PARTIAL EULER PRODUCTS AS A NEW APPROACH TO
RIEMANN HYPOTHESIS

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Abstract. In this paper, we show that Riemann hypothesis (concerning zeros of
the zeta function in the critical strip) is equivalent to the analytic continuation of
Euler products obtained by restricting the Euler zeta product to suitable subsets
$M_k$, $k \geq 1$ of the set of prime numbers. Each of these Euler product defines so
a partial zeta function $\zeta_k(s)$ equal to a Dirichlet series of the form
$\sum \epsilon(n)/n^s$, with coefficients $\epsilon(n)$ equal to 0 or 1 as $n$ belongs or not to the population of integers
generated by $M_k$. We show that usual formulas of the arithmetic adapt themselves
to such populations (Möbius, Mertens, Lambert series,...). We envisage also the
study of summations inside these populations and new functions (generalizations of
the integer part function, of the harmonic series) directly connected to the existence
of analytical continuations.

1. Introduction

The number $\pi(x)$ of prime numbers less or equal to $x$ is known to be equivalent to
$\frac{x}{\ln(x)}$ and prime number theorem has been proved with the estimation:

$$\pi(x) = \text{Li}(x) + O(x e^{-c \sqrt{\ln(x)}})$$

for some suitable positive constant $c$: an improvement (Vinogradov and Korobov) of
the error term and associated comments can be found in [Nar] p.236. One importance
of Riemann hypothesis (which states that the non-real zeros of $\zeta(s)$ have real part equal
to $\frac{1}{2}$) lies in the fact that the estimation

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \ln(x))$$

is probably true. Many arithmetical conjectures and various problems are directly
connected to Riemann hypothesis. The zeta function has been studied intensively with
developments in various directions. Literature on Riemann zeta function is known to
be huge and very diversified and this work hopes not to find results already published.

One purpose of this work is to show that analytic continuation of partial zeta functions

$$\zeta_k(s) = \prod_{p \in M_k} \frac{1}{1 - \frac{1}{p^s}} \quad \text{with} \quad M_k = \{p_1, p_{k+1}, p_{2k+1}, p_{3k+1}, \cdots\}$$

is equivalent to Riemann hypothesis: here, $k$ is an integer and $\mathbb{P} = \{p_n\}_{n \in \mathbb{N}^*}$ is the set
of all primes. The obtaining of the analytical continuation of functions $\zeta_k(s)$ is in direct
relation with the study of subsets of $\mathbb{N}^*$ generated by an arbitrary subset of $\mathbb{P}$: this
approach is a new facet of arithmetic. So, another purpose of this work is to adapt
classic formulas of arithmetic to such subsets.

Every subset $M$, included in $\mathbb{P}$ (the set of prime numbers) generates, by successive
products of the different elements of $M$, a subset of $\mathbb{N}^*$, that is a sub-population of $\mathbb{N}^*$.
As a consequence, we use notation $\text{pop}(M)$ to indicate such a set, $\text{pop}$ term being the
abbreviation of population. When there is no ambiguity on the choice of $M$, one writes
simply $\text{pop}$ in the place of $\text{pop}(M)$. By definition, $\text{pop}(\mathbb{P}) = \mathbb{N}^*$ and each function $\zeta_k(s)$
is constructed by choosing for $M$ the prime numbers taken by jumps of $k$ inside the set of index
of the descriptive formula $\mathbb{P} = (p_n)_{n \in \mathbb{N}^*}$. Such an $M$ is called an arithmetical
list of reason $k$ (see Conventions of section [2]): naturally, this does not mean that the
$p_i$’s chosen constitute an arithmetical sub-sequence of $\mathbb{N}^*$.

The integer part function $[x]$ of $x$, appropriate for the analysis in $\mathbb{N}^*$, is then replaced
by the function

$$N_{\text{pop}}(x) \equiv \text{card}([1; x] \cap \text{pop})$$

and another useful function is

$$S_{\text{pop}}(x) \equiv \sum_{n \in \text{pop} \cdot n \leq x} \frac{1}{n}$$

(generalization of the harmonic series)

The rigorous evaluation of $N_{\text{pop}}(x)$ (or of $S_{\text{pop}}(x)$) insures the existence of the ana-
tycal continuation of partial zeta functions $\zeta_k(s)$ to the open set $\{\Re(s) > \frac{1}{2}\} - [\frac{1}{2}; 1]$. More precisely, for $M$ an arithmetical list of reason $k$, these unproved evaluations are

$$N_{\text{pop}}(x) = Cte \frac{x \ln(x)}{\sqrt{x \ln(x)}} + \psi(x)$$

with $\psi(x) = O(x^{\frac{1}{2} + \epsilon})$ or

$$S_{\text{pop}}(x) = Cte \frac{\ln(x)}{\sqrt{x \ln(x)}} + \phi(x)$$

with $\phi(x) = O(x^{\frac{1}{2} - \epsilon})$. Formula 3.29 is the natural equivalent of $N_{\text{pop}}(x)$ and we indicate
in last section some consequences connected to this equivalence. In summary, when one
passes through evaluation of $N_{\text{pop}}(x)$ (to obtain the analytical continuation of $\zeta_k(s)$),
Riemann’s problem is returned to an arithmetical problem. One can also tempt others
techniques to obtain the existence of the analytical continuation of $\zeta_k(s)$.

2. Preliminaries

In the following, $n$ stands for an integer, $p$ stands for a prime number. We denote by
$\mathbb{P} = (p_n)_{n \in \mathbb{N}^*}$ the set of all primes $p$ (with $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7 \cdots$ ), by $\pi(x)$
the number of $p \in \mathbb{P}$ satisfying $2 \leq p \leq x$. As usual, $\zeta(s) = \sum_{n \in \mathbb{N}^*} \frac{1}{n^s}$ is the Riemann
zeta function, and $Z_\zeta$ the set of zeros of $\zeta(s)$ in the half space $\Re(s) > \frac{1}{2}$. It is a known
fact that zeros of $\zeta(s)$ in the critical strip are displayed symetrically relatively to the
line $\sigma = \frac{1}{2}$: thus, we may restrict some considerations to $\Re(s) > \frac{1}{2}$.

We shall need in this paper, partial zeta functions $Z_M$ defined by an Euler product:

$$Z_M(s) = \prod_{p \in M} \frac{1}{1 - \frac{1}{p^s}}$$

where $M$ is a subset of $\mathbb{P}$. Expanding each function:

$$\frac{1}{(1 - \frac{1}{p^s})} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots$$
in power series, with \( p \in M \), and multiplying all expansions, we develop the Euler product as a Dirichlet series \( \sum_{n \in \text{pop}_M} \frac{n^{-s}}{n^{s}} \), clearly analytic for \( \Re(s) > 1 \): here, \( n \) moves in the set \( \text{pop}_M \) of all integers of the form \( n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots \), with all \( \alpha_i \geq 0 \), \( \alpha_i \in \mathbb{N}^* \) and all \( q_i \in M \). The characteristic function of \( \text{pop}_M \) will be denoted \( 1_{\text{pop}_M} \), and equals 1 when \( n \in \text{pop}_M \) and 0 elsewhere. Thus, an equality of the form

\[
\sum_{n \in \text{pop}_M} \alpha_n \frac{n^{-s}}{n^{s}} = \sum_{n \in \mathbb{N}^*} 1_{\text{pop}_M}(n) \frac{\alpha_n}{n^{s}}
\]

means that the Dirichlet series considered on left side satisfies \( \alpha_n = 0 \) whenever \( n \notin \text{pop}_M \).

We also need:

\[
\eta_M(s) = \sum_{p \in M} \frac{1}{p^{s}}
\]

clearly analytic for \( \Re(s) > 1 \).

**Lemma 2.1.** Let \( M \subset \mathbb{P} \) and

\[
f(s) = \sum_{n \in \text{pop}_M} \frac{\alpha_n}{n^{s}}, \quad g(s) = \sum_{n \in \text{pop}_M} \frac{\beta_n}{n^{s}}
\]

be two absolutely convergent Dirichlet series. Then, the Dirichlet product

\[
(fg)(s) = \sum_{n \in \mathbb{N}^*} \frac{\gamma_n}{n^{s}}
\]

of \( f(s) \) and \( g(s) \) satisfies \( \gamma_n = 0 \) whenever \( n \notin \text{pop}_M \).

The lemma follows from equality

\[
\gamma_n = \sum_{kl=n, k \geq 1, l \geq 1} \alpha_k \beta_l
\]

and the fact that \( k|n \) with \( n \in \text{pop}_M \) implies \( k \in \text{pop}_M \) and \( l \in \text{pop}_M \).

Let \( Z_M(s) = \sum_{n \in \text{pop}_M} \frac{1}{n^{s}} \) be the associated partial zeta function (defined and analytic for \( \Re(s) > 1 \)). One verifies that some of known formulas concerning \( \zeta(s) \) become now:

\[
\frac{Z_M(s-1)}{Z_M(s)} = \sum_{n \in \text{pop}_M} \frac{\Phi(n)}{n^{s}} \quad \text{for} \quad \Re(s) > 2
\]

\[
Z_M(s) Z_M(s-1) = \sum_{n \in \text{pop}_M} \frac{\sigma(n)}{n^{s}} \quad \text{for} \quad \Re(s) > 2
\]

\[
Z_M^2(s) = \sum_{n \in \text{pop}_M} \frac{\tau(n)}{n^{s}} \quad \text{for} \quad \Re(s) > 1
\]

\[
\frac{1}{Z_M(s)} = \sum_{n \in \text{pop}_M} \frac{\mu(n)}{n^{s}} \quad \text{for} \quad \Re(s) > 1
\]

\[
(1 + \eta_M(s)) \ Z_M(s) = \sum_{n \in \text{pop}_M} \frac{\nu(n)}{n^{s}} \quad \text{for} \quad \Re(s) > 1
\]

\[
\sum_{n \in \text{pop}_M} \frac{\ln(n)}{n^{s}} = Z_M(s) \sum_{p \in M} \frac{\ln(p)}{p^{s-1}} \quad \text{for} \quad \Re(s) > 1
\]

where, as usual:

\( \Phi(n) = \) the Euler Phi-function:

\( \sigma(n) = \sum_{d|n} d \), the sum of the divisors of \( n \);
\( \tau(n) = \sum_{d|n} 1 \), the number of divisors of \( n \);
\( \nu(n) = \sum_{p|n} 1 \), the number of distinct prime factors of \( n \);
\( \mu(n) \) the Möbius function, \( \mu(1) = 1 \), \( \mu(n) = 0 \) if \( n \) is divisible by a square \( > 1 \), 
\( = (-1)^{\nu(n)} \) in others cases.

Subsets \( M \) of main interest will be arithmetical lists obtained from the list of all primes:
\[ \mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, \cdots \} \]

Conventions:
An arithmetical list of \( \mathbb{P} \) having a reason \( r > 0 \) is a subset \( M \subset \mathbb{P} \) of the form
\[ M = \{p_{r_0}, p_{r_0+r}, p_{r_0+2r}, \cdots \} \]
for some \( r_0 > 0 \) with \( r_0 \leq r \).
Whenever a function \( f_M(\cdots) \) is defined or depends on an arithmetical list determined
by its two first elements \( a \in M \) and \( b \in M \) with \( a = p_{r_0} \) and \( b = p_{r_0+r} \), we shall write
\[ M = M_{a,b} \]
\[ f_M(\cdots) = f_{a,b}(\cdots) \]
Moreover, when the arithmetical list \( M_{a,b} \) starts with the first prime (ie one has \( a = p_1 = 2 \)), we write simply
\[ M_{a,b} = M_r \]
\[ f_{a,b}(\cdots) = f_r(\cdots) \]
where \( r \) is the reason of the list.

For example, \( M = \{5, 13, 23, 37, 47, 61, 73, \cdots \} \) is an arithmetical list with first element \( p_3 = 5 \) and reason \( r = 3 \).

We observe that many results concerning functions built with an arithmetical list
are easily adapted to functions built with subsets \( M \subset \mathbb{P} \) which are finite union of
arithmetical lists having all the same reason \( r > 0 \) (such an \( M \) has the following property:
\( p_k \in M \) implies \( p_{k+r} \in M \) and \( p_{k-r} \in M \) whenever \( k-r \geq 1 \)).
With our conventions, choosing an \( M \) of the form \( M_{a,b} \), one may write \( Z_M(s) = Z_{a,b}(s) \)
and \( 1_{pop,M}(n) = 1_{pop,a,b}(n) \). In fact,
\[
\zeta_k(s) = \prod_{p \in M_k} \frac{1}{(1 - \frac{1}{p^s})} \quad \text{with} \quad M_k = \{p_1, p_{k+1}, p_{2k+1}, p_{3k+1}, \cdots \}
\]
\[
\zeta_2(s) = \prod_{n \in \mathbb{N} \ n \geq 0} \frac{1}{(1 - \frac{1}{p_{2n+1}^s})} = \frac{1}{(1 - \frac{1}{2^s})} \frac{1}{(1 - \frac{1}{5^s})} \frac{1}{(1 - \frac{1}{7^s})} \cdots = Z_{2,5}(s)
\]
\[
\zeta_3(s) = \prod_{n \in \mathbb{N} \ n \geq 0} \frac{1}{(1 - \frac{1}{p_{3n+1}^s})} = \frac{1}{(1 - \frac{1}{2^s})} \frac{1}{(1 - \frac{1}{7^s})} \frac{1}{(1 - \frac{1}{13^s})} \cdots = Z_{2,7}(s)
\]
\[
Z_{3,7}(s) = \prod_{n \in \mathbb{N} \ n \geq 0} \frac{1}{(1 - \frac{1}{p_{3n+2}^s})} = \frac{1}{(1 - \frac{1}{3^s})} \frac{1}{(1 - \frac{1}{7^s})} \frac{1}{(1 - \frac{1}{13^s})} \cdots = Z_{3,7}(s)
\]
\[
Z_{2,5}(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{11^s} + \frac{1}{16^s} + \frac{1}{17^s} + \frac{1}{20^s} + \frac{1}{22^s} + \frac{1}{23^s} + \frac{1}{25^s} + \frac{1}{31^s} + \frac{1}{32^s} + \cdots
\]
\[
Z_{3,7}(s) = 1 + \frac{1}{3^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{13^s} + \frac{1}{19^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{29^s} + \frac{1}{37^s} + \frac{1}{39^s} + \frac{1}{43^s} + \frac{1}{49^s} + \cdots
\]
\[
\eta_{3,7}(s) = \frac{1}{3^s} + \frac{1}{7^s} + \frac{1}{13^s} + \frac{1}{19^s} + \frac{1}{29^s} + \frac{1}{37^s} + \frac{1}{43^s} + \cdots
\]
Properties are equivalent:

We now show that there exists a function \( f \), which satisfy formula of 2). Assuming \( \Re \eta \leq \alpha_n \leq \beta_n \leq \alpha_{n+1} \) for all \( n, \alpha_1 > 1 \) and \( \sum_n \frac{1}{\alpha_n^\lambda} < +\infty \) for every \( \lambda > 1 \).

Then, the following properties holds:

1. Euler products

\[
E_1(s) = \prod_n \frac{1}{(1 - \frac{1}{\alpha_n^s})} \quad E_2(s) = \prod_n \frac{1}{(1 - \frac{1}{\beta_n^s})}
\]

are well defined and analytic for \( \Re(s) > 1 \).

2. There exists a function \( f(s) \), defined and analytic for \( \Re(s) > 0 \), which never vanishes on this set, such that \( E_1(s) = E_2(s)f(s) \) when \( \Re(s) > 1 \).

If one knows besides that \( E_1(s) \) has a meromorphic (resp. analytic) continuation for \( \Re(s) > a \) for some \( 0 < a < 1 \) (for example, \( E_1 \) may be the Riemann zeta function as in [Gr-Sc]), our lemma implies that \( E_2 \) has also a meromorphic (resp. analytic) continuation for \( \Re(s) > a \). Therefore, functions \( E_1 \) and \( E_2 \) have the same zeros and same poles for \( \Re(s) > a \).

Proof: It is known that \( |(1 - (1 - z)\varepsilon^z)| \leq |z|^2 \) for \( |z| \leq 1 \), hence the infinite product \( \prod_n (1 - \frac{1}{\alpha_n^s}) e^{\frac{1}{\alpha_n^s}} \) defines an analytic function for \( \Re(s) > \frac{1}{2} \). The equation \( 1 - \frac{1}{\alpha_n^s} = 0 \) has no zeros in variable \( s \), hence the infinite product never vanishes. Clearly,

\[
E_1(s) = e^{\sum_n \frac{1}{\alpha_n^s} \prod_n \frac{1}{(1 - \frac{1}{\alpha_n^s}) e^{\frac{1}{\alpha_n^s}}}}
\]

for \( \Re(s) > 1 \) and this formula clearly defines an analytic function on that set. Since \( \sum_n \frac{1}{\alpha_n^\lambda} < +\infty \) for every \( \lambda > 1 \), we get same conclusions for \( \prod_n (1 - \frac{1}{\beta_n^s}) e^{\frac{1}{\beta_n^s}} \) : this proves 1).

We now show that there exists a function \( f(s) \), defined and analytic for \( \Re(s) > \frac{1}{2} \), which satisfy formula of 2). Assuming \( n \geq 1 \), one has, for \( \Re(s) > 1 \):

\[
E_1(s) = E_2(s) \frac{\prod_n (1 - \frac{1}{\alpha_n^s}) e^{\frac{1}{\alpha_n^s}}}{\prod_n (1 - \frac{1}{\beta_n^s}) e^{\frac{1}{\beta_n^s}}}
\]
We put:

\[ f(s) = \prod_n \left( 1 - \frac{1}{\sigma_n} \right) e^{\frac{1}{\sigma_n}} = \prod_n \left( 1 - \frac{1}{\alpha_n} + \frac{1}{\alpha_n} - \frac{1}{\beta_n} + \frac{1}{\beta_n} - \frac{1}{\alpha_n} + \cdots \right) e^{\frac{1}{\alpha_n}} e^{\frac{1}{\beta_n}} e^{\frac{1}{\alpha_n}} e^{\frac{1}{\beta_n}} \]

From Dirichlet series, we see that the series \( \frac{1}{\alpha_i} - \frac{1}{\beta_i} + \cdots + \frac{1}{\alpha_n} - \frac{1}{\beta_n} + \cdots \) is analytic in the half-plane \( \Re(s) > 0 \). Thus, \( f(s) \) has the required properties for \( \Re(s) > \frac{1}{2} \).

In the general case, taking \( k \in \mathbb{N}^* \), we use the Weierstrass factor

\[ W_k(z) = (1 - z) e^{\frac{z}{2} + \cdots + \frac{z^k}{k}} \]

having the property \( |1 - W_k(z)| \leq |z|^{k+1} \) for \( |z| \leq 1 \). Writing:

\[ E_1(s) = E_2(s) \frac{\prod_n W_k(\frac{1}{\alpha_n})}{\prod_n W_k(\frac{1}{\alpha_n})} e^{\frac{1}{\alpha_1} - \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_n} - \frac{1}{\beta_n} + \cdots} e^{\frac{1}{\beta_1} - \frac{1}{\beta_1} + \cdots + \frac{1}{\alpha_n} - \frac{1}{\beta_n} + \cdots} \]

our conclusion follows for \( \Re(s) > \frac{1}{1+1} \). Now, \( f(s) \) is the analytic continuation of \( \frac{E_1(s)}{E_2(s)} \) for \( \Re(s) > \frac{1}{1+1} \) and all \( k \in \mathbb{N}^* \): this proves the lemma.

**Proposition 3.2.** Let \( k \geq 2, k \in \mathbb{N}^* \) and \( \Omega \) be a simply connected open set contained in \( \Re(s) > 0 \) in which the Riemann function \( \zeta(s) \) never vanishes. The following properties hold:

1. \( \zeta_k(s) \) extends analytically for \( s \in \Omega \).
2. There exists a function \( g_k(s) \) defined and analytic for \( \Re(s) > 0 \), having no zeros on this half-plane, such that

\[ \zeta(s) = g_k(s) \zeta_k(s)^k \quad \text{for} \quad s \in \Omega \]

It follows that

\[ \zeta_k(s) \sim \frac{A_k}{\sqrt{1-s}} \quad \text{as} \quad s \to 1 \]

for some constant \( A_k \).

**Proof:** One has \( \zeta_k(s) = \prod_{p \in M_k} \frac{1}{1 - \frac{1}{p^s}} \). For \( i \geq 1, i \leq k \), we define

\[ A_i = \{ p_i, p_{k+i}, p_{2k+i}, p_{3k+i}, \ldots \} \text{ and } Z_{p_i,p_{k+i}}(s) = \prod_{p \in A_i} \frac{1}{1 - \frac{1}{p^s}} \]

Each \( A_i \) is an arithmetical list of \( \mathbb{P} \) with \( A_1 = M_k \) and \( \mathbb{P} = \bigcup_{i=1}^{k} A_i \). One has \( \zeta(s) = \prod_{i=1}^{k} Z_{p_i,p_{k+i}}(s) \) and \( \zeta_k(s) = Z_{p_1,p_{k+i}}(s) \). Using lemma 3.1, we get for every \( i > 1 \) an analytic function \( f_i(s) \) having no zeros for \( \Re(s) > 0 \) such that \( Z_{p_i,p_{k+i}}(s) = f_i(s) \zeta_k(s) \) for \( \Re(s) > 1 \). Putting \( g_k(s) = f_1(s)f_2(s) \cdots f_{k-1}(s) \), we get \( \zeta(s) = g_k(s) \zeta_k(s)^k \) for \( \Re(s) > 1 \).

Let us now assume \( s \in \Omega \). There exists an analytic function \( h(s) \) analytic in \( \Omega \) such that \( \frac{\zeta_k(s)}{g_k(s)} = e^{h(s)} \) for \( s \in \Omega \). Thus

\[ \left( \frac{\zeta_k(s)}{g_k(s)} \right)^k = 1 \quad \text{for} \quad \Re(s) > 1 \]

therefore, for some fixed \( \alpha \in \mathbb{Z} \), \( \zeta_k(s) = e^{2i\pi \frac{\alpha}{k}} e^{\frac{h(s)}{k}} \) for \( \Re(s) > 1 \). This formula defines now the analytic continuation of \( \zeta_k(s) \) for \( s \in \Omega \) and formula 3.1 holds for \( s \in \Omega \) by uniqueness of analytic continuations.
The following lemma is contained in the proof of lemma [3.1].

**Lemma 3.3.** Let $M \subset \mathbb{P}$. The function $W_M(s) = \prod_{p \in M} \frac{1}{1 - \frac{s}{p}}$ is analytic for $\Re(s) > \frac{1}{2}$ and has no zeros on this open set. It satisfies:

\[
Z_M(s) = W_M(s) e^{\eta_M(s)} \quad \text{for} \quad \Re(s) > 1
\]

4. **Functions $\eta_M(s)$**

As defined previously, $\eta_M(s) = \sum_{p \in M} \frac{1}{p^s}$ and $\eta_1(s) = \sum_{p \in \mathbb{P}} \frac{1}{p^s} = \eta_{2,3}(s)$.

**Proposition 4.1.** Let $M = \{p_{r_0}, p_{r_0+r}, p_{r_0+2r}, \cdots \}$ be an arithmetical list of $\mathbb{P}$ with a reason $r > 1$ and $\Omega$ be an open connected set contained in $\Re(s) > 0$. Then, one has:

\[
\eta_1(s) = r \eta_M(s) + w(s) \quad \text{for} \quad \Re(s) > 1
\]

where $w(s)$ is analytic for $\Re(s) > 0$.

Therefore, $\eta_M(s)$ has an analytic (resp. meromorphic) continuation to $\Omega$ if and only if $\eta_1(s)$ has the same property.

**Proof:** Let $M_r = \{p_1, p_{1+r}, p_{1+2r}, \cdots, p_{1+nr}, \cdots \} \subset \mathbb{P}$. For every $0 \leq j < r$, we put:

$M^{(j)} = \{p_{1+j}, p_{1+j+r}, p_{1+j+2r}, \cdots, p_{1+j+nr}, \cdots \}$

to be the shifted list of $M_r$, hence $M^{(0)} = M_r$. One has

\[
\eta_{M^{(0)}}(s) - \eta_{M^{(j)}}(s) = \sum_{p \in M_r} \frac{1}{p^s} - \sum_{p \in M^{(j)}} \frac{1}{p^s}
\]

for $\Re(s) > 1$, thus

\[
\eta_{M^{(0)}}(s) - \eta_{M^{(j)}}(s) = \frac{1}{p_1^s} - \frac{1}{p_{1+j}^s} + \frac{1}{p_{1+r}^s} - \frac{1}{p_{1+j+r}^s} + \cdots + \frac{1}{p_{1+nr}^s} - \frac{1}{p_{1+j+nr}^s} + \cdots
\]

The Dirichlet series on the right hand side is an analytic function $w_j(s)$ in $\Re(s) > 0$ for every $0 \leq j < r$. From

\[
\eta_1(s) = \eta_{M^{(0)}}(s) + \eta_{M^{(1)}}(s) + \cdots + \eta_{M^{(r-1)}}(s)
\]

we get

\[
\eta_1(s) = \eta_{M^{(0)}}(s) + (\eta_{M^{(0)}}(s) - w_0(s)) + \cdots + (\eta_{M^{(0)}}(s) - w_{r-1}(s))
\]

which leads to

\[
\eta_1(s) = r \eta_{M^{(0)}}(s) - W(s)
\]

with $W(s) \equiv w_0(s) + \cdots + w_{r-1}(s)$. This proves [4.1] for $M = M^{(0)}$.

The arithmetical sublist $M = \{p_{r_0}, p_{r_0+r}, p_{r_0+2r}, \cdots \}$ is one of the $M^{(j)}$ (with $j = r_0-1$) and relation $\eta_{M^{(0)}}(s) - \eta_{M^{(j)}}(s) = w_j(s)$ implies formula [4.1]. It follows from this formula that $\eta_M(s)$ extends as an analytic (resp. meromorphic) function to $\Omega$ if and only if $\eta_1(s)$ has the same property.

We recall that $Z_\zeta$ stands for the set of zeros of $\zeta(s)$ in the half space $\Re(s) > \frac{1}{2}$.

**Proposition 4.2.** Let $M$ be an arithmetical list.

1. The derivative $\frac{d}{ds} \eta_M(s) = -\sum_{p \in M} \frac{\ln(p)}{p^s}$ extends as a meromorphic function in $\Re(s) > 0$ with simple poles for singular points.
2. \( \eta_M(s) \) can be continued to an analytic function in the open simply connected

\[
\Omega_c = \{ \Re(s) > 0 \} - \bigcup_{\alpha \in \mathbb{Z}_c} [0; \alpha] - [0; 1]
\]

Assertion 2) is more precise than [Tit] page 182.

Proof: Let \( r > 0 \) be the reason of the list. By proposition 4.1, \( \eta_1(s) = r \eta_M(s) + w(s) \) for \( \Re(s) > 1 \), where \( w(s) \) is analytic for \( \Re(s) > 0 \). Taking the derivative, we are reduced to the case \( \eta_M = \eta_1 \) for which this property is true for \( \Re(s) > 0 \). This proves 1) since our formula now defines the holomorphic continuation.

For 2), we note that \( \Omega_c \) has the property that \( s \in \Omega_c \) and \( \lambda \geq 1 \) implies that \( \lambda s \in \Omega_c \). Since the Riemann function \( \zeta(s) \) never vanishes on \( \Omega_c \), it follows from proposition 3.2 that

\[
\zeta(s) = g_k(s) Z_M(s)^k \quad \text{for} \quad s \in \Omega_c
\]

for some suitable analytic function \( g_k \) defined on \( \Omega_c \) which never vanishes on this set.

Now, the function:

\[
W_M(s) = \prod_{p \in M} \frac{1}{1 - \frac{1}{p^s}}
\]

is analytic for \( \Re(s) > \frac{1}{2} \), has no zeros on this half-plane and satisfies \( Z_M(s) = W_M(s) \eta_M(s) \) for \( \Re(s) > 1 \). Thus, \( \frac{Z_M(s)}{W_M(s)} = \eta_M(s) \) has no zeros in \( \Omega_c \) and this leads to the analytic continuation on \( \Omega_c \cap \{ \Re(s) > \frac{1}{p+1} \} \) with \( p = 1 \). We now iterate this process as follows. For \( p \in \mathbb{P} \) with \( p \geq 1 \)

\[
W_M^{(p)}(s) = \prod_{p \in M} \frac{1}{1 - \frac{1}{p^s} e^{\frac{1}{p^s} + \frac{1}{2} + \frac{1}{p^s} + \frac{1}{p^s} + \cdots}}
\]

is analytic for \( \Re(s) > \frac{1}{p+1} \), has no zeros on this half-plane and satisfies:

\[
Z_M(s) = W_M^{(p)}(s) e^{\eta_M(s) + \frac{1}{2} \eta_M(2s) + \cdots + \frac{1}{p} \eta_M(ps)} \quad \text{for} \quad \Re(s) > 1
\]

It follows that \( \eta_M(s) + \frac{1}{2} \eta_M(2s) + \cdots + \frac{1}{p} \eta_M(ps) \) has an analytic continuation \( f_{p+1}(s) \) on \( \Omega_c \cap \{ \Re(s) > \frac{1}{p+1} \} \). By induction, the analytic continuation \( \eta_M(s) \) of \( \eta_M(s) \) being obtained on \( \Omega_c \cap \{ \Re(s) > \frac{1}{p} \} \), one has, for \( \Re(s) > \frac{1}{p+1} \)

\[
f_{p+1}(s) = \eta_M(s) + \frac{1}{2} \eta_M(2s) + \cdots + \frac{1}{p} \eta_M(ps)
\]

since \( \Re(2s) > \frac{1}{p} \), \( \Re(ps) > \frac{1}{p} \). Thus,

\[
f_{p+1}(s) - \frac{1}{2} \eta_M(2s) - \cdots - \frac{1}{p} \eta_M(ps)
\]

defines the analytic continuation \( \eta_M(s) \) of \( \eta_M(s) \) to the open half-plane \( \Re(s) > \frac{1}{p+1} \).

The following proposition is known for \( M = \mathbb{P} \).

\textbf{Proposition 4.3.} Let \( M \subset \mathbb{P} \).

1. one has, for \( \Re(s) > 1 \),

\[
\sum_{p \in M} \frac{1}{p^s} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \ln(Z_M(ns))
\]
2. Let \( l_{\text{pop}_M}(x) = \sum_{k \in \text{pop}_M} \frac{x^k}{k} \) for \( |x| < 1 \). One has:

\[
\sum_{n \in \text{pop}_M} \frac{\mu(n)}{n} \left( \sum_{k \in \text{pop}_M} x^n \right) = x
\]

\[
l_{\text{pop}_M}(x) = \sum_{n \in \text{pop}_M} \frac{\mu(n)}{n} \ln \left( \frac{1}{1-x^n} \right)
\]

It follows that

\[
\sum_{n \in \text{pop}_M} \frac{\mu(n)}{n} \left( \sum_{p \in M} l_{\text{pop}_M} \left( \frac{1}{p^n s} \right) \right) = \sum_{p \in M} \frac{1}{p^s}
\]

The function \( l_{\text{pop}_M}(x) \) coincide with \( \ln \left( \frac{1}{1-x^n} \right) \) when \( M = \mathbb{P} \) (because \( \text{pop}_M = \mathbb{N}^* \) in this case). Taking \( \text{pop}_M = \text{pop}_{2,5} \) in 2), one has:

\[
l_{\text{pop}}(x) = x + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^8}{8} + \cdots
\]

and our first summation is

\[
\frac{\mu(1)}{1} l_{\text{pop}}(x) + \frac{\mu(2)}{2} l_{\text{pop}}(x^2) + \frac{\mu(4)}{4} l_{\text{pop}}(x^4) + \frac{\mu(5)}{5} l_{\text{pop}}(x^5) + \frac{\mu(8)}{8} l_{\text{pop}}(x^8) + \cdots
\]

**Proof:** For 1), we use formula \( \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \ln \left( \frac{1}{1-x^n} \right) = x \) valid for \( |x| < 1 \). Choosing \( x = \frac{1}{p^s} \) for \( p \in M \) and \( \Re(s) > 1 \), it remains to sum expansions for all \( p \). For 2), one has

\[
\sum_{n \in \text{pop}_M} \frac{\mu(n)}{n} l_{\text{pop}_M}(x^n) = \sum_{n \in \text{pop}_M} \frac{\mu(n)}{n} \sum_{k \in \text{pop}_M} \frac{x^{nk}}{nk} = \sum_{n \in \text{pop}_M} \frac{x^n}{n} \sum_{k \mid n} \frac{\mu(k)}{k}
\]

Since conditions \( k \mid n \) with \( k \in \text{pop}_M \) and \( k \mid n \) are identical when \( n \in \text{pop}_M \), using for \( n > 1 \)

\[
\sum_{d \mid n} \frac{\mu(d)}{d^\alpha} = \frac{1}{n} \sum_{d \mid n} \mu(d) = 0
\]

one has, from \( \mu(1) = 1 \)

\[
\sum_{n \in \text{pop}} \frac{\mu(n)}{n} l_{\text{pop}}(x^n) = x
\]

This proves first formula of 2). Formula

\[
l_{\text{pop}_M}(x) = \sum_{n \in \text{pop}_{(P-M)}} \frac{\mu(n)}{n} \ln \left( \frac{1}{1-x^n} \right)
\]

follows from inclusion-exclusion principle. Last formula is obtained with \( x = \frac{1}{p^s} \) in first formula of 2) and summation for all \( p \in M \).
5. Equivalences between analytic continuations and Riemann hypothesis

As seen previously, $\Omega_{\frac{1}{2}} = \{\Re(s) > \frac{1}{2}\} - [\frac{1}{2}; 1]$.

**Proposition 5.1.** The following properties are equivalent:

1. $\eta_1(s)$ extends as an analytic function in $\Omega_{\frac{1}{2}}$.
2. $\eta_1(s)$ extends as a meromorphic function in $\Omega_{\frac{1}{2}}$.
3. The Riemann function $\zeta(s)$ never vanishes in $\Omega_{\frac{1}{2}}$.
4. For one $i \in \mathbb{N}^*$, $i \geq 2$ (resp. for all $i \in \mathbb{N}^*$), the function $\zeta_i(s)$ extends as an analytic function in $\Omega_{\frac{1}{2}}$ and never vanishes in that open set.
5. $\eta_i^n(s)$ extends as a meromorphic function in $\Omega_{\frac{1}{2}}$, for one $n \in \mathbb{N}^*$, $n \geq 1$ (resp. for all $n \in \mathbb{N}^*$).

**Proof:** Let $M$ be an arithmetical list of $\mathbb{P}$. From proposition 5.1, $\eta_M(s)$ has an analytic (resp. meromorphic) continuation to $\Omega$ if and only if $\eta_1(s)$ has the same property: and one has $Z_M(s) = W_M(s) e^{\eta_M(s)}$ for $\Re(s) > 1$, where $W_M$ is an analytic function for $\Re(s) > \frac{1}{2}$ and has no zeros on this half plane.

Assume that 1) is satisfied. Taking $M = M_i$ with $i \geq 1$, one has $\zeta_i(s) = W_{M_i}(s) e^{\eta_i(s)}$ for $\Re(s) > 1$, thus the right side hand has an analytic continuation to $\Omega_{\frac{1}{2}}$: hence, $1 \Rightarrow 3$ and $1 \Rightarrow 4$. Conversely, if 4) is satisfied for one $i \geq 2$, then $\frac{\zeta_i(s)}{W_{M_i}(s)} = eh(s)$ for some analytic function $h = h(s)$ defined on $\Omega_{\frac{1}{2}}$. It follows that $\eta_i(s) - h(s)$ is constant for $\Re(s) > 1$, hence $\eta_i(s)$ has an analytic continuation in $\Omega_{\frac{1}{2}}$. Hence, $3 \Rightarrow 1$ and $4 \Rightarrow 1$.

Let us assume 2). Let $\alpha$ be a pole of $\eta(s)$ in $\Omega_{\frac{1}{2}}$. Formula $\zeta(s) = W(s) e^{\eta(s)}$ has an analytic continuation in $\Omega_{\frac{1}{2}} - \{\text{the set of poles of } \eta\}$. Taking a small neighborhood $V(\alpha)$ of $\alpha$ not containing $\alpha$, we see that the right (resp. left) hand side our the formula is unbounded (resp. bounded) on $V(\alpha)$ leading to a contradiction: hence $1 \Leftrightarrow 2$.

Let us assume 5) for some $n \geq 2$. The derivative $\frac{d}{ds}\eta^n(s) = n\eta^{n-1}(s)\frac{d}{ds}\eta(s)$ is meromorphic in $\Omega_{\frac{1}{2}}$. Since $\frac{d}{ds}\eta(s)$ is meromorphic in $\Re(s) > 0$, it follows that $\eta^{n-1}(s)$ is meromorphic in $\Omega_{\frac{1}{2}}$: hence by iteration, one has $5 \Rightarrow 1$ and $1 \Rightarrow 5$ is clear.

**Theorem 5.2.** The following properties are equivalent:

1. The Riemann hypothesis holds.
2. The function $\zeta_i(s)$ has a meromorphic continuation in $\Omega_{\frac{1}{2}}$, for all $i \in \mathbb{N}^*$.
3. The function $\zeta_i(s)$ has an analytic continuation in $\Omega_{\frac{1}{2}}$, for all $i \in \mathbb{N}^*$ (resp. for $i$ moving in an infinite subsequence of $\mathbb{N}^*$).

**Proof:** By proposition 5.1, proposition 5.3 and lemma 3.3, one has $1 \Rightarrow 2$ and $1 \Rightarrow 3$.

Let us assume 2). With notations of proposition 5.2, one has $\zeta(s) = g_i(s) \zeta_i(s)^i$ for $\Re(s) > 1$ and this holds by analytic continuation in $\Omega_{\frac{1}{2}} - \{\text{the set of poles of } \zeta_i\}$. Let $\alpha$ be a pole of $\zeta_i$ in $\Omega_{\frac{1}{2}}$. Taking a small compact neighborhood $V(\alpha)$ of $\alpha$ not containing $\alpha$, we get that $\zeta_i(s)$ is bounded on $V(\alpha)$ hence is analytic at $\alpha$ (ie the singularity is removable). Hence, $\zeta(s) = g_i(s) \zeta_i(s)^i$ for $s \in \Omega_{\frac{1}{2}}$, with $\zeta_i(s)$ analytic for all $s \in \Omega_{\frac{1}{2}}$.

Let $s_0$ be a possible root of the equation $\zeta(s) = 0$ in $\Omega_{\frac{1}{2}}$. It follows that $\zeta_i(s) = 0$ and $\zeta(s_0) = g_i(s) \zeta_i(s_0)^i$ implies that $N_0$ has multiplicity $i$. Since this holds for all $i \in \mathbb{N}^*$, we get a contradiction. Hence $2 \Rightarrow 1$. The proof of $3 \Rightarrow 1$ is similar.

**Theorem 5.3.** The following properties are equivalent:
1. The Riemann hypothesis holds.
2. For any arithmetical list \( M \subset \mathbb{P} \), the function \( \frac{d}{ds} \eta_M(s) \) extends analytically in \( \Omega_\frac{1}{2} \).
3. The function of
   \[
   \int_2^{+\infty} \frac{\pi(t) \ln(t)}{t^{s+1}} dt
   \]
   extends as an analytic function in \( \Omega_\frac{1}{2} \).

Proof: One has 1 \( \Rightarrow \) 2 by proposition 5.1. Conversely, as \( \Omega_\frac{1}{2} \) is simply connected, one has 2 \( \Rightarrow \) 1 by lemma 3.3.

Using Abel summation, one has, for \( \Re(s) > 1 \),

\[
\frac{d}{ds} \eta_1(s) = \int_2^{+\infty} \frac{\pi(t)dt}{t^{s+1}} - s \int_2^{+\infty} \frac{\pi(t) \ln(t)dt}{t^{s+1}}
\]

Therefore, for \( \Re(s) > 1 \)

\[
\frac{d}{ds} \eta_1(s) - \frac{1}{s} \eta_1(s) = -s \int_2^{+\infty} \frac{\pi(t) \ln(t)dt}{t^{s+1}}
\]

Thus, if 3) holds, \( \frac{d}{ds} \eta_1(s) - \frac{1}{s} \eta_1(s) \) extends as an analytic function in \( \Omega_\frac{1}{2} \), thus \( \eta_1(s) \) has a meromorphic continuation to \( \Omega_1 \), since \( \frac{d}{ds} \eta_1 \) is always meromorphic in that set. From proposition 4.3, we get 3 \( \Rightarrow \) 1. The implication 1 \( \Rightarrow \) 3 is similar.

6. Summations relative to \( \text{pop}_M \)

In the previous considerations, we met subsets \( A \) of \( \mathbb{N}^* \) having the following properties:

1). conditions \( a \in A \) and \( b \in A \) imply \( ab \in A \)
2). conditions \( a \in A \) and \( d \mid a \) with \( d \in \mathbb{N}^* \) imply \( d \in A \)

Obviously, 1 \( \in A \) when \( A \neq \emptyset \) and:

Lemma 6.1. There exists a unique subset \( M \subset \mathbb{P} \) such that \( A \) consists of all products (with multiplicities) of elements of \( M \), i.e. one has \( A = \text{pop}_M \).

We need to consider summations or formulas involving \( \text{pop}_M \). In some cases, formulas involving \( \text{pop}_M \) can be viewed as the restriction to \( \text{pop}_M \) of a known formula assumed classic on \( \mathbb{N}^* \): this fact often appears for arithmetical functions \( f \) expanding some value \( f(n) \) as a finite summation of others values depending on all \( d \mid n \). Thus, proposition 7 is the direct adaptation of Moebius formulas.

Another kind of formulas rest on the known principle of inclusion-exclusion sketched in next lemma.

Lemma 6.2. Let \( f : \mathbb{N}^* \to \mathbb{C} \) be a function, \( M \subset \mathbb{P} \) with \( A = \text{pop}_M \). Then, putting \( M^* = \mathbb{P} - M \) and \( B = \text{pop}_{M^*} \), one has:

\[
\sum_{i \in A} f(i) = \sum_{k \in B} \mu(k) \sum_{i \in \mathbb{N}^*} f(ki)
\]

(questions of convergence are not considered here)

Let \( M \subset \mathbb{P} \) (we abbreviate \( \text{pop}_M = \text{pop} \)) and \( G_{\text{pop}} \) be the ”geometrical” series defined for \( |q| < 1 \) by:

\[
G_{\text{pop}}(q) \equiv \sum_{k \in \text{pop}} q^k
\]
Taking $f(i) = e^{-it}$ for $t > 0$ fixed, we get, with $G_{\text{pop}}(e^{-t}) \equiv \sum_{n \in \text{pop}} e^{-nx}$ that:

$$G_{\text{pop}}(e^{-t}) = \frac{1}{1 - e^x} - \sum_{p \in M^*} \frac{1}{1 - e^{px}} + \sum_{p \quad q \in M^*} \frac{1}{1 - e^{pqx}} - \sum_{p \quad q \quad r \in M^*} \frac{1}{1 - e^{pqr x}} + \cdots$$

This summation is essential in formula

$$\Gamma(s) Z_M(s) = \int_0^{+\infty} G_{\text{pop}}(e^{-t}) t^{s-1} \frac{dt}{t}$$

valid for $\Re(s) > 1$.

**Proposition 6.3.** Let $M \subset \mathbb{P}$ and $\text{pop} = \text{pop}_M$.

1. The identity

$$Z_M(s) \sum_{n \in \text{pop}} \frac{a_n}{n^s} = \sum_{n \in \text{pop}} A_n \frac{x}{n^s}$$

is equivalent to

$$\sum_{n \in \text{pop}} a_n G_{\text{pop}}(x^n) = \sum_{n \in \text{pop}} A_n x^n$$

2. Let

$$Z_M(s) \sum_{n \in \text{pop}} \frac{a_n}{n^s} = \sum_{n \in \text{pop}} A_n \frac{x}{n^s} \quad \text{and} \quad \zeta(s) \sum_{n \in \text{pop}} \frac{a_n}{n^s} = \sum_{n \in \mathbb{N}^*} B_n \frac{x}{n^s}$$

Then, $B_n = A_n$ when $n \in \text{pop}$ and $B_n = \sum_{|n| \in \text{pop}} a_i$ when $n \notin \text{pop}$.

Taking $\alpha_n = \mu(n)$ for $n \in \text{pop}$, we get, in 1):

$$\sum_{n \in \text{pop}} \mu(n) G_{\text{pop}}(x^n) = x$$

Taking $\alpha_n = \Phi(n)$ for $n \in \text{pop}$, we get, in 1):

$$\sum_{n \in \text{pop}} \Phi(n) G_{\text{pop}}(x^n) = \sum_{n \in \text{pop}} n x^n = x \frac{d}{dx} G_{\text{pop}}(x)$$

For 2), we deduce from Lambert series that:

$$\sum_{n \in \text{pop}} a_n \frac{x^n}{1 - x^n} = \sum_{n \in \text{pop}} B_n x^n + \sum_{n \in \mathbb{N}^* - \text{pop}} B_n x^n$$

With $a_n = 1$ for $n \in \text{pop}$, we get:

$$\sum_{n \in \text{pop}} \frac{x^n}{1 - x^n} = \sum_{n \in \text{pop}} \tau(n) x^n + \sum_{n \in \mathbb{N}^* - \text{pop}} B_n x^n$$

where $B_n = \text{card}(\text{Div}(n) \cap \text{pop})$, and $\text{Div}(n)$ stands for the set of divisors of $n$.

In the same way,

$$\sum_{n \in \text{pop}} n \frac{x^n}{1 - x^n} = \sum_{n \in \text{pop}} \sigma(n) x^n + \sum_{n \in \mathbb{N}^* - \text{pop}} B_n x^n$$

where $B_n = \sum_{i \in \text{Div}(n) \cap \text{pop}} i$.

In the case $\text{pop} = \mathbb{N}^*$, first assertion is a classical calculation met in Lambert series with $G_{\text{pop}}(x) = \frac{x}{1 - x}$. 
Proof: To get a visual proof, we choose \( \text{pop} = \text{pop}_{2,5} = \{1, 2, 4, 5, 8, 10, 11, 16, \ldots \} \).

Then \( G_{\text{pop}} \) is the "geometrical" series
\[
G_{\text{pop}}(q) = q + q^2 + q^4 + q^5 + q^8 + q^{10} + q^{11} + \cdots
\]

One has:
\[
\begin{align*}
a_1 G_{\text{pop}}(x) &= a_1 x + a_1 x^2 + a_1 x^4 + a_1 x^5 + a_1 x^8 + a_1 x^{10} + \cdots \\
a_2 G_{\text{pop}}(x^2) &= a_2 x^2 + a_2 x^4 + a_2 x^5 + a_2 x^8 + a_2 x^{10} + \cdots \\
a_4 G_{\text{pop}}(x^4) &= a_4 x^4 + a_4 x^8 + \cdots \\
a_5 G_{\text{pop}}(x^5) &= a_5 x^5 + a_5 x^{10} + \cdots \\
a_8 G_{\text{pop}}(x^8) &= \cdots \\
a_{10} G_{\text{pop}}(x^{10}) &= \cdots
\end{align*}
\]

Summing columns, we get our formula since \( A_n = 0 \) when \( n \notin \text{pop} \) and \( A_n = \sum_{d|n} a_d \).

Assertion 2) is a simple verification.

An important function is, for \( x > 0 \):
\[
N_{\text{pop}}(x) \equiv \text{card}(\{1; x \cap \text{pop}\}) = \sum_{n \leq x \atop n \in \text{pop}} 1_{n \in \text{pop}}(n)
\]

which agree, by Perron’s formula, with
\[
N_{\text{pop}}(x) = \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} Z_M(s) \frac{x^s}{s} ds + O\left(\frac{x^{\sigma + \epsilon}}{T}\right)
\]
(with \( \sigma > 1 \) and \( \epsilon > 0 \) when \( x \) is not an integer).

**Proposition 6.4.** Let \( M \subset \mathbb{P} \) and \( \text{pop} = \text{pop}_M \).

1. Given a function \( f = f(x) \), we put \( F(x) = \sum_{k \in \text{pop}} f(kx) \). Then,
\[
\sum_{n \in \text{pop}} a_n F(nx) = \sum_{n \in \text{pop}} A_n f(nx)
\]
where \( a_n \) and \( A_n \) are related by formula \( Z_M(s) \sum_{n \in \text{pop}} \frac{a_n}{n^s} = \sum_{n \in \text{pop}} \frac{A_n}{n^s} \).

2. Formulas
\[
F(x) = \sum_{k \in \text{pop}} f(kx)
\]
and
\[
f(x) = \sum_{k \in \text{pop}} \mu(k) F(kx)
\]
are equivalent.

Proof: One has
\[
\sum_{n \in \text{pop}} a_n F(nx) = \sum_{n \in \text{pop}} \sum_{k \in \text{pop}} a_n f(knx) = \sum_{n \in \text{pop}} \sum_{k \mid n} a_k f(nx)
\]

Since
\[
A_n = \sum_{j \mid n} a_j
\]
this proves 1), noting that \( k \mid n \) with \( n \in \text{pop} \) implies \( k \in \text{pop} \).

For 2), taking \( a_n = \mu(n) \) for \( n \in \text{pop} \) (zero elsewhere), one has \( A_n = 0 \) for all
n except $A_1 = 1$, hence $f(x) = \sum_{k \in \text{pop}} \mu(k)F(kx)$. Conversely, from $f(x) = \sum_{k \in \text{pop}} \mu(k)F(kx)$, we obtain

$$\sum_{n \in \text{pop}} f(nx) = \sum_{n \in \text{pop}} \sum_{k \in \text{pop}} \mu(k)F(knx) = \sum_{n \in \text{pop}} \sum_{k \mid n} \frac{n}{k} F(nx)$$

and our assertion follows from $\mu(1) = 1$ and $\sum_{k \mid n} \mu\left(\frac{n}{k}\right) = 0$ for $n > 1$ with the observation that

$$\sum_{k \mid n} \mu\left(\frac{n}{k}\right)$$

Proposition 6.5.

1. For $k \in \mathbb{N}^*$ let $G(k) = \sum_{i \mid k} i \in \text{pop} g(i)$ where $g$ is a function defined on $\text{pop}$. Then, for $x \geq 1$

$$\sum_{i \leq x} i \in \text{pop} G(i) = \sum_{i \leq x} \text{N}_{\text{pop}}\left(\frac{x}{i}\right) g(i) \quad (6.6)$$

2. For $k \in \mathbb{N}^*$ and $a \in \mathbb{C}$, let

$$H(k) = \sum_{i \mid k} i \in \text{pop} h(i) \frac{i^a}{k^a} \quad \text{and} \quad S_{\text{pop}}(u; a) = \sum_{i \leq u} \frac{1}{i^a}$$

where $h$ is a function defined on $\text{pop}$. Then

$$\sum_{i \leq x} i \in \text{pop} H(i) = \sum_{i \leq x} \text{S}_{\text{pop}}\left(\frac{x}{i}; a\right) h(i) \quad (6.7)$$

When $a = 1$, the series $S_{\text{pop}}(u; a)$ can be considered as the harmonic series associated to a given pop.

Proof: The formula (6.6) can be considered as a particular case of (6.7) with $a = 0$. For (6.7), we expand successively $H(k)$ with $k \in \text{pop}$ and $k \leq x$. The term $h(i)$ appears exactly one time in each expansion of $H(k)$, $k \leq x$, when $i \mid k$: when it is it so, the coefficient of $h(i)$ is $\frac{i^a}{k^a}$. As $\frac{k}{i} \in \text{pop}$, we see that the total coefficient of $h(i)$ is of the form

$$\sum_{j \leq C(i)} \frac{1}{j^a}$$

with $C(i)$ to be clarified. The last term of this summation is $\frac{x^a}{y^a}$ where $y$ is the biggest element of $\text{pop} \cap [1; x]$. Thus our summation consists of all $\frac{1}{j^a}$ with $j \in \text{pop}$ such that $ij \leq x$ with $i \in \text{pop}$. Using definition (6.3), this gives $C(i) = N_{\text{pop}}\left(\frac{x}{i}\right)$. From

$$\sum_{j \leq N_{\text{pop}}\left(\frac{x}{i}\right)} \frac{1}{j^a} = S_{\text{pop}}\left(\frac{x}{i}; a\right)$$

we get our formula (6.7).

In formula (6.6), we choose for $g$ the Euler $\Phi$-function. Since $\sum_{i \mid k} \Phi(i) = k$, we get

$$\sum_{i \leq x, i \in \text{pop}} i = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}\left(\frac{x}{i}\right) \Phi(i) \quad (6.8)$$
Taking for \( g \) the function \( g(i) = |\mu(i)| \), we get
\[
\sum_{i \leq x, i \in \text{pop}} 2^{\nu(i)} = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i}) \ |\mu(i)|
\]
where \( \nu(n) \) is the number of distinct prime factors of \( k \): this follows from \( \sum_{i | k} |\mu(i)| = 2^{\nu(k)} \). Thus
\[
\sum_{i \leq x, i \in \text{pop}} 2^{\nu(i)} = N_{\text{pop}}(x) + \sum_{p \in \text{pop}} N_{\text{pop}}(\frac{x}{p}) + \sum_{p < q, p, q \in \text{pop}} N_{\text{pop}}(\frac{x}{pq}) + \cdots
\]

If we choose for \( g \) the Moebius \( \mu \)-function, from \( G(i) = \sum_{i | k} \mu(i) = 0 \) for \( i > 1 \) and \( G(1) = 1 \), we get for \( x > 1 \)
\[
\sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i}) \mu(i) = 1
\]

Taking for \( g \) the function \( g(i) = \frac{\mu(i)}{i} \), from formula \( \sum_{i | k} \frac{\mu(i)}{i} = \frac{\Phi(k)}{k} \), we get
\[
\sum_{i \leq x, i \in \text{pop}} \frac{\Phi(i)}{i} = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i}) \ \frac{\mu(i)}{i}
\]

Taking for \( g \) the function \( g(i) = 1 \) when \( i \in \text{pop} \), from formula \( \sum_{i | k} 1 = \tau(k) \) (the number of positive divisors of \( k \)), we get
\[
\sum_{i \leq x, i \in \text{pop}} \tau(i) = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i})
\]

Replacing \( g \) by the function \( g(i) = 1 \) when \( i \in \text{pop} \) and \( i \) is a prime number, we get
\[
\sum_{i \leq x, i \in \text{pop}, \text{i prime}} \nu(i) = \sum_{i \leq x, i \in \text{pop}, \text{i prime}} N_{\text{pop}}(\frac{x}{i})
\]

Taking for \( g \) the function \( g(i) = i^{a} \) when \( i \in \text{pop} \), from formula \( \sum_{i | k} i^{a} = \sigma_{a}(k) \) (the sum of powers of positive divisors of \( k \)), we get
\[
\sum_{i \leq x, i \in \text{pop}} \sigma_{a}(i) = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i}) \ i^{a}
\]

Noting that \( \sum_{i | k} i^{-a} = \frac{1}{i^{a}} \sigma_{a}(k) \), we get for \( g(i) = i^{-a} \)
\[
\sum_{i \leq x, i \in \text{pop}} \frac{\sigma_{a}(i)}{i^{a}} = \sum_{i \leq x, i \in \text{pop}} N_{\text{pop}}(\frac{x}{i}) \ \frac{1}{i^{a}}
\]

We now use formula \( \text{[5.7]} \). With \( h(i) = 1 \) for all \( i \), one has \( H(k) = \frac{\sigma_{a}(k)}{i^{a}} \), hence
\[
\sum_{i \leq x, i \in \text{pop}} \frac{\sigma_{a}(i)}{i^{a}} = \sum_{i \leq x, i \in \text{pop}} S_{\text{pop}}(\frac{x}{i}; a)
\]

With \( h(i) = \frac{1}{i^{a}} \) for all \( i \), one has \( H(k) = \frac{\tau(k)}{i^{a}} \), hence
\[
\sum_{i \leq x, i \in \text{pop}} \frac{\tau(i)}{i^{a}} = \sum_{i \leq x, i \in \text{pop}} S_{\text{pop}}(\frac{x}{i}; a) \ \frac{1}{i^{a}}
\]
With $h(i) = \frac{1}{i^2}$ when $i$ is a prime number and 0 elsewere, one has $H(k) = \frac{\nu(k)}{k^\tau}$, hence

$$
\sum_{i \leq x \in \text{pop}} \frac{\mu(i)}{i^a} = \sum_{i \leq x \in \text{pop} \text{ prime}} S_{\text{pop}}(\frac{x}{i} ; a) \frac{1}{i^a}
$$

(6.18)

More generally, for $k \in \mathbb{N}^*$, let $F(k) = \sum_{i|k \in \text{pop}} f(i)g(\frac{x}{i})$ where $h$ and $g$ are functions defined on pop. Then, the same calculation yields

$$
\sum_{i \leq x \in \text{pop}} F(i) = \sum_{i \leq x \in \text{pop}} f(i) \sum_{j \leq \frac{x}{i} \in \text{pop}} g(j)
$$

(6.19)

Thus, with $x$ infinite, $f(i) = a^i$, $g(i) = b^i$, we obtain, for $|a| < 1$ and $|b| < 1$

$$
G_{\text{pop}}(a) G_{\text{pop}}(b) = \sum_{n \in \text{pop}} n \geq 1 \left( \sum_{d|n} a^d b^{\frac{n}{d}} \right)
$$

(6.20)

One also has the formula

$$
\sum_{i \leq x \in \text{pop}} i = \sum_{i \leq x \in \text{pop}} \mu(i) \sum_{j \leq \frac{x}{i} \in \text{pop}} \sigma(j) = \sum_{i \leq x \in \text{pop}} \sigma(i) \sum_{j \leq \frac{x}{i} \in \text{pop}} \mu(j)
$$

(6.21)

due to $\sum_{i|k} \sigma(i)\mu(\frac{x}{i}) = k$ for $k \in \mathbb{N}^*$. And $\sum_{i|k} \tau(i)\mu(\frac{x}{i}) = 1$ for $i \in \mathbb{N}^*$ yields

$$
N_{\text{pop}}(x) = \sum_{i \leq x \in \text{pop}} \mu(i) \sum_{j \leq \frac{x}{i} \in \text{pop}} \tau(j) = \sum_{i \leq x \in \text{pop}} \tau(i) \sum_{j \leq \frac{x}{i} \in \text{pop}} \mu(j)
$$

(6.22)

$\tau(n)$ being the divisor function.

Let $A$ be a subset of $\mathbb{N}^*$. We put $A^+ = \bigcup_{i \in A} [i; i+1] = A + [0; 1]$ and, for $n \in \mathbb{N}^*$, $A_n^+ = \bigcup_{i \in A} [i; i+1]$. 

The comparison of a series with an integral takes here the following aspect

**Proposition 6.6.** Let $A$ be a subset of $\mathbb{N}^*$ and $f : [1; +\infty[ \rightarrow \mathbb{R}$ be a continuous positive decreasing function on the interval $[1; +\infty[ \text{ such that } \lim_{t \rightarrow +\infty} f(t) = 0$. Then, the sequence:

$$
u_n = \sum_{i \leq x \in A} f(i) - \int_{A_n^+} f(t)dt
$$

has a finite limit $C_A$ when $n \rightarrow +\infty$.

**Proof:** We modify the function $f$ on interval $[i; i+1]$ whenever $i \notin A$ making it affine on such intervals (values $f(i)$ and $f(i+1)$ being unaltered). Let $h : [1; +\infty[ \rightarrow \mathbb{R}$ be the continuous positive decreasing function so obtained. It is known that the sequence:

$$
v_n = \sum_{i=1}^{i=n} h(i) - \int_{1}^{n+1} h(t)dt
$$

$$
= h(1) - \int_{1}^{2} h(t)dt + \cdots + h(i) - \int_{i}^{i+1} h(t)dt + \cdots + h(n) - \int_{n}^{n+1} h(t)dt
$$

has a finite limit when $n \rightarrow +\infty$. Whenever $i \notin A$, one has:

$$
h(i) - \int_{i}^{i+1} h(t)dt = \frac{1}{2} \left( f(i) - f(i+1) \right)
$$

thus

$$
u_n = v_n - \frac{1}{2} \sum_{i \notin A} f(i) - f(i+1))
$$
Our conclusion follows since \( \sum_{i \in \mathbb{A}} (f(i) - f(i + 1)) \) satisfies hypothesis of the theorem of alternating convergent series.

**Proposition 6.7.** Let \( \mathbb{A} \) be a subset of \( \mathbb{N}^* \) and \( Z(s) \) be the function \( Z(s) = \sum_{n \in \mathbb{A}} \frac{1}{n^s} \) for \( \Re(s) > 1 \). One has, for \( \Re(s) > 1 \):}

\[
Z(s) = \int_{\mathbb{A}^+} \frac{dt}{t^s} + h(s)
\]

where \( h(s) \) is analytic in the half-plane \( \Re(s) > 0 \).

**Proof:** Classic demonstration (see [Nar] p.209 when \( \mathbb{A} = \mathbb{N}^* \) ) adapts itself without any trouble. One has, for \( n \in \mathbb{N}^* \) and \( \Re(s) > 1 \):

\[
\int_0^1 \frac{dt}{(t + n)^s} = \frac{1}{(1 + n)^s} + s \int_0^1 \frac{t dt}{(t + n)^{s+1}}
\]

Summing for \( n \in \mathbb{A} \), we get

\[
Z(s) = \sum_{n \in \mathbb{A}} \left( \frac{1}{n^s} - \frac{1}{(1 + n)^s} \right) - \sum_{n \in \mathbb{A}} s \int_0^1 \frac{t dt}{(t + n)^{s+1}} + \sum_{n \in \mathbb{A}} \int_0^1 \frac{dt}{(t + n)^s}
\]

The first expression of the right-hand side is an alternate Dirichlet series clearly analytic for \( \Re(s) > 0 \). We define, for \( t \in [0; 1] \):

\[
g_{\mathbb{A}}(t) = \sum_{n \in \mathbb{A}} \frac{t}{(t + n)^{s+1}}
\]

The series of \( g_{\mathbb{A}}(t) \) converges uniformly for \( t \in [0; 1] \) and \( s \) in any given compact of \( \Re(s) > 0 \), hence

\[
\sum_{n \in \mathbb{A}} s \int_0^1 \frac{t dt}{(t + n)^{s+1}} = s \int_0^1 g_{\mathbb{A}}(t) dt
\]

defines an analytic function in that half-plane.

Since

\[
\sum_{n \in \mathbb{A}} \int_0^1 \frac{dt}{(t + n)^s} = \int_{\mathbb{A}^+} \frac{dt}{t^s}
\]

this proves the lemma.

One has, for \( \Re(s) > 1 \):

\[
(6.23) \quad Z_M(s) = \sum_{n \in \mathbb{N}^*} N_{\text{pop}_M}(n) \left[ \frac{1}{n^s} - \frac{1}{(n + 1)^s} \right] = s \int_1^{+\infty} N_{\text{pop}_M}(t) \frac{dt}{t^{s+1}}
\]

and, by Abel summation, for \( t > 0 \):

\[
G_{\text{pop}}(e^{-t}) = \sum_{k=1}^{+\infty} N_{\text{pop}}(k) t \int_k^{k+1} e^{-tu} du = t \int_1^{+\infty} N_{\text{pop}}(u) e^{-tu} du
\]

Now, from formula [6.3] with \( g(i) = 1 \) for all \( i \leq n \) and 0 elsewhere, in the case \( \text{pop}_M = \text{pop}_{2,5} \), we obtain

\[
N_{\text{pop}}(n) = n - \left[ \frac{n}{3} \right] - \left[ \frac{n}{7} \right] - \cdots + \left[ \frac{n}{21} \right] + \cdots
\]

which is close to

\[
n(1 - \frac{1}{3})(1 - \frac{1}{7})(1 - \frac{1}{13}) \cdots \approx n \prod_{p \leq n, p \in \text{pop}_{1,7}} (1 - \frac{1}{p}) \sim \alpha_{3,7} \frac{n}{\sqrt{\ln(n)}}
\]
for some constant \( \alpha_{3,7} \) (by proposition below). Numerical computations directly checked on \( \text{pop}_{2,5} = \{1, 2, 4, 5, 8, 10, 11, 16, 17, \cdots \} \) gives \( \alpha_{3,7} \sim 0.736 \cdots \) with an oscillating limit.

More generally, it is highly probable that, for an arithmetical list \( M \) having a reason \( r > 0 \), one has, for some suitable constant \( A_M \):

\[
N_{\text{pop}}(x) \sim A_M x \frac{\sqrt{\ln(x)}}{\ln(x)} \quad \text{as} \quad x \to +\infty
\]

This property is related to formula \( Z_M(s) \sim \frac{\text{Cte}}{(s-1)^r} \) associated with Perron’s formula \( \text{p.} 278 \) or a possible improvement of Ikehara-Wiener theorem.

Mertens’s formulas takes the form:

**Proposition 6.8.** Let \( M \) be an arithmetical list of \( \mathbb{P} \), \( r > 0 \) being its reason. There exists a constant \( \gamma_M \) such that one has:

\[
\prod_{p \leq x, \, p \in M} \left(1 - \frac{1}{p}\right) = e^{-\gamma_M} + O\left(\frac{1}{\ln(x) \sqrt{\ln(x)}}\right) \quad \text{(6.24)}
\]

\[
\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = e^{\gamma_M} \sqrt{\ln(x)} + O\left(\frac{\sqrt{\ln(x)}}{\ln(x)}\right) \quad \text{(6.25)}
\]

When \( M = \mathbb{P} \), one has \( r = 1 \) and \( \gamma_M \) is the Euler constant \( \gamma \).

**Proof:** Demonstration proposed here is inspired, in the main lines, by that of [Nat] p. 278. One has

\[
\sum_{p \leq x, \, p \in M} \frac{1}{p} = \sum_{p \leq x, \, p \in M} \frac{\ln(p)}{p} \frac{1}{\ln(p)} = \sum_{2 \leq n \leq x} f(n)g(n)
\]

with \( g(t) = \frac{1}{\ln(t)} \) for \( t > 2 \), \( f(n) = \frac{\ln(p)}{p} \) if \( n = p \in M \) and \( f(n) = 0 \) otherwise.

From known formula

\[
\sum_{p \leq x} \frac{\ln(p)}{p} = \ln(x) + O(1) \quad \text{(6.26)}
\]

we first deduce that

\[
\sum_{p \leq x, \, p \in M} \frac{\ln(p)}{p} = \frac{1}{r} \ln(x) + O(1) \quad \text{(6.27)}
\]

Indeed, if

\[
p_1 \leq q_1 \leq p_2 \leq q_2 \leq \cdots \leq p_n \leq q_n \leq p_{n+1} \leq \cdots
\]

are real numbers tending to infinity as \( n \to +\infty \), one must have

\[
\sum_{p_i \leq x} f(p_i) - \sum_{q_i \leq x} f(q_i) = O(1)
\]

since

\[
f(p_1) - f(q_1) + f(p_2) - f(q_2) + \cdots + f(p_n) - f(q_n) + \cdots
\]

is an alternating convergent sequence (and partial summations gives rise to two monotonic sequences having the same limit): thus, we adapt the proof of proposition [4.1]
taking \( s = 1 \), \( w_j(s) = O(1) \) and formula \( 6.27 \) follows.

Let

\[
F(t) = \sum_{p \leq t, \ p \in M} \frac{\ln(p)}{p} = \sum_{p \leq t, \ p \in M} f(n)
\]

Then \( F(t) = \frac{1}{r} \ln(t) + r(t) \) with \( r(t) = O(1) \). Our next goal is to get formula

\[
\sum_{p \leq x, \ p \in M} \frac{1}{p} = \frac{1}{r} \ln(\ln(x)) + b + O\left(\frac{1}{\ln(x)}\right) \tag{6.28}
\]

for some fixed \( b \in \mathbb{R} \). This is a consequence of partial summation since

\[
\int_{x}^{+\infty} \frac{r(t)}{t(\ln(t))^2} \, dt = O\left(\frac{1}{\ln(x)}\right)
\]

and

\[
\sum_{2 \leq n \leq x} f(n)g(n) = F(x)g(x) - \int_{2}^{+\infty} F(t)g'(t) \, dt
\]

\[
= \frac{1}{r} + O\left(\frac{1}{\ln(x)}\right) + \frac{1}{r} \int_{2}^{x} \frac{1}{t \ln(t)} \, dt + \frac{1}{r} \int_{2}^{x} \frac{r(t)}{t(\ln(t))^2} \, dt
\]

\[
= \frac{1}{r} + \frac{1}{r} \ln(\ln(x)) - \frac{1}{r} \ln(\ln(2)) + \frac{1}{r} \int_{2}^{+\infty} \frac{r(t)}{t(\ln(t))^2} \, dt + O\left(\frac{1}{\ln(x)}\right)
\]

leading to formula \( 6.28 \) with

\[
b = \frac{1}{r} \ln(\ln(2)) - \frac{1}{r} \ln(\ln(2)) + \frac{1}{r} \int_{2}^{+\infty} \frac{r(t)}{t(\ln(t))^2} \, dt
\]

It follows that

\[
\ln\left(\prod_{p \leq x, \ p \in M} \left(1 - \frac{1}{p}\right)^{-1}\right) = - \sum_{p \leq x, \ p \in M} \ln(1 - \frac{1}{p}) = - \sum_{p \leq x, \ p \in M} \frac{1}{p} + \sum_{p \in M} \sum_{k=2}^{\infty} \frac{1}{kp^k} - \sum_{p \in M, p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k}
\]

Noting that

\[
\sum_{p \in M} \sum_{k=2}^{\infty} \frac{1}{kp^k} \leq \sum_{p \in \mathbb{P}} \sum_{k=2}^{\infty} \frac{1}{kp^k} < +\infty
\]

\[
\sum_{p \in M, p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k} \leq \sum_{p \in \mathbb{P}, p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k} \leq \frac{2}{x}
\]

we find a constant \( \gamma_M \) such that:

\[
\ln\left(\prod_{p \leq x, \ p \in M} \left(1 - \frac{1}{p}\right)^{-1}\right) = \frac{1}{r} \ln(\ln(x)) + \gamma_M + O\left(\frac{1}{\ln(x)}\right)
\]

Thus,

\[
\prod_{p \leq x, \ p \in M} \left(1 - \frac{1}{p}\right) = e^{\gamma_M \sqrt{\ln(x)}} e^{O\left(\frac{1}{\ln(x)}\right)}
\]

\[
= e^{\gamma_M \sqrt{\ln(x)}} \left(1 + O\left(\frac{1}{\ln(x)}\right)\right)
\]

which is formula \( 6.25 \). Taking the inverse, we get formula \( 6.24 \).
Proposition 6.9. Let $M$ be an arithmetical list of $\mathbb{P}$ with a reason $r > 0$. We assume that, for large $t$:

$$N_{pop}(t) \sim A_M \frac{\sqrt{\ln(t)}}{\ln(t)}$$

for some suitable constant $A_M$. Then, one has:

$$\sum_{n \in \text{pop}_M, n \leq x} \frac{1}{n} \sim r A_M \sqrt{\ln(x)} \quad \text{as} \quad x \to +\infty$$

Proof: With Abel summation, putting $m = [x]$, one has:

$$\sum_{n \in \text{pop}_M, n \leq x} \frac{1}{n} = \int_1^m N_{pop}(t) \frac{dt}{t^2} + \frac{1}{m} N_{pop}(m)$$

Clearly, $\frac{1}{m} N_{pop}(m) \sim A_M \frac{\sqrt{\ln(m)}}{\ln(m)} \to 0$ as $m \to +\infty$. For large $A$, one may write, for $t \geq A$, $N_{pop}(t) = (A_M + o(1)) t \frac{\sqrt{\ln(t)}}{\ln(t)}$, hence for $m \geq A$:

$$\int_A^m N_{pop}(t) \frac{dt}{t^2} = (A_M + o(1)) \int_A^m \frac{\sqrt{\ln(t)}}{\ln(t)} \frac{dt}{t} = (A_M + o(1)) \left[ r \sqrt{\ln(t)} \right]_{t=A}^{t=m}$$

$$= r (A_M + o(1)) \left( \sqrt{\ln(m)} - \sqrt{\ln(A)} \right) \sim r A_M \sqrt{\ln(m)} \quad \text{as} \quad m \to +\infty$$

Now, with $A$ fixed, as $1 \leq N_{pop}(t) \leq t$ for any $t \geq 1$, one has:

$$0 \leq \int_1^A \frac{N_{pop}(t) \ dt}{r A_M \sqrt{\ln(m)}} \leq \frac{\ln(A)}{r A_M \sqrt{\ln(m)}} \to 0 \quad \text{when} \quad m \to +\infty$$

Hence,

$$\int_1^m N_{pop}(t) \frac{dt}{t^2} \sim r A_M \sqrt{\ln(m)} \quad \text{as} \quad m \to +\infty$$

This proves the proposition.

Proposition 6.10. Let $M$ be an arithmetical list of $\mathbb{P}$ with a reason $r > 0$. We assume that

$$N_{pop}(t) \sim A_M \frac{\sqrt{\ln(t)}}{\ln(t)} \quad \text{as} \quad t \to +\infty$$

Then, one has, for $x > 0$:

$$\sum_{n \in \text{pop}_M} e^{-nx} = x \int_1^{+\infty} N_{pop}(t) e^{-tx} \ dt$$

For $x$ near 0, one has

$$\sum_{n \in \text{pop}_M} e^{-nx} \sim \frac{A_M}{x} \frac{\sqrt{\ln(\frac{1}{x})}}{\ln(\frac{1}{x})}$$

In formula 6.3, using $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$, we obtain that the function

$$\varphi : s \mapsto \int_1^{+\infty} \sum_{n \in \text{pop}_M} e^{-nt} t^s \frac{dt}{t}$$
clearly defines an analytic function \( \varphi(s) \) for \( \Re(s) > 0 \). The remaining integral can be written:

\[
\int_0^1 \sum_{n \in \text{pop}_M} e^{-nt} \frac{t^s dt}{t} = \int_0^1 \frac{A_M}{t} \frac{\sqrt{\ln(\frac{1}{t})}}{\ln(\frac{1}{t})} t^s\frac{dt}{t} + \int_0^1 h(t) t^s\frac{dt}{t}
\]

for some unknown function \( h(t) \), weak near \( t = 0 \). Since one has, for \( \Re(s) > 1 \)

\[
\int_0^1 \frac{1}{t} \frac{\sqrt{\ln(\frac{1}{t})}}{\ln(\frac{1}{t})} t^{s-1}\frac{dt}{t} = \frac{1}{\sqrt{s-1}} \Gamma(\frac{1}{r})
\]

we get that:

\[
\Gamma(s) Z_M(s) = \frac{A_M}{\sqrt{s-1}} \Gamma(\frac{1}{r}) + \int_0^1 h(t) t^s\frac{dt}{t} + \varphi(s)
\]

which is consistent with the property 3.2.

It is also known that \( \sum_{n \in \mathbb{N}^*} \frac{1}{n} e^{-nx} = \frac{1}{x} - \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{2x}{x^2 + 4n^2 \pi^2} \) and important features of \( \zeta(s) \) are obtained from this formula. When \( M \) is an arithmetical list with a reason \( r \), simple arguments seems to indicate that the corresponding development of \( \sum_{n \in \text{pop}_M} e^{-nx} \) looks like

\[
\sum_{n \in \text{pop}_M} e^{-nx} = \frac{B_r}{x} \frac{\sqrt{\ln(\frac{1}{x})}}{\ln(\frac{1}{x})} + \sum_{n \in \text{pop}_M} \frac{2x}{x^2 + 4n^2 \pi^2} + A(x)
\]

where \( A(x) \) is a new expression to be found. A complete knowledge of this expansion may be essential to the solution of Riemann’s conjecture.

**Proof of proposition 6.10:** Let \( m \in \mathbb{N}^* \), \( m \geq 2 \). By Abel summation,

\[
\sum_{1 \leq n \leq m} e^{-nx} = x \int_1^m N_{\text{pop}}(t) e^{-tx} dt + e^{-mx} N_{\text{pop}}(m)
\]

Clearly, for \( x > 0, \)

\[
e^{-mx} N_{\text{pop}}(m) \sim m \frac{A_M}{\ln(m)} e^{-mx} \rightarrow 0 \quad \text{when} \quad m \rightarrow +\infty
\]

As \( 1 \leq N_{\text{pop}}(t) \leq t \) for any \( t \geq 1 \), the function \( t \rightarrow N_{\text{pop}}(t) e^{-tx} \) is integrable on the interval \([1; +\infty[\), hence letting \( m \rightarrow +\infty \), one has, for fixed \( x > 0 \)

\[
\sum_{1 \leq n \leq m} e^{-nx} = x \int_1^{+\infty} N_{\text{pop}}(t) e^{-tx} dt
\]

since \( \sum_{1 \leq n \in \text{pop}} e^{-nx} \) is a convergent series.

Let \( A > 1 \). We put \( f(t) = t \frac{\sqrt{\ln(t)}}{\ln(t)} \) for \( t \geq A \). One has, for large \( t \):

\[
\frac{f'(t)}{f(t)} = \frac{1}{t} + \left( \frac{1}{r} - 1 \right) \frac{1}{t \ln(t)} \sim \frac{1}{t}
\]

and it follows from Laplace’s method that, for \( x \) near 0:

\[
\int_A^{+\infty} f(t) e^{-tx} dt \sim \frac{\Gamma(2)}{x} f\left(\frac{1}{x}\right) = \frac{1}{x^2} \frac{\sqrt{\ln(\frac{1}{x})}}{\ln(\frac{1}{x})}
\]
For any $0 < \epsilon < 1$, one can find $A = A(\epsilon)$ such that, for $t \geq A$, one has

$$(1 - \epsilon) A \, f(t) \leq N_{\text{pop}}(t) \leq (1 + \epsilon) A \, f(t)$$

leading to

$$(1 - \epsilon) A \int_{A}^{\infty} f(t) \, e^{-tx} dt \leq x \int_{A}^{\infty} N_{\text{pop}}(t) \, e^{-tx} dt \leq (1 + \epsilon) A \int_{A}^{\infty} f(t) \, e^{-tx} dt$$

for any $x > 0$, thus it remains to show that

$$\frac{x \int_{1}^{A} N_{\text{pop}}(t) \, e^{-tx} dt}{A \, \sqrt{\ln(\frac{A}{x})} / \ln(\frac{A}{x})} \to 0 \text{ as } x \to 0$$

for $A$ fixed, which easily follows from $\int_{1}^{A} N_{\text{pop}}(t) \, e^{-tx} dt \leq A^2$. This proves the proposition.

References

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