Compensation process and generation of chirped femtosecond solitons and double-kink solitons in Bose–Einstein condensates with time-dependent atomic scattering length in a time-varying complex potential

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Abstract We consider the one-dimensional (1D) cubic-quintic Gross–Pitaevskii (GP) equation, which governs the dynamics of Bose–Einstein condensate matter waves with time-varying scattering length and loss/gain of atoms in a harmonic trapping potential. We derive the integrability conditions and the compensation condition for the 1D GP equation and obtain, with the help of a cubic-quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift, exact analytical solitonlike solutions with the corresponding frequency chirp which describe the dynamics of femtosecond solitons and double-kink solitons propagating on a vanishing background. Our investigation shows that under the compensation condition, the matter wave solitons maintain a constant amplitude, the amplitude of the frequency chirp depends on the scattering length, while the motion of both the matter wave solitons and the corresponding chirp depend on the external trapping potential. More interesting, the frequency chirps are localized and their feature depends on the sign of the self-steepening parameter. Our study also shows that our exact solutions can be used to describe the compression of matter wave solitons when the absolute value of the s-wave scattering length increases with time.

Keywords Bose–Einstein condensate · Gross–Pitaevskii equation · Compensation process · Chirped femtosecond solitons · Double-kink solitons

1 Introduction

First realized experimentally in 1995 for rubidium [1], lithium [2, 3] and sodium [4], Bose–Einstein condensate is a significant, rapidly growing research area at the forefront of soft condensed matter physics and nonlinear physics. The observation of modulated matter waves such as the bright and dark solitons in BECs has opened up a lot of avenues in the fields of nonlinear physics and condensed matter physics [5–7]. Providing unique opportunities for exploring quantum phenomena on a macroscopic scale and allowing us to model the quantum fields that underlie nearly all of modern physics in systems where we have unparalleled experimental control, BECs have virtually become a natural experimental playground for nonlinear waves. In general, the evolution of the macroscopic wave function of the condensates is described a nonlinear Schrödinger (NLS) equation with an external trapping potential, alias the Gross–Pitaevskii equation, the two-body inter-
action, corresponding to the cubic nonlinear term in the GP equation [8].

At very low temperatures, the dynamics of BECs trapped in an external potential \( V \) can usually be described by the time-dependent, nonlinear, mean-field Gross–Pitaevskii equation for the wave function \( \Psi(r, t) \) at position \( r \) and time \( t \) [9–15]

\[
i\hbar \frac{\partial \Psi(r, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r, t) + g_0 |\Psi(r, t)|^2 + \chi_0 |\Psi(r, t)|^4 \right] \Psi(r, t). \tag{1}
\]

Here, \( g_0 = 4\hbar^2 a_s(t)/m \) and \( \chi_0 \) are the strengths of time-dependent two-body and three-body interatomic interactions, respectively, \( \hbar \) is the Planck constant, \( m \) is the atomic mass of the single bosonic atom, and \( a_s(t) \) is the time-dependent \( s \)-wave scattering length which can be tuned to any desired value by using the Feshbach resonance technique. In this paper, we consider a cigar-shaped harmonic trapping potential with the elongated axis in the \( x \)–direction as used in experiment [16], having the form \( V(r, t) = \frac{\omega_0^2}{2} \left[ \omega^2 x^2 + \omega_\perp^2 (y^2 + z^2) \right] + i\gamma \) in which \( \omega << \omega_\perp \) so that the variation of the profile of the wave function (order parameter) can be expected to be in the elongated direction. (Here, \( \gamma \) is a small parameter added to take into account the loss or the gain of atoms in the condensate.) As pointed out by Gammal et al. [14], the strength of the three-body interaction is usually very small in comparison with the strength of the two-body interactions. We will therefore consider for this study that \( \chi_0 = \tilde{\chi}_0 g_0, 0 \leq \tilde{\chi}_0 < 1 \) being a real parameter. Thus, both the two- and three-body interactions can be controlled by the tuning of \( s \)-wave scattering length [14]. Introducing the transformation

\[
\Psi(r, t) = \frac{1}{\sqrt{2\pi a_B a_\perp}} \exp \left[ -i\omega_\perp t - \frac{\gamma^2 + z^2}{2a_\perp^2} \right] \psi \left( \frac{x}{a_\perp}, \omega_\perp t \right)
\]

reduces, in the physically important case of the cigar-shaped BECs, the GP Eq. (1) into the one-dimensional (1D) distributed dissipative cubic-quintic NLS equation with an external harmonic trapping potential

\[
i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + g |\psi|^2 \psi + \tilde{\chi}_0 g |\psi|^4 \psi - (\alpha x^2 + i\gamma) \psi = 0, \tag{2}
\]

where \( a_B \) is the Bohr radius. In Eq. (2), \( t \) and \( x \) are the temporal and spatial coordinates measured in \( 1/\omega_\perp \) and \( a_\perp = \sqrt{\hbar/(m\omega_\perp)} \), respectively. \( g = -2a_s/(3a_B) \) is the nonlinearity parameter which is negative (positive) for repulsive (attractive) interatomic interactions. \( \alpha \) expresses the trapping frequency in the \( x \)–direction [18]. \( \gamma \) is a small parameter related to the feeding \( (\gamma > 0) \) or loss \( (\gamma < 0) \) of atoms in the condensate resulting from the contact with the thermal cloud and three-body recombination [19–21]. In this paper, we assume that the \( s \)-wave scattering length is a function of time \( t \) [22] so that parameter \( g \) of the nonlinearity and strength \( \alpha \) of the magnetic trap are time-dependent; also, parameter \( \gamma \) of the gain/loss of atoms is assumed to be time-varying. Therefore, Eq. (2) can be used for describing the control and management of BECs by properly choosing these three time-dependent parameters \( g, \alpha, \) and \( \gamma \) and the time-independent parameter \( \tilde{\chi}_0 \).

The main purpose of this paper is to study via the GP Eq. (2) the generation of chirped femtosecond solitons and double-kink solitons in BECs with time-dependent atomic scattering length in a time-varying complex potential. Due to their wide applications in many different areas of physics and engineering such as ultrahigh-bit-rate optical communication systems, ultrafast physical processes, infrared time-resolved spectroscopy, as well as optical sampling systems, ultrashort (femtosecond) pulses have been extensively studied [23–26]. Femtosecond sources such as broadband solid-state and fiber lasers become tools for optical metrology, as well as for environment and industrial monitoring including trace gas detection. They offer higher spectral brightness, permitting more rapid measurements in the broader spectral range, covering several absorption lines simultaneously. One of their possible implementations, especially that of laser spectroscopy, is the intracavity absorption spectroscopy allowing direct measurement of important molecular gases with high resolution and good signal-to-noise ratio [27,28]. The limits of applicability and/or implementation of femtosecond sources lie at the issues of operation stability, spectrum and pulse deformation becoming important whenever femtosecond sources are to be used for quantitative spectroscopic measurements or femtosecond-pulse applications [29,30].

It has been established that physical systems whose dynamics can be described by the NLS equation with...
higher-order terms such as third-order dispersion, self-steepening, and self-frequency shift may support the propagation of ultrashort pulses [31,32]. Although the effect of the third-order dispersion is significant for femtosecond pulses when the group velocity group (GVD) is close to zero, Vyas et al. [33] showed that it can be neglected for pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero. However, the effects of self-steepening and self-frequency shift terms are still dominant and cannot be ignored. Alka et al. [34] demonstrated that the competing cubic-quintic non-linearity induces propagating chirped solitonlike dark (bright) solitons and double-kink solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift. Extensive research work has been performed on chirped matter wave solitons and pulses because of their application in pulse compression or amplification, and thus, they are particularly useful in the design of fiber-optic amplifiers, optical pulse compressors, BECs, solitary-wave-based communications links, and observer based control [35–43]. In the context of the observer-based control, the block pulse functions (BPFs) parametrization techniques are generally used for the synthesis of the state observer with optimal control. This tool transforms the optimal control problem to a mathematical programming problem that can then be solved numerically [44]. Combined with the linear matrix inequality technique, such a parametrization technique for the synthesis of the state observer with optimal control has been illustrated through a numerical simulation study on a spacecraft electromagnetic docking system model [40,41].

This paper deals with the dynamics of chirped femtosecond solitons and double-kink solitons of BECs with time-varying atomic scattering length described by the GP Eq. (2). To facilitate this study, we employ a modified phase-imprint method which consists of reducing the governing Eq. (2) into a higher-order nonlinear Schrödinger (HO-NLS) equation with self-steepening and self-frequency shift terms for which we can directly construct the exact solutions. By adopting a nonlinear chirping ansatz which differs from that used by many authors [34,38], we derive families of chirped solitonlike solutions for the HO-NLS equation which are then used to investigate analytically the dynamics of chirped femtosecond solitons and double-kink solitons in BECs under consideration. The rest of this paper is organized as follows: The main transformation leading to the integrability conditions of Eq. (2) and the compensation condition is presented in Sect. 2 and the HO-NLS equation whose chirped solitonlike solutions lead to chirped femtosecond soliton and double-kink soliton solutions of the GP Eq. (2) is derived and integrated. With the use of exact chirped solitonlike solutions of the HO-NLS equation, we investigate in Sect. 3 the dynamics of chirped femtosecond solitons and double-kink solitons in the BECs under consideration. The main results are summarized in Sect. 4.

2 Higher-order nonlinear Schrödinger equation for the GP Eq. (2)

2.1 Integrability and compensation conditions for the GP Eq. (2)

In order to obtain the integrable conditions of Eq. (2) and to reduce the GP Eq. (2) into a HO-NLS equation, we perform a lens-type transformation of the form:

\[
\psi(x, t) = \phi(X, T) \exp \left[ i \left( \gamma x^2 + \theta \right) \right]
\]

in which \( T \) is a real function of time \( t \), \( X = \sqrt{\frac{g}{g_0}} x \) for any real constant \( g_0 \) having the same sign as \( g \), and \( \theta = \theta(X, T) \) is a real function of variables \( X \) and \( T \). We then demand that

\[
\begin{align*}
\frac{dT}{dt} &= \frac{g}{g_0}, \\
\frac{1}{g} \frac{dg}{dt} + 4\gamma &= 0, \\
\frac{d\gamma}{dt} + 2\gamma^2 + \alpha &= 0, \\
\frac{\partial \theta}{\partial X} &= \alpha_0 |\phi|^2, \\
\frac{\partial \theta}{\partial T} &= i \left( \frac{3}{2} \alpha_0 - \beta_0 \right) \left( \phi \frac{\partial \phi^*}{\partial X} + \phi^* \frac{\partial \phi}{\partial X} \right) + \left[ g_0 \beta_0 - \frac{1}{2} \beta_0^2 + \frac{3}{2} \alpha_0 (\beta_0 - \alpha_0) \right] |\phi|^4,
\end{align*}
\]

where \( \alpha_0 \neq 0 \) and \( \beta_0 \) two arbitrary real parameters. The choice of Eq. (4a) is made to preserve the scaling. The real function \( \theta \) that satisfies the conditions (4d) and (4e) is called the phase-imprint on the old order parameter \( \psi(x, t) \) [45]. Ansatz (3) under the conditions (4a)–
(4e) converts the GP Eq. (2) to the following HO-NLS equation with self-steepening and self-frequency shift

\[ i \frac{\partial \phi}{\partial T} + \frac{1}{2} \frac{\partial^2 \phi}{\partial X^2} + g_0 |\phi|^2 \phi + i \alpha_0 \frac{\partial}{\partial X} \left( |\phi|^2 \phi \right) + i (\beta_0 - 2\alpha_0) \frac{\partial |\phi|^2}{\partial X} + \frac{1}{2} (\beta_0 - \alpha_0)(\beta_0 - 2\alpha_0) |\phi|^4 \phi = 0. \] (5)

In the context of nonlinear optics, the HO-NLS Eq. (5) models the propagation of ultrashort (femtosecond) pulses in a single-mode optical fiber [46–49]. In this context, the normalized complex envelope \( |\phi|^2 \) and the HO-NLS equation (5) is rather general, as it can reduce to a series of well-established integrable equations of Schrödinger type such as Chen–Lee–Liu-type NLS (CLL–NLS) equation if \( \beta_0 - \alpha_0 = 0 \) [51], the Kaup–Newell-type NLS (KN–NLS) equation if \( \beta_0 - 2\alpha_0 = 0 \) [52], and the Gerdjikov–Ivanov (GI) equation if \( \beta_0 = 0 \) [53]. It is important to point out that the GP Eq. (2) is reduced into the HO-NLS (5) when the nonlinearity parameter \( g(t) \), the harmonic trapping potential parameter \( \alpha(t) \), and the gain/loss parameter \( \gamma(t) \) satisfy Eqs. (4b) and (4c). Throughout this work, Eqs. (4b) and (4c) will be referred to as the integrable conditions of the GP Eq. (2). Thus, the virtue of the lens-type transformation (3) is that, without much complicated calculation, we not only find the integrable conditions (4b) and (4c) for the GP Eq. (2), but we also retrieve a higher-order nonlinear Schrödinger equation with self-steepening and self-frequency shift terms whose solitons solutions and double-kink solitons solutions with nonlinear chirp can be obtained under certain parametric conditions.

Integrating Eq. (4b) yields \( g(t) = \lambda_0 \exp \left[ -4 \int_0^t \gamma(\tau) d\tau \right] \). This means that the absolute value of the nonlinearity parameter \( g(t) \) will be an increasing function of time \( t \) if and only, if \( \gamma(t) \) is negative. Therefore, BECs with loss of atoms are associated with increasing (in absolute value) nonlinearity parameter \( g(t) \), while BECs with feeding of atoms correspond to decreasing (in absolute value) nonlinearity parameter \( g(t) \). As well as we know, the density \( |\psi(x, t)|^2 \) of a BEC with the increasing the absolute value of the s-wave scattering length (with loss of atoms) has an increase (decrease) in the peak value; also, the density \( |\psi(x, t)|^2 \) of a BEC with the decreasing of the absolute value of the s-wave scattering length (with feeding of atoms) has a decrease (increase) in the peak value [17,19]. We thus conclude that Eq. (4b) can lead to a compensation process so that for BECs that satisfy condition (4b), the density \( |\psi(x, t)|^2 \) will have a constant in the peak value for all time \( t \). The compensation process consists of a balance of loss or gain effects with the effects of the s-wave scattering length on the condensate. Such a compensation process ensures the stability of the condensates over a longer interval of time. Equation (4b) can thus be referred to as the “compensation condition”.

It is important to notice that parameter \( \tilde{\chi}_0 \) will not have any effect of the density \( |\psi(x, t)|^2 \). As we can see from Eq. (4e), our above ansatz is made so that the factor \( \tilde{\chi}_0 \) of the three-body interatomic interaction will affect the wave phase through the phase-imprint \( \theta \).

2.2 Chirped femtosecond and double-kink solitonlike solutions of the HO-NLS Eq. (5)

Because Eq. (5) in the context of nonlinear optics models the propagation of ultrashort (femtosecond) pulses, its solitonlike solutions will be referred to as the femtosecond or double-kink solitonlike solutions. In this subsection, we focus ourselves to the traveling wave solutions of the HO-NLS Eq. (5), that is, the solutions of the form

\[ \phi(X, T) = \rho(\xi) \exp \left[ i \left( \chi(\xi) - \Omega T \right) \right], \] (6)
where $\rho$ and $\chi$ are reals functions of the traveling coordinate $\xi = X - \nu T$; here, $\nu$ is a real parameter given in terms of the group velocity of the wave packet. The corresponding frequency chirp is given by $\delta \omega (X, T) = - \frac{d}{dX} (\chi (\xi) - \Omega T) = - \frac{d}{dX} \chi (\xi) = - \chi' (\xi)$. Inserting Eq. (6) into ansatz (3) yields the following solution of the GP Eq. (2)

$$
\psi (x, t) = \rho (\xi) \exp \left[ i \left( \gamma x^2 + \theta (X, T) + \chi (\xi) - \Omega T \right) \right].
$$

(7)

with the corresponding chirp $\delta \omega (x, t) = - \frac{d}{dX} [\gamma x^2 + \theta (X, T) + \chi (\xi) - \Omega T]$. Using Eq. (4d), the frequency chirp associated with solution (7) is found to be

$$
\delta \omega (x, t) = - \left[ 2 \gamma x + \sqrt{\frac{8}{g_0}} (\alpha_0 \rho^2 (\xi) + \chi' (\xi)) \right]_{\xi = \sqrt{\frac{8}{g_0}} - \frac{\beta_0}{\alpha_0} g(t) \nu t}.
$$

(8)

Substituting Eq. (6) in Eq. (5) and separating the real and imaginary parts yield

$$
\Omega \rho + \nu \frac{d \rho}{d \xi} - \frac{1}{2} \left( \frac{d \chi}{d \xi} \right)^2 \rho + \frac{1}{2} \frac{d^2 \rho}{d \xi^2} - \alpha_0 \frac{d \chi}{d \xi} \rho^3 + g_0 \rho^3 + \frac{1}{2} (\beta_0 - \alpha_0) (\beta_0 - 2 \alpha_0) \rho^5 = 0,
$$

(9a)

and

$$
- \nu \frac{d \rho}{d \xi} + \frac{1}{2} \frac{d^2 \rho}{d \xi^2} \rho + \frac{d \chi}{d \xi} \frac{d \rho}{d \xi} + (2 \beta_0 - \alpha_0) \rho^2 \frac{d \rho}{d \xi} = 0.
$$

(9b)

One easily verifies that

$$
\frac{d \chi}{d \xi} = \frac{\alpha_0 - 2 \beta_0}{2} \rho^2 + \nu.
$$

(10)

satisfies Eq. (9b). Inserting Eq. (10) into Eqs. (8) and (9a) yields

$$
\delta \omega (x, t) = - \left[ 2 \gamma x + \sqrt{\frac{8}{g_0}} \left( \nu + \frac{3 \alpha_0 - 2 \beta_0}{2} \rho^2 (\xi) \right) \right]_{\xi = \sqrt{\frac{8}{g_0}} - \frac{\beta_0}{\alpha_0} g(t) \nu t},
$$

(11)

and

$$
\frac{d^2 \rho}{d \xi^2} + b_5 \rho^5 + b_3 \rho^3 + b_1 \rho = 0,
$$

(12a)

$$
b_5 = \frac{3}{2} \alpha_0 (\alpha_0 - \beta_0), \ b_3 = 2 (g_0 - \alpha_0 \nu), \ b_1 = 2 \Omega + \nu^2.
$$

(12b)

respectively. It is seen from Eq. (11) that the frequency chirp $\delta \omega (x, t)$ depends on different coefficients of the HO-NLS Eq. (5) such as Kerr nonlinearity, self-steepening, and self-frequency shift, as well as on the nonlinearity parameter $g(t)$ and the gain/loss parameter $\gamma(t)$. This means that for given either the nonlinearity parameter $g(t)$ or the gain/loss parameter $\gamma(t)$, the amplitude of chirping can be controlled by varying the three free parameters $g_0, \alpha_0$, and $\beta_0$. It is also seen from Eq. (11) that the frequency chirp evolves in space and time.

Rewriting Eq. (12a) in the form

$$
\left( \frac{d r}{d \xi} \right)^2 = \alpha_1 r^4 + 4 \beta_1 r^3 + 6 \gamma_1 r^2 + 4 \delta_1 r, \ r = \rho^2,
$$

(13a)

$$
\alpha_1 = - \frac{8}{3} b_5, \ \beta_1 = - \frac{b_3}{2}, \ \gamma_1 = - \frac{2}{3} b_1,
$$

(13b)

all its traveling-wave solutions can be expressed in a generic form by means of the Weierstrass function $\wp$ [54–56]. (Here, $\delta_1$ is a constant of integration.) In this paper, we limit ourselves to localized solutions of Eq. (12a). In the special case of the integrable CLL–NLS ($\beta_0 - \alpha_0 = 0$), Eq. (12a) reduces to a cubic nonlinear equation that admits bright and dark soliton solutions. In the special case when $\nu = g_0/\alpha_0$, Eq. (12a) can be solved for localized solutions by using a Eq. (13a). For $2 \Omega + \nu^2 = 0$, Eq. (12a) will admit a Lorentzian-type solution. When $\alpha_0 (\alpha_0 - \beta_0) (g_0 - \alpha_0 \nu) (2 \Omega + \nu^2) \neq 0$, Eq. (13a) is useful to find double-kink-type and bright- and dark-soliton solutions [56]. To apply Eq. (13a) for finding localized solutions of Eq. (12a), we first introduce the invariants $g_2$ and $g_3$ of Weierstrass’ elliptic function $\wp$ which are related to the coefficients of the polynomial $R(r) = \alpha_1 r^4 + 4 \beta_1 r^3 + 6 \gamma_1 r^2 + 4 \delta_1 r$ according to

$$
g_2 = -4 \beta_1 \delta_1 + 3 \gamma_1^2, \ g_3 = 2 \beta_1 \gamma_1 \delta_1 - \alpha_1 \delta_1^2 - \gamma_1^3.
$$

If the discriminant $\Delta = g_2^3 - 27 g_3^2$ of $\wp$ and $R$ is zero, $g_2 \geq 0$ and $g_3 \leq 0$, then $r(\xi)$ is solitary wavelike and given [55, 56]

$$
r(\xi) = r_0 + \frac{R'(r_0)}{4} \frac{\sinh^2 \left[ \sqrt{3} e_1 \xi \right]}{3 e_1 + \left( e_1 - \frac{R'(r_0)}{24} \right) \sinh^2 \left[ \sqrt{3} e_1 \xi \right]},
$$

(14)
where \( e_1 = \sqrt[3]{-g_3} \) and \( r_0 \) is any simple zero of polynomial \( R(r) \). In what follows, we discuss, dependent on the behavior of the parameters \( b_5, b_3, \) and \( b_1 \), the solitonlike solutions of Eq. (12a). We will mainly build such solutions with the help of solitonlike solutions of Eq. (13a). From the relationship \( r(\xi) = \rho^2(\xi) \) yields. \( \rho(\xi) = \pm \sqrt{r(\xi)} \). Since for the GP Eq. (2) we are interesting in the density \( \rho(\xi) \) we are building such solutions with the help of solitonlike solutions of Eq. (13a).

2.2.1 Case \( b_5 = 0 \) and \( b_3 b_1 \neq 0 \)

In the situation when \( b_5 = 0 \) and \( b_3 b_1 \neq 0 \), we consider the two interesting situations \( \delta_1 = 0 \) and \( \delta_1 = \frac{9\gamma_1^2}{16b_1} \), leading to \( \Delta = 0 \). When \( b_5 = 0 \), \( R(r) \) admits one simple zero \( r_0 = -\frac{3\gamma_1}{4b_1} = \frac{2\Omega + \nu^2}{\alpha_0 \nu - g_0} \), if \( \delta_1 = 0 \). In this special case, \( \Delta = 0 \). Inserting \( r_0 \) into Eq. (14) yields

\[
r(\xi) = -\frac{3}{2} \frac{\Omega + \nu^2}{g_0 - \alpha_0 \nu}
\]

\[
\rho(\xi) = \frac{1}{\cosh\left(\sqrt{(2\Omega + \nu^2)\xi}\right)} - 2\Omega + \nu^2 < 0, \quad g_0 - \alpha_0 \nu > 0.
\]

Inserting \( \rho(\xi) \) into Eqs. (7) and (11) yields, under the conditions that \( 2\Omega + \nu^2 < 0 \) and \( g_0 - \alpha_0 \nu > 0 \), the following solution of the GP Eq. (2) with the corresponding frequency chirp

\[
\psi(x, t) = \sqrt{-\frac{2\Omega + \nu^2}{g_0 - \alpha_0 \nu}} \cosh\left[\sqrt{-\left(2\Omega + \nu^2\right)}\left(\frac{\text{g}(t)}{g_0} x - \frac{\nu}{g_0} \int_0^t g(\tau) d\tau\right)\right] \exp\left[i \left(\nu x^2 + \theta(X, T) + \chi(\xi) - \Omega T\right)\right],
\]

\[
\delta \omega(x, t) = -\left[2\nu(t)x + \sqrt{\frac{\text{g}(t)}{g_0}}\left(\nu - \frac{3\alpha_0 - 2\beta_0}{2g_0 (g_0 - \alpha_0 \nu)} \cosh^2\left[\sqrt{-\left(2\Omega + \nu^2\right)}\left(\frac{\text{g}(t)}{g_0} x - \frac{\nu}{g_0} \int_0^t g(\tau) d\tau\right)\right]\right)\right].
\]

respectively.

If \( \delta_1 = \frac{9\gamma_1^2}{16b_1} \), then \( \Delta = 0 \) and polynomial \( R(r) \) admits one double root \( r_0 = -\frac{3\gamma_1}{4b_1} \) and one simple zero \( r_0 = 0 \). Seeking a localized solution of Eq. (13a) associated with the double zero \( r_0 = -\frac{3\gamma_1}{4b_1} \) in the form \( r = A \tan^2 [B \xi] \) yields

\[
\rho(\xi) = \sqrt{\frac{2\Omega + \nu^2}{2 (\alpha_0 \nu - g_0)}} \cosh\left(\sqrt{\frac{2\Omega + \nu^2}{2}}\xi\right).
\]

\[
2\Omega + \nu^2 > 0, \quad g_0 - \alpha_0 \nu < 0.
\]

(16a)

For the solution of Eq. (13a) corresponding to the simple zero \( r_0 = 0 \), we use Eq. (14) and obtain the following kink solitonlike solution of Eq. (12a)

\[
\rho(\xi) = \frac{1}{2} \sqrt{\frac{3 (2\Omega + \nu^2)}{\alpha_0 \nu - g_0}} \sinh\left(\sqrt{\frac{2\Omega + \nu^2}{3}} \xi\right),
\]

\[
2\Omega + \nu^2 > 0, \quad g_0 - \alpha_0 \nu < 0.
\]

(16b)

Using now Eqs. (7) and (11), we obtain from Eqs. (16a) and (16b) the following exact solutions of the GP Eq. (2) with the corresponding chirp

\[
\psi(x, t) = \sqrt{\frac{2\Omega + \nu^2}{2 (\alpha_0 \nu - g_0)}} \tanh\left[\sqrt{\frac{2\Omega + \nu^2}{2}}\xi\right] \left[\exp\left[i \left(\nu x^2 + \theta(X, T) + \chi(\xi) - \Omega T\right)\right]\right],
\]

\[
\delta \omega(x, t) = -\left[2\nu(t)x + \sqrt{\frac{\text{g}(t)}{g_0}}\left(\nu - \frac{3\alpha_0 - 2\beta_0}{4 (\alpha_0 \nu - g_0)} \cosh^2\left[\sqrt{\frac{2\Omega + \nu^2}{3}}\xi\right]\right)\right].
\]

(17b)
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2.2.2 Case b

Following exact solitonlike solution of the GP Eq. (2) with $\rho(\xi)$

\[\begin{align}
\delta\omega(x, t) &= \left[2 \gamma(t)x + \frac{g(t)}{g_0} \left(\nu + 3 \beta \frac{\Omega}{\gamma^2} \right) \left(\frac{2 \Omega + \nu^2}{3 \alpha_0 (\alpha_0 - \beta_0)} \right) \right] - \frac{1}{3} \left(2 \Omega + \nu^2 \right) \left(\frac{2 \Omega + 2 \nu^2}{3 \alpha_0 (\alpha_0 - \beta_0)} \right) \\
&\times \exp \left[i \left(\gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T\right)\right].
\end{align}\]

respectively, where $\alpha_0 \neq 0$, $g_0 \neq 0$, $\nu$, and $\Omega$ are four arbitrary real parameters satisfying the conditions $2\Omega + \nu^2 > 0$ and $g_0 - \alpha_0 \nu < 0$.

2.2.2 Case $b_3 = 0$, $b_5 b_1 \neq 0$

When $b_3 = 0$ and $b_5 b_1 \neq 0$, we have $\Delta = -27 \alpha_1 (\alpha_1^2 + 2 \gamma^2) \delta_1^2 = 0$ if and only, if either $\delta_1 = 0$ or $\delta_1 = \pm \gamma \sqrt{-2 \frac{\nu^2}{\nu^2} = \pm \frac{\gamma}{2} (2 \Omega + \nu^2) - \frac{2 \Omega + \nu^2}{3 \alpha_0 (\alpha_0 - \beta_0)} \right]$. Here, we focus ourselves to the special case when $\delta_1 = \pm \frac{\gamma}{2} (2 \Omega + \nu^2) - \frac{2 \Omega + \nu^2}{3 \alpha_0 (\alpha_0 - \beta_0)}$, under the condition that $\alpha_0 (\alpha_0 - \beta_0) (2 \Omega + \nu^2) < 0$. In this case, $r_0 = 0$ is a simple zero of $R(r)$. Inserting $r_0 = 0$ into Eq. (14) and using the relationship $\rho^2 = r(\xi)$ lead to the kink solution

\[\begin{align}
\rho^2(\xi) &= \frac{2}{3} \left(2 \Omega + \nu^2 \right) \sinh^2 \left[\frac{2}{3} \left(2 \Omega + \nu^2 \right) \xi \right] \\
&> 0 \text{ and } \alpha_0 (\alpha_0 - \beta_0) < 0.
\end{align}\]

Inserting $\rho(\xi)$ into Eqs. (7) and (11) leads to the following exact solitonlike solution of the GP Eq. (2) with the corresponding frequency chirp

\[\begin{align}
\psi(x, t) &= \sqrt{\frac{2}{3} \left(2 \Omega + \nu^2 \right) \sinh^2 \left[\frac{2}{3} \left(2 \Omega + \nu^2 \right) \xi \right]} \\
&\times \exp \left[i \left(\gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T\right)\right].
\end{align}\]

where $\alpha_0 \neq 0$, $\beta_0$, $\Omega$, and $\nu$ are four arbitrary real parameters satisfying the conditions $2\Omega + \nu^2 > 0$ and $\alpha_0 (\alpha_0 - \beta_0) < 0$.

2.2.3 Case $b_1 = 0$ and $b_5 b_3 \neq 0$

We now focus on the situation when $b_1 = 0$ and $b_5 b_3 \neq 0$. In this special case, we have $\Delta = - \left(27 \alpha_1^2 \delta_1 + 64 \beta^2 \right) \delta_1^2 = 0$ if and only, if either $\delta_1 = 0$ or $\delta_1 = - \frac{64 \beta_1}{3 \alpha_1}$, the interesting possibility being $\delta_1 = - \frac{64 \beta_1}{3 \alpha_1}$ (since $\delta_1 = 0$ leads to the constant solution $r = \frac{g_0 - \alpha_0 \nu}{\alpha_0 (\alpha_0 - \beta_0)}$, if $\alpha_0 (\beta_0 - \alpha_0) (g_0 - \alpha_0 \nu) > 0$). In this special case, $r_0 = 0$ is a simple zero of polynomial $R(r)$. Inserting $r_0 = 0$ into Eq. (14) yields

\[\begin{align}
r(\xi) &= \frac{\alpha_0 \nu - g_0}{3 \alpha_0 (\alpha_0 - \beta_0)} \\
&\times \sinh^2 \left[\frac{2}{3} \left(2 \Omega + \nu^2 \right) \xi \right] \\
&\times \exp \left[i \left(\gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T\right)\right].
\end{align}\]

To obtain the corresponding exact solution of the GP Eq. (2) with the corresponding chirp, we recall that $r(\xi) = \rho(\xi)$ and use Eqs. (7) and (11). We then obtain

\[\begin{align}
\psi(x, t) &= \sqrt{\frac{2}{3} \left(2 \Omega + \nu^2 \right) \sinh^2 \left[\frac{2}{3} \left(2 \Omega + \nu^2 \right) \xi \right]}
\end{align}\]
Now, we consider the general case of Eq. (12a) when \( b \neq 0 \). Comparing Eqs. (12b) and (13b), condition \( b \neq 0 \) leads to \( a_1 b_1 \gamma_1 
eq 0 \). In this general case,

\[
\Delta = -\left[ 27a_1^2 \delta_1^2 + 4 \beta_1 \left( 16 \beta_1^2 - 27a_1 \gamma_1 \right) \delta_1 + 18 \gamma_1^2 \left( 3a_1 \gamma_1 - 2 \beta_1^2 \right) \right] \delta_1^2.
\]

To the solution \( \delta_1 \neq 0 \) of equation \( \Delta = 0 \), we associate the simple zero \( r_0 = 0 \) of polynomial \( R(t) \). Let \( \delta_{10} \) be any positive root of equation \( \Delta = 0 \) that satisfies the conditions \( \gamma_1^3 - 2 \beta_1 \gamma_1 \delta_{10} + a_1 \delta_{10}^2 > 0 \) and \( \gamma_1^3 - 2 \beta_1 \gamma_1 \delta_{10} + a_1 \delta_{10}^2 - 1/4 > 0 \). Inserting \( r_0 = 0 \) into Eq. (14) yields

\[
\rho(\xi) = \sqrt{\frac{\delta_{10}}{e_1 - \frac{1}{2} \gamma_1}} \frac{\sinh \left[ \sqrt{3e_1 \xi} \right]}{\sqrt{\frac{3e_1}{e_1 - \frac{1}{2} \gamma_1}} + \sinh \left[ \sqrt{3e_1 \xi} \right]},
\]

where \( a_0 \neq 0, \beta_0, g_0 \neq 0 \), and \( \nu \) are real four parameters satisfying the conditions \( a_0 (a_0 - \beta_0) < 0 \) and \( g_0 - a_0 \nu > 0 \).

2.2.4 Case \( b_5 b_3 b_1 \neq 0 \)

Using the relationship \( r(\xi) = \rho^2(\xi) \) leads to the following double-kink-type soliton solution of Eq. (12a)

\[
\rho(\xi) = \sqrt{\frac{\delta_{10}}{e_1 - \frac{1}{2} \gamma_1}} \frac{\sinh \left[ \sqrt{3e_1 \xi} \right]}{\sqrt{\frac{3e_1}{e_1 - \frac{1}{2} \gamma_1}} + \sinh \left[ \sqrt{3e_1 \xi} \right]},
\]

It is important to point out that the interesting double-kink feature of the solution given by Eq. (21) exists only for sufficiently large values of \( \frac{3e_1}{e_1 - \frac{1}{2} \gamma_1} \). The amplitude profile of the soliton solution (21) for different \( \alpha_1, \beta_1, \gamma_1, \) and \( \delta_{10} \) is shown in Fig. 1. For Fig. 1a, \( \frac{3e_1}{e_1 - \frac{1}{2} \gamma_1} = \frac{6}{5} \), while for Fig. 1b, \( \frac{3e_1}{e_1 - \frac{1}{2} \gamma_1} = 1003 \). The data used in Fig. 1b mean that \( \delta_1 = 1, \nu = \frac{g_0 + 0.56137}{a_0}, \Omega = -\frac{3 + 2\nu^2}{4}, \) and \( a_0 (a_0 - \beta_0) + 0.62218 = 0 \).

Going back to Eqs. (7) and (11) and using Eq. (21), we obtain the following double-kink-type soliton solution of the GP Eq. (2) with the corresponding chirp...
Fig. 1 Amplitude profile of the soliton solution in Eq. (21) for different values of the solution parameters. a: $\alpha_1 = -1, \beta_1 = 1, \gamma_1 = \frac{2\beta_1^2}{3\alpha_1}$, and $\delta_{10} = \frac{8}{27} \frac{\beta_1^2}{\alpha_1}$, b: $\gamma_1 = \delta_{10} = 1, \alpha_1 = 0.248872$, and $\beta_1 = 0.56137$.

\[
\psi(x, t) = \sqrt{\frac{3\delta_{10}}{3\epsilon_1 + 2\Omega + \upsilon^2} \sinh \left[ \sqrt{3\epsilon_1} \left( \sqrt{\frac{g(t)}{g_0}} x - \frac{\upsilon}{g_0} \int_0^t g(\tau) d\tau \right) \right]}
\times \exp \left[ i \left( \gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T \right) \right],
\]

\[
\delta\omega(x, t) = - \left[ 2\gamma(t)x + \frac{g(t)}{g_0} \left( \nu + \frac{3\delta_{10} (3\alpha_0 - 2\beta_0)}{2(3\epsilon_1 + 2\Omega + \upsilon^2)} \right) \right] \times \sinh^2 \left[ \sqrt{3\epsilon_1} \left( \sqrt{\frac{g(t)}{g_0}} x - \frac{\upsilon}{g_0} \int_0^t g(\tau) d\tau \right) \right] + \sinh^2 \left[ \sqrt{3\epsilon_1} \left( \sqrt{\frac{g(t)}{g_0}} x - \frac{\upsilon}{g_0} \int_0^t g(\tau) d\tau \right) \right].
\] (22a)

(22b)

here, $\epsilon_1 = -i \sqrt{(2\Omega + \upsilon^2)^3 + \frac{3}{2}(g_0 - \alpha_0 \upsilon)(2\Omega + \upsilon^2)} \delta_{10} + \frac{2}{3\alpha_0} (g_0 - \delta_{10}) \delta_{10}^2$, $\delta_{10}$ is any root of equation $27\epsilon_1^2 \delta_{10}^2 + 4\beta_1 (16\beta_1^2 - 27\alpha_1 \gamma_1) \delta_{10} + 18\gamma_1^2 (3\alpha_1 \gamma_1 - 2\beta_1^2) = 0$, and $\epsilon_1 \neq 0, \gamma_0 \neq 0, g_0 \neq 0$, $\beta_0$, $\upsilon$, and $\Omega$ are five real parameters to be taken from conditions $\gamma_1^3 - 2\beta_1 \gamma_1 \delta_{10} + \alpha_1 \delta_{10}^2 > 0$ and $\sqrt{\gamma_1^3} - 2\beta_1 \gamma_1 \delta_{10} + \alpha_1 \delta_{10}^2 - \frac{1}{2} \gamma_1 > 0$, $\alpha_1$, $\beta_1$, and $\gamma_1$ being given by Eq. (13b).

When $\delta_{10} = 0$, polynomial $R$ then takes the form $R(r) = \alpha_1 r^4 + 4\beta_1 r^3 + 6\gamma_1 r^2$ and admits two double zeros, $r_0 = 0$ and, under the condition $\gamma_1 = \frac{2\beta_1^2}{3\alpha_1}$, $r_0 = -\frac{2\beta_1}{\alpha_1}$; for $\gamma_1 \neq \frac{2\beta_1^2}{3\alpha_1}$, $R(r)$ admits two simple zeros, $r_0 = -2\beta_1 \pm \sqrt{2(2\beta_1^2 - 3\alpha_1 \gamma_1)}$. Interesting solitonlike solutions of Eq. (13a) with $\delta_{10} = 0$ can be sought using the direct method. Here, we focus ourselves on the kink and bright solitonlike solutions of Eq. (13a). As an example of kink solitonlike solution of Eq. (13a), we obtain, under the conditions $\alpha_0 (\beta_0 - \alpha_0) > 0, g_0 - \alpha_0 \upsilon > 0, (g_0 - \alpha_0 \upsilon)^2 - 4\alpha_0 (2\Omega + \upsilon^2) (\alpha_0 - \beta_0) = 0$,

\[
r(\xi) = \frac{g_0 - \alpha_0 \upsilon}{4\alpha_0 (\beta_0 - \alpha_0)} \left( 1 \pm \tanh \left[ \frac{g_0 - \alpha_0 \upsilon}{2\sqrt{\alpha_0 (\beta_0 - \alpha_0)} \xi} \right] \right).
\]

Going back in Eqs. (7) and (11) and knowing that $r(\xi) = \rho^2(\xi)$ yield the following kink solitonlike solution of the GP Eq. (2) and the corresponding frequency chirp.
The dynamics of a double-kink soliton in a time-independent harmonic trapping potential given by equation (22a) and the corresponding frequency chirp given by Eq. (22b).

\[ \psi(x, t) = \frac{1}{2} \left[ \frac{g_0 - \alpha_0 \nu}{\alpha_0 (\beta_0 - \alpha_0)} \right] ^{1 \pm \tanh \left[ \frac{g_0 - \alpha_0 \nu}{2 \sqrt{\alpha_0 (\beta_0 - \alpha_0)}} \left( \sqrt{\frac{g(t)}{g_0}} x - \frac{\nu}{g_0} \int_0^t g(\tau) d\tau \right) \right]} \times \exp \left[ i \left( \gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T \right) \right], \]  

\[ \delta \omega(x, t) = - \left[ 2 \gamma x + \sqrt{\frac{g}{g_0}} \left( \nu + \frac{3 \alpha_0 - 2 \beta_0}{2} \right) \frac{g_0 - \alpha_0 \nu}{4 \alpha_0 (\beta_0 - \alpha_0)} \right] \times \left[ 1 \pm \tanh \left( \frac{g_0 - \alpha_0 \nu}{2 \sqrt{\alpha_0 (\beta_0 - \alpha_0)}} \left( \sqrt{\frac{g(t)}{g_0}} x - \frac{\nu}{g_0} \int_0^t g(\tau) d\tau \right) \right) \right], \]  

(23a)  

(23b)

Here, \( \alpha_0 \neq 0, g_0 \neq 0, \beta_0, \nu, \) and \( \Omega \) are five real parameters satisfying the conditions \( \alpha_0 (\beta_0 - \alpha_0) > 0, g_0 - \alpha_0 \nu > 0, 4 \alpha_0 (\alpha_0 - \beta_0) (2\Omega + \nu^2) - (g_0 - \alpha_0 \nu)^2 = 0. \)

If \( \gamma_1 > 0, \beta_1 < 0, 2\beta_1^2 - 3 \gamma_1 \alpha_1 > 0, \) then Eq. (13a) admits a bright solitonlike solution.
Compensation process and generation of chirped

\[ r(\xi) = \frac{3\gamma_1}{\beta_1} \frac{1}{1 + \sqrt{\frac{6b_2^2 - 9\gamma_1\omega_1}{6b_1^2}}} \cosh[\sqrt{6\gamma_1\xi}] \]

that leads to the below bright solitonlike solution of the GP Eq. (2) with the associated frequency chirp

\[ \psi(x, t) = 2 \sqrt{\frac{2\Omega + \nu^2}{2(\alpha_0 \nu - g_0 - \tilde{\lambda}_0 \cosh[2\sqrt{-(2\Omega + \nu^2)}(\sqrt{\frac{g(t)}{g_0}} x - \frac{\nu}{g_0} \int_0^t \tau g(\tau) d\tau)]}} \times \exp\left[i(\gamma x^2 + \theta(X, T) + \chi(\xi) - \Omega T)\right], \]

\[ \delta\omega(x, t) = -2\gamma(t)x + \frac{g(t)}{g_0}(\nu + (2\beta_0 - 3\alpha_0) \times \frac{(2\Omega + \nu^2)}{g_0 - \alpha_0 \nu + \tilde{\lambda}_0 \cosh[2\sqrt{-(2\Omega + \nu^2)}(\sqrt{\frac{g(t)}{g_0}} x - \frac{\nu}{g_0} \int_0^t \tau g(\tau) d\tau)]}] \]

where \( \tilde{\lambda}_0 = \sqrt{(g_0 - \alpha_0 \nu)^2 - 4\alpha_0 (\alpha_0 - \beta_0) (2\Omega + \nu^2)} \), and \( \alpha_0 \neq 0, g_0 \neq 0, \beta_0, \nu, \) and \( \Omega \) are five real parameters satisfying the conditions \( 2\Omega + \nu^2 < 0, (g_0 - \alpha_0 \nu)^2 - 4\alpha_0 (\alpha_0 - \beta_0) (2\Omega + \nu^2) > 0. \)

3 Dynamics of chirped femtosecond solitons and double-kink solitons in BECs with time-dependent atomic scattering length in a complex potential

3.1 Results

With the help of the above exact solitonlike solutions of Eq. (2), we now turn to the analytical investigation of the dynamics of chirped femtosecond solitons and double-kink solitons in BECs described by the GP Eq. (2). It follows from different expressions for the wave function \( \psi(x, t) \) and the corresponding frequency chirp \( \delta\omega(x, t) \) found in the previous section that the center of different solitons corresponding to \( \psi(x, t) \) and \( \delta\omega(x, t) \) is \( \xi = \frac{\nu}{g_0} \sqrt{\frac{g_0}{g(t)}} \int_0^t \tau g(\tau) d\tau, \) which satisfies the equation \( \frac{d^2 \xi}{dt^2} + 2\alpha \xi = 0. \) This means that the center of mass of the macroscopic wave packet behaves like a classical particle, and allows us to manipulate the motion of chirped femtosecond solitons and double-kink solitons in BEC systems by controlling the external harmonic trapping potential. It follows from the above equation of the center of matter waves and the integrability conditions that (i) the found chirped femtosecond solitons and double-kink solitons move with the speed \( \frac{d\xi}{dt} = \frac{\nu}{g_0} (2\gamma \int_0^t \tau g(\tau) d\tau + g) \sqrt{\frac{g_0}{g(t)}} \) which depends on both the s-scattering length and the feeding/loss parameter, while the width is proportional to \( \sqrt{\frac{g_0}{g(t)}}. \) Therefore, the soliton width decreases (increases) during the wave propagation when \( g(t) \) increases (decreases) as time \( t \) increases; also, the soliton width increases when the nonlinearity parameter \( g_0 \) increases.

Because the width of obtained femtosecond solitons is proportional to \( \sqrt{\frac{g_0}{g(t)}} \), we also that solitonlike solutions found in the previous section can be used to describe the compression of chirped femtosecond solitons when the absolute value of the s-wave scattering length \( a_s(t) \) increases with time. An obvious analysis of different exact solitonlike solutions derived in the previous section shows that under the integrability conditions (4b) and (4c), the amplitude of the femtosecond pulses is independent of both the nonlinearity parameter \( g(t) \) (that is, on the s-wave scattering length \( a_s(t) \)) and the gain/loss parameter \( \gamma(t) \). Thus, contrary to BECs with either time-varying s-wave scattering length or/and with gain/loss of atoms in which the amplitude of the matter waves generally vary with time during the wave propagation, our results show that under the compensation condition (4b), the pulse amplitude in a BEC with time-varying s-wave scattering length and with gain/loss of atoms remains constant during the wave propagation. It is also important to note that the total number of BEC atoms remains unchanged.
Analyzing each of the chirping corresponding to the found solitonlike solutions in the previous section, it is clearly seen that the frequency chirp $\delta \omega(x,t)$ depends both on the loss/gain parameter $\gamma(t)$ and the s-wave scattering length $a_s(t)$ through the nonlinearity parameter $g(t)$. Indeed, as we can see from the expression of the chirping, $\delta \omega(x,t)$ is composed of two terms, one linear term $2\gamma(t)x = -\frac{1}{2} g(t) \frac{dt}{dx} x^2$ (here, we have used the compensation condition (4b)) which is proportional to the gain/loss parameter $\gamma(t)$, and one nonlinear term which is proportional to $\sqrt{\frac{g(t)}{\rho_0}}$. This means that the chirp amplitude varies both with time $t$ and with the spatial variable $x$. It is important to note that the linear part of the chirp will not contribute to the absolute depth/amplitude if it is reached at $x = 0$. It is important to note that the linear part $2\gamma(t)x + \nu \sqrt{\frac{g(t)}{\rho_0}}$ of the chirp acts as a linear wave background, so that the frequency chirp of this work will be referred to as “frequency chirps embedded on a linear wave background”. 

In what follows, we take some examples to demonstrate the dynamics of femtosecond solitons and double-kink solitons in one-dimensional BEC systems with different kinds of scattering length, harmonic trapping potential, and feeding or loss parameter. For the demonstration of the dynamics of femtosecond solitons and double-kink solitons in the BEC systems, we will limit ourselves to the use of the double-kink soliton solution (22a) and the bright femtosecond soliton solution (24a) and their corresponding chirps (22b) and (24b), respectively. For the double-kink soliton solution (22a) with the associating chirp (22b), we will use the parameters $\delta_1 = 1$, $\nu = \frac{80\pi + 0.562137}{a_0}, \Omega = -\frac{3 + 2\nu^2}{4}$, and $\beta_0 = \frac{2 + 6.2218}{a_0}$, while the dynamics of bright femtosecond solitons and the corresponding chirp associated with solution (24a) with chirp (24b) will be investigated with the use of either the parameters $a_0^2 = 1$, $\beta_0 = \frac{1}{6a_0}, \frac{1}{a_0} > 1$, $(g_0 - 1)^2 + 5 > 0$, and $\Omega = -\frac{3 + 2
u^2}{4}$, or the parameters $a_0^2 = 1$, $g_0 > -1$, $\beta_0 = \frac{1}{a_0}, \nu = -\frac{1}{a_0}$, and $\Omega = -\frac{3 + 2
u^2}{4a_0^2}$. As we will see in the below examples, frequency chirp corresponding to femtosecond solitons and double-kink solitons will be localized; moreover, chirp associated with double-kink soliton will have a double-kink feature dark or bright dependent on the sign of the self-steepening coefficient $a_0$, while that corresponding to bright femtosecond soliton will have, dependent on the sign of the self-steepening coefficient $a_0$, either a bright or dark soliton feature.

3.2 BECs with time-independent harmonic potential

As the first example, we consider the time-independent harmonic potential which was used in the creation of bright BEC solitons [57]. For that experience, the strength of the harmonic potential was $\alpha = -2\lambda^2$ with $\lambda \approx 0.05$. From the integrability conditions (4b) and (4c), we obtain $\gamma(t) = \pm \lambda$ and $g(t) = \lambda_0 \exp[-4\lambda t]$, where $\lambda_0 \neq 0$ is an arbitrary real parameter having the same sign as $g_0$, that is, $\lambda_0 g_0 > 0$. The dynamics of a double-kink soliton in the harmonic trapping potential and the corresponding chirp are shown in Fig. 2. Figure 2a and c is obtained with $g(t) = \lambda_0 \exp[4\lambda t]$ and corresponds to BECs with the loss of atoms associating with the loss parameter $\gamma(t) = -\lambda$, while Fig. 2b and d corresponds to BECs with the gain of atoms for the feeding parameter $\gamma(t) = \lambda$ corresponding to a time decreasing s-wave scattering length with $g(t) = \lambda_0 \exp[-4\lambda t]$. Figure 2c and d reveals that the frequency chirp associated with the double-kink soliton has double-kink feature. We can see from plots of Fig. 2 that with the increasing (decreasing) of the absolute value of the s-wave scattering length, the double-kink soliton keeps the same absolute depth and has a compression (broadening) in its width, while the corresponding frequency chirp has an increase (decrease) in the absolute depth value and a compression (broadening) in its width. It is seen from plots of Fig. 2 that the absolute depths of the double-kink soliton and that of the corresponding chirp are reached at $x = 0$. Therefore, the linear part of the frequency chirp of Fig. 2 has not affect the chirp absolute depth. For increasing (decreasing) nonlinearity parameter $g(t)$, the wave and the corresponding chirp propagate in the +x—direction (−x—direction), as we can see from Fig. 2a and c, b and d.

For the set of parameters $a_0^2 = 1$, $\beta_0 = \frac{1}{6a_0}, \nu = \frac{1}{a_0}, g_0 > 1$, $(g_0 - 1)^2 + 5 > 0$, $\Omega = -\frac{3 + 2
u^2}{4}$, with $g_0 = 1.8$ and $a_0 = \pm 1$, we show in Fig. 3 the effects of the self-steepening coefficient $a_0$ on the dynamics of the femtosecond soliton in the harmonic trapping potential defined by Eq. (24a) and the corresponding chirp given by Eq. (24b). Figure 3a and b shows the evolution of, respectively, a bright femtosecond soliton…
and the corresponding frequency chirp, which reveal that for the chirp has bright soliton feature for a positive self-steepening coefficient \( \alpha_0 > 0 \). Figure 3c shows the dynamics of the frequency chirp associated with the bright femtosecond soliton shown in Fig. 3c with a negative self-steepening coefficient \( \alpha_0 < 0 \) having dark soliton feature. Independent of the sign of the self-steepening coefficient \( \alpha_0 \), the bright femtosecond soliton has an constant in the peak value although the absolute value of the s-wave scattering length increases as a function of time (Fig. 3a and d). As a consequence of the increasing of the absolute value of the s-wave scattering length, the bright femtosecond soliton and the corresponding chirp during their propagation have a compression in their width. It is seen from Fig. 3b, d that the frequency chirp associated with a positive [negative] self-steepening coefficient \( \alpha \) has an increase in the peak value [absolute depth]. It is also seen from plots of Fig. 3 that for positive (negative) self-steepening coefficient \( \alpha_0 \), the wave propagates in the \(-x\) direction \((+x\) direction). Therefore, the self-steepening coefficient \( \alpha_0 \) can be used to manipulate the motion of the matter wave solitons in the BEC systems. As we can see from Fig. 3b and d that the absolute depths of the chirp corresponding to femtosecond solitons of Fig. 3a and c are obtained for \( x \neq 0 \), this means that the linear part of the chirp \( \delta \omega(x, t) \) has contributed to the absolute depth of the frequency chirp for both \( \alpha_0 = \pm 1 \).

3.3 BECs with a temporal periodic modulation of the s-wave scattering length

As the second example, we consider the temporal periodic modulation of the s-wave scattering length \([58]\) and the nonlinearity parameter takes the form \( g(t) = \lambda_0 (1 + m \sin [\omega t]) \) with \( 0 < m < 1 \), \( \lambda_0 \neq 0 \) being any real constant. Using the integrability condi-
Figures 4 and 5 show the dynamics of, respectively, a double-kink soliton and a bright femtosecond soliton with the corresponding frequency chirp for BEC system with a temporal periodic modulation of the s-wave scattering length in the temporal periodic modulation of the trapping potential a temporal periodic loss and gain of atoms. Plots of the top panels correspond to positive self-steepening coefficient $\alpha_0$, while those of the bottom panels are obtained with negative self-steepening coefficient $\alpha_0$. It is seen from the chirping dynamics (plots of the bottom panels) that due to the temporal periodic modulation of the s-wave scattering length and trapping potential, the trajectories of the frequency chirp oscillate and the direction of the wave motion depends on the sign of the self-steepening coefficient $\alpha_0$. For positive self-steepening coefficient $\alpha_0$ (plots of the top panels), the frequency chirp corresponding to the double-kink soliton has bright double-kink feature, while for the negative self-steepening coefficient $\alpha_0$, the chirp has double-kink feature, as we can see from Fig. 4d. It is seen from Fig. 5b that the frequency chirp associated with the bright femtosecond soliton has dark (bright) soliton feature for positive (negative) self-steepening coefficient $\alpha_0$. Due to the temporal periodic modulation of the s-wave scattering length and trapping potential, the wave trajectories oscillate, as we can see from Figs. 4 and 5. Plots of Figs. 4 and 5 reveal that the direction of the wave propagation depends on the sign of the self-steepening coefficient $\alpha_0$. Indeed, waves of top panels, generated with positive $\alpha_0$ and those of the bottom panels, obtained with negative self-steepening coefficient $\alpha_0$, propagate, respectively, in the $-x-$ direction and in $+x-$ direction.
Compensation process and generation of chirped

Fig. 5 Spatiotemporal evolution of the bright femtosecond soliton with the corresponding chirp associated with solution (24a) with chirp (24b) of the GP Eq. (2) for the self-steepening coefficient having different sign. Plots a and b are obtained with $\alpha_0 = 1$, while plots c and d are generated with $\alpha_0 = -1$. Other parameters used in different plots are $\lambda_0 = 2.1$, $g_0 = 1.8$, $m = 0.4$, and $\sigma = 1$

3.4 BECs with a time-varying hyperbolic s-wave scattering length

Following Xue [59], we consider for the last example a BEC in the time-dependent trapping potential with the time-varying hyperbolic s-wave scattering length that corresponds to $g(t) = \lambda_0 (1 + m \tanh [\mu t])$ for any real constants $\lambda_0 \neq 0$ and $\mu \neq 0$, and $0 < m < 1$. Using the integrability conditions (4b) and (4c) yields $\gamma(t) = -\frac{\mu}{4} \frac{1}{(1+\tanh[\mu t]) \cosh^2[\mu t]}$ and $\alpha(t) = -\frac{\mu^2}{8} \frac{3+4 \cosh[\mu t] \sinh[\mu t] (1+\tanh[\mu t])}{\cosh^4[\mu t] (1+\tanh[\mu t])^2}$. Figures 6 and 7 report the spatiotemporal evolution of, respectively, the double-kink soliton and the femtosecond soliton in BEC with a time-varying hyperbolic s-wave scattering length for, respectively, a positive and negative nonlinearity parameter $g_0$. For the double-kink soliton and the femtosecond soliton, we have used, respectively, the above set of data and the set of data $\alpha_0^2 = 1$, $g_0 > -1$, $\beta_0 = \frac{1}{\alpha_0}$, $\nu = -\frac{1}{\alpha_0}$, and $\Omega = -\frac{3\alpha_0^2 + 2}{4\alpha_0^2}$. As in the previous two examples, the direction of the wave motion and the feature of the frequency chirpdepend on the sign of the self-steepening coefficient $\alpha_0$. The chirp has a dark feature for positive
Fig. 6 Spatiotemporal evolution of (left panels) the double-kink soliton given by Eq. (22a) and (right panels) the corresponding chirp defined by Eq. (22b) for BEC systems with a time-varying hyperbolic s-wave scattering length with $g(t) = \lambda_0 (1 + \tanh [\mu t])$. Plots a and b are generated with a positive self-steepening coefficient $\alpha_0$ ($\alpha_0 = 0.5$), while plots c and d are obtained with a negative self-steepening coefficient $\alpha_0$ ($\alpha_0 = -0.5$). Other parameter used in plots a–d are $\lambda_0 = 0.3$, $g_0 = 1.8$, and $\mu = 2$, and $m = 0.99$. Other parameters are given in the text.

Self-steepening coefficient $\alpha_0$ (plots of the top panels) and a bright feature for negative self-steepening coefficient $\alpha_0$, as we can see from plots of the bottom panels. Figures 6 and 7 show that the width of the waves decreases at the beginning of the wave propagation and then reaches its minimum value, which no longer changes. It is sure that for negative $\mu$, the width of the waves will increase at the beginning of the wave propagation and then reaches its maximum value, which no longer changes as the wave propagates. For a better understanding, we show in Fig. 8 the evolution plot of both double-kink soliton (top panels) and femtosecond soliton (bottom panels) and their corresponding chirp for a negative $\mu$. Plots of the left panels in Figs. 6 and 7, showing the evolution of the frequency chirp, reveal that the chirp linear wave background is strong enough in the direction of the wave propagation.

4 Conclusion

In this paper, we have considered a cubic-quintic GP equation which describes the dynamics of BECs with both two- and three-body interactions in a time-varying complex potential consisting of parabolic and complex terms. With the help of a modified phase-imprint trans-
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Fig. 7 The dynamics of (left panels) a bright femtosecond soliton defined by Eq. (24a) and (right panel) the corresponding frequency chirp given by Eq. (24b) for BEC systems with a time-varying hyperbolic s-wave scattering length with \( g(t) = \lambda_0 \left( 1 + \tanh \left[ \mu t \right] \right) \). Plots (a) and (b) of the top panels are obtained with a positive self-steepening coefficient \( \alpha_0 (\alpha_0 = 1) \), while (c) and (d) are generated with a negative self-steepening coefficient \( \alpha_0 (\alpha_0 = -1) \). Other parameter used in plots (a)–(d) are \( \lambda_0 = -1.1, g_0 = -0.95, \) and \( \mu = 2, \) and \( m = 0.99. \) Other parameters are given in the text.

formation, we have obtained both the integrability conditions and the compensation condition, and converted the GP equation under consideration to a NLS equation with self-steepening and self-frequency shift. We have demonstrated that the competing cubic-quintic non-linearity induces propagating solitonlike dark (bright) solitons and double-kink solitons in the NLS equation with self-steepening and self-frequency shift. Parameter domains are delineated in which these BEC solitons exist. Our results have showed that the nonlinear chirp associated with each of the derived BEC solitons is formed of two main terms, one linear term proportional to the loss/gain parameter \( \gamma(t) \) and the other term proportional to the absolute value of the s-wave scattering length through the quantity \( \sqrt{g_0} \); this latter term is composed of one constant (with respect to the spatial variable \( x \)) term and one nonlinear term which is directly proportional to the intensity of the wave. Such nonlinear chirps in this work have been referred to as frequency chirp embedded in a linear wave background. The results of this work reveal that the chirp associated with each BEC soliton is localized and its feature (dark or bright) depends on the sign of the self-steepening coefficient \( \alpha_0 \). We showed that the motion of both the waves and the corresponding chirp depend on the self-steepening coefficient \( \alpha_0 \). More interesting, our results reveal that the solitons amplitude is constant during the wave propagation (this result comes from the compensation process), while the amplitude of the chirp strongly depends on the s-wave scattering length. It is also showed that for BECs having an increasing in time absolute s-wave scattering length, our exact solutions can be used to describe the compression of solitons in the BECs under consideration.
Fig. 8  

The methodology presented in this work is powerful for systematically finding an infinite number of BEC bright and dark solitonlike solutions by exactly matching the s-wave scattering length, the strength of the loss/feeding parameter and external harmonic trapping potential.

**Author contributions**  
EK contributed to conceptualization, project administration, methodology, software, writing—original draft, investigation, visualization, and writing–review and editing. AL contributed to writing—review and editing, investigation, validation.

**Data availability**  
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Declarations**  
**Conflict of interest**  
The authors declare that they have no known conflict of interest or personal relationships that could have appeared to influence the work reported in this paper.
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