BMAP Two Phase Retrial Queue with Feedback

K. Prakash Rani¹, K. Udaya Chandrika², U. Elayarani³

¹Department of Mathematics, A.D.M. College for Women, Nagapattinam
²Professor of Mathematics, Avinashilingam Deemed University, Coimbatore
³Department of Mathematics, A.D.M. College for Women, Nagapattinam

Abstract: This paper is concerned with the analysis of a single server batch arrival retrial queue with general retrial times. The server provides two phases of service – essential and optional. A customer after receiving essential service may leave the system or rejoin the orbit and begin requesting service again. The model for the system is derived and the joint distribution of the server state and the orbit length in steady state is obtained. Numerical examples are presented to illustrate the effect of the parameters on several performance measures.

Keywords: Retrial queues, batch arrival, probability generating function, steady state, feedback

1. Introduction

Retrial queueing system are characterized by the feature that the arrivals who find a free server enters into service immediately; otherwise the customer enters into an orbit. An orbiting customer competes for service by sending out signals at random times until a free server is captured. Retrial queues are widely used in mathematical models of several computer and telecommunication systems.

There have been several contributions considering queueing systems of M/G/1 type in which the server may provide a second phase of service. Medhi [9] has studied an M/G/1 queue where the server provides the first essential service to all the arriving customers, whereas only some of them receive a second option service. Choudhury [4] has obtained the waiting time distribution for the queueing model discussed by Medhi [10]. Krishna Kumar et al. [8, 9] have analyzed an M/G/1 retrial queue with two phase services, preemptive priority and single server feedback retrial queue respectively.

Feedback queues relate to those queues in which a customer served once, when his service becomes unsuccessful and is served again and again till his service becomes successful. Many authors Takacs [11], Folely et al. [6] and Workman et al. [12] have analyzed queueing model with feedback. These models are motivated mainly by application in data transmission, manufacturing processes where quality control inspections are performed and so on. In this paper an M/G/1 queue with additional optional service and customers feedback is analyzed for its performance measures.

2. Model Description

Consider a single server retrial queue where optional service is provided after the essential service. The server provides essential service to all the arriving customers. Customers arrive in groups according to a Poisson process with rate λ. The batch size Y is a random variable and $P(Y=k) = C_k^y$ is provided after the essential service. The server provides the first essential service to all the arriving customers, whereas only some of them receive a second option service. The functions $B(.)$ and $b(.)$ be respectively the cumulative distribution and the probability density function of the batch size distribution with first two moments $h_1$ and $h_2$.

Denote by $C(z) = \sum C_k z^k$ the generating function of the batch size distribution with first two moments $\tau_1$ and $\tau_2$. If the server is free then the essential service commences for one of the arriving customers and the others join the orbit. While at the essential service, the server may push out (with probability $\alpha$) the customer under going service to the orbit and start serving an arriving customer or continue the ongoing service (with probability $1-\alpha$) so that all the arriving customers join the orbit. Successive inter retrial times of any customer are governed by an arbitrary probability distribution function $A(.)$ with corresponding density function $a(.)$ and Laplace Stieltjes transform $A^*(.)$.

The server provides the essential service to all arriving customers. Let $B(.)$ and $b(.)$ be respectively the cumulative distribution and the probability density function of the essential service time with Laplace Stieltjes transform $B^*(.)$. The server under going essential service can return back to the orbit with probability $\gamma (= 1-\beta-\delta)$. The optional service times of customers are independent random variables with common distribution function $B_1(.)$ and density function $b_1(.)$, Laplace Stieltjes transform $B_1^*(.)$ and first two moments $h_1$ and $h_2$.

The stochastic behaviour of this retrial queueing system can be described by the Markov process

\[ \{N(t), t \geq 0\} = \{(C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), t \geq 0)\} \] where $C(t)$ denotes the server state, 0, 1 or 2 according as the server being idle, busy with essential service, busy with optional service and $X(t)$ corresponds to the number of customers in the orbit at time $t$. If $C(t) = 0$ and $X(t) > 0$, then $\tau(t)$ represents the elapsed retrial time. If $C(t) = 1$ and $X(t) \geq 0$, then $\xi(t)$ corresponds to the elapsed time of the customer being provided phase 1 service. If $C(t) = 2$ and $X(t) \geq 0$, then $\xi(t)$ represents the elapsed time of the customer being provided optional service. The functions $\eta(x), \mu(x)$ and $\mu_1(x)$ are the conditional completion rates for repeated attempts, essential service and for optional service respectively. Then $\eta(x) = a(x) / [1 - A(x)];$ $\mu(x) = b(x) / [1 - B(x)]$ and $\mu_1(x) = b_1(x) / [1 - B_1(x)].$
3. Theorem

Let $X_n$, $n \geq 1$ be the orbit length at the time of the $n^{th}$ customers departure. Then $\{X_n, n \geq 1\}$ is ergodic if and only if $\tau_1 [1 - B^*(\alpha \lambda)] / (\alpha B^*(\alpha \lambda)) + \gamma + \beta \lambda \tau_1 + \tau_1 [1 - A^*(\lambda)] < 1$.

The theorem can be proved along similar lines as in Gomez-Corral [7].

As the arrival stream is a Poisson process with mean batch size $\tau_1$, it can be shown from Burke’s theorem [5] that the steady state probabilities of $\{C(t), X(t), t \geq 0\}$ exist and they are positive if and only if $\tau_1 [1 - B^*(\alpha \lambda)] / (\alpha B^*(\alpha \lambda)) + \gamma + \beta \lambda \tau_1 + \tau_1 [1 - A^*(\lambda)] < 1$.

From the mean drift $\chi_j = \tau_1 [1 - B^*(\alpha \lambda)] / (\alpha B^*(\alpha \lambda)) + \gamma + \beta \lambda \tau_1 + \tau_1 [1 - A^*(\lambda)] - 1$, for $j \geq 1$, we have the reasonable conclusion that the first two terms represents the mean number of customers leaving for orbit due to the decision of the server to push out or continue the ongoing service or due to feedback of customers. The term $\tau_1 \beta \lambda \tau_1$ represents the batch arrival during service time in the optional service leaving for the orbit. The last term $\tau_1 [1 - A^*(\lambda)] - 1$ refers to the contribution to the orbit size due to a batch arrival during the retrial time excluding the arbitrary customer of the arriving batch whose service commences immediately. Thus, the condition $\chi_j < 0$ ensures that the orbit size does not grow indefinitely in course of time.

4. Steady State Distribution

In this section the steady state distribution of the system are derived. For the process $\{N(t), t \geq 0\}$, define the probability

$$I_0(t) = P\{C(t) = 0, X(t) = 0\}$$

and the probability densities

$$I_n(t) = P\{C(t) = 0, X(t) = n, x \leq \xi_n(t) < x + dx\}, \quad t \geq 0, x \geq 0$$

$$W_n(x, t) = P\{C(t) = 1, X(t) = n, x \leq \xi_n(t) < x + dx\}, \quad t \geq 0, x \geq 0$$

$$S_n(x, t) = P\{C(t) = 2, X(t) = n, x \leq \xi_n(t) < x + dx\}, \quad t \geq 0, x \geq 0, n \geq 1$$

By supplementary variable technique, the system of equations that governs the model are given by

$$\frac{d}{dt}I_0(t) = -\lambda I_0(t) + \delta \int_0^\infty W_0(x, t) \mu(x) \, dx$$

$$+ \int_0^\infty S_0(x, t) \mu_1(x) \, dx \quad (1)$$

$$\frac{d}{dt}W_n(x, t) = -\lambda W_n(x, t)$$

$$+ \lambda (1 - \alpha) (1 - \delta_{0n}) \sum_{k=1}^n C_k W_{n-k}(x, t), \quad n \geq 0$$

$$\frac{d}{dt}S_n(x, t) = -\lambda S_n(x, t)$$

$$+ \lambda (1 - \delta_{0n}) \sum_{k=1}^n C_k S_{n-k}(x, t), \quad n \geq 1 \quad (4)$$

with boundary conditions

$$I_0(0) = \delta \int_0^\infty W_0(x, t) \mu(x) \, dx$$

$$+ \int_0^\infty S_0(x, t) \mu_1(x) \, dx, \quad n \geq 1 \quad (5)$$

$$W_0(0, t) = \lambda C_1 I_0(t) + \int_0^\infty I_1(x, t) \eta(x) \, dx \quad (6)$$

$$S_0(0, t) = \beta \int_0^\infty W_0(x, t) \mu(x) \, dx, \quad n \geq 0 \quad (8)$$

The steady state equations corresponding to the equations (1) through (8) are given by

$$\lambda I_n = \delta \int_0^\infty W_n(x, t) \mu(x) \, dx + \int_0^\infty S_n(x, t) \mu_1(x) \, dx \quad (9)$$

$$\frac{d}{dx}I_n(x) = -\lambda I_n(x) + \delta \int_0^\infty W_n(x, t) \mu(x) \, dx$$

$$+ \int_0^\infty S_n(x, t) \mu_1(x) \, dx, \quad n \geq 1 \quad (10)$$

$$\frac{d}{dx}W_n(x, t) = -\lambda W_n(x, t)$$

$$+ \lambda (1 - \alpha) (1 - \delta_{0n}) \sum_{k=1}^n C_k W_{n-k}(x, t), \quad n \geq 0 \quad (11)$$

$$\frac{d}{dx}S_n(x, t) = -\lambda S_n(x, t)$$

$$+ \lambda (1 - \delta_{0n}) \sum_{k=1}^n C_k S_{n-k}(x, t), \quad n \geq 0 \quad (12)$$

With boundary conditions

$$I_0(0) = \delta \int_0^\infty W_0(x, t) \mu(x) \, dx + \gamma \int_0^\infty W_{n-1}(x, t) \mu(x) \, dx$$

$$+ \int_0^\infty S_n(x, t) \mu_1(x) \, dx, \quad n \geq 1$$

$$W_0(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x, t) \eta(x) \, dx \quad (13)$$

$$S_0(0) = \beta \int_0^\infty W_0(x, t) \mu(x) \, dx + \lambda C_{n-1} I_0$$

$$+ \int_0^\infty I_{n-1}(x) \eta(x) \, dx + \lambda C_{n-1} I_0 \quad (14)$$

Volume 6 Issue 12, December 2017

www.ijsr.net

License Under Creative Commons Attribution CC BY
The probability that the server is idle during retrial time I(1), the probability that the server is idle during the time is given by

\[
I(1) = [1 - A^*(\lambda)] [\tau_1 [1 - B^*(\alpha\lambda) (1 - \alpha - \alpha\beta\lambda h_1)]
\]

for any probability \( p(x) \) during retrial time \( I(1) \), the probability that the server is idle during

\[
-\alpha(1-\gamma)B^*(\alpha\lambda) / [\alpha(1-\gamma)A^*(\lambda)B^*(\alpha\lambda)]
\]

The probability that the server is busy for providing essential service is given by

\[
W(1) = \tau_1 [1 - B^*(\alpha\lambda)] / [\alpha (1 - \gamma) B^*(\alpha\lambda)]
\]

The probability that the server is busy for providing optional service is given by

\[
S(1) = \tau_1 / (1 - \gamma)
\]

Take

\[
D_1 = \tau_1 [B^*(\alpha\lambda)] [1 - (1 - A^*(\lambda) + \beta\lambda h_1)] - 1
\]

\[
+ \alpha (1 - \gamma) B^*(\alpha\lambda)
\]

\[
D_2 = \lambda \tau_1 (1 - \alpha) A^*(\alpha\lambda) \gamma + \beta\tau_1 h_1
\]

\[
- \tau_1 (1 - \alpha) - \alpha - \alpha\tau_1 A^*(\lambda)
\]

\[
+ \tau_1 [B^*(\lambda) \gamma + \beta\lambda\tau_1 h_1]
\]

\[
+ B^*(\alpha\lambda) [\tau_2 (1 - A^*(\lambda) - \gamma + \beta\lambda\tau_1 h_1)]
\]

\[
= \tau_1 [1 - B^*(\alpha\lambda)] / [\alpha (1 - \gamma) B^*(\alpha\lambda)]
\]

The mean number of customers in the system is

\[
L_s = K'(1)
\]

\[
= N_1 / [\alpha (1 - \gamma) B^*(\alpha\lambda)] + D_2 / D_1
\]

The mean number of customer in the orbit is

\[
L_q = H'(1)
\]

\[
= N_2 / [\alpha (1 - \gamma) B^*(\alpha\lambda)] + D_2 / D_1
\]

6. Numerical Results

Table 1 presents the values of expected system size \( L_s \), expected queue size \( L_q \), the probability that the server is idle during retrial time \( I(1) \), the probability that the server is busy for providing essential service W(1) and the probability that the server is busy for providing optional service S(1) for fixed values of \( C_1=0.5, C_2=0.5, \alpha=0.2, \beta=0.1, \lambda=1 \) and various values of \( \gamma, \eta, \mu_1 \) and \( \mu_2 \). The following results are observed from the table 1.

- System size \( L_s \) directly proportional to the feedback service rates of essential and optional.
- The same trend is observed with respect to the orbit size \( L_q \).
- The probability to have the server idle during retrial time increases with increase of feedback and decreases with increase in \( \eta, \mu_1, \mu_2 \).
- W(1), the probability that the server is busy for providing essential service increases for increase in \( \gamma \) decreases for increase in \( \mu_1 \) and constant for the variation in \( \mu_2 \) and \( \eta \).
- The influence of the two parameters \( \eta \) and \( \mu_1 \) are not felt in S(1).

| Table 1: Parameters influence on performance measures |
|----------------|----------------|------------------|----------------|----------------|
| \( \gamma \)  | \( L_s \)  | \( L_q \)  | \( I(1) \) | \( W(1) \)  | \( S(1) \) |
| 0.2            | 0.4910  | 0.3973  | 0.1938      | 0.0750  | 0.0188      |
| 0.4            | 1.0226  | 0.8976  | 0.3250      | 0.1000  | 0.0214      |
| 0.6            | 4.3620  | 4.1745  | 0.5875      | 0.1500  | 0.0375      |
Author Profile

K. Prakash Rani is an Associate Professor & Head in PG and Research Department of Mathematics, A.D.M. College for Women (Autonomous), Nagapattinam. She received her B.Sc. degree in Mathematics from Madras University, M.Sc. Applied Mathematics from Bharathidasan University and M.Phil. degree from (MIT), Anna University. In 1986, she joined as a Research Assistant in the Department of Mathematics, REC.Tiruchirappalli then in 1987 she joined as a Lecturer in A.D.M. College. She received her Ph.D.degree from Bharathidasan University in 2008. Her Research interests include Stochastic Processes, Queues, Retrial Queues and Graph theory. She has more than 30 years of teaching, 16 years of Research experience and she published 15 articles. Presented paper in different conferences held at Vijayawada, New Delhi and Calcutta University. She has finished one UGC sponsored Minor Research Project and she is a co-Guide for five Research Scholars.

Volume 6 Issue 12, December 2017

www.ijsr.net

References

[1] Artalejo, J.R. Accessible bibliography on retrial queues. Mathematical and Computer Modelling 1999; 30, 1-6.
[2] Artalejo, J.R. A classified bibliography of research on retrial queues : Progress in 1990-1999. 1999;Top, 7, 187-211.
[3] Artalejo, J.R. and Falin, G.I. Standard and retrial queueing systems : A comparative analysis. Revista Mathematica Complutense, 2002; 15, 101-129.
[4] Choudhury, G. Some aspects of an M/G/1 queueing system with optional second service. 2003; Top, 11, 141-150.
[5] Cooper, R.B. Queues served in cyclic order : Waiting times. Bell system Technical Journal, (1970); 49, 339-413.
[6] Fioley, Robert, D., Disney, Ralph L. Queues with delayed feedback. Adv. In Appl. Probab., 1983;15, 162-182.
[7] Gomez-Corral, A. Stochastic analysis of a single server retrial queue with general retrial times. Naval Research Logistics, 1999; 46, 561-581.
[8] Krishna Kumar, B., Vijay Kumar, A. and Arivudainambi, D. An M/G/1 retrial queueing system with two-phase service and preemptive resume. Annals of Operations Research, 2002; 113, 61-79.
[9] Krishna Kumar, B., Vijaylakshmi, G., Krishnamoorthy, A. and Sadiq Basha, S. A single server retrial queue with collision. Computers and Operations Research, 2010;37(7), 1247-1255.
[10] Medhi, J. A single server Poisson input queue with a second operational channel. Queueing Systems, 2002; 42, 239-242.
[11] Takacs, L. A single-server queue with feedback. Bell System Tech. J., 1963); 42, 505-519.
[12] Wortman, M.A., Disney, Ralph, L., Kiessler, Peter, C. The M/G/1 Bernoulli feedback queue with vacations. Queueing Systems Theory Appl., 1991;9(4), 353-363.

Author Profile

K. Udayachandrika is a Professor and Head of the Department of Mathematics, Avinashilingam University, Coimbatore. She received her Ph.D. degree from Bharathiar University. In 1978, she joined in the Department of Mathematics, Avinashilingam University, Coimbatore. Her Research interests include Queues, Retrial and Fuzzy Mathematics. She has more than 39 years of teaching, 30 years of Research experience and she published 64 articles. She has done 2 projects and organized 2 conferences.