A LEMMA OF LAZARSFELD AND THE JACOBIAN BLOW UP

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Abstract. For a complex analytic function \( f \), the exceptional divisor of the jacobian blow-up is of great importance. In this paper, we show what a lemma from the thesis of Lazarsfeld tells one about the structure of this exceptional divisor.

1. Introduction

Let \( \mathcal{U} \) be an open subset of \( \mathbb{C}^{n+1} \) and let \( f: (\mathcal{U}, 0) \to (\mathbb{C}, 0) \) be a nowhere locally constant complex analytic function. Near the origin, the critical locus \( \Sigma f \) of \( f \) is contained in the hypersurface \( V(f) \) defined by \( f \); we assume that \( \mathcal{U} \) is chosen small enough so that this is true throughout \( \mathcal{U} \). We use \( z := (z_0, \ldots, z_n) \) for the coordinates on \( \mathbb{C}^{n+1} \) and so on \( \mathcal{U} \).

We let \( \pi: \text{Bl}_j(\mathcal{U}) \to \mathcal{U} \) be the projection map of the blow-up of \( \mathcal{U} \) along the jacobian ideal \( j(f) := \langle \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \rangle \), where \( \text{Bl}_j(\mathcal{U}) \subseteq \mathcal{U} \times \mathbb{P}^n \). Let \( E = \pi^{-1}(\Sigma f) \) denote the exceptional divisor, which is purely \( n \)-dimensional. Let \( W_0, W_1, \ldots, W_r \) denote the distinct irreducible components of \( E \) over \( 0 \), i.e., the irreducible components \( W \) of \( E \) such that \( 0 \in \pi(W) \).

Now let us identify the \( \mathcal{U} \times \mathbb{P}^n \) which contains the jacobian blow-up with the projectivized cotangent space \( \mathbb{P}(T^*\mathcal{U}) \). Under this identification, \( \text{Bl}_j(\mathcal{U}) \) is equal to the projectivized closure of the relative conormal of \( f \), that is,

\[
\text{Bl}_j(\mathcal{U}) = \mathbb{P}(T^*_{\mathcal{U}}/\Sigma f).
\]

For \( 0 \leq k \leq r \), we let \( Y_k \) denote the irreducible analytic set \( \pi(W_k) \). Then the fact that \( E \) is purely \( n \)-dimensional, combined with the existence of an \( a_f \) stratification, tells us that \( W_k \) is equal to the closure of the conormal space of the regular part \( Y^o_k \) of \( Y_k \), that is, \( W_k = \mathbb{P}(T^*_{Y^o_k}) \).

We refer to the \( Y_k \) as the Thom varieties of \( f \) at \( 0 \).

Clearly, the Thom varieties are important for understanding limiting relative conormals and the \( a_f \) condition. In addition, the result of [1] and [4] tells us that, as sets, the exceptional divisor \( E \) is equal to the projectivized characteristic cycle of the sheaf of vanishing cycles of \( C_\mathcal{U} \) (or \( Z_\mathcal{U} \)) along \( f \). Thus, the Thom varieties are also closely related to the topology of the Milnor fibers of \( f \) at points in \( \Sigma f \).

While we prove a more general result in Theorem 2.2, a special case is much easier to state:

Corollary. Suppose that \( \Sigma f \) is smooth at \( 0 \). Then, at \( 0 \), either

(1) there is a Thom variety of \( f \) of codimension 1 in \( \Sigma f \), or

(2) \( \Sigma f \) itself is the only Thom variety, and \( f \) defines a family of isolated singularities with constant Milnor number (that is, a simple \( \mu \)-constant family as given in Definition 1.1 of [3]).

2020 Mathematics Subject Classification. 32S25, 32S05, 32S30, 32S50.

Key words and phrases. hypersurface singularities, jacobian blow up, vanishing cycles, \( a_f \) condition.
So, if $\Sigma f$ is smooth and there are any proper sub-Thom varieties in $\Sigma f$, then there must be one of codimension 1. We find this somewhat surprising.

The crux of the proof of our theorem lies in a lemma from the Ph.D. thesis [2] of Lazarsfeld, which we recall in the next section.

2. The Lemma and the Theorem

We now state Lemma 2.3 of [2] (with some changes in notation):

**Lemma 2.1.** (Lazarsfeld) Let $Z$ be an irreducible normal variety of dimension $n + 1$, and let $X \subseteq Z$ be a subvariety which is locally defined (set-theoretically) by $n + 1 - e$ equations. Fix an irreducible component $V$ of $X$. Then, for all $x \in V \cap X \setminus V$, $\dim_x (V \cap X \setminus V) \geq e - 1$.

The proof of our theorem below uses the above lemma in a crucial way. Our proof also uses, as sets, our notation and results on relative polar cycles and Lê cycles as presented in \cite{1}. We should mention that our theorem and its proof are closely related to Proposition 1.31 of \cite{5}, which we recall in the next section.

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\( \Gamma_{f, z} \) is purely \( e \)-dimensional, every component of \( \Sigma f \) of dimension \( \geq e \) must be an irreducible component of \( X \). In particular, \( V \) is an irreducible component of \( X \).

We now apply Lazarsfeld’s Lemma. It tells us that, if \( 0 \in V \cap (\Gamma_{f, z} \cup \Sigma f \setminus V) \), then either 
\[ \dim_0 V \cap \Gamma_{f, z} \geq e - 1 \quad \text{or} \quad \dim_0 V \cap \Sigma f \setminus V \geq e - 1. \]

Suppose that we are not in case (1) of the theorem, i.e., suppose that \( \dim_0 V \cap \Sigma f \setminus V < e - 1 \). Then either \( \dim_0 V \cap \Gamma_{f, z} \geq e - 1 \), or \( 0 \notin \Gamma_{f, z} \) and \( 0 \notin \Sigma f \setminus V \). We claim that these correspond to cases (2) and (3), respectively.

Case 2:

Suppose that \( \dim_0 V \cap \Sigma f \setminus V < e - 1 \) and \( \dim_0 V \cap \Gamma_{f, z} \geq e - 1 \).

By Proposition 1.15 of [6], as sets,
\[ V \cap \Gamma_{f, z} = V \cap \bigcup_{j \leq e - 1} \Lambda_{j, f, z}, \]
where \( \Lambda_{j, f, z} \) is the purely \( j \)-dimensional Lê cycle. Thus, \( \dim_0 V \cap \Gamma_{f, z} \geq e - 1 \) implies that \( V \) contains an \((e - 1)\)-dimensional irreducible component \( Y \) of \( \Lambda_{e-1, f, z} \) at the origin. By Corollary 10.15 and/or Theorem 10.18 of [6], \( Y \) is a component of an absolute polar variety of a Thom variety \( T \) (which necessarily must have dimension \( \geq e - 1 \)) of \( f \); however, since we are assuming that \( \dim_0 V \cap \Sigma f \setminus V < e - 1 \), \( T \) cannot be contained in another irreducible component of \( \Sigma f \), but rather must be contained in \( V \). But by our hypotheses, the Thom varieties in \( V \) of dimension \( > e - 1 \) are smooth and have no absolute polar varieties of dimension \( (e - 1) \). Thus \( T \) must be \((e - 1)\)-dimensional and so \( Y = T \), and we have the conclusion of Case 2.

Case 3:

Now suppose that \( 0 \notin \Gamma_{f, z} \) and \( 0 \notin \Sigma f \setminus V \). First, \( 0 \notin \Sigma f \setminus V \) immediately implies that \( V = \Sigma f \) at the origin. As we saw in Case 2, but using that \( V = \Sigma f \), we have
\[ V \cap \Gamma_{f, z} = \bigcup_{j \leq e - 1} \Lambda_{j, f, z}, \]
and, as we are assuming that \( 0 \notin \Gamma_{f, z} \), this implies that, at the origin, \( \Lambda_{j, f, z} = \emptyset \) for all \( j \leq e - 1 \). But \( \Lambda_{j, f, z} \) includes any Thom variety of dimension \( j \), and so there are none for \( j \leq e - 1 \). Therefore, we have the conclusion of Case 3. \( \square \)

Letting \( e = d \) in the theorem above, we obtain the corollary from the introduction:

**Corollary 2.3.** Suppose that \( \Sigma f \) is smooth at \( 0 \). Then, at \( 0 \), either

1. there is a Thom variety of \( f \) of codimension 1 in \( \Sigma f \), or

2. \( \Sigma f \) itself is the only Thom variety, and \( f \) defines a family of isolated singularities with constant Milnor number, that is, a simple \( \mu \)-constant family as given in Definition 1.1 of [3].

**Proof.** If we let \( e = d \) in Theorem 2.2, we obtain essentially the whole corollary. If \( \Sigma f \) is smooth, then it is irreducible, and Case 1 from the theorem cannot occur. Cases 2 and 3 from the theorem correspond to Cases 1 and 2, respectively, of the corollary. The only thing that requires further proof is that Case 3 of the theorem implies that \( f \) defines a family of isolated singularities with constant Milnor number.

However, Case 3 of Theorem 2.2 is the case where \( 0 \notin \Gamma_{d, f, z} \), that is, \( \Gamma_{d, f, z} \) is empty near the origin or, with its cycle structure, is 0. Then one applies the equivalence of Conditions 3 and 5 from Theorem 2.3 of [3] to conclude that \( f \) defines a family of isolated singularities with constant Milnor number (a simple \( \mu \)-constant family). \( \square \)
We can use Theorem 2.2 to prove a version of itself which refers to super-Thom varieties rather than sub-Thom varieties.

**Theorem 2.4.** (Thom Going Up) Let $T$ be an $r$-dimensional Thom variety of $\Sigma f$ at $0$. Let $V \supseteq T$ be an irreducible component of $\Sigma f$ at $0$. Then, at $0$, one of the following must hold:

1. $T = V$, or
2. $T \subseteq V \cap \overline{\Sigma f \setminus V}$, or
3. there exists a Thom variety $T' \subseteq V$ of $f$ at $0$ such that $T \subseteq \Sigma T'$, or
4. there exists a Thom variety $T' \subseteq V$ of $f$ at $0$ such that $\dim T' = r + 1$ and $T \subseteq T'$.

**Proof.** Suppose that we are not in Cases 1, 2, or 3. Then, let

$$X := \overline{T \setminus \overline{\Sigma f \setminus V}} \cup \bigcup_{T'} \Sigma T' \cup \bigcup_{T'' \supset T} T''$$

where the unions are over all Thom varieties $T'$ and $T''$ contained in $V$, and $T'' \not\supset T$. Since we are not in Cases 2 or 3, $X$ is an open, dense subset of $T$. Let $x \in X$. Then, $x \not\in \overline{\Sigma f \setminus V}$ and, at $x$, every Thom variety $T' \subseteq V$ contains $T$ and is smooth at $x$.

We apply Theorem 2.2 at $x$ in place of $0$. Let $e$ be the smallest dimension of a Thom variety $T' \subseteq V$ at $x$ such that $T'$ properly contains $T$; there is such an $e$ since we are not in Case 1, i.e., $V$ itself is a Thom variety in $V$ which properly contains $T$. Then we must be in Case 2 of Theorem 2.2, and there must be a Thom variety $\widetilde{T}$ of dimension $(e - 1)$ in $V$ at $x$. But this $\widetilde{T}$ must contain $T$ (by the choice of $x$), and we would have a contradiction of the definition of $e$ unless $\widetilde{T}$ does not properly contain $T$. Thus we must have $\widetilde{T} = T$ at $x$ and $e = r + 1$. Since this is true for $x$ in an open, dense subset of $T$, the conclusion of Case 4 follows. \[\square\]

### 3. Examples

**Example 3.1.** Suppose that the irreducible components of $\Sigma f$ at the origin are a line $L$ and a plane $P$. Is it possible that $L$ and $P$ are the only Thom varieties of $f$ at $0$? The answer is “no”, and one might suspect that that is because $\{0\}$ must also be a Thom variety. However, Theorem 2.2 with $e = 2$ tells us that, in fact, the plane $P$ must contain a 1-dimensional Thom variety.

Let us look at a specific example. Let $f = w^2 + xyz^2$. Then,

$$\Sigma f = V(2w, yz^2, xz^2, 2xyz) = V(w, z) \cup V(w, y, x) = P \cup L.$$ 

Of course, $P$ and $L$ are Thom varieties, but Theorem 2.2 tells us that there must be a 1-dimensional Thom variety contained in $P$.

The reader is invited to calculate the blow-up of the jacobian ideal to show that the other Thom varieties are, in fact, $V(w, z, x)$, $V(w, z, y)$ and $\{0\}$.

**Example 3.2.** Is the smoothness requirement in Theorem 2.2 and Corollary 2.3 really necessary? Yes. Consider $f = w^2 + (x^2 + y^2 + z^2)^2$. Then,

$$\Sigma f = V(2w, 4(x^2 + y^2 + z^2)x, 4(x^2 + y^2 + z^2)y, 4(x^2 + y^2 + z^2)z) = V(w, x^2 + y^2 + z^2).$$

Now $V(w, x^2 + y^2 + z^2)$ is a Thom variety. However, by symmetry, there cannot be a 1-dimensional Thom variety inside $V(w, x^2 + y^2 + z^2)$ and yet, by direct calculation, one can show that $\{0\}$ is a Thom variety.

Thus, if $\Sigma f$ is irreducible, but not smooth, the conclusions of Corollary 2.3 need not hold.
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