Hiring Expert Consultants in E-Healthcare: A Two Sided Matching Approach

Vikash Kumar Singh\textsuperscript{a}, Sajal Mukhopadhyay\textsuperscript{a}, Fatos Xhafa\textsuperscript{b}, Aniruddh Sharma\textsuperscript{a}, Arpan Roy\textsuperscript{c,∗}

\textsuperscript{a}Department of Computer science and Engineering, NIT, Durgapur, India
\textsuperscript{b}Department of Computer science, Polytechnic University of Catalonia, Barcelona, Catalonia, Spain
\textsuperscript{c}Applications Engineer, Oracle, Bangalore, India

Abstract

Very often in some censorious healthcare scenario, there may be a need to have some expert consultancies (especially by doctors) that are not available in-house to the hospitals. Earlier, this interesting healthcare scenario of hiring the ECs (mainly doctors) from outside of the hospitals had been studied with the robust concepts of mechanism design with or without money. In this paper, we explore the more realistic two sided matching in our set-up, where the members of the two participating communities, namely patients and doctors are revealing the strict preference ordering over all the members of the opposite community for a stipulated amount of time. We assume that patients and doctors are strategic in nature. With the theoretical analysis, we demonstrate that the TOMHECs, that results in stable allocation of doctors to the patients is strategy-proof (or truthful) and optimal. The proposed mechanisms are also validated with exhaustive experiments.

Keywords: E-Healthcare, hiring ECs, DSIC, mechanism design, stable allocation.

∗This research was done while the author was a graduate student at Department of Information Technology, NIT Durgapur, India.

Email addresses: vikas.1688@gmail.com (Vikash Kumar Singh), sajmure@gmail.com (Sajal Mukhopadhyay), fatos.xhafa@gmail.com (Fatos Xhafa), annirudhsharma189@gmail.com (Aniruddh Sharma), arpanroy1994@gmail.com (Arpan Roy)
1. Introduction

The expert advices or consultancies provided by the expert consultants (ECs) mainly doctors can be thought of as one of the most indispensable events that occurs in the hospital(s) or medical unit(s) on a regular basis. Over the past few years, there had been a perplexing growth in the demand of ECs (especially doctors) during some critical surgical processes (or operations) that are taking place in the operation theatres (OTs) of the hospitals. The unprecedented growth in the demand of the ECs, has made ECs busy and scarce in nature. It is to be noted that, this unique nature (i.e. busy and scarce) of ECs in the healthcare lobby provides an edge to the research community in the healthcare domain to think of: How to manage or schedule these limited (or scarce) ECs in the OTs of the hospitals, during some censorious healthcare situation? In order to answer the above coined question, previously, there had been a spate of research work in the direction of handling the issues of scheduling the in-house ECs especially doctors [1, 2] and nurses [3] in an efficient and effective manner. In [1, 2, 4, 5] different techniques are discussed and presented to schedule the physicians that are in-house to the hospitals in an efficient way for some critical operations that are taking place in the OTs of that hospitals. In past, the work had been also done in the direction of managing the OTs and the hospitals during the patient congestion scenario. The work in [6, 7, 8] focuses on the question of: how to effectively and efficiently plan and schedule the OTs? In [9, 10, 8] the work has been done for allocating OTs on time to increase operating room efficiency.

More importantly, in healthcare domain, one scenario that may be thought of as a challenging issue is, say; in certain critical medical cases, there may be a requirement of some external manpower in the form of ECs (mainly doctors) that are not available in-house to the hospitals. Now, the immediate natural question that came in the mind is that, how to have some external expertise mainly in the form of doctors that are not available in-house to the hospitals? Surprisingly, literature is very limited for this problem in healthcare domain.

This interesting situation of taking expert consultancy from outside of the in-house medical unit during some censorious medical scenario (mainly surgical process) was taken care by Starren et. al. [11]. Moreover, the introduction of such a pragmatic field of study in the healthcare domain by Starren et. al. [11] has given rise to several open questions for the researchers, such as: (a) which ECs are to be considered as the possible expertise provider in
the consultancy arena? (b) What incentives policies in the form of perks and facilities are to be presented in-front of the ECs, so as to drag as many ECs as possible in the consultancy arena?

In [12], the problem of hiring one or more doctors for a patient from outside of the admitted hospital for some critical operation under monetary environment (experts are charging for their services) with the infinite budget are addressed. With the consideration that, ECs are having some social connections in real life, Singh et. al. [13] considered a budgeted setting of the problem in [12] motivated by [14] for hiring $k$ doctors out of the available $n$ doctors ($k < n$), such that the total payment made to the ECs do not exceed the total budget of a patient. As opposed to the money involved hiring of ECs as mentioned in [12, 13], another market of hiring ECs can be thought of where the ECs are providing their expertise free of cost. Recently, Singh et. al. [15] have addressed this idea, where the expert services are distributed free of cost. For hiring ECs in money free environment (i.e. money is not involved in any sense) they have utilized the idea of one sided strict preference (in this case strict preference from patient side) over the available doctors in the consultancy arena.

In this paper, we have tried to model the ECs hiring problem as a two sided preference market in healthcare domain motivated by [16, 17, 18, 19]. The idea behind studying the ECs hiring problem as a more appealing two sided preference market is that, in this environment, the members present in two different communities have the privilege to provide the strict preference ordering over all the available members of the opposite community. For example, in our case, we have two communities (or parties) in the consultancy arena: (a) Patient party (b) Doctor party. So, the members of the patient party provide strict preference ordering over all the available members (or the subset of available members) in the doctor party and vice versa.

1.1. Our Contributions

The main contributions of our work are as follows.

- We have tried to model the ECs hiring problem as a two sided matching problem in healthcare domain.
- We propose two mechanisms: a naive approach i.e. randomized mechanism for hiring expert consultants (RAMHECs) and a truthful and optimal mechanism; namely truthful optimal mechanism for hiring expert consultants (TOMHECs).
- We have also proved that for any instance of $n$ patients and $n$ doctors the
allocation done by TOMHECs results in stable, truthful, and optimal allocation for requesting party.

- TOMHECs establish an upper bound of $O(kn^2)$ on the number of iterations required to determine a stable allocation for any instance of $n$ patients and $n$ doctors.
- A substantial amount of analysis and simulation are done to validate the performance of RAMHECs and TOMHECs via optimal allocation measure.

The remainder of the paper is structured as follows. Section 2 describes our proposed model. Some required definitions are discussed in section 3. Section 4 illustrates the proposed mechanisms. Further analytic-based analysis of the mechanisms are carried out in section 5. A detailed analysis of the experimental results is carried out in section 6. Finally, conclusions are drawn and some future directions are depicted in section 7.

2. System model

We consider the scenario, where there are multiple hospitals say $n$ given as $h = \{h_1, h_2, \ldots, h_n\}$. In each hospital $h_i \in h$, there exists several patients with different diseases (in our case patients and doctors are categorized based on the diseases and areas of expertise respectively.) belonging to different income group that requires somewhat partial or complete expert consultancies from outside of the admitted hospitals. By partial expert consultancies it is meant that, the part of expertise from the overall required expert consultancies. The set of $k$ different categories is given as: $C = \{c_1, c_2, \ldots, c_k\}$. The set of all the admitted patients in different categories to different hospitals is given as: $P = \bigcup_{h_k \in h} \bigcup_{c_i \in C} \bigcup_{j=1}^{h_k} p_{i(j)}^{h_k}$ where $p_{i(j)}^{h_k}$ is the patient $j$ belonging to $c_i$ category admitted to $h_k$ hospital. The expression $h_k^i$ in term $p_{i(h_k^i)}^{h_k}$ indicates the total number of patients in hospital $h_k$ belonging to $c_i$ category. The patients who need consultancy may belong to different income bars. So, in this scenario, each hospital tries to select the patient from the lowest income bar in a particular category (say $c_i$ category) who will get the free consultation. On the other hand, there are several doctors having different expertise associated with different hospitals say $H = \{H_1, H_2, \ldots, H_n\}$. The set of all the available doctors in different categories associated with different hospitals is given as: $D = \bigcup_{H_k \in H} \bigcup_{c_j \in C} \bigcup_{i=1}^{H_k} d_{i(j)}^{H_k}$ where $d_{i(j)}^{H_k}$ is the doctor $j$ belonging
to $c_i$ category associated to $\mathcal{H}_k$ hospital. The expression $\tilde{\mathcal{H}}^j_k$ in term $d^j_{\mathcal{H}_k}$ indicates the total number of doctors associated with hospital $\mathcal{H}_k$ in $c_j$ category. Our model captures only a single category say $c_i$ but it is to be noted that our proposed model works well for the system considering multiple categories simultaneously. Only thing is that we have to repeat the process $k$ times as $k$ categories are existing. For simplicity purpose, in any category $c_i$ we have considered the number of patients and number of doctors are same i.e. $n$ along with an extra constraint that each of the members of the participating parties provides a strict preference ordering over all the available members of the opposite party. But, one can think of the situation where there are $n$ number of patients and $m$ number of doctors in a category such that $m \neq n$ ($m > n$ or $m < n$). Moreover, the condition that every members of the participating party is providing the strict preference ordering over all the available members of the opposite community is not essential and can be relaxed for all the three cases (i.e. $m = n$, $m < n$, and $m > n$).

By relaxation, it is meant that the members of the participating parties may provide the strict preference ordering over the subset of the members of the opposite party. At a particular time several doctors (> n) are providing their willingness to impart free consultancy to some patients present in the consultancy arena in a particular category as shown in the right side of the opposite party. At a particular time several doctors (> n) are providing their willingness to impart free consultancy to some patients present in the consultancy arena in a particular category as shown in the right side

![System model](image-url)
of the Figure 1. In the schematic diagram shown in Figure 1, for representation purpose one doctor is selected from all the interested doctors from each hospital belonging to a particular category \( c_i \). But in general one can think of the situation where multiple doctors can be selected from the available doctors from a particular hospital in a particular category \( c_i \) such that 

\[ |\mathcal{P}_i| = \sum_{\mathcal{H}_j} |\mathcal{H}_j| \]

where \( 0 \leq \mathcal{H}_j \leq n \) is the number of doctors selected from hospital \( \mathcal{H}_j \) in \( c_i \) category and placed into the consultancy arena. Following the above discussed criteria, the third party selects \( n \) doctors out of all the doctors in a particular category \( c_i \) as a possible expert consultant and is given as \( \mathcal{D}_i = \{d^{H_1}_{i(1)}, d^{H_2}_{i(2)}, \ldots, d^{H_n}_{i(n)}\} \) and a set of selected patients from \( c_i \) category is given as \( \mathcal{P}_i = \{p^{h_1}_{i(1)}, p^{h_2}_{i(2)}, \ldots, p^{h_n}_{i(n)}\} \). If not specified explicitly, \( n \) denotes the total number of patients and the total number of doctors that are participating in the consultancy arena in any category \( c_i \). For placing \( n \) doctors in the consultancy arena from the available doctors, the third party can take the help of the qualification of the doctors and number of successful operations or consultancies given so far by that doctor. Each patient \( p^{h_{i(k)}}_{i(j)} \) reveals a strict preference ordering over the participating set of doctors \( \mathcal{D}_i \) in a category \( c_i \) and also each doctor \( d^{H_{k(i)}}_{i(l)} \) provides the strict preference ordering over the set of participating patients of category \( c_i \) in the consultancy arena.

The strict preference ordering of the patient \( p^{h_{i(k)}}_{i(j)} \) over the set of doctors \( \mathcal{D}_i \) is denoted by \( \succ_k \). More formally, the significance of \( d^{H_{j(i)}}_{i(l)} \succ_k d^{H_{k(i)}}_{i(m)} \) is that the patient \( p^{h_i}_{i(k)} \) ranks doctor \( d^{H_{j(i)}}_{i(l)} \) above the doctor \( d^{H_{k(i)}}_{i(m)} \). The preference profile of all the patients for \( k \) different categories is denoted as \( \succ = \{\succ^1, \succ^2, \ldots, \succ^k\} \), where \( \succ^i \) denotes the preference profile of all the patients in category \( c_i \) over all the doctors in set \( \mathcal{D}_i \) represented as \( \succ^i = \{\succ^i_1, \succ^i_2, \ldots, \succ^i_n\} \). The preference profile of all the patients in \( c_i \) category except the patient \( r \) is given as \( \succ^i_{r-1} = \{\succ^i_1, \succ^i_2, \ldots, \succ^i_{r-1}, \succ^i_{r+1}, \ldots, \succ^i_n\} \). On the other hand, the doctors may give preferences based on the location where he/she (henceforth he) and the patients are located. The strict preference ordering of the doctor \( d^{H_{j(i)}}_{j(i)} \) is denoted by \( \succ_j \) over the set of patients \( \mathcal{P}_j \), where \( p^{h_{j(i)}}_{j(i)} \succ_j p^{h_{j(m)}}_{j(m)} \) means that doctor \( d^{H_{j(i)}}_{j(i)} \) ranks \( p^{h_{j(i)}}_{j(i)} \) above \( p^{h_{j(m)}}_{j(m)} \). The set of preferences of all the doctors in \( k \) different categories is denoted as \( \succ = \{\succ_1, \succ_2, \ldots, \succ_k\} \), where \( \succ_j \) contains the strict preference ordering of all the doctors in \( c_j \) category over all the patients in set \( \mathcal{P}_j \) represented as \( \succ_j = \{\succ^1_j, \succ^2_j, \ldots, \succ^n_j\} \). The strict preference ordering of all the doctors in \( c_j \) category except the doctor \( s \) is represented as \( \succ^{-s}_j = \{\succ^1_j, \succ^2_j, \ldots, \succ^{s-1}_j, \succ^{s+1}_j, \ldots, \succ^n_j\} \). It is to be noted that the allo-
cation of the doctors to the patients for category $c_i$ under consideration is captured by the allocation function $A_i: \succ \times \succ \rightarrow \mathcal{P}_i \times \mathcal{D}_i$. The resulting allocation vector is given as $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$. Each allocation vector $A_i \in \mathcal{A}$ denotes the patient-doctor pairs belonging to the $c_i$ category denoted as $\mathcal{A}_i = \bigcup_{j=1}^{m_i,n_i} a_i^{j,k}$, where each $a_i^{j,k} \in \mathcal{A}_i$ is a pair $\{p_i^{b_k}, d_i^{R_k}\}$. The matching between the patients and doctors for any category $c_i$ is captured by the mapping function $\mathcal{M}: \mathcal{P}_i \cup \mathcal{D}_i \rightarrow \mathcal{D}_i \cup \mathcal{P}_i$. More formally, $\mathcal{M}(p_i^{b_k}) = d_i^{R_k}$ means that patient $p_i^{b_k}$ is matched to $d_i^{R_k}$ doctor and $\mathcal{M}(d_i^{R_k}) = p_i^{b_k}$ means that doctor $d_i^{R_k}$ is matched to $p_i^{b_k}$.

3. Required definitions

**Definition 1.** **Blocking pair:** Fix a category $c_k$. We say that a pair $p_k^{b_i}$ and $d_k^{R_i}$ form a blocking pair for matching $\mathcal{M}$, if the following three conditions holds: (i) $\mathcal{M}(p_k^{b_i}) \neq d_k^{R_i}$, (ii) $d_k^{R_i} \succ_k \mathcal{M}(p_k^{b_i})$, and (iii) $p_k^{b_i} \succ_j \mathcal{M}(d_k^{R_i})$.

**Definition 2.** **Stable matching:** Fix a category $c_k$. A matching $\mathcal{M}$ is stable if there is no pair $p_k^{b_i}$ and $d_k^{R_i}$ such that it satisfies the conditions mentioned in (i)-(iii) in Definition 1.

**Definition 3.** **Perfect matching:** Fix a category $c_k$. A matching $\mathcal{M}$ is perfect matching if there exists one-to-one matching between the members of $\mathcal{P}_k$ and $\mathcal{D}_k$.

**Definition 4.** **Patient-optimal stable allocation:** Fix a category $c_k$. A matching $\mathcal{M}$ is patient optimal, if there exists no stable matching $\mathcal{M}'$ such that $\mathcal{M}'(p_k^{b_i}) \succ_k \mathcal{M}(p_k^{b_i})$ or $\mathcal{M}'(p_k^{b_i}) = \mathcal{M}(p_k^{b_i})$ for at least one $p_k^{b_i} \in \mathcal{P}_i$. Similar is the situation for doctor-optimal stable allocation.

**Definition 5.** **Strategy-proof for requesting party:** Fix a category $c_k$. Given the preference profile $\succ_k$ and $\succ_k$ of the patients and doctors in $c_k$ category, a mechanism $\mathcal{M}$ is strategy-proof (truthful) for the requesting party if for each members of the requesting party $A_k$ is preferred over $A_k$; where $A_k$ is the allocation when at least one member in requesting party is misreporting.
4. Proposed mechanisms

The idea behind proposing randomized mechanism i.e. RAMHECs is to better understand the more robust and philosophically strong optimal mechanism TOMHECs. The further illustration of the mechanisms are done under the consideration that patient party is requesting. Moreover, one can utilize the same road map of the mechanisms by considering doctors as the requesting party. This can easily be done by just interchanging their respective roles in the mechanisms.

4.1. RAMHECs

The idea lies behind the construction of initialization phase is to handle the system consisting \( k \) different categories. The algorithm is depicted in Algorithm 1.

4.1.1. Upper bound analysis of RAMHECs

The overall running time of RAMHECs is \( O(1) + O(kn) = O(kn) \).

4.1.2. Correctness of RAMHECs

The correctness of RAMHECs is proved with the loop invariant technique [20, 21]. The loop invariant that we have to prove is that at the end of the \( i^{th} \) iteration each of the patients in \( c_j \) category gets one distinct doctor allocated. We must show three things for the loop invariant technique to be true.

**Initialization:** It is true prior to the first iteration of the while loop. Just before the first iteration of the while loop \( A_j \leftarrow \phi \). This confirms that \( A_j \) contains no patient-doctor pair prior to the first iteration of the while loop.

**Maintenance:** The loop invariant to be true, we have to show that if it is true before each iteration of while loop, it remains true before the next iteration. The body of the while loop allocates a doctor to a patient in a particular category i.e. each time \( A_j \) is incremented by 1. Just before the \( i^{th} \) iteration, the \( A_j \) data structure contains \((i - 1)\) number of patient-doctor pairs. After the \( i^{th} \) iteration, the \( A_j \) data structure contains \( i \) patient-doctor pairs. This way at the end of the \( i^{th} \) iteration all the \( i \) patients gets a distinct doctor and the patient-doctor pairs are stored in \( A_j \) \([1..i]\).

**Termination:** The third property is to check, what happens when the while loop terminates. The condition causing the while loop to terminate is that, for any category \( c_j \), each of the patients are allocated with one distinct doctor leading to \( n \) patient-doctor pairs in \( A_j \) data structure. Because each loop
iteration increments $A_j$ by 1, we must have $|A_j| = n$ when all $n$ patients are already processed. So, when the loop terminates we have a data structure $A_j[1..n]$ that is already processed and it consists of $n$ patient-doctor pairs. If the RAMHECs is true for a particular category $c_j \in C$ it will remain true when all category in $C$ taken simultaneously. Hence, the RAMHECs is correct.

**Algorithm 1** RAMHECs $(\mathcal{D}, \mathcal{P}, \mathcal{C}, \succ, \succeq)$

**Output:** $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$

1: begin
   /* Initialization phase */
2: $\mathcal{A} \leftarrow \emptyset$
3: for each $c_i \in C$ do
4:    $k \leftarrow 0, i \leftarrow 0, d^* \leftarrow \emptyset, p^* \leftarrow \emptyset, A_i \leftarrow \emptyset, \mathcal{P}^* \leftarrow \emptyset, \mathcal{D}^* \leftarrow \emptyset$
5:    $\mathcal{P}^* \leftarrow \mathcal{P}_i$  // return the index of patient from patient set.
6:    $i \leftarrow \text{select}(\mathcal{D})$  // return the index of doctor from doctor set.
7:    $\mathcal{D}^* \leftarrow \mathcal{D}_i$
   /* Allocation phase */
8:   while $|A_i| \neq n$ do
9:      $t \leftarrow \text{rand}(\mathcal{P}^*)$  // return index of randomly selected patient.
10:     $p^* \leftarrow p_{i(t)}^{k}$
11:     $k \leftarrow \text{rand}(\succ i, \mathcal{D}^*)$  // returns the index of randomly selected
12:        doctor from the patient $t$ preference list.
13:     $d^* \leftarrow d_{i(k)}^{R_t}$
14:     $A_i \leftarrow A_i \cup \{(p^*, d^*)\}$
15:     $\mathcal{P}_i \leftarrow \mathcal{P}_i \setminus p^*$  // Removes the allocated patients from available
16:     patient list.
17:     $\mathcal{D}_i \leftarrow \mathcal{D}_i \setminus d^*$  // Removes the allocated doctors from available
18:        doctor list.
19:   end while
20: $\mathcal{A} \leftarrow \mathcal{A} \cup A_i$
21: end for
22: return $\mathcal{A}$
23: end
4.1.3. Illustrative example

For understanding purpose, let the category be $c_3$ (say eye surgery). The set of patient from 4 different hospitals $h = \{h_1, h_2, h_3, h_4\}$ is given as: $\mathcal{P}_3 = \{p_{3(1)}, p_{3(2)}, p_{3(3)}, p_{3(4)}\}$. The set of available doctors engaged to 4 different hospitals $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$ is given as: $\mathcal{D}_3 = \{d_{3(1)}^{\mathcal{H}_1}, d_{3(2)}^{\mathcal{H}_1}, d_{3(3)}^{\mathcal{H}_1}, d_{3(4)}^{\mathcal{H}_1}\}$. The preference profile of doctor set $\mathcal{D}_3$ is given as: $d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1} = d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1} = d_{3(4)}^{\mathcal{H}_1}$. Similarly, the preference profile of patient set $\mathcal{P}_3$ is given as: $p_{3(1)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(2)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(3)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(4)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$. The TOMHECs is illustrated in Algorithm 2. For understanding purpose, let the category be $c_3$ (say eye surgery). The set of patient from 4 different hospitals $h = \{h_1, h_2, h_3, h_4\}$ is given as: $\mathcal{P}_3 = \{p_{3(1)}, p_{3(2)}, p_{3(3)}, p_{3(4)}\}$. The set of available doctors engaged to 4 different hospitals $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4\}$ is given as: $\mathcal{D}_3 = \{d_{3(1)}^{\mathcal{H}_1}, d_{3(2)}^{\mathcal{H}_1}, d_{3(3)}^{\mathcal{H}_1}, d_{3(4)}^{\mathcal{H}_1}\}$. The preference profile of patient set $\mathcal{P}_3$ is given as: $p_{3(1)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(2)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(3)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$, $p_{3(4)} = [d_{3(1)}^{\mathcal{H}_1} = d_{3(2)}^{\mathcal{H}_1} = d_{3(3)}^{\mathcal{H}_1}]$. Similarly, the preference profile of doctor set $\mathcal{D}_3$ is given as: $d_{3(1)} = [p_{3(1)} = 3, p_{3(2)} = 3, p_{3(3)} = 3, p_{3(4)} = 3]$.

Given the preference profiles, while loop in line 9-17 randomly selects patient $p_{3(3)}$ from the available patients list $\mathcal{P}_3$. Line 12 of the RAMHECs randomly selects doctor $d_{3(3)}$ from the available preference ordering of $p_{3(3)}$. At the end of first iteration of the while loop, the RAMHECs captures $(p_{3(3)}^{H_1}, d_{3(3)}^{H_1})$ pair in the $\mathcal{A}_3$ data structure. In the similar fashion, the remaining allocation is done. The final patient-doctor allocation pair done by the mechanism is $\mathcal{A}_3 = \{(p_{3(3)}^{H_1}, d_{3(3)}^{H_1}), (p_{3(2)}^{H_1}, d_{3(3)}^{H_1}), (p_{3(1)}^{H_1}, d_{3(3)}^{H_1}), (p_{3(4)}^{H_1}, d_{3(3)}^{H_1})\}$.

4.2. TOMHECs

Our main focus is to propose a mechanism that satisfy the two important economic properties: truthfulness, and optimality. The TOMHECs is illustrated in Algorithm 2.

4.2.1. Running time

The total running time of TOMHECs is given as: $T(n) = \sum_{i=1}^{k}(O(1) + (\sum_{i=1}^{n} O(n))) = O(kn^2)$

| Algorithm 2 | TOMHECs $(\mathcal{D}, \mathcal{P}, \mathcal{C}, >, \sim)$ |
|-------------|-------------------------------------------------------------|
| Output: $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$ |
| 1: begin | /* Initialization phase */ |
| 2: $i \leftarrow 0$, $\mathcal{A} \leftarrow \phi$ | /* Allocation phase */ |
| 3: for each $c_i \in \mathcal{C}$ do | |
4: \[ \mathcal{A}_i \leftarrow \phi \]
5: \[ i \leftarrow \text{select}(\mathcal{P}) \]
6: \[ \mathcal{P}^* \leftarrow \mathcal{P}_i \]
7: \[ i \leftarrow \text{select}(\mathcal{D}) \]
8: \[ \mathcal{D}^* \leftarrow \mathcal{D}_i \]
9: \[ \text{for each } d_{i(j)}^{R_k} \in \mathcal{D}^* \text{ do} \]
10: \[ \Pi(d_{i(j)}^{H_k}) \leftarrow \phi \quad \triangleright \quad \Pi(d_{i(j)}^{R_k}) \text{ data structure keeps track of set of} \]
11: \[ p_{i(j)}^{h_k} \in \mathcal{P}_i \text{ requesting to } d_{i(j)}^{R_k}. \]
12: \[ \text{end for} \]
13: \[ \text{while } |\mathcal{A}_i| \neq n \text{ do} \]
14: \[ \text{for each free patient } p_{i(j)}^{h_k} \in \mathcal{P}_i \text{ do} \]
15: \[ \Pi(d^*) \leftarrow \Pi(d^*) \cup p_{i(j)}^{h_k} \]
16: \[ \text{end for} \]
17: \[ \text{for each engaged doctor } d_{i(j)}^{R_k} \in \mathcal{D}_i \text{ do} \]
18: \[ \text{if } |\Pi(d_{i(j)}^{H_k})| > 1 \text{ then} \]
19: \[ p^* \leftarrow \text{select}\_\text{best}(\succ^j, \Pi(d_{i(j)}^{H_k})) \]
20: \[ \text{if } (p_{i(j)}^{h_k}, d_{i(j)}^{H_k}) \in \mathcal{A}_i \text{ and } p^* \succ^j_i p_{i(j)}^{h_k} \text{ then} \]
21: \[ \mathcal{A}_i \leftarrow \mathcal{A}_i \setminus (p_{i(j)}^{h_k}, d_{i(j)}^{H_k}) \]
22: \[ \mathcal{A}_i \leftarrow \mathcal{A}_i \cup (p^*, d_{i(j)}^{H_k}) \]
23: \[ \Pi(d_{i(j)}^{H_k}) \leftarrow \Pi(d_{i(j)}^{H_k}) \setminus \Pi(d_{i(j)}^{H_k}) - \{p^*\} \]
24: \[ \text{else if } (p_{i(j)}^{h_k}, d_{i(j)}^{H_k}) \notin \mathcal{A}_i \text{ then} \]
25: \[ \mathcal{A}_i \leftarrow \mathcal{A}_i \cup (p^*, d_{i(j)}^{H_k}) \]
26: \[ \Pi(d_{i(j)}^{H_k}) \leftarrow \Pi(d_{i(j)}^{H_k}) \setminus \Pi(d_{i(j)}^{H_k}) - \{p^*\} \]
27: \[ \text{end if} \]
28: \[ \text{else if } |\Pi(d_{i(j)}^{R_k})| == 1 \text{ then} \]
29: \[ \text{if } (\Pi(d_{i(j)}^{H_k}), d_{i(j)}^{R_k}) \notin \mathcal{A}_i \text{ then} \]
30: \[ \mathcal{A}_i \leftarrow \mathcal{A}_i \cup (\Pi(d_{i(j)}^{H_k}), d_{i(j)}^{R_k}) \]
31: \[ \text{end if} \]
32: \[ \text{end if} \]
33: \[ \text{end for} \]
34: \[ \text{end while} \]
35: \[ \mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{A}_i \]
4.2.2. Correctness of the TOMHECs

The correctness of the TOMHECs is proved with the loop invariant technique [20, 21].

The loop invariant: Fix a category $c_i$. At the start of $\ell^{th}$ iteration of the while loop, the number of temporarily processed patient-doctor pairs or in other words the number of patient-doctor pairs held by $A_i$ is given as: $|\bigcup_{j=1}^{\ell-1} A'_j|$, where $A'_j$ is the net patient-doctor pairs temporarily maintained in the set $A'_j$ at the $j^{th}$ iteration. So, on average the number of patients or doctors (whomsoever is greater) that are to be explored in further iterations are $n - |\bigcup_{i=1}^{\ell-1} A'_i|$. From the construction of the TOMHECs it is clear that after any $\ell^{th}$ iteration this condition holds: $0 \leq n - |\bigcup_{i=1}^{\ell-1} A'_i| \leq n$; where $1 \leq \ell \leq n^2$. The net minimum number of patient-doctor pairs that can be processed temporarily at any iteration may be zero. Hence, inequality $0 \leq n - |\bigcup_{i=1}^{\ell-1} A'_i| \leq n$ is always true. We must show three things for this loop invariant to be true.

Initialization: It is true prior to the first iteration of the while loop. Just before the first iteration of the while loop, in TOMHECs the inequality $0 \leq n - |\bigcup_{i=1}^{\ell-1} A'_i| \leq n$ blows down to $0 \leq n - 0 \leq n \Rightarrow 0 \leq n \leq n$ i.e. no patient-doctor pair is temporarily added to $A_i$ prior to the first iteration of while loop. This confirms that $A_i$ contains no patient-doctor pair.

Maintenance: For the loop invariant to be true, if it is true before each iteration of the while loop, it remains true before the next iteration of the while loop. The body of while loop allocates doctor(s) to the patient(s) with each doctor is allocated to a patient; i.e. each time the cardinality of $A_i$ is either incremented by some amount or remains similar to previous iteration. Just before the $\ell^{th}$ iteration the patient-doctor pairs temporarily added to $A_i$ are $\bigcup_{i=1}^{\ell-1} A'_i$. So, one can conclude from here that the number of patient-doctor pairs that are left is given by inequality: $0 \leq n - |\bigcup_{i=1}^{\ell-1} A'_i| \leq n$. After the $(\ell - 1)^{th}$ iteration, the available number of patient-doctor pair $n - |\bigcup_{i=1}^{\ell-1} A'_i| \geq 0$ can be captured under two cases:

Case 1: If $|A_i| = n$: This case will lead to exhaust all the remaining patient-doctor pair in the current $\ell^{th}$ iteration and no patient-doctor pair is left for the next iteration. The inequality $n - (|\bigcup_{i=1}^{\ell-1} A'_i \cup A'_i|) = n - (|\bigcup_{i=1}^{\ell-1} A'_i|) = n - |A_i| = 0$. Hence, it means that all the remaining patient-doctor is absorbed in
this iteration and no patient-doctor pair is left for processing.

**Case 2:** If $|A_i| < n$: This case captures the possibility that there may be the scenario when few patient-doctor pairs from the remaining patient-doctor pairs may still left out; leaving behind some of the pairs for further iterations. So, the inequality $n - (|\bigcup_{i=1}^{\ell-1} A'_i \cup A'_\ell|) > 0 \Rightarrow n > n - (|\bigcup_{i=1}^{\ell} A'_i|) > 0$ is satisfied.

From Case 1 and Case 2, at the end of $\ell^{th}$ iteration the loop invariant is satisfied.

**Termination:** It is clear that in each iteration the cardinality of output data structure i.e. $A_i$ either incremented by some amount or remains as the previous iteration. This indicates that at some $\ell^{th}$ iteration the loop terminates by dissatisfying the condition of the while loop $|A_i| \neq n$ at line 12. When the loop terminates it is for sure that $|A_i| = n$. We can say $n - |\bigcup_{i=1}^{\ell} A'_i| = 0 \Rightarrow 0 \leq n$. Thus, this inequality indicates that all the $n$ patient and doctors in $c_i$ category are processed and each patient allocated a best possible doctor when the loop terminates.

If the TOMHECs is true for the $c_i \in C$ category it will remain true when all category in $C$ taken simultaneously. Hence, the TOMHECs is correct.

### 4.2.3. Example

Considering the initial set-up discussed in section 4.1.3. According to line 13-16 of TOMHECs each of the patients $p_{3(1)}^{h_2}$, $p_{3(2)}^{h_3}$, $p_{3(3)}^{h_4}$, and $p_{3(4)}^{h_1}$ are requesting to the most preferred doctor from their respective preference list i.e. $d_{3(4)}^{H_2}$, $d_{3(3)}^{H_4}$, $d_{3(4)}^{H_2}$, and $d_{3(2)}^{H_1}$, respectively. In the next step, we will check if any requested doctor among $d_{3(1)}^{H_3}$, $d_{3(2)}^{H_1}$, $d_{3(3)}^{H_4}$, and $d_{3(4)}^{H_2}$ has got the multiple request from the patients in $P_3$. Now, it can be seen that, in the first iteration of TOMHECs doctor $d_{3(4)}^{H_2}$ have got requests from patients $p_{3(1)}^{h_2}$, and $p_{3(3)}^{h_4}$. As each doctor can be assigned to only one patient, so this competitive environment between patient $p_{3(1)}^{h_2}$, and $p_{3(3)}^{h_4}$ can be resolved by considering the strict preference ordering of doctor $d_{3(4)}^{H_2}$ over the available patients in $P_3$. From the strict preference ordering of doctor $d_{3(4)}^{H_2}$ it is clear that patient $p_{3(3)}^{h_4}$ is preferred over patient $p_{3(1)}^{h_2}$. Hence, patient $p_{3(1)}^{h_2}$ is rejected. So, for the meanwhile $p_{3(2)}^{h_3}$ gets a doctor $d_{3(3)}^{H_4}$; $p_{3(3)}^{h_4}$ gets a doctor $d_{3(4)}^{H_2}$; and $p_{3(4)}^{h_1}$ gets a doctor $d_{3(2)}^{H_1}$. Now, as the patient $p_{3(1)}^{h_2}$ do not get his/her (henceforth his) most preferred doctor i.e. $d_{3(4)}^{H_2}$ from his preference list. So, he will request the second best doctor i.e. $d_{3(3)}^{H_4}$ from his preference list. As
doctor $d_{3(3)}^{4}$ is already been requested by $p_{3(2)}^{3}$, the similar situation now occurs in case of doctor $d_{3(3)}^{4}$ where patients $p_{3(1)}^{2}$ and $p_{3(2)}^{3}$ are simultaneously requesting to doctor $d_{3(3)}^{4}$. Looking at the preference list of $d_{3(3)}^{4}$, we get, patient $p_{3(1)}^{2}$ is preferred over $p_{3(2)}^{3}$. So, patient $p_{3(2)}^{3}$ is rejected. Now, $p_{3(2)}^{3}$ request the second best doctor i.e. $d_{3(4)}^{4}$ from his preference list. In the similar fashion, the remaining allocation is done. The final allocation is:

\[
\{(p_{3(1)}^{1}, d_{3(3)}^{4}), (p_{3(2)}^{2}, d_{3(1)}^{3}), (p_{3(3)}^{3}, d_{3(2)}^{4}), (p_{3(4)}^{4}, d_{3(2)}^{4})\}.
\]

4.3. Several properties

The proposed TOMHECs has several compelling properties. These properties are discussed next.

**Proposition 1.** The matching computed by the Gale-Shapley mechanism [18, 19] results in a stable matching.

**Proposition 2.** A stable matching computed by Gale-Shapley mechanism [18, 19] is requesting party optimal.

**Proposition 3.** Gale-Shapley mechanism [18, 19] is truthful for the requesting party.

Following the above mentioned propositions and motivated by [18, 19] we have proved that the TOMHECs results in stable, optimal, and truthful allocation when all the $k$ different categories are taken simultaneously.

**Lemma 1.** TOMHECs results in a stable allocation for the requesting party (patient party or doctor party).

**Proof.** Fix a category $c_i \in C$. Let us suppose for the sake of contradiction there exists a blocking pair $(p_{i(j)}^{k}, d_{i(j)}^{H})$ that results in an unstable matching $\mathcal{M}$ for the requesting party. As their exists a blocking pair $(p_{i(j)}^{k}, d_{i(j)}^{H})$ it may be due to the case that $(p_{i(j)}^{k}, d_{i(k)}^{H})$ and $(p_{i(k)}^{b}, d_{i(k)}^{H})$ are their in the resultant matching $\mathcal{M}$. This situation will arise only when $d_{i(j)}^{H} \succ_i d_{i(k)}^{H}$ i.e. in the strict preference ordering of patient $p_{i(j)}^{b}$ doctor $d_{i(j)}^{H}$ is preferred over doctor $d_{i(k)}^{H}$. From the matching result $\mathcal{M}$ obtained, it can be seen that in-spite the fact that $d_{i(j)}^{H} \succ_i d_{i(k)}^{H}$; $d_{i(j)}^{H}$ is not matched with $p_{i(j)}^{b}$ by the TOMHECs. So, this upset may happen only when doctor $d_{i(j)}^{H}$ received a proposal from
a patient \( p_{i(k)} \) to whom \( d_{i(j)}^k \) prefers over \( p_{i(j)}^h \) i.e. \( p_{i(k)}^h \succ_i^j p_{i(j)}^h \). Hence, this contradicts the fact that the \( (p_{i(j)}^h, d_{i(j)}^k) \) is a blocking pair. As there exists no blocking pair, it can be said that the resultant matching by TOMHECs is stable.

From our claim that, the TOMHECs results in a stable matching in a particular category \( c_i \), it must be true for any category. Hence, it must be true for the system considering the \( k \) categories simultaneously. \( \square \)

**Lemma 2.** A stable allocation resulted by TOMHECs is requesting party (patient or doctor) optimal.

**Proof.** Fix a category \( c_i \). Let us suppose for the sake of contradiction that the allocation set \( \mathcal{M} \) obtained using TOMHECs is not an optimal allocation for requesting party (say patient party). Then, from Lemma 1 there exists a stable allocation \( \mathcal{M}' \) such that \( \mathcal{M}'(p_{i(j)}^h) \succ_i^j \mathcal{M}(p_{i(j)}^h) \) or \( \mathcal{M}'(p_{i(j)}^h) =_i^j \mathcal{M}(p_{i(j)}^h) \) for at least one patient \( p_{i(j)}^h \). Therefore, it must be the case that, some patient \( p_{i(j)}^h \) proposes to \( \mathcal{M}'(p_{i(j)}^h) \) before \( \mathcal{M}(p_{i(j)}^h) \) since \( \mathcal{M}'(p_{i(j)}^h) \succ_i^j \mathcal{M}(p_{i(j)}^h) \) and is rejected by \( \mathcal{M}'(p_{i(j)}^h) \). Since doctor \( \mathcal{M}'(p_{i(j)}^h) \) rejects patient \( p_{i(j)}^h \), the doctor \( \mathcal{M}'(p_{i(j)}^h) \) must have received a better proposal from a patient \( p_{i(k)}^h \) to whom doctor \( \mathcal{M}'(p_{i(j)}^h) \) prefers over \( p_{i(j)}^h \) i.e. \( p_{i(k)}^h \succ_i^j p_{i(j)}^h \). Since, this is the first iteration at which a doctor rejects a patient under \( \mathcal{M}' \). It follows that the allocation \( \mathcal{M} \) is preferred over allocation \( \mathcal{M}' \) for the patient \( p_{i(j)}^h \). Hence, this contradicts the fact that the allocation set \( \mathcal{M} \) obtained using TOMHECs is not an optimal allocation. As their exists an optimal allocation \( \mathcal{M} \).

Form our claim that, the TOMHECs results in optimal allocation in a particular category \( c_i \), it must be true for any category. Hence, it must be true for the system considering the \( k \) categories simultaneously. \( \square \)

**Lemma 3.** A stable allocation resulted by TOMHECs is requesting party (patient or doctor) truthful.

**Proof.** Fix a category \( c_i \). Let us suppose for the sake of contradiction that the matching set \( \mathcal{M} \) obtained using TOMHECs is not a truthful allocation for requesting party (say patient party). The TOMHECs results in stable matching \( \mathcal{M} \) when all the members of the proposing party reports their true preferences. Now, let’s say a patient \( p_{i(j)}^h \) misreport his preference list \( \succ_i^j \) and getting better off in the resultant matching \( \mathcal{M}' \). Let \( \mathcal{P}'_i \) be the set of patients
who are getting better off in $\mathcal{M}'$ as against $\mathcal{M}$. Let $\mathcal{D}'_i$ be the set of doctors matched to patients in $\mathcal{P}'_i$ in matching $\mathcal{M}'$. Let $d^H_{i(k)}$ be the doctor that $p^h_{i(j)}$ gets in $\mathcal{M}'$. Since $\mathcal{M}$ is stable, we know that $d^H_{i(k)}$ cannot prefer $p^h_{i(j)}$ to the patient got in $\mathcal{M}$, because this would make $(p^h_{i(j)}, d^H_{i(k)})$ a blocking pair in $\mathcal{M}$ (see Lemma 1). In other words, doctor $\mathcal{M}(d^H_{i(k)}) \preceq p^h_{i(j)}$. Now, if $\mathcal{M}(d^H_{i(k)})$ patient would not improve in $\mathcal{M}'$ then $\mathcal{M}(d^H_{i(k)}) \preceq p^h_{i(j)}$. Hence, $d^H_{i(k)}$ can not be matched with $p^h_{i(j)}$ in $\mathcal{M}'$, a contradiction. Therefore, patient in $\mathcal{M}$ also improves in $\mathcal{M}'$. That is, $\mathcal{D}'_i$ is not the only set of doctors in $\mathcal{M}'$ of those patient who are getting better off in $\mathcal{M}$; but also the set of doctors where patient in $\mathcal{M}$ improve in $\mathcal{M}'$. In other words, each doctor in $\mathcal{D}_i$ is matched to two different patient from $\mathcal{P}_i$ in match $\mathcal{M}$ and $\mathcal{M}'$, being better off in $\mathcal{M}$ than in $\mathcal{M}'$. It can also be proved using Lemma 1 that $\mathcal{M}'$ is not stable; a contradiction that terminates the proof.

From our claim that, the TOMHECs results in a truthful matching in a particular category $c_i$, it must be true for any category. Hence, it must be true for the system considering the $k$ categories simultaneously.

\section{Further analytics-based analysis}

In order to provide sufficient reasoning to our simulation results presented in section 6, the two proposed mechanisms are in general analyzed on the ground of the expected distance of allocation done by the mechanisms from the top most preference. As a warm up, first the the analysis is done for any patient $j$, to estimate the expected distance of allocation from the top most preference. After that the analysis is extended to more general setting where all the patients present in the system are considered. It is to be noted that the results revealed by the simulations can easily be verified by the lemmas below.

\textbf{Lemma 4.} The allocation resulted by RAMHECs for any patient (or doctor) $j$ being considered first is on an average $\frac{n}{2}$ distance away from its most preferred doctor (or patient) i.e. $E[Z] \simeq \frac{n}{2}$; where $Z$ is the random variable measuring the distance from the top most preference.

\textbf{Proof.} Fix a category $c_i \in \mathcal{C}$, and an arbitrary patient $j$ being considered first. In RAMHECs, for any arbitrary patient (AP) being considered first are allotted a random doctor from his preference list. The index position of
the doctor in the preference list is decided by $k$, where $k = 1, 2, \ldots, n$. Now, when a doctor is selected randomly from the preference list any of these $k$ ($1 \leq k \leq n$) may be selected. So any index $k$ could be the outcome of the experiment (allocation of a doctor) and it is to be noted that selection of any such $k$ is equally likely. Therefore, for each $k$ such that $1 \leq k \leq n$, any $k^{th}$ doctor can be selected with probability $\frac{1}{n}$. For $k = 1, 2, \ldots, n$, we define indicator random variable $X_k$ where

$$X_k = I\{k^{th} \text{ doctor selected from patients' preference list}\}$$

$$X_k = \begin{cases} 1, & \text{if } k^{th} \text{ doctor is selected} \\ 0, & \text{otherwise} \end{cases}$$

Taking expectation both side, we get;

$$E[X_k] = E[I\{k^{th} \text{ doctor selected from patients' preference list}\}]$$

As always with the indicator random variable, the expectation is just the probability of the corresponding event [20]:

$$E[X_k] = 1 \cdot Pr\{X_k = 1\} + 0 \cdot Pr\{X_k = 0\} = 1 \cdot Pr\{X_k = 1\} = \frac{1}{n}$$

For a given call to RAMHECs, the indicator random variable $X_k$ has the value 1 for exactly one value of $k$, and it is 0 for all other $k$. For $X_k = 1$, we can measure the distance of $k^{th}$ allocated doctor from the most preferred doctor in the patient $j$'s preference list. So, let $d_k$ be the distance of $k^{th}$ allocation from the best preference. More formally, it can be represented in the case analytic form as:

$$Z = \begin{cases} d_0 : & \text{If } 1^{st} \text{ agent is selected from the preference list } (k = 1) \\ d_1 : & \text{If } 2^{nd} \text{ agent is selected from the preference list } (k = 2) \\ \vdots & \vdots \\ d_{n-1} : & \text{If } n^{th} \text{ agent is selected from the preference list } (k = n) \end{cases}$$

Where $Z$ is the random variable measuring the distance of the allocation from the patient’s top most preference. Here, $d_0 = 0, d_1 = 1, d_2 = 2, \ldots, d_{n-1} = n - 1$. It is to be observed that, once the doctor $k$ is selected from the patient $j$’s preference list, the value calculation of $d_k$ is no way dependent on
Now, observe that the random variable $Z$ that we really care about can be formulated as:

$$Z = \sum_{k=1}^{n} X_k \cdot d_{k-1}$$

Taking expectation both side. We get:

$$E[Z] = E\left[\sum_{k=1}^{n} X_k \cdot d_{k-1}\right]$$

$$= \sum_{k=1}^{n} E[X_k \cdot d_{k-1}] \quad \text{(by linearity of expectation)}$$

$$= \sum_{k=1}^{n} E[X_k] \cdot E[d_{k-1}] \quad (X_k \text{ and } d_{k-1} \text{ are independent})$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot E[d_{k-1}] = \frac{1}{n} \sum_{k=1}^{n} E[d_{k-1}]$$

$$= \frac{1}{n} \sum_{k=1}^{n} d_{k-1} \quad \text{(once } k \text{ is fixed } d_{k-1} \text{ becomes constant)}$$

$$= \frac{1}{n} \cdot \frac{(n-1)(n)}{2}$$

$$= \frac{(n-1)}{2} \approx \frac{n}{2},$$

as claimed. \qed

**Lemma 5.** In RAMHECs, $E[D] \approx \frac{n^2}{16}$; where $D$ is the total distance of all the patients in the system from the top most preference.

**Proof.** Fix a category $c_i \in C$. We are analyzing, the expected distance of the allocations done to the patients by RAMHECs from the top most preferences. For this purpose, as there are $n$ patients, the index of these patients are captured by $i$ such that $i = 1, 2, \ldots, n$. Without loss of generality, the patients are considered in some order. The index position of the doctor in any patient $j$’s preference list is decided by $k$, where $k = 1, 2, \ldots, n$. For any patient $i$ ($1 \leq i \leq n$) selected first, when a doctor is selected randomly from the preference list any of the available $k$ ($1 \leq k \leq n$) doctors can be
selected. So, any index $k$ could be the outcome of the experiment (allocation of doctor) and any such $k$ is equally likely. But what could the case, if instead of considering the patient in the first place, say a patient is selected in $i^{th}$ iteration. In that case, from the construction of RAMHECs the length of the preference list of the patient under consideration would be $n - i + 1$. So, when a doctor is selected randomly from the preference list, any of the $(n - i + 1)$ doctors may be selected. It is to be noted that the selection of any of the $(n - i + 1)$ doctors is equally likely. Therefore, for a patient under consideration in $i^{th}$ iteration, for each $k$ such that $1 \leq k \leq n - i + 1$ any $k^{th}$ doctor can be selected with probability $\frac{1}{n - i + 1}$. Here, we are assuming that each agent’s top preferences are still remaining when that agent is considered by the RAMHECs. To get the lower bound this is the best possible setting. If an agent is not provided that list, he will be further away from his top most preference. For each patient $i$ and for $k = 1, 2, \ldots, n$, we define indicator random variable $X_{ik}$ where

$$X_{ik} = \begin{cases} 1, & \text{if $k^{th}$ doctor is selected from patient i's preference list} \\ 0, & \text{otherwise} \end{cases}$$

Taking expectation both side, we get;

$$E[X_{ik}] = E[I\{k^{th} \text{ doctor selected from patient i's preference list}\}]$$

As always with the indicator random variable, the expectation is just the probability of the corresponding event:

$$E[X_{ik}] = 1 \cdot Pr\{X_{ik} = 1\} + 0 \cdot Pr\{X_{ik} = 0\} = 1 \cdot Pr\{X_{ik} = 1\} = \frac{1}{n - i + 1}$$

For a given call to RAMHECs, the indicator random variable $X_{ik}$ has the value 1 for exactly one value of $k$, and it is 0 for all other $k$. For $X_{ik} = 1$, we can measure the distance of $k^{th}$ allocated doctor from the most preferred doctor in the patient $j$’s preference list. So, let $d_{ik}$ be the distance of $k^{th}$ allocation from the best preference. More formally, it can be represented in the case analytic form as:

$$D = \begin{cases} d_{i0} : & \text{If 1^{st} agent is selected from the preference list (k = 1)} \\ d_{i1} : & \text{If 2^{nd} agent is selected from the preference list (k = 2)} \\ \vdots & \vdots \\ d_{i(n-1)} : & \text{If n^{th} agent is selected from the preference list (k=n)} \end{cases}$$
Where $D$ is the total distance of all the patients in the system from the top most preference. It is to be observed that, once the doctor $k$ is selected from the patient $j$’s preference list, the value calculation of $d_k$ is no way dependent on $k$. Now, observe that the random variable $D$ that we really care about is given as:

$$D \geq \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} X_{ik} \cdot d_{ik}$$

Taking expectation both side. We get:

$$E[D] \geq E[\sum_{i=1}^{n} \sum_{k=1}^{n-i+1} X_{ik} \cdot d_{ik}]$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} E[X_{ik} \cdot d_{ik}] \quad \text{(by linearity of expectation)}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} E[X_{ik}] \cdot E[d_{ik}] \quad \text{($X_{ik}$ and $d_{ik}$ are independent)}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{1}{n-i+1} \cdot d_{ik}$$

$$\geq \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{1}{n} \cdot d_{ik}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} d_{ik}$$

$$= \frac{1}{n} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=1}^{n-i+1} d_{ik} + \sum_{i=\frac{n}{2}}^{n} \sum_{k=1}^{n-i+1} d_{ik} \right]$$

$$\geq \frac{1}{n} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=1}^{n-i+1} 0 + \sum_{i=\frac{n}{2}}^{n} \sum_{k=1}^{n-i+1} 0 \right] \quad (1)$$

\footnote{discarding the lower order terms}
as claimed. It is to be observed that for each agent, the expected distance of allocation done by RAMHECs from the top preference in an amortized sense is \( \frac{n}{16} \).

**Lemma 6.** The expected number of rejections for any arbitrary patient (or doctor) \( j \) resulted by TOMHECs is constant. If the probability of any \( k \) length rejection is considered as \( \frac{1}{2} \) i.e. \( \Pr\{Y_k = 1\} = \frac{1}{2} \) then \( E[Y] = 2 \); where \( Y \) is the random variable measuring the total number of rejections made to the patient (or doctor) under consideration.

**Proof.** Fix a category \( c_i \in \mathcal{C} \), and an arbitrary patient \( j \). To analyze the expected number of rejections suffered by the patient under consideration in case of TOMHECs, we capture the total number of rejections done to any patient \( j \) by a random variable \( Y \). So, the expected number of rejections

\[
\geq \frac{1}{n} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} d_{ik} \right]
\]

\[
\geq \frac{1}{n} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} d_{i\frac{n}{2}} \right] \quad (2)
\]

\[
= \frac{1}{n} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} \frac{n}{2} \right]
\]

\[
= \frac{1}{2} \left[ \sum_{i=1}^{\frac{n}{2}} \sum_{k=\frac{n}{2}}^{n-i+1} 1 \right]
\]

\[
\geq \left( \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} j \right) - 1
\]

\[
= \frac{1}{2} \left[ \frac{n^2}{2} \frac{(n+1)}{2} \right] - 1
\]

\[
= \frac{n^2 + 2n - 16}{16} \approx \frac{n^2}{16}
\]

\[2\text{replacing each term of the series by its first term}\]
suffered by any patient $j$ is given as $E[Y]$. It is considered that the rejection by any member $k = 0, \ldots, n - 1$, present on the patients’ $j$ preference list is an independent experiment. It means that, the $m$ length rejections suffered by an arbitrary patient $j$ is no way dependent on any of the previous $m - 1$ rejections. Let us suppose for each $0 \leq k \leq n - 1$, the probability of rejection by any $k^{th}$ doctor be $\frac{1}{2}$ (it can be any value between 0 and 1 depending on the scenario). For $k = 0, \ldots, n - 1$, we define indicator random variable $Y_k$ where

$$Y_k = I\{k \text{ length rejection}\}$$

$$Y_k = \begin{cases} 1, & \text{if } k \text{ length rejection} \\ 0, & \text{otherwise} \end{cases}$$

Taking expectation both side, we get;

$$E[Y_k] = E[I\{k \text{ length rejection}\}]$$

As always with the indicator random variable, the expectation is just the probability of the corresponding event:

$$E[Y_k] = 1 \cdot Pr\{Y_k = 1\} + 0 \cdot Pr\{Y_k = 0\} = 1 \cdot Pr\{Y_k = 1\} = \left(\frac{1}{2}\right)^k$$

Observe that the random variable $Y$ that we really care about is given as,

$$Y = \sum_{k=0}^{n-1} Y_k$$

Taking expectation both side. We get;

$$E[Y] = E\left[\sum_{k=0}^{n-1} Y_k\right]$$

$$= \sum_{k=0}^{n-1} E[Y_k] \quad (\text{by linearity of expectation})$$

$$= \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k < \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

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\[
\frac{1}{1 - \left(\frac{1}{2}\right)} = 2
\]
as claimed. Moreover, if we consider the probability of \(k^{th}\) rejection as \(\frac{2}{3}\) then, the expected number of rejections will be given as \(3\) i.e \(E[Y] = 3\). Similarly, \(E[Y] = 10\) if the probability of \(k^{th}\) rejection is taken as \(\frac{9}{10}\). It means that, even with the high probability of rejection to any arbitrary patient \(j\) by the members of the proposed party, there is a chance that after constant number of rejections patient \(j\) will be allocated a good doctor according to his choice. Hence, we can say that each agent’s allocation is not far away from his top most preference.

Lemma 7. In TOMHECs, \(E[R] = 2n\), where \(R\) is the random variable measuring the total number of rejections made to all the patients.

Proof. Fix a category \(c_i \in C\). We are analyzing the total number of rejections suffered by all the patients in expectation. For this purpose, as there are \(n\) patients, the index of these patients are captured by \(i\) such that \(i = 1, 2, \ldots, n\). The index position of the doctor in any patient \(j\)’s preference list is decided by \(k\), where \(k = 1, 2, \ldots, n\). We capture the total number of rejections done to all patients by a random variable \(R\). So, the expected number of rejections suffered by all the patients is given as \(E[R]\). It is considered that the rejection by any member \(k = 1, \ldots, n - 1\), present on the patients’ \(i\) preference list is an independent experiment. It means that, the \(m\) length rejections suffered by an arbitrary patient \(i\) is no way dependent on any of the previous \(m - 1\) rejections. Let us suppose for each patient \(i\) and for each \(1 \leq k \leq n - 1\), the probability of rejection by any \(k^{th}\) doctor be \(\frac{1}{2}\) (it can be any value between 0 and 1 depending on the scenario). For \(k = 1, \ldots, n - 1\), we define indicator random variable \(R_{ik}\) where

\[
R_{ik} = \begin{cases} 
1, & \text{if } k \text{ length rejection of } i^{th} \text{ patient} \\
0, & \text{otherwise}
\end{cases}
\]

Taking expectation both side, we get;

\[
E[R_{ik}] = E[I\{k \text{ length rejection of } i^{th} \text{ patient}\}] = E[I] = 1
\]
As always with the *indicator random variable*, the expectation is just the probability of the corresponding event:

\[ E[R_{ik}] = 1 \cdot Pr\{R_{ik} = 1\} + 0 \cdot Pr\{R_{ik} = 0\} = 1 \cdot Pr\{R_{ik} = 1\} = \left(\frac{1}{2}\right)^k \]

Observe that the random variable \( R \) that we really care about is given as,

\[ R = \sum_{i=1}^{n} \sum_{k=1}^{n-i} R_{ik} \]

Taking expectation both side. We get;

\[ E[R] = E\left[\sum_{i=1}^{n} \sum_{k=1}^{n-i} R_{ik}\right] \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{n-i} E[R_{ik}] \quad \text{(by linearity of expectation)} \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{n-i} \left(\frac{1}{2}\right)^k \]

\[ < \sum_{i=1}^{n} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \]

\[ = \sum_{k=0}^{n-1} \frac{1}{1 - \left(\frac{1}{2}\right)} = 2n \]

as claimed.

**Corollary 1.** *It is to be observed that for each patient, the expected number of rejections in case of TOMHECs in an amortized sense is \( O(1) \). As we have shown that for all \( n \) agents, the expected number of rejection are \( O(n) \).*

6. Experimental findings

The experiments are carried out in this section to compare the efficacy of the TOMHECs based on the preference lists of the doctors and patients generated randomly using *Random* library in Python. RAMHECs is considered as the benchmark mechanism.
6.1. Simulation setup

For creating a real world healthcare scenario we have considered 10 different categories of patients and doctors for our simulation purpose. One of the scenarios that is taken into consideration is, say there are equal number of patients and doctors present in each of the categories along with the assumption that each of the patients are providing strict preference ordering (generated randomly) over all the available doctors and also each of the doctors are providing strict preference ordering over all the available patients. Second scenario is the case where, there are equal number of patients and doctors are present in the market. Each of the members in the respective parties are providing the strict preference ordering over the subset of the members of the opposite community. The other two scenarios i.e. \( m < n \) and \( m > n \) with partial preference are not shown due to page limit.

6.2. Performance metrics

The efficacy of TOMHECs is measured under the banner of two important parameters: (a) **Satisfaction level** (\( \eta_{\ell} \)): It is defined as the sum over the difference between the index of the doctor (patient) allocated from the patient’s (doctor’s) preference list to the index of the most preferred doctor (patient) by the patient (doctor) from his/her preference list. Considering the requesting party, the \( \eta_{\ell}^j \) for \( c_j \) category is defined as: 
\[
\eta_{\ell}^j = \sum_{i=1}^{n} (\xi_i - \xi_i);
\]
where, \( \xi_i \) is the index of the doctor (patient) allocated from the initially provided preference list of the patients (doctors) \( i \), and \( \xi_i \) is the index of the most preferred doctor (patient) in the initially provided preference list of patient (doctor) \( i \). For \( k \) categories, \( \eta_{\ell} = \sum_{j=1}^{k} \sum_{i=1}^{n} (\xi_i - \xi_i) \). It is to be noted that lesser the value of satisfaction level higher will be the satisfaction of patients or doctors. (b) **Number of preferable allocation** (\( \zeta \)): The term "preferable allocation" refers to the allocation of most preferred doctor or patient from the revealed preference lists by the patients or the doctors respectively. For a particular patient or doctor the preferable allocation is captured by the function \( f : \mathcal{P}_i \rightarrow \{0, 1\} \). For the category \( c_i \), the number of preferable allocation (NPA) is defined as the number of patients (doctors) getting their first choice from the initially provided preference list. So, \( \zeta_i = \sum_{j=1}^{n} f(p_i^{h_j}) \). For \( k \) categories \( \zeta = \sum_{i=1}^{k} \sum_{j=1}^{n} f(p_i^{h_j}) \).
6.3. Simulation directions

The three directions are seen for measuring the performance of TOMHECs, they are: (1) All the patients and doctors are reporting their true preference list. (2) When fraction of total available members of the requesting party are misreporting their preference lists. (3) When fraction of total available members of the requested party are misreporting their preference lists.

6.4. Result analysis

In this section, the result is simulated for the above mentioned three cases and discussed.

Table 1: Abbreviations used in simulation

| Abbreviation | Description |
|--------------|-------------|
| RAMHECs-P    | Patients allocation using RAMHECs without variation. |
| TOMHECs-P    | Patients allocation using TOMHECs without variation. |
| RAMHECs-D    | Doctors allocation using RAMHECs without variation. |
| TOMHECs-D    | Doctors allocation using TOMHECs without variation. |
| TOMHECs-PS   | Patients allocation using TOMHECs with small variation. |
| TOMHECs-DS   | Doctors allocation using TOMHECs with small variation. |
| TOMHECs-PM   | Patients allocation using TOMHECs with medium variation. |
| TOMHECs-DM   | Doctors allocation using TOMHECs with medium variation. |
| TOMHECs-PL   | Patients allocation using TOMHECs with large variation. |
| TOMHECs-DL   | Doctors allocation using TOMHECs with large variation. |

Expected amount of patients/doctors deviating. The following analysis motivated by [20] justifies the idea of choosing the parameters of variation. Let $\chi_j$ be the random variable associated with the event in which $j^{th}$ patient in $c_i$ category varies its true preference ordering. Thus, $\chi_j = \{j^{th} \text{ patient varies preference ordering}\}$. $\chi = \sum_{j=1}^{n} \chi_j$. We can write $E[\chi] = \sum_{j=1}^{n} E[\chi_j] = \sum_{j=1}^{n} 1/8 = n/8$. Here, $\Pr\{j^{th} \text{ patient varies preference ordering}\}$ is the probability that given a patient whether he will vary his true preference ordering. The probability of that is taken as $1/8$ (small variation).

Our result analysis is broadly classified into four categories:

- **Case 1a: Requesting party with full preference (FP)** In Figure 2a and Figure 2b, it can be seen that the satisfaction level of the requesting party in case of TOMHECs is more as compared to RAMHECs. As TOMHECs always allocates the most preferred member from the preference list. Under the manipulative environment of the requesting party, it can be seen in Figure 2a and Figure 2b that, the satisfaction level of the system
in case of TOMHECs with large variation is less than the *satisfaction level* of the system in case of TOMHECs with medium variation is less than the *satisfaction level* of the system in case of TOMHECs with small variation is less than the *satisfaction level* of the system in case of TOMHECs. It is natural from the construction of TOMHECs. Considering the second parameter *i.e. number of preferable allocation*, it can be seen in Figure 3a and Figure 3b that the NPA of the requesting party in case of TOMHECs is more as compared to RAMHECs.

Under the *manipulative* environment of the *requesting party*, it can be seen in Figure 3a and Figure 3b that, the NPA of the system in case of TOMHECs...
with large variation is less than the NPA of the system in case of TOMHECs with medium variation is less than the NPA of the system in case of TOMHECs with small variation is less than the NPA of the system in case of TOMHECs. It is natural from the construction of TOMHECs.

- **Case 1b: Requesting party with partial preference (PP)** In Figure 4a and Figure 4b, it can be seen that the satisfaction level of the requesting party in case of TOMHECs is more as compared to RAMHECs. As TOMHECs always allocates the most preferred member from the preference list.

Under the manipulative environment of the requesting party, it can be seen in Figure 4a and Figure 4b that, the satisfaction level of the system in case of TOMHECs with large variation is less than the satisfaction level.
Figure 5: ζ of requesting party with $m = n$

- **Case 2a: Requested party with full preference (FP)** In Figure 6a, Figure 6b and Figure 7a, Figure 7b, it can be seen that the satisfaction level and the NPA respectively of the requested party in case of TOMHECs is more as compared to RAMHECs. It can be seen from Figure 2a-3b and Figure 6a-7b that the TOMHECs is requesting party optimal. It is natural from the construction of TOMHECs.
• Case 2b: Requested party with partial preference (PP) In Figure 8a, Figure 8b and Figure 9a, Figure 9b, it can be seen that the satisfaction level and the NPA respectively of the requested party in case of TOMHECs is more as compared to RAMHECs.
7. Conclusions and future works

We have tried to model the ECs hiring problem as a two sided matching problem in healthcare domain. This paper proposed an optimal and truthful mechanism, namely TOMHECs to allocate the ECs to the patients. The more general settings are of $n$ patients and $m$ doctors ($m \neq n$ or $m = n$) with the constraint that members of the patient party and doctor party can provide the preference ordering (not necessarily strict) over the subset of the members of the opposite party can be thought of as our future work.
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