Computing sums in terms of beta, polygamma, and Gauss hypergeometric functions

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Abstract
In the paper, by virtue of the binomial inversion formula, a general formula of higher order derivatives for a ratio of two differentiable function, and other techniques, the authors compute several sums in terms of the beta function and its partial derivatives, polygamma functions, the Gauss hypergeometric function, and a determinant. These results generalize known ones in combinatorics.

Keywords
Sum · Identity · Beta function · Polygamma function · Gauss hypergeometric function · Determinant · Binomial inversion formula · Derivative formula for a ratio of two differential functions

Mathematics Subject Classification
Primary 33C05; Secondary 05A10 · 11A25 · 11B65 · 33B15

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Dedicated to people facing and fighting COVID-19

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1 Background and motivations

In [14, Theorem 1] and [23, p. 80, Eq. (7.5)], it was obtained that

\[
\sum_{q=0}^{n} \binom{n}{q} (-1)^q \frac{1}{q+k} = \frac{1}{k^{k+n}} \quad (1.1)
\]

for \( k \geq 1 \) and \( n \geq 0 \). The binomial inversion formula in [4, p. 192, (5.48)] reads that

\[
g(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell f(\ell) \iff f(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell g(\ell). \quad (1.2)
\]

Applying (1.2) into (1.1) yields

\[
\sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{q+k} = \frac{k}{k+n} \quad (1.3)
\]

for \( k, n \geq 0 \). This can be rewritten as

\[
\sum_{q=0}^{n} \binom{n}{q} [(-1)^q B(k, q + 1)] = \frac{1}{k+n} \quad (1.4)
\]

for \( k \geq 1 \) and \( n \geq 0 \), where

\[
B(z, w) = \int_{0}^{1} t^{z-1} (1 - t)^{w-1} dt, \quad \Re(z), \Re(w) > 0
\]

denotes the classical Euler beta function. The beta function \( B(z, w) \) and the classical Euler gamma function

\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0
\]

have the relation

\[
B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(x), \Re(y) > 0.
\]

The logarithmic derivative \([\ln \Gamma(z)]' = \frac{\Gamma(z)}{\Gamma(z)} \psi(z)\) is denoted by \( \psi(z) \) and the derivatives \( \psi^{(k)}(z) \) for \( k \geq 0 \) are called polygamma functions. For very recent results on the beta, gamma, and polygamma functions, please refer to the papers [22,25–29] and closely related references therein.

In this paper, we will extend the identities (1.1), (1.3), and (1.4), compute generalized sums in terms of the beta function \( B(z, w) \) and its partial derivatives, polygamma functions \( \psi^{(k)}(z) \) for \( k = 0, 1 \), the Gauss hypergeometric function \( {}_{2}F_{1}(-n, z; 1 + z; -1) \), a determinant, and an identity.

2 Computing sums in terms of beta and polygamma functions

In this section, via replacing corresponding integers \( k \) by a complex variable \( \Re(z) \), we generalize identities (1.1), (1.3), and (1.4) in terms of the beta function and its partial derivatives, polygamma functions, and integrals as follows.
Theorem 2.1 Let $n \geq 0$ be an integer and $\Re(z) > 0$. Then

$$
\sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{q+z} = B(z, n+1) \tag{2.1}
$$

and

$$
\sum_{q=0}^{n} \binom{n}{q} [(-1)^q B(z, q+1)] = \frac{1}{z+n}. \tag{2.2}
$$

Proof Let

$$
f(t) = \sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{q+z} t^{q+z}, \quad t \geq 0.
$$

It is obvious that $f(0) = 0$ and

$$
f'(t) = \sum_{q=0}^{n} (-1)^q \binom{n}{q} t^{q+z-1} = t^{z-1} \sum_{q=0}^{n} (-1)^q \binom{n}{q} t^q = t^{z-1}(1-t)^n.
$$

Integrating on both sides of the above equality with respect to $t$ over $[0, 1]$ gives $f(1) = B(z, n+1)$. The formula (2.1) is thus proved.

The identity (2.2) follows from applying the binomial inversion formula (1.2) to (2.1). The proof of Theorem 2.1 is complete. $\square$

Remark 2.2 The identity (2.1) recovers [23, p. 82, Eq. (7.7)].

Theorem 2.3 Let $m, n \geq 0$ be an integer and $\Re(z) > 0$. Then

$$
\sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{(q+z)^{m+1}} = \frac{(-1)^m}{m!} \frac{\partial^m B(z, n+1)}{\partial z^m}
$$

and

$$
\sum_{q=0}^{n} (-1)^q \binom{n}{q} \frac{\partial^m B(z, n+1)}{\partial z^m} = \frac{(-1)^m m!}{(z+n)^{m+1}}.
$$

In particular, we have

$$
\sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{(q+z)^2} = \left[\psi(z+n+1) - \psi(z)\right] B(z, n+1),
$$

$$
\sum_{q=0}^{n} \binom{n}{q} \frac{(-1)^q}{(q+z)^3} = \frac{\left[\psi(z) - \psi(z+n+1)\right]^2 + \psi'(z) - \psi'(z+n+1)}{2} B(z, n+1),
$$

$$
\sum_{q=0}^{n} \binom{n}{q} (-1)^q [\psi(q+z+1) - \psi(z)] B(z, q+1) = \frac{1}{(z+n)^2},
$$

and

$$
\sum_{q=0}^{n} \binom{n}{q} (-1)^q \left[\psi(z) - \psi(q+z+1)\right]^2 + \psi'(z) - \psi'(q+z+1) B(z, q+1) = \frac{2}{(z+n)^3}.
$$
Proof These identities follow from differentiating with respect to \( z \) on both sides of (2.1) and (2.2) and employing partial derivatives
\[
\frac{\partial B(z, w)}{\partial z} = B(z, w)[\psi(z) - \psi(z + w)]
\]
and
\[
\frac{\partial^2 B(z, w)}{\partial z^2} = B(z, w)[(\psi(z) - \psi(z + w))^2 + \psi'(z) - \psi'(z + w)].
\]
The proof of Theorem 2.3 is complete.

Theorem 2.4 Let \( m, n \geq 0 \) be an integer and \( \Re(z) > 0 \). Then
\[
\sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z} = \left(\frac{-1}{m}\right)^m \int_{0}^{1} t^{z-1} (1 + t)^m (\ln t)^m dt
\]  
(2.3)
and
\[
\sum_{q=0}^{n} \binom{n}{q} \left(\frac{-1}{m}\right)^m \int_{0}^{1} t^{z-1} (1 + t)^q (\ln t)^m dt \right] = \frac{(-1)^m+n!}{(z+n)^{m+1}}.
\]  
(2.4)
Proof Let
\[ h(t) = \sum_{q=0}^{n} \frac{1}{q + z} \binom{n}{q} t^{q+z}, \quad t \geq 0. \]
It is apparent that \( h(0) = 0 \) and
\[ h'(t) = \sum_{q=0}^{n} \binom{n}{q} t^{q+z-1} = t^{z-1} \sum_{q=0}^{n} \binom{n}{q} t^q = t^{z-1} (1 + t)^n. \]
Integrating the above equation with respect to \( t \) over \([0, 1]\) arrives at
\[ h(1) = \int_{0}^{1} t^{z-1} (1 + t)^n dt. \]
Consequently, it follows that
\[ \sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z} = \int_{0}^{1} t^{z-1} (1 + t)^n dt. \]  
(2.5)
Further differentiating \( m \) times with respect to \( z \) on both sides of (2.5) leads to the formula (2.3).

The identity (2.4) follows from applying the binomial inversion formula (1.2) to (2.3). The proof of Theorem 2.4 is complete.

3 Computing a sum in terms of the Gauss hypergeometric function and a determinant

The Gauss hypergeometric function \( {}_2F_1(\alpha, \beta; \gamma; z) \) are defined by
\[
{}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}
\]
for $|z| < 1$, for complex numbers $\alpha, \beta \in \mathbb{C}$, for $\gamma \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, and for
\[
(c)_k = \prod_{\ell=0}^{k-1}(c + \ell) = \begin{cases} 
     c(c + 1) \cdots (c + k - 1), & k \geq 1 \\
     1, & k = 0
\end{cases}
\]
which is called the rising factorial of $c \in \mathbb{C}$.

In the above sections, we mainly consider summability of
\[
\sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z}, \quad n \geq 0, \quad \mathfrak{R}(z) > 0
\]
and their inversion formulas in terms of beta functions and integrals. In this section, we compute the sum
\[
\sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z}, \quad n \geq 0, \quad \mathfrak{R}(z) > 0
\]
in terms of the Gauss hypergeometric function $2F_1$, a determinant, and an identity.

**Theorem 3.1** Let $n \geq 0$ be an integer and $\mathfrak{R}(z) > 0$. Then
\[
\sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z} = \frac{2F_1(-n, z; 1+z; -1)}{z}. \tag{3.1}
\]

**Proof** In [3, p. 320, Item 8], it is listed that
\[
\int_0^{\mu} t^{z-1}(t + a)^{\lambda}(u - t)^{\mu-1} dt = a^\lambda u^{\mu+\lambda-1} B(\mu, z) \, 2F_1\left(-\lambda, z; \mu + z; -\frac{u}{a}\right) \tag{3.2}
\]
for $\mathfrak{R}(\mu), \mathfrak{R}(z) > 0$ and $|\arg \frac{u}{a}| < \pi$. Taking $u = \mu = a = 1$ and $\lambda = n$ in (3.2) leads to
\[
\int_0^{1} t^{z-1}(1+t)^{\mu} dt = \frac{2F_1(-n, z; 1+z; -1)}{z}, \quad \mathfrak{R}(z) > 0.
\]
Hence, it follows that
\[
2F_1(-n, z; 1+z; -1) = z \int_0^{1} t^{z-1}(1+t)^{\mu} dt = z \int_0^{1} t^{z-1} \sum_{q=0}^{n} \binom{n}{q} t^q dt
\]
\[
= z \sum_{q=0}^{n} \binom{n}{q} \int_0^{1} t^{q+z-1} dt = z \sum_{q=0}^{n} \binom{n}{q} \frac{1}{q+z}.
\]
The proof of Theorem 3.1 is complete. \qed

**Theorem 3.2** Let $n \geq 0$ be an integer and $\mathfrak{R}(z) > 0$. Then
\[
\sum_{q=0}^{n} \binom{n}{q} \frac{1}{q + z} = \frac{1}{z} \begin{pmatrix}
\binom{n+z}{2n} & 4 & 0 & \ldots & 0 & 0 & 0 \\
\binom{n+z}{2n} & 1 & 4 & \ldots & 0 & 0 & 0 \\
\binom{n+z}{2n} & 0 & 2 & 4 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\binom{n+z}{2n} & 0 & 0 & \cdots & n-2 & 4 & 0 \\
\binom{n+z}{2n} & 0 & 0 & \cdots & 0 & n-1 & 4 \\
\binom{n+z}{2n} & 0 & 0 & \cdots & 0 & 0 & n
\end{pmatrix}. \tag{3.3}
\]
where
\[
\langle c \rangle_k = \prod_{\ell=0}^{k-1} (c - \ell) = \begin{cases} 
  c(c - 1) \cdots (c - k + 1), & k \geq 1 \\
  1, & k = 0
\end{cases}
\]

is the falling factorial of \( c \in \mathbb{C} \).

**Proof** By virtue of the formulas

\[
\phantom{\textit{2F1}(\ldots)} = \frac{1}{(1-w)^{\alpha+1}(1+w)^{\beta}} \frac{d^n}{dw^n} \left[ (1-w)^n (1+w)^n \right]
\]

in [1, p. 561, Item 15.4.6] or [5, p. 442, Item 18.5.7], we obtain

\[
\phantom{\textit{2F1}(\ldots)} = \frac{(-1)^n}{(z+1)_n} \frac{1}{2^n n! (1-w)^\alpha (1+w)^\beta} \frac{d^n}{dw^n} \left[ (1-w)^n (1+w)^n \right]
\]

In [2, p. 40, Exercise 5], it is stated that the \( n \)-th derivative of the ratio \( \frac{u(t)}{v(t)} \) can be computed by

\[
\frac{d^n}{dt^n} \left[ \frac{u(t)}{v(t)} \right] = (-1)^n \frac{W_{(n+1),(n+1)}(t)}{v^{n+1}(t)},
\]

where \( u(t) \) and \( v(t) \neq 0 \) are differentiable functions, \( W_{(n+1),(n+1)}(t) \) denotes the determinant of the \((n+1) \times (n+1)\) matrix

\[
W_{(n+1),(n+1)}(t) = \left( U_{(n+1)}(t) \right) V_{(n+1)}(t),
\]

the elements of the \((n+1) \times 1\) matrix \( U_{(n+1)}(t) \) are \( u_{k,1}(t) = u^{(k-1)}(t) \) for \( 1 \leq k \leq n+1 \), and the elements of the \((n+1) \times n\) matrix \( V_{(n+1)}(n) \) are

\[
v_{ij}(t) = \begin{cases} 
  \binom{i-1}{j-1} v^{(i-j)}(t), & i-j \geq 0 \\
  0, & i-j < 0
\end{cases}
\]
for \(1 \leq i \leq n+1\) and \(1 \leq j \leq n\). This conclusion has been employed in the papers [6–13,15–18,20,21,24] and closely related references. Applying the formula (3.5) to \(u(w) = (w-1)^{a+z}\) and \(v(w) = w + 1\) results in

\[
\frac{d^n}{dw^n} \left[ \frac{(1-w)^{a+z}}{1+w} \right] = (-1)^{a+z} \frac{d^n}{dw^n} \left[ \frac{(w-1)^{a+z}}{w+1} \right] = \frac{(-1)^z}{(w+1)^{a+1}}
\]

\[
\begin{array}{c|cccccccc}
& n + z & 1 + w & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
(n+z)_{0} & (w-1)^{a+z} & 1 + w & 0 & 0 & \cdots & 0 & 0 & 0 \\
(n+z)_{1} & (w-1)^{a+z-1} & 1 + w & 0 & 0 & \cdots & 0 & 0 & 0 \\
(n+z)_{2} & (w-1)^{a+z-2} & 0 & 2 & 1 + w & \cdots & 0 & 0 & 0 \\
(n+z)_{3} & (w-1)^{a+z-3} & 0 & 0 & 0 & \cdots & 0 & n - 1 & 1 + w \\
\end{array}
\]

\[
\Rightarrow \frac{(-1)^z}{4^{a+1}} \sum_{k} \begin{array}{c|cccccccc}
& n + z & 2^{a+z} & 4 & 0 & \cdots & 0 & 0 & 0 \\
\hline
(n+z)_{0} & 2^{a-1+z} & 1 & 4 & 0 & \cdots & 0 & 0 & 0 \\
(n+z)_{2} & 2^{a-2+z} & 0 & 2 & 4 & \cdots & 0 & 0 & 0 \\
(n+z)_{3} & 2^{a-3+z} & 0 & 0 & 0 & \cdots & 0 & n - 2 & 4 \\
(n+z)_{4} & 2^{a-4+z} & 0 & 0 & 0 & \cdots & 0 & n - 1 & 4 \\
\end{array}
\]

\[
= \frac{2F_1(-n, z; 1 + z; -1)}{(z+1)n} = (-1)^n \frac{n!}{(z+1)n} \left[ 1 + \sum_{k=1}^{n} (-1)^k \frac{2^k}{k!} (n+z)_k \right].
\]  

**Theorem 3.3** Let \(n \geq 0\) be an integer and \(\Re(z) > 0\). Then

\[
2F_1(-n, z; 1 + z; -1) = (-1)^n \frac{n!}{(z+1)n} \left[ 1 + \sum_{k=1}^{n} (-1)^k \frac{2^k}{k!} (n+z)_k \right].
\]  

**Proof** Let \(n \geq 2\) and

\[
P_n = \begin{pmatrix}
\alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_3 & \beta_3 & \gamma_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_4 & 0 & \beta_4 & \gamma_4 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\
\alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\
\alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\
\alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n
\end{pmatrix} = (p_{ij}s)_{1 \leq i, j \leq n},
\]

where

\[
p_{ij}s = \begin{cases}
\alpha_i, & 1 \leq i \leq n, \; j = 1; \\
\beta_i, & 2 \leq i = j \leq n; \\
\gamma_i, & 1 \leq i = j - 1 \leq n - 1; \\
0, & \text{otherwise}.
\end{cases}
\]
Theorem 2.2 in [19] reads that the determinant $|P_n|$ of the tridiagonal matrix $P_n$ for $n \geq 2$ can be computed explicitly by

$$|P_n| = \alpha_1 \prod_{k=2}^{n} \beta_k - \sum_{k=2}^{n} (-1)^k \left( \prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n} \beta_m \right) \alpha_k. \quad (3.8)$$

Setting $\alpha_k = \frac{(n+z)_{k-1}}{k!}$ for $1 \leq k \leq n+1$, $\beta_k = k - 1$ for $2 \leq k \leq n+1$, and $\gamma_k = 4$ for $1 \leq k \leq n$ in (3.8) arrives at

$$|P_{n+1}| = \alpha_1 \prod_{k=2}^{n+1} \beta_k - \sum_{k=2}^{n+1} (-1)^k \left( \prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n+1} \beta_m \right) \alpha_k$$

$$= n! - \sum_{k=2}^{n+1} (-1)^k 4^{k-1} \frac{n!}{(k-1)!} \frac{(n+z)_{k-1}}{2^{k-1}}$$

$$= n! \left[ 1 + \sum_{k=1}^{n} (-1)^k \frac{2^k}{k!} (n+z)_{k} \right].$$

Substituting this into (3.3) leads to (3.7). The proof of Theorem 3.3 is complete. \hfill \square

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**References**

1. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 10th printing. Dover Publications, New York and Washington (1972)
2. Bourbaki, N.: Functions of a Real Variable, Elementary Theory, Translated from the 1976 French original by Philip Spain, Elements of Mathematics (Berlin). Springer, Berlin (2004). https://doi.org/10.1007/978-3-642-59315-4
3. Gradshteyn, I.S.: Ryzhik, Table of Integrals, Series, and Products, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the Seventh Edition, 8th edn. Academic Press, Amsterdam (2015). https://doi.org/10.1016/B978-0-12-384933-5.00013-8
4. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics—A Foundation for Computer Science, 2nd edn. Addison-Wesley Publishing Company, Reading (1994)
5. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark C.W. (eds.), NIST Handbook of Mathematical Functions. Cambridge University Press, New York (2010). http://dlmf.nist.gov/
6. Qi, F.: Derivatives of tangent function and tangent numbers. Appl. Math. Comput. 268, 844–858 (2015). https://doi.org/10.1016/j.amc.2015.06.123
7. Qi, F., Čeřňanová, V., Semenov, Y.S.: Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 81(1), 123–136 (2019)
8. Qi, F., Čeřňanová, V., Shi, X.-T., Guo, B.-N.: Some properties of central Delannoy numbers. J. Comput. Appl. Math. 328, 101–115 (2018). https://doi.org/10.1016/j.cam.2017.07.013
9. Qi, F., Chapman, R.J.: Two closed forms for the Bernoulli polynomials. J. Number Theory 159, 89–100 (2016). https://doi.org/10.1016/j.jnt.2015.07.021
10. Qi, F., Guo, B.-N.: A diagonal recurrence relation for the Stirling numbers of the first kind. Appl. Anal. Discret. Math. 12(1), 153–165 (2018). https://doi.org/10.2298/AADM170405004Q
11. Qi, F., Guo, B.-N.: Some determinantal expressions and recurrence relations of the Bernoulli polynomials. Mathematics 4(4), 11 (2016). https://doi.org/10.3390/math4040065. (Article 65)
12. Qi, F., Kızılateş, C., Du, W.-S.: A closed formula for the Horadam polynomials in terms of a tridiagonal determinant. Symmetry 11(6), 8 (2019). https://doi.org/10.3390/sym11060782
13. Qi, F., Lim, D., Guo, B.-N.: Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 113(1), 1–9 (2019). https://doi.org/10.1007/s13398-017-0427-2
14. Qi, F., Lim, D., Yao, Y.-H.: Notes on two kinds of special values for the Bell polynomials of the second kind. Miskolc Math. Notes 20(1), 465–474 (2019). https://doi.org/10.18514/MMN.2019.2635
15. Qi, F., Mahmoud, M., Shi, X.-T., Liu, F.-F.: Some properties of the Catalan–Qi function related to the Catalan numbers. SpringerPlus 5(1126), 20 (2016). https://doi.org/10.1186/s40064-016-2793-1
16. Qi, F., Niu, D.-W., Guo, B.-N.: Some identities for a sequence of unnamed polynomials connected with the Bell polynomials. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. 113(2), 557–567 (2019). https://doi.org/10.1007/s13398-018-0494-z
17. Qi, F., Shi, X.-T., Liu, F.-F., Kruchinin, D.V.: Several formulas for special values of the Bell polynomials of the second kind and applications. J. Appl. Anal. Comput. 7(3), 857–871 (2017). https://doi.org/10.11948/2017054
18. Qi, F., Wang, J.-L., Guo, B.-N.: A determinantal expression for the Fibonacci polynomials in terms of a tridiagonal determinant. Bull. Iran. Math. Soc. 45(6), 1821–1829 (2019). https://doi.org/10.1007/s41980-019-00232-4
19. Qi, F., Wang, W., Guo, B.-N., Lim, D.: Several explicit and recurrent formulas for determinants of tridiagonal matrices via generalized continued fractions. Nonlinear Analysis: Problems, Applications and Computational Methods, Proceedings of the 6th International Congress of the Moroccan Society of Applied Mathematics (SM2A 2019) organized by Sultan Moulay Slimane University, Faculté des Sciences et Techniques, BP 523, Béni-Mellal, Morocco, during 7th–9th November 2019, Springer Book Series Lecture Notes in Networks and Systems, 2020; HAL preprint (2019). https://hal.archives-ouvertes.fr/hal-02372394
20. Qi, F., Zhao, J.-L.: Some properties of the Bernoulli numbers of the second kind and their generating function. Bull. Korean Math. Soc. 55(6), 1909–1920 (2018). https://doi.org/10.4134/BKMS.b180039
21. Qi, F., Zhao, J.-L., Guo, B.-N.: Closed forms for derangement numbers in terms of the Hessenberg determinants. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 112(4), 933–944 (2018). https://doi.org/10.1007/s13398-017-0401-z
22. Qian, W.-M., He, Z.-Y., Chu, Y.-M.: Approximation for the complete elliptic integral of the first kind. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 114(2), 12 (2020). https://doi.org/10.1007/s13398-020-00784-9. (Paper No. 57)
23. Quaintance, J., Gould, H.W.: Combinatorial Identities for Stirling Numbers, The Unpublished Notes of H. W. Gould, with a Foreword by George E. Andrews. World Scientific Publishing Co Pte Ltd., Singapore (2016)
24. Wei, C.-F., Qi, F.: Several closed expressions for the Euler numbers. J. Inequal. Appl. 2015(219), 8 (2015). https://doi.org/10.1186/s13660-015-0738-9
25. Yang, Z.-H., Tian, J.-F.: A class of completely mixed monotonic functions involving the gamma function with applications. Proc. Am. Math. Soc. 146(11), 4707–4721 (2018). https://doi.org/10.1090/proc/14199
26. Yang, Z.-H., Tian, J.-F.: A comparison theorem for two divided differences and applications to special functions. J. Math. Anal. Appl. 464(1), 580–595 (2018). https://doi.org/10.1016/j.jmaa.2018.04.024
27. Yang, Z.-H., Tian, J.-F.: Monotonicity rules for the ratio of two Laplace transforms with applications. J. Math. Anal. Appl. 470(2), 821–845 (2019). https://doi.org/10.1016/j.jmaa.2018.10.034
28. Yang, Z.-H., Tian, J.-F., Ha, M.-H.: A new asymptotic expansion of a ratio of two gamma functions and complete monotonicity for its remainder. Proc. Am. Math. Soc. 148(5), 2163–2178 (2020). https://doi.org/10.1090/proc/14917
29. Yang, Z.-H., Tian, J.-F., Wang, M.-K.: A positive answer to Bhatia-Li conjecture on the monotonicity for a new mean in its parameter. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 114(3), 22 (2020). https://doi.org/10.1007/s13398-020-00856-w. (Paper No. 126)

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