On Bipartite Operators Defined by Completely Different Permutations

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We introduce a class of bipartite operators acting on $\mathcal{H} \otimes \mathcal{H}$ ($\mathcal{H}$ being an $n$-dimensional Hilbert space) defined by a set of $n$ Completely Different Permutations CDP. Bipartite operators are of particular importance in quantum information theory to represent states and observables of composite quantum systems. It turns out that any set of CDPs gives rise to a certain direct sum decomposition of the total Hilbert space which enables one simple construction of the corresponding bipartite operator. Interestingly, if set of CDPs defines an abelian group then the corresponding bipartite operator displays an additional property – the partially transposed operator again corresponds to (in general different) set of CDPs. Therefore, our technique may be used to construct new classes of so called PPT states which are of great importance for quantum information. Using well known relation between bipartite operators and linear maps one use also construct linear maps related to CDPs.

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I. INTRODUCTION

Quantum entanglement is a basic physical resource for modern quantum technologies like quantum teleportation, quantum computation, quantum communication and quantum cryptography \cite{1,2}. It is therefore clear that detailed analysis of the mathematical structure of quantum states represented by bipartite density operators is of great importance for quantum information theory. However, in general it is very hard to check whether a given density matrix describing a quantum state of the composite system is separable or entangled (so called separability problem). It was shown by Gurvits that the separability problem is NP hard \cite{3}.

There are several operational criteria which provide necessary conditions for separability and sufficient conditions for entanglement \cite{2,4,5}. The most simple is the celebrated Peres-Horodecki criterion \cite{2} based on the operation of partial transposition: if a state $\rho_{AB}$ is separable then its partial transposition $(\mathbb{I} \otimes T)\rho$ is positive. States which are positive under partial transposition are called PPT states. Clearly each separable state is necessarily PPT but the converse is not true. These two sets coincide only for $2 \otimes 2$ and $2 \otimes 3$ systems \cite{2}.

The paper is organised as follows. In Section II we introduce the concept of Completely Different Permutations CDP and we derive the basic properties of maximal sets of Completely Different Permutations. It appears that these stets have many interesting and useful properties, in particular it is shown that any commutative set of CDP is necessarily an abelian group. In next Section we consider groups of CDP and shown in particular that wellknown Cayley construction of permutational representations of finite groups leads to the groups of CDP. In Section IV we define a class of tensor product matrices, which we call CDP matrices, whose construction is based on the properties of sets of CDPs. The construction is a generalisation of construction given in paper \cite{6}, where the cyclic group of order $n$, a particular case of CDP, where used. Using the properties of sets of CDPs we derive the basic properties tensor product matrices. In particular we derive the direct sum decomposition of the carrier space of CDP matrices and the corresponding decomposition of CDP matrices into a direct sum of orthogonally supported operators. Further we derive some properties of partially transposed CDP matrices. It appears that, under assumptions of commutativity of the set of CDPs defining the CDP matrix, the partially transposed CDP matrix is again a CDP matrix with, in general, a transformed set of CDP. This allows formulate a conditions for PPT property of the CDP matrix. In next subsections we give a realignment and majorisation criteria for tensor product matrices. In the last V section we consider the properties of some linear maps: particular Irreducibly Covariant Quantum Channels, the Reduction map and of its generalisation the Breuer-Hall map \cite{12,13} and we describe their relation, via Choi-Jamiolkowski isomorphism, to CDP matrices.

II. COMPLETELY DIFFERENT PERMUTATIONS

In order to generalize the idea of circulant states introduced in \cite{6} we introduce a concept of Completely Different Permutations CDP. In what follows we denote by $S(n)$ the symmetric group $S(n)$ and $\sigma \in S(n)$ denotes a permutation with the corresponding matrix representation $m(\sigma) = (\delta_{\sigma^{-1}(i)\jmath})$. A cyclic group generated by a cycle permutation $c$ of length $n$ will be denoted $C(n) = \{ c^i = (c^i) : i = 0, 1, \ldots, n-1 \} \subset S(n)$ where $c^0 = \text{id}, c^j(j) = j+n, i \equiv i+j \mod(n)$.
Definition 1 Two permutations $\sigma, \rho \in S(n)$ are Completely Different CDP iff $\sigma(i) \neq \rho(i)$ for any $i = 1, \ldots, n$.

It is easy to check that

Proposition 1 Permutations $\sigma, \rho \in S(n)$ are CDP iff $\text{tr}(m(\sigma^{-1})m(\rho)) = 0$.

Hence $\sigma, \rho \in S(n)$ are CDP iff $m(\sigma)$ and $m(\rho)$ are mutually orthogonal with respect to the Frobenius scalar product in $M(n, \mathbb{C})$. Therefore, one may equivalently call a set of CDP a set of Mutually Orthogonal Permutations (MOP).

One can check that

Proposition 2 Any set of CDP in $S(n)$ contain at most $n$ elements.

In what follows we consider only maximal sets of CDPs containing $n$ permutations and they will be denoted $\Sigma_n = \{\sigma_i\}_{i=1}^n$. The sets of CDPs have many interesting properties and the first one is the possibility to enumerate the permutations from CDP $\Sigma_n$ in a very convenient and useful way. The structure of a set of CDPs allows to enumerate a set $\{\sigma_1, \ldots, \sigma_n\}$ as follows

$$\sigma_i(1) = i, \quad i = 1, \ldots, n.$$ (1)

Using this convention one finds

Proposition 3 Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be an abelian set of CDP. Then

$$\forall i, j \quad \sigma_i(j) = \sigma_j(i),$$ (2)

and

$$\forall k \quad \forall i, j \quad \sigma_{\sigma_k(i)}(j) = \sigma_{\sigma_k(j)}(i).$$ (3)

Proof. If $\Sigma_n = \{\sigma_i\}_{i=1}^n$ is an abelian CDP, then

$$\forall k \quad \forall i, j \quad \sigma_i \sigma_j(k) = \sigma_j \sigma_i(k),$$ (4)

in particular it holds for $k = 1$ which in our enumeration $\sigma_i(1) = i$ gives

$$\forall i, j \quad \sigma_i \sigma_j(1) = \sigma_j \sigma_i(1) \Rightarrow \sigma_i(j) = \sigma_j(i).$$ (5)

The second statement simply follows. \hfill \blacksquare

Let us consider some examples of sets of CDPs. The simplest one is the group $S(2) = \{\text{Id}, (12)\}$ which is obviously a commutative set of CDPs. The next example is also simple but less trivial

Example 1 In the group $S(3)$ we have two sets of CDPs

$$\Sigma_3 = \{\sigma_1 = (23), \sigma_2 = (12), \sigma_3 = (13)\}, \quad C(3) = \{c^i = (012)^i : i = 0, 1, 2\}.$$ (6)

Example 2 Any cyclic group generated by a cycle permutation $c = (01\ldots n-1)$ of length $n$

$$C(n) = \{c^i = (c)^i : i = 0, 1, \ldots, n-1\}$$ (7)

is a set of CDPs.

Example 3 The abelian group

$$V(4) = \{\sigma_1 = \text{id}, \sigma_2 = (12)(34), \sigma_3 = (13)(24), \sigma_4 = (14)(23)\} \subset S(4)$$ (8)

defines a set of CDPs. However, the following sets of CDPs

$$\Sigma_4 = \{\sigma_1 = (34), \sigma_2 = (12), \sigma_3 = (13)(24), \sigma_4 = (14)(23)\} \subset S(4),$$ (9)

and

$$\Sigma'_4 = \{\sigma_1 = (234), \sigma_2 = (124), \sigma_3 = (132), \sigma_4 = (143)\} \subset S(4)$$ (10)

are not groups.
The CDP is denoted \( m \in E \) \( \equiv \) \( \sigma \in M \). The matrices are mutually orthogonal e.i. in particular, the matrices \( \{ \sigma_i \}_{i=1}^n \) are permutation matrices, which are natural matrix representations of corresponding permutations \( \sigma \in \Sigma_n \). Due to our enumeration convention we have always \( \sigma_1(1) = 1 \). If the identity permutation \( id = \sigma_1 \) is in a set of CDPs \( \Sigma_n \) then obviously all \( j = 1, \ldots, n \) are fixed points of \( \sigma_1 \) and all remaining \( \sigma_i \in \Sigma_n \) have no fixed points.

**Definition 2** For any set of CDPs \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) we define a set of matrices \( m(E) = \{ m(E_j) \} \), \( j = 1, \ldots, n \)

\[
\forall j = 1, \ldots, n \quad m(E_j) = \sum_{k=1}^n e_{\sigma_k(1)\sigma_k(j)} = \sum_{k=1}^n e_{k\sigma_k(j)} \in \mathbb{M}(n, \mathbb{C})
\]

where \( \{ e_{ij} \}_{i,j=1}^n \) is a natural basis of the linear space \( \mathbb{M}(n, \mathbb{C}) \). In particular we have \( m(E_1) = 1_n \in \mathbb{M}(n, \mathbb{C}) \).

It is not difficult to check that the structure of the set of CDPs \( \Sigma_n \) implies

**Proposition 6** Let \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) be a set of CDPs then for any \( j = 1, \ldots, n \) the matrices \( m(E_j) = \sum_{k=1}^n e_{k\sigma_k(j)} \) are mutually orthogonal e.i.

\[
\text{tr}(m(E_i) \cdot m(E_j)) = \delta_{ij} n.
\]

The matrices \( m(E_j) \) are permutation matrices, which are natural matrix representations of corresponding permutations denoted \( E_j \in S(n) \) which are of the form

\[
E_j = \begin{pmatrix}
\sigma_1(j) & \sigma_2(j) & \cdots & \sigma_n(j)
\end{pmatrix},
\]

\( E_j(\sigma_k(j)) = k \), \( \sigma_k(1) = 1 \). Moreover the set of permutations \( E \equiv \{ E_j : j = 1, \ldots, n \} \) is also a set of CDPs, which is a direct consequence of the mutual orthogonality of the matrices \( m(E_j) \).

**Definition 3** The CDP \( E \equiv \{ E_j : j = 1, \ldots, n \} \) will be called the set of permutations conjugated to the set of CDPs \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \), which generates the matrices \( m(E_j) \).
The set of matrices \( m(E) \equiv \{ m(E_j) : j = 1, \ldots, n \} \), defined in Def. \( \square \), which is a matrix representation of the set \( E \) has the following nice property

**Proposition 7** Let \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) is an abelian set of CDPs. Then the set of matrices \( m(E) \equiv \{ m(E) : j = 1, \ldots, n \} \) is an abelian group with the following composition law

\[
\forall i, j = 1, \ldots, n \quad m(E_i) m(E_j) = m(E_{\sigma_i(j)}),
\]

which follows directly from the definition of \( m(E_i) \). From Def. \( \square \) it follows that \( m(E_1) = 1_n \in M(n, \mathbb{C}) \).

Now because the natural matrix representation of \( S(n) \) is faithful, we get

**Corollary 1** If \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) is an abelian set of CDPs, then the set of CDPs \( E \equiv \{ E_j : j = 1, \ldots, n \} \) is an abelian group with composition law of the form

\[
\forall i, j = 1, \ldots, n \quad E_i E_j = E_{\sigma_i(j)}.
\]

On the other hand the set \( E \) is related to the abelian set of CDPs \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \) in the following way

**Proposition 8** Suppose that \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) is an abelian set of CDPs, then the permutations in the set \( E \equiv \{ E_j : j = 1, \ldots, n \} \) are related to the permutations of the set \( \Sigma_n \) in a very simple way

\[
\forall i = 1, \ldots, n \quad E_i = \sigma_i^{-1} \Rightarrow E = \Sigma_n^{-1},
\]

which follows from the relation for its matrix representations

\[
\forall i = 1, \ldots, n \quad m(E_i) = m(\sigma_i^{-1}) \Rightarrow M(E) = M(\Sigma_n^{-1})
\]

and this is a simple consequence of Prop. \( \square \).

Finally, one arrives at the following

**Theorem 1** Let \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) be a set of CDPs. Then the set \( \Sigma_n \) is abelian if and only if it is an abelian group of permutations realised as a set of CDPs.

**Remark 1** Note that a finite groups may be realised as a groups of permutations, which are not CDPs. For example the group \( V(4) \) in Example \( \square \) is isomorphic to the following group of permutations \( G = \{ id, \ (12), \ (34), \ (12)(34) \} \), which are not CDP.

The correspondence between the sets \( \Sigma_n \) and \( E = \{ E_j : j = 1, \ldots, n \} \) is not unique, in fact we have

**Proposition 9** Let \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \subset S(n) \) be a set of CDPs with \( m(E) \equiv \{ m(E_j) : j = 1, \ldots, n \} \), defined in Def. \( \square \). Suppose that \( \rho \in S(n) : \rho(1) = 1 \) and \( \delta \in S(n) \) is such that

\[
m(\delta) M(E) m(\delta)^{-1} = M(E),
\]

i.e. \( m(\delta) \) belongs to the centraliser of the set \( m(E) \), then the set of CDPs \( \Sigma'_n = \{ \sigma'_i = \delta \sigma_i \rho \}_{i=1}^n \) is such that

\[
m(E'_i) = m(\delta) m(E_{\rho(i)}) m(\delta)^{-1} \Rightarrow m(E'_i) = (E),
\]

so the set of CDPs \( \Sigma_n \) and \( \Sigma'_n \) generates the same set \( M(E) \) and consequently the same conjugated set of CDPs \( E \).

Let us consider some examples.

**Example 5** In the group \( S(3) \) we have two sets of CDPs

\[
\Sigma_3 = \{\sigma_1 = (23), \sigma_2 = (12), \sigma_3 = (13)\},
\]

and

\[
\Sigma'_3 = \{\sigma'_1 = id, \sigma'_2 = (123), \sigma'_3 = (132)\} = \Sigma_3(23),
\]

such that

\[
M(E') = M(E) \iff E' = E = \Sigma_3.'
\]
Example 6 For \( \Sigma_4 = \{\sigma_1 = (34), \sigma_2 = (12), \sigma_3 = (13)(24), \sigma_4 = (14)(23)\} \subset S(4) \) we have
\[
E_1 = id, \quad E_2 = (12)(34), \quad E_3 = (1324), \quad E_4 = (1423),
\]
so in this case we have
\[
E = \{E_j : j = 1, \ldots, 4\} = C(4) = \{(1324)^k : k = 1, \ldots, 4\},
\]
so it is a cyclic group and the matrices \( \{m(E_j) : j = 1, \ldots, n\} \) are simply natural matrix representation of this group.

Example 7 Let \( \Sigma_4 = \{\sigma_1 = (234), \sigma_2 = (124), \sigma_3 = (132), \sigma_4 = (143)\} \subset S(4) \), then
\[
E_1 = id, \quad E_2 = (13)(24), \quad E_3 = (14)(23), \quad E_4 = (12)(34),
\]
and we have
\[
E = \{E_j : j = 1, \ldots, 4\} = V(4),
\]
so again it is an abelian group although the set \( \Sigma_4 \) is neither group nor a commutative set.

Example 8 The set
\[
\Sigma_5 = \{\sigma_1 = id, \sigma_2 = (12)(345), \sigma_3 = (13)(542), \sigma_4 = (14)(352), \sigma_5 = (15)(243)\} \subset S(5)
\]
is such that
\[
E = \{E_j : j = 1, \ldots, 5\} = \Sigma_5.
\]

In the next section we will consider sets of CDPs which are groups.

III. GROUPS OF CDPs

In previous section we have presented some examples of sets of CDPs which were groups and we have proved a remarkable property of abelian sets of CDPs, which are always groups. In general finite groups are very rich source of sets of CDPs, which follows from well-known construction of permutation representations of finite groups. In fact we have

Proposition 10 Let \( G = \{g_i : i = 1, \ldots, n\} \) be a finite group. Then its regular representation, which is in fact a permutation representation
\[
R(g_i) \equiv \sigma_i \equiv \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_i g_1 & g_i g_2 & \cdots & g_i g_n \end{pmatrix} \in S(n)
\]
is such that the group of permutations \( R(G) = \{R(g_i) : i = 1, \ldots, n\} \subset S(n) \) is a set of CDPs.

Proof. Suppose that for some \( k = 1, \ldots, n \)
\[
R(g_i)(g_k) = R(g_j)(g_k) \iff g_ig_k = g_jg_k \iff g_i = g_j,
\]
so the permutations \( R(g_i) = \sigma_i \) and \( R(g_j) = \sigma_j \) are completely different for any \( i \neq j = 1, \ldots, n \).

The cyclic groups in Examples 2, 3 were, in fact regular representations. The smallest nonabelian group is \( S(3) \), of rank 6 and its regular representation may be presented as a set of CDPs as follows

Example 9 \( R(S(3)) = \{\sigma_1 = id, \sigma_2 = (123)(456), \sigma_3 = (132)(465), \sigma_4 = (14)(26)(53), \sigma_5 = (15)(24)(36), \sigma_6 = (16)(25)(34)\} \).

Proposition 11 If \( \Sigma_n = \{\sigma_i\}_{i=1}^n \subset S(n) \) is a group of CDPs, then

1. the composition rule in the group \( \Sigma_n \) has remarkable simple form
\[
\sigma_i \sigma_j = \sigma_{\sigma_i(j)}
\]
i.e. the matrix \( M = (\sigma_i(j)) \) describe the table of composition of the group \( \Sigma_n = \{\sigma_i\}_{i=1}^n \),
2. the inverse elements are the following
\[(σ_i)^{-1} = σ_j \iff σ_i(j) = 1 = σ_j(i), \]  
(38)

3. \(σ_1 = id\) and all the remaining elements \(σ_i \in Σ_n\) have no fixed points.

**Proof.** a) For arbitrary \(σ_i, σ_j\) belonging to a set of CDPs \(Σ_n\) there exist \(σ_k \in Σ_n\) such that
\[σ_iσ_j = σ_k \Rightarrow σ_iσ_j(1) = σ_k(1) \Rightarrow σ_i(j) = k, \]  
(39)

because in our enumeration, we have \(∀i = 1, \ldots, n\). The remaining statements of the Proposition follow easily from the structure of a set CDPs. ■

If \(Σ_n = \{σ_i\}_{i=1}^n \subset S(n)\) is a group of CDPs, then its group structure strongly determines the structure of the set of CDPs \(E = \{E_j : j = 1, \ldots, n\}\).

**Proposition 12** If \(Σ_n = \{σ_i\}_{i=1}^n \subset S(n)\) is a group of CDPs, then

1. the set of CDPs \(E = \{E_j : j = 1, \ldots, n\}\) is a group isomorphic with the group \(Σ_n\). The isomorphism is given by the following map
\[f(σ_i) = E_i, \quad i = 1, \ldots, n \]  
(40)

which satisfy the isomorphism relation \(f(σ_i)f(σ_j) = f(σ_iσ_j)\) because, one can check that, we have
\[m(E_i)m(E_j) = m(E_{σ_iσ_j}), \]  
(41)

so the composition rule for the set of CDPs \(E = \{E_j : j = 1, \ldots, n\}\) is the same as for the group \(Σ_n = \{σ_i\}_{i=1}^n\).

If the group \(Σ_n\) is abelian then \(Σ_n = E\).

2. the matrix groups \(M(Σ_n) = \{m(σ_i)\}_{i=1}^n\) and \(M(E) = \{m(E_j) : j = 1, \ldots, n\}\) are mutually commutant e.i. we have
\[∀i, j = 1, \ldots, n \quad m(σ_i)m(E_j) = m(E_j)m(σ_i). \]  
(42)

note that these groups mutually commute even if the set of CDPs \(Σ_n\) is not commutative.

**Proof.** 2. We have \(∀i, j = 1, \ldots, n\)
\[m(σ_i)m(E_j) = \sum_{k=1}^n e_{kσ^{-1}_i(k)} \sum_{l=1}^n e_{σ_l(j)} = \sum_{k,l=1}^n δ_{σ_l^{-1}(k)}e_{kσ_l^{-1}(k)j} = \sum_{k=1}^n e_{kσ_l^{-1}(k)j} \]  
(43)

and from composition rule in the group \(Σ_n = \{σ_i\}_{i=1}^n\) we get
\[m(σ_i)m(E_j) = \sum_{k=1}^n e_{kσ_l^{-1}(k)j}. \]  
(44)

On the other hand we have
\[m(E_j)m(σ_i) = \sum_{k=1}^n e_{kσ_l(j)} \sum_{l=1}^n e_{σ_l^{-1}(l)} = \sum_{k,l=1}^n δ_{σ_l(j)}e_{kσ_l^{-1}(l)} = \sum_{k=1}^n e_{kσ_l^{-1}(l)j}. \]  
(45)

Let us check this on some examples. Let us check this on some examples.

**Example 10** If
\[Σ_3 = S(3) = \{σ_1 = id, \quad σ_2 = (123)(456), \quad σ_3 = (132)(465), \quad σ_4 = (14)(26)(53), \quad σ_5 = (15)(24)(36), \quad σ_6 = (16)(25)(34)\}, \]
then
\[E = \{E_1 = id, \quad E_2 = (132)(456), \quad E_3 = (123)(465), \quad E_1 = (14)(25)(63), \quad E_5 = (15)(26)(34), \quad E_6 = (16)(24)(35)\}, \]
so it is again the group \(S(3)\) but differently embedded in \(S(6)\).
Example 11 Let $\Sigma_n = C(n) = \{c^i = c^i : i = 0, 1, \ldots, n - 1\} \subset S(n)$ where $c = (012\ldots n-1) \Rightarrow c^k(t) = t +_n k$. Then one can check that

$$E_i = (c^i)^{-1},$$

and therefore in this case we have $E = \Sigma_n = C(n)$, in agreement with Prop. 8.

From Prop. 9 and Example 5 we know that a given set $M(E)$ may be generated by different sets of CDPs, we have however

Proposition 13 Let $M(E) \equiv \{m(E_j) : j = 1, \ldots, n\}$ be a an abelian set of matrices, so in fact an abelian group (see Th. 7) generated by a set of CDPs $\Sigma_n = \{\sigma_i\}_{i=1}^n$, which may be neither abelian nor group. Then there exists an abelian set of CDPs $\Sigma_n' = \{\sigma_i'\}_{i=1}^n$ such that

$$M(E') = M(E),$$

in particular, when $id \in \Sigma_n$ then $\Sigma_n$ is an abelian CDP and if $id \notin \Sigma_n$ then $\Sigma_n' = \Sigma_n\sigma_1^{-1}$, where $\sigma_1 \in \Sigma_n$ and $\sigma_1(1) = 1$ and such an element always exist in the set of CDPs $\Sigma_n$. Thus any abelian $M(E)$ is always generated by some abelian set of CDPs $\Sigma_n'$. 

Remark 2 The Example 5 illustrates this in case of the group $S(3)$.

The group structure of a set of CDPs allows to define a permutation, which is in natural way connected with its group properties

Definition 4 Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs with $\sigma_1 = id$, then we define

$$\xi^\Sigma \in S(n) : \xi^\Sigma(i) = j \iff \sigma_i(j) = 1 = \sigma_j(i),$$

which is well defined because any $\sigma_i \in \Sigma_n$ has only one inverse. For a given permutation $\sigma_i \in \Sigma_n$, the permutation $\xi^\Sigma$ shows what is the index of the inverse permutation $\sigma_j \in \Sigma_n$ e.i.

$$(\sigma_i)^{-1} = \sigma_j = \sigma_{\xi^\Sigma(i)}.$$

Proposition 14 Let $\xi^\Sigma \in S(n)$ be a permutation defined as above and $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs, then

a) 

$$\xi^\Sigma(1) = 1, \quad \xi^\Sigma(i) = j \iff \xi^\Sigma(j) = i, \quad \sigma_i\xi^\Sigma(i) = 1,$$

b) 

$$\xi^\Sigma = \left( \begin{array}{ccc} 1 & i & \ldots \sigma_i^{-1}(1) & \ldots \sigma_n^{-1}(1) \\ 1 & (i_1) & \ldots (i_k) & \ldots (i_{k+1}j_{k+1}) \ldots (i_t) \end{array} \right) = (1)(i_1)\ldots(i_k)\ldots(i_{k+1}j_{k+1})\ldots(i_t),$$

where $i_1, \ldots, i_k$ are such that $(\sigma_{i_p})^{-1} = \sigma_{i_p}$ and $(\sigma_{i_l})^{-1} = \sigma_{j_l}$.

From these properties of the permutation $\xi^\Sigma \in S(n)$ and from Cor. 1 one deduce easily

Corollary 2 Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $E = \{E_i : i = 1, \ldots, n\}$ the conjugated set of permutations, then

$$E_i^{-1} = E_{\xi^\Sigma(i)},$$

so the permutation $\xi^\Sigma$ shows also what is the index of the inverse permutation $E_i \in E$.

Let look on some examples.

Example 12 Let $\Sigma_n = C(n) = \{c^i = c^i : i = 0, 1, \ldots, n - 1\} \subset S(n)$ where $c = (012\ldots n-1) \Rightarrow c^k(t) = t +_n k$. Then $(c^k)^{-1} = c^{n-k}$ and

$$\xi^\Sigma(k) = n - k.$$

Example 13 It is clear that in case of the group $S(3)$, realised as a set of CDPs in regular representation

$$\Sigma_n = R(S(3)) = \{\sigma_1 = id, \sigma_2 = (123)(456), \sigma_3 = (132)(465), \sigma_4 = (1 4)(26)(53), \sigma_5 = (15)(24)(36), \sigma_6 = (16)(25)(34)\},$$

the permutation $\xi^\Sigma$ has very simple form $\xi^\Sigma = (23)$. 

IV. CDP MATRICES. GENERALISATION OF THE CIRCULANT MATRICES.

A. CDP Matrices.

Observe, that any set of CDPs induces a decomposition of $\mathbb{C}^n \otimes \mathbb{C}^n = \text{span}\{e_i \otimes e_j\}$ into a direct sum on $n$-dimensional subspaces $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-1}$, where $\mathcal{H}_k = \text{span}\{e_i \otimes e_{\sigma_k(i)}\}$. The facts, that permutations are pairwise completely different guarantees that $\mathcal{H}_k \perp \mathcal{H}_l$ for $k \neq l$.

We define the following class of operators over tensor product which we will call CDP matrices.

**Definition 5** Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $A = \{A^k = (a_{ij}^k) \in M(n, \mathbb{C}) : k = 1, ..., n\}$ a set of matrices. Then we define

$$
\rho[A, \Sigma_n] = \sum_{i,j=1}^n \sum_{k=1}^n a_{ij}^k e_{ij} \otimes e_{\sigma(i)\sigma(j)} = \sum_{k=1}^n \rho[A^k, \Sigma_n] = \sum_{i,j=1}^n e_{ij} \otimes B_{ij}(A, \Sigma_n) \in M(n^2, \mathbb{C}),
$$

where

$$
\rho[A^k, \sigma_k] = \sum_{i,j=1}^n a_{ij} e_{ij} \otimes e_{\sigma_k(i)\sigma_k(j)} \in M(n^2, \mathbb{C}), \quad B_{ij}(A, \Sigma_n) = \sum_{k=1}^n a_{ij} e_{\sigma_k(i)\sigma_k(j)} \in M(n, \mathbb{C}).
$$

Operators $\rho[A^k, \sigma_k]$ are supported on $\mathcal{H}_k$. When a set of CDPs $\Sigma_n$ is a cyclic group $C(n) = \{c^i = (c)^i : i = 0, 1, ..., n-1\} \subset S(n)$ where $c = (012...n-1)$, we recognize the definition of the circulant matrices from the paper \[.]

**Example 14** Take $n = 4$ and consider two sets of CDPs: circulant one $C(4) = \{\text{id}, c, c^2, c^3\}$, with $c = (0123)$, and $V(4) = \{\text{id}, (01)(23), (02)(13), (03)(12)\}$. For $C(4)$ one finds the following decomposition of the total Hilbert space

$$
\mathbb{C}^4 \otimes \mathbb{C}^4 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,
$$

with

$$
\mathcal{H}_0 = \text{span}\{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\},
$$

$$
\mathcal{H}_1 = \text{span}\{e_0 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_0\},
$$

$$
\mathcal{H}_2 = \text{span}\{e_0 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_3 \otimes e_2\},
$$

$$
\mathcal{H}_3 = \text{span}\{e_0 \otimes e_3, e_1 \otimes e_0, e_2 \otimes e_1, e_3 \otimes e_2\}.
$$

One finds the corresponding bipartite operator

$$
A_1 = \begin{pmatrix}
a_{00} & \cdots & a_{01} & \cdots & a_{02} & \cdots & a_{03} \\
\cdot & b_{00} & \cdots & \cdot & b_{01} & \cdots & b_{02} \\
\cdot & \cdot & c_{00} & \cdots & \cdot & c_{01} & \cdots & c_{03} \\
\cdot & \cdot & \cdot & d_{00} & \cdots & d_{01} & \cdots & d_{03} \\
\cdot & \cdot & \cdot & \cdot & d_{10} & d_{11} & \cdots & d_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdots & a_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{11} & \cdots & b_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_{11} & \cdots & c_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & c_{20} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_{21} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_{22} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_{23} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_{30} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{31} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{32} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{33} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & c_{30} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & c_{31} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & c_{32} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & c_{33} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & d_{30} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & d_{31} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & d_{32} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & d_{33} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & a_{33}
\end{pmatrix}.
$$
For $V(A)$ one finds the following decomposition of the total Hilbert space
\[
\mathbb{C}^4 \otimes \mathbb{C}^4 = \mathcal{H}_0' \oplus \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \mathcal{H}_3',
\] (58)
with
\[
\begin{align*}
\mathcal{H}_0' &= \text{span} \{e_0 \otimes e_0, e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3\}, \\
\mathcal{H}_1' &= \text{span} \{e_0 \otimes e_1, e_1 \otimes e_0, e_2 \otimes e_3, e_3 \otimes e_2\}, \\
\mathcal{H}_2' &= \text{span} \{e_0 \otimes e_2, e_2 \otimes e_0, e_3 \otimes e_1, e_1 \otimes e_3\}, \\
\mathcal{H}_3' &= \text{span} \{e_0 \otimes e_3, e_1 \otimes e_2, e_2 \otimes e_1, e_3 \otimes e_0\}.
\end{align*}
\]

One finds the corresponding bipartite operator

\[
A_k = \begin{pmatrix}
a_{00} & a_{01} & \cdots & a_{03} \\
\cdot & b_{01} & \cdots & a_{03} \\
\cdot & \cdot & c_{01} & \cdots & a_{03} \\
\cdot & \cdot & \cdot & d_{01} & \cdots \\
\cdot & \cdot & \cdot & a_{10} & b_{11} & \cdots \\
\cdot & \cdot & \cdot & \cdot & a_{11} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{12} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{13} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{20} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{21} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{22} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{23} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{30} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{31} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{32} \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_{33}
\end{pmatrix}.
\] (59)

Due to the CDP structure of the set $\Sigma_n = \{\sigma_i\}_{i=1}^n$ the matrices $\rho[A, \Sigma_n]$ have very the following spectral properties.

**Proposition 15** Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $A = \{A^k = (a_{ij}^k) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ a set of arbitrary matrices, then

a) The matrix $\rho[A, \Sigma_n] = \sum_{k=1}^n \rho[A^k, \sigma_k]$ is in fact a direct sum of operators, because we have

\[
k \neq l \Rightarrow \rho[A^k, \sigma_k] \rho[A^l, \sigma_l] = 0,
\] (60)

so in particular they commute.

b) If the matrices $A = \{A^k \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ are diagonalizable e.i.

\[
A^k x^k(q) = \lambda^k_q x^k(q), \quad x^k(q) = (x^k_j(q)) \in \mathbb{C}^n, \quad q, j = 1, \ldots, n
\] (61)

then the matrices $\rho[A^k, \sigma_k]$ are also diagonalizable and we have

\[
\rho[A^k, \sigma_k] w^l(q) = \delta_{kl} \lambda^k_q w^k(q),
\] (62)

where

\[
w^k(q) = \sum_{j=1}^n x^k_j(q) c_j \otimes e_{\sigma_k(j)}
\] (63)

is an eigenvector of the matrix $\rho[A^k, \sigma_k]$ corresponding to the eigenvalues $\lambda^k_q$ and all remaining eigenvectors $w^l(q) : l \neq k$ have eigenvalues 0.

c) For the matrix $\rho[A, \Sigma_n]$ we have the following eigen-equation

\[
\rho[A, \Sigma_n] w^k(q) = \lambda^k_q w^k(q),
\] (64)

so eigenvalues of the matrices $A = \{A^k \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ are eigenvalues of the matrix $\rho[A, \Sigma_n]$. 

From this structure of CDP matrices $\rho[A, \Sigma_n]$ one easily deduce the following norm properties

**Proposition 16** Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $A = \{A^k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ a set of arbitrary matrices, then

$$\|\rho[A, \Sigma_n]\|_{HS} = \sum_{i=1}^{n} \|A^k\|_{HS}, \quad \|\rho[A, \Sigma_n]\|_{tr} = \sum_{i=1}^{n} \|A^k\|_{tr},$$

(65)

where

$$\|X\|_{HS} = [tr X^+ X]^\frac{1}{2}, \quad \|X\|_{tr} = tr [X^+ X]^\frac{1}{2},$$

(66)

so these norms of $\rho[A, \Sigma_n]$ depends directly on the norms of matrices only and does not depend on a set of CDPs $\Sigma_n = \{\sigma_i\}_{i=1}^n$.

The matrix $\rho[A, \Sigma_n]$ may be written also using the set of matrices $M(E) \equiv \{m(E_j) : j = 1, \ldots, n\}$, defined in Def. 2, namely we have

**Proposition 17** Under assumptions of the Prop. 16 the block structure of the matrix $\rho[A, \Sigma_n]$ is the following

$$\rho[A, \Sigma_n] = \sum_{i,j=1}^{n} e_{ij} \otimes B_{ij}(A, \Sigma_n) \in M(n^2, \mathbb{C}),$$

(67)

where

$$B_{ij}(A, \Sigma_n) = \sum_{k=1}^{n} a_k e_{\sigma_k(i)\sigma_k(j)} = m(E_i)^+ A_{ij} m(E_j),$$

(68)

so, we have

$$\rho[A, \Sigma_n] = \sum_{i,j=1}^{n} e_{ij} \otimes m(E_i)^+ A_{ij} m(E_j) : A_{ij} = (a^k_{ij} \delta_{kl}),$$

(69)

which is similar to formula in [2].

**Definition 6** The CDP matrices

$$\rho[A, \Sigma_n] = \sum_{i,j=1}^{n} e_{ij} \otimes m(E_i)^+ A_{ij} m(E_j) : A_{ij} = (a^k_{ij} \delta_{kl}),$$

(70)

where the set $M(E) \equiv \{m(E_j) : j = 1, \ldots, n\}$ is abelian (equivalently the $\Sigma_n = \{\sigma_k\}$ is abelian) we will call abelian or commutative CDP matrices.

From this structure of the matrix $\rho[A, \Sigma_n]$ we deduce easily that

**Proposition 18** Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $A = \{A^k \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ a set of matrices. Then the matrix $\rho[A, \Sigma_n]$ is hermitian iff the matrices $A$ are hermitian and similarly the matrix $\rho[A, \Sigma_n]$ is semipositive definite iff all matrices $A$ are semi-positive definite.

Unfortunately the good properties of the matrices $\rho[A, \Sigma_n]$ disappear after partial transpose of second part of the tensor product, in fact we have

**Proposition 19** Let $\Sigma_n = \{\sigma_i\}_{i=1}^n$ be a set of CDPs and $A = \{A^k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ be a set of hermitian matrices, then a partial transposed matrices $\rho[A^k, \sigma_k]T_2 = \text{id} \otimes T \rho[A^k, \sigma_k]$ where $T$ is the transpose in $M(n, \mathbb{C})$ have the following properties

a) the matrices $\rho[A^k, \sigma_k]$ are hermitian but in general they do not commute,

b) the matrix $\rho[A^k, \sigma_k]T_2$ has $n^2 - n$ eigenvectors

$$w^k_{pq} = a^k_{p\sigma_k^{-1}(q)} e_p \otimes e_q \pm |a^k_{p\sigma_k^{-1}(q)}| e_{\sigma_k^{-1}(q)} \otimes e_{\sigma_k(q)} : p \neq \sigma_k^{-1}(q),$$

(71)
corresponding to the eigenvalues
\[ \gamma^k_{pq} = \pm |a^k_{pq}| \] (72)
and n eigenvectors and eigenvalues of the form
\[ u^k_{pp} = e_p \otimes e_{\sigma_k(p)}, \quad \gamma^k_{pp} = a^{pp}. \] (73)

What can be checked by a direct calculation. From this it follows that

**Corollary 3** If \( \Sigma_n = \{\sigma_i\}_{i=1}^n \) be a set of CDPs and \( A = \{A_k \in M(n, \mathbb{C}) : k = 1, \ldots, n\} \) is a set of hermitian matrices, then

a) if the matrix \( A_k \) is not diagonal, then a partially transposed matrices \( \rho[A_k, \sigma_k] T_2 \) is not semi-positive definite and therefore the matrix \( \rho[A_k, \sigma_k] \) is NPT state.

b) if the matrices \( A = \{A_k \in M(n, \mathbb{C}) : k = 1, \ldots, n\} \) are not diagonal, then the matrix \( \rho[A, \Sigma_n] = \sum_{k=1}^n \rho[A_k, \sigma_k] \) is a direct sum of NPT matrices.

c) Two unitarily equivalent matrices \( A^k, \ A'^k = U A^k U^+ : U \in U(n) \) may define CDP matrices \( \rho[A_k, \sigma_k] \) and \( \rho[A'^k, \sigma_k] \) with non-equivalent separability properties. For example if \( A^k \) is a non-diagonal matrix and \( A'^k = U A^k U^+ \) is a diagonal form of the matrix \( A^k \), then \( \rho[A'^k, \sigma_k] \) is a PPT matrix, whereas \( \rho[A^k, \sigma_k] \) is a NPT matrix.

**Remark 3** In general the matrix eigenvalues depends in a complicated way on matrix elements and generally there is no formulae that describe this dependence. In particular the eigenvalues \( \lambda^k_{ij} \) of the matrices \( \rho[A, \Sigma_n] \) depends on the matrix elements \( (a^k_{ij}) \) in such a complicated, in general way. It remarkable that after partial transpose of the matrix the eigenvalues of matrix \( \rho[A, \Sigma_n] T_2 \) depends in a very simple way given by Eq. (72).

We have one more useful property of the matrix \( \rho[A, \Sigma_n] \)

**Proposition 20** Let \( \Sigma_n = \{\sigma_i\}_{i=1}^n \) be a set of CDPs and \( \Sigma'_n = \{\sigma'_i = \delta \sigma_i \eta \}_{i=1}^n \), where \( \delta, \eta \in S(n) \), so \( \Sigma'_n \) is also a set of CDPs and \( A = \{A_k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\} \) a set of matrices. Then

\[ \rho[A, \Sigma'_n] = \rho[A, \Sigma_n] = m(\eta^{-1}) \otimes m(\delta) \rho[A^k, \sigma_k] T_2 m(\eta^{-1}) A^k m(\eta) \otimes m(\delta^{-1}), \] (74)

so, the elementary transformation of a set of CDPs \( \Sigma_n \rightarrow \Sigma'_n \) induces a local unitary transformations of the matrix \( \rho[A, \Sigma'_n] = \rho[A, \Sigma_n] \) together with a similarity transformation of the matrices \( A^k \rightarrow m(\eta^{-1}) A^k m(\eta) \).

This Proposition and the Proposition [33] give us the following statement concerning CDP matrices \( \rho[A^k, \Sigma_n] \).

**Corollary 4** Suppose that a commutative CDP matrix

\[ \rho[A^k, \Sigma_n] = \sum_{i=1}^n e_i \otimes m(E_i)^+ A_{ij} m(E_j) : A_{ij} = (a^k_{ij}) \] (75)

is generated by a set of CDPs \( \Sigma_n = \{\sigma_i\}_{i=1}^n \) (so the set of matrices \( M(E) = \{m(E) : j = 1, \ldots, n\} \) is abelian), then there exist an abelian set of CDPs \( \Sigma'_n = \{\sigma'_i = \sigma_1^{-1} \sigma_i \sigma_1 : \sigma_1 \in \Sigma_n, \quad \sigma_1(1) = 1 \} \) such that

\[ \rho[A^k, \Sigma'_n] = m(\sigma_1^{-1}) \otimes id \rho[A^k, \sigma_1] \otimes m(\sigma_1) \otimes id, \] (76)

so the commutative CDP matrices \( \rho[A^k, \Sigma_n] \) are in fact generated by abelian set of CDPs.

### B. The Partial Transpose of CDP Matrices.

Let us consider the partial transpose of the CDP matrix \( \rho[A, \Sigma_n] T_2 \). It is clear that, in general the matrix \( \rho[A, \Sigma_n] T_2 \) has different structure in comparison with the matrix \( \rho[A, \Sigma_n] \). It appears however that if the matrix \( \rho[A, \Sigma_n] \) is commutative e.i. if a set of CDP \( \Sigma_n = \{\sigma_i\} \) is abelian, then the matrix \( \rho[A, \Sigma_n] T_2 \) is also a CDP matrix from the set of CDPs e.i. \( \rho[A, \Sigma_n] T_2 = \rho[A, \Sigma'_n] \), where \( \Sigma'_n \) is a set of CDP. In fact from Prop. [17] we have

\[ \rho[A, \Sigma_n] = \sum_{i,j=1}^n e_{ij} \otimes \sum_{k=1}^n a^k_{ij} e_{\sigma_k(i)\sigma_k(j)} = \rho[A, \Sigma_n] T_2 = \sum_{i,j=1}^n e_{ij} \otimes \sum_{k=1}^n a^k_{ij} e_{\sigma_k(j)\sigma_k(i)}. \] (77)
On the other hand let us consider the matrix
\[ \rho[\tilde{A}, \Sigma_n \xi^\Sigma] : \tilde{A}^k = (\tilde{a}_{ij}^k) \equiv (a_{ij}^{-1} \sigma_j^{-1}(k)), \] (78)
where the permutation \( \xi^\Sigma \) is defined in Def. 4. From Prop. 17 we have
\[ \rho[\tilde{A}, \Sigma_n \xi^\Sigma] = \sum_{i,j=1}^n e_{ij} \otimes \sum_{k=1}^n \tilde{a}_{ij}^k e_{\xi^\Sigma(j) \sigma_k \xi^\Sigma(i)} = \sum_{i,j=1}^n e_{ij} \otimes \sum_{k=1}^n a_{ij}^k e_{\xi^\Sigma(i)(k) \sigma_i \xi^\Sigma(j)(k)}, \] (79)
where in the last step we have used the commutativity of the set of CDPs \( \Sigma_n = \{ \sigma_i \} \). Next using the definition of \( \tilde{A}^k = (\tilde{a}_{ij}^k) \) and Def. 4 we get
\[ \rho[\tilde{A}, \Sigma_n \xi^\Sigma] = \sum_{i,j=1}^n e_{ij} \otimes \sum_{k=1}^n a_{ij}^{-1} \sigma_j^{-1}(k) e_{\sigma_i^{-1}(k) \sigma_i^{-1}(k)}. \] (80)
Making substitution \( l = \sigma_i^{-1} \sigma_j^{-1}(k) \), we obtain
\[ \rho[\tilde{A}, \Sigma_n \xi^\Sigma] = \sum_{i,j=1}^n e_{ij} \otimes \sum_{l=1}^n a_{ij}^l e_{\sigma_l(i) \sigma_l(j)} = \sum_{i,j=1}^n e_{ij} \otimes \sum_{l=1}^n a_{ij}^l e_{\sigma_l(i) \sigma_l(j)} = \rho[A, \Sigma_n]^T_2. \] (81)
So we may formulate the main result of this section

**Theorem 2** Suppose that a CDP matrix \( \rho[A, \Sigma_n] \) is a commutative and is generated by an abelian set of CDPs \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \) and the matrices \( A = \{ A^k \in M(n, \mathbb{C}) : k = 1, \ldots, n \} \) are semi-positive definite, then
\[ \rho[A, \Sigma_n]^T_2 = \rho[\tilde{A}, \Sigma_n \xi^\Sigma] : \tilde{A}^k = (\tilde{a}_{ij}^k) = (a_{ij} E_i E_j(k)) = (a_{ij}^i \sigma_j^{-1}(k)), \] (82)
where \( \xi^\Sigma \) is defined in Def. 4. The matrix \( \rho[A, \Sigma_n] \) is then a PPT state iff all matrices \( \tilde{A}^k \) are semi-definite.

This theorem is a generalisation of corresponding result concerning cyclic groups [6], to arbitrary abelian groups.

**Remark 4** Note that for the abelian group \( \Sigma_4 = V(4) \) Example 3, for which \( \xi^\Sigma = \text{id} \) we have
\[ \rho[A, V(4)]^T_2 = \rho[\tilde{A}, V(4)] : \tilde{A}^k = (\tilde{a}_{ij}^k) = (a_{ij}^i \sigma_j(k)), \] (83)
so in this case, after partial transpose, the group remains the same.

**C. Realignment Criterion for CDP Matrices.**

It appears that matrices of the form \( \rho[A, \Sigma_n] \) are friendly for Realignment Criterion derived in [19]. Namely we have

**Theorem 3** Suppose that \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \) is an abelian set of CDPs, then
\[ \rho[A, \Sigma_n]^\text{RL} = \rho[\tilde{A}, \Sigma_n] : \tilde{A}^k = (\tilde{a}_{ij}^k) = (a_{ij}^i \sigma_i^{-1}(j)), \] (84)
where RL means Realignment. So after this transformation the set of CDPs \( \Sigma_n \) in matrix \( \rho[A, \Sigma_n]^\text{RL} \) remains the same.

From this theorem, from Proposition 16 and Realignment Criterion it follows immediately the following necessary condition for separability of commutative matrices \( \rho[A, \Sigma_n] \).

**Proposition 21** Let \( \rho[A, \Sigma_n] \) be a commutative CDP matrix (e.i. \( \Sigma_n = \{ \sigma_i \}_{i=1}^n \) is an abelian) then \( \rho[A, \Sigma_n] \) may be separable only if
\[ \sum_{k=1}^n ||\tilde{A}^k||_{tr} \leq 1. \] (85)
D. Majorisation Criterion for Tensor matrices from sets of CDPs.

The matrices $\rho[A, \Sigma_n]$ have, by construction, traceless off diagonal blocks and the diagonal blocks are diagonal, which may imply that a majorisation criteria of entanglement could be easier to application for such a matrices. We will need

**Definition 7** Let $A \in M(n, \mathbb{C})$ and $A^+ = A$. Then $\lambda(A) \in \mathbb{C}^n$ is the vector whose components are the eigenvalues of $A$ arranged in decreasing order e.i. we have

$$\lambda(A) = (\lambda_k(A)), \quad \lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A).$$

We say that a matrix $A$ is majorised by a matrix $B$, which is denoted $A \prec B$ if

$$\lambda_1(A) \leq \lambda_1(B),$$

$$\lambda_1(A) + \lambda_2(A) \leq \lambda_1(B) + \lambda_2(B),$$

\[ \ldots \]

$$\lambda_1(A) + \ldots + \lambda_{n-1}(A) \leq \lambda_1(B) + \ldots + \lambda_{n-1}(B),$$

$$\lambda_1(A) + \ldots + \lambda_n(A) = \lambda_1(B) + \ldots + \lambda_n(B),$$

e.i. if $\lambda(A) \prec \lambda(B)$, so majorisation of the hermitian matrices is defined as majorisation of its vectors of eigenvalues.

Now let us consider an arbitrary CDP matrix $\rho[A, \Sigma_n]$ where $\Sigma_n = \{\sigma_i\}_{i=1}^n$ is an arbitrary set of CDPs and $A = \{A^k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ a set of hermitian positive matrices, then

**Proposition 22**

$$\rho_1[A, \Sigma_n] \equiv (id \otimes \text{tr})\rho[A, \Sigma_n] = \sum_{i=1}^{n} \sum_{k=1}^{n} e_{ii}(\sum_{k=1}^{n} a^k_{ii}),$$

$$\rho_2[A, \Sigma_n] \equiv (\text{tr} \otimes id)\rho[A, \Sigma_n] = \sum_{i=1}^{n} \sum_{k=1}^{n} a^k_{i\sigma_k(i)} e_{\sigma_k(i)\sigma_k(i)},$$

so both these matrices are diagonal with positive entries on the diagonal. Note that $\rho_1[A, \Sigma_n]$ depends on the matrices $A = \{A^k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ only (in fact on their diagonals) and not on the set of CDPs $\Sigma_n$.

Now we have the following Majorisation Criterion (MC)

**Theorem 4** If a CDP state $\rho_{12}$ is separable then

$$\rho_{12} \prec \rho_1 \equiv (id \otimes \text{tr})\rho_{12} \land \rho_{12} \prec \rho_2 \equiv (\text{tr} \otimes id)\rho_{12}.$$ 

From this we get

**Proposition 23** The CDP matrix $\rho[A, \Sigma_n]$ where $\Sigma_n = \{\sigma_i\}_{i=1}^n$ is an arbitrary set of CDPs and $A = \{A^k = (a^k_{ij}) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ a set of hermitian positive matrices may be separable only if

$$\rho[A, \Sigma_n] \prec \sum_{i=1}^{n} e_{ii}(\sum_{k=1}^{n} a^k_{ii}) \land \rho[A, \Sigma_n] \prec \sum_{i=1}^{n} \sum_{k=1}^{n} a^k_{i\sigma_k(i)} e_{\sigma_k(i)\sigma_k(i)},$$

where the matrices on RHS are diagonal e.i. their eigenvalues are given explicitly so it simplify calculation of majorisation.
From the above Definition of majorisation of hermitian matrices we know that it is in fact majorisation of corresponding vectors of eigenvalues. From Prop. 15 we know also that the eigenvalues of the matrix $\rho[\mathcal{A}, \Sigma_n]$ are exactly the eigenvalues of the matrices $\mathcal{A} = \{A^k = (a_{jk}^k) \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$. So the relations in the last Proposition show in what a way the eigenvalues of the matrices $\mathcal{A}$ should be majorised by sums of their diagonal elements. On the other hand we have famous theorem by Shurr

**Theorem 5** Let $A \in M(n, \mathbb{C})$ be a hermitian matrix, then

$$d(A) \prec \lambda(A),$$

(96)

where $d(A) \in \mathbb{C}^n$ is the vector whose components are diagonal elements of $A$ arranged in decreasing order.

Thus we see that the majorisation necessary conditions for separability of the matrix $\rho[\mathcal{A}, \Sigma_n]$ from the last Proposition give the majorisation of the eigenvalues of the matrices $\mathcal{A} = \{A^k \in M(n, \mathbb{C}) : k = 1, \ldots, n\}$ by sums of its diagonal elements and on the other hand, from Schur Theorem, the eigenvalues of the matrices $\mathcal{A}$ majorises its diagonal elements. So we see that we have non-trivial conditions for separability for the matrices $\rho[\mathcal{A}, \Sigma_n]$ and in this case for arbitrary set of CDPS $\Sigma_n = \{\sigma_i\}_{i=1}^n$, not only for groups.

**V. EXAMPLES OF LINEAR MAPS RELATED TO SETS OF CDPS**

In the paper [17] (see also [18]) the Irreducible Covariant Quantum Channels were introduced, which are defined in the following way.

**Definition 8** Let

$$u : G \to M(n, \mathbb{C}), \quad u(g) = (u_{ij}(g)) \in M(n, \mathbb{C})$$

be an unitary irreducible representation (IRREP) of a given finite group $G$. A quantum channel $\Phi$, which is by definition completely positive and trace preserving map is called irreducible and invariant (ICQC) with respect to IRREP $U : G \to M(n, \mathbb{C})$ if

$$\forall g \in G \quad \forall x \in M(n, \mathbb{C}) \quad Ad_{U(g)}[\Phi(x)] = \Phi[Ad_{U(g)}(x)],$$

where

$$Ad_{U(g)}(x) \equiv U(g)xU^+(g),$$

so $\Phi$ commute with $Ad_{U(g)}$.

It has been shown that under, assumption that the tensor product is simply reducible e.i. $U \otimes \overline{U} = \sum_{\alpha \in \hat{G}} m_{\alpha} \varphi^\alpha : m_{\alpha} = 0, 1$, $\hat{G}$ is the set of all IRREP’s, the (ICQC) have the following structure

**Proposition 24** A quantum channel $\Phi \in End[M(n, \mathbb{C})]$, which is irreducible and invariant with respect to IRREP $U : G \to M(n, \mathbb{C})$ is necessarily of the form

$$\Phi = l_{id} \Pi^{id} + \sum_{\alpha \in G, \alpha \neq id} l_{\alpha} \Pi^\alpha : \quad l_{\alpha} \in \mathbb{C},$$

where

$$\Pi^\alpha = \frac{\dim \varphi^\alpha}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1})Ad_{U(g)} : \chi^\alpha(g^{-1}) = tr \varphi^\alpha(g^{-1}).$$

It appears that the the value of Choi-Jamiolkowski isomorphism on (ICQC) for $S(3)$ and quaternion groups have the structure of a CDP matrix. In fact we have

**Example 15** For the group $S(3)$ we have

$$\Phi = \Pi^{id} + l_{sgn} \Pi^{sgn} + l_{\lambda} \Pi^\lambda,$$
where λ denotes the two-dimensional IRREP of $S(3)$. The corresponding Choi-Jamiolkowski matrix is of the form

$$J(\Phi) = \begin{bmatrix}
\frac{1}{2}(1 + l_{\text{sgn}}) & 0 & 0 & l_{\lambda} \\
0 & \frac{1}{2}(1 - l_{\text{sgn}}) & 0 & 0 \\
0 & 0 & \frac{1}{2}(1 - l_{\text{sgn}}) & 0 \\
l_{\lambda} & 0 & 0 & \frac{1}{2}(1 + l_{\text{sgn}})
\end{bmatrix} = \rho[A, \Sigma_2],$$

where $\Sigma_2 = S(2)$ and

$$A = \{A^1, A^2\}: A^1 = \left(\begin{array}{cccc}
\frac{1}{2}(1 + l_{\text{sgn}}) & \frac{1}{2}l_{\lambda} \\
\frac{1}{2}l_{\lambda} & \frac{1}{2}(1 - l_{\text{sgn}})
\end{array}\right), \quad A^2 = \left(\begin{array}{cccc}
\frac{1}{2}(1 - l_{\text{sgn}}) & 0 \\
0 & \frac{1}{2}(1 - l_{\text{sgn}})
\end{array}\right).$$

**Example 16** The quaternion group $Q = \{\pm Q_e, \pm Q_1, \pm Q_2, \pm Q_3\}$ is a non-abelian group of order eight satisfying

$$Q = \left\langle Q_e, Q_1, Q_2, Q_3 \mid (-Q_e)^2 = Q_e, Q_1^2 = Q_2^2 = Q_3^2 = Q_1Q_2Q_3 = -Q_e\right\rangle. \quad (97)$$

It possesses five inequivalent irreducible representations which we label by $\text{id}, t_1, t_2, t_3, t_4$, respectively. However, only one of them, labeled by $t_4$, has dimension greater than one and its dimension is equal to two. It is known that the quaternion group can be represented as a subgroup of $GL(2, \mathbb{C})$. The matrix representation $R : Q \rightarrow GL(2, \mathbb{C})$ is given by

$$Q_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (98)$$

where $i^2 = -1$. In Table I we present values of the characters for all irreducible representations of the group $Q$.

| $Q$ | $Q_e$ | $Q_1$ | $Q_2$ | $Q_3$ | $Q_eQ_1$ | $Q_eQ_2$ | $Q_eQ_3$ |
|-----|-------|-------|-------|-------|---------|---------|---------|
| $\chi_\text{id}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^{t_1}$ | 1 | 1 | -1 | 1 | -1 | 1 | -1 |
| $\chi^{t_2}$ | 1 | 1 | 1 | -1 | -1 | 1 | -1 |
| $\chi^{t_3}$ | 1 | 1 | -1 | 1 | -1 | -1 | 1 |
| $\chi^{t_4}$ | 2 | -2 | 0 | 0 | 0 | 0 | 0 |

**TABLE I:** Table of characters for the quaternion group $Q$.

$$\Phi^{t_4} = l_{t_4}\Pi^{\text{id}} + l_{t_1}\Pi^{t_1} + l_{t_2}\Pi^{t_2} + l_{t_3}\Pi^{t_3} \quad (99)$$

The corresponding Choi-Jamiolkowski matrix is of the form

$$J(\Phi^{t_4}) = \frac{1}{2} \begin{pmatrix}
1 + l_{t_2} & 0 & 0 & l_{t_1} + l_{t_3} \\
0 & 1 - l_{t_2} & l_{t_3} - l_{t_1} & 0 \\
l_{t_1} - l_{t_3} & l_{t_2} & 1 - l_{t_2} & 0 \\
l_{t_1} + l_{t_3} & 0 & 0 & 1 + l_{t_2}
\end{pmatrix} = \rho[A, \Sigma_2], \quad (100)$$

where $\Sigma_2 = S(2)$ and

$$A = \{A^1, A^2\}: A^1 = \left(\begin{array}{cccc}
\frac{1}{2}(1 + l_{t_2}) & \frac{1}{2}l_{t_1} + l_{t_3} \\
\frac{1}{2}l_{t_1} + l_{t_3} & \frac{1}{2}(1 + l_{t_2})
\end{array}\right), \quad A^2 = \left(\begin{array}{cccc}
\frac{1}{2}(1 - l_{t_2}) & \frac{1}{2}(l_{t_3} - l_{t_1}) \\
\frac{1}{2}(l_{t_3} - l_{t_1}) & \frac{1}{2}(1 - l_{t_2})
\end{array}\right).$$

Now we consider the reduction map and its generalisation to the Breuer-Hall map. It is not difficult to check that the reduction map is related to CDP matrices in the following simple and non unique way.

**Proposition 25** Let us consider the reduction map

$$R : M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}); \quad R(A) = \text{tr}(A)\text{id}_n - A, \quad A \in M(n, \mathbb{C}). \quad (101)$$
Then we have

$$R^\otimes \equiv \sum_{ij} R(e_{ij}) \otimes e_{ij} = \rho[A, \Sigma_n],$$

(102)

where

$$A = \{ A^1 = id_n - J, \quad A^k = id_n : k = 2, \ldots, n \}$$

(103)

and $$\Sigma_n = \{ \sigma_i \}_{i=1}^n$$ is an arbitrary set of CDPs such that $$\sigma_1 = id$$. So in fact, due to the structure of the matrices $$A$$, the reduction map weakly depends on the set of CDPs $$\Sigma_n = \{ \sigma_i \}_{i=1}^n$$.

Let us consider now the Breuer-Hall map which is a generalisation the reduction map

$$B : M(n, \mathbb{C}) \to M(n, \mathbb{C}); \quad B(X) = \text{tr}(X)id_n - A - UX^TU^+, \quad X \in M(n, \mathbb{C}),$$

(104)

$$X \in M(n, \mathbb{C}), \quad U \in U(n), \quad U^T = -U.$$  

(105)

So in order to construct a Breuer-Hall map one have to construct an unitary and anti-symmetric matrix. It appears that one can construct a large class of unitary anti-symmetric matrices $$U \in U(n)$$ (in fact orthogonal), using permutations from $$S(n)$$. We have

**Proposition 26** Let $$n = 2k$$. We divide the set $$\{1, \ldots, n\}$$ into two disjoint subsets $$O$$ and $$P$$, where the first one contains all odd numbers from $$\{1, \ldots, n\}$$ and the second one contains all even numbers from $$\{1, \ldots, n\}$$. The permutation $$\sigma = (o_1p_1), \ldots, (o_{n-1}p_{n-1}) \in S(n)$$, where $$o_i \in O$$ and $$p_i \in P$$ is involutive and the matrix

$$U^\sigma = ((-1)^j \delta_{\sigma(ij)})$$

(106)

is unitary (orthogonal) and anti-symmetric.

**Remark 5** So we have a large class of such unitary and antisymmetric matrices, which are however, orthogonally similar. Note that the permutations $$\sigma = (o_1p_1), \ldots, (o_{n-1}p_{n-1}) \in S(n)$$, where $$o_i \in O$$ and $$p_i \in P$$ belongs to the regular representation (i.e. permutational) of the group $$(\mathbb{Z}_2)^n$$, so it is an element of a group of CDPs. In the following we will use a particular, more convenient form of such a permutations, which looks $$\sigma = (1p_1), \ldots, (n-1p_{n-1})$$.

Now we are to formulate the main result of this section, which may be checked by a direct calculation

**Theorem 6** Let $$\Sigma_{2n}, n = 2l$$ is a regular representation of the group $$(\mathbb{Z}_2)^n$$, so it is CDP, whose elements are compositions of disjoint transpositions only i.e. $$\Sigma_{2n} = \{ \sigma_1 \}^{2n}_{i=1} : \sigma_i = (i_1j_1)\ldots(i_{l}j_{l})$$. We choose the permutation $$\sigma_{p_1} = (1p_1), \ldots, (n-1p_{n-1}) \in \Sigma_{2n}$$, where $$p_i \in P$$. Next let $$A = \{ A^k, k = 1, \ldots, n \}$$ be such that

$$A^1 = (a^1_{ij}) : a^1_{ii} = a^1_{2k-1p_{2k-1}} = a^1_{p_{2k-1}2k-1} = 0, \quad k = 1, \ldots, l, i = 1, \ldots, n;$$

(107)

$$a^1_{ij} = -1, \quad i, j \neq 2k-1, p_{2k-1},$$

(108)

$$A^{p_1} = 0.$$  

(109)

and

$$\forall k \neq 1, p_1 \quad A^k = (a^k_{ij}) = a^k_{ii} = \left\{ \begin{array}{ll}
(-1)^{\sigma_{p_1}(i)+j} \delta_{\sigma_{p_1}(i)k} & : i \neq j \\
1 & : i = j
\end{array} \right.$$  

(110)

then we have

$$\rho[A, \Sigma_n] = \sum_{ij=1}^{n} e_{ij} \otimes B(e_{ij}),$$

(111)

where

$$B(A) = \text{tr}(A)id_n - A - U^{\sigma_{p_1}} A^T (U^{\sigma_{p_1}})^+, \quad A \in M(n, \mathbb{C}),$$

(112)

and $$U^{\sigma_{p_1}}$$ is an orthogonal, anti-symmetric matrix defined in the last Proposition.
Interestingly, for \( n = 4 \) the operator corresponding to the Breuer-Hall map belongs to both classes

\[
\sum_{i,j=0}^3 e_{ij} \otimes B(e_{ij}) = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix},
\]

which is (113).

VI. CONCLUSIONS

In this paper we provide a class of sets of Completely Different Permutations (CDPs) which define a substantial generalization of the circular group \( C(n) = \{\text{id}, c, c^2, \ldots, c^{n-1}\} \), where \( c = (0, 1, \ldots, n-1) \). A class of CDPs enjoys several interesting properties analysed in Section II. This class is used to construct a bipartite operators acting on \( \mathcal{H} \otimes \mathcal{H} \), with \( \mathcal{H} \) being an \( n \) dimensional Hilbert space. The crucial observation shows that if \( A \) is a bipartite operator corresponding to some abelian group of CDPs, then its partial transposition \((I \otimes T)A\) corresponds to another abelian group of CDPs. Therefore, it may be used to construct and classify some classes of PPT states. Interestingly, several well known linear maps (reduction or Breuer-Hall maps) are related to sets of CDPs as well via Choi-Jamiołkowski isomorphism.

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