infinitesimal generator of the chains in $C$, we assume:

(A5) $\#\Omega < \infty$.

In particular, (A5) implies a general assumption in the literature, namely, that $\sum_{y \in \mathcal{Y}\setminus \{x\}} q_{x,y} < \infty$ for all $x \in \mathcal{Y}$. Moreover, (A5) also implies $\sum_{z \in T} \sum_{y \in \mathcal{Y}\setminus T} q_{y,z} < \infty$ (relevant in Section 4.4), and hence $\sup_{y \in \mathcal{Y}\setminus T} \sum_{z \in T} q_{y,z} < \infty$.

We restrict to polynomial transition rates, which are common in applications.
(A6) For \( \omega \in \Omega \), \( \lambda_\omega \) is a polynomial for large \( x \).

It follows that \( \lambda_\omega \) is increasing for large \( x \). Let \( R = \max\{\deg(\lambda_\omega) : \omega \in \Omega\} \), where \( \deg(\cdot) \) denotes the degree of a polynomial, and define the following finite parameters:

\[
\alpha = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega}{x^R}, \quad \gamma = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega - \alpha x^R}{x^{R-1}},
\]

\[
\vartheta = \frac{1}{2} \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega^2}{x^R}, \quad \beta = \gamma - \vartheta.
\]

Trivially, \( \gamma = 0 > \beta \) if \( R = 0 \).

We provide threshold criteria for dynamical properties in terms of \( R, \alpha, \beta \) and \( \gamma \) (\( \vartheta \) is used later). In comparison with the previous section, results in this section not only rely on the structure of the state space but also on the strength of the connection between states in terms of the transition rates. As we will see, the chains with infinite state space in \( C \) have consistent dynamical properties, that is, the dynamical behaviour is independent of the initial state as well as the invariant subspace. In contrast, for one-dimensional CTMCs on \( \mathbb{N}_0 \), \( d > 1 \), such consistency disappears, i.e., the dynamics confined to different invariant subspaces might differ (paper in preparation).

4.1. **Explosivity and non-explosivity**. Given \( Y_t \in C \), the sequence \( J = (J_n)_{n \in \mathbb{N}_0} \) of jump times are defined by \( J_0 = 0 \), and \( J_n = \inf\{t \geq J_{n-1} : Y_t \neq Y_{J_{n-1}}\} \), \( n \geq 1 \), where \( \inf \emptyset = \infty \). The life time is denoted by \( \zeta = \sup_n J_n \). The process \( Y_t \) is said to explode (with positive probability) at \( Y_0 = y \in \mathcal{Y}_0 \) if \( P_y(\{\zeta < \infty\}) > 0 \). In particular, \( Y_t \) explodes almost surely (a.s.) if \( P_y(\{\zeta < \infty\}) = 1, \) and does not explode at \( y \) if \( P_y(\{\zeta < \infty\}) = 0 \) [39]. Hence, \( E_y(\zeta < \infty) \) implies \( Y_t \) explodes at \( y \) a.s. The class \( C \) is non-explosive if no CTMC in \( C \) explodes, and \( C \) is explosive if it is not non-explosive i.e., there exists some CTMC in \( C \) that explodes.

Let \( E_T = \bigcup_{k \in \Sigma^*} E^{(k)} \) be the set of escaping states eventually leading to a trapping state, and \( E_P = \bigcup_{k \in \Sigma^*} E^{(k)} \) be the set of escaping states leading to a PIC, see Corollary 3.12(v). Furthermore, we have \( E_P = E \) if \( T = \emptyset \), and \( E_P = \emptyset \) if \( T \neq \emptyset \). Since \#E < \( \infty \), \( Y_t \in C \) does not explode provided \( Y_0 \in N \cup T \cup E_T \).

Next, we present necessary and sufficient conditions for explosivity and non-explosivity for infinite CTMCs in \( C \).

**Theorem 4.1.** Assume (A1)-(A6). Let \( Y_t \in C \) with \( Y_0 \notin N \cup T \cup E_T \). Then \( Y_t \) explode if and only if one of the following conditions holds: (i) \( R > 1 \) and \( \alpha > 0 \), (ii) \( R > 2, \beta > 0 \) and \( \alpha = 0 \). Moreover, \( Y_t \) explodes a.s. if \( Y_0 \in P \cup E_p \).

In particular, \( C \) is explosive if and only if (i) or (ii) hold. The assumption (A6) is crucial: if \( Y_0 \in P \cup E_p \), then explosion never happens with probability smaller than one. In contrast, this might occur with non-polynomial transition rates [36]. Reuter’s criterion and its generalizations provide necessary and sufficient conditions for explosivity for general CTMCs [33, 45], though they are not easy to check. In comparison, for CTMCs with polynomial transition rates (and \( d = 1 \)), Theorem 4.1 provides an explicit necessary and sufficient condition.

4.2. **Moments of passage times.** For a non-empty subset \( A \subseteq \mathcal{Y}_0 \), let \( \tau_A = \inf\{t \geq 0 : Y_t \in A\} \) be the hitting time of \( A \), with the convention that \( \inf \emptyset = \infty \). Hence \( \tau_A = 0 \), whenever \( Y_0 \notin A \). Let \( \tau_A^+ = \inf\{t \geq J_1 : Y_t \in A\} \) be the time of the first recurrence to \( A \). Obviously, \( \tau_A = \tau_A^+ \) if and only if \( Y_0 \notin A \).

Below we present threshold results on the existence of moments of passage times, particularly for recurrent states, since transience of a state implies non-existence of all
finite moments of passage times. Moreover, limited by the tools we apply, we do not discuss existence and non-existence of the moments of the time to extinction whenever there is a trapping state. Hence, we assume \( P \neq \emptyset \) \((i - o < \omega_a)\) by Corollary 3.12 and provide existence and non-existence of moments of passage times for states in \( P \cup E_P \).

**Theorem 4.2.** Assume (A1)-(A6), and \( i - o < \omega_a \). (i) Then there exists a finite non-empty subset \( B \subseteq \Omega \) such that

\[
\mathbb{E}_x(\tau_B^t) < +\infty, \quad \forall x \in \Omega, \forall \epsilon > 0,
\]

if one of the following conditions holds: (i-1) \( \alpha < 0 \), (i-2) \( R > 1, \alpha = 0, \beta < 0 \), (i-3) \( R > 2, \alpha = \beta = 0 \). Furthermore,

\[
\mathbb{E}_x(\tau_B^t) < +\infty, \quad \forall x \in \Omega, \forall \epsilon > 0,
\]

for some \( \delta > 0 \) if one of the following conditions holds: (i-4) \( R = 1, \alpha = 0, \gamma < 0 \), (i-5) \( R = \alpha = 0 \).

(ii) There exists a finite non-empty subset \( B \subseteq \Omega \) such that

\[
\mathbb{E}_x(\tau_B^t) = +\infty, \quad \forall x \in \Omega \setminus B, \forall \epsilon > 0,
\]

if \( R = 2, \alpha = \beta = 0 \). Furthermore,

\[
\mathbb{E}_x(\tau_B^t) = +\infty, \quad \forall x \in \Omega \setminus B, \forall \epsilon > \delta,
\]

for some \( 0 < \delta \leq 1 \) if one of the following conditions holds: (ii-1) \( R = \alpha = 0 \) with \( \delta = 1 \), (ii-2) \( R = 1, \alpha = 0, \gamma > 0 \) with \( \delta < 1 \).

There is a subtle difference between (i) and (ii). In (ii), one can always find a finite set \( B' \), containing \( B \) and all states in \( \Omega \) one jump away from \( E_P \), such that \( \mathbb{E}_x(\tau_{B'}^t) < \infty \), for all \( x \in E_P \) and \( \epsilon > 0 \). Hence, the initial state cannot be in \( E_P \).

**4.3. Recurrence and transience.** Since the states in \( Q \cup E \) are transient, and the states in \( N \cup T \) are (positive) recurrent, it suffices to discuss recurrence and transience for states in \( \Omega \). Hence as in the previous subsection, we assume \( i - o < \omega_a \). Note that \( Y_t \) is irreducible (with state space being one PIC) if \( Y_0 \in \Omega \).

**Theorem 4.3.** Assume (A1)-(A6), and \( i - o < \omega_a \). Let \( Y_t \in \mathcal{C} \) with \( Y_0 \in \Omega \). Then \( Y_t \) is recurrent if one of the following conditions holds: (i) \( \alpha < 0 \), (ii) \( \alpha = 0 \) and \( \beta \leq 0 \), while transient otherwise.

**4.4. Positive recurrence and null recurrence.** We provide threshold criteria for positive and null recurrence as well as exponential ergodicity of SDs and QSDs.

Let us first recall a standard setup for QSDs of a CTMC. Assume \( \Sigma^+ \neq \emptyset \). Let \( \partial = T^{(k)} \cup E^{(k)} \) be the absorbing set, and \( \mathcal{E} = Q^{(k)} \), for some \( k \in \Sigma^+ \). For a probability measure \( \nu \) on \( \partial \cup \mathcal{E} \), define

\[
\mathbb{P}_\nu(\cdot) = \int_{\partial \cup \mathcal{E}} \mathbb{P}_x(\cdot) d\nu(x).
\]

Note that \( Y_t \in \partial \) implies \( Y_t \in \partial \) for all \( t \geq s \), since \( \partial \) is closed. If \( \tau_\partial < \infty \) \( \mathbb{P}_x \)-a.s. for all \( x \in \mathcal{E} \) (certain absorption), then the process (associated with \( Y_t \)) conditioned to never be absorbed is called a \( Q \)-process [12].

A probability measure \( \nu \) on \( \mathcal{E} \) is a QSD for \( Y_t \) if for all \( t \geq 0 \) and all sets \( B \subseteq \mathcal{E} \),

\[
\mathbb{P}_\nu(Y_t \in B | \tau_\partial > t) = \nu(B).
\]
The existence of a QSD implies certain absorption [49]. A probability measure $\eta$ on $\partial^c$ is a quasi-ergodic distribution if, for any $x \in \partial^c$ and any bounded function $f$ on $\partial^c$ [9, 25]:

$$\lim_{t \to \infty} \mathbb{E}_x \left( \frac{1}{t} \int_0^t f(Y_s) ds \mid \tau_\partial > t \right) = \int_{\partial^c} f d\eta.$$

**Theorem 4.4.** Assume (A1)-(A6).

(i) Assume $i - o < \omega_k$, and $Y_t \in \mathcal{C}$ with $Y_0 \in \mathcal{P}(k)$ for some $k \in \Sigma^-$. Then $Y_t$ is positive recurrent and there exists a unique SD $\pi^{(k)}$ on $\mathcal{P}(k)$, if one of the following conditions holds: (i-1) $\alpha < 0$, (i-2) $R > 1$, $\alpha = 0$, and $\beta < 0$, (i-3) $R > 2$ and $\alpha = \beta = 0$, (i-4) $R = 1$, $\alpha = 0$, and $\gamma < 0$.

Moreover, if (i-1)' $R \geq 1$ and $\alpha < 0$, or (i-2)' $R > 2$, $\alpha = 0$, and $\beta \leq 0$, then $\pi^{(k)}$ is exponentially ergodic in the sense that there exists $0 < \delta < 1$ such that for all probability measures $\mu$ on $\mathcal{P}(k)$, there exists a constant $C_{\mu} > 0$ such that

$$\|\mathbb{P}_\mu(Y_t \in \cdot) - \pi^{(k)}(\cdot)\|_{TV} \leq C_{\mu}\delta^t.$$

Furthermore, $Y_t$ is null recurrent if one of the following conditions holds: (i-5) $R = 2$ and $\alpha = \beta = 0$, (i-6) $R = 1$, $\alpha = 0$, $\beta \leq 0$, and $\gamma > 0$.

(ii) Assume $i - o > 0$, and $Y_t \in \mathcal{C}$ with $Y_0 \in \mathcal{Q}(k)$ for some $k \in \Sigma^+$. Then there exists a unique uniform exponential (in total variation norm) ergodic QSD $\nu^{(k)}$ supported on $\mathcal{Q}(k)$ if one of the following conditions holds: (ii-1) $R > 1$ and $\alpha < 0$, (ii-2) $R > 2$ and $\alpha = 0$, and $\beta \leq 0$. More precisely, there exist constants $\delta \in (0, 1)$ and $C > 0$ such that for all probability measures $\mu$ on $\mathcal{Q}(k)$,

$$\|\mathbb{P}_\mu(Y_t \in \cdot | t < \tau_\partial) - \nu^{(k)}(\cdot)\|_{TV} \leq C\delta^t.$$

Moreover, there exists a unique quasi-ergodic distribution for the process $Y_t$, which is the unique SD of the Q-process. There exists no QSD if one of the following conditions holds: (ii-3) $R > 1$ and $\alpha > 0$, (ii-4) $R > 2$, $\alpha = 0$ and $\beta > 0$.

We make a few remarks.

- The convergence (or ergodicity) in (ii) is uniform w.r.t. the initial distribution, while in contrast, the convergence is not uniform in (i). Indeed, the subcritical linear BDP is exponentially ergodic but not uniformly so [6].
- Indeed, one can obtain uniform exponential ergodicity in Theorem 4.4(i) with one of the conditions (ii-1)-(ii-3) by choosing the absorption set such that it is not reachable, hence imposing that the time to extinction is infinite. In such cases, the QSD degrades to an SD [13].
- The subtle difference between the conditions for positive recurrence and that for ergodicity of QSDs lies in the fact that we have no a priori estimate of the decay parameter

$$\psi_0 = \inf \{ \psi > 0: \lim_{t \to \infty} e^{\psi t} \mathbb{P}_x(X_t = x) > 0 \},$$

which is independent of $x$ [13]. We cannot compare $\psi_0$ with $-\alpha$ when $R > 1$ or with $-\beta$ when $R > 2$ and $\alpha = 0$. Refer to the constructive proofs (using Lyapunov functions) in Appendix B for details.
- If $\partial$ is chosen to be the trapping states only, and the set of escaping states leading to $\partial$ is non-empty, then a QSD still exists but it is concentrated on the escaping states. See Example 3.13 for a situation where the QSD is concentrated on the unique escaping state 2 with $\partial = \{0, 1\}$. 


The only cases not covered in Theorem 4.4(i) are \( R = 1, \alpha = \gamma = 0 \) and \( R = \alpha = 0 \).

When \( i - o < \omega_* \), it seems that \( Y_t \in \mathcal{C} \) with \( Y_0 \in \mathcal{P} \) is null recurrent if either of these two conditions holds, as illustrated in the example below.

**Example 4.5.** Consider \( \omega_* = 1, \Omega = \{\omega_*, -\omega_*\} \), \( \lambda_{\omega_*}(x) = (x - 4)1_{\mathbb{N}_{\geq 5}}(x) \) and \( \lambda_{-\omega_*}(x) = (x - \theta)1_{\mathbb{N}_{\geq 6}}(x) \), for some \( 0 \leq \theta < 6 \). Here \( R = 1, \alpha = 0 \), and \( \gamma = \theta - 4 \).

Moreover, by Theorem 4.1, this class of BDPs is non-explosive, and there exists a stationary measure on the unique PIC \( \mathbb{N}_{\geq 5} \),

\[
\pi(x) = \pi(5) \prod_{j=5}^{x} \frac{j - 4}{j - 4 - \gamma + 1}, \quad \forall x \in \mathbb{N}_{\geq 5}.
\]

In particular, \( \pi \) is an SD if \( \gamma < 0 \), while an infinite stationary measure if \( \gamma = 0 \). Hence the process (starting from the unique PIC) is null recurrent if \( \gamma = 0 \) and positive recurrent if \( \gamma < 0 \). Indeed, one can even show for all BDPs on a subset of \( \mathbb{N}_0 \) with polynomial transition rates, the process is null recurrent if \( R = 1 \) and \( \alpha = \gamma = 0 \).

**Example 4.6** (Symmetric simple random walk with left reflecting barrier). Consider \( \omega_* = 1, \Omega = \{\omega_*, -\omega_*\} \), \( \lambda_{\omega_*}(x) = \frac{1}{BD}(x) \) and \( \lambda_{-\omega_*}(x) = \frac{1}{BD}(x) \). Then \( R = \alpha = 0 \). Moreover, by Theorem 4.1, this class of BDPs is non-explosive, and all uniform measures on \( \mathbb{N}_0 \) are stationary measures. Hence, the process is null recurrent.

Analogously, the only cases not covered in Theorem 4.4(ii) are \( R \leq 1 \), and \( R > 1, \alpha > 0 \), where there might exist no or not a unique QSD. For instance, consider the BDP on \( \mathbb{N}_0 \) with birth rate \( \lambda_j = j \) and death rate \( \mu_j = 2j \) for \( j \in \mathbb{N} \). Here \( R = 1 \) and \( \alpha < 0 \). Nevertheless, this CTMC admits a continuum family of QSDs.

**Table 1.** Summary of parameter regions for dynamics. Respective properties hold in connected regions. Positive recurrence (red), null recurrence (blue), recurrence of unknown type (gray), transience (green), and empty set (black). ES=exponential ergodicity of SD, UQ=uniform exponential ergodicity of QSD, NQ=non-existence of QSD, NS=no ergodic SD.

|        | \( \alpha < 0 \) | \( \alpha = 0 \) | \( \alpha > 0 \) |
|--------|------------------|------------------|------------------|
| \( R = 0 \) | \( \gamma < 0 \) | \( \gamma = 0 \) | \( \beta < 0 < \gamma \) |
| \( R = 1 \) | ES | NS | NS/UQ |
| \( R = 2 \) | ES | NS | NS/NQ |
| \( R > 2 \) | ES/UQ | NS | NS/NQ |

4.5. **Implosivity.** Let \( Y_t \in \mathcal{C} \) with \( Y_0 \in \mathcal{P} \cup \mathcal{E}_p \). Then \( \mathcal{P}^{(k)} \subseteq \mathcal{Y}_0 \subseteq (\mathcal{P} \cup \mathcal{E}_p) \cap \Gamma^{(k)} \) for some \( k \in \Sigma^- \). Let \( B \subseteq \mathcal{Y}_0 \) be a non-empty proper subset. Then \( Y_t \) implodes towards \( B \) [36] if there exists \( t_* > 0 \) such that

\[
\mathbb{E}_y(\tau_B) \leq t_*, \quad \forall y \in B^c = \mathcal{Y}_0 \setminus B.
\]

Implosion towards a single state \( x \) implies finite expected first return time, and thus positive recurrence of \( x \). Indeed, assume w.o.l.g. that \( x \) is not an absorbing state. Then

\[
\mathbb{E}_x(\tau_x^+) \leq \mathbb{E}_x J_1 + \sup_{y \neq x} \mathbb{E}_y(\tau_x) < \infty,
\]
where \( \tau^+_x = \tau^+_{\{x\}} \), \( \tau_x = \tau_{\{x\}} \), and \( J_1 \) has finite expectation since \( x \) is not absorbing. Hence \( Y_t \) does not implode towards any transient state.

The process \( Y_t \) is \textit{implosive} if \( Y_t \) implodes towards any state of \( \mathcal{Y} \cap \mathcal{P} \neq \emptyset \), otherwise, \( Y_t \) is \textit{non-implosive}. By Theorem 4.3, impllosivity implies \( \mathcal{P} \) is recurrent. If \( Y_t \) implodes towards a finite non-empty subset of \( \mathcal{Y} \), then \( Y_t \) is implosive (see Proposition S5.8).

**Theorem 4.7.** Assume (\( A_1 \))-(\( A_6 \)), \( i-o < \omega_* \), and \( Y_t \in \mathcal{C} \) with \( Y_0 \in \mathcal{P} \cup \mathcal{E}_\mathcal{P} \). Then \( Y_t \) is implosive, and there exists \( \epsilon > 0 \) such that for every non-empty finite subset \( B \subseteq Y_0 \) and every \( x \in B^c \),

\[
E_x(\exp(\tau^B_\epsilon)) < \infty,
\]

if one of the following conditions holds: (i) \( R > 1 \) and \( \alpha < 0 \), (ii) \( R > 2 \), \( \alpha = 0 \), and \( \beta \leq 0 \), while \( Y_t \) is non-implosive otherwise.

Implosivity is indeed stronger than positive recurrence, as shown in the following example.

**Example 4.8.** Let \( \mathcal{C} \) be associated with \( \Omega = \{1,-1\} \) and

\[
\lambda_{-1}(x) = x, \quad \lambda_1(x) = 1, \quad x \in \mathbb{N}_0.
\]

In this case, \( R = 1 \) and \( \alpha = -1 \). By Theorems 4.4 and 4.7, all CTMCs in \( \mathcal{C} \) are positive recurrent while non-implosive.

### 5. Stationary distributions and quasi-stationary distributions

An explicit SD for a CTMC is known only in few cases, e.g., for BDPs [6], complex-balanced reaction networks [4, 10], autocatalytic reaction networks [26], and open networks of reversible queues [28]. However, any probability distribution on \( \mathbb{N}_0 \) can be approximated by a sequence of ergodic SDs of one-dimensional BDPs on \( \mathbb{N}_0 \) with polynomial transition rates (paper in preparation). Hence, CTMCs can be cooked up with diverse multimodal SDs.

In contrast, the tail distributions of SDs/QSDs of the CTMCs in our setting are of a few simple forms, see Theorem 5.6 and 5.8 below. We prove this by establishing identities for SDs/QSDs.

#### 5.1. Stationary distributions

Let \( R_\pm = \max_{\omega \in \Omega_{\pm}} |\omega| \), \( \omega_\pm = \max_{\omega \in \Omega_{\pm}} |\omega| \), and define

\[
\alpha_\pm = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega_{\pm}} \lambda_{\omega}(x)|\omega|}{xR_\pm}, \quad \alpha^*_\pm = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega_{\pm}} \lambda_{\omega}(x)}{xR_{\pm}},
\]

\[
\gamma_\pm = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega_{\pm}} \lambda_{\omega}(x)|\omega| - \alpha_\pm xR_\pm}{xR_{\pm-1}}.
\]

In particular, \( \alpha = \alpha_+ - \alpha_- \), provided \( R_- = R_+ \).

We first preview some possible types of tail distributions.

**Example 5.1** (Conley Maxwell Poisson (CMP) distribution). Let \( \mathcal{C} \) be associated with \( \Omega = \{1,-1\} \) and

\[
\lambda_{-1}(x) = \sum_{j=1}^h S(b,j)x^j, \quad \lambda_1(x) = a, \quad x \in \mathbb{N}_0,
\]
where \( a \in \mathbb{R}_{>0}, b \in \mathbb{N} \), and \( S(i,j) \) is the Stirling numbers of the second kind [1]. The unique SD is the CMP distribution with parameter \( (a, b) \) [27]:

\[
\pi(x) = \frac{a^x}{(x!)^b} \left( \sum_{j=0}^{\infty} \frac{a^j}{(j!)^b} \right)^{-1}, \quad x \in \mathbb{N}_0.
\]

In particular, for \( b = 1 \), \( \pi \) is a Poisson distribution.

**Example 5.2** (Geometric distribution). Let \( \mathcal{C} \) be associated with \( \Omega = \{1, -1\} \) and

\[
\lambda_1(x) = x + a^{-1}, \quad \lambda_{-1}(x) = x, \quad x \in \mathbb{N}_0,
\]

where \( 0 < a < 1 \). The unique SD is geometric with parameter \( a \):

\[
\pi(x) = a^x(1-a), \quad x \in \mathbb{N}_0.
\]

**Example 5.3** (Zeta distribution). Let \( \mathcal{C} \) be associated with \( \Omega = \{1, -1\} \) and

\[
\lambda_1(x) = \sum_{j=1}^{b+1} S(b+1,j)(x+1)^j, \quad \lambda_{-1}(x) = \sum_{j=2}^{b+1} (S(b+1,i) - S(b,i))(x+1)^j, \quad x \in \mathbb{N}_0,
\]

where \( b \in \mathbb{N} \setminus \{1\} \). The unique SD is of Zeta type:

\[
\pi(x) = (x+1)^{-b} \left( \sum_{j=1}^{\infty} j^{-b} \right)^{-1}, \quad x \in \mathbb{N}_0.
\]

From the above examples, the SD can be heavy- and light-tailed (for their definitions, c.f., [18]). Moreover, there exists a unique SD, since \( \omega_* = 1 \). This is however not guaranteed in general (Corollary 3.12). For BDPs the above picture is complete, as demonstrated in the next statement.

For any two real-valued functions \( f, g \) on \( \mathbb{R} \), \( f(x) \sim g(x) \) if there exists \( C > 1 \) such that \( C^{-1} \leq \frac{f(x)}{g(x)} \leq C \) for all \( x \in \mathbb{R} \).

**Proposition 5.4.** Assume (A1)-(A6) with \( \Omega = \{\omega_*, -\omega_*\} \). Let \( \pi \) be an SD with infinite support. Then for large \( x \in \text{supp} \pi \),

\[
T_\pi(x) \sim \begin{cases} 
\Gamma(\omega_*^{-1}x)^{(R_+ - R_-)} \cdot \left( \frac{\omega_*^{(R_+ - R_-)} x^{\omega_*^{-1}x}}{\alpha^\omega_*^{(R_+ - R_-)}} \right)^{\omega_*^{-1}x} x \left( \frac{\omega_*}{\alpha} \right)^{\omega_*^{-1}x}, \quad & \text{if } \alpha < 0, \\
x^{2 + \beta \gamma^{-1} - R}, \quad & \text{if } \alpha = 0,
\end{cases}
\]

where \( \Gamma(\cdot) \) denotes the Gamma function.

When the chain is not a BDP, the asymptotics can be established in some cases.

**Proposition 5.5.** Assume (A1)-(A6), \( \alpha = 0 \), \( \beta + 2\gamma < 0 \), and \( \lambda - \delta < \omega_* \). Let \( \pi \) be an SD with infinite support. Then for large \( x \in \text{supp} \pi \),

\[
T_\pi(x) \sim x^{2 + \beta \gamma^{-1} - R}.
\]

The proofs of Proposition 5.4 and 5.5 are provided in the supplement. In the following, we provide asymptotics of the tail distributions for chains in \( \mathcal{C} \). To obtain the asymptotics, we first come up with an identity for stationary measures, which was first stated in an alternative form in [24]. Let \( \omega_+ = \max \Omega_+ \) and \( \omega_- = -\min \Omega_- \).

**Theorem 5.6.** Assume (A1)-(A5). Then the following statements are equivalent.

(i) \( \pi \) is a stationary measure for \( \mathcal{C} \).
(ii) For \( x \in \mathbb{N}_0 \),
\[
\sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega/\omega_+} \lambda_\omega(x + (j-1)\omega_+) \pi(x + (j-1)\omega_+) = \sum_{\omega \in \Omega_+} \sum_{k=1}^{\omega/\omega_+} \lambda_\omega(x - k\omega_+) \pi(x - k\omega_+),
\]
where \( \pi(-\omega_+) = \cdots = \pi(-1) = 0 \).

(iii) For \( x \in \mathbb{N}_0 \),
\[
\sum_{j=1}^{\omega_-/\omega_+} \pi(x + (j-1)\omega_+) \sum_{-j\omega_+ \geq \omega \in \Omega_-} \lambda_\omega(x + j\omega_+) - \sum_{-j\omega_+ \geq \omega \in \Omega_-} \lambda_\omega(x + (j-1)\omega_+))
\]
\[
= \sum_{k=1}^{\omega_-/\omega_+} \pi(x - k\omega_+) \left( \sum_{k\omega_+ \leq \omega \in \Omega_+} \lambda_\omega(x - k\omega_+) - \sum_{(k+1)\omega_+ \leq \omega \in \Omega_+} \lambda_\omega(x - (k+1)\omega_+) \right).
\]

The standard result about the form of the SD for a BDP [6] can be derived from Theorem 5.6. Regarding the tail distributions, we have the following.

**Corollary 5.7.** Assume (A1)-(A5) and let \( \pi \) be an SD. Then, for \( x \in \mathbb{N}_0 \),
\[
T_\pi(x) \left( \sum_{\omega \in \Omega_-} \lambda_\omega(x) + \sum_{\omega \in \Omega_+} \lambda_\omega(x - \omega_+) \right) = \sum_{j=1}^{\omega_-/\omega_+} \pi(x + j\omega_+)
\]
\[
\cdot \left( \sum_{-(j+1)\omega_+ \geq \omega \in \Omega_-} \lambda_\omega(x + j\omega_+) - \sum_{-j\omega_+ \geq \omega \in \Omega_-} \lambda_\omega(x + (j-1)\omega_+) \right)
\]
\[
= \sum_{k=1}^{\omega_-/\omega_+} \pi(x - k\omega_+) \left( \sum_{k\omega_+ \leq \omega \in \Omega_+} \lambda_\omega(x - k\omega_+) - \sum_{(k+1)\omega_+ \leq \omega \in \Omega_+} \lambda_\omega(x - (k+1)\omega_+) \right).
\]

It is noteworthy that the identities hold for general CTMCs, not restricted to those with polynomial transition rates. However, the estimates of the tail asymptotics do rely on the latter.

**Theorem 5.8.** Assume (A1)-(A6). Let \( \pi \) be an SD on \( \mathbb{N}_0 \) with infinite support.

(i) (CMP tail) Assume \( R_- > R_+ \). Then there exist \( \delta_+, \delta_- \) in \( \mathbb{R} \) such that for \( \delta \in (\delta_+, +\infty) \), \( \tilde{\delta} \in (-\infty, \delta_-) \),
\[
\lim_{x \to \infty} T_\pi(x) \Gamma(\omega_-^{-1}x)^{R_- - R_+} \left( \frac{\alpha_-^{\omega_-}}{\alpha_+^{\omega_+}} \right)^{R_- - R_+} x^{\omega_-^{-1}x} \mid_{x^{\tilde{\delta}}} = +\infty,
\]
\[
\lim_{x \to \infty} T_\pi(x) \Gamma(\omega_+^{-1}x)^{R_- - R_+} \left( \frac{\alpha_+^{\omega_+}}{\alpha_-^{\omega_-}} \right)^{R_- - R_+} x^{\omega_+^{-1}x} \mid_{x^{\delta}} = 0.
\]

In particular,
\[
\omega_-^{-1}(R_+ - R_-) \leq \lim \inf_{x \to \infty} \frac{\log T_\pi(x)}{x \log x}, \quad \lim \sup_{x \to \infty} \frac{\log T_\pi(x)}{x \log x} \leq \omega_+^{-1}(R_+ - R_-).
\]

(ii) (Geometric tail) Assume \( R_- = R_+ = R \) and \( \alpha < 0 \). Then there exist \( \delta_+, \delta_- \) in \( \mathbb{R} \) such that for \( \delta \in (\delta_+, +\infty) \), \( \tilde{\delta} \in (-\infty, \delta_-) \),
\[
\lim_{x \to \infty} T_\pi(x) \left( \min \left\{ \frac{\alpha_-^{\omega_-} + \alpha_+^{\omega_+}}{\alpha_+}, \frac{\alpha_-^{\omega_-}}{\alpha_+} \right\} \right)^{\omega_-^{-1}x} \mid_{x^{\tilde{\delta}}} = +\infty,
\]
\[
\lim_{x \to \infty} T_\pi(x) \left( \frac{\alpha_-^{\omega_-}}{\alpha_+} \right)^{(\omega_+ + \omega_- - \omega_+)^{-1}x} \mid_{x^{\delta}} = 0.
\]
In particular,
\[
\frac{1}{\omega_*} \log \left( \max \left\{ \frac{\alpha^*_+}{\alpha^*_+ + \alpha^*_+}, \frac{\alpha^*_+}{\alpha^*_+} \right\} \right) \leq \liminf_{x \to \infty} \frac{\log T_\pi(x)}{x},
\]
\[
\limsup_{x \to \infty} \frac{\log T_\pi(x)}{x} \leq \frac{1}{\omega_+ + \omega_- - \omega_*} \log \frac{\alpha^*_+}{\alpha^*_+}
\]

(iii) (Zeta tail) Assume \( \alpha = 0 \) and that one of the conditions (i-2)-(i-4) in Theorem 4.4 holds. Then
\[
\liminf_{x \to \infty} T_\pi(x)(x\omega_*^{-1})^{(R\theta - \gamma)}\alpha^*_+^{-1}\omega_*^{-1} > 0,
\]
and
\[
\limsup_{x \to \infty} T_\pi(x)(x\omega_*^{-1})^{(R\theta - \gamma)}\alpha^*_+^{-1}(\omega_+ + \omega_- - \omega_*)^{-1} < \infty.
\]
Moreover, \( R\theta - \gamma > \alpha_+\omega_* \),
\[
1 - (R\theta - \gamma) \alpha^*_+^{-1}\omega_*^{-1} \leq \liminf_{x \to \infty} \frac{\log T_\pi(x)}{\log x},
\]
and
\[
\limsup_{x \to \infty} \frac{\log T_\pi(x)}{\log x} \leq 1 - (R\theta - \gamma) \alpha^*_+^{-1}(\omega_+ + \omega_- - \omega_*)^{-1}.
\]

From this result, generically, no SD can decay faster than a CMP distribution nor slower than a Zeta distribution.

In the light of Proposition 5.4, the tail asymptotics of an SD for a BDP is sharp up to the leading order. Moreover, if \( R_- > R_+ \), then the asymptotics is sharp up to the leading order for upwardly skip-free processes [6] (i.e., when \( \omega_+ = \omega_* \)).

**Example 5.9.** Consider \( \mathcal{C} \) associated with \( \Omega = \{1, -2\} \), and
\[
\lambda_1(x) = x + x^2 = x^2, \quad \lambda_{-2}(x) = x^2 = x(x - 1)(x - 2).
\]
By Theorem 5.8, the unique SD decays as fast as the Poisson distribution:
\[
\lim_{x \to \infty} \frac{\log T_\pi(x)}{x \log x} = 1,
\]
since \( \omega_+ = \omega_* = 1, R_+ = 2, R_- = 3 \). However, it is not Poisson (see also Example 6.2).

Assumption (A5) is crucial for Theorem 5.8. Consider a model of protein production [48], given by \( \Omega = \{-1\} \cup \mathbb{N} \) and
\[
\lambda_{-1}(x) = \mathbb{1}_\mathbb{N}(x), \quad \lambda_j(x) = ab_{j-1}, \quad \forall j \in \mathbb{N}, \quad \forall x \in \mathbb{N}_0,
\]
where \( a > 0 \), and \( (b_j)_{j \in \mathbb{N}_0} \) refers to the burst size distribution of proteins. When \( b_j = (1 - \delta)\delta^j \) for \( 0 < \delta < 1 \), the SD is the negative binomial distribution [48]:
\[
\pi(x) = \frac{\Gamma(x + a)}{\Gamma(x + 1)b_0(a)}\delta^x(1 - \delta)^a, \quad x \in \mathbb{N}_0.
\]
In particular, when \( a = 1 \), \( \pi \) is also geometric. While Theorem 5.8, if it did apply, would seem to suggest the tail of \( \pi \) is CMP, since \( 1 = R_- > R_+ = 0 \). Hence it may not be possible to directly extend the result in Theorem 5.8 to a CTMC with infinitely many possible jumps.
5.2. Quasi-stationary distributions. In this part, we do a parallel study for QSDs, namely, we establish identities for QSDs, and derive asymptotic estimates for tail distributions thereof.

Recall from subsection 4.4 that \( \partial = T^{(k)} \cup E^{(k)} \) is the absorbing set, and \( \partial^c = Q^{(k)} \), for some \( k \in \Sigma_+ \). If \( \nu \) is a QSD on \( \partial^c \) for some \( Y_t \in \mathcal{C} \) killed at \( \partial \) then \( \nu \) is said to be a QSD for \( \mathcal{C} \). Let

\[
\theta_{\nu} = \sum_{\omega \in \Omega_-} \sum_{y \in \partial \cap (\partial - \omega)} \nu(y) \lambda_{\omega}(y).
\]

The following necessary condition (called the residual equations) is classical.

**Proposition 5.10.** ([14, 35, 41]) Assume (A1)-(A5), and \( o < i \). Let \( \nu \) be a QSD for \( \mathcal{C} \). Then for all \( x \in \partial^c \)

\[
\sum_{\omega \in \Omega_-} (\lambda_{\omega}(x - \omega)\nu(x - \omega) - \lambda_{\omega}(x)\nu(x)) + \theta_{\nu}\nu(x) = \sum_{\omega \in \Omega_+} (\lambda_{\omega}(x)\nu(x) - \lambda_{\omega}(x - \omega)\nu(x - \omega)).
\]

Based on the above proposition, an identity for QSDs follows.

**Theorem 5.11.** Assume (A1)-(A5), and \( o < i \). Then the following three statements are equivalent.

(i) \( \nu \) is a QSD for \( \mathcal{C} \) supported on \( \partial^c \).

(ii) For all \( x \in \partial^c \),

\[
\sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega/\omega_+} \lambda_{\omega}(x+(j-1)\omega_x)\nu(x+(j-1)\omega_x) = \sum_{\omega \in \Omega_+} \sum_{k=1}^{\omega/\omega_+} \lambda_{\omega}(x-k\omega_x)\nu(x-k\omega_x) + \theta_{\nu}T_{\nu}(x).
\]

(iii) For all \( x \in \partial^c \),

\[
\sum_{j=1}^{\omega_-/\omega_*} \nu(x+(j-1)\omega_x) \sum_{-j+1 \omega_x \geq \omega \in \Omega_-} \lambda_{\omega}(x+(j-1)\omega_x) = \sum_{k=1}^{\omega_+/\omega_*} \nu(x-k\omega_x) \sum_{k\omega_x \leq \omega \in \Omega_+} \lambda_{\omega}(x-k\omega_x) + \theta_{\nu}T_{\nu}(x).
\]

Another result for BDPs follows from Theorem 5.11.

**Corollary 5.12** ([11, 43]). Consider a BDP \( Y_t \) on \( \mathbb{N}_0 \) with absorbing state \( 0 \) and birth and death rates (at state \( j \)) being \( d_j \) and \( b_j \). Then \( (\nu(j))_{j \geq 1} \) is a QSD for \( Y_t \) if and only if

(i) \( \nu(j) \geq 0, \forall j \geq 1 \) and \( \sum_{j \geq 1} \nu(j) = 1 \).

(ii) \( d_j \nu(j) = b_{j-1} \nu(j-1) + d_1 \nu(1) \left( 1 - \sum_{i=1}^{j-1} \nu(i) \right), \forall j \geq 2 \).
Corollary 5.13. Assume \((A1)-(A6), o < i\). Let \(\nu\) be a QSD. Then for all \(x \in \mathbb{N}_0\),

\[
T_\nu(x) \left( \sum_{\omega \in \Omega_-} \lambda_\omega(x) \right) + \sum_{\omega \in \Omega_+} \lambda_\omega(x - \omega) - \theta_\nu + \sum_{j=1}^{\omega_-/\omega_*} T_\nu(x + j\omega_*).
\]

Before presenting the asymptotics of tail distributions of QSDs, we review a few BDPs with an explicit QSD.

Example 5.14. Consider the linear BDP on \(\mathbb{N}_0\) with \(b_j = bj\) for \(j \in \mathbb{N}\) and \(d_j = (d_1/2 + b)(j + 1)\) for \(j \in \mathbb{N}_{\geq 2}\); where \(b > 0\) and \(d_1 > 0\) [43]. Then a QSD \(\nu\) is given by

\[
\nu(x) = \frac{1}{x(x + 1)}, \quad x \in \mathbb{N},
\]
a Zeta distribution.

Example 5.15. Consider the linear BDP on \(\mathbb{N}_0\) with \(b_j = bj\) and \(d_j = d_1j\) with \(0 < b < d_1\) for \(j \in \mathbb{N}\) [49]. The extremal QSD \(\nu\) in the continuum family of QSDs is given by

\[
\nu(x) = \left( \frac{b}{d_1} \right)^{-1} \left( 1 - \frac{b}{d_1} \right), \quad x \in \mathbb{N},
\]
a geometric distribution.

An artificial example also illustrates that a QSD can be a convex combination of Poisson distributions of different parameters [43].

The above examples may drive us to believe that tails of QSDs can also be categorized into the three types (CMP, geometric, and Zeta) as in Theorem 5.8. Indeed, employing similar arguments as in the proof of Theorem 5.8, the asymptotics of QSDs are provided below. We mention that there seems no literature on the estimates or the tail asymptotics of QSDs, possibly due to the nonlinearity in the residual equations as well as the scarce examples with explicit QSDs [43].

Theorem 5.16. Assume \((A1)-(A6), o < i, R > 1, and let \(\nu\) be a QSD. Then the results of Theorem 5.8 hold with \(\pi\) replaced by \(\nu\) except for (5.3) and (5.4) in (iii). In (iii), the remaining results hold regardless of the assumptions in (iii).

The proof of this result is in the same spirit as that of Theorem 5.8 and thus is omitted. We conclude the section with a few remarks.

(1) When \(\alpha = 0\), although it is not proved in Theorem 5.16, it is still believed that a QSD cannot decay slower than a Zeta distribution.

(2) There is a vast literature on the tail asymptotics of SDs for DTMCs on the integers, cf. [7, 15, 16, 20, 29, 37, 38]. From the relationship between an SD of a CTMC and that of its embedded DTMC [40, 42], one can readily estimate the tail of the SD of the DTMC based on Theorem 5.8, when both SDs exist.
6. Applications to reaction networks

A stochastic reaction network (SRN) with mass-action kinetics is a CTMC given by a directed graph and propensities as in (1.1) [5]. SRNs are used to describe interactions of constituent chemical species, though the area of application extends beyond (bio)chemistry [17, 19, 44]. In this section, we apply the results developed in Sections 3-5 to some SRN examples.

Example 6.1. Recall the two reaction networks (1.1) in the Introduction. For the first reaction network, \( R = 4, \alpha = 0 \) and \( \beta = 1 \); for the second, \( R = 3, \alpha = 0 \) and \( \beta = 0 \). By Theorem 4.1, the first is explosive and the second is non-explosive.

Example 6.2. (i) Consider a strongly connected reaction network:

\[
\begin{align*}
\kappa_3 & \quad S \xrightarrow{\kappa_1} 2S \xrightarrow{\kappa_2} 3S
\end{align*}
\]

The associated class of CTMCs are given in Example 5.9. In this case, \( R = R_\rightarrow = 3 > R_\leftarrow = 2 \), and \( \alpha = -2\kappa_3 \). By Theorem 4.4, there exists a unique exponentially ergodic SD. Moreover, this SD is Poisson if and only if \( \kappa_1\kappa_3 = \kappa_2^2 \) (corresponding to the network being complex balanced [10]). Nevertheless, by Theorem 5.8, the SD has Poisson tail regardless of the (positive) rate constants \( \kappa_i, i = 1, 2, 3 \).

(ii) Consider a similar reaction network including direct degradation of S:

\[
\begin{align*}
\varnothing \xleftarrow{\kappa_4} & \quad S \xrightarrow{\kappa_1} 2S \xrightarrow{\kappa_2} 3S
\end{align*}
\]

The threshold parameters are the same as in (i). By Theorems 4.4 and 5.16, the network has a uniform exponentially ergodic QSD with a Poisson tail.

Beyond these examples, one can show that all upwardly skip-free weakly reversible (finite union of strongly connected) one-dimensional reaction networks have a unique exponentially ergodic SD with Poisson tails on every PIC (paper in preparation).

7. From polynomials to power-laws

The main results of Sections 4 and 5 hold more generally under a power-law assumption.

\((A6')\) For \( \omega \in \Omega \), \( \lambda_\omega \) is of power-law for large \( x \).

Hence for every \( \omega \in \Omega \), let \( \lambda_\omega(x) = \sum_{k=1}^{m_\omega} a_\omega^{(k)} x^{e_\omega^{(k)}} \) for large \( x \) with \( e_\omega^{(1)} > \cdots > e_\omega^{(m_\omega)}, a_\omega^{(k)} \neq 0 \) for all \( k = 1, \ldots, m_\omega \), and some \( m_\omega \in \mathbb{N} \). Let

\[ R = \max \left\{ e_\omega^{(k)} : k = 1, \ldots, m_\omega, \ \omega \in \Omega \right\}, \]

and \( \left\{ R_\omega^{(k)} > R - 1 : k = 1, \ldots, m_\omega, \ \omega \in \Omega \right\} = \{ R_j \}_{j=1}^M \) for some \( M \in \mathbb{N} \) with \( R = R_1 > R_2 > \cdots > R_M > R - 1 \).

Analogous to Section 4, one can define the critical parameters,

\[
\alpha^{(j)} = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x)^\omega - \sum_{1 \leq i < j} \alpha^{(i)} x^{R_i}}{x^{R_j}}, \quad j = 1, \ldots, M,
\]
\[
\gamma = \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega - \sum_{j=1}^{M} \alpha^{(j)} x R_j}{x^{R-1}}, \quad \vartheta = \frac{1}{2} \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega^2}{x^R}, \quad \beta = \gamma - \vartheta.
\]

For example, the conditions (i) and (ii) of Theorem 4.3 regarding transience and recurrence are translated to “if the first non-zero \(\alpha_j\) is negative . . . ”, and “if all \(\alpha_j = 0\) and \(\beta \leq 0\) then . . . ”, respectively.

**Appendices**

**A. Proofs of results in Section 3**

We say \(\omega \in \Omega\) is active on a state \(x \in \mathbb{N}_0^d\) if \(\lambda_\omega(x) > 0\). A state \(y \in \mathbb{N}_0^d\) is one-step reachable from \(x \in \mathbb{N}_0^d\), denoted \(x \rightarrow_\omega y\), if \(\omega\) is active on \(x\) for some \(\omega = y - x \in \Omega\).

An ordered set of states \(\{x^{(j)}\}_{j=1}^m\) for \(m > 1\) is a path from \(x^{(1)}\) to \(x^{(m)}\) if

\[
x^{(1)} \rightarrow_{\omega(1)} \cdots \rightarrow_{\omega(m-1)} x^{(m)}.
\]

In particular, if \(x^{(1)} = x^{(m)}\), then (A.1) is called a cycle connecting the states \(x^{(i)}\), \(i = 1, \ldots, m\). A state \(x\) is reachable from itself if and only if there exists a cycle through \(x\).

**Proof of Theorem 3.2.** (i) None of the jumps in \(\Omega\) are active on a state in \(\mathbb{N}_0^d \setminus I_\geq\), i.e., \(\mathbb{N}_0^d \setminus I_\geq \subseteq \mathbb{T} \cup \mathbb{N}\). On the other hand, if \(x \in \mathbb{T} \cup \mathbb{N}\), then \(\lambda_\omega(x) = 0\) for \(\omega \in \Omega\), hence \(\mathbb{T} \cup \mathbb{N} \subseteq \mathbb{N}_0^d \setminus I_\geq\), and equality holds. It now suffices to show that \(N = \mathbb{N}_0^d \setminus (O_\geq \cup I_\geq)\), based on the basic property \(A_\geq \setminus B_\geq = (A \setminus B)_\geq \setminus B_\geq\). First, it is obvious that \(N \subseteq \mathbb{N}_0^d \setminus (O_\geq \cup I_\geq)\). Conversely, suppose there exists \(x \in \mathbb{N}_0^d \setminus (O_\geq \cup I_\geq)\) such that \(x \rightarrow y\) for some \(y \in \mathbb{N}_0^d\). Then there must exist a path from \(x\) to \(y\), which implies that there exist a jump active on \(x\), i.e., \(x \in I_\geq\), a contradiction. Analogously, one can show that none of the states in \(\mathbb{N}_0^d \setminus (O_\geq \cup I_\geq)\) are accessible from any state in \(\mathbb{N}_0^d\). This shows that \(\mathbb{N}_0^d \setminus (O_\geq \cup I_\geq) \subseteq \mathbb{N}\). Hence \(E \cup P \cup Q = I_\geq\). Next, we verify the expressions for \(E\) and \(P \cup Q\). By the definition of escaping states, \(E = \{x \in I_\geq \colon x \rightarrow y \text{ implies } y \neq x\}\), and for any \(x \in I_\omega^\ast\), \(x \leftrightarrow x + \omega\), we have \(E \subseteq I_\geq \cup \bigcup_{\omega \in \Omega} I_\omega^\ast\). Conversely, for any \(x \in I_\geq \cup \bigcup_{\omega \in \Omega} I_\omega^\ast\), suppose \(x \leftrightarrow y\) for some \(y \in \mathbb{N}_0^d\). Then there exists a cycle through both \(x\) and \(y\), and thus there exists \(\omega \in \Omega\) such that \(x \leftrightarrow x + \omega\), i.e., \(x \in I_\omega^\ast\), a contradiction. Since \(E \cup P \cup Q = I_\geq\) and \(\bigcup_{\omega \in \Omega} I_\omega^\ast \subseteq I_\geq\), we have \(P \cup Q = \bigcup_{\omega \in \Omega} I_\omega^\ast\).

For any \(x \in (I_\omega)_\geq \setminus I_\omega^\ast\) for some \(\omega\), \(x \rightarrow x + \omega\) but \(x + \omega \neq x\). Hence \(x \in E \cup Q\), within an open communicating class, i.e., \(\bigcup_{\omega \in \Omega} ((I_\omega)_\geq \setminus I_\omega^\ast) \subseteq E \cup Q\).

(i) We first prove the sufficiency. It suffices to show \(E \cup Q = \emptyset\), which further implies that \(T = \emptyset\), and thus \(P = I_\geq\) and \(N = \mathbb{N}_0^d \setminus I_\geq\). Suppose \(E \cup Q \neq \emptyset\).

Then there exist \(x, y \in I_\geq\) such that \(x \rightarrow y\), \(y \neq x\). Assume

\[
x = x^{(1)} \rightarrow_{\omega(1)} x^{(2)} \rightarrow_{\omega(2)} \cdots \rightarrow_{\omega(m-2)} x^{(m-1)} \rightarrow_{\omega(m-1)} x^{(m)} = y.
\]

By the condition, \(x^{(j)} \leftrightarrow x^{(j+1)}\) for all \(j = 1, \cdots, m - 1\). This further implies that \(x \leftrightarrow y\), a contradiction. Next we prove the necessity. Since \(\bigcup_{\omega \in \Omega} ((I_\omega)_\geq \setminus I_\omega^\ast) \subseteq E \cup Q = \emptyset\), it follows that \(\Omega_\omega = \emptyset\).

(ii) It follows from the expression for \(P \cup Q\) as well as the definition of \(I_\omega^\ast\).

(iii) It suffices to show \(I_\omega^\ast \subseteq O_\geq\) for all \(\omega \in \Omega\). Let \(x \in I_\omega^\ast\), then there exists a path from \(x + \omega\) to \(x\), and thus we have \(x \in (O_\omega)_\geq\) for some \(\omega\), i.e., \(I_\omega^\ast \subseteq O_\geq\).
Proof of Theorem 3.4. It suffices to show the equivalence of accessibility of one state to another state for $C$ and $\tilde{C}$. For any $x, y \in \mathbb{N}_0^d$, assume $x \rightarrow y$ for $C$. Then there exists $\{\omega^{(j)}\}_{j=1}^m \subseteq \Omega$ such that $x + \sum_{j=1}^i \omega^{(j)} \in (I_{\omega^{(i)}}^\geq)$ for all $i = 1, \ldots, m - 1$ and $y = x + \sum_{j=1}^m \omega^{(j)}$. Construct $\{\tilde{\omega}^{(j)}\}_{j=1}^m \subseteq \tilde{\Omega}$ from $\{\omega^{(j)}\}_{j=1}^m$ in the following way: For every $j = 1, \ldots, m$, if $\omega^{(j)} \in \tilde{\Omega}$, then maintain it. Otherwise, by the condition, there must exist $\tilde{\omega}^{(j)} \in \Omega$ such that $\omega^{(j)} = k_j \tilde{\omega}^{(j)}$ for some $k_j \in \mathbb{N}$ and $(I_{\omega^{(i)}}^\geq) \subseteq (I_{\tilde{\omega}^{(i)}}^\geq)$. In this case, replace $\omega^{(j)}$ by $k_j$ copies of $\tilde{\omega}^{(j)}$.

By (i)-(iii), $x + \sum_{j=1}^i \tilde{\omega}^{(j)} \in (\tilde{I}_{\omega^{(i)}}^\geq)$, $\forall i = 1, \ldots, m - 1$ for some $\tilde{m} \in \mathbb{N}$ and $y = x + \sum_{j=1}^{\tilde{m}} \omega^{(j)}$. Hence $x \rightarrow y$ for $\tilde{C}$. The reverse implication is trivial by (i).

Proof of Theorem 3.5. From Theorem 3.2, $T = \emptyset$ is equivalent to $O_\geq' \subseteq I_\geq'$. Hence, if $T = \emptyset$, choose any $x \in O \subseteq O_\geq' \subseteq I_\geq'$, then there also exists $y \in I$ such that $x \geq y$. To see the converse, by definition of $O_\geq'$, we find the condition implies $O_\geq' \subseteq I_\geq'$.

Proof of Theorem 3.6. From Theorem 3.2(i), $\# T < \infty$ if and only if $O_\geq' \setminus I_\geq'$ is bounded, if and only if ($\ast$) there exists $M > 0$ such that if $x \in O_\geq'$ with $x_j > M$ for some $j \in \{1, \ldots, d\}$, then $x \in I_\geq'$.

It then suffices to show (iii) is equivalent to both ($\ast$) and (ii). We first show that ($\ast$) is equivalent to (iii), and then (ii) is equivalent to (iii).

($\ast$) $\Rightarrow$ (iii). Choose $M$ satisfying ($\ast$). Fix $j \in \{1, \ldots, d\}$ and $x \in O$. Let $M = M + 1 + x_j$, and define $\tilde{x}$ by

$$\tilde{x}_k = \begin{cases} M, & \text{if } k = j, \\ x_k, & \text{if } k \in \{1, \ldots, d\} \setminus \{j\}. \end{cases}$$

Obviously $\tilde{x} \geq x$, and $\tilde{x} \in O_\geq'$ with $\tilde{x}_j > M$. Hence, by ($\ast$), $\tilde{x} \in I_\geq'$, i.e., there exists $y \in I$ such that $\tilde{x} \geq y$, which yields that $x_k = \tilde{x}_k \geq y_k$ for all $k \in \{1, \ldots, d\} \setminus \{j\}$, that is, (iii) holds.

(iii) $\Rightarrow$ ($\ast$). Let $M = \max_{x \in I} \|x\|_\infty$. Let $x \in O_\geq$ with $x_j > M$ for some $j \in \{1, \ldots, d\}$. There exists $z \in O$ such that $x \geq z$. By (iii), there exists $y \in I$ such that $x_k \geq z_k \geq y_k$ for all $k \in \{1, \ldots, d\} \setminus \{j\}$. Since $x_j > M \geq y_j$, $x \in I_\geq$, which yields ($\ast$).

(ii) $\Rightarrow$ (iii). Fix $j = 1, \ldots, d$ and $x \in O$. Since

$$\tilde{x}_j \in (\tilde{O}_j)_{\geq} \subseteq (\tilde{I}_j)_{\geq},$$

there exists $y \in I$ such that $x_k \geq y_k$, for all $k \in \{1, \ldots, d\} \setminus \{j\}$, which implies (iii).

(iii) $\Rightarrow$ (ii). Fix $j = 1, \ldots, d$ and $x = (x_1, \ldots, x_{d-1}) \in (\tilde{O}_j)_{\geq}$. Then there exists $\tilde{x} \in O$ with

$$x_k \geq \begin{cases} \tilde{x}_k & \text{if } k = 1, \ldots, j - 1, \\ \tilde{x}_{k+1} & \text{if } k = j, \ldots, d - 1. \end{cases}$$

By (iii), there exists $y \in I$ such that

$$x_k \geq \begin{cases} \tilde{x}_k \geq y_k & \text{if } k = 1, \ldots, j - 1, \\ \tilde{x}_{k+1} \geq y_{k+1} & \text{if } k = j, \ldots, d - 1, \end{cases}$$

that is, $x \in (\tilde{I}_j)_{\geq}$. Since $j = 1, \ldots, d$ and $x \in (\tilde{O}_j)_{\geq}$ were arbitrary, (ii) holds.

Proof of Theorem 3.7. It follows readily from Theorem 3.6(iii).
Proof of Theorem 3.8. By Theorem and 3.7, it suffices to show that (3.1) implies Theorem 3.6(iii). Let \( y^{(j)} \in \mathcal{I} \) such that \( y^{(j)} = \min \tilde{I}_j \) for \( j = 1, 2 \). From (3.1), for all \( x \in \mathcal{O} \) and \( j = 1, 2 \), we have
\[
x_{3-j} \geq \min \tilde{O}_{3-j} \geq \min \tilde{I}_j = y^{(3-j)}_j,
\]
with \( y^{(3-j)} \in \mathcal{I} \). Since \( j \in \{1, 2\} \) is arbitrary, Theorem 3.6(iii) holds.

Proof of Theorem 3.11. To prove Theorem 3.11, we need three lemmata provided in the supplement.

We only prove Theorem 3.11 for the case \( \omega \in \mathbb{Z}^d \setminus \mathbb{N}^d_0 \). The proof can readily be adapted to the (simpler) case \( \omega \in \mathbb{N}^d_0 \) with \( b = 0 \).

Let \( M = \max_{\omega \in \Gamma} \frac{d_{\hat{\omega}}}{\omega} \), and define \( b \in \mathbb{N}^d_0 \) by
\[
b_j = M |(\omega_*)_j| + \max \tilde{I}_j \cup \tilde{O}_j, \quad \text{for} \quad j = 1, \ldots, d.
\]
Obviously,
\[
\omega_*[-M, M]_1 + b \subseteq \bigcap_{y \in \mathcal{I} \cup \mathcal{O}} y_{\geq},
\]
which implies that all jumps in \( \Omega \) are active on all states in \( \omega_*[-M, M]_1 + b \). Let \( c \in \mathbb{N}^d_0 + b \). Within this proof we slightly abuse \( \mathcal{I}_\geq \) to mean \( \mathcal{I}_\geq \cap \mathcal{L}_c \) for convenience to ignore the dependence on \( c \). Analogously for \( (\mathcal{I}_\pm)_\geq \cap \mathcal{L}_c \), etc. Let
\[
D_k = \omega_*[-M + 1, M - 1]_1 + \frac{k - 1}{\omega_*} \omega_* + c, \quad \text{for} \quad k = 1, \ldots, \omega_*.
\]
Given \( k \in [1, \omega_*]_1 \). Since \( c \geq b \), all jumps in \( \Omega \) are active on every state in \( D_k \).

Moreover, in the light of the definition of \( \omega_* \), there exist \(-m_1 \omega_* \in \Omega_-\) and \( m_2 \omega_* \in \Omega_+\) with \( m_1, m_2 \in \mathbb{N} \) coprime. By Lemma S4.1, \( D_k \) is communicable.

Moreover, by Lemma S4.3, the sets \( (\mathcal{I}_\pm)_\geq \), \( (\mathcal{O}_\pm)_\geq \) are non-empty lattice intervals, and so are their finite intersections, and as well as their finite unions due to the non-emptiness of intersections. In particular, \( F = (\mathcal{I}_+)_\geq \cap (\mathcal{O}_-)_\geq \cup (\mathcal{I}_-)_\geq \cap (\mathcal{O}_+)_\geq \) is also a lattice interval, and thus \( F = [\mathcal{I}, \mathcal{O}]_{\omega_*} \), by the definition of \( \mathcal{I}, \mathcal{O} \).

For every \( k \in [1, \omega_*]_1 \), let \( G_k = F \cap \mathcal{I}^{(k)}_c \). In Step I and Step II below, we will show that \( G_k \) is a communicating class with at least two distinct states, which in turn implies that \( G_k \) is either a PIC or a QIC. In Step III, we show that \( G_k \) is a QIC trapped into \( T^{(k)}_c \) for all \( k \in \Sigma^+_c \), and a PIC for all \( k \in \Sigma^-_c \).

Step I. \( G_k \) is communicable with \( \#G_k \geq 2 \). Indeed, since \( M \geq 2 \) and \( c \geq b \),
\[
\left(c \pm (M - 1)\omega_* + \frac{k - 1}{\omega_*} \omega_*\right)_j \geq \max \tilde{I}_j \cup \tilde{O}_j, \quad \forall j = 1, \ldots, d,
\]
which implies that
\[
c \pm (M - 1)\omega_* + \frac{k - 1}{\omega_*} \omega_* \in G_k.
\]
This shows \( \#G_k \geq 2 \).

Next we prove that \( G_k \) is communicable. Let \( D = \bigcup_{i=1}^{\omega_*} D_i \). By Lemma S4.3, \( D = [\mathcal{I}, \mathcal{O}]_{\omega_*} \), and is a lattice interval with \( \mathcal{O} = c + (M + 1)\omega_* \) and \( \mathcal{I} = c + (M - 1)\omega_* + \frac{k - 1}{\omega_*} \omega_* \). By Lemma S4.2, for any \( x \in D_i \) and any \( y \in D_j \) with \( i \neq j \), \( i, j \in [1, \omega_*]_1 \), \( x \) neither is accessible from nor leads to \( y \). Since \( D_k \) is communicable,
it then suffices to show for all \( x \in F \setminus D \), there exists \( y \in D \) such that \( x \leftrightarrow y \). Since both \( D \) and \( F \) are lattice intervals, \( F \setminus D = \min \{ D \} \setminus D \cup [D, \pi] \). 

In the following, we prove that for all \( x \in [\min \{ D \}] \), there exists \( y \in D \) such that \( x \leftrightarrow y \). The analogous property holds for \( [\pi, \overline{D}] \).

Recall that \( F = (I_+ \cap (O_-) \cup (I_-) \cap (O_+) \), and both \( (I_+) \cap (O_-) \) and \( (I_-) \cap (O_+) \) are lattice intervals. Assume w.l.o.g. that \( \emptyset \neq [\min \{ D \}] \subseteq (I_+) \cap (O_-) \). Then \( \exists \min (I_+ \cap (O_-) \cup (I_-) \cap (O_+) \). Let \( x \in [\min \{ D \}] \). On the one hand, since \( x \in (I_+) \), and \( \overline{D} > \max_{\omega \in \Omega} |\omega| \), one can show by induction that there exists \( y \in D \) such that \( x \rightarrow y \), realized by a finite ordered set of jumps in \( \Omega_+ \), with \( y - x \in \omega_N \). On the other hand, since \( x \in (O_-) \), in an analogous manner, one can show that there exists \( z \in D \) such that \( z \rightarrow x \), realized by a finite ordered set of jumps in \( \Omega_- \), with \( x - z \in \omega_N \). Hence \( y - z = y - x + x - z \in \omega_N \). By Lemma S4.2, \( y, z \in D_k \) for some \( k \in [1, \omega_N] \), i.e., \( y = z \) or \( y \leftrightarrow z \). By transitivity, \( x \leftrightarrow y \in D \).

**Step II.** \( I_+ \setminus F = \emptyset \). From Theorem 3.2, \( T_c \cup N_c = L_c \setminus I_+ \), and thus it suffices to show that \( L_c \setminus F \) is composed of singleton communicating classes assuming that \( I_+ \setminus F \neq \emptyset \). 

Since \( F \subseteq I_+ \), and both \( F \) and \( I_+ \) are lattice intervals, we have \( I_+ \setminus F = [\min (I_+ \cap (O_-) \cup (I_-) \cap (O_+) \] \]. Assume w.l.o.g. that \( [\min (I_+ \cap (O_-) \cup (I_-) \cap (O_+) \) \). It then suffices to show that \( [\min (I_+) \cap (O_-) \cup (I_-) \cap (O_+) \) is composed of singleton communicating classes. It is easy to see that

\[
\min (I_+) < \omega, \min (O_-) > \omega, \min (O_+) > \omega, \min (O_-) > \omega,
\]

from which it readily yields from the definition of \( \overline{D} \) that

\[
[\min (I_+ \cap (O_-) \cup (I_-) \cap (O_+) \] \] \( \cup \) \[\min (I_- \cap (O_-) \cup (I_-) \cap (O_+) \] \] \( \cup \) \[\min (I_+) \cap (O_-) \cup (I_-) \cap (O_+) \] \].

Further assume w.l.o.g. that \( [\min (I_+) \cap (O_-) \cup (I_-) \cap (O_+) \] \]. Let \( x \in [\min (I_+) \cap (O_-) \cup (I_-) \cap (O_+) \] \]. Now we only show that no other state communicates with \( x \) by contradiction. Suppose there exists \( y \neq x \) such that \( x \leftrightarrow y \). Then there exists a cycle connecting \( x \) and \( y \), denoted by

\[x = y^{(0)} \rightarrow \ldots \rightarrow y^{(m)} \rightarrow y^{(0)} \]

Let \( z = \min y^{(j)} \) for some \( k \in [0, m] \). Since \( k \neq (k + 1) \mod (m + 1) \), \( z < \omega \), \( y^{(k+1)} \mod (m+1) \), and thus \( z \in (I_+) \), for \( z \rightarrow y^{(k+1)} \mod (m+1) \) must be realized by a jump in \( \Omega_+ \). On the one hand, since \( x \notin (I_+) \), \( \exists \min (I_+) \cap (O_-) \), we have \( z \leq \omega, x < \omega, \min (I_+) \) \( \cup \) \( (I_-) \cap (O_-) \), and \( (I_-) \cup (O_-) \) \( \subseteq \) \( L_c \) is an interval, we have \( z \notin (I_+) \). This is a contradiction.

**Step III.** For every \( k \in \Sigma^+_c \), every \( x \in Q^{(k)}_c \), and \( y \in T^{(k)}_c \), \( x \rightarrow y \).

Based on Steps I and II, by Lemma S4.2, for every \( k \in \Sigma^+_c \), \( Q^{(k)}_c \) is a QIC, and for every \( k \in \Sigma^-_c \), \( P^{(k)}_c \) is a PIC. In particular, there are precisely \( \#\Sigma^+_c \) QICs ultimately leading only to trapping states, and there are \( \#\Sigma^-_c \) PICs.

In the light of Lemma S4.2, it suffices to show that:

\( \forall x \in T \), there exists \( y \in F \) such that \( y \rightarrow x \).


Again on account of Lemma S4.2, assuming w.o.l.g. that $T \neq \emptyset$ and $\omega_{*} = 1$. Then it is enough to show: For any $x \in T$, there exists some $z \in F$ such that $z \rightarrow x$.

Given $x \in T$, there exists $y \in E \cup F$ such that $y \rightarrow x$. Assume w.o.l.g. that $y \in E$. From Step II,

$$y \in I_{z} \setminus F = [\min_{\omega_{*}} I_{z}, \xi_{[\omega_{*}]} \cup \tau, \max_{\omega_{*}} I_{z}]_{\omega_{*}}.$$ 

Furthermore assume w.o.l.g. that $y \in [\min_{\omega_{*}} I_{z}, \xi_{[\omega_{*}]}]$. By the analysis in Step II, it suffices to prove that there exists $z \in F$ such that $z \rightarrow y$ under the further assumption w.o.l.g. that $y \in [\min_{\omega_{*}} I_{z}, \xi_{[\omega_{*}]}] \subseteq (I_{-})_{\geq} \cup (O_{+})_{\geq}$. Since $\min_{\omega_{*}} (I_{-})_{\geq} > \omega_{*}$, we have $y \in (O_{-})_{\geq}$, and $\max_{\omega_{*}} \xi_{[\omega_{*}]} > \max_{\omega_{*}} \xi_{[\omega_{*}]}$, similar as in Step I, one can show by induction that there exists $z \in F$ such that $z \rightarrow y$, realized by a finite ordered set of jumps in $\Omega_{-}$.

### B. Proofs of results in Section 4

For the reader’s convenience, we restate various Lyapunov-Foster criteria for various types of dynamical properties in the supplement. Since the proofs are mainly based on the specific construction of Lyapunov functions, we only provide the specific Lyapunov function in the respective proofs without tedious but straightforward verification against the corresponding criteria. Asymptotic expansion of $Qf$ is provided in Section S8, where $Q$ is the infinitesimal generator of the CTMC in $\mathcal{C}$ (note $Q$ is determined by $(\Omega, \mathcal{F})$), and $f$ is the Lyapunov function. Straightforward verification that a constructed function is a Lyapunov function (with additional properties required for respective criteria) is left to the interested reader.

**Proof of Theorem 4.1.** From Corollary 3.12, $E = E_{T}$ whenever $T \neq \emptyset$. Hence $E_{P} = E$ or $E_{P} = \emptyset$. Assume w.o.l.g. that $\omega_{*} = 1$. Hence $Q = \emptyset$ or $P = \emptyset$. It suffices to prove the result in the following cases:

(a) $T = \emptyset$ and $Y_{0} \in P$. Then $Y_{t}$ is irreducible. We can directly apply the propositions on explosivity and non-explosivity.

(b) $T = \emptyset$ and $Y_{0} \in E$. In this case, since $\#E = i_{+} - i < \infty$, for the embedded DTMC $(\tilde{Y}_{n})$, the chain ended in $P$ within $(i_{+} - i)$ jumps. Since holding times are exponentially distributed, the embedded DTMC cannot have infinitely many jumps within finite time before entering $P$, and thus have the same dynamical property regarding non-explosivity (and thus explosivity) as in case (a).

(c) $T \neq \emptyset$ and $Y_{0} \in Q$. Let $Z_{t}$ be the (irreducible) CTMC on $Q$ with $Z_{0} = Y_{0} = x$ and transition operator $\tilde{Q}$ with $\tilde{q}_{x,y} = q_{x,y}$ for all $x$, $y \in Q$ and $x \neq y$. It suffices to show that $Z_{t}$ is explosive if and only if $Y_{t}$ is explosive. By Reuter’s criterion [45], $Z_{t}$ is explosive if and only if there exists a solution $v$ such that $\tilde{Q}v = v$. Suppose first that $Z_{t}$ is explosive. Then $\tilde{Q}v = v$ for some bounded non-negative non-zero $v$. Define $u_{x} = v_{x} 1_{Q}(x)$, $\forall x \in T \cup E \cup Q$. It is straightforward to verify that $Qu = u$. Hence $Y_{t}$ is also explosive. Conversely, if $Y_{t}$ is explosive, then $Qu = u$ for some bounded non-negative non-zero $u$. Let $w = u|_{T \cup E}$, i.e., $w_{x} = u_{x}$ for all $x \in T \cup E$. Since $Q|_{T \cup E}$ is a lower-triangular matrix with non-positive diagonal entries, and $w \geq 0$, it is readily deduced that $w = 0$. This implies from $Qu = u$ that $\tilde{Q}v = v$ with $v = u|_{Q}$. Hence $Z_{t}$ is explosive. To sum up, $Z_{t}$ is explosive if and only if $Y_{t}$ is explosive. Hence it reduces to investigate necessary and sufficient conditions for $Z_{t}$ to be explosive, and it reduces to case (a).
Based on the above analysis, it suffices to show case (a), in other words, we have the irreducibility. Now we can apply Proposition S5.1 and S5.2.

(i) Fix $\delta > 0$ as well as $A = [1, x_0]$ for some $x_0 > 0$, with both $\delta$ and $x_0$ to be determined. Let $f$ be non-increasing and bounded such that $f(x) = x^{-\delta}$ for all $x \geq x_0$. Obviously, Proposition S5.1(i) is satisfied for the set $A$. Next we verify the conditions in Proposition S5.1(ii). It is easy to verify that for all $x > x_0$, $Qf(x) < -\epsilon$ for all $x \in A^c$ where $\epsilon = \delta \alpha / 2$ provided $\alpha > 0$ and $R > 1$ with $\delta < R - 1$; $\alpha = \delta$ $(\beta - \delta \vartheta) / 2$ provided $\alpha = 0$, $\beta > 0$, $R > 2$ with $\delta < \min \{\beta / \vartheta, R - 2\}$, and $x_0$ is chosen accordingly. Since $\delta > 0$ can be arbitrarily small, in either case, there exist $\delta$ and $\epsilon$ such that conditions in Proposition S5.1 are fulfilled, and thus $E_x \zeta < +\infty$ for all $x \in \mathcal{X}$, i.e., $Y_t$ explodes a.s. for all $Y_0 = x \in \mathcal{X}$.

(ii) Now we prove non-explosivity using Proposition S5.2. Let $f(x) = x + M$ for some large $M > 0$ to be determined. Choose $g(x) = (\alpha + |\gamma| + 1)(x + 1)$. One can show that all conditions in Proposition S5.2 are satisfied for some constant $M > 0$, provided one of the following conditions holds: (ii-1) $\alpha < 0$ (ii-2) $\alpha = 0 \geq \beta$, (ii-3) $R - 2 \leq \alpha < 0$, (ii-4) $\alpha > 0 \geq R - 1$. Hence $Y_t$ is non-explosive.

**Proof of Theorem 4.2.** Assume w.o.l.g. that $\omega_+ = 1$. Otherwise, take $B$ as the finite union of $B^{(k)}$ for each $P^{(k)}$, and $\tau_B = \tau_{B^{(k)}}$ for $x \in P^{(k)}$, for $k \in \Sigma^-$.

We first prove the existence of moments of passage times. It suffices to prove the conclusion for $x \in \mathcal{X}$. Indeed, since the holding time is exponentially distributed and $\#E < \infty$, for all $x \in E$, $\sup_{x \in E} \mathbb{E}_x(\tau^s_p) < \infty$ for all $\epsilon > 0$. Hence $\tau_B(x) \leq \tau_p(x) + \tau_B(Y_{\tau_p})$, where the argument emphasises the initial condition. Let $\omega_+ = \max_{\omega \in \Omega} \omega$.

Since $Y_{\tau_p} \in [\min P, \min P + \omega_+]|_1$ a.s., then

$$\tau_B(Y_{\tau_p}) \leq \sup_{z \in [\min P, \min P + \omega_+]|_1} \tau_B(z) \leq \sum_{z = \min P}^{\min P + \omega_+} \tau_B(z).$$

If $\epsilon \geq 1$, by Minkowski’s inequality,

$$\mathbb{E}_x(\tau_B^\epsilon)^{1/\epsilon} \leq \mathbb{E}_x(\tau^s_p)^{1/\epsilon} + \sum_{z = \min P}^{\min P + \omega_+} \mathbb{E}_x(\tau_B^\epsilon)^{1/\epsilon} \mathbb{E}_x(\tau_B^\epsilon)^{1/\epsilon} < \infty,$$

provided

$$\mathbb{E}_x(\tau_B^\epsilon)^{1/\epsilon} < \infty, \quad \forall z \in \mathcal{X}.$$  

(B.1)

If $\epsilon < 1$,

$$\mathbb{E}_x(\tau^s_p)^{1/\epsilon} \leq 2^\epsilon \mathbb{E}_x(\tau^s_p)^{1/\epsilon} + 2^\epsilon (\omega_+ + 1)^\epsilon \sum_{z = \min P}^{\min P + \omega_+} \mathbb{E}_x(\tau_B^\epsilon)^{1/\epsilon} < \infty,$$

provided (B.1) holds, where the inequality $\left(\sum_{j=1}^m a_j\right)^\epsilon \leq m^\epsilon \sum_{j=1}^m a_j^\epsilon$ for $a_j \geq 0$, $j = 1, \ldots, m$, is applied.

Based on the above analysis, we assume w.o.l.g. that $x \in \mathcal{X}$, and hence $Y_t$ is irreducible. Assume either $R > 1 > 0 > \alpha$ or $R - 2 > \alpha = 0 \geq \beta$. In either case, choose $f(x) = \log \log x$ for large $x$. One can directly verify that for every $\sigma > 0$ there exists $0 < \epsilon < +\infty$ such that $Qf^\sigma(x) \leq -ef^{\sigma-2}(x)$ for all large $x$. By Proposition S5.3(i), there exists $a > 0$ such that

$$\mathbb{E}_x(\tau^s_{\{f \leq a\}})^{1/\epsilon} < \infty, \quad \forall x \in \mathcal{X}, \forall 0 < \epsilon < \sigma/2.$$  

(B.2)
Since \( \sigma > 0 \) is arbitrary, so is \( \epsilon \). Moreover, \( \{ f \leq a \} \) is finite. Here \( a \) can be chosen independently of \( \epsilon \).

Analogous arguments are valid for \( R = 0 > \alpha \) with \( f(x) = x^{1/2} \) and \( R - 1 = 0 > \alpha \) with \( f(x) = \log x \).

For \( R - 2 = \alpha = 0 > \beta \), let \( f(x) = x^\rho \) for some \( \rho > 0 \). Since \( \beta < 0 < \vartheta \), one can choose \( \sigma < -\frac{\beta}{\rho} \) and show in a similar way that (B.2) holds. Moreover, \( \sigma, \epsilon > 0 \) are arbitrary since \( \rho > 0 \) is arbitrary.

For \( R = \alpha = 0 \), let \( f(x) = x^\rho \) with \( \rho \geq 1 \) and \( \sigma < \rho^{-1} \). Similar arguments yield that (B.2) holds with \( \sigma \) replaced by 1.

For \( R - 1 = \alpha = 0 > \gamma \), let \( f(x) = x^{1/2} \) and \( \sigma < -2\beta/\vartheta \). Then (B.2) holds with \( \sigma \) replaced by \(-2\beta/\vartheta > 2 \).

Next we prove the non-existence of passage times.

For \( R - 1 = \alpha = 0 < \gamma \), let \( f(x) = g(x) = x \) with \( \delta > 1 \) arbitrary as well as \( \sigma \in (\max\{1 - \gamma/\vartheta, 0\}, 1) \) in Proposition S5.3(ii-3). By Proposition S5.3(ii), one can show that there exists \( a > 0 \) such that

\[
(B.3) \quad \mathbb{E}_x(\tau_{x \leq a}) = \infty, \quad \forall x > a,
\]

for all \( \epsilon > \max\{1 - \gamma/\vartheta, 0\} \), where \( 1 - \gamma/\vartheta < 1 \). For \( R = \alpha = 0 \), choose the same \( f \) and \( g \). Again by Proposition S5.3(ii), one can show (B.3) holds with arbitrary \( \epsilon > 1 \).

For \( R - 2 = \alpha = \beta = 0 \), let \( f(x) = g(x) = x^\rho \) for \( \rho > 1 \). Then one can show (B.3) holds with arbitrary \( \epsilon > 0 \).

**Proof of Theorem 4.3.** Choose \( Y_0 \in \mathcal{P} \). Then \( Y_t \) is irreducible with \( \mathcal{Y}_0 \) being one PIC. Define the following two auxiliary functions used in the proof afterwards:

\[
m_1(x) = \sum_{\omega \in \Omega} \omega \lambda_\omega(x) \omega, \quad m_2(x) = \sum_{\omega \in \Omega} \frac{\lambda_\omega(x) \omega^2}{\lambda_\omega(x)}, \quad x \in \mathcal{Y}_0,
\]

Note that \( m_1, m_2 \) are well-defined since \( x \geq i_+ \). By (A3), these two functions have the following asymptotics:

\[
m_1(x) = \frac{\alpha x^R + \gamma x^{R-1} + \bar{m}_1(x)}{\sum_{\omega \in \Omega} \lambda_\omega(x)}, \quad m_2(x) = \frac{2\vartheta x^R + \bar{m}_2(x)}{\sum_{\omega \in \Omega} \lambda_\omega(x)},
\]

with \( \bar{m}_1(x) = \bar{m}_2(x) = O(x^{R-2}) \) and \( \bar{m}_1(x) \geq 0 \geq \bar{m}_2(x) \) for all large \( x \) (c.f., the proof of [32, Theorem 3.1]).

Under (A5) and (A6), \( \liminf_{x \to \infty} m_2(x) > 0 \). Indeed,

\[
\liminf_{x \to \infty} m_2(x) = \frac{2\vartheta}{\alpha^*_+ \mathbb{1}_{\{R_+\}}(R) + \alpha^*_- \mathbb{1}_{\{R_-\}}(R)} \geq \frac{2\vartheta}{\alpha^*_+ + \alpha^*_-},
\]

where \( \alpha^*_+, \alpha^*_- \in \mathbb{R}_{>0} \) are defined in the proof of Theorem 5.8. Then the conclusion directly follows from Proposition S5.4.

**Proof of Theorem 4.4.** Assume w.o.l.g. that \( \omega_* = 1 \). We prove this theorem by Propositions S5.5, S5.6 and S5.7.

(i) First, we show positive recurrence. For either \( R - 1 > 0 > \alpha \) or \( R - 2 > \alpha = 0 \), let \( f(x) = \log \log x \) for all large \( x \in \mathcal{P} \). For \( R - 1 = 0 > \alpha \), let \( f(x) = x \). Then one can verify that there exists \( \epsilon > 0 \) such that \( Qf(x) \leq -\epsilon f(x) \) for all large \( x \).

By Proposition S5.5, \( X_t \) is positive recurrent and there exists a unique *exponentially* ergodic SD on \( \mathcal{P} \).
For $R = 0 > \alpha$, let $f(x) = x$. For $R - 2 = \alpha = 0 > \beta$, let $f(x) = \log x$. For $R - 1 = \alpha = 0 > \gamma$, let $f(x) = x^\delta$ with some $\delta > 1$ satisfying $\gamma < -(\delta - 1)\vartheta$. Then one can verify that there exists $\epsilon > 0$ such that $Qf(x) \leq -\epsilon$ for all large $x$. By Proposition S5.6, $X_t$ is positive recurrent and there exists a unique SD on $\mathbb{P}$.

Next, we show null recurrence. By Theorem 4.3, it suffices to show that there exists $x \in \mathbb{P}$ such that $\mathbb{E}_x(\tau^+_x) = \infty$. Let $B \subseteq \mathbb{P}$ be as in Theorem 4.2(ii) and $x = \max B$. By Bayes’ Theorem,

$$\mathbb{E}_x(\tau^+_x) = \sum_{z \in \Omega^+} \mathbb{P}_x(Y_{J_1} = z) \left(\mathbb{E}_x(J_1|Y_{J_1} = z) + \mathbb{E}_z(\tau_x)\right)$$

$$\geq \sum_{z \in \Omega^+} \mathbb{P}_x(Y_{J_1} = z) \mathbb{E}_z(\tau_x) \geq \sum_{z \in (\mathbb{P} \setminus B) \cap (\Omega^+)} \mathbb{P}_x(Y_{J_1} = z) \mathbb{E}_z(\tau_x)$$

$$\geq \mathbb{P}_x(Y_{J_1} \notin B) \min_{z \in (\mathbb{P} \setminus B) \cap (\Omega^+)} \mathbb{E}_z(\tau_x)$$

$$\geq \sum_{\omega \in \Omega^+} \lambda_\omega(x) \min_{z \in (\mathbb{P} \setminus B) \cap (\Omega^+)} \mathbb{E}_z(\tau_B) = \infty,$$

since $\sum_{\omega \in \Omega^+} \lambda_\omega(x) > 0$, and $\mathbb{E}_z(\tau_B) = \infty$ for all $z \in \mathbb{P} \setminus B$, by Theorem 4.2(ii), under respective conditions.

(ii). For either $R - 1 > 0 > \alpha$ or $R - 2 > 0 = \alpha \geq \beta$, let $f(x) = 2 - x^{-\delta}$ with $0 < \delta < R - 1 - 1_{\{0\}}(\alpha)$. By the proof of Theorem 4.1, under the assumptions made in Theorem 4.4, it is easy to see that

$$\lim_{x \to \infty} \frac{Qf(x)}{f(x)} = -\infty,$$

and one can choose the finite set $D = \{x \in E : \frac{Qf(x)}{f(x)} \geq -\psi_0 - 1\}$. Then the conclusion follows from Proposition S5.7. Note that $\text{supp} \nu^{(k)} = Q^{(k)}$ comes from the fact that the support of the ergodic SD of the Q-process is $Q^{(k)}$ by the irreducibility.

Since existence of a QSD implies almost sure extinction events, which further implies non-explosivity, we arrive at the non-existence of QSDs provided either (ii)-3 or (ii-4) holds by Theorem 4.1.

**Proof of Theorem 4.7.** Let $Y_t$ be a non-absorbed CTMC on $\mathcal{Y}_0$. Since $\iota - \omega < \omega_+$, $\mathbb{P} \neq \emptyset$. First we prove implosivity. Assume condition (i) or (ii) holds. Choose $f = x^{-1}$ for large $x$, one can show that the conditions in Proposition S5.9(i-1) are fulfilled, and implosivity is achieved.

Next we turn to non-implosivity. Assume both (i) and (ii) fail. Since $Y_t$ does not implode towards any transient state, it suffices to prove non-implosivity assuming recurrence, i.e., $\alpha < 0$, or $\alpha = 0$ and $\beta \leq 0$, by Theorem 4.3. Let $f(x) = \log x$ for large $x$ if $1 - R \geq 0 > \alpha$, and $f(x) = \log \log x$ if $2 - R \geq 0 = \alpha \geq \beta$. Again by Proposition S5.9(ii), $Y_t$ is non-implosive.

**Proof of Theorem 5.6.** Assume w.o.l.g. $\omega_+ = 1$. (i)$\Rightarrow$(ii). Prove by induction.

**Base case.** The LHS of (ii) is zero, since $\lambda_\omega(j) = 0$, for all $0 \leq j < -\omega$, $\omega \in \Omega_-$. Similarly, the RHS of (ii) also vanishes, since $\pi(-k) = 0$ for all $1 \leq k \leq \omega_+$.

**C. Proofs of results in Section 5**

We use LHS (RHS) as shorthand for left (right) hand side of an equation.
**Induction.** Assume
\[
\sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}(x + (j - 1)) \pi(x + j - 1) = \sum_{\omega \in \Omega_+} \sum_{k=1}^{\omega} \lambda_{\omega}(x - k) \pi(x - k)
\]
holds for some \(x \in \mathbb{N}_0\). Since \(\pi\) is a stationary measure, from (2.1), it follows that
\[
\sum_{\omega \in \Omega_-} (\lambda_{\omega}(x - \omega) \pi(x - \omega) - \lambda_{\omega}(x) \pi(x)) = \sum_{\omega \in \Omega_+} (\lambda_{\omega}(x) \pi(x) - \lambda_{\omega}(x - \omega) \pi(x - \omega)).
\]
Hence
\[
\sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}((x + 1) + j - 1) \pi((x + 1) + j - 1) = \sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}(x + j) \pi(x + j)
\]
\[
= \sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}(x + j - 1) \pi(x + j - 1) + \sum_{\omega \in \Omega_-} \lambda_{\omega}(x - \omega) \pi(x - \omega) - \lambda_{\omega}(x) \pi(x)
\]
\[
= \sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}(x - k) \pi(x - k) + \sum_{\omega \in \Omega_+} \lambda_{\omega}(x) \pi(x) - \lambda_{\omega}(x - \omega) \pi(x - \omega)
\]
\[
= \sum_{\omega \in \Omega_-} \sum_{j=1}^{\omega} \lambda_{\omega}(x - k) \pi(x - k) + \sum_{\omega \in \Omega_+} \sum_{k=1}^{\omega} \lambda_{\omega}(x + 1 - k) \pi(x + 1 - k).
\]
(ii)\(\Rightarrow\)(iii). We have
\[
\sum_{j=1}^{\omega} \pi(x + j - 1) \sum_{j \geq 1} \lambda_{\omega}(x + j - 1) = \sum_{\omega} \sum_{j=1}^{\omega} \lambda_{\omega}(x + j - 1) \pi(x + j - 1), \quad x \in \mathbb{N}_0,
\]
\[
\sum_{k=1}^{\omega} \pi(x - k) \sum_{k \leq \omega} \lambda_{\omega}(x - k) = \sum_{\omega} \sum_{k=1}^{\omega} \lambda_{\omega}(x - k) \pi(x - k), \quad x \in \mathbb{N}_0.
\]
(iii)\(\Rightarrow\)(i). Since \(\#\Omega < \infty\), then by induction, subtracting (ii) for \(x\) from that for \(x + 1\) yields the desired conclusion.

**Proof of Corollary 5.7.** Assume w.l.o.g. \(\omega_*=1\). From Theorem 5.6(iii),
\[
\text{LHS} = \sum_{j=1}^{\omega_+} (T_{\pi}(x + j) - T_{\pi}(x)) \sum_{-j \geq \omega} \lambda_{\omega}(x + j - 1)
\]
\[
= \sum_{j=0}^{\omega_+} T_{\pi}(x + j) \sum_{-j \geq \omega} \lambda_{\omega}(x + j) - \sum_{j=1}^{\omega} T_{\pi}(x + j) \sum_{-j \geq \omega} \lambda_{\omega}(x + j - 1)
\]
\[
= \sum_{j=1}^{\omega_+} T_{\pi}(x + j) \left( \sum_{-j \geq \omega} \lambda_{\omega}(x + j) - \sum_{-j \geq \omega} \lambda_{\omega}(x + j - 1) \right) + T_{\pi}(x) \sum_{-1 \geq \omega} \lambda_{\omega}(x),
\]
\[
\text{RHS} = \sum_{k=1}^{\omega_+} (T_{\pi}(x - k) - T_{\pi}(x)) \sum_{\omega \geq k} \lambda_{\omega}(x - k)
\]
\[
= \sum_{k=1}^{\omega_+} T_{\pi}(x - k) \left( \sum_{\omega \geq k} \lambda_{\omega}(x - k) - \sum_{\omega \geq k+1} \lambda_{\omega}(x - k - 1) \right) - T_{\pi}(x) \sum_{\omega \geq 1} \lambda_{\omega}(x - 1),
\]
which together yields the desired identity.
Proof of Theorem 5.8. Let

\[ \alpha_{\ell} = \begin{cases} 
\lim_{x \to \infty} \frac{\sum_{k \geq \ell} \lambda_{k\omega_*}(x)}{x^{R_+}}, & \text{if } \ell \in [1, \omega_+ / \omega_*], \\
\lim_{x \to \infty} \frac{\sum_{k < \ell} \lambda_{k\omega_*}(x)}{x^{R_-}}, & \text{if } -\ell \in [1, \omega_- / \omega_*], \\
0, & \text{otherwise.}
\end{cases} \]

\[ \gamma_{\ell} = \begin{cases} 
\lim_{x \to \infty} \frac{\sum_{k > \ell} \lambda_{k\omega_*}(x) - \alpha_{\ell} x^{R_+}}{x^{R_+ - 1}}, & \text{if } \ell \in [1, \omega_+ / \omega_*], \\
\lim_{x \to \infty} \frac{\sum_{k \leq \ell} \lambda_{k\omega_*}(x) - \alpha_{\ell} x^{R_-}}{x^{R_- - 1}}, & \text{if } -\ell \in [1, \omega_- / \omega_*], \\
0, & \text{otherwise.}
\end{cases} \]

Note that \( \alpha^*_+ = \alpha_1 \) and \( \alpha^*_- = \alpha_{-1} \). Throughout the proof, we use Lemma S6.1, the identities in Theorem 5.6 as well as Corollary 5.7 to establish the estimates of the tails of the stationary distributions. For expository purpose, we only present the proof of case (i), and leave the detailed proofs of cases (ii) and (iii) in the supplement.

Assume \( R_- > R_+ \). Then (5.2) follows directly from (5.1). Next we prove (5.1).

First Limit in (5.1). Under (A5), there exists \( N \in \mathbb{N} \) such that for all \( \omega \in \Omega \), \( \lambda_\omega \) is a non-negative non-decreasing polynomial on \( \mathbb{N}_{\geq N} \).

Recall from Corollary 5.7 that for all \( x \in \mathbb{N}_0 \),

\[ T_\pi(x) \left( \sum_{\omega \in \Omega_-} \lambda_\omega(x) + \sum_{\omega \in \Omega_+} \lambda_\omega(x - \omega_*) \right) + \sum_{j=1}^{\omega_+ / \omega_*} T_\pi(x + j\omega_*) \]

\[ \cdot \left( \sum_{-\omega_+ - 1 \omega \geq \omega \in \Omega_-} \lambda_\omega(x + j\omega_*) - \sum_{-j\omega_* \geq \omega \in \Omega_-} \lambda_\omega(x + (j - 1)\omega_*) \right) \]

\[ = \sum_{k=1}^{\omega_+ / \omega_*} T_\pi(x - k\omega_*) \left( \sum_{k \omega_* \leq \omega \in \Omega_+} \lambda_\omega(x - k\omega_*) - \sum_{(k+1)\omega_* \leq \omega \in \Omega_+} \lambda_\omega(x - (k+1)\omega_*) \right). \]

Then we have

\[ \text{LHS} = T_\pi(x) \left( \sum_{\omega \in \Omega_-} \lambda_\omega(x) + \sum_{\omega \in \Omega_+} \lambda_\omega(x - \omega_*) \right) + \sum_{j=1}^{\omega_+ / \omega_*} T_\pi(x + j\omega_*) \]

\[ \cdot \left( -\lambda_{-j\omega_*}(x + (j - 1)\omega_*) + \sum_{-\omega_+ - 1 \omega \geq \omega} \left( \lambda_\omega(x + j\omega_*) - \lambda_\omega(x + (j - 1)\omega_*) \right) \right) \]

\[ \leq T_\pi(x) \left( \sum_{j=1}^{\omega_+ / \omega_*} \sum_{-\omega_+ - 1 \omega \geq \omega} \left( \lambda_\omega(x + j\omega_*) - \lambda_\omega(x + (j - 1)\omega_*) \right) \right) \]

\[ + \sum_{\omega \in \Omega_-} \lambda_\omega(x) + \sum_{\omega \in \Omega_+} \lambda_\omega(x - \omega_*) \]

\[ = T_\pi(x) \left( \sum_{j \geq 1} (\alpha_{j} - \alpha_{j-1}) x^{R_+} + \sum_{k \geq 1} (\alpha_k - \alpha_{k+1}) x^{R_+} + O\left(x^{R_+ - 1}\right) \right) \]

\[ = T_\pi(x) \left( \alpha_1 x^{R_-} + O\left(x^{R_- - 1}\right) \right). \]
and
\[
\text{RHS} = \sum_{k=1}^{\omega_k/\omega_s} T_\pi(x - k\omega_s) \left( \sum_{k\omega_s \leq \omega \leq \omega_s} \lambda_\omega(x - k\omega) - \sum_{(k+1)\omega_s \leq \omega \leq \omega_s} \lambda_\omega(x - (k+1)\omega_s) \right) \\
\geq T_\pi(x - \omega_s) \sum_{k=1}^{\omega_s/\omega_s} \lambda_k \omega_s(x - k\omega_s) \\
\geq T_\pi(x - \omega_s) \sum_{k=1}^{\omega_s/\omega_s} \left( (\alpha_k - \alpha_{k+1}) x^{R_\pi} + O \left( x^{R_\pi - 1} \right) \right) \\
= T_\pi(x - \omega_s) \left( \alpha^*_+ x^{R_\pi} + O \left( x^{R_\pi - 1} \right) \right).
\]

Then it holds for all \( x \geq N \) (possibly a larger \( N \)) that
\[
\frac{T_\pi(x)}{T_\pi(x - \omega_s)} \geq x^{R_\pi - R_-} \frac{\alpha^*_+ + O \left( x^{-1} \right)}{\alpha^*_- + O \left( x^{-1} \right)} = x^{R_\pi - R_-} \left( \frac{\alpha^*_+}{\alpha^*_-} + O \left( x^{-1} \right) \right)
\]
\[
\geq x^{R_\pi - R_-} \frac{\alpha^*_+}{\alpha^*_-} \left( 1 - N_0 x^{-1} \right),
\]
for some \( N_0 \in \mathbb{N} \), which further implies that
\[
T_\pi(x) \geq \prod_{j=0}^{(x-N)/\omega_s} \left( \frac{\alpha^*_+(x - j\omega_s)}{\alpha^*_-(x - j\omega_s)} \right)^{R_\pi - R_-} \left( 1 - \frac{N_0}{x - j\omega_s} \right) \\
\geq T_\pi(N - \omega_s) \left( \frac{\alpha^*_+}{\alpha^*_-} \right)^{1+(x-N)/\omega_s} \left( \omega_s \right)^{(1+(x-N)/\omega_s)(R_\pi - R_-)} \frac{\Gamma \left( x\omega_s^{-1} + 1 \right)}{\Gamma \left( x\omega_s^{-1} + 1 - N_0\omega_s^{-1} \right)} \\
\cdot \frac{\Gamma(N)}{\Gamma(N - N_0\omega_s^{-1})} \\
\geq C \Gamma(x\omega_s^{-1})^{R_\pi - R_-} \left( \frac{\alpha^*_+}{\alpha^*_-} \right)^{x\omega_s^{-1}} \left( x\omega_s^{-1} \right)^{R_\pi - R_-} \left( x\omega_s^{-1} + N_0\omega_s^{-1} \right),
\]
for some \( C = C_{N,N_0} > 0 \), where we employ the fact that \( T_\pi(N - \omega_s) > 0 \) for some appropriate \( N \) since \( \text{supp} \pi \supseteq \mathcal{P}^{(k)} \) for some \( k \in \Sigma^- \). Hence for all \( \delta \in (R_- - R_+ + N_0\omega_s^{-1}, +\infty) \),
\[
\lim_{x \to \infty} T_\pi(x) \Gamma(x\omega_s^{-1})^{R_\pi - R_-} \left( \frac{\alpha^*_+}{\alpha^*_-} \right)^{x\omega_s^{-1}} \left( x\omega_s^{-1} \right)^{\delta} = \infty.
\]

Second Limit in (5.1). Let
\[
f_\ell(x) = \begin{cases} \sum_{k \geq \ell} \lambda_k \omega_s(x - k\omega_s), & \text{if } \ell \in [1, \omega_+/\omega_s], \\ \sum_{k \leq \ell} \lambda_k \omega_s(x - (k+1)\omega_s), & \text{if } -\ell \in [1, \omega_-/\omega_s]. \end{cases}
\]
Denote
\[
f_\ell(x) = (\alpha_\ell + \beta_\ell(x - \ell\omega_s)) \cdot \begin{cases} x^{R_+}, & \text{if } \ell \in [1, \omega_+/\omega_s], \\ x^{R_-}, & \text{if } -\ell \in [1, \omega_-/\omega_s]. \end{cases}
\]
where by definition of $\alpha$, we have $\beta(x) = O(x^{-1})$. Hence from Theorem 5.6(ii),

(C.1) \[ \sum_{j=1}^{\infty} (\alpha_{-j} + \beta_{-j}(x + j\omega_\pm)) \pi(x + (j-1)\omega_\pm) = x^{R_+ - R_-} \sum_{k=1}^{\omega_+ / \omega_\pm} (\alpha_k + \beta_k(x - k\omega_\pm)) \pi(x - k\omega_\pm), \]

where there exist $M, N > 0$ such that

(C.2) \[ |\beta_k(y - \ell\omega_\pm)| \leq M y^{-\ell}, \quad \forall y \geq N, \quad \forall \ell \in [1, \omega_+ / \omega_\pm] \cup [-\omega_+ / \omega_\pm, -1]. \]

Since $T_{\pi}(x) < \infty$, summing up (C.1) from $x$ to infinity yields

\[ \sum_{y \in \omega_\pm N_0 + x} \sum_{j=1}^{\omega_+ / \omega_\pm} (\alpha_{-j} + \beta_{-j}(y + j\omega_\pm)) \pi(y + (j-1)\omega_\pm) \]

\[ = \sum_{y \in \omega_\pm N_0 + x} \sum_{k=1}^{\omega_+ / \omega_\pm} y^{R_+ - R_-} (\alpha_k + \beta_k(y - k\omega_\pm)) \pi(y - k\omega_\pm). \]

By monotonicity of $T_{\pi}(x)$ and $x^{R_+ - R_-}$, there exists $C > 0$ such that for a possibly larger $N$, it follows from (C.2) that for all $x \geq N$,

\[ \sum_{y \in \omega_\pm N_0 + x} \sum_{j=1}^{\omega_+ / \omega_\pm} (\alpha_{-j} + \beta_{-j}(y + j\omega_\pm)) \pi(y + (j-1)\omega_\pm) \]

\[ \geq \sum_{y \in \omega_\pm N_0 + x} \sum_{j=1}^{\omega_+ / \omega_\pm} (\alpha_{-j} - Cy^{-1}) \pi(y + (j-1)\omega_\pm) \]

\[ \geq \sum_{j=1}^{\omega_+ / \omega_\pm} (\alpha_{-j} - Cx^{-1}) \sum_{y \in \omega_\pm N_0 + x} \pi(y + (j-1)\omega_\pm) \]

\[ = \sum_{j=1}^{\omega_+ / \omega_\pm} (\alpha_{-j} - Cx^{-1}) T_\pi(x + (j-1)\omega_\pm) \]

\[ \geq \left( \alpha_* - Cx^{-1} \right) T_\pi(x). \]

Similarly,

\[ \sum_{y \in \omega_\pm N_0 + x} \sum_{k=1}^{\omega_+ / \omega_\pm} y^{R_+ - R_-} (\alpha_k + \beta_k(y - k\omega_\pm)) \pi(y - k\omega_\pm) \]

\[ \leq \sum_{y \in \omega_\pm N_0 + x} \sum_{k=1}^{\omega_+ / \omega_\pm} y^{R_+ - R_-} (\alpha_k + Cy^{-1}) \pi(y - k\omega_\pm) \]

\[ \leq x^{R_+ - R_-} \sum_{k=1}^{\omega_+ / \omega_\pm} (\alpha_k + Cx^{-1}) \sum_{y \in \omega_\pm N_0 + x} \pi(y - k\omega_\pm) \]

\[ = x^{R_+ - R_-} \sum_{k=1}^{\omega_+ / \omega_\pm} (\alpha_k + Cx^{-1}) T_\pi(x - k\omega_\pm) \]

\[ \leq x^{R_+ - R_-} T_\pi(x - \omega_\pm) \sum_{k=1}^{\omega_+ / \omega_\pm} (\alpha_k + Cx^{-1}). \]
\[ \leq x^{R_1 - R_*} T_\pi(x - \omega_+) \left( \alpha_+ \omega_+^{-1} + C \omega_+ x^{-1} \right), \]

which further implies that

\[ \frac{T_\pi(x)}{T_\pi(x - \omega_+)} \leq x^{R_1 - R_*} \frac{\alpha_+ \omega_+^{-1} + C \omega_+ x^{-1}}{\alpha_* - C x^{-1}}. \]

Similar as the argument for the first limit, one can derive the second limit.

**S1. Well-posedness of the definitions of states**

**Proposition S1.1.** Let \( x \in \mathbb{N}_0^d \) and \( A \subseteq \mathbb{N}_0^d \). Then

(i) \( x \) is a neutral state for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \).

(ii) \( x \) is a trapping state for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \) with at least two states.

(iii) \( x \) is an escaping state for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \).

(iv) \( A \) is a PIC for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \).

(v) \( A \) is a QIC for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \).

(vi) \( A \) is a communicating class for some \( Y_\ell \in \mathcal{C} \) if and only if it is so for all CTMCs in \( \mathcal{C} \).

Consequently, \( \{N, T, E, P, Q\} \) is a decomposition of \( \mathbb{N}_0^d \).

**Proof.** The conclusion (vi) can be deduced from the first five conclusions. (i) is obvious since \( x \) is a neutral state for \( Y_\ell \in \mathcal{C} \) if and only if \( Y_0 = \{Y_0\} = \{x\} \). We only prove case (ii), and the remaining cases can be argued analogously.

It suffices to show the necessity (by contradiction). Let \( x \) be a trapping state for \( Y_\ell \in \mathcal{C} \). Then \( x \neq Y_0 \), and \( x \) is reachable from \( Y_0 \). Suppose \( x \) is not a trapping state for \( Y_\ell \) with the state space \( \bar{Y}_0 \) and \( \# \bar{Y}_0 > 1 \). Then there exists \( y \in \bar{Y}_0 \setminus \{x\} \) such that \( y \) is reachable from \( x \), and hence also reachable from \( Y_0 \). Hence \( y \in Y_0 \). Hence \( x \) is not a trapping state for \( Y_\ell \), a contradiction.

**S2. A property on greatest common divisor**

**Proposition S2.1.** Let \( A \subseteq \mathbb{Z}^d \setminus \{0\} \) be non-empty. Then \( \dim \text{span} A = 1 \) if and only if \( A \) has a common divisor if and only if \( \text{gcd}(A) \) exists.

**Proof.** If \( \text{gcd}(A) \) exists, then clearly \( A \) has a common divisor. If \( A \) has a common divisor \( a \), then necessarily \( \text{span} A \subseteq a\mathbb{Z} \). Hence, \( \dim \text{span} A \leq 1 \). On the other hand, \( A \neq \{0\} \) and \( A \) is non-empty, hence \( \dim \text{span} A \geq 1 \). So \( \dim \text{span} A = 1 \). If \( \dim \text{span} A = 1 \), then there exists \( a \in \mathbb{Z}^d \setminus \{0\} \) such that \( \text{span} A = a\mathbb{Z} \). Clearly, \( a|b \), \( b \in \text{span} A \), in particular \( a|b \), \( b \in A \). Hence, \( A \) has a common divisor. Since \( A \neq \{0\} \) and \( A \) is non-empty, then the set of common divisors is finite \( \{a_1, \ldots, a_k\} \). We will show by contradiction that \( A \) has a gcd. Assume \( a_i \leq a_k \) for \( 1 \leq i \leq k \) and that \( A \) does not have a gcd. Then \( \frac{a_i}{a_k} = \frac{q}{p} \), \( p, q \in \mathbb{Z} \), for some \( i \), such that the fraction is irreducible. Since \( a_i|b \) and \( a_k|b \) for \( b \in A \), then it must be that also \( pa_k|b \in \mathbb{Z} \), contradicting that \( a_k \) is the largest common divisor. Hence \( A \) has a gcd. \( \square \)
S3. A property on minimal set

**Proposition S3.1.** Let $B$ be a non-empty subset of $\mathbb{N}_0^d$. Then $B$ has a finite non-empty minimal set $E$, and $B \subseteq E_\geq$.

**Proof.** We first prove the former part of the conclusion. That $E \neq \emptyset$ follows directly from Zorn’s lemma. In the following we show $\#E < \infty$. Suppose $\#E = \infty$. Then there exists $k \in [1, d_1]$ and a sequence $(x^{(j)})_{j \geq 1}$ such that $\{x^{(j)}_k\}$ is unbounded. Assume w.o.l.g. $k = 1$ and $x^{(j)}_1 \uparrow \infty$ as $j \to \infty$. If the remaining sets $\{x^{(j)}_i, i \in [2, d_1]\}$ are bounded, say $x^{(j)}_i \leq M$ for $j \geq 1$, then there will be two points such that $x^{(j_1)} \geq x^{(j_2)}$, contradicting minimality. Hence, we might assume w.l.o.g. that $x^{(j)}_i \uparrow \infty$, for $i = 1, 2$, as $j \to \infty$. Continuing in this fashion yields a sequence $(x^{(j)})_{j \geq 1}$ such that $x^{(j)}_1 \leq x^{(j_1)}_1$, $j \geq 1$, contradicting minimality. Hence, $E$ is finite.

Next we prove $B \subseteq E_\geq$. Suppose there exists $x^{(1)} \in B \setminus E_\geq \subseteq B \setminus E$. By minimality, there exists $x^{(2)} \in B \setminus \{x^{(1)}\}$ such that $x^{(1)} \geq x^{(2)}$. Since $x^{(1)} \notin E_\geq, x^{(2)} \in B \setminus E_\geq$. By induction one can get a decreasing sequence $(x^{(j)})_{j \geq 1} \subseteq B \setminus E_\geq$ of distinct elements. This is impossible since $\#(x^{(j)})_{j \geq 1} \leq \prod_{k=1}^d (x^{(1)}_k + 1)$. \hfill \Box

S4. Lemmata for Theorem 3.11

The first lemma establishes a result on communicability of two states. Recall that for any $c \in \mathbb{N}_0^d$, $\dim L_c = 1$ and $L_c$ is a (finite or infinite) lattice interval. Two vectors $u, v \in \mathbb{Z}^d \setminus \{0\}$ are coprime if either $\gcd\{\{u, v\}\}$ does not exist or $\|\gcd\{\{u, v\}\}\|_\infty = 1$.

**Lemma S4.1.** Assume $\omega^{(1)} = -m_1 \omega_* \in \Omega_-$ and $\omega^{(2)} = m_2 \omega_* \in \Omega_+$ with coprime $m_1, m_2 \in \mathbb{N}$. Let $x \in \mathbb{N}_0^d$. If $\omega^{(1)}$ is active on $x + (m_1 - 1 + j)\omega_*$ for all $j \in [1, m_2]$ and $\omega^{(2)}$ is active on $x + (j - 1)\omega_*$ for all $j \in [1, m_1]$, then $\omega_*[0, m_1 + m_2 - 1] + x$ is communicable. In particular, if $\omega^{(1)}$ and $\omega^{(2)}$ are both active on $x$ and $x + (m_1 + m_2 - 1)\omega_*$, then $[0, M]\omega_* + x$ is communicable for all integers $M \geq m_1 + m_2 - 1$.

**Proof.** Assume w.o.l.g. that $x = 0$ and $\omega_* = 1$. It then suffices to show that $[0, m_1 + m_2 - 1]$ is communicable. We first show $[0, \max\{m_1, m_2\} - 1]$ is communicable, and then $[0, m_1 + m_2 - 1]$ is communicable.

**Step I.** $[0, \max\{m_1, m_2\} - 1]$ is communicable. To see this, we first show that $[0, m_2 - 1]$ is communicable. Then in an analogous way, one can show $[0, m_1 - 1]$ is also communicable. This implies that $[0, \max\{m_1, m_2\} - 1]$ is communicable.

Indeed, there exists a cycle connecting all states in $[0, m_2 - 1]$, which immediately yields that $[0, m_2 - 1]$ is communicable. To prove this, one needs the following elementary identity,

(S4.1) \[
\left\{ \left\lfloor \frac{jm_1}{m_2} \right\rfloor m_2 - jm_1 : j \in \mathbb{N} \right\} = \left\{ \left\lfloor \frac{jm_1}{m_2} \right\rfloor m_2 - jm_1 : j \in [1, m_2] \right\} = [0, m_2 - 1],
\]

(a proof is included below) from which it immediately follows that

\[
0 \leq \left\lfloor \frac{jm_1}{m_2} \right\rfloor m_2 - jm_1 \leq \left\lfloor \frac{jm_1}{m_2} \right\rfloor m_2 - (j - 1)m_1 \leq m_1 + m_2 - 1,
\]

which further yields the desired cycle by repeated jumps of $\omega^{(1)}$ and $\omega^{(2)}$, connecting the states in $\left\{ \left\lfloor \frac{jm_1}{m_2} \right\rfloor m_2 - jm_1 : j \in [1, m_2] \right\}$. 


\[ 0 \rightarrow \left[ \frac{m_1}{m_2} \right] m_2 \rightarrow \left[ \frac{m_1}{m_2} \right] m_2 - m_1 \rightarrow \left[ \frac{2m_1}{m_2} \right] m_2 - m_1 \rightarrow \left[ \frac{2m_1}{m_2} \right] m_2 - 2m_1 \]
\[ \rightarrow \cdots \rightarrow \left[ \frac{jm_1}{m_2} \right] m_2 - (j - 1)m_1 \rightarrow \left[ \frac{jm_1}{m_2} \right] m_2 - jm_1 \]
\[ \rightarrow \cdots \rightarrow \left[ \frac{(m_2 - 1)m_1}{m_2} \right] m_2 - (m_2 - 2)m_1 \rightarrow \left[ \frac{(m_2 - 1)m_1}{m_2} \right] m_2 - (m_2 - 1)m_1 \]
\[ \rightarrow \left[ \frac{m_2m_1}{m_2} \right] m_2 - (m_2 - 1)m_1 \rightarrow \left[ \frac{m_2m_1}{m_2} \right] m_2 - m_2m_1 (= 0), \]
connecting all states \([0, m_2 - 1]_1\), which must be communicable.

For the reader’s convenience, we here give a proof of the elementary identity (S4.1).

On the one hand,
\[
0 \leq \left[ \frac{jm_1}{m_2} \right] m_2 - jm_1 < m_2, \quad \text{for all } j \in \mathbb{N},
\]
which yields
\[(S4.2) \quad \left\{ \left[ \frac{jm_1}{m_2} \right] m_2 - jm_1 : j \in \mathbb{N} \right\} \subseteq [0, m_2 - 1]_1.\]

On the other hand, for all \(1 \leq i < j \leq m_2\),
\[(S4.3) \quad \left[ \frac{im_1}{m_2} \right] m_2 - im_1 \neq \left[ \frac{jm_1}{m_2} \right] m_2 - jm_1.\]

If this is not so, then
\[
(j - i)m_1 = \left( \left[ \frac{jm_1}{m_2} \right] m_2 - \left[ \frac{im_1}{m_2} \right] m_2 \right),\]
for some \(i < j\). Since \(0 < j - i < m_2\), and \(m_1, m_2\) are coprime, \(m_2 \nmid (j - i)m_1\), while
\(m_2 \mid \left( \left[ \frac{im_1}{m_2} \right] - \left[ \frac{im_1}{m_2} \right] \right) m_2\). This is an obvious contradiction.

From (S4.3), it follows that
\[
\# \left\{ \left[ \frac{jm_1}{m_2} \right] m_2 - jm_1 : j \in [1, m_2]_1 \right\} \geq m_2,
\]
which together with (S4.2) yields (S4.1).

Analogous to (S4.1), one can show that
\[
\left\{ - \left[ \frac{jm_2}{m_1} \right] m_1 + jm_2 : j \in \mathbb{N} \right\} = \left\{ - \left[ \frac{jm_2}{m_1} \right] m_1 + jm_2 : j \in [1, m_1]_1 \right\}
= \{-j : j \in [0, m_1 - 1]_1\},
\]
from which it follows that \([0, m_1 - 1]_1\) is communicable, since it is also within a cycle. Hence \([0, \max\{m_1, m_2\} - 1]_1\) is communicable.

**Step II.** Then \([0, m_1 + m_2 - 1]_1\) is communicable, since
\[
x \mod m_1, y \mod m_2 \in [0, \max\{m_1, m_2\} - 1]_1
\]
for \(x, y \in [0, m_1 + m_2 - 1]_1\), and
\[
x = (x \mod m_1) + \left[ \frac{x}{m_1} \right] m_1, \quad y = (y \mod m_2) + \left[ \frac{y}{m_2} \right] m_2.
\]
Then, similarly to Step I, a path from \(x\) to \(y\) is constructed. The proof is complete. \(\square\)
The next lemma shows when two states are not accessible from each other.

**Lemma S4.3.** Let $c \in \mathbb{N}_0^d$. For any $x, y \in L_c$, if $x - y \notin \omega_* \mathbb{Z}$, then $x \not\sim y$ and $y \not\sim x$.

**Proof.** We prove $x \not\sim y$ by contradiction. By symmetry, $y \not\sim x$. Suppose $x \rightarrow y$, then there exists a path from $x$ to $y$ realized by a finite ordered set of (possibly repeated) jumps $(\omega^{(j)})_{1 \leq j \leq m}$ for some $m \in \mathbb{N}$ such that $y - x = \sum_{j=1}^{m} \omega^{(j)}$. Since $\omega^{(j)} \in \omega_* \mathbb{Z}$, then $y - x \in \omega_* \mathbb{Z}$, which is a contradiction. \[\square\]

The last lemma guarantees connectedness of a discrete set.

**Lemma S4.3.** Let $U = \{u^{(j)}\}_{j \in [1,m]} \subseteq \mathbb{Z}^d \setminus \{0\}$ for some $m \in \mathbb{N}$ and define $\pi$ by

$$\pi_c = \max_{1 \leq j \leq m} u^{(j)}_i, \quad \forall i \in [1,d].$$

Then for every $c \geq \pi$, $L_c \cap U_\geq$ is a non-empty lattice interval.

**Proof.** Recall that if $A, B \subseteq \mathbb{L}_c$ are both lattice intervals, and $A \cap B \neq \emptyset$, then $A \cap B$ and $A \cup B$ are also lattice intervals. We use this property to prove the conclusion. Indeed,

$$U_\geq = \bigcup_{1 \leq j \leq m} u^{(j)}_\geq.$$

If $c \geq \pi$, then $c \in L_c \cap u^{(j)}_\geq$ for all $j \in [1,d]$. Moreover, $L_c \cap u^{(j)}_\leq$ is a lattice interval, since $L_c$ is a lattice interval, and $u^{(j)}_\leq$ is a convex set in $\mathbb{N}_0^d$. Hence $L_c \cap U_\geq = \bigcup_{1 \leq j \leq m} (L_c \cap u^{(j)}_\geq)$ is also a (non-empty) lattice interval containing $c$. \[\square\]

**S5. Propositions used in the proofs of results in Section 4**

Let $Y_t \in \mathcal{C}$ be a CTMC with state space $\mathcal{Y} \subseteq \mathbb{N}_0$ and transition operator $Q$. Let $(\tilde{Y}_n)_{n \in \mathbb{N}_0}$ be the embedded DTMC of $Y_t$.

Let $\mathcal{F}$ be the set of all nonnegative (Borel) measurable functions on $\mathcal{Y}$. The associated infinitesimal generator, also denoted by $Q$, is

$$Qf(x) = \sum_{\omega \in \Omega} \lambda_\omega(x) (f(x + \omega) - f(x)), \quad \forall x \in \mathcal{Y}, \quad f \in \mathcal{F}.$$

Let $\text{Dom} = \{f \in \mathcal{F}: \sum_{\omega \in \Omega} \lambda_\omega(x)|f(x + \omega)| < +\infty, \forall x \in \mathcal{Y}\}$ be the domain of $Q$.

**S5.1. Criteria for explosivity and non-explosivity.**

**Proposition S5.1.** [36, Theorem 1.12, Remark 1.13] Assume $Y_t$ is irreducible. Suppose that there exists a triple $(\epsilon, A, f)$ with $\epsilon > 0$, $A$ a proper finite subset of $\mathcal{Y}$ such that $A^c$ is infinite, and $f \in \text{Dom}$ such that

(i) there exists $x_0 \in A^c$ with $f(x_0) < \min_{A} f$,

(ii) $Qf(x) \leq -\epsilon$ for all $x \in A^c$.

Then the expected explosion time fulfills $E_x(\zeta) < +\infty$ for all $x \in \mathcal{Y}$.

**Proposition S5.2.** [36, Theorem 1.14] Assume $Y_t$ is irreducible. Let $f \in \text{Dom}$. If

(i) $\lim_{x \to \infty} f(x) = +\infty$,

(ii) there exists a non-decreasing function $g: [0, \infty[ \rightarrow [0, \infty]$ such that $G(z) = \int_0^z g(\theta) d\theta < +\infty$ for all $z \geq 0$ but $\lim_{z \to \infty} G(z) = +\infty$,

(iii) $Qf(x) \leq g(f(x))$ for all $x \in \mathcal{Y}$,

then $P_x(\zeta = +\infty) = 1$ for all $x \in \mathcal{Y}$. 

S5.2. Criteria for existence and non-existence of moments of passage times.

The following condition provides sufficient conditions for existence and non-existence of moments of passage times.

**Proposition S5.3.** [36, Theorem 1.5] Let \( f \in \text{Dom} \) be such that \( \lim_{x \to \infty} f(x) = +\infty \).

(i) If there exist constants \( c_1, c_2 > 0 \), and \( \sigma > 0 \) such that \( f^\sigma \in \text{Dom} \) and

\[
Qf^\sigma(x) \leq -c_2 f^{\sigma-2}(x), \quad \forall x \in \{ f > c_1 \},
\]

then \( \mathbb{E}_x \left( \tau_{f \leq c_1} \right) < +\infty \) for all \( 0 < \epsilon < \sigma/2 \) and all \( x \in \mathcal{X} \).

(ii) Let \( g \) be a non-negative measurable function on \( \mathcal{X} \). If there exist

(iii-1) a constant \( c_1 > 0 \) such that \( f \leq c_1 g \),

(ii-2) constants \( c_2, c_3 > 0 \) such that \( Qg(x) \geq -c_3 \) for all \( g(x) > c_2 \),

(iii-3) constants \( c_4 > 0 \) and \( \delta > 1 \) such that \( g^\delta \in \text{Dom} \) and \( Qg^\delta(x) \leq c_4 g^{\delta-1}(x) \)

for \( g(x) > c_2 \), and

(ii-4) a constant \( \sigma > 0 \) such that \( f^\sigma \in \text{Dom} \) and \( Qf^\sigma(x) \geq 0 \) for all \( f(x) > c_1 c_2 \),

then \( \mathbb{E}_x \left( \tau_{f \leq c_1} \right) = +\infty \) for all \( \epsilon > \sigma \) and all \( f(x) > c_2 \).

S5.3. Criteria for recurrence and transience. Let \( m_1(\cdot) \) (\( i = 1, 2 \)) be defined in the proof of Theorem 4.3. From [38, Corollary 2.1.10] (see also [15, p.3032]), [42, Theorem 3.4.1] and [32, Theorem 3.2], we have the following criterion on recurrence and transience of an irreducible CTMC.

**Proposition S5.4.** Assume \( Y_t \) is irreducible, \( \liminf_{x \to \infty} m_2(x) > 0 \), and \( \sum_{\omega \in \Omega} \lambda_\omega(x) \eta^{1+\epsilon_0} < +\infty \) for some \( \epsilon_0 > 0 \). Then \( Y_t \) is recurrent if \( m_1(x) \leq m_2(x)/(2x) \) for all large \( x \) while transient if \( m_1(x) > \theta m_2(x)/(2x) \) for all large \( x \), with some constant \( \theta > 1 \).

S5.4. Criteria for positive recurrence and ergodicity of SDs/QSDs.

**Proposition S5.5.** [39, Theorem 7.1] Assume \( Y_t \) is irreducible. Then \( Y_t \) is positive recurrent and there exists an exponentially ergodic stationary distribution if there exists a triple \( (\epsilon, A, f) \) with \( \epsilon > 0 \), \( A \) a finite subset of \( \mathcal{Y} \) and \( f \in \text{Dom} \) with \( \lim_{x \to \infty} f(x) = \infty \) verifying \( Qf(x) \leq -\epsilon f(x) \) for all \( x \notin A \).

**Proposition S5.6.** [36, Theorem 1.7] Assume \( Y_t \) is irreducible and recurrent. Then the following are equivalent:

(i) \( Y_t \) is positive recurrent.

(ii) There exists a triple \( (\epsilon, A, f) \), with \( \epsilon > 0 \), \( A \) a finite non-empty subset of \( \mathcal{X} \) and \( f \in \text{Dom} \) verifying \( Qf(x) \leq -\epsilon f(x) \) for all \( x \notin A \).

**Proposition S5.7.** [13, Theorem 5.1, Remark 11], [25, Theorem 2.1] Let \( Y_t \) be a CTMC with transition matrix \( Q = (q_{x,y}) \) on \( \mathcal{Y} \) absorbed at the absorbing set \( \partial \), and let \( \partial^c = \mathcal{Y} \setminus \partial \). Then there exists a finite subset \( D \subseteq \partial^c \) such that \( P_x(X_1 = y) > 0 \) for all \( x, y \in D \), so that the constant

\[
\psi_0 := \inf \{ \psi \in \mathbb{R} : \liminf_{t \to \infty} e^{\psi t} P_x(X_t = x) > 0 \}
\]

is finite and independent of \( x \in D \). If in addition there exists \( \psi_1 > \max \{ \psi_0, \sup_{x \in \partial^c} \sum_{z \in \partial} |q_{x,z}| \} \), a function \( f : \mathcal{Y} \to \mathbb{R}_{\geq 0} \) such that \( f|_{\partial^c} \geq 1, f|_{\partial^c} = 0, \sup_{x \in \partial^c} f < \infty, \sum_{y \in \partial^c \setminus \{x\}} q_{x,y} f(y) < \infty \) for all \( x \in \partial^c \) and

\[
Qf(x) \leq -\psi_1 f(x), \quad \forall x \in \mathcal{Y} \setminus D,
\]
then there exists a unique QSD $\nu$ with constants $\delta \in (0,1)$ and $C \in \mathbb{R}_{>0}$ such that for all Borel probability measures $\mu$ on $\partial \mathcal{C}$,

$$\left\| \mathbb{P}_\mu(Y_t \in \cdot | t < \tau_0) - \nu \right\|_{TV} \leq C \delta^t, \quad \forall t \geq 0.$$ 

In addition, $\eta(y)\nu(dy)$ is the unique quasi-ergodic distribution for $Y_t$, and the unique stationary distribution of the $Q$-process, where the nonnegative measurable function

$$\eta(x) = \lim_{t \to \infty} e^{\zeta \nu \tau}(t < \tau_0), \quad x \in \partial \mathcal{C}.$$ 

S5.5. Criterion for implosivity and non-implosivity.

**Proposition S5.8.** Let $Y_t \in \mathcal{C}$ be a non-absorbed CTMC on $\mathcal{Y}$. If there exists a non-empty proper subset $B \subset \mathcal{Y}$ such that $Y_t$ implodes towards $B$, then $Y_t$ is implosive.

**Proof.** Assume w.o.l.g. that $\omega_* = 1$. Since $Y_t$ is non-absorbed, $P \subseteq \mathcal{Y} \subseteq P \cup E$. According to [36, Proposition 2.14], the conclusion holds for an irreducible CTMC (when $E = \emptyset$). Assume $E \neq \emptyset$. Since $Y_t$ implodes towards $B$, $B \cap P \neq \emptyset$, simply because there is zero probability to jump from any state in $P$ to any state in $E$, which ends in an infinite expected passage time. Hence assume w.o.l.g. that $B \subseteq P$ (otherwise choose $B \cap P$ since $\tau_B^+ = \tau_B^+P$ for $\mathbb{P}(\tau_B^+P = \infty) = 1$ for all $x \in P$). Since $Y_t$ is irreducible provided $Y_0 \in P$, by [36, Proposition 2.14], for every $x \in P$, there exists $t_x < \infty$ such that

$$E_y(\tau_x^+) < t_x, \quad \forall y \in P.$$ 

Since $\#E < \infty$, $\sup_{Y_t \in E} \tau_x^+ < \infty$ a.s. Moreover there exists $t_* < \infty$ such that $\sup_{x \in E} E_x(\tau_x^+) < t_*$, since the holding time is exponentially distributed. This shows that

$$E_y(\tau_x^+) < t_x + t_*, \quad \forall y \in \mathcal{Y} \setminus \{x\}. \quad \square$$

A criterion for implosivity and non-implosivity follows from Proposition S5.8 and [36, Theorem 1.15, Proposition 1.16].

**Proposition S5.9.** Assume $Y_t$ is irreducible and recurrent.

(i) The following are equivalent:

(i-1) There exists a triple $(\epsilon, F, f)$ with $\epsilon > 0$, $F$ a finite set and $f \in \text{Dom}$ such that $\sup_{x \in \mathcal{Y}} f(x) < +\infty$ and $Qf(x) \leq -\epsilon$ whenever $x \in \mathcal{Y} \setminus F$.

(i-2) There exists $c > 0$, and for every finite $A \subseteq \mathcal{Y}$, there exists a positive and finite constant $C = C_A$ such that $E_x(\tau_A) \leq C$ and $E_x(\exp(c\tau_A)) < \infty$ whenever $x \in \mathcal{Y} \setminus A$. (Hence there is implosion towards $A$ and subseqeuntly the chain is implosive.)

(ii) Let $f \in \text{Dom}$ be such that $\lim_{x \to \infty} f(x) = +\infty$ and assume there exist positive constants $a$, $c$, $\epsilon$ and $\delta > 1$ such that $f^\delta \in \text{Dom}$. If further

(ii-1) $Qf(x) \geq -\epsilon$ whenever $x \in \{f > a\}$, and

(ii-2) $Qf^\delta(x) \leq cf^{\delta-1}(x)$ whenever $x \in \{f > a\}$.

Then the chain does not implode towards $\{f \leq a\}$.
S6. A lemma for Theorem 5.8

Let $\alpha_l$ and $\gamma_l$ be as in the proof of Theorem 5.8.

Lemma S6.1.  

(i) $\alpha_{-1} \geq \alpha_{-2} \geq \cdots \geq \alpha_{-\omega_*/\omega_*}$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\omega_*/\omega_*}$.  

(ii) For $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$,

$$\lambda_{k\omega_*}(x-\ell \omega_*) = (\alpha_k - \alpha_{k+1})x^{R_+} + (\gamma_k - \gamma_{k+1} - R_+ \ell \omega_*(\alpha_k - \alpha_{k+1}))x^{R_+-1} + O(x^{R_+ - 2}).$$

For $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$,

$$\lambda_{-k\omega_*}(x-\ell \omega_*) = (\alpha_{-k} - \alpha_{-(k+1)})x^{R_-} + (\gamma_{-k} - \gamma_{-(k+1)} - R_- \ell \omega_* (\alpha_{-k} - \alpha_{-(k+1)}))x^{R_- - 1} + O(x^{R_- - 2}).$$

(iii) It holds that

$$\alpha_+ = \omega_* \sum_{\ell=1}^{\omega_*/\omega_*} \alpha_{\ell}, \quad \alpha_- = \omega_* \sum_{\ell=1}^{\omega_*/\omega_*} \alpha_{-\ell}, \quad \gamma_+ = \omega_* \sum_{\ell=1}^{\omega_*/\omega_*} \gamma_{\ell}, \quad \gamma_- = \omega_* \sum_{\ell=1}^{\omega_*/\omega_*} \gamma_{-\ell},$$

In particular, $\alpha_+^{*} = \alpha_{+1} > 0$.

(iv) If $\alpha = 0$, then

$$\sum_{k=-\omega_*/\omega_*}^{\omega_*/\omega_*} \omega_* |k| \alpha_k = \frac{1}{2} \alpha_+^{*} \omega_* - \theta.$$  

Proof. Assume w.o.l.g. that $\omega_* = 1$. We only prove the “+” cases. Analogous arguments apply to “−” cases.

(i)-(ii). The first two properties follow directly from the definitions.

(iii) By Fubini’s theorem,

$$\sum_{k=1}^{\omega_*/\omega_*} k \lambda_k(x) = \sum_{k=1}^{\omega_*/\omega_*} \sum_{l \geq k} \lambda_k(x) = \sum_{1 \leq k} k \lambda_k(x) = \sum_{\omega \in \Omega_+} \lambda_\omega(x) \omega,$$

comparing the coefficients before the highest degree of the polynomials on both sides, and the definition of $\alpha_\ell$ as well as $\alpha_+$ yields

$$\alpha_+ = \sum_{1 \leq \ell \leq \omega_+} \alpha_\ell.$$

(iv) Due to Fubini’s theorem again,

$$\sum_{k=1}^{\omega_*/\omega_*} k \lambda_k(x) = \sum_{k=1}^{\omega_*/\omega_*} \frac{k(k+1)}{2} \lambda_k(x).$$

Since $\alpha_- = \alpha_+$ and $R_+ = R$, it yields that

$$\sum_{k=1}^{\omega_*/\omega_*} k \alpha_k = \frac{1}{2} \left( \alpha_+ + \lim_{x \to \infty} \frac{\sum_{\omega \in \Omega_+} \lambda_\omega \omega^2}{x^{R_+}} \right).$$

Then the conclusion follows from the definition of $\theta$.  

$\Box$
S7. Asymptotics of stationary distributions

S7.1. Proof of Proposition 5.4. Since \#\text{supp } \pi = \infty, we have \( P \neq \emptyset \), i.e., \( i - o < \omega_* \). Otherwise, there only exists degenerate stationary distributions with probability concentrated on \( T \). Hence there exists \( k \in \Sigma^- \) such that \( \text{supp } \pi \supseteq P^{(k)} \). Moreover, there exists \( \Xi \subseteq \Sigma^- \) such that \( \text{supp } \pi = \bigcup_{k \in \Xi} P^{(k)} \). Observe that the asymptotics of \( T_\pi(x) \) given in

\[
T_\pi(x) \sim \begin{cases} 
\Gamma(\omega_*^{-1}x)^{R+} \cdot \left( \frac{\alpha_+ \omega_*^{(R+ - R_-)}}{\alpha_- \omega_*^{(R_+ - R_-)}} \right)^{\omega_*^{-1}x} \cdot x \left( \frac{\gamma_+ - \gamma_-}{\alpha_+ - \alpha_-} \right) \omega_*^{-1} - R_-, & \text{if } \alpha < 0, \\
x^{2 + \beta \theta^{-1} - R}, & \text{if } \alpha = 0,
\end{cases}
\]

is independent of \( k \). Thus assume w.o.l.g. that \( \text{supp } \pi = P^{(k)} \) for some \( k \in \Sigma^- \). Let \( x_\ast = \min P^{(k)} \).

Since \( \Omega = \{ \pm \omega_* \} \), and \( \pi \) is a stationary measure on \( P^{(k)} \) for some \( k \in \Sigma^- \),

\[
\pi(x) = \pi(x_\ast^{(k)}) \prod_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} \frac{\lambda_{\omega_*}(j - 1)\omega_* + x_\ast^{(k)}}{\lambda_{-\omega_*}(j\omega_* + x_\ast^{(k)})}, \quad \forall x \in P^{(k)}.
\]

By the Stirling formula,

\[
(x-x_\ast^{(k)})\omega_*^{-1} \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} \log \lambda_{\omega_*}(j - 1)\omega_* + x_\ast^{(k)})
\]

\[
= \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} \log((\alpha_+ (j - 1)\omega_* + x_\ast^{(k)})R_+ + \gamma_+ (j - 1)\omega_* + x_\ast^{(k)})^{R_+ - 1} + O(j^{R_+ - 2}))
\]

\[
= \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} \log(\alpha_+ \omega_*^{R_+}) + R_+ \log((j - 1) + x_\ast^{(k)}\omega_*^{-1}) + \log \left( 1 + \frac{\gamma_+}{\alpha_+} j\omega_* \right) + O(j^{-2})
\]

\[
= (x - x_\ast^{(k)})\omega_*^{-1} \log(\alpha_+ \omega_*^{R_+}) + R_+ \log \left( \frac{\Gamma(x \cdot \omega_*^{-1})}{\Gamma(x_\ast^{(k)}\omega_*^{-1})} + \gamma_+ (\alpha_+ \omega_*)^{-1} \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} j^{-1} \right)
\]

\[
+ O \left( \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} j^{-2} \right)
\]

\[
= R_+ \log \Gamma(x \omega_*^{-1}) + x \omega_*^{-1} \log \alpha_+ \omega_*^{R_+} + \frac{\gamma_+}{\alpha_+} \omega_*^{-1} \log x \omega_*^{-1} + O(1).
\]

Indeed, with more effort one can even show that for some \( C_+ \in \mathbb{R} \)

\[
(x-x_\ast^{(k)})\omega_*^{-1} \sum_{j=1}^{(x-x_\ast^{(k)})\omega_*^{-1}} \log \lambda_{-\omega_*}(j\omega_* + x_\ast^{(k)})
\]

\[
= R_- \log \Gamma(x \omega_*^{-1}) + x \omega_*^{-1} \log \alpha_+ \omega_*^{R_-} + \frac{\gamma_+}{\alpha_+} \omega_*^{-1} \log x \omega_*^{-1} + C_+ + O(x^{-1}).
\]
Analogously, for some $C_\in \mathbb{R}$,
\[
(x-x_\star^{(k)}) \omega_\star^{-1} \sum_{j=1} \log \left( \alpha_- \left( (j-1) \omega_\star + x_\star^{(k)} \right)^{R_-} + \gamma_+ \left( (j-1) \omega_\star + x_\star^{(k)} \right)^{R_-} + \text{O} \left( j^{R_- - 2} \right) \right)
= R_- \log \Gamma \left( \omega_\star^{-1} x \right) + \frac{\log \alpha_- \omega_\star^{R_-}}{\omega_\star} x + \left( R_- + \frac{\gamma_- \omega_\star^{-1}}{\alpha_-} \right) \log x \omega_\star^{-1} + C_- + \text{O}(x^{-1}).
\]
Hence for some $C \in \mathbb{R}$,
\[
\log \pi(x) = \log \pi(x_\star^{(k)}) + (x-x_\star^{(k)}) \omega_\star^{-1} \sum_{j=1} \left( \log \lambda_\star \left( (j-1) \omega_\star + x_\star^{(k)} \right) - \log \lambda_- \left( j \omega_\star + x_\star^{(k)} \right) \right)
= (R_+ - R_-) \log \Gamma \left( \omega_\star^{-1} x \right) + x \omega_\star^{-1} \log \left( \frac{\alpha_- \omega_\star^{R_-}}{\alpha_- \omega_\star} \right)
+ \left( \frac{\gamma_+ \omega_\star^{-1}}{\alpha_+} - \frac{\gamma_- \omega_\star^{-1}}{\alpha_-} - R_- \right) \log x + C + \text{O}(x^{-1}).
\]
This implies that for some positive constant $C$,
\[
(S7.2)
\pi(x) = \overline{\Gamma} \left( \omega_\star^{-1} x \right) R_+ - R_- \cdot \alpha_- \omega_\star \left( \frac{\alpha_- \omega_\star^{R_-}}{\alpha_- \omega_\star} \right) \omega_\star^{-1} x \cdot \left( \frac{\gamma_+ \omega_\star^{-1}}{\alpha_+} - \frac{\gamma_- \omega_\star^{-1}}{\alpha_-} \right) \omega_\star^{-1} \left( R_+ - R_- \right)
\cdot \left( 1 + \text{O}(x^{-1}) \right).
\]
In the following, we show (S7.1) case by case.
(i) $R_- > R_+$. For all large $x \in \mathbb{P}^{(k)}$,
\[
\frac{\pi(x + \omega_\star)}{\pi(x)} \leq 2(x \omega_\star^{-1})^R_+ - R_- \cdot \frac{\alpha_- \omega_\star^{R_-}}{\alpha_-}.
\]
Hence
\[
T_\pi(x + \omega_\star) = \sum_{y \in \omega_\star \cdot \mathbb{N} + x} \pi(y) = \pi(x) \sum_{j=1}^{\infty} \prod_{l=1}^{j} \frac{\pi(x + l \omega_\star)}{\pi(x + (l-1) \omega_\star)}
\leq \pi(x) \sum_{j=1}^{\infty} \prod_{l=1}^{j} \left( 2(x \omega_\star^{-1} + l - 1)^{R_+ - R_-} \frac{\alpha_- \omega_\star^{R_-}}{\alpha_-} \right)
\leq \pi(x) \sum_{j=1}^{\infty} \left( 2 \frac{\alpha_- \omega_\star^{R_-}}{\alpha_-} \right)^j \left( \frac{\Gamma(x \omega_\star^{-1} + j)}{\Gamma(x \omega_\star^{-1})} \right)^{R_+ - R_-}
\leq \pi(x) \sum_{j=1}^{\infty} \left( 2 \frac{\alpha_- \omega_\star^{R_-}}{\alpha_-} \right)^j \left( x \omega_\star^{-1} \right)^j
= \pi(x) \sum_{j=1}^{\infty} \left( 2 \frac{\alpha_- \omega_\star^{R_-}}{\alpha_-} \right)^j
\leq \pi(x) \frac{2 \alpha_- \omega_\star^{R_-}}{\alpha_-} \sum_{j=1}^{\infty} 2^{-(j-1)} = \frac{4 \alpha_- \omega_\star^{R_-}}{\alpha_-} = \frac{\alpha_- x R_+ - R_- \pi(x)}{\pi(x)},
\]
which implies that
\[
\lim_{x \to \infty} \frac{T_\pi(x)}{\pi(x)} = 1;
\]
by (S7.2),
\[ T_\pi(x) \sim \Gamma(\omega_*^{-1}x) R_+ R_- \left( \frac{\alpha_+}{\alpha_-} \omega_* R_+ - R_- \right) \omega_*^{-1}x \cdot x \left( \frac{2\alpha_+ - \gamma_-}{\alpha_+} \right) \omega_*^{-1}R_- . \]

(ii) \( R_- = R_+ \) and \( \alpha < 0 \). Since \( \alpha_- > \alpha_+ \), for all large \( x \in \mathcal{P}(k) \),
\[ \frac{\pi(x + \omega_*)}{\pi(x)} \leq \frac{\alpha_+}{\alpha_-} (1 + x\omega_*^{-1}) \left( \frac{\gamma_+}{\alpha_+} - \frac{\gamma_-}{\alpha_-} \right) \omega_*^{-1}R_- \leq \frac{\alpha_+}{\alpha_-} \left( 1 + \frac{\alpha_- - \alpha_+}{\alpha_- + \alpha_+} \right) = \frac{2\alpha_+}{\alpha_- + \alpha_+} < 1, \]
which implies that
\[ T_\pi(x) \leq \pi(x) \sum_{j=0}^{\infty} \left( \frac{2\alpha_+}{\alpha_- + \alpha_+} \right)^j = \pi(x) \frac{1}{1 - \frac{2\alpha_+}{\alpha_- + \alpha_+}} = \frac{\alpha_- + \alpha_+}{\alpha_- - \alpha_+} \pi(x), \]
which further yields
\[ \limsup_{x \to \infty} \frac{T_\pi(x)}{\pi(x)} < \infty. \]
Analogously, one can show that
\[ \liminf_{x \to \infty} \frac{T_\pi(x)}{\pi(x)} > 0. \]
Hence
\[ T_\pi(x) \sim \left( \frac{\alpha_+}{\alpha_-} \right) \omega_*^{-1}x \cdot x \left( \frac{\gamma_+}{\alpha_+} - \frac{\gamma_-}{\alpha_-} \right) \omega_*^{-1}R_- . \]

(iii) \( \alpha = 0 \). Recall that \( R_- = R \) and \( \gamma = \gamma_+ - \gamma_- \). From (S7.2) it follows that
\[ \pi(x) = C x^{\gamma (\alpha_+ \omega_*)^{-1} - R} (1 + O(x^{-1})), \]
which yields that \( \sum_{x \in \mathcal{P}(k)} \pi(x) < \infty \) if and only if
\[ \gamma (\alpha_+ \omega_*)^{-1} < R - 1. \]
This results in
\[ T_\pi(x) = \sum_{y \in \omega_* N_0 + x} \pi(y) \leq 2C \int_{x-\omega_*}^{\infty} y^{\gamma (\alpha_+ \omega_*)^{-1} - R} dy \]
\[ = \frac{2C}{\gamma (\alpha_+ \omega_*)^{-1} - R + 1} (x - \omega_*)^{\gamma (\alpha_+ \omega_*)^{-1} - R + 1}, \]
which implies that
\[ \limsup_{x \to \infty} T_\pi(x) x^{R - \gamma (\alpha_+ \omega_*)^{-1} - 1} < \infty. \]
Analogously, one can show that
\[ \liminf_{x \to \infty} T_\pi(x) x^{R - \gamma (\alpha_+ \omega_*)^{-1} - 1} > 0. \]
Hence
\[ T_\pi(x) \sim x^{\gamma (\alpha_+ \omega_*)^{-1} + 1 - R}. \]
Since \( \vartheta = \alpha_+ \omega_* \), and \( \gamma = \beta + \vartheta \), the conclusion is obtained.
S7.2. **Proof of Proposition 5.5.** We apply [15, Theorem 1] to obtain asymptotics for tails of the stationary distribution, together with the relation between stationary distributions of CTMC and those of its embedded chain [42, Theorem 3.5.1] (see also [40, Theorem 3]).

Assume w.o.l.g. that \( \omega_* = 1 \). Hence supp \( \pi = \mathbb{P} \).

Let \( \tilde{\pi}(x) = \pi(x) \sum_{\omega \in \Omega} \lambda_\omega(x), x \in \mathbb{N}_0 \). By [42, Theorem 3.4.1, Theorem 3.5.1] (see also [40, Theorem 3]), \( \tilde{\pi} \) is a stationary distribution of the embedded chain.

We first obtain the asymptotics of \( \tilde{\pi}(x) \). Under the assumptions in Theorem 5.5, it is easy to verify that all assumptions in [15, Theorem 1] are satisfied. Since \( \alpha = 0 \) implies \( R_+ = R_- = R \), the \( i \)-th moments of jumps of the embedded chain is given by:

\[
m_i(x) = \frac{\sum_{\omega \in \Omega} \lambda_\omega(x) \omega^i}{\sum_{\omega \in \Omega} \lambda_\omega(x)} = \begin{cases} \frac{\gamma x^{R-1+O(x^{R-2})}}{(\alpha_+^* + \alpha_-^*)x^{R+O(x^{R-1})}}, & \text{if } i = 1, \\ \frac{2R \lambda_{\omega_1}(x) + O(x^{R-1})}{(\alpha_+^* + \alpha_-^*)x^{R+O(x^{R-1})}}, & \text{if } i = 2, \end{cases} \forall x \in \mathbb{P}.
\]

Hence

\[
h(x) = \frac{2m_1(x)}{m_2(x)} = \frac{2 \sum_{\omega \in \Omega} \lambda_\omega(x) \omega}{\sum_{\omega \in \Omega} \lambda_\omega(x)} = \frac{\gamma}{\vartheta^2} x^{-1} + O\left(x^{-2}\right),
\]

which implies that

\[
\int_1^x h(y) dy = \frac{\gamma}{\vartheta} \log x + O(x^{-1}).
\]

By [15, Theorem 1],

\[
T_{\tilde{\pi}}(x + 1) = C' x \exp\left(-\int_1^x h(y) dy\right) = C' x^{1-\frac{\gamma}{\vartheta}} \theta x^{-1},
\]

for some \( C', \theta > 0 \). It is straightforward to verify that

\[
\tilde{\pi}(x) = T_{\tilde{\pi}}(x) - T_{\tilde{\pi}}(x + 1) = C' \left(\frac{\gamma}{\vartheta} - 1\right) x^{\frac{\gamma}{\vartheta}} \left(1 + O\left(x^{-1}\right)\right).
\]

Hence

\[
\pi(x) = \frac{\tilde{\pi}(x)}{\sum_{\omega \in \Omega} \lambda_\omega(x)} = \frac{C' \left(\frac{\gamma}{\vartheta} - 1\right)}{\alpha_+^* + \alpha_-^*} x^{\frac{\gamma}{\vartheta}} \left(1 + O\left(x^{-1}\right)\right).
\]

Applying similar arguments as in the proof of Proposition 5.4 as well as \( \gamma = \beta + \vartheta \),

\[
T_x(x) \sim \int_x^\infty y^{\frac{\gamma}{\vartheta} - R} dy \sim x^{1+\frac{\gamma}{\vartheta} - R} = x^{2+\frac{\beta}{\vartheta} - R}.
\]

S7.3. **Remaining proofs of Theorem 5.8.** Here we provide the proofs of cases (ii) and (iii) in Theorem 5.8.

(ii) \( R_- = R_+ \) and \( \alpha_- > \alpha_+ \). Since

\[
\frac{1}{\omega_*} \log \left(\max\left\{ \frac{\alpha_-^*}{\alpha_+^* + \alpha_-^*}, \frac{\alpha_+}{\alpha_+^* + \alpha_-} \right\} \right) \leq \liminf_{x \to \infty} \frac{\log T_\pi(x)}{x},
\]

\[
\limsup_{x \to \infty} \frac{\log T_\pi(x)}{x} \leq \frac{1}{\omega_+ + \omega_- - \omega_*} \log \frac{\alpha_+}{\alpha_-},
\]

is a direct consequence of

\[
\lim_{x \to \infty} T_\pi(x) \left(\min\left\{ \frac{\alpha_-^* + \alpha_+^*}{\alpha_+^* + \alpha_-^*}, \frac{\alpha_+}{\alpha_+^* + \alpha_-^*} \right\} \right)^{-\omega_*^{-1} x} x^{\frac{\beta}{\omega_+ + \omega_- - \omega_*} + 1} = +\infty,
\]

\[
\lim_{x \to \infty} T_\pi(x) \left(\frac{\alpha_+}{\alpha_+^* + \alpha_-^*} \right)^{(\omega_+ + \omega_- - \omega_*)^{-1} x} x^{\frac{\beta}{\omega_+ + \omega_- - \omega_*}} = 0,
\]
Recall that

$$f_\ell(x) = x^R(\alpha_\ell + \beta_\ell(x - \ell \omega_\ell)),$$ \quad \forall \ell \in [-\omega_-/\omega_+, -1] \cup [1, \omega_+ / \omega_+]_1,$$

$$\sum_{j=1}^{\omega_-/\omega_+} \alpha_{-j} = \alpha_{-\omega_+^{-1}} > \sum_{k=1}^{\omega_+/\omega_+} \alpha_k = \alpha_{+\omega_+^{-1}},$$

and

$$\beta_\ell(x) = O(x^{-1}), \quad \forall \ell \in [-\omega_-/\omega_+, -1] \cup [1, \omega_+ / \omega_+]_1.$$

From (C.1) it follows that,

$$\sum_{j=1}^{\omega_-/\omega_+} \pi (x + (j - 1)\omega_+) \alpha_{-j} + \sum_{j=1}^{\omega_-/\omega_+} \pi (x + (j - 1)\omega_+) \beta_{-j}(x + j\omega_+)$$

$$= \sum_{k=1}^{\omega_+/\omega_+} \pi (x - k\omega_+) \alpha_k + \sum_{k=1}^{\omega_+/\omega_+} \pi (x - k\omega_+ \beta_k(x - k\omega_+).$$

Summing up from $x$ to $\infty$ yields

$$\sum_{j=1}^{\omega_-/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi (y + (j - 1)\omega_+) \alpha_{-j} + \sum_{j=1}^{\omega_-/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi (y + (j - 1)\omega_+) \beta_{-j}(y + j\omega_+)$$

$$= \sum_{k=1}^{\omega_+/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi (y - k\omega_+) \alpha_k + \sum_{k=1}^{\omega_+/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi (y - k\omega_+) \beta_k(y - k\omega_+).$$

Since every double sum in the latter equation is convergent,

$$0 = \sum_{j=1}^{\omega_-/\omega_+} \alpha_{-j} \sum_{y \in \omega_+ N_0 + x} (\pi(y) - \pi(y + (j - 1)\omega_+)) + \sum_{k=1}^{\omega_+/\omega_+} \alpha_k \sum_{y \in \omega_+ N_0 + x} (\pi(y - k\omega_+) - \pi(y))$$

$$+ \left(\sum_{k=1}^{\omega_+/\omega_+} \alpha_k - \sum_{j=1}^{\omega_-/\omega_+} \alpha_{-j}\right) \sum_{y \in \omega_+ N_0 + x} \pi(y)$$

$$+ \sum_{j=1}^{\omega_-/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi(y) \beta_{-j}(y + \omega_+) - \pi(y + (j - 1)\omega_+) \beta_{-j}(y + j\omega_+)$$

$$+ \sum_{k=1}^{\omega_+/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi(y - k\omega_+) \beta_k(y - k\omega_+) - \pi(y) \beta_k(y)$$

$$+ \sum_{k=1}^{\omega_+/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi(y) \beta_k(y) - \sum_{j=1}^{\omega_-/\omega_+} \sum_{y \in \omega_+ N_0 + x} \pi(y) \beta_{-j}(y + \omega_+).$$
which yields
\[
\sum_{j=2}^{\omega_-/\omega_s} \sum_{\ell=0}^{(j-2)\omega_s} \pi(x + \ell \omega_s) (\alpha_{-j} + \beta_{-j}(x + (\ell + 1)\omega_s)) + \sum_{k=1}^{\omega_+/\omega_s} \sum_{\ell=1}^{k} \pi(x - \ell \omega_s) (\alpha_k + \beta_k(x - \ell \omega_s))
\]
\[= (\alpha_+ - \alpha_+) \omega_s^{-1} \sum_{y \in \omega_s N_0 + x} \pi(y) + \sum_{j=1}^{\omega_-/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_{-j}(y + \omega_s) - \sum_{k=1}^{\omega_+/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_k(y).
\]
(S7.4)

Note that there exist $C, N > 0$ such that
\[
\left| \sum_{k=1}^{\omega_-/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_k(y) - \sum_{j=1}^{\omega_-/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_{-j}(y + \omega_s) \right| \leq C \sum_{y \in \omega_s N_0 + x} \pi(y) y^{-1} \leq C x^{-1} T_\pi(x), \quad \forall x \geq N.
\]

Estimate both sides of (S7.4) by further choosing larger $N$ and $C$, $\forall x \geq N$,
\[
\text{LHS} = \sum_{j=2}^{\omega_-/\omega_s} \sum_{\ell=0}^{(j-2)\omega_s} \pi(x + \ell \omega_s) (\alpha_{-j} + \beta_{-j}(x + (\ell + 1)\omega_s))
\]
\[+ \sum_{k=1}^{\omega_+/\omega_s} \sum_{\ell=1}^{k} \pi(x - \ell \omega_s) (\alpha_k + \beta_k(x - \ell \omega_s)) \geq (\alpha_+ + \omega_s^{-1} - C x^{-1}) \pi(x - \omega_s),
\]

while
\[
\text{RHS} = (\alpha_+ - \alpha_+) \sum_{y \in \omega_s N_0 + x} \pi(y) + \sum_{j=1}^{\omega_-/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_{-j}(y + \omega_s)
\]
\[+ \sum_{k=1}^{\omega_+/\omega_s} \sum_{y \in \omega_s N_0 + x} \pi(y) \beta_k(y) \leq ((\alpha_+ - \alpha_+) \omega_s^{-1} + C x^{-1}) T_\pi(x).
\]

Again by further choosing possibly larger $N$ and $C$, for all $x \geq N$,
\[
((\alpha_+ - \alpha_+) \omega_s^{-1} + C x^{-1}) T_\pi(x) \geq (\alpha_+ \omega_s^{-1} - C x^{-1}) (T_\pi(x - \omega_s) - T_\pi(x)),
\]

which implies that
\[
\frac{T_\pi(x)}{T_\pi(x - \omega_s)} \geq \frac{\alpha_+ \omega_s^{-1} - C x^{-1}}{\alpha_- \omega_s^{-1}}.
\]

Using similar arguments as in the proof of (i), one immediately obtains that there exists $\delta_+ \in \mathbb{R}$ such that the first limit in (S7.3) holds for all $\delta \in (\delta_+, +\infty)$.

Second Limit in (S7.3). From (C.2) it follows that for some constant $C > 0$,
\[
\text{LHS} \leq (\alpha_+ \omega_s^{-1} + C x^{-1}) (T_\pi(x - \omega_+) - T_\pi(x))
\]
\[+ (\alpha_- \omega_s^{-1} - \alpha^*_+ + C x^{-1}) (T_\pi(x) - T_\pi(x + \omega_- - \omega_+)) \leq (\alpha_+ \omega_s^{-1} + C x^{-1}) T_\pi(x - \omega_+) + (\alpha_- \omega_s^{-1} - \alpha_+ \omega_s^{-1} - \alpha^*_+) T_\pi(x)
\]
\[+ (\alpha_- \omega_s^{-1} - \alpha^*_+ + C x^{-1}) T_\pi(x + \omega_- - \omega_+),
\]
and
\[ \text{RHS} \geq (\alpha_{-} \omega_{-}^{-1} - \alpha_{+} \omega_{+}^{-1} - C x^{-1}) T_{\pi}(x). \]
This implies that for a possibly larger \( C \),
\[ \frac{T_{\pi}(x)}{T_{\pi}(x - (\omega_{+} + \omega_{-} - \omega_{s}))} \leq \frac{\alpha_{+} \omega_{+}^{-1} + C x^{-1}}{\alpha_{-} \omega_{-}^{-1}}, \]
The remaining arguments are the same as those in (i).
(iii) \( \alpha = 0 \). Note that
\[ 1 - ((R - 1) \vartheta - \beta) \alpha_{+} \omega_{+}^{-1} \leq \liminf_{x \to \infty} \frac{\log T_{\pi}(x)}{\log x}, \]
\[ \limsup_{x \to \infty} \frac{\log T_{\pi}(x)}{\log x} \leq 1 - (R \vartheta - \gamma) \alpha_{+} \omega_{+}^{-1}(\omega_{+} + \omega_{-} - \omega_{s})^{-1}, \]
can be deduced from
\begin{align*}
\liminf_{x \to \infty} T_{\pi}(x) (x \omega_{+}^{-1})^{(R \vartheta - \gamma) \alpha_{+} \omega_{+}^{-1} - 1} &> 0, \\
\limsup_{x \to \infty} T_{\pi}(x) (x \omega_{+}^{-1})^{(R \vartheta - \gamma) \alpha_{+} \omega_{+}^{-1}} &< \infty.
\end{align*}
(S7.5)
In the following, we only prove the latter.
First we define an auxiliary functional. For every \( \mu \in \mathcal{M}_{b} \), define its associated weighted tail distribution as
\[ S_{\mu}: \mathbb{N} \to [0, 1], \quad x \mapsto \sum_{y \in \omega_{s} \mathbb{N}_{0} + x} y^{-1} \mu(y). \]
Let \( \Delta = (R \vartheta - \gamma) \omega_{s}^{-1} \). It is easy to verify that any of the conditions (i-2)-(i-4) in
Theorem 4.4 yields \( \Delta > 0 \).
In the following, we first show
\begin{align*}
\liminf_{x \to \infty} S_{\pi}(x) (x \omega_{+}^{-1})^{-(R \vartheta - \gamma)(\alpha_{+} \omega_{+})^{-1}} &> 0, \\
\limsup_{x \to \infty} S_{\pi}(x) (x \omega_{+}^{-1})^{-(R \vartheta - \gamma)(\alpha_{+} (\omega_{+} + \omega_{-} - \omega_{s}))^{-1}} &< \infty,
\end{align*}
(S7.6)
First Limit in (S7.6). For \( x \geq 2 \omega_{s} \), from Lemma S6.1 and the definition of \( \beta_{\ell} \), let
\[ \beta_{\ell}(x) = \begin{cases} 
(\gamma_{\ell} - R \ell \omega_{s} \alpha_{\ell}) x^{-1} + \vartheta_{\ell}(x), & \text{if } \ell > 0, \\
(\gamma_{\ell} - R (\ell + 1) \omega_{s} \alpha_{\ell}) (x - \omega_{s})^{-1} + \vartheta_{\ell}(x), & \text{if } \ell < 0,
\end{cases} \]
with \( \vartheta_{\ell}(x) = O(x^{-2}) \), for all \( \ell \in [-\omega_{-} / \omega_{s}, -1] \cup [1, \omega_{+} / \omega_{s}]. \) From (S7.4) it follows that
(S7.7)
LSH
\[ \omega_{-} / \omega_{s} \sum_{j=2}^{j-2} \sum_{\ell=0}^{\ell} \pi(x + \ell \omega_{s}) \left( \alpha_{-j} + (\gamma_{-j} + R(j - 1) \omega_{s} \alpha_{-j}) \cdot (x + \ell \omega_{s})^{-1} + \vartheta_{-j}(x + (\ell + 1) \omega_{s}) \right) \]
\[ + \sum_{k=1}^{\omega_{+} / \omega_{s}} \sum_{\ell=1}^{k} \pi(x - \ell \omega_{s}) \left( \alpha_{k} + (\gamma_{k} - R k \omega_{s} \alpha_{k}) \cdot (x - \ell \omega_{s})^{-1} + \vartheta_{k}(x - \ell \omega_{s}) \right). \]
Similarly, by Lemma S6.1, and \( \beta = -\gamma + \vartheta \),

\[
\text{RHS} = \sum_{j=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \beta_{-j} (y + \omega_s) - \sum_{k=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \beta_k (y)
\]

\[
= \sum_{j=1}^{\omega_+ / \omega_*} (\gamma_{-j} + R(j-1) \omega_s \alpha_{-j}) \sum_{y \in \omega_* N_0 + x} \pi(y)y^{-1}
\]

\[
- \sum_{k=1}^{\omega_+ / \omega_*} (\gamma_k - Rk \omega_s \alpha_k) \sum_{y \in \omega_* N_0 + x} \pi(y)y^{-1}
\]

\[
+ \sum_{j=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_{-j} (y + \omega_s) - \sum_{k=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_k (y)
\]

\[
= (-\gamma \omega_*^{-1} - R \alpha_- + R \left( \alpha_+ + \vartheta \omega_*^{-1} \right)) \sum_{y \in \omega_* N_0 + x} \pi(y)y^{-1}
\]

\[
+ \sum_{j=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_{-j} (y + \omega_s) - \sum_{k=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_k (y)
\]

\[
= (-\beta \omega_*^{-1} + (R-1) \vartheta \omega_*^{-1}) \sum_{y \in \omega_* N_0 + x} \pi(y)y^{-1}
\]

\[
+ \sum_{j=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_{-j} (y + \omega_s) - \sum_{k=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_k (y).
\]

Note that there exist constants \( C > 0 \), \( N \in \mathbb{N} \) such that for all \( x \geq N \),

\[
\left| \sum_{k=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_k (y) - \sum_{j=1}^{\omega_+ / \omega_*} \sum_{y \in \omega_* N_0 + x} \pi(y) \vartheta_{-j} (y + \omega_s) \right| 
\]

\[
\leq C \sum_{y \in \omega_* N_0 + x} \pi(y)y^{-2} \leq Cx^{-1} S_\pi(x).
\]

Hence for possibly larger \( N \) and \( C \), \( \forall x \geq N \),

\[
\text{LHS} \geq (\alpha_+ \omega_*^{-1} - Cx^{-1}) x \pi (x - \omega_s) (x-\omega_s)^{-1} = (\alpha_+ \omega_*^{-1} x - C) (S_\pi(x-\omega_s) - S_\pi(x)),
\]

while

\[
\text{RHS} \leq (\Delta + Cx^{-1}) S_\pi(x).
\]

Choosing possibly larger \( N \) and \( C \), by the monotonicity of \( S_\pi \), for all \( x \geq N \),

\[
(\alpha_+ \omega_*^{-1} x - C) S_\pi(x-\omega_s) \leq (\alpha_+ \omega_*^{-1} x - C + \Delta + Cx^{-1}) S_\pi(x),
\]
which implies that,
\[
\frac{S_\pi(x)}{S_\pi(x-\omega_*)} \geq \frac{x - C\alpha_+^{-1}\omega_*}{x + (\Delta - C)\alpha_+^{-1}\omega_* + C\alpha_+^{-1}\omega_* x^{-1}} \geq \frac{x - C\alpha_+^{-1}\omega_*}{x + (\Delta - C)\alpha_+^{-1}\omega_* + C\alpha_+^{-1}\omega_* x^{-1}} \geq \frac{x - C\alpha_+^{-1}\omega_*}{x + (\Delta - C)\alpha_+^{-1}\omega_* + C\alpha_+^{-1}\omega_* x^{-1}} \geq \frac{1}{x + (\Delta - C)\alpha_+^{-1}\omega_* + \frac{\omega_*}{2\alpha_+} x^{-2}}.
\]

Hence
\[
S_\pi(x) \geq S_\pi(N - \omega_*) \prod_{j=0}^{(x-N)\omega_*^{-1}} \frac{1}{1 + \frac{C}{2\alpha_+\omega_*} (x\omega_*^{-1} - j)^2}.
\]

It is easy to verify that \( \prod_{j=0}^{(x-N)\omega_*^{-1}} \frac{1}{1 + \frac{C}{2\alpha_+\omega_*} (x\omega_*^{-1} - j)^2} = O(1) \) is bounded below by a positive constant. Similar as in the proof of (i), for some \( \overline{C} = \overline{C}(N) > 0, \)
\[
S_\pi(x) \geq \overline{C}(x\omega_*^{-1})^{-\Delta\alpha_+^{-1}}.
\]

This shows that
\[
\liminf_{x \to \infty} (x\omega_*^{-1})^{\Delta\alpha_+^{-1}} S_\pi(x) > 0.
\]

Second Limit in (S7.6). From (S7.7), there exists \( N \) and positive constants \( C_i \) \( (i = 0, 1, 2, 3) \) such that \( \forall x \geq N, \)

(S7.8) \[
\text{LHS} \leq \left( \alpha_+\omega_*^{-1} + C_0 x^{-1} + C_1 x^{-2} \right) \sum_{\ell = -\omega_*/\omega_*}^{\omega_/\omega_* - 2} \pi(x + \ell\omega_*) \leq \left( \alpha_+\omega_*^{-1} x + C_2 + C_3 x^{-1} \right) \sum_{\ell = -\omega_*/\omega_*}^{\omega_/\omega_* - 2} \pi(x + \ell\omega_*) (x + \ell\omega_*)^{-1} = \left( \alpha_+\omega_*^{-1} x + C_2 + C_3 x^{-1} \right) (S_\pi(x - \omega_*) - S_\pi(x + \omega_+ - \omega_*)) \]

whereas
\[
\text{RHS} \geq (\Delta - C x^{-1}) S_\pi(x),
\]
where

\[ C_0 = \max_{1 \leq \ell \leq \omega_+} \left\{ \frac{\omega_+}{\omega_+} \left( \sum_{k=\ell}^{\omega_+} (\gamma_k - Rk\alpha_k\omega_+), \sum_{j=2}^{\omega_-} (\gamma_{-j} + R(j+1)\alpha_-j\omega_+) \right) \right\}, \]

\[ C_2 = C_0 + \alpha_+\omega_*^{-1}(\omega_-/\omega_+ - 2) \]

Hence for all \( x \geq N \),

\[ \left( \alpha_+\omega_*^{-1}x + C_2 + C_3x^{-1} \right) S_\pi(x - \omega_+) \]

\[ \geq \left( \alpha_+\omega_*^{-1}x + C_2 + \Delta + (C_3 - C)x^{-1} \right) S_\pi(x + \omega_- - \omega_+). \]

This implies that

\[ \frac{S_\pi(x + \omega_- - \omega_+)}{S_\pi(x - \omega_+)} \leq \frac{\alpha_+\omega_*^{-1}x + C_2 + C_3x^{-1}}{\alpha_+\omega_*^{-1}x + C_2 + \Delta + (C_3 - C)x^{-1}}. \]

Similar as the argument for the first limit, one can show that the second limit in (S7.6) holds.

Next, we show

\[ \beta\alpha_+^{-1}\omega_*^{-1} < (R - 1) \left( \theta\alpha_+^{-1}\omega_*^{-1} + 1 \right). \]

Let \( \delta = (R\theta - \gamma)\alpha_+^{-1}(\omega_+ + \omega_- - \omega_*)^{-1} \). It suffices to show \( \delta > 1 \). From (S7.6), there exists \( C' > 0 \) such that for all \( x \geq N \),

\[ S_\pi(x) \geq C'x^{-\delta}. \]

From (S7.8),

\[ \text{LHS} \leq \left( \alpha_+\omega_*^{-1}x + C_0x^{-1} + C_1x^{-2} \right) (T_\pi(x - \omega_+) - T_\pi(x + \omega_- - \omega_+)), \]

\[ \text{RHS} \geq (\Delta - Cx^{-1})C'x^{-\delta}, \quad \forall x \geq N. \]

Hence for a possibly larger \( N \),

\[ T_\pi(x - \omega_+) - T_\pi(x + \omega_- - \omega_+) \geq C'x^{-\delta} \frac{\Delta - Cx^{-1}}{\alpha_+\omega_*^{-1}x + C_0x^{-1} + C_1x^{-2}} \geq C' \frac{\Delta}{2\alpha_+\omega_*^{-1}}x^{-\delta}, \]

for \( x \geq N \). Let \( \overline{\Delta} = \omega_+ + \omega_- - \omega_* \). For a possibly bigger \( N \), and for all \( x \geq N \), by summation we have

\[ 1 \geq T_\pi(x - \omega_+) = \sum_{j=1}^\infty \left( T_\pi \left( x - \omega_+ + (j - 1)\overline{\Delta} \right) - T_\pi \left( x - \omega_+ + j\overline{\Delta} \right) \right) \]

\[ \geq \frac{\Delta}{2\alpha_+\omega_*^{-1}}C'x^{-\delta} \sum_{j=1}^\infty \left( (j - 1)\overline{\Delta}x^{-1} \right)^{-\delta}, \]

which implies \( \delta > 1 \).

Moreover, since \( S_\pi(x) \leq x^{-1}T_\pi(x) \), from (S7.6) it follows that the lower limit in (S7.5) is obtained.

Next, we prove the second limit in (S7.5). Let \( \tilde{\delta} = (R\theta - \gamma)\alpha_+^{-1}(\omega_+ + \omega_- - \omega_*)^{-1} \).

From (S7.6) there exists \( C(\tilde{\delta}) > 0 \) such that for all \( x \in \mathbb{N}, \)

\[ S_\pi(x) \leq C(\tilde{\delta})x^{-\tilde{\delta}}. \]
Hence for some positive constant $C(\delta)$,
\[
\text{LHS} \geq \pi (x - \omega_*) \left( \alpha + \omega_*^{-1} - C(\delta) x^{-1} \right), \quad \text{RHS} \leq C(\delta) x^{\delta} \left( \Delta + Cx^{-1} \right)
\]
Thus for some $\tilde{C} > 0$,
\[
\pi (x - \omega_*) \leq \tilde{C} x^{-\delta}.
\]
Assume w.l.o.g. that $\tilde{\delta} > 1$. Otherwise,
\[
\limsup_{x \to \infty} T_\pi(x) x^{\tilde{\delta} - 1} = 0 < \infty,
\]
since
\[
\lim_{x \to \infty} T_\pi(x) = 0.
\]
Hence for a possibly larger $\tilde{C}$,
\[
T_\pi(x) = \sum_{y \in \omega_* N+x} \pi(y - \omega_*) \leq \tilde{C} \sum_{y \geq x+\omega_*} y^{-\tilde{\delta}} \leq \tilde{C} \int_x^{\infty} y^{-\tilde{\delta}} dy = \frac{\tilde{C}}{\tilde{\delta} - 1} x^{-\tilde{\delta} + 1}.
\]

S8. Asymptotic expansion of $Qf$ for all Lyapunov functions $f$ used in the proofs

In the following, we provide the asymptotic expansion of $Qf(x)$ for all large $x$, for various Lyapunov functions $f$. Let $\delta \in \mathbb{R}$.

For $f(x) = x^\delta$,
\[
Qf(x) = \delta x^\delta \left\{ \alpha x^{R-1} + (\beta + \delta \vartheta) x^{R-2} + O(x^{R-3}) \right\}.
\]

For $f(x) = (x \log x)^{-1})^\delta$,
\[
Qf(x) = \delta (x \log x)^{-1})^\delta \left\{ \alpha \left( 1 - (\log x)^{-1} \right) x^{R-1} + \left( (\beta + \delta \vartheta) - (\beta + 2 \delta \vartheta) (\log x)^{-1} \right) x^{R-2} + O(x^{R-2}(\log x)^{-2}) \right\}.
\]

For $f(x) = (x \log x)^{\delta}$,
\[
Qf(x) = \delta (x \log x)^{\delta} \left\{ \alpha \left( 1 + (\log x)^{-1} \right) x^{R-1} + (\beta + \delta \vartheta) x^{R-2} + \gamma \delta \vartheta x^{R-2}(\log x)^{-1} + O(x^{R-2}(\log x)^{-2}) \right\}.
\]

For $f(x) = (\log x)^{\delta}$,
\[
Qf(x) = \delta (\log x)^{\delta - 1} \left\{ \alpha x^{R-1} + \beta x^{R-2} + (\delta - 1) \vartheta x^{R-2}(\log x)^{-1} + O(x^{R-3}) \right\}.
\]

For $f(x) = (\log \log x)^{\delta}$,
\[
Qf(x) = \delta (\log \log x)^{\delta - 1} (\log x)^{-1} \left\{ \alpha x^{R-1} + \beta x^{R-2} - \vartheta x^{R-2}(\log x)^{-1} + O \left( x^{R-2}(\log x)^{-1}(\log \log x)^{-1} \right) \right\}.
\]
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