Polynomials related to chromatic polynomials

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Abstract
For a simple graph $G$, let $\chi(G, x)$ denote the chromatic polynomial of $G$. This manuscript introduces some polynomials which are related to chromatic polynomial and their relations.

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1 The Potts model partition function

(1.1). Let $G = (V, E)$ be a multigraph, which may have loops and parallel edges, with a weight $w_e$ for each edge $e \in E$.

The partition function of the $q$-state Potts model of $G$, or multivariate Tutte polynomial of $G$, is defined as:

$$Z_G(q, \{w_e\}) = \sum_{\sigma \in \Delta_G(q)} \prod_{e \in E} [1 + w_e \delta(\sigma(x_1(e)), \sigma(x_2(e)))],$$

(1.1)

i.e.,

$$Z_G(q, \{w_e\}) = \sum_{\sigma \in \Delta_G(q)} \prod_{e \in E} (1 + w_e),$$

(1.2)

where $x_1(e)$ and $x_2(e)$ are the two ends of edge $e$, $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ otherwise, and $\Delta_G(q)$ is the set of mappings $\sigma : V \to \{1, 2, \cdots, q\}$.

(1.2). In statistical physics, the expression (1.1) arises as follows:
In the Potts model, an “atom” (or “spin”) at a site $u \in V$ can exist in any one of $q$ different states, i.e., $\sigma(u) \in \{1, 2, \ldots, q\}$ for any $u \in V$ and any $\sigma \in \Delta_G(q)$.

A configuration is a mapping $\sigma : V \to \{1, 2, \ldots, q\}$.

The energy of a configuration $\sigma$, denoted by $H_\sigma$, is the sum, over all edges $e \in E$, of $0$ if the spins at the two endpoints of that edge are unequal and $-J_e$ if they are equal. Thus

$$H_\sigma = \sum_{e \in E} (-J_e) \delta(\sigma(x_1(e)), \sigma(x_2(e))).$$

The Boltzmann weight of a configuration $\sigma$ is $\exp(-\beta H_\sigma)$, where $\beta \geq 0$ is the inverse temperature.

The partition function is the sum, over all configurations, of their Boltzmann weights:

$$\sum_{\sigma \in \Delta(q)} \exp(-\beta H_\sigma)$$

$$= \sum_{\sigma \in \Delta(q)} \exp \left(-\beta \sum_{e \in E} (-J_e) \delta(\sigma(x_1(e)), \sigma(x_2(e))) \right)$$

$$= \sum_{\sigma \in \Delta(q)} \prod_{e \in E} \exp \left[ \beta J_e \cdot \delta(\sigma(x_1(e)), \sigma(x_2(e))) \right]$$

$$= \sum_{\sigma \in \Delta(q)} \prod_{e \in E(G)} \left[ 1 + w_e \delta(\sigma(x_1(e)), \sigma(x_2(e))) \right],$$

where $w_e = e^{\beta J_e} - 1$.

A parameter value $J_e$ (or $w_e$) is called ferromagnetic if $J_e \geq 0$ ($w_e \geq 0$), as it is then favored for adjacent spins to take the same value; antiferromagnetic if $-\infty \leq J_e \leq 0$ ($-1 \leq w_e \leq 0$), as it is then favored for adjacent spins to take different values; and unphysical if $w_e \not\in [-1, \infty)$, as the weights are then no longer nonnegative.

(1.3). Potts model is named after Renfrey Potts, who described the model near the end of his 1951 Ph.D. thesis. The model was suggested to him by his advisor, Cyril Domb.

Special case $q = 2$: Ising model, named after the physicist Ernst Ising.
The Ising model was invented by the physicist Wilhelm Lenz in 1920, who gave it as a problem to his student Ernst Ising. The one-dimensional Ising model has no phase transition and was solved by Ising himself in his 1924 thesis. The two-dimensional square lattice Ising model is much harder, and was given an analytic description much later, by Lars Onsager (1944).

More details on Potts model can be found in [22, 36, 39, 40, 41, 62, 61].

(1.4) Proposition 1.1 For any multigraph \( G \) and \( q \in \mathbb{N} \),

\[
Z_G(q, \{w_e\}) = \sum_{A \subseteq E} q^{c(A)} \prod_{e \in A} w_e,
\]

where \( c(A) \) is the number of components of the spanning subgraph of \( G \) with edge set \( A \).

(1.5) By expression (1.4), \( Z_G(q, \{w_e\}) \) is a polynomial in \( q \) of degree \(|V|\):

\[
Z_G(q, \{w_e\}) = \sum_{1 \leq i \leq |V|} \left( \sum_{c(A) = i} \prod_{e \in A} w_e \right) q^i.
\]

Thus \( Z_G(q, \{w_e\}) \) can be considered as a function with variable \( q \) which is a complex number.

(1.6) Examples.

For the empty graph \( N_n \), by expression (1.4),

\[
Z_{N_n}(q, \{w_e\}) = q^n.
\]

For the complete graph \( K_2 \) with edge \( e \),

\[
Z_{K_2}(q, \{w_e\}) = q(q + w_e).
\]

For the graph \( L \) with only one vertex and only one loop \( e \),

\[
Z_L(q, \{w_e\}) = q(1 + w_e).
\]

(1.7) Special cases. Assume that \( Z_G(q, y) = Z_G(q, \{w_e\}) \), where \( w_e = y \) for all \( e \in E \).
(i) \( Z_G(x, -1) \) is the chromatic polynomial \( \chi(G, x) \), as
\[
\chi(G, x) = \sum_{A \subseteq E} (-1)^{|A|} x^{c(A)}.
\]

(ii) The Whitney rank generating function is defined as
\[
R_G(x, y) = \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-r(A)}
\]
where \( r(A) = |V| - c(A) \). Thus
\[
R_G(x, y) = x^{-c(G)} y^{-|V|} Z_G(xy, y).
\]

Proof. Note that
\[
R_G(x, y) = \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-r(A)}
\]
\[
= \sum_{A \subseteq E} x^{c(A)-c(E)} y^{|A|-|V|+c(A)}
\]
\[
= x^{-c(G)} y^{-|V|} \sum_{A \subseteq E} (xy)^{c(A)} y^{|A|}.
\]

\( \square \)

(iii) The Tutte polynomial of \( G \) is defined as
\[
T_G(x, y) = R_G(x - 1, y - 1) = \sum_{A \subseteq E} (x - 1)^{r(E)-r(A)} (y - 1)^{|A|-r(A)}.
\]
Thus
\[
T_G(x, y) = (x - 1)^{-c(G)} (y - 1)^{-|V|} Z_G((x - 1)(y - 1), y - 1).
\]

(1.8). Factorizations.

(i) If \( G \) is disconnected with components \( G_1, G_2, \ldots, G_k \),
\[
Z_G(q, \{ w_e \}) = \prod_{i=1}^{k} Z_{G_i}(q, \{ w_e \}).
\]

(ii) If \( G \) is connected with blocks \( G_1, G_2, \ldots, G_k \),
\[
Z_G(q, \{ w_e \}) = \frac{1}{q^{k-1}} \prod_{i=1}^{k} Z_{G_i}(q, \{ w_e \}).
\]
(1.9). **Computation.** For any $G = (V, E)$, $Z_G(q, \{w_e\})$ can be determined by the following rules:

(i) If $G$ is empty, then $Z_G(q, \{w_e\}) = q^{|V|}$;

(ii) If $e'$ is a loop of $G$, then

$$Z_G(q, \{w_e\}) = (1 + w_{e'})Z_{G\setminus e'}(q, \{w_e\});$$

(iii) If $e'$ is a bridge of $G$, then

$$Z_G(q, \{w_e\}) = (q + w_{e'})Z_{G/e'}(q, \{w_e\});$$

(iv) If $e'$ is a normal edge in $G$, i.e., $e$ is not a loop nor a bridge of $G$, then

$$Z_G(q, \{w_e\}) = Z_{G\setminus e'}(q, \{w_e\}) + w_{e'}Z_{G/e'}(q, \{w_e\}),$$

where $G/e'$ is the multigraph obtained from $G\setminus e'$ by identifying the two ends $x_1(e')$ and $x_2(e')$ of $e'$, where all edges in $G\setminus e'$ parallel to $e'$ become loops of $G/e'$ with the weights unchanged.

(1.10). **Example 1.1** For any tree $T$, we have

$$Z_T(q, \{w_e\}) = q \prod_{e \in E(T)} (q + w_e).$$

**Example 1.2** For any cycle $C$, we have

$$Z_C(q, \{w_e\}) = \prod_{e \in E(C)} (q + w_e) + (q - 1) \prod_{e \in E(C)} w_e.$$

(1.11). **Parallel-reduction identity.**

If $G$ contains edges $e_1, e_2$ connecting the same pair of vertices $u$ and $v$, they can be replaced, without changing the value of $Z_G(q, \{w_e\})$, by a single edge $e = uv$ with weight

$$w_e = (1 + w_{e_1})(1 + w_{e_2}) - 1.$$
(1.12). **Series-reduction identity.**

We say that edges \( e_1, e_2 \in E \) are in series if there exist vertices \( x, y, z \in V \) with \( x \neq y \) and \( y \neq z \) such that \( e_1 \) connects \( x \) and \( y \), \( e_2 \) connects \( y \) and \( z \), and \( y \) has degree 2 in \( G \). In this case the pair of edges \( e_1, e_2 \) can be replaced, without changing the value of \( Z_G(q, \{ w_e \}) \), by a single edge \( e' = xz \) with weight

\[
 w_{e'} = \frac{w_{e_1} w_{e_2}}{q + w_{e_1} + w_{e_2}}
\]

provided that we then multiply \( Z \) by the prefactor \( q + w_{e_1} + w_{e_2} \).

(1.13). **Question 1.1** Let \(-1 \leq w_e < 0\) for all \( e \in E \). Show that \((-1)^{|V|} Z_G(q, \{ w_e \}) > 0\) whenever \( q < 0 \).

(1.14). **(Multivariate) Independent-set polynomial.**

For any graph \( H \) with a mapping \( w : V(H) \to \mathbb{R} \), the (multivariate) independent-set polynomial of \( H \) is defined as

\[
 I(H, w) = \sum_{V' \subseteq \mathcal{I}(H)} \prod_{u \in V'} w(u),
\]

where \( \mathcal{I}(H) \) is the family of independent sets of \( H \). Note that \( \emptyset \) is also a member of \( \mathcal{I}(H) \) and it contributes 1 to the above summation.

Clearly, if \( w(u) = x \) for all \( u \in V(H) \), then \( I(H, w) \) is the independence polynomial of \( H \).

For example, if \( H \) is \( K_3 \) with vertex set \( \{ u_1, u_2, u_3 \} \), then

\[
 I(K_3, w) = 1 + w(u_1) + w(u_2) + w(u_3).
\]

If \( H \) is a path \( P_3 \) with vertex set \( \{ u_1, u_2, u_3 \} \) but \( u_1 u_3 \) is not an edge in \( P_3 \), then

\[
 I(P_3, w) = 1 + w(u_1) + w(u_2) + w(u_3) + w(u_1)w(u_3).
\]

(1.15). For any \( u \in V(H) \), we denote by \( N_H(u) \) (or simply \( N(u) \)) the set of vertices in \( H \) that are adjacent to \( u \) and write \( N[u] = N(u) \cup u \). More generally, for any \( S \subseteq V(H) \), we write \( N[S] = \cup_{u \in S} N[u] \).
Theorem 1.1 (Fernández and Procacci [13]) Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with vertex weight $w(u)$ for each $u \in \mathcal{V}$. If there exists a mapping $\mu : \mathcal{V} \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers, such that

$$|w(u)|I(G[N[u]], \mu) \leq \mu(u)$$

holds for all $u \in \mathcal{V}$, then $I(G[S], w) \neq 0$ for each $S \subseteq \mathcal{V}$, where $G[S]$ is the subgraph of $G$ induced by $S$.

(1.16). From partition function to independent-set polynomial.

Let $G = (\mathcal{V}, \mathcal{E})$ be the graph constructed from $G$ with vertex set

$$\mathcal{V} = \{S \subseteq V(G) : \mid S \mid \geq 2, G[S] \text{ is connected} \},$$

where any two vertices $S_1, S_2$ of $G$ are adjacent if and only if $S_1 \cap S_2 \neq \emptyset$.

For any $S \in \mathcal{V}$, define

$$w(S) = q^{1-\mid S \mid} \sum_{E' \subseteq E(G[S]) \text{ connected}} \prod_{e \in E'} w_e,$$

where $(S, E')$ is the subgraph of $G$ with vertex set $S$ and edge set $E'$.

Then

$$Z_G(q, \{w_e\})/q^{\mid \mathcal{V} \mid} = I(G, w).$$

Note that for $q \neq 0$,

$$Z_G(q, \{w_e\}) = 0 \iff I(G, w) = 0.$$

(1.17). Some results on $Z_G(q, \{w_e\})$.

(i) Theorem 1.2 (Sokal 2001 [41]) If $G$ is loopless and $w_e$ is complex with $|1 + w_e| \leq 1$ for all $e \in \mathcal{E}$, then all zeros of $Z_G(q, \{w_e\})$ lie in the disc

$$|q| < K \max_{v \in \mathcal{V}} \sum_{e \in E_v} |w_e|$$

where $K \leq 7.963907$ and $E_v$ is the set of edges incident with $v$.

Sokal’s result implies that for any graph $G$ with maximum degree $D$, the zeros of $\chi(G, z)$ are within the disc $|z| < 7.963907D$.  

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(ii) **Theorem 1.3 (Jackson and Sokal [22])** Let \( G \) be a graph with \( n \) vertices and \( c \) components, and let \( q \in (0, 1) \). Suppose that:

(a) \( w_e > -1 \) for every loop \( e \);
(b) \( w_e < -q \) for every bridge \( e \); and
(c) \( -1 - \sqrt{1-q} < w_e < -1 + \sqrt{1-q} \) for every normal (i.e., non-loop non-bridge) edge \( e \).

Then \((-1)^{n+c} Z_G(q, \{w_e\}) > 0\).

## 2 Tutte polynomial \( T_G(x, y) \)

(2.1). For any multigraph \( G = (V,E) \), the Tutte polynomial \( T_G(x,y) \) of \( G \) is defined as

\[
T_G(x,y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)},
\]

where \( r(A) = |V| - c(A) \) and \( c(A) \) is the number of components of the spanning subgraph \((V,A)\).

(2.2). The Tutte polynomial can also be obtained by the following rules (Tutte 1947 [54]):

(i) \( T_G(x,y) = 1 \) if \( E = \emptyset \);
(ii) \( T_G(x,y) = y T_{G\setminus e}(x,y) \) if \( e \) is a loop of \( G \);
(iii) \( T_G(x,y) = x T_{G/e}(x,y) \) if \( e \) is a bridge of \( G \);
(iv) \( T_G(x,y) = T_{G/e}(x,y) + T_{G\setminus e}(x,y) \) if \( e \) is not a bridge or loop of \( G \).

(2.3). Some examples.

(i) If \( G \) is a tree of order \( n \), then \( T_G(x,y) = x^{n-1} \);

(ii) If \( G \) is a cycle of order \( n \), then

\[
T_G(x,y) = x + x^2 + \cdots + x^{n-1} + y.
\]

(2.4). **Expression in terms of spanning trees** (Tutte 1947 [52, 54]):

\[
T_G(x,y) = \sum_T x^{\text{in}(T)} y^{\text{ex}(T)},
\]

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where the sum runs over all spanning trees \( T \) of \( G \) and \( \text{in}(T) \) (resp. \( \text{ex}(T) \)) is the number of internally active members (resp. externally active members) with respect to \( T \).

Let \( w \) be an injective weight function \( w : E \to \mathbb{Z} \).

For any spanning tree \( T \) and \( e \in E(T), T \setminus e \) has two components, say \( T_1 \) and \( T_2 \). If \( w(e) \leq w(e') \) holds for all \( e' \in E(G) \) joining a vertex in \( V(T_1) \) to a vertex in \( V(T_2) \), \( e \) is called an \textit{internally active edge} with respect to \( T \).

For an edge \( e \in E(G) - E(T) \), \( e \) is called an \textit{externally active edge} with respect to \( T \) if \( w(e) \leq w(e') \) holds for all edges \( e' \) on the unique cycle in the spanning subgraph \( (V, E(T) \cup \{e\}) \).

(2.5). \textbf{Tutte Polynomial for a matroid} \( M = (E, r) \).

Let \( M = (E, r) \) be a matroid with ground set \( E \) and rank function \( r \).

The Tutte polynomial \( T_M(x, y) \) of \( M \) is defined as follows:

\[
T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)}(y - 1)^{|A| - r(A)}.
\]

Note that \( T_M(x, y) \) can also be determined by (2.1) or (2.4).

(2.6). \textbf{Special polynomials}.

\[
\begin{aligned}
\text{Tutte polynomial } T_G(x, y) \\
\text{chromatic polynomial} &\quad \text{flow polynomial}
\end{aligned}
\]

(i) \( x^{c(G)}T_G(1 - x, 0) = (-1)^{|V| - c(G)}\chi(G, x) \), where \( \chi(G, x) \) is the chromatic polynomial of \( G \);

(ii) \( T_G(0, 1 - y) = (-1)^{|E| - |V| + c(G)}F(G, y) \), where \( F(G, x) \) is the flow polynomial of \( G \).

(2.7). \textbf{Basic properties}.

(i) Dual property.
Proposition 2.1 If $G$ is a connected plane graph and $G^*$ is its dual, then

$$T_G(x, y) = T_{G^*}(y, x).$$

More generally,

Proposition 2.2 If $M = (E, r)$ is a matroid and $M^*$ is its dual, then

$$T_M(x, y) = T_{M^*}(y, x).$$

Note that $M^*$ is the matroid $(E, r^*)$ with its rank function $r^*(A)$ determined by

$$|A| - r^*(A) = r(E) - r(E - A)$$

for any $A \subseteq E$, i.e., $r^*(A) = |A| - \min_{B \in \mathcal{B}(M)} |B \cap A|$, where $\mathcal{B}(M)$ is the family of bases of $M$.

(ii) Factorization.

If $G$ is disconnected with components $G_1, G_2, \ldots, G_k$ or $G$ is connected with blocks $G_1, G_2, \ldots, G_k$, then

$$T_G(x, y) = \prod_{i=1}^{k} T_{G_i}(x, y).$$

(iii) Coefficients $t_{i,j}$.

Let $M = (E, r)$ be a matroid with ground set $E$.

Proposition 2.3 ([5]) If $t_{i,j}$ is the number of bases $B$ of $M$ with $\text{in}(B) = i$ and $\text{ex}(B) = j$, then

$$T_M(x, y) = \sum_{i,j} t_{i,j} x^i y^j,$$

and if $M$ has neither loops nor coloops for statements (ii) to (iv) below,

(i) $t_{1,0} = t_{0,1}$ when $|E| \geq 2$;

(ii) $t_{i,j} = 0$ whenever $i > r(M)$ or $j > |E| - r(M)$;

(iii) $t_{r(M),0} = 1$ and $t_{0,|E|-r(M)} = 1$. 

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(iv) \( t_{r(M),j} = 0 \) for all \( j > 0 \) and \( t_{i,|E|-r(M)} = 0 \) for all \( i > 0 \);

(v) \( \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{i,j} = 0 \)

Note that if \( M \) is replaced by a connected graph \( G = (V,E) \), then Proposition 2.3 holds with \( r(M) = |V| - 1 \) and \( t_{i,j} \) to be the number of spanning trees \( T \) of \( G \) with \( in(T) = i \) and \( ex(T) = j \).

(2.8). An important identity on \( T_M(x,y) \).

**Theorem 2.1** For any matroid \( M = (E,r) \),

\[
T_M((v + 1)/v, v + 1) = \frac{(v + 1)^{|E|}}{v^{r(M)}}.
\]

In particular, for any connected graph \( G \) of order \( n \) and size \( m \),

\[
T_G((v + 1)/v, v + 1) = \frac{(v + 1)^m}{v^{n-1}}.
\]

**Proof.** Let \( f_M(u,v) = \sum_{A \subseteq E} u^{r(M) - r(A)} v^{|A|} \). Then

\[
f_M(1,v) = (v + 1)^{|E|}.
\]

It can be shown that

\[
f_M(u,v) = u^{r(M)} T_M(u/v + 1, v + 1).
\]

Thus the result follows. \( \square \)

By Theorem 2.1 it can be shown that

\[
(v + 1)^{|E|} = \sum_{i,j \geq 0} t_{i,j} v^{r(M) - i} (v + 1)^{i+j}.
\]

Let \( w = v + 1 \). Then

\[
w^{|E|} = \sum_{i,j \geq 0} t_{i,j} w^{i+j} (w - 1)^{r(M) - i} = \sum_{i,j \geq 0} t_{i,j} \sum_{k=0}^{r(M) - i} (-1)^k \binom{r(M) - i}{k} w^{r(M) + j - k}.
\]
Thus

\[ w^{\mid E \mid - r(M)} = \sum_{i,j \geq 0} t_{i,j} \sum_{k=0}^{r(M)-i} (-1)^k \binom{r(M)-i}{k} w^{j-k}. \]

If \( \mid E \mid > r(M) \), then the constant term is 0, implying that

\[ \sum_{i=0}^{r(M)-i} (-1)^j \binom{r(M)-i}{j} = 0. \]

More identities can be obtained by taking \( k \) such that \( j - k \neq \mid E \mid - r(M) \).

(2.9). A Convolution Formula for the Tutte Polynomial by Kook, Reiner and Stanton in 1999.

**Theorem 2.2 (Kook et al 1999 [25])** For any matroid \( M = (E, r) \),

\[ T_M(x, y) = \sum_{A \subseteq E} T_{M/A}(x, 0)T_{M\mid A}(0, y). \]

Note that \( T_{M/A}(x, 0) = 0 \) if \( M/A \) has a loop and \( T_{M\mid A}(0, y) = 0 \) if \( M\mid A \) has a bridge.

Thus Theorem 2.2 can be revised as follows:

**Theorem 2.3** For any matroid \( M = (E, r) \),

\[ T_M(x, y) = \sum_{A \subseteq F^*(M)} T_{M/A}(x, 0)T_{M\mid A}(0, y). \]

where \( F^*(M) \) is the family of those flats \( F \) which contain no bridge.

For a graph \( G = (V, E) \), let \( P^*(G) \) be the family of those partitions \( P = \{V_1, V_2, \cdots, V_r\} \) of \( V \) such that \( V_i \neq \emptyset \) and the induced subgraph \( G[V_i] \) is connected and bridgeless for each \( i \). For any \( P = \{V_1, V_2, \cdots, V_r\} \in P^*(G) \), let \( G[P] \) denote the spanning subgraph of \( G \) which is the disjoint union of \( G[V_i]'s \) for \( i = 1, 2, \cdots, r \), and let \( G/P \) be the graph obtained from \( G \) by contracting all edges in \( G[V_i] \) for all \( i = 1, 2, \cdots, r \). Thus \( G/P \) is a graph of order \( r \).

Then Theorem 2.3 implies that
Theorem 2.4  For any graph $G = (V, E)$,
\[
T_G(x, y) = \sum_{P \subseteq P^*(G)} T_{G/P}(x, 0)T_{G/P}(0, y).
\]

(2.10). Interpretation on some values of $T_G(x, y)$, where $G = (V, E)$ is connected.

(i) $T_G(0, 0) = 0$ if $E \neq \emptyset$.
(ii) $T_G(2, 2) = 2|E|$.
(iii) $T_G(1, 2)$ is the number of spanning connected subgraphs of $G$.
(iv) $T_G(2, 1)$ is the number of spanning forests of $G$.
(v) $T_G(1, 1)$ is the number of spanning trees of $G$, denoted by $\tau(G)$.
(vi) $T_G(0, 1)$ is the number of those spanning trees $T$ of $G$ with $\text{in}(T) = 0$.
(vii) $T_G(1, 0)$ is the number of those spanning trees $T$ of $G$ with $\text{ex}(T) = 0$.
(viii) $T_G(0, 2) = \text{is}$ is the number of totally cyclic orientations of $G$, denoted by $\alpha^*(G)$.
(ix) $T_G(2, 0)$ is the number of acyclic orientations of $G$, denoted by $\alpha(G)$.

(x) (Stanley [43]) for any integer $k \geq 1$, $T_G(k + 1, 0)$ is equal to
\[
\frac{1}{k}|\chi(G, -k)| = \frac{1}{k} \sum_{j=1}^{k} (k)_j |\Upsilon_j|,
\]
where $\Upsilon_j$ is the set of order pairs $(P, O)$, where $P$ is a partition of $V$ into exactly $j$ non-empty subsets $V_1, V_2, \cdots, V_j$ and $O$ is an acyclic orientation of the spanning subgraph of $G$ with edge set $\bigcup_{1 \leq i \leq j} E(G[V_i])$.

Proof. By Stanley’s result in [43],
\[
|\chi(G, -k)| = \tilde{\chi}(G, k),
\]
where $\tilde{\chi}(G, k)$ is the number of order pairs $(f, D)$, where $D$ is an acyclic orientation of $G$ and $f$ is a mapping $f : V \to \{1, 2, \cdots, k\}$ such that $f(u) \leq f(v)$ whenever $u \rightarrow v$ in $D$.
For any $j$ with $1 \leq j \leq k$, let $\psi_j$ be the number of order pairs $(f, D)$ such that
(a) $D$ is an acyclic orientation of $G$;
(b) $f$ is a mapping $f : V \to \{1, 2, \cdots, k\}$ such that $f(u) \leq f(v)$ whenever $u \to v$ in $D$;
(c) $|f(V)| = j$.
Thus
$$\bar{\chi}(G, k) = \sum_{j=1}^{k} \psi_j = \sum_{j=1}^{k} (k)_j |\Upsilon_j|.$$
The directed medial graph of $G$, denoted by $\tilde{G}_m$, is obtained by assigning a direction to each edge of the medial graph so that the black face is on the left.

An example of $\tilde{G}_m$ is shown in Figure 1.

(ii) For any directed graph $H$, let $\mathcal{D}_n(H)$ be the family of ordered partitions $(D_1, \ldots, D_n)$ of $E(H)$ such that $H$ restricted to $D_i$ is 2-regular and consistently oriented for all $i$.

**Theorem 2.5 (Martin 1977 [30])** Let $\tilde{G}_m$ be the directed medial graph of a plane graph $G$. Then, for any positive integer $n$,

$$(-n)^{c(G)}T_G(1 - n, 1 - n) = \sum_{(D_1, \ldots, D_n) \in \mathcal{D}_n(\tilde{G}_m)} (-1)^{\sum_{1 \leq i \leq n} c(D_i)}.$$

(iii) **Theorem 2.6 (Martin [30])** Let $\tilde{G}_m$ be the directed medial graph of a plane graph $G$. Then, for any positive integer $n$,

$$n^{c(G)}T_G(1 + n, 1 + n) = \sum_{\phi} 2^{\mu(\phi)},$$

where the sum runs over all edge colorings $\phi$ of $\tilde{G}_m$ with $n$ colors so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph, and where $\mu(\phi)$ is the number of monochromatic vertices in the coloring $\phi$. 
(iv) An *anticircuit* in a digraph is a closed trail so that the directions of the edges alternate as the trail passes through any vertex of degree greater than 2.

In 4-regular Eulerian digraph, an anticircuit can be obtained by choosing the two incoming edges or the two outgoing edges at each vertex.

**Theorem 2.7** Let $G$ be a connected plane graph. Then

(a) (Martin 1978\[29\])

$$T_G(-1, -1) = (-1)^{|E(G)|}(-2)^{a(\tilde{G}_m) - 1}$$

where $a(\tilde{G}_m)$ is the number of anticircuits in $\tilde{G}_m$;

(b) (Vergnas 1988\[55\])

$$T_G(3, 3) = K2^{a(\tilde{G}_m) - 1},$$

where $K$ is some odd integer.

Note that $a(\tilde{G}_m)$ is actually equal to the number of components of the link diagram $D(G)$.

(2.12). **Universality of the Tutte Polynomial.**

**Theorem 2.8** (Brylawski and Oxley) Let $\mathcal{G}$ be a minor closed class of graphs. If a graph invariant $f$ from $\mathcal{G}$ to a commutative ring $R$ with unity satisfying all conditions below for any $G, H \in \mathcal{G}$:

(i) $f(N_1) = 1$;

(ii) $f(G \cup_0 H) = f(G)f(H)$;

(iii) $f(G) = x_0f(G/e)$ if $e$ is a bridge;

(iv) $f(G) = y_0f(G\backslash e)$ if $e$ is a loop;

(v) $f(G) = af(G\backslash e) + bf(G/e)$ for each edge $e$ which is not a loop nor a bridge, where $a, b$ are non-zero constants;

then

$$f(G) = a^{|E(G)| - r(E(G))}b^{r(E(G))}T_G \left( \frac{x_0}{b}, \frac{y_0}{a} \right).$$
(2.13). **Codichromatic graphs** (or $T$-equivalent graphs).

Two graphs $G_1$ and $G_2$ having the same Tutte polynomial are called *codichromatic graphs* by Tutte [49] and also called *$T$-equivalent graphs*.

It is trivial that two isomorphic graphs are $T$-equivalent. If two non-isomorphic graphs have isomorphic cyclic matroids, then they are also $T$-equivalent [3].

A well-known operation for constructing such a pair of graphs is the Whitney twist [60] which changes a graph to another one by flipping a subgraph at a vertex-cut of size 2. An example for such a pair of graphs $G$ and $G'$ is shown in Figure 2, where $\{u_1, u_2\}$ is the cut-set chosen from $G$.

![Figure 2: $G'$ is obtained from $G$ by a Whitney twist.](image)

(2.14). A pair of $T$-equivalent graphs.

Mentioned by Tutte [49], the two graphs $G_0$ and $H_0$ in Figure 3 were found by Dr. Marion C. Gray in 1930s. These two graphs are not isomorphic and even have non-isomorphic cyclic matroids, because $H_0$, unlike $G_0$, contains a triangle having no common edge with any other triangle [49]. However, $G_0 \setminus e \cong H_0 \setminus f$ and $G_0 / e \cong H_0 / f$, where $e$ and $f$ are the edges in $G_0$ and $H_0$ which are expressed by dashed lines in Figure 3.

(2.15). Invariants for Tutte polynomial.

**Proposition 2.4** Let $M_1$ and $M_2$ be connected matroids. If $T_{M_1}(x, y) = T_{M_2}(x, y)$, then

(i) $r(M_1) = r(M_2)$;
(ii) \(|E(M_1)| = |E(M_2)|\);

(iii) for each \(i\) with \(0 \leq i \leq r(M_1)\), the number of independent sets of \(M_1\) of cardinality \(i\) is equal to the number of independent sets of \(M_2\) of cardinality \(i\);

(iv) the girth \(g(M_1) = g(M_2)\);

(v) the number of circuits of \(M_1\) of cardinality \(g(M_1)\) is equal to the number of circuits of \(M_2\) of cardinality \(g(M_2)\);

(vi) for each \(i\) with \(0 \leq i \leq r(M_1)\), if \(f_i(M_j)\) is the largest cardinality among all flats of \(M_j\) of rank \(i\), then \(f_i(M_1) = f_i(M_2)\);

(vii) the number of rank-\(i\) flats \(F_1\) of \(M_1\) with \(|F_1| = f_i(M_1)\) is equal to the number of rank-\(i\) flats \(F_2\) of \(M_2\) with \(|F_2| = f_i(M_2)\).

**Proposition 2.5** Let \(G\) be a simple 2-connected graph. The following parameters of \(G\) are determined by its Tutte polynomial \(T_G(x, y)\):

(i) the edge-connectivity \(\lambda(G)\); in particular, a lower bound for the minimum degree \(\delta(G)\);

(ii) the number of cliques of each order and the clique-number \(\omega(G)\);

(iii) the number of cycles of length three, four and five, and the number of cycles of length four with exactly one chord.

(2.16). \(T\)-equivalent graphs produced by flipping a rotor.

Assume that \(R\) is a graph and \(\psi\) is an automorphism of \(R\). For any vertex \(x\) in \(R\), the set \(\{\psi^i(x) : i \geq 0\}\) is called a vertex orbit of \(\psi\) and \(x\) is called a fixed vertex of \(\psi\) if \(\psi(x) = x\).
If $R$ is a subgraph of a graph $G$, a subset $B$ of $V(R)$ is called a border of $R$ in $G$ if every edge in $G$ incident with some vertex $V(R) - B$ must be an edge in $R$. We call $R$ a rotor of $G$ with a border $B$ if $B$ is a vertex orbit of some automorphism $\psi$.

Tutte [49] showed that if $G$ is a graph containing a rotor $R$ with a border $B$ of size at most 5, then $G$ and $G'$ are $T$-equivalent, where $G'$ is the graph obtained from $G$ by flipping $R$ along its border $B$, i.e., by replacing $R$ by its mirror image. We will express Tutte’s result below.

Given any vertex-disjoint graphs $G$ and $W$ with $\{u_1, u_2, \cdots, u_k\} \subseteq V(G)$ and $\{w_1, w_2, \cdots, w_k\} \subseteq V(W)$, let $G(u_1, u_2, \cdots, u_k) \sqcup W(w_1, w_2, \cdots, w_k)$ denote the graph obtained from $G$ and $W$ by identifying $u_i$ and $w_i$ as a new vertex for all $i = 1, 2, \cdots, k$. An example of $G(u_1, u_2, u_3) \sqcup W(w_1, w_2, w_3)$ is shown in Figure 4.

\begin{itemize}
\item[(a)] $G$
\item[(b)] $W$
\item[(c)]
\end{itemize}

Figure 4: Graph $G(u_1, u_2, u_3) \sqcup W(w_1, w_2, w_3)$

**Theorem 2.9 (Tutte [49])** Let $R$ be a connected graph with an automorphism $\psi$. If $\{u_1, u_2, \cdots, u_k\}$ is a vertex orbit of $\psi$ (i.e., $\psi(u_i) = u_{i+1}$ for all $i = 1, 2, \cdots, k$), where $k \leq 5$, then the two graphs $R(u_1, \cdots, u_k) \sqcup W(w_1, \cdots, w_k)$ and $R(u_k, \cdots, u_1) \sqcup W(w_1, \cdots, w_k)$ are $T$-equivalent for an arbitrary graph $W$, where $w_1, \cdots, w_k$ are distinct vertices in $W$.

(2.17). $T$-equivalent class.

**Theorem 2.10 ([14])** If $G$ is a simple outerplanar graph and $T_G(x, y) = T_H(x, y)$, then $H$ is also outerplanar.
A graph $G$ is said to be $T$-unique if for any graph $H$, $G \cong H$ whenever $T_H(x,y) = T_G(x,y)$.

**Theorem 2.11 (56)** The following graphs are $T$-unique:

(i) for every set of positive integers $p_1, p_2, \cdots, p_k$, the complete multipartite graph $K_{p_1, p_2, \cdots, p_k}$ is $T$-unique, with the only exception of $K_{1,p}$;

(ii) $C_n^2$, where $n \geq 3$ and $C_n^2$ is obtained from the cycle graph $C_n$ by adding edges joining any two vertices in $C_n$ with distance 2;

(iii) graph $C_n \times K_2$;

(iv) the Möbius ladder $M_n$, where $n \geq 2$, which is constructed from an even cycle $C_{2n}$ by joining every pair of vertices at distance $n$;

(v) The $n$-cube $Q_n$, $n \geq 2$, which is defined as the product of $n$ copies of $K_2$.

(2.18). Results on inequalities.

**Theorem 2.12 (Merino et al 32)** If a matroid $M$ has neither loops nor isthmuses, then

$$\max\{T_M(4,0), T_G(0,4)\} \geq T_M(2,2).$$

It can be proved by applying the fact that $T_M(2,2) = 2^{|E|}$ and $T_M(4,0) \geq 4^r(M)$ and $T_M(0,4) \geq 4^{|E|-r(M)}$.

**Theorem 2.13 (Merino et al 32)** If a matroid $M = (E,r)$ contains two disjoint bases, then

$$T_M(0,2a) \geq T_M(a,a),$$

for all $a \geq 2$. Dually, if its ground set $E$ is the union of two bases of $M$, then

$$T_M(2a,0) \geq T_M(a,a),$$

for all $a \geq 2$. 
Conjecture 2.1 (Merino and Welsh [33]) Let $G$ be a 2-connected graph with no loops. Then
\[ \max\{T_G(2,0), T_G(0,2)\} \geq T_G(1,1). \]

Merino and Welsh also mentioned the following stronger conjecture.

Conjecture 2.2 (Merino and Welsh [33]) Let $G$ be a 2-connected graph with no loops. Then
\[ T_G(2,0)T_G(0,2) \geq T_G(1,1)^2. \]

Theorem 2.14 (Thomassen [47]) If $G$ is a simple graph on $n$ vertices with $m \leq 16n/15$ edges, then
\[ T_G(2,0) > T_G(1,1), \]
and if $G$ is a bridgeless graph on $n$ vertices with $m \geq 4n - 4$ edges, then
\[ T_G(0,2) > T_G(1,1), \]

A graph is called a series-parallel graph if it is obtained from a single edge by repeatedly duplicating or subdividing edges in any fashion.

Theorem 2.15 (Noble and Royle [34]) Conjecture 2.1 holds for all series-parallel graphs.

Theorem 2.16 (Jackson [19]) Let $G$ be a graph without loops or bridges and $a, b$ be positive real numbers with $b \geq a(a + 2)$. Then
\[ \max\{T_G(b,0), T_G(0,b)\} \geq T_G(a,a). \]

(2.20). Identities.

(i) Merino [31] proved the following identity,
\[ T_{K_n+2}(1,-1) = T_{K_n}(2,-1). \]
(ii) Merino’s result was generalized by Goodall et al [15]:
A graph is called a threshold graph if the vertices can be ordered so that each vertex is adjacent to either all or none of the previous vertices. Threshold graphs are also the graphs with no induced $P_4, C_4$ or $2P_2$.
If $G$ is a threshold graph and $u$ and $v$ are the first and last vertex in an ordering of the vertices of $G$ such that each vertex is adjacent to either all or none of the previous ones, then

$$T_G(1, -1) = T_{G_{-u-v}}(2, -1).$$

3 Characteristic polynomial of a matroid

(3.1). Characteristic polynomials of matroids were first studied by Rota [38].
The characteristic polynomial $C(M, x)$ of a matroid $M = (E, r)$ is defined as

$$C(M, x) = \sum_{A \subseteq E} (-1)^{|A|} x^{r(M) - r(A)}.$$  (3.7)

$r(A)$ is the rank function of the matroid $M$ is a function $r : 2^E \to \mathbb{N}_0$ satisfying the following conditions:

(i) $0 \leq r(A) \leq |A|$ for all $A \subseteq E$;
(ii) $r(A) \leq r(B)$ if $A \subseteq B$;
(iii) (submodularity) for any $A, B \subseteq E$,

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

The rank of the matroid $M$ is $r(E)$.

(3.2). Relation with Tutte polynomial:

$$C(M, x) = (-1)^{r(E)} T_M(1 - x, 0).$$

(3.3). Example. Let $U_{k,n}$ be the uniform matroid, where $k \leq n$, i.e., $|E| = n$ and $r(A) = |A|$ if $|A| \leq k$, and $r(A) = k$ otherwise. Then

$$C(U_{1,1}, x) = x - 1;$$
\[ C(U_{2\cdot 4}, x) = x^2 - 4x + 3; \]
\[ C(U_{2\cdot n}, x) = x^2 - nx + (n - 1); \]
\[ C(U_{3\cdot n}, x) = x^3 - nx^2 + \binom{n}{2} x - \binom{n - 1}{2}. \]

(3.4). **Question 3.1** Find \( C(U_{4\cdot 6}, x). \)

(3.5). **Question 3.2** Show that for any \( 0 \leq k \leq n, \)
\[ C(U_{k\cdot n}, x) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} x^{k-i} + \sum_{i=k}^{n} (-1)^i \binom{n}{i}. \]

Observe that if \( k = n-1, \) then \( C(U_{n-1\cdot n}, x) = \chi(C_n, x)/x, \) where \( n \geq 3 \) and \( C_n \) is the cycle graph of order \( n. \)

(3.6). For \( A \subseteq E, \) \( A \) is called a flat of \( M \) if \( r(A \cup \{e\}) > r(A) \) for any \( e \in E - A. \)

Let \( \mathcal{F}(M) \) be the set of flats of \( M \) and \( \mu \) be the möbius function \( \mu(A, B) \)
on flats \( A, B \) in \( \mathcal{F}(M). \)

Note that \( \mu(A, A) = 1 \) for all \( A \in \mathcal{F}(M), \) and for each pair of flats \( A, B \in \mathcal{F}(M) \) with \( A \subseteq B: \)
\[ \mu(A, B) = - \sum_{B' \subseteq B, B' \in \mathcal{F}(M)} \mu(A, B'). \]

(3.7). **Lemma 3.1** For any flat \( F \in \mathcal{F}(M), \)
\[ \sum_{A \subseteq F, r(A) = r(F)} (-1)^{|A|} = \begin{cases} \mu(\emptyset, F), & M \text{ is loopless;} \\ 0, & \text{otherwise.} \end{cases} \]

**Proof.** Assume that \( M \) is loopless. Define
\[ U_F = \sum_{A \subseteq F, r(A) = r(F)} (-1)^{|A|}. \]

It is clear that if \( F = \emptyset, \) then \( U_F = \mu(\emptyset, F). \)
Now assume that \( r(F) \geq 1 \). By induction, for any flat \( F' < F \) (i.e., \( r(F) < r(F') \)), the following holds:

\[
U_{F'} = \mu(\emptyset, F').
\]

Then

\[
\mu(\emptyset, F) = - \sum_{\emptyset \leq F' < F} \mu(\emptyset, F') = - \sum_{\emptyset \leq F' < F} U_{F'},
\]

\[
= - \sum_{\emptyset \leq F' < F} \sum_{A \subseteq F', r(A) = r(F')} (-1)^{|A|}.
\]

\[
= \sum_{A \subseteq F} (-1)^{|A|} - \sum_{\emptyset \leq F' < F} \sum_{A \subseteq F', r(A) = r(F')} (-1)^{|A|} = U_F.
\]

If \( M \) has a loop \( e \), then \( e \in F \) and the power set \( 2^F \) is partitioned into \( 2^{F - \{e\}} \) and \( \{A \cup \{e\} : A \in 2^{F - \{e\}}\} \). Thus

\[
U_F = \sum_{A \subseteq F - \{e\}} (-1)^{|A|} + \sum_{A \subseteq F - \{e\}} (-1)^{|A \cup \{e\}|} = 0.
\]

(3.8). Assume that \( M \) is loopless, i.e., \( r(A) = 0 \) implies that \( A = \emptyset \).

**Proposition 3.1** If \( M \) is loopless, then \( C(M, x) \) has another expression:

\[
C(M, x) = \sum_{A \subseteq F(M)} \mu(\emptyset, A)x^{r(M) - r(A)}.
\]

**Proof.** Note that

\[
C(M, x) = \sum_{A \subseteq E} (-1)^{|A|}x^{r(M) - r(A)}
\]

\[
= \sum_{F \in F(M)} \sum_{A \subseteq F, r(A) = r(F')} (-1)^{|A|}x^{r(M) - r(A)}
\]

\[
= \sum_{F \in F(M)} \mu(\emptyset, F)x^{r(M) - r(F)},
\]

where the last equality follows from Lemma 3.1. \( \square \)
(3.9). $C(M, x)$ can be determined by the following properties:

(i) if $M$ has a loop, then $C(M, x) = 0$;
(ii) the characteristic polynomial of the uniform matroid $U_{1,1}$ is $C(U_{1,1}, x) = x - 1$;
(iii) if $M = M_1 \oplus M_2$ then,

$$C(M, x) = C(M_1, x)C(M_2, x);$$

(iv) if $e$ is not a loop or coloop of $M$, then

$$C(M, x) = C(M \setminus e, x) - C(M/e, x).$$

(3.10). Multiplication identity.

Theorem 3.1 (Kung 2004 [26]) For any matroid $M = (E, r),$

$$C(M, x_1x_2) = \sum_{F \in F(M)} C(M/F, x_1)x_2^{r(M) - r(M|F)}C(M|F, x_2),$$

where $G|F$ is the restriction of $M$ to $F$ and $G/F$ is the contraction of $M$ by $F$.

Proof. If $F$ is not a flat, then $C(M/F, x) = 0$. Thus the right-hand side can be changed to

$$RHS = \sum_{S \subseteq E} C(M/S, x_1)x_2^{r(M) - r(M|S)}C(M|S, x_2).$$

By definition,

$$C(M/S, x) = \sum_{S \subseteq A \subseteq E} (-1)^{|A-S|}x^{r(M)/s - r_{M/S}(A-S)}$$

$$= \sum_{S \subseteq A \subseteq E} (-1)^{|A-S|}x^{r(M) - r(A)},$$

as $r(M/S) = r(M) - r(S)$ and $r_{M/S}(A - S) = r(A) - r(S)$, while

$$C(M|S, x) = \sum_{B \subseteq S} (-1)^{|B|}x^{r(S) - r(B)}.$$
Thus
\[
RHS = \sum_{S \subseteq E} \sum_{B \subseteq S \subseteq A \subseteq E} (-1)^{|A|+|B|-|S|} x_1^{r(M)-r(A)} x_2^{r(M)-r(B)}
\]
\[
= \sum_{B \subseteq A \subseteq E} \sum_{B \subseteq S \subseteq A} (-1)^{|A|+|B|-|S|} x_1^{r(M)-r(A)} x_2^{r(M)-r(B)}
\]
\[
= \sum_{B=A \subseteq E} (-1)^{|A|} x_1^{r(M)-r(A)} x_2^{r(M)-r(A)} = C(M, x_1 x_2),
\]
where the second last equality follows from the following fact that if \(A\) and \(B\) are fixed with \(B \subset A\), then
\[
\sum_{B=A \subseteq E} (-1)^{|A|+|B|-|S|} x_1^{r(M)-r(A)} x_2^{r(M)-r(B)} = 0.
\]

\(\square\)

(3.11). Relation with chromatic polynomial and flow polynomial.

Observe that for any graph \(G\), if \(M_G\) and \(M^*_G\) are the cycle matroid and the cocycle matroid of \(G\) respectively, then
\[
C(M_G, x) = x^{-c} \chi(G, x), \quad C(M^*_G, x) = F(G, x).
\]

where \(c\) is the number of components of \(G\).
Thus this polynomial \(C(G, x)\) is an extension of both \(\chi(G, x)\) and \(F(G, x)\).

(3.12). By Proposition 3.1 we have:

**Corollary 3.1** For any simple graph \(G = (V, E)\),
\[
\chi(G, xy) = \sum_{E' \subseteq E} \chi(G/E', x)\chi(G|E', y),
\]
where \(G/E'\) is the graph obtained from \(G\) by contracting all edges in \(E'\) and \(G|E'\) is the graph with edge set \(E'\) and vertex set \(V_{E'} = \{u \in V : N_u \cap E' \neq \emptyset\}\).
(3.13). Oxley \cite{35} showed that if every cocircuit of $M$ has size at most $d$, then $C(M, x) > 0$ holds for all real numbers $r \geq d$. Jackson \cite{21} pointed out that the idea in Oxley’s proof can be applied to get a more general result.

A simple minor of $M$ is a minor which contains no loops or circuits of length two.

**Theorem 3.2 \cite{21}** Let $M$ be a matroid. If every simple minor of $M$ has a cocircuit of size at most $d$, then $C(M, x) > 0$ for all real numbers $x \geq d$.

As $F(G, x) = C(M^*_G, x)$, Theorem 3.2 implies that for any bridgeless graph $G$, if every 3-edge-connected minor of $G$ has a circuit of length at most $d$, then $F(G, x) > 0$ holds for all real numbers $t \geq d$. It is not difficult to show that every 3-connected graph $G$ of order $n$ has a circuit of length at most $2 \log_2 n$. Thus every bridgeless graph of order $n$ has all real flow roots less than $2 \log_2 n$.

4 Flow polynomial $F(G, x)$

(4.1). Let $D$ be any orientation of a graph $G$ and $\Gamma$ be any Abelian group.

Let $f$ be a mapping $f : A(D) \Rightarrow \Gamma$, where $A(D)$ is the set of arcs in $D$. $f$ is called a $\Gamma$-flow of $D$ if at every vertex $u$ of $D$:

$$\sum_{a \in A^+(u)} f(a) = \sum_{a \in A^-(u)} f(a),$$

where $A^+(u)$ (resp. $A^-(u)$) is the set of arcs with head (resp. tail) at $u$. $f$ is called a nowhere-zero $\Gamma$-flow of $D$ if it is a $\Gamma$-flow and $f(a) \neq 0$ for each $a \in A(D)$.

An example is shown in Figure 3.

(4.2). Nowhere-zero flows were introduced by Tutte \cite{53} as a dual concept to proper colourings.

(4.3). For any two orientations $D_1, D_2$ of $G$ and any two Abelian groups $\Gamma_1, \Gamma_2$ with the same order,
An undirected graph is said to have a nowhere-zero $\Gamma$-flow if some of its orientation has such a flow.

(4.4). For any positive integer $q$, a nowhere-zero $q$-flow is a nowhere-zero $\mathbb{Z}$-flow $g$ such that $|g(a)| < q$ for all arcs $a$ in $D$.

A nowhere-zero 3-flow is shown below in Figure 6.

(4.5). Tutte [53] showed that $G$ has a nowhere-zero $q$-flow if and only if it has a nowhere-zero $\mathbb{Z}_q$-flow.
But the number of nowhere-zero $q$-flows may be not equal to the number of nowhere-zero $Z_q$-flows.

(4.6). If $G$ has a bridge, then $G$ does not have a nowhere-zero $Z_q$-flow for all $q \geq 2$.

(4.7). Theorem 4.1 (Tutte 1954[52]) A plane graph $G$ is $k$-face-colourable if and only if it has a nowhere-zero $k$-flow.

Proof. Let $G$ be a plane graph and $\vec{G}$ be an orientation of $G$.
From a face colouring of $G$ with colouring $0, 1, \cdots, k - 1$, we can get a nowhere-zero $k$-flow by assigning each arc the difference of the two values of its two sides: the right-hand side to the arrow minus the other side.

Example:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{align*}
$$

(4.8). Tutte’s flow conjectures.

Tutte’s 5-flow Conjecture [1954]:
Every bridgless graph has a nowhere-zero 5-flow.

Tutte’s 4-flow Conjecture [1966]:
Every bridgless graph with no Petersen minor has a nowhere-zero 4-flow.

Tutte’s 3-flow Conjecture [1970s]:
Every 4-edge-connected graph has a nowhere-zero 3-flow.

(4.9). Jaeger’s weak 3-flow Conjecture [1988]:
There exists a fixed integer $k$ so that every $k$-edge-connected graph has a nowhere-zero 3-flow.
C. Thomassen [106] proved Jaeger’s weak 3-flow conjecture for $k = 8$. 30
(4.10). For any graph $G$ and any positive integer $t$, let $F(G, t)$ be the number of distinct nowhere-zero $\mathbb{Z}_t$-flows of $G$ for any positive integer $t$.

The function $F(G, t)$ is called the flow polynomial of $G$.

Tutte’s 5-flow conjecture is equivalent to the statement that $F(G, 5) > 0$ for all bridgeless graph $G$.

(4.11). The flow polynomial $F(G, x)$ of a graph $G$ can be obtained from the following rules (see Tutte [48]):

$$F(G, x) = \begin{cases} 
1, & \text{if } E = \emptyset; \\
0, & \text{if } G \text{ has a bridge;} \\
F(G_1, x)F(G_2, x), & \text{if } G = G_1 \cup_0 G_2; \\
(x - 1)F(G \setminus e, x), & \text{if } e \text{ is a loop;} \\
F(G/e, x) - F(G \setminus e, x), & \text{if } e \text{ is not a loop nor a bridge},
\end{cases}$$

(4.8)

where $G_1 \cup_0 G_2$ is the disjoint union of graphs $G_1$ and $G_2$.

(4.12). Examples.

(i) If $G$ is a cycle, then $F(G, x) = x - 1$.

(ii) If $G = L_k$ is a graph with two vertices $u$ and $v$ and $k$ edges joining them, then

$$F(G, x) = ((x - 1)^k + (-1)^k(x - 1)) / x.$$ 

(iii) $F(L_3, x) = (x - 1)(x - 2)$.

(iv) If $G = K_4$, then

$$F(G, x) = (x - 1)(x - 2)(x - 3).$$

Thus $K_4$ has no nowhere-zero 3-flow.

(4.13). Dual polynomials.

Let $G^*$ be the dual of a plane graph $G$. Then

$$\chi(G, x) = xF(G^*, x),$$

where $\chi(G, x)$ is the chromatic polynomial of $G$. 

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(4.14). Interpretation.

(i) For a connected graph $G = (V, E)$,

$$F(G, x) = \sum_{A \subseteq E} (-1)^{|E| - |A||x|^{|A| - |V|} + c(A)}, \quad (4.9)$$

where $c(A)$ is the number of components in the subgraph $(V, A)$.

(ii) Assume that $G$ is bridgeless and connected. By expression (4.9),

$$F(G, t) = x^{m-n+1} - b_1 x^{m-n} + b_2 x^{m-n-1} + \cdots + (-1)^{m-n+1} b_{m-n+1},$$

where $m = |E|$ and

$$b_i = \nu_{i,1} - \nu_{i+1,2} + \nu_{i+2,3} - \cdots = \sum_{j \geq 1} (-1)^{j-1}\nu_{i+j-1,j},$$

where $\nu_{k,j}$ is the number of subsets $A$ of $E$ such that $|A| = k$ and $c(G - A) = j$.

(iii) $b_i$’s are positive (can be proved by induction);

(iv) If $G$ is connected without 2-edge-cut, then $b_1 = m$ and $b_2 = m - \gamma$, where $\gamma$ is the number of 3-edge-cuts of $G$.

(4.15). Basic properties.

(i) If $G_1, G_2, \cdots, G_k$ are components of $G$, then

$$F(G, x) = \prod_{1 \leq i \leq k} F(G_i, x).$$

(ii) If $G$ is connected and $G_1, G_2, \cdots, G_k$ are blocks of $G$, then

$$F(G, x) = \prod_{1 \leq i \leq k} F(G_i, x).$$

(iii) If $\delta(G) = 1$, then $F(G, x) = 0$.

(iv) If $N(w) = \{u, v\}$ for $w \in V(G)$, then

$$F(G, x) = F((G - w) \cdot uv, x),$$

where $(G - w) \cdot uv$ is the graph obtained from $G - w$ by identifying $u$ and $v$.

Thus, for flow polynomials, we may just consider connected graphs with minimum degree at least 3.
(v) (Jackson 2007) Let $G$ be a bridgeless connected graph, $v$ be a vertex of $G$, $e = u_1u_2$ be an edge of $G$, and $H_1$ and $H_2$ be edge-disjoint subgraphs of $G$ such that $E(H_1) \cup E(H_2) = E(G \setminus e)$, $V(H_1) \cap V(H_2) = \{v\}$, $V(H_1) \cup V(H_2) = V(G)$, $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$, as shown blow. Then

$$F(G, x) = \frac{F(G_1, x)F(G_2, x)}{x - 1}.$$  

where $G_i = H_i + vu_i$ for $i \in \{1, 2\}$.

(vi) (Jackson 2007) Let $G$ be a bridgeless connected graph, $S$ be a 2-edge-cut of $G$, and $H_1$ and $H_2$ be the sides of $S$, as shown blow. Let $G_i$ be obtained from $G$ by contracting $E(H_{3-i})$, for $i \in \{1, 2\}$. Then

$$F(G, x) = \frac{F(G_1, x)F(G_2, x)}{x - 1}.$$  

(vii) (Jackson 2007) Let $G$ be a bridgeless connected graph, $S$ be a 3-edge-cut of $G$, and $H_1$ and $H_2$ be the sides of $S$. Let $G_i$ be obtained from $G$ by contracting $E(H_{3-i})$, for $i \in \{1, 2\}$. Then

$$F(G, x) = \frac{F(G_1, x)F(G_2, x)}{(x - 1)(x - 2)}.$$
(4.16). Relation with the Tutte polynomial \( T_G(x, y) \) of \( G = (V, E) \):

\[
F(G, x) = (-1)^{|E| - |V| + c(E)} T_G(0, 1 - x)
\]

where \( c(E') \) is the number of components of the spanning subgraph \((V, E')\).

(4.17). Known facts.

(i) **Theorem 4.2** (Waklin 1994 [58]) Let \( G = (V, E) \) be a bridgeless connected graph with block number \( b(G) \). Then

(a) \( F(G, x) \) is non-zero with sign \((-1)^{|E| - |V| + 1}\) for \( x \in (-\infty, 1)\);
(b) \( F(G, x) \) has a zero of multiplicity \( b(G) \) at \( x = 1 \);
(c) \( F(G, x) \) is non-zero with sign \((-1)^{|E| - |V| + b(G) - 1}\) for \( x \in (1, 32/27] \).

(ii) **Theorem 4.3** (Jackson 2007[20]) If \( G \) has at most one vertex of degree larger than 3, then \( F(G, x) \) is non-zero in the interval \((1, 2)\).

(iii) **Theorem 4.4** (Dong [6, 7]) If \( G \) has at most two vertices of degrees larger than 3, then \( F(G, x) \) is non-zero in the interval \((1, 2)\).

More generally, if all vertices in \( W := \{ u \in V(G) : d(u) \geq 4 \} \) are dominated by one component of \( G - W \), then \( F(G, x) \) is non-zero in the interval \((1, 2)\).

(iv) **Theorem 4.5** (Kung and Royle [27]) If \( G \) is a bridgeless graph, then its flow roots are integral if and only if \( G \) is the dual of a chordal and plane graph.

(v) **Theorem 4.6** (Dong [8]) For any bridgeless graph \( G \), if \( F(G, x) \) has real roots only, then either all roots of \( F(G, x) \) are integral or \( F(G, x) \) has at least 9 roots in \((1, 2)\).

(vi) **Theorem 4.7** (Jackson [20]) Let \( G \) be a 3-connected cubic graph with \( n \) vertices and \( m \) edges. Then

(a) \( F(G, x) \) is non-zero in the interval \((1, 2)\) with sign \((-1)^{m-n}\);
(b) \( F(G, x) \) has a zero of multiplicity 1 at \( x = 2 \);
(c) \( F(G, x) \) is non-zero with sign \((-1)^{m-n+1}\) for \( x \in (2, d) \), where \( d \approx 2.546 \) is the flow root of the cube in \((2, 3)\).
(vii) Jackson [21] showed that for any bridgeless graph $G$ of order $n$, all real roots of $F(G, q)$ are smaller than $2 \log_2 n$.

(4.18). A survey on the study of real roots of flow polynomials is provided in [9].

(4.19). Open problems.

(i) **Conjecture 4.1 (Welsh [59])** For any bridgeless graph $G$, $F(G, q) > 0$ for all real numbers $q \in (4, \infty)$.

Haggard, Pearce and Royle [17] showed that the generalised Petersen graph $G_{16,6}$ has real flow roots around $4.0252205$ and $4.2331455$, where the generalized Petersen graph $G_{n,k}$ for $n \geq 3$ and $1 \leq k \leq \lfloor (n - 1)/2 \rfloor$ is the graph with vertex set $\{u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+k} : 1 \leq i \leq n\}$, where $v_s$ for $s > n$ is considered as $v_t$, where $t$ is the integer with $1 \leq t \leq n$ such that $s - t$ is a multiple of $n$.

(ii) **Conjecture 4.2 (Haggard, Pearce and Royle [17])** For any bridgeless graph $G$, $F(G, q) > 0$ for all real numbers $q \in [5, \infty)$.

The above conjecture was recently disproved by Jacobsen and Salas [23] who found counter-examples by studying the subfamily of generalised Petersen graphs $G_{nr,r}$ for $n \geq 2$ and $r \geq 2$.

**Theorem 4.8 (Jacobsen and Salas [23])** The value $q = 5$ is an isolated accumulation point of real zeros of the flow polynomial $F(G, q)$ for the families of bridgeless graphs $G_{6n,6}$ and $G_{7n,7}$ with $n \geq 3$. Moreover:

(a) There is a sequence of real zeros $\{q_n\}$ of the flow polynomials $F(G_{6n,6}, q)$ that converges to $q = 5$ from below.

(b) There is a sequence of real zeros $\{q_n\}$ of the flow polynomials $F(G_{7n,6}, q)$ that converges to $q = 5$. The sub-sequence with odd (resp. even) $n$ converges to $q = 5$ from above (resp. below).

**Theorem 4.9 (Jacobsen and Salas [23])** (a) The bridgeless graph $G_{119,7}$ has flow roots at $q \approx 5.00002$ and $q \approx 5.16534$ (where $\approx$ means “within $10^{-5}$”).
(b) The value \( q' \approx 5.235261 \) (where \( \approx \) means “within \( 10^{-6} \)”) is an accumulation point of real zeros of the flow polynomials \( F(G_{7n}, q) \). In particular, the sub-sequence for odd \( n \) of the real zeros \( \{q_n\} \) of the flow polynomials \( F(G_{7n}, q) \) converges to \( q' \) from below.

(iii) **Conjecture 4.3** (Jacobsen and Salas [23]) For any bridgeless graph \( G \), \( F(G, q) > 0 \) for all real numbers \( q \in [6, \infty) \).

(iv) **Conjecture 4.4** (Dong [8, 10]) For any bridgeless graph \( G \), if \( F(G, q) \) has real roots only, then all roots of \( F(G, q) \) are integral.

5 Order polynomial \( \Omega(D, x) \)

(5.1). In this section, let \( D \) be a digraph of order \( p \) unless stated otherwise.

(5.2). For any positive integer \( k \), let \( \Omega(D, k) \) (or resp. \( \Omega(D, k) \)) be the number of strictly order-preserved mappings (or resp. order-preserved mappings) \( \theta : V(D) \to \{1, 2, \cdots, k\} \) with respect to \( D \), i.e., \( \theta(u) < \theta(v) \) (or resp. \( \theta(u) \leq \theta(v) \)) whenever \( u \to v \) in \( D \).

(5.3). \( \Omega(D, k) \) is called the order polynomial of \( D \).

(5.4). A digraph is said to be acyclic if it does not contain directed cycles.

The order polynomial was first defined for a poset by Stanley in 1970. For an acyclic digraph \( D \), let \( \bar{D} \) be the poset which is the reflexive transitive closure of \( D \), i.e., the poset with element set \( V(D) \) and binary relation \( u \preceq v \) whenever there exists an path in \( D \) from \( u \) to \( v \). Thus, the order polynomial for poset \( \bar{D} \) is actually the polynomial \( \Omega(D, x) \).

(5.5). **Example 5.1** Let \( D \) be a digraph of order \( p \).

(i) \( \Omega(D, k) = 0 \) if \( D \) is not acyclic;

(ii) \( \Omega(D, k) = \binom{k}{p} \) if \( D \) is an acyclic tournament, i.e., an orientation of a complete graph which contains no directed cycles;

(iii) \( \Omega(D, k) = k^p \) if \( D \) contains no arcs.

\(^{\dagger}\)In some articles or books, \( \Omega(D, k) \) denotes the the number of strictly order-preserved mappings \( \theta : V(D) \to \{1, 2, \cdots, k\} \) with respect to \( D \), while \( \overline{\Omega}(D, k) \) denotes the the number of order-preserved mappings \( \theta : V(D) \to \{1, 2, \cdots, k\} \).
(5.6). **Theorem 5.1 (Stanley [44])** If $u, v$ are distinct vertices in $D$ with $u \not\rightarrow v$ and $v \not\rightarrow u$, then

$$
\Omega(D, k) = \Omega(D_{u\rightarrow v}, k) + \Omega(D_{v\rightarrow u}, k) + \Omega(D_{uv}, k),
$$

where $D_{u\rightarrow v}$ is the digraph obtained from $D$ by adding a new arc $u \rightarrow v$ and $D_{uv}$ is the digraph obtained from $D$ by identifying $u$ and $v$.

(5.7). **Corollary 5.1** If $u \rightarrow v \rightarrow w$ and $u \not\rightarrow w$ in $D$, then

$$
\Omega(D, k) = \Omega(D_{u\rightarrow w}, k).
$$

(5.8). Computing $\Omega(D, k)$ by applying Theorem 5.1

(i) apply Theorem 5.1 repeatedly until all digraphs are tournaments;
(ii) in each step of applying Theorem 5.1 remove every digraph which is not acyclic;
(iii) let $t_i$ be the total number of acyclic tournaments of order $i$ that are left after Steps (i) and (ii). Then

$$
\Omega(D, k) = \sum_{i \leq p} t_i \binom{k}{i}.
$$

(5.9). **Example 5.2** Let $D_1, D_2$ be the digraphs below. Find $\Omega(D_i, k)$ for $i = 1, 2$.

\begin{figure}[h]
\centering
\includegraphics{example52.png}
\caption{Two digraphs $D_1$ and $D_2$}
\end{figure}

By Theorem 5.1

$$
\Omega(D_1, k) = \binom{k}{2} + 2 \binom{k}{3}, \quad \Omega(D_2, k) = \binom{k}{3},
$$

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by the operations shown below.

\[
\begin{align*}
\text{Not acyclic} & \quad \Rightarrow \\
\text{Not acyclic} & \quad \Rightarrow
\end{align*}
\]

Note that \( \overline{\Omega}(D_2, k) = \binom{k}{3} \) also follows from Corollary 5.1.

(5.10). From now on, assume that \( D \) is acyclic.

**Theorem 5.2 (Stanley [45])**

\[
\overline{\Omega}(D, k) = \sum_{i \leq p} e_i \binom{k}{i},
\]

where \( e_i \) is the number of surjective strictly order-preserved mappings \( \theta : V(D) \to \{1, 2, \ldots, i\} \) with respect to \( D \).

Thus \( \overline{\Omega}(D, k) \) is a polynomial in \( k \) of degree \( p \).

(5.11). Multiplication identity.

**Proposition 5.1 (Tugger 1978)** For any acyclic digraph \( D \),

\[
\overline{\Omega}(D, x + y) = \sum_{D'} \overline{\Omega}(D', x) \overline{\Omega}(D \setminus V(D'), y),
\]

where the sum runs over all order ideals \( D' \) of \( D \); i.e., \( D' \) is a subdigraph of \( D \) such that \( x \to y \) in \( D \) and \( y \in V(D') \) imply that \( x \in V(D') \).
Fengming Dong  Polynomials related to chromatic polynomials (Sect. 5)

Mentioned in [26], this result was obtained by Tugger in 1978. Such an identity was first obtained by Tutte [50] in 1967 on chromatic polynomials (see Theorem 6.2).

(5.12). Note that $\Omega(D, k) \leq \Omega(D, k)$.

In the following, we shall show that for any positive integer $k$,

$$\Omega(D, k) = (-1)^p \Omega(D, -k).$$

(5.13). By Theorem 5.2, $\Omega(D, k)$ is independent of the labels of its vertices.

Suppose that $D$ is acyclic and of order $p$. Assume vertices in $D$ are labelled by numbers $1, 2, \cdots, p$ such that $i < j$ whenever $i \rightarrow j$ in $D$. It works as $D$ is acyclic.

As example for $D$ and such a label is shown below:

```
    3
   /\n  1   2
   \/
    4
```

(5.14). Let $OP(D)$ be the set of those order preserved permutations $(i_1, i_2, \cdots, i_p)$ of $1, 2, \cdots, p$, i.e., $i_s$ appears before $i_t$ (i.e., $s < t$) whenever $i_s \rightarrow i_t$ in $D$.

For example, for the above digraph $D$, $OP(D)$ contains 5 permutations:

- $\pi_1 : (1, 2, 3, 4)$
- $\pi_2 : (2, 1, 3, 4)$
- $\pi_3 : (1, 2, 4, 3)$
- $\pi_4 : (2, 1, 4, 3)$
- $\pi_5 : (2, 4, 1, 3)$

(5.15). For any permutation $\pi = (i_1, i_2, \cdots, i_p)$ of $1, 2, \cdots, p$, we say a map $\theta : \{1, 2, \cdots, p\} \rightarrow \{1, 2, \cdots, k\}$ is compatible with $\pi$ if the two conditions below are satisfied:

(a) $\theta(i_1) \leq \theta(i_2) \leq \cdots \leq \theta(i_p)$ and

(b) $\theta(i_j) < \theta(i_{j+1})$ whenever $i_j < i_{j+1}$.
Lemma 5.1 (Stanley [45]) Let \( \theta : \{1, 2, \cdots, p\} \rightarrow \{1, 2, \cdots, k\} \) be a strictly order-preserved mapping with respect to \( D \), i.e., \( \theta(i) < \theta(j) \) whenever \( i \rightarrow j \) in \( D \). Then \( \theta \) is compatible with exactly one member in \( OP(D) \).

Proof. There exist a unique permutation \( \pi : (i_1, i_2, \cdots, i_p) \) of \( 1, 2, \cdots, p \) and a unique subset \( \{j_1, j_2, \cdots, j_s\} \) of \( 1, 2, \cdots, p-1 \) with \( j_1 < j_2 < \cdots < j_s \) such that the three conditions below are all satisfied:

(i) \( \theta(i_1), \theta(i_2), \cdots, \theta(i_p) \) is in non-decreasing order and \( \theta(i_q) < \theta(i_{q+1}) \) if and only if \( q \in \{j_1, j_2, \cdots, j_s\} \), i.e.,

\[
\theta(i_1) = \cdots = \theta(i_{j_1}) < \theta(i_{j_1+1}) = \cdots = \theta(i_{j_2}) < \theta(i_{j_2+1}) = \cdots = \theta(i_{j_s}) < \theta(i_{j_s+1}) = \cdots = \theta(i_p); \quad (5.11)
\]

(ii) for each pair \( s, t: 1 \leq s < t \leq p, \ i_s > i_t \) whenever \( \theta(i_s) = \theta(i_t) \).

As \( \theta \) is a strictly order-preserved with respect to \( D \), for any two vertices \( i_s \) and \( i_t \) in \( D \), \( s \leq j_r < j_{r+1} \leq t \) holds for some \( 1 \leq r \leq s \) whenever \( i_s \rightarrow i_t \) in \( D \).

Thus the permutation \( \pi = (i_1, i_2, \cdots, i_p) \) is order preserved, i.e., \( \pi \in OP(D) \).

Also observe that \( \theta \) is compatible with \( \pi \), as conditions (i) and (ii) imply that

\( \{1 \leq q \leq p-1 : i_q < i_{q+1}\} \subseteq \{j_1, j_2, \cdots, j_s\} \).

Suppose that \( \theta \) is also compatible with another order preserved permutation \( \pi': (i'_1, i'_2, \cdots, i'_p) \) of \( 1, 2, \cdots, p \). By definition of the compatibility, \( \theta(i'_1), \theta(i'_2), \cdots, \theta(i'_p) \) is in non-decreasing order. As the sequence \( \theta(1), \theta(2), \cdots, \theta(p) \) produces a unique sequence in non-decreasing order, we have \( \theta(i_q) = \theta(i'_q) \) for all \( q = 1, 2, \cdots, p \). Then, by \((5.11)\), we have

\[
\{i_q : j_t < q \leq j_{t+1}\} = \{i'_q : j_t < q \leq j_{t+1}\}
\]

for all \( t = 0, 1, 2, \cdots, s \), where \( j_0 = 0 \) and \( j_{s+1} = p \). Furthermore, for each \( t = 0, 1, 2, \cdots, s \), \( \theta(i'_q) \) is a constant for all \( q : j_t < q \leq j_{t+1} \). As \( \theta \)
is compatible with \( \pi' \), by definition, \( i'_q > i'_{q+1} \) holds for all \( q : j_t < q \leq j_{t+1} - 1 \).
Therefore \( \pi' = \pi \), a contradiction. \( \square \)

(5.18). For each order-preserved permutation \( \pi \in OP(D) \), let \( \mathcal{OM}(\pi, k) \) be the set of strictly order-preserved mappings \( \tau : \{1, 2, \cdots, p\} \to \{1, 2, \cdots, k\} \) with respect to \( D \) that are compatible with \( \pi \).

(5.19). Given any permutation \( \pi = (i_1, i_2, \cdots, i_p) \), let \( \rho(\pi) \) denote the size of the following set
\[
\{1 \leq j < p : i_j < i_{j+1}\}.
\]
For example, \( \rho(\pi) = 2 \) if \( \pi \) is \((2, 1, 3, 4)\).
In general, \( 0 \leq \rho(\pi) \leq p - 1 \).

(5.20). **Lemma 5.2 (Stanley [45])** For any \( \pi \in OP(D) \),
\[
|\mathcal{OM}(\pi, k)| = \binom{k + p - 1 - \rho(\pi)}{p}.
\]

**Proof.** Let \( \pi = (i_1, i_2, \cdots, i_p) \in OP(D) \). Assume that \( \rho(\pi) = s \geq 0 \).
Thus there are exactly \( s \) numbers \( j_1, j_2, \cdots, j_s \) in the set \( \{1, 2, \cdots, p - 1\} \) such that \( j_1 < j_2 < \cdots < j_s \) and \( i_q < i_{q+1} \) holds for all \( q \in \{j_1, j_2, \cdots, j_s\} \).
Note that \( \mathcal{OM}(\pi, k) \) is the set of those mappings \( \theta : \{1, 2, \cdots, p\} \to \{1, 2, \cdots, k\} \) such that
(a) \( \theta(i_1) \leq \theta(i_2) \leq \cdots \leq \theta(i_p) \);
(b) \( \theta(i_q) < \theta(i_{q+1}) \) for all \( q \in \{j_1, j_2, \cdots, j_s\} \).
The two conditions (a) and (b) above on \( \theta \) is equivalent to the following inequality:
\[
0 < \theta(i_1) \leq \cdots \leq \theta(i_{j_1}) \leq \theta(i_{j_1+1}) \leq \cdots \leq \theta(i_{j_2}) < \theta(i_{j_2+1}) \leq \cdots \leq \theta(i_{j_s}) \leq \theta(i_{j_s+1}) \leq \cdots \leq \theta(i_p) \leq k. \tag{5.12}
\]
Let \( x_0, x_1, x_2, \cdots, x_p \) be numbers defined by \( x_p = k - \theta(i_p) \),
\[
x_q = \theta(i_{q+1}) - \theta(i_q) - 1, \quad \forall q \in \{0, j_1, j_2, \cdots, j_s\},
\]
where $\theta(i_0) = 0$, and
$$x_q = \theta(i_{q+1}) - \theta(i_q)$$
for all $q \in \{1, 2, \cdots, p - 1\} - \{j_1, j_2, \cdots, j_s\}$.

Observe that each $x_q$ is non-negative. There is a bijection between the set of vectors $(\theta(i_1), \cdots, \theta(i_p))$ satisfying (5.12) and the set of vectors $(x_0, x_1, \cdots, x_p)$, where each $x_q$ is a non-negative integer and
$$x_0 + x_1 + \cdots + x_p = k - (s + 1).$$

Thus $|\mathcal{OM}(\pi, k)|$ is equal to the number of non-negative integer solutions $(x_0, x_1, \cdots, x_p)$ of the following equation:
$$x_0 + x_1 + \cdots + x_p = k - (s + 1).$$

Hence
$$|\mathcal{OM}(\pi, k)| = \binom{k - (s + 1) + p}{p} = \binom{k + p - \rho(\pi) - 1}{p}.$$  \hfill $\square$

(5.21). By Lemmas 5.1 and 5.2 the following result is obtained.

**Theorem 5.3 (Stanley [45])**

$$\Omega(D, k) = \sum_{\pi \in OP(D)} |\mathcal{OM}(\pi, k)| = \sum_{\pi \in OP(D)} \binom{k + p - 1 - \rho(\pi)}{p}.$$  

(5.22). The generating function of the sequence $\{\Omega(D, k)\}_{k \geq 0}$ is
$$\sum_{k=0}^{\infty} \Omega(D, k)x^k = \sum_{\pi \in OP(D)} x^{\rho(\pi)+1}/(1 - x)^{p+1}.$$  

**Proof.** By Theorem 5.3

$$\sum_{k=0}^{\infty} \Omega(D, k)x^k = \sum_{k=0}^{\infty} x^k \sum_{\pi \in OP(D)} \binom{k + p - 1 - \rho(\pi)}{p}$$
$$= \sum_{\pi \in OP(D)} x^{1+\rho(\pi)} \sum_{k=0}^{\infty} \binom{k + p - 1 - \rho(\pi)}{p} x^{k-\rho(\pi)}$$
$$= \sum_{\pi \in OP(D)} x^{\rho(\pi)+1}/(1 - x)^{p+1}.$$
(5.23). **Theorem 5.4 (Stanley [45])**

\[ \Omega(G, k) = \sum_{\pi \in \text{OP}(D)} \binom{k + \rho(\pi)}{p}. \]

This result can be obtained similarly as Theorem 5.3 by the following steps:

(i) For any permutation \( \pi = (i_1, i_2, \ldots, i_p) \) of 1, 2, \ldots, \( p \), we say a map \( \theta : \{1, 2, \ldots, p\} \rightarrow \{1, 2, \ldots, k\} \) is *anti-compatible* with \( \pi \) if the two conditions below are satisfied:

(a) \( \theta(i_1) \leq \theta(i_2) \leq \cdots \leq \theta(i_p) \) and
(b) \( \theta(i_j) < \theta(i_{j+1}) \) whenever \( i_j > i_{j+1} \).

For example, a map \( \theta : \{1, 2, 3, 4\} \rightarrow \{1, 2, \ldots, k\} \) is anti-compatible with the permutation \( (2, 1, 3, 4) \) if \( \theta(2) < \theta(1) \leq \theta(3) \leq \theta(4) \).

(ii) **Lemma 5.3 (Stanley [45])** Let \( \theta : \{1, 2, \ldots, p\} \rightarrow \{1, 2, \ldots, k\} \) be an order-preserved mapping with respect to \( D \), i.e., \( \theta(i) \leq \theta(j) \) whenever \( i \rightarrow j \) in \( D \). Then \( \theta \) is anti-compatible with exactly one member in \( \text{OP}(D) \).

(iii) For any \( \pi \in \text{OP}(D) \), let \( \mathcal{AOP}(\pi, k) \) be the set of order-preserved mappings \( \theta : \{1, 2, \ldots, p\} \rightarrow \{1, 2, \ldots, k\} \) with respect to \( D \) that are anti-compatible with \( \pi \).

(iv) By Stanley [45],
\[ |\mathcal{AOP}(\pi, k)| = \binom{k + \rho(\pi)}{p}. \]

(v) Then
\[ \Omega(D, k) = \sum_{\pi \in \text{OP}(D)} |\mathcal{AOP}(\pi, k)| = \sum_{\pi \in \text{OP}(D)} \binom{k + \rho(\pi)}{p}. \]

(5.24). Example. Let \( D \) be the digraph shown in (5.13). Then \( \text{OP}(D) = \{\pi_i : i = 1, 2, \ldots, 5\} \)

\[ \rho(\pi_1) = 3, \rho(\pi_2) = \rho(\pi_3) = \rho(\pi_5) = 2, \rho(\pi_4) = 1. \]
Thus
\[\Omega M(\pi_1, k) = \binom{k + 3 - 3}{4} = \binom{k}{4};\]
\[\Omega M(\pi_i, k) = \binom{k + 3 - 2}{4} = \binom{k + 1}{4}, \quad i = 2, 3, 5\]
and
\[\Omega M(\pi_4, k) = \binom{k + 3 - 1}{4} = \binom{k + 2}{4}.\]
Hence, by Theorem 5.3,
\[\Omega(D, k) = \sum_{i=1}^{5} \Omega M(\pi_i, k) = \left(\binom{k}{4}\right) + 3\left(\binom{k + 1}{4}\right) + \left(\binom{k + 2}{4}\right)\]
and by Theorem 5.4
\[\Omega(D, k) = \sum_{i=1}^{5} |AOP(\pi_i, k)| = \left(\binom{k + 3}{4}\right) + 3\left(\binom{k + 2}{4}\right) + \left(\binom{k + 1}{4}\right).\]

(5.25). The generating function of the sequence \{\Omega(D, k)\}_{k \geq 0} is
\[\sum_{k=0}^{\infty} \Omega(D, k)x^k = \sum_{\pi \in OP(D)} x^{p-\rho(\pi)}/(1 - x)^{p+1}.\]

(5.26). Theorems 5.3 and 5.4 imply that
\[\Omega(D, k) = (-1)^p \Omega(D, -k),\]
as for all integers \(k > 0,\)
\[\binom{k + \rho(\pi)}{p} = (-1)^p \binom{-k + p - 1 - \rho(\pi)}{p}.\]
Note that for any real number \(\alpha, \binom{\alpha}{p}\) is defined to be
\[\binom{\alpha}{p} = \frac{\alpha(\alpha - 1)\cdots(\alpha - p + 1)}{p!} = (\alpha)_p/p!.
\]
(5.27). A new expression for order polynomials is given by Dong \[10\].

Let $D = (V, A)$ be an acyclic digraph with $V = [n] = \{1, 2, \cdots, n\}$.

Note that $\mathcal{OP}(D)$ is the set of orderings $(v_1, v_2, \cdots, v_n)$ of $1, 2, \cdots, n$ which are order-preserved by $D$, i.e., for any $i < j$, $v_i \to v_j$ in $A$ implies that $v_i < v_j$.

For any $\pi \in \mathcal{OP}(D)$, let $\rho(\pi)$ be the size of the set \{1 \leq j \leq n-1 : a_j < a_{j+1} or $(a_j, a_{j+1}) \in A$\}.

Let $W(D)$ be the family of subsets $\{a, b, c\}$ of $V$ with $a < b < c$ such that $(c, a) \in A$ but $b \not\in R_D(c)$ and $a \not\in R_D(b)$, where $R_D(c)$ is the set of vertices in $D$ which are reachable from $c$ in $D$.

Stanley's work Theorem 5.4 is extended as follows.

**Theorem 5.5 \([10]\)** Let $D$ be an acyclic digraph of order $p$. Then $W(D) = \emptyset$ if and only if $\Omega(D, x) = \sum_{\pi \in \mathcal{OP}(D)} \binom{x + \delta(\pi)}{n}$.

6 Express $\chi(G, x)$ in terms of $\Omega(D, x)$

(6.1). In this section, let $G$ be a simple graph of order $p$. Let $\chi(G, x)$ be the chromatic polynomial $G$, i.e., $\chi(G, x)$ is the number of proper $x$-colourings whenever $x$ is a positive integer.

(6.2). **Proposition 6.1 (Stanley \([43]\))** For a non-negative integer $k$, $\chi(G, k)$ is equal to the number of pairs $(\theta, \mathcal{O})$, where $\theta$ is any map $\theta : V \to \{1, 2, \cdots, k\}$ and $\mathcal{O}$ is an orientation of $G$, subject to the two conditions:

(a) the orientation $\mathcal{O}$ is acyclic;

(b) if $u \to v$ in the orientation $\mathcal{O}$, then $\theta(u) < \theta(v)$.

**Proof.** Define a mapping $\psi$ with $\psi(f) = (\theta, \mathcal{O})$ from the set of proper $k$-colourings $f$ of $G$ to the set of ordered pairs $(\theta, \mathcal{O})$, where for any $k$-colouring $f$ of $G$, let $\theta = f$ and let $\mathcal{O}$ be the orientation of $G$ such that $u \to v$ whenever $uv \in E(G)$ and $f(u) < f(v)$. 

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Clearly, for any given \( f \), the ordered pair \((\theta, \mathcal{O})\) defined above satisfies conditions (a) and (b).

It is obvious that \( \psi \) is a bijection and thus the result holds. \( \square \)

(6.3). Define \( \tilde{\chi}(G, k) \) as the number of pairs \((\theta, \mathcal{O})\), where \( \theta \) is any map \( \theta : V \to \{1, 2, \cdots, k\} \) and \( \mathcal{O} \) is an orientation of \( G \), subject to the two conditions:

(a) The orientation \( \mathcal{O} \) is acyclic;

(b) If \( u \to v \) in the orientation \( \mathcal{O} \), then \( \theta(u) \leq \theta(v) \).

(6.4). The relationship between \( \chi(G, x) \) and \( \tilde{\chi}(G, x) \) is somewhat analogous to the relationship between combinations of \( n \) things taken \( k \) at a time without repetition, enumerated by \( \binom{n}{k} \), and with repetition, enumerated by \( \binom{n+k-1}{k} = (-1)^k \binom{-n}{k} \). (Note that \( \binom{n+k-1}{k} \) is the number of non-negative integer solutions of \( x_1 + x_2 + \cdots + x_n = k \).)

(6.5). **Theorem 6.1 (Stanley [43])** For all non-negative integers \( x \),

\[
\tilde{\chi}(G, x) = (-1)^p \chi(G, -x), \quad \text{i.e.,} \quad \chi(G, x) = (-1)^p \tilde{\chi}(G, -x),
\]

where \( p \) is the order of \( G \).

**Proof.** It suffices to show that

(i) \( \tilde{\chi}(N_1, x) = x \);

(ii) \( \tilde{\chi}(G + H, x) = \tilde{\chi}(G, x) \tilde{\chi}(H, x) \), where \( G + H \) is the disjoint union of \( G \) and \( H \);

(iii) \( \tilde{\chi}(G, x) = \tilde{\chi}(G \setminus e, x) + \tilde{\chi}(G/e, x) \) holds for any edge \( e \). \( \square \)

(6.6). Theorem 6.1 provides a combinatorial interpretation of the positive integer \((-1)^p \chi(G, -k)\), where \( k \) is a positive integer. In particular, when \( k = 1 \), every orientation of \( G \) is automatically compatible with every map \( \theta : V \to \{1\} \).

(6.7). **Corollary 6.1** If \( G \) is a graph with \( p \) vertices, then \((-1)^p \chi(G, -1)\) is equal to the number of acyclic orientations of \( G \).
(6.8). Let $G$ be a $p$-vertex graph and let $\omega$ be a labeling of $G$, i.e., a bijection $\omega : V(G) \to \{1, 2, \ldots, p\}$. Define an equivalence relation $\sim$ on the set of all $p!$ labelings $\omega$ of $G$ by the condition that $\omega_1 \sim \omega_2$ if whenever $\{u, v\} \in E(G)$, then $\omega_1(u) < \omega_1(v) \iff \omega_2(u) < \omega_2(v)$.

How many equivalence classes of labelings of $G$ are there?

Answer: the number of equivalence classes is $(-1)^p\chi(G, -1)$, i.e., the number of acyclic orientations of $G$.

(6.9). Let $A(G)$ be the set of acyclic orientations of $G$.

(6.10). By Proposition 6.1

$$
\chi(G, k) = \sum_{D \in A(G)} \Omega(D, k),
$$

(6.13)

where $\Omega(D, k)$ is the number of strictly order-preserved mappings $\theta : V(G) \to \{1, 2, \ldots, k\}$ with respect to $D$, i.e., $\theta(u) < \theta(v)$ whenever $u \to v$ in $D$.

(6.11). **Theorem 6.2 (Tutte 1967 [50])** For any graph $G$,

$$
\chi(G, x + y) = \sum_{S \subseteq V(G)} \chi(G[S], x)\chi(G - S, y).
$$

Note that $G - S = G[V - S]$.

It can be proved by applying expression (6.13) and Proposition 5.1.

A direct proof by induction is shown below.

**Proof.** Let

$$
Q(G, x, y) = \sum_{S \subseteq V(G)} \chi(G[S], x)\chi(G - S, y).
$$

First, if $G = N_p$, the null graph of order $p$, then $Q(G, x, y) = (x + y)^p = \chi(N_p, x + y)$.

Let $e$ be any edge with distinct ends $v_1, v_2$ in $G$. By induction, $Q(G \setminus e, x, y) = \chi(G \setminus e, x + y)$ and $Q(G/e, x, y) = \chi(G/e, x + y)$.

The power set $2^{V(G)}$ is partitioned into three subfamilies:
(a) \( S_1 = \{ S \subseteq V(G) : \{v_1, v_2\} \cap S = \emptyset \} \);
(b) \( S_2 = \{ S \subseteq V(G) : \{v_1, v_2\} \subseteq S \} \);
(c) \( S_3 = \{ S \subseteq V(G) : |\{v_1, v_2\} \cap S| = 1 \} \).

Observe that
\[
\sum_{S \in S_3} \chi((G \setminus e)[S], x)\chi((G \setminus e) - S, y) = \sum_{S \in S_3} \chi(G[S], x)\chi(G - S, y)
\]
and
\[
Q(G/e, x, y) = \sum_{S \subseteq V(G/e)} \chi(G/e[S], x)\chi(G/e - S, y)
= \sum_{S \in S_1} \chi(G[S], x)\chi(G/e - S, y) + \sum_{S \in S_2} \chi(G[S]/e, x)\chi(G - S, y).
\]

Thus, applying deletion-contraction formula for \( \chi(G, x) \),
\[
Q(G/e, x, y) - Q(G/e, x, y)
= \sum_{S \subseteq V(G)} \chi((G \setminus e)[S], x)\chi((G \setminus e) - S, y) - \sum_{S \subseteq V(G/e)} \chi((G/e)[S], x)\chi((G/e) - S, y)
= \sum_{S \in S_1} [\chi(G[S], x)\chi(G \setminus e - S, y) - \chi(G[S], x)\chi(G/e - S, y)]
+ \sum_{S \in S_2} [\chi(G[S] \setminus e, x)\chi(G - S, y) - \chi(G[S]/e, x)\chi(G - S, y)]
+ \sum_{S \in S_3} \chi(G[S] \setminus e, x)\chi(G \setminus e - S, y)
= \sum_{i=1}^{3} \sum_{S \in S_i} \chi(G[S], x)\chi(G - S, y)
= Q(G, x, y).
\]

As \( \chi(G, x) = \chi(G \setminus e, x) - \chi(G/e, x) \), the result holds. \( \square \)

(6.12) By expression (6.13) and Theorem 5.3, the following result is obtained.

**Theorem 6.3 (Stanley [45])**

\[
\chi(G, x) = \sum_{D \in A(G)} \sum_{\pi \in OP(D)} \left( x + p - 1 - \rho(\pi) \right) \frac{1}{p}.
\]
(6.13). By Theorem 5.4 and definition of $\tilde{\chi}(G, x)$,

**Theorem 6.4 (Stanley [45])**

$$\tilde{\chi}(G, x) = \sum_{D \in A(G)} \Omega(D, x) = \sum_{D \in A(G)} \sum_{\pi \in OP(D)} \left( x + \rho(\pi) \right).$$

(6.14). By Theorems 6.3 and 6.4 and

$$\Omega(D, k) = (-1)^p \Omega(D, -k), \quad \forall D \in A(G),$$

Theorem 6.1 follows, i.e.,

$$\tilde{\chi}(G, x) = (-1)^p \chi(G, -x).$$

(6.15). Let $G = (V, E)$ be a simple graph with $V = \{1, 2, \cdots, p\}$. Let $L$ denote the labeling of vertices in $G$. For an ordering $\pi = (v_1, v_2, \cdots, v_p)$ of all elements of $V$, let $\delta_G(\pi)$ be the number of $i$'s, where $1 \leq i \leq p - 1$, with either $v_i < v_{i+1}$ or $v_i v_{i+1} \in E$. Let $W_L(G)$ be the set of subsets $\{a, b, c\}$ of $V$, where $a < b < c$, which induces a subgraph of $G$ with $ac$ as its only edge.

By applying Theorem 5.5, the following result follows.

**Theorem 6.5 ([10])** For any simple graph of order $p$, $W_L(G) = \emptyset$ if and only if $(-1)^p \chi(G, -x) = \sum_{\pi} \left( x + \delta_G(\pi) \right)$, where the sum runs over all $n!$ orderings $\pi$ of $V$.

Let $\mathcal{N}W$ denote the set of graphs $G$ which has a label $L$ of its vertices by different numbers in $\{1, 2, \cdots, n\}$, where $n = |V(G)|$, such that $W_L(G) = \emptyset$. Theorem 6.5 can be applied to all graphs in $\mathcal{N}W$.

**Problem 6.1 ([10])** Determine the set $\mathcal{N}W$.
7 $\sigma$-polynomial $\sigma(G, x)$

(7.1). Let $G$ be a graph of order $p$ and its chromatic polynomial be written as

$$\chi(G, x) = \sum_{0 \leq i \leq p} a_i(G) \cdot (x)_i,$$

where $(x)_i = x(x-1) \cdots (x-i+1)$.

(7.2). For any non-adjacent pair of vertices $u$ and $v$ in $G$,

$$a_i(G) = a_i(G + uv) + a_i(G \cdot uv),$$

where $G + uv$ is the graph obtained from $G$ by adding a new edge joining $u$ and $v$ and $G \cdot uv$ is the graph obtained from $G$ by identifying $u$ and $v$.

(7.3). $a_i(G) = 0$ for $i < \chi(G)$, and $a_i(G)$ is positive integer for $\chi(G) \leq i \leq p$.

(7.4). Actually $a_i(G)$ is the number of partitions of $V(G)$ into $i$ non-empty independent sets.

(7.5). $a_i(G)$ is also the number of copies of $K_i$ obtained by repeating the following operations starting from $G$ until all graphs are complete:

if $u$ and $v$ are not adjacent in $H$, then replace $H$ by $H + uv$ and $H \cdot uv$.

(7.6). Define

(i) $$\sigma(G, x) = \sum_{0 \leq i \leq p} a_i(G)x^i, $$

(ii) $$\bar{\sigma}(G, x) = \sum_{0 \leq i \leq p} i!a_i(G)x^i.$$ 

(7.7). $\sigma(G, x)$ was defined by Korfhage in 1978, although the original function he introduced was actually $\sigma(G, x)/x^{\chi(G)}$.
The adjoint polynomial $h(G, x)$ was defined by Liu Ruying \[28\] in 1987:

$$h(G, x) = \sum_i h_i x^i,$$

where $h_i$ is the number of partitions of $V(G)$ into exactly $i$ subsets each of which is a clique.

Thus $h(G, x) = \sigma(\overline{G}, x)$, where $\overline{G}$ is the complement of $G$.

Examples:

(i) $\sigma(K_p, x) = x^p$, as $\chi(K_p, x) = (x)_p$;

(ii) For the empty graph $N_p$ of order $p$,

$$\sigma(N_p, x) = \sum_{1 \leq k \leq p} S(p, k)x^k,$$

as

$$\chi(N_p, x) = x^p = \sum_{1 \leq k \leq p} S(p, k)(x)_k,$$

where $S(p, k)$, called the Sterling number of second kind, is the number of partitions of $\{1, 2, \cdots, p\}$ into $k$ non-empty subsets.

Note that

$$\sum_{k \leq p} S(p, k)x^k = e^{-x} \sum_{i=0}^{\infty} \frac{i^p}{i!} x^i = B_p(x)$$

is called a Bell polynomial.

Also note that

$$\sum_{k \leq p} S(p, k)(x)_k = x^p;$$

$$\sum_{p \geq k} \frac{S(p, k)}{p!} x^p = \frac{1}{k!}(e^x - 1)^k$$

and

$$\sum_{p \geq k} S(p, k)x^p = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$  

(iii) If $G$ is the complete $r$-partite graph $K_{m_1,m_2,\cdots,m_r}$, then

$$\sigma(G, x) = \prod_{i=1}^{r} B_{m_i}(x).$$
(iv) (Liu RY, 1987)

\[ \sigma(\bar{P}_n, x) = \sum_{i \leq n} \binom{i}{n-i} x^i. \]

(v) (Dong et al, 2002)

\[ \sigma(\bar{P}_n, x) = x^{[n/2]} \prod_{s=1}^{[n/2]} \left( x + 2 + 2 \cos \frac{2s\pi}{n+1} \right) \]

and

\[ \sigma(\bar{C}_n, x) = x^{[n/2]} \prod_{s=1}^{[n/2]} \left( x + 2 + 2 \cos \frac{(2s-1)\pi}{n+1} \right). \]

(7.10). Basic properties on computation:

(i) The joint of disjoint graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \vee G_2 \), is obtained from the disjoint union of graphs \( G_1 \) and \( G_2 \) by adding edges joining each \( u \in V(G_1) \) to each \( v \in V(G_2) \), then

\[ a_k(G_1 \vee G_2) = \sum_{i+j=k} a_i(G_1) a_j(G_2), \quad \forall k \geq 1 \]

and

\[ \sigma(G_1 \vee G_2, x) = \sigma(G_1, x) \sigma(G_2, x). \]

(ii) If \( u, v \) are non-adjacent vertices in \( G \), then

\[ \sigma(G, x) = \sigma(G + uv, x) + \sigma(G \cdot uv, x). \]

(7.11). Some coefficients. Let \( G \) be of order \( p \) and size \( q \).

(i) \( a_p(G) = 1; \)

(ii) \( a_{p-1} = \binom{p}{2} - q; \)

(iii) (Brenti [II])

\[ a_{p-2} = \binom{q}{2} - q \binom{q-1}{2} + \binom{p}{3} \binom{3p-5}{4} - t(G), \]

where \( t(G) \) is the number of triangles in \( G \).
A graph $G$ is said to be $\sigma$-real if $\sigma(G, x)$ has real zeros only, and it is said to be $\sigma$-unreal if it is not $\sigma$-real.

The $w$-unreal and $\tau$-unreal graphs are defined similarly with respect to $w$-polynomial and $\tau$-polynomial respectively. These polynomials will be introduced in the following sections.

**Theorem 7.1 (Brenti [1])** $G$ is $\sigma$-real if one of the following conditions is satisfied:

(i) $\bar{G}$ is a comparability graph, where a graph $H$ is called a comparability graph if there exists a partial order $\preceq$ such that $uv \in E(H)$ if and only if $u \not= v$ and $u \preceq v$ or $v \preceq u$;

(ii) $\chi(G) \geq |V(G)| - 2$;

(iii) $\bar{G}$ is $K_3$-free;

(iv) there exists a simplicial vertex $u$ in $G$ such that $G - u$ is $\sigma$-real;

(v) $G = G_1 \cup G_2^2$, where each $G_i$ is $\sigma$-real and $G_1 \cap G_2$ is complete;

(vi) $\bar{\sigma}(G, x)$ has real zeros only;

(vii) $w(G, x)$ has real zeros only.

(7.14). The $\sigma$-unreal, $w$-unreal and $\tau$-unreal connected graphs on up to 9 vertices were determined by Cameron, Colbourn, Read and Wormald [4], and the numbers of $\sigma$-unreal, $w$-unreal and $\tau$-unreal connected graphs of orders from 3 to 9 are shown below:

| order | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-------|----|----|----|----|----|----|----|
| no. $\sigma$-unreal con. graphs | 0  | 0  | 0  | 0  | 2  | 42 |    |
| no. $w$-unreal con. graphs | 0  | 1  | 3  | 16 | 116| 1237| 22515|
| no. $\tau$-unreal con. graphs | 0  | 0  | 0  | 0  | 0  | 0  | 0  |

Note that, in the above table, $w$-unreal (resp. $\tau$-unreal) con. graphs refer to connected graphs whose $w$-polynomials (resp. $\tau$-polynomials) have unreal roots. $w$-polynomials and $\tau$-polynomials are introduced in Section 8 and Section 9 respectively.

(7.15). The two $\sigma$-unreal connected graphs on 8 vertices are shown below [2]:

$G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, but for any $u, v \in V(G_1) \cap V(G_2)$, $uv \in E(G_1)$ if and only if $uv \in E(G_2)$.
Their $\sigma$-polynomials are $x^8+11x^7+38x^6+36x^5+11x^4+x^3$ and $x^8+10x^7+30x^6+31x^5+10x^4+x^3$ respectively and both contain non-real zeros [2].

The $\sigma$-unreal connected graphs on 9 vertices are listed in [2].

(7.16). Conjecture 7.1 (Brenti [1]) Let $G$ be a simple graph of order $p$. Then $G$ is $\sigma$-real if $\chi(G) \geq p-3$.

(7.17). $\sigma(G, x) = \xi(F, x)$ if $F$ is the family of independent sets in $G$, where $\xi(F, x)$ is the partition polynomial of a set system $F$:

(i) given a finite set $V$, a set system $F$ is a collection of subsets of $V$ with $\emptyset \in F$ and $\bigcup_{X \in F} X = V$;

(ii) the partition polynomial of a set system $F$ (see [57]) of a set $V$ is defined below:

$$\xi(F, x) = \sum_{k \geq 0} p_k(F)x^k,$$

where $p_k(F)$ is the number of partitions of $V$ into $k$ non-empty subsets which belong to $F$.

8 w-polynomial $w(G, x)$

(8.1). Let $G$ be a graph of order $p$. Assume that

$$\chi(G, x) = \sum_{0 \leq i \leq p} w_i \binom{x + p - i}{p}.$$ 

Define

$$w(G, x) = \sum_{0 \leq i \leq p} w_ix^i.$$
(8.2). By Theorem 6.3, \( w_i \) is the number of order pairs \((D, \pi)\), where \( D \in A(G) \), \( A(G) \) is the set of acyclic orientations of \( G \) and \( \pi \) is an order-preserved permutation in \( OP(D) \) with \( \rho(\pi) = i - 1 \).

(8.3). Example. As
\[
\chi(K_p, x) = (x)_p = p! \binom{x}{p},
\]
we have
\[
w(K_p, x) = p!x^p.
\]

(8.4). Lemma 8.1 If \( u,v \) are not adjacent in \( G \), then
\[
w_i(G) = w_i(G + uv) + w_i(G \cdot uv) - w_{i-1}(G \cdot uv).
\]

Proof. Observe that
\[
\chi(G, x) = \chi(G + uv, x) + \chi(G \cdot uv, x)
\]
and
\[
\chi(G \cdot uv, x)
\]
\[
= \sum_{0 \leq i \leq p-1} w_i(G \cdot uv) \binom{x + p - 1 - i}{p - 1}
\]
\[
= \sum_{0 \leq i \leq p-1} w_i(G \cdot uv) \binom{x + p - i}{p} - \sum_{0 \leq i \leq p-1} w_i(G \cdot uv) \binom{x + p - 1 - i}{p}
\]
\[
= \sum_{0 \leq i \leq p-1} w_i(G \cdot uv) \binom{x + p - i}{p} - \sum_{1 \leq i \leq p} w_{i-1}(G \cdot uv) \binom{x + p - i}{p}
\]

Thus the result holds. \( \square \)

(8.5). Lemma 8.2 If \( u,v \) are not adjacent in \( G \), then
\[
w(G, x) = w(G + uv, x) + (1 - x)w(G \cdot uv, x).
\]

It follows from Lemma 8.1.
(8.6). Examples. Applying Lemma 8.2 yields that
\[ w(P_3, x) = w(K_3, x) + (1 - x)w(K_2, x) = 3!x^3 + (1 - x)2!x^2 = 4x^3 + 2x^2 \]
and
\[ w(N_3, x) = w(K_2 \cup_0 K_1, x) + (1 - x)w(N_2, x) \]
\[ = w(P_3, x) + (1 - x)w(K_2, x) + (1 - x)w(K_2, x) + (1 - x)^2w(K_1, x) \]
\[ = 4x^3 + 2x^2 + 2(1 - x)2!x^2 + (1 - x)^2x \]
\[ = x^3 + 4x^2 + x, \]
where \( K_2 \cup_0 K_1 \) is the disjoint union of \( K_2 \) and \( K_1 \).

(8.7). Theorem 8.1 (Brenti [1]) For any graph \( G \) of order \( p \),
\[ \sum_{i \geq 0} \chi(G, i)x^i = \frac{w(G, x)}{(1 - x)^{p+1}}. \]

Proof. Observe that
\[ \frac{w(G, x)}{(1 - x)^{p+1}} = (w_0 + w_1 x + \cdots + w_p x^p) \sum_{j \geq 0} \binom{p+j}{p} x^j. \]
Thus the coefficient of \( x^i \) is
\[ \sum_{0 \leq j \leq i} w_{i-j} \binom{p+j}{p} = \sum_{0 \leq k \leq i} w_k \binom{p+i-k}{p} \]
\[ = \sum_{0 \leq k \leq p} w_k \binom{p+i-k}{p} \]
\[ = \chi(G, i). \]

(8.8). Proposition 8.1 (Brenti [1]) Let \( G \) be a graph of order \( p \). Then
(a) \( w_i = 0 \) for \( i < \chi(G) \);
(b) \( w(G, 1) = \sum_{i \leq p} w_i = p! \);
(c) \( w_i \) is positive for \( \chi(G) \leq i \leq p \);
(d) \(w_p\) is the number of acyclic orientations of \(G\).

**Proof.** (a) By Theorem 8.1

\[ w(G, x) = (1 - x)^p \sum_{i \geq \chi(G)} \chi(G, i)x^i, \]

implying that (a) holds. (a) also follows from Lemma 8.1.

(b) It holds when \(G\) is \(K_p\), as

\[ w(K_p, x) = p!x^p. \]

Then, by Lemma 8.2, \(w(G, 1) = p!\) for any graph \(G\) of order \(p\).

(c) It directly follows from Theorem 6.3.

(d) Taking \(x = -1\) yields that

\[ \chi(G, -1) = \sum_{0 \leq i \leq p} w_i \binom{p - i}{p} = w_p \binom{-1}{p} = (-1)^p w_p. \]

As \((-1)^p \chi(G, -1)\) is the number of acyclic orientations of \(G\), (d) holds.

\[ \blacksquare \]

(8.9). **Theorem 8.2 (Brenti [3])** For any graph \(G\) of order \(p\),

\[ w(G, x) = (1 - x)^p \bar{\sigma} \left( G, \frac{x}{1 - x} \right). \]

**Proof.** Let \(z = x/(1 - x)\), i.e., \(x = z/(1 + z)\). Then the identity is equivalent to the following one:

\[ w(G, z/(1 + z)) = (1 + z)^{-p} \bar{\sigma} (G, z); \]

\[ \sum_{i \leq p} w_iz^i(1 + z)^{p-i} = \sum_{0 \leq i \leq p} i!a_iz^i; \]

\[ k!a_k = \sum_{i \leq k} w_i \binom{p - i}{k - i} = \sum_{i \leq k} w_i \binom{p - i}{p - k}, \quad \forall k \leq p. \quad (8.14) \]
By definition,
\[ \chi(G, x) = \sum_i w_i \left( x + p - i \right) = \sum_k a_k(x)_k. \]

As
\[ \left( x + p - i \right) = \sum_{i \leq k \leq p} \left( \begin{array}{c} x \\ k \end{array} \right) \left( \begin{array}{c} p - i \\ p - k \end{array} \right) = \sum_{i \leq k \leq p} \left( \begin{array}{c} p - i \\ p - k \end{array} \right) (x)_k / k!, \]

we have
\[ a_k = \sum_{i \leq k \leq p} w_i \left( \begin{array}{c} p - i \\ p - k \end{array} \right) / k!, \]

implying that identity (8.14) holds. \qed

(8.10). Theorem 8.2 is equivalent to

**Theorem 8.3** For any graph $G$ of order $p$, if
\[ \chi(G, x) = \sum_{i \leq p} a_i \cdot (x)_i, \]

then
\[ w(G, x) = \sum_{i \leq p} a_i i! x^i (1 - x)^{p-i}. \]

(8.11). **Corollary 8.1** Show that for any graph $G$,
\[ w_k(G) = \sum_{i \leq k} (-1)^{k-i} \left( \begin{array}{c} p - i \\ p - k \end{array} \right) i! a_i(G). \]

**Proof.** By Theorem 8.2,
\[ w(G, x) = (1 - x)^p \sigma(G, x/(1 - x)) = (1 - x)^p \sum_{i \leq p} i! a_i(x/(1 - x))^i 
= \sum_{i \leq p} i! a_i x^i (1 - x)^{p-i}. \]
Thus
\[ w_k = \sum_{i \leq p} a_i \binom{p - i}{k - i} (-1)^{k-i} = \sum_{i \leq k} (-1)^{k-i} \binom{p - i}{p - k} i!a_i. \]
(8.12).

Corollary 8.2
\[ w_k(N_p) = \sum_{i \leq k} (-1)^{k-i} \binom{p - i}{p - k} i!S(p, i) \]
and
\[ w(N_p, x) = \sum_{i \leq p} i!S(p, i)x^i(1-x)^{p-i}. \]
Proof. As
\[ a_i(N_p) = S(p, i), \]
By Corollary 8.1,
\[ w_k(N_p) = \sum_{i \leq k} (-1)^{k-i} \binom{p - i}{p - k} i!S(p, i). \]
By Theorem 8.3
\[ w(N_p, x) = \sum_{i \leq p} i!S(P, i)x^k(1-x)^{p-i}. \]
(8.13).

Theorem 8.4 (Brenti [1]) G is w-real (i.e., \( w(G, x) \) has real zeros only) if one of the following conditions is satisfied:

(i) G contains a simplicial vertex \( u \) such that \( G - u \) is w-real;
(ii) (a special case of (i)) G is chordal; or
(iii) G is the disjoin union of \( G_1 \) and \( G_2 \), where each \( G_i \) is w-real.

(8.14). \( C_4 \) is the w-unreal graph with the minimal order:
\[ w(C_4, x) = 2x^2(7x^2 + 4x + 1). \]
(8.15). Question 8.1 Find w-unreal graphs of order 5.
(8.16). Conjecture 8.1 (Brenti [2]) If both G and H are w-real and \( G \cap H \) is complete, then \( G \cup H \) is w-real.
9 \(\tau\)-polynomial \(\tau(G, x)\)

(9.1) \(\tau(G, x)\) and \(\bar{\tau}(G, x)\) are defined as
\[
\tau(G, x) = \sum_{0 \leq i \leq p} c_i x^i
\]
and
\[
\bar{\tau}(G, x) = \sum_{0 \leq i \leq p} i! c_i x^i;
\]
where \(c_i\)'s are determined by
\[
\chi(G, x) = \sum_{0 \leq i \leq p} (-1)^{p-i} c_i \langle x \rangle_i,
\]
where \(\langle x \rangle_i = x(x+1) \cdots (x+i-1)\).

(9.2) Example.
(a) \(\tau(K_1, x) = x\), as \(\chi(K_1, x) = x\).
(b) \(\tau(K_2, x) = x^2 + 2x\), as
\[
\chi(K_2, x) = x(x-1) = x(x+1-2) = x(x+1) - 2x.
\]
(c) \(\tau(N_p, x) = B_p(x)\), as \(\chi(N_p, x) = x^p = \sum_{1 \leq i \leq p} (-1)^{p-i} S(p, i) \langle x \rangle_i\), where
\[
B_p(x) = \sum_{1 \leq i \leq p} S(p, i) x^i
\]
is called a Bell polynomial and \(S(p, i)\) is a Stirling number of the second kind, counting the number of partitions of \(\{1, 2, \cdots, p\}\) into \(i\) non-empty subsets.

Proof. It is well known that
\[
x^p = \sum_{i \leq p} S(p, i)(x)_i.
\]
Letting \(x = -z\) gives that
\[
(-z)^p = \sum_{i \leq p} S(p, i)(-z)_i = \sum_{i \leq p} (-1)^i S(p, i)(z)_i.
\]
Thus
\[ \chi(N_p, x) = x^p = \sum_{1 \leq i \leq p} (-1)^{p-i} S(p, i) \langle x \rangle_i. \]

\(\ast\)

(9.3). If \(u\) is an isolated vertex of \(G\), then
\[ c_i(G) = c_{i-1}(G - u) + ic_i(G - u). \]

**Proof.**
\[
\begin{align*}
\chi(G, x) &= x\chi(G - u, x) \\
&= x \sum_{i \leq p-1} (-1)^{p-1-i} c_i(G - u) \langle x \rangle_i \\
&= \sum_{i \leq p-1} (-1)^{p-1-i} c_i(G - u)(x + i - i) \langle x \rangle_i \\
&= \sum_{i \leq p-1} (-1)^{p-1-i} c_i(G - u) \langle x \rangle_{i+1} - \sum_{i \leq p-1} (-1)^{p-1-i} c_i(G - u)i \langle x \rangle_i \\
&= \sum_{j \leq p} (-1)^{p-j} c_{j-1}(G - u) \langle x \rangle_j + \sum_{i \leq p-1} (-1)^{p-i} c_i(G - u)i \langle x \rangle_i.
\end{align*}
\]

\(\ast\)

(9.4). If \(u\) is an isolated vertex of \(G\), then
\[ \tau(G, x) = x\tau(G - u, x) + x(\tau(G - u, x))'. \]

(9.5). More general.

**Proposition 9.1** Let \(u\) be a simplicial vertex of \(G\) with degree \(k\). Then
\[ \tau(G, x) = x\tau'(G - u, x) + (x + k)\tau(G - u, x). \]

**Proof.** Observe that
\[ \chi(G, x) = (x - k)\chi(G - u, x). \]
Assume that
\[ \chi(G - u, x) = \sum_{i=0}^{p-1} (-1)^{p-1-i} b_i \langle x \rangle_i. \]

Then
\[ (x - k)\chi(G - u, x) = \sum_{i=0}^{p-1} (-1)^{p-1-i} b_i (x - k) \langle x \rangle_i \]
\[ = \sum_{i=0}^{p-1} (-1)^{p-1-i} b_i ((x + i) - k - i) \langle x \rangle_i \]
\[ = \sum_{i=0}^{p-1} (-1)^{p-1-i} b_i \langle x \rangle_{i+1} + \sum_{i=0}^{p-1} (-1)^{p-1-i} b_i (k + i) \langle x \rangle_i \]
\[ = \sum_{j=1}^{p} (-1)^{i-j} b_{j-1} \langle x \rangle_j + \sum_{i=0}^{p-1} (-1)^{i-p} b_i (k + i) \langle x \rangle_i. \]

Thus for \( i = 0, 1, \ldots, p - 1, p, \)
\[ c_i = b_{i-1} + b_i (k + i). \]

where \( b_p = 0. \) Hence
\[ \tau(G, x) = \sum_{i=0}^{p} c_i x^i = \sum_{i=0}^{p} (b_{i-1} + b_i (k + i)) x^i \]
\[ = \tau(G - u, x) + k \sum_{i=0}^{p-1} b_i x^i + x \sum_{i=1}^{p-1} ib_i x^{i-1} \]
\[ = x\tau(G - u, x) + k\tau(G - u, x) + x(\tau(G - u, x))^\prime \]
\[ = (x + k)\tau(G - u, x) + x(\tau(G - u, x))^\prime. \]

(\( \square \))

(9.6). For any graph \( G \) with \( e \in E(G), \)
\[ c_i(G) = c_i(G \setminus e) + c_i(G / e), \]

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where \( G \setminus e \) (or resp. \( G/e \)) is the graphs obtained from \( G \) by removing \( e \) (or resp. contracting \( e \) and removing parallel edges but one).

**Proof.** The result follows from the following identity:

\[
\sum_{i \leq p} (-1)^{p-i} c_i(x)_i = \chi(G, x) = \chi(G \setminus e, x) - \chi(G/e, x) = \sum_{i \leq p} (-1)^{p-i} c_i(G \setminus e)(x)_i - \sum_{i \leq p-1} (-1)^{p-1-i} c_i(G/e)(x)_i.
\]

\( \Box \)

(9.7). **Proposition 9.2** For any graph \( G \) with \( e \in E(G) \),

\[
\tau(G, x) = \tau(G \setminus e, x) + \tau(G/e, x).
\]

(9.8). For any given graph \( G \), \( \tau(G, x) \) can be determined by applying Proposition 9.2 repeatedly until all graphs obtained are empty graphs and the result that \( \tau(N_p, x) = B_p(x) \).

**Example 9.1** Let \( P_2 \cup_0 K_1 \) be the disjoint union of \( P_2 \) and \( K_1 \). Then applying Proposition 9.2 yields that

\[
\tau(P_3, x) = \tau(P_2 \cup_0 K_1, x) + \tau(P_2, x) = \tau(N_3, x) + 2\tau(N_2, x) + \tau(N_1, x) = B_3(x) + 2B_2(x) + B_1(x).
\]

**Example 9.2** Let \( T \) be a tree of order \( p \). Then

\[
\tau(T, x) = \sum_{1 \leq k \leq p} t_{p,k} B_k(x),
\]

where \( t_{p,k} = 0 \) if \( k > p \) or \( k = 0 \), and

\[
t_{p,k} = t_{p-1,k} + t_{p-1,k-1}.
\]

Thus, it can be shown that \( t_{p,k} = \binom{p-1}{k-1} \) and

\[
\tau(T, x) = \sum_{1 \leq k \leq p} \binom{p-1}{k-1} B_k(x).
\]
(9.9). **Proposition 9.3** For any simple graph $G$ of order $p$, if

$$
\chi(G, x) = \sum_{k \leq p} (-1)^{p-k} b_k x^k,
$$

then

$$
\tau(G, x) = \sum_{k \leq p} b_k B_k(x)
$$

and

$$
\sigma(G, x) = \sum_{k \leq p} (-1)^{p-k} b_k B_k(x).
$$

Note that $b_k = \sum_j (-1)^{p-k+j} N_{j,k}$, where $N_{j,k}$ is the number of spanning subgraphs of $G$ which have exactly $j$ edges and $k$ components.

**Proof.** It holds when $G = N_p$. Then it can be proved by induction and applying Proposition 9.2.

Or there is a direct proof by the definition of $\tau(G, x)$. Note that

$$
\chi(G, x) = \sum_{k \leq p} (-1)^{p-k} b_k x^k = \sum_{k \leq p} (-1)^{p-k} b_k \sum_{i \leq k} S(k, i) (-1)^{k-i} \langle x \rangle_i
$$

$$
= \sum_{i \leq p} (-1)^{p-i} \langle x \rangle_i \sum_{i \leq k \leq p} b_k S(k, i).
$$

Thus, by the definition of $\tau(G, x)$,

$$
\tau(G, x) = \sum_{i \leq p} \sum_{i \leq k \leq p} b_k S(k, i) x^i = \sum_{i \leq p} b_k \sum_{i \leq k} S(k, i) x^i = \sum_{k \leq p} b_k B_k(x).
$$

For the expression $\sigma(G, x)$, the proof is similar as

$$
x^k = \sum_{i \leq k} S(k, i) \langle x \rangle_i.
$$
Corollary 9.1
\[
\tau(K_p, x) = \sum_{k \leq p} \left[ \begin{array}{c} p \\ k \end{array} \right] B_k(x) = \sum_{i \leq p} \sum_{i \leq k \leq p} S(k, i) \left[ \begin{array}{c} p \\ k \end{array} \right] x^i,
\]
where \[\left[ \begin{array}{c} p \\ k \end{array} \right] \] is the Stirling number of the first kind, counting the number of permutations of \(p\) elements with \(k\) disjoint cycles.

Remarks:
(a) \[\left[ \begin{array}{c} p \\ k \end{array} \right] \] is determined by the following identity:
\[
\langle x \rangle_p = \sum_{k \leq p} \left[ \begin{array}{c} p \\ k \end{array} \right] x^k \quad \text{or} \quad (x)_p = \sum_{k \leq p} (-1)^{p-k} \left[ \begin{array}{c} p \\ k \end{array} \right] x^k.
\]

(b) \[\left[ \begin{array}{c} p \\ k \end{array} \right] \] is also determined by the recursive expression:
\[
\left[ \begin{array}{c} p+1 \\ k \end{array} \right] = p \left[ \begin{array}{c} p \\ k \end{array} \right] + \left[ \begin{array}{c} p \\ k-1 \end{array} \right]
\]
for \(k \geq 1\), with the following initial conditions:
\[
\left[ \begin{array}{c} 0 \\ k \end{array} \right] = 1, \quad \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} k \\ 0 \end{array} \right] = 0
\]
for \(k \geq 1\).

Lemma 9.1 (Brenti [1])
For any \(i \leq p\),
\[
c_i(G) = \sum_{P \in \Pi(G)} |A(G(P))|,
\]
where \(A(H)\) is the set of acyclic orientations of a graph \(H\).

Proof. The result follows from (9.6)(c), Proposition 9.2 and the fact that
\[
|A(H)| = |A(H\setminus e)| + |A(H/e)|
\]
holds for any graph \(H\) and edge \(e\) in \(H\) which is not a loop.

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(9.13) $c_0 = 0$, $c_1 = |A(G)|$, $c_i \geq 1$ for $i = 2, 3, \cdots, p-2$, $c_{p-1} = \binom{p}{2} + q$, $c_p = 1$.

(9.14) **Question 9.1** Show that for any simple graph $G$ of order $p$ and size $q$,

$$c_{p-2}(G) = \binom{p}{3} + 3 \binom{p}{4} + q \binom{p-2}{2} + m_2 + \sum_{i=1}^{3} l_i (2i - 1),$$

where $m_2$ is the number of matchings of $G$ with two edges and $l_i$ is the number of induced subgraphs of $G$ with $3$ vertices and $i$ edges.

(9.15) **Theorem 9.1 (Brenti [1])** For any graph $G$,

$$\tau(G,x) = \sum_{P \in \Pi(G)} |A(G(P))| x^{|P|}.$$ 

(9.16) **Corollary 9.2**

$$\chi(G,x) = \sum_{P \in \Pi(G)} |A(G(P))| (-1)^{|P|-|P'|} x^{|P'|}.$$ 

(9.17) Let $u \in V(G)$. By Theorem 9.1

$$\tau(G,x) = x \sum_{u \in V' \subseteq V(G)} |A(G[V'])| \tau(G - V', x).$$

(9.18) For any positive integer $k$,

$$(-1)^p \chi(G, -k) = \sum_{\sigma : V \rightarrow [k]} |A(G(P_\sigma))|.$$ 

where $[k] = \{1, 2, \cdots, k\}$ and $P_\sigma$ is the partition of $V(G)$ induced by $\sigma$, i.e., two vertices $u$ and $v$ in $V(G)$ are in the same set if and only if $\sigma(u) = \sigma(v)$.

**Proof.** By (4.1) and Lemma 9.1

$$(-1)^p \chi(G, -k) = \sum_{i \leq p} c_i(G) \binom{k}{i} i! = \sum_{i \leq p} \binom{k}{i} i! \sum_{P \in \Pi(G)} |A(G(P))|.$$
Also note that
\[
\sum_{\sigma: V \to [k]} |A(G(\mathcal{P}_\sigma))| = \sum_{i \leq p} \sum_{\sigma \text{ onto } V \to [i]} |A(G(\mathcal{P}_\sigma))| \binom{k}{i}
\]
\[
= \sum_{i \leq p} \binom{k}{i} i! \sum_{\mathcal{P} \in \Pi(G)} |A(G(\mathcal{P}))|,
\]
where the last equality follows from the fact that for each partition \( \mathcal{P} \in \Pi(G) \), the summation \( \sum_{\sigma: V \to [i]} \) has exactly \( i! \) partitions \( \mathcal{P}_\sigma \) each of which is the same as \( \mathcal{P} \). \( \blacksquare \)

(9.19). **Theorem 9.2 (Brenti [1])** For any graph \( G \) of order \( p \),
\[
w(G, x) = (x - 1)^p x \bar{\tau} \left( G, \frac{1}{x - 1} \right).
\]

**Proof.** It is equivalent to each of the following identities:
\[
w(G, x + 1) = x^p (x + 1) \bar{\tau} \left( G, \frac{1}{x} \right);
\]
\[
\sum_k w_k (x + 1)^{k - 1} = \sum_k k! c_k x^{p - k};
\]
\[
(p - i)! c_{p - i} = \sum_k \binom{k - 1}{i} w_k, \quad \forall i \leq p;
\]
\[
i! c_i = \sum_k \binom{k - 1}{p - i} w_k, \quad \forall i \leq p. \quad (9.15)
\]
By definition,
\[
\sum_{i \leq p} (-1)^{p - i} c_i(x)_i = \chi(G, x) = \sum_k w_k \binom{x + p - k}{p}.
\]
Replacing \( x \) by \( -x \) yields that
\[
\sum_{i \leq p} i! c_i(x)_i = \sum_{k \leq p} w_{k+1} \binom{x + k}{p};
\]

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\[
\sum_{i \leq p} i! c_i \binom{x}{i} = \sum_{k \leq p} \sum_{i \leq p} \binom{x}{i} \binom{k}{p-i} c_k;
\]
\[
\sum_{i \leq p} i! c_i \binom{x}{i} = \sum_{i \leq p} \sum_{k \leq p} w_{k+1} \binom{x}{i} \binom{k}{p-i};
\]
implying that identity (9.15) holds. \(\square\)

(9.20). **Corollary 9.3** Relation between \(\bar{\sigma}(G, x)\) and \(\bar{\tau}(G, x)\):

\[
(-1)^p(x+1)\bar{\tau}(G, y) = y \bar{\sigma}(G, -1 - y).
\]

**Proof.** By Theorems 9.2 and 8.2 we have

\[
x(x-1)^p \bar{\tau}(G, \frac{1}{x-1}) = (1-x)^p \bar{\sigma}(G, \frac{x}{1-x});
\]
\[
x(-1)^p \bar{\tau}(G, \frac{1}{x-1}) = \bar{\sigma}(G, \frac{x}{1-x});
\]
\[
y + 1 \frac{1}{y} (-1)^p \bar{\tau}(G, y) = \bar{\sigma}(G, -1 - y).
\]

Then the result follows. \(\square\)

(9.21). What is the relation between \(\sigma(G, x)\) and \(\tau(G, x)\)?

**Proposition 9.4**

\[
a_i = \sum_{i \leq k \leq p} (-1)^{p-k}(k-i)! \binom{k}{i} \binom{k-1}{k-i} c_k.
\]

**Proof.** By the definitions of \(a_k\) and \(c_k\),

\[
\chi(G, x) = \sum_{k \leq p} a_k(x)_k = \sum_{k \leq p} (-1)^{p-k} c_k(x)_k.
\]

Note that

\[
\langle x \rangle_k/k! = \binom{x+k-1}{k} = \sum_{0 \leq i \leq k} \binom{x}{i} \binom{k-1}{k-i} = \sum_{0 \leq i \leq k} \binom{k-1}{k-i} (x)_i/i!.
\]
Thus
\[ a_i = \sum_{i \leq k \leq p} (-1)^{p-k}k!c_k \binom{k-1}{k-i} / i! = \sum_{i \leq k \leq p} (-1)^{p-k}(k-i)! \binom{k}{i} \binom{k-1}{k-i} c_k. \]
\[\square\]

(9.22). **Theorem 9.3** (Brenti [2]) \(G\) is \(\tau\)-real if one of the following conditions is satisfied:

(i) \(G\) is chordal;
(ii) \(G\) is a cycle \(C_p\), \(p \geq 3\);
(iii) \(G = H \lor K_m\), where \(H\) is \(\tau\)-real;
(iv) \(G = H \lor K_m\), where \(m\) is sufficiently large;
(v) \(G = G_1 \cup G_2\), where each \(G_i\) is \(\tau\)-real and \(|V(G_1) \cap V(G_2)| \leq 1\).

(9.23). **Conjecture 9.1** (Brenti [2]) If both \(G\) and \(H\) are \(\tau\)-real and \(G \cap H\) is complete, then \(G \cup H\) is \(\tau\)-real.

(9.24). **Conjecture 9.2** (Brenti [2]) Let \(G\) and \(H\) be vertex-disjoint graphs. If both \(G\) and \(H\) are \(\tau\)-real, then the join \(G \lor H\) is also \(\tau\)-real.

(9.25). **Problem 9.1** (Brenti [1]) Is every graph \(G\) \(\tau\)-real? So far no \(\tau\)-unreal graphs are known.

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