Higher-order geodesic deviations
applied to the Kerr metric

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Abstract

Starting with an exact and simple geodesic, we generate approximate geodesics by summing up
higher-order geodesic deviations within a General Relativistic setting, without using Newtonian
and post-Newtonian approximations.

We apply this method to the problem of closed orbital motion of test particles in the Kerr
metric space-time. With a simple circular orbit in the equatorial plane taken as the initial geodesic
we obtain finite eccentricity orbits in the form of Taylor series with the eccentricity playing the
role of small parameter.

The explicit expressions of these higher-order geodesic deviations are derived using successive
systems of linear equations with constant coefficients, whose solutions are of harmonic oscillator
type. This scheme gives best results when applied to the orbits with low eccentricities, but with
arbitrary values of \( (GM/Rc^2) \), smaller than \(1/6\) in the Schwarzschild limit.

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1 Introduction

In two recently published articles, [1] and [2], the approximate low-eccentricity relativistic trajectories
of planets with small mass \(m\) (as compared to the central body’s mass \(M\)) have been constructed
in Schwarzschild and Reissner-Nordstrøm space-time metrics. In the latter case, the motion of
electrically charged particles have been investigated, too.

The two-body problem in General Relativity has been the object of many excellent studies; one
of the first checks of this theory has been the very precise value of the perihelion advance calculated
by Einstein [3] for the planet Mercury. The calculus was based on the solution of the geodesic
equation in Schwarzschild’s metric, using the first integrals; the solution was obtained in the form

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of a quadrature, with the proper time $\tau$ expressed as a quasi-elliptic integral. Such an integral can not be evaluated analytically; instead, Einstein has developed the integrand into a power series with respect to the small parameter $GM/rc^2$, which led to simple integrations that could be easily performed. The approximate formula for the perihelion advance after one revolution is then

$$\Delta \varphi = \frac{6\pi GM}{a(1-e^2)} = \frac{6\pi GM}{a} (1 + e^2 + e^4 + e^6 + \ldots) \quad (1)$$

where $a$ is the major semi-axis and $e$ the eccentricity of the orbit, and $G$ stands for Newton’s gravitational constant divided by $c^2$, so that in this notation the quantity $GM/r$ becomes dimensionless.

We shall adopt this notation from now on, equivalent of using the units in which $c = 1$. The above approximation is acceptable only if the value of the parameter $GM/a$ is negligibly small.

In our previous articles [2] and [1] we have proposed an alternative way of determining the value of perihelion advance and finding an explicit (parametrized by the proper time) form of trajectory and the law of motion as a series of successive approximations, without supposing that the ratio $GM/a$ (or equivalently, the ratio $v^2/c^2$) is very small, or using the Newtonian approximation. Instead, we start from a very simple particular solution of geodesic equation in Schwarzschild (or Reissner-Nordstrøm) metric: a perfect circular orbit along which the small mass $m$ is advancing with a constant angular velocity. It is very easy to check that such a motion is a geodesic curve in the aforementioned space-times. The fact that all geometrical quantities, such as the Christoffel coefficients, or the components of Riemann’s tensor, take on constant values on this trajectory, leads to a particularly simple form of the geodesic deviation equations: they reduce themselves to a system of second-order linear differential equations with constant coefficients, and the solution is just a collection of harmonic oscillators.

Here we apply the same method to the case of the axially-symmetric Kerr metric. Although the motion of test particles along closed orbits in Kerr’s metric has been analyzed in a very exhaustive manner in many papers [4]-[10], our method gives results in an explicit form, and is very well adapted for computer-based calculations. In the case of orbits with low eccentricity it converges very quickly even for the non-negligible values of the ratio $GM/r$.

In the following Sections we shall briefly recall the essential features of our approximation method best suited for higher-order deviations. Using the Kerr space-time, we choose a circular orbit in the equatorial plane as the initial geodesic. Then, the first, second and third deviations are obtained, the latter one with the help of the Poincaré’s method [11] enabling us to obtain higher-order corrections to the basic frequencies. The explicit form of the perihelion advance in the field of Kerr metric displays interesting features as a combined result of the influence of two essential parameters, the mass $M$ and the angular momentum density $a$ of the central body.

In the last section, we discuss the physical content of the results and consider some future applications of higher-order deviations, including the effects of finite mass $m$ and internal spin (angular momentum) of the planet. In contrast with our previous article [1] we shall not consider here the problem of gravitational radiation; it will be left to a detailed future work.

2 Geodesic deviations using small deformations

The previous article [1] was based on the deviation vectors $n^\mu$, $b^\mu$, $h^\mu$ and their deviation equations. However, as the order of the deviation increases, it becomes harder to calculate the deviation equations for the deviation vectors.

So, for our purpose here, which is the effective calculation of deformations of circular orbits in Kerr metric, we need the explicit coordinate-dependent expressions for the deviations that we
shall add to given functions of proper time $s$ which define the relativistic trajectory and the law of motion. This is why we need to consider an alternative approach, which deals with small deviations of arbitrary order of coordinate functions, thus deforming the trajectories directly.

Consider an infinitesimal deformation of the geodesic curve $x^\mu (s)$:

$$ x^\mu (s) \Rightarrow \tilde{x}^\mu (s) = x^\mu (s) + \delta x^\mu (s). \quad (2) $$

Suppose that we want the new curve $\tilde{x}^\mu (s)$ to satisfy the geodesic equation, too:

$$ \frac{d^2 \tilde{x}^\mu}{ds^2} + \Gamma^\mu_{\lambda \rho} (\tilde{x}^\nu) \frac{d\tilde{x}^\lambda}{ds} \frac{d\tilde{x}^\rho}{ds} = 0. \quad (3) $$

Expanding the Christoffel coefficients into power series of $\delta x^\mu$,

$$ \Gamma^\mu_{\lambda \rho} (\tilde{x}^\nu) = \Gamma^\mu_{\lambda \rho} (x^\nu) + \delta x^\sigma \partial_\sigma \Gamma^\mu_{\lambda \rho} (x^\nu) + \frac{1}{2!} \delta x^\sigma \delta x^\tau \partial_\sigma \partial_\tau \Gamma^\mu_{\lambda \rho} (x^\nu) + \ldots \quad (4) $$

Substituting the expressions (2) and (4) into Eq. (3) and expanding in consecutive powers of deviations $\delta x^\mu$, we get in the zeroth-order the initial geodesic equation (3) satisfied by $x^\mu (s)$; collecting then all the terms linear in $\delta x^\mu$ and their derivatives, we get

$$ \frac{d^2 \delta x^\mu}{ds^2} + 2 \Gamma^\mu_{\lambda \rho} u^\lambda \frac{d\delta x^\rho}{ds} + (\partial_\sigma \Gamma^\mu_{\lambda \rho}) u^\lambda u^\rho \delta x^\sigma = 0 \quad (5) $$

which coincides with the first-order deviation equation of the Ref. [1] if we replace the vector $n^\mu$ by the infinitesimal deviation $\delta x^\mu$. The geometrical meaning of this equation is now very clear: it gives the conditions to be satisfied by infinitesimal functions $\epsilon n^\mu (s) = \delta x^\mu$ (with $\epsilon$ being an infinitesimal parameter) defined along a given geodesic curve $x^\lambda (s)$, in order to ensure that the new curve, infinitesimally close to it and defined by $\tilde{x}^\lambda (s) = x^\lambda (s) + \epsilon n^\lambda (s)$, is also a geodesic one, up to the first order in $\epsilon$.

The fact that $n^\mu$ is a vector is in agreement with the transformation properties of infinitesimal deviations $\delta x^\mu$, which under an arbitrary change of coordinates $x^\mu = x^\mu (y^\rho)$ transform as

$$ \delta x^\mu = \frac{\partial x^\mu}{\partial y^\lambda} \delta y^\lambda \quad (6) $$

up to higher-order terms, neglected at the linear approximation level.

However, the higher-order terms in the expansion are still there: collecting all the second-order terms in $\delta x^\mu$ from the Taylor expansion of Eq. (3), we get

$$ \Gamma^\mu_{\lambda \rho} \frac{d\delta x^\lambda}{ds} \frac{d\delta x^\rho}{ds} + 2 \delta x^\nu (\partial_\nu \Gamma^\mu_{\lambda \rho}) u^\lambda \frac{d\delta x^\rho}{ds} + \frac{1}{2} \delta x^\nu \delta x^\sigma (\partial_\nu \partial_\sigma \Gamma^\mu_{\lambda \rho}) u^\lambda u^\rho, \quad (7) $$

and there is no reason for it to vanish even if the deviations $\delta x^\mu (s)$ satisfy Eq. (3). The vanishing of expression (7) would impose too many conditions, a priori incompatible with Eq. (3) on the same set of functions $\delta x^\mu (s)$, and we need extra degrees of freedom if we want to cancel also all the second-order terms.

This means that from the very beginning, infinitesimal deviations of higher-order must be introduced:

$$ \tilde{x}^\mu (s) = x^\mu (s) + \delta x^\mu (s) + \frac{1}{2!} \delta^2 x^\mu (s) + \frac{1}{3!} \delta^3 x^\mu (s) + \ldots \quad (8) $$
so that two new second-order terms will add up to the expression (9), namely
\[
\frac{d^2(\delta^2 x^\mu)}{ds^2} + \delta^2 x^\nu \partial_\nu \Gamma^\mu_{\lambda\rho} u^\lambda u^\rho
\]
and the sum of all these terms represents a new set of second-order differential equations imposed on the independent functions \(\delta^2 x^\mu\), which may be solved after we insert the solutions for \(u^\mu(s)\) and \(\delta x^\mu(s)\) obtained previously.

At this point, another problem arises: not only the coefficients in these differential equations are not covariant objects, but also the functions \(\delta^2 x^\mu\) do not behave as vectors under the change of coordinates. As a matter of fact, they will mix up with terms quadratic in \(\delta x^\mu\) as follows:
\[
\delta^2 x^\mu = \delta(\delta x^\mu) = \frac{\partial x^\mu}{\partial y^\lambda} \delta^2 y^\lambda + \frac{\partial^2 x^\mu}{\partial y^\lambda \partial y^\rho} \delta y^\lambda \delta y^\rho.
\] (10)

This non-homogeneous transformation law suggests that we can introduce a covariant quantity \(D^2 x^\mu\) defined as
\[
D^2 x^\mu = \delta^2 x^\mu + \Gamma^\mu_{\lambda\rho} \delta x^\lambda \delta x^\rho.
\] (11)

Defining the infinitesimal vector \(b^\mu\) as \(\epsilon^2 b^\mu = D^2 x^\mu\) and expressing the Taylor expansion (8) in terms of \(n^\mu\) and \(b^\mu\) as
\[
\delta x^\mu = x^\mu + \epsilon n^\mu + \frac{1}{2!} \epsilon^2 \left( b^\mu - \Gamma^\mu_{\lambda\rho} n^\lambda n^\rho \right) + \ldots
\] (12)

and requiring the geodesic equation for \(\delta x^\mu\) to be satisfied up to the second order in \(\epsilon\), we arrive at the same second-order deviation equation of the Ref. [1] satisfied by \(b^\mu\) that is non-manifestly covariant and equivalent to the manifestly covariant equation.

In Ref. [1], the non-manifestly covariant deviation equations could be easier obtained using this Taylor expansion approach than transforming the manifestly covariant deviation equations. In practical calculations the non-manifestly covariant equations turn out to be of much use, i.e., it is easier to obtain the solutions of \(n^\mu\) and \(b^\mu\) using them instead of the manifestly covariant equations.

Similar corrections are needed to define the higher-order deviations, like
\[
\epsilon^3 h^\mu = D^3 x^\mu = \delta^3 x^\mu + 3 \Gamma^\mu_{\nu\sigma} \delta x^\nu \delta^2 x^\sigma + \left( \partial_\nu \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\nu\rho} \Gamma^\rho_{\lambda\sigma} \right) \delta x^\lambda \delta x^\nu \delta x^\sigma,
\] (13)

so the real coordinate deviation \(\delta^3 x^\mu\) reads
\[
\delta^3 x^\mu = \epsilon^3 \left[ h^\mu - 3 \Gamma^\mu_{\rho\sigma} n^\rho b^\sigma - \left( \partial_\nu \Gamma^\mu_{\nu\sigma} - 2 \Gamma^\mu_{\lambda\rho} \Gamma^\rho_{\nu\sigma} \right) \Gamma^\rho_{\nu\sigma} \right].
\] (14)

A study of higher-order differentials and their covariant generalizations can be found in recent papers [12]-[13].

But there is even another easier form to calculate higher-order geodesic deviations: we indeed need \(\delta^m x^\mu\) to obtain the geodesic \(\delta x^\mu\), and the differential equations for \(\delta^m x^\mu\) are simpler than their counterparts \(n^\mu\), \(b^\mu\), \(h^\mu\), etc. For example, requiring again the geodesic equation for \(\delta x^\mu\) to be satisfied up to the second order in \(\epsilon\), the following second-order deviation equation for \(\delta^2 x^\mu\) is obtained
\[
\frac{d^2 \delta^2 x^\mu}{ds^2} + \left( \partial_\rho \Gamma^\mu_{\lambda\rho} \right) u^\lambda u^\sigma \delta^2 x^\rho + 2 \Gamma^\mu_{\lambda\rho} \right) u^\lambda \frac{d \delta^2 x^\sigma}{ds} = -2 \Gamma^\mu_{\lambda\rho} \frac{d \delta^2 x^\lambda}{ds} \frac{d \delta x^\rho}{ds} - 4 \left( \partial_\nu \Gamma^\mu_{\lambda\rho} \right) u^\lambda \delta x^\sigma \frac{d \delta x^\rho}{ds} - \left( \partial_\nu \Gamma^\mu_{\lambda\rho} \right) u^\lambda u^\sigma \delta x^\rho \delta x^\nu.
\] (15)
where we see that the l.h.s. is unchanged, but the r.h.s. has only 3 terms instead of 10 found in the non-manifestly covariant second-order deviation equation for $b^\mu$ (see Ref. [1]).

The non-manifestly covariant third-order deviation equation for $h^\mu$ is not shown here, but has 60 terms in the r.h.s., while the third-order deviation equation for $\delta^3 x^\mu$ has only 7 terms:

$$\frac{d^2 \delta^3 x^\mu}{ds^2} + (\partial_\mu \Gamma^\mu_{\lambda\sigma}) u^\lambda u^\sigma \delta^3 x^\rho + 2 \Gamma^\mu_{\lambda\sigma} u^\lambda \frac{d\delta^3 x^\sigma}{ds} =$$

$$-6 \Gamma^\mu_{\lambda \rho} \frac{d \delta^2 x^\lambda}{ds} \frac{d \delta x^\rho}{ds} - 6(\partial_\sigma \Gamma^\sigma_{\lambda \rho}) \left( \delta x^\alpha \frac{d \delta x^\lambda}{ds} \frac{d \delta x^\rho}{ds} + u^\lambda \delta^2 x^\rho \frac{d \delta x^\rho}{ds} + u^\lambda \delta x^\sigma \frac{d \delta^2 x^\sigma}{ds} \right)$$

$$-3(\partial_\nu \partial_\sigma \Gamma^\mu_{\lambda \rho}) u^\lambda \delta x^\nu \left( 2 \delta x^\sigma \frac{d \delta x^\rho}{ds} + u^\rho \delta^2 x^\rho \right) - (\partial_\tau \partial_\nu \partial_\sigma \Gamma^\mu_{\lambda \rho}) u^\lambda u^\rho \delta x^\sigma \delta x^\nu \delta x^\tau. \quad (16)$$

The fourth-order deviation equation for $\delta^4 x^\mu$ has 15 terms, and the fifth-order deviation equation for $\delta^5 x^\mu$ has 26 terms. We have developed a symbolic computer program to calculate $n^{th}$-order deviation equations for $\delta^n x^\mu$.

The non-manifestly covariant geodesic deviation equations are well suited to deriving successive approximations for geodesics close to an initial one. Starting from a given geodesic $x^\mu (s)$ we can solve Eq. (3) and find the first-order deviation vector $\delta x^\mu (s)$. Now, with $u^\mu (s)$ and $\delta x^\mu (s)$, the system (15) can be solved and we obtain the second-order deviation $\delta^2 x^\mu (s)$. Then, using $u^\mu (s)$, $\delta x^\mu (s)$ and $\delta^2 x^\mu (s)$ into the system (16) the third-order deviation $\delta^3 x^\mu (s)$ is calculated, and so forth.

The literature about geodesic deviations includes a rigorous mathematical study of geodesic deviations up to the second-order, as well as geometric interpretation, but using different derivation, presented in [4]. Also, a Hamilton–Jacobi formalism had been derived in [15], which was applied to the problem of free falling particles in the Schwarzschild space-time [16]. Anyway, the resulting expressions are not well optimized for successive calculations of higher-order geodesic deviations. Interesting effects resulting from the analysis of first-order geodesic deviations of test particles suspended in hollow spherical satellites have been discussed in [17].

### 3 Circular orbits in the Kerr metric

We choose as initial geodesic a circular orbit in the axisymmetric gravitational field created by a massive body with rotation, i.e., in the Kerr metric. This metric and their circular orbits have been studied in several papers [4]-[8] and books [9]-[10].

The gravitational field is described by the line-element (in natural coordinates with $c = 1$ and $G = 1$)

$$ds^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{2Mr - \rho^2}{\rho^2} dt^2 - \frac{4Mr \rho}{\rho^2} \sin^2 \theta \, dt \, d\phi + \frac{\sin^2 \theta}{\rho^2} (-\Delta a^2 \sin^2 \theta + \rho^2 (r^2 + a^2)^2) d\phi^2 \quad (17)$$

with $\Delta = r^2 + a^2 - 2Mr$ and $\rho^2 = r^2 + a^2 \sin^2 \theta$, where $M$ and $a = \frac{J}{Mr}$ are the mass and the angular momentum density of the central body rotating in opposite $\phi$ direction.

The circular orbit of radius $R$ in the equatorial plane (which is a geodesic in the background Kerr metric) is described by a simple 4-velocity vector:

$$u^r = \frac{dr}{ds} = 0, \quad u^\theta = \frac{d\theta}{ds} = 0,$$
frequency of this circular motion is $\omega_c = \frac{\sqrt{M}}{R^{3/2}} \sqrt{1 - \frac{3M}{R} + \frac{2a\sqrt{M}}{R^{3/2}}}$.

Likewise the calculations for the Schwarzschild case in Ref. [1], we use the non-manifestly first-order differential equation for the components $\delta r$, $\delta \theta$, $\delta \phi$ and $\delta t$ (or $n^r$, $n^\theta$, $n^\phi$ and $n^t$), as we can see in a matrix form:

\[
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{21} & m_{22} & m_{23} & m_{24} \\
  m_{31} & m_{32} & m_{33} & m_{34} \\
  m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix}
\begin{pmatrix}
  \delta r \\
  \delta \theta \\
  \delta \phi \\
  \delta t
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix},
\]

where the matrix elements are:

\[
m_{11} = \frac{d^2}{ds^2} - 3M \frac{f_2}{R^3 f_1}, \quad m_{12} = 0, \quad m_{13} = -2\sqrt{\frac{M}{R}} \left(1 + \frac{a\sqrt{M}}{R^{3/2}}\right) \frac{f_2}{\sqrt{f_1}} \frac{d}{ds}, \quad m_{14} = \frac{2M f_2}{R^2 \sqrt{f_1}} \frac{d}{ds},
\]

\[
m_{21} = m_{23} = m_{24} = 0, \quad m_{22} = \frac{d^2}{ds^2} + \frac{M f_3}{R^3 f_1},
\]

\[
m_{31} = \frac{2\sqrt{M}}{R^{3/2}} \frac{f_4}{f_2 \sqrt{f_1}} \frac{d}{ds}, \quad m_{32} = m_{34} = 0, \quad m_{33} = \frac{d^2}{ds^2},
\]

\[
m_{41} = \frac{2M}{R^2} \frac{f_5}{f_2 \sqrt{f_1}} \frac{d}{ds}, \quad m_{42} = m_{43} = 0, \quad m_{44} = \frac{d^2}{ds^2},
\]

using the functions:

\[
f_1 = \left(1 - \frac{3M}{R}\right) + \frac{2a\sqrt{M}}{R^{3/2}}, \quad f_2 = \left(1 - \frac{2M}{R}\right) + \frac{a^2}{R^2}, \quad f_3 = 1 - \frac{4a\sqrt{M}}{R^{3/2}} + \frac{3a^2}{R^2},
\]

\[
f_4 = \left(1 - \frac{2M}{R}\right) + \frac{a\sqrt{M}}{R^{3/2}}, \quad f_5 = 1 - \frac{2a\sqrt{M}}{R^{3/2}} + \frac{a^2}{R^2}.
\]

The harmonic oscillator equation for $n^\theta = \delta \theta$ has an angular frequency $\omega_\theta$:

\[
\omega_\theta = \frac{\sqrt{M}}{R^{3/2}} \sqrt{\frac{f_3}{f_1}} = \frac{\sqrt{M}}{R^{3/2}} \sqrt{\left(1 - \frac{3M}{R}\right) \frac{1 - \frac{4a\sqrt{M}}{R^{3/2}} + \frac{3a^2}{R^2}}{\left(1 - \frac{3M}{R}\right) + \frac{2a\sqrt{M}}{R^{3/2}}}.
\]
One possible choice of solution is

\[ n^\theta = \delta \theta = -n^\theta_0 \cos(\omega \theta s). \]  

(26)

In the Schwarzschild limit \((a \to 0)\), \(\omega_0 = \omega_c\), so in this case we can neglect this solution \((n^\theta_0 = 0)\) because the new plane of orbit is a new one inclined, or just a change of coordinate system.

Using the differential equation for \(n^r = \delta r\) we can eliminate the derivatives of \(\delta \phi\) and \(\delta t\), yielding the harmonic oscillator equation

\[ \frac{d^2 \delta r}{ds^2} + \omega^2 \delta r = 0, \]  

(27)

with the characteristic frequency

\[ \omega = \sqrt{\frac{M}{R^{3/2}}} \sqrt{\frac{(1 - \frac{6M}{R}) + \frac{8a\sqrt{M}}{R^{3/2}} - \frac{3a^2}{R^2}}{(1 - \frac{3M}{R}) + \frac{2a\sqrt{M}}{R^{3/2}}}} = \sqrt{\frac{M}{R^{3/2}}} \sqrt{\frac{f_6}{f_1}}, \]  

(28)

where:

\[ f_6 = \left(1 - \frac{6M}{R}\right) + \frac{8a\sqrt{M}}{R^{3/2}} - \frac{3a^2}{R^2}. \]  

(29)

We shall choose the initial phase to have (with \(n^r_0 > 0\)):

\[ n^r = \delta r = -n^r_0 \cos(\omega s) \]  

(30)

so the perihelion occurs when \(s = 0\).

The calculation of \(\delta \phi\) and \(\delta t\) is now simple:

\[ n^\phi = \delta \phi = n^\phi_0 \sin(\omega s), \]  

(31)

\[ n^t = \delta t = n^t_0 \sin(\omega s), \]  

(32)

where the amplitudes depend on \(n^r_0\):

\[ n^\phi_0 = \frac{2n^r_0 f_4}{R f_2 \sqrt{f_6}}, \]  

(33)

\[ n^t_0 = 2n^r_0 \sqrt{\frac{M}{R f_2 \sqrt{f_6}}}. \]  

(34)

Adding this first-order deviation to the circular orbit, the new trajectory and the law of motion are given by

\[ r = R - n^r_0 \cos(\omega s), \]  

(35)

\[ \theta = \frac{\pi}{2} - n^\theta_0 \cos(\omega \theta s), \]  

(36)

\[ \varphi = \omega_c s + n^\phi_0 \sin(\omega s), \]  

(37)

\[ t = u^t s + n^t_0 \sin(\omega s), \]  

(38)

and this solution is a geodesic up to the first-order in \(\epsilon\). It is important to note once again that the coefficient \(n^r_0\), which also fixes the values of the two remaining amplitudes, \(n^t_0\) and \(n^\phi_0\), defines the size of the actual deviation, so that the ratio \(\frac{n^r_0}{R}\) becomes the dimensionless infinitesimal parameter \(\epsilon\) controlling the approximation series with consecutive terms proportional to the consecutive powers of \(\frac{n^r_0}{R}\) (or \(\frac{n^\theta_0}{R}\)).
One easily checks that the Schwarzschild limit ($a \to 0$) of the solution above yields the results of the Ref. [1], including the perihelion advance and the generalized epicycle [18], where we identify the major semi-axis $a$ with $R$ and the eccentricity $e$ with $n_0^2$:

$$r(t) = \frac{a(1 - e^2)}{1 + e \cos(\omega_0 t)} \simeq a \left[ 1 - e \cos(\omega_0 t) \right],$$

But it is more interesting to show the perihelion advance for the Kerr case, even using the first-order deviation, because the Kerr parameter $a$ appears with positive and negative coefficients, i.e., the angular momentum density $a$ can increase or decrease the perihelion advance. Note that the post-Newtonian limit matches the Eq. (1) for small eccentricities.

Despite the limitations of the first-order geodesic deviation, we have already obtained a generalized perihelion advance valid for high values of $\frac{n_0^2}{R}$ and $a$, but low values of the “eccentricity” $e$ (or $\frac{n_0^2}{R}$). The high-order deviations will, for example, allow the calculation of $\Delta \varphi$ for higher values of the “eccentricity” $\frac{n_0^2}{R}$.

5 The second-order geodesic deviation

Inserting the complete solution for the first-order deviation $\delta x^\mu = n^\mu$, Eqs. (26), (30)–(32) into the second-order deviation equation for $\delta^2 x^\mu$ [15], we find the same matrix with a new non-homogeneous vector $C^\mu$:

$$\begin{pmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{21} & m_{22} & m_{23} & m_{24} \\
  m_{31} & m_{32} & m_{33} & m_{34} \\
  m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix}
\begin{pmatrix}
  \delta^2 r \\
  \delta^2 \theta \\
  \delta^2 \phi \\
  \delta^2 t
\end{pmatrix}
= e^2
\begin{pmatrix}
  C^r \\
  C^\theta \\
  C^\phi \\
  C^t
\end{pmatrix},$$

where we have put into evidence the common factor $e^2$, which shows the explicit quadratic dependence of the second-order deviation $\delta^2 x^\mu$ on the first-order deviation amplitude $n_0^\mu$ (or $n_0^\theta$). The functions $C^r$, $C^\theta$, $C^\phi$ and $C^t$ are expressions depending on $M$, $R$, $a$, and on the functions $\sin(2\omega s)$, $\cos(2\omega s)$, $\sin(2\omega_0 s)$, $\cos(2\omega_0 s)$, $\cos[(\omega - \omega_0)s]$ and $\cos[(\omega + \omega_0)s]$:

$$C^r = C_{00}^r + C_{0\theta}^r \cos(2\omega s) + C_{2\theta}^r \cos(2\omega_0 s),$$

$$C^\theta = C_{-\theta}^\theta \cos[(\omega - \omega_0)s] + C_{+\theta}^\theta \cos[(\omega + \omega_0)s],$$

$$C^\phi = C_{2\phi}^\phi \sin(2\omega s) + C_{2\theta}^\phi \sin(2\omega_0 s),$$

$$C^t = C_{2\phi}^t \sin(2\omega s) + C_{2\theta}^t \sin(2\omega_0 s).$$

The solution of the above matrix for $\delta^2 x^\mu(s)$ has the same characteristic equations of the matrix (13) for $\delta x^\mu(s) = n^\mu(s)$, and the general solution containing oscillating terms with angular frequency $\omega$ and $\omega_0$ is of no interest because it is already accounted for by $n^\mu(s)$. But the particular solution
The second-order deviation order in \( n \) approximations beginning with \( g \) gives more compact results. The exact equation of an ellipse is obtained in the limit \( M \rightarrow 0 \), and the shape of the orbit described by \( x^\mu \) including second-order deviations is not an ellipse due to the General Relativity effects of \( M \), but we can match the perihelion and aphelion distances of the Keplerian, i.e., elliptical orbit, with the same perihelion and aphelion distances of the orbit described by \( x^\mu \):

\[
\delta^2 r = \delta^2 r_0 + \delta^2 r_2 \cos(2\omega s), \quad \delta^2 \theta = \delta^2 \theta_0 \cos[(\omega - \omega_0)s] + \delta^2 \theta_\omega \cos[(\omega + \omega_0)s],
\]

\[
\delta^2 \phi = (\delta^2 \phi_0)s + \delta^2 \phi_2 \sin(2\omega s) + \delta^2 \phi_\omega \sin(2\omega_0 s),
\]

\[
\delta^2 t = (\delta^2 t_0)s + \delta^2 t_2 \sin(2\omega s) + \delta^2 t_\omega \sin(2\omega_0 s).
\]

The constants \( \delta^2 r_0, \delta^2 \phi_0 \) and \( \delta^2 t_0 \) depend on two arbitrary constants, so we can choose the initial conditions of the differential solutions so that the constants \( \delta^2 r_0 \) and \( \delta^2 \phi_0 \) are null, and \( \delta^2 t_0 \) is simplified. The Appendix 1 shows the explicit values of the above coefficients.

In the Schwarzschild limit, the solution for the second-order geodesic deviation \( \delta^2 x^\mu(s) \) is:

\[
\delta^2 r = -\frac{(n_0^r)^2}{R} \left( 1 - \frac{7M}{R} \right) \cos(2\omega s), \quad \delta^2 \theta = 0, \quad \delta^2 \phi = -\frac{2(n_0^\phi)^2}{R^2} \left( 1 - \frac{5M}{R} \right) \sin(2\omega s),
\]

\[
\delta^2 t = \frac{(n_0^t)^2}{R} \left[ -\frac{3}{2} \frac{1 + \frac{M}{R}}{1 - \frac{2M}{R}} s + \sqrt{\frac{M}{R}} \frac{2 - \frac{15M}{R} + \frac{14M^2}{R^2}}{1 - \frac{2M}{R}} \sin(2\omega s) \right].
\]

The second-order deviation \( \delta^2 x^\mu \) computed in the Ref. [1] used another choice for the constants \( \delta^2 r_0, \delta^2 \phi_0 \) and \( \delta^2 t_0 \), i.e., equivalent to different initial conditions for the initial geodesic. So, using the initial conditions chosen above, the comparison with an ellipse in the Schwarzschild limit also gives more compact results.

The trajectory described by \( x^\mu \) including second-order deviations is not an ellipse due to the General Relativity effects of \( M \), but we can match the perihelion and aphelion distances of the Keplerian, i.e., elliptical orbit, with the same perihelion and aphelion distances of the orbit described by \( x^\mu \):

\[
a = R - \frac{(n_0^r)^2}{2R} \left( 1 - \frac{7M}{R} \right),
\]

\[
e = \frac{2n_0^r \left( 1 - \frac{6M}{R} \right)}{2R \left( 1 - \frac{2M}{R} \right) - \frac{(n_0^t)^2}{R} \left( 1 - \frac{7M}{R} \right)} = \frac{n_0^r}{R} + O \left( \frac{(n_0^r)^3}{R^3} \right).
\]

The shape of the orbit described by \( r(\varphi) \) can be obtained from \( \varphi(s) \), then \( s(\varphi) \) by means of successive approximations beginning with \( \omega s = \frac{\omega}{\omega_c} \varphi \), and \( s \) is replaced in \( r(s) \) giving \( r(\varphi) \) up to the second order in \( n_0^r/R \):

\[
r = 1 - \frac{n_0^r}{R} \cos \left( \frac{\omega}{\omega_c} \varphi \right) + \left( \frac{n_0^r}{R} \right)^2 \left[ 1 + \frac{1 - \frac{5M}{R}}{2 \left( 1 - \frac{6M}{R} \right)} \cos \left( \frac{2\omega}{\omega_c} \varphi \right) \right] + ...
\]

The exact equation of an ellipse is obtained in the limit \( M/R \rightarrow 0 \), up to the second order in \( e = n_0^r/R \):

\[
r = \frac{r_0}{1 + e \cos \varphi} = \frac{1 - \frac{3}{2} e^2}{1 + e \cos \varphi} R = R \left[ 1 - e \cos \varphi + e^2 \left( -1 + \frac{1}{2} \cos 2\varphi \right) + ... \right].
\]
In the ellipse equation (39) we have \( r_0 = a(1 - e^2) \), so
\[
a = R \left( 1 - \frac{3}{2}e^2 \right) \approx R \left( 1 - \frac{e^2}{2} \right),
\]
(56)

These values for \( a \) and \( e \) agree with Eqs. (52)–(53).

In order to improve the comparison of the perihelion advance in the post-Newtonian limit with Eq. (11), the \( \Delta \varphi \) should include \( \frac{n_0}{\pi} \) terms, which is not yet the case using second-order deviations due to the imposed initial conditions.

6 Third-order deviation and Poincaré’s method

Using the solutions for \( \delta x^\mu = n^\mu \) and \( \delta^2 x^\mu \) into third-order deviation equation for \( \delta^3 x^\mu \) (62), we again find the same matrix with a new non-homogeneous vector \( D^\mu \):
\[
\begin{pmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix}
\begin{pmatrix}
\delta^3 r \\
\delta^2 \theta \\
\delta^3 \phi \\
\delta^3 t
\end{pmatrix}
= \epsilon^3
\begin{pmatrix}
D^r \\
D^\theta \\
D^\phi \\
D^t
\end{pmatrix},
\]
(57)

where the common factor \( \epsilon^3 \) shows the explicit cubic dependence of the third-order deviation \( \delta^3 x^\mu \) on the first-order deviation amplitude \( n_0^\mu \) (or \( n_0^\phi \)). The functions \( D^r, D^\theta, D^\phi \) and \( D^t \) are expressions depending on \( M, R, a, \) and sin and cos functions of \( \omega s, \omega gs, 3\omega, 3\omega gs, (\omega - 2\omega g), (\omega + 2\omega g), (2\omega - \omega g) \) and \( (2\omega + \omega g) \):
\[
\begin{align*}
D^r &= D^r_1 \cos(\omega s) + D^r_3 \cos(3\omega s) + D^r_- \cos[(\omega - 2\omega g)s] + D^r_+ \cos[(\omega + 2\omega g)s], \\
D^\theta &= D^\theta_1 \cos(\omega gs) + D^\theta_3 \cos(3\omega gs) + D^\theta_- \cos[(2\omega - \omega g)s] + D^\theta_+ \cos[(2\omega + \omega g)s], \\
D^\phi &= D^\phi_1 \cos(\omega s) + D^\phi_3 \cos(3\omega s) + D^\phi_- \cos[(\omega - 2\omega g)s] + D^\phi_+ \cos[(\omega + 2\omega g)s], \\
D^t &= D^t_1 \cos(\omega s) + D^t_3 \cos(3\omega s) + D^t_- \cos[(\omega - 2\omega g)s] + D^t_+ \cos[(\omega + 2\omega g)s].
\end{align*}
\]
(58–61)

The functions \( \cos(\omega s) \) and \( \cos(\omega gs) \) represent a new problem for the third-order deviation, as they are resonance terms whose angular frequency \( \omega \) (or \( \omega g \)) is the same as the eigenvalue of the matrix-operator acting on the left-hand side, yielding secular terms, proportional to \( s \). To avoid unbounded deviations, we can apply the Poincaré’s method (11) to take into account possible perturbation of the basic frequency itself, replacing \( \omega \) (or \( \omega g \)) by an infinite series in powers of the infinitesimal parameter, which in our case can be the “eccentricity” \( \epsilon = \frac{n_0^\phi}{\pi} \):
\[
\omega \to \omega_p = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \ldots,
\]
(62)

where the new \( \omega \) is renamed \( \omega_p \) and \( \omega_0 \) is the old \( \omega \), and
\[
\omega \to \omega_{g p} = \omega_{g 0} + \epsilon \omega_{g 1} + \epsilon^2 \omega_{g 2} + \epsilon^3 \omega_{g 3} + \ldots,
\]
(63)

where the new \( \omega g \) is renamed \( \omega_{g p} \) and \( \omega_{g 0} \) is the old \( \omega g \).

We shall build the complete differential equation for \( x^\mu \), taking together the harmonic oscillator equations for \( \delta r, \delta^2 r \) and \( \delta^3 r \):
\[
\frac{d^2}{ds^2} \left( \delta r + \frac{\delta^2 r}{2} + \frac{\delta^3 r}{6} \right) + \omega_0^2 \left( \delta r + \frac{\delta^2 r}{2} + \frac{\delta^3 r}{6} \right) = \frac{\Delta^3 r_0}{6} + \frac{\Delta^3 r_1}{6} \cos(\omega ps) + \frac{\Delta^2 r_2}{2} \cos(2\omega ps)
\]
We can choose the initial conditions of the differential solutions so that the constants $\delta \phi_0$ and $\delta t_0$ are null. The Appendix 2 lists the explicit values of the above coefficients.

Then, developing both sides into a series of powers of the parameter $\epsilon$, we can not only recover the former differential equations for the vectors $\delta x^\mu$, $\delta^2 x^\mu$, $\delta^3 x^\mu$, but get also some algebraic relations defining the corrections $\omega_1$, $\omega_2$, $\omega_{\theta 1}$ and $\omega_{\theta 2}$, see the Appendix 2. In the Schwarzschild limit, we have:

$$\omega_1 = 0, \quad \omega_2 = \frac{3M^3/2}{4R^{5/2}} \sqrt{1 - \frac{2M}{R}} (1 - \frac{6M}{R})^{3/2}. \quad (66)$$

so the new frequency corrected by the Poincaré’s method is simply:

$$\omega_p = \frac{\sqrt{M}}{R^{3/2}} \sqrt{1 - \frac{6M}{R}} - \frac{3(n_0^0)^2 M^{3/2}}{4R^9/2} \frac{(6 - \frac{37M}{R})}{\sqrt{1 - \frac{3M}{R}} (1 - \frac{6M}{R})^{3/2}}. \quad (67)$$

Finally, we can obtain that the first and second-order deviations are the same, but with the new $\omega_p$ and $\omega_{\theta p}$; and the third-order deviation $\delta^3 x^\mu$ is given by:

$$\delta^3 r = \delta^3 r_0 + \delta^3 r_3 \cos(3\omega_p s) + \delta^3 r_+ \cos[(\omega_p - 2\omega_{\theta p}) s] + \delta^3 r_- \cos[(\omega_p + 2\omega_{\theta p}) s], \quad (68)$$

$$\delta^3 \theta = \delta^3 \theta_3 \cos(3\omega_{\theta p} s) + \delta^3 \theta_+ \cos[(2\omega_p - \omega_{\theta p}) s] + \delta^3 \theta_- \cos[(2\omega_p + \omega_{\theta p}) s], \quad (69)$$

$$\delta^3 \phi = (\delta^3 \phi_0) s + \delta^3 \phi_1 \sin(\omega_p s) + \delta^3 \phi_3 \sin(3\omega_p s) + \delta^3 \phi_- \sin[(\omega_p - 2\omega_{\theta p}) s] + \delta^3 \phi_+ \sin[(\omega_p + 2\omega_{\theta p}) s], \quad (70)$$

$$\delta^3 t = (\delta^3 t_0) s + \delta^3 t_1 \sin(\omega_p s) + \delta^3 t_3 \sin(3\omega_p s) + \delta^3 t_- \sin[(\omega_p - 2\omega_{\theta p}) s] + \delta^3 t_+ \sin[(\omega_p + 2\omega_{\theta p}) s]. \quad (71)$$

We can choose the initial conditions of the differential solutions so that the constants $\delta^3 r_0$, $\delta^3 \phi_0$ and $\delta^3 t_0$ are null. The Appendix 2 lists the explicit values of the above coefficients.

The long expressions of the Kerr case are well simplified in the Schwarzschild limit:

$$\delta^3 \theta = 0, \quad \delta^3 r_- = \delta^3 r_+ = \delta^3 \phi_- = \delta^3 \phi_+ = \delta^3 t_- = \delta^3 t_+ = 0, \quad (72)$$

and the non-null coefficients are:

$$\delta^3 r_3 = \frac{-9(n_0^0)^3 (2 - \frac{28M}{R} + \frac{97M^2}{R^2})}{8R^2 (1 - \frac{6M}{R})^2}, \quad (73)$$

$$\delta^3 \phi_1 = \frac{9(n_0^0)^3}{R^3} \frac{1 - \frac{7M}{R}}{(1 - \frac{2M}{R})^{3/2}}, \quad \delta^3 \phi_3 = \frac{(n_0^0)^3 (26 - \frac{336M}{R} + \frac{1083M^2}{R^2})}{4R^3 (1 - \frac{6M}{R})^{3/2}}, \quad (74)$$

$$\delta^3 t_1 = \frac{3(n_0^0)^3}{R^2} \sqrt{\frac{M (2 - \frac{19M}{R} + \frac{40M^2}{R^2} - \frac{36M^3}{R^3})}{(1 - \frac{2M}{R})^3 (1 - \frac{6M}{R})^{3/2}}}, \quad (75)$$
\[ \delta^3 t_3 = \frac{(n_0^r)^3}{4R^3} \sqrt{\frac{M}{R} \left( 18 - 276\frac{M}{R} + 1339\frac{M^2}{R^2} - 2172\frac{M^3}{R^3} + 1164\frac{M^4}{R^4} \right)} \left( 1 - \frac{2M}{R} \right)^{5/2}. \]  

(76)

The same approach could be used in the second-order deviation calculations, but it is not necessary because there are no resonances in the second-order deviation equations.

Now we can compare the perihelion advance with the post-Newtonian limit, Eq. (1). The Schwarzschild limit gives the perihelion advance as

\[ \Delta \varphi = \left( \frac{6\pi M}{R} + \frac{27\pi M^2}{R^2} + \frac{135\pi M^3}{R^3} + \ldots \right) + \frac{(n_0^r)^2}{R^2} \left( \frac{9\pi M}{R} + \frac{159\pi M^2}{2R^2} \right) + \frac{585\pi M^3}{R^3} + \ldots \]  

(77)

which agrees with the \( \Delta \varphi \) of Eq. (1) after replacing the major semi-axis \( a \) by the value of Eq. (56):

\[ \Delta \varphi = \frac{6\pi M}{R} + \frac{9\pi e^2 M}{R} + \ldots \]  

(78)

The perihelion advance of \( \Delta \varphi \) in the Kerr case depends on \( M, R, a, n_0^r \) and \( n_0^\theta \), and can be described with high accuracy by a long explicit expression, not shown here.

### 7 Discussion

We have further developed a new method for calculating geodesics in a completely relativistic setting, using higher-order deviations without introducing the Newtonian or post-Newtonian approximations.

The computation of first, second and third-order deviations for the Kerr metric have shown that this method can be reduced to a straightforward iteration of solving linear systems of differential equations with constant coefficients.

The only complexity resides in the simplification of symbolic coefficients of the deviations, which is successfully performed by means of symbolic computing softwares \[19\] - \[20\].

It is interesting to observe how at the very first level of approximation the angular momentum density \( a \) of the central body influences the perihelion advance via two different effects, which are linear and quadratic in \( a \), respectively. The expressions linear in \( a \) depend on the sign of this parameter, i.e. on the relative sign of two rotations: that of the central body, and the direction of the orbital motion – a kind of spin-orbital coupling. This is the so-called dragging effect characteristic for General Relativity, which tends to raise the perihelion advance if the rotation of the planet is in the same direction as the rotation of the central body itself, and tends to decrease the perihelion advance if these two rotations are opposite to each other. The terms quadratic in \( a \) represent an additional perihelion advance which is due to the fact that the non-vanishing angular momentum of the central body is perceived from the exterior as an extra energy, which by the equivalence principle, may be considered as an extra mass \( \delta M \) added to the central mass \( M \); therefore, it always tends to produce higher perihelion advance.

It is worth stressing that these effects are absent in the first-order post-Newtonian approximation. In this sense, our method gives a shorter way enabling one to display certain effects, than the commonly used post-Newtonian approach. Its convergence properties are very good, too, so that there are physical situations when it is more appropriate. Consider a small mass rotating quite close to a black hole, so that the quantity \( GM/rc^2 \simeq v^2/c^2 \) is of the order of 0.1; then the second post-Newtonian effects are of the order 0.01. Now, if the eccentricity of the orbit is of the same
order, i.e. $e = 0.1$ then our third-order terms give the precision of 0.001, keeping the quasi exact functional dependence on physical parameters $GM/r$ and $a$.

There are many possible applications and further developments. The computation of fourth and higher-order deviations in Schwarzschild and Kerr metrics can improve the accuracy for practical calculations, and is just a matter of spending more time and computer resources because we have developed a semi-automatic program for explicit calculation of higher-order geodesic deviations.

The gravitomagnetic clock effect, i.e., the time-difference between the orbits of two freely counter-revolving test particles around a central rotating mass $M$, is an ideal target for high-order geodesic deviations, as the usual approaches are limited to circular orbits [22] or slowly rotating mass $M$ [21], i.e., small values of $a$. So the high-order geodesic deviations method has the potential to compute the gravitomagnetic clock effect in the case of strong general relativistic effects (large values of $M$ and $a$).

For practical applications involving the X-ray or gravitational radiation, the dynamics of accretion disks, etc, it would be useful to generalize our method to the case of initial orbits inclined w.r.t. the equatorial plane of the rotating central mass $M$. For example, the Ref. [23] considers inclined orbits in the Kerr metric, with the constraint of low-eccentricity orbits, i.e., up to the first-order deviation.

Still within the test particle concept, we can extend it for test bodies carrying charge and/or internal spin, see Ref. [24] where the first-order geodesic deviation is derived in the Reissner-Nordstrøm background field. We foresee that higher-order geodesic deviations for test particles with spin can provide useful results to compare with the experimental data of satellite gyroscopes.

We can also replace the background metric by some cylindrical or axially-symmetric metric to investigate approximated models of star and galaxy orbits, accretion disks, etc.

Almost all the work available in the domain of gravitational radiation is based on post-Newtonian approximations [25]-[29]. Within the higher-order geodesic deviations approach, one possibility is to maintain the test particle mass $m$ negligible compared to $M$, and compute the emission of gravitational radiation [30] with some formula better suited for Schwarzschild and Kerr metrics than the quadrupole formula [31]. Another possibility, more challenging, is to cope with the finite-size of the mass $m$ by taking it into account with appropriate perturbation of the background metric, then trying to repeat the higher-order geodesic deviations calculations and finally employing a modified gravitational radiation formula based on the perturbed metric.

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Appendix 1

The coefficients of the solution, in the Kerr metric case, for the second-order geodesic deviation $\delta^2 x^{\mu}(s)$ are:

$$\delta^2 r_0 = 0, \quad \delta^2 r_2 = -\frac{(n_0^r)^2}{R f_6} \left( 1 - \frac{7M}{R} + \frac{10a\sqrt{M}}{R^{3/2}} - \frac{4a^2}{R^2} \right),$$

$$\delta^2 \theta_- = -\delta^2 \theta_+ = \frac{2n_0^r n_0^\theta}{R} \frac{\sqrt{f_3}}{\sqrt{f_6}},$$

(79) (80)
\[ \delta^2 \phi_{2r} = -\frac{2(n_0^r)^2}{R^2 f_6^{3/2} f_2^2} \left[ \left( 5 - \frac{32M}{R} \right) \left( 1 - \frac{2M}{R} \right)^2 - \frac{2a\sqrt{M}}{R^{3/2}} \left( 26 - \frac{119M}{R} + \frac{126M^2}{R^2} \right) \right. \\
+ \frac{2a^2}{R^2} \left( 8 - \frac{61M}{R} + \frac{66M^2}{R^2} \right) + \frac{2a^3 \sqrt{M}}{R^{7/2}} \left( 5 + \frac{21M}{R} \right) + \frac{a^4}{R^4} \left( 5 - \frac{58M}{R} \right) + \frac{14a^5 \sqrt{M}}{R^{11/2}} \right], \]

\[ \delta^2 \phi_0 = 0, \quad \delta^2 \phi_{2\theta} = \left( \frac{n_0^\theta}{\sqrt{f_3}} \right)^2 \left( 1 - \frac{2a\sqrt{M}}{R^{3/2}} \right), \]

\[ \delta^2 t_0 = -\frac{3}{\sqrt{f_1}} \left[ \frac{(n_0^r)^2}{R^2} \left( 1 + \frac{M}{R} - \frac{2a^2}{R^2} \right) + \left( \frac{a\sqrt{M}}{R^{3/2}} - \frac{a^2}{R^2} \right) \right] + \frac{(n_0^\theta)^2}{R^2} \left( 1 - \frac{2a\sqrt{M}}{R^{3/2}} \right), \]

\[ \delta^2 t_{2r} = \frac{(n_0^r)^2 \sqrt{M}}{R^{3/2} f_6^{3/2} f_2^2} \left[ \left( 2 - \frac{15M}{R} + \frac{14M^2}{R^2} \right) + \frac{a\sqrt{M}}{R^{3/2}} \left( 11 + \frac{34M}{R} - \frac{64M^2}{R^2} \right) \right. \\
- \frac{3a^2}{R^2} \left( 1 + \frac{28M}{R} - \frac{42M^2}{R^2} \right) + \frac{2a^3 \sqrt{M}}{R^{7/2}} \left( 28 - \frac{33M}{R} \right) - \frac{3a^4}{R^4} \left( 4 + \frac{7M}{R} \right) + \frac{29a^5 \sqrt{M}}{R^{11/2}} - \frac{27a^6}{R^6} \left], \]

\[ \delta^2 t_{2\theta} = \left( \frac{n_0^\theta}{\sqrt{f_3}} \right)^2 \frac{a^2 \sqrt{M}}{R^{3/2} \sqrt{f_3}}, \]  

**Appendix 2**

With the third-order deviation in the Kerr metric, we can determine the values of:

\[ \omega_1 = \omega_{\theta_1} = 0, \]

\[ \omega_2 = -\frac{3 \left( \frac{n_0^r}{\sqrt{R}} \right)^2 a \sqrt{M} \left( \frac{\sqrt{M}}{\sqrt{R}} - \frac{a}{R} \right) f_2}{(n_0^r)^2 \sqrt{R}} - \frac{3 \sqrt{M} \left( \frac{\sqrt{M}}{\sqrt{R}} - \frac{a}{R} \right)^2}{4 R^{3/2} \sqrt{f_1 f_6^{3/2}}} \left[ \left( 1 - \frac{2M}{R} \right) \left( 6 - \frac{37M}{R} \right) \right. \\
+ \frac{2a \sqrt{M}}{R^{3/2}} \left( 27 - \frac{62M}{R} \right) - \frac{a^2}{R^2} \left( 17 - \frac{57M}{R} \right) + \frac{6a^3 \sqrt{M}}{R^{7/2}} \frac{7a^4}{R^4}, \]

\[ \omega_{\theta_2} = -\frac{3a \sqrt{M}}{2 R^{3/2}} \left( \frac{\sqrt{M}}{\sqrt{R}} - \frac{a}{R} \right) \left( 3 - \frac{3M}{R} + \frac{2a^2}{R^2} \right) - \frac{3 \left( \frac{n_0^r}{\sqrt{R}} \right)^2 a^2 \sqrt{M} \left( 1 - \frac{4M}{R} + \frac{4a \sqrt{M}}{R^{3/2}} - \frac{a^2}{R^2} \right)}{4 (n_0^r)^2 R^{3/2} \sqrt{f_1 \sqrt{f_3}}}, \]

therefore the new frequencies corrected by the Poincaré's method, which are exact up to the second order w.r.t. the small parameter \( \epsilon = \frac{n_0^r}{\sqrt{R}} \), are given by:

\[ \omega_p = \omega_0 + \frac{(n_0^r)^2}{R^2} \omega_2, \quad \omega_{\theta_2} = \omega_{\theta_0} + \frac{(n_0^r)^2}{R^2} \omega_{\theta_2}. \]

The coefficients of the solution, in the Kerr metric case, for the third-order geodesic deviation \( \delta^3 x^i(s) \) are:

\[ \delta^3 r_0 = 0, \quad \delta^3 r_3 = -\frac{9 (n_0^r)^3}{8 R^2 f_6^{3/2} f_2^2} \left[ \left( 2 - \frac{28M}{R} + \frac{97M^2}{R^2} \right) + \frac{4a \sqrt{M}}{R^{3/2}} \left( 10 - \frac{69M}{R} \right) \right. \\
- \frac{2a^2}{R^2} \left( 8 - \frac{153M}{R} \right) - \frac{156a^3 \sqrt{M}}{R^{7/2}} + \frac{31a^4}{R^4}], \]
\begin{align}
\delta^3 r_- &= \frac{3n_0}{4R^2} \left( \frac{n_0^2}{R} \right)^2 a^2 \sqrt[3]{R} \left( f_6 + \sqrt{f_3} \sqrt{f_6} \right), \\
\delta^3 r_+ &= \frac{3n_0^*}{4R^2} \left( \frac{n_0^2}{R} \right)^2 a^2 \left( f_6 + \sqrt{f_3} \sqrt{f_6} \right), \\
\delta^3 \theta_3 &= \frac{(n_0^2)^3}{8f_3} \left[ 2 - 8a\sqrt{M} R^{3/2} - 3a^2 R^2 \left( 1 - 4M \frac{R}{R} \right) + \frac{12a^3 \sqrt{M}}{R^{7/2}} - \frac{15a^4}{R^4} \right], \\
\delta^3 \theta_- &= -\delta^3 \theta_+ = \frac{3(n_0^2)^2 n_0^2}{4R^2} \left\{ \frac{4f_3}{f_6} - \frac{\sqrt{f_3}}{f_6^{3/2}} \left[ \left( 5 - 32M \frac{R}{R} \right) + \frac{44a\sqrt{M}}{R^{3/2}} - \frac{17a^2}{R^2} \right] \right\}, \\
\delta^3 \phi_1 &= \frac{(n_0^2)^3}{R^3 f_6^{3/2} f_2^{3/2}} \left[ 3 \left( 1 - 2M \frac{R}{R} \right)^3 \left( 1 - 7M \frac{R}{R} \right) + \frac{a\sqrt{M}}{R^{3/2}} \left( 39 - \frac{269M}{R} + \frac{580M^2}{R^2} - \frac{412M^3}{R^3} \right) \right. \\
&\quad - \frac{a^2}{R^2} \left( 14 - \frac{169M}{R} + \frac{424M^2}{R^2} - \frac{332M^3}{R^3} \right) - \frac{a^3 \sqrt{M}}{R^{7/2}} \left( 41 - \frac{66M}{R} + \frac{12M^2}{R^2} \right) \\
&\quad + \frac{a^4}{R^4} \left( 3 + \frac{63M}{R} - \frac{134M^2}{R^2} \right) - \frac{a^5 \sqrt{M}}{R^{11/2}} \left( 31 - \frac{79M}{R} \right) + \frac{a^6}{R^6} \left( 4 - \frac{17M}{R} \right) + \frac{a^7 \sqrt{M}}{R^{15/2}} \left\}, \\
\delta^3 \phi_3 &= \frac{(n_0^2)^3}{4R^3 f_6^{3/2} f_2^{3/2}} \left[ \left( 1 - 2M \frac{R}{R} \right)^3 \left( 26 - \frac{336M}{R} + \frac{1083M^2}{R^2} \right) + \frac{a\sqrt{M}}{R^{3/2}} \left( 518 - \frac{6624M}{R} + \frac{27939M^2}{R^2} \right) \right. \\
&\quad - \frac{48140M^3}{R^5} - \frac{29484M^4}{R^4} - \frac{2a^2}{R^2} \left( 84 - \frac{2479M}{R} + \frac{14109M^2}{R^2} - \frac{27456M^3}{R^3} + \frac{17604M^4}{R^4} \right) \\
&\quad - \frac{2a^3 \sqrt{M}}{R^{7/2}} \left( 822 - \frac{6251M}{R} + \frac{10737M^2}{R^2} - \frac{4422M^3}{R^3} \right) + \frac{3a^4}{R^4} \left( 69 - \frac{410M}{R} - \frac{2971M^2}{R^2} \right) \\
&\quad + \frac{5750M^3}{R^5} - \frac{a^5 \sqrt{M}}{R^{11/2}} \left( 777 - \frac{11664M}{R} + \frac{15625M^2}{R^2} \right) + \frac{2a^6}{R^6} \left( 95 - \frac{1965M}{R} + \frac{1176M^2}{R^2} \right) \\
&\quad + \frac{6a^7 \sqrt{M}}{R^{15/2}} \left( 55 + \frac{463M}{R} \right) + \frac{a^8}{R^8} \left( 45 - \frac{1474M}{R} \right) + \frac{225a^9 \sqrt{M}}{R^{19/2}} \right\], \\
\delta^3 \phi_0 &= 0, \quad \delta^3 \phi_- = \frac{3n_0}{2R} \left( \frac{n_0^2}{R} \right)^2 \left[ \left( 2 - 4a\sqrt{M} \frac{R}{R^2} \right)^2 \frac{a^2 f_4}{f_2 \sqrt{f_3}} \right], \\
\delta^3 \phi_+ &= \frac{3n_0^*}{2R} \left( \frac{n_0^2}{R} \right)^2 \left[ \frac{a^2 f_4}{f_2 \sqrt{f_3}} \right], \\
\delta^3 t_1 &= \frac{3(n_0^2)^3 \sqrt{M}}{R^{5/2} f_6^{3/2} f_2^{3/2}} \left[ \left( 2 - \frac{19M}{R} + \frac{40M^2}{R^2} - \frac{36M^3}{R^3} \right) + \frac{a\sqrt{M}}{R^{3/2}} \left( 13 + \frac{26M}{R} - \frac{140M^2}{R^2} + \frac{168M^3}{R^3} \right) \right. \\
&\quad - \frac{a^2}{R^2} \left( 5 + \frac{103M}{R} - \frac{396M^2}{R^2} + \frac{412M^3}{R^3} \right) + \frac{a^3 \sqrt{M}}{R^{7/2}} \left( 73 - \frac{332M}{R} + \frac{332M^2}{R^2} \right) - \frac{a^4}{R^4} \left( 15 - \frac{91M}{R} \right) \\
&\quad + \frac{12M^2}{R^2} + \frac{a^5 \sqrt{M}}{R^{11/2}} \left( 11 - \frac{134M}{R} \right) - \frac{a^6}{R^6} \left( 7 - \frac{79M}{R} \right) - \frac{17a^7 \sqrt{M}}{R^{15/2}} + \frac{a^8}{R^8} \right]\right] \right],
\end{align}
\[
\delta^3 t_3 = \left( \frac{n_0^3}{R^3} \right)^{\sqrt{f}} \left( \frac{n_0^3}{R^3} \right)^{\sqrt{f}} \left( \frac{n_0^3}{R^3} \right)^{\sqrt{f}} \left[ \left( 18 - \frac{276M^2}{R^2} + \frac{1339M^4}{R^4} - \frac{2172M^3}{R^3} + \frac{1164M^4}{R^4} \right) + \frac{2a\sqrt{M}}{R^{3/2}} \left( 120 - \frac{762M}{R} \right) \right.
\]
\[- \frac{755M^2}{R^2} + \frac{5124M^3}{R^3} - \frac{4324M^4}{R^4} \) \right] - \frac{a^2}{R^2} \left( 74 - \frac{210M^2}{R^2} + \frac{1385M^4}{R^4} + \frac{38736M^3}{R^3} - \frac{29484M^4}{R^4} \right) \]
\[+ \frac{4a^3\sqrt{M}}{R^{7/2}} \left( \frac{85 - \frac{4404M}{R^2} + \frac{12069M^2}{R^2} - \frac{8802M^3}{R^3} + \frac{105}{R^3} \right) - \frac{10426M}{R} + \frac{21699M^2}{R^2} \]
\[\left. - \frac{8844M^3}{R^3} \right] - \frac{2a^5\sqrt{M}}{R^{11/2}} \left( 1461 + \frac{2902M}{R^2} - \frac{8625M^2}{R^3} + \frac{105}{R^3} \right) + \frac{a^6}{R^6} \left( 309 + \frac{1058M}{R} - \frac{15625M^2}{R^2} \right) \]
\[\left. - \frac{4a^7\sqrt{M}}{R^{15/2}} \left( 83 - \frac{49M}{R} \right) + \frac{a^8}{R^8} \left( 547 + \frac{2778M}{R} \right) - \frac{1474a^9\sqrt{M}}{R^{19/2}} + \frac{225a^{10}}{R^{11/2}} \right] ,
\]
\[
\delta^3 t_0 = 0, \quad \delta^3 t_- = \frac{3n_0^3}{2R^{5/2}} \left[ \frac{-4}{\sqrt{f_6}} + \frac{f_5}{f_2\sqrt{f_3}} \right],
\]
\[
\delta^3 t_+ = \frac{3n_0^3}{2R^{5/2}} \left[ \frac{-4}{\sqrt{f_6}} - \frac{f_5}{f_2\sqrt{f_3}} \right].
\]

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