Distributed Regularized Primal-Dual Method: Convergence Analysis and Trade-offs

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Abstract—We study deterministic and stochastic primal-dual subgradient methods for distributed optimization of a separable objective function with global inequality constraints. In both algorithms, the norms of dual variables are controlled by augmenting the corresponding Lagrangian function with smooth regularization. Specifically, for each underlying algorithm we show that as long as its step size satisfies a certain restriction, the norm of dual variables is upper bounded by an expression that is inversely proportional to the regularizer curvature. Based on this result, we establish upper bounds on the subgradients of the Lagrangian function. In the deterministic optimization case, we leverage the bounds on the subgradients to analyze the consensus terms and subsequently establish the convergence rate of the distributed primal-dual algorithm. In the stochastic optimization case, the bounds are further used to derive a high probability bound on the convergence rate via the method of bounded martingale difference. For both algorithms, we exhibit a tension between the convergence rate of the objective value and the decay rate associated with the constraint violation. In particular, we show that improving the convergence rate results in a slower decay rate for the constraint violation bound and vice versa. We verify the convergence of our proposed algorithms numerically for distributed regression with hinge and logistic loss functions over different graph structures.

Index Terms—Primal-dual method, consensus algorithm, randomized algorithm.

I. INTRODUCTION

Network optimization is a framework for distributing the computational complexity of solving an optimization problem among many nodes in a network. In such a framework, each node $i$ in the network is assigned with a local objective function $f_i(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ with the domain $\text{dom}(f_i) \equiv \{x \in \mathbb{R}^n : f_i(x) < \infty\}$. Further, each node coordinates its actions with other nodes by exchanging local information with adjacent nodes in the network. In this paper, we study a distributed primal-dual algorithm to optimize a separable convex objective function subject to a set of global inequality constraints

$$\min_{x \in X} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

subject to: $g(x) \preceq 0,$

where $g(x) \equiv (g_1(x), \cdots, g_m(x))$ and $g_k : \mathbb{R}^d \rightarrow \mathbb{R}$ are convex constraints with $\text{dom}(g_i) \equiv \{x \in \mathbb{R}^n : g_i(x) < \infty\}$, $X \subseteq \cap_{i=1}^{m} \text{dom}(f_i) \cap \cap_{k=1}^{m} \text{dom}(g_k)$ is a non-empty, convex, and closed subset, and $\preceq$ denotes the element-wise inequality. Including some constraints in the explicit form as in Eq. (1b) can simplify the constraint set $X$ which in turn simplifies the projection step of the algorithm. In this paper, we use dual variables to handle the constraints (1b) and we will examine the effect of inequality constraints in Eq. (1b) on the convergence rate of the distributed algorithm.

Distributed optimization has found applications in large-scale problems arising in statistics, machine learning, sensor networks, and related areas [1]. In particular, a prevalent problem in statistical machine learning is to learn and make predictions by minimizing a loss function on a large data set. However, processing data in a centralized fashion on large data sets is often infeasible. An alternative approach is to spread the computation task among many computing nodes, where each node has only access to a subset of data set. Other application domains for distributed optimization include trajectory optimization for formation control of vehicles [2], [3], [4], decentralized control of power systems [5], packet routing [6], and estimation problems in sensor networks [7]. Most of these applications indeed deal with a special case of distributed optimization, namely the consensus problem. Therefore, there is a fairly good understanding of the behavior of the consensus protocols as well as the conditions under which these protocols converge, e.g., see [8], [9].

A. Related Works

There has been a growing interest in developing efficient algorithms for the distributed constrained optimization. In most of existing frameworks, the constraints take an implicit form [10], [11], [12]. For instance, in [10] a dual averaging algorithm is proposed where there is a global constraint set $X$ on agents’ actions, but without explicit inequality constraints. When the constraint set has a further structure that can be written as the intersection of finite simple convex constraints, a distributed random projection algorithm is studied [11], where projections are performed locally by each agent based on the random observations of the local constraint components.

In the case of optimization with coupled linear equality constraints, i.e., when decision variables of agents must jointly satisfy a set of linear equality constraints, penalty and barrier function methods are studied [13]. Moreover, based on a game theoretic argument, the convergence of those methods are established. For distributed optimization with a set of global nonlinear inequality constraints as in this paper, distributed primal-dual (PD) methods are studied in [14], [15]. A variation of this method is also studied in [16] for the case where
each agent has a local inequality constraint set. However, the proposed methods require projection of the dual variables onto a simplex at each algorithm iteration whereas in our proposed method in this paper the projection is onto the non-negative orthant of the Euclidean space. As a result, the projection is greatly simplified in our proposed scheme. More importantly, the projection simplex and the error bound of the PD algorithm in [14], [15], [16] depend on a Slater vector $\hat{x}$ for the inequality constraints, i.e., $g(\hat{x}) < 0$. However, dependency on the Slater vector is unappealing as it can be computationally expensive to find such a vector and it also ties the algorithm performance to the structure of the feasible set. We resolve this issue by regularizing the Lagrangian with a smooth and strongly convex function of dual variables.

B. Our Contributions

We study a distributed primal-dual (PD) subgradient method for optimization over a network of fixed topology subject to a set of inequality constraints. Our study is inspired by the work of [17] where it has been shown that a quadratic regularization of dual variables in a centralized online PD algorithm yields a sublinear ‘regret’ and also achieves a vanishing long-term constraint violation. However, the approach in [17] is not easily applicable to the distributed settings as it does not provide upper bounds on the subgradient of the Lagrangian function. It turns out that the bounds on the subgradients are essential in analyzing the ‘consensus’ terms in the distributed primal-dual method. Moreover, in the distributed stochastic primal-dual method, the bounds on subgradients further play a crucial role in deriving a high probability bound for the convergence rate through the concentration inequalities.

Therefore, herein we take a different approach from the work of [17]. In particular, we establish an upper bound on the norm of dual variables that is related inversely to regularizer curvature. In turn, this upper bound allows us to bound the subgradient of the Lagrangian function. Moreover, we characterize our results for a general form of regularizer subsuming the quadratic regularizer as a special case. Our approach also reveals the tension between the convergence rate of the distributed PD algorithm and the corresponding constraint violation performance.

We summarize our contributions as follows:

- In both the deterministic and stochastic optimizations, we establish an upper bound on the norm of dual variables that is related to the inverse of regularizer curvature.

- We determine the convergence rate in the objective value of the distributed regularized primal-dual method. We also derive asymptotic bounds on the constraint violation.

- We characterize the trade-off between the convergence rates of the objective value and the constraint violation. In particular, we show that that a faster convergence rate in the objective value results in a slower decaying rate of the constraint violation and vice versa.

- We develop a distributed stochastic primal-dual method to reduce the computational complexity of the deterministic optimization method. Specifically, we show that by randomizing the distributed PD algorithm, each node in the network only needs to compute the subgradient of its local objective function and only one constraint at each algorithm iteration.

- We use the method of bounded margingale difference to derive a high probability bound on the convergence rate of the stochastic PD method.

C. Organization

The rest of this paper is organized as follows. In Section [1] we define the problem settings and specify admissible conditions for the regularizer we use. In Section [III] we describe a deterministic PD algorithm and state our main results. In Section [IV] we study a stochastic PD algorithm and study its convergence properties. In Section [V] we compare our convergence results with relevant distributed algorithms for constrained optimization problem. In Section [VI] we verify our theoretical studies with numerical simulations. Lastly, in Section [VII] we discuss our results and conclude the paper.

Notation. For ease of notation, we denote the $l_2$-norm by $\|\cdot\|$. However, we use the standard notation $\|\cdot\|_1$ for the $l_1$-norm. Furthermore, we denote the dual norm by $\|\cdot\|_*$, which is defined as $\|x\|_* ≡ \sup_{\|y\|_1=1} \langle x,y \rangle$. We also use standard asymptotic notation. If $a_n$ and $b_n$ are positive sequences, then $a_n = O(b_n)$ means that $\limsup_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$. Similarly, $a_n = o(b_n)$ denotes $\liminf_{n \rightarrow \infty} \frac{a_n}{b_n} > 0$. When, $a_n = O(b_n)$ and $a_n = O(b_n)$, we write $a_n = \Theta(b_n)$. We denote the vectors by $x = (x_1, x_2, \cdots, x_n)$ and matrices by $X = [X]_{ij}$. For two vectors $x$ and $y$, the vector inequality $a \leq b$ means the element-wise inequality, i.e., $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$. For a scalar $x \in \mathbb{R}$ we define $|x|_+ = \max\{0, x\}$. Lastly, we denote the projection of the vector $x$ onto the closed set $\mathcal{X}$ by $\Pi_{\mathcal{X}}(x) ≡ \arg \min_{y \in \mathcal{X}} \|x - y\|_2$.

II. Preliminaries

We consider a multi-agent optimization problem, consisting of $n$ nodes that exchange information on the edges of the graph $G = (\mathcal{V}, \mathcal{E})$ with a fixed topology, where $\mathcal{V} = \{1, 2, \cdots, n\}$ denotes the set of vertices, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges defined on the vertices. At each iteration $t \in [T] \equiv \{1, 2, \cdots, T\}$ of the distributed algorithm, agent $i \in \mathcal{V}$ takes an action $x_i^t \in \mathcal{X} \subset \mathbb{R}^d$ based on knowledge of a local objective function $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$. We also consider a set of global inequality constraints $g_k(x) \leq 0, k \in [m] \equiv \{1, 2, \cdots, m\}$ on the actions of each agent. The objective of agents is to cooperatively minimize the global loss function $f(x) \equiv \frac{1}{n} \sum_{i=1}^n f_i(x)$ while satisfying the inequality constraints.

More concretely, we study a distributed primal-dual algorithm for the optimization problem in Eqns. (1a)- (1b), where we assume the subspace $\mathcal{X}$ is known at each node of the network and has a finite diameter $R \equiv \sup_{x,y \in \mathcal{X}} \|x - y\|$. Without loss of generality, we further assume that $0 \in \mathcal{X}$. This last requirement is always attainable by a simple translation $\varphi : \mathcal{X} \rightarrow \mathcal{X} + \Delta, x \mapsto x + \Delta$ for some $\Delta \in \mathbb{R}^d$ and optimizing the composite functions $\tilde{f}_i \equiv f_i \circ \varphi^{-1}$ and $\tilde{g}_k \equiv g_k \circ \varphi^{-1}$.

We assume that $f_i : \mathbb{R}^d \rightarrow \mathbb{R}, i \in \mathcal{V}$ and $g_k : \mathbb{R}^d \rightarrow \mathbb{R}, k \in [m]$ are convex functions. Furthermore, we assume that $f_i$ and
and the regularizer’s curvature \( \gamma \).

**Definition 1.** An admissible regularizer \( \psi(\cdot) : \mathbb{R}^m \to \mathbb{R}, t \in [T] \) is characterized by the following three conditions:

(i) \( \psi(\lambda) \geq 0, \psi(0) = 0 \) and \( \langle \nabla \psi(0), \lambda \rangle \geq 0 \) for all \( \lambda \in \mathbb{R}^m_+ \),

(ii) \( \psi(\lambda) \) is \( \eta \)-strongly convex with respect to the induced norm \( \| \cdot \| \),

\[
\psi(\lambda) - \psi(\hat{\lambda}) - \langle \nabla \psi(\hat{\lambda}), \lambda - \hat{\lambda}\rangle \geq \frac{\eta}{2} \| \lambda - \hat{\lambda} \|^2,
\]

for all \( \lambda, \hat{\lambda} \in \mathbb{R}^m_+ \),

(iii) \( \psi(\lambda) \) is \( \gamma \)-smooth function with respect to the induced norm \( \| \cdot \| \),

\[
\psi(\lambda) - \psi(\hat{\lambda}) - \langle \nabla \psi(\hat{\lambda}), \lambda - \hat{\lambda}\rangle \leq \frac{\gamma}{2} \| \lambda - \hat{\lambda} \|^2,
\]

for all \( \lambda, \hat{\lambda} \in \mathbb{R}^m_+ \).

**Definition 2.** The condition number associated with the regularizer \( \psi \) is defined as the ratio of the smoothness constant \( \gamma \) and the regularizer’s curvature \( \eta \), i.e., \( Q_\psi \equiv \gamma/\eta \).

Conditions (i) and (ii) in Definition 1 are standard requirements of a regularizer (e.g., see [10]). Condition (iii), however, is an additional restriction that we put in order to provide an upper bound on the norm of the Lagrangian dual variables \( \| \lambda \| \) (cf. Thm. [2]). It is easy to verify that the squared \( \ell_2 \)-norm regularizer \( \psi(\lambda) = \theta \| \lambda \|_2^2 / 2 \) satisfies the specified conditions with \( \eta = \gamma = \theta \) and is thus admissible. We also note that in the case that the regularizer \( \psi \) is twice continuously differentiable, the condition number \( Q_\psi \) in Definition 1 corresponds to the ratio of the largest and smallest eigenvalues of the Hessian matrix of \( \psi \). For example, for the quadratic function \( \psi(\lambda) = \theta \| \lambda \|_2^2 / 2 \) the condition number is \( Q_\psi = 1 \).

To simplify the analysis, in the following we assume the regularizer satisfies \( \nabla \psi(0) = 0 \). However, the more general case \( \langle \nabla \psi(0), \lambda \rangle \geq 0 \) for \( \lambda \in \mathbb{R}^m_+ \) can be treated similarly.

Based on the definition of the admissible regularizer \( \psi \), we define the augmented Lagrangian as follows

\[
\Omega_i(x, \lambda) \equiv f_i(x) + \langle \lambda, g(x) \rangle - \psi(\lambda),
\]

(5)

Furthermore,

\[
\nabla \lambda \Omega_i(x, \lambda) \equiv \nabla f_i(x) + \langle \lambda, \nabla g(x) \rangle,
\]

(6a)

\[
\nabla x \Omega_i(x, \lambda) \equiv g(x) - \nabla \psi(\lambda).
\]

(6b)

Note that in the case that functions \( f_i \) and \( g_k \) are not differentiable, we use their corresponding subgradients. However, for ease of notation, we use \( \nabla f_i(x) \) and \( \nabla g_k(x) \) to denote both gradient and subgradient when \( f_i \) and \( g_k \) are differentiable and non-differentiable, respectively. In the latter case, we define the set of subgradients of \( f_i \) and \( g_k \) as follows

\[
\partial f_i \equiv \{ \nu \in \mathbb{R}^d : f_i(x) - f_i(x_0) \geq \langle \nu, x - x_0 \rangle, \forall x, x_0 \in \text{dom } f_i \},
\]

\[
\partial g_k \equiv \{ \tilde{\nu} \in \mathbb{R}^d : g_k(x) - g_k(x_0) \geq \langle \tilde{\nu}, x - x_0 \rangle, \forall x, x_0 \in \text{dom } g_k \},
\]

for all \( i \in \mathcal{V} \) and \( k \in [m] \), respectively.

Based on the definition of \( \Omega_i(\cdot, \cdot) \), we solve the regularized min-max problem characterized below

\[
\max_{\lambda \in \mathbb{R}^m_+} \min_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} \Omega_i(x, \lambda),
\]

(7)

To describe the distributed primal-dual algorithm, we consider a weight matrix \( W \equiv [W]_{ij} \) that fulfills the following conditions:

- (Doubly stochastic) The weight matrix is doubly stochastic,

\[
W \times 1_n = 1_n, \quad 1_n^T \times W = 1_n^T,
\]

where \( 1_n \in \mathbb{R}^n \) is the column vector with all elements equal to one.

- (Connectivity) The weight matrix respects the graph topology

\[
W_{ij} > 0 \quad \text{if} \quad (i, j) \in \mathcal{E}
\]

\[
W_{ij} = 0 \quad \text{if} \quad (i, j) \notin \mathcal{E}.
\]

**Remark 1.** For \( n \times n \) doubly stochastic matrices, the singular values can be sorted in a non-increasing fashion \( \sigma_1(W) \geq \sigma_2(W) \geq \cdots \geq \sigma_n(W) \geq 0 \), where \( \sigma_1(W) = 1 \) (cf. [19]). This is due to the fact that for a doubly stochastic matrix \( 1_n \)
Algorithm 1  DISTRIBUTED DETERMINISTIC PRIMAL-DUAL METHOD

1: Initialize: $x_i^0 = 0$, $\lambda_i^0 = 0$, $\forall i \in \mathcal{V}$ and a constant step size $\alpha \in \mathbb{R}_+$.
2: for $t = 0, 1, 2, \ldots, T$ at the $i$-th node do
3: Update the primal and dual variables
\[
\hat{x}_i^t = x_i^t - \alpha \nabla_x \ell_i(x_i^t, \lambda_i^t)
\]
\[
\hat{\lambda}_i^t = \lambda_i^t + \alpha \nabla_\lambda \ell_i(x_i^t, \lambda_i^t).
\]
4: Run the consensus steps
\[
x_i^{t+1} = \Pi_{\mathcal{X}} \left( \sum_{j=1}^n [W]_{ij} \hat{x}_j^t \right),
\]
\[
\lambda_i^{t+1} = \Pi_{\mathbb{R}_+^m} \left( \sum_{j=1}^n [W]_{ij} \hat{\lambda}_j^t \right).
\]
5: end for
6: Output: $\tilde{x}_i^T = \frac{1}{T} \sum_{t=1}^T x_i^t$ for all $i \in \mathcal{V}$.

is both the left and right eigenvector, i.e., $WX_n = \mathbb{I}_n$ and $\mathbb{I}_n^T W = \mathbb{I}_n^T$. Throughout the paper, we refer to $1 - \sigma_2(W)$ as the spectral gap of the matrix $W$.

REMARK 2. The intuition underlying using augmented Lagrangian in Eq. (5) is as follows. Since the vector of Lagrangian dual variables $\lambda_i$ take values from $\mathbb{R}^m_+$, the subgradients defined in Eqs. (6a) and (6b) can be unbounded which imposes a challenge for the convergence analysis of centralized and distributed primal-dual method. By penalizing the lagrangian variables via regularizer, we can guarantee that the norm of dual variables $\lambda_i$ are bounded.

III. DISTRIBUTED DETERMINISTIC PRIMAL-DUAL ALGORITHM

We are now in position to describe the distributed algorithm for solving the regularized min-max formulation in Eq. (7); see Algorithm 1.

As our first result, we prove a theorem that supplies us with an upper bound on the norm of the Lagrangian dual variables:

**Proposition 2.** Under the restriction $0 < \alpha \leq \frac{1}{2Q^2 \eta}$ on the step size of Algorithm 1 the norm of the Lagrangian dual variables $\lambda_i^t$ is bounded by
\[
\|\lambda_i^t\| \leq \frac{2L(1 + R)\sqrt{mn}}{\eta},
\]
for all $t \in [T]$. Specifically, for the choice of $\eta = \frac{2L(1 + R)\sqrt{mn}}{\beta}$, $\beta > 0$ we have
\[
\|\lambda_i^t\| \leq \beta.
\]

The proof of Proposition 2 can be found in Appendix B. Proposition 2 highlights the role that the regularizer $\psi$ plays in the augmented Lagrangian $\ell_i(\cdot, \cdot)$. Specifically, the curvature $\eta$ of regularizer $\psi$ provides a degree of freedom to control the norm of the dual variable $\lambda_i^t$ in the primal dual method. The upper bound in Eq. (8) is also intuitive. As $\eta$ becomes larger, the cost associated with choosing a large lagrangian dual variable $\lambda_i^t$ increases which results in a smaller norm $\|\lambda_i^t\|$.

We now use the result of Theorem 1 to compute upper bounds on the norm of subgradients of $\ell_i(x_i^t, \lambda_i^t)$.

**Corollary 3.** For all $t \in [T],\n\[
\|\nabla_\lambda \ell_i(x_i^t, \lambda_i^t)\| \leq 3L(1 + R)Q \sqrt{mn},
\]
\[
\|\nabla_x \ell_i(x_i^t, \lambda_i^t)\| \leq L\beta,
\]
where $\beta = 1 + \beta \sqrt{m}$.

Next, we leverage the result of Corollary 3 to bound the ‘consensus’ terms $\|x_i^t - x_j^t\|$ which is a measure of deviation between agents’ decision variables. Specifically, the next theorem provide an upper bound on the consensus terms in Algorithm 1.

**Proposition 4.** For all $i, j \in \mathcal{V}$, the deviation in the primal variables of nodes is bounded by
\[
\|x_i^t - x_j^t\| \leq 10\alpha \beta L \frac{\log(T\sqrt{n})}{1 - \sigma_2(W)},
\]
for all $n \geq 2$.

We note that the separation in the primal variables of a pair of nodes is governed by the inverse of the spectral gap $1 - \sigma_2(W)$ which itself is dictated by the choice of the weight matrix $W$ as well as the structure of underlying graph. For example, an admissible choice of the weight matrix $W$ is given by the lazy Metropolis matrix [20] that is characterized as $W = \frac{1}{2} I + \frac{1}{2} M$, where given the degrees $d(i)$ and $d(j)$ of nodes $i$ and $j$, respectively, the matrix $[M]$ has the following elements
\[
[M]_{ij} = \begin{cases} 
\frac{1}{\max(d(i), d(j))} & \text{if } (i, j) \in \mathcal{E} \\
0 & \text{if } (i, j) \not\in \mathcal{E}.
\end{cases}
\]

It is easy to verify that the lazy Metropolis matrix $W$ is stochastic, symmetric, and diagonally dominant. Further, due to symmetry, the singular values are simply the absolute value of the eigenvalues. More importantly, the inverse of the spectral gap has an upper bounded proportional to $n^2$ [20]. Specifically,
\[
\frac{1}{1 - \sigma_2(W)} \leq 71n^2.
\]

By putting together Propositions 2 and 4 we arrive at the following result:

**Theorem 5.** For all $j \in \mathcal{V}$, the following holds
\[
\frac{1}{n} \sum_{i=1}^n f_i(\tilde{x}_j^T) - f_i(x^*) \leq \frac{R^2}{2T\alpha} + \alpha m L^2 (1 + R)^2 + 13\alpha L^2 \beta^2 \frac{\log(T\sqrt{n})}{1 - \sigma_2(W)}.
\]

Specifically, suppose $\alpha = \frac{1}{4L\sqrt{mn}}$ and $\eta = \frac{2L(1 + R)\sqrt{mn}}{\sigma_2(W)}$, where $r \in (0, 1)$ and $\psi$ is such that $\alpha \eta < \frac{1}{2Q^2}$ (cf. Remark 4).
Then,
\[
\frac{1}{n} \sum_{i=1}^{n} f_i(\bar{x}_i^T) - f_i(x^*) \leq \frac{3(1 + R)^2L\sqrt{m}}{\sqrt{T}} + \frac{13g^2L\sqrt{m}}{1 - \sigma_2(W)} \cdot \frac{\log(T\sqrt{n})}{T^{\frac{1}{2}}} ,
\]
for all \( j \in \mathcal{V} \) and \( n \geq 2 \).

In the next theorem, we characterize two asymptotic bounds for the constraint violation of Algorithm 1

**Theorem 6.** Consider the step size \( \alpha \) and the regularizer's curvature \( \eta \) as defined in Thm. 3 and \( r \in \{0, 1\} \). The norm of the constraint violation has the following asymptotic for all \( i \in \mathcal{V} \),
\[
\| \Pi_{R_{\alpha}^m} g(x_i^T) \|_2 \leq O \left( \frac{(1 + R)Q_{\psi} n\sqrt{m}}{\sqrt{T}} \right) .
\]
Furthermore, if the optimal solution \( x^* \) is strictly feasible \( g(x^*) < 0 \), we have
\[
\| \Pi_{R_{\alpha}^m} g(x_i^T) \|_2 \leq O \left( \frac{(1 + R)Q_{\psi} n\sqrt{m}}{\sqrt{T}^{1/2}} \right) .
\]
In particular, when the optimal solution is strictly feasible \( g(x^*) < 0 \), the value of \( r = 0 \) provides the optimal rates in both Eqs. 12 and 14.

**Remark 3.** The case of \( r = 1 \) in Theorem 5 and Theorem 6 is excluded since it creates an error term in the upper bound in Eq. 12 that grows unboundedly as \( T \to \infty \). The case of \( r = 0 \) is, however, more subtle as it can cause a non-vanishing term in the constraint violation bound under the condition that \( g_k(x^*) = 0 \) for at least one coordinate \( k \in [m] \), see Eq. 13.

**Remark 4.** The constant parameter \( \varrho \) incorporated in \( \eta \) in Theorem 5 and Theorem 6 satisfies
\[
\varrho \geq \frac{(1 + R)Q_{\psi} \sqrt{m}}{\sqrt{T}^{1/2}} .
\]
It is easy to verify that with such a choice of \( \varrho \), the condition \( \alpha \eta < \frac{1}{\sqrt{T^{1/2}}} \) is satisfied. However, in most cases of interest, \( T \) is a large number in which case we can comfortably put \( \varrho = 1 \).

From Theorem 5 and Theorem 6, we observe that when one of the constraints is binding at the optimal solution, i.e., \( g_k(x^*) = 0 \) for at least one coordinate \( k \in [m] \), there is a tension between the convergence rate of the distributed primal-dual algorithm and the decay rate of the constraint violation bound. More specifically, adopting a small value for \( r \in (0, 1] \) improves the convergence rate in Eq. 12 while deteriorates the constraint violation bound in Eq. 13.

This tension can be explained by inspecting the role that the regularizer \( \psi \) plays in Algorithm 1. We observe that by selecting a regularizer with a large curvature \( \eta \), the norm of dual variables \( ||\lambda^*_i|| \) can be reduced arbitrarily. We already noted this point in the discussion after Proposition 2. In turn, a small norm \( ||\lambda^*_i|| \) results in small subgradients of \( \mathcal{L}_i(-, \cdot) \) which render a fast consensus between agents in the network and thus a fast convergence rate in Theorem 5. Nevertheless, a small norm \( ||\lambda^*_i|| \) also reduces the penalty of constraint violation and hence worsens the first asymptotic bound in Theorem 6.

**Algorithm 2** DISTRIBUTED STOCHASTIC PRIMAL-DUAL METHOD

1: **Initialize:** \( x_i^0 = 0, \lambda_i^0 = 0 \) for all \( i \in \mathcal{V} \) and a constant step size \( \alpha \in \mathbb{R}_+ \). Select \( p_i^0 = \text{Uniform}[1, 2, \cdots, m] \).
2: **for** \( t = 0, 1, 2, \cdots, T \) at the \( i \)-th node \( i \in \mathcal{V} \) do
3: **Draw** a random index \( K_i^t \in \{1, 2, \cdots, m\} \) according to the distribution \( K_i^t \sim p_i^t \).
4: **Update** the primal and dual variables
\[
\bar{x}_i^t = x_i^t - \alpha \nabla \mathcal{L}_i(x_i^t, \lambda^*_i; K_i^t)
\]
\[
\lambda_i^t = \lambda_i^t + \alpha \nabla \mathcal{L}_i(x_i^t, \lambda^*_i).
\]
5: **Run** the consensus step
\[
x_i^{t+1} = \Pi_{\mathcal{X}} \left( \sum_{j=1}^{n} |W|_{ij} \bar{x}_j^t \right)
\]
\[
\lambda_i^{t+1} = \Pi_{\mathcal{R}_m} \left( \sum_{j=1}^{n} |W|_{ij} \lambda_j^t \right) .
\]
6: **Update** \( p_i^t(k) = \frac{\lambda_i^t(k)}{||\lambda^*_i||} \) for \( k = 1, 2, \cdots, m \). Set \( p_i^t = \text{Uniform}[1, 2, \cdots, m] \) if \( \lambda_i^t = 0 \).
7: **end for**
8: **Output:** \( \bar{x}^T = \frac{1}{T} \sum_{i=1}^{T} x_i^T \) for all \( i \in \mathcal{V} \).

**IV. DISTRIBUTED STOCHASTIC PRIMAL-DUAL METHOD**

Here, we devise a stochastic regularized primal-dual (SPD) algorithm for solving the min-max problem in Eq. 7. The motivation for studying a randomized PD algorithm is due to the observation that during each iteration \( t \in [T] \) of Algorithm 1 the full subgradient vector \( \{\nabla g_k(x^t[i])\}_{k=1}^{m} \) must be computed in Eq. 6(a) at each node \( i \in \mathcal{V} \). But for high dimensional data sets (large \( d \)), the computation of subgradient vector is expensive. In particular, the complexity of computing gradient vector for functions defined by an explicit sequence of standard operations is proportional (with a constant proportionality coefficient) to the computational complexity of the value of corresponding function \( \mathcal{F}_i \). Even for medium size data sets, evaluation of each iteration of the standard PD algorithm is prohibitive when the number of constraints \( m \) is large.

To reduce the complexity associated with computing subgradients of constraint functions \( \{g_k\}_{k=1}^{m} \), we randomize the PD algorithm based on a distribution that is updated according to the dual variables computed at each algorithm iteration. Consequently, at each step of SPD, one constraint \( k \in \{1, 2, \cdots, m\} \) is selected randomly and its associated subgradient \( \nabla g_k(\cdot) \) is computed at each node.

To make this statement more rigorous, let \( K_i^t \in \{1, 2, \cdots, m\} \) denotes a random variable distributed as \( P[K_i^t = k] = K_i^t = \frac{1}{m} \). To have a well-defined formulation for \( p_i^t \), we assume that when the dual parameters are all zero \( \lambda_i^t = 0 \in \mathbb{R}_+^m \), the distribution \( p_i^t \) is uniform, i.e., \( p_i^t(k) = 1/m \) for all \( k \in \{1, 2, \cdots, m\} \).
With a slight abuse of notation, we define
\[
\nabla \hat{x}_t(x^i_t, \lambda^i_t; K^i_t) \equiv \nabla f_i(x^i_t) + \|\lambda^i_t\| \cdot \nabla g_{K^i_t}(x^i_t) \\
\nabla \hat{\lambda}_t(x^i_t, \lambda^i_t) \equiv g(\lambda^i_t) - \nabla \psi(\lambda^i_t).
\]
(16a)
(16b)

The randomization step in Eq. (16a) resembles to the incremental gradient methods which has been notably used in the training of neural networks where they are known as ‘backpropagation’ methods; see Refs. [22] and [23]. In contrast to the incremental methods, however, our proposed strategy uses an adaptive distribution that is updated based on the observed dual variables at each algorithm iteration.

Equipped with these definitions, we outline the pseudocode for the distributed stochastic primal-dual method in Algorithm 2.

Our first observation about Algorithm 2 is that the bound- edness of dual variables in Proposition 2 extends to the stochastic optimization setting. This is due to the fact that for any realization of random variables \(x^i_t \in X, t \in [T]\) the function \(g(\lambda^i_t)\) is bounded by Eq. (2). Therefore, the proof of Proposition 2 can be carried over without modification to the stochastic case. Consequently, under the restriction \(0 < \alpha \leq \frac{1}{4L\sqrt{m}}\) on the step size of Algorithm 2 and with \(\eta = \frac{2L(1+R)\sqrt{m}}{\beta}\), any realization of random variable \(\lambda^i_t\) satisfies
\[
\|\lambda^i_t\| \leq \beta,
\]
(17)
for all \(t \in [T]\).

Moreover, analogous to Corollary 3, the inequality in Eq. (17) provides upper bounds on the norm of the subgradients in Eqs. (16a)–(16b). Specifically,
\[
\|\nabla \hat{x}_t(x^i_t, \lambda^i_t; K^i_t)\| \leq 3L(1+R)Q \sqrt{m}\sqrt{T} \\
\|\nabla \hat{\lambda}_t(x^i_t, \lambda^i_t)\| \leq L\beta,
\]
almost surely for all \(t \in [T]\), where we recall \(\beta = 1 + \sqrt{m}\).

Due to the boundedness of subgradients, we can invoke the method of bounded martingale difference to derive a high probability bound. Therefore, in the stochastic optimization case the regularization \(\psi\) is required for two reasons:

i) It provides us with an almost sure bound on the consensus terms \(\|x^i_t - x_j\|\) similar to Proposition 3

ii) It allows us to bound the tail of the difference between the deterministic and stochastic Lagrangian functions
\[
\nabla \hat{x}_t(x^i_t, \lambda^i_t) - \nabla \hat{x}_t(x^i_t, \lambda^i_t; K^i_t),
\]
using Azuma’s martingale inequality [24].

The first result determines the convergence rate of Algorithm 2. To prove the following theorem, let \(\mathcal{F}_t\) denotes the \(\sigma\)-field such that the processes \((x^\tau)^{t+1}\) and \((\lambda^\tau)^{t+1}\) defined in Algorithm 2 are \(\mathcal{F}_t\)-measurable.

**Theorem 7.** The following inequality holds for Algorithm 2

(a) With the probability of at least \(1 - \varepsilon\),
\[
\frac{1}{n} \sum_{i=1}^n f_i(x^i_T) - f_i(x^*) \leq \frac{R^2}{2T} + \alpha mL^2(1 + R)^2 \\
+ 13\alpha L^2\beta^2 \frac{\log(T \sqrt{n})}{1 - \sigma_2(W)} + 3LR\beta \sqrt{\frac{\log \frac{1}{\varepsilon}}{T}},
\]
(18)

for all \(j \in \mathcal{V}\).

In particular, let \(\alpha = \frac{1}{4L\sqrt{m}}\) and \(\eta = \frac{2L(1+R)\sqrt{m}}{\sqrt{T}}\), where \(r \in [0, 1)\) and \(\psi\) is specified in Eq. (15). With the probability of at least \(1 - \frac{1}{T}\),
\[
\frac{1}{n} \sum_{i=1}^n f_i(x^i_T) - f_i(x^*) \leq \frac{3(1+R)L\sqrt{m}}{\sqrt{T}T} + 13\alpha L^2\beta^2 \frac{\log(T \sqrt{n})}{1 - \sigma_2(W)} + 3LR\beta \sqrt{\frac{\log \frac{1}{\varepsilon}}{T}}.
\]
(19)

(b) The expected value of the convergence rate is given by
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E}[f_i(x^i_T)] - f_i(x^*)
\leq \frac{R^2}{2T} + \alpha mL^2(1 + R)^2 + 13\alpha L^2\beta^2 \frac{\log(T \sqrt{n})}{1 - \sigma_2(W)}.
\]
(20)

In particular, with the same set of parameters of part (a), we have
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E}[f_i(x^i_T)] - f_i(x^*)
\leq \frac{3(1+R)L\sqrt{m}}{\sqrt{T}T} + 13\alpha L^2\beta^2 \frac{\log(T \sqrt{n})}{1 - \sigma_2(W)}.
\]
(21)

It is immediate from Eq. (19) that \(f(x^i_T) \to f(x^*)\) almost surely as \(T \to \infty\). Moreover, by comparing Eq. (19) with the deterministic bound in Eq. (12) we observe that both Algorithm 1 and Algorithm 2 supply the same asymptotic convergence rate \(O(\log(T)/T^{\frac{1}{2}})\). This is due to the fact that in both algorithms, the consensus step (Steps 4 of Alg. 1 and Step 5 of Alg. 2) is the bottleneck in the convergence speed.

In the next theorem, we address the constraint violation performance of Algorithm 2. The proof is omitted since it borrows similar ideas from the proofs of Theorems 6 and 7.

**Theorem 8.** Consider \(\alpha\) and \(\beta\) as specified in Theorem 7, then with the probability of at least \(1 - \frac{1}{T}\)
\[
\left\|\Pi_{\mathbb{R}^m}(g(\hat{x}^T))\right\|^2 = O\left(\frac{L(1+R)Q \sqrt{m}}{\sqrt{T}T} \frac{\sqrt{n}}{T} \right).
\]
(22)

Furthermore, if the optimal solution \(x^*\) satisfies the inequality constraints strictly \(g(x^*) < 0\), we have
\[
\left\|\Pi_{\mathbb{R}^m}(g(\hat{x}^T))\right\|^2 = O\left(\frac{L(1+R)Q \sqrt{m}}{\sqrt{T}T} \frac{\sqrt{n}}{T} \right),
\]
(23)

with the probability of at least \(1 - \frac{1}{T}\).

V. COMPARISON WITH RELATED METHODS

To put our work into the context of other distributed constrained optimization methods, here we compare our results from Section III with the dual averaging algorithm in [10] and the primal-dual methods without regularization in [14], [15], [16].

The distributed dual averaging algorithm aims at solving the following constrained optimization problem
\[
\min_{x \in X} \frac{1}{n} \sum_{i=1}^n f_i(x),
\]
(24)
where the inequality constraints \( g(x) \leq 0 \) in Eq. (1b) are absent in this formulation.

The distributed dual averaging algorithm with the time varying size \( \alpha(t) \) consists of two steps:

1. (Averaging Step):
   \[
   x_{i}^{t+1} = \frac{1}{\alpha(t)} \sum_{j=1}^{n} [W]_{ij} x_{j}^{t} + \nabla f_{i}(x_{i}^{t}).
   \]

2. (Projection Step):
   \[
   x_{i}^{t+1} = \arg \min_{x \in X} \left\{ \langle \hat{x}_{i}^{t}, x \rangle + \frac{1}{\alpha(t)} \Psi(x) \right\}.
   \]

The initial value is \( x_{0}^{t} \in X, \forall i \in \mathcal{V} \), and \( \Psi : \mathbb{R}^{d} \to \mathbb{R}_{+} \) is a proximal function that stabilizes each step. To compare the convergence rate of our algorithm with that of dual averaging we consider two different regimes:

- **Binding constraints**: In this case \( g_{k}(x^{*}) = 0 \) for at least one coordinate \( k \in [m] \). We observe that in this regime including the inequality constraints \( g(x) \leq 0 \) in the optimization problem (10)-(1b) provides the convergence rate of \( O(\log(T)/T^{2}) \) in the objective value and the decaying rate of \( O(1/T^{2}) \) in the constraint violation. In the dual averaging algorithm, the convergence rate is given by \( O(\log(T)/T^{2}) \).

- **Non-binding constraints**: In this regime \( g(x^{*}) < 0 \) and we can plug \( r = 0 \) in Eq. (12) to obtain the same asymptotic convergence rate \( O(\log(T)/T^{2}) \) as the distributed dual averaging algorithm. Moreover, from Eq. (14) with \( r = 0 \), the constraint violation decays at the rate of \( O(1/T^{2}) \).

Despite having a slower convergence rate for optimization problems with active constraints, [1] we must note that computing the projection in the primal-dual method can be significantly more efficient due to presence of explicit inequality constraints. In particular, by separating the constraints into the explicit and implicit forms, the geometry of the projection space \( \mathcal{X} \) can be simplified. To demonstrate this with an example, consider the following semidefinite least square problem

\[
\min_{X} \frac{1}{2} \|X - Y\|_{F}^{2}
\]

subject to:
\[
\langle A_{k}, X \rangle \leq b_{k}, \quad k = 1, 2, \cdots, m
\]

\[
0 \preceq X,
\]

where \( X, Y, A_{k} \in \mathbb{S}^{n} \) are symmetric matrices, \( \langle A_{k}, X \rangle \equiv \text{Tr}(A_{k}X^{T}) \), and \( \| \cdot \|_{F} \) is the Frobenius norm. By applying the primal-dual method with \( g_{k}(X) = \langle A_{k}, X \rangle - b_{k} \), we observe that the projection in Step 4 of Algorithm 1 is onto the cone of positive semidefinite matrices \( \mathbb{S}^{n}_{+} \equiv \{ X \in \mathbb{R}^{n \times n} : X \succeq 0, X^{T} = X \} \). In this case, the projection has a closed form expression. In particular, consider the eigendecomposition of a matrix \( X \),

\[
X = U \text{diag}(\lambda_{1}, \cdots, \lambda_{n}) U^{T}
\]

1While this statement is true for non-smooth functions, our numerical experiments with smooth objective functions have shown that the primal-dual method indeed converges faster than the dual averaging algorithm, cf. Section VI.

where \( \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n} \) are the eigenvalues of \( X \) and \( U \) is the corresponding orthonormal matrix of eigenvectors of \( X \). Then the projection of \( X \) onto \( \mathbb{S}^{n}_{+} \) is given by [25]

\[
X = U \text{diag}(\lambda_{1}^{+}, \cdots, \lambda_{n}^{+}) U^{T},
\]

where we recall the notation \( [x]_{+} = \max\{0, x\} \). The projection in the dual averaging method, however, involves a non-trivial region \( \mathcal{X} = \mathbb{S}_{+}^{n} \cap \{ X \in \mathbb{S}^{n} : \langle A_{k}, X \rangle \leq b_{k}, k = 1, 2, \cdots, m \} \). In this case, computing the projection requires solving a minimization problem.

As another simple example, consider a convex optimization problem involving a set of box constraints as well as a norm constraint,

\[
\min_{x} \frac{1}{n} \sum_{i=1}^{n} f_{i}(x)
\]

subject to \( l_{k} \leq x_{k} \leq u_{k}, \quad k = 1, 2, \cdots, d, \quad \|x\| \leq r. \)

By applying the distributed primal-dual method to this problem with \( g_{k}(x) = l_{k} - x_{k}, k = 1, 2, \cdots, d \) and \( g_{k+d}(x) = x_{k} - u_{k}, k = 1, 2, \cdots, d \), we observe that the projection \( \Pi_{B}(\cdot) \) in Step 4 of Algorithm 1 is onto the \( \ell_{2} \)-ball of radius \( r \) which corresponds to a rescaled vector

\[
\Pi_{B_{r}}(x) = \frac{r \cdot x}{\max\{r, \|x\|\}}.
\]

In contrast, the projection step of the dual averaging algorithm is onto \( \mathcal{X} = \{ x \in \mathbb{R}^{d} : x \in \bigcap_{k=1}^{m} [l_{k}, u_{k}] \} \cap B_{r}(r) \).

In [14, 15, 16], primal-dual methods similar to this paper are studied. However, these methods rely on the boundedness of the dual solution set under Slater’s constraint qualification. More specifically, suppose there exists a vector \( \hat{x} \) such that

\[
g(\hat{x}) \prec 0.
\]

Furthermore, define

\[
\hat{\lambda}(\hat{\lambda}) \equiv \inf_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} f_{i}(x) + \langle \hat{\lambda}, g(x) \rangle.
\]

Then it is shown that [26],

\[
\|\hat{\lambda}\|_{1} \leq \mu^{-1} \cdot (f(\hat{x}) - \inf_{\lambda \in \mathbb{R}_{+}^{m}} \hat{\lambda}(\hat{\lambda})),
\]

where \( \mu \equiv \min_{k=1,2,\cdots,m} \{-g_{k}(\hat{x})\} \) and \( \hat{\lambda}^{*} \) is the optimal dual variable. The approach adopted in [14] has two main drawbacks in the distributed systems:

- In the proposed primal-dual algorithm in [14], each agent projects the local computation of dual variables \( \lambda_{i}^{t} \) onto the following simplex

\[
\Lambda \equiv \{ \lambda \in \mathbb{R}^{m} : \|\lambda\|_{1} \leq 1 \} \}

where \( \hat{\lambda} \in \mathbb{R}_{+}^{m} \) is an arbitrary vector. In contrast, the projection in our proposed algorithm (cf. Algorithm 1) is onto the non-negative orthant \( \mathbb{R}_{+}^{m} \) which simply corresponds to replacing each negative component of the dual vector \( \lambda_{i}^{t} \) with zero. Since the projection step must be executed at each algorithm iteration, our proposed strategy is significantly more efficient.
The performance of the algorithm is tied to the structure of feasible set. Firstly, for many problems, it is computationally expensive to find such Slater vector $\hat{x}$ that $g(\hat{x}) < 0$. Moreover, the size of the projection set $\Lambda$ is governed by the inverse of $\mu$ and the value of $\mu$ can be small when $g_k(\hat{x})$ is small for at least one coordinate $k \in \{1, 2, \ldots, m\}$. This, in turn, results in a loose upper bound on the norm of the gradients in Eqs. (6a) and (6b) and hence a loose convergence bound for the underlying primal-dual algorithm in [14]. The upper bound on the constraint violation in [14] also depends on $\mu^{-1} \cdot (f(\hat{x}) - \inf_{\lambda \in \mathbb{R}^m} \tilde{\mathcal{G}}(\lambda))$ which can be loose by the same reasoning.

VI. NUMERICAL EXPERIMENTS

In this section, we report the numerical simulations studying the convergence of the regularized primal-dual method for the distributed regression on synthetic data. We demonstrate the performance of Algorithm [1] we use examples of smooth and non-smooth classifiers.

- **Smooth case**: we consider a logistic loss function with a norm constraint as well as a set of box constraints

$$\min_{x} f(x) \equiv \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(b_i \langle a_i, x \rangle)) \quad (29a)$$

subject to $g_k(x) = -l - x_k \leq 0$, $g_{k+d}(x) = x_k - u \leq 0$, $k = 1, \ldots, d$, $x \in \mathcal{X} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, (29b)

where $(a_i, b_i) \in \mathbb{R}^d \times \{-1, +1\}$.

- **Non-smooth case**: we consider the following hinge loss function with a norm constraint as well as a set of box constraints

$$\min_{x} f(x) \equiv \frac{1}{n} \sum_{i=1}^{n} \max\{1 - b_i \langle a_i, x \rangle, 0\} \quad (30a)$$

subject to $g_k(x) = -l - x_k \leq 0$, $g_{k+d}(x) = x_k - u \leq 0$, $k = 1, \ldots, d$, $x \in \mathcal{X} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, (30b)

where $(a_i, b_i) \in \mathbb{R}^d \times \{-1, +1\}$.

The optimization problems in Eqs. (29a)-(29b) and (30a)-(30b) are exact in the context of classification in supervised learning, where $\{a_1, b_1, \ldots, a_n, b_n\}$ is the set of $n$ training data such that $a_i$ is the feature vector (a.k.a. explanatory variables in regression) and $b_i$ is its associated label. In the case of logistic classifier, to make a prediction given a new vector $a$, the classifier outputs $b = \pm 1$ with the probability

$$P(b = \pm 1|a, x) = \frac{1}{1 + \exp(\pm(x, a))}.$$  

In the case of hinge loss function, the goal is to obtain a linear classifier of the form $a \mapsto \text{sign}(\langle a, x \rangle)$ for some vector $x \in \mathbb{R}^d$.

In our simulations with the logistic classifier, we generate $a_i$ from a uniform distribution on the unit sphere. We then choose a random vector from Gaussian distribution $w \sim \mathcal{N}(0, I_{d \times d})$ and generate the labels $b_i \sim \text{Bernoulli}(p)$, where $p = \frac{1}{1 + \exp(\langle w, a_i \rangle)}$. It is straightforward to verify that $L = \max_{i=1,2,\ldots,n} \|a_i\| = 1$ and $R = 1$. Note that the solution of the optimization problem in Eq. (29a) approximates $w$ under the restrictions specified in Eqs. (29b). To perform the projection onto $\mathcal{X}$ in our primal-dual method, we employ the closed form expression in Eq. (25). We use the ridge penalty function $\psi(\lambda) = \frac{1}{2} \|\lambda\|^2$ in Eq. (5) in all of our simulations. Furthermore, we consider vectors of the dimension $d = 5$ (thus $m = 10$) and study three different network sizes, $n \in \{50, 100, 150\}$ and two different upper/lower limits $l = u \in \{0.1, 0.001\}$.

To investigate the performance of Algorithm [1] on different networks, we consider three different classes of graphs in our simulations: (a) Watts-Strogatz small-world graph model [27], (b) Erdős-Rényi random graph [28], (c) unwrapped 8-connected neighbors lattice. See Fig. 1.

A graph is characterized as a small-world if it is highly clustered locally (like regular lattices) and with a small separation globally. Social networks is an example where each person is only five or six people away from anyone else. Watts-Strogatz model is a framework to generate random graphs with small-world properties based on two structural features, namely the clustering and average path length. These features are captured by two parameters, the mean degree $K$ and a parameter $\theta$ that interpolates between a lattice ($\theta = 0$) and a random graph ($\theta = 1$).

In the Erdős-Rényi random graph, an edge between a pair of nodes is included to the graph with probability $p$ independent from other edges. Note that the Watts-Strogatz small-world graph model reduces to the Erdős-Rényi random graph model when $\theta = 1$, where $p = K / N - 1$.

Figure [1] shows the average of maximum error gap $\max_{e \in V} f(x_e^f) - f(x^*)$ over three different classes of graphs demonstrated in Fig. 1(a,b,c) and for 10 trials for $R = 1$ and $l = u = 1$. Each blue curve in Figure 1 illustrates the average values over the trials. From figure we observe that in the case of non-smooth hinge loss function, the dual averaging method generally has a faster convergence rate to the optimal objective value. However, for the smooth logistic loss function, the primal-dual method provides a faster convergence rate.

Our convergence analysis of the PD method does not use the smoothness assumption of the objective function. Therefore, it is an interesting to see if the convergence rate in Theorem 2 can be improved by further restricting the objective function and constraints to the set of smooth functions. We also observe that since the step size of our algorithm does not incorporate information about the graph structure, the performance of the algorithm is less sensitive to the underlying graphical model compared to the dual averaging method.

Give a connectivity graph $\mathcal{G}$ with $n$ nodes, let $\varepsilon(T; n)$ denotes the maximum relative error of the network, i.e.,

$$\varepsilon_{\mathcal{G}}(T; n) \equiv \max_{i=1,2,\ldots,n} \frac{f(x_i^f) - f(x^*)}{f(x^*)}$$

for every node in the graph $i \in V$. Further, we define $\delta_{\mathcal{G}}(T; n) \equiv \max_{i=1,2,\ldots,n} \|g(x_i^f)\|$ as the maximum constraint violation among all the nodes in the network. In the case of the centralized PD method, we similarly use $\varepsilon(T, n)$ and $\delta(T; n)$ to denote the relative error gap and the constraint violation,
respectively. In our simulations, we use MATLAB convex programming toolbox CVX [29] to compute $f(x^*)$ as well as to compute the projection in the dual averaging algorithm.

Figure 2 shows the constraint violation as well as convergence rate in the centralized PD algorithm without regularization and decentralized regularized PD algorithms for various iterations $T$ and the value of $u = l = 0.1$ for upper/lower limits. In this particular example, we observe that in the decentralized PD algorithm, the algorithm output $\tilde{x}_i^T$ is feasible for all $T \in \mathbb{N}$ and $i \in \mathcal{V}$, whereas in the centralized PD algorithm, the outputs are infeasible. Here, we thus clearly observe that the addition of regularizer can mitigate the constraint violation.

By tightening the upper and lower bounds in the box constraints (29b), we further observe a constraint violation in both the decentralized and centralized methods; see Figure 3. However, even in this case the amplitude of $\delta_G(T; n)$ in the decentralized method is significantly smaller than $\delta(T; n)$ in the centralized case. Interestingly, we also observe that the error in the objective value after $T$ iterations decreases monotonically in the decentralized method while this is not the case in the centralized PD method.

Lastly, Figure 4 provides a comparison between the performance of the stochastic regularized primal-dual method and its deterministic counterpart. In particular, 20 sample paths of the stochastic algorithm are generated where each solid line corresponds to one sample path. Further, the dashed line shows the performance of the deterministic algorithm.
Fig. 2: Distributed regression with logistic loss on synthetic data using Watts-Strogatz graph with $K = 20$, $\vartheta = 0.02$, $\eta \propto T^{-1/5}$, $\alpha \propto T^{-1/2}$ and $l = u = 0.1$. Panel (a): Constraint violation $\delta(T; n)$ of the centralized PD algorithm without regularization. Panel (b): Convergence rate $\varepsilon(T; n)$ of the centralized PD algorithm without regularization. Panel (c) Constraint violation $\delta_G(T; n)$ of the decentralized regularized PD algorithm. Panel (d): Convergence rate $\varepsilon_G(T; n)$ of the decentralized regularized PD algorithm.

VII. CONCLUSION AND DISCUSSION

In this paper, we studied a distributed primal-dual method for solving convex optimization problems with inequality constraints over a network. In the proposed distributed framework, dual variables are regularized with a smooth and strongly convex function. As a result, the norm of dual variables, and hence the subgradients of the Lagrangian function, are bounded. Based on this regularization, we obtained an upper bound on the consensus terms and subsequently an upper bound on the convergence rate of the proposed algorithm. Furthermore, we presented asymptotic results for the diminishing rate of the constraint violation. Our results demonstrates an interesting transition in the behavior of the distributed regularized PD algorithm in the sense that when one of the inequality constraints is active at the optimal solution, there is a tension between the convergence rate speed and the diminishing rate associated with the constraint violation. Nevertheless, this tension vanishes when the constraints are satisfied strictly at the optimal solution. We also studied the convergence rate of the distributed stochastic primal-dual method. We showed that in the distributed case, the stochastic algorithm enjoys the same asymptotic convergence rate as its deterministic counterpart.

There are several interesting questions that can be addressed to supplement the result of this paper. Clearly, our experiments with the smooth logistic loss function hints at a faster convergence rate than those predicted by our theoretical analysis. Therefore, characterizing the convergence rate of the primal-dual method under smoothness assumption of the objective function and constraints is an interesting open problem. Another research direction is to have a comprehensive analysis of penalty/barrier function methods in the distributed setting and compare the result with the algorithm we developed in this paper.

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Fig. 3: Distributed regression with logistic loss on synthetic data using Watts-Strogatz graph with $K = 20$, $\vartheta = 0.02$, $\eta \propto T^{-1/3}$, $\alpha \propto T^{-1/2}$ and $l = u = 0.001$. Panel (a): Constraint violation $\delta(T; n)$ of the centralized PD algorithm without regularization. Panel (b): Convergence rate $\varepsilon(T; n)$ of the centralized PD algorithm without regularization. Panel (c): Constraint violation $\delta_G(T; n)$ of the decentralized PD algorithm. Panel (d): Convergence rate $\varepsilon_G(T; n)$ of the decentralized PD algorithm.

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Appendix

A. Proof of Proposition 2

From the update rule of $\lambda_i^j$ in Algorithm 1 we have

$$\|\lambda_i^{i+1}\| = \left\| \Pi_{B^\alpha} \left( \sum_{j=1}^{n} |W|_{ij} \lambda_j^j + \alpha \nabla \lambda \xi_j(x_j^?, \lambda_j^j) \right) \right\|$$

$$\leq \sum_{j=1}^{n} |W|_{ij} \|\lambda_j^j + \alpha \nabla \lambda \xi_j(x_j^?, \lambda_j^j)\|$$

$$\leq \sum_{j=1}^{n} |W|_{ij} \|\lambda_j^j + \alpha \nabla \lambda \xi_j(x_j^?, \lambda_j^j)\|,$$

where the last step follows by the triangular inequality. We now square both sides of the inequality to obtain

$$\|\lambda_i^{i+1}\|^2 \leq \left( \sum_{j=1}^{n} |W|_{ij} \|\lambda_j^j + \alpha \nabla \lambda \xi_j(x_j^?, \lambda_j^j)\| \right)^2$$

$$= \left( \sum_{j=1}^{n} |W|_{ij} \|\lambda_j^j + \alpha \nabla \lambda \xi_j(x_j^?, \lambda_j^j)\| \right)^2$$

$$\leq \sum_{j=1}^{n} |W|_{ij} \left( (1 + \delta^{-1}) \|\alpha \nabla \xi_j(x_j^?)\|^2 + (1 + \delta) \|\lambda_j^j - \alpha \nabla \xi_j(x_j^?)\|^2 \right)$$

$$= (1 + \delta^{-1}) \alpha^2 \sum_{j=1}^{n} |W|_{ij} \|g(x_j^?)\|^2$$

$$+ \left( 1 + \delta \right) \sum_{j=1}^{n} |W|_{ij} \left( \|\lambda_j^j\|^2 - 2\alpha \langle \nabla \xi_j(x_j^?), \lambda_j^j \rangle + \alpha^2 \|\nabla \xi_j(x_j^?)\|^2 \right),$$

where (a) follows from Jensen’s inequality, and (b) follows from Young’s inequality which is valid for any $\delta > 0$ (cf. Appendix G for proof). In particular, the addition of the parameter $\delta$ in (b) allows us to have a tight control over terms in the bound.

Taking the summation with respect to $i = 1, 2, \cdots, n$ results in

$$\sum_{i=1}^{n} \|\lambda_i^{i+1}\|^2 \leq (1 + \delta^{-1}) \alpha^2 m^2 L^2 (1 + R)^2$$

$$+ \left( 1 + \delta \right) \sum_{i=1}^{n} \left( \|\lambda_i^i\|^2 - 2\alpha \langle \nabla \xi_i(x_i^?), \lambda_i^i \rangle + \alpha^2 \|\nabla \xi_i(x_i^?)\|^2 \right),$$

where we used the fact that $\sum_{i=1}^{n} |W|_{ij} = 1$ and $\|g(x_i^?)\|^2 \leq m L^2 (1 + R)^2$. We now recall that $\psi(\lambda_i^i)$ is $\eta$-strongly convex (cf. Def. L). Therefore,

$$\langle \nabla \psi(\lambda_i^i) - \nabla \psi(\hat{\lambda}_i^i), \lambda_i^i - \hat{\lambda}_i^i \rangle \geq \eta \|\lambda_i^i - \hat{\lambda}_i^i\|^2.$$
Setting $\lambda^t_i = 0$ gives us
\begin{equation}
(\nabla \psi(\lambda^t_i) - \nabla \psi(0), \lambda^t_i) \geq \eta \|\lambda^t_i\|^2. \tag{32}
\end{equation}
Since $\nabla \psi(0) = 0$, after a sign change and multiplication by $2\alpha$ we obtain
\begin{equation}
-2\alpha (\nabla \psi(\lambda^t_i), \lambda^t_i) \leq -2\alpha \eta \|\lambda^t_i\|^2. \tag{33}
\end{equation}
Moreover, from $\gamma$-smoothness assumption, we have
\begin{equation}
\|\nabla \psi(\lambda^t_i)\| \leq \gamma \|\lambda^t_i\|. \tag{34}
\end{equation}
Combining Eqs. (33), (34) and (31) results in
\begin{equation}
\sum_{i=1}^n \|\lambda^{t+1}_i\|^2 \leq (1 + \delta^{-1}) n m \alpha^2 L^2 (1 + R)^2 + (1 + \delta) (1 - 2\alpha \eta + \alpha^2 \gamma^2) \sum_{i=1}^n \|\lambda^t_i\|^2. \tag{35}
\end{equation}
Now, we note that $0 < 1 - 2\alpha \eta + \alpha^2 \gamma^2 < 1$ provided $\alpha < 2\eta/\gamma^2$. This is due to the fact that $1 - 2\alpha \eta + \alpha^2 \gamma^2 \geq (1 - \gamma \alpha)^2 > 0$ as $\eta \leq \gamma$. Moreover, the constraint $\alpha < 2\eta/\gamma^2$ ensures that $1 - 2\alpha \eta + \alpha^2 \gamma^2 < 1$.

Therefore, we can choose $\delta = \frac{\varepsilon}{1 - 2\alpha \eta + \alpha^2 \gamma^2} - 1$, where $\varepsilon \in (1 - 2\alpha \eta + \alpha^2 \gamma^2, 1)$. This choice of $\delta$ results in $(1 + \delta) (1 - 2\alpha \eta + \alpha^2 \gamma^2) = \varepsilon$ and we can proceed from Eq. (35) as below
\begin{equation}
\sum_{i=1}^n \|\lambda^{t+1}_i\|^2 \leq \frac{\varepsilon}{1 - \varepsilon} \frac{\varepsilon \alpha^2}{\varepsilon - (1 - 2\alpha \eta + \alpha^2 \gamma^2)} \sum_{i=1}^n \|\lambda^t_i\|^2 \leq \frac{\varepsilon \alpha^2}{1 - \varepsilon} \sum_{i=1}^n \|\lambda^t_i\|^2. \tag{36}
\end{equation}
The upper bound in Eq. (36) is minimized by $\varepsilon^* = \sqrt{1 - 2\alpha \eta + \alpha^2 \gamma^2}$. Substituting $\varepsilon^*$ in Eq. (36) gives
\begin{equation}
\sum_{i=1}^n \|\lambda^{t+1}_i\|^2 \leq \frac{\alpha^2}{1 - 2\alpha \eta + \alpha^2 \gamma^2} \sum_{i=1}^n \|\lambda^t_i\|^2 \leq \frac{mm^2 L^2 (1 + R)^2}{\eta^2 (1 - \alpha^2 \gamma^2)}. \tag{37}
\end{equation}
The last step is due to the Bernoulli inequality $(1 + x)^r \leq 1 + rx$ for $x \geq -1$ and $r \in [0, 1]$ which results in
\begin{align*}
1 - \sqrt{1 - 2\alpha \eta + \alpha^2 \gamma^2} \geq \alpha \eta - \frac{1}{2} \alpha^2 \gamma^2.
\end{align*}
By further restricting the step size $\alpha < \eta/(2\gamma^2) = 1/(2Q^2 \eta)$, we can simplify Eq. (37) as
\begin{equation}
\|\lambda^{t+1}_i\|^2 \leq \frac{4mm^2 L^2 (1 + R)^2}{\eta^2}. \tag{38}
\end{equation}

\[\text{B. Proof of Corollary 3}\]
Based on the upper bound in Proposition 2, we compute
\begin{align*}
\|\nabla \lambda \mathcal{L}_i(x^i_t, \lambda^t_i)\| &\leq \|g(x^i_t) - \nabla \psi(\lambda^t_i)\| \\
&\leq \|g(x^i_t)\| + \|\nabla \psi(\lambda^t_i)\| \\
&\leq \|g(x^i_t)\| + \gamma \|\lambda^t_i\| \\
&\leq L (1 + R) \sqrt{\frac{m^2}{(1 + 2\sqrt{mQ})}} \\
&\leq 3L (1 + R) \sqrt{\frac{m^2}{m^2 Q}}, \tag{39}
\end{align*}
where in (a) we used the $\gamma$-smoothness assumption of $\psi(\cdot)$. Similarly,
\begin{align*}
\|\nabla \mathcal{L}_i(x^i_t, \lambda^t_i)\| &\leq \|\nabla f_i(x^i_t)\| + \sum_{k=1}^m \lambda^t_{ik} \cdot \|\nabla g_k(x^i_t)\| \\
&\leq L (1 + ||\lambda^t_i||) \\
&\leq L (1 + \sqrt{m} ||\lambda^t_i||) \tag{39}
\end{align*}

\[\text{C. Proof of Proposition 4}\]
We begin by obtaining a recursive formula for $||x^t_i - x^{t-1}_i||$. From the update rule in Algorithm 1 and the non-expansive property of the projection we have
\begin{align*}
||x^t_i - x^{t-1}_i|| &= \Pi_x \left( \sum_{k=1}^n [W]_{ik} \tilde{x}^{t-1}_k \right) - \Pi_x \left( \sum_{k=1}^n [W]_{ik} \tilde{x}^{t-1}_k \right) \\
&\leq \sum_{k=1}^n ||[W]_{ik} - [W]_{jk}|| \tilde{x}^{t-1}_k \tag{a} \\
&\leq \sum_{k=1}^n ||[W]_{ik} - [W]_{jk}|| \tilde{x}^{t-1}_k \tag{a} \\
&\leq \sum_{k=1}^n ||[W]_{ik} - [W]_{jk}|| \tilde{x}^{t-1}_k \tag{b} \\
&+ \alpha \sum_{k=1}^n ||[W]_{ik} - [W]_{jk}|| \|\nabla \mathcal{L}_i(x^t_i - 1, \lambda^t_{i-1})\|, \tag{41}
\end{align*}
where in steps (a) and (b) we used the triangle inequality. Further, since $\mathcal{X}$ contains the origin $x = 0$, once again from
the non-expansive property of projection we have
\[
\|x_t^i - x_j^j\| = \|\Pi x \left( \sum_{t=1}^{n} [W]_{kt} x_t^{i-2} \right) \|
\leq \left\| \sum_{t=1}^{n} [W]_{kt} x_t^{i-2} \right\|
\leq \sum_{t=1}^{n} \| [W]_{kt} \| \cdot \| x_t^{i-2} \|
\leq \sum_{t=1}^{n} \| [W]_{kt} \| \cdot \| x_t^{i-2} \| + \alpha \sum_{t=1}^{n} \| \nabla_x L_k(x_t^{i-2}, \lambda_t^{i-2}) \|.
\]
(42)

Plugging (42) in Eq. (41) yields
\[
\| x_t^i - x_j^j \| \leq \sum_{t=1}^{n} \| [W]_{kt} \| \cdot \| x_t^{i-2} \|
+ \alpha \sum_{t=1}^{n} \| [W]_{kt} \| \cdot \| \nabla_x L_k(x_t^{i-2}, \lambda_t^{i-2}) \|
+ \alpha \sum_{t=1}^{n} \| x_t^{i} - x_j^{j} \| \cdot \| x_t^{i-1} \|.
\]
(43)

Pursuing this recursive analysis in conjunction with the state transition matrix \( \Phi(t, r) \equiv W^{t-r} \) gives us a more compact form of inequality,
\[
\| x_t^i - x_j^j \| \leq \alpha \sum_{r=0}^{t-1} \sum_{k=1}^{n} \| \Phi(t, r) \| \cdot \| x_k^{i} \|.
\]
(44)

where we used the fact that \( x_0^i = 0 \) for all \( i \in \mathcal{V} \). We combine the upper bound in Eq. (44) with the expression in Eq. (41) to compute
\[
\| x_t^i - x_j^j \| \leq \alpha \sum_{r=0}^{t-1} \sum_{k=1}^{n} \| \Phi(t, r) \| \cdot \| x_k^{i} \|.
\]
(45)

From the triangle inequality for \( \ell_1 \)-norm we have
\[
\| \Phi(t, r) \| \leq 1 / \sqrt{n} \| x_k^{i} \|.
\]
(46)

for all \( i, j \in \{1, 2, \ldots, n\}, i \neq j \). We now closely follow the argument of Duchi, et al. [10] to bound the terms on the r.h.s. of Eq. (46). We notice that \( \Phi(t, r) \) is a doubly stochastic matrix. Borrowing the analysis from [10] we therefore have
\[
\| \Phi(t, r) \| \leq 1 / \sqrt{n} \| x_k^{i} \| \leq \sqrt{n} \sigma_2(W)^{t-r},
\]
(47)

for all \( i \in \{1, 2, \ldots, n\} \). Consequently, to achieve the accuracy of \( \| \Phi(t, r) \| \leq 1 / \sqrt{n} \| x_k^{i} \| \leq \frac{1}{\sqrt{n}} \), we need
\[
t - r \geq \frac{1}{\log \sigma_2(W)} \log \sigma_2(W^{-1}).
\]
(48)

Otherwise, the deviation can be bounded as \( \| \Phi(t, r) \| \leq 1 / \sqrt{n} \| x_k^{i} \| \leq 2 \). We now define the cutoff time \( \tau \equiv \frac{1}{\log \sigma_2(W)} \).

Then, we break the sum in Eq. (45) as below
\[
\| x_t^i - x_j^j \| \leq \alpha \beta L \left( \sum_{r=t-r+1}^{t} \| \Phi(t, r) \| \cdot \| x_k^{i} \| + \sum_{r=0}^{t-r} \| \Phi(t, r) \| \cdot \| x_k^{i} \| \right)
\]
\[
\leq 4 \alpha \beta L \log \frac{\log(T \sqrt{n})}{\log \sigma_2(W)} + 2 \alpha \beta L
\]
\[
\leq 10 \alpha \beta L \log \frac{\log(T \sqrt{n})}{\log \sigma_2(W)},
\]
(49)

where the last step follows from the inequality \( \log(x^{-1}) \geq 1 - x, x > 0 \), and the fact that \( \frac{\log(T \sqrt{n})}{\log \sigma_2(W)} \geq 1 \) for \( n \geq 2 \) and all \( T \in \mathbb{N} \) since \( \log(\sqrt{2}) \approx 1.04 \).

D. Proof of Theorem 5

First, we state a proposition.

Proposition 9. Let \( f(\cdot) = \frac{1}{n} \sum_{i=1}^{n} f_i(\cdot) \). For the optimal solution \( x^* \in X \) the following inequality holds
\[
\sum_{t=1}^{T} \left( f(x_t^i) - f(x^*) \right)
\leq \frac{\| x^* \|}{2 \alpha} + \frac{\alpha}{2n} \sum_{t=1}^{T} \sum_{i=1}^{n} \| \nabla_x L_i(x_t^i, \lambda_t^i) \|^2 + \| \nabla_x L_i(x_t^i, \lambda_t^i) \|^2
\]
\[
+ \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \| \nabla f_i(x_t^i) \| \cdot \| x_t^i - x^* \| - \frac{\eta}{2n} \sum_{t=1}^{T} \sum_{i=1}^{n} \| \lambda_t^i \|^2.
\]

Proof: See Appendix [H].

Now, due to the second upper bound in Corollary [3] we obtain
\[
\| \nabla_x L_i(x_t^i, \lambda_t^i) \|^2 \leq \tilde{\beta}^2 L^2.
\]
(51)

Also, from Eq. (38) we have
\[
\| \nabla x L_i(x_t^i, \lambda_t^i) \|^2 \leq L \cdot 10 \alpha \beta L \frac{\log(T \sqrt{n})}{\log \sigma_2(W)}
\]
\[
\leq 10 \alpha \beta^2 L^2 \log \frac{\log(T \sqrt{n})}{\log \sigma_2(W)},
\]
(54)

Moreover, by employing the upper bound on the consensus term in Proposition [3] we establish the following inequality for the third term in the r.h.s. of Eq. (50).
\[
\| \nabla f_i(x_t^i) \| \cdot \| x_t^i - x^* \| \leq L \cdot 10 \alpha \beta L \frac{\log(T \sqrt{n})}{\log \sigma_2(W)}
\]
\[
\leq 10 \alpha \beta^2 L^2 \log \frac{\log(T \sqrt{n})}{\log \sigma_2(W)},
\]
(54)

where the last step is true since by definition \( \tilde{\beta} \geq 1 \).

Substituting Eqs. (51)-(54) into Eq. (50) and dividing both sides by \( T \) we derive
\[
\frac{1}{T} \sum_{t=1}^{T} \left( f(x_t^i) - f(x^*) \right) \leq \frac{\| x^* \|}{2T \alpha} + \alpha \lambda L^2 (1 + R)^2 + \alpha L^2 \tilde{\beta}^2
\]
\[
+ 10 \alpha \beta^2 L^2 \frac{\log(T \sqrt{n})}{\log \sigma_2(W)} + \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \left( \alpha \gamma^2 - \frac{2}{2 \alpha} \right) \| \lambda_t^i \|^2.
\]
(55)
Due to the constraint on the step size $\alpha \eta < \frac{1}{2\alpha T}$ in Proposition 2, we have $\alpha \gamma^2 = \alpha \eta^2 Q^2 \psi \leq \frac{1}{2} \eta$. Therefore, we can eliminate the last term,

$$\frac{1}{T} \sum_{i=1}^{T} \left( f(x_i^j) - f(x^*) \right) \leq \frac{\|x^*\|^2}{2T\alpha} + \alpha m L^2 (1 + R)^2 + \alpha L^2 \beta^2 + 10\alpha L^2 \beta^2 \frac{\log(T\sqrt{n})}{1 - \sigma_2(W)}.$$  

(56)

Since $\|x^*\| \leq R$ and $\frac{3\log(T\sqrt{n})}{1 - \sigma_2(W)} \geq 1$ for $n \geq 2$ we further have

$$\frac{1}{T} \sum_{i=1}^{T} \left( f(x_i^j) - f(x^*) \right) \leq \frac{R^2}{2T\alpha} + \alpha m L^2 (1 + R)^2 + 13\alpha L^2 \beta^2 \frac{\log(T\sqrt{n})}{1 - \sigma_2(W)}.$$  

(57)

Now, recall the running local average $\bar{x}_T^j \equiv \frac{1}{T} \sum_{t=1}^{T} x_i^j$. Due to convexity of the function $f(\cdot)$ we have

$$f(\bar{x}_T^j) \leq \frac{1}{T} \sum_{t=1}^{T} f(x_i^j).$$

We thus establish

$$f(\bar{x}_T^j) - f(x^*) \leq \frac{R^2}{2T\alpha} + \alpha m L^2 (1 + R)^2 + 13\alpha L^2 \beta^2 \frac{\log(T\sqrt{n})}{1 - \sigma_2(W)}.$$  

(58)

Let $\alpha = \frac{1}{4L\sqrt{mT}}$ and $\beta = gT^\frac{r}{2}$, where $r \in [0,1)$ and $g$ specified in Eq. (15). Note that since $g \geq 1$, we have $\tilde{\beta} = 1 + \beta \sqrt{m} \leq 2\sqrt{m}gT^\frac{r}{2}$. From Eq. (58) we thus obtain

$$f(\bar{x}_T^j) - f(x^*) \leq \frac{3(1 + R)^2 L \sqrt{m}}{\sqrt{T}} + \frac{13L \sqrt{m} \sigma^2 \log(T\sqrt{n})}{2T\alpha \sqrt{T}}.$$  

E. Proof of Theorem 6

We state a proposition first (cf. Appendix B):

**Proposition 10.** For all $x \in \mathcal{X}$ and $\lambda \in \mathbb{R}^m_+$ the following inequality holds

$$\frac{1}{n} \sum_{i=1}^{n} \left( \mathcal{L}_i(x_i^j, \lambda) - \mathcal{L}_i(x_i^j, \lambda_i^j) \right) \leq \frac{\|x^*\|^2 + \|\lambda\|^2}{2\alpha} + \frac{\alpha}{2n} \sum_{i=1}^{n} \left( \|\nabla_x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2 + \|\nabla_{\lambda} \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2 \right).$$  

(59)

Let $x = x^*$ and use the inequalities of Corollary 3 to obtain

$$\frac{1}{n} \sum_{i=1}^{n} \left( \mathcal{L}_i(x_i^j, \lambda) - \mathcal{L}_i(x_i^j, \lambda_i^j) \right) \leq \frac{\|x^*\|^2 + \|\lambda\|^2}{2\alpha} + \frac{1}{2T\alpha} \left( L^2 \beta^2 + 9L^2 (1 + R)^2 Q^2 \psi mn \right).$$  

(60)

By expanding the l.h.s. of Eq. (60) and dividing by $T$ we derive

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} (f_i(x_i^j) - f_i(x^*)) + \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \langle \lambda_i g(x_i^j) \rangle - \psi(\lambda) + \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \psi(\lambda_i^j) \leq \frac{R^2 + \|\lambda\|^2}{2T\alpha} + \frac{1}{2} \left( L^2 \beta^2 + 9L^2 (1 + R)^2 Q^2 \psi mn \right),$$

where we used the fact that $\|x^*\| \leq R$. Due to the positivity of the dual variables $\lambda_i^j \geq 0$ as well as $g(x^*) \leq 0$ we have $\langle \lambda_i^j, g(x^*) \rangle \leq 0$. Furthermore, $\psi(\lambda) \geq 0$ for an admissible regularizer (cf. Def. 1). Hence, we can eliminate these terms which leaves us

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} (f_i(x_i^j) - f_i(x^*)) \leq \frac{R^2 + \|\lambda\|^2}{2T\alpha} + \frac{1}{2} \left( L^2 \beta^2 + 9L^2 (1 + R)^2 Q^2 \psi mn \right),$$  

(61)

where we used the definition $\bar{x}_T^j = \frac{1}{T} \sum_{t=1}^{T} x_i^j$ as well as the convexity of the functions $f_i$ and $g_i$.

Due to $\gamma$-smoothness assumption of $\psi$, for all $\lambda \in \mathbb{R}^m$,

$$\psi(\lambda) \leq \frac{\eta Q^2 \psi}{2} \|\lambda\|^2.$$  

(62)

Recall the definition $\lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Using the inequalities (62) and (61), we get

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} (f_i(x_i^j) - f_i(x^*)) + \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \langle \lambda_k g_k(x_i^j) \rangle - \left( \frac{Q\psi \eta}{2} + \frac{1}{2\alpha T} \right) \lambda_k^2 \leq R^2 + \frac{R^2}{2T\alpha} + \frac{1}{2} \left( L^2 \beta^2 + 9L^2 (1 + R)^2 Q^2 \psi mn \right),$$  

(63)

Let $F \in \mathbb{R}^+$ be a constant such that $-F \leq \frac{1}{n} \sum_{i=1}^{n} (f_i(x_i^j) - f_i(x^*))$. By maximizing the l.h.s. of Eq. (63) with respect to $\lambda_k, k = 1, 2, \ldots, m$ we derive

$$\left( \frac{Q\psi \eta}{2} + \frac{1}{2\alpha T} \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \left[ \Pi_{R+} (g_k(x_i^j)) \right]^2 \leq F + \frac{R^2}{2T\alpha} + \frac{1}{2} \left( L^2 \beta^2 + 9L^2 (1 + R)^2 Q^2 \psi mn \right).$$  

(64)

Substituting $\alpha = \frac{1}{4L\sqrt{nm}}$ and $\beta = gT^\frac{r}{2}$ from Theorem 6 results

$$\frac{1}{n} \sum_{i=1}^{n} \left\| \Pi_{R_{\psi \sqrt{nm}}} (g(x_i^j)) \right\|^2 \leq \left( \frac{L(1 + R)Q\psi \sqrt{nm}}{gT^\frac{r}{2}} + \frac{L}{2m\sqrt{T}} \right) \left( F + O \left( \frac{1}{T^{\frac{r}{2}}} \right) \right),$$  

whence,

$$\left\| \Pi_{R_{\psi \sqrt{nm}}} (g(x_i^j)) \right\|^2 \leq O \left( \frac{L(1 + R)Q\psi \sqrt{nm}}{gT^\frac{r}{2}} \right).$$

To prove the remaining part of Theorem 6 we have the following lemma:
Lemma 11. Suppose the optimal solution \( x^* \) satisfies the inequality constraints strictly \( g_k(x^*) < 0, k = 1, 2, \ldots, m \). Then,
\[
\frac{1}{n} \sum_{i=1}^{n} (f_i(\tilde{x}_i^T) - f_i(x^*)) \geq 0,
\]
for all \( i \in \mathcal{V} \) and \( t \in [T] \), i.e., \( F = 0 \).

Proof: Due to convexity of \( f = \frac{1}{n} \sum_{i=1}^{n} f_i \) for all \( i = 1, 2, \ldots, n \) we have
\[
f(\tilde{x}_i^T) - f(x^*) \geq \langle \nabla f(x^*), \tilde{x}_i^T - x^* \rangle.
\]
(66)

We now write the Karush-Kuhn-Tucker (KKT) conditions for the optimal solution \( x^* \) and the vector of optimal Lagrangian multipliers \( \lambda^* \equiv (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \).

[C1] \( \langle \xi, x - x^* \rangle \geq 0, \forall \xi \in \partial f(x^*) + \sum_{k=1}^{m} \lambda_k \partial g_k(x^*) \), \( \forall x \in \mathcal{X} \).

[C2] \( \lambda_k \partial g_k(x^*) = 0, k = 1, 2, \ldots, m \).

[C3] \( g(x^*) \geq 0 \) and \( \lambda^* \geq 0 \).

From [C2] we note that \( \lambda_k^* = 0 \) since \( g_k(x^*) < 0 \) for \( k = 1, 2, \ldots, m \). Consequently, the condition in [C1] turns into
\[
\langle \xi, x - x^* \rangle \geq 0, \forall \xi \in \partial f(x^*), \forall x \in \mathcal{X}.
\]
By definition \( \nabla f(x^*) \in \partial f(x^*) \). Therefore, by choosing \( \xi = \nabla f(x^*) \) we have the following inequality
\[
\langle \nabla f(x^*), x - x^* \rangle \geq 0,
\]
for all \( x \in \mathcal{X} \). This proves the claim since \( \tilde{x}_i^T \in \mathcal{X} \).

Equation (13) in Theorem 6 now follows from Eq. (65) by equating \( F = 0 \).

F. Proof of Theorem 7

Analogous to the derivation in Eq. (77), for any realization of random variables \((x_i^t, \lambda_i^t, K_i^t)_{t=1}^{T} \) it can be shown that for all \( x \in \mathcal{X} \),
\[
\frac{T}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} 2\alpha \langle \nabla \widehat{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x \rangle \leq \frac{n}{2} \| x \|^2 + \sum_{i=1}^{n} \alpha^2 \| \nabla \widehat{L}_i(x_i^t, \lambda_i^t; K_i^t) \|^2.
\]
(67)

Recall the definition
\[
\widehat{L}_i(x_i^t, \lambda_i^t) \equiv f_i(x_i^t) + \langle \lambda_i^t, g(x_i^t) \rangle - \psi(\lambda_i^t).
\]
(69)

By putting together the inequality (67) and the definition (69) we obtain
\[
\frac{T}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \langle \nabla \widehat{L}_i(x_i^t, \lambda_i^t), x_i^t - x \rangle \leq \frac{n}{2} \| x \|^2 + \frac{\alpha}{2} \sum_{i=1}^{n} \| \nabla \widehat{L}_i(x_i^t, \lambda_i^t; K_i^t) \|^2
\]
(70)

Further, similar to Eq. (78) and due to the fact that \( \nabla \lambda \mathcal{L}(x^t, \lambda^t) = \nabla \lambda \mathcal{L}(x^t, \lambda^t) \), we derive
\[
\frac{T}{n} \sum_{i=1}^{n} \langle \nabla \lambda \mathcal{L}(x_i^t, \lambda_i^t), \lambda - \lambda_i^t \rangle \leq \frac{1}{2\alpha} \| \lambda \|^2 + \frac{\alpha}{2} \sum_{i=1}^{n} \| \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t) \|^2.
\]
(72)

Thus, following the footsteps of the proof of Proposition 1, we derive for \( x = x^* \in \mathcal{X} \) that
\[
f(\tilde{x}_i^T) - f(x^*) \leq \frac{\| x^* \|^2}{2T\alpha}
\]
(73)

\[
+ \frac{\alpha}{2nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left( \| \nabla \widehat{L}_i(x_i^t, \lambda_i^t; K_i^t) \|^2 + \| \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t) \|^2 \right)
\]
(74)

\[
+ \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \| \nabla f_i(x_i^t) \| \| x_i^t - x^* \| - \frac{n}{2nT} \sum_{i=1}^{n} \| \lambda_i^t \|^2
\]
(75)

\[
+ \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle.
\]

We now obtain a high probability bound for the last term of the expression in Eq. (73). First notice that
\[
\nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t) = \mathbb{E}_p \left[ \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t) | F_{t-1} \right].
\]

Consequently, we can compute the expectation of the last term with \( x = x^* \). Specifically, since \( x^t \in F_{t-1} \) we can write
\[
\mathbb{E} \left[ \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle \right]
\]
(76)

\[
= \mathbb{E} \left[ \mathbb{E} \left[ \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle | F_{t-1} \right] \right]
\]
(77)

\[
= \mathbb{E} \left[ \mathbb{E} \left[ \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle | F_{t-1} \right] , x_i^t - x^* \right]
\]
(78)

\[
= 0.
\]

Moreover, from the upper bound on the dual variables in Eq. (17), we can obtain that
\[
\langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle
\]
(79)

\[
\leq \| \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t) \| \| x_i^t - x^* \|
\]
(80)

\[
\leq 2LR \| \lambda_i^t \|.
\]
(81)

where the last step follows from the fact that \( \| \lambda_i^t \| \leq \sqrt{m} \| \lambda \| \). Applying the Azuma-Hoeffding inequality yields the tail bound,
\[
\mathbb{P} \left[ \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x^* \rangle \geq \delta \right]
\]
(82)

\[
\leq \exp \left( -\frac{\delta^2}{8mTL^2R^2 \lambda^2} \right).
\]
(83)

Hence, with probability of at least \( 1 - \varepsilon \) we have
\[
\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \langle \nabla \lambda \mathcal{L}_i(x_i^t, \lambda_i^t; K_i^t), x_i^t - x \rangle \leq 3LR \beta \sqrt{\frac{m \log \frac{1}{\varepsilon}}{T}}.
\]
(84)

The rest of the proof can be carried out similar to the proof in Appendix D.
To prove the second part of Theorem 4, we compute the expectation of both sides of inequality (73) and use the fact that the expectation of the last term in Eq. (75) is zero due to Eq. (74).

G. Proof of Young’s Inequality

For any $\delta > 0$ and $a, b \in \mathbb{R}^m$ we have

\[ \|a + b\|^2 \leq (\|a\| + \|b\|)^2 \]

\[ \leq \|a\|^2 + \|b\|^2 + 2\|a\|\|b\| \]

\[ = \|a\|^2 + \|b\|^2 + (\sqrt{2/\delta}\|a\|)(\sqrt{2\delta}\|b\|) \]

\[ \leq \|a\|^2 + \|b\|^2 + \frac{1}{2}(2\delta^{-1}\|a\|^2 + 2\delta\|b\|^2) \]

\[ = (1 + \delta^{-1})\|a\|^2 + (1 + \delta)\|b\|^2. \]

H. Proofs of Proposition 9 and Proposition 10

From the non-expansive property of the projection, we obtain that for all $x \in \mathcal{X}$,

\[ \|x_{j+1} - x\|^2 = \|\Pi_{\mathcal{X}} \left( \sum_{j=1}^n [W]_{ij} x_i^j \right) - x\|^2 \]

\[ \leq \sum_{j=1}^n [W]_{ij} x_i^j - x \|^2 \]

\[ \leq \sum_{j=1}^n [W]_{ij} [(x_j^i - x) - \alpha \nabla x \mathcal{L}_i(x_j^i, \lambda_j^i)] \|^2 \]

\[ = \sum_{j=1}^n [W]_{ij} \left( \|x_j^i - x\|^2 - 2\alpha \langle \nabla x \mathcal{L}_i(x_j^i, \lambda_j^i), x_j^i - x \rangle \right) \]

\[ + \alpha^2 \|\nabla x \mathcal{L}_i(x_j^i, \lambda_j^i)\|^2 \],

where (a) follows from the convexity of the norm square, $\| \cdot \|_a$ is the dual norm. Computing the summation over $i = 1, 2, \cdots, n$ results

\[ \sum_{i=1}^n \|x_{i+1}^j - x\|^2 \]

\[ \leq \sum_{i=1}^n \sum_{j=1}^n [W]_{ij} \left( \|x_j^i - x\|^2 - 2\alpha \langle \nabla x \mathcal{L}_i(x_j^i, \lambda_j^i), x_j^i - x \rangle \right) \]

\[ + \alpha^2 \|\nabla x \mathcal{L}_i(x_j^i, \lambda_j^i)\|^2 \]

\[ = \sum_{i=1}^n \left( \|x_i^j - x\|^2 - 2\alpha \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), x_i^j - x \rangle \right) \]

\[ + \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2 \],

where we used the fact that $\sum_{i=1}^n [W]_{ij} = 1$ as $W$ is a doubly stochastic matrix. From this recursion and by noting that $x_i^0 = 0$, we derive

\[ \sum_{i=1}^n \|x_{i+1}^j - x\|^2 \leq \|x\|^2 \]

\[ - \sum_{i=1}^T 2\alpha \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), x_i^j - x \rangle + \sum_{i=1}^T \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \]

Since the left hand side is non-negative, we obtain the following inequality

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), x_i^j - x \rangle \]

\[ \leq n\|x\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (77) \]

We can similarly show that

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), \lambda - \lambda_i^j \rangle \]

\[ \leq n\|\lambda\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (78) \]

Due to convexity of $\mathcal{L}_i(\cdot, \lambda_i)$ and concavity of $\mathcal{L}_i(x_i, \cdot)$, the following pair of inequalities hold respectively,

\[ \mathcal{L}_i(x_i^j, \lambda_i^j) - \mathcal{L}_i(x, \lambda_i^j) \leq \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), x_i^j - x \rangle \]

\[ \mathcal{L}_i(x_i^j, \lambda) - \mathcal{L}_i(x_i^j, \lambda_i^j) \leq \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), \lambda - \lambda_i^j \rangle. \quad (79) \]

Combining Eqs. (77) and (78) coupled with the inequalities (79) results in

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle \mathcal{L}_i(x_i^j, \lambda_i^j) - \mathcal{L}_i(x, \lambda_i^j) \rangle \]

\[ + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \langle \nabla x \mathcal{L}_i(x_i^j, \lambda_i^j), \lambda - \lambda_i^j \rangle \]

\[ \leq n\|x\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (81) \]

This completes the proof of Proposition 10.

To prove Proposition 9, we first combine Eq. (77) with Eq. (79) to obtain

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle \mathcal{L}_i(x_i^j, \lambda_i^j) - \mathcal{L}_i(x, \lambda_i^j) \rangle \]

\[ \leq n\|x\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (82) \]

Let $x = x^*$. Expanding $\mathcal{L}_i(x_i^j, \lambda_i^j)$ and $\mathcal{L}_i(x^*, \lambda_i^j)$ on the l.h.s. gives us

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle f_i(x_i) - f_i(x^*) \rangle \]

\[ + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \langle \lambda_i^j, g(x_i) \rangle - \langle \lambda_i^j, g(x^*) \rangle \]

\[ \leq n\|x^*\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (83) \]

Since $\lambda_i^j \geq 0$ and $g(x^*) \leq 0$, we have $-\langle \lambda_i^j, g(x^*) \rangle \geq 0$. Therefore, we can eliminate it from the l.h.s. of Eq. (83),

\[ \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle f_i(x_i) - f_i(x^*) \rangle + \sum_{i=1}^T \sum_{j=1}^n 2\alpha \langle \lambda_i^j, g(x_i^j) \rangle \]

\[ \leq n\|x^*\|^2 + \sum_{i=1}^T \sum_{j=1}^n \alpha^2 \|\nabla x \mathcal{L}_i(x_i^j, \lambda_i^j)\|^2. \quad (84) \]
To obtain an upper bound in terms of 
\[ \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha (f_i(x^*_t) - \sum_{i=1}^{n} f_i(x^*_t)), \]
we use the convexity of \( f_i(\cdot) \),

\[ f_i(x^*_t) + \langle \nabla f_i(x^*_t), x^*_t - x^*_j \rangle \leq f_i(x^*_j). \] (85)

Substituting the inequality (85) combined with the Cauchy-Schwarz inequality and the definition \( f(\cdot) = \frac{1}{n} \sum_{i=1}^{n} f_i(\cdot) \) gives us

\[
\begin{align*}
\sum_{t=1}^{T} 2\alpha (f(x^*_t) - f(x^*)) + \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha \langle \lambda^*_t, g(x^*_t) \rangle \\
\leq \|x^*\|^2 + \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \alpha^2 \|\nabla \mathcal{L}_i(x^*_t, \lambda^*_t)\|^2 \\
+ \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha \|\nabla f_i(x^*_t)\|_* \cdot \|x_j - x_i\|. \quad (86)
\end{align*}
\]

In the remainder of the proof, we establish a lower bound on \( \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha \langle \lambda^*_t, g(x^*_t) \rangle \). To do so, we combine Eq. (80) with Eq. (78) to derive

\[
\sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha (\mathcal{L}_i(x^*_t, \lambda) - \mathcal{L}_i(x^*_t, \lambda^*_t)) \leq n\|\lambda\|^2 + \sum_{t=1}^{T} \sum_{i=1}^{n} \alpha^2 \|\nabla \lambda \mathcal{L}_i(x^*_t, \lambda^*_t)\|^2. \] (87)

We let \( \lambda = 0 \) and then expand the l.h.s. of Eq. (87),

\[
\begin{align*}
\sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha (\psi(\lambda^*_t) - \psi(0) - (\lambda^*_t, g_i(x^*_t))) \\
\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \alpha^2 \|\nabla \lambda \mathcal{L}_i(x^*_t, \lambda^*_t)\|^2. \quad (88)
\end{align*}
\]

From Condition (i) in Definition [1] we have \( \psi(0) = 0 \) and \( \psi(\lambda^*_t) \geq 0 \). Hence,

\[
\begin{align*}
\sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha \psi(\lambda^*_t) - \sum_{t=1}^{T} \sum_{i=1}^{n} \alpha^2 \|\nabla \lambda \mathcal{L}_i(x^*_t, \lambda^*_t)\|^2 \\
\leq \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha (\lambda^*_t, g_i(x^*_t)). \quad (89)
\end{align*}
\]

By the strong convexity condition, for all \( \lambda \in \mathbb{R}^m \) we have

\[ \psi(\lambda) \geq \frac{\eta}{2} \|\lambda\|^2. \] (90)

Based on inequality (90) we obtain

\[
\begin{align*}
\sum_{t=1}^{T} \sum_{i=1}^{n} \alpha \eta \|\lambda^*_t\|^2 - \sum_{t=1}^{T} \sum_{i=1}^{n} \alpha^2 \|\nabla \lambda \mathcal{L}_i(x^*_t, \lambda^*_t)\|^2 \\
\leq \sum_{t=1}^{T} \sum_{i=1}^{n} 2\alpha (\lambda^*_t, g_i(x^*_t)). \quad (91)
\end{align*}
\]

Substituting this lower bound in Eq. (86) gives us Eq. (50).