**Abstract**

By considering a $C^r$ structure on the ordered non-increasing of elements of $R^n$, we show that it is a differentiable manifold. By using of Lie groups, we show that eigenvalue function is a submersion. This fact is used to prove some results. These results are applied to prove a few facts about spectral manifolds and spectral functions. Orthogonal matrices act on the real symmetric matrices as a Lie transformation group. This fact, also, is used to prove the results. For example, the set $S(n,R)$ of real symmetric $n\times n$ matrices is a submanifold (with dimension $n(n+1)/2$) of the set of all real matrices $M(n\times n,R)$.

When the topology on $M'$ induced by its $C^r$ structure is its topology as a subset of $M$, $M'$ is said to be a regular submanifold of $M$.

Next proposition is about submersions and it plays key role in the differentiability of eigenvalue function [8], propositions 6.1.2; 6.1.4; 6.2.1.

**Proposition 1.1**

1. $\psi$ is a submersion of $M$ onto $M'$. If $\psi; M' \rightarrow M''$ is such that $\psi \circ \varphi$ is differentiable then $\psi$ also is differentiable.

2. A differentiable function $\varphi; M \rightarrow M'$ is a submersion if and only if for each $m$ in its domain there exists a differentiable section of $\varphi$ containing $m$ in its range.

3. Let $\varphi; M \rightarrow M'$ be a submersion. If $dim(M) > dim(M')$, each set $\varphi^{-1}(s)$ has the structure of a regular submanifold of $M$ of dimension $dim(M) - dim(M')$.

**Definition 1.2**

1. A Lie group $G$ is a group which has the structure of a differentiable manifold and for which the group function $\theta; G \times G \rightarrow G$ defined by $(g,h) \mapsto gh$ is differentiable.

2. A Lie group is said to act on a differentiable manifold $M$ as a Lie transformation group if we are given a global surjection $\Phi; G \times M \rightarrow M$ which is differentiable and such that if $g,h \in G$ and $m \in M$ $\Phi(g,\Phi(h,m)) = \Phi(gh,m)$.

If $g \in G$, the function $\Phi; M \rightarrow M$ defined by $m \mapsto \Phi(g,m)$ is a diffeomorphism of $M$ onto itself.
Let \( GL(n, R) \) be non-singular real matrices. The set \( O(n, R) \) of real orthogonal \( n \times n \) matrices can be given the structure of a Lie subgroup of \( GL(n, R) \) of dimension \( \frac{1}{2}n(n-1) \). This Lie subgroup is a regular submanifold.

\( GL(n, R) \) acts on \( R^n \) as a Lie transformation group with the function \( \Phi \) defined by \( (A, z) \mapsto Az \).

Also, \( O(n, R) \) acts on \( S^{n-1} \), unit sphere of \( R^n \), as a Lie transformation group. This action is transitive.

**Differentiability of \( \lambda(.) \)**

In this section our aim is to prove that \( \lambda \) is a differentiable function between two differentiable manifolds. For this work, we will consider a \( C^\infty \) structure on the ordered non-increasing of elements of \( R^n \). We start with a remark:

**Remark 2.1**

1. Let \( f: A \mapsto M \) be a bijection on a set \( A \) with values on a differentiable manifold \( M \). Then \( f \) defines a complete \( C^\infty \) atlas of \( A \). With this structure on \( A, f \) is a diffeomorphism \([8]\).

2. Let \( n \in \mathbb{N} \) (natural numbers) is a finite cardinal number and \( c \) be the cardinal of real number of \( R \) \([9]\). We recall that
   
   \( n+c=c \)
   \( c+c=c \)

(c) For each \( A, B \) sets

\[ \text{card}A + \text{card}B = \text{card}(A \cup B) + \text{card}(A \cap B). \]

Where \( A, B \) are arbitrarily sets, \( \text{card}A \) and \( \text{card}B \) are their cardinal numbers.

Denote by \( R'_c \) the ordered non-increasing of elements of \( R^n \). That is,

\[ R'_c = \{ (x_1, \ldots, x_n) \in R^n \mid x_1 \geq \cdots \geq x_n \}. \]

It is clear that \( R'_c \) is a convex, closed subset of \( R^n \). The next result show that \( R'_c \) is bijective with \( R^n \).

**Lemma 2.1:** Let

\[ R'_c = \{ (x_1, \ldots, x_n) \in R^n \mid x_1 \geq \cdots \geq x_n \}. \]

Then \( R'_c \) is bijective with \( R^n \).

**Proof 2:** Let \( x=(x_1, \ldots, x_n) \in R^n \). There are \( n! \) (factorial) possible orders for \( x \) components. So there are \( n! \) permutation matrices as \( P_1, \ldots, P_{n!} \), gives rise to these orders in matrix notation. If \( P_s \) shows identity matrix and \( P_i K'_c = \{ P_s x \mid x \in R^n \} \), \( i=0, \ldots, n! \) then we have:

\[ R^n = P_1 R'_c \cup P_2 R'_c \cup \ldots \cup P_{n!} R'_c. \]

(Where \( P_{i,j} \) is a permutation, \( R^n \) is the set of \( n \)-tuple of ordered real numbers, \( c \) is cardinal of real number set)

But \( P_i R'_c \) is bijective with \( R'_c \). So \( \text{card}(P_i R'_c) = \text{card}(R'_c) = c \). By the above remark, part 2

\[ c + c + \ldots + c = n\times c = c = \text{card}(P_1 R'_c) + \text{card}(P_2 R'_c) + \ldots \]

\[ + \text{card}(P_i R'_c) + \text{card}(P_i R'_c \cap P_2 R'_c) + \ldots + \text{card}(P_i R'_c \cap \ldots \cap P_{n!} R'_c). \]

But

\[ k = \text{card}(P_1 R'_c \cap P_2 R'_c \cap \ldots \cap P_{n!} R'_c) \leq \text{card} R'_c = c. \]

So

\[ \text{card}(P_1 R'_c \cup P_2 R'_c \cup \ldots \cup P_{n!} R'_c) + \text{card}(P_1 R'_c \cap P_2 R'_c \cap \ldots \cap P_{n!} R'_c) = c \]

\[ \text{card}(R'_c) = c + k = c. \]

Therefore \( \text{card}(R'_c) = \text{card}(R^n) \). That is \( R'_c \) bijective with \( R^n \).

So if \( P \in O(n, R) \), the function \( \Phi \), \( \text{card}(R'_c) \rightarrow \text{card}(R^n) \), defined by \( A \mapsto \text{diag}(A) \), is a diffeomorphism of \( S(n, R) \) onto itself and so it is a submersion. This follows that the function \( T : O(n, R) \times \text{diag}(R'_c) \rightarrow S(n, R) \) defined by

\[ (P, \text{diag}(s)) \mapsto \text{diag}(s)P \]

is a submersion. Let \( \rho \) be projection function on second factor

\[ \rho : O(n, R) \times R^n \rightarrow S(n, R) \]

\[ (P, \text{diag}(s)) \mapsto \text{diag}(s), \]

then \( \rho \) is differentiable and \( \rho = \lambda \circ T \). Hence, by the Proposition 1.1, part 1, \( \lambda \) is differentiable.

Proposition 2.5 has some important results about the properties of \( \lambda \):

**Corollary 2.7**

1. \( \lambda \) is a submersion.
2. Let \( A \in S(n, R) \) with spectral decomposition \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \).
   \( P \) the Jacobian of \( \lambda \), \( \forall \lambda \), is given by

\[ \nabla \lambda \ |_h = \text{Ph}P \]

Where \( h \in S(n, R) \).
3. \( \lambda \) is an open mapping.

**Proof 2.8**

1. Let \( A \in S(n, R) \) and \( A = \text{diag}(\lambda)P \) its spectral decomposition. If \( \psi \) be Lie transformation group on Proposition 2.5, then
The state follows from Proposition 1.1, part 2.

2. It is well known that: If A is any point in the domain of a submersion \( \lambda : S(n,R) \to R^n \) there exist charts \( x,y \) of \( S(n,R) \to R^n \) at \( A, \lambda(A) \) respectively such that the function \( y \circ x^{-1} = (z,w) \to z \) is it clear that \( y \) is identity function on \( R^n \) and \( x^{-1} \) is a \( P \in O(n,R) \) such that \( A = P \lambda(A) P^{-1} \). We can write \( \lambda \) such that, on some neighborhood of \( A \in S(n,R) \) \( (P \circ \lambda) \circ (P^{-1} \circ \lambda) \). In other words \( PAP^{-1} = \lambda(A) \). By multiplying both sides of this equality by \( P \), \( P \) from the right and left, respectively, we have \( P \lambda(A) P^{-1} = A \) (1.1)

Suppose that \( A \) is fixed and \( h \) belong to a small neighborhood of \( A \). With differentiating from both sides of (1.1) we have \( P \lambda P^{-1} = I \), where \( \lambda \) is identity matrix in \( S(n,R) \). Therefore \( P \lambda P^{-1} = h \). This yields \( \lambda = P \lambda P^{-1} \).

3. Any submersion is an open mapping.

**Example 2.1:** The set \( M \) of all real symmetric matrices of order \( 2 \times 2 \) with distinct eigenvalues is an open subset of \( S(2,R) \). So, by the above Corollary, \( \lambda(M) \) is an open subset of \( R_+ \).

**More Spectral Manifolds (Isotropic)**

For fixed subset \( M \subseteq R^n \), some properties on \( M \) remain true on the corresponding set \( \lambda^{-1}(M) \) (spectral set). If \( M \) is differentiable manifold of \( R^n \), then \( \lambda^{-1}(M) \) will be called spectral manifold. The spectral manifolds are entirely defined by their eigenvalues [10].

For example, if the set \( M \) is symmetric, then properties such as closedness, convexity, prox-regular are transferred between \( M \) and \( \lambda^{-1}(M) \) [11,12].

Also, transfer of differentiable structure of a submanifold \( M \subseteq R^n \) has been studied. We speculate that most of these results do not depend on the property of symmetry. We must rewrite some of these results in the other work. In this section, we study fibers and orbits of \( \lambda \) as two types of spectral manifolds.

\( A \) determines an equivalence relation \( \rho \) on \( S(n,R) \), defined by \( (A,B) \in \rho \) if and only if \( \lambda(A) = \lambda(B) \). The equivalence classes of \( \rho \) are symmetric matrices. In fact, the following Lemma holds for any submersion .

**Lemma 3.1:** There is an equivalence relation \( \rho \) on \( S(n,R) \) such that \( \frac{S(n,R)}{\rho} \) has a differentiable structure and it becomes a quotient manifold diffeomorphic to \( R^n \).

This lemma has an important result about spectral manifolds. Let \( L \) be the diffeomorphism in the lemma and \( M \subseteq R^n \) be a submanifold. Then \( \lambda^{-1}(M) \) is a submanifold of \( \frac{S(n,R)}{\rho} \).

**Theorem 3.3:** For \( s \in R^n \), \( \lambda(s) \) (fiber of \( \lambda \)) has the structure of a regular submanifold of \( S(n,R) \) of dimension \( \frac{n(n-1)}{2} \).

**Proof 3.4:** Since \( \lambda \) is a submersion, Lemma follows from Proposition 1.4.

Let \( \varphi \) given by Proposition 2.5. This Lie transformation group sets up an equivalence relation on \( S(n,R) \). The equivalence class containing a point \( A \) is the range of the function \( \varphi : O(n,R)S(n,R) \) (orbit of \( A \) and \( \lambda^{-1}(\lambda(A)) = \{A \} \).

The set \( H_A = \{h^\circ(A) \) is called stabilizer of \( A \) (or isotropy group). It is well known that if \( H_A \) is not an open subgroup of \( O(n,R) \), then the orbit of \( A \) can be given the structure of a regular submanifold of \( S(n,R) \) diffeomorphic to the quotient manifold \( \frac{H_A}{H} \). Thus, by the following Lemma, \( \lambda^{-1}(\lambda(A)) \) is a regular submanifold of \( S(n,R) \) of dimension \( \text{dim} \varphi - \text{dim} O(n,R) + \text{dim} H_A \).

**Lemma 3.5:** For \( A \in S(n,R) \), the stabilizer of \( A \) is not an open subset of \( O(n,R) \).

Let \( H_A \) be an open subgroup of \( O(n,R) \). Then the quotient set topology on \( \frac{O(n,R)}{H} \) is discrete. Therefore \( \lambda^{-1}(\lambda(A)) \) is discrete. This contradicts with Lemma 3.2.

**Spectral Functions**

Let \( A = P \lambda \{\lambda_1,\ldots,\lambda_n\} P^{-1} \), for some \( P \in O(n,R) \). In Chen, Qi, and Tseng showed that for any function \( f : R^n \to R \), one can define a spectral function \( f : S(n,R) \to S(n,R) \).

\[ f(A) = P \lambda \{\lambda_1,\ldots,\lambda_n\} P^{-1}, \quad \lambda \in O(n,R) \]

which are constant on the orbit \( A \) of the properties of continuity, directional differentiability, differentiability, and continuous differentiability are inherited by \( f \) from \( f \). As we shall see, \( f \) does not play a large role.

In this section we examine only differentiability and continuous properties by using Lie group tools.

**Proposition 4.1:** Let \( A \in S(n,R) \) and \( M \) be the orbit of \( A \). Then differentiability of \( f : M \) depends on the differentiability of \( f \).

**Proof 4.2:** For each \( B \in M \), \( \lambda(B) = \lambda(B) \). Let \( \lambda = \lambda(B) \).

If \( M = \{B \in S(n,R) \mid B = P \lambda \{\lambda_1,\ldots,\lambda_n\} P^{-1} \} \) be the orbit of \( C = \text{diag} \{\lambda_1,\ldots,\lambda_n\} \) then the function \( f : M \to M \) induced by \( f \) is defined as

\[ B = \varphi(P,C), \quad \forall P \in O(n,R) \]

where \( \varphi \) is Lie transformation action on the Proposition 2.5. Therefore \( f \) is differentiable.

**Proposition 4.3:** For function \( f : R^n \to R \), the following results hold:

1. \( f \) is differentiable at \( A \in S(n,R) \) with eigenvalues \( \lambda_1,\ldots,\lambda_n \) if and only if \( f \) is differentiable at \( \lambda_1,\ldots,\lambda_n \).
2. \( f \) is continuous at \( A \in S(n,R) \) with eigenvalues \( \lambda_1,\ldots,\lambda_n \) if and only if \( f \) is continuous at \( \lambda_1,\ldots,\lambda_n \).

**Proof 4.4**

1. According to the second part of the Corollary 2.7, there exists an open neighborhood \( U \) of \( A \in S(n,R) \) and open neighborhood \( V \) of \( \lambda(A) \) in \( R^n \) such that \( \lambda(U) = V \) and \( \lambda \) in these local coordinates is the standard projection.
$\lambda(x_1, \ldots, x_n, \ldots, x_m) = (x_1, \ldots, x_m)$.

Therefore we can write

$f'(A) = f(x_1, \ldots, x_m) = \text{diag}(f(x_1), \ldots, f(x_m))$

That is $f'$ differentiable if and only if $f$ differentiable.

2. This part is a consequence of part 1.

In the following proposition we have removed symmetric condition and other extra conditions [16,17].

**Proposition 4.5**: Suppose $f: R^n \to R$ be a function. Then the following assertions are true:

1. $f$ is continuous at $\lambda \in R^n$ if and only if $f$ is continuous at $P\lambda$ for any $P \in O(n,R)$.

2. $f$ is differentiable at $\lambda \in R^n$ if and only if $f$ is differentiable $P\lambda$ for any $P \in O(n,R)$.

**Proof 4.6**

1. $O(n,R)$ acts on $R^n$ as a transitively Lie transformation group (with matrix notation). Then $\Phi_{P}\lambda: R^n \to R^n$ by $\Phi_{P}(\lambda) = P\lambda$

   is a diffeomorphism. But $f(\lambda) = f(P\lambda)$. Therefore two side of state are clear.

2. Proof of this part is similar to part (1).

   In the sequel, corresponding with (1.1), for each function $f: R^n \to R^n$ with

   $f(x) = (f_1(x), \ldots, f_n(x))'$ ($x \in R^n$),

   we define $\bar{f}(A): S(n, R) \to S(n, R)$ by

   $\bar{f}(A) = f(P^T \lambda(A), \ldots, f_n(P^T \lambda(A)))' = f(P^T \lambda(A), \ldots, f_n(P^T \lambda(A)))' P^{-1}$

   and similar to the Proposition 4.3 we can show that if $f: R^n \to R^n$ has the property of continuity (respectively, differentiability), then do the symmetric-matrix function $\bar{f}$ is defined by (1.3).

   Note that $\bar{f}$ is independent of choice of $P \in O(n,R)$ and that $\bar{f}(A)$ belongs to $S(n,R)$. [18].

**Proposition 4.7**: Let $f: R^n \to R^n$ be a function with $f(x) = (f_1(x), \ldots, f_n(x))'$ ($x \in R^n$), and let $\bar{f}$ be defined by (1.3). Let $A \in S(n, R)$ with eigenvalue $\lambda_1(A), \ldots, \lambda_n(A)$ and $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))^T$. Then the following assertions are equivalent:

1. $f$ is continuous at $\lambda(A)$

2. $f$ is continuous at $A$.

**Proposition 4.8**: Let $f: R^n \to R^n$ be a function with $f(x) = (f_1(x), \ldots, f_n(x))'$ ($x \in R^n$). Let $A \in S(n, R)$ with eigenvalues $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))'$ and $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))'$. Then the following assertions are equivalent:

1. $f$ is differentiable at $\lambda(A)$

2. $f$ is differentiable at $A$.

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