Constructing 2- and 3-connected graphs

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Abstract

This work re-examines a classical construction of a 2-connected (simple) graph where every intermediate graph is 2-connected before detailing an analogous construction for 3-connected graphs which requires a graph equivalence relation \( \sim_2 \) and a related concept of the \( \sim_2 \)-core of a graph. The case of \( k \)-connected graphs for \( k \geq 4 \) is also addressed.

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1. Introduction

There exists a well-known result [3, p.44] (stated as Theorem 3.2 in the sequel) that every 2-connected graph can be constructed from a cycle by successively adding a path at each step, where the endpoints of the path are identified with two distinct vertices in the graph constructed in the previous step. This construction has the property that every intermediate graph (between the initial cycle and the 2-connected graph which is being constructed) is also 2-connected.

An obvious question to ask is can this result be restated in the 3-connected (or indeed, any higher connectivity) case. Clearly 3-connected analogs of the initial cycle and the path added at each step of the construction would be required, however it quickly becomes apparent (see Example 3.5) that the property of all intermediate graphs being 3-connected cannot be directly extended. It should be noted however that there exists an analogous construction for 3-connected planar graphs which is used in the constructive proof of Steinitz’ theorem, see [6] as well as [1].

As the aim of this paper is to outline a construction pertaining to all 3-connected graphs then it is necessary to introduce the notion of a graph being 3-connected “from a distance,” by using a variant of the core of a graph [4, p.104].

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The main result of this work is a 3-connected analog of Theorem 3.2. By considering Definitions 2.3, 2.4, 3.1 and 3.4, as well as the notion of a 3-admissible union which is detailed in Section 4, then it is possible to state the aforementioned main result.

Theorem 4.3 A graph $G$ is 3-connected if and only if $G$ can be constructed from some $G_0 \in [K_4]_{\sim 2}$ (i.e. $\epsilon(G_0) \simeq K_4$) by successive 3-admissible unions of either a $H$-path $P$ or a $H$-$Y$-graph $Q$ and subgraphs $H$ which have already been constructed and every intermediate graph has a 3-connected $\sim_2$-core.

In relation to the proof of Theorem 4.3, sufficiency is shown by utilising a detailed case-by-case analysis of the $H$-path/$H$-$Y$-graph construction which is distilled into Lemmas 4.1 and 4.2, necessity is proved by contradiction in a manner similar to the proof of Theorem 3.2. The possibility of establishing $k$-connected analogs for $k \geq 4$ of either Theorem 3.2 or Theorem 4.3 is examined in Section 5.

2. Preliminaries and graph equivalences

Let $G = (V_G, E_G)$ be a graph where $V_G$ denotes the vertex set of $G$ and $E_G \subseteq [V_G]^2$ denotes the edge set of $G$ (where $[V_G]^2$ is the set of all 2-element subsets of $V_G$). An edge $\{a, b\}$ is denoted $ab$ in the sequel. The union of graphs $G$ and $H$ i.e. $(V_G \cup V_H, E_G \cup E_H)$, is denoted $G \cup H$ in the sequel. All graphs $G$ to which this work pertains are undirected, finite i.e. $|V_G|, |E_G| < \infty$, and simple i.e. contain no loops ($aa \notin E_G$) or multiple edges ($\{ab, ab\} \notin E_G$). The degree of a vertex $v$ in a graph $G$ is the number of edges in $G$ which contain $v$. A path of length $n - 1$, where $n \geq 2$, is a graph with $n$ vertices in which two vertices, known as the endpoints, have degree 1 and $n - 2$ vertices have degree 2. Observe that if $n = 2$, then the resulting path is an edge. A graph is connected if there exists at least one path between every pair of vertices in the graph. All graphs to which this work pertains are connected. Given distinct vertices $a, b \in V_G$, then two paths $P_1$ and $P_2$ with endpoints $a$ and $b$ are openly disjoint if $V_{P_1} \setminus \{a, b\}$ and $V_{P_2} \setminus \{a, b\}$ are disjoint sets. A graph $G$ is $k$-connected if between any two vertices $a$ and $b$ in $G$ there exist $k$ paths which are openly disjoint (this characterisation is a consequence of Menger’s theorem). In the sequel, given an edge $ab$ and a path $P$ then $ab \cup P$ is understood to be the graph $(\{a, b\}, \{ab\}) \cup (V_P, E_P)$. A connected graph all of whose vertices have degree two is called a cycle, and $K_n$ denotes the complete graph on $n$ vertices. All basic graph theory definitions can be found in standards text such as [2], [5] or [7].

Definition 2.1 Given a graph $G = (V_G, E_G)$, vertices $a, b, c \in V_G$ where $b$ has degree 2, edges $ab, bc \in E_G$ and that the edge $ac \notin E_G$, then a series-contraction is an operation applied to $G$ whereby the edges $ab$ and $bc$ are replaced by a single edge $ac$. 

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This definition of a series-contraction is similar to what are termed a series-reduction in [8, p.106].

**Definition 2.2** Given a graph \( G = (V_G, E_G) \) such that the edge \( ac \in E_G \) and \( b \notin V_G \), then a series-expansion is an operation applied to \( G \) whereby the edge \( ac \) is replaced by the edges \( ab \) and \( bc \).

\[ G \xrightarrow{\text{SC}} G' \quad \xrightarrow{\text{SE}} \quad G' \]

Figure 1: A series-contraction (SC) operation applied to \( G \) resulting in \( G' \) and a series-expansion (SE) operation applied to \( G' \) resulting in \( G \)

From the definitions of series-contractions and series-expansions it is now possible to define the relation \( \sim_2 \).

**Definition 2.3** Given graphs \( G \) and \( H \) then \( G \sim_2 H \) whenever \( G \) and \( H \) differ by a sequence of series-contractions and/or series-expansions.

Observe that \( \sim_2 \) is reflexive, symmetric and transitive, hence \( \sim_2 \) is an equivalence relation. Informally, \( G \sim_2 H \) whenever \( G \) and \( H \) are “identical” if both graphs are “viewed from a distance” i.e. paths in \( G \) or \( H \), all of whose internal vertices have degree 2 (in \( G \) or \( H \)), and edges are essentially indistinguishable.

\[ G \xrightarrow{\sim_2} \sim_2 \quad H \]

Figure 2: \( G \) and \( H \) differ by a sequence of series-contractions and series-expansions i.e. \( G \sim_2 H \)

Informally, one of the principal purposes of the equivalence relation \( \sim_2 \) is that it identifies graphs which have the same cycle structure (this is the reason for the caveat in Definition 2.1 that \( ac \notin E_G \)). Therefore, the series-contraction and series-expansion operations do not change the underlying structural features of a graph, but merely change the cardinalities of edge and vertex sets.

A core of a graph \( G \) is a graph \( H \) such that there exists a homomorphism from \( G \) to \( H \), there exists a homomorphism from \( H \) to \( G \), and \( H \) is minimal with respect to this property. See [4] for details. A slight variation on the concept of a core of a graph is now introduced.
Definition 2.4 Given a graph \( G \) and the equivalence relation \( \sim_2 \), then a graph \( H \) is called the \( \sim_2 \)-core of \( G \) if \( H \) has the minimum number of vertices within the equivalence class determined by \( \sim_2 \) which contains \( G \).

The \( \sim_2 \)-core of a graph \( G \) is denoted \( c(G) \) in the sequel. Referring to Figure 2, observe that the \( \sim_2 \)-core of \( G \) and \( H \) is \( K_4 \).

Given a graph \( G \) which has a \( \sim_2 \)-core \( H \), then the equivalence class containing \( G \) is termed a \( \sim_2 \)-class and is denoted \( [H]_{\sim_2} \). Observe that it is possible to speak of the \( \sim_2 \)-core of a graph \( G \) as \( c(G) \) is unique up to relabelling. Note that the \( \sim_2 \)-core of a cycle is the complete graph \( K_3 \) and that a graph \( G \) is a \( \sim_2 \)-core if and only if is not possible to perform any series-contractions to \( G \).

3. Constructing 2-connected and 3-connected graphs

Definition 3.1 Given a graph \( H \), then a path \( P \) with endpoints \( a \) and \( b \) is called a \( H \)-path whenever \( |E_P| \geq 1 \) and \( P \cap H = \{a, b\} \).

Theorem 3.2 now states how \( H \)-paths can be used to iteratively construct all 2-connected graphs with the additional caveat that all intermediate graphs are also 2-connected. The proof of Theorem 3.2, which can be found in [3, p.45], is not reproduced here as the necessity portion of the proof of Theorem 4.3 is essentially a modified version of the aforementioned proof.

Theorem 3.2 A graph \( G \) is 2-connected if and only if it can be constructed from a cycle by successively adding \( H \)-paths to subgraphs \( H \) which have already been constructed.

It is worth reiterating that every intermediate graph in this iterative construction is also 2-connected. An attempt is now made to derive some analog of Theorem 3.2 when the graph under construction is 3-connected.

Definition 3.3 A \( Y \)-graph is a graph \( Q \) such that \( c(Q) \) is isomorphic to the graph \((\{u, a, b, c\}, \{ua, ub, uc\})\).

The (three) vertices which have degree one in a \( Y \)-graph are called the endpoints of the \( Y \)-graph. A 3-connected analog of the \( H \)-path concept is now introduced.

Definition 3.4 Given a graph \( H \), then a \( Y \)-graph \( Q \) with endpoints \( a, b \) and \( c \) is a \( H-Y \)-graph whenever \( H \cap Q = \{a, b, c\} \).

In the \( H \)-path construction of Theorem 3.2, there is no restriction on the length of the \( H \)-paths which are added at each step of the construction as the resulting graph is always (at least) 2-connected. However, when considering the 3-connected case, the addition of any graph with a vertex of degree 2 would automatically deem the graph to be (at most) 2-connected. Considering the obvious analog of a \( H \)-path in the 3-connected case i.e. a \( H-Y \)-graph
(\{u, a, b, c\}, \{ua, ub, uc\}) then clearly the three arms of the Y-graph must all have length one as we desire the graph to be 3-connected. Similarly, any H-path added can only have length one. However, Example 3.5 shows that it is straightforward to construct a small example of a 3-connected graph which cannot be constructed from a graph whose ∼₂-core is isomorphic to \(K_4\) by successively adding \(H-Y\)-graphs which contain only three edges, or \(H\)-paths containing a single edge, to subgraphs already constructed, such that each intermediate graph is 3-connected.

**Example 3.5** Consider the iterated construction of the graph \(G\) shown in Figure 3. Observe that \(c(G_0)\) is isomorphic to \(K_4\), the union of \(G_0\) and a \(H\)-path of length 1 results in \(G_1\) and that \(G_1\) is 3-connected. However, \(G_2\) is not 3-connected (as \(G_2\) contains a degree 2 vertex) and note that there is no possible alternative to \(G_2\) from which \(G\) could be constructed. It follows that in any iterated construction of a 3-connected graph using \(H-Y\)-graphs (and/or \(H\)-paths), the length of the arms of the relevant \(H-Y\)-graphs (and similarly \(H\)-paths) must be allowed to have lengths greater than 1.

The observation contained in Example 3.5 makes it necessary to undertake a detailed analysis of the iterative addition of \(H\)-paths and \(H-Y\)-graphs and this analysis now follows in Section 4. After this analysis it is possible to state and prove Theorem 4.3 which is the main result of this work.

### 4. Admissible unions

The aforementioned analysis of the iterative addition of \(H\)-paths and \(H-Y\)-graphs which now follows is distilled into Lemmas 4.1 and 4.2. These lemmas are then used in the proof of Theorem 4.3 which concludes the current section. Some relevant terminology is now introduced.

- the union of a graph \(H\) and a \(H\)-path \(P\) (resp. a \(H-Y\)-graph \(Q\)) which results in \(c(H \cup P)\) (resp. \(c(H \cup Q)\) ) containing \(\lambda\) vertices and \(\mu\) edges more than \(c(H)\), is referred to as a \((\lambda, \mu)\)-operation, where \(\lambda, \mu \in \mathbb{N}_0\).

Some properties associated with a \((\lambda, \mu)\)-operation are now introduced.
• The union of a 2-connected graph $H$ and a $H$-path $P$ is 2-admissible if and only if $H \cup P$ is 2-connected (equivalently, the union of a graph $H$ where $c(H)$ is 2-connected and a $H$-path $P$ is 2-admissible if and only if $c(H \cup P)$ is 2-connected.)

• The union of a graph $H$ where $c(H)$ is 3-connected and a $H$-path $P$ (resp. a $H$-$Y$-graph $Q$) is 3-admissible if and only if $c(H \cup P)$ (resp. $c(H \cup Q)$) is 3-connected.

All the $(\lambda, \mu)$-operations which are 2-admissible are classified in Lemma 4.1 and all the $(\lambda, \mu)$-operations which are 3-admissible are classified in Lemma 4.2. The strategy utilised in the proof of Lemma 4.1 (resp. Lemma 4.2) to show that each $(\lambda, \mu)$-operation is 2-admissible (resp. 3-admissible) is to explicitly construct two (resp. three) openly disjoint paths between two vertices in $c(H \cup P)$ or $c(H \cup Q)$ i.e. use Menger’s theorem.

Note that in Figures 4 - 10 each “edge” in $H$, $P$ and/or $Q$, respectively, represents a path which series-contracts to an edge in $c(H)$, $c(P)$ and/or $c(Q)$, respectively.

**Lemma 4.1** Given a 2-connected graph $H$ and a $H$-path $P$, then the only 2-admissible $(\lambda, \mu)$-operations are

(a) the $(0, 1)$-operation
(b) the $(1, 2)$-operations
(c) the $(2, 3)$-operation
(d) the $(3, 4)$-operation.

**Proof** The first part of the proof is to show that the $(\lambda, \mu)$-operations contained in cases (a) – (d) are 2-admissible $(\lambda, \mu)$-operations.

(a) the $(0, 1)$-operation: this occurs whenever both endpoints of a $H$-path $P$ are identified with two (distinct) vertices $a$ and $b$ in $c(H)$ such that the edge $ab$ is not contained in $c(H)$. There are no additional vertices contained in $c(H \cup P)$ as a path series-contracts to an edge $ab$ in $c(H \cup P)$ whenever $ab \notin H$; hence $c(H \cup P)$ has one additional edge. As $c(H)$ is 2-connected, $c(H \cup P)$ and $c(H)$ have identical vertex sets and $|E_{c(H \cup P)}| = |E_{c(H)}| + 1$, then $c(H \cup P)$ is also 2-connected.

(b) this case is split into cases (b1) and (b2);

(b1) the $(1, 2)_A$-operation: this occurs whenever one endpoint of a $H$-path $P$ is identified with a vertex in $c(H)$ and the other endpoint of $P$ is identified with a vertex which is not in $c(H)$. As one endpoint of $P$ is identified with a vertex which is not contained in $c(H)$ then this vertex must have degree 2 in $H$. This vertex $a$, see Figure 4, has degree 3 in both $H \cup P$ and $c(H \cup P)$ hence $c(H \cup P)$ has one additional vertex. The vertex $a \in H$ is contained in a path which series-contracts to an edge $xy$ in $c(H)$. Observe that $c(H \cup P)$ does not
contain the edge $xy$ but does contain the edges $ab, ax$ and $ay$ (assuming $b \in P$ and some $u \in c(H)$ are identified), hence $c(H \cup P)$ has two additional edges. Referring to Figure 4, as $c(H)$ is 2-connected then there exists a path $P_1$ (resp. $P_2$) between $x$ (resp. $y$) and $b$ which does not contain $y$ (resp. $x$) such that $P_1$ and $P_2$ are openly disjoint. The paths $ax \cup P_1$ and $ay \cup P_2$ are two openly disjoint paths between the vertex $a \notin V_{c(H)}$ and the (arbitrarily chosen) vertex $b \in V_{c(H)}$. It follows that $c(H \cup P)$ is 2-connected.

(b2) the $(1, 2)_{H}$-operation: this occurs whenever both endpoints of a $H$-path $P$ are identified with two (distinct) vertices $a$ and $b$ in $c(H)$, see Figure 4, such that the edge $ab$ is contained in $c(H)$. There is necessarily an additional vertex in $c(H \cup P)$ as the path $P$ cannot be series-contracted to an edge $ab$ in $c(H \cup P)$ since $ab \in H$, hence $c(H \cup P)$ has one additional vertex $u$, say, and two additional edges, viz., $au$ and $bu$. Referring to Figure 4, as $c(H)$ is 2-connected then there exists a path $P_1$ (resp. $P_2$) between $a$ (resp. $b$) and an arbitrary vertex $v$ which does not contain $b$ (resp. $a$) such that $P_1$ and $P_2$ are openly disjoint. The paths $ua \cup P_1$ and $ub \cup P_2$ are two openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. It follows that $c(H \cup P)$ is 2-connected.

![Figure 4: The cases (b1) and (b2), respectively](image)

(c) the $(2, 3)$-operation: this occurs whenever the two endpoints $a$ and $b$ of a $H$-path $P$ are identified with two distinct vertices that are not contained in $c(H)$ and are in fact contained in paths in $H$ which series-contract to the distinct edges $xy$ and $wz$, respectively, in $c(H)$, see Figure 5. Clearly vertices in $H$ which are identified with $a$ and $b$ have degree 2 in $H$ but have degree 3 in both $H \cup P$ and $c(H \cup P)$, hence $c(H \cup P)$ has two additional vertices. Observe that $c(H \cup P)$ does not contain the edges $xy$ or $wz$ but does contain the edges $ax, ay, bw, bz$ and $ab$, hence $c(H \cup P)$ has three additional edges. Referring to Figure 5, as $c(H)$ is 2-connected then there exists a path $P_1$ (resp. $P_2$) between $x$ (resp. $y$) and an arbitrary vertex $v$ which does not contain $y$ (resp. $x$) such that $P_1$ and $P_2$ are openly disjoint. The paths $ax \cup P_1$ and $ay \cup P_2$ are two openly disjoint paths between the vertex $a \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. Using a similar argument for the vertex $b$, it follows that $c(H \cup P)$ is 2-connected.

(d) the $(3, 4)$-operation: this occurs whenever both endpoints $a$ and $b$ of a $H$-path $P$ are identified with two (distinct) vertices which are not contained in $c(H)$ but are contained in a path in $H$ which series-contract to the same edge
As per the previous case, the vertices in $H$ which are identified with $a$ and $b$ have degree 2 in $H$ but have degree 3 in both $H \cup P$ and $c(H \cup P)$. However, as there is a path in $H \cup P$ which series-contracts to an edge in $c(H \cup P)$, then the $H$-path $P$ cannot series-contrace to a single edge in $c(H \cup P)$. This means that there must be a third additional vertex, $u$, say, which is contained in $c(H \cup P)$. Observe that $c(H \cup P)$ does not contain the edges $xy$ but does contain the edges $ax, au, ab, bu$ and $by$, hence $c(H \cup P)$ has four additional edges. Referring to Figure 5, as $c(H)$ is 2-connected then there exists a path $P_1$ (resp. $P_2$) between $x$ (resp. $y$) and an arbitrary vertex $v$ which does not contain $y$ (resp. $x$) such that $P_1$ and $P_2$ are openly disjoint. The paths $ua \cup ax \cup P_1$ and $ub \cup by \cup P_2$ are two openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. Using a similar argument for the vertices $a$ and $b$, it follows that $c(H \cup P)$ is 2-connected.

The second part of the proof is to show that the $(\lambda, \mu)$-operations contained in cases (a) – (d) are in fact the only 2-admissible $(\lambda, \mu)$-operations.

At each step of the construction contained in Theorem 3.2 the endpoints of a $H$-path $P$ must be identified with two distinct vertices $a$ and $b$ in $H$. The vertices $a$ and $b$ are either:

- contained in $c(H)$, denoted “C” (Core), or not contained in $c(H)$, denoted “nC” (non-Core), or
- adjacent, denoted “A” (i.e. $a$ and $b$ are contained in a path in $H$ which series-contracts to the same edge in $c(H)$) or non-adjacent, denoted “nA” (i.e. $a$ and $b$ are contained in paths in $H$ which series-contract to different edges in $c(H)$).

As a result of these observations regarding the properties of the vertices $a$ and $b$, it is possible to construct the following (unordered) pairs of properties possessed by $a$ and $b$:

1. $(C+nA, C+nA)$
2. $(C+nA, nC+nA)$
3. $(C+A, C+A)$
4. $(nC+nA, nC+nA)$
5. $(nC+A, nC+A)$

The cases $(C+nA, C+A)$, $(nC+nA, C+A)$, $(nC+A, C+nA)$ and $(nC+nA, nC+A)$ cannot exist as adjacency/non-adjacency is a property which must be possessed
by both vertices simultaneously, and so, there are exactly five possible cases, 1, 2, 3, 4 and 5 which correspond to cases (a), (b1), (b2), (c) and (d), respectively.

The 3-connected analog of Lemma 4.1 is now presented.

**Lemma 4.2** Given a graph $H$ where $c(H)$ is 3-connected, a $H$-path $P$ and a $H$-$Y$-graph $Q$, then the only 3-admissible $(\lambda, \mu)$-operations are

(a) the (0,1)-operation
(b) the (1,2)-operation
(c) the (1,3)-operation
(d) the (2,3)-operation
(e) the (2,4)-operation
(f) the (3,5)-operations
(g) the (4,6)-operations.

**Proof** The first part of the proof is to show that each of the $(\lambda, \mu)$-operations described in the cases (a)-(g) is 3-admissible.

(a) the (0,1)-operation: this operation occurs whenever the endpoints of a $H$-path $P$ are identified with two distinct (non-adjacent) vertices in $c(H)$, see Lemma 4.1 (a) for details. As $c(H)$ is 3-connected, $c(H \cup P)$ and $c(H)$ have identical vertex sets and $|E_{c(H \cup P)}| = |E_{c(H)}| + 1$, hence $c(H \cup P)$ is also 3-connected.

(b) the (1,2)-operation: this occurs whenever one endpoint of a $H$-path $P$ is identified with a vertex $b \in c(H)$, see Figure 6, and the other endpoint of $P$ is identified with a vertex $a \notin c(H)$ which is contained in a path that series-contracts to an edge $xy$ in $c(H)$, such that $b \neq x, y$. The vertex $a$ has degree 3 in both $H \cup P$ and $c(H \cup P)$ hence $c(H \cup P)$ has one additional vertex. Observe that $c(H \cup P)$ does not contain the edge $xy$ but does contain the edges $ab, ax$ and $ay$, hence $c(H \cup P)$ has two additional edges. This is similar to the (1,2)$_A$-operation outlined in Lemma 4.1 (b1) but with the additional caveat that $b \neq x, y$ (clearly the (1,2)$_B$-operation outlined in Lemma 4.1 (b2) is not valid in this instance as the additional vertex has degree 2). Referring to Figure 6, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$) between $x$ (resp. $y$) and $b$ which does not contain $y$ (resp. $x$) such that $P_1$ and $P_2$ are openly disjoint. The paths $ax \cup P_1$ and $ay \cup P_2$ along with the edge $ab$ are three openly disjoint paths between the vertex $a \notin V_{c(H)}$ and the (arbitrarily chosen) vertex $b \in V_{c(H)}$. It follows that $c(H \cup P)$ is 3-connected.

(c) the (1,3)-operation: this occurs whenever the three endpoints $a, b$ and $c$ of a $H$-$Y$-graph $Q$ are identified with three distinct vertices in $c(H)$, see Figure 6. As $Q$ series-contracts to the graph $\{(u, a, b, c), \{ua, ub, uc\}\}$, then it follows that $c(H \cup Q)$ contains one additional vertex $u$ and three additional edges $au, bu$ and $cu$. Referring to Figure 6, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$ and $P_3$) between $a$ (resp. $b$ and $c$) and $v$ which does not contain $b$ or
c (resp. a or c and a or b) such that $P_1$ and $P_2$ and $P_3$ are openly disjoint. The paths $ua \cup P_1$, $ub \cup P_2$ and $uc \cup P_3$ are three openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the (arbitrarily chosen) vertex $v \in V_{c(H)}$. It follows that $c(H \cup P)$ is 3-connected.

(d) the $(2, 3)$-operation: this occurs whenever both endpoints $a$ and $b$ of a $H$-path $P$ are identified with two distinct vertices which are not contained in $c(H)$ and are contained in paths in $H$ which series-contract to two distinct edges, $xy$ and $wz$, in $c(H)$, see Lemma 4.1 (c) for details. Referring to Figure 7, as $c(H)$ is 3-connected then there exists a path $P_1$ between $x$ and $w$ which does not contain $y$ or $z$ and a path $P_2$ between $y$ and $z$ which does not contain $x$ or $w$ such that $P_1$ and $P_2$ are openly disjoint (as $xy \neq wz$ then it can be assumed w.l.o.g. that $x \neq w$ which means that $|E_{P_1}| \geq 1$ and $|E_{P_2}| \geq 0$). The paths $ax \cup P_1 \cup wb$ and $ay \cup P_2 \cup zb$ along with the edge $ab$ are three openly disjoint paths between the vertices $a$ and $b$ and so $c(H \cup P)$ is 3-connected. Clearly this is equivalent to constructing three openly disjoint paths between the vertex $a$ (resp. $b$) and an arbitrary vertex $v \in c(H)$, hence, $c(H \cup P)$ is 3-connected.

(e) the $(2, 4)$-operation: this occurs whenever two of the three endpoints of a $H$-$Y$-graph $Q$, $a$ and $b$ say, see Figure 7, are identified with two distinct vertices in $c(H)$ while the third endpoint of $Q$, $c$ say, is identified with a vertex contained in a path in $H$ which series-contracts to the edge $xy$ in $c(H)$ such that $\{a, b\} \neq \{x, y\}$. Observe that $c(H)$ contains the additional vertices $c$ and $u$, does not contain the edge $xy$ but does contain the additional edges $cx$, $cy$, $au$, $bu$ and $cu$. Referring to Figure 7, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$ and $P_3$) between $a$ (resp. $b$ and $x$) and (an arbitrarily chosen) $v$ which does not contain $b$ or $x$ (resp. $a$ or $x$ and $a$ or $b$) such that $P_1$ and $P_2$ and $P_3$ are openly disjoint. The paths $ua \cup P_1$, $ub \cup P_2$ and $uc \cup cx \cup P_3$ are three openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. A similar argument applies to the second additional vertex $c$. It follows that $c(H \cup P)$ is 3-connected.
this case is split into cases (f1) and (f2);

(f1) the $(3,5)_A$-operation: this occurs whenever one of the three endpoints of a $H$-$Y$-graph $Q$, $a$ say, see Figure 8, is identified with a vertex in $c(H)$ while the remaining two endpoints of $Q$, $b$ and $c$ say, are identified with two distinct vertices (of degree two) which are not in $c(H)$ and are contained in a path in $H$ which series-contracts to the edge $xy$ in $c(H)$ such that $a \neq x, y$. Clearly $b$ and $c$ have degree 3 in $c(H \cup Q)$ hence, along with $u$, $c(H \cup Q)$ has three additional vertices. Observe that $c(H \cup Q)$ does not contain the edge $xy$ but does contain the edges $bx, bc, cy, au, bu$ and $cu$, (or equivalently $cx, bc, by, au, bu$ and $cu$) hence $c(H \cup Q)$ has five additional edges. Referring to Figure 8, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$ and $P_3$) between $a$ (resp. $x$ and $y$) and (an arbitrarily chosen vertex) $v \in V_{c(H)}$ which does not contain $x$ or $y$ (resp. $a$ or $y$ and $a$ or $x$) such that $P_1$ and $P_2$ and $P_3$ are openly disjoint. The paths $ua \cup P_1, ub \cup bx \cup P_2$ and $uc \cup cy \cup P_3$ (or equivalently $ua \cup P_1, ub \cup by \cup P_3$ and $uc \cup cx \cup P_2$) are three openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. Using a similar argument for vertices $b$ and $c$, then it follows that $c(H \cup P)$ is 3-connected.

(f2) the $(3,5)_B$-operation: this occurs whenever one of the three endpoints of $Q$, $a$ say, see Figure 8, is identified with a vertex in $c(H)$ while the remaining two endpoints of $Q$, $b$ and $c$ say, are identified with two distinct vertices which are not contained in $c(H)$ and are contained in paths in $H$ that series-contract to distinct edges $xy$ and $wz$, respectively, in $c(H)$. Clearly $b$ and $c$ have degree 3 in $H \cup Q$ and $c(H \cup P)$ hence, along with $u$, $c(H \cup Q)$ has three additional vertices. Observe that $c(H \cup Q)$ does not contain the edges $xy$ or $wz$ but does contain the edges $bx, by, cz, cw, au, bu$ and $cu$, hence $c(H \cup Q)$ has five additional edges. Referring to Figure 8, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$ and $P_3$) between $a$ (resp. $x$ and $w$) and (an arbitrarily chosen vertex) $v$ which does not contain $x$ or $w$ (resp. $a$ or $w$ and $a$ or $x$) such that $P_1$ and $P_2$ and $P_3$ are openly disjoint (as $xy \neq wz$ then it can be assumed w.l.o.g. that $x \neq w$). The paths $ua \cup P_1, ub \cup bx \cup P_2$ and $uc \cup cw \cup P_3$ are three openly disjoint paths between the vertex $u \notin V_{c(H)}$ and the vertex $v \in V_{c(H)}$. Using a similar argument for vertices $b$ and $c$, then it follows that $c(H \cup P)$ is 3-connected.
(g) this case is split into cases (g1) and (g2):

(g1) the $(4,6)_A$-operation: this occurs whenever each of the three endpoints of a $H$-$Y$-graph $Q$, $a$, $b$ and $c$ say, see Figure 9, are identified with three distinct vertices which are not in $c(H)$ and are contained in paths in $H$ which series-contract to two distinct edges in $c(H)$. Assume that $a$ is identified with a vertex contained in the path in $H$ which series-contracts to the edge $xy$ in $c(H)$ and that $b$ and $c$ are identified with two distinct vertices contained in the path in $H$ which series-contracts to the edge $wz$ in $c(H)$ such that $xy \neq wz$. Clearly $a$, $b$ and $c$ have degree 3 in $H \cup Q$ and $c(H \cup P$ hence, along with $u$, $c(H \cup Q)$ has four additional vertices. Observe that $c(H \cup Q)$ does not contain the edges $xy$ or $wz$ but does contain the edges $ax$, $ay$, $bw$, $bc$, $cz$, $au$, $bu$ and $cu$ (or equivalently $ax$, $ay$, $cw$, $bc$, $bz$, $au$, $bu$ and $cu$) hence $c(H \cup Q)$ has six additional edges. Referring to Figure 9, as $c(H)$ is 3-connected then there exists a path $P_1$ (resp. $P_2$ and $P_3$) between $x$ (resp. $w$ and $z$) and (an arbitrarily chosen vertex) $v$ which does not contain $w$ or $z$ (resp. $x$ or $z$ and $x$ or $w$) such that $P_1$ and $P_2$ and $P_3$ are openly disjoint (as $xy \neq wz$ then it can be assumed w.l.o.g. that $x \neq w, z$). The paths $ua \cup ax \cup P_1$, $ub \cup bw \cup P_2$ and $uc \cup cz \cup P_3$ are three openly disjoint paths between the vertex $u \notin V(c(H))$ and the vertex $v \in V(c(H))$. Using a similar argument for vertices $a$, $b$ and $c$, then it follows that $c(H \cup P)$ is 3-connected.

(g2) the $(4,6)_B$-operation: this occurs whenever each of the three endpoints of a $H$-$Y$-graph $Q$, $a$, $b$ and $c$ say, see Figure 9, are identified with three distinct vertices which are not in $c(H)$ and are contained in paths in $H$ which series-contract to three distinct edges $xy$, $wz$ and $pq$, respectively, in $c(H)$. Clearly $a$, $b$ and $c$ have degree 3 in $H \cup Q$ and $c(H \cup Q)$ hence, along with $u$, $c(H \cup Q)$ has
four additional vertices. Observe that \( c(H \cup Q) \) does not contain the edges \( xy, wz \) or \( pq \) but does contain the edges \( ax, ay, bw, bz, cp, cq, au, bu \) and \( cu \), hence \( c(H \cup Q) \) has six additional edges. Referring to Figure 9, as \( c(H) \) is 3-connected then there exists a path \( P_1 \) (resp. \( P_2 \) and \( P_3 \)) between \( x \) (resp. \( w \) and \( p \)) and (an arbitrarily chosen vertex) \( v \) which does not contain \( w \) or \( p \) (resp. \( x \) or \( p \) and \( x \) or \( w \)) such that \( P_1 \) and \( P_2 \) and \( P_3 \) are openly disjoint (As \( xy \neq wz \neq pq \) then it can be assumed w.l.o.g. that \( x \neq w \neq p \)). The paths \( ua \cup ax \cup P_1, ub \cup bw \cup P_2 \) and \( uc \cup cp \cup P_3 \) are three openly disjoint paths between the vertex \( u \notin V_{c(H)} \) and the vertex \( v \in V_{c(H)} \). Using a similar argument for vertices \( a, b \) and \( c \), then it follows that \( c(H \cup P) \) is 3-connected.

The second part of the proof is to show that the \((\lambda, \mu)\)-operations contained in cases \((a) - (g2)\) are in fact the only valid 3-admissible \((\lambda, \mu)\)-operations.

Recalling the proof of Lemma 4.1:

- a vertex is either Core (C) or non-Core (nC), and
- two vertices are either Adjacent (A) or non-Adjacent (nA).

It is now possible to construct the following (unordered) pairs of properties possessed by the distinct vertices \( a \) and \( b \) which are identified with the endpoints of a \( H \)-path \( P \):

1. \((C+nA, C+nA)\) 2. \((C+nA, nC+nA)\) 3. \((nC+nA, nC+nA)\)

These are, respectively, the cases \((a),(b),(d)\). Note \((C+A,nC+A)\) and \((nC+A, nC+A)\) are disallowed as both cases would result in the introduction of a vertex of degree 2, hence rendering \( c(H \cup P) \) at most 2-connected.

Similarly, it is possible to construct the following (unordered) triples of properties possessed by the distinct vertices \( a, b \) and \( c \), which are identified with the endpoints of a \( H-Y \)-graph \( Q \). Note that if a vertex is not assigned an A or nA, then that vertex’s adjacency with the other vertices is irrelevant.

4. \((C, C, C)\) 5. \((C, C+nA, nC+nA)\) 6. \((C, nC+A, nC+A)\)

7. \((C, nC+nA, nC+nA)\) 8. \((nC+nA, nC+A, nC+A)\) 9. \((nC+nA, nC+nA, nC+nA)\)

The cases 4, 5, 6, 7, 8 and 9, respectively are the cases \((c),(e),(f1),(f2),(g1)\) and \((g2)\), respectively. Note \((nC+A, nC+A, nC+A)\) is disallowed as \( c(H \cup P) \) would become disconnected by removing some pair of the endpoints of \( Q \), thus rendering \( c(H \cup P) \) at most 2-connected. Hence, there are exactly nine possible cases, 1, 2, 3, 4, 5, 6, 7, 8 and 9 which correspond to cases \((a),(b),(c),(d),(e),(f1),(f2),(g1)\) and \((g2)\), respectively.
It is now possible to state and prove the main result of this work.

**Theorem 4.3 (Main Result)** A graph $G$ is 3-connected if and only if $G$ can be constructed from some $G_0 \in [K_4]_{\sim_2}$ (i.e. $c(G_0) \simeq K_4$) by successive 3-admissible unions of either a $H$-path $P$ or a $H$-$Y$-graph $Q$ and subgraphs $H$ which have already been constructed and every intermediate graph has a 3-connected $\sim_2$-core.

**Proof** It is first necessary to show that given a graph $H$ where $c(H)$ is 3-connected and a $H$-path $P$ (resp. a $H$-$Y$-graph $Q$), then following a 3-admissible union the graph $c(H \cup P)$ (resp. $c(H \cup Q)$) is 3-connected. This fact has just been shown in Lemma 4.2.

And now for the opposite direction. Assume that $c(G)$ is 3-connected, then $c(G)$ contains a graph which is contained in the $\sim_2$-class whose $\sim_2$-core is isomorphic to $K_4$ and hence also contains a maximal subgraph $H$ which is constructible as per the statement of the result. Suppose that $G \neq H$ and consider the subgraph $G \setminus H$ contained in $G$. Observe that $G \setminus H$ cannot contain an edge $ab$ where $ab \notin E_H$ and $a, b \in V_G$ as this would contradict the maximality of $H$. It follows therefore that $G \setminus H$ must contain at least one vertex $u$ such that $u \notin V_H$. As $G$ is connected then it is possible to choose a vertex $u \in V_{G \setminus H}$ such that there is at least one path in $G$ which series-contracts to the edge $ua$ (in $c(G)$) where $a \in V_H$ and $ua \in E_{c(G \setminus H)}$. As $c(G)$ is 3-connected then there must exist (at least) three openly disjoint paths between $u \in V_{G \setminus H}$ and $a \in V_H$. One of these paths has endpoints $u$ and $a$ and series-contracts to the edge $ua$ in $c(G)$, by assumption. There exists (at least) two other openly disjoint paths $P_1$ and $P_2$ which, respectively, contain sub-paths $P_1^*$ with endpoints $u$ and $b$, and $P_2^*$ with endpoints $u$ and $c$, where $b$ and $c$ are the only vertices in $P_1^*$ and $P_2^*$, respectively, that are contained in $H$. This is illustrated in Figure 10.

![Figure 10: Deriving the contradiction that $H$ is maximal](image)

It now follows that all interior vertices (i.e. non-endpoints) of $P_1^*$ and $P_2^*$ have degree 2, otherwise there would be at least one $H$-path joining an interior vertex in $P_1^*$ (resp. $P_2^*$) with either a vertex in $P_2^*$ (resp. $P_1^*$) or in $H$. If such a $H$-path were to exist then this would contradict the maximality of $H$ with respect to the
construction outlined in the statement. Hence $P_1^*$ and $P_2^*$ must series-contract
to edges $ub$ and $uc$, respectively, in $c(G)$. Observe however, that the union of
$ua, ub$ and $uc$ is a $H$-$Y$-graph which contradicts the assumption that $H$ is max-
imal with respect to the construction outlined in the statement, hence $H = G$. □

5. The $k$-connected case when $k \geq 4$

The possibility of deriving analogs of Theorems 3.2 and 4.3 for $k$-connected
graphs whenever $k \geq 4$ is now examined.

**Definition 5.1** A $k$-star-graph is a graph $S$ such that $c(S)$ is isomorphic to the
graph $(\{u, a_1, ..., a_k\}, \{ua_i | i \in 1, ..., k\})$.

Observe that a $Y$-graph is a 3-star-graph and analogously, the vertices $a_i$ for
$i = 1, ..., k$ in a $k$-star-graph $S$ are called the endpoints of $S$. A $k$-connected
analog of the $H$-$Y$-graph concept is now introduced.

**Definition 5.2** Given a graph $H$, then a $k$-star-graph $S$ with endpoints $a_i$ for
$i = 1, ..., k$ is a $H$-$k$-star-graph whenever $H \cap S = \{a_i | i = 1, ..., k\}$.

In a similar fashion to the previous section the union of a graph $H$ where $c(H)$
is $k$-connected and a $H$-path $P$ (resp. a $H$-$k$-star-graph $Q$) is $k$-admissible if
and only if $c(H \cup P)$ (resp. $c(H \cup Q)$) is $k$-connected.

Observe that it is not possible to derive analogs of Theorem 3.2 for $k$-connected
graphs with $k \geq 4$ as degree 3 vertices may be introduced in any similar con-
struction and clearly a graph with a degree 3 vertex cannot have a $\sim_2$-core which
is more than 3-connected. Moreover, Example 5.3 illustrates the fact that even if
the condition that “the $\sim_2$-core of every intermediate graph in the construction
of a $k$-connected graph is $k$-connected” is dropped, there still does not exist a
$k$-connected analog of Theorems 3.2 and 4.3 for $k \geq 4$.

**Example 5.3** It is not possible to construct $K_{2,2,2}$ from $K_5$ by successively
adding either a $H$-path $P$ or a 4-star-graph $S$ to subgraphs $H$ which have already
been constructed. Observe that $|V_{K_5}| = 5$ and $|V_{K_{2,2,2}}| = 6$, and that $|E_{K_5}| = 10$
and $|E_{K_{2,2,2}}| = 12$, meaning that a $H$-path $P$ of length 2 is the only possible
addition which can be made to $K_5$ which could possibly result in $K_{2,2,2}$. As all
vertices in both $K_5$ and $K_{2,2,2}$ have degree 4, then clearly $K_5 \cup P \neq K_{2,2,2}$.
A $k$-connected analog of Lemmas 4.1 and 4.2, for $k \geq 4$, is now stated.

**Lemma 5.4** Given a graph $H$ where $c(H)$ is $k$-connected with $k \geq 4$, a $H$-path $P$ and a $H$-$k$-star-graph $S$ then the only $k$-admissible $(\lambda, \mu)$-operations are

(a) the $(0,1)$-operation

(b) the $(1, k)$-operation.

**Proof** The first part of the proof is to show that the $(0,1)$-operation and the $(1,k)$-operation are both $k$-admissible.

(a) the $(0,1)$-operation: this occurs whenever the endpoints of a $H$-path $P$ are identified with two distinct and non-adjacent vertices in $c(H)$. The $H$-path $P$ series-contracts to a single edge in $c(H \cup P)$ and so, as $c(H)$ is $k$-connected, $c(H \cup P)$ and $c(H)$ have identical vertex sets and $|E_{c(H \cup P)}| = |E_{c(H)}| + 1$ then $c(H \cup P)$ is also $k$-connected.

(b) the $(1,k)$-operation: this occurs whenever each of the $k$ endpoints $a_1, \ldots, a_k$ of $S$ are identified with $k$ distinct vertices contained in $c(H)$. As $c(S)$ is isomorphic to the graph $(\{u, a_1, \ldots, a_k\}, \{ua_i \mid i \in 1, \ldots, k\})$, and $H$ is $k$-connected, by assumption, then it follows that there exists $k$ openly disjoint paths in $c(H \cup S)$ between the additional vertex $u$ and an arbitrary vertex $v \in V_{c(H \cup S)}$ (where $v \neq u$), hence $c(H \cup S)$ is also $k$-connected.

The second part of the proof is to show that the $(0,1)$-operation and the $(1, k)$-operation are in fact the only $k$-admissible $(\lambda, \mu)$-operations when $k \geq 4$. Clearly the union of $H$ and a $H$-$d$-star graph $S$ for $d < k$ is not $k$-admissible as $c(H \cup S)$ would contain a vertex of degree $d$ rendering $c(H \cup S)$ at most $d$-connected. Hence, only the unions of $H$ with a $H$-$k$-star graph $S$ or a $H$-path $P$ can be $k$-admissible. Furthermore, the endpoints of the $H$-path $P$ or the $H$-$k$-star graph $S$ which is added to $H$ at each step, must be identified with vertices contained in $c(H)$ or else $c(H \cup S)$ would contain a vertex of degree 3, thus rendering $c(H \cup S)$ at most 3-connected. Hence, the $(0,1)$-operation and the $(1, k)$-operation are the only $k$-admissible $(\lambda, \mu)$-operations. \qed
6. Comments

The author is currently utilising the iterated construction of 2-connected and 3-connected graphs which has just been described to construct a framework within which it is possible to give upper bounds on the number of non-isomorphic 2-connected and 3-connected $\sim_2$-cores with a given number of vertices and edges.

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