Self-consistent geodesic equation and quantum tunneling from charged AdS black holes

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Abstract. Some urgent shortcomings in previous derivations of geodesic equations are remedied in this paper. In contrast to the unnatural and awkward treatment in previous works, here we derive the geodesic equations of massive and massless particles in a unified and self-consistent manner. Furthermore, we extend to investigate the Hawking radiation via tunneling from charged black holes in the context of AdS spacetime. Of special interest, the application of the first law of black hole thermodynamics in tunneling integration manifestly simplifies the calculation.

1. Introduction
It is widely believed that Hawking radiation \cite{1,2} via tunneling could be advanced as a promising tool to have a closer insight into puzzling black holes. Nowadays, research on Hawking radiation has captured popular attention \cite{3–8}. It is worth mentioning that Parikh-Wilczek’s semi-classical tunneling method \cite{9} has always prevailed in further investigating tunneling radiation from various black holes \cite{10, 11}. With respect to Parikh-Wilczek’s semi-classical tunneling picture, the derivation of particles’ geodesic equations plays an important role in the study of black hole radiation. However, there existed some notable shortcomings in previous derivations. For the case of massive particles, the geodesics were awkwardly derived by employing the relation $v_p = v_g/2$ between the group velocity and the phase velocity, which is inconsistent with the variational principle of action. According to the variation principle, in General Relativity, geodesic equations should be defined by applying the variational principle on the Lagrangian action. What’s more, the above treatment is not applicable for defining the geodesics of massless particles. Geodesic equations of massless particles used to be derived by setting $ds^2 = 0$ instead. Even worse, previous derivation involved inconsistent foundation, i.e. mixing together relativistic and non relativistic descriptions. Motivated by these facts, this paper concentrates on remedying the urgent shortcomings and extending to investigate the Hawking radiation via tunneling from charged black holes in anti-de Sitter (AdS) spacetime.

2. Charged AdS black holes and the Painlevé-like form
The spherical Reissner-Nordström AdS black hole in $(3+1)$-dimensional spacetime is usually described by the line element and the U(1) gauged potential

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),$$

(1)
\[ F = dA, \quad A = -\frac{Q}{r} dt, \quad (2) \]

with

\[ \Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2, \quad (3) \]

where the parameters \( M \) and \( Q \) represent the ADM mass and the total charge of the black hole respectively, the cosmological constant \( \Lambda \) takes a negative value, and \( t_R \) is the time coordinate. The event horizon of the black hole is located at \( r = r_+ \), where \( r_+ \) is determined by the largest root of the equation \( \Delta(r) = 0 \).

The key trick to apply Parikh-Wilczek’s semi-classical tunneling analysis is to introduce a coordinate system which is well-behaved at the horizon. Therefore, to remove the coordinate singularity in the metric (1), we perform a generalized Painlevé coordinate transformation

\[ dt_R = dt + \frac{\sqrt{\frac{1}{\Delta}}}{} dr, \quad (4) \]

and obtain the following Painlevé-like line element

\[ ds^2 = \Delta dt^2 + 2\sqrt{1-\Delta} dtdr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (5) \]

Obviously, the new form (5) is regular at the horizon and displays the stationary, nonstatic nature of the spacetime. Besides, more importantly, it satisfies the Landau’s condition of the coordinate clock synchronization [12,13]. It should be emphasized that these attractive features are very advantageous for us to explore the tunneling process of the charged massive particles across the horizon.

### 3. Geodesic equations of massive and massless tunneling particles

As mentioned above, the derivation of geodesics is an important aspect in Parikh-Wilczek’s semi-classical tunneling picture. However, in previous works, the geodesic equations were unnaturally and awkwardly derived. In this section, our attentions are focused on remedying the urgent shortcomings and attempting to define the geodesic equations of massive and massless particles in a unified and self-consistent way.

Generally speaking, both the null and timelike geodesics can be derived from the Lagrangian action with the help of the variational principle. Associated with a black hole solution, there often exists three conserved integration constants, namely, the energy \( E \) and angular momentum \( L \) which correspond to the Killing vector \( \partial_t, \partial_\phi \) respectively, as well as the Hamiltonian \( \mathcal{H} \) that can be restricted by the 4-velocity normalization condition \( \mathcal{H} = -mK/2, (K = 0, 1) \). Particularly, \( K = 0, 1 \) correspond to the cases of null and timelike geodesics. With these integration constants in hand, it’s enough to determine the equations of motion for massive particles. Of special interest, the massless particles’ geodesics can be derived by setting \( K = 0 \) or the rest mass \( m = 0 \) from the geodesic equations of massive particles. As a consequence, we can write out the geodesic equations of massive and massless particles in a unified and self-consistent way. In what follows, we first utilize this attractive improvement to derive the geodesics of massive and massless particles tunneling across the Reissner-Nordström black hole in AdS spacetime.

Working with the metric (5) and considering a charged massive particle, the corresponding Lagrangian [14] is given by

\[ L = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + qA_\mu \dot{x}^\mu \]

\[ = \frac{m}{2} \left( -\Delta \dot{t}^2 + 2\sqrt{1-\Delta} \dot{t} \dot{r} + \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{qQ}{r} \dot{r}, \quad (6) \]
in which \( m, q \) are the rest mass and electric charge of the particle. Subsequently, the generalized momentum \( P_\alpha = \partial L / \partial \dot{x}^\alpha \) can be obtained,

\[
P_t = m \left( -\Delta \dot{t} + \sqrt{1-\Delta^2} \dot{r} \right) - \frac{qQ}{r},
\]

\[
P_r = m \left( \sqrt{1-\Delta} \dot{t} + \dot{r} \right),
\]

\[
P_\theta = m \dot{r} \dot{\theta},
\]

\[
P_\phi = m r^2 \sin^2 \theta \dot{\phi}.
\]

(7) \hspace{1cm} (8) \hspace{1cm} (9) \hspace{1cm} (10)

Due to the fact that

\[
\frac{dP_t}{d\tau} = \frac{\partial L}{\partial t} = 0, \quad \frac{dP_\phi}{d\tau} = \frac{\partial L}{\partial \phi} = 0,
\]

(11)

where \( \tau \) is the affine parameter, apparently, \( t \) and \( \phi \) act as the cyclic coordinates which correspond to the conserved generalized momenta \( P_t \) and \( P_\phi \), i.e.

\[
P_t = E = \text{Const}, \quad P_\phi = L = \text{Const}.
\]

(12)

Moreover, with the help of the Legendre transformation, we get the Hamiltonian as follows

\[
\mathcal{H} = iP_t + \dot{r} P_r + \dot{\theta} P_\theta + \dot{\phi} P_\phi - L
\]

\[
= \frac{m}{2} \left( -\Delta \dot{t}^2 + 2\sqrt{1-\Delta^2} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)
\]

\[
= \frac{m}{2} \ 4 \mu x^\mu x^\nu .
\]

(13)

Considering the 4-velocity normalization condition

\[
-\frac{2}{m} \mathcal{H} = K = \begin{cases} 0 & \text{the massless particle's case} \\ 1 & \text{the massive particle's case} \end{cases}
\]

(14)

and combining with Eq. (12), \( \dot{r}, \dot{t} \) can be worked out. It should be pointed out that the definition for the massless particle’s geodesics is equivalent to its previous derivation by means of \( ds^2 = 0 \). In consequence, the geodesic equation can be calculated as

\[
\ddot{r} = \frac{dr}{dt} = \frac{\dot{r}}{\dot{t}}
\]

\[
= \Delta \left\{ \sqrt{1-\Delta} \pm \frac{(E + qQ/r)}{(E + qQ/r)^2 - \Delta Y} \right\}^{-1},
\]

(15)

in which

\[
Y = \frac{L^2}{r^2 \sin^2 \theta} + m^2 \left( K + r^2 \dot{\theta}^2 \right),
\]

(16)

and the signs “±” correspond to the geodesics of ingoing and outgoing particles from the event horizon. To better understand the geodesic equation (15), without loss of generality, we take the geodesics of the plane \( \theta = \pi/2 \) as an example.

i) If setting \( K = 1 \) and considering the s-wave approximation \( (L = 0) \), it’s easy to get the massive particle’s geodesics equation in the plane \( \theta = \pi/2 \) as

\[
\ddot{r} = \Delta \left\{ \sqrt{1-\Delta} \pm \frac{(E + qQ/r)}{(E + qQ/r)^2 - m^2 \Delta} \right\}^{-1},
\]

(17)

in which the signs “±” correspond to the cases of ingoing and outgoing particles.
ii) For the case of massless particles, considering the \( s \)-wave approximation \( (L = 0) \) and setting \( K = 0 \) yields
\[
\bar{r} = \Delta \left( \sqrt{1 - \Delta} \pm 1 \right)^{-1} = \pm 1 - \sqrt{1 - \Delta} .
\] (18)

Remarkably, the massless particle’s geodesics equation (18) coincides with the result in Ref. [9].

So far, the previous shortcomings in deriving the geodesics have been tamed, and we have achieved the geodesic equations of the charged massive and massless particles tunneling across the Reissner-Nordström AdS black hole in a unified and self-consistent way.

4. Tunneling radiation across the event horizon

Hawking describes black hole radiation as a quantum tunneling process triggered by vacuum fluctuations [15]. Based on this scenario, a pair of particles is created near the horizon. The positive energy particle tunnels out to infinity while the negative energy “partner” is absorbed into the black hole and arrives at the singularity point after passing through the one-way membrane. The energy of the black hole decreases and the horizon shrinks. Here we take the self-gravitation between particles into consideration and permit the mass fluctuation of black holes under the conditions of energy conservation. Assuming that a particle with the energy \( \omega \) and charge \( q \) tunnels out the black hole, the mass and electric charge of the black hole system would change to \((M - \omega)\) and \((Q - q)\), respectively. Meanwhile, the line element and the vector potential switch to
\[
ds^2 = -\bar{\Delta} dt^2 + 2\sqrt{1 - \bar{\Delta}} dt dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 ,
\] (19)
\[
F = d\bar{A} , \quad \bar{A} = -\frac{Q - q}{r} dt ,
\] (20)

where \( \bar{\Delta} = 1 - 2 \frac{(M - \omega)}{r} + (Q - q)^2 / r^2 - (\Lambda / 3) r^2 \).

According to the WKB approximation, the emission rate of the tunneling particles is closely related to the action of the tunneling process, that is
\[
\Gamma \sim e^{-2\text{Im} S} .
\] (21)

As alluded before, the generalized coordinate \( A_t \) is an ignorable one. Therefore, to eliminate this degree of freedom, we can express the action as
\[
S = \int_{T_i}^{T_f} \left( \mathcal{L} - P_{A_t} \dot{A}_t \right) dt = \int_{r_i}^{r_f} \left( P_r \bar{r} - P_{A_t} \dot{A}_t \right) \frac{dr}{\bar{r}}
\]
\[
= \int_{r_i}^{r_f} \left[ \int_{(0,0)}^{(P_r, P_{A_t})} \left( \bar{r} dP'_r - \dot{A}_t dP'_{A_t} \right) \right] \frac{dr}{\bar{r}} ,
\] (22)

where, \( r_i \) and \( r_f \) denote the horizon radii before and after the shrinkage, \( P_r \) and \( P_{A_t} \) are the canonical momenta conjugated to \( r \) and \( A_t \). Proceeding with an explicit calculation, we substitute the Hamilton’s equations of motion
\[
\bar{r} = \frac{dH}{dP_r} \bigg|_{(r; A_t, P_{A_t})} = \frac{d(M - \omega)}{dP_r} ,
\] (23)
\[
\dot{A}_t = \frac{dH}{dP_{A_t}} \bigg|_{(A_t; r, P_r)} = \Phi \frac{d(Q - q)}{dP_{A_t}} ,
\] (24)
into Eq. (22) and have
\[ S = \int_{\bar{r}_i}^{\bar{r}_f} \int_{(M,Q)} [(M - \omega') - \Phi' (Q - q')] \frac{d\bar{r}}{\bar{r}}. \] \hspace{1cm} (25)

In general, it is a crucial challenge to finish the tunneling integration in the study of black hole radiation via tunneling. As a highlight, here we tactfully utilize the thermodynamic properties of the black hole to simplify the calculation.

To facilitate completing the above integration, we need the help of the asymptotic behavior of the outgoing particles near the event horizon
\[ \bar{r} \approx \kappa' (r - r'_+), \] \hspace{1cm} (26)
where $\kappa'$ represents the surface gravity on the horizon after the particle emission. Moreover, it can be verified that thermodynamical quantities, such as the entropy and electric potential
\[ S' = \pi r'_+^2, \quad \Phi' = \frac{Q - q'}{r'_+}, \] \hspace{1cm} (27)
satisfy the differential form of the first law of black hole thermodynamics as below
\[ d(M - \omega') = \frac{\kappa'}{2\pi} dS' + \Phi' d(Q - q'). \] \hspace{1cm} (28)

With these relations in hand, the imaginary part of the action can be calculated as
\[
\text{Im} S \approx \text{Im} \int_{\bar{r}_i}^{\bar{r}_f} \int_{(M,Q)} \frac{d(M - \omega') - \Phi' (Q - q')}{\kappa' (r - r'_+)} d\bar{r}
= -\frac{1}{2} \int_{S_{BH}(M,Q)} dS' = -\frac{1}{2} \Delta S_{BH}. \]

Note that the above integral was worked out by deforming the contour around the single pole $r = r'_+$. We switched the order of integration and first carried out the integral with respect to $r$. Eventually, the emission rate of the charged massive particles tunneling from the Reissner-Nordstrom black hole in the context of AdS spacetime is
\[ \Gamma \sim e^{-2\text{Im} S} = e^{\Delta S_{BH}}, \] \hspace{1cm} (30)
in which $\Delta S_{BH} = S_{BH}(M - \omega, Q - q) - S_{BH}(M, Q)$. As a result, the tunneling probability of a Reissner-Nordstrom AdS black hole is closely linked with the change of Bekenstein-Hawking entropy and the real radiation spectrum deviates from the pure thermal spectrum.

5. Summary
In this work, we have remedied the urgent shortcomings in deriving the geodesics of tunneling particles. By applying the Lagrangian analysis on the action, utilizing the conserved constants and the 4-velocity normalization condition, we have redefined the geodesic equations. Eventually, the geodesic equations of massive and massless particles have been worked out in a unified and self-consistent manner. Furthermore, by employing the prevailing Parikh-Wilczek’s semi-classical tunneling method, we have clarified that, even if in the context of AdS spacetime, the tunneling probability of Reissner-Nordstrom black holes is still closely related to the change of Bekenstein-Hawking entropy and the real radiation spectrum deviates from the pure thermal one. It is noteworthy that, in contrast to the conventional treatment frequently used, the superiority of the calculation resorting to the first law of black hole thermodynamics during the tunneling integration is highlighted.
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