A PROOF OF SENDOV’S CONJECTURE

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Abstract. The Theorem of Gauß-Lucas states that the zeros of the derivative of a complex polynomial $p$ are contained in the closed convex hull of its zeros. For a polynomial $p$ having all its zeros in the closed unit disk $B$. Sendov conjectured that the distance of an arbitrary zero to the closest derivative zero is at most 1. In this article we will give a proof.

The zeros of the derivative of a complex polynomial $p$ are functions of the zeros of $p$ itself. In general we do not know explicit expressions for these functions. So approximate localizations of the derivative zeros in terms of the given zeros of $p$ are of interest. A question of this type is the famous conjecture of Bl. Sendov which goes back to 1959 and took place in Hayman’s booklet on problems in Complex Analysis (1967, [2], by a misunderstanding, there it was named after Ilieff). This conjecture states:

Let $p \in \mathbb{C}[z]$ be a polynomial of degree $n > 1$ having all zeros $z_1, \ldots, z_n$ in the closed unit disk $\overline{E}$. Does there exist for every $z_j$ some $\zeta$ with $|z_j - \zeta| \leq 1$ and $p'(\zeta) = 0$?

For a history of the conjecture and a list of the numerous (about 100) publications on it, most of them in famous international journals, see the recent article of Bl. Sendov [6]. In this paper we give a proof of this question.

By $\mathcal{P}_n$ we denote the class of all monic polynomials of degree $n > 1$ having all its zeros in $\overline{E}$.

For the following we fix some polynomial $p \in \mathcal{P}_n$ with the factorization

$$p(z) = \prod_{j=1}^{n}(z - z_j).$$

Definition 1. Let $p \in \mathcal{P}_n$ and $w_0 \in \mathbb{C}$ a zero of $p$. The disk $|z - w_0| \leq \rho$ is called critical with respect to $w_0$ if $p'$ has no zero in the open disk but at least one on the boundary (the critical circle). In this case $\rho = \rho(p, w_0) \geq 0$ is called the critical radius for $w_0$ and the derivative zeros of $p$ on the critical circle are called to be essential (with respect to $w_0$). The polynomial $p \in \mathcal{P}_n$ is maximal with respect to the point $w_0 \in \overline{E}$ if among all polynomials $q \in \mathcal{P}_n$ with $0 = q(w_1)$ the critical radii fulfill $\rho(p, w_0) \geq \rho(q, w_1)$.

Of course the term $\rho(p, w_0)$ also makes sense for polynomials $p$ with $p(w_0) = 0$, which are not necessarily in $\mathcal{P}_n$. But in this general case one will not find a maximal polynomial as this is true in the compact class $\mathcal{P}_n$.

Now Sendov’s conjecture can be formulated as $\rho(p, w_0) \leq 1$ for all $p \in \mathcal{P}_n, p(w_0) = 0$. In order to prove the conjecture it would be enough to check maximal polynomials in $\mathcal{P}_n$. But which $p \in \mathcal{P}_n$ are maximal? Phelps and Rodriguez [4] guessed that these...
are only the polynomials \( p_n(z) = z^n - 1 \) and their rotations \( p_n(ze^{ia})e^{-ina} \). In the following we will confirm this extension of Sendov’s conjecture.

This will come out as a consequence of

**Theorem 1.** Let \( p \in \mathcal{P}_n \) have the zero \( z_1 \in E \). Then there is some \( p^* \in \mathcal{P}_n \) which has a zero \( w_0 \) on the unit circle and fulfills \( \rho(p, z_1) \leq \rho(p^*, w_0) \).

1. **The basic idea**

We start with some elementary formulas. If \( p \in \mathcal{P}_n \) is a polynomial with the zeros \( z_1, \ldots, z_n \) and the derivative zero \( \zeta \) with \( p(\zeta) \neq 0 \), then

\[
\frac{p'}{p}(\zeta) = 0 = \sum_{j=1}^{n} \frac{1}{\zeta - z_j}.
\]

We let the zeros \( z_2, \ldots, z_n \) be fixed and vary \( z_1 \), i.e., we consider the polynomials

\[
Q(z, u) = (z - u) \prod_{j=2}^{n} (z - z_j) = (z - u) q(z).
\]

We assume for the moment that \( \zeta \) is a zero of \( p' \), but not a zero of \( p'' \). The implicit function theorem (cf. [3]) shows the existence of a holomorphic function \( \zeta(u) \) with

\[
\zeta(z_1) = \zeta \quad \text{and} \quad \frac{\partial Q}{\partial z}(\zeta(u), u) \equiv 0,
\]

defined in a neighborhood of \( z_1 \). If we move \( u \) along a path \( \gamma \) in \( \mathbb{C} \) starting in \( \gamma(0) = z_1 \) then we have an unrestricted analytic continuation of \( \zeta(\gamma(t)) \) if \( \frac{\partial^2 Q}{\partial z^2}(\zeta(\gamma(t)), \gamma(t)) \neq 0 \) for all \( t \). If the path would meet these exceptional points, we would have at least a continuation of \( \zeta(\gamma(t)) \) which is at least continuous in such points. Note that the values of \( \zeta(\gamma(t)) \) with respect to this continuation move on the Riemann surface \( R \), which is defined by the equation \( Q'(z, u) = 0 \) (derivative with respect to \( z \)). We will discuss this surface in section 2.

Note that \( \ln |\zeta(u) - u| = \Re \log(\zeta(u) - u) \).

Let \( p \in \mathcal{P}_n \) and \( \zeta \) be a (not necessarily essential) derivative zero of \( p \). As above let \( z_1, z_2, \ldots, z_n \in \overline{E} \) be the zeros of \( p \) and \( |z_1| < 1 \). If \( \gamma : [0, 1] \to \mathbb{C} \) is a path with \( \gamma(0) = z_1, \gamma(1) = u \in \mathbb{C} \), we see

\[
\frac{d}{dt} \ln |\zeta(\gamma(t)) - \gamma(t)| = \frac{d}{dt} \Re \log(\zeta(\gamma(t)) - \gamma(t)) = \Re \frac{\frac{d}{dt}(\zeta(\gamma(t)) - \gamma(t))}{(\zeta(\gamma(t)) - \gamma(t))}.
\]

Note that \( \zeta(\gamma(t)) \) depends on the path \( \gamma \). Again we assume that \( \zeta(\gamma(0)) = \zeta \). So we have

\[
\ln |\zeta(u) - u| - \ln |\zeta - z_1| = \int_{0}^{1} \frac{d}{dt} \ln |\zeta(\gamma(t)) - \gamma(t)| dt
\]

\[
= \Re \int_{0}^{1} \frac{d}{dt} \log(\zeta(\gamma(t)) - \gamma(t)) dt = \Re \int_{0}^{1} \gamma'(t) \frac{\zeta'(\gamma(t)) - 1}{\zeta(\gamma(t)) - \gamma(t)} dt = \Re \int_{\gamma} \frac{\zeta'(v) - 1}{\zeta(v) - v} dv.
\]

The right hand side can be written as

\[
\Re \int_{\gamma} \frac{\zeta'(v)}{\zeta(v) - v} dv - \frac{1}{\zeta(v) - v} dv.
\]
From (1) we obtain
\[ 0 = \frac{Q'(\zeta(v), v)}{Q(\zeta(v), v)} = \frac{1}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)). \]
So it comes out
\[ \ln |\zeta(u) - u| = \ln |\zeta - z_1| + \Re \int_{\gamma} \frac{\zeta'(u)}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)) \, dv, \]
and therefore
\[ (3) \quad |\zeta(u) - u| = |\zeta - z_1| \cdot |\exp \left( \int_{\gamma} \frac{\zeta'(u)}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)) \, dv \right)|. \]

2. The Riemann surface \( R \)

It is enough to consider polynomials \( p \in P_n \) with the property that \( p \) has no multiple zeros and no multiple derivative zeros. If we succeed to prove theorem \( \Box \) under this restriction, the general statement is clear by a continuity argument. By the same argument we may assume that \( q \) has no multiple zeros and no multiple derivative zeros.

The Riemann surface \( R \) of the derivative zeros of \( Q \) is given by the equation
\[ Q'(w) = q(w) + (w - u)q'(w) = 0. \]
This (actually compact) manifold \( R \) consists of the points \( w \) (which are the derivative zeros of \( Q(., u) \), and the equation gives local uniformizations of \( R \), if the derivative of \( u = \varphi(w) := w + \frac{q}{Q}(w) \) with respect to \( w \) does not vanish (note that these branch points are also described by \( \frac{\partial^2 Q}{\partial^2 w}(w, u) = 0 \)). So the points \( w \) where \( 2q'(w)^2 = q(w)q''(w) \) are branch points of the surface. this branch points play in fact no special role on the Riemann surface, their appearance depend on the special local coordinates, which are given by the defining equation (example: the surface of the square root is defined by \( w^2 = u \) with 0 as a branch point; if we add this point, it is conformally equivalent to the plane resp. \( \mathbb{C} \)). They can actually added as "normal" points to the surface and have simply connected neighborhoods on which local coordinates can be found.

\( R \), as a compact surface, may be regarded as a \((n - 1)\)-sheeted covering of \( \mathbb{C} \), and \( \varphi \) gives a canonically projection \( R \rightarrow \mathbb{C} \).

We define
\[ f(u, \zeta(u)) := \exp \left( \int_{\gamma} \frac{\zeta'(u)}{\zeta(v) - v} + \frac{q'}{q}(\zeta(v)) \, dv \right), \]
where \( \gamma_u : [0, 1] \rightarrow \mathbb{C} \) with \( \gamma_u(0) = z_1, \gamma_u(1) = u \in \mathbb{C} \) and \( \zeta(\gamma_u(0)) = \zeta \) (some fixed derivative zero of \( p \)), \( \zeta(\gamma_u(1)) = \zeta(u) \). By (3) we have
\[ (6) \quad |\zeta(u) - u| = |\zeta - z_1| \cdot |f(u, \zeta(u))|. \]
\( f \) is, up to isolated singularities, a holomorphic function on \( R \), because it has this property in the local coordinate \( u \in \mathbb{C} \) (the case \( u = \infty \) we discuss separately) . The holomorphy is not obviously clear in the following cases.

(i) \( Q(w_1, u_1) = 0 \) (this includes the case \( u = \zeta(u) \), or
(ii) \(2q'(w_2)^2 = q(w_2) \cdot q''(w_2)\) (branch points)

We discuss this two cases.

Case (i): The assumption implies that \(Q(., u_1)\) has a multiple zero in the point \(w_1\). This is only possible if \(u_1\) is one of the zeros \(z_1, \ldots, z_{n-1}\) of \(p\) and \(u_1 = w_1\). We have \(|\zeta(u_1) - u_1| = 0\) if \(\zeta(u_1) = w_1\). So this singularities of \(f\) is removable by (6). Moreover we have \(\rho(Q(., u_1)) = 0\) in this case.

Case (ii): If \(2q'(w_2)^2 = q(w_2)q''(w_2)\), then \(w_2 \notin \{z_1, z_2, \ldots, z_{n-1}\}\), because \(q\) has only simple zeros in the points \(z_2, \ldots, z_n\). Especially \(\frac{q}{w}(w_2)\) is finite. So (6) shows that \(f\) is bounded in a neighborhood of the branch point \(w_2\) on \(R\). Again we conclude that \(f\) has a removable singularity in this case.

We summarize: The function \(f\) as defined in (5) is holomorphic on the Riemann surface \(R' := \{w \in R : \varphi(w) \in \mathbb{C}\}\).

From (3) we obtain that \(f(u, \zeta(u))\) equals \(\frac{\zeta(u) - u}{\zeta - z_1}\), up to a possible factor of modulus one. For \(u = z_1\) we see that this factor is one. By (1) we receive the representation:

\[
(7) \quad f(u, \zeta(u)) = \frac{q'}{q}(\zeta) \cdot \frac{q}{q}(\zeta(u)).
\]

Finally we investigate the structure of \(R\) close to \(u = \infty\). The point infinity is no branch point of \(R\), because the function \(1/\varphi(1/w)\) has in \(w = 0\) the expansion \(w^{\frac{n-1}{n}} + a_1 w + \ldots\). For \(u \in \overline{E}\) all zeros of \(Q(., u)\) are contained in \(E\). By the Gauß-Lucas theorem we know that the zeros of the derivative \(Q'(z, u) = \frac{\partial Q}{\partial z}(z, u)\) lie in the convex hull \(C\) of the zeros. They are inner points of \(C\) with the only exception of multiple zeros of \(Q\). None of these derivative zeros in our case is of bigger order than 1. So the same argument gives that the zeros of the second order derivative \(Q''(z, u) = \frac{\partial^2 Q}{\partial z^2}(z, u)\) are points the open unit disk \(E\). So the same is true for the branch points of \(R\). To be more precise, all branch points \(w\) of \(R\) fulfill \(|\varphi(w)| < 1\).

The subset \(D_1\) of \(R\) with \(\varphi(D_1) = \overline{E}\) therefore contains all branch points.

As a consequence, the complement \(R \setminus D_1\) (including \(\infty\)) consists of \(n - 1\) simply connected domains \(G_1, \ldots, G_{n-1}\). Let \(\zeta(u)\) be the function which is defined on \(G_k\) with respect to a fixed start point \(\zeta_0\) with \(\varphi(\zeta_0) = z_1\). Then the mappings \(\Phi_k := \varphi|G_k = \varphi|G_k : G_k \to \{u \in \mathbb{C} : |u| > 1\}\) are conformal.

The boundaries of the domains \(G_j\) are pairwise disjoint. Each \(\partial G_j\) is mapped homeomorphically by \(\varphi\) on the unit circle.

It holds \(P(z, u) := \frac{Q(z, u)}{u} = (\frac{z}{u} - 1)q(z)\). The derivative zeros of \(P\) with respect to \(z\) are the same as those of \(Q\). For \(u \to \infty\) the polynomials \(P(z, u)\) tend locally uniformly to \(q(z)\). So, in this case, \(\zeta(u)\) tends to \(\infty\) on one \(G_k\), let us say on \(G_1\). For \(k = 2, \ldots, n - 1\) it follows that each \(\zeta_j(u) \in G_k\) tends to some derivative zero \(\xi_k\) of \(q'\) if \(u \to \infty\).

If \(\zeta(u) \in G_1\) we see from (7) that \(\zeta(u)\) has a pole of first order in \(\infty_k \in G_k\) for all \(k = 1, \ldots, n - 1\).

Now let \(\zeta(u) \in G_k\) with some \(k > 1\). In this cases \(\zeta(u) \to \xi_k\) for \(u \to \infty\). Again we obtain that \(f\) has a simple pole in \(\infty_k \in G_k\) by (7), because \(q\) has only simple
derivative zeros which are no zeros of $q$ (compare the remark at the beginning of section 2).

3. Blowing up and pulling back

Let $r > 0$ and $p_r(z) = r^n p(z/r)$. If we start the considerations of the preceding section with $p_r$ instead of $p$ we have to replace the zeros $z_1, \ldots, z_n$ of $p$ by $rz_1, \ldots, rz_n$ and the derivative zeros $\zeta(u)$ by $r \zeta(u)$ as well as $q(z)$ by $r^{n-1} q(z/r)$. The variation is then

$$Q_r(z, u) := r^n Q(z/r, u) = (z - ur) \cdot r^{n-1} q(z/r) = (z - ur) \prod_{j=2}^n (z - z_j r).$$

Let $w_0$ be an arbitrary complex number on the unit circle. If $r$ is large enough we may provide that $|f(rw_0, r\zeta(w_0))| > 1$, because of the poles of $f$ in $\infty_k \in G_k$ for all $k = 1, \ldots, n - 1$. From (6) we conclude that $|r\zeta(w_0) - rw_0| > |r\zeta - rz_1|$ for all sufficiently large $r$ and all derivative zeros of $Q_r(\cdot, w_0)$. This gives $|\zeta(w_0) - w_0| > |\zeta - z_1|$. We define $Q(\cdot, w_0) = : p^*$. If $\zeta$ has been taken above as an essential derivative zero of $p$ this says that $|\zeta(w_0) - w_0| > \rho(p, z_1)$ for all derivative zeros of $p^*$. This gives $\rho(p^*, w_0) > \rho(p, z_1)$ and theorem 1 is proved.

4. Proof of Sendov’s conjecture

The following Theorem has been proved by Goodman, Rahman, Ratti 1 and independently by Schmeisser 5.

**Theorem 2.** Let $n \geq 2$ and $p \in \mathcal{P}_n$. If $p(1) = 0$ then there is some $\zeta$ with $p'(\zeta) = 0$ and $|2\zeta - 1| \leq 1$. Moreover, there is such some $\zeta$ in the open disk $|2z - 1| < 1$ unless all derivative zeros of $p$ lie on the circle $|2z - 1| = 1$.

The polynomial $z^n - 1$ shows that $p(z_1) \geq 1$ in order that $p$ is maximal with respect to its zero $z_1$. So the only point in the closed disk $|2z - 1| \leq 1$ where $p$ may have a derivative zero is 0. So we obtain from Theorem 2 that $p'$ has a single zero, located in the origin. Now $p(1) = 0$ implies $p(z) = z^n - 1$. We see that the only maximal polynomials in $\mathcal{P}_n$ are given by $z^n + a$ with $|a| = 1$. This has been conjectured 1972 by Phelps and Rodriguez 4.

**References**

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