Diquark Condensate in QCD with Two Colors
at Next-to-Leading Order

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Abstract

We study QCD with two colors and quarks in the fundamental representation at finite baryon density in the limit of light quark masses. In this limit the free energy of this theory reduces to the free energy of a chiral Lagrangian which is based on the symmetries of the microscopic theory. In earlier work this Lagrangian was analyzed at the mean field level and a phase transition to a phase of condensed diquarks was found at a chemical potential of half the diquark mass (which is equal to the pion mass). In this article we analyze this theory at next-to-leading order in chiral perturbation theory. We show that the theory is renormalizable and calculate the next-to-leading order free energy in both phases of the theory. By deriving a Landau-Ginzburg theory for the order parameter we show that the finite one-loop contribution and the next-to-leading order terms in the chiral Lagrangian do not qualitatively change the phase transition. In particular, the critical chemical potential is equal to half the next-to-leading order pion mass, and the phase transition is second order.

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1 Introduction

QCD with two colors and quarks in the fundamental representation of the gauge group is in many respects similar to QCD with three colors and quarks in the fundamental representation. Both theories are confining and asymptotically free, both exhibit a spontaneous breaking of chiral symmetry, and both are believed to be in a different phase at high temperature and density. However, the mechanism of the phase transition at nonzero baryon chemical potential has to be different. The nucleon in QCD with three colors is a fermion with mass much larger than the pion mass. At finite baryon density a phase transition may occur if the Fermi-surface becomes unstable. Instead, for two colors, the lightest baryon is a boson with the same mass as the usual pionic Goldstone bosons. It is a diquark state. For a quark-chemical potential equal to half the mass of the lightest baryon a phase transition to a Bose-Einstein condensate of diquarks takes place very much like the transition to a pion condensate for increasing pion chemical potential in ordinary QCD. This phase transition can be described completely in terms of a low-energy effective theory for the Goldstone modes associated with the spontaneous breaking of chiral symmetry \[1, 2\]. This low-energy effective theory essentially relies on the symmetries of the underlying microscopic theory. In earlier work \[1, 2\] this theory was studied at the mean field level. A phase transition from the normal phase to a phase of condensed diquarks was found at a chemical potential equal to half the pion mass. This phase is a superfluid phase with massless Goldstone bosons. One of them results from the spontaneous breaking of the \(U(1)\) group associated with baryon number.

For QCD with three colors and fundamental quarks, the fermion determinant at nonzero chemical potential has a complex phase making first principle Monte Carlo simulations impossible. On the other hand, for QCD with two colors the fermion determinant remains real at nonzero chemical potential and Monte Carlo simulations can be performed for an even number of flavors \[3, 4\]. We are thus in a unique situation where both analytical results and Monte-Carlo simulations are available in the nonperturbative domain of QCD at nonzero baryon density. The hope is that this ultimately will lead to a better understanding of dense quark matter in QCD with three colors and fundamental quarks. What already has been achieved is that the mean field results have been confirmed by lattice QCD simulations of several independent groups \[5\].

The same analysis can be made in several other cases. They can be classified according to the Dyson index of the Dirac operator of the underlying microscopic theory \[6, 2\]. For QCD with two colors and \(N_f\) quarks in the fundamental representation the Dyson index is \(\beta = 1\) with Goldstone manifold given by \(SU(2N_f)/Sp(2N_f)\). For QCD with two or more colors and quarks in the adjoint representation the Dyson index is \(\beta = 4\) and the Goldstone manifold is given by \(SU(2N_f)/O(2N_f)\). The third case with Dyson index \(\beta = 2\) is QCD with three or more colors and quarks in the fundamental representation. Although chiral perturbation theory is irrelevant for full QCD at nonzero baryon chemical potential, the low-energy effective theory for \(\beta = 2\) can be applied to phase-quenched QCD \[7\], i.e. QCD at nonzero chemical potential with the fermion determinant replaced by its absolute value. Phase quenched QCD has been reinterpreted as QCD at finite isospin.
density [3] and both theories are described by the same low-energy effective theory. This theory has been generalized to include chemical potentials for the different quark species [4] [10]. The theories for $\beta = 1$, $\beta = 2$ and $\beta = 4$ display a similar phase diagram at finite baryon chemical potential. They can be studied within very similar effective field theories [2]. It is also possible to write down effective theories for the Goldstone sector of microscopic theories with an imaginary vector potential but without an involutive automorphism such as the axial symmetry of the Dirac operator. The main difference is the structure of the Goldstone manifold which is determined by the pattern of spontaneous symmetry breaking. Such theories enter in the context of disordered condensed matter systems [11, 12] and the effective theory and its phase diagram are similar to that of QCD-like theories. They are also relevant to QCD in 3 dimensions at nonzero chemical potential [13]. As is the case in chiral perturbation theory at zero chemical potential [14, 15, 16, 17, 18, 19, 20] these mean field results can also be obtained from a chiral Random Matrix Model [21] at nonzero chemical potential.

Up to now the above effective theories have only been analyzed at the mean field level [1, 2, 3, 8, 9, 10]. In this work we perform a one-loop calculation for the case $\beta = 1$. Our first goal is to show that the theory is renormalizable in both phases, i.e. that for an arbitrary background field the one-loop divergences can be absorbed into the coupling constants of the next order Lagrangian. Our second goal is to investigate the one-loop corrections to the mean field results for the phase transition. Our expectation is that the critical chemical potential of the next-to-leading order theory is equal to half the renormalized pion mass at this order. Related questions such as the effect of one-loop corrections on the order of the phase transition and the critical exponents will be addressed as well.

The paper is organized as follows. In section 2 we give a brief review of the low-energy effective theory for QCD with two colors and nonzero chemical potential. The next-to-leading order effective Lagrangian is introduced in section 3. The one-loop corrections of the leading order effective Lagrangian are calculated for an arbitrary background field and it is shown that the divergences can be absorbed into the coupling constants of the next-to-leading order effective Lagrangian. In section 4 we calculate the one-loop corrections to the free energy in the normal phase and the phase of condensed diquarks. A Landau-Ginzburg model for the phase transition is derived from the effective theory in section 5, and its parameters are calculated from chiral perturbation theory. In section 6, we discuss the number density, the chiral condensate and the diquark condensate. Concluding remarks are made in section 7. Some of the technical details are worked out in two appendices. In Appendix A we calculate the next-to-leading order corrections to the pion mass and the pion decay constant. In Appendix B the one-loop integrals for the baryonic Goldstone modes in the diquark phase are evaluated.
2 Low-Energy Limit of QCD with Two Colors

In this section we review the low-energy effective theory of QCD with two colors and fundamental quarks. More details can be found in the original literature [22, 23, 1, 4].

2.1 The Lagrangian of QCD with Two Colors

The QCD partition function is given by the ensemble average determinant of the Dirac operator. For two colors the ensemble average is over $SU(2)$-valued gauged fields weighted by the usual gluonic action. The Dirac operator for quark mass $m$ and chemical potential $\mu$ is given by

$$D = \gamma_\nu D_\nu + m + \mu \gamma_0.$$

Here, $D_\nu \equiv \partial_\nu + A_\nu$ is the covariant derivative, $\gamma_\mu$ are the Euclidean Dirac matrices, and $A_\nu$ are the $SU(2)$-valued gauge fields. Many of the differences between QCD with gauge group $SU(2)$ and fundamental quarks and QCD with three or more colors and fundamental quarks originates from the pseudo-reality of $SU(2)$. It manifests itself through the anti-unitary symmetry of the Dirac operator $D$ ($C$ is the charge conjugation operator and $\tau_2$ acts in color space)

$$D\tau_2 C \gamma_5 = \tau_2 C \gamma_5 D^*.$$  \hspace{1cm} (2.2)

The reality of the fermion determinant is a consequence of this symmetry. However in our context a more important consequence is that for $N_f$ fundamental quarks the flavor symmetry group is enlarged to $SU(2N_f)$ which is sometimes referred to as the Pauli-Gürsey symmetry [24]. In a basis given by left-handed quarks, $q_L$, and conjugate right-handed anti-quarks, $\sigma_2 \tau_2 q_R^*$, ($\sigma_2$ acts in flavor space) this enlarged symmetry group acts on the flavor indices of the spinors

$$\Psi \equiv \begin{pmatrix} q_L \\ \sigma_2 \tau_2 q_R^* \end{pmatrix} \quad \text{as} \quad \Psi \rightarrow V\Psi, \text{ with } V \in SU(2N_f). \hspace{1cm} (2.3)$$

This symmetry becomes manifest by writing the Lagrangian for QCD with two colors in terms of these spinors:

$$L_{QCD} = i\Psi^\dagger \sigma_\nu (D_\nu - \mu B\delta_{0\nu}) \Psi - \frac{1}{2} \Psi^T \sigma_2 \tau_2 M \Psi + \text{h.c.},$$  \hspace{1cm} (2.4)
where $\sigma = (-i, \sigma_k)$. The baryon charge matrix denoted by $B$ is given by

$$B \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

(2.5)

and the matrix that includes the mass term and the diquark source term is defined by

$$\mathcal{M} \equiv \sqrt{m^2 + j^2} (\hat{M} \cos \phi + \hat{J} \sin \phi) ,$$

(2.6)

where $\tan \phi = j/m$ and

$$\hat{M} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \text{ and } \hat{J} \equiv \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix} \text{ with } I \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

(2.7)

Here and in the rest of the paper we consider only even $N_f$. The matrices $\hat{M}$ and $\hat{J}$ correspond to the mass term and to the diquark source, respectively. It is clear that only the kinetic term has the full $SU(2N_f)$ symmetry whereas it is broken explicitly by the chemical potential and the source terms.

What is more important is the spontaneous breaking of chiral symmetry by the formation of a chiral condensate. It breaks the flavor symmetry in the same way as the mass terms, i.e. according to $SU(2N_f) \rightarrow Sp(2N_f)$. Along with the chiral condensate it is possible to form a diquark condensate. This condensate preserves the same symmetries as its source term, i.e. $Sp(N_f) \times Sp(N_f)$. At nonzero chemical potential and mass the symmetry is $SU(N_f) \times U_B(1)$. The diquark condensate breaks this symmetry spontaneously according to $SU(N_f) \times U_B(1) \rightarrow Sp(N_f)$. The baryon number is no longer a conserved quantity.

Although the source terms and the chemical potential break the flavor symmetry, the full flavor symmetry can be retained if the source terms are transformed according to

$$\mathcal{M} \rightarrow V^* \mathcal{M} V^\dagger , \quad B \rightarrow V B V^\dagger .$$

(2.8)

This global $SU(2N_f)$ symmetry of the Lagrangian can be extended to a local flavor symmetry by introducing the vector field $B_\nu = (B,0,0)$ with transformation properties

$$B_\nu \rightarrow V B_\nu V^\dagger - \frac{1}{\mu} V \partial_\nu V^\dagger .$$

(2.9)
Because of the spontaneous breaking of chiral symmetry, the low-energy limit of QCD with two colors is a theory of weakly interacting Goldstone bosons. The Goldstone manifold is given by $SU(2N_f)/Sp(2N_f)$. If the chiral condensate is denoted by $\Sigma$ it can be parameterized by

$$\Sigma = U\bar{\Sigma}U^T,$$

(2.10)

where

$$U = \exp(i\Pi/2F) \quad \text{and} \quad \Pi = \pi_a X_a / \sqrt{2N_f}. \quad (2.11)$$

Here, $F$ is the pion decay constant, the fields $\pi_a$ are the Goldstone modes and the $X_a$ are the $2N_f^2 - N_f - 1$ generators of the coset $SU(2N_f)/Sp(2N_f)$. They obey the relation $X_a\bar{\Sigma} = \bar{\Sigma}X_a^T$ and are normalized according to $\text{Tr}X_aX_b = 2N_f\delta_{ab}$.

The Lagrangian of the Goldstone modes is obtained by the requirement that it has the same symmetry properties as the underlying microscopic theory. The Lagrangian for $\Sigma$ should be invariant under the local transformations

$$\Sigma \to V\Sigma V^T, \quad V \in SU(2N_f) \quad (2.12)$$

if the source terms and $B_\nu$ are transformed according to (2.8,2.9). We adopt the usual power counting of chiral perturbation theory that the square root of the quark mass, the square root of the diquark source, the chemical potential and the momenta are of the same order. Then to order $p^2$ it is simple to write down a Lagrangian that is invariant under the global transformations (2.8). Invariance under the local transformation (2.9) can be achieved by introducing the covariant derivative

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - \mu (B_\nu \Sigma + \Sigma B_\nu^T),$$

$$\nabla_\nu \Sigma^\dagger = \partial_\nu \Sigma^\dagger + \mu (B_\nu^\dagger \Sigma^\dagger + \Sigma B_\nu). \quad (2.13)$$

The effective Lagrangian to leading order in the momentum expansion having the required invariance properties is thus given by

$$\mathcal{L}^{(2)} = \frac{F^2}{2} \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger - \chi^\dagger \Sigma - \chi \Sigma^\dagger \right], \quad (2.14)$$

where we have introduced the source term

$$\chi = \frac{G}{F^2} \mathcal{M}^\dagger. \quad (2.15)$$
To leading order in the chiral expansion $G$ is related to the quark-antiquark condensate at $j = \mu = T = 0$ according to $\langle \bar{\psi} \psi \rangle_0 = 2N_f G$, and the pion mass is given by the Gell-Mann–Oakes–Renner relation: $M^2 = mG/F^2$.

For zero diquark source, the leading-order Lagrangian (2.14) describes two different phases: a normal phase and a phase of condensed diquarks. They are separated by a second order phase transition at $\mu_c = M/2$. The chiral condensate in both phases can be parameterized as

$$\bar{\Sigma}(\alpha) = \cos \alpha \Sigma_c + \sin \alpha \Sigma_d,$$  \hspace{1cm} (2.16)

where

$$\alpha = 0 \quad \text{if} \quad \mu < \mu_c$$
$$\cos \alpha = \frac{M^2}{4\mu^2} \quad \text{if} \quad \mu > \mu_c,$$  \hspace{1cm} (2.17)

and

$$\Sigma_c \equiv I \quad \text{and} \quad \Sigma_d \equiv \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}.$$

(2.18)

A non-zero value of $\alpha$ corresponds to diquark condensation [2]. The orientation of the condensate (2.16) minimizes the static part of the leading-order effective Lagrangian (2.14). At nonzero $\alpha$ it is sometimes useful to introduce rotated generators of $SU(2N_f)/Sp(2N_f)$ defined by

$$X_a(\alpha) = V_\alpha X_a(\alpha = 0)V_\alpha^\dagger \quad \text{with} \quad V_\alpha = \exp(-\frac{\alpha}{2} \Sigma_d \Sigma_c).$$

(2.19)

At nonzero $j$ the diquark condensate is always non-vanishing. In this case, the saddle point is still given by $\bar{\Sigma} = \cos \alpha \Sigma_c + \sin \alpha \Sigma_d$, but $\alpha$ is determined by the saddle point equation

$$4\mu^2 \cos \alpha \sin \alpha = \tilde{M}^2 \sin(\alpha - \phi),$$

(2.20)

where $\tilde{M}^2 = G\sqrt{m^2 + j^2}/F^2$. Also in this case it may be useful to introduce rotated generators.

The spectrum contains $2N_f^2 - N_f - 1$ pseudo-Goldstone bosons. In the normal phase, $\alpha = 0$, they can be distinguished by their baryon charge. There are $N_f^2 - 1$ “usual” pions with charge zero, denoted by $P$, $N_f(N_f - 1)/2$ diquarks with charge +2, denoted by $Q$, etc.
and $N_f(N_f - 1)/2$ antidiquarks with charge $-2$, denoted by $Q^\dagger$. The inverse of their propagator is respectively given by [2],

$$D^P = p^2 + M^2,$$

$$D^Q = \begin{pmatrix} p^2 + M^2 - 4\mu^2 & 4i\mu p_0 \\ 4i\mu p_0 & p^2 + M^2 - 4\mu^2 \end{pmatrix}. \quad (2.21)$$

In the phase of condensed diquarks the pseudo-Goldstone modes of the normal phase are mixed into four different types. We now have $N_f(N_f + 1)/2 P_S$-modes, $(N_f^2 - N_f - 2)/2 P_A$-modes and $N_f(N_f - 1) Q$-modes. The inverse of their respective propagators reads [2]

$$D^{P_S} = p^2 + M_1^2 + \frac{1}{4}M_3^2,$$

$$D^{P_A} = p^2 + M_2^2 + \frac{1}{4}M_3^2, \quad (2.22)$$

$$D^Q = \begin{pmatrix} p^2 + M_1^2 & iM_3 p_0 \\ iM_3 p_0 & p^2 + M_2^2 \end{pmatrix},$$

where

$$M_1^2 = \tilde{M}^2 \cos(\alpha - \phi) - 4\mu^2 \cos 2\alpha$$

$$M_2^2 = \tilde{M}^2 \cos(\alpha - \phi) - 4\mu^2 \cos^2 \alpha$$

$$M_3^2 = 16\mu^2 \cos^2 \alpha. \quad (2.23)$$

By diagonalizing the inverse propagator for the $Q$-modes we find $N_f(N_f - 1)/2 \tilde{Q}$-modes and $N_f(N_f - 1)/2 \tilde{Q}^\dagger$-modes. The $\tilde{Q}$-modes are the true massless Goldstone modes of the superfluid phase at $j = 0$ and $\mu > M/2$ [4].

3 Next-to-Leading Order Effective Theory

We wish to examine the effect of all one-loop diagrams to the free energy. Such contributions are of $O(p^4)$ in the power counting of Chiral Perturbation Theory, that is next-to-leading order in the momentum expansion [26]. In this section we shall construct and renormalize Chiral Perturbation Theory to $O(p^4)$ for QCD with two colors and fundamental quarks. We will closely follow the work of Gasser and Leutwyler for QCD with three colors and fundamental quarks [28].

At next-to-leading order in chiral perturbation theory, the partition function contains four different contributions: the tree graphs of the $O(p^2)$ effective Lagrangian $\mathcal{L}^{(2)}$, the
one-loop diagrams from $\mathcal{L}^{(2)}$, the tree graphs of the $O(p^4)$ effective Lagrangian $\mathcal{L}^{(4)}$ and
the contribution from the axial anomaly [26]. The free energy at next-to-leading order can thus be written as

$$ S = S_2 + S_{1 \text{-loop}} + S_4 + S_A, \quad (3.24) $$

where $S_2 = \int dx \mathcal{L}^{(2)}$, $S_{1 \text{-loop}}$ is the one-loop contribution to the free energy from the
one-loop diagrams of $\mathcal{L}^{(2)}$, $S_4 = \int dx \mathcal{L}^{(4)}$, and $S_A$ is the Wess-Zumino-Witten functional
which reproduces the axial anomaly.

### 3.1 Operators at Next-to-Leading Order

For QCD with three colors and $N_f$ quarks in the fundamental representation the most
general effective Lagrangian to $O(p^4)$ has already been constructed in the literature [26]
[27]. The $O(p^4)$ effective Lagrangian for QCD with two colors and $N_f$ quarks in the
fundamental representation can be found along the same lines. Although the Goldstone
manifold is different in the two cases, the same operators appear in the Lagrangian. The
$O(p^4)$ effective Lagrangian contains all the Lorentz invariant operators of $O(p^4)$ that are
invariant under local $SU(2N_f)$ flavor transformations. Keeping only the terms that are
relevant to our case, we deduce from [26, 27] the effective Lagrangian

$$\begin{align*}
\mathcal{L}^{(4)} &= -L_0 \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\tau \Sigma^\dagger \nabla_\nu \Sigma \nabla_\tau \Sigma^\dagger \right] - L_1 \left( \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger \right] \right)^2 \\
&- L_2 \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\tau \Sigma^\dagger \right] \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\tau \Sigma^\dagger \right] - L_3 \text{Tr} \left[ \left( \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger \right)^2 \right] \\
&+ L_4 \text{Tr} \left[ \chi \Sigma^\dagger + \Sigma \chi^\dagger \right] \text{Tr} \left[ \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger \right] + L_5 \text{Tr} \left[ \left( \chi \Sigma^\dagger + \Sigma \chi^\dagger \right) \left( \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger \right) \right] \\
&- L_6 \left( \text{Tr} \left[ \chi \Sigma^\dagger + \Sigma \chi^\dagger \right] \right)^2 - L_7 \left( \text{Tr} \left[ \chi^\dagger \Sigma - \chi \Sigma^\dagger \right] \right)^2 \\
&- L_8 \text{Tr} \left[ \chi \Sigma^\dagger \chi^\dagger + \Sigma \chi^\dagger \Sigma \chi^\dagger \right] - H_2 \text{Tr} \left[ \chi \chi^\dagger \right].
\end{align*}$$

The low-energy coupling constants $L_i$, $i = 1, \ldots, 8$, and $H_2$ are ‘bare’ coupling constants.
The term related to the coupling constant $H_2$ is a contact term that, in our case, will
enter only in the vacuum energy and in the quark-antiquark condensate. Because we only
consider quarks with equal masses, the coupling constant $L_7$ will not appear in any of our
results.

### 3.2 Renormalization to One Loop

In order to compute the one-loop contributions from $\mathcal{L}^{(2)}$, we expand $\mathcal{L}^{(2)}$ in the fluctuations around the solution of the classical equation of motion following the analysis of
Gasser and Leutwyler [26]. The one-loop contributions from $L^{(2)}$ are UV divergent. They have to be regularized in some way, and the regularized divergences must be canceled by a renormalization of the coupling constants that appear in $L^{(4)}$. If the theory is renormalizable the values of the counter-terms must be independent of $\mu$, $M$, and $j$, and, especially, the renormalization constants must be identical in the two phases. We show that this is indeed the case.

The solution of the classical equation of motion in the normal phase can be written as

$$\tilde{\Sigma} = U_{\alpha} \Sigma U_{\alpha}^T,$$

where $U_{\alpha}$ belongs to the Goldstone manifold $SU(2N_f)/Sp(2N_f)$ with $Sp(2N_f)$ subgroup that leaves $\Sigma$ invariant. Using that the $\bar{\Sigma}$ can be viewed as a rotation of $\Sigma_c = I$, i.e. $\tilde{\Sigma} = V_{\alpha} \Sigma_{\alpha} V_{\alpha}^T$ with $V = \exp(-\alpha \Sigma_{c} \Sigma_{c}/2)$, we can also parameterize the Goldstone manifold as follows

$$\tilde{\Sigma} = V_{\alpha} U V_{\alpha}^T U^T V_{\alpha}^T = V_{\alpha} U I U^T V_{\alpha}^T = V_{\alpha} U^2 I V_{\alpha}^T.$$ (3.27)

The last equality holds because $U = U_{\alpha=0}$. The expansion around $\tilde{\Sigma}$ is obtained by the substitution

$$U \to U e^{i\xi},$$ (3.28)

where $\xi$ is a linear combination of the generators $X_{\alpha}$ of the Goldstone manifold $SU(2N_f)/Sp(2N_f)$ satisfying $X_{\alpha} I = IX_{\alpha}^T$.

$$\xi = \sum_{a=1}^{2N_f^2-N_f-1} \xi^a X^a,$$ (3.29)

In other words, $\xi$ is a $2N_f \times 2N_f$ hermitian traceless matrix that satisfy $\xi I = I\xi^T$. The expansion about $\tilde{\Sigma}$ can thus can be written as

$$\Sigma = V_{\alpha} U \left(1 + i \left(\frac{\xi}{2} - \frac{1}{2} \left(\frac{\xi}{2}\right)^2 + \cdots\right) I \left(1 + i \left(\frac{\xi^T}{2} - \frac{1}{2} \left(\frac{\xi^T}{2}\right)^2 + \cdots\right) U^T V_{\alpha}^T\right)$$

$$= V_{\alpha} U \left(1 + i\xi - \frac{1}{2} \xi^2 + \cdots\right) U I V_{\alpha}^T.$$ (3.30)

The $O(p^2)$ effective Lagrangian is given by (2.14). Therefore, to second order in the fluctuations, we find

$$\mathcal{L}^{(2)} = \frac{F^2}{2} \text{Tr}[\nabla_{\nu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger - \chi \tilde{\Sigma}^\dagger - \tilde{\Sigma} \chi^\dagger]$$

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\[ + \frac{F^2}{2} Tr[\nabla_\nu (U^2)\nabla_\nu (U^{\dagger} \nabla_\mu U^{\dagger}) - \frac{1}{2} \nabla_\nu (U^2)\nabla_\nu (U \nabla_\mu U^{\dagger}) - \frac{1}{2} \nabla_\nu (U^{\dagger} \nabla_\mu U^{\dagger})] \tag{3.31} \]

where the superscript \( \alpha \) indicates that the original sources have been replaced by the rotated sources:

\[ \chi^\alpha = V_\alpha^\dagger \chi V_\alpha^* \quad \text{and} \quad B_\nu^\alpha = V_\alpha^\dagger B_\nu V_\alpha, \tag{3.32} \]

and the covariant derivative \( \nabla_\nu^\alpha \) is given by \( \nabla_\nu \) defined in (2.13) with \( B_\nu \) replaced by \( B_\nu^\alpha \). Notice that \( B_\nu^\alpha = I B_\nu I \).

Let us denote the classical action by \( \tilde{S}_2 \). Up to second order in the fluctuations the action can be written as

\[ \int dx L^{(2)} = \tilde{S}_2 + \frac{F^2}{2} \int dx Tr\left(d_\nu \xi d_\nu \xi - [\Delta_\nu, \xi][\Delta_\nu, \xi] - \sigma \xi^2\right), \tag{3.33} \]

where

\[
\begin{align*}
    d_\nu \xi &= \partial_\nu \xi + [\Gamma_\nu, \xi] \\
    \Gamma_\nu &= \frac{1}{2} U^{\dagger} \partial_\nu U - \frac{1}{2} U \partial_\nu U^{\dagger} - \frac{\mu}{2} U^{\dagger} B_\nu^\alpha U - \frac{\mu}{2} U B_\nu^\alpha T U^{\dagger} \\
    \Delta_\nu &= \frac{1}{2} U^{\dagger} \nabla_\nu (U^2) U^{\dagger} - \frac{1}{2} U \nabla_\nu (U^{\dagger} U) \\
    \sigma &= \frac{1}{2} (U^{\dagger} \chi^\alpha I U^{\dagger} - U I \chi^\alpha T). \tag{3.34}
\end{align*}
\]

Notice that with the above conventions \( \nabla_\nu (U \xi U) = U (d_\nu \xi + [\Delta_\nu, \xi]) U \), and that the field strength associated with \( \Gamma_\nu \) reduces to \( \Gamma_{\mu \nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = -[\Delta_\mu, \Delta_\nu] \). In terms of the components of \( \xi = \sum_a \xi^a X^a \) where \( X^a \) are the generators of \( SU(2N_f)/Sp(2N_f) \) with \( Tr X^a X^b = 2N_f \delta^{ab} \), we obtain

\[ \int dx L^{(2)} = \tilde{S}_2 - N_f F^2 \xi, D\xi, \tag{3.35} \]

with scalar product defined by \( (f, g) = \sum_a \int dx f^a(x) g^a(x) \), and the differential operator \( D \) is given by

\[
\begin{align*}
    D^{ab}_{\xi^b} &= d_\nu d_\nu \xi^a + \dot{\sigma}^{ab} \xi^b \tag{3.36} \\
    d_\nu \xi^a &= \partial_\nu \xi^a + \dot{\Gamma}^{ab} \xi^b \tag{3.37} \\
    \dot{\Gamma}^{ab} &= -\frac{1}{2N_f} Tr\left([X^a, X^b] \Gamma_\nu\right) \tag{3.38} \\
    \dot{\sigma}^{ab} &= \frac{1}{2N_f} Tr\left([X^a, \Delta_\nu] [X^b, \Delta_\nu]\right) + \frac{1}{4N_f} Tr\left(\sigma \{X^a, X^b\}\right). \tag{3.39}
\end{align*}
\]
Notice that the field strength associated with $\hat{\Gamma}_\mu$ is simply given by
$$
\hat{\Gamma}_{\mu\nu}^{ab} = -\frac{1}{2} \text{Tr}(X^a, X^b) \Gamma_{\mu\nu}.
$$

The one-loop correction to the free energy is obtained by performing a Gaussian integral resulting in

$$
S_{1\text{-loop}} = \frac{1}{2} \ln \det D. \quad (3.40)
$$

As is usual in ChPT [26] we choose to regularize the determinant by means of dimensional regularization [28]. There are ultraviolet divergences that produce poles in $d$. The standard calculation leads to [26, 27]

$$
S_{1\text{-loop}} = \sum_{a,b} \int d^d x \left( \frac{1}{d} \delta^{ab} - \frac{1}{4\pi(d-2)} \hat{\delta}^{ab} \delta^{ab} + \frac{1}{4\pi^2(d-4)} \left\{ \frac{1}{12} \Gamma_{\mu\nu}^{ab} \Gamma_{\mu\nu}^{ba} + \frac{1}{2} \hat{\sigma}^{ab} \hat{\sigma}^{ba} \right\} + \cdots \right) \quad (3.41)
$$

In order to perform the sum over the $SU(2N_f)/Sp(2N_f)$ generators, we need the following two formulae

$$
\sum_{a=1}^{2N_f^2-1} \text{Tr}[X^a A] \text{Tr}[X^a B] = N_f \text{Tr}[AB] - N_f \text{Tr}[AIB^T I] - \text{Tr}[A] \text{Tr}[B] \quad (3.42)
$$

and

$$
\sum_{a=1}^{2N_f^2-1} \text{Tr}[X^a AX^a B] = -\text{Tr}[AB] + N_f \text{Tr}[AIB^T I] + N_f \text{Tr}[A] \text{Tr}[B]. \quad (3.43)
$$

From these two formulae we easily derive the identities

$$
\sum_{a,b} \hat{\Gamma}_{\mu\nu}^{ab} \hat{\Gamma}_{\mu\nu}^{ba} = 2(N_f - 1) \text{Tr}[\Gamma_{\mu\nu} \Gamma_{\mu\nu}] \quad (3.44)
$$

and

$$
\sum_{a,b} \hat{\delta}^{ab} \hat{\delta}^{ba} = 2N_f \text{Tr}[(\Delta_\mu \Delta_\nu)^2] + \left( \text{Tr}[\Delta_\mu \Delta_\mu] \right)^2 + 2\text{Tr}[\Delta_\mu \Delta_\nu] \text{Tr}[\Delta_\mu \Delta_\nu]
$$

$$
- \text{Tr}[\sigma] \text{Tr}[\Delta_\mu \Delta_\mu] - 2N_f \text{Tr}[\sigma \Delta_\mu \Delta_\mu]
$$

$$
+ \frac{N_f^2 + 1}{4N_f^2} (\text{Tr}[\sigma])^2 + \frac{(N_f + 1)(N_f - 2)}{2N_f} \text{Tr}[\sigma^2], \quad (3.46)
$$

where we have used that $X^a I = IX^a$, $I \Delta_\mu^{T} I = -\Delta_\mu$, $I \sigma^{T} I = -\sigma$, and $I \Gamma_{\mu\nu}^{T} I = -\Gamma_{\mu\nu}^{T} = \Gamma_{\mu\nu}$. Therefore using (3.34) we find

$$
\sum_{a,b} \left\{ \frac{1}{12} \hat{\Gamma}_{\mu\nu}^{ab} \hat{\Gamma}_{\mu\nu}^{ba} + \frac{1}{2} \hat{\delta}^{ab} \hat{\delta}^{ba} \right\} = \frac{N_f - 1}{48} \text{Tr}[\nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^{\dagger} \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^{\dagger}] + \frac{1}{32} \left( \text{Tr}[\nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^{\dagger}] \right)^2
$$
\[ + \frac{1}{16} \text{Tr} \left[ \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger \right] \text{Tr} \left[ \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger \right] + \frac{2 N_f + 1}{48} \text{Tr} \left[ \left( \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger \right)^2 \right] \quad (3.47) \]

\[- \frac{1}{16} \text{Tr} \left[ \chi \tilde{\Sigma}^\dagger + \tilde{\Sigma} \chi^\dagger \right] \text{Tr} \left[ \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger \right] - \frac{N_f}{8} \text{Tr} \left[ \left( \chi \tilde{\Sigma}^\dagger + \tilde{\Sigma} \chi^\dagger \right) \left( \nabla_{\mu} \tilde{\Sigma} \nabla_{\nu} \tilde{\Sigma}^\dagger \right) \right] \]

\[+ \frac{N_f^2 + 1}{32 N_f^2} \left( \text{Tr} \left[ \chi \tilde{\Sigma}^\dagger + \tilde{\Sigma} \chi^\dagger \right] \right)^2 + \frac{(N_f + 1)(N_f - 2)}{16 N_f} \text{Tr} \left[ \chi \tilde{\Sigma}^\dagger \chi \tilde{\Sigma}^\dagger + \tilde{\Sigma} \chi^\dagger \chi \chi^\dagger + 2 \chi \chi^\dagger \right]. \]

The \( d = 4 \) pole in \( S_{1-\text{loop}} \) can therefore be absorbed by the following renormalization of the coupling constants

\[
L_0 = L_0^r + \frac{N_f - 1}{2} \lambda \\
L_3 = L_3^r + \frac{2 N_f + 1}{48} \lambda \\
L_6 = L_6^r + \frac{N_f^2 + 1}{32 N_f^2} \lambda \\
L_1 = L_1^r + \frac{1}{32} \lambda \\
L_4 = L_4^r + \frac{1}{16} \lambda \\
L_7 = L_7^r \\
L_2 = L_2^r + \frac{1}{96} \lambda \\
L_5 = L_5^r + \frac{N_f}{8} \lambda \\
L_7 = L_7^r + \frac{(N_f + 1)(N_f - 2)}{16 N_f} \lambda \quad (3.48)
\]

where

\[
\lambda = -\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} \Lambda^{d-4} + \frac{1}{64\pi^2} \Lambda^{d-4}. \quad (3.49)
\]

The finite counter terms in \( \lambda \) are introduced to partially cancel the finite contributions of the one-loop integrals. This is the subtraction scheme traditionally used in chiral perturbation theory \[26\]. The cancellation of the finite counterterms will be discussed in detail below. In terms of the renormalized coupling constants \( L_i^r \) and \( H_2^r \), the sum \( S_{1-\text{loop}} + S_4 \) remains finite for \( d \to 4 \) both in the normal phase and in superfluid phase. Notice that the renormalized low-energy constants depend on the renormalization scale \( \Lambda \). The numerical values of these coupling constants are not yet known, but they can be obtained, at least in principle, from lattice QCD simulations. For QCD with three colors and quarks in the fundamental representation, the experimental result for the physical value of these coupling constants, i.e. \( L_i^r \) at \( \Lambda = m_\pi \), is that they are of the order of \( 10^{-3} \). We expect that this is a reasonable estimate for their value for QCD with two colors and quarks in the fundamental representation as well.

### 3.2.1 Wess-Zumino Term

To account for the chiral anomaly the \( \mathcal{O}(p^4) \) Lagrangian must be supplemented by a Wess-Zumino term. At nonzero baryon density this term can be obtained by gauging the Wess-Zumino term at zero baryon density \[29, 30\]. The latter is given by

\[
\Gamma_{WZ}[\Sigma] = \frac{i}{120 \pi^2} \int_{M^5} \text{Tr} \alpha^5, \quad (3.50)
\]
where

\[ \alpha = (\partial_\nu \Sigma) \Sigma^{-1} dx^\nu, \quad (3.51) \]

and \( M^5 \) is a five dimensional domain with space-time, \( M^4 \), as boundary.

At non-zero chemical potential the vector field \( B_\nu = \delta_{\nu,0} B \) has only one nonzero component and all terms in the gauged Wess-Zumino term with more than one external vector field vanish. We thus find that

\[ \Gamma_{WZ}[\Sigma, B_\nu = \delta_{\nu,0} B] = \mu \frac{i}{12\pi^2} \int_{M^4} Tr [B^{\alpha_3}] = \mu \frac{i}{12\pi^2} \int_{M^4} Tr [B^{\alpha_4}] e^{0ijk} d^4 x. \quad (3.52) \]

This term is proportional to \( 2i \) times the winding number of the field. The factor 2 arise because \( \mu \) is the quark chemical potential equal to half the baryon number chemical potential for QCD with two colors. The lightest excitation with nonzero winding number is expected to be of the order of the mass of the nucleon for QCD with three colors. Such terms are subleading in the low-energy limit we are considering and can be ignored.

4 Free Energy

In the previous section we have shown that the 1-loop contribution to the free energy given by to the logarithm of the determinant of the propagator matrix is renormalizable. In this section, we obtain an explicit expression for the free energy to one-loop order. The Feynman graphs that enter the free energy at this order are given in fig. 1.

\[ \bullet \quad \boxed{4} \]

Figure 1: Feynman diagrams that enter into the free energy at next-to-leading order. The dot denotes the contribution from \( \mathcal{L}^{(2)} \), and the boxed 4 the contribution from \( \mathcal{L}^{(4)} \).
4.1 Free Energy of the Normal Phase

In the normal phase the one-loop free energy is readily found to be given by

\[ \Omega = -2N_f F^2 M^2 - \frac{1}{2}(N_f^2 - 1)\Delta_0^P - \frac{1}{4}N_f(N_f - 1)\Delta_0^Q \]
-2N_f M^4 \left(8N_f L_6 + 2L_8 + H_2\right),

(4.53)

where \( M^2 = mG/F^2 \) is the leading order pion mass. The terms \( \Delta_0^P \) and \( \Delta_0^Q \) represent the one-loop graphs with a \( P \)-mode or both \( Q \)-modes respectively. As we shall see shortly, the coupling constants \( L_i \) derived in previous section (3.48) contain the counter-terms that regularize these divergent one-loop integrals.

The one-loop diagrams with a \( P \)-mode, or both \( Q \)-modes are respectively given by

\[
\Delta_0^P = -\int \frac{d^d p}{(2\pi)^d} \ln(p^2 + M^2) = \frac{1}{(4\pi)^{d/2}} \Gamma\left(-\frac{d}{2}\right) M^d
\]

\[
\Delta_0^Q = -\int \frac{d^d p}{(2\pi)^d} \ln \left((p^2 + M^2 - 4\mu^2)^2 + 16\mu^2 p_0^2\right) = \frac{2}{(4\pi)^{d/2}} \Gamma\left(-\frac{d}{2}\right) M^d.
\]

(4.54)

(4.55)

For the \( Q \)-modes, we simplified the integral using that

\[
(p^2 + M^2 - 4\mu^2)^2 + 16\mu^2 p_0^2 = \left((p_0 - 2i\mu)^2 + \vec{p}^2 + M^2\right)\left((p_0 + 2i\mu)^2 + \vec{p}^2 + M^2\right).
\]

(4.56)

Notice that the divergent part of the one-loop diagrams is exactly canceled by the renormalized coupling constants (3.48). The free energy of the normal phase reads

\[
\Omega = -2N_f F^2 M^2 \left(1 + \frac{M^2}{F^2} \left[8N_f L_6^r + 2L_8^r + H_2^r + \frac{2N_f^2 - N_f - 1}{256\pi^2 N_f} \left(1 - 2\ln \frac{M^2}{\Lambda^2}\right)\right]\right).
\]

(4.57)

As expected at zero temperature, the free energy of the normal phase does not depend on the chemical potential. Therefore the baryon density is zero in the normal phase. The free energy does not depend on the renormalization scale \( \Lambda \): The logarithmic dependence of the renormalized coupling constants in \( \Lambda \) cancels the explicit logarithm in (4.57). Using the free energy (4.57) the chiral condensate in the normal phase at next-to-leading order in chiral perturbation theory is given by

\[
\langle \bar{\psi}\psi \rangle_0 = -\frac{\partial \Omega}{\partial m} = 2N_f G \left(1 + \frac{M^2}{F^2} \left[8N_f L_6^r + 2L_8^r + H_2^r - \frac{2N_f^2 - N_f - 1}{128\pi^2 N_f} \ln \frac{M^2}{\Lambda^2}\right]\right).
\]

(4.58)
4.2 Free Energy of the Phase of Condensed Diquarks

In the phase of condensed diquarks, the free energy is found to be given by

$$\Omega = -2N_f F^2 \left( \frac{1}{2}(M_1^2 + M_2^2) + \frac{1}{4} M_3^2 \right)$$

$$- \frac{1}{4} N_f (N_f + 1) \Delta_0^{P_S} - \frac{1}{4} (N_f (N_f - 1) - 2) \Delta_0^{P_A} - \frac{1}{4} N_f (N_f - 1) \Delta_0^Q$$

$$- 2N_f (M_1^2 - M_2^2)^2 \left( L_0 + 2N_f L_1 + 2N_f L_2 + L_3 \right)$$

$$- 4N_f (M_1^2 - M_2^2)(M_2^2 + \frac{1}{4} M_3^2) \left( 2N_f L_4 + L_5 \right)$$

$$- 8N_F (M_2^2 + \frac{1}{4} M_3^2)^2 \left( 2N_f L_6 + L_8 \right) - 2N_f \tilde{M}^4 \left( -2L_8 + H_2 \right),$$

where $\tilde{M}^2 = G\sqrt{m^2 + j^2/F^2}$ and $\tan \phi = j/m$.

The one-loop diagrams that contribute to the free energy contain a $P_S$, a $P_A$ or the mixed $Q$ modes. Their contributions are given by

$$\Delta_0^{P_S} = - \int \frac{d^d p}{(2\pi)^d} \ln \left( p^2 + M_1^2 + \frac{1}{4} M_3^2 \right) = \frac{1}{(4\pi)^{d/2}} \Gamma \left( \frac{-d}{2} \right) (M_1^2 + \frac{1}{4} M_3^2)^{d/2}$$

$$\Delta_0^{P_A} = - \int \frac{d^d p}{(2\pi)^d} \ln \left( p^2 + M_2^2 + \frac{1}{4} M_3^2 \right) = \frac{1}{(4\pi)^{d/2}} \Gamma \left( \frac{-d}{2} \right) (M_2^2 + \frac{1}{4} M_3^2)^{d/2}$$

$$\Delta_0^Q = - \int \frac{d^d p}{(2\pi)^d} \ln \left( (p^2 + \frac{3}{4} M_1^2)(p^2 + M_2^2 + \frac{1}{4} p_0^2 M_3^2) \right)$$

$$= 2 \frac{\Gamma \left( \frac{-d}{2} \right)}{(4\pi)^{d/2}} \left[ \frac{1}{2} \right] (M_1^2 + M_2^2) + \frac{1}{4} M_3^2 \right]^{d/2} + \frac{\Gamma \left( \frac{-d}{2} \right) M^{d-4}}{8} \frac{d}{(4\pi)^{d/2}} (M_1^2 - M_2^2)^2$$

$$\left[ \frac{1}{2} \right] \ln \left( \frac{\sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2 + M_3^2}}}{2M} \right)$$

$$+ \left[ \frac{1}{2} \right] \ln \left( \frac{\sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2 + M_3^2}}}{M_3^2} \right)$$

$$\left[ \frac{1}{2} \right] \ln \left( \frac{\sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2 + M_3^2}}}{M_3^2} \right)$$

$$\left[ \frac{1}{2} \right] \ln \left( \frac{\sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2 + M_3^2}}}{M_3^2} \right)$$

$$+ O((M_1^4 - M_2^4)^2) + O((M_4^4 - M_3^4)^2) + O(d - 4),$$

respectively. The masses $M_{1,2,3}$ have been defined in (2.23). A derivation of the power series expansion in $M_1^2 - M_2^2$ of $\Delta_0^Q$ is given in appendix B. The divergent term can be computed exactly, but we succeeded only to compute the finite corrections near the leading-order saddle point for $j \ll m$. One easily shows that the sum of the divergent one-loop terms given by

$$- \frac{\Gamma \left( \frac{-d}{2} \right)}{4(4\pi)^{d/2}} \left[ N_f^2 (2M_1^4 + 2M_2^4 + \frac{1}{4} M_3^4 + M_3^2 (M_1^2 + M_2^2) - 2(N_f + 1)(M_2^2 + \frac{1}{4} M_3^2)^2 \right],$$
is exactly canceled by the divergent contributions of the counterterms.

In addition to the finite terms already given in the expression for $\Delta_0^Q$, there are several more finite terms that arise from the $d \to 4$ limit of the one-loop diagrams. The first type of contributions we consider are of the form

$$\Delta_I \equiv \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} M^{d-4}(1 + S)^{d/2},$$

and are present in $\Delta_0^{P_k}$, $\Delta_0^{P^\lambda}$ and $\Delta_0^Q$. To extract the finite terms contributing to $\Delta_I$ we express $\Delta_I$ as

$$\Delta_I = \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} M^{d-4}(1 + S)^{2} + \frac{1}{32\pi^2} M^{d-4}(1 + S)^{2} (\frac{1}{2} - \ln(1 + S)) + O(d - 4)$$

$$\Delta_I = -\lambda \left( \frac{M}{\Lambda} \right)^{d-4} (1 + S)^2 + \frac{1}{64\pi^2} M^{d-4}(1 - S)^2 + O(S^3) + O(d - 4). \quad (4.62)$$

The second type of one-loop contributions we have to consider in the limit $d \to 4$ are of the form

$$\Delta_{II} \equiv \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} \frac{d}{8} M^{d-4}(M_1^2 - M_2^2)^2.$$

It occur only in $\Delta_0^Q$. By expanding in powers of $d - 4$ we find that

$$\Delta_{II} = \left[ \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} - \frac{1}{64\pi^2} \right] \frac{1}{2} M^{d-4}(M_1^2 - M_2^2)^2 + O(d - 4)$$

$$\Delta_{II} = -\frac{1}{2} \lambda \left( \frac{M}{\Lambda} \right)^{d-4} (M_1^2 - M_2^2)^2 + O(d - 4). \quad (4.64)$$

Thus, in our subtraction scheme, there are no finite terms resulting from such contributions.

Using (4.62), one easily finds the following finite contributions from the one-loop diagrams to second order in $M_1^2 - M_2^2$ and $M_3^2/M^2 - M^2$,

$$- \frac{2N_f^2 - N_f - 1}{128\pi^2} + \frac{N_f^2 + N_f}{128\pi^2} (M_1^2 - M_2^2)^2 + \frac{2N_f^2 - N_f - 1}{64\pi^2} (M_2^2 + \frac{1}{4} M_3^2 - M^2)^2$$

$$+ \frac{N_f^2}{32\pi^2} (M_1^2 - M_2^2)(M_2^2 + \frac{1}{4} M_3^2 - M^2). \quad (4.65)$$

The second term also includes the finite contribution to $\Delta_0^Q$ given in (4.60).
5 Landau Ginzburg Model

In this section we study the phase transition that separates the normal phase from the phase of condensed diquarks. At the mean field level and for zero diquark source, it was found that there is a second order phase transition at $\mu = M/2$, where $M$ is the pion mass to leading order in chiral perturbation theory. The order parameter of this phase transition is the diquark condensate. To leading order it was found that $\langle \bar{\psi}\psi \rangle = 2N_fG\sin \alpha$, with $\alpha = \arccos(M^2/4\mu^2)$ [2].

Because of the contributions of the $Q$-integrals, the next-to-leading order analytical expressions for the free energy are quite complicated. However, the neighborhood of the critical point can be easily studied by means of a Landau-Ginzburg theory for the order parameter. From the leading order result we find that $\alpha$ is a suitable order parameter near the critical point. This is quite natural since $\alpha$ is the rotation angle of the condensate. Therefore if we expand the free energy in powers of $\alpha$ for $\alpha \ll 1$, we should obtain a Landau-Ginzburg model describing the phase transition. Notice, however, that this Landau-Ginzburg model results from an exact calculation within the effective theory. Furthermore in this approach $\alpha$ has to be an independent variable so that the saddle point equation cannot be used to express $\alpha$ in terms of the pion mass and the chemical potential. We first illustrate the usefulness of the expansion of the free energy in powers of $\alpha$ by analyzing the theory at leading order and then consider the free energy at next-to-leading order.

At the mean field level, the free energy in the diquark condensation phase just above the critical chemical potential, i.e. for $\alpha \ll 1$, is given by

$$\Omega = -N_fF^2\left(2M^2 + 2M^2\phi\alpha + (4\mu^2 - M^2)\alpha^2 - \frac{1}{12}(16\mu^2 - M^2)\alpha^4\right) + O(\alpha^6), \quad (5.66)$$

where the diquark source enters through $\phi = \tan(j/m) \simeq j/m$ for $j \ll m$. This free energy can be studied as a Landau-Ginzburg model. At $\phi = 0$ there is a second-order phase transition where the coefficient of the $\alpha^2$-term vanishes, that is at $\mu = M/2$. For zero diquark source, the order parameter obtained from the minimum of the effective potential is given by

$$\alpha = \sqrt{\frac{6(4\mu^2 - M^2)}{16\mu^2 - M^2}}. \quad (5.67)$$

At the critical point for nonzero diquark source we find the usual mean field value of the critical exponent with $\alpha$ given by

$$\alpha^3 = 2\phi, \quad (5.68)$$
Notice that this relation implies that $\phi \ll \alpha \ll 1$. Therefore the diquark condensate near the critical point is given by

$$
\langle \psi \psi \rangle = -\frac{\partial \Omega}{\partial j}\bigg|_{j=0} = -\frac{1}{m} \frac{\partial \Omega}{\partial \phi}\bigg|_{\phi=0} = 2N_f G \alpha
$$

where we have used that to leading order $mG = M^2 F^2$. To leading order in a $(\mu - M/2)$ expansion this result agrees with the exact mean field theory result.

At next-to-leading order the situation is more complicated. We study the free energy for a chemical potential near the leading-order critical point and define $\bar{\mu}$ by $\mu = M/2 + \bar{\mu}$. The leading order results provides us with power counting rules for $\alpha$, $\bar{\mu}$ and $\phi$ near the critical chemical potential: if $\alpha \sim \epsilon$, then $\bar{\mu} \sim \epsilon^2$ and $\phi \sim \epsilon^3$. To obtain an expansion to fourth order in $\epsilon$, we use

$$
\frac{M^2 - M_2^2}{M^2} = \frac{4\mu^2 \sin^2 \alpha}{M^2} = \alpha^2 - \frac{1}{3}\alpha^4 + 4\alpha^2 \bar{\mu} + \cdots,
$$

$$
\frac{M_2^2 + \frac{1}{4}M_3^2}{M^2} - 1 = \cos(\alpha - \phi) - 1 = -\frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4 + \phi \alpha + \cdots.
$$

To this order the free energy is given by

$$
\frac{\Omega}{-2N_f F^2 M^2} = 1 + a_0 M^2 \frac{F^2}{F^2} + \left(1 + a_1 M^2 \frac{F^2}{F^2}\right) \alpha \phi + a_2 M^2 \frac{F^2}{F^2} \alpha^2
$$

$$
+ \left(2 + a_3 M^2 \frac{F^2}{F^2}\right) \bar{\mu} \alpha^2 + \left(-\frac{1}{8} + a_4 M^2 \frac{F^2}{F^2}\right) \alpha^4,
$$

where the coefficients of the next to leading order corrections are given by

$$
a_0 = 8N_f L_6^r + 2L_8^r + H_2^r + \frac{2N_f^2 - N_f - 1}{256\pi^2 N_f} (1 - 2 \ln \frac{M^2}{\Lambda^2}),
$$

$$
a_1 = 8(2N_f L_6^r + L_8^r) - \frac{2N_f^2 - N_f - 1}{64\pi^2 N_f} \ln \frac{M^2}{\Lambda^2},
$$

$$
a_2 = 2(2N_f L_4^r + L_5^r - 4N_f L_6^r - 2L_8^r) - \frac{N_f + 1}{128\pi^2 N_f} \ln \frac{M^2}{\Lambda^2},
$$

$$
a_3 = 8(2N_f L_4^r + L_5^r) - \frac{N_f}{16\pi^2} \ln \frac{M^2}{\Lambda^2},
$$

$$
a_4 = \frac{5}{3}(2N_f L_4^r + L_5^r) + \frac{4}{3}(2N_f L_6^r + L_8^r) + (L_6^r + 2N_f L_1^r + 2N_f L_2^r + L_3^r)
$$

$$
- \frac{N_f - 1}{512\pi^2 N_f} + \frac{N_f + 1}{384\pi^2 N_f} \ln \frac{M^2}{\Lambda^2}.
$$

(5.72)
This free energy is typical of a system exhibiting a second order phase transition.

For zero diquark source, the phase transition occurs when the coefficient of the \( \alpha^2 \)-term in the free energy vanishes. Therefore we find that there is a second order phase transition at

\[
\bar{\mu}_c = -\frac{1}{2} a_2 \frac{M^2}{F^2}
\]

\[
= \frac{M^2}{F^2} \left[-2N_f L_4^r - L_5^r + 4N_f L_6^r + 2L_8^r + \frac{N_f + 1}{256\pi^2 N_f} \ln \frac{M^2}{\Lambda^2} \right] + O\left(\frac{M^4}{F^4}\right). \tag{5.73}
\]

We compare this result to the pion mass at zero chemical potential at next to leading order in chiral perturbation theory (see Appendix A),

\[
m_\pi^2 = M^2 \left(1 + \frac{M^2}{F^2} \left[4(-2N_f L_4^r - L_5^r + 4N_f L_6^r + 2L_8^r) + \frac{N_f + 1}{64\pi^2 N_f} \ln \frac{M^2}{\Lambda^2} \right] \right). \tag{5.74}
\]

Therefore, the critical chemical potential is given by

\[
\mu_c = M/2 + M\bar{\mu}_c = m_\pi/2. \tag{5.75}
\]

For a nonzero diquark source we find that at the critical point

\[
\alpha^3 = 2\phi \left(1 + a_1 + 8a_4\right) \left(\frac{M^2}{F^2}\right)
\]

\[
= 2\phi \left(1 + \frac{M^2}{F^2} \left[-\frac{40}{3}(2N_f L_4^r + L_5^r) + \frac{56}{3}(2N_f L_6^r + L_8^r) \right.ight.
\]

\[
\left. + 8(L_5^r + 2N_f L_4^r + 2N_f L_2^r + L_3^r) - \frac{N_f - 1}{64\pi^2 N_f} - \frac{6N_f - 7N_f}{192\pi^2 N_f} \ln \frac{M^2}{\Lambda^2} \right) \right) \tag{5.76}
\]

There are no logarithmic corrections in \( \phi \) or \( \alpha \) so that the critical exponent remains at its mean field value. This is somewhat surprising. For zero diquark source, half of the \( Q^- \)-modes are true massless Goldstone modes. They result from the breaking of the remaining symmetries of the microscopic theory at nonzero chemical potential and nonzero quark mass by the diquark condensate \[2\]. For nonzero diquark source, these modes become massive. They are pseudo-Goldstone modes, with a square mass equal to \( M^2\phi(3\alpha + 8\phi)/32 \) at leading order at \( \mu = M/2 \) \[2.22\]. Usually loop-graphs with pseudo-Goldstone modes produce logarithmic terms of the type found for instance in the free energy of the normal phase \( (4.57) \). In the superfluid phase, we would have therefore expected one-loop corrections to the free energy of the form \( \alpha\phi \ln(\phi) \). These terms would produce a deviation from mean-field critical indices in \( (5.76) \). However, we do not find
any logarithmic term neither in $\phi$ nor in $\alpha$. This is a non-trivial result, and one should expect that to one-loop order the critical exponents of the microscopic theory are given by mean field theory.

For zero diquark source, the value of the order parameter $\alpha$ is given by the minimum of the free energy. It reads

\[
\alpha^2 = 4 \left( 2\bar{\mu} + \frac{M^2}{F^2}(a_2 + a_3\mu) \right) \left( 1 + 8\frac{M^2}{F^2}a_4 \right)
\]

\[
= \frac{8}{m_\pi} \frac{\mu - m_\pi}{2} \left( 1 + \frac{m_\pi^2}{F^2} \left\{ -\frac{34}{3}(2N_fL_4^r + L_5^r) + \frac{44}{3}(2N_fL_6^r + L_8^r) \right. \\
+ 8(L_0^r + 2N_fL_1^r + 2N_fL_2^r + L_3^r) - \frac{N_f - 1}{64\pi^2N_f} - \frac{12N_f^2 - 11N_f - 11}{384\pi^2N_f} \ln \frac{M^2}{\Lambda^2} \right\} \right)
\]

(5.77)

Therefore we find that the order parameter $\alpha$ vanishes for $\mu \leq m_\pi/2$ and that it increases continuously for higher chemical potential. The phase transition is thus second order.

\section{Number Density and Condensates}

The phase structure of QCD with two colors at nonzero baryon chemical potential can be characterized by three important observables: the quark number density, the chiral condensate and the diquark condensate. We compute them at zero diquark source. The number density is obtained from the free energy as follows

\[
\frac{d}{d\mu} = -\frac{1}{M} \frac{\partial \Omega}{\partial \bar{\mu}}
\]

\[
= 4N_fMF^2 \left( 1 + \frac{1}{2a_3} \frac{M^2}{F^2} \right) \alpha^2
\]

\[
= 4N_fMF^2 \left( \frac{M^2}{F^2} \left\{ 4(2N_fL_4^r + L_5^r) - \frac{N_f - 1}{32\pi^2N_f} \ln \frac{M^2}{\Lambda^2} \right\} \right)
\]

\[
= 4N_fMF^2 \pi^2 \alpha^2.
\]

(6.78)

The chiral condensate is given by

\[
\langle \bar{\psi}\psi \rangle = -\frac{d}{dm} = -\frac{G}{F^2} \left( \frac{\partial \Omega}{\partial M^2} + \frac{d\bar{\mu}}{dM^2} \frac{\partial \Omega}{\partial \bar{\mu}} \right) = -\frac{G}{F^2} \left( \frac{\partial \Omega}{\partial M^2} - \frac{1 + 2\bar{\mu}}{4M^2} \frac{\partial \Omega}{\partial \bar{\mu}} \right)
\]

\[
= 2N_f \left( 1 - \frac{1}{2} \alpha^2 + \frac{M^2}{F^2} \left[ 2a_0 - \frac{2N_f^2 - N_f - 1}{128\pi^2N_f} + \left( 2a_2 - \frac{1}{4}a_3 - \frac{N_f + 1}{128\pi^2N_f} \right) \alpha^2 \right] \right)
\]
\[ \langle \bar{\psi} \psi \rangle_0 \left( 1 - \frac{1}{2} \alpha^2 + \frac{M^2}{F^2} \left[ 4N_f L_6^\ast + 2L_8^\ast - 8N_f L_6^r - 6L_8^r + H_2^r \right. \right. \]
\[ \left. \left. - \frac{N_f + 1}{128\pi^2 N_f} - \frac{N_f + 1}{128\pi^2 N_f} \ln \frac{M^2}{\Lambda^2} \right] \alpha^2 \right). \]

Finally the diquark condensate is given by

\[ \langle \bar{\psi} \psi \rangle = \langle \bar{\psi} \psi \rangle_0 \left( 1 - \frac{1}{2} \alpha^2 + \frac{M^2}{F^2} \right) \].

The baryon number density and the diquark condensate vanish at \( \mu = m_\pi/2 \). They increase continuously for \( \mu \geq m_\pi/2 \), whereas the quark-antiquark condensate diminishes. At leading order it was found that in the superfluid phase \( \langle \bar{\psi} \psi \rangle^2 + \langle \bar{\psi} \psi \rangle^2 = \langle \bar{\psi} \psi \rangle_0^2 \), where \( \langle\bar{\psi} \psi\rangle_0 \) is the quark-antiquark condensate in the normal phase. This relation does not hold at next-to-leading order. Other than that, the qualitative behavior of these observables is the same at leading order and at next-to-leading order.

7 Conclusions

As is the case for QCD with three colors, the low-energy sector of QCD with two colors and quarks in the fundamental representation is a theory of weakly interacting Goldstone bosons. What distinguishes QCD with two colors from QCD with three colors is that we have both mesonic and baryonic Goldstone bosons (also known as diquarks). Therefore, the phase transition to a phase of condensed diquarks can be described entirely by means of a low-energy effective theory which is completely determined by the symmetries of the microscopic theory. In earlier work, in which this theory was analyzed at the mean field level, it was found that such phase transition takes place at a baryonic chemical potential equal to half the mass of the Goldstone bosons. Above this transition point the chiral condensates rotates into a diquark condensate with a rotation angle that is determined by the chemical potential. Meanwhile, this result has been observed in lattice QCD simulations.
In this article we have analyzed the one-loop corrections of the leading order effective Lagrangian as well as the tree-graph contributions of the next-to-leading order terms in the effective Lagrangian. We have shown that the theory is renormalizable, i.e. all infinities generated by the one-loop diagrams can be absorbed into a background field independent redefinition of the coupling constants of the next-to-leading order terms in the effective Lagrangian. We have derived the counter-terms for a general background field, and by an explicit calculation, we have verified that they cancel the one-loop divergences both in the normal phase and in the phase of condensed diquarks.

The one-loop corrections of the effective theory do not qualitatively change the predictions obtained from the mean field analysis. In particular we have obtained the physically satisfying result that a second order phase transition to a phase of condensed diquarks takes place at a chemical potential equal to half the one-loop renormalized pion mass. From the effective theory, we have derived a Landau-Ginsburg theory for this phase transition with the rotation angle of the chiral condensate as order parameter. The next-to-leading order corrections do not qualitatively change the mean field coefficients. The normal phase is characterized by a nonzero quark-antiquark condensate, a zero diquark condensate and a zero baryon number density. A second order phase transition at \( \mu = m_\pi/2 \) separate the normal phase from the diquark superfluid phase, where the quark-antiquark condensate, the diquark condensate and the baryon number density are nonzero. However, they do affect the magnitude of chiral condensate and the diquark condensate in a different way so that they are no longer related by the tangent of the rotation angle.

At the phase transition point, half of the baryonic mesons become massless. The Goldstone bosons associated with the formation of a diquark condensate remain massless above the phase transition point. At the one-loop level these massless modes do not produce any infrared singularities. Neither do we find any chiral logarithms related the modes that become massive above the critical chemical potential. A nonzero diquark source does not lead to chiral logarithms either. These results corroborate with the absence of one-loop corrections to the mean field critical exponents. The only nonanalytic behavior in the free energy that we find is of the form \( \alpha^4 \sqrt{\alpha^2 - 2\bar{\mu}} \) (with \( \alpha \) the rotation angle of the chiral condensate and \( \bar{\mu} \) the deviation from the critical chemical potential). This statement should also hold at the two-loop level. The analytic structure of the propagator near the critical point suggests that critical exponents remain at their mean field value to all (finite) orders in chiral perturbation theory.

Our results have been derived for QCD with two colors and quark in the fundamental representation. Very similar low-energy effective theories have been derived for QCD with quarks in the adjoint representation and for QCD with three or more colors and fundamental quarks but with the fermion determinant replaced by its absolute value. The latter theory can be interpreted as QCD at nonzero isospin chemical potential and can be generalized to a theory with a chemical potential for each quark flavor again resulting in a similar low-energy effective theory. A closely related effective theory has been derived in condensed matter physics for disordered systems with an imaginary vector
potential. We expect that our conclusions for QCD with two colors will also be valid for each of these theories. In particular we expect the next-to-leading order corrections do not qualitatively change the mean field behavior.

All our results have been derived for zero temperature. As should be the case for a renormalizable theory, no additional ultraviolet divergencies appear at nonzero temperature. Just above the critical chemical potential we expect a second order phase transition to the normal phase at a critical temperature that is in the domain of validity of our chiral Lagrangian. We thus are able to obtain exact results for the critical temperature within the framework of chiral perturbation theory. These results as well as other properties of the diquark phase at nonzero temperature will be discussed in a forthcoming publication.

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Appendix A: One-loop expressions for $m_\pi$ and $F_\pi$

In QCD with three colors and quarks in the fundamental representation, the usual way to compute the pion mass and the pion decay constant is to extract them from the axial two-point correlation function.

For the pion mass, the Feynman diagrams of fig. 2 have to be evaluated.

![Feynman diagrams](image)

Figure 2: Feynman diagrams that enter into the axial-vector two-point correlation function that contribute to the pion mass at next-to-leading order. The dot denotes a vertex from $\mathcal{L}^{(2)}$, and the boxed 4 a vertex from $\mathcal{L}^{(4)}$.

The pion mass is given by the pole of this two-point function in momentum space. However, as will be explained next, we do not need an explicit calculation of this two-point function. The key observation is that the only finite one-loop contributions originate from the integral in the second diagram. At zero chemical potential, all the Goldstone modes
have the same mass. Therefore the one-loop diagram is proportional to

\[
\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2} = -\frac{d}{2} M^{d-2} \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} = 2M^2 \lambda \left( \frac{M}{\Lambda} \right)^{d-4}. \tag{A.81}
\]

Because our theory is renormalizable, in our subtraction scheme divergent contributions of the type given in the last line are cancelled by contributions from the counter terms. The pion mass is thus completely determined by the tree graphs of \( L^{(2)} \) and \( L^{(4)} \).

The correction to the pion mass from the tree-graphs \( L^{(4)} \) is given by twice the coefficient of the quadratic term in the pion fields in \( L^{(4)} \). It is essential to impose the Euclidean on-shell condition \( p^2 = -M^2 \) in the \( O(p^4) \) terms of the effective Lagrangian [20]. For example,

\[
\text{Tr} [\partial_\nu \Sigma \partial_\nu \Sigma^\dagger] = -M^2 \pi^a \pi^a + \ldots \tag{A.82}
\]

The on-shell condition results in a contribution from the \( L_4 \) and \( L_5 \) terms in \( L^{(4)} \) even though \( \mu = 0 \). Calculating all traces in \( L^{(4)} \) to order \( \Pi^2 \) we find

\[
L^{(4)} = \ldots + \frac{2M^4}{F^2} \left[ -2N_f L_4 - L_5 + 4N_f L_6 + 2L_8 \right] \pi^a \pi^a + \ldots \tag{A.83}
\]

The pion mass is therefore given by

\[
m^2_\pi = M^2 \left( 1 + \frac{4M^2}{F^2} \left[ -2N_f L_4 - L_5 + 4N_f L_6 + 2L_8 \right] \right) + \text{tadpoles}. \tag{A.84}
\]

As argued before, our subtraction scheme is such that the tadpoles and the divergent counter terms in the combination (3.49) cancel. We thus find the renormalized pion mass

\[
m^2_\pi = M^2 \left( 1 + \frac{M^2}{F^2} \left[ 4(-2N_f L_4^r - L_5^r + 4N_f L_6^r + 2L_8^r) + \frac{N_f + 1}{64\pi^2 N_f} \ln \left( \frac{M^2}{\Lambda^2} \right) \right] \right). \tag{A.85}
\]

with renormalized coupling constants defined in (3.48).

We can proceed in the same way for the pion decay constant. The contributions from the tree diagrams of \( L^{(2)} \) and \( L^{(4)} \) are easy to obtain. They are given by the coefficient of the \( \partial_\mu \pi^a \partial_\nu \pi^a \)-term in the effective Lagrangian. The contribution from the one-loop diagrams are again proportional to (A.81). We can therefore apply the same method as the one we used to determine the pion mass. We find that the pion decay constant at next-to-leading order reads

\[
F^2_\pi = F^2 \left( 1 + \frac{M^2}{F^2} \left[ 4(2N_f L_4^r + L_5^r) - \frac{N_f}{32\pi^2} \ln \left( \frac{M^2}{\Lambda^2} \right) \right] \right). \tag{A.86}
\]
Appendix B: Q-Modes in the Diquark Condensation Phase

In this appendix we evaluate the one-loop contribution of the $Q$-modes to the free energy. We thus analyze the following integral

$$\Delta_0^Q = - \int \frac{d^d p}{(2\pi)^d} \ln \left( (p^2 + M_1^2)(p^2 + M_2^2) + p_0^2 M_3^2 \right),$$

where $M_1^2$, $M_2^2$, and $M_3^2$ are defined in (2.23). The pole part of this integral at $d = 4$ can be easily obtained by expanding the logarithm in inverse powers of the momentum,

$$\ln \left( (p^2 + M_1^2)(p^2 + M_2^2) + p_0^2 M_3^2 \right) = 2 \ln p^2 + \ln \left( 1 + \frac{M_1^2 + M_2^2}{p^2} + \frac{M_1^2 M_2^2}{p^4} + \frac{p_0^2 M_3^2}{p^4} \right)$$

$$= 2 \ln p^2 + 1 + \frac{M_1^2 + M_2^2}{p^2} + \frac{M_1^2 M_2^2}{p^4} + \frac{p_0^2 M_3^2}{p^4}$$

$$- \frac{1}{2} \frac{(M_1^2 + M_2^2)^2}{p^4} - \frac{1}{2} \frac{p_0^4 M_3^4}{p^8} - \frac{p_0^2 M_3^2 (M_1^2 + M_2^2)}{p^6} + \cdots$$

(B.87)

The higher orders in the expansion in inverse powers of the momentum vanish in dimensional regularization. The pole term in $d - 4$ is thus given by the integral

$$PP(\Delta_0^Q) = - \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty p^{d-5} dp \int_0^\pi d\theta \sin^{d-2} \theta$$

$$\times \left[ M_1^2 M_2^2 - \frac{1}{2} (M_1^2 + M_2^2)^2 - M_3^2 (M_1^2 + M_2^2) \cos^2 \theta - \frac{1}{2} M_3^4 \cos^4 \theta \right]$$

(B.88)

where we have introduced polar coordinates with $p_0 = p \cos \theta$, and the area of the $d$-dimensional unit sphere given by

$$\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma \left( \frac{d-1}{2} \right)}. \quad (B.89)$$

The integrals in (B.88) can be easily calculated using the formulae

$$\int_0^\infty dx \frac{x^\beta}{(x^2 + M^2)^\alpha} = \frac{1}{2} \frac{\Gamma \left( \frac{\beta+1}{2} \right) \Gamma \left( \alpha - \frac{\beta+1}{2} \right)}{\Gamma(\alpha)(M^2)^{\alpha-(\beta+1)/2}}$$

(B.90)

$$\int_0^{\pi/2} d\theta \sin^\alpha \theta \cos^\beta \theta = \frac{\Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\beta}{2} \right)}{2\Gamma \left( 1 + \frac{\alpha+\beta}{2} \right)}.$$  

(B.91)
For the $p$-integral one can derive the limiting relation for $d \to 4$, 

$$ PP \left( \int_0^\infty p^{d-5} dp \right) = \Gamma\left( -\frac{d}{2} \right). \quad (B.92) $$

The final result for the pole part of $\Delta_0^Q$ is thus given by 

$$ PP(\Delta_0^Q) = \frac{\Gamma\left( -\frac{d}{2} \right)}{(4\pi)^{d/2}} \left( M_1^4 + M_2^4 + \frac{1}{2} (M_1^2 + M_2^2) M_3^2 + \frac{1}{8} M_3^4 \right). \quad (B.93) $$

The finite contributions to $\Delta_0^Q$ can only be obtained close to the transition point ($\alpha \to 0$, $\mu \to M/2$ and $j \to 0$). As starting point the integral is rewritten as 

$$ \Delta_0^Q = - \int \frac{d^4p}{(2\pi)^d} \ln \left( (p^2 + \frac{M_1^2 + M_2^2}{2})^2 + p_0^2 M_3^2 - \frac{(M_1^2 - M_2^2)^2}{4} \right). $$

We wish to compute the $Q$-mode contribution to the free energy up to $O(\alpha^6)$ and to first order in the diquark source $j$, i.e. $\phi$. To this order the integral is given by 

$$ \Delta_0^Q = - \int \frac{d^4p}{(2\pi)^d} \ln \left( (p^2 + \frac{M_1^2 + M_2^2}{2})^2 + p_0^2 M_3^2 \right) + \int \frac{d^4p}{(2\pi)^d} \frac{(M_1^2 - M_2^2)^2}{4[(p^2 + \frac{M_1^2 + M_2^2}{2})^2 + p_0^2 M_3^2]}, \quad (B.94) $$

The first integral can be simplified by shifting the $p_0$-integration. Introducing $d$-dimensional polar coordinates as in the calculation of the pole part we then find to order $\alpha^6$, 

$$ \Delta_0^Q = \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dp \int_0^\pi d\theta \sin^{d-2}\theta \left[ -2 \ln(p^2 + \frac{1}{2}(M_1^2 + M_2^2) + \frac{1}{4} M_3^2) + \frac{(M_1^2 - M_2^2)^2}{4[(p^2 + \frac{1}{2}(M_1^2 + M_2^2))^2 + p_0^2 M_3^2]} \right] + O((M_1^2 - M_2^2)^4), \quad (B.95) $$

$$ = \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dp \int_0^\pi d\theta \sin^{d-2}\theta \left[ -2 \ln(p^2 + \frac{1}{2}(M_1^2 + M_2^2) + \frac{1}{4} M_3^2) + \frac{(M_1^2 - M_2^2)^2}{4p^2[p^2 + M_1^2 + M_2^2 + \cos^2\theta M_3^2]} \right] + O((M_1^2 - M_2^2)^4) + O((M_1^4 - M_2^4)^2) $$

The final result is 

$$ \Delta_0^Q = 2 \frac{\Gamma\left( -\frac{d}{2} \right)}{(4\pi)^{d/2}} \left[ \frac{1}{2} (M_1^2 + M_2^2) + \frac{1}{4} M_3^2 \right]^{d/2} + \frac{\Gamma\left( -\frac{d}{2} \right) d}{(4\pi)^{d/2}} M^{d-4}(M_1^2 - M_2^2)^2 $$

26
\begin{align}
  &+ \frac{(M_1^2 - M_2^2)^2}{32\pi^2} \left[ \frac{1}{2} - \ln \sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2 + M_3^2}} \right] \\
  &+ \frac{M_1^2 + M_2^2}{M_3^2} - \sqrt{M_1^2 + M_2^2 + \sqrt{M_1^2 + M_2^2}} \sqrt{M_1^2 + M_2^2} \\
  &+ O((M_1^2 - M_2^2)^4) + O((M_1^4 - M_2^4)^2) + O(d - 4) \tag{B.96}
\end{align}

This result correctly reproduces the pole part of $\Delta_Q^0$ which was obtained at the beginning of this section.

References

[1] J.B. Kogut, M.A. Stephanov, and D. Toublan, Phys. Lett. B 464 (1999) 183-191.
[2] J.B. Kogut, M.A. Stephanov, D. Toublan, J.J.M. Verbaarschot, and A. Zhitnitsky, Nucl. Phys. B 582 (2000) 477.
[3] E. Dagotto, F. Karsch, and A. Moreo, Phys. Lett. B 169 (1986) 421; E. Dagotto, A. Moreo, and U. Wolff, Phys. Rev. Lett. 57 (1986) 1292; Phys. Lett. B 186 (1987) 395.
[4] S. Hands, J. B. Kogut, M. Lombardo and S. E. Morrison, Nucl. Phys. B 558 (1999) 327; S. Hands and S.E. Morrison, [hep-lat/9902012, hep-lat/9905021].
[5] S. Hands, I. Montvay, S. Morrison, M. Oevers, L. Scorzato, and J. Skullerud, Eur. Phys. J. C 17 (2000) 285-302; R. Aloisio, V. Azcoiti, G. Di Carlo, A. Galante, A.F. Grillo, [hep-lat/0007018] and Phys. Lett. B 493 (2000) 189-196; Y. Liu, O. Miyamura, A. Nakamura, and T. Takaishi, [hep-lat/0009009]; S.J. Hands, J.B. Kogut, S.E. Morrison, and D.K. Sinclair, [hep-lat/0010028]; J. B. Kogut, D. Toublan and D. K. Sinclair, [hep-lat/0104010]; B. Alles, M. D’Elia, M. P. Lombardo and M. Pepe, Nucl. Phys. Proc. Suppl. 94 (2001) 441; E. Bittner, M. Lombardo, H. Markum and R. Pullirsch, Nucl. Phys. Proc. Suppl. 94 (2001) 445; B. A. Berg, E. Bittner, H. Markum, R. Pullirsch, M. P. Lombardo and T. Wettig, [hep-lat/0007008]; E. Bittner, M. Lombardo, H. Markum and R. Pullirsch, [hep-ph/0009192]; J. B. Kogut, D. K. Sinclair, S. J. Hands and S. E. Morrison, [hep-lat/0105026].
[6] M.A. Halasz, J.C. Osborn and J.J.M. Verbaarschot, Phys. Rev. D56 (1997) 7059.
[7] D. Toublan and J. J. Verbaarschot, Int. J. Mod. Phys. B 15 (2001) 1404.
[8] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 86 (2001) 592.
[9] K. Splittorff, D. T. Son and M. A. Stephanov, Phys. Rev. D 64 (2001) 016003.

[10] J. B. Kogut and D. Toublan, Phys. Rev. D 64 (2001) 034007.

[11] Y.V. Fyodorov, B.A. Khoruzenko and H.-J. Sommers, Phys. Lett. A226 (1997) 46; Y.V. Fyodorov, M. Titov and H.-J. Sommers, Phys. Rev. E58 (1998) 1195.

[12] K.B. Efetov, Phys. Rev. Lett. 79 (1997) 491; Phys. Rev. B56 (1996) 9630; A.V. Kolesnikov and K.B. Efetov, Waves in Random Media 9 (1999) 71.

[13] G. Akemann, hep-th/0106053.

[14] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. B540 (1999) 317.

[15] P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. B547 (1999) 305.

[16] M. E. Berbenni-Bitsch, M. Gockeler, H. Hehl, S. Meyer, P. E. Rakow, A. Schafer and T. Wettig, Phys. Lett. B 466 (1999) 293.

[17] P. H. Damgaard and K. Splittorff, Nucl. Phys. B 572 (2000) 478.

[18] K. Takahashi and S. Iida, Nucl. Phys. B573 (2000) 685.

[19] T. Guhr and T. Wilke, Nucl. Phys. B 593 (2001) 361.

[20] P. H. Damgaard, hep-lat/0105010.

[21] B. Vanderheyden and A. D. Jackson, hep-ph/0102064.

[22] A. Smilga and J. J. Verbaarschot, Phys. Rev. D 51 (1995) 829.

[23] D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. B560 (1999) 259.

[24] W. Pauli, Nuovo Cimento 6 (1957) 205 ; Gürsey, Nuovo Cimento 7 (1958) 411.

[25] R. F. Alvarez-Estrada and A. Gomez Nicola, Phys. Lett. B 355 (1995) 288 [Erratum-ibid. B 380 (1995) 491].

[26] J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142; Nucl. Phys. B 250 (1985) 465.

[27] J. Bijnens, G. Colangelo and G. Ecker, Annals Phys. 280 (2000) 100.

[28] M.J.G. Veltman, Diagrammatica, Cambridge Lecture Notes in Physics 4 (1994).

[29] Z. Duan, P.S. Rodrigues da Silva and F. Sannino, Nucl. Phys. B 592 (2001) 371.

[30] J.T. Lenaghan, F. Sannino, and K. Splittorff, hep-ph/0107099.