Reductions of self-dual Einstein equations

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Reductions of self-dual Einstein equations

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Abstract. Transformations between the Plebański equation and other reductions of the self-dual Einstein equations are given. A new reduction is presented. It consists of two equations describing canonical transformations in a 2-dimensional phase space. Symmetries and examples of linearizations of these equations are given.

1. Introduction
Metrics giving rise to the self-dual Riemann tensor were first considered in 1935 in the framework of so-called wave geometry [1, 2]. The self-duality conditions were reduced to an equation known now as the first Plebański equation [3]. A concept similar to that of Penrose’s nonlinear graviton [4] was proposed. These results were unnoticed by relativists who became interested in self-dual metrics in 1970’s [3, 4, 5, 6].

An application of Ashtekar’s variables [7, 8] allowed to find several equations equivalent to the Plebański equations [9, 10, 11]. The Husain-Park equation is similar to the completely integrable chiral equation and the Grant equation is especially convenient in studying the initial value problem. These different versions of the self-duality conditions can play an important role in a description of completely integrable systems [12, 13, 14].

In this communication we shortly present relations between the first Plebański equation and other formulations and we propose a new approach leading to a pair of equations describing canonical transformations in classical mechanics.

2. Equivalent reductions of SDE
In a preferred system of coordinates

\[ q, \ p, \ \tilde{q}, \ \tilde{p} \] (1)

complex solutions of the self-dual Einstein equations (SDE)

\[ R_{\alpha\beta\gamma\delta} \sim R^{\mu}_\alpha \epsilon^{\mu\nu\gamma\delta} \] (2)

are given by solutions \( \Omega \) of the first Plebański equation [3]

\[ \Omega_{\tilde{q}q} \Omega_{\tilde{p}p} - \Omega_{\tilde{q}p} \Omega_{\tilde{p}q} = 1. \] (3)

The corresponding self-dual metric reads

\[ g = \Omega_{\tilde{q}q}dq\tilde{q} + \Omega_{\tilde{p}p}dp\tilde{p} + \Omega_{\tilde{q}p}dq\tilde{p} + \Omega_{\tilde{p}q}dp\tilde{q}. \] (4)
Equation (3) is equivalent to another reduction of SDE equation known as the second Plebański equation [3]. Let Ω satisfies (3). Consider a new system of coordinates \( p, q, x, y \) such that
\[
x = \Omega_{p}, \quad y = \Omega_{q}
\] (5)
and define a function \( \theta \) as a solution of the following differential equations
\[
\theta_{xx} = \Omega_{qq}, \quad \theta_{xy} = -\Omega_{qp}, \quad \theta_{yy} = \Omega_{pp}
\] (6)
(these equations admit solutions due to (3)). Then \( \theta \) has to satisfy the second Plebański equation
\[
\theta_{xp} + \theta_{yp} + \theta_{xx}\theta_{yy} - \theta_{xy}\theta_{xy} = 0.
\] (7)
Equation (7) is equivalent to (3) and metric (4) can be written exclusively in terms of \( \theta \) and coordinates \( p, q, x, y \).

Now, instead of (1) let us consider coordinates \( t, q, \tilde{q}, p, \tilde{p} \), where \( t \) is given by
\[
t = \Omega_{p}.
\] (8)
If we perform a partial Legendre transformation
\[
\chi = \Omega - p\Omega_{p}
\] (9)
then equation (3) converts into the Grant equation [11] (modulo a gauge transformation)
\[
\chi_{tt} + \chi_{qp}\chi_{\tilde{q}\tilde{p}} - \chi_{q\tilde{q}}\chi_{\tilde{p}p} = 0.
\] (10)
Since equation (10) can be solved with respect \( \chi_{tt} \) it is suitable for studying evolutionary problems.

Another version of the self-dual Einstein equations is the Husain-Park equation [9, 10]
\[
\Lambda_{p'p'} + \Lambda_{q'q'} + \Lambda_{q'\tilde{q}'}\Lambda_{p'\tilde{p}'} - \Lambda_{q'\tilde{q}'}\Lambda_{p'\tilde{p}'} = 0
\] (11)
Here space-time coordinates are denoted by \( q', p', \tilde{q}', \tilde{p}' \). In terms of the potentials \( A_{q'} = -\Lambda_{p'}, A_{p'} = \Lambda_{q'} \) equation (11) takes the form
\[
\partial_{q'} A_{p'} - \partial_{p'} A_{q'} + \{A_{p'}, A_{q'}\} = 0
\] (12)
where \( \{ \} \) denotes the Poisson bracket with respect to \( \tilde{p}' \) and \( \tilde{q}' \). Equations (12) are analogous to the chirial equations (see e.g. [12]). Due to this an infinite number of conservation laws was deduced [10, 15].

Equation (11) can be obtained from (3) by the following transformation [15]. Let
\[
p' = \tilde{p}, \quad q' = \tilde{q}
\] (13)
and define new coordinates \( q', p' \) as solutions of the linear equations
\[
q'_{p'} = -U(p'), \quad p'_{p'} = V(q')
\] (14)
\[
q'_{q'} = V(p'), \quad p'_{q'} = -U(q')
\] (15)
where \( V \) and \( U \) are the following differential operators
\[
V = \Omega_{p\tilde{q}} \partial_{q} - \Omega_{q\tilde{q}} \partial_{p}, \quad U = -\Omega_{pp} \partial_{q} + \Omega_{qq} \partial_{p}.
\] (16)
Equations (14), (15) are integrable due to the Plebański equation (3). Now define a function \( \Lambda \) as a solution of the following integrable linear system
\[
\Lambda_{p'} = \Omega_{p}, \quad \Lambda_{q'} = \Omega_{q}.
\] (17)
Modulo a gauge transformation the function \( \Lambda \) must satisfy the Husain-Park equation [10, 9] and the transformation given by (13)-(17) is reversible.
3. New version of SDE

It was shown in [16] that the Plebański equation (3) is equivalent to the following pair of equations for functions $u$ and $v$

\begin{align}
\{u, v\}_{qp} &= 1 \\
\{u, v\}_{q\bar{q}} &= 1.
\end{align}

Equations (18), (19) are preserved provided $\{u, v\}_{q\bar{q}}$ are related to $\Omega$ via equations

\begin{align}
\{u, v\}_{q\bar{q}} &= \Omega_{q\bar{q}} \\
\{u, v\}_{p\bar{q}} &= \Omega_{p\bar{q}}.
\end{align}

It follows from (18), (19) that $u$ and $v$ are related to $q, p$ and to $\bar{q}, \bar{p}$ by canonical transformations of classical mechanics. In terms of a generating function $S(v, q, \bar{q}, \bar{p})$, where $\bar{q}, \bar{p}$ appear as parameters, solutions $u, v$ of (18) are given implicitly by

\begin{align}
p &= S_q, \quad u = S_v.
\end{align}

Similarly, equation (19) can be solved by means of a generating function $\tilde{S}(v, q, \bar{q}, \bar{p})$,

\begin{align}
\tilde{p} &= \tilde{S}_{\bar{q}}, \quad u = \tilde{S}_v.
\end{align}

Functions $S$ and $\tilde{S}$ should be correlated in such a way that solutions $u, v$ obtained from (23) or (24) coincide. Below we give examples of such correlations [16].

Assume that

\begin{align}
S &= F_q + \tilde{p}q(v + q),
\end{align}

where function $F(v, q, \bar{q})$ is subject to the linear equation

\begin{align}
F_{vq} + F_{v\bar{q}} - F_{q\bar{q}} &= 0.
\end{align}

In this case the corresponding metric (21) possesses the translational Killing vector $\partial_p - \partial_{\bar{p}}$. For this reason it must belong to the class of generalized Hawking metrics [5].

Another reductions to linear equations can be obtained when

\begin{align}
S &= F_v - e^q F_{\bar{q}} + \tilde{p}(q\bar{q} + e^q v + f(q))
\end{align}

or

\begin{align}
S &= F_{\bar{q}} + \tilde{p}(v\bar{q} + e^{-v} q + f(v)),
\end{align}

Given a solution of (18), (19) the corresponding self-dual metric reads

\begin{align}
g = \{u, v\}_{qq} dq d\bar{q} + \{u, v\}_{pq} dp d\bar{p} + \{u, v\}_{q\bar{q}} dq p d\bar{p} + \{u, v\}_{v\bar{p}} dq d\bar{p}.
\end{align}

Equations (18), (19) admit continuous point symmetries. Let new variables $u', v'$ be functions of $u$ and $v$, new coordinates $\bar{q}', \bar{p}'$ be functions of $q$ and $p$ and $\bar{q}', \bar{p}'$ be functions of $\bar{q}$ and $\bar{p}$. Equations (18), (19) are preserved provided

\begin{align}
\{u', v'\}_{uv} = \{\bar{q}', \bar{p}'\}_{qq} = \{\bar{q}', \bar{p}'\}_{p\bar{q}} = c,
\end{align}

where $c$ is a nonvanishing constant. These transformations form the maximal group $G$ of point symmetries of equations (18), (19) modulo discrete symmetries [17]. Assuming invariance of a solution with respect to one or two-dimensional subgroup of $G$ one can obtain new self-dual metrics [17].

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where \( f \) is an arbitrary function of one variable. In these cases we obtain the following linear equations for \( F \), respectively,

\[
F_{,vq} + (\tilde{q} + e^q v + f_q)F_{,vq} - e^q F_{,\tilde{q}q} - e^q F_{,\tilde{q}q} = 0, \tag{29}
\]

\[
e^{-v} F_{,v\tilde{q}} + (e^{-v} q - \tilde{q} - f_{,v})F_{,v\tilde{q}} + F_{,vq} = 0. \tag{30}
\]

Given a solution \( F \) of equation (29) or (30) one can find functions \( u, v \) from relations (23) and then construct the corresponding self-dual metric (21).

4. Conclusions

We have shortly reviewed different reductions of the self-dual Einstein equations to a single second order equation of the Monge-Ampere type (the first and the second Plebański equation, the Grant equation, the Husain-Park equation). We have presented a new reduction equivalent to a pair of equations (18), (19) describing canonical transformation in classical mechanics. These equations admit an infinite dimensional group of symmetries which can be used to obtain symmetric solutions. Under some ansatz the equations reduce to linear equations (26), (29), (30). We hope that this new formulation of the self-dual Einstein equations can be useful in the theory of completely integrable systems.

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