Continuous-time multi-type Ehrenfest model and related Ornstein-Uhlenbeck diffusion on a star graph

Antonio Di Crescenzo†  Barbara Martinucci‡  Serena Spina§

Abstract
We deal with a continuous-time Ehrenfest model defined over an extended star graph, defined as a lattice formed by the integers of $d$ semiaxis joined at the origin. The dynamics on each ray are regulated by linear transition rates, whereas the switching among rays at the origin occurs according to a general stochastic matrix. We perform a detailed investigation of the transient and asymptotic behavior of this process. We also obtain a diffusive approximation of the considered model, which leads to an Ornstein-Uhlenbeck diffusion process over a domain formed by semiaxis joined at the origin, named spider. We show that the approximating process possesses a truncated Gaussian stationary density. Finally, the goodness of the approximation is discussed through comparison of stationary distributions, means and variances.

Keywords: Branching processes; Diffusion processes; Ehrenfest model; Ornstein-Uhlenbeck process; Stationary distribution

1 Introduction
The celebrated Ehrenfest model is a Markov chain over a finite state space, with linearly state-depending transition rates and reflecting endpoints, that was suitably proposed to describe the diffusion of gas molecules in a container. It is widely studied as a prototype for random motions in physics and in applied sciences, and for modeling random phenomena in thermodynamics and chemistry (see, for instance, Balaji et al. [4] and Flegg et al. [16]). Modified versions of the basic model have been considered such that (i) a general probabilistic rule holds for the system state change (cf. Hauert et al. [23]), (ii) the presence of additional large jumps is used to explain certain features emerging in finance for returns in stock index prices and exchange rates for currencies (cf. Takahashi [38]), (iii) catastrophes occurring at constant rate force the system to reset into state 0 (cf. Dharmaraja et al. [13]).

In this paper we investigate a multi-type extension of the continuous-time Ehrenfest model, and its diffusive approximation based on the Ornstein-Uhlenbeck process. The state space of the extended model is a finite lattice, say $S$, formed by the integers $0, 1, 2, \ldots, N$ of $d$ lines joined at the origin, thus constituting an extended star graph. The evolution of the stochastic process over each line of $S$ evolves as a classical Ehrenfest model, i.e. as a continuous-time skip-free Markov chain (as a birth-death process) with linear decreasing upward transition rate $\lambda(N - k)$ and increasing downward transition rate $\mu(N + k)$ at $k$. The state $k = N$ is reflecting since the upward transition rate vanishes therein. Moreover, the transitions from the state 0 to each of the $d$ lines are governed by rates depending on the elements of a stochastic matrix. The transitions of the process from a line to another one correspond to the type changes of the considered multi-type Ehrenfest model. The case $d = 2$ corresponds to the one-dimensional Ehrenfest model. Our analysis, based on the probability-generating-function approach, allows to determine the explicit expression of the transient probabilities (cumulative on the
rays) when $\lambda = \mu$, and the asymptotic probabilities for any choice of parameters $\lambda$ and $\mu$. In particular, when $\lambda = \mu$, the asymptotic distribution is strictly related to the binomial distribution with parameters $(2N, \frac{1}{2})$.

A similar process describing the dynamics of a multi-type birth-death-immigration process has been analyzed in Di Crescenzo et al. [14], where the transitions on the states of a star graph with various semiaxis are regulated by linear increasing transition rates. This process was also studied under certain limit conditions that lead to a diffusion process on the star graph with linear drift and infinitesimal variance on each ray. In the realm of mathematical biology, other investigations devoted to birth-death processes on graphs are due to Allen et al. [2], Kaveh et al. [26], and Sui et al. [37], for instance. Furthermore, the analysis of birth-death processes on networks and lattice structures viewed as graphs have been performed to model evolutionary systems also by means of the mean-field methods (cf. Granovsky and Zeifman [21], and Peliti [81]).

The difficulties related to the analysis of discrete evolution models on star graph stimulated several authors to consider alternative models consisting in diffusion processes on the state space formed by semiaxis joined at the origin (also known as spider). In this framework, we recall the contribution by Benichou and Desbois [4], where a Brownian particle diffusing along the links of a general graph is considered and relevant quantities are computed for different kinds of graphs, such as for star graphs. Other investigations concerning the dynamics of the Brownian motion on the spider are due to Csáki et al. [10] and Kostrykin et al. [27], also with care to the possible boundary conditions at the vertex in view of important applications. Furthermore, see Dassios and Zhang [12] for the analysis of the reflected Brownian motion with drift on a finite collection of rays, in view of possible applications in risk theory finalized to price the Parisian type options. In addition, Papanicolaou et al. [30] also pointed out that diffusion processes of this kind can be applied to spread of toxic particles in a system of channels or vessels, or to propagation of information in networks. In this framework, we recall that one of the first contributions on diffusion processes on graphs was given by Freidlin and Wentzell [17]. Occupation time functionals for birth-death processes and diffusion processes on graphs were studied by Weber [39].

Along the line of the above mentioned researches, after investigating the transient and the asymptotic behavior we employ a scaling procedure that leads to a diffusive approximation of the considered model. The resulting process is an Ornstein-Uhlenbeck diffusion on the spider with special reflecting-type conditions on the vertex at the origin. In the papers by Csáki et al. [10], and Dassios and Zhang [12], the switching of the Brownian motion between the semiaxis is regulated by independent general distributions, whereas in the contribution by Papanicolaou et al. [30] it follows a uniform distribution over the rays. In the present paper we provide the explicit expression of the stationary probabilities for the diffusive approximation in the cases such that when the diffusive particle reaches the vertex then the choice of the next line occurs
1. uniformly to any of the $d$ lines,
2. uniformly to any of the $d - 1$ lines different from the originating one,
3. toward the next line, from $l$ to $l + 1$, and from $d$ to $1$, thus visiting cyclically any line,
4. toward the next line, from $l$ to $l + 1$, until it reaches the last line, i.e. line $d$,
5. toward one of the adjacent lines, according to a random walk scheme.

It is worth mentioning that the Ornstein-Uhlenbeck process, obtained here through a diffusive approximation, has been largely investigated for its important applications in several fields, in particular in the context of neuronal activity. Ricciardi and Sacerdote [38] provided one of the first contributions in this area, by studying mean and variance of the first-passage time through a constant boundary. We recall also Lansky et al. [28] for the analysis of an optimum signal in the related neuronal model, and Buonocore et al. [7] for applications in neuronal models with periodic input signals through an Ornstein-Uhlenbeck process in the presence of a reflecting boundary. The membrane potential is also modeled by an Ornstein-Uhlenbeck process on the star graph with linear drift and infinitesimal variance on each ray. In the realm of evolutionary systems also by means of the mean-field methods (cf. Granovsky and Zeifman [21], and Peliti [81]).

**Plan of the paper:** In Section 2 we provide a thorough description of the stochastic model and the differential-difference equations for the transient probabilities. We also describe some possible fields of application of the considered model. Section 3 is devoted to the analysis of the stochastic process, with special attention to the determination of the probability generating functions, which allow to obtain a closed-form expression of the probabilities in the special case when $\lambda = \mu$. Comparisons between exact probabilities and their estimates based on simulation are also provided. Various asymptotic results are then investigated in Section 4 as time tends to infinity, including the asymptotic probability generating function and the corresponding stationary probabilities, with mean, variance and coefficient of variation. We also investigate the (Shannon) entropy of the...
system in the stationary phase, and show its maximum over the ratio $\lambda/\mu$ of rates, which depends on the number $N$. Section 5 is concerning the diffusion approximation that leads to a diffusion process on the spider through a suitable scaling procedure. We determine the partial differential equation for the transient probability density of the process, with the reflecting/switching condition at the vertex of the spider. The equations of the cumulative density on the rays of the spider correspond to those of the Ornstein-Uhlenbeck process in the presence of a reflecting boundary at 0. Thus, we obtain the joint asymptotic probability distribution of the process, which is formed by two independent laws: (i) the density of the location on the ray of the spider, which has a truncated Gaussian form, and (ii) the distribution of the occupied ray, which depends strictly on the probabilities that govern the switching mechanism between the rays. Some possible choices of the switching probabilities are studied in order to come to a complete description of the asymptotic distribution of the diffusion process. Some comparisons between the distributions of the discrete model and the diffusive approximating process are provided to illustrate the goodness of the approximation. Finally, concluding remarks on possible future developments are given in Section 6.

Throughout the paper, $N$ denotes the set of positive integers, and $N_0 = \{0\} \cup \mathbb{N}$.

2 The multi-type Ehrenfest model

We consider a system that can accommodate at most $N$ particles, with $N \in \mathbb{N}$, and such that $d$ types of particles are allowed, for $d \in \mathbb{N}$. The set of possible types is denoted by $D := \{1, 2, \ldots, d\}$. Moreover, the particles accommodated simultaneously in the system must be of the same type. The particle dynamics is regulated by the following assumptions, where $h > 0$ is sufficiently small:

(a) If the system at time $t$ contains $k = 1, 2, \ldots, N$ particles of type $j \in D$, then during the time interval $(t, t + h]$ either one particle leaves the system with probability $\mu(N + k)h + o(h)$, or a new particle of the same type joins the system with probability $\lambda(N - k)h + o(h)$, or the particle number is unchanged with probability $1 - [\mu(N + k) + \lambda(N - k)]h + o(h)$.

(b) If the system is empty at time $t$, then during the time interval $(t, t + h]$ either the system is occupied by a particle of type $j \in D$, with probability $c_{l,j}Nh + o(h)$, assuming that the last particle in the system was of type $l \in D$, or the system remains empty with probability $1 - \lambda Nh + o(h)$.

From the above assumptions we have that $\lambda$ and $\mu$ are positive parameters that regulate the joining and leaving intensities of the particles, respectively. Moreover, assumption (a) implies that the arrivals of new particles are inhibited if the system contains $N$ particles. The coefficients $c_{l,j}$ actually form the discrete probability distribution that regulates the switching mechanism for the particle types, that acts when the system empties. We have

$$c_{l,j} \geq 0, \quad \sum_{j \in D} c_{l,j} = 1, \quad \forall l, j \in D, \quad (1)$$

so that $C := \{c_{l,j}\}_{l,j \in D}$ is a stochastic matrix.

Let us now introduce the continuous-time Markov chain $\{(N(t), L(t)), t \geq 0\}$ that describes the system dynamics, such that, at time $t$, $N(t) = k$ gives the number of particles in the system, and $L(t) = j$ gives the type of such particles. The state space of the process is the set $S_0 = \{(0,0)\} \cup (\mathbb{N} \times D)$, with $\mathbb{N} := \{1, 2, \ldots, N\}$, consisting of the integers of $d$ segments $S_1, S_2, \ldots, S_d$ ($d \in \mathbb{N}$) with a common extreme $(0,0)$ (see Figure 1).

We denote $S = S_0 \setminus \{(0,0)\}$ and, for simplicity, we write 0 instead of $(0,0)$.

Formally, the system dynamics is regulated by the transition rates

$$q(\alpha; \beta) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}[(N(t + h), L(t + h)) = \beta | (N(t), L(t)) = \alpha], \quad \alpha \in S, \; \beta \in S_0,$$

$$q(0; 1, j; l) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}[(N(t + h), L(t + h)) = (1, j) | (N(t), L(t)) = 0, J(t) = l], \quad j, l \in D,$$

where $J(t)$ is the last state visited by the Markov chain before arriving in 0. According to the assumptions (a) and (b), the following relations hold, for $l, j \in D$,

$$q(k, j; k - 1, j) = \mu(N + k), \quad k \in \mathbb{N},$$

$$q(k, j; k + 1, j) = \lambda(N - k), \quad k \in \mathbb{N},$$

$$q(0; 1, j; l) = c_{l,j} \lambda N, \quad (2)$$

$$\lambda/\mu$$
where $\lambda, \mu > 0$, and $c_{l,j}$ satisfy the conditions [1]. Moreover, for $i, j, k, r \in \mathbb{N}$ one has

$$q(k, j; r, j) = 0 \quad \text{if } |k - r| > 1,$$

$$q(k, i; r, j) = 0 \quad \text{if } i \neq j,$$

$$q(0, 0; r, j) = q(r, j; 0, 0) = 0 \quad \text{if } r \neq 1.$$

Note that $c_{l,j}\lambda N$ represents the intensity of the arrival of a new particle of type $j$, given that the system is empty and the last previous particle in the system was of type $l$. We remark that the considered Markov chain is a skip-free process and that $(0, 0)$ is a non-absorbing state. Moreover, the given process is bounded, and hence uniquely determined by the transition rates (cf. Chen et al. [8]).

Let us now introduce the transition probabilities of the process $\{(N(t), L(t)), t \geq 0\}$. Assuming that the initial condition is given by $(N(0), L(0)) = 0, J(0) = l_0$, with $l_0 \in D$, we consider

$$p(0, l, \cdot) := P\{ (N(\cdot), L(\cdot)) = 0, J(\cdot) = l \mid (N(0), L(0)) = 0, J(0) = l_0 \}, \quad l \in D,$$

$$p(k, j, \cdot) := P\{ (N(\cdot), L(\cdot)) = (k, j) \mid (N(0), L(0)) = 0, J(0) = l_0 \}, \quad k \in \mathbb{N}, \ j \in D,$$

with initial conditions expressed as

$$p(0, l, 0) = \delta_{l,l_0},$$

(4)

where $\delta_{l,l_0}$ is the Kronecker delta, and

$$p(k, j, 0) = 0, \quad k, j \in \mathbb{N}.$$

(5)

We are now able to provide the Kolmogorov forward equations governing the transition probabilities [3].
Recalling the rates (2), the following system of differential-difference equations holds, for \( j \in D \), and \( t > 0 \):

\[
\begin{align*}
\frac{d}{dt} p(0,j,t) &= \mu(N + 1) p(1,j,t) - \lambda N p(0,j,t), \\
\frac{d}{dt} p(1,j,t) &= \mu(N + 2) p(2,j,t) + \sum_{l \in D} c_{l,j} \lambda N p(0,l,t) - [\lambda(N - 1) + \mu(N + 1)] p(1,l,t), \\
\frac{d}{dt} p(k,j,t) &= \mu(N + k + 1) p(k + 1,j,t) + \lambda(N - k + 1)p(k - 1,j,t) \\
&\quad - [\lambda(N - k) + \mu(N + k)] p(k,t), \quad k \in \mathbb{N} \setminus \{1, N\} \\
\frac{d}{dt} p(N,j,t) &= \lambda p(N - 1,j,t) - \mu 2N p(N,j,t).
\end{align*}
\]

Moreover, we can express the marginal probabilities for the number of particles in the system in terms of probabilities (3) as follows:

\[
p(0, \cdot) := \mathbb{P}\{(N(\cdot), L(\cdot)) = 0 \mid (N(0), L(0)) = 0, J(0) = l_0\} = \sum_{l \in D} p(0,l,\cdot) \tag{7}
\]

and

\[
p(k, \cdot) := \mathbb{P}\{N(\cdot) = k \mid (N(0), L(0)) = 0, J(0) = l_0\} = \sum_{j \in D} p(k,j,\cdot), \quad k \in \mathbb{N}. \tag{8}
\]

Taking into account the conditions (1), from the system (6) it follows that the probabilities (7) and (8) satisfy the following Kolmogorov forward equations, for \( t > 0 \):

\[
\begin{align*}
\frac{d}{dt} p(0,t) &= \mu(N + 1) p(1,t) - \lambda N p(0,t), \\
\frac{d}{dt} p(1,t) &= \mu(N + 2) p(2,t) + \lambda N p(0,t) - [\lambda(N - 1) + \mu(N + 1)] p(1,t), \\
\frac{d}{dt} p(k,t) &= \mu(N + k + 1) p(k + 1,t) + \lambda(N - k + 1)p(k - 1,t) \\
&\quad - [\lambda(N - k) + \mu(N + k)] p(k,t), \quad k \in \mathbb{N} \setminus \{1, N\} \\
\frac{d}{dt} p(N,t) &= \lambda p(N - 1,t) - \mu 2N p(N,t).
\end{align*}
\]

Due to (4) and (5), the related initial conditions are given by

\[
p(0,0) = 1, \quad p(k,0) = 0, \quad k \in \mathbb{N}. \tag{9}
\]

We pinpoint that the present model deserves interest in several contexts. For instance, the review of Crawford and Suchard [9] points out how various kinds of birth-death processes can be applied in ecology, genetics, and evolution. Moreover, the paper by Giorno et al. [18] shows that a process with linear decreasing birth rate and linear increasing death rate can be used to describe the number of customers in a finite-capacity queue. In this setting, the process with rates (2) can also be viewed as a model for the evolution of a multi-type queueing system, where the following rules hold:

- new customers are discouraged from joining long queues,
- the system can accommodate at most \( N \) customers,
- the server adapts the service rate to the number of customers,
- there are \( d \) types of customers,
- all customers in the system belong to the same type,
- the jockeying mechanism governed by the stochastic matrix \( C \) allows to switch possibly from a type to another type of customers when the system is empty.
3 Analysis of the model

In this section we use the generating function-based approach to investigate the transient dynamics of the considered system. To this aim, let us consider the probability generating function

\[ F(z, t) := \mathbb{E} \left[ z^{N(t)} \mid (N(0), L(0)) = (0, 0) \right] = p(0, t) + \sum_{k \in \mathbb{N}} z^k p(k, t), \quad z \in [0, 1], \quad t \geq 0, \quad (10) \]

where the state probabilities \( p(0, t) \) and \( p(k, t) \) have been defined in Eqs. (7) and (8), respectively. By virtue of (9), the following initial condition holds:

\[ F(z, 0) = 1, \quad z \in [0, 1]. \quad (11) \]

Moreover, from (10) one has the boundary conditions

\[ F(1, t) = 1, \quad F(0, t) = p(0, t), \quad t \geq 0. \quad (12) \]

Proposition 3.1 The generating function (10) satisfies the following partial differential equation for \( z \in [0, 1] \) and \( t \geq 0 \):

\[ \frac{\partial}{\partial t} F(z, t) = (1 - z) \left[ -\mu z^{-1} p(0, t) + \frac{N}{z} (\mu - \lambda z) F(z, t) + (\mu + \lambda z) \frac{\partial}{\partial z} F(z, t) \right]. \quad (13) \]

Proof. Recalling Eqs. (7) and (8), the probability generating function (10) can be expressed in terms of (3) as

\[ F(z, t) = p(0, t) + \sum_{k \in \mathbb{N}} z^k \sum_{j \in D} p(k, j, t) = \sum_{j \in D} [p(0, j, t) + G_j(z, t)], \quad (14) \]

where we have set

\[ G_j(z, t) := \sum_{k \in \mathbb{N}} z^k p(k, j, t), \quad z \in [0, 1], \quad t \geq 0. \]

From the system (4), for every \( j \in D \), the probability generating function \( G_j(z, t) \) satisfies the following differential equation:

\[ \frac{\partial}{\partial t} G_j(z, t) = (1 - z)(\mu + \lambda z) \frac{\partial}{\partial z} G_j(z, t) + \frac{N}{z} (1 - z)(\mu - \lambda z) G_j(z, t) + (N + 1) \mu p(1, j, t) + z \sum_{l \in D} c_{l,j} \lambda N p(0, l, t). \]

Hence, the equation (13) follows making use of (6), (14) and condition (1).

Hereafter, the result given in Proposition 3.1 is used to obtain an integral form of \( F(z, t) \).

Proposition 3.2 For all \( \lambda, \mu > 0 \), Eq. (13), with conditions (11) and (12), admits the following solution for \( z \in [0, 1] \) and \( t \geq 0 \):

\[ F(z, t) = \left[ \frac{(\mu(z - 1) + (z\lambda + \mu) e^{(\lambda+\mu)t}) (\lambda(1 - z) + (z\lambda + \mu) e^{(\lambda+\mu)t})}{(\lambda + \mu)^2 e^{2(\lambda+\mu)t} z} \right]^N \frac{\mu N (1 - z)}{z^N (\lambda + \mu)^{2N-1}} \times \int_0^t p(0, y) e^{-2N(t-y)(\lambda+\mu)} \left[ (z\lambda + \mu) e^{(t-y)(\lambda+\mu)} - \lambda (z - 1) \right]^N \times \left[ (z\lambda + \mu) e^{(t-y)(\lambda+\mu)} + \mu(z - 1) \right]^{N-1} dy. \quad (15) \]
By solving this linear first order differential equation, we obtain

\[
\frac{d\lambda}{ds} = (\mu + \lambda z)(z - 1), \quad \frac{dt}{ds} = 1, \quad \frac{dF}{ds} = -\frac{\mu N(1 - z)}{z}p(0, t) - \frac{N}{z}(1 - z)(\lambda z - \mu)F(z, t). \tag{16}
\]

From Eqs. (16), along the characteristic curves

\[
z = \frac{\mu + \lambda \tau + \mu(\tau - 1)e^{s(\lambda + \mu)}}{\mu + \lambda \tau - \lambda(\tau - 1)e^{s(\lambda + \mu)}}, \quad t = s, \quad \tau \in \mathbb{R}, \tag{17}
\]

the partial differential equation (13) yields

\[
F(s) = \left(\frac{\mu + \lambda}{\mu + \lambda \tau - \lambda(\tau - 1)e^{s(\lambda + \mu)}}\right)^N \left[\mu + \lambda \tau + \mu(\tau - 1)e^{s(\lambda + \mu)}\right]^N \mu N(\lambda + \mu)(\tau - 1) \left[\mu + \lambda \tau - \lambda(\tau - 1)e^{s(\lambda + \mu)}\right]^N \int_0^{s} p(0, y)e^{y(\lambda + \mu)} \left[\mu + \lambda \tau - \lambda(\tau - 1)e^{y(\lambda + \mu)}\right]^{N-1} dy. \tag{18}
\]

From (17) one has:

\[
\tau = \frac{\mu(z - 1) + (z\lambda + \mu)e^{t(\lambda + \mu)}}{\lambda(1 - z) + (z\lambda + \mu)e^{t(\lambda + \mu)}}, \quad s = t;
\]

so, by substituting in (18), after some calculations and due to (11) we obtain the solution (15).

The integral form of \(F(s, t)\) obtained in Proposition 3.2 is expressed in terms of \(p(0, t)\). Hence, determining the latter function is a relevant problem. In the following proposition we obtain its Laplace transform

\[
H(\eta) := \mathcal{L}_\eta[p(0, t)] = \int_0^\infty e^{-\eta t}p(0, t) \, dt, \quad \eta \geq 0
\]

in terms of the Gauss hypergeometric function

\[
\pFq21{(a, b; c; z)}{+\infty}{(a)_n(b)_n}{c\rangle_n}{z^n}{n!}. \tag{19}
\]

**Proposition 3.3** For all \(\lambda, \mu > 0\), the Laplace transform of \(p(0, t)\) is given by

\[
H(\eta) = \left[\mu_2F_1\left(-N, \frac{\eta}{\lambda + \mu}; 1 + N; \frac{\eta}{\lambda + \mu}, \frac{-\lambda}{\mu}\right)\right] \left[\eta \mu_2F_1\left(-N + 1 + \frac{\eta}{\lambda + \mu}; 1 + N + \frac{\eta}{\lambda + \mu}; \frac{-\lambda}{\mu}\right)\right]^{-1}
\]

Moreover, if \(\lambda = \mu\) then

\[
H(\eta) = \frac{2}{\eta} \frac{\Gamma\left(1 + \frac{\eta}{4\mu}\right)}{\Gamma\left(1 + \frac{\eta}{4\mu}\right)} \frac{\Gamma\left(N + \frac{1}{2} + \frac{\eta}{4\mu}\right)}{\Gamma\left(N + 1 + \frac{\eta}{4\mu}\right)} \quad \eta \geq 0. \tag{20}
\]
Proof. By requiring that \( \lim_{z \to 0^+} z^N F(z, t) = 0 \), from \( \text{[15]} \) we obtain, for all \( \lambda, \mu \succ 0 \),
\[
\left[ \frac{\mu (e^t(\lambda+\mu) - 1) (e^t(\lambda+\mu) + \lambda)^N}{(\lambda+\mu)^2 e^{2t(\lambda+\mu)}} \right]^N - \frac{\mu^N N}{(\lambda+\mu)^2} \int_0^t p(0, y) e^{-2N(t-y)(\lambda+\mu)}
\times \left[ \frac{1}{\mu} e^{(t-y)(\lambda+\mu) + \lambda} \right]^N \left[ e^{(t-y)(\lambda+\mu) - 1} \right]^{N-1} dy = 0,
\]
so that
\[
\left[ (1 - e^{-t(\lambda+\mu)}) (\mu + \lambda e^{-t(\lambda+\mu)}) \right]^N = N(\lambda + \mu) \int_0^t p(0, y) \frac{[(1 - e^{-(t-y)(\lambda+\mu)}) (\mu + \lambda e^{-(t-y)(\lambda+\mu)})]^N}{e^{(t-y)(\lambda+\mu) - 1}} dy.
\]
Applying the Laplace transform \( \mathcal{L}_\eta \) on both sides, one has
\[
\frac{\mu^N N}{\eta} \frac{\Gamma(N) \Gamma(1 + \frac{\eta}{\lambda+\mu})}{\Gamma(N + 1 + \frac{\eta}{\lambda+\mu})} 2F_1 \left( -N, \frac{\eta}{\lambda+\mu}; 1 + N + \frac{\eta}{\lambda+\mu}; -\frac{\lambda}{\mu} \right)
= \left[ (1 - e^{-t(\lambda+\mu)}) (\mu + \lambda e^{-t(\lambda+\mu)}) \right]^N \left[ e^{-(t-y)(\lambda+\mu)} \right]^{N-1}
= N(\lambda + \mu) \left\{ \mu \mathcal{L}_{\eta + \lambda+\mu} \left[ \left[ (1 - e^{-(t-y)(\lambda+\mu)}) (\mu + \lambda e^{-(t-y)(\lambda+\mu)}) \right]^N \right] \right\}
+ \lambda \mathcal{L}_{\eta + 2(\lambda+\mu)} \left[ \left[ (1 - e^{-(t-y)(\lambda+\mu)}) (\mu + \lambda e^{-(t-y)(\lambda+\mu)}) \right]^N \right]^{N-1},
\]
(22)
where \( H(\eta) \) denotes the Laplace transform of \( p(0, t) \), and (cf. Eq. (28) of Prudnikov et al. [33])
\[
\mathcal{L}_\rho \left[ \left[ (1 - e^{-(t-y)(\lambda+\mu)}) (\mu + \lambda e^{-(t-y)(\lambda+\mu)}) \right]^N \right]^{N-1}
= \frac{\mu^{N-1}}{\rho} \frac{\Gamma(N) \Gamma(1 + \frac{\rho}{\lambda+\mu})}{\Gamma(N + \frac{\rho}{\lambda+\mu})} 2F_1 \left( 1 - N, \frac{\rho}{\lambda+\mu}; N + \frac{\rho}{\lambda+\mu}; -\frac{\lambda}{\mu} \right).
\]
(23)
Hence, from \( \text{[22]} \) and \( \text{[23]} \) we obtain the expression given in \( \text{[20]} \) for \( \lambda \neq \mu \). Moreover, if \( \lambda = \mu \), then making use of (see Eq. (15.1.21) of Abramowitz and Stegun[1])
\[
2F_1 (a, b; a-b+1; -1) = \frac{2^{-a} \sqrt{\pi} \Gamma(a-b+1)}{\Gamma \left( \frac{a+1}{2} \right) \Gamma \left( \frac{a-b+1}{2} \right)}
\]
the expression given in \( \text{[21]} \) thus follows from \( \text{[20]} \).

Aiming to obtain the inverse Laplace transform of \( H(\eta) \), we first provide the following lemma, whose proof is given in Appendix A.

Lemma 3.1 The \((N + 1)\)-degree polynomial
\[
P(x) = x \left[ \prod_{r=0}^{N-1} (x + 2\mu(2r + 1)) + \prod_{r=0}^{N-1} (x + 2\mu(2r + 2)) \right]
\]
(24)
has one root equal to 0 and \( N \) distinct negative roots.

Hereafter we obtain the expression of the probability \( \text{[7]} \) by inverting the Laplace transform \( H(\eta) \) when \( \lambda = \mu \). We set \( \rho(0) := \lim_{t \to +\infty} p(0, t) \), so that we shall express \( p(0, t) \) as the sum of a time-dependent term and the asymptotic value \( \rho(0) \).
Proposition 3.4 If $\lambda = \mu$, for all $t \geq 0$ one has

$$p(0, t) = \rho(0) + 2 \sum_{k=2}^{N+1} \frac{Q(\alpha_k)}{\beta_k} e^{\alpha_k t},$$

(25)

with

$$\rho(0) = \frac{2(2N)^N}{(2N)^N + 4N} = \frac{2}{1 + \sqrt{\pi} N! / \Gamma(N + 1/2)},$$

(26)

and

$$\beta_k = \lim_{\eta \to \alpha_k} \frac{P(\eta)}{\eta - \alpha_k} = \prod_{s=1}^{N+1} (\alpha_k - \alpha_s), \quad k = 1, 2, \ldots, N + 1,$$

(27)

where $0 = \alpha_1 > \alpha_2 > \ldots > \alpha_{N+1}$ are the roots of the polynomial $P(\eta)$, and

$$Q(x) = \prod_{r=0}^{N-1} [x + 2\mu(2r + 1)].$$

(28)

Proof. Expanding the gamma functions in the right-hand-side of (21), one has

$$H(\eta) = 2 \frac{Q(\eta)}{P(\eta)},$$

for $Q$ and $P$ given in (28) and (24), respectively. The roots of the $N$-degree polynomial defined in (28) are all distinct and negative, given by $-2\mu, -6\mu, \ldots, -2(2N - 1)\mu$. Hence, by taking the inverse Laplace transform and making use of Eq. 2.1.4.7 of Prudnikov et al. [34] we obtain

$$p(0, t) = 2 \sum_{k=1}^{N+1} \frac{Q(\alpha_k)}{\beta_k} e^{\alpha_k t}, \quad t \geq 0,$$

where $0 = \alpha_1 > \alpha_2 > \ldots > \alpha_{N+1}$ are the roots of the polynomial $P$, due to Lemma 3.1 and where $\beta_k$ is defined in (27). Finally, after straightforward calculations one obtains the expression (25).

The knowledge of $p(0, t)$ when $\lambda = \mu$, obtained in Proposition 3.4, allows to determine the expression of the probabilities (8) in terms of the polynomials (24) and (28), and of the hypergeometric function (19).

Proposition 3.5 If $\lambda = \mu$, for all $t \geq 0$ one has

$$p(r, t) = \frac{1}{4N} \left( \frac{2N}{N + r} \right) \sum_{l=0}^{N} \binom{N}{l} (-e^{-4\mu t})^l 2F_1(-2l, -N + r, -2N, 2) + \frac{\mu N}{22N - 2} (-1)^{N-r} \sum_{j=0}^{N-1} \binom{N - 1}{j} (-1)^{N-1-j}

\times \left\{ \binom{2N - 1}{N + r} 2F_1(-2j, -N + r + 1, -2N + 1, 2) - \binom{2N - 1}{N + r - 1} 2F_1(-2j, -N + r, -2N + 1, 2) \right\}

\times \left\{ \sum_{k=1}^{N+1} \frac{R(\alpha_k)e^{-|\alpha_k|t}}{|\alpha_k| - 2\mu(2N - 1 - 2j)} - e^{-2\mu t(2N - 1 - 2j)} \sum_{k=1}^{N+1} \frac{R(\alpha_k)}{|\alpha_k| - 2\mu(2N - 1 - 2j)} \right\}

+ \frac{\mu N}{22N - 2} (-1)^{N-r} \binom{2N}{N + r} \sum_{j=0}^{N-1} \binom{N - 1}{j} (-1)^{N-1-j} 2F_1(-2j, -N + r, -2N, 2)

\times \left\{ \sum_{k=1}^{N+1} \frac{R(\alpha_k)e^{-|\alpha_k|t}}{|\alpha_k| - 4\mu(N - j)} - e^{-4\mu t(N - j)} \sum_{k=1}^{N+1} \frac{R(\alpha_k)}{|\alpha_k| - 4\mu(N - j)} \right\}, \quad r = 1, 2, \ldots, N.

(29)

where

$$R(\alpha_k) = \frac{Q(\alpha_k)}{\beta_k}, \quad k = 1, 2, \ldots, N + 1,$$

with $\beta_k$ defined in (27), and where $0 = \alpha_1 > \alpha_2 > \ldots > \alpha_{N+1}$ are the roots of the polynomial (24).
Figure 2: The probabilities $p(r, t)$, given in (29), are plotted for $N = 2$ (left) and $N = 3$ (right), with $\lambda = \mu = 1$.

**Proof.** For $\lambda = \mu$, making use of (29) in the right-hand-side of Eq. (15) we have

$$
F(z, t) = \left(\frac{(z+1)^2 - e^{-4\mu t}(1-z)^2}{4z}\right)^N + \frac{\mu N}{(2z)^N} \left(\frac{1}{2N-2} \sum_{j=0}^{N-1} \binom{N-1}{j} (-1)^{N-1-j} \left[\frac{z+1}{z-1}\right]^{2j}\right) \\
\times \sum_{k=1}^{N+1} R(\alpha_k) \frac{1}{|\alpha_k| - 2\mu(2N-1-2j)} \left[e^{-2\mu(2N-1-2j)t} - e^{-|\alpha_k| t}\right] - \frac{\mu N}{z^{N-2N-2}} \sum_{j=0}^{N-1} \binom{N-1}{j} (-1)^{N-1-j} \\
\times (z-1)^{2N-2j}(z+1)^{2j} \sum_{k=1}^{N+1} R(\alpha_k) \frac{1}{|\alpha_k| - 4\mu(N-j)} \left[e^{-4\mu(N-j)t} - e^{-|\alpha_k| t}\right].
$$

Hence, by employing series expansion techniques one obtains Eq. (29).

Figure 2 shows the transient probabilities obtained in Proposition 3.5 for two choices of $N$. Unfortunately, for $\lambda \neq \mu$ the expression of $p(r, t)$ is very hard to be computed. However, in this case we adopt a Monte Carlo simulation approach to obtain estimates of the probabilities defined in (7) and (8). Some plots of estimates of such probabilities based on simulation and the corresponding exact values, when available, are provided in Figures 3, 4, 5 and 6. In all cases, the estimates provide a quite good correspondence with the exact probabilities.

The following section will be devoted to determine the asymptotic probability law of the process under investigation in the limit as $t \to \infty$. Note that some values of $\lim_{t \to \infty} p(r, t)$, shown in the Figures 3, 4, 5 and 6, have been evaluated by means of Eq. (34) below.

### 4 Asymptotic results

A typical problem of interest in the analysis of stochastic systems is the determination of the existence of a steady-state behavior when $t$ tends to $+\infty$. For instance, it is well known that the asymptotic distribution of the classical continuous-time Ehrenfest model is of binomial type (see, e.g. Section 2.1 of Dharmaraja et al. [14]). Aiming to analyze the steady state of the present multi-type extension of the model, now we introduce the stationary probabilities

$$
\rho(k) := \mathbb{P}(N = k) = \lim_{t \to +\infty} p(k, t), \quad k = 0, 1, \ldots, N,
$$

where $N$ denotes the discrete random variable describing the stationary state of the system, with $p(0, t)$ and $p(k, t)$ defined respectively in (7) and (8). The corresponding asymptotic probability generating function is given by

$$
F(z) := \mathbb{E}[z^N] = \lim_{t \to +\infty} F(z, t) = \rho(0) + \sum_{k \in \mathbb{N}} z^k \rho(k), \quad z \in [0, 1],
$$
Figure 3: On the left: the probability $p(0, t)$ (dashed line) given in (25) compared with its estimation (continuous line) performed via $10^4$ Monte Carlo simulations, for $\lambda = \mu = 1$ and $N = 3$. On the right: the estimates of $p(0, t)$ for various choices of $\lambda$, $\beta$ and $t$, with $N = 3$.

where $F(z, t)$ is defined in (10). In the following proposition we obtain the explicit expression of $F(z)$, given in terms of the hypergeometric function (19). We shall see that it depends on the rates $\lambda$ and $\mu$ only through their ratio. Hence, now we set

$$\varrho = \frac{\lambda}{\mu}. \quad (31)$$

**Proposition 4.1** The probability generating function of $N$, for $z \in [0, 1]$ results:

$$F(z) = \frac{(1 + \varrho z)^2}{z^N(1 + \varrho)^2N} \left[ 1 + g(\varrho, N) \sum_{j=0}^{N-1} \binom{N}{j+1} \left( \frac{z - 1}{1 + \varrho z} \right)^{j+1} \text{ hypergeom} \left( -N, j+1, j+2, \frac{\varrho(z-1)}{1 + \varrho z} \right) \right], \quad (32)$$

where $\varrho$ is defined in (31), and

$$g(\varrho, N) := \frac{1}{2 \text{ hypergeom}(-N, 1, 1 + N, -\varrho)}, \quad (33)$$

**Proof.** The proof is given in Appendix A.

Note that, due to (32), it is not hard to see that $F(1) = 1$. We are now able to obtain the steady-state distribution of the multi-type extension of the continuous-time Ehrenfest model.

**Proposition 4.2** The stationary probabilities defined in (30) are given by

$$\rho(k) = \frac{g(\varrho, N)}{\binom{2N}{k}} \varrho^k \binom{2N}{N + k}, \quad k = 0, 1, \ldots, N, \quad (34)$$

where the function $g$ has been introduced in (33), and $\varrho$ is defined in (31).

**Proof.** The proof is given in Appendix A.

The stationary probabilities given in Proposition 4.2 are plotted in Figure 7 for various choices of $N$ and $\varrho$.

A relevant role is played by the stationary probability $\rho(0)$, which is the probability that the system is asymptotically empty. The case $\varrho = 1$, i.e. $\lambda = \mu$, has been already considered in Proposition 3.4, where it is shown that $p(0, t)$ tends to $\rho(0)$ exponentially.
Remark 4.1 We note that, from the formula (15.1.23) of Abramowitz and Stegun[1] and properties of the Gamma function, Eq. (34) gives

\[ g(1, N) = \frac{2(2N)}{(2N)^2 + 4N}, \]

(35)

Hence, if \( \rho = 1 \) then the stationary probabilities (34) can be written as

\[ \rho(k) = \mathbb{P}(N = k) = \frac{2(2N+k)}{(2N)^2 + 4N}, \quad k = 0, 1, \ldots, N. \]

(36)

In this case, given a random variable \( B \sim \text{Bin}(2N, \frac{1}{2}) \), from (36) the following decomposition holds, for \( k = 0, 1, \ldots, N \),

\[ \rho(k) = c [\mathbb{P}(B = N - k) + \mathbb{P}(B = N + k)], \quad \text{with} \quad c = \frac{4N}{4N + \binom{2N}{N}}. \]
Figure 6: Same as Figure 3, for $\rho(3, t)$.

Figure 7: The stationary probabilities $\rho(k)$ given in (34) are plotted for $N = 10$ on the left, $N = 20$ on the right, and for $\varrho = 1$ (empty circle), $\varrho = 1/3$ (full circle), $\varrho = 3$ (square).

**Remark 4.2** Note that, when $\varrho = 1$, we can verify explicitly that $\sum_{k=0}^{N} \rho(k) = 1$. Indeed, noting that
\[
2^{2N} = \left(\frac{2N}{N} \right) + 2 \sum_{k=0}^{N} \left(\frac{2N}{N+k} \right),
\]
from (36) we have
\[
\sum_{k=0}^{N} \rho(k) = 2 \frac{2^{2N}}{(2N/4^N + 4^N) \sum_{k=0}^{N} \left(\frac{2N}{N+k} \right)} = 1.
\]

**Remark 4.3** It is worth mentioning that, in the case $\lambda = \mu$, we can disclose the explicit relationship between the stationary probabilities $\rho(k)$ given in (34) and the stationary probabilities of the classical Ehrenfest model. Indeed, it is well known that, for $\lambda = \mu$, the stationary probabilities of the Ehrenfest model are given by (see, for instance, Eq. (16) of Dharmaraja et al. [13])
\[
\tilde{q}_k := \left(\frac{2N}{N-k} \right) 2^{-2N}, \quad k \in \{-N, -N + 1, \ldots, -1, 0, 1, \ldots, N\}.
\]

In order to compare the probabilities $\tilde{q}_k$ with $\rho(k)$, we first determine a suitable normalization constant $c(\lambda, N)$ such that
\[
c(\lambda, N) \cdot \left(\sum_{j=-N}^{0} \tilde{q}_j + \sum_{j=0}^{N} \tilde{q}_j \right) = 1,
\]
Figure 8: The stationary mean, variance and coefficient of variation are plotted for \( N = 20 \) and for \( \varrho = 1 \) (empty circle), \( \varrho = 1/3 \) (full circle), \( \varrho = 3 \) (square).

and thus
\[
c(\lambda, N) = 4^{-N} \left\{ 2 \left( \frac{2N}{N} \right) + \left( \frac{2N}{N-1} + \frac{2N}{N+1} \right) \right\} _2 F_1(1, 1-N; N+2; -1).
\]

Hence, the following identity holds
\[
\rho(k) = \frac{q_k + q_{-k}}{c(\lambda, N)}, \quad k = 0, 1, \ldots, N.
\]

Note that the special role of the state 0 in the multi-type Ehrenfest model yields \( \rho(0) = 2q_0/c(\lambda, N) \).

Now we provide the asymptotic mean, variance and coefficient of variation of \( N \).

**Proposition 4.3** The asymptotic mean, the asymptotic variance and the the asymptotic coefficient of variation of \( N \) are given respectively by:

- **Mean**: \( E[N] = \frac{N}{1 + \varrho} \left[ \varrho - 1 + g(\varrho, N) \right] \),
- **Variance**: \( Var[N] = \frac{N}{(1 + \varrho)^2} \left[ \varrho(2 - g(\varrho, N) - g(\varrho, N)N + N(1 - g(\varrho, N))g(\varrho, N)) \right] \),
- **Coefficient of Variation**: \( CV[N] = \sqrt{\frac{\varrho(2 - g(\varrho, N))}{N(\varrho - 1 + g(\varrho, N))^2}} - \frac{g(\varrho, N)}{\varrho - 1 + g(\varrho, N)} \),

where the function \( g \) is provided in (33).

**Proof.** The given results follow from the probability generating function given in (32).

In Figure 8 the stationary mean, variance and coefficient of variation given in Proposition 4.3 are plotted for \( N = 20 \), and for different choices of \( \varrho \).

**Remark 4.4** If \( \varrho = 1 \), i.e. \( \lambda = \mu \), making use of (35) we can see that the quantities provided in Proposition 4.3 become respectively

- **Mean**: \( E[N] = \frac{N(2^N)}{(2^N + 4^N)} + 4^N \),
- **Variance**: \( Var[N] = \frac{N}{2} \left[ 1 - \frac{(2^N) + 4^N}{(2^N + 4^N) + 4^N} \right] \),
- **Coefficient of Variation**: \( CV[N] = \sqrt{\frac{2^{2N-1}}{N(2^N)}} \left( 1 + \frac{2^{2N}}{(2^N)^2} \right) - 1. \)
In order to investigate the behaviour of the mean, the variance and the coefficient of variation of $N$ when $N$ is large, let us now discuss the behavior of $g(\rho, N)$ for $N$ large. In spite of the difficulty in managing the Gauss hypergeometric function in the denominator of Equation (33), in the following Lemma we disclose an useful asymptotic result, whose proof is given in Appendix A.

**Lemma 4.1** If $\rho < 1$, then for $N$ large the function $g(\rho, N)$ defined in (33) can be approximated as

$$
g(\rho, N) \approx \frac{2^{-2N}(2N)!\sqrt{\pi}N^2(\rho - 1)^3 \left(\log \left(\frac{(\rho+1)^2}{4\rho}\right)\right)^{\frac{3}{2}}}{(N!)^2 \left[3(\rho - 1)^3 + N \left(\log \left(\frac{(\rho+1)^2}{4\rho}\right)\right)^{\frac{3}{2}} \left[(3\rho + 1)^2 - 8N(\rho - 1)^2\right]\right]}.
$$

**Proposition 4.4** The asymptotic mean, the asymptotic variance and the asymptotic coefficient of variation given in Proposition 4.3 for $N \to +\infty$ admit the following behaviour:

- if $\rho > 1$, then both $E[N]$ and $Var[N]$ tend to $+\infty$, whereas $CV[N]$ tends to 0;
- if $\rho = 1$, then both $E[N]$ and $Var[N]$ tend to $+\infty$, whereas $CV[N]$ tends to $\sqrt{\frac{3}{2}} - 1$;
- if $\rho < 1$, then following limits hold:

$$
\lim_{N \to \infty} E[N] = \frac{\rho}{1 - \rho},
$$

$$
\lim_{N \to \infty} Var[N] = \frac{3(1 - \rho)^3}{8(1 + \rho)^2 \left(\log \left(\frac{(1+\rho)^2}{4\rho}\right)\right)^{\frac{5}{2}}} - \frac{145\rho^4 + 492\rho^3 + 374\rho^2 + 12\rho + 1}{128(1 - \rho^2)^2},
$$

$$
\lim_{N \to \infty} CV[N] = \frac{1}{8\rho(1 + \rho)} \sqrt{\frac{24(1 - \rho)^5}{\left(\log \left(\frac{(1+\rho)^2}{4\rho}\right)\right)^{\frac{5}{2}}} - \frac{145\rho^4 + 492\rho^3 + 374\rho^2 + 12\rho + 1}{2}}.
$$

**Proof.** If $\rho > 1$, the function $g$ defined in (33) is a divergent series as $N \to \infty$. Hence, in this case the mean and variance given in Proposition 4.3 diverge, whereas the corresponding coefficient of variation tends to 0.

When $\rho = 1$, the mean and variance given in Proposition 4.3 diverge by comparing infinities. For the related coefficient of variation, making use of $\frac{(2N)}{N} = \frac{4N}{\Gamma(N + 1/2)\sqrt{\pi}} \Gamma(N + 1/2)$, it results

$$
CV[N] = \sqrt{\frac{\Gamma(N + 1)\sqrt{\pi}}{2N\Gamma(N + 1/2)}} + \frac{\Gamma(N + 1)}{N^{-1/2}\Gamma(N + 1/2)} \frac{\pi}{2} 1 - \frac{N}{\Gamma(N + 1)\sqrt{\pi}} \frac{N\sqrt{\pi}}{2N\Gamma(N + 1/2)}
$$

since $\frac{N\sqrt{\pi}}{2N\Gamma(N + 1/2)}$ goes to zero by comparing infinities, and $\frac{\Gamma(N + 1)}{N^{-1/2}\Gamma(N + 1/2)}$ tends to 1 due to formula (6.1.46) of Abramowitz and Stegun [1].

For $\rho < 1$, by substituting (37) in the asymptotic mean and variance given in Proposition 4.3 as $N \to \infty$, one obtains the results (38) and (39), and thus the limit (40).

See also the details provided in Eq. (66) below for the case $\rho = 1$.

In order to appreciate the goodness of the numerical approximation provided for $g(\rho, N)$ in Lemma 4.1, in Table 1 we compare the exact stationary probabilities given in Proposition 4.2 with the corresponding quantities approximated by means of (37). The considered cases include three choices of $\rho < 1$, and confirm that the approximation is satisfactory when $N$ is large.

We conclude this section by investigating the (Shannon) entropy of the system in the steady state, i.e.

$$
H(N) = E[-\log \rho(N)] = -\sum_{k=0}^{N} \rho(k) \log \rho(k),
$$

where $\rho(k)$ is given in (34). As well known, it is a measure of the amount of information provided by $N$. Figure 2 presents the plot of $H(N)$ as a function of $\rho = \lambda/\mu$, for some choices of $N$. It is clear that $H(N)$ is increasing in $N$. Moreover, we see that $H(N)$ is unimodal in $\rho$. The maxima $m = \arg\max_{\rho > 0} H(N)$ are reported in Table 2, where it is shown that $m$ is not monotonic in $N$. The considered cases show that the entropy of the system in the steady state reaches the maximum when $\lambda$ is close to the double of $\mu$, depending on $N$. 


Table 1: Comparisons between the closed form of $\rho(k)$ given in [34], with its approximation obtained by means of [37], with three choices of $N$, three choices of $\theta$, and various choices of $k$.

| $N = 100$ | $\theta = 0.25$ | $\theta = 0.5$ | $\theta = 0.75$ |
|---|---|---|---|
| $k$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ |
| 0 | 0.75044 | 0.75298 | 0.51290 | 0.28840 |
| 10 | 2.6543\times10^{-7} | 2.6506\times10^{-7} | 0.00185182 | 0.00018463 |
| 20 | 1.2476\times10^{-14} | 1.2456\times10^{-14} | 8.9135\times10^{-9} | 8.8851\times10^{-9} |
| 30 | 7.3537\times10^{-23} | 7.3433\times10^{-23} | 5.3796\times10^{-14} | 5.3645\times10^{-14} |
| 40 | 4.8497\times10^{-32} | 4.8491\times10^{-32} | 3.6302\times10^{-20} | 3.6293\times10^{-20} |
| 50 | 2.9815\times10^{-42} | 2.9775\times10^{-42} | 2.2871\times10^{-27} | 2.2806\times10^{-27} |
| 60 | 1.2844\times10^{-53} | 1.2826\times10^{-53} | 1.0089\times10^{-35} | 1.0060\times10^{-35} |
| 70 | 2.4477\times10^{-66} | 2.4427\times10^{-66} | 9.6988\times10^{-45} | 9.6332\times10^{-45} |
| 80 | 1.9104\times10^{-81} | 1.9111\times10^{-81} | 7.5728\times10^{-57} | 7.5517\times10^{-57} |
| 90 | 1.2199\times10^{-97} | 1.2182\times10^{-97} | 1.0289\times10^{-70} | 1.0268\times10^{-70} |
| 100 | 5.1822\times10^{-120} | 5.1749\times10^{-120} | 4.4757\times10^{-90} | 4.4631\times10^{-90} |

| $N = 500$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ |
|---|---|---|---|
| $k$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ |
| 0 | 0.75082 | 0.75082 | 0.50298 | 0.2956 |
| 10 | 3.9775\times10^{-33} | 3.9775\times10^{-33} | 2.9998\times10^{-18} | 9.8730\times10^{-10} |
| 20 | 8.5836\times10^{-70} | 8.5836\times10^{-70} | 7.2844\times10^{-40} | 1.5295\times10^{-22} |
| 30 | 5.4892\times10^{-111} | 5.4892\times10^{-111} | 5.2479\times10^{-66} | 7.0221\times10^{-40} |
| 40 | 5.8394\times10^{-157} | 5.8394\times10^{-157} | 6.2856\times10^{-97} | 5.3628\times10^{-62} |
| 50 | 4.0927\times10^{-208} | 4.0927\times10^{-208} | 4.9601\times10^{-133} | 2.6952\times10^{-89} |
| 60 | 4.4298\times10^{-265} | 4.4298\times10^{-265} | 6.0445\times10^{-175} | 2.9687\times10^{-122} |
| 70 | 7.9392\times10^{-329} | 7.9392\times10^{-329} | 1.0897\times10^{-223} | 2.0964\times10^{-122} |
| 80 | 2.6599\times10^{-401} | 2.6599\times10^{-401} | 6.4010\times10^{-281} | 2.4103\times10^{-162} |
| 90 | 3.1096\times10^{-486} | 3.1096\times10^{-486} | 6.0548\times10^{-351} | 6.4886\times10^{-211} |
| 100 | 2.9294\times10^{-601} | 2.9294\times10^{-601} | 5.9845\times10^{-451} | 5.4446\times10^{-272} |

| $N = 1000$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ |
|---|---|---|---|
| $k$ | $\rho(k)$ | $\rho(k)$ | $\rho(k)$ |
| 0 | 0.750415 | 0.750415 | 0.501485 | 0.255005 |
| 10 | 2.09542\times10^{-65} | 2.09542\times10^{-65} | 1.7751\times10^{-35} | 3.6698\times10^{-18} |
| 20 | 9.60904\times10^{-139} | 9.60904\times10^{-139} | 1.0319\times10^{-78} | 8.6371\times10^{-44} |
| 30 | 3.8264\times10^{-221} | 3.8264\times10^{-221} | 5.2089\times10^{-131} | 1.7799\times10^{-78} |
| 40 | 4.1605\times10^{-313} | 4.1605\times10^{-313} | 7.1709\times10^{-193} | 9.9747\times10^{-123} |
| 50 | 1.9313\times10^{-415} | 1.9313\times10^{-415} | 4.2248\times10^{-265} | 2.3863\times10^{-177} |
| 60 | 2.0902\times10^{-529} | 2.0902\times10^{-529} | 5.7964\times10^{-349} | 1.3309\times10^{-243} |
| 70 | 4.7854\times10^{-657} | 4.7854\times10^{-657} | 1.6824\times10^{-446} | 1.5709\times10^{-323} |
| 80 | 2.5561\times10^{-802} | 2.5561\times10^{-802} | 2.5204\times10^{-561} | 9.5655\times10^{-421} |
| 90 | 5.6205\times10^{-972} | 5.6205\times10^{-972} | 3.1749\times10^{-701} | 4.8950\times10^{-543} |
| 100 | 3.1911\times10^{-1203} | 3.1911\times10^{-1203} | 2.2850\times10^{-902} | 1.4336\times10^{-726} |

5 The diffusion approximation

Diffusion processes are largely adopted in the literature to model the dynamics of randomly fluctuating systems, and for the mean-field description of interacting particle systems and multi-agents modeling. In particular, the Ornstein-Uhlenbeck process is often used as it provides a fruitful compromise between the need to describe the dynamics of phenomena subject to fluctuations in the presence of an equilibrium point and the opportunity to have closed-form expressions of interest in applications, such as transition density and first-passage-time density through the equilibrium point. For instance, the recent papers by Ascione et al. [3] and Hongler and Filliger [24] and Ratanov [35] deal with suitable generalizations of the Ornstein-Uhlenbeck process. In various contexts, such as queueing and mathematical neurobiology, generalized Ornstein-Uhlenbeck processes arise through a scaling of continuous-time processes on a discrete state space.

Along this line, in this section we construct a diffusion approximation for the process $\{\mathcal{N}(t), \mathcal{L}(t), t \geq 0\}$ that leads to an Ornstein-Uhlenbeck process on the spider. Before adopting a scaling procedure, we perform a different parameterization of the model studied in Section 2 by setting

$$\lambda = \frac{\alpha}{2} + \frac{\gamma}{2} \epsilon, \quad \mu = \frac{\alpha}{2} - \frac{\gamma}{2} \epsilon, \quad \text{for } \alpha > 0, \epsilon > 0, |\gamma| < \frac{\alpha}{\epsilon}. \quad (41)$$
Since \( N \) and (5) thus correspond to the spider, i.e. the star graph \( x \) for \( N \), note that the process is shown to converge weakly to a diffusion process \( X \).

Under the limit conditions Proposition 5.1, \( \epsilon \to 0^+ \), for all \( t > 0 \), consider the position \( N^\epsilon(t) = N(t) \epsilon \), so that \( \{N^\epsilon(t), \mathcal{L}(t); \ t \geq 0\} \) is a continuous-time stochastic process having state space \( S^\epsilon = \{0\} \cup (N_\epsilon \times D) \), where \( N_\epsilon = \{\epsilon, 2\epsilon, \ldots, N\epsilon\} \). Let \( l_0 \in D \); recalling (3), the transient probabilities of the scaled process, for \( \epsilon > 0, t \geq 0, k, l \in N \), and \( j, l \in D \), are given by

\[
\begin{align*}
p^\epsilon_\ast(0, l, t) & := \mathbb{P}\{(N^\epsilon(t), \mathcal{L}(t)) = 0, \mathcal{J}(t) = l | (N^\epsilon(0), \mathcal{L}(0)) = 0, \mathcal{J}(0) = l_0\}, \\
p^\epsilon_\ast(k, j, t) & := \mathbb{P}\{(N^\epsilon(t), \mathcal{L}(t)) = (k \epsilon, j) | (N^\epsilon(0), \mathcal{L}(0)) = 0, \mathcal{J}(0) = l_0\}.
\end{align*}
\]

(42)

Since \( N^\epsilon(t) = N(t) \epsilon \), we have \( p^\epsilon_\ast(0, l, t) = p(0, l, t) \) and \( p^\epsilon_\ast(k, j, t) = p(k \epsilon, j, t) \). In the limit as \( \epsilon \to 0^+ \), the scaled process is shown to converge weakly to a diffusion process \( \mathcal{X} := \{(X(t), \mathcal{L}(t)); \ t \geq 0\} \), whose state space is the spider, i.e. the star graph \( \mathcal{S}_\mathcal{X} := \{0\} \cup (\mathbb{R}^+ \times D) \). When \( \epsilon \) tends to 0, then the probabilities \( p^\epsilon_\ast(0, l, t) \) and \( p^\epsilon_\ast(k, j, t) \) given in (42) correspond respectively to

\[
\begin{align*}
\mathbb{P}\{0 \leq X(t) < \epsilon, \mathcal{L}(t) = 0, \mathcal{J}(t) = l | (X(0), \mathcal{L}(0)) = 0, \mathcal{J}(0) = l_0\} & =: f(0, l, t) \epsilon + o(\epsilon), \\
\mathbb{P}\{x \leq X(t) < x + \epsilon, \mathcal{L}(t) = j | (X(0), \mathcal{L}(0)) = 0, \mathcal{J}(0) = l_0\} & =: f(x, j, t) \epsilon + o(\epsilon),
\end{align*}
\]

for \( t \geq 0, x = k \epsilon \in \mathbb{R}^+ \), and \( j, l \in D \). Hence, \( f(0, l, t) \) and \( f(x, j, t) \) denote the probability density of the process \( \mathcal{X} \) at time \( t \) in the state 0 and in the state \( x \) along the ray \( \mathcal{S}_j \), respectively. Moreover, the initial conditions (3) and (5) thus correspond to

\[
f(x, j, 0) = \delta(x) \delta_{j, l_0}, \quad x \in \{0\} \cup \mathbb{R}^+, \ j \in D,
\]

where \( \delta(x) \) is the delta-Dirac function.

We are now able to obtain the equations satisfied by the probability density of the diffusion process \( \mathcal{X} \).

**Proposition 5.1** Under the limit conditions

\[
\epsilon \to 0^+, \quad N \to +\infty, \quad N \epsilon = N_\epsilon \to +\infty, \quad N \epsilon^2 = N_\epsilon \nu \to \nu > 0,
\]

(43)

for \( x \in \mathbb{R}^+, t > 0 \) and \( j \in D \), the density \( f(x, j, t) \) satisfies the following partial differential equation:

\[
\frac{\partial}{\partial t} f(x, j, t) = -\frac{\partial}{\partial x} \left\{[-\alpha(x - \beta)] f(x, j, t)\right\} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} f(x, j, t),
\]

(44)
with boundary conditions

\[ \sum_{i \in D} \left\{ \alpha \beta f(0, l, t) - \frac{\sigma^2}{2} \frac{\partial}{\partial x} \epsilon f(x, l, t) \right\}_{x=0} = 0, \quad (45) \]

\[ f(0, j, t) = \sum_{i \in D} c_{i,j} f(0, l, t), \quad j \in D, \quad (46) \]

\[ \lim_{x \to + \infty} f(x, j, t) = 0, \quad \forall j \in D, \quad (47) \]

where, for \( \nu > 0 \),

\[ \sigma^2 = \alpha \nu > 0, \quad \beta = \frac{\gamma \nu}{\alpha} \in \mathbb{R}. \quad (48) \]

**Proof.** Since \( p(k, j, t) = p^*_k(k, j, t) \approx f(k \epsilon, j, t) \epsilon \), for \( \epsilon \) close to 0, in analogy with the second equations of system \( \text{(6)} \), for \( x = k \epsilon \) with \( k = 2, 3 \ldots N - 1 \), \( j \in D \) and \( t \geq 0 \) we have

\[ \frac{\partial}{\partial t} \sum_{i \in D} f(0, j, t) \cdot \epsilon = \mu \frac{N_x}{\epsilon} + \frac{\sigma^2}{\epsilon} f(x, l, t) \cdot \epsilon - \frac{\lambda}{\epsilon} f(0, l, t) \cdot \epsilon, \quad (49) \]

\[ \frac{\partial}{\partial t} f(x, j, t) \cdot \epsilon = \mu \frac{N_x + 2 \epsilon}{\epsilon} f(x, l, t) \cdot \epsilon + \sum_{i \in D} c_{i,j} \frac{N_x}{\epsilon} f(0, l, t) \cdot \epsilon \]

\[ - \left( \lambda + \mu \right) N_x - \left( \lambda - \mu \right) \frac{\epsilon}{\epsilon} f(x, j, t) \cdot \epsilon, \quad (50) \]

\[ \frac{\partial}{\partial t} f(x, j, t) \cdot \epsilon = \mu \frac{N_x + x + \epsilon}{\epsilon} f(x + \epsilon, j, t) \cdot \epsilon + \frac{\lambda}{\epsilon} N_x f(x - \epsilon, j, t) \cdot \epsilon \]

\[ - \left( \lambda + \mu \right) N_x - \left( \lambda - \mu \right) \frac{x}{\epsilon} f(x, j, t) \cdot \epsilon, \quad (51) \]

\[ \frac{\partial}{\partial t} f(N_x, j, t) \cdot \epsilon = \lambda f(N_x - \epsilon, j, t) \cdot \epsilon - \mu \frac{2 N_x}{\epsilon} f(N_x, j, t) \cdot \epsilon. \quad (52) \]

where \( N_x = N \epsilon \). Expanding \( f \) as Taylor series, from equation \( \text{(51)} \) we obtain

\[ \frac{\partial}{\partial t} f(x, j, t) = (\mu + \lambda) f(x, j, t) + [(\mu - \lambda) N_x + (\mu + \lambda) x + (\mu - \lambda) \epsilon] \frac{\partial}{\partial x} f(x, j, t) \]

\[ + \epsilon \left( (\mu + \lambda) N_x + (\mu - \lambda) x + (\mu + \lambda) \epsilon \right) \frac{\partial^2}{\partial x^2} f(x, j, t) + o(\epsilon^2). \]

Due to \( \text{(41)} \) one has \( \lambda - \mu = \gamma \epsilon \) and \( \lambda + \mu = \alpha \), so that

\[ \frac{\partial}{\partial t} f(x, j, t) = \alpha f(x, j, t) + (-\gamma \epsilon N_x + \alpha x - \gamma \epsilon^2) \frac{\partial}{\partial x} f(x, j, t) \]

\[ + \frac{\epsilon}{2} (\alpha N_x - \gamma \epsilon x + \alpha \epsilon) \frac{\partial^2}{\partial x^2} f(x, j, t) + o(\epsilon^2). \]

Making use of the limit conditions \( \text{(43)} \), as \( \epsilon \to 0^+ \) we get

\[ \frac{\partial}{\partial t} f(x, j, t) = \alpha f(x, j, t) + (-\gamma \nu + \alpha x) \frac{\partial}{\partial x} f(x, j, t) + \frac{\alpha \nu}{2} \frac{\partial^2}{\partial x^2} f(x, j, t), \]

that coincides with \( \text{(44)} \) thanks to positions \( \text{(48)} \). Similarly, Eq. \( \text{(49)} \) yields

\[ \frac{\partial}{\partial t} \sum_{i \in D} f(0, j, t) \epsilon = \sum_{i \in D} \left\{ \left[ N_x (\mu - \lambda) + \mu \epsilon \right] f(0, l, t) + (\mu \epsilon^2 + \mu N_x) \epsilon \right\} \frac{\partial}{\partial x} f(x, l, t) \bigg|_{x=0}, \]

and thus for \( \epsilon \to 0^+ \), we come to condition \( \text{(45)} \). Finally, following an analogous procedure, from \( \text{(50)} \) and \( \text{(52)} \) we obtain the relations \( \text{(46)} \) and \( \text{(47)} \), respectively.
From Proposition 5.1, it is clear that the considered scaling procedure leads to a diffusion process that follows Ornstein-Uhlenbeck dynamics along the semi-infinite rays of the star graph. The corresponding drift and infinitesimal variance are given respectively by

\[ A_1(x) = -\alpha (x - \beta), \quad A_2(x) = \sigma^2, \quad x \in \mathbb{R}^+, \]  

(53)

with \( \alpha > 0, \beta \in \mathbb{R} \) and \( \sigma > 0 \). We point out that (45) represents the reflection condition in the state 0. Moreover, recalling that \( C = (c_{l,j})_{l,j \in D} \) is a stochastic matrix, the relation (46) expresses the switching mechanism in the origin of the state space. Finally, (47) is a regularity condition on the endpoint \( +\infty \).

Remark 5.1 Equation (46) is equivalent to

\[ \sum_{l \neq j} c_{l,j} f(0, l, t) = \sum_{l \neq j} c_{j,l} f(0, j, t), \quad \forall \, t > 0 \text{ and } j \in D. \]  

(54)

This relation expresses a conservation of probability in the state 0. Namely, the left-hand-side of (54) expresses the intensity that the process enters the line \( S_j \) at time \( t \) arriving from any different line, whereas the right-hand-side of (54) gives the intensity that the process exits from the line \( S_j \) at time \( t \) moving toward any different line, so that Eq. (54) provides an identity between the entrance and exit probability current for the line \( S_j \) through the state 0.

Let us now introduce the density

\[ h(x, t) := \sum_{j=1}^d f(x, j, t), \quad x \in \mathbb{R}^+, \quad t \geq 0. \]  

(55)

Proposition 5.2 For \( x \in \mathbb{R}^+ \) and \( t \geq 0 \), the transition density (55) satisfies the following differential equation:

\[ \frac{\partial}{\partial t} h(x, t) = -\frac{\partial}{\partial x} \left\{ -\alpha (x - \beta) h(x, t) \right\} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} h(x, t), \]  

(56)

with conditions

\[ \alpha \beta h(0, t) - \sigma^2 \frac{\partial}{\partial x} h(x, t) \bigg|_{x=0} = 0, \quad \lim_{x \to +\infty} h(x, t) = 0. \]  

(57)

Proof. The proof of Eqs. (56) and (57) follows immediately from Proposition 5.1 and recalling position (55).

Note that Eq. (56) is the Fokker-Planck equation for a Ornstein-Uhlenbeck diffusion process on \( \mathbb{R}^+ \) with drift and infinitesimal variance given in (53), where (57) gives the reflection condition at the regular endpoint \( x = 0 \) and the regularity condition for the nonattracting-natural endpoint \( x = +\infty \). We remark that in general there is no explicit form for the corresponding transition density. However, if \( \beta = 0 \) then the transition density can be expressed as a combination of two transition densities of the unrestricted process (for details see, for instance, Appendix A of Giorno et al. [19]).

5.1 Asymptotic behavior

In order to investigate the steady state of the approximating diffusion process, we denote by \((X, \mathcal{L})\) the two-dimensional random variable describing the asymptotic behavior of \( X \). The support of \((X, \mathcal{L})\) is the spider, i.e. \( \mathbb{R}_+ \cup \{0\} \times D \). Hereafter we determine the probability law of \((X, \mathcal{L})\). Specifically, we show that \( X \) and \( \mathcal{L} \) are independent, where \( X \) has a truncated normal distribution and \( \mathcal{L} \) is distributed as the stationary distribution of the Markov chain characterized by the transition matrix \( C \) treated in (1) and (2). To this aim, the (sub)density related to the \( j \)-th ray of the spider is denoted as

\[ w(x, j) := \lim_{t \to +\infty} f(x, j, t), \quad x \in \mathbb{R}^+ \cup \{0\}, \quad j \in D. \]  

(58)
Moreover, the probability density function of $X$ is
\[ w(x) = \sum_{j \in D} w(x, j), \quad x \in \mathbb{R}^+ \cup \{0\}, \quad (59) \]
whereas $\pi = (\pi_1, \ldots, \pi_d)$ is the vector of the stationary probabilities of the Markov chain having transition matrix $C$.

**Proposition 5.3** For all $\alpha > 0$, $\beta \in \mathbb{R}$ and $\sigma > 0$, the asymptotic density \((58)\) satisfies
\[ w(x, j) = w(x) \pi_j, \quad \forall x \in \mathbb{R}^+ \cup \{0\}, \quad j \in D, \]
with
\[ w(x) = \frac{1}{Q} \exp \left\{ -\frac{2\alpha x}{\sigma^2} \left( \frac{x}{2} - \beta \right) \right\}, \quad x \in \mathbb{R}^+ \cup \{0\}, \quad (60) \]
where $Q$ is the normalizing constant given by
\[ Q = \frac{\sigma \sqrt{\pi}}{2\sqrt{\alpha}} \left( 1 + \text{Erf} \left( \frac{\sqrt{\alpha} \sigma \beta}{\sigma^2} \right) \right) \exp \left\{ \frac{\alpha \beta^2}{\sigma^2} \right\}, \quad (61) \]
and $\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function.

**Proof.** As $t \to +\infty$, Eq. \((44)\) becomes
\[ 0 = -\frac{\partial}{\partial x} \left\{ -\alpha(x - \beta) w(x, j) \right\} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} w(x, j), \]
whose solution for $x \in \mathbb{R}^+$ and $j \in D$ is given by
\[ w(x, j) = w(0, j) \exp \left\{ -\frac{2\alpha x}{\sigma^2} \left( \frac{x}{2} - \beta \right) \right\}. \quad (62) \]
By letting $t \to +\infty$ in \((46)\), due to \((58)\) one has
\[ w(0, j) = \sum_{l \in D} c_{l,j} w(0, l), \quad j \in D. \quad (63) \]
Hence,
\[ w(0, j) = \frac{\pi_j}{Q}, \quad j \in D, \]
where $(w(0, j); j \in D) \equiv \pi$ is the vector of the stationary probabilities of the Markov chain characterized by the transition matrix $C$ treated in \((1)\) and \((2)\). From \((62)\) and \((59)\) one thus obtains $w(x, j) = w(x) \pi_j$, with $w(x)$ given in \((60)\). Finally, by integrating on $x$ and summing on all $j \in D$, from \((60)\) one has \((61)\) after a straightforward calculation.

**Remark 5.2** From Eq. \((63)\) it is not hard to see that (cf. \((54)\))
\[ \sum_{l \neq j} c_{l,j} w(0, l) = \sum_{l \neq j} c_{j,l} w(0, j), \quad \forall j \in D. \]

**Remark 5.3** The asymptotic density \((60)\) is unimodal, with mode in the equilibrium point $x = \beta$. When $\lambda = \mu$, from \((47)\) and \((48)\) we have $\beta = 0$. In this case, we can compare the density $w(x)$ with the asymptotic density of the Ehrenfest model. Indeed, it can be easily proven that
\[ w(x) = 2 \tilde{W}(x), \quad x \in \mathbb{R}^+ \cup \{0\}, \]
where (see, for instance, Eq. \((31)\) of Dharmaraja et al. \[13\]) $\tilde{W}(x)$ is the steady-state density of the diffusion approximation of the discrete-time Ehrenfest model. Clearly, this result is in agreement with the comparison given in Remark \[3.3\] for the discrete models.
Making use of Eqs. (60) and (61), we are now able to recover the asymptotic mean and variance of $X$.

**Proposition 5.4** If $\beta \neq 0$, the mean and the variance of $X$ are given respectively by

$$\mathbb{E}[X] = \beta \left( 1 + \frac{1}{\sqrt{\pi}} \frac{\sigma}{\sqrt{\alpha \beta}} \exp \left\{ -\frac{\alpha \beta^2}{\sigma^2} \right\} \right),$$

and

$$\text{Var}[X] = \frac{\sigma^2}{2\alpha} \left( 1 - \frac{2}{\pi} \left( 1 + \text{Erf} \left( \frac{\sqrt{\alpha \beta}}{\sigma} \right) \right) - \frac{2\beta}{\sqrt{\pi}} \frac{\sigma}{\sqrt{\alpha}} \exp \left\{ -\frac{\alpha \beta^2}{\sigma^2} \right\} \right),$$

whereas if $\beta = 0$ then

$$\mathbb{E}[X] = \frac{\sigma}{\sqrt{\pi} \alpha}, \quad \text{Var}[X] = \frac{\sigma^2}{2\alpha} \left( 1 - \frac{2}{\pi} \right).$$

With reference to the parameters $\alpha > 0$, $\beta \in \mathbb{R}$ and $\sigma > 0$, we point out the following.

(i) The mean of $X$ is increasing in $\beta$, with $\mathbb{E}[X] \to 0$ for $\beta \to -\infty$, and $\frac{\mathbb{E}[X]}{\beta} \to 1$ for $\beta \to \infty$. Furthermore, $\mathbb{E}[X]$ is decreasing with respect to $\alpha/\sigma^2$, such that $\mathbb{E}[X] \to \infty$ if $\alpha/\sigma^2 \to 0^+$. Moreover, if $\alpha/\sigma^2 \to \infty$ then $\mathbb{E}[X] \to \beta$ if $\beta > 0$, and $\mathbb{E}[X] \to 0$ if $\beta \leq 0$.

(ii) The variance of $X$ is increasing in $\beta$, with $\text{Var}[X] \to 0$ for $\beta \to -\infty$, and $\text{Var}[X] \to \frac{\sigma^2}{2\alpha}$ for $\beta \to \infty$. Moreover, $\text{Var}[X]$ is decreasing with respect to $\alpha/\sigma^2$, such that $\text{Var}[X] \to \infty$ when $\alpha/\sigma^2 \to 0^+$, and $\text{Var}[X] \to 0$ when $\alpha/\sigma^2 \to \infty$, for all $\beta \in \mathbb{R}$.

**Example 5.1** Recalling (1) and (2), let us now consider some examples of the matrix $C$, which regulates the switching mechanism for the particle types, and the corresponding vector $\vec{\pi} = (\pi_1, \ldots, \pi_d)$ of the stationary probabilities.

1. The transitions from line $l$ to line $j$ occur uniformly:

$$c_{l,j} = \frac{1}{d}, \quad \forall l, j \in D.$$

2. The transitions occur uniformly on any line different from the previous one:

$$c_{l,j} = \begin{cases} \frac{1}{d-1}, & l \neq j, \\ 0, & \text{otherwise} \end{cases} \quad (\forall l, j \in D).$$

3. The transitions occur cyclically clockwise:

$$c_{l,j} = \begin{cases} 1, & j = l + 1, \\ 0, & \text{otherwise} \end{cases} \quad (l = 1, 2, \ldots, d - 1), \quad c_{d,j} = \begin{cases} 1, & j = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Under the assumptions of the first three cases, one obtains the stationary uniform distribution $\vec{\pi} = \left( \frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d} \right)$.

4. The transitions occur sequentially, until line $d$ is reached:

$$c_{l,j} = \begin{cases} 1, & j = l + 1, \\ 0, & \text{otherwise} \end{cases} \quad (l = 1, 2, \ldots, d - 1), \quad c_{d,j} = \begin{cases} 1, & j = d, \\ 0, & \text{otherwise}. \end{cases}$$

In this case, since $d$ is an absorbing line, the stationary probability vector is $\vec{\pi} = (0, 0, \ldots, 0, 1)$. 

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5. The transitions occur on adjacent lines, according to a random-walk scheme:

\[ c_{1,j} = \begin{cases} 1, & j = 2, \\ 0, & \text{otherwise} \end{cases} \quad c_{l,j} = \begin{cases} 1 - p, & j = l - 1, \\ p, & j = l + 1, \\ 0, & \text{otherwise} \end{cases} \quad (l = 2, 3, \ldots, d-1), \quad c_{d,j} = \begin{cases} 1, & j = d - 1, \\ 0, & \text{otherwise}. \end{cases} \]

If \( p \neq \frac{1}{2} \), then the stationary vector \( \vec{\pi} \) has components

\[ \pi_1 = \left( \frac{1}{p} - 1 \right)^{d-2} \pi_d, \quad \pi_j = \frac{1}{1 - p} \left( \frac{1}{p} - 1 \right)^{d-j} \pi_d, \quad j = 2, 3, \ldots, d-1, \quad \pi_d = \frac{(p-1)(2p-1)}{2p \left[ p - 1 + \left( \frac{1}{p} - 1 \right)^d \right]} ; \]

on the other hand, if \( p = \frac{1}{2} \), then the components of \( \vec{\pi} \) are

\[ \pi_1 = \frac{1}{2(d-1)}, \quad \pi_j = \frac{1}{d-1}, \quad j = 2, 3, \ldots, d-1. \]

5.2 Some comparisons

Let us now discuss the goodness of the continuous approximation derived so far. Since the approximation is performed under the limit conditions (43), we expect that it improves as \( \epsilon \) tends to 0 and as \( N \) grows larger.

We first assess the correspondence between the stationary distributions of the Ehrenfest model and its continuous approximation. Hence, we refer to the stationary probabilities \( \rho(k) \) introduced in (30) and to the probability density function \( w(x) \) specified in (59). By considering the case \( \varrho = 1 \), i.e. \( \lambda = \mu \) and thus \( \beta = 0 \), due to the Stirling approximation one has

\[ \left( \frac{2N}{N+k} \right)^{\frac{2N}{N+k}} \sim \frac{1}{\sqrt{\pi N}} \quad \text{as } N \rightarrow \infty, \]

and thus Eq. (36) yields

\[ \rho(k) \sim \frac{2}{\sqrt{\pi N}} \quad \text{as } N \rightarrow \infty, \]

whereas Eq. (60) becomes

\[ w(ke^\epsilon) = \frac{2}{\sqrt{\pi N}} \exp \left\{ - \frac{k^2}{N} \right\}, \]

so that we finally obtain, for any \( \epsilon > 0 \) and \( k \in \mathbb{N}_0 \),

\[ \rho(k) \sim w(ke^\epsilon) \epsilon \quad \text{as } N \rightarrow \infty. \]

This confirms the agreement between the stationary distributions of the considered processes. See also Table 3, where the quantities of interest are shown for some choices of the parameters, together with the relative difference

\[ \Delta(k) := \frac{w(ke^\epsilon) \epsilon - \rho(k)}{\rho(k)}, \]

and according to the limiting procedure considered above. Again, the given values confirm that the approximation improves as \( N \rightarrow \infty \).

Moreover, the agreement is also confirmed by comparing the mean and the variance of \( N \) and \( X/\epsilon \). Indeed, if \( \varrho = 1 \), and \( \beta = 0 \), making use of (63) from Remark 4.4 one has

\[ E[N] \sim \frac{\sqrt{N}}{\sqrt{\pi}}, \quad \text{Var}[N] \sim N \left( \frac{1}{2} - \frac{1}{\sqrt{\pi}} \right) - \frac{1}{2} \sqrt{\pi} \quad \text{as } N \rightarrow \infty, \]

so that recalling (64) under the considered scaling one immediately has

\[ E[N] \sim E \left[ \frac{X}{\epsilon} \right], \quad \text{Var}[N] \sim \text{Var} \left[ \frac{X}{\epsilon} \right], \quad \text{as } N \rightarrow \infty. \]
Table 3: For \( \lambda = \mu = 1, \alpha = 2 \) and \( \beta = \gamma = 0 \) the quantities \( w(k) \epsilon, \rho(k) \) are \( \Delta(k) \) are shown for \( \epsilon = 0.1 \) and for various choices of \( k \) and \( N, \sigma^2 \) such that \( \sigma^2 = \alpha N \epsilon^2 \).

6 Concluding remarks

Nowadays many researchers are interested in the analysis of random motions on star graphs and related structures. Up to now various efforts have been devoted mainly to the cases of birth-death processes and Brownian diffusion on such domains. This contribution is among the first studies concerning birth-death processes with state-dependent rates and the approximating Ornstein-Uhlenbeck process over a spider. It is noteworthy that the present investigation leads to closed-form results for the transient analysis, at least in the case \( \lambda = \mu \), and to the complete asymptotic analysis of the multi-type Ehrenfest model, as well as to a detailed study of the asymptotic behavior of the approximating Ornstein-Uhlenbeck process.

Possible future developments can be oriented to the analysis of
(i) the first-passage-time problem for the considered processes through the origin of the spider or other fixed states,
(ii) suitable modifications of the stochastic system, such as after the inclusion of the possibility of instantaneous transitions as due to the effect of catastrophes occurring randomly in time,
(iii) extension to the multidimensional version, in which the various branches of the state-space can be occupied at the same time,
(iv) modification in the transition rates leading to a birth-death process with quadratic birth and death rates, similar as in Section 5 of Di Crescenzo et al. [15], leading to a diffusion approximation expressed by a lognormal diffusion process.

Finally, we remark that the multi-type Ehrenfest model introduced in Section 2 can be modeled as a finite non homogeneous quasi-birth-death (QBD) process (see, for instance, the book by Latouche and Ramaswami [24]). Such QBD process has a two-dimensional state space \( \bigcup_{k=0}^{N} l(k) \), where \( l(0) = \{(0,1), (0,2), \ldots, (0,d)\} \) and \( l(k) = \{(k,1), (k,2), \ldots, (k,d)\} (k = 1, 2, \ldots, N; d \in D) \); the subset of the states \( l(k) \) is called level \( k \). In our context, the states \( (0) \) \( (j = 1, 2, \ldots, d) \) of \( l(0) \) correspond to the state \( 0 \) (the origin of the graph), whereas the second element of the couple \( j \) represents the last visited line. Hence, numerical techniques from matrix-analytic methods could therefore be applied to obtain e.g. the stationary distribution of the model. This approach allows also to construct suitable generalizations of the process. This can be the object of a further prosecution of the present investigation.

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Conflict of interest

This work does not have any conflicts of interest.

\[\begin{array}{cccccc}
N = 5000, \sigma^2 = 100 & N = 10000, \sigma^2 = 200 & N = 15000, \sigma^2 = 300 \\
\hline
k & w(k) \epsilon & \rho(k) & \Delta(k) & w(k) \epsilon & \rho(k) & \Delta(k) & w(k) \epsilon & \rho(k) & \Delta(k) \\
0 & 0.0159577 & 0.015831 & 0.00800385 & 0.0112838 & 0.0112203 & 0.0065544 & 0.00921318 & 0.00917085 & 0.00461492 \\
1 & 0.0159545 & 0.0158278 & 0.00800383 & 0.0112827 & 0.0112192 & 0.00655439 & 0.00921256 & 0.00917024 & 0.00461492 \\
2 & 0.0159449 & 0.0158183 & 0.00800377 & 0.0112793 & 0.0112159 & 0.00655438 & 0.00921072 & 0.00916841 & 0.00461491 \\
3 & 0.015929 & 0.0158025 & 0.00800366 & 0.0112736 & 0.0112103 & 0.00655435 & 0.00920765 & 0.00916535 & 0.0046149 \\
4 & 0.0159067 & 0.0157804 & 0.00800352 & 0.0112658 & 0.0112024 & 0.00655432 & 0.00920336 & 0.00916108 & 0.0046149 \\
5 & 0.0158781 & 0.015752 & 0.00800344 & 0.0112556 & 0.0111923 & 0.00655427 & 0.00919783 & 0.00915558 & 0.00461487 \\
10 & 0.0156417 & 0.0155175 & 0.00800184 & 0.0111715 & 0.0111087 & 0.00655389 & 0.009151196 & 0.00910992 & 0.0046147 \\
20 & 0.0147308 & 0.014614 & 0.00800093 & 0.0108413 & 0.0107804 & 0.00655241 & 0.00897074 & 0.00892954 & 0.00461404 \\
30 & 0.013329 & 0.0132234 & 0.00798679 & 0.0103126 & 0.0102547 & 0.00655001 & 0.00867664 & 0.00863668 & 0.00461295 \\
40 & 0.0115877 & 0.011496 & 0.00797503 & 0.00961541 & 0.00956142 & 0.00654678 & 0.00828104 & 0.00824302 & 0.00461148 \\
50 & 0.00967883 & 0.00960238 & 0.00796185 & 0.00878783 & 0.00873852 & 0.00654287 & 0.00779879 & 0.007763 & 0.00460965 \\
\end{array}\]
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A

**Proof of Lemma 3.1**

Before providing the proof of Lemma 3.1, we recall the following useful conditions about the Gamma function, for \( i \in \mathbb{N} \):

\[
\begin{align*}
\Gamma \left( \frac{1}{4} - 2i \right) &> 0, \quad \Gamma \left( \frac{3}{4} - 2i \right) > 0, \quad \Gamma \left( \frac{9}{4} - 2i \right) > 0, \quad \Gamma \left( \frac{11}{4} - 2i \right) > 0 \quad (67) \\
\Gamma \left( \frac{5}{4} - 2i \right) &< 0, \quad \Gamma \left( \frac{7}{4} - 2i \right) < 0. \quad (68)
\end{align*}
\]

With reference to (24), to show that \( P(x) \) has \( N \) distinct negative roots in addition to 0, we deal with two cases: \( N \) even and \( N \) odd.

(i) Let \( N \) be even, i.e. \( N = 2n \), with \( n \in \mathbb{N} \). In this case we apply the Intermediate Zero Theorem to the following intervals of negative numbers:

\[
\begin{align*}
( -4(n + k)\mu - \mu, -4(n + k) ), & \quad k = 1, 2, \ldots, n, \quad (69) \\
( -4(n - k + 1)\mu - \mu, -4(n - k)\mu - \mu ), & \quad k = 1, 2, \ldots, n. \quad (70)
\end{align*}
\]

Evaluating \( P(x) \) in the left-hand extreme of the interval (69) we obtain:

\[
\begin{align*}
P(-4(n + k)\mu - \mu) = -16^n \mu^{2n+1}(4n + 4k + 1) \left\{ \frac{\Gamma \left( \frac{1}{4} - k + n \right)}{\Gamma \left( \frac{1}{4} - k - n \right)} + \frac{\Gamma \left( \frac{3}{4} - k + n \right)}{\Gamma \left( \frac{3}{4} - k - n \right)} \right\}.
\end{align*}
\]

Note that \(-16^n \mu^{2n+1}(4n + 4k + 1) < 0\), with \( \Gamma \left( \frac{1}{4} - k + n \right) > 0 \) and \( \Gamma \left( \frac{3}{4} - k + n \right) > 0 \). Moreover, discussing various cases on the basis of the parity of \( n \) and \( k \) it can be shown that \( \Gamma \left( \frac{1}{4} - k - n \right) \Gamma \left( \frac{3}{4} - k - n \right) > 0 \). Consequently, the polynomial \( P(x) \) takes opposite signs in the interval’s extremes, so that it has at least one root in each interval (69). Similarly, the same result can be shown for the interval (70) since

\[
\begin{align*}
P(-4(n - k)\mu - \mu) = -16^n \mu^{2n+1}(4n - 4k + 1) \left\{ \frac{\Gamma \left( \frac{1}{4} + k + n \right)}{\Gamma \left( \frac{1}{4} + k - n \right)} + \frac{\Gamma \left( \frac{3}{4} + k + n \right)}{\Gamma \left( \frac{3}{4} + k - n \right)} \right\}.
\end{align*}
\]

In conclusion, for \( N \) even, the polynomial \( P(x) \) defined in (24) has \( N \) distinct (negative) roots.

(ii) Let \( N \) be odd, with \( N = 2n - 1 \), \( n \in \mathbb{N} \). The polynomial \( P(x) \) has a root given by

\[
- (2N + 1)\mu = -(4n - 1)\mu, \quad (71)
\]

since

\[
P(-4n - 1)\mu = -4^n \mu^{2n-1}(4n - 1)\Gamma \left( \frac{3}{2} \right) \sin(n\pi) \pi^{-1/2} = 0.
\]

So, for \( n = 1 \) (i.e. \( N = 1 \)) the unique root of \( P(x) \) is (71). We now focus on the case \( n = 2, 3, \ldots, M \). In addition to the solution (71), the remaining \( 2n - 2 \) roots can be obtained by applying the Intermediate Zero Theorem to the following intervals, having negative extremes:

\[
\begin{align*}
( -(2N + 1)\mu - 2(2k + 1)\mu, -(2N + 1)\mu - 2(2k - 1) + 1)\mu ), & \quad k = 1, 2, \ldots, n - 1, \\
( -(2N + 1)\mu + 2(2k + 1)\mu, -(2N + 1)\mu + 2(2k - 1) + 1)\mu ), & \quad k = 1, 2, \ldots, n - 1.
\end{align*}
\]

Following the same procedure adopted for \( N \) even, we can conclude that the polynomial \( P(x) \) has \( N \) distinct (negative) solutions also when \( N \) is odd. This concludes the proof of Lemma 3.1. \( \square \)

**Proof of Proposition 4.1**
Recall that the Laplace transform of \( p(0, t) \), denoted by \( H(\eta) \), is given in (20). Hence, due to (15), the Laplace transform of \( F(z, t) \) can be expressed as

\[
\mathcal{L}_\eta [F(z, t)] = \int_0^\infty e^{-\eta t} F(z, t) dt = \frac{1}{(\lambda + \mu)^2N^2 N z N} \mathcal{L}_\eta \left[ \left( (\mu z - \mu) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^N \left( (\lambda - \lambda) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^N \right]
- \sum_{N=1}^{\infty} \frac{\mu N (1 - z)}{z^N (\lambda + \mu)^2 N - 1} H(\eta) \left[ e^{-2N t (\lambda + \mu)} (\mu z - \mu) + (\lambda \mu + \mu) e^{t(\lambda + \mu)} \right]^{N-1}\left( (\lambda - \lambda) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^N
- \frac{1}{(\lambda + \mu)^2 N z N} \mathcal{L}_\eta \left[ \left( (\mu z - \mu) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^N \left( (\lambda - \lambda) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^N \right]
- \frac{1}{(\lambda + \mu)^2 N} \sum_{N=1}^{\infty} \frac{\mu N (1 - z)}{z^N (\lambda + \mu)^2 N - 1} H(\eta) \left( (\lambda + \mu) \mathcal{L}_{\eta + \lambda + \mu} \left[ \left( (\mu z - \mu) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^{N-1}\left( (\lambda - \lambda) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^{N-1} \right) \right].
\]  

(72)

We need to compute the following Laplace transform, for \( n \in \mathbb{N} \):

\[
\mathcal{L}_\eta \left[ \left( (\mu z - \mu) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^n \left( (\lambda - \lambda) e^{-t(\lambda + \mu)} + (z \lambda + \mu) \right)^n \right]
= (\lambda \mu)^n (z - 1)^n \left( \frac{\lambda z + \mu}{\lambda z - \mu} \right)^n \mathcal{L}_\eta \left[ \left( 1 + \frac{e^{-t(\lambda + \mu)}}{\frac{\lambda z + \mu}{\lambda z - \mu}} \right)^n \left( 1 - \frac{e^{-t(\lambda + \mu)}}{\frac{\lambda z + \mu}{\lambda z - \mu}} \right)^n \right]
= (\lambda z + \mu)^n \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{j} \binom{n}{k} \left( \frac{\mu z - \mu}{\lambda z + \mu} \right)^j \left( \frac{\lambda - \lambda}{\lambda z + \mu} \right)^k \mathcal{L}_\eta \left[ e^{-t(\lambda + \mu)(k+j)} \right]
= (\lambda z + \mu)^n \sum_{j=0}^{\infty} \binom{n}{j} \left( \frac{\mu z - \mu}{\lambda z + \mu} \right)^j \frac{1}{\eta + j(\lambda + \mu)} F_2 F_1 \left( -n, j + \frac{\eta}{\lambda + \mu}, j + 1 + \frac{\eta}{\lambda + \mu}, \frac{\lambda z - \lambda}{\lambda z + \mu} \right); \]

so the expression (72), for (20), becomes:

\[
\mathcal{L}_\eta [F(z, t)] = \frac{(\lambda z + \mu)^2 N}{(\lambda + \mu)^2 N z N} \mathcal{L}_\eta \left[ \sum_{n=1}^{\infty} \binom{N}{j} \mu^j \left( \frac{z - 1}{\lambda z + \mu} \right)^j \frac{1}{\eta + j(\lambda + \mu)} F_2 F_1 \left( -N, j + \frac{\eta}{\lambda + \mu}, j + 1 + \frac{\eta}{\lambda + \mu}, \frac{\lambda z - \lambda}{\lambda z + \mu} \right) \right]
+ \mu N(\lambda z + \mu)^2 N \sum_{z N (\lambda + \mu)^2 N - 1}
\]

(73)

\[
\frac{1}{\eta \mu^2 F_1 \left( 1 - N, 1 + \frac{\eta}{\lambda + \mu} \right)} \frac{1}{\eta + 1 + \frac{\eta}{\lambda + \mu}} F_2 F_1 \left( -N, j + 1 + \frac{\eta}{\lambda + \mu}, j + 2 + \frac{\eta}{\lambda + \mu}, \frac{\lambda z - \lambda}{\lambda z + \mu} \right)
\]

Hence, recalling that \( F(z) = \lim_{\eta \to 0} \eta \mathcal{L}_\eta [F(z, t)] \) by the Tauberian theorem (see Chapter VIII of Bhattacharya...
and Waymire \cite{6}, and making use of \cite{75}, we have

\[
F(z) = \frac{(\lambda z + \mu)^{2N}}{(\lambda + \mu)^{2N} z^N} + \frac{\mu N (\lambda z + \mu)^{2N}}{\lambda N (\lambda + \mu)^{2N}}
\]
\[
\times \frac{\mu (N + 1)}{\mu (N + 1) + \frac{1}{z^N}} + \lambda \cdot \frac{1}{1 - \frac{\mu}{N}} + \lambda \cdot \frac{1}{1 + \frac{\mu}{N}}
\]
\[
\times \left[ \sum_{j=0}^{N-1} \binom{N-1}{j} \mu^j \left( \frac{z - 1}{\lambda z + \mu} \right)^{j+1} \frac{1}{1+j} \right] \frac{1}{2+j} 2F1 \left( -N + 1, j + 1, j + 2, \frac{\lambda z - \lambda}{\lambda z + \mu} \right)
\]
\[
- \lambda \sum_{j=0}^{N-1} \frac{\binom{N-1}{j} \mu^j \left( \frac{z - 1}{\lambda z + \mu} \right)^{j+2}}{2+j} \frac{1}{2+j} 2F1 \left( -N + 1, j + 2, j + 3, \frac{\lambda z - \lambda}{\lambda z + \mu} \right).
\]  

(74)

Due to the definition of the Hypergeometric function, after some calculation it is possible to simplify \cite{74} to obtain \cite{32}.

\textbf{Proof of Proposition 4.2}

In Eq. \cite{32} we make use of the following series expansions:

\[
\frac{(1 + \varrho z)^{2N}}{z^N} = \sum_{r=1}^{N} \binom{2N}{N-r} \varrho^{N-r} z^{-r} + \varrho^N \binom{2N}{N} \varrho^{N+r} z^r,
\]

and

\[
\frac{(1 + \varrho z)^{2N}}{z^N} \left( \frac{z - 1}{1 + \varrho z} \right)^{j+1} \frac{1}{1+j} \frac{1}{2+j} 2F1 \left( -N + 1, j + 1, j + 2, \frac{\varrho (z - 1)}{1 + \varrho z} \right)
\]

\[
= (j + 1) \sum_{h=0}^{N-1} \sum_{r=0}^{N-r} \sum_{s=0}^{h+r+1} \binom{N-1}{r} \binom{h+j+1}{h} \binom{2N-r}{N} \frac{1}{\varrho^r} z^{-N}
\]

Hence, after some calculations, the series expansion of \cite{32} becomes

\[
F(z) = \rho(0) + \sum_{s=1}^{N} z^s \varrho(s)
\]

\[
= \varrho^N \left( \frac{2N}{N} \right) + \frac{N \varrho^{N-1}}{(1 + \varrho)^{2N}} \varrho(\varrho, N)
\]
\[
\times \sum_{j=0}^{N-1} \sum_{h=0}^{N} \sum_{r=0}^{h+j+1} \binom{N-1}{r} \binom{h+j+1}{h} \binom{2N-r}{N} \frac{1}{\varrho^r}
\]
\[
+ \sum_{s=1}^{N} z^s \varrho(s) \left\{ \varrho^N \left( \frac{2N}{N+s} \right) + \frac{N \varrho^{N-1}}{(1 + \varrho)^{2N}} \varrho(\varrho, N)
\right. \]
\[
\left. \times \sum_{j=0}^{N-1} \sum_{h=0}^{N} \sum_{r=0}^{h+j+1} \binom{N-1}{r} \binom{h+j+1}{h} \binom{2N-r}{N+s} \frac{1}{\varrho^r} \right\},
\]

where the function \( \varrho \) is defined in \cite{33}, so that

\[
\rho(k) = \frac{\varrho^{N+k}}{(1 + \varrho)^{2N}} \left( \frac{2N}{N+k} \right) + \frac{N \varrho^{N+k-1}}{(1 + \varrho)^{2N}} \varrho(\varrho, N)
\]
\[
\times \sum_{j=0}^{N-1} \sum_{h=0}^{N-k} \sum_{r=0}^{h+j+1} \binom{N-1}{r} \binom{h+j+1}{h} \binom{2N-r}{N+k} \frac{1}{\varrho^r}.
\]  

(75)

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Noting that
\[ \sum_{r=0}^{N-k} (-1)^r \binom{h+j+1}{r} \binom{2N-r}{N+k} (1+\varrho)^r = \binom{2N}{k+N} \frac{2F_1(-1-h-j,k-N;-2N;1+\varrho)}{2F_1(-h-j,k-N;-2N;1+\varrho)}, \]
the expression (75) becomes
\[ \rho(k) = \frac{\varrho^{N+k}}{(1+\varrho)^{2N}} \binom{2N}{N+k} + \frac{N\varrho^{N+k-1}}{(1+\varrho)^{2N}} g(\varrho,N) \times \sum_{j=0}^{N-1} \sum_{h=0}^{N} \frac{(-1)^h}{j+h+1} \binom{N-1}{j} \binom{2N}{k+N} \frac{1}{\varrho} \frac{1}{(j+1+N)!} \frac{N!j!}{\varrho^j} \binom{2F_1(-1-h-j,k-N;-2N;1+\varrho)}{2F_1(-h-j,k-N;-2N;1+\varrho)}. \tag{76} \]
Moreover, due to Eq. (5.92.12) of Prudnikov et al. \[32\], after some manipulations one has
\[ \sum_{h=0}^{N} \frac{(-1)^h}{j+h+1} \binom{N}{h} 2F_1(-1-h-j,k-N;-2N;1+\varrho) = \frac{N!j!}{(j+1+N)!} \]
and thus the expression (76) becomes
\[ \rho(k) = \frac{\varrho^{N+k}}{(1+\varrho)^{2N}} \binom{2N}{N+k} + \frac{N\varrho^{N+k-1}}{(1+\varrho)^{2N}} g(\varrho,N) \binom{2N}{N+k} \frac{1}{(j+1+N)!} \frac{N!j!}{\varrho^j} \binom{2F_1(1,1-N;N+2;-\frac{1}{\varrho})}{N+1}. \tag{77} \]
Finally, recalling Eq. (33) we obtain
\[ \rho(k) = \frac{\varrho^{N+k}}{(1+\varrho)^{2N}} \binom{2N}{N+k} \left[ 1 + \frac{N}{N+1} \cdot \frac{1}{\varrho} \frac{2F_1(1,1-N;N+2;-\frac{1}{\varrho})}{2F_1(1,-N;N+1;-\varrho)} \right]. \tag{78} \]
Hereafter we show that, if \( k = 0 \), then the stationary probability \(30\) is given by
\[ \rho(0) = g(\varrho,N). \tag{79} \]
For Equation 7.3.1.143 of Prudnikov et al.\[32\], one has
\[ 2F_1(1,1-N;2+N;-\frac{1}{\varrho}) = \frac{(N-1)!(N+1)!}{(2N)!} \left( 1 + \frac{1}{\varrho} \right)^{N-1} P_{N-1}^{(N+1,-N)} \left( \frac{1-\varrho}{1+\varrho} \right), \]
where \( P_{n}^{(\alpha,\beta)}(z) \) is the Jacobi Polynomial. Therefore, from this last equality and (78) with \( k = 0 \), it results:
\[ \rho(0) = g(\varrho,N) \left[ \frac{\varrho^N}{g(\varrho,N)(1+\varrho)^{2N}} \binom{2N}{N} + \frac{1}{(1+\varrho)^{N+1}} P_{N-1}^{(N+1,-N)} \left( \frac{1-\varrho}{1+\varrho} \right) \right]. \tag{80} \]
The thesis (79) thus follows by proving that the quantity in square brackets in (80) is equal to 1. Indeed, by using Equation 7.3.1.143 of Prudnikov et al.\[32\] for \( g(\varrho,N) = 1/2F_1(-N,1,1+N;-\varrho) \), one has
\[ \frac{1}{g(\varrho,N)(1+\varrho)^{2N}} \binom{2N}{N} + \frac{1}{(1+\varrho)^{N+1}} P_{N-1}^{(N+1,-N)} \left( \frac{1-\varrho}{1+\varrho} \right) = \frac{\varrho^N}{(1+\varrho)^{N+1}} P_{N-1}^{(N+1,-N)} \left( \frac{1-\varrho}{1+\varrho} \right) + \frac{1}{(1+\varrho)^{N+1}} P_{N-1}^{(N+1,-N)} \left( \frac{1-\varrho}{1+\varrho} \right) = 4^N \Gamma \left( N + \frac{1}{2} \right) \frac{B_{\frac{N}{\sqrt{\varrho}}} (N,N+1)}{\sqrt{\pi} \Gamma(N)} + B_{\frac{1}{\sqrt{\varrho}}} (N+1, N) \]
\[ = 4^N \Gamma \left( N + \frac{1}{2} \right) B(N,N+1) = 1, \]
where \( B(a, b) \) is the Beta function and \( B_z(a, b) \) is the Incomplete Beta function, and where use of the formula in Section C of Chapter 1 of Gupta and Nadarajah\[22\] has been made. Finally, by comparing (79) with (78) evaluated at \( k = 0 \), Eq. (34) immediately follows.

**Proof of Lemma 4.1**

For Theorem 1.1 of Daalhuis \[11\], if \( q < 1 \), for \( N \) large one has

\[
\, _2F_1(1, -N, N + 1, -q) \approx \frac{2^N (1 + q)^{-N} (N!)^2}{\theta^{N/2} (2N)! \sqrt{2\pi}} \left\{ D_{-1} \left( \sqrt{2N} \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) (1 + q) \right. \\
+ N^{-\frac{1}{2}} D_0 \left( \sqrt{2N} \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) \left[ (q + 1) + \frac{2(1 + q^2)}{q - 1} \sqrt{\log \left[ \frac{(1 + q)^2}{4\theta} \right]} \right] - \sqrt{2} \sqrt{\log \left[ \frac{(1 + q)^2}{4\theta} \right]} \\
+ \frac{1}{N} \left[ 2(q - 1) + 2 \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right] 3^{3/2} \right\},
\]

(81)

where \( D_r(z) \) is the Parabolic Cylinder function. Since \( D_0(z) = e^{-\frac{z^2}{4}} \) and \( D_r(z) \approx z^r e^{-\frac{z^2}{4}} \left[ 1 - \frac{(r-1)x}{2z^2} + \frac{(r-3)(r-2)(r-1)x}{8z^4} \right] \) for \( z \to \infty \), in our case it results

\[
D_0 \left( \sqrt{2N} \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) = 2^N \left[ \frac{(1 + q)^2}{\theta} \right]^{-N/2},
\]

and

\[
D_{-1} \left( \sqrt{2N} \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) \approx 2^{-N} - \frac{1}{2} \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) - \frac{1}{2} \left( \frac{(1 + q)^2}{\theta} \right)^{-\frac{1}{2}} \\
	imes \left[ \frac{4N^2 \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right)^2 - 2N \log \left[ \frac{(1 + q)^2}{4\theta} \right] + 3}{4N^2 \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right)^3} \right].
\]

Hence, the expression (81) becomes

\[
\, _2F_1(1, -N, N + 1, -q) \approx \frac{2^{2N} (N!)^2}{(2N)! \sqrt{2\pi N} (1 + q)} \times \left\{ \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right) - \frac{1}{2} \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right)^2 - 2N \log \left[ \frac{(1 + q)^2}{4\theta} \right] + 3 \right\} (q + 1) \\
+ 1 \frac{2(1 + q^2)}{q - 1} \sqrt{\log \left[ \frac{(1 + q)^2}{4\theta} \right]} + \frac{1}{N} \left[ 2(q - 1) + 2 \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right] 3^{3/2} \right\} \\
- \frac{1}{N} \left[ 2(q - 1) + 2 \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right] 3^{3/2} \right\} \\
= \frac{2^{2N-3} (N!)^2}{(2N)! \sqrt{\pi N^2}} \left[ 3(q - 1)^3 + N \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right)^3 \left[ (3q + 1)^2 - 8N(q - 1)^2 \right] \right] \\
\times \left[ (q - 1)^3 \left( \log \left[ \frac{(1 + q)^2}{4\theta} \right] \right)^3 \right],
\]

so that the result (37) finally holds.