ON WEAK LIE 3-ALGEBRAS

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Abstract. In this article, we introduce a category of weak Lie 3-algebras with suitable weak morphisms. The definition is based on the construction of a partial resolution over \( \mathbb{Z} \) of the Koszul dual cooperad \( \text{Lie}^! \) of the Lie operad, with free symmetric group action. Weak Lie 3-algebras and their morphisms are then defined via the usual operadic approach—as solutions to Maurer–Cartan equations. As 2-term truncations we recover Roytenberg’s category of weak Lie 2-algebras. We prove a version of the homotopy transfer theorem for weak Lie 3-algebras. A right homotopy inverse to the resolution is constructed and leads to a skew-symmetrization construction from weak Lie 3-algebras to 3-term \( L_\infty \)-algebras. Finally, we give two applications: the first is an extension of a result of Rogers comparing algebraic structures related to \( n \)-plectic manifolds; the second is the construction of a weak Lie 3-algebra associated to an CLWX 2-algebroid leading to a new proof of a result of Liu–Sheng.

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Introduction

The study of algebraic structures up to homotopy combines the fields of algebra and homotopy theory. The objects of study are types of algebras and their invariance properties with respect to certain homotopy operations on their underlying spaces. In our setting, the underlying base category of spaces is a symmetric monoidal model category \( \mathcal{C} \), and the algebraic structures considered are algebras over an operad \( \mathcal{P} \) in \( \mathcal{C} \). Operads model many input, single output operations and their composition and are therefore suitable to describe many of the classical types of algebras, e.g. associative, commutative and Lie algebras (see \([21, 22]\)).

In general, algebras over an operad are rigid structures, meaning they do not play nice with homotopy operations on the underlying space. However, for some operads \( \mathcal{Q} \) their algebras do have good homotopy properties. This is the case in particular for those operads \( \mathcal{Q} \) that are cofibrant in the model structure on operads in \( \mathcal{C} \) (see \([4, 12, 13]\)). This model structure exists under some assumptions on the underlying model category \( \mathcal{C} \) and some restrictions on the operads, see op. cit. for details. For such a cofibrant operad \( \mathcal{Q} \), we can also equip its category \( \mathcal{Q}\text{-Alg} \) of \( \mathcal{Q} \)-algebras with a model structure and in this category a version of the Boardman–Vogt homotopy invariance property holds: given a homotopy equivalence of cofibrant-fibrant spaces \( X, Y \) in \( \mathcal{C} \), a structure of \( \mathcal{Q} \)-algebra on either induces a homotopy equivalent \( \mathcal{Q} \)-algebra structure on the other \( [4, \text{Theorem 3.5}] \).

We will often be interested in the homotopy category of \( \mathcal{Q} \)-algebras, which is defined as the localization \( \text{Ho} \mathcal{Q}\text{-Alg} = \mathcal{Q}\text{-Alg}[W^{-1}] \) with respect to the class \( W \) of weak equivalences. An isomorphism \( A \to A' \) in the homotopy category is a zigzag of weak equivalences in \( \mathcal{Q}\text{-Alg} \),

\[
A \xleftarrow{\sim} \bullet \xrightarrow{\sim} \cdots \xleftarrow{\sim} \bullet \xrightarrow{\sim} A'.
\]

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Given cofibrant resp. fibrant replacement functors $Q$ resp. $R$ on $Q$-$\text{Alg}$, it is a consequence of a general result on model categories (see e.g. [15, Theorem 1.2.10]) that

$$\text{Hom}_{\text{Ho}} Q$-$\text{Alg}(A, A') \cong \text{Hom}_{Q$-$\text{Alg}}(QA, RA')/\sim_h,$$

where the relation $\sim_h$ is homotopy of morphisms. We call a morphism $QA \to RA'$ of $Q$-algebras a homotopy morphism from $A$ to $A'$ and denote it by $A \sim A'$.

In this paper, we work in the differential graded framework. The category $\mathcal{C}$ will be the differential graded $\mathcal{C}$-algebras equipped with the standard projective model structure, i.e. weak equivalences are quasi-isomorphisms, fibrations are degree-wise epimorphisms, and cofibrations are determined by the left lifting property w.r.t. acyclic fibrations. This model structure can be transferred to give model structures on $\otimes$-modules, operads, and algebras over an operad, by defining the weak equivalences resp. fibrations to be those maps that are weak equivalences resp. fibrations on all underlying chain complexes. The cofibrations are then again determined by their lifting property. Note that the lifting property defining cofibrations depends on the structures, not just the underlying chain complexes. In particular, while it is clear from the above definition that all operads in $\mathcal{C}$ are fibrant, cofibrancy is an entirely different question.

On the category of operads in $\mathcal{C}$, we have functorial cofibrant resolutions given by the counit $\Omega \circ \text{Bar} \cong \text{P}$ of the cobar-bar adjunction for any operad $\mathcal{P}$ that is already $\otimes$-cofibrant, i.e. cofibrant in the underlying category of $\otimes$-modules. When working with chain complexes over a field $k$ of characteristic 0, this $\otimes$-cofibrancy condition is always satisfied. When $k$ is an arbitrary unital commutative ring, one way to obtain an $\otimes$-cofibrant resolution for any operad $\mathcal{P}$ is to take the arity-wise tensor product with the algebraic Barratt–Eccles operad $\varepsilon$ (see e.g. [3]). In this way, one obtains a cofibrant resolution $\Omega \circ \text{Bar}(\mathcal{P} \otimes \varepsilon) \cong \mathcal{P} \otimes \varepsilon \cong \text{P}$ of any operad. We give a more conceptual understanding of this resolution in joint work with Bruno Vallette [7], where we model non-unital operads as algebras over a particular colored operad $\mathcal{O}$. A choice of Koszul presentation for this operad $\mathcal{O}$, leads to a new type of cobar-bar resolution resolving simultaneously operadic composition as well as the symmetric group actions. We denote the new cobar and bar constructions by $\breve{\text{O}}$ resp. $\breve{\text{B}}$. It is then shown in loc. cit. that there is an isomorphism of operads $\mathcal{P} \otimes \breve{\text{O}} \breve{\text{B}} \cong \Omega \circ \text{Bar}(\mathcal{P} \otimes \varepsilon)$, where $\breve{\text{P}}$ denotes the operadic augmentation ideal, $\mathcal{P} = \mathcal{P} \otimes \mathcal{I}$.

For the classical types of algebras Ass, Com, and Lie, there are well known homotopy invariant analogues. For associative algebras, these are the $A_\infty$-algebras introduced by Stasheff [26]. For commutative algebras, there exist the notions of $C_\infty$-algebras, introduced by Kadeishvili [16], and of $E_\infty$-algebras, going back to May [23] and Boardman–Vogt [6]. For Lie algebras, the homotopy invariant notion of $L_\infty$-algebras was introduced by Hinich–Schechtman [14], see also [8]. The operad $A_\infty$ is cofibrant over any unital commutative ring $k$ since the operad Ass is already $\otimes$-cofibrant. The operads $C_\infty$ and $L_\infty$ are only cofibrant over fields $k$ of characteristic 0. The notion of $E_\infty$-algebras describes algebras over any $\otimes$-cofibrant resolution of Com, e.g. over the Barratt–Eccles operad $\varepsilon$. A cofibrant $E_\infty$ operad is then given by the cobar-bar resolution of the Barratt–Eccles operad, $\Omega \circ \varepsilon \cong \varepsilon \otimes \breve{\text{B}} \text{Com}$. In the case of the $A_\infty$, $L_\infty$ and $C_\infty$ operads, these can be obtained as resolutions $\mathcal{P}_\infty = \Omega \circ \mathcal{P}_1 \cong \mathcal{P}$ for some cooperad $\mathcal{P}_1$ weakly equivalent to $\mathcal{B}$ in the Quillen-type model structure on dg cooperads (see [18]). The cooperad $\mathcal{P}_1$ is given by a presentation dual to a choice of presentation for $\mathcal{P}$. The presentation is called Koszul, if in fact $\mathcal{P}_1 \Rightarrow \mathcal{P}$ is a weak equivalence and therefore $\Omega \circ \mathcal{P}_1 \Rightarrow \Omega \circ \mathcal{B} \mathcal{P} \Rightarrow \mathcal{P}$ is a quasi-isomorphism. By this Koszul duality approach, it is possible to obtain much smaller resolutions for many operads. The restriction that $\mathcal{P}$ needs to be $\otimes$-cofibrant, however, still holds.

Our interest in this article lies in finding a small cofibrant replacement $\text{EL}_\infty \cong \text{Lie}$ for the Lie operad over any unital commutative ring $k$. Since the operad Lie is not $\otimes$-cofibrant, we cannot use the classical Koszul duality methods. Within the context of our new cobar-bar adjunction $\breve{\text{O}} \Rightarrow \breve{\text{B}}$, however, a Koszul duality approach is not yet available. We will therefore resort to a more ad hoc approach to build our resolution: we start from the usual Koszul dual cooperad $\text{Lie}_1$ and build an $\otimes$-free resolution step by step. Assume for a moment that we had completed this process, i.e. we have an $\otimes$-free resolution $\psi: \text{Lie}_\psi \Rightarrow \text{Lie}_1$ of dg cooperads. Since $\text{Lie}_\psi$ is $\otimes$-free, $\Omega \circ \text{Lie}_\psi$ is then a cofibrant operad. To show that the composition $g_{\psi} \circ \Omega \psi: \Omega \circ \text{Lie}_\psi \Rightarrow \Omega \circ \text{Lie}_1 \Rightarrow \text{Lie}$ forms a cofibrant resolution, we need to verify that $\Omega \psi$ is a quasi-isomorphism or, equivalently, that the twisted composite product $\text{Lie}_\psi \circ_{\text{Lie}_1} \text{Lie}$ is acyclic. In this case, we can apply the standard machinery of algebraic operads to obtain a category
of $\mathbb{L}_{\infty}$-algebras with homotopy morphisms satisfying a version of the homotopy transfer theorem. We recall the relevant background material in Section 1.

While we do not have a complete $\mathbb{S}$-free resolution of dg cooperads as described above, in Section 2 we do construct such a resolution in low degrees and show that at least truncated versions of the relevant statements hold. As a first step, we introduce an explicit $\mathbb{S}$-free resolution $\psi: \text{Lie}^3_\mathbb{S} \to \text{Lie}^3$ of dg $\mathbb{S}$-modules in low degrees, i.e. such that $H_r(\psi)$ are isomorphisms for $r \leq 3$. In the second step, we equip $\text{Lie}^3$ with a decomposition map turning it into a dg cooperad. Since Leibniz algebras are essentially non-symmetric Lie algebras, it makes sense to use their Koszul dual cooperad $\text{Leib}^1$ as a starting point for both steps. The higher degrees of $\text{Lie}^3$ can be viewed as a coherent system of higher homotopies for the (missing) skew-symmetry. We extend the decomposition map of $\text{Leib}^1$ to the higher degrees in a way that is compatible with the differential by solving systems of linear diophantine equations and verify that it is actually coassociative. Finally we prove the following result, which—while of no immediate consequences—is a necessary condition if we intend to extend our low degree resolution to a full cofibrant resolution.

**Proposition 2.3** The twisted composite product $\text{Lie}^3_\mathbb{S} \circ_\psi \text{Lie}$ satisfies

$$H_r(\text{Lie}^3_\mathbb{S} \circ_\psi \text{Lie})(n) = 0,$$

for all $r \leq 3$ in all arities $n$.

In Section 3 we use the resolution $\text{Lie}^3_\mathbb{S}$ to define weak Lie 3-algebras as $\Omega \text{Lie}^3_\mathbb{S}$-algebras on a 3-term complex and introduce the corresponding notion of weak morphisms. Since we used $\text{Leib}^1$ as a starting point for our resolution, weak Lie 3-algebras and their morphisms consist of extra structure on top of Leibniz 3-algebras and morphisms of such. We make the definitions explicit in terms of structure maps and equations. We proceed to show explicitly that, given a deformation retract of 3-term chain complexes, i.e. chain maps $p$ and $i$ and a chain homotopy $h$ as in

$$h \begin{array}{c} \psi \circ p \end{array} (L, d) \begin{array}{c} \psi \circ i \end{array} (L', d') ,$$

such that

$$\begin{cases} \text{id}_L - i \circ p = [d, h], \\ \text{id}_{L'} - p \circ i = 0, \end{cases}$$

the following homotopy transfer property holds.

**Proposition 3.6** Let $(L, d, \lambda)$ be a weak Lie 3-algebra and let $(L', d')$ be a deformation retract of $(L, d)$. Then $(L', d')$ can be equipped with a transferred weak Lie 3-algebra structure in such a way, that the map $i$ admits an extension to a weak morphism of weak Lie 3-algebras.

In 1 the notion of a Lie 2-algebra is introduced using a very different approach known as categorification. It is then shown that the definition is equivalent to that of a 2-term $\mathbb{L}_{\infty}$-algebra, i.e. a 2-term chain complex with a binary graded skew-symmetric bracket satisfying the Jacobi identity up to homotopy. In 2, the definition of a weak Lie 2-algebra is introduced, again as a categorification of Lie algebras, this time with the skew-symmetry of the Lie bracket relaxed up to homotopy in addition to the Jacobi identity. Truncating the complexes underlying our weak Lie 3-algebras to 2-term complexes, we recover Roytenberg’s definitions of weak Lie 2-algebras and their weak morphisms. Similarly, we recover his homotopy transfer theorem 2 Theorem 4.1 for weak Lie 2-algebras as a truncation of Proposition 3.6.

By construction of $\text{Lie}^3_\mathbb{S}$, we have morphisms of dg cooperads $\text{Leib}^3_\mathbb{S} \hookrightarrow \text{Lie}^3_\mathbb{S} \twoheadrightarrow \text{Lie}^3$ and therefore functors

$$\text{Leibniz 3-algebras} \hookrightarrow \text{Weak Lie 3-algebras} \twoheadrightarrow \text{Lie 3-algebras} .$$

While (homotopy) Lie algebras are precisely (homotopy) Leibniz algebras with skew-symmetric structure maps, in general skew-symmetrizing the bracket(s) of a (homotopy) Leibniz algebra does not give a (homotopy) Lie algebra. Operadically speaking, this means that there is no simple non-trivial morphism of cooperads $\text{Lie}^1 \to \text{Leib}^1$. Using the higher degree terms in $\text{Lie}^3_\mathbb{S}$ we can, however, construct a morphism $\text{Lie}^3_\mathbb{S} \hookrightarrow \text{Leib}^1_\mathbb{S}$ of dg cooperads up to homotopy. Precisely, we will prove the following result in Section 4.

**Lemma 4.1** The morphism $\Omega \psi$ admits a right inverse, i.e. a morphism $\Phi$ of dg operads

$$\Omega \psi: \Omega \text{Lie}^3_\mathbb{S} \hookrightarrow \Omega \text{Lie}^3_\mathbb{S} : \Phi ,$$

such that $\Omega \psi \circ \Phi = \text{id}$.  

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3
This leads to a skew-symmetrization construction producing for each weak Lie 3-algebra \((L, d, \lambda)\) a (semi-strict) Lie 3-algebra \((L, d, \overline{\lambda})\). We then introduce an ad hoc definition of skew-symmetrization for morphisms of weak Lie 3-algebras and proceed to show that this construction—while not strictly functorial—is in some sense functorial up to homotopy.

We end this article with a discussion of applications of our results in higher differential geometry in Section 5. In \([2, 24]\) the concept of \(n\)-plectic manifolds is introduced as a higher analogue for symplectic manifolds. To any \(n\)-plectic manifold \((M, \omega)\), two algebraic structures on an \(n\)-term truncation of the de Rham complex are associated: an \(L_\infty\)-algebra \(L_\infty(M, \omega)\) and a dg Leibniz algebra \(\text{Leib}(M, \omega)\) with a certain hidden skew-symmetry. In the case of \(3\)-plectic manifolds, both are examples of weak 3-algebras. We show that \(L_\infty(M, \omega)\) and \(\text{Leib}(M, \omega)\) are isomorphic as such. The analogous result for 2-plectic manifolds is due to Rogers \([24, \text{Appendix A}]\).

In recent work of Liu–Sheng \([19]\), the notion of a CLWX 2-algebroid is introduced as a higher Courant algebroid, and it is shown that a 3-algebra can be assigned to any CLWX 2-algebroid. We refine the construction to give a weak Lie 3-algebra and show that the Lie 3-algebra constructed in op. cit. is in fact the skew-symmetrization of our weak Lie 3-algebra, thereby giving a new proof for \([19, \text{Theorem 3.10}]\).

Acknowledgements. The author would like to thank Chris Rogers and Dmitry Roytenberg for their insightful comments and notation we follow in the remainder of this article. In Section 1.2, we introduce some conventions, fix notation, and recall basic definitions and results of the theory of algebraic operads. For a detailed introduction to the theory of algebraic operads we refer the reader to \([21]\), and in Section 1.3, we give the basic definitions of operads and cooperads. In Section 1.5, we define the operadic bar and cobar constructions. In Section 1.6, we recall the twisting morphism bifunctor. Section 1.7 serves as a quick reminder of Koszul duality for operads. Finally, Section 1.8 deals with homotopy algebras and morphisms in general.

1. Preliminaries

In this section, we introduce some conventions, fix notation, and recall basic definitions and results of the theory of algebraic operads. For a detailed introduction to the theory of algebraic operads we refer the reader to \([21]\), upon which this section is heavily based. This section is organized as follows. In Section 1.1, we introduce some conventions and notation we follow in the remainder of this article. In Section 1.2, we introduce \(S\)-modules and equip them with the structure of a monoidal category. In Section 1.3, we give the basic definitions of operads and cooperads. In Section 1.5, we define the operadic bar and cobar constructions. In Section 1.6, we recall the twisting morphism bifunctor. Section 1.7 serves as a quick reminder of Koszul duality for operads. Finally, Section 1.8 deals with homotopy algebras and morphisms in general.

1.1. Conventions and notation. We denote by \(\mathbf{k}\) an arbitrary unital commutative ring. For any computations we will work over the integers \(\mathbf{k} = \mathbb{Z}\). Since \(\mathbb{Z}\) is the initial object in the category of unital commutative rings, this ensures that our results hold over any such ring \(\mathbf{k}\). Our chain complexes are \(\mathbb{Z}\)-graded complexes of \(\mathbf{k}\)-modules. We follow the Koszul sign rule, i.e. whenever symbols \(x, y\) of homological degree \(|x|\) resp. \(|y|\) change their relative order, a factor \((-1)^{|x||y|}\) is introduced.

1.1.1. Suspension. We denote by \(sk\) the chain complex that is \(\mathbf{k}\) in degree 1 and zero in all other degrees. For any chain complex \((V, d^V)\), we define its suspension to be \(sV := sk \otimes V\) with \(d^sV = 1 \otimes d^V\). We denote by \(s: V \rightarrow sV\), \(v \mapsto sv = s \otimes v\) the suspension isomorphism. Desuspension \(s^{-1}\) is defined similarly.

1.1.2. Symmetric group. We denote by \(S_n\) the symmetric group on \(n\) elements, i.e. the group of bijections of the set \(\{1, \ldots, n\}\). We use the notation \(\mathbf{k}[S_n]\) for the group algebra and the (right) regular representation of \(S_n\). By \(\mathbf{k} \cdot \text{sgn}_n\) we denote the one-dimensional signature representation of \(S_n\), i.e. its underlying module is \(\mathbf{k}\) and the adjacent transpositions \(\sigma_i = (i \ i + 1)\) act by multiplication with \(-1\). We implicitly extend the group representations to representations of the group algebra and write e.g. \(x^{-\sigma_\tau} = -x^\sigma + x^\tau\).

1.1.3. Shuffle permutations. Let \(n_1, \ldots, n_m\) be natural numbers, s.t. \(n = n_1 + \cdots + n_m\). We call \(\sigma \in S_n\) an \((n_1, \ldots, n_m)\)-shuffle if \(|\sigma(i) - \sigma(i+1)| = 1\) for all \(1 \leq i < n\) except when \(i = n_i + \cdots + n_j\) for some \(1 \leq j < m\). We denote by \(\text{Sh}(n_1, \ldots, n_m) \subset S_n\) the subset of these \((n_1, \ldots, n_m)\)-shuffles. The shuffles \(\text{Sh}(n_1, \ldots, n_m)\) form a set of representatives for the cosets \(S_n/(S_{n_1} \times \cdots \times S_{n_m})\).
We call an \((n_1, \ldots, n_m)\)-shuffle \(\sigma\) reduced, if \(\sigma(n_j) < \sigma(n_{j+1})\) for all \(1 \leq j < m\). The set of these reduced shuffles is denoted as \(\text{Sh}(n_1, \ldots, n_m)\). The inverse of a shuffle is called an unshuffle and the set of these is denoted by \(\text{Sh}^{-1}(n_1, \ldots, n_m)\).

1.2. \(S\)-Modules. A dg \(S\)-module \(M\) consists of a (right) dg \(k[S_n]\)-module \(M(n)\) for each arity \(n \in \mathbb{N}\). We sometimes write a dg \(S\)-module \(M\) as a sequence \((M(0), M(1), \ldots)\). A morphism of dg \(S\)-modules \(f: M \to N\) consists of a morphism \(f(n): M(n) \to N(n)\) of dg \(k[S_n]\)-modules for each arity \(n \in \mathbb{N}\). We denote the category of dg \(S\)-modules by \(\text{dg} S\text{-Mod}\). When \(M(0) = 0\), we call \(M\) reduced.

1.2.1. Monoidal structure. For \((M, dM), (N, dN) \in \text{dg} \text{-}\text{S}\text{-Mod}\), we define their composite product \((M \circ N, d_{M \circ N})\) by

\[
(M \circ N)(n) := \bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} M(m) \otimes_{\text{S}_m} \text{Ind}_{\text{S}_{n_1} \times \cdots \times \text{S}_{n_m}}^\text{S}_n N(n_1) \otimes \cdots \otimes N(n_m),
\]

with the differential given by

\[
d_{M \circ N} := \bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} d^M \otimes_{\text{S}_m} 1 \otimes_{\text{S}_m} \left( \sum_{i=1}^m 1 \otimes (i-1) \otimes d^N \otimes 1 \otimes (m-i) \right).
\]

The composite product of morphisms \(f: M \to M', g: N \to N'\) is defined to be the morphism \(f \circ g: M \circ N \to M' \circ N'\) given by

\[
f \circ g := \bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} f \otimes_{\text{S}_m} g \otimes_{\text{S}_m}.
\]

Note that, since the shuffles \(\text{Sh}(n_1, \ldots, n_m)\) form a set of representatives for the cosets \(S_m/(S_{n_1} \times \cdots \times S_{n_m})\), this composite product admits an expansion

\[
(M \circ N)(n) = \bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} M(m) \otimes_{\text{S}_m} \left( N(n_1) \otimes \cdots \otimes N(n_m) \otimes k[\text{Sh}(n_1, \ldots, n_m)] \right),
\]

and when the \(S\)-module \(N\) is reduced, the \(S_m\)-action on the right is free and therefore we obtain

\[
\bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} M(m) \otimes \left( N(n_1) \otimes \cdots \otimes N(n_m) \otimes k[\text{Sh}(n_1, \ldots, n_m)] \right).
\]

We denote an element \(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_m \otimes \sigma^{-1}\) of \(M \circ N\) by \(\mu \circ (\nu_1, \ldots, \nu_m)\).

The \(S\)-module \(I = (0, k, 0, 0, \ldots)\) acts as a (two-sided) unit w.r.t. the composite product. This structure turns the category of dg \(S\)-modules into a monoidal category (dg \(S\text{-Mod}\), \(\circ, I\)).

1.2.2. The linearized composite product. Consider the composite product \(M \circ (N_1 \oplus N_2)\). We denote by \(M \circ (N_1; N_2)\) the sub dg \(S\)-module that is linear in \(N_2\), i.e. that is spanned by elements \(\mu \circ (\nu_1, \ldots, \nu_m)\) where \(\nu_i \in N_2\) for exactly one of \(\nu_1, \ldots, \nu_m\) and \(\nu_j \in N_1\) for \(j \neq i\). We use the notation \(M \circ_{(1)} N\) instead of \(M \circ_{(1)} (I; N)\) so it appears so frequently. We write \(\mu \otimes \nu\) for \(\mu \circ_{(1; I; \nu_1, \ldots, \nu_m)}\) with \(\nu\) in \(i\)-th place.

We introduce two types of linearized composite products of morphisms. Given \(f: M \to M', g_1: N_1 \to N_1', g_2: N_2 \to N_2'\), we define

\[
f \circ (g_1; g_2) := \left( M \circ (N_1; N_2) \xrightarrow{f \circ (g_1 \oplus g_2)} M' \circ (N_1' \oplus N_2') \right).
\]

Consider now \(f: M \to M'\) and \(g: N \to N'\). We denote by \(f \circ_{(1)} g\) the morphism \(f \circ (1; g)\), i.e. \((f \circ_{(1)} g)(\mu \otimes \nu) = (-1)^{|g||\mu|} f(\mu) \otimes g(\nu)\). Given the same data, we can also define a morphism

\[
f' \circ g: M \circ N \to M' \circ (N; N'), \quad f' \circ g := \bigoplus_{m \in \mathbb{N}} \bigoplus_{n_1 + \cdots + n_m = n} f \otimes_{\text{S}_m} \left( \sum_{i=1}^m 1 \otimes (i-1) \otimes g \otimes 1 \otimes (m-i) \right).
\]

When \(N' = N\) we implicitly postcompose with \(M \circ (N; N) \to M \circ N\) s.t. \(f' \circ g: M \circ N \to M' \circ N\). With this notation, we can write the differential of the full composite product as \(d_{M \circ N} = d^M \circ 1 + 1 \circ d^N\).

1.3. Operads and cooperads. A dg operad is a monoid \((\mathcal{P}, \gamma, \eta)\) in dg \(S\text{-Mod}\), i.e. an \(S\)-module \(\mathcal{P}\) with composition map \(\gamma: \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}\) and unit \(\eta: I \to \mathcal{P}\) satisfying associativity and left and right unit axioms. A morphism of dg operads \(f: \mathcal{P} \to \mathcal{P}'\) is a monoid morphism, i.e. a morphism of the underlying dg \(S\)-modules commuting with the structure maps. An augmentation for \(\mathcal{P}\) is a morphism of dg operads \(\varepsilon: \mathcal{P} \to I\) s.t. \(\varepsilon \eta = \text{id}\). We denote by \(\overline{\mathcal{P}}\) the
augmentation ideal \( \overline{P} := \ker \varepsilon \) and the restriction of \( \gamma \) to it by \( \overline{\gamma} : \overline{P} \circ \overline{P} \to \overline{P} \). We call \( P \) reduced, if its underlying \( S \)-module is reduced.

Dually, a dg cooperad is a comonoid \((C, \Delta, \varepsilon)\) in \( \text{dg} \ S\text{-Mod} \), i.e. an \( S \)-module \( C \) with decomposition map \( \Delta : C \to C \circ C \) and counit \( \varepsilon : C \to I \) satisfying coassociativity and left and right counit axioms. A morphism of dg cooperads \( f : C \to C' \) is a comonoid morphism, i.e. a morphism of the underlying \( S \)-modules commuting with the structure maps. A coaugmentation for \( C \) is a morphism of dg cooperads \( \eta : I \to C \) s.t. \( \varepsilon \circ \eta = \id \). We denote by \( \overline{C} \) the coaugmentation coideal \( \overline{C} := \ker \eta \) and by \( \overline{\Sigma} : \overline{C} \to \overline{C} \circ \overline{C} \) the corestriction of \( \Delta \), i.e. \( \Delta(\mu) = 1 \circ \mu + \overline{\Sigma}(\mu) + \mu \circ (1, \ldots, 1) \).

A coaugmented dg cooperad is called conilpotent if for any element its successive decompositions stabilize, see [21, §5.8.6] for the technical details. We call \( C \) reduced, if its underlying \( S \)-module is reduced.

1.3.1. Infinitesimal (de)composition. By restriction, we obtain for an operad \((P, \gamma, \eta)\) the infinitesimal or partial composition map \( \gamma_1 : P \circ_1 P \to P \). Similarly, we obtain the infinitesimal of partial decomposition map \( \Delta_{1} : C \to C \circ_1 C \) for a cooperad \((C, \Delta, \varepsilon)\) by corestriction. Note that the (co)operad (de)composition maps are completely determined by their partial (de)composition counterparts. We use the classical notation \( \mu_i \circ \nu := \gamma_1(\mu \otimes \nu) \).

\[
\begin{align*}
\gamma : \quad & P \times P \to P \\
\gamma(1) : \quad & P \times P \to P
\end{align*}
\]

(A) Monoid composition. (B) Partial composition.

**Figure 1.** Comparison of operadic composition maps.

1.3.2. The Hadamard tensor product. Given operads \( P, P' \), we can equip the arity-wise tensor product of their underlying \( S \)-modules \( (P \otimes P')(n) := P(n) \otimes P'(n) \) with an operad structure as follows. For elements \( \mu \otimes \mu' \in (P \otimes P')(n) \) and \( \nu \otimes \nu' \in (P \otimes P')(n) \), define their \( i \)-th partial composition by

\[
(\mu \otimes \mu') \circ_1 (\nu \otimes \nu') := (\mu \circ \nu) \otimes (\mu' \circ \nu').
\]

We call the resulting operad \( P \otimes P' \) the Hadamard product of \( P \) and \( P' \).

1.3.3. The endomorphism operad. Given a chain complex \((V, d)\), we define its endomorphism operad \( \text{End}_V \) as follows. Let \( \text{End}_V(n) := \text{hom}(V^\otimes n, V) \) be the space of \( n \)-multilinear maps on \( V \). Note that hom denotes internal homomorphisms, i.e. elements of \( \text{End}_V(n) \) need not be chain maps. With the symmetric group action given by permutation inputs

\[
\mu^\sigma(v_1, \ldots, v_n) := \varepsilon(\sigma; v_1, \ldots, v_n) \cdot \mu(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}),
\]

where \( \varepsilon \) denotes the Koszul sign, and the usual differential \( \partial(\mu) = d^V \circ \mu - (-1)^{|\mu|} \mu \circ d^V \circ n, \text{End}_V \) becomes a dg \( S \)-module. For \( \mu \in \text{End}_V(n) \) and \( \nu \in \text{End}_V(k) \), we define their partial composition by

\[
(\mu \circ \nu)(v_1, \ldots, v_{n+k-1}) := (-1)^{|\nu|(|v_1| + \cdots + |v_{n-1}|)} \cdot \mu(v_1, \ldots, v_{i-1}, \nu(v_i, \ldots, v_{i+k-1}), v_{i+k}, \ldots, v_{n+k-1}).
\]

We introduce the shorthand notation \( S := \text{End}_{V^k} \) and call this operad the suspension operad. Similarly, we introduce the desuspension operad \( S^{-1} := \text{End}_{V^1} \). The operad \( u\text{Com} := \text{End}_{k^k} \) governs unital commutative algebras. The operad \( \text{Com} \) differs from \( u\text{Com} \) only in arity 0 where \( u\text{Com}(0) = \text{hom}(k, k) \cong k \) while \( \text{Com}(0) = 0 \).

1.3.4. The (co)free (co)operad. For any \( M \in \text{dg} \ S\text{-Mod} \), we denote by \( T(M) \) the free augmented dg operad on \( M \) and by \( T^c(M) \) the cofree conilpotent dg cooperad on \( M \). The underlying dg \( S \)-module is the same in both cases. It consists of planar trees with vertices labeled by elements of \( M \) of arity corresponding to the number of incoming edges at the vertex and with a total ordering on the set of leaves. Free composition is then given by grafting of such trees and reordering the leaves. The (co)free (co)operad carries an extra grading we call weight-grading. The component \( T(M)(w) \) resp. \( T^c(M)(w) \) of weight \( w \) is spanned by the trees with precisely \( w \) vertices. We refer the reader to [21] for more details.
1.4. **The category of \( \mathcal{P} \)-algebras.** By definition of operads as monoids in dg \( \mathbb{S} \)-modules, they come with the usual notions of modules. Of particular interest is the notion of a \( \mathcal{P} \)-algebra, i.e. a left \( \mathcal{P} \)-module structure on a chain complex viewed as an \( \mathbb{S} \)-module concentrated in arity 0. Such a module structure is given by a morphism \( \gamma_A: \mathcal{P} \circ A \to A \). Under the currying isomorphism, this map corresponds to a collection of morphisms \( \mathcal{P}(n) \to \text{hom}(A^\otimes n, A) \). In fact, a \( \mathcal{P} \)-algebra \((A, d, \gamma_A)\) is equivalently a triple \((A, d, g)\) consisting of a chain complex \((A, d)\) equipped with a morphism of operads \(g: \mathcal{P} \to \text{End}_A\).

1.4.1. **Morphisms of \( \mathcal{P} \)-algebras.** A morphism \( f: (A, d, \gamma_A) \to (A', d', \gamma_A') \) of \( \mathcal{P} \)-algebras is a morphism of the underlying chain complexes \( f: (A, d) \to (A', d') \) commuting with the structure maps, i.e. such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P} \circ A & \xrightarrow{\gamma_A} & A \\
\downarrow{1 \circ f} & & \downarrow{f} \\
\mathcal{P} \circ A' & \xrightarrow{\gamma_A'} & A'.
\end{array}
\]

Note that, under the currying isomorphism, the diagonal \( \mathcal{P} \circ A \to A' \) corresponds to a collection of morphisms \( \mathcal{P}(n) \to \text{hom}(A^\otimes n, A') \).

Given chain complexes \((V, d)\) and \((V', d')\), one may consider the sub dg \( \mathbb{S} \)-module \( \text{End}^V_{V'} \subset \text{End}_{V' \otimes V'} \) given by \( \text{End}^V_{V'}(n) = \text{hom}(V^\otimes n, V') \). While \( \text{End}^V_{V'} \) is not a sub dg operad, it does have an obvious \((\text{End}^V_{V'}, \text{End}^V_{V'})\)-bimodule structure given by restriction of the composition map in \( \text{End}_{V' \otimes V'} \). Using this notation, commutativity of the diagram in equation (2) can equivalently be expressed as commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{g} & \text{End}_A \\
\downarrow{g'} & & \downarrow{f_*} \\
\text{End}_{A'} & \xrightarrow{f^*} & \text{End}^A_{A'}.
\end{array}
\]

where \( f_* \) resp. \( f^* \) denote post- resp. pre-composition with \( f \).

1.5. **The cobar-bar resolution.** The category of reduced augmented dg operads over a commutative ring \( k \) admits a model category structure \([4, 12, 13]\), where the weak equivalences are the arity-wise quasi-isomorphisms and the fibrations are the arity-wise degree-wise epimorphisms. In this context, we will be interested in cofibrant replacements \( Q \buildrel{Q} \longrightarrow \mathcal{P} \) for a given operad \( \mathcal{P} \). Algebras over such a cofibrant replacement satisfy a certain homotopy invariance property \([4, \text{Theorem 3.5}]\). One way to obtain such a cofibrant resolution for an operad is by means of the cobar-bar resolution \([5, 10]\), i.e. the counit of the adjunction

\[
\Omega: \text{conil dg Coop} \longrightarrow \text{aug dg Op} : B,
\]

which we recall here.

1.5.1. **The bar construction.** The bar construction of an augmented dg operad \( \mathcal{P} \) is the cofree conilpotent dg cooperad on the arity-wise suspension \( s\mathcal{P} \) with an additional term in the codifferential, \( B \mathcal{P} := (T^c(s\mathcal{P}), d = d_1 - d_2) \). The component \( d_1 \) of the codifferential \( d \) is the coextension of the internal differential of \( \mathcal{P} \) with appropriate shifting, i.e. \( d_1 \) coextends

\[
T^c(s\mathcal{P}) \longrightarrow T^c(s\mathcal{P})^{(1)} \cong s^1\mathcal{P} \xrightarrow{d} s\mathcal{P}.
\]

Similarly, \( d_2 \) coextends the multiplication of \( \mathcal{P} \) with appropriate shifting, i.e. the following composition,

\[
T^c(s\mathcal{P}) \longrightarrow T^c(s\mathcal{P})^{(2)} \cong s\mathcal{P} \otimes_{(1)} s\mathcal{P} \xrightarrow{s^{-1}\circ_{(1)}s^{-1}} \mathcal{P} \otimes_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P} \longrightarrow s\mathcal{P}.
\]

In plain english, the codifferential amounts to the sum of applying \( d\mathcal{P} \) at each vertex and contracting each internal edge while composing its source and target vertex by the partial composition map \( \gamma_{(1)} \) (with appropriate signs), as shown in Figure 2.
1.5.2. The cobar construction. The cobar construction of a conilpotent dg cooperad \( C \) is defined as the dg operad \( \Omega C := (\mathcal{T}(s^{-1}C), d = d_1 - d_2) \). The differential again has two terms. The \( d_1 \) term is the extension of

\[
\begin{array}{c}
s^{-1}C \xrightarrow{\delta} s^{-1}C \cong \mathcal{T}(s^{-1}C)^{(1)} \\
\end{array}
\]

and \( d_2 \) extends

\[
\begin{array}{c}
s^{-1}C \xrightarrow{s} C \xrightarrow{\Delta(1)} C \circ (1) C \cong s^{-1}C \circ (1) s^{-1}C \cong \mathcal{T}(s^{-1}C)^{(2)} \xrightarrow{} \mathcal{T}(s^{-1}C) .
\end{array}
\]

1.5.3. The adjunction. The fact that the cobar and bar constructions form an adjunction \( \Omega \dashv B \) is most easily seen from equation (4) below after introducing the twisting morphism bifunctor.

For the counit of the cobar-bar adjunction to give a cofibrant resolution of an operad \( \mathcal{P} \), one needs the assumption that the operad \( \mathcal{P} \) is \( S \)-cofibrant. This condition means that the underlying \( S \)-module is in fact cofibrant, i.e. the \( k[S_n] \)-modules \( \mathcal{P}(n) \) are projective in each arity \( n \). This is always true when \( k \) is a field of characteristic 0 and for this special case the above result is contained in [10, 11].

**Theorem 1.1** ([8, §8.5]). The counit of the cobar-bar adjunction gives a cofibrant resolution \( \Omega B \mathcal{P} \xrightarrow{\sim} \mathcal{P} \), provided the operad \( \mathcal{P} \) is \( S \)-cofibrant.

1.6. Twisting morphisms. Let \( \mathcal{P} \) be an augmented dg operad and \( C \) a conilpotent dg cooperad. Since the operad underlying the cobar construction is the free operad \( \mathcal{T}(s^{-1}C) \), any morphism of augmented operads \( \Omega C \rightarrow \mathcal{P} \) is uniquely determined by its value on generators \( s^{-1}C \). This means morphisms of augmented operads \( \mathcal{T}(s^{-1}C) \rightarrow \mathcal{P} \) are in one to one correspondence with morphisms of \( S \)-modules \( s^{-1}C \rightarrow \mathcal{P} \) or, equivalently, degree \(-1\) morphisms of \( S \)-modules \( C \rightarrow \mathcal{P} \). A similar argument holds for the bar construction and we thus obtain

\[
\text{Hom}_{\text{aug Op}}(\mathcal{T}(s^{-1}C), \mathcal{P}) \cong \text{hom}_{S\text{-Mod}}(C, \mathcal{P})_{-1} \cong \text{Hom}_{\text{conil Coop}}(C, \mathcal{T}^n(s\mathcal{P})).
\]

We introduce the subset of twisting morphisms \( \text{Tw}(C, \mathcal{P}) \subset \text{hom}_{S\text{-Mod}}(C, \mathcal{P})_{-1} \) that correspond (under the above isomorphisms) to morphisms of dg (co)operads. Given \( f, g \in \text{hom}_{S\text{-Mod}}(C, \mathcal{P}) \), we define their pre-Lie convolution product to be the composition

\[
f \star g = \left( C \xrightarrow{\Delta(1)} C \circ (1) C \xrightarrow{f \circ (1) g} \mathcal{P} \circ (1) \mathcal{P} \xrightarrow{\gamma(1)} \mathcal{P} \right).
\]

The differential for \( f \in \text{hom}_{S\text{-Mod}}(C, \mathcal{P}) \) is defined as usual by \( \partial(f) = d^\mathcal{P} \circ f - (-1)^{|f|} f \circ d^C \). We consider the subset of degree \(-1\) elements satisfying the Maurer–Cartan equation,

\[
(3) \quad \text{Tw}(C, \mathcal{P}) := \{ \alpha \in \text{hom}_{S\text{-Mod}}(C, \mathcal{P})_{-1} | \partial \alpha + \alpha \star \alpha = 0 \}.
\]

These elements are called twisting morphisms. A simple computation shows that the twisting morphism bifunctor is represented by the cobar and bar functors in the following sense:

\[
\xymatrix{ \text{Hom}_{\text{aug Op}}(C, \mathcal{P}) \ar[r]^-{\sim} & \text{Tw}(C, \mathcal{P}) \ar[r]^-{\sim} & \text{Hom}_{S\text{-Conil Coop}}(C, B \mathcal{P}) }.
\]

1.6.1. Koszul twisting morphisms. The differential of the composite product of \( S \)-module \( C \circ \mathcal{P} \) is defined as \( d^C \circ_{\mathcal{P}} \mathcal{P} = d^C \circ 1 + 1 \circ d^\mathcal{P} \). Given a twisting morphism \( \alpha : C \rightarrow \mathcal{P} \), we may use it to modify this differential by adding an additional term,

\[
(5) \quad d_\alpha = \left( C \circ \mathcal{P} \xrightarrow{\Delta(1) \circ (1)} (C \circ (1) C) \circ \mathcal{P} \xrightarrow{(1 \circ (1) \alpha) \circ 1} (C \circ (1) \mathcal{P}) \circ \mathcal{P} \cong C \circ (\mathcal{P} \circ \mathcal{P}) \xrightarrow{\text{Id}(1) \gamma} C \circ (\mathcal{P} \circ \mathcal{P}) \right).
\]
We denote by $C \circ_\alpha P$ the composite product $C \circ P$ with the twisted differential $d_{C \circ_\alpha P} = d_{C \circ P} + d_\alpha$ and call it the twisted composite product.

We call a twisting morphism $\alpha$ Koszul if the twisted composite product $C \circ_\alpha P$ is acyclic. The set of Koszul twisting morphisms is denoted by $\text{Kos}(C, P)$.

**Proposition 1.2.** Assume that $(C(n)$ and $P(n)$ are projective $k$-modules for all $n$. Under the isomorphisms of equation (2), the Koszul twisting morphisms correspond to the quasi-isomorphisms,

$$\text{qIso}_{\text{dg aug Op}}(\Omega C, P) \cong \text{Kos}(C, P) \cong \text{qIso}_{\text{conil Coop}}(C, B P).$$

**Proof.** See [21, Theorem 2.1.15] or [21, Theorem 6.6.1]. □

### 1.7. Koszul duality

In Section 1.5 we have introduced the (functorial) cobar-bar resolution for any operad $P$. In this section, we consider a type of operads for which a much smaller resolution can be produced.

Assume that we have a quadratic presentation $(E, R)$, i.e. an $S$-module of generators $E$, and a sub $S$-module $R \subset T(E)^{(2)}$ of relations. We denote by $(R)$ the operadic ideal generated by $R$, i.e. the smallest sub $S$-module $R \subset (R) \subset T(E)$ s.t. $T(E)/(R)$ with the induced structure is an operad. This operad is denoted $P = P(E, R)$. The same quadratic presentation also determines a cooperad $C(E, R)$; it is defined as the largest subcooperad of $T^c(E)$ for which the composition

$$C(E, R) \longrightarrow T^c(E) \longrightarrow T^c(E)/R$$

vanishes, i.e. any other such subcooperad inclusion $C \hookrightarrow T^c(E)$ factors through $C(E, R)$.

We define the Koszul dual cooperad of $P(E, R)$ to be $P^! := C(sE, s^2R)$. It comes with a canonical twisting morphism

$$\kappa = \left( P^! = C(sE, s^2R) \longrightarrow T^c(sE) \longrightarrow sE \overset{s^{-1}}{\longrightarrow} E \longrightarrow P(E, R) \right).$$

An operad $P(E, R)$ is called Koszul if the twisted composite $P^! \circ_\kappa P$ is acyclic or, equivalently, $g_\kappa : \Omega P^! \to P$ is a quasi-isomorphism. The classical operads $\text{Assoc}$ of associative algebras, $\text{Com}$ of commutative associative algebras, and $\text{Lie}$ of Lie algebras are examples of Koszul operads.

Given a finite-dimensional presentation $(E, R)$, we can also produce a dual presentation. We restrict ourselves to the case of a binary quadratic presentation, i.e. when $E = E(2)$ is concentrated in arity 2. The general case is essentially the same but requires some degree shifts. Consider the $S$-module $E^c := E \otimes k \cdot \text{sgn}_2$ with the orthogonal space of relations $R^c \subset T(E^c)^{(2)}$. This gives a new binary quadratic presentation $(E^c, R^c)$ and an associated operad $P^c = P(E^c, R^c)$ which we call the Koszul dual operad. One can show that the Koszul dual operad and cooperad are related by the equation $P^c = (S \otimes P)^*$. For the classical operads we have $\text{Ass}^c = \text{Ass}$, $\text{Lie}^c = \text{Com}$, and $\text{Com}^c = \text{Lie}$. Our interest lies in particular in the case of the Lie operad. Here we obtain $\text{Lie}^c = (S \otimes \text{Lie})^* = (S \otimes \text{Com})^* = S^*$ and therefore $\text{Lie}^c(0) = 0$ and $\text{Lie}^c(n) = (I_n \cdot k \cdot \text{sgn}_n)/[n-1]$ for $n \geq 1$ as an $S$-module. We will make its cooperad structure explicit in Section 1.7.1.

### 1.8. Homotopy algebras and morphisms

Given any cooperad $C$, we can consider the category of homotopy algebras over $\Omega C$. The objects in this category are the usual $\Omega C$-algebras, however we introduce a different notion of morphisms. Recall that algebras over any operad $P$ come with a natural notion of morphism. Such a morphism of $P$-algebras is a map of the underlying chain complexes commuting with all structure maps, see Section 1.4.1.

In the context of homotopy algebras, this notion of morphism is too strict. A better behaved type of morphism is introduced below.

#### 1.8.1. Weak morphisms of homotopy algebras

Consider two $\Omega C$-algebras $V$ and $V'$, i.e. two chain complexes $(V, d)$, $(V', d')$ equipped with structure maps given by twisting morphisms

$$\alpha \in \text{Tw}(C, \text{End}_V), \quad \alpha' \in \text{Tw}(C, \text{End}_{V'}).$$

We apply the general theory as described in [21, §10.2.4]. Even though we are not working with Koszul operads, the relevant proofs still hold for general cooperads $C$. In particular, a weak morphism as defined below corresponds to a morphism of quasi-cofree codifferential $C$-coalgebras $C(V) \to C(V')$. 

\[ \]
A homotopy morphism or weak morphism of $\Omega C$-algebras is a degree 0 solution to the following Maurer–Cartan equation:
\begin{equation}
 f : C \to \text{End}_{V'}, \\
 \partial(f) - f * \alpha + \alpha' \circ f = 0,
\end{equation}
where
\begin{align}
 f * \alpha &= \left( C \xrightarrow{\Delta} C \otimes (C \xrightarrow{\varnothing}) \text{End}_{V'} \otimes (\text{End}_{V'} \to \text{End}_{V'}) \right), \\
 \alpha' \circ f &= \left( C \xrightarrow{\Delta} C \circ C \xrightarrow{\varnothing' \circ f} \text{End}_{V'} \circ \text{End}_{V'} \to \text{End}_{V''} \right).
\end{align}

1.8.2. Composition of weak morphisms. Given weak morphisms $V \xrightarrow{\ell_1} V' \xrightarrow{\ell_2} V''$, their composition $f' \circ f$ is defined to be
\begin{equation}
 f' \circ f = \left( C \xrightarrow{\Delta} C \circ C \xrightarrow{f' \circ f} \text{End}_{V''} \circ \text{End}_{V'} \to \text{End}_{V''} \right).
\end{equation}
Note the double use of the notation $f' \circ f$; the meaning should be clear from the context.

It is simple to verify that $f' \circ f$ satisfies equation (6), i.e. $f' \circ f$ is indeed a weak morphism. This definition for the composition is equivalent to the composition of the corresponding quasi-cofree codifferential $C$-coalgebras $C(V) \to C(V') \to C(V'')$.

## 2. An $S$-free resolution of the Koszul dual cooperad of the Lie operad

In this section, we attempt to construct an $S$-free resolution $\psi : \text{Lie}^0 \xrightarrow{\sim} \text{Lie}^i$ of dg cooperads over $\mathbb{Z}$ for the Koszul dual operad of the Lie operad. Assume for a moment that we have such a resolution. If we can show that $\psi$ is in fact a weak equivalence, i.e. $\Omega \psi$ is a quasi-isomorphism, then this gives us a small cofibrant resolution $\text{Lie}_{\infty} := \Omega \otimes \text{Lie}^0 \xrightarrow{\sim} \text{Lie}$ of the Lie operad over $\mathbb{Z}$, and therefore over any unital commutative ring $k$. Unfortunately, we do not yet have a general method to obtain such an $S$-free resolution. Instead, we proceed degree-wise to construct a dg cooperad $\text{Lie}_3^0$ that satisfies the desired conditions in low degrees (see the following sections for the precise meaning of “in low degrees.”)

**Remark 2.1.** While our notation below may suggest that we are working with truncations of a dg cooperad $\text{Lie}_3^0$ and $\text{Leib}_1^0$ of the Lie and Leibniz operads, respectively. In Section 2.2 we construct an explicit $S$-free resolution $\psi : \text{Lie}_3^0 \to \text{Lie}^i$ as a dg $S$-module in low degrees. In Section 2.4 we equip $\text{Lie}_3^0$ with a decomposition map and show that our definition turns it into a dg cooperad. Finally, in Section 2.4. we prove that the homology of the twisted composite product $\text{Lie}_3^0 \otimes (\text{Lie}^0)$ Lie vanishes in low degrees.

### 2.1. The Koszul dual cooperads of the Lie and Leibniz operads

In Section 1.7 we saw that the Koszul dual operad for Lie is $\text{Lie}^i = S^*$. Explicitly, this means the $S$-module underlying $\text{Lie}^i$ is given by $\text{Lie}^i(0) = 0$ and $\text{Lie}^i(n) = \{ l_n, k \cdot \text{sgn}_n \}[n - 1]$ for $n \geq 1$, and its decomposition map is given by
\begin{equation}
 \Delta(\bar{l}_n) = \sum_{i_1 + \cdots + i_j = n} (-1)^{(j - 1)(n - i)} \cdot (-1)^{\sum_{p=1}^{j-1}(i_{p-1} - 1)} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_j)} (-1)^{|\sigma|} \cdot \bar{l}_{\sigma} \circ (\bar{l}_{i_1}, \ldots, \bar{l}_{i_j}).
\end{equation}
From this we obtain, by projection to $\text{Lie}^i \otimes (\text{Lie}^0)$, the partial decomposition map
\begin{equation}
 \Delta(1)(\bar{l}_n) = \sum_{i+j = n+1} (-1)^{(j - 1)(i - 1)} \sum_{p=1}^j (-1)^{(p-1)(i-1)} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_j)} (-1)^{|\sigma|} \cdot (\bar{l}_j \circ (\bar{l}_{i_1}, \ldots, \bar{l}_{i_j}))^\sigma.
\end{equation}
Using the skew-symmetry of the $\tilde{l}_n$, it can be rewritten as
\begin{equation}
\Delta((l_n) = \sum_{i+j=n+1} (-1)^{(i-1)(j-1)} \sum_{\sigma \in \text{Sh}_{i,j}(i,j-1)} (-1)^{\vert \sigma \vert} \cdot (\tilde{l}_j \otimes \tilde{l}_i)^{\sigma}.
\end{equation}

The Koszul dual operad $\text{Leib}^! = \text{Zinb}$ for Leib was first introduced by Loday in [20]. Algebras over $\text{Leib}^!$ were originally referred to as dual Leibniz algebras, but are now more commonly known as Zinbiel algebras. As before, the Koszul dual operad $\text{Leib}^!$ can be computed as $\text{Leib}^! = (S \otimes \text{Zinb})^*$. Explicitly, it is given by $\text{Leib}^!(0) = 0$ and $\text{Leib}^!(n) = (\tilde{l}_n \cdot \mathbf{k}[S_n])[n-1]$ for $n \geq 1$, and with (partial) decomposition map given as in equations (9) and (10), substituting $l_n$ for $\tilde{l}_n$. Note however, that equations (10) and (11) are not equivalent in this case, since the $l_n$ are not skew-symmetric.

2.1.1. Homotopy Lie algebras are homotopy Leibniz algebras. There is an obvious morphism of dg cooperads,
\begin{equation}
\psi: \text{Leib}^! \to \text{Lie}^!,
\end{equation}
\begin{equation}
\psi(l_n^! = (-1)^{\sigma} \cdot \tilde{l}_n.
\end{equation}
Since $\text{Leib}^!$ is $\mathbb{S}$-free and this morphism is surjective, this provides us with a good starting point for our resolution of $\text{Lie}^!$.

2.2. The resolution as an $\mathbb{S}$-free dg $\mathbb{S}$-module. In this section, we fix an $\mathbb{S}$-free resolution of $\text{Lie}^!$ as an $\mathbb{S}$-module in low degrees. We describe, in general, a way to obtain an $\mathbb{S}$-free $\mathbb{S}$-module $\text{Lie}^!_{\mathbb{S}}$ with a morphism $\psi: \text{Lie}^!_{\mathbb{S}} \to \text{Lie}^!$ satisfying $H_r(\psi) = 0$ for $r \leq k$. We make such an $\mathbb{S}$-module explicit for $k = 3$.

We construct, for each arity $n \geq 1$, an exact augmented complex $0 \leftarrow \text{Lie}^!(n)_{n-1} \leftarrow \text{Lie}^!_{\mathbb{S}}(n)_{\mathbb{S}} \to \mathbf{k}[S_n]-modules up to degree $k + 1$. As indicated in Section 2.1.1, we may choose $\text{Lie}^!_{\mathbb{S}}(n)_{n-1} := \text{Leib}^!(n)_{n-1}$ and the augmentation map to be $\psi$ as defined by equation (12), i.e. our complexes are of the following shape:
\[ 0 \leftarrow \text{Lie}^!(n)_{n-1} \leftarrow \text{Lie}^!_{\mathbb{S}}(n)_{\mathbb{S}} \leftarrow d_n \text{Lie}^!(n) \leftarrow d_{n+1} \text{Lie}^!(n)_{n+1}. \]

Our general approach to determining the higher degrees of $\text{Lie}^!_{\mathbb{S}}(n)$ is as follows. For $r = n-1, \ldots, k$, successively, we extend the complex using the following steps:

(i) compute $\ker(d_r)$,

(ii) choose generators $\{x_i\}_{i \in I}$ for $\ker(d_r)$ as a $\mathbf{k}[S_n]$-module, and

(iii) define $\text{Lie}^!_{\mathbb{S}}(n)_{r+1} := \langle \hat{x}_i \rangle_{i \in I}$ to be the free $\mathbf{k}[S_n]$-module generated by symbols $\{\hat{x}_i\}_{i \in I}$ and the differential by $d_{r+1}(\hat{x}_i) := x_i$.

Obviously, any complex $\text{Lie}^!_{\mathbb{S}}(n)$ constructed in this way will be exact. We describe the beginning of these computations explicitly for general $n$.

Degree $r = n-1$: Since
\[ d_{n-1}(l_n - (-1)^{\sigma} l_n^!) = \psi(l_n - (-1)^{\sigma} l_n^!) \tilde{l}_n - (-1)^{\sigma} l_n^! = 0 \]
and $\dim_k (\text{im}(d_{n-1})) = 1$, we obtain $\ker(d_{n-1}) = \text{span}_k \{l_n - (-1)^{\sigma} l_n^! \mid \sigma \in S_n \setminus \text{id}\}$. As a $\mathbf{k}[S_n]$-module, the kernel is generated by the set $\{l_n + l_n^! \}_{i=1}^{n-1}$ for the adjacent transpositions $\sigma_i = (i \ i+1)$. We define
\[ \text{Lie}^!_{\mathbb{S}}(n) := \langle l_n \rangle_{1 \leq i \leq n}, \]
\[ d_n(l_n) = -l_n^! \].

Degree $r = n$: Clearly, the following hold
\[ d_n(l_{n;i}^! - l_{n;i}^! \sigma_i) = (l_{n;i} - l_{n;i}^! \sigma_i) = 0, \]
\[ d_n(l_{n;i} - l_{n;i}^! \sigma_i + l_{n;i+1}^! \sigma_i - l_{n;i+1}^! + l_{n;i+1}^! \sigma_i) = (l_{n;i} - l_{n;i}^!) \sigma_i + l_{n;i+1}^! \sigma_i, \]
\[ d_n(l_{n;i} - l_{n;i}^! \sigma_i) = (l_{n;i} - l_{n;i}^!) \sigma_i - (l_{n;i} - l_{n;i}^!) \sigma_i = 0. \]

In fact, these elements generate the kernel of $d_n$ under the $\mathbf{k}[S_n]$-action. We omit the general proof here, since we will only need this result in arity $3$ where it is a trivial computation of the rank of a $12 \times 18$ matrix over $\mathbf{k}$.
We define
\[
\text{Lie}_k^3(n)_{n+1} := \langle l_{n;i,j} \mid 1 \leq i \leq j < n \rangle,
\]
and
\[
d_{n+1}(l_{n;i,j}) = \begin{cases} -l_{n;i} + l_{n;i}, & j = i, \\ l_{n;i} - l_{n;i+1} + l_{n;i+1}, & j = i + 1, \\ l_{n;i} - l_{n;i+1} + l_{n;i+1}, & j > i + 1. \end{cases}
\]

The above computations are already enough for our purpose. In summary, we obtain the following explicit dg \(S\)-module \(\text{Lie}_k^3\).
\[
\begin{array}{c}
\langle l_2 \rangle \\
\langle l_2, l_1 \rangle \\
\langle l_3 \rangle \\
\langle l_2, l_1, l_1 \rangle \\
\langle l_3, l_1 \rangle \\
\langle l_4 \rangle \\
\langle l_3, l_2 \rangle \\
\langle l_4, l_2 \rangle \\
\langle l_5 \rangle
\end{array}
\]

Note that we left out arity 1 so far. We will need to define \(\text{Lie}_k^3(1) := \text{Lie}_A^3(1) = k\) for the counit of the cooperad structure introduced in the following section.

2.3. The resolution as a dg cooperad. In this section, we describe how to equip a dg \(S\)-module \(\text{Lie}_k^3\), as obtained in the previous section, with a decomposition map \(\Delta\) in such a way that

(i) \((\text{Lie}_k^3, d, \Delta)\) becomes a dg cooperad, and

(ii) \(\psi: \text{Lie}_k^3 \rightarrow \text{Lie}^d\) becomes a morphism of dg cooperads.

As before, we begin by explaining the general approach and then proceed to make such a structure explicit for the case \(k = 3\), i.e. on the dg \(S\)-module \(\text{Lie}_k^3\) of the previous section.

Consider condition \(\psi\) first. Since we started our resolution of dg \(S\)-modules by \(\psi: \text{Lie}_k^3(n)_{n-1} = \text{Lie}^d(n) \rightarrow \text{Lie}^d(n)\), and this is in fact a morphism of dg cooperads, we define the decomposition map \(\Delta\) on \(\text{Lie}_k^3(n)_{n-1}\) in the same way as on \(\text{Lie}^d(n)\), i.e. by equation \((9)\). It remains to define the decomposition map for the higher degree terms of \(\text{Lie}_k^3(n)\).

Next, consider condition \((i)\). Note that it implies, in particular, that the decomposition map be a map of dg \(S\)-modules, i.e. \(\Delta\) has to commute with the differential as in the diagram
\[
\begin{array}{c}
\text{Lie}_k^3(n)_{r+1} \xrightarrow{\text{d}} (\text{Lie}_k^3 \circ \text{Lie}_k^3)(n)_{r+1} \\
\text{Lie}_k^3(n)_r \xrightarrow{\Delta} (\text{Lie}_k^3 \circ \text{Lie}_k^3)(n)_r.
\end{array}
\]

We use this condition to define \(\Delta\) as follows: In each arity \(n\), proceed degree-wise for \(r = n - 1, \ldots, k + 1\). Given a \(k[S_n]\)-basis \(\{x_i\}_{i \in I}\) for \(\text{Lie}_k^3(n)_{r+1}\), solve the equations \(d(y_i) = \Delta(dx_i)\) for \(y_i\) and define \(\Delta\) by \(\Delta(x_i) := y_i\) and \(k[S_n]\)-linearity. Finally, it remains to check that our decomposition map \(\Delta\) satisfies the coassociativity condition.

The remainder of this section consists of the explicit computations for the case of \(\text{Lie}_k^3\). As mentioned in Section \((11)\) we work over the integers \(\mathbb{Z}\). Finding a decomposition map as described above amounts to solving systems of linear diophantine equations. For \(\text{Lie}_k^3\) these are still manageable by hand; for \(\text{Lie}_k^3\) for \(k \geq 4\), a computer can be used to solve them. To save a bit of space, we work with the reduced decomposition map \(\overline{\Delta}\) instead of the full decomposition \(\Delta\) below.

Arity \(n = 2\): The (reduced) decomposition \(\overline{\Delta}\) vanishes for degree reasons, i.e. we have
\[
\overline{\Delta}(l_2) := \overline{\Delta}(l_{2,1}) := \overline{\Delta}(l_{2,1,1}) := 0.
\]
Arity $n = 3$: For $l_3$, we have defined $\overline{\Delta}$ by equation (9), i.e.

$$\overline{\Delta}(l_3) = -l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)},$$

as in Leibniz (3). To extend the decomposition map $\overline{\Delta}$ to the next degree, we compute for $l_{3,1}$,

$$\overline{\Delta}(dl_{3,1}) = \overline{\Delta}(-l_3 + l_{3}^{(12)}) = -\overline{\Delta}(l_3) - \overline{\Delta}(l_3)^{(12)}$$

$$= - \left( -l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)} \right) - \left( -l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)} \right)^{(12)}$$

$$= l_2 \circ (l_2, 1) + l_2 \circ (1, l_2)^{(12)} - l_2 \circ (1, l_2)^{(12)^{(12)}}$$

then solve for a preimage under $d$, 

$$= -l_2 \circ (dl_{2,1}, 1) = d(l_2 \circ (l_{2,1}, 1)).$$

We proceed for $l_{3,2}$ in the same way,

$$\overline{\Delta}(dl_{3,2}) = \overline{\Delta}(-l_3 + l_{3}^{(23)}) = -\overline{\Delta}(l_3) - \overline{\Delta}(l_3)^{(23)}$$

$$= - \left( -l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)} \right) - \left( -l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)} \right)^{(23)}$$

$$= l_2 \circ (l_2, 1) - l_2 \circ (1, l_2) + l_2 \circ (1, l_2)^{(12)} + l_2 \circ (1, l_2)^{(12)^{(12)}} + l_2 \circ (1, l_2)^{(12)^{(23)}} - l_2 \circ (1, l_2)^{(12)^{(12)^{(12)}}}$$

$$(132)$$

$$= l_2 \circ (l_2, 1) - l_2 \circ (1, l_2) + l_2 \circ (1, l_2)^{(12)} + l_2 \circ (1, l_2)^{(12)^{(12)}} + l_2 \circ (1, l_2)^{(12)^{(23)}} - l_2 \circ (1, l_2)^{(12)^{(12)^{(12)}}}$$

$$(12)$$

$$= -l_2 \circ (dl_{2,1}, 1) - (dl_{2,1}) \circ (1, l_2)^{(12)} - (dl_{2,1}) \circ (1, l_2)^{(12)^{(12)}} - (dl_{2,1}) \circ (1, l_2)^{(12)^{(23)}} - (dl_{2,1}) \circ (1, l_2)^{(12)^{(12)^{(12)}}}$$

$$(123)$$

$$= l_2 \circ (l_2, 1) - l_2 \circ (1, l_2) + l_2 \circ (1, l_2)^{(12)} + l_2 \circ (1, l_2)^{(12)^{(12)}} + l_2 \circ (1, l_2)^{(12)^{(23)}} - l_2 \circ (1, l_2)^{(12)^{(12)^{(12)}}}$$

$$(132)$$

$$= d(-l_2 \circ (l_{2,1}, 1) - l_2 \circ (1, l_2) + l_2 \circ (1, l_2)^{(12)} - l_2 \circ (1, l_2)^{(12)^{(12)}} - l_2 \circ (1, l_2)^{(12)^{(23)}} - l_2 \circ (1, l_2)^{(12)^{(12)^{(12)}}})$$

$$(132)$$

This gives us candidates for the definition of $\overline{\Delta}$ for which the diagram [14] commutes. We define

$$\overline{\Delta}(l_{3,1}) := l_2 \circ (l_{2,1}, 1),$$

$$\overline{\Delta}(l_{3,2}) := -l_2 \circ (l_{2,1}, 1) - l_2 \circ (1, l_{2,1}) - l_2 \circ (1, l_2)^{(12)}.$$

Using these definitions, we continue in the next degree and find

$$\overline{\Delta}(dl_{3,1,1}) = \overline{\Delta}(-l_3 + l_{3,1}^{(12)}) = -\overline{\Delta}(l_{3,1}) + \overline{\Delta}(l_{3,1})^{(12)}$$

$$= l_2 \circ (l_2, 1) + l_2 \circ (l_{2,1}, 1)^{(12)} - l_2 \circ (l_2, 1) - l_2 \circ (l_{2,1}, 1)^{(12)}$$

$$= l_2 \circ (dl_{2,1,1}, 1) = d(-l_2 \circ (l_{2,1,1}, 1)).$$

$$\overline{\Delta}(dl_{3,1,2}) = \overline{\Delta}(l_{3,1} + l_{3,1}^{(23)}) + l_{3,1}^{(132)} - l_{3,2} - l_{3,2}^{(12)} - l_{3,2}^{(12)^{(12)}}$$

$$= l_2 \circ (l_2, 1) + l_2 \circ (l_{2,1}, 1)^{(23)} + l_2 \circ (l_{2,1}, 1)^{(132)} - l_2 \circ (l_{2,1}, 1) - l_2 \circ (l_{2,1}, 1)^{(12)}$$

$$= -l_2 \circ (dl_{2,1,2}) - (dl_{2,1,2}) \circ (1, l_{2,1})^{(12)} - (dl_{2,1,2}) \circ (1, l_{2,1})^{(12)^{(12)}} - (dl_{2,1,2}) \circ (1, l_{2,1})^{(12)^{(23)}} - (dl_{2,1,2}) \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}}$$

$$= d(-l_2 \circ (l_{2,1,2}, 1) - l_2 \circ (1, l_{2,1}) - l_2 \circ (1, l_{2,1})^{(12)} - l_2 \circ (1, l_{2,1})^{(12)^{(12)}} - l_2 \circ (1, l_{2,1})^{(12)^{(23)}} - l_2 \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}})$$

$$\overline{\Delta}(dl_{3,2,1}) = \overline{\Delta}(-l_3 + l_{3,2}^{(12)}) = -\overline{\Delta}(l_{3,2}) + \overline{\Delta}(l_{3,2})^{(23)}$$

$$= -l_2 \circ (l_2, 1) - l_2 \circ (l_{2,1}, 1) - l_2 \circ (1, l_{2,1}) - l_2 \circ (1, l_2)^{(12)}$$

$$= l_2 \circ (dl_{2,1,1} + dl_{2,1,1}) \circ (1, l_{2,1})^{(12)}$$

$$= -l_2 \circ (dl_{2,1,1} + dl_{2,1,1}) \circ (1, l_{2,1})^{(12)^{(12)}} - l_2 \circ (1, l_{2,1})^{(12)^{(23)}} - l_2 \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}}$$

$$= l_2 \circ (l_2, 1) + l_2 \circ (1, l_{2,1}) + l_2 \circ (1, l_{2,1})^{(12)} - l_2 \circ (1, l_{2,1})^{(12)^{(12)}} - l_2 \circ (1, l_{2,1})^{(12)^{(23)}} - l_2 \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}}$$

$$= d(-l_2 \circ (dl_{2,1,1} + dl_{2,1,1}) - l_2 \circ (1, l_{2,1,1}) - l_2 \circ (1, l_{2,1})^{(12)} - l_2 \circ (1, l_{2,1})^{(12)^{(12)}} - l_2 \circ (1, l_{2,1})^{(12)^{(23)}} - l_2 \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}})$$

$$= d(-l_2 \circ (dl_{2,1,1} + dl_{2,1,1}) - l_2 \circ (1, l_{2,1,1}) - l_2 \circ (1, l_{2,1})^{(12)} - l_2 \circ (1, l_{2,1})^{(12)^{(12)}} - l_2 \circ (1, l_{2,1})^{(12)^{(23)}} - l_2 \circ (1, l_{2,1})^{(12)^{(12)^{(12)}}}).$$
This again gives us candidates for the definition of $\Delta$ and we define

\begin{align}
\Delta(l_{3,1,1}) &= -l_2 \circ (l_{2,1,1}, 1), \\
\Delta(l_{3,1,2}) &= -l_2 \circ l_{2,1} \circ (1, l_{2,1}) - l_2 \circ l_2 \circ (1, l_{2,1})^{(12)} + l_{2,1} \circ (1, l_{2,1})^{(12)}, \\
\Delta(l_{3,2,2}) &= -l_2 \circ l_{2,1} \circ (1, l_{2,1}) + l_2 \circ (1, l_{2,1,1}) - l_{2,1} \circ (1, l_{2,1})^{(12)}. 
\end{align}

**Arity $n = 4$:** For $l_4$ we have

$$\Delta(l_4) = l_2 \circ (l_3, 1) + l_2 \circ (l_1, l_3) - l_2 \circ (1, l_3)^{(12)} + l_2 \circ (1, l_3)^{(123)} - l_2 \circ (l_2, l_2) + l_2 \circ (l_2, l_2)^{(23)}$$

and analogous to the arity 3 computations, we obtain (indicating by the ellipses $\cdots$ terms that cancel on the nose)

$$\Delta(dl_{4,1}) = \Delta(-l_4 - l_4^{(12)}) = -\Delta(l_4) - \Delta(l_4)^{(12)}$$

$$\Delta(dl_{4,2}) = \Delta(-l_4 - l_4^{(23)}) = -\Delta(l_4) - \Delta(l_4)^{(23)}$$

$$\Delta(dl_{4,3}) = \Delta(-l_4 - l_4^{(34)}) = -\Delta(l_4) - \Delta(l_4)^{(34)}$$
As before, this gives us candidates for the definition of $\bar{\Sigma}$. We define

$$\bar{\Sigma}(l_{(4,1)}):= -l_2 \circ (l_{(3,1),1}) - l_2 \circ (1, l_{(3,1)}) + l_2 \circ (l_{(2,1),1}) + l_3 \circ (l_{(2,1),1}) + l_3 \circ (1, 1, l_2)$$

$$\bar{\Sigma}(l_{(4,2)}):= -l_2 \circ (l_{(3,2),1}) - l_2 \circ (1, l_{(3,1)}) + l_2 \circ (l_{(2,1),2}) + l_3 \circ (l_{(2,1),1}) + l_3 \circ (1, 1, l_2)$$

$$\bar{\Sigma}(l_{(4,3)}):= l_{2,1} \circ (l_{(1,1),1}) - l_2 \circ (1, l_{(3,2),1}) - l_2 \circ (1, l_{(3,2),1}) + l_2 \circ (l_{(2,2),1}) + l_2 \circ (l_{(2,2),2}) + l_2 \circ (1, 1, l_2)$$

$$+ l_3 \circ (1, 1, l_2) - l_2 \circ (1, l_{(2,2),1}) + l_3 \circ (1, 1, l_2)$$

$$+ l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2)$$

$$\bar{\Sigma}(l_{(5)}):= -l_2 \circ (l_{(4,1)} + l_2 \circ (l_{(3,1),1}) - l_2 \circ (1, l_{(3,1)}) + l_2 \circ (l_{(2,1),1}) + l_3 \circ (l_{(2,1),1}) + l_3 \circ (1, 1, l_2) + l_2 \circ (l_{(2,1),1}) + l_2 \circ (1, 1, l_2)$$

$$+ l_3 \circ (1, 1, l_2) - l_2 \circ (1, l_{(2,2),1}) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2)$$

$$- l_3 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2) + l_3 \circ (1, 1, l_2)$$

$$+ l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2) + l_4 \circ (1, 1, l_2)$$

Since there are no higher degree terms in $\text{Lie}_5^\circ(5)$, we are done here.

**Lemma 2.2.** The triple $(\text{Lie}_5^\circ, d, \Delta)$ consisting of the dg $\mathbb{S}$-module $(\text{Lie}_5^\circ, d)$ defined in equation (13) and the decomposition map as defined by equations (15) and (20) is a dg cooperad.

The decomposition structure map we defined is compatible with the differential by its construction. What is left to do is to show that it defines a dg cooperad structure on the dg $\mathbb{S}$-module $\text{Lie}_5^\circ$ defined in the previous section, is to do is that it satisfies the coassociativity condition $(\Delta \circ \text{id}) \Delta = (\text{id} \circ \Delta) \Delta$. Note that coassociativity is automatic in arities $n \leq 3$ and we already know that it holds for $l_n \in \text{Lie}_n^\circ(n)_{n-1} = \text{Leib}^l(n)$. Thus, it is sufficient to check it for the elements $l_{4,1}$, $l_{4,2}$, and $l_{4,3}$. See Appendix $\bar{\Sigma}$ for this long and tedious computation.

2.4. The twisted composite product. As explained in the introduction to this section, if we could show that for an $\mathbb{S}$-free resolution $\psi$: $\text{Lie}^\circ \rightarrow \text{Lie}^\circ$ of dg cooperads the map $\psi$ is a quasi-isomorphism, this would give us a cofibrant resolution of the Lie operad over $\mathbb{Z}$. One way to do this would be to show, that the twisted composite product $\text{Lie}_n^\circ \circ_k \text{Leib}^l$ is acyclic for $k = k \circ \psi$. Since we do not have the full dg cooperad $\text{Lie}_n^\circ$, we will show the following truncated statement. This is of course a necessary condition for our dg cooperad $\text{Lie}_n^\circ$ to be a truncation of an $\mathbb{S}$-free resolution $\text{Lie}_0^\circ$ with the desired properties.

**Proposition 2.3.** The twisted composite product $\text{Lie}_n^\circ \circ_k \text{Leib}^l$ satisfies

$$H_r((\text{Lie}_n^\circ \circ_k \text{Leib}^l)(n)) = 0,$$

for all $r \leq 3$ in all arities $n$.

**Proof.** Since the result actually holds for any $\text{Lie}_n^\circ$ constructed as in the previous sections, we phrase the proof for arbitrary $k$ instead of just $k = 3$.

Consider the bigrading on the composite product $\text{Lie}_n^\circ \circ_k \text{Leib}^l$ given by

$$(\text{Lie}_n^\circ \circ \text{Leib}^l)(n)_{p,q} = \left(\text{Lie}_k^\circ(p + 1) \otimes_{\mathbb{S}_{p+1}} (\text{Lie}_n^\circ)^{(p+1)}(n)\right)_{p+q}.$$

The differential of this composite product, $d \text{Lie}_n^\circ \circ \text{Leib}^l = d \text{Lie}_n^\circ \circ 1$, is of bidegree $(0, -1)$. On the twisted composite product $\text{Lie}_n^\circ \circ_k \text{Leib}^l$, the differential has another term, $d_k$, defined as in equation (5). Note that $d_k$ is of bidegree
(−1, 0), since \( \tilde{\kappa} := \kappa \circ \psi \) vanishes everywhere except on \( \text{Lie}_k^\circ(2)_1 \). Thus, with the above bigrading, the twisted composite product becomes a first quadrant bicomplex. Below, we consider its spectral sequence \( E(n) \) for each arity \( n \).

Since \( \text{Lie}(0) = 0 \), the action of \( S_{p+1} \) on \( \text{Lie}^\otimes_{p+1}(n) \) is free and hence \( E(n)^0_{p,q} \) admits the following expansion,

\[
E(n)^0_{p,q} = (\text{Lie}_k^\circ \circ \text{Lie})(n)_{p,q} \\
= \left( \text{Lie}_k^\circ (p+1) \otimes_{S_{p+1}} (\text{Lie}^\otimes_{p+1})(n) \right)_{p+q} \\
= \left( \bigoplus_{n=k_1+\cdots+k_{p+1}} \text{Lie}_k^\circ (p+1) \otimes_{S_{p+1}} \text{Ind}_{\text{Sh}_{k_1}}^{\text{Sh}_{k_{p+1}}} \text{Lie}(k_1) \otimes \cdots \otimes \text{Lie}(k_{p+1}) \right)_{p+q} \\
= \left( \bigoplus_{n=k_1+\cdots+k_m} \text{Lie}_k^\circ (p+1) \otimes \text{Lie}(k_1) \otimes \cdots \otimes \text{Lie}(k_{p+1}) \otimes k[\text{Sh}(k_1, \ldots, k_{p+1})] \right)_{p+q}.
\]

Since \( \text{Lie} \) is \( k \)-projective, i.e. \( \text{Lie}(n) \) are projective \( k \)-modules for all arities \( n \), this implies that the first page \( E(n)^1_{p,q} \) is given by

\[
E(n)^1_{p,q} = H_{p+q} \left( \text{Lie}_k^\circ (p+1) \otimes_{S_{p+1}} (\text{Lie}^\otimes_{p+1})(n), d_{\text{Lie}_k^\circ} \right) \\
= H_{p+q} \left( \text{Lie}_k^\circ (p+1), d_{\text{Lie}_k^\circ} \right) \otimes_{S_{p+1}} (\text{Lie}^\otimes_{p+1})(n).
\]

By the construction of Section 2.2 we have for \( n \leq k + 1 \):

\[
H_r \left( \text{Lie}_k^\circ(n) \right) = \begin{cases} 
\text{Lie}^i(n)_r & \text{if } r \leq k, \\
\ker d_r & \text{if } r = k + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Hence, for \( p + q \leq k \) we obtain

\[
E(n)^1_{p,q} = \left( \text{Lie}^i(p+1) \otimes_{S_{p+1}} (\text{Lie}^\otimes_{p+1})(n) \right)_{p+q} = \begin{cases} 
(\text{Lie}^i \times \text{Lie})(n)^p & \text{if } q = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \psi \) is a morphism of dg cooperads and \( \tilde{\kappa} = \kappa \circ \psi \), it follows from the definition of the twisted differential in equation \( \Box \), that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Lie}_k^\circ \circ \text{Lie} & \xrightarrow{d^x} & \text{Lie}_k^\circ \circ \text{Lie} \\
\downarrow \psi \circ \text{id} & & \downarrow \psi \circ \text{id} \\
\text{Lie}^i \circ \text{Lie} & \xrightarrow{d_x} & \text{Lie}^i \circ \text{Lie}.
\end{array}
\]

Since \( \psi \) is a quasi-isomorphism, this shows that \( d^1_{1,0} = (d_{\kappa})_0 \) for \( p \leq k + 1 \). Thus we obtain \( E^2_{p,q} = 0 \) for \( p + q \leq k \), since the operad \( \text{Lie} \) is Koszul.

The proof for \( n > k + 1 \) is essentially the same, except for the computation of \( E(n)^2_{k,0} \). We have

\[
H_r \left( \text{Lie}_k^\circ(k+2) \right) = \text{Lie}_k^\circ(k+2)_r = \begin{cases} 
\text{Lie}_k^\circ(k+2)_{k+1} & \text{if } r = k + 1, \\
0 & \text{otherwise,}
\end{cases}
\]

and hence \( d^1_{k+1,0} \neq (d_{\kappa})_{k+1} \). However, we do find

\[
\text{im} \left( \left( E(n)^1_{k+1,0} \bigoplus_{\text{Sh}_{k}}^{\text{Sh}_{k+1}} \right) \xrightarrow{d_{k+1,0}} E(n)^1_{k,0} \right) = \text{im} \left( \left( \text{Lie}_k^\circ \circ \text{Lie}\right)(n)_{k+1} \xrightarrow{(\psi \circ \text{id}) \circ \text{id}} \left( \text{Lie}^i \circ \text{Lie}\right)(n)_{k} \right)
\]

\[
= \text{im} \left( \left( \text{Lie}^i \circ \text{Lie}\right)(n)_{k+1} \xrightarrow{d_x} \left( \text{Lie}^i \circ \text{Lie}\right)(n)_{k} \right),
\]

since \( \psi \) is surjective. Thus we obtain \( E^2_{p,q} = 0 \) for \( p + q \leq k \) again.

\[
\square
\]

3. The category of weak Lie 3-algebras

Assume again that we had an \( \Box \)-free resolution \( \psi : \text{Lie}^\circ \to \text{Lie}^i \) of dg cooperads, and in addition that \( \text{Lie}_k^\circ \circ (\kappa \circ \psi) \text{Lie} \) is acyclic and therefore \( E_{\infty} = \Omega \text{Lie}^\circ \xrightarrow{\sim} \text{Lie} \) is a cofibrant resolution. Now consider a 3-term complex \( (L, d) = \)
Let \( L_0 \xleftarrow{d} L_1 \xleftarrow{d} L_2 \) and note that its endomorphism operad \( \text{End}_L \) vanishes in degrees \( r > 2 \). Since the Maurer–Cartan equation for a twisting morphism \( \lambda : \text{Lie}^\circ \to \text{End}_L \) is of degree \(-2\), only degrees \( \leq 4 \) of \( \text{Lie}^\circ \) play a role in the definition of 3-term \( \text{EL}_\infty \)-algebras. A homotopy transfer theorem for such 3-term \( \text{EL}_\infty \)-algebras holds as a special case of the general HTT for algebras over a cofibrant operad. For our resolution \( \text{Lie}^\circ_3 \), however, we do not know that it is the truncation of such a dg cooperad \( \text{Lie}^\circ \). Nonetheless we show constructively that our weak Lie 3-algebras satisfy a homotopy transfer theorem (Proposition [3.4.])

This section is organized as follows. In Section 3.1.1 we recall the definition of a homotopy Leibniz algebra and of its homotopy morphisms. We spell out the details for the case of a homotopy Leibniz algebra on a 3-term complex or Leibniz 3-algebra, and for morphisms between such. In Section 3.2 we introduce the notion of a weak Lie 3-algebra as extra structure on a Leibniz 3-algebra. In Section 3.3 the accompanying notion of morphisms is made explicit. Finally, in Section 3.4 we prove a version of the homotopy transfer theorem for weak Lie 3-algebras.

### 3.1. Homotopy Leibniz algebras

From the description of the Koszul dual cooperad \( \text{Leib}^i \) in Section 2.1, we obtain the following explicit definitions of algebras and morphisms over \( \text{Leib}_\infty = \Omega \text{Leib}^i \) via the Maurer–Cartan equations (9) and (10). For a more thorough exposition, we refer the reader to [17].

**Definition 3.1.** A homotopy Leibniz algebra or \( \text{Leib}_\infty \)-algebra \( (L, d, \lambda) \) consists of a chain complex \( (L, d) \) with structure maps

\[
\lambda_n : L^\otimes n \to L[n - 2], \quad \forall n \geq 2,
\]

satisfying the following generalized Jacobi identities:

\[
\partial(\lambda_n) = - \sum_{i + j = n + 1} (-1)^{(j-1)n} \sum_{p=1}^j (-1)^{(p-1)(i-1)} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \cdot (\lambda_j \circ_p \lambda_i)^\sigma, \quad \forall n \geq 3.
\]

Let \( (L, d, \lambda), (L', d', \lambda') \) be \( \text{Leib}_\infty \)-algebras. A homotopy morphism or \( \text{Leib}_\infty \)-morphism \( f : L \to L' \) consists of maps

\[
f_n : L^\otimes n \to L'[n - 1], \quad \forall n \geq 1,
\]

satisfying the following equations:

\[
\partial(f_n) = \sum_{i + j = n + 1} (-1)^{(j-1)n} \sum_{p=1}^j (-1)^{(p-1)(i-1)} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \cdot (f_j \circ_p \lambda_i)^\sigma
\]

\[
- \sum_{1 \leq j \leq n} (-1)^{(j-1)(n-j)} \cdot (-1)^{\sum_{p=1}^j (p-1)(i_p-1)} \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \cdot \lambda_j^\sigma \circ (f_{i_1}, \ldots, f_{i_j})^\sigma.
\]

### 3.1.1. Leibniz 3-algebras

A Leibniz 3-algebra is just a homotopy Leibniz algebra on a 3-term complex. Since this is the foundation for our definition of weak Lie 3-algebra, we spell out the definition here.

**Definition 3.2.** A Leibniz 3-algebra \( (L, d, \lambda) \) consists of a 3-term chain complex \( (L, d) = L_0 \xleftarrow{d} L_1 \xleftarrow{d} L_2 \), equipped with structure maps

\[
\lambda_2 : L^\otimes 2 \to L, \quad \lambda_3 : L^\otimes 3 \to L[1], \quad \lambda_4 : L^\otimes 4 \to L[2],
\]

satisfying the following generalized Jacobi identities:

\[
\partial(\lambda_2) = 0, \quad (27)
\]

\[
\partial(\lambda_3) = \lambda_1 \circ_2 \lambda_2 - \lambda_3 \circ_1 \lambda_2 - (\lambda_2 \circ_2 \lambda_2)^{(12)}, \quad (28)
\]

\[
\partial(\lambda_4) = \lambda_2 \circ_1 \lambda_3 + \lambda_2 \circ_2 \lambda_3 - (\lambda_2 \circ_2 \lambda_3)^{(12)} + (\lambda_2 \circ_2 \lambda_3)^{(123)} - \lambda_3 \circ_1 \lambda_2
\]

\[
+ \lambda_3 \circ_2 \lambda_2 - (\lambda_3 \circ_2 \lambda_2)^{(12)} - \lambda_2 \circ_3 \lambda_2 + (\lambda_3 \circ_3 \lambda_2)^{(23)} - (\lambda_3 \circ_3 \lambda_2)^{(132)}, \quad (29)
\]

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0 = \lambda_2 \circ_1 \lambda_4 - \lambda_2 \circ_2 \lambda_4 + (\lambda_2 \circ_2 \lambda_4)^{(12)} - (\lambda_2 \circ_2 \lambda_4)^{(123)} + (\lambda_2 \circ_2 \lambda_4)^{(1234)} \\
+ \lambda_1 \circ_1 \lambda_3 + \lambda_1 \circ_2 \lambda_3 - (\lambda_3 \circ_2 \lambda_3)^{(12)} + (\lambda_3 \circ_2 \lambda_3)^{(123)} + \lambda_3 \circ_3 \lambda_3 \\
- (\lambda_3 \circ_3 \lambda_3)^{(23)} + (\lambda_3 \circ_3 \lambda_3)^{(132)} + (\lambda_3 \circ_3 \lambda_3)^{(234)} - (\lambda_3 \circ_3 \lambda_3)^{(1342)} + (\lambda_3 \circ_3 \lambda_3)^{(13)(24)} \\
+ \lambda_4 \circ_1 \lambda_2 - \lambda_4 \circ_2 \lambda_2 + (\lambda_4 \circ_2 \lambda_2)^{(12)} + \lambda_4 \circ_3 \lambda_2 - (\lambda_4 \circ_3 \lambda_2)^{(23)} \\
+ (\lambda_4 \circ_3 \lambda_2)^{(132)} - \lambda_4 \circ_4 \lambda_2 + (\lambda_4 \circ_4 \lambda_2)^{(34)} - (\lambda_4 \circ_4 \lambda_2)^{(234)} + (\lambda_4 \circ_4 \lambda_2)^{(1432)}.

3.1.2. **Morphisms of Leibniz 3-algebras.** A morphism of Leibniz 3-algebras is just a homotopy morphism between 3-term homotopy Leibniz algebras.

**Definition 3.3.** Let \((L, d, \lambda), (L', d', \lambda')\) be Leibniz 3-algebras. A weak morphism \(f: L \to L'\) consists of maps

\begin{align*}
f_1 &: L \to L', & f_2 &: L^\otimes 2 \to L'[1], & f_3 &: L^\otimes 3 \to L'[2],
\end{align*}

satisfying the following equations,

\begin{align*}
\partial(f_1) &= 0, \\
\partial(f_2) &= f_1 \circ_1 \lambda_2 - \lambda_2 \circ (f_1, f_1), \\
\partial(f_3) &= f_1 \circ_1 \lambda_3 - f_2 \circ_2 \lambda_3 + f_2 \circ_1 \lambda_2 + (f_2 \circ_2 \lambda_2)^{(12)} \\
&- \lambda_3 \circ (f_1, f_1, f_1) - \lambda_2 \circ (f_1, f_2) + \lambda_2 \circ (f_2, f_1) + \lambda_2 \circ (f_1, f_2)^{(12)}, \\
&= f_2 \circ_1 \lambda_3 + f_2 \circ_2 \lambda_3 - (f_2 \circ_2 \lambda_3)^{(12)} + (f_2 \circ_2 \lambda_3)^{(123)} - f_3 \circ_1 \lambda_2 \\
&+ f_3 \circ_2 \lambda_3 - (f_3 \circ_2 \lambda_3)^{(12)} - f_3 \circ_3 \lambda_3 + (f_3 \circ_3 \lambda_3)^{(23)} - (f_3 \circ_3 \lambda_3)^{(132)} \\
&+ \lambda_3 \circ (f_3, f_1) + \lambda_2 \circ (f_1, f_3) - \lambda_2 \circ (f_1, f_3)^{(12)} + \lambda_2 \circ (f_1, f_3)^{(123)} \\
&- \lambda_3 \circ (f_2, f_2) + \lambda_3 \circ (f_2, f_2)^{(23)} - \lambda_3 \circ (f_2, f_2)^{(132)} + \lambda_3 \circ (f_2, f_2)^{(1323)} \\
&- \lambda_3 \circ (f_1, f_1, f_2) + \lambda_3 \circ (f_1, f_2, f_1)^{(12)} + \lambda_3 \circ (f_1, f_2, f_1)^{(123)} \\
&- \lambda_3 \circ (f_1, f_1, f_2)^{(23)} + \lambda_3 \circ (f_1, f_1, f_2)^{(132)}.
\end{align*}

Let \(L \xrightarrow{f} L' \xrightarrow{f'} L''\) be weak morphisms of Leibniz 3-algebras. Their composition is defined by the following components:

\begin{align*}
(f' \circ f)_1 &= f'_1 \circ f_1, \\
(f' \circ f)_2 &= f'_2 \circ (f_1, f_1) + f'_1 \circ f_2, \\
(f' \circ f)_3 &= f'_3 \circ (f_1, f_1, f_1) - f'_2 \circ (f_2, f_1) + f'_1 \circ (f_1, f_2) - f'_2 \circ (f_1, f_2)^{(12)} + f'_1 \circ f_3.
\end{align*}

3.2. **Weak Lie 3-algebras.** In this section, we spell out the definition of weak Lie 3-algebras as solutions to the Maurer–Cartan equation \([3]\) on a 3-term complex. To do this, we evaluate the Maurer–Cartan equation for a twisting morphism \(\lambda: \text{Lie}_3^3 \to \text{End}_L\) on the \(k[S]_n\)-generators introduced in Section \([2.2]\) and use the shorthand notation \(\lambda_\ast\) for \(\lambda(l_\ast)\). E.g. for \(l_3\), we find

\begin{align*}
0 &= (\partial \lambda + \lambda * \lambda)(l_3) \\
&= \partial(\lambda(l_3)) + \lambda(d_3) + (\gamma(l_3) \circ (\lambda \circ_{(1)} \lambda) \circ \Delta_1)(l_3) \\
&= \partial(\lambda(l_3)) + \lambda(-l_3 - l_3^{(12)}) + (\gamma(l_3) \circ (\lambda \circ_{(1)} \lambda))(l_2 \circ l_2) \\
&= \partial(\lambda(l_3)) - \lambda_3 - \lambda_3^{(12)} - \lambda_2 \circ_1 \lambda_2, \\
\end{align*}

which is equivalent to equation \([11]\). Doing this for all generators of \(\text{Lie}_3^3\) and restricting to 3-term complexes leads to the following Definition.

**Definition 3.4.** A weak Lie 3-algebra \((L, d, \lambda)\) is a 3-term chain complex \((L, d) = L_0 \xleftarrow{d} L_1 \xleftarrow{d} L_2\), equipped with structure maps

\begin{align*}
\lambda_2 &: L^\otimes 2 \to L, & \lambda_2.lock(1) &= L[1], & \lambda_2.lock(2) &= L[2], \\
\lambda_3 &: L^\otimes 3 \to L[l], & \lambda_3.lock(1) &= L[1], & \lambda_3.lock(2) &= L[2], \\
\lambda_3.lock(3) &= L[2],
\end{align*}

and
We require these to satisfy the equations (27)–(30), i.e. \((L,d,\lambda)\) to be a Leibniz 3-algebra, and in addition we require the following equations to hold,

\[
\begin{align*}
\partial(\lambda_{2,1}) &= \lambda_2 + \lambda_{2,1}^{(12)}, \\
\partial(\lambda_{2,1,1}) &= \lambda_{2,1} - \lambda_{2,1}^{(12)}, \\
\partial(\lambda_{3,1}) &= \lambda_3 + \lambda_{3,1}^{(12)} + \lambda_2 \circ_1 \lambda_{2,1}, \\
\partial(\lambda_{3,2}) &= \lambda_3 + \lambda_{3,2}^{(23)} - \lambda_2 \circ_2 \lambda_{2,1} + \lambda_{2,1} \circ_1 \lambda_2 + (\lambda_{2,1} \circ_2 \lambda_2)^{(12)}, \\
\lambda_{2,1,1} + \lambda_{2,1,1}^{(12)} &= 0, \\
\lambda_{3,1} - \lambda_{3,1}^{(12)} &= \lambda_2 \circ_1 \lambda_{2,1,1}, \\
\lambda_{3,1} - \lambda_{3,1}^{(12)} + \lambda_{3,2}^{(123)} - \lambda_{3,2} + \lambda_{3,1}^{(23)} - \lambda_{3,2}^{(123)} \\
&= \lambda_{2,1} \circ_2 \lambda_{2,1} + \lambda_{2,1} \circ_1 \lambda_{2,1} + (\lambda_{2,1} \circ_2 \lambda_{2,1})^{(12)} + (\lambda_{2,1,1} \circ_2 \lambda_2)^{(132)}, \\
\lambda_{3,2} - \lambda_{3,2}^{(23)} &= -\lambda_2 \circ_2 \lambda_{2,1,1} + \lambda_{2,1,1} \circ_1 \lambda_2 + (\lambda_{2,1,1} \circ_2 \lambda_2)^{(12)}, \\
\lambda_4 + \lambda_4^{(12)} &= \lambda_2 \circ_1 \lambda_{3,1} + \lambda_{3,1} \circ_3 \lambda_2 + (\lambda_2 \circ_2 \lambda_{3,1})^{(123)} + \lambda_3 \circ_1 \lambda_{2,1}, \\
\lambda_4 + \lambda_4^{(23)} &= \lambda_2 \circ_1 \lambda_{3,2} - \lambda_2 \circ_2 \lambda_{2,1} + \lambda_2 \circ_2 \lambda_{3,1} - \lambda_{3,3} \circ_1 \lambda_2 - (\lambda_{3,3} \circ_2 \lambda_2)^{(12)} - (\lambda_{3,1} \circ_4 \lambda_2)^{(132)}, \\
\lambda_4 + \lambda_4^{(34)} &= \lambda_{2,1} \circ_1 \lambda_3 - \lambda_{2,2} \circ_1 \lambda_2 + \lambda_{2,2} \circ_2 \lambda_2 - (\lambda_{2,2} \circ_2 \lambda_2)^{(12)} + \lambda_3 \circ_3 \lambda_{2,1} + (\lambda_{3,3} \circ_2 \lambda_2)^{(23)}, \\
(\lambda_3 \circ_3 \lambda_{2,2})^{(123)} + \lambda_2 \circ_2 \lambda_{3,2} - (\lambda_2 \circ_2 \lambda_{3,3})^{(12)} + (\lambda_{2,1} \circ_2 \lambda_3)^{(123)}. \\
\end{align*}
\]

When in the above definition we assume \(L_2 = 0\), this forces the structure maps \(\lambda_{3,1}, \lambda_{3,2}, \lambda_4\) to vanish for degree reasons. The equations (30) and (43)–(49) hold for degree reasons in this case, and the left-hand sides of equations (29) and (41)–(42) become 0. In this way, we recover Roytenberg’s definition of a 2-term EL\(_\infty\)-algebra [27] Definition 2.16] (which we will call weak Lie 2-algebras.)

### 3.3. Morphisms of weak Lie 3-algebras.

Since weak Lie 3-algebras are algebras over the operad \(\Omega \text{Lie}_3^\circ\), they come with a general notion of morphism of operadic algebras. Such a morphism of weak Lie 3-algebras \(L \to L’\) is a morphisms of the underlying chain complexes \((L,d) \to (L’,d’)\) commuting with all structure maps. This notion is of limited use however and we will introduce another type of morphisms below.

Consider again weak Lie 3-algebras \(L\) and \(L’\), i.e. chain complexes \((L,d)\), \((L’,d’)\) equipped with structure maps given by twisting morphisms

\[
\lambda \in \text{Tw}(\text{Lie}_3^\circ, \text{End}_L), \quad \lambda' \in \text{Tw}(\text{Lie}_3^\circ, \text{End}_{L'}). 
\]

Following the general theory described in Section 1.8.1 a weak morphism of weak Lie 3-algebras is a degree 0 solution to the following Maurer–Cartan equation:

\[
(50) \quad f : \text{Lie}_3^\circ \to \text{End}_{L'}, \quad \partial(f) - f \ast \lambda + \lambda' \circ f = 0.
\]

We again evaluate this Maurer–Cartan equation for the \(k[\mathbb{S}_n]\)-generators of \(\text{Lie}_3^\circ\), using the short-hand notation \(f\) for \(f(l_n)\). E.g. for \(l_{2,1}\), this amounts to

\[
0 = (\partial(f) - f \ast \lambda + \lambda' \circ f)(l_{2,1}) = \partial(f(l_{2,1})) - f(dl_{2,1}) - (f \ast \lambda)(l_{2,1}) + (\lambda' \circ f)(l_{2,1})
\]

\[
= \partial(f_{2,1}) + f(l_{2,1}) + f(l_{2,1})^{(12)} - f(id) \circ_1 \lambda(l_{2,1}) + \lambda'(l_{2,1}) \circ (f(id), f(id))
\]

\[
= \partial(f_{2,1}) + f_{2,1} + f_{2,1}^{(12)} - f_1 \circ \lambda_{2,1} + \lambda'_2 \circ (f_1, f_1),
\]

which is equivalent to equation (51). In summary, we obtain the following definition.

**Definition 3.5.** A weak morphism of weak Lie 3-algebras \(f : (L,d,\lambda) \to (L',d',\lambda')\) consists of a collection of \(k\)-linear maps,

\[
\begin{align*}
 f_1 & : L \to L', & f_2 & : L^{\otimes 2} \to L'[1], & f_{2,1} & : L^{\otimes 2} \to L'[2], \\
 f_3 & : L^{\otimes 3} \to L'[2].
\end{align*}
\]
We assume that the equations \((52) - (55)\) hold, i.e. \(f\) is a morphism of Leibniz 3-algebras. In addition the maps are required to satisfy the following equations:

\[
\begin{align*}
(51) & \quad \partial(f_{2;1}) = - f_2 - f_2^{(12)} + f_1 \circ \lambda_{2;1} - \lambda_{2;1}^\prime \circ (f_1, f_1), \\
(52) & \quad f_1 \circ \lambda_{2;1;1} - \lambda_{2;1;1}^\prime \circ (f_1, f_1) = f_{2;1} - f_2^{(12)}, \\
(53) & \quad f_1 \circ \lambda_{3;1} - \lambda_{3;1}^\prime \circ (f_1, f_1, f_1) = f_3 + f_3^{(12)} + f_2 \circ \lambda_{2;1} + \lambda_{2;1}^\prime \circ (f_2, f_1), \\
(54) & \quad - \lambda_2^\prime \circ (f_1, f_{2;1}) - \lambda_{2;1}^\prime \circ (f_2, f_1) - \lambda_{2;1}^\prime \circ (f_1, f_2)^{(12)}. \\
\end{align*}
\]

Let \(L \xrightarrow{f} L' \xrightarrow{f'} L''\) be weak morphisms of weak Lie 3-algebras. Their composition is defined to be the composition of the underlying weak morphisms of Leibniz 3-algebras, i.e. by equations \((56) - (58)\), with the additional component

\[
(55) \quad (f' \circ f)_{2;1} := f'_{2;1} \circ (f_1, f_1) + f'_{1} \circ f_{2;1}.
\]

If we assume that \(L\) and \(L'\) in the above definition are in fact weak Lie 2-algebras, then for degree reasons \(f_{2;1}\) and \(f_3\) must vanish. Equations \((55)\) and \((52) - (54)\) hold again for degree reasons, and the left-hand sides of equations \((54)\) and \((51)\) are zero. In this way we recover Roytenberg’s notion of morphism of 2-term EL_∞-algebras [22, Definition 2.18].

3.4. The homotopy transfer theorem. While we explicitly prove a homotopy transfer result for weak Lie 3-algebras below, let us remark again that for a cofibrant resolution of any operad such a result is automatic.

Let \((L, d, \lambda)\) be a weak Lie 3-algebra. Assume that we are given a deformation retract of the underlying chain complex, i.e. chain maps \(p\) and \(i\), and a chain homotopy \(h\) as follows:

\[
(56) \quad \xymatrix{ h: (L, d) \ar[r]^i & (L', d') \ar[l]_p }, \quad \text{such that } \begin{cases} id_L - i \circ p = [d, h], \\
    id_{L'} - p \circ i = 0. \end{cases}
\]

In this setting, the following homotopy transfer theorem for weak Lie 3-algebras holds.

**Proposition 3.6.** Let \((L, d, \lambda)\) be a weak Lie 3-algebra and let \((L', d')\) be a deformation retract of \((L, d)\) as in equation \((56)\). Then \((L', d')\) can be equipped with a transferred weak Lie 3-algebra structure in such a way, that the map \(i\) admits an extension to a weak morphism of weak Lie 3-algebras.

**Proof.** This follows directly from Lemma \([57]\) and Lemma \([58]\) below. \(\square\)

By the homotopy transfer theorem for Leibniz_∞-algebras we know that one can define a Leibniz 3-algebra structure on \((L', d')\) by the structure maps

\[
(57) \quad \lambda_2^\prime := p \circ \lambda_2 \circ (i, i), \\
(58) \quad \lambda_3^\prime := p \circ \lambda_2 \circ (i, i, i) + p \circ \left( \lambda_2 \circ (h \circ \lambda_2) - (\lambda_2 \circ (h \circ \lambda_2)^{(1)} \circ (i, i, i) \right), \\
(59) \quad \lambda_4^\prime := p \circ \lambda_2 \circ (i, i, i, i) - p \circ \left( \begin{align*}
\lambda_2 \circ (h \circ \lambda_3) + (\lambda_2 \circ (h \circ \lambda_3) + (\lambda_3 \circ (h \circ \lambda_2)^{(1)} \circ (i, i, i, i), \\
\lambda_3 \circ (h \circ \lambda_2)^{(1)} \circ (i, i, i, i, i), \\
\end{align*} \right)
\]

and that the following components define an extension of \(i\) to a homotopy morphism of Leibniz 3-algebras \(i: (L', d', \lambda') \to (L, d, \lambda)\):

\[
(60) \quad i_1 = i,
\]
Lemma 3.7. The Leibniz 3-algebra \((L', d')\) with structure maps defined by equations \((57) - (59)\) admits an extension to a weak Lie 3-algebra. Explicitly, such an extension is given by the structure maps

\[
\begin{align*}
\lambda'_{2,1} &:= p \circ \lambda_{2,1} \circ (i, i), \\
\lambda'_{2,1,1} &:= p \circ \lambda_{2,1,1} \circ (i, i), \\
\lambda_{3,1} &:= p \circ \lambda_{3,1} \circ (i, i, i) - p \circ (\lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i), \\
\lambda_{3,2} &:= p \circ \lambda_{3,2} \circ (i, i, i) + p \circ (\lambda_{2,1} \circ (h \circ \lambda_2) + \lambda_2 \circ (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ (h \circ \lambda_2))) \circ (i, i, i).
\end{align*}
\]

The proof is a straight-forward verification of equations \((59) - (61)\), which we postpone to Appendix A.2.

Lemma 3.8. The Leibniz 3-algebra morphism \(i: (L', d') \rightarrow (L, d)\) with components defined by equations \((60) - (62)\) extends to a morphism of weak Lie 3-algebras with the additional component

\[
i_{2,1} = -h \circ \lambda_{2,1} \circ (i, i).
\]

Proof. We verify that with this definition of the additional component \(i_{2,1}\) equations \((51) - (53)\) hold:

\[
\begin{align*}
\partial(i_{2,1}) &= -\partial(h \circ \lambda_{2,1} \circ (i, i)) \\
&= -(\text{id} - i\partial) \circ \lambda_{2,1} \circ (i, i) + h \circ (\lambda_2 + \lambda_2^{(12)}) \circ (i, i) \\
&= -i_2 - i_2^{(12)} + i_1 \circ \lambda'_{2,1} - \lambda_{2,1} \circ (i_1, i_1), \\
i_1 \circ \lambda'_{2,1,1} - \lambda_{2,1,1} \circ (i_1, i_1) &= i \circ (p \circ \lambda_{2,1,1} \circ (i, i)) - \lambda_{2,1,1} \circ (i, i) \\
&= -\partial h \circ \lambda_{2,1,1} \circ (i, i) \\
&= -h \circ (\lambda_{2,1} - \lambda_{2,1}^{(12)}) \circ (i, i) \\
&= i_{2,1} - i_{2,1}^{(12)}. \\
i_1 \circ \lambda'_{3,1} - \lambda_{3,1} \circ (i_1, i_1, i_1) &= i \circ (p \circ (\lambda_{3,1} - \lambda_{2,1} \circ (h \circ \lambda_{2,1})) \circ (i, i, i)) - \lambda_{3,1} \circ (i, i, i) \\
&= -\partial h \circ \lambda_{3,1} \circ (i, i, i) - (\text{id} - \partial h) \circ (\lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \\
&= -h \circ (\lambda_3 + \lambda_2^{(12)} + \lambda_2 \circ \lambda_{2,1}) \circ (i, i, i) + \lambda_2 \circ (i_{2,1}, i_1) \\
&+ h \circ (\lambda_2 \circ (\partial h \circ \lambda_{2,1})) \circ (i, i, i) - h \circ (\lambda_2 \circ (h \circ (\lambda_2 + \lambda_2^{(12)}))) \circ (i, i, i) \\
&= -h \circ (\lambda_3 + \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) - h \circ (\lambda_3 + \lambda_2 \circ (h \circ \lambda_{2,1}))^{(12)} \circ (i, i, i) \\
&+ \lambda_2 \circ (i_{2,1}, i_1) - h \circ (\lambda_2 \circ (i \circ \lambda_{2,1})) \circ (i, i, i) \\
&= i_3 + i_3^{(12)} + \lambda_2 \circ (i_{2,1}, i_1) + i_2 \circ \lambda'_{2,1}. \\
i_1 \circ \lambda'_{3,2} - \lambda_{3,2} \circ (i_1, i_1, i_1) &= i \circ (p \circ (\lambda_{3,2} + \lambda_{2,1} \circ (h \circ \lambda_2) + \lambda_2 \circ (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ (h \circ \lambda_2)))^{(12)}) \circ (i, i, i) - \lambda_{3,2} \circ (i, i, i) \\
&= i \circ (p \circ \lambda_{3,2} \circ (i, i, i) + p \circ (\lambda_{2,1} \circ (h \circ \lambda_2) + \lambda_2 \circ (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ (h \circ \lambda_2)))^{(12)}) - \lambda_{3,2} \circ (i, i, i) \\
&= -\partial h \circ \lambda_{3,2} \circ (i, i, i) + (\text{id} - \partial h) \circ (\lambda_{2,1} \circ (h \circ \lambda_2) + \lambda_2 \circ (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ (h \circ \lambda_2)))^{(12)} \circ (i, i, i)
\end{align*}
\]
This concludes the proof.

4. Skew-symmetrization

Since an $\text{L}_\infty$-algebra is just a Leib$_\infty$-algebra with skew-symmetric structure maps, it seems natural to try and construct an $\text{L}_\infty$-algebra by skew-symmetrizing the structure maps of a Leib$_\infty$-algebra. One way to put this more formally is in terms of the Koszul dual cooperads: Assume for a moment that we can construct a right inverse to the morphism $\psi$ defined by equation (12), i.e. a morphism $\phi$ of dg cooperads

$$\psi : \text{Leib}^i \xrightarrow{\sim} \text{Lie}^i : \phi ,$$

such that $\psi \circ \phi = \text{id}$.

In this case, we obtain for any Leib$_\infty$-algebra $(L, d, \lambda)$ given by a twisting morphism $\lambda : \text{Leib}^i \to \text{End}_L$, an $\text{L}_\infty$-algebra $(L, d, \tilde{\lambda})$ via precomposition of the twisting morphism with $\phi$, i.e. $\tilde{\lambda} := \lambda \circ \phi$. We shall make a naive attempt at defining such a morphism $\phi$ below to see how it fails.

Define $\phi$ on $k[S_n]$-generators by

$$\phi(l_n) := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\vert \sigma \vert} \cdot l_n^\sigma .$$

Clearly, this is a well-defined morphism of dg $S$-modules and satisfies $\psi \circ \phi = \text{id}$, provided the $1/n!$ exist. The calculation below, however, shows that the map $\phi$ does not define a morphism of dg cooperads since $\phi$ does not commute with the decomposition map $\Delta$ already in arity 3. We find

$$\Delta(\phi(l_3)) = \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot (l_2 \circ (l_2, 1) + l_2 \circ (1, l_2) - l_2 \circ (1, l_2)^{(12)})^\sigma$$

$$= -\frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot l_2 \circ (l_1, 1)^\sigma + \frac{1}{3} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot l_2 \circ (1, l_2)^\sigma ,$$

while

$$\phi(\Delta(l_3)) = -\frac{1}{4} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot (l_2 \circ (l_2, 1) - l_2 \circ (1, l_2))^\sigma .$$

Note that the difference

$$(\Delta \circ \phi - \phi \circ \Delta)(l_3) = \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot (l_2 \circ (l_2, 1) + l_2 \circ (1, l_2))^\sigma$$

is actually a coboundary when we view Leib$^3_i \subset \text{Lie}^3_o$ as a dg subcooperad,

$$= -\frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\vert \sigma \vert} \cdot (dl_2 \circ (l_2, 1))^\sigma .$$
This suggests that (i) while Leib$_\infty$-algebras do not admit a skew-symmetrization in general, for weak Lie 3-algebras (and more generally El$_\infty$-algebras) such a construction may exist, and (ii) we should try to extend $\phi$ to a homotopy morphism of dg cooperads.

The remainder of this section is organized as follows. In Section 4.1 we construct a right inverse $\Phi$ for $\Omega \psi: \Omega \text{Lie}_\infty^3 \to \Omega \text{Lie}_3^3$. In Section 4.2 we use $\Phi$ to define a skew-symmetrization construction for weak Lie 3-algebras. In Section 4.3 we define an ad hoc skew-symmetrization for morphisms of weak Lie 3-algebras and show that it is functorial up to homotopy.

In this entire section, we assume that $2, 3 \in k^\times$ are units.

### 4.1. A right inverse homotopy morphism for the cooperad resolution.

Below we construct a right inverse for $\Omega \psi: \Omega \text{Lie}_\infty^3 \to \Omega \text{Lie}_3^3$. We think of such a map as a homotopy morphism of dg cooperads.

**Lemma 4.1.** The morphism $\Omega \psi$ admits a right inverse, i.e. a morphism $\Phi$ of dg operads

$$
\Omega \psi: \Omega \text{Lie}_\infty^3 \rightleftharpoons \Omega \text{Lie}_3^3: \Phi ,
$$

such that $\Omega \psi \circ \Phi = \text{id}$.

One such morphism $\Phi$ is defined by

$$
\Phi(s^{-1}l_2) = \frac{1}{2} \sum_{\sigma \in S_2} (-1)^{|\sigma|} s^{-1}l_2^\sigma,
$$

$$
\Phi(s^{-1}l_3) = \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{|\sigma|} s^{-1}l_3^\sigma - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{|\sigma|} (s^{-1}l_{2,1} o_1 s^{-1}l_2 + s^{-1}l_{2,1} o_2 s^{-1}l_2)^\sigma,
$$

$$
\Phi(s^{-1}l_4) = \frac{1}{24} \sum_{\sigma \in S_4} (-1)^{|\sigma|} s^{-1}l_4^\sigma + \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \left( s^{-1}l_{2,1} o_1 s^{-1}l_3 - s^{-1}l_{3,1} o_1 s^{-1}l_2 + s^{-1}l_{3,2} o_2 s^{-1}l_2 \right)^\sigma
$$

Proof. Any morphism of free dg operads is completely determined by its value on generators, i.e. it is sufficient to define $\Phi|_{\text{free Lie}_\infty}$ and its extension $\Phi$ is then automatically a morphism of operads. It remains to verify that $\Phi$ commutes with the differential, which we do below:

$$
d\Phi(s^{-1}l_2) = \frac{1}{2} \sum_{\sigma \in S_2} (-1)^{|\sigma|} ds^{-1}l_2^\sigma = 0 = \Phi(0) = \Phi(ds^{-1}l_2),
$$

$$
d\Phi(s^{-1}l_3) = -\frac{1}{6} \sum_{\sigma \in S_3} (-1)^{|\sigma|} d_2 s^{-1}l_3^\sigma - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{|\sigma|} (d_1 s^{-1}l_{2,1} o_1 s^{-1}l_2 + d_1 s^{-1}l_{2,1} o_2 s^{-1}l_2)^\sigma
$$

$$
= -\frac{1}{6} \sum_{\sigma \in S_3} (-1)^{|\sigma|} \left( -s^{-1}l_2 o_1 s^{-1}l_2 + s^{-1}l_2 o_2 s^{-1}l_2 - (s^{-1}l_2 o_2 s^{-1}l_2)^{(12)} \right)^\sigma
$$

$$
- \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{|\sigma|} \left( (s^{-1}l_2 + s^{-1}l_2^{(12)}) o_1 s^{-1}l_2 \right)^\sigma
$$

$$
= -\Phi(s^{-1}l_2) o_1 \Phi(s^{-1}l_2) + \Phi(s^{-1}l_2) o_2 \Phi(s^{-1}l_2) - (\Phi(s^{-1}l_2) o_2 \Phi(s^{-1}l_2))^{(12)}
$$

$$
= \Phi(-s^{-1}l_2 o_1 s^{-1}l_2 + s^{-1}l_2 o_2 s^{-1}l_2 - (s^{-1}l_2 o_2 s^{-1}l_2)^{(12)}) = \Phi(-d_2 s^{-1}l_3) = \Phi(ds^{-1}l_3),
$$

$$
d\Phi(s^{-1}l_4) = -\frac{1}{24} \sum_{\sigma \in S_4} (-1)^{|\sigma|} d_2 s^{-1}l_4^\sigma + \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \left( d_1 s^{-1}l_{2,1} o_1 s^{-1}l_3 + s^{-1}l_{2,1} o_1 (d_2 s^{-1}l_3) \right)^\sigma
$$

$$
- \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \left( d_1 s^{-1}l_{3,1} o_1 s^{-1}l_2 - (d_2 s^{-1}l_{3,1}) o_1 s^{-1}l_2 \right)^\sigma
$$

$$
+ \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \left( (d_1 s^{-1}l_{3,2}) o_2 s^{-1}l_2 - (d_2 s^{-1}l_{3,2}) o_2 s^{-1}l_2 \right)^\sigma
$$

$$
= -\frac{1}{24} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \left( -s^{-1}l_2 o_1 s^{-1}l_4 - 3s^{-1}l_2 o_2 s^{-1}l_3 + s^{-1}l_4 o_2 s^{-1}l_2 + 3s^{-1}l_3 o_3 s^{-1}l_2 \right)^\sigma
$$

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we obtain a morphism
\( \Phi \).

4.2. Skew-symmetrization for weak Lie 3-algebras. In Section 3.2 we defined a weak Lie 3-algebra \((L, d, \lambda)\) as a 3-term complex \((L, d)\) with a twisting morphism \(\lambda: \Omega \text{Lie}_\delta^3 \to \text{End}_L\). By equation (14), such a twisting morphism corresponds to a morphism of dg operads \(g_\lambda: \Omega \text{Lie}_\delta^3 \to \text{End}_L\) via \(g_\lambda|_{s^{-1}I_4} = \lambda_*\). By precomposition with \(\Phi\) we obtain a morphism
\[ \Phi^*g_\lambda: \Omega \text{Lie}_d^3 \to \Omega \text{Lie}_\delta^3 \to \text{End}_L \]
of dg operads, which in turn corresponds to a 3-term \(L_\infty\)-algebra or (semi-strict) Lie 3-algebra. We make the result of this construction explicit below.

Definition 4.2. Let \((L, d, \lambda)\) be a weak Lie 3-algebra. We define its skew-symmetrization to be the (semi-strict) Lie 3-algebra \((L, d, \Lambda)\) given by the following structure maps:

\[ \Lambda_2 := \frac{1}{2} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \lambda_2^\sigma, \]
\[ \Lambda_3 := \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \lambda_3^\sigma - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot (\lambda_{2,1} \circ_1 \lambda_2 + \lambda_{2,2} \circ_2 \lambda_2)^\sigma, \]
\[ \Lambda_4 := \frac{1}{24} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot \lambda_4^\sigma + \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot (\lambda_{2,1} \circ_1 \lambda_3 - \lambda_{3,1} \circ_1 \lambda_2 + \lambda_{3,2} \circ_2 \lambda_2)^\sigma. \]

Note that for weak Lie 2-algebras, \(\lambda_{2,1}\) is symmetric and we recover Roytenberg’s skew-symmetrization construction in this case.

4.3. Skew-symmetrization for morphisms of weak Lie 3-algebras. A morphism \(f: (L, d, \lambda) \to (L', d', \Lambda')\) of weak Lie 3-algebras was defined in Section 3.3 as a morphism \(\Phi: \text{Lie}_\delta^3 \to \text{End}_{L'}^3\) satisfying a certain Maurer–Cartan type equation (15). Such a morphism in general does not correspond to a morphism \(\Omega \text{Lie}_\delta^3 \to \text{End}_{L'}^3\), and there is no obvious way to precompose \(f\) with \(\Phi\). Below we give an ad hoc construction for a skew-symmetrization of morphisms instead.
Lemma 4.3. Let \( f : (L, d, \lambda) \to (L', d', \lambda') \) be a morphism of weak Lie 3-algebras. The following components define a morphism of (semi-strict) Lie 3-algebras \( \tilde{f} : (L, d, \lambda) \to (L', d', \lambda') \):

\[
\tilde{f}_1 := f_1,
\]

\[
\tilde{f}_2 := \frac{1}{2} \sum_{\sigma \in S_2} (-1)^{\abs{\sigma}} \cdot f_2^\sigma,
\]

\[
\tilde{f}_3 := \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot f_3^\sigma - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot \left( f_{2,1}^1 \lambda_2 + f_{2,1}^2 \lambda_2 - \lambda_{2,1}^2 \circ (f_2, f_1) - \lambda_{2,1}^2 \circ (f_1, f_2) \right)^\sigma.
\]

We call \( \tilde{f} \) the skew-symmetrization of \( f \).

The proof is a direct verification of equations (32)–(34) and is given in Appendix A.3.

4.3.1. Functoriality. In low degrees, skew-symmetrization commutes with composition of morphisms,

\[
(f' \circ f)_1 = (f' \circ f)_1 = f'_1 \circ f_1 \circ f_1 = (f' \circ f)_1 = (f' \circ f)_1,
\]

\[
(f' \circ f)_2 = \frac{1}{2} \left( (f' \circ f)_2 - (f' \circ f)_2^{(12)} \right)
\]

\[
= \frac{1}{2} \left( f'_2 \circ (f_1, f_1) + f'_1 \circ f_2 - f'_2 \circ (f_1, f_1)^{(12)} - f'_1 \circ f_2^{(12)} \right)
\]

\[
= \tilde{f}_2 \circ (\tilde{f}_1, \tilde{f}_1) + \tilde{f}_1 \circ \tilde{f}_2
\]

\[
= (\tilde{f} \circ \tilde{f})_2.
\]

In particular, this implies that skew-symmetrization of weak Lie 2-algebras forms a functor [25, Theorem 3.2]. However, for weak Lie 3-algebras this is no longer the case as the following computation shows:

\[
(f' \circ f - \tilde{f} \circ \tilde{f})_3 = \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot (f' \circ f)_3^\sigma - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot \left( f_{2,1}^1 \lambda_2 + f_{2,1}^2 \lambda_2 - \lambda_{2,1}^2 \circ (f_2, f_1) - \lambda_{2,1}^2 \circ (f_1, f_2) \right)^\sigma
\]

\[
- \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot \left( f_{2,1}^1 \circ (f_2, f_1) + f_{2,1}^2 \circ (f_1, f_1) \right)^\sigma
\]

\[
- \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot \left( f_{2,1}^1 \circ (f_2, f_1) + f_{2,1}^2 \circ (f_1, f_1) \right)^\sigma
\]

\[
- \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot (f' \circ f - \tilde{f} \circ \tilde{f})_3^\sigma
\]

\[
= \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\abs{\sigma}} \cdot (f' \circ f - \tilde{f} \circ \tilde{f})_3^\sigma
\]
\[-\frac{1}{24} \sum_{\sigma \in S_3} (-1)^{|\sigma|} \cdot \begin{pmatrix}
    f_{2,1}^f \circ (f_1 \circ \lambda_2, f_1) + f_{2,1}^f \circ (f_1, f_1 \circ \lambda_2) \\
    - f_{2,1}^f \circ (\lambda_2 \circ (f_1, f_1), f_1) - f_{2,1}^f \circ (f_1, \lambda_2 \circ (f_1, f_1)) \\
    - \lambda_{2,1}'' \circ (f_1 \circ f_2, f_1 \circ f_1) - \lambda_{2,1}'' \circ (f_1 \circ f_1, f_1 \circ f_2) \\
    + f_1 \circ \lambda_{2,1}'' \circ (f_2, f_1) + f_1 \circ \lambda_{2,1}'' \circ (f_1, f_2)
\end{pmatrix} \cdot (f_{2,1}^f \circ (f_2, f_1) + f_{2,1}^f \circ (f_1, f_2)) \right\}
\]

Note that the defect of functoriality is a coboundary. We say that skew-symmetrization is functorial up to homotopy.

5. Applications

In this section, we give two examples of applications of the theory developed in the earlier sections. The first is an extension of a result of [24] on algebraic structures on \( n \)-plectic manifolds. The second is a construction of a weak Lie 3-algebra associated to an CLWX 2-algebroid, whose skew-symmetrization is precisely the Lie 3-algebra of [19, Theorem 3.10]. We thereby give an alternative proof for the theorem cited.

5.1. Higher symplectic geometry. In [2, 24] the concept of \( n \)-plectic manifolds is introduced. We recall some of the basic definitions here. We then consider the case of a 3-plectic manifold and compare two associated algebraic structures, an \( L_\infty \)-algebra and a dg Leibniz algebra with a certain hidden skew-symmetry. Both structures are examples of weak Lie 3-algebras and turn out to be isomorphic as such. The analogous result for 2-plectic manifolds was shown in [24, Appendix A].

Definition 5.1. An \( n \)-plectic manifold \((M, \omega)\) is a smooth manifold \( M \) with a closed, non-degenerate \((n + 1)\)-form \( \omega \in \Omega^{n+1}(M) \), i.e. \( d\omega = 0 \) and \( \iota(v)\omega = 0 \) implies \( v = 0 \). An \((n - 1)\)-form \( \alpha \in \Omega^{n-1}(M) \) is called Hamiltonian, if there exists a vector field \( v_\alpha \in \mathfrak{X}(M) \) such that \( d\alpha = -\iota(v_\alpha)\omega \). In order to simplify notation, we let \( v_\alpha = 0 \) whenever \( \alpha \) is not Hamiltonian, in particular when \( \alpha \) is not an \((n - 1)\)-form.

Let \((M, \omega)\) be an \( n \)-plectic manifold. Two algebraic structures are introduced on the chain complex

\[
L_i := \begin{cases}
\Omega^\text{ Hamilton,}^{n-1}(M), & i = 0, \\
\Omega^{n-1-i}(M), & 0 < i \leq n - 1,
\end{cases}
\]

with differential the usual de Rham differential. Note that this is well-defined since closed forms are always Hamiltonian. The first structure we introduce is that of an \( n \)-term \( L_\infty \)-algebra or (semi-strict) Lie \( n \)-algebra \( L_\infty(M, \omega) \) given by the brackets

\[
\lambda_k(\alpha_1, \ldots, \alpha_k) := \pm \iota(v_{\alpha_1}, \ldots, v_{\alpha_k})\omega.
\]

We will provide the sign in low degrees in the proof of Proposition 5.2. For more details, including the general definition of the sign, we refer the reader to loc. cit. The second structure on the complex of equation (74) is a dg Leibniz algebra given by

\[
\lambda_2^L(\alpha, \beta) := \mathcal{L}(v_\alpha)\beta,
\]

denoted by \( \text{Leib}(M, \omega) \). This dg Leibniz algebra actually satisfies a certain symmetry up to homotopy, and in the case \( n = 2 \) is shown to be a weak Lie 2-algebra with alternator bracket

\[
\lambda_{2,1}''(\alpha, \beta) := \iota(v_\alpha)\beta + \iota(v_\beta)\alpha.
\]

This result can be extended to the case \( n = 3 \), i.e. the same binary bracket and alternator form a weak Lie 3-algebra on the underlying 3-term chain complex in that case.

Proposition 5.2. Let \((M, \omega)\) be a 3-plectic manifold. Then \( L_\infty(M, \omega) \) and \( \text{Leib}(M, \omega) \) are isomorphic as weak Lie 3-algebras.

Proof. For a 3-plectic manifold \((M, \omega)\), the chain complex underlying both \( L_\infty(M, \omega) \) and \( \text{Leib}(M, \omega) \) is

\[
(L, d^L) := \begin{array}{c}
\Omega^\text{ Hamilton,}^0(M) \\
\Omega^1(M) \\
\Omega^2(M) \rightarrow C^\infty(M)
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{d^L} \\
\xrightarrow{\iota} \\
\xrightarrow{d^L}
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{d^L} \\
\xrightarrow{\iota} \\
\xrightarrow{d^L}
\end{array}
\]

\[= C^\infty(M) \]


Note that for a Hamiltonian form $\alpha$, $d\alpha$ need not vanish, while $d^2\alpha = 0$ for degree reasons. The structure maps for $L_\infty(M, \omega)$ are explicitly given by

$$\lambda_2(\alpha, \beta) := \iota(v_\alpha, v_\beta)\omega, \quad \lambda_3(\alpha, \beta, \gamma) := -\iota(v_\alpha, v_\beta, v_\gamma)\omega, \quad \lambda_4(\alpha, \beta, \gamma, \eta) := -\iota(v_\alpha, v_\beta, v_\gamma, v_\eta)\omega.$$ 

We define a weak morphism $f : L_\infty(M, \omega) \rightarrow \text{Leib}(M, \omega)$ of weak Lie 3-algebras by

$$f_1(\alpha) = \alpha, \quad f_2(\alpha, \beta) = -\iota(v_\alpha, v_\beta), \quad f_{2,1}(\alpha, \beta) = 0, \quad f_3(\alpha, \beta, \gamma) = -\iota(v_\alpha, v_\beta, v_\gamma).$$

To show that these maps define a weak morphism of weak Lie 3-algebras, we begin by verifying that they indeed satisfy equations (32)–(35), i.e., define a morphism of Leibniz 3-algebras.

Clearly $\partial f_1 = 0$ and so equation (32) holds. When $|\alpha| > 0$, equation (33) is trivially satisfied. In case $|\alpha| = |\beta| = 0$ we obtain

$$\partial f_2 - f_1 \circ \lambda_2 + \lambda_2' \circ (f_1, f_1))(\alpha, \beta) = d^2 f_2(\alpha, \beta) - \lambda_2(\alpha, \beta) + \lambda_2'(\alpha, \beta) = d^2(-\iota(v_\alpha, v_\beta) - \iota(v_\alpha, v_\beta)\omega + \mathcal{L}(v_\alpha)\beta),$$

which, using the fact that $\beta$ is Hamiltonian and Cartan’s formula, can be written as

$$= -d\iota(v_\alpha, v_\beta) - \iota(v_\alpha, v_\beta)d\beta + d\iota(v_\alpha, v_\beta) + \iota(v_\alpha)d\beta = 0,$$

while for $|\alpha| = 0, |\beta| > 0$ we have

$$\partial f_2 - f_1 \circ \lambda_2 + \lambda_2' \circ (f_1, f_1))(\alpha, \beta) = d^2 f_2(\alpha, \beta) + f_2(\alpha, d^2\beta) - \lambda_2(\alpha, \beta) + \lambda_2'(\alpha, \beta) = d^2(-\iota(v_\alpha, v_\beta) - \iota(v_\alpha, v_\beta)d\beta - \iota(v_\alpha, v_\beta)\omega + \mathcal{L}(v_\alpha)\beta),$$

which, using that $\beta$ is not Hamiltonian, reduces to

$$= -d\iota(v_\alpha, v_\beta) - \iota(v_\alpha, v_\beta)d\beta + \mathcal{L}(v_\alpha)\beta = 0.$$ 

Evaluating equation (34) for $|\alpha| = |\beta| = 0$, we obtain—omitting $f_1 = 1$ for brevity from now on

\[
\begin{align*}
\partial (\mathcal{L}(\alpha) - \lambda_3 + f_2 \circ \lambda_2 - f_2 \circ 1 \lambda_2 - (f_2 \circ 2 \lambda_2)^{(12)} + \lambda_3' \circ 2 f_2 - \lambda_2' \circ 1 f_2 - \lambda_2' \circ 2 f_2^{(12)}) & (\alpha, \beta, \gamma) \\
= d^2 \mathcal{L}(\alpha, \beta, \gamma) - f_3(\alpha, \beta, d^2\gamma) - \lambda_3(\alpha, \beta, \gamma) + f_2(\alpha, \lambda_2(\beta, \gamma)) - f_2(\lambda_2(\alpha, \beta, \gamma) - f_2(\alpha, \lambda_2(\alpha, \gamma)) \\
& + \lambda_2'(\alpha, \lambda_2(\beta, \gamma)) - \lambda_2' f_2(\alpha, \beta, \gamma) - \lambda_2' f_2(\alpha, \beta, \gamma) \\
= d^2(-\iota(v_\alpha, v_\beta) - \iota(v_\alpha, v_\beta)d^2\gamma + \iota(v_\alpha, v_\beta, v_\gamma)\omega - \iota(v_\alpha, \iota(v_\beta, v_\gamma)\omega + \iota(v_\lambda(\alpha, \beta, \gamma) + \iota(v_\beta)\iota(v_\alpha, v_\gamma)\omega \\
& - \mathcal{L}(v_\alpha)\iota(v_\gamma) + \mathcal{L}(v_\beta)\iota(v_\alpha)\gamma
\end{align*}
\]

which, using the identities $v_\lambda(\alpha, \beta, \gamma) = [v_\alpha, v_\beta, v_\gamma] = [\mathcal{L}(v_\alpha), \iota(v_\beta)]\omega$, becomes

$$= -d\iota(v_\alpha, v_\beta) \gamma + \iota(v_\alpha, v_\beta)d^2\gamma - d\iota(v_\alpha, v_\beta, v_\gamma)\omega - \iota(v_\beta)\mathcal{L}(v_\alpha)\gamma + \mathcal{L}(v_\beta)\iota(v_\alpha)\gamma.$$

Now consider two cases: when $|\gamma| = 0$, $d^2\gamma = 0$ and we have

$$= -d\iota(v_\alpha, v_\beta) \gamma - \iota (v_\alpha, v_\beta, v_\gamma)\omega - \iota(v_\alpha, v_\beta)d\gamma + d\iota(v_\alpha, v_\beta)\gamma = 0,$$

while for $|\gamma| > 0$, $\iota (v_\alpha, v_\beta, v_\gamma)\omega = 0$ and $v_\gamma = 0$, and we obtain

$$= \iota (v_\alpha, v_\beta)d\gamma - \iota(v_\alpha, v_\beta)d\gamma = 0.$$

To verify equation (35), it is sufficient to consider the case $|\alpha| = |\beta| = |\gamma| = |\eta| = 0$. Leaving out the terms vanishing for degree reasons, we obtain

\[
\begin{align*}
(\lambda_4 - f_2 \circ 2 \lambda_3 + (f_2 \circ 2 \lambda_3)^{(12)} - (f_2 \circ 2 \lambda_3)^{(123)} + f_3 \circ 1 \lambda_2 - f_3 \circ 2 \lambda_2 + (f_3 \circ 2 \lambda_2)^{(12)} \\
+ f_3 \circ 3 \lambda_2 - (f_3 \circ 3 \lambda_2)^{(23)} + (f_3 \circ 3 \lambda_2)^{(123)} - \lambda_2' \circ 2 f_3 + (\lambda_2' \circ 2 f_3)^{(12)} - (\lambda_2' \circ 2 f_3)^{(123)}) & (\alpha, \beta, \gamma, \eta) \\
= \lambda_4(\alpha, \beta, \gamma, \eta) - f_2(\alpha, \lambda_3(\beta, \gamma, \eta)) + f_2(\beta, \lambda_3(\alpha, \gamma, \eta)) - f_2(\alpha, \lambda_3(\alpha, \gamma, \eta)) + f_3(\lambda_2(\alpha, \beta, \gamma, \eta) \\
- f_3(\alpha, \lambda_2(\beta, \gamma, \eta) + f_3(\beta, \lambda_2(\alpha, \gamma, \eta) + f_3(\alpha, \beta, \lambda_2(\gamma, \eta)) - f_3(\alpha, \gamma, \lambda_2(\beta, \eta)) + f_3(\beta, \gamma, \lambda_2(\alpha, \eta)) \\
- \lambda_2'(\alpha, \beta, \gamma, \eta)) + \lambda_2'(\beta, f_3(\alpha, \gamma, \eta)) - \lambda_2'(\gamma, f_3(\alpha, \beta, \eta)).
\end{align*}
\]
Note that the terms $f_2 \circ \lambda_3$ cancel with $f_3 \circ \lambda_2$, leaving us with
\[
\begin{align*}
&= -\iota(v_\alpha, v_\beta, v_\gamma, v_\eta) \omega - \iota(v_{x_2(\alpha, \beta)}, v_\gamma) \eta + \iota(v_\alpha, v_{x_2(\beta, \gamma)}) \eta - \iota(v_\beta, v_{x_2(\alpha, \gamma)}) \eta \\
&\quad + \mathcal{L}(v_\alpha) \iota(\mathcal{E}) v_\beta) \eta - \mathcal{L}(v_\alpha) \iota(\mathcal{E}) v_\beta) \eta + \mathcal{L}(v_\gamma) \iota(v_\alpha, v_\beta) \eta,
\end{align*}
\]

\[
\begin{align*}
&= -\iota(v_\alpha, v_\beta, v_\gamma, v_\eta) \omega + \iota(v_\beta) \mathcal{L}(v_\alpha) \iota(\mathcal{E}) v_\gamma) \eta - \iota(v_\beta) \mathcal{L}(v_\alpha) \iota(v_\gamma) \eta + \mathcal{L}(v_\gamma) \iota(v_\alpha, v_\beta) \eta \\
&\quad - \iota(v_\alpha, v_\beta, v_\gamma, v_\eta) \omega + \iota(v_\beta) \mathcal{L}(v_\alpha) \iota(v_\gamma) \eta + \mathcal{L}(v_\gamma) \iota(v_\alpha, v_\beta) \eta = 0.
\end{align*}
\]

We proceed to show that $f$ additionally satisfies equations (51)–(54) and hence defines a morphism of weak Lie 3-algebras. For equation (51) we find
\[
(f_2 + f_2^{(12)} + \lambda_2^{(1)}) (\alpha, \beta) = -\iota(v_\alpha) \beta - \iota(v_\beta) \alpha + \iota(v_\alpha) \beta + \iota(v_\beta) \alpha = 0.
\]

Since all involved maps vanish, equation (52) is trivially satisfied. Finally we consider equations (53) and (54),
\[
(f_3 + f_3^{(12)} - \lambda_2^{(2)}) (\alpha, \beta, \gamma) = -\iota(v_\alpha, v_\beta, v_\gamma) \gamma - \iota(v_\beta, v_\gamma) \gamma = 0,
\]

\[
(f_3 + f_3^{(23)} - \lambda_2^{(2;1)} \circ f_2 - (\lambda_2^{(2;1)} \circ f_2)^{(12)}) (\alpha, \beta, \gamma) = -\iota(v_\alpha, v_\beta, v_\gamma) \gamma - \iota(v_\beta, v_\gamma) \gamma + \iota(v_\gamma) \iota(v_\alpha) \beta + \iota(v_\beta) \iota(v_\alpha) \gamma = 0.
\]

This concludes the proof. \hfill \Box

5.2. Higher Courant algebroids. In [19], the notion of an CLWX 2-algebroid is introduced as a higher analogue of a Courant algebroid. A Lie 3-algebra is associated to any CLWX 2-algebroid in [19, Theorem 3.10]. In this section, we give a construction of a weak Lie 3-algebra for any CLWX 2-algebroid. We then show that the Lie algebra associated to any CLWX 2-algebroid in [19, Theorem 3.10]. In this section, we give a construction of a weak Lie 3-algebra for any CLWX 2-algebroid. We then show that the Lie 3-algebra constructed in loc. cit. is in fact the skew-symmetrization of our weak Lie 3-algebra.

Definition 5.3. Let $E = (E_0 \overset{\partial}{\leftarrow} E_1)$ be a 2-term dg vector bundle over $M$ equipped with a morphism $\rho: E \to TM$ of dg vector bundles, a (graded) bilinear map $\circ: \Gamma E \otimes \Gamma E \to \Gamma E$ which is skew-symmetric on $\Gamma E_0 \otimes \Gamma E_0$, a (graded) 3-form $\Omega: \Gamma E \otimes \Gamma E \otimes \Gamma E \to \Gamma E[1]$, and a non-degenerate symmetric bilinear form $S: \Gamma E \otimes \Gamma E \to C^\infty(M)$. Using these data, we define a map $\mathcal{D}: C^\infty(M) \to \Gamma E_1$ by
\[
\begin{align*}
S(e, \mathcal{D}f) &= \rho(e)(f), \quad \forall e \in \Gamma E.
\end{align*}
\]

We call $E = (E_0 \overset{\partial}{\leftarrow} E_1, \rho, \circ, \Omega, S)$ an CLWX 2-algebroid, if the following conditions are satisfied:

(i) $E_0, E_1$ are isotropic, i.e. $S(E_i, E_i) = 0$ for $i = 0, 1$,

(ii) $(\Gamma E_0 \overset{\partial}{\leftarrow} \Gamma E_1, \circ, \Omega)$ is a Leibniz 2-algebra,

(iii) $e \circ e = \frac{\mathcal{D}}{2} S(e, e)$ for all $e \in \Gamma E$,

(iv) $S(\partial e_1, e_2) = S(e_1, \partial e_2)$ for all $e_i \in \Gamma E$,

(v) $\rho(e_1) S(e_2, e_3) = S(e_1 \circ e_2, e_3) + S(e_2, e_1 \circ e_3)$ for all $e_i \in \Gamma E$, and

(vi) $S(\Omega(e_1, e_2, e_3), e_4) = -S(e_3, \Omega(e_1, e_2, e_4))$ for all $e_i \in \Gamma E$.

Proposition 5.4. For any CLWX 2-algebroid, there is an associated complex
\[
(L, d) := \left( \begin{array}{c} \Gamma E_0 \ \overset{\partial}{\leftarrow} \ \Gamma E_1 \ \overset{\mathcal{D}}{\leftarrow} C^\infty(M) \end{array} \right),
\]

which, equipped with structure maps
\[
\begin{align*}
\lambda_2 &= (\circ -) + S \circ \mathcal{D}, \\
\lambda_2^{(1)} &= S, \\
\lambda_3 &= \Omega,
\end{align*}
\]

\[
\begin{align*}
\lambda_{2,1} &= 0, \\
\lambda_{3,1} &= 0, \\
\lambda_{3,2} &= 0, \\
\lambda_4 &= 0,
\end{align*}
\]

defines a weak Lie 3-algebra.

Proof. The fact that $(L, d)$ is indeed a complex is equivalent to $\partial \mathcal{D} = 0$, which is shown in [19, Lemma 3.6].

We begin by showing that $(L, d, \lambda)$ is a Leibniz 3-algebra, i.e. by checking equations (24)–(29). Since, by condition (i), we already know that the 2-term truncation of $L$ forms a Leibniz 2-algebra, it is sufficient to verify that these equations hold on tuples containing at least one degree 2 element. To keep notation as simple as possible,
where the last equality holds by definition of $D$, which, by equation (76), can be written as

\begin{equation}
\partial(\lambda_2)(e^0, f) = d(\lambda_2(e^0, f)) - \lambda_2(e^0, Df) = DS(e^0, Df) - e^0 \circ Df = 0,
\end{equation}

and

\begin{equation}
\partial(\lambda_2)(f, e^0) = -\lambda_2(Df, e^0) = -Df \circ e^0 = 0,
\end{equation}

which are both shown to hold in [19] Lemma 3.6. In addition we have

\begin{equation}
\partial(\lambda_2)(e^1, f) = -\lambda_2(\partial e^1, f) = -S(\partial e^0, Df) = -S(e^0, \partial Df) = 0,
\end{equation}

which holds since $\partial D = 0$, as we have seen above. In the remaining cases, i.e. for $(f, e^1)$ and $(f_1, f_2)$, the equation holds trivially since all structure maps vanish. For equation (28) we obtain

\begin{equation}
(\partial \lambda_3 - \lambda_2 \circ_2 \lambda_2 + \lambda_2 \circ_1 \lambda_2 + (\lambda_2 \circ_2 \lambda_2)(12))(e^0, e^2, f)
\end{equation}

\begin{align}
&= -\lambda_3(e^0, \lambda_2(e^0, f)) + \lambda_2(e^0, \lambda_2(e^0, f)) + \lambda_2(e^2, \lambda_2(e^0, f)) \\
&= -S(e^0, DS(e_2^0, Df)) + S(e^0, Df) + S(e^0, DS(e_1^0, Df)),
\end{align}

which, by equation (70), can be written as

\begin{equation}
= -S(e_1^0, DS(e^0, Df)) + S(e_0^0, Df) + S(e_2^0, e_1^0 \circ Df),
\end{equation}

and using condition (37) becomes

\begin{equation}
= -S(e_0^0, DS(e^0, Df)) + \rho(e_1^0)S(e^0, Df) = 0,
\end{equation}

where the last equality holds by definition of $D$. Since equation (29) is of degree 1, it holds for degree reasons when evaluated on any tuple containing a degree 2 element. Similarly, equation (30) is always satisfied for degree reasons. This completes the proof that $L$ is a Leibniz 3-algebra.

We proceed to show that $L$ is in fact a weak Lie 3-algebra. Consider equation (39). By isotropy of $E_0$ and skew-symmetry of $\circ$ on $L_0 \otimes L_0$ we see that

\begin{equation}
(\partial(\lambda_2;1) - \lambda_2 - \lambda_2^{(12)})(e_1^0, e_2^0) = \partial S(e_1^0, e_2^0) - e_1^0 \circ e_2^0 - e_2^0 \circ e_1^0 = 0.
\end{equation}

In addition we find

\begin{equation}
(\partial(\lambda_2;1) - \lambda_2 - \lambda_2^{(12)})(e^0, e^1) = DS(e^0, e^1) - e^0 \circ e^1 - e^1 \circ e^0
\end{equation}

\begin{equation}
= \frac{1}{2}DS(e^0 + e^1, e^0 + e^1) - (e^0 + e^1) \circ (e^0 + e^1) = 0,
\end{equation}

which holds using condition (44) and

\begin{equation}
(\partial(\lambda_2;1) - \lambda_2 - \lambda_2^{(12)})(e^0, e^1) = S(\partial e_1^0, e_2^0) - S(e_1^0, \partial e_2^0) = 0,
\end{equation}

which holds by condition (39). In the remaining cases the equation is trivially satisfied, e.g.

\begin{equation}
(\partial(\lambda_2;1) - \lambda_2 - \lambda_2^{(12)})(e^0, f) = S(e^0, Df) - S(e^0, Df) = 0.
\end{equation}

Next, we consider equation (40). Since $\lambda_{2,1,1} = 0$, the only non-trivial case is

\begin{equation}
(\partial(\lambda_2;1,1) + \lambda_2;1 - \lambda_2^{(12)})(e^0, e^1) = S(e^0, e^1) - S(e^0, e^1) = 0,
\end{equation}

which holds by symmetry of $S$. For equations (41) and (42) we obtain

\begin{equation}
(\partial(\lambda_3;1) - \lambda_3 - \lambda_3^{(12)})(e_1^0, e_2^0, e_3^0) = -\Omega(e_1^0, e_2^0, e_3^0) - \Omega(e_2^0, e_1^0, e_3^0) - \Omega(e_3^0, e_1^0, e_2^0) = 0,
\end{equation}

which holds by skew-symmetry of $\Omega$, and

\begin{equation}
(\partial(\lambda_3;2) - \lambda_3 - \lambda_3^{(23)} + \lambda_2 \circ_2 \lambda_2 - \lambda_2 \circ_1 \lambda_2 - (\lambda_2 \circ_2 \lambda_2)(12))(e_1^0, e_2^0, e_3^0)
\end{equation}

\begin{align}
&= -\Omega(e_1^0, e_2^0, e_3^0) - \Omega(e_1^0, e_3^0, e_2^0) + S(e_1^0, DS(e_2^0, e_3^0)) - S(e_2^0 \circ e_1^0, e_3^0) - S(e_3^0 \circ e_2^0, e_1^0) = 0,
\end{align}
which follows again from skew-symmetry of $\Omega$ and using condition (v) Equations (13)–(18) hold trivially since each term vanishes for degree reasons or because one of the relevant structure maps is zero. Finally, equation (19) reduces to

$$(\lambda_{2,1} \circ_1 \lambda_3 + (\lambda_{2,1} \circ_2 \lambda_3)^{(123)})(e_1^0, e_2^0, e_3^0, e_4^0) = S(\Omega(e_1^0, e_2^0, e_3^0, e_4^0) + S(e_3^0, \Omega(e_1^0, e_2^0, e_4^0)) = 0,$$

which is precisely condition (vi).

\[\square\]

**Corollary 5.5.** The skew-symmetrization $\overline{L}$ of $L$ is a Lie 3-algebra. Its structure maps are explicitly given by

$$\overline{x}_2 = \frac{1}{2} \left( (\Delta \circ \Delta) - (\Delta \circ \Delta) \right),$$

$$\overline{x}_3 = \Omega - \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\sigma_1} \cdot (S \circ_1 (\Delta \circ \Delta) \sigma),$$

$$\overline{x}_4 = S \circ_1 \Omega.$$

**Proof.** Taking into account the skew-symmetry of $\Omega$, the symmetry of $S$, and condition (vi) it is clear that the given terms are precisely the skew-symmetrized structure maps of Definition 4.2.

The structure maps of Corollary 5.5 are precisely the ones given in [12, Theorem 3.10]. Hence, our construction in Proposition 5.4 gives an alternative proof of the cited theorem, based on the methods developed in Section 3.

**Appendix A. Computations**

This appendix contains some long and tedious computations that were removed from the main text as to not disturb its flow.

**A.1. Proof of Lemma 2.2.** We verify that $(\Delta \circ \text{id}) \Delta = (\text{id} \circ \Delta) \Delta$ when evaluated on $l_{4,1}, l_{4,2}$, or $l_{4,3}$ by computing both sides of the equation individually:

$$(\Delta \circ \text{id}) \Delta(l_{4,1}) = 1 \circ (1 \circ (l_{4,1}) - (1 \circ (l_2) + l_2 \circ (1,1))) \circ (l_{3,1}, 1) - (1 \circ (l_2) + l_2 \circ (1,1)) \circ (1, l_{3,1})^{(123)} + (l_3 \circ (1, 1, 1)) \circ (l_{2,1}, 1, 1) + (l_2 \circ (l_{2,1}, 1)) + l_2 \circ (l_{2,1}, 1) + l_2 \circ (1, l_{2,1}) - l_2 \circ (1, l_{2,1})$$

$$= 1 \circ (1 \circ (l_{4,1}) - l_2 \circ (l_{3,1}, 1) - l_2 \circ (1, l_{3,1}))^{(123)} + l_2 \circ (l_{2,1}, 1) + l_2 \circ (1, l_{2,1}) + l_2 \circ (1, l_{2,1})$$

and see (using the associativity of the composite product $\circ$) that they agree. In the same way we compute

$$(\Delta \circ \text{id}) \Delta(l_{4,2}) = 1 \circ (1 \circ (l_{4,2} - (1 \circ (l_2) + l_2 \circ (1,1))) \circ (l_{3,2}, 1) - (1 \circ (l_2) + l_2 \circ (1,1)) \circ (1, l_{3,2}) + (1 \circ (l_2)$$

$$+ l_2 \circ (1, l_{2,1})) \circ (l_{2,1}, 1)^{(123)} + (1 \circ (l_{3,2}) + l_2 \circ (l_{2,1}, 1) + l_{3,1} \circ (1, l_{2,1}) + (l_2 \circ (1, l_{2,1})$$

and see (using the associativity of the composite product $\circ$) that they agree.
\[ l_{3,1} \circ (1, 1, 1) + l_2 \circ (l_{2,1}) + l_{2,1} \circ (1, 1, 1) = \lambda_2 + \lambda_2^{(12)} \]
\[ l_3 \circ (l_{2,1}) + l_2 \circ (1, 1, 1) = \lambda_2 + \lambda_2^{(12)} \]
\[ l_3 \circ (1, 1, 1) = \lambda_2 + \lambda_2^{(12)} \]

and

\[ (\Delta \circ \text{id}) \Delta(l_{3,3}) = 1 \circ (1 \circ (l_{4,3}) + (1 \circ (l_{2,1}) + l_{2,1} \circ (1, 1) \circ (l_{3,2})) + (1 \circ (l_{2,1}) + (1 \circ (l_{2,1}) + l_{2,1} \circ (1, 1) \circ (l_{3,2})) + (1 \circ (l_{2,1}) + l_{2,1} \circ (1, 1) \circ (l_{3,2})) + l_2 \circ (l_{2,1}) + l_2 \circ (l_{2,1}) \]
\[ + (1 \circ (l_{3,2}) - l_2 \circ (l_{2,1}) - l_2 \circ (l_{2,1}) - l_2 \circ (l_{2,1}) = \lambda_2 + \lambda_2^{(12)} + 1 \circ (l_{3,2}) - l_2 \circ (l_{2,1}) - l_2 \circ (l_{2,1}) - l_2 \circ (l_{2,1}) \]

and again we see that both sides agree. This concludes the proof of Lemma 2.2.

A.2. \textbf{Proof of Lemma 3.7} We verify that the structure maps defined in equations (63)–(66) satisfy equations (59)–(60) and hence form a weak Lie 3-algebra:

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) = p \circ (\lambda_2 + \lambda_2^{(12)}) \circ (i, i) \]
\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) = p \circ (\lambda_2 + \lambda_2^{(12)}) \circ (i, i) \]
\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ = p \circ (\lambda_2 + \lambda_2^{(12)} + \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

where \( \lambda_{2,1} \) is the structure map defined in equation (63).

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]

\[ \partial(\lambda_{2,1}) = p \circ (\partial \lambda_{2,1}) \circ (i, i) - p \circ (\partial \lambda_2 \circ (h \circ \lambda_{2,1})) \circ (i, i, i) \]
\[\begin{align*}
\lambda'_4 &= \lambda'_3 + \lambda^{(12)}_4 + \lambda'_2 \circ \lambda'_2, \\
\partial(\lambda'_{3,2}) &= p \circ (\lambda_3 + \lambda_2 \circ_1 (h \circ \lambda_2)) \circ (i, i, i) + p \circ (\lambda^{(12)}_4 + (\lambda_2 \circ_1 (h \circ \lambda_2))^{(12)}) \circ (i, i, i) \\
&= \lambda'_3 + \lambda^{(12)}_4 + \lambda'_2 \circ_1 \lambda'_2, \\
\partial(\lambda'_{3,2}) &= p \circ (\lambda_3 \circ_1 (h \circ \lambda_2) + \lambda_2 \circ_2 (h \circ \lambda_{2,1}) + (\lambda_2 \circ_2 (h \circ \lambda_2))^{(12)}) \circ (i, i, i) \\
&= \lambda'_3 + \lambda^{(12)}_4 + \lambda'_2 \circ_1 \lambda'_2, \\
\lambda'_{3,1} - \lambda^{(12)}_{3,1} &= p \circ (\lambda_3 \circ_1 - \lambda^{(12)}_3 \circ_1) \circ (i, i, i) - p \circ (\lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_1 (h \circ \lambda_2))^{(12)}) \circ (i, i, i) \\
&= \lambda'_3 - \lambda^{(12)}_3 \circ_1 \lambda'_2, \\
\lambda'_{3,1} - \lambda^{(12)}_{3,1} &= p \circ (\lambda_3 \circ_1 - \lambda^{(12)}_3 \circ_1) \circ (i, i, i) - p \circ (\lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_1 (h \circ \lambda_2))^{(12)}) \circ (i, i, i) \\
&= \lambda'_3 - \lambda^{(12)}_3 \circ_1 \lambda'_2, \\
\lambda'_{3,2} - \lambda^{(23)}_{3,2} &= p \circ (\lambda_3 \circ_2 + \lambda_2 \circ_2 (h \circ \lambda_{2,1}))^{(1+12)} + (\lambda_2 \circ_2 (h \circ \lambda_2))^{(12)} \circ (i, i, i) \\
&= \lambda'_3 + (\lambda_2 \circ_2 (h \circ \lambda_{2,1}))^{(12)} + (\lambda_2 \circ_2 (h \circ \lambda_2))^{(12)} \circ (i, i, i) \\
&= \lambda'_3 + \lambda^{(12)}_4 + \lambda'_2 \circ_1 \lambda'_2 - \lambda^{(23)}_4, \\
\lambda'_{3,1} - \lambda^{(12)}_{3,1} &= p \circ (\lambda_3 \circ_1 - \lambda^{(12)}_3 \circ_1) \circ (i, i, i) - p \circ (\lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_1 (h \circ \lambda_2))^{(12)}) \circ (i, i, i) \\
&= \lambda'_3 - \lambda^{(12)}_3 \circ_1 \lambda'_2, \\
\lambda'_{3,2} - \lambda^{(23)}_{3,2} &= p \circ (\lambda_3 \circ_2 + \lambda_2 \circ_2 (h \circ \lambda_{2,1}))^{(1+12)} + (\lambda_2 \circ_2 (h \circ \lambda_2))^{(12)} \circ (i, i, i) \\
&= \lambda'_3 + (\lambda_2 \circ_2 (h \circ \lambda_{2,1}))^{(12)} + (\lambda_2 \circ_2 (h \circ \lambda_2))^{(12)} \circ (i, i, i) \\
&= \lambda'_3 + \lambda^{(12)}_4 + \lambda'_2 \circ_1 \lambda'_2 - \lambda^{(23)}_4.
\end{align*}\]
\[
\begin{align*}
&= p \circ (\lambda_2 \circ_1 \lambda_3;_1 + (\lambda_2 \circ_2 \lambda_3;_1)^{(123)} + \lambda_3 \circ_1 \lambda_2;_1 - \lambda_3;_1 \circ_3 \lambda_2) \circ (i, i, i, i) \\
&= p \circ \left( \lambda_2 \circ_1 (h \circ (\lambda_3 + \lambda_3^{(12)} + \lambda_2 \circ_1 \lambda_2;_1)) \\
&+ (\lambda_2 \circ_2 (h \circ (\lambda_3 + \lambda_3^{(12)} + \lambda_2 \circ_1 \lambda_2;_1)))^{(123)} \\
&- (\lambda_2 \circ_1 \lambda_2 - (\lambda_2 \circ_2 \lambda_2;_1)^{(1-(12)}) \circ_1 (h \circ \lambda_2;_1) + \lambda_3 \circ_1 (h \circ (\lambda_2 + \lambda_2^{(12)})) \\
&+ (\lambda_3 + \lambda_3^{(12)} + \lambda_2 \circ_1 \lambda_2;_1) \circ_3 (h \circ \lambda_2) \right) \circ (i, i, i, i) \\
&= p \circ \left( \lambda_2 \circ_1 \lambda_2 (h \circ \lambda_2;_1) - \lambda_2 \circ_1 ((id - \partial h) \circ \lambda_2) \circ_1 (h \circ \lambda_2;_1) \\
&+ (\lambda_2 \circ_2 (id - \partial h) \circ \lambda_2) (\lambda_3;_1) - (\lambda_3;_1 \circ_3 (id - \partial h) \circ \lambda_2) \circ_1 (h \circ \lambda_2;_1) \\
&- (\lambda_2 \circ_2 ((id - \partial h) \circ \lambda_2) \circ_1 (h \circ \lambda_2;_1))^{(123)} \\
&+ \lambda_3 \circ_1 ((id - \partial h) \circ \lambda_2;_1) + \lambda_2 \circ_1 (h \circ \lambda_2) \circ_1 (id - \partial h) \circ \lambda_2;_1 \\
&- \lambda_2 \circ_2 (h \circ \lambda_2;_1)^{(1-(12))} \circ_1 (id - \partial h) \circ \lambda_2;_1 \\
&- \lambda_3;_1 \circ_3 (id - \partial h) \circ \lambda_2 + \lambda_2 \circ_1 (h \circ \lambda_2;_1) \circ_3 ((id - \partial h) \circ \lambda_2) \right) \circ (i, i, i, i) \\
&= \lambda_2 \circ_1 \lambda_3;_1 \circ_1 \lambda_2;_1 \circ_1 (\lambda_3;_1 - \lambda_2 \circ_1 (h \circ \lambda_2;_1)) \\
&+ (\lambda_2 \circ_2 (i \circ p \circ (\lambda_3;_1 - \lambda_2 \circ_1 (h \circ \lambda_2;_1)))^{(12)} \\
&+ (\lambda_3 + \lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_2 (h \circ \lambda_2))^{(1-(12))} \circ_1 (i \circ p \circ \lambda_2;_1) \\
&- (\lambda_3;_1 - \lambda_2 \circ_1 (h \circ \lambda_2;_1)) \circ_3 (i \circ p \circ \lambda_2) \right) \circ (i, i, i, i) \\
&= \lambda_2 \circ_1 \lambda_3;_1 + (\lambda_2 \circ_2 \lambda_3;_1)^{(123)} + \lambda_2 \circ_2 \lambda_2;_1 \circ_1 \lambda_3;_1 \circ_3 \lambda_2, \\
\lambda_4^{(24)} + \lambda_4^{(23)} &\circ (i, i, i, i) - p \circ \left( \lambda_2 \circ_1 (h \circ \lambda_3;_1) + \lambda_2 \circ_2 (h \circ \lambda_3) + \lambda_3 \circ_1 (h \circ \lambda_2) \\
&- (\lambda_3 \circ_2 (h \circ \lambda_2;_1)^{(1-(12))} + (\lambda_3;_1 \circ_3 (h \circ \lambda_2))^{(123)}) \circ (i, i, i, i) \\
&= p \circ \left( \lambda_2 \circ_1 (h \circ \lambda_3;_1 - \lambda_2 \circ_1 (h \circ \lambda_2;_1) + \lambda_2 \circ_2 (h \circ \lambda_2;_1) - (\lambda_3;_1 \circ_3 \lambda_2) \circ (i, i, i, i) \\
&+ \lambda_2 \circ_2 ((h \circ \lambda_3;_1)^{(23)} + \lambda_2 \circ_1 \lambda_2 - \lambda_2 \circ_2 \lambda_2;_1 + (\lambda_2;_1 \circ_2 \lambda_2;_2))^{(12)} \\
&+ \lambda_2 \circ_2 ((h \circ \lambda_3;_1 + \lambda_3;_1 \circ_2 \lambda_2;_1) + \lambda_2 \circ_1 \lambda_2;_1) \circ_1 (h \circ \lambda_2) + (\lambda_3;_1 + \lambda_2 \circ_1 \lambda_2;_1) \circ_1 (h \circ \lambda_2;_1) \\
&+ \lambda_2 \circ_1 \lambda_2 - (\lambda_2 \circ_2 \lambda_2;_1)^{(1-(12))} \circ_2 (h \circ \lambda_2;_1) - \lambda_3 \circ_2 (h \circ (\lambda_2 + \lambda_2^{(12)})) \\
&+ ((\lambda_3;_1 + \lambda_2 \circ_1 \lambda_2;_1) \circ_2 (h \circ \lambda_2;_1))^{(12)} \\
&+ ((\lambda_3;_1 + \lambda_2 \circ_1 \lambda_2;_1) \circ_3 (h \circ \lambda_2;_1))^{(132)} \right) \circ (i, i, i, i) \\
&= p \circ \left( \lambda_2 \circ_1 (h \circ \lambda_3;_1 + \lambda_2 \circ_2 (h \circ \lambda_2;_1) + (\lambda_2;_1 \circ_2 \lambda_2;_2) \circ (i, i, i, i) \\
&- \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_1) + \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_2))^{(12)} \\
&+ \lambda_2 \circ_2 (h \circ (\lambda_2;_1)) + \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_2))^{(12)} \\
&+ \lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_2 (h \circ \lambda_2))^{(1-(12))} \circ_2 \lambda_2;_1 \\
&+ (\lambda_2;_1 \circ_2 (h \circ \lambda_2;_1)) \circ_2 (h \circ \lambda_2;_1) - (\lambda_2 \circ_1 (h \circ \lambda_2;_1) \circ_2 (h \circ \lambda_2))^{(12)} \\
&+ ((\lambda_2 \circ_1 (h \circ \lambda_2;_1)) \circ_3 (h \circ \lambda_2))^{(132)} - ((\lambda_2 \circ_1 (h \circ \lambda_2;_1)) \circ_3 (h \circ \lambda_2))^{(132)} \right) \circ (i, i, i, i) \\
&+ p \circ \left( \lambda_2 \circ_1 (h \circ \lambda_2;_1) + \lambda_2 \circ_2 (h \circ \lambda_2;_1) + (\lambda_2;_1 \circ_2 \lambda_2;_2) \circ (i, i, i, i) \\
&- \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_1) + \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_2))^{(12)} \\
&+ \lambda_2 \circ_2 (h \circ (\lambda_2;_1)) + \lambda_2 \circ_2 ((\lambda_2;_1 \circ_2 \lambda_2;_2))^{(12)} \\
&+ \lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_2 (h \circ \lambda_2))^{(1-(12))} \circ_2 \lambda_2;_1 \\
&+ (\lambda_2;_1 \circ_2 (h \circ \lambda_2;_1)) \circ_2 (h \circ \lambda_2;_1) - (\lambda_2 \circ_1 (h \circ \lambda_2;_1) \circ_2 (h \circ \lambda_2))^{(12)} \\
&+ ((\lambda_2 \circ_1 (h \circ \lambda_2;_1)) \circ_3 (h \circ \lambda_2))^{(132)} - ((\lambda_2 \circ_1 (h \circ \lambda_2;_1)) \circ_3 (h \circ \lambda_2))^{(132)} \right) \circ (i, i, i, i)
\end{align*}
\]
\[
\begin{align*}
\lambda_2 \circ_1 ((\text{id} - \partial h) \circ \lambda_{3,2}) \\
+ \lambda_2 \circ_1 \left( (\text{id} - \partial h) \circ \left( \lambda_{2,1} \circ_1 (h \circ \lambda_2) + \lambda_2 \circ_2 (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ_2 (h \circ \lambda_2))^{(12)} \right) \right) \\
+ \lambda_2 \circ_2 ((\text{id} - \partial h) \circ \lambda_{3,1}) - \lambda_2 \circ_2 ((\text{id} - \partial h) \circ (\lambda_2 \circ_1 (h \circ \lambda_{2,1}))) \\
- \lambda_3 \circ_1 ((\text{id} - \partial h) \circ \lambda_2) + (\lambda_2 \circ_1 (h \circ \lambda_{2,1})) \circ_1 ((\text{id} - \partial h) \circ \lambda_2) \\
- \lambda_3 \circ_2 ((\text{id} - \partial h) \circ \lambda_{2,1}) \\
- (\lambda_2 \circ_1 (h \circ \lambda_2) - (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)}) \circ_2 ((\text{id} - \partial h) \circ \lambda_{2,1}) \\
- (\lambda_{3,1} \circ_2 ((\text{id} - \partial h) \circ \lambda_2))^{(132)} + ((\lambda_2 \circ_1 (h \circ \lambda_{2,1})) \circ_2 ((\text{id} - \partial h) \circ \lambda_2))^{(12)} \\
- (\lambda_{3,1} \circ_3 ((\text{id} - \partial h) \circ \lambda_2))^{(132)} + ((\lambda_2 \circ_1 (h \circ \lambda_{2,1})) \circ_3 ((\text{id} - \partial h) \circ \lambda_2))^{(132)} = \lambda_4 \circ_1 \lambda_{3,2} + \lambda_2 \circ_2 \lambda_{3,1} - \lambda_{3,1} \circ_1 \lambda'_2 + \lambda_2 \circ_2 \lambda'_{2,1} - (\lambda_{3,1} \circ_2 \lambda'_2)^{(12)} - (\lambda_{3,1} \circ_3 \lambda''_{2})^{(132)} \\
\lambda'_4 + \lambda''_4 = \circ (\lambda_4 + \lambda_4^{(34)}) \circ (i, i, i, i) \\
- \circ (\lambda_2 \circ_1 (h \circ \lambda_3) + (\lambda_2 \circ_2 (h \circ \lambda_3))^{1-(12)} + (\lambda_3 \circ_3 (h \circ \lambda_2))^1-(12)) \circ (i, i, i, i) \\
- \circ (\lambda_2 \circ_2 (h \circ \lambda_2) - (\lambda_2 \circ_1 (h \circ \lambda_2) \circ_2 (h \circ \lambda_2))^1-(12)) \circ (i, i, i, i) \\
- \circ (\lambda_{3,2} \circ_2 \lambda_2)^1-(12) + (\lambda_{3,2} \circ_3 \lambda_{2,1})^{1-(12)} = \circ (i, i, i, i) \\
\end{align*}
\]
If we denote

\[
\begin{align*}
&- \lambda_{2,1} \circ_1 \left( - \lambda_2 \circ_1 (h \circ \lambda_2) + (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \\
+ &\lambda_{2,1} \circ_1 \left( \partial h \circ ( - \lambda_2 \circ_1 (h \circ \lambda_2) + (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)}) \right) \\
+ &\left( \lambda_2 \circ_2 \left( \lambda_{2,1} \circ_1 (h \circ \lambda_2) + \lambda_2 \circ_2 (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \right) \\
+ &\left( \left( \lambda_{2,1} \circ_1 (h \circ \lambda_2) + \lambda_2 \circ_2 (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \right) \\
- &\left( \left( \lambda_{2,1} \circ_1 (h \circ \lambda_2) + \lambda_2 \circ_2 (h \circ \lambda_{2,1}) + (\lambda_{2,1} \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \right) \\
- &\left( - \lambda_2 \circ_1 (h \circ \lambda_2) + (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \\
- &\lambda_{3,2} \circ_1 \left( (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \\
+ &\lambda_{3,2} \circ_2 \left( - \lambda_2 \circ_1 (h \circ \lambda_2) + (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \right) \\
- &\lambda_3 \circ_1 \circ (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \\
+ &\lambda_3 \circ_2 \circ (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \\
+ &\lambda_3 \circ_3 \circ (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \\
- &\lambda_3 \circ_1 \circ (\lambda_2 \circ_2 (h \circ \lambda_2))^{1-(12)} \\
\end{align*}
\]

This proves Lemma 3.7.
A.3. Proof of Lemma 4.3 Since the components of \( \mathbf{f} \) are clearly skew-symmetric, the proof boils down to checking equations (32)–(35). The first three are easily verified:

\[
\begin{align*}
\partial(f_1) &= \partial f_1 = 0, \\
\partial(f_2) &= \frac{1}{2} \partial (f_2 - f_2^{(12)}) \\
&= \frac{1}{2} \left( f_1 \circ \lambda_2 - \lambda_2 \circ (f_1, f_1) - f_1 \circ \lambda_2^{(12)} + \lambda_2 \circ (f_1, f_1)^{(12)} \right) \\
&= f_1 \circ X_2 - X_2 \circ (f_1, f_1),
\end{align*}
\]

\[
\partial(f_3) = \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \partial(f_3)^{\sigma} - \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \left( \partial(f_2, 1) \circ \lambda_2 - \partial(\lambda_2, 1) \circ (f_2, f_1) + \lambda_2^{(12)} \circ (f_1, \partial(f_2)) \right) \\
&= \frac{1}{6} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \left( f_1 \circ \lambda_3 + f_2 \circ \lambda_2 - f_2 \circ \lambda_2 + (f_2 \circ \lambda_2^{(12)}) \\
&\quad + \lambda_2^{(12)} \circ (f_1, f_2) + \lambda_2 \circ (f_1, f_2)^{(12)} - \lambda_2 \circ (f_1, f_1) \right) \\
&= f_1 \circ X_3 - X_3 \circ (f_1, f_1, f_1) \\
&+ \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot (2 f_2 \circ \lambda_2 - 4 f_2 \circ \lambda_2)^{\sigma} + \frac{1}{24} \sum_{\sigma \in S_3} (-1)^{\sigma} \cdot \left( f_2 + f_2^{(12)} \circ \lambda_2 \right)^{\sigma} \\
&= f_1 \circ X_3 + f_2 \circ \lambda_2 - f_2 \circ \lambda_2 + (f_2 \circ \lambda_2^{(12)}) \\
&\quad + \lambda_2^{(12)} \circ (f_1, f_1) - \lambda_2 \circ (f_1, f_1)^{(12)} - X_3 \circ (f_1, f_1, f_1).
\]

A trick to showing equation (35) is to define

\[
\mathbf{f}_4 := \frac{1}{24} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot f_4^{\sigma} + \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot \left( f_2 \circ \lambda_3 - f_2 \circ \lambda_3 - \lambda_2 \circ (f_2, f_1) + \lambda_2 \circ (f_1, f_3) \\
&\quad + f_3 \circ \lambda_2 + f_3 \circ \lambda_2 + \lambda_3 \circ (f_1, f_2) + \lambda_3 \circ (f_1, f_2)^{(12)} \right),
\]

which for degree reasons must vanish, and verify

\[
0 = \partial(f_4) = \frac{1}{24} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot \partial(f_4)^{\sigma} \\
\quad \left( \partial(f_2, 1) \circ \lambda_2 + f_2 \circ \lambda_3 - f_2 \circ \lambda_3 - \lambda_2 \circ (f_2, f_1) + \lambda_2 \circ (f_1, f_3) \\
&\quad + f_3 \circ \lambda_2 + f_3 \circ \lambda_2 + \lambda_3 \circ (f_1, f_2) + \lambda_3 \circ (f_1, f_2)^{(12)} \right),
\]

\[
= \frac{1}{24} \sum_{\sigma \in S_4} (-1)^{\sigma} \cdot \left( f_2 \circ \lambda_3 + 3 f_2 \circ \lambda_3 \right)^{\sigma}.
\]
\[
\begin{align*}
&+ \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \cdot \left( \begin{array}{c}
- f_2 - f_2^{(12)} + f_1 \circ \lambda_2, \quad (f_1, f_1, f_1) \\
\end{array} \right) \\
&+ \frac{1}{48} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \cdot \left( \begin{array}{c}
- f_2 - f_2^{(12)} + f_1 \circ \lambda_2, \quad (f_1, f_1, f_1) \\
\end{array} \right)
\end{align*}
\]
This concludes the proof of Lemma 4.3.

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