Asymptotic behavior of singular values of the acoustic observation problem

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Abstract

We consider the problem of recovering of initial data in the IBVP for the wave-type equation in the half-space by the solution restricted to the boundary. The singular value decomposition of this problem is concerned: the asymptotics of singular values is obtained.

Keywords: observation problem, singular value decomposition, spectral asymptotics.

1 Introduction

Fix an integer \( d \geq 2 \) and put \( \mathbb{R}_d^+ = \{ x = (x_1, \cdots, x_d) \in \mathbb{R}^d | x_d > 0 \} \). Suppose \( q \in C_0^\infty(\mathbb{R}_d^+) \) is a real-valued function. Consider the following initial boundary value problem for scalar function \( u(x,t) \), \((x,t) \in \mathbb{R}_d^+ \times \mathbb{R}\):

\[
\partial_t^2 u - \Delta u + qu = 0, \quad \partial_{x_d} u |_{x_d=0} = 0,
\]

\[
u |_{t=0} = 0, \quad \partial_t u |_{t=0} = v, \tag{1}
\]

where \( v \in C_0^\infty(\mathbb{R}_d^+) \). Introduce the operator

\[
\mathcal{O} : v \mapsto u|_{\Sigma_0},
\]

where \( \Sigma_0 = \partial \mathbb{R}_d^+ \times \mathbb{R} \) (the time-space boundary) and \( u \) is the solution of (1) for the given initial data \( v \). The operator \( \mathcal{O} \) is well defined as the solution \( u \) is regular.

Next we introduce a restriction of the operator \( \mathcal{O} \). Let \( \Omega \subset \mathbb{R}_d^+ \) be a bounded open set that satisfies \( \overline{\Omega} \subset \mathbb{R}_d^+ \). Also let \( \Sigma \) be a bounded relatively open subset of \( \Sigma_0 \), such that \( \overline{\Sigma} \subset \partial \mathbb{R}_d^+ \times (0,\infty) \). Consider the operator \( \mathcal{O}_\Sigma^\Omega : C_0^\infty(\Omega) \rightarrow C^\infty(\Sigma) \) acting as follows

\[
\mathcal{O}_\Omega^\Sigma v = (\mathcal{O} v)|_{\Sigma}.
\]

The operator \( \mathcal{O}_\Omega^\Sigma \) can be continued as compact operator acting from \( L_2(\Omega) \) to \( L_2(\Sigma) \) (see sec. 5).

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We consider the problem of recovering \( v \) by \( O^\Sigma_\Omega v \) (the coefficient \( q \) is given), which we call the observation problem after [1]. Invertibility of \( O^\Sigma_\Omega \) depends on the geometry of \( \Omega \) and \( \Sigma \). An example of the case when \( O^\Sigma_\Omega \) is invertible (more precisely \( \text{Ker}O^\Sigma_\Omega = \{0\} \)) give \( \Sigma = \Gamma \times (0, T) \) and \( \Omega \) satisfying \( \Omega \subset R^d_+ \cap (\cup_{\gamma \in \Gamma} B_\gamma(T)) \). Here \( T > 0 \), \( \Gamma \) is an open bounded subset of \( \partial R^d_+ \), \( B_\gamma(T) \) is an open ball of radius \( T \) centered in \( \gamma \) [3]. Note that similar problems were also considered in [6, 11], where the method based on Carleman estimates was developed. In this paper we are interested in the procedure of recovering \( v \) based on the singular value decomposition (SVD) for \( O^\Sigma_\Omega \). Recall that for a compact linear operator \( A \) acting in Hilbert spaces \( H_0 \to H_1 \) there exist orthonormal basis \( \{v_n\} \subset H_0 \) and \( \{f_n\} \subset H_1 \) such that

\[
Av = \sum_{n \geq 1} s_n(A)(v,v_n)_{H_0}f_n.
\]

Here \( s_n(A) \) are singular values of \( A \) defined as \( s_n(A) = (\lambda_n(A^*A))^{1/2} \), where \( \lambda_n(A^*A) \) are eigenvalues of the compact operator \( A^*A \) being numbered in non-increasing order with the multiplicity taken into account. If \( \text{Ker}A = \{0\} \) and \( f \) belongs to the range of \( A \) then

\[
A^{-1}f = \sum_{n \geq 1} s_n(A)^{-1}(f,f_n)_{H_1}v_n.
\]  

(2)

It can be seen that the stability of inversion (in particular) depends on the behavior of \( s_n(A) \). In this paper we prove that \( s_n(O^\Sigma_\Omega) \) decrease as a negative power of \( n \) (relation (4)). Note that we do not need the invertibility of \( O^\Sigma_\Omega \) and thus no geometric conditions on \( \Omega \) and \( \Sigma \) are imposed.

To study the asymptotics of \( s_n(O^\Sigma_\Omega) \) we approximate the operator \( O^\Sigma_\Omega \) by Fourier integral operators (FIO), which yields approximation of \((O^\Sigma_\Omega)^*O^\Sigma_\Omega\) by pseudodifferential operators (ΨDO) of order \(-2\). Then we apply the result on asymptotics of singular values for non-elliptic ΨDOs of negative order [3]. The theory of FIOs is a natural tool for analysis of operator \( O^\Sigma_\Omega \). Furthermore, FIOs were applied to the problems of integral geometry [3] [8, 9, 15] [17, 20] (the list of references is not complete), which are similar to the problem of inverting \( O^\Sigma_\Omega \).

The principal symbols of ΨDOs that approximate \((O^\Sigma_\Omega)^*O^\Sigma_\Omega\) vanish on the subset of \( T^*\Omega \setminus 0 \) of nonzero measure. If \((O^\Sigma_\Omega)^*O^\Sigma_\Omega\) was precisely an elliptic ΨDO (i.e. operator with nonvanishing principal symbol) then \((O^\Sigma_\Omega)^*O^\Sigma_\Omega)^{-1}\) would be a ΨDO of order \(-2\); if additionally \( \text{Ker}O^\Sigma_\Omega = \{0\} \) then the operator \((O^\Sigma_\Omega)^{-1}\) would be bounded in certain Sobolev spaces since

\[
(O^\Sigma_\Omega)^{-1} = ((O^\Sigma_\Omega)^*O^\Sigma_\Omega)^{-1}(O^\Sigma_\Omega)^*.
\]

In our case the operators that approximate \((O^\Sigma_\Omega)^*O^\Sigma_\Omega\) do not have even parametrices, so the observation problem is ill-posed. The principal symbols of approximating ΨDOs vanish outside of the conic set \( AZ \subset T^*\Omega \setminus 0 \) which is defined in sec. [2] The same situation arises in problems of integral geometry with limited data – see [2] [8] [15] [16] [20]. The conic set \( AZ \) is called an audible zone (or visible zone) there. The results of these papers show that it is possible to reconstruct some part of singularities of unknown function, namely – the intersection of the wavefront set of the function with the audible zone.
This gives a hope that series (2) applied to $O_{\Sigma}^\Omega$ allows to reconstruct $v$ microlocally in the audible zone, the reconstruction being stable in some reasonable sense. However, in the present paper we only obtain the asymptotics of $s_n(O_{\Sigma}^\Omega)$.

Note that for some problems of integral geometry singular values and singular basis (i.e. $v_n$ and $f_n$ in (2)) were found in a nearly explicit form. It concerns some particular (though very important) cases such as limited angle tomography [12, 14] and exterior Radon transform [18]. It seems that there are no such results for $O_{\Sigma}^\Omega$ even for $q = 0$ and some certain $\Omega$ and $\Sigma$.

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2 The audible zone and the asymptotics of $s_n(O_{\Sigma}^\Omega)$

To define the audible zone $AZ$ and formulate the main result we introduce some notation. A point in the plane $\Sigma_0$ will be identified by $(x', t)$, $x' \in \mathbb{R}^{d-1}$, $t \in \mathbb{R}$ (we use identification of $\partial \mathbb{R}^{d+}$ and $\mathbb{R}^{d-1}$). For $x \in \mathbb{R}^d$ we put $x' = (x_1, \ldots, x_{d-1})$. We also use natural identification of cotangent bundle $T^*\Omega$ and $\Omega \times \mathbb{R}^d$. Introduce the following mapping from $T^*\Omega$ to $\Sigma_0$:

$$\gamma(y, \eta) = \left( y' - y_d \frac{\eta'}{\eta_d}, y_d \frac{|\eta|}{|\eta_d|} \right), \quad y \in \Omega, \eta \in T_y^*\Omega$$

(the function $\gamma$ is defined if $\eta_d \neq 0$, i.e. almost everywhere in $T^*\Omega$). Note that the first component of $\gamma(y, \eta)$ is an intersection point of the line $\{y + s\eta, s \in \mathbb{R}\}$ and $\partial \mathbb{R}^d_+$, while the second component is the distance between this intersection point and $y$. The audible zone is defined as follows

$$AZ = \{(y, \eta) \in T^*\Omega \setminus 0 \mid \eta_d \neq 0, \gamma(y, \eta) \in \Sigma \}.$$ 

We say that $K \subset T^*\Omega \setminus 0$ is a conic set if it is invariant with respect to the mapping $(y, \eta) \mapsto (y, r\eta)$ for any $r > 0$. The set $AZ$ is conic, and besides, it is invariant with respect to the mapping $(y, \eta) \mapsto (y, -\eta)$.

Denote by $n(A, \lambda)$ the number of singular values $s_n(A)$ greater than $\lambda > 0$. The main result of the paper is the following Theorem.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^d_+$, $\Sigma \subset \partial \mathbb{R}^d_+ \times (0, \infty)$ be open bounded subsets satisfying $\overline{\Omega} \subset \mathbb{R}^d_+$, $\Sigma \subset \partial \mathbb{R}^d_+ \times (0, \infty)$, $\mu_d(\partial \Sigma) = 0$ ($\mu_d$ is the $d$-dimensional Lebesgue measure on $\Sigma_0$, $\partial \Sigma$ is taken in the topology of $\Sigma_0$). Then

$$\lim_{\lambda \to 0} \lambda^d n(O_{\Sigma}^\Omega, \lambda) = (2\pi)^{-d} \int_{AZ} dy d\eta \theta \left( \frac{1}{|\eta| |\eta_d|} - 1 \right),$$

(3)

$\theta$ is the Heaviside function.
Note that the integral in (3) is finite. Indeed, the Heaviside function vanishes if $\eta$ is large and $|\eta| \geq 1$; if $\eta$ is large and $|\eta| < 1$ then $(y, \eta) \notin AZ$ since $y \in \Omega$ and $\Omega$ is separated from $\partial \mathbb{R}_+^d$. We conclude that the Heaviside function is nonzero only on the bounded set of $(y, \eta)$.

The result (3) implies the following behavior of singular values:

$$s_n(O_{\Sigma\Omega}^\Sigma) \sim \left(\frac{\sigma}{n}\right)^{1/d}, \quad n \to \infty,$$

(4)

where $\sigma$ is the r.h.s. of (3). Note that the asymptotics (4) has sense if $\sigma \neq 0$. Also relation (3) may be written in the equivalent form in terms of $(O_{\Sigma\Omega}^\Sigma)^*O_{\Sigma\Omega}^\Sigma$:

$$\lim_{\lambda \to 0} \lambda^{d/2}n((O_{\Sigma\Omega}^\Sigma)^*O_{\Sigma\Omega}^\Sigma, \lambda) = \sigma.$$

(5)

Note that here $n$ coincides with the counting function of (positive) eigenvalues of $(O_{\Sigma\Omega}^\Sigma)^*O_{\Sigma\Omega}^\Sigma$, i.e. the number of eigenvalues greater than $\lambda$. The relation (5) is a Weyl-type asymptotics for positive operator $(O_{\Sigma\Omega}^\Sigma)^*O_{\Sigma\Omega}^\Sigma$.

In the rest of the paper we prove Theorem 1.

3 Operators $I, I_\pm$

Here we introduce and investigate FIOs connected with the problem (1). Some basic concepts and facts of the theory of FIOs are used (canonical relations, composition of FIOs, principal symbol of composition of FIO with its adjoint) – we send the reader to books [7, 10, 21] for details. First we express $O_{\Sigma\Omega}^\Sigma v$ in terms of the following Cauchy problem for $\tilde{u}(x', t)$ in $\mathbb{R}^d \times \mathbb{R}$ (i.e. in the whole space):

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} + \tilde{q} \tilde{u} = 0,$$

$$\tilde{u}|_{t=0} = 0, \quad \partial_t \tilde{u}|_{t=0} = \tilde{v}.$$

(6)

Here $\tilde{q}(x) = q(x', |x_d|)$ is an even continuation of $q$ to the whole space ($\tilde{q}$ is smooth as $q$ is compactly supported in $\mathbb{R}_+^d$), and $\tilde{v} \in C_0^\infty(\mathbb{R}^d)$. Suppose $v \in C_0^\infty(\mathbb{R}_+^d)$ and take $\tilde{v}$ such that $\tilde{v} = v$ in $\mathbb{R}_+^d$ and $\tilde{v} = 0$ in $\mathbb{R}^d \setminus \mathbb{R}_+^d$. Then the solutions of (1) and (6) are related as follows

$$u(x, t) = \tilde{u}(x', x_d, t) + \tilde{u}(x', -x_d, t), \quad x_d > 0, \quad t \in \mathbb{R}.$$

(7)

Indeed, $\tilde{u}(x', -x_d, t)$ satisfies the wave-type equation in (1) for $x_d > 0$ and has zero Cauchy data for $t = 0$ in $\mathbb{R}_+^d$. The boundary condition for $x_d = 0$ is obviously satisfied. It follows from (7) that

$$O_{\Sigma\Omega}^\Sigma v = 2 \tilde{u}|_{\Sigma_\Omega}.$$

(8)

Next we represent the solution of (6) in terms of FIOs:

$$\tilde{u} = \tilde{I}_+ \tilde{v} + \tilde{I}_- \tilde{v} \text{ (modulo a smoothing operator),}$$

4
\[(\tilde{I}_\pm \tilde{v})(x, t) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} dy d\eta e^{i\tilde{\varphi}_\pm(x, t, y, \eta)} \tilde{a}_\pm(x, t, y, \eta) \tilde{v}(y),\]

where \(\tilde{\varphi}_\pm(x, t, y, \eta) = (x - y)\eta \pm t|\eta|\) is a nondegenerate phase function with phase variable \(\eta\) and \(\tilde{a}_\pm\) is an amplitude of order \(-1\). The amplitude \(\tilde{a}_\pm\) is classical:

\[\tilde{a}_\pm \sim \sum_{j \geq 1} \zeta(|\eta|) \tilde{a}_\pm^{(j)}, \quad (9)\]

\(\zeta\) is a smooth function such that \(\zeta(s) = 1\) for \(s > 2\) and \(\zeta(s) = 0\) for \(s < 1\), functions \(\tilde{a}_\pm^{(j)}\) are homogeneous of degree \(-j\) in \(\eta\). Functions \(\tilde{a}_\pm^{(j)}\) are solutions of certain transport equations \([13]\), but we need only \(\tilde{a}_\pm^{(1)}\):

\[\tilde{a}_\pm^{(1)} = \pm (2i|\eta|)^{-1}. \quad (10)\]

Note that the definition of a FIO in \([21]\) implies the normalization factor \((2\pi)^{-d-1/4}\) rather than \((2\pi)^{-d}\) in formula for \(\tilde{I}_\pm\). Nevertheless our choice will be more convenient in further considerations.

The operators \(\tilde{I}_\pm\) are continuous from \(C^\infty_0(\mathbb{R}^d)\) to \(C^\infty(\mathbb{R}^{d+1})\) (as well as from \(E'(\mathbb{R}^d)\) to \(D'(\mathbb{R}^{d+1})\)). Hence we may consider the restriction \(\tilde{I}_\pm v|\Sigma_0\) for \(\tilde{v} \in C^\infty_0(\mathbb{R}^d)\). The restriction can be written in the form of FIO as well:

\[(\tilde{I}_\pm \tilde{v})(x', t) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} dy d\eta e^{i\varphi_\pm(x', t, y, \eta)} a_\pm(x', t, y, \eta) \tilde{v}(y), \quad (x', t) \in \Sigma_0,\]

where \(\varphi_\pm\) and \(a_\pm\) are just the restrictions of \(\tilde{\varphi}_\pm\) and \(\tilde{a}_\pm\). Note that

\[\varphi_\pm(x', t, y, \eta) = x'\eta' - y\eta \pm t|\eta|\]

is a nondegenerate phase function. The mapping

\[\tilde{v} \mapsto (\tilde{I}_+ \tilde{v} + \tilde{I}_- \tilde{v} - \tilde{u})|\Sigma_0 = \tilde{I}_+ \tilde{v} + \tilde{I}_- \tilde{v} - \tilde{u}|\Sigma_0\]

is smoothing. In view of \([8]\) we obtain the following representation of \(O\):

\[O = 2(\tilde{I}_+ + \tilde{I}_-)|_{C^\infty_0(\mathbb{R}_d^+)} \quad \text{(modulo a smoothing operator)}. \quad (11)\]

Now let \(\chi \in C^\infty_0(\Sigma_0)\) be a real-valued function, and put

\[I_\pm = \chi \tilde{I}_\pm|_{C^\infty_0(\mathbb{R}_d^+)}. \quad (12)\]

We consider \(I_\pm\) as a FIO mapping functions in \(\mathbb{R}^d_+\) to functions in \(\Sigma_0\). The phase function \(\varphi_\pm\) and the amplitude \(\chi a_\pm\) of the operator \(I_\pm\) are functions of \((x', t, y, \eta)\), where \((x', t)\) and \(\eta\) vary over \(\Sigma_0\) and \(\mathbb{R}^d \setminus \{0\}\) correspondingly (just as for \(\tilde{I}_\pm\)), while \(y\) varies over \(\mathbb{R}^d_+\).

For the operator \(I = 2(I_+ + I_-)\) due to \([11]\) and \([12]\) we have

\[\chi O = I \quad \text{(modulo a smoothing operator)}. \quad (13)\]
In sec. 5 some approximation of the characteristic function of the set Σ will be chosen as the multiplier χ in (12). This will provide that the restriction of the operator χO to functions supported in Ω will approximate the operator $O^\Sigma_{\Omega}$, which means that the analogous restriction of $I$ will approximate $O^\Sigma_{\Omega}$.

The canonical relation of $I_\pm$ looks as follows

$$C_\pm = \{(x', t, \partial_{x'}, \varphi_\pm, y, -\partial_y \varphi_\pm) | \partial_\eta \varphi_\pm = 0\} \subset (T^*\Sigma \setminus 0) \times (T^*\mathbb{R}_+^d \setminus 0).$$

Note that $-\partial_y \varphi_\pm = \eta$, hence

$$C_\pm = \{(x', t, \partial_{x'}, \varphi_\pm, y, \eta) | \partial_\eta \varphi_\pm = 0\}$$

and the phase variable η may be treated as an element of $T^*\mathbb{R}_+^d$. Introduce an open conic subset of $T^*\mathbb{R}_+^d \setminus 0$:

$$K = \{(y, \eta) \in T^*\mathbb{R}_+^d \setminus 0 | \eta_d \neq 0\}.$$

Consider the pair of canonical transformations from $K$ to $T^*\Sigma_0 \setminus 0$:

$$(y, \eta) \mapsto (x', t, \xi), \quad (x', t) = \gamma_\pm(y, \eta), \quad \xi' = \eta', \quad \xi_d = \pm |\eta|,$$

where

$$\gamma_\pm(y, \eta) = \left(y' - y_d \frac{\eta'}{\eta_d}, \pm y_d \frac{|\eta|}{\eta_d}\right).$$

Analysis of $\varphi_\pm$ shows that the canonical relation $C_\pm$ is a graph of the mapping (14).

Due to the multiplier χ in the definition (12) we may consider the compositions $I^*_\pm I_\pm$ and $I^*_\mp I_\pm$. Since $C_\pm$ is a canonical graph the composition $I^*_\pm I_\pm$ is a ΨDO (see [21]):

$$(I^*_\pm I_\pm v)(\overline{\gamma}) = (2\pi)^{-d} \int_{\mathbb{R}_+^d \times \mathbb{R}^d} dy d\eta e^{i(y - y_\pm)\eta} b_\pm(\gamma, \eta) v(y).$$

Since $I_\pm$ and $I^*_\pm$ are classical FIOs (i.e. Schwartz kernels can be represented by oscillatory integrals with classical amplitudes) the composition $I^*_\pm I_\pm$ is a classical ΨDO:

$$b_\pm \sim \sum_{j \geq 2} \zeta(|\eta|) b^{(j)}_\pm,$$

where $b^{(j)}_\pm$ are homogeneous of degree $-j$ in η. The adjoint operator $I^*_\pm$ is a FIO with a canonical relation $C^*_\pm$. Composition $I^*_\pm I_\pm$ is a smoothing operator in $\mathbb{R}_+^d$ since $C^*_\pm \circ C_\pm = 0$.

Now we may write

$$I^* I = 4(I^*_+ I_+ + I^*_- I_-) \text{ (modulo a smoothing operator)}$$

and $I^* I$ is a classical ΨDO in $\mathbb{R}_+^d$ of order $-2$ with symbol

$$b = 4(b_+ + b_-).$$
From (13) it follows that
\[ (\chi \mathcal{O})^* \chi \mathcal{O} = I^* I \text{ (modulo a smoothing operator).} \] (17)

We now calculate the principal symbol of the ΨDO \( I^*_\pm I_\pm \), i.e. the function \( b^{(2)}_\pm (\varphi, \eta) \) in (15). We use the formula from [21, sec. 8.6] to express the principal symbol in terms of \( \varphi \pm \) and \( a \pm \). Consider the map
\[ (x', t, y, \eta) \mapsto (y, -\partial_y \varphi_\pm (x', t, y, \eta), \partial_\eta \varphi_\pm (x', t, y, \eta)) \]
and denote by \( \Delta_\pm \) the absolute value of its Jacobian determinant. Then the principal symbol of \( I^*_\pm I_\pm \) in \( y, \eta \in K \) equals
\[ (|a_\pm^{(1)}|^2 \Delta_\pm^{-1})(\gamma_\pm (\varphi, \eta), \varphi, \eta), \]
where \( a_\pm^{(1)} \) is the leading term of the amplitude \( a_\pm \), due to (10) we have \( a_\pm^{(1)} = \pm (2i|\eta|)^{-1} \).

Direct calculation shows that for \( (\varphi, \eta) \in K \) we have \( \Delta_\pm = |\eta_d|/|\eta| \), so
\[ b^{(2)}_\pm (\varphi, \eta) = \frac{\chi(\gamma_\pm (\varphi, \eta))^2}{4|\eta| |\eta_d|}. \]

The principal symbol \( b^{(2)}_\pm \) should be continued with zero to \( (T^* R^d_+ \setminus 0) \setminus K \). Due to (16) for the principal symbol \( b^{(2)} \) in the expansion
\[ b \sim \sum_{j \geq 2} \zeta(|\eta|) b^{(j)} \]
we have
\[ b^{(2)} (\varphi, \eta) = \frac{\chi(\gamma_\pm (\varphi, \eta))^2 + \chi(\gamma_\mp (\varphi, \eta))^2}{|\eta| |\eta_d|}. \] (19)

Further we will apply operators \( \mathcal{O}, I \) to functions in \( L^2(\Omega) \), which is possible since these operators can be expressed in terms of FIOs that act on compactly supported distributions in \( \mathbb{R}^d_+ \) and \( \Omega \) is separated from \( \partial \mathbb{R}^d_+ \).

### 4 Spectral Asymptotics of ΨDOs

We use the following result on spectral asymptotics of a ΨDO with symbol \( \zeta(|\eta|) b^{(m)} (\varphi, \eta) \)
\[ (Bv)(\varphi) = (2\pi)^{-d} \int_{\Omega \times \mathbb{R}^d} \, dyd\eta \, e^{i(\varphi - y)\eta} \zeta(|\eta|) b^{(m)} (\varphi, \eta) v(y), \]
where \( b^{(m)} \) is homogeneous of degree \( -m \) \( (m > 0) \) in \( \eta \) and \( b^{(m)} \in C^\infty(\Omega \times (\mathbb{R}^d \setminus \{0\})) \), function \( \zeta \) is the same as in (9). \( B \) is a compact operator in \( L^2(\Omega) \). The result of [5] claims that
\[ \lim_{\lambda \to 0} \lambda^{d/m} n(B, \lambda) = (2\pi)^{-d} \int_{\Omega \times \mathbb{R}^d} \, dyd\eta \, \theta(|b^{(m)} (\varphi, \eta)| - 1). \] (20)
Note that $B$ is not supposed to be self-adjoint or elliptic in \([5]\). Also note that \([5]\) concerns the more general case when $b^{(m)}$ is anisotropically homogeneous and is only continuous in $\overline{\Omega} \times (\mathbb{R}^d \setminus \{0\})$.

Now let $I_\Omega$ be the restriction of $I$ to $L_2(\Omega)$. The result \((20)\) implies the following asymptotics for $I_\Omega^* I_\Omega$ (the operator $I_\Omega^* I_\Omega$ is compact since it is the restriction of the $\Psi DO$ $I^*I$ of order $-2$ to $L_2(\Omega)$)

$$\lim_{\lambda \to 0} \lambda^{d/2} n(I_\Omega^* I_\Omega, \lambda) = (2\pi)^{-d} \int_{\Omega \times \mathbb{R}^d} d\eta d\zeta \theta(|b^{(2)}(\eta, \zeta)| - 1),$$  \tag{21}

where $b^{(2)}$ is given by \((19)\). To prove \((21)\) we need to show that

$$\lim_{\lambda \to 0} \lambda^{d/2} n(I_\Omega^* I_\Omega, \lambda) = \lim_{\lambda \to 0} \lambda^{d/2} n(B, \lambda),$$  \tag{22}

where $B$ is a $\Psi DO$ in $\Omega$ with symbol $\zeta(|\eta|) b^{(2)}(\eta, \zeta)$. The relation \((22)\) claims merely that terms $b^{(j)}$, $j \geq 3$, in the expansion \((18)\) do not influence the asymptotics of $n(I_\Omega^* I_\Omega, \lambda)$. Although this fact seems to be trivial, the author could not find an appropriate reference, so a short proof is provided here. We establish the estimate

$$\lambda^{d/3} n(I_\Omega^* I_\Omega - B, \lambda) \leq C \quad \forall \lambda > 0,$$  \tag{23}

which means that $n(I_\Omega^* I_\Omega - B, \lambda)$ is estimated by less power of $1/\lambda$ than $(1/\lambda)^{d/2}$ and \((22)\) will then follow (see \([4]\) sec. 11.6).

Choose an open bounded set $\Omega'$ containing $\overline{\Omega}$, such that $\overline{\Omega'} \subset \mathbb{R}^d_+$, $\partial \Omega' \in C^\infty$, and denote by $I_{\Omega'}$ the restriction of $I$ to $L_2(\Omega')$. Let $B'$ be the $\Psi DO$ in $\Omega'$ with symbol $\zeta(|\eta|) b^{(2)}(\eta, \zeta)$. Denote by $P_{\Omega}$ the projector onto $L_2(\Omega)$ acting in $L_2(\Omega')$. We have

$$n(I_\Omega^* I_\Omega - B, \lambda) = n(P_{\Omega}(I_{\Omega'}^* I_{\Omega'} - B')P_{\Omega}, \lambda).$$  \tag{24}

Choose $\rho \in C_0^\infty(\Omega')$, $\rho|_{\Omega} = 1$, and put $F = \rho(I_{\Omega'}^* I_{\Omega'} - B')\rho$. Since

$$P_{\Omega}(I_{\Omega'}^* I_{\Omega'} - B')P_{\Omega} = P_{\Omega}FP_{\Omega}$$

we have

$$n(P_{\Omega}(I_{\Omega'}^* I_{\Omega'} - B')P_{\Omega}, \lambda) \leq n(F, \lambda).$$  \tag{25}

Here we used the following inequality

$$s_n(AL), s_n(LA) \leq \|L\| s_n(A)$$

for a compact operator $A$ and a bounded operator $L$. Put $D = (I - \Delta)^{3/2}$, where $\Delta$ is the self-adjoint Laplace operator in $\Omega'$ with Dirichlet boundary condition. $D$ is a $\Psi DO$ in $\Omega'$ \([19]\). The composition $FD$ is also well-defined as a $\Psi DO$ of order $\leq 0$, hence it is a bounded operator in $L_2(\Omega')$. The operator $D^{-1}$ is also bounded in $L_2(\Omega')$. For $v \in L_2(\Omega')$ we have

$$(F v, F v)_{L_2(\Omega')} = (FD^{-1} v, FD^{-1} v)_{L_2(\Omega')} \leq \|FD\|^2_{L_2(\Omega')} (D^{-2} v, v)_{L_2(\Omega')}.$$
Hence \( F^*F \leq \|FD\|_{L_2(\Omega')}^2 D^{-2} \) and
\[
n(F^*F, \lambda) \leq n(\|FD\|_{L_2(\Omega')}^2 D^{-2}, \lambda) = n(D^{-2}, \lambda/\|FD\|_{L_2(\Omega')}^2).
\]
The well known asymptotics of eigenvalues of \(-\Delta\) yields
\[
n(D^{-2}, \lambda) \leq C\lambda^{-d/6}.
\]
We arrive at
\[
n(F, \lambda) \leq C\lambda^{-d/3}.
\]
In view of (24), (25) this leads to (23).

5 Proof of Theorem 1

In this section we prove the relation (5).

First show that \( O_{\Omega}^\Sigma \) is a compact operator from \( L_2(\Omega) \) to \( L_2(\Sigma) \). Since the operator \( I_{\Omega}^* I_{\Omega} \) is compact (see sec. 4) the operator \( I_{\Omega} \) acting from \( L_2(\Omega) \) to \( L_2(\Sigma_0) \) is also compact. Denote by \( O_{\Omega}^\Sigma \) the restriction of \( O \) to \( L_2(\Omega) \). Now due to (13) the composition \( \chi O_{\Omega}^\Sigma \) is a compact operator, which implies that \( O_{\Omega}^\Sigma \) is also compact since we can choose \( \chi \) such that \( \chi|_{\Sigma} = 1 \).

To prove (5) we will make different choices of function \( \chi \) in the definition (12). First suppose that \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) on \( \Sigma \) and \( \text{supp} \chi \subset \partial \mathbb{R}_+^d \times (0, \infty) \). For \( v \in L_2(\Omega) \) we have
\[
(\chi O_{\Omega}^\Sigma v, \chi O_{\Omega}^\Sigma v)_{L_2(\Sigma)} \leq (\chi O_{\Omega}^\Sigma v, \chi O_{\Omega}^\Sigma v)_{L_2(\Sigma_0)}.
\]
Therefore \( (\chi O_{\Omega}^\Sigma)^* O_{\Omega}^\Sigma \leq (\chi O_{\Omega})^* \chi O_{\Omega} \) and so
\[
n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda) \leq n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda).
\]
It follows that
\[
\lim_{\lambda \to 0} \frac{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)}{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)} \leq \lim_{\lambda \to 0} \frac{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)}{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)}.
\]
Due to (17) the “principal parts” of operators \((\chi O_{\Omega})^* \chi O_{\Omega}\) and \(I_{\Omega}^* I_{\Omega}\) coincide, hence
\[
\lim_{\lambda \to 0} \frac{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)}{\lambda^{d/2} n((\chi O_{\Omega})^* \chi O_{\Omega}, \lambda)} = \lim_{\lambda \to 0} \frac{\lambda^{d/2} n(I_{\Omega}^* I_{\Omega}, \lambda)}{\lambda^{d/2} n(I_{\Omega}^* I_{\Omega}, \lambda)} = \sigma_2,
\]
where \( \sigma_2 \) is the r.h.s. of (21). To prove this one should repeat the proof of (22).

Now put
\[
\kappa(y, \eta) = \chi(\gamma_+(y, \eta))^2 + \chi(\gamma_-(y, \eta))^2, \quad \kappa_\Sigma(y, \eta) = \chi_\Sigma(\gamma(y, \eta)),
\]
\( \chi_\Sigma \) is a characteristic function of \( \Sigma \). We have
\[
\kappa_\Sigma \leq \kappa \leq 1.
\]
Indeed, the first inequality is obvious if \(\gamma(y, \eta) \notin \Sigma\). Suppose \(\gamma(y, \eta) \in \Sigma\) in case \(\eta_d > 0\) and \(\gamma_-(y, \eta) \in \Sigma\) in case \(\eta_d < 0\). Now the inequality \(\kappa_{\Sigma} \leq \kappa\) follows from \(\chi|_{\Sigma} = 1\). The second inequality follows from the fact that conditions \(\gamma_+(y, \eta) \in \text{supp}\ \chi\) and \(\gamma_-(y, \eta) \in \text{supp}\ \chi\) cannot hold simultaneously since \(\text{supp}\ \chi \subset \partial \mathbb{R}^d_+ \times (0, \infty)\).

Due to the first inequality in (29) and formula (19) we have \(\sigma \leq \sigma_2\). Now we need to estimate \(\sigma_2 - \sigma\).

Since \(\overline{\Omega} \subset \mathbb{R}^d\), there exists \(\varepsilon > 0\) such that if \(y \in \Omega\) and \(|\eta_d|/|\eta| < \varepsilon\) then \(\kappa(y, \eta) = 0\). If \(|\eta| > \varepsilon^{-1/2}\) and \(y \in \Omega\) then \(\kappa(y, \eta)/(|\eta| |\eta_d|) < 1\). Indeed, if \(|\eta_d|/|\eta| \geq \varepsilon\) (otherwise \(\kappa(y, \eta) = 0\)) then due to (29)

\[
\frac{\kappa(y, \eta)}{|\eta| |\eta_d|} < \frac{1}{|\eta| |\eta_d|} < 1.
\]

This means that the integrals in (21) and in (3) are taken over the set \(y \in \Omega, |\eta| < \varepsilon^{-1/2}\).

The difference \(\sigma_2 - \sigma\) can be estimated by the measure of the set of \((y, \eta)\) such that

\[
|\eta| < \varepsilon^{-1/2}, \quad \kappa_{\Sigma}(y, \eta) < \kappa(y, \eta).
\]

Due to (29) the second condition implies that \(\kappa_{\Sigma}(y, \eta) = 0\), and so \(\gamma(y, \eta) \notin \Sigma\), which means that neither \(\gamma_+(y, \eta)\) nor \(\gamma_-(y, \eta)\) belongs to \(\Sigma\). However, on the set (30) we have \(\kappa(y, \eta) > 0\), and so \(\gamma_+(y, \eta)\) or \(\gamma_-(y, \eta)\) belongs to \(\text{supp}\ \chi\). We conclude that every \((y, \eta)\) from the set (30) satisfies

\[
|\eta| < \varepsilon^{-1/2}, \quad \gamma_+(y, \eta) \text{ or } \gamma_-(y, \eta) \text{ belongs to } \text{supp}\ \chi \setminus \Sigma.
\]

Now choosing \(\chi\) such that \(\text{supp}\ \chi\) shrinks to \(\overline{\Sigma}\) (note that in this case \(\varepsilon\) does not depend on \(\chi\)) we make the set (31) shrink to the set of \((y, \eta)\) such that \(|\eta| < \varepsilon^{-1/2}\) and \(\gamma_+(y, \eta)\) or \(\gamma_-(y, \eta)\) belongs to \(\partial \Sigma\). This set has zero measure in \(\Omega \times \mathbb{R}^d\) since \(\mu_d(\partial \Sigma) = 0\).

This means that choosing appropriate \(\chi\) we can provide that the difference \(\sigma_2 - \sigma\) is arbitrarily small. Together with (27) and (28) this means that

\[
\lim_{\lambda \to 0} \lambda^{d/2} n((\mathcal{O}_\Omega^\Sigma)^* \mathcal{O}_\Omega^\Sigma, \lambda) \leq \sigma.
\]

Taking \(\chi \in C_0^\infty(\Sigma), 0 \leq \chi \leq 1\), we obtain the inequality reverse to (26), therefore we have the relation (which is a counterpart of (27))

\[
\lim_{\lambda \to 0} \lambda^{d/2} n((\mathcal{O}_\Omega^\Sigma)^* \mathcal{O}_\Omega^\Sigma, \lambda) \geq \lim_{\lambda \to 0} \lambda^{d/2} n((\chi \mathcal{O}_\Omega)^* \chi \mathcal{O}_\Omega, \lambda).
\]

Then arguing the same way as in proof of (32) we obtain that

\[
\lim_{\lambda \to 0} \lambda^{d/2} n((\mathcal{O}_\Omega^\Sigma)^* \mathcal{O}_\Omega^\Sigma, \lambda) \geq \sigma.
\]

Together with (32) this yields (5) and thus Theorem 1 is proved.

Now we expose the reason why we used the smooth multiplier \(\chi\) in the definition (12) of the operator \(I_\Omega\). Instead of this we could try to deal with \(I_\Omega' := 2(I_+ + I_-)\) considered
as an operator from $L_2(\Omega)$ to $L_2(\Sigma)$ (see formula (11)). It seems to be possible to consider $(I_{\Omega}')^*I_{\Omega}'$ as a ΨDO with the principal symbol

$$\frac{\chi_\Sigma(\gamma(\mathbf{y},\eta))}{|\eta||\eta_d|}$$

($\chi_\Sigma$ is the characteristic function of $\Sigma$), which is discontinuous in $\Omega \times (\mathbb{R}^d \setminus \{0\})$. However, in this case the result (20) of [5] can not be applied since the principal symbol is required to be continuous there. Note that classical results on spectral asymptotics of ΨDOs with discontinuous symbols by H. Widom and their improvements also can not be applied to $(I_{\Omega}')^*I_{\Omega}'$ – we do not go into details here.

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