Leutwyler–Smilga sum rules for Ginsparg–Wilson lattice fermions*

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Abstract

We argue that lattice QCD with Ginsparg–Wilson fermions satisfies the Leutwyler–Smilga sum rules for the eigenvalues of the chiral Dirac operator. The result is obtained in the one flavor case, by rephrasing Leutwyler and Smilga’s original analysis for the finite volume partition function. This is a further evidence that Ginsparg–Wilson fermions, even if breaking explicitly the chirality on the lattice in accordance to the Nielsen–Ninomiya theorem, mimic the main features of the continuum theory related to chiral symmetry.

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1 Introduction

According to the Nielsen–Ninomiya theorem \[1\] there is no way to avoid the explicit breaking on the lattice of the chiral symmetry of the continuum QCD (at least if one wants to keep the fundamental locality property). In the simplest discretization, given by the Wilson action, the explicit breaking introduces many annoying artifacts in the lattice theory, the most popular being the quark mass renormalization with related fine tuning problem; more in general, features and mechanisms of the continuum theory associated to the chirality find no correspondence in the lattice theory. For example, the Atiyah–Singer theorem \[2\], relating the chirality of the zero modes of the Dirac operator in a finite volume to the topological charge of the background configuration, has no lattice counterpart; the study of the spontaneous breaking of the symmetry is awkward since the explicit breaking introduces spurious effects which are not under full theoretical control (e.g. an order parameter analogous to the fermion condensate is missing).

Ginsparg and Wilson provided the condition \[3\] under which the breaking of the chirality of the lattice action is the mildest possible, compatibly with the Nielsen–Ninomiya theorem. This is the so called Ginsparg–Wilson condition (GWC) for the lattice Dirac operator.

Recently, it has been realized that the Dirac operator associated to a fixed point of a renormalization group transformation \[4\], and the one coming from the ‘overlap formalism’ \[5\], both satisfy the GWC \[4, 6\]. Further analysis has shown that the main mechanisms of QCD related to chiral symmetry find correspondence on the lattice with Ginsparg–Wilson fermions \[7, 8, 9\]. Formulation of chiral gauge theories describing Weyl fermions is also possible \[10\]. All this is somehow miraculous, taking place in a scenario where the chirality is explicitly broken.

Here we concentrate on one particular aspect, i.e. the Leutwyler–Smilga sum rules \[11\] for the eigenvalues of the Dirac operator of chiral QCD. These can be explicitly derived in the continuum (regularized) theory, from the analysis of the finite volume partition function in a limit (large volumes and small quark masses) where QCD reduces to a simple matrix model and chiral symmetry plays the fundamental role. Starting from this, Shuryak and Verbaarschot put forward the hypothesis \[12\] that the sum rules are an universal feature, i.e. model independent, the only precondition being chiral symmetry. The simplest chiral model is a chiral Random Matrix Theory \[12\]; here Leutwyler–Smilga sum rules have been verified, enabling also a
systematic classification of the different universal behaviors \[13\]. First checks of the predictions from chiral Random Matrix Theory have been done in the framework of lattice gauge theory with staggered fermions \[14\], which however restore the chirality on the lattice only partially.

Ginsparg–Wilson fermions, which effectively restore chiral symmetry, appear the ideal environment for studying Shuryak and Verbaarschot’s chirality–induced universality on the lattice. Equivalence of a Random Matrix Theory complying with the GWC to the ordinary chiral Random Matrix Theory has been proved in \[16\].

Here, we explicitly show how Leutwyler–Smilga sum rules are recovered in lattice QCD, in a framework of explicitly broken chiral symmetry, with Ginsparg–Wilson fermions.\footnote{After the completion of the work, the author realized that this result was already derived, with a slightly different approach, by F. Niedermayer in his talk \[17\] at Lattice 98 conference.}

We consider the simplest case of just one quark flavor, where the absence of massless excitations allows to avoid the complications related to the management of finite–size effects. We follow the line of reasoning of the seminal paper \[11\].

## 2 Finite volume partition function

We start from the lattice action

\[
S = S_G(U) + \sum_x \bar{\psi}_x (D_{x,x'}(U) + \mathcal{M}) \psi_{x'} \ ,
\]

(1)

(Dirac and color indices are omitted) where

\[
\mathcal{M} = m \frac{1 + \gamma^5}{2} + m^* \frac{1 - \gamma^5}{2}
\]

(2)

(m is a complex mass); \(D\) is any Ginsparg–Wilson Dirac operator satisfying

\[
\{ D_{x,x'}, \gamma^5 \} = (D \gamma^5 D)_{x,x'} \ .
\]

(3)

The latter relation is a particular case of a broader (but still equivalent) condition

\[
\{ D_{x,x'}, \gamma^5 \} = (2D \gamma^5 RD)_{x,x'} \ .
\]

(4)
where $R$ is an operator depending only on color and space–time (not Dirac) indices, whose matrix elements have finite range in space–time indices, or exponentially decay with the distance. Our case corresponds to $R_{x,x'} = (1/2) \delta_{x,x'}$. We assume for $D$ also the $\gamma^5$–hermiticity property $D^\dagger = \gamma^5 D \gamma^5$ which implies in particular that the spectrum is invariant under complex conjugation $\lambda \rightarrow \bar{\lambda}$. We consider the theory on a finite lattice of extension $L$, with periodic (anti–periodic) boundary conditions for gauge (fermionic) degrees of freedom. All quantities are expressed in lattice units.

As a consequence of (3) the spectrum of $D \{\lambda\}$ is constrained on a unit circle in the complex plane centered in (1,0)

$$|\lambda - 1|^2 = 1 ;$$

the set of its eigenvectors $\{v_\lambda\}$ form an orthonormal complete basis, and have definite chiral properties:

$$\gamma^5 v_\lambda = \begin{cases} v_\overline{\lambda} & \text{if } \lambda \neq \overline{\lambda} \\ \pm v_\lambda & \text{if } \lambda \in \mathbb{R} \end{cases} .$$

In particular, the index of $D(U), \nu(U)$, can be defined [7] as the number of zero modes counted with their chirality. [3]

In analogy with the continuum, the partition function $Z(\theta, m)$ is defined as:

$$Z(\theta, m) = \sum_{\nu=-\infty}^{\infty} e^{i\theta\nu} Z_\nu(m) ,$$

where $Z_\nu(m)$ is obtained by integrating just on configurations with associated index $\nu$:

$$Z_\nu(m) = \int[dU^{(\nu)}] e^{-\beta S_G} \det(D(U^{(\nu)}) + \mathcal{M}) .$$

As a consequence of (6) the matrix $\mathcal{M}$ is (as in the continuum) block–diagonal in the basis $\{v_\lambda\}$, each block living in the 2–dimensional subspace spanned by $v_\lambda$ and $v_{\overline{\lambda}}$:

$$\left( \begin{array}{cc} \Re(m) & i \Im(m) \\ i \Im(m) & \Re(m) \end{array} \right) ;$$

\[^2\text{We assume [11] that different chiralities cannot mix.}\]
in the case \( \lambda \in \mathbb{R} \) (i.e., because of (5), \( \lambda = 0, 2 \)) \( v_\lambda \) is also an eigenvector of \( \mathcal{M} \) with eigenvalue \( m \) for positive chirality and \( m^* \) for negative chirality.

Using these properties and exploiting the constraint (5) as well, we can write an explicit expression for the fermion determinant in (8) in terms of the eigenvalues of \( \mathcal{D} \); for \( \nu > 0 \):

\[
\det(\mathcal{D}(U(\nu)) + \mathcal{M}) = (2 + m)^{N_{rm}^+} (2 + m^*)^{N_{rm}^-} m^\nu \prod_i'' \left( (1 + \text{Re}(m)) |\lambda_i|^2 + |m|^2 \right), \tag{10}
\]

where the double–primed product indicates the product over half of the complex eigenvalues and \( N_{rm}^+ (N_{rm}^-) \) is the number of positive (negative) real modes; for \( \nu < 0 \), \( m^\nu \to (m^*)^{-\nu} \) in the above formula.

Following [11] we now consider the problem of finding an explicit representation for \( Z(\theta,m) \) in the limit

\[
L \to \infty, \quad m \to 0, \quad mL^d = \text{const} \tag{11}
\]

(\( d \) is the number of space–time dimensions). We rewrite the determinant in (10):

\[
\det(\mathcal{D}(U(\nu)) + \mathcal{M}) = (2 + m)^{N_{rm}^+} (2 + m^*)^{N_{rm}^-} \left[ (1 + \text{Re}(m))^{\frac{1}{2}(N_c N_D L^d - \nu - N_{rm})} \right] m^\nu \prod_i'' \left( |\lambda_i|^2 + \frac{|m|^2}{(1 + \text{Re}(m))} \right), \tag{12}
\]

where \( N_c \) and \( N_D \) denote the number of color and Dirac degrees of freedom respectively and \( N_{rm} = N_{rm}^+ + N_{rm}^- \). In the limit (11) we can replace

\[
(1 + \text{Re}(m))^{\frac{1}{2}(N_c N_D L^d - \nu - N_{rm})} \to e^{\frac{4}{2}N_c N_D L^d \text{Re}(m)},
\]

\[
\frac{|m|^2}{(1 + \text{Re}(m))} \to |m|^2 \tag{13}
\]

and the factor related to the real modes reduces to \( 2^{N_{rm}} \); we obtain

\[
\det(\mathcal{D}(U(\nu)) + \mathcal{M}) \to e^{\frac{4}{2}N_c N_D L^d \text{Re}(m)} 2^{N_{rm}} m^\nu \prod_i'' \left( |\lambda_i|^2 + |m|^2 \right). \tag{14}
\]

Apart from the multiplicative factor \( e^{\frac{4}{2}N_c N_D L^d \text{Re}(m)} \), the expression of the continuum is recovered (except that the \( \lambda \)s are complex and not purely imaginary); in particular, if we write:

\[
Z(\theta,m) = e^{\frac{1}{2}N_c N_D L^d \text{Re}(m)} Z'(\theta,m); \tag{15}
\]
we see that in the limit $Z'(\theta, m)$ is invariant under the symmetry (applying in the continuum for $Z(\theta, m)$):

$$
m \rightarrow e^{i\phi} m
\theta \rightarrow \theta - \phi ,
$$

which implies

$$Z'(\theta, m) = Z''(m e^{i\theta}) .$$

If the theory has a mass gap non–vanishing for $m \rightarrow 0$ (which is true in the one flavor case), in the infinite volume limit we can assume

$$Z(\theta, m) = \exp \left\{ -L^d \epsilon_0(\theta, m) \right\} ,$$

where $\epsilon_0(m, \theta)$ is the lattice analogous of the vacuum energy density. The corrections to this relation are exponentially small, $O(e^{-m_0 L})$, where $m_0$ corresponds to the mass–gap of the theory $Z$.

Using the factorization property $Z''(me^{i\theta})$ we parametrize $\epsilon_0(\theta, m)$ as

$$\epsilon_0(\theta, m) = C - \Sigma \mathrm{Re}(m e^{i\theta}) - \frac{1}{2} N_c N_D \mathrm{Re}(m) + O(m^2) ;
$$

the last but one term is an ultraviolet divergent and topology–independent contribution to the vacuum energy density $\sim 1/a^{D-1}$. It is a lattice artifact (absent in the continuum $Z$) appearing because of the explicit breaking of the chirality; it must be subtracted in order to get the correct continuum limit. The parameter $\Sigma$ is expected to scale as a physical quantity of dimension $D – 1$ and gives a lattice definition of the fermion condensate in the infinite volume limit. From (19) it follows:

$$\Sigma = -\langle \bar{\psi}_x \psi_x \rangle_{m=0,L\rightarrow\infty} - \frac{1}{2} N_c N_D ;$$

observe that in the case of absence of massless excitations the two limits $L \rightarrow \infty$ and $m \rightarrow 0$ can be interchanged.

In the framework of Ginsparg–Wilson fermions a subtracted fermion condensate can be defined $\langle\rangle$ (see also the discussion in $\langle\rangle$), which for $N_f > 1$ represents an order parameter for spontaneous breaking of the $SU(N_f)_A$ symmetry:

$$\langle \bar{\psi}_x \psi_x \rangle_{\text{sub}} = \langle \mathrm{tr}^{D FC}(-D^{-1}_{x,x} + R_{x,x}) \rangle_{\mathrm{gauge}} ,$$
where \( \text{tr}^{DFC} \) is a trace over Dirac, flavor and color indices and \( \langle \cdots \rangle_{\text{gauge}} \) denotes the gauge average (including the fermion determinant); in our case, where \( R_{x,x'} = (1/2) \delta_{x,x'} \) and \( N_f = 1 \), we argue from (20)

\[
\Sigma = - \langle \bar{\psi}_x \psi_x \rangle_{\text{sub}} . \tag{22}
\]

Using (19) the calculation of \( Z_{\nu}(m) \) is readily worked out:

\[
Z_{\nu}(m) = \int d\theta e^{-i\nu\theta} Z(\theta, m) = \left( \frac{m}{|m|} \right)^\nu e^{\frac{1}{2} N_c N_D L^d} \text{Re}(m) I_\nu(L^d\Sigma|m|) . \tag{23}
\]

3 Leutwyler–Smilga sum rules

Now we exploit Leutwyler and Smilga’s idea to obtain the wanted sum rules; we write (from now on we take \( m \) real and positive and \( \theta = 0 \)):

\[
Z_{\nu}(m) = m^{\nu} \int [dU(\nu)] e^{-\beta S_G} \left( \prod_{i} \lambda_i \right) \left[ \prod_{i} \left( 1 + \frac{m}{\lambda_i} \right) \right] , \tag{24}
\]

where the primed product is intended over non–zero modes. We rewrite

\[
\prod_{i} \left( 1 + \frac{m}{\lambda_i} \right) = (1 + \frac{m}{2})^{N_c N_D L^d - \nu} \prod_{i} \left[ 1 + \left( \frac{m}{1 + \frac{m}{2}} \right)^2 \frac{1}{\lambda_i^2} \right] \tag{25}
\]

where we have introduced the new real variable:

\[
\hat{\lambda}(\lambda) = \frac{-i \lambda}{1 - \frac{i}{2} \lambda} ; \tag{26}
\]

geometrically, \( \hat{\lambda}(\lambda) \) is obtained from \( \lambda \) by the stereographic projection of the unit circle centered in \((1,0)\) (where \( \lambda \) lives) onto the imaginary axis; clearly \( \hat{\lambda}(\bar{\lambda}) = -\hat{\lambda}(\lambda) \).

In the limit \((11)\) the r.h.s. of (25) may be replaced by

\[
e^{\frac{1}{2} N_c N_D L^d m} \prod_{i} \left[ 1 + \frac{m^2}{\lambda_i^2} \right] , \tag{27}
\]

again an expression analogous to the continuum one apart from the multiplicative factor \( e^{\frac{1}{2} N_c N_D L^d m} \). Inserting (27) in the r.h.s. of (24), we come to the relation, exact in the limit \((11)\):

\[
\langle \langle \prod_{i} \left[ 1 + \frac{m^2}{\lambda_i^2} \right] \rangle \rangle_{\nu} = e^{\frac{1}{2} N_c N_D L^d m} \frac{m^{-\nu} Z_{\nu}(m)}{\lim_{m \to 0} (m^{-\nu} Z_{\nu}(m))} \tag{28}
\]

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where \( \langle \langle \cdots \rangle \rangle_\nu \) is the average over gauge configurations with associated index \( \nu \) in the massless case, the fermion determinant being replaced by the product of non–zero eigenvalues of the Dirac operator. Using the explicit representation \( 28 \) and

\[
I_\nu(x) \simeq \frac{1}{|\nu|!} \left( \frac{x}{2} \right)^{|\nu|}, \quad x \ll 1, \tag{29}
\]

we obtain (for any sign of \( \nu \))

\[
\langle \langle \prod_{i}^{''} \left( 1 + \frac{m_i^2}{\lambda_i^2} \right) \rangle \rangle_\nu = |\nu|! \left( \frac{2}{x} \right)^{|\nu|} I_\nu(x), \tag{30}
\]

where \( x = L^d \Sigma m \). So we recover formally the same result of the continuum, from which Leutwyler–Smilga sum rules originate, with the only difference that in the sums the original eigenvalue \( \lambda \) is replaced by the projected one \( \tilde{\lambda} \). For example from \( 30 \) it follows \( 11 \)

\[
\lim_{L \to \infty} \frac{1}{(L^d \Sigma)^2} \langle \langle \sum_{i}^{''} \frac{1}{\lambda_i^2} \rangle \rangle_\nu = \frac{1}{4(\nu + 1)}. \tag{31}
\]

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\( ^{3} \)Or any other definition \( \tilde{\lambda} ' \) such that \( \tilde{\lambda} ' = \tilde{\lambda}(1 + O(\tilde{\lambda})) \)
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