Entanglement distance for an arbitrary state of $M$ qubits

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We propose a measure of entanglement that can be computed for any pure state of an $M$-qubit system. The entanglement measure has the form of a distance that we derive from an adapted application of the Fubini-Study metric. This measure is invariant under local unitary transformations and defined as trace of a suitable metric that we derive, the entanglement metric $\tilde{g}$. Furthermore, the analysis of the eigenvalues of $\tilde{g}$ gives information about the robustness of entanglement.

INTRODUCTION

Entanglement is an essential resource for progressing in the field of quantum-based technologies. Quantum information has confirmed its importance in quantum cryptography and computation, in teleportation, in the frequency standard improvement problem and metrology based on quantum phase estimation [1]. The rapid experimental progress on quantum control is driving the interest in entanglement theory. Nevertheless, despite its key role, entanglement remains elusive and the problem of its characterisation and quantification is still open [2, 3]. We propose: i) is invariant under local unitary transformations; ii) has the structure of a distance such that the action of $SU(2)$ local unitary operators $U^j$ ($j = 0, \ldots, M - 1$) on a given state $|s\rangle$, generates a class of states

$$|U, s\rangle = \prod_{j=0}^{M-1} U^j |s\rangle$$

that share the same degree of entanglement. For each $j$, $U^j$ operates on the $j$th qubit. We define an infinitesimal variation of state (2) as

$$|dU, s\rangle = \sum_{j=0}^{M-1} dU^j |U, s\rangle,$$

where

$$dU^j = -i (n^j \cdot \sigma^j) d\xi^j / 2$$

rotates the $j$th qubit by an infinitesimal angle $d\xi^j$ around the unitary vector $n^j$. We denote by $\sigma^j_1$, $\sigma^j_2$ and $\sigma^j_3$ the Pauli matrices operating on the $j$-th qubit ($j = 0, \ldots, M - 1$) where the index $j$ numerates the spins from right to left. From Eq. (1), with this choice, we get the following expression for the Fubini-Study metric $g$

$$g_{\mu\nu}(v^\nu) d\xi^\mu d\xi^\nu = \frac{1}{4} \left( |s|(v^\mu \cdot \sigma^\mu)(v^\nu \cdot \sigma^\nu)|s\rangle + \langle s|(v^\mu \cdot \sigma^\mu)|s\rangle (s|(v^\nu \cdot \sigma^\nu)|s\rangle) d\xi^\mu d\xi^\nu. \right.$$ (5)

The unitary vectors $v^\nu$ in the latter equation are derived by a rotation of the original ones according to

$$v^\nu \cdot \sigma^\nu = U^{+t} n^\nu \cdot \sigma^\nu U^\nu,$$

where there is no summation on the index $\nu$. The proposed entanglement measure of the state $|s\rangle$ is

$$E(|s\rangle) = \inf_{\mathcal{M}} \text{tr}(g),$$

where

$$\mathcal{M} = \{v^\nu\}_\nu.$$
where $\text{tr}$ is the trace operator and where the inf is taken “in measure” over all the possible orientations of the unitary vectors $\psi^{\nu}$. With the term “in measure”, we mean that possible pathologies, similar to the one of the Dirichlet function, are eliminated. The inf operation, makes the measure (7) independent from the operators $U_j$, hence, its numerical value is associated to the class of states generated by local unitary transformations and not to the specific element chosen inside the class. This is a necessary condition for a good entanglement measure definition. The unitary vectors $\tilde{\psi}^{\nu}$ corresponding to the inf of $\text{tr}(g)$, identify a metric

$$\tilde{g} = g(\tilde{\psi}^{\nu})$$ (8)

that we name entanglement metric (EM). The off-diagonal elements of $\tilde{g}$ provide the quantum correlations between qubits. In addition, states that differ another for local unitary transformations, have the same form of $\tilde{g}$. In this way, the expression of EM identifies the classes of equivalence. Remarkably, the analysis of the eigenvalues and eigenvectors of $\tilde{g}$ allows one to check the existence of states with super-Heisenberg sensitivity, i.e. beyond Heisenberg limit.

**EXAMPLES**

In order to verify the efficacy of the proposed entanglement measure, we have first considered two families of one-parameter states depending on a real parameter. The degree of entanglement of each state depends on this parameter and the configuration corresponding to the maximally entangled states for each of the families is known. The first family of states we consider has been introduced by Briegel and Raussendorf in Ref. [5]. For this reason, we will name the elements in this family Briegel-Raussendorf states (BRS). The second family of states is related to the Greenberger-Horne-Zeilinger states [6]. We will name the elements of such family Greenberger-Horne-Zeilinger–like states (GHZLS). It is worth emphasizing that in Ref. [5] it has been shown that the maximally entangled states of these two families are not equivalent if $M \geq 4$, whereas they are equivalent if $M \leq 3$. This fact offers us a further test for our approach to entanglement estimation. In fact, we have found that i) the entanglement measure (7) provides the same value for the maximally entangled states of both the families; ii) in the case $M \leq 3$ the entanglement metric (8) has the same form for the maximally entangled states of the two families, whereas, if $M \geq 4$, the EMs of the maximally entangled states of the two families are not equivalent.

The last case we have considered is a family of three-qubit states depending on two real parameters. With a suitable choice of these parameters, the state can be fully separable or bi-separable, whereas in the generic case it is a genuine tripartite entangled state. We will show that the proposed entanglement measure provides an accurate description of all these cases.

**Briegel Raussendorf states**

We denote with $\Pi_{0}^{j} = (I + \sigma_{3}^{j})/2$ and $\Pi_{1}^{j} = (I - \sigma_{3}^{j})/2$ the projector operators onto the eigenstates of $\sigma_{3}^{j}, |0\rangle_{j}$ (with eigenvalue +1) and $|1\rangle_{j}$ (with eigenvalue −1), respectively. Each $M$ qubit state of the BRS class is derived by applying to the fully separable state

$$|r, 0\rangle = \bigotimes_{j=0}^{M-1} \frac{1}{\sqrt{2}}(|0\rangle_{j} + |1\rangle_{j}),$$ (9)

the non local unitary operator

$$U_{0}(\phi) = \exp(-i\phi H_{0}) = \prod_{j=1}^{M-1} \left( I + \alpha \Pi_{0}^{j} \Pi_{1}^{j+1} \right),$$ (10)

where $H_{0} = \sum_{j=1}^{M-1} \Pi_{0}^{j} \Pi_{1}^{j+1}$ and $\alpha = (e^{-i\phi} - 1)$.

The full operator (10) is diagonal on the states of the standard basis \{0\ldots0, 0\ldots01, \ldots, 1\ldots1\}. In fact, each vector of the latter basis is identified by $M$ integers $0_{0}, \ldots, n_{M-1} = 0, 1$ as \{n\} = $|n_{M-1} \ldots n_{M-2} \ldots n_{0}\rangle$ and we can enumerate such vectors according to the binary integers representation \{k\} = \{|n^{k}\rangle\}, with $k = \sum_{j=0}^{M-1} n_{j}2^{j}$, where $n_{k}$ is the $\nu$-th digit of the number $k$ in binary representation and $k = 0, \ldots, 2^{M} - 1$. Then, the eigenvalue $\lambda_{k}$ of operator (10), corresponding to a given eigenstate $|k\rangle$ of this basis, results

$$\lambda_{k} = \sum_{j=0}^{\frac{n(k)}{2}} \left( \frac{n(k)}{j} \right) \alpha^{j},$$ (11)

where $n(k)$ is the number ordered couples 01 inside the sequence of the base vector $|k\rangle$. For the initial state (9) we consistently get

$$|r, 0\rangle_{M} = 2^{-M/2} \sum_{k=0}^{2^{M-1}} |k\rangle,$$ (12)

and, under the action of $U_{0}(\phi)$ one obtains

$$|r, \phi\rangle_{M} = 2^{-M/2} \sum_{k=0}^{2^{M-1}} \sum_{j=0}^{n(k)} \left( \frac{n(k)}{j} \right) \alpha^{j} |k\rangle.$$ (13)

For $\phi = 2\pi k$, with $k \in \mathbb{Z}$, this state is separable, whereas, for all the other choices of the value $\phi$, it is entangled. In particular, in [5] it is argued that the values $\phi = (2k+1)\pi$, where $k \in \mathbb{Z}$, give the maximally entangled states.
Fubini-Study metric for the Briegel Raussendorf states $M = 2,3$

In the case of two-qubit BRS the trace of the Fubini-Study metric is

$$
\text{tr}(g) = \frac{1}{4} \left[ 1 - c^2 \left( (c v_1^\nu + (-1)^{\nu+1} s v_2^\nu)^2 \right) \right],
$$

where $c = \cos(\phi/2)$ and $s = \sin(\phi/2)$. (14) is minimised with the choice $\tilde{\nu}_k^\nu = (c, (-1)^{\nu+1}s, 0)$. Consistently, the EM results in

$$
\bar{g} = \frac{1}{4} \left( \frac{s^2 1}{1 s^2} \right)
$$

and

$$
E(|r, \phi\rangle) = \frac{s^2}{2}.
$$

We have already mentioned that in the case $M = 2,3$, the maximally-entangled BRS $|r, 2\pi k + \pi\rangle$, where $k \in \mathbb{Z}$, and the maximally entangled GHZLS are equivalent because differing just for local unitary transformations. In the following, we will show that the EM for these states have the same forms in the case $M = 2,3$ in accordance to the results of Ref. [5]. In the case $M = 3$ and $\phi \neq (2k+1)\pi$, with $k \in \mathbb{Z}$, the trace of $g$,

$$
\text{tr}(g) = \frac{1}{4} \left[ 3 - c^2 \left( (c v_0^\nu + v_1^\nu + v_2^\nu) + s(v_2^\nu - v_0^\nu)^2 \right) \right],
$$

is minimised with the choices $\tilde{\nu}_0^\nu = (c, -s, 0)$, $\tilde{\nu}_1^\nu = (1, 0, 0)$ and $\tilde{\nu}_2^\nu = (c, s, 0)$. The EM and the entanglement measure in this case results to be

$$
\bar{g} = \frac{s^2}{4} \left( \frac{1}{c} \frac{c}{1 + s^2} \frac{-2s^2 c^2}{c} \right)
$$

and

$$
E(|r, \phi\rangle) = \frac{s^2}{4} (3 + c^2),
$$

respectively. By direct calculation, one can verify that in the case of the maximally entangled BRS ($M = 3$), the choice $\nu_0^\nu = (-1, 0, 0)$, $\nu_1^\nu = (0, 0, 1)$ and $\nu_2^\nu = (1, 0, 0)$ makes the EM equivalent to the one of the three-qubit Greenberger-Horne-Zeilinger state. This agrees with the results of Ref. [5].

Fubini-Study metric for the Briegel Raussendorf states $M > 3$

In the general case, the trace of $g$ results

$$
\text{tr}(g) = \frac{1}{4} \left\{ M - \sum_{\nu=0}^{M-1} \nu_3^\nu \nu_3^\nu + \nu_4^\nu \nu_4^\nu + \nu_5^\nu \nu_5^\nu \right\}.
$$

where $v_\nu^\nu = v_1^\nu \pm iv_2^\nu$, $c_k = 2^{-M/2}\lambda_k$, and

$$
\begin{align*}
\nu_3^\nu &= \sum_{k=0}^{M-1} \delta \nu_4^0 \nu_5^c + 2c_k \\
\nu_4^\nu &+ \nu_5^\nu = \sum_{k=0}^{M-1} \delta \nu_5^0 \nu_4^c - 2c_k \\
\nu_5^\nu &+ \nu_3^\nu = \sum_{k=0}^{M-1} (-1)^{\nu_5^c} |c_k|^2
\end{align*}
$$

The trace is minimised by setting $\tilde{\nu}_+^\nu = \nu^\nu /\|\nu^\nu\|$, $\tilde{\nu}_-^\nu = \nu^\nu /\|\nu^\nu\|$ and $\tilde{\nu}_0^\nu = \nu^\nu /\|\nu^\nu\|$. From the latter we get the entanglement measure for the BRS that is

$$
E(|r, \phi\rangle) = \frac{1}{4} \left( M - \sum_{\nu=0}^{M-1} \|\nu^\nu\|^2 \right).
$$

Greenberger-Horne-Zeilinger–like states

Now, we consider a second class of states (GHZLS) defined according to

$$
|GHZ, \theta\rangle_M = \cos(\theta)|0\rangle + \sin(\theta)e^{i\nu}|2^M - 1\rangle.
$$

For $\theta = k\pi/2$, where $k \in \mathbb{Z}$, these states are fully separable, whereas $\theta = k\pi/2 + \pi/4$ selects the maximally entangled states. In this case, the trace for the Fubini-Study metric,

$$
\text{tr}(g) = \frac{1}{4} \left[ M - \cos^2(2\theta) \sum_{\nu=0}^{M-1} |\nu^\nu|^2 \right],
$$

is minimised by the values $v_\nu^\nu = 1$. Consistently, we have

$$
\bar{g} = \frac{1}{4} \sin^2(2\theta) J_M
$$

where $J_M$ is the $M \times M$ matrix of ones. The entanglement measure for the GHZLS results

$$
E(|GHZ, \theta\rangle) = \frac{M}{4} \sin^2(2\theta).
$$

Three-qubit states depending on two parameters

The last class of states we consider is

$$
|\varphi, \gamma, \tau\rangle_3 = \cos(\gamma)|00\rangle + \sin(\gamma)|11\rangle,
$$

These states are fully separable for $\gamma = 0, \pi/2$ and $\tau = 0, \pi/2$ whereas they are bi-separable for $\tau = \pi/4$. In this case, the trace of the Fubini-Study metric is

$$
\text{tr}(g) = \frac{1}{4} \left\{ 3 - \cos^2(2\varphi) \cos^2(2\varphi) \right\} + \frac{1}{4} \left\{ (v_0^3)^2 + (v_1^3)^2 \right\}
$$

(28)
and it is minimised by the values $\hat{v}_3 = (0, 0, 1)$, $\nu = 0, 1$

\[
\hat{v}_1^3 = \frac{\sin(2\gamma) \sin(2\tau)}{\sqrt{\sin^2(2\gamma) \sin^2(2\tau) + \cos^2(2\gamma)}},
\]

\[
\hat{v}_2^3 = 0,
\]

\[
\hat{v}_3^3 = \frac{\cos(2\gamma)}{\sqrt{\sin^2(2\gamma) \sin^2(2\tau) + \cos^2(2\gamma)}}.
\]

(29)

Consistently, the entanglement measure for these states results to be

\[
E(|\phi, \gamma, \tau\rangle_3) = \frac{1}{4}[2 \sin^2(2\tau) + 3 \sin^2(2\gamma) \cos^2(2\tau)].
\]

(30)

RESULTS

Entanglement measure

In Fig. 1 we plot the measure $E(|r, \phi\rangle_M)/M$ vs $\phi/(2\pi)$ according to Eq. (22), for the states (13) in the case $M = 3, 4, 7, 9$. Figure 1 show that the proposed entanglement measure provides in all these cases a correct estimation of the degree of entanglement for the BRS. In particular, for the fully separable states ($\phi = 0$) it gives a vanishing value, whereas for the maximally entangled states ($\phi = \pi$) it provides the maximum possible value for the trace, that is $E(|r, \pi\rangle_M)/M = 1/4$. This implicitly indicates that on the maximally entangled states the expectation values for all $\hat{v}_\nu \cdot \sigma_\nu$ ($\nu = 0, \ldots, M - 1$) vanish. The entanglement measure (7) successfully passes

also the second test of the GHZLS for which it provides zero in the case of fully separable states ($\theta = 0$) and the maximum value $(1/4)$ in the case of the maximally entangled state ($\theta = \pi/2$). In figure 2 we compare the curves $E(|r, \phi\rangle_M)/M$ vs $\phi/(2\pi)$ in continuous line and $E(|GHZ, \theta\rangle_M)/M$ vs $2\theta/\pi$ in dashed line for the case $M = 3$. Also in this case, for the maximally entangled states the expectation value for the operators $\hat{v}_\nu \cdot \sigma_\nu$ ($\nu = 0, \ldots, M - 1$) is zero.

In Fig. 3, we report, with a 3D plot, the measure $E(|\phi, \gamma, \tau\rangle_3)/3$ as a function of $\gamma/(2\pi)$ and $\tau/(2\pi)$ according to Eq. (30), for the states (27). The measure (7) catches, in a surprisingly clear way, the entanglement properties of this family of states. In particular, $E(|\phi, \gamma, \tau\rangle_3)/3$ is null in the case of fully separable states ($\gamma = 0, \pi/2, \pi$ and $\tau = 0, \pi/2, \pi$) and it is max-
imum (with value 1/4) in the case of maximally entangled states \((\gamma = \pi/4, 3\pi/4\) and \(\tau = 0, \pi/2, \pi\)). In addition, the case of bi-separable states \((\tau = \pi/4)\) results in \(0 < E(|\varphi, \gamma, \tau\rangle) < 1/4\).

### Eigenvalues analysis

Other interesting characteristics of the entanglement measure come from the analysis of the metric’s eigenvalues. In fig. 4, we plot the eigenvalues of \(\tilde{g}\) for the state \(|r, \phi\rangle_M\) vs \(\phi/(2\pi)\) for the case \(M = 7\). In the general case \(\phi \neq 0, 2\pi\) the BRS \(\tilde{g}\) have \(M\) not null eigenvalues. This fact makes the class of the BRS robust, concerning entanglement, inasmuch the minimum distance between states in a direction randomly chosen is greater than the minimum eigenvalue. On the contrary, the GHZLS have only one non-vanishing eigenvalue. Although the value of the latter is greater than the eigenvalues of the BRS (see Fig. 5), the GHZLS appear weak, in the sense of entanglement, since there exist \(M - 1\) directions with null minimum distance between states. In fig. 5, we compare the plots of the eigenvalues of \(\tilde{g}\) for \(|r, \phi\rangle_M\) vs \(\phi/(2\pi)\) (dotted lines), with the plot of the unique not vanishing eigenvalue of \(\tilde{g}\) for GHZLS vs \(2\theta/\pi\) (continuous line), in the case \(M = 7\).

This shows that the minimum density distance \(ds^2/dr^2\), obtained by varying the vectors \(v\), is bounded from below by the entanglement measure \(E(|s\rangle)\). For fully separable states, the minimum density distance is zero whereas for maximally entangled states, it results \(M/4\) at the very best. It is worth emphasizing that \(ds^2\) can overcome the value of \(E(|s\rangle)/dr^2\).

### CONCLUDING REMARKS

In this paper, we have introduced a new measure of entanglement for the case of an arbitrary pure state of \(M\) qubits \((7)\). We verified the invariance under local unitary transformations identifying classes of equivalence of states, a demanded property of a good entanglement measure. Furthermore, the measure has the characteristics of a distance and assumes the intuitive physical interpretation of an obstacle to the minimum distance between infinitesimally close states. Finally, the analysis of the eigenvalues allows one to determine if there are any states which are more sensitive to small variations than others. For instance, Fig. 4 shows that, in the case of \(|r, \pi/2\rangle_7\) state, a small variation along the eigenvector's direction of the maximum eigenvalue of \(\tilde{g}\) brings a greater distance than the one derived in the case of the maximally entangled state \(|r, \pi\rangle_7\). This analysis is a possible useful mean in the task of determining states with super-Heisenberg sensitivity.

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