INEQUALITIES FOR FREE MULTI-BRAID ARRANGEMENTS

MICHAEL DIPASQUALE

Abstract. We prove that, on a large cone containing the constant multiplicities, the only free multiplicities on the braid arrangement are those identified in work of Abe, Nuida, and Numata (2009). We also give a conjecture on the structure of all free multiplicities on braid arrangements.

1. Introduction

Let $V \cong \mathbb{K}^{\ell+1}$ be a vector space over a field $\mathbb{K}$ of characteristic zero, $V^*$ its dual space and $S = \text{Sym}(V^*) \cong \mathbb{K}[x_0, \ldots, x_\ell]$. Given a polynomial $f \in S$ denote by $V(f)$ the zero-locus of $f$ in $V$. The braid arrangement of type $A_\ell \subset V$ is defined as $A_\ell = \bigcup_{0 \leq i < j \leq \ell} H_{ij}$, where $H_{ij} = V(x_i - x_j)$. A multiplicity on $A_\ell$ is a map $\mathbf{m} : \{H_{ij}\} \to \mathbb{Z}_{>0}$; we will set $m_{ij} = \mathbf{m}(H_{ij})$. The pair $(A_\ell, \mathbf{m})$ is called a multi-arrangement. The multi-arrangement $(A_\ell, \mathbf{m})$ is free if the corresponding module $D(A_\ell, \mathbf{m})$ of multi-derivations (i.e., vector fields tangent to $A_\ell$ with multiplicities prescribed by $\mathbf{m}$) is a free module over the polynomial ring $\mathbb{K}[x_0, \ldots, x_\ell]$.

Free multiplicities on braid arrangements have been studied since the introduction of the module of logarithmic differentials by Saito [12], largely due to their importance in the theory of Coxeter arrangements and later in connection with a conjecture of Athanasiadis [6]. Terao made a major breakthrough in [14], showing that the constant multiplicity on any Coxeter arrangement is free and determining the corresponding exponents. Subsequently, many authors studied freeness of ‘almost-constant’ multiplicities on Coxeter and braid arrangements [13, 14, 16, 5].

In the setting of the braid arrangement, this line of inquiry resulted in a paper of Abe-Nuida-Numata [2], where the authors classify what we shall call ANN multiplicities. Given non-negative integers $n_0, \ldots, n_\ell$ and integers $\epsilon_{ij} \in \{-1, 0, 1\}$ for all $0 \leq i < j \leq \ell$, an ANN multiplicity is a multiplicity satisfying

1. $m_{ij} = n_i + n_j + \epsilon_{ij}$ and
2. $m_{ij} \leq m_{ik} + m_{jk} + 1$ for every triple $i, j, k$.

In [2] ANN multiplicities are classified as free if and only if a corresponding signed graph is signed-eliminable; we will describe this precisely in Section 5. We shall refer to the set of multiplicities satisfying the inequalities in (2) as the balanced cone of multiplicities. The reason for this name will be explained in Section 3.

In this note we prove that a multiplicity in the balanced cone is free if and only if it is a free ANN multiplicity. This partially generalizes the recent classification of all free multiplicities on the $A_3$ braid arrangement [8], which is joint work of the author with Francisco, Mermin, and Schweig. To state our result more concretely we shall associate to the multi-braid arrangement $(A_\ell, \mathbf{m})$ an edge-labeled complete graph $(K_{\ell+1}, \mathbf{m})$. The vertices of $K_{\ell+1}$ are labeled in bijection with the variables $x_0, \ldots, x_\ell \in S$. An edge $\{v_i, v_j\}$ corresponds to $H_{ij} = V(x_i - x_j)$ and is furthermore...
labeled by \( m(H_{ij}) = m_{ij} \). Now suppose \( C \) is a four-cycle in \( K_{\ell+1} \) which traverses the vertices \( v_i, v_j, v_s, v_t \) in order. Define \( m(C) = |m_{ij} - m_{js} + m_{st} - m_{ti}| \) since we take absolute value, \( m(C) \) is independent of orientation, depending only on the four cycle and the multiplicity. Let \( C_4(K_{\ell+1}) \) be the set of all four cycles of \( K_{\ell+1} \). Given a subset \( U \subset \{v_0, \ldots, v_\ell\} \) of size at least four, the deviation of \( m \) over \( U \) is

\[
DV(m_U) = \sum_{C \in C_4(K_{\ell+1})} m(C)^2.
\]

Our main result is the following.

**Theorem 1.1.** Suppose \((A_\ell, m)\) is a multi-braid arrangement with \( m \) in the balanced cone of multiplicities. For a subset \( U \subset \{v_0, \ldots, v_\ell\} \) let \( m_U = \{m_{ij} | \{v_i, v_j\} \subset U\} \) and denote by \( q_U \) the number of integers \( \{m_{ij} + m_{ik} + m_{jk} | \{v_i, v_j, v_k\} \subset U\} \) that are odd. Then the following are equivalent.

1. \((A_\ell, m)\) is free
2. \( DV(m_U) \leq q_U(|U| - 1) \) for every subset \( U \subset \{v_0, \ldots, v_\ell\} \) where \(|U| \geq 4\).
3. \( m \) is a free ANN multiplicity. In other words, there exist non-negative integers \( n_0, \ldots, n_\ell \) and \( \epsilon_{ij} \in \{-1, 0, 1\} \) (for \( 0 \leq i < j \leq \ell \)) so that
   a. \( m_{ij} = n_i + n_j + \epsilon_{ij} \)
   b. the signed graph \( G \) on \( \{v_0, \ldots, v_\ell\} \) with \( E^-_G = \{(v_i, v_j) : \epsilon_{ij} < 0\} \), \( E^+_G = \{(v_i, v_j) : \epsilon_{ij} > 0\} \) is signed-eliminable in the sense of [2].

**Remark 1.2.** Notice that \( DV(m_U) = 0 \) if and only if \( m_{ij} - m_{js} + m_{st} - m_{ti} = 0 \) for every four-tuple \( (v_i, v_j, v_s, v_t) \) of distinct vertices in \( U \). These equations cut out the linear space \( L_U \) parametrized by \( m_{ij} = n_i + n_j \) for \( \{v_i, v_j\} \subset U \). Thus \( DV(m_U) \) can be viewed as a measure of how far \( m_U \) is from the linear space \( L_U \); which in turn measures how far \( m_U \) ‘deviates’ from being an ANN multiplicity on the sub-braid arrangement corresponding to \( U \). This is why we call it the deviation of \( m \) over \( U \).

The implication \( (3) \implies (1) \) in Theorem 1.1 is the result of Abe-Nuida-Numata [2].

The quantity \( DV(m_U) \) in Theorem 1.1(2) arises from studying the local and global mixed products (introduced in [3]) of \((A_\ell, m)\). The main point of our note is to show that these ‘deviations’ not only detect freeness in the balanced cone but also interact well with the notion of signed-eliminable graphs.

The structure of the paper is as follows. In Section 2 we give some background on arrangements. The proof of Theorem 1.1 is split across sections 3 and 4. The implication \( (1) \implies (2) \) is proved in Theorem 3.3. We split the proof of \( (2) \implies (3) \) into two parts. The first part, establishing that \( m \) is an ANN multiplicity if the inequalities in (2) are satisfied, is Proposition 4.2. The second part, showing that the inequalities in (2) detect when the associated signed graph is not signed-eliminable, is Proposition 5.4. The final implication \( (3) \implies (1) \) is proved in [2].

We finish in Section 6 by introducing the notion of a free vertex and presenting a conjecture about the structure of all free multiplicities on braid arrangements.

1.1. **Examples.** We provide some computations using Theorem 1.1. For the braid arrangement \( A_\ell \), corresponding to the complete graph \( K_{\ell+1} \), we label the vertices of \( K_{\ell+1} \) by \( v_0, \ldots, v_\ell \) and, given a multiplicity \( m \), we denote by \( m_{ij} \) the value of \( m \) on the hyperplane \( H_{ij} = V(x_i - x_j) \). If \( U \subset \{v_0, \ldots, v_\ell\} \), we denote by \( A_U \) the corresponding sub-braid arrangement of \( A_\ell \).
**Example 1.3.** First we consider a family of multiplicities on the $A_3$ arrangement. Given positive integers $s, t$, define the multiplicity $m_{s,t}^3$ by $m_{01} = m_{12} = m_{23} = s$ and $m_{02} = m_{03} = m_{12} = t$ (this assigns different multiplicities along two edge-disjoint paths of length three).

The multiplicity $m_{s,t}^3$ is in the balanced cone of multiplicities if and only if $s \leq 2t + 1$ and $t \leq 2s + 1$. Assuming $m_{s,t}^3$ is in the balanced cone of multiplicities, we now compute the deviation $DV(m_{s,t}^3)$. There are three four cycles: one of these has $m(C) = |2s - 2t|$ while the other two have $m(C) = |s - t|$. Hence $DV(m_{s,t}^3) = 6(s - t)^2$.

Now consider the sums $m_{ijk}$ around three cycles. There are four such sums, two of the form $2s + t$ and two of the form $2t + s$. So, applying Theorem 1.1 if $m_{s,t}^3$ is in the balanced cone of multiplicities, it is free if and only if $6(s - t)^2 < 4 \cdot 3$, or $|s - t| \leq 1$. In fact, using the classification from [8], it follows that $m_{s,t}^3$ is a free multiplicity if and only if $|s - t| \leq 1$ (regardless of whether $m_{s,t}^3$ is in the balanced cone or not).

**Example 1.4.** Next we consider a similar family of multiplicities on the $A_4$ braid arrangement. Let $C_1$ be the five-cycle traversing the vertices $v_0, v_1, v_2, v_3, v_4$ in order and $C_2$ be the five-cycle traversing the vertices $v_0, v_2, v_1, v_3, v_6$ in order; $C_1$ and $C_2$ are edge-disjoint and every edge of $K_5$ is contained in either $C_1$ or $C_2$. Given positive integers $s, t$ we define the multiplicity $m_{s,t}^4$ by $m_{s,t}^4|_{C_1} = s$ and $m_{s,t}^4|_{C_2} = t$.

Any closed sub-arrangement of $(A_4, m_{s,t}^4)$ of rank three has the form $(A_3, m_{s,t}^3)$ considered in Example 1.3. It follows that $(A_4, m_{s,t}^4)$ is not free if $|s - t| > 1$. So we consider the case when $|s - t| \leq 1$. If $|s - t| \leq 1$ then $m_{s,t}^4$ is in the balanced cone of multiplicities. We compute $DV(m_{s,t}^4)$ as follows. A four-cycle of $K_5$ lies in a unique complete sub-graph on four vertices and each complete sub-graph on four vertices contains three such four-cycles. As we saw in Example 1.3 one of these satisfies $m(C) = |2s - 2t|$ while the other two have $m(C) = |s - t|$. So the contribution to $DV(m_{s,t}^4)$ from each complete sub-graph on four vertices is $6(s - t)^2$. As there are five such sub-graphs, we have $DV(m_{s,t}^4) = 30(s - t)^2$.

Now we consider the sums $m_{ijk}$ around three cycles. There are ten such sums, five of the form $2s + t$ and five of the form $2t + s$. If $|s - t| = 1$, then exactly one of $s, t$ is odd so there are precisely five sums around three cycles that are odd. Hence $DV(m_{s,t}^4) = 30 > 4 \cdot 5$ and $m_{s,t}^4$ is not free by Theorem 1.1. So we conclude that $m_{s,t}^4$ is free if and only if $|s - t| = 1$.

We also consider why $m_{s,t}^4$ is not free when $|s - t| = 1$ using the criterion of Abe-Nuida-Numata (which is the third statement of Theorem 1.1). Without loss, suppose $t = s + 1$ and let $n_i = \lfloor s/2 \rfloor$ for $i = 0, 1, 2, 3, 4$. If $s$ is even then $m_{ij} = n_i + n_j = s$ for $\{i,j\} \subset C_1$ while $m_{ij} = n_i + n_j + 1 = s + 1$ for $\{i,j\} \subset C_2$. In this case the graph $G$ on the vertices $v_0, v_1, v_2, v_3, v_4$ is the (positive) five-cycle given by $C_2$ and is hence not signed-eliminable by the characterization in [2] (see also Corollary 5.3). Similarly, if $s$ is odd, then $G$ is the negatively signed five-cycle $C_1$.

**Example 1.5.** The following example shows that criterion (2) in Theorem 1.1 really does need to be checked on all proper subsets of size at least four. Consider the $A_4$ arrangement with the multiplicity $m$ defined by $m_{01} = m_{02} = m_{03} = m_{12} =
There are four odd sums around three cycles (so in the notation of Theorem 1.1, \( q = 4 \)). Also, we compute \( \text{DV}(\mathbf{m}) = 16 \). From Theorem 1.1 we cannot conclude that \( (A_4, \mathbf{m}) \) is not free since \( q \ell = 16 \) also in this case. However, let us consider the \( A_4 \) sub-arrangement \( A_U \) where \( U = \{v_0, v_1, v_3, v_4\} \). Let \( \mathbf{m}_U \) be the restricted multiplicity; it also lies in the balanced cone of multiplicities on \( A_U \). All sums around three-cycles are even, and \( \text{DV}(\mathbf{m}_U) = 8 \). Since \( 8 > 0 \), it follows from Theorem 1.1 that \( (A_U, \mathbf{m}_U) \) is not free, hence \( (A_4, \mathbf{m}) \) is also not free.

2. Notation and preliminaries

Let \( V = \mathbb{K}^\ell \) be a vector space over a field \( \mathbb{K} \) of characteristic zero. A central hyperplane arrangement \( \mathcal{A} = \bigcup_{i=1}^k H_i \) is a union of hyperplanes \( H_i \subset V \) passing through the origin in \( V \). In other words, if we let \( \{x_1, \ldots, x_\ell\} \) be a basis for the dual space \( V^* \) and \( S = \text{Sym}(V^*) \cong \mathbb{K}[x_1, \ldots, x_\ell] \), then \( H_i = V(\alpha_{H_i}) \) for some choice of linear form \( \alpha_{H_i} \in V^* \), unique up to scaling. We will use the language of graphic arrangements for referring to the braid arrangement \( \mathcal{A}_\ell \) and its sub-arrangements. Namely, suppose \( G = (V_G, E_G) \) is a graph with vertices ordered as \( V_G = \{v_0, v_1, \ldots, v_\ell\} \), and let \( S = \mathbb{K}[x_0, \ldots, x_\ell] \). If \( \{v_i, v_j\} \) is an edge in \( E_G \) then let \( H_{ij} = V(x_i - x_j) \). The graphic arrangement associated to \( G \) is \( \mathcal{A}_G = \bigcup_{(i,j) \in E_G} H_{ij} \). Clearly \( \mathcal{A}_G \) is a sub-arrangement of the full braid arrangement \( \mathcal{A}_\ell \), which may be identified with the graphic arrangement corresponding to the complete graph \( K_{\ell+1} \) on \( (\ell + 1) \) vertices.

A multi-arrangement is a pair \((\mathcal{A}, \mathbf{m})\) of a central arrangement \( \mathcal{A} = \bigcup_{i=1}^k H_i \) and a map \( \mathbf{m} : \{H_1, \ldots, H_k\} \to \mathbb{Z}_{\geq 0} \), called a multiplicity. If \( \mathbf{m} \equiv 1 \), then \((\mathcal{A}, \mathbf{m})\) is denoted \( \mathcal{A} \) and is called a simple arrangement. If \( \mathcal{A}_G \) is a graphic arrangement then the multi-arrangement \((\mathcal{A}_G, \mathbf{m})\) is equivalent to the information of the edge-labeled graph \((G, \mathbf{m})\), where \( \{v_i, v_j\} \) is labeled by \( \mathbf{m}(H_{ij}) = m_{ij} \). We will frequently move back and forth between these notations. We will always assume that a graph \( G \) comes with some ordering \( V_G = \{v_0, v_1, \ldots, v_\ell\} \) of its vertices and may refer to the vertices simply by their integer labels \( \{0, \ldots, \ell\} \).

The module of derivations on \( S \) is defined by \( \text{Der}_S(S) = \bigoplus_{i=1}^\ell S \partial_{x_i} \), the free \( S \)-module with basis \( \partial_{x_i} = \partial/\partial x_i \) for \( i = 1, \ldots, \ell \). The module \( \text{Der}_S(S) \) acts on \( S \) by partial differentiation. Given a multi-arrangement \((\mathcal{A}, \mathbf{m})\), our main object of study is the module \( D(\mathcal{A}, \mathbf{m}) \) of logarithmic derivations of \((\mathcal{A}, \mathbf{m})\):

\[
D(\mathcal{A}, \mathbf{m}) := \{ \theta \in \text{Der}_S(S) : \theta(\alpha_H) \in \langle \alpha_H^{m_H} \rangle \text{ for all } H \in \mathcal{A} \},
\]

where \( \langle \alpha_H^{m_H} \rangle \subset S \) is the ideal generated by \( \alpha_H^{m_H} \). If \( D(\mathcal{A}, \mathbf{m}) \) is a free \( S \)-module, then we say \((\mathcal{A}, \mathbf{m})\) is free or \( \mathbf{m} \) is a free multiplicity of the simple arrangement \( \mathcal{A} \). For a simple arrangement, \( D(\mathcal{A}, \mathbf{m}) \) is denoted \( D(\mathcal{A}) \); if \( D(\mathcal{A}) \) is free we say \( \mathcal{A} \) is free.

The intersection lattice of \( \mathcal{A} \) is the ranked poset \( L = L(\mathcal{A}) \) consisting of all intersections of hyperplanes of \( \mathcal{A} \) ordered with respect to reverse inclusion (the vector space \( V \) is included as the ‘empty’ intersection). We denote by \( L_k \) the intersections of rank \( k \), where the rank of an intersection is its codimension. If \( X \in L_k \), \( \mathcal{A}_X \) denotes the sub-arrangement consisting of hyperplanes which contain \( X \), \( L_X \) denotes the lattice of \( \mathcal{A}_X \), and \( \mathbf{m}_X \) denotes the multiplicity function restricted to hyperplanes containing \( X \). If \( \mathcal{A}_G \) is a graphic arrangement with lattice \( L \) and \( H \subset G \)
is a connected induced sub-graph of $G$ on $(k + 1)$ vertices, then $H$ corresponds to an intersection $X(H) \in L_k$, and the graphic arrangement $A_H$ is the same as $(A_G)_{X(H)}$. In this setting, if $m$ is a multiplicity on $A_G$, we denote by $m_H$ the restriction of $m$ to the sub-arrangement $A_H$.

**Proposition 2.1.** [3 Proposition 1.7] If $(A, m)$ is a free multi-arrangement, then so is $(A_X, m_X)$.

If $D(A, m)$ is free then it has $\ell$ minimal generators as an $S$-module whose degrees are an invariant of $D(A, m)$. These degrees are called the exponents of $(A, m)$ and we will list them as a non-increasing sequence $(d_1, \ldots, d_\ell)$. Put $|m| = \sum_{H \in \ell} m(H)$. Then $\sum_{i=1}^\ell d_i = |m|$ (this follows for instance by an extension of Saito’s criterion to multi-arrangements [17]). For a free multi-arrangement, define the $k$th **global mixed product** by

$$GMP(k) = \sum d_1^{d_1} d_2^{d_2} \cdots d_k^{d_k},$$

where the sum runs across all $k$-tuples satisfying $1 \leq i_1 < \cdots < i_k \leq \ell$. Furthermore, define the $k$th **local mixed product** by

$$LMP(k) = \sum_{X \subseteq \ell} d_1^X d_2^X \cdots d_k^X,$$

where $d_1^X, \ldots, d_k^X$ are the (non-zero) exponents of the rank $k$ sub-arrangement $A_X$. We make use of the following result for $k = 2$.

**Theorem 2.2.** [3 Corollary 4.6] If $(A, m)$ is free then $GMP(k) = LMP(k)$ for every $2 \leq k \leq \ell$.

### 3. Deviations and Mixed Products in the Balanced Cone

In this section we study the local and global mixed products of multiplicities in the balanced cone. In particular, we prove the implication $(1) \implies (2)$ of Theorem [11]. Recall the **balanced cone** of multiplicities on a braid arrangement $A_k$ is the set of multiplicities satisfying the three inequalities $m_{ij} + m_{jk} + 1 \geq m_{ik}, m_{ij} + m_{ik} + 1 \geq m_{jk}$, and $m_{ik} + m_{jk} + 1 \geq m_{ij}$ for every triple $0 \leq i < j < k \leq \ell$. The following proposition (due to Wakamiko) explains why we call this the balanced cone; it is because the exponents of every sub-$A_2$ arrangement are as balanced as possible.

**Proposition 3.1.** [15] Suppose $H$ is a three-cycle on the vertices $i, j, k$ of the edge-labeled complete graph $(K_{\ell+1}, m)$. Put $m_{ijk} = m_{ij} + m_{ik} + m_{jk}$. If $m$ is in the balanced cone of multiplicities then the (non-zero) exponents of $(A_H, m_H)$ are $([m_{ijk}/2], [m_{ijk}/2]).$

If $\{i, j, k\}$ are vertices of $K_{\ell+1}$ so that $m_{ij} + m_{ik} + m_{jk}$ is odd then we will call $\{i, j, k\}$ an **odd three-cycle**.

**Proposition 3.2.** Let $(A_\ell, m)$ be a multi-braid arrangement so that $m$ is in the balanced cone of multiplicities. Set $|m| = \sum m_{ij}$ and $m_{ijk} = m_{ij} + m_{jk} + m_{ik}$. If $q$ is the number of odd three cycles of $m$, then

$$LMP(2) = \sum_{0 \leq i < j < k \leq \ell} (m_{ijk}/2)^2 + \sum_{\{i,j\} \cap \{s,t\} = \emptyset} m_{ij}m_{st} - q/4$$
and

\[ G\text{MP}(2) \leq \left( \frac{\ell}{2} \right) \frac{|\bm{m}|^2}{\ell^2}. \]

\textbf{Proof.} We prove the formula for LMP(2) first. If \( X \in L_2 \), then either (1) : \( X = H_{ij} \cap H_{st} \) for a pair of non-adjacent edges \( \{i, j\} \) and \( \{s, t\} \) or (2) : \( X = H_{ij} \cap H_{jk} \cap H_{ik} \) corresponds to a triangle. In the first case the arrangement is boolean with (non-zero) exponents \((m_{ij}, m_{st})\), contributing \(m_{ij}m_{st}\) to LMP(2). In the second case the arrangement is an \( A_2 \) braid arrangement with exponents \((m_{ijk}/2, m_{ijk}/2)\) if \( m_{ijk} \) is even and \((m_{ijk} - 1)/2, (m_{ijk} + 1)/2\) if \( m_{ijk} \) is odd from Proposition 3.1. The former contributes \(m_{ijk}^2/4\) to LMP(2) while the latter contributes \(m_{ijk}^2/4 - 1/4\).

This yields the expression for LMP(2) = \( \sum_{X \in L_2} d_X^2 \).

Now consider the inequality for GMP(2). Supposing \((A_t, \bm{m})\) is free, let \((d_1, \ldots, d_\ell)\) be its (non-zero) exponents, ordered so that \(d_1 \geq \cdots \geq d_\ell\). By an extension of Saito’s criterion to multi-arrangements \([17]\). \( \sum_{i=1}^\ell d_i = |\bm{m}| \). Now, following remarks just after \([3]\ Corollary 4.6\), we say that \((b_1, \ldots, b_\ell)\) with \(b_1 \geq \cdots \geq b_\ell\) and \(\sum b_i = |\bm{m}|\) is ‘more balanced’ than \((d_1, \ldots, d_\ell)\) if \(\sum (b_{i+1} - b_i) \leq \sum (d_{i+1} - d_i)\).

Then

\[ \sum b_1 \cdots b_\ell \geq \sum d_1 \cdots d_\ell = G\text{MP}(k). \]

Let \(|\bm{m}| = k\ell + p\) be the result of dividing \(|\bm{m}|\) by \(\ell\), so \(k\) is a positive integer and \(0 \leq p < \ell\). The ‘most balanced’ distribution of exponents occurs when

\[ b_1 = \cdots = b_p = k + 1 \quad \text{and} \quad b_{p+1} = \cdots = b_\ell = k, \]

so \(\sum (b_{i+1} - b_i) = b_{p+1} - b_p\) is zero if \(p = 0\) and one if \(p > 0\). Some algebra yields that, for this choice of exponents,

\[ \sum_{1 \leq i < j \leq \ell} b_i b_j = \left( \frac{\ell}{2} \right) \frac{|\bm{m}|^2}{\ell^2} - \frac{p(\ell - p)}{2\ell} \leq \left( \frac{\ell}{2} \right) \frac{|\bm{m}|^2}{\ell^2}, \]

which gives the result. \(\square\)

\textbf{Definition 3.3.} Fix a multiplicity \(\bm{m}\) on the braid arrangement \(A_t\) and a subset \(U \subset \{v_0, \ldots, v_\ell\}\). Denote by \(C_4(K_{\ell+1})\) the set of four-cycles in \(K_{\ell+1}\) and by \(C_4(U)\) the set of four-cycles in \(K_{\ell+1}\) whose vertices are contained in \(U\). If \(C \in C_4(K_{\ell+1})\) traverses the vertices \(i,j,s,t\) in order, set \(m(C) = |m_{ij} - m_{js} + m_{st} - m_{it}|\). The deviation of \(\bm{m}\) over \(U\) is the sum of squares

\[ \text{DV}(\bm{m}_U) = \sum_{C \in C_4(U)} m(C)^2. \]

If \(U\) consists of all vertices of \(K_{\ell+1}\), then we write \(\text{DV}(\bm{m})\) instead of \(\text{DV}(\bm{m}_U)\).

\textbf{Theorem 3.4.} Suppose \((A_t, \bm{m})\) is a multi-braid arrangement, \(\bm{m}\) is in the balanced cone of multiplicities, and \(q\) is the number of odd three cycles of \(\bm{m}\). If \(\text{DV}(\bm{m}) > q\ell\), then \(\bm{m}\) is not free.

\textbf{Remark 3.5.} The inequality \(\text{DV}(\bm{m}) > q\ell\) in Theorem [3.4] can be strengthened to \(\text{DV}(\bm{m}) > q\ell - 2p(\ell - p)\), where \(p\) is the remainder of \(|\bm{m}|\) on division by \(\ell\). We will see that the simpler inequality \(\text{DV}(\bm{m}) > q\ell\) suffices to detect non-freeness.
Proof. By Proposition 3.2 we know that
\[
\text{LMP}(2) - \text{GMP}(2) \geq \sum_{0 \leq i < j < k \leq \ell} \frac{(m_{ijk})}{2} + \sum_{\{i,j\} \cap \{s,t\}} m_{ij}m_{st} - \left( \frac{\ell}{2} \right) \frac{|m|^2}{\ell^2} - q/4.
\]
Our primary claim is
\[
(1) \quad 4\ell \left( \sum_{0 \leq i < j < k \leq \ell} \frac{(m_{ijk})}{2} + \sum_{\{i,j\} \cap \{s,t\}} m_{ij}m_{st} - \left( \frac{\ell}{2} \right) \frac{|m|^2}{\ell^2} \right) = \text{DV}(m).
\]
Once Equation (1) is proved, notice that
\[
4\ell (\text{LMP}(2) - \text{GMP}(2)) \geq \text{DV}(m) - q\ell.
\]
Then Theorem 2.2 immediately yields Theorem 3.4. So we prove Equation (1). We first consider the right hand side, namely the sum \(\text{DV}(m)\). Since every edge of \(K_{\ell+1}\) is contained in \(2\binom{\ell - 1}{2}\) four-cycles, every pair of disjoint edges is contained in two four-cycles, and every pair of adjacent edges is contained in \((\ell - 2)\) four-cycles,

\[
(2) \quad \text{DV}(m) = 2\binom{\ell - 1}{2} \sum_{0 \leq i < j < k \leq \ell} m_{ij}^2 + 4 \sum_{\{i,j\} \cap \{s,t\}} m_{ij}m_{st} - 2(\ell - 2) \sum_{0 \leq i < j < k \leq \ell} (m_{ijk} + m_{imj} + m_{ikm}).
\]

Now consider the left hand side of Equation (1). Distributing \(4\ell\) yields
\[
\ell \sum_{0 \leq i < j < k \leq \ell} (m_{ijk})^2 + 4\ell \sum_{\{i,j\} \cap \{s,t\}} m_{ij}m_{st} - 2(\ell - 1)|m|^2.
\]
This expression can be re-written in the form of Equation (2) using the following two expressions and simplifying:

\[
|m|^2 = \sum_{0 \leq i < j \leq \ell} m_{ij}^2 + 2 \sum_{\{i,j\} \cap \{s,t\}} m_{ij}m_{st} + \sum_{0 \leq i < j \leq k \leq \ell} (m_{ijk} + m_{imj} + m_{ikm}) \quad + 2 \sum_{0 \leq i < j < k \leq \ell} (m_{ijk} + m_{imj} + m_{ikm}).
\]

As an immediate corollary we obtain (1) \(\Longleftrightarrow\) (2) in Theorem 1.1.

**Corollary 3.6.** If \((A_{\ell}, m)\) is free then \(\text{DV}(m_U) \leq q_U(|U| - 1)\) for every subset \(U \subset \{v_0, \ldots, v_\ell\}\).

Proof. By Proposition 2.1 freeness of \((A_{\ell}, m)\) implies freeness of \((A_U, m_U)\) for every subset \(U \subset \{v_0, \ldots, v_\ell\}\). By Theorem 3.4 we must have \(\text{DV}(m_U) \leq q_U(|U| - 1)\) for every subset \(U \subset \{v_0, \ldots, v_\ell\}\) as well. \(\Box\)
4. From deviations to ANN multiplicities

In this section we prove the first part of the implication $(2) \implies (3)$ in Theorem \ref{thm:main}. Namely, we prove that for a multiplicity $m$ in the central cone, the inequalities $DV(m) \leq qv(|U| - 1)$ on deviations are enough to guarantee that $m$ is an ANN multiplicity. In fact, we show that it is enough to have these inequalities on subsets of size four.

Recall from the introduction that we call $m$ an ANN multiplicity on $A_4$ if $m$ is in the balanced cone of multiplicities and there exist non-negative integers $n_0, \ldots, n_\ell$ and $\epsilon_{ij} \in \{-1, 0, 1\}$ so that $m_{ij} = n_i + n_j + \epsilon_{ij}$ for $0 \leq i < j \leq \ell$.

**Lemma 4.1.** Suppose $m$ is a multiplicity on $A_3$ and $DV(m) \leq 3q$. Then $m(C) \leq 2$ for each four-cycle $C$ in $K_4$.

**Proof.** There are three four-cycles. Set
\[
T_1 = m_{01} - m_{12} + m_{23} - m_{03},
T_2 = m_{13} - m_{01} + m_{02} - m_{23},
T_3 = m_{13} - m_{12} + m_{02} - m_{03}.
\]
Notice $T_1 + T_2 = T_3$, and $DV(m) = T_1^2 + T_2^2 + T_3^2$. Now, suppose without loss that $|T_3| \geq 3$. Then either $|T_1| \geq 2$ or $|T_2| \geq 2$. But then $P(m) \geq 13$, contradicting that $DV(m) \leq 3q \leq 12$ (since $q \leq 4$). \hfill \Box

**Proposition 4.2.** Let $(A_4, m)$ be a multi-braid arrangement so that $m$ is in the balanced cone of multiplicities and $DV(m_U) \leq 3qU$ for every subset $U \subset \{v_0, \ldots, v_\ell\}$ with $|U| = 4$. Then $m$ is an ANN multiplicity.

**Proof.** We need only show that there exist non-negative integers $n_i$ for $i = 0, \ldots, \ell$ and integers $\epsilon_{ij} \in \{-1, 0, 1\}$ for $0 \leq i < j \leq \ell$ so that $m_{ij} = n_i + n_j + \epsilon_{ij}$. By Lemma 4.1, we must have $m(C) \leq 2$ for every four-cycle $C \in C_4(K_{\ell+1})$. We use this condition to provide an inductive algorithm producing the integers $n_0, \ldots, n_\ell$.

If $\ell = 2$, set $n_0 = \left\lceil \frac{m_{01} + m_{02} - m_{12}}{2} \right\rceil$, $n_1 = \left\lceil \frac{m_{01} + m_{12} - m_{02}}{2} \right\rceil$, and $n_2 = \left\lceil \frac{m_{02} + m_{12} - m_{01}}{2} \right\rceil$. Since $m$ is in the balanced cone, $n_i \geq 0$ for $i = 0, 1, 2$. Moreover, $m_{ij} = n_i + n_j + \epsilon_{ij}$, where $\epsilon_{ij} \in \{-1, 0\}$.

Now assume $\ell > 2$. We make an initial guess at what the non-negative integers $n_0, \ldots, n_\ell$ and $\epsilon_{ij}$ should be, and then adjust as necessary. By induction on $\ell$, there exist non-negative integers $\hat{n}_0, \ldots, \hat{n}_{\ell-1}$ and $\hat{\epsilon}_{ij} \in \{-1, 0, 1\}$ such that $m_{ij} = \hat{n}_i + \hat{n}_j + \hat{\epsilon}_{ij}$ for $0 \leq i < j \leq \ell - 1$. Let $\tilde{n}_\ell$ be a non-negative integer satisfying $\hat{n}_i + \hat{n}_j \geq m_{i\ell} - 1$ and set $\tilde{\epsilon}_{i\ell} = m_{i\ell} - (\hat{n}_i + \hat{n}_\ell)$ for every $i < \ell$, so $m_{i\ell} = \hat{n}_i + \hat{n}_\ell + \tilde{\epsilon}_{i\ell}$. By the choice of $\tilde{n}_\ell$, we have $\tilde{\epsilon}_{i\ell} \leq 1$ for all $i < \ell$.

Now suppose there is an index $0 \leq j < \ell$ so that $\tilde{\epsilon}_{j\ell} \leq -2$. Our goal is to decrease either $\hat{n}_\ell$ or $\hat{n}_j$ by one, thereby increasing $\tilde{\epsilon}_{j\ell}$, without disturbing any of the hypotheses made so far, namely
\[
\hat{n}_i \geq 0 \quad \text{for all} \ 0 \leq i \leq \ell,
\]
\[
\tilde{\epsilon}_{i\ell} \leq 1 \quad \text{for all} \ 0 < i < \ell,
\]
\[
\epsilon_{st} \in \{-1, 0, 1\} \quad \text{for all} \ 0 \leq s < t \leq \ell - 1.
\]

First we assume $\hat{n}_\ell > 0$ and try to decrease $\hat{n}_\ell$ by one. We can do this without disturbing assumptions \(\Box\) provided there is no index $s$ so that $\epsilon_{st} = 1$. So, assume
that there is an index $0 \leq s < \ell$ so that $\epsilon_{st} = 1$. We claim that in this situation, $\epsilon_{st} \geq 0$ for every $t \neq s$. Suppose to the contrary that there is an index $t$ so that $\epsilon_{st} = -1$ and consider the four-cycle $C: \ell \rightarrow s \rightarrow t \rightarrow j \rightarrow \ell$. Then

$$m(C) = |\tilde{\epsilon}_{st} - \tilde{\epsilon}_{jt} + \tilde{\epsilon}_{jt} - \tilde{\epsilon}_{st}|$$

$$\geq 1 + 2 + \tilde{\epsilon}_{jt} + 1$$

$$\geq 3,$$

since $\tilde{\epsilon}_{jt} \in \{-1, 0, 1\}$ by the inductive hypothesis. This contradicts our assumption that $m(C) \leq 2$. So it follows that $\tilde{\epsilon}_{st} \in \{0, 1\}$ for all $t$. Thus we may increase $\tilde{n}_s$ by one, thereby decreasing $\tilde{\epsilon}_{st}$ by one for every $t \neq s$, without disturbing the hypothesis that $\tilde{\epsilon}_{st} \in \{-1, 0, 1\}$. Since we can apply this argument at every index $s$ so that $\tilde{\epsilon}_{st} = 1$, we may assume $\tilde{\epsilon}_{st} \leq 0$ for every $0 \leq s < \ell$. Hence, if $\tilde{n}_\ell > 0$, it is now clear that we can decrease $\tilde{n}_\ell$ by one without disturbing assumptions (3).

Now assume that $\tilde{n}_\ell = 0$. Then, for any $s < \ell$,

$$m_{st} + m_{jt} - m_{js} = (\tilde{n}_s + \tilde{\epsilon}_{st}) + (\tilde{n}_j + \tilde{\epsilon}_{jt}) - (\tilde{n}_j + \tilde{n}_s + \tilde{\epsilon}_{js})$$

$$= \tilde{\epsilon}_{st} + \tilde{\epsilon}_{jt} - \tilde{\epsilon}_{js}$$

$$\leq 0 - 2 - \tilde{\epsilon}_{js}$$

$$\leq -1,$$

since $\tilde{\epsilon}_{js} \in \{-1, 0, 1\}$ by the inductive hypothesis. Since $m$ is in the balanced cone, we must have equality for all of these, so $\epsilon_{js} = -1$ for every $s \neq j, s < \ell$. If $\tilde{n}_j = 0$ as well, then $m_{jt} = \tilde{n}_j + \tilde{n}_\ell + \epsilon_{jt} \leq -2$, contradicting that $m_{jt}$ is non-negative. Hence $\tilde{n}_j > 0$ and we can decrease $\tilde{n}_j$ by one without disturbing any of assumptions (3).

In either case, we have shown how to increase $\tilde{\epsilon}_{jt}$ if $\tilde{\epsilon}_{jt} \leq -2$ without disturbing assumptions (3). So we iterate the above arguments until $\tilde{\epsilon}_{jt} \geq -1$ for every $j < \ell$, then set $n_i = \tilde{n}_i$ for $0 \leq i < \ell$ and $\tilde{\epsilon}_{ij} = \epsilon_{ij}$ for $0 \leq i < j < \ell$. This completes the algorithm and the proof. □

With Proposition 4.2 we now prove (1) $\iff$ (3) in Theorem 1.1. Most of the heavy lifting is done by Abe-Nuida-Numata in [2].

Corollary 4.3. Suppose $m$ is in the balanced cone of multiplicities on the $A_\ell$ braid arrangement. Then $(A_\ell, m)$ is free if and only if $m$ is an ANN multiplicity and the signed graph with $E_G^+ = \{\{v_i, v_j\} : \epsilon_{ij} = -1\}$ and $E_G^- = \{\{v_i, v_j\} : \epsilon_{ij} = 1\}$ is signed-eliminable.

Proof. If $(A_\ell, m)$ is free and $m$ is in the balanced cone, then $\text{DV}(m_U) \leq 3q_U$ for every subset $U \subset \{v_0, \ldots, v_\ell\}$ of size four by Corollary 3.6. By Proposition 4.2, $m$ is an ANN multiplicity. By [2, Theorem 0.3], the signed graph with $E_G^+ = \{\{v_i, v_j\} : \epsilon_{ij} = -1\}$ and $E_G^- = \{\{v_i, v_j\} : \epsilon_{ij} = 1\}$ is signed-eliminable. For the converse, if $m$ is an ANN multiplicity associated to a signed-eliminable graph, then $(A_\ell, m)$ is free by [2, Theorem 0.3]. □

Remark 4.4. In the result [2, Theorem 0.3], Abe-Nuida-Numata do not have the condition that $m$ is in the balanced cone. However, this turns out to be a necessary condition for their arguments [1]. Furthermore their arguments, using addition-deletion techniques for multi-arrangements from [4], work for any ANN multiplicity as we have defined it [1].
5. Detecting signed-eliminable graphs

In this section we finish the proof of Theorem 1.1. We have already shown in Proposition 4.2 that if the inequalities of Theorem 1.1(2) are satisfied then \( m \) is an ANN multiplicity. Now we show that these inequalities also detect when the associated signed graph is not signed-eliminable. We follow the presentation of signed-eliminable graphs from [11, 2].

Let \( G \) be a signed graph on \( \ell + 1 \) vertices. That is, each edge of \( G \) is assigned either a + or a −, and so the edge set \( E_G \) decomposes as a disjoint union \( E_G = E^+_G \cup E^-_G \).

Define

\[
m_G(ij) = \begin{cases} 
1 & \{i, j\} \in E^+_G \\
-1 & \{i, j\} \in E^-_G \\
0 & \text{otherwise}.
\end{cases}
\]

The graph \( G \) is signed-eliminable with signed-elimination ordering \( \nu : V(G) \rightarrow \{0, \ldots, \ell\} \) if \( \nu \) is bijective and, for every three vertices \( v_i, v_j, v_k \in V(G) \) with \( \nu(v_i), \nu(v_j) < \nu(v_k) \), the induced sub-graph \( G|_{v_i, v_j, v_k} \) satisfies the following conditions.

- For \( \sigma \in \{+, -\} \), if \( \{v_i, v_k\} \) and \( \{v_j, v_k\} \) are edges in \( E^\sigma_G \) then \( \{v_i, v_j\} \in E^\sigma_G \)
- For \( \sigma \in \{+, -\} \), if \( \{v_k, v_i\} \in E^\sigma_G \) and \( \{v_i, v_j\} \in E^{-\sigma}_G \) then \( \{v_k, v_j\} \in E_G \)

These two conditions generalize the notion of a graph possessing an elimination ordering, which is equivalent to the graph being chordal. A graph is chordal if and only if it has no induced sub-graph which is a cycle of length at least four. In [11], Nuida establishes a similar characterization for signed-eliminable graphs, to which we now turn.

**Definition 5.1.**

(1) A graph with \( \ell + 1 \) vertices \( v_0, v_1, \ldots, v_\ell \) with \( \ell \geq 3 \) is a \( \sigma \)-mountain, where \( \sigma \in \{+, -\} \), if \( \{v_0, v_i\} \in E^\sigma_G \) for \( i = 2, \ldots, \ell - 1 \), \( \{v_i, v_{i+1}\} \in E^{-\sigma}_G \) for \( i = 1, \ldots, \ell - 1 \), and no other pair of vertices is joined by an edge. (See Figure 1: edges of sign \( \sigma \) are denoted by a single edge and edges of sign \( -\sigma \) are denoted by a doubled edge.)

(2) A graph with \( \ell + 1 \) vertices \( v_0, v_1, v_2, \ldots, v_\ell \) with \( \ell \geq 3 \) is a \( \sigma \)-hill, where \( \sigma \in \{+, -\} \), if \( \{v_0, v_1\} \in E^\sigma_G \), \( \{v_0, v_i\} \in E^\sigma_G \) for \( i = 2, \ldots, \ell - 1 \), \( \{v_1, v_i\} \in E^-_G \) for \( i = 3, \ldots, \ell \), \( \{v_i, v_{i+1}\} \in E^-_G \) for \( i = 2, \ldots, \ell - 1 \), and no other pair of vertices is connected by an edge. (See Figure 1)

(3) A graph with \( \ell + 1 \) vertices \( v_0, \ldots, v_\ell \) with \( \ell \geq 2 \) is a \( \sigma \)-cycle if \( \{v_i, v_{i+1}\} \in E^\sigma_G \) for \( i = 0, \ldots, \ell - 1 \), \( \{v_0, v_\ell\} \in E^\sigma_G \), and no other pair of vertices is connected by an edge.

**Theorem 5.2.** [11, Theorem 5.1] Let \( G \) be a signed graph. Then \( G \) is signed-eliminable if and only if the following three conditions are satisfied.

(C1) Both \( G_+ \) and \( G_- \) are chordal.
(C2) Every induced sub-graph on four vertices is signed eliminable.
(C3) No induced sub-graph of $G$ is a $\sigma$-mountain or a $\sigma$-hill.

All signed-eliminable graphs on four vertices are listed (with an elimination ordering) in [2, Example 2.1], along with those which are not signed-eliminable. For use in the proof of Corollary 5.3, we also list those graphs which are not signed-eliminable in Table 1. The property of being signed-eliminable is preserved under interchanging $+$ and $-$. Consequently, we list these graphs in Table 1 up to automorphism with the convention that a single edge takes one of the signs $+, -$, while a double edge takes the other sign.

Corollary 5.3. Let $G$ be a signed graph. Then $G$ is signed-eliminable if and only if the following three conditions are satisfied.

(C1) No induced sub-graph of $G$ is a $\sigma$-cycle of length $> 3$.
(C2) Every induced sub-graph on four vertices is signed eliminable.
(C3) No induced sub-graph of $G$ is a $\sigma$-mountain or a $\sigma$-hill.

Proof. Clearly (C1) from Theorem 5.2 implies (C1'). We show that (C1') and (C2) imply condition (C1). Assume for contradiction that $E^\sigma_G$, $\sigma \in \{-, +\}$, is a cycle of length $\ell + 1 > 3$ and $V_G = \{v_0, \ldots, v_\ell\}$ where $\{v_i, v_{i+1}\} \in E^\sigma_G$ for $i = 0, \ldots, \ell - 1$ and $\{v_0, v_\ell\} \in E^\sigma_G$. If $E^\sigma_G = \emptyset$ then $G$ is a $\sigma$-cycle which is forbidden by (C1'), so we assume $E^\sigma_G \neq \emptyset$. Let $m$ be the maximal integer such that there is a sequence of consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+m-1}$ so that the induced sub-graph on these consecutive vertices consists only of edges in $E^\sigma_G$. Since $E^\sigma_G \neq \emptyset$, $m < \ell + 1$.

Relabel the vertices so that $v_0, \ldots, v_{m-1}$ are the vertices of a maximal induced sub-graph with edges only in $E^\sigma_G$. If $m = 2$ or $m = 3$, then the induced sub-graph on $v_0, v_1, v_2, v_3$ consists of the three $\sigma$ edges $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$ along with at least one $-\sigma$ edge. No such graph is signed eliminable (see Table 1). So $m \geq 4$.

Now consider the induced sub-graph $H$ on $v_0, v_1, \ldots, v_{m-1}, v_m$. By definition of $m$, $H$ has exactly one $-\sigma$ edge, namely $\{v_0, v_m\}$. But then the induced sub-graph on $v_0, v_1, v_{m-1}, v_m$ consists of the two $\sigma$ edges $\{v_0, v_1\}, \{v_{m-1}, v_m\}$ and the $-\sigma$ edge $\{v_0, v_m\}$, which is not signed-eliminable. It follows that $E^\sigma_G$ cannot have a cycle of length $> 3$, so $E^\sigma_G$ is chordal.

The following proposition proves the implication $(2) \implies (3)$ in Theorem 1.1, thereby completing the proof of Theorem 1.1.

Proposition 5.4. Suppose $n_0, \ldots, n_\ell$ are non-negative integers, $G$ is a signed graph on $v_0, \ldots, v_\ell$, and let $m$ be the multiplicity on $A_\ell$ given by $m_{ij} = n_i + n_j + m_G(ij)$. If $G$ is not signed-eliminable, then there is a subset $U \subset \{0, \ldots, \ell\}$ so that $DV(m_U) > q_U \cdot (|U| - 1)$.
Proof. Notice that, for a four-cycle traversing $i,j,s,t$ in order,
\[ m(C) = |mi_j - m_j s + m_s t - m_t i| = |m_G(ij) - m_G(js) + m_G(st) - m_G(it)|. \]
Furthermore, for a three-cycle $i,j,k$,
\[ m_{ij} + m_{ik} + m_{jk} = 2(n_i + n_j + n_k) + m_G(ij) + m_G(ik) + m_G(jk). \]
It follows that the values of $DV(m) = \sum m(C)^2$ and $q\ell = (\# \text{ odd three cycles}) \cdot \ell$ from Theorem 5.4 may be determined after replacing $m_{ij}$ by $m_G(ij)$, which takes values only in $\{-1, 0, 1\}$. Hereafter we write $DV(G)$ for $DV(m)$ and $q_G$ for $q$ to emphasize their dependence only on the signed graph $G$. If $U \subset \{v_0, \ldots, v_\ell\}$, we let $DV(G_U)$ represent $DV(m_U)$ to emphasize dependence only on $G$ and the subset $U$. As usually, $q_U$ denotes the number of odd three cycles contained in $U$.

Now, if $G$ is not signed eliminable then by Corollary 5.3 $G$ contains an induced sub-graph $H$ which is
- a signed graph on four vertices which is not signed-eliminable,
- a $\sigma$-cycle of length $> 3$,
- a $\sigma$-hill,
- or a $\sigma$-mountain.
We assume $G = H$ and show that $DV(G) > q_G \ell$ in each of these cases, where $\ell$ is one less than the number of vertices of $G$. The inequality $DV(G) > 3q_G$ can easily be verified by hand for each of the twelve graphs on four vertices which are not signed-eliminable (see Table 1; this is also done in [8, Corollary 6.2]). If $G$ is a $\sigma$-cycle, $\sigma$-mountain, or $\sigma$-hill on $(\ell + 1)$ vertices we will show that $DV(G)$ and $q_G$ are given by the formulas:

(3) \[ DV(G) = \ell^3 - 2\ell^2 - \ell + 2 \]
(4) \[ q_G = \ell^2 - 2\ell - 3. \]

Given these formulas, note that $DV(G) = q\ell + 2(\ell + 1) > q\ell$, thus proving the result. We prove Equations (3) and (4) for the $\sigma$-cycle directly, relying on the two additional formulas:

(5) \[ DV(G) = \sum_{U \subset V_G, |U| = 4} DV(G_U) \]
(6) \[ q_G = (\sum_{U \subset V_G, |U| = 4} q_U)/(\ell - 2). \]

Equation (5) follows since each four-cycle is contained in a unique induced sub-graph on four vertices and Equation (6) follows since each three-cycle appears in $(\ell - 2)$ sub-graphs on four vertices. Using these equations, it suffices to identify all possible types of induced sub-graphs of the $\sigma$-cycle on four vertices, how many of each type there are, and compute $DV(G_U)$ and $q_U$ for each of these. Then we use Equation (5) to compute $DV(G)$ and Equation (6) to compute $q_G$. The list of all possible induced sub-graphs with four vertices of a $\sigma$-cycle on $(\ell + 1)$ vertices are listed in Table 2. The number of sub-graphs of each type is listed in the second column, while the third and fourth columns record $q_U$ and $DV(G_U)$, respectively, for each type of sub-graph. The final row records the total number of sub-graphs on four vertices, the number of odd three-cycles, and the deviation of $m$, $DV(m) = \sum_{C \in \mathcal{G}(K_{\ell+1})} m(C)^2$. We find that $DV(m) = \ell^3 - 2\ell^2 - \ell + 2$ and $q = \ell^2 - 2\ell - 3$, proving Equations (3) and (4) for the $\sigma$-cycle. The same
\( \sigma \)-cycle of length \((\ell + 1)\)

| Type of sub-graph | Count | \(q_U\) | \(DV(G_U)\) |
|-------------------|-------|---------|-------------|
| \( \cdot \cdot \) | \(\binom{\ell - 4}{2} + \binom{\ell - 3}{2}\) | 0 | 0 |
| \( \cdot \cdot \) | \((\ell + 1)\binom{\ell - 4}{2}\) | 2 | 2 |
| \( \cdot \cdot \) | \(\frac{(\ell + 1)(\ell - 4)}{2}\) | 4 | 8 |
| \( \cdot \cdot \) | \((\ell + 1)(\ell - 4)\) | 2 | 2 |
| \( \cdot \cdot \) | \(\ell + 1\) | 2 | 6 |
| Total | \(\binom{\ell + 1}{4}\) | \(q = \ell^2 - 2\ell - 3\) | \(DV = \ell^3 - 2\ell^2 - \ell + 2\) |

Table 2. Computing \(DV(G)\) where \(G\) is a \(\sigma\)-cycle

computations can be done to prove Equations (3) and (4) for the \(\sigma\)-hill and \(\sigma\)-mountain; for the convenience of the reader we collect these in Appendix A. □

6. FREE VERTICES AND A CONJECTURE

In this final section we discuss free vertices of a multiplicity on a graphic arrangement and present a conjecture on the structure of free multiplicities on braid arrangements.

**Definition 6.1.** Suppose \(G\) is a graph. A vertex \(v_i \in V_G\) is a simplicial vertex if the sub-graph of \(G\) induced by \(v_i\) and its neighbors is a complete graph. Given a multi-arrangement \((\mathcal{A}_G, m)\) and the corresponding edge-labeled graph \((G, m)\), a vertex \(v_i\) is a free vertex of \((G, m)\) if it is a simplicial vertex and for every triangle with vertices \(v_i, v_j, v_k\) we have \(m_{ij} + m_{ik} \leq m_{jk} + 1\).

**Theorem 6.2.** Suppose \(G\) is a graph, \(v_i\) is a free vertex of \((G, m)\), and \(G'\) is the induced sub-graph on the vertex set \(V_G \setminus \{v_i\}\). Then \((\mathcal{A}_G, m)\) is free if and only if \((\mathcal{A}_{G'}, m_{G'})\) is free.
Proof of Theorem 6.2. We use a result whose proof we omit since it is virtually identical to the proof of [1] Theorem 5.10. Recall that a flat $X \in L(A)$ is called modular if $X + Y \in L(A)$ for every $Y \in L(A)$, where $X + Y$ is the linear span of $X, Y$ considered as linear sub-spaces of $V = \mathbb{K}^\ell$.

**Theorem 6.3.** Suppose $(A, m)$ is a central multi-arrangement of rank $\ell \geq 3$ and $X$ is a modular flat of rank $\ell - 1$. Suppose $(A_X, m_X)$ is free with exponents $(d_1, \ldots, d_{\ell - 1}, 0)$ and for all $H \in A \setminus A_X$ and $H' \in A_X$, set $Y := H \cap H'$. If one of the following two conditions is satisfied:

1. $A_Y = H \cup H'$ or
2. $m(H') \geq \sum_{H \in A \setminus A'} m(H) - 1.$

Then $(A, m)$ is free with exponents $(d_1, \ldots, d_{\ell - 1}, |m| - |m'|)$.

Now suppose $G$ is a graph on $\ell + 1$ vertices $\{v_0, \ldots, v_\ell\}$ and $A_G$ is the associated graphic arrangement. Further suppose that $v_i$ is a free vertex of $(G, m)$, and $G'$ is the induced sub-graph on the vertex set $V_G \setminus \{v_i\}$. Set $m' = m|_{G'}$. By Proposition 2.1 if $(A_G', m')$ is not free, then neither is $(A_G, m)$.

Suppose now that $(A_G', m')$ is free. We show that $(A_G, m)$ is free using Theorem 6.3. Write $H_{ij} = V(x_i - x_j)$. Since $v_i$ is a simplicial vertex of $G$, the flat $X = \cap_{v_j, v_k \neq v_i} H_{jk}$ is modular and has rank $\ell - 1$. The sub-arrangement $(A_G)_X$ is the graphic arrangement $A_G'$. Suppose $H = H_{ij} \in A_G \setminus A_G', H' = H_{st} \in A_G$, and set $Y = H_{ij} \cap H_{st}$. If $\{s, t\} \cap \{i, j\} = \emptyset$, then $A_Y = H_{ij} \cup H_{st}$. Otherwise, suppose $s = j$. Since $v_i$ is a simplicial vertex, $\{i, t\} \in E_G$, so $A_Y = H_{ij} \cup H_{it} \cup H_{jt}$. Since $v_i$ is a free vertex, $m_{ij} + m_{it} \leq m_{jt} + 1$, which is condition (2) from Theorem 6.3. Hence $(A_G, m)$ is free by Theorem 6.3.

**Remark 6.4.** Theorem 6.2 can also be proved using homological techniques from [7].

We use Theorem 6.2 to inductively construct two types of free multiplicities. Given a graph $G$, an elimination ordering is an ordering $v_0, \ldots, v_\ell$ of the vertices $V_G$ so that $v_i$ is a simplicial vertex of the induced sub-graph on $v_0, \ldots, v_i$ for every $i = 1, \ldots, \ell$. It is known that $V_G$ admits an elimination ordering if and only if $G$ is chordal [9].

**Corollary 6.5.** Suppose $(G, m)$ is an edge-labeled chordal graph with elimination ordering $v_0, \ldots, v_\ell$ satisfying that $v_i$ is a free vertex of the induced sub-graph on $\{v_0, \ldots, v_i\}$ for every $i \geq 2$. Then $(A_G, m)$ is free.

**Corollary 6.6.** Let $(A_{\ell}, m)$ be a multi-braid arrangement corresponding to the complete graph $K_{\ell + 1}$ on $(\ell + 1)$ vertices. Suppose that $K_{\ell + 1}$ admits an ordering $\{v_0, \ldots, v_\ell\}$ so that:

1. For some integer $0 \leq k \leq \ell$, the induced sub-graph $G'$ on $\{v_0, \ldots, v_k\}$ satisfies that $m_{G'}$ is a free ANN multiplicity.
2. For $k + 1 \leq i \leq \ell$, $v_i$ is a free vertex of the induced graph on $\{v_0, \ldots, v_i\}$. Then $(A_{\ell}, m)$ is free.

We conjecture that all free multi-braid arrangements take the form of Corollary 6.6.

**Conjecture 6.7.** The multi-braid arrangement $(A_{\ell}, m)$ is free if and only if it is one of the multi-braid arrangements constructed in Corollary 6.6. Equivalently, by
Theorem 6.2. If \((A, m)\) is free then either \(m\) is a free ANN multiplicity or \(m\) has a free vertex. Using Theorem 1.1, this is equivalent to the following statement: if \(m\) is a free multiplicity which is not in the balanced cone of multiplicities, then \(m\) has a free vertex.

Remark 6.8. Conjecture 6.7 is proved for the \(A_3\) braid arrangement in [8]. Using Macaulay2 [10], we have verified Conjecture 6.7 for many multiplicities on the \(A_4\) arrangement.

7. Acknowledgements

I would especially like to thank Jeff Mermin, Chris Francisco, and Jay Schweig for their collaboration on the analysis of free multiplicities on the \(A_3\) braid arrangement. The current work would not be possible without their help. I am very grateful to Takuro Abe for freely corresponding and offering many suggestions. Computations in Macaulay2 [10] were indispensable for this project.

References

[1] Takuro Abe. personal communication.
[2] Takuro Abe, Koji Nuida, and Yasuhide Numata. Signed-eliminable graphs and free multiplicities on the braid arrangement. J. Lond. Math. Soc. (2), 80(1):121–134, 2009.
[3] Takuro Abe, Hiroaki Terao, and Max Wakefield. The characteristic polynomial of a multiarrangement. Adv. Math., 215(2):825–838, 2007.
[4] Takuro Abe, Hiroaki Terao, and Max Wakefield. The Euler multiplicity and addition-deletion theorems for multiarrangements. J. Lond. Math. Soc. (2), 77(2):335–348, 2008.
[5] Takuro Abe and Masahiko Yoshinaga. Coxeter multiarrangements with quasi-constant multiplicities. J. Algebra, 322(8):2839–2847, 2009.
[6] Christos A. Athanasiadis. Deformations of Coxeter hyperplane arrangements and their characteristic polynomials. In Arrangements—Tokyo 1998, volume 27 of Adv. Stud. Pure Math., pages 1–26. Kinokuniya, Tokyo, 2000.
[7] M. DiPasquale. Generalized Splines and Graphic Arrangements. J. Algebraic Combin., 2016. doi:10.1007/s10801-016-0704-8
[8] M. DiPasquale, C. A. Francisco, J. Mermin, and J. Schweig. Free and Non-free Multiplicities on the \(A_3\) Arrangement. ArXiv e-prints, September 2016.
[9] G. A. Dirac. On rigid circuit graphs. Abh. Math. Sem. Univ. Hamburg, 25:71–76, 1961.
[10] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/]
[11] Koji Nuida. A characterization of signed graphs with generalized perfect elimination orderings. Discrete Math., 310(4):819–831, 2010.
[12] Kyoji Saito. On the uniformization of complements of discriminant loci. In Conference Notes. Amer. Math. Soc. Summer Institute, Williamstown, 1975.
[13] Louis Solomon and Hiroaki Terao. The double Coxeter arrangement. Comment. Math. Helv., 73(2):237–258, 1998.
[14] Hiroaki Terao. Multiderivations of Coxeter arrangements. Invent. Math., 148(3):659–674, 2002.
[15] Atsushi Wakamiko. On the exponents of 2-multiarrangements. Tokyo J. Math., 30(1):99–116, 2007.
[16] Masahiko Yoshinaga. The primitive derivation and freeness of multi-Coxeter arrangements. Proc. Japan Acad. Ser. A Math. Sci., 78(7):116–119, 2002.
[17] Günter M. Ziegler. Multiarrangements of hyperplanes and their freeness. In Singularities (Iowa City, IA, 1986), volume 90 of Contemp. Math., pages 345–359. Amer. Math. Soc., Providence, RI, 1989.
### Appendix A. Computations for mountains and hills

**σ-mountain on (ℓ + 1) vertices**

| Type of subgraph | Count          | $q_U$ | $DV(G_U)$ |
|------------------|----------------|-------|-----------|
| * * *            | $(\ell - 3)^4$ | 0     | 0         |
| * * *            | $3(\ell - 3)^3$ | 2     | 2         |
| * * *            | $2(\ell - 3)^2$ | 2     | 2         |
| * * *            | $2\ell - 9 + (\ell - 5)^2$ | 4     | 8         |
| * * *            | $(\ell - 3)^2$  | 2     | 6         |
| * * *            | $\ell - 4$      | 2     | 2         |
| * * *            | 2               | 2     | 6         |
| * * *            | $2(\ell - 4)^2$ | 2     | 2         |
| * * *            | $2(\ell - 4)$   | 4     | 8         |
| * * *            | $2(\ell - 4)$   | 2     | 2         |
\[
\begin{array}{c|ccc}
\text{Type of subgraph} & q_U & DV(G_U) & \text{Count} \\
\hline
\bullet \bullet & \binom{\ell - 4}{4} & 0 & 2 \\
\hline
\bullet \bullet & 3 \binom{\ell - 4}{3} & 2 & 2 \\
\hline
\bullet & 2 \binom{\ell - 4}{2} & 2 & 2 \\
\hline
\bullet \bullet & 2(\ell - 11) + \binom{\ell - 6}{2} & 4 & 6 \\
\hline
\end{array}
\]

**Table 3:** Computing DV(G) where G is a \(\sigma\)-mountain

\[
\begin{align*}
q_G &= \ell^2 - 2\ell - 3 \\
DV &= \ell^3 - 2\ell^2 - \ell + 2
\end{align*}
\]
| Diagram | Expression | Value 1 | Value 2 |
|---------|------------|---------|---------|
|         | $\ell - 4$ | 2       | 6       |
|         | $2\binom{\ell - 4}{2}$ | 2       | 2       |
|         | $2(\ell - 4)$ | 4       | 8       |
|         | $2(\ell - 4)$ | 2       | 2       |
|         | $2(\ell - 4)$ | 2       | 2       |
|         | $2(\ell - 4)$ | 2       | 6       |
|         | $2\binom{\ell - 4}{3}$ | 0       | 0       |
|         | $4\binom{\ell - 4}{2}$ | 2       | 2       |
|         | $2(\ell - 4)$ | 2       | 2       |
|         | $1$ | 2       | 6       |
|         | $2(\ell - 4)$ | 2       | 2       |
|         | $2(\ell - 4)$ | 2       | 2       |
|         | $2$ | 2       | 6       |
|         | $\binom{\ell - 4}{2}$ | 2       | 2       |
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Total} & \ell - 4 & 4 & 8 \\
\hline
\left(\frac{\ell + 1}{4}\right) & q_G = \ell^2 - 2\ell - 3 & \text{DV} = \ell^3 - 2\ell^2 - \ell + 2 \\
\hline
\end{array}
\]

Table 4: Computing DV(G) where G is a \(\sigma\)-hill

Michael DiPasquale, Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-1058, USA

E-mail address: mdipasq@okstate.edu

URL: [http://math.okstate.edu/people/mdipasq/](http://math.okstate.edu/people/mdipasq/)