A statistical mechanics approach to the factorization problem

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We map the problem of finding the prime factors of an integer to the statistical physics problem of finding the ground state of a long-range Ising-like model. As in the strongly disordered Newman-Stein (NS) spin-glass model, the bond distribution is exponentially wide and grows with system size, but unlike the NS model we find that it is not wide enough for a greedy algorithm to be applicable.

On the other hand, we also find that the frustration and exponential width of the bond distribution renders classical and quantum annealing and tempering methods no faster than a random search for this challenging model.

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I. INTRODUCTION

Since the security of public-key cryptography depends on the presumed difficulty of computing the factors of large integers, there is very strong incentive to find efficient factorization algorithms. Shor’s quantum algorithm\textsuperscript{1,2} which can find the factors in a time that scales polynomially with the system size has motivated a massive research effort into quantum computing. While there are no known classical algorithms that scale polynomially, the state-of-the-art general number field sieve scales sub-exponentially.\textsuperscript{3,4} The time required to factor a composite integer \( q \) scales as \( \exp[(\log \log q)^3(\log \log q)^3] \), and this has enabled factoring of, for example, the RSA-200 (200 digits, 663 bits) challenge in 2005.

Our aim in this paper is to view the problem from a statistical physics point of view. We show that the problem of finding two prime factors of a large composite integer can be transformed to an optimization or statistical mechanics problem in which Ising spins are used to represent the numbers in binary format. The resulting expression can be treated as a system of interacting two-state spins, similar to an Ising model in statistical physics with frustrated long-range multi-spin interactions. The model can be formulated so that the desired prime factors are given by the lowest energy configuration.\textsuperscript{5,6}

For statistical mechanics problems without frustration and with moderate or no disorder, a number of very efficient Monte Carlo algorithms have been developed. On the other hand, certain special classes of disordered optimization problems, such as the minimum spanning tree problem,\textsuperscript{7} are exactly solved by simple "greedy" algorithms, where locally optimal choices lead to a global solution. Somewhat paradoxically, a broader class of problems is approximately soluble by greedy algorithms in the limit of extreme disorder. For example, the resistance of a random resistor network\textsuperscript{7} can be solved to a good approximation by selecting the single cheapest path that percolates across the cluster. This is asymptotically exact in the limit of infinite disorder. Finding the ground state of certain models of classical spin glasses\textsuperscript{8,9} are also minimum spanning tree problems in the limit of very broad disorder. For quantum spin problems, asymptotically exact renormalization group methods based on analogous ideas have been developed\textsuperscript{10,11}. Here the pair of spins with the single strongest bond is integrated out which perturbatively modifies the remaining bonds. As long as the bond distribution continues to broaden under repetition of this renormalization procedure, the system flows to an infinite-disorder fixed point and this treatment is asymptotically exact.

As we will see below, the factoring problem presents several extreme difficulties. First we require the exact ground state. Nearby solutions are of no use. Second, the problem contains multi-spin interactions and the bond strengths, while not truly random as in a spin-glass model, are frustrated and very broadly distributed, but not so broadly that a greedy algorithm is even approximately applicable. Third, the energy landscape is extremely rough and there is no concept of nearness in spin space. Two nearly identical spin configurations can have vastly different energies and conversely two very different configurations can have nearly identical energies.

Our main result is the formulation of the problem in a spin language which sheds light on why this is such a difficult optimization problem. We explore a number of numerical methods for attacking this model, including greedy algorithms, single and multi-spin updates, parallel tempering and quantum annealing. However we find that all exhibit exponential cost and, being essentially no faster than a random search, are completely defeated by this difficult model.

II. THE ISING MODEL

In this section we discuss the form of the two-state model representing the factorization problem. Assume that we want to find two (unknown) prime factors \( p_1 \) and \( p_2 \) of a composite integer \( q \). We can write the factors in binary form, \( p_1 = \sum_{i=0}^{n-1} 2^i S_i \) and \( p_2 = \sum_{j=0}^{n-1} 2^j S'_j \), where the spin variables \( S \in \{0,1\} \). Considering two \( n\)-
spin factors, the factorization problem, \( q = p_1 p_2 \), can be written
\[
q = \sum_{i,j=0}^{n-1} 2^{i+j} S_i^1 S_j^2,
\]
(1)

The right hand side is of the same form as a long-range Ising model, \( H = \sum_{i,j} J_{i,j} S_i^1 S_j^2 \), with the exponential coupling constant \( J_{i,j} = 2^{i+j} \). Note that only spins belonging to separate integers interact. We can now use Eq. (1) to construct a cost function that can be used as an effective Hamiltonian in a Monte Carlo simulation. The ground state should be given by the desired prime factors and first we consider
\[
H_m = \frac{1}{m} \left| q - \sum_{i,j=1}^{n} 2^{i+j} S_i^1 S_j^2 \right|^m,
\]
(2)

where \( m \) is a positive number. The ground state is doubly degenerate due to a simple permutation of \( p_1 \) and \( p_2 \). We will consider the cases \( m = 1/2, 1, 2 \). The case \( m = 2 \) has the simplest interpretation as a statistical mechanics problem since it corresponds to a model with two- and four-spin interactions. A similar model has been studied recently in the context of a quantum adiabatic algorithm and a branch and bond optimization method\textsuperscript{57}.

Notice for this problem that the bond strengths are powers of two and therefore cover an enormous range. On the other hand, we also note that the strongest bond approximately equals the sum of the other bonds, \( \sum_{i=1}^{n-1} 2^i = 2^n - 1 \), a limit to which we will return in the section on greedy algorithms. Flipping all other spins in an integer can therefore approximately cancel the effect of flipping the highest spin. As a simple example consider the drastic effect of "carrying" in the following addition \( 01111111 + 1 = 10000000 \), which clearly demonstrates the lack of a simple connection between Hamming distance in spin space and the distance in energy.

Introducing a fictitious temperature \( T = 1/\beta \) the thermal expectation value of the energy is given by \( \langle H \rangle = \sum_{i} E_i e^{-\beta E_i} / \sum_{i} e^{-\beta E_i} \), where the sum is over all states, and \( E_i \) is the energy of state \( i \). It is worth noting that the factorization problem differs from the spin glass problem in that we know the lowest energy eigenvalue, namely zero, and we need to find only the lowest energy eigenstate. During a stochastic simulation one can therefore immediately interrupt the execution of the program when the true ground state is found, which makes the factorization problem an ideal testing case for ground state algorithms. We have generated several instances of the factorization problem for increasing system size, and in Table I we list the series of composite integers, along with the prime factors, used in this work.

| \( n \) | \( p_1 \) | \( p_2 \) | \( q \) |
|---|---|---|---|
| 10 | 601 | 911 | 547511 |
| 12 | 2081 | 3329 | 692764 |
| 14 | 10007 | 15091 | 151015637 |
| 16 | 40093 | 60013 | 2406101209 |
| 18 | 150011 | 140007 | 36003690077 |
| 20 | 700057 | 900001 | 630052000057 |
| 22 | 2500339 | 3500227 | 8751754076953 |
| 24 | 11600489 | 14000083 | 162407880840587 |
| 26 | 4161528 | 61614749 | 2564187087815599 |
| 28 | 150243361 | 220293523 | 33090127134000803 |
| 30 | 800000087 | 90000083 | 72000144700007221 |

### III. GREEDY ALGORITHMS

The Edward-Anderson model, \( H = \sum_{i<j} J_{i,j} S_i S_j \), is the archetypical Ising spin glass model. The coupling parameters \( J_{i,j} \) are quenched random variables, for example Gaussian distributed with mean zero. The combination of disorder and frustration makes spin glass models very challenging to solve, and there is no known general solution. However, in the limit of very broadly distributed coupling constants \( J_{i,j} \), finding the ground state of this model maps to a minimum spanning tree problem, solvable by a greedy algorithm\textsuperscript{58}. The criterion is that the magnitude of each coupling is greater than the absolute sum of all smaller couplings, which, as a consequence, means that the width of the distribution increases with system size. This may, at first, seem to hold also for the factorization model, Eq. (1), since \( \sum_{i=0}^{n-1} 2^i = 2^n - 1 < 2^n \). However, there are two caveats. First, all spin pairs for which \( i + j = k \) share the same coupling constant \( J_k \), so there are many coupling constants with the same magnitude. Second, when the model is formulated so that the prime factors are given by the ground state, like \( H_2 \), the resulting model contains multi-spin interactions.

Next we consider the limit of an even broader bond distribution and show that it leads to important simplifications also for the factorization model. As mentioned above there are, in general, many couplings of a given magnitude \( J_k \). They all contribute to the \( k \)-th digit of the composite integer \( q \). In addition a carry bit must be added. If we can avoid the carry bit the problem simplifies in that the bonds of magnitude \( J_k \) directly determine the \( k \)-th digit of \( q \). This means that we can, in principle, satisfy the bonds (and corresponding digits in \( q \)) in decreasing order without running the risk of weaker bonds upsetting already satisfied stronger bonds.

As an illustration, we consider prime factors of the form \( p = \sum_{i=0}^{n-1} k^i S_i \), where \( S \in \{0, 1\} \), but \( k > 2 \). If \( k \) is a no carry operation is needed, and here we consider the case of \( k = 10 \). Examples of such prime factors include (in base 10 representation): 11, 101, 1011, 101111, 1011001, 1100101. The simplest case is \( 11 \times 101 = 1111 \). Using long multiplication we work out the factors simul-
taneously from the high (since there are no carry bits) and low end:

\[
\begin{array}{cccc}
\times & c & b & a \\
& f & e & d \\
& cd & be & ad \\
& ce & bd & a \\
& cf & be & af \\
\end{array}
\]

At the high end \( cf = 0 \) and \( ce + bf = 1 \) imply that \( c=1, f=0 \) (or vice versa) and \( e=1 \). At the low end \( ad = 1 \) implies that \( a = d = 1 \). Both remaining conditions yield \( b=1 \) rendering the factors \( 11 \) and \( 101 \). Solving the problem decade by decade is therefore much simpler in the limit of a bond distribution broad enough to prevent the carry operation. Thus, while the distribution of bond strengths in the full factorization problem is exponentially broad, it is not broad enough to allow us to apply a greedy algorithm in which we satisfy the bonds one at a time.

IV. STOCHASTIC ALGORITHMS

Stochastic Monte Carlo algorithms have been used to calculate thermal expectation values, as well as ground state properties, of statistical models. Depending on the model, and method, large systems can often be studied to high precision. Using cluster algorithms the critical temperature and other properties for three-dimensional Ising-like models have been determined up to 6 decimals by performing simulations of systems containing \( 128^3 \) spins.\textsuperscript{12} and the ground state energy of the two-dimensional Heisenberg model has been determined to 5 decimals using quantum Monte Carlo methods.\textsuperscript{13,14} There are also many examples in which the use of efficient algorithms can change the functional dependence of the scaling with respect to computational effort. The use of cluster updates at the critical point for the Ising model in effect eliminates the problem of critical slowing down, and the computational effort decreases from \( L^{d+2} \) for single-spin updates to \( L^d \) for cluster updates, where \( d \) is the dimensionality of the model and \( L \) is the linear system size.\textsuperscript{15} In some cases use of the loop algorithms can make the exponentially difficult sign problem tractable in polynomial time.\textsuperscript{16,17}

This suggests that Monte Carlo methods developed for statistical physics problems could be useful in analyzing the factorization problem, since finding factors containing a few hundred bits is considered a hard problem.\textsuperscript{18} However, the Ising model that arises from the factorization problem has a very complex energy landscape, with multiple local minima separated by very high barriers. Factorization is but one of many such difficult optimization problems whose solution requires finding a global minimum in a very complex landscape. Other examples include various aspects of circuit design in electronics,\textsuperscript{24} protein folding in life science,\textsuperscript{19} spin-glass behavior in materials science,\textsuperscript{20} and the traveling salesman problem in computer science and mathematics. In an attempt to alleviate the problems associated with the complicated energy landscape the methods of thermal annealing and parallel tempering have been developed. A fictitious or real temperature is introduced and as this parameter is lowered, the system settles in a local minimum. If the cooling is sufficiently slow the ground state is found.\textsuperscript{21} However, for many complex systems it is practically impossible to reduce the temperature slowly enough.

Classical simulated annealing relies on thermal fluctuations to find the ground state. If the energy barriers separating different minima are high and narrow, quantum tunneling can be a more efficient way to equilibrate. Quantum mechanical systems are able to tunnel through barriers, instead of traversing the barriers. In quantum annealing the minimum is found using the quantum mechanical tunneling effect instead of thermal fluctuations. The efficiency of quantum annealing has been studied both experimentally and computationally, in which case quantum annealing can be realized by introducing off-diagonal terms into the classical model, which cause tunneling between the diagonal, classical states.

Next we consider a number of different approaches to a Monte Carlo simulation of the factorization problem in order of increasing complexity.

A. Random search

The simplest stochastic method for solving the factorization problem is to generate random integers \( p \) and interrupt the search when \( q \equiv p \mod 0 \). The probability of finding a factor is \( \mu = 2/2^n \) for a single attempt (since there are two factors). Since the probability is constant for each attempt the number of trials before success, \( P(N) \), is Poisson distributed, \( P(N) = \mu \exp(-\mu N) \), with an average of \( N = \mu^{-1} = 2^{n-1} \).

We use the random search as a point of reference and next we consider Monte Carlo techniques using importance sampling. If the weight function for a problem is fairly smooth and dominated by a few pronounced maxima, then it may be possible to generate states distributed according to the weight function, with a substantial gain in performance. However, the weight function for the factorization problem is very complex and the potential gain less certain. To demonstrate the complicated structure of the factorization problem we plot the function \( q \mod p \) for the case of \( q = 547511 = 601 \times 911 \) in Fig. [1]. We note an average linear increase of \( q \mod p \), but otherwise there is no apparent ordered structure. In the next sections we investigate whether importance sampling can nevertheless still be used to improve convergence to the ground state.
FIG. 1: The remainder after dividing $q=547511$ by $p$. The remainder is equal to zero for $p=601$ and 911.

**B. Simple spin flips and local temperature**

Using the Boltzmann weight for state $i$, $W_i = \exp(-\beta E_i)$, we implement a single-spin flip Metropolis algorithm for the model defined by Eq. (2). As a single spin flip is attempted the new weight function $W'$ is calculated, and the spin flip is accepted with probability $p = \max(W'/W, 1) = \max(\exp(-\beta \Delta E), 1)$, where $\Delta E = E' - E$. To investigate the performance of the Metropolis algorithm we start with two randomly chosen trial factors $p_1$ and $p_2$. During the simulation we choose spins at random and attempt to flip single spins with the above probability. When one of the factors is found the execution is halted, and the number of attempts, $N$ is recorded. Note that all the states visited during the simulation are counted, independently of whether the state is accepted or not. This is repeated until a reliable average, $\bar{N}$ can be calculated (typically one thousand runs). We restrict the search to odd factors, by locking the lowest spin in the 1-state. This is to prevent one of the factors becoming zero, in which case the energy is independent of the value of the other factor. In Fig. 2 we display the results obtained for some of the integers listed in Tab. I.

For each system size there is a minimum at an intermediate temperature, and as the temperature is lowered the number of attempts increases dramatically, while it levels out at as the temperature is increased. This behavior is understood considering Fig. 3, where the acceptance probability for the different spins are displayed for the five temperatures recorded in Fig. 2 for $n=22$. At the highest temperature the acceptance rate approaches unity for all the spins. This means that the spins are freely fluctuating, and the results approach the random search described above (consecutive configurations are still correlated, so the random search is faster than the high temperature limit of the Metropolis algorithm). As the temperature is lowered, the high spins feel the effect first. For the nearest neighbor Ising model the change in energy, $\Delta E = E' - E$ is of the order $J$, the uniform coupling constant, for all spins considered. However, for the model described by Eq. (2), the change in energy is of order $2^j \times p_2$ when an attempt is made to flip the $j$:th spin in $p_1$. As the temperature is lowered, the probability of flipping the high spins decreases quickly. The minimum in Fig. 2 occurs when the probability of flipping the highest spin is about 0.05, just before it freezes. Lowering the temperature further causes the number of attempts to increase dramatically.

This indicates that one could adjust the model, given by Eq. (2) so that the probability distribution for accepting a spin flip is more even. Any alterations are allowed as long as the ground state is unchanged, and the aim is to decrease the level spacing at higher energies. There-
Therefore we consider the following forms of the Hamiltonian,
\[
H_\frac{1}{2} = \left| q - \sum_{i,j=1}^{n} 2^{i+j} S_i^1 S_j^2 \right|^2,
\]
and
\[
H_{\ln} = \ln \left( \left| q - \sum_{i,j=1}^{n} 2^{i+j} S_i^1 S_j^2 \right| + 1 \right).
\]
Both the square root and the logarithm are monotonically increasing functions that do not change the order of the states. The logarithm function, in particular, approximately cancels the exponential factor $2^i \times p_2$ and allows all the spins to be updated in a more even fashion.

Yet another way to increase the fluctuations of the higher spins is to introduce a site-dependent temperature, $T_i$. A higher temperature at the higher spins ensures more even fluctuations. We therefore define a local temperature $T_i = T \times k^i$, with a parameter $k$ whose value can vary from 1 (no change) to 2 (cancels the exponential factor $2^i \times p_2$). In Fig. 4, we compare the efficiency of the different approaches. We display the number of states visited before a prime factor is found as a function of system size. The temperature is adjusted for each data point to optimize the search. All methods are compared to the random search, for which $N = 2^{n-2}$, since we restrict the search to odd integers, and there are two distinct prime factors. We notice that all methods represent slight improvements over the random search, but no model works better than $H_1$ (absolute value), which scales like $0.58 \times 2^{n-2}$ and requires about half the number of visited states compared to a random search.

In Fig. 5, we compare the acceptance rates for the different methods. The acceptance rate for $H_1$ (absolute value) and $H_\frac{1}{2}$ (square root) quickly saturate to unity, while $H_{\ln}$ (logarithm) and a local temperature scaling like $T_i = T \times 1.5^i$ result in a much more even acceptance rate. The results clearly show that obtaining a more even acceptance rate does not necessarily speed up convergence to the ground state.

C. Parallel tempering

In the previous section the behavior of the model was investigated while the temperature was held constant. More efficient algorithms for converging to the ground state rely on a temperature that evolves during the simulation. The main methods are annealing, where the temperature is decreased as the simulation progresses, and tempering methods, where the temperature fluctuates between a maximum and a minimum during the simulation. Next we implement a parallel tempering method for the factorization problem. In the factorization problem, the energy scale and the position of the spins are closely tied together. Once the energy has been lowered sufficiently, the higher spins are entirely frozen, and if they are not in the correct positions the ground state cannot be found. Compared to annealing methods the tempering method offers the advantage that the system can return to higher temperatures and explore several local minima.

In the method of parallel tempering several copies, or replicas, of the system are simulated concurrently. Each replica is initially assigned a temperature, $T_i$, and after performing a number of Monte Carlo updates at the assigned temperatures attempts are made to swap nearby temperatures with a probability $P(T_i, T_j) = \exp(E_i - E_j)/(T_i - T_j)$, which preserves detailed balance. The
attempted temperature swaps are repeated at regular intervals, and in this manner the temperature of a given replica varies during the simulation. The method has been very successful in equilibrating disordered spin-glass systems at low temperatures as well as studying phase transitions. Given the high energy barriers between low lying states of the factorization model one could expect parallel tempering to allow the replicas to transverse the barriers and not get stuck in a given local minimum so easily.

We have implemented a parallel tempering algorithm for the factorization problem, based on the Metropolis algorithm of $H_1$. One attempt is made, on average, to flip every spin in all the replicas, and thereafter an attempt is made to switch all neighboring temperatures. The maximum temperature is set to where the Metropolis acceptance rate for flipping the highest spin is about 0.9, and the lowest temperature when the acceptance rate for the lowest spin is about 0.1. In Fig. 6 we show the temperature fluctuations for a single replica of a 40-spin factorization problem with $p_1 = 1000000000003$, $p_2 = 5000000000023$, $q = p_1 * p_2 = 500000000003800000000069$. The temperature varies from a maximum of $T_{\text{max}} = 1 \times 10^{24}$ to a minimum of $T_{\text{min}} = 1 \times 10^{11}$. In between there are 40 temperatures that ensure that the swap rates remain above 0.5. As can be seen from the figure the temperature of the replica wanders repeatedly back and forth between the highest and lowest temperatures.

![Graph showing temperature fluctuations for replica during parallel tempering](image)

FIG. 6: The temperature fluctuations for replica during parallel tempering

However, counting the total number of attempted spin updates in all the replicas until a prime factor is encountered we find that the method is not more efficient than the Metropolis algorithm considered in the last section. This indicates that although the tempering method improves equilibration at low and finite temperatures, it is not, in this case, superior in picking out the ground state itself. There are many low-lying energy states, and finding precisely the right one is a difficult problem.

### D. Classical SSE cluster update

So far we have discussed single spin flips, but one goal of this investigation is to determine whether cluster updates, which have proved highly useful for many difficult problems, can be of use for the factorization problem. Since flipping the $j$:th spin in factor $p_1$ changes the energy by $2^j \times p_2$ one could, in principle, offset the large energy change by simultaneously flipping several spins. Cluster updates for long-range classical models have been developed within the framework of the Swendsen-Wang update\cite{35} and the stochastic series expansion\cite{36}. However, the factorization models lacks the up-down symmetry of the standard Ising model, which cluster updates usually rely on. If the spin variables assume the values $\pm 1$ the product $S_i S_j$ is unchanged if both spins are flipped. This is not the case for the factorization model of Eq. (2) since $S$ takes the values 0 and 1. By a transformation $S' = 2S - 1$ we can introduce a new variable $S'$ that takes the values $\pm 1$, but this introduces single spin operators $S'_i$ in Eq. (2), which also destroy the up-down symmetry. Therefore we have not been able to introduce a large-scale cluster update, but instead we implement a “small-cluster” update within the SSE method, which we describe next.

The SSE method is based on a Taylor expansion of the partition function $Z$,

$$Z = \sum_{\alpha} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \langle \alpha | H^n | \alpha \rangle,$$

where $|\alpha\rangle$ is a complete set of basis states. The SSE method has been applied to Ising model with arbitrary interactions, and we refer to Ref. 34 for a detailed description. Here we only describe modifications that arise when applying the method to the factorization model.

Expressing the multiplication of two integers in binary form we obtain the model

$$H = \sum_{i,j=1}^{n} J_{i,j} S_i S_j,$$

with $J_{i,j} = 2^{i+j}$. Defining the bond operator $H_{i,j} = J_{i,j} (1 - S_i S_j)$ this can be written as

$$H = \sum_{i,j=1}^{n} -H_{i,j} + \sum_{i,j=1}^{n} J_{i,j}.$$

Including additional unit operators $I$, the Taylor expansion can be written as

$$Z = \frac{1}{L!} \sum_{\alpha} \sum_{n=0}^{\infty} \sum_{S_L} \frac{\beta^n (L-n)!}{L!} \langle \alpha | S_L | \alpha \rangle,$$

where we have introduced a cut-off at $n = L$, and $S_L$ is the operator string $S_L = \prod_{p=1}^{L} H_p$ with $H_p \in \{H_{i,j}, I\}$.
The matrix element in Eq. (7) can be written as a product of elements of the form
\[
\langle S^z_i S^z_j | H_{i,j} | S^z_i S^z_j \rangle,
\]
and for the model we consider only the matrix elements
\[
\langle 00 | H_{i,j} | 00 \rangle = \langle 01 | H_{i,j} | 01 \rangle = \langle 10 | H_{i,j} | 10 \rangle = J_{ij}
\]
contribute since \(\langle 11 | H_{i,j} | 11 \rangle = 0\).

In order to sample the configuration space of all operator sequences \(S_L\) and all states \(|\alpha\rangle\), two types of updates are necessary. The first update changes the number of non-identity operators in the sequence by attempting to exchange identity and bond operators. The probability of exchanging a unit operator for a bond operator is
\[
P_{\text{bond}} = \frac{\beta \sum_{i,j} J_{ij}}{L - n + \beta \sum_{i,j} J_{ij}}
\]
and the reverse probability of exchanging a bond operator for a unit operator is
\[
P_{\text{unit}} = \frac{L - n + 1}{L - n + 1 + \beta \sum_{i,j} J_{ij}}
\]
If a bond operator is to be inserted, the particular bond is chosen according to the relative weight \(J_{ij}\) of the bond, as described in Ref. [34].

The update described above only changes the operator sequence \(S_L\), and the state \(|\alpha\rangle\) is not affected. The classical cluster update described in Ref. [34] flips all the spins that are interconnected by bond operators; since the Ising operators depend on the relative orientation of the spins this does not change the weight and is always allowed. For the factorization model flipping two spins in the state \(|0\rangle\) connected by a bond leads to the forbidden \(\langle 11 | H_{i,j} | 11 \rangle = 0\) vertex. Hence a different update is needed, and so we implement a "small-cluster" move.

In the small-cluster move a spin is chosen at random and an attempt is made to flip it. First we consider the case of a 1-spin. The spin in question may be connected to other spins (in the other integer) through bond operators. Let us denote these spins nearest-neighbor (nn) spins. The nn-spins, in turn, may be connected to spins in the same integer as the spin we are attempting to flip. We call these spins next nearest neighbor (nnn) spins. An attempt to flip a 1-spin is always accepted, but in order to satisfy detailed balance with the reverse update, to be described below, we also need to consider all nn spins. If there is a nn spin connected to nnn spins, all of which take the value 0, then these nn spins are assigned values 0 and 1 with equal probability as the original spin is flipped.

The reverse move is flipping a 0-spin to a 1-spin. This move is accepted with probability \(2^{-nn_0}\), where \(nn_0\) is the number of nn spins connected to nn spins, all of which necessarily assume the value 0. If the move is accepted all the nn spins are set to 0. The factor \(2^{-nn_0}\) corresponds to the number of ways the nn spins can be assigned states 0 and 1 in the above reverse move and ensures that detailed balance is satisfied. An illustration of this move (with \(nn_0 = 1\)) is shown in Fig. 7. The advantage of this update, compared to a single-spin flip, is that it allows 0-spins to be updated even though they may be connected to 1-spins.

The two updates together ensure ergodicity, satisfy detailed balance and demonstrate that it is possible to use an update procedure that flips more than one spin at a time. However, it only works for the model Eq. (5), which does not have the prescribed integer as a ground state. One way to still use the small-cluster update is to allow only updates that do not lead to a state with an energy lower than the prescribed composite integer \(q\). This we have implemented, but unfortunately the scaling, once again, is not better than simple Metropolis updates.

E. Transverse Field

The above algorithms are based on thermal fluctuations. However, it is also possible to modify the model and introduce quantum mechanical terms. Since quantum mechanical systems are able to penetrate through energy barriers, instead of going over energy barriers this method can, in certain cases be superior. We have therefore added a a transverse magnetic field to the model,
\[
H_3 = \left| p - \sum_{i,j} 2^{i+j} S_{1,i} S_{2,j} \right| + h \sum_i (S_i^+ + S_i^-),
\]
in order to compare the efficiency of the quantum and thermal fluctuations. First we consider a transverse field that is constant in time, and find the temperature and field strength combination that minimizes the number of states visited before finding the ground state. We implement the transverse field using a continuous time
algorithm. Individual spins now fluctuate in imaginary time, $S(t)$, and imaginary-time segments of individual spins can be flipped. However, due to the difficulties explained in the above section we do not use a cluster update, but an imaginary-time spin segment $\{t_0, t_1\}$ is flipped with probability $\exp(\int_{t_0}^{t_1} dt \Delta E(t))$.

In Fig. 8 we show the effect of increasing the transverse field for the $N = 14$ spin system. Interestingly it appears that, in this case, the quantum fluctuations are not more efficient in finding the ground state than thermal fluctuations. The smallest number of steps are found for the case of zero transverse field. We have also used an exponentially increasing field, $h_i = 1.5^i$, which leads to more fluctuations in imaginary time for the higher spins, but we find that this does not improve the scaling.

F. Quantum annealing

While in the last subsection we considered the effect of a time-independent transverse field it is more common to gradually reduce the quantum mechanical terms during the simulation. A commonly used annealing method defines a Hamiltonian

$$H = sH_c + (s - 1)H_q,$$

where $H_c$ is the classical model with the desired ground state. Quantum fluctuations are introduced through $H_q$, which, for Ising systems, usually is a transverse field. The control parameter $s$ is slowly reduced from the initial value $s = 0$ to the final value $s = 1$ as the system evolves according to the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t)|\psi(t)\rangle.$$

The process is performed at a very low temperature, and, if the process is sufficiently slow as to be adiabatic, the system evolves from the ground state of $H_q$ to the ground state of $H_c$, which is the solution to the problem. The time required to find the correct ground state with a significant probability is called the complexity of the problem, and numerical simulations of small systems indicate that the complexity may (in some cases) increase polynomially in system size.

In Fig. 9 we show the complexity, or number of MC steps, necessary to find the correct prime factors using an imaginary time quantum annealing method.

![FIG. 8: The average number of attempts, $\bar{N}$ before finding a prime factor as a function of temperature and transverse field for a 14-spin factor.](image8)

![FIG. 9: The complexity, or number of MC steps, necessary to find the correct prime factors using an imaginary time quantum annealing method.](image9)

![FIG. 10: The 20 lowest energy levels as function of transverse field for an instance of the factorization problem ($q=551$) with two five bit integers.](image10)
could limit the applicability of quantum annealing methods to smaller system sizes. A recent study shows that, for small system sizes, real-time quantum annealing of the factorization model does indeed scale polynomially in system size. In order to find out whether the scaling persists to larger system sizes it would be necessary to study the scaling of the ground state energy gap with system size.\(^{31,32}\) Here we only use imaginary-time dynamics to study larger system sizes. This does not constitute the real time dynamics of the Schrödinger equation, but limited success has nevertheless been reported.\(^{33}\) In order to test this method on the factorization problem we study the model

\[
H = s \left| q - \sum_{i, j=1}^{n} 2^{i+j} S_i^1 S_j^2 \right| + (s-1) \hbar \sum_i \left( S_i^+ + S_i^- \right). \tag{10}
\]

The temperature \(T\) is set so that virtually no spin flips are accepted when \(s=1\), and the strength of the transverse field is set to \(\hbar = 10 T\). We have determined the number of sweeps of the whole lattice (MC steps) necessary to find the correct ground state with a probability of about 30\%, and display the result in Fig. 9. From the figure we see that the system size dependence is still exponential, and it appears that the standard quantum annealing method as applied in an imaginary-time path-integral simulation does not improve the scaling.

V. DISCUSSION AND CONCLUSION

We have developed a statistical mechanics Monte Carlo approach to the factorization problem. The resulting model is highly complex with exponentially large frustrated long-range multi-spin interactions. We found that importance sampling is only very weakly effective in improving the convergence to the ground state. The fastest method we found remains exponential and beats random sampling by only a factor two.

We believe that the challenge to standard statistical methods is threefold. First, as for other difficult optimization problems, there is a complex energy landscape with many low-lying states separated by high energy barriers. In this case the global minimum is required and therefore tempering and annealing methods that have been so successful in determining the low-temperature properties of spin glasses are of only limited use.

Second, unlike in standard short-range spin glass models the energy change resulting from a single spin update has an exponentially broad distribution, implying that most single-spin acceptance rates are either close to unity or close to zero. This gives the importance sampling the character of a random search. Rescaling the temperature and transverse fields for individual spins did not ameliorate this problem. We also found that while the distribution of bond strengths was exponentially broad with a width increasing with system size, it was not broad enough to permit solution via a greedy algorithm.

Third, as a direct consequence of the broadly distributed energy change following a single spin flip it follows that there is no concept of nearness in spin space. Two states with nearly identical spin configurations may have vastly different energies. Standard importance sampling methods are based on small changes in energy, a requirement not easily satisfied for the factorization problem.

The statistical physics model resulting from the integer factorization problem is a good benchmark model for ground state algorithms since the ground state energy is known by construction. We hope that this work may encourage further investigations in the efficiency of possible cluster algorithms and annealing methods for the integer factorization problem. All methods considered in this work require \(O(q^4)\) operations, to factor a composite integer \(q\). Whether it is possible to find a stochastic method that scales with an exponent less than \(\frac{1}{2}\) remains an open question. Another interesting question which we have not addressed is the existence of a finite-temperature spin-glass transition. The factorization model is frustrated and, to some extent, disordered. These are considered two necessary conditions for the existence of a stable spin-glass phase. A freezing of the spins would certainly protect the ground state and further explain the difficulty of solving the factorization problem.

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