A twistor formulation of the non-heterotic superstring with manifest worldsheet supersymmetry

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Abstract

We propose a new formulation of the $D = 3$ type II superstring which is manifestly invariant under both target-space $N = 2$ supersymmetry and worldsheet $N = (1, 1)$ super reparametrizations. This gives rise to a set of twistor (commuting spinor) variables, which provide a solution to the two Virasoro constraints. The worldsheet supergravity fields are shown to play the rôle of auxiliary fields.

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1 Introduction

During the last few years a new formulation of the superparticle and the heterotic superstring with a $D = 3, 4, 6, 10$ target space has been developed \cite{1}-\cite{10}. It has $N = 1$ target supersymmetry and at the same time manifest worldline (or worldsheet) $N = D-2$ local supersymmetry. The latter replaces the well-known kappa-symmetry \cite{11}, \cite{12} of the superparticle (string). Thus kappa-symmetry finds its natural explanation as an on-shell version of the off-shell local supersymmetry of the worldsheet.

The key to such formulations is the use of commuting spinor (“twistor”) variables, as proposed in the pioneering work of Sorokin, Tkach, Volkov and Zheltukhin \cite{1}. These variables emerge in a natural way as the worldsheet supersymmetry superpartners of the target superspace Grassmann coordinates. In this context one obtains a twistor-like solution for the null momentum of the massless superparticle (or for one of the Virasoro vectors of the heterotic superstring) as a bilinear combination of the twistors. Thus, the twistor variables turn out to parametrize the sphere $S^{D-2}$ associated with the above null vector. All of this is achieved as a consequence of one of the equations of motion of the twistor-like superparticle (string), the so-called geometro-dynamical constraint. It specifies the way the worldsheet superspace $\mathcal{M}$ is embedded in the target superspace $\mathcal{M}$. Namely, one requires that the odd part of the tangent space to $\mathcal{M}$ lies entirely within the odd part of the tangent space to $\mathcal{M}$ at any point of $\mathcal{M}$. The conditions for this particular embedding are generated dynamically from a Lagrange multiplier term in the action.

It is very natural to try to extend the above results to the non-heterotic superstring. This means to solve both Virasoro constraints in terms of twistor variables and to interpret the kappa-symmetry of the theory as non-heterotic $N = (D-2, D-2)$ worldsheet supersymmetry. However, changing from one dimension (the case of the superparticle) or essentially one dimension (the case of the heterotic superstring) to two dimensions of the worldsheet is far from trivial. An attempt in this direction has recently been made by Chikalov and Pashnev \cite{13}. There only the first half of this program was achieved. Considering an $N = 2$ target superspace, but still only $N = (1, 0)$ worldsheet supersymmetry, Chikalov and Pashnev obtained two twistor variables and solved both Virasoro constraints. At the same time, their worldsheet possessed only one supersymmetry, which could not explain the full kappa-symmetry of the theory and in addition broke two-dimensional Lorentz invariance. An interesting feature of their formulation was the absence of any worldsheet supergravity fields.

In the present paper we shall make a step further towards the realization of the full twistor program. We present a twistor formulation of the $D = 3$ type II (i.e. with $N = 2$ target space supersymmetry) non-heterotic superstring. On the worldsheet we have $N = (1, 1)$ local supersymmetry and thus are able to completely eliminate kappa-symmetry. At first sight our construction closely resembles the one in the heterotic case \cite{10}. However, significant differences appear in the analysis of the twistor constraints, which follow from the geometro-dynamical embedding of $\mathcal{M}$ in $\mathcal{M}$. If in the heterotic case it was relatively easy to show that as a result of these constraints the twistor variables parametrized the space $S^{D-2}$, here a careful study is needed. The solution to the twistor constraints now consists of two sectors, a regular one, which corresponds to non-trivial superstring motion and a singular one, in which the superstring collapses into a superparticle. Another
big difference is that one needs the full set of worldsheet supergravity fields in order to make the superfield action super reparametrization invariant. However, at the level of components one discovers that the worldsheet gravitino is in fact an auxiliary field and can be eliminated via its algebraic field equation. The worldsheet metric and the twistor variables compete for the rôle of reparametrization gauge fields. Algebraic elimination of the twistor variables leads to the familiar Green-Schwarz action. Finally, in the heterotic case the formalism worked equally well in all the cases $D = 3, 4, 6, 10$. However, in the non-heterotic case the attempt to go beyond $D = 3$ (thus having extended worldsheet supersymmetry) causes a problem, namely, the geometro-dynamical constraint starts producing equations of motion. Understanding this crucial point will probably give us some new non-trivial insight into the geometric nature of the superstring. It will also help us achieve a complete twistor formulation in all the dimensions where the classical superstring exists.

The paper is organized as follows. In section 2 we explain the notation and introduce the basic geometric objects of the worldsheet and target superspaces. In section 3 we present the twistor formulation of the $D = 3$ superparticle with $N = 2$ target-space and $N = (1, 1)$ worldsheet supersymmetry. This is a simplified version of the superstring theory, which helps illustrate some of the new features encountered here. In section 4 the two terms of the superstring action, the geometro-dynamical and the Wess-Zumino ones are given and it is shown how the former allows one to establish the consistency of the latter. In section 5 we study in detail the component structure of the action. We find out which component fields are auxiliary and by eliminating them arrive at the standard Green-Schwarz action. This analysis crucially depends on which solution of the twistor constraints we use, the regular or the singular one. In the latter case we observe the string shrinking to a particle. In the Appendix we find the general solution to the algebraic twistor constraints, which consists of a regular and a singular sector.

2 Two- and three-dimensional supergeometry

In this section we shall introduce some basic concepts concerning $N = 2$ superspaces in two and three dimensions. These superspaces will serve as the worldsheet and the target space of the superstring, respectively.

The worldsheet of the type II $D = 3$ superstring is a 2|2-dimensional superspace parametrized by two even and two odd real coordinates $Z^M = (\xi^m, \eta^\mu)$, where $m = (0, 1)$ and $\mu = (1, 2)$. We assume that it is endowed by $N = (1, 1)$ two-dimensional supergravity. The latter is described by a vielbein $E_A^M$ and a $SO(1, 1)$ Lorentz connection $\omega_A$ which satisfy the following constraint

$$\{D_\alpha, D_\beta\} = 2i(\gamma^c C)_{\alpha\beta} D_c + R_{\alpha\beta}. \quad (1)$$

1In the case of the superparticle no worldline supergravity fields appear in the action [3]. For the heterotic superstring only one component of the two-dimensional metric is needed to generate the second Virasoro constraint [4, 5].

2We use the following conventions for the gamma and charge conjugation matrices: $\gamma^0 = C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ with $\eta^{00} = 1$, $\eta^{11} = -1$. 

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Here \( A = (a, \alpha) \) with \( a = (0, 1) \) and \( \alpha = (1, 2) \) being \( SO(1, 1) \) vector and spinor indices, and the covariant derivatives are

\[
D_A = E_A^M \partial_M + \omega_A. \tag{2}
\]

Equation (1) means that we have imposed the following torsion constraints:

\[
T_{\alpha\beta}^\gamma = 2i(\gamma^\gamma C)_{\alpha\beta}, \quad T_{\alpha\beta}^\gamma = 0. \tag{3}
\]

The complete set of \( N = (1, 1) \) supergravity constraints and their consistency have been studied in [14]. There it has been shown that two-dimensional \( N = (1, 1) \) supergravity can be considered superconformally flat (ignoring moduli problems). This means that there exist superconformal transformations of the vielbeins and connections which leave (1) invariant and which can gauge away all the torsion and curvature tensors. For our purposes we shall need the infinitesimal form of the super-Weyl transformations of the vielbeins:

\[
\begin{align*}
\delta E_M^a &= \Lambda E_M^a, \\
\delta E_M^\alpha &= \frac{1}{2} \Lambda E_M^\alpha - \frac{i}{2} E_M^a (C^{-1} \gamma_a)^{\alpha\beta} D_\beta \Lambda.
\end{align*} \tag{4}
\]

In what follows we shall often use two-dimensional light-cone notation. There one employs \( \gamma^{\pm\pm} = \frac{1}{2}(\gamma^0 \pm \gamma^1) \) as projection operators for the two irreducible halves of the spinor. Then the light-cone form of (1) is

\[
\{D_{++}, D_{++}\} = 2i D_{++}, \quad \{D_{--}, D_{--}\} = 2i D_{--}, \quad \{D_{+-}, D_{-+}\} = R_{+-}. \tag{5}
\]

Our final point about two-dimensional supergravity concerns the structure of the covariant derivatives (2) taken at the point \( \eta = 0 \). They will be used in section 5 for evaluating the superstring component action. From [14] we learn that in a certain gauge for the super-Weyl and tangent Lorentz groups one has

\[
D_\alpha|_{\eta=0} = \partial_\alpha + \omega_\alpha, \quad D_a|_{\eta=0} = e_a^m(\xi) \partial_m + \psi_a^{\mu}(\xi) \partial_\mu + \omega_a, \tag{6}
\]

where \( e_a^m(\xi) \) and \( \psi_a^{\mu}(\xi) \) are the two-dimensional graviton and gravitino fields.

Now we pass to the target superspace of the \( D = 3 \) \( N = 2 \) superstring. It is a 3|4-dimensional superspace parametrized by \( Z^{\underline{M}} = (X^m, \Theta^{\underline{\mu}}, \overline{\Theta}^{\underline{\mu}}) \), where \( m = (0, 1, 2) \) and \( \mu = (1, 2) \). Note that the Grassmann variables are combined here into a complex doublet \( \overline{\Theta}^{\underline{\mu}} \) and its conjugate \( \Theta^{\underline{\mu}} \) (hence \( N = 2 \)). Our formulation of the superstring will be of a sigma model type. \( ^3 \) In such a context one treats the target space coordinates as worldsheet superfields \( Z^{\underline{M}}(\overline{Z}^{\underline{M}}) \). Then one can define the following differential one-forms

\[
E_{\underline{A}} = dZ^{\underline{M}} F_{\underline{M}\underline{A}}(Z). \tag{7}
\]

\( ^3 \)To avoid the proliferation of indices we use the same letters to denote similar types of indices on the worldsheet and in the target space. The distinction is made by underlining the target space ones.

\( ^4 \)The sigma model nature of the superstring was revealed in [13].
Here $E_M^\Lambda(Z)$ are the vielbeins of the target supergeometry. The flat $D = 3$ $N = 2$ superspace is characterized by the one-forms:

$$E^\alpha = dX^\alpha - i d\Theta \gamma^\alpha \bar{\Theta} - i d\bar{\Theta} \gamma^\alpha \Theta,$$

$$E^\bar{\alpha} = d\Theta^\bar{\alpha} - i d\bar{\Theta} \gamma^\bar{\alpha} \Theta - i d\Theta \gamma^\bar{\alpha} \bar{\Theta}. \tag{8}$$

These forms are invariant under target space $D = 3$ $N = 2$ supersymmetry and with respect to the worldsheet local symmetries. The pull-backs of these forms onto the worldsheet are

$$E^A_\Lambda = D_A Z^M E_M^\Lambda(Z). \tag{9}$$

Acting on (9) with the covariant derivative $D_B$ and performing graded antisymmetrization in $A, B$ we obtain an important relation which involves the worldsheet and target superspace torsions:

$$D_A E^C_B - (-)^{AB} D_B E^C_A = T^C_A B^A_B E_B^C E_C^A T^B_A C. \tag{10}$$

The explicit form of the flat target superspace torsion is

$$T_{\alpha\bar{\beta}}^\phi = T_{\alpha\bar{\beta}}^\phi = 2i(\gamma^\phi)_{\alpha\bar{\beta}}, \quad \text{the rest} = 0. \tag{11}$$

A characteristic feature of the superstring considered as a sigma model is the presence of a Wess-Zumino term in the action. It is based on another target superspace geometric object, the super two-form $B_{MN}(Z^\Lambda)$. In the flat case it is given by

$$B_{\mu} = \frac{i}{2} (\gamma_{\mu} \Theta)_{\mu}, \quad B_{\bar{\mu}} = \frac{i}{2} (\gamma_{\bar{\mu}} \bar{\Theta})_{\bar{\mu}}, \quad B_{\mu\nu} = -\frac{1}{2} (\gamma_{\mu} \Theta)_{\mu}(\gamma_{\nu} \Theta)_{\nu}, \quad B_{\bar{\mu}\bar{\nu}} = -\frac{1}{2} (\gamma_{\bar{\mu}} \bar{\Theta})_{\bar{\mu}}(\gamma_{\bar{\nu}} \bar{\Theta})_{\bar{\nu}}, \tag{12}$$

the rest $= 0$.

Its field strength is a three-form,

$$H_{MNK} = \partial_M B_{NK}, \tag{13}$$

where $[M N K]$ means graded antisymmetrization. Using the $D = 3$ gamma matrix identity

$$(\gamma^m)_{\mu\nu}(\gamma^m)_{\nu\mu} = 0, \tag{14}$$

one can show that

$$H_{m\mu\lambda} = H_{m}\phi\lambda = i(\gamma_m)_{\mu\lambda}. \quad \text{the rest} = 0. \tag{15}$$

In what follows we shall also need the expression for the three-form with tangent space indices

$$H_{ABC} = (-)^{(B+N)(A+C)} (A+B) E_C^M E_B^N E_A^M H_{MNK}. \tag{16}$$

Its projections are similar to those in (13):

$$H_{a\beta\gamma} = H_{a\beta\bar{\gamma}} = i(\gamma_a)_{\beta\gamma}, \quad \text{the rest} = 0. \tag{17}$$

\footnote{Here $\gamma^\Lambda$ are the ordinary $D = 3$ gamma matrices times the charge conjugation matrix; we use the representation $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.}
Note that the part of the target space geometry involving only the pull-backs (one-forms) and the torsion respects the automorphism group $U(1)$ of $D = 3 N = 2$ supersymmetry. The latter rotates $\Theta$ by a phase factor. A peculiarity of the type II superstring is that its Wess-Zumino term violates this $U(1)$ symmetry, as can be seen from (12). As shown in [10], this is the only way to have a closed three-form in a type II superspace. An interesting geometric interpretation of this fact has been given in [17].

3 The $D = 3 N = 2$ superparticle

In this section we shall present a twistor formulation of the superparticle moving in $D = 3 N = 2$ superspace. It is the one-dimensional simplified version of the superstring. It shares some new features with the superstring and can thus serve as an introduction to the superstring. Moreover, as we shall see in subsection 5.3 a specific solution to the superstring twistor constraints leads to a degenerate form of the superstring, which is just the superparticle.

In the traditional Brink-Schwarz formulation [18] of the $D = 3 N = 2$ superparticle one finds two kappa-symmetries which gauge away half of the target space Grassmann coordinates. In a twistor formulation one expects to have two worldline local supersymmetries. So, we consider a superworldline parametrized by $\tau, \eta^i$, where $i = 1, 2$ is a doublet index of the $SO(2)$ automorphism of the $N = 2$ supersymmetry algebra

$$\{D_i, D_j\} = 2i\delta_{ij}\partial_\tau. \tag{18}$$

The action we propose for the $D = 3 N = 2$ superparticle is given by

$$S = \int d\tau d^2\eta P_{i\underline{a}}(D_i X^\underline{a} - iD_i \Theta \gamma^\underline{a}\bar{\Theta} - iD_i \bar{\Theta} \gamma^\underline{a}\Theta). \tag{19}$$

It contains the pull-back $E^\underline{a}$ of the invariant one form of target space supersymmetry (cf. (8)) and a Lagrange multiplier. All superfields in (19) are unconstrained. As a kinematical restriction we require that the pull-back $E^\underline{a}$ defining the commuting spinor (twistor) variables be a non-vanishing matrix,

$$D_i \Theta^\underline{a} \neq 0. \tag{20}$$

We note that this action is invariant under the $N = 2$ superconformal group. 6 Our aim is to study the component content of the above action and to show its equivalence to the Brink-Schwarz superparticle action [18]. Integrating over the worldline Grassmann coordinates we get

$$S = \int d\tau [D_1 D_2 P_{i\underline{a}}(D_i X^\underline{a} - iD_i \Theta \gamma^\underline{a}\bar{\Theta} - iD_i \bar{\Theta} \gamma^\underline{a}\Theta)$$

$$+ iD_1 P_{i\underline{a}}(\partial_\tau X^\underline{a} - i\partial_\tau \Theta \gamma^\underline{a}\bar{\Theta} - i\partial_\tau \bar{\Theta} \gamma^\underline{a}\Theta - 2D_1 \Theta \gamma^\underline{a}D_1 \bar{\Theta})$$

$$- iD_1 P_{i\underline{a}}(\partial_\tau X^\underline{a} - i\partial_\tau \Theta \gamma^\underline{a}\bar{\Theta} - i\partial_\tau \bar{\Theta} \gamma^\underline{a}\Theta - 2D_2 \Theta \gamma^\underline{a}D_2 \bar{\Theta})$$

$$+ D_1 P_{i\underline{a}}(D_1 D_2 X^\underline{a} - iD_1 D_2 \Theta \gamma^\underline{a}\bar{\Theta} - iD_1 D_2 \bar{\Theta} \gamma^\underline{a}\Theta + iD_1 \Theta \gamma^\underline{a}D_2 \bar{\Theta} + iD_1 \bar{\Theta} \gamma^\underline{a}D_2 \Theta) \tag{21}$$

For a discussion of the extended one-dimensional superconformal groups see [8].
\[ + D_2 P_{2a} (D_1 D_2 X^a - i D_1 D_2 \Theta \bar{\gamma} \bar{\Theta} - i D_1 D_2 \bar{\Theta} \gamma \Theta - i D_1 \Theta \gamma \bar{D}_2 \bar{\Theta} - i D_1 \bar{\Theta} \gamma \bar{D}_2 \Theta) \]
\[ - i P_{1a} \left( \frac{1}{2} \partial_\tau D_2 X^a - i \partial_\tau D_2 \Theta \bar{\gamma} \bar{\Theta} + i \partial_\tau \bar{\Theta} \gamma \Theta + 2 D_1 D_2 \Theta \gamma \bar{D}_2 \bar{\Theta} + h.c. \right) \]
\[ + i P_{2a} \left( \frac{1}{2} \partial_\tau D_1 X^a - i \partial_\tau D_1 \Theta \bar{\gamma} \bar{\Theta} + i \partial_\tau \bar{\Theta} \gamma \Theta - 2 D_1 D_2 \Theta \gamma \bar{D}_2 \bar{\Theta} + h.c. \right) \}_{\eta=0}. \]

The variation with respect to the component \( D_1 D_2 P_{i\bar{a}} \) produces the auxiliary field equations (we omit the subscript \( \eta = 0 \))
\[ D_1 X^a - i D_1 \Theta \bar{\gamma} \bar{\Theta} - i D_1 \bar{\Theta} \gamma \Theta = 0. \] (22)

The variation with respect to the components \( D_1 P_{1\bar{a}} \) and \( D_2 P_{2\bar{a}} \) defines the auxiliary component \( D_1 D_2 X^a \) and also leads to one of the twistor constraints
\[ D_1 \Theta \gamma \bar{D}_2 \bar{\Theta} + D_1 \bar{\Theta} \gamma \bar{D}_2 \Theta = 0. \] (23)

Further, varying with respect to the sum \( D_2 P_{1\bar{a}} + D_1 P_{2\bar{a}} \) we get the other twistor constraint
\[ D_1 \Theta \gamma \bar{D}_2 \bar{\Theta} - D_2 \Theta \gamma \bar{D}_2 \bar{\Theta} = 0. \] (24)

The difference \( i (D_2 P_{1\bar{a}} - D_1 P_{2\bar{a}}) \) is identified with the particle’s momentum \( p_{\bar{a}} \).

Finally, the last two terms in the component action (21) can be simplified by using the auxiliary field equations (22) and the resulting action takes the form
\[ S = \int d\tau [p_{\bar{a}} (\partial_\tau X^a - i \partial_\tau \Theta \bar{\gamma} \bar{\Theta} - i \partial_\tau \bar{\Theta} \gamma \Theta - D_1 \Theta \gamma \bar{D}_2 \bar{\Theta}) \]
\[ + 2 P_{1\bar{a}} (D_2 \Theta \gamma \bar{D}_2 \bar{\Theta} + D_2 \bar{\Theta} \gamma \bar{D}_2 \Theta - i D_1 \bar{\Theta} \gamma \bar{D}_2 \Theta + i D_1 \bar{\Theta} \gamma \bar{D}_2 \Theta)] \]
\[ - 2 P_{2\bar{a}} (D_1 \Theta \gamma \bar{D}_2 \bar{\Theta} + D_1 \bar{\Theta} \gamma \bar{D}_2 \Theta + i D_2 \Theta \gamma \bar{D}_2 \Theta + i D_2 \Theta \gamma \bar{D}_2 \Theta)]. \] (25)

Here the twistor variables \( D_1 \Theta \bar{\Phi} \) and \( D_2 \bar{\Phi} \) are restricted by the constraints (23) and (24). Our next step is to solve these constraints. Using the explicit representation for the \( D = 3 \) gamma matrices in the light-cone basis
\[ \gamma^{++} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^{-+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{+-} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (26)
we easily find the general solution to the twistor constraints (23) and (24) under the assumption (21):
\[ D_1 \Theta \bar{\Phi} = \lambda \bar{\Phi}, \quad D_2 \bar{\Phi} = i s \lambda \bar{\Phi}; \quad s = \pm 1, \] (27)
where \( \lambda \bar{\Phi} \) is an arbitrary complex nonvanishing spinor.

Substituting this solution into the action (25) we get
\[ S = \int d\tau [p_{\bar{a}} (\partial_\tau X^a - i \partial_\tau \Theta \bar{\gamma} \bar{\Theta} - i \partial_\tau \bar{\Theta} \gamma \Theta - 2 \lambda \bar{\Phi} \bar{\chi} \lambda) - 2 P_{2\bar{a}} \bar{\gamma} \bar{\Phi} \lambda - 2 P_{\bar{a}} \bar{\gamma} \bar{\Phi} \lambda], \] (28)
where
\[ P_{\bar{a}} = P_{2\bar{a}} - i s P_{1\bar{a}}, \quad \bar{\chi} \bar{\Phi} = \partial_\tau \Theta \bar{\Phi} + s D_1 D_2 \Theta \bar{\Phi}. \]
Now we remark that the tri-linear term in this action is purely auxiliary. Indeed, the variation with respect to \( P^a \) leads to the equation
\[
\bar{\chi} \gamma^a \lambda = 0.
\] (29)
Since the commuting spinor \( \lambda \) is non-vanishing, the only solution to this equation is \( \bar{\chi} = 0 \). The variation with respect to \( \bar{\chi} \) in (28) implies
\[
P_\alpha (\gamma^a \lambda)_{\alpha} = 0.
\] (30)
The general solution of this twistor equation is
\[
P_\alpha = \rho(\tau) \lambda \gamma^a \lambda
\] (31)
with an arbitrary complex function \( \rho(\tau) \). However, the right hand side of (31) is an obvious gauge invariance of the action (28) due to the gamma matrix identity (14). Hence the two last terms in (28) vanish.

Finally, we vary with respect to the twistor variable \( \lambda \):
\[
p_\alpha (\gamma^a \bar{\lambda})_{\alpha} = 0 \Rightarrow p^\alpha = \bar{\mu}(\tau) \bar{\lambda} \gamma^a \bar{\lambda}.
\] (32)
The fact that the particle momentum is real and non-vanishing then implies
\[
\bar{\mu}(\tau) \bar{\lambda} \gamma^a \bar{\lambda} = \mu(\tau) \lambda \gamma^a \lambda.
\] (33)
The solution to this equation is
\[
\bar{\lambda}^\alpha = e^{i\phi} \lambda^\alpha, \quad \bar{\mu} = e^{-2i\phi} \mu.
\] (34)
It implies that on shell the complex spinor \( \lambda^\alpha \) becomes real modulo a phase. The arbitrary phase \( \phi \) corresponds to the \( SO(2) \) subgroup of the superconformal invariance of the action (19) and can be completely gauged away. Then we can replace the twistor combination \( \lambda \gamma^a \bar{\lambda} \) in (28) by \( |\mu|^{-1} \eta^\alpha \) and obtain the standard Brink-Schwarz superparticle action.

The conclusion is that the action (19) is equivalent to the Brink-Schwarz action upon eliminating the auxiliary fields (including the twistor variables) and fixing certain gauges. An unusual feature compared to the twistor superparticle of [1] is the presence of two twistors (the real and imaginary parts of \( \lambda \)) instead of only one. As we shall see in subsection 5.3, it is this form of the superparticle which appears as a degenerate case of the non-heterotic superstring.

4 The \( D = 3 N = 2 \) superstring action

The twistor superstring action consists of two terms
\[
S = S_{GD} + S_{WZ}.
\] (35)
The first one resembles very much the superparticle action of section 3:
\[
S_{GD} = \int d^2 \xi d^2 \eta P_\alpha^a E_{a}^\alpha.
\] (36)
The variation with respect to the Lagrange multiplier $P_{\underline{\underline{\alpha}}} \alpha$ leads to the following geometro-dynamical constraint on the target space coordinates treated as worldsheet superfields $Z^M(Z^M)$:

$$E_{\alpha a} = 0.$$ \hfill (37)

The meaning of eq. (37) is that the pull-back of the target superspace vector vielbein onto the spinor directions of the worldsheet must vanish. In other words, if one considers the worldsheet as a 2|2-dimensional hypersurface embedded in the 3|4 target superspace, then there should be no projections of the target space even directions onto the worldsheet odd ones. Using (10), (3) and (11), we obtain the important consequence

$$2(\gamma^c C)_{\alpha\beta} E_{\alpha} = E_\alpha \gamma^a \bar{E}_\beta + \bar{E}_\alpha \gamma^a E_\beta.$$ \hfill (38)

In two-dimensional light-cone notation (38) reads

$$E_+ \gamma^a \bar{E}_+ = E_{++} a,$$ \hfill (39)

$$E_- \gamma^a \bar{E}_- = E_{--} a,$$ \hfill (40)

$$E_+ \gamma^a \bar{E}_- + E_- \gamma^a \bar{E}_+ = 0.$$ \hfill (41)

These equations are constraints on the superfields $Z^M(Z^M)$. In particular, they imply algebraic restrictions on the first components in the $\eta$ expansion of the spinor-spinor pull-backs

$$E_{\alpha a} \equiv E_{\alpha a}|_{\eta=0}.$$ \hfill (42)

These are commuting spinors (with respect to the two- and three-dimensional Lorentz groups), which we shall call “twistor variables”. In section 3 we shall show that as a result of these restrictions the first components in the $\eta$ expansion of the vectors $E_{\pm \pm a}$ defined by (39) and (40) are lightlike,

$$E_{\pm \pm a} = E_{\pm \pm a}|_{\eta=0} : \quad (E_{++ a})^2 = (E_{-- a})^2 = 0.$$ \hfill (43)

In other words, one of the main purposes of the geometro-dynamical constraint (37) is to provide a solution to the Virasoro constraints of the superstring in terms of the twistor variables (42).

The presence of the geometro-dynamical term (36) makes it possible to introduce the leading term in the superstring action in the form of a generalized Wess-Zumino term. The latter requires certain consistency conditions, and we shall show that they are satisfied as a consequence of the geometro-dynamical constraint (37).

The Wess-Zumino term has the following form

$$S_{WZ} = \int d^2 \xi d^2 \eta \left( - \right)^{MN+M+N} P^{MN} (B_{MN} - E^a_M E^b_N \epsilon_{ab} A - \partial_M Q_N).$$ \hfill (44)

Here $P^{MN}$ is a graded-antisymmetric Lagrange multiplier, $Q_M$ is another Lagrange multiplier, $B_{MN} = (-)^{(N+N)M} E^N_M E^M_N B_{MN}$ is the pull-back of the target superspace two-form, $E^a_M$ are worldsheet vielbeins. The quantity $A$ is to be found from the consistency conditions below. Varying with respect to $P^{MN}$ leads to the equation of motion

$$B_{MN} - E^a_M E^b_N \epsilon_{ab} A = \partial_M Q_N.$$ \hfill (45)
The meaning of this equation is that the pull-back of the two-form becomes almost “pure gauge” on shell. The consistency condition following from (43) is that the graded curl of the left-hand side of eq. (42) must vanish. Thus we obtain

$$H_{M NK} = \partial_K B_{MN} = 2\partial_i (E_M a E_N b) \epsilon_{ab} A + (-)^{(M+N)} E_{(M} a E_{N)} b \epsilon_{ab} \partial_K A,$$  \hspace{1cm} (46)

where $H_{M NK}$ is the pull-back of the three-form. It is convenient to pass to tangent space indices in eq. (46), i.e. to multiply it by two-dimensional vielbeins. Using the expression for the worldsheet torsion

$$T_{AB}^C = -(-)^{A(B+N)} E_B^N E_A^M \partial_M E_N^C + \omega_{AB}^C - (-)^{AB} (A \leftrightarrow B)$$  \hspace{1cm} (47)

one can rewrite (46) as follows:

$$H_{ABC} = (-T_{BA}^d + 2\omega_{[BA}^d \delta_{C]}^e \epsilon_{de} A + \delta_{[B}^d \delta_{C]}^e \epsilon_{de} D_A) A,$$  \hspace{1cm} (48)

where $H_{ABC}$ is the pull-back of the three-form

$$H_{ABC} = (-)^{(B+E)(A+C)(A+B)} E_C^E E_B^B E_A^A \delta H_{ABC}.$$  \hspace{1cm} (49)

Let us study the different projections of the condition (48). First we consider the projections of its left-hand side. If we take all the indices spinor and use the geometrodynamical equation (i), we find

$$H_{\alpha\beta\gamma} = E_\alpha E_\beta E_\gamma \delta H_{\alpha\beta\gamma} + 3E_\alpha E_\beta E_\gamma \delta H_{\alpha\beta\gamma} + \text{c.c.} = 0$$  \hspace{1cm} (50)

as a consequence of (17). In accordance with this the right-hand side of eq. (48) vanishes identically for this choice of the indices.

Further, let us take one vector and two spinor indices in (48). Substituting (17) into (49) and using (39)-(41) and (14), we find

$$H_{++|--} = i(E_+ \gamma a E_+) E_+ \delta + \text{c.c.} = i(E_+ \gamma a E_+) (E_+ \gamma a \bar{E}_+) + \text{c.c.} = 0;$$  \hspace{1cm} (51)

$$H_{++|--} = i(E_- \gamma a E_-) E_- \delta + \text{c.c.} = i(E_- \gamma a E_-) (E_- \gamma a \bar{E}_-) + \text{c.c.} = 0;$$  \hspace{1cm} (52)

Similarly, we obtain

$$H_{--|--} = H_{++|--} = 0.$$

The only non-vanishing pull-backs with one vector and two spinor indices are

$$H_{--|--} = i(E_- \gamma a E_-) E_- \delta + \text{c.c.} = i(E_- \gamma a E_-) (E_- \gamma a \bar{E}_-) + \text{c.c.}$$  \hspace{1cm} (53)

$$H_{--|--} = -2i(E_+ \gamma a E_-) (E_+ \gamma a \bar{E}_-) + \text{c.c.};$$  \hspace{1cm} (54)

$$H_{++|--} = i(E_- \gamma a E_-) E_+ \delta + \text{c.c.} = i(E_- \gamma a E_-) (E_+ \gamma a \bar{E}_+) + \text{c.c.}$$  \hspace{1cm} (55)

Here and in what follows we frequently use the constraint (37) or its corollary (38) in the Wess-Zumino action term. This actually means that we produce terms proportional to the constraint, which can be absorbed into a redefinition of the Lagrange multiplier in (36).
\[ = -2i(E_+ \gamma_2 E_+)(E_+ \gamma_2 \bar{E}_+) + \text{c.c.} = -H_{-|+|+} \]

If we compare these expressions for the pull-backs of the three-form with the right-hand side of eq. (48) and use (3), we see complete agreement, provided that the quantity \( A \) is given by

\[ A = \frac{1}{8} \left( E_\alpha \gamma_2 E_+ + \text{c.c.} \right) \epsilon^{ab} E_{b}^\alpha. \quad (56) \]

The remaining possibility in eq. (48) is to have two vector and one spinor index. This does not lead to new relations, since the component \( H_{ab\gamma} \) of the pull-back of the three-form is determined by the Bianchi identity

\[ D_{[a} H_{\alpha\beta\gamma]} + T_{[a\beta} E^{H_{\alpha\gamma}\delta]} = 0 \quad (57) \]

and thus automatically agrees with the right-hand side of (48).

This concludes the verification of the consistency of our Wess-Zumino term. We have seen that the pull-back of the two-form itself is not closed \((dB \neq 0)\), but this can be corrected by an appropriately chosen term with \( A \) given in (56).

We note that the action term (44) is invariant under the superconformal transformations (4). Indeed, we see that the vielbein factor in front of \( A \) in (44) transforms as a density of weight +2. The twistor vector \( E_\alpha \gamma_2 E_+ \) and its conjugate are densities of weight -1. As to the vectors \( E_a^\alpha \), their transformation laws are less trivial. Take, for instance,

\[ \delta E_{++}^a = -\Lambda E_{++}^a + iD_+ \Lambda E_{+}^a. \quad (58) \]

The first term in (58) provides the weight -1 needed to compensate the other two factors. The second term is proportional to \( E_a^\alpha \), which vanishes according to (37). In other words, this second term can be compensated by a suitable transformation of the Lagrange multiplier \( P_a^\alpha \) in the action term (36). As to the term (36) itself, its super Weyl invariance is assured by ascribing a certain weight to the Lagrange multiplier.

Another remark concerns the \( U(1) \) automorphism of \( D = 3 \ N = 2 \) supersymmetry. The term \( S_{GD} \) of our action respects this symmetry, whereas \( S_{WZ} \) does not.

5 The component action

In this section we shall obtain the component expression of the two terms (36) and (44) of the superstring action. We shall show that the Wess-Zumino term (44) is reduced to the usual superstring action of Green-Schwarz type. The geometro-dynamical term (36) will turn out to be purely auxiliary.

5.1 The Wess-Zumino term

We begin with the Wess-Zumino term (44). The variation with respect to the Lagrange multiplier \( Q_M \) produces the equation

\[ \partial_N P^{NM} = 0. \quad (59) \]
Its general solution is:

\[ P^{MN} = \partial_K \Sigma^{MNK} + \delta^M_N \partial_l \epsilon^{kl} T \eta^2. \]  

(60)

It consists of two parts. The first one has the form of a pure gauge transformation with
graded antisymmetric parameter \( \Sigma^{MNK} \). We are sure that this is an invariance of the
action because the consistency condition (48) holds. It is easy to see that almost all of
the components of \( P^{MN} \) can be gauged away by a \( \Sigma \) transformation without using any
parameters with space-times derivatives. Hence, one is allowed to use such a gauge in the
action. The only remaining non-trivial part of \( P^{MN} \) is the second term in the solution
(60). It contains a coefficient \( T \), which is restricted by (59) to be an arbitrary constant,

\[ \partial_{++} T = \partial_{--} T = 0. \]  

(61)

Inserting the solution (60) back into the action term (44) and doing the \( \eta \) integral
with the help of the Grassmann delta function \( \eta^2 \) present in (60) we obtain

\[ S_{WZ} = \int d^2 \xi T \epsilon^{mn} \left( B_{mn} - \frac{1}{2} E_m^a E_n^b \epsilon^{ab} A \right) \eta_0. \]  

(62)

To evaluate the various objects at \( \eta = 0 \) we use eq. (6). Thus, \( E_m^a \big|_0 = e_m^a(\xi) \), so the
factor in front of \( A \) becomes simply \( \det e \). The quantity \( A \) from (59) takes the form

\[ A|_{\eta=0} = \frac{1}{8} \left( \mathcal{E}_{\alpha}^a \gamma_2 \mathcal{E} + \text{c.c.} \right) \epsilon^{ab} \mathcal{E}_{\beta}^b, \]  

(63)

where

\[ \mathcal{E}_{\alpha}^a = E_{\alpha}^a \big|_0, \quad \mathcal{E}_{a}^a = E_{a}^a \big|_0 = \left[ (e_m^a \partial_m + \psi^a_{\mu} D_\mu) Z^M E_{M}^a \right]_0 = \left[ e_m^a \partial_m Z^M E_{M}^a \right]_0. \]  

(64)

Note that the only place in (62) where the gravitino field \( \psi^a_{\mu} \) could occur is in (64), but
even there it dropped out as a consequence of the geometro-dynamical equation (37).

At this point we are going to use the solution to the \( \eta = 0 \) part of the constraint (38).
This is the constraint on the twistor variables \( \mathcal{E}_{\alpha}^a \) and appears as a component of the
geometro-dynamical term (36) (see subsection 5.2). As explained in the Appendix, the
twistor constraint has two solution, a regular and a singular one. The regular solution is
obtained under the assumption that the twistor matrix \( \mathcal{E}_{\alpha}^a \) is non-degenerate,

\[ \det \| \mathcal{E}_{\alpha}^a \| \neq 0 \]  

(65)

and has the form

\[ \mathcal{E}_{\alpha}^a = e^{i\phi} \left( \begin{array}{c} \mathcal{\lambda}_{+}^a \\ i\mathcal{\lambda}_{-}^a \end{array} \right). \]  

(66)

Here \( \phi(\xi) \) is an arbitrary phase. The spinors \( \lambda \) in (66) are real,

\[ \bar{\mathcal{\lambda}}_{+}^a = \mathcal{\lambda}_{+}^a, \quad \bar{\mathcal{\lambda}}_{-}^a = \mathcal{\lambda}_{-}^a \]  

(67)
and satisfy two further relations:

\[ \lambda_+ \gamma^a \lambda_+ = E^{++}_a, \quad \lambda_- \gamma^a \lambda_- = E^{--}_a. \]  

These equations give expressions for the vectors \( \mathbf{E}^{\pm \pm}_a \) in terms of the twistor variables \( E_a \). Using (14), one sees that the vectors in (68) are lightlike,

\[ (E^{++}_a)^2 = (E^{--}_a)^2 = 0. \]  

Thus, we see that the lowest-order component of the twistor constraint (38) has reduced the \( 2 \times 2 \) complex twistor matrix to two independent real twistors \( \lambda_\pm \), in terms of which the superstring Virasoro constraints (69) are solved.

In this subsection we shall restrict ourselves to the regular solution (66). The case of the singular one (which corresponds to the case of a string collapsed into a particle) will be treated in subsection 5.3. So, putting the above expressions in (63) and then in (62), we obtain

\[ S_{WZ} = T \int d^2 \xi \left( \epsilon^{mn} B_{mn} - \frac{1}{2} \det e \cos 2\phi \mathbf{E}^{++}_a \mathbf{E}^{--}_a \right). \]  

We see that this expression is almost identical with the usual Green-Schwarz type II superstring action, if the constant \( T \) is interpreted as the string tension. The only difference is in the factor containing the auxiliary scalar field \( \phi(\xi) \). In subsection 5.2 we shall see that \( \phi \) does not appear in the geometro-dynamical term (36) of the superspace superstring action. This is not surprising, since \( \phi \) appears as the parameter of a \( U(1) \) transformation, and \( S_{GD} \) respects this symmetry. Therefore we can vary with respect to \( \phi \) in (70) and obtain the following field equation

\[ \sin 2\phi \mathbf{E}^{++}_a \mathbf{E}^{--}_a = 0. \]  

Using the twistor expressions (68), it is not hard to show that

\[ \mathbf{E}^{++}_a \mathbf{E}^{--}_a = (\lambda_+ \gamma^a \lambda_+) (\lambda_- \gamma^a \lambda_-) = (\det || \lambda_a ||)^2 = -e^{-4i\phi} (\det || E_a ||)^2. \]  

Note that the first term in (72) is in fact proportional to the determinant of the induced two-dimensional metric of the superstring. Since in this subsection we assume that the twistor matrix is non-singular (see (63)), we conclude that the solution to (71) is

\[ \phi = 0. \]  

Putting this solution back into the action (70) we obtain

\[ S_{WZ} = T \int d^2 \xi \left( \epsilon^{mn} B_{mn} - \frac{1}{2} \det e \mathbf{E}^{++}_a \mathbf{E}^{--}_a \right). \]  

This is the action of a type II \( D = 3 \) Green-Schwarz superstring. In it the twistor variables are not present any more, they have been eliminated through the algebraic relations (68). \[ \text{10The mechanism where the string tension appears as an integration constant was proposed in a different context in [ ]} \]

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We have seen that in the process of derivation of the action (74) the two-dimensional gravitino field dropped out (see (64)), although in the original superfield form (44) it was needed to maintain the invariance with respect to the local supersymmetry transformations on the worldsheet. Despite the absence of the gravitino, this local supersymmetry is still present in the component term (74), but now in the non-manifest form of kappa symmetry. In order to see how local supersymmetry is transformed into kappa symmetry on shell let us consider the supersymmetry variation of the target superspace coordinates 

\[ z^M = Z^M |_{\eta = 0} \]

\[ \delta z^A \equiv \delta z^M E_M^A = \epsilon^\alpha D_\alpha Z^M E_M^A |_{\eta = 0} = \epsilon^\alpha \mathcal{E}_\alpha^A. \]  

(75)

From the geometro-dynamical equation (37) follows

\[ \delta z^a = 0, \quad \delta z^\alpha = \epsilon^\alpha \mathcal{E}_\alpha^a. \]  

(76)

Let us now introduce an anticommuting parameter \( \kappa^a_\alpha \) carrying a \( D = 2 \) vector and a \( D = 3 \) spinor index by substituting

\[ \epsilon_\alpha = (\gamma_a)_\alpha^\beta \mathcal{E}_\beta^\beta \kappa^a_\alpha. \]  

(77)

Putting this in (76), using the solution to the constraints on the twistor matrix, eq. (73) and the Fierz identity for the three-dimensional gamma matrices, we obtain

\[ \delta z^\alpha = (\gamma_a)^{\alpha\beta}(\mathcal{E}_{++} \kappa^a_\alpha^{++} - \mathcal{E}_{--} \kappa^a_\alpha^{--}). \]  

(78)

Equation (78) coincides with the kappa symmetry transformations of the type II superstring [12].

Note an interesting feature of the transition from the action term (70) to the final form (74). The former is not invariant under the \( U(1) \) automorphism of \( D = 3 \) \( N = 2 \) supersymmetry. In the first term in (70) this is due to the \( U(1) \) non-invariant two-form. In the second term in (70) the only object which breaks \( U(1) \) is the phase \( \phi \). Indeed, looking at (66), one sees that \( \phi \) is shifted by the \( U(1) \) transformation of the index \( \alpha \). However, once this field has been eliminated from the action, one obtains the peculiar mixture of a non-invariant and an invariant term in (74), characteristic for the type II superstring.

5.2 The geometro-dynamical term

In the previous subsection we saw that the usual superstring action is essentially contained in the Wess-Zumino term. Here we shall show that the rôle of the geometro-dynamical term (36) is purely auxiliary, i.e. that it only leads to algebraic constraints on the component fields. Among them are the twistor constraints reducing the twistor matrix to the two light-like vectors from the Virasoro constraints. Another of these constraints will allow us to express the two-dimensional gravitino field in terms of the derivatives of the Grassmann coordinates \( \theta^a_\alpha \) of the target superspace. Other equations will put the Lagrange multipliers \( P_a^\alpha \) to zero on shell. We shall also show that the scalar field \( \phi \) appearing in \( S_{WZ} \) is not present in \( S_{GD} \), so the derivation of its field equation from \( S_{WZ} \) in subsection 5.1 was correct. Thus the superstring action (35) will be reduced to the Green-Schwarz one (74).
In two-dimensional light-cone notation the term $S_{GD}$ becomes (see (8))

$$S_{GD} = \int d^2 \xi D_+ D_- \left[ P_{-\underline{a}} (D_+ X_{\underline{a}} - iD_+ \Theta \gamma_{\underline{a}} \bar{\Theta}) + \text{c.c.} + (+ \leftrightarrow -) \right]_{\eta=0}$$

$$= \int d^2 \xi \left[ D_+ D_- P_{-\underline{a}} (D_+ X_{\underline{a}} - iD_+ \Theta \gamma_{\underline{a}} \bar{\Theta}) \right.$$  
$$- D_+ P_{-\underline{a}} (D_- D_+ X_{\underline{a}} - iD_- D_+ \Theta \gamma_{\underline{a}} \bar{\Theta} - iD_+ \Theta \gamma_{\underline{a}} D_- \bar{\Theta})$$  
$$+ D_- P_{-\underline{a}} (iD_+ X_{\underline{a}} + D_+ \Theta \gamma_{\underline{a}} \bar{\Theta} - iD_+ \Theta \gamma_{\underline{a}} D_- \bar{\Theta})$$  
$$+ P_{-\underline{a}} (D_- D_- D_+ X_{\underline{a}} - iD_- D_- D_+ \Theta \gamma_{\underline{a}} \bar{\Theta})$$  
$$+ D_+ \Theta \gamma_{\underline{a}} D_- \bar{\Theta} + 2i D_- \Theta \gamma_{\underline{a}} D_+ \bar{\Theta} \right] + \text{c.c.} - (+ \leftrightarrow -) \right]_{\eta=0} \quad (79)$$

The variation with respect to the components $D_+ D_- P_{-\underline{a}}$ and $D_+ D_- D_+ \Theta$ gives two equations for the auxiliary odd components of the superfield $X_{\underline{a}}$:

$$D_\pm X_{\underline{a}} = iD_\pm \Theta \gamma_{\underline{a}} \bar{\Theta} + iD_\pm \bar{\Theta} \gamma_{\underline{a}} \Theta \quad (80)$$

(from here on we shall drop the indication $\eta = 0$). The variation with respect to the components $D_+ P_{-\underline{a}}$ and $D_- P_{-\underline{a}}$ gives equations for the auxiliary even component $D_- D_+ X_{\underline{a}}$ and also leads to the twistor constraint

$$D_+ \Theta \gamma_{\underline{a}} D_- \bar{\Theta} + D_- \bar{\Theta} \gamma_{\underline{a}} D_+ \Theta = 0. \quad (81)$$

This is just the lowest-order component of the constraint (111). The other two constraints, the $\eta = 0$ components of (89) and (112), follow from the terms with $D_- P_{-\underline{a}}$ and $D_+ P_{+\underline{a}}$. This set of constraints was discussed in subsection 5.1. Postponing once again the investigation of the singular solution of the constraint till subsection 5.3, we consider only the regular one (66):

$$D_+ \Theta \underline{a} = e^{i\phi} \lambda _+ \underline{a}, \quad D_- \Theta \underline{a} = ie^{i\phi} \lambda _- \underline{a}. \quad (82)$$

Next we shall simplify the term with $P_{-\underline{a}}$ in (89). Using (80) and the anticommutation relations (8), we find that the first two terms after $P_{-\underline{a}}$ equal the third term. Further, the covariant derivative $D_+$ in this third term can be written out in detail according to (8):

$$D_+ \Theta \gamma_{\underline{a}} D_- \bar{\Theta} + D_- \bar{\Theta} \gamma_{\underline{a}} D_+ \Theta \quad (83)$$

where we have used (81) and (82). Note that the covariant derivative $D_+$ in the $D_- P_{-\underline{a}}$ term in (79) does not contain the gravitino field, as a consequence of (80) (see also (74)).

Now we are going to put all this back into the action term (79). The purely auxiliary terms drop out and $S_{GD}$ is reduced to

$$S_{GD} = \int d^2 \xi \left\{ P_{-\underline{a}}'(\epsilon_{++} \underline{a} - 2 \lambda _+ \gamma_{\underline{a}} \lambda _+) \right.$$  
$$+ 2P_{-\underline{a}}(i\chi \gamma_{\underline{a}} \lambda _+ + ie_{++}m(-e^{-i\phi} \partial_m \Theta + e^{i\phi} \partial_m \bar{\Theta}) \gamma_{\underline{a}} \lambda _+ + 2\psi_{++}\lambda _+ \gamma_{\underline{a}} \lambda _-) - (+ \leftrightarrow -) \right\}, \quad (84)$$

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where $E_{\pm \pm a}$ was defined in (74) and we have introduced the notation

$$iD_+ P_{-\underline{a}} = P_{-\underline{a}}, \quad \chi = e^{-i\phi} D_- \Theta + e^{i\phi} D_- \Theta$$

(and similarly in the $(+ \leftrightarrow -)$ sector). The field equation for $\chi$ is a typical twistor equation,

$$P_{-\underline{a}}(\gamma^2 \lambda_+) = 0,$$  

which has the general solution

$$P_{-\underline{a}} = P_{-3}(\lambda + \gamma^2 \lambda_+),$$  

with an arbitrary odd scalar field $P_{-3}(\xi)$. Further, the gravitino field $\psi_{++-}$ appears only in (84) (see (70)), so its field equation is

$$P_{-\underline{a}}(\lambda_- \gamma^2 \lambda_-) = P_{-3}(\lambda + \gamma^2 \lambda_+) (\lambda_- \gamma^2 \lambda_-) = 0.$$  

As explained in (72), under the current assumption of non-singularity of the twistor matrix the twistor factor in (88) is non-vanishing, so we conclude

$$P_{-3} = 0 \implies P_{-\underline{a}} = 0.$$  

Before inserting (89) back into (84) and thus eliminating the $P_{-\underline{a}}$ term from the action, we have to study the field equation for $P_{-\underline{a}}$ itself:

$$i\chi \gamma^2 \lambda_+ + i e^{++ m} (-e^{-i\phi} \partial_m \Theta + e^{i\phi} \partial_m \bar{\Theta}) \gamma^2 \lambda_- + 2 \psi_{++-} \lambda_- \gamma^2 \lambda_- = 0.$$  

These are three equations (as many as the projections of the vector index $\underline{a}$). Two of them can be used to solve for the auxiliary field $\chi$ (because here we assume that the matrix $\lambda_{\pm \underline{a}}$ is invertible). The third one enables us to solve for the gravitino field $\psi_{++-}$ (to this end one multiplies eq.(81) by $\lambda + \gamma^2 \lambda_+$ and uses the non-singularity of the twistor factor $(\lambda + \gamma^2 \lambda_+) (\lambda_- \gamma^2 \lambda_-)$). Thus we see that the gravitino field is an auxiliary field. It is expressed in terms of the derivative $e^{++ m} \partial_m \theta$ (where $\theta = \Theta |_0$). This is possible since $\theta$ transforms inhomogeneously under the worldsheet local supersymmetry, $\delta \theta^2 = e^a \lambda_\alpha^2$ (see (76)).

So far we have shown that the term with $P_{-\underline{a}}$ in (84) is purely auxiliary and drops out of the action. Now we shall show that the term with $P_{-\underline{a}}$ vanishes on shell as well. First we shall vary with respect to the twistor field $\lambda_+$. It appears only once (we have already put $P_{-\underline{a}} = 0$ and in the Wess-Zumino term (84) we have eliminated the twistors in favor of the vectors $E_{\pm \pm a}$, so we get an equation similar to (80):

$$P_{-\underline{a}}(\gamma^2 \lambda_+) = 0 \implies P_{-\underline{a}} = P_{-4} (\lambda + \gamma^2 \lambda_+).$$  

Further, the variation with respect to $P_{-\underline{a}}$ gives

$$E_{++ \underline{a}} = 2 \lambda + \gamma^2 \lambda_+.$$  

Finally, we vary with respect to the vielbein fields $e_{a}^m$. They appear both in $S_{GD}$ (84) and in $S_{WZ}$ (74). The variational equation for $e_{m}^{--}$ is

$$P_{-\underline{a}} E_{m}^{a} \sim e_{m}^{++} E_{++ \underline{a}} - E_{m}^{a} E_{-- \underline{a}}.$$  

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Multiplying eq. (93) by $e_{\pm m}$ we find
\[ P_{--}^{\pm} \mathcal{E}_{++} = 0, \quad P_{--}^{\pm} \mathcal{E}_{--} \sim \mathcal{E}_{--} \mathcal{E}_{--}. \quad (94) \]

Inserting the solution (91) and the analog of (92) into (94), we finally obtain
\[ P_{-4}(\lambda_{+} \gamma_{-} \lambda_{+})(\lambda_{-} \gamma_{-} \lambda_{-}) = 0 \Rightarrow P_{-4} = 0 \Rightarrow P_{--} = 0. \quad (95) \]

Once again, we see that the zweibeins play the role of auxiliary fields (like the gravitino above). In the standard superstring theory they produce the Virasoro constraints (69). In the twistor theory these constraints are already solved in terms of twistors. Therefore the zweibeins just give rise to auxiliary equations like (93), which help eliminate some of the Lagrange multipliers.

This concludes our demonstration that the term $S_{GD}$ (36) in the superstring action is purely auxiliary. It does not lead to any new equations of motion for the physical fields $x$ and $\theta$ and thus the on-shell component action is just the Green-Schwarz one (74).

5.3 The case of a degenerate twistor matrix

In subsections 5.1 and 5.2 we studied the component content of the twistor superstring action under the assumption that the twistor algebraic constraint (38) (taken at $\eta = 0$) has the regular solution (66). Here we shall investigate the alternative singular solution. We shall show that in this case the string collapses into a particle. For simplicity we shall only consider the bosonic fields in the action.

As explained in the Appendix, the singular solution, for which $\det \| \mathcal{E}_{\alpha \dot{\alpha}} \| = 0$, has the form
\[ \mathcal{E}_{+}^{\alpha} = \lambda^{\alpha}, \quad \mathcal{E}_{--}^{\alpha} = i r \lambda^{\alpha}. \quad (96) \]

Here $\lambda^{\alpha}$ is an arbitrary complex spinor and $r$ is an arbitrary real factor. Let us insert this solution into the Wess-Zumino term of our string action. The quantity $A$ (63) vanishes due to the gamma matrix identity (14):
\[ A \sim (\mathcal{E}_{-} \gamma_{-}^{\alpha} \mathcal{E}_{+}^{\alpha}) (\mathcal{E}_{-} \gamma_{-}^{\dot{\alpha}} \mathcal{E}_{+}^{\dot{\alpha}}) + \text{c.c.} = r^2 (\lambda \gamma_{-} \lambda) (\lambda \gamma_{-} \lambda) + \text{c.c.} = 0. \quad (97) \]

Further, the two-form term in (52) is proportional to $\theta$, so it does not contribute to the bosonic terms in the action. Thus, $S_{WZ}$ vanishes in this case.

Let us now turn to the geometro-dynamical term (36). Dropping the fermion fields and using the solution (96), we see that the component expansion in (79) is reduced to two terms only:
\[ S_{GD} = \int d^2 \xi \left[ P_{++}^{\alpha} (D_{++} X^{\alpha} - \lambda \gamma_{-} \lambda) + P_{--}^{\alpha} (D_{--} X^{\alpha} - r^2 \lambda \gamma_{-} \lambda) \right]. \quad (98) \]

The variation with respect to the following combination of Lagrange multipliers, $\delta P_{--} = r^2 \delta P_{++}$ shows that the two vectors $D_{++} X^{\alpha}$ and $D_{--} X^{\alpha}$ tangent to the string surface are linearly dependent
\[ r^2 D_{++} X^{\alpha} - D_{--} X^{\alpha} = 0. \quad (99) \]
This means that the dependence on the one of the worldsheet coordinates drops out and the string collapses into a one-dimensional object (particle). The degeneracy of the worldsheet leads to an additional gauge invariance. For instance, the action (98) has the following gauge invariance

\[ \delta e^{+m} = \rho(\xi) r^2 e^{-m}, \quad \delta \lambda^a = \frac{1}{2} \rho r^2 \lambda^a, \quad \delta e^{-m} = \rho r^2 e^{-m}, \]
\[ \delta P^{--}_a = -\rho P^{++}_a - \rho r^2 P^{--}_a, \quad \delta P^{++}_a = 0. \]

The appearance of new gauge invariances is observed in the fermionic part of the superstring action too. Thus, for example, the worldsheet gravitino drops out from the action. This is in agreement with previous twistor formulations of the superparticle (see, e.g., [3]), where one does not need a gravitino field to achieve the local worldsheet supersymmetry invariance.

Using all these gauge invariances along with worldsheet reparametrizations, tangent space Lorentz and Weyl transformations, we can gauge away the zweibeins and the field \( r \). Then with the help of (99) we find

\[ S_{GD} = \int d^2 \xi \, P_{\underline{a}} (\partial_{\tau} X^{\underline{a}} - \lambda \gamma^2 \overline{\lambda}), \]

where \( P_{\underline{a}} \) corresponds to an orthogonal combination of the Lagrange multipliers. Integrating out the inessential worldsheet coordinate (\( \sigma \)), we see that this is a twistor particle action of the type described in section 3.

The conclusion of this subsection is that when one employs the singular solution of the twistor constraint (38), the superstring action becomes degenerate. The gauge invariance widens, leaving a number of component fields arbitrary. The remaining physical fields do not depend on \( \sigma \) any more, so the superstring becomes a superparticle. Since the ordinary Green-Schwarz superstring formulation does contain the superparticle as a certain singular limit, we see that both the regular and singular solutions to the twistor constraints have to be taken into account.

6 Conclusions

In this paper we have shown how the non-heterotic \( D = 3 \) type II superstring can be formulated with manifest \( N = (1,1) \) worldsheet supersymmetry. The central point in the construction was the geometro-dynamical constraint (37) and its corollary (38). In particular, they reduced the initial \( 2 \times 2 \) complex twistor matrix \( E_{\alpha \alpha} \) to the two null vectors from the Virasoro constraints. The rest of (37) gave rise to purely auxiliary equations.

The geometro-dynamical principle is common for the twistor formulations of the superparticle [3], the heterotic superstring [9], [10] and, as we have seen here, the non-heterotic \( D = 3 \) type II superstring. One would be tempted to extrapolate this to the non-heterotic type II superstring in higher dimensions as well. Indeed, analysing the lowest-order component of eq. (38), one can show that the \( D = 3 \) situation is reproduced. For instance, in \( D = 10 \) the \( 16 \times 16 \) complex twistor matrix is once again reduced to the two null vectors from the Virasoro constraints. However, starting from \( D = 4 \) (and \( N = (2,2) \)) there is
an unexpected difficulty at the next level in the \( \eta \) expansion of eq. (38). One can show (most easily in the linearized approximation) that some of the constraints are equations of motion for \( \theta \). This is inadmissible, since the geometro-dynamical constraint is produced by a Lagrange multiplier, which implies that some of the components of the latter will propagate as well. One clearly sees that the case \( D = 3 \) is the only exception, due to the trivial algebra of the transverse gamma matrices in \( D = 3 \). In fact, the same problem is also encountered in the framework of the type II superparticle discussed in section 3. So, the main open problem now is to find a modification of the geometro-dynamical constraint such that it would not imply equations of motion in \( D > 3 \). We hope to be able to report progress in this direction elsewhere.

**Note added** After this paper has been completed, we received a new preprint by Pasti and Tonin [20], in which they claim that a similar construction applies to the \( D = 11 \) supermembrane with full \( N = 8 \) \( D = 3 \) worldsheet supersymmetry. This would be very surprising, since they impose the same type of geometro-dynamical constraint. As we mentioned above, in the case of extended (\( N > 1 \)) worldsheet supersymmetry this constraint is most likely to produce equations of motion and the corresponding Lagrange multiplier will contain new propagating degrees of freedom. One simple argument explaining this phenomenon has been proposed to us by P. Howe. The supermembrane theory of [20] could be truncated to a \( D = 11 \) superparticle with \( N = 16 \) worldline supersymmetry. There the geometro-dynamical constraint reduces the twistor variables (i.e. the bosonic physical fields) to the sphere \( S^9 \) (modulo gauge transformations). At the same time, the 32 components of the fermion \( \theta^\alpha \) are brought down to 16 after taking into account the 16 local worldline supersymmetries. It is clear that 9 bosons and 16 fermions do not form an off-shell supermultiplet, therefore the geometro-dynamical constraint must involve equations of motion.

## 7 Appendix. Solution to the twistor constraints

In section 4 we derived the geometro-dynamical constraint (38) or, in light-cone notation, (33)-(41). The lowest-order terms in the \( \eta \) expansion of this constraint gives restrictions on the twistor matrix \( \| \mathcal{E}_{\alpha \bar{\alpha}} \| \):

\[
\mathcal{E}_{+} \gamma^{a} \mathcal{E}_{+} = \mathcal{E}_{++}^{a},
\]

(102)

\[
\mathcal{E}_{-} \gamma^{a} \mathcal{E}_{-} = \mathcal{E}_{--}^{a},
\]

(103)

\[
\mathcal{E}_{+} \gamma^{a} \mathcal{E}_{-} + \mathcal{E}_{-} \gamma^{a} \mathcal{E}_{+} = 0.
\]

(104)

In fact, the first two equations define two vectors \( \mathcal{E}_{++}^{a} \) and only the third equation constrains the twistor variables. Here we are going to solve (104) in a general way.

We start by writing out the components of the twistor matrix

\[
\mathcal{E}_{\alpha \bar{\alpha}} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

(105)
A basic kinematic assumption about the twistor variables is that they can never vanish identically. This means that at least one element of the matrix (105) is non-vanishing. It is convenient to write down the constraint (104) using the light-cone basis (26) for the gamma matrices. There the three projections read

\begin{align}
(++) : & \quad A \bar{C} + C \bar{A} = 0, \\
(--) : & \quad B \bar{D} + D \bar{B} = 0, \\
(+-) : & \quad B \bar{C} + C \bar{B} + A \bar{D} + D \bar{A} = 0.
\end{align}

The general solution to equations (106), (107) is given by

\begin{align}
A = ae^{i\alpha}, \quad C = ice^{i\alpha}, \quad B = be^{i\beta}, \quad D = ide^{i\beta},
\end{align}

where \(a, b, c\) and \(d\) are real. Substituting this into (108) one gets

\begin{align}
(ad - bc) \sin(\alpha - \beta) = 0.
\end{align}

Now, there are two possibilities: the matrix \(E_{\alpha \alpha}\) can be either degenerate or non-degenerate. With the help of (109) we evaluate the determinant of this matrix

\begin{align}
\det \| E_{\alpha \alpha} \| = i(ad - bc) e^{i(\alpha + \beta)}.
\end{align}

Hence if the matrix \(E_{\alpha \alpha}\) is non-degenerate, \(ad - bc \neq 0\) and (110) implies in turn \(\alpha = \beta\). In the degenerate case \(ad - bc = 0\) (and hence \((c,d) \sim (a,b)\)) and the phases \(\alpha\) and \(\beta\) are independent.

In summary, the general solution to (104) consists of two sectors. In the first sector, the matrix \(E_{\alpha \alpha}\) is non-degenerate and represented as follows

\begin{align}
E_{\alpha \alpha} = e^{i\phi} \begin{pmatrix} a & b \\ ic & id \end{pmatrix} \equiv e^{i\phi} \begin{pmatrix} \lambda_+^\alpha \\ i\lambda_-^\alpha \end{pmatrix},
\end{align}

where the spinors \(\lambda_+^\alpha\) and \(\lambda_-^\alpha\) are real and restricted by the condition

\begin{align}
\lambda_+^\alpha \lambda_-^\alpha \equiv ad - bc \neq 0.
\end{align}

The second sector consists of the degenerate matrix \(E_{\alpha \alpha}\)

\begin{align}
E_{\alpha \alpha} = \begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ irae^{i\alpha} & irbe^{i\beta} \end{pmatrix} \equiv \begin{pmatrix} \lambda^\alpha \\ ir\lambda^\alpha \end{pmatrix}
\end{align}

where \(\lambda^\alpha\) is now an arbitrary complex spinor, and \(r\) is real.

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