A Trigonometric Cubic B-spline Finite Element Method for Solving the Nonlinear Coupled Burger Equation

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Abstract

The coupled Burgers equation is solved by way of the trigonometric B-spline collocation method. The unknown of the coupled Burgers equation is integrated in time by aid of the Crank-Nicolson method. Resulting time-integrated coupled Burgers equation is discretized using the trigonometric cubic B-spline collocation method. Fully-integrated coupled Burgers equation which is a system of nonlinear algebraic equation is solved with a variant of Thomas algorithm. The three model test problems are studied to illustrate the accuracy of the suggested method.

1 Introduction

One of the flow equations, derived by Episov [1], is the coupled Burgers equation (CBE) describing sedimentation or evaluation of the scaled volume concentration of two kinds particles in fluid suspensions or colloids under the effect of gravity [4]. The CBE is known as the simple case of the Navier-Stokes equation due to including the nonlinear convection term and viscosity term

The following nonlinear partial differential equation describes the CBE

\[
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + k_1 U \frac{\partial U}{\partial x} + k_2 (UV)_x = 0
\]

\[
\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} + k_1 V \frac{\partial V}{\partial x} + k_3 (UV)_x = 0
\]

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where $k_1$, $k_2$ and $k_3$ are real constants and subscripts $x$ and $t$ denote differentiation, $x$ distance and $t$ time, is considered. Boundary conditions

\begin{align*}
U(a, t) &= f_1(a, t), \quad U(b, t) = f_2(b, t) \\
V(a, t) &= g_1(a, t), \quad V(b, t) = g_2(b, t), \quad t > 0
\end{align*}

(2)

and initial conditions

\begin{align*}
U(x, 0) &= f(x) \\
V(x, 0) &= g(x), \quad a \leq x \leq b
\end{align*}

(3)

will be decided in the later sections according to test problem. $U(x, t)$ and $V(x, t)$ are the unknown to be determined, $U_t$ and $V_t$ are the unsteady terms, $UU_x$ and $VV_x$ are the nonlinear terms and $U_{xx}$ and $V_{xx}$ are diffusive terms.

The approximate-analytical and numerical solutions of the CBE are in need because precise analytical solution of the CBE does not exist in the literature for wide class of boundary and initial conditions. The engineers and scientists are studying the CBE to discover more properties of the CBE due to its applicability in relevant fields. The approximate-analytical methods such as decomposition method [6], variational iteration method [7], Tanh-method [9], Adomian-pade technique [10], differential transform method [16], homotopy perturbation method [12, 13, 24] are applied to find the approximate functional solutions of the CBE.

Researcher have set up numerical methods to reveal more properties of the CBE for specific solutions which does not found with analytical and approximate analytical solutions. Episov presented numerical solution of the one dimensional CBE and gave comparison of the results with the experimental data [11, 2]. Rashid et al. constructed the Fourier pseudo-spectral method for providing an approximate solutions for the CBE of a set of initial and periodic boundary condition. A fully-implicit finite difference method [26], a composite numerical scheme based on finite difference [27] and an implicit logarithmic finite difference method have been presented for the CBE [28]. The Differential quadrature [22] and the generalized differential quadrature method have been set up to obtain numerical solutions of the CBE. The collocation algorithms based on chebyshev functions [11] and B-splines [18] are written to study solutions of the CBE. The numerical solutions of CBE are given by local discontinuous Galerkin method [21], the Galerkin $H^2$-Galerkin mixed method [23] and Galerkin quadratic B-spline finite element method [25]. Taylor Collocation-Extended Cubic B-spline method is presented in the doctoral dissertation of Aksoy [33]. The Modified cubic B-spline collocation method is presented by R. C. Mittal and A. Tripathi [29].

The B-splines are popular in science and engineering for solving the differential equations. Main advantages of using the B-splines in the numerical method is to provide simple and economical algorithms. The trigonometric B-splines (TB) is an alternative base functions to the well-known polynomial B-spline base functions. TB, introduced by I. J. Schoenberg in 1964, is known as a non-polynomial B-splines including sine function. These spline functions have started to use as an approximate function for constructing the
numerical methods to get numerical solutions of the differential equations. The numerical methods, based on quadratic and TB, for solving types of ordinary differential equations were given in the studies [3, 8, 15, 17]. Very recently a collocation finite difference scheme based on new TB is developed for the numerical solution of a one-dimensional hyperbolic equation (wave equation) with non-local conservation condition [30]. A new two-time level implicit technique based on TB is proposed for the approximate solution of the nonclassical diffusion problem with nonlocal boundary condition in the study [31].

Different kinds of numerical methods have been developed to deal with finding the solutions of the Burgers equation. It is known that solutions of the Burgers type equations include the steep front and sharpness. Therefore modelling of the solutions of the Burgers equation having sharp behavior is of interest for the numerical analyst. In this paper, TB are used to establish a collocation method for obtaining the numerical solutions of the CBE. The fully-integration of the CBE is going to give a nonlinear algebraic equation. The time and space integrations have been managed by employing the Crack-Nicolson method and the trigonometric B-spline approximation respectively. After getting solution of the algebraic system, the numerical solutions of the GBE will be found in terms of combination of the cubic TB and the solutions of the algebraic equation. We will present easy and simple algorithm with the use of the cubic TB in the collocation method for solving CBE.

2 Trigonometric Cubic B-spline Collocation Method

Consider a uniform partition of the problem domain \([a = x_0, b = x_N]\) at the knots \(x_i, i = 0, ..., N\) with mesh spacing \(h = (b-a)/N\). On this partition together with additional knots \(x_{-1}, x_0, x_{N+1}, x_{N+2}, x_{N+3}\) outside the problem domain, \(CTB_i(x)\) can be defined as

\[
CTB_i(x) = \frac{1}{\theta} \begin{cases} 
\omega^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}] \\
\omega(x_{i-2})\omega(x_{i-1})\phi(x_i) + \phi(x_{i+1})\omega(x_{i-1}) + \phi(x_{i+2})\omega^2(x_{i-1}), & x \in [x_{i-1}, x_i] \\
\omega(x_{i-2})\phi^2(x_{i+1}) + \phi(x_{i+2})(\omega(x_{i-1})\phi(x_{i+1}) + \phi(x_{i+2})\omega(x_i)), & x \in [x_i, x_{i+1}] \\
\phi^2(x_{i+2}), & x \in [x_{i+1}, x_{i+2}] \\
0, & \text{otherwise}
\end{cases}
\]

where

\[
\omega(x_i) = \sin(\frac{x - x_i}{2}), \quad \phi(x_i) = \sin(\frac{x_i - x}{2}), \quad \theta = \sin(\frac{h}{2}) \sin(h) \sin(\frac{3h}{2}).
\]

\(CTB_i(x)\) are twice continuously differentiable piecewise trigonometric cubic B-spline on the interval \([a, b]\). The iterative formula

\[
T_i^k(x) = \frac{\sin(\frac{x-x_i}{2})}{\sin(\frac{x_{i+k-1}-x_{i+1}}{2})} T_{i+1}^{k-1}(x) + \frac{\sin(\frac{x_{i+k-x}{2}}{2})}{\sin(\frac{x_{i+k-x_{i+1}}}{2})} T_i^{k-1}(x), \quad k = 2, 3, 4, ...
\]
gives the cubic B-spline trigonometric functions starting with the CTB-splines of order 1

\[ T_1^i(x) = \begin{cases} 
1, & x \in [x_i, x_{i+1}) \\
0, & \text{otherwise}.
\end{cases} \]

Each \( CTB_i(x) \) is twice continuously differentiable and the values of \( CTB_i(x), CTB'_i(x) \) and \( CTB''_i(x) \) at the knots \( x_i \)'s can be computed from Eq. (4) as

\[
CTB_i(x), \quad i = -1, ..., N + 1 \quad \text{are a basis for the trigonometric spline space. An approximate solution} \quad U_N \quad \text{to the unknown} \quad U \quad \text{is written in terms of the expansion of the CTB as}
\]

\[
U_N(x, t) = \sum_{i=-1}^{N+1} \delta_i CTB_i(x), \quad V_N(x, t) = \sum_{i=-1}^{N+1} \phi_i CTB_i(x)
\]

where \( \delta_i \) are time dependent parameters to be determined from the collocation points \( x_i, i = 0, ..., N \) and the boundary and initial conditions. The first and second derivatives also can be defined by

\[
U'_N(x, t) = \sum_{i=-1}^{N+1} \delta_i CTB'_i(x), \quad V'_N(x, t) = \sum_{i=-1}^{N+1} \phi_i CTB'_i(x)
\]

\[
U''_N(x, t) = \sum_{i=-1}^{N+1} \delta_i CTB''_i(x), \quad V''_N(x, t) = \sum_{i=-1}^{N+1} \phi_i CTB''_i(x)
\]

Using the Eq. (6), (7) and Table 1, we see that the nodal values \( U_i, V_i \), their first derivatives \( U'_i, V'_i \) and second derivatives \( U''_i, V''_i \) at the knots are given in terms of parameters by the following relations

\[
U_i = \alpha_1 \delta_{i-1} + \alpha_2 \delta_i + \alpha_1 \delta_{i+1} \\
U'_i = \beta_1 \delta_{i-1} + \beta_2 \delta_i + \beta_1 \delta_{i+1} \\
U''_i = \gamma_1 \delta_{i-1} + \gamma_2 \delta_i + \gamma_1 \delta_{i+1} \\
V_i = \alpha_1 \phi_{i-1} + \alpha_2 \phi_i + \alpha_1 \phi_{i+1} \\
V'_i = \beta_1 \phi_{i-1} + \beta_2 \phi_i + \beta_1 \phi_{i+1} \\
V''_i = \gamma_1 \phi_{i-1} + \gamma_2 \phi_i + \gamma_1 \phi_{i+1}
\]

\[
\text{Table 1: Values of } B_i(x) \text{ and its principle two derivatives at the knot points}
\]

| \( x_{i-2} \) | \( T_1^i(x_k) \) | \( T'_1(x_k) \) | \( T''_1(x_k) \) |
|---|---|---|---|
| \( x_{i-1} \) | \( \sin^2 \left( \frac{h}{2} \right) \csc(h) \csc \left( \frac{3h}{4} \right) \) | \( \frac{3}{4} \csc \left( \frac{3h}{4} \right) \) | \( \frac{3(1+3 \cos(h)) \csc^2 \left( \frac{h}{2} \right)}{16 \left[ 2 \cos \left( \frac{h}{2} \right) + \cos \left( \frac{3h}{2} \right) \right]} \) |
| \( x_i \) | \( \frac{2}{1+2 \cos(h)} \) | \( 0 \) | \( -\frac{3 \cos^2 \left( \frac{h}{2} \right)}{2+4 \cos(h)} \) |
| \( x_{i+1} \) | \( \sin^2 \left( \frac{h}{2} \right) \csc(h) \csc \left( \frac{3h}{4} \right) \) | \( -\frac{3}{4} \csc \left( \frac{3h}{4} \right) \) | \( \frac{3(1+3 \cos(h)) \csc^2 \left( \frac{h}{2} \right)}{16 \left[ 2 \cos \left( \frac{h}{2} \right) + \cos \left( \frac{3h}{2} \right) \right]} \) |
| \( x_{i+2} \) | \( 0 \) | \( 0 \) | \( 0 \) |
where

\[
\begin{align*}
\alpha_1 &= \sin^2\left(\frac{b}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right) \\
\beta_1 &= -\frac{3}{4} \csc\left(\frac{3h}{2}\right) \\
\gamma_1 &= \frac{3((1 + 3 \cos(h)) \csc^2\left(\frac{h}{2}\right))}{16(2 \cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right))} \\
\alpha_2 &= \frac{2}{1 + 2 \cos(h)} \\
\beta_2 &= \frac{3}{4} \csc\left(\frac{3h}{2}\right) \\
\gamma_2 &= -\frac{3 \cot^2\left(\frac{h}{2}\right)}{2 + 4 \cos(h)}
\end{align*}
\]

The Crank-Nicolson scheme is used to discretize time variables of the unknown \( U \) and \( V \) in the Coupled Burger equation. We obtain the time discretized form of the equation as

\[
\frac{U^{n+1} - U^n}{\Delta t} - \frac{U_{xx}^{n+1} + U_{xx}^n}{2} + k_1 \frac{(UU_x)^{n+1} + (UU_x^n)}{2} + k_2 \frac{(UV_x)^{n+1} + (UV_x^n)}{2} = 0
\]

\[
\frac{V^{n+1} - V^n}{\Delta t} - \frac{V_{xx}^{n+1} + V_{xx}^n}{2} + k_1 \frac{(VV_x)^{n+1} + (VV_x^n)}{2} + k_3 \frac{(UV_x)^{n+1} + (UV_x^n)}{2} = 0
\]

where \( U^{n+1} = U(x, t_n + \Delta t) \) and \( V^{n+1} = V(x, t_n + \Delta t) \). The nonlinear term \((UU_x)^{n+1}\), \((VV_x)^{n+1}\) and \((UV_x)^{n+1}\) in Eq. (9) is linearized by using the following forms [32]:

\[
\begin{align*}
(UU_x)^{n+1} &= U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n \\
(VV_x)^{n+1} &= V^{n+1}V_x^n + V^nV_x^{n+1} - V^nV_x^n \\
(UV_x)^{n+1} &= (U_xV)^{n+1} + (UV_x^n)^{n+1} \\
&= U^{n+1}V^n + U^nV^{n+1} - U^nV^n + U^{n+1}V^n + V^{n+1}V_x^n + U^nV_x^{n+1} - U^nV_x^n
\end{align*}
\]

Substitution the expressions (8) into (9) and evaluating the resulting equations at the knots yields the system of the fully-discretized equations

\[
\begin{align*}
\nu_{m1}\delta_{m-1}^{n+1} + \nu_{m2}\phi_{m-1}^{n+1} + \nu_{m3}\delta_{m}^{n+1} + \nu_{m4}\phi_{m}^{n+1} + \nu_{m5}\delta_{m+1}^{n+1} + \nu_{m6}\phi_{m+1}^{n+1} &= \nu_{m7}\delta_{m-1}^{n} + \nu_{m8}\delta_{m}^{n} + \nu_{m9}\delta_{m+1}^{n} \tag{11}
\end{align*}
\]

and

\[
\begin{align*}
\nu_{m10}\delta_{m-1}^{n+1} + \nu_{m11}\phi_{m-1}^{n+1} + \nu_{m12}\delta_{m}^{n+1} + \nu_{m13}\phi_{m}^{n+1} + \nu_{m14}\delta_{m+1}^{n+1} + \nu_{m15}\phi_{m+1}^{n+1} &= \nu_{m7}\phi_{m-1}^{n} + \nu_{m8}\phi_{m}^{n} + \nu_{m9}\phi_{m+1}^{n} \tag{12}
\end{align*}
\]
where

\[ \nu_{m1} = \left( \frac{2}{\Delta t} + k_1K_2 + k_2L_2 \right) \alpha_1 + (k_1K_1 + k_2L_1) \beta_1 - \gamma_1 \]
\[ \nu_{m2} = (k_1K_2) \alpha_1 + (k_2K_1) \beta_1 \]
\[ \nu_{m3} = \left( \frac{2}{\Delta t} + k_1K_2 + k_2L_2 \right) \alpha_2 - \gamma_2 \]
\[ \nu_{m4} = (k_1K_2) \alpha_2 \]
\[ \nu_{m5} = \left( \frac{2}{\Delta t} + k_1K_2 + k_2L_2 \right) \alpha_1 - (k_1K_1 + k_2L_1) \beta_2 - \gamma_1 \]
\[ \nu_{m6} = (k_1K_2) \alpha_1 - (k_2K_1) \beta_2 \]
\[ \nu_{m7} = \frac{2}{\Delta t} \alpha_1 + \gamma_1 \]
\[ \nu_{m8} = \frac{2}{\Delta t} \alpha_2 + \gamma_2 \]
\[ \nu_{m9} = \frac{2}{\Delta t} \alpha_1 + \gamma_1 \]

\[ \nu_{m10} = (k_3L_2) \alpha_1 + (k_3L_1) \beta_1 \]
\[ \nu_{m11} = \left( \frac{2}{\Delta t} + k_1L_2 + k_3K_2 \right) \alpha_1 + (k_1L_1 + k_3K_1) \beta_1 - \gamma_1 \]
\[ \nu_{m12} = (k_3L_2) \alpha_2 \]
\[ \nu_{m13} = \left( \frac{2}{\Delta t} + k_1L_2 + k_3K_2 \right) \alpha_2 - \gamma_2 \]
\[ \nu_{m14} = (k_3L_2) \alpha_1 - (k_3L_1) \beta_2 \]
\[ \nu_{m15} = \left( \frac{2}{\Delta t} + k_1L_2 + k_3K_2 \right) \alpha_1 - (k_1L_1 + k_3K_1) \beta_2 - \gamma_1 \]

\[ K_1 = \alpha_1 \delta^n_{m-1} + \alpha_2 \delta^n_m + \alpha_3 \delta^n_{m+1} \]
\[ L_1 = \alpha_1 \phi^n_{m-1} + \alpha_2 \phi^n_m + \alpha_3 \phi^n_{m+1} \]
\[ K_2 = \beta_1 \delta^n_{m-1} + \beta_2 \delta^n_{m+1} \]
\[ L_2 = \beta_1 \phi^n_{m-1} + \beta_2 \phi^n_{m+1} \]

The system with (11) and (12) can be expressed in the following matrices system:

\[ \textbf{A} \textbf{d}^{n+1} = \textbf{B} \textbf{d}^n \]  \hspace{1cm} (13)

where

\[
\begin{bmatrix}
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
\nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15} \\
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
\nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15} \\
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
\nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 \\
0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} \\
\nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 \\
0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} \\
\nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 \\
0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9}
\end{bmatrix}
\]
The system \((13)\) consist of \(2N + 2\) linear equation in \(2N + 6\) unknown parameters \(d^{n+1} = (\delta^{n+1}_1, \phi^{n+1}_1, \delta^{n+1}_0, \phi^{n+1}_0, \ldots, \delta^{n+1}_{N+1}, \phi^{n+1}_{N+1})\). To obtain a unique solution an additional four constraints are needed. By imposing the Dirichlet boundary conditions this will lead us to the following relations;

\[
\begin{align*}
\delta_{-1} &= (f_1(a, t) - \alpha_2 \delta_0 - \alpha_1 \delta_1)/\alpha_1 \\
\phi_{-1} &= (g_1(a, t) - \alpha_2 \phi_0 - \alpha_1 \phi_1)/\alpha_1 \\
\delta_{N+1} &= (f_2(b, t) - \alpha_1 \delta_{N-1} - \alpha_2 \delta_N)/\alpha_1 \\
\phi_{N+1} &= (g_2(b, t) - \alpha_1 \phi_{N-1} + \alpha_2 \phi_N)/\alpha_1 \\
\end{align*}
\]

Elimination of the parameters \(\delta_{-1}, \phi_{-1}, \delta_{N+1}, \phi_{N+1}\) from the system \((13)\) gives us a solvable system. Once the initial parameters are determined, parameters \(\delta^n_i, i = -1\ldots N+1\) can be found from the iterative system \((13)\), so that using the equations solutions of the equations at the defined time steps can be found iteratively.

3 The Initial State

Initial parameters \(\delta^0_{-1}, \phi^0_{-1}, \delta^0_0, \phi^0_0, \ldots, \delta^0_{N+1}, \phi^0_{N+1}\) can be determined from the initial condition and first space derivative of the initial conditions at the boundaries as the following:

\[
\begin{align*}
U^0(a, 0) &= \alpha_1 \delta^0_{-1} + \alpha_2 \delta^0_0 + \alpha_1 \delta^0_1 \\
U^0(x_m, 0) &= \alpha_1 \phi^0_{m-1} + \alpha_2 \phi^0_m + \alpha_1 \phi^0_{m+1}, \quad m = 1, 2, \ldots, N - 1 \\
U^0(b, 0) &= U^0_N = \alpha_1 \delta^0_{N-1} + \alpha_2 \delta^0_N + \alpha_1 \delta^0_{N+1} \\
V^0(a, 0) &= \alpha_1 \phi^0_{-1} + \alpha_2 \phi^0_0 + \alpha_1 \phi^0_1 \\
V^0(x_m, 0) &= \alpha_1 \phi^0_{m-1} + \alpha_2 \phi^0_m + \alpha_1 \phi^0_{m+1}, \quad m = 1, 2, \ldots, N - 1 \\
V^0(b, 0) &= \alpha_1 \phi^0_{N-1} + \alpha_2 \phi^0_N + \alpha_1 \phi^0_{N+1} \\
\end{align*}
\]

and

\[
\begin{align*}
\delta^0_{-1} &= (U'_0 - \beta_2 \delta^0_1)/\beta_1, \quad \delta^0_{N+1} = (U'_N - \beta_1 \delta^0_{N-1})/\beta_2, \\
\end{align*}
\]

The system \((15)\) consists \(N + 1\) equations and \(N + 3\) unknown, so we have to eliminate \(\delta^0_{-1}\) and \(\delta^0_{N+1}\) for solving this system using following derivatives conditions

\[
\begin{align*}
\delta^0_{-1} &= (U'_0 - \beta_2 \delta^0_1)/\beta_1, \quad \delta^0_{N+1} = (U'_N - \beta_1 \delta^0_{N-1})/\beta_2, \\
\end{align*}
\]

and if the equations system is rearranged for the above conditions, then following form is obtained

\[
\begin{bmatrix}
\alpha_2 & 2\alpha_1 \\
\alpha_1 & \alpha_2 & \alpha_1 \\
& \ddots & \ddots \\
& & \alpha_1 & \alpha_2 & \alpha_1 \\
& & & 2\alpha_1 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
\delta^0_0 \\
\delta^0_1 \\
\vdots \\
\delta^0_{N-1} \\
\delta^0_N
\end{bmatrix}
= 
\begin{bmatrix}
U'_0 - \frac{\alpha_1}{\beta_1} U'_0 \\
U'_1 \\
\vdots \\
U'_{N-1} \\
U'_N - \frac{\alpha_2}{\beta_1} U'_N
\end{bmatrix}
\]
which can also be solved using a variant of the Thomas algorithm. With the same way, from the system (16), \( \phi_{0-1}^{0} \) and \( \phi_{N+1}^{0} \) can be eliminated using

\[
\phi_{0-1}^{0} = \left( V_0' - \beta_2 \phi_1^0 \right) / \beta_1, \quad \phi_{N+1}^{0} = \left( V_N' - \beta_1 \phi_{N-1}^0 \right) / \beta_2,
\]

conditions and the following three bounded matrix is obtained.

\[
\begin{bmatrix}
\alpha_2 & 2\alpha_1 \\
\alpha_1 & \alpha_2 & \alpha_1 \\
& & \ddots \\
& & & \alpha_1 & \alpha_2 & \alpha_1 \\
& & & & & 2\alpha_1 & \alpha_2 \\
\end{bmatrix}
\begin{bmatrix}
\phi_0^0 \\
\phi_1^0 \\
\vdots \\
\phi_{N-1}^0 \\
\phi_N^0 \\
\end{bmatrix}
= \begin{bmatrix}
V_0' - \frac{\alpha_1}{\beta_1} V_0' \\
V_1' \\
\vdots \\
V_{N-1}' \\
V_N' - \frac{\alpha_1}{\beta_1} V_N' \\
\end{bmatrix}
\]

4 Numerical Tests

In this section, numerical results of the three test problems will be presented for the coupled Burgers equation. The accuracy of suggested method problem will be shown by calculating the error norm

\[
L_{\infty} = |U - U_N|_{\infty} = \max_j |U_j - (U_N)_j^n|
\]

The obtained results will compare with results of [18], [22], [29] and [33].

**Problem 1** Consider the coupled Burgers equation (11) with the following initial and boundary conditions

\[
U(x, 0) = \sin(x), \quad V(x, 0) = \sin(x)
\]

and

\[
U(-\pi, t) = U(\pi, t) = V(-\pi, t) = V(\pi, t) = 0
\]

The exact solution is

\[
U(x, t) = V(x, t) = e^{-t} \sin(x)
\]

We compute the numerical solutions using the selected values \( k_1 = -2, k_2 = 1 \) and \( k_3 = 1 \) with different values of time step length \( \Delta t \). In our first computation, we take \( t = 0.1, \Delta t = 0.001 \) while the number of partition \( N \) changes. The corresponding results are presented in Table 2 a. In our computation, we compute the maximum absolute errors at time level \( t = 1 \) for the parameters with different decreasing values of \( t \). The corresponding results are reported in Table 2 b. In both computations, the results are same for \( U(x, t) \) and \( V(x, t) \) because of symmetric initial and boundary conditions. And also we correspond the obtained numerical solutions by different settings of parameters, specifically for those taken by [29] in Table 2 c for \( N = 50, \Delta t = 0.01 \) and increasing \( t \). And also in Table 2, we present the space rate of convergence for \( t = 3 \) which is clearly of second order.
The corresponding graphical illustrations are presented in Figures 2 for $k_1 = -2$, $k_2 = 1$, $k_3 = 1$, $N = 400$ and $\Delta t = 0.001$ at different $t$ for best parameter $p = 0.0002166$. In Figure 2-4, computed solutions of $V$ different time levels for $k_1$, $k_2$ fixed, $k_1$, $k_3$ and $k_2$, $k_3$
fixed respectively.

Figure 2: Computed solutions of $U$ Problem 1 for $t = 3$, $k_1 = -2$, $k_2 = 1$, $k_3 = -8$

Figure 3: Computed solutions of $U$ Problem 1 for $t = 3$, $k_1 = -2$, $k_2 = 1$, $k_3 = -4$
Problem 2) Numerical solutions of considered coupled Burgers’ equations are obtained for $k_1 = 2$ with different values of $k_2$ and $k_3$ at different time levels. In this situation the exact solution is

$$U(x, t) = a_0 - 2A\left(\frac{2k_2}{4k_2k_3 - 1}\right)\tanh(A(x - 2At))$$

$$V(x, t) = a_0\left(\frac{2k_3}{2k_2 - 1}\right) - 2A\left(\frac{2k_2}{4k_2k_3 - 1}\right)\tanh(A(x - 2At))$$

Thus, the initial and boundary conditions are taken from the exact solution is

$$U(x, 0) = a_0 - 2A\left(\frac{2k_2}{4k_2k_3 - 1}\right)\tanh(Ax)$$

$$V(x, 0) = a_0\left(\frac{2k_3}{2k_2 - 1}\right) - 2A\left(\frac{2k_2}{4k_2k_3 - 1}\right)\tanh(Ax)$$

Thus, the initial and boundary conditions are extracted from the exact solution. Where $a_0 = 0.05$ and $A = \frac{1}{2}(\frac{a_0(4k_2k_3 - 1)}{2k_2 - 1})$. The numerical solutions have been computed for the domain $x \in [-10, 10]$, $\Delta t = 0.01$ and number of partition $N = 100$. The maximum error norms have been computed and compared in Tables 3 a-3 b for $t = 0.5$ and $t = 1.0$ with those available in the literature [18], [11] and [34]. For $N = 21$ the maximum error norms have been computed and compared in Tables 3 c-3 d for $t = 0.5$, ...
$t = 1.0$ and $t = 3.0$ with those available in the literature \[11], \[34], \[22] and \[29].

**Table 3 a: Maximum error norms for $U(x, t)$, $k_1 = 2$, $\Delta t = 0.01$, $N = 100$**

| $t$ | $k_2$ | $k_3$ | Present | \[18\] | \[11\] | \[34\] |
|-----|-------|-------|---------|------|------|------|
| 0.5 | 0.1   | 0.3   | $0.4707 \times 10^{-4}$ | $0.4167 \times 10^{-4}$ | $0.438 \times 10^{-4}$ | $0.9619 \times 10^{-3}$ |
|     | 0.3   | 0.03  | $0.2709 \times 10^{-4}$ | $0.4590 \times 10^{-4}$ | $0.458 \times 10^{-4}$ | $0.4310 \times 10^{-3}$ |
| 1.0 | 0.1   | 0.3   | $0.2831 \times 10^{-4}$ | $0.8258 \times 10^{-4}$ | $0.866 \times 10^{-4}$ | $0.1152 \times 10^{-2}$ |
|     | 0.3   | 0.03  | $0.4988 \times 10^{-4}$ | $0.9182 \times 10^{-4}$ | $0.916 \times 10^{-4}$ | $0.1268 \times 10^{-2}$ |

**Table 3 b: Maximum error norms for $V(x, t)$, $k_1 = 2$, $\Delta t = 0.01$, $N = 100$**

| $t$ | $k_2$ | $k_3$ | Present | \[18\] | \[11\] | \[34\] |
|-----|-------|-------|---------|------|------|------|
| 0.5 | 0.1   | 0.3   | $0.1247 \times 10^{-4}$ | $0.1480 \times 10^{-3}$ | $0.499 \times 10^{-4}$ | $0.3332 \times 10^{-3}$ |
|     | 0.3   | 0.03  | $0.7641 \times 10^{-4}$ | $0.5729 \times 10^{-3}$ | $0.181 \times 10^{-3}$ | $0.1148 \times 10^{-2}$ |
| 1.0 | 0.1   | 0.3   | $0.2474 \times 10^{-4}$ | $0.4770 \times 10^{-4}$ | $0.992 \times 10^{-4}$ | $0.1162 \times 10^{-2}$ |
|     | 0.3   | 0.03  | $0.1523 \times 10^{-4}$ | $0.3617 \times 10^{-3}$ | $0.362 \times 10^{-3}$ | $0.1638 \times 10^{-2}$ |

**Table 3 c: Maximum error norms for $U(x, t)$ in Problem 2 $k_1 = 2$, $\Delta t = 0.01$, $N = 21$**

| $t$ | $k_2$ | $k_3$ | Present | \[11\] | \[34\] | \[22\] | \[29\] |
|-----|-------|-------|---------|------|------|------|------|
| 0.5 | 0.1   | 0.3   | $0.13145 \times 10^{-2}$ | $0.144 \times 10^{-2}$ | $0.9619 \times 10^{-3}$ | $0.4173 \times 10^{-4}$ | $0.4189217417 \times 10^{-4}$ |
|     | 0.3   | 0.03  | $0.15686 \times 10^{-2}$ | $0.668 \times 10^{-3}$ | $0.4310 \times 10^{-3}$ | $0.4585 \times 10^{-4}$ | $0.4584830094 \times 10^{-4}$ |
| 1.0 | 0.1   | 0.3   | $0.25126 \times 10^{-2}$ | $0.127 \times 10^{-2}$ | $0.1153 \times 10^{-2}$ | $0.8275 \times 10^{-4}$ | $0.8269641708 \times 10^{-4}$ |
|     | 0.3   | 0.03  | $0.29666 \times 10^{-2}$ | $0.130 \times 10^{-2}$ | $0.1268 \times 10^{-2}$ | $0.9167 \times 10^{-4}$ | $0.9147335667 \times 10^{-4}$ |
| 3.0 | 0.1   | 0.3   | $0.68877 \times 10^{-2}$ | $0.2408 \times 10^{-3}$ | $0.2401202768 \times 10^{-3}$ | $0.2747 \times 10^{-3}$ | $0.2704203611 \times 10^{-3}$ |
|     | 0.1   | 0.03  | $0.70013 \times 10^{-2}$ | $0.2408 \times 10^{-3}$ | $0.2401202768 \times 10^{-3}$ | $0.2747 \times 10^{-3}$ | $0.2704203611 \times 10^{-3}$ |

**Table 3 d: Maximum error norms for $V(x, t)$ in Problem 2 $k_1 = 2$, $\Delta t = 0.01$, $N = 21$**

| $t$ | $k_2$ | $k_3$ | Present | \[11\] | \[34\] | \[22\] |
|-----|-------|-------|---------|------|------|------|
| 0.5 | 0.1   | 0.3   | $0.25322 \times 10^{-5}$ | $0.78143 \times 10^{-3}$ | $0.542 \times 10^{-3}$ | $0.3332 \times 10^{-3}$ |
|     | 0.3   | 0.03  | $0.10429 \times 10^{-4}$ | $0.29775 \times 10^{-2}$ | $0.120 \times 10^{-2}$ | $0.1148 \times 10^{-2}$ |
| 1.0 | 0.1   | 0.3   | $0.25602 \times 10^{-5}$ | $0.14757 \times 10^{-2}$ | $0.129 \times 10^{-2}$ | $0.1162 \times 10^{-2}$ |
|     | 0.3   | 0.03  | $0.10512 \times 10^{-4}$ | $0.57049 \times 10^{-2}$ | $0.235 \times 10^{-2}$ | $0.1638 \times 10^{-2}$ |
| 3.0 | 0.1   | 0.3   | $0.25980 \times 10^{-5}$ | $0.39769 \times 10^{-2}$ | $0.3119 \times 10^{-3}$ | $0.45054 \times 10^{-3}$ |
|     | 0.1   | 0.03  | $0.10542 \times 10^{-4}$ | $0.80324 \times 10^{-2}$ | $0.1663 \times 10^{-3}$ | $0.14983 \times 10^{-3}$ |
Figure 5: Numerical Solutions for $U(x, t)$ and $V(x, t)$, $N = 21$, $\Delta t = 0.001$, $t = 1$, $0 \leq x \leq 1$, $k_2 = 0.1$ and $k_3 = 0.1$

**Problem 3)** Consider the Coupled Burger equation (1) with the following initial conditions

$$U(x, 0) = \begin{cases} \sin(2\pi x), & x \in [0, 0.5] \\ 0, & x \in (0.5, 1) \end{cases}$$

$$V(x, 0) = \begin{cases} 0, & x \in [0, 0.5] \\ -\sin(2\pi x), & x \in (0.5, 1) \end{cases}$$

and zero boundary conditions. In the Problem 3, the solutions have been carried out on $x \in [0, 1]$ with $\Delta t = 0.001$ and number of partitions as 50. Maximum values of $u$ and $v$ at different time levels for $k_2 = k_3 = 10$ have been given in Table 4a and 4b, while the Tables 4c and 4d represent the maximum values for $k_2 = k_3 = 100$.

| Table 4 a: Maximum values of $U$ at different time levels for $k_2 = k_3 = 10$ |
|--------------------------|-----------------|
| $t$ | Present | [18] | [29] | at point |
|---|---|---|---|---|
| 0.1 | 0.142427 | 0.14456 | 0.144491495800 | 0.58 |
| 0.2 | 0.051716 | 0.05237 | 0.052356151890 | 0.54 |
| 0.3 | 0.019087 | 0.01932 | 0.019318838080 | 0.52 |
| 0.4 | 0.007099 | 0.00718 | 0.007184856672 | 0.50 |

| Table 4 b: Maximum values of $V$ at different time levels for $k_2 = k_3 = 10$ |
|--------------------------|-----------------|
| $t$ | Present | [18] | [29] | at point |
|---|---|---|---|---|
| 0.1 | 0.144178 | 0.14306 | 0.143141957500 | 0.66 |
| 0.2 | 0.049030 | 0.04697 | 0.047006446750 | 0.56 |
| 0.3 | 0.018049 | 0.01725 | 0.017260356430 | 0.52 |
| 0.4 | 0.006711 | 0.00641 | 0.006416614856 | 0.50 |
Table 4 c: Maximum values of $U$ at different time levels for $k_2 = k_3 = 100$

| $t$ | Present [18] | [29] at point |
|-----|--------------|---------------|
| 0.1 | 0.039322     | 0.04175       | 0.041682987260 | 0.46     |
| 0.2 | 0.013495     | 0.01479       | 0.014770415340 | 0.58     |
| 0.3 | 0.004874     | 0.00534       | 0.005337325631 | 0.54     |
| 0.4 | 0.001808     | 0.00198       | 0.001978065014 | 0.52     |

Table 4 d: Maximum values of $V$ at different time levels for $k_2 = k_3 = 100$

| $t$ | Present [18] | [29] at point |
|-----|--------------|---------------|
| 0.1 | 0.053927     | 0.05065       | 0.050737669860 | 0.76     |
| 0.2 | 0.011531     | 0.01033       | 0.010356602970 | 0.64     |
| 0.3 | 0.003970     | 0.00350       | 0.003517189432 | 0.56     |
| 0.4 | 0.001464     | 0.00129       | 0.001294450199 | 0.52     |

Figs. 6, 7 and 8 show the numerical results obtained for different time levels $t \in [0, 1]$ at $k_2 = k_3 = 10$ for $U$ and $V$ with different values of $k_1$. From the Figs. 7-9, it can be easily seen that the numerical solutions $U^n$ and $V^n$ decay to zero as $t$ and $k_1$ increased.

Fig 6: Num. Sol. $U(x,t)$ and $V(x,t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 1$
Fig 7: Num. Sol. $U(x,t)$ and $V(x,t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 50$

Fig 8: Num. Sol. $U(x,t)$ and $V(x,t)$ of Problem 3 at different time levels for $k_2 = k_3 = 10$ while $k_1 = 100$
5 Conclusion

Evaluation of the suggested algorithm has done by studying three test problem. The first two of them have analytical solution. For the problem 1, the B-spline($\lambda = 0$) collocation algorithm provides the same error with the suggested algorithm whereas, for the problem 2, errors of the suggested algorithm is smaller than that of the B-spline collocation method. The cubic B-spline with additional term defines the extended B-spline functions. The extended B-spline collocation method with a free parameters gives smaller error than both the cubic B-spline and cubic trigonometric cubic B-spline collocation methods. Shock propagation is studied in the text problem 3. Tabulated results and graphical solutions cohere with results of both the Galerkin quadratic B-spline finite element method and the modified B-spline collocation method. As a result, the trigonometric B-spline based numerical algorithm can be used for solving the system of partial differential equation reliably.

References

[1] S. E. Esipov, Coupled Burgers Equations- A Model of Polydispersive sedimentation, James Franck Institute and Department of Physics, University of Chicago, 1995.

[2] S. E. Esipov, Coupled Burgers equation :A Model of Polydispersive sedimentation, Phys. Rev E, 52(1995), 3711-3718.

[3] A. Nikolis, “Numerical solutions of ordinary differential equations with quadratic trigonometric splines,” Applied Mathematics E-Notes, 4(1995), 142-149.

[4] J. Nee and J. Duan, Limit set of trajectories of the coupled viscous Burgers’equation, Appl. Math. Lett., 11(1998),57-61.

[5] S. F. Radwan, On the Fourth-Order Accurate Compact ADI Scheme for Solving the Unsteady Nonlinear Coupled Burgers’ Equations, Journal of Nonlinear Mathematical Physics, 6(1)(1999),13-34.

[6] D Kaya, An explicit solution of coupled viscous Burger’s equation by decomposition method, I.J.N. M. S. 27(11)(2011), 675-650.

[7] M. A. Abdou and A. A. Soliman, Variational iteration method for solving Burgers and coupled Burgers, J. Comput. Appl. Meth. 181(2)(2005), 245-251.

[8] A. Nikolis and I. Seimenis, Solving dynamical systems with cubic trigonometric splines, Applied Mathematics E-notes, 5(2005), 116-123.

[9] A. A. Soliman, Modified-extended tanh-function method for solving Burgers-type equations, Physica A, 361(2006), 394-404.
10. M. Degham, A. Hamidi and M Shakourifar, The solution of the coupled Burgers equation using Adomian-Pade technique, Appl. Math. Comput., 189(2007), 1034-1047.

11. A. H. Khater, R. S. Temsah and M. M. Hassan, A Chebyshev spectral collocation method for solving Burgers’-type equations, Journal of Computational and Applied Mathematics, 222(2008), 333–350.

12. A. R. Ghotbi, A. Avaei and A. Barari, M. A. Mohammadzade, Assessment of He’s homotopy perturbation method in Burgers and coupled Burgers equations, Journal of Applied Sciences, 8(2)(2008), 322-327.

13. N. H. Sweilam and M. M. Khader, Exact solutions of some coupled nonlinear differential equations using the homotopy perturbation method, Computer and mathematics with applications,58(2000), 2134-2141.

14. A. Rashid and A. I. B. MD. Ismail, A Fourier Pseudospectral Method for Solving Coupled Viscous Burgers Equations, Computational Methods in Applied Mathematics, 9(4)(2009),412-420.

15. Nur Nadiah Abd Hamid , Ahmad Abd. Majid, and Ahmad Izani Md. Ismail , Cubic Trigonometric B-Spline Applied to Linear Two-Point Boundary Value Problems of Order, World Academy of Science, Engineering and Technology 70 798-803, 2010.

16. J. Liu and G. Hou, Numerical solutions of the space- and time-fractional coupled Burgers equations by generalized differential transform method, Applied Mathematics and Computation Vol. 217, pp. 7001–7008, 2011.

17. Yogesh Gupta and Manoj Kumar A Computer based Numerical Method for Singular Boundary Value Problems, International Journal of Computer Applications Vol. 30, No 1, pp. 21-25, 2011.

18. R. C. Mittal and G. Arora, Numerical solution of the coupled viscous Burgers’ equation, Commun Nonlinear Sci Numer Simulat, Vol. 16, pp. 1304–1313, 2011.

19. R. Mokhtari, A. S. Toodar and N. G. Chengini, Application of the Generalized Differential Quadrature Method in Solving Burgers’ Equations, Commun. Theor. Phys. Vol. 56, pp. 1009–1015, 2011.

20. I. Sadek and I. Kucuk, A robust technique for solving optimal control of coupled Burgers’ equations, IMA Journal of Mathematical Control and Information Vol:28, 239-250, 2011.

21. Z. Rong-Pei, Y. Xi-Jun and Z. Guo-Zhong, Local discontinuous Galerkin method for solving Burgers and coupled Burgers equations, Chinese Physics B, 2011.

22. R.C. Mittal and Ram Jiwar, A differential quadrature method for numerical solutions of Burgers’-type equations, International Journal of Numerical Methods for Heat & Fluid, Vol. 22 No. 7, pp. 880 - 895, 2012.
[23] X Jia, H Li, Y Liu and Z. Fang, An $H^1$-Galerkin mixed method for the coupled Burgers equation, International Journal of Computational and Mathematical Sciences, Vol 6, 163-166, 2012.

[24] K. R. Desai V. H. Pradhan, Solution of Burgers equation and coupled Burgers equation by Homotopy perturbation method, International journal of Engineering Research and applications, Vol 2, No 3, 2033-2040, 2012.

[25] S. Kutluay and Y. Ucar, Numerical solutions of the coupled Burgers’ equation by the Galerkin quadratic B-spline finite element method, Math. Meth. Appl. Sci. 2013, 36 2403–2415.

[26] V.K. Srivastava, M. K. Awasthi, and M. Tamsir, A fully implicit Finite-difference solution to one dimensional Coupled Nonlinear Burgers’ equations, International Journal of Mathematical, Computational, Physical and Quantum Engineering Vol. 7 No:4, pp. 417-422, 2013.

[27] M. Kumar and S. Pandit, A composite numerical scheme for the numerical simulation of coupled Burgers’ equation, Computer Physics Communications Vol. 185, pp. 809–817, 2014.

[28] V. K. Srivastava, M. Tamsir, M. K. Awasthi and S. Sing, One-dimensional coupled Burgers’ equation and its numerical solution by an implicit logarithmic finite-difference method, Aip Advances, Vol. 4, 037119, 2014.

[29] R. C. Mittal and A. Tripathi, A Collocation Method for Numerical Solutions of Coupled Burgers’ Equations, International Journal for Computational Methods in Engineering Science and Mechanics, Vol 15, pp. 457–471, 2014.

[30] M. Abbas, A. A. Majid, A. I. M İsmail and A. Rashid, The application of the cubic trigonometric B-spline to the numerical solution of the hyperbolic problems, Applied Mathematica and Computation, Vol 239, 74-88, 2014.

[31] M. Abbas, A. A. Majid, A. I. M İsmail and A. Rashid, Numerical method using cubic trigonometric B-spline technique for nonclassical diffusion problems, Abstract and applied analysis, 2014.

[32] S. G. Rubin and R. A. Graves, Cubic spline approximation for problems in fluid mechanics, Nasa TR R-436, Washington, DC, 1975.

[33] A. M. Aksoy, Numerical Solutions of Some Partial Differential Equations Using the Taylor Collocation-Extended Cubic B-spline Functions, Department of Mathematics, Doctoral Dissertation, 2012.

[34] Siraj-ul-Islam S, Haq M, Uddin. A meshfree interpolation method for the numerical solution of the coupled nonlinear partial differential equations. Eng. Anal Boundary Elem 2009;33:399–409.