STRONG LAW OF LARGE NUMBERS ON GRAPHS AND GROUPS

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ABSTRACT. We consider (graph-)group-valued random element $\xi$, discuss the properties of a mean-set $E(\xi)$, and prove the generalization of the strong law of large numbers for graphs and groups. Furthermore, we prove an analogue of the classical Chebyshev's inequality for $\xi$ and Chernoff-like asymptotic bounds. In addition, we prove several results about configurations of mean-sets in graphs and discuss computational problems together with methods of computing mean-sets in practice and propose an algorithm for such computation.

Key words and phrases: Probability measures on graphs and groups, average, expectation, mean-set, strong law of large numbers, Chebyshev inequality, Chernoff bound, configuration of mean-sets, free group, shift search problem.

1. Introduction

Random objects with values in groups and graphs are constantly dealt with in many areas of mathematics and theoretical computer science. In particular, such objects are very important in group-based cryptography (see [23] or [9] for introduction to the subject). Having the notion of the average for random group elements, generalized laws of large numbers for groups with respect to this average together with results on the rate of convergence in these laws would broaden the range of applications of random group objects from both theoretical and practical point of view. With a continuing development of group-based cryptography, availability of such tools for analysis of probability measures and their characteristics on groups becomes especially important. In this paper, we develop these probabilistic tools for finitely generated groups and propose practical algorithms for computing mean values (or expectations) of group/graph-valued random elements. The results of this paper form a new mathematical framework for group-based cryptography and find applications to security analysis of Sibert type authentication protocols ([24]).

The classical strong law of large numbers (SLLN) states that for independent and identically distributed (i.i.d.) real-valued random variables $\{\xi_i\}_{i=1}^{\infty}$

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \to E(\xi_1)$$

almost surely (with probability one) as $n \to \infty$, provided that expectation $E(\xi_1)$ is finite (see [5], [11], or [30]). It is natural to pose a question about the existence of
counterparts of this result for different topological spaces and/or algebraic structures, including groups. Starting from the middle of the last century, there has been ongoing research, following different trends, concerning the existence of such generalizations of the SLLN. One line of this research investigates random walks on groups (see Section 1.1.5 for a brief list of relevant literature sources). The present work follows another direction of that research—the one which is concerned with the problem of averaging in arbitrary metric spaces. We generalize classical probability results to groups starting with the concept of expectation (mean value) for group elements. Then we prove the almost sure (with probability one) convergence, in some appropriate sense, of sample (empirical) means for group/graph random elements to the actual (theoretical) mean, thus, generalizing the classical law (1) and preserving its fundamental idea. We supplement our results with the analogues of Chebyshev and Chernoff-like bounds on the rate of convergence in the SLLN for random graph/group elements.

1.1. Historical Background. Below we give a brief account of some developments concerning probabilities and mean-values for various spaces as well as some already existing generalizations of the strong law of large numbers in order to highlight several stages of research that preceded our work. The reader willing to proceed to the core of our work right away may skip this section and move on to Section 1.2.

1.1.1. Linear spaces. In 1935, Kolmogorov [22] proposed to study probabilities in Banach spaces. Later, the interpretation of stochastic processes as random elements in certain function spaces inspired the study of laws of large numbers for random variables taking values in linear topological spaces. Banach spaces fit naturally into the context of the strong law of large numbers because the average of $n$ elements $x_1, \ldots, x_n$ in a Banach space is defined as $n^{-1}(x_1 + \ldots + x_n)$. In addition, Banach space provides convergence, and Gelfand–Pettis integration provides the notion of expectation of a random element (see [15] and [27]). It goes as follows. Let $X$ be a linear space with norm $\| \cdot \| : X \to \mathbb{R}$ and $X^*$ is the topological dual of $X$. A random $X$-element $\xi$ is said to have the expected value $\mathbb{E}(\xi) \in X$ if

$$\mathbb{E}(f(\xi)) = f(\mathbb{E}(\xi))$$

for every $f \in X^*$. Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of random $X$-elements. Without loss of generality, we may assume that $\mathbb{E}\xi_i = 0$ for every $i$. The strong law of large numbers in a separable Banach space $X$ for a sequence of i.i.d. random $X$-elements $\{\xi_i\}_{i=1}^\infty$ is first proved in [25]. It states that

$$\lim_{n \to \infty} \|n^{-1}(\xi_1 + \ldots + \xi_n)\| = \mathbb{E}(\xi_1) = 0$$

with probability one. The strong law of large numbers for i.i.d. random elements in a Fréchet space was proved in [1]. A few other works discussing generalizations of the strong law of large numbers in linear spaces are [2], [3], [33].

1.1.2. Metric spaces. Unlike in linear spaces, in a general (non-linear) topological space $X$, one has to do find some other ways to introduce the concept of averaging and expectation. In 1948, Fréchet, [12], proposed to study probability theory in general metric spaces and introduced a notion of a mean (sometimes called Fréchet mean) of a probability measure $\mu$ on a complete metric space $(X, d)$ as the minimizer...
SSLN FOR GRAPHS AND GROUPS

of $Ed^2(x, y)$, where

$$Ed^2(x, y) = \int_X d^2(x, y) \mu(dy)$$

when it exists and is unique. If the minimizer is not unique, then the set of mini-
mizers can be considered. If $\xi : \Omega \to X$ on a given probability space $(\Omega, \mathcal{F}, P)$ (see [5, 11]) is a random element in $(X, d)$ and if for some $x \in X$,

$$(2) \quad Ed^2(\xi, x) = \inf_{y \in X} Ed^2(\xi, y) < \infty,$$

then $x$ is called an expected element of $X$. These generalizations were not met with much enthusiasm at the time (see historical remarks on probabilities in infinite dimensional vector spaces in [16]), and Fréchet’s suggestions, due to the luck of their applications, underwent rather slow developments in the middle of the last century.

Let us briefly mention some existing works on generalizing the classical SLLN to a metric space $X$. Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of i.i.d. random elements with values in $X$. Let the expectation be defined as in (2), written as a set

$$E(X) = \left\{ x \in X \mid \frac{1}{n} \sum_{i=1}^n d^2(\xi_i, x) = \inf_{y \in X} \frac{1}{n} \sum_{i=1}^n d^2(y, \xi_i) \right\}.$$ 

Define an empirical mean (average) of elements $\xi_1(\omega), \ldots, \xi_n(\omega)$ to be the set

$$M(\xi_1, \ldots, \xi_n) = \left\{ x \in X \mid \frac{1}{n} \sum_{i=1}^n d^2(x, \xi_i(\omega)) = \inf_{y \in X} \frac{1}{n} \sum_{i=1}^n d^2(y, \xi_i(\omega)) \right\}.$$ 

One of the first works on generalization of the SLLN for metric spaces is given in 1977 by Ziezold ([36]). Ziezold considers a separable quasi-metric space $X$ with a finite quasi-metric $d$. For a sequence of i.i.d. random $X$-elements $\{\xi_i\}_{i=1}^\infty$ such that $Ed^2(\xi, x)$ is finite for at least one $x \in X$, he proves that inclusion

$$(3) \quad M(\omega) = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty M(\xi_1(\omega), \ldots, \xi_n(\omega)) \subseteq E(\xi_1)$$

holds with probability one. Here, $\bigcup_{n=k}^\infty M(\xi_1(\omega), \ldots, \xi_n(\omega))$ is the closure of the union of $M(\xi_1(\omega), \ldots, \xi_n(\omega))$’s. He also shows that, in general, the equality does not hold (for a finite quasi-metric space). In 1981, Sverdrup-Thygeson (32) proves inclusion (3) for compact connected metric spaces and shows that the equality does not hold in general (for a metric space) when the minimizer in (2) is not unique. In 2003, Bhattacharya and Patrangenaru in [4, Theorem 2.3] prove equality in (3) for the unique minimizer in (2) for metric spaces $X$ such that every closed bounded subset of $X$ is compact, improving Ziezold’s and Sverdrup-Thygeson’s results.

**Manifolds.** As the need for statistical analysis for spaces with differential geometric structure was arising, statistical inference on Riemannian manifolds started to develop rapidly, especially due to applications in statistical theory of shapes and image analysis. These applications evolve around the concept of averaging. See [21] for an introduction into shape theory. The interested reader may also refer to [18], for instance.

There are two main approaches to averaging of elements on a manifold. Every Riemannian manifold $X$ is a metric space and hence one can use constructions from the previous section to define the notion of a mean. Fréchet mean of a probability
measure on a manifold is also known as an *intrinsic mean* [4]. Non-uniqueness of the intrinsic mean is a source of different technical problems. Also, the intrinsic mean, even when unique, is often very difficult to compute in practice.

On the other hand, a manifold $X$ can also be looked at as a submanifold of some Euclidean space $\mathbb{R}^k$ and one can define a mean relative to this inclusion. Let $\tau : X \to \mathbb{R}^k$ be an embedding of $X$ into Euclidean space $(\mathbb{R}^k, d_0)$. A point $p \in \mathbb{R}^k$ is called *nonfocal* if there exists a unique $x \in \tau(X)$ such that $d_0(p, x) = d_0(p, \tau(X))$.

Let $\mu$ be a probability measure on $X$, $\mu'$ a probability measure on $\mathbb{R}^k$ induced by $\tau$, and $x^* \in \mathbb{R}^k$ the expectation of $\mu'$. We say that the measure $\mu$ is nonfocal if $x^*$ is a nonfocal point. For a nonfocal probability measure $\mu$ on $X$ we define the mean as $\tau^{-1}(x^*)$. In [4] the authors prove the strong law of large numbers for the intrinsic and extrinsic means on manifolds.

1.1.3. **K-means.** A notion of a mean (or a mean-set) can be generalized into $k$-mean. Let $B$ be a Banach space with a norm $\| \cdot \|$ and $k \in \mathbb{N}$. For a set $H = \{h_1, \ldots, h_k\}$ we define a partition of $B$ as follows

$$S_i = \{x \in B \mid \|x - h_i\| \leq \|x - h_j\| \text{ for every } j = 1, \ldots, k\} \setminus (S_1 \cup \ldots \cup S_{i-1})$$

where $i = 1, \ldots, k$ and a function $\pi_H : B \to B$

$$\pi_H(x) = \sum_{i=1}^k h_i \cdot 1_{S_i}(x)$$

where $1_{S_i}$ is the indicator function of $S_i$. Fix a suitable non-decreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ (e.g., $\Phi(x) = x$ or $\Phi(x) = x^2$) and for a probability measure $\mu$ on $B$ define a number

$$M(H) = \int_{x \in B} \Phi(\|x - \pi_H(x)\|) d\mu(x).$$

A set $H_0$ of $k$ elements that minimizes the value of $M$ is called a $k$-**mean** of a probability distribution $\mu$. In general, there can be several minimizers, which leads to technical complications. In 1988, Cuesta and Matran [8] proved that empirical $k$-means converge to the $k$-mean $H_0$ of $\mu$ under the assumption that the $k$-mean is unique.

It is straightforward to generalize a notion of a $k$-mean to a general metric space $(X, d)$. Indeed, if we put

$$S_i = \{x \in B \mid d(x, h_i) \leq d(x, h_j) \text{ for every } j = 1, \ldots, k\} \setminus (S_1 \cup \ldots \cup S_{i-1})$$

and

$$M(H) = \int_{x \in B} \Phi(d(x, \pi_H(x))) d\mu(x),$$

then we get a similar notion. This type of $k$-means, with $\Phi(x) = x$, was considered by Rubinshtein in 1995 in [29] where it was called the $k$-center.

As we can see, in general, depending on the research goals, one can define mean values on a given metric space $(X, d)$ using any powers of $d$, i.e., instead of dealing with minimization of $Ed^2(\xi, x)$ in [4], one can work with a very similar functional by minimizing $Ed^r(\xi, x)$ for any $r > 0$ if necessary.
1.1.4. Probabilities on algebraic structures. Metrics and probabilities on algebraic structures have been studied from different perspectives. One source to look at is the book of M. Gromov [17]. The reader can find some applications of Fréchet mean in statistical analysis of partially ranked data (such as elements of symmetric groups and homogeneous spaces) in the book of Diaconis ([10]). An extensive historical background of the studies of probabilities on algebraic structures is given in [16], where the author considers probabilities for stochastic semi-groups, compact and commutative stochastic groups, stochastic Lie groups, and locally compact stochastic groups employing the techniques of Fourier analysis to obtain limit theorems for convolutions of probability distributions. The reader interested in the question of defining probabilities on groups can find several approaches to this issue in [6].

1.1.5. Random walks on groups. One way to generalize the strong law of large numbers for groups is to study the asymptotic behavior of the products $g_1 g_2 \ldots g_n$, where $\{g_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random group elements, the so-called random walk on a group. The reader can consult [35] for an introduction to random walks on groups. In 1960, Furstenberg and Kesten ([14]) prove the generalization of the strong law of large numbers for random matrices. Namely, they show that the limit
$$
\lim_{n \to \infty} \frac{\log ||g_1 g_2 \ldots g_n||}{n}
$$
exists with probability one, with some restrictive conditions on the entries of $g_i$, without computing the limit explicitly. In 1963, Furstenberg solved this problem for normalized products of random matrices in terms of stationary measures ([13]). Computational techniques that would allow to compute these measures are investigated in [28]. A concise account of a number of illuminating results in the direction of the generalization of the SLLN to groups can be found in [20], where the authors prove the theorem about the directional distribution of the product $g_1 g_2 \ldots g_n$, thus, proving a general law of large numbers for random walks on general groups. The authors call it a multiplicative ergodic theorem or a general, non-commutative law of large numbers (see [20] for the precise statement).

1.2. The core of our work. Motivated by applications to group-based cryptanalysis, we study Fréchet type mean values and their properties in graph/group theoretic settings.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a locally finite graph and $(\Omega, \mathcal{F}, P)$ a given probability space. A random $\Gamma$-element $\xi$ is a measurable function $\xi : \Omega \to V(\Gamma)$. This random $\Gamma$-element $\xi$ induces an atomic probability measure $\mu : V(\Gamma) \to [0, 1]$ on $V(\Gamma)$ in a usual way:
$$
\mu(v) = P(\{\omega \in \Omega \mid \xi(\omega) = v\}), \; v \in V(\Gamma).
$$
Next, we introduce a weight function $M_\xi : V(\Gamma) \to \mathbb{R}$ by
$$
M_\xi(v) = \mathbb{E}d^2(v, \xi) = \sum_{s \in V(\Gamma)} d^2(v, s)\mu(s),
$$
where $d(v, s)$ is the distance between $v$ and $s$ in $\Gamma$, and note that, trivially, the domain of definition of $M_\xi(\cdot)$ is either the whole $V(\Gamma)$ (in which case we say that $M$ is totally defined) or $\emptyset$. The domain of $M$ is the set
$$
domain(M) = \left\{ v \in V(\Gamma) \mid \sum_{s \in V(\Gamma)} d^2(v, s)\mu_\xi(s) < \infty \right\}.
$$
In the case when \( \text{domain}(M_\xi) = V(\Gamma) \), we define the mean-set of \( \xi \) to be

\[
E(\xi) = \{ v \in V(\Gamma) \mid M_\xi(v) \leq M_\xi(u), \forall u \in V(\Gamma) \}.
\]

The above definition of \( E(\xi), \xi : \Omega \to V(\Gamma) \), provides the corresponding notion of a mean (average, expectation) for finitely generated groups via their Cayley graphs.

Once we have the notion of mean-set for group-valued random elements, we notice that it satisfies the so-called “shift” property; namely,

\[
E(g_\xi) = gE(\xi), \forall g \in G
\]

which is analogous to the linearity property of a classical expectation for real-valued random variables.

Next, for a sample \( \xi_1(\omega), \ldots, \xi_n(\omega) \) of i.i.d. random \( \Gamma \)-elements we define a relative frequency \( \mu_n(u; \omega) = \mu_n(u) \) with which the value \( u \in V(\Gamma) \) occurs in the sample above:

\[
\mu_n(u) = \frac{1}{n} |\{ i \mid \xi_i = u \}|.
\]

Relative frequency \( \mu_n \) is a probability measure on \( \Gamma \), and we can define empirical (sampling) weight function (random weight) as

\[
M_n(v) = \sum_{s \in V(\Gamma)} d^2(v, s) \mu_n(s).
\]

Going further, we define an empirical mean or sample mean-set of the sample \( \xi_1, \ldots, \xi_n \) to be the set of vertices

\[
S(\xi_1, \ldots, \xi_n) = \{ v \in V(\Gamma) \mid M_n(v) \leq M_n(u), \forall u \in V(\Gamma) \}.
\]

The function \( S(\xi_1, \ldots, \xi_n) \) on graphs is an analogue of the average function \( (x_1, \ldots, x_n) \mapsto (x_1 + \ldots + x_n)/n \) for \( x_1, \ldots, x_n \in \mathbb{R} \). We let \( S_n = S(\xi_1, \ldots, \xi_n) \). With these notions at hand, we first formulate and prove the following generalization of the strong law of large numbers for graphs and groups with one-point mean sets.

**Theorem A. (Strong Law of Large Numbers for graphs)** Let \( \Gamma \) be a locally-finite connected graph and \( \{\xi_i\}_{i=1}^\infty \) a sequence of i.i.d. random \( \Gamma \)-elements. If the weight function \( M_{\xi_1}(\cdot) \) is totally defined and \( E(\xi_1) = \{v\} \) for some \( v \in V(\Gamma) \), then

\[
\lim_{n \to \infty} S_n(\xi_1, \ldots, \xi_n) = E(\xi_1)
\]

holds with probability one.

Next, we improve this result and prove the generalized law of large numbers for groups when \( |E(\xi)| > 1 \) (see Section 3.1). The simplest version of multi-vertex SLLN in terms of \( \limsup \) is as follows:

**Theorem B. (Multi-vertex SLLN for graphs)** Let \( \Gamma \) be a locally-finite connected graph and \( \{\xi_i\}_{i=1}^\infty \) a sequence of i.i.d. random \( \Gamma \)-elements. Assume that the weight function \( M \) is totally defined and \( E(\xi) = \{v_1, \ldots, v_k\} \), where \( k \geq 4 \). If the random walk \( \mathbb{R}^k \) associated to \( v_1 \) is genuinely \( (k-1) \)-dimensional, then

\[
\limsup_{n \to \infty} S_n = E(\xi)
\]

holds with probability one.

In addition, we prove analogues of classical Chebyshev’s inequality and Chernoff-like bounds for a graph-(group-) random element \( \xi \).
Theorem C. (Chebyshev’s inequality for graphs) Let $\Gamma$ be a locally-finite connected graph and $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi_1}(\cdot)$ is totally defined then there exists a constant $C = C(\Gamma, \xi_1) > 0$ such that

$$P\left(\overline{S}(\xi_1, \ldots, \xi_n) \not\subseteq E(\xi)\right) \leq \frac{C}{n}.$$ 

Theorem D. (Chernoff-like bounds for graphs) Let $\Gamma$ be a locally-finite connected graph and $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi_1}(\cdot)$ is totally defined and $\mu_{\xi_1}$ has finite support, then for some constant $C > 0$

$$P\left(\overline{S}(\xi_1, \ldots, \xi_n) \not\subseteq E(\xi)\right) \leq O(e^{-Cn}).$$

1.3. Outline. In Section 2 we give basic definitions and discuss some properties of the newly defined objects. Next, we turn to the formulation and the proof of the strong law of large numbers on graphs and groups. These tasks are carried out in Section 3. Chebyshev’s inequality and Chernoff-like bounds for graphs are proved in Section 4. In Section 5 we consider configurations of mean-sets in graphs and their applications to trees and free groups. Section 6 deals with computational problems and methods of computing $E(\xi)$. In particular, we propose an algorithm and prove that this algorithm finds a central point for trees. Finally, in Section 7 we perform series of experiments in which we compute the sample mean-sets of randomly generated samples of $n$ random elements and observe the convergence of the sample mean-set to the actual mean.

2. Mean (expectation) of a group-valued random element

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $G$ a finitely generated group. In this section, we define the notion of expectation for graph random elements in the sense of Fréchet mean set, which is one of the possible ways to look at mean values (see Section 1.1.2). The same definition will hold for random elements $\xi : \Omega \rightarrow G$ on groups. We also discuss properties of the mean sets on groups. In particular, we prove that for our expectation $E$, we have $E(g_{\xi}) = gE(\xi)$.

2.1. The mean set in a graph. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a locally finite connected graph. A random $\Gamma$-element $\xi$ is a measurable function $\xi : \Omega \rightarrow V(\Gamma)$. The random element $\xi$ induces an atomic probability measure $\mu : V(\Gamma) \rightarrow [0, 1]$ on $V(\Gamma)$ in a usual way:

$$\mu(v) = \mu_{\xi}(v) = P(\{\omega \in \Omega \mid \xi(\omega) = v\}), \ v \in V(\Gamma).$$

Next, we introduce a weight function $M_{\xi} : V(\Gamma) \rightarrow \mathbb{R}$ by

$$M_{\xi}(v) = \mathbb{E}d^2(v, \xi) = \sum_{s \in V(\Gamma)} d^2(v, s)\mu(s),$$

where $d(v, s)$ is the distance between $v$ and $s$ in $\Gamma$. If $M_{\xi}(v)$ is finite, then we say that the weight function $M_{\xi}$ is defined at $v$. The domain of $M$ is the set

$$\text{domain}(M) = \left\{v \in V(\Gamma) \mid \sum_{s \in V(\Gamma)} d^2(v, s)\mu_{\xi}(s) < \infty \right\}.$$
The case of interest of course, is when \( M_\xi(v) \) is \textit{totally defined}, meaning that \( M_\xi(v) \) is finite at every \( v \in V(\Gamma) \).

**Definition 2.1.** Let \( \xi \) be a random \( \Gamma \)-element such that \( M_\xi(\cdot) \) is totally defined. The set of vertices \( v \in \Gamma \) that minimize the value of \( M_\xi \)

\[
\mathbb{E}(\xi) = \{ v \in V(\Gamma) \mid M_\xi(v) \leq M_\xi(u), \quad \forall u \in V(\Gamma) \},
\]

is called the \textit{mean-set} (or the \textit{center-set}, or \textit{average}) of \( \xi \).

Very often we leave the random element \( \xi \) in the background to shorten the notation and write \( M(v) \) instead of \( M_\xi(v) \). Moreover, we write \( \mathbb{E}(\mu) \) instead of \( \mathbb{E}(\xi) \) sometimes and speak of the mean set of distribution \( \mu \) induced by \( \xi \) on \( V(\Gamma) \).

**Lemma 2.2.** Let \( \Gamma \) be a connected graph, \( \xi \) a random \( \Gamma \)-element, and \( u, v \) adjacent vertices in \( \Gamma \). If \( M(u) < \infty \), then \( M(v) < \infty \).

\textit{Proof.} Easily follows from the definition of \( M \) and the triangle inequality. \( \square \)

**Corollary 2.3.** Let \( \Gamma \) be a connected graph and \( \xi \) a random \( \Gamma \)-element. Then either \( \text{domain}(M) = V(\Gamma) \) or \( \text{domain}(M) = \emptyset \).

**Lemma 2.4.** Let \( \Gamma \) be a connected locally finite graph and \( \xi \) a random \( \Gamma \)-element. If \( M_\xi \) is totally defined, then \( 0 < |\mathbb{E}(\xi)| < \infty \).

\textit{Proof.} Let \( \mu \) be a measure of \( \mathbb{E}(\xi) \) induced on \( \Gamma \) by \( \xi \). For an arbitrary but fixed vertex \( v \in \Gamma \), the weight function

\[
M(v) = \sum_{i \in V(\Gamma)} d^2(v, i) \mu(i) = \sum_{n=0}^{\infty} \left( n^2 \sum_{i \in V(\Gamma), d(v, i) = n} \mu(i) \right)
\]

is defined at \( v \) by assumption. Choose \( r \in \mathbb{N} \) such that

\[
\frac{1}{2} M(v) \leq \sum_{n=0}^{r} \left( n^2 \sum_{i \in V(\Gamma), d(v, i) = n} \mu(i) \right) = \sum_{i \in B_v(r)} d^2(v, i) \mu(i),
\]

where

\[
B_v(r) = \{ i \in V(\Gamma) \mid d(v, i) \leq r \}
\]

is the \textit{ball} in \( \Gamma \) of radius \( r \) centered at \( v \). If we take a vertex \( u \) such that \( d(u, v) \geq 3r \), then using the triangle inequality, we obtain the following lower bound:

\[
M(u) = \sum_{i \in V(\Gamma)} d^2(u, i) \mu(i) \geq \sum_{i \in B_v(r)} [2r]^2 \mu(i) + \sum_{i \notin B_v(r)} d^2(u, i) \mu(i) \geq 4 \sum_{i \in B_v(r)} d^2(v, i) \mu(i) \geq 2M(v).
\]

Thus, \( d(v, u) \geq 3r \) implies \( u \notin \mathbb{E}(\xi) \) and, hence, \( \mathbb{E}(\xi) \subseteq B_v(3r) \). Since the graph \( \Gamma \) is locally finite, it follows that the sets \( B_v(3r) \) and \( \mathbb{E}(\xi) \) are finite. This implies that the function \( M \) attains its minimal value in \( B_v(3r) \) and hence \( \mathbb{E}(\xi) \neq \emptyset \). \( \square \)

### 2.2. The mean set in a group

Let \( G \) be a group and \( X \subseteq G \) a finite generating set for \( G \). The choice of \( X \) naturally determines a distance \( d_X \) on \( G \) via its Cayley graph \( C_G(X) \). Hence Definition 2.1 gives us a notion of a mean set for a random \( G \)-element. It follows from the definition of the distance \( d_X \) that for any \( a, b, g \in G \) the equality

\[
d_X(a, b) = d_X(ga, gb)
\]
Proposition 2.5 ("Shift" Property). Let $G$ be a group and $g \in G$. Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a given probability space and $\xi : \Omega \to G$ a $G$-valued random element on $\Omega$. Then for the random element $\xi_g$ defined by $\xi_g(\omega) = g\xi(\omega)$ we have $\mathbb{E}(\xi_g) = g\mathbb{E}(\xi)$.

Proof. Let $\mu_{\xi_g}$ be the measure induced on $G$ by $\xi_g$, in the manner of §B. It follows from the definition of $\xi_g$ that for any $h \in G$

$$
\mu_{\xi_g}(h) = \mathbf{P}(\{\omega \mid \xi_g(\omega) = h\}) = \mathbf{P}(\{\omega \mid g\xi(\omega) = h\}) = \mathbf{P}(\{\omega \mid \xi(\omega) = g^{-1}h\}) = \mu_{\xi}(g^{-1}h).
$$

This, together with (9), implies that for any $h \in G$

$$
M_{\xi_g}(h) = \sum_{i \in G} d^2(h, i)\mu_{\xi_g}(i) = \sum_{i \in G} d^2(g^{-1}h, g^{-1}i)\mu_{\xi}(g^{-1}i) = \sum_{i \in G} d^2(g^{-1}h, i)\mu_{\xi}(i) = M_{\xi}(g^{-1}h).
$$

Hence, the equality $M_{\xi_g}(h) = M_{\xi}(g^{-1}h)$ holds for any random element $\xi$ and $g, h \in G$. Therefore, for any $h_1, h_2 \in G$, $M_{\xi_g}(h_1) < M_{\xi_g}(h_2) \iff M_{\xi}(g^{-1}h_1) < M_{\xi}(g^{-1}h_2)$ and

$$
\mathbb{E}(\xi_g) = \left\{ h \in G \mid M_{\xi_g}(h) \leq M_{\xi_g}(f), \forall f \in G \right\} = \left\{ h \in G \mid M_{\xi}(g^{-1}h) \leq M_{\xi}(g^{-1}f), \forall f \in G \right\} = \left\{ gh \in G \mid M_{\xi}(h) \leq M_{\xi}(f), \forall f \in G \right\} = g\mathbb{E}(\xi).
$$

The equality $d_X(a, b) = d_X(ag, bg)$ does not hold for a general group $G = \langle X \rangle$. It holds for abelian groups.

Proposition 2.6. Let $G$ be an abelian group and $g \in G$. Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and $\xi : \Omega \to G$ a $G$-valued random element on $\Omega$. Then for the random element $\xi_g$ defined by $\xi_g(\omega) = \xi(\omega)g$ we have $\mathbb{E}(\xi_g) = (\mathbb{E}(\xi))g$.

2.3. Other possible definitions of $\mathbb{E}$. There are other possible definitions of $\mathbb{E}$ for which the statement of Proposition 2.3. (and other results of Section 1) holds. Let $c$ be a positive integer. By analogy to the function $M_{\xi}(v)$, define a weight function $M_{\xi}^{(c)}(v)$ of class $c$ by

$$
M_{\xi}^{(c)}(v) = \sum_{i \in V(\Gamma)} d^c(v, i)\mu(i)
$$

and the mean-set $\mathbb{E}^{(c)}(\xi)$ of class $c$ to be

$$
\mathbb{E}^{(c)}(\xi) = \{ v \in V(\Gamma) \mid M^{(c)}(v) \leq M^{(c)}(u), \forall u \in V(\Gamma) \}.
$$

It is straightforward to check that all the statements of the previous section hold for $M_{\xi}^{(c)}(\cdot)$ and $\mathbb{E}^{(c)}(\xi)$. In fact, it is not hard to see that when $c = 1$, we have a counterpart of the median of the distribution $\mu$. Next proposition shows that our $\mathbb{E}$ agrees with the classical definition of the expectation on $\mathbb{Z}$ in the following sense.

Proposition 2.7. Let $\xi : \Omega \to \mathbb{Z}$ be an integer-valued random variable with classical expectation $m = \sum_{n \in \mathbb{Z}} n \mathbf{P}(\xi = n)$. Assume that $M \equiv M_{\xi}^{(2)}$ is defined on $\mathbb{Z}$. Then $1 \leq |E^{(2)}(\xi)| \leq 2$ and for any $v \in E^{(2)}(\xi)$, we have $|m - v| \leq \frac{1}{2}$.

Proof. Straightforward.
Remark 2.8. Observe that \( E^{(2)} \) does not coincide with the classical mean in \( \mathbb{R}^2 \). Recall that the classical mean in \( \mathbb{R}^2 \) is defined coordinate-wise, i.e., the mean of \( (x_1, y_1), \ldots, (x_n, y_n) \) is a point in \( \mathbb{R}^2 \) defined by \( (E X, E Y) \). For example, consider the distribution on \( \mathbb{Z}^2 \) such that \( \mu(0, 0) = \mu(0, 3) = \mu(3, 0) = 1/3 \) and for all other points \( \mu = 0 \). Then the classical mean is the point \((1, 1)\), and the mean-set \( E^{(2)} \) is the point \((0, 0)\).

3. Strong Law of Large Numbers

Let \( \xi_1, \ldots, \xi_n \) be a sample of independent and identically distributed (i.i.d.) graph-valued random elements \( \xi : \Omega \rightarrow V(\Gamma) \) defined on a given probability space \( (\Omega, \mathcal{F}, P) \). For every \( \omega \in \Omega \), let \( \mu_n(u; \omega) \) be the relative frequency

\[
\mu_n(u; \omega) = \frac{|\{i \mid \xi_i(\omega) = u, \ 1 \leq i \leq n\}|}{n}
\]

with which the value \( u \in V(\Gamma) \) occurs in the random sample \( \xi_1(\omega), \ldots, \xi_n(\omega) \). We shall suppress the argument \( \omega \in \Omega \) to ease notation, and let

\[
M_n(v) = \sum_{i \in V(\Gamma)} d^2(v, i) \mu_n(i)
\]

be the sampling weight, corresponding to \( v \in V(\Gamma) \), and \( M_n(\cdot) \) the resulting sampling weight function.

Definition 3.1. The set of vertices

\[
S_n = S(\xi_1, \ldots, \xi_n) = \{v \in V(\Gamma) \mid M_n(v) \leq M_n(u), \ \forall u \in V(\Gamma)\}
\]

is called the sample mean-set (or sample center-set, or average) of the vertices \( \xi_1, \ldots, \xi_n \).

Lemma 3.2. Let \( \Gamma \) be a locally-finite connected graph, \( v \in V(\Gamma) \), and \( \{\xi_i\}_{i=1}^\infty \) a sequence of i.i.d. random \( \Gamma \)-elements such that \( M_{\xi_i}(v) \) is defined. Then

\[
P\left(M_n(v) \rightarrow M(v) \text{ as } n \rightarrow \infty\right) = 1.
\]

Proof. For every \( v \in V(\Gamma) \), \( M(v) \) is the expectation of the random variable \( d^2(v, \xi_1) \). The result follows by the strong law of large numbers applied to \( \{d^2(v, \xi_i)\}_{i=1}^\infty \). \( \square \)

It is important to notice that in general the convergence in Lemma 3.2 is not uniform in a sense that, for some distribution \( \mu \) on a locally finite (infinite) graph \( \Gamma \) and some \( \epsilon > 0 \), it is possible that

\[
P\left(\exists N \text{ s.t. } \forall n > N \ \forall v \in V(\Gamma), \ |M_n(v) - M(v)| < \epsilon\right) < 1.
\]

In other words, the convergence for every vertex, as in Lemma 3.2, is insufficient to prove the strong law of large numbers, stated in introduction. Next lemma is a key tool in the proof of our strong law of large numbers.

Lemma 3.3 (Separation Lemma). Let \( \Gamma \) be a locally-finite connected graph and \( \{\xi_i\}_{i=1}^\infty \) a sequence of i.i.d. random \( \Gamma \)-elements. If the weight function \( M_{\xi_i}(\cdot) \) is totally defined, then

\[
P\left(\exists N \text{ s.t. } \forall n > N, \ \max_{v \in \mathbb{R}(\xi_i)} M_n(v) < \inf_{u \in V(\Gamma) \setminus \mathbb{R}(\xi_i)} M_n(u)\right) = 1.
\]
Proof. Our goal is to prove that for some $\delta > 0$

(12) $P\left(\exists N \forall n > N \forall v \in \mathbb{E}(\xi_1), \forall u \in V(\Gamma) \setminus \mathbb{E}(\xi_1), \ M_n(u) - M_n(v) \geq \delta \right) = 1.$

We prove the formula above in two stages. In the first stage we show that for some fixed $v_0 \in \mathbb{E}(\xi_1)$ and for sufficiently large number $m > 0$ the following holds

(13) $P\left(\exists N \text{ s.t.} \forall n > N \forall v \in \mathbb{E}(\xi_1), \forall u \in V(\Gamma) \setminus B_{v_0}(m), \ M_n(u) - M_n(v) \geq \delta \right) = 1$

in the notation of (8). In the second stage we prove that

(14) $P\left(\exists N \text{ s.t.} \forall n > N \forall v \in \mathbb{E}(\xi_1), \forall u \in B_{v_0}(m) \setminus \mathbb{E}(\xi_1), \ M_n(u) - M_n(v) \geq \delta \right) = 1$

Having the formulae above proved we immediately deduce that (12) holds using $\sigma$-additivity of measure.

Let $v_0 \in \mathbb{E}(\xi_1)$ and $\mu$ be the probability measure on $\Gamma$ induced by $\xi_1$, as in (9). Since the weight function $M(\cdot) = \mu(\cdot)$ is defined at $v_0$, we can choose $r \in \mathbb{R}$ as in Lemma 2.4 such that $\frac{1}{2}M(v_0) = \sum_{i \in B_{v_0}(r)} d^2(v_0, i)\mu(i)$. Put $m = 3r$. In Lemma 2.4 we proved that, if a vertex $u$ is such that $d(u, v_0) \geq 3r$, then

(15) $M(u) = \sum_{i \in V(\Gamma)} d^2(u, i)\mu(i) \geq 4 \sum_{i \in B_{v_0}(r)} d^2(u, i)\mu(i) \geq 2M(v_0)$

It implies that $\mathbb{E}(\xi_1) \subseteq B_{v_0}(3r)$.

Since $\Gamma$ is locally finite, the set $B_{v_0}(r)$ of (8) is finite. We also know from the SLLN for the relative frequencies $\mu_n(u)$ that $\mu_n(u) \stackrel{a.s.}{\to} \mu(u)$ as $n \to \infty$. These facts imply that for any $\varepsilon > 0$, the event

(16) $C_\varepsilon = \left\{ \exists N = N(\varepsilon), \forall n > N, \forall u \in B_{v_0}(r), \ |\mu_n(u) - \mu(u)| < \varepsilon \right\}$

has probability one. In particular, this is true for $\varepsilon = \varepsilon^* = \frac{1}{2} \min\{\mu(u) \mid u \in B_{v_0}(r), \mu(u) \neq 0\}$, and the event $C_\varepsilon$ is a subset of

(17) $\left\{ \exists N = N(\varepsilon^*), \forall n > N, \forall u \in V(\Gamma) \setminus B_{v_0}(3r), \ M_n(u) \geq \frac{3}{2}M(v_0) \right\}$

Indeed, on the event $C_\varepsilon$, as in (16), we have $\mu_n(i) \geq \frac{1}{2}\mu(i), i \in B_{v_0}(r)$. Using this fact together with (15), we can write

$M_n(u) = \sum_{i \in V(\Gamma)} d^2(u, i)\mu_n(i) \geq 4 \sum_{i \in B_{v_0}(r)} d^2(u, i)\mu_n(i) \geq 3 \sum_{i \in B_{v_0}(r)} d^2(u, i)\mu(i) \geq \frac{3}{2}M(v_0)$.

Thus we have

(18) $P\left(\exists N \text{ s.t.} \forall n > N, \forall u \in V(\Gamma) \setminus B_{v_0}(3r), \ M_n(u) \geq \frac{3}{2}M(v_0) \right) = 1.$

By Lemma 3.2 for any $v \in V(\Gamma)$ and any $\varepsilon > 0$, we have

$P\left(\exists N = N(\varepsilon), \forall n > N, \ |M_n(v) - M(v)| < \varepsilon \right) = 1$

and, since $B_{v_0}(3r)$ is a finite set, we have simultaneous convergence for all vertices in $B_{v_0}(3r)$, i.e.,

(19) $P\left(\exists N = N(\varepsilon), \forall n > N, \forall v \in B_{v_0}(3r), \ |M_n(v) - M(v)| < \varepsilon \right) = 1.$
In particular, remembering that $\mathbb{E}(\xi_1) \subseteq B_{v_0}(3r)$, for $\varepsilon = M(v_0)/4$,

$$ \frac{3}{4} M(v) < M_n(v) < \frac{5}{4} M(v) $$

(20) $\mathbb{P}\left( \exists N = N(\varepsilon), \ \forall n > N, \ \forall v \in \mathbb{E}(\xi_1), \ \frac{3}{4} M(v) < M_n(v) < \frac{5}{4} M(v) \right) = 1.$$

Finally, we notice that on the intersection of the events in (18) and (20), we have

$$ M_n(u) - M_n(v) \geq \frac{3}{2} M(v) - \frac{5}{4} M(v) = \frac{1}{4} M(v) = \frac{1}{4} M(v_0), $$

by the virtue of the fact that $M(v_0) = M(v)$ (as both $v_0, v \in \mathbb{E}(\xi_1)$), and formula (13) holds for any $\delta$ such that $\delta \leq \frac{1}{7} M(v_0)$.

For the second part of our proof we use statement (19) that holds, in particular, for

$$ \varepsilon = \varepsilon' = \frac{1}{4} \min\{M(u) - M(v_0) \mid u \in B_{v_0}(3r), \ M(u) - M(v_0) > 0\}. $$

It means that, with probability 1, there exists $N = N(\varepsilon')$ such that for any $n > N$ and all $u \in B_{v_0}(3r)$, we have $|M_n(u) - M(u)| < \varepsilon'$. Moreover, since $\mathbb{E}(\xi_1) \subseteq B_{v_0}(3r)$, we can assert the same for any $v \in \mathbb{E}(\xi_1)$; namely, $|M_n(v) - M(v)| < \varepsilon'$. Together with the fact that $M(u) - M(v_0) > 0$, the obtained inequalities imply that, with probability 1, there exists number $N = N(\varepsilon')$ such that for any $n > N$ and all $u \in B_{v_0}(3r) \setminus \mathbb{E}(\xi_1)$,

$$ M_n(v_0) < M(v_0) + \varepsilon' \leq M(v_0) + \frac{1}{4} (M(u) - M(v_0)) $$

$$ M(u) - \frac{1}{4} (M(u) - M(v_0)) \leq M(u) - \varepsilon' < M_n(u), $$

and, hence,

$$ M_n(u) - M_n(v_0) \geq M(u) - \frac{1}{4} (M(u) - M(v_0)) - M(v_0) - \frac{1}{4} (M(u) - M(v_0)) = $$

$$ = \frac{1}{2} (M(u) - M(v_0)) \geq 2\varepsilon', \text{ i.e.,} $$

$$ \mathbb{P}\left( \exists N = N(\varepsilon), \ \forall n > N, \ \forall u \in B_{v_0}(3r) \setminus \mathbb{E}(\xi_1) : \ M_n(u) - M_n(v_0) \geq 2\varepsilon' \right) = 1. $$

Therefore, (14) holds for any $\delta \leq 2\varepsilon'$. Choosing $\delta = \min(\frac{1}{4} M(v_0), 2\varepsilon')$ finishes the proof. 

□

**Corollary 3.4** (Inclusion Lemma). Let $\Gamma$ be a locally-finite connected graph, $(\xi_i)_{i=1}^\infty$ a sequence of i.i.d. random $\Gamma$-elements, and $\mu = \mu_{\xi_1}$. Suppose that the weight function $M_{\xi}(\cdot)$ is totally defined. Then

$$ \mathbb{P}\left( \limsup_{n \to \infty} S(\xi_1, \ldots, \xi_n) \subseteq \mathbb{E}(\xi_1) \right) = 1. $$

**Proof.** Lemma 3.3 implies that $\mathbb{P}\left( u \notin \limsup S_n, \text{ for every } u \in V(\Gamma) \setminus \mathbb{E}(\xi_1) \right) = 1$. □

**Theorem 3.5.** (SLLN for graph-valued random elements with a singleton mean-set.) Let $\Gamma$ be a locally-finite connected graph and $(\xi_i)_{i=1}^\infty$ a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi}(\cdot)$ is totally defined and $\mathbb{E}(\xi_1) = \{v\}$ for some $v \in V(\Gamma)$, then

$$ \lim_{n \to \infty} S(\xi_1, \ldots, \xi_n) = \mathbb{E}(\xi_1) $$
almost surely (with probability one).

Proof. \( P\left( \exists N \text{ s.t. } \forall n > N, \ M_n(v) < \inf_{u \in V(\Gamma) \setminus \{v\}} M_n(u) \right) = 1, \) by Lemma 3.3 and, hence, \( P\left( \exists N \text{ s.t. } \forall n > N, \ \{\xi_1, \ldots, \xi_n\} = \{v\} \right) = 1. \)

3.1. Case of multi-vertex mean-sets. In this section we investigate a multi-vertex mean-set case and conditions under which the strong law of large numbers holds for such set. We reduce this problem to the question of recurrence of a certain subset in \( \mathbb{Z}^n \) relative to a random walk on this integer lattice. If \( 2 \leq |E(\xi)| \leq 3, \) no restrictive assumptions are required; we formulate and prove the law for these special instances separately. The case \( |E(\xi)| > 3 \) requires more technical assumptions, and, thus, more work to handle it.

3.1.1. Preliminaries. Assume \( E(\xi) = \{v_1, v_2, \ldots, v_k\} \). Our goal is to find conditions that would guarantee the inclusion \( E(\xi) \subseteq \limsup_{n \to \infty} S_n \) or, without loss of generality, conditions for \( v_1 \in \limsup_{n \to \infty} S_n \).

By Lemma 3.3 it follows that, with probability one, for a sequence of random \( \Gamma \)-elements \( \{\xi_n\}_{n=1}^{\infty} \), there exists a number \( N \) such that for any \( n > N \) we have

\[
\max\{M_n(v_1), M_n(v_2), \ldots, M_n(v_k)\} < \inf_{u \in \Gamma \setminus \{v_1, v_2, \ldots, v_k\}} M_n(u).
\]

Hence, for any \( n > N, v_1 \in S_n \) if and only if \( M_n(v_1) \leq M_n(v_i) \) for every \( i = 2, \ldots, k \). Thus, to achieve our goal, we need to show that the system of inequalities

\[
\begin{cases}
M_n(v_2) - M_n(v_1) &\geq 0, \\
\vdots \\
M_n(v_k) - M_n(v_1) &\geq 0,
\end{cases}
\]

is satisfied for infinitely many \( n \in \mathbb{N} \).

For \( i = 1, \ldots, k - 1 \) and \( n \in \mathbb{N} \), define

\[
R_i(n) = n(M_n(v_{i+1}) - M_n(v_i)) = \sum_{s \in \Gamma} (d^2(v_{i+1}, s) - d^2(v_1, s)) \cdot |\{i \mid \xi_i = s, 1 \leq i \leq n\}|
\]

and observe that

\[
R_i(n+1) - R_i(n) = \sum_{s \in \Gamma} [d^2(v_{i+1}, s) - d^2(v_1, s)] \mathbf{1}_{\{\xi_{n+1} = s\}},
\]

i.e., every \( R_i(n) \) represents a random walk on \( \mathbb{Z} \) starting at \( 0 \). Consider a random walk \( \overline{R} \), associated with \( v_1 \), in \( \mathbb{Z}^{k-1} \), starting at the origin \((0, \ldots, 0)\) with the position of the walk after \( n \) steps given by

\[
\overline{R}(n) = (R_1(n), R_2(n), \ldots, R_{k-1}(n)).
\]

An increment step for \( \overline{R} \) is defined by a vector \( \overline{\zeta}(s) = (\zeta_1(s), \ldots, \zeta_{k-1}(s)) \), \( s \in V(\Gamma) \), with probability \( \mu(s) \), where

\[
\zeta_i(s) = d^2(v_{i+1}, s) - d^2(v_1, s).
\]

The following lemma shows the significance of this random walk.

Lemma 3.6. In the notation of this section, \( v_1 \in \limsup_{n \to \infty} S_n \) if and only if the random walk \( \overline{R} \) visits the set \( \mathbb{Z}^{k-1}_+ = \{(a_1, \ldots, a_{k-1}) \mid a_i \geq 0\} \) infinitely often. Therefore,

\[
P(v_1 \in \limsup_{n \to \infty} S_n) = P(\overline{R}(n) \in \mathbb{Z}^{k-1}_+, \text{ i.o.}).
\]
Proof. Follows from the discussion preceding the lemma. \hfill \Box

It is worth redefining $\overline{R}$ in the terms of transition probability function, as in [31]. Let $\overline{0} \in \mathbb{Z}^{k-1}$ be the zero vector and $x_i = \zeta_i(s)$, $s \in V(\Gamma)$. For every $x = (x_1, \ldots, x_{k-1}) \in \mathbb{Z}^{k-1}$, we define a function $P(\overline{0}, x)$ by
\begin{equation}
P(\overline{0}, x) = \mu\{s \mid x_i = d^2(v_{i+1}, s) - d^2(v_1, s) \text{ for every } i = 1, \ldots, k-1\}.
\end{equation}
It is trivial to check that this is, indeed, the transition probability for $\overline{R}$. To continue further, we investigate some properties of our random walk $\overline{R}$.

Lemma 3.7. Let $\overline{R}$ be a random walk defined above. Then
\[ m_1 = \sum_{x \in \mathbb{Z}^{k-1}} \pi P(\overline{0}, x) = \overline{0} \quad \text{and} \quad m_2 = \sum_{x \in \mathbb{Z}^{k-1}} |x|^2 P(\overline{0}, x) < \infty. \]

Proof. The first equality trivially holds. For the second one, we get
\begin{align*}
\sum_{x \in \mathbb{Z}^{k-1}} |x|^2 P(\overline{0}, x) &= \sum_{s \in V(\Gamma)} \sum_{i=1}^{k-1} \left(d^2(v_{i+1}, s) - d^2(v_1, s)\right)^2 \mu(s) \\
&\leq \sum_{i=1}^{k-1} d^2(v_1, v_{i+1}) \sum_{s \in V(\Gamma)} \left(d(v_1, s) + d(v_{i+1}, s)\right)^2 \mu(s) \\
&\leq \sum_{i=1}^{k-1} d^2(v_1, v_{i+1})(4M(v_1) + 4M(v_{i+1})) < \infty.
\end{align*}

Clearly, conditions under which this random walk is recurrent would guarantee that $v_1 \subseteq \limsup_{n \to \infty} S_n$ (see [31] page 30, Proposition 3.3). A general (not simple, not symmetric) one-dimensional random walk is recurrent if its first moment is zero and its first absolute moment is finite (see [31], pg. 23). Sufficient conditions for the recurrence of two-dimensional random walk involve the finiteness of its second moment and can be found in [31] page 83. The result stated there indicates that genuinely 2-dimensional random walk is recurrent if its first moment is zero, and its second moment is finite. Let us recall some important notions before we go on.

Consider an arbitrary random walk $\overline{R}$ on $\mathbb{Z}^n$ given by a transition probability $P$, as in (22). The support, $\text{supp}(P)$, of the probability measure $P$ is defined to be the set $\text{supp}(P) = \{x \in \mathbb{Z}^n \mid P(x) \neq 0\}$ of all possible one-step increments of $\overline{R}$. Further, with $\overline{R}$, one can associate an abelian subgroup $A_{\overline{R}}$ of $\mathbb{Z}^n$ generated by the vectors in $\text{supp}(P)$. It is well-known in group theory that any subgroup $A_{\overline{R}}$ of $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^k$, where $k \leq n$ (the reader can also check [31] Proposition 7.1 on pg 65) for details), in which case we write $\text{dim}(A_{\overline{R}}) = k$ and say that $\overline{R}$ is genuinely $k$-dimensional. Let us stress that we speak of an $n$-dimensional random walk on $\mathbb{Z}^n$ when $P(0, x)$ is defined for all $x \in \mathbb{Z}^n$; this walk is genuinely $n$-dimensional if $\text{dim}(A_{\overline{R}}) = n$. We say that $\overline{R}$ is aperiodic if $A_{\overline{R}} = \mathbb{Z}^n$. Observe that genuinely $n$-dimensional random walk does not have to be aperiodic. A standard simple random walk, which we denote by $S = S(n)$, is an example of an aperiodic random walk on $\mathbb{Z}^n$. It will be convenient to define a vector space $V_{\overline{R}} \subset \mathbb{R}^n$ spanned by the vectors in $\text{supp}(P)$. It is easy to see that the genuine dimension of $\overline{R}$ is equal to the dimension of $V_{\overline{R}}$. We shall need another notion for our developments. Assume that $D$ is an $k \times n$ matrix (not necessarily integer valued) which maps $A_{\overline{R}}$ onto $\mathbb{Z}^k$. 
Then $D$ naturally induces a random walk $\mathbf{R}^D$ on $\mathbb{Z}^k$ with transition probability $P^D$ given by $P^D(\pi) = P(\pi \in \mathbb{Z}^n | D(\pi) = \pi)$ for every $\pi \in \mathbb{Z}^k$.

3.1.2. Strong law of large numbers for two or three vertices mean-sets. Now, we can easily prove our strong law of large numbers for mean-sets with two or three elements.

**Theorem 3.8** (SLLN for graph random elements with two or three point mean-set). Let $\Gamma$ be a locally-finite connected graph and $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi_1}(\cdot)$ is totally defined and $2 \leq |E(\xi)| \leq 3$, then

$$\limsup_{n \to \infty} S_n = \mathbb{E}(\xi_1)$$

holds with probability one.

**Proof.** Assume that $\mathbb{E}(\xi) = \{v_1, v_2\}$. Then the random walk $\mathbf{R}$ is one-dimensional. It is recurrent if

$$\sum_{s \in \Gamma} |\xi_1(s)|\mu(s) < \infty \quad \text{and} \quad \sum_{s \in \Gamma} \xi_1(s)\mu(s) = 0$$

(see [31], pg. 23). The equality $M(v_1) = M(v_2)$ implies the second conditions and

$$\sum_{s \in \Gamma} |\xi_1(s)|\mu(s) = \sum_{s \in \Gamma} |d^2(v_2, s) - d^2(v_1, s)|\mu(s)$$

$$\leq \sum_{s \in \Gamma} (d^2(v_2, s) + d^2(v_1, s))\mu(s) = M(v_1) + M(v_2) < \infty$$

implies the first condition. Hence, $\mathbf{R}$ is recurrent, and takes on positive and negative values infinitely often. We conclude that almost always $\limsup_{n \to \infty} S_n = \{v_1, v_2\} = \mathbb{E}\xi$.

Assume that $\mathbb{E}(\xi) = \{v_1, v_2, v_3\}$. Then the random walk $\mathbf{R}$ can be genuinely 0, 1, or 2-dimensional. The first case is trivial, the second can be considered as the case when $|\mathbb{E}(\xi)| = 2$. So, assume $\mathbf{R}$ is genuinely 2-dimensional. By Lemma 3.7, the first moment of $\mathbf{R}$ is $(0, 0)$ and the second moment is finite. Now, it follows from [31, Theorem 8.1] that $\mathbf{R}$ is recurrent. In particular, $\mathbb{Z}^{k-1}_+$ is visited infinitely often with probability 1.

In both cases, it follows from Lemma 3.6 that $P(v_1 \in \limsup_{n \to \infty} S_n) = 1$. Hence the result. \qed

Recall that a subset of $\mathbb{Z}^n$ is called **recurrent** if it is visited by a given random walk infinitely often with probability one, and it is transient otherwise. A criterion for recurrence of a set for a simple random walk was obtained in [19] for $n = 3$ (it can also be found in [31, Theorem 26.1]). It turns out that the criterion does not depend on a random walk in question. This is the subject of the extension of the Wiener’s test, proved in [34], that we state below. This invariance principle is one of the main tools we use in our investigation of the recurrence properties of the positive octant in $\mathbb{Z}^n$ for $\mathbf{R}$.

**Theorem.** (Extension of Wiener’s test, [34]) Let $n \geq 3$. An infinite subset $A$ of $\mathbb{Z}^n$ is either recurrent for each aperiodic random walk $\mathbf{R}$ on $\mathbb{Z}^n$ with mean zero and a finite variance, or transient for each of such random walks.
For a positive constant \( \alpha \in \mathbb{R} \) and a positive integer \( m \leq n \) define a subset of \( \mathbb{R}^n \)
\[
\text{Cone}_\alpha^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0, \ldots, x_{n-m} = 0, \sqrt{x_{n-m+1}^2 + \ldots + x_{n-1}^2} \leq \alpha x_n \right\}
\]
called an \( m \)-dimensional cone in \( \mathbb{R}^n \). If \( m = n \), then we omit the superscript in \( \text{Cone}_\alpha^n \).
For an \( n \times n \) matrix \( D \) and a set \( A \subseteq \mathbb{R}^n \), define a set \( A^D = \{ D \cdot \overline{v} \mid \overline{v} \in A \} \), which is a linear transformation of \( A \). If \( D \) is an orthogonal matrix, then the set \( (\text{Cone}_\alpha)^D \) is called a rotated cone. As in [19], for any non-decreasing function \( i : \mathbb{N} \to \mathbb{R}_+ \) define a set
\[
\text{Thorn}_i = \{ \overline{v} \in \mathbb{Z}^n \mid \sqrt{v_1^2 + \ldots + v_{n-1}^2} \leq i(v_n) \}.
\]
Observe that \( \text{Cone}_\alpha \cap \mathbb{Z}^n = \text{Thorn}_i \) where \( i(t) = \alpha t \).

**Theorem 3.9.** For any \( \alpha > 0 \) and any orthogonal matrix \( D \),
\[
\mathbb{P}\left( S(n) \in (\text{Cone}_\alpha)^D, \ i.o. \right) = 1,
\]
i.e., the probability that the simple random walk on \( \mathbb{Z}^n \) visits \( (\text{Cone}_\alpha)^D \) infinitely often is 1.

**Proof.** Direct consequence of (6.1) and (4.3) in [19], where the criterion for recurrence of \( \text{Thorn}_i \) is given. \( \square \)

Next two lemmas are obvious

**Lemma 3.10.** Assume that a set \( A \subseteq \mathbb{R}^n \) contains a rotated cone. Then for any invertible \( n \times n \) matrix \( D \), the set \( A^D \) contains a rotated cone.

**Lemma 3.11.** If \( S_1 \subseteq S_2 \subseteq \mathbb{Z}^n \) and \( S_1 \) is visited by the simple random walk infinitely often with probability 1 then \( S_2 \) is visited by the simple random walk infinitely often with probability 1.

Now, we return to our strong law of large numbers for multi-vertex mean-sets. Assume that \( \mathbb{E} \xi = \{v_1, \ldots, v_k\} \), where \( k \geq 4 \). Let \( \overline{R} \) be a random walk on \( \mathbb{Z}^{k-1} \), associated with \( v_i \), where \( i = 1, \ldots, k \) (in our notation, \( \overline{R} = \overline{R}^1 \)). This is a \((k-1)\)-dimensional random walk which, in general, is not aperiodic. In fact, \( \overline{R} \) is not even genuinely \((k-1)\)-dimensional. Fortunately, it turns out that it does not matter to what vertex \( v_i \) we associate our random walk, since the choice of the vertex does not affect the dimension of the corresponding walk, as the following lemma shows.

**Lemma 3.12.** Let \( \mu \) be a probability measure on a locally finite graph \( \Gamma \) such that \( \mathbb{E} \mu = \{v_1, \ldots, v_k\} \), where \( k \geq 2 \). Then the random walks \( \overline{R}^1, \ldots, \overline{R}^k \), associated with vertices \( v_1, \ldots, v_k \) respectively, all have the same genuine dimension.

**Proof.** We prove that random walks \( \overline{R}^1 \) and \( \overline{R}^2 \) have the same genuine dimension. Recall that the subgroup \( A_{\overline{R}^1} \) is generated by the set of vectors \( \overline{v}^1 \in \mathbb{Z}^{k-1} \) such that for some \( s \in \text{supp}(\mu) \), \( \overline{v}^1 = \overline{v}^1(s) = (d^2(v_2, s) - d^2(v_1, s), d^2(v_3, s) - d^2(v_1, s), \ldots, d^2(v_k, s) - d^2(v_1, s)) \) and the subgroup \( A_{\overline{R}^2} \) is generated by the set of vectors \( \overline{v}^2 \in \mathbb{Z}^{k-1} \) such that for some \( s \in \text{supp}(\mu) \), \( \overline{v}^2 = \overline{v}^2(s) = (d^2(v_1, s) - d^2(v_2, s), d^2(v_3, s) - d^2(v_2, s), \ldots, d^2(v_k, s) - d^2(v_1, s)) \). Observe that for every \( s \in \text{supp}(\mu) \)},
supp(μ) the equality \( \overline{\pi}^D(s) = D \cdot \overline{\pi}^1(s) \) holds, where \( D \) is a \((k - 1) \times (k - 1)\) matrix

\[
D = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
\cdots
\end{pmatrix}
\]

Therefore, \( A_{\overline{\mathbb{R}}^i} = (A_{\overline{\mathbb{R}}^i})^D \). Since the matrix \( D \) is invertible it follows that \( A_{\overline{\mathbb{R}}^1} \) and \( A_{\overline{\mathbb{R}}^i} \) have the same dimension. \( \square \)

**Theorem 3.13.** Let \( \Gamma \) be a locally-finite connected graph and \( \{\xi_k\}_{k=1}^\infty \) a sequence of i.i.d. random \( \Gamma \)-elements. Assume that the weight function \( M \) is totally defined and \( \mathbb{E}(\xi) = \{v_1, \ldots, v_k\} \), where \( k \geq 4 \). If the random walk \( \overline{\mathbb{R}}^1 \) associated to \( v_1 \) is genuinely \((k - 1)\)-dimensional, then

\[
\limsup_{n \to \infty} S_n = \mathbb{E}(\xi_1)
\]

holds with probability 1.

**Proof.** Since \( \overline{\mathbb{R}}^1 \) is genuinely \((k - 1)\)-dimensional it follows that the subgroup \( A_{\overline{\mathbb{R}}^1} \) is isomorphic to \( \mathbb{Z}^{k-1} \) and there exists an invertible matrix \( D \) that isomorphically maps \( A_{\overline{\mathbb{R}}^1} \subseteq \mathbb{Z}^{k-1} \) onto \( \mathbb{Z}^{k-1} \). Consider a set \( \mathbb{R}^{k-1}_+ = \{(x_1, \ldots, x_{k-1}) \mid x_i \geq 0\} \). Obviously, \( \mathbb{P}(\overline{\mathbb{R}}^1 \in \mathbb{Z}^{k-1}_+ \text{ i.o.}) = \mathbb{P}(\overline{\mathbb{R}}^1 \in \mathbb{R}^{k-1}_+ \text{ i.o.}) \).

Let \( (\overline{\mathbb{R}}^1)^D \) be the random walk on \( \mathbb{Z}^{k-1} \) induced by \( D \) by application of \( D \) to \( \overline{\mathbb{R}}^1 \). The random walk \( (\overline{\mathbb{R}}^1)^D \) is aperiodic since \( D \) maps \( A_{\overline{\mathbb{R}}^1} \) onto \( \mathbb{Z}^{k-1} \) and, by construction of \( (\overline{\mathbb{R}}^1)^D \),

\[
\mathbb{P}(\overline{\mathbb{R}}^1 \in \mathbb{R}^{k-1}_+ \text{ i.o.}) = \mathbb{P}(\overline{\mathbb{R}}^1 \in (\mathbb{R}^{k-1}_+)^D \text{ i.o.})
\]

Let \( S \) be the simple random walk on \( \mathbb{Z}^{k-1} \). Since \( (\overline{\mathbb{R}}^1)^D \) and \( S \) are both aperiodic random walks on \( \mathbb{Z}^{k-1} \), it follows from the Invariance Principle (Extension of Wiener’s test) that

\[
\mathbb{P}(\overline{\mathbb{R}}^1^{(D)} \in (\mathbb{R}^{k-1}_+)^D \text{ i.o.}) = \mathbb{P}(S \in (\mathbb{R}^{k-1}_+)^D \text{ i.o.})
\]

Clearly, the set \( \mathbb{R}^{k-1}_+ \) contains a rotated cone and, hence, by Lemma 3.10 its image under an invertible linear transformation \( D \) contains a rotated cone too. Now, by Theorem 3.9 \( \mathbb{P}(S \in (\mathbb{R}^{k-1}_+)^D \text{ i.o.}) = 1 \). Thus, \( \mathbb{P}(\overline{\mathbb{R}}^1 \in \mathbb{Z}^{k-1}_+ \text{ i.o.}) = 1 \) and by Lemma 3.6

\[
\mathbb{P}(v_1 \in \limsup_{n \to \infty} S_n) = 1
\]

Finally, it follows from Lemma 3.12 that for any \( i = 2, \ldots, k \) the random walk \( \overline{\mathbb{R}}^i \) is genuinely \((k - 1)\)-dimensional. For any \( i = 2, \ldots, k \) we can use the same argument as for \( v_1 \) to prove that \( \mathbb{P}(v_i \in \limsup_{n \to \infty} S_n) = 1 \). Hence the result. \( \square \)
3.1.3. The case when random walk is not genuinely \((k - 1)\)-dimensional. The case when \(\mathbb{R}^k\) is not genuinely \((k - 1)\)-dimensional is more complicated. To answer the question whether \(v_1\) belongs to \(\lim \sup_{n \to \infty} S_n\) (namely, how often \(v_1 \in S_n\)), we need to analyze how the space \(V_{\mathbb{R}^k}\) “sits” in \(\mathbb{R}^{k-1}\). We know that the subgroup \(A_{\mathbb{R}^k} \subset \mathbb{Z}^{k-1}\) is isomorphic to \(\mathbb{Z}^m\), where \(m < k - 1\) in the case under consideration. Therefore, there exists a \(m \times (k - 1)\) matrix \(D\) which maps the subgroup \(A_{\mathbb{R}^k}\) onto \(\mathbb{Z}^m\) which is injective onto \(A_{\mathbb{R}^k}\). Furthermore, the mapping \(D\) maps the subspace \(V_{\mathbb{R}^k}\) bijectively onto \(\mathbb{R}^m\). The linear mapping \(D\) induces an aperiodic random walk \((D^{\mathbb{R}^k})^D\) on \(\mathbb{Z}^m\) in a natural way and \(P\left(D^{\mathbb{R}^k} \in (\mathbb{R}^{k-1}_+)^{i.o.}\right) = P\left(D^{\mathbb{R}^k} \in (\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k})^{i.o.}\right) = P\left(D^{\mathbb{R}^k} \cap (\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k})^D\right).\) The main problem here is to understand the structure of the set \((\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k})^D\) and, to be more precise, the structure of the set \(B_{\mathbb{R}^k} = \mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k}\). Clearly \(B_{\mathbb{R}^k}\) is a monoid, i.e., contains the trivial element and a sum of any two elements in \(B_{\mathbb{R}^k}\) belongs to \(B_{\mathbb{R}^k}\). We can define dimension of \(B_{\mathbb{R}^k}\) to be the maximal number of linearly independent vectors in \(B_{\mathbb{R}^k}\).

**Theorem 3.14.** Suppose \(A_{\mathbb{R}^k} \cong \mathbb{Z}^m\) and the set \(B_{\mathbb{R}^k}\) has dimension \(m\). Then \(P(v_1 \in \lim \sup_{n \to \infty} S_n) = 1\).

**Proof.** Since \(B_{\mathbb{R}^k}\) is a monoid of dimension \(m\) it is not hard to see that \(B_{\mathbb{R}^k}\) contains an \(m\)-dimensional rotated cone. Since \(D\) is a linear isomorphism from \(V_{\mathbb{R}^k}\) onto \(\mathbb{R}^m\) it follows by Lemma 3.10 that \((B_{\mathbb{R}^k})^D\) contains an \(m\)-dimensional rotated cone in \(\mathbb{R}^m\). If \(S\) is a simple random walk in \(\mathbb{Z}^m\) then \(P(S \subseteq (B_{\mathbb{R}^k})^D)^{i.o.} = 1\) and since \(S\) and \((\mathbb{R}^k)^D\) are both aperiodic, by the extension of Wiener’s test (Invariance Principle), we see that \(P(D^{\mathbb{R}^k} \subseteq (B_{\mathbb{R}^k})^D)^{i.o.}\) = 1. Hence, \(P(D^{\mathbb{R}^k} \subseteq (\mathbb{R}^{k-1}_+)^{i.o.}) = 1\) and by Lemma 3.10, \(P(v_1 \in \lim \sup_{n \to \infty} S_n) = 1\).

Below we investigate under what conditions the subgroup \(A_{\mathbb{R}^k}\) and the set \(B_{\mathbb{R}^k}\) have the same dimension \(m\).

**Lemma 3.15.** Assume that \(A_{\mathbb{R}^k}\) contains a positive vector. Then \(A_{\mathbb{R}^k}\) and the set \(\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k}\) have the same dimension.

**Proof.** Straightforward. \(\square\)

**Lemma 3.16.** Assume that \(\mu(v_1) \neq 0\). Then \(A_{\mathbb{R}^k}\) and the set \(\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k}\) have the same dimension.

**Proof.** Observe that if \(\mu(v_1) \neq 0\) then \(A_{\mathbb{R}^k}\) contains the vector \((d^2(v_2, v_1), \ldots, d^2(v_k, v_1))\) which has all positive coordinates. Therefore, by Lemma 3.12 the set \(A_{\mathbb{R}^k}\) and \(\mathbb{R}^{k-1}_+ \cap V_{\mathbb{R}^k}\) have the same dimension. \(\square\)

**Corollary 3.17.** Let \(\Gamma\) be a locally-finite connected graph and \(\{\xi_i\}_{i=1}^\infty\) be a sequence of i.i.d. random \(\Gamma\)-elements. Assume that the weight function \(M_{\xi_i}(\cdot)\) is totally defined and \(E(\xi) = \{v_1, \ldots, v_k\}\), where \(k \geq 4\). If \(E(\xi_1) \subseteq \text{supp}(\mu)\) then \(\lim \sup_{n \to \infty} S_n = E(\xi_1)\) holds with probability one.

**Proof.** Follows from Lemma 3.10, 3.15, and 3.14. \(\square\)
4. Concentration of measure inequalities

Concentration inequalities are upper bounds on the rate of convergence (in probability) of sample (empirical) means to their ensemble counterparts (actual means). Chebyshev inequality and Chernoff-Hoeffding exponential bounds are classical examples of such inequalities in probability theory. In this section, we prove analogues of the classical Chebyshev’s inequality and Chernoff-Hoeffding like bounds - the concentration of measure inequalities for a graph- (group-)valued random elements.

4.1. Chebyshev’s inequality for graphs/groups. The classical Chebyshev’s inequality asserts that if \( \xi \) is a random variable with \( \mathbb{E}(\xi^2) < \infty \), then for any \( \varepsilon > 0 \), we have

\[
\mathbb{P}(|\xi - \mathbb{E}(\xi)| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2},
\]

where \( \sigma^2 = \text{Var}(\xi) \), see [5].

Chebyshev discovered it when he was trying to prove the law of large numbers, and the inequality is widely used ever since. Chebyshev’s inequality is a result concerning the concentration of measure, giving a quantitative description of this concentration. Indeed, it provides a bound on the probability that a value of a random variable \( \xi \) with finite mean and variance will differ from the mean by more than a fixed number \( \varepsilon \). In other words, we have a crude estimate for concentration of probabilities around the expectation, and this estimate has a big theoretical significance.

The inequality (23) applied to the sample mean random variable \( \overline{X} = \frac{S_n}{n} \), where \( S_n = \xi_1 + \ldots + \xi_n \), \( \mathbb{E}(\xi_i) = m \), \( \text{Var}(\xi_i) = \sigma^2 \), \( i = 1, \ldots, n \) results in

\[
\mathbb{P}(|\overline{X} - m| \geq \varepsilon) \leq \frac{\sigma^2}{n \varepsilon^2}
\]

The goal is to prove a similar inequality for a graph-valued random element \( \xi \).

**Lemma 4.1.** Let \( \mu \) be a distribution on a locally finite graph \( \Gamma \) such that \( M \equiv M^{(2)} \) is defined. If for some \( r \in \mathbb{N} \) and \( v_0 \in V(\Gamma) \) the inequality

\[
\sum_{s \in V(\Gamma) \setminus B_{v_0}(r/2)} d(v_0, s)\mu(s) - \frac{r}{2} \mu(v_0) < 0
\]

holds, then \( M(u) > M(v_0) \) for any \( u \in V(\Gamma) \setminus B_{v_0}(r) \).

**Proof.** Indeed, pick any \( u \in V(\Gamma) \setminus B_{v_0}(r) \) and put \( d = d(v_0, u) \). Then

\[
M(u) - M(v_0) = \sum_{s \in V(\Gamma)} (d^2(u, s) - d^2(v_0, s))\mu(s)
\]

\[
\geq d^2 \mu(v_0) - \sum_{d(v_0, s) > d(u, s)} (d^2(v_0, s) - d^2(u, s))\mu(s)
\]

\[
\geq d^2 \mu(v_0) - 2d \sum_{d(v_0, s) > d(u, s)} d(v_0, s)\mu(s) \geq d^2 \mu(v_0) - 2d \sum_{s \in V(\Gamma) \setminus B_{v_0}(r/2)} d(v_0, s)\mu(s).
\]

Since \( d > r \) it follows that the last sum is positive. Thus \( M(u) > M(v_0) \) as required. \( \square \)
**Theorem 4.2.** Let \( \Gamma \) be a locally-finite connected graph and \( \{\xi_i\}_{i=1}^{\infty} \) a sequence of i.i.d. random \( \Gamma \)-elements. If the weight function \( M_{\xi_i} \) is totally defined and \( \mathbb{E}(\xi_1) = \{v\} \) for some \( v \in V(\Gamma) \) then there exists a constant \( C = C(\Gamma, \xi_1) > 0 \) such that

\[
P\left( S(\xi_1, \ldots, \xi_n) \neq \{v\} \right) \leq \frac{C}{n}.
\]

**Proof.** It follows from the definition of the sample mean-set that

\[
\{S_n \neq \{v\}\} = \{ \exists u \in V(\Gamma) \setminus \{v\}, \ M_n(u) \leq M_n(v) \}.
\]

Hence, it is sufficient to prove that \( P\left( \exists u \in V(\Gamma) \setminus \{v\}, \ M_n(u) \leq M_n(v) \right) \leq \frac{C}{n} \), for some constant \( C \). We do it in two stages. We show that for some \( v_0 \in V(\Gamma) \) and constants \( r \in \mathbb{N}, C_1, C_2 \in \mathbb{R} \) such that \( v \in B_{v_0}(r) \) and inequalities

\[
P\left( \exists u \in B_{v_0}(r) \setminus \{v\}, \ M_n(u) \leq M_n(v_0) \right) \leq \frac{C_1}{n}
\]

and

\[
P\left( \exists u \in V(\Gamma) \setminus B_{v_0}(r), \ M_n(u) \leq M_n(v_0) \right) \leq \frac{C_2}{n}
\]

hold. Clearly, for any \( u, v_0, v \in V(\Gamma) \) if \( M_n(u) \leq M_n(v) \) then either \( M_n(u) \leq M_n(v_0) \) or \( M_n(v_0) \leq M_n(v) \). It is not hard to see that if we find \( C_1 \) and \( C_2 \) satisfying (27) and (28) respectively, then (26) holds for \( C = C_1 + C_2 \) and the theorem is proved.

First we argue (28). Choose any \( v_0 \in V(\Gamma) \) such that \( \mu(v_0) > 0 \) and \( r \in \mathbb{N} \) such that the inequality (25) holds. We can choose such \( r \) since \( M^{(1)}(v_0) \) is defined. Observe that the left hand side of the inequality above is the expectation of a random variable \( \eta: V \to \mathbb{R} \) defined as \( \eta(s) = d(v_0, s)\mathbf{1}_{V(\Gamma) \setminus B_{v_0}(r/2)}(s) - \frac{r}{2}\mathbf{1}_{v_0}(s) \), \( s \in V(\Gamma) \), where \( \mathbf{1} \) is an indicator function. Since by our assumption \( M = M^{(2)} \) is defined, it follows that \( \sigma^2(\eta) \) is defined, and, applying Lemma 4.1 and the Chebyshev inequality with \( \varepsilon = |\mathbb{E}\eta|/2 \), we obtain

\[
P\left( \exists u \in V(\Gamma) \setminus B_{v_0}(r), \ M_n(u) \leq M_n(v_0) \right) \leq \frac{\sigma^2(\eta)}{n|\mathbb{E}\eta|}.
\]

Hence, inequality (28) holds for \( C_2 = C_2(r, v_0, \mu) = \frac{4\sigma^2(\eta)}{|\mathbb{E}\eta|^2} \). To prove (27) we notice that for any \( u \in V(\Gamma) \setminus \{v\} \), \( M(u) - M(v) = \sum_{s \in V(\Gamma)} (d(u, s) - d(v, s))(d(u, s) + d(v, s))\mu(s) \), i.e., \( M(u) - M(v) \) is the expectation of a random variable \( \tau: V \to \mathbb{R} \) defined as

\[
\tau_{u,v}(s) = (d(u, s) - d(v, s))(d(u, s) + d(v, s)), \ s \in V(\Gamma).
\]

Furthermore, since \( M_{\xi_1}(\cdot) \) is defined and for every \( s \in V(\Gamma) \), \( d(u, s) - d(v, s) \leq d(v, u) \), it is easy to see that \( \sigma^2(\tau_{u,v}(s)) < \infty \). Thus, by the Chebyshev inequality for the sample average of \( \tau_{u,v}(s) \),

\[
P\left( |M_n(u) - M_n(v) - (M(u) - M(v))| \geq \varepsilon \right) \leq \frac{\sigma^2(\tau_{u,v}(s))}{n\varepsilon^2}.
\]
holds. Now, if $0 < \varepsilon < M(u) - M(v)$, then
\[
P(M_n(u) < M_n(v)) \leq P\left| \frac{M_n(u) - M_n(v) - (M(u) - M(v))}{\varepsilon} \right| \geq \varepsilon.
\]
Finally, we choose $\varepsilon$ to be $\frac{1}{2} \inf \{M(u) - M(v) \mid u \in B_{v_0}(r) \setminus \{v\}\}$ and using $\sigma$-additivity of measure we see that inequality $\text{(27)}$ holds for the constant $C_1 = \varepsilon^{-2} \sum_{u \in B_{v_0}(r)} \sigma^2(\tau_{u,v}(s))$. □

In fact, one can easily generalize the previous theorem to the following statement.

**Theorem 4.3.** Let $\Gamma$ be a locally-finite connected graph and $\{\xi_i\}_{i=1}^\infty$ a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi_1}$ is totally defined then there exists a constant $C = C(\Gamma, \xi_1) > 0$ such that
\[
P(S(\xi_1, \ldots, \xi_n) \not\subseteq E(\xi)) \leq \frac{C}{n}.
\]

4.2. **Chernoff-Hoeffding like bound for graphs/groups.** Let $x_i$ be independent random variables. Assume that each $x_i$ is almost surely bounded, i.e., assume that for every $i \in \mathbb{N}$ there exists $a_i, b_i \in \mathbb{R}$ such that $P(x_i - E x_i \in [a_i, b_i]) = 1$. Then for $S_n = \sum_{i=1}^n x_i$ and for any $\varepsilon > 0$ we have the inequality (called the Hoeffding’s inequality)
\[
P(|S_n - E S_n| \geq n\varepsilon) \leq 2 \exp\left(-\frac{2n^2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).
\]
If $x_i$ are identically distributed then we get the inequality
\[
P\left(\left| \frac{1}{n}(x_1 + \ldots + x_n) - E x_1 \right| \geq \varepsilon \right) \leq 2 \exp\left(-\frac{2\varepsilon^2}{(b - a)^2 n}\right).
\]

Techniques of the previous section can be used to find a similar bound on $P(S(\xi_1, \ldots, \xi_n) \not\subseteq E(\xi))$ for a sequence of iid graph-valued $\xi_i$ satisfying some additional assumptions.

**Theorem 4.4.** Let $\Gamma$ be a locally-finite connected graph and $\{\xi_i\}_{i=1}^\infty$ a sequence of i.i.d. random $\Gamma$-elements. If the weight function $M_{\xi_1}(\cdot)$ is totally defined and $\mu_{\xi_1}$ has finite support then for some constant $C > 0$
\[
P\left(S(\xi_1, \ldots, \xi_n) \not\subseteq E(\xi)\right) \leq O(e^{-Cn}).
\]

**Proof.** Proof is similar to the proof of Theorem 4.2. We find $v_0 \in V(\Gamma)$, $r \in \mathbb{N}$, and constants $C_1, C_2 > 0$ such that inequalities
\[
P\left(\exists u \in B_{v_0}(r) \setminus \{v\}, \; M_n(u) \leq M_n(v)\right) \leq O(e^{-C_1 n})
\]
and
\[
P\left(\exists u \in V(\Gamma) \setminus B_{v_0}(r), \; M_n(u) \leq M_n(v_0)\right) \leq O(e^{-C_2 n})
\]
hold.

Choose $v_0 \in V(\Gamma)$ and $r \in \mathbb{N}$ exactly the same way as in Theorem 4.2. Note that a random variable $\eta(s) = d(v_0, s) \mathbf{1}_{V(\Gamma) \setminus B_{v_0}(r/2)}(s) - \frac{1}{2} \mathbf{1}_{v_0}(s)$ (where $s \in V(\Gamma)$) is almost surely bounded. Choose a lower and an upper bounds for $\eta$ and denote them
by $a$ and $b$ respectively. Now, applying Hoeffding’s inequality to $\eta$ with $\varepsilon = |\mathbb{E}\eta|/2$ we obtain
\[
P\left( \exists u \in V(\Gamma) \setminus B_{v_0}(r), \ M_n(u) \leq M_n(v_0) \right) 
\leq P\left( \left| \sum_{s \in V(\Gamma) \setminus B_{v_0}(r/2)} d(v_0, s)\mu_n(s) - \frac{r}{2}\mu_n(v_0) - \mathbb{E}\eta \right| \geq |\mathbb{E}\eta|/2 \right) 
\leq 2 \exp\left( -\frac{|\mathbb{E}\eta|^2}{2(b-a)^2} n \right).
\]
Therefore, $|\mathbb{E}\eta|^2 \leq \frac{\mathbb{E}\eta^2}{2(b-a)^2}$.

To prove (61) we notice that for any $v \in \mathbb{E}(\xi)$ and $u \in V(\Gamma) \setminus \mathbb{E}(\xi)$ we have
\[
M(u) - M(v) = \sum_{s \in V(\Gamma)} (d(u, s) - d(v, s))(d(u, s) + d(v, s))\mu(s),
\]
i.e., $M(u) - M(v)$ is the expectation of a random variable $	au_{u,v} : V \rightarrow \mathbb{R}$ defined as
\[
\tau_{u,v}(s) = (d(u, s) - d(v, s))(d(u, s) + d(v, s)), \ s \in V(\Gamma).
\]
Furthermore, since $\xi_1$ has finite support it follows that the random variable $	au_{u,v}(s)$ almost surely belongs to $[a_{u,v}, b_{u,v}]$. Thus, by the Hoeffding’s inequality for the sample average of $\tau_{u,v}(s)$,
\[
P\left( |M_n(u) - M_n(v) - (M(u) - M(v))| \geq \varepsilon \right) \leq 2 \exp\left( -\frac{2\varepsilon^2}{(b_{u,v} - a_{u,v})^2} n \right).
\]
holds. Now, if $0 < \varepsilon < M(u) - M(v)$, then
\[
P\left( M_n(u) < M_n(v) \right) \leq P\left( |M_n(u) - M_n(v) - (M(u) - M(v))| \geq \varepsilon \right).
\]
Choose $\varepsilon$ to be $\frac{1}{2}\inf\{M(u) - M(v) \mid v \in \mathbb{E}(\xi), \ u \in B_{v_0}(r) \setminus \mathbb{E}(\xi)\}$ and $\delta = \max\{b_{u,v} - a_{u,v} \mid v \in \mathbb{E}(\xi), \ u \in B_{v_0}(r) \setminus \mathbb{E}(\xi)\}$. Finally, using $\sigma$-additivity of measure we see that inequality (61) holds for the constant $C_1 = \frac{2\varepsilon^2}{\delta^2}$.

5. Configurations of mean-sets with applications

In this section, we discuss several configurations of mean-sets on graphs and, in particular, on trees and free groups. First, we make a simple observation stated in the lemma below.

Lemma 5.1. Let $\Gamma$ be a connected graph. Then for any $v \in V(\Gamma)$ there exists a measure $\mu$ such that $\mathbb{E}(\mu) = \{v\}$.

Proof. Indeed, the statement of the lemma holds for the distribution defined by
\[
\mu(u) = \begin{cases} 
1, & \text{if } u = v; \\
0, & \text{otherwise}.
\end{cases}
\]

On the other hand, it is easy to see that not any subset of $V(\Gamma)$ can be realized as $\mathbb{E}(\mu)$. For instance, consider a graph as in Figure 1. Let $\mu_0 = \mu(v_0), \mu_1 = \mu(v_1), \mu_2 = \mu(v_2)$.

Figure 1. Impossible configuration of centers (gray vertices).
\( \mu_2 = \mu(v_2), M_0 = M(v_0), M_1 = M(v_1), M_2 = M(v_2) \) Then \( M_1 = \mu_0 + 4\mu_2, M_0 = \mu_1 + \mu_2, M_2 = 4\mu_1 + \mu_0 \). Clearly, for no values of \( \mu_0, \mu_1, \) and \( \mu_2 \) both inequalities \( M_0 > M_1 \) and \( M_0 > M_2 \) can hold simultaneously (since we can not have \( 2M_0 > M_1 + M_2 \)). Thus, \( v_1 \) and \( v_2 \) can not comprise \( \mathbb{E}_t \). In fact, a tree can have only a limited configuration of centers as proved in Proposition 5.5 below.

Let \( \Gamma \) be a graph. We say that \( v_0 \in V(\Gamma) \) is a cut-point if removing \( v_0 \) from \( \Gamma \) results into a disconnected graph. The same definition holds for any metric space. It turns out that existence of a cut-point in \( \Gamma \) affects configurations of mean-sets. The following lemma provides a useful inequality that holds for any metric space with a cut-point.

**Lemma 5.2 (Cut-point inequality).** Let \( (\Gamma, d) \) be a metric space and \( v_0 \) a cut point in \( \Gamma \). If \( v_1, v_2 \) belong to distinct connected components of \( \Gamma \setminus \{v_0\} \) then for any \( s \in V(\Gamma) \) the inequality

\[
\frac{d(v_0, v_2)(d^2(v_1, s) - d^2(v_0, s)) + d(v_0, v_1)(d^2(v_2, s) - d^2(v_0, s))}{C} \geq 0
\]

holds, where \( C = C(v_0, v_1, v_2) = d(v_0, v_2)d(v(v_0, v_1) + d(v_0, v_2)). \)

**Proof.** Denote the left hand side of (33) by \( g(s) \). There are 3 cases to consider.

**Case 1.** Assume that \( s \) does not belong to the components of \( v_1 \) and \( v_2 \). Then

\[
d(v_0, v_2)(d^2(v_1, s) - d^2(v_0, s)) + d(v_0, v_1)(d^2(v_2, s) - d^2(v_0, s))
\]

\[
= d(v_0, v_2)d(v_0, v_1)(2d(v_0, s) + d(v_0, v_1)) + d(v_0, v_1)d(v_0, v_2)(2d(v_0, s) + d(v_0, v_2))
\]

\[
= d(v_0, v_2)d(v_0, v_1)(4d(v_0, s) + d(v_0, v_1)) + d(v_0, v_2) \geq d(v_0, v_2)d(v_0, v_1) + d(v_0, v_2)
\]

and hence (33) holds.

**Case 2.** Assume that \( s \) belongs to the component of \( v_1 \). Define

\[
x = x(s) = d(v_1, s) \text{ and } y = y(s) = d(v_0, s).
\]

In this notation we get

\[
g(s) = g(x, y) = d(v_1, v_2)(x^2 - y^2) + d(v_0, v_1)(2yv_0, v_2) + 2(v_0, v_2).
\]

Dividing by a positive value \( d(v_0, v_2), \) we get

\[
g(s) \geq 0 \text{ if and only if } \frac{g(x, y)}{d(v_0, v_2)} = x^2 - y^2 + d(v_0, v_1)(2y + d(v_0, v_2)) > 0.
\]

Now, observe that the numbers \( x, y, \) and \( d(v_0, v_1) \) satisfy triangle inequalities

\[
\begin{align*}
x + y &\geq d(v_0, v_1); \\
x + d(v_0, v_1) &\geq y; \\
y + d(v_0, v_1) &\geq x;
\end{align*}
\]

that bound the area visualized in Figure 2. The function of two variables \( \frac{g(x, y)}{d(v_0, v_2)} \) attains the minimal value \( d^2(v_0, v_1) + d(v_0, v_1)d(v_0, v_2) \) on the boundary of the specified area. Hence, the inequality \( g(s) \geq d(v_0, v_2)d(v_0, v_1)(d(v_0, v_1) + d(v_0, v_2)) \) holds for any \( s \in C \) of the component of \( v_1 \).

**Case 3.** If \( s \) belongs to the component of \( v_2 \) then using same arguments as for the previous case one shows that (33) holds.

**Corollary 5.3.** Let \( \Gamma \) be a connected graph, \( v_0 \) a cut-point in \( \Gamma \), and \( v_1, v_2 \) belong to distinct components of \( \Gamma \setminus \{v_0\} \). Then the inequality

\[
d(v_0, v_2)(M(v_1) - M(v_0)) + d(v_0, v_1)(M(v_2) - M(v_0)) \geq C > 0
\]

holds, where \( C = C(v_0, v_1, v_2) = d(v_0, v_2)d(v_0, v_1)(d(v_0, v_1) + d(v_0, v_2)). \)
Proof. Indeed,
\[
d(v_0, v_2)(M(v_1) - M(v_0)) + d(v_0, v_1)(M(v_2) - M(v_0))
= \sum_{s \in V(\Gamma)} (d(v_0, v_2)(d^2(v_1, s) - d^2(v_0, s)) + d(v_0, v_1)(d^2(v_2, s) - d^2(v_0, s))) \mu(s)
\geq \sum_{s \in V(\Gamma)} C\mu(s) = C = d(v_0, v_2)d(v_0, v_1)(d(v_0, v_1) + d(v_0, v_2)),
\]
by Lemma 5.2.
\[\Box\]

**Corollary 5.4 (Cut Point Lemma).** Let $\Gamma$ be a connected graph, $v_0$ a cut-point in $\Gamma$. If $v_1$ and $v_2$ belong to distinct connected components of $\Gamma \setminus \{v_0\}$, then the inequalities $M(v_0) \geq M(v_1)$ and $M(v_0) \geq M(v_2)$ cannot hold simultaneously.

**Proof.** Assume to the contrary that $M(v_0) \geq M(v_1)$ and $M(v_0) \geq M(v_2)$ hold simultaneously which is equivalent to $M(v_1) - M(v_0) \leq 0$ and $M(v_2) - M(v_0) \leq 0$. Then, multiplying by positive constants and adding the inequalities above, we get $d(v_0, v_2)(M(v_1) - M(v_0)) + d(v_0, v_1)(M(v_2) - M(v_0)) \leq 0$ which is impossible by Corollary 5.3. This contradiction finishes the proof.
\[\Box\]

**Corollary 5.5 (Mean-set in a graph with a cut-point).** Let $v_0$ be a cut-point in a graph $\Gamma$ and $\Gamma \setminus \{v_0\}$ a disjoint union of connected components $\Gamma_1, \ldots, \Gamma_k$. Then for any distribution $\mu$ on $\Gamma$ there exists a unique $i = 1, \ldots, k$ such that $\mathbb{E}\mu \subseteq V(\Gamma_i) \cup \{v_0\}$.

**Corollary 5.6 (Mean-set in a graph with several cut-points).** Let $v_1, \ldots, v_n$ be cut-points in a graph $\Gamma$ and $\Gamma \setminus \{v_1, \ldots, v_n\}$ a disjoint union of connected components $\Gamma_1, \ldots, \Gamma_k$. Then for any distribution $\mu$ on $\Gamma$ there exists a unique $i = 1, \ldots, k$ such that $\mathbb{E}\mu \subseteq V(\Gamma_i) \cup \{v_1, \ldots, v_n\}$.

**Corollary 5.7.** Let $G_1$ and $G_2$ be finitely generated groups and $G = G_1 \ast G_2$ a free product of $G_1$ and $G_2$. Then for any distribution $\mu$ on $G$ the set $\mathbb{E}\mu$ is a subset of elements of the forms $gG_1$ or $gG_2$ for some element $g \in G$.

**Proposition 5.8.** Let $\Gamma$ be a tree and $\mu$ a probability measure on $V(\Gamma)$. Then $|\mathbb{E}\mu| \leq 2$. Moreover, if $\mathbb{E}\mu = \{u, v\}$ then $u$ and $v$ are adjacent in $\Gamma$. 

**Figure 2.** Area of possible triangle side lengths.
Proof. Observe that any points \( v_1, v_0, v_2 \) such that \( v_0 \) is connected to \( v_1 \) and \( v_2 \) satisfy the assumptions of Cut Point Lemma (Corollary 5.4). Assume that \( v_0 \in \mathcal{E}_\mu \). At most one of the neighbors of \( v_0 \) can belong to \( \mathcal{E}_\mu \), otherwise we would have 3 connected vertices with equal \( M \) values which contradicts Cut Point Lemma.

\[ \square \]

Corollary 5.9. Let \( \mu \) be a probability distribution on a free group \( F \). Then \( |\mathcal{E}_\mu| \leq 2 \).

In general, the number of central points can be unlimited. To see this, consider the complete graph \( K_n \) on \( n \) vertices and let \( \mu \) be a uniform probability distribution on \( V(K_n) \). Another example of the same type is a cyclic graph \( C_n \) on \( n \) vertices with a uniform probability distribution \( \mu \) on \( V(C_n) \). Clearly \( \mathcal{E}_\mu = V(C_n) \). In all previous examples, the centers in a graph formed a connected subgraph. This is not always the case. One can construct graphs with as many centers as required and property that distances between centers are very large (as large as one wishes).

6. Computation of mean-sets in graphs

In this section we discuss computational issues that we face in practice. One of the technical difficulties is that, unlike the average value \( S_n/n \) for real-valued random variables, the sample mean-set \( S_n \equiv S(\xi_1, \ldots, \xi_n) \) is hard to compute. Let \( G \) be a group and \( \{\xi_i\}_{i=1}^n \) a sequence of random i.i.d. elements taking values in \( G \). Several problems arise when trying to compute \( S_n \):

- Computation of the set \( \{M(g) \mid g \in G\} \) requires \( O(|G|^2) \) steps. This is computationally infeasible for large \( G \), and simply impossible for infinite groups. Hence we might want to reduce the search of a minimum to some small part of \( G \).
- There exist infinite groups in which the distance function \(|\cdot|\) is very difficult to compute. The braid group \( B_\infty \) is one of such groups. The computation of the distance function for \( B_\infty \) is an NP-hard problem, see [26]. Such groups require special treatment. Moreover, there exist infinite groups for which the distance function \(|\cdot|\) is not computable. We omit consideration of such groups.

On the other hand, we can try to devise some heuristic procedure for this task. As we show below, if the function \( M \) satisfies certain local monotonicity properties, then we can achieve good results. The next algorithm is a simple direct descent heuristic which can be used to compute the minimum of a function \( f \).

Algorithm 6.1. (Direct Descent Heuristic)

**Input:** A graph \( \Gamma \) and a function \( f : V(\Gamma) \to \mathbb{R} \).

**Output:** A vertex \( v \) that locally minimizes \( f \) on \( \Gamma \).

**Computations:**

A. Choose a random \( v \in V(\Gamma) \).

B. If \( v \) has no adjacent vertex with smaller value of \( f \), then output current \( v \).

C. Otherwise put \( v \leftarrow u \) where \( u \) is any adjacent vertex such that \( f(u) < f(v) \). Go to step B.
2.4. Choose an arbitrary value \( b \).

The algorithm, at Step C, chooses any vertex \( u \) such that \( f(u) < f(v) \). We say that a function \( f \) is locally finite if for any \( a, b \in \mathbb{R} \) the set \( f(V(\Gamma)) \cap [a, b] \) is finite.

**Lemma 6.2.** Let \( \Gamma \) be a graph and \( f : V(\Gamma) \to \mathbb{R} \) a real-valued function that attains its minimum on \( \Gamma \). If \( f \) is locally decreasing and locally finite, then Algorithm 6.1 for \( \Gamma \) and \( f \) finds the vertex that minimizes \( f \) on \( \Gamma \).

**Proof.** Let \( v \in V(\Gamma) \) be a random vertex chosen by Algorithm 6.1 at Step A. If \( v \) is a minimum of \( f \), then the algorithm stops with the correct answer \( v \). Otherwise, the algorithm, at Step C, chooses any vertex \( u \) adjacent to \( v \) such that \( f(u) < f(v) \). Such a vertex \( u \) exists, since the function \( f \) is locally decreasing by assumption.

Next, Algorithm 6.1 performs the same steps for \( u \). Essentially, it produces a succession of vertices \( v_0, v_1, v_2, \ldots \) such that \( v_0 = v \) and, for every \( i = 0, 1, 2, \ldots \), the vertices \( v_i, v_{i+1} \) are adjacent in \( \Gamma \) with the property \( f(v_i) > f(v_{i+1}) \).

We claim that the constructed succession cannot be infinite. Assume, to the contrary, that the chain \( v_0, v_1, v_2, \ldots \) is infinite. Let \( m \) be the minimal value of \( f \) on \( \Gamma \). Then \( f(V(\Gamma)) \cap [m, f(v)] \) is infinite, and, \( f \) cannot be locally finite. Contradiction. Hence the sequence is finite, and the last vertex minimizes \( f \) on \( V(\Gamma) \). \( \square \)

**Lemma 6.3.** Let \( \mu \) be a distribution on a locally finite graph \( \Gamma \) such that a weight function \( M(\cdot) \) is defined. Then the function \( M(\cdot) \) is locally finite on \( \Gamma \).

**Proof.** Since the function \( M \) is non-negative, it suffices to prove that for any \( b \in \mathbb{R}_+ \) the set \( M(V(\Gamma)) \cap [0, b] \) is finite. Let \( v \in \mathbb{E}(\mu) \), i.e., \( v \) minimizes the value of \( M \), and \( r \in \mathbb{N} \) such that \( 0 < \frac{b}{r} M(v) \leq \sum_{i \in B_v(r)} d^2(v, i) \mu(i) \), as in the proof of Lemma 2.4.

Choose an arbitrary value \( b \in \mathbb{R}_+ \) and put \( \alpha = \max\{2, b/M(v)\} \). Then one can prove (as in Lemma 2.4) that for any \( u \in \Gamma \setminus B_v((\alpha + 2)r) \), we have \( M(u) > (\alpha + 1)M(v) > b \). Therefore, \( M(V(\Gamma)) \cap [0, b] \subset M(B_v((\alpha + 2)r)) \) and the set \( B_v((\alpha + 2)r) \) is finite. \( \square \)

**Theorem 6.4.** Let \( \mu \) be a distribution on a locally finite tree \( T \) such that a function \( M \) is totally defined. Then Algorithm 6.1 for \( T \) and \( M \) finds a central point (mean-set) of \( \mu \) on \( T \).

**Proof.** Follows from Lemmata 6.2, 6.4, 6.3 and 2.4. \( \square \)

Note, the function \( M \) is not locally decreasing for every graph, and a local minimum, computed by Algorithm 6.1, is not always a global minimum.

7. Experiments

In this section we demonstrate how the technique of computing mean-sets, employing the Direct Descent Algorithm 6.1 described in section 6 works in practice and produces results supporting our SLLN for graphs and groups. More precisely, we arrange series of experiments in which we compute the sample mean-sets of randomly generated samples of \( n \) random elements and observe a universal phenomenon: the greater the sample size \( n \), the closer the sample mean gets to the actual mean of a given distribution. In particular, we experiment with free groups,
in which the length function is easily computable. All experiments are done using
the CRAG software package, see [7].

One of the most frequently used distributions on the free groups is a uniform
distribution \( \mu_L \) on a sphere of radius \( L \) defined as \( S_L = \{ w \in F(X) \mid |w| = L \} \). Clearly, \( S_L \) is finite. Therefore, we can easily define a uniform distribution \( \mu_L \) on it as

\[
\mu_L(w) = \begin{cases} 
\frac{1}{|S_L|} & \text{if } |w| = L; \\
0 & \text{otherwise.}
\end{cases}
\]

The reader interested in the question of defining probabilities on groups can
find several approaches to this issue in [6]. One of the properties of \( \mu_L \) is that its
mean-set is just the trivial element of the free group \( F(X) \). Observe also that the
distance of any element of \( F(X) \) to the mean-set is just the length of this element
(or length of the corresponding word).

Table 1 below contains the results of experiments for the distributio ns \( \mu_5, \mu_{10}, \mu_{20}, \mu_{50} \) on the group \( F_4 \). The main parameters in our experiments are the rank \( r \) of the free group, the length \( L \), and the sample size \( n \). For every particular triple of parameter values \((r, L, n)\), we perform series of 1000 experiments to which we refer (in what follows), somewhat loosely, as series \((r, L, n)\). Each cell in the tables below corresponds to a certain series of experiments with parameters \((r, L, n)\). In each experiment from the series \((r, L, n)\), we randomly generate \( n \) words \( w_1, \ldots, w_n \), according to distribution \( \mu_L \), compute the sample mean-set \( S_n \), and compute the displacement of the actual center of \( \mu_L \) from \( S_n \). The set \( S_n \) is computed using Algorithm 6.1 which, according to Theorem 6.4, always produces correct answers for free groups. Every cell in the tables below contains a pair of numbers \((d, N)\); it means that in \( N \) experiments out of 1000 the displacement from the real mean

| \( L \) | 2   | 4   | 6   | 8   | 10  | 12  | 14  | 16  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \mu_5 \) | 0.885 (1.101) | 0.865 (1.22) | 0.928 (1.12) | 0.909 (1.1) | 0.909 (1.1) | 0.1000 (1.1) | 0.999 (1.1) |
| \( \mu_{10} \) | 0.864 (1.117) | 0.930 (1.24) | 0.993 (1.6) | 0.994 (1.1) | 0.999 (1.1) | 0.1000 (1.1) | 0.1000 (1.1) |
| \( \mu_{20} \) | 0.859 (1.116) | 0.940 (1.25) | 0.984 (1.15) | 0.991 (1.9) | 0.1000 (1.1) | 0.999 (1.1) | 0.999 (1.1) |
| \( \mu_{50} \) | 0.872 (1.108) | 0.928 (1.16) | 0.991 (1.9) | 0.998 (1.2) | 0.997 (1.3) | 0.998 (1.2) | 0.999 (1.1) |

**Table 1.** The results of experiment for \( F_4 \).

By doing experiments for free groups of higher ranks, one can easily observe
that as the rank of the free group grows, we get better and faster convergence.
Intuitively, one may think about this outcome as follows: the greater the rank is,
the more branching in the corresponding Cayley graph we have, which means that
more elements are concentrated in a ball, and the bigger growth (in that sense)
causes the better and faster convergence.
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