Three-pion decays of the tau lepton, the $a_1(1260)$ properties, and the $a_1 \rho \pi$ Lagrangian

Martin Vojík$^1$ and Peter Lichard$^{1,2}$

$^1$Institute of Physics, Silesian University in Opava, Bezručovo nám. 13, 746 01 Opava, Czech Republic
$^2$Institute of Experimental and Applied Physics, Czech Technical University, Horská 3/a, 120 00 Prague, Czech Republic

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We show that the $a_1 \rho \pi$ Lagrangian is a decisive element for obtaining a good phenomenological description of the three-pion decays of the $\tau$ lepton. We choose it in a two-component form with a flexible mixing parameter $\sin \theta$. In addition to the dominant $a_1 \rightarrow \pi \rho$ intermediate states, the $a_1 \rightarrow \pi \pi$ ones are included. When fitting the three-pion mass spectra, three data sets are explored: (1) ALEPH 2005 $\pi^- \pi^+ \pi^-$ data, (2) ALEPH 2005 $\pi^- \pi^0 \pi^0$ data, and (3) previous two sets combined and supplemented with the ARGUS 1993, OPAL 1997, and CLEO 2000 data. The corresponding confidence levels are (1) 28.3%, (2) 100%, and (3) 7.7%. After the inclusion of the $a_1(1640)$ resonance, the agreement of the model with data greatly improves and the confidence level reaches 100% for each of the three data sets. From the fit to all five experiments [data set (3)], the following parameters of the $a_1(1260)$ are obtained: $m_{a_1} = (1233 \pm 18)$ MeV, $\Gamma_{a_1} = (431 \pm 20)$ MeV. The optimal value of the Lagrangian mixing parameter $\sin \theta = 0.459 \pm 0.004$ agrees with the value obtained recently from the $e^+ e^-$ annihilation into four pions.

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I. INTRODUCTION

The dominance of the $a_1(1260)$ meson, hereafter referred as $a_1$, and its $\pi \rho$ decay mode in the three-pion decays of the $\tau$-lepton is firmly established experimentally [1–16]. A convincing demonstration was provided by the ARGUS Collaboration [7, 16], who compared the distribution of unlike- and like-sign two-pion masses. The $\pi \rho$ intermediate state is the core of several models of the three-pion decays of the $\tau$-lepton [12, 17–26]. It should be mentioned that the $a_1$ dominance in heavy lepton decays was proposed in 1971 [27], about five years before the $\tau$-lepton was actually discovered.

The $a_1(1260)$ resonance, discovered almost fifty years ago [28], plays an important role in many phenomena of the nuclear and particle physics. Its properties have been studied in many processes, but even its basic parameters are not very well known. The values of the $a_1(1260)$ resonance mass determined from different processes or by different experimental groups often contradict one another. The same applies, even to a larger extent, to the $a_1$ width. Very little improvement has been achieved over the last thirty years, see Table I. The origin of those problems lies in the very nature of the $a_1$ resonance with its short lifetime and large width. The usual definition of the mass and usual procedures for its measurement are not applicable. The $a_1$ mass and width enter the formulas for experimentally accessible quantities via the assumed form of the resonance propagator, which generates a specific Breit-Wigner formula. Those formulas are further modified in different ways when modeling the dynamics of the processes in which the $a_1$ participates.

As a result, we do not have a unique definition of the $a_1$ mass $m_{a_1}$ and width $\Gamma_{a_1}$. In fact, every formula represents a specific definition of $m_{a_1}$ and $\Gamma_{a_1}$. Given this, it is not surprising that different models yielded different results even when being applied to the same data. It would be natural to accept as the $a_1$ canonical parameters the results of a model that best describes a broad class of data on various processes and from various experiments. Unfortunately, we are not in such a situation yet.

The situation of the heavier meson states with $J^{PC} = 1^{++}$ is a little unclear and none of them has found its place in the Summary Table of the Review of Particle Properties [39]. The first indication of the state with mass of 1.65 GeV and width of 0.4 GeV appeared already in 1978 [40]. The later experimental evidence, which comes mainly from hadronic reactions, is summarized in [39], where this resonance is listed as $a_1(1640)$ and assigned the mass of $(1647 \pm 22)$ MeV and the width of $(254 \pm 27)$ MeV. The three-pion decay of the $\tau$-lepton is less convenient for studying the $a_1(1640)$ resonance (often denoted as $a_1'$ in what follows) because of fundamental limitations due to the $\tau$ mass which is not big enough to provide sufficient phase space for three-pion final states with the needed invariant mass. Nevertheless, the DELPHI Collaboration [11] performed the analysis.

| PDG   | $m_{a_1}$ (MeV) | $\Gamma_{a_1}$ (MeV) | $a_1 \rightarrow \rho \pi$ |
|-------|-----------------|----------------------|---------------------------|
| 1978  | $\approx 1100$  | $\approx 300$        | $\approx 100\%$          |
| 1980  | 1100 to 1300    | $\approx 300$        | dominant                  |
| 1982  | 1275±30         | 315±45               | dominant                  |
| 1986  | 1275±28         | 316±45               | dominant                  |
| 1988  | 1260±30         | 300 to 600           | dominant                  |
| 1990  | 1260±30         | 350 to 500           | dominant                  |
| 1992  | 1260±30         | $\approx 400$        | dominant                  |
| 1994  | 1230±40         | $\approx 400$        | dominant                  |
| 1998  | 1230±40         | 250 to 600           | dominant                  |
| 2000  | 1230±40         | 250 to 600           | seen                      |
| 2008  | 1230±40         | 250 to 600           | seen                      |
of the Dalitz plots for different 3-pion mass ranges and observed an enhancement that “could be explained by a decay mode of the \( \tau \) to a resonance of mass similar to or greater than the \( \tau \) mass which then decays to three pions through the intermediate state of a pion plus a particle of mass 1.25 GeV or greater.” They interpreted this as an evidence for the \( a_1' \). In our opinion, this observation need not signify the existence of the \( a_1' \). It may also be a decay of the \( a_1(1260) \), produced with a larger than nominal mass, into \( \pi \) and \( \rho \). A more convincing proof of the \( a_1' \) in the decay of the \( \tau \) lepton comes from the CLEO Collaboration [12]. They showed that adding the \( a_1' \) term into the Breit-Wigner function improved significantly the agreement with the data.

On the theoretical side, a radial excitation of the quark-antiquark system with a mass of 1.82 GeV appeared in a relativized quark model with chromodynamics of Godfrey and Isgur [41]. Its decay width into the \( \pi \rho \) channel was calculated in the flux-tube-breaking model by Kokoski and Isgur [42] with result \( \lesssim 70 \) MeV (our estimate is based on their Table II). The seminal analysis of Barnes, Close, Page, and Swanson [43] has shown that the experimentally observed dominance of the D-wave decay mode of the \( \rho \) meson system was calculated in the flux-tube-breaking model by Kokoski and Isgur [42] with result \( \lesssim 70 \) MeV (our estimate is based on their Table II). The seminal analysis of Barnes, Close, Page, and Swanson [43] has shown that the experimentally observed dominance of the D-wave over S-wave [44, 45] excludes the hybrid meson nature of the \( a_1' \) and confirms it as a radial excitation of the quark-antiquark system.

A few \( F^G(J^{PC}) = 1^- (1^{++}) \) meson states above the \( a_1(1640) \) have been observed by a single group, mainly in the \( pp \) annihilation. They still need confirmation. For details, see [39].

Another important ingredient that defines a particular model of the three-pion decay of the \( \tau \)-lepton is, besides the \( a_1(1640) \) propagator, the \( a_1 \pi \rho \) vertex. The models assembled by different authors are based on different \( a_1 \pi \rho \) vertexes. These are sometimes simply constructed as allowed combinations of the metric tensor and participating four-momenta. A more rigorous way lies in deriving them from the interaction Lagrangians among the axial, vector and pseudoscalar fields. Unfortunately, here the situation is unclear yet. Various theoretical concepts provide different effective Lagrangians [46]. This is probably the reason why the model builders preferred trivial Lagrangians or \textit{ad hoc} vertexes. However, recent articles [24–26] are different. Dumm, Pich, and Portolés [24] got their Lagrangian from the resonance chiral theory. Their work was revised in the light of later developments in [25]. Achasov and Kozhevnikov [26] used the Generalized Hidden Local Symmetry model.

Several models of the three-pion decay of the tau lepton have been proposed. With some simplification one can say that each of them gives compatible results when applied to different sets of data, but the results of different models are incompatible. Also the agreement of many models with data (often verbally claimed as satisfactory) is poor when judged by usual statistical criteria. The most popular models were those of Isgur, Morningstar, and Reader (IMR) [19] and of Kühn and Santamaria (KS) [20]. Other models were much less successful in fitting the data. As an example we recall the results from [7], where the ARGUS Collaboration compared various models with their data. Using the \( \chi^2 \)'s and the numbers of degrees of freedom (NDF) from their Table 4, we are getting the confidence level (C.L.) of \( \approx 10^{-4} \) for Bowler’s model [47] and 2.2% for the model of Ivanov, Osipov, and Volkov [48]. The KS and IMR models look better with C.L. 10.7% and 79.0%, respectively. However, in a later article [49] the ARGUS Collaboration used an enlarged set of data (integrated luminosity of 445 pb\(^{-1}\) against 264 pb\(^{-1}\) in [7]) and found that the KS model is rejected on a 7.4 \( \sigma \) level. The IMR model with parameters as given in [19] was incompatible with the data on the same level [49].

Up to now, the best results have been obtained by the CLEO model [12] and by the model of Achasov and Kozhevnikov [26]. The former obtained, when fitting the CLEO \( \pi^- \pi^0 \pi^0 \) data [12], C.L. of 54.6% without the \( a_1' \) resonance and 88.2% with it. The latter fitted the ALEPH \( \pi^- \pi^\mp \pi^- \) data [15] assuming two heavier axial mesons \( a_1' \) and \( a_2' \) and got \( \chi^2/NDF=79/102 \), which corresponds to C.L. of 95.6%. Unfortunately, each of these two successful models has been applied only to one data set.

The finding of an \( a_1 \pi \rho \) Lagrangian that leads to a satisfactory description of the three-pion production in the tau decays would have important consequences for other areas of the high energy and nuclear physics. For example, the \( a_1 \) resonance and its coupling to the \( \rho \pi \) system play important role in the evaluation of the dilepton and photon production rates from a hadronic fireball presumably created in the relativistic heavy ion collisions. The calculations performed so far, see, e.g., Refs. [50], have shown that the yield of electromagnetic signals strongly depends on the choice of the \( a_1 \pi \rho \) Lagrangian. Fixing its correct form is thus important for distinguishing the electromagnetic radiation of the Quark-Gluon Plasma (QGP) from the hadronic sources.

The outline for this paper is as follows. In Sec. II we describe our model and mention briefly its similarities and differences with other models. The experimental data used for testing our model and fixing its parameters are listed in Sec. III. Some details about our calculations and the results are presented in Sec. IV. We summarize our results and conclude in Sec. V. Two Appendixes contain technical details. The present work supersedes an earlier paper [51].

II. MODEL OF THE THREE–PION DECAYS OF THE TAU LEPTON

In this section we present our model, which will be used for fitting the three–pion mass spectra of the decays \( \tau^- \to \nu_\tau \pi^- \pi^0 \) and \( \tau^- \to \nu_\tau \pi^- \rho^0 \). The basic information can be obtained by inspecting Figs. 1 and 2. In addition to the standard \( a_1 \to \pi \rho \) intermediate states we include also the states in which the \( a_1 \) couples to a pion and an \( f_0(600) \) (hereafter called \( \sigma \)). The \( \pi \sigma \)
intermediate states improve the behavior of the differential decay width at small masses of the three-pion system and bring the difference between $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ and $\tau^- \rightarrow \nu_\tau \pi^- \pi^0 \pi^0$ decays. Their importance has been pointed out by the CLEO Collaboration [12].

To show what is specific for our model, what differs it from other existing models of the three-pion decays of the tau lepton, we have to provide more information. This is done in the following subsections.

A. Phenomenological $a_1\rho\pi$ Lagrangian

The interaction Lagrangian among the $a_1$, $\rho$, and pion fields implies the form of the $a_1\rho\pi$ vertex in the Feynman diagrams. But sometimes a vertex is postulated that can hardly be related to any effective Lagrangian. In the literature, one can find several prescriptions for the $a_1\rho\pi$ vertex used in the calculation of the decay rate of the tau lepton into three pions and neutrino. The simplest one is $X^{a\mu} \propto g^{a\mu}$, where index $a$ ($\mu$) couples to the $a_1$ ($\rho$) line. It can be derived from the interaction Lagrangian among the $a_1$, $\rho$, and $\pi$ fields without derivatives. It was used, e.g., in Ref. [17]. On the opposite pole of complexity is a two-component vertex used in the IMR model [19]. Both its components are transversal both to the $a_1$ and $\rho$ four-momenta. The relative weight of the two components can vary, what gives the IMR model more flexibility. This is probably the main reason why this model sometimes fits the data a little better than the KS model [20], see, for example, [8].

To maintain both the flexibility and the correspondence with the effective field theory, we use a two-component Lagrangian of the $a_1\rho\pi$ interaction in the form

$$L_{a_1\rho\pi} = \frac{g_{a_1\rho\pi}}{\sqrt{2}} (L_1 \cos \theta + L_2 \sin \theta),$$

where

$$L_1 = A^\mu \cdot (V_{\mu\nu} \times \partial^\nu P),$$

$$L_2 = V_{\mu\nu} \cdot (\partial^\mu A^\nu \times P),$$

and $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The isovectors $A^\mu$, $V_\mu$, and $P$ denote the operators of the $a_1$, $\rho$ and $\pi$ fields, respectively.

Our Lagrangian differs from that derived by Wess and Zumino, see Eq. (67) in [52], only by notation. We will consider the mixing angle $\theta$ a free parameter that has to be determined by fitting the experimental three-pion mass distribution. For each $\theta$, the coupling constant $g_{a_1\rho\pi}$ can be determined from the $a_1 \rightarrow \rho\pi$ decay width. The Lagrangian (1) implies the following $a_1\rho\pi$ vertex

$$X^{a\mu} = \frac{ig_{a_1\rho\pi}}{\sqrt{2}} \left\{ \cos \theta \left[ p_\rho^\mu p_\pi^\mu - (p_\pi p_\rho) g^{a\mu} \right] - \sin \theta \left[ p_\rho^\mu p_\pi^\mu - (p_\rho p_\pi) g^{a\mu} \right] \right\},$$

where $p$'s denote the four-momenta of the corresponding mesons (incoming $a_1$, outgoing $\rho$ and $\pi$).

Lagrangian (1) has recently been used [53, 54] in a model of the electron–positron annihilation into four pions. Value of the mixing parameter $\sin \theta$ was obtained by fitting the excitation function (dependence of the annihilation cross section on the invariant collision energy). From the $\pi^+\pi^-\pi^+\pi^-$ channel the value of $0.460 \pm 0.003$ has been obtained [53]. In [54], a combined fit to $\pi^+\pi^-\pi^+\pi^-$ and $\pi^+\pi^-\pi^0\pi^0$ channels has provided the value of $0.466 \pm 0.005$.

B. Other effective Lagrangians and their parameters

The Lagrangian describing the interaction of the $a_1$ triplet with strange pseudoscalar and vector mesons is not required when calculating the amplitudes of the three-pion decays of the $\tau$, see Figs. 1 and 2. It is needed for evaluating the strange channel contribution to the total decay width of the $a_1(1260)$. The latter enters the $a_1$ propagator discussed below. The Lagrangian is chosen in a form analogous to Eq. (1)

$$L_{a_1K^*K} = \frac{g_{a_1K^*K}}{\sqrt{2}} (L_1^' \cos \theta + L_2^' \sin \theta),$$

with

$$L_1^' = \partial^K K^* K^\mu + \text{H.c.},$$

$$L_2^' = K^\dagger \partial^K K^* + \text{H.c.},$$
Matrix notation is now used, in which

\[
K = \begin{pmatrix} K^+ & K^0 \end{pmatrix}, \quad K^*_\mu = \begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix},
\]

\[
A^\mu = \begin{pmatrix} (a_1^0)^\mu + \sqrt{2}(a_1^\pm)^\mu \\ (a_1^-)^\mu - (a_1^0)^\mu \end{pmatrix},
\]

and \(K^*_\mu = \partial_\mu K^*_\nu - \partial_\nu K^*_\mu\). As usual [55], a particle symbol denotes the field operator which annihilates that particle and creates its antiparticle. In the spirit of the SU(3) symmetry, we assume the same mixing angle \(\theta\) as in the \(a_1\rho\pi\) case \((1)\). The coupling constant \(g_{a_1K^*K}\) cannot be reliably extracted from the experimental data yet because of conflicting information about the \(a_1 \to K\bar{K}\pi\) branching fractions.\(^1\) We will therefore use the SU(3) symmetry relation

\[
g^2_{a_1K^*K} = \frac{1}{4}g^2_{a_1\rho\pi}. \tag{2}\]

In order to evaluate the amplitudes of the Feynman diagrams depicted in Figs 1 and 2, we also need to specify the interaction Lagrangian among the \(a_1\), \(\pi\), and \(\sigma\) fields. We write it in the form

\[
\mathcal{L}_{a_1\sigma\pi} = g_1 (A^\mu \cdot \partial_\mu P) S + g_2 (A^\mu \cdot P) \partial_\mu S,
\]

where \(S\) is the operator of the \(\sigma\) field. The Lorentz condition for the \(a_1\) field implies that the amplitude of the decay \(a_1 \to \sigma + \pi\) is proportional to the difference

\[
g_{a_1\sigma\pi} = g_1 - g_2. \tag{3}\]

In the \(\tau\) decay diagrams, where the off-mass-shell \(a_1\) resonance is represented by its propagator \((5)\), also the terms proportional to

\[
h_{a_1\sigma\pi} = g_1 + g_2 \tag{4}\]

contribute. There is no way of inferring the \(g_{a_1\sigma\pi}\) and \(h_{a_1\sigma\pi}\) from the hadron decay data. We will return to this problem later in this article.

The interaction Lagrangian between the \(\sigma\) and \(\pi\) fields is given by

\[
\mathcal{L}_{\sigma\pi\pi} = g_{\sigma\pi\pi} (P \cdot P) S.
\]

The coupling constant \(g_{\sigma\pi\pi}\) could be estimated from the data on the \(\sigma\) mass and width [56, 57]. But because this constant enters the amplitudes of the three-pion decays of the taon multiplied by \(g_{a_1\sigma\pi}\) or \(h_{a_1\sigma\pi}\), which are both unknown, it does not have much sense.

\(^1\) See [39] and the discussion on p. 253 in [15].

\(^2\) In Refs. [12, 18] an unsubtracted dispersion relation was used.
with coupling constant $g_{K^*K\pi}$. Moreover, the empirical widths [39] of the $\rho(770)$ and $K^*(892)$ provide the ratio

$$\frac{g_{K^*K\pi}^2}{g_{\rho\pi\pi}^2} = 0.883 \pm 0.035,$$

where the error has been enlarged to absorb the difference between the charged and neutral $K^*(892)$. Value (13) is a little higher than the SU(3) value of 3/4. Relations (2) and (13) enable us to express the product of coupling constants squared acting in (11) as a multiple of

$$G^2 = g_{a_1\rho\pi}^2 g_{\rho\pi\pi}^2,$$

which determines the partial decay width of (10). We introduce the ratio

$$x = \frac{g_{a_1\rho\pi}^2 g_{K^*K\pi}^2}{G^2},$$

the value of which is given by multiplying (2) by (13).

As we have already mentioned, there is no way of getting the product $g_{a_1\rho\pi} g_{\rho\pi\pi}$ from the data, as the partial decay width of (12) is unknown. We therefore proceed in another way. We define the parameter

$$y = \frac{g_{a_1\rho\pi} g_{\rho\pi\pi}}{G}.$$  

If we insert the parameters $x$ and $y$ into the formula for the $a_1$ total decay width, it becomes proportional to $G^2$. So does the derivative of the running mass squared (9). When the condition (7) is applied, $G^2$ can be canceled. With known $x$, the condition (7) thus becomes an equation for the unknown $y^2$. As we neglect the possible interference between the Feynman diagrams containing the $\rho$ with those containing the $\sigma$, the sign of $y$ is not essential and we choose $y \geq 0$. The dimension of $y$ is (energy)$^2$ because the two Lagrangians that describe the decay $a_1 \rightarrow 3\pi$ via $\rho\pi$ have together three derivatives, while those via $\sigma\pi$ just one.

An important note concerns the hadron vertexes. The effective Lagrangian approach takes hadrons as elementary quanta of the corresponding fields, ignoring thus their internal structure. As a consequence, the interaction strength is overestimated at higher momentum transfers. To describe the interaction among participating mesons more realistically, we explore the chromoelectric flux-tube breaking model of Kokoski and Isgur [42], as it was done already in the IMR model [19]. Each strong interaction vertex is modified by the factor

$$F(q) = \exp \left\{-\frac{g^2}{12\beta^2}q^2 \right\},$$

where $q$ is the three-momentum magnitude of a daughter meson in the rest frame of the parent one (virtual masses are taken in the intermediate states). In the original paper [42], the value $\beta = 0.4$ GeV/c was established. We will use this value in all our calculations, as we found that moving from it did not bring statistically significant improvement of the agreement of our model with data. A cutoff similar to (17) was used in the model by CLEO Collaboration [12]. Their parameter $R = 1.2$ corresponds to $\beta = 0.340$.

During the course of development of our model we tried various versions of the $a_1$ propagators, from the most primitive one with the constant mass and width to the most sophisticated and best physically justified one (5). The best fit to data has been provided by the latter.

When investigating the presence of the suspected radial recurrence of the $a_1(1260)$, denoted as $a_1'$, we supplement the $a_1$ propagator (5) with the term

$$-iG_{a_1'}^{\mu\nu}(p) = \alpha -\frac{g^{\mu\nu} + p^{\mu}p^{\nu}/m_{a_1'}^2}{s - m_{a_1'}^2 + im_{a_1'}\Gamma_{a_1'}(s)},$$

where $\alpha$ is a complex parameter. We assume that the energy dependent total decay width $\Gamma_{a_1'}(s)$ exhibits the same energy behavior as that of $a_1(1260)$ and write

$$\Gamma_{a_1'}(s) = \frac{\Gamma_{a_1}(m_{a_1}')}{\Gamma_{a_1}(m_{a_1})}\Gamma_{a_1'}(s),$$

where $\Gamma_{a_1'} = \Gamma_{a_1}(m_{a_1}')$ is the assumed width of the $a_1'$ resonance.

What concerns the relation to the previous models, our $a_1$ propagator is closest to that used by CLEO Collaboration [12]. If we ignored the momentum dependent terms in the numerators of (5) and (18), we would recover their Breit-Wigner function.

**D. Propagators of the $\rho$, $K^*$, and $\sigma$ resonances**

In order to calculate the amplitudes of the taon’s three-pion decays we need also the $\rho$ and $\sigma$ propagators. They play a role also in decays (10) and (12). In addition, the evaluation of the decay width (11) requires the knowledge of the $K^*$ propagator.

We choose the propagator of both the charged and neutral rho resonances in the form

$$-iG_{\rho}^{\mu\nu}(p) = -\frac{g^{\mu\nu} + p^{\mu}p^{\nu}/m_{\rho}^2}{s - M_{\rho}^2(s) + im_{\rho}\Gamma_{\rho}(s)},$$

which uses the running mass squared $M_{\rho}^2(s)$ and the energy dependent total width $\Gamma_{\rho}(s)$ from Ref. [59]. The denominator of propagator (19) is an analytic function in the $s$-plane with a cut running from $4m_{\rho}^2$ to infinity, as required by general principles. The real function $M_{\rho}^2(s)$ is calculated from $\Gamma_{\rho}(s)$ using a once-subtracted dispersion relation, which guarantees that the condition $M_{\rho}^2(m_{\rho}^2) = m_{\rho}^2$ is satisfied. The condition

$$\frac{dM_{\rho}^2}{ds}(m_{\rho}^2) = 0$$

is not fulfilled automatically and serves as a check that all important contributions to the total $\rho$-meson width $\Gamma_{\rho}(s)$
have properly been taken into account. They include, in addition to the basic two-pion decay channel, the $\omega\pi^0$, $K^+K^-$, $K^0\bar{K}$, and $\eta\pi^+\pi^-$, which get open as the $\rho$ resonance goes above its nominal mass. The structure of the participating mesons is taken into account by means of the Kokoski-Isgur form factor (17).

The running mass description of the $\rho$ propagator [59] differs from other approaches that appeared in the literature [60–62]. Gounaris and Sakurai [60] considered only the two-pion contribution to the total width of the $\rho^0$ resonance and ignored structure effects. The result is a simple analytic formula, the main reason why their approach is so popular. Vaughn and Wali [61] took into account the running mass description of the $\rho$ meson and ignored structure effects. The result is a simpler form, with the constant of the Kokoski-Isgur form factor (17).

The decay width includes only the contribution from the $\pi^0\pi^0$ channel, and is normalized to the nominal width $\Gamma_K^*$ at $s = m_K^2$. The corresponding formula, taking into account also the kokoski-Isgur form factor (17), is

$$\Gamma_K^*(s) = \frac{m_K^2}{m} \left[ \frac{q(s)}{q(m_K^2)} \right]^3 \frac{F(q(s))}{F(q(m_K^2))} \Gamma_K^*, \quad (20)$$

The decay width includes only the contribution from the $K^* \rightarrow K + \pi$ channel and is normalized to the nominal width $\Gamma_K^*$ at $s = m_K^2$. The formula (20) contains the momentum of a daughter particle in the rest frame of the parent $K^*$ with the mass $\sqrt{s}$. The $K^*(892)$ is a narrow resonance and we experienced numerical instabilities when calculating integrals containing the square of (20). To get rid of problems, we have used the procedure described in Appendix B.

Also for the $\sigma$ propagator we use the form with fixed mass and energy dependent width

$$-iG_\sigma(p) = \frac{1}{s - m_\sigma^2 + im_\sigma \Gamma_\sigma(s)},$$

where $\Gamma_\sigma(s)$ includes only the contribution from the two-pion decay channel and is equal to

$$\Gamma_\sigma(s) = \frac{m_\sigma^2}{s} \sqrt{s - 4m_\sigma^2} \frac{F(q(s))}{F(q(m_\sigma^2))} \Gamma_\sigma.$$

As the current Review of Particle Physics [39] is not very specific about the $f_0(600)$ mass and width, we rely on the mutually compatible values obtained by the Fermilab E791 Collaboration [56] and the CLEO Collaboration [57], who both analyzed the $D$ mesons decays. The results (in MeV) of E791 are $m_\sigma = 478_{-24}^{+24} \pm 17$, $\Gamma_\sigma = 324_{-46}^{+46} \pm 21$, whereas those of CLEO are $m_\sigma = 513 \pm 32$, $\Gamma_\sigma = 335 \pm 67$. We adopt the weighted averages $m_\sigma = 500$ MeV and $\Gamma_\sigma = 329$ MeV.

### III. EXPERIMENTAL DATA

We will compare the calculated three-pion mass distribution in the $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ and $\tau^- \rightarrow \nu_\tau \pi^- \pi^0 \pi^0$ decays with the outcome of the five experiments.

1. The ARGUS Collaboration [7] used the ARGUS detector at the DORIS II $e^+e^-$ storage ring at DESY and studied the $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ decay. Their background and acceptance corrected three pion mass distribution is given in twenty-eight bins within the mass range 0.425–1.775 GeV.

2. The OPAL Collaboration published their results on the three-pion-mass squared distribution in the charged-pion channel in two papers. The first of them [8] was based on the data collected with the OPAL detector at the CERN Large Electron-Positron Collider (LEP) during 1992 and 1993. We used it in our recent publication [51]. In this work we explore the updated version [9], in which also the data of 1994 were included. The three-pion-mass-squared plot is corrected for background and efficiency and consists of twenty-three bins with much smaller statistical errors than in [8].

3. The $\tau$-lepton decay into three pions and neutrino was also investigated by the CLEO Collaboration at the Cornell Electron Storage Ring (CESR). Their results on the all-charged-pions channel still exist only in a preliminary form [14]. We can therefore use only the data on the $\pi^- \pi^0 \pi^0$ channel [12]. The background-subtracted, efficiency corrected three-π mass spectrum is given in 47 bins.

4.5. The ALEPH Collaboration at CERN LEP have published an article summarizing their results about the branching ratios and spectral functions of the $\tau$ decays [15]. It is based on the data collected with the ALEPH detector during 1991-1995 but processed by an improved method. We use the tables of the corrected three-pion mass squared spectra both in the $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ and $\tau^- \rightarrow \nu_\tau \pi^- \pi^0 \pi^0$ decays, which are publicly accessible at the website [63]. The all-charged pion spectrum contains 116 bins 0.025 GeV$^2$ wide starting at 0.225 GeV$^2$. In the two-neutral pion case the spectrum starts at 0.2 GeV$^2$, but we discard the bin centered at 0.2375 GeV$^2$ with a zero value and, comparing to its neighbors, an unrealistically small error. We are thus left again with 116 bins. In both cases we ignore the correlation matrices among the errors in different bins and add statistical and systematic errors linearly.
IV. CALCULATIONS AND RESULTS

To be sure that our results are free of programming errors, we have written two independent computer codes, one in C++ (M.V.), another in FORTRAN 95 (P.L.) and debugged them until they produced identical results.

We found that the parity-violating term in the $\tau$ decay amplitude influences the three-pion-mass distribution only negligibly and have not considered it any longer in our calculations. The decay amplitude $M$ in (A1) then depends only on relativistic invariants $s_{ij} = (p_i + p_j)^2$, which are symmetric against transformation $\varphi' \rightarrow 2\pi - \varphi'$. Using this symmetry when calculating the innermost integral in (A1) enables us to speed up the computing by a factor of two.

In the diagrams with the $\sigma$ in the intermediate states, depicted in Figs. 1 and 2, also the product $h_{a_1\sigma\sigma\sigma\sigma\pi\pi}$ plays a role (for definitions, see Sec. II B). We introduce additional free parameter

$$z = \frac{h_{a_1\sigma\pi}}{g_{a_1\pi\pi}},$$

which allows us to express that unknown product as a multiple of $g_{a_1\sigma\sigma\sigma\sigma\pi\pi}$, which is determined by the method described in Sec. II C. The parameter $z$ itself will be obtained by minimalization of $\chi^2$ when fitting the experimental three-pion mass distributions in the threepion decays of the $\tau$ lepton.

To be closer to experimental conditions, we do not calculate the (unnormalized) differential decay rate at certain values of the three-pion mass $W$ or its square $Q^2 \equiv W^2$, but its averages over the experimentally given bins in $W$ [7, 12] or $Q^2$ [9, 15].

When the contribution of the $a_1'$ to the $a_1$ propagator is not considered, the calculated differential decay rates, and thus also the $\chi^2$ evaluated from them and data, depend on the following four parameters: (1) the nominal $a_1$ mass $m_{a_1}$, (2) the nominal $a_1$ width $\Gamma_{a_1}$, (3) the $a_1\rho\pi$ Lagrangian mixing parameter $\sin \theta$, and (4) the off-shell-mass coupling constant ratio $z$ defined by Eq. (21). Quantity $y$ (16) is not an extra parameter, condition (7) determines it as an implicit function of $m_{a_1}$ and $\sin \theta$.

The ARGUS, OPAL, and CLEO experiments present the three-pion mass spectra in the acceptance corrected number of events. Both ALEPH spectra are normalized to the integrated branching fractions. As the outcome of our model is not normalized (the coupling of the $a_1$ meson to the $W$ boson is not fixed by meson dominance [64]), we opt to compare just shape of the mass distribution and introduce five multiplicative constants. The values of them are obtained by minimizing the individual $\chi^2$'s for each experiment while keeping the common parameters ($m_{a_1}$, $\Gamma_{a_1}$, $\sin \theta$, and $z$) fixed.

To get a quick insight into the dependence of the quality of the fit on the Lagrangian mixing parameter $\sin \theta$, we first fix the basic $a_1$ parameters at the “standard” values, frequently used in theoretical considerations, namely, $m_{a_1} = 1.23$ GeV/$c^2$ and $\Gamma_{a_1} = 0.4$ GeV.

We also set $z = 0$ and calculate the ratio of the usual $\chi^2$ to the number of experimental points $N$ for each of the five data sets as a function of $\sin \theta$. The results are shown in Fig. 3. It is clear that the choice of the correct $a_1\rho\pi$ Lagrangian is of the utmost importance for obtaining a good agreement with the data. The fact that all five experiments point to the same narrow region in $\sin \theta$ is extremely important. In addition, this region overlaps with the interval (0.457, 0.471) based on the results of the model [53, 54] of the electron-positron annihilation into four pions built around the same Lagrangian (1). It indicates the soundness both of the present model and of the $e^+e^- \rightarrow 4\pi$ annihilation model.

The region $\sin \theta \gtrsim 0.5$ is not shown in Fig. 3 because for those values of $\sin \theta$ it is impossible to satisfy condition (7) by procedure described in Sec. II C. The square of parameter $y$, defined by Eq. (16), acquires negative, i.e. unphysical, values, which mean the negative branching ratio of decay (12).

In the next step we allow all four parameters to vary and use the CERN computer library program MINUIT of James and Roos [65] for finding their values that minimize $\chi^2$ for the three data sets defined in Sec. III. The results are summarized in Table II. As always, the assessment of errors of the parameters is a difficult task. We combined the errors provided by MINUIT, which reflect the errors of experimental data, with our estimates of the errors induced by the uncertainties of the input parameters (the $\sigma$ mass, various coupling constants). The agreement of our model with the ALEPH $\pi^-\pi^0\pi^0$ data

![Figure 3](image-url)
TABLE II. Results of fitting various data sets. Only $a_1(1260)$ considered. Parameter $y$ is defined by Eq. (16), parameter $z$ by Eq. (21). All= ARGUS [7] + OPAL [9] + CLEO [12] + both ALEPH [15] data sets. For comparison, the values of Lagrangian mixing parameter from the $e^+e^-$ annihilation into four pions are also shown.

| Data       | Type  | $\chi^2$/NDF | C.L. (%) | $m_{a_1}$ (MeV) | $\Gamma_{a_1}$ (MeV) | $\sin \theta$ | $y$ (GeV$^2$) | $z$     |
|------------|-------|---------------|----------|-----------------|----------------------|--------------|--------------|---------|
| ALEPH [15] | $\pi^+\pi^+\pi^-$ | 119.1/111 | 28.25    | 1220 ± 20       | 418 ± 40             | 0.460 ± 0.004 | 0.094 ± 0.010 | 0.31 ± 0.03 |
| ALEPH [15] | $\pi^-\pi^0\pi^0$ | 51.5/111  | 100.00   | 1256 ± 10       | 443 ± 15             | 0.466 ± 0.004 | 0.111 ± 0.022 | 0.12 ± 0.16 |
| All Mixed  |        | 357.7/321    | 7.74     | 1232 ± 25       | 431 ± 25             | 0.463 ± 0.005 | 0.099 ± 0.009 | 0.30 ± 0.05 |

$e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$ \cite{53}  
$e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^- \& \pi^+\pi^-\pi^0\pi^0$ \cite{54}  

The agreement of the model with data in all cases. For the all-argon, CLEO, ALEPH $\pi^-\pi^+\pi^-$, and ALEPH $\pi^-\pi^0\pi^0$ data. The mass and width of the $a_1$ as well as other two free parameters (sin $\theta$ and $z$) are very stable against the inclusion of $a_1'$. Their new values (Table III) differ only very little from the corresponding old ones (Table II). Also the values obtained from different data sets are mutually compatible. This is true also for two new parameters Re $\alpha$ and Im $\alpha$.

The calculated three-pion-mass distribution is compared to the $\pi^-\pi^+\pi^-$ data of ALEPH Collaboration [15] in Fig. 4. We have just learned that the $a_1(1640)$ resonance greatly improves the agreement with data. It is therefore a little surprising that there is no bump or shoulder corresponding to this resonance visible in

Fig. 4. Three-pion-mass-squared distribution calculated assuming both $a_1$ and $a_1'$ contributions and compared to the ALEPH [15] $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ data. The model parameters taken from Table III.
TABLE III. Results of fitting various data sets. Both $a_1(1260)$ and $a_1(1640)$ are considered. For definition of $\alpha$, see Eq. (18).

| Data          | Type | $\chi^2$/NDF | C.L. | $m_{a_1}$ (MeV) | $\Gamma_{a_1}$ (MeV) | $\sin \theta$ | $y$ (GeV$^2$) | $z$ | $\text{Re } \alpha$ | $\text{Im } \alpha$ |
|---------------|------|--------------|------|----------------|----------------------|----------------|---------------|-----|---------------------|---------------------|
| ALEPH [15]    | $\pi^- \pi^+ \pi^-$ | 30.7/109 | 100% | 1218 ± 19 | 418 ± 30 | 0.457(4) | 0.106 ± 0.019 | 0.34 ± 0.03 | −0.30 ± 0.10 | 0.31 ± 0.06 |
| ALEPH [15]    | $\pi^- \pi^0 \pi^0$ | 12.3/109 | 100% | 1255 ± 18 | 455 ± 15 | 0.457(6) | 0.148 ± 0.025 | 0.36 ± 0.14 | −0.34 ± 0.13 | 0.29 ± 0.10 |
| All           | Mixed | 219.5/318 | 100% | 1233 ± 18 | 431 ± 20 | 0.459(4) | 0.114 ± 0.014 | 0.34 ± 0.05 | −0.31 ± 0.10 | 0.32 ± 0.09 |

![FIG. 5](image1.png)

FIG. 5. Investigating the role of the $a_1'$ resonance in fitting the ALEPH $\pi^- \pi^+ \pi^-$ data. Full curve: the complete calculation shown in Fig. 4; Dotted curve: parameters unchanged, but only the $a_1(1260)$ term (5) in the $a_1$ propagator; Dashed curve: parameters unchanged, but only the $a_1'$ term (18) in the $a_1$ propagator. Note the change of scale against Fig. 4.

![FIG. 6](image2.png)

FIG. 6. Investigating the role of the $a_1'$ resonance in fitting the ALEPH $\pi^- \pi^0 \pi^0$ data. Full curve (buried in data): the complete calculation; Dotted curve: parameters unchanged, but only the $a_1(1260)$ term (5) in the $a_1$ propagator; Dashed curve: parameters unchanged, but only the $a_1(1640)$ term (18) in the $a_1$ propagator.

V. SUMMARY AND CONCLUSIONS

The main message of this study is that the form of the $a_1 \rho \pi$ Lagrangian is the decisive factor for achieving a good model description of the three-pion decays of the tau lepton. This is again illustrated in Fig. 7 where the total $\chi^2$, which is calculated as a sum of the individual $\chi^2$’s from the five experiments, is divided by the total number of experimental points and plotted as a function of the Lagrangian mixing parameter $\sin \theta$. Two different cases are considered: (1) only $a_1(1260)$ included in the $a_1$ propagator, (2) both $a_1(1260)$ and $a_1(1640)$ included. In contrast to Fig. 3, the other parameters are fixed at their optimal values taken from the appropriate tables (Tabs. II and III). Even if the curve (2) is shifted a little toward smaller values of $\sin \theta$, the minima of both curves fall to the interval found in the model of the $e^+e^-$ annihilation into four pions [53, 54].

Our further finding, even not documented in this work in detail, concerns the form of the $a_1$ propagator. We have found that the running mass form (5), suggested and already used in several papers [12, 18, 19], provides a better fit to the tauon three-pion decay data than simpler forms with a constant $a_1$ mass and a constant or energy dependent $a_1$ total decay width.

The $a_1$ running mass squared is given by the dispersion relation. In this work we have chosen a once-subtracted version (9). As the input for the dispersion relation, the energy dependent total decay width of the $a_1$ for all $s$ above the three-pion threshold is required (9). We approximated it as a sum of the decay widths to the three pion final states (via the $\pi \rho$ and $\pi \sigma$ intermediate states) and the $K \bar{K} \pi$ final states (via $K^*(892)\bar{K} + c.c.$). A typical behavior of the energy dependent total width is shown in Fig. 8. The hump centered around $\sqrt{s} \approx 1$ GeV develops as the mass of the two-pion subsystem falls predominantly first on the ascending and then on the descending side of the rho propagator. We ignored the channels $\rho(1450)\pi$, $f_0(1370)\pi$, and $f_2(1270)\pi$, which have been
FIG. 7. $\sin \theta$ dependence of the sum of $\chi^2$s from all five experiments divided by the total number of experimental points. The other parameters are kept at the optimal values from the “All” row of the corresponding tables. Dotted curve: only $a_1(1260)$ (Table II) included in the $a_1$ propagator; Full curve: both $a_1(1260)$ and $a_1(1640)$ included (Table III). The range of $\sin \theta$ from the electron-positron annihilation into four pions [53, 54] is shown as a short abscissa.

FIG. 8. Energy dependent width of the $a_1(1260)$ as a function of $\sqrt{s}$. Parameters $m_{a_1}$, $\Gamma_{a_1}$, and $\sin \theta$ taken from Table III, row “All”.

FIG. 9. Running mass of the $a_1(1260)$ as a function of $\sqrt{s}$ for the same parameters as Fig. 8. The cross marks the point in which condition (6) is satisfied.

seen in the $a_1$ decays [39] and which open at higher $s$. This is probably the reason why the running mass behaves wildly, see Fig. 9, and does not have a nice plateau around the nominal mass, as it did in the case of the $\rho(770)$ [59].

Another important ingredient of our model are the $\pi\sigma$ intermediate states. On one side, they enter the calculation of the total decay width of the $a_1$ resonance, which is necessary for constructing the running mass propagator (5). On the other side, they contribute to the decay rates of the three-pion decays of the tau lepton, Figs. 1 and 2.

To investigate the role of the $\pi\sigma$ intermediate states in the evaluation of the $\tau^- \to \nu_\tau \pi^- \pi^0 \pi^0$ decay width, we split the distribution depicted in Fig. 6 by the full curve into its $\pi\rho$ and $\pi\sigma$ components. The result is shown in Fig. 10. It is obvious that the $\pi\sigma$ intermediate states play a unique role in describing the behavior of the differential decay width at small three-pion masses. What is a little suspicious, is the large magnitude at the intermediate masses. To see whether it is reasonable or not, we integrate the distributions to get the branching ratio

$$B = \frac{\Gamma(\tau^- \to \nu_\tau \pi^- \pi^- \pi^0 \pi^0)}{\Gamma(\tau^- \to \nu_\tau \pi^- \pi^0 \pi^0)}$$

with the result $B \approx 41\%$. This number is more than twice higher than the experimental value of $(16.18 \pm 3.85 \pm 3\%$).

The inclusion of them would bring additional free parameters, what we wanted to avoid.
The source of this obvious deficiency of our model is the following: The important parameter $y$, which regulates the rate of the $a_1 \rightarrow \pi\sigma \rightarrow 3\pi$ transition, was obtained from the condition that the derivative of the running mass at the nominal-mass point should vanish (7). But the absence of higher decay channels, which influence the values of the running mass at all $s$, may modify the resulting value of $y$ significantly. The larger than correct $y$ may mimic the missing channels.

A new feature of our work, which, to our knowledge, has not appeared in the literature yet, is that we fit the data from several experiments simultaneously. We intend to continue in this approach and include not only the data concerning the three-pion decays of the tau lepton, but also the experimental results from other weak, electromagnetic, and perhaps also strong interaction processes. The natural candidate is the electron-positron annihilation into four pions, for which a model based on the same Lagrangian as here has already been built. The value of the Lagrangian mixing parameter $\sin \theta$ perfectly agrees with the value obtained from the $e^+e^-$ annihilation into four pions.

(2) Our confirmation of the existence of the $a_1(1640)$ resonance with the mass and width compatible with the PDG [39] values is based on the increase of the confidence level from 7.7% to 100% after the $a_1(1640)$ has been included.

(3) We have explained why the $a_1(1640)$ resonance, which is important for getting a good agreement with data, is not visible in the three-pion-mass spectrum as a bump or shoulder.

(4) From the common fit to the data from five experiments we have obtained the following results:

- Mass of the $a_1(1260)$ $m_{a_1} = (1233 \pm 18)$ MeV;
- Width of the $a_1(1260)$ $\Gamma_{a_1} = (431 \pm 20)$ MeV;
- Lagrangian mixing parameter $\sin \theta = 0.459 \pm 0.004$.

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Appendix A: Differential decay rate formula

We use the following formula for the differential decay rate in the invariant three-particle mass $W = \sqrt{(p_2 + p_3 + p_4)^2}$ in a four-body decay $a \rightarrow 1 + 2 + 3 + 4$:

$$\frac{d\Gamma}{dW} = \frac{|p_1|}{16(2\pi)^6m_2^2} \int_{m_3+m_4}^{W-m_2} dm_{34} |p_2'| |p_3'| \times \int^{-1}_1 d\cos \theta_2^* \int^{-1}_1 d\cos \theta_3 \int^{2\pi}_0 d\varphi_3' |\mathcal{M}|^2.$$ (A1)

The asterisk denotes the (2,3,4) rest frame, the prime the (3,4) rest frame. $m_{34}$ is the mass of the system consisting of particles 3 and 4, $E_{34} = E_3 + E_4$ and $\mathbf{P}_{34} = \mathbf{p}_3^* + \mathbf{p}_4 = -\mathbf{p}_2^*$ are its energy and momentum, respectively, in the (2,3,4) rest frame. In the rest frame of the parent particle $a$ the momentum of particle 1 points along the negative $z$-axis. In the (2,3,4) rest frame, the momentum of particle 2 lies in the $(xz)$ plane.

Appendix B: Integrating over a narrow peak

Let us assume that we need to evaluate an integral over an interval that includes a narrow resonance peak

$$Q = \int_{s_1}^{s_2} \frac{f(s)}{(s-M^2(s))^2 + m^2\Gamma^2(s)} ds,$$ (B1)

where $f(s)$ is a slowly varying function. Further, let the two functions in the denominator satisfy conditions
$M^2(m^2) = m^2$ and $\Gamma(m^2) = \gamma$. If $\gamma \ll m$ then the integrand is rapidly varying function of $s$ and a numerical quadrature of very high order is required to get reliable results. After introducing a new variable $\xi$ by substitution $s = m^2 + m \gamma \tan(e \xi + d)$, where $e = (a_2 - a_1)/2$, $d = (a_1 + a_2)/2$, $a_1 = \arctan((s_1 - m^2)/(m \gamma))$, and $a_2 = \arctan((s_2 - m^2)/(m \gamma))$, the integral (B1) becomes

$$Q = \frac{a_2 - a_1}{2m\gamma} \int_{-1}^{1} \frac{(s - m^2)^2 + m^2\gamma^2}{(s - M(s))^2 + m^4\Gamma^2(s)} f(s) \, d\xi,$$

which can be safely evaluated using, e.g., the Gauss-Legendre quadrature. We apply this method for calculating the integrals containing the square of the $K^*(892)$ propagator (20). In that case $M(s) \equiv m = m_{K^*}$.

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