Spectral boundary of positive random potential in a strong magnetic field

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Abstract: We consider the problem of randomly distributed positive delta-function scatterers in a strong magnetic field and study the behavior of density of states close to the spectral boundary at $E = \hbar \omega_c / 2$ in both two and three dimensions. Starting from dimensionally reduced expression of Brezin et al. and using the semiclassical approximation we show that the density of states in the Lifshitz tail at small energies is proportional to $e^{-f^2}$ in two dimensions and to $\exp\left(-3.14f \ln(3.14f / \pi e) / \sqrt{2me}\right)$ in three dimensions, where $e$ is the energy and $f$ is the density of scatterers in natural units.

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1 Introduction

The quantum mechanical problem of particle’s motion in random external potential and in strong magnetic field has been a subject of lot of interest in past years, primarily because of the connection to the problem of quantum Hall effect. If a particle is confined to two-dimensional (2d) motion perpendicular to a sufficiently strong magnetic field the Hilbert space available to it is only the lowest Landau level (LLL). The average density of states (DOS) has then been calculated exactly for the white-noise (Gaussian) random potential by Wegner [1]. Subsequently, it has been shown that the expression for DOS in the LLL and in uncorrelated random potential undergoes a spectacular D→D-2 dimensional reduction [2], so that Wegner’s result can be generalized to any external potential of this type. For a particle in 3d, the problem of calculating DOS therefore reduces to solving a 1d field theory. The exact results are not available in that case, but the leading behavior of DOS in the tail of the distribution for the white-noise potential has been obtained by several authors using approximate methods [3, 4, 5].

In this paper we study the spectrum of quantum-mechanical particle moving in a random distribution of positive δ-function scatterers in strong magnetic field. The main difference between this and the problem of Gaussian random potential studied before is the presence of fixed lower boundary in the spectrum at $E = \hbar \omega_c / 2$, where $\omega_c$ is the cyclotron energy. Because of the analytic constraint on the wave-functions in the LLL spectrum the structure of DOS close to the boundary is quite intricate even in 2d case. The additional interest in this type of random potential comes from the fact that it can be used to model disorder in studies of the superconducting “glassy” transition in the materials with columnar or point defects [6]. The behavior of DOS and structure of the eigenstates close to the spectral boundary has direct consequences for the critical behavior in the model. In this paper we therefore concentrate on DOS close to the spectral boundary and apply a semiclassical approximation to the problem to obtain the leading term [7, 8, 9]. Our main results are the
following: in 2d the saddle-point approximation and the direct numerical diagonalization both confirm the result of ref. 2 that the DOS is proportional to \(e^{f-2}\) where \(f\) is the density of scatterers in appropriate units; in 3d the leading term in the expression for logarithm of DOS is proportional to \(\ln(e)/\sqrt{2me}\) which is similar but distinct behavior from purely 1d problem.

The paper is organized in the following manner: in the next section we introduce the basic concepts and a toy-problem: particle in 2d and in the LLL where the expression for DOS reduces to a simple integral. In section III we perform the instanton calculation of the DOS in 3d case and in the last section we discuss the obtained results. Finally, in appendix we prove that the obtained instanton solution is indeed the requisite negative mode of the action.

## 2 Two dimensions

We study the spectrum of the Hamiltonian:

\[
\hat{H} = (-i\hbar \nabla - \vec{A})^2/2m + \lambda \sum_i \delta(\vec{r} - \vec{r}_i),
\]

where \(\vec{A} = (-By/2, Bx/2, 0)\), \(\lambda > 0\) and \(m\) is the mass of the particle. Coordinates of the scatterers \(\{\vec{r}_i\}\) are independent random variables and the average density of scatterers is \(\rho\).

It is assumed that the magnetic field \(B\) is strong enough so that the Hilbert space for the motion orthogonal to the field is restricted to the LLL. Under that condition, if the particle is confined to the two dimensions orthogonal to the field, DOS per unit area is given by the expression [2, 10]:

\[
\rho(e) = \frac{1}{\pi \lambda} Im \frac{\partial}{\partial e} \ln Z
\]

where ”the average partition function” \(Z\) is

\[
Z = \int \int_{-\infty}^{+\infty} d\phi_1 d\phi_2 \exp (ie(\phi_1^2 + \phi_2^2)) - f \int_{0}^{\phi_1^2 + \phi_2^2} dx \frac{dx}{x} (1 - \exp (-ix))
\]
and we rescaled the energy as \( e = (E - \hbar \omega_c/2)2\pi l^2/\lambda, \ f = \rho 2\pi l^2 \) and \( l \) is the magnetic length. We assume that \( f > 1 \), since otherwise DOS will have a delta-function singularity at \( e = 0 \) as a consequence of the nature of the LLL wave-functions. The fact that \( Z \) is an ordinary integral follows from the hidden supersymmetry in the problem uncovered by Brezin et al. (ref. [2]) which led to the dimensional reduction by two. We are interested in the behavior of \( \rho(e) \) as \( e \) approaches zero from the positive side. First, we rotate the lines of integration over variables \( \phi_1, \phi_2 \) from the real axes for \( \pi/4 \) in the complex plane. The DOS is then given by eq.2 with

\[
Z = \int \int d\phi_1 d\phi_2 \exp(-S),
\]

and the exponentiated action is

\[
S = -e(\phi_1^2 + \phi_2^2) + f \int_0^{(\phi_1^2+\phi_2^2)} dx (1 - \exp(-x)).
\]

If \( e < 0 \) the integrand goes to zero fast when the variables of integration tend to infinity; the integral is a finite real number and the density of states vanishes. When \( e > 0 \) however, the integral diverges and we use a saddle point method to extract the imaginary piece. Saddle-points of the action \( S \) are determined by \( \partial S/\partial \phi_{1/2} = 0 \), i.e. :

\[
\phi_{1/2}(-e + f \frac{1 - \exp(-\phi_1^2 - \phi_2^2)}{\phi_1^2 + \phi_2^2}) = 0.
\]

First, there is a trivial saddle-point \( \phi_1 = \phi_2 = 0 \) where the action vanishes. At this saddle point we have \( \partial^2 S/\partial \phi_{1/2}^2 = f - e \) and the mixed derivative is zero. Thus the contribution of this saddle-point and the quadratic fluctuations around it to the integral \( Z \) is real for \( e < f \) and it equals \( 2\pi/(f - e) \). The second set of saddle-points is determined by:

\[
\frac{e}{f}(\phi_1^2 + \phi_2^2) = 1 - \exp(-\phi_1^2 - \phi_2^2).
\]

The last equation admits a simple solution in the limit \( e << f \): \( \phi_1 = \sqrt{f/e}, \phi_2 = 0 \). Other solutions are related to this one by a rotation around the origin in \( (\phi_1, \phi_2) \) plane. Note that in the limit of interest \( (e << f) \) this saddle point is infinitely far from the trivial one. The value
of the action at this saddle point is $S = -0.42f + f \ln(f/e)$, and $\partial^2 S/\partial \phi_1^2 = -2e$ with all other second derivatives vanishing. Thus, fluctuations in $\phi_1$ around this saddle-point represent the "negative mode" and we need to rotate the line of integration over this variable by $\pi/2$ in the complex plane. This rotation makes the contribution of this saddle-point and fluctuations around it to the integral purely imaginary. Fluctuations in $\phi_2$ represent the "zero mode" in the problem; the manifestation of breaking of U(1) symmetry by picking a saddle-point. The integration over this variable has to be transformed into integration over "collective coordinate" with the appropriate Jacobian \[11\], which takes into account the contributions of all saddle-points related by the symmetry. Including the trivial saddle-point, the result for the partition function when $e/f << 1$ becomes:

$$Z = \frac{2\pi}{f} + i \frac{\exp(0.42f)\pi^{3/2}}{f^{1/2}} e^{f - 1}$$ (8)

which leads to the result for the DOS when $f > 1$:

$$\lambda \rho(e) = \frac{\exp(0.42f)(f - 1)}{2\pi^{1/2} f^{(f - 3/2)}} e^{f - 2}.$$ (9)

This simple analysis yields the correct behavior of DOS at small energies, and even the coefficient of proportionality is numerically close to the exact value \[2\]. The $e^{f-2}$ dependence is somewhat unexpected since it is not obvious why the number of states at the boundary should change from diverging to vanishing at the density of scatterers $f = 2$. It would be interesting to have some intuitive understanding of this feature of DOS. Also, we note here that the semiclassical analysis of DOS starting from the full field theory (and not from its dimensionally reduced form like it has been done here) using either the replica trick \[8\] or the supersymmetry \[12\] leads to a wrong power law: $\rho(e) \propto e^{f-1}$. This comes as a surprise when one recalls that for Gaussian disorder for instance, both methods give the same behavior in the tail of DOS as found in the exact solution \[8\], \[9\]. We suspect that this is related to the fact that we are not dealing with the true tail of the distribution here, but we are close to the fixed edge of the spectrum instead. Since DOS vanishes only as a power law in our case (in
contradistinction to the Gaussian disorder DOS in the tail) the correct power could easily be missed by semi-classical treatment. As an independent check of the validity of the result from dimensionally reduced expression for DOS we performed numerical diagonalization of the Hamiltonian 1. On Figure 1 we have shown the result for DOS obtained by taking 30 different realizations of the random potential with degeneracy of the LLL being 100 and density of scatterers $f = 1.5$. We used the basis of angular momentum eigenstates in the numerical diagonalization. The abundance of states close to $e = 0$ for $1 < f < 2$ comes from the fact that the LLL wave functions efficiently use their zeroes to cover sparse scatterers, so our choice of the basis is crucial for revealing the right behavior of DOS when $e \to 0$. The result clearly shows that the number of states at the spectral boundary $e = 0$ still diverges at $f = 1.5$ in agreement with the result 9, although the number of different realizations of the random potential is too small for a more quantitative comparison. Numerical diagonalization at $f = 2.5$ (Fig. 2) shows that the DOS remains flat down to the lowest energies, and no divergence is seen at $e = 0$. At energies $e < 0.01$ essentially no states are found.

In the above calculation we assumed that the strengths of all scatterers are the same whilst only their positions are the random variables. This however is irrelevant for the obtained power law behavior of DOS close to the boundary; using the same reasoning as in this section one can easily show that as long as all strengths are positive the result $\rho(e) \propto e^{f-2}$ for $e << f$ holds even if the strengths of the scatterers are allowed to fluctuate.

3 Three dimensions

We now assume that the particle moves through a full 3d space and that it’s mass is anisotropic with different values along and orthogonal to the field. Density of states is then given by:

$$\rho(e) = \frac{d}{\pi L^3} \Im \frac{\partial}{\partial e} \ln Z$$ (10)
and the partition function $Z$ is expressed as a functional integral

$$Z = \int D[\phi^*(z), \phi(z)] \exp(-S),$$  \hspace{1cm} (11)$$

with the action

$$S = \int_{-L/2d}^{L/2d} dz (-e|\phi(z)|^2 + \frac{|\partial_z \phi(z)|^2}{2m} + f \int_0^{|\phi(z)|^2} dx \frac{1 - \exp(-x))}{x}).$$  \hspace{1cm} (12)$$

We chose the unit of length $d = \hbar^2 / 2\pi l^2 / 2\lambda m_\parallel$ and $e = (E - \hbar\omega_c)/2 \pi l^2 d / \lambda$, $m = m_\parallel \lambda d / 2\pi l^2$ and $f = \rho_2 \pi l^2 d$ are dimensionless. The cyclotron energy is determined by the mass orthogonal to the field, and we take the length of the box $L \to \infty$. In the limit $m \to \infty$ the expression for DOS reduces to its 2d limiting form from the previous section.

Field configuration which minimizes the action is the solution of the equation:

$$-e\phi(z) - \frac{\partial_z^2 \phi(z)}{2m} + f \frac{1 - \exp(-|\phi(z)|^2)}{|\phi(z)|^2} \phi(z) = 0.$$  \hspace{1cm} (13)$$

There is again the trivial solution $\phi(z) = 0$ which contributes to the real part of the partition function. To obtain the imaginary part one needs to find a non-trivial solution (instanton) of the above equation. In the region where $\phi^2(z) >> f/e$ the instanton is proportional to $\cos(z\sqrt{2me})$ and where $\phi^2(z) << 1$ it decays to zero exponentially. Instead of solving the nonlinear differential equation 13 we propose an anzats for the instanton: $\phi(z) = a \cos(zb)$ if $-\pi/2b < z < \pi/2b$ and zero otherwise. The parameters $a$ and $b$ are to be chosen to minimize the action 12. If we find that $a^2 >> f/e$ when $e/f \to 0$, there will be a wide region where our anzats will approximate the actual solution of the above differential equation very well. This variational procedure is similar to the one used in ref.7 to obtain the tail of DOS in the same disorder potential but without the magnetic field.

Inserting the proposed anzats into the action we get

$$S = -\frac{ea^2 \pi}{2b} + \frac{a^2 b \pi}{4m} + \frac{f}{b} I_2(a),$$  \hspace{1cm} (14)$$

and minimizing it with respect to $a^2$ and $b$ leads to the equations:

$$\frac{\pi}{2} \left( \frac{b^2}{2m} - e \right) + \frac{f}{a^2} (\pi - I_1(a)) = 0$$  \hspace{1cm} (15)$$
and

\[ \frac{a^2 \pi}{2} (e + \frac{b^2}{2m}) = f I_2(a). \] (16)

When \( a \) is very large the integrals appearing in the previous lines can be simplified:

\[
I_1(a) \equiv \int_{\pi/2}^{\pi/2} dz \exp(-a^2 \cos^2(z)) \approx \frac{\sqrt{\pi}}{a} \]

and

\[
I_2(a) \equiv \int_{\pi/2}^{\pi/2} d\zeta \int_0^1 dx \frac{dx}{x} (1 - \exp(-x a^2 \cos^2(z))) \approx -2.24 + 3.14 \ln(a^2) \]

Eliminating \( b \) from the equation 16 leads to the equation for parameter \( a \):

\[
a^2 = \frac{f (3.14 \ln(a^2) - 2.24)}{\pi (e - \frac{f}{a^2})}, \]

which for \( e << f \) has an approximate solution

\[
a^2 \approx \frac{3.14 f}{\pi e} \left(1 - \frac{2.24}{\pi} + \ln\left(\frac{3.14 f}{\pi e}\right) + \ln(\ln\left(\frac{3.14 f}{\pi e}\right))\right). \]

Then from equation 15 it follows:

\[
b^2 \approx 2me - \frac{4mf}{a^2}. \]

The comparison between the anzats determined by these parameters and numerically determined instanton is shown on Fig. 3. Note that \( a^2 \propto (f/e) \ln(f/e) \) so that for \( e/f \rightarrow 0 \) the variational parameter indeed increases faster than \( f/e \).

The value of the action at the instanton saddle-point is:

\[
S_0 \approx \frac{3.14 f}{\sqrt{2me}} \ln\left(\frac{3.14 f}{\pi e}\right) \]

and we kept only the leading, most diverging term in the limit \( e << f \). Besides the slower diverging terms which enter \( S_0 \) from our anzats there will be additional terms coming from the region where the anzats deviates appreciably from the exact instanton. These terms can be systematically investigated starting from the proposed anzats. In appendix we prove that
our variational ansatz is indeed a negative mode of the action, as it is necessary to get the imaginary part of the partition function.

Since we consider here only the leading term of the logarithm of DOS when \( e << f \) we ignore the quadratic fluctuations around both trivial and instanton saddle-point of the action 12. Then from 22, 10 and 11 one obtains

\[
\ln \rho(e) = -\frac{3.14f}{\sqrt{2me}} \ln\left(\frac{3.14f}{\pi e}\right) + ....
\]  

(23)

The leading behavior is similar as in the corresponding purely 1d problem where one obtains the Lifshitz tail \( \ln \rho(e) \propto -1/\sqrt{2me} \) when \( e \to 0 \). Note however that DOS in our case vanishes faster at small energies than in 1d.

4 Discussion

In 2d the behavior of DOS in the limit of small energies \( e/f << 1 \) depends critically on the density of scatterers: it diverges for \( 1 < f < 2 \), goes to constant when \( f = 2 \) and to zero when \( 2 < f \). In contrast, in 3d DOS resembles more the familiar 1d case; irrespectively of \( f \) DOS vanishes exponentially fast in the limit \( e/f \to 0 \). One expects that the states at the bottom of the band in 3d are localized in the rare large regions free of impurities, which would roughly correspond to the found behavior of DOS [13]. To quantitatively study the relation between 3d DOS in the limit when \( m \to \infty \) (i.e. when the mass parallel to the field becomes very large) and the 2d result one needs to include the fluctuations around the saddle points studied in the previous section and the next order terms in the instanton action 22. We will not dwell on this here, since it is already possible to see qualitatively what happens. Ignoring the energy dependence that comes from the quadratic fluctuations, we may write \( Z \approx 1 + i \exp(-S_0) \), where the first term is the contribution from the trivial saddle-point and the imaginary peace comes from the instanton. Differentiating the logarithm of \( Z \) with respect to the energy and taking the imaginary peace leads to DOS:

\[
\rho(e) \propto \ln(1/e) \exp(-S_0)/\sqrt{2me^3}.
\]

At fixed density of scatterers \( f \), the energy at which DOS
reaches its maximum value goes to zero when $m \rightarrow \infty$ as illustrated on Fig 4. On the other hand, when $m$ is constant increasing the density of scatterers $f$ makes the peak of DOS flatter (Fig. 5). Thus, having a finite (as opposed to infinite) mass along the field basically shifts the maximum of DOS from being at $e \approx 0$ for $f \approx 2$ to reside at some finite energy $e_{\text{max}} \propto 1/m$. The behavior of DOS close and right to the maximum resembles its 2d counterpart, while left to it DOS drops sharply to zero, so that for $e << 1/m$ there are essentially no states.

The described behavior resembles very much the situation in 1d in the limit of weak disorder [13, 14]. In that case DOS differs from the one in an ideal system only in the narrow region between zero and $c^2$, where $c = \rho/\lambda$ is the parameter characterizing the strength of disorder.

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6 Appendix

For the partition function 11 to have an imaginary part it is essential that the instanton saddle-point discussed in section 3 has a mode with a negative eigenvalue. The rotation of the line of integration over this mode by $\pi/2$ in the complex plane then makes the contribution of this saddle-point to the integral purely imaginary. Here we show that the proposed anzats is indeed such a mode. To study fluctuations around the non-trivial saddle-point configuration $\phi_0(z)$ one needs to diagonalize the operator

$$\hat{O} = \frac{\delta^2 S}{\delta \phi(z') \delta \phi^*(z)}|_{\phi_0} = -\frac{\partial^2}{2m} - (e - f \exp(-|\phi_0(z)|^2))\delta(z - z').$$

(24)

We take the operator at $\phi_0(z) = a \cos(zb)$ with $a$ and $b$ given as in eqs. 20 and 21 and calculate the matrix element $\langle \phi_0 | \hat{O} | \phi_0 \rangle$. In the limit of large $a$ it is straightforward to
obtain:

$$\langle \phi_0 | \hat{O} | \phi_0 \rangle = -\frac{f \pi}{b} (1 - \frac{1}{2a \sqrt{\pi}})$$  \hspace{1cm} (25)

In the limit $e/f \to 0$ we have $a \to \infty$ so from the last equation it follows that in the same limit $\langle \phi_0 | \hat{O} | \phi_0 \rangle < 0$. Thus our ansatz has a non-zero overlap with the exact negative mode of the operator $\hat{O}$ taken at $\phi_0$. This is sufficient to make the contribution of the corresponding saddle-point to the partition function purely imaginary.
Captions:

Figure 1. DOS in 2d at $f = 1.5$ obtained by taking 30 different realizations of random potential in LLL with degeneracy 100. In the inset DOS close to zero energy is shown.

Figure 2. Same is in Figure 1 but at $f=2.5$ and for 10 realizations of random potential.

Figure 3. Numerical solution of eq. 13 (dots) and the anzats determined by the parameters from eq. 20 and 21 (full line) at energy $e/f = 10^{-4}$ and $2m = 1$.

Figure 4. Approximate expression for DOS $\rho(e) \approx \ln(1/e) \exp(-S_0)/\sqrt{2me^3}$ (see the text) at $f = 1.5$ for three values of the mass $m = 1000, 2000, 4000$. The peak shifts to the left as the mass increases.

Figure 5. The same expression for DOS as in figure 3. but with the mass fixed at $m = 1000$ and density of scatterers varied: $f=1.5, 2, 2.5$ from top to bottom.
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