Progress on NNLO subtraction

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We review standard subtraction, as a method to compute cross sections at NNLO accuracy.

1. Introduction

High-energy physics will enter a new era of discovery with the start of LHC operations in 2007. The LHC is a proton-proton collider that will function at the highest energy ever attained in the laboratory, and will probe a new realm of high-energy physics. The use of a high-energy hadron collider as a research tool makes substantial demands upon the theoretical understanding and predictive power of QCD, the theory of the strong interactions within the Standard Model.

At high $Q^2$ any production rate can be expressed as a series expansion in $\alpha_S$. Because QCD is asymptotically free, the simplest approximation is to evaluate any series expansion to leading order in $\alpha_S$. However, for most processes a leading-order evaluation yields unreliable predictions. The next simplest approximation is the inclusion of radiative corrections at the NLO accuracy, which usually warrants a satisfying assessment of the production rates. In the previous decade, a lot of effort was devoted to devise process-independent methods and compute rates to NLO accuracy. In some cases, though, the NLO corrections may not be accurate enough. Specimen cases are: the extraction of $\alpha_S$ from fitting the predictions to the data, where in order to avoid that the main source of uncertainty be due to the NLO evaluation of some production rates, like the event shapes of jet production in $e^+e^-$ collisions, only observables evaluated to next-to-next-to-leading order (NNLO) accuracy are considered \cite{1}; open $b$-quark production at the Tevatron, where the NLO uncertainty bands are too large to test the theory \cite{2} vs. the data \cite{3}; Higgs production from gluon fusion in hadron collisions, where it is known that the NLO corrections are large \cite{4,5}, while the NNLO corrections \cite{6,7,8}, which have been evaluated in the large-$m_t$ limit, display a modest increase, of the order of less than 20%, with respect to the NLO evaluation; Drell-Yan productions of $W$ and $Z$ vector bosons at LHC, which can be used as “standard candles” to measure the parton luminosity at the LHC \cite{9,10,11,12}.

In the last few years the NNLO corrections have been computed to the total cross section \cite{6,13} and to the rapidity distribution \cite{14,15} of Drell-Yan production, to the total cross section for the production of a scalar \cite{6,7,8} and a pseudoscalar \cite{16,17} Higgs from gluon fusion as well as to a fully differential cross section \cite{18,19}, and to jet production in $e^+e^-$ collisions \cite{20,21,22}. The methods which have been used are disparate: analytic integration, which is the first method to have been used \cite{13}, cancels the divergences analytically, and is flexible enough to include a limited class of acceptance cuts by modelling cuts as propagators \cite{7,14,15,17}; sector decomposition \cite{18,20,23,24,25,26,27}, which is flexible enough to include any acceptance cuts and for which the cancellation of the divergences is performed numerically; subtraction \cite{22,26,28,29,30,31,32,33,34}, for which the
cancellation of the divergences is organised in a process-independent way by exploiting the universal structure of the infrared divergences of a gauge theory, in particular the universal structure of the three-parton tree-level splitting functions [33,36,37,38,39,10] and the two-parton one-loop splitting functions [11,12,13,14,15].

The standard approach of subtraction to NNLO relies on defining approximate cross sections which match the singular behaviour of the QCD cross sections in all the relevant unresolved limits. For processes without coloured partons in the initial state, we constructed in [34] subtraction terms which regularise the kinematical singularities of the squared matrix element in all singly- and doubly-unresolved parts of the phase space. Thus, the regularised squared matrix element is integrable over all the phase space regions where at most two partons become unresolved.

2. Subtraction scheme at NNLO

The NNLO correction to any \( m \)-jet cross section reads

\[
\sigma^{\text{NNLO}} = \int_{m+2} \sigma^{\text{NNLO}}_{m+2} + \int_{m+1} \sigma^{\text{NNLO}}_{m+1} + \int_m \sigma^{\text{NNLO}}_m, \tag{1}
\]

The three integrals in Eq. (1) are separately divergent, but their sum is finite for infrared-safe observables. As explained in [34], we rewrite Eq. (1) as

\[
\sigma^{\text{NNLO}} = \int_{m+2} \sigma^{\text{NNLO}}_{m+2} + \int_{m+1} \sigma^{\text{NNLO}}_{m+1} + \int_m \sigma^{\text{NNLO}}_m, \tag{2}
\]

where each integral in Eq. (2) is finite by construction. Here we will focus on the subtractions that regularise doubly real emission, so we recall only that

\[
\sigma^{\text{NNLO}}_{m+2} = \left[ \sigma^{\text{RR}}_{m+2} J_{m+2} - \sigma^{\text{RR,A}_1}_{m+2} J_m - \sigma^{\text{RR,A}_2}_{m+2} J_m \right]_{\varepsilon=0}, \tag{3}
\]

where \( \sigma^{\text{RR,A}_1}_{m+2} \) and \( \sigma^{\text{RR,A}_2}_{m+2} \) regularise the doubly- and singly- unresolved limits of \( \sigma^{\text{RR}}_{m+2} \) respectively and \( \sigma^{\text{RR,A}_2}_{m+2} \) accounts for their overlap.

3. Subtractions for doubly-real emission

The cross section \( \sigma^{\text{RR}}_{m+2} \) is the integral of the tree-level squared matrix element for \( (m+2) \)-parton production over the \( (m+2) \)-parton phase space

\[
\sigma^{\text{RR}}_{m+2} = \sigma^{(m+2)} [M^{(0)}_{m+2}]^2. \tag{4}
\]

The counterterms may be written symbolically as

\[
\begin{align*}
\sigma^{\text{RR,A}_1}_{m+2} &= \sigma^{(m+1)} [\sigma^{(1)} A_1 |M^{(0)}_{m+2}]^2, \tag{5} \\
\sigma^{\text{RR,A}_2}_{m+2} &= \sigma^{(m+1)} [\sigma^{(1)} A_2 |M^{(0)}_{m+2}]^2, \tag{6} \\
\sigma^{\text{RR,A}_2}_{m+2} &= \sigma^{(m)} [\sigma^{(1)} A_2 |M^{(0)}_{m+2}]^2. \tag{7}
\end{align*}
\]

In this contribution we define explicitly the \( A_1 |M^{(0)}_{m+2}|^2 \) term, the terms \( A_2 |M^{(0)}_{m+2}|^2 \) and \( A_1 |M^{(0)}_{m+2}|^2 \) will be presented elsewhere.

The singly-singular subtraction \( A_1 |M^{(0)}_{m+2}|^2 \) is

\[
A_1 |M^{(0)}_{m+2}|^2 = \sum_{i,r} \left( \frac{1}{2} C_{ir} - C_{ir} S_{ir} \right) + \sum_r S_{ir}. \tag{8}
\]

The singly-collinear term in Eq. (5) reads

\[
C_{ir} = 8 \pi \alpha_s \mu^2 \frac{1}{s_{ir}} \left( M^{(0)}_{m+1} |\hat{P}^{(0)}_{i,r} |M^{(0)}_{m+1} \right), \tag{9}
\]

where \( \hat{P}^{(0)}_{i,r} = \hat{P}^{(0)}_{i,r} (z_{i,r}, \tilde{z}_{i,r}) \) is the Altarelli-Parisi splitting function. The momentum fractions \( \tilde{z}_{i,r} \) and \( \tilde{z}_{i,r} \) are defined as

\[
\tilde{z}_{i,r} = \frac{s_{ir}}{s_{ir} Q}, \quad \text{and} \quad \tilde{z}_{i,r} = \frac{s_{ir}}{s_{ir} Q}, \tag{10}
\]

and the transverse momentum \( \tilde{k}_{i,ir} \) is given by

\[
\tilde{k}_{i,ir}^\mu = \zeta_{i,r} p_{i,r}^\mu - \zeta_{i,r} p_{i,r}^\mu + (\tilde{z}_{i,r} - \tilde{z}_{i,r}) p_{i,r}^\mu, \tag{11}
\]

with

\[
\zeta_{i,r} = \frac{s_{ir}}{\alpha_s s_{ir} Q}, \quad \zeta_{i,r} = \frac{s_{ir}}{\alpha_s s_{ir} Q}. \tag{12}
\]
We used the abbreviations \( s_{iQ} = 2p_i \cdot Q \), \( s_{rQ} = 2p_r \cdot Q \) and \( s_{i(r)Q} = s_{iQ} + s_{rQ} \) above.

The \( m + 1 \) momenta entering the matrix elements on the right hand side of Eq. (15) are

\[
\tilde{p}_{ir}^\mu = \frac{p_{ir}^\mu + p_{rQ}^\mu - \alpha_{ir} Q^\mu}{1 - \alpha_{ir}}, \quad \tilde{p}_{irn}^\mu = \frac{p_{irn}^\mu}{1 - \alpha_{ir}}.
\]

In Eq. (13) \( n \neq i, r \) and

\[
\alpha_{ir} = s_{i(r)Q} \sqrt{(s_{i(r)Q})^2 - 4 s_{ir} s} \quad (14)
\]

with \( Q^\mu \) the total four-momentum of the incoming electron and positron and \( s = Q^2 \).

The singly soft term is

\[
S_r = -8\pi\alpha_s \mu^2 \sum_{i,k \neq r} \frac{1}{2} S_{ik}(r) |\mathcal{M}_{m+1,(i,k)}^{(0)}|^2,\quad (15)
\]

if \( r \) is a gluon, and \( S_r = 0 \) if \( r \) is a quark or antiquark. The \( m + 1 \) momenta entering the matrix element on the right hand side of Eq. (15) are

\[
\tilde{p}_{ir}^\mu = \Lambda_{ir}^\mu [Q, (Q - p_i)/\lambda_r] (p_{ir}^\mu/\lambda_r), \quad n \neq r, \quad (16)
\]

where \( \lambda_r = \sqrt{1 - s_{rQ}/s} \) and

\[
\Lambda_{ir}^\mu [K, \tilde{K}] = g_{ir}^\mu \frac{2 (K + \tilde{K})_{\nu} (K + \tilde{K})_{\nu} + 2 K_{\nu} \tilde{K}_{\nu} (K + \tilde{K})^2}{(K + \tilde{K})^2}, \quad (17)
\]

The matrix \( \Lambda_{ir}^\mu [K, \tilde{K}] \) generates a (proper) Lorentz transformation, provided \( K^2 = \tilde{K}^2 \neq 0 \).

In Eq. (15) \( S_{ik}(r) \) denotes the eikonal factor

\[
S_{ik}(r) = \frac{s_{ik}}{s_{ir}s_{rik}}, \quad (18)
\]

and the sum in Eq. (15) runs over the external partons of the \((m+1)\)-parton matrix element.

The soft-collinear subtraction is defined by

\[
C_{ir} S_r = 8\pi\alpha_s \mu^2 \frac{1}{8\pi^2} \frac{2 \hat{s}_{ir}}{\hat{s}_{ir} - \hat{z}_{ri}} \mathcal{T}_i^2 |\mathcal{M}_{m+1}^{(0)}|^2, \quad (19)
\]

if \( r \) is a gluon, and \( C_{ir} S_r = 0 \) if \( r \) is a quark or antiquark. The momentum fractions are given by Eq. (16). The correct variables in the matrix element in the soft-collinear limit are those that appear in the soft limit [34]. Thus the \( m + 1 \) momenta entering the matrix elements on the right hand side are given by Eq. (16).

The momentum mappings introduced in Eqs. (13) and (16) in addition to conserving total four-momentum, both lead to exact phase-space factorisation in the form

\[
d\phi^{(m+2)} = d\phi^{(m+1)} [d\phi^{(1)}]^2, \quad (20)
\]

where the \( m + 1 \) momenta in \( d\phi^{(m+1)} \) are precisely those defined in Eq. (15) or Eq. (16).

The explicit expressions for \( [d\phi^{(1)}] \) read

\[
[d\phi^{(1)}] = \mathcal{J} \frac{d^d p_r}{(2\pi)^{d-1}} \delta_+(p_r^2), \quad (21)
\]

and the Jacobian factors are

\[
\mathcal{J} = \frac{(1 - \alpha_{ir})}{}^{(d-2)-1} \Theta(1 - \alpha_{ir}) s_{irQ} \sqrt{(s_{ir} + s_{irQ} - s_{rQ})^2 + 4 s_{ir} (s - s_{irQ})} \quad (22)
\]

\[
\mathcal{J} = \lambda_r^{(d-2)-2} \Theta(\lambda_r) \quad (23)
\]

for the collinear and soft phase-space factorisations of Eqs. (13) and (16) respectively.

In Eq. (22) \( \alpha_{ir} \) is understood to be expressed in terms of the variable \( \tilde{p}_{ir} \).

\[
\alpha_{ir} = [\sqrt{(s_{ir} + s_{irQ} - s_{rQ})^2 + 4 s_{ir} (s - s_{irQ})} - (s_{ir} + s_{irQ} - s_{rQ})] \frac{2(s - s_{irQ})}{1} \quad (24)
\]

The analytical integration of the counterterms over the factorised one-parton phase-space \([d\phi^{(1)}]\) is now possible. Starting with the collinear subtraction, notice that \( k_{L,ir} \) as defined by Eq. (11) is orthogonal to \( \tilde{p}_{ir} \), therefore, the spin correlations generally present in Eq. (9) vanish after azimuthal integration [30]. Thus when evaluating the integral of the subtraction term \( C_{ir} \) over the factorised phase space \([d\phi^{(1)}]\), we may replace the Altarelli–Parisi splitting functions \( \hat{F}_{i,f}^{(1)} \) by their azimuthally averaged counterparts \( F_{i,f}^{(1)} \), so

\[
\int [d\phi^{(1)}] C_{ir} = a_c C_{ir} (y_{irQ}; m, \varepsilon) \mathcal{T}_{ir}^2 |\mathcal{M}_{m+1}^{(0)}|^2, \quad (25)
\]
where \( y_{ir} = 2 \bar{p}_{ir} \cdot q/Q^2 \),

\[
a_{\varepsilon} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left( 4\pi \mu^2 \right)^{-\varepsilon} \tag{26}
\]

and

\[
a_{\varepsilon} C_{ir}(y_{ir}; m, \varepsilon) = 8\pi\alpha_s \mu^{2\varepsilon} \int \left[ \frac{dp^{(1)}}{s_{ir}} \right] \frac{P_{i,k}^{(0)}}{S_{ir}}. \tag{27}
\]

The function \( C_{ir}(y_{ir}; m, \varepsilon) \) depends on the momentum of the parent parton and the flavours of the daughter partons. Explicitly

\[
C_{qq}(x; m, \varepsilon) = x^{-2\varepsilon} \left[ 2 \left( I_{m}^{(1)}(x; \varepsilon) - I_{m}^{(0)}(x; \varepsilon) \right) + (1-\varepsilon) \left( I_{m}^{(1)}(x; \varepsilon) - I_{m}^{(2)}(x; \varepsilon) \right) \right], \tag{28}
\]

\[
C_{qg}(x; m, \varepsilon) = \frac{T_R}{C_A} x^{-2\varepsilon} \left[ I_{m}^{(0)}(x; \varepsilon) - \frac{2}{1-\varepsilon} \right] \times \left( I_{m}^{(1)}(x; \varepsilon) - I_{m}^{(2)}(x; \varepsilon) \right), \tag{29}
\]

and

\[
C_{gg}(x; m, \varepsilon) = 2x^{-2\varepsilon} \left[ 2 \left( I_{m}^{(1)}(x; \varepsilon) - I_{m}^{(0)}(x; \varepsilon) \right) + I_{m}^{(1)}(x; \varepsilon) - I_{m}^{(2)}(x; \varepsilon) \right]. \tag{30}
\]

The analytical formulae for the \( I_{m}^{(k)}(x; \varepsilon) \) functions involve Beta functions, the \( \mathcal{F}_1 \) hypergeometric function as well as the first Appell function \( F_1 \) and a certain generalisation of the last, \( F_1^{(2)} \), see Table \[\text{I} \]. The expansion of \( I_{m}^{(k)}(x; \varepsilon) \) in powers of \( \varepsilon \) is performed using the techniques of \[\text{[7,13,14]} \] to obtain

\[
C_{qq}(x; m, \varepsilon) = \left[ \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} - \frac{2}{\varepsilon} \ln(x) + O(\varepsilon^0) \right], \tag{31}
\]

\[
C_{qg}(x; m, \varepsilon) = \frac{T_R}{C_A} \left[ - \frac{2}{3\varepsilon} + O(\varepsilon^0) \right], \tag{32}
\]

\[
C_{gg}(x; m, \varepsilon) = \left[ \frac{2}{\varepsilon^2} + \frac{11}{3\varepsilon} - \frac{4}{3} \ln(x) + O(\varepsilon^0) \right]. \tag{33}
\]

The finite parts, not shown here, depend on \( m \) and can be easily found for any given \( m \) using the program of Ref. \[\text{[38]} \].

Integrating the soft subtraction, the color correlations of Eq. \[\text{[10]} \] are still present in the integrated expression

\[
\int \left[ dp^{(1)} \right] S_r = \sum_{i,k \neq k} a_\varepsilon S_{ik}(y_{ik}, y_{ik}, y_{ik}; m, \varepsilon) \times |\mathcal{M}_{m+1,(i,k)}|^2, \tag{34}
\]

where

\[
a_\varepsilon S_{ik}(y_{ik}, y_{ik}, y_{ik}; m, \varepsilon) = -8\pi\alpha_s \mu^{2\varepsilon} \int \left[ dp^{(1)} \right] \frac{1}{2} S_{ik}(r). \tag{35}
\]

For \( S_{ik}(y_{ik}, y_{ik}, y_{ik}; m, \varepsilon) \) we find

\[
S_{ik}(y_{ik}, y_{ik}, y_{ik}; m, \varepsilon) = -\frac{m(1-\varepsilon)(1-2\varepsilon)}{\varepsilon^2} \times B(1-\varepsilon, 1-\varepsilon) B(1-2\varepsilon, m(1-\varepsilon)) \times \kappa \mathcal{F}_1(1, 1, 1-\varepsilon, 1-\varepsilon), \tag{36}
\]

where \( \kappa = y_{ik}/(y_{ik} y_{ik}) \) and the hypergeometric function can be expanded using

\[
\kappa \mathcal{F}_1(1, 1, 1-\varepsilon, 1-\varepsilon) = \kappa^{-\varepsilon} \left[ 1 + \varepsilon^2 \text{Li}_2(1-\kappa) + O(\varepsilon^3) \right]. \tag{37}
\]

The integral of the soft-collinear counterterm, Eq. \[\text{[19]} \], reads

\[
\int \left[ dp^{(1)} \right] C_{ir} S_r = a_\varepsilon C_{ir} S_r(m, \varepsilon) \mathcal{F}_2 \mathcal{M}^{(0)}_{m+1}^2 \tag{38}
\]

with

\[
a_\varepsilon C_{ir} S_r(m, \varepsilon) = 8\pi\alpha_s \mu^{2\varepsilon} \int \left[ dp^{(1)} \right] \frac{1}{s_{ir}} \frac{2 \varepsilon}{2 \varepsilon} \tag{39}
\]

Performing the integrations, we obtain

\[
C_{ir} S_r(m, \varepsilon) = \left[ \frac{m(1-\varepsilon)(1-2\varepsilon)}{\varepsilon^2} + 2 \right] \times B(1-\varepsilon, 1-\varepsilon) B(1-2\varepsilon, m(1-\varepsilon)). \tag{40}
\]

In order to show that the integrated subtraction term \( \int \mathcal{A}_1 |\mathcal{M}_{m+2}^{(0)}|^2 \) has the correct pole structure \[\text{[10]} \], we exploit colour conservation in Eqs. \[\text{[2a]} \] and \[\text{[38]} \]. Indeed, using \( T_i^2 = -\sum_{k \neq k} T_i T_k \), we can combine the soft and soft-collinear subtractions in Eqs. \[\text{[2a]} \] and \[\text{[38]} \].

\[
\int \left( \sum_{r \neq r, i \neq k} S_r - \sum_{r, i \neq r} C_{ir} S_r \right) = \sum_{r, i \neq r} \left( S_{ik} + C_{ir} S_r \right) |\mathcal{M}_{m+1,(i,k)}|^2, \tag{41}
\]

\[
\int \left( \sum_{r \neq r, i \neq k} S_r - \sum_{r, i \neq r} C_{ir} S_r \right) = \sum_{r, i \neq r} \left( S_{ik} + C_{ir} S_r \right) |\mathcal{M}_{m+1,(i,k)}|^2, \tag{42}
\]
Table 1: Definitions of the \( I_m^{(k)}(x; \varepsilon) \) functions, \((a)_m = \Gamma(m + a)/\Gamma(a)\) denotes the Pochhammer symbol.

| Function | Expression |
|----------|------------|
| \( I_m^{(-1)}(x; \varepsilon) \) | \( B(-2\varepsilon, 1 - \varepsilon)B(-\varepsilon, 1 - \varepsilon) \times F_1^{(2)}(-2\varepsilon, -\varepsilon, 2m(1 - \varepsilon) - 1 - 2\varepsilon, -2(m - 1)(1 - \varepsilon); 1 - 3\varepsilon, 1 - 2\varepsilon; 1 - x, 1) + B(1 - 2\varepsilon, 1 - \varepsilon)B(1 - \varepsilon, 1 - \varepsilon) F_1^{(2)}(1 - 2\varepsilon, 1 - \varepsilon, 2m(1 - \varepsilon) - 1 - 2\varepsilon, -2(m - 1)(1 - \varepsilon); 2 - 3\varepsilon, 2 - 2\varepsilon; 1 - x, 1) \) |
| \( I_m^{(0)}(x; \varepsilon) \) | \( B(1 - \varepsilon, 1 - \varepsilon)B(-\varepsilon, 2m(1 - \varepsilon))B(2m(1 - \varepsilon) - 1 - 2\varepsilon, -\varepsilon; 2m(1 - \varepsilon) - \varepsilon; 1 - x) \) |
| \( I_m^{(k)}(x; \varepsilon) \) | \( \sum_{i=0}^{k} \binom{k}{i} B(k - i + 1 - \varepsilon, 1 - \varepsilon)B(i - \varepsilon, 2m(1 - \varepsilon) + k - i) F_1(i - \varepsilon, 2m(1 - \varepsilon) - 1 - 2\varepsilon, k; 2m(1 - \varepsilon) + k - \varepsilon; 1 - x, -1), \quad k = 1, 2 \) |

Expanding \((S_{ik} + C_{ir}S_r)\) in powers of \( \varepsilon \), we find

\[
S_{ik} + C_{ir}S_r = \frac{1}{\varepsilon} \ln \frac{y_{ik}}{y_{iq}y_{kq}} + O(\varepsilon^0). \tag{42}
\]

We can combine these contributions with the collinear functions in Eqs. \((31) - (33)\) after using colour conservation in Eq. \((25)\) and find that the \( \ln y_{ij}/\varepsilon \) terms in Eqs. \((31) - (33)\) and Eq. \((42)\) exactly cancel and the known structure of the one-loop squared matrix elements is reproduced.

4. Conclusion

We outlined a subtraction scheme for computing cross sections at NNLO accuracy using the known singly- and doubly-singular limits of squared matrix elements. A way of disentangling these overlapping limits was presented in \([34]\). Here we discussed how to define the singly-singular subtraction terms. We presented exact phase space factorisations that allow us the integration of the singular factors and the results of these integrations. We have also coded Eq. \((3)\) for the case when \( d\sigma^{\text{RR}}_{m+2} \) is the fully differential cross section for the process \( e^+e^- \rightarrow q\bar{q}ggg \) \((m = 3)\) and \( J_i \) defines the \( C \)-parameter. We found that the integral of \( d\sigma_n^{\text{NNLO}}_{m+2} \) is finite and integrable in four dimensions using standard Monte Carlo methods.

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