Higher-dimensional Hasimoto transform for vortex membranes: counterexamples and generalizations

Boris Khesin* and Cheng Yang†

Abstract

The Hasimoto transform takes the binormal equation to the NLS and barotropic fluid equations. We show that in higher dimensions its existence would imply the conservation of the Willmore energy in skew-mean-curvature flows and present a counterexample to the latter for vortex membranes. Furthermore, such examples, based on products of spheres, provide new explicit solutions of the localized induction approximation for the higher-dimensional incompressible Euler equation that collapse in finite time.

These (counter)examples imply that there is no straightforward generalization to higher dimensions of the 1D Hasimoto transform. We derive its replacement, the evolution equations for the mean curvature and the torsion form for membranes, thus generalizing the barotropic fluid and Da Rios equations.

Contents

1 Introduction 2

2 Skew-mean-curvature flows 3
  2.1 The vortex filament equation 3
  2.2 Higher-dimensional generalization of the 1D binormal flow 4

3 (Non)invariance of the Willmore energy 5
  3.1 Motivation: the Hasimoto and Madelung transforms 5
  3.2 Counterexamples: Clifford tori and sphere products in binormal flows 7

4 Torsion forms and torsion vector fields for membranes 9

5 Generalized Da Rios equations 10
  5.1 Gradient of the Willmore energy 10
  5.2 Evolution of the Willmore energy in the skew-mean-curvature flows 12
  5.3 The continuity equation and generalized Da Rios equations 14

References 15

*Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada; e-mail: khesin@math.toronto.edu
†Department of Mathematics and Statistics, McMaster University, Hamilton, ON L8S 4K1, Canada, and the Fields Institute, Toronto, ON M5T 3J1, Canada; e-mail: yangc74@math.mcmaster.ca
1 Introduction

The vortex filament equation describes the motion of a curve in $\mathbb{R}^3$ under the binormal flow: each point on the curve is moving in the binormal direction with a speed equal to the curvature at that point. This equation is a “local” approximation of the 3D Euler equation for vorticity supported on a curve. It is known that the filament equation is a Hamiltonian equation, whose Hamiltonian function is the length of the curve.

The skew-mean-curvature flow is a natural higher-dimensional generalization of the 1D binormal flow \cite{14, 7}: instead of curves in $\mathbb{R}^3$, one traces the evolution of codimension 2 submanifolds in $\mathbb{R}^n$ (called vortex membranes), where the velocity of each point on the membrane is given by the skew-mean-curvature vector. The latter is the mean curvature vector to the membrane rotated in the normal plane by $\pi/2$. Similar to the filament equation, the skew-mean-curvature flows are Hamiltonian with respect to the so-called Marsden-Weinstein symplectic structure and the Hamiltonian functional given the volume of the membrane. Unlike the 1D case, such flows are apparently nonintegrable.

For a curve moving according to the binormal flow, its curvature $\kappa$ and torsion $\tau$ have interesting properties. In 1906 Da Rios derived the evolution equations for $\kappa$ and $\tau$. At that time, he was the first mathematician to bring the idea of localized induction approximation (LIA) of the Euler equation of an ideal fluid and use it to study the vortex dynamics. Da Rios’ work did not draw much attention when it was published, and it is known today mostly thanks to his advisor Levi-Civita, who introduced, promoted, and extended Da Rios’ work (see \cite{12} for a historical survey of the Da Rios equations). The LIA method was reconsidered in the 1960s and Betruchov \cite{1} rediscovered the Da Rios equations in 1965.

The Da Rios equations also appear in several other contents \cite{2}. For example Lakshmanan \cite{10} derived the Da Rios equations when studying the 1-dimensional classical spin system. By considering the evolution of the density $\rho = \kappa^2$ and the velocity $v = 2\tau$ one obtains the equations of barotropic-type (quantum) 1D fluids. Furthermore, in 1972 Hasimoto discovered a transformation allowing a complex-valued wave function $\psi$ from the pair of real functions $(\kappa, \tau)$ such that this wave function satisfies the nonlinear Schrödinger equation, see Figure 1 for the relations between these equations.

A natural question to ask is whether the higher-dimensional skew-mean-curvature flow possesses similar relations to barotropic- and Schrödinger-type equations, as well as what are its implications for the Euler hydrodynamics. Finding a higher-dimensional version of the Hasimoto transform was a folklore problem for quite a while, and various mentioning of that can be found, e.g. in \cite{5, 7, 13, 14}. It is showed in \cite{15} that the Gauss map of the SMCF satisfies a Schrödinger flow equation. A key observation proposed in \cite{8} was that to construct a higher dimensional generalization of the Hasimoto transformation one needs to prove a conservation law for the Willmore energy. Namely, the conjectural invariance of the Willmore energy would imply the simplest of the two barotropic fluid equations, the continuity equation, which is a necessary condition for the existence of a Hasimoto transformation.

In the present paper, we give a counterexample to the energy invariance conjecture by describing explicitly the motion of Clifford tori under the skew-mean-curvature flow and show that their Willmore energy is not conserved. Essentially, these counterexamples imply that there is no straightforward generalization of the Hasimoto transform to relate the binormal and barotropic (and hence Schrödinger) equations, and it must be necessarily complicated. Some of the presented explicit solutions of the LIA for the Euler equation exist for finite time (the simplest case is the motion of a three-dimensional vortex membrane in $\mathbb{R}^5$). This could shed some light on the singularity problem for the higher-dimensional Euler equation, as the
skew-mean-curvature flow is an approximation of the Euler equation for vorticity supported on a membrane. (To the best of our knowledge, it is the first example of an explicit solution of the LIA existing for finite time.) Finally, we introduce a natural generalization of the torsion for codimension 2 membranes and derive the evolution equations for the mean curvature and the torsion form, thus replacing equations of a barotropic-type fluid in Hasimoto transform and generalizing the Da Rios equations. These counterexamples emphasize the difference between the 1D and higher-dimensional skew-mean-curvature flows and might be particularly useful to prove the vortex filament conjecture for membranes, cf. [5].

Figure 1: Diagram of relations between equations in 1D and in higher dimensions

Acknowledgments. We are indebted to R. Jerrard and B. Shashikanth for many fruitful discussions. B.K. was partially supported by an NSERC research grant. A part of this work was done while C.Y. was visiting the Fields Institute in Toronto and the Instituto de Ciencias Matemáticas (ICMAT) in Madrid. C.Y. is grateful for their supports and kind hospitality.

2 Skew-mean-curvature flows

2.1 The vortex filament equation

Consider the space of (nonparametrized) knots $S\in\mathbb{R}^3$, which is the set of images of all smooth embeddings $\gamma:S^1\rightarrow\mathbb{R}^3$.

Definition 2.1. The vortex filament equation is

$$\dot{\gamma} = \gamma' \times \gamma'',$$

where $\gamma' := \dot{\gamma}/\dot{s}$ with respect to the arc-length parameter $s$ of the curve $\gamma$. Alternatively, the filament equation can be rewritten in the binormal form as $\dot{\gamma} = \kappa b$, where, respectively, $\kappa$ is the curvature and $b = t \times n$ is the binormal vector, the cross-product of the the tangent and normal unit vectors, at the corresponding point of the curve $\gamma$. Thus according to this equation each point on the curve $\gamma$ is moving in the binormal direction with a speed equal to the curvature at that point.
The space of knots $\mathcal{K}$ is equipped with a natural symplectic structure.

**Definition 2.2.** Let $\gamma \in \mathcal{K}$ be an oriented space curve in $\mathbb{R}^3$, then the Marsden-Weinstein symplectic structure $\omega^{MV}$ on the space $\mathcal{K}$ is given by

$$
\omega^{MV}(\gamma)(u,v) = \int_{\gamma} i_u i_v \mu = \int_{\gamma} \mu(u,v,\gamma') \, ds,
$$

(2)

where $u$ and $v$ are two vector fields attached to $\gamma$, and $\mu$ is the volume form in $\mathbb{R}^3$.

**Remark 2.3.** The definition of the Marsden-Weinstein symplectic structure $\omega^{MV}$ implies that it does not depend on the parametrization of the curve $\gamma$, and hence one can take any parameter $s$ in the equation (2).

It is known that the filament equation is Hamiltonian with respect to the Marsden-Weinstein symplectic structure on the space $\mathcal{K}$, the corresponding Hamiltonian function is the length functional $L = \int_{\gamma} |\gamma'(s)| \, ds$ of the curve.

The vortex filament equation also serves as an approximation for the 3D incompressible Euler equation for the vorticity confined to the curve $\gamma$, where only local interaction is taken into account [2].

### 2.2 Higher-dimensional generalization of the 1D binormal flow

The higher-dimensional generalization of the 1D binormal flow is called the skew-mean-curvature flow, and it is defined as follows:

**Definition 2.4.** Let $P^n \subset \mathbb{R}^{n+2}$ be a codimension 2 membrane (i.e., a compact oriented submanifold of codimension 2 in the Euclidean space $\mathbb{R}^{n+2}$), the skew-mean-curvature flow is described by the equation:

$$
\partial_t P(p) = -J(H(p)),
$$

(3)

where $p \in P$, $H(p)$ is the mean curvature vector to $P$ at the point $p$, $J$ is the operator of positive $\pi/2$ rotation in the normal space $N_p P$ to $P$.

The skew-mean-curvature flow (3) is a natural generalization of the binormal equation: in dimension $n = 1$ the mean curvature vector of a curve $\gamma$ at a point is $H = \kappa \mathbf{n}$, where $\kappa$ is the curvature of the curve $\gamma$ at that point, hence the skew-mean-curvature flow becomes the binormal equation: $\partial_t \gamma = -J(\kappa \mathbf{n}) = \kappa \mathbf{b}$. It was studied for codimension 2 membranes in $\mathbb{R}^4$ in [14] and in any dimension in [7].

It turns out that on the infinite-dimensional space $\mathfrak{M}$ of codimension 2 membranes, one can also define the Marsden-Weinstein symplectic structure in a similar way:

**Definition 2.5.** The Marsden-Weinstein symplectic structure $\omega^{MV}$ on the space $\mathfrak{M}$ of codimension 2 membranes is

$$
\omega^{MV}(P)(u,v) = \int_P i_u i_v \mu,
$$

(4)

where $u$ and $v$ are two vector fields attached to the membrane $P \in \mathfrak{M}$, and $\mu$ is the volume form in $\mathbb{R}^{n+2}$.

Define the Hamiltonian functional $L(P) := \text{vol}(P)$ on the space $\mathfrak{M}$ which associates the $n$-dimensional volume to a compact $n$-dimensional membrane $P \subset \mathbb{R}^{n+2}$.
Proposition 2.6. The skew-mean-curvature flow \( \mathfrak{M} \) is the Hamiltonian flow on the membrane space \( \mathfrak{M} \) equipped with the Marsden-Weinstein structure and with the Hamiltonian given by the volume functional \( L \).

Proof. In a nutshell, the Marsden-Weinstein symplectic structure is the averaging of the symplectic structures in all 2-dimensional normal planes \( N_pP \) to \( P \), hence the skew-gradient for any functional on submanifolds \( P \) is obtained from its gradient field attached at \( P \subset \mathbb{R}^{n+2} \) by applying the \( \pi/2 \)-rotation operator \( J \). On the other hand, the fact that minus the mean curvature vector field \(-H\) is the gradient for the volume functional \( L(p) \) is well-known. Hence the Hamiltonian field on \( \mathfrak{M} \) for the Hamiltonian functional \( L(p) \) is given by \(-JH(p)\) at any point \( p \in P \).

3 (Non)invariance of the Willmore energy

3.1 Motivation: the Hasimoto and Madelung transforms

In this section we describe relations between three different avatars of the skew-mean-curvature flows in one- and higher-dimensional settings. These relations motivate the conjecture on the Willmore energy conservation.

In [4], Hasimoto introduced the following transformation.

Definition 3.1. To a parametrized curve \( \gamma : \mathbb{R} \to \mathbb{R}^3 \) with curvature \( \kappa \) and torsion \( \tau \), the Hasimoto transformation assigns the wave function \( \psi : \mathbb{R} \to \mathbb{C} \) according to the formula

\[
(k(s), \tau(s)) \mapsto \psi(s) = \kappa(s)e^{i\int_{s_0}^s \tau(x) \, dx},
\]

where \( s_0 \) is some fixed point on the curve. (The ambiguity in the choice of \( s_0 \) defines the wave function \( \psi \) up to a phase.)

This map takes the vortex filament equation (1) to the 1D nonlinear Schrödinger (NLS) equation:

\[
i\dot{\psi} + \psi'' + \frac{1}{2} |\psi|^2 \psi = 0
\]

for \( \psi : \mathbb{R} \to \mathbb{C} \), see e.g. [2]. On the other hand, by considering separately the curvature \( \kappa(t, \cdot) \) and torsion \( \tau(t, \cdot) \) of the curve \( \gamma(t, \cdot) \in \mathbb{R}^3 \) moving by the binormal flow, the evolution of \( \kappa \) and \( \tau \) satisfies the following system of equations discovered by Da Rios [3]:

\[
\begin{aligned}
\dot{\kappa} + 2\kappa'\tau + \kappa\tau' &= 0, \\
\dot{\tau} + 2\tau'\tau - \left(\frac{\kappa''}{\kappa} + \frac{\kappa^2}{2}\right)' &= 0.
\end{aligned}
\]

By introducing the density \( \rho = \kappa^2 \) and the velocity \( v = 2\tau \), the Da Rios equations turn into the following system of compressible fluid equations:

\[
\begin{aligned}
\dot{\rho} + \text{div}(\rho v) &= 0, \\
\dot{v} + vv' - \left(2\sqrt{\rho} + \rho\right)' &= 0.
\end{aligned}
\]

One could ask what part of the above can be generalized to higher dimensions. It turns out that long before the discovery of the Hasimoto transform, Madelung [11] gave a hydrodynamical formulation of the Schrodinger equation in 1927, which is called the Madelung transform.
Definition 3.2. Let $\rho$ and $\theta$ be real-valued functions on an $n$-dimensional manifold $M$ with $\rho > 0$. The Madelung transform is the mapping $\Phi : (\rho, \theta) \mapsto \psi$ defined by
\[ \psi = \sqrt{\rho}e^{i\theta}. \] (8)

The Madelung transform maps the system of equations for a barotropic-type fluid to the Schrödinger equation. More specifically, let $(\rho, \theta)$ satisfy the following barotropic-type fluid equations:
\[
\begin{align*}
\dot{\rho} + \text{div}(\rho v) &= 0, \\
\dot{v} + \nabla_v v + \nabla \left( 2V + 2f(\rho) - \frac{2\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) &= 0
\end{align*}
\] (9)
with potential velocity field $v = \nabla \theta$, and functions $V : M \to \mathbb{R}$ and $f : (0, \infty) \to \mathbb{R}$. Then the complex-valued function $\psi = \Phi(\rho, \theta)$ obtained by the Madelung transform satisfies the nonlinear Schrödinger equation
\[ i\dot{\psi} = -\Delta \psi + V\psi + f(|\psi|^2)\psi. \] (10)

One can see that for $V = 0$ and $f(z) = -z/2$ the Madelung transform in the 1D case reduces to the Hasimoto transform \cite{Hasimoto}. A corollary from this hydrodynamic point of view is that the “total mass” $\int_\gamma k^2$ of a compact curve $\gamma$ is preserved under the binormal flow: indeed, the curvature density $\rho = k^2$ satisfies the continuity equation.

It is natural to ask if there exists an analogue of the Hasimoto map, which can send an $n$-dimensional binormal equation \cite{Hasimoto} to an NLS-type equation for any dimension $n$ \cite{Hasimoto}? Note that the Madelung transform provides an identification of the NLS and the barotropic equations in any dimension. Thus a search for a higher-dimensional Hasimoto analogue can be reduced to a search for a relation of the skew-mean-curvature flow and barotropic-type fluid equations \cite{Hasimoto} in higher-dimensions.

In view of the binormal equation \cite{Hasimoto} and continuity equations \cite{Hasimoto}-\cite{Hasimoto}, the square of the mean curvature vector $|H|^2$ can be regarded as a natural analogue of the density $\rho$. Therefore an analogue of the total mass of the fluid is the Willmore energy:

Definition 3.3. For an immersed submanifold $F : \Sigma^k \to \mathbb{R}^n$, the Willmore energy is defined as
\[ W(F) = \int_\Sigma |H(F(p))|^2 \, dvol_g = \int_{F(\Sigma)} |H(p)|^2 \, dvol_{g_e}, \] (11)
where $g = F^*g_e$ denotes the pull-back metric of the Euclidean metric $g_e$ on $\mathbb{R}^n$ and $H$ is the mean curvature vector at point $p = F(q)$ on the submanifold $F(\Sigma) \subset \mathbb{R}^n$.

Assuming the existence of a relation between the skew-mean-curvature flow and a barotropic fluid, one arrives at the following conjecture:

Conjecture 3.4. \cite{Hasimoto} For a codimension 2 submanifold $F_t : \Sigma^k \to \mathbb{R}^{n+2}$ moving by the skew-mean-curvature flow $\dot{q} = -JH(q)$ for $q \in \Sigma$ the following equivalent properties hold:

i) its Willmore energy $W(F_t)$ is invariant,

ii) its square mean curvature $\rho = |H|^2$ evolves according to the continuity equation
\[ \dot{\rho} + \text{div}(\rho v) = 0 \]
for some vector field $v$ on $\Sigma$. 

Remark 3.5. The equivalence of the two statements is a consequence of Moser’s theorem: if the total mass on a surface is preserved, the corresponding evolution of density can be realized as a flow of a time-dependent vector field.

Proposition 3.6. Conjecture 3.4 is true in dimension 1.

Proof. For a curve \( \gamma \subset \mathbb{R}^3 \), the conservation of the Willmore energy means the time invariance of the integral \( \mathcal{W}(\gamma) = \int \kappa^2 \, ds \) or, equivalently, in the arc-length parameterization, of the integral \( \int |\gamma''|^2 \, ds \). The latter invariance follows from this straightforward computation:

\[
\frac{d}{dt} \mathcal{W}(\gamma) = 2 \int_{\gamma} (\dot{\gamma}''', \gamma''') \, ds = -2 \int_{\gamma} (\dot{\gamma}'', \gamma''') \, ds = -2 \int_{\gamma} ((\gamma' \times \gamma''')', \gamma''') \, ds = 0.
\]

However, it turns out that unlike the 1D case, the Willmore energy is not necessarily an invariant in dimension \( n \geq 2 \), and this provides an interesting distinction between the 1D and higher-dimensional binormal flows.

3.2 Counterexamples: Clifford tori and sphere products in binormal flows

It turns out, for submanifolds of dimension \( n \geq 2 \), Conjecture 3.4 does not necessarily hold. Here we describe a family of counterexamples to the conjecture by considering the evolution of the Clifford tori and their generalizations in the skew-mean-curvature flow.

Theorem 3.7. Let \( F : \Sigma = S^m(a) \times S^l(b) \to \mathbb{R}^{m+1} \times \mathbb{R}^{l+1} = \mathbb{R}^{m+l+2} \) be the product of two spheres of radiuses \( a \) and \( b \). Then the evolution \( F_t \) of this surface \( \Sigma \) in the binormal flow is the product of spheres \( F_t(\Sigma) = S^m(a(t)) \times S^l(b(t)) \) at any \( t \) with radiuses changing monotonically according to the ODE system:

\[
\begin{align*}
\dot{a} &= -l/b, \\
\dot{b} &= +m/a.
\end{align*}
\]

The corresponding Willmore energy is not preserved for any initial values of \( a \) and \( b \), and it explicitly changes as

\[
\mathcal{W}(F_t) = C_{m,l} \left( \frac{m^2}{a(t)^2} + \frac{l^2}{b(t)^2} \right) \cdot \text{vol}(\Sigma)
\]

for a constant \( C_{m,l} \) and \( \text{vol}(\Sigma) = \text{vol}(F_1(\Sigma)) := a^m b^l \).

Corollary 3.8. The Willmore energy of the Clifford torus \( F : \mathbb{T}^2 = S^1(a) \times S^1(b) \to \mathbb{R}^4 \) evolves in the binormal flow as follows:

\[
\mathcal{W}(F_t) = 4\pi^2 \left( \frac{b(t)}{a(t)} + \frac{a(t)}{b(t)} \right) = 4\pi^2 \left( \frac{a}{b} e^{2t/(ab)} + \frac{a}{b} e^{-2t/(ab)} \right).
\]

Corollary 3.9. In the general case of sphere products \( \Sigma = S^m(a) \times S^l(b) \) the radiuses of \( F_t(\Sigma) \) change as follows: \( a(t) = ae^{-lt/(ab)} \) and \( b(t) = be^{mt/(ab)} \) for \( m = l \) and

\[
a(t) = a^{m/(m-l)} (a - (l - m)b^{-1}) t^{l/(l-m)} \text{ and } b(t) = b^{l/(l-m)} (b + (m - l)a^{-1}t) t^{m/(m-l)},
\]

for \( m \neq l \) and initial conditions \( a(0) = a, b(0) = b \).
Remark 3.10. For $0 < m < l$ (e.g., $m = 1, l = 2$ in $\mathbb{R}^5$) the corresponding solution $F_t$ exists only for finite time and collapses at $t = ab/(l - m)$. Since the skew-mean-curvature flow is the localized induction approximation of the Euler equation, this explicit solution might be useful to study the Euler singularity problem in higher dimensions. Note also that the odd-dimensional Euler equation has fewer invariants (generalized helicities) than the even-dimensional one (generalized enstrophies). The existence of many invariants help control solutions, so it is indicative that the first example with a finite life-span occurs in odd 5D.

Proof. For a point $q = (q_1, q_2) \in S^m(a) \times S^l(b) \to \mathbb{R}^{m+1} \times \mathbb{R}^{l+1}$, let $n_1$ and $n_2$ be the outer unit normal vectors to the corresponding spheres at the points $q_1$ and $q_2$ respectively. Then the mean curvature vectors of $S^m(a)$ and $S^l(b)$ as hypersurfaces in $\mathbb{R}^{m+1}$ and $\mathbb{R}^{l+1}$ are $-\frac{m}{a} n_1$ and $-\frac{l}{b} n_2$ respectively. Therefore the mean curvature vector of $F : S^m(a) \times S^l(b) \to \mathbb{R}^{m+l+2}$ is $H = -\frac{m}{a} n_1 - \frac{l}{b} n_2$, and the skew-mean-curvature vector is $-JH = -\frac{m}{a} n_1 + \frac{l}{b} n_2$.

This implies that for the skew-mean-curvature flow $\hat{c}_t q = -JH(q)$ given by the above linear combination of the normals on the product of spheres, $\Sigma_t$ remains the product of two spheres $S^m(a(t)) \times S^l(b(t))$ for all times, where one of the spheres is shrinking, while the other is expanding.

The explicit form of the $-JH$ vectors implies the system of ODEs \([12]\) on the evolution of radii. Rewriting this as one first order ODE one can solve this explicitly, as in Corollary 3.9. The system \([12]\) is Hamiltonian on the $(a, b)$-plane with the Hamiltonian function given by

$$H(a, b) := \ln(a^m b^l),$$

which is the logarithm of the volume of the product of two spheres: $\text{vol}(\Sigma) = Ca^m b^l$. (Note that the invariance of this Hamiltonian is consistent with conservation of the volume of $\Sigma$, as the latter is the Hamiltonian function of the skew-mean-curvature flow.)

To be invariant, the Willmore energy has to be a function of $\text{vol}(\Sigma)$ as well. But we obtain

$$W(F_t) = \int_{\Sigma_t} |H|^2 d\text{vol}_g = \left( \frac{m^2}{a(t)^2} + \frac{l^2}{b(t)^2} \right) \cdot \text{vol}(\Sigma_t) = C_{m, l} \left( \frac{m^2}{a(t)^2} + \frac{l^2}{b(t)^2} \right) \cdot a^m b^l,$$

where $C_{m, l}$ is a constant depending on the dimensions $m, l$. One observes that factor $(m^2/a(t)^2 + l^2/b(t)^2)$ in the Willmore energy cannot be a function of the area $a^m b^l$, hence $W(F_t)$ is not preserved.

Remark 3.11. One can give a simple parametrization to a Clifford torus and derive explicitly its second fundamental form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a} n_1 & 0 \\ 0 & -\frac{1}{b} n_2 \end{bmatrix}.$$

This example might be particularly interesting to prove the filament conjecture for membranes, cf. [53] [54].

Remark 3.12. We will quantify the measure of noninvariance of the Willmore energy in the skew-mean-curvature flow by deriving (in Section 3.3) the continuity equation with source term governing the density $\rho = |H|^2$ of the mean curvature:

$$\hat{c}_t \rho + \text{div}(\rho \chi) = -2g^{ik} g^{jl} (A_{ij}, H) (A_{kl}, JH),$$

where $A_{ij}$ are the second fundamental forms in local coordinates, $(g^{ij})$ is the inverse matrix of the induced metric $(g_{ij})$, and $\chi(q) = 2g^{ij} \left( \nabla_j \frac{H}{\sqrt{H}}, \frac{H}{\sqrt{H}} \right) e_i$ is the torsion vector field discussed in the next section. Here and below we assume the sum over repeated indices.
4 Torsion forms and torsion vector fields for membranes

While torsion is a classical intrinsic notion in Riemannian geometry, for codimension 2 membranes one can introduce a natural torsion of their embedding into the ambient Euclidean space, similar to that of curves in $\mathbb{R}^3$. Namely, for such curves, according to the Frenet–Serret formulas, the curvature vector $\kappa n$ is described by its magnitude and the angle of rotation in the normal plane as a function of the curve parametrization. Similarly, for codimension 2 membranes, one can define the vector of the mean curvature $H$ in the normal plane, while its “angle of rotation” leads to the following definition of the torsion connection form in the $S^1$ (normal) bundle over the membrane.

For an immersed submanifold $F : \Sigma^n \to \mathbb{R}^{n+2}$ consider the principal $S^1$-bundle $N$ of unit normal vectors over $\Sigma$:

\[
\begin{array}{ccc}
S^1 & \longrightarrow & N \\
\pi & & \downarrow \\
\Sigma^n & \to & \\
\end{array}
\]

Figure 2: The $S^1$-bundle of unit normal vectors over $\Sigma$.

Let $H$ be the field of mean curvature vectors over $\Sigma$, and we assume that $|H| \neq 0$ everywhere (otherwise we pass to the open part $\Sigma_0 \subset \Sigma$ where $H$ is nonvanishing: our consideration is local). Then the normalized vectors $h := \frac{H}{|H|}$ define a smooth section of the $S^1$-bundle $N$.

**Definition 4.1.** The (normal) reference connection $A_0$ on $N$ is defined by setting the tangent space to $\Sigma$ to be its horizontal space. Then any other connection $A$ in $N$ can be expressed as a connection (and hence a 1-form) on the base by comparing it to the connection $A_0$.

Recall that for any principal $G$-bundle $\pi : P \to B$, all its connections $A \in \Omega^1(P, g)$ form an affine space. Upon fixing a reference connection $A_0 \in \Omega^1(P, g)$, any other connection $A \in \Omega^1(P, g)$ can be expressed via the difference $A - A_0 \in \Omega^1(B, g)$, which is a $g$-valued 1-form on the base. In the case of $S^1$, this difference becomes a real-valued 1-form.

**Definition 4.2.** The (mean) curvature connection $A$ on the principal $S^1$-bundle $N$ is defined by declaring the section $h := \frac{H}{|H|} : \Sigma \to \mathbb{R}^{n+2}$ be its horizontal section. The generalized torsion form of the submanifold $F : \Sigma \to \mathbb{R}^{n+2}$ is the 1-form $\tau = A - A_0 \in \Omega^1(\Sigma, \mathbb{R})$, where $A_0 \in \Omega^1(N, \mathbb{R})$ is the normal connection described above.

The torsion vector field $\chi := 2\tau^2$ is defined as metric dual to the (double) torsion form, i.e. for any vector $v \in T_q\Sigma$ one sets $(\chi, v) = 2\tau(v)$ at any point $q \in \Sigma$.

**Proposition 4.3.** The 2-form $-d\tau$ is equal to the normal curvature of the submanifold $F : \Sigma \to \mathbb{R}^{n+2}$.

**Proof.** The exterior covariant derivative in $N$ is just the exterior derivative, since $S^1$ is abelian. Hence the curvature of the connection $A$ is $\Omega = dA = d\tau + dA_0$. Furthermore, $A$ is a flat connection, since $\frac{H}{|H|}$ is a global section. We obtain that $d\tau = -dA_0$, which means $-d\tau$ coincides with the normal curvature, since $A_0$ is induced by the normal connection. 

**Proposition 4.4.** For a local frame $\{e_1, e_2, \cdots, e_n\}$ in the tangent space of $\Sigma$ an explicit expression for the torsion vector field in the corresponding local coordinates is $\chi = 2g^{ij}\left(\nabla_i \frac{H}{|H|} \cdot \frac{\partial H}{|H|}\right) e_i$, where $(g^{ij})$ is the inverse matrix of the metric $(g_{ij})$ induced on $\Sigma$ from $\mathbb{R}^{n+2}$.
Proof. For a vector $v \in T_q\Sigma$ and the section $h := \frac{H}{|H|}$, the tangent map $Dh : T_q\Sigma \to T_{h(q)}N$ maps $v$ to a vector $Dh(v)$ in the tangent space of the section $h$, then the normal component of $Dh(v)$ is equal to $\tau(v) = (A - A_0)(v)$.

For $e_i \in TX$, denote by $\nabla_i h$ the vector in the tangent space of the smooth section $h : \Sigma \to \mathbb{R}^{n+2}$ and $(\nabla_i h, Jh) Jh$ is its normal component. Let $v = v^i e_i$, then the normal component of $Dh(v)$ is $v^i (\nabla_i h, Jh) Jh$, hence the torsion form is

$$\tau(v) = v^i (\nabla_i h, Jh) = \frac{1}{2}(\chi, v),$$

where $\chi = 2g^{ij} \left( \nabla_i h, Jh \right) e_j$ is the torsion vector field. \hfill \Box

Remark 4.5. Proposition 4.3 emphasizes an important difference of the higher-dimensional and 1D cases: in higher dimensions the torsion form $\tau$ is not exact in general, which partially explains the absence of the Hasimoto transform: one cannot introduce the “phase” of the would-be wave function, i.e. the “angle of rotation” of the mean curvature vector $H$, as it depends not only on a point $q \in \Sigma$, but also on a path along the membrane $\Sigma$ from a reference point $q_0$ to $q$.

Furthermore, unlike the 1D case, in higher dimensions the density $\rho := |H|^2$ is not transported by the torsion vector field $\chi$ related to $\tau$, but satisfies the continuity equation with a source term, as we will see below.

5 Generalized Da Rios equations

The evolution of the codimension 2 membranes according to the skew-mean-curvature flow satisfies a system of equations on its mean curvature vector $H$ and generalized torsion form $\tau$. Here we derive those generalized Da Rios-type equations. Due to their similarity with the compressible fluid equations, we will call the equation on the mean curvature $H$ the continuity equation, while the evolution of the torsion form $\tau$ is the momentum equation.

5.1 Gradient of the Willmore energy

We start by deriving the gradient of the Willmore energy in any dimension, which is of independent interest. For this we generalize the derivation of the gradient of the Willmore functional done in [9] for 2-dimensional, compact immersed surfaces in $\mathbb{R}^m$ to the case of compact immersed submanifolds of any dimension.

More specifically, consider an immersed submanifold $F : \Sigma^n \to \mathbb{R}^{n+k}$. Recall that the Willmore energy is defined as

$$W(F) = \int_{\Sigma} |H|^2 \, d\text{vol}_g,$$

where $g = F^* g_e$ denotes the pull back metric of the Euclidean metric $g_e$ on $\mathbb{R}^{n+k}$ and $H$ is the corresponding mean curvature vector field.

In local coordinates $(x_1, ..., x_n)$ on the manifold $\Sigma$ the pull-back metric $g$ on $\Sigma$ is

$$g_{ij} = (\partial_i F, \partial_j F),$$

and the corresponding volume element is the $n$-form $d\text{vol} = \sqrt{\det g_{ij}} \, dx_1 \wedge \cdots \wedge dx_n$. \hfill \Box
We have the following splitting of the pull-back bundle $F^*T\mathbb{R}^n = \bigcup_{q\in\Sigma} T_{F(q)}\mathbb{R}^n$

$$T_{F(q)}\mathbb{R}^n = DF|_q(T_q\Sigma) \oplus N_p\Sigma,$$

where $DF$ is the tangent map of $F$. The second fundamental form $A_{ij} = (\partial_i \partial_j F)^\perp$ is the projection of the second derivatives of $F$ to the normal bundle $N_p\Sigma$. Then the mean curvature vector at any point is $H = g^{ij} A_{ij}$, where $(g^{ij})$ is the inverse matrix of the induced metric $(g_{ij})$.

Now we give the formula of the normal gradient of the Willmore energy.

**Theorem 5.1.** The normal part of the gradient of the Willmore energy is

$$\frac{1}{2} \nabla^\perp W = \Delta^\perp H + g^{ik} g^{jl} (A_{ij}, H) A_{kl} - \frac{1}{2} |H|^2 H,$$

where $\Delta^\perp = g^{ij} \nabla^\perp_i \nabla^\perp_j$ denotes the Laplacian in the normal bundle, and $\nabla^\perp_i = \nabla^\perp_{\partial_i}$ is the normal connection.

To prove this theorem, we need the following two lemmas, which we include for a self-contained proof.

**Lemma 5.2.** For a smooth family of immersions $F_t : \Sigma^n \to \mathbb{R}^{n+k}$ with $\partial_t F_t|_{t=0} = V$ be a normal variation along $F_t$, the time derivative of the volume element is

$$\partial_t \text{vol}_g = - (H, V) \text{ vol}_g.$$  

**Proof.** One has $\partial_t \det(g_{ml}) = (g^{ij} \partial_t g_{ij}) \det(g_{ml})$, and

$$\partial_t g_{ij} = (\partial_t \partial_i F, \partial_j F) + (\partial_i F, \partial_t \partial_j F) - (\partial_t \partial_i F, \partial_t \partial_j F) = -2 (A_{ij}, V).$$

From this we obtain

$$\partial_t \det(g_{ml}) = -2 g^{ij} (A_{ij}, V) \det(g_{ml}) = -2 (H, V) \det(g_{ml}).$$

Therefore,

$$\partial_t \sqrt{\det(g_{ml})} = \frac{1}{2 \sqrt{\det(g_{ml})}} \partial_t \det(g_{ml}) = -(H, V) \sqrt{\det(g_{ml})},$$

i.e.

$$\partial_t \text{vol}_g = - (H, V) \text{ vol}_g$$

Define the normal derivative $\partial_t^\perp H$ of the mean curvature vector $H$ as the projection of the time derivative $\partial_t H$ to the normal bundle to $\Sigma$.

**Lemma 5.3.** For a smooth family of immersions $F_t : \Sigma^n \to \mathbb{R}^{n+k}$ with a normal field $\partial_t F_t|_{t=0} = V$ along $F_t$, the normal time derivative $\partial_t^\perp H$ of $H$ is

$$\partial_t^\perp H = \Delta^\perp V + g^{im} g^{jl} (A_{ij}, V) A_{ml},$$

11
Proof. We know that $A_{ij} = (\partial_i \partial_j F)^\perp = \nabla_i^\perp \partial_j F$, hence

$$\partial_t^\perp A_{ij} = \nabla_i^\perp \partial_j \partial_t^\perp F = \nabla_i^\perp \partial_j V = \nabla_i^\perp \nabla_j^\perp V + \nabla_i^\perp ((\partial_j V, \partial_m F) g^{ml} \partial_l F)$$

$$= \nabla_i^\perp \nabla_j^\perp V - (A_{jm}, V) g^{ml} \partial_l F = \nabla_i^\perp \nabla_j^\perp V - (A_{jm}, V) g^{ml} A_{lt}.$$

Since $H = g^{ij} A_{ij}$, we have

$$\partial_t^\perp H = g^{ij} (\partial_t^\perp A_{ij}) + (\partial_t^\perp g^{ij}) A_{ij}$$

$$= g^{ij} (\nabla_i^\perp \nabla_j^\perp V - (A_{jm}, V) g^{ml} A_{lt}) + 2 g^{im} g^{jl} (A_{ml}, V) A_{ij}$$

$$= \Delta^\perp V + g^{im} g^{jl} (A_{ij}, V) A_{ml}.$$

Now we can complete the proof.

Proof of Theorem 5.1. Consider a smooth family of immersions $F_t : \Sigma^n \to \mathbb{R}^{n+k}$ with $\partial_t F_t|_{t=0} = V$ normal along $F_t$, then the time derivative of the Willmore energy is

$$\frac{d}{dt} W(F_t) = \int_{\Sigma} \partial_t (|H|^2 \, d\text{vol}_g) = 2 \int_{\Sigma} (\partial^\perp H, H) \, d\text{vol}_g + \int_{\Sigma} |H|^2 \partial_t d\text{vol}_g$$

$$= 2 \int_{\Sigma} (\Delta^\perp V, H) + g^{im} g^{jl} (A_{ij}, V) (A_{ml}, H) - \frac{1}{2} |H|^2 (H, V) \, d\text{vol}_g$$

$$= \int_{\Sigma} (\Delta^\perp V + g^{im} g^{jl} (A_{ij}, H) A_{ml} - \frac{1}{2} |H|^2 H, V) \, d\text{vol}_g.$$

Therefore, the normal gradient of the Willmore energy is

$$\frac{1}{2} \nabla^\perp W = \Delta^\perp H + g^{im} g^{jl} (A_{ij}, H) A_{ml} - \frac{1}{2} |H|^2 H.$$

\[ \square \]

5.2 Evolution of the Willmore energy in the skew-mean-curvature flows

Consider now a smooth family of immersions $F_t : \Sigma^n \to \mathbb{R}^{n+2}$ evolved by the skew-mean-curvature flow: $\partial_t F_t|_{t=0} = -JH$, where $J$ is the operator of rotation by $\pi/2$ in the positive direction in every normal space to $\Sigma$.

Proposition 5.4. The Willmore energy of $\Sigma$ changes in time in the binormal flow as follows:

$$\frac{d}{dt} W(F_t) = -2 \int_{\Sigma} (A_{ij}, H) (A_{ij}, JH) \, d\text{vol}_g.$$

Proof. Employing the formula of the gradient of the Willmore energy we obtain

$$\frac{d}{dt} W(F_t) = -2 \int_{\Sigma} (\nabla W, JH) \, d\text{vol}_g = -2 \int_{\Sigma} (\nabla W^\perp, JH) \, d\text{vol}_g$$

$$= -2 \int_{\Sigma} (\Delta^\perp H + g^{im} g^{jl} (A_{ij}, H) A_{ml} - \frac{1}{2} |H|^2 H, JH) \, d\text{vol}_g.$$

The third term $(\frac{1}{2} |H|^2 H, JH)$ is pointwise zero on $\Sigma$, since $H \perp JH$.

For the first term, we need the following lemma.

Lemma 5.5. $\nabla^\perp J = J \nabla^\perp$. 

12
Proof. Let’s prove that $\nabla^\perp JV = J\nabla^\perp V$ for an arbitrary unit normal vector field $V$. Note that $\{V, U = JV\}$ form a local orthonormal frame. Hence for any tangential vector field $X$, we have

$$J\nabla^\perp_X V = J(\partial_X V)^\perp = J((\partial_X V, V) V + (\partial_X V, U) U) = J(\partial_X V, U) U,$$

and

$$\nabla^\perp_X (JV) = \nabla^\perp_X (U) = (\partial_X U)^\perp = (\partial_X U, V) V + (\partial_X U, U) U = (\partial_X U, V) V = J(\partial_X V, U) U.$$

Therefore $J\nabla^\perp_X V = \nabla^\perp_X (JV)$. □

To complete the proof of the proposition, we integrate by parts the first term of $\frac{d}{dt}W(F_t)$:

$$-\int_{\Sigma} (\Delta^\perp H, JH)\ d\text{vol}_g = \int_{\Sigma} (\nabla^\perp H, \nabla^\perp JH)\ d\text{vol}_g,$$

and by Lemma 5.5 $\int_{\Sigma} (\nabla^\perp H, \nabla^\perp JH) = (\nabla^\perp H, J\nabla^\perp H) = 0$ pointwise on $\Sigma$, i.e., $-\int_{\Sigma} (\Delta^\perp H, JH)\ d\text{vol}_g$ vanishes on $\Sigma$. Thus we conclude that

$$\frac{d}{dt}W(F_t) = -2\int g^{im} g^{jl} (A_{ij}, H) (A_{ml}, JH)\ d\text{vol}_g = -2\int_{\Sigma} (A_i^l, H) (A_i^l, JH)\ d\text{vol}_g.$$

□

**Corollary 5.6.** The Willmore energy of a closed submanifold $\Sigma$ is invariant under the skew-mean-curvature flow, if and only if

$$\int_{\Sigma} (A_i^l, H) (A_i^l, JH)\ d\text{vol}_g = 0$$

for all times $t$.

**Remark 5.7.** For a 1-dimensional $\Sigma$, i.e. for vortex filaments $\gamma$, the second fundamental form reduces to the mean curvature of a curve: $A = H = \kappa$. Hence $(A, H) (A, JH) = 0$ at every point on $\Sigma$. However, for a higher-dimensional $\Sigma$, in general, the pointwise identity $(A_i^l, H) (A_i^l, JH) = 0$ does not hold on $\Sigma$, and hence the Willmore energy might not conserve.

**Remark 5.8.** For a Clifford torus the computation of $\int_{\Sigma^2} (A_{ij}, H) (A_{ij}, JH)\ d\text{vol}_g$ is straightforward: since

$$(A_{ij}, H) (A_{ij}, JH) = -\frac{1}{a^2} \frac{1}{ab} + \frac{1}{b^2} \frac{1}{ab} = -\frac{1}{a^3b} + \frac{1}{ab^3},$$

we obtain

$$\int_{\Sigma^2} (A_{ij}, H) (A_{ij}, JH)\ d\text{vol}_g = \int_0^{2\pi} \int_0^{2\pi} \left(-\frac{1}{a^3b} + \frac{1}{ab^3}\right) ab\, d\theta\, d\phi = 4\pi^2 \left(-\frac{1}{b^2} + \frac{1}{a^2}\right).$$

So if $a \neq b$ at time $t$, we have $\int_{\Sigma} (A_{ij}, H) (A_{ij}, JH)\ d\text{vol}_g \neq 0$. Furthermore, the tori with equal radiiuses $a = b$ do not form an invariant set, since under the skew-mean-curvature flow one radius of the Clifford torus is increasing, while the other is decreasing. Thus the Willmore energy is not preserved under the skew-mean-curvature flow for any initial values $a$ and $b$. 

13
5.3 The continuity equation and generalized Da Rios equations

Let $F_t : \Sigma^n \to \mathbb{R}^{n+2}$ be a codimension 2 vortex membrane moving by the skew-mean-curvature flow. Let $(x_1, x_2, \cdots, x_n)$ be local coordinates on $\Sigma^n$, and $\{e_1, e_2, \cdots, e_n\}$ the corresponding local frame in the tangent space.

According to Lemma 5.3, the time derivative of the square of the mean curvature is
\[
\partial_t |H|^2 = -2 \left( \Delta_H H + g^{ik} g^{jl} (A_{ij}, H) A_{kl}, JH \right). \tag{16}
\]

It turns out that the first term in the right-hand side can be expressed as the divergence of certain vector field.

**Lemma 5.9.** For a vector field $\sigma = (g^{ik} \nabla^i_k H, JH) e_i$ on the submanifold $\Sigma$ its divergence is
\[
\text{div}_\Sigma \sigma = (\Delta^i H, JH). \tag{17}
\]

**Proof.** Recall that for an arbitrary vector field $X = X^i e_i$ on $\Sigma$ its divergence is as follows:
\[
\text{div}_\Sigma X = \text{tr}(\nabla X) = g^{ij} \nabla_i X_j = \nabla^i X_i. \tag{18}
\]
Furthermore, define $T^i = g^{ik} \nabla^i_k H$. Then we have $\nabla^i_i T^i = \nabla^i_i g^{ij} \nabla^j_i H = \Delta^i H$. This implies that
\[
\text{div}_\Sigma \sigma = \nabla^i (T^i, JH) = \left( \nabla^i_i T^i, JH \right) + (T^i, \nabla^i_i JH) = (\Delta^i H, JH) + (g^{ij} \nabla^i_j H, \nabla^i_i JH) = (\Delta^i H, JH). \tag{19}
\]

Recall that the torsion form $\tau = \tau_i dx^i$ has components $\tau_i = \left( \nabla^i_i \frac{H}{|H|^2}, \frac{H}{|H|^2} \right)$ for the mean curvature vectors $H$ on a membrane $\Sigma$ with the second fundamental form $A_{ij}$. The corresponding torsion vector field is $\chi = 2g^{ij} \left( \nabla^i_i \frac{H}{|H|^2}, \frac{H}{|H|^2} \right) e_i$. An analogue of the equations of a barotropic-type fluid (9) related to vortex membranes in higher dimensions is the following system of equations, also generalizing the Da Rios system (4).

**Theorem 5.10.** The skew-mean curvature evolution of the membrane $\Sigma$ implies the following continuity equation with a source on the “curvature density” is $\rho = |H|^2$
\[
\partial_t \rho + \text{div}(\rho \chi) = -2g^{ik} g^{jl} (A_{ij}, H) (A_{kl}, JH), \tag{18}
\]
and the momentum equation on the torsion $\tau = \tau_i dx^i$
\[
\partial_t \tau_i + \nabla_i |\tau|^2 - \nabla_i \frac{\Delta|H|}{|H|^2} = -\nabla_i g^{mk} g^{jl} (A_{mj}, H)(A_{kl}, JH) \frac{A_{mk}, JH}{|H|^2} + \frac{\rho^k l}{|H|^2} \left( (A_{ik}, H) (\nabla_j JH, JH) - (A_{kl}, JH) (\nabla^i JH, H) \right). \tag{19}
\]

**Corollary 5.11.** The continuity equation (18) can be written in the form
\[
\partial_t \Delta^i H + 2g^{ij} \tau_i \nabla^i_j H + (\nabla^i \tau_i) H = -g^{ij} g^{jl} (A_{kl}, JH) A_{ij}. \tag{20}
\]
Proof. Equation (18) follows from (16) and Lemma 5.9, since
\[ \chi^2 g^{ij} \left( \nabla_j^H \frac{H}{|H|^2}, \frac{H}{|H|^2} \right) e_i = 2g^{ij} \frac{\nabla_j^H H, JH}{|H|^2} e_i. \]
Plugging \( \rho = |H|^2 \) into (18), we get
\[ 2(H, \partial_t H) + |H|^2 \text{div} \chi + 2(H, \nabla \chi H) = -2g^{ik} g^{jl} (A_{ij}, H) (A_{kl}, JH). \]
Then plugging in \( \chi = 2g^{ij} \tau_j e_i \) we obtain
\[ 2(H, \partial_t H) + 4(H, g^{ij} \tau_i \nabla_j H) + 2|H|^2 \nabla_i \tau_i = -2g^{ik} g^{jl} (A_{ij}, H) (A_{kl}, JH), \]
i.e.
\[ \partial_t^2 H + 2g^{ij} \tau_i \nabla_j^H + (\nabla_i \tau_i) H = -g^{ik} g^{jl} (A_{kl}, JH) A_{ij}. \]

The momentum equation is obtained by using \( \partial_t \left( \nabla_i H \frac{H}{|H|^2} \right) = \nabla_i \left( \partial_t H \frac{H}{|H|^2} \right) \) via a direct but tedious computation comparing the corresponding coefficients. \( \square \)

Remark 5.12. Recall that for the normalized mean curvature \( h := H/|H| \) the orthonormal frame \( \{h, Jh\} \) is a basis of the normal bundle to the membrane \( \Sigma \). The torsion form \( \tau = \tau_i dx^i \) for \( \tau_i = (\nabla_i^H h, Jh) \) measures how much this frame rotates when one moves along the tangent vector \( e_i \) on the surface \( \Sigma \). One can also compare Equation (20) with the continuity equation of the Da Rios system (6).

References

[1] R. Betchov. On the curvature and torsion of an isolated vortex filament. J. Fluid Mech. 22 (1965), 471-479.
[2] A. Calini. Integrable dynamics of knotted vortex filaments. Geometry, integrability and quantization, Softex, Sofia (2004), 11-50.
[3] L. S. Da Rios. Sul moto d’un liquido indefinito con un filetto vorticoso di forma qualunque. Rend. Circ. Mat. Palermo 22 (1906), 117-135.
[4] H. Hasimoto. A soliton on a vortex filament. Journal of Fluid Mechanics 51 (1972), 477-485.
[5] R. L. Jerrard. Vortex filament dynamics for Gross–Pitaevsky type equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1 (2002), no.4, 733-768.
[6] R. L. Jerrard, D. Smets. Vortex dynamics for the two-dimensional non-homogeneous Gross-Pitaevskii equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XIV (2015), 1-38.
[7] B. Khesin. Symplectic structures and dynamics on vortex membranes. Moscow Math. Journal 12(2) (2012), 413-434.
[8] B. Khesin, G. Misiolek and K. Modin. Geometry of the Madelung transform. preprint, 28pp., arXiv:1807.07172, (2018).
[9] E. Kuwert and R. Schätzle. Gradient flow for the Willmore functional. Communications in Analysis and Geometry 10(2) (2002), 307-339.
[10] M. Lakshmanan. Continuum spin system as an exactly solvable dynamical system. Phys. Lett. A 61 (1977), 53-54.
[11] E. Madelung. Quantentheorie in hydrodynamischer form. Zeitschrift Phys 40 (1927), 322-326.

[12] R. L. Ricca. Rediscovery of Da Rios equations. Nature 352 (1991), 561-562.

[13] J. A. Sanders, J. P. Wang. Integrable systems in $n$-dimensional Riemannian geometry. Moscow Math. Journal 3(4), (2003), 1369-1393.

[14] B. N. Shashikanth. Vortex dynamics in $\mathbb{R}^4$. J. Math. Phys. 53:1 (2012), 013103.

[15] C. Song. Gauss map of the skew mean curvature flow. Proc. Amer. Math. Soc. 145 (2017), 4963-4970.