Anomalous scaling in statistical models of passively advected vector fields

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Abstract. The field theoretic renormalization group and the operator product expansion are applied to the stochastic model of passively advected vector field with the most general form of the nonlinear term allowed by the Galilean symmetry. The advecting turbulent velocity field is governed by the stochastic Navier–Stokes equation. It is shown that the correlation functions of the passive vector field in the inertial range exhibit anomalous scaling behaviour. The corresponding anomalous exponents are determined by the critical dimensions of tensor composite fields (operators) built solely of the passive vector field. They are calculated (including the anisotropic sectors) in the leading order of the expansion in $y$, the exponent entering the correlator of the stirring force in the Navier–Stokes equation (one-loop approximation of the renormalization group). The anomalous exponents exhibit an hierarchy related to the degree of anisotropy: the less is the rank of the tensor operator, the less is its dimension. Thus the leading terms, determined by scalar operators, are the same as in the isotropic case, in agreement with the Kolmogorov’s hypothesis of the local isotropy restoration.

1. Introduction

In the past two decades, much attention has been attracted by turbulent advection of passive scalar fields; see the review paper [1] and references therein. Being of practical importance in itself, the problem of passive advection can be viewed as a starting point for studying intermittency and anomalous scaling in the fluid turbulence on the whole [2]. Most progress was achieved for the so-called Kraichnan’s rapid-change model: for the first time, the anomalous exponents were derived on the basis of a microscopic model and within controlled approximations [3]. In Kraichnan’s model the advecting velocity field $v_i(x)$ with $x = \{t, x\}$ is modelled by a Gaussian statistics with vanishing correlation time and prescribed correlation function $\langle vv \rangle \propto \delta(t - t')k^{-d - \xi}$, where $k$ is the wave number, $d$ is the dimension of space and $\xi$ is an arbitrary exponent with the most realistic (Kolmogorov) value $\xi = 4/3$.

The “zero-mode approach,” developed in [3], is based on the fact that, due to the temporal decorrelation of the Kraichnan’s velocity field, some closed differential equations can be derived for the equal-time correlation functions of the passive fields.
In this sense, the model is equivalent to a certain quantum-mechanical problem and appears “exactly solvable.” Although such equations cannot be explicitly solved, the anomalous exponents can be extracted from the analysis of their infrared (IR) asymptotic behaviour; see [1] for a detailed review and the references.

In [4] and subsequent papers, the field theoretic renormalization group (RG) and the operator product expansion (OPE) were applied to Kraichnan’s model; see [5] for a review and the references. In that approach, the anomalous scaling emerges as a consequence of the existence in the corresponding OPE of certain composite fields (“operators” in the quantum-field terminology) with negative dimensions, which are identified with the anomalous exponents. This allows one to construct a systematic perturbation expansion for the anomalous exponents and to calculate them up to the order $\xi^2$ [4] and $\xi^3$ [6]. Besides the calculational efficiency, an important advantage of the RG+OPE approach is its relative universality: it can also be applied to the case of finite correlation time or non-Gaussian advecting fields.

For passively advected vector fields, any calculation of the exponents for higher-order correlations calls for the RG techniques already in the $O(\xi)$ approximation.

Owing to the presence of a new stretching term in the dynamic equation, the behavior of the passive vector field appears much richer than that of the scalar field: “...there is considerably more life in the large-scale transport of vector quantities” (p. 232 of Ref. [2]). Indeed, passive vector fields reveal anomalous scaling already on the level of the pair correlation function [7, 8, 9]. They also exhibit interesting large-scale instabilities that can be interpreted as manifestation of the dynamo effect [7, 10, 11]. Some special case ($A = 0$, see below) reveals a close formal resemblance to the NS turbulence [12, 13, 14]. From the physics viewpoints, passive vector fields can have different physical meaning: magnetic field, weak perturbation of the prescribed background flow, concentration or density of the impurity particles with an internal structure.

In this paper, we study anomalous scaling of a passive vector quantity, advected by a non-Gaussian velocity field with finite correlation time, governed by the stirred Navier–Stokes (NS) equation.

The plan of the paper is the following. In sec. 2 we give detailed description of the stochastic problem and explain the motivation of our study and its relation to previous work. In sec. 3 we give the field theoretic formulation of the model. In sec. 4 we establish its renormalizability, derive the corresponding RG equations and present the explicit one-loop results for the renormalization constants and RG functions. Possible IR attractive fixed points are discussed in sec. 5. In sec. 6 the operator product expansion is employed to establish the anomalous scaling of the correlation functions in the inertial-range. The corresponding anomalous exponents are determined by the critical dimensions of tensor composite operators built solely of the passive field. The practical calculation is performed in the leading (one-loop) approximation; the results are presented in sec. 7. Section 8 is reserved for a brief conclusion; in particular, we mention an hierarchy demonstrated by the anisotropic contributions.
2. Description of the model

We confine ourselves with the case of transverse (divergence-free) passive $\theta_i(x)$ and advecting $v_i(x)$ vector fields. Then the general advection-diffusion equation has the form

$$\nabla_t \theta_i - A_0(\theta_k \partial_k) v_i + \partial_i \mathcal{P} = \kappa \partial^2 \theta_i + \eta_i,$$

where $\nabla_t$ is the Lagrangian (Galilean covariant) derivative, $\mathcal{P}(x)$ is the pressure, $\kappa_0$ is the diffusivity, $\partial^2$ is the Laplace operator and $\eta_i(x)$ is a transverse Gaussian stirring force with zero mean and covariance

$$\langle \eta_i(x) \eta_k(x') \rangle = \delta(t - t') C_{ik}(r/L).$$

The parameter $L$ is an integral scale related to the stirring, and $C_{ik}$ is a dimensionless function with the condition $\partial_i C_{ik} = 0$, finite at $r = 0$ and rapidly decaying for $r \to \infty$; its precise form is unessential. Due to the transversality conditions $\partial_i \theta_i = 0$, $\partial_i v_i = 0$, the pressure can be expressed as the solution of the Poisson equation:

$$\partial^2 \mathcal{P} = (A_0 - 1) \partial_i \partial_k \theta_i.$$

Thus the pressure term makes the dynamics (2.1) consistent with the transversality. The amplitude factor $A_0$ in front of the “stretching term” $(\theta_k \partial_k) v_i$ is not fixed by the Galilean symmetry and thus can be arbitrary. Such general “$A$ model” was introduced and studied in refs. [15]–[17]; it can be naturally justified within the so-called multiscale techniques.

From the physics viewpoints most interesting is the special case $A_0 = 1$, where the pressure term disappears: it corresponds to magnetohydrodynamic (MHD) turbulence [18]. It was studied earlier in numerous papers; see e.g. refs. [7, 8, 9, 19, 20] and references therein.

In earlier studies, the velocity field in (2.1) was usually described by the Kraichnan’s rapid-change model: Gaussian statistics with vanishing correlation time and prescribed power-like correlation function. In this paper, we employ the stochastic NS equation:

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \varphi + f_i,$$

where $\nabla_t$ is the same Lagrangian derivative, $\varphi$ and $f_i$ are the pressure and the transverse random force per unit mass. We assume for $f$ a Gaussian distribution with zero mean and correlation function

$$\langle f_i(x) f_j(x') \rangle = \frac{\delta(t - t')}{(2\pi)^d} \int_{k \geq m} dk \ P_{ij}(k) \ d_f(k) \ \exp \left[ ik \cdot (x - x') \right],$$

where $P_{ij}(k) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $d_f(k)$ is some function of $k \equiv |k|$ and model parameters. The momentum $m = 1/L$, the reciprocal of the integral scale related to the velocity, provides IR regularization. For simplicity, we do not distinguish it from the integral scale related to the scalar noise in (2.2).

The standard RG formalism is applicable to the problem (2.4), (2.5) if the correlation function of the random force is chosen in the power form [21]

$$d_f(k) = D_0 k^{4-d-y},$$

where $D_0$ is a constant.
where $D_0 > 0$ is the positive amplitude factor and the exponent $0 < y \leq 4$ plays the role of the RG expansion parameter. The most realistic value of the exponent is $y = 4$: with an appropriate choice of the amplitude, the function (2.6) for $y \to 4$ turns to the delta function, $d_f(k) \propto \delta(k)$, which corresponds to the injection of energy to the system owing to interaction with the largest turbulent eddies; for a more detailed justification see e.g. [22, 23].

3. Field theoretic formulation

According to the general theorem (see e.g. [22]), the full-scale stochastic problem (2.1)–(2.6) is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{v, v', \theta, \theta'\}$ with the action functional

$$S(\Phi) = S_v(v', v) + \theta' D_0 \theta'/2 + \theta' \left\{ -\nabla_t - \mathcal{A}_0(\theta_k \partial_k) v_i + \kappa_0 \partial^2 \right\} \theta,$$

where $D_\theta$ is the correlation function (2.2) of the random force $\eta_i$ in (2.1) and $S_v$ is the action for the problem (2.4)–(2.6):

$$S_v(v', v) = v' D_v v'/2 + v' \left\{ -\nabla_t + \nu_0 \partial^2 \right\} v,$$

where $D_v$ is the correlation function (2.5) of the random force $f_i$. All the integrations over $x = \{t, x\}$ and summations over the vector indices are understood. The auxiliary vector fields $v', \theta'$ are also transverse, $\partial_i v'_i = \partial_i \theta'_i = 0$, which allows to omit the pressure terms on the right-hand sides of expressions (3.1), (3.2), as becomes evident after the integration by parts. For example,

$$\int dt \int dx \ v'_i \partial_i \varphi = - \int dt \int dx \ \varphi (\partial_i v'_i) = 0.$$ 

Of course, this does not mean that the pressure contributions are unimportant: the fields $v', \theta'$ act as transverse projectors and select the transverse parts of the expressions to which they are contracted.

The part of the coupling constants is played by the three parameters $g_0 \equiv D_0/\nu_0^3$, $\mathcal{A}_0$ and $u_0 = \kappa_0/\nu_0$, the analog of the inverse Prandtl number in the scalar case. By dimension,

$$g_0 \propto \Lambda^y, \quad \mathcal{A}_0 \text{ and } u_0 \propto \Lambda^0,$$

(3.3)

where $\Lambda$ is the characteristic ultraviolet (UV) momentum scale. Thus the model (3.1), (3.2) becomes logarithmic (all the coupling constants become dimensionless) at $y = 0$, and the UV divergences manifest themselves as poles in $y$.

4. Renormalization and RG equations

The renormalization and RG analysis of the model (3.1), (3.2) are similar to that of the scalar advection by the NS velocity field [24], and here we discuss them only briefly. Dimensional analysis shows that superficial UV divergences can be present only in
the 1-irreducible Green functions $\langle v'v \rangle$, $\langle v'vv \rangle$, $\langle \theta'\theta \rangle$ and $\langle \theta'v\theta \rangle$. The corresponding counterterms reduce to the forms $v'\partial_tv$, $v'\partial^2v$, $v'(v\partial)v$, $\theta'\partial_t\theta$, $\theta'\partial^2\theta$ and $\theta'(v\partial)v$ and $\theta'(v\partial)\theta$.

The spatial derivative $\partial$ at the vertices $v'(v\partial)v$, $\theta'(\theta\partial)v$ and $\theta'(v\partial)\theta$ in (3.1) can be moved, using the integration by parts, onto the auxiliary fields $v'$ and $\theta'$. Thus any counterterm must include one spatial derivative per each auxiliary field. This excludes the counterterms $v'\partial_tv$ and $\theta'\partial_t\theta$ without a spatial derivative. Then the Galilean symmetry excludes the structures $v'(v\partial)v$ and $\theta'(v\partial)\theta$ because they must enter the counterterms only in the form of Galilean invariant combinations $v'\nabla_tv$ and $\theta'\nabla_t\theta$.

The remaining three counterterms $v'\partial^2v$, $\theta'\partial^2\theta$ and $\theta'(v\partial)v$ can be reproduced by multiplicative renormalization of the parameters

$$v_0 = \nu Z_\nu, \quad \kappa_0 = \kappa Z_\kappa, \quad A_0 = AZ_A, \quad g_0 = g\mu^\nu Z_g, \quad Z_\nu = Z_\nu^{-\nu}; \quad (4.1)$$

no renormalization of the fields $\Phi = \{v, v', \theta, \theta'\}$ and the IR scale $m$ is needed. Here $\nu$, $g$, $\kappa$ and $A$ are renormalized analogs of the bare parameters $v_0$, $g_0$, $\kappa_0$ and $A_0$, while the reference scale $\mu$ is an additional parameter of the renormalized theory. The last relation in (4.1) follows from the absence of renormalization of the amplitude $D_0 = g_0\nu_0^3 = g\mu^\nu\nu^3$ in the first term of the action $S_{vR}$. The renormalization constants $Z_i = Z_i(g, u, A, d, y)$ absorb all the UV divergences, so that the Green functions are UV finite (that is, finite at $y = 0$) when expressed in renormalized parameters.

The corresponding renormalized action has the form

$$S_R(\Phi) = S_{vR}(v', v) + \theta' D_\theta \theta'/2 + \theta' \{-\nabla_t - AZ_A(\theta\partial_k)\nu_i + \kappa Z_\kappa \theta^2\} \theta,$$

$$S_{vR}(v', v) = v'D_v v' / 2 + v' \{-\nabla_t + \nu Z_\nu \theta^2\} v; \quad (4.2)$$

where $D_v$ is expressed in renormalized parameters using (4.1). It differs from the original (unrenormalized) action (3.1), (3.2) only by the choice of parameters, $S_R(\Phi, e, \mu) = S_0(\Phi, e_0)$, where $e_0$ is the full set of bare parameters and $e$ are their renormalized counterparts. Thus the original $G = (\Phi \ldots \Phi)$ and the renormalized $G_R$ Green functions are also related as $G(e_0, \ldots) = G_R(e, \mu, \ldots)$; the ellipsis stands for the other arguments (times/frequencies and coordinates/momenta). We use $\bar{D}_\mu$ to denote the differential operation $\mu \partial_\mu$ at fixed bare parameters $e_0$ and operate on both sides of the last relation with it. This gives the basic RG differential equation:

$$\{D_\mu - \gamma_\nu D_\nu + \beta_\nu \partial_\nu + \beta_\mu \partial_\mu + \beta_A \partial_A\} G_R = 0. \quad (4.3)$$

Here $u = \kappa/\nu$ and $D_s = s \partial_s$ for any variable $s$. The RG functions (the $\beta$ functions and the anomalous dimensions $\gamma$) are defined as

$$\gamma_F = \bar{D}_\mu \ln Z_F \quad (4.4)$$

for any quantity $F$ and

$$\beta_\nu = \bar{D}_\mu g = g[-y + 3\gamma_\nu],$$

$$\beta_u = \bar{D}_\mu u = u[\gamma_\nu - \gamma_\kappa],$$

$$\beta_A = \bar{D}_\mu A = -A \gamma_A \quad (4.5)$$

for completely dimensionless variables (coupling constants). Here $\bar{D}_\mu$ is the operation $D_\mu$ at fixed bare parameters and the second relations in (4.5) follow from the definitions.
Anomalous scaling of passively advected vector fields

and the relations (4.1). It remains to note that the differential operator in (4.3) is nothing other than $\tilde{D}_\mu$ expressed in renormalized variables.

The one-loop calculation gives:

$$Z_\nu = 1 - g \tilde{S}_d \frac{(d - 1)}{4(d + 2)} \frac{1}{y} + O(g^2), \quad Z_A = 1 + O(g^2),$$

$$Z_\kappa = 1 - \frac{g S_d}{u(u + 1)^2} \frac{Q}{2d(d + 2)} \frac{1}{y} + O(g^2),$$

where

$$Q = (u + 1)(3A^2 + Ad - 2A + d^2 - 3) - 2A(A - 1),$$

$$\tilde{S}_d \equiv S_d/(2\pi)^d$$

and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $d$-dimensional space. Of course, due to the passivity of the field $\theta$, the constant $Z_\nu$ is the same as in the model (3.2); it does not depend on the parameters $u$ and $A$ related to the passive field. It is noteworthy that the expression (4.7) for $Q$ simplifies for the aforementioned special values of $A$:

$$Q = (u + 1)(d - 1)(d + 2) \text{ for } A = 1,$$

$$Q = (u + 1)(d^2 - 3) \text{ for } A = 0.$$

From (4.6) we obtain the following explicit one-loop expressions for the anomalous dimensions:

$$\gamma_\nu = g \tilde{S}_d \frac{(d - 1)}{4(d + 2)} + O(g^2), \quad \gamma_A = O(g^2),$$

$$\gamma_\kappa = \frac{g S_d}{u(u + 1)^2} \frac{Q}{2d(d + 2)} + O(g^2).$$

In the rapid-change version of our model [15], $Z_A = 1$ and $\gamma_A = 0$ identically because all nontrivial Feynman diagrams of the 1-irreducible Green function $\langle \theta' v \theta \rangle$ contain closed circuits of retarded propagators and therefore vanish. In the present case, the absence of the $O(g)$ term in $Z_A$ and $\gamma_A$ is a result of the cancellation of the (nontrivial!) contributions from the three one-loop diagrams in the 1-irreducible Green function $\langle \theta' v \theta \rangle$. For the counterterm $\theta' (v \partial) \theta$ such a cancellation is guaranteed by the Galilean symmetry (to all orders in $g$; see the discussion above). For $\theta' (\partial v) v$ the cancellation looks accidental and can be explained by a rather simple form of the one-loop diagrams: the structures corresponding to the counterterm $\theta' (v \partial) \theta$ cancel each other due to the Galilean symmetry, while the structures corresponding to $\theta' (\partial v) v$ enter all the one-loop diagrams with the same coefficients and cancel out into the bargain. This mechanism is not expected to work beyond the one-loop approximation; thus nontrivial contributions of the order $g^2$ and higher in $Z_A$ and $\gamma_A$ are not forbidden.

5. Fixed points

It is well known that IR asymptotic behaviour of a multiplicatively renormalizable field theory is governed by IR attractive fixed points of the corresponding RG equations.
Their coordinates are found from the requirement that all the $\beta$ functions vanish, $\beta_i(g_*) = 0$, while the type of the point is determined by the matrix $\Omega = \{\Omega_{ik} = \partial \beta_i/\partial g_k | g = g_*\}$: for an IR attractive fixed points it is positive, that is, the real parts of all its eigenvalues are positive. Here $g = \{g_i\}$ is the full set of couplings and $\beta_i$ is the full set of the corresponding $\beta$ functions.

From the explicit expressions (4.5) and (4.9) for $\beta_g$ it follows that the model (3.2) has a nontrivial fixed point $g_* \bar{S}_d = y (4 + 2d)/3(d - 1) + O(y^2)$, (5.1) which is positive and IR attractive ($\partial g \beta_g > 0$) for $y > 0$ (of course, this fact is well known, see e.g. [22, 23]). Substituting (5.1) into the equation $\beta_u = 0$ and using the explicit expressions (4.5) and (4.9) gives

$$2Q = u(u + 1)^2 d(d - 1),$$

where $Q$ from (4.7) and corrections of order $O(y)$. The last equation is $\beta_A = 0$. From eqns. (4.5) and (4.9) one finds that it is satisfied automatically up to the order $O(g)$. Thus there are two possibilities that cannot be distinguished within the one-loop approximation:

The first one is that $Z_A = 1$ to all orders in $g$, as it happens in the rapid-change version of our model [15]. Then the equation $\beta_A = 0$ becomes an identity and imposes no restriction on the coordinates of the fixed points. Then eq. (5.2) determines the coordinate $u_*$ as a function of the remaining free parameter $A$.

In particular, for the most interesting physical case $d = 3$, the simple numerical analysis shows that the positive solution $u_*$ of eq. (5.2) is unique and exists for all $A$. As a function of $A$, it achieves a minimum $u_* \simeq 0.94$ for $A \simeq -0.5$ and grows as $u_* = |A| + O(1)$ for $A \to \pm \infty$. Some special values are $u_* \simeq 1.393$ for $A = 1$ in agreement with the kinematic fixed point of the full-scale MHD problem [25], $u_* = 1$ for $A = 0$ in agreement with [13] and $u_* = 1$ for $A = -1$. The simple inspection shows that this fixed point is IR attractive: $\partial_u \beta_u > 0$, $\partial_A \beta_A = \partial_u \beta_A = 0$.

A very similar behaviour of the solution $u_*(A)$ takes place for all $d > 2$. As a function of $d$, it decreases monotonically and tends to unity as $d$ tends to infinity. The explicit analytic solution of the cubic equation (5.2) for general $d$ looks rather cumbersome and we do not present it here. For $d \leq 2$ our results become inapplicable because the renormalization of the NS model (3.2) itself must be revisited [23].

Another possibility is that the function $\beta_A$ has a nonvanishing contribution of order $g^2$ or higher. Then the equations $\beta_u = \beta_A = 0$ determine the possible values of the coordinates $u_*$ and $A_*$. To find all their values, the two-loop calculation of $Z_A$ is needed. However, it is clear without any calculation that $A_* = 0$ and $A_* = 1$ are among the possible fixed-point values of $A$ to all orders in $g$: the first case possesses additional symmetry with respect to the shift $\theta \to \theta + \text{const}$ (only derivatives of $\theta$ enter the stochastic equation (2.1)), while for the second case the nonlinearity $V_i = (v_k \theta_k) \theta_1 - (\theta_k v_k) v_i = \partial_k (v_k \theta_i - \theta_k v_i)$ in (2.1) is transverse: $\partial_i V_i = 0$, so that
the nonlocal pressure term \(2.3\) vanishes. The both properties are preserved by the renormalization procedure.

Existence of the fixed points different from \(\mathcal{A}_*=0\) and 1 and their stability cannot be established without the explicit two-loop calculation. This issue lies beyond the scope of the present paper; here we only can say that for the passive vector field advected by the compressible Kraichnan’s ensemble such points do exist; see \[16\].

6. Inertial-range anomalous scaling of the correlation functions, composite fields and operator product expansion

The key role in the following is played by the critical dimensions \(\Delta_{n,l}\) associated with the irreducible tensor composite fields ("local composite operators" in the field theoretic terminology) built solely of the fields \(\theta\) at a single space-time point \(x=(t,\mathbf{x})\). They have the forms

\[
F_{n,l} \equiv \theta_{i_1}(x) \cdots \theta_{i_l}(x) \left(\theta_i(x)\theta_i(x)\right)^p + \ldots ,
\]

(6.1)

where \(l \leq n\) is the number of the free vector indices and \(n = l + 2p\) is the total number of the fields \(\theta\) entering into the operator; the tensor indices and the argument \(x\) of the symbol \(F_{n,l}\) are omitted. The ellipsis stands for the appropriate subtractions involving the Kronecker delta symbols, which ensure that the resulting expressions are traceless with respect to contraction of any given pair of indices, for example, \(\theta_i\theta_j - \delta_{ij}(\theta_k\theta_k/d)\) and so on.

The quantities of interest are, in particular, the equal-time pair correlation functions of the operators (6.1). For these, solving the corresponding RG equations gives the following asymptotic expression

\[
\langle F_{n,l}(t,\mathbf{x})F_{k,j}(t,\mathbf{x}')\rangle \simeq r^{-\Delta_{n,l}-\Delta_{k,j}} \zeta_{n,l;k,j}(mr)
\]

(6.2)

with \(r = |\mathbf{x} - \mathbf{x}'|\) and certain scaling functions \(\zeta(mr)\). To simplify the notation, here and below in similar expressions we omit the tensor indices and the labels of the scaling functions; the IR irrelevant parameters (like \(\Lambda\) or \(\nu_0\)) are also not shown.

The last expression in (6.2) is valid for \(\Lambda r \gg 1\) and arbitrary values of \(mr\). The inertial-convective range corresponds to the additional condition that \(mr \ll 1\). The forms of the functions \(\zeta(mr)\) are not determined by the RG equations themselves; their behavior for \(mr \to 0\) is studied using Wilson’s OPE.

According to the OPE, the equal-time product \(F_1(x)F_2(x')\) of two renormalized composite operators at \(x=(\mathbf{x} + \mathbf{x}')/2 = \text{const}\) and \(r = \mathbf{x} - \mathbf{x}' \to 0\) can be represented in the form

\[
F_1(x)F_2(x') \simeq \sum_F C_F(r)F(t,\mathbf{x}),
\]

(6.3)

where the functions \(C_F\) are the Wilson coefficients, regular in \(m^2\), and \(F\) are, in general, all possible renormalized local composite operators allowed by symmetry. More precisely, the operators entering the OPE are those which appear in the
corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. If these operators have additional vector indices, they are contracted with the corresponding indices of the coefficients $C_F$.

It can always be assumed that the expansion in Eq. (6.3) is made in operators with definite critical dimensions $\Delta_F$. The correlation functions (6.2) are obtained by averaging equation of the type (6.3) with the weight $\exp S$, where $S$ is the action functional (4.2); the quantities $\langle F \rangle$ appear on the right hand sides. Their asymptotic behavior for $m \to 0$ is found from the corresponding RG equations and has the form $\langle F \rangle \propto m^{\Delta_F}$.

From the expansion (6.3) we therefore find the following asymptotic expression for the scaling function $\zeta(mr)$ in the representation (6.2) for $mr \ll 1$:

$$\zeta(mr) \simeq \sum_F A_F (mr)^{\Delta_F}, \quad (6.4)$$

where the coefficients $A_F = A_F(mr)$ are regular in $(mr)^2$.

### 7. Anomalous scaling and the exponents in the one-loop approximation

The feature specific of the models of turbulence is the existence of composite operators with negative critical dimensions. Such operators are termed “dangerous,” because their contributions to the OPE diverge at $mr \to 0$ [22, 23].

Obviously, most dangerous are the operators (6.1) with the critical dimensions $\Delta_{n,l} = -n + O(y)$. Like in the Kraichnan’s case, the analysis shows that their anomalous dimensions can be calculated in the simplified model without the random forcing in the stochastic equation (2.1) because the correlator (2.2) does not enter the relevant Feynman diagrams [9]. Then those operators become multiplicatively renormalizable, $F_{n,l} = Z_{n,l} F_{R,n,l}$. The practical one-loop calculation of the renormalization constants $Z_{n,l}$ is similar to the case of Kraichnan’s velocity field, discussed in [9, 16] in detail, so that here we give only the result:

$$Z_{n,l} = 1 - \frac{gS_d}{u(u+1)} \frac{A^2 Q_{nl}}{4d(d+2)} \frac{1}{y} + O(g^2), \quad (7.1)$$

where

$$Q_{n,l} = 2n(n-1) - (d+1)(n-l)(d+n+l-2) = - (d-1)(n-l)(d+n+l) + 2l(l-1) \quad (7.2)$$

and $S_d$ is defined below equation (4.7). Note that the same polynomial $Q_{n,l}$ arises in the scalar case [26] and in Kraichnan’s MHD model [9].

The corresponding anomalous dimension is

$$\gamma_{n,l} = \frac{gS_d}{u(u+1)} \frac{A^2 Q_{nl}}{4d(d+2)} + O(g^2). \quad (7.3)$$

Substituting the fixed-point value (5.1) gives

$$\gamma_{n,l} = \frac{A^2}{u(u+1)} \frac{Q_{n,l}}{3d(d-1)} y + O(y^2). \quad (7.4)$$
If the function $\beta_A$ vanishes identically, the solution $u_\ast(A)$ of the equation (5.2) should be substituted into (7.4); then the dependence on the free parameter $A$ persists in $\gamma_{n,l}$. Otherwise, the fixed-point values $u_\ast$, $A_\ast$ should be used; see discussion in section 5. In particular, for $A_\ast = 1$ the fixed-point value of $u_\ast$ is the positive solution of the quadratic equation $u(u + 1) = 2(d + 2)/d$. Then eq. (7.4) becomes

$$\gamma_{n,l} = \frac{Q_{n,l}}{6(d - 1)(d + 2)} y + O(y^2),$$

which agrees with the result derived in [20] for the MHD case. Note that (7.5) coincides with its analog in the Kraichnan’s case [9] up to the substitution $\xi \to y/3$.

For $A = 0$, the anomalous dimensions $\gamma_{n,l}$ vanish to all orders in $y$, because the operators $F_{n,l}$ become UV finite and are not renormalized. In that case, interesting quantities are structure functions (rather than plain correlation functions); their inertial-range behaviour is determined by the operators built of the derivatives of the fields $\theta$; see [12, 13].

With this exception, the amplitude $A^2/u(u + 1)$ in (7.4) is positive for any physical fixed point. Thus the dimension $\gamma_{n,l}$ is negative for the most interesting case of the scalar operator with $l = 0$ and increases monotonically with $l$ (for a fixed $n$).

From the relation $\Delta_{n,l} = -n + O(y)$ it follows that the critical dimensions satisfy the same hierarchy relations: $\Delta_{n,l} > \Delta_{n,l'}$ if $l > l'$, which are conveniently expressed as inequalities $\partial \Delta_{n,l}/\partial l > 0$.

This fact, first established in [8] for the Kraichnan’s MHD model, has a deep physical meaning: in the presence of large-scale anisotropy, the leading contribution in the inertial-range behavior $mr \to 0$ of the correlation function like (6.2) is given by the isotropic “shell” ($l = 0$). The corresponding anomalous exponent is the same as for the purely isotropic case. The anisotropic contributions give only corrections which vanish for $mr \to 0$, the faster the higher the degree of anisotropy $l$ is. This effect gives some quantitative support for Kolmogorov’s hypothesis of the local isotropy restoration and appears rather robust, being observed for the real fluid turbulence [27] and the passive scalar model [26].

8. Conclusion

We have studied a model of a divergence-free (transverse) vector quantity $\theta$, passively advected by a random non-Gaussian velocity field with finite correlation time, governed by the stochastic NS equation. The model is described by an advection-diffusion equation with a random large-scale stirring force, nonlocal pressure term and the most general form of the inertial nonlinearity, “controlled” by the parameter $A \propto A_0$.

We have shown that, in the inertial range of scales, the correlation functions of the field $\theta$ exhibit anomalous scaling behaviour. The corresponding anomalous exponents are determined by the critical dimensions of tensor composite fields (6.1) built solely of the passive vector field. They are calculated (including the anisotropic sectors) to the
Anomalous scaling of passively advected vector fields

leading order of the expansion in $y$, the exponent entering the correlation function of the stirring force in the NS equation (the one-loop approximation in the RG terminology).

Like in the special MHD case $A = A_0 = 1$, the exponents exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank of the tensor operator, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behaviour. Thus in the presence of large-scale anisotropy the leading terms, determined by the scalar operators, remain the same as in the purely isotropic case, in agreement with the phenomenological hypothesis of the local isotropy restoration.

The question that remains open is whether the amplitude $A$ in front of the “stretching term” $(\theta \partial) v$ in the advection-diffusion equation tends to some fixed-point values, or it remains a free parameter which the anomalous exponents depend upon. The analysis of that alternative lies beyond the scope of the one-loop approximation. We plan to perform it in the nearest future.

Acknowledgments

The authors are indebted to Loran Adzhemyan, Michal Hnatich and Juha Honkonen for discussion. The authors thank the Organizers of the IV International Conference “Models in Quantum Field Theory” dedicated to A.N. Vasiliev (St. Petersburg–Petrodvorez, 24–27 September 2012) for the possibility to present the results of this work. The work was supported in part by the Russian Foundation for Fundamental Research (project 12-02-00874-a).

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