Educational potential of studying recurrence relations in the preparing of prospective mathematics teachers

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Abstract. The substantive preparing of prospective mathematics teachers includes studying many branches of mathematics. Students encounter a very large number of various mathematical concepts and facts from different disciplines, so it is difficult for them to detect connections between these concepts and facts and to have an idea of the general system of mathematical knowledge. This makes it complicated to successfully study at the university, since many mathematical concepts and facts are assimilated by them formally. In the last three decades the section “recurrence relations” have appeared in almost all discrete mathematics textbooks. However, students encounter recurrent formulas when studying other mathematical disciplines. The experience of studying recurrence relations was analyzed in the article and it was demonstrated that studying recurrence relations has great educational potential for solving these problems and contributes to the higher level of preparing of prospective mathematics teachers. We propose to realize this potential through a system of reference tasks for practical exercises with students. The article presents both well-known problems from various branches of mathematics, and new problems, that we made by ourselves. These tasks can be used as a basis for the development of such a system of reference tasks.

The process of preparing of prospective mathematics teachers includes studying many branches of mathematics, such as linear and vector algebra, analytical and projective geometry, differential and integral calculus, probability theory, discrete mathematics and others.

Already in the first year of study, a very large number of various mathematical concepts and facts faced by students. As students concern, it is difficult to find connections between concepts and facts of different disciplines and to have a general idea of the general system of mathematical knowledge. Therefore, many mathematical concepts and facts are assimilated formally.

Subsequently, students experience difficulties in using different concepts and facts of one mathematical discipline to solve problems in another. This makes it complicated to successfully study at the university and to succeed in future professional activities.

There are a large number of reasons explaining this formal assimilation of mathematical concepts and facts and the lack of understanding of their interconnected system among students - prospective mathematics teachers.

Consider some obvious reasons related to the content and teaching of mathematical disciplines.

1. The narrow field of teachers’ specialization in a specific mathematical discipline.

The teacher does not always have the opportunity to demonstrate the connections of the facts and concepts of the discipline he is teaching with the concepts and facts from other disciplines.
2. Isolation and disunity of training programs of mathematical disciplines in curricula.

Training programs for some math disciplines rarely include relevant questions and facts from other disciplines. For example, not all concepts and facts of mathematical analysis are provided with their geometric interpretation.

3. There are not enough problems that require the use of connections between concepts and facts from various mathematical disciplines.

Problem books that are used to study a specific discipline rarely contain such tasks that require finding connections with concepts and facts from another discipline at first so that the student can solve these tasks.

Notably that at present, new requirements related to a practical approach are imposed on the prospective mathematics teachers’ preparing [1]. The practical approach of preparing prospective mathematics teachers cannot be implemented in the learning process without establishing close links between the concepts and facts of various branches of mathematics.

One of the most important disciplines for preparing prospective teachers of mathematics in the modern era of digital technologies is discrete mathematics.

This discipline provides students with the essential concepts and methods for studying real-world situations and objects that are discrete in nature. Accordingly, students can create and study models of various processes that differ from the classical continuous ones.

Discrete mathematics includes several branches that are important for the training of prospective teachers of mathematics: mathematical logic, combinatorics, recurrence relations, graph theory, etc.

In the last three decades, the content and study on recurrence relations have been changing greatly [2]. This topic appeared in almost all discrete mathematics textbooks [3, 4, 5, 6, 7, 8, 9, 10]. However, students encounter recurrent formulas when studying other mathematical disciplines, starting from the first year.

On the one hand, recurrence relations are a way of describing or specifying an object (for example, for a numerical sequence, for a sequence of functions, etc.) On the other hand, recurrence relations are a method to solve various problems (combinatorial, computational, etc.).

The article [2] notes that: «the recurrences are a very powerful tool (sometimes a unique one) for solving many counting problems, where it is difficult (or impossible) to count the objects by using the known combinatorial techniques. So, the recurrence relations and their solutions are important and useful complements to the knowledge in combinatorics and thence in probability theory and statistics».

It is mentioned in the article [11] that drawing analogies in the study on recurrence relations and linear differential equations contributes to the continuity and unity in the construction of the entire course of mathematics. Comparison of the theory of recurrence relations and the theory of differential equations also affects other areas of mathematics: linear algebra, series theory, etc. A consistent presentation of the structural correspondence between different branches of mathematics reveal the unity of mathematics for students, which, from a propaedeutic point of view, is very important for the preparing of a prospective teacher.

Some practical problems that lead to recurrent relations of different orders (for example, about building trains made up of cars of different lengths) and give students better understanding of the method of recurrence relations are considered in the article [12].

Moreover, the study on recurrent relations generalizes students’ hands-on experience of using recurrent formulas in other branches of mathematics. For example, students encounter recursive formulas in mathematical analyses while studying numerical sequences and the integration by parts, or in algebra, while calculating the determinants of special matrices. Furthermore, solving recurrent relations establishes links with other branches of mathematics.

To solve the recurrent relation, a student must be able to find the roots of polynomials, perform operations with complex numbers, and solve systems of linear equations.

In this regard, in the article we discuss the educational potential of studying recurrence relations for the preparing of prospective teachers of mathematics. To implement this approach, a system of reference
tasks is needed for practical exercises with students. The reference point for the development of such a system of problems will be the problems arising in various branches of mathematics.

1. Using of integration by parts when finding the indefinite integral leads to the recurrent formulas. Let us consider the examples.

- Let \( J_n = \int x^n e^x \, dx \). Then we have \( J_n = x^n e^x - n \cdot \int x^{n-1} e^x \, dx \). Thus
  \[
  J_n = x^n e^x - n \cdot J_{n-1}.
  \]

- 2) Let \( J_n = \int \frac{dx}{\sin^n x} \). Then we have \( J_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x} \). Thus
  \[
  J_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} J_{n-2}.
  \]

2. Calculation of the determinant of a tridiagonal matrix of order \( n \)

\[
\begin{vmatrix}
  a & b & 0 & \cdots & 0 & 0 \\
  c & a & b & \cdots & 0 & 0 \\
  0 & c & a & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a & b \\
  0 & 0 & 0 & \cdots & c & a
\end{vmatrix}
\]

is resolved to solving the recurrence relation. If we denote the determinant by \( x_n \), then, factoring it along the first row, we obtain the recurrence relation \( x_n = a x_{n-1} - b c x_{n-2} \), \( x_1 = a, x_2 = \left| \begin{array}{cc} a & b \\ c & a \end{array} \right| = a^2 - bc \).

3. It is well known in number theory that the Wallis-Euler recurrence relations generate sequences of numerators and denominators for convergents of continued fractions. Consider the continued fraction

\[
b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n + \cdots}}}
\]

with \( n \)th approximant \( c_n = p_n/q_n \). Then \( n \)th partial numerator \( p_n \) and the \( n \)th partial denominator \( q_n \) of the continued fraction satisfy, respectively, the recurrence relations

\[
p_n = b_n \cdot p_{n-1} + a_n \cdot p_{n-2}, \quad p_{-1} = 1, p_0 = b_0,
\]

\[
q_n = b_n \cdot q_{n-1} + a_n \cdot q_{n-2}, \quad q_{-1} = 0, q_0 = 1.
\]

4. In the probability theory, when solving some problems associated with the law of total probability, recurrence relations can also be obtained. Let us consider the example.

Problem 1. A player tossing a coin is to score one point for every head he turns up and two for every tail, and is to play on until his score reaches or passes a total \( n \). Show that his chance of making exactly the total \( n \) is \( \frac{1}{3} \left( 2 + \left( -\frac{1}{2} \right)^n \right) \).

Let \( p_n \) denotes the probability of scoring exactly \( n \) points. Then, using the law of total probability, we obtain the recurrence relation

\[
p_n = \frac{1}{2} p_{n-1} + \frac{1}{2} p_{n-2}
\]

with initial conditions \( p_0 = 1, p_1 = \frac{1}{2} \).

5. When studying discrete mathematics, it is possible to offer contextual problems to students, so they need to construct recurrent relations.
For example, the plot of the path tiling problem enables us to formulate various specific problems leading to recurrent relations of the third, fourth, and higher orders. Accordingly, we suggest the following general problem statement:

General problem statement. Let a given path be 2 ft. in width and \( n \) ft. in length. It must be paved with multi-colored tiles of sizes \( 2 \times 1, 2 \times 2, \ldots, 2 \times k \). Tiles sized \( 2 \times 1 \) can be painted using no more than \( a_1 \) colors, tiles sized \( 2 \times 2 \) can be painted using no more than \( a_2 \) colors (other than \( a_1 \) used), and tiles sized \( 2 \times k \) can be painted using no more than \( a_k \) colors (other than all previously used ones; i.e., in total, no more than \( a_1 + a_2 + \cdots + a_k \) different colors can be used). In how many ways can a multi-colored pavement be made with such a path?

Remark. Without loss of generality, we can consider the problem in the case when tiles sized \( 2 \times m \) (\( m \leq k \)) should not be used. In this case, we will assume that the corresponding value \( a_m = 0 \).

For example, to study fourth-order recurrence relations, the problem formulation can be written explicitly with specific numerical values.

Problem 2. Let a given path be 2 ft. in width and \( n \) ft. in length. It must be paved with multi-colored tiles of sizes \( 2 \times 1, 2 \times 2, 2 \times 3, \) and \( 2 \times 4 \). Tiles sized \( 2 \times 1 \) can be painted using no more than \( a \) colors, tiles sized \( 2 \times 2 \) can be painted using no more than \( b \) colors (other than \( a \) used), tiles sized \( 2 \times 3 \) can be painted using no more than \( c \) colors (other than \( a+b \) used), and tiles sized \( 2 \times 4 \) can be painted using no more than \( d \) colors (other than all previously used ones; i.e., in total, no more than \( a + b + c + d \) different colors can be used). In how many ways can a multi-colored pavement be made with such a path?

Let \( x_n \) be the number of ways to cover the path sized \( 2 \times n \) with different colors. Consider all possible positions of the last tile (or several tiles) in the cover of the path sized \( 2 \times n \).

The total number of ways for a path to be tiled gives the following fourth-order linear recurrence relation with constant coefficients

\[
x_n = ax_{n-1} + (a^2 + b)x_{n-2} + cx_{n-3} + dx_{n-4}.
\]

The initial conditions are

\[
x_1 = a, x_2 = 2a^2 + b, x_3 = 3a^3 + 2ab + c, x_4 = 5a^4 + 5a^2b + b^2 + 2ac + d.
\]

By setting the specific values of \( a, b, c \) and \( d \) it is possible to obtain various cases of roots (real and complex) for the characteristic equation.

If in Problem 2, we let \( a = 1, b = 0, c = 1 \) and \( d = 2 \), then we obtain the following fourth-order linear recurrence relation

\[
x_n = x_{n-1} + x_{n-2} + x_{n-3} + 2x_{n-4}.
\]

and the initial conditions \( x_1 = 1, x_2 = 2, x_3 = 4, \) and \( x_4 = 9 \).

The characteristic equation takes the form

\[
\lambda^4 - \lambda^3 - \lambda^2 - \lambda - 2 = 0.
\]

This equation has real roots \( \lambda_1 = -1, \lambda_2 = 2 \) and complex conjugate roots \( \lambda_3, \lambda_4 = \pm i \). Thus, we find \( |\lambda_3| = 1, \varphi = \text{arg} \lambda_3 = \frac{\pi}{2} \). We obtain the following general solution:

\[
x_n = C_1(-1)^n + C_22^n + C_3 \cos\left(\frac{\pi}{2}n\right) + C_4 \sin\left(\frac{\pi}{2}n\right).
\]

To determine the constants \( C_1, C_2, C_3 \) and \( C_4 \), we use the initial conditions, which give

\[
\begin{align*}
-C_1 + 2C_2 + C_4 &= 1 \\
C_1 + 4C_2 - C_3 &= 2 \\
-C_1 + 8C_2 - C_4 &= 4 \\
C_1 + 16C_2 + C_3 &= 9
\end{align*}
\]
We obtain $C_1 = \frac{1}{6}$, $C_3 = \frac{3}{10}$, $C_3 = \frac{3}{10}$, and $C_4 = \frac{1}{10}$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $(x_n)$ with

$$x_n = \frac{1}{6} (-1)^n + \frac{8}{15} 2^n + \frac{3}{10} \cos \left( \frac{\pi}{2} n \right) + \frac{1}{10} \sin \left( \frac{\pi}{2} n \right).$$

The above-mentioned examples show that students use concepts and methods from various mathematical disciplines when solving recurrent relations. As follows the activities of students in the study of recurrent relations will contribute to the establishment of links between these concepts and methods and the development of an integrated system of mathematical knowledge, which is fundamentally important in the preparing process of prospective mathematics teachers.

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