COLORED KAC-MOODY ALGEBRAS, PART I

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ABSTRACT. We introduce a parametrization of formal deformations of Verma modules of $\mathfrak{sl}_2$. A point in the moduli space is called a coloring. We prove that for each coloring $\psi$ satisfying a regularity condition, there is a formal deformation $U_h(\psi)$ of $U(\mathfrak{sl}_2)$ acting on the deformed Verma modules. We retrieve in particular the quantum algebra $U_h(\mathfrak{sl}_2)$ from a coloring by $q$-numbers. More generally, we establish that regular colorings parametrize a broad family of formal deformations of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$. As a corollary, we obtain a new rigidity result for $U(\mathfrak{sl}_2)$. This paper is the first of a series aimed to lay the foundations of a new approach to deformations of Kac-Moody algebras. Colored Kac-Moody algebras have been originally devised by the author as an attempt to solve conjectures formulated by Frenkel and Hernandez in [6] on Langlands duality for quantum groups. A positive answer to these conjectures will appear in a forthcoming paper.

1. Introduction

1.1. Deformation by Tannaka duality. The Lie algebra $\mathfrak{sl}_2$ formed by 2-by-2 matrices with zero trace is the easiest example of a semisimple Lie algebra, or more generally of a Kac-Moody algebra. The Chevalley-Serre presentation \cite{13} of $\mathfrak{sl}_2$ consists of the Chevalley generators $X^-, H, X^+$ and of the relations

\begin{align}
[H, X^\pm] &= \pm 2X^\pm, \\
[X^-, X^+] &= H.
\end{align}

We present in this paper a new approach, both elementary and systematic, to deformations of the universal enveloping algebra $U(\mathfrak{sl}_2)$, over a ground field $K$ of characteristic zero. Deformations here are formal, i.e. they are considered over the power series ring $K[[h]]$. We shall give a precision. It follows from a cohomological rigidity criterion of Gerstenhaber \cite{8} that formal deformations of the structure of associative algebra of $U(\mathfrak{sl}_2)$ are all trivial, i.e. they are conjugate to the constant formal deformation. In this paper though we are interested in deforming a slightly richer structure, which consists of the algebra $U(\mathfrak{sl}_2)$ together with the Chevalley generators. In other words, when considering a formal deformation of $U(\mathfrak{sl}_2)$, we want to specify within it a deformation of the generators $X^-, H, X^+$. Equivalently, we may say that we are looking at formal deformations of the Chevalley-Serre presentation $\mathfrak{h}$ of $U(\mathfrak{sl}_2)$.

Representations of $\mathfrak{sl}_2$ carry all the information of the algebra $U(\mathfrak{sl}_2)$, in the sense that $U(\mathfrak{sl}_2)$ can be reconstructed by Tannaka duality from the category $\text{Rep}(\mathfrak{sl}_2)$ of representations of $\mathfrak{sl}_2$. More specifically, $U(\mathfrak{sl}_2)$ can be defined as the algebra of

\section*{Date:} December 31, 2014.
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endomorphisms (namely the natural transformations) of the forgetful functor from Rep($\mathfrak{sl}_2$) to the category of vector spaces.

We propose to construct deformations of $U(\mathfrak{sl}_2)$ via Tannaka duality. In our view, the category Rep($\mathfrak{sl}_2$) would be too large to be deformed in one go. We need to look for a more modest subcategory to start with. One first candidate that comes easily in mind is the subcategory of finite-dimensional representations of $\mathfrak{sl}_2$. On the one hand, the subcategory is rich enough to distinguish by Tannaka duality every finite-dimensional representations of $\mathfrak{sl}_2$ semisimple, and irreducible finite-dimensional representations of $\mathfrak{sl}_2$, on every finite-dimensional representations of $\mathfrak{sl}_2$. On the other hand, the category of finite-dimensional representations of $\mathfrak{sl}_2$ is notably elementary: all objects are semisimple, and irreducible finite-dimensional representations of $\mathfrak{sl}_2$ are classified by their dimensions. There is however a slightly larger category which appears more suited to our purpose. This category is generated by Verma modules, in a sense which we will make precise in a next paper. Note that we consider here only integral Verma modules, that is to say Verma modules for which the action of $H$ has integral eigenvalues. One reason to prefer Verma modules rather than finite-dimensional representations is that the former are all equal when forgetting the action of $\mathfrak{sl}_2$ (they share the same underlying vector space). This makes deformation and Tannaka duality easier to deal with. Another reason is that Verma modules are induced by one-dimensional representations of a Borel subalgebra of $U(\mathfrak{sl}_2)$. We will use this property in a next paper to induce deformations of the whole category Rep($\mathfrak{sl}_2$) from deformations of Verma modules.

1.2. Summary of the main results.

**Definition 1.** A coloring is a sequence $\psi = (\psi^p(n))_{p \geq 0}$ with values in $K[[h]]$ depending on $n \in \mathbb{Z}$ and satisfying

(C1) $\psi^p(n) \equiv (p + 1)(n - p) \pmod{h}$ for all $p \in \mathbb{Z}_{\geq 0}$ and for all $n \in \mathbb{Z}$,

(C2) $\psi^p(n) = 0$ for all $p \in \mathbb{Z}_{\geq 0}$,

(C3) $\psi^{n+p+1}(n) = \psi^p(-n - 2)$ for all $p, n \in \mathbb{Z}_{\geq 0}$.

For $n \in \mathbb{Z}$, we denote by $M(n)$ the integral Verma module of $\mathfrak{sl}_2$ of highest weight $n$. Forgetting the action of $X^+$, integral Verma modules of $\mathfrak{sl}_2$ become representations of the Borel subalgebra $\mathfrak{b}$ spanned by $X^-$ and $H$. The action of $X^+$ can be retrieved from the natural coloring $N$, defined by $N^p(n) = (p + 1)(n - p)$. In view of the axiom (C1), colorings can then be regarded as formal deformations of the action of $X^+$ on the integral Verma modules of $\mathfrak{sl}_2$.

**Definition 2.** We denote by $M_{h}(n, \psi)$ the $K[[h]]$-module $M(n)[[h]]$ endowed with the constant deformation of the action of $\mathfrak{b}$ on $M(n)$, together with the deformation of the action of $X^+$ that yields the coloring $\psi$.

Here is where Tannaka duality comes into the picture.

**Definition 3.** We denote by $U_h(\psi)$ the $K[[h]]$-algebra generated by $X^-, H, X^+$ and subject to the relations satisfied in every representation $M_{h}(n, \psi)$.

We prove that $U_h(\psi)$ deforms the algebra $U(\mathfrak{sl}_2)$, provided that the coloring $\psi$ satisfies a regularity condition. We call it a colored Kac-Moody algebra.

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1For the reader who may find unclear why this definition involves Tannaka duality, let us mention that there is a category built directly from the representations $M_{h}(n, \psi)$ and whose Tannaka dual algebra is canonically isomorphic to $U_h(\psi)$. Details will appear in a next paper.
Theorem 1. The algebra $U_h(\psi)$ is a formal deformation of the algebra $U(\mathfrak{sl}_2)$ if and only if the coloring $\psi$ is regular, i.e.

(R1) $\psi^p(n) = \sum_{m \geq 0} \psi^p_m(n) h^m$ where $\psi^p_m(n)$ is a polynomial function of $n$,
(R2) for each $m$ the degree of $\psi^p_m(n)$ is a function of $p$ bounded above.

A colored Kac-Moody algebra defines not only a formal deformation of the algebra $U(\mathfrak{sl}_2)$, but also a formal deformation of the Chevalley generators of $U(\mathfrak{sl}_2)$. As a result, it defines unambiguously – once we have fixed a basis of $U(\mathfrak{sl}_2)$, e.g., the canonical PBW basis of $U(\mathfrak{sl}_2)$ – a formal deformation of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$.

Theorem 2. When the coloring $\psi$ is regular, the $K[[h]]$-algebra $U_h(\psi)$ is generated by $X^-, H, X^+$ and subject to the relations

$$[H, X^\pm] = \pm 2X^\pm,$$
$$X^+ X^- = \sum_{a=0}^{\infty} (X^-)^a \xi^a(H) (X^+)^a \quad \text{with } \xi^a(H) \in K[H][[h]],$$

where the structure constants $\xi^a$ form the regular solution of an infinite-dimensional linear equation (see section 3).

Let $\mathfrak{a}$ be the Lie algebra generated by $X^-, H, X^+$ and subject to the relations (1a) of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$. The Lie algebra $\mathfrak{a}$ is a cover of the Lie algebra $\mathfrak{sl}_2$. There is in particular an algebra homomorphism from $U(\mathfrak{a})$ to $U(\mathfrak{sl}_2)$; we say that $U(\mathfrak{sl}_2)$ is an $\mathfrak{a}$-algebra. The relations (1a) hold in $U_h(\psi)$ for every coloring $\psi$. Put in other words, the colored Kac-Moody algebra $U_h(\psi)$ is a $U_h(\mathfrak{a})$-algebra, i.e. there is an algebra homomorphism from $U_h(\mathfrak{a})$ to $U_h(\psi)$, where $U_h(\mathfrak{a})$ designates the $K[[h]]$-algebra $U(\mathfrak{a})[[h]]$. We may then regard $U_h(\psi)$ as a formal deformation of the structure of $\mathfrak{a}$-algebra of $U(\mathfrak{sl}_2)$.

For any symmetrizable Kac-Moody algebra $\mathfrak{g}$, Drinfel’d [4] and Jimbo [9] have defined a formal deformation $U_h(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$.\footnote{The structure of associative algebra of $U_h(\mathfrak{sl}_2)$ was first discovered by Kulish-Sklyanin [10].} We prove in this paper that $U_h(\mathfrak{sl}_2)$ is an example of a colored Kac-Moody algebra. More precisely, we show that $U_h(\mathfrak{sl}_2)$ arises from a coloring $N_q$, defined from the natural coloring $N$ by replacing natural numbers with $q$-numbers.

Theorem 3. The quantum algebra $U_h(\mathfrak{sl}_2)$ is isomorphic as $U_h(\mathfrak{a})$-algebra to the colored Kac-Moody algebra $U_h(N_q)$.

It has been proved by Drinfel’d [5] that for $\mathfrak{g}$ semisimple, $U_h(\mathfrak{g})$ is a $\mathfrak{h}$-trivial formal deformation of $U(\mathfrak{g})$, i.e. there exists an equivalence of formal deformation between $U(\mathfrak{g})[[h]]$ and $U_q(\mathfrak{g})$ fixing the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We establish that regular colorings classify all $\mathfrak{h}$-trivial formal deformations of the structure of $\mathfrak{a}$-algebra of $U(\mathfrak{sl}_2)$.

Theorem 4. Every $\mathfrak{h}$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is isomorphic as $U_h(\mathfrak{a})$-algebra to a unique colored Kac-Moody algebra $U_h(\psi)$.

As a corollary, we obtain a new\footnote{To the best of the author’s knowledge.} rigidity result for $U(\mathfrak{sl}_2)$. 

Reference [4].
Theorem 5. Every $\mathfrak{h}$-trivial formal deformation $A$ of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial, i.e. there exists an equivalence of formal deformation between $U(\mathfrak{sl}_2)[[\hbar]]$ and $A$ fixing both $X^-$ and $H$.

1.3. Colored Kac-Moody algebras. The present paper is the first of a series. We present here results in the rank one case, focusing on one-parameter formal deformations of the structure of $\mathfrak{a}$-algebra of $U(\mathfrak{sl}_2)$. We will investigate in a next paper formal deformations of the structure of Hopf algebra of $U(\mathfrak{sl}_2)$. It will be proved that regular (di)colorings provide a classification of formal deformations of the Hopf algebra $U(\mathfrak{sl}_2)$ (together with formal deformations of the Chevalley generators). More generally, we will show that there is a natural group action on the set of colorings and that the resulting action groupoid is equivalent to the groupoid formed by formal deformations of the Hopf algebra $U(\mathfrak{sl}_2)$. These results will be generalized in subsequent papers from $\mathfrak{sl}_2$ to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. Let us precise that we won’t be concerned with all the deformations of the Hopf algebra $U(\mathfrak{g})$, as we will restrict ourselves to those deformations which preserve the grading of $U(\mathfrak{g})$ by the weight lattice of $\mathfrak{g}$.

Colored Kac-Moody algebras are defined by Tannaka duality. In a next paper, we will explain how a coloring $\psi$ induces in an elementary way a closed monoidal category $\text{Rep}(\mathfrak{g}, \psi)$ and we will show that this category is a deformation of the category of all representations of $\mathfrak{g}$. The colored Kac-Moody algebra $U(\mathfrak{g}, \psi)$ will be defined as the Hopf algebra corresponding by Tannaka duality to the category $\text{Rep}(\mathfrak{g}, \psi)$. Note that whereas the construction of the category $\text{Rep}(\mathfrak{g}, \psi)$ is aimed to be as elementary as possible, the colored Kac-Moody algebra $U(\mathfrak{g}, \psi)$ itself may be in general difficult to describe explicitly (consider for example the Chevalley-Serre presentation of $U_\hbar(\psi)$, see theorem 4.2).

Colored Kac-Moody algebras should be understood as multi-parameters deformations of usual Kac-Moody algebras, with as many deformation parameters as there are degrees of freedom in the choice of a coloring – an example of a colored Kac-Moody algebra of rank one with two deformation parameters has been developed in [1]. We will show in a next paper that all constructions and results obtained over the power series ring $K[[h]]$ hold over more general rings. We will show that there is a generic colored Kac-Moody algebra, which is universal in the sense that every other colored Kac-Moody algebra can be obtained from it by specializing generic deformation parameters. Specializations will be a key feature of colored Kac-Moody algebras, with several applications, as for example crystallographic Kac-Moody algebras to name one – we will show that there is a natural correspondence between representations of crystallographic Kac-Moody algebras and crystals of representations of quantum Kac-Moody algebras.

1.4. Langlands interpolation. Littelmann [11] and McGerty [12] have revealed the existence of relations between representations of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and representations of its Langlands dual $^L\mathfrak{g}$ (the Kac-Moody algebra defined by transposing the Cartan matrix of $\mathfrak{g}$). They have proved that the action of the quantum algebra $U_q(\mathfrak{g})$ on certain representations interpolates between an action of $\mathfrak{g}$ and an action of $^L\mathfrak{g}$; namely, they have shown that the actions of $\mathfrak{g}$ and $^L\mathfrak{g}$ can be retrieved from the action of $U_q(\mathfrak{g})$ by specializing the parameter.

\footnote{The $\mathfrak{b}$-triviality of the quantum algebra $U_\hbar(\mathfrak{sl}_2)$ had been already proved by Jimbo [9] and Chari-Pressley [3].}
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$q$ to 1 and to some root of unity $\epsilon$, respectively. For $\mathfrak{g}$ semisimple, Frenkel and Hernandez introduced in [6] an algebra depending on an additional parameter $t$ and they conjectured the existence of representations for this algebra interpolating between representations of the quantum algebras $U_q(\mathfrak{g})$ and $U_t(\mathfrak{l})$. They besides conjectured that the constructions could be extended to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. They lastly suggested a Langlands duality for crystals.

We will give in a forthcoming paper a positive answer to these conjectures. More precisely, we will show that for any symmetrizable Kac-Moody algebra $\mathfrak{g}$ there exists a colored Kac-Moody algebra $U(\mathfrak{g},N_q,t)$ whose representations possess the predicted interpolation property. Using crystallographic Kac-Moody algebras, we will moreover confirm manifestations of Langlands duality at the level of crystals.

Let us precise that the colored Kac-Moody algebra $U(\mathfrak{g},N_q,t)$ is related to Frenkel-Hernandez’s algebra only when $\mathfrak{g} = \mathfrak{sl}_2$. For $\mathfrak{g}$ of higher ranks, the two algebras differ significantly.

Let us mention that Langlands duality for quantum groups might have promising connections with the geometric Langlands correspondence, see [7] and [6].

1.5. Organization of the paper. In section 2, we introduce colorings and we construct the deformed Verma modules $M_h(n,\psi)$ induced by a coloring $\psi$. We briefly recall the notion of formal deformation of associative algebras and we give several definitions suited to the context of this paper. We define the algebra $U_h(\psi)$ and we prove that $U_h(\psi)$ is “almost” a formal deformation of $U(\mathfrak{sl}_2)$. In section 3, we express the action of $U_h(\psi)$ on $M_h(n,\psi)$ in terms of infinite-dimensional linear equations (proposition 3.2). We prove that these equations always admit regular solutions if and only if $\psi$ is regular (proposition 3.3); this is the key technical result of this paper. In section 4, we prove that $U_h(\psi)$ is a formal deformation of $U(\mathfrak{sl}_2)$ if and only if the coloring $\psi$ is regular (theorem 4.1). We give a Chevalley-Serre presentation of the colored Kac-Moody algebra $U_h(\psi)$ (theorem 4.2). We show that the constant formal deformation $U(\mathfrak{sl}_2)[[h]]$ and the quantum algebra $U_h(\mathfrak{sl}_2)$ can be realized as colored Kac-Moody algebras (theorem 4.3). We prove that regular colorings classify $\mathfrak{h}$-trivial formal deformations of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.4). We prove that $U_h(\psi)$ is a $\mathfrak{h}$-trivial deformation of $U(\mathfrak{sl}_2)$ for all regular colorings (theorem 4.5). As a corollary, we obtain that every $\mathfrak{h}$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial (corollary 4.6).

2. Preliminaries

2.1. Notations and conventions.

2.1.1. The integers are elements of $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The non-negative integers are elements of $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. We recall that $K$ designates a field of characteristic zero. We denote by $K^\mathbb{Z}$ the $K$-vector space formed by functions from $\mathbb{Z}$ to $K$.

2.1.2. We denote by $K[[h]]$ the power series ring in the variable $h$ over the field $K$. An element $\lambda$ in $K[[h]]$ is of the form $\lambda = \sum_{m \geq 0} \lambda_m h^m$ with $\lambda_m \in K$, we denote by $\lambda_{h=0}$ the scalar $\lambda_0$. 
2.1.3. Associative algebras and their homomorphisms are unital. Representations are left. Let $\mathfrak{g}$ be a Lie algebra over $K$, its universal enveloping algebra is denoted by $U(\mathfrak{g})$. We identify as usual representations of $\mathfrak{g}$ with representations of $U(\mathfrak{g})$.

2.1.4. Let $B$ be a $R$-algebra ($R = K, K[[h]]$). A $B$-algebra is a $R$-algebra $A$, together with a structural $R$-algebra homomorphism $f : B \to A$. Let $A'$ be another $B$-algebra, a $B$-algebra homomorphism from $A$ to $A'$ is a $R$-algebra homomorphism $g : A \to A'$ such that $g \circ f = f'$, where $f'$ designates the structural homomorphism from $B$ to $A'$. Let $\mathfrak{g}$ be a Lie algebra over $K$, a $\mathfrak{g}$-algebra is a $U(\mathfrak{g})$-algebra.

2.1.5. For $V_0$ a $K$-vector space, we denote by $V_0[[h]]$ the $K[[h]]$-module formed by series of the form $\sum_{m \geq 0} v_m h^m$ with $v_m \in V_0$. A structure of $K$-algebra on a $K$-vector space $B_0$ induces a structure of $K[[h]]$-algebra on the $K[[h]]$-module $B_0[[h]]$. Similarly, a representation $V_0$ of the $K$-algebra $B_0$ induces a representation $V_0[[h]]$ of the $K[[h]]$-algebra $B_0[[h]]$. More generally, a $B_0$-algebra $A_0$ induces a $B_0[[h]]$-algebra $A_0[[h]]$.

2.1.6. For $V$ a $K[[h]]$-module, we denote by $V_{h=0}$ the $K$-vector space $V/hV$. Let $f : V \to W$ be a $K[[h]]$-linear map, we denote by $f_{h=0}$ the $K$-linear map induced by $f$ from $V_{h=0}$ to $W_{h=0}$. A structure of $K[[h]]$-algebra on a $K[[h]]$-module $B$ induces a structure of $K$-algebra on the $K$-vector space $A_{h=0}$. Similarly, a representation $V$ of the $K[[h]]$-algebra $B$ induces a representation $V_{h=0}$ of the $K$-algebra $B_{h=0}$, and a $B$-algebra $A$ induces a $(B_{h=0})$-algebra $A_{h=0}$.

2.1.7. The $h$-adic topology of a $K[[h]]$-module $V$ is the linear topology whose local base at zero is formed by the $K[[h]]$-submodules $h^m V$ ($m \in Z_{\geq 0}$). A $K[[h]]$-module isomorphic to $V_0[[h]]$ for some $K$-vector space $V_0$ is said topologically free.

2.2. Colorings. We call the relations

$$[H, X^\pm] = \pm 2X^\pm$$

the non-deformable relations of $\mathfrak{sl}_2$. We denote by $\mathfrak{a}$ the Lie algebra over $K$ generated by $X^-, H, X^+$ and subject to the non-deformable relations of $\mathfrak{sl}_2$. We are interested in this paper in deformations of the algebra $U(\mathfrak{sl}_2)$ where the non-deformable relations of $\mathfrak{sl}_2$ still hold. Representations of such deformations of $U(\mathfrak{sl}_2)$ are in particular representations of the constant formal deformation $U(\mathfrak{a})[[h]]$. In order to simplify the notation, we will denote this algebra by $U_h(\mathfrak{a})$.

We denote by $\mathfrak{b}^+$ the Borel subalgebra of $\mathfrak{sl}_2$ spanned by $H$ and $X^+$. We recall that a Verma module of $\mathfrak{sl}_2$ is a representation of $\mathfrak{sl}_2$ induced from a one-dimensional representation of $\mathfrak{b}^+$. For $n \in Z$, the (integral) Verma module $M(n)$ of highest weight $n$ is equal to $\bigoplus_{p \geq 0} K h_p$ as vector space, together with the action of $\mathfrak{sl}_2$ defined by

$$H.b_p = (n - 2p)b_p,$$

$$X^-b_p = b_{p+1},$$

$$X^+.b_p = \begin{cases} 0 & \text{if } p = 0, \\ p(n - p + 1)b_{p-1} & \text{if } p > 0. \end{cases}$$

We remark that $X^-, H$ and $X^+$ act on $M(n)$ as scalar multiplication operators between $Z_{\geq 0}$ copies of $K$. 

$$\tag{2}$$
Let $B^+$ designate the quiver formed by the bottom arrows of the previous graph. We can think of the action of $X^+$ on integral Verma modules of $\mathfrak{sl}_2$ as a $\mathbb{Z}$-graded representation of the quiver $B^+$. This representation, which we denote by $N$, assigns to each vertex of $B^+$ the $K$-vector space $\bigoplus_{n \in \mathbb{Z}} K$, and assigns to the $(p+1)$th arrow $(p \in \mathbb{Z}_{\geq 0})$ the $K$-linear map represented by the diagonal matrix $\text{Diag}(N^p(n); n \in \mathbb{Z})$ where $N^p(n) = (p+1)(n-p)$.

Fixing the actions of $X^-$ and $H$, a formal deformation of the action of the Lie algebra $\mathfrak{a}$ on the integral Verma modules of $\mathfrak{sl}_2$ then corresponds to a formal deformation of the $\mathbb{Z}$-graded representation $N$ of the quiver $B^+$. Such a deformation is specified by a deformation $\psi^p(n)$ in $K[[h]]$ of the scalar $N^p(n)$, for each $p \in \mathbb{Z}_{\geq 0}$ and for each $n \in \mathbb{Z}$.

Let us recall that for each $n \in \mathbb{Z}_{\geq 0}$ there is a nonzero morphism from the Verma module $M(n-2)$ to the Verma module $M(n)$. At the level of the representation $N$ of the quiver $B^+$, the property becomes: $N^0(n) = 0$ and $N^{n+p+1}(n) = N^p(-n-2)$ for all $n, p \in \mathbb{Z}_{\geq 0}$. A formal deformation of the representation $N$ which preserves these conditions is called a coloring.

**Definition 2.1.** A coloring is a sequence $\psi = (\psi^p)_{p \geq 0}$ where $\psi^p$ is a formal power series of the form $\sum_{p \geq 0} \psi^p_m h^m$ with $\psi^p_m \in K^Z$, satisfying

\begin{align*}
(C1) & \quad \psi^p_0(n) = (p+1)(n-p) \quad \text{for all } p \in \mathbb{Z}_{\geq 0} \text{ and for all } n \in \mathbb{Z}, \\
(C2) & \quad \psi^p(n) = 0 \quad \text{for all } n \in \mathbb{Z}_{\geq 0}, \\
(C3) & \quad \psi^{n+p+1}(n) = \psi^p(-n-2) \quad \text{for all } p, n \in \mathbb{Z}_{\geq 0}.
\end{align*}

As explained before, a coloring encodes a formal deformation of the action of $X^+$ on the integral Verma modules of $\mathfrak{sl}_2$, in such a way that the non-deformed relations of $\mathfrak{sl}_2$ remain satisfied.

**Definition 2.2.** Let $\psi$ be a coloring and let $n \in \mathbb{Z}$. We denote by $M_h(n, \psi)$ the representation of $U_h(\mathfrak{a})$ whose underlying $K[[h]]$-module is $\left(\bigoplus_{p \geq 0} K b_p\right)[[h]]$ and where the action of $U_h(\mathfrak{a})$ is defined by

$$
H b_p = (n-2p)b_p,
$$

$$
X^- b_p = b_{p+1},
$$

$$
X^+ b_p = \begin{cases} 
0 & \text{if } p = 0, \\
\psi^{p-1}(n)b_{p-1} & \text{if } p \geq 1.
\end{cases}
$$

**Example 2.3.** There is a unique coloring $\psi$ which satisfies $\psi^p_m = 0$ for all $p \in \mathbb{Z}_{\geq 0}$ and for all $m \in \mathbb{Z}_{\geq 1}$. We denote it again by $N$ and we call it the natural coloring. The natural coloring encodes the action of $\mathfrak{a}$ on the integral Verma modules of $\mathfrak{sl}_2$: $M_h(n, N) = M(n)[[h]]$ as representations of $U_h(\mathfrak{a})$ for all $n \in \mathbb{Z}$.
Example 2.4. The quantum algebra $U_h(\mathfrak{sl}_2)$ is the $K[[h]]$-algebra generated topologically by $X^-, H, X^+$ and subject to the relations

$$[H, X^\pm] = \pm 2X^\pm,$$

$$[X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

with $q = \exp(h)$ and $q^H = \exp(hH)$, i.e. $U_h(\mathfrak{sl}_2)$ is the quotient of the $K[[h]]$-algebra $K\langle X^-, H, X^+\rangle[[h]]$ by the smallest closed (for the $h$-adic topology) two-sided ideal containing the previous relations.

We denote by $N_q$ the coloring defined by

$$N_q(n) = [p + 1]_q[n - p]_q$$

where $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ for $k \in \mathbb{Z}$.

We call it the $q$-coloring. The $q$-coloring encodes the action of $X^+$ on the integral Verma modules of $U_h(\mathfrak{sl}_2)$: for all $n \in \mathbb{Z}$, the representation $M_h(n, N_q)$ is the Verma module of $U_h(\mathfrak{sl}_2)$ of highest weight $n$, when viewed as representation of $U_h(a)$.

2.3. Formal deformation of associative algebras. A formal deformation of a $K$-algebra $A_0$ usually designates a $K[[h]]$-algebra $A$ such that the $K[[h]]$-modules $A$ and $A_0[[h]]$ are isomorphic, together with a $K$-algebra isomorphism $f_0$ from $A_{h=0}$ to $A_0$. Two formal deformations $(A, f_0)$ and $(A', f'_0)$ are said equivalent if there exists a $K[[h]]$-algebra isomorphism $g$ from $A$ to $A'$ such that $f'_0 \circ g_{h=0} = f_0$.

As already mentioned, we are interested in this paper in formal deformations of $U(\mathfrak{sl}_2)$ where the non-deformable relations of $\mathfrak{sl}_2$ still hold. In other words, the deformations of interest will be the formal deformations of $U(\mathfrak{sl}_2)$ within the category of $\mathfrak{a}$-algebras – $U(\mathfrak{sl}_2)$ has a canonical structure of $\mathfrak{a}$-algebra, induced by the projection map from $\mathfrak{a}$ to $\mathfrak{sl}_2$. Remark that specifying a $B$-algebra structure on an algebra $A$ not only forces every relation in the algebra $B$ to be satisfied in $A$, but also fixes pointwise in $A$ the image of $B$. A formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ should then be understood as a formal deformation of the $K$-algebra $U(\mathfrak{sl}_2)$, together with a formal deformation of the Chevalley generators $X^-, H, X^+$ within the deformed algebra, in such a way that the non-deformable relations of $\mathfrak{sl}_2$ are preserved.

Definition 2.5. Let $B_0$ be a $K$-algebra and let $A_0$ be a $B_0$-algebra. We suppose that the structural homomorphism from $B_0$ to $A_0$ is surjective. A formal deformation of the $B_0$-algebra $A_0$ is a $B_0[[h]]$-algebra $A$ such that

(D1) the $K[[h]]$-modules $A$ and $A_0[[h]]$ are isomorphic,

(D2) the $B_0$-algebras $A_{h=0}$ and $A_0$ are isomorphic.

Axioms (D2) may need a precision: the structure of $B_0[[h]]$-algebra on $A$ induces a structure of $(B_0[[h]]_{h=0})$-algebra on $A_{h=0}$, and thus a structure of $B_0$-algebra (the $K$-algebras $B_0[[h]]_{h=0}$ and $B_0$ are canonically isomorphic).

As the structural homomorphism from $B_0$ to $A_0$ is surjective, there is a unique way to identify the $B_0$-algebras $A_{h=0}$ and $A_0$. This shows that definition 2.5 extends the usual definition of formal deformation of $K$-algebras.

Remark also that in view of axiom (D1) the formal deformation $A$ is in particular Hausdorff and complete for the $h$-adic topology. This implies that the structural homomorphism from $B_0[[h]]$ to $A$ is necessarily surjective.

Example 2.6. The quantum algebra $U_h(\mathfrak{sl}_2)$ (as defined in example 2.4) is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. 
With formal deformations of algebras come naturally formal deformations of representations.

**Definition 2.7.** Let $B_0$ be a $K$-algebra and let $V_0$ be a representation of $B_0$. A formal deformation of $V_0$ along $B_0[[h]]$ is a representation $V$ of $B_0[[h]]$ such that

1. the $K[[h]]$-modules $V$ and $V_0[[h]]$ are isomorphic,
2. the representations $V_{h=0}$ and $V_0$ are isomorphic.

Let $A$ be a formal deformation of a $B_0$-algebra $A_0$. We suppose that the structural homomorphism from $B_0$ to $A_0$ is surjective and we suppose that the action of $B_0$ on $V_0$ factorizes through $A_0$. We say that $V$ is a formal deformation of $V_0$ along $A$ if the action of $B_0[[h]]$ on $V$ factorizes through $A$.

Axiom [D2] may need again a precision: the action of $B_0[[h]]$ on $V$ induces an action of $(B_0[[h]]|_{h=0})$ on $V_{h=0}$, and thus an action of $B_0$ on $V_{h=0}$.

**Example 2.8.** For every $n \in \mathbb{Z}$, the representation $M_h(n,N_q)$ is a formal deformation along $U_h(\mathfrak{sl}_2)$ of the Verma module $M(n)$ (see example 2.4).

The following lemma gives an example of formal deformations along $U_h(a)$.

**Lemma 2.9.** Let $\psi$ be a coloring. For every $n \in \mathbb{Z}$, the representation $M_h(n,\psi)$ is a formal deformation along $U_h(\mathfrak{a})$ of the Verma module $M(n)$ when viewed as representation of $U(\mathfrak{a})$.

**Proof.** Immediate from the coloring axiom [C1] \hfill $\square$

2.4. **The algebra $U_h(\psi)$.** The integral Verma modules of $\mathfrak{sl}_2$ distinguish elements in the algebra $U(\mathfrak{sl}_2)$: an element in $U(\mathfrak{sl}_2)$ is zero if and only if it acts by zero on $M(n)$ for all $n \in \mathbb{Z}$. This remains true if we replace integral Verma modules of $\mathfrak{sl}_2$ with irreducible finite-dimensional representations of $\mathfrak{sl}_2$. These are known facts. We give a proof of them, for the reader’s convenience.

**Proposition 2.10.** Let $x \in U(\mathfrak{sl}_2)$. The following assertions are equivalent.

1. The element $x$ is zero.
2. The element $x$ acts by zero on all the integral Verma modules of $\mathfrak{sl}_2$.
3. The element $x$ acts by zero on all the irreducible finite-dimensional representations of $\mathfrak{sl}_2$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is immediate. For $n \in \mathbb{Z}_{\geq 0}$ we denote by $L(n)$ the unique (up to isomorphism) irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$. The representation $L(n)$ is a quotient of the integral Verma module $M(n)$. Hence, assertion (ii) implies assertion (iii). Let $x$ be a nonzero element in the algebra $U(\mathfrak{sl}_2)$ and suppose that $x$ acts by zero on $L(n)$ for all $n \in \mathbb{Z}$. The $K$-algebra $U(\mathfrak{sl}_2)$ has a $\mathbb{Z}$-gradation, defined by $\deg(H) = 0$ and $\deg(X^\pm) = \pm 1$. Without loss of generality, we can assume that $x$ is a homogeneous element of $U(\mathfrak{sl}_2)$. Let $d$ be the degree of $x$. The algebra is spanned by monomials $(X^-)^a H^b (X^+)^c$ with $a, b, c \in \mathbb{Z}_{\geq 0}$ (this is an immediate consequence of the Serre-Chevalley relations). It follows that there are nonzero polynomials $\xi_{a_1}, \xi_{a_1+1}, \ldots, \xi_{a_2}$ in $K[H]$ with $a_1 = \max(0,d)$ and $a_2 \geq a_1$, such that $x = \sum_{a=a_1}^{a_2} (X^-)^{a-d} \xi_a(H)(X^+)^a$. According to the definition (2) of the Verma module $M(n)$, the action of $x$ on $b_{a_1}$ then satisfies

$$x.b_{a_1} = \xi_{a_1}(H)(X^+)^{a_1}.b_{a_1} = \xi_{a_1}(n) \frac{a_1! n!}{(n-a_1)!} b_{a_1}.$$
The image of $b_{a_1}$ in the quotient $L(n)$ being nonzero for all $n \geq a_1$, it follows that $\xi^{a_1}(n)$ is zero for all $n \geq a_1$. The polynomial $\xi^{a_1}$ is therefore zero, which is a contradiction. \hfill \Box

In other words, one can define the algebra $U(\mathfrak{s\ell}_2)$ as the quotient of $\mathfrak{u}(a)$ by $I$, where $I$ designates the two-sided ideal of $\mathfrak{u}(a)$ formed of the elements acting by zero on all the integral Verma modules of $\mathfrak{s\ell}_2$, when viewed as representations of $\mathfrak{u}(a)$. This construction of $U(\mathfrak{s\ell}_2)$ may be viewed as an expression of a Tannaka duality between the algebra $U(\mathfrak{s\ell}_2)$ on the one side, and the Verma modules $M(n)$ on the other. We propose to consider the same construction, where integral Verma modules of $\mathfrak{s\ell}_2$ now carry a “colored” action of $X^+$.

**Definition 2.11.** Let $\psi$ be a coloring.

1. We denote by $I_h(\psi)$ the two-sided ideal of $U_h(a)$ formed of the elements acting by zero on all the representations $M_h(n, \psi)$ $(n \in \mathbb{Z})$.
2. We denote by $U_h(\psi)$ the quotient of $U_h(a)$ by $I_h(\psi)$.

The algebra $U_h(\psi)$ has a natural structure of $U_h(a)$-algebra, given by the projection map from $U_h(a)$ to $U_h(\psi)$, and it follows from the definition that the action of $U_h(a)$ on $M_h(n, \psi)$ factorizes through $U_h(\psi)$. The algebra $U_h(\psi)$ is universal for this property.

**Proposition 2.12.** Let $\psi$ a coloring. If $A$ is a $U_h(a)$-algebra such that

- the structural homomorphism from $U_h(a)$ to $A$ is surjective,
- the action of $U_h(a)$ on $M_h(n, \psi)$ factorizes through $A$ for all $n \in \mathbb{Z}$,

then there is a unique $U_h(a)$-algebra homomorphism from $A$ to $U_h(\psi)$. 

**Proof.** Let $A$ be a $U_h(a)$-algebra which satisfies the two conditions of the proposition, and let $f$ be the structural homomorphism from $U_h(a)$ to $A$. As $f$ is surjective, a $U_h(a)$-algebra homomorphism from $A$ to $U_h(\psi)$ is necessarily unique. Let $x$ be an element in $U_h(a)$ such that $f(x) = 0$. Since the action of $U_h(a)$ on $M_h(n, \psi)$ factorizes through $A$ for all $n \in \mathbb{Z}$, it follows that $x$ acts by zero on $M_h(n, \psi)$ for every $n \in \mathbb{Z}$. It implies by definition of the algebra $U_h(\psi)$ that the image of $x$ in $U_h(\psi)$ is zero. Put in other words, the projection map from $U_h(a)$ to $U_h(\psi)$ factorizes through $A$. \hfill \Box

The algebra $U_h(\psi)$ is not in general a formal deformation of $U(\mathfrak{s\ell}_2)$. We will give in section 3 a sufficient and necessary condition on the coloring $\psi$ for $U_h(\psi)$ to be a formal deformation of $U(\mathfrak{s\ell}_2)$. However, the algebra $U_h(\psi)$ always satisfies one of the axioms of a formal deformation. Namely, $U_h(\psi)$ is topologically free (recall that a $K[[h]]$-module is said topologically free if it is isomorphic to $V_0[[h]]$ for some $K$-vector space $V_0$).

**Lemma 2.13.** For any coloring $\psi$, the $K[[h]]$-module $U_h(\psi)$ is topologically free.

**Proof.** The $K[[h]]$-module $U_h(\psi)$ is by definition topologically free. It is in particular complete for the $h$-adic topology. As $U_h(\psi)$ is a quotient of $U_h(a)$, it is also complete. For $n \in \mathbb{Z}$ we denote by $E(n)$ the $K[[h]]$-algebra $\text{End}_{K[[h]]}(M_h(n, \psi))$ and we denote by $f_n$ the $K[[h]]$-algebra homomorphism from $U_h(a)$ to $E(n)$ given by the representation $M_h(n, \psi)$. We denote by $f$ the product of the $f_n$’s $(n \in \mathbb{Z})$. Remark that by definition $I_h(\psi)$ is equal to $\ker(f)$. The $K[[h]]$-modules $M_h(n, \psi)$ are topologically free. They are in particular Hausdorff for the $h$-adic topology and torsion-free. Hence so is $E = \prod_{n \in \mathbb{Z}} E(n)$. In view of the equality $I_h(\psi) = \ker(f)$,
Proof. The structural homomorphisms from $U$ are topologically free; see for example [2].

Therefore, proving that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ consists from now in proving that $U_h(\psi)_{h=0}$ is isomorphic as $\mathfrak{a}$-algebra to $U(\mathfrak{sl}_2)$. As mentioned earlier, this is not true for a general coloring $\psi$. Nevertheless, we can show that the algebra $U_h(\psi)_{h=0}$ is always a cover of $U(\mathfrak{sl}_2)$.

Lemma 2.14. For any coloring $\psi$, there is a unique surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$.

Proof. The structural homomorphisms from $U(\mathfrak{a})$ to the $\mathfrak{a}$-algebras $U_h(\psi)_{h=0}$ and $U(\mathfrak{sl}_2)$ are surjective. It implies that an $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$ is necessarily unique and surjective. Consider the functor $(\bullet)_{h=0}$ from the category of $K[[h]]$-modules to the category of $K$-vector spaces. It is is a right-exact functor. Hence, there is a natural isomorphism between $U_h(\psi)_{h=0}$ and the quotient of $U_h(\mathfrak{a})_{h=0}$ by the two-sided ideal $I_h(\psi)_{h=0}$. As the algebra $U_h(\mathfrak{a})_{h=0}$ is canonically isomorphic to $U(\mathfrak{a})$, proving the lemma then reduces to proving that every element of $I_h(\psi)_{h=0}$ is zero in $U(\mathfrak{sl}_2)$. For every $n \in \mathbb{Z}$, the representation $M_h(n, \psi)$ is a formal deformation along $U_h(\mathfrak{a})$ of the Verma module $M(n)$ when viewed as representation of $U(\mathfrak{a})$ (lemma 2.9). This implies that every element $x$ in $I_h(\psi)_{h=0}$ acts by zero and all the integral Verma modules of $\mathfrak{sl}_2$, and in consequence that $x$ is zero in $U(\mathfrak{sl}_2)$ (proposition 2.10). \hfill \Box

3. THE EQUATION $M(\psi), \xi = \theta$

We proved in section 2 that for every coloring $\psi$ the algebra $U_h(\psi)$ is “almost” a formal deformation of the $\mathfrak{a}$-algebra $U_h(\psi)$; namely $U_h(\psi)$ is a topologically free $K[[h]]$-module and there is a surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$. It follows that $U_h(\psi)$ is an actual formal deformation of $U(\mathfrak{sl}_2)$ if and only if the aforementioned homomorphism is also injective. It is equivalent to prove that there is a relation in $U_h(\psi)$ which deforms the relation $[X^+, X^-] = H$ of $U(\mathfrak{sl}_2)$, or that the element $X^+X^-$ can be expressed in $U_h(\psi)$ as a linear combination of the monomials $(X^-)^a H^b (X^+)^c$ ($a, b, c \in \mathbb{Z}_{\geq 0}$); see section 4. In order to address this problem, we introduce in the present section a family of infinite-dimensional linear equations which encode the action of $U_h(\psi)$ on $M_h(n, \psi)$ (proposition 3.2).

We prove that these equations always admit regular solutions if and only if the coloring $\psi$ is regular (proposition 3.2); this is the key technical result of this paper.

3.1. Definition. We denote by $\Xi$ the $K[[h]]$-module formed by sequences of the form $\xi = (\xi^a)_{a \geq 0}$ with $\xi^a = \sum_{n \geq 0} \xi^a_n h^n$ and $\xi^a_n \in K^Z$. For $d \in \mathbb{Z}$ we denote by $\Xi_d$ the $K[[h]]$-submodule of $\Xi$ formed of the sequences $\xi$ such that $\xi^a$ is zero for all $a < d$. We say that a sequence $\xi = (\xi^a)_{a \geq 0}$ in $\Xi$ is regular if $\xi^a(n)$ is a polynomial function of $n$ for all $a, m \in \mathbb{Z}_{\geq 0}$ and if for each $m$ the polynomial $\xi^a(n)$ is zero for all but finitely many $a$.

We denote by $\Theta$ the $K[[h]]$-module formed by sequences of the form $\theta = (\theta^p)_{p \geq 0}$ with $\theta^p = \sum_{m \geq 0} \theta^p_m h^m$ and $\theta^p_m \in K^Z$, satisfying the third coloring axiom

\[ \theta^{n+p-1}(n) = \theta^p (-n - 2) \quad \text{for all } n, p \in \mathbb{Z}_{\geq 0}. \]

For $d \in \mathbb{Z}$ we denote by $\Theta_d$ the $K[[h]]$-submodule of $\Theta$ formed of the sequences $\theta$ such that $\theta^p$ is zero for all $p < d$. We say that a sequence $\theta = (\theta^p)_{p \geq 0}$ in $\Theta$ is
regular if \( \theta_{m,n}^p(n) \) is a polynomial function of \( n \) for all \( p, m \in \mathbb{Z}_{\geq 0} \) and if for each \( m \) the degree of \( \theta_{m,n}^p(n) \) is a function of \( p \) bounded above.

Let \( \psi \) be a coloring. We denote by \( M(\psi) \) the \( K[[h]] \)-linear map from \( \Xi \) to \( \Theta \) defined by (an empty product is by convention equal to 1)

\[
(M(\psi)\xi)(n) = \sum_{a=0}^{p} \left( \prod_{b=p-a}^{p-1} \psi_b \right) \xi^a(n - 2p + 2a)
\]

for \( \xi \in \Xi \) and for \( n \in \mathbb{Z} \).

**Remark 3.1.** A sequence \( \theta \) in \( \Theta \) is regular if and only if for all \( m \in \mathbb{Z}_{\geq 0} \), \( \theta_{m,n}^p(n) \) is a polynomial function of the two variables \( p \) and \( n \). Whereas we do not use this fact in the present paper, the author believes it is interesting in its own.

**Proof of remark 3.1.** Let \( \theta \) be a regular vector in \( \Theta \) and let \( m \in \mathbb{Z}_{\geq 0} \). There are functions \( c_0, \ldots, c_l \) \((l \geq 0)\) from \( \mathbb{Z}_{\geq 0} \) to \( K \) such that \( \theta_{m,n}^p(n) = \sum_{b=0}^{l} c_b(p)n^b \). Let \( f^p \in K^N \) defined by \( f^p(n) = \theta_{m,n}^p(n) - \theta_{m,n}^p(n-1) \). The coloring axiom (C3) for \( \theta \) implies that \( \theta_{m,n}^{p+1}(n) \) is a polynomial function of \( n \) for every \( p \in \mathbb{Z}_{\geq 0} \). When \( l = 0 \) it implies that \( c_0(p) \) is a polynomial function of \( p \). When \( l \geq 1 \) it implies that there are functions \( c_0', \ldots, c_{l-1}' \) from \( \mathbb{Z}_{\geq 0} \) to \( K \) with \( c_{l-1}' = lc_l \), such that \( f^p(n) = \sum_{b=0}^{l-1} c'_b(p)n^b \) and such that \( f^{n+p+1}(n) \) is a polynomial function of \( n \) for every \( p \in \mathbb{Z}_{\geq 0} \). It follows by induction on \( l \) that \( c_0(p), \ldots, c_l(p) \) are polynomial functions of \( p \). This proves that if \( \theta \) is regular in \( \Theta \), then \( \theta_{m,n}^p(n) \in K[p,n] \). The converse implication is immediate. \( \square \)

### 3.2. Interpretation

Let \( \psi \) be a coloring. We say that an element \( x \) in \( U_h(\psi) \) is of degree \( d \) \((d \in \mathbb{Z})\) if for every \( p \geq d \) and for every \( n \in \mathbb{Z} \), \( x.b_p \) is equal to \( \theta^p(n)b_{p-d} \) in \( M_h(n,\psi) \) for some \( \theta^p(n) \in K[[h]] \). It follows from the definition of \( M_h(n,\psi) \) that the values \( \theta^p(n) \) define a sequence in \( \Theta_d \). We denote this sequence by \( \psi(x) \). Remark that the degree \( d \) of \( x \) is unique (by definition of \( U_h(\psi) \) the element \( x \) is zero if and only if it acts by zero on \( M_h(n,\psi) \) for all \( n \in \mathbb{Z} \)). Let us also remark that \( H \) and \( X^\pm \) are of degrees 0 and \( \pm 1 \) in \( U_h(\psi) \).

**Proposition 3.2.** Let \( \psi \) be a coloring. Let \( \theta \in \Theta_d \) \((d \in \mathbb{Z})\) and let \( \xi \) be a regular sequence in \( \Xi_d \). The following two assertions are equivalent.

(i) The sequence \( \theta \) is equal to \( \psi(x) \) for \( x = \sum_{a=\max(0,d)}^{\infty} (X^-)^{a-d}\xi^a(H)(X^+)^a \).

(ii) The sequence \( \xi \) is a solution of the equation \( M(\psi)\xi = \theta \).

**Proof.** The sequence \( \xi \) is regular in \( \Xi \). In other words \( \xi^a(n) \) belongs to \( K[n][[h]] \) and tends to zero for the \( h \)-adic topology as \( a \) goes to infinity. Therefore, \( \xi^a(H) \) defines an element in \( U_h(\psi) \) which tends to zero for the \( h \)-adic topology as \( a \) goes to infinity. As \( U_h(\psi) \) is Hausdorff and complete for the \( h \)-adic topology (lemma 2.13), the series \( \sum_{a=\max(0,d)}^{\infty} (X^-)^{a-d}\xi^a(H)(X^+)^a \) then converges to a unique element \( x \) in \( U_h(\psi) \). Remark that \( x \) is of degree \( d \). By definition of \( \psi(x) \), assertion (i) holds if and only if the equality

\[
\sum_{a=\max(0,d)}^{\infty} (X^-)^{a-d}\xi^a(H)(X^+)^a b_p = \theta^p(n)b_{p-d}
\]
holds in the representation $M_h(n, \psi)$ for every $p \geq d$ and for every $n \in \mathbb{Z}$. In other words, assertion (i) is true if and only if the equality

$$\sum_{a=\max(0,d)}^{p} \left( \prod_{b=p-a}^{p-1} \psi^b \right) \xi^a(n-2p+2a) = \theta^p(n)$$

holds for all $p \geq d$ and for all $n \in \mathbb{Z}$. \hfill \Box

### 3.3. Regular solutions.

**Proposition 3.3.** Let $\psi$ be coloring.

1. The equation $M(\psi)\xi = \psi$ admits a regular solution $\xi$ in $\Xi$ if and only if $\psi$ is regular in $\Theta$.

2. Suppose that $\psi$ is regular in $\Theta$. For each $d \in \mathbb{Z}$ the map $\xi \mapsto M(\psi)\xi$ defines a $K[[h]]$-linear isomorphism from regular sequences in $\Xi_d$ to regular sequences in $\Theta_d$.

**Proof of proposition 3.3.**

**Step 1.** If the equation $M(\psi)\xi = \psi$ admits a solution $\xi$ in $\Xi$ such that $\xi^a(n)$ belongs to $K[n][h]$ for all $a \in \mathbb{Z}_{\geq 0}$, then $\psi^p(n)$ belongs to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$.

**Proof.** Let $\xi$ be a sequence in $\Xi$ such that $\xi^a(n)$ belongs to $K[n][\Theta]$ for all $a \in \mathbb{Z}_{\geq 0}$. If $M(\psi)\xi = \psi$ holds, i.e. if $\xi$ satisfies

$$\sum_{a=\max(0,d)}^{p} \left( \prod_{b=p-a}^{p-1} \psi^b \right) \xi^a(n-2p+2a) = \psi^p(n)$$

for all $p \in \mathbb{Z}_{\geq 0}$ and for all $n \in \mathbb{Z}$, then it follows by induction on $p$ that $\psi^p(n)$ belongs to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$. \hfill \Box

**Step 2.** Let $\xi$ be a regular sequence in $\Xi$. If $\psi^p(n)$ belongs to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$, then the sequence $M(\psi)\xi$ is regular in $\Theta$.

**Proof.** Let $\theta$ be the sequence $M(\psi)\xi$ and suppose that $\psi^p(n)$ belongs to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$. As $\xi^p(n)$ also belongs to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$, so does $\theta^p(n)$. Let $m \in \mathbb{Z}_{\geq 0}$. As $\xi$ is a regular in $\Xi$, there exists $a(m)$ such that $\xi^{a(m)}(n) \in h^{m+1} K[n][\Theta]$ for all $a > a(m)$. Therefore, the equality

$$\sum_{a=0}^{a(m)} \left( \prod_{b=p-a}^{p-1} \psi^b(n) \right) \xi^a(n-2p+2a) = \theta^p(n)$$

holds in $K[n][\Theta]/h^{m+1} K[n][\Theta]$ for all $p \geq a(m)$. It follows that the degree of the polynomial $\theta^p(n)$ in $K[n]$ is a function of $p$ bounded above. \hfill \Box

**Step 3.** Let $\theta \in \Theta$. If $\psi^p(n)$ and $\theta^p(n)$ belong to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$, then the equation $M(\psi)\xi = \theta$ admits a solution $\xi$ in $\Xi$ such that $\xi^a(n)$ belongs to $K[n][\Theta]$ for all $a \in \mathbb{Z}_{\geq 0}$.

**Proof.** Suppose that $\psi^p(n)$ and $\theta^p(n)$ belong to $K[n][\Theta]$ for all $p \in \mathbb{Z}_{\geq 0}$. Let us fix $p \in \mathbb{Z}_{\geq 0}$ and let $\xi^0(n), \xi^1(n), \ldots, \xi^p(n)$ in $K[n][\Theta]$ satisfying

$$\sum_{a=0}^{l} \left( \prod_{b=p-a}^{p-1} \psi^b(n) \right) \xi^a(n-2l+2a) = \theta^p(n)$$

for every $l \geq 0$. Then $\sum_{a=0}^{\max(0,d)} \left( \prod_{b=p-a}^{p-1} \psi^b \right) \xi^a(n-2p+2a) = \psi^p(n)$ holds. \hfill \Box
for all $0 \leq l \leq p$. As $\psi$ satisfies the coloring axioms $\text{[C1]}$ and $\text{[C2]}$, $\psi^b(n)$ is equal to $(n - b)^f(n)$ for some invertible element $f^b(n)$ in $K[[h]]$. Therefore, there exists $\xi^{p+1}(n)$ in $K[[h]]$ such that

$$
\sum_{a=0}^{p} \left( \prod_{b=p+1-a}^{p} \psi^b(n) \right) \xi^a(n - 2(p + 1) + 2a) + \left( \prod_{b=0}^{p} \psi^b(n) \right) \xi^{p+1}(n) = \theta^{p+1}(n)
$$

if only if the equality

$$
\sum_{a=0}^{p} \left( \prod_{b=p+1-a}^{p} \psi^b(n') \right) \xi^a(n' - 2(p + 1) + 2a) = \theta^{p+1}(n')
$$

holds in $K[[h]]$ for all $n' \in \{0, 1, \ldots, p\}$. For such $n'$, the left-hand side of (4) is equal to

$$
\sum_{a=0}^{p-n'} \left( \prod_{b=p+1-a}^{p} \psi^b(n') \right) \xi^a(n' - 2(p + 1) + 2a)
$$

by $\text{[C2]}$

$$
= \sum_{a=0}^{p-n'} \left( \prod_{b=p+1-a}^{p} \psi^{b-n'-1}(-n' - 2) \right) \xi^a(n' - 2(p + 1) + 2a)
$$

by $\text{[C3]}$

$$
= \sum_{a=0}^{p-n'} \left( \prod_{b=p-n'-a}^{p-n'-1} \psi^b(-n' - 2) \right) \xi^a(-n' - 2 - 2(p - n') + 2a).
$$

As $\theta$ satisfies the coloring axiom $\text{[C3]}$, it follows that for all $n' \in \{0, 1, \ldots, p\}$ equality (4) is equivalent to the equality

$$
\sum_{a=0}^{p-n'} \left( \prod_{b=p-n'-a}^{p-n'-1} \psi^b(-n' - 2) \right) \xi^a(-n' - 2 - 2(p - n') + 2a) = \theta^{p-n'}(-n' - 2),
$$

which is equality (3) for $l = p - n'$ and for $n = -n' - 2$. We therefore have proved by induction on $p$ that the equation $M(\psi), \xi = \theta$ admits a solution $\xi$ in $\Xi$ such that $\xi^a(n)$ belongs to $K[[h]]$ for all $a \in \mathbb{Z}_{\geq 0}$.

**Step 4.** Let $\theta \in \Theta$. If $\psi$ and $\theta$ are regular sequences in $\Theta$, then the equation $M(\psi), \xi = \theta$ admits a regular solution $\xi$ in $\Xi$.

**Proof.** Suppose that $\psi$ and $\theta$ are regular sequences in $\Theta$ and let $\xi$ be a solution of the equation $M(\psi), \xi = \theta$ such that $\xi^a(n)$ belongs to $K[[h]]$ for all $a \in \mathbb{Z}_{\geq 0}$ (step 3). Let us prove by induction on $m$ that $\xi^a_m(n)$ is zero for sufficiently large $a$. Let $m \in \mathbb{Z}_{\geq 0}$ and suppose that there exists $a(m)$ such that $\xi^a_m(n)$ is zero for all $a > a(m)$ and for all $m' \leq m$. We denote by $\bar{\theta}$ the sequence in $\Theta$ defined by

$$
\bar{\theta}^p(n) = \theta^p(n) - \sum_{a=0}^{a(m)} \left( \prod_{b=p-a}^{p-1} \psi^b(n) \right) \xi^a(n - 2p + 2a)
$$

for $p \in \mathbb{Z}_{\geq 0}$ and for $n \in \mathbb{Z}$. As $\psi$ and $\theta$ are regular, there is $l \geq 0$ such that the degree of the polynomial $\bar{\theta}^p_{m+1}(n)$ in $K[n]$ is less than or equal to $l$ for all $p \in \mathbb{Z}_{\geq 0}$.
Since $\zeta^a_m(n)$ is zero for all $a > a(m)$ and for all $m' \leq m$, the equality $M(\psi).\xi = \theta$ implies

$$\sum_{a=a(m)+1}^p \left( \prod_{b=p-a}^{p-1} \psi^b_0(n) \right) \zeta^a_{m+1}(n-2p+2a) = \theta^p_{m+1}(n)$$

for all $p \geq a(m) + 1$ and for all $n \in \mathbb{Z}$. According to the coloring axiom $\text{(C1)}$, the polynomial $\psi^b_0(n)$ is of degree 1 in $K[n]$. Equality $\text{(5)}$ therefore implies that if $l(a) (a \in \mathbb{Z}_{\geq 0})$ designates the degree of the polynomial $\zeta^a_{m+1}(n)$ in $K[n]$, then

$$\sum_{a=a(m)+1}^p l(a) + a \leq l \text{ for all } p \geq a(m) + 1.$$  

Hence, it follows from $\text{(5)}$ again, with $p$ replaced by $p+1$, that $l(p+1) + (p+1) \leq l$ for all $p \geq a(m) + 1$. As a consequence, the polynomial $\zeta^a_{m+1}(n)$ is zero for sufficiently large $a$.

\textbf{Step 5.} Let $\theta \in \Theta$. The equation $M(\psi).\xi = \theta$ admits at most one solution $\xi$ in $\Xi$ such that $\xi^a(n)$ belongs to $K[n][[h]]$ for all $a \in \mathbb{Z}_{\geq 0}$.

\textit{Proof.} Let $\xi$ be a solution in $\Xi$ of the equation $M(\psi).\xi = 0$, i.e.

$$\sum_{a=0}^p \left( \prod_{b=p-a}^{p-1} \psi^b(n) \right) \xi^a(n-2p+2a) = 0$$

for all $p \in \mathbb{Z}_{\geq 0}$ and for all $n \in \mathbb{Z}$. Suppose that $\xi^a(n)$ belongs to $K[n][[h]]$ for all $a \in \mathbb{Z}_{\geq 0}$. It follows by induction on $p \in \mathbb{Z}_{\geq 0}$ that $(\prod_{b=0}^{p-1} \psi^b(n))\xi^p(n)$ is zero for all $n \in \mathbb{Z}$. It follows from the coloring axiom $\text{(C1)}$ that $\psi^b(n)$ is zero only if $n = b$. Therefore, $\xi^p(n)$ is zero for infinitely many values of $n$. As $\xi^p(n)$ belongs to $K[n][[h]]$, it follows that $\xi^p$ is zero.

\textbf{Step 6 (Conclusion).}

Step 1 together with step 2 for $\theta = \psi$, prove that if the equation $M(\psi).\xi = \psi$ admits a regular solution $\xi$ in $\Xi$, then $\psi$ is a regular sequence in $\Theta$. Conversely, step 2 proves that if $\psi$ is a regular sequence in $\Theta$, then the equation $M(\psi).\xi = \psi$ admits a regular solution $\xi$ in $\Xi$.

Suppose that $\psi$ is a regular sequence in $\Theta$. Step 2 proves that $\xi \mapsto M(\psi).\xi$ defines a map from regular sequences in $\Xi$ to regular sequences in $\Theta$. Step 2 proves that the map is surjective and step 5 proves it is injective.

\textbf{Remark 3.4.} Let $\psi$ a coloring and let $\theta \in \Theta$ (we do not assume here that $\psi$ and $\theta$ are regular). We see from the proof of proposition 3.3 (step 5) that the equation $M(\psi).\xi = \theta$ admits as many solutions $\xi$ in $\Xi$ as they are choices for the values $\xi^p(n)$ ($0 \leq n \leq p$) – we see in particular that if not regular, a solution of $M(\psi).\xi = \theta$ is not unique. It may be interesting to find a condition on these values characterizing regularity for the solution $\xi$, when $\psi$ and $\theta$ are regular.

4. Colored Kac-Moody algebras of rank one

We present here the main results of this paper. We prove that $U_h(\mathfrak{sl}_2)$ is a formal deformation of $U(\mathfrak{sl}_2)$ if and only if the coloring $\psi$ is regular (theorem 4.1), and we give a Chevalley-Serre presentation of $U_h(\psi)$ for $\psi$ regular (theorem 4.2). We show that the constant formal deformation $U(\mathfrak{sl}_2)[[h]]$ and the quantum algebra $U_h(\mathfrak{sl}_2)$ can be realized as colored Kac-Moody algebras (theorem 4.3). We prove that regular colorings classify $\mathfrak{g}$-trivial formal deformations of the $\mathfrak{g}$-algebra $U(\mathfrak{sl}_2)$.
(theorem 1.2). We prove that $U_h(\psi)$ is a $h$-trivial deformation of $U(\mathfrak{sl}_2)$ for all regular colorings $\psi$ (theorem 3.4). As a corollary, we obtain that every $h$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $h$-trivial (corollary 4.4).

4.1. Formal deformations of $U(\mathfrak{sl}_2)$. We say that a coloring $\psi$ is regular if $\psi$ is a regular sequence in $\Theta$, i.e. if $\psi$ satisfies

(R1) $\psi_m^p(n)$ is a polynomial function of $n$ for all $p, m \in \mathbb{Z}_{\geq 0}$,

(R2) for each $m$ the degree of $\psi_m^p(n)$ is a function of $p$ bounded above.

For $\psi$ regular, we call the algebra $U_h(\psi)$ a colored Kac-Moody algebra.

Theorem 4.1. Let $\psi$ be coloring. The following three assertions are equivalent.

(i) The coloring $\psi$ is regular.

(ii) The algebra $U_h(\psi)$ is a formal deformation of the $K$-algebra $U(\mathfrak{sl}_2)$.

(iii) The algebra $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$.

Proof of theorem 4.1. We denote by $f$ the surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$ (lemma 2.14). Remark that as $U_h(\psi)$ is topologically free (lemma 2.13), it is sufficient to prove that $f$ is injective in order to prove that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$.

Step 1. Assertion (i) implies assertion (iii).

Proof. Suppose that the coloring $\psi$ is regular. Then the equation $M(\psi)\xi = \psi$ admits a regular solution $\xi = (\xi^a)_{a \geq 0}$ in $\Xi$ (proposition 3.3). It implies that the series $\sum_{a \geq 0}(X^-)^a \xi^a(H)(X^+)^a$ converges in $U_h(\psi)$ (for the $h$-adic topology) to a unique element $x$ such that $\psi(x) = \psi(X^+X^-)$ (proposition 3.2). It means that for every $n \in \mathbb{Z}$ the elements $X^+X^-$ and $x$ act identically on $M_h(n, \psi)$ when viewed as representation of $U_h(\psi)$. By definition of $U_h(\psi)$, the relation $X^+X^- = x$ therefore holds in $U_h(\psi)$. It follows that $U_h(\psi)_{h=0}$ is spanned by the monomials $(X^-)^a H^b(X^+)^c$ ($a, b, c \in \mathbb{Z}_{\geq 0}$). These monomials form a basis in $U(\mathfrak{sl}_2)$ (namely, the canonical PBW basis). In other words, $f$ sends a spanning subset to a basis. This implies that $f$ is injective.

Step 2. Assertions (ii) and (iii) are equivalent.

Proof. Suppose that $U_h(\psi)$ is a formal deformation of the $K$-algebra $U(\mathfrak{sl}_2)$. Then there is a $K$-algebra isomorphism $f'$ from $U(\mathfrak{sl}_2)$ to $U_h(\psi)_{h=0}$. We therefore have a surjective $K$-algebra endomorphism $g = f \circ f'$ of $U(\mathfrak{sl}_2)$. We denote by $L(n)^g$ the pullback by $g$ of the irreducible representation $L(n)$ of $\mathfrak{sl}_2$ of dimension $n$ ($n \geq 0$). The pullback $L(n)^g$ is a representation of $U(\mathfrak{sl}_2)$ of dimension $n$, irreducible again as $g$ is surjective. The representation $L(n)^g$ is thus isomorphic to $L(n)$. Consider now an element $x$ in $U(\mathfrak{sl}_2)$ such that $g(x) = 0$. The element $x$ then acts by zero on the pullback $L(n)^g$ for every $n \geq 0$. As a consequence, $x$ acts by zero on $L(n)$ for every $n$, and is thus equal to zero (proposition 2.10). In other words, $g$ is injective. It implies that $f$ is injective. We hence have proved that assertion (ii) implies assertion (iii). The converse implication is immediate.

Step 3. Assertion (iii) implies assertion (i).

Proof. We denote by $T$ the subset $\{(X^-)^a H^b(X^+)^c; a, b, c \geq 0\}$ of $U_h(\psi)$. As $U_h(\psi)$ is Hausdorff and complete for the $h$-adic topology (lemma 2.13), the inclusion map from $T$ to $U_h(\psi)$ induces a $K[[h]]$-linear map $j$ from $(KT)[[h]]$ to $U_h(\psi)$. Suppose that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. Then the image of $T$
in \( U_h(\psi)_{h=0} \) is a basis of \( U_h(\psi)_{h=0} \) – it corresponds to the canonical PBW basis of \( U(\mathfrak{sl}_2) \) via the unique \( \mathfrak{a} \)-algebra isomorphism between \( U_h(\psi)_{h=0} \) and \( U(\mathfrak{sl}_2) \). The map \( f_{h=0} \) is therefore surjective. It follows that \( j \) is surjective, since \( U_h(\psi) \) is Hausdorff and complete for the \( h \)-adic topology. In other words, there exists a regular sequence \( \xi = (\xi_a)_{a \geq 0} \) in \( \Xi \) such that the relation

\[
X^+ X^- = \sum_{a=0}^{\infty} (X^-)^a \xi^a (H)(X^+)^a
\]

holds in \( U_h(\psi) \) (the series converges to a unique element in \( U_h(\psi) \) for the \( h \)-adic topology since \( U_h(\psi) \) is Hausdorff and complete). It implies that \( \xi \) is a solution of the equation \( M(\psi) \xi = \psi \) (proposition \ref{prop:regular_solution}). The coloring \( \psi \) is therefore regular (proposition \ref{prop:regularity}). \( \square \)

4.2. Generators and relations.

**Theorem 4.2.** Let \( \psi \) be a regular coloring. The \( K[[h]] \)-algebra \( U_h(\psi) \) is generated topologically by \( X^-, H, X^+ \) and subject to the relations

\[
\begin{align*}
(6a) & \quad [H, X^\pm] = \pm 2 X^\pm, \\
(6b) & \quad X^+ X^- = \sum_{a=0}^{\infty} (X^-)^a \xi^a (H)(X^+)^a,
\end{align*}
\]

where \( \xi = (\xi_a)_{a \geq 0} \) is the regular solution in \( \Xi \) of the equation \( M(\psi) \xi = \psi \).

**Proof.** Since \( \psi \) is regular the equation \( M(\psi) \xi = \psi \) admits a unique regular solution \( \xi = (\xi^a)_{a \geq 0} \) in \( \Xi \) (proposition \ref{prop:regular_solution}). The element \( \xi^a \) tends to zero in \( K[H][[h]] \) for the \( h \)-adic topology when \( a \) goes to infinity. It implies that the series in \( 6b \) converges to a unique element in the \( K[[h]] \)-algebra \( K\langle X^-, H, X^+ \rangle[[h]] \) (which is Hausdorff and complete for the \( h \)-adic topology). Let \( U \) be the quotient of \( K\langle X^-, H, X^+ \rangle[[h]] \) by the smallest closed (for the \( h \)-adic topology) two-sided ideal containing the relations \( 6b \). Since \( \xi \) is a regular solution in \( \Xi \) of the equation \( M(\psi) \xi = \psi \), it follows from proposition \ref{prop:regular_solution} (see the proof of theorem \ref{thm:principal_ideal}) that the relation \( 6b \) holds in \( U_h(\psi) \). Therefore, as \( U_h(\psi) \) is complete and Hausdorff for the \( h \)-adic topology (lemma \ref{lem:complete_hausdorff}), there is a canonical \( K \)-algebra homomorphism \( f \) from \( U \) to \( U_h(\psi) \). It follows from the relation \( 6b \) that \( U_{h=0} \) is spanned by the monomials \( (X^-)^a H^b (X^+)^c \) with \( a, b, c \in \mathbb{Z}_{\geq 0} \). These monomials form a basis in \( U_h(\psi)_{h=0} \) – it corresponds to the canonical PBW basis of \( U(\mathfrak{sl}_2) \) via the unique \( \mathfrak{a} \)-algebra isomorphism between \( U_h(\psi)_{h=0} \) and \( U(\mathfrak{sl}_2) \). In other words, the map \( f_{h=0} \) sends a spanning subset to a basis. This implies that \( f_{h=0} \) is bijective. As \( U \) is Hausdorff and complete (for the \( h \)-adic topology), and since \( U_h(\psi) \) is Hausdorff and torsion-free (lemma \ref{lem:torson-free}), it follows that \( f \) is bijective. \( \square \)

4.3. Classical and quantum realizations. Let us recall that the quantum algebra \( U_h(\mathfrak{sl}_2) \) designates the \( K[[h]] \)-algebra generated topologically by \( X^-, H, X^+ \) and subject to the relations

\[
\begin{align*}
(7a) & \quad [H, X^\pm] = \pm 2 X^\pm, \\
(7b) & \quad [X^+, X^-] = \frac{q^H - q^{-H}}{q^{-1} - q} \quad & \text{with } q = \exp(h) \text{ and } q^H = \exp(hH),
\end{align*}
\]
i.e. $U_h(\mathfrak{sl}_2)$ is the quotient of the $K[[h]]$-algebra $K\langle X^-, H, X^+ \rangle[[h]]$ by the smallest closed (for the $h$-adic topology) two-sided ideal containing the previous relations.

We recall that $N$ and $N_q$ designate the natural coloring and the $q$-coloring, defined by

\[ N^p(n) = (p+1)(n - p), \]
\[ N_q^p(n) = [p + 1]_q[n - p], \quad \text{where } [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} \text{ for } k \in \mathbb{Z}. \]

**Theorem 4.3.** The colored Kac-Moody algebras $U_h(N)$ and $U_h(N_q)$ are isomorphic as $U_h(a)$-algebras to $U(\mathfrak{sl}_2)[[h]]$ and $U_h(\mathfrak{sl}_2)$, respectively.

**Proof.** The algebra $U(\mathfrak{sl}_2)[[h]]$ has a canonical structure of $U_h(a)$-algebra. Remark that the structural homomorphism from $U_h(a)$ to $U(\mathfrak{sl}_2)[[h]]$ is surjective. The relation $[X^+, X^-] = H$ holds in the representation $M_h(n,N)$ for all $n \in \mathbb{Z}$. In view of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$, and since $M_h(n,N)$ is Hausdorff for the $h$-adic topology, it implies that the action of $U_h(a)$ on $M_h(n,N)$ factorizes through $U(\mathfrak{sl}_2)[[h]]$ for all $n \in \mathbb{Z}$. It follows by the universal property of $U_h(N)$ (proposition 2.12) that there is a $U_h(a)$-algebra homomorphism $f$ from $U(\mathfrak{sl}_2)[[h]]$ to $U_h(N)$. Since the structural homomorphism from $U_h(a)$ to $U_h(N)$ is surjective, so is $f$. The $a$-algebra $U(\mathfrak{sl}_2)[[h]]_{h=0}$ is isomorphic to $U(\mathfrak{sl}_2)$. Therefore, there is an $a$-algebra homomorphism $g_0$ from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)[[h]]_{h=0}$. Consider then the map $g_0 \circ f_{h=0}$. It is an $a$-algebra endomorphism of $U(\mathfrak{sl}_2)[[h]]_{h=0}$. Hence it is equal to the identity map. This implies that $f_{h=0}$ is injective. Since $U_h(N)$ is torsion-free (lemma 2.13) and since $U(\mathfrak{sl}_2)[[h]]$ is Hausdorff for the $h$-adic topology, it follows that $f$ is injective, and therefore bijective.

The proof for $U_h(\mathfrak{sl}_2)$ is similar. Namely, the algebra $U_h(\mathfrak{sl}_2)$ has a canonical structure of $U_h(a)$-algebra. Remark that the structural homomorphism from $U_h(a)$ to $U_h(\mathfrak{sl}_2)$ is surjective. The relation 2.13 holds in the representation $M_h(n,N_q)$ for all $n \in \mathbb{Z}$. In view of the presentation (7a) of $U_h(\mathfrak{sl}_2)$, and since $M_h(n,N_q)$ is Hausdorff for the $h$-adic topology, it implies that the action of $U_h(a)$ on $M_h(n,N_q)$ factorizes through $U_h(\mathfrak{sl}_2)$ for all $n \in \mathbb{Z}$. It follows by the universal property of $U_h(N_q)$ (proposition 2.12) that there is a $U_h(a)$-algebra homomorphism $f$ from $U_h(\mathfrak{sl}_2)$ to $U_h(N_q)$. Since the structural homomorphism from $U_h(a)$ to $U_h(N_q)$ is surjective, so is $f$. The $a$-algebra $U_h(\mathfrak{sl}_2)_{h=0}$ is isomorphic to $U(\mathfrak{sl}_2)$ (the functor $(\bullet)_{h=0}$ from the category of $K[[h]]$-modules to the category of $K$-vector spaces is is a right-exact functor). Therefore, there is an $a$-algebra homomorphism $g_0$ from $U_h(\psi)_{h=0}$ to $U_h(\mathfrak{sl}_2)_{h=0}$. Consider then the map $g_0 \circ f_{h=0}$. It is an $a$-algebra endomorphism of $U_h(\mathfrak{sl}_2)_{h=0}$. Hence it is equal to the identity map. This implies that $f_{h=0}$ is injective. Since $U_h(N_q)$ is torsion-free (lemma 2.13) and since $U_h(\mathfrak{sl}_2)$ is Hausdorff for the $h$-adic topology, it follows that $f$ is injective, and therefore bijective.

**4.4. Classification.** Let $A$ be a formal deformation of the $a$-algebra $U(\mathfrak{sl}_2)$. The elements $X^-, H$ and $X^+$ in $A$ designate the images of $X^-$, $H$ and $X^+$ in $U_h(a)$ by the structural homomorphism from $U_h(a)$ to $A$.

We say that the formal deformation $A$ is $h$-trivial if there exists a $K[[h]]$-algebra homomorphism $g$ from $U_h(N)$ to $A$ such that $g(H) = H$ and such that $g_{h=0}$ is an $a$-algebra homomorphism. The homomorphism $g$ is called a $h$-trivialization of $A$. 
Theorem 4.4. For every \( h \)-trivial formal deformation \( A \) of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{sl}_2) \), there is a unique regular coloring \( \psi \) such that \( A \) and \( U_h(\psi) \) are isomorphic as \( U_h(\mathfrak{a}) \)-algebras.

Proof of theorem 4.4. Let \( n \in \mathbb{Z} \) and let \( \varphi = (\varphi^p)_{p \geq 0} \) be a sequence with values in \( K[[h]] \). We denote by \( M_h(n, \varphi) \) the representation of \( U_h(\mathfrak{a}) \) whose underlying \( K[[h]] \)-module is \( (\bigoplus_{p \geq 0} K b_p)[[h]] \) and where the action of \( U_h(\mathfrak{a}) \) is defined by

\[
H b_p = (n - 2p) b_p, \\
X^+ b_p = b_{p+1}, \\
X^- b_p = \begin{cases} 0 & \text{if } p = 0, \\ \varphi^{p-1} b_{p-1} & \text{if } p \geq 1. \end{cases}
\]

Step 1. Let \( A \) be a formal deformation of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{sl}_2) \) and let \( n \in \mathbb{Z} \). There is at most one sequence \( \varphi \) with values in \( K[[h]] \) such that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \varphi) \) factorizes through \( A \).

Proof. Let \( V(n) \) be the representation of \( A \) generated topologically by \( v \) and subject to the relations \( H.v = nv \) and \( X^+.v = 0 \) - i.e. the representation \( V(n) \) is the quotient of the left regular representation \( A.v \) of \( A \) by the smallest closed (for the \( h \)-adic topology) subrepresentation containing the elements \( H.v - nv \) and \( X^+.v \). Let \( \varphi \) be a sequence with values in \( K[[h]] \) such that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \varphi) \) factorizes through \( A \). We regard from now \( M_h(n, \varphi) \) as a representation of \( A \). By definition of \( V(n) \), and since \( M_h(n, \varphi) \) is Hausdorff for the \( h \)-adic topology, there is a morphism \( f \) from \( V(n) \) to \( M_h(n, \varphi) \) such that \( f(v) = b_0 \). The representation \( V(n)_{h=0} \) of \( A_{h=0} \) is generated by the image \( v_0 \) of \( v \), and the relations \( H.v_0 = nv_0 \) and \( X^+.v_0 = 0 \) hold in \( V(n)_{h=0} \). Since \( A \) is a formal deformation of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{sl}_2) \), the relations \( [H, X^{-}] = -2X^{-} \) and \( [X^+, X^{-}] = H \) hold in \( A_{h=0} \). It follows by induction on \( p \) that the vectors \( (X^{-})^p.v_0 \ (p \in \mathbb{Z}_{\geq 0}) \) span the \( K \)-vector space \( V(n)_{h=0} \). As a consequence, the \( \mathfrak{a} \)-algebra homomorphism \( f_{h=0} \) sends a spanning subset of \( V(n)_{h=0} \) to a basis of \( M_h(n, \varphi)_{h=0} \). The map \( f_{h=0} \) is therefore bijective. As \( V(n) \) is Hausdorff and complete (for the \( h \)-adic topology), and since \( M_h(n, \varphi) \) is Hausdorff and torsion-free, it follows that \( f \) is bijective. Suppose that there is another sequence \( \varphi' \) with values in \( K[[h]] \) such that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \varphi') \) factorizes through \( A \). Then, as for \( \varphi \), there is an \( A \)-isomorphism \( f' \) from \( V(n) \) to \( M_h(n, \varphi') \) such that \( f'(v) = b_0 \). It follows that there is an \( A \)-isomorphism \( g \) from \( M_h(n, \varphi) \) to \( M_h(n, \varphi') \) such that the image of \( b_0 \) by \( g \) is \( b_0 \). Since \( g \) commutes with the action of \( X^- \), the image of \( b_p \) by \( g \) is \( b_p \) for all \( p \in \mathbb{Z}_{\geq 0} \). Since \( g \) commutes with the action of \( X^+ \), it follows that \( \varphi \) and \( \varphi' \) are equal.

Step 2. Let \( \psi, \psi' \) be two regular colorings. If \( U_h(\psi) \) and \( U_h(\psi') \) are isomorphic as \( U_h(\mathfrak{a}) \)-algebras, then \( \psi = \psi' \).

Proof. Let \( n \in \mathbb{Z} \). By definition, the actions of \( U_h(\mathfrak{a}) \) on the representations \( M_h(n, \psi) \) and \( M_h(n, \psi') \) factorize through \( U_h(\psi) \) and \( U_h(\psi') \), respectively. Suppose that \( U_h(\psi) \) and \( U_h(\psi') \) are isomorphic \( U_h(\mathfrak{a}) \)-algebras. It implies that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \psi') \) also factorizes through \( U_h(\psi) \). It follows from step 1 that \( \psi^p(n) = (\psi')^p(n) \) for all \( p \in \mathbb{Z}_{\geq 0} \).

Step 3. Let \( A \) be a formal deformation of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{sl}_2) \). If \( \psi \) is a coloring such that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \psi) \) factorizes through \( A \) for all \( n \in \mathbb{Z} \), then \( A \) and \( U_h(\psi) \) are isomorphic as \( U_h(\mathfrak{a}) \)-algebras.
and since \( A \) is Hausdorff and complete for the \( h \)-adic topology, it follows that \( f \) is surjective. Let then \( \psi: A \to U_h(\mathfrak{a}) \) be a coloring such that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \psi) \) factorizes through \( A \) for all \( n \in \mathbb{Z} \). It follows by the universal property of \( U_h(\psi) \) (proposition \( \ref{2.12} \)) that there is a \( U_h(\mathfrak{a}) \)-algebra homomorphism \( \tilde{f} \) from \( A \) to \( U_h(\psi) \). The map \( \tilde{f} \) is surjective, since the structural homomorphism from \( U_h(\mathfrak{a}) \) to the \( U_h(\mathfrak{a}) \)-algebra \( U_h(\psi) \) is. Since \( A_{h=0} \) and \( U(\mathfrak{a}_2) \) are isomorphic as \( \mathfrak{a} \)-algebras, there is an \( \mathfrak{a} \)-algebra homomorphism \( g_0 \) from \( U_h(\psi)_{h=0} \) to \( A_{h=0} \) (lemma \( \ref{2.14} \)). Consider then the map \( g_0 \circ \tilde{f}_{h=0} \). It is an \( \mathfrak{a} \)-algebra endomorphism of \( A_{h=0} \). Hence it is equal to the identity map. This implies that \( \tilde{f}_{h=0} \) is injective. Since \( U_h(\psi) \) is torsion-free (lemma \( \ref{2.13} \)) and since \( A \) is Hausdorff for the \( h \)-adic topology, it follows that \( \tilde{f} \) is injective, and therefore bijective.

\( \square \)

**Step 4.** For every \( \mathfrak{b} \)-trivial formal deformation \( A \) of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{a}_2) \), there exists a regular coloring \( \psi \) such that \( A \) and \( U_h(\psi) \) are isomorphic as \( U_h(\mathfrak{a}) \)-algebras.

**Proof.** Let \( A \) be a \( \mathfrak{b} \)-trivial formal deformation of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{a}_2) \) and let \( f \) be a \( \mathfrak{b} \)-trivialization of the \( \mathfrak{a} \)-algebra \( A \). We denote by \( V(n) \) (\( n \in \mathbb{Z} \)) the pullback of the representation \( M(n)[[h]] \) by \( f^{-1} \). Consider the grading of \( U(\mathfrak{a}_2) \) defined by \( \deg(H) = 0 \) and \( \deg(X^\pm) = \pm 1 \). An element \( x \) in \( U(\mathfrak{a}_2) \) has degree \( d \) if and only if \( \deg(x) = 2d \). Since \( \deg(H, X^\pm) = \pm 2X^\pm \) holds in \( A \) and since \( f \) is a \( \mathfrak{b} \)-trivialization, it follows that there exist \( \alpha^p(n) = \sum_{m \geq 0} \alpha^p_m(n) h^m \) and \( \beta^p(n) = \sum_{m \geq 0} \beta^p_m(n) h^m \) in \( K([[h]]) \) such that

\[
H.b_p = (n-2p)b_p, \quad X^-b_p = \alpha^p(n)b_{p+1}, \quad X^+b_{p+1} = \beta^p(n)b_p
\]

holds in \( V(n) \) for all \( p \in \mathbb{Z}_{\geq 0} \), with \( \alpha^0_p(n) = 1 \) and \( \beta^0_p(n) = (p+1)(n-p) \). If \( b'_p \) \((p \in \mathbb{Z}_{\geq 0})\) designates the vector \((X^-)^p.0 \) in \( V(n) \), then

\[
H.b'_p = (n-2p)b'_p, \quad X^-b'_p = b'_{p+1}, \quad X^+b'_{p+1} = \psi^p(n)b'_p
\]

holds in \( V(n) \) for all \( p \in \mathbb{Z}_{\geq 0} \), with \( \psi^p(n) = \alpha^p(n)\beta^p(n) \). When \( n \geq 0 \) the \( K \)-vector subspace \( \bigoplus_{p \geq n+1} K b_p \) is a subrepresentation of \( M(n) \), and \( \bigoplus_{p \geq n+1} K b_p[[[h]]] \) is thus subrepresentation of \( V(n) \). It implies that \( \beta^n(n) \) is zero for all \( n \in \mathbb{Z}_{\geq 0} \). It then follows from step \( \ref{3.1} \) that \( \psi^{p+n+1}(n) = \psi^p(-n-2) \) for all \( p, n \in \mathbb{Z}_{\geq 0} \). The values \( \psi^p(n) \) hence define a coloring \( \psi \) such that for every \( n \in \mathbb{Z} \) the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \psi) \) factorizes through \( A \). It follows from step \( \ref{3.3} \) that \( A \) and \( U_h(\psi) \) are isomorphic as \( U_h(\mathfrak{a}) \)-algebras. It implies in particular that \( U_h(\psi) \) is a formal deformation of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{a}_2) \). The coloring \( \psi \) is therefore necessarily regular (theorem \( \ref{2.11} \)).

\( \square \)

4.5. **\( \mathfrak{b} \)-triviality.** Recall that for a formal deformation \( A \) of the \( \mathfrak{a} \)-algebra \( U(\mathfrak{a}_2) \), the elements \( X^- \), \( H \) and \( X^+ \) in \( A \) designate the images of \( X^- \), \( H \) and \( X^+ \) in \( U_h(\mathfrak{a}) \) by the structural homomorphism from \( U_h(\mathfrak{a}) \) to \( A \).

We say that the formal deformation \( A \) is **\( \mathfrak{b} \)-trivial** if there exists a \( K[[[h]]] \)-algebra homomorphism \( g \) from \( U(\mathfrak{a}_2)[[[h]]] \) to \( A \) such that \( g(H) = H \), \( g(X^-) = X^- \) and such that \( g = 0 \) is an \( \mathfrak{a} \)-algebra homomorphism. The homomorphism \( g \) is called a **\( \mathfrak{b} \)-trivialization** of \( A \).

**Theorem 4.5.** Let \( \psi \) be a regular coloring. The colored Kac-Moody algebra \( U_h(\psi) \) is \( \mathfrak{b} \)-trivial and admits a unique \( \mathfrak{b} \)-trivialization.
Every \( \mathfrak{h}\)-trivial formal deformation of the \( \mathfrak{a}\)-algebra \( U(\mathfrak{s}l_2) \) is as \( U_h(\mathfrak{a}) \)-algebra isomorphic to \( U_h(\psi) \) for a regular coloring \( \psi \) (theorem 4.5). Theorem 4.5 then implies the following result.

**Corollary 4.6.** A formal deformation of the \( \mathfrak{a}\)-algebra \( U(\mathfrak{s}l_2) \) is \( \mathfrak{h}\)-trivial if and only if it is \( \mathfrak{b}\)-trivial, and it admits at most one \( \mathfrak{b}\)-trivialization.

**Proof of the theorem 4.5.**

**Step 1.** There exists a \( K[[\mathfrak{h}]]\)-algebra homomorphism \( g \) from \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \) to \( U_h(\psi) \) such that \( g(X^-) = X^- \) and \( g(H) = H \).

**Proof.** Consider the natural coloring \( N \) and consider the sequence \( N(X^+) \) in \( \Theta \geq 1 \) defined by \( N(X^+)^p = N^{p-1} \) for \( p \geq 1 \). As the coloring \( N \) is regular, so is the sequence \( N(X^+) \). As \( \psi \) is also regular, it follows that the equation \( M(\psi).\xi = N(X^+) \) admits a regular solution \( \xi \) in \( \Xi_{\mathfrak{h}} \geq 1 \) (proposition 3.3). According to proposition 3.2, the equality \( M(\psi).\xi = N(X^+) \) implies that the sequences \( N(X^+) \) and \(\psi(x^+)\) are equal, where \( x^+ \) is the element in \( U_h(\psi) \) defined by

\[
x^+ = \sum_{a=1}^{\infty}(X^-)^{a-1}\xi^a(H)(X^+)^a
\]

Let then \( f \) be the surjective \( K[[\mathfrak{h}]]\)-algebra homomorphism from \( U_h(\mathfrak{a}) \) to \( U_h(N) \) defined by \( f(X^-) = X^- \), \( f(H) = H \) and \( f(X^+) = x^+ \). By definition of the sequences \( N(X^+) \) and \(\psi(x^+)\), the equality \( N(X^+) = \psi(x^+) \) implies that the action of \( U_h(\mathfrak{a}) \) on \( M_h(n, \psi) \) factorizes through \( f \) for every \( n \in \mathbb{Z} \). Hence, there exists, by the universal property of \( U_h(\psi) \) (proposition 2.12), a \( U_h(\mathfrak{a})\)-algebra homomorphism \( g \) from \( U_h(N) \) to \( U_h(\psi) \), with \( U_h(N) \) endowed with the \( U_h(\mathfrak{a})\)-algebra structure defined by \( f \). In particular, \( g \) satisfies \( g(X^-) = X^- \) and \( g(H) = H \). As \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \) and \( U_h(\mathfrak{a}) \)-algebras (theorem 4.3), step 1 follows.

**Step 2.** The identity map is the unique \( K[[\mathfrak{h}]]\)-algebra endomorphism of \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \) which fixes both \( X^- \) and \( H \).

**Proof.** Let \( g \) be a \( K[[\mathfrak{h}]]\)-algebra endomorphism of \( U(\mathfrak{s}l_2)[N] \) which fixes both \( X^- \) and \( H \), and let \( x^+ \) be the image of \( X^+ \) by \( g \). The Chevalley-Serre relations \( [H,X^+] = 2X^+ \) and \( [X^+,X^-] = H \) hold in \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \), hence so do the relations \( [H,x^+] = 2x^+ \) and \( [x^+,X^-] = H \). Let \( n \in \mathbb{Z} \) and consider the representation \( M(n)[[\mathfrak{h}]] \) of \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \). The relation \( [H,x^+] = 2x^+ \) implies that \( H.(x^+.b_0) = (n+2)(x^+.b_0) \) holds in \( M(n)[[\mathfrak{h}]] \). Hence, \( x^+.b_0 = 0 \). The relation \( [x^+,X^-] = H \) then implies by induction on \( p \) that \( x^+.b_p = p(n-p+1)b_{p+1} \) for all \( p \geq 1 \). As a consequence, \( x^+ - X^+ \) acts by zero on \( M(n)[[\mathfrak{h}]] \) for every \( n \in \mathbb{Z} \). Since \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \) is isomorphic as \( U_h(\mathfrak{a})\)-algebra to \( U_h(N) \), and as \( M(n)[[\mathfrak{h}]] \) and \( M_h(n,N) \) are equal as representations of \( U_h(\mathfrak{a}) \), it follows by definition of the algebra \( U_h(\mathfrak{a}) \) that \( x^+ - X^+ = 0 \).

**Step 3.** (Conclusion)

The unicity of a \( \mathfrak{b}\)-trivialization for \( U_h(\psi) \) follows from step 2. Let then \( g \) be a \( K[[\mathfrak{h}]]\)-algebra homomorphism from \( U(\mathfrak{s}l_2)[[\mathfrak{h}]] \) to \( U_h(\psi) \) such that \( g(X^-) = X^- \) and \( g(H) = H \) (step 1). The map \( g_{n=0} \) satisfies in particular \( g_{n=0}(X^-) = X^- \) and \( g_{h=0}(H) = H \). Denote by \( \bar{g} \) the \( K[[\mathfrak{h}]]\)-algebra homomorphism induced by \( g_{h=0} \) from \( (U(\mathfrak{s}l_2)[[\mathfrak{h}]]_{h=0})[[\mathfrak{h}]] \) to \( (U_h(\psi)_{h=0})[[\mathfrak{h}]] \). As \( U_h(\mathfrak{a})\)-algebra, \( (U(\mathfrak{s}l_2)[[\mathfrak{h}]]_{h=0})[[\mathfrak{h}]] \)
is isomorphic to $U(\mathfrak{sl}_2)[[\hbar]]$. The same holds for $U_h(\psi)$, since $U_h(\psi)$ is a formal deformation of the $\alpha$-algebra $U(\mathfrak{sl}_2)$ (theorem [4]). Step 2 therefore implies that $\tilde{g}$ is a $U_h(\alpha)$-algebra homomorphism, hence that $g_h=0$ is an $\alpha$-algebra homomorphism.

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