A q-deformed Weyl-Heisenberg algebra is used to define a deformed displacement operator giving rise to a naturally normalized nonlinear coherent states type. Robust maximally entangled deformed coherent states are studied and the effect of such a deformation on the amount of the entanglement is discussed. The analogy between environment decoherence and algebra deformation is made through the deformation parameter.

Keywords: q-deformed Weyl-Heisenberg algebra; coherent states; entanglement; concurrence; decoherence.

1. Introduction

The interaction of quantum systems involved in quantum information processing with the surrounding environment may affect their physical properties, resulting in decoherence. From a mathematical point of view, decoherence effects affecting a quantum system may be represented by a modification of its symmetry. This could be performed through a correction on the mathematical formalism consisting in a small deformation introduced to the commutation relation defining the algebra that describes the system.
The notion of algebra deformation is very familiar to mathematical physicist. Many studies devoted to the investigation of a deformed quantum oscillator algebra appeared as early as seventies, involving deformed creation and annihilation operators known as the $q$—oscillator algebra or the $q$—deformed algebra. From a mathematical point of view, the $q$—oscillators ladder operators are shown to have a structure of a non-trivial Hopf algebra. However, the physical relevance of $q$—deformed creation and annihilation operators is not always very transparent in the studies that have been published on the subject so far. Therefore it is important to emphasize that there are — from our point of view — at least, two main properties which make $q$—oscillators interesting objects for physics. The first is the fact that they constitute a fundamental tools of completely integrable theories. The second, which is one of the points we are interested in, concerns the connection between the $q$—deformation and nonlinearity in the context of coherent states (CS).

Since 1926, at the beginning of quantum mechanics, Schrödinger was interested to the study of quantum states that mimic their classical analogs and defined CS as the states minimizing the Heisenberg uncertainty relation. They were rediscovered by Klauder at the beginning of 1960s, then by Sudarshan and Glauber in a series of papers for the description of the coherence phenomenon in lasers. A decade later, Barut and Girardello have introduced coherent states defined as eigenstates of the annihilation operator, then Gilmore and Parelomov have constructed coherent states by the application of a displacement operator on the vacuum state.

In this paper, we use a weak deformation approximation to construct a deformed CS displacement operator. In section two, we present our mathematical formalism and highlight some interesting properties of our deformed coherent states especially the analogy with their nonlinear counterpart. In the third section, the bipartite concurrence of the maximally entangled deformed coherent states is discussed. In section four we present the results of our numerical study of the bipartite entangled coherent states concurrence before drawing our conclusions.

2. Mathematical formalism

In the Fock representation, Schrödinger-Klauder-Glauber coherent states called standard coherent states or also canonical coherent states are given by

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle ,$$

where $\alpha = |\alpha|e^{i\phi}$ is a complex parameter such that $\langle \alpha |\alpha\rangle = 1$ and $|n\rangle$ is the number operator eigenstate. The quantum harmonic oscillator CS $|\alpha\rangle$ reproduce in averaging the same classical behavior. This unique specificity is due to numerous properties of CS:
(i) The CS $|\alpha\rangle$ minimizes the Heisenberg uncertainty relation:

$$\langle \Delta Q \rangle_{\alpha} \langle \Delta P \rangle_{\alpha} = \frac{\hbar}{2},$$

with $\langle \Delta X \rangle_{\alpha} = \sqrt{\langle |X|^2 \rangle_{\alpha} - \langle X \rangle_{\alpha}^2}$, $(X = Q, P)$ where $Q$ and $P$ are the oscillator position and momentum operators respectively.

(ii) $|\alpha\rangle$ is eigenstate of the annihilation operator $a$ with eigenvalue $\alpha$:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C},$$

where $a = \frac{m\omega Q + iP}{\sqrt{2m\hbar \omega}}$ satisfies the commutation relation

$$[a, a^+] = aa^+ - a^+a = 1,$$

(iii) The CS $|\alpha\rangle$ is obtained from the fundamental state $|0\rangle$ with a unitary transformation of the Weyl-Heisenberg group $D(\alpha) = e^{\alpha a^+ - \bar{\alpha}a}$ called displacement operator:

$$|\alpha\rangle = e^{\alpha a^+ - \bar{\alpha}a} |0\rangle.$$  

(iv) The coherent states $\{|\alpha\rangle\}$ form a complete set in the Hilbert space, such that

$$\frac{1}{\pi} \int_{\mathbb{C}} d\{\text{Re}(\alpha)\} \ d\{\text{Im}(\alpha)\} \ |\alpha\rangle\langle\alpha| = I,$$

where $I$ stands for the identity operator.

(v) The probability $p(n)$ to be in the number operator eigenstate $|n\rangle$ is time independent and has a Poisson distribution

$$p(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^2}{n!} e^{-|\alpha|^2}.$$
In what follows we introduce a type of deformation where the related creation and annihilation operators denoted by $b$ and $b^+$ respectively define a modified Weyl-Heisenberg algebra satisfying the $q$–deformation commutation relation\cite{6,7}
\begin{equation}
[b, b^+]_q = bb^+ - q b^+ b = 1,
\end{equation}
where the deformation parameter $q$ is taken to be real. In our approach, we use a week deformation approximation where $q = 1 + \varepsilon$ with $\varepsilon \ll 1$, such that we recover the non-deformed case for $\varepsilon \rightarrow 0$. In this case one can show that up to $O(\varepsilon^2)$ we have:
\begin{equation}
[b, b^+] = bb^+ - b^+ b = 1 + \varepsilon b^1 b.
\end{equation}
As a representation of the deformed operators $b$ and $b^+$ satisfying the commutation relation \cite{9} one has :
\begin{align}
b &= a + \frac{1}{4} \varepsilon a^+ a^2 + O(\varepsilon^2), \quad (10) \\
b^+ &= a^+ + \frac{1}{4} \varepsilon a^2 a + O(\varepsilon^2). \quad (11)
\end{align}
In this representation the deformed number operator will be defined as:
\begin{equation}
\hat{n}_d = b^+ b = \hat{n} + \varepsilon (\hat{n} + \frac{1}{2} \hat{n}^2) + O(\varepsilon^2), \quad (12)
\end{equation}
where $\hat{n} = a^+ a$ is the non-deformed number operator.

It is worth to mention that the so called $f$–deformed oscillators (generalization of $q$–deformed oscillators), were interpreted as nonlinear oscillators corresponding classically to a frequency dependence of the oscillation amplitude. In the framework of nonlinear coherent states (NCS), Man’ko defined an $f$–deformed oscillator with the creation and annihilation operators\cite{11,22,23}
\begin{align}
b &= a f(\hat{n}) = f(\hat{n} + 1) a, \quad (13) \\
b^+ &= f(\hat{n}) a^+ = a^+ f(\hat{n} + 1), \quad (14)
\end{align}
satisfying :
\begin{equation}
[\hat{n}, b] = -b, \\
[\hat{n}, b^+] = b^+, \\
[b, b^+] = (\hat{n} + 1) f^2(\hat{n} + 1) - \hat{n} f^2(\hat{n}). \quad (15)
\end{equation}
Comparing Eqs.\cite{10,11} and Eqs.\cite{12,13}, one can show that in our case the $f$-deformation function has the following expression
\begin{equation}
f(\hat{n}) = 1 + \frac{1}{4} \varepsilon (\hat{n} - 1), \quad (16)
\end{equation}
such as the non-deformed limit is recovered for $f(\varepsilon \rightarrow 0) = 1$. Thus, the CS which we will construct starting from our deformed algebra could be interpreted as NCS.
We remind the reader the major importance of NCS in the study of nonlinear potentials systems where their non-classical state description of the electromagnetic field, quantum optics and the atomic center of mass displacement was successful\cite{25-30}. In fact, the interest for the generalization of CS pertinent for the nonlinear potentials have started since the beginning of 1970s\cite{20,21,31}. Nieto and co-workers constructed CS corresponding to the Pöschl-Teller one dimensional potential as the states minimizing the uncertainty relation between the canonical coordinates $Q$ and $P$\cite{32}. Two decades later Gazeau and Klauder proposed a generalization to one dimensional systems with discrete and continue spectrum\cite{33}. The CS corresponding to the trigonometric and modified Pöschle-Teller potentials were derived in Ref.\cite{34} and those concerning the Morse potential in Ref.\cite{35}. Man’ko et al\cite{36} and Filho\cite{37} have introduced NCS as eigenstates of the deformed annihilation operator whereas the displacement operator NCS were derived in Ref.\cite{38,39}.

The derivation of CS from displacement operator was generalized to deformed oscillators at the end of 1990s. In the same spirit as in Ref.\cite{40} we introduce a deformed displacement operator $D_d(\alpha)$ such that:

$$D_d(\alpha) = \exp \left( \alpha b^+ - \bar{\alpha} b \right), \quad (17)$$

generating deformed coherent states (DCS) $|\alpha\rangle_d$ defined as

$$|\alpha\rangle_d = D_d(\alpha)|0\rangle_d, \quad (18)$$

where $|0\rangle_d$ is the deformed vacuum state. Starting from the fact that $\exp(\bar{\alpha} b)|0\rangle_d = |0\rangle_d$ and using the BCH formula, straightforward simplifications (see appendix (Appendix A)) lead to the following expression for the DCS defined in Eq.(18)

$$|\alpha\rangle_d = \left[ 1 + \left( \frac{|\alpha|^4}{24} - \frac{|\alpha|^2}{6} \bar{\alpha} \alpha^+ + \frac{1}{8} \alpha^2 \alpha^2 \right) \frac{1}{2} \right] |\alpha\rangle, \quad (19)$$

where $|\alpha\rangle$ is the non-deformed CS.

Furthermore, using the relations

$$e^{\bar{\beta}a^+} = (a^+ + \bar{\beta})e^{\beta a}, \quad \beta \in \mathbb{C}, \quad (20)$$

$$a e^{\beta a^+} = e^{\beta a^+}(a + \beta), \quad (21)$$

we can derive the following overlaps :

$$d\langle\beta|\alpha\rangle_d = \left[ 1 + \varepsilon \left( \frac{|\alpha|^4}{24} - \frac{|\alpha|^2}{6} |\beta|^2 \bar{\beta} \alpha + \frac{1}{4} |\beta|^2 \alpha^2 \right) \right] \langle\beta|\alpha\rangle, \quad (22)$$

$$\langle\beta|\alpha\rangle_d = \left[ 1 + \varepsilon \left( \frac{|\alpha|^4}{24} - \frac{|\alpha|^2}{6} \bar{\beta} \alpha + \frac{1}{8} |\beta|^2 \alpha^2 \right) \right] \langle\beta|\alpha\rangle, \quad (23)$$
\[ d\langle \beta | \alpha \rangle = \left[ 1 + \varepsilon \left( \frac{|\beta|^4}{24} - \frac{|\beta|^2}{6} - \bar{\beta} \alpha + \frac{1}{8} \bar{\beta}^2 \alpha^2 \right) \right] \langle \beta | \alpha \rangle , \] 

(24)

where \( \langle \beta | \alpha \rangle = \exp\left[ \frac{1}{2}(2\alpha^* - |\alpha|^2 - |\beta|^2) \right] \).

Note that from Eq. (22) one has

\[ d\langle \alpha | \alpha \rangle_d = \langle \alpha | \alpha \rangle = 1 \] 

(25)

that is our DCS are naturally normalized without any need to an additional normalization constant. This is a specific feature compared to many other constructions of DCS. Also, Eq. (22) shows that our \( q \)-deformed coherent state are non-orthogonal as in the non-deformed case.

It is very important to mention, as it is pointed out in Ref. [22] that the DCS constructed from deformed displacement operator and those constructed as annihilation operators eigenstates are not the same even though the two approaches are equivalent in the non deformed formalism.

The analogy between our DCS and NCS reveals the physical interpretation of the algebra deformation introduced in the previous section. Indeed, we can understand the small correction introduced to the commutation relation (9), implying a symmetry deformation, as a decoherence effect due to the system environment and acting on its physical properties. In the next section, we will study the effect of such a deformation on some properties of a system of entangled coherent states.

3. Entanglement of deformed coherent states

The relevance of our DCS constructed from a deformed symmetry (deformed Weyl-Heisenberg algebra) resides in their interpretation as NCS which are originated from a certain kind of nonlinear potential describing an external environment action like.

In other words, the weak \( q \)-deformation of the original symmetry can be explained as a small perturbation acting on the system characterized by an order parameter \( \varepsilon \) and leading to a sort of decoherence phenomenon due to the environment affecting the entanglement between physical states as it will be emphasized in what fellows.

Since the seminal works of Tombesi and Mecozzi [43, 44] in which entangled coherent states (ECS) have been first studied as entities of physical interest in their own right, ECS have continued to be of a large interest in quantum information processing like quantum teleportation, superdense coding, quantum key distribution and telecloning [45–49]. Entangled nonorthogonal states have attracted much attention in quantum cryptography. Bosonic, SU(2) and SU(1,1) ECS are typical examples of such states [51].
In what follows we study a bipartite entanglement of DCS quantified using the entanglement concurrence. Let us first take a deformed state
\[
|\psi_d\rangle = \mu |\alpha_d\rangle \otimes |\gamma_d\rangle + \nu |\gamma_d\rangle \otimes |\delta_d\rangle ,
\]
where \(|\alpha_d\rangle, |\gamma_d\rangle\) (resp. \(|\beta_d\rangle\) and \(|\delta_d\rangle\)) are normalized deformed coherent states of system 1 (resp. system 2) with complex coefficients \(\mu\) and \(\nu\). Following Ref.56 we define a deformed orthogonal basis \(\{|0\rangle_d, |1\rangle_d\}\) as:
\[
|0\rangle_d = |\alpha_d\rangle, \quad |1\rangle_d = (|\gamma_d\rangle - \mu^2 |\alpha_d\rangle) / N_1 \quad \text{for system 1},
\]
\[
|0\rangle_d = |\beta_d\rangle, \quad |1\rangle_d = (|\delta_d\rangle - \nu^2 |\beta_d\rangle) / N_2 \quad \text{for system 2},
\]
where
\[
p_1 = d \langle \alpha | \gamma \rangle, \quad N_1 = \sqrt{1 - |p_1|^2},
\]
\[
p_2 = d \langle \beta | \delta \rangle, \quad N_2 = \sqrt{1 - |p_2|^2}.
\]
After deriving the reduced density matrix \(\rho_{1(2)}\) (resp. \(\rho_{2(1)}\)) for system 1 (resp. system 2)57 the concurrence \(C\) takes the well known form58, 59
\[
C = \frac{2 \mu \nu \sqrt{1 - |p_1|^2} \sqrt{1 - |p_2|^2}}{\mu^2 + |\nu|^2 + \mu \nu^2 p_1^* p_2 + \mu^* \nu p_1 p_2^*}.
\]

For general bipartite non-orthogonal pure states the necessary and sufficient conditions for a maximal entanglement, i.e. \(C = 1\), have been found and discussed in details in Refs.56, 60, 61. For the case of our interest one can show that the maximal entanglement conditions hold too. That is, for the deformed state \(|\psi_d\rangle\) to be maximally entangled state the following conditions must be satisfied:
\[
\mu = \nu e^{i\theta} \quad (\theta \in \mathbb{R}),
\]
and
\[
|\alpha|^2 + |\gamma|^2 - 2 \alpha^* \gamma = |\beta|^2 + |\delta|^2 - 2 \beta^* \delta - 2i(\theta + \pi).
\]
This highlights the fact that if non-deformed CS are maximally entangled states they remain maximally entangled states in the \(q\)-deformed case. That is maximally entangled coherent states are robust against algebra deformation.

Furthermore, in comparison with the non-deformed case one can construct more maximally entangled deformed coherent states. As an example one has the states
\[
|\alpha_d\rangle \otimes |-\alpha_d\rangle - | - \alpha_d\rangle \otimes | 3\alpha_d\rangle,
\]
\[
|\alpha_d\rangle \otimes |-\alpha_d\rangle - | - i\alpha_d\rangle \otimes |i\alpha_d\rangle,
\]
\[
|\varepsilon \alpha_d\rangle \otimes | - \alpha_d\rangle - | - \alpha_d\rangle \otimes |\varepsilon \alpha_d\rangle,
\]
and for \(\alpha, z, z' \in \mathbb{R}\), one can find
\[
|\alpha_d\rangle \otimes | - \alpha + z \varepsilon\rangle_d - | - \alpha + (z' - z) \varepsilon\rangle_d \otimes | 3\alpha + z' \varepsilon\rangle_d,
\]
etc...
4. Numerical study

For deeper understanding of the entanglement between two entangled deformed coherent states (EDCS) we consider states of the form

$$|\psi_1\rangle_d = |\alpha\rangle_d \otimes |\beta\rangle_d + e^{i\theta} |\beta\rangle_d \otimes |\alpha\rangle_d ,$$

for which the concurrence is given by

$$C = \frac{1 - |d(\alpha|\beta\rangle_d|^{2}}{1 + \cos \theta |d(\alpha|\beta\rangle_d|^{2}} .$$

Then, the state $|\psi_1\rangle_d$ satisfies one ebit of entanglement, i.e. $C = 1$, if one of the following conditions is satisfied:

a) The state $|\psi_1\rangle_d$ is an antisymmetric state, i.e. $\theta = \pi$, then $C = 1$ independently from the parameters $\alpha$ and $\beta$ involved.

b) The deformed coherent states $|\alpha\rangle_d$ and $|\beta\rangle_d$ are almost orthogonal, i.e. $d(\alpha|\beta\rangle_d \sim 0$ (for large $\alpha, \beta$), then $C = 1$ independently from the phase $\theta$.

To be more explicit let us take the special case

$$|\psi_2\rangle_d = |\alpha\rangle_d \otimes | - \alpha\rangle_d + e^{i\theta} | - \alpha\rangle_d \otimes |\alpha\rangle_d .$$

For such an entangled state, in the weak deformation approximation limit, the concurrence given by Eq. (39) takes the form

$$C = \frac{1 - \left(1 + \frac{1}{2}|\alpha|^{4} |e| \right) e^{-4|\alpha|^{2}}}{1 + \cos \theta \left(1 + \frac{1}{2}|\alpha|^{4} |e| \right) e^{-4|\alpha|^{2}}} .$$

Numerical calculations performed using the above concurrence of the bipartite entangled deformed coherent states $|\psi_2\rangle_d$ reveal that:
Fig. 2. Concurrence of the state $|\psi_2\rangle_d$ given by Eq.(40) as function of the phase $\theta$ for small negative and positive values of deformation parameter $\varepsilon$ corresponding to an intermediate coherence parameter $|\alpha| = 1$.

(i) Maximally ECS ($C = 1$) are robust against algebra deformation, confirming the analytical calculations.

As a result, if non-deformed ECS ($\varepsilon = 0$) are maximally entangled, either for the antisymmetric states i.e. $\theta = \pi$ (\forall $\alpha$) or for the case $\alpha \gg 1$ (\forall $\theta$), they remain maximally entangled independently from the deformation parameter $\varepsilon$ (see Fig.1 and Fig.2).

(ii) There exists an intermediate regime ($|\alpha| \sim 1$) where $C$ increases with the absolute value of the coherence parameter $|\alpha|$ and is sensitive to the symmetry deformation parameter $\varepsilon$.

As Fig.1 shows, the concurrence increases with increasing $|\alpha|$ approaching a maximally entanglement with $C = 1$ for a sufficiently large $|\alpha|$ (depending on the value of $\varepsilon$). Notice that for a given value of $|\alpha|$ before the concurrence saturation, the entanglement between deformed coherent states is very sensitive to the algebra deformation. Indeed, when $\varepsilon$ increases in the interval $[-0.4, 0.4]$, we notice e.g. a 4.7% decrease of the concurrence given by Eq.(41) for $|\alpha| = 1$, and a 6.3% decrease for $|\alpha| = 0.9$ (see Fig.1).

(iii) As a function of the phase parameter $\theta$ and the deformation parameter $\varepsilon$, the concurrence which has a maximum ($C = 1$) at $\theta = \pi$ independently from the value of $\varepsilon$ (see Fig.2), decreases faster as $\varepsilon$ get larger when $\theta \neq \pi$.

Indeed, for the case $\theta = 0$ exhibited in Fig.1 we notice that if $\varepsilon$ lies in the interval $[-0.4, 0.4]$, the concurrence significantly decreases by $\sim 6.3\%$ for $|\alpha| = 0.9$, $\sim 4.7\%$ for $|\alpha| = 1$ and $\sim 3\%$ for $|\alpha| = 1.1$. Similarly, we observe in Fig.2 a reduction of $\sim 4.7\%$ when $\varepsilon \in [-0.4, 0.4]$ and $|\alpha| = 1$ at $\theta = 2\pi$.

The decrease of the concurrence as a function of the deformation parameter observed above can be interpreted as an effect of the environment decoherence since
Fig. 3. Allowed values (green area) of the coherence parameter $\alpha$ and deformation parameter $\varepsilon$.

$\varepsilon$ plays the role of a perturbation and dissipation parameter as it is pointed out in Sec. 3.

It is worth to mention that in order to have a convergent perturbation series with respect to the algebra deformation parameter $\varepsilon$ and then reliable conclusions one has from Eq. (41) the constraint

$$\frac{1}{2}|\alpha^4 \varepsilon| \ll 1.$$  

Fig. 3 displays the allowed regime for the coherence parameter $\alpha$ and the deformation parameter $\varepsilon$.

5. Conclusion

Throughout this paper we have constructed normalized deformed coherent states in the weak deformation approximation and shown their relationship to their nonlinear analogues. We have studied the effect of such a deformation on the coherent states entanglement. It turns out that in comparison with the non-deformed case, the number of maximally bipartite entangled deformed coherent states is larger as new maximally entangled states can be found. In this paper we have quantified the entanglement using the concurrence $C$ showing that maximally entangled coherent states are robust against algebra $q$–deformation. Moreover, numerical results reveal an important effect of the algebra $q$–deformation on the entanglement. Indeed, for bipartite entangled deformed coherent states, except the case of the antisymmetric state, the concurrence is shown to be a decreasing function of the deformation parameter for a given values of the phase and the coherence parameters. This can be interpreted as a reliable argument that the algebra deformation parameter $\varepsilon$ can play the role of a decoherence order parameter representing the effect of the environment on the entangled coherent states system. Further more, preliminary results show that bipartite entanglement of three modes entangled coherent states is also affected by algebra deformation even when those later are maximally entangled states (under investigation).
Acknowledgments

We are grateful to the Algerian ministry of higher education and research for the financial support. M.T.R. is partially supported by an Averroes exchange program.

Appendix A. Derivation of deformed coherent states given by Eq. (19)

We define a normalized deformed coherent states by the application of the deformed displacement operator $D_d(\alpha)$ upon the vacuum state

$$|\alpha\rangle_d = D_d(\alpha)|0\rangle_d.$$  \hfill (A.1)

Then, we introduce an additional term $\exp(\bar{\alpha}b)$ using the fact that $e^{\bar{\alpha}b}|0\rangle_d = |0\rangle_d$. Eq. (A.1) can be rewritten as

$$|\alpha\rangle_d = \exp\left(\alpha b^+ - \bar{\alpha}b\right) \exp(\bar{\alpha}b)|0\rangle_d.$$  \hfill (A.2)

We point out that the deformed and non-deformed vacuum states $|0\rangle_d$ and $|0\rangle$ are equivalent (see Ref. [53]). This can be checked easily in our case. Let us consider the state $|0\rangle_d = \kappa (|0\rangle + \varepsilon |\xi\rangle)$ where $|\xi\rangle$ is an arbitrary ket and $\kappa$ is a normalization constant. Applying the deformed bosonic annihilation operator on the deformed bosonic vacuum states gives

$$b|0\rangle_d = \kappa \left(a + \frac{\varepsilon}{4}a^2\right) (|0\rangle + \varepsilon |\xi\rangle),$$  \hfill (A.3)

This leads to $|0\rangle_d = \kappa (1 + \varepsilon)|0\rangle$. Imposing the normalization condition $d<|0\rangle_d = 1$, we deduce that

$$|0\rangle_d = |0\rangle.$$  \hfill (A.4)

To derive the expression of the deformed displacement operator coherent states given by Eq. (19), we need to use the BCH formula [54]

$$\log \left(e^X e^Y\right) = X + Y + \frac{1}{2!}[X,Y] + \frac{1}{3!}[X,[X,Y]]/2$$

$$- \frac{1}{3!}[Y,[X,Y]]/2 - \frac{1}{4!}[Y,[X,[X,Y]]]$$

$$+ \frac{1}{5!}([[[X,Y],Y],X] + [[[Y,X],X],Y] + [[[Y,X],Y],X]),$$

$$+ \frac{1}{360} ([[[[X,Y],Y],Y],Y] + [[[Y,X],X],X],$$

$$- \frac{1}{720} ([[[[X,Y],Y],Y],Y] + [[[Y,X],X],X],$$

$$+ \cdots$$  \hfill (A.5)

where

$$X = \alpha b^+ - \bar{\alpha}b,$$

$$Y = \bar{\alpha}b.$$  \hfill (A.6)
Straightforward but tedious calculations give

\[ [X, Y] = -|\alpha|^2 (1 + \varepsilon b^+ b) + \mathcal{O}(\varepsilon^2), \]

\[ [X, [X, Y]] = |\alpha|^2 \varepsilon (ab^+ + \bar{\alpha}b) + \mathcal{O}(\varepsilon^2), \]

\[ [Y, [X, Y]] = -|\alpha|^2 \bar{\alpha} \varepsilon b + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^2), \]

\[ [Y, [X, [X, Y]]] = |\alpha|^4 \varepsilon + \mathcal{O}(\varepsilon^2), \]

where we used the identity \([A, BC] = [A, B]C + B[A, C]\) and neglected all terms of second and higher order in \(\varepsilon\) (\(\varepsilon \ll 1\)). Notice that the last commutator above corresponding to the 6th term in BCH formula is a c-number”. That is, all next commutators vanish.

Using the commutation relations above, the deformed coherent states of Eq.(A.2) become

\[ |\alpha\rangle_d = e^{-|\alpha|^2/2} e^{-\varepsilon |\alpha|^4/4} \times \exp \left[ \alpha b^+ + \varepsilon \left( \frac{|\alpha|^2}{12} \alpha b^+ + \frac{|\alpha|^2}{6} \bar{\alpha} b - \frac{|\alpha|^2}{2} b^+ b \right) \right] |0\rangle. \]

Using a Taylor series expansion of the third exponential in Eq.(A.11) and the relations

\[ a^n a^+ = na^{(n-1)} + a^+ a^n, \]

\[ a a^+ = na^{(n-1)} + a^+ a^n, \]

as well as

\[ b^+ = a^+ + \frac{1}{4} \varepsilon a^2 a + \mathcal{O}(\varepsilon^2), \]

\[ b = a + \frac{1}{4} \varepsilon a^2 + \mathcal{O}(\varepsilon^2). \]

Direct simplifications lead to

\[ |\alpha\rangle_d = e^{-|\alpha|^2/2} e^{-\varepsilon |\alpha|^4/4} \sum_{n=0}^{\infty} \frac{|A + \varepsilon B|^n}{n!} |0\rangle, \]

\[ = \sum_n \frac{1}{n!} A^n |0\rangle + \sum_{p=0}^{n-1} \sum_{n} \frac{1}{n!} A^{n-p-1} \varepsilon B A^p |0\rangle + \mathcal{O}(\varepsilon^2), \]

where \(A = ab^+\) and \(B = \frac{|\alpha|^2}{2} b^+ b + \frac{|\alpha|^2}{12} \alpha b^+ + \frac{|\alpha|^2}{6} \bar{\alpha} b\).

Now it is easy to see that

\[ \sum_n \frac{A^n}{n!} |0\rangle_d = e^{-|\alpha|^2/2} \left( 1 + \varepsilon \frac{1}{8} \alpha^2 a^+ a^2 \right) |\alpha \rangle, \]
and
\[ \sum_{n=0}^{n-1} \sum_{p=0}^{n} \frac{1}{n!} A^{n-p-1} \varepsilon B A^p = \varepsilon \left( \frac{|\alpha|^2}{6} \alpha a^+ + \frac{|\alpha|^4}{12} \right) e^{-\varepsilon |\alpha|^2} |\alpha\rangle . \]  
(A.17)

Combining
\[ e^{-\varepsilon |\alpha|^2} = 1 - \varepsilon |\alpha|^4 \frac{1}{24} + \mathcal{O}(\varepsilon^2) , \]  
(A.18)

with results (A.16) and (A.17), our deformed coherent states take the form
\[ |\alpha\rangle_d = \left[ 1 + \varepsilon \left( \frac{|\alpha|^4}{24} - \frac{|\alpha|^2}{6} \alpha a^+ + \frac{1}{8} \alpha^2 a^+ a^2 \right) \right] |\alpha\rangle . \]  
(A.19)

Furthermore, using the relations
\[ e^{\beta a^+} = (a + \beta) e^{\beta a} , \]  
(A.20)
\[ a e^{\beta a^+} = e^{\beta a^+} (a + \beta) , \]  
(A.21)

the overlaps \(d\langle \beta | \alpha \rangle_d\) writes
\[ d\langle \beta | \alpha \rangle_d = \left[ 1 + \varepsilon \left( \frac{1}{4} \beta^2 a^2 \right) \right] \langle \beta | \alpha \rangle , \]  
(A.22)

leading to
\[ d\langle \alpha | \alpha \rangle_d = \left[ 1 + \varepsilon \left( \frac{2|\alpha|^4}{24} - \frac{2|\alpha|^4}{6} + \frac{|\alpha|^4}{4} \right) \right] \langle \alpha | \alpha \rangle , \]
\[ = \langle \alpha | \alpha \rangle , \]
\[ = 1 . \]  
(A.23)

References

1. R. I. Karasik, K.-P. Marzlin, B. C. Sanders and K. B. Whaley, Phys. Rev. A 76 (Jul 2007) p. 012331.
2. M. Arik and D. D. Coon, Journal of Mathematical Physics 17 (1976) 524.
3. P. Kulish and N. Reshetikhin, Journal of Soviet Mathematics 23 (1983) 2435.
4. M. Jimbo, Letters in Mathematical Physics 10 (1985) 63.
5. V. G. Drinfeld, Quantum groups, in Proceedings of the International Congress of Mathematicians Barkley, California USA, (Amer. Math. Soc., Providence, RI, 1986).
6. L. C. Biedenharn, Journal of Physics A: Mathematical and General 22 (1989) p. L873.
7. A. J. Macfarlane, Journal of Physics A: Mathematical and General 22 (1989) p. 4581.
8. T. Curtright and C. Zachos, Physics Letters B 243 (1990) 237 .
9. A. P. Polychronakos, Modern Physics Letters A 05 (1990) 2325.
10. M. Jimbo, Letters in Mathematical Physics 11 (1986) 247.
11. V. I. Man’ko and R. V. Mendes, Journal of Physics A: Mathematical and General 31 (1998) p. 6037.
12. E. Schrödinger, Naturwissenschaften 14 (1926) 664.
13. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, Cambridge, 1995).
14. J. R. Klauder, Journal of Mathematical Physics 4 (1963) 1055.
15. J. R. Klauder, *Journal of Mathematical Physics* **4** (1963) 1058.
16. E. C. G. Sudarshan, *Phys. Rev. Lett.* **10** (Apr 1963) 277.
17. R. J. Glauber, *Phys. Rev. Lett.* **10** (Feb 1963) 84.
18. R. J. Glauber, *Phys. Rev.* **130** (Jun 1963) 2529.
19. R. J. Glauber, *Phys. Rev.* **131** (Sep 1963) 2766. Cited 2994.
20. A. Barut and L. Girardello, *Communications in Mathematical Physics* **21** (1971) 41.
21. A. Perelomov, *Communications in Mathematical Physics* **26** (1972) 222.
22. V. I. Man’ko, G. Marmo, S. Solimeno and F. Zaccaria, *International Journal of Modern Physics A* **08** (1993) 3577.
23. V. Man’ko, G. Marmo, S. Solimeno and F. Zaccaria, *Physics Letters A* **176** (1993) 173.
24. D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano and D. J. Wineland, *Phys. Rev. Lett.* **76** (Mar 1996) 1796.
25. C. Monroe, D. M. Meekhof, B. E. King and D. J. Wineland, *Science* **272** (1996) 1131.
26. V. Man’ko, G. Marmo, A. Porzio, S. Solimeno and F. Zaccaria, *Phys. Rev. A* **62** (Oct 2000) p. 053407.
27. S. Sivakumar, *Journal of Optics B: Quantum and Semiclassical Optics* **2** (2000) p. R61.
28. V. Sunilkumar, B. A. Bambah, R. Jagannathan, P. K. Panigrahi and V. Srinivasan, *Journal of Optics B: Quantum and Semiclassical Optics* **2** (2000) p. 126.
29. Z. Kis, W. Vogel and L. Davidovich, *Phys. Rev. A* **64** (Aug 2001) p. 033401.
30. D. S. Freitas and M. C. Nemes, *Modern Physics Letters B* **28** (2014) p. 1450082.
31. S. M. Roy and V. Singh, *Phys. Rev. D* **25** (Jun 1982) 3413.
32. M. M. Nieto and L. M. Simmons, *Phys. Rev. Lett.* **41** (Jul 1978) 207.
33. J. P. Gazeau and J. R. Klauder, *Journal of Physics A: Mathematical and General* **32** (1999) p. 123.
34. O. de los Santos-Sanchez and J. Recamier, *Journal of Physics A: Mathematical and Theoretical* **44** (2011) p. 145307.
35. O. de los Santos-Sanchez and J. Recamier, *Journal of Physics A: Mathematical and Theoretical* **45** (2012) p. 415310.
36. V. I. Man’ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria, *Physica Scripta* **55** (1997) p. 528.
37. R. L. de Matos Filho and W. Vogel, *Phys. Rev. A* **54** (Nov 1996) 4560.
38. B. Roy and P. Roy, *Journal of Optics B: Quantum and Semiclassical Optics* **2** (2000) p. 65.
39. B. Roy and P. Roy, *Journal of Optics B: Quantum and Semiclassical Optics* **2** (2000) p. 505.
40. P. Aniello, V. Man’ko, G. Marmo, S. Solimeno and F. Zaccaria, *Journal of Optics B: Quantum and Semiclassical Optics* **2** (2000) p. 718.
41. J. Recamier, M. Gorayeb, W. Mochn and J. Paz, *International Journal of Theoretical Physics* **47** (2008) 673.
42. N. Boucherredj, N. Mebarki and A. Benslama, *Canadian Journal of Physics* **83** (2005) 929.
43. A. Mecozzi and P. Tombesi, *Phys. Rev. Lett.* **58** (Mar 1987) 1055.
44. P. Tombesi and A. Mecozzi, *J. Opt. Soc. Am. B* **4** (Oct 1987) 1700.
45. K. Berrada, S. Abdel-Khalek, H. Eleuch and Y. Hassouni, *Quantum Information Processing* **12** (2013) 69.
46. C. H. Bennett, G. Brassard, C. Créepeau, R. Jozsa, A. Peres and W. K. Wootters, *Phys. Rev. Lett.* **70** (Mar 1993) 1895.
47. C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69** (Nov 1992) 2881.
48. A. K. Ekert, *Phys. Rev. Lett.* 67 (Aug 1991) 661.
49. M. Murao, D. Jonathan, M. B. Plenio and V. Vedral, *Phys. Rev. A* 59 (Jan 1999) 156.
50. C. A. Fuchs, *Phys. Rev. Lett.* 79 (Aug 1997) 1162.
51. B. C. Sanders, *Phys. Rev. A* 45 (May 1992) 6811.
52. S. J. van Enk and O. Hirota, *Phys. Rev. A* 64 (Jul 2001) p. 022313.
53. O. Hirota, S. J. van Enk, K. Nakamura, M. Sohma and K. Kato, Entangled nonorthogonal states and their decoherence properties, quant-ph/010106v1 (Jan, 2001).
54. H. Jeong, M. S. Kim and J. Lee, *Phys. Rev. A* 64 (Oct 2001) p. 052308.
55. X. Wang, B. C. Sanders and S. Hua Pan, *Journal of Physics A: Mathematical and General* 33 (2000) p. 7451.
56. X. Wang, *Journal of Physics A: Mathematical and General* 35 (2002) p. 165.
57. A. Mann, B. C. Sanders and W. J. Munro, *Phys. Rev. A* 51 (Feb 1995) 989.
58. S. Hill and W. K. Wootters, *Phys. Rev. Lett.* 78 (Jun 1997) 5022.
59. W. K. Wootters, *Phys. Rev. Lett.* 80 (Mar 1998) 2245.
60. H. Fu, X. Wang and A. I. Solomon, *Physics Letters A* 291 (2001) 73.
61. X. Wang, *Phys. Rev. A* 64 (Jul 2001) p. 022302.
62. M. T. Rouabah and N. Mebarki, Monogamy and decoherence from algebra $q$-deformation, (in preparation).
63. G. Vinod, K. B. Joseph and V. C. Kuriakose, *Pramana* 42 (1994) 299.
64. K. F. Riley, M. P. Hobson and S. J. Bence, *Mathematical Methods for Physics and Engineering* (Cambridge University Press, New York, 2006).