On optimum multi-input multi-output radar signal design: Ambiguity function, manifold structure and duration-bandwidth

Yuqing Liu1 | Jia Xu2 | Kon Max Wong1,2 © | Timothy N. Davidson2

1School of Information Engineering, Zhengzhou University, Zhengzhou, China
2Department of ECE, McMaster University, Hamilton, Canada

Correspondence
Kon Max Wong, Dept. of Electrical & Computer Engineering, McMaster University, Hamilton, ON L8S 4K1, Canada.
Email: wongkm@mcmaster.ca

Abstract
A design technique is developed for the probing signals of a Multi-Input Multi-Output (MIMO) radar. The concentration of the energy of the signal in its essential duration and essential bandwidth is achieved through the use of a class of time-frequency concentrated functions called the WLJ functions as the synthesizing signal set. The goal is to design a signal vector having a pre-specified desired covariance (CoV) matrix while ensuring that the side-lobes of the ambiguity functions are small. Since CoV matrices are structurally constrained, they form a manifold in the signal space. Hence, we argue that the difference between these matrices should not be measured in terms of the conventional Euclidean distance (ED); rather, the distance should be measured along the surface of the manifold, that is, in terms of a Riemannian distance (RD). In either case, the signal optimisation problem is non-convex in the design variables, involving, respectively, a quartic and a square-root objective function. An efficient algorithm based on successive convex approximation is developed in which the original non-convex problems are transformed so that they can be approximated by a convex quadratically constrained quadratic problem at each stage, resulting in good approximate solutions. Comparing the designs using ED and RD, we find that the convergence of the algorithm can be significantly faster when optimising over the manifold (RD) than when optimising over the whole space (ED). More importantly, for tight constraints, the use of RD yields solutions which satisfy the constraints far better than the use of ED.

1 | INTRODUCTION
The multi-input multi-output (MIMO) concept was proposed for radar systems around the turn of the 21st century. A MIMO radar [1,2] utilises multiple antennas to transmit independent waveforms and multiple antennas to receive the returned signals. The availability of multiple transmitting and receiving antennas offers the opportunity to mitigate fading, enhance the resolution and suppress interference [3]. As a result, MIMO radar has the potential to provide substantial improvements over traditional single-input single output (SISO) radar, especially in the areas of target detection, parameter estimation, and target tracking and recognition.

In order to harness the potential of MIMO radar, considerable effort has been dedicated to the design and synthesis of the transmission signals. For example, the inter-transmitter waveform covariance can be optimised for specific channel realisations [4,5], and waveforms that maximise the mutual information [6,7] or minimise the mean square error [6,8] have been proposed. (In the case of uncorrelated noise, maximising the mutual information and minimising the mean square error yield equivalent designs.) Different methods [9] aiming at matching given transmission beam patterns and minimising the cross-correlation of reflected signals have also been investigated. In particular, algorithms to design a unimodulus signal set matching beam pattern specifications and suppressing sidelobes of both cross- and auto-correlations have been developed [10]. Furthermore, related designs that have signal-to-interference-plus-noise (SINR) objectives [11–13], sidelobe-suppression objectives [14], or seek to optimize the peak of certain estimation metrics [15] are also available. In contrast to these design objectives, algorithms were presented in References [16,17] to synthesise a waveform directly such that its covariance (CoV) matrix is close to a desired matrix R which has good cross- and
auto-correlation properties. Using various forms of weighting, the design problem was formulated as different mathematical expressions optimised under a low peak-to-average power ratio (PAR) constraint. Several computationally efficient cyclic algorithms (CA) were presented to design unimodular MIMO waveforms, minimising the distance between the CoV matrix and the desired matrix.

The problem of optimum signal design and synthesis in a MIMO radar is also addressed in this paper. We first examine the choice of an orthogonal function set for the synthesis of the transmitted waveforms from the viewpoint of maximum signal energy concentration. Then, we formulate the problem of designing a radar transmission signal that has a CoV matrix close to a pre-specified desired matrix. Our approach to the latter part is not unlike those taken in Reference [16,17]. However, the design objectives in those papers measure the difference between the designed and desired CoV matrices in terms of the Euclidean distance (ED), which is often called [18] the Frobenius distance (FD). Our key observation is that since the CoV matrices are positive semi-definite and Hermitian symmetric, there are structurally constrained and thus form a manifold \( M \) in the linear vector space \( H \) of all \( M \times M \) matrices [19,20]. Therefore, even though the ED is commonly used, it may not be appropriate for measuring the distance between two CoV matrices. Instead, we should measure the distance along the surface of the manifold.

Thus, we formulate the problem of designing the radar transmission signal so that its CoV matrix is close to a pre-specified desired matrix, with the distance between the two matrices being measured by a Riemannian distance (RD) on the manifold of positive semidefinite Hermitian symmetric matrices. In addition, we seek to sharpen the main lobe and suppress the sidelobes of the MIMO ambiguity function [21–24] so that the accuracy of estimating the delay and Doppler shift will be enhanced.

Unfortunately, like the existing ED-based designs, the resulting RD-based design problem is not convex and can be difficult to solve directly. In order to efficiently generate good solutions for both RD- and ED-based designs, we develop (see also [25,26]) tailored successive convex approximation algorithms; for example, see [27,28]. In each algorithm, we approximate the original non-convex problem by a (different) sequence of convex quadratically constrained quadratic problems. Each of those approximate problems can be efficiently solved.

As we demonstrate in our numerical results, the RD-based design exhibits much faster convergence than the ED-based design. Furthermore, the RD-based design yields higher accuracy in estimating the ranges and velocities of the targets. Under tighter constraints, we also find cases in which the RD-based design converges properly while the ED-based design fails.

**Notations:** The imaginary unit is denoted by \( j = \sqrt{-1} \) and complex conjugate by \( (\cdot)^* \). Vectors and matrices are represented by lower and upper cases of bold letters respectively, for example, \( \mathbf{a} \) and \( \mathbf{A} \); \( \text{tr}(\cdot) \) and \( (\cdot)^H \) denote, respectively, the trace and the Hermitian conjugate of a square matrix; \( \langle \cdot, \cdot \rangle \) denotes the inner product of vectors or matrices; and \( ||A||_2 \) denotes the Frobenius norm of the matrix \( A \). Vector spaces and subspaces,

![Figure 1](image-url) A bistatic multi-input multi-output (MIMO) radar

as well as manifolds are denoted by upper case script letters; in particular, a Euclidean space by \( H \), a manifold by \( M \).

## 2 | MULTI-INPUT MULTI-OUTPUT RADAR SIGNAL MODEL

We will consider a bistatic MIMO radar system, which has a single transmitter array and a single receiver array, and these arrays do not share any sensor elements. A schematic diagram is shown in Figure 1. We consider the case of uniform linear arrays at the transmitter and receiver, with \( M_T \) and \( M_R \) sensors, respectively. The inter-sensor spacings are assumed equal and are denoted by \( d_s \) such that \( d_s \leq \lambda_c/2 \), with \( \lambda_c \) being the carrier wavelength of the transmitted signal. The range of the target is assumed to be much larger than the aperture of the transmitter and receiver arrays. The transmitter array simultaneously transmits \( M_T \) narrowband waveforms. These signals are reflected by \( P \) moving targets. The direction of departure (DOD) (i.e. the angle subtended by the target direction and the array normal) and direction of arrival (DOA) of the \( p \)-th target, \( \rho = 1, 2, \ldots, P \), with respect to the transmitter and receiver arrays are denoted by \( \theta_p \) and \( \phi_p \). Thus, the DOD and DOA vectors are given by \( \theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \) and \( \phi = [\phi_1, \phi_2, \ldots, \phi_P]^T \), respectively.

Let \( s_m(t) = y_m(t)e^{j2\pi f_0 t} \) be the complex signal [29–31] transmitted by the \( m \)-th sensor of the transmitter array. Each of these signals is a bandpass signal having a complex envelope \( y_m(t) \) modulated to carrier frequency \( \omega_c \). Thus, the vector of the transmission waveforms from the \( M_T \) sensors of the transmitter array can be written as

\[
s(t) = [s_1(t) \ s_2(t) \ \ldots \ s_{M_T}(t)]^T \quad (1)
\]

The signals from the transmitter array reach the \( p \)-th target as a linear combination of all the \( M_T \) signals, each weighted by the corresponding element of the array directional vector...
μ = \frac{1}{2} \sin \theta_p is often called the electrical angle of departure of the p-th target. Hence, the transmitted signal in the direction of the p-th target can be written as

\[ x_p(t) = a_1^T(\theta_p) s(t) = \sum_{m=1}^{M_1} y_m(t) e^{j\omega t} \cdot e^{-j(m-1)\phi_p} \tag{3} \]

This signal hits the p-th target and is reflected back with a (complex) reflection coefficient \( \beta_p \), which is proportional to the radar cross-section (RCS) of the target. Here, the reflection coefficients are assumed to be under the Swerling II target model [32,33] such that the RCS fluctuations remain unchanged for the duration of a pulse but may vary independently from pulse to pulse and from target to target. Thus, \( \beta_p \) is a complex Gaussian variable with zero-mean and variance \( \sigma^2 \), and \( \beta_1, \ldots, \beta_p \) are uncorrelated.

The returned signal from the p-th target is a time-delayed (\( t_\Delta \)) and Doppler-shifted (\( \omega_\Delta \)) version of the transmitted signal such that

\[ y_p(t) = \beta_p x_p(t - t_\Delta) e^{j\omega_\Delta t} \tag{4} \]

This signal is received by an array having a directional vector

\[ a_p(\phi_{p1}) = [1 e^{-j\phi_p} \ldots e^{-j(M_1-1)\phi_p}]^T \tag{5} \]

where \( \phi_{p1} = \frac{1}{2} \sin \phi_p \) is the electrical angle of arrival of the p-th sensor. The signal \( y_p(t) \) arriving at the \( \mu \)-th sensor, \( \mu = 1, \ldots, M_R \), of the receiver array after the appropriate directional shift as indicated by Equation (5), is demodulated by \( e^{-j\omega t} \) giving

\[ y_{pu}(t) = y_p(t) e^{-j(\omega-\omega_\Delta) t} = \beta_p^* e^{-j(\omega-\omega_\Delta) t} a_1^T \cdot s(t; t_\Delta, \omega_\Delta) \tag{6} \]

where \( \beta_p^* = \beta_p e^{-j\omega_\Delta t_\Delta} \) and

\[ s(t; t_\Delta, \omega_\Delta) = [y_1(t-t_\Delta) e^{j\omega_\Delta t} \ldots y_{M_1}(t-t_\Delta) e^{j\omega_\Delta t}]^T \tag{7} \]

Using a matching signal \( s(t, \tau, \omega) \) such that

\[ s(t, \tau, \omega) = [y_1(t+\tau) e^{j\omega t} \ldots y_{M_1}(t+\tau) e^{j\omega t}]^T \tag{8} \]

a matched filter [22] is then employed to process this returned signal, and all \( M_R \) filter outputs are subsequently summed up giving

\[ r_p(t) = \sum_{\mu=1}^{M_R} \int y_{pu}(t) y_{pu}^*(t) dt \tag{9} \]

where \( y_{pu}^*(t) \) is given by

\[ y_{pu}^*(t) = \beta_p^* e^{-j(\omega-\omega_\Delta) t} a_1^T(\theta_p) \tag{10} \]

with \( \beta_p^*, \theta_p^*, \phi_p^* \) being, respectively, the matching reflection coefficient, and the matching DOD and DOA shifts.

Central to the evaluation of the output signal \( r_p(t) \) is the integral which forms the \( M_T \times M_f \) ambiguity matrix

\[ F(\tau, \omega) = \int s(t; \tau, \omega_\Delta) s^T(t; \tau, \omega) dt \]

\[ = \begin{bmatrix} f_{11}(\tau, \omega) & f_{12}(\tau, \omega) & \cdots & f_{1M_f}(\tau, \omega) \\ f_{21}(\tau, \omega) & f_{22}(\tau, \omega) & \cdots & f_{2M_f}(\tau, \omega) \\ \vdots & \vdots & \ddots & \vdots \\ f_{M_f1}(\tau, \omega) & f_{M_f2}(\tau, \omega) & \cdots & f_{M_fM_f}(\tau, \omega) \end{bmatrix} \tag{11} \]

where \( f_{m,m_2}(\tau, \omega) \), the \( m_1m_2 \)-th element of \( F(\tau, \omega) \), is given by:

\[ f_{m,m_2}(\tau, \omega) \triangleq \int y_{m_1}(t-t_\Delta) y_{m_2}(t+\tau) e^{-j(\omega-\omega_\Delta) t} dt \tag{12} \]

and is called the cross-ambiguity function. Assuming \( \beta_p^* \) and \( \beta_\Delta \) are perfectly matched, then, for the p-th target, the receiver output in Equation (9) can be written as

\[ r_p(t) \approx \left[ \sum_{\mu=1}^{M_1} \beta_p^* e^{-j(\omega-\omega_\Delta) t} \right] \cdot f(\tau, \omega, \theta_p^*, \phi_p^*) \tag{13} \]

with

\[ f(\tau, \omega, \theta_p^*, \phi_p^*) = a_1^T(\theta_p^*) \cdot F(\tau, \omega) \cdot a_1^*(\phi_p^*) \tag{14a} \]

\[ = \sum_{m_1=1}^{M_1} \sum_{m_2=1}^{M_1} f_{m_1,m_2}(\tau, \omega) \cdot e^{-j(\omega_{m_1} - \omega_{m_2}) t} \tag{14b} \]

Equation (13) represents the receiver output for the reflected signal from the p-th target. The first factor of the right side describes spatial processing at the receiver and is not affected by the waveforms \( y_p(t) \). However, the second factor, \( f(\tau, \omega, \theta_p^*, \phi_p^*) \), which is often called [22] the MIMO radar ambiguity function and which is written more explicitly in Equation (14b), shows how the waveforms affect not only the spatial direction resolution, but also the delay and Doppler resolutions.
3 | MIMO RADAR SIGNAL DESIGN

3.1 | Time-bandwidth product

For the signal transmitted from a radar system, having minimum duration and bandwidth occupation often is advantageous [29,30]. Narrow bandwidth tends to mitigate the effects of signal distortion caused by transmission over a dispersive channel. Short pulse duration, on the other hand, tends to improve range resolution.

Now, let us make some mild assumptions on \( y(t) \), the complex envelope of the transmitted radar signal:

1. The signal envelope is of finite energy (normalised to unity), that is,
   \[
   \int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Gamma(\omega)|^2 d\omega = E_T = 1.
   \]
2. As is often true in practice, the envelope is real and symmetric
   \[
   y(t) = y(-t)
   \]
3. The signal envelope possesses a finite essential duration \( T \), that is, \( |y(t)| \) is negligible for \( |t| > T/2 \).
4. The signal envelope possesses a finite essential bandwidth \( \Omega \), that is, \( |\Gamma(\omega)| \) is negligible for \( |\omega| > \Omega \), \( \Gamma(\omega) \) being the Fourier transform of the signal \( y(t) \), and \( \Omega \) being the essential bandwidth.

We now define the time and frequency signal energy concentration coefficients, respectively, as:

\[
E_T(\gamma) = \int_{-T/2}^{T/2} |y(t)|^2 dt, \quad E_\Omega(\Gamma) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\Gamma(\omega)|^2 d\omega
\]

(15)

which can be interpreted as the percentage of energy of the signal \( y(t) \) in the designated duration \([-T/2, T/2]\) and the bandwidth \([-\Omega, \Omega]\). Clearly, from Assumption 1 above, \( 0 \leq E_T(\gamma) \leq 1 \) and \( 0 \leq E_\Omega(\Gamma) \leq 1 \). Ideally, we would like to have a signal \( y(t) \) that satisfies both \( E_T(\gamma) = 1 \) and \( E_\Omega(\Gamma) = 1 \), so that all the energy of the signal concentrates within the allotted duration and bandwidth. Unfortunately, due to a fundamental limit [30], a signal cannot be both time-limited and band-limited simultaneously. Therefore, we seek signals that have the property [34].

\[
\max_{\gamma \in S} E_T(\gamma) \cdot E_\Omega(\Gamma)
\]

(16)

where \( S \) is the set of essentially time-limited and band-limited signals satisfying Assumptions 1, 2, 3, and 4 above.

We also note that the signal \( y(t) \) can be synthesised using a linear combination of a set of orthonormal functions, such as the time-limited sinusoids. To facilitate the design of an optimum signal having the property of Equation (16), however, we propose to synthesise \( y(t) \) using a function set called WLJ functions [34,35] presented in References [34–38] such that

\[
\psi_k(t) = \alpha_{1k} \phi_k(t) + \alpha_{2k} \varphi_{kT}(t)
\]

(17)

where \( \phi_k(t) \) and \( \varphi_{kT}(t) \) are, respectively, the prolate spheroidal wave (PSW) functions and the truncated PSW functions [36–38]. By definition, each \( \psi_k(t) \) satisfies the following eigen-equation

\[
\lambda_k \varphi_{kT}(t) = \int_{-2}^{2} \varphi_{kT}(t) \frac{\sin \omega \tau}{\pi(t - \tau)} d\tau, \quad \lambda_k > 0
\]

(18)

and each

\[
\varphi_{kT} = \begin{cases} \varphi_k(t) & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}
\]

(19)

The coefficients in Equation (17) are given by

\[
\alpha_{1k} = (2 + 2\sqrt{\lambda_k})^{-1/2}, \quad \alpha_{2k} = (2\sqrt{\lambda_k} + 2\lambda_k)^{-1/2}
\]

(20)

The above choice of \( \alpha_{1k} \) and \( \alpha_{2k} \), together with the PSW functions and the truncated PSW functions, ensures that \( \psi_k(t) \) has the maximum energy concentration within a given time duration \( T \) and a given bandwidth \( \Omega \) [34]. The function set \( \{\psi_k(t)\} \) in Equation (17) offers the following advantages over that proposed in [38]:

a. The function set \( \{\psi_k(t)\} \) in Equation (17) is orthonormal and is complete in the space containing all the signals that are a linear combination of a band-limited signal with bandwidth \([-\Omega, \Omega]\) and a time-limited signal with duration \([-T/2, T/2]\), and thus forms a basis in that signal space.

b. The basis \( \{\psi_k(t)\} \) is arranged in decreasing order of maximum energy concentration in \([-\Omega, \Omega]\) and \([-T/2, T/2]\).

Detailed properties of this basis can be found in Reference [34]. To synthesise a signal to be transmitted from the \( m \)-th transmission sensor, we let the signal complex envelope \( y_m(t) \) be made up of a linear combination of the first \( K \) terms of \( \{\psi_k(t)\} \), that is,

\[
y_m(t) = \sum_{k=0}^{K-1} c_{mk} \psi_k(t) = c_{m}^T \psi(t)
\]

(21)

where \( c_m = [c_{m0} \cdots c_{m(K-1)}]^T \) and \( \psi(t) = [\psi_0(t) \cdots \psi_{K-1}(t)]^T \). The coefficients \( \{c_{mk}\} \) will be determined by other optimality criteria discussed in the ensuing sections.

[WLJ functions are so designated because the set was first developed and denoted as \{wlj\} (0), \( \ell = 1, \text{two2} \), \( j = 0, 1, \ldots \) in [34]. Here, we adopt the single-subscripted notation [33].]
3.2 | Cross- and multi-input multi-output ambiguity functions

Another factor which prescribes the optimality of a radar signal is the ambiguity function. For a MIMO radar, because of the multi-sensor transmission, we have the $M_T \times M_T$ ambiguity matrix of Equation (11). Each element of this matrix denotes the cross-ambiguity function [Equation (12)] between the $m_1$-th transmitted signal and the $m_2$-th returned signal. A simple change of variables shifting the origin to $(t_d, \omega_d)$ renders Equation (12) to be re-written as:

$$f_{m_1m_2}(\tau, \omega) = \int_{-\infty}^{\infty} \gamma_{m_1}(t) \gamma_{m_2}^*(t + \tau) e^{j\omega t} dt$$  \hspace{1cm} (22)

Thus, from Equation (14b), we notice that for fixed values of $\tau$ and $\omega$, the MIMO radar ambiguity function $f(\tau, \omega, \theta_p, \theta_p')$ is a two-dimensional Fourier transform of the cross-ambiguity function $f_{m_1m_2}$ in the parameters $m_1$, $m_2$. If the transmitted signals are orthonormal, that is, $\int f_{m_1m_2}(\tau, \omega) dt = \delta_{m_1m_2}$, then the following properties of the ambiguity function hold [22]:

1. $f(0, 0, \theta_p, \theta_p') = M_T$
2. $\int \int |f_{m_1m_2}(\tau, \omega)|^2 d\tau d\omega = 1$
3. $f_{m_1m_2}(-\tau, -\omega) = f_{m_2m_1}^*(\tau, \omega) e^{-j\omega \tau}$

A major task of a radar system is to estimate the range and velocity of a target, which can be calculated from the time-delay $t_d$ and the Doppler frequency shift $\omega_d$ of the returned signal [31]. Both these parameters can be measured from the ambiguity function. We therefore, seek to design a set of signals for a MIMO radar so that the MIMO ambiguity function enables accurate estimation of the range and velocity of a target. To facilitate this, we first assume that the target direction has been determined so that the MIMO ambiguity function in Equation (14a) becomes

$$f(\tau, \omega, \theta_p, \theta_p') = a^T(\theta_p') \cdot F(\tau, \omega) \cdot a(\theta_p)$$  \hspace{1cm} (23)

Furthermore, as is often true in practice and imposes no harsh condition, we also assume $\gamma_m(t)$ to be real and symmetric, with unit energy, that is,

$$\gamma_m(t) = \gamma_m^*(t) = \gamma_m(\tau - t), \quad \int_{-\infty}^{\infty} |\gamma_m(t)|^2 dt = 1$$  \hspace{1cm} (24)

From Equation (21), the envelope is synthesised using a set of orthonormal signals, $Y(t) = [y_1(t) \ y_2(t) \ \ldots \ y_K(t)]^T$, such that $\int_{-\infty}^{\infty} y_i(t) y_j^*(t) dt = \delta_{ij}$, $1 \leq i, j \leq K$, thus we can write the set of envelope signals of Equation (21) as a vector:

$$\gamma(t) = [\gamma_1(t) \ \gamma_2(t) \ \ldots \ \gamma_M(t)]^T = \mathbf{C} \psi(t)$$  \hspace{1cm} (25)

where $\mathbf{C} = [c_1 \ c_2 \ \ldots \ c_M]^T$ is the $M_T \times K$ (usually $K > M_T$) coefficient matrix. Also, from Equation (8),

$$\mathbf{s}(\tau, \omega) = [\gamma_1(\tau) e^{j\omega \tau} \ \ldots \ \gamma_M(\tau) e^{j\omega \tau}]^T = \mathbf{C} \psi(t)$$  \hspace{1cm} (26)

where $\psi(t) \triangleq \psi(t + \tau) e^{j\omega \tau}$. Hence, having shifted the origin to $(t_d, \omega_d)$, the cross ambiguity matrix in Equation (11) can be expressed as

$$\mathbf{F}(\tau, \omega) = \mathbf{C} \mathbf{R}(\tau, \omega) \mathbf{C}^H$$  \hspace{1cm} (27)

such that its $m_1 m_2$-th element is given by

$$f_{m_1m_2}(\tau, \omega) = c_{m_1}^T \mathbf{R}(\tau, \omega) c_{m_2}^* = \sum_{k=1}^{K} \sum_{j=1}^{K} c_{m_1} c_{m_2}^* \rho_{ij}(\tau, \omega)$$  \hspace{1cm} (28)

Here, $\mathbf{R}(\tau, \omega) = \int \psi(t)^H \psi^*(t) dt$ is the ambiguity matrix between $\psi(t)$ and $\psi(t)$, having the $ij$-th element given by

$$\rho_{ij}(\tau, \omega) = \int \psi_i(t) \psi_j^*(t + \tau) e^{-j\omega \tau} dt$$  \hspace{1cm} (29)

We note that, due to the orthonormality between $\psi(t)$ and $\psi(t)$, for $\tau = 0$ and $\omega = 0$, $\rho_{ij}(0,0) = \int \psi_i(t) \psi_j^*(t) dt = \delta_{ij}$. Thus,

$$\mathbf{R}(0,0) = \mathbf{I} \Rightarrow \mathbf{F}(0,0) = \mathbf{CC}^H$$  \hspace{1cm} (30)

3.2.1 | Signal design in terms of the ambiguity matrix

From the above observations, we infer that

1. Since $f_{m_1m_2}(0,0)$ correlates signals from $m_1$-th and $m_2$-th transmitters, good correlation property means, ideally,

$$f_{m_1m_2}(0,0) = \begin{cases} 1 & m_1 = m_2 \\ 0 & m_1 \neq m_2 \end{cases}$$

2. For the case of $m_1 = m_2$, $f_{mm}(\tau, \omega)$ should be concentrated in a 'main lobe' around the origin, increasing and decreasing sharply in magnitude around the point $(0,0)$ so that the time-delay and Doppler shift can be located accurately. Outside the main lobe (i.e. in the 'sidelobes'), the magnitude of $f_{mm}(\tau, \omega)$ should be much smaller than that of $f_{mm}(0,0)$.

3. For $m_1 \neq m_2$, the magnitude of $f_{m_1m_2}(\tau, \omega)$ should be much smaller than that of $f_{mm}(0,0)$ for all $(\tau, \omega)$.

4 | DISTANCE MEASURES AND OPTIMUM SIGNAL DESIGN

Equipped with an appropriate set of orthonormal functions $\{\psi(t)\}$, our goal is to find a matrix $\mathbf{C}$ that synthesises the
transmitted signal having an ambiguity function as described above. As indicated by Equations (30) and (31), F(0, 0) = CC* should be close to an identity matrix. As well, each element of F(τ, ω) should be small for values of (τ, ω) that lie outside the main lobe. As explained in more detail in the subsections below, combining these principles, together with additional control of the shape of the main lobe and the usual power constraint, our particular formulation of the signal design problem can be stated as follows: Given a sidelobe shaping function c(τ, ω) that bounds the norm squared of the sidelobes of the MIMO ambiguity matrix in Equation (27), with sidelobe sample points (τ,ω), , ∈ 1, 2, ..., L, and a mainlobe shaping function α(τ, ω) that bounds the norm squared of the main lobe, with main lobe sampling points (τ,ω), , ∈ 1, 2, ..., K0.

\[
\min_c \ d^2(\mathbf{CC}^H, \mathbf{I}) \quad (32a)
\]
subject to:

\[
\|\mathbf{CR}(\tau, \omega)\mathbf{C}^H\|^2 \leq c(\tau, \omega), \quad \ell' = 1, 2, \ldots, L, \quad (32b)
\]

\[
\|\mathbf{CR}(\tau, \omega)\mathbf{C}^H\|^2 \leq \alpha(\tau, \omega), \quad \kappa = 1, 2, \ldots, K_0, \quad (32c)
\]

\[
\text{trace}(\mathbf{CC}^H) \leq M_T \quad (32d)
\]

Equation (33), induced by the inner product norm, is sometimes also called the Frobenius distance [18].

### 4.1.2 Riemannian distance

On the other hand, HPD matrices are structurally constrained and form a manifold \( M \) in the signal space \( H \). Thus, the distance between two such matrices should be measured along the surface of the manifold. This distance is known as a RD. Measurement of RD between HPD matrices \( \mathbf{P} \in M \) can be facilitated [19,20] by establishing a mapping \( \pi : M \to H \). This mapping \( \pi \) associates each point \( \mathbf{P} \in M \) with a \( M_T \times M_T \) complex matrix but may no longer be HPD, that is, \( \mathbf{P} \notin H \). By choosing a particular mapping \( \pi \), together with an appropriate Riemannian metric, we can find an Euclidean subspace \( U_H \) in \( H \) that contains \( \mathbf{P} \) and is isometric with \( T_M(\mathbf{P}) \) the tangent space at \( \mathbf{P} \) on the manifold \( M \). Thus, the RD on the manifold can be expressed directly in the Euclidean subspace \( U_H \) in which 1D is the distance measure. Following this approach, three different closed-form expressions of RD for the power spectral density (PSD) matrix manifold have been obtained and studied [20]. We employ the more readily manipulable RD \( d_{R_2} \), which, between two HPD matrices \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \), is given by:

\[
d_{R_2}(\mathbf{P}_1, \mathbf{P}_2) = \|\mathbf{P}_1^\dagger - \mathbf{P}_2^\dagger\|^2 = \text{tr}\left[(\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{P}_1 - \mathbf{P}_2)^H\right] \quad (34)
\]

We will use both Equations (33) and (34) in our design and examine the effects on the performance of the MIMO radar.

### 4.2 Constraints of design

As outlined in the second and third design principles, each element of \( \mathbf{F}(\tau, \omega) \) should be small for all \( (\tau, \omega) \) pairs outside the main lobe. A natural implementation of that constraint is to impose this small amplitude on representative values of \( \tau \) and \( \omega \). However, in Equation (32), we have opted to reduce the computational cost of controlling the sidelobes, at the price of coarser control, by constraining the norm of \( \mathbf{F}(\tau, \omega) \) to be less than a pre-specified level \( \epsilon(\tau, \omega) \) at each sample point, rather than constraining the magnitude of each element of \( \mathbf{F}(\tau, \omega) \); see constraint (32b).

While Constraint (32b) provides norm-based control of the shape of the maximum sidelobe level through \( \epsilon(\tau, \omega) \), constraint (32c) uses the pre-specified function \( \alpha(\tau, \omega) \) to provide additional control of the shape of the main lobe of the scalar ambiguity functions as they drop from the peak at \( (\tau, \omega) = (0, 0) \) to the sidelobes. In Equation (32c), the sample points \( (\tau, \omega) \) are within the beam width of the main lobe.

The last constraint (32d) bounds the overall transmitted power of the system.
4.3 Optimum signal design

Now, having selected a suitable set of orthonormal functions, we are ready to tackle the optimum signal design problem of Equation (32). We will use both the ED and the RD as respectively expressed in Equations (33) and (34) to be our objective function and examine the properties and performance of each of the two resulting optimised signals.

- Using Equation (33), the objective function becomes:
  \[ d_E^2(CC^H, I) = \|CC^H - I\|_2^2, \]
  which involves a quartic term.

- Using Equation (34), the objective function becomes:
  \[ d_R^2(CC^H, I) = \|\{CC^H\}^{1/2} - 1^{1/2}\|_2^2, \]
  which involves square-root terms.

In each case, the objective function is not convex. Furthermore, the first constraint in Equation (32b) is quartic in \( C \), as is the second in Equation (32c). Finally, the third constraint is a quadratic constraint. As a result, the problem in Equation (32) is not convex and is difficult to tackle directly.

Instead, we will develop an efficient algorithm for solving the problem of Equation (32) by employing successive convex approximation [26–28], the principal steps of which are outlined as follows:

1. The approximation: At the \( n \)th iteration, we approximate the product term \( CC^H \) by \( CC^{(n-1)H} \), where \( C^{(n-1)} \) is the solution from the previous iteration.

2. The approximate objective functions:
   (a) ED Objective—Replacing \( CC^H \) by \( CC^{(n-1)H} \), the ED objective becomes:
   \[ d_E^2(CC^{(n-1)H}, I) = \|CC^{(n-1)H} - I\|_2^2 \] (35)
   This is now a convex quadratic function of \( C \).

   (b) RD objective—we can apply the singular value decomposition (SVD) to the \( M_T \times K \) matrix \( C \) at any stage, so that
   \[ C = V\Sigma U^H = V[\tilde{\Sigma} \begin{bmatrix} \tilde{U}^H & 0 \end{bmatrix}] = V\tilde{\Sigma}U^H \] (36)
   where the diagonal matrix \( \tilde{\Sigma} \) is of size \( M_T \times M_T \), giving
   \[ C\tilde{U} = V\tilde{\Sigma} \] (37)
   Thus, from Equation (36), we have
   \[ CC^H = V\Sigma U^H U\Sigma V^H = V\tilde{\Sigma}U^H \tilde{\Sigma}U^H \]
   yielding,
   \[ (CC^H)^{1/2} = V\tilde{\Sigma}V^H = C\tilde{U} \]
   where the last step of Equation (38) is from Equation (37). Thus, at the \( n \)th stage, replacing \( CC^H \) by \( CC^{(n-1)H} \), Equation (38) can be used to yield an approximate RD objective such that,
   \[ d_R^2(CC^{(n-1)H}, I) = \|C\tilde{U}(n-1)V(n-1)^H - I\|_2^2 \] (39)
   This approximate objective is a convex quadratic function of \( C \).

1. Approximate constraints: The sidelobe shaping constraints in Equation (32b) are quartic functions of \( C \) and are not convex. Similarly, the main lobe shaping constraints in Equation (32c) are also quartic and not convex. However, the power constraint in Equation (32d) is a convex quadratic constraint. To approximate the non-convex constraints in Equations (32b) and (32c), we will approximate \( F(\tau, \omega) = CR(\tau, \omega)C^{H} \) in Equation (27) by \( CR(\tau, \omega)C^{(n-1)H} \). Doing so converts Constraints (32b) and (32c) into convex quadratic constraints on \( C \), and hence, the feasible set of the approximated problem is convex.

2. Refining the solution: We solve the convex approximation for \( C^* \), and then update the current estimate of the optimal \( C \) for the original problem in Equation (32) using \( C^{(n)} = C^{(n-1)} + \sigma^{(n)}(C^* - C^{(n-1)}) \), where \( \sigma^{(n)} \) is decreasing, yet persistently exciting step size; for example, [28].

The optimum signal design algorithms for both ED and RD objective functions are summarised in Algorithm 1.

4.3.1 Computational complexity

To analyse the computational cost of the proposed SCA method, we observe that the total number of operations is the number of operations required to complete a single SCA iteration multiplied by the number of SCA iterations required for convergence. To analyse the computational cost of a single SCA iteration, we note that if interior point methods are used to solve the convex quadratically-constrained quadratic program that arises at each SCA iteration, then that problem can be solved in \( O((\xi M_T K)^{3.5} \log(1/\epsilon)) \) operations, where \( \xi = L + M_T K_0 + 2 \) and \( \epsilon \) is the desired value of the duality gap; for example, [39]. In the case of the RD-based design, we also have to perform an SVD at each SCA iteration. The
computational cost of doing so is $O(MTK_{\text{min}}(M_T, K))$. Thus, the typical computational cost of one iteration of the SCA method grows approximately as $O(MTK^{3.5})$. The analysis of the number of the required number of SCA iterations is significantly more complicated and is often examined by numerical experiment. In the experiments below we will observe that the RD-based design converges in far fewer iterations than the ED-based design.

Those experiments, and others not reported here, have shown good convergence behaviour of the proposed algorithm for the RD-based designs. However, the approximations of the non-convex constraints described in point 3 above are generic approximations, and do not necessarily constitute an inner approximation of those constraints. As a result, Algorithm 1 does not necessarily satisfy the sufficient conditions in Reference [28] that theoretically guarantee convergence to a stationary point of (32). Nevertheless, the algorithm has performed well in our RD-based numerical examples and in most of our ED-based numerical examples; see, for example, Figures 2 and 5, below.

**Algorithm 1** Successive Convex Quadratic Approximation Algorithm for Equation (32)

**Initialisation:** Select an initial matrix $C^{(0)}$ with elements $c_{mk}$. Set $a^{(0)} = 1$, $n = 0$, and select a value for $a$.

**Step 1:** Update the iteration index $n \leftarrow n + 1$.

**Step 2:** Compute the SVD of $C^{(n-1)}$ to obtain $\tilde{U}^{(n-1)}$ and $\tilde{V}^{(n-1)}$.

**Step 3:** Given $C^{(n-1)}$, and for RD-based designs $\tilde{U}^{(n-1)}$ and $\tilde{V}^{(n-1)}$, solve the convex quadratic approximation of (32) to obtain $C^*$.

**Step 4:** Compute the step size $a^{(n)} = a^{(n-1)} (1 - a)$.

**Step 5:** Calculate $C^{(n)} = C^{(n-1)} + a^{(n)} (C^* - C^{(n-1)})$.

**Step 6:** Test for convergence and if the test fails return to Step 1.

## 5 | NUMERICAL EXPERIMENTS

We now examine the effectiveness of the proposed MIMO radar signal design using different orthonormal synthesis signal sets and using different distance measures.

### 5.1 | Sinusoidal signal set: Euclidean and Riemannian designs

Our test examples focus on a MIMO radar with two co-located linear arrays of transmission and reception antennas, each having four sensors. Each transmission waveform $y_m(t)$ is a linear combination of $K$ orthonormal signals, that is, $y_m(t) = \sum_{k=1}^{K} c_{mk} \psi_k(t)$. In our numerical experiments in this section we choose $K = 16$ and the orthonormal signal set to be a set of sinusoidal pulses: $\psi_k(t) = \cos(\omega_0 t)$ for $|t| \leq T/2$ with $\omega_0 = 2\pi \times 5$ rad/\(\mu s\) and $T = 1\mu s$. For the side-lobe constraints on the AF in Equation (32b), the bound on the norm-based side-lobe level is chosen to be $c(\tau, \omega) = 0.2 \|F(0, 0)\|^2$, and we will enforce a

| Case 1 | Case 2 |
|-------|-------|
| Main lobe | $|\tau| \leq 0.05 \mu s$ | $|\tau| \leq 0.033 \mu s$ |
| | $|\omega| \leq 2\pi \times 0.167 \text{ rad/}\mu s$ | $|\omega| \leq 2\pi \times 0.083 \text{ rad/}\mu s$ |
| Main lobe | $(\pm 0.03 \mu s, 0)$ | $(\pm 0.02 \mu s, 0)$ |
| Shaping points | $(0, \pm 2\pi \times 0.1 \text{ rad/}\mu s)$ | $(0, \pm 2\pi \times 0.05 \text{ rad/}\mu s)$ |

**TABLE 1** Characteristics of Cases 1 and 2

![Convergence of design algorithms in iterations](image.png)  
(a) Case 1  
(b) Case 2

**FIGURE 2** Convergence of design algorithms in iterations (sinusoids)
main lobe shaping constraint in Equation (32c) at four points, with \(\alpha(\tau, \omega) = 0.4\|F(0, 0)\|^2\). We will consider the two cases in Table 1, with Case 2 corresponding to a narrower main lobe.

5.1.1 Convergence rate

Figures 2(a) and (b) show the convergence rates of the designs in Case 1 and Case 2, respectively. The black and blue lines show the convergence of the ED objective function measured in ED and RD, respectively, whereas the green and red lines shows the convergence of the RD objective function measured respectively in ED and RD. It can be observed that in both Cases 1 and 2, the ED design takes more than 30 times the number of iterations to converge than the RD design. The slower convergence of the ED design can be explained partly by examining the approximate objective functions in Equations (35) and (39), respectively. We can see that \((\hat{U}^{(n-1)} \Sigma^{(n-1)} \hat{V}^{(n-1)H})\) in \(\hat{d}\) is a general matrix that is not necessarily unitary, whereas \(\hat{U}^{(n-1)} \Sigma^{(n-1)} \hat{V}^{(n-1)H}\) in \(\tilde{d}\) is always unitary. Since it is desired to minimise the difference between the designed CoV matrix and \(I\), then the matrix \(C\) in \(\tilde{d}\) should be close to unitary. Hence in the search for \(C\) under RD, the search space will be constrained to a relevant space smaller than the more general space in the case of searching under ED. As a result, we would expect the design of the CoV matrix under RD to converge faster than that under ED. For Case 1, both ED and RD objective functions converge to a low value, yielding synthesised signals that satisfy the requirements. As the constraint becomes tighter as in Case 2 (in particular, the required width of the main lobe of the AF becomes narrower), it is observed that the objective function using ED does not converge properly anymore, that is, the final value is no longer stable. However, in this case, the use of RD still converges well, showing greater robustness.

5.1.2 The MIMO ambiguity functions

Figures 3(a) and (b), respectively, show the sections of the MIMO AF \(|f(\tau, \omega, \theta^r, \phi^r)|\) of Equation (23) evaluated at \(\theta^r = \pi/2\) along the time-delay and the Doppler shift axes for the optimised ED and RD designs for Case 1. Here, it can be observed that both ED and RD designs satisfy the constraints almost equally, both having almost equally wide mainlobes and almost equally low sidelobes.

Figures 4(a) and (b) show respectively the sections of the MIMO AF along the time-delay and the Doppler shift axes for the optimised ED and RD designs for Case 2 in which the width of the AF mainlobe is constrained to be substantially narrower in both axes than in Case 1. (The differences in the main lobe widths can be seen by comparing Figures 3 and 4.) It is observed that while the AF of the RD design still maintains the amplitudes of its sidelobes below 20% of the peak of its mainlobe, the AF sidelobes of the ED design have overshot that fraction by a significant margin. This is due to the poor convergence behaviour of the algorithm in the ED design, as can be seen in Figure 2(b).

5.1.3 Distance and velocity estimation errors

We set the scenario of having \(P\) targets at distances \(d_1, \ldots, d_P\) from the reference point of the array, travelling, respectively, with velocities \(v_1, \ldots, v_P\) and all arriving at the same angle \(\theta = 30^\circ\) to the normal of the transmission antenna. (Here, the spacing between transmission antenna elements is taken to be \(d_c = \lambda_c/2\), thus rendering \(\theta = \pi/2\). The receiver signal-to-noise ratio (SNR) is assumed to be 3 dB. A threshold is set on the magnitude of the AF of the returned signal and time-frequency shifted local signal so that any local maximum greater than the threshold is considered a target candidate. The value of the threshold is determined by the Neyman–Pearson criterion fixing the false
alarm rate at 0.08. For an SNR of 3 dB, this stipulates the maximum sidelobe/mainlobe ratio to be 0.2. We now carry out the estimation of the distances and velocities of targets using the MIMO AF of Equation (14a) generated by the different optimally designed signals. For each set of transmitted signals estimating the distances and velocities of the $P$ targets, we, respectively, define the average normalised square errors of distance and velocity estimation as:

$$e_d^2 = \frac{1}{P} \sum_{p=1}^{P} \left( \frac{e_{dp}}{d_p} \right)^2,$$

$$e_v^2 = \frac{1}{P} \sum_{p=1}^{P} \left( \frac{e_{vp}}{v_p} \right)^2,$$  \hspace{1cm} (40)

where $e_{dp}$ and $e_{vp}$ are, respectively, the errors in the estimation of the distance and velocity of the $p$th target.

Table 2 shows the sum of the normalised squared errors ($e_d^2 + e_v^2$) in the estimation of the distances and velocities of two ($p = 2$) targets using the signal AF in Case 1. Recall that in Case 1, the objective functions of both ED and RD designs converge properly and the AF of both design are rather close. Thus, it is not surprising to observed here that, for both designs, the estimation errors are similar in magnitude with the RD design yielding slightly higher accuracies.

Table 3 shows the total squared error in the estimation of the distances and velocities of the same two targets using the signal AF in Case 2. Here, since the objective function of design algorithm for the ED design did not converge properly (see Figure 2(b)), we expect that the estimations carried out using the ED design to be inferior to those using the RD design. Indeed, we can observe that the accuracies of the estimations using the RD design are substantially higher. We can also observe that the estimations using the RD design in Case 2 is higher in accuracy than the corresponding estimations using the RD design in Case 1. This is because the AF in Case 2 has a narrower mainlobe thus the mutual interference of the two target mainlobes is reduced.

### 5.2 | WLJ signal set: Euclidean distance and Riemannian distance designs

In this series of experiments, we again focus on a MIMO radar with two co-located linear arrays of transmission and reception antennas, each having four sensors. Each of the transmitter antennas transmits a waveform which is made up of a linear combination of $K = 16$ orthonormal even WLJ signal pulses, that is, $y_m(t) = \sum_{k=0}^{K-1} C_m 2k \psi_{2k}(t)$ for $|t| \leq T/2$, where $\psi_{2k}(t)$ is given by Equation (17), the signal pulse duration is $T = 1 \mu s$, and the time-bandwidth product is $T \Omega / 2 \pi = 5$. The two cases of the constraints on the AF are the same as those in the previous example in Section 5.1, with both cases having the same sidelobe level, but Case 2 having a narrower main lobe; see Table 1.
5.2.1 Convergence rate

Figures 5(a) and (b) show the convergence rates of the designs in Case 1 and Case 2, respectively. The black and blue lines show the convergence of the ED objective function measured in ED and RD, respectively, whereas the green and red lines show the convergence of the RD objective function measured, respectively, in ED and RD. In both Cases 1 and 2, the ED design takes much longer to converge than the RD design. The explanation in Section 5.1 for the difference in convergence rates also applies here.

Unlike in the example in Section 5.1, the ED and RD design objectives converge to low values in both Cases 1 and 2. This shows that if the signals are synthesised by WLJ functions, even when the required width of the main lobe of the AF is stipulated to be narrower as in Case 2, the ED design becomes more robust than that in the sinusoidal synthesis, and, like that of the RD design, its objective function also converges properly.

5.2.2 The multi-input multi-output ambiguity functions

As in the example in Section 5.1, we specify the width of the main lobe differently in Case 1 and Case 2—being narrower in Case 2. Figure 6 shows the sections of the MIMO AF at \( \theta = \pi/2 \) along the time-delay and the Doppler shift axes for the optimised ED and RD designs. Here, it can be observed that both ED and RD designs satisfy the AF requirements easily.

Figure 7 shows Case 2 in which the width of the mainlobe of the AF is substantially narrower in both the time-delay and the Doppler shift axes. It is observed that while the AF of the RD design satisfies easily the mainlobe stipulations and has the sidelobes well below 20\% of the mainlobe peak, the AF of the ED design has some sidelobe amplitudes exceeding the 20\% level by a sizable margin along the delay-axis. On the other hand, the AF of both designs remains close to each other along the Doppler frequency-axis. Comparing the results here to those in the AF of Case 2 in Section 5.1, it can be observed that, because of the higher time-bandwidth energy concentration, using WLJ functions as the synthesising elements may result in a signal AF fitting better to tighter constraints than using sinusoids.

5.2.3 Distance and velocity estimation errors

Here, we repeat the scenario of the example in Section 5.1 such that \( P \) targets at distances \( d_1, \ldots, d_P \) from the reference point of the arrays, travelling with velocities \( v_1, \ldots, v_P \), respectively, all arriving at the same angle \( \theta = 30^\circ (\theta' = \pi/2) \) to the normal of the transmission antenna. The receiver signal-to-noise ratio (SNR) is set at 3dB. We now carry out the estimation of the distances and velocities of targets using the MIMO AF generated by the different optimally designed signals synthesised with the WLJ function set. The average normalised
square errors of distance and velocity estimation [as defined in Equation (40)] are then evaluated.

Table 4 shows the sum of the normalised squared errors \( \bar{e}_d^2 + \bar{e}_v^2 \) in the estimation of the distances and velocities of two targets using the signal AF in Case 1. Here, just like in Case 1 of Section 5.1, it can be observed that using the RD design yields moderately higher accuracies in the estimation than using the ED design.

Table 5 shows the total squared error in the estimation of the distances and velocities of two targets using the signal AF in Case 2. Here, since the objective functions of both the ED and RD designs converge properly, we expect that the estimations carried out are not much different from those in Case 1 above. Indeed, the estimation accuracies in Case 2 are marginally better than those in Case 1 due to the difference in mainlobe width.

5.3 | Bandwidth comparison

We now examine the time-bandwidth energy concentration of the optimal signals, respectively, synthesised by sinusoids and WLJ functions. Since in our examples, the transmission signal pulse is stipulated to have a duration of \( T = 1\, \mu s \), all the designed signals will have their energy accumulated within the duration, that is, \( \int_{-\infty}^{\infty} |\gamma(t)|^2 dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} |\gamma(t)|^2 dt = 1 \). Thus, to compare the time-bandwidth signal energy concentration, we

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**FIGURE 6** Case 1: multi-input multi-output (MIMO) AF sections

**FIGURE 7** Case 2: multi-input multi-output (MIMO) AF sections
just have to examine the bandwidth of the PSD of the designed signals; the narrower is the bandwidth, the more concentrated is the signal energy.

Figure 8 shows the PSD of the optimum signals at the four transmission antennas, respectively, synthesised by the sinusoidal set (blue) and the WLJ functions (red). Both sets of optimum signals are designed using the measure of RD for Case 1. As can be observed here, the –40dB bandwidth of the PSD of each of the sinusoid-synthesised optimum signals is approximately 90 Hz. On the other hand, the –40dB bandwidth of the PSD of each of the WLJ-synthesised optimum signals occupies less than half of the bandwidth of

| Locations (km) & velocities (km/h) | (30, 800) | (30.5, 800) | (30, 800) | (30.1, 800) | (30.05, 800) | (30.02, 800) |
|-----------------------------------|-----------|-------------|-----------|-------------|-------------|-------------|
| ED                                | 0         | 8.70e-7     | 7.30e-5   | 1.98e-4     |             |             |
| RD                                | 0         | 8.01e-7     | 7.69e-6   | 5.70e-5     |             |             |

| Locations (km) & velocities (km/h) | (30, 800) | (30.5, 800) | (30, 800) | (30.1, 800) | (30.05, 800) | (30.02, 800) |
|-----------------------------------|-----------|-------------|-----------|-------------|-------------|-------------|
| ED                                | 0         | 8.54e-7     | 1.36e-5   | 1.90e-4     |             |             |
| RD                                | 0         | 3.21e-7     | 4.38e-6   | 4.26e-5     |             |             |

**Figure 8** Power spectral density (PSD) of RD-designed signals (sinusoid synthesised) for case 1
the signal synthesised with sinuosids. Thus, we can infer that using WLJ functions as the signal synthesising set yields the advantage of having higher time-bandwidth signal energy concentration.

The designed signals in the other cases are also examined. It is observed that the PSD of the designed signal is independent of distance measure (ED or RD) in the design. Neither is the bandwidth of the signal PSD dependent on AF design specifications. Thus, for a fixed signal duration, the signal bandwidth only depends on the choice of the synthesising orthonormal signal set. Therefore, we will omit showing the PSD of the signals designed for the other cases here.

6 | CONCLUSION

We examine the problem of the design of optimum transmission signals for a MIMO radar. From the energy concentration point of view, we seek a signal vector having minimum duration-bandwidth occupancy. For this purpose, we suggest the use of the WLJ functions as the synthesising signal set. From the target estimation point of view, we seek a signal vector whose ambiguity matrix is closest to the ideal, that is, the identity matrix, while suppressing the AF sidelobes in time-delay and Doppler frequency shift. In the latter consideration, we argue that since covariance matrices are not freely structured but are Hermitian and positive semi-definite, the distance from the ideal should be measured using an RD on the manifold formed by these matrices rather than the traditional use of ED. By analysing the optimisations that use ED and RD, it is argued that the RD algorithm only needs to search for the optimum in a smaller space, resulting in faster convergence. The RD signal design also proves to have higher accuracy in both the range and velocity estimation of the targets. As well, in comparison to the ED design, the RD design yields signals which satisfy tighter constraints (such as a narrower mainlobe) on the AF. It is also observed that the use of WLJ functions as the synthesising signal set results in signals having only about half the bandwidth of that exhibited by signals synthesised from a sinusoidal signal set. Hence, we may conclude that employing the WLJ functions as the synthesising set and using RD as the distance measure for covariance optimisation are attractive measures conducive to improvements in MIMO radar signal design.

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ORCID

Kon Max Wong © https://orcid.org/0000-0002-2242-8070

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