Abstract. Determining the index of the Simon congruence is a long outstanding open problem. Two words \( u \) and \( v \) are called Simon congruent if they have the same set of scattered factors, which are parts of the word in the correct order but not necessarily consecutive, e.g., \( \text{oath} \) is a scattered factor of \( \text{logarithm} \). Following the idea of scattered factor \( k \)-universality, we investigate \( m \)-nearly \( k \)-universality, i.e., words where \( m \) scattered factors of length \( k \) are absent, w.r.t. Simon congruence. We present a full characterisation as well as the index of the congruence for \( m = 1 \). For \( m \neq 1 \), we show some results if in addition \( w \) is \((k - 1)\)-universal as well as some further insights for different \( m \).

1 Introduction

Given a word \( w \), a scattered factor (also known as (scattered) subsequence or subword) of \( w \) is a word, that is obtained by deleting letters from \( w \) while preserving the order, i.e., formally \( u \) of length \( n \in \mathbb{N}_0 \) is a scattered factor of \( w \) (denoted by \( u \in \text{ScatFact}(w) \)) if \( w = v_1u[1]v_2u[2]...v_nu[n]v_{n+1} \) for existing (possibly empty) words \( v_1, ..., v_{n+1} \). For instance, \text{power\_flower} \ is scattered factors of \( \text{power\_flow\_poor\_wow} \) \text{rope\_loop} \ are not scattered factors since the letters do not occur in the correct order in \( w \). Therefore, scattered factors can be seen as a representation of \( w \) with some lost data. Hence, scattered factors are not only of a theoretical interest, but a practical, too. When examining discrete data, e.g., protein sequences or incomplete or faulty transmissions of signals, scattered factors can be used as a representation (cf., [7, 29]). For instance, \text{power\_flow\_poor\_wow} are scattered factors of \text{power\_flower} \ but \text{rope\_loop} \ are not scattered factors since the letters do not occur in the correct order in \( w \). Therefore, scattered factors can be seen as a representation of \( w \) with some lost data. Hence, scattered factors are not only of a theoretical interest, but a practical, too. When examining discrete data, e.g., protein sequences or incomplete or faulty transmissions of signals, scattered factors can be used as a representation (cf., [7, 29]). For instance, a faulty transmission may be reconstructed using scattered factors as described in [12, 8, 25]. Scattered factors are also useful in sign language recognition [22] and to get alignment-free phylogeny of whole genomes or with biological subwords to detect protein S-sulfenylation sites [5, 7]. Moreover, scattered factors can be found in some famous algorithmic problems like searching for longest (increasing) subsequences [4, 3, 1], shortest common supersequences [24], string-to-string correction problems [28], most unusual time series subsequence [20], fast subsequence matching in time-series databases [9]. Furthermore, there exist neural machine translations, which use rare words with subword units [26] or byte-level subwords [29].
The foundations of scattered factors research were introduced by Higman [15], where it is shown that an infinite set of words always contains words $u$ and $w$ with $u \in \text{ScatFact}(w)$. Applications of these results can be found in [13]. In 1972, Simon defined the famous congruence relation regarding scattered factors in the context of piecewise testable events [27], today known as Simon congruence: two words $x$ and $y$ are called congruent w.r.t. $k \in \mathbb{N}$ ($x \sim_k y$), iff $x$ and $y$ have the same set of scattered factors of length $k$, i.e., $\text{ScatFact}_k(x) = \text{ScatFact}_k(y)$ with the index denoting the length of the considered scattered factors. Thus, we have $\text{aba} \sim_2 \text{aabaa}$ since $\text{ScatFact}_2(\text{aba}) = \{\text{aa}, \text{ab}, \text{ba}\} = \text{ScatFact}_2(\text{aabaa})$ and $\text{aba} \not\sim_2 \text{abab}$ since $\text{bb}$ is a scattered factor of $\text{abab}$ but not of $\text{aba}$. A profound introduction into scattered factors and Simon congruence can be found in [23, Section 6] by Sakarovitch and Simon.

Although $\sim_k$ is well studied from different perspectives with deep insights (cf. [27, 23, 10]), determining its index, i.e., determining $|\Sigma^*/\sim_k|$ for a given alphabet $\Sigma$ and given $k \in \mathbb{N}$, is still an open problem. First, in [2] a special class of words, the $k$-universal words, were investigated. A word is called $k$-universal if $\text{ScatFact}_k(w) = \Sigma^k$, i.e., $w$ has all the possible scattered factors of length $k$. By the definitions of $\sim_k$, we have that all these words are in one congruence class. These words were further investigated and characterised in [6, 2, 11]. Notice that the idea of $k$-universality coincides with the notion of $k$-richness (cf. [2] for explanations) investigated in the context of piecewise testable languages [16–18].

One of the main insights of $k$-universal words is that a word $w$ is $k$-universal iff $w$'s arch factorisation [14] has $k$ arches.

Pursuing the idea of $k$-universality, where the main focus is on the cardinality of a word's scattered factors set rather than on the question whether two words are congruent, one can define the sets $M_{i,k} = \{L \subseteq \Sigma^* : w \in \Sigma^* : \text{ScatFact}_k(w) = L, |L| = i\}$ for all $1 \leq i \leq |\Sigma|^k$, i.e. $M_{i,k}$ contains all languages of cardinality $i$ which occur as a scattered factor set of some word $w$ w.r.t. a length $k$. Notice that each such $L$ is a congruence class of $\sim_k$ and $M_{|\Sigma|^k,k} = \{\Sigma^k\}$ is built by the $k$-universal words. In this work, we investigate the sets $M_{i,k}$ for $i < |\Sigma|^k$. Since our main results are for words where exactly one scattered factor from the possible scattered factor set is absent, we call a word $m$-nearly $k$-universal if $|\text{ScatFact}_k(w)| = |\Sigma|^k - m$, i.e. $k$-universal words are 0-nearly $k$-universal in the new notion. For instance, the word $\text{aabb}$ is 1-nearly 2-universal since $\text{ba}$ is absent and $\text{aab}$ is 2-nearly 2-universal since $\text{ba}$ and $\text{bb}$ are absent. A special subclass of $m$-nearly $k$-universal words has recently been studied from an algorithmic point of view in [21]. There the authors investigated shortest absent scattered factors of words, i.e., for a given $(k-1)$-universal word $w$ the set of words with length $k$ that are not scattered factors of $w$. If this set has cardinality $m$, we obtain a subset of $m$-nearly $k$-universal words. This subset may be proper since there exists words with $m$ absent scattered factors of length $k$ without $w$ being $(k-1)$-universal, witness by the word $\text{aabbb}$ which is 13-nearly 4-universal but not 3-universal.

**Our contribution.** In this work, we give a full characterisation of 1-nearly $k$-universal words as well as all congruence classes occurring in this subset of $\Sigma^*$. 
The latter result is obtained by an algorithm that computes in linear time for a given \(u\) of a length \(k\) a word \(w\) such that \(u\) is the only absent scattered factor of \(w\). Moreover, we present an algorithm which decides in linear time whether a word is 1-nearly \(k\)-universal. Afterwards, we give some first insights into \(m\)-nearly \(k\)-universal words for \(m > 1\). Our main result in this part is built on the algorithm in [21] by putting this algorithmic result into a combinatorial context, i.e., we are able to determine the number of absent scattered factors and giving the congruence classes w.r.t. \(\sim_k\) for these sets.

**Structure of the work.** In Section 2 we give the basic definitions and notations regarding scattered factors and \(m\)-nearly \(k\)-universality. In Section 3 we present the results on 1-nearly \(k\)-universal words including the characterisation and the congruence classes w.r.t. \(\sim_k\). The result for \(m > 1\) are presented in Section 4.

## 2 Preliminaries

Let \(\mathbb{N}\) be the set of all natural numbers, \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\), \([n] = \{1, \ldots, n\}\), and \([n]_0 := [n] \cup \{0\}\).

An alphabet \(\Sigma\) is a non empty finite set whose elements are called letters. Set \(\sigma = |\Sigma|\). A word is a finite sequence of letters from \(\Sigma\). Let \(\Sigma^*\) be the set of all finite words over \(\Sigma\) with concatenation and the empty word \(\varepsilon\) as neutral element. Set \(\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}\). Let \(w \in \Sigma^*\). For all \(n \in \mathbb{N}_0\) define inductively, \(w^0 = \varepsilon\) and \(w^n = w w^{n-1}\). The length of \(w\) is the number of \(w\)'s letters; thus \(|\varepsilon| = 0\). For all \(k \in \mathbb{N}_0\) set \(\Sigma^k := \{w \in \Sigma^* \mid |w| = k\}\) and denote \(w\)'s \(i\)th letter by \(w[i]\) and by \(w[i..j]\) denote \(w[i] \cdots w[j]\) if \(i < j\), \(w[i]\) if \(i = j\), and \(\varepsilon\) if \(i > j\) for all \(i, j \in [|w|]\). Set \(\text{alph}(w) = \{a \in \Sigma \mid \exists i \in [|w|]: w[i] = a\}\) as \(w\)'s alphabet and for each \(a \in \Sigma\) set \(|w|_a = |\{i \in [|w|] \mid w[i] = a\}|\). The word \(u \in \Sigma^*\) is called a factor of \(w\) if there exist \(x, y \in \Sigma^*\) such that \(w = xuy\). In the case \(x = \varepsilon\), we call \(u\) a prefix of \(w\) and suffix if \(y = \varepsilon\). Let \(\text{Fact}(w)\), \(\text{Pref}(w)\) and \(\text{Suff}(w)\), respectively, be the sets of all factors, prefixes and suffixes of \(w\). Define the reverse of \(w\) by \(w^R = w|[w]| \cdots w[1]\) and if \(w = x_1^{k_1} x_2^{k_2} \cdots x_{\ell}^{k_{\ell}} \in \Sigma^*\) with \(k_i, \ell \in \mathbb{N}\), \(i \in [\ell]\), the condensed form (print) of \(w\) is defined by \(\text{cond}(w) = x_1 \cdots x_\ell\) assumed that \(x_j \neq x_{j+1}\) for \(j \in [\ell - 1]\). Let \(\triangleq\) be a total order on \(\Sigma\). We extend this order to the lexicographical order on \(\Sigma^*\) by \(u < v\) for \(u, v \in \Sigma^*\) iff \(u \in \text{Pref}(v)\) or \(u = xu'\) and \(v = xw'\) with \(a < b\) for \(a, b \in \Sigma\) and some \(u', v', x \in \Sigma^*\). Define \(w_\Sigma^*\) as the word in \(\Sigma^*\) with \(w_\Sigma^*[i] < \Sigma w_\Sigma^*[i + 1]\) and \(\text{alph}(w) = \Sigma\). For further definitions see [23].

After fixing the basic notations, we introduce the scattered factors.

**Definition 1.** Let \(w \in \Sigma^*\) and \(n \in \mathbb{N}_0\). A word \(u \in \Sigma^n\) is called a scattered factor of \(w\) (\(u \in \text{ScatFact}(w)\)) if there exist \(v_1, \ldots, v_{n+1} \in \Sigma^*\) such that \(w = v_1 u[1] v_2 u[2] \cdots v_n u[n] v_{n+1}\). Set \(\text{ScatFact}_k(w) = \{u \in \text{ScatFact}(w) \mid |u| = k\}\).

To give an example \(\text{cau}, \text{flower}, \text{cafe}, \text{life}\) and \(\text{ufo}\) are all scattered factors of \(\text{cauliflower}\) but neither \(\text{flour}\) nor \(\text{row}\).
Tightly related to the notion of scattered factors is the famous Simon congruence. Two words are congruent modulo $k \in \mathbb{N}_0$ if they have the same set of scattered factors of length $k$, e.g., $aba$ and $aabaa$ are congruent w.r.t. 2 since $\text{ScatFact}_2(aba) = \{aa, ab, ba\} = \text{ScatFact}_2(aabaa)$.

**Definition 2.** Two words $w, v \in \Sigma^*$ are Simon congruent w.r.t. $k \in \mathbb{N}_0$ ($w \sim_k v$) if $\text{ScatFact}_k(w) = \text{ScatFact}_k(v)$.

Since $\text{ScatFact}_k(w) \subseteq \Sigma^k$ holds for all $k \in \mathbb{N}_0$, determining the index of the Simon congruence can be split into the parametrised problem on determining how many scattered factor sets - or equivalently how many different words - exist with $|\text{ScatFact}_k(w)| = \sigma^k - m$ for all $m \in \mathbb{N}_0$. In [2, 11, 6] the scattered factor universality was investigated, which describes the problem for $m = 0$.

**Definition 3.** A word $w \in \Sigma^*$ is called $k$-universal if $|\text{ScatFact}_k(w)| = \sigma^k$. Let $i(w)$ denote the universality index, i.e. the largest $k \in \mathbb{N}_0$ such that $w$ is $k$-universal. We call a 1-universal word $w$ just universal. Denote by $\text{Univ}_{\Sigma,k}$ the set of all words $w$ with $i(w) = k$.

**Remark 4.** By definition, all $k$-universal words are congruent modulo $k$ and a $k$-universal word $w \in \Sigma^*$ is also $k'$-universal for all $k' \leq k$.

In this work, we are investigating $m$-nearly $k$-universal words. These are words, where in comparison to $\Sigma^k$, $m$ words of length $k$ are absent from the scattered factor set. A special case of these words was investigated in [21] where the shortest absent scattered factors of a word are determined. In the unary alphabet $\varepsilon$ is the only word which has $|\Sigma|^k - 1 = 0$ scattered factors and the notion is not well-defined for $m > 1$. Therefore we only consider at least binary alphabets. Moreover, we assume $\Sigma = \text{alph}(w)$ for a given $w$, if not stated otherwise.

**Definition 5.** Let $k, m \in \mathbb{N}_0$. A word $w \in \Sigma^*$ is called $m$-nearly $k$-universal if $|\text{ScatFact}_k(w)| = \sigma^k - m$. Let $\text{NUniv}_{\Sigma,m,k}$ denote the set of all $m$-nearly $k$-universal words in $\Sigma^*$. We call a 1-nearly $k$-universal word simply nearly $k$-universal.

**Remark 6.** Unlike the $k$-universality, $w \in \text{NUniv}_{\Sigma,m,k}$ does not imply $w \in \text{NUniv}_{\Sigma,m,k-1}$ for $m > 0$; we have $aba \in \text{NUniv}_{\Sigma,1,2}$ but by $i(aba) = 1$, $aba$ is not $m$-nearly 1-universal for all $m > 0$.

One of the main tools for the investigation of $m$-nearly $k$-universal words is the arch factorisation which was introduced by Hebrard [14]. In this factorisation a word is factorised into universal factors and a rest.

**Definition 7.** For a word $w \in \Sigma^*$ the arch factorisation is given by $w = ar_1(w) \cdots ar_k(w)r(w)$ for $k \in \mathbb{N}_0$ with
(a) $i(ar_i(w)) = 1$ for all $i \in [k]$,
(b) $|ar_i(w)||ar_i(w)| \notin \{1, 2, \ldots, |ar_i(w)| - 1\}$ for all $i \in [k]$, and
Remark 11. Theorem 10 implies that the length \( a \) of nearly

\[
\text{Corollary 12. Given } w \in \text{NUniv}_{\Sigma,1,k} \text{, we have } \text{ScatFact}_k(w) = \Sigma^k \setminus \{m(w)a_w\}.
\]
Proof. Follows directly by Theorem 10.

The conditions of Theorem 10 do not suffice for a characterisation of nearly $k$-universal words. Consider the word $w = (acb)\cdot ba$ with $v(w) = 1$ and alph$(r(w)) = \{a, b\}$. We have $|\text{ScatFact}_2(w)| = |\Sigma^2\setminus\{cc, bc\}|$ and thus $w \notin \text{NUniv}_{\Sigma,1.2}$. The first, naïve characterisation uses Corollary 12: all words of length $k$ ending in $a_w$, but $m(w)a_w$, have to appear within the word (all others appear necessarily).

**Proposition 13.** A word $w \in \text{NUniv}_{\Sigma,1,k}$ iff $v(w) = k - 1$, alph$(r(w)) = \Sigma\setminus\{a_w\}$, and for all $v \in \Sigma^k$ with $v[1..k-1] \neq m(w)$ and $v[k] = a_w$ there exists $i \in [k-2]$ with $v[i]v[i+1] \in \text{ScatFact}_2(ar_i(w))$ or $v[k-1]a_w \in \text{ScatFact}_2(ar_{k-1}(w))$.

**Proof.** First, let $w$ be nearly $k$-universal. The first two claims follow immediately from Theorem 10 and Corollary 12. Moreover, we know $m(w)a_w \notin \text{ScatFact}_k(w)$. Let $v \in \Sigma^k$ with $v[1..k-1] \neq m(w)$ and $v[k] = a_w$. If $v[k-1]a_w \notin \text{ScatFact}_k(ar_{k-1}(w))$, we are done. Thus, assume that $v[k-1]a_w \notin \text{ScatFact}_k(ar_{k-1}(w))$. Since $w \in \text{NUniv}_{\Sigma,1,k}$, $v \neq m(w)a_w$, and $a_w \notin \text{alph}(r(w))$, we get immediately by the pigeon hole principle that there exists $i \in [k-2]$ with $v[i]v[i+1] \in \text{ScatFact}_2(ar_i(w))$.

Second assume the three constraints to hold true and suppose that $w$ is not nearly $k$-universal. We get immediately $m(w)a_w \notin \text{ScatFact}_k(w)$. Since by supposition $w$ is not nearly $k$-universal there exists $u \notin \text{ScatFact}_k(w)$ with $u \neq m(w)a_w$. Since $v(w) = k-1$, we have for all $u \in \Sigma^{k-1}$ and $a \in \text{alph}(r(u))$ immediately $ua \in \text{ScatFact}_k(w)$. This implies $u[k] = a_w$ and $u[1..k-1] \neq m(w)$. Thus, there exists $i \in [k-2]$ with $u[i]u[i+1] \in \text{ScatFact}_2(ar_i(w))$ or $u[k-1]a_w \in \text{ScatFact}_2(ar_{k-1}(w))$. In the first case $u[1..i-1]$ can be chosen from the first $i-1$ arches (which build a $(i-1)$-universal word) and $u[i+2..k-1]a$ can be chosen from $ar_{i+1}(w)\cdots ar_{k-1}(w)$ (which is a $(k-i-1)$-universal word and $|u[i+2..k-1]| = k-i-1$). In the second case $u[1..k-2]$ be chosen from the first $k-2$ arches. Thus, in both cases we have $u \notin \text{ScatFact}_k(w)$ - a contradiction.

Since $ac, cc \in \text{ScatFact}_2(ar_1(w))$ and $ab, bc \in \text{ScatFact}_2(ar_2(w))$, we have $w = (accb) \cdot (bac) \cdot ab \in \text{NUniv}_{\Sigma,1.3}$. This characterisation is not very helpful since checking whether a word is nearly $k$-universal means to check all $\sigma^{k-1}$ options for $v$. The following characterisation does not only provide an efficient way to check whether $w \in \text{NUniv}_{\Sigma,1,k}$ but also builds the basis for an efficient algorithm regarding $\sim_k$. In beforehand, we prove that cutting off $\ell$ arches at the beginning of a nearly $k$-universal word, leads to a nearly $(k-\ell)$-universal word.

**Lemma 14.** Let $\ell \leq k-1$. If $w \in \text{NUniv}_{\Sigma,1,k}$ with $w = ar_1(w)\cdots ar_{k-1}(w)r(w)$, then $ar_{t+1}(w)\cdots ar_{k-1}(w)r(w) \in \text{NUniv}_{\Sigma,1,k-\ell}$.

**Proof.** It suffices to prove the claim for $\ell = 1$; the main statement follows inductively. Set $w' = ar_2(w)\cdots ar_{k-1}(w)r(w)$. Let $x = m(w)[1]$. We have exactly $\frac{1}{2} \cdot \sigma^k$ scattered factors of length $k$ with first letter $a$ for all $a \in \Sigma \setminus \{x\}$. The number of scattered factors starting with $x$ is $\frac{1}{2} \cdot \sigma^k - 1$ because $m(w)a_w$ is not a scattered factor of $w$. Set $S_a = \{u|a \in \text{ScatFact}(u)\}$ for all $a \in \Sigma$. Thus, for all $a, b \in \Sigma \setminus \{x\}$ we have $S_a = S_b$. Moreover, we have $S_x = S_a \setminus \{m(w)[2..k-1]a_w\}$. 
By $x = m(w)[1]$ follows that $u \in \text{ScatFact}_{k-1}(w')$ for all $u \in S_x$. These are $\sigma^{k-1} - 1$ many and by $m(w)[2..k-1]a_w \notin \text{ScatFact}_{k-1}(w')$, the claim is proven. 

\[ \Box \]

**Remark 15.** Notice that Lemma 14 is not applicable for arches in the middle: 
(a) $\alpha b \cdot (aab) \cdot b \in \text{NUniv}_{\Sigma,1,3}$ but $(ab) \cdot b \notin \text{NUniv}_{\Sigma,1,2}$. Moreover, Lemma 14 does not hold for $m > 1$: $w = (abc) \cdot (bca) \cdot bb \in \text{NUniv}_{\Sigma,7,3}$ but $(bca) \cdot bbb \notin \text{NUniv}_{\Sigma,3,2} \neq \text{NUniv}_{\Sigma,7,2}.$

Now we present a more suitable characterisation for nearly $k$-universal words. Here, $\text{ScatFact}_k(w^R) = \{u^R \mid u \in \text{ScatFact}_k(w)\}$ plays an important role.

**Theorem 16.** For $w \in \Sigma^*$ the following statements are equivalent 
\begin{enumerate}
\item $w \in \text{NUniv}_{\Sigma,1,k}$,
\item $\iota(w) = k - 1$, $|\alpha\beta(w)| = |\alpha\beta(w^R)|$, and 
(a) if $k$ is even then there exists $u_1,\ldots,u_v \in \text{PUniv}_{\Sigma,1,v}$, $x_1 \in \Sigma^+$ with $\alpha\beta(x_1) = \sigma - 1$ with $w = u_i x_i v^R$ for $i \in [2]$.
(b) if $k$ is odd then there exist $u,\ldots,u_v \in \text{PUniv}_{\Sigma,1,v}$, and $x \in \Sigma^+$ with $\alpha\beta(x) = \sigma - 1$ with $w = u x v^R$.
\item $\iota(w) = k - 1$, $|\alpha\beta(w)| = |\alpha\beta(w^R)|$, and for all $k,\tilde{k} \in \mathbb{N}$ with $k + \tilde{k} + 1 = k$ there exist $u \in \text{PUniv}_k, v \in \text{PUniv}_{\tilde{k}}$, and $x \in \Sigma^+$ with $\alpha\beta(x) = \sigma - 1$ such that $w = u x v^R$.
\end{enumerate}

**Proof.** First, we prove (2) implies (1). We have to show that $w$ is nearly $k$-universal under the three constraints. We know $m(w)a_w \notin \text{ScatFact}_k(w)$. Let $y \in \Sigma^k \setminus \{m(w)a_w\}$. If $y[k] \neq a_w$, we have immediately $y \in \text{ScatFact}_k(w)$ by the second condition. Thus, assume $y[k] = a_w$.

**Case 1: $k$ is even**
Choose $u_1, u_2, v_1, v_2, x_1, x_2$ according to condition (2). Since $u_1$ and $v_2$ are perfect $(\frac{k}{2})$-universal and $u_2$ and $v_1$ are perfect $(\frac{k}{2}) - 1$-universal, we have 
\[
 y \left[ \frac{1}{2} \right] \in \text{ScatFact}(u_1), \quad y \left[ \frac{k}{2} + 2..k \right] \in \text{ScatFact}(v^R_1), \\
 y \left[ \frac{1}{2} \right] \in \text{ScatFact}(u_2), \quad y \left[ \frac{k}{2} + 1..k \right] \in \text{ScatFact}(v^R_2).
\]
Thus, if $y[\frac{k}{2} + 1] \in \alpha\beta(x_1)$ or $y[\frac{k}{2}] \in \alpha\beta(x_2)$, we have $y \in \text{ScatFact}_k(w)$. Assume $y[\frac{k}{2} + 1] \notin \alpha\beta(x_1)$ and $y[\frac{k}{2}] \notin \alpha\beta(x_2)$. Since we have also proven the claim if two consecutive letters of $y$ are in one arch of $u_1, u_2, v_1, \text{or } v_2$, we may assume that $y[1..\frac{k}{2} - 1] = m(u_2)$ and $y[\frac{k}{2} + 2..k] = m(v_1)^R$. By $|\alpha\beta(x_1)| = \sigma - 1$, we have $y[\frac{k}{2} + 1] = m(v_2)[\frac{k}{2}]$, and analogously by $|\alpha\beta(x_2)| = \sigma - 1$, we have $y[\frac{k}{2}] = m(u_1)[\frac{k}{2}]$. Choose $i_1, i_2 \in \{|w|\}$ with $w[i_1] = m(v_2)[\frac{k}{2}]$ and $w[i_2] = m(u_1)[\frac{k}{2}]$. If $i_1 \geq i_2$, $w$ would have at least $\iota(u_1) + \iota(v^R_2) = k$ arches - a contradiction. Thus we have $i_1 < i_2$. This implies that $y[\frac{k}{2} + 1]$ has be chosen before $y[\frac{k}{2}]$ in $w$. This implies $\iota(w) < k - 1$ - a contradiction.
Lemma 14 with $u, v \in \Sigma$.

Proof. We have $N_{\text{Univ}}(\alpha(r(w))) = \sigma - 1$ and $\alpha(r(v)) = \sigma - 1$ implies that $N_{\text{Univ}}(\alpha(r(w))) = \sigma - 1$ and $\alpha(r(v)) = \sigma - 1$.

Again we can also assume $y[1..k] = m(u)$ and $y[1..k] = m(v)$ if $|\alpha(r(x))| = \sigma - 1$, we have $y[1..k] = m(w)[k-1] + 1$ which occurs after $m(v)[k-1] + 1$ in $w$. Again we obtain $\ell(w) < k - 1$, a contradiction.

Now we prove that (1) implies (3). Consider firstly $w$ be nearly $k$-universal. Then the first two claims follow immediately by Theorem 10 and the fact that $w^R$ is nearly $k$-universal.

By $\ell(w) = k - 1$, we have $w, w^R \in \text{Univ}_{\Sigma, k'}$ for all $k' < k - 1$. Let $k, k \in \mathbb{N}_{< k}$ with $k + k = k$. Thus, there exist $u \in \text{PUniv}_{\Sigma, k}$ and $v \in \text{PUniv}_{\Sigma, k}$ with $u \in \text{Pref}(w)$ and $v \in \text{Pref}(w^R)$. Choose $x \in \Sigma^*$ with $w = uxv^R$. By Lemma 14, we get $x^R \in \text{Univ}_{\Sigma, 1, k-1}$. Thus, $w^R \in \text{Univ}_{\Sigma, 1, k-1}$. Applying Lemma 14 again, we obtain $x^R \in \text{Univ}_{\Sigma, 1, 1}$. By Theorem 10 we get $|\alpha(r(x))| = \sigma - 1$.

Since (3) implies (2) immediately, the claim is proven.

We have $w = (\text{aab}) \cdot (\text{ba}) \cdot (\text{ab}) \not\in \text{Univ}_{\Sigma, 1.4}$ since we have the factorisation $(\text{aab})(\text{ba}) \cdot (\text{ab})^R$ meeting the requirements but also the factorisation $(\text{aab}) \cdot (\text{ba}) \cdot (\text{ab})^R$ not meeting them, witnessing that both factorisations are needed.

**Corollary 17.** We have $ww^R \in \text{Univ}_{\Sigma, 1.2k-1}$ if $w \in \text{Univ}_{\Sigma, 1,k}$ as well as $ww^R \in \text{Univ}_{\Sigma, 1,2k-1}$ with $a \in \Sigma$ iff $w \in \text{Univ}_{\Sigma, 1,k}$ and $a \in \alpha(r(w))$.

**Proof.** Consider first $ww^R$ with $w \in \text{Univ}_{\Sigma, 1,k}$. Since $2k-1$ is odd, $|\alpha(r(x))| = |\alpha(r(w))r(w^R)| = \sigma - 1$ shows that $ww^R \in \text{Univ}_{\Sigma, 1,2k-1}$. Now let $ww^R \in \text{Univ}_{\Sigma, 1,2k-1}$. Since $|w| = |w^R|$ we receive a factorisation $ww^R = uxv^R$ with $u, v \in \text{PUniv}_{\Sigma, k-1}, x \in \Sigma^+$ with $|\alpha(r(x))| = \sigma - 1$ as well as $u \in \text{Pref}(w)$, $v^R \in \text{Suff}(w^R)$. Since $ww^R$ is a palindrome, we have $u = v$. Thus, $ww^R = uv^R$. Moreover, $x = yy$ for $y \in \Sigma^+$ with $|\alpha(r(y))| = \sigma - 1$ holds. Applying Lemma 14 with $\ell = k - 1$ we obtain that $x^R \in \text{Univ}_{\Sigma, 1, k-1}$. Thus, $xx \in \text{Univ}_{\Sigma, 1, k-1}$.

**Fig. 1.** The factorisation of $w$ for even $k$ where $y$’s letters occur as the modus.
NUniv$_{Σ,1,k-1}$. Since multiple occurrences of letters in the rest do not have an impact on the absent scattered factor of nearly $k$-universal words, $w = uyv, w^R = yu \in \text{NUniv}_{Σ,1,k}$ follows.

Considering palindromes of odd length, we can also apply Theorem 16 directly, choosing $k = k = k$, we get $|\text{alph}(x)| = |\text{alph}(r(w) r(w^R))| = \sigma - 1$. The same argumentation as in the even case proves the claim. \hfill \Box

With Theorem 16 we are able to solve the following two problems (for a given $k$) efficiently: decide whether a word is nearly $k$-universal and find for a given $u \in Σ^k$ a $w \in \text{NUniv}_{Σ,1,k}$ such that $u \notin \text{ScatFact}_k(w)$. The latter one leads immediately to the index of the Simon congruence restricted to nearly $k$-universal words. Notice that for the first problem, a linear time algorithm is implicitly given in [21]: if $w$ is a word of length $n$, the SAS tree can be constructed in time $O(n)$ and in time $O(k)$ the lexicographically smallest shortest absent scattered factors can be determined; if there is only one shortest absent scattered factor, we have $w \in \text{NUniv}_{Σ,1,k}$. The following algorithm can only check whether a word is nearly $k$-universal but therefore does not need any additional data structures.

**Proposition 18.** Given $w \in Σ^*$ and $k \in \mathbb{N}$, we can decide whether $w \in \text{Univ}_{Σ,1,k}$ in time $O(|w|)$. In the positive, the absent scattered factor is also computed (see Algorithm 1).

**Proof.** By [2] we know that the arch factorisation can be computed in time $O(|w|)$. While computing the arch factorisation of $w$, store the end of the $(\frac{k}{2} - 1)\text{th}$ arch in $i_1$ and the end of the $\frac{k}{2} \text{th}$ in $i_2$. Analogously, while computing the arch factorisation of $w^R$, store the end of the $(\frac{k}{2} - 1)\text{th}$ arch in $j_2$ and the end of the $\frac{k}{2} \text{th}$ in $j_1$. These four values can be obtained in $O(|w|)$. Now we have to check that $w[i_1, j_2]$ and $w[i_2, j_1]$ both contain each all letters from Σ but one. This can be done in $O(\sigma)$. By $\sigma \leq n$, the claim is proven. For $k$ odd, we only need to check one factorisation.

While checking the conditions of Theorem 16, we also computed $m(w)_{\Sigma, w}$ and thus the algorithm also determines the absent scattered factor. \hfill \Box

**Remark 19.** Theorem 16 can also be used to construct nearly $k$-universal words: if $k$ is odd choose $u, v \in \text{PUniv}_{Σ,1,k}$ as well as an $x$ with $|\text{alph}(x)| = \sigma - 1$ and $uxv^R$ is nearly $k$-universal. In the case that $k$ is even, choose $u, v \in \text{PUniv}_{Σ,1,k}$ as well $x_1, x_2$ such that $|\text{alph}(x_1)| = |\text{alph}(x_2)| = \sigma - 1$. Now, we have $uxy^Rv \in \text{NUniv}_{Σ,1,k}$ iff $y[|y|] \notin \text{alph}(x_2)$ and $y[1] \notin \text{alph}(x_1)$.

Now, we present an algorithm for the second problem. Please recall that $Σ_a = Σ\backslash\{a\}$ and $w_{Σ_a}$ is the word containing all letters of $Σ_a$ w.r.t. a predefined order $\prec_Σ$ on $Σ$. These words can be preprocessed in time $O(\sigma)$ for all $a \in Σ$.

**Theorem 20.** Given $u \in Σ^k$ for $k \in \mathbb{N}$, one can compute $w \in Σ^*$ with $\text{ScatFact}_k(w) = Σ^k \backslash\{u\}$ in time $O(k)$. More precisely, there exists an algorithm needing $k$ steps computing $w \in \text{NUniv}_{Σ,1,k}$ of minimal length (see Algorithm 2).
**Data:** Given \( w \in \Sigma^* \) with arch factorisation and \( k \in \mathbb{N} \).

**Result:** True, if \( w \in \text{NU}u \Sigma_{1,k} \). False, otherwise.

if \( \varepsilon(w) \neq k - 1 \) \( \land \ |\text{alph}(r(w))| \neq \sigma - 1 \) \( \land \ |\text{alph}(r(w^R))| \neq \sigma - 1 \) then

\[
\text{return false;}
\]
else

if \( k \mod 2 = 0 \) then

\[
w_{v_1} := (r_k(w^R) \cdots ar_{k-1}(w^R)) R; \quad /* \text{The index denotes the deleted archs of } w \text{'s factorisation */}
w_{v_2} := (r_{k+1}(w^R) \cdots ar_{k-1}(w^R)) R;
\]

\[
\text{return} \ |\text{alph}(r(w_{v_1}))| = \sigma - 1 \quad \&\& \quad |\text{alph}(r(w_{v_2}))| = \sigma - 1;
\]
else

\[
w_u := (ar_{k+1}(w^R) \cdots ar_{k-1}(w^R)) R;
\]

\[
\text{return} \ |\text{alph}(r(w_u))| = \sigma - 1;
\]
end

**Algorithm 1:** Testing nearly \( k \)-universality (cf. Proposition 18)

**Proof.** Given \( k \in \mathbb{N} \) and \( u \in \Sigma^k \) the following inductive algorithm constructs a word \( w \) with \( \text{ScatFact}(w) = \Sigma^k \setminus \{u\} \); for all \( i \in [k-1] \) set iteratively \( x_i := w_{\Sigma_{u[i]} \cdot v_i \cdot u[i]} \) for \( v_i = \varepsilon \) if \( u[i] = u[i + 1] \) and \( v = u[i + 1] \) otherwise. Lastly, set \( w = x_1 \cdots x_{k-1} \cdot w_{\Sigma_{u[k]}} \).

Firstly, we want to prove that \( u \) is an absent scattered factor of the returned word \( w \). By the construction of \( x_i \), we get \( ar_i(w) = x_i \); the prefix of \( x_i \) of length \( \sigma - 1 \) contains all letters of \( \Sigma \) but \( u[i] \), then \( u[i + 1] \) is appended iff \( u[i] \neq u[i + 1] \) and lastly \( u[i] \) is appended which is therefore unique in \( x_i \). Since \( w_{\Sigma_{u[i]}} \) contains all letters of \( \Sigma \) but \( u[k] \), we get \( u[1..k-1] = m(w) \) and \( u \notin \text{ScatFact}_k(w) \).

To prove \( w \in \text{NU}u \Sigma_{1,k} \) we show the three conditions of Theorem 16. We already showed \( \varepsilon(w) = k - 1 \). Moreover, \( \text{alph}(w_{\Sigma_{u[i]}}) = \Sigma \setminus \{u[i]\} \) follows by definition. Set \( y_i = v_i u[i] w_{\Sigma_{u[i] + 1}} \) for all \( i \in [k-1] \). By the definition of \( y_i \) we get that \( y_k y_{k-1} \cdots y_i \cdots y_2 y_1 \) is the arch factorisation of \( w^R \). This implies \( \text{alph}(w^R) = \Sigma \setminus \{u[1]\} \) and thus the second condition is fulfilled. Hence, only the third conditions remains to be proven.

**case 1:** \( k \) even

Set

\[
u_1 = y_k \cdots y_{\frac{1}{2} + 1} \quad \quad v_1 = y_k \cdots y_{\frac{1}{2} + 1} \quad \quad w_{v_1} := (r_{\frac{1}{2}}(w^R) \cdots ar_{\frac{1}{2} - 1}(w^R)) R; \quad /* \text{The index denotes the deleted archs of } w \text{'s factorisation */}
w_{v_2} := (r_{\frac{1}{2} + 1}(w^R) \cdots ar_{\frac{1}{2} - 1}(w^R)) R;
\]

Thus, we get \( x_1 = w_{\Sigma_{u[1/2 + 1]}} \) and \( x_2 = w_{\Sigma_{u[1/2]}} \). Both fulfil by definition the required property.

**case 2:** \( k \) odd

Set

\[
u = x_1 \cdots x_{k-1} \quad \land \quad v = y_k \cdots y_{\frac{1}{2} + 1} \quad \land \quad x_1 = w_{\Sigma_{u[1/2 + 1]}} \quad \land \quad x_2 = w_{\Sigma_{u[1/2]}}.
\]
Thus, we get $x = w_{\Sigma}^{\Sigma^k_{w[i+1],[i]}}$ which fulfils by definition the required property.

Hence, in both cases all three properties are fulfilled and by Theorem 16, we have $w \in N\text{Univ}_{\Sigma,1,k}$.

Assuming that all $w_{\Sigma_a}$ for all $a \in \Sigma$ are precalculated, we just have to compare $u[i]$ with $u[i+1]$ and append the appropriate words for obtaining $w$. Since $u \in \Sigma^k$, we have $k$ of those comparisons and extensions of the word.

It remains to show that $w$ is of minimal length among all nearly $k$-universal words where $u$ is the only absent scattered factor. Let $w' \in N\text{Univ}_{\Sigma,1,k} \setminus \{w\}$ with $u \in \text{ScatFact}_k(w')$. By Theorem 10 we know $\iota(w') = k - 1$ and by Corollary 12 we have $m(w')a_{w'} = u$. This implies immediately $m(w) = m(w')$ and $a_{w'} = a_u$. Since each arch has to contain the complete alphabet, $w'$ has at least one arch which is shorter than a corresponding arch in $w$. By the definition of $w$ we know that this arch $\alpha$ contains each letter of $\Sigma$ exactly once. Thus, in the arch factorisation of $(w')^R$ the changed arch goes further to the left. The corresponding $x$ from Theorem 16 does not contain all letters from $\Sigma$ but one and we can conclude that $w'$ is not nearly $k$-universal.

This concludes the proof. \hfill \qed

**Data:** Given $u \in \Sigma^k$ with $\Sigma = \{a_1, \ldots, a_\sigma\}$.

**Result:** nearly $k$-universal word $w \in \Sigma^*$ with $\text{ScatFact}_k(w) = \Sigma^k \setminus \{u\}$

```plaintext
w := ε;

w_{\Sigma} = a_1 \cdots a_\sigma;

for $i = 1$ to $k - 1$ do

if $u[i] \neq u[i+1]$ then

$w := w \cdot w_{\Sigma_{a[u]}} \cdot u[i+1] \cdot u[i]$

else

$w := w \cdot w_{\Sigma_{a[u]}} \cdot u[i]$

end

end

w := w \cdot w_{\Sigma_{u[k]}}

return $w$;
```

**Algorithm 2:** Computing $w \in N\text{Univ}_{\Sigma,1,k}$ for $u \in \Sigma^k$ absent (cf. Theorem 20).

---

**Fig. 2.** An illustration for the construction for the absent scattered factor $u = \text{abccab}$. 
As illustrated in Figure 2, let \( u = abccab \) and \( \bullet \) represent placeholder. Since \( u[1..5] = m(w) \), we get \((\bullet a) \cdot (\bullet b) \cdot (\bullet c) \cdot (\bullet c) \cdot (\bullet a)\). By \( \text{alph}(r(w)) = \Sigma \setminus \{b\} \), we get \((\bullet a) \cdot (\bullet b) \cdot (\bullet c) \cdot (\bullet c) \cdot (\bullet a) \cdot \alpha \cdot \beta \cdot \gamma \cdot \delta \). Including the arches of \( w^R \) we obtain \((\bullet b) \cdot (\bullet c) \cdot (\bullet c) \cdot (\bullet a) \cdot \alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \gamma \cdot \delta \). Now, the \( \bullet \) are replaced by the missing letters from each arch of \( w \). Thus, we finally get \((\bullet b \cdot \alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \gamma \cdot \delta \cdot \gamma \cdot \delta) \cdot \beta \cdot \gamma \cdot \delta \cdot \gamma \cdot \delta \). We are now able to give a characterisation of the congruence classes of \( \sim_k \) in \( \text{NUlv}_{\Sigma,1,k} \). Since we know that for each \( u \in \Sigma^k \) there exists one congruence class, we fix \( u \in \Sigma^k \). We know so far that for each \( u \) obtained by the application of Lemma 24, we have \( w \in [u]_{\sim_k} \). Notice that Lemma 24 cannot be generalised to an equivalence, since deleting letters from arches may violate the nearly \( k \)-universality: considering \( w = (aab) \cdot b \) and deleting one \( a \) in the first arch, indeed does not change the modus, but it deletes \( aa \) and therefore we have \( |\text{ScatFact}_2(abbb)| < 3 \). Recall that the output \( w_u \) of the algorithm in Theorem 20 is w.r.t. a given order \( \prec \) on \( \Sigma \), in particular \( \text{Pref}_{\sigma_n} \), for all \( i \in [k-1] \), is the lexicographically smallest word containing all letters of \( \Sigma \) but \( m[i] \).
Analogously, \( r(w_u) \) is the lexicographically smallest word containing all letters but \( u[k] \). If we change this order, we obtain other words of the same length, which are all by Theorem 20 of minimal length. Moreover, if we choose different orders for each arch and for the rest, we still obtain a nearly \( k \)-universal word since the crucial point of Theorem 20 still holds. Thus, each such word can be obtained from \( w_u \) by applying some morphic permutation of \( \Sigma \) on \( \text{Pref}_{\sigma-1}(a_{i}(w_u)) \) and \( r(w_u) \) for all \( i \in [k-1] \).

**Definition 26.** Let \( \pi_1, \ldots, \pi_{\sigma!} \) be the different morphic permutations on \( \Sigma \), set \( p_i = \text{Pref}_{\sigma-1}(a_{i}(w_u)) \) for all \( i \in [k-1] \), and choose \( s_1, \ldots, s_{k-1} \in \Sigma^* \) with \( w_u = p_1s_1 \cdots p_{k-1}s_{k-1}r(w_u) \). Define the basis of \( [w_u]_{\sim_k} \) by \( B_u = \{ w \in \Sigma^* | \exists i_1, \ldots, i_k \in [\sigma!] : w = \pi_{i_1}(p_1)s_1 \cdots \pi_{i_{k-1}}(p_{k-1})s_{k-1}\pi_{i_k}(r(w_u)) \} \).

**Remark 27.** For \( u \in \Sigma^k \), we have \( |B_u| = ((\sigma - 1)!)^{k-1}(\sigma - 1)! \).

Based on this \( B_u \) and Lemma 24, we can characterise \( [w_u]_{\sim_k} \).

**Theorem 28.** Given \( u \in \Sigma^k \), we have \( [w_u]_{\sim_k} = \{ w \in \Sigma^* | \exists v \in B_u : w \in P(v) \} \).

**Proof.** If \( w \in P(v) \) for some \( v \in B_u \), we have immediatly \( w \in [w_u]_{\sim_k} \). Assume \( w \in [w_u]_{\sim_k} \). Thus \( w \in \text{N Univ}_{\Sigma,1,k} \) and \( m(w)a_w = u \). Now, we examine \( w \)'s \( i \)th arch for a fixed \( i \in [k-1] \). We know \( \text{alph}(i_n(w)) = \Sigma \setminus \{ u[i] \} \). Let \( u[i] \neq u[i+1] \). Suppose that \( |i_n(w)|_{u[i+1]} = 1 \). The application of Theorem 16 with \( k = i-1 \) and \( k = k-i \) implies that the \( (k-i) \)th arch from \( v^R \) ends in this occurrence of \( u[i+1] \), i.e. \( u[i], u[i+1] \notin \text{alph}(x) \). Since this is a contradiction to \( w \in \text{N Univ}_{\Sigma,1,k} \) we not only have \( |i_n(w)|_{u[i+1]} \geq 2 \) but also Theorem 16 leads to \( \text{ar}_2(w) = \alpha_1u[i+1][\Sigma] \) with \( \text{alph}(\alpha_1) = \Sigma \setminus \{ u[i] \} \) and \( \beta_1 \in (\Sigma \setminus \{ u[i] \})^* \). Thus, there exists \( v_1 \in \text{ScatFact}_{\sigma-1}(\alpha_1) \) with \( \text{alph}(v_1) = \Sigma \setminus \{ u[i] \} \). Hence there exists a permutation \( \pi \) on \( \Sigma \) which morphically applied yields \( \text{ar}_2(w) = \text{Pref}_{\sigma-1}(u_{i}(w_u)). \) Since \( \text{alph}(r(v)) = \Sigma \setminus \{ u[k] \} \) we get by the same argument an \( r \) which is a permutation of \( r(w_u) \). This leads to \( v = v_1u[1] \cdots v[k-1]u[k-1]r \in B_u \). Adding all letters of \( \alpha_1, \beta_1 \) and \( r(w) \), resp., which are not in \( v_1 \) and \( r \), implies \( w \in P(v) \). \( \square \)

Let \( u = \text{abc} \) and \( a < b < c \). By Theorem 20 we get \( w_u = (\text{bcba}) \cdot (\text{acb}) \cdot (\text{acbb}) \cdot \text{ab} \) and \( w \in B_u \) iff \( w = w_1w_2w_3w_4 \) with \( w_1 \in \{ \text{bcba}, \text{cbba} \} \), \( w_2 \in \{ \text{acb}, \text{cab} \} \), \( w_3 \in \{ \text{acbb}, \text{cabc} \} \), \( w_4 \in \{ \text{ab}, \text{ba} \} \). Thus, we have 16 basis elements for \( u \). Each word this can be enriched by additional letters in the inner of an arch and the rest w.r.t. Lemma 24 to obtain all elements equivalent to \( w_u \).

We finish this section with a third characterisation of nearly \( k \)-universal words that relies on Theorem 16 and Lemma 14 and illustrates the relation of \( w \) and \( w^R \) in \( \text{N Univ}_{\Sigma,1,k} \).

**Theorem 29.** We have \( w \in \text{N Univ}_{\Sigma,1,k} \) iff \( l(w) = k-1 \), \( |\text{alph}(r(w))| = \sigma - 1 \), and \( (\text{ar}_2(w^R)) \cdots (\text{ar}_{k-1}(w^R))r(w^R) \in \text{N Univ}_{\Sigma,1,k-1} \).

**Proof.** Consider first \( w \in \text{N Univ}_{\Sigma,1,k} \). The first two conditions follow by Theorem 10. Since \( w \in \text{N Univ}_{\Sigma,1,k} \), we have \( w^R \in \text{N Univ}_{\Sigma,1,k} \) and Lemma 14.
implies $\hat{w} = a_2(w_R) \cdots a_{k-1}(w_R) r(w_R) \in \text{NUuni}_k$. Thus, we have $\hat{w} R \in \text{NUuni}_{k-1}$.

Consider now $w \in \Sigma^*$ with $i(w) = k - 1$, $|\text{alph}(r(w))| = \sigma - 1$, and $\hat{w} = (a_2(w_R) \cdots a_{k-1}(w_R) r(w_R)) R \in \text{NUuni}_{k-1}$. Thus, we have $a_i(\hat{w}) = a_i(w)$ for all $i \in [k - 2]$. By $\hat{w} \in \text{NUuni}_{k-2}$ we get $|\text{alph}(r(\hat{w}))| = \sigma - 1$. Thus, $w = \hat{w} a_1(w_R)$ fulfills the conditions of Theorem 16 and the claim is proven. \[\square\]

Notice that only the deletion of a reversed arch from the beginning leads to an equivalence. Deleting the first arch of $w$ does not suffice for a characterisation as witnessed by $w = bcaabcbab$: indeed, we have $i(bcab) = 1$, $\text{alph}(r(w)) = \Sigma \setminus \{e\}$, and $abcabab \in \text{NUuni}_{1,2}$ but we get $(bca) \cdot (abc) \cdot ab \not\in \text{NUuni}_{1,3}$.

In this section, we presented a characterisation for nearly $k$-universal words as well as the index of $\sim_k$ and a characterisation of its congruence classes.

4 $m$-Nearly $k$-Universal Words

In this section, we consider $m$-nearly $k$-universal words, where $m$ is not necessarily 1, i.e., we are interested in $w \in \Sigma^*$ with $|\text{ScatFact}_k(w)| = \sigma^k - m$. Implicitly, a subset of these words was investigated in [21]. There, the authors determine all shortest absent scattered factors, i.e. if $i(w) = k - 1$ and $|\text{ScatFact}_k(w)| = \sigma^k - m$, we have that $w \in \text{NUuni}_{k,m,k}$. In contrast to 1-nearly $k$-universal words, for $m > 1$, $i(w) = k - 1$ does not necessarily hold as witnessed by $ababac \in \text{NUuni}_{14,3}$ with $i(ababca) = 1 \neq 2$. Thus, a thorough characterisation of $\text{NUuni}_{k,m,k}$ is still open. Unfortunately, we cannot give such a characterisation but we present some first insights for $m \in \{\sigma^k, \sigma^{k-1}, 2\}$ as well as a full characterisation of the subclass established in [21] including the congruence classes of $\sim_k$ in this case.

Remark 30. Similar to $\text{NUuni}_{0,k} = \text{Univ}_{k}$, the set $\text{NUuni}_{k,\sigma^k,k}$ provides exactly one equivalence class for $\sim_k$, since exactly the words strictly shorter than $k$ do not have any scattered factor of length $k$.

Now, we have a look at $m \in \{\sigma^k - 1, \sigma^k - 2\}$. Since $w \in \text{NUuni}_{k,\sigma^k-1,k}$ for all $w \in \Sigma^k$, we have $|u| \geq k + 1$ for all $u \in \text{NUuni}_{k,m,k}$ with $m < \sigma^k - 1$.

Proposition 31. For each $k \in \mathbb{N}$, we have $|\text{NUuni}_{k,\sigma^k-1,k}/\sim_k| = \sigma^k$.

Proof. First, we can observe that for $u, v \in \Sigma^k$ with $u \neq v$, we have $u, v \in \text{NUuni}_{k,\sigma^k-1,k}$ and $|u|_{\sim_k} \neq |v|_{\sim_k}$. Now let $w \in \Sigma^*$ with $|u| \geq k$. If $\text{alph}(w) = \{a\} \subseteq \Sigma$, we have immediately that all scattered factors but $a^k$ are absent. Thus, we have $|a^k|_{\sim_k} = |a|^k$ for all $a \in \Sigma$. If, on the other hand, we have $|\text{alph}(w)| \geq 2$, we can factorise $w = w_1 a w_2 b w_3$ with $a \not\in \text{alph}(w_1)$ and $b \not\in \text{alph}(w_3)$ and obtain that $w[1..k]$ and $w[[|w| - k + 1..|w|]$ are different scattered factors of $w$. Thus, all $w \in \text{NUuni}_{k,\sigma^k-1,k}$ are unary. This proves the claim. \[\square\]

Lemma 32. If $w \in \text{NUuni}_{k,\sigma^k-2,k}$ then $|\text{alph}(w)| = 2 = |\text{cond}(w)|$. 
Proof. If $|\text{alph}(w)| \leq 1$, we have $|\text{ScatFact}_k(w)| \in \{0, 1\}$. Suppose $|\text{alph}(w)| \geq 3$. Then there exists $a_1, a_2, a_3 \in \Sigma$. Choose words $w_1, w_2, w_3, w_4 \in \Sigma^*$ and $r_1, r_2, r_3 \in \mathbb{N}$ such that

$$w = w_1a_1^{r_1}w_2a_2^{r_2}w_3a_3^{r_3}w_4$$

and $w_1||w_2|, w_2|, w_3|, w_4| \neq a_1, a_2, a_3$. Moreover, choose $v_1 \in \text{Pref}(w_1a_1^{r_1-1})$, $v_2 \in \text{Pref}(w_2a_2^{r_2-1})$, $v_3 \in \text{Pref}(w_3a_3^{r_3-1})$, and $v_4 \in \text{Pref}(w_4)$ such that $|v_1| + |v_2| + |v_3| + |v_4| = k - 2$. Set

$$u_1 = v_1a_1^{r_1}v_3a_3^{r_3}v_4, u_2 = v_1a_1^{r_1}v_3a_3^{r_3}v_4, u_3 = v_1a_1^{r_1}a_2^{r_2}v_3v_4.$$

Then we have $u_1, u_2, u_3 \in \text{ScatFact}_k(w)$. With $\ell_1 = |v_1| + 1$. By $u_1[\ell_1] = v_2[|v_1|] \neq a_1 = u_2[|v_1|], u_3[|v_1|]$, we get $u_1 \neq u_2$. Let $a^*_s \in \text{Suff}(v_1)$ maximal for $s \in \mathbb{N}_0$. Then $a_3^{s+1}v_4 \in \text{Suff}(u_2)$ but $a_3^{s+1}v_4 \notin \text{Suff}(u_3)$. Thus, $u_2 \neq u_3$ - a contradiction to $w \in \text{NUuv}_{\Sigma, a^k-2, k}$. Suppose there exist $r_1, r_2, r_3 \in \mathbb{N}$ and $w_1 \in \Sigma^*$ with $w = a^r b^s a^r w_1$, i.e., $|\text{cond}(w)| > 2$ (we assume w.l.o.g. that $w[1] = a$). Set $u_1 = a^{r_1-1}b^s a^{r_1}v_1$, $u_2 = a^r b^s-1 a^{r_1}v_1$, $u_3 = a^r b^s a^{r_1-1}v_1$ such that $s_1 - 1 + s_2 + s_3 + |v_1| = k$ for $s_i \leq r_i, i \in [3]$, $v_1 \in \text{Pref}(w_1)$. Then $u_1, u_2$, and $u_3$ are different scattered factors of $w$ - a contradiction.

\textbf{Proposition 33.} For each $k \in \mathbb{N}$, we have $|\text{NUuv}_{\Sigma, a^k-2, k} / \sim_k| = 2\binom{n}{2}(k+2)$.

\textbf{Proof.} Let $w \in \text{NUuv}_{\Sigma, a^k-2, k}$. By Lemma 32 we have $|\text{alph}(w)| = 2$ and there exist $r_1, r_2 \in \mathbb{N}$ such that w.l.o.g. $w = a^r b^s w_1$ with $r_1 + r_2 \geq k + 1$. Thus all $a^r b^s$ are scattered factors of $w$ of length $k$ with $s_1 + s_2 = k, s_1 \leq r_1, s_2 \leq r_2$.

\textbf{case 1:} $r_1 + r_2 = k + 1$

In this case, we have exactly two scattered factors, namely $a^{r_1-1}b^s$ and $a^{r_1}b^{s-1}$ (choosing less $a$ required more $b$ than available and v.v.). Thus, for fixed $a_1, a_2 \in \Sigma$, all classes $[a_1^{r_1}a_2^{s-1}]_k$ are different, for all possible $t_1, t_2 \in \mathbb{N}$ with $t_1 + t_2 = k + 1$. These are $2k$ congruence classes for each choice of $a_1, a_2$. Thus, we have $2k\binom{n}{2}$ classes. Notice that for all these congruence classes $[a_1^{r_1}a_2^{s-1}]_k$, we have $|\text{alph}(w)| = 2$ for all $w \in \text{ScatFact}_k(a_1^{r_1}a_2^{s-1})$.

\textbf{case 2:} $r_1 + r_2 > k + 1, r_1, r_2 < k$

In this case, we have $\min\{r_1, r_2\}$ different scattered factors and by the choice of $w$, we have $\min\{r_1, r_2\} = 2$ leading to $w \in \{a^r b^s, a^r b^t\}$ with $a^r b^{k-2}, a^r b^{k-1}$ and $a^r b^s, a^r b^{s-1}$, resp., as scattered factors. By $a^r b^{k-2} \in [a^r b^{k-2}]_{-k}$ and $a^r b^{s-1} \in [a^r b^{s-1}]_{-k}$, we do not obtain different classes.

\textbf{case 3:} $r_1 + r_2 > k + 1$ and $r_1 \geq k$ or $r_2 \geq k$

In this case, we have $|\text{ScatFact}_k(w)| = \min\{r_1, r_2\} + 1$. By the choice of $w$, we have $2 = \min\{r_1, r_2\} + 1$, thus $1 = \min\{r_1, r_2\}$ leading to $w \in \{a^r b^s, a^r b^t\}$ with the scattered factors $a^r b^{k-1}, a^r b^s$. This implies that in this case we get new congruence classes, since all words have one unary and one binary scattered factor. Thus, for fixed $a_1, a_2 \in \Sigma$, the four classes $[a_1^{r_1}a_2^{s-1}]_{-k}, [a_1^{r_1}a_2^{s-1}]_{-k}, [a_2a_1^{r_2}]_{-k}$, and $[a_2^s]_{-k}$, for a fixed $t > k$ are different. Thus, we have $4\binom{n}{2}$ new classes.

Summing up, we get $|\text{NUuv}_{\Sigma, a^k-2, k} / \sim_k| = 2\binom{n}{2}(k+2)$. \qed
Proposition 33 shows that the formula determining the index of \( \alpha \) gets more complicated the farther \( m \) is from 0 or \( \sigma^k \), resp. Now we show a similar result to Theorem 10 for \( \text{NUniv}_{\Sigma,2,k} \) backing the observation that the conditions on \( w \) get more complicated. Notice that Theorem 34 does not hold for \( \sigma = 2 \) witnessed by \( w = aabaa \in \text{NUniv}_{\Sigma,4,3} \) but \( \iota(w) = 1 \). Moreover, \( w \in \text{NUniv}_{\Sigma,\sigma-1,k} \) implies \( \iota(w) = k-1 \).

**Theorem 34.** Let \( w \in \text{NUniv}_{\Sigma,2,k} \) with \( \sigma > 2 \). Then \( \iota(w) = k-1 \) and either \( |\text{alph}(r(w))| = |\text{alph}(r(w^R))| = \sigma - 1 \), or \( |\text{alph}(r(u))| = \sigma - 1 \) and \( |\text{alph}(r(u^R))| = \sigma - 2 \) for all \( u \in \{w, w^R\} \).

**Proof.** Suppose \( \iota(w) < k-1 \). Then there exits \( v \in \Sigma^{k-1} \) with \( v \notin \text{ScatFact}_k(w) \). Since \( \sigma > 2 \) there exist \( a_1, a_2, a_3 \in \Sigma \). This implies \( va_i \notin \text{ScatFact}_k(w) \) for all \( i \in [3] \). Thus, we know \( \iota(u) = k-1 \).

If we had \( w \in \text{PUniv}_{\Sigma, \Sigma/k-1} \), each \( m(w)a \) for \( a \in \Sigma \) would be an absent scattered factor. Thus, we have \( r(w) \neq \varepsilon \). Analogously we get \( r(w^R) \neq \varepsilon \). Similarly to the previous argumentation, if \( |\text{alph}(r(w))| < \sigma - 2 \) or \( |\text{alph}(r(w^R))| < \sigma - 2 \), we would have at least three absent scattered factors. In the case of \( \text{alph}(r(w)) = \Sigma \setminus \{a, b\} \) for \( a, b \in \Sigma \) with \( a \neq b \) we know that \( m(w)a, m(w)b \notin \text{ScatFact}_k(w) \). Thus, \( m(w)^R, b m(w)^R \notin \text{ScatFact}_k(w^R) \). Thus, \( a, b \notin \text{alph}(r(w^R)) \). If we had \( m(w)^R \neq m(w^R) \), then \( m(w)^R a \) would be a third absent scattered factor. Thus, we have \( m(w)^R = m(w^R) \). This leads to the following contradiction: by \( ar_1(w) = r(w^R)m(w^R)[k-1] = r(w^R)m(w)[1] \) and \( a, b \notin \text{alph}(r(w^R)) \) we would get that either \( a \) or \( b \) cannot be in \( ar_1(w) \).

We finish this section by characterising \( \text{NUniv}_{\Sigma, m,k} \cap \text{Univ}_{\Sigma, k-1} \). Let from now on \( w \in \text{Univ}_{\Sigma, k-1} \). By \( |\text{alph}(r(w^R))| < \sigma \), we have \( r(w^R) \in \text{Pref}(ar_1(w^R)) \) and \( m(w^R)[1] \in \text{alph}(ar_1(w^R)) \). Thus, choose \( \alpha_{k-1}, \beta_{k-1} \in \Sigma^* \) with \( ar_{k-1}(w) = \alpha_{k-1} \beta_{k-1} \) and \( ar_1(w^R) = (\beta_{k-1} r(w^R))^R \). With \( \text{alph}(\beta_{k-1}) \subseteq \Sigma \), inductively there exist \( \alpha_i, \beta_i \in \Sigma^* \) such that \( ar_{k-i}(w) = \alpha_i \beta_i \) and \( ar_i(w^R) = (\beta_i \alpha_{i+1})^R \) with \( \alpha_i = r(w) \) and \( \alpha_1 = r(w^R) \), for all \( i \in [k-1] \).

![Fig. 3. α-β factorisation of w.](image)

**Proposition 35.** Let \( u \in \Sigma^k \). Then \( u \notin \text{ScatFact}_k(w) \) iff \( u[1] \notin \text{alph}(\beta_1) \setminus \text{alph}(\alpha_1) \), \( u[i] \in \text{alph}(\beta_i) \), \( u[i] u[i+1] \notin \text{ScatFact}_2(\beta_i \alpha_{i+1}) \) for all \( i \in [k-1], \{1\} \), and \( u[k] \notin \text{alph}(r(w)) \).
Proof. Let \( w \in \text{NUinv}_{\Sigma,m,k} \) with \( \iota(w) = k - 1 \).

Next, we want to characterise each length \( k \) absent scattered factor of \( w \). Consider \( u \in \Sigma^k \) with \( u \notin \text{ScatFact}_k(w) \). Suppose \( u[1] \in \text{alph}(r(w^R)) \). By \( \iota(w^R) = k - 1 \), we may choose \( u[i] \) from \( \text{ar}_{k-i+1}(w^R) \) and get \( u \in \text{ScatFact}_k(w) \). Thus, \( u[1] \notin \text{alph}(r(w^R)) \) but since \( u[1] \in \text{alph}(ar_1(w)) \) we have \( u[1] \notin \text{alph}(\beta_1) \setminus \text{alph}(\alpha_1) \).

If \( u[1]u[2] \in \text{ScatFact}_2(\text{ar}_1(w)) \), we could choose \( u[i] \in \text{ar}_{i-1}(w) \) for all \( i \in [k-1]\{1,2\} \), and would get \( u \in \text{ScatFact}_k(w) \). Analogously to the argumentation for \( u[1] \notin \text{alph}(\alpha_1) \), we get \( u[1]u[2] \notin \text{ScatFact}_2(\alpha_1\beta_2) \). This concludes the induction basis.

As induction hypothesis, assume for one fixed \( i \in [k-1] \) and all \( \ell \in [i-1] \)

1. \( u[\ell] \in \text{alph}(\beta_\ell) \),
2. \( u[\ell]u[\ell+1] \notin \text{ScatFact}_2(\alpha_\ell\beta_{\ell+1}) \)

Consider \( u[i] \). Suppose that \( u[i] \in \text{alph}(\alpha_i) \). Then, we have \( u[i-1]u[i] \) is a scattered factor of \( \text{ar}_{k-i+1}(w^R) \). As there are exactly \( k - i \) arches preceding \( \alpha_i \), in \( w^R \), we have \( u \in \text{ScatFact}_k(w) \), a contradiction. Thus, \( u[i] \notin \text{alph}(\beta_i) \). With \( \text{ar}_{k-i}(w^R) = (\beta_\ell \alpha_{\ell+1})^R \), we have \( u[i+1] \in \text{alph}(\alpha_i) \). Suppose \( u[i]u[i+1] \in \text{ScatFact}_2(\beta_\ell \alpha_{\ell+1}) \), thus \( u[i+1] \in \text{ScatFact}_{i+1}(\text{ar}_1(w) \cdots \text{ar}_i(w) \alpha_{i+1}) \) and as there are exactly \( k - i - 1 \) arches preceding in \( w^R \) it follows that \( u \in \text{ScatFact}_k(w) \), a contradiction. Thus, \( u[i]u[i+1] \notin \text{ScatFact}_2(\beta_\ell \alpha_{\ell+1}) \). This proves 1. and 2. for \( u[i] \) and concludes the induction.

Additionally, \( u[k] \notin \text{alph}(r(w)) \) as this would contradict \( u \notin \text{ScatFact}_k(w) \) and, analogously, \( u[k-1]u[k] \notin \text{ScatFact}_2(\beta_{k-1}r(w^R)) \).

So far, we have proven that \( u \notin \text{ScatFact}(w) \) then \( u[i] \in \text{alph}(\beta_\ell) \), \( u[i]u[i+1] \notin \text{ScatFact}_2(\beta_\ell \alpha_{\ell+1}) \) for all \( i \in [k-1] \), and \( u[k] \notin \text{alph}(r(w)) \).

Now, we prove the other direction, i.e., if the conditions hold for some \( u \in \Sigma^k \), we have \( u \notin \text{ScatFact}_k(w) \). Thus, consider \( u \in \Sigma^k \) such that \( u[1] \in \text{alph}(\beta_1) \setminus \text{alph}(\alpha_1) \), \( u[i] \in \text{alph}(\beta_\ell) \), \( u[i]u[i+1] \notin \text{ScatFact}_2(\beta_\ell \alpha_{\ell+1}) \) for all \( i \in [k-1]\{1\} \), and \( u[k] \notin \text{alph}(r(w)) \). Suppose that \( u \in \text{ScatFact}_k(w) \). Then, \( u[1] \) occurs first in \( \beta_1 \), and the letters \( u[i] \) can only be chosen first from \( \text{ar}_i(w) \) for every \( i \in [k-1] \). But \( u[k] \notin \text{alph}(r(w)) \) leads to a contradiction. Thus, \( u \notin \text{ScatFact}_k(w) \). \( \square \)

Define \( f_w : [n[w]] \to [k] \) such that \( f_w(i) = \ell \) iff \( w \)'s \( i \)th letter belongs to \( \text{ar}_\ell(w) \), for \( i \in [n[w] - 1] \). Setting \( M_{w,1} = \text{alph}(\beta_1) \setminus \text{alph}(\alpha_1) \) and \( M_{w,j} = \text{alph}(\beta_j) \setminus \text{alph}(\beta_{j-1} \cdots \beta_1 |_{\alpha_{j+1}}) \setminus \text{alph}(\beta_{1,\ell}) \) where \( f_w(j) = i \), \( j = j - (\sum_{i=1}^{\ell} | \text{ar}_i(w) | + | \alpha_i |) \) and \( \text{alph}(\beta_k) = \Sigma \setminus \text{alph}(r(w)) \), as well as \( M'_{w,1} = g_{w,1}(M_{w,1}) \) and \( M'_{w,j} = g_{w,f(j)+1}(M_{w,j}) \) for all \( 2 \leq j < \max\{m \mid f_w(m) < k\} \). Let \( h_w(i) = \sum_{j \in M'_{w,1}} h_w(j) \) for all \( i \in \{ \ell \mid f_w(\ell) < k - 1 \} \) and \( h_w(i) = | \Sigma \setminus \text{alph}(r(w)) | \) otherwise.

Remark 36. Notice that by the definition of \( m(w) \) and Proposition 35, we have \( m(w)[i+1] \in M_{w,g_{w,f(i)}(m(w)[i])} \) for all \( i \in [k-2] \).

Proposition 37. If \( w \in \text{NUinv}_{\Sigma,m,k} \cap \text{NUinv}_{\Sigma,k-1} \) then \( m = h_w(1) \).
Proof. Choose a sequence of numbers $\mathcal{I}_u \in \mathbb{N}^{k-1}$ such that $\mathcal{I}_u[1] \in M'_{w,1}$ and $\mathcal{I}_u[i + 1] \in M'_{w,\mathcal{I}_u[i]}$ for all $i \in [k - 1]$. Then for the word $u \in \Sigma^k$ such that $u[i] = w[\mathcal{I}_u[i]]$ for $i \in [k - 1]$ and $u[k] \in \Sigma \setminus \text{alph}(r(w))$ we have

- $u[1] \in \text{alph}(\beta_i) \setminus \text{alph}(\alpha_1)$,
- $u[i] \in \beta_i$,
- $u[i]u[i + 1] \not\in \text{ScatFact}_2(\beta_i \alpha_{i+1})$ for all $i \in [k - 2]\setminus\{1\}$, and
- $u[k] \not\in \text{alph}(r(w))$.

Thus, by Proposition 35, we have $u \not\in \text{ScatFact}_k(w)$. Then, calculating $h_w(1)$ recursively equals the number of possibilities to choose such sequences $\mathcal{I}_u$ and extend them with any letter $a_w \not\in r(w)$. Each such sequence is associated to a different absent scattered factor $u$, i.e., $h(1)$ equals exactly the number $m$ of length $k$ absent scattered factors in $w$. \hfill \square

The following lemma shows that $u \in \Sigma^k$ is absent in $w, w'$ iff the sets of possible candidates for positions in $\beta_i$ coincide for $w$ and $w'$ resp.

**Lemma 38.** Let $w, w' \in \text{Univ}_{\Sigma,k-1}$ with $\text{alph}(r(w')) \subseteq \text{alph}(r(w))$ and $u \in \Sigma^k$ with $u \not\in \text{ScatFact}_k(w)$. Choose $\mathcal{I}[1] \in M'_{w,1}$ and $\mathcal{I}[i + 1] \in M'_{w,\mathcal{I}[i]}$ such that $u[1,k - 1] = w[\mathcal{I}[1]] \cdots w[\mathcal{I}[k - 1]]$. Then $u \not\in \text{ScatFact}_k(w')$ iff there exist $\mathcal{I}'[1] \in M'_{w,1}$ and $\mathcal{I}'[i + 1] \in M'_{w,\mathcal{I}'[i]}$ with $u[1,k - 1] = w'[\mathcal{I}'[1]] \cdots w'[\mathcal{I}'[k - 1]]$ and $u[i] \in M_{w,\mathcal{I}[i]} \cap M_{w',\mathcal{I}'[i]}$ for all $i \in [k - 1]$.

**Proof.** First, consider $u \not\in \text{ScatFact}_k(w')$. By the definition of the sets $M'_{w,1},$ $M'_{w,}$, and $M'_{w'}$, the claim follows immediately.

The second direction follows by $\text{alph}(r(w')) \subseteq \text{alph}(r(w))$. \hfill \square

For $w, w' \in \text{Univ}_{\Sigma,k-1}$ and $u \in \Sigma^k$, let $C(u, w, w')$ be the predicate of the iff-conditions for $u \not\in \text{ScatFact}_k(w)$.

**Theorem 39.** For all $w, w' \in \text{Univ}_{\Sigma,k} \cap \text{Univ}_{\Sigma,k-1}$, we have $w \sim_k w'$ iff $C(u, w, w')$ and $C(u, w', w)$ for all $u \in \Sigma^k$.

**Proof.** Follows directly by Lemma 38. \hfill \square

Notice that $w \sim_k w'$ is equivalent to $M_{w,j} = M_{w',j'}$ for all $j, j'$ according to appropriate sequences $\mathcal{I}$ and $\mathcal{I}'$ - illustrated in the following example.

**Example 40.** To give an example, consider the word $w = (aabc) \cdot (bcca) \cdot b \in \text{NUniv}_{\Sigma,4,3}$. Applying Proposition 35 results in the absent scattered factors $\text{baa, bac, caa, cac}$. Considering the appropriate factorisation in $\alpha_i, \beta_i$ for $i \in [k - 1]$, we get $\alpha_1 = a, \beta_1 = abc, \alpha_2 = bc, \beta_2 = ca$ and $\alpha_3 = b$.

Now, we want to calculate $h_w(1)$ as in Proposition 37. Thus, we need to consider $f_w$ first and have $f_w(i) = 1$ for $i \in [4], f_w(i) = 2$ for $i \in [8]\setminus[4]$ and $f_w(9) = 3$. Now, $g_w(j)(a)$ defines the index of the leftmost occurrence of $a$ in the $\ell^\text{th}$ arch. Here we give an example for the leftmost occurrence of $a$ in the first arch, described by $g_{w,1}(a) = \min\{i \mid w[i] = a \wedge f_w(i) = 1\} = \min\{1, 2\} = 1$, and
in the second arch respectively, i.e., \( g_w,2(a) = \min \{ i \mid w[i] = a \wedge f_w(i) = 2 \} = \min \{ 8 \} = 8 \). By definition we have

\[
M_{w,1} = \text{alph}(\beta_1) \setminus \text{alph}(\alpha_1) = \text{alph}(abc) \setminus \text{alph}(a) = \{ b, c \},
\]

\[
M_{w,3} = (\text{alph}(\beta_{1+1}) \setminus \text{alph}(\beta_1[2+1..|\beta_1||\alpha_1+1])) \cap \text{alph}(\beta_1[1..2])
\]

\[
= (\{ c, a \} \setminus \{ c, b \}) \cap \{ a, b \} = \{ a \},
\]

\[
M_{w,4} = (\text{alph}(\beta_{1+1}) \setminus \{ \text{alph}(\beta_1[3+1..|\beta_1||\alpha_1+1])) \cap \text{alph}(\beta_1[1..3])
\]

\[
= (\{ c, a \} \setminus \{ c, b \}) \cap \{ a, b, c \} = \{ a \},
\]

\[
M_{w,8} = (\text{alph}(\beta_{2+1}) \setminus \{ \text{alph}(\beta_2[2+1..|\beta_2||\alpha_2+1])) \cap \text{alph}(\beta_2[1..2])
\]

\[
= (\{ a, c \} \setminus \{ b \}) \cap \{ a, c \} = \{ a, c \}.
\]

Further, we get

\[
M'_{w,1} = g_w,1(M_{w,1}) = g_w,1(\{ b, c \}) = \{ 3, 4 \},
\]

\[
M'_{w,3} = g_w,f(3)+1(M_{w,3}) = g_w,2(\{ a \}) = \{ 8 \},
\]

\[
M'_{w,4} = g_w,f(4)+1(M_{w,4}) = g_w,2(\{ a \}) = \{ 8 \}.
\]

Notice that \( M'_{w,j} \) for all \( j < 4 \) is not defined since \( f_w(j) > 1 \), thus \( g_w,f(j)+1 \) is not defined. Now, it is easy to see how the sequences \( \mathcal{I}_w \) belong to the absent scattered factors of \( w \). With \( \mathcal{I}_w[1] \in M'_{w,1} \) and \( \mathcal{I}_w[2] \in M'_{w,2} \cap \mathcal{I}_w[1] \), the possible sequences are \( (3, 8) \) and \( (4, 8) \). Since \( w[3] = b, w[4] = c \) and \( w[8] = a \) all absent scattered factors of \( w \) have either one of them as prefix and end in one of the letters \( a \) or \( c \) (missing in \( r(w) \)).

To determine \( h_w(1) \), we have with \( h_w(8) = |\Sigma| - |\text{alph}(r(w))| = 2 \)

\[
h_w(1) = \sum_{j \in M'_{w,1}} h_w(j)
\]

\[
= h_w(3) + h_w(4)
\]

\[
= \sum_{j \in M'_{w,3}} h_w(j) + \sum_{j \in M'_{w,4}} h_w(j)
\]

\[
= h_w(8) + h_w(8)
\]

\[
= 4.
\]

Fig. 4. Factorisation of \( w = aabcccab \)
Moreover, we have \((aabc) \cdot (bcca) \cdot b \sim_k (aabc) \cdot (bcca) \cdot b\) since the letters occurring in \(\alpha'_i, \beta'_i\) of the factorisation of \(aabcbcbbcab\) for \(i \in [k-1]\) are pairwise equal to those in \(w\).

\[
\begin{array}{c}
\text{Fig. 5. Factorisation of } w' = aabcbbcab \\
\end{array}
\]

Similarly, we have \(aabcbbcab \not\sim_k aabcbbcab \in \mathbb{N}^{\text{Univ}}_{\Sigma, 5, 3}\), assuming a given factorisation into \(\alpha''_i, \beta''_i\) for \(i \in [k-1]\), since \(\alpha_2 \neq \alpha''_2\) as illustrated in Figure 6. Thus, \(ccc\) is absent as well.

\[
\begin{array}{c}
\text{Fig. 6. Factorisation of } w'' = aabcbbcab \\
\end{array}
\]

In this section we showed for some \(m\) how \(\mathbb{N}^{\text{Univ}}_{\Sigma, m, k}\) looks like and determined \(m\) for \(w \in \mathbb{N}^{\text{Univ}}_{\Sigma, m, k} \cap \mathbb{N}^{\text{Univ}}_{\Sigma, k-1}\) as well as \([w] \sim_k \).

## 5 Conclusion

In this work, we pursued the approach to partition \(\Sigma^*\) w.r.t. the number of absent scattered factors of a given length \(k\). This lead to the notion of \(m\)-nearly \(k\)-universal words, which are words where exactly \(m\) scattered factors of length \(k\) are absent. We haven chosen this perspective to investigate the index of the Simon congruence \(\sim_k\) and indeed we were able to fully characterise \(1\)-nearly \(k\)-universal words and give the index as well as a characterisation of \(\sim_k\) restricted to this subclass. Moreover, we gave some insights for \(m > 1\), especially for \(m \in \{2, |\Sigma|^k - 1, |\Sigma|^k - 2\}\) (notice that \(m = 0\) is fully investigated in [2]). Additionally in Section 4, we followed the idea from [21] from a combinatorial
point of view, showing for instance that letters have the same dist-value in [21]
iff they are in the same arch of $w\mathcal{R}$. By this approach we showed that $m$
can be determined recursively for $w$ with $i(w) = k − 1$ by investigating the overlaps
of the arches from $w$ and $w\mathcal{R}$. Moreover, we proved when to words $w_1, w_2$ with
$i(w_1) = i(w_2) = k − 1$ fulfill $w_1 \sim_k w_2$.
Unfortunately, we were not able to give a full characterisation of $\text{NUinv}_{\Sigma,m,k}$
for arbitrary $m$. A first step could be to determine $i(w)$ for $w \in \text{NUinv}_{\Sigma,m,k}$.
We conjecture that choosing $i \in [\sigma^k]$ such that $\sigma^i \leq m \leq \sigma^{i+1} − 1$
leads to $k − \lfloor \frac{m}{\sigma^i} \rfloor − 1 \leq i(w) \leq k − \lfloor \frac{m}{\sigma^{i+1}} \rfloor$. A subpartition
of $\text{NUinv}_{\Sigma,m,k}$ depending on $i(w)$ (as introduced in [21] and used in Section 4) could prove useful.

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