Mixed Boundary Value Problems of Semilinear Elliptic PDEs and BSDEs with Singular Coefficients

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Abstract

In this paper, we prove that there exists a unique weak solution to the mixed boundary value problem for a general class of semilinear second order elliptic partial differential equations with singular coefficients. Our approach is probabilistic. The theory of Dirichlet forms and backward stochastic differential equations with singular coefficients and infinite horizon plays a crucial role.

Keywords: Dirichlet forms; Quadratic forms; Fukushima’s decomposition; Mixed boundary value problem; Backward stochastic differential equations; Reflecting diffusion processes.

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1 Introduction

In this paper, our aim is to use probabilistic methods to solve the mixed boundary value problem for semilinear second order elliptic partial differential equations (called PDEs for short) of the following form:

\[
\begin{aligned}
&Lu(x) = -F(x,u(x),\nabla u(x)), \quad \text{on } D \\
&\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) - \hat{B} \cdot n(x) u(x) = \Phi(x) \quad \text{on } \partial D
\end{aligned}
\]  

(1.1)

The elliptic operator \(L\) is given by:

\[
L = \frac{1}{2} \nabla \cdot (A \nabla) + B \cdot \nabla - \nabla \cdot (\hat{B} \cdot \cdot) + Q
\]  

(1.2)

on a \(d\)-dimensional smooth bounded Euclidean domain \(D\).

\(A(x) = (a_{ij})_{1 \leq i,j \leq d}: \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d\) is a smooth, symmetric matrix-valued function which is uniformly elliptic. That is, there is a constant \(\lambda > 1\) such that

\[
\frac{1}{\lambda} I_{d \times d} \leq A(\cdot) \leq \lambda I_{d \times d}.
\]  

(1.3)

Here \(B = (B_1,\ldots,B_d)\) and \(\hat{B} = (\hat{B}_1,\ldots,\hat{B}_d) : \mathbb{R}^d \to \mathbb{R}^d\) are Borel measurable functions, which could be singular, and \(Q\) is a real-valued Borel measurable function defined on \(\mathbb{R}^d\) such that, for some \(p > \frac{d}{2}\),

\[
I_D(|B|^2 + |\hat{B}|^2 + |Q|) \in L^p(D).
\]

\(L\) is rigorously determined by the following quadratic form:

\[
Q(u,v) := (-Lu,v)_{L^2(D)} = \frac{1}{2} \sum_{i,j} \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_i \int_D B_i(x) \frac{\partial u}{\partial x_i} v(x) dx
\]

\[
- \sum_i \int_D \hat{B}_i(x) \frac{\partial v}{\partial x_i} u(x) dx - \int_D Q(x) u(x) v(x) dx.
\]

Details about the operator \(L\) can be found in [9], [16] and [20].

The function \(F(\cdot,\cdot,\cdot)\) in (1.1) is a nonlinear function defined on \(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d\) and \(\Phi(x)\) is a bounded measurable function defined on the boundary \(\partial D\) and \(\gamma = An\), where \(n\) denotes the inward normal vector field defined on the boundary \(\partial D\).

To solve the problem (1.1), it turns out that we need to establish the existence and uniqueness of solutions of backward stochastic differential equations (BSDEs) with singular coefficients and infinite horizon, which is of independent interest.

Probabilistic approaches to boundary value problem of second order differential operators have been adopted by many authors and the earliest work went back as early as 1944 in [12].
There has been a lot of study on the Dirichlet boundary problem (see [1], [8], [3], [6], [11] and [22]). However, there are not many articles on the probabilistic approaches to the Neumann boundary problem.

When \( A = I \), \( B = 0 \) and \( \hat{B} = 0 \), the following Neumann boundary problem

\[
\begin{align*}
\frac{1}{2} \Delta u(x) + qu(x) &= 0, \quad \text{on } D \\
\frac{1}{2} \frac{\partial u(x)}{\partial n} &= \phi(x) \quad \text{on } \partial D
\end{align*}
\]

was solved in [1] and [11], which also gives the solution the following representation:

\[
u(x) = \mathbb{E}_x \left[ \int_0^\infty e^{\int_0^t q(B_u) du} \phi(B_t) dL_t^0 \right],
\]

where \( (B_t)_{t>0} \) is the reflecting Brownian motion on the domain \( D \) associated with the infinitesimal generator

\[
G = \frac{1}{2} \Delta,
\]

and \( L_t^0, t > 0 \) is the boundary local time satisfying \( L_t^0 = \int_0^t \nu_0(B_s) dL_s^0 \).

But when \( \hat{B} \neq 0 \), the term \( \nabla \cdot (\hat{B} \cdot \cdot) \) is just a formal way of writing because the divergence does not exist as \( \hat{B} \) is only a measurable vector field. It should be interpreted in the distributional sense. For this reason, the term \( \nabla \cdot (\hat{B} \cdot \cdot) \) can not be handled by Girsanov transform or Feyman-Kac transform.

The study of the boundary value problems for the general operator \( L \) in the PDE literature (see e.g. [9], [20]) was always carried out under the extra condition:

\[-div(\hat{B}) + Q(x) \leq 0\]

in the sense of distribution in order to use the maximum principle.

When \( F = 0 \), i.e. the linear case, problem (11) was studied in [4] (see also [3] for the Dirichlet boundary problem). The term \( \nabla \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \) is tackled using the time-reversal of Girsanov transform of the symmetric reflecting diffusion \( (\Omega, P_x^0, X^0_t, t > 0) \) associated with the operator

\[
L_t = \frac{1}{2} \nabla \cdot (A \nabla).
\]

The semigroup \( S_t \) associated with the operator \( L \) has the following representation (see [5]):

\[
S_t f(x) = \mathbb{E}_x^0 [f(X_t^0) \exp(\int_0^t (A^{-1}B)^*(X_s^0) dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0) dM_s^0) \circ \gamma_t^0 \\
- \frac{1}{2} \int_0^t (B - \hat{B}) A^{-1}(B - \hat{B})^*(X_s^0) ds + \int_0^t Q(X_s^0) ds)],
\]

where \( M_t \) is the martingale part of the diffusion \( X^0_t \) and \( \gamma_t \) is the reverse operator.
The main purpose of this paper is to study the nonlinear equation (1.1) (i.e., \( F \neq 0 \)), which cannot be handled by the methods used for the linear case. Our approach is first to solve a backward stochastic differential equation (BSDE) with singular coefficients and infinite horizon to produce a candidate for the solution of the boundary value problem and then to show that the candidate is indeed a solution. The results we obtained for BSDEs with infinite horizon are of independent interest.

We would like to mention that the first results on BSDEs and probabilistic interpretation of solutions of semilinear parabolic PDEs via BSDEs were obtained by Peng and Pardoux in [19], [17] and [18]. There the operator \( L \) is smooth and the solution is a viscosity solution. We stress that the solutions we considered for PDEs in this paper are Sobolev (also called weak) solutions, not viscosity solutions.

In [22], the corresponding Dirichlet problem for the semilinear elliptic PDEs:

\[
\begin{cases}
Lu(x) = -F(x, u(x), \nabla u(x)), & \text{on } D \\
u(x) = \Phi(x), & \text{on } \partial D
\end{cases}
\]

was solved. The strategy in [3], [22] is to transform the general operator \( L \) by a kind of \( h \)-transform to an operator of the form:

\[
L_2 = \frac{1}{2} \nabla (A \nabla) + b \cdot \nabla + q
\]

which does not have the "bad" term such as \( \nabla (\hat{B} \cdot) \). This idea is used in current paper too.

The BSDEs we studied are inspired by the ones in [10] where the author gave a probabilistic interpretation of the solution to the following Neumann problem:

\[
\begin{cases}
\frac{1}{2} \Delta - \nu u(x) = 0, & \text{on } D \\
\frac{\partial u}{\partial n} = \phi, & \text{on } \partial D
\end{cases}
\]

The content of the paper as follows. In Section 2, we study the following BSDEs with infinite horizon:

\[
dY(t) = -F(X(t), Y(t), Z(t))dt + e^{\int_0^t q(X(u))du} \Phi(X(s))dL_t + (Z(t), dM(t)),
\]

\[
\lim_{t \to \infty} e^{\int_0^t q(X(u))du} Y_t = 0 \quad \text{in } L^2(\Omega),
\]

where \( (X(t))_{t>0} \) is the reflecting diffusion associated with an infinitesimal generator of the form: \( A = \frac{1}{2} \nabla (A \nabla) + b \cdot \nabla \), \( M(t) \) is the martingale part of \( X(t) \), \( L_t \) is the boundary local time of \( X \) and \( d(\cdot) \) is an appropriate measurable function. The existence and uniqueness of an \( L^2 \)-solution \((Y, Z)\) is obtained.

In Section 3, we solve the linear PDEs of the form:

\[
\begin{cases}
\frac{1}{2} \nabla (A \nabla u)(x) + b \cdot \nabla u(x) + qu(x) = F(x), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial n}(x) = \phi(x), & \text{on } \partial D
\end{cases}
\]

under the condition:

\[
E_{x_0} \left[ \int_0^\infty e^{\int_0^t q(X(u))du} dL_t \right] < \infty
\]
for some $x_0 \in \bar{D}$. Useful estimates for local time and Girsanov density are proved which will also be used in subsequent sections.

In Section 4, we obtain the solution of the semilinear PDE:

$$\begin{cases}
\frac{1}{2} \nabla (A \nabla u)(x) + b \cdot \nabla u(x) + qu(x) = G(x, u(x), \nabla u(x)), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D.
\end{cases}$$ (1.7)

To this end, we first use the solution $(Y_x(t), Z_x(t))$ of the BSDE (1.5) to produce a candidate $u_0(x) = E_x[Y_x(0)]$ and then find a solution $u$ of an equation like (1.6) with a given $F(x) := G(x, u_0(x), v_0(x))$. Finally we identify $u$ with $u_0$. In Section 5, we consider the general problem:

$$\begin{cases}
Lu(x) = -F(x, u(x)), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) - \hat{B} \cdot n(x)u(x) = \Phi(x) & \text{on } \partial D
\end{cases}.$$ (1.8)

We apply the transformation introduced in [3] to transform the problem (1.8) to a problem like (1.7). An inverse transformation will yield the solution of the problem (1.8) under the condition that the $L^p$ norm of $\hat{B}$ is sufficiently small.

To remove some of the restrictions imposed on $\hat{B}$ in Section 5, in Section 6, we study the $L^1$-solutions of the BSDEs (1.5) under appropriate conditions. Our approach is inspired by the one in [2]. The study of $L^2$-solutions and $L^1$-solutions of the BSDEs (1.5) are carried out in Section 2 and Section 6 separately because the methods used for these two cases are quite different.

### 2 BSDEs with Singular Coefficients and Infinity Horizon

Consider the operator

$$L_1 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}$$

on the domain $D$ equipped with the Neumann boundary condition:

$$\frac{\partial}{\partial \gamma} := \langle An, \nabla \cdot \rangle = 0, \quad \text{on } \partial D.$$

By [13], there exists a unique reflecting diffusion process denoted by $(\Omega, \mathcal{F}_t, X_x(t), P_x, \theta_t, x \in D)$ associated with the generator $L_1$.

Here $\theta : \Omega \to \Omega$ is the shift operator defined as follows:

$$X_x(s)(\theta_t) = X_x(t + s), \quad s, t \geq 0.$$

Let $E_x$ denote the expectation under the measure $P_x$.

Set $\tilde{b} = \{\tilde{b}_1, ..., \tilde{b}_d\}$, where $\tilde{b}_i = \frac{1}{2} \sum_j \frac{\partial a_{ij}}{\partial x_j} + b_i$.

Then the process $X_x(t)$ has the following decomposition:

$$X_x(t) = X_x(0) + M_x(t) + \int_0^t \tilde{b}(X_x(s))ds + \int_0^t A_n(X_x(s))dL_s, \quad P_x - a.s..$$ (2.1)
Here $M_x(t)$ is a $\mathcal{F}_t$ square integrable continuous martingale additive functional. And $L_t$ is a positive increasing continuous additive functional satisfying $L_t = \int_0^t I_{\{X_x(s) \in \partial D\}} dL_s$. We write $X_x(t)$ as $X(t)$ for short in the following discussion.

In this section, we will study the backward stochastic differential equations with singular coefficients and infinite horizon associated with the martingale part $M_x(t)$ and the local time $L_t$. A unique $L^2$ solution of such BSDEs is obtained.

Let $g(\omega, t, y, z) : \Omega \times R^+ \times R \times R^d \to R$ be a progressively measurable function. Consider the following conditions:

(A.1) $(y_1 - y_2)(g(t, y_1, z) - g(t, y_2, z)) \leq -a_1(t)|y_1 - y_2|^2$,
(A.2) $|g(t, y, z_1) - g(t, y, z_2)| \leq a_2|z_1 - z_2|$,
(A.3) $|g(t, y, z)| \leq |g(t, 0, 0)| + a_3(t)(1 + |y|)$.

Here $a_1(t)$ and $a_3(t)$ are two progressively measurable processes and $a_2$ is a constant.

Set $a(t) = -a_1(t) + \delta a_2^2$, for some constant $\delta > \frac{1}{2\lambda}$, where $\lambda$ is the constant appeared in (1.3).

**Lemma 2.1** Assume the conditions (A.1)-(A.3) and

$$E_x[\int_0^\infty e^{2\int_0^r a(u)du}|g(t, 0, 0)|^2 dt] < \infty.$$  

Then there exists a unique solution $(Y_x(t), Z_x(t))$ to the following backward stochastic differential equation:

$$Y_x(t) = Y_x(T) + \int_t^T g(s, Y_x(s), Z_x(s))ds - \int_t^T <Z_x(s), dM_x(s)>, \quad t < T;$$

$$\lim_{t \to \infty} e^{\int_0^t a(u)du}Y_x(t) = 0, \quad \text{in} \quad L^2(\Omega).$$  

Moreover,

$$E_x[\sup_t e^{2\int_0^r a(u)du}|Y_x(t)|^2] < \infty \quad \text{and} \quad E_x[\int_0^\infty e^{2\int_0^r a(u)du}|Z_x(s)|^2 ds] < \infty.$$  

**Proof.**

Existence:

The proof of this lemma is similar to that of Theorem 3.2 in [22], but the terminal conditions here are different. By Theorem 3.1 in [22], the following BSDE has a unique solution $(Y^n_x(t), Z^n_x(t))$:

$$Y^n_x(t) = \int_t^n g(s, Y^n_x(s), Z^n_x(s))ds - \int_t^n <Z^n_x(s), dM_x(s)>, \quad t \leq n;$$

and moreover,

$$Y^n_x(t) = 0, \quad Z^n_x(t) = 0, \quad t > n.$$
Fix $t > 0$ and $n > m > t$. It follows that
\[
e^2\int_0^t a(u)du |Y^n_x(t) - Y^m_x(t)|^2 + \int_t^\infty e^2\int_0^s a(u)du \langle A(X(s)) (Z^n_x(s) - Z^m_x(s)), (Z^n_x(s) - Z^m_x(s)) \rangle ds
\]
\[
= -2 \int_t^n a(s)e^{2\int_0^s a(u)du} |Y^n_x(s) - Y^m_x(s)|^2 ds
\]
\[
+ 2 \int_t^n e^{2\int_0^s a(u)du} (Y^n_x(s) - Y^m_x(s))(g(s, Y^n_x(s), Z^n_x(s)) - g(s, Y^m_x(s), Z^m_x(s))) ds
\]
\[
+ 2 \int_m^n e^{2\int_0^s a(u)du} (Y^n_x(s) - Y^m_x(s)) g(s, 0, 0) ds
\]
\[
- 2 \int_t^n e^{2\int_0^s a(u)du} (Y^n_x(s) - Y^m_x(s)) < Z^n_x(s) - Z^m_x(s), dM_x(t) >
\]
Choose two positive numbers $\delta_1$ and $\delta_2$ such that $\delta_1 > \frac{1}{2\lambda}$ and $\delta_1 + \delta_2 < \delta$. Then from
\[
2 \int_t^n e^{2\int_0^s a(u)du} (Y^n_x(s) - Y^m_x(s))(g(s, Y^n_x(s), Z^n_x(s)) - g(s, Y^m_x(s), Z^m_x(s))) ds
\]
\[
\leq -2 \int_t^n a_1(s)e^{2\int_0^s a(u)du} |Y^n_x(s) - Y^m_x(s)|^2 ds
\]
\[
+ 2\delta_1 a_2^2 \int_t^n e^{2\int_0^s a(u)du} |Y^n_x(s) - Y^m_x(s)|^2 ds
\]
\[
+ \frac{1}{2\lambda\delta_1} \int_t^n e^{2\int_0^s a(u)du} (A(X(s)) (Z^n_x(s) - Z^m_x(s)), (Z^n_x(s) - Z^m_x(s))) ds
\]
and
\[
2 \int_m^n e^{2\int_0^s a(u)du} (Y^n_x(s) - Y^m_x(s)) g(s, 0, 0) ds
\]
\[
\leq 2\delta_2 a^2 \int_m^n e^{2\int_0^s a(u)du} |Y^n_x(s) - Y^m_x(s)|^2 ds + \frac{1}{2\delta_2 a^2} \int_m^n e^{2\int_0^s a(u)du} |g(s, 0, 0)|^2 ds,
\]
it follows that
\[
E_x[e^{2\int_0^t a(u)du} |Y^n_x(t) - Y^m_x(t)|^2] + \frac{1}{\lambda}(1 - \frac{1}{2\lambda\delta_1}) E_x[\int_t^\infty e^{2\int_0^s a(u)du} |Z^n_x(s) - Z^m_x(s)|^2 ds]
\]
\[
\leq \frac{1}{2\delta_2 a^2} E_x[\int_m^n e^{2\int_0^s a(u)du} |g(s, 0, 0)|^2 ds].
\]
This implies that
\[
E_x[\int_0^\infty e^{2\int_0^s a(u)du} |Z^n_x(s) - Z^m_x(s)|^2 ds] \to 0, \quad as \quad m, n \to \infty.
\]
Hence there exists $\hat{Z}_x$ such that
\[
\hat{Z}_x = \lim_{n \to \infty} e^{\int_0^s a(u)du} Z^n_x \quad in \quad L^2([0, \infty) \times \Omega).
\]
At the same time, we also obtain the following estimates:

\[
\sup_t e^{f_0^a(u)du} |Y^*_x(t) - Y^m_x(t)|^2 \\
\leq \frac{1}{2\delta_2 a_2^2} \int_m^n e^{f_0^a(u)du} |g(s,0,0)|^2 ds \\
+ 2 \sup_t \int_t^n e^{f_0^a(u)du} (Y^n_x(s) - Y^m_x(s)) < Z^n_x(s) - Z^m_x(s), dM_x(t) > |
\]

Taking expectation on both sides of the above inequality, by BDG inequality, we obtain

\[
E_x [\sup_t e^{f_0^a(u)du} |Y^*_x(t) - Y^m_x(t)|^2 ] \\
\leq \frac{1}{2\delta_2 a_2^2} E_x [\int_m^n e^{f_0^a(u)du} |g(s,0,0)|^2 ds] \\
+ C_1 E_x [\int_t^n e^{f_0^a(u)du} |Y^n_x(s) - Y^m_x(s)|^2 |Z^n_x(s) - Z^m_x(s)|^2 ds]^{\frac{1}{2}} \\
\leq \frac{1}{2\delta_2 a_2^2} E_x [\int_m^n e^{f_0^a(u)du} |g(s,0,0)|^2 ds] + \frac{1}{2} E_x [\sup_t e^{f_0^a(u)du} |Y^*_x(t) - Y^m_x(t)|^2 ] \\
+ C_2 E_x [\int_0^\infty e^{f_0^a(u)du} |Z^n_x(s) - Z^m_x(s)|^2 ds]
\]

Thus

\[
E_x [\sup_t e^{f_0^a(u)du} |Y^*_x(t) - Y^m_x(t)|^2 ] \\
\leq \frac{1}{\delta_2 a_2^2} E_x [\int_m^n e^{f_0^a(u)du} |g(s,0,0)|^2 ds] + 2C_2 E_x [\int_0^\infty e^{f_0^a(u)du} |Z^n_x(s) - Z^m_x(s)|^2 ds] \\
\rightarrow 0, \quad \text{as} \quad m, n \rightarrow \infty.
\]

So, there exists \( \{\hat{Y}_x(t)\} \) such that

\[
\lim_{n \rightarrow \infty} E_x [\sup_t e^{f_0^a(u)du} |Y^*_x(t) - e^{f_0^a(u)du} Y^*_x(t)|^2 ] = 0.
\]

For any \( \varepsilon > 0 \), there exist a positive number \( N \) such that for any \( n \geq N \),

\[
E_x [\sup_t |\hat{Y}_x(t) - e^{f_0^a(u)du} Y^*_x(t)|^2 ] < \frac{\varepsilon}{2}.
\]

For \( t > N \), noticing \( Y^N_x(t) = 0 \), it follows that

\[
E_x [\hat{Y}_x(t)|^2 ] \leq 2E_x [\hat{Y}_x(t) - e^{f_0^a(u)du} |Y^N_x(t)|^2 ] + 2E_x [e^{f_0^a(u)du} |Y^N_x(t)|^2 ] \\
\leq 2E_x [\sup_t |\hat{Y}_x(t) - e^{f_0^a(u)du} |Y^N_x(t)|^2 ] + 2E_x [e^{f_0^a(u)du} |Y^N_x(t)|^2 ] \\
< \varepsilon.
\]
Thus we have \( \lim_{t \to 0} E_x[|\tilde{Y}_x(t)|^2] = 0 \).

By chain rule, it is easy to see from (2.4) that

\[
Y_x(t) = e^{-\int_0^t a(u)du} \tilde{Y}_x(t) \quad \text{and} \quad Z_x(t) = e^{-\int_0^t a(u)du} \tilde{Z}_x(t)
\]
satisfy the equation (2.2) and

\[
\lim_{t \to 0} E_x[e^{2\int_0^t a(u)du}|Y_x(t)|^2] = \lim_{t \to \infty} E_x[|\tilde{Y}_x(t)|^2] = 0.
\]

From the above proof, we also see that (2.3) holds.

Uniqueness:

Suppose that \((Y_x^1, Z_x^1)\) and \((Y_x^2, Z_x^2)\) are two solutions of the equation (2.2).

Set \(\tilde{Y}_x(t) = Y_x^1(t) - Y_x^2(t)\) and \(\tilde{Z}_x(t) = Z_x^1(t) - Z_x^2(t)\).

Then

\[
d(e^{\int_0^t a(u)du} \tilde{Y}_x(t)) = -e^{\int_0^t a(u)du}(g(t, Y_x^1(t), Z_x^1(t)) - g(t, Y_x^2(t), Z_x^2(t)))dt + a(t)e^{\int_0^t a(u)du} \tilde{Y}_x(t)dt + e^{\int_0^t a(u)du} \tilde{Z}_x(t, dM_x(t)). \tag{2.5}
\]

By Ito’s formula, we get, for any \(t < T\),

\[
e^{2\int_0^t a(u)du} |\tilde{Y}_x(t)|^2 + \int_t^T e^{2\int_0^u a(v)du} \langle A(X(s)) \tilde{Z}_x(s), \tilde{Z}_x(s) \rangle ds
\]

\[
= e^{2\int_0^T a(u)du} |\tilde{Y}_x(T)|^2 + 2 \int_t^T e^{2\int_0^u a(v)du} \langle \tilde{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds
\]

\[
-2 \int_t^T a(s)e^{2\int_0^u a(v)du} |\tilde{Y}_x(s)|^2 ds
\]

\[
-2 \int_t^T a(s)e^{2\int_0^u a(v)du} \langle \tilde{Z}_x(s), dM_x(s) \rangle. \tag{2.6}
\]

By condition (A.1) and (A.2), we have

\[
2 \int_t^T e^{2\int_0^u a(v)du} \langle \tilde{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds
\]

\[
= 2 \int_t^T e^{2\int_0^u a(v)du} \langle \tilde{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^1(s)))ds
\]

\[
+ 2 \int_t^T e^{2\int_0^u a(v)du} \langle \tilde{Y}_x(s)(g(s, Y_x^2(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds \leq -2 \int_t^T a(s)e^{2\int_0^u a(v)du} |\tilde{Y}_x(s)|^2 ds + a_2 \int_t^T e^{2\int_0^u a(v)du} |\tilde{Z}_x(s)|^2 ds
\]

\[
-2 \int_t^T a(s)e^{2\int_0^u a(v)du} |\tilde{Y}_x(s)|^2 ds + c'a_2 \int_t^T e^{2\int_0^u a(v)du} |\tilde{Z}_x(s)|^2 ds
\]

\[+a_2 \frac{1}{c'\lambda} \int_t^T e^{2\int_0^u a(v)du} |\tilde{Z}_x(s)|^2 ds. \tag{2.7}
\]
Choosing \( c' = 2\delta a_2 \), we obtain
\[
|e^{\int_0^t a(u)du}Y_x(t)|^2 + (1 - \frac{1}{2\delta \lambda})\int_t^T e^{\int_0^t a(u)du}A(X(s))\bar{Z}_x(s), \bar{Z}_x(s))ds
\leq e^{\int_0^T a(u)du}Y_x(T)^2 - 2\int_t^T a(s)e^{\int_0^s a(u)du}\bar{Y}_x(s)\bar{Z}_x(s), dM_x(s) \tag{2.8}
\]

Taking expectation on both sides of the above inequality, we get that, for any \( Y \)
\[
E_x[e^{\int_0^t a(u)du}|\bar{Y}_x(t)|^2] \leq E_x[e^{\int_0^T a(u)du}|\bar{Y}_x(T)|^2].
\]

For both \( Y^1 \) and \( Y^2 \) satisfy the terminal condition in (2.2), so that
\[
\lim_{T \to \infty} E_x[e^{\int_0^T a(u)du}|\bar{Y}_x(T)|^2] = 0,
\]
which leads to \( E_x[e^{\int_0^T a(u)du}|\bar{Y}_x(t)|^2] = 0 \).
We conclude that \( Y^1_x(t) = Y^2_x(t) \) and \( Z^1_x(t) = Z^2_x(t) \). □

We now want to apply Lemma 2.1 to a particular situation.
Let \( F(x, y, z) : R^d \times R \times R^d \to R \) be a Borel measurable function. Consider the following conditions:

(D.1) \( (y_1 - y_2)(F(x, y_1, z) - F(x, y_2, z)) \leq -d_1(x)|y_1 - y_2|^2 \);
(D.2) \( |F(x, y, z_1) - F(x, y, z_2)| \leq d_2|z_1 - z_2| \);
(D.3) \( |F(x, y, z)| \leq |F(x, 0, z)| + K(x)(1 + |y|) \).

Set \( d(x) = -d_1(x) + \delta d_2^2 \) for some constant \( \delta > \frac{1}{2K} \).
The follows result follows from Lemma 2.1

**Lemma 2.2** Assume the conditions (D.1)-(D.3) and
\[
E_x[\int_0^\infty e^{\int_0^t a(u)du}F(X(t), 0, 0)dM(t)] < \infty.
\]

Then there exists a unique solution \((Y_x(t), Z_x(t))\) to the following equation:
\[
Y_x(t) = Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s))ds - \int_t^T < Z_x(s), dM_x(s) >, \quad t < T;
\]
\[
\lim_{t \to \infty} e^{\int_0^t a(u)du}Y_x(t) = 0, \quad \text{in } L^2(\Omega). \tag{2.9}
\]

Consider the following condition instead of (D.3).

(D.3)' \( |F(X(t), y, z)| \leq K(t) \), for any \( y \in R \) and \( z \in R^d \).

Let \( \Phi \) be a bounded measurable function defined on \( \partial D \), and function \( \tilde{q} \in L^p(D) \), for \( p > \frac{d}{2} \).
The following theorem is the main result in this section.
Theorem 2.1 Assume the conditions (D.1), (D.2) and (D.3)',
\[ E_{x_0} \left[ \int_0^\infty e^{\int_0^t q(X(u))du} dL_s \right] < \infty \]
for some \( x_0 \in D \) and for \( x \in D \),
\[ E_x \left[ \int_0^\infty e^{2 \int_0^t d(X(u))du} \{ e^{2 \int_0^t q(X(u))du} \ + \ |K(t)|^2 \} dt \right] < \infty. \tag{2.10} \]

Then there exists a unique solution \((Y_x, Z_x)\) to the following BSDE:
\[
Y_x(t) = Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s)) ds - \int_t^T e^{\int_s^T q(X(u))du} \Phi(X(s)) dL_s \\
- \int_t^T \langle Z_x(s), dM_x(s) \rangle, \quad \text{for } t < T, \tag{2.11}
\]
and
\[
\lim_{t \to \infty} e^{\int_0^t d(X(u))du} Y_t = 0 \quad \text{in} \quad L^2(\Omega). \tag{2.12}
\]

Proof.
Uniqueness:
Suppose that \((Y_x^1, Z_x^1)\) and \((Y_x^2, Z_x^2)\) are two solutions of the equation \((2.11)\) satisfying \((2.12)\).
Set \( \bar{Y}_x(t) = Y_x^1(t) - Y_x^2(t) \) and \( \bar{Z}_x(t) = Z_x^1(t) - Z_x^2(t) \). Then
\[
d(e^{\int_0^t d(X(u))du} \bar{Y}_x(t)) = -e^{\int_0^t d(X(u))du} \left( F(X(t), Y_x^1(t), Z_x^1(t)) - F(X(t), Y_x^2(t), Z_x^2(t)) \right) dt \\
+ d(X(t)) e^{\int_0^t d(X(u))du} \bar{Y}_x(t) dt \\
+ e^{\int_0^t d(X(u))du} \langle \bar{Z}_x(t), dM_x(t) \rangle. \tag{2.13}
\]

By Ito's formula, we get, for any \( t < T \),
\[
e^{2 \int_0^t d(X(u))du} |\bar{Y}_x(T)|^2 + \int_t^T e^{2 \int_0^s d(X(u))du} \langle A(X(s)) \bar{Z}_x(s), \bar{Z}_x(s) \rangle ds
= e^{2 \int_0^t d(X(u))du} |\bar{Y}_x(T)|^2 \\
+ 2 \int_t^T e^{2 \int_0^s d(X(u))du} \bar{Y}_x(s) (F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s))) ds \\
- 2 \int_t^T d(X(s)) e^{2 \int_0^s d(X(u))du} |\bar{Y}_x(s)|^2 ds \\
- 2 \int_t^T d(X(s)) e^{2 \int_0^s d(X(u))du} \bar{Y}_x(s) \langle \bar{Z}_x(s), dM_x(s) \rangle \tag{2.14}
\]
By (D.1) and (D.2), we have
\[
2 \int_t^T e^{2 \int_0^s d(X(u))du} \bar{Y}_x(s) (F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s))) ds
\]
Choosing \( c = 2\delta d_2 \), we obtain from (2.15)

\[
|e^{-\int_0^t d(X(u))du} \tilde{Y}_x(t)|^2 + (1 - \frac{1}{2\delta\lambda}) \int_t^T e^{-\int_0^t d(X(u))du} (A(X(s))\tilde{Z}_x(s), \tilde{Z}_x(s)) ds
\]

\[
\leq e^{\int_t^T d(X(u))du} \int_0^t e^{-\int_0^t d(X(u))du} (A(X(s))\tilde{Z}_x(s), \tilde{Z}_x(s)) ds
\]

Taking expectation on both sides of the above inequality and letting \( T \) tend to infinity, we obtain that

\[
E_x [e^{\int_0^T d(X(u))du} |\tilde{Y}_x(T)|^2] = 0
\]

We conclude that \( Y_x^1(t) = Y_x^2(t) \) and hence from (2.16), \( Z_x^1(t) = Z_x^2(t) \).

Existence:

First of all, the assumption (2.10) implies (see [?])

\[
\sup_x E_x \left[ \int_0^\infty e^{\int_0^t \tilde{q}(X(u))du} dL_s \right] < \infty.
\]

1°: There exists \((p_x(t), q_x(t))\) such that

\[
dp_x(t) = e^{\int_t^T \tilde{q}(X(u))du} \Phi(X(t)) dL_t + \langle q_x(t), dM_x(t) \rangle,
\]

and \( e^{\int_0^t d(X(u))du} p_x(t) \to 0 \) as \( t \to \infty \) in \( L^2(\Omega) \).

In fact, let

\[
p_x(t) := -E_x \left[ \int_t^\infty e^{\int_0^s \tilde{q}(X(u))du} \Phi(X(s)) dL_s | F_t \right]
\]

\[
= \int_0^t e^{\int_0^s \tilde{q}(X(u))du} \Phi(X(s)) L_s - E_x \left[ \int_0^\infty e^{\int_0^s \tilde{q}(X(u))du} \Phi(X(s)) dL_s | F_t \right].
\]

By the martingale representation theorem in [22], there exists a process \( q_x(t) \), such that

\[
- E_x \left[ \int_0^\infty e^{\int_0^s \tilde{q}(X(u))du} \Phi(X(s)) dL_s | F_t \right] = - E_x \left[ \int_0^\infty e^{\int_0^s \tilde{q}(X(u))du} \Phi(X(s)) dL_s \right] + \int_0^t < q_x(s), dM_x(s) >.
\]
Then \((p_x, q_x)\) satisfies the equation (2.17).
Moreover,

\[
P_x(t) := -E_x\left[\int_t^\infty e^{\int_0^t \tilde{q}(X(u))\,du} \Phi(X(s))\,dL_s|\mathcal{F}_t\right]
= -e^{\int_0^t \tilde{q}(X(u))\,du} E_x\left[\int_t^\infty e^{\int_t^s \tilde{q}(X(u))\,du} \Phi(X(s))\,dL_s|\mathcal{F}_t\right]
= -e^{\int_0^t \tilde{q}(X(u))\,du} E_x\left[\int_0^{t+s} e^{\int_s^t \tilde{q}(X(u))\,du} \Phi(X(s + t))\,dL_{s+t}|\mathcal{F}_t\right]
= -e^{\int_0^t \tilde{q}(X(u))\,du} E_x\left[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} \Phi(X(t))\,dL_t\right]
\]  

The last equality follows from the fact that \(L_{t+s} = L_t + L_s \circ \theta_t\). Therefore,

\[
\sup_x |p_x(t)| \leq e^{\int_0^t \tilde{q}(X(u))\,du} \sup_{x \in D} |\Phi(x)| \cdot \sup_{x \in \mathcal{D}} E_x\left[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} \,dL_t\right].
\]

Set \(M = \sup_{x \in \mathcal{D}} |\Phi(x)| \cdot \sup_{x \in \mathcal{D}} E_x\left[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} \,dL_t\right].\)

In view of (2.10), we have \(\lim_{t \to \infty} e^{\int_0^t (d+\tilde{q})(X(u))\,du} = 0\) in \(L^2(\Omega)\).

Hence,

\[
e^{\int_0^t d(X(u))\,du} p_x(t) \leq Me^{\int_0^t (d+\tilde{q})(X(u))\,du} \to 0 \quad \text{as} \quad t \to \infty, \quad \text{in} \quad L^2(\Omega).
\]  

2°: Set \(g(t, y, z) = F(X(t), p_x(t) + y, q_x + z)\). Then

\[
(y_1 - y_2)(g(t, y_1, z) - g(t, y_2, z))
= (y_1 - y_2)(F(X(t), p_x(t) + y_1, q_x + z) - F(X(t), p_x(t) + y_2, q_x + z))
\leq -d_1(X(t))|y_1 - y_2|^2.
\]

and

\[
|g(t, y, z_1) - g(t, y, z_2)| = |F(X(t), p_x(t) + y, q_x + z_1) - F(X(t), p_x(t) + y, q_x + z_1)| 
\leq d_2|z_1 - z_2|.
\]

Moreover,

\[
E_x\int_0^\infty e^{2\int_0^t d(X(u))\,du} |g(X(t), 0, 0)|^2 \,dt 
\leq E_x\int_0^\infty e^{2\int_0^t d(X(u))\,du} |F(X(t), p_x(t), q_x(t))|^2 \,dt 
\leq E_x\int_0^\infty e^{2\int_0^t d(X(u))\,du} |K(t)|^2 \,dt 
< \infty.
\]
Proving Theorem 2.1, we find that
\[ t = \frac{1}{2} \int_0^t q(X(u)) du, \]
and
\[ e^{\int_0^t (X(u)) du} k_x(t) \rightarrow 0, \]
as \( t \rightarrow \infty. \)

Putting \( Y_x(t) = p(t) + k_x(t) \) and \( Z_x(t) = q(t) + l_x(t) \), we find that \((Y_x(t), Z_x(t))\) satisfies the following equation
\[ dY_x(t) = e^{\int_0^t \tilde{q}(X(u)) du} \phi(X(t)) dL_t - F(t, Y_x(t), Z_x(t)) dt + < Z_x(t), dM_x >. \]

**Corollary 2.1** Suppose all the assumptions in Theorem 2.1 hold. If, in addition,
\[ \sup_x E_x \int_0^\infty e^{\int_0^t (X(u)) du} |K(t)|^2 dt < \infty, \]
then it follows that
\[ \sup_{x \in D} |Y_x(0)| < \infty. \]

**Proof.**
As shown in the proof of Theorem 2.1, \( Y_x(t) \) has the decomposition: \( Y_x(t) = p(t) + k_x(t) \).
Setting \( t = 0 \) in (2.20), it follows that
\[
|p_x(0)| \leq E_X(t) \int_0^\infty e^{\int_0^t \tilde{q}(X(u)) du} \Phi(X(t)) dL_t
\leq \|\Phi\|_\infty \sup_x E_x \int_0^\infty e^{\int_0^t \tilde{q}(X(u)) du} dL_t
< \infty.
\]

By Itô's formula, we obtain
\[
de^2 \int_0^t (X(u)) du \left| k_x(t) \right|^2 = -2e^2 \int_0^t (X(u)) du k_x(t) g(t, k_x(t), l_x(t)) dt + \frac{1}{2} \left( 1 - \frac{1}{2\lambda_0} \right) E_x \int_t^T e^{2 \int_0^s (X(u)) du} |l_x(s)|^2 ds + \frac{1}{2\delta_2^2} E_x \int_t^T e^{2 \int_0^s (X(u)) du} |g(s, 0, 0)|^2 ds.
\]
Choosing two positive numbers \( \delta_1 \) and \( \delta_2 \) such that \( \delta_1 > \frac{1}{2\lambda_0} \) and \( \delta_1 + \delta_2 < \delta \), similar calculations as in the proof of Theorem 2.1 yield that, for any \( t < T \),
\[
E_x \left[ e^{2 \int_0^T (X(u)) du} \left| k_x(T) \right|^2 \right] \leq E_x \left[ e^{2 \int_0^T (X(u)) du} \left| k_x(t) \right|^2 \right] + \frac{1}{2\delta_2^2} E_x \int_t^T e^{2 \int_0^s (X(u)) du} |g(s, 0, 0)|^2 ds.
\]
Setting $t = 0$, we have
\[
|k_x(0)|^2 = E_x[|k_x(0)|^2] \leq E_x[e^{2\int_0^T d(X(u))du}|k_x(T)|^2] \\
+ \frac{1}{2\delta_2 d_2^2} E_x[\int_0^T e^{2\int_0^s d(X(u))du} |g(s,0,0)|^2 ds].
\]
Let $T \to \infty$ to obtain that
\[
\sup_x |k_x(0)| \leq \left( \frac{1}{2\delta_2 d_2^2} \sup_x E_x[\int_0^\infty e^{2\int_0^s d(X(u))du} |g(s,0,0)|^2 ds] \right)^{\frac{1}{2}} < \infty,
\]
where the fact that $e^{2\int_0^T d(X(u))du}k_x(T) \to 0$ as $T \to \infty$, has been used. Hence, we have
\[
\sup_x |Y_x(0)| \leq \sup_x |p_x(0)| + \sup_x |k_x(0)| < \infty.
\]

3 Linear PDEs

Set
\[
L_2 = \frac{1}{2} \nabla \cdot (A \nabla) + b \cdot \nabla + q
\]
where $b = (b_1, \ldots, b_d)$ is a $\mathbb{R}^d$-valued Borel measurable function, and $q$ is a Borel measurable function on $\mathbb{R}^d$ such that:
\[
I_D(|b|^2 + |q|) \in L^p(D), \quad p > \frac{d}{2}.
\]
In this section, we solve the following linear boundary value problem:
\[
\begin{cases}
\frac{1}{2} \nabla \cdot (A \nabla u) + b \cdot \nabla u(x) + q(x)u(x) = F(x), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) = \phi & \text{on } \partial D,
\end{cases} \tag{3.1}
\]
where $F$ and $\phi$ are bounded measurable functions on $D$.

It is well known that operator $L_2$ defined on a bounded domain $D$ with Neumann boundary condition $\frac{\partial u}{\partial \gamma}(x) = 0$ is associated with the quadratic form:
\[
\mathcal{E}(f,g) = -\int_D L_2 f(x)g(x)dx \\
= \frac{1}{2} \int_D \langle A \nabla f, \nabla g \rangle dx - \int_D b \cdot \nabla f(x)g(x)dx - \int_D q(x)f(x)g(x)dx
\]

**Definition 3.1** A bounded continuous function $u(x)$ defined on $D$ is a weak solution of the problem (3.1) if $u \in W^{1,2}(D)$, and for any $g \in C^\infty(\overline{D})$,
\[
\mathcal{E}(u,g) = \int_{\partial D} \phi(x)g(x)\sigma(dx) - \int_D F(x)g(x)dx,
\]
where $\sigma$ denotes the $d - 1$ dimensional Lebesgue measure on $\partial D$. 

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Consider the operator
\[ L_0 = \frac{1}{2} \nabla \cdot (A \nabla u) \] (3.2)
on domain \( D \) with boundary condition \( \frac{\partial u}{\partial \gamma} = 0 \) on \( \partial D \).

\( L_0 \) is associated with a reflecting diffusion process \( (X^0, P^0_x) \). By [13], \( X^0 \) has the following decomposition:
\[ dX^0_t = \sigma(X^0_t) dW_t + \frac{1}{2} \nabla A(X^0_t) dt + \gamma(X^0_t) dL^0_t, \]
\[ L^0_t = \int_0^t I\{X^0_s \in \partial D\} dL^0_s, \] (3.3)
where the matrix \( \sigma(x) \) is the positive definite symmetric square root of the matrix \( A(x) \) and \( \{W_t\}_{t>0} \) is a \( d \)-dimensional standard Brownian motion.

It is well known that operator \( L_0 \) is associated with the regular Dirichlet form:
\[ \mathcal{E}^0(u,v) = \frac{1}{2} \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx \]
and the domain of \( \mathcal{E}^0 \) is \( W^{1,2}(D) := \{ u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D) \} \).

The following lemma can be proved similarly as the Corollary 3.8 in [11] using the heat kernel estimates in [21].

**Lemma 3.1** There exists a constant \( K > 0 \), such that
\[ \sup_{x \in \overline{D}} E^0_x[L^0_t] \leq K \sqrt{t} \quad \text{and} \quad \inf_{x \in \overline{D}} E^0_x[L^0_t] > 0. \]
Moreover, we have \( \sup_{x \in \overline{D}} E^0_x[(L^0_t)^n] \leq K_n t^{n/2} \), for some constant \( K_n > 0 \).

Set \( M^0_t = \int_0^t \sigma(X^0_s) dW_s \) and
\[ Z_t = e^{\int_0^t \langle A^{-1} b(X^0_s), \sigma M^0_s \rangle \, ds} = \frac{1}{2} \int_0^t b A^{-1} b^*(X^0_s) ds + \int_0^t q(X^0_s) ds, \] (3.4)
where \( b^* \) is the transpose of the row vector \( b \).

The proof of the following two lemmas are inspired by that of the Lemma 2.1 and Theorem 2.2 in [11].

**Lemma 3.2** For \( t > 0 \), there are two strictly positive functions \( M_1(t) \) and \( M_2(t) \) such that, for any \( x \in \overline{D} \), \( M_1(t) \leq E^0_x[\int_0^t Z_s dL^0_s] \leq M_2(t) \). Furthermore, \( M_2(t) \to 0 \) as \( t \to 0 \).
\[ \tilde{M}(t) = e^{\int_0^t A^{-1}b(X_s^0) \cdot dM_s^0} - \frac{1}{2} \int_0^t bA^{-1}b^*(X_s^0) \, ds, \]  
\[ e_q(t) = e^{\int_0^t q(X_s^0) \, ds}, \]  
\[ M_q(t) = \sup_{x \in \mathcal{D}} E_x^0 \left[ \left| \int_0^t q(X_s^0) \, ds \right| \right]. \]

Then we have
\[ \sup_{x \in \mathcal{D}} E_x^0 \left[ \int_0^t Z_s \, dL_s^0 \right] = \sup_{x \in \mathcal{D}} E_x^0 \left[ \int_0^t \tilde{M}(s)e_q(s) \, dL_s^0 \right] \]
\[ \leq \sup_{x \in \mathcal{D}} E_x^0 \left[ \max_{0 \leq s \leq t} \left| \tilde{M}(s) \right|^2 \right] \cdot \sup_{x \in \mathcal{D}} E_x^0 \left[ e_{2q}(t) \right] \frac{1}{2} \]
\[ \leq \sup_{x \in \mathcal{D}} E_x^0 \left[ |\tilde{M}(t)|^2 \right] \cdot \sup_{x \in \mathcal{D}} E_x^0 \left[ e_q(t) \right] \frac{2}{3} \cdot \sup_{x \in \mathcal{D}} E_x^0 \left[ (L_0^t)^\frac{1}{2} \right] \]  
\[ \text{(III)} \]

By Khas’Min斯基’s lemma and Theorem 2.1 in [15], (I) and (II) are bounded if \( t \) belongs to a bounded interval. Because of \( E_x^0[(L_0^t)^n] \leq K_n \sqrt{t} \), we see that \( M_2(t) := K(I)(II)\sqrt{t} \) is the required upper bound.

\[ \text{2°: Since} \]
\[ E_x^0[L_0^t]^2 \leq E_x^0 \left[ \int_0^t \tilde{M}^{-1}(s)e_{-q}(s) \, dL_s^0 \right] \cdot E_x^0 \left[ \int_0^t \tilde{M}(s)e_q(s) \, dL_s^0 \right], \]  
we obtain
\[ E_x^0 \left[ \int_0^t \tilde{M}(s)e_q(s) \, dL_s^0 \right] \geq \frac{E_x^0[L_0^t]^2}{E_x^0 \left[ \int_0^t \tilde{M}^{-1}(s)e_{-q}(s) \, dL_s^0 \right]}. \]

Here
\[ \tilde{M}^{-1}(t) = e^{-\int_0^t A^{-1}b(X_s^0) \cdot dM_s^0} - \frac{1}{2} \int_0^t bA^{-1}b^*(X_s^0) \, ds \]
\[ = e^{-\int_0^t A^{-1}b(X_s^0) \cdot dM_s^0} - \frac{1}{2} \int_0^t bA^{-1}b^*(X_s^0) \, ds \cdot \inf_0^t bA^{-1}b^*(X_s^0) \, ds \]
\[ = N(t) \cdot \inf_0^t bA^{-1}b^*(X_s^0) \, ds \]  
\[ \text{(3.9)} \]

By the proof of the first part, replacing \( \tilde{M}_t, q \) by \( N_t \) and \( bA^{-1}b^* - q \) respectively, it is seen that there exists \( K(t) > 0 \) such that \( \sup_{x \in \mathcal{D}} E_x^0 \left[ \int_0^t \tilde{M}^{-1}(s)e_{-q}(s) \, dL_s^0 \right] \leq K(t). \)

As \( \inf_{x \in \mathcal{D}} E_x^0[L_0^t] > 0 \), we complete the proof of the lemma by setting \( M_1(t) = \frac{\inf_{x \in \mathcal{D}} E_x^0[L_0^t]^2}{K(t)} \). \( \square \)

Set \( G(x) := E_x^0 \left[ \int_0^\infty Z_s \, dL_s^0 \right]. \)

**Lemma 3.3** If there is a point \( x_0 \in \mathcal{D} \), such that \( G(x_0) < \infty \), then there are two positive constants \( K \) and \( \beta \) such that \( \sup_{x \in \mathcal{D}} E_x^0[Z_t] \leq K e^{-\beta t} \).
Proof.
By Girsanov Theorem and Feymann-Kac formula, \( L_2 = \frac{1}{2} \nabla \cdot (A \nabla) + b \cdot \nabla + q \) is associated with the semigroup \( \{ T_t \}_{t > 0} \), where \( T_t f(x) = E^0_x[Z_t f(X_0^t)] \) for \( f \in L^2(D) \).

By the upper and lower bound estimates of the heat kernel \( p_2(t, x, y) \) associated with \( T_t \) in [21], the following inequality holds,

\[
c^{-1} \int_D f(x) dx \leq E^0_x[Z_t f(X_0^t)] \leq c \int_D f(x) dx,
\]

where \( c \) is a positive constant. Since \( G(x) = \sum_{n=0}^{\infty} E^0_x[Z_n E^0_x[Z_1 L_1^0(ds)]] \geq M_1(1) \sum_{n=0}^{\infty} E^0_x[Z_n] \)

and \( G(x_0) < \infty \), there is a positive integer number \( N \) such that

\[
\frac{1}{2c^2} \geq E^0_{x_0}[Z_N] = E^0_{x_0}[Z_1 E^0_x[Z_{N-1}]] \geq c^{-1} \int_D E^0_x[Z_{N-1}] m(dx).
\]

This implies

\[
\int_D E^0_x[Z_{N-1}] m(dx) \leq \frac{1}{2c}.
\]

Thus

\[
\sup_{x \in D} E^0_x[Z_N] = \sup_{x \in D} E^0_x[Z_1 E^0_x[Z_{N-1}]] \leq c \int_D E^0_x[Z_{N-1}] m(dx) \leq \frac{1}{2}.
\]

For any \( t > 0 \), there exists a positive number \( n \) such that \( \frac{1}{N} \in [n-1, n) \). Then by (3.11), it follows that

\[
E^0_x[Z_t] \leq \frac{1}{2^n-1} E^0_x[Z_{t-N}(n-1)] \leq \left( \sup_{x \in D, 0 \leq t \leq N} E^0_x[Z_t] \right) \frac{1}{2^n-1} \leq 2 \sup_{x \in D, 0 \leq t \leq N} E^0_x[Z_t] e^{-\frac{\ln 2}{N} t}. \square
\]

Theorem 3.1 If there exists \( x_0 \in \overline{D} \) such that \( G(x_0) < \infty \), then there exists a unique bounded continuous weak solution of the problem (3.1):

Proof.
Existence:
Due to Theorem 3.2 in [4], there exists a unique, bounded, continuous weak solution \( u_2 \) of the following problem:

\[
\begin{cases}
L_2 u_2(x) = 0, & \text{on } D \\
\frac{1}{2} \frac{\partial u_2}{\partial n}(x) = \phi & \text{on } \partial D.
\end{cases}
\]

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Thus by the linearity of the problem (3.1), we only need to show that the following problem has a bounded continuous weak solution:

\[
\begin{cases}
L_2u_1(x) = F(x), & \text{on } D \\
\frac{\partial u_1}{\partial \gamma}(x) = 0, & \text{on } \partial D
\end{cases}
\] (3.14)

The semigroup associated with operator \(L_2\) is \(\{T_t, t > 0\}\). By Lemma 3.3, we have

\[
\sup_{x \in D} |T_t F(x)| = \sup_{x \in D} |E_x^0[Z_t F(X_t^0)]| \leq K e^{-\beta t} \|F\|_{\infty}.
\]

Then

\[
u_1(x) := \int_0^\infty T_t F(x) dt
\]

is well defined and has the following bound:

\[
\sup_{x \in D} |\nu_1(x)| \leq \frac{K}{\beta} \|F\|_{\infty}.
\]

The function \(\nu_1(x)\) is also continuous on \(D\).

In fact, fixing any \(x \in D\) and \(\epsilon > 0\), we can firstly choose a constant \(t_0 > 0\), such that

\[
\sup_{x \in D} |\int_0^{t_0} T_s F(z) ds| < \frac{\epsilon}{3}.
\]

And because \(T_{t_0} u_1(x)\) is continuous, there exists a constant \(\delta > 0\), such that for any \(y\) with \(|y - x| < \delta\), \(|T_{t_0} u_1(x) - T_{t_0} u_1(y)| \leq \frac{\epsilon}{3}\).

We find that

\[
T_t u_1(x) = E_x^0[Z_t u_1(X_t^0)] = E_x^0[Z_t \int_0^\infty E_{X_s^0} [Z_s F(X_s^0)] ds]
\]

\[
= \int_0^t E_x^0[Z_{t+s} u_1(X_{t+s}^0)] ds
\]

\[
= \int_t^\infty T_s F(x) ds
\]

\[
= u_1(x) - \int_0^t T_s F(x) ds.
\] (3.15)

For any \(y\) satisfying \(|y - x| < \delta\), it follows that

\[
|\nu_1(x) - \nu_1(y)| \leq |T_{t_0} u_1(x) - T_{t_0} u_1(y)| + |\int_0^{t_0} T_s F(x) ds| + |\int_0^{t_0} T_s F(y) ds| \leq \epsilon.
\] (3.16)

This implies that the function \(\nu_1\) is continuous on domain \(D\).

Denote the resolvents associated with operator \(L_2\) by \(\{G_\beta, \beta > 0\}\). Note that

\[
G_\beta u_1(x) = \int_0^\infty e^{-\beta t} T_t u_1(x) dt
\]

\[
= \int_0^\infty e^{-\beta t} u_1(x) dt - \int_0^{t_0} e^{-\beta t} \int_0^t T_s F(x) ds dt
\]
We have

\[ \beta(u_1(x) - G_\beta u_1(x)) = \beta G_\beta F(x). \]  

Therefore,

\[ \lim_{\beta \to \infty} \int_D \beta(u_1(x) - G_\beta u_1(x))u_1(x)dx = \lim_{\beta \to \infty} \int_D \beta G_\beta F(x)u(x)dx = \int_D F(x)u_1(x)dx < \infty. \]

This implies that \( u_1 \in D(\mathcal{E}) \) (see [16]) and \( u_1 \) is a weak solution of equation (3.14). By the linearity, \( u = u_1 + u_2 \) is a bounded continuous weak solution of equation (3.1).

**Uniqueness:**

Let \( v_1 \) and \( v_2 \) be two bounded continuous weak solutions of the equation (3.1). Then \( v_1 - v_2 \) is the solution of equation (3.13) with \( \phi = 0 \). Then by the uniqueness of the equation (3.13) proved in [4], we know that \( v_1 = v_2 \). \( \square \)

### 4 Semilinear PDEs

Recall that

\[ L_1 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} \]

and \( L_2 = L_1 + q \) are two operators both defined on the domain \( D \) and equipped with the Neumann boundary condition \( \frac{\partial}{\partial \gamma} = 0 \) on \( \partial D \).

\( \Omega, \mathcal{F}_t, X(t), P_x, x \in D \) is the reflecting diffusion process associated with the operator \( L_1 \) with the decomposition introduced in (2.1).

In this section, we solve the following semilinear boundary value problem:

\[
\begin{cases}
L_2 u(x) = -G(x, u(x), \nabla u(x)), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D
\end{cases}
\]

Let \( \mathcal{E}(\cdot, \cdot) \) be the quadratic form associated with the operator \( L_2 \):

\[ \mathcal{E}(u, v) = \frac{1}{2} \int_D < A \nabla u, \nabla v > dx - \int_D < b, \nabla u > vdx - \int_D quvdx. \]
Definition 4.1 A bounded continuous function $u(x)$ defined on $D$ is called a weak solution of the equation \[(4.1)\] if $u \in W^{1,2}(D)$, and for any $g \in C_\infty(D)$,
\[
E(u, g) = \int_{\partial D} \phi(x)g(x)\sigma(dx) + \int_D G(x,u(x),\nabla u(x))g(x)dx.
\]
Recall that $L_t$ is the boundary local time of $X(t)$ defined in (2.1) and $L^0_t$ is the boundary local time of $X^0_t$ in (3.3).

As a consequence of the Girsanov theorem, we have:

Lemma 4.1 Suppose that the function $f$ satisfies $E_x[\int_0^T e^{\int_0^t f(X(u))du} dL_t] < \infty$. Then it holds that
\[
E_x[\int_0^T e^{\int_0^t f(X(u))du} dL_t] = \tilde{E}_x[\int_0^T \tilde{M}_t e^{\int_0^t f(X^0(u))du} dL^0_t],
\]
where $\tilde{M}_t$ was defined in (3.5).

The following lemma is deduced from Theorem 3.2 in [4].

Lemma 4.2 Suppose that the function $\tilde{q} \in L^p(D)$ and $p > \frac{d}{2}$. If there exists some point $x_0 \in D$, such that
\[
E_{x_0}[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du} dL_t] < \infty, \tag{4.2}
\]
then it holds that
\[
\sup_x E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du} dL_t] < \infty.
\]

Let $G(x,y,z) : R^d \times R \times R^d \to R$ be a bounded Borel measurable function. Introduce the following conditions:

(H.1) $|y_1 - y_2| (G(x,y_1,z) - G(x,y_2,z)) \leq -h_1(x)|y_1 - y_2|^2$,
(H.2) $|G(x,y,z_1) - G(x,y,z_2)| \leq h_2|z_1 - z_2|$.

Set $h(t) = -h_1(X(t)) + \delta h_2^2 + q(X(t))$ and $\tilde{h}(t) = -h_1(X(t)) + \delta h_2^2$ for some constant $\delta > \frac{1}{2X}$.

Theorem 4.1 Suppose that the conditions (H.1) and (H.2) are satisfied. Assume
\[
E_{x_1}[\int_0^\infty e^{\int_0^t (q(X(u)) + \tilde{h}(u))du} dt] < \infty, \text{ for some } x_1 \in D, \tag{4.3}
\]
and there exists some point $x_0 \in D$, such that
\[
E_{x_0}[\int_0^\infty e^{\int_0^t q(X(u))du} dL_t] < \infty. \tag{4.4}
\]
Then the semilinear Neumann boundary value problem (4.1) has a unique continuous weak solution.
by Ito’s formula, we have that, for any $t < T$

$$
(y_1 - y_2)(\tilde{G}(X(t), y_1, z) - \tilde{G}(X(t), y_2, z)) \leq -h_1(x)|y_1 - y_2|^2 \quad (4.5)
$$

and

$$
|\tilde{G}(X(t), y, z_1) - \tilde{G}(X(t), y, z_2)| \leq h_2|z_1 - z_2|. \quad (4.6)
$$

Note that

$$
\tilde{G}(X(t), y, z) \leq e^{\int_0^t q(X(u))du}G(t, y, z).
$$

By Theorem 2.1 there exists a unique process $(\tilde{Y}_x, \tilde{Z}_x)$ satisfying

$$
d\tilde{Y}_x(t) = -\tilde{G}(X(t), \tilde{Y}_x(t), \tilde{Z}_x(t))dt + e^{\int_0^t q(X(u))du}dL(t) + <\tilde{Z}_x(t), dM_x(t)\rangle
$$

$$
e^{\int_0^t h(u)du}\tilde{Y}_x(t) \to 0 \quad as \quad t \to \infty.
$$

Furthermore, Corollary 2.1 implies that $\sup_{x} \tilde{Y}_x(0) < \infty$.

From Ito’s formula, it follows that

$$
d(e^{-\int_0^t q(X(u))du}\tilde{Y}_x(t))
$$

$$
= -q(X(t))e^{-\int_0^t q(X(u))du}\tilde{Y}_x(t)dt - e^{-\int_0^t q(X(u))du}\tilde{G}(X(t), \tilde{Y}_x(t), \tilde{Z}_x(t))dt
$$

$$
+ \phi(X(t))dL_t + \langle e^{-\int_0^t q(X(u))du}\tilde{Z}_x(t), dM_x(t)\rangle >.
$$

Setting $Y_x(t) := e^{-\int_0^t q(X(u))du}\tilde{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))du}\tilde{Z}_x(t)$, we obtain

$$
dY_x(t) = -(q(X(t))Y_x(t) + G(X(t), Y_x(t), Z_x(t)))dt + \phi(X(t))dL_t + <Z_x(t), dM_x(t)>.
$$

Moreover,

$$
e^{\int_0^t h(u)du}Y_x(t) = e^{\int_0^t h(u)du}e^{-\int_0^t q(X(u))du}\tilde{Y}_x(t) = e^{\int_0^t h(u)du}\tilde{Y}_x(t) \to 0 \quad as \quad t \to \infty. \quad (4.7)
$$

So by Ito’s formula, we have that, for any $t < T$,

$$
e^{\int_0^t h(u)du}Y_x(t)
$$

$$
= e^{\int_0^T h(u)du}Y_x(T) + \int_t^T e^{\int_s^T h(u)du} (G(X_x(s), Y_x(s), Z_x(t)) + q(X_x(s))Y_x(s)) ds
$$

$$
- \int_t^T e^{\int_s^T h(u)du} \phi(X(s))dL_s - \int_t^T h(s)e^{\int_0^s h(u)du}Y_x(s)ds
$$

$$
- \int_t^T e^{\int_s^T h(u)du} <Z_x(t), dM_x(t)\rangle. \quad (4.8)
$$
Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.

Since $Y_x(0) = \bar{Y}_x(0)$, we know that $u_0$ is a bounded function on domain $D$. By the Markov property of $X$ and the uniqueness of $(Y_x, Z_x)$, it is easy to see that

$$Y_x(t) = u_0(X(t)), \quad Z_x(t) = v_0(X(t)).$$

So that $\sup_{x \in D, t > 0} |Y_x(t)| \leq \|u_0\|_\infty < \infty$.

Now consider the following problem:

$$\left\{ \begin{array}{ll}
L_2 u(x) = -G(x, u_0(x), v_0(x)), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial n}(x) = \phi(x), & \text{on } \partial D
\end{array} \right. \tag{4.9}$$

By Theorem 3.1, problem (4.9) has a unique continuous weak solution $u(x)$. Next we will show that $u = u_0$.

Since $u$ belongs to the domain of the Dirichlet form associated with the process $X(t)$, it follows from the Fukushima’s decomposition that:

$$du(X(t)) = -[G(X(t), u_0(X(t)), v_0(X(t))) + q(X(t))u(X(t))]dt + \phi(X(t))dL(t) + \langle \nabla u(X(t)), dM_x(t) \rangle$$

From the condition (4.9) and the boundedness of $u(x)$, it follows that

$$\lim_{t \to \infty} E_x[e^{2 \int_0^t h(u)u_0du}u^2(X(t))] \leq \|u\|_\infty^2 \lim_{t \to \infty} E_x[e^{2 \int_0^t (h + q)(u)du}] = 0.$$

By Ito’s formula, it follows that, for any $t < T$,

$$e^{\int_0^t h(u)u_0du}u(X(t))$$

$$= e^{\int_0^T h(u)u_0du}u(X(T)) + \int_t^T e^{\int_t^s h(u)u_0du}[G(X(s), Y_s(s), Z_x(s)) + q(X(s))u(X(s))]ds$$

$$- \int_t^T e^{\int_t^s h(u)u_0du}\phi(X(s))dL(s) - \int_t^T h(s)e^{\int_t^s h(u)u_0du}u(X(s))ds$$

$$- \int_t^T e^{\int_t^s h(u)u_0du}\langle \nabla u(X(t)), dM_x(t) \rangle. \tag{4.10}$$

Set

$$v_x(t) = u(X(t)) - Y_x(t) \quad \text{and} \quad R_x(t) = \nabla u(X(t)) - Z_x(t).$$

Subtracting the equations (4.8) from (4.10), we obtain the following equation: for any $t < T$,

$$e^{\int_0^t h(u)u_0du}v(X(t))$$

$$= e^{\int_0^T h(u)u_0du}v(X(T)) + \int_t^\infty (q(X(u)) - h(u))e^{\int_u^t h(u)u_0du}v(X(s))ds$$

$$- \int_t^\infty e^{\int_u^t h(u)u_0du} < R_x(t), dM_x(t) >.$$
where

\[ g(t) = e^{\int_0^T h(u) du} v(X(T)) - \int_t^T \tilde{h}(s) e^{\int_0^s h(u) du} v(X(s)) ds - \int_t^T e^{\int_0^s h(u) du} < R_x(t), dM_x(t) >. \]

Set \( g(t) = e^{\int_0^T h(u) du} v(t) \). Taking conditional expectation on both sides of (4.11), we find that

\[ g(t) = E_x[g(T) - \int_t^T \tilde{h}(s) g(s) ds | F_t] \]

\[ = E_x[g(T)(1 - \int_t^T \tilde{h}(s) ds) + \int_t^T \int_s^T \tilde{h}(s) \tilde{h}(s_1) g(s_1) ds_1 ds | F_t] \]

\[ = E_x[g(T)(1 - \int_t^T \tilde{h}(s) ds) + \frac{1}{2} (\int_t^T \tilde{h}(s) ds)^2) \]

\[ + (-1)^3 \int_t^T \int_s^T \int_{s_1}^T \int_{s_2}^T \tilde{h}(s) \tilde{h}(s_1) \tilde{h}(s_2) g(s_2) ds_2 ds_1 ds | F_t]. \]

Keeping iterating, we obtain

\[ g(t) = E_x[g(T) \sum_{k=0}^{n} (- \int_t^T \tilde{h}(s) ds)^n \]

\[ + (-1)^{n+1} \int_t^T \int_s^T \int_{s_1}^T ... \int_{s_{n-1}}^T \tilde{h}(s) \tilde{h}(s_1) ... \tilde{h}(s_n) g(s_n) ds_n ... ds_1 ds | F_t]. \]

Since \( E_x[|g(T)| e^{\int_0^T |\tilde{h}(s)| ds}] < \infty \), letting \( n \to \infty \), by dominated convergence theorem, it follows that

\[ g(t) = E_x[g(T)e^{-\int_t^T \tilde{h}(s) ds} | F_t]. \]

Then

\[ v(t) = E_x[v(T)e^{\int_t^T \tilde{h}(s) - \tilde{h}(s)) ds} | F_t] \leq (\|u_0\|_\infty + \|u\|_\infty) E_x[e^{\int_0^T q(X(s)) ds} | F_t]. \]

(4.11)

Hence, it follows that

\[ 0 \leq e^{\int_0^T q(X(s)) ds} | v(t) \| \leq (\|u_0\|_\infty + \|u\|_\infty) \lim_{T \to \infty} E_x[e^{\int_0^T q(X(s)) ds} | F_t]. \]

(4.12)

Since the condition (4.4) implies

\[ \lim_{T \to \infty} E_x[e^{\int_0^T q(X(s)) ds}] = 0, \]

we deduce that \( E_x[e^{\int_0^T q(X(s)) ds} | v(t) \| = 0 \) and hence \( v(t) = 0, P_x - a.s. \).

Therefore, for any \( t > 0 \), we have \( u(X(t)) = Y_x(t) \) and \( \nabla u(X(t)) = Z_x(t) \) by the uniqueness of the Doob-Meyer decomposition of semimartingales. In particular, \( u(x) = E_x[u(X_x(0))] = \)
This shows that $u(x)$ is a weak solution of the equation (4.1). If $\tilde{u}$ is another solution of the problem (4.1). Then the processes $\tilde{Y}_x(t) := \tilde{u}(X(t))$ and $\tilde{Z}_x(t) := \nabla \tilde{u}(X(t))$ satisfy the following equation

$$d\tilde{Y}_x(t) = -G(X(t), \tilde{Y}_x(t), \tilde{Z}_x(t))dt - \phi(X(t))dL_t + \langle \tilde{Z}_x(t), dM_t \rangle.$$

(4.13)

Set $\bar{Y}_x(t) = e^{\int_0^t h(u)du} \tilde{Y}_x(t)$ and $\bar{Z}_x(t) = e^{\int_0^t h(u)du} \tilde{Z}_x(t)$. By chain rule, it follows that

$$d\bar{Y}_x(t) = -\tilde{G}(X(t), \bar{Y}_x(t), \bar{Z}_x(t))dt + e^{\int_0^t h(u)du} \phi(X(t))dL_t + \langle \bar{Z}_x(t), dM_t \rangle.$$

Moreover, because $\tilde{u}$ is bounded, we have

$$\lim_{t \to \infty} e^{\int_0^t h(u)du} \tilde{Y}_x(t) = \lim_{t \to \infty} e^{\int_0^t h(u)du} \tilde{u}(X(t)) = 0.$$

Therefore, from the uniqueness of the solution of the BSDE in Theorem 2.1, we have

$$\tilde{Y}_x(t) = Y_x(t) \quad \tilde{Z}_x(t) = Z_x(t).$$

In particular,

$$\tilde{u}(x) = E_x[\tilde{Y}_x(t)] = E_x[Y_x(t)] = u(x).$$

5 Semilinear Elliptic PDEs with Singular Coefficients

Recall the operator

$$L = \frac{1}{2} \nabla \cdot (A \nabla) + B \cdot \nabla - \nabla \cdot (\hat{B}) + Q$$

on the domain $D$ equipped with the mixed boundary condition on $\partial D$:

$$\frac{1}{2} \frac{\partial u}{\partial \gamma} - \langle \hat{B}, n \rangle u(x) = 0.$$

The quadratic form associated with $L$ is given by:

$$Q(u, v) := (-Lu, v) = \frac{1}{2} \sum_{i,j} \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_i \int_D B_i(x) \frac{\partial u}{\partial x_i} v(x) dx - \sum_i \int_D \hat{B}_i(x) \frac{\partial v}{\partial x_i} u(x) dx - \int_D Q(x) u(x) v(x) dx,$$

where $(\cdot, \cdot)$ stands for the inner product in $L^2(D)$. The domain of the quadratic form is

$$\mathcal{D}(Q) = W^{1,2}(D) := \{ u : u \in L^2(D), \frac{\partial u}{\partial x_i} \in L^2(D), i = 1, \ldots, d \}.$$
Let \( \{S_t, t \geq 0\} \) denote the semigroup generated by \( L \).

In this section, our aim is to solve the following equation:

\[
\begin{cases}
    L f(x) = -F(x, f(x)), & \text{on } D \\
    \frac{1}{2} \partial f(x) - <\hat{B}, n> (x)f(x) = \Phi(x) & \text{on } \partial D
\end{cases}
\]  

(5.1)

**Definition 5.1** A bounded continuous function \( f(x) \) defined on \( D \) is called a weak solution of the equation (5.1) if \( f \in W^{1,2} \), and for any \( g \in C^\infty(D) \),

\[
Q(u, g) = \int_{\partial D} \Phi(x)g(x)\sigma(dx) + \int_D F(x, u(x))g(x)dx.
\]

Here the function \( F : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \) is a bounded measurable function and satisfies the following condition:

\((E.1)\) \((y_1 - y_2)(F(x, y_1) - F(x, y_2)) \leq -r_1(x)|y_1 - y_2|^2.\)

Recall the following regular Dirichlet form

\[
\begin{cases}
    \mathcal{E}^0(u, v) = \frac{1}{2} \sum_{i,j} \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \\
    D(\mathcal{E}^0) = W^{1,2}(D)
\end{cases}
\]

(5.2)

associated with the operator \( L_0 = \frac{1}{2} \nabla (A \nabla) \) equipped with the Neumann boundary condition \( \frac{\partial}{\partial \gamma} = 0 \) on \( \partial D \).

The associated reflecting diffusion process is denoted by \( \{\Omega, \bar{\mathcal{F}}_t, X^0_t, \theta^0_t, \gamma^0_t, P^0_t\} \). Here \( \theta^0_t \) and \( \gamma^0_t \) are the shift and reverse operators defined by

\[
X^0_s(\theta^0_t(\omega)) = X^0_{t+s}(\omega), s, t \geq 0 \\
X^0_s(\gamma^0_t(\omega)) = X^0_{t-s}(\omega), s, t.
\]

The process \( (X^0_t)_{t \geq 0} \) has the decomposition in (3.3). The martingale part of \( X^0_t \) is \( M^0_t = \int_0^t \sigma(X^0_s)dW_s \).

The following probabilistic representation of semigroup \( S_t \) was proved in [5]

\[
S_t f(x) = E^0_x[f(X^0_t)] \exp(\int_0^t (A^{-1}B)^*(X^0_s)dM^0_s + (\int_0^t (A^{-1}\hat{B})^*(X^0_s)dM^0_s) \circ \gamma^0_t \\
- \frac{1}{2} \int_0^t (B - \hat{B})A^{-1}(B - \hat{B})^*(X^0_s)ds + \int_0^t Q(X^0_s)ds] \]

(5.3)

\( E^0_x \) denotes the expectation under \( P^0_x \).

Set

\[
\hat{Z}_t = \exp(\int_0^t (A^{-1}B)^*(X^0_s)dM^0_s + (\int_0^t (A^{-1}\hat{B})^*(X^0_s)dM^0_s) \circ \gamma^0_t \\
- \frac{1}{2} \int_0^t (B - \hat{B})A^{-1}(B - \hat{B})^*(X^0_s)ds + \int_0^t Q(X^0_s)ds).
\]

(5.4)
Moreover, $v$ satisfies the following equations: for $g \in C^1(\hat{D})$,

$$\int_D <A\nabla v, \nabla g > (x)dx = \int_D <\hat{B}, \nabla g > (x)dx. \tag{5.6}$$

Thus the representation of $S_t$ becomes:

$$S_tf(x) = e^{-v(x)}E_x^0[f(X_t)]e^{v(X_t)} \exp(\int_0^t (A^{-1}(B - \hat{B} - A\nabla v))^*dM^0_s - \frac{1}{2} \int_0^t (B - \hat{B} - A\nabla v)^*A^{-1}(B - \hat{B} - A\nabla v)(X^0_s)ds)
+ \int_0^t (Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v \rangle)(X^0_s)ds)]$$

$$= e^{-v(x)}\tilde{S}_t[f e^v](x). \tag{5.7}$$

Here, setting $b := B - \hat{B} - (A\nabla v)$ and $q := Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v \rangle$, we see that $\tilde{S}_t$ is the semigroup generated by the following operator:

$$L_2 = \frac{1}{2} \nabla \cdot (A\nabla) + (B - \hat{B} - (A\nabla v)) \cdot \nabla + (Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v \rangle)$$

$$= \frac{1}{2} \nabla \cdot (A\nabla) + b \cdot \nabla + q$$

equipped with the boundary condition $\frac{\partial}{\partial \nu} = 0$.

In this section, we will stick to this particular choice of $b$ and $q$.

Recall that

$$\tilde{M}(t) = e^{\int_0^t A^{-1}b(X^0_s)dM^0_s - \frac{1}{2} \int_0^t bA^{-1}b^*(X^0_s)ds}$$

and set $Z_t = \tilde{M}(t)e^{\int_0^t q(X^0_s)ds}$.

Then from (5.7), it follows that $\hat{Z}(t) = Z_t e^{v(X^0_t) - v(X^0_0)}$.

Recall the operator $L_1 = \frac{1}{2} \nabla \cdot (A\nabla) + b \cdot \nabla$ with Neumann boundary condition, which is associated with the reflecting diffusion $(X(t), P_x)$. It is known from [14] that

$$dP_x|_{\mathcal{F}_t} = \tilde{M}_t dP^0_x|_{\mathcal{F}_t},$$

and

$$X(t) = x + \int_0^t \sigma(X(s))dW_s + \int_0^t (\frac{1}{2} \nabla A + b)(X(s))ds + \int_0^t \gamma(X(s))dL_s, \quad P_x - a.s.$$
where \( \{W_t\} \) is a d-dimensional Brownian motion and \( L_t \) is the local time satisfying that \( L_t = \int_0^t \mathbb{I}_{\partial D}(X(s))dL_s \).

**Lemma 5.1** Assume that there exists \( x_0 \in D \), such that

\[
E_{x_0}^0 \left[ \int_0^\infty |\dot{Z}_t|^2 e^{\int_0^t (2Q-4r_1)(X_u^0)du}dL_t^0 \right] < \infty. \tag{5.8}
\]

Then there exists a positive number \( \varepsilon > 0 \), if \( \|\hat{B}\|_{L^p} \leq \varepsilon \), the following inequality holds:

\[
\sup_{x \in D} E_x \left[ \int_0^\infty e^{\int_0^t (-r_1+q)(X_u)du} dt \right] < \infty. \tag{5.9}
\]

**Proof.**

\[
E_x \left[ e^{\int_0^t (-r_1+q)(X_u)du} \right] = E_x^0 \left[ \tilde{M}(t)e^{\int_0^t (-r_1+q)(X_u)du} \right] = E_x^0 \left[ Z(t)e^{\int_0^t (-2r_1+q)(X_u)du} \right] \leq C_1 E_x^0 \left[ \tilde{Z}(t)e^{\int_0^t Q(Q+\frac{1}{2})<A\nabla v-2(B-\hat{B}),\nabla v>(X_u^0)du} \right] \leq C_1 E_x^0 \left[ \tilde{Z}^2(t)e^{\int_0^t Q(Q-2r_1)(X_u^0)du} \right] \frac{1}{2} E_x^0 \left[ e^{\int_0^t <A\nabla v-2(B-\hat{B}),\nabla v>(X_u^0)du} \right] \frac{1}{2}
\]

By Lemma 3.3 and condition (5.8), there exists two constant \( c_2, \beta > 0 \) such that

\[
\sup_{x \in D} E_x \left[ \tilde{Z}^2(t)e^{\int_0^t Q(Q-2r_1)(X_u^0)du} \right] < c_2 e^{-\beta t}.
\]

Moreover, for \( p > d \), by the Theorem 2.1 in [15], there exist two positive constants \( c_3 \) and \( c_4 \) such that

\[
E_x^0 \left[ e^{\int_0^t <A\nabla v-2(B-\hat{B}),\nabla v>(X_u^0)du} \right] \leq c_3 e^{c_4 t},
\]

where \( c_4 = c \| <A\nabla v-2(B-\hat{B}),\nabla v> \|_{L^p/2} \).

Since \( \|\nabla v\|_{L^p} \leq C \|\hat{B}\|_{L^p(D)} \) (see [21]), there exists \( \varepsilon > 0 \), such that \( \|\hat{B}\|_{L^p(D)} \leq \varepsilon \) implies \( c_4 < \beta \). Thus (5.9) holds. \( \square \)

**Theorem 5.1** Assume (5.8) and for some point \( x_0 \in D \)

\[
E_{x_0}^0 \left[ \int_0^\infty \dot{Z}_s dL_t^0 \right] < \infty \tag{5.10}
\]

Then there exists \( \varepsilon > 0 \) such that if \( \|\hat{B}\|_{L^p} \leq \varepsilon \), the problem (5.1) has a unique, bounded, continuous weak solution \( u(x) \).

**Proof.**

Existence: Set \( \hat{F}(x, y) = e^{v(x)} F(x, e^{-v(x)} y) \) and \( \phi(x) = e^{v(x)} \Phi(x) \).

From the boundedness of \( v \), \( \hat{F} \) is also bounded.

And \( \hat{F} \) satisfies

\[
(y_1 - y_2)(\hat{F}(x, y_1) - \hat{F}(x, y_2)) \leq -r_1(x)|y_1 - y_2|^2.
\]
Moreover, there is a constant $c > 0$, such that
\[ \infty > E_{x_0}^0 \left[ \int_0^\infty \hat{Z}_s dL_s^0 \right] = E_{x_0}^0 \left[ \int_0^\infty Z_s e^{v(X_0^s) - v(X_0^0)} dL_s^0 \right] \]
\[ \geq cE_{x_0}^0 \left[ \int_0^\infty Z_s dL_s^0 \right] = cE_{x_0}^0 \left[ \int_0^\infty \tilde{M}_s e^{\int_0^s q(X_u^0) du} dL_s^0 \right] \]  
(5.11)

By Lemma 4.1, we know that, at $x_0 \in D$,
\[ E_{x_0} \left[ \int_0^\infty e^{\int_0^s q(X_u^0) du} dL_s \right] < \infty. \]  
(5.12)

Furthermore, by Lemma 4.2, it follows that
\[ \sup_x E_x \left[ \int_0^\infty e^{\int_0^s q(X_u) du} dL_s \right] < \infty. \]  
(5.13)

By Lemma 5.1, the following condition is satisfied:
\[ E_x \left[ \int_0^\infty e^{2 \int_0^s (q - r_1)(X_u) du} dt \right] < \infty, \]  
(5.14)

So $\tilde{F}$ satisfies all of the conditions in Theorem 4.1 replacing $G$ by $\tilde{F}$. Thus the following problem
\[ \begin{cases} L_2 u(x) = -\tilde{F}(x, u(x)), & \text{on } D \\ \frac{1}{2} \frac{\partial u}{\partial \gamma}(x) = \phi & \text{on } \partial D \end{cases} \]  
(5.15)

has a unique bounded continuous weak solution $u(x)$.

Set $f(x) = e^{-v(x)} u(x)$. Then we claim the function $f(x)$ is the weak solution of the equation (5.1).

Because function $v$ is continuous and bounded, $f(x)$ is also continuous. From the fact that function $u$ is the weak solution of the problem (5.15), we obtain, for any function $\psi \in C^\infty(D),
\[ \mathcal{E}(u, e^{-v} \psi) = \frac{1}{2} \int_D < A\nabla u, \nabla (e^{-v} \psi) > - < b, \nabla u > e^{-v} \psi - e^{-v} q u \psi dx \]
\[ = \int_{\partial D} e^{-v} \psi d\sigma + \int_D \tilde{F}(x, u(x)) e^{-v} \psi dx. \]  
(5.16)

As in the proof of Theorem 5.1 in [22], we can show that the left side of the equation (5.16) equals to
\[ Q(f, \psi) = \frac{1}{2} \int_D [ < A\nabla f, \nabla \psi > - < B, \nabla u > \psi - < \tilde{B}, \nabla \psi > f - Qf \psi] dx. \]

At the same time, by the definition of the function $\phi$ and $\tilde{F}$, the right side of the equation (5.16) equals to
\[ \int_{\partial D} \Phi \psi d\sigma + \int_D F(x, f(x)) \psi dx. \]
Thus it follows that, for any $\psi \in C^\infty(D)$,

$$Q(f, \psi) = \int_{\partial D} \Phi \psi d\sigma + \int_D F(x, f(x)) \psi dx.$$ 

which proves that function $f$ is a weak solution of the problem (6.1).

Uniqueness:
If $\bar{f}$ is another solution of the problem (5.1), then $\bar{u} := e^\bar{f}$ can be shown to be the solution of the equation (5.15). Then by the uniqueness of the problem (5.15) proved in the Theorem 4.1, we find $\bar{u} = u$. Therefore, $f = \bar{f}$.$\square$

6 \text{ } L^1 \text{ solutions of the BSDE and Semilinear PDEs}

Recall the operator

$$L_1 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}$$

on the domain D equipped with the Neumann boundary condition $\frac{\partial}{\partial n} = 0$, on $\partial D$.

And $(\Omega, \mathcal{F}_t, X(t), P_x, x \in D)$ is the reflecting diffusion process associated with the generator $L_1$.

Then the process $X(t)$ has the following decomposition:

$$X(t) = X(0) + M(t) + \int_0^t \tilde{b}(X(s)) ds + \int_0^t A_n(X(s)) dL_s, \quad P_x - a.s.$$ 

Here $\tilde{b} = \{\tilde{b}_1, ..., \tilde{b}_d\}$ with $\tilde{b}_i = \frac{1}{2} \sum_j \frac{\partial a_{ij}}{\partial x_j} + b_i$. $M(t)$ is the $\mathcal{F}_t$ square integrable continuous martingale additive functional.

In this section, we will consider the $L^1$ solutions of the BSDEs in Section 2 and use this result to solve the nonlinear elliptic partial differential equation with the mixed boundary condition.

Let $f : \Omega \times R^+ \times R \to R$ be progressively measurable. Consider the following conditions:

(1.1) $\langle y - y'(t, y) - f(t, y') \rangle \leq d(t)|y - y'|^2$, where $d(t)$ is a progressively measurable process;

(1.2) $E[\int_0^\infty e^{\int_0^t d(w) dw} |f(s, 0)| ds] < \infty$;

(1.3) $P_x - a.s.$, for any $t > 0$, $y \to f(t, y)$ is continuous;

(1.4) $\forall r > 0$, $T > 0$, $\psi_r(t) := \sup_{|y| \leq r} |f(t, y) - f(t, 0)| \in L^1([0, T] \times \Omega, dt \times dP_x)$.

The following lemma is deduced from Corollary 2.3 in [2].

**Lemma 6.1** Suppose a pair of progressively measurable processes $(Y, Z)$ with values in $R \times R^d$ such that $t \to Z_t$ belongs to $L^2([0, T])$ and $t \to f(t, Y_t)$ belongs to $L^1([0, T])$, $P_x - a.s.$.

If

$$Y_t = \xi + \int_t^T f(r, Y_r) dr - \int_t^T < Z_r, dM_r >,$$  \hspace{1cm} (6.1)
then the following inequality holds, for $0 \leq t < u \leq T$,

$$|Y_t| \leq |Y_u| + \int_t^u \hat{Y}_s f(s, Y_s) ds - \int_t^u \hat{Y}_s \langle Z_r, dM_r \rangle.$$

where $\hat{y} = \frac{y}{|y|} I_{\{y \neq 0\}}$.

The following lemma can be proved by modifying the proof of Proposition 6.4 in [2].

**Lemma 6.2** Assume that conditions (I.1)-(I.4) with $d(t) \equiv 0$. Then there exists a unique solution $(Y, Z)$ of the BSDE

$$Y_t = \int_t^T f(r, Y_r) dr - \int_t^T \langle Z_r, dM_r \rangle, \quad \text{for } t \leq T. \quad (6.2)$$

Moreover, for each $\beta \in (0, 1)$, $E[\sup_{t \leq T} |Y_t|^\beta] + E[(\int_0^T |Z_r|^2 dr)^{\frac{\beta}{2}}] < \infty$.

Suppose $\beta \in (0, 1)$. $S^\beta$ denotes the set of real-valued, adapted and continuous process $\{Y_t\}_{t \geq 0}$ such that

$$\|Y\|^\beta := E[\sup_{t > 0} |Y_t|^\beta] < \infty.$$

It is known that $\| \cdot \|^\beta$ deduces a complete metric on $S^\beta$.

$M^\beta$ denotes the set of $R^d$-valued predictable processes $\{Z_t\}$ such that

$$\|Z\|_{M^\beta} := E[(\int_0^\infty |Z_t|^2 dt)^{\frac{\beta}{2}}] < \infty.$$

$M^\beta$ is also a complete metric space with the distance deduced by $\| \cdot \|_{M^\beta}$.

**Lemma 6.3** Under the same assumption as the Lemma 6.2, there exists a unique solution $(Y, Z)$ of the BSDE

$$Y_t = Y_T + \int_t^T f(r, Y_r) dr - \int_t^T \langle Z_r, dM_r \rangle, \quad \text{any } t \leq T; \quad \lim_{t \to \infty} Y_t = 0, \quad \text{P-a.s.} \quad (6.3)$$

**Proof.** Existence:

By the Lemma 6.2 above, there exists $(Y^n, Z^n)$ such that, for $0 \leq t \leq n$,

$$Y^n_t = \int_t^n f(r, Y^n_r) dr - \int_t^n \langle Z^n_r, dM_r \rangle,$$

and $Y^n_T = Z^n_T = 0$, for $t \geq n$.

Fix $t > 0$ and $t < n < n + i$, then

$$Y^{n+i}_t - Y^n_t = \int_t^{n+i} (f(r, Y^{n+i}_r) - f(r, Y^n_r)) dr - \int_t^{n+i} \langle (Z^{n+i}_r - Z^n_r), dM_r \rangle + \int_n^{n+i} f(r, 0) dr$$
Set \( F^n(r, y) = f(r, y + Y^n_r) - f(r, Y^n_r) + f(r, 0)I_{\{r > n\}} \), \( y^n_t = Y^{n+i}_t - Y^n_t \) and \( z^n_t = Z^{n+i}_t - Z^n_t \). Then \((y^n_t, z^n_t)\) is the solution of the following BSDE:

\[
y^n_t = \int_t^{n+i} F(r, y^n_r) dr - \int_t^{n+i} \langle z^n_r, dM_r \rangle.
\]

(6.4)

So that by the condition (I.1) with \(d(t) \equiv 0\), it follows from Lemma 6.1 that

\[
|y^n_t| \leq \int_t^{n+i} \langle \hat{y}^n_r, F^n(r, y^n_r) \rangle dr - \int_t^{n+i} \langle \hat{y}^n_r, z^n_r dM_r \rangle
\]

\[
\leq \int_t^{n+i} \frac{I(y^n_r \neq 0)}{|y^n_r|} \langle y^n_r, f(r, y^n_r + Y^n_r) - f(r, Y^n_r) \rangle dr + \int_t^n |f(s, 0)| ds
\]

\[
- \int_t^{n+i} \langle \hat{y}^n_r, z^n_r dM_r \rangle
\]

\[
\leq \int_t^n |f(s, 0)| ds - \int_t^{n+i} \langle \hat{y}^n_r, z^n_r dM_r \rangle.
\]

(6.5)

Taking conditional expectation on both side of the inequality, we got

\[
|y^n_t| \leq E[\int_n^{n+i} |f(s, 0)| ds | F_t ] := M^n_t,
\]

where \( M^n_t \) is a martingale. Then by Doob’s inequality and condition (I.2), it follows that, for \( \beta \in (0, 1) \),

\[
E[\sup_t |y^n_t|^\beta] \leq E[\sup(M^n_t)^\beta] \leq \frac{1}{1-\beta} E[\int_n^{n+i} |f(s, 0)| ds]^\beta
\]

\[
\rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.
\]

(6.6)

Therefore, \( \{Y^n\} \) is a Cauchy sequence under the norm \( \| \cdot \|_\infty^\beta \). So that there is a process \( Y \) such that \( E[\sup_t |Y_t - Y^n_t|^\beta] \rightarrow 0 \).

This also implies that \( Y_t \rightarrow 0 \), as \( t \rightarrow \infty \), \( P_x - a.s. \).

Moreover, by the equation (6.4), Ito’s formula and the condition (I.1), it follows that

\[
|y^n_t|^2 + \int_t^{n+i} \langle A(X(r))z^n_r, z^n_r \rangle dr
\]

\[
= 2 \int_t^{n+i} \langle y^n_r, F^n(r, y^n_r) \rangle dr - 2 \int_t^{n+i} \langle y^n_r, z^n_r dM_r \rangle
\]

\[
\leq 2 \int_t^n \langle y^n_r, f^n(r, 0) \rangle dr + 2 | \int_t^{n+i} \langle y^n_r, z^n_r dM_r \rangle |
\]

\[
\leq \sup_r |y^n_r|^2 + (\int_t^n |f(r, 0)| dr)^2 + 2 | \int_t^\infty \langle y^n_r, z^n_r dM_r \rangle |,
\]

and thus that

\[
(\int_t^{n+i} |z^n_r|^2 dr)^\beta \leq c_1 [\sup_r |y^n_r|^\beta + (\int_t^{n+i} |f(r, 0)| dr)^\beta + \int_t^{n+i} |y^n_r, z^n_r dM_r > |^\beta].
\]

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Taking expectation on both sides of the inequality and applying the BDG inequality, we obtain

\[
E[(\int_t^{n+i} |z^n_r|^2 dr)^{\beta/2}] \\
\leq c_1(E[\sup_r |y^n_r|^\beta] + E[(\int_t^{n+i} |f(r,0)|dr)^\beta] + c_2E[(\int_t^{n+i} |y^n_r|^\beta \int_t^{n+i} |z^n_r|^2 dr)^{\beta/4}]) \\
\leq (c_1 + \frac{c_2}{2})(E[\sup_r |y^n_r|^\beta] + E[(\int_n^{n+i} |f(r,0)|dr)^\beta]) + \frac{1}{2}E[(\int_t^{n+i} |z^n_r|^2 dr)^{\beta/2}].
\]

Therefore, we know that there is a constant \( C > 0 \), such that

\[
E[(\int_0^{\infty} |z^n_s|^2 ds)^{\beta/2}] \\
\leq CE[\sup_t |y^n_t|^\beta + (\int_n^{n+i} |f(s,0)|ds)^\beta] \\
\leq CE[\sup_t |y^n_t|^\beta] + CE[\int_n^{n+i} |f(s,0)|ds]^\beta \\
\to 0 \quad \text{as} \quad n \to \infty.
\]

So that \( \{Z^n_t\} \) is a Cauchy sequence in \( M^\beta \). Let \( Z \) denote the limit of \( \{Z^n\} \).

At last, by the condition (I.3), we find that

\[
\int_0^T f(t,Y^n_t)dt \to \int_0^T f(t,Y_t)dt, \quad P_x - a.s. 
\]

Therefore, \((Y,Z)\) is the solution satisfies the BSDE (6.3).

Uniqueness: Consider \((Y, Z)\) and \((Y', Z')\) are two solutions to (6.3). Then by the same method as in the proof of Lemma 2.1, we can show that,

\[
\forall t > 0, \quad |Y_t - Y'_t| = 0, \quad P - a.s.. \quad \square
\]

(I.5) The process \( d(t) \) is a progressively measurable process satisfying

\[
d(\cdot) \in L^1([0,T] \times \Omega, dt \otimes P), \quad \text{for any} \quad T > 0.
\]

**Theorem 6.1** Assume the conditions (I.1)-(I.4). Then there exists a unique process \((Y,Z)\) such that,

\[
Y_t = Y_T + \int_t^T f(r,Y_r)dr - \int_t^T <Z_r,dM_r>, \quad \text{for any} \quad t < T;
\]

\[
\lim_{t \to \infty} e^{\int_0^t d(u)du}Y_t = 0, \quad P - a.s. \quad (6.8)
\]
Proof.

Existence:
Set \( \hat{f}(t,y) = e^{\int_0^t d(u)du} f(t, e^{-\int_0^t d(u)du} y) - d(t)y \). Then

1. \((y - y')(\hat{f}(t,y) - \hat{f}(t,y')) \leq 0\);
2. \(\hat{f}(t,0) = e^{\int_0^t d(u)du} f(t,0)\).

So \( E[\int_0^T |\hat{f}(s,0)|ds] = E[\int_0^\infty e^{\int_0^t d(u)du} f(t,0)ds] < \infty \).

Therefore, \( \hat{f} \) satisfies all the conditions of the Lemma 6.3. So there exists a pair of processes \((\hat{Y}, \hat{Z})\) satisfying the equation:

\[
\hat{Y}_t = \hat{Y}_T + \int_t^T \hat{f}(r, \hat{Y}_r)dr - \int_t^T \langle \hat{Z}_r, dM_r \rangle,
\]

and obviously \( \lim_{t \to \infty} \hat{Y}_t = 0 \).

By the chain rule and the definition of the function \( \hat{f} \), it follows that

\[
d e^{-\int_0^t d(u)du} \hat{Y}_t = -f(t, e^{-\int_0^t d(u)du} \hat{Y}_t) dt + \langle e^{-\int_0^t d(u)du} \hat{Z}_t, dM_t \rangle.
\]

Set \( Y_t = e^{-\int_0^t d(u)du} \hat{Y}_t \) and \( Z_t = e^{-\int_0^t d(u)du} \hat{Z}_t \). Then the process \((Y, Z)\) is the solution to the equation (6.8).

Uniqueness:
The uniqueness of the solution to (6.8) follows from the uniqueness of the solution to equation (6.3). □

Let \( G(x, y) : R^d \times R \to R \) be a bounded Borel measurable function. Consider the following conditions:

(H.1)' \( (y_1 - y_2)(G(x, y_1, z) - G(x, y_2, z)) \leq -h_1(x)\|y_1 - y_2\|^2 \), where \( h_1 \in L^p(D) \) for \( p > \frac{d}{2} \).

(H.2)' \( y \to G(x, y) \) is continuous.

**Theorem 6.2** Assume the Conditions (H.1)' and (H.2)' and that there is some point \( x_0 \in D \), such that

\[
E_{x_0} \left[ \int_0^\infty e^{\int_0^s q(X(u))du}dL_s \right] < \infty.
\]

Then the semilinear Neumann boundary value problem

\[
\begin{cases}
L_2u(x) = -G(x, u(x)), & \text{on } D \\
\frac{\partial u}{\partial n}(x) = \phi(x), & \text{on } \partial D
\end{cases}
\]

has a unique continuous weak solution.
Moreover, the condition (I.4) is also satisfied. In fact, for any $t < T$, obviously condition (I.3) is satisfied, i.e., $y$. Furthermore, as the function $G$ is bounded, we see that condition (I.2) is satisfied:

$$
E_x \left[ \int_0^\infty e^{-\int_0^s h_1(X(u))du} g(X(s), 0) ds \right] < \infty.
$$

Furthermore, as the function $G$ is bounded, we see that condition (I.2) is satisfied:

$$
E_x \left[ \int_0^\infty e^{-\int_0^s h_1(X(u))du} |g(X(s), 0)| ds \right] < \infty.
$$

Obviously condition (I.3) is satisfied, i.e., $y \rightarrow g(x, y)$ is continuous. Moreover, the condition (I.4) is also satisfied. In fact, for any $x$, $y$ we have

$$
\psi_x(t) = \sup_y |G(X(t), y) - G(X(t), 0)| \leq 2 \|G\|_\infty e^{\int_0^t q(X_u)du}.
$$

Proof.

Step 1

Set $G(X(t), y) = e^{\int_0^t q(X(u))du} G(x, e^{-\int_0^t q(X(u))du} y)$. Then there exists a unique solution $(\hat{Y}_x, \hat{Z}_x)$ to the following BSDE:

for any $T > 0$ and $0 < t < T$,

$$
\hat{Y}_x(t) = \hat{Y}_x(T) + \int_t^T \tilde{G}(X(t), \hat{Y}_x(s)) ds - \int_t^T e^{\int_0^s q(X(u))du} \phi(X(s)) dL_s
$$

and

$$
\lim_{t \rightarrow \infty} e^{-\int_0^t h_1(X(u))du} \hat{Y}_x(t) = 0 \quad P_x - a.s.
$$

The uniqueness follows from the uniqueness proved in Theorem 6.1. Only the existence of solution $(\hat{Y}_x, \hat{Z}_x)$ needs to be proved:

(a) Similarly as the proof of Theorem 2.1, we can show that there exists $(p_x(t), q_x(t))$ such that

$$
dp_x(t) = e^{\int_0^t q(X(u))du} \phi(X(t)) dL_t + q_x(t), dM_x(t),
$$

$$
e^{-\int_0^t h_1(X(u))du} p_x(t) \rightarrow 0, \quad as \quad t \rightarrow \infty, \quad P_x - a.s. \quad (6.11)
$$

(b) Set $g(x, y) = \tilde{G}(x, y + p_x(t))$. Then it follows that

$$
(y - y')(g(x, y) - g(x, y')) \leq -h_1(x)|y - y'|^2.
$$

The condition (6.9) and Lemma 3.3 imply, for $x \in D$,

$$
E_x \left[ \int_0^\infty e^{\int_0^s (-h_1 + q)(X(u))du} ds \right] < \infty.
$$

Furthermore, as the function $G$ is bounded, we see that condition (I.2) is satisfied:

$$
E_x \left[ \int_0^\infty e^{-\int_0^s h_1(X(u))du} |g(X(s), 0)| ds \right] < \infty.
$$

Moreover, for any $r > 0$, we have

$$
\psi_r(t) = \sup_r |\tilde{G}(X(t), r) - \tilde{G}(X(t), 0)| \leq 2 \|G\|_\infty e^{\int_0^t q(X_u)du}.
$$

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and for any $T > 0$, by the fact that $q \in L^p(D)$ with $p > \frac{d}{2}$ and Theorem 2.1 in [15], 
$E_x \left[ \int_0^T e^{\int_0^t q(X_u)du} dt \right] < \infty$.
Therefore, the function $g(x,y)$ satisfies all of the conditions of Theorem 6.1. There exists a pair of processes $(y_x(t), z_x(t))$ such that for any $T > 0$ and $0 < t < T$,

$$y_x(t) = y_x(T) + \int_t^T g(X(s), y_x(s))ds - \int_t^T \langle z_x(s), dM_x(s) \rangle$$  \hspace{1cm} (6.13)

and

$$\lim_{t \to \infty} e^{-\int_0^t h_1(X(u))du} y_x(t) = 0 \quad P_x - a.s.$$  \hspace{1cm} (6.14)

Put $\hat{Y}_x(t) = p_x(t) + y_x(t)$ and $\hat{Z}_x(t) = q_x(t) + z_x(t)$. It follows that $(\hat{Y}_x(t), \hat{Z}_x(t))$ satisfies the following equation

$$d\hat{Y}_x(t) = e^{\int_0^t q(X(u))du} \phi(X(t))dL_t - \hat{G}(t, \hat{Y}_x(t))dt + \langle \hat{Z}_x(t), dM_x(t) \rangle,$$

$$\lim_{t \to \infty} e^{-\int_0^t h_1(X(u))du} \hat{Y}_x = 0 \quad P_x - a.s.$$  \hspace{1cm} (6.15)

Step 2.
Put $Y_x(t) := e^{-\int_0^t q(X(u))du} \hat{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))du} \hat{Z}_x(t)$, we have

$$dY_x(t) = -F(X(t), Y_x(t)) + \phi(X(t))dL_t + \langle Z_x(t), dM_x(t) \rangle,$$

where $F(x, y) = q(x)y + G(x, y)$. Moreover,

$$e^{\int_0^t (-h_1+q)(X(u))du} Y_x(t) = e^{\int_0^t (-h_1+q)(X(u))du} e^{-\int_0^t q(X(u))du} \hat{Y}_x(t)$$

$$= e^{-\int_0^t h_1(X(u))du} \hat{Y}_x(t) \to 0 \quad as \quad t \to \infty.$$

Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.
Now as in the proof of Theorem 4.1, we can solve the following equation

$$\begin{cases}
L_2 u(x) = -G(x, u_0(x)), & \text{on } D \\
\frac{1}{2} \nabla u(x) = \phi(x), & \text{on } \partial D
\end{cases} \hspace{1cm} (6.15)$$

and prove that the solution $u$ coincides with $u_0(x)$. This completes the proof of the theorem. \(\square\)

Suppose that $F : R^d \times R \to R$ is a bounded measurable function and $r_1 \in L^p(D)$. Consider the following conditions:

**E.1** $(y_1 - y_2)\langle G(x, y_1, z) - G(x, y_2, z) \rangle \leq -r_1(x)|y_1 - y_2|^2$;

**E.3** $y \to F(x, y)$ is continuous;

Now, after establishing Theorem 6.2, following the same proof as that of Theorem 5.1, we finally have the following main result:
Theorem 6.3  Suppose that the function $F$ satisfies the condition (E.1) and (E.2), and there exists $x_0 \in D$ such that

$$E_{x_0}^0 \left[ \int_0^\infty \hat{Z}_s \, dL_t^0 \right] < \infty. \quad (6.16)$$

Then the following problem

$$\begin{cases} 
Lu(x) = -F(x,u(x)), & \text{on } D \\
\frac{1}{2} \frac{\partial u}{\partial \gamma}(x) - \langle \hat{B}, n \rangle (x) u(x) = \Phi(x) & \text{on } \partial D 
\end{cases} \quad (6.17)$$

has a unique, bounded, continuous weak solution.
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