Noncommutative Dipole QED

Néda Sadooghi † and Masoud Soroush ‡

†‡ Department of Physics, Sharif University of Technology
P.O. Box 11365-9161, Tehran-Iran

and

† Institute for Studies in Theoretical Physics and Mathematics (IPM)
School of Physics, P.O. Box 19395-5531, Tehran-Iran

Abstract

The noncommutative dipole QED is studied in detail for the matter fields in the adjoint representation. The axial anomaly of this theory is calculated in two and four dimensions using various regularization methods. The Ward-Takahashi identity is proved by making use of a non-perturbative path integral method. The one-loop $\beta$-function of the theory is calculated explicitly. It turns out that the value of the $\beta$-function depends on the direction of the dipole length $\vec{L}$, which defines the noncommutativity. Finally using a semi-classical approximation a non-perturbative definition of the form factors is presented and the anomalous magnetic moment of this theory at one-loop order is computed.

PACS No.: 11.15.Bt, 11.10.Gh, 11.25.Db

Keywords: Noncommutative Dipole Field Theory, Axial Anomaly, Ward-Takahashi Identity, Renormalization Constant, $\beta$-Function, Form Factor

Electronic address: sadooghi@theory.ipm.ac.ir

Electronic address: soroush@mehr.sharif.ac.ir
1 Introduction

In the past few years, the noncommutative theories are studied intensively by many authors. Especially Moyal noncommutative gauge theories are interesting due to their realization in the String Theory. As it turns out, in the decoupling limit, noncommutative gauge theories can occur in the world volume of $Dp$-branes in the presence of constant background $B_{\mu\nu}$ field, with $\mu$ and $\nu$ lying both on the branes \[1\]. For a review of Moyal noncommutativity see Refs. \[2, 3\].

In Refs. \[4, 5\], however, a new type a noncommutative product is introduced by considering a system of $Dp$-branes in the presence of a constant background $B_{\mu\nu}$ field with one index along the branes' world volume and the other index transverse to it. The gauge theories defined on these $Dp$-branes are also noncommutative, but, in contrast to the Moyal case, the space-time remains commutative. The noncommutativity appears only in the product of functions and has its origin in the finite dipole length $\vec{L}$ associated to each field. A supergravity description of these so called noncommutative dipole theories is presented in \[6\]. As in the Moyal case, the noncommutative dipole Field Theory can also be defined by replacing the ordinary product of function by a noncommutative dipole $\ast$-product. The noncommutative dipole Field Theory is studied first in Ref. \[7\].

In this paper, a detailed study of the noncommutative dipole QED is presented. In this theory, in analogy to the Moyal case, the matter fields appear in fundamental, antifundamental and adjoint representations. Here, we will restrict ourselves to noncommutative dipole QED with matter fields in the adjoint representation. In this theory point-like charged particles are absent. The only interacting objects are multipoles. Hence noncommutative dipole QED with adjoint matter fields is an appropriate candidate to study the interaction of neutral particles with finite dipole moments, like neutrinos, with gauge particles like photons. There are some experimental evidences of such interactions, which cannot be described by the commutative version of the standard model of particles \[8\]. This is not the only ground to study this adjoint theory. In the framework of perturbative calculations planar as well as nonplanar Feynman diagrams appear, which make the theory non-trivial. As in the noncommutative Moyal gauge theory, UV/IR mixing effects \[9\] can appear for small dipole length and large momentum cutoff.

In Sect. 2, a brief description of the algebraic structure of noncommutative dipole Field Theory is presented. The action of the noncommutative dipole QED in the adjoint representation is introduced in Sect. 3, where its global symmetries are also studied. As in the noncommutative Moyal case \[10, 11\], the theory possesses three different currents. We will show that only two of them correspond to finite conserved axial charges. In Sect. 4, the axial anomalies of these currents are calculated in two and four dimensions using point split and dimensional regularization methods. In Ref. \[12\], the axial
anomaly arising from only one of the currents of the theory is calculated using the Fujikawa’s path integral method \[13\]. Our results coincides with the results presented in this paper.

In Sect. 5, a one-loop perturbative analysis is carried out to compute the one-loop contributions to the fermion-self energy, vacuum polarization tensor and the vertex function. We will show that one-loop fermion-self energy and vertex function include planar and nonplanar parts, whereas the one-loop vacuum polarization tensor exhibits only a planar Feynman integral. We will calculate the one-loop contributions to the renormalization constants \(Z_i, i = 1, 2, 3\), and show that \(Z_1 = Z_2\) and that \(Z_3\) is proportional to the tensorial structure \((p_\mu p_\nu - p^2 \eta_{\mu\nu})\). To show that these identities are also valid in all higher orders of perturbative expansion, we will prove the Ward-Takahashi identity of the noncommutative dipole QED in the adjoint representation in Sect. 6. This will be done first in the framework of perturbation theory and then using the non-perturbative path integral method. In Sect. 7, the Ward-Takahashi identity will be used to obtain the general forms for the renormalization constant \(Z_i, i = 1, 2, 3\).

We will then calculate explicitly the one-loop \(\beta\)-function of noncommutative QED with adjoint matter fields [see Sect. 7.3]. It is shown that, in contrast to the commutative QED, this theory is asymptotically free, and that the one-loop \(\beta\)-function is proportional to a non-negative factor \((\vec{p} \cdot \vec{L})^2 \equiv |\vec{p}| |\vec{L}| \cos \vartheta\), where \(\vec{p}\) is a small external momentum, \(\vec{L}\) is the dipole length associated to each matter fields, and \(\vartheta\) is the relative angle between these two vectors. The value of the \(\beta\)-function depends therefore on the lengths of two vectors \(\vec{p}\) and \(\vec{L}\), as well as on the relative direction of \(\vec{L}\) and \(\vec{p}\).

The factor \((\vec{p} \cdot \vec{L})^2\) appearing in the one-loop \(\beta\) function of the theory, has in fact an interesting physical origin: As is known, for small external momentum the scattering amplitudes has to coincide with the classical results. According to the Born approximation, the scattering amplitude is proportional to the Fourier transformed of the scattering potential energy. In a theory where point-like charged particles exist, like in ordinary commutative QED, this potential energy is the Coulomb potential. But in our noncommutative dipole QED in the adjoint representation, this potential energy is the energy between multipoles. We have found that the factor \((\vec{p} \cdot \vec{L})\), which appears in the one-loop \(\beta\)-function of the theory, arises in fact from the Fourier transformed of the potential energy between two dipoles with the dipole moments \(g_0 \vec{L}\) (see Appendix A for a derivation), defining a modified bare coupling constant \(\bar{g}_0 = (\vec{p} \cdot \vec{L}) g_0\).

Using a semi-classical approximation in Sect. 8, we have shown that the form factors of noncommutative dipole QED with adjoint matter fields, can be defined by the Fourier transformed of the potential energy between a dipole \(g\vec{L}\) and external electric and magnetic potentials. The anomalous magnetic moment is then calculated in one-loop order. Sect. 9 is devoted to discussions.
2 Algebraic Structure of Dipole Field Theory

Let us establish a noncommutative Field Theory, by defining a \( \star \)-product in the linear space \( \mathcal{A} \) of quantum fields and forming an associative complex \( C^* \) algebra with respect to this product. A noncommutative space is then defined by an automorphism \( Q : \mathcal{A} \rightarrow \mathcal{A} \) which defines a derivative on the algebra \( \mathcal{A} \) and a linear map from \( \mathcal{A} \) to the field \( C \). This map is then given by \( f : \mathcal{A} \rightarrow C \) which acts as a trace on this algebra. If the following three properties

1. the Leibnitz Rule: \( \forall \Phi_a, \Phi_b \in \mathcal{A} \)
   \[ Q(\Phi_a \star \Phi_b) = (Q\Phi_a) \star \Phi_b + \Phi_a \star (Q\Phi_b), \tag{2.1} \]

2. integration by part, \( i.e. \forall \Phi_a \in \mathcal{A} \)
   \[ \int Q(\Phi_a) = 0, \tag{2.2} \]

3. and the cyclicity, \( i.e. \forall \Phi_a, \Phi_b \in \mathcal{A} \)
   \[ \int (\Phi_a \star \Phi_b) = \int (\Phi_b \star \Phi_a), \tag{2.3} \]

are satisfied for the above two maps, then the collection \( (\mathcal{A}, Q, f) \) forms a noncommutative space (for a review see \[2\]). The noncommutative Field Theory is then defined on this noncommutative space and its action is built using the above maps \( Q \) and \( f \). To construct a noncommutative dipole Field Theory a constant dipole \( L_a^\mu = (0, L_a^i) \) is assigned to each element \( \Phi_a \) of the algebra. The dipole \( \star \)-product is then defined as follows:

\[ \star : \quad C^\infty(\mathcal{R}^4) \otimes C^\infty(\mathcal{R}^4) \rightarrow C^\infty(\mathcal{R}^4) \quad (\Phi_a \star \Phi_b)(x) \equiv \Phi_a(x - L_b/2)\Phi_b(x + L_a/2). \tag{2.4} \]

The dipole length corresponding to the product \( \Phi_a \star \Phi_b \) is, due to the associativity of the algebra, given by the sum of the dipoles of \( \Phi_a \) and \( \Phi_b \) \[7\]. The derivative on the noncommutative dipole space is defined by the ordinary derivative of functions and satisfies automatically the Leibnitz rule.

As in the Moyal case, the trace on the algebra is given by the integration over all space-time components, so that the second property [Eq. 2.2)] of the noncommutative space is easily satisfied. The third property (2.3), however, is satisfied only for the kernel of the following map \( S_n \):

\[ S_n : \bigotimes_{i=1}^n C^\infty(\mathcal{R}^4) \rightarrow \mathcal{R}^4, \quad S_n(\Phi_1, \Phi_2, \cdots, \Phi_n) = \sum_{i=1}^n L_i. \tag{2.5} \]
For physical purposes, we restrict ourselves to this kernel. For the fields $\Phi$ belonging to the $C^*$-algebra, the product $(\Phi \star \Phi^\dagger)(x)$ is real valued. The dipole length assigned to the self-adjoint of a field is therefore given by the negative value of the dipole length associated to the field itself. As a consequence, the dipole length corresponding to Hermitian fields is zero.

If the sum of the dipole lengths of two fields $\Phi_a$ and $\Phi_b$ vanishes, i.e. if $(\Phi_a, \Phi_b) \in \text{Ker} \, S_2$, we have:

$$\int d^4x \, (\Phi_a \star \Phi_b)(x) = \int d^4x \, \Phi_a(x)\Phi_b(x).$$

(2.6)

This can be shown by a simple change of integration variable. Note that the same property is also valid for the Moyal $\star$-product, only if both fields $\Phi_a$ and $\Phi_b$ have trivial boundary conditions.

3 Dipole QED in the Adjoint Representation

Consider the action of noncommutative dipole QED, that includes the pure gauge part and the matter field part:

$$S_{QED}[\psi, \bar{\psi}, A] = \int F_{\mu\nu} \star F^{\mu\nu} + \int \bar{\psi} \star (i\partial - m)\psi.$$  

(3.1)

Here, the field strength tensor is defined as in the ordinary commutative QED by $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. Note that in the Moyal case, the field strength tensor includes a Moyal bracket of two gauge fields $A_\mu$. This nonlinear term leads to new three and four gauge vertices. As it turns out these vertices are absent in the present noncommutative dipole QED.

The matter field part includes the covariant derivative, defined by:

$$D_\mu \equiv \partial_\mu + igA_\mu.$$  

(3.2)

Due to the noncommutativity of the dipole $\star$-product, the matter fields have, in analogy to the Moyal case, three different representations: the fundamental, antifundamental and the adjoint representations. The action of the fermions in the adjoint representation is given by:

$$S_{QED}^{\text{adj}}[\psi, \bar{\psi}, A] = \int F_{\mu\nu} \star F^{\mu\nu} + \int \bar{\psi} \star (i\partial - m)\psi - g \int \bar{\psi} \gamma^\mu \star [A_\mu, \psi].$$  

(3.3)

The action (3.3) is invariant under the following transformation of matter and gauge fields:

$$\psi \longrightarrow (U \star \psi \star U^{-1}), \quad U \in U(1)$$

$$A \longrightarrow U \star A \star U^{-1} + \frac{i}{g}(\partial U \star U^{-1}) = A + \frac{i}{g}(\partial U)U^{-1}.$$  

(3.4)

As is known, in noncommutative dipole Field Theory, the space-time coordinates are still commutative. The space and time are therefore homogeneous and this means that the Lagrangian densities of QFTs are, in general, translational invariant. As a consequence, in addition to a energy-momentum conservation, the sum of the dipole lengths associated to each term in the Langrangian must vanish.
The dipole $\star$-product in the expression $U \star A \star U^{-1}$ could be removed, because both $A_\mu$ and $U(x)$ are dipoleless. Using Eq. (2.6), the dipole $\star$-product can be removed from the free part of the action (3.3), too. Hence, the fermion and photon propagators are exactly the same as in the ordinary commutative QED. The vertex of two fermions and one gauge field must, however, be modified and is given by:

$$V_\mu (p_1, p_2; k) = -(2\pi)^4 \delta^4(p_1 + p_2 + k) 2 g\gamma_\mu \sin \left( \frac{k \cdot L}{2} \right). \quad (3.5)$$

### 3.1 Noether Currents and Conserved Charges

In this section the Noether currents and the corresponding conserved charges of the noncommutative dipole U(1) gauge theory with adjoint matter fields will be derived explicitly.

Consider first the action $S^{adj} [\psi, \bar{\psi}, A]$ of a noncommutative Field Theory with adjoint matters, which is invariant under an arbitrary global and continuous symmetry transformation of matter fields

$$\psi \to \psi + \epsilon F(\psi), \quad \text{and} \quad \bar{\psi} \to \bar{\psi} + \epsilon F^*(\bar{\psi}). \quad (3.6)$$

Here, $\epsilon$ is a constant real valued number. The variation of the action under a local infinitesimal transformation is given by:

$$S^{adj} [\psi + [\epsilon, F(\psi)]^\star, \bar{\psi} + [\epsilon, F^*(\bar{\psi})^\star, A] - S^{adj} [\psi, \bar{\psi}, A] = - \int d^4x \ J^\mu (\psi(x), \bar{\psi}(x)) \partial_\mu \epsilon(x), \quad (3.7)$$

where $\epsilon(x) \in C^\infty(\mathbb{R}^4)$. For the classical path in which the fields satisfy the equation of motion, the r.h.s. of the above equation vanishes for all $\epsilon(x)$. According to the property (2.6), the most general form for the divergence of the current is given by:

$$\partial_\mu J^\mu = [f(\psi, \bar{\psi}), g(\psi, \bar{\psi})]^\star, \quad (3.8)$$

where $f$ and $g$ are arbitrary functions of $\psi$ and $\bar{\psi}$. To find the conserved charge corresponding to the current $J_\mu$, let us consider the three dimensional volume integral over the divergence of the current. Using the equation (3.8), the continuity equation is given by:

$$\frac{dQ}{dt} - \int d^3x \ \partial_i J_i(x) = \int d^3x \ [f(\psi, \bar{\psi}), g(\psi, \bar{\psi})] = \int d^3x \ [f(\psi, \bar{\psi}), g(\psi, \bar{\psi})]^\star, \quad (3.9)$$

---

This infinitesimal transformation is an arbitrary one. It can be equivalently given by $\psi \to \psi + \epsilon \star F(\psi)$ or $\psi \to \psi + F(\psi) \star \epsilon$. 

---

This page is numbered 5.
where the charge $Q$ is defined by $Q \equiv \int d^3x J^0(x)$, as in the ordinary commutative Field Theory. Since the time component of the dipole length corresponding to the fermionic fields $\psi$ and $\bar{\psi}$ is defined to be zero, the r.h.s. of the Eq. (3.3) vanishes, defining a conserved charge.

This general formulation can be used to find the local vector current of the noncommutative dipole QED with adjoint matter fields. The action (3.3) is invariant under the following global transformation of matter fields:

$$
\psi \rightarrow e^{i\alpha} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha},
$$

(3.10)

where $\alpha$ is a real valued number. Considering now the Langrangian density corresponding to this action:

$$
L_{QED}^{adj.}(\psi, \bar{\psi}, A) = \bar{\psi} \star (i\partial - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 - g\bar{\psi} \gamma^\mu \star [A_\mu, \psi]_*.
$$

(3.11)

and a local infinitesimal transformation:

$$
\psi \rightarrow (1 + i\alpha) \star \psi \star (1 - i\alpha) \simeq \psi + i[\alpha, \psi]_*,
$$

(3.12)

the Lagrangian density transforms as:

$$
L_{QED}^{adj.} \rightarrow L_{QED}^{adj.} - g\bar{\psi} \gamma^\mu \star [\partial_\mu \alpha, \psi]_*.
$$

(3.13)

Going through the same formulation described before, the vector current of this theory can be given by:

$$
J_\mu(x) = -g(\gamma_\mu)^{\alpha\beta}\{\psi_\beta, \bar{\psi}_\alpha\}_*(x).
$$

(3.14)

As in the Moyal case, this theory possesses two other global vector currents [11]:

$$
J'_\mu(x) = -g(\gamma_\mu)^{\alpha\beta}(\psi_\beta \star \bar{\psi}_\alpha)(x)
$$

$$
J''_\mu(x) = -g(\gamma_\mu)^{\alpha\beta}(\bar{\psi}_\alpha \star \psi_\beta)(x),
$$

(3.15)

which correspond to the local infinitesimal transformations $\psi \rightarrow (1 + i\alpha) \star \psi$ and $\psi \rightarrow \psi \star (1 + i\alpha)$, respectively. Using the equation of motion corresponding to the Lagrangian density (3.12), it can be easily shown that all these currents are classically conserved:

$$
\partial^\mu J_\mu = 0, \quad \text{for} \quad J_\mu \in \{J_\mu, J'_\mu, J''_\mu\}.
$$
4 Axial Anomaly

In this section we will study the axial anomaly of noncommutative dipole QED with matter fields in the adjoint representation. First we study the axial anomaly corresponding to the global axial vector current

\[ J_{\mu(5)}(x) = -(\gamma_\mu \gamma_5)^{\alpha\beta} \{ \bar{\psi}_\beta, \psi_\alpha \}(x). \]  

Here we will use two different regularization methods: the point split and dimensional regularization in two and four dimensions\(^6\). We then calculate the axial anomaly corresponding to the two other global axial vector currents of the theory \( J'_{\mu(5)} \) and \( J''_{\mu(5)} \):

\[ J'_{\mu(5)}(x) = -(\gamma_\mu \gamma_5)^{\alpha\beta}(\bar{\psi}_\beta \ast \bar{\psi}_\alpha)(x) \]
\[ J''_{\mu(5)}(x) = -(\gamma_\mu \gamma_5)^{\alpha\beta}(\bar{\psi}_\alpha \ast \psi_\beta)(x), \]

using only dimensional regularization.

4.1 The Axial Anomaly of \( J_{\mu(5)} \)

4.1.1 Point Splitting Regularization

i) Two Dimensions

Consider the axial vector current \( J_{\mu(5)} \) from Eq. (4.1), whose point splitted version reads:

\[ J_{\mu(5)}(x) = \text{symm } \lim_{\epsilon \to 0} (\gamma_\mu \gamma_5)^{\alpha\beta} \left[ \psi_\beta(x - \frac{\epsilon}{2}) \ast U^\dagger(x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2}) \ast \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) + \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) \ast U(x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2}) \ast \psi_\beta(x - \frac{\epsilon}{2}) \right]. \]  

Here, we have introduced the link variables:

\[ U(y, x) \equiv \exp \left( -ig \int_x^y dz^\mu A_\mu(z) \right), \]

with the known gauge transformation property:

\[ U(x, y) \to U(x) \ast U(x, y) \ast U^\dagger(y), \]

where \( U \in U(1) \)-gauge group. Using the transformation (3.4) of the matter fields and Eq. (4.4) for the link variables, it can be shown that \( J_{\mu(5)}(x) \) from Eq. (4.3) is gauge invariant. This is because the product of \( \psi, \bar{\psi} \) and \( U \), appearing on the r.h.s. of Eq. (4.3), is dipoleless. Similar

\(^6\)When this study was almost done an article appeared \[12\], where the axial anomaly of noncommutative dipole QED corresponding to the axial vector current \( J_{\mu(5)} \) was calculated using the Fujikawa’s path integral method.
The calculation is also performed in [10] for the Moyal case, where, however, the regularized currents is only gauge covariant. An integration over all (Moyal) noncommutative space-time coordinates has to be performed to preserve the gauge invariance.

After this remark, we use the expansion of the link variable in the first order of $\epsilon$:

$$U(x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2}) = 1 - i g e^\mu A_\mu(x) + O(\epsilon^2).$$  \hspace{1cm} (4.6)

The VEV of the divergence of the axial vector current is then given by:

$$\langle \partial^\mu J_{\mu(5)}(x) \rangle = \text{symm} \lim_{\epsilon \to 0} g e^\nu (\gamma^\mu \gamma^5)^{\alpha \beta} \left( 2 \left[ \psi_\beta(x - \frac{\epsilon}{2}), \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) \right] \right) \partial_\nu A_\mu(x)$$

$$+ \left( \psi_\beta(x - \frac{\epsilon}{2}) \star \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) F_{\mu \nu}(x + L) - \left( \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) \star \psi_\beta(x - \frac{\epsilon}{2}) \right) F_{\mu \nu}(x - L) \right).$$  \hspace{1cm} (4.7)

In the zeroth order of perturbative expansion, using the definition of dipole $\star$-product, the expectation value of matter fields can be easily calculated:

$$(\gamma^\mu \gamma^5)^{\alpha \beta} \langle \psi_\beta(x - \frac{\epsilon}{2}) \star \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) \rangle = (\gamma^\mu \gamma^5)^{\alpha \beta} \left( \psi_\beta(x + \frac{L}{2} - \frac{\epsilon}{2}) \bar{\psi}_\alpha(x + \frac{L}{2} + \frac{\epsilon}{2}) \right)$$

$$= (\gamma^\mu \gamma^5)^{\alpha \beta} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \langle \bar{\psi}_\beta(p) \bar{\psi}_\alpha(q) \rangle e^{i p.(x + \frac{L}{2} - \frac{\epsilon}{2})} e^{-i q.(x + \frac{L}{2} + \frac{\epsilon}{2})}$$

$$= (\gamma^\mu \gamma^5)^{\alpha \beta} \int \frac{d^2 p}{(2\pi)^2} (\frac{i p}{p^2})^\beta_\alpha e^{-i p. \epsilon}$$

$$= -i \frac{\epsilon}{2\pi} \text{Tr}(\gamma^\rho \gamma^\mu \gamma^5 \epsilon^\rho).$$  \hspace{1cm} (4.8)

Using as next \( \lim_{\epsilon \to 0} \frac{\epsilon \epsilon^\mu}{\epsilon^2} = \frac{1}{d} \delta^\mu_\rho \), where $d$ is the space-time dimension, and \( \text{Tr}(\gamma^\rho \gamma^\mu \gamma^5) = 2 \epsilon^\rho_\mu \), and replacing the result from Eq. (4.8) in the expression on the r.h.s. of Eq. (4.7), we arrive at the axial anomaly of two dimensional noncommutative dipole QED with adjoint matters, which reads:

$$\langle \partial^\mu J_{\mu(5)}(x) \rangle = \frac{g}{2\pi} \epsilon^{\mu \nu} \left[ F_{\mu \nu}(x + L) + F_{\mu \nu}(x - L) - 2 F_{\mu \nu}(x) \right].$$  \hspace{1cm} (4.9)

Since the field strength tensor is dipoleless, the above result is gauge invariant. Further the anomaly vanishes, if we integrate both sides over one space coordinate $x^1$:

$$\int dx^1 \langle \partial^\mu J_{\mu(5)}(x) \rangle = 0.$$  \hspace{1cm} (4.10)

Note that classically the axial charge $Q_5$,

$$Q_5 \equiv \int dx^1 J_{0(5)}(x),$$  \hspace{1cm} (4.11)

corresponding to $J_{\mu(5)}$ from Eq. (4.1) vanishes. The above result shows that in the quantum level $Q_5$ is still conserved.
ii) Four Dimensions

To obtain the axial anomaly in four dimensions using the point split regularization method, the same steps leading from Eq. (4.1) to Eq. (4.7) must be repeated. It turns out that in the zeroth order of the perturbative expansion the contribution to the VEV of two matter fields from Eq. (4.7) vanishes. In the first order of perturbative expansion we have therefore

$$\langle \partial^\mu J_{\mu(5)}(x) \rangle =$$

$$= \text{symm} \lim_{\epsilon \to 0} g e^{\nu} \left\{ 2 \left[ \tau_1^\mu(x, \epsilon) - \tau_1'^\mu(x, \epsilon) \right] \partial_\nu A_\mu(x) + \tau_1^\mu(x, \epsilon) F_{\mu\nu}(x + L) - \tau_1'^\mu(x, \epsilon) F_{\mu\nu}(x - L) \right\},$$

(4.12)

where two functions $\tau_1^\mu(x, \epsilon)$ and $\tau_1'^\mu(x, \epsilon)$ are defined by:

$$\tau_1^\mu(x, \epsilon) \equiv (\gamma^\mu \gamma^5)^{\alpha\beta} \left\langle \psi_\beta(x - \frac{\epsilon}{2}) \bar{\psi}_\alpha(x + \frac{\epsilon}{2})(ig) \int d^4 z \left( \bar{\psi} \gamma^\lambda \Lambda_\psi \right)(z) \right\rangle,$$

(4.13)

and

$$\tau_1'^\mu(x, \epsilon) \equiv (\gamma^\mu \gamma^5)^{\alpha\beta} \left\langle \bar{\psi}_\alpha(x + \frac{\epsilon}{2}) \psi_\beta(x - \frac{\epsilon}{2})(ig) \int d^4 z \left( \bar{\psi} \gamma^\lambda \Lambda_\psi \right)(z) \right\rangle.$$

(4.14)

The function $\tau_1(x, \epsilon)$ can be evaluated using standard perturbative methods and is given by:

$$\tau_1^\mu(x, \epsilon) = -ig \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \int d^4 z e^{ip.(x + \frac{\epsilon}{2} - \frac{z}{2})} e^{-iq.(x + \frac{\epsilon}{2} + \frac{z}{2})} e^{ik.z}$$

$$\times (\gamma^\lambda)^{\rho\sigma} (\gamma^\mu \gamma^5)^{\alpha\beta} \tilde{A}_\lambda(k) \left[ \left\langle \bar{\psi}_\beta(p) \bar{\psi}_\alpha(q) \bar{\psi}_{\sigma}(k_2) \bar{\psi}_\rho(k_1) \right\rangle e^{ik_2.(z + \frac{\epsilon}{2})} e^{-ik_1.(z + \frac{\epsilon}{2})} \right.$$

$$\left. + \left\langle \bar{\psi}_\beta(p) \bar{\psi}_\alpha(q) \bar{\psi}_\rho(k_1) \bar{\psi}_{\sigma}(k_2) \right\rangle e^{-ik_1.(z - \frac{\epsilon}{2})} e^{ik_2.(z - \frac{\epsilon}{2})} \right]$$

$$= -ig(\gamma^\lambda)^{\rho\sigma} (\gamma^\mu \gamma^5)^{\alpha\beta} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_1}{(2\pi)^4} (k + k_2)^\rho (k_1)^{\alpha} \tilde{A}_\lambda(k) e^{ik.(x - \frac{\epsilon}{2})} e^{-ik_2.\epsilon (1 - e^{ik.L})}.$$  (4.14)

For the limit $\epsilon \to 0$ and for large $k_2$ the last integral reads

$$\tau_1^\mu(x, \epsilon) = -ig(\gamma^\lambda)^{\rho\sigma} (\gamma^\mu \gamma^5)^{\alpha\beta} \left( \int \frac{d^4 k}{(2\pi)^4} \frac{k_2}{{k}_2^2} (\gamma^5)^{\sigma\alpha} e^{-ik_2.\epsilon} \right.$$

$$\left. \times \left[ \int \frac{d^4 k}{(2\pi)^4} \delta_{\beta\rho} \tilde{A}_\lambda(k) e^{ik.(x - \frac{\epsilon}{2})} - \int \frac{d^4 k}{(2\pi)^4} \delta_{\beta\rho} \tilde{A}_\lambda(k) e^{ik.(x + L - \frac{\epsilon}{2})} \right] \right.$$  

$$= \frac{g}{8\pi^2} \bar{\epsilon} \left( \partial_\delta A_\lambda(x - \frac{\epsilon}{2}) - \partial_\delta A_\lambda(x + L - \frac{\epsilon}{2}) \right) \text{Tr}(\gamma^\delta \gamma^\lambda \gamma^\gamma \gamma^5).$$ 

(4.14)

Similarly $\tau_1'^\mu(x, \epsilon)$ can be easily evaluated and is given by:

$$\tau_1'^\mu(x, \epsilon) = -ig \frac{e^{\mu\lambda\delta}}{2\pi^2} \left( \partial_\delta A_\lambda(x - \frac{\epsilon}{2}) - \partial_\delta A_\lambda(x - L - \frac{\epsilon}{2}) \right).$$ 

(4.14)
where $\text{Tr}(\gamma^5\gamma^\mu\gamma^5\gamma^\delta\gamma^\lambda\gamma^\zeta) = -4i\varepsilon^{\mu\delta\lambda\zeta}$ is used. Replacing now the results from Eqs. (4.16) and (4.17) in Eq. (4.12), we arrive at the axial anomaly of the four dimensional noncommutative dipole QED with adjoint matters:

\[
\left\langle \partial^\mu J_{\mu(5)}(x) \right\rangle = \frac{g^2}{16\pi^2} \varepsilon^{\mu\lambda\delta\zeta} \left[ F_{\mu\nu}(x-L)F_{\delta\lambda}(x-L) - F_{\mu\nu}(x+L)F_{\delta\lambda}(x+L) - 2\left( F_{\mu\nu}(x+L) - F_{\mu\nu}(x-L) \right) F_{\delta\lambda}(x) \right].
\]

The same result is also obtained in [12] where the Fujikawa’s path integral method is used. As in two dimensional case, the above result turns out to be gauge invariant and vanishes after integrating over three spatial coordinates:

\[
\int d^3x \left\langle \partial^\mu J_{\mu(5)}(x) \right\rangle = 0.
\]

### 4.1.2 Triangle Anomaly

In this section the triangle diagrams will be calculated in two and four dimensions using the dimensional regularization method.

**i) Two Dimensions**

Let us consider the two point function:

\[
\Gamma^{\mu\nu}(x, y) \equiv \left\langle T\left( J_{\mu(5)}(x)J^\nu(y) \right) \right\rangle,
\]

with $J^\nu(x)$ and $J_{\mu(5)}(y)$ defined in Eqs. (3.14) and (4.1), respectively. After an appropriate shift of integration variable the dimensional regulated Feynman integral corresponding to the above two point function is given by:

\[
\Gamma^{\mu\nu}(x, y) = g \int \frac{d^d q}{(2\pi)^d} e^{i q.(x-y)} (2 - e^{i q.L} - e^{-i q.L}) \int \frac{d^d \ell}{(2\pi)^d} \text{Tr}\left( \frac{1}{f} \gamma^\mu \gamma^5 \frac{1}{f + \ell} \gamma^\nu \right).
\]

The divergence of $\Gamma^{\mu\nu}$ with respect to $y^\nu$ can be easily calculated. As it turns out the vector Ward identity vanishes:

\[
\left\langle \frac{\partial}{\partial y^\nu} J^\nu(y) \right\rangle = 0.
\]

What concerns the axial vector Ward identity, the divergence of $\Gamma^{\mu\nu}(x, y)$ with respect to $x$ is to be calculated. It is given by:

\[
\frac{\partial}{\partial x^\mu} \Gamma^{\mu\nu}(x, y) \propto \int \frac{d^d \ell}{(2\pi)^d} \text{Tr}\left( \frac{1}{f} \gamma^5 \frac{1}{f + \ell} \gamma^\nu \right).
\]
Using the standard 't Hooft’s definition of $\gamma_5$ in d-dimensions and the following identity:

$$\not{q}\gamma_5 = -\gamma_5(\not{q} + \not{f}) - \not{f}\gamma_5 + 2\gamma_5\not{f}_\perp,$$

we arrive at:

$$\frac{\partial}{\partial x^\mu} \Gamma^{\mu\nu}(x, y) = -\frac{g}{\pi} \epsilon^{\alpha\nu} \int \frac{d^2 q}{(2\pi)^2} e^{iq.(x-y)} q_\alpha (2 - 2\cos(q.L)).$$

(4.25)

Now using the relation

$$\langle \partial^\mu J_{\mu(5)}(x) \rangle = \int d^2 y \frac{\partial}{\partial x^\mu} \Gamma^{\mu\nu}(x, y) A_\nu(y),$$

(4.26)

and the result from Eq. (4.25) we obtain:

$$\langle \partial^\mu J_{\mu(5)}(x) \rangle = \frac{g}{2\pi} \epsilon^{\alpha\beta} \left( F_{\alpha\beta}(x + L) + F_{\alpha\beta}(x - L) - 2F_{\alpha\beta}(x) \right),$$

(4.27)

which coincides with our result from the point split regularization [see Eq. (4.9)].

**ii) Four Dimensions**

To calculate the anomaly in four dimensions, the three point function of one axial vector and two vector currents must be considered:

$$\Gamma_{\mu\lambda\nu}(x, y, z) = \langle T(J_{\mu(5)}(x)J_\lambda(y)J_\nu(z)) \rangle.$$

(4.28)

The triangle Feynman integrals corresponding to the $\Gamma_{\mu\lambda\nu}$ is then given by:

$$\Gamma_{\mu\lambda\nu}(x, y, z) =$$

$$= 2g^2 \int \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} e^{-i(k_2 + k_3).x} e^{i(k_2 + k_3).y} e^{ik_3.z} \left[ \sin(k_2.L) + \sin(k_3.L) - \sin((k_2 + k_3).L) \right]$$

$$\times \int \frac{d^d \ell}{(2\pi)^d} \left[ \text{Tr} \left( \frac{1}{\ell + k_3} \gamma^\mu \gamma^5 \frac{1}{\ell - k_2} \gamma^\lambda \gamma^\nu \right) + \left( (k_2, \lambda) \leftrightarrow (k_3, \nu) \right) \right].$$

(4.29)

The divergence of the above integral with respect to $y^\lambda$ can be easily calculated, leading to vanishing vector Ward identity:

$$\langle \partial_{y^\lambda} J^\lambda(y) \rangle = 0.$$

Further, the divergence of $\Gamma_{\mu\lambda\nu}$ with respect to $x^\mu$ is given by:

$$\frac{\partial}{\partial x^\mu} \Gamma_{\mu\lambda\nu}(x, y, z) =$$

$$= 2ig^2 \int \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} e^{-i(k_2 + k_3).x} e^{i(k_2 + k_3).y} e^{ik_3.z} \left[ \sin(k_2.L) + \sin(k_3.L) - \sin((k_2 + k_3).L) \right]$$

$$\times \left[ R_{\lambda\nu}(k_2, k_3) + A_{\lambda\nu}(k_2, k_3) \right].$$

(4.30)
where two functions $R^{\lambda\nu}$ and $A^{\lambda\nu}$ are defined by:

$$
R^{\lambda\nu}(k_2, k_3) = \int \frac{d^d q}{(2\pi)^d} \left[ \text{Tr} \left( \gamma^5 \frac{1}{\ell - \frac{k_2}{2}} \gamma^\lambda \frac{1}{\ell} \gamma^\nu \right) + \text{Tr} \left( \gamma^5 \frac{1}{\ell + \frac{k_3}{3}} \gamma^\nu \frac{1}{\ell} \gamma^\lambda \right) \right],
$$

and

$$
A^{\lambda\nu}(k_2, k_3) = \int \frac{d^d q}{(2\pi)^d} \left[ \text{Tr} \left( \gamma^5 \frac{1}{\ell - \frac{k_3}{3}} \gamma^\nu \frac{1}{\ell + \frac{k_2}{2}} \gamma^\lambda \right) + \text{Tr} \left( (k_2, \lambda) \leftrightarrow (k_3, \nu) \right) \right].
$$

Here, we have used the identity $(\ell + k_2)\gamma^5 = (\ell + k_3)\gamma^5 + \gamma^5 (\ell - k_3) - 2\gamma^5 \ell_\perp$. As it turns out the function $R^{\lambda\nu}$ vanishes using the cyclic permutation symmetry of the trace. The anomaly is then entirely given by $A^{\lambda\nu}$, which reads

$$
A^{\lambda\nu}(k_2, k_3) = \frac{1}{4\pi^2} \epsilon^{\alpha\beta\lambda\nu} k_{2\alpha} k_{3\beta}.
$$

Replacing this results in Eq. (4.31) and using the identity

$$
\langle \partial^\mu J_{\mu(5)}(x) \rangle = \frac{1}{2} \int d^4 y d^4 z \frac{\partial}{\partial x^\mu} \Gamma^{\mu\lambda\nu}(x, y, z) A_\lambda(y) A_\nu(z),
$$

the VEV of the divergence of $J_{\mu(5)}$ is given by the same Eq. (4.18), which we have obtained by making use of point splitting regularization method.

4.2 The Axial Anomaly of $J'_{\mu(5)}$ and $J''_{\mu(5)}$

As we have seen above, the noncommutative QED with adjoint matters consists three different vector and axial vector currents. In this section we will calculate the axial anomaly corresponding to two currents $J'_{\mu(5)}$ and $J''_{\mu(5)}$ from Eq. (4.2) in two and four dimensions by making use of dimensional regularization.

i) Two Dimensions

Let us indicate the two-point function corresponding to these currents by $\Gamma'_{\mu\nu}$ and $\Gamma''_{\mu\nu}$ respectively:

$$
\Gamma'_{\mu\nu}(x, y) = \langle T \left( J'_{\mu(5)}(x) J_\nu(y) \right) \rangle, \quad \Gamma''_{\mu\nu}(x, y) = \langle T \left( J''_{\mu(5)}(x) J_\nu(y) \right) \rangle.
$$

In a dimensional regularization, the Feynman integrals corresponding to these two-point functions are given by:

$$
\Gamma'_{\mu\nu}(x, y) = g \int \frac{d^d q}{(2\pi)^d} e^{i q \cdot (x-y)} (1 - e^{i q \cdot L}) \int \frac{d^d \ell}{(2\pi)^d} \text{Tr} \left( \frac{1}{\ell + \frac{k_2}{2}} \gamma^5 \frac{1}{\ell + \frac{k_3}{3}} \gamma^\nu \right),
$$

(4.36)
and
\[ \Gamma^{\mu\nu}(x, y, z) = g \int \frac{d^d q}{(2\pi)^d} e^{iq.(x-y)} \left( 1 - e^{-iq.L} \right) \int \frac{d^d \ell}{(2\pi)^d} \text{Tr} \left( \frac{1}{\ell} \gamma^5 \frac{1}{\ell + \not{r}} \gamma^\nu \right). \] (4.37)

Deriving the above integrals with respect to \( x^\mu \) and going through the same standard procedure leading to the divergence of the axial vector currents, we arrive at:
\[ \langle \partial^\mu J'_\mu(5) \rangle = \frac{g}{2\pi} \varepsilon^{\alpha\beta} \left( F_{\alpha\beta}(x + L) - F_{\alpha\beta}(x) \right), \quad \langle \partial^\mu J''_\mu(5) \rangle = \frac{g}{2\pi} \varepsilon^{\alpha\beta} \left( F_{\alpha\beta}(x - L) - F_{\alpha\beta}(x) \right). \] (4.38)

Integrating now these results over \( x^1 \) we obtain:
\[ \int dx^1 \langle \partial^\mu J'_\mu(5) \rangle = 0, \quad \text{and} \quad \int dx^1 \langle \partial^\mu J''_\mu(5) \rangle = 0. \] (4.39)

The corresponding axial charges are therefore conserved.

**ii) Four Dimensions**

Let us now consider the three-point function corresponding to the axial vector currents \( J'_\mu(5) \) and \( J''_\mu(5) \) from Eq. (4.2):
\[ \Gamma_{\mu\lambda
u}(x, y, z) = \left\langle T \left( J'_\mu(5)(x) J_\lambda(y) J_\nu(z) \right) \right\rangle, \quad \Gamma''_{\mu\lambda\nu}(x, y, z) = \left\langle T \left( J''_\mu(5)(x) J_\lambda(y) J_\nu(z) \right) \right\rangle. \] (4.40)

The dimensional regularized Feynman integrals are given by:
\[ \Gamma^{\mu\nu\lambda}(x, y, z) = \left( 1 - e^{-ik_L} \right) \left( 1 - e^{-ik_L} \right) \text{Tr} \left[ \frac{1}{\not{r} + \not{k}_3} \gamma^\mu \gamma^5 \frac{1}{\not{r} - \not{k}_2} \gamma^\lambda \gamma^\nu \right] \] (4.41)
\[ \times \int \frac{d^d \ell}{(2\pi)^d} \left[ (k_2, \lambda) \leftrightarrow (k_3, \nu) \right], \]
and
\[ \Gamma''^{\mu\nu\lambda}(x, y, z) = \left( 1 - e^{+ik_L} \right) \left( 1 - e^{+ik_L} \right) \text{Tr} \left[ \frac{1}{\not{r} + \not{k}_3} \gamma^\mu \gamma^5 \frac{1}{\not{r} - \not{k}_2} \gamma^\lambda \gamma^\nu \right] \] (4.42)
\[ \times \int \frac{d^d \ell}{(2\pi)^d} \left[ (k_2, \lambda) \leftrightarrow (k_3, \nu) \right]. \]

The axial vector anomalies corresponding to these currents can be calculated using the standard dimensional regularization procedure and read
\[ \left\langle \partial^\mu J'_\mu(5) \right\rangle = \frac{g^2}{16\pi^2} \varepsilon^{\alpha\beta\lambda\nu} \left[ F_{\alpha\lambda}(x) F_{\beta\nu}(x) + F_{\alpha\lambda}(x + L) F_{\beta\nu}(x + L) - 2 F_{\alpha\lambda}(x) F_{\beta\nu}(x + L) \right], \] (4.41)
and
\[ \left\langle \partial^\mu J''_\mu(5) \right\rangle = -\frac{g^2}{16\pi^2} \varepsilon^{\alpha\beta\lambda\nu} \left[ F_{\alpha\lambda}(x) F_{\beta\nu}(x) + F_{\alpha\lambda}(x - L) F_{\beta\nu}(x - L) - 2 F_{\alpha\lambda}(x) F_{\beta\nu}(x - L) \right], \] (4.42)
respectively. Integrating over three spatial coordinates and using an appropriate shift of integration variable, we arrive at:

\[
\int d^3x \left\langle \partial^\mu J'_\mu(5) \right\rangle = - \int d^3x \left\langle \partial^\mu J''_\mu(5) \right\rangle = \frac{g^2}{8\pi^2} \epsilon^{\alpha\beta\lambda\nu} \left( \int d^3xF_{\alpha\lambda}(x)F_{\beta\nu}(x) - \int d^3xF_{\alpha\lambda}(x - \frac{L}{2})F_{\beta\nu}(x + \frac{L}{2}) \right),
\]

which, in contrary to the result for \( J_{\mu(5)} \) from Eq. (4.19) does not vanish. This means that the corresponding axial charges to \( J'_{\mu(5)} \) and \( J''_{\mu(5)} \) are anomalous.

5 One-Loop Perturbative Calculation

In this section, a perturbative calculation is performed to obtain the one-loop contribution to the renormalization constants \( Z_i, i = 1, 2, 3 \) of a dipole U(1) gauge theory with matter fields in the adjoint representation.

i) Fermion Self Energy

Using the vertex of two Fermion and one gauge field of the noncommutative dipole U(1) gauge theory from Eq. (3.5), the Feynman integral corresponding to the one-loop fermion self energy diagram can be given by

\[
-i\Sigma(p) \equiv \int d^4k \frac{\gamma^\mu}{k^2[(p-k)^2-m^2]} \gamma^\mu \sin^2\left(\frac{kL}{2}\right).
\]

Using now the identity \( \sin^2 x = \frac{1}{2}(1 - \cos(2x)) \), it can be shown that the above integral includes planar and nonplanar parts. The planar part is exactly twice the value of the one-loop contribution of fermion self energy in the commutative U(1) gauge theory. In the minimal subtraction (MS) scheme, the mass and wave function renormalization constants \( \delta m \) and \( Z_2 \) are therefore given by:

\[
\delta m = \left. \Sigma(p) \right|_{p=m} = -\frac{3mg^2}{4\pi^2} \frac{1}{(4-d)}, \quad Z_2 \equiv 1 + \delta Z_2 \quad \text{with} \quad \delta Z_2 = \frac{g^2}{4\pi^2} \frac{1}{(4-d)}. \]

What concerns the nonplanar part, it is considered to be finite for finite dipole length \( L \). In the \( L \to 0 \) limit, a UV/IR mixing will occur. In this case, the infrared divergences which arise from the nonplanar
part cancel the ultraviolet divergences from the planar part, leading to a vanishing renormalization constant and eventually to a vanishing $\beta$-function. This is indeed consistent with the observation that the theory is free in the limit $L \to 0$.

ii) Vertex Function

The one-loop vertex function in noncommutative dipole QED, receives contribution from only one vertex diagram:

$$\bar{u}(p') \left(-2g \delta \Gamma^\mu(p', p) \sin \left(\frac{q.L}{2}\right)\right) u(p) \equiv$$

![Figure 2: The diagram contributing to the one-loop vertex function.](image)

where $\delta \Gamma^\mu$ is the one-loop contribution to the vertex function. Remember that in the Moyal case, due to the appearance of additional three gauge vertex, a second diagram contributes to the one-loop vertex function of the theory. Using the Feynman rules of noncommutative dipole U(1) gauge theory with adjoint matters, the Feynman integral corresponding to the above diagram can be given by:

$$\delta \Gamma^\mu(p', p) = -4ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\rho(p' - k + m)\gamma^\mu(p - k + m)\gamma^\rho}{k^2[(p' - k)^2 - m^2][(p - k)^2 - m^2]} \sin^2 \left(\frac{k.L}{2}\right). \quad (5.3)$$

Replacing the factor $\sin^2\left(\frac{k.L}{2}\right)$ by $\frac{1}{2}(1 - \cos(k \cdot L))$ in the above integral the planar and the nonplanar parts of the above integral can be separated. Its planar part is twice the vertex integral in the commutative case and its nonplanar part turns out to be finite for finite value of $L$. If we use the standard minimal subtraction (MS) scheme, where only UV divergences from the planar diagrams are to be taken into account, the wave function renormalization constant can be obtained and reads:

$$Z_1 \equiv 1 + \delta Z_1 \quad \text{with} \quad \delta Z_1 = \frac{g^2}{4\pi^2} \frac{1}{(4 - d)}. \quad (5.4)$$

Comparing this result with the result of $Z_2$ from Eq. (5.2), it turns out that the identity $Z_1 = Z_2$ is satisfied in the MS scheme.

Now let us consider again the Feynman integral corresponding to the fermion self-energy from Eq. (5.1). Using the usual definition of $\delta Z_2 \equiv \frac{d\Sigma(p)}{dp} \bigg|_{p=m}$ we have

$$\delta Z_2 = 8ig^2 \left[ \int \frac{d^4k}{(2\pi)^4} \frac{(k - p)^2 - 4p.(k - p) + m^2}{k^2[(k - p)^2 - m^2]^2} \sin^2 \left(\frac{k.L}{2}\right) \right]_{p=(m,0)}, \quad (5.5)$$
which turns out to be equal to $\delta Z_1$ coming from the vertex function. The identity $Z_1 = Z_2$ seems therefore to be valid for both planar and nonplanar parts. This is one of the results from the Ward-Takahashi (WT) identity, which will be proved in the next section. We will show its validity in any order of perturbative expansion and any renormalization prescription scheme.

**iii) Photon Self-Energy**

The Feynman integral corresponding to the photon self-energy diagram

$$i\Pi_{\mu\nu}(p) \equiv \int \frac{d^d k}{(2\pi)^d} \text{Tr}\left[ \gamma_\mu k_\nu - m \gamma_\nu k_\mu - p_\mu - m \right].$$ (5.6)

This integral involves only a planar part. The planar phase factor appears only as a multiplicative factor before the Feynman integral which is exactly the same integral from the commutative U(1) and turns out to be proportional to usual tensorial structure ($\eta_{\mu\nu} p^2 - p_\mu p_\nu$). As in the commutative U(1) the coupling constant renormalization parameter is defined for small external momenta $p$. In this limit, it is given by:

$$Z_3 = 1 + \delta Z_3, \quad \text{with} \quad \delta Z_3 = \frac{g^2}{6\pi^2} \frac{1}{(4-d)(p \cdot L)^2}.$$ (5.7)

In the next section, after proving the Ward-Takahashi identity for noncommutative dipole QED with adjoint matters, the general form of $Z_3$ will be presented. We will show that the above result is valid for all orders of perturbative expansion.

### 6 Ward-Takahashi Identity of Noncommutative Dipole QED

In this section the Ward-Takahashi identity of the noncommutative dipole QED associated to the gauge invariance of the theory is proved for all order of perturbative expansion. The matter fields are still in the adjoint representations. As in the commutative QED, there are two different methods to
show this identity: the perturbative and the path integral methods. Let us begin with the perturbative method:

**Perturbative Approach**

This identity will be shown using the LSZ formalism for the correlation functions. As in the ordinary commutative QED an arbitrary diagram is to be considered which involves an arbitrary number of fermion and photon propagators, that are connected by vertices of two fermions and one gauge field. Let us insert a photon line into this diagram. According to the Feynman rules of noncommutative dipole U(1) gauge theory, the only possibility to insert this photon to the diagram is either to attach it to an open fermion line or to an fermion loop. Consider first a part of the correlation function corresponding to the \( j \)-th fermion line in the momentum space before inserting the photon line to it:

\[
\mathcal{M}_{0j}(k) = \left[ \prod_{\ell=1}^{n} \frac{i}{p_\ell - m} \left( -2g\gamma^\mu \sin\left( \frac{q_\ell \cdot L}{2} \right) \right) \right] \frac{i}{p - m}. \tag{6.1}
\]

Now let us attach our photon first between the \( r \)-th and the \( r + 1 \)-th vertices:

\[
\mathcal{M}_{j,r}(k) \equiv \left[ \prod_{\ell=1}^{n} \frac{i}{p_\ell - m} \left( -2g\gamma^\mu \sin\left( \frac{q_\ell \cdot L}{2} \right) \right) \right] \frac{i}{p - m}. \tag{6.2}
\]

Figure 4: A photon with the momentum \( k \) is attached to a part of the correlation function corresponding to the \( j \)-th fermion line.

Here, \( p_i = p_{i-1} + q_i \) and the index \( r \) labels the new vertex, which is created by this attachment. We arrive at:

\[
\mathcal{M}_{j,r}(k) = \epsilon_\mu \mathcal{M}_{j,r}^\mu(k),
\]

with

\[
k_\mu \mathcal{M}_{j,r}^\mu(k) = \frac{i}{p_n + k - m} \left( -2g\gamma^\mu \sin\left( \frac{q_n \cdot L}{2} \right) \right) \cdots \frac{i}{p_r + k - m} \left( -2g\gamma^\mu \sin\left( \frac{k \cdot L}{2} \right) \right) \frac{i}{p_r - m} \cdots \frac{i}{p - m}. \tag{6.2}
\]
Using now the identity $k^2 = (p^2 + k^2 - m^2) - (p^2 - m^2)$ we obtain:

$$k_\mu M_{j,r}^\mu(k) = 2g \sin \left( \frac{kL}{2} \right)$$

$$\times \left[ \prod_{\ell=1}^n \left( \frac{i}{\tilde{p}_\ell + \theta(\ell - r - 1)k - m} - \frac{i}{\tilde{p}_\ell + \theta(\ell - r)k - m} \right) \left( -2g\gamma^\mu \sin \left( \frac{q_\ell L}{2} \right) \right) \right] \frac{i}{p - m}, \quad (6.3)$$

where $\theta(n)$ is the step function defined by

$$\theta(n) \equiv \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases}, \quad \forall n \in \mathbb{Z}.$$

There are indeed $n + 1$ possibilities to attach the photon to the fermion line. Summing over all these insertion points $r$, we arrive at:

$$\sum_{r=0}^n k_\mu M_{j,r}^\mu(k) =$$

$$= \sum_{r=0}^n \left[ \prod_{\ell=1}^n \left( \frac{i}{\tilde{p}_\ell + \theta(\ell - r - 1)k - m} - \frac{i}{\tilde{p}_\ell + \theta(\ell - r)k - m} \right) \left( -2g\gamma^\mu \sin \left( \frac{q_\ell L}{2} \right) \right) \right]$$

$$\times \left( 2g \sin \left( \frac{kL}{2} \right) \right) \frac{i}{p - m} + \frac{i}{\tilde{p}_n + k - m} \left( -2g\gamma^\mu \sin \left( \frac{q_n L}{2} \right) \right) \cdots$$

$$\times \cdots \frac{i}{\tilde{p} + k - m} \left( -2g\gamma^\mu \sin \left( \frac{kL}{2} \right) \right) \frac{i}{p - m}. \quad (6.4)$$

After an appropriate redefinition of the summation index in the second term of the above expression and after a long but straightforward calculation, we arrive at:

$$\sum_{r=0}^n k_\mu M_{j,r}^\mu(k) = 2g \sin \left( \frac{kL}{2} \right) \left[ \prod_{\ell=0}^n \left( \frac{i}{\tilde{p}_\ell - m} - \frac{i}{\tilde{p}_\ell + k - m} \right) \left( -2g\gamma^\mu \sin \left( \frac{q_\ell L}{2} \right) \right) \right]. \quad (6.5)$$

As next, let us sum over all possible fermion lines, (i.e. sum over all $j$'s). We arrive at the Ward-Takahashi identity for the correlation functions, which reads:

$$k_\mu M_\mu(k; \{p_i, q_i\}; L) = 2g \sin \left( \frac{kL}{2} \right)$$

$$\times \sum_j \left[ M_0(\{p_i\}, q_1, \ldots, q_j - k, \ldots, q_n; L) - M_0(p_1, \ldots, p_j + k, \ldots, p_n; \{q_i\}; L) \right]. \quad (6.6)$$

Using now the standard LSZ formalism, and considering only the on-shell external momenta $\{p_i\}$ and $\{q_i\}$, the r.h.s. of the above equation vanishes by usual argumentation from the commutative U(1) gauge theory. The WT identity for the $S$-matrix elements is therefore given by:

$$k_\mu M_\mu(k; \{p_i, q_i\}; L) = 0. \quad (6.7)$$

The same expression is indeed valid if the photon is inserted to a fermion loop. This can be seen by putting $p_0 = p_n$ in the Eq. (6.5).
Path Integral Approach

In this section, the WT identity of noncommutative dipole U(1) gauge theory with matter fields in the adjoint representation will be calculated using the non-perturbative path integral method. Using the local gauge transformation of the matter fields from Eq. (3.12), it can be shown that the inner product $\bar{\psi} \star \psi$ is invariant:

$$\bar{\psi} \star \psi \rightarrow (U \star (\bar{\psi} \star \psi)) \star U^{-1} = (U^{-1}U)(\bar{\psi} \star \psi) = \bar{\psi} \star \psi.$$  \hspace{0.5cm} (6.8)

Note that both $U \in U(1)$ and the product $\bar{\psi} \star \psi$ are dipoleless. Let us now consider the following expression

$$F_{\{\mu_1, \nu_i\}}(\{x_i, x_i'\}) = \int D\bar{\psi} \int D\psi \int d^4x \mathcal{L}_{\text{QED}}^\text{adj} \mathcal{T}\left(\prod_{i=1}^{n} \bar{\psi}_{\mu_i}(x_i)\bar{\psi}_{\nu_i}(x_i')\right).$$  \hspace{0.5cm} (6.9)

Under an infinitesimal gauge transformation of the matter fields in the form given in Eq. (3.12), we have:

$$\begin{align*}
0 &= \int D\bar{\psi} D\psi DA e^{i \int d^4x \mathcal{L}_{\text{QED}}^\text{adj}} \left[\prod_{i=1}^{n} \bar{\psi}_{\mu_i}(x_i)\bar{\psi}_{\nu_i}(x_i')\right] \int d^4x (\bar{\psi}\gamma^\mu \star [\partial_\mu \alpha, \psi])(x) \\
&-\left(\sum_{i=1}^{n} \bar{\psi}_{\mu_i}(x_1) \cdots [\alpha, \psi_{\mu_i}](x_i) \cdots \bar{\psi}_{\mu_n}(x_n)\right) \left(\prod_{j=1}^{n} \bar{\psi}_{\nu_j}(x'_j)\right) \\
&-\left(\prod_{j=1}^{n} \bar{\psi}_{\nu_j}(x_j)\right) \left(\sum_{i=1}^{n} \bar{\psi}_{\nu_i}(x'_i) \cdots [\alpha, \psi_{\nu_i}](x'_i) \cdots \bar{\psi}_{\nu_n}(x'_n)\right). \hspace{0.5cm} (6.10)
\end{align*}$$

Here, Eq. (3.13) and the invariance of the measure of the integral under the infinitesimal transformation from Eq. (3.12) are used. After a partial integration in the first term on the r.h.s. of Eq. (6.10) and using the relation:

$$[\alpha, \psi_{\mu_i}](x_i) = -\int d^4x \psi_{\mu_i}(x_i)\alpha(x)\left(\delta(x-x_i - \frac{L}{2}) - \delta(x-x_i + \frac{L}{2})\right),$$  \hspace{0.5cm} (6.11)

and the corresponding relation for $\bar{\psi}$ in the second and third terms of Eq. (6.10), we have:

$$ik_\mu \langle \Omega | T \tilde{J}^\mu(k) \prod_{i=1}^{n} \bar{\psi}_{\mu_i}(\ell_i)\bar{\psi}_{\nu_i}(\ell_i') | \Omega \rangle =$$

$$\times 2ig \sin\left(\frac{k \cdot L}{2}\right) \left[\langle \Omega | T \left(\sum_{i=1}^{n} \bar{\psi}_{\mu_i}(\ell_1) \cdots \bar{\psi}_{\mu_i}(\ell_i - k) \cdots \bar{\psi}_{\mu_n}(\ell_n)\right) \left(\prod_{j=1}^{n} \bar{\psi}_{\nu_j}(\ell_j')\right) | \Omega \rangle \\
-\langle \Omega | T \left(\prod_{j=1}^{n} \bar{\psi}_{\nu_j}(\ell_j)\right) \left(\sum_{i=1}^{n} \bar{\psi}_{\nu_i}(\ell_i') \cdots \bar{\psi}_{\nu_n}(\ell_i' + k) \cdots \bar{\psi}_{\nu_n}(\ell_n')\right) | \Omega \rangle \right].$$  \hspace{0.5cm} (6.12)

This relation can be finally rewritten in the form given in Eq. (6.6), that we obtained in the previous section using the perturbative method. Eq. (6.7) can therefore be given for the S-matrix elements, too.
7 Renormalization Constants and the one-loop $\beta$-Function

In this section the renormalization constants of the noncommutative dipole QED with adjoint matter fields will be studied non-perturbatively and the one-loop $\beta$-function of the noncommutative QED with adjoint matters will be derived explicitly. Using the WT identity, which is proved in the previous section, we will first show that $Z_1 = Z_2$ in all orders of perturbative expansion. We then turn to $Z_3$ and argue that due to the gauge invariance arising from WT identity, the vacuum polarization amplitude is in all order of the perturbative expansion proportional to the usual tensorial structure $(p_\mu p_\nu - \eta_{\mu\nu}p^2)$, where $p$ is the external photon momentum. Note that the same situation occurs in the ordinary commutative QED but in no way in the Moyal case.

7.1 $Z_1$ and $Z_2$

Let us consider the vertex function $\Gamma^\mu(p, q)$ in the momentum space with the incoming external momentum $q$ for the photon and outgoing momentum $p$ for the fermion [See Fig. 5]:

\[ p + q = p' \]

\[ = 2g \sin \left( \frac{q_L}{2} \right) \]

\[ S(p + q)(−2ig\gamma_\mu\Gamma^\mu(p + q, p))S(p) = 2g \sin \left( \frac{q_L}{2} \right)(S(p) − S(p + q)), \]

(7.1)

where $S(p)$ is the exact fermion propagator. Taking as next the zero momentum limit $q \to 0$, the vertex function (7.1) can be given by:

\[ \Gamma^\mu(p + q, p) = Z_1^{-1}\gamma^\mu \sin \left( \frac{q_L}{2} \right), \]

(7.2)

which indeed gives an appropriate definition for $Z_1$. Using, as usual the LSZ formalism, the only relevant term in the exact fermion propagator is given by:

\[ S(p) = \frac{iZ_2}{p - m}. \]

(7.3)

Replacing as next the vertex function and the fermion propagator from Eq. (7.2) and (7.3) in the Eq. (7.1), we obtain $Z_1 = Z_2$, as expected. This result is obtained without any use of perturbation theory, and is therefore exact for all orders of perturbative expansion.
7.2 The General Structure of Vacuum Polarization Tensor and $Z_3$

Let us consider the full photon propagator $D_{\mu\nu}(p,L)$ with $p$ the external photon momentum and $L$ the dipole length. As in the ordinary QED $D_{\mu\nu}(p,L)$ is given as a series of 1PI vacuum polarization tensor $i\Pi_{\mu\nu}(p,L)$, which includes all loop corrections [see Fig. 6].

$$D_{\mu\nu} \equiv \mu^{\mu} \nu^{\nu} = \mu^{\mu} \nu^{\nu} + \mu^{\mu} i\Pi^{\mu \nu} + \mu^{\mu} i\Pi^{\mu \nu} i\Pi^{\mu \nu} + \cdots$$

with

$$\mu^{\mu} i\Pi^{\mu \nu} \equiv i\Pi_{\mu\nu}(p,L)$$

Figure 6: The full photon propagator can be written as a series of one particle irreducible (1PI) diagrams.

According to the WT identity $D_{\mu\nu}(p,L)$ must satisfy the relation:

$$p^\mu D_{\mu\nu}(p,L) = 0 = p^\nu D_{\mu\nu}(p,L).$$

(7.4)

The same relation must therefore be satisfied by each $i\Pi_{\mu\nu}(p,L)$. The only two index object can be made from $p_\mu$, $\eta_{\mu\nu}$ and $L_\mu$ which satisfies both sides of the above equation is given by

$$i\Pi_{\mu\nu}(p,L) = 4i \sin^2\left(\frac{p \cdot L}{2}\right)(p_\mu p_\nu - \eta_{\mu\nu} p^2)\Pi(p,L).$$

(7.5)

Here we have separated the contribution of two vertices $V_\mu$ and $V_\nu$ [Eq. (3.5)] as a factor $-4 \sin^2\frac{p \cdot L}{2}$ before $\Pi(p,L)$. Note that the above tensor structure does not include $L$. Remember that in the Moyal case no such simple tensor structure can be obtained for the vacuum polarization tensor [14].

Now, summing up the 1PI diagrams from the r.h.s. of the Fig. 6, we obtain:

$$D_{\mu\nu}(p,L) = \frac{-i \eta_{\mu\nu}}{p^2[1 + 4 \sin^2\left(\frac{p \cdot L}{2}\right) \Pi(p,L)]},$$

(7.6)

where a term proportional to $\frac{p_\mu p_\nu}{p^2}$ on the r.h.s. of the above equation is neglected, because it does not contribute in the $S$-matrix elements. Here $\Pi(p,L)$ includes all radiative corrections.

The general structure of $Z_3$ can now be defined using the Eq. (7.6) where the small momentum limit $p_i \to 0, i = 1, 2, 3$ is to be considered

$$Z_3 = \frac{1}{1 + (p \cdot L)^2 \Pi(0,L)},$$

(7.7)

The factor $(p \cdot L)^2$ which appears before $\Pi(0,L)$ in the denominator of the above equation arises from the $p_i \to 0$ limit of $\sin^2\left(\frac{p \cdot L}{2}\right)$ in the denominator of the expression on the r.h.s of the Eq. (7.6). As
it is known, in the ordinary QED, where no $\sin^2\left(\frac{p \cdot L}{2}\right)$ factor appears in the denominator of $Z_3$, the renormalization constant $Z_3$ is defined by the function $\Pi(p^2)$ evaluated at $p^2$ exactly equal to zero. It can be shown that in the commutative case the function $\Pi(p^2)$ coming from the vacuum polarization tensor includes both $p$-dependent and $p$-independent parts. In the limit $p_i \to 0$ the $p$-dependent part can be neglected comparing to the $p$-independent part because of its polynomial structure. This is the same as putting $p_i$ exactly equal zero, from the begin on, in the original commutative $\Pi(p^2)$.

In the noncommutative dipole Field Theory, however, $\Pi(p, L)$ is $ab initio$ multiplied by $\sin^2\left(\frac{p \cdot L}{2}\right)$. This means that no $p$-independent terms appear in the denominator of $\Pi_{\mu\nu}(p, L)$ and $Z_3$. The factor $(p \cdot L)^2$ survives therefore the $p_i \to 0$ $(i = 1, 2, 3,)$ limit. Using now the Eq. (5.7), $\delta Z_3$ can be given in all order of perturbative expansion by:

$$\delta Z_3 = -(p \cdot L)^2 \Pi(0, L), \quad \text{with} \quad p_i \to 0.$$  \hspace{1cm} (7.8)

### 7.3 One-loop $\beta$-Function of Noncommutative Dipole QED

In this section, we will first calculate the one-loop $\beta$-function of noncommutative dipole QED with adjoint matters explicitly. We will show that it is proportional to the same factor $(p \cdot L)^2$, which appears also in $\delta Z_3$ from Eq. (7.8). Using a semi-classical analysis we will then explain the physical origin of this factor.

Following the standard procedure from commutative QED, we calculate the one-loop $\beta$-function, by separating the Lagrangian of the theory in two parts, the renormalized part and the counterterm part. The renormalized part is in terms of renormalized fields $\psi_r, A_{\mu,r}$ and renormalized coupling constant $g$. They are related to bare fields $\psi$ and $A_{\mu}$ and bare coupling constant $g_0$ through standard commutative relations:

$$\psi_r = Z_2^{-1/2} \psi, \quad A_{\mu,r} = Z_3^{-1/2} A_{\mu}, \quad \text{and} \quad g_0 = g \ Z_3^{-1/2} Z_1 Z_2^{-1} = g \ Z_3^{-1/2}. \quad (7.9)$$

In the last expression, the WT identity from previous section is used ($Z_1 = Z_2$). All loop diagrams which arise from the perturbative expansion using this Lagrangian, depend therefore on renormalized coupling constant $g$, which must be taken as a function of the renormalization (energy) scale $\mu$. The $\beta$-function of the theory is then defined by the variation of $g$ with respect to $\mu$:

$$\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu}. \quad (7.10)$$

According to the last expression in Eq. (7.9) the renormalized coupling constant is proportional to $Z_3$. In the one-loop order it is given by

$$Z_3 = 1 + \delta Z_3 = 1 + \frac{g^2(\mu)}{(4\pi)^{d/2}} \sin^2\left(\frac{\vec{p} \cdot \vec{L}}{2}\right) \int_0^1 dx \ (8x(1 - x)) \frac{\Gamma(2)}{-x(1 - x)p^2} x^{d/2} \bigg|_{(p_0 = \mu, \vec{p}^2 < \mu^2)}. \quad (7.11)$$

22
The factor \( \sin(\vec{p} \cdot \vec{L}/2) \) comes from the vertex of two fermion and one gauge field from Eq. (3.5). Taking \( p_0 = \mu \) and \( \vec{p}^2 << \mu^2 \) the renormalization condition \( p^2 = p_0^2 - \vec{p}^2 = \mu^2 \) is guaranteed. Putting now \( p = (p_0 = \mu, \vec{p}) \) with small \( \vec{p}^2 << \mu^2 \) and \( L = (0, \vec{L}) \), the factor \( \sin(\vec{p} \cdot \vec{L}/2) \) can be replaced by \( (\vec{p} \cdot \vec{L}/2) \).

The one-loop \( \beta \)-function of the noncommutative dipole QED is then by:

\[
\beta(g) = -\frac{g^3(\mu)}{12\pi^2} \alpha^2(\vec{p}, \vec{L}),
\]

(7.12)

where we have introduced the function \( \alpha(\vec{p}, \vec{L}) \equiv \vec{p} \cdot \vec{L} \) with \( \vec{p}^2 << \mu^2 \). The minus sign of \( \beta(g) \) indicates that the noncommutative QED with adjoint matters is asymptotically free \[^{[\text{7}]\text{f}}\]. According to this result the one-loop \( \beta \)-function of the theory depends on the product \( \vec{p} \cdot \vec{L} \equiv |\vec{p}| |\vec{L}| \cos \vartheta \) with \( \vartheta \) the relative angle between the external momentum \( \vec{p} \) and the dipole length \( \vec{L} \). This means that the function \( \beta(g) \) vanishes not only at \( |\vec{L}| = 0 \), as expected\[^{[\text{7}]\text{f}}\], but also when either \( |\vec{p}| = 0 \) (forward scattering) or when \( \vartheta = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \). The theory, however, remains always asymptotically free, because the factor \( \alpha^2(\vec{p}, \vec{L}) \) in the one-loop \( \beta \)-function is always positive. The appearance of the scalar product \( (\vec{p} \cdot \vec{L}) \) in the argument of the sine is due to the broken Lorentz symmetry of the noncommutative dipole QED, which has its origin in the appearance of a fixed vector \( \vec{L} \) with a definite length and direction.

In the following we would like to give a physical interpretation of the factor \( \alpha(\vec{p}, \vec{L}) \) in the one-loop \( \beta \)-function from Eq. (7.12):\[^{[\text{7}]\text{f}}\]

As is well-known, in the QFT, in general, the results of the scattering amplitudes in the small momentum limit \( (p_i \to 0, i = 1, 2, 3) \) must coincide with the classical results. In the Born approximation the scattering amplitudes are proportional to the Fourier transformed of the scattering potential energy. In a theory with finite point like charges, the potential \( V_e(\vec{r}) \) is a central Coulomb potential \( V_e(r) = \frac{g}{4\pi\epsilon_0 r} \). Hence in the framework of QED for the small momentum limit, the scattering amplitudes of the electrons are proportional to \( \tilde{V}_e(p) = \frac{g^2}{p^2} \) in the momentum space.

In the noncommutative dipole QED with matter fields in the adjoint representation, however, no point like charged particles are present. The theory includes only finite multipoles. In the lowest order of the dipole length \( L \), the scattering potential energy is therefore the energy between two dipoles. According to the Born approximation, the classical scattering amplitudes are proportional to the Fourier transformed of the potential energy of two dipoles \( g_0L_1 \) and \( g_0L_2 \) at large distances. For \( L_1 = L_2 = L \) this classical potential energy in the momentum space is given by [see Appendix A for a derivation]:

\[
\tilde{V}_L(p) \bigg|^{\text{classical}} = -\frac{\tilde{g}_0^2}{p^2}, \quad \text{with} \quad \tilde{g}_0 \equiv g_0\alpha(\vec{p}, \vec{L}),
\]

(7.13)

\[^{[\text{7}]\text{f}}\text{For } |\vec{L}| = 0 \text{ noncommutative dipole QED with adjoint matters turns out to be free.}
and \( \alpha(\vec{p}, \vec{L}) \equiv \vec{p} \cdot \vec{L} \). Hence comparing to the potential of a point like charged particle \( \tilde{V}_e(p) \), the bare coupling constant must be replaced by \( \tilde{g}_0 \), and is therefore proportional to the factor \( \alpha(\vec{p}, \vec{L}) \). This replacement is only true for a process where two photons with the momentum \( p \) are involved as in Fig. 7.

![Figure 7](image)

The new renormalized coupling constant can then be given by \( \tilde{g} \equiv \alpha(\vec{p}, \vec{L})g = \tilde{g}_0 Z_3^{1/2} \). This is exactly the same factor which appears in the expression (7.11) for \( Z_3 \). Going now through the same procedure as described above the one-loop \( \beta \)-function of noncommutative dipole QED turns out to be given by the same Eq. (7.12).

### 8 Form Factors and One-Loop Anomalous Magnetic Moment

In this section we will study the form factors of noncommutative dipole QED with adjoint matter fields and derive explicitly the anomalous magnetic moment of this theory.

The general form of the vertex function \( \Gamma^\mu(p', p) \) of noncommutative QED is given by:

\[
\Gamma^\mu(p', p) = \left[ \gamma^\mu A_1 + \frac{(p + p')^\mu}{m} A_2 + \frac{(p' - p)^\mu}{m} A_3 + mL^\mu A_4 \right] \sin \left( \frac{q \cdot L}{2} \right),
\]

where according to the diagram on the l.h.s. of the Fig. 2, \( p' \) and \( p \) are the momenta of the external fermions and \( q \) is the photon momentum. The functions \( A_i, i = 1, 2, 3, 4 \) depend in general on \( \frac{q^2}{m^2}, p.L, p'.L, \frac{p}{m}, \frac{p'}{m}, mL \) which appear only in polynomial structures. Here \( L \) is the dipole length of the theory and for our physical purposes is a small parameter. The last term on the r.h.s. of Eq. (8.1) can therefore be neglected. Besides the parameter \( mL \) does not appear in the functions \( A_i, i = 1, 2, 3 \) anymore. Following the standard arguments form commutative QED and using the WT identity \( q_\mu \Gamma^\mu(p', p) = 0 \), we arrive at:

\[
\Gamma^\mu(p', p) = \left[ \gamma^\mu F_1(q^2, p \cdot L, p' \cdot L) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2, p \cdot L, p' \cdot L) \right] \sin \left( \frac{q \cdot L}{2} \right),
\]

where \( \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \) and \( F_i, i = 1, 2 \) are the form factors of the theory. Here, in comparison to the commutative QED, they have new physical origins. Remember that in the noncommutative dipole
QED, no point like charged particles are present. The theory includes only multipoles. In a semi-classical approximation, the form factors are therefore defined by the interaction of a finite dipole moment $g \bar{L}$ with external electric and magnetic fields. This is in contrast to commutative QED, where the interaction of point like charged particles with external fields defines the form factors.

To compute $F_{1}(q^{2}, p \cdot L, p' \cdot L)$ let us take, as in the commutative case, a classical external potential $A_{\mu}^{\text{cl}}(x) = (\phi(\bar{x}), \vec{0})$, where $\phi(\bar{x})$ is an arbitrary electrical potential. In the momentum space, it can be given by

$$\tilde{A}_{\mu}^{\text{cl}}(q) = (2\pi\delta(q^{0})\tilde{\phi}(\vec{q}), \vec{0}).$$

Using $\tilde{A}_{\mu}^{\text{cl}}(q)$, the scattering amplitude of a dipole moment can be calculated and reads:

$$iM = ig \bar{u}(p') \Gamma^{0}(p', p) u(p) \tilde{\phi}(\vec{q}).$$

(8.3)

Now going back to the Eq. (8.2) and taking the small momentum limit $q_{i} \to 0, i = 1, 2, 3$, we arrive at:

$$\Gamma^{0}(p', p) = 2g\gamma^{0} \frac{q \cdot \bar{L}}{2} F_{1}(0, L).$$

(8.4)

Putting now this expression on the r.h.s. of the Eq. (8.3), we obtain:

$$iM = -iF_{1}(0, L) (g \bar{L} \cdot q) \tilde{\phi}(\vec{q}) (2m\xi^{\dagger} \xi),$$

(8.5)

where $\xi$ and $\xi^{\dagger}$ are two component spinors. Going back to the position space and comparing this relation with the classical potential energy of an external electric field $\vec{E}$ acting on a dipole moment $g \bar{L}$

$$V(\bar{x}) = -g F_{1}(0, L) \bar{L} \cdot \vec{E},$$

we have

$$F_{1}(0, L) = 1.$$  

(8.6)

This is in contrast with the commutative QED where, for the limit of small momenta, the scattering amplitude is to be compared with the Fourier transform of the classical potential energy of a point like charged particle in the external field $\phi(\bar{x})$.

Taking now $A_{\mu}^{\text{cl}}(x) = (0, \tilde{A}_{\text{cl}}(\bar{x}))$ as an arbitrary magnetic potential with $\tilde{A}_{\mu}^{\text{cl}}(q) = (0, 2\pi\delta(q^{0})\tilde{A}_{\mu}^{\text{cl}}(\vec{q}))$ in the momentum space, the scattering amplitude of a dipole moment can be first given by:

$$iM = 2ig \frac{\vec{q} \cdot \bar{L}}{2} \bar{u}(p') \left[ \gamma^{i} F_{1}(0, L) + \frac{i \sigma^{i} q_{\nu}}{2m} F_{2}(0, L) \right] u(p) A_{\mu}^{\text{cl}}(\vec{q}).$$

(8.7)

Using now the standard relation:

$$\bar{u}(p') \left[ \gamma^{i} F_{1}(0, L) + \frac{i \sigma^{i} q_{\nu}}{2m} F_{2}(0, L) \right] u(p) = 2m \xi^{\dagger} \left\{ -\frac{i}{2m} \varepsilon^{ijk} q^{j} \sigma^{k} \left[ F_{1}(0, L) + F_{2}(0, L) \right] \right\} \xi,$$

(8.8)
with the Pauli matrices $\sigma^k, k = 1, 2, 3$, we arrive at:

$$i\mathcal{M} = -(g\vec{L} \cdot \vec{q}) (2m) \xi^i (\frac{\sigma^k}{2}) \xi [e^{kij}q^j \vec{A}^i_d(q)] \left[ 1 + F_2(0, L) \right],$$  \hspace{1cm} (8.9)

where we have used the previous result from Eq. \[8.6\]. Going again back to the position space and comparing this relation with the classical potential energy of an external magnetic field $\vec{B}$ acting on a dipole moment $g\vec{L}$:

$$V(\vec{x}) = -\langle \vec{\mu} \rangle \cdot \left( \frac{\vec{L}}{|L|} \vec{\nabla} \right) \vec{B}(\vec{x}),$$

where the magnetic dipole moment $\vec{\mu} \equiv g_{\text{landé}} \left( \frac{g\vec{L}}{2m} \right) \vec{S}$ is introduced, we arrive at:

$$\langle \vec{\mu} \rangle = \frac{g|\vec{L}|}{m} \left[ F_1(0, L) + F_2(0, L) \right] \xi^i \vec{S} \xi,$$  \hspace{1cm} (8.10)

with $\vec{S} = \vec{\sigma}/2$. The Landé factor is therefore given by: $g_{\text{landé}} = 2[1 + F_2(0, L)]$, where $2F_2(0, L)$ is the anomalous part of this factor. It is now possible to find the one-loop contribution to $F_2(0, L)$ from the one-loop Feynman diagram corresponding to the vertex function (5.3). We obtain

$$F_2(0, L) = \frac{g^2}{\pi^2} m^2 \vec{L}^2 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_0^{1-\alpha-\beta} d\gamma \delta(\alpha + \beta + \gamma - 1) \frac{\alpha(\beta + \gamma)}{(\alpha + \beta + \gamma)^4} \exp \left( \frac{(\beta + \gamma)^2}{\alpha + \beta + \gamma} \right)$$

$$= \frac{g^2}{2\pi^2} m^2 \vec{L}^2.$$  \hspace{1cm} (8.11)

The anomalous magnetic moment is therefore proportional to $\vec{L}^2$ and vanishes by taking the limit $L \to 0$, as expected from a free Field Theory.

## 9 Conclusion

In this paper, a detailed study of the noncommutative dipole QED with matter fields in the adjoint representation is presented. After introducing the action of the theory in Sect. 3, the Noether currents and the corresponding conserved charges are derived using the noncommutative version of the Noether procedure. In analogy to the noncommutative Moyal case [11], noncommutative dipole QED possesses three different global vector and axial vector currents. In Sect. 4, the axial anomaly for all these currents are calculated in two and four dimensions using the point split and dimensional regularization methods. In [12], the axial anomaly was calculated for only one of the three global currents of the theory using the Fujikawa’s path integral method [13]. Our result coincides with the result presented in this paper. We have further shown, that in two dimensions the axial anomalies corresponding to all three currents of the theory vanish after integrating over one space component. The axial charges corresponding to these currents are therefore conserved. In four dimensions, however,
the axial charges corresponding to two currents $J_{\mu(5)}'$ and $J_{\mu(5)}''$ from Eq. (4.2) are anomalous, whereas the axial charge of $J_{\mu(5)}$ from Eq. (4.1) is still conserved.

In Sect. 5, the fermion- and photon-self energy and the vertex function are calculated up to one-loop order. Comparing the one-loop Feynman integrals of the fermion self-energy and the vertex function, which include both planar and nonplanar parts, we have shown that $Z_1 = Z_2$, where $Z_i, i = 1, 2$ are the standard renormalization constants. Further the vacuum polarization tensor has the usual tensorial structure $(\eta_{\mu\nu}p^2 - p_\mu p_\nu)$. These properties are also valid for all higher orders of perturbative expansion. This could be shown in a detailed analysis of the Ward-Takahashi identity of the noncommutative QED with adjoint matter fields using two different methods in Sect. 6. It shall be noticed, that the noncommutative Moyal case has none of these two properties [14]. This is mainly so, because in the Moyal noncommutativity, even in one-loop order, additional diagrams appear, which arise from additional three and four gauge vertices. These vertices are absent in the noncommutative dipole QED.

The general structure of the renormalization constant $Z_3$ is presented in Sect. 7, and the one-loop $\beta$-function of the theory is then calculated explicitly. We have found that comparing to the commutative QED, the noncommutative dipole QED with adjoint matters is asymptotically free. Further the $\beta$-function of the theory is proportional to a factor $\alpha(\vec{p}, \vec{L}) \equiv (\vec{p} \cdot \vec{L})$ with small momentum $\vec{p}$. In fact, it could be shown that $\vec{p}^2 << \mu^2$, where $\mu$ is the renormalization scale parameter. As in the semi-classical approximation of commutative QED, we expect that for the small momenta $\vec{p}$, the scattering amplitudes must coincide with the classical results of scattering processes. According to the Born approximation, the scattering amplitude must be proportional to the Fourier transformed of the potential energy. Since in a noncommutative dipole theory with adjoint matters, point like charged particles are absent, the potential energy can only be defined between multipoles. In the lowest order of multipole expansion, we calculated the potential energy between two dipoles [see Appendix A], and found out that the bare coupling constant of the noncommutative dipole QED must indeed be modified by the factor $\alpha(\vec{p}, \vec{L})$. This is the same factor which appears in the one-loop $\beta$-function of the theory.

Since the one-loop $\beta$-function of the theory depends on the product $\vec{p} \cdot \vec{L} \equiv |\vec{p}| \ |\vec{L}| \ \cos(\theta)$ with $\theta$ the relative angle between the momentum $\vec{p}$ and the dipole length $\vec{L}$, it vanishes not only at $|\vec{L}| = 0$, as expected from a free theory, but also when either $|\vec{p}| = 0$, i.e. in a forward scattering or when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. The theory, however, remains always asymptotically free, because the factor $\alpha^2(\vec{p} \cdot \vec{L})$ in the one-loop $\beta$-function is always positive. Hence the value of the one-loop $\beta$-function depends on the direction of the dipole length $\vec{L}$. This is mainly because introducing a constant vector $\vec{L}$ with a definite length and direction in the theory breaks the Lorentz symmetry of the theory.
In Sect. 8, the form factors of the theory are defined using a semi-classical approximation. In contrary to the commutative QED, where the form factors can be defined by studying the effect of external electric and magnetic fields on point-like charged particles, the form factors in the noncommutative QED are to be defined by the Fourier transformed of the potential energy between the dipole and external electric and magnetic fields. The anomalous magnetic moment is then calculated up to one-loop order. We have shown, that it is proportional to \((m\vec{E})^2\) and vanishes by taking \(L \to 0\).

10 Acknowledgement

Both authors thank F. Ardalan and H. Arfaei for useful discussions.

References

[1] M. R. Douglas, and C. Hull, D-branes and the Noncommutative Torus, JHEP 9802 (1998) 008, hep-th/9711165.

F. Ardalan, H. Arfaei, and M.M. Sheikh-Jabbari, Mixed Branes and M(atrix) Theory on Noncommutative Torus, hep-th/9803067. Noncommutative Geometry From Strings and Branes, JHEP 9902 (1999) 016, hep-th/9810072. Dirac Quantization of Open Strings and Noncommutativity in Branes, Nucl. Phys. B576, 578 (2000), hep-th/9906161.

Y.-K. E. Cheung, and M. Krogh, Noncommutative Geometry from 0-Branes in a Background B Field, Nucl. Phys. B528, 185 (1998).

C.-S. Chu, and P.-M. Ho, Noncommutative Open String and D-brane, Nucl. Phys. B550, 151 (1999), hep-th/9812219. Constrained Quantization of Open String in Background B Field and Noncommutative D-brane, Nucl. Phys. B568, 447 (2000), hep-th/9906192.

V. Shomerus, D-branes and Deformation Quantization, JHEP 9906 (1999) 030, hep-th/9903203.

N. Seiberg, and E. Witten, String Theory and Noncommutative Geometry, JHEP 9909 (1999) 032, hep-th/9908142.

[2] I. Ya. Aref’eva, D. M. Belov, A. A. Giryavets, A. S. Koshelev, and P. B. Medvedev, Noncommutative Field Theories and (Super)String Field Theories, hep-th/0111208.

[3] M. R. Douglas, and N. A. Nekrasov, Noncommutative Field Theory, Rev. Mod. Phys. 73, 977 (2002), hep-th/0106048.
A Classical Potential of Two Electric Dipole Moments

Let us consider two electric dipole moments $g\vec{L}_1$ and $g\vec{L}_2$. The potential energy of the second dipole arising from the first one is given by:

$$V_L(\vec{r}) = g\vec{L}_2 \cdot \nabla \phi(\vec{r})$$

with

$$\phi(\vec{r}) = \frac{g}{4\pi} \frac{\vec{L}_1 \cdot \vec{r}}{r^3}.$$  \hspace{1cm} (A.1)
where $r$ is the distance between the center of two dipoles. In this section we would like to calculate the Fourier transformed of this potential energy explicitly. In the momentum space we have

$$\tilde{V}(\vec{p}) = \int d^3r e^{-i\vec{p} \cdot \vec{r}} V(\vec{r}). \quad (A.2)$$

Taking $\vec{p}$ along the $z$-axis and putting Eq. (A.1) in Eq. (A.2), we arrive at:

$$\tilde{V}(\vec{p}) = g^2 \frac{\vec{L}_2 \cdot \vec{L}_1}{4\pi} \left[ \int d^3r \nabla \left( \frac{\vec{L}_1 \cdot \vec{r}}{r^3} e^{-i\vec{p} \cdot \vec{r}} \right) + i\vec{p} \int d^3r \frac{\vec{L}_1 \cdot \vec{r}}{r^3} e^{-i\vec{p} \cdot \vec{r}} \right], \quad (A.3)$$

where an integration by part is performed. Using the well-known vector algebra identities, the first term is given by

$$g \int d^3r \nabla \left( \frac{\vec{L}_1 \cdot \vec{r}}{r^3} e^{-i\vec{p} \cdot \vec{r}} \right) = g \oint_S d\vec{s} \frac{\vec{L}_1 \cdot \vec{r}}{r^3} e^{-i\vec{p} \cdot \vec{r}} = g \lim_{r \to \infty} \int d\Omega \frac{\vec{L}_1 \cdot \vec{r}}{r} e^{-ipr \cos \theta}. \quad (A.4)$$

This term vanishes due to the factor $e^{-ipr \cos \theta}$ and the symmetric integration interval of $\cos \theta \in [-1, 1]$.

Now let us consider the second term on the r.h.s of the Eq. (A.3). In the spherical coordinates, we obtain

$$\tilde{V}(\vec{p}) = \frac{ig^2}{2} \vec{L}_2 \cdot \vec{p} \left( \vec{L}_1 \cdot \hat{z} \right) \int dr \frac{\partial}{-i\vec{p} \partial r} \int_{-1}^{1} d\cos \theta e^{-pr \cos \theta} = -\frac{g^2 \vec{L}_2 \cdot \vec{p} \vec{L}_1 \cdot \hat{z}}{p^2} \lim_{r \to 0} \frac{\sin(pr)}{r}. \quad (A.5)$$

We finally arrive at:

$$\tilde{V}(\vec{p}) = -\frac{g^2 \vec{L}_1 \cdot \vec{p} \vec{L}_2 \cdot \vec{p}}{p^2}. \quad (A.6)$$

This result is used in the Eq. (7.13).