MATRX PERMANENT AND QUANTUM ENTANGLEMENT
OF PERMUTATION INVARIANT STATES

TZU-CHIEH WEI AND SIMONE SEVERINI

Abstract. We point out that a geometric measure of quantum entanglement is related to the
matrix permanent when restricted to permutation invariant states. This connection allows us
to interpret the permanent as an angle between vectors. By employing a recently introduced
permanent inequality [Carlen, Loss and Lieb, Meth. and Appl. of Analysis, 13 (2006), no.
1, 1–17], we can write combinatorial formulas for quantifying the entanglement of permutation
invariant basis states. When applying the geometric measure to permutation invariant states
with nonnegative coefficients, we show that the overlap with a product state is maximized by a
tensor product of the same single-party state. This extends some observations in [Hayashi et al.,
Phys. Rev. A 77, 012104 (2008)].

1. Introduction

In the Editor’s statement forewording the 1982 monograph Permanents by Minc [28], Gian-Carlo
Rota wrote the following words:

“A permanent is an improbable construction to which we might have given little
chance of survival fifty years ago. Yet numerous appearances it has made in physics
and in probability betoken the mystifying usefulness of the concept, which has a
way of recurring in the most disparate circumstances.”

The present paper highlights a connection between the permanent and entanglement of certain
quantum states, therefore putting in evidence a further appearance of the permanent in physics.
Some background is useful for delineating the context. The permanent and the determinant of an
n × n matrix A (with entries in a commutative ring) are respectively defined as

\[ \text{perm}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{i,\pi(i)} \quad \text{and} \quad \det(A) = \sum_{\pi \in S_n} (-1)^{\text{sgn}(\pi)} \prod_{i=1}^{n} A_{i,\pi(i)}, \]

where \( S_n \) denotes the full symmetric group on a set of \( n \) symbols. In light of the definitions, it
is reasonable that there are cases, as it was originally observed in 1859 by Cayley [7], in which
permanents can be computed by means of determinants (see also, e.g., Kasteleyn [22], Godsil and
Gutman [12], and Frieze and Jerrum [11]). Geometrically, the determinant is the volume of the
parallelepiped defined by the lines of the matrix; algebraically, it is the product of all eigenvalues
including their multiplicities. It is curious that despite its striking similarity with the determinant,
the permanent does not have any known geometric or algebraic interpretation. Moreover, while standard Gaussian elimination provides an efficient technique for computing the determinant, the exact computation of the permanent remains a notoriously difficult problem.

The best known algorithm for an \( n \times n \) matrix, due to Ryser in 1963 [33], needs \( \Theta(n^2) \) operations. By a seminal result of Valiant [38], computing the permanent is indeed “The \#P-hard problem”. Thus computing the permanent on worst case inputs cannot be done in polynomial time unless \( P = \#P \) and in particular \( P = NP \). It follows that algorithms for the permanent acquired a special position in computational complexity to the point of becoming a fertile ground for many approaches, including iterative balancing (Linial et al. [26]), elementary recursive algorithms (Rasmussen [32]) and, most relevantly, Markov chain Monte Carlo methods (Broder [4], Jerrum and Sinclair [24]). These efforts contributed to a deeper understanding of the permanent and produced important results as fully polynomial randomized approximation schemes for several types of matrices (see Jerrum, Sinclair and Vigoda [25], Barvinok [2], and the references contained therein).

From the mathematical perspective, there are two main lines of research centered on the permanent: the study of permanents and probability (see, e.g., Friedland et al. [10]); min/max questions concerning a number of inequalities, peaking with the proofs by Egorychev [8] and Falikman [9] (see also [30]) of the 1926 van der Waerden’s conjecture about permanents of doubly stochastic matrices.

When focusing our attention on physics, we may divide into at least four groups the known applications of the permanent: the many aspects of the dimer problem, some uses involving the hafnian, Monte Carlo generators, and, more pertinent to our discussion, linear optical networks for quantum information processing. Intuitively, the permanent tends to be related to bosons while the determinant to fermions. A list of applications follows.

The problem of computing the permanent of a \((0, 1)\)-matrix is the same as the problem of counting the number of perfect matchings in a bipartite graph. This translates into the dimer problem [22, 19]. The problem is traditionally related to models for adsorption of diatomic molecules on crystal surfaces, mixtures of molecules of different sizes and the cell-cluster theory of the liquid state (see, e.g., Welsh [42]). New applications occur in the study of configurations of melting crystals (Okounkov et al. [27]), BPS black holes (Heckman and Vafa [18]), and quiver gauge theories (Hanany and Kennaway [14]).

The hafnian of a matrix \( A \), denoted by \( \text{hf}(A) \), is a polynomial generalizing the permanent as the pfaffian generalizes the determinant. This notion was originally introduced by Caianiello to express the perturbation expansions of boson field theories [5].

Bia\l as and Krzywicki [3] (see also Wosiek [43]) introduced a procedure to include Bose-Einstein correlations in Monte Carlo event generators. The procedure makes use of the generalized Wigner functions and it requires to compute the permanent of a correlation matrix depending on particle momenta and other parameters.

Sheele et al. [35] have proved that matrix elements of unitarily transformed photonic multi-mode states can be written as permanents associated with the symmetric tensor power of the beam splitter matrix (see also Kok et al. [23]). The result implies that computing matrix elements in the Fock basis is not an easy task. Permanents count the ways of redistributing \( n \) single photons through an SU(\( n \)) network to yield exactly \( n \) single photons at the outputs and then allow to compute probability amplitudes [34, 36]; optimal networks are obtained by maximizing the permanent under given constraints.

The present paper is concerned with quantum entanglement. It is now established that entanglement is an important physical quantity whose presence appears to be necessary in many applications
of quantum information processing [31]. Given a generic quantum state, detecting and measuring its entanglement is a challenge from both the mathematical and the experimental point of view. It is computationally hard [13] and furthermore there is no general agreement on how to quantify entanglement. A number of different entanglement measures have been therefore introduced using a variety of approaches (see the Horodecki’s review [20] and its references).

Here we consider a geometric measure. This quantifies the angle between two states, subject to an optimization problem: the closest state with zero entanglement and the state under analysis. It turns out that, when we restrict the analysis to permutation invariant states, such a measure can be described in terms of the permanent of a certain matrix. A recent permanent inequality of Carlen, Loss and Lieb [6] can be employed to bypass the optimization problem. Hence, maximizing the permanent of a matrix with certain constraints, reduces to a simpler problem. As we have mentioned earlier, the permanent does not have any known geometric interpretation. The present work contributes in this direction, by interpreting the permanent as an angle between vectors (or a cosine, to be more precise). Similar results concerning entanglement of permutation invariant states are obtained independently by Hayashi et al. [17] without the use of permanents.

The rest of the paper is structured as follows. In Section 2, we define the geometric measure of entanglement. In Section 3, we state and prove our results. For the sake of clarity, we shall also include a few examples

2. Geometric measure of entanglement

The geometric measure of entanglement used here was firstly introduced by Shimony [37] in the setting of bipartite pure states. It was generalized to the multipartite states by Barnum and Linden [1], and further extended by Wei and Goldbart [41]. The intuition beyond the measure consists of thinking about entanglement as an angle between two states: namely, the state in analysis and a product state, i.e., a state with zero entanglement. Crucially, the product state is chosen over all possible product states so that it minimizes the angle with the state in analysis.

Let

\[ |\psi \rangle = \sum_{p_1 \ldots p_n} \chi_{p_1p_2\ldots p_n} |e^{(1)}_{p_1} e^{(2)}_{p_2} \cdots e^{(n)}_{p_n} \rangle \]

be a generic multipartite pure state in a Hilbert space \( \mathcal{H} \cong \bigotimes_{i=1}^k C_{k_i} \) of dimensionality \( \dim(\mathcal{H}) = \prod_{i=1}^n k_i \). Each set \( \{ |e^{(i)}_{p_i} \rangle : p_i = 1, 2, \ldots, k_i \} \) is a local basis for the \( i \)-th subspace \( C_{k_i} \). A pure state in \( \mathcal{H} \) is said to be a product state if it can be written in the form

\[ |\phi \rangle = \bigotimes_{i=1}^n |\phi^{(i)} \rangle \equiv \bigotimes_{i=1}^n \sum_{i=1}^{k_i} \left( e^{(i)}_{p_i} |e^{(i)}_{p_i} \rangle \right), \]

where \( |\phi^{(i)} \rangle \in C_{k_i} \) is some pure state; \( |\psi \rangle \) is said to be entangled, otherwise. Given a state \( |\psi \rangle \) as in Eq. (2.1), let us define

\[ \Lambda_{\max}(\psi) := \max_{|\phi \rangle} \langle \phi |\psi \rangle \]

where the maximization is performed over all product states \( |\phi \rangle \in \mathcal{H} \). This formula tells us how well the possibly entangled state \( |\psi \rangle \) can be approximated by a product state. The formula provides a method that satisfies the desiderata for a well-defined measure of entanglement [20]. Such a method is usually called geometric measure. The terminology is justified since \( \Lambda_{\max}(\psi) \) is an angle between two vectors. Hence, notice that the amount of entanglement increases while \( \Lambda_{\max}(\psi) \) decreases and
therefore the quantity of entanglement depends essentially on $\Lambda_{\text{max}}(\psi)$. Given a state $|\psi\rangle$, concrete geometric measures are

$$E_{\text{sin}^2}(\psi) := 1 - \Lambda_{\text{max}}^2(\psi) \quad \text{and} \quad E_{\log}(\psi) := -2 \log_2 \Lambda_{\text{max}}(\psi),$$

introduced in [41] and [40], respectively. The relation of the geometric measure to other measures has been studied in [15, 15, 39]. In the next section, we will show how the geometric measure can be related to the permanent.

### 3. Permanent and Entanglement

A **permuation invariant basis state** is a pure state of the form

$$|S(n, \vec{k})\rangle = \frac{\sqrt{C_n^n k}}{n!} \sum_{\pi \in S_n} |\pi_i(1, \ldots, 2, \ldots, d, \ldots, d)\rangle,$$

where $C_n^n := n! \prod_{i=1}^d k_i!$.

As an example, we consider the permutation invariant basis state $\ket{n, \vec{k}} := \ket{\Psi_n}$, where $\vec{k} := (k_1, k_2, \ldots, k_d)$.

$$\ket{\Psi_n} = \frac{1}{\sqrt{6!}} \sum_{\pi \in S_4} (\pi(1, 1, 2, 2)) = \frac{1}{\sqrt{6}} (|1212\rangle + |1212\rangle + |1221\rangle + |2112\rangle + |2112\rangle + |2211\rangle).$$

Here $n = 4$ and $d = 2$. Then Theorem 1 is our main result concerning these class of states:

**Theorem 1.** Let $|S(n, \vec{k})\rangle$ be a permutation invariant basis state. Then

$$\Lambda_{\text{max}}(S(n, \vec{k})) = \sqrt{\frac{n!}{\prod_{i=1}^d k_i!}} \prod_{i=1; k_i \neq 0}^d \left(\frac{k_i^n}{n}\right)^{\frac{1}{2}}.$$

**Proof.** We prove the statement by comparing the possibly entangled state $|S(n, \vec{k})\rangle$ to the general product states

$$|\phi\rangle = \bigotimes_{j=1}^n \left(\sum_{l=1}^d \alpha_{j,l} |l_j\rangle \right) \quad \text{with} \quad \sum_{l=1}^d |\alpha_{j,l}|^2 = 1.$$

According to the definition of geometric measure, the first step is to evaluate the overlap

$$\phi_{\vec{k}} := \langle S(n, \vec{k}) | \phi \rangle,$$

which gives

$$\phi_{\vec{k}} = \frac{\sqrt{C_n^n}}{n!} \sum_{\pi \in S_n} \alpha_{\pi,1(1)} \cdots \alpha_{\pi,1(d)} \alpha_{\pi,2(1)} \cdots \alpha_{\pi,2(d)} \cdots \alpha_{\pi,n(1)} \cdots \alpha_{\pi,n(d)} = \frac{\sqrt{C_n^n}}{n!} \text{per}(A_{\vec{k}}),$$

where $A_{\vec{k}}$ is an $n \times n$ matrix defined as follows:

$$A_{\vec{k}} := \begin{bmatrix}
(\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,d} \\
\alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,d}
\end{bmatrix}_{k_1 \times k_2 \times \cdots \times k_d}.$$
The matrix $A_{k}$ has $k_{i}$ identical columns $\vec{v}_{i} = (\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{n,i})^{T}$ with $i = 1, \ldots, d$ and $\sum_{i=1}^{d} k_{i} = n$. The next step consists of maximizing the absolute value of the overlap, i.e., $|\phi_{\vec{A}}|$, over the set of all $\alpha_{j,i}$’s such that $\sum_{i=1}^{d} |\alpha_{j,i}|^2 = 1$. On the basis of Eq. (3.1), we can write

$$\Lambda_{\text{max}} \left( S(n, \vec{k}) \right) = \max_{\alpha_{j,i}} \frac{1}{n!} \left| \text{per}(A_{k}) \right|.$$ 

To deal with this equation, we will use a recent result by Carlen, Loss and Lieb [6]. For any matrix $F$ defined as

$$F := [\vec{f}_{1}, \vec{f}_{2}, \ldots, \vec{f}_{n}],$$

where $\vec{f}_{1}, \vec{f}_{2}, \ldots, \vec{f}_{n}$ are arbitrary column vectors of dimension $n$, they have shown that

$$|\text{per}(F)| \leq \frac{n!}{n^{n/2}} \prod_{i=1}^{n} ||\vec{f}_{i}||_{2},$$

where $||\vec{f}_{i}||_{2}$ denotes the $L_{2}$-norm. The r.h.s. of the inequality can also be regarded as the permanent of a matrix whose $i$-th column contains only identical entries $||\vec{f}_{i}||_{2}/\sqrt{n}$. For example,

$$\frac{1}{\sqrt{n}} \left[ \begin{array}{ccc}
|f_{1}| & |f_{2}| & \cdots \\
|f_{1}| & |f_{2}| & \cdots \\
\vdots & \vdots & \ddots \\
|f_{1}| & |f_{2}| & \cdots 
\end{array} \right].$$

By applying the inequality in Eq. (3.2), we obtain

$$|\phi_{\vec{A}}| \leq \frac{\sqrt{C_{n}^{\vec{k}}}}{n!} \text{per}(A_{k}) = \sqrt{C_{n}^{\vec{k}}} \prod_{i=1}^{n} \bar{\alpha}_{i}^{{k}_{i}},$$

where

$$\bar{\alpha}_{i} := \frac{1}{n} \sum_{j=1}^{n} |\alpha_{j,i}|^2,$$

with the property that $\sum_{i=1}^{d} \bar{\alpha}_{i}^2 = 1$ and

$$A_{\vec{k}} := \begin{bmatrix}
\bar{\alpha}_{1} & \bar{\alpha}_{1} & \cdots & \bar{\alpha}_{1} \\
\bar{\alpha}_{1} & \bar{\alpha}_{1} & \cdots & \bar{\alpha}_{1} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{\alpha}_{1} & \bar{\alpha}_{1} & \cdots & \bar{\alpha}_{1}
\end{bmatrix}_{k_{1}} \begin{bmatrix}
\bar{\alpha}_{2} & \bar{\alpha}_{2} & \cdots & \bar{\alpha}_{2} \\
\bar{\alpha}_{2} & \bar{\alpha}_{2} & \cdots & \bar{\alpha}_{2} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{\alpha}_{2} & \bar{\alpha}_{2} & \cdots & \bar{\alpha}_{2}
\end{bmatrix}_{k_{2}} \begin{bmatrix}
\bar{\alpha}_{d} & \bar{\alpha}_{d} & \cdots & \bar{\alpha}_{d} \\
\bar{\alpha}_{d} & \bar{\alpha}_{d} & \cdots & \bar{\alpha}_{d} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{\alpha}_{d} & \bar{\alpha}_{d} & \cdots & \bar{\alpha}_{d}
\end{bmatrix}_{k_{d}}. $$

By Eq. (3.3), we have

$$\Lambda_{\text{max}} \left( S(n, \vec{k}) \right) = \max_{|\phi|} \phi_{\vec{A}} \leq \max_{\bar{\alpha}_{i} \in \mathbb{R}^{+}} \sqrt{C_{n}^{\vec{k}}} \prod_{i=1}^{n} \bar{\alpha}_{i}^{{k}_{i}} = \max_{\phi_{S}} \langle \phi_{S} | S(n; \vec{k}) \rangle,$$
where
\[
(3.5) \quad |\phi_S\rangle = \bigotimes_{j=1}^{n} \left( \sum_{l=1}^{d} \bar{\alpha}_l |l\rangle \right) = \sum_{k} \sqrt{C_n^k} \bar{\alpha}_1^{k_1} \ldots \bar{\alpha}_d^{k_d} |S(n, \vec{k})\rangle.
\]

The interpretation is that there is a product state constructed from $|\phi\rangle$ by appropriately averaging the coefficients as in Eq. (3.4) and that the derived product state has a larger overlap. The resulting $|\phi_S\rangle$ is a tensor product of $n$ copies of the same state. The last step is a simple maximization procedure. We need to maximize the function
\[
f(x_1, x_2, \ldots, x_d) = \sqrt{C_n^k} \prod_{i=1}^{d} x_i^{k_i}
\]
with nonnegative domain, under the constraint $\sum_{i=1}^{d} x_i^2 = 1$. This gives
\[
\Lambda_{\text{max}}(S(n, \vec{k})) = \sqrt{C_n^k} \prod_{i=1, k_i \neq 0}^{d} \left( \frac{k_i}{n} \right)^{\frac{k_i}{2}},
\]
which verifies the statement. □

Note that in order to find $\Lambda_{\text{max}}(|S(n, \vec{k})\rangle)$, it is sufficient to use the state $|\phi\rangle$ to be the product of $n$ identical copies of an arbitrary single-party state $|\alpha\rangle$, i.e.,
\[
(3.6) \quad |\phi\rangle = \bigotimes_{i=1}^{n} |\alpha\rangle.
\]

When the number of levels $d$ is equal to the number of parties $n$ and $k_i = 1$, for every $i = 1, 2, \ldots, n$, we have
\[
A_{\vec{k}} := \left[\begin{array}{ccccc}
\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,d} \\
\alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{d,1} & \alpha_{d,2} & \cdots & \alpha_{d,d}
\end{array}\right].
\]
The form of the matrix $A_{\vec{k}}$ is generic. The only constraint is that each column has unit norm. In this case,
\[
\Lambda_{\text{max}}(S(d, (1, 1, \ldots, 1))) = \sqrt{d!} \left( \frac{1}{d} \right)^{\frac{d}{2}}.
\]

For the case of qubits, namely when $d = 2$, we can prove the corresponding result without using the Carlen-Lieb-Loss inequality, but the older Schwarz and McClaurin inequalities. In this case, the permutation invariant states have the form
\[
|S(n, k)\rangle = \frac{1}{\sqrt{C_n^k}} \sum_{\pi \in S_n} \prod_{k} \pi(0, \ldots, 0, 1, \ldots, 1), \quad \text{where} \quad C_n^k := \frac{n!}{k!(n-k)!}.
\]
The theorem below states the connected result:

**Theorem 2.** Let $|S(n, k)\rangle$ be a permutation invariant basis state for qubits. Then
\[
\Lambda_{\text{max}}(n, k) = \sqrt{\frac{n!}{k!(n-k)!}} \left( \frac{k}{n} \right)^{\frac{k}{2}} \left( \frac{n-k}{n} \right)^{\frac{n-k}{2}}
\]
Proof. We want to find the maximal overlap between \(|S(n,k)|\) with product states

\[ |\phi\rangle = \otimes_{j=1}^{n} (\sqrt{q_j}|0\rangle + \sqrt{1-q_j}e^{i\beta_j}|1\rangle). \]

As the coefficients in \(|S(n,k)|\) are nonnegative, we can set \(\beta_j = 0\). We then evaluate

\[ \phi_k \equiv \langle S(n,k)|\phi\rangle = \frac{\sqrt{C_k^n}}{n!} \sum_{\pi_i \in S_n} \left( \prod_{l=1}^{k} \sqrt{q_{\pi_i(l)}} \prod_{l=k+1}^{n} \sqrt{1-q_{\pi_i(l+1)}} \right). \]

Using the Cauchy-Schwarz inequality, we have

\[ |\phi_k|^2 \leq \frac{C_k^n}{(n!)^2} \left( \sum_{\pi_i \in S_n} \prod_{l=1}^{k} q_{\pi_i(l)} \right) \left( \sum_{\pi_i \in S_n} \prod_{l=k+1}^{n} 1-q_{\pi_i(l+1)} \right). \]

By the Maclaurin inequality

\[ \frac{1}{n!} \prod_{l=1}^{k} x_{\pi_i(l)} \leq \left( \frac{1}{n} \sum_{l=1}^{n} x_l \right)^k, \]

we arrive at

\[ |\phi_k|^2 \leq C_k^n (\bar{q})^k (1-\bar{q})^{n-k} = C_k^n \cos^k \theta \sin^{2(n-k)} \theta, \]

for \(\cos^2 \theta = \bar{q}\). This means that

\[ |\phi_k| \leq \sqrt{C_k^n (\bar{q})^{k/2} (1-\bar{q})^{(n-k)/2}} = \sqrt{C_k^n \cos^k \theta \sin^{(n-k)} \theta}. \]

Maximizing the expression on the r.h.s. over the angle \(\theta\), we obtain

\[ \Lambda_{\text{max}}(n,k) = \max_{\theta} \sqrt{C_k^n \cos^k \theta \sin^{(n-k)} \theta} = \sqrt{\frac{n!}{k!(n-k)!}} \left( \frac{k}{n} \right)^{\frac{k}{2}} \left( \frac{n-k}{n} \right)^{\frac{n-k}{2}}. \]

This concludes the proof. \(\square\)

For the case of three qubits, some examples of permutation invariant states are the following ones:

\[ |\text{GHZ}\rangle \equiv (|000\rangle + |111\rangle)/\sqrt{2}, \quad |W\rangle \equiv (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}, \quad |\bar{W}\rangle \equiv (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}. \]

The states \(|\text{GHZ}\rangle\) and \(|W\rangle\) have extremal properties and have particularly important roles in quantum mechanics [20]: the Greenberger–Horne–Zeilinger state, \(|\text{GHZ}\rangle\), was used to test Bell’s inequalities; the \(W\) state, \(|W\rangle\), exhibits genuine three-party entanglement, in a different way from \(|\text{GHZ}\rangle\).

By applying Theorem 2 we have

\[ \Lambda_{\text{max}}(\text{GHZ}) = 1/\sqrt{2} \quad \text{and} \quad \Lambda_{\text{max}}(W) = \Lambda_{\text{max}}(\bar{W}) = 2/3. \]

Let us now consider examples for three-party 4-level systems, namely \(n = 3\) and \(d = 4\). The chosen vectors are \(\vec{a} = (2,0,0,1)\) and \(\vec{b} = (1,1,1,0)\) and their corresponding states are

\[ |\vec{a}\rangle = \frac{1}{\sqrt{3}}(|114\rangle + |141\rangle + |411\rangle), \quad \text{and} \quad |\vec{b}\rangle = \frac{1}{\sqrt{6}}(|123\rangle + |132\rangle + |213\rangle + |231\rangle + |312\rangle + |321\rangle). \]

With the use of Theorem 1 we have

\[ \Lambda_{\text{max}}(|\vec{a}\rangle) = \Lambda_{\text{max}}(W) = 2/3 \quad \text{and} \quad \Lambda_{\text{max}}(|\vec{b}\rangle) = \sqrt{2}/3. \]

Note that the states \(|\vec{a}\rangle\) and \(|W\rangle\) have the same structure.

We conclude by asking the following question and then providing a partial answer:
**Problem 1.** Is it true that in order to obtain the maximal overlap of any permutation invariant state, we can assume the product state to be a tensor product of the same single-party state?

Hayashi et al. [16] have attempted to answer this question. However, it still remains an open problem. Here we prove that the question has an affirmative answer for certain classes of permutation invariant states, beyond the basis states discussed above.

The first class we consider is the case where the coefficients $c_{\vec{k}}$’s are nonnegative. To obtain the maximal overlap for the corresponding state $|\psi\rangle$, we can as well set the coefficients in the unentangled state $|\phi\rangle$ to be nonnegative, as the goal is to maximize the overlap between $|\psi\rangle$ and $|\phi\rangle$. Thus we have

$$
\langle \phi | \psi \rangle = \sum_k c_k \bar{c}_k = \sum_k c_k \sqrt{\frac{C^n_k}{n!}} \text{per}(A_k) \leq \sum_k c_k \sqrt{\frac{C^n_k}{n!}} \text{per}(\bar{A}_k) = \sum_k c_k \sqrt{\frac{C^n_k}{n!}} \bar{c}_{\vec{k}} \ldots \bar{c}_{\vec{d}} = \langle \phi_S | \psi \rangle,
$$

where we have used again the inequality proved in [6]. This means that for nonnegative $c_{\vec{k}}$’s, in order to maximize the overlap, we can use the product state $|\phi_S\rangle$ in Eq. (3.6), consisting of a direct product of identical single-party states. An example of this is given by the states

$$
|W \bar{W}(s)\rangle \equiv \sqrt{s}|W\rangle + \sqrt{1-s}|\bar{W}\rangle, \quad \text{where} \quad s \in [0, 1].
$$

The way to use the product state in the form of Eq. (3.6) is justified. This was the ansatz used to calculate the entanglement for this family of states in [41]. In particular, the product state can be written as

$$
|\phi_S(\theta)\rangle \equiv (\cos \theta |0\rangle + \sin \theta |1\rangle)^{\otimes 3}
$$

and then, by maximizing the inner product $\langle \phi_S(\theta) | W \bar{W}(s) \rangle$ over $\theta$, it is straighforward to obtain the expression

$$
A_{\max} (W \bar{W}(s)) = \frac{1}{2} \left( \sqrt{s} \cos \theta(s) + \sqrt{1-s} \sin \theta(s) \right) \sin 2\theta(s),
$$

where $\theta(s)$ is the solution of the equation

$$
\sqrt{1-s} \tan^2 \theta + 2\sqrt{s} \tan \theta - 2\sqrt{1-s} \tan \theta - \sqrt{s} = 0, \quad \text{where} \quad \tan \theta \in [1/\sqrt{2}, \sqrt{2}].
$$

We can also approach a more general class of states. When the coefficient $c_{\vec{k}}$’s are arbitrary but nonnegative, the above consideration of using states as in Eq. (3.6) holds; this gives us the corresponding state

$$
|\psi\rangle = \sum_{\vec{k}} c_{\vec{k}} |S(n, \vec{k})\rangle.
$$

Now, we perform a basis change on $|\psi\rangle$, i.e.,

$$
|\psi'\rangle \equiv (U \otimes U \otimes \ldots \otimes U) |\psi\rangle = \sum_{\vec{k}} b_{\vec{k}} |S(n, \vec{k})\rangle,
$$

where $U$ is any unitary transformation in $U(d)$. Specifically, the transformation $U$ acts on a single party, and $b_{\vec{k}}$’s are the resulting coefficients for $|\psi'\rangle$ expanded in the basis of $|S(n, \vec{k})\rangle$. The resulting coefficients $b_{\vec{k}}$’s are in general complex. Since we have shown that to calculate the entanglement for $|\psi\rangle$ we can assume the product state to be as in Eq. (3.6) and $|\psi'\rangle$ is simply given by a local change of basis, to calculate the entanglement for $|\psi'\rangle$, we can take a fiducial state of the same form. To illustrate this fact, we consider a generic element of $SU(2)$:

$$
U = \begin{pmatrix} u & v \\ -\bar{v}^* & \bar{u}^* \end{pmatrix}, \quad \text{where} \quad |u|^2 + |v|^2 = 1.
$$
The effect of $U$ on the parties of $|W\rangle$ is given by

$$|W\rangle \mapsto \frac{1}{\sqrt{3}} \left( -3u^2v^*|000\rangle + \sqrt{3}u \left( |u|^2 - 2|v|^2 \right) |W\rangle + \sqrt{3}v \left( 2|u|^2 - |v|^2 \right) |\bar{W}\rangle + 3u^*v^2|111\rangle \right)$$

and similarly for $|\bar{W}\rangle$, $|000\rangle$ and $|111\rangle$, our basis states in the symmetric subspace. The corresponding coefficients are in general complex. When $U$ is diagonal, the coefficients $c\vec{k}$ are transformed as $d\vec{k} = c\vec{k}e^{i\vec{k} \cdot \vec{\theta}}$, where $\vec{\theta}$ is an arbitrary real $d$-component vector characterizing the matrix $U$.

It takes 8 real parameters to describe the generic permutation invariant states for three qubits. If we start with 4 arbitrary nonnegative coefficients and supplement with arbitrary $U(2)$ transformations (which have 4 real parameters), we have then 8 real parameters in total. The number boils down to 6 in both cases, if we take into account normalization and global phase. This counting suggests that the statement in Problem 1 may well be true for three qubits.

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