Deviations from Tribimaximal and Golden Ratio mixings under radiative corrections of neutrino masses and mixings.

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Abstract

The impact of renormalization group equations(RGEs) on neutrino masses and mixings at high energy scales in Minimal Supersymmetric Standard Model(MSSM) is studied using two different mixing patterns such as Tri-Bimaximal(TBM) mixing and Golden Ratio(GR) mixing in consistent with cosmological bound of the sum of three neutrino masses, \( \sum_i |m_i| \). Magnifications of neutrino masses and mixing angles at low energy scale, are obtained by giving proper input masses, and mixing angles from TBM mixing matrix and GR mixing matrix at high energy scales. High energy scales, \( M_R \) such as \( 10^{13}\text{GeV}, 10^{14}\text{GeV}, 10^{15}\text{GeV} \) are employed in the analysis. The large solar(\( \theta_{12} \)) and atmospheric(\( \theta_{23} \)) neutrino mixing angles with zero reactor angle (\( \theta_{13} \)) from both TBM mixing matrix and GR mixing matrix at high scale, can magnify the reactor angle(\( \theta_{13} \)) at low energy scale in \( 3\sigma \) confidence level. Both cases of normal hierarchy(NH) and inverted hierarchy(IH ) are addressed here. In normal hierarchical case, it is found that \( \theta_{23} \simeq 51.1^\circ \) and that in inverted hierarchical case is \( \theta_{23} \simeq 39.1^\circ \) in both mixing patterns. Possibility of \( \theta_{23} > 45^\circ \) or \( \theta_{23} < 45^\circ \) is observed at low scale. The analysis shows the validity of the two mixing patterns at high energy scale.

Keywords: renormalization group equations, Minimal Supersymmetric Standard Model(MSSM), Tribimaximal(TBM) mixing, Golden ratio(GR) mixing.

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1 Introduction

The renormalization group equations (RGEs) \cite{1,2,3} in neutrino physics serve as an essential way to magnify neutrino mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) from high energy scales to low energy scale. In order to obtain a radiative correction to the neutrino masses and the three mixing angles, the RGEs are made to run in terms of neutrino mass eigenvalues and sine of the mixing angles at different high energy scales, $M_R(10^{13} GeV, 10^{14} GeV, 10^{15} GeV)$. The TBM mixing \cite{4,5,6} which was regarded as a faithful candidate of the PMNS mixing \cite{7}, is ruled out after the discovery of $\theta_{13} \neq 0$ \cite{8,9}. The observation of $\theta_{13} \neq 0$ has made an important contribution in $\delta_{CP}$ in leptonic sector. Both the TBM mixing and GR mixing \cite{10,11,12} predict exact $\theta_{13} = 0$. However, there is a possibility of obtaining non-zero $\theta_{13}$ at low scale by giving mixing angles from TBM mixing and GR mixing as input with proper choices of $m_i (i=1,2,3)$, consistent with cosmological bound on $\sum_i m_i$.

In the work, we are dealing mostly on the validity of the TBM and GR mixing matrix in both normal hierarchy (NH) and inverted hierarchy (IH) at high energy scale along with radiative magnification of $\theta_{13}$ at low energy scale. To obtain these, we consider the evolution equations of neutrino masses and sine of mixings taking into account the scale dependent vacuum expectation value (vev). We consider large tan$\beta$ value which in fact gives a large contribution in obtaining precise values of mixing angles particularly in the magnification of $\theta_{13}$. We notice that with small tan$\beta$ value, radiative magnification of $\theta_{13}$ is not possible. We assign neutrino mass eigenvalues in the form $(m_1, -m_2, m_3)$ with magnitude of $m_2 > m_1$ and $m_3 \neq 0$. With very small $m_3$ value in IH, the value of $\theta_{13}$ cannot be magnified at low scale. An updated global analysis of neutrino oscillation measurements provided by NuFit is given in \cite{13}.

The paper is organised as follows. In section 2, we briefly outline the lepton mixing including the Tribimaximal mixing and Golden Ratio mixing. In section 3, we present the renormalization group equations for neutrino mass eigenvalues and mixing angles. The numerical analysis and results are presented in section 4. We summarise our results in section 5.
2 Tribimaximal mixing and Golden ratio mixing

The PMNS mixing matrix

\[
U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}
\]  

(1)

after parametrisation in terms of three rotations \( R(\theta_{12}), R(\theta_{13}), R(\theta_{23}) \), is given by \[14–16\]

\[
U_{PMNS} = \begin{pmatrix} c_{13} c_{12} & c_{13} s_{12} & s_{13} \\ -c_{23} s_{12} - c_{12} s_{13} & c_{12} c_{23} - s_{12} s_{13} s_{23} & c_{13} s_{23} \\ s_{12} s_{23} - c_{12} c_{13} c_{23} & -c_{12} s_{23} - c_{23} s_{13} s_{12} & c_{13} c_{23} \end{pmatrix},
\]

(2)

where \( s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij} \). Here, CP violating Dirac \( \delta_{CP} \) phase is neglected for simplicity.

The Tri-bimaximal mixing matrix is given by \[17\]

\[
U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 1 & 1 & 1 \\ -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{2}} & \sqrt{2} \end{pmatrix},
\]

(3)

which predicts that \( s_{13}^2 = 0, s_{23}^2 = 1/2 \) and \( s_{12}^2 = 1/3 \).

Another approachable pattern of neutrino mixing matrix deals with the golden ratio \( \phi = \frac{1 + \sqrt{5}}{2} \), with the assumption \( \tan \theta_{12} = 1/\phi \), \( s_{13}^2 = 0 \) and \( s_{23}^2 = 1/2 \). The golden ratio mixing matrix is thus given by \[18\]

\[
U_{GR} = \begin{pmatrix} \phi & 1 & 0 \\ \frac{1}{\sqrt{2 + \phi}} & \frac{1}{\sqrt{2 + \phi}} & \phi \\ \frac{1}{\sqrt{4 + 2\phi}} & \frac{1}{\sqrt{4 + 2\phi}} & \frac{1}{\sqrt{2}} \end{pmatrix},
\]

\[
= \begin{pmatrix} 0.850639 & 0.525735 & 0 \\ -0.371750 & 0.601491 & 0.707107 \\ 0.371750 & -0.601491 & 0.707107 \end{pmatrix}.
\]

(4)
In both TBM and GR mixing matrices, it is predicted that $\theta_{13}=0$. However, recent neutrino oscillation data denied $\theta_{13}$ to be zero.

In the present paper, we analyse how the TBM mixing and GR mixing at high energy scale of $10^{13} GeV, 10^{14} GeV, 10^{15} GeV$, can magnify the $\theta_{13}$ value to a required value at low energy scale along with other necessary neutrino parameters such as $\theta_{12}, \theta_{23}, \Delta m_{12}^2, \Delta m_{23}^2$. While doing this, the present work also focuses on the consistency with the cosmological bound on $\sum_i |m_i|$ where $i = 1, 2, 3$.

3 RGEs for neutrino mass eigenvalues and mixing angles

The neutrino mass matrix taking into account the running of vev is given by [19]

$$m_i(t) = v_u^2(t)K_i(t)$$

or

$$m_i(t) = v^2 \frac{\tan^2 \beta}{(1 + \tan^2 \beta)} K_i(t),$$

where $t=\ln(\mu/1GeV)$, $v_u = v\sin\beta$ with $v=174GeV$, $\tan\beta$ is the ratio of the two vev’s of two Higgs doublets in MSSM and $K_i$ is coefficient of dim-5 neutrino mass operator in scale-dependent manner.

This implies that

$$\frac{d(lnm_i)}{dt} = \frac{d(lnK_i)}{dt} + 2\frac{d(lnv_u)}{dt},$$

where the evolution equations for $K_i$ and $v_u$ in MSSM are given by

$$
\frac{dK_i}{dt} = \frac{1}{16\pi^2} \sum_{f=e,\mu,\tau} \left[ \left( - \frac{9}{5} g_1^2 - 6 g_2^2 + 6 Tr(h_u^2) \right) + 2 h_T^2 U_{f_i}^2 \right] K_i, $$

$$
\frac{dv_u}{dt} = \frac{1}{16\pi^2} \left[ \frac{3}{20} g_1^2 + \frac{3}{4} g_2^2 - 3 h_t^2 \right] v_u,$$

where $g_1, g_2$ are gauge couplings, and $h_u, h_t$ are up-quark and top-quark Yukawa coupling respectively.

Now, the RGEs for neutrino mass eigenvalues are given by

$$
\frac{d}{dt}m_i = \frac{1}{16\pi^2} \left[ - \frac{9}{10} g_1^2 - \frac{9}{2} g_2^2 + 2 h_T^2 U_{\tau_i}^2 \right] m_i,$$
where \( h_\tau \) is tau-lepton Yukawa coupling.

Also, the evolution equations for the PMNS matrix elements \( U_{fi} \) are given by

\[
\frac{d}{dt} U_{fi} = -\frac{1}{16\pi^2} \sum_{k \neq i} \frac{m_k + m_i}{m_k - m_i} U_{fk} (U^T H^2 e U)_{ki},
\]

where \( f = e, \mu, \tau \); \( i, k = 1, 2, 3 \) respectively and \( H_e \) is the Yukawa coupling matrices of the charged-leptons in the diagonal basis.

Now, we have

\[
(U^T H^2 e U)_{12} = h^2_e (U^T_{1e} U_{2e}) + h^2_\mu (U^T_{1\mu} U_{2\mu}) + h^2_\tau (U^T_{1\tau} U_{2\tau})
\]

\[
(U^T H^2 e U)_{13} = h^2_e (U^T_{1e} U_{3e}) + h^2_\mu (U^T_{1\mu} U_{3\mu}) + h^2_\tau (U^T_{1\tau} U_{3\tau})
\]

\[
(U^T H^2 e U)_{23} = h^2_e (U^T_{2e} U_{3e}) + h^2_\mu (U^T_{2\mu} U_{3\mu}) + h^2_\tau (U^T_{2\tau} U_{3\tau})
\]

As \( h_\tau > h_e, h_\mu \), we neglect \( h_e, h_\mu \). Then, the above equations become

\[
(U^T H^2 e U)_{12} \approx h^2_\tau (U^T_{1\tau} U_{2\tau}), \quad (U^T H^2 e U)_{13} \approx h^2_\tau (U^T_{1\tau} U_{3\tau}),
\]

\[
(U^T H^2 e U)_{23} \approx h^2_\tau (U^T_{2\tau} U_{3\tau})
\]

Denoting \( \frac{m_k + m_i}{m_k - m_i} = A_{ki} \) and using the above three equations together with eqns.10, we have

\[
\frac{dU_{e2}}{dt} \approx -\frac{1}{16\pi^2} \left[ U_{e2} h^2_\tau (A_{32} U_{e3} U^\dagger_{3\tau} + A_{12} U_{e1} U^\dagger_{1\tau}) \right],
\]

\[
\frac{dU_{e3}}{dt} \approx -\frac{1}{16\pi^2} \left[ U_{e3} h^2_\tau (A_{13} U_{e1} U^\dagger_{1\tau} + A_{23} U_{e2} U^\dagger_{2\tau}) \right],
\]

\[
\frac{dU_{\mu3}}{dt} \approx -\frac{1}{16\pi^2} \left[ U_{\mu3} h^2_\tau (A_{13} U_{\mu1} U^\dagger_{1\tau} + A_{23} U_{\mu2} U^\dagger_{2\tau}) \right].
\]

Upon solving the above three equations using eqn.(2), we have

\[
\frac{ds_{12}}{dt} \approx \frac{1}{16\pi^2} h^2_\tau c_{12} \left[ c_{23} s_{13} s_{12} U_{\tau 1} A_{31} - c_{23} c_{13} c_{12} U_{\tau 2} A_{32} + U_{\tau 1} U_{\tau 2} A_{21} \right],
\]

\[
\frac{ds_{13}}{dt} \approx \frac{1}{16\pi^2} h^2_\tau c_{23} c_{13} \left[ c_{12} U_{\tau 1} A_{31} + s_{12} U_{\tau 2} A_{32} \right],
\]

\[
\frac{ds_{23}}{dt} \approx \frac{1}{16\pi^2} h^2_\tau c_{23} \left[ -s_{12} U_{\tau 1} A_{31} + c_{12} U_{\tau 2} A_{32} \right].
\]

The equations \([9], [14], [15], [16]\) are used for numerical analysis in this paper.
4 Numerical Analysis and Result

For numerical analysis, different high scales such as $10^{13} GeV$, $10^{14} GeV$, $10^{15} GeV$ and $\tan\beta = 58$ are considered. Corresponding to these different scales and $\tan\beta$, different values of gauge couplings and the third-family Yukawa couplings are taken. Running of these couplings in MSSM is done by bottom-up method i.e from top-quark mass scale, $m_t$ (low energy scale) to $M_R$ (high energy scale) where the B-L symmetry breaks down. For simplicity, we consider $t_o = \ln m_t$ where $m_t$ is mass of top-quark and $t_R = \ln M_R$. At electroweak scale $M_Z = 91.18 GeV$, we have

$$\alpha_S(M_Z) = \alpha_3(M_Z) = 0.1179 \pm 0.009$$
$$\alpha_{em}^{-1}(M_Z) = 127.952 \pm 0.009$$
$$\sin^2\theta_W(M_Z) = 0.23121 \pm 0.00017$$

The matching condition at $M_Z$ scale gives

$$\frac{1}{\alpha_{em}(M_Z)} = \frac{5}{3\alpha_1(M_Z)} + \frac{1}{\alpha_2(M_Z)}$$  \hspace{1cm} (17)

From the definition of Weinberg mixing angle, we have

$$\sin^2\theta_W(M_Z) = \frac{\alpha_{em}(M_Z)}{\alpha_2(M_Z)}$$  \hspace{1cm} (18)

where $\alpha_i (i = 1, 2, 3)$ are electromagnetic, weak, strong coupling constants respectively and $\theta_W$ is the Weinberg angle.

Using the values of $\alpha_3(M_Z)$, $\alpha_{em}^{-1}(M_Z)$, $\sin^2\theta_W(M_Z)$ together with eqns (17) and (18), we have

$$\alpha_{1,2,3}^{-1}(M_Z) = (59.021, 29.584, 8.482)$$

Using the relation $g_i = \sqrt{4\pi\alpha_i}$ and the values of $\alpha_i^{-1}(M_Z)$ where $i = 1, 2, 3$, the values of the gauge couplings at $M_Z$ scale are

$$g_{1,2,3}(M_Z) = (0.461469, 0.6514018, 1.21740)$$

The values of the coupling constants at $m_t$ scale can be obtained by using the following one-loop RGE for gauge couplings(Non-SUSY) in the energy scale, $M_Z \leq \mu \leq m_t$ [21]

$$\alpha_i^{-1}(m_t) = \alpha_i^{-1}(M_Z) - \frac{b_i}{2\pi} \ln \left( \frac{m_t}{M_Z} \right)$$  \hspace{1cm} (19)
where $\mu = m_t$, $i = 1, 2, 3$ and $b_i = \left( \frac{53}{10^4} - \frac{1}{2}, -4 \right)$ for $n_f = 5$ and $n_H = 1$ 

With this, we have

$$\alpha_{1,2,3}^{-1}(m_t) = (58.482, 29.635, 8.89)$$

Correspondingly,

$$g_{1,2,3}(m_t) = (0.463547, 0.651186, 1.189021)$$

The third family charged-lepton masses at $m_t$ scale are given by

$$m_\ell(m_t) = m_\ell(m_t) ; m_b(m_t) = \frac{m_b(m_b)}{\eta_b} ; m_\tau(m_t) = \frac{m_\tau(m_\tau)}{\eta_\tau}$$

where the physical masses are $m_\ell(m_t) = 172.76 GeV$, $m_b(m_b) = 4.18 GeV$, $m_\tau(m_\tau) = 1.777 GeV$ \textsuperscript{20} and $\eta_b, \eta_\tau$ are QCD-QED rescaling factors with $\eta_b \simeq 1.53, \eta_\tau \simeq 1.015$ \textsuperscript{23}.

Now, the values of top quark, bottom quark and tau lepton Yukawa couplings at $m_t$ scale with $\tan \beta = 58$ are obtained by using the relations from MSSM as given below \textsuperscript{24}

$$h_\ell(m_t) = \frac{m_\ell(m_t)}{v \sin \beta} = 0.9930211$$

$$h_b(m_t) = \frac{m_b(m_b)}{v \eta_b \cos \beta} = 0.910811$$

$$h_\tau(m_t) = \frac{m_\tau(m_\tau)}{v \eta_\tau \cos \beta} = 0.583666$$

where $v=174 GeV$ is the vacuum expectation value in non-SUSY. We provide the above values of gauge couplings and Yukawa couplings as inputs. The values of the gauge couplings and Yukawa couplings at different $M_R$ scales are computed using the following RGEs \textsuperscript{24,26} for gauge couplings and Yukawa couplings(SUSY) in energy scale, $m_t \leq \mu \leq M_R$

$$\frac{dg_i}{dt} = \frac{b_i}{16\pi^2} g_i^3 + \frac{1}{(16\pi^2)^2} \left[ \sum_{j=1}^{3} b_{ij} g_i^3 g_j^2 - \sum_{j=1, \nu, \tau} a_{ij} g_i^3 h_j^2 \right]$$

where $t = \ln \left( \mu / 1 GeV \right)$, $b_i = (6.6, 1.6, -3.0)$;

$$b_{ij} = \begin{pmatrix} 7.96 & 5.40 & 17.60 \\ 1.80 & 25.00 & 24.00 \\ 2.20 & 9.00 & 14.00 \end{pmatrix} ; a_{ij} = \begin{pmatrix} 5.2 & 2.8 & 3.6 \\ 6.0 & 6.0 & 2.0 \\ 4.0 & 4.0 & 0.0 \end{pmatrix}$$
\[
\frac{d h_t}{dt} = \frac{h_t}{16\pi^2} \left( 6h_t^2 + h_b^2 - \sum_{i=1}^{3} c_i g_i^2 \right)
\]
\[
\frac{d h_b}{dt} = \frac{h_b}{16\pi^2} \left( 6h_b^2 + h_t^2 + h_t^2 - \sum_{i=1}^{3} c'_i g_i^2 \right)
\]
\[
\frac{d h_\tau}{dt} = \frac{h_\tau}{16\pi^2} \left( 4h_\tau^2 + 6h_b^2 - \sum_{i=1}^{3} c''_i g_i^2 \right)
\]

where \( c_i = \left( \frac{13}{15}, 3, \frac{16}{3} \right) \); \( c'_i = \left( \frac{7}{15}, 3, \frac{16}{3} \right) \); \( c''_i = \left( \frac{9}{5}, 3, 0 \right) \)

\[
\begin{array}{cccc}
 t_o &=& 5.1519 & t_R = 29.93 \\
g_1 &=& 0.463547 & g_1 = 0.626687 \\
g_2 &=& 0.651186 & g_2 = 0.708076 \\
g_3 &=& 1.189021 & g_3 = 0.783018 \\
h_t &=& 0.993021 & h_t = 0.706396 \\
h_b &=& 0.910811 & h_b = 0.765436 \\
h_\tau &=& 0.583666 & h_\tau = 0.818337 \\
\end{array}
\]

Table 1: Values of gauge couplings and Yukawa couplings at \( t_o = \ln m_t, t_R = \ln M_R \) where \( m_t = 172.76 \text{GeV}, M_R = 10^{13}, 10^{14}, 10^{15} \text{GeV} \).

Figs. 1-2 show the three gauge couplings and third-generation Yukawa couplings unification at \( M_U = 2 \times 10^{16} \text{GeV} \) and \( 1.25 \times 10^{11} \text{GeV} \) respectively. With the values above, we put arbitrary mass eigenvalues \((m_1, -m_2, m_3)\) in the range \(0.03-0.059 \text{GeV}(0.04-0.03 \text{GeV})\) for normal(inverted)mass ordering along with the three fixed mixing angles of TBM mixing matrix in one case and along with the three fixed mixing angles of GR mixing matrix in the other case as input. Using these several values at different high scales, we obtain the neutrino parameters at low energy by simultaneously solving the RGEs 9, 14, 15, 16 mentioned above.

At first, we consider neutrino mass eigenvalues following pure normal and inverted mass ordering as input. Here, we find the output values of \( \sin \theta_{13} \) beyond the experimentally allowed range even though we vary the \( \tan \beta \) value from very small to large value. We consider degenerate NH(IH) in the present work.
In Table 2-3, $m^0_i$ and $s^0_{ij}$ are input values with fixed mixing angles of TBM mixing matrix, at different high scales while $m_i$ and $s_{ij}$ $(i,j=1,2,3; \ i \neq j)$ are output values at $m_t$ scale. For different high scales, we take arbitrary neutrino mass eigenvalues as input but with same mixing angles. If we take same mass eigenvalues for different high scales, then we cannot obtain the experimentally allowed neutrino parameters. In normal hierarchy, we find that $\theta_{12} = 35.8^o$, $\theta_{13} \approx 8.5^o$, $\theta_{23} \approx 51.5^o$ while in inverted hierarchy, we find that $\theta_{12} = 35.7^o$, $\theta_{13} \approx 8.2^o$, $\theta_{23} \approx 38.8^o$ which is slightly less than that from experimental data. The mass squared differences $\Delta m^2_{21}, \Delta m^2_{31}, \Delta m^2_{32}$ lie in the allowed range as given by [13] except for the case of $t_R = 29.93$ (TBM-IH) which is slightly large. Here, we obtain an important result that in NH, $\theta_{23} > 45^o$ and in IH, $\theta_{23} < 45^o$. Figs. 3-4 represent evolution of mass eigenvalues and mixing angles of TBM mixing matrix with high energy scale in both NH and IH at $t_R = 29.93$ . For $t_R = 32.236, 34.54$, the evolutions are almost the same with that of $t_R = 29.93$.

In Table 4-5, we put arbitrary mass eigenvalues with fixed mixing angles of GR mixing matrix as input for both the normal and inverted mass orderings. Then, we compute the output for different high scales $M_R$. In this case also, we find that the neutrino oscillation parameters $\theta_{12}$, $\theta_{13}$, $\Delta m^2_{21}, \Delta m^2_{31}, \Delta m^2_{32}$ all lie in the experimentally allowed range while $\theta_{23}$ for inverted hierarchy is slightly smaller than that in the experimental data. In NH, we find that $\theta_{23} > 45^o$ and in IH, $\theta_{23} < 45^o$ which is similar to what we have found in case of TBM. Figs. 5-6 represent evolution of mass eigenvalues and mixing angles of GR mixing matrix with high energy scale in both NH and IH at $t_R = 29.93$ . For $t_R = 32.236, 34.54$, the evolutions are almost same with that of $t_R = 29.93$. It is generally observed that both the $\theta_{12}$ and $\theta_{13}$ always magnify with decrease of energy scale unlike $\theta_{23}$. While neutrino masses $m_i$ also increase with decrease of energy scale.
Table 2: Input values \((m^0_i, s^0_{ij})\) with fixed mixing angles of TBM mixing matrix and output values \((m_i, s_{ij})\) with three different high energy scales, \(t_R(=\ln M_R/1\text{GeV})\) for normal hierarchy (NH).

| TBM - NH | \(t_R = 29.93\) | \(t_R = 32.236\) | \(t_R = 34.54\) |
|----------|-----------------|-----------------|-----------------|
| \(m^0_1(\text{eV})\) | 0.04349 | 0.03899 | 0.03399 |
| \(m^0_2(\text{eV})\) | -0.04479 | -0.04029 | -0.03529 |
| \(m^0_3(\text{eV})\) | 0.05873 | 0.05453 | 0.04977 |
| \(s^0_{12}\) | 0.5773503 | 0.5773503 | 0.5773503 |
| \(s^0_{13}\) | 0.0 | 0.0 | 0.0 |
| \(s^0_{23}\) | 0.707107 | 0.707107 | 0.707107 |
| \(m_1(\text{eV})\) | 0.060482 | 0.056103 | 0.050666 |
| \(m_2(\text{eV})\) | -0.061126 | -0.056760 | -0.051362 |
| \(m_3(\text{eV})\) | 0.078716 | 0.075298 | 0.070839 |
| \(s_{12}\) | 0.58443 | 0.58456 | 0.584327 |
| \(s_{13}\) | -0.14985 | -0.14981 | -0.145484 |
| \(s_{23}\) | 0.782026 | 0.782830 | 0.781903 |
| \(\Delta m^2_{21}(10^{-5}\text{eV}^2)\) | 7.8 | 7.42 | 7.1 |
| \(\Delta m^2_{31}(10^{-3}\text{eV}^2)\) | 2.538 | 2.52 | 2.451 |
| \(\sum_i |m_i|(i = 1, 2, 3)(\text{eV})\) | 0.20 | 0.188 | 0.173 |

Table 3: Input values \((m^0_i, s^0_{ij})\) with fixed mixing angles of TBM mixing matrix and output values \((m_i, s_{ij})\) with three different high energy scales, \(t_R(=\ln M_R/1\text{GeV})\) for inverted hierarchy (IH).

| TBM-IH | \(t_R = 29.93\) | \(t_R = 32.236\) | \(t_R = 34.54\) |
|--------|-----------------|-----------------|-----------------|
| \(m^0_1(\text{eV})\) | 0.05323 | 0.04856 | 0.04529 |
| \(m^0_2(\text{eV})\) | -0.05463 | -0.05095 | -0.04774 |
| \(m^0_3(\text{eV})\) | 0.04728 | 0.03881 | 0.03535 |
| \(s^0_{12}\) | 0.5773503 | 0.5773503 | 0.5773503 |
| \(s^0_{13}\) | 0.0 | 0.0 | 0.0 |
| \(s^0_{23}\) | 0.707107 | 0.707107 | 0.707107 |
| \(m_1(\text{eV})\) | 0.074070 | 0.07129 | 0.069014 |
| \(m_2(\text{eV})\) | -0.074599 | -0.071837 | -0.069561 |
| \(m_3(\text{eV})\) | 0.0562921 | 0.052484 | 0.049161 |
| \(s_{12}\) | 0.5839937 | 0.584154 | 0.584323 |
| \(s_{13}\) | 0.1433709 | 0.143462 | 0.143469 |
| \(s_{23}\) | 0.626464 | 0.625306 | 0.624038 |
| \(\Delta m^2_{21}(10^{-5}\text{eV}^2)\) | 7.86 | 7.87 | 7.59 |
| \(\Delta m^2_{31}(10^{-3}\text{eV}^2)\) | -2.396 | -2.41 | -2.422 |
| \(\sum_i |m_i|(i = 1, 2, 3)(\text{eV})\) | 0.205 | 0.196 | 0.188 |
| $t_R$ | $m_1^0$ (eV) | $m_2^0$ (eV) | $m_3^0$ (eV) | $s_{12}$ | $s_{13}$ | $s_{23}$ | $\Delta m_{21}^2$ \(10^{-5} \text{eV}^2\) | $\Delta m_{32}^2$ \(10^{-3} \text{eV}^2\) | $\sum_i |m_i|(i = 1, 2, 3)$ (eV) |
|-------|-------------|-------------|-------------|---------|---------|--------|----------------|----------------|----------------|
| 29.93 | 0.04354     | -0.04531    | 0.05899     | 0.525739| 0.0     | 7.52   | 7.381          | -2.57           | 0.201           |
| 32.236| 0.04058     | -0.04239    | 0.05633     | 0.525739| 0.0     | 7.65   | 7.04           | -2.56           | 0.196           |
| 34.54 | 0.03551     | -0.03735    | 0.05179     | 0.525739| 0.0     | 7.14   | 7.14           | -2.56           | 0.181           |

Table 4: Input values \((m_1^0, s_{ij})\) with fixed mixing angles of Golden Ratio (GR) mixing matrix and output values \((m_i, s_{ij})\) with three different high energy scales, \(t_R=\ln M_R/1\text{GeV}\) for normal hierarchy (NH).

| $t_R$ | $m_1^0$ (eV) | $m_2^0$ (eV) | $m_3^0$ (eV) | $s_{12}$ | $s_{13}$ | $s_{23}$ | $\Delta m_{21}^2$ \(10^{-5} \text{eV}^2\) | $\Delta m_{32}^2$ \(10^{-3} \text{eV}^2\) | $\sum_i |m_i|(i = 1, 2, 3)$ (eV) |
|-------|-------------|-------------|-------------|---------|---------|--------|----------------|----------------|----------------|
| 29.93 | 0.05352     | -0.05633    | 0.04449     | 0.525739| 0.0     | 7.52   | 7.381          | -2.57           | 0.201           |
| 32.236| 0.05149     | -0.05445    | 0.04211     | 0.525739| 0.0     | 7.65   | 7.04           | -2.56           | 0.196           |
| 34.54 | 0.048353    | -0.05146    | 0.03933     | 0.525739| 0.0     | 7.14   | 7.14           | -2.56           | 0.181           |

Table 5: Input values \((m_1^0, s_{ij})\) with fixed mixing angles of Golden Ratio (GR) mixing matrix and output values \((m_i, s_{ij})\) with three different energy scales, \(t_R=\ln M_R/1\text{GeV}\) for inverted hierarchy (IH).
From the tables above, it is seen that both the case of NH and IH of TBM and GR mixing matrix are valid at high energy scale considering the bound $\sum_i |m_i| < 0.23\text{eV}$. This result contradicts the claim in earlier analysis [27], although the analysis is discussed with the inclusion of CP violating phases.

## 5 Conclusion and Discussion

We have presented a detailed analysis of radiative corrections of neutrino masses and mixings in MSSM with TBM mixing and GR mixing at three values of high energy $M_R$ scales $10^{13}, 10^{14}, 10^{15}$ GeV. In all cases, we have used large value of $\tan \beta = 58$. We take arbitrary values on the three neutrino mass eigenvalues at high energy scales for both NH and IH, and obtain the experimentally allowed neutrino oscillation parameters at low energy scale, $m_t$. The neutrino mass eigenvalues $|m_i|(i=1,2,3)$ in the range $(0.03-0.079)\text{eV}$ provide correct values of the neutrino oscillation parameters at low energy scale. We have successfully obtained almost all the mixing angles and mass squared differences in the $3\sigma$ range of NuFit data. The value of the sum of absolute neutrino masses, $\sum_i |m_i|$ is in the range $(0.17-0.20)\text{eV}$ for normal hierarchy and $(0.188-0.21)\text{eV}$ for inverted hierarchy. These values are within the upper bound on $\sum_i |m_i| < 0.23\text{eV}$ at the 95% confidence level from Planck 2015 [28]. However, these values are still higher as compared with the latest Planck bound $\sum_i |m_i| < 0.12\text{eV}$ [29–31]. There are a lot of uncertainties in the process of observation of cosmological bound on the sum of the absolute three neutrino masses [32,33]. It may still requires further analysis to achieve the most correct cosmological bound in future measurements.

In the present analysis, we have not included the effects of running of the Dirac and Majorana CP phases. We also consider $m_t = 172.76\text{GeV}$ as the SUSY breaking scale $m_s$ for simplicity. The analysis shows the validity of the TBM and GR mixings at high seesaw scale $M_R$, and the deviations are the effects of radiative corrections at low energy scale. For future investigations, it would be very interesting to consider higher SUSY breaking scale, $m_s$ as well as higher $M_R$ scale($2 \times 10^{16}\text{GeV}$) for different $\tan \beta$ values [34].
Figure 1: Three gauge couplings unification is observed at $M_U = 2 \times 10^{16}\text{GeV}$.

Figure 2: Third-generation Yukawa couplings unification is observed at $\mu = 1.25 \times 10^{11}\text{GeV}$.
Figure 3: Evolution of mass eigenvalues and mixing angles of TBM mixing matrix with energy scale in NH for $t_R = 29.93$.

Figure 4: Evolution of mass eigenvalues and mixing angles of TBM mixing matrix with energy scale in IH for $t_R = 29.93$. 
Figure 5: Evolution of mass eigenvalues and mixing angles of GR mixing matrix with energy scale in NH for $t_R = 29.93$.

Figure 6: Evolution of mass eigenvalues and mixing angles of GR mixing matrix with energy scale in IH for $t_R = 29.93$. 
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