Singular cycles connecting saddle periodic orbit and saddle equilibrium in piecewise smooth systems

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Abstract For flows, the singular cycles connecting saddle periodic orbit and saddle equilibrium can potentially result in the so-called singular horseshoe, which means the existence of a non-uniformly hyperbolic chaotic invariant set. However, it is very hard to find a specific dynamical system that exhibits such singular cycles in general. In this paper, the existence of the singular cycles involving saddle periodic orbits is studied by two types of piecewise smooth systems: One is the piecewise smooth systems having an admissible saddle point with only real eigenvalues and an admissible saddle periodic orbit, and the other is the piecewise smooth systems having an admissible saddle-focus and an admissible saddle periodic orbit. Several kinds of sufficient conditions are obtained for the existence of only one heteroclinic cycle or only two heteroclinic cycles in the two types of piecewise smooth systems, respectively. In addition, some examples are presented to illustrate the results.

Keywords Singular cycles · Periodic orbits · (Un)stable manifolds · Piecewise smooth systems

1 Introduction

Singular cycles (a singular cycle generally refers to a homoclinic orbit connecting a singularity to itself, or a heteroclinic cycle connecting different singular elements at least one of which is a singularity [2]) are one of the most important mechanisms leading to complicated dynamic behaviors and bifurcations [12, 20, 24, 30, 31]. For example, the well-known Shil’nikov theory [24, 31] shows that the existence of singular cycles involving only equilibrium points implies the existence of a countable number of chaotic invariant sets under some conditions. In addition, another type of singular cycle connecting a saddle equilibrium point and a saddle periodic orbit can generate a kind of so-called singular horseshoe which means the existence of a non-uniformly hyperbolic chaotic invariant set, and the so-called singular horseshoe is also the basis for the theoretical study of the famous Lorenz system and the geometrical Lorenz model [1, 2, 6, 10, 16, 21, 28]. As we know, the preconditions for the Shil’nikov theory and the singular horseshoe theory are both the existence of the corresponding types of singular cycles, while proving their existence in a concrete smooth system is very difficult, which has been a hot research topic for many years [8, 9, 17, 18, 27].

Recently, the dynamics in piecewise smooth systems have received much attention due to the wide applications in various fields of science, such as electrical circuits [7, 13, 25, 26, 32], walking robots [11, 19], con-
control system [3, 15] and economics and finance [4, 5].
Especially, to some extent, the existences of some
types of singular cycles can be implemented more easily
in piecewise smooth systems than smooth systems
[13, 14, 23, 29, 32]. Consequently, based on the
generalized Shil’nikov chaotic theory in piecewise smooth
systems, some chaotic circuits have been designed
successfully by using the piecewise smooth models
possessing the corresponding types of singular cycles
[13, 14, 23, 32]. However, these singular cycles in these ref-

erences involve only equilibrium points but no periodic
orbits. To the best of our knowledge, the study on the
existence of singular cycles having a periodic orbit as
one of the critical elements in piecewise smooth sys-
tems has hardly appeared in the literature.

The main interest of this paper is the existence of
singular cycles involving saddle periodic orbits in a
class of piecewise smooth systems. Two types of such
singular cycles are considered: One is the heteroclinic
cycle connecting a saddle point with only real eigen-
values and a saddle periodic orbit, and the other is the
heteroclinic cycle connecting a saddle-focus point and
a saddle periodic orbit. For each type of singular cycles,
we obtain the sufficient conditions under which there
exists only one heteroclinic cycle or only two hete-

clinic cycles involving saddle periodic orbits in a
class of piecewise smooth systems. Two main results (Theorems 1,
2) of this paper. Section 3 gives some preliminaries.
Sections 4 and 5 present the proof of Theorems 1 and
2, respectively. Section 6 introduces some examples.
Section 7 gives some conclusions and prospects.

2 Systems and main results

Consider the following piecewise smooth system:

\[ \dot{x} = \begin{cases} (A - \text{diag}(x_1^2 + x_2^2, x_1^2 + x_2^2, 0))x, & \text{if } x \in \Sigma \cup \Sigma^- \\ B(x - q), & \text{if } x \in \Sigma^+ \end{cases}, \]  

where \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \) is a vector of state vari-
ables, \( q = (q_1, q_2, q_3)^T \in \mathbb{R}^3 \),

\[ \Sigma = \{ x \in \mathbb{R}^3 | c^T x = d \}, \quad \Sigma^- = \{ x \in \mathbb{R}^3 | c^T x < d \}, \quad \Sigma^+ = \{ x \in \mathbb{R}^3 | c^T x > d \}, \]  

with \( d > 0 \) being a constant and \( c = (1, 0, 1)^T \in \mathbb{R}^3 \), and

\[
A = \begin{pmatrix} \rho & -\omega & 0 \\ \omega & \rho & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & \lambda \end{pmatrix}
\]

with \( \rho > 0, \omega > 0, \mu > 0, \lambda > 0 \) and \( b_{ij} \in \mathbb{R} \) \( (i, j = 1, 2) \). Moreover, matrix \( B \) satisfies the fol-
lowing hypotheses:

\[ (H1) \] the eigenvalues of \( B: \lambda_{1,2} < 0 \) and \( \lambda > 0 \),
or \[ (H2) \] the eigenvalues of \( B: \alpha \pm \beta i \) and \( \lambda \) where \( \alpha < 0, \beta > 0 \) and \( \lambda > 0 \).

And matrix \( A \) and \( q \) satisfy the following hypothesis

\[ (H3) \] \( 0 < \sqrt{\rho} < d, \ c^T q > d, \ q_1 = d. \]

Denote by

\[ \phi_A(t, \cdot) \] and \( \phi_B(t, \cdot) \)

the flows generated by the left subsystem

\[ \dot{x} = (A - \text{diag}(x_1^2 + x_2^2, x_1^2 + x_2^2, 0))x, \quad x \in \mathbb{R}^3 \]  

and right subsystem

\[ \dot{x} = B(x - q), \quad x \in \mathbb{R}^3, \]

respectively. Obviously, \( q \) is the only saddle equilib-
rium of (4) with its stable manifold and unstable man-
ifold being

\[ W^s(q) = \{ x \in \mathbb{R}^3 | x_3 = q_3 \}, \]

\[ W^u(q) = \{ x \in \mathbb{R}^3 | x_1 = q_1, x_2 = q_2 \}, \]

respectively. Furthermore, by the polar coordinates
transformation, (3) can be transformed to

\[
\begin{cases}
\dot{r} = r(\rho - r^2) \\
\dot{\theta} = \omega \\
\dot{x}_3 = \mu x_3
\end{cases}
\]

Let

\[ \gamma = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 = \rho, x_3 = 0 \}. \]

It is not hard to see that \( \gamma \) is the only saddle periodic
orbit of (3) with its stable manifold and unstable man-
ifold being (refer the analysis for (20) in Sect. 3),

\[ W^s(\gamma) = \{ 0 \neq x \in \mathbb{R}^3 | x_3 = 0 \}, \]

\[ W^u(\gamma) = \{ x \in \mathbb{R}^3 | x_1^2 + x_2^2 = \rho \}, \]

respectively.

From \( H(3) \), it is easy to see that

\[ q \in \Sigma^+ \text{ and } \gamma \subseteq \Sigma^- \]

which shows \( q \) and \( \gamma \) are, respectively, the admissible
saddle equilibrium and admissible saddle periodic orbit
of system (1).
For simplifying the statements of the following theorems, let
\[
p_0 = (\sqrt{\rho}, 0, d - \sqrt{\rho})^T \in \Sigma, \\
p_1 = (-\sqrt{\rho}, 0, d + \sqrt{\rho})^T \in \Sigma, \\
q_0 = (d, q_2, 0)^T \in \Sigma, \\
q_1 = (d, q_2, 0)^T \in \Sigma, \\
L_1 = \{x \in \mathbb{R}^3 \mid x_1 = d, x_3 = 0\} \subset \Sigma, \\
L_2 = \{x \in \mathbb{R}^3 \mid x_1 = d - q_3, x_3 = q_3\} \subset \Sigma, \\
\sigma_{\pm} = -\omega \pm \sqrt{\omega^2 - 4d^2(d^2 - \rho)}.
\]

\[
v_+ = (d, \sigma_+, 0)^T \in L_1, \\
x_- = \frac{d - c^T q}{c^T B^{-1} c^\perp} B^{-1} c^\perp + q \quad \text{with} \quad c^\perp = (0, 1, 0)^T.
\]

It is readily achieved that $c^T x_- = d$. Thus, $x_- \in \Sigma$. Moreover, denote the closed line segment, the open line segment and the half-closed line segment between $x_1$ and $x_2$ $(x_1, x_2 \in \mathbb{R}^3$ (or $\mathbb{R}^2$)) by
\[
[x_1, x_2] = \{x \mid x = \lambda x_1 + (1 - \lambda) x_2, 0 \leq \lambda \leq 1\}, \\
(x_1, x_2) = \{x \mid x = \lambda x_1 + (1 - \lambda) x_2, 0 < \lambda < 1\}, \quad (13) \\
[x_1, x_2] = [x_1, x_2] - [x_2] \quad \text{and} \quad (x_1, x_2) = [x_1, x_2] - [x_1],
\]
respectively.

Two main results on the existence of heteroclinic cycles connecting $\mathcal{Y}$ and $q$ can be presented in the following two theorems, which will be proved in Sects. 4 and 5, respectively.

**Theorem 1** For system (1) with hypotheses (H1) and (H3)

(i) When $d^2 - \rho \geq \frac{\alpha^2}{4d^2}$,

(a) there exists only one heteroclinic cycle connecting $\mathcal{Y}$ and $q$ if $q_3 = d - \sqrt{\rho}$ and $c^T B (p_0 - q) \geq 0$;
(b) there exists only one heteroclinic cycle connecting $\mathcal{Y}$ and $q$ if $q_3 = d + \sqrt{\rho}$, $\frac{\alpha^2}{4} \rho < \mu^2 (d^2 - \rho)$ and $c^T B (p_1 - q) \geq 0$;
(c) there exist only two heteroclinic cycles each of which connects $\mathcal{Y}$ and $q$ if $q_3 \in (d - \sqrt{\rho}, d + \sqrt{\rho})$, $\frac{\alpha^2}{4} \rho < \mu^2 (d^2 - \rho)$ and $c^T B (p_{\pm} - q) \geq 0$, where
\[
p_{\pm} = \left( d - q_3, \pm \sqrt{\rho} - (d - q_3)^2, q_3 \right), \quad (14)
\]
$\mathcal{Y}$ is given as (7), $p_0$ and $p_1$ are given in (9).

(ii) When $0 < d^2 - \rho < \frac{\alpha^2}{4d^2}$, then $L_1 \cap \{\phi_A (t, v_+) \mid t < 0\} \neq \emptyset$ with $v_+ \in L_1$ is given in (11). Denote by
\[
v_+ = (d, \sigma_+, 0)^T \in L_1 \\
\text{the first intersection of} \ L_1 \text{and backward-time orbit} \ \phi_A (t, v_+) \mid t < 0.
\]

Then, of all the three conclusions in (a), (b) and (c) in (i) still hold if $\sigma_+ > \sigma_+$ and $q_2 \in [\sigma_+, \sigma_-]$, or if $\sigma_- < \sigma_- \quad \text{and} \quad q_2 \in (-\infty, \sigma_-) \cup [\sigma_+, +\infty)$. Here, $\sigma_{\pm}$ are given by (11).

**Theorem 2** Suppose that system (1) satisfies hypotheses (H2) and (H3); then, $L_2 \cap \{\phi_B (t, x_-) \mid t < 0\} \neq \emptyset$ where $x_-$ and $L_2$ are defined as (12) and (10), respectively. Denote by $x_+$ the first intersection of $L_2$ and backward-time orbit $\{\phi_B (t, x_-) \mid t < 0\}$.

(i) When $d^2 - \rho \geq \frac{\alpha^2}{4d^2}$,

(a) there exists only one heteroclinic cycle connecting $\mathcal{Y}$ and $q$ if $q_3 = d - \sqrt{\rho}$ and $p_0 \in [x_-, x_+]$;
(b) there exists only one heteroclinic cycle connecting $\mathcal{Y}$ and $q$ if $q_3 = d + \sqrt{\rho}$, $\frac{\alpha^2}{4} \rho < \mu^2 (d^2 - \rho)$ and $p_1 \in [x_-, x_+]$;
(c) there exists only two heteroclinic cycles each of which connects $\mathcal{Y}$ and $q$ if $q_3 \in (d - \sqrt{\rho}, d + \sqrt{\rho})$, $\frac{\alpha^2}{4} \rho < \mu^2 (d^2 - \rho)$ and $p_{\pm} \in [x_-, x_+]$, where $p_{\pm}$ is given in (14) and $[x_-, x_+]$ is defined as (13).

(ii) When $0 < d^2 - \rho < \frac{\alpha^2}{4d^2}$, all of the three conclusions in (a), (b) and (c) in (i) still hold if $\sigma_+ > \sigma_+ \quad \text{and} \quad q_2 \in [\sigma_+, \sigma_-]$, or if $\sigma_- < \sigma_- \quad \text{and} \quad q_2 \in (-\infty, \sigma_-) \cup [\sigma_+, +\infty)$. Here, $\sigma_{\pm}$ are given by (11) and $\sigma_+$ is given in (15).

To prove Theorems 1 and 2, some preliminaries are needed in Sect. 3.

**3 Preliminaries: some important results in planar smooth systems**

3.1 A result on a nonlinear planar system with one stable periodic trajectory

Consider the following planar smooth system
\[
\dot{x} = (A_0 - ||x||^2 I)x, \quad (16)
\]
where \( \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 \), \( A_0 = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \) with \( \rho > 0 \) and \( \omega > 0 \), \( ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2} \) and \( I \) denotes the identity matrix of order 2.

For \( \mathbf{x}_0 \in \mathbb{R}^2 \), denote by \( O(\mathbf{x}_0) \), \( O_+(\mathbf{x}_0) \) and \( O_-(\mathbf{x}_0) \) the whole orbit, the positive semi-orbit and the negative semi-orbit of \( \mathbf{x}_0 \), respectively, i.e.,

\[
O(\mathbf{x}_0) = \{ \phi(t, \mathbf{x}_0) | t \in \mathbb{R} \}, \\
O_+(\mathbf{x}_0) = \{ \phi(t, \mathbf{x}_0) | t > 0 \} \\
O_-(\mathbf{x}_0) = \{ \phi(t, \mathbf{x}_0) | t < 0 \},
\]

where \( \phi(t, \cdot) \) denotes the flow generated by system (16). Let

\[
L = \{ \mathbf{x} \in \mathbb{R}^2 | x_1 = k \}
\]

with \( k > 0 \). Then, \( L \) is perpendicular to \( x_1 \)-axis, does not pass through the origin and divides the plane into three disjoint subsets \( L, L^+ \) and \( L^− \), where

\[
L^+ = \{ \mathbf{x} \in \mathbb{R}^2 | x_1 > k \}, \\
L^− = \{ \mathbf{x} \in \mathbb{R}^2 | x_1 < k \}.
\]

Obviously, the origin is in \( L^− \). In addition, let

\[
\varrho_\pm = -\omega \pm \sqrt{\omega^2 - 4k^2(k^2 - \rho)}. \tag{17}
\]

**Proposition 1** For system (16)

(i) When \( k^2 - \rho \geq \frac{\omega^2}{4k^2} \)

\( O_+(\mathbf{x}) \subset L^− \) for any \( \mathbf{x} \in L \).

(ii) When \( 0 < k^2 - \rho < \frac{\omega^2}{4k^2} \), let

\[
\mathbf{u}_\pm = (k, \varrho_\pm)^T.
\]

Then, \( O_−(\mathbf{u}_+) \) intersects with \( L \). Denote by \( \mathbf{u}_\pm = (k, \varrho_\pm)^T \in L \cap O_−(\mathbf{u}_+) \) the first intersection of \( L \) and backward-time orbit \( O_−(\mathbf{u}_+) \).

(a) If \( \varrho_+ > \varrho_− \), then

\( O_+(\mathbf{x}) \subset L^− \cup L \iff \mathbf{x} \in [\mathbf{u}_+, \mathbf{u}_−], \) for \( \mathbf{x} \in L \)

and

\( O_+(\mathbf{x}) \subset L^− \iff \mathbf{x} \in (\mathbf{u}_−, \mathbf{u}_+), \) for \( \mathbf{x} \in L \).

(b) If \( \varrho_+ < \varrho_− \), then

\( O_+(\mathbf{x}) \subset L^− \cup L \iff \mathbf{x} \in L−(\mathbf{u}_+, \mathbf{u}_−), \) for \( \mathbf{x} \in L \)

and

\( O_+(\mathbf{x}) \subset L^− \iff \mathbf{x} \in L−(\mathbf{u}_+, \mathbf{u}_−), \) for \( \mathbf{x} \in L \).

**Proof** In \( L \), the points at which the vector field is tangent to \( L \) have to meet the following equations on \( x_1 \) and \( x_2 \).

\[
\begin{cases}
x_1 = k \\
\rho x_1 - \omega x_2 - x_1(x_1^2 + x_2^2) = 0,
\end{cases}
\]

from which we have

\[
kx_2^2 + \omega x_2 + k(k^2 - \rho) = 0. \tag{18}
\]

\( \square \)

**Case (i)** When \( k^2 - \rho \geq \frac{\omega^2}{4k^2} \)

This case is trivial. In fact, since \( \dot{x}_1 = -kx_2^2 - \omega x_2 + k(\rho - k^2) \leq 0 \) in \( L \), the vector field at any point of \( L \) either points to the interior of \( L^− \) or is tangent to \( L \) at only one point (i.e., \( (k, \frac{-\omega}{2k})^T \)). Hence, for any \( \mathbf{x} \in L \), \( O_+(\mathbf{x}) \subset L^− \).

**Case (ii)** When \( 0 < k^2 - \rho < \frac{\omega^2}{4k^2} \)

In the case, \( \varrho_\pm \) defined by (17) are just two negative real roots of (18). Thus, \( \mathbf{u}_\pm = (k, \varrho_\pm)^T \) are the only two points in \( L \) at which the vector fields are tangent to \( L \), and divide \( L \) into the following three segments: \( L_u, L_m, L_d \) (see Figs. 1, 2) where

\[
L_u = \{ \mathbf{x} \in L | x_2 > \varrho_+ \}, \\
L_m = \{ \mathbf{x} \in L | \varrho_- < x_2 < \varrho_+ \}, \\
L_d = \{ \mathbf{x} \in L | x_2 < \varrho_- \}.
\]

Obviously, \( L = L_u \cup \mathbf{u}_+ \cup L_m \cup \mathbf{u}_- \cup L_d \). Now, we analyze the directions of vector fields of system (16) at \( L_u, L_m, L_d \), respectively.

(1) Provided, \( \mathbf{x} = (x_1, x_2)^T \in L_u \).

Since \( x_2 > \varrho_+ \), we have \( \dot{x}_1 = -kx_2^2 - \omega x_2 - k(k^2 - \rho) < 0 \) by the general nature of quadratic function, which shows that in \( L_u \) the vector field of system (16) must be transverse to the direction of \( L \) and point to the interior of \( L^− \), see Fig. 1.

(2) Provided, \( \mathbf{x} = (x_1, x_2)^T \in L_m \).

Since \( \varrho_- < x_2 < \varrho_+ \), \( \dot{x}_1 = -kx_2^2 - \omega x_2 - k(k^2 - \rho) > 0 \) in \( L_m \). Therefore, in \( L_m \) the vector field of (16) must be transverse to the direction of \( L \) and point to the interior of \( L^+ \), see Fig. 1.

(3) Provided, \( \mathbf{x} = (x_1, x_2)^T \in L_d \).

Similarly, it can be readily got that in \( L_d \) the vector field is transverse to the direction of \( L \) and points to the interior of \( L^− \), see Fig. 1.
Furthermore, simple calculation shows that

\[ x_2|_{u_\pm} = (\omega x_1 + \rho x_2 - x_2(x_1^2 + x_2^2))|_{(x_1, x_2) = (k, \varrho_{\pm})} \]
\[ = ok + \rho \varrho_{\pm} - k^2 \varrho_{\pm} - \varrho_{\pm}^3 \]
\[ > ok > 0. \tag{19} \]

Combining (19) with the discussions in (1), (2) and (3) shows \( \phi(t, u_+) \in L^- \) and \( \phi(t, u_-) \in L^+ \) for small \( t \neq 0 \) by continuity.

In addition, by the polar coordinates transformation, (16) can be transformed to

\[ \begin{aligned}
\dot{r} &= r(\rho - r^2) \\
\dot{\theta} &= \omega
\end{aligned} \tag{20} \]

As we all know, (20) has an asymptotically stable limit cycle \( \Gamma : r = \sqrt{\rho} \) with its attracting region being \( \mathbb{R}^2 - \{O\} \), and the flow \( r(t, r_0) \) with \( r_0 > \sqrt{\rho} \) will tend to positive infinity with clockwise rotation around \( \Gamma \) as \( t \to -\infty \).

Thus, \( O_-(u_+) \) must intersect \( L \) infinite times. Denote by \( u_+ = (k, \varrho_2)^T = \phi(t_0, u_+) \) the first intersection of \( L \) and backward-time orbit \( O_-(u_+) \). Then, from the direction of vector field in \( L \) shown in (1), (2) and (3), \( u_+ \) must be contained either in \( L_u \) or in \( L_d \).

(a) If \( u_+ \in L_u \) (i.e., \( \varrho_2 > \varrho_+ \)), see Fig. 1.

Consider the open region \( D_1 \) surrounded by \( \{u_+, u_+\} \cup \{\phi(t, u_+)\}_{t_0 < t < 0} \) as shown in Fig. 1. Obviously, \( D_1 \) is a positively invariant set contained in \( L^- \). Therefore, for any \( x \in (u_+, u_+) \), \( \phi(x) \subset L^- \) and \( \phi(t, x) \to \Gamma \) as \( t \to +\infty \). On the other hand, since the attracting region of \( \Gamma \) is \( \mathbb{R}^2 - \{O\} \), for any \( x \in (L_u - [u_+, u_+]) \cup L_d \), the flow \( \phi(t, x) \) under the forward-time must first enter into \( L^- - D_1 \) and then leave \( L^- \) by intersecting \( L_m \), and enter into \( D_1 \) ultimately by intersecting \( (u_+, u_+) \). Moreover, it is obvious that for \( x \in L_u \), under the forward-time \( \phi(t, x) \) will first enter into \( L^+ \) and eventually enter into \( D_1 \) by intersecting \( (u_+, u_+) \). In conclusion, \( O_+(x) \subset L^- \Leftrightarrow x \in (u_+, u_+) \), for \( x \in L \).

Furthermore, since \( O_+(u_+) \) is tangent to \( L \) at \( u_+ \), \( O_+(x) \subset L^- \cup L \Leftrightarrow x \in [u_+, u_+] \), for \( x \in L \).

(b) If \( u_+ \in L_d \) (i.e., \( \varrho_2 < \varrho_- \)), see Fig. 2.

Consider the open region \( D_2 \) surrounded by \( \{u_+, u_+\} \cup \{\phi(t, u_+)\}_{t_0 < t < 0} \). Obviously, \( L^- - \tilde{D}_2 \) is a positively invariant set contained in \( L^- \), where \( \tilde{D}_2 \) is the closure of \( D_2 \), i.e., \( \tilde{D}_2 = D_2 \cup \{u_+, u_+\} \cup \{\phi(t, u_+)\}_{t_0 < t < 0} \). Furthermore, for any \( x \in (u_-, u_-) \), under the forward-time \( \phi(t, x) \) must enter first into \( D_2 \) and then enter into \( L^+ \) by intersecting \( L_m \), and eventually enter into \( L^- \) by intersecting \( L_u \). Combining the directions of the vector fields in \( L \) shows \( O_+(x) \subset L^- \cup L \Leftrightarrow x \in L - (u_+, u_+) \), for \( x \in L \) and \( O_+(x) \subset L^- \Leftrightarrow x \in L - (u_+, u_+) \), for \( x \in L \).

□

From the proof of Proposition 1, it is not hard to get the following conclusion if omitting the points in \( L \) at which the vector fields are tangent to \( L \) in all cases.
Corollary 1 For system (16) and any \( x \in L \)

(i) When \( 0 < k^2 - \rho < \frac{\omega^2}{4k^2} \)
   (a) If \( \varphi_0 > \varphi_* \), \( O_+(x) \subset L^- \) and \( O(x) \) intersect \( L \) at \( x \) transversely if and only if \( x \in (u_+, u_-) \).
   (b) If \( \varphi_0 < \varphi_* \), \( O_+(x) \subset L^- \) and \( O(x) \) intersect \( L \) at \( x \) transversely if and only if \( x \arrows_{\varphi} \in \bigcup_{t\varphi \in [u_+, u_-]} \bigcup_{\varphi_0 \in \varphi_*} \bigcup_{L} \).

(ii) When \( k^2 - \rho = \frac{\omega^2}{4k^2} \), \( O_+(x) \subset L^- \) and \( O(x) \) intersect \( L \) at \( x \) transversely if and only if

\[
x \in L - \left\{ \left( k, \frac{-\omega}{2k} \right)^T \right\}.
\]

(iii) When \( k^2 - \rho > \frac{\omega^2}{4k^2} \), \( O_+(x) \subset L^- \) and \( O(x) \) intersect \( L \) at \( x \) transversely.

3.2 Two useful results on planar linear systems

Consider general planar linear system as follows:

\[
\dot{x} = A_0 x, \quad x \in \mathbb{R}^2.
\]  
(21)

Let

\[
0 \neq k = (k_1, k_2)^T \in \mathbb{R}^2,
\]
and

\[
L = \{x \in \mathbb{R}^2 | k^T x = 1\}, \quad L^- = \{x \in \mathbb{R}^2 | k^T x < 1\}.
\]

Proposition 2 [29] For system (21), suppose that the eigenvalues of \( A_0 \) are given by \( \mu_{1,2} < 0 \), and then, for any \( x \in L^- \),

(i) \( O_+(x_0) \subset L^- \) if and only if \( k^T A_0 x_0 \leq 0 \),
   (ii) \( O_+(x_0) \subset L^- \) and \( O_+(x_0) \) intersect \( L \) at \( x_0 \) transversely if and only if \( k^T A_0 x_0 < 0 \).

Proposition 3 [29] For system (21), suppose that the eigenvalues of \( A_0 \) are given by

\[
\alpha \pm \beta i
\]
with \( \alpha < 0, \beta > 0 \) and \( i = \sqrt{-1} \). Let

\[
x_* = \frac{1}{k^T A_0^{-1} k} A_0^{-1} k^T
\]

with \( k^T = (-k_2, k_1)^T \). Obviously, \( x_* \in L \) is the only point in \( L \) at which the vector field of system (21) is tangent to \( L \), and \( L \cap O_-(x_*) \neq \emptyset \). Moreover, denote by \( x^* \) the first intersection of \( L \) and the backward-time orbit \( O_-(x_*) \).

Then, for \( x_0 \in L^- \),

(i) \( O_+(x_0) \subset L^- \cup L \) if and only if \( x_0 \in [x_*, x^*] \),
   (ii) \( O_+(x_0) \subset L^- \) if and only if \( x_0 \in [x_*, x^*] \),
   (iii) \( O_+(x_0) \subset L^- \) and \( O(x_0) \) intersect \( L \) at \( x_0 \) transversely if and only if \( x_0 \in (x_*, x^*) \), where \( [x_*, x^*], [x_*, x^*] \) and \( (x_*, x^*) \) are defined as (13).

4 Proof of Theorem 1

4.1 When \( d^2 - \rho \geq \frac{\omega^2}{4d^2} \)

To prove the existence of a heteroclinic cycle, the key is to prove the existence of two heteroclinic orbits which can form a cycle connecting \( \Upsilon \) and \( q \) each other.

From (5), (8) and (9), we have

\[ q_0 = W^s(\Upsilon) \cap \Sigma \cap W^u(q). \]  
(22)

Then,

\[ \phi_B(t, q_0) \rightarrow q(t \rightarrow -\infty), \phi_A(t, q_0) \rightarrow \Upsilon(t \rightarrow +\infty). \]  
(23)

Lemma 1 \( \Gamma_1 \) is the only heteroclinic orbit from \( q \) to \( \Upsilon \) for system (1), where

\[ \Gamma_1 = \{ \phi_B(t, q_0) | t < 0 \} \cup q_0 \cup \{ \phi_A(t, q_0) | t > 0 \}. \]  
(24)

Proof From (22) and (23), it is sufficient to prove

\[ \{ \phi_B(t, q_0) | -\infty < t < 0 \} \subset \Sigma^+ \cup \Sigma, \]  
(25)

\[ \{ \phi_A(t, q_0) | 0 < t < +\infty \} \subset \Sigma^- \cup \Sigma. \]  
(26)

Obviously, \( \{ \phi_B(t, q_0) | -\infty < t < 0 \} = (q, q_0) \) which belongs to \( \Sigma^+ \) since \( q_0 \in \Sigma \) and \( q \in \Sigma^+ \). Thus, (25) holds. In addition, we have

\[ q_0 \in L_1 \]  
and \( \{ \phi_A(t, q_0) | 0 < t < +\infty \} \subset W^s(\Upsilon), \) where \( L_1 \) is defined by (10). Obviously, \( L_1 = W^s(\Upsilon) \cap \Sigma \). From system (3), we know that in \( W^s(\Upsilon) \) the flow of (3) is determined absolutely by the planar system (16).

Since \( d^2 - \rho \geq \frac{\omega^2}{4d^2} \), according to the conclusion (i) in Proposition 1, \( [\phi_A(t, x) | 0 < t < +\infty] \) is contained in \( L_1^- = \{x \in \mathbb{R}^3 | x_1 < d, x_3 = 0 \} \subset \Sigma^- \) for any \( x \in L_1 \) which shows (26) holds. Thus, \( \Gamma_1 \) is indeed a heteroclinic orbit from \( q \) to \( \Upsilon \).

Lemma 1 gives the existence of a heteroclinic orbit from \( q \) to \( \Upsilon \) for (1). To show the existence of heteroclinic cycles, we now need to show the existence of other heteroclinic orbits of (1) from \( \Upsilon \) to \( q \).
Lemma 2 If \( q_3 = d - \sqrt{\rho} \) and \( e^T B(p_0 - q) \geq 0 \), then \( I_2 \) is the only heteroclinic orbit from \( \mathcal{Y} \) to \( q \), where

\[
I_2 = \{ \phi_A(t, p_0) | -\infty < t < 0 \} \cup \{ p_0 \}
\cup \{ \phi_B(t, p_0) | 0 < t < +\infty \}. \tag{27}
\]

Proof From (5), (8) and (9),

\[
\{ p_0 \} = W^u(\mathcal{Y}) \cap \Sigma \cap W^s(q). \tag{28}
\]

Then,

\[
\phi_A(t, p_0) \to \mathcal{Y}(t \to -\infty), \phi_B(t, p_0) \to q(t \to +\infty).
\]

To show that \( I_2 \) is a heteroclinic orbit from \( \mathcal{Y} \) to \( q \), it is sufficient to show that

\[
\{ \phi_A(t, p_0) | -\infty < t < 0 \} \subset \Sigma^- \cup \Sigma, \tag{29}
\]

\[
\{ \phi_B(t, p_0) | 0 < t < +\infty \} \subset \Sigma^+ \cup \Sigma. \tag{30}
\]

For convenience, let

\[
\phi_A(t, p_0) = (x_1(t, p_0), x_2(t, p_0), x_3(t, p_0))^T.
\]

Then, \( (x_1(0, p_0), x_2(0, p_0), x_3(0, p_0))^T = p_0 = (\sqrt{\rho}, 0, d - \sqrt{\rho})^T \). Since \( p_0 \in W^u(\mathcal{Y}) \), it follows that for any \( t < 0 \),

\[
(x_1(t, p_0), x_2(t, p_0), x_3(t, p_0))^T \subset W^u(\mathcal{Y}).
\]

Therefore, \( x_1^2(t, p_0) + x_2^2(t, p_0) = \rho \) for \( t < 0 \) by (8). Additionally, from the third equation in (6) we have \( x_3(t, p_0) = \mu x_3(t, p_0) \) with \( \mu > 0 \) which shows that \( x_3(t, p_0) = (d - \sqrt{\rho})e^{\mu t} \). Thus, \( 0 < x_3(t, p_0) < d - \sqrt{\rho} \) for \( t < 0 \). Hence, for \( t < 0 \),

\[
x_3(t, p_0) + x_1(t, p_0) \leq x_3(t, p_0)
+ \sqrt{x_1^2(t, p_0) + x_2^2(t, p_0)} < d - \sqrt{\rho} + \sqrt{\rho} = d,
\]

which means that (29) holds.

Furthermore, since \( p_0 \in W^s(q) \cap \Sigma, \{ \phi_B(t, p_0) | 0 < t < +\infty \} \subset W^s(q) \). From (4), in \( W^s(q) \), the flow of (4) is absolutely determined by the following planar system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = B_0 \begin{bmatrix}
x_1 - q_1 \\
x_2 - q_2
\end{bmatrix},
\]

where \( B_0 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \) with eigenvalues \( \lambda_{1,2} < 0 \) from (H1). Since \( e^T B(p_0 - q) \geq 0 \), by normal coordinate transformation, it is not hard to know that (30) holds by the conclusion (i) in Proposition 2. Therefore, \( I_2 \) given by (27) is indeed the only heteroclinic orbit from \( \mathcal{Y} \) to \( q \). \( \square \)

Remark 1 From Lemmas 1 and 2, the conclusion (a) in the case (i) of Theorem 1 holds with the only one heteroclinic cycle connecting \( \mathcal{Y} \) and \( q \) being

\[\gamma \cup I_2 \cup q \cup I_1.\]

Lemma 3 If \( q_3 = d + \sqrt{\rho} \), \( \omega^2 \rho < \mu^2(d^2 - \rho) \) and \( e^T B(p_1 - q) \geq 0 \), then \( I_3 \) is the only heteroclinic orbit from \( \mathcal{Y} \) to \( q \), where

\[
I_3 = \{ \phi_A(t, p_1) | -\infty < t < 0 \} \cup \{ p_1 \}
\cup \{ \phi_B(t, p_1) | 0 < t < +\infty \}. \tag{31}
\]

Proof In this case,

\[
\{ p_1 \} = W^u(\mathcal{Y}) \cap \Sigma \cap W^s(q). \tag{32}
\]

Therefore,

\[
\phi_A(t, p_1) \to \mathcal{Y}(t \to -\infty), \phi_B(t, p_1) \to q(t \to +\infty).
\]

To prove that \( I_3 \) is the only heteroclinic orbit from \( \mathcal{Y} \) to \( q \), we need only to show that

\[
\{ \phi_B(t, p_1) | 0 < t < +\infty \} \subset \Sigma^+ \cup \Sigma, \tag{33}
\]

\[
\{ \phi_A(t, p_1) | -\infty < t < 0 \} \subset \Sigma^- \cup \Sigma. \tag{34}
\]

Since \( p_1 \in W^s(q) \cap \Sigma \) and \( e^T B(p_1 - q) \geq 0 \), the proof of (33) can be easily carried out similarly to the proof of (30) by conclusion (i) in Proposition 2.

Now, we prove (34). Let

\[
E = W^u(\mathcal{Y}) \cap \Sigma.
\]

Geometrically, \( E \) is an elliptic secant line of \( W^u(\mathcal{Y}) \) (see Fig. 3) and can be parameterized by
Furthermore, it is not hard to see that $p_1 = p(\pi)$ is one of the endpoints of the large axis of $E$. Moreover, for any $\varsigma \in [0, 2\pi)$, the tangent vector of $p(\varsigma)$ can be calculated as

$$p'(\varsigma) = (-\sqrt{\rho} \sin \varsigma, -\sqrt{\rho} \cos \varsigma, \sqrt{\rho} \sin \varsigma)^T.$$ 

Let

$$\theta_1(\varsigma) \in [0, \pi]$$

be the angle between $p'(\varsigma)$ and $-e_3 = (0, 0, -1)^T$ (see Fig. 3). Then,

$$\cos(\theta_1(\varsigma)) = \frac{-\sqrt{\rho} \sin \varsigma}{\sqrt{\rho} + \rho \sin^2 \varsigma} = -\frac{\sin \varsigma}{\sqrt{1 + \sin^2 \varsigma}}.$$ 

Moreover, for (3) the vector field at $p(\varsigma)$ can be formulated by

$$f = (\omega \sqrt{\rho} \sin \varsigma, \omega \sqrt{\rho} \cos \varsigma, \mu (d - \sqrt{\rho} \cos \varsigma))^T.$$ 

Then,

$$\cos(\theta_2(\varsigma)) = \frac{\mu (d - \sqrt{\rho} \cos \varsigma)}{\sqrt{\omega^2 \rho + \mu^2 (d - \sqrt{\rho} \cos \varsigma)^2}},$$

where $\theta_2(\varsigma) \in [0, \pi]$ denotes the angle between $-f$ and $-e_3 = (0, 0, -1)^T$ (see Fig. 3). Obviously,

$$\det(p'(\varsigma), -f, -e_3) = 0$$

which shows that the three vectors are always coplanar for any $\varsigma \in [0, 2\pi)$. Furthermore, from $d > \sqrt{\rho}$ and $\mu > 0$, we have that $\cos(\theta_2(\varsigma)) > 0$ for any $\varsigma \in [0, 2\pi)$. Thus,

$$\theta_2(\varsigma) \in \left(0, \frac{\pi}{2}\right).$$

Now, we consider the sign of $\cos^2(\theta_2(\varsigma)) - \cos^2(\theta_1(\varsigma))$. Simple calculation shows that

$$\cos^2(\theta_2(\varsigma)) - \cos^2(\theta_1(\varsigma)) = \frac{\mu^2 (d - \sqrt{\rho} \cos \varsigma)^2 - \omega^2 \rho \sin^2 \varsigma}{(1 + \sin^2 \varsigma)(\omega^2 \rho + \mu^2 (d - \sqrt{\rho} \cos \varsigma)^2)}.$$ 

(37)

$$= \frac{(\mu - \mu \sqrt{\rho} \cos \varsigma - \omega \sqrt{\rho} \sin \varsigma)(\mu - \mu \sqrt{\rho} \cos \varsigma + \omega \sqrt{\rho} \sin \varsigma)}{(1 + \sin^2 \varsigma)(\omega^2 \rho + \mu^2 (d - \sqrt{\rho} \cos \varsigma)^2)}.$$ 

Since $\omega^2 \rho < \mu^2 (d^2 - \rho)$, we have $\sqrt{\mu^2 + \omega^2} < \frac{\mu d}{\sqrt{\rho}}$. Thus,

$$\mu \sqrt{\rho} \cos \varsigma + \omega \sqrt{\rho} \sin \varsigma \leq \sqrt{\rho} \sqrt{\mu^2 + \omega^2} < \mu d$$

and $\mu \sqrt{\rho} \cos \varsigma - \omega \sqrt{\rho} \sin \varsigma \leq \sqrt{\rho} \sqrt{\mu^2 + \omega^2} < \mu d$.

Combining these with (37) shows $\cos^2(\theta_2(\varsigma)) - \cos^2(\theta(\varsigma)) > 0$. Thus, it can be readily achieved that

$$\theta_2(\varsigma) < \theta_1(\varsigma)$$

for any $\varsigma \in [0, 2\pi)$.

From (35) and (36). This shows that the reverse field $-f$ at any point $p(\varsigma) \in E$ points into the interior of $\{(x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 = \rho, 0 < z < d - x\} \subset W^u(\mathcal{Y}) \cap \Sigma^-$, which means that the flow with any initial condition $p(\varsigma) \in E$ will tend to $\mathcal{Y}$ along $W^u(\mathcal{Y})$ without touching $\Sigma$ once again as $t \to -\infty$. Hence, (34) holds. 

Remark 2 From Lemmas 1 and 3, the conclusion (b) in the case (i) of Theorem 1 holds with the only one heteroclinic cycle connecting $\mathcal{Y}$ and $q$ being

$$\mathcal{Y} \cup \Gamma_3 \cup \{q\} \cup \Gamma_1.$$ 

Lemma 4 If $d - \sqrt{\rho} < q_3 < d + \sqrt{\rho}, \omega^2 \rho < \mu^2 (d^2 - \rho)$ and $c^T B(p_\pm - q) \geq 0$, then $\Gamma_\pm$ are the only two heteroclinic orbits from $\mathcal{Y}$ to $q$.

where

$$\Gamma_\pm = \{\phi_A(t, p_\pm) | -\infty < t < 0\} \cup \{p_\pm\}$$

$$\cup \{\phi_B(t, p_\pm) | 0 < t < +\infty\},$$

and $p_\pm$ being given in (14).

Proof It is easy to get

$$W^u(\mathcal{Y}) \cap \Sigma \cap W^s(q) = \{p_+, p_-\}$$

from (2), (5), (8) and (14). Hence,

$$\phi_A(t, p_\pm) \to \mathcal{Y}(t \to -\infty), \phi_B(t, p_\pm) \to q(t \to +\infty).$$

Since $\omega^2 \rho < \mu^2 (d^2 - \rho)$, by the similar analysis for the proof of (34) in Lemma 3, it is easy to obtain

$$\{\phi_A(t, p_\pm) | -\infty < t < 0\} \subset \Sigma^- \cup \Sigma$$

by conclusion (i) in Proposition 2. Then, $\Gamma_\pm$ are two different heteroclinic orbits from $\mathcal{Y}$ to $q$ from (39–41).

Remark 3 From Lemmas 1 and 4, the conclusion (c) in the case (i) of Theorem 1 holds with the two heteroclinic cycles connecting $\mathcal{Y}$ and $q$ being

$$\mathcal{Y} \cup \Gamma_\pm \cup \{q\} \cup \Gamma_1.$$ 

Combining Remarks 1, 2 and 3, the proof of (i) is accomplished.
4.2 When $0 < d^2 - \rho < \frac{\omega^2}{4d^2}$

The proof is similar to the proof of statement (i) in Theorem 1. Thus, we state it briefly.

**Lemma 5** If $\Gamma_1$ defined in (24) is still a heteroclinic orbit from $q$ to $\Upsilon$ for system (1) if $\sigma_+ > \sigma_-$ and $q_2 \in [\sigma_+, \sigma_*]$, or if $\sigma_+ < \sigma_-$ and $q_2 \in (-\infty, \sigma_*] \cup [\sigma_+, +\infty)$.

**Proof** To carry out the proof, it is still sufficient to prove that both (25) and (26) hold. (25) is obviously true by using the same discussion in Lemma 1. Now, it is crucial to prove (26). Since $0 < d^2 - \rho < \frac{\omega^2}{4d^2}$, \{\phi_A(t, v_+)|t < 0\} will intersect with $L_1$ according to Proposition 1, where $v_+$ is defined in (11). Since $v_* = (d, \sigma_+, 0)^T$ is the first intersection of $L_1$ and the backward-time orbit $\phi_B(t, v_+, t < 0)$,

(1) if $\sigma_+ > \sigma_-$ and $q_2 \in [\sigma_+, \sigma_*]$, then $q_0 \in [v_+, v_*]$. By the conclusion a) in the case ii) in Proposition 1, (26) holds.

(2) if $\sigma_+ < \sigma_-$ and $q_2 \in (\infty, -\sigma_*] \cup [\sigma_+, +\infty)$, then $q_0 \in L_1 - (v_+, v_*)$ which means that (26) still holds from the conclusion (b) in the case (ii) in Proposition 1.

Thus, $\Gamma_1$ is still a heteroclinic orbit from $\Upsilon$ to $q$ in this case.

In addition, it is readily to see that all of the conclusions in Lemmas 2, 3 and 4 are still right, combining which with Lemma 5 the proof of statement (ii) is accomplished. Then, the proof of Theorem 1 is finished.

5 Proof of Theorem 2

Under hypotheses (H2) and (H3), the fundamental difference between Theorems 1 and 2 is that $q$ is a saddle equilibrium with only purely real eigenvalues in Theorem 1, while $q$ is a saddle-focus in Theorem 2. In addition, we will see that, when proving Theorem 2, Proposition 3 can play the similar role as Proposition 2 in the proof of Theorem 1. Thus, to shorten the writing, we present only a brief proof of Theorem 2.

5.1 When $d^2 - \rho \geq \frac{\omega^2}{4d^2}$

All of the discussions for (22), (23), (24), (25) and (26) can be carried out in this case. Thus, $\Gamma_1$ defined as (24) is still the only heteroclinic orbit from $q$ to $\Upsilon$.

(a) If $q_3 = d - \sqrt{\rho}$ and $p_0 \in [x_-, x_+]$.

Then, (28) holds still. Let $\Gamma_2$ be defined as (27). Since the left subsystem in Theorem 2 is same as Theorem 1, (29) holds still. Now, we prove (30). Obviously, $\{\phi_B(t, p_0)|t > 0\}$ is contained in $W^s(q)$. From $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and hypothesis (H2), we know, in $W^s(q)$, the flow of (4) is determined only by the following planar system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = B_0 \begin{pmatrix} x_1 - q_1 \\ x_2 - q_2 \end{pmatrix},$$

where $B_0 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ with its eigenvalues being $\alpha \pm \beta i$ ($\alpha < 0$, $\beta > 0$). From the conclusion (i) in Proposition 3 and the definitions of $x_-$ (see (12)) and $x_+$ (see the statement of Theorem 2), it is readily obtained that (30) holds for $p_0 \in [x_-, x_+]$. Thus, $\Gamma_2$ is the only heteroclinic orbit from $\Upsilon$ to $q$, and $\Upsilon \cup \Gamma_2 \cup q \cup \Gamma_1$ is the only heteroclinic cycle connecting $\Upsilon$ and $q$.

(b) If $q_3 = d + \sqrt{\rho}$, $\omega^2 \rho < \mu^2 (d^2 - \rho)$ and $p_1 \in [x_-, x_+]$.

In this case, (32) holds still. Similarly, to the discussions in a) and the proof of Lemma 3, the only heteroclinic cycle connecting $\Upsilon$ and $q$ in this subcase can be expressed by $\Upsilon \cup \Gamma_3 \cup q \cup \Gamma_1$, where $\Gamma_1$ and $\Gamma_3$ are defined as (24) and (31), respectively.

(c) If $d - \sqrt{\rho} < q_3 < d + \sqrt{\rho}$, $\omega^2 \rho < \mu^2 (d^2 - \rho)$ and $p_\pm \in [x_-, x_+]$.

In this case, (39) holds still. Similarly, to the discussions in a) and the proof of Lemma 4, the only two heteroclinic cycles connecting $\Upsilon$ and $q$ in this subcase can be expressed by $\Upsilon \cup \Gamma_\pm \cup q \cup \Gamma_1$, where $\Gamma_1$ and $\Gamma_\pm$ are defined as (24) and (38), respectively.

5.2 When $0 < d^2 - \rho < \frac{\omega^2}{4d^2}$

Combining the proof presented in Sect. 4.2 and the proof presented in Sect. 5.1, this proof is easy to achieve here. We omit it for the sake of simplicity.
From Sects. 5.1 and 5.2, the proof of Theorem 2 is accomplished.

Remark 4 We can see that Theorems 1 and 2 do not eliminate the situation under which the heteroclinic cycles obtained in these two theorems are tangent to the switching plane at some points. However, it is not hard to get the sufficient conditions under which the heteroclinic cycles intersect the switching plane transversely from Corollary 1, Propositions 2 and 3. We omit the detailed discussions here to shorten the writing.

6 Examples

6.1 Example 1: Heteroclinic cycle connecting a saddle equilibrium with only real eigenvalues and a saddle periodic orbit

For system (1), let

\[
A = \begin{pmatrix}
\rho & -\omega & 0 \\
\omega & \rho & 0 \\
0 & 0 & \mu
\end{pmatrix} = \begin{pmatrix}
1 & -10 & 0 \\
10 & 1 & 0 \\
0 & 0 & 5
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
b_{11} & b_{12} & 0 \\
b_{21} & b_{22} & 0 \\
0 & 0 & \lambda
\end{pmatrix} = \begin{pmatrix}
-2 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{pmatrix},
\]

\[
q = (q_1, q_2, q_3)^T = (1.2, 0, 0.2)^T, \quad d = 1.2.
\]

Then, \( p_0 = (\sqrt{\rho}, 0, d - \sqrt{\rho})^T = (1, 0, 0.2)^T, \)

\[
q_0 = (d, q_2, 0)^T = (1.2, 0, 0)^T,
\]

\[
\sigma_+ = -\omega + \sqrt{\omega^2 - 4d^2(d^2 - \rho)} = -0.05314,
\]

\[
v_+ = (d, \sigma_+, 0)^T = (1.2, -0.05314, 0)^T,
\]

\[
\rho = 1 < 1.44 = d^2, \quad c^T q = 1.4 > 1.2 = d, \quad q_1 = d = 1.2.
\]

Moreover, the eigenvalues of \( B \): \( \lambda_1 = -2 < 0, \lambda_2 = -1 < 0 \) and \( \lambda = 2 > 0 \), and thus, (H1) and (H3) hold. And

\[
0 < d^2 - \rho = 0.44 < \frac{625}{36} = \frac{\omega^2}{4d^2}.
\]

In addition, \( v_* = (d, \sigma_*, 0)^T = (1.2, 2.363, 0)^T \) by numerical calculation. Thus,

\[
\sigma_* = 2.363 > 0.05314 = \sigma_+, \quad \sigma_+ < q_2 = 0 < \sigma_*,
\]

\[
q_3 = d - \sqrt{\rho} = 0.2, \quad c^T B(p_0 - q) = 0.4 \geq 0.
\]

According to conclusion (a) in case (ii) in Theorem 1, there exists only one heteroclinic cycle connecting the periodic orbit \( \gamma = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, z = 0\} \) and the equilibrium \( q \) as shown in Fig. 4.

6.2 Example 2: Heteroclinic cycle connecting a saddle-focus and a saddle periodic orbit

For system (1), let

\[
A = \begin{pmatrix}
1 & -35 \frac{1}{2} & 0 \\
35 \frac{1}{2} & 1 & 0 \\
0 & 0 & 5
\end{pmatrix}, \quad B = \begin{pmatrix}
-0.5 & 4 & 0 \\
-4 & -0.5 & 0 \\
0 & 0 & 2
\end{pmatrix},
\]

\[
d = (35/11)^2, \quad q = (d, -4.5, d + 1)^T.
\]

Then,

\[
p_1 = (-\sqrt{\rho}, 0, d + \sqrt{\rho})^T = (-1, 0, d + 1),
\]

\[
\sigma_+ = -\omega + \sqrt{\omega^2 - 4d^2(d^2 - \rho)} = -0.9045,
\]

\[
\sigma_- = -\omega - \sqrt{\omega^2 - 4d^2(d^2 - \rho)} = -2.4121,
\]

\[
\rho = 1 < 35 \frac{1}{11} = d^2, \quad c^T q = 2d + 1 > d,
\]

\[
q_1 = d = (35/11)^2,
\]

\[
v_+ = (d, \sigma_+, 0)^T = (d, -0.9045, 0)^T.
\]

Then, the eigenvalues of \( B \) are \( -0.5 \pm 4i \) and 2. Thus, (H2) and (H3) hold. And

\[
0 < d^2 - \rho = \frac{24}{11} < \frac{11}{4} = \frac{\omega^2}{4d^2}.
\]

By numerical calculation, \( v_* = (d, \sigma_*, 0)^T = (d, -4.4162, 0)^T \). Then,
Fig. 5 A heteroclinic cycle connecting a periodic orbit and a saddle-focus in Example 2

\[ \sigma_s = -4.4162 < -2.4124 = \sigma_- , \]
\[ q_2 = -4.5 < \sigma_s = -4.4162 . \]

In addition,
\[ q_3 = d + \sqrt{\rho} = d + 1 , \quad \omega^2 \rho \]
\[ = 35 < \frac{600}{11} = \mu^2(d^2 - \rho) . \]

Meanwhile, \[ x_- = \frac{d - \sqrt{\rho} e^T q_3 B^{-1} e^{\pm}}{d + 1} = (-1, -4.848) , \]
\[ x_+ = (0.0476, d + 1)^T \] by numerical calculation. Thus,
\[ p_1 = (-\sqrt{\rho}, 0, d + \sqrt{\rho})^T = (-1, 0, d + 1) \in (x_-, x_+) . \]

According to conclusion b) in case (ii) of Theorem 2, there exists only one heteroclinic cycle connecting the periodic orbit \( \Upsilon = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \rho = 1 , z = 0 \} \) and the saddle-focus \( q \) as shown in Fig. 5.

6.3 Example 3: A pair of heteroclinic cycles each connecting a saddle-focus and a saddle periodic orbit

For system (1), let
\[ A = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} , \quad B = \begin{pmatrix} -3.5 & 6 & 0 \\ -6 & -3.5 & 0 \\ 0 & 0 & 2 \end{pmatrix} , \]
\[ d = 2 , \quad q = (2, 0, 2)^T . \]

Then,
\[ p_-(0, -1, 2)^T , \quad p_+(0, 1, 2)^T , \quad q_0 = (2, 0, 0)^T , \]
\[ \rho = 1 < 4 = d^2 , \quad \epsilon^T q = 4 > 2 = d , \quad q_1 = d = 2 . \]

Moreover, the eigenvalues of \( B : -3.5 \pm 6i \), thus (H2) and (H3) are satisfied. And
\[ d^2 - \rho = 3 > 2.5 = \frac{\omega^2}{4d^2} . \]

Furthermore,
\[ d - \sqrt{\rho} = 1 < q_3 = 2 < 3 = d + \sqrt{\rho} , \quad \omega^2 \rho \]
\[ = 36 < 48 = \mu^2(d^2 - \rho) . \]

and \( x_- = (0, -1.1667, 2)^T \) and \( x_+ = (0, 27.6586, 2)^T \) by numerical calculation. Thus,
\[ p_\pm \in (x_-, x_+) . \]

According to Theorem 2, there exist only two heteroclinic cycles each connecting the periodic orbit \( \Upsilon = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \rho = 1 , z = 0 \} \) and the saddle-focus \( q \) as shown in Fig. 6.

7 Conclusions

As mentioned in the Introduction section, the dynamical phenomenon that the singular cycles involving only equilibrium points can lead to the topological horseshoes in smooth systems has been extended to piecewise smooth systems, based on which many new chaotic attractors are generated and some chaotic circuits are designed by various piecewise linear systems \([13,22,23,29,32,33]\). Along this idea of thought, in this paper we have obtained some interesting results on the existence of heteroclinic cycles connecting saddle periodic orbit and saddle equilibrium in a class of piecewise smooth systems. What’s more, the main results are convenient to be used in the construction of the piecewise
smooth systems possessing such heteroclinic cycles. Unfortunately, no applications of this type of singular cycles appeared in the literature to the knowledge of the authors, due to its hard analysis. Thus, we expect that our analyses may pave the way for the study of the singular horseshoes and the non-uniformly hyperbolic chaotic invariant sets in piecewise smooth systems and the application of singular cycles to other applied systems.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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