Approximation of Functions over Manifolds: 
A Moving Least-Squares Approach

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Abstract

We present an algorithm for approximating a function defined over a $d$-dimensional manifold utilizing only noisy function values at locations sampled from the manifold with noise. To produce the approximation we do not require any knowledge regarding the manifold other than its dimension $d$. The approximation scheme is based upon the Manifold Moving Least-Squares (MMLS) presented in [25]. The proposed algorithm is resistant to noise in both the domain and function values. Furthermore, the approximant is shown to be smooth and of approximation order of $O(h^{m+1})$ for non-noisy data, where $h$ is the mesh size with respect to the manifold domain, and $m$ is the degree of a local polynomial approximation utilized in our algorithm. In addition, the proposed algorithm is linear in time with respect to the ambient-space’s dimension. Thus, in case of extremely large ambient space dimension, we are able to avoid the curse of dimensionality without having to perform non-linear dimension reduction, which introduces distortions to the manifold data. Using numerical experiments, we compare the presented method to state-of-the-art algorithms for regression over manifolds and show its potential.

1 Introduction

Approximating a function defined over an extremely large dimensional space from scattered data is a very challenging task. First, from the sample-set perspective, to achieve a constant sampling resolution, the number of points grows exponentially with respect to the number of dimensions. For example, a uniform grid on $[0, 1]^n$ with resolution of 0.1 requires $10^n$ samples. Second, the high dimensionality of the domain introduces serious computational issues. Thus, the performance of both parametric and non-parametric approximations (or regressions) deteriorates sharply as the dimension increases [6, 9, 10]. These types of problems, sometimes referred to as the curse of dimensionality, occur frequently in many scientific disciplines since data, originating from various sources and of various types, is becoming more and more available.

In the last two decades, there has been a rapid development of mathematical frameworks to deal with this problem setting. Among the various approaches are the utilization of Neural Networks [13], Dimensionality Reduction techniques [2, 11, 17, 22], kernel based methods [5, 8, 23, 26], Support Vector Regression [24] and a variety of other methods targeted at performing regression directly over manifold data [4, 7, 18, 19]. In many of these approaches, there exists an underlying assumption that the high dimensional domain of the point cloud has a lower intrinsic dimension. In other words, the given point cloud $\{r_i\}^N_{i=1} \subset \mathbb{R}^n$, for an extremely large
\( n \), is a sample set of a lower dimensional manifold \( \mathcal{M}^d \), where \( d \) is the intrinsic dimension of \( \mathcal{M} \) and \( d \ll n \). It is, therefore, their aim to harvest this geometrical connection between the points, in order to reduce the effective number of parameters needed to be optimized.

In this work, we utilize the Moving Least-Squares (MLS) framework to approximate a function based upon scattered samples. The MLS approximation was originally designed for the purpose of smoothing and interpolating scattered data, sampled from some multivariate function \([12, 14, 16, 20]\). Then, it evolved to deal with surfaces (i.e., \( n - 1 \) dimensional manifolds in \( \mathbb{R}^n \)), which can be viewed as a function locally rather than globally \([3, 15]\). This has been generalized lately in \([25]\) to the Manifold Moving Least-Squares (MMLS), which deals with manifolds of an arbitrary dimension \( d \) embedded in \( \mathbb{R}^n \).

The MMLS framework, which will be described formally in Section 2, aims at an implicit construction of the manifold’s atlas of charts. Explicitly, for each point \( p \in \mathcal{M} \) a local coordinate chart (mapping a neighborhood of \( p \) into a Euclidean \( d \)-dimensional linear space) is constructed. In this paper, we utilize the MMLS atlas of charts to provide an approximation of a function defined on the manifold. Since we approximate the function through its atlas of charts, on a local level the approximation is from \( \mathbb{R}^d \) to \( \mathbb{R}^n \). Thus, our approximation framework avoids the curse of dimensionality without having to globally project the sample set into a lower dimensional Euclidean space.

We show in Theorem 3.1 that our approximant is a smooth function defined on a neighborhood of the manifold domain. In addition, in Theorem 3.2 we show that in case of clean samples the approximation yields an \( O(h^{m+1}) \) approximation order, where \( h \) is the fill distance with respect to the manifold domain. Our algorithmic approach has linear complexity with respect to the ambient space’s dimension \( n \), which makes the utilization of the proposed method feasible in cases where \( n \) is extremely large.

The rest of the paper is organized as follows: in Section 2 we describe the MMLS approximation framework; in Section 3 we describe the proposed approach of function approximation over manifolds; and in Section 4 we give some numerical examples showing the potential of the proposed method.

2 Preliminaries - The MMLS Framework

2.1 MLS For Function Approximation

The moving least-squares for function approximation was first presented by McLain in \([16]\). Let \( \{x_i\}_{i=1}^N \) be a set of distinct scattered points in \( \mathbb{R}^d \) and let \( \{f(x_i)\}_{i=1}^N \) be the corresponding sampled values of some function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \). Then, the \( m^{th} \) degree moving least-squares approximation to \( f \) at a point \( x \in \mathbb{R}^d \) is defined as \( p_x(x) \), where

\[
p_x = \arg\min_{p \in \Pi_m^d} \sum_{i=1}^N (p(x_i) - f(x_i))^2 \theta(\|x - x_i\|),
\]

\( \theta(t) \) is a non-negative weight function (rapidly decreasing as \( t \rightarrow \infty \)), \( \| \cdot \| \) is the Euclidean norm and \( \Pi_m^d \) is the space of polynomials of total degree \( m \) in \( \mathbb{R}^d \). Notice, that if \( \theta(t) \) is of finite support then the approximation is made local, and if \( \theta(0) = \infty \) the MLS approximation interpolates the data.

We wish to quote here two previous results regarding the resulting approximation presented in \([14]\). In Section 3 we will prove properties extending these theorems to the general case of approximation of functions over a \( d \)-dimensional manifold residing in \( \mathbb{R}^n \).
**Theorem 2.1.** Let \( \theta(t) \in C^\infty \) and let the distribution of the data points \( \{x_i\}_{i=1}^N \) be such that the problem is well conditioned (i.e., the least-squares matrix is invertible). Then the MLS approximation is a \( C^\infty \) function interpolating the data points \( \{f(x_i)\}_{i=1}^N \).

The second result, dealing with the approximation order, necessitates the introduction of the following definition:

**Definition 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^d \). We say that a set \( X = \{x_i\}_{i=1}^N \subset \Omega \) is an \( h\)-\( \rho\)-\( \delta \) set (a set of fill distance \( h \), density \( \leq \rho \), and separation \( \geq \delta \)) if:

1. \( h \) is the fill distance with respect to the domain \( \Omega \):
   \[
   h = \sup_{x \in \Omega} \min_{x_i \in X} \|x - x_i\|, \tag{2}
   \]

2. \[
   \# \{X \cap \overline{B}(y,kh)\} \leq \rho \cdot k^d, \quad k \geq 1, \quad y \in \mathbb{R}^d. \tag{3}
   \]

Here \( \#Y \) denotes the number of elements in a given set \( Y \), while \( \overline{B}(x,r) \) is the closed ball of radius \( r \) around \( x \).

3. \( \exists \delta > 0 \) such that
   \[
   \|x_i - x_j\| \geq \frac{h}{\delta}, \quad 1 \leq i < j \leq N \tag{4}
   \]

**Remark 2.2.** Note, that in [14], the fill distance \( h \) was defined slightly different. However, the two definitions are equivalent.

**Theorem 2.3.** Let \( f \) be a function in \( C^{m+1}(\Omega) \) with an \( h\)-\( \rho\)-\( \delta \) sample set. Then for fixed \( \rho \) and \( \delta \), there exists a fixed \( k > 0 \), independent of \( h \), such that the approximant given by equation (1) is well conditioned for \( \theta \) with a finite support of size \( s = kh \). In addition, the approximant yields the following error bound:

\[
\|\tilde{p}(x) - f(x)\|_{\Omega,\infty} < M \cdot h^{m+1} \tag{5}
\]

for some \( M \) independent of \( h \).

**Remark 2.4.** Although both Theorem 2.1 and Theorem 2.3 are stated with respect to an interpolatory approximation (i.e., the weight function satisfies \( \theta(0) = \infty \)), the proofs articulated in [14] are still valid taking any compactly supported non-interpolatory weight function.

**Remark 2.5.** Notice that the weight function \( \theta \) in the definition of the MLS for function approximation is applied on the distances in the domain. In what follows, we will apply \( \theta \) on the distances between points in \( \mathbb{R}^n \) as we aim at approximating manifolds rather than functions. In order for us to be able to utilize Theorems 2.1 and 2.3 the distance in the weight function of equation [1] should be \( \theta(||(x,0) - (x_i,f(x_i))||) \) instead of \( \theta(||x - x_i||) \) (see Fig. 1). Nevertheless, the proofs of both theorems as presented in [14] are still valid even if we take the new weights.

**Remark 2.6.** The approximation order remains the same even if the weight function is not compactly supported in case the weight function decays fast enough (e.g., by taking \( \theta(t) \overset{\Delta}{=} e^{-t^2/k^2} \)).
Figure 1: The effect of remote points when taking $\theta(\| (x, 0) - (x_i, f(x_i)) \|)$ instead of $\theta(\| x - x_i \|)$. Assuming that the green line represents a given coordinate system around the point $x$ (marked by the blue $\times$), by taking the weights $\theta(\| x - x_i \|)$ the contribution of both the red and blue samples to the weighted cost function would be $O(h^{m+1})$. Alternatively, by taking $\theta(\| (x, 0) - (x_i, f(x_i)) \|)$ with a fast decaying weight function the contribution of the red points would be negligible. Thus, the approximation (in purple) would fit the behavior of the blue points alone.

2.2 The MMLS Projection

The MMLS projection, introduced in [25], is expressed in the following procedure. Let $\mathcal{M}$ be a $C^{m+1}$ manifold of dimension $d$ embedded in $\mathbb{R}^n$, and let $\{ r_i \}_{i=1}^N$ be points situated near $\mathcal{M}$ (i.e., samples of $\mathcal{M}$ with zero-mean additive noise). We wish to approximate the projection of a given point $r$ situated near these samples onto $\mathcal{M}$.

The projection procedure comprises two steps:

step 1. Find a local $d$-dimensional affine space $H = H(r)$ approximating the sampled points ($H \simeq \mathbb{R}^d$). This affine space will be used later as a local coordinate system.

step 2. Define the projection of $r$ using $p : H \rightarrow \mathbb{R}^n$, a local polynomial approximation of $\mathcal{M}$ over the new coordinate system $H$. Explicitly, we denote by $x_i$ the projections of $r_i$ onto $H$ and then define $f : H \rightarrow \mathbb{R}^n$ with noisy samples $f_i = r_i$. Accordingly, the $d$-dimensional polynomial $p$ is an approximation of the vector valued function $f$.

Remark 2.7. Since $\mathcal{M}$ is a differentiable manifold it can be viewed locally as a function from the tangent space to $\mathbb{R}^n$. Thus, it is plausible to find a coordinate system $H$ and refer to the manifold $\mathcal{M}$ locally as a function $f : H \rightarrow \mathbb{R}^n$.

A more elaborate description of the steps follows:

Step 1 - The local Coordinates

For a given point $r$, find a local $d$-dimensional affine space $H = H(r)$, and a point $q = q(r)$ on $H$, such that the following constrained problem is minimized:

$$J(r; q, H) = \sum_{i=1}^N d(r_i, H)^2 \theta(\| r_i - q \|)$$

s.t.

$$r - q \perp H \quad \text{i.e.,} \quad r - q \in H^\perp,$$

(6)

where $d(r_i, H)$ is the Euclidean distance between the point $r_i$ and the subspace $H$. $H^\perp$ is the $n - d$ dimensional orthogonal complement of $H$ with respect to the origin $q$. Explicitly, for $x \in H$ we have:

$$x = q + \sum_{k=1}^d \alpha_k e_k,$$
where \( \{e_k\}_{k=1}^d \) is some basis of the linear space \( H - q \).

**Assumption 2.8 (Uniqueness domain).** We assume that there exists a subset \( U \subset \mathbb{R}^n \) such that for any \( r \in U \) the minimization problem (6) has a unique minimum \( q(r) \in U \), and a unique affine subspace \( H(r) \). Furthermore, we demand that \( \mathcal{M} \subset U \), and that for any \( r \in U \) there is a unique Euclidean projection onto \( \mathcal{M} \).

Note, that the demand \( r - q(r) \perp H(r) \) implies that \( q(r) \) would be the same for all \( r \in U \) such that \( r - q(r) \in H^\perp \), which is an \( n - d \) dimensional subspace.

**Step 2 - The MLS projection.** Let \( x_i \) be the orthogonal projections of \( r_i \) onto \( H(r) \). As before, we note that \( r \) is orthogonally projected to the origin \( q \). Now we would like to approximate \( f : \mathbb{R}^d \to \mathbb{R}^n \), such that \( f_i = f(x_i) = r_i \). The approximation of \( f \) is performed by a weighted least-squares vector valued polynomial function \( \tilde{g}(x) = (g_1(x), ..., g_n(x))^T \) where, for \( 1 \leq k \leq n \), \( g_k(x) \in \Pi_m^d \) is a \( d \)-dimensional polynomial of total degree \( m \).

\[
\tilde{g} = \arg\min_{\tilde{p} = (p_1, ..., p_n)^T, p_i \in \Pi_m^d} \sum_{i=1}^N \|p(x_i) - \tilde{f}_i\|^2 \theta(\|r_i - q\|).
\]

(7)

The projection \( P_m(r) \) is then defined as:

\[
P_m(r) = \tilde{g}(0)
\]

3 Extending The MMLS to Function Approximation

In the following section we address the case where \( \mathcal{M}^d \subset \mathbb{R}^n \) is a \( d \)-dimensional manifold, and \( \psi : \mathcal{M}^d \to \mathbb{R}^n \) is a function sampled with noise at noisy locations. Explicitly, let \( \{p_i\}_{i=1}^N \subset \mathcal{M} \), \( r_i = p_i + \epsilon_i \), and \( \psi_i = \psi(p_i) + \delta_i \), then the sample-set at hand is \( \{(r_i, \psi_i)\}_{i=1}^N \). Given some point \( r \) adjacent to \( \mathcal{M} \) (i.e., \( r = p + \epsilon \), and \( p \in \mathcal{M} \)) we wish to approximate \( \psi(p) \). For simplicity, in what follows we assume that \( \psi : \mathcal{M}^d \to \mathbb{R} \) (i.e., scalar valued function). The extension to the multidimensional case is immediate.

Below we suggest an approximation framework and algorithm for this case, based upon the MMLS procedure described in the preliminaries. The main theoretical results of this section are the smoothness and approximation properties portrayed in Theorems 3.1, 3.2. Following this, we describe how one can utilize our proposed framework to produce interpolatory approximation. We conclude the section with a concise description of the algorithm.

3.1 Constructing The Function Approximation

For the purpose of discussion let us assume for a moment that our samples of \( \mathcal{M} \) are without noise, that is \( \{(p_i, \psi(p_i))\}_{i=1}^N \). Then, the most natural way to obtain an approximation to a differentiable function defined over a manifold, is to utilize a chart. More precisely, for any \( r \in \mathcal{M} \), given some coordinate chart \((V, \phi)\), where \( V \subset \mathcal{M} \) is an open neighborhood of \( r \) and \( \phi : V \to \mathbb{R}^d \), we would have liked to approximate the following function:

\[
g \triangleq \psi \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R},
\]

at \( x \in \mathbb{R}^d \) such that \( \phi^{-1}(x) = r \in \mathcal{M} \). This way, instead of trying to approximate a function from \( \mathbb{R}^n \) to \( \mathbb{R} \) we can approximate, on a local level, a function from \( \mathbb{R}^d \) to \( \mathbb{R} \). Since we assume
Let \( \mathcal{M} \) be a smooth manifold, it can be viewed, locally, as a graph of a differentiable function \( \eta : T_r \mathcal{M} \to \mathbb{R}^n \), where \( T_r \mathcal{M} \simeq \mathbb{R}^d \) is the tangent space of \( \mathcal{M} \) at point \( r \). This gives us a valid option to produce a chart around \( r \) through taking \( \phi \triangleq \eta^{-1} \). Then, if we had \( T_r \mathcal{M} \) we could generalize the Moving Least-Squares, described in Section 2.1, in a natural way to:

\[
p_r = \arg\min_{p \in \Pi^d} \sum_{i=1}^{N} \| p(x_i) - g_i \|^2 \theta(\| x_i - r \|),
\]

where \( x_i = \phi(p_i) \) are the projections of \( p_i \) onto \( T_r \mathcal{M} \) shifted around zero (i.e., \( \phi(r) = 0 \in \mathbb{R}^d \)), and \( g_i = \psi(p_i) \). And the approximating value of \( f(r) \) would be

\[
\psi(r) \approx \tilde{\psi}(r) = p_r(0).
\]

Unfortunately, to obtain the tangent space we need to know the manifold analytically, or at least have infinite sampling resolution. Moreover, in our problem-setting, the points \( \{r_i\}_{i=1}^{N} \) as well as \( r \) are sampled with noise. Thus, \( r \notin \mathcal{M} \) and it is meaningless to have a tangent space around it. Nevertheless, taking an in-depth look at the twofold approximation method of the MMLS (described in Section 2.2), we can utilize its first step to produce an alternative moving coordinate system for the manifold. Explicitly, for any given \( r \) near \( \mathcal{M} \) we can apply step 1 of the MMLS procedure to obtain an approximating affine space \( H(r) \) around an origin \( q(r) \). As shown in [25], since \( \{r_i\}_{i=1}^{N} \) are noisy samples of \( \mathcal{M} \), they can be viewed as noisy samples of a function \( \eta \) defined over \( H(r) \). That is,

\[
\eta(x) : H(r) \to \mathcal{M},
\]

and \( H(r) \simeq \mathbb{R}^d \). As have been stated above, instead of approximating \( \psi \) directly we can aim at approximating

\[
g \triangleq \psi \circ \eta : \mathbb{R}^d \to \mathbb{R},
\]

at \( x \in \mathbb{R}^d \) such that \( \eta(x) = r \). Similar to what have been stated above, the generalization of the MLS for function approximation is:

\[
p_r = \arg\min_{p \in \Pi^d} \sum_{i=1}^{N} \| p(x_i) - g_i \|^2 \theta(\| r_i - q(r) \|),
\]

where \( x_i \) are the projections of \( r_i \) onto \( H(r) \) shifted around zero (i.e., \( q(r) \mapsto 0 \in \mathbb{R}^d \)) and \( g_i = \psi(p_i) \). The approximating value of \( \tilde{\psi}(r) \) would then be

\[
\tilde{\psi}(r) \triangleq p_r(0) \approx \psi(r).
\]

As explained in detail in [25], under some mild assumptions \( H(r) \) is a valid moving coordinate system and is an indirect approximation to \( T_p \mathcal{M} \), where \( p \) is the projection of \( r \) onto \( \mathcal{M} \). Furthermore, it possesses a certain desired property of varying smoothly with respect to \( r \) (see Definition 2 below), as shown in [25]. In any case, as we shall see below, any valid choice of a smoothly varying coordinate system should suffice for the approximation defined in Equations (10) and (11) to be smooth as well.

**Definition 2.** Let \( H(r) \) be a parametric family of \( d \)-dimensional affine sub-spaces of \( \mathbb{R}^n \) centered at \( q(r) \). Explicitly,

\[
w = q(r) + \sum_{k=1}^{d} c_k e_k(r), \quad \forall w \in H(r),
\]

where \( e_k \) are

\[
e_k(r) = \begin{cases} 1 & \text{if } k = i \setminus 1 \leq d, \\ 0 & \text{otherwise}. \end{cases}
\]
where \( \{ e_k(r) \}_{k=1}^d \) is an orthonormal basis of the linear sub-space \( H(r) - q(r) \). We say that the family \( \langle H(r), q(r) \rangle \) changes smoothly with respect to \( r \) if for any vector \( v \in \mathbb{R}^n \) the function

\[
w(r) = q(r) + \sum_{k=1}^d \langle v - q(r), e_k(r) \rangle e_k(r),
\]

describing the Euclidean projections of \( v \) onto \( H(r) \), vary smoothly with respect to \( r \).

Accordingly, a smoothly varying coordinate system would be a family of affine sub-spaces which vary smoothly with respect to our parameter \( r \), such that our manifold can be viewed locally as a graph of a function over it. Using this notion we can arrive at the following conclusion.

**Theorem 3.1.** Assume \( \mathcal{M} \) is a \( d \)-dimensional boundaryless manifold embedded in \( \mathbb{R}^n \), \( \psi \) is a function from \( \mathcal{M} \) to \( \mathbb{R} \), and \( \theta(t) \) is a \( C^\infty \) radial weight function. Let \( \{(r_i, \psi_i)\}_{i=1}^N \) be noisy samples of \( \mathcal{M} \) and \( \psi \) respectively, and let \( r \in U \) be a point in the uniqueness domain of assumption \( 2.8 \) at which we wish to approximate \( \psi \). And, let \( \langle H(r), q(r) \rangle \) be a smoothly varying coordinate system such that \( \mathcal{M} \) can be viewed locally as a graph of a function over \( H(r) \). Furthermore, we demand that the Least-Squares problem of Equation (10) be well conditioned (i.e., that there are enough independent constraints). Then the approximation \( \hat{\psi}(r) \) derived from equations (10) and (11) is a \( C^\infty \) function from an open neighborhood of \( \mathcal{M} \) (with respect to the topology of \( \mathbb{R}^n \)) to \( \mathbb{R} \).

**Proof.** We begin with looking at a fixed coordinate system \( \langle H, q \rangle \)

\[
p_r = \arg\min_{p \in \Pi^d_m} \sum_{i=1}^N (p(x_i) - \psi_i)^2 \theta(\|r_i - q\|),
\]

where \( x_i \) are the projections of \( r_i \) onto \( H \). Let \( \mathcal{B} = \{ b_j(x) \} \) for \( j = 1, \ldots, \binom{m+d}{d} \) be a basis of \( \Pi^d_m \). As shown in [14], if we expand \( p_r \) in the basis \( \mathcal{B} \) then the coefficients vector of \( p_r \) is given by

\[
\bar{a} = DE(E^T DE)^{-1}\bar{c},
\]

where \( D = 2\text{diag}\{\theta(\|r_1 - q\|), \ldots, \theta(\|r_N - q\|)\} \), \( \bar{c} = (\psi_1^1, \ldots, \psi_N^d)^T \), and \( E_{i,j} = b_j(x_i) \). Now, as \( \theta \in C^\infty \) it follows that in the case of a fixed coordinate system the minimizing polynomials \( p_r \) will vary smoothly with respect to \( r \). This result is articulated in Theorem 2.1.

In our case, the coordinate system \( \langle H(r), q(r) \rangle \) depends on the parameter \( r \). Thus, we now obtain

\[
D(r) = 2\text{diag}\{\theta(\|r_1 - q(r)\|), \ldots, \theta(\|r_N - q(r)\|)\},
\]

and

\[
E(r)_{i,j} = b_j(x_i(r)).
\]

Since \( \langle H(r), q(r) \rangle \) vary smoothly with respect to \( r \) we achieve that \( x_i(r) \) vary smoothly as well. Combining this with the fact that \( \theta \in C^\infty \) we get that the right hand side of Equation (13) will still vary smoothly as \( r \) changes. Therefore, our local polynomial approximation \( p_r \) changes smoothly with respect to \( r \) and so does our MLS approximation given by:

\[
\tilde{\psi}(r) \triangleq p_r(0)
\]
After obtaining the smoothness property, we turn to the case of clean samples, for which we achieve the desired approximation order property.

**Theorem 3.2.** Let \( \psi \) be a function in \( C^{m+1}(\mathcal{M}) \) with a sample set \( \{(p_i, \psi(p_i))\}_{i=1}^N \subset \mathcal{M} \times \mathbb{R} \) such that \( \{p_i\}_{i=1}^N \) are an \( h\)-\( \rho\)-\( \delta \) set, let \( p \in \mathcal{M} \), and let \((H(p), q(p))\) be the coordinate system resulting from the minimization of Equation (6). Then, for fixed \( \rho \) and \( \delta \), there exists a fixed \( k > 0 \), independent of \( h \), such that the approximant given by equations (10)-(11) for \( \theta \) with a finite support of size \( s = kh \) yields the following error bound, for \( h \) small enough:

\[
\left\| \tilde{\psi}(p) - \psi(p) \right\|_{\mathcal{M}, \infty} < M \cdot h^{m+1}
\]  

**Proof.** Let \( p \in \mathcal{M} \), then, the approximating affine space \( H(p) \) minimizing Equation (6) approximates the sample set \( \{r_i\} \) up to the order of \( O(h^2) \) in an \( O(h) \) neighborhood of \( q(r) \) (see 25). Therefore, the projections of \( p_i \) onto \( H(p) \) in the support of \( \theta \) around \( p \) are also an \( h\)-\( \rho\)-\( \delta \) for sufficiently small \( h \) with respect to the domain \( H(p) \). Now, the approximation of \( \psi(p) \) is equivalent to the approximation of \( g(\phi_p(p)) \), given \( g(x_i) = \psi(p_i) \), where \( \phi_p : \mathcal{M} \to \mathbb{R}^d \) is the map to the coordinate system \( H(p) \), \( g \triangleq \psi \circ \phi_p^{-1} : H(p) \to \mathbb{R} \) and \( x_i \triangleq \phi_p(p_i) \).

Explicitly, we now approximate a function \( g : \mathbb{R}^d \to \mathbb{R} \) based upon the sample set \( \{(x_i, g(x_i))\}_{i=1}^N \) around a point \( x = \phi_p(p) \in \mathbb{R}^d \) using a weighted least-squares polynomial approximation of degree \( \leq m \). The only difference between our current situation and the conditions of Theorem 2.3 is that our coordinate system vary with every point. However, by looking at the proof of Theorem 2.3 given in [14] (numbered as Theorem 5 there), we can see that if \( H(p) \) vary continuously the proof remains the same. From [25] we know that \( (H(p), q(p)) \) varies smoothly with respect to \( p \), thus, the desired bound is achieved. \( \square \)

### 3.2 Interpolation Scheme

The above-mentioned mechanism can be utilized to provide an interpolatory function approximation. However, it is important to note that our approximant is not defined over the original manifold \( \mathcal{M} \) but on a neighborhood \( U \) of \( \mathcal{M} \) (note that this neighborhood is the uniqueness domain of Assumption 2.8). Furthermore, this function is not injective since all points producing the same \( q(r) \) will yield the same approximated value. Nevertheless, we can limit ourselves to the definition of \( \tilde{\psi} \) over \( \mathcal{M} \) alone, and the interpolation property we discuss below will hold.

As mentioned above, in Section 2.1 when dealing with function approximation over a flat domain, if we wish the approximation to become interpolatory at the samples \( \{p_i\}_{i=1}^N \) all we need is to choose a weight function \( \theta \) such that \( \theta(0) = \infty \). Evidently, the fact that \( \lim_{t \to 0} \theta(t) = \infty \) forces the moving least-squares approximation \( \tilde{\psi} \) to reach \( \lim_{r \to r_i} \tilde{\psi}(r) = \psi(r_i) \).

**Proposition 3.3.** *(Interpolation)* Let \( \{(p_i, \psi(p_i))\}_{i=1}^N \subset \mathcal{M} \times \mathbb{R} \) be our sample set where \( \psi : \mathcal{M} \to \mathbb{R} \) and \( \mathcal{M} \) is a d-dimensional submanifold of \( \mathbb{R}^n \). For any \( p \) in the uniqueness domain of Assumption 2.8 such that for all \( i \neq p \), let the approximation \( \tilde{\psi}(p) \approx \psi(p) \) be defined by Equation (11) with a decreasing weight function \( \theta \) satisfying \( \lim_{t \to 0} \theta(t) = \infty \) (for \( p = p_i \) we define \( \tilde{\psi}(p) \triangleq \psi(p_i) \)). Then, \( \tilde{\psi} \) is an interpolatory scheme.

**Proof.** Let \( x_i \) be the projection of \( p_i \) onto the coordinate system \((H(p), q(p))\), then the condition \( \lim_{t \to 0} \theta(t) = \infty \) enforces the local polynomial approximation of Equation (10) \( p_r \) to satisfy \( \lim_{r \to x_i} p_r(0) = p_j \). \( \square \)

**Remark 3.4.** The extension of the interpolatory scheme to the multidimensional case is immediate.
3.3 Algorithm Description

As a result of the theoretical discussion, our procedure will go along the lines of the MMLS procedure described above. For the sake of generality, we refer to the multidimensional case where $\psi : \mathcal{M}^d \rightarrow \mathbb{R}^n$. Explicitly,

1. Find a local $d$-dimensional affine space $H(r)$ approximating the sampled points ($H \simeq \mathbb{R}^d$). This affine space will be used in the following step as a local coordinate system.

2. Approximate the function $g : H(r) \rightarrow \mathbb{R}^n$ through weighted least-squares, based upon the samples $\{(x_i, g_i)\}_{i=1}^N$, where $x_i$ are the projections of $r_i$ onto $H(r)$ and $g_i = \psi_i$.

In what follows, we discuss in more details both steps as well as their implementation. The implementation of step 2 is trivial as it is a standard least-squares problem. Thus, after explaining it, we give a short description of it in Algorithm 2. However, the implementation of step 1 is a bit more tricky and will be discussed here in detail with a concise summary in Algorithm 1.

**Step 1 - Finding The Local Coordinates**

Find a $d$-dimensional affine space $H$, and a point $q$ on $H$, such that the following constrained problem is minimized:

$$J(r; q, H) = \sum_{i=1}^{N} d(r_i, H)^2 \theta(\|r_i - q\|)$$

s.t.

$$r - q \perp H \quad i.e., \quad r - q \in H^\perp$$

We find the affine space $H$ by an iterative procedure. Assuming we have $q_j$ and $H_j$ at the $j^{th}$ iteration, we compute $H_{j+1}$ by performing a linear approximation over the coordinate system $H_j$. In view of the constraint $r - q \perp H$, we define $q_{j+1}$ as the orthogonal projection of $r$ onto $H_{j+1}$. We initiate the process by taking $q_0 = r$ and choose $d$ basis vectors $\{u_k^1\}_{k=1}^d$ randomly.

This first approximation is denoted by $H_1$. Thence, we compute:

$$q_1 = \sum_{k=1}^{d} (r - q_0, u_k^1)u_k^1 + q_0 = q_0.$$ 

Upon obtaining $q_1, H_1$ we continue with the iterative procedure as follows:

- Assuming we have $H_j, q_j$ and its respective basis $\{u_k^j\}_{k=1}^d$ w.r.t the origin $q_j$, we project our data points $r_i$ onto $H_j$ and denote the projections by $x_i$. Then, we find a linear approximation of the samples $f_i^j = f^j(x_i) = r_i$:

$$\tilde{P}(x) = \arg\min_{p = (p_1, ..., p_n), p_i \in \Pi_1^d} \sum_{i=1}^{N} \|p(x_i) - f_i^j\|_2^2 \theta(\|r_i - q_j\|).$$

Note, that this is a standard weighted linear least-squares as $q_j$ is fixed!

- Given $\tilde{P}(x)$ we obtain a temporary origin:

$$\tilde{q}_{j+1} = \tilde{P}(0).$$
Then, around this temporary origin we build a basis \( \hat{B} = \{v_{k}^{j+1}\}_{k=1}^{d} \) for \( H_{j+1} \) with:

\[
v_{k}^{j+1} \triangleq \tilde{v}(u_{k}^{j}) - \tilde{q}_{j+1}, \quad k = 1, \ldots, d
\]

We then use the basis \( \hat{B} \) in order to create an orthonormal basis \( B = \{u_{k}^{j+1}\}_{k=1}^{d} \) through a \( d \)-dimensional \( QR \) decomposition, which costs \( O(nd^{2}) \) flops. Finally we derive

\[
q_{j+1} = \sum_{k=1}^{d} (r - \tilde{q}_{j+1}, u_{k}^{j+1}) u_{k}^{j+1} + \tilde{q}_{j+1}.
\]

This way we ensure that \( r - q_{j+1} \perp H_{j+1} \).

Remark 3.5. In [25], the initialization of the basis vectors of \( H_{1} \) is based upon a local PCA of the data points. In theory, this ought to improve the number of iterations needed for convergence, at the expense of significantly increasing the computational time. Alternatively, we could have utilized a low-rank approximation of the PCA such as proposed in [1] to do this more efficiently. Nevertheless, we found that a random initialization requires a similar number of iterations, and thus reduces the computation time even further. For this reason, we used a random initialization of \( H_{1} \).

See Figure 2 for the approximated local coordinate systems \( H \) obtained by Step 1 on noisy samples of a sphere.

![Figure 2](image)

*Figure 2: An approximation of the local coordinates \( H(r) \) (blue plain) resulting from Step 1 implementation after three iterations, where \( r \) is marked by the blue ×. Marked in green are the points affecting the approximation.*

**Step 2 - The approximation of \( \psi \)**

Let \( \{e_{k}\}_{k=1}^{d} \) be an orthonormal basis of \( H(r) \) (taking \( q(r) \) as the origin), and let \( x_{i} \) be the orthogonal projections of \( r_{i} \) onto \( H(r) \) (i.e., \( x_{i} = q(r) + \sum_{k=1}^{d} (r_{i} - q(r), e_{k}) e_{k} \)). As before, we note that \( r \) is orthogonally projected to the origin \( q \). Now we would like to approximate \( g : \mathbb{R}^{d} \to \mathbb{R}^{\tilde{n}} \), such that \( g_{i} = g(x_{i}) = \psi_{i} \). The vector-valued approximation of \( g \) is performed by minimizing
the weighted least-squares cost function, using a polynomial \( \tilde{p}_r(x) = (p^1_r(x), \ldots, p^\tilde{n}_r(x))^T \) where \( p^k_r(x) \in \Pi^d_m \), for \( 1 \leq k \leq \tilde{n} \).

\[
\tilde{p}_r(x) = \arg\min_{p^k_r \in \Pi^d_m, 1 \leq k \leq \tilde{n}} \sum_{i=1}^N \| p_r(x_i) - \psi_i \|^2 \, \theta(||r_i - q||).
\]

The approximation \( \tilde{\psi}(r) \) is then defined as:

\[
\tilde{\psi}(r) = \tilde{p}_r(0)
\]

**Remark 3.6.** The weighted least-squares approximation is invariant to the choice of an orthonormal basis of \( \mathbb{R}^d \).

**Remark 3.7.** Note, that the demand \( r - q(r) \perp H(r) \) along with Assumption 2.8 implies that \( \tilde{\psi}(r) \) would be the same for all \( r \) such that \( r - q(r) \in H^\perp \) and \( r \in U \) the uniqueness domain.

**Remark 3.8.** To save computation time, note that the normal equations of (17) are the same for all \( \tilde{n} \) coordinates just with a different right hand side. Thus, the least-squares matrix should be inverted only once.

**Algorithm 1** Finding The Local Coordinate System \((H(r), q(r))\)

1: **Input:** \( \{r_i\}_{i=1}^N, r \)
2: **Output:** \( q \) - an \( n \) dimensional vector  
   \( U \) - an \( n \times d \) matrix whose columns are \( \{u_j\}_{j=1}^d \)
3: define \( R \) to be an \( n \times N \) matrix whose columns are \( r_i \)
4: initialize \( U \) randomly
5: \( q \leftarrow r \)
6: repeat
   7: \( \tilde{R} = R - \text{repmat}(q, 1, N) \)
   8: \( X_{N\times d} = \tilde{R}^TU \) \( \triangleright \) find the representation of \( r_i \) in \( \text{Col}(U) \)
   9: define \( \tilde{X}_{N\times(d+1)} = \left[ (1, \ldots, 1)^T, X \right] \)
10: solve \( \tilde{X}^T\tilde{X}\alpha = \tilde{X}^TR^T \) for \( \alpha \in M_{(d+1)\times n} \) \( \triangleright \) solving the LS minimization of \( \tilde{X}\alpha \approx R^T \)
11: \( \tilde{q} = q + \alpha(:,1) \)
12: \( Q = qr(\alpha(:,2:end) - \text{repmat}(q, 1, d)) \)
13: \( U \leftarrow Q \)
14: \( q = \tilde{q} + UU^T(r - \tilde{q}) \)
15: until \( ||q - q_{\text{prev}}|| < \text{threshold} \)

**Algorithm 2** Function Approximation

1: Input: \( \{(r_i, \psi_i)\}_{i=1}^N, r \)
2: Output: \( \tilde{\psi}(r) \)
3: build a coordinate system \( H \) around \( r \) using \( \{r_i\}_{i=1}^N \) (e.g., via Algorithm 1)
4: project each \( r_i \in \mathbb{R}^n \) onto \( H \rightarrow x_i \in \mathbb{R}^d \)
5: \( \tilde{\psi}(r) \) is the solution of the weighted least-squares problem using the samples \( \{(x_i, \psi_i)\}_{i=1}^N \) around \( r \)
4 Numerical Examples

Generally, it is desirable for an algorithm to have a few, but not too many, parameters for tuning purposes. In our case, in order to fine tune the application of the algorithm, one needs to decide how to set the weight function $\theta$ of equations (15)-(17). In all of the examples below we have chosen to use the single parametric family of weight functions. For a given choice of the parameter $k$ we define

$$\theta_k(t) \triangleq \exp\left(\frac{-t^2}{(t-kh)^2}\right) \cdot \chi_{kh},$$

where $\chi_{kh}$ is an indicator function of the interval $[-kh,kh]$. This function is $C^\infty$ and compactly supported. The minimal requirement for the support size is such that the local least-squares matrix would be invertible. We chose $k$ such that the support would contain about 3 times the minimal required amount of points.

4.1 Approximation Order

In Theorem 3.2 we show that, given clean $h$-$\rho$-$\delta$ sample sets (for fixed $\rho$ and $\delta$), our function approximation scheme yields an approximation order of $O(h^{m+1})$, where $m$ is the total degree of the local polynomial. Denote the error of approximation of a point using a $h$-$\rho$-$\delta$ set by $err_h$. In this experiment, we show numerically that

$$err_h \approx M h^{m+1},$$

or, in other words, for $h_1$ and $h_2$:

$$\log \frac{err_{h_1}}{err_{h_2}} \approx (m+1) \log \frac{h_1}{h_2}. \quad (19)$$

In order to show that Equation (19) holds, we take $N$ points on the unit sphere $S^2$ chosen on an equispaced grid in the spherical coordinate system (excluding the $r$ coordinate), for $N = 20^2, 30^2, \ldots, 80^2$. Samples from this distribution is a good-enough approximation for $h$-$\rho$-$\delta$ sets with fixed $\rho, \delta$ parameters. The function that we approximate, $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, match any point on the sphere with its spherical coordinates $(\phi, \theta) \in [0, 2\pi) \times [0, \pi)$. For any pair $\{N_i, N_j\} \subset \{20^2, 30^2, \ldots, 80^2\}$ we estimate $h_k$ by $1/\sqrt{N_k}$. Then, in order to estimate the slope, we perform a least-squares linear fit using the points

$$\left(\log \frac{h_i}{h_j}, \log \frac{err_{h_i}}{err_{h_j}}\right)$$

In Figure 3 the small blue dots represent $\left(\log \frac{h_i}{h_j}, \log \frac{err_{h_i}}{err_{h_j}}\right)$ for an approximation using a first degree polynomial ($m = 1$), and similarly, the larger green dots correspond to $m = 3$. The dashed line and the full line are the linear fits for the $m = 1$ and $m = 3$ data points respectively. The slopes of the lines are 1.933 and 4.081, which is similar to $m + 1$ in both cases.

4.2 Large dimensional ambient space

The dataset in this experiment included a set of 72 gray-scale images of size $448 \times 416$ pixels. The images are taken from the unprocessed dataset of [21]. They are 2D projections of a 3D piggy bank obtained through rotating the object by 72 equispaced angles on a single axis.
Figure 3: Estimating the approximation order for $m = 1, 3$. For any pair $\{N_i, N_j\} \subset \{20^2, 30^2, \ldots, 80^2\}$, we plotted the points $\left(\log_N h_i, \log \frac{\text{err}_{h_i}}{\text{err}_{h_j}}\right)$. The case of $m = 1$ is represented by the small blue dots and $m = 3$ is represented by the larger green dots. The slope of the linear fit, to each set of points, gives an estimate to the approximation order, which is nearly $m + 1$ in both cases.

An example of the images is given in Figure 4. The approximated function $\psi$ is the angle of rotation. Therefore our dataset consists of 72 samples of a 1-dimensional manifold embedded in $\mathbb{R}^{186368}$ along with scalar values representing the angle of rotation.

In order to assess the presented algorithm, we used the leave-one-out cross-validation scheme. In each iteration, one image, chosen at random, is taken out of the dataset, and its angle is estimated using the angles of the other images.

Using $m = 1$, with 50 experiments, the average error is 0.0066 and the variance is $5.7 \cdot 10^{-5}$. However, when using $m = 3$ the average error is 0.06 and the variance is 0.02. This decrease in accuracy can be explained by the fact that higher order approximations require more data points, which means that the locality of the approximation is compromised.

Figure 4: A part of the dataset of [21], consisting of 72 images of a rotating piggy bank. Each image is of size $448 \times 416$ pixels. Thus, we approximate a function $\psi : \mathbb{R}^{448 \times 416} \to [0, 2\pi)$, that returns the rotation angle for each given image.

4.3 Regression Over a Klein Bottle

In the following example we compare the algorithm presented here to the algorithms presented in [4, 7]. The setting is taken from [7] section 5.1. Let $\mathcal{M}$ be the Klein bottle, a two-dimensional closed and smooth manifold, embedded in $\mathbb{R}^4$, which is parametrized by $\phi_{\text{Klein}} : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^4$ as

$$(u, v) \to ((2 \cos v + 1) \cos u, (2 \cos v + 1) \sin u, 2 \sin v \cos(u/2), 2 \sin v \sin(u/2)).$$
We sample $n$ points $(u_i, v_i)$ uniformly from $[0, 2\pi) \times [0, 2\pi)$ and obtain the corresponding points $p_i = \phi_{Klein}(u_i, v_i)$. Our sampled points are based on $p_i$ with added noise. Explicitly, $r_i = p_i + \sigma_r \eta$ where $\eta$ is a four dimensional normal random variable with zero-mean and identity covariance matrix, and $\sigma_r$ is a parameter that changes in the experiment.

The function $\psi$ that we approximate is defined as

$$
\psi(p) = 7 \sin(4u) + 5 \cos^2(2v) + 6e^{-32((u-\pi)^2+(v-\pi)^2)},
$$

where $(u, v) = \phi_{Klein}^{-1}(p)$. The samples that we have of $\psi$ are $\psi_i = \psi(p_i) + \sigma(p)\epsilon$ where $\epsilon \sim \mathcal{N}(0, 1)$ and $\sigma(p) = \sigma_0(1 + 0.1 \cos(u) + 0.1 \sin(v))$, where $\sigma_0$ determines the signal-to-noise ratio, defined by:

$$
\text{snrdb} = 10 \log_{10} \left( \frac{\text{var} \psi(p_i i = 1 \ldots n)}{\sigma_0^2} \right)
$$

We follow the same eight experiments done in [7] and compare the results to [4, 7]. The experiments parameters are: $n = 1000$ or $1500$, snrdb = 5 or 2, and $\sigma_r = 0$ or $0.2$, utilizing all the combinations. As reported in [7] the MALLER method yielded significantly better results than all the other tested algorithms. Thus, we show here a performance comparison of our approach and MALLER alone (Table 1). For more details regarding the performance of the algorithms designed in [4] see the original tables at [7].

It is easy to see that for $m = 1, 3$ and 5, the MMLS algorithm achieves more accurate results (the results for $m = 2$ and 4 are similar).

For running time measurements of the MMLS, we used a laptop with Intel i7 -6700HQ core with 16 GB RAM. We compared our timing against the fastest method reported in [7], which is NEDE [4] (Table 2). The timing of NEDE, quoted from [7], based upon a server with 96 GB of RAM, two Intel Xeon X5570 CPUs, each with four cores running at 2.93GHz.

| Alg       | $\sigma_r = 0$ | $\sigma_r = 0.2$ |
|-----------|----------------|-----------------|
|           | $n = 1500$    | $n = 1000$      |
| MALLER    | 2.36 ± 0.68   | 3.85 ± 0.77     |
| MMLS ($m = 1$) | 1.51 ± 0.34   | 3.11 ± 0.82     |
| MMLS ($m = 3$) | 1.41 ± 0.35   | 2.97 ± 0.72     |
| MMLS ($m = 5$) | 1.51 ± 0.37   | 2.87 ± 0.70     |

Table 1: Accuracy of approximation
| Alg       | $n = 1500$  | $n = 1000$ |
|-----------|-------------|------------|
| NEDE      | 6.04 ± 0.16 | 5.59 ± 0.15|
| MMLS ($m = 1$) | 2.18 ± 0.02 | 1.47 ± 0.02 |
| MMLS ($m = 3$) | 2.86 ± 0.02 | 1.95 ± 0.02 |
| MMLS ($m = 5$) | 3.73 ± 0.02 | 2.63 ± 0.03 |

Table 2: Time for computing the approximation

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References

[1] Yariv Aizenbud and Amir Averbuch. Matrix decompositions using sub-gaussian random matrices. arXiv preprint arXiv:1602.03360, 2016.

[2] Yariv Aizenbud, Amit Bermanis, and Amir Averbuch. PCA-based out-of-sample extension for dimensionality reduction. arXiv preprint arXiv:1511.00831, 2015.

[3] Marc Alexa, Johannes Behr, Daniel Cohen-Or, Shachar Fleishman, David Levin, and Claudio T Silva. Computing and rendering point set surfaces. Visualization and Computer Graphics, IEEE Transactions on, 9(1):3–15, 2003.

[4] Anil Aswani, Peter Bickel, and Claire Tomlin. Regression on manifolds: Estimation of the exterior derivative. The Annals of Statistics, pages 48–81, 2011.

[5] Mikhail Belkin and Partha Niyogi. Laplacian eigenmaps for dimensionality reduction and data representation. Neural computation, 15(6):1373–1396, 2003.

[6] Richard Bellman. Dynamic Programming. Princeton University Press, Princeton, NJ, USA, 1 edition, 1957.

[7] Ming-Yen Cheng and Hau-tieng Wu. Local linear regression on manifolds and its geometric interpretation. Journal of the American Statistical Association, 108(504):1421–1434, 2013.

[8] Ronald R Coifman and Stéphane Lafon. Diffusion maps. Applied and computational harmonic analysis, 21(1):5–30, 2006.

[9] David L Donoho et al. High-dimensional data analysis: The curses and blessings of dimensionality. AMS Math Challenges Lecture, pages 1–32, 2000.

[10] G Hughes. On the mean accuracy of statistical pattern recognizers. Information Theory, IEEE Transactions on, 14(1):55–63, 1968.

[11] Ian Jolliffe. Principal component analysis. Wiley Online Library, 2002.

[12] Peter Lancaster and Kes Salkauskas. Surfaces generated by moving least squares methods. Mathematics of computation, 37(155):141–158, 1981.
[13] Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. Nature, 521(7553):436–444, 2015.

[14] David Levin. The approximation power of moving least-squares. Mathematics of Computation of the American Mathematical Society, 67(224):1517–1531, 1998.

[15] David Levin. Mesh-independent surface interpolation. In Geometric modeling for scientific visualization, pages 37–49. Springer, 2004.

[16] Dermot H McLain. Drawing contours from arbitrary data points. The Computer Journal, 17(4):318–324, 1974.

[17] Roy Mitz, Nir Sharon, and Yoel Shkolnisky. Symmetric rank one updating from partial spectrum with an application to out-of-sample extension. arXiv preprint arXiv:1710.02774, 2017.

[18] Amit Moscovich, Ariel Jaffe, and Boaz Nadler. Minimax-optimal semi-supervised regression on unknown manifolds. arXiv preprint arXiv:1611.02221, 2016.

[19] Sayan Mukherjee, Qiang Wu, and Ding-Xuan Zhou. Learning gradients on manifolds. Bernoulli, pages 181–207, 2010.

[20] Andrew Nealen. An as-short-as-possible introduction to the least squares, weighted least squares and moving least squares methods for scattered data approximation and interpolation. URL: http://www.nealen.com/projects, 130:150, 2004.

[21] Sameer A Nene, Shree K Nayar, Hiroshi Murase, et al. Columbia object image library (coil-20). 1996.

[22] Sam T Roweis and Lawrence K Saul. Nonlinear dimensionality reduction by locally linear embedding. Science, 290(5500):2323–2326, 2000.

[23] Bernhard Schölkopf, Alexander Smola, and Klaus-Robert Müller. Nonlinear component analysis as a kernel eigenvalue problem. Neural computation, 10(5):1299–1319, 1998.

[24] Alex J Smola and Bernhard Schölkopf. A tutorial on support vector regression. Statistics and computing, 14(3):199–222, 2004.

[25] Barak Sober and David Levin. Manifolds’ projective approximation using the moving least-squares (mmls). arXiv preprint arXiv:1606.07104, 2016.

[26] Joshua B Tenenbaum, Vin De Silva, and John C Langford. A global geometric framework for nonlinear dimensionality reduction. Science, 290(5500):2319–2323, 2000.