VON NEUMANN ALGEBRAS ARISING FROM BOST-CONNES TYPE SYSTEMS

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Abstract. We show that the KMS$_\beta$-states of Bost-Connes type systems for number fields in the region $0 < \beta \leq 1$, as well as of the Connes-Marcolli GL$_2$-system for $1 < \beta \leq 2$, have type III$_1$. This is equivalent to ergodicity of various actions on adelic spaces. For example, the case $\beta = 2$ of the GL$_2$-system corresponds to ergodicity of the action of GL$_2(\mathbb{Q})$ on Mat$_2(\hat{A})$ with its Haar measure.

Introduction

The Bost-Connes system [4] is a C$^*$-dynamical system such that for inverse temperatures $\beta > 1$ the extremal KMS$_\beta$-states carry a free transitive action of the Galois group of the maximal abelian extension of $\mathbb{Q}$, have type I and partition function $\zeta(\beta)$, while for every $\beta \in (0,1]$ there exists a unique KMS$_\beta$-state of type III$_1$. The uniqueness and the type of the KMS$_\beta$-states in the critical interval $(0,1]$ are the most difficult parts of the analysis of the system. The result is equivalent to ergodicity of certain measures on the space $\hat{A}$ of adeles with respect to the action of $\mathbb{Q}^*$. In particular, the case $\beta = 1$ corresponds to a Haar measure $\mu_1$. To see why ergodicity of $\mu_1$ is nontrivial, observe that as $\mathbb{Q}$ is discrete in $\hat{A}$, the orbit of any point in $\hat{A}^*$ is discrete in $\hat{A}$, while the ergodicity implies that almost every orbit in $\hat{A}$ is dense. There is of course no contradiction since $\hat{A}^*$ is a subset of $\hat{A}$ of measure zero, but one does realize that it is difficult to immediately see a single dense orbit in $\hat{A}$.

Recently the construction of Bost and Connes has been generalized first to imaginary quadratic fields [8] and then to arbitrary number fields [11]. The crucial step of imaginary quadratic fields was achieved by introducing a universal system of quadratic fields, the so called GL$_2$-system of Connes and Marcolli [7]. In [18] and [16] we analyzed these systems in the critical intervals $0 < \beta \leq 1$ for number fields and $1 < \beta \leq 2$ for the GL$_2$-system, and showed that there exist unique KMS$_\beta$-states. The aim of the present paper is to prove that these KMS$_\beta$-states have type III$_1$. For number fields the proof is similar to the one for the original Bost-Connes system [4, 20]. The interesting case is that of the GL$_2$-system. It amounts to proving that the action of GL$_2(\mathbb{Q})$ on PGL$_2(\mathbb{R}) \times$ Mat$_2(\hat{A}_f)$ has type III$_1$ with respect to certain product-measures. After passing to the quotient space GL$_2(\mathbb{Z}) \backslash (\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\hat{A}_f))/\text{GL}_2(\hat{Z})$, we essentially use an argument showing that a nonzero element of the asymptotic ratio set is contained in the ratio set, written in terms of $L^2$-spaces rather than measure spaces and using a representation of the Hecke algebra $\mathcal{H}(\text{GL}_2(\mathbb{Q}), \text{GL}_2(\mathbb{Z}))$ instead of group actions. An additional difficulty is that we do not have a product decomposition of the representation of the Hecke algebra. In other words, the Hecke operators defined by elements of GL$_2(\mathbb{Z}[p^{-1}])$ act nontrivially on GL$_2(\mathbb{Z}) \backslash (\text{PGL}_2(\mathbb{R}) \times \prod_{q \neq p} \text{Mat}_2(\mathbb{Z}_q))/\text{GL}_2(\hat{Z})$. As a side remark, a similar problem would not arise for the finite part of the GL$_2$-system [17]. What saves the day for the full system is that this action is mixing on large subsets, which is a consequence of a variant of equidistribution of Hecke points [6].

Apart from some trivial cases when there is a subgroup of measure preserving transformations acting ergodically (for example, for GL$_2(\mathbb{Q})$ acting on $\mathbb{R}^2$), computations of ratio sets are usually quite hard, see e.g. [5, 12, 13]. A large class of type III$_1$ actions can be obtained as follows [22]. Let $G$ be a connected non-compact simple Lie group with finite center, $\Gamma \subset G$ a lattice and $P \subset G$
a parabolic subgroup. Then the action of $\Gamma$ on $G/P$ has type $\text{III}_1$. This is proved by identifying the underlying space of the associated flow with the measure-theoretic quotient $\Gamma\backslash G/P_0$, where $P_0$ is the kernel of the modular function of $P$, and using that the action of $P_0$ on $\Gamma\backslash G$ is mixing by Howe-Moore’s theorem [22]. With these examples in mind, it seems only natural that to compute the type of the states of the $\text{GL}_2$-system a form of adelic mixing is needed.

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1. Actions of Type $\text{III}_1$

Assume a countable group $G$ acts ergodically on a measure space $(X, \mu)$. The ratio set of the action [15] consists of all numbers $\lambda \geq 0$ such that for any $\varepsilon > 0$ and any subset $A \subset X$ of positive measure there exists $g \in G$ such that

$$\mu\left(\left\{x \in gA \cap A : \left|\frac{dg\mu}{d\mu}(x) - \lambda\right| < \varepsilon\right\}\right) > 0,$$

where $g\mu$ is the measure defined by $g\mu(Z) = \mu(g^{-1}Z)$. The ratio set depends only on the orbit equivalence relation $R = \{(x, gx) \mid x \in X, \ g \in G\} \subset X \times X$. We will denote it by $r(R)$. The set $r(R) \setminus \{0\}$ is a closed subgroup of $\mathbb{R}_+^*$. The action is said to be of type $\text{III}_1$ if this subgroup coincides with the whole group $\mathbb{R}_+^*$.

Denote by $\lambda_\infty$ the Lebesgue measure on $\mathbb{R}$. We have two commuting actions of $\mathbb{R}$ and $G$ on $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu)$,

$$g(t, x) = \left(\frac{dg\mu}{d\mu}(gx)t, gx\right) \quad \text{for} \quad g \in G, \ s(t, x) = (e^{-s}t, x) \quad \text{for} \quad s \in \mathbb{R}.$$

The flow of weights [9] of the von Neumann algebra $W^*(R)$ is the flow induced by the above action of $\mathbb{R}$ on the measure-theoretic quotient of $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu)$ by the action of $G$. The original action of $G$ on $(X, \mu)$ has type $\text{III}_1$ if and only if the flow of weights is trivial, that is, the action of $G$ on $(\mathbb{R}_+ \times X, \lambda_\infty \times \mu)$ is ergodic.

Let $\{(X_n, \mu_n)\}_{n=1}^\infty$ be a sequence of at most countable probability spaces. Put $(X, \mu) = \prod_n (X_n, \mu_n)$, and define an equivalence relation $R$ on $X$ by

$$x \sim y \quad \text{if} \quad x_n = y_n \quad \text{for all} \quad n \quad \text{large enough}.$$

For a finite subset $I \subset \mathbb{N}$ and $a \in \prod_{n \in I} X_n$ put

$$Z(a) = \{x \in X \mid x_n = a_n \text{ for } n \in I\}.$$  

The asymptotic ratio set $r_\infty(R)$ consists by definition [1] of all numbers $\lambda \geq 0$ such that for any $\varepsilon > 0$ there exist a sequence $\{I_n\}_{n=1}^\infty$ of mutually disjoint finite subsets of $\mathbb{N}$, disjoint subsets $K_n, L_n \subset \prod_{k \in I_n} X_k$ and bijections $\varphi_n : K_n \to L_n$ such that

$$\left|\frac{\mu(Z(\varphi_n(a)))}{\mu(Z(a))} - \lambda\right| < \varepsilon \quad \text{for all} \quad a \in K_n \text{ and } n \geq 1, \quad \text{and} \quad \sum_{n=1}^\infty \sum_{a \in K_n} \mu(Z(a)) = \infty.$$

It is known that $r_\infty(R) \setminus \{0\} = r(R) \setminus \{0\}$. We will only need the rather obvious inclusion $\subset$. 
2. Bost-Connes type systems for number fields

Suppose $K$ is an algebraic number field with subring of integers $O$. Denote by $V_K$ the set of places of $K$, and by $V_{K,f} \subset V_K$ the subset of finite places. For $v \in V_K$ denote by $K_v$ the corresponding completion of $K$. If $v$ is finite, let $O_v$ be the closure of $O$ in $K_v$. Denote also by $K_v = \prod_{v\in V_K} K_v$ the completion of $K$ at all infinite places. The adele ring $\mathbb{A}_K$ is the restricted product of the rings $K_v$ with respect to $O_v \subset K_v$, $v \in V_K$. When the product is restricted to $v \in V_{K,f}$, we get the ring $\mathbb{A}_{K,f}$ of finite adeles. The ring of finite integral adeles is $\hat{O} = \prod_{v \in V_{K,f}} O_v \subset \mathbb{A}_{K,f}$. We identify $\mathbb{A}_{K,f}^*$ with the subgroup of $\mathbb{A}_K^*$ consisting of elements with coordinates 1 for all infinite places.

Consider the topological space $G(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $G(K^{ab}/K)$ is the Galois group of the maximal abelian extension of $K$. On this space there is an action of the group $\mathbb{A}_{K,f}^*$ of finite ideles, via the Artin map $s: \mathbb{A}_K^* \to G(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm) \text{ for } j \in \mathbb{A}_{K,f}^*, \gamma \in G(K^{ab}/K), \ m \in \mathbb{A}_{K,f}.$$ 

Consider the quotient space $G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}$ by the action of $\hat{O}^* \subset \mathbb{A}_{K,f}^*$. On this space we have a quotient action of the group $\mathbb{A}_{K,f}^*/\hat{O}^*$, which is isomorphic to the group $J_K$ of fractional ideals.

One can define a Bost-Connes type system for $K$ [8, 11, 18] as the corner $pAp$ of the crossed product

$$A := C_0(G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}) \rtimes J_K,$$

where $p$ is the characteristic function of the clopen set $G(K^{ab}/K) \times \hat{O} \subset G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}$. We denote the algebra $pAp$ by $C_r^*(J_K \bowtie (G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}))$.

The dynamics is defined by

$$\sigma_t(fu_g) = N(g)^{it}fu_g \text{ for } f \in C_0(G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}) \text{ and } g \in J_K,$$

where $u_g$ denotes the element of the multiplier algebra of the crossed product corresponding to $g$, and $N: J_K \to (0, +\infty)$ is the absolute norm.

In [18] we showed that for every $\beta \in (0, 1]$ there exists a unique KMS$_\beta$-state $\varphi_\beta$ of the system. It is defined by the measure $\mu_\beta$ on $G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}$ which is the push-forward of the product measure $\mu_\beta \times \prod_{v \in V_{K,f}} \mu_{\beta,v}$ on $G(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mu_\beta$ is the normalized Haar measure on $G(K^{ab}/K)$, and $\mu_{\beta,v}$ is the unique measure on $K_v$ such that $\mu_{\beta,v}(O_v) = 1$ and

$$\mu_{\beta,v}(gZ) = \|g\|_v^\beta \mu_{\beta,v}(Z) \text{ for } g \in K_v^*,$$

where $\| \cdot \|_v$ is the normalized valuation in the class $v$, so $\|\pi\|_v = |O_v/p_v|^{-1} \text{ for any element } \pi$ generating the maximal ideal $p_v \subset O_v$.

**Theorem 2.1.** The KMS$_\beta$-states $\varphi_\beta$, $\beta \in (0, 1]$, have type III$_1$. In other words, for every $\beta \in (0, 1]$ the action of $J_K$ on $(G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}, \mu_\beta)$ is of type III$_1$.

By considering the flow of weights we can reformulate the result as follows. Denote by $K^*_+ \subset K^*$ the subgroup of totally positive elements, that is, elements $k \in K^*$ such that $\alpha(k) > 0$ for any real embedding $\alpha: K \hookrightarrow \mathbb{R}$. Denote by $\mu_{\beta,f}$ the measure $\prod_{v \in V_{K,f}} \mu_{\beta,v}$ on $\mathbb{A}_{K,f}$. Observe that for $\beta = 1$ we get a Haar measure on the additive group $\mathbb{A}_{K,f}$.

**Corollary 2.2.** For every $\beta \in (0, 1]$, the action of $K^*_+$ on $(\mathbb{R}_+ \times \mathbb{A}_{K,f}, \lambda_\infty \times \mu_{\beta,f})$ defined by $k(t, x) = (N(k)t, kx)$, is ergodic.

**Proof.** For any $g \in J_K$ we have $dg\mu_{\beta}/d\mu_\beta = N(g)^\beta$. Since the action of $J_K$ on $(G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}, \mu_\beta)$ is of type III$_1$, it follows that the action of $J_K$ on $(\mathbb{R}_+ \times (G(K^{ab}/K) \times \hat{O} \times \mathbb{A}_{K,f}), \lambda_\infty \times \mu_\beta)$ defined by $g(t, x) = (N(g)t, g(x))$ is ergodic. In other words, the action of $\mathbb{A}_{K,f}^*$ on

$$(\mathbb{R}_+ \times G(K^{ab}/K) \times \mathbb{A}_{K,f}, \lambda_\infty \times \mu_\beta \times \mu_{\beta,f})$$

defined by $g(t, x, y) = (N(g)t, s(g)^{-1}x, gy)$, is ergodic.
The Artin map $s: K^*_K \to \mathcal{G}(K^{ab}/K)$ is surjective with kernel $K^*/(K^*_\infty)^{0}$, where $(K^*_\infty)^{0}$ is the connected component of the identity in $K^*_\infty$. Since $K^*_K = K^*_\infty \times K^*_f$, it follows that the action of $K^* \times K^*_f$ on $(K^*_\infty/\mathcal{G}(K^{ab}/K) \times K^*_f, \lambda^\infty \times \nu \times \mu \times \mu_\beta)$, respectively. In other words, the action of $K^*$ on $(\mathbb{R}_+ \times (K^*_\infty/K^*_\infty)^{0}) \times K^*_f, \lambda^\infty \times \nu \times \mu_\beta)$, defined by $k(t, x, z) = (N(k)^{-1}t, k^{-1}x, k^{-1}g^{-1}y, g)$, is ergodic, where $\nu$ and $\mu$ are Haar measures on $K^*_\infty/K^*_\infty)^{0}$ and $K^*_f$, respectively. By (2.1) we also have $B((x, y, z, g)) = (t, x, y, z)$, where one can also find an estimate of the error term.

The action of $K^*_f$ on $(\mathbb{R}_+ \times K^*_f, \lambda^\infty \times \mu_\beta)$ is ergodic by the proof of [18, Theorem 2.1]. The computation of the ratio set will be based on the following lemma.

**Lemma 2.3.** For any $\beta \in (0, 1]$, $\lambda > 1$ and $\varepsilon > 0$ there exists a set $\{p_n, q_n\}_{n \geq 1}$ of prime ideals in $\mathcal{O}$ such that

$$\left| \frac{N(q_n)^\beta}{N(p_n)^\beta} - \lambda \right| < \varepsilon \quad \text{for all} \quad n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} N(q_n)^{-\beta} = \infty.$$ 

**Proof.** The proof is the same as in [2, 3] for $K = \mathbb{Q}$, the only difference is that instead of the prime number theorem one has to use the prime ideal theorem.

It suffices to consider the case $\beta = 1$. For $x > 0$ denote by $\pi(x)$ the number of prime ideals $p$ in $\mathcal{O}$ with $N(p) \leq x$. Then

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty,$$

see e.g. [19, Theorem 1.3], where one can also find an estimate of the error term.

Choose $\delta > 0$ such that $1 + \delta < \lambda$ and $\lambda \delta < \varepsilon$. Since

$$\frac{(1 + \delta)x}{\log((1 + \delta)x)} - \frac{x}{\log x} \sim \frac{\delta x}{\log x},$$

we get

$$\pi((1 + \delta)x) - \pi(x) \sim \frac{\delta x}{\log x}. \quad (2.1)$$

In other words, if we put $B(x) = \{p \mid x < N(p) \leq (1 + \delta)x\}$, then $|B(x)| \sim \frac{\delta x}{\log x}$. In particular, there exists $x_0 > 0$ such that $|B(\lambda x)| > |B(x)|$ for all $x \geq x_0$. Let $p_1, p_2, \ldots$ be an enumeration of the set $\cup_{m \geq 0} B(\lambda^{2m}x_0)$ such that $N(p_1) \leq N(p_2) \leq \ldots$. For each $m \geq 0$ choose a subset $C_m \subset B(\lambda^{2m+1}x_0)$ such that $|C_m| = |B(\lambda^{2m}x_0)|$. Let $q_1, q_2, \ldots$ be an enumeration of the set $\cup_{m \geq 0} C_m$ such that $N(q_1) \leq N(q_2) \leq \ldots$. Then, for every $n \geq 1$, if $p_n \in B(\lambda^{2m}x_0)$ then $q_n \in B(\lambda^{2m+1}x_0)$, so that

$$\lambda - \varepsilon < \frac{\lambda}{1 + \delta} < \frac{N(q_n)}{N(p_n)} < \lambda(1 + \delta) < \lambda + \varepsilon.$$

By (2.1) we also have

$$\sum_{n=1}^{\infty} N(q_n)^{-1} \geq \sum_{m=0}^{\infty} \frac{|B(\lambda^{2m+1}x_0)|}{(1 + \delta)\lambda^{2m+1}x_0} = \infty.$$
Proof of Theorem 2.1. Since \( \mathcal{G}(K^{ab}/K) \) is compact and totally disconnected, it suffices to show that the action of \( J_K \) on

\[
(\mathcal{G}(K^{ab}/K) \times \hat{\mathcal{O}}, \hat{\mathbb{A}}_{K,f}/\mathcal{G}(K^{ab}/K), \mu_\beta) = (\hat{\mathbb{A}}_{K,f}/\hat{\mathcal{O}}^*, \mu_{\beta,f})
\]

is of type \( \text{III}_1 \), see the proof of the main theorem in [20], as well as [16, Proposition 4.6] for a more general statement.

The measure space \((\hat{\mathcal{O}}/\hat{\mathcal{O}}^*, \mu_{\beta,f})\) can be identified with \( \prod_{v \in V_{K,f}} (\mathbb{Z}_+, \nu_{\beta,v}) \), where \( \nu_{\beta,v} \) is the measure defined by

\[
\nu_{\beta,v}(n) = N(p_v)^{-n\beta}(1 - N(p_v)^{-\beta}) \quad \text{for} \ n \geq 0,
\]

and the equivalence relation \( \mathcal{R} \) induced on \( \hat{\mathcal{O}}/\hat{\mathcal{O}}^* \) by the action of \( J_K \) on \( \hat{\mathbb{A}}_{K,f}/\hat{\mathcal{O}}^* \) is exactly the equivalence relation considered in our discussion of the asymptotic ratio set.

To compute \( r_\infty(\mathcal{R}) \), fix \( \lambda > 1 \) and \( \varepsilon > 0 \). Let \( \{p_n, q_n\}_{n \geq 1} \) be the set of prime ideals given by Lemma 2.3. Let \( v_n \) and \( w_n \) be the places corresponding to \( p_n \) and \( q_n \), respectively. Then we define the sets \( I_n \subset V_{K,f} \) and \( K_n, L_n \subset \prod_{v \in I_n} \mathbb{Z}_+ \) required by the definition of the asymptotic ratio set by

\[
I_n = \{v_n, w_n\}, \quad K_n = \{(0, 1)\}, \quad L_n = \{(1, 0)\},
\]

and denote by \( \varphi_n : K_n \to L_n \) the unique bijection. For \( a = (0, 1) \in K_n \) and \( b = (1, 0) \in L_n \) we have

\[
\frac{\mu_{\beta,f}(Z(b))}{\mu_{\beta,f}(Z(a))} = \frac{\nu_{\beta,v_n}(1) \nu_{\beta,w_n}(0)}{\nu_{\beta,v_n}(0) \nu_{\beta,w_n}(1)} = \frac{N(p_n)^{-\beta}(1 - N(p_n)^{-\beta})(1 - N(q_n)^{-\beta})}{N(q_n)^{-\beta}(1 - N(p_n)^{-\beta})(1 - N(q_n)^{-\beta})},
\]

which for large \( n \) is arbitrarily close to \( N(q_n)^{-\beta}/N(p_n)^{-\beta} \), which in turn is close to \( \lambda \) up to \( \varepsilon \). We also have

\[
\sum_{n=1}^{\infty} \sum_{a \in K_n} \mu_{\beta,f}(Z(a)) = \sum_{n=1}^{\infty} N(q_n)^{-\beta}(1 - N(p_n)^{-\beta})(1 - N(q_n)^{-\beta}) = \infty.
\]

It follows that \( \lambda \in r_\infty(\mathcal{R}) \). Since this is true for all \( \lambda > 1 \), we conclude that the action is of type \( \text{III}_1 \). \( \square \)

Remark 2.4. In view of Corollary 2.2 it is natural to ask whether the action of \( K^* \) on \( (K_\infty \times \hat{\mathbb{A}}_{K,f}, \lambda_\infty \times \mu_{\beta,f}) \) is ergodic, where \( \lambda_\infty \) is a Haar measure on \( K_\infty \cong \mathbb{R}^{[K: \mathbb{Q}]} \); see also Remark 3.7(ii) below for a more general question. Assume for simplicity that \( K \) is an imaginary quadratic field of class number one. Then one can try to prove that the action of \( K^* \) on \( \mathbb{T} \times \hat{\mathbb{A}}_{K,f} \) given by \( k(z, x) = (k|z|^{-1}z, kx) \), is ergodic, e.g. by adapting the strategy in [20, 18], and then compute the ratio set of this action. However, for the latter one would need an information about the distribution of the angles of prime ideals, that is, of the values of the homomorphism \( J_K \to \mathbb{T}/\mathcal{O}^*, (k) \to k|k|^{-1} \). We are not aware of any result of this sort.

3. The Connes-Marcolli \( \text{GL}_2 \)-system

Let \( G \) be a discrete group and \( \Gamma \) be a subgroup of \( G \). Recall that \( (G, \Gamma) \) is called a Hecke pair if every double coset of \( \Gamma \) contains finitely many right cosets of \( \Gamma \), so that

\[
R_\Gamma(g) := |\Gamma \backslash \Gamma g \Gamma| < \infty \quad \text{for any} \quad g \in G.
\]

Then the space \( \mathcal{H}(G, \Gamma) \) of finitely supported functions on \( \Gamma \backslash G/\Gamma \) is a \( * \)-algebra with product

\[
(f_1 * f_2)(g) = \sum_{h \in \Gamma \backslash G} f_1(gh^{-1})f_2(h)
\]

and involution \( f^*(g) = \overline{f(g^{-1})} \). Denote by \([g] \in \mathcal{H}(G, \Gamma)\) the characteristic function of the double coset \( \Gamma g \Gamma \).
If $G$ acts on a space $X$ then every element $g \in G$ defines a Hecke operator $T_g$ acting on functions on $\Gamma \backslash X$, which we also consider as $\Gamma$-invariant functions on $X$:

$$(T_g f)(x) = \frac{1}{R_{\Gamma}(g)} \sum_{h \in \Gamma \backslash \Gamma} f(hx).$$

Then $[g^{-1}] \mapsto R_{\Gamma}(g)T_g$ is a representation of the Hecke algebra $\mathcal{H}(G, \Gamma)$ on the space of $\Gamma$-invariant functions.

If $X$ is locally compact and the action of $\Gamma$ on $X$ is proper, one can define a C*-algebra $C_r^*(\Gamma \backslash G \times \Gamma X)$ which can be thought of as a crossed product of $C_0(\Gamma \backslash X)$ by $\mathcal{H}(G, \Gamma)$, see [7, 16]. It is a completion of the algebra $C_r(\Gamma \backslash G \times \Gamma X)$ of continuous compactly supported functions on $\Gamma \backslash G \times \Gamma X$ with convolution product

$$(f_1 * f_2)(g, x) = \sum_{h \in \Gamma \backslash G} f_1(gh^{-1}, hx)f_2(h, x)$$

and involution $f^*(g, x) = \overline{f(g^{-1}, gx)}$. If the action of $\Gamma$ is free then $\Gamma \backslash G \times \Gamma X$ is a groupoid and $C_r^*(\Gamma \backslash G \times \Gamma X)$ is the usual groupoid C*-algebra.

Consider now the Hecke pair $(\text{GL}_2^+ (\mathbb{Q}), \text{SL}_2(\mathbb{Z}))$, where $\text{GL}_2^+ (\mathbb{Q})$ is the group of rational matrices with positive determinant. The group $\text{GL}_2^+ (\mathbb{Q})$ acts by multiplication on $\text{Mat}_2(\mathbb{Q}_p)$ for every prime $p$. It also acts by Möbius transformations on the upper half-plane $\mathbb{H}$. The $\mathbb{GL}_2$-system of Connes and Marcolli [7] is the corner of the C*-algebra

$$C_r^*(\text{SL}_2(\mathbb{Z}) \backslash \text{GL}_2^+ (\mathbb{Q}) \times \text{SL}_2(\mathbb{Z})) (\mathbb{H} \times \text{Mat}_2(\hat{A}_f))$$

defined by the projection corresponding to the subspace $\mathbb{H} \times \text{Mat}_2(\hat{Z}) \subset \mathbb{H} \times \text{Mat}_2(\hat{A}_f)$, where $\hat{A}_f = \hat{A}_{Q, f}$. We denote this algebra by $C_r^*(\text{SL}_2(\mathbb{Z}) \backslash \text{GL}_2^+ (\mathbb{Q}) \boxtimes \text{SL}_2(\mathbb{Z})) (\mathbb{H} \times \text{Mat}_2(\hat{Z}))$. The dynamics on it is defined by

$$\sigma_t(f)(g, x) = \det(g)^{it} f(g, x).$$

In [16] we showed that for every $\beta \in (1, 2]$ there exists a unique KMS$_\beta$-state $\varphi_\beta$ on the $\mathbb{GL}_2$-system. It is defined by the product-measure $\mu_{\mathbb{H}} \times \prod_p \mu_{\beta, p}$ on $\mathbb{H} \times \text{Mat}_2(\hat{A}_f)$, where $\mu_{\mathbb{H}}$ is the unique $\text{GL}_2^+(\mathbb{R})$-invariant measure on $\mathbb{H}$ such that $\mu_{\mathbb{H}}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = 2$, and $\mu_{\beta, p}$ is the unique measure on $\text{Mat}_2(\mathbb{Q}_p)$ such that $\mu_{\beta, p}(\text{Mat}_2(\mathbb{Z}_p)) = 1$ and

$$\mu_{\beta, p}(gZ) = |\det(g)|^\beta p \mu_{\beta, p}(Z) \quad \text{for} \quad g \in \text{GL}_2(\mathbb{Q}_p).$$

Denote the measure $\prod_p \mu_{\beta, p}$ by $\mu_{\beta, f}$. Observe that $\mu_{2, f}$ is the Haar measure of the additive group $\text{Mat}_2(\hat{A}_f)$ normalized so that $\mu_{2, f}(\text{Mat}_2(\hat{Z})) = 1$.

The definition of the $\mathbb{GL}_2$-system required a new type of crossed product construction because of non-freeness of the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H} \times \text{Mat}_2(\hat{A}_f)$. However, the set of points with nontrivial stabilizers is $\mathbb{H} \times \{0\}$, which has measure zero with respect to $\mu_{\mathbb{H}} \times \mu_{\beta, f}$. As a result the von Neumann algebra generated by the $\mathbb{GL}_2$-system in the GNS-representation of $\varphi_\beta$ is much easier to describe. It is the reduction of the von Neumann algebra crossed product $L^\infty(\mathbb{H} \times \text{Mat}_2(\hat{A}_f), \mu_{\mathbb{H}} \times \mu_{\beta, f}) \times \text{GL}_2^+(\mathbb{Q})$ by the projection defined by a fundamental domain of the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H} \times (\text{Mat}_2(\hat{Z}) \backslash \{0\})$. Therefore to compute the type of the algebra we have to compute the type of the action of $\text{GL}_2^+(\mathbb{Q})$ on $\mathbb{H} \times \text{Mat}_2(\hat{A}_f)$.

It is natural to consider a slightly more general problem. Namely, replace $\mathbb{H} = \text{PGL}_2(\mathbb{R})/\text{PSO}_2(\mathbb{R})$ by $\text{PGL}_2(\mathbb{R})$ and $\text{GL}_2^+(\mathbb{Q})$ by $\text{GL}_2(\mathbb{Q})$. Denote by $\mu_\infty$ the Haar measure of $\text{PGL}_2(\mathbb{R})$ normalized so that $\mu_\infty(\text{GL}_2(\mathbb{Z}) \backslash \text{PGL}_2(\mathbb{R})) = 2$. Put $\mu_\beta = \mu_\infty \times \mu_{\beta, f}$. The action of $\text{GL}_2(\mathbb{Q})$ on $\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\hat{A}_f), \mu_\beta)$ is ergodic by [16, Corollary 4.7].

**Theorem 3.1.** For every $\beta \in (1, 2]$, the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\hat{A}_f), \mu_\beta)$ has type $\text{III}_1$. In particular, the KMS$_\beta$-states $\varphi_\beta$, $\beta \in (1, 2]$, of the Connes-Marcolli $\text{GL}_2$-system have type $\text{III}_1$. 

As we already remarked in [16], the flows of weights of the above actions are easy to describe, and then the result takes the following essentially equivalent form. Denote by $\lambda_\infty$ the usual Lebesgue measure on $\text{Mat}_2(\mathbb{R}) \cong \mathbb{R}^4$, and put $\lambda_\beta = \lambda_\infty \times \mu_\beta$. For $\beta = 2$ we get a Haar measure on the additive group $\text{Mat}_2(\mathbb{A}) = \text{Mat}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f)$, where $\mathbb{A} = \mathbb{A}_Q$.

**Corollary 3.2.** For every $\beta \in (1, 2]$, the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{Mat}_2(\mathbb{A}), \lambda_\beta)$ is ergodic.

**Proof.** If the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{PGL}_2^+(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f), \mu_\beta)$ is of type III$_1$ then clearly also the action of $\text{GL}_2^+(\mathbb{Q})$ on $(\text{PGL}_2^+(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f), \mu_\beta)$ is of type III$_1$. As we discussed in [16, Remark 4.9], using the isomorphism $\text{GL}_2^+(\mathbb{R})/\{\pm 1\} \cong \mathbb{R}_+ \times \text{PGL}_2^+(\mathbb{R})$ we can identify the underlying space of the flow of weights of this action with the quotient of $((\text{GL}_2^+(\mathbb{R})/\{\pm 1\}) \times \text{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ by the diagonal action of $\text{GL}_2^+(\mathbb{Q})$. Therefore this diagonal action is ergodic. Since $\text{GL}_2^+(\mathbb{R})$ is connected and $\{\pm 1\}$ is finite, by [16, Proposition 4.6] we conclude that the action of $\text{GL}_2^+(\mathbb{Q})$ on $(\text{GL}_2^+(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ is ergodic. But then the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{GL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ is also ergodic.

To simplify notation from now on we write $G$ for $\text{GL}_2(\mathbb{Q})$, $\Gamma$ for $\text{GL}_2(\mathbb{Z})$ and $X$ for $\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f)$.

Recall, see e.g. [14], that the group $G$ is generated by $\Gamma$ and the matrices $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of prime numbers. We have

$$\{m \in \text{Mat}_2(\mathbb{Z}) : |\det(m)| = p\} = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma \quad \text{and} \quad R_\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = p + 1.$$  \hspace{1cm} (3.1)

Recall also, see [16, Section 3], that

$$\mu_{\beta, p}(\text{GL}_2(\mathbb{Z}_p)) = (1 - p^{-\beta})(1 - p^{-\beta + 1}),$$  \hspace{1cm} (3.2)

Using the scaling property of $\mu_{\beta, p}$ and (3.1) we then conclude that

$$\mu_{\beta, p}(\{m \in \text{Mat}_2(\mathbb{Z}_p) : |\det(m)|_p = p^{-1}\}) = p^{-\beta}(p + 1)(1 - p^{-\beta})(1 - p^{-\beta + 1}).$$  \hspace{1cm} (3.3)

One of the key ingredients of the proof of the theorem will be the following version of equidistribution of Hecke points. I am indebted to Hee Oh for explaining me how to put it in the general setup of equidistribution of Hecke points for reductive groups.

Denote by $G_p$ the group generated by $\Gamma$ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Equivalently, $G_p$ is the group $\text{GL}_2(\mathbb{Z}[p^{-1}])$. For a nonempty subset $F$ of primes denote by $G_F$ the group generated by $G_p$ for all $p \in F$. For finite $F$ denote by $X_F$ the space $\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{Q}_F)$, where $\mathbb{Q}_F = \prod_{p \in F} \mathbb{Q}_p$, and by $\tilde{\mu}_{\beta, F}$ the measure $\mu_\infty \times \prod_{p \in F} \mu_{\beta, p}$. Consider also the measure $\tilde{\nu}_{\beta, F}$ on $\Gamma \setminus X_F$ defined by $\tilde{\nu}_{\beta, F}$. We shall write $\mathbb{Z}_F$ for $\prod_{p \in F} \mathbb{Z}_p$.

**Lemma 3.3.** Let $F$ be a finite set of primes, $r \in \text{GL}_2(\mathbb{Q}_F)$. Assume $f$ is a compactly supported continuous right $\text{GL}_2(\mathbb{Z}_F)$-invariant function on $Z = \Gamma \setminus ((\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}_F)r\text{GL}_2(\mathbb{Z}_F)) \alpha \setminus X_F)$. Then for any $\varepsilon > 0$ and any compact subset $C$ of $Z$ there exists $M > 0$ such that if $g \in G_{F^c} \quad (F^c = \mathcal{P} \setminus F)$ and $R_\Gamma(g) \geq M$ then

$$|T_g f(x) - \tilde{\nu}_{\beta, F}(Z)^{-1} \int_Z f \, d\tilde{\nu}_{\beta, F}| < \varepsilon \quad \text{for all} \quad x \in C.$$  

**Proof.** The $\text{GL}_2(\mathbb{Z}_F)$-space $\text{GL}_2(\mathbb{Z}_F)r\text{GL}_2(\mathbb{Z}_F)/\text{GL}_2(\mathbb{Z}_F)$ can be identified with $\text{GL}_2(\mathbb{Z}_F)/H$, where $H = r\text{GL}_2(\mathbb{Z}_F)r^{-1} \cap \text{GL}_2(\mathbb{Z}_F)$. Therefore $f$ can be considered as a function on $\Gamma \setminus ((\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}_F))/H$. Consider the compact open subgroup $U = H \times \prod_{p \in F^c} \text{GL}_2(\mathbb{Z}_p)$ of $\text{GL}_2(\mathbb{Z})$. By considering $\text{GL}_2(\mathbb{Z}_F)$ as the subgroup of $\text{GL}_2(\mathbb{Z})$ consisting of elements with coordinates 1 for $p \in F^c$, we get a homeomorphism

$$\Gamma \setminus ((\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}_F))/H) \to G \setminus ((\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{A}_f))/U,$$
since $G \cap \text{GL}_2(\hat{\mathbb{Z}}) = \Gamma$ and $\text{GL}_2(\mathbb{A}_f) = G\text{GL}_2(\hat{\mathbb{Z}})$. Furthermore, we have $\text{GL}_2(\mathbb{A}_f) = GU$. To see this recall that $\text{SL}_2(\mathbb{Q})$ is dense in $\text{SL}_2(\mathbb{A}_f)$ by the strong approximation theorem. It follows that $GU$ contains $\text{SL}_2(\mathbb{A}_f)$, and in particular $\text{SL}_2(\hat{\mathbb{Z}})$. Since $\text{GL}_2(\mathbb{A}_f) = G\text{GL}_2(\hat{\mathbb{Z}})$, it is therefore enough to check that $\text{SL}_2(\hat{\mathbb{Z}})U = \text{GL}_2(\hat{\mathbb{Z}})$, that is, $\text{SL}_2(\mathbb{Z}_F)H = \text{GL}_2(\mathbb{Z}_F)$. For this we have to show that the determinant map $\det: H \to \mathbb{Z}_F^*$ is surjective. Since every double coset of $\text{GL}_2(\mathbb{Z}_F)$ contains a diagonal matrix, without loss of generality we may assume that $r$ is diagonal. But then $H$ contains all the diagonal matrices of $\text{GL}_2(\mathbb{Z}_F)$, and surjectivity of the determinant is immediate.

We can then proceed as in [6], see Remark (3) following [6, Theorem 1.7], as well as Sections 2 and 3 in [10]. □

For a $\Gamma$-invariant measurable subset $A$ of $\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f)$, denote by $m(A)$ the operator of multiplication by the characteristic function of $A$ on $L^2(\Gamma \setminus X, d\nu_\beta)$, where $\nu_\beta$ is the measure on $\Gamma \setminus X$ defined by $\mu_\beta$.

**Lemma 3.4.** For a prime $p$, consider the sets
\[
A_p = \{ x \in \text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\hat{\mathbb{Z}}) : x_p \in \text{GL}_2(\mathbb{Z}_p) \},
\]
\[
B_p = \{ x \in \text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\hat{\mathbb{Z}}) : |\det(x_p)|_p = p^{-1}, \}
\]
and the element $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Then for the operator $m(A_p)T_g m(B_p) = m(A_p)T_g$ on $L^2(\Gamma \setminus X, d\nu_\beta)$ we have
\[
\| m(A_p)T_g m(B_p) \| = p^{\beta/2}(p + 1)^{-1/2} = \nu_\beta(\Gamma \setminus A_p)^{1/2} \nu_\beta(\Gamma \setminus B_p)^{-1/2}.
\]

**Proof.** Since $B_p = \Gamma g A_p$, $R_\Gamma(g) = p + 1$ and $|T_g(f)|^2 \leq T_g(|f|^2)$ pointwise by convexity, this follows from [16, Lemma 2.7], but we will sketch a proof for the reader’s convenience.

Fix representatives $h_1, \ldots, h_{p+1}$ of $\Gamma \setminus \Gamma g \Gamma$. Choose a fundamental domain $D$ for the action of $\Gamma$ on $A_p$. Using that the action of $\Gamma$ on $A_p$ is free and that $G_p \cap \text{GL}_2(\mathbb{Z}_p) = \Gamma$ one can easily check that the sets $\Gamma h_i C$ are mutually disjoint and the factor-map $p: X \to \Gamma \setminus X$ is injective on the sets $h_i C$.

Consider the operators $S_i$ defined by
\[
(S_i f)(p(x)) = \begin{cases} f(p(h_i x)), & \text{if } x \in C, \\ 0, & \text{if } x \notin A_p. \end{cases}
\]

Then $p^{-\beta/2}S_i$ is a partial isometry with initial space $L^2(p(h_i C), d\nu_\beta)$ and range $L^2(p(C), d\nu_\beta)$, and $m(A_p)T_g m(B_p) = (S_1 + \cdots + S_{p+1})/(p+1)$. Since the spaces $L^2(p(h_i C), d\nu_\beta)$ are mutually orthogonal, we have
\[
\| p^{-\beta/2}S_1 + \cdots + p^{-\beta/2}S_{p+1} \| = (p + 1)^{1/2},
\]
which gives the first equality in the statement. The second equality follows from (3.2) and (3.3). □

For $x \in X$ denote by $\bar{x}_F$ the image of $x$ under the factor-map $X \to X_F$. For a function $f$ on $X_F$ consider the function $f_F$ on $X$ defined by
\[
(f_F(x)) = \begin{cases} f(\bar{x}_F), & \text{if } x_p \in \text{Mat}_2(\mathbb{Z}_p) \text{ for all } p \notin F, \\ 0, \text{ otherwise.} \end{cases}
\]

**Lemma 3.5.** For any $\beta \in (1, 2]$ and $\lambda > 1$ there exists $c > 0$ such that for any $\varepsilon > 0$, any finite set $F$ of primes and any positive compactly supported continuous right $\text{GL}_2(\mathbb{Z}_F)$-invariant function $f$ on $\Gamma \setminus (\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{Z}_F)) \subset \Gamma \setminus X_F$ with $\int_{\Gamma \setminus X_F} f d\nu_\beta, F = 1$, there exist a set $\{p_n, q_n\}_{n \geq 1}$ of $F^c$ and $\Gamma$-invariant measurable subsets $X_{1n}, X_{2n}, Y_{1n}$ and $Y_{2n}$, $n \geq 1$, of $X$ such that
(i) $\left| \frac{q_n}{p_n} - \lambda \right| < \varepsilon$ for all $n \geq 1$;
(ii) the sets $Y_{1n}, n \geq 1$, as well as the sets $Y_{2n}, n \geq 1$, are mutually disjoint;
(iii) \( \sum_{n=1}^{\infty} \left( \frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|} f_F, \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|} f_F \right)_{L^2(\Gamma \setminus X, d\nu_\beta)} > c, \) \( \text{where } g_n = \begin{pmatrix} 1 & 0 \\ 0 & p_n \end{pmatrix} \) and \( h_n = \begin{pmatrix} 1 & 0 \\ 0 & q_n \end{pmatrix} \).

Proof. Fix \( \delta \in (0,1) \). Choose representatives \( r_k, k \geq 1, \) of the double cosets
\[
\text{GL}_2(\mathbb{Z}_F) \backslash (\text{GL}_2(\mathbb{Q}_F) \cap \text{Mat}_2(\mathbb{Z}_F)) / \text{GL}_2(\mathbb{Z}_F),
\]
and put \( Z_k = \Gamma \setminus (\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Z}_F) r_k \text{GL}_2(\mathbb{Z}_F)) \). Let \( N \) be such that
\[
\sum_{k=1}^{N} \int_{Z_k} f d\tilde{\nu}_{\beta,F} > \int_{\Gamma \setminus X_F} f d\tilde{\nu}_{\beta,F} - \delta = 1 - \delta.
\]

Next choose compact subsets \( C_k \) of \( Z_k \) such that
\[
\tilde{\nu}_{\beta,F}(C_k) > (1 - \delta) \tilde{\nu}_{\beta,F}(Z_k) \text{ for } k = 1, \ldots, N.
\]

By Lemma \( 3.3 \) there exists \( M \) such that for any element \( g \in G_{F^c} \) with \( R_\Gamma(g) \geq M \) we have
\[
T_g f(x) \geq \frac{1 - \delta}{\tilde{\nu}_{\beta,F}(Z_k)} \int_{Z_k} f d\tilde{\nu}_{\beta,F} \text{ for } x \in C_k, \ k = 1, \ldots, N.
\]

It follows that if we take two elements \( g, h \in G_{F^c} \) with \( R_\Gamma(g), R_\Gamma(h) \geq M \), then
\[
\int_{\Gamma \setminus X_F} T_g f T_h f d\tilde{\nu}_{\beta,F} \geq \sum_{k=1}^{N} \int_{C_k} T_g f T_h f d\tilde{\nu}_{\beta,F} \geq \sum_{k=1}^{N} \left( \frac{1 - \delta}{\tilde{\nu}_{\beta,F}(Z_k)} \int_{Z_k} f d\tilde{\nu}_{\beta,F} \right)^2 \tilde{\nu}_{\beta,F}(C_k) \geq (1 - \delta)^3 \sum_{k=1}^{N} \left( \int_{Z_k} f d\tilde{\nu}_{\beta,F} \right)^2 \tilde{\nu}_{\beta,F}(Z_k) \geq (1 - \delta)^3 \left( \sum_{k=1}^{N} \int_{Z_k} f d\tilde{\nu}_{\beta,F} \right)^2,
\]

since the function \( t \mapsto t^2 \) is convex and \( \sum_{k=1}^{N} \tilde{\nu}_{\beta,F}(Z_k) \leq \tilde{\nu}_{\beta,F}(\Gamma \setminus (\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{Z}_F))) = 1. \)

Therefore
\[
\int_{\Gamma \setminus X_F} T_g f T_h f d\tilde{\nu}_{\beta,F} \geq (1 - \delta)^5.
\]

Using Lemma \( 2.3 \) choose a subset \( \{p_n, q_n\}_{n \geq 1} \) of \( F^c \) such that \( q_n > p_n \geq M \) and \( |q_n^\beta/p_n^\beta - \lambda| < \varepsilon \) for all \( n \), and
\[
\sum_{n} p_n^{1-\beta} = \infty. \tag{3.5}
\]

Consider the sets \( A_p \) and \( B_p \) from Lemma \( 3.4 \) and put
\[
X_{1n} = A_{p_n} \setminus (B_{p_1} \cup \cdots \cup B_{p_{n-1}}), \quad Y_{1n} = B_{p_n} \setminus (B_{p_1} \cup \cdots \cup B_{p_{n-1}}),
\]
\[
X_{2n} = A_{q_n} \setminus (B_{q_1} \cup \cdots \cup B_{q_{n-1}}), \quad Y_{2n} = B_{q_n} \setminus (B_{q_1} \cup \cdots \cup B_{q_{n-1}}).
\]

Let \( g_n = \begin{pmatrix} 1 & 0 \\ 0 & p_n \end{pmatrix} \) and \( h_n = \begin{pmatrix} 1 & 0 \\ 0 & q_n \end{pmatrix} \).

We claim that if \( g \in \Gamma g_n \Gamma \) and \( x \in X_{1n} \) then \( gx \in Y_{1n} \). Indeed, we clearly have \( gA_{p_n} \subset B_{p_n} \), so that \( gx \in B_{p_n} \). Furthermore, if \( gx \in B_{p_k} \) for some \( k < n \) then \( p_k^{-1} = |\det(gx|_{p_k})|_{p_k} = |\det(x|_{p_k})|_{p_k} \), since \( g_n \in \text{GL}_2(\mathbb{Z}_{p_k}) \). Therefore \( x \in B_{p_k} \), which contradicts the assumption that \( x \in X_{1n} \). Hence \( gx \in Y_{1n} \).
It follows that \( m(X_1)T_m, m(Y_1)f_F = (T_m, f)_F \Gamma \backslash X_1 \). For similar reasons, \( m(X_2)T_m, m(Y_2)f_F = (T_m, f)_F \Gamma \backslash X_2 \). Therefore
\[
\begin{align*}
(m(X_1)T_m, m(Y_1)f_F, m(X_2)T_m, m(Y_2)f_F)_{L^2(\Gamma \backslash X_1, d\nu)} \\
= (T_m, f)_F L^2(\Gamma \backslash X_2, d\nu, f_F) \nu_\beta(\Gamma \backslash (X_1 \cap X_2)) \geq (1 - \delta)^5 \nu_\beta(\Gamma \backslash (X_1 \cap X_2))
\end{align*}
\]
by (3.4).

By Lemma 3.4 we have
\[
\|m(X_1)T_m, m(Y_1)\| \leq \nu_\beta(\Gamma \backslash B_{p_n})^{-1/2} \quad \text{and} \quad \|m(X_2)T_m, m(Y_2)\| \leq \nu_\beta(\Gamma \backslash B_{q_n})^{-1/2}.
\]
If follows that
\[
\sum_{n=1}^\infty \left( \frac{m(X_1)T_m, m(Y_1)}{\|m(X_1)T_m, m(Y_1)\|} f_F, \frac{m(X_2)T_m, m(Y_2)}{\|m(X_2)T_m, m(Y_2)\|} f_F \right)_{L^2(\Gamma \backslash X_1, d\nu)} \geq (1 - \delta)^5 \sum_{n=1}^\infty (\nu_\beta(\Gamma \backslash B_{p_n}) \nu_\beta(\Gamma \backslash B_{q_n}))^{1/2} \nu_\beta(\Gamma \backslash (X_1 \cap X_2)). \tag{3.6}
\]
We have
\[
\nu_\beta(\Gamma \backslash (X_1 \cap X_2)) = \nu_\beta(\Gamma \backslash A_{p_n}) \nu_\beta(\Gamma \backslash A_{q_n}) \prod_{k=1}^{n-1} (1 - \nu_\beta(\Gamma \backslash (B_{p_k} \cup B_{q_k}))) \tag{3.7}
\]
We may assume that \( M \) is so large that
\[
\nu_\beta(\Gamma \backslash A_{p_n}) \nu_\beta(\Gamma \backslash A_{q_n}) = (1 - p_n^{-\beta})(1 - p_n^{-\beta + 1})(1 - q_n^{-\beta})(1 - q_n^{-\beta + 1}) \geq 1 - \delta, \tag{3.8}
\]
see (3.2). Since
\[
\nu_\beta(B_p) = p^{-\beta}(p + 1)(1 - p^{-\beta})(1 - p^{-\beta + 1}) \sim p^{1-\beta}
\]
by (3.3), we may also assume that \( M \) is so large and the ratios \( q_n^{1/p_n^{1/\beta}} \) are so close to \( \lambda \) that
\[
(\nu_\beta(\Gamma \backslash B_{p_n}) \nu_\beta(\Gamma \backslash B_{p_n}))^{1/2} > (1 - \delta)c_0 \left( \nu_\beta(\Gamma \backslash B_{p_n}) + \nu_\beta(\Gamma \backslash B_{q_n}) - \nu_\beta(\Gamma \backslash B_{p_n}) \nu_\beta(\Gamma \backslash B_{q_n}) \right) = (1 - \delta)c_0 \nu_\beta(\Gamma \backslash (B_{p_n} \cup B_{q_n})), \tag{3.9}
\]
where
\[
c_0 = \frac{\lambda^{(1 - \beta)/2} \beta}{1 + \lambda^{(1 - \beta)/2}}.
\]
Combining (3.7)-(3.9) we conclude that the right hand side of (3.6) is not smaller than
\[
(1 - \delta)^7 c_0 \sum_{n=1}^\infty \nu_\beta(\Gamma \backslash (B_{p_n} \cup B_{q_n})) \prod_{k=1}^{n-1} (1 - \nu_\beta(\Gamma \backslash (B_{p_k} \cup B_{q_k}))) = (1 - \delta)^7 c_0,
\]
because
\[
\sum_{n=1}^\infty \nu_\beta(\Gamma \backslash (B_{p_n} \cup B_{q_n})) \geq \sum_{n=1}^\infty \nu_\beta(\Gamma \backslash B_{p_n}) = \infty
\]
by (3.5). Since \( \delta \) can be arbitrarily small, we see that we can take any \( c < c_0 \).

\[\square\]

**Proof of Theorem 3.1.** Similarly to the proof of Theorem 2.1, since \( \text{GL}_2(\mathbb{Z}) \) is compact and totally disconnected, it suffices to show that the action of \( \text{GL}_2(\mathbb{Q}) \) on \( (\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_f)) / \text{GL}_2(\mathbb{Z}), \mu_\beta) \) is of type III. In other words, in computing the ratio set it suffices to consider right \( \text{GL}_2(\mathbb{Z}) \)-invariant sets.

Let \( \lambda > 1, \varepsilon > 0 \) and \( Y \) be a measurable right \( \text{GL}_2(\mathbb{Z}) \)-invariant subset of \( X \) such that \( \mu_\beta(Y) > 0 \). Let \( c > 0 \) be given by Lemma 3.5. There exists \( g_0 \in \text{GL}_2(\mathbb{Q}) \) such that the intersection \( Y_0 := g_0 Y \cap (\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{Z})) \) has positive measure. Let \( \varphi \) be the function \( \nu_\beta(\Gamma \backslash Y_0)^{-1} \mathds{1}_{\Gamma \backslash Y_0} \) on \( \Gamma \backslash X \).
We can find a finite set $F$ of primes and a positive compactly supported continuous right $\text{GL}_2(\mathbb{Z}_F)$-invariant function $f$ on $\Gamma \setminus \text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{Z}_F)$ such that

$$\int_{\Gamma \setminus \mathcal{X}_F} f \, d\nu_{\beta,F} = 1 \quad \text{and} \quad \|\varphi - f\|_2(\|\varphi\|_2 + \|f\|_2) < c.$$ 

Let $p_n, q_n \in F^c, X_1, X_2, Y_1, Y_2 \subset X$ and $g_n, h_n \in G$ be given by Lemma 3.5.

Denote by $T'_n$ and $T''_n$ the contractions

$$\frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|} \quad \text{and} \quad \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|}$$

respectively, and by $e'_n$ and $e''_n$ the projections $m(Y_{1n})$ and $m(Y_{2n})$. We have

$$\langle T'_n \varphi, T''_n \varphi \rangle \geq \langle T'_n f, T'_n f \rangle - \|T'_n \varphi - T'_n f\|_2^2 - \|T''_n \varphi - T''_n f\|_2^2 - \|T''_n f\|_2^2 \geq \langle T'_n f, T'_n f \rangle - \|e'_n(\varphi - f)\|_2^2 - \|e''_n(\varphi - f)\|_2^2 - \|e'_n f\|_2^2.$$

Since the projections $e'_n$, as well as the projections $e''_n$, are mutually orthogonal, we have

$$\sum_n \|e'_n(\varphi - f)\|_2^2 \leq \left( \sum_n \|e'_n(\varphi - f)\|_2^2 \right)^{1/2} \leq \|\varphi - f\|_2(\|\varphi\|_2 + \|f\|_2),$$

and similarly

$$\sum_n \|e''_n(\varphi - f)\|_2 \leq \|\varphi - f\|_2\|f\|_2.$$ 

It follows that

$$\sum_n \langle T'_n \varphi, T''_n \varphi \rangle \geq \sum_n \langle T'_n f, T'_n f \rangle - \|\varphi - f\|_2(\|\varphi\|_2 + \|f\|_2) > c - c = 0.$$ 

Hence there exists $n$ such that $\langle T'_n \varphi, T''_n \varphi \rangle > 0$. Then

$$\int_{\Gamma \setminus \mathcal{X}} T_{g_n} \varphi T_{h_n} \varphi \, d\nu_{\beta} > 0,$$

which means that the set $\Gamma g_0^{-1} \Gamma Y_0 \cap \Gamma h_0^{-1} \Gamma Y_0$ has positive measure. It follows that there exist $g \in \Gamma g_0 \Gamma$ and $h \in \Gamma h_0 \Gamma$ such that $g_0^{-1} \Gamma Y_0 \cap h^{-1} \Gamma Y_0$ has positive measure, and hence $g_0^{-1} h g^{-1} g_0 Y \cap Y$ has positive measure. Since

$$\frac{d(g_0^{-1} h g^{-1} g_0 \mu_\beta)}{d\mu_\beta} = |\det(g_0^{-1} h g^{-1} g_0)|^\beta = \det(h \gamma n^{-1})^\beta = \frac{g_0^\beta}{p_n^\beta},$$

and $|g_0^\beta/p_n^\beta - \lambda| < \varepsilon$, we conclude that $\lambda$ belongs to the ratio set of the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_F)/\text{GL}_2(\mathbb{Z}_F), \mu_\beta)$. Since this is true for all $\lambda > 1$, the action is of type III$_1$. 

We finish our discussion of the GL$_2$-system with the following simple observation.

**Proposition 3.6.** For every $\beta \in (1, 2]$, the action of $\text{GL}_2(\mathbb{Q})$ on $(\text{PGL}_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{A}_F), \mu_\beta)$ is amenable. Therefore the von Neumann algebra generated by the GL$_2$-system in the GNS-representation of $\varphi_\beta$, $\beta \in (1, 2]$, is the injective factor of type III$_1$.

**Proof.** If $F$ is a finite set of primes then $G_F$ is a discrete subgroup of $\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Q}_F)$, since $G_F \cap \text{GL}_2(\mathbb{Z}_F) = \Gamma$ and the homomorphism $\Gamma \to \text{PGL}_2(\mathbb{R})$ has discrete image and finite kernel. Since $\text{PGL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{Q}_F)$ is a subset of $X_F$ of full measure, it follows that $L^\infty(X_F, \mu_{\beta,F}) \rtimes G_F$ is a type I von Neumann algebra. Since the algebra $L^\infty(X, \mu_{\beta}) \rtimes G$ is the closure of an increasing union of algebras of the form $L^\infty(X, \mu_{\beta, F}) \rtimes G_F$, it is injective, and therefore the action of $G$ on $(X, \mu_{\beta})$ is amenable [21]. 


Remark 3.7.

(i) Since the $C^*$-algebra $A$ of the $GL_2$-system is a subalgebra of the crossed product of $C_0(SL_2(\mathbb{Z}) \setminus (\mathbb{H} \times \text{Mat}_2(\mathbb{A}_f)))$ by the Hecke algebra $\mathcal{H}(\text{GL}_2^+(\mathbb{Q}), SL_2(\mathbb{Z}))$, which is abelian, one might expect that not only the von Neumann algebras $\pi_{\varphi}(A)'n$ are injective, but that $A$ is nuclear. To see that this is not the case, consider the state $\varphi_0$ on $A$ defined by the measure $\frac{1}{2}\mu_H$, considered as a measure supported on $\mathbb{H} \times \{0\} \subset \mathbb{H} \times \text{Mat}_2(\mathbb{A}_f)$. The algebra $\pi_{\varphi_0}(A)'n$ is a reduction of the crossed product $L^\infty(\mathbb{H}, \mu_H) \rtimes (\text{GL}_2^+(\mathbb{Q})/(\pm 1))$. Therefore if $A$ were nuclear, the action of $\text{GL}_2^+(\mathbb{Q})/(\pm 1)$ on $(\mathbb{H}, \mu_H)$ would be amenable, which would contradict [23, Corollary 1.2].

(ii) It is apparently straightforward to extend the above results to $GL_n$ (with the interval $(1, 2]$ replaced by $(n - 1, n]$). One can however formulate a more general problem. Let $K$ be a number field, $M$ a finite dimensional central simple $K$-algebra, $G$ the group of invertible elements in $M$. Is the action of $G(K)$ on $M(\mathbb{A}_K)$ with its Haar measure ergodic?

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