APPLICATION OF A UNIFIED KENMOTSU-TYPE FORMULA
FOR SURFACES IN EUCLIDEAN OR LORENTZIAN
THREE-SPACE

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Abstract. Kenmotsu’s formula describes surfaces in Euclidean 3-space by
their mean curvature functions and Gauss maps. In Lorentzian 3-space, Akuta-
gawa-Nishikawa’s formula and Magid’s formula are Kenmotsu-type formulas
for spacelike surfaces and for timelike surfaces, respectively. We apply them
to a few problems concerning rotational or helicoidal surfaces with constant
mean curvature. Before that, we show that the three formulas above can be
written in a unified single equation.

1. Introduction

In surface theory, the Weierstrass representation for minimal surfaces is one of
the fundamental tools. There are also Weierstrass-type representation formulas for
spacelike maximal surfaces and for timelike minimal surfaces in Lorentzian 3-space
$L^3$, and they play an important role as well. On the other hand, in the case where
surfaces are not minimal, Kenmotsu [13] gave a formula which describes surfaces
in Euclidean 3-space $E^3$ in terms of the Gauss map and mean curvature function
$H$ (under the assumption that $H \neq 0$). After that, Akutagawa-Nishikawa [1] and
Magid [20] (see also [2]) gave such formulas for spacelike surfaces and timelike sur-
faces in $L^3$, respectively. These three formulas, although established independently,
have created a common understanding that they all come from essentially the same
principle. One of the aims of this paper is to unify these three formulas (see Sections
2, 3).

Kenmotsu’s, Akutagawa-Nishikawa’s and Magid’s formulas have been consid-
ered as important ones in Euclidean or Lorentzian surface theory similar to the
Weierstrass representation, whereas it seems that they do not have much direct
application so far. Another aim is to present a practical aspect of these three for-
mulas. We give applications of a Kenmotsu-type formula to surfaces of constant
mean curvature from the classical differential geometric viewpoint. Although there
have been several modern theories nowadays, our argument is independent of them.
More precisely, we provide representations of (Euclidean or Lorentzian) Delaunay
surfaces and helicoidal surfaces with constant mean curvature ($cmc-H$ helicoids,
for short) in $E^3$ as explicitly as possible (see Sections 4, 5). It has been pointed
out in many literatures that Delaunay surfaces can be described in terms of elliptic

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functions and elliptic integrals. In fact, for example, we can find in [9] an explicit description of unduloids. We provide in this paper all (Lorentzian) Delaunay surfaces in terms of Jacobi’s elliptic functions. We show that cmc-$H$ helicoids can also be expressed with Jacobi’s elliptic functions. Jacobi’s elliptic functions play an important role in this paper.

The remaining aim of this paper is to give further applications (see Section 6). To make the most of the explicit representation here, we solve the periodicity condition when a cmc-$H$ helicoid is cylindrical, and as a result, we give another proof of Burstall-Kilian’s theorem [5]. Also, we introduce the radius of a cmc-$H$ helicoid, and then we notice that a cmc-$H$ helicoid is determined by the pitch and radius. As a consequence, the periodicity of a cmc-$H$ helicoid is determined by the pitch and radius. We also give a criterion by the pitch and radius whether two cmc-$H$ helicoids belong to the same associated family (i.e. a family of non-congruent but locally isometric surfaces with the same mean curvature). As a corollary, we can show that an unduloid and a nodoid whose mean curvatures are the same value are associated (i.e., locally isometric) if and only if the ratios $\rho/R$ of the inner radius $\rho$ and the outer radius $R$ are coincident.

2. Kenmotsu-type formula

We give a coordinate-free formula which describes surfaces by their mean curvature functions and Gauss maps. As a consequence, formulas due to Kenmotsu, Akutagawa-Nishikawa and Magid can be considered within a unified single formula. (The original formulas [13], [1], [20] are given in terms of local coordinate systems, respectively.)

In fact, our statement is as follows:

**Theorem 2.1** (cf. [13], [1], [20]). Let $M$ be a connected, oriented 2-dimensional manifold, and suppose that $x: M \to N$ is either

(i) an immersion to $N = \mathbb{E}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{E})$,

(ii) a spacelike immersion to $N = L^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{L})$, or

(iii) a timelike immersion to $N = L^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{L})$

with non-zero mean curvature function $H$ and Gauss map $n$. Then $x$ can be represented by $H$ and $n$, as

$$x = -\int \frac{1}{2H} \{dn + n \times (\ast dn)\}$$

(2.1)

where $\times$ denotes the Euclidean (resp. Lorentzian) vector product for (i) (resp. for (ii), (iii)), and $\ast$ denotes the Riemannian (resp. Lorentzian) Hodge $\ast$-operator on $T^*M$ with respect to the induced metric for (i), (ii) (resp. for (iii)).

Conversely, suppose that $n$ is a unit vector-valued function and $H$ is a function on a simply-connected Riemannian or Lorentzian 2-manifold $M$ satisfying

$$d \left( \frac{dn + n \times (\ast dn)}{H} \right) = 0 \quad (dn + n \times (\ast dn) \neq 0).$$

(2.2)

Then (2.1) gives

(i) a surface in $\mathbb{E}^3$ if $M$ is Riemannian, $n$ is unit Euclidean vector-valued and $\times = \times_E$,

(ii) a spacelike surface in $L^3$ if $M$ is Riemannian, $n$ is negative unit Lorentzian vector-valued and $\times = \times_L$, or
(iii) a timelike surface in $L^3$ if $M$ is Lorentzian, $n$ is positive unit Lorentzian vector-valued and $\times = \times_L$, whose Gauss map and mean curvature function are $n$ and $H$.

In the statement of Theorem 2.1, the following are utilized:

- For $\mathbf{a} = \mathbf{t}(a_1, a_2, a_3), \mathbf{b} = \mathbf{t}(b_1, b_2, b_3) \in \mathbb{R}^3$,
  \[(\mathbf{a}, \mathbf{b})_E = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad (\mathbf{a}, \mathbf{b})_L = a_1 b_1 + a_2 b_2 - a_3 b_3,
  \]
  and
  \[
  \mathbf{a} \times_E \mathbf{b} = \begin{pmatrix}
  a_2 b_3 - b_2 a_3 \\
  a_3 b_1 - b_3 a_1 \\
  -a_1 b_2 + b_1 a_2
  \end{pmatrix}, \quad \mathbf{a} \times_L \mathbf{b} = \begin{pmatrix}
  a_2 b_3 - b_2 a_3 \\
  a_3 b_1 - b_3 a_1 \\
  -(a_1 b_2 - b_1 a_2)
  \end{pmatrix}.
  \]

- The Gauss map $n$ is regarded as a vector-valued function, indeed, $n: M \to S^2 = \{x \mid \langle x, x \rangle_E = 1\} \subset E^3$ in the case (i), $n: M \to H^2 = \{x \mid \langle x, x \rangle_L = -1\} \subset L^3$ in the case (ii), and $n: M \to S^2_1 = \{x \mid \langle x, x \rangle_L = 1\} \subset L^3$ in the case (iii).

**Remark 2.2.** (1) As a pre-equal to (2.1), we have
\[-2Hd\mathbf{x} = dn + n \times (\ast dn), \tag{2.3}\]
so the additional condition in (2.2) is that $\mathbf{x}$ does not have zero mean curvature.

Note that the formula (2.3) for the Euclidean case (i) was also known in \cite{6} via a quaternionic description of surfaces.

(2) During the preparing this paper, the author found a paper \cite{23} which discusses a unification of representation formulas from the different viewpoint from ours.

As a consequence of Theorem 2.1 for surfaces with non-zero constant mean curvature $H$ (called cmc-$H$ for short), we can assert the following:

**Corollary 2.3.** Let $M$ be a connected, oriented 2-dimensional manifold, and suppose that $\mathbf{x}: M \to N$ is either

(i) a cmc-$H$ immersion to $N = E^3$,
(ii) a spacelike cmc-$H$ immersion to $N = L^3$, or
(iii) a timelike cmc-$H$ immersion to $N = L^3$

with the Gauss map $n$. Then $\mathbf{x}$ is represented as
\[
\mathbf{x} = -\frac{1}{2H} \left\{ n + \int n \times (\ast dn) \right\}, \tag{2.4}
\]
where $\times$ denotes the Euclidean (resp. Lorentzian) vector product for (i) (resp. for (ii), (iii)), and $\ast$ denotes the Riemannian (resp. Lorentzian) Hodge $\ast$-operator on $T^*M$ with respect to the induced metric for (i), (ii) (resp. for (iii)).

Conversely, suppose that $n$ is a unit vector-valued function on a simply-connected Riemannian or Lorentzian 2-manifold $M$ satisfying
\[
d \ast dn \text{ is parallel to } n, \quad (dn + n \times \ast dn \neq 0). \tag{2.5}
\]

Then (2.4) with a non-zero constant $H$ gives

(i) a cmc-$H$ surface in $E^3$ if $M$ is Riemannian and $n$ is unit Euclidean vector-valued,
(ii) a spacelike cmc-H surface in $L^3$ if $M$ is Riemannian and $n$ is negative unit Lorentzian vector-valued, or

(iii) a timelike cmc-H surface in $L^3$ if $M$ is Lorentzian and $n$ is positive unit Lorentzian vector-valued,

whose Gauss map and mean curvature are $n$ and $H$.

It is a well known fact that the condition (2.5) is equivalent to the harmonicity of the map $n: M \to S^2$, $H^2$ or $S^1$ in each case (i), (ii) or (iii), and it is also well known that a surface has constant mean curvature if and only if its Gauss map is harmonic (cf. [22], [14]).

We call the formula (2.1) (and its special case (2.4)) the Kenmotsu-type formula.

**Remark 2.4.**

1. Given a cmc-$H$ surface whose Gauss map is $n$, the map $-n$ is also harmonic. By the formula (2.4) with $-n$, we obtain another cmc-$H$ surface which is known as the parallel surface of $x$ with constant mean curvature.

2. Another well-known fact is that a parallel surface $\tilde{x} := x + \frac{1}{2H} n$ of a cmc-$H$ surface $x$, i.e.,

$$\tilde{x} = -\frac{1}{2H} \left\{ \int n \times (dn) \right\}$$

has constant Gaussian curvature

$$\begin{cases} 4H^2 \quad \text{in the case (i) or (iii)}, \\ -4H^2 \quad \text{in the case (ii)}. \end{cases}$$

We call $\tilde{x}$ the \textit{cgc-companion} of $x$.

Incidentally, a representation formula for negative constant Gaussian curvature surface in $E^3$ is known as \textit{Lelieuvre’s formula}. We thus have obtained a unified \textit{Lelieuvre-type formula} (2.6) for positive constant Gaussian curvature surfaces in $E^3$, for spacelike surfaces with negative constant Gaussian curvature in $L^3$, and for timelike surfaces with positive constant Gaussian curvature in $L^3$.

We also remark that there are formulas due to Sym [24] and Bobenko [3], which create surfaces with negative constant Gaussian curvature and surfaces with positive constant Gaussian curvature (simultaneously cmc-$H$) from harmonic maps to $S^2$, respectively. Sym-Bobenko’s formula is well-known as one of the most powerful tool nowadays. We refer to [21] for details.

### 3. Proof of Kenmotsu-type Formula

We shall provide proofs by the moving frame method. Notations here are standard (cf. [1], etc.).

**3.1. The case (i).** Let $e_1, e_2$ be a local orthonormal frame tangent to $x(M) \subset E^3$. We regard $e_1, e_2$ as vector-valued functions. Define a unit normal field $e_3$ by

$$e_3 := e_1 \times e_2.$$ 

Define local 1-forms $\omega^i$ ($i = 1, 2$) and $\omega^a_\beta$ ($\alpha, \beta = 1, 2, 3$) by

$$d\mathbf{x} = e_i \omega^i, \quad d\epsilon_\alpha = e_\beta \omega^\alpha_\beta.$$ 

(3.1)
Note that \((\omega_3^2)\) is \(\mathfrak{so}(3)\)-valued and
\[ d\omega^j = -\omega_j^j \wedge \omega^j, \quad d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega_\beta^\gamma. \]
are satisfied. Moreover, defining \(h_{ij}\) by
\[ \omega_i^3 = h_{ij} \omega^j, \]
then \(h_{ij} = h_{ji}\) and the mean curvature \(H\) is, by definition,
\[ H = \frac{h_{11} + h_{22}}{2}. \]

We shall also use complex-number notation. Set
\[ e := \frac{1}{2}(e_1 - ie_2), \quad \omega := \omega^1 + i\omega^2, \quad \pi := \omega_3^1 - i\omega_3^2. \]
Then we have
\[ \pi = q\omega + H\bar{\omega}, \quad \text{where} \quad q = \frac{1}{2}(h_{11} - h_{22} - 2ih_{12}). \quad (3.2) \]
The equation \((3.1)\) can be rewritten as
\[ dx = e\omega + \bar{e}\bar{\omega}, \quad de = -ie\omega_2 + \frac{1}{2}e_3\pi, \quad de_3 = -e\bar{\pi} - \bar{e}\pi. \quad (3.3) \]
Since the Hodge \(*\)-operator acts as \(*\omega = -i\omega\), it follows from \((3.2), (3.3)\) that
\[ *de_3 = i\{2H(e\omega - \bar{e}\bar{\omega}) - e\bar{\pi} + \bar{e}\pi\}. \]
Moreover, since \(e_3 \times e = ie\), we have
\[ e_3 \times (*de_3) = -2H(e\omega + \bar{e}\bar{\omega}) + e\bar{\pi} + \bar{e}\pi = -2Hdx - de_3. \]
The map \(e_3\) is nothing but the Gauss map \(n\), and henceforth
\[ dx = -\frac{1}{2H}\{dn + n \times (*dn)\}, \]
another assertion. Integrating this, we have the former assertion.

The latter assertion (i.e., converse assertion) follows from Poincaré’s lemma.

**Remark 3.1.** The completely integrable condition \((2.2)\) can be written as
\[ \frac{dH}{H} \wedge \{dn + n \times (*dn)\} = n \times (d*dn). \]
Hence, in the case where \(H\) is non-zero constant, \((2.2)\) is equivalent to \(n \times (d*dn) = 0\), i.e., \(d*dn\) is parallel to \(n\). This observation completes the proof of Corollary 2.3.

3.2. **The case (ii).** Although there are some points to pay attention to, e.g., using \(e_3 = -e_1 \times L e_2\), the 1-form \((\omega_3^2)\) is \(\mathfrak{so}(2,1)\)-valued, etc, the proof is quite similar to the cases (i) above and (iii) below. So the proof is omitted here and is left to the reader.
3.3. The case (iii). Let $e_1, e_2$ be a local orthonormal frame tangent to the timelike surface $\mathbf{x}(M) \subset \mathbb{L}^3$ such that
\[
\langle e_1, e_1 \rangle_L = 1, \quad \langle e_2, e_2 \rangle_L = -1, \quad \langle e_1, e_2 \rangle_L = 0.
\]
We regard $e_1, e_2$ as $\mathbb{R}^3$-valued functions. Define a unit normal field $e_3$ by
\[
e_3 := -e_1 \times_L e_2.
\]
It should be noted that $e_2 \times_L e_3 = -e_1$ and $e_3 \times_L e_1 = e_2$.

Define local 1-forms $\omega^i$ ($i = 1, 2$) and $\omega^\alpha_\beta$ ($\alpha, \beta = 1, 2, 3$) by
\[
d\mathbf{x} = e_i \omega^i, \quad de_\alpha = e_\beta \omega^\beta_\alpha.
\]
(3.4)

Note that $(\omega^\alpha_\beta)$ is $(1, 2)$-valued, i.e., $\omega^\alpha_\alpha = 0$, $\omega^1_2 = \omega^2_1$, $\omega^1_3 = -\omega^3_1$, $\omega^2_3 = \omega^3_2$.

The following equations hold:
\[
d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta.
\]
Moreover, define $h_{ij}$ by
\[
\omega^3_i = h_{ij} \omega^j,
\]
then $h_{ij} = h_{ji}$ and the mean curvature $H$ is, by definition,
\[
H = \frac{h_{11} - h_{22}}{2}.
\]

We shall also use the paracomplex-number notation. Recall that
\[
\tilde{\mathbb{C}} = \{ x + jy \mid x, y \in \mathbb{R} \},
\]
with the rules of addition and multiplication given by
\[
(x + jy) + (u + jv) = (x + u) + j(y + v),
\]
\[
(x + jy)(u + jv) = (xu + yv) + j(xy + xu) \quad \text{(in particular, } j^2 = 1),
\]
is a commutative algebra whose elements are called paracomplex numbers (also called split complex numbers).

Set
\[
e := \frac{1}{2}(e_1 + je_2), \quad \omega := \omega^1 + j\omega^2, \quad \pi := \omega^3 + j\omega^3.
\]
It is easily verified that
\[
e_3 \times_L e = je.
\]
(3.5)
The Hodge $*$-operator with respect to $I = (\omega^1)^2 - (\omega^2)^2$ satisfies, by definition,
\[
*\omega^1 = \omega^2, \quad *\omega^2 = \omega^1,
\]
hence
\[
*\omega = j\omega.
\]
(3.6)
On the other hand, we have
\[
\pi = q\omega + H\tilde{\omega} \quad \text{where } q = \frac{1}{2}(h_{11} + h_{22} + 2jh_{12}).
\]
(3.7)
The equation (3.4) can be rewritten as
\[
d\mathbf{x} = e\omega + \tilde{e}\omega, \quad de_3 = -e\pi - \tilde{e}\pi.
\]
(3.8)
It follows from (3.6), (3.7), (3.8) that
\[
*de_3 = -j\{2H(e\omega - \tilde{e}\omega) - e\pi + \tilde{e}\pi\}.
\]
Moreover, by (3.5), we have
\[ e_3 \times_L (\ast de_3) = -2H(e\omega + e\bar{\omega}) + e\pi + e\bar{\pi} = -2Hd\mathbf{x} - de_3. \]
The \( e_3 \) is nothing but the Gauss map \( n \), and henceforth
\[ dx = -\frac{1}{2H}\{dn + n \times_L (\ast dn)\}. \]
Integrating this, we have the former assertion.

The latter assertion (i.e., converse assertion) follows from Poincaré’s lemma.

4. Lorentzian Delaunay Surfaces

Corollary 2.3 asserts that a cmc-\( H \) surface is determined by its intrinsic Riemannian (Lorentzian) structure and a harmonic map to \( S^2, H^2 \) or \( S^1_2 \).

A Delaunay surface is, by definition, a rotational surface with non-zero constant mean curvature in \( \mathbb{E}^3 \), after the work \[8\]. Also, by the terminology (Lorentzian) Delaunay surface we mean a rotational surface with non-zero constant mean curvature in \( L^3 \).

(Lorentzian) Delaunay surfaces have been studied by many authors. We revisit them in relation to Kenmotsu-type formulas. Euclidean Delaunay surfaces will be treated in Section 5 as a special case of helicoidal cmc-\( H \) surfaces. We consider Lorentzian Delaunay surfaces in this section. Lorentzian Delaunay surfaces are classified into six types by their causality and rotation axes: whether it is a spacelike surface or a timelike surface, whether the rotation axis is timelike, spacelike or lightlike. Moreover, the classes of timelike Delaunay surfaces with spacelike axis can be divided into two kinds. Therefore, one can say that Lorentzian Delaunay surfaces are classified into seven types in a slightly finer sense.

Applying a Lorentzian motion on a Delaunay surface \( \mathbf{x} \) if necessary, we may assume the rotation axis of \( \mathbf{x} \) is the line through the origin in the direction
\[ \{0, 0, 1\} \text{ if } \mathbf{x} \text{ has timelike rotation axis}, \]
\[ \{1, 0, 0\} \text{ if } \mathbf{x} \text{ has spacelike rotation axis}, \]
\[ \{1, 0, 1\} \text{ if } \mathbf{x} \text{ has lightlike rotation axis}. \]

Moreover we can choose parameters \((u, v)\) so that \( v \) is the rotation parameter and the induced metric is conformal either to \( du^2 + dv^2 \) or to \( du^2 - dv^2 \).

According to these normalizations, the Gauss map \( \mathbf{n} \) of \( \mathbf{x} \) is written in one of the following forms:

1. (for a spacelike rotational surface \( \mathbf{x} \) with timelike axis)
\[ \mathbf{n}: (u, v) \mapsto \begin{pmatrix} e^{iv} & 0 & \sigma \end{pmatrix} \begin{pmatrix} \gamma \end{pmatrix} \in H^2 \subset \mathbb{C} \times \mathbb{R} \cong (L^3, ++ -), \tag{4.1} \]

2. (for a timelike rotational surface \( \mathbf{x} \) with timelike axis)
\[ \mathbf{n}: (u, v) \mapsto \begin{pmatrix} e^{iv} & 0 & \gamma \end{pmatrix} \begin{pmatrix} \sigma \end{pmatrix} \in S^1 \subset \mathbb{C} \times \mathbb{R} \cong (L^3, ++ -), \tag{4.2} \]

3. (for a spacelike rotational surface \( \mathbf{x} \) with spacelike axis)
\[ \mathbf{n}: (u, v) \mapsto \begin{pmatrix} 0 & 1 & \sigma \end{pmatrix} \begin{pmatrix} j\gamma \end{pmatrix} \in H^2 \subset \mathbb{R} \times \hat{\mathbb{C}} \cong (L^3, ++ -), \tag{4.3} \]
(4) (for a timelike rotational surface \( \mathbf{x} \) with spacelike axis of the first kind)

\[
\mathbf{n}: (u,v) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{-jv} \end{pmatrix} \begin{pmatrix} \gamma \\ j \sigma \end{pmatrix} \in S^2_1 \subset \mathbb{R} \times \hat{\mathbb{C}} \cong (L^3, ++ -),
\]

(4.4)

(5) (for a timelike rotational surface \( \mathbf{x} \) with spacelike axis of the second kind)

\[
\mathbf{n}: (u,v) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{-jv} \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix} \in S^2_1 \subset \mathbb{R} \times \hat{\mathbb{C}} \cong (L^3, ++ -),
\]

(4.5)

(6) (for a spacelike rotational surface \( \mathbf{x} \) with lightlike axis)

\[
\mathbf{n}: (u,v) \mapsto \exp vA \bigg( \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \bigg) \begin{pmatrix} \sigma \\ \gamma \end{pmatrix} \in H^2,
\]

(4.6)

(7) (for a timelike rotational surface \( \mathbf{x} \) with lightlike axis)

\[
\mathbf{n}: (u,v) \mapsto \exp vA \bigg( \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \bigg) \begin{pmatrix} s \\ c \end{pmatrix} \in S^2_1 \subset \mathbb{R} \times \hat{\mathbb{C}} \cong (L^3, ++ -),
\]

(4.7)

where \( \sigma = \sigma(u), \gamma = \gamma(u) \) are functions with \( \gamma^2 - \sigma^2 = 1 \), and \( s = s(u), c = c(u) \) are functions with \( c^2 + s^2 = 1 \), and

\[
A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{so}(2,1).
\]

We remark that one can start (4.3)–(4.5) with \( e^{jv} \) instead of \( e^{-jv} \). However, we prefer \( e^{-jv} \) for the matching with Remark 4.2 stated later.

As mentioned above, we provide a conformal structure \([du^2 + dv^2]\) on the domain \((U; u,v)\) for (4.1), (4.3), (4.6), and a Lorentzian conformal structure \([du^2 - dv^2]\) for (4.2), (4.4), (4.5), (4.7). It should be noted that each harmonic map equation for (4.1)–(4.4) is the same one, indeed,

\[
(\sigma' \gamma - \sigma \gamma')' - \sigma \gamma = 0 \quad (\text{with } \gamma^2 - \sigma^2 = 1).
\]

(4.8)

Moreover, the equation (4.8) can be explicitly solved in terms of Jacobi’s elliptic functions as follows:

\[
\sigma = cs(u,k), \quad \gamma = ns(u,k) \quad (-\infty < k^2 < \infty),
\]

(4.9)

where \( u \) can be replaced by \( \pm u + C \) for any constant \( C \), and \( -\infty < k^2 < \infty \) means that the modulus \( k \) can be any real or pure imaginary number.

Note that the harmonic map equation for (4.5) is

\[
(s'c - sc')' + sc = 0 \quad (\text{with } c^2 + s^2 = 1),
\]

(4.10)

which has the solution

\[
s = k \sinh(u,k), \quad c = \pm \cosh(u,k) \quad (0 \leq k < \infty),
\]

(4.11)

where \( u \) can be replaced by \( \pm u + C \) for any constant \( C \).

Note also that each harmonic map equation for (4.6) or for (4.7) is

\[
(\sigma' \gamma - \sigma \gamma')' + (\sigma - \gamma)^2 = 0 \quad (\text{with } \gamma^2 - \sigma^2 = 1).
\]

(4.12)

With the setting

\[
\sigma = \frac{1}{2} \left( \phi - \frac{1}{\phi} \right), \quad \gamma = \frac{1}{2} \left( \phi + \frac{1}{\phi} \right),
\]

the equation (4.12) is equivalent to

\[
\phi \phi'' - (\phi')^2 + \frac{1}{8} = 0,
\]

(4.13)
which has solutions
\[ \phi = u \quad \text{or} \quad \phi = \frac{\sin ku}{k}, \quad (-\infty < k^2 < \infty, k \neq 0), \quad (4.14) \]
where \( u \) can be replaced by \( \pm u + C \) for any constant \( C \).

Without loss of generality, the value of mean curvature \( H \) is assumed to be \(-1/2\).

As an application of (4.1)–(4.14) and Corollary 2.3, we have simple and explicit expressions of Lorentzian Delaunay surfaces as follows:

- **Delaunay surfaces with timelike axis**
  - spacelike ones
    \[ x(u, v) = \left( e^{iu} 0 1 \right) \left\{ \left( \begin{array}{c} \sigma' \\ \gamma \\ \end{array} \right) + \left( \int \frac{\sigma'}{\sigma^2} \right) \right\} \in \mathbb{C} \times \mathbb{R} \cong \mathbb{L}^3, \quad (4.15) \]
  - timelike ones
    \[ x(u, v) = \left( e^{iu} 0 1 \right) \left\{ \left( \begin{array}{c} \gamma' \\ \sigma \\ \end{array} \right) + \left( -\int \frac{\sigma'}{\gamma^2} \right) \right\} \in \mathbb{C} \times \mathbb{R} \cong \mathbb{L}^3, \quad (4.16) \]

- **Delaunay surfaces with spacelike axis**
  - spacelike ones
    \[ x(u, v) = \left( 1 0 e^{-jv} \right) \left\{ \left( \begin{array}{c} \sigma' \\ j\gamma \\ \end{array} \right) + \left( -\int \frac{\sigma'}{j\gamma^2} \right) \right\} \in \mathbb{R} \times \mathbb{C} \cong \mathbb{L}^3, \quad (4.17) \]
  - timelike ones of the first kind
    \[ x(u, v) = \left( 1 0 e^{-jv} \right) \left\{ \left( \begin{array}{c} \gamma' \\ j\sigma \\ \end{array} \right) + \left( \int \frac{\sigma^2}{j\gamma'\gamma} \right) \right\} \in \mathbb{R} \times \mathbb{C} \cong \mathbb{L}^3, \quad (4.18) \]
  - timelike ones of the second kind
    \[ x(u, v) = \left( 1 0 e^{-jv} \right) \left\{ \left( \begin{array}{c} s' \\ c \\ \end{array} \right) + \left( -\int \frac{c^2}{s'c} \right) \right\} \in \mathbb{R} \times \mathbb{C} \cong \mathbb{L}^3, \quad (4.19) \]

- **Delaunay surfaces with lightlike axis**
  - spacelike ones
    \[ x(u, v) = \exp vA \cdot \left\{ \left( \begin{array}{c} \sigma' \\ \gamma \\ \end{array} \right) + \left( -\int \frac{\sigma'}{\gamma^2} \right) \right\} \quad (4.20) \]
  - timelike ones
    \[ x(u, v) = \exp vA \cdot \left\{ \left( \begin{array}{c} \gamma' \\ \sigma \\ \end{array} \right) + \left( \int \frac{\gamma^2}{\gamma'\gamma} \right) \right\}. \quad (4.21) \]

where \( (\sigma, \gamma) = (\sigma(u), \gamma(u)), (s, c) = (s(u), c(u)) \) and \( \phi = \phi(u) \) are solutions to (4.8), (4.10) and (4.13), respectively, i.e., functions given in (4.9), (4.11) and (4.14).

**Remark 4.1.**
(1) In (4.15)–(4.19), each term including an integral can be also expressed by Jacobi’s elliptic functions and the elliptic integral \( E \) of the second kind:

\[
\int \sigma^2 du = \begin{cases} 
- \left\{ \frac{cn}{sn} + E \circ am \right\} (u, k) & \text{(if } 0 \leq k \leq 1) \\
(k^2 - 1)u - k \left\{ \frac{cn}{sn} + E \circ am \right\} (ku, 1/k) & \text{(if } k > 1) \\
- \alpha \left\{ \frac{cn}{sn} + E \circ am \right\} (\alpha u, \beta) & \text{(if } k \in i\mathbb{R})
\end{cases}
\]
where \( \alpha = \sqrt{1-k^2} \), \( \beta = \sqrt{-k^2/(1-k^2)} \).

\[
\int \gamma^2 du = u + \int \sigma^2 du,
\]

\[
\int c^2 du = \begin{cases} 
E \circ \text{am}(u, k) & \text{if } 0 \leq k < 1 \\
(1-k^2)u + kE \circ \text{am}(ku, 1/k) & \text{if } k > 1.
\end{cases}
\]

On the other hand, all terms in (4.20), (4.21) are elementary functions.

(2) Some of the Lorentzian Delaunay surfaces have conical singularities, namely, it can happen that \( x \) is not a regular surface.

Remark 4.2.  (1) Observing (4.15) and (4.18), one notices that the generating curves of a spacelike Delaunay surface with timelike axis and a timelike Delaunay surface with spacelike axis of the first kind are coincident, after the reflection \( R : x \leftrightarrow z \) in the \( xz \)-plane with respect to the line \( x = z \). Note that \( R \not\in O(2,1) \). In other words, if a curve \( \Gamma(u) = (\sigma+\sigma'/\gamma, 0, \gamma + \int \sigma^2 du) \)

is given in the \( xyz \)-space, then the trajectory of \( \Gamma(u) \) by \( \left( e^{iv} 0 \\
0 1 \right) \)

is a spacelike Delaunay surface with timelike axis in \( L^3 \), whereas the trajectory of \( \Gamma(u) \) by \( \left( 1 0 \\
0 e^{-jv} \right) \circ R \) is a timelike Delaunay surface with spacelike axis in \( L^3 \). The same phenomenon holds for cgc-companions \( \bar{x} \) of (4.15) and (4.18).

We note that the situation stated above also occurs for the pair (4.16) and (4.17).

(2) The profile curve of a timelike Delaunay surface with spacelike axis of the second kind is the same as for a Delaunay surface in \( E^3 \) (ignoring the underlying Riemannian or Lorentzian structure). See Figures 1–4.

For previous works on details for Lorentzian Delaunay surfaces, we refer to [10], [11], [12], [16], [17], [18], [19] and so on.
5. CMC HELICOIDS

We call a helicoidal surface with non-zero constant mean curvature \( H \) in \( E^3 \) a \textit{cmc-}H \textit{helicoid}. Cmc-H helicoids were studied and already classified by do Carmo-Dajczer [7]. However, in this section, we investigate them again by another approach using the Kenmotsu formula (2.4).

5.1. Formulation. We consider helicoidal motions in \( E^3 \). (A helicoidal motion is also called a screw motion.) Without loss of generality, we always suppose the axis of any helicoidal motion is the \( z \)-axis. Thus a group of helicoidal motions of pitch \( \lambda \) is a one-parameter group of Euclidean motions

\[
M_\lambda(t): \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda t \end{pmatrix}
\]

for a real constant \( \lambda \). Note that a helicoidal motion of pitch 0 is a rotational motion.

A trajectory of a space curve \( \mathbf{c}: I \to E^3 \) by a group of helicoidal motions of pitch \( \lambda \) is called a helicoidal surface of pitch \( \lambda \) with generating curve \( \mathbf{c} \). A generating curve is not unique for a helicoidal surface. We can choose a generating curve \( \mathbf{c}(u) = (x(u), y(u), z(u)) \) so that \( (u, v) \mapsto M_\lambda(v)\mathbf{c}(u) \) is an orthogonal parametrization. Then, the first fundamental form is given by

\[
I = \{(x')^2 + (y')^2 + (z')^2\} du^2 + \{\lambda^2 + x^2 + y^2\} dv^2.
\]

Moreover, after a suitable parameter change of \( u \), we have an isothermal parametrization \( (u, v) \mapsto M_\lambda(v)\mathbf{c}(u) \), that is, \( I \) is proportional to \( du^2 + dv^2 \). We suppose this
parametrization on any helicoidal surface \( x = M_\lambda(v)e(u) \). The Gauss map is defined to be \( n = (x_u \times x_v)/|x_u| \). It has the form

\[
 n = \begin{pmatrix}
 \cos v & -\sin v & 0 \\
 \sin v & \cos v & 0 \\
 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
 \alpha(u) \\
 \beta(u) \\
 \gamma(u)
\end{pmatrix}, \quad \alpha^2 + \beta^2 + \gamma^2 = 1. \tag{5.1}
\]

For simplicity, we frequently identify \( E^3 \) with \( \mathbb{C} \times \mathbb{R} \) similarly to the identification in Section 4. For example, (5.1) is also expressed as

\[
 n = \begin{pmatrix}
 e^{iv} & 0 & X \\
 0 & 1 & \gamma
\end{pmatrix} \in \mathbb{C} \times \mathbb{R} = E^3, \quad |X|^2 + \gamma^2 = 1, \quad (X = \alpha + i\beta). \tag{5.2}
\]

The following formulas will often be needed.

**Lemma 5.1.** For \( x = \begin{pmatrix} X \\ x \end{pmatrix}, y = \begin{pmatrix} Y \\ y \end{pmatrix} \in \mathbb{C} \times \mathbb{R} \cong E^3, \)

\[
 \langle x, y \rangle = \text{Re}(XY) + xy, \quad x \times y = \begin{pmatrix}
 -i|X Y| \\
 \text{Im}(\bar{X}Y)
\end{pmatrix}.
\]

First of all, we find the harmonic map equation for the map \( n \) given by (5.2). Paying attention to the isothermality of \((u, v)\), we have

\[
 \begin{aligned}
 d\ n & = \left( e^{iv} \ 0 \right) \begin{pmatrix} X' \\
 \gamma
\end{pmatrix} du + \left( iX \right) dv, \\
 *d\ n & = \left( e^{iv} \ 0 \right) \begin{pmatrix} -iX' \\
 \gamma'
\end{pmatrix} du + \left( X' \gamma' \right) dv, \\
 n \times (d^*n) & = \left( e^{iv} \ 0 \right) \begin{pmatrix} X \gamma \\
 -|X|^2 \\
 \text{Im}(X'X)
\end{pmatrix} du + \left( \text{Im}(X'X) \right) dv.
\end{aligned}
\]

It follows that

\[
 d\ n + n \times (d^*n) = 0 \iff \begin{cases} X' + X\gamma = 0 \\
 \gamma' - 1 + \gamma^2 = 0 \\
 X + X'\gamma - X\gamma' = 0 \\
 \text{Im}(X'X) = 0 \end{cases} \iff \begin{cases} X' + X\gamma = 0 \\
 \gamma' = 1 - \gamma^2 \\
 X + X'\gamma - X\gamma' = 0 \\
 \text{Im}(X'X) = 0 \end{cases}.
\]

We have noted that the equation \( d\ n + n \times (d^*n) = 0 \) is a condition that \( H = 0 \), in other words, \( n \) is an orientation-reversing conformal map.

On the other hand,

\[
 d * d\ n = \left( e^{iv} \ 0 \right) \begin{pmatrix} X'' \\
 \gamma''
\end{pmatrix} du \wedge dv,
\]

and hence \( d * d\ n \) is proportional to \( n \) if and only if

\[
 \left| \begin{array}{cc}
 X'' & -X \\
 \gamma'' & \gamma
\end{array} \right| = 0, \quad \text{Im}(X''X) = 0,
\]

equivalently,

\[
 (X'\gamma - X\gamma')' = X\gamma, \quad \text{Im}(X'X) = c_1 (\text{constant}).
\]

We have thus the following lemma.
Lemma 5.2. Let a map \( n: (U, [du^2 + dv^2]) \to S^2 \subset \mathbb{C} \times \mathbb{R} \cong \mathbb{E}^3 \) be given in the form
\[
 n = \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X(u) \\ \gamma(u) \end{pmatrix}, \quad |X|^2 + \gamma^2 = 1.
\] (5.3)
Then the map \( n \) is
- orientation-reversing and conformal if and only if
  \[
  X' + X\gamma = 0, \quad \gamma' = 1 - \gamma^2,
  \] (5.4)
- harmonic if and only if
  \[
  (X'\gamma - X\gamma')' = X\gamma, \quad \text{Im}(X'\bar{X}) = c_1 (= \text{constant}).
  \] (5.5)

Lemma 5.2 leads to the following proposition.

Proposition 5.3. Let
\[
 n = \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X(u) \\ \gamma(u) \end{pmatrix}: (U, [du^2 + dv^2]) \to S^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{E}^3
\] be a harmonic map satisfying \( dn + n \times^* dn \neq 0 \). The cmc-\( H \) surface whose Gauss map is \( n \) is given by
\[
 -2Hx = \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} X' \\ \gamma \end{pmatrix} \left( X'\gamma - X\gamma' \right) \right\} + \begin{pmatrix} 0 \\ c_1 v \end{pmatrix},
\]
where \( c_1 = \text{Im}(\bar{X}X') (= \text{constant}) \).

The cgc-companion \( \check{x} \) of \( x \) is
\[
 \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} X' \\ \gamma \end{pmatrix} \left( X'\gamma - X\gamma' \right) \right\} + \begin{pmatrix} 0 \\ c_1 v \end{pmatrix}.
\]

Proof. By (5.5), we have
\[
 n \times (*dn) = \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} X' \\ \gamma \end{pmatrix} \left( X'\gamma - X\gamma' \right) \right\} + \begin{pmatrix} 0 \\ c_1 v \end{pmatrix}
\] = \( d\{e^{iv}(X'\gamma - X\gamma')\}/(\gamma^2 - 1)du + c_1 dv \).

Thus
\[
 \int n \times (*dn) = \begin{pmatrix} e^{iv}(X'\gamma - X\gamma')/\int(\gamma^2 - 1)du + c_1 v \\ 0 \end{pmatrix} = \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} X' \\ \gamma \end{pmatrix} \left( X'\gamma - X\gamma' \right) \right\} + \begin{pmatrix} 0 \\ c_1 v \end{pmatrix}.
\]

Hence the assertion follows from Corollary 2.3.

The cmc-\( H \) surface obtained in Proposition 5.3 is, of course, a cmc-\( H \) helicoid.

5.2. Parametrization by the explicit solution. We wish to find an explicit description of a harmonic map \( n \) which is not orientation-reversing conformal. For this purpose, we first determine the solution to the orientation-reversing conformal map equation (5.4).

Lemma 5.4. Let a map \( n: (U, [du^2 + dv^2]) \to S^2 \) be given in the form (5.3). Then the map \( n \) solves the equation (5.4) if and only if
\[
 n = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} e^{iv} & 0 \\ 0 & 1 \end{pmatrix} \left( \frac{1}{\cosh u} \begin{pmatrix} 1 \\ \tanh u \end{pmatrix} \right)
\] (5.6)
up to a translation \((u, v) \mapsto (u + c_1, v + c_2)\) in the uv-plane.
Proof. The assertion is immediately verified and is left to the reader. □

Other than (5.6), we seek the solution to (5.5). Since $|X|^2 + \gamma^2 = 1$, we set

$$\begin{pmatrix} X \\ \gamma \end{pmatrix} = \begin{pmatrix} e^{ig \cos f} \\ \sin f \end{pmatrix}$$

for $f = f(u), g = g(u)$. Then we have

\[
\begin{align*}
(f'') + (1 + (g')^2) \cos f \sin f &= 0 \\
fg' - (g' \cos f \sin f)' &= 0 \\
g' \cos^2 f &= c_1
\end{align*}
\]

(5.7)

If a solution to (5.7) consists of a constant function $f$ other than (5.6), then $\sin f = 0$ and $g = c_1 u + C$ ($c_1 \neq 0$). In other words, the solution to (5.5) which has constant $\gamma$ is

$$\begin{pmatrix} X \\ \gamma \end{pmatrix} = \begin{pmatrix} e^{ic_1 u} \\ 0 \end{pmatrix}, \quad c_1 \neq 0.$$ (5.8)

Applying Proposition 5.3 with (5.8), we have

\[
-2Hx = \begin{pmatrix} e^{iv} \\ 0 \\ 0 \\ 1 \end{pmatrix} \left\{ \begin{pmatrix} \pm e^{ic_1 u} \\ 0 \\ 0 \\ 1 \end{pmatrix} + \left( \begin{pmatrix} 0 \\ f(-1)du \end{pmatrix} \right) + \begin{pmatrix} 0 \\ c_1 v \end{pmatrix} \right\} = e^{i(v \pm c_1 u)} - u + c_1 v.
\]

(5.9)

Note that (5.9) represents a circular cylinder.

Next, we consider the system (5.7) of equations under the assumption that $f$ is non-constant. Substituting the second equation into the first, we have

\[
f'' + c_1^2 \frac{\sin f}{\cos^3 f} + \sin f \cos f = 0.
\]

Multiplying by $f'$ on both sides and integrating them, we have

\[
(f')^2 + \frac{c_1^2}{\cos^2 f} - \cos^2 f = c_2 + 1
\]

for some constant $c_2$. (On the right hand side, we have used $c_2 + 1$ instead of just $c_2$ for later convenience.)

\[
\{\cos f \cdot f'\}^2 + c_1^2 - \cos^4 f = (c_2 + 1) \cos^2 f
\]

Recalling that $\sin f = \gamma$, we have

\[
(\gamma')^2 = (1 - \gamma^2)^2 + (c_2 + 1)(1 - \gamma^2) - c_1^2
= \gamma^4 - (c_2 + 1)\gamma^2 + (c_2 - c_1^2).
\]

(5.10)

So the solution $\gamma$ must satisfy the ordinary differential equation (5.10). To analyze (5.10), we may assume $c_1 \geq 0$. At the same time, we have to take it into consideration that the range of $\gamma$ must $-1 \leq \gamma \leq 1$. Hence, the constants $c_1, c_2$ receive restrictions on their values. Indeed, we have the following lemma.
Lemma 5.5. Let $^t(X, \gamma)$ be a solution to (5.5) with $|X|^2 + \gamma^2 = 1$ which is neither (5.6) nor (5.8). Then $\gamma$ satisfies the differential equation (5.10) for constants $c_1, c_2$ with

$$c_2 > c_1^2.$$ 

Proof. We have only to show that $c_2 > c_1^2$.

For a solution $\gamma = \gamma(u)$, there exists $u = u_0$ such that

$$(\gamma')^2 > 0, \text{ i.e., } \gamma^4 - (c_2 + 1)\gamma^2 + (c_2 - c_1^2) > 0,$$

and thus the following assertion necessarily holds: there exists $x \in [0, 1]$ such that

$$F(x) = x^2 - (c_2 + 1)x + (c_2 - c_1^2) > 0.$$ 

Since $F(0) = c_2 - c_1^2$, $F(1) = -c_1^2 \leq 0$ and $F(x)$ is convex below, $F(0) = c_2 - c_1^2$ must be positive. \hfill $\square$

We continue to investigate (5.10) with $c_2 > c_1^2$. Let

$$C := \{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq 0, \ c_2 > c_1^2\}.$$ 

For $(c_1, c_2) \in C$, the roots of the quadric equation $x^2 - (c_2 + 1)x + (c_2 - c_1^2) = 0$ are

$$\alpha = \frac{c_1 + 1 - \sqrt{c_2 - c_1^2} + 4c_1^2}{2}, \quad \beta = \frac{c_1 + 1 + \sqrt{c_2 - c_1^2} + 4c_1^2}{2}.$$ 

By these, we have a bijective correspondence

$$C = \{c_1 \geq 0, \ c_2 > c_1^2\} \longleftrightarrow \mathcal{A}' = \{(\alpha, \beta) \mid 0 < \alpha \leq 1, 1 \leq \beta < \infty\}.$$ 

The inverse is

$$\begin{cases} c_1 = \sqrt{(1 - \alpha)(\beta - 1)} \\ c_2 = \alpha + \beta - 1. \end{cases}$$ 

Moreover, we set

$$a := \sqrt{\alpha}, \quad b := \sqrt{\beta}$$

and have a bijective correspondence

$$C \longleftrightarrow \mathcal{A} := \{(a, b) \mid a \in (0, 1), b \in [1, \infty)\}.$$ 

Therefore, the differential equation (5.10) with $(c_1, c_2) \in C$ is rewritten as

$$(\gamma')^2 = (a^2 - \gamma^2)(b^2 - \gamma^2), \quad (a, b) \in \mathcal{A}.$$ 

The general solution is

$$\gamma = a \operatorname{sn}(bu, \frac{a}{b}) \left( b \operatorname{sn}(au, \frac{b}{a}) \right),$$

where $u$ can be replaced by $\pm u + C$ for an arbitrary constant $C$. Note that the range of $\gamma = a \operatorname{sn}(bu, \frac{a}{b})$ is included in $[-1, 1]$ for each fixed $(a, b) \in \mathcal{A}$.

Without loss of generality, we may assume $C = 0$. Thus we think of $\gamma = \pm \operatorname{sn}(bu, \frac{a}{b})$, It will be explained later that the choice of the sign of $\gamma = \pm \operatorname{sn}(bu, \frac{a}{b})$ does not create any essential difference. So we concentrate on $\gamma = -a \operatorname{sn}(bu, \frac{a}{b})$ for a while. The second equation of (5.7) leads us to

$$g = \int \frac{c_1}{\cos^2 \frac{u}{2}} \, du = c_1 \int \frac{du}{1 - a^2 \operatorname{sn}^2(bu, a/b)} = c_1 \pi \left( \operatorname{am}(bu, \frac{a}{b}) \right), \quad c_1 = \sqrt{(1 - a^2)(b^2 - 1)},$$

(5.11)
where $\Pi$ denotes the elliptic integral of the third kind

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}.$$ 

The argument up to here proves the following lemma:

**Lemma 5.6.** Let a map $\mathbf{n}: (U, [du^2 + dv^2]) \to S^2$ be given in the form (5.3) with non-constant $\gamma$. Then $\mathbf{n}$ is a harmonic map not being (5.6) if and only if it is given by (5.8) or

$$\mathbf{n} = \left( e^{i\psi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \sqrt{1 - a^2 \sin^2 (bu, \frac{a}{b})} \\ -a \sin (bu, \frac{a}{b}) \end{pmatrix} \right), \quad (5.12)$$

where $a, b$ are constants with $0 < a \leq 1 \leq b$ and $g$ is a function given by (5.11).

We have obtained Lemma 5.6 under the assumption that $\gamma$ is non-constant. However, we can include the case that $\gamma$ is constant in Lemma 5.6, by allowing that the number $a$ can be 0 in (5.12). Hence, Lemma 5.6, Proposition 5.3 and (5.9) lead us to the following proposition.

**Proposition 5.7.** For constants $a, b$ with $0 \leq a \leq 1 \leq b$, set

$$c_1 = \sqrt{(1 - a^2)(b^2 - 1)}, \quad g(u) = \frac{c_1}{b} \Pi(am(bu, \frac{a}{b}), a^2, \frac{a}{b}).$$

Then, for non-zero constant $H$,

$$-2H \mathbf{x} = \left( e^{i\psi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \sqrt{1 - a^2 \sin^2 (bu, \frac{a}{b})} \\ -a \sin (bu, \frac{a}{b}) \end{pmatrix} \right) + \left( \begin{pmatrix} \frac{a(b \cos (bu, \frac{a}{b}) - i c_1 \sin (bu, \frac{a}{b}) \sqrt{1 - a^2 \sin^2 (bu, \frac{a}{b})}}{(b^2 - 1)u - b \cos (bu, \frac{a}{b})} \end{pmatrix} \right) + \left( \begin{pmatrix} 0 \\ c_1 v \end{pmatrix} \right) \right) \quad (5.13)$$

gives a cmc-$H$ helicoid $\mathbf{x}: U \to \mathbb{E}^3$.

Conversely, any cmc-$H$ helicoid can be parametrized in this manner.

**Remark 5.8.**

(1) As do Carmo-Dajczer [7] pointed out, for a fixed non-zero constant $H$, all cmc-$H$ helicoids form a two-parameter family. Proposition 5.7 asserts that $a, b$ can play a role of parameters of it.

(2) For the value $a = 1$, (5.13) reduces to

$$\mathbf{x} = \left( e^{i\psi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \cos (bu, \frac{1}{b}) \\ \sin (bu, \frac{1}{b}) \end{pmatrix} \right),$$

which is a nodoid.

For the value $b = 1$, (5.13) reduces to

$$\mathbf{x} = \left( e^{i\psi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \cos (bu, \frac{1}{b}) \\ \sin (bu, \frac{1}{b}) \end{pmatrix} \right),$$

which is an unduloid.

5.3. **Fundamental forms.** We have obtained an explicit formula for cmc-$H$ helicoids. Now we calculate their fundamental forms.

First, we state a general proposition.

**Proposition 5.9.** Let $\mathbf{x}: M \to \mathbb{E}^3$ be an immersion which is not a minimal surface. Let $\mathbf{n}$ be the Gauss map of $\mathbf{x}$. Then its fundamental forms and Hopf differential
are given by

\[
I = \frac{1}{4H^2} \left\{ |\mathbf{dn}|^2 + 2|\mathbf{n}, *\mathbf{dn}, \mathbf{dn}| + |*\mathbf{dn}|^2 \right\}, \\
\begin{align*}
\mathbb{II} &= \frac{1}{2H} \left\{ |\mathbf{dn}|^2 + |\mathbf{n}, *\mathbf{dn}, \mathbf{dn}| \right\}, \\
\mathbb{III} &= |\mathbf{dn}|^2, \\
Q &= |(1 + i*)\mathbf{dn}|^2.
\end{align*}
\]

Moreover, assume that \( \mathbf{x} \) is a \( \text{cmc-}H \) surface. Then, for the cgc-companion of \(-2H\mathbf{x}\), i.e., \( \mathbf{x} = \int \mathbf{n} \times (\ast \mathbf{dn}) \), the first and second fundamental forms are given by

\[
\begin{align*}
\mathbf{I} &= | \ast \mathbf{dn} |^2, & \mathbb{II} &= | \mathbf{n}, \mathbf{dn}, \ast \mathbf{dn} |,
\end{align*}
\]

respectively.

A proof of Proposition 5.9 is straightforward by the definitions \( I = \langle dx, dx \rangle \), \( \mathbb{II} = -\langle dx, \mathbf{dn} \rangle \), \( \mathbb{III} = \langle \mathbf{dn}, \mathbf{dn} \rangle \), the formula \( Q = \mathbb{II} - HI + |\mathbf{n}, dx, \ast \mathbf{dn}| \), and \((2.3)\) or \((2.6)\). So the detail is left to the reader.

The following proposition follows from Proposition 5.9 and Proposition 5.7.

**Proposition 5.10.** For a cmc-\(H\) helicoid \( \mathbf{x} \) \((5.13)\),

\[
4H^2 I = \left( (a \cos b \mathbf{dn}) \left( \frac{a}{b} \right)^2 \right) (du^2 + dv^2), \\
4H (\mathbb{II} - HI) = (c_2 - 1)(du^2 - dv^2) + 4c_1 dudv, \\
\mathbb{III} = (c_2 - a^2 \sin^2(bu, \frac{a}{b})) du^2 + 2c_1 dudv + (1 - a^2 \sin^2(bu, \frac{a}{b})) dv^2, \\
Q = \frac{(\sqrt{b^2 - 1} - i\sqrt{1 - a^2})^2}{4H} (du + i dv)^2
\]

and, for the cgc-companion of \(-2H\mathbf{x}\),

\[
\begin{align*}
\mathbf{I} &= (1 - a^2 \sin^2(bu, \frac{a}{b})) du^2 - 2c_1 dudv + (c_2 - a^2 \sin^2(bu, \frac{a}{b})) dv^2, \\
\mathbb{II} &= -ab \cos \mathbf{dn} \left( \frac{a}{b} \right)^2 (du^2 + dv^2),
\end{align*}
\]

where \( c_1 = \sqrt{(1 - a^2)(b^2 - 1)} \) and \( c_2 = a^2 + b^2 - 1 \).

**Remark 5.11** (Isothermic nets). One application of these formulas in Proposition 5.10 is that we can obtain an isothermic net on any cmc-\(H\) helicoidal surface explicitly. The word ‘isothermic’ means ‘isothermal and curvature line’, in other words, the first and second fundamental forms can be simultaneously diagonalized. In fact, if we introduce a new parameter \((x, y)\) by

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \frac{1}{\sqrt{b^2 - a^2}} \begin{pmatrix}
\sqrt{1 - a^2} & -\sqrt{b^2 - 1} \\
\sqrt{b^2 - 1} & \sqrt{1 - a^2}
\end{pmatrix} \begin{pmatrix}
u \\
w
\end{pmatrix},
\]

then \((x, y)\) is isothermic.

**Remark 5.12** (Cgc-companions are wave fronts.). Another application of Proposition 5.10 is to prove that any cgc-companion \( \mathbf{x} \) fails to be an immersion. In fact, one can show that the first fundamental form \( \mathbf{I} \) degenerates where \( \mathrm{sn}(bu, \mu) = \pm 1 \). This happens for \( u = (2k - 1)K(a/b)/b \), where \( k \in \mathbb{Z} \) and \( K(a/b) \) is the complete elliptic integral of the first kind. Thus \( \mathbf{x} \) is a wave front because it is a parallel surface of a regular surface \( \mathbf{x} \).
6. FURTHER INVESTIGATION ON cmc-\(H\) HECICOIDs

We shall further investigate cmc-\(H\) helicoids and their cgc-companions based on the explicit expression (5.13).

6.1. Modification. For the sake of later argument, we start by modifying the formula (5.13) slightly. We introduce
\[ \mu = a/b, \]
and use \( \mu, b \) instead of \( a, b \). The range is
\[ \mathfrak{M} = \{ (\mu, b) \mid 0 \leq \mu \leq 1/b, \ 1 \leq b \} = \{ (\mu, b) \mid 0 \leq \mu \leq 1, \ 1 \leq b \leq 1/\mu \}. \]
Namely, \( \mathfrak{M} \) can be considered as a parameter space of all cmc-\(H\) helicoids. \( b = 1 \) and \( b = 1/\mu \) correspond to unduloids and nodoids, respectively. In particular, \( (\mu, b) = (1, 1) \) corresponds to the round sphere. \( \mu = 0 \) with arbitrary \( b \) corresponds to the same circular cylinder. Note that a circular cylinder can be considered as a helicoidal surface whose value of pitch is arbitrary.

The first fundamental form \( I \) and Hopf differential \( Q \) are rewritten as
\begin{align*}
4H^2 I &= \{(\mu \cn - \dn)(bu, \mu)\}^2 b^2 (du^2 + dv^2), \\
4HQ &= (1 - \mu^2)e^{i\bar{\theta}(u,b)} (du + iv)^2,
\end{align*}
where \( \theta = \bar{\theta}(\mu, b) \) is determined by
\[ \cos \theta = \frac{\sqrt{b^2 - 1}}{b^2(1 - \mu^2)}, \quad \sin \theta = -\frac{1 - \mu^2b^2}{b^2(1 - \mu^2)}. \]
Setting \( bu = \tilde{u}, \ bv = \tilde{v} \), we have
\begin{align*}
4H^2 I &= \{(\mu \cn - \dn)(\tilde{u}, \mu)\}^2 (d\tilde{u}^2 + d\tilde{v}^2), \\
4HQ &= (1 - \mu^2)e^{i\bar{\theta}(u,b)} (d\tilde{u} + id\tilde{v})^2. \quad (6.1)
\end{align*}
The formulas (6.1) imply that two cmc-\(H\) helicoids having the same \( \mu \) and distinct \( b \) are isometric but non-congruent. In other words, two cmc-\(H\) helicoids belong to the same associated family if and only if they have the same value of \( \mu \). Moreover, for any fixed \( \mu \), the family \( \{ x \mid 1 \leq b \leq 1/\mu \} \) includes an unduloid \( (b = 1) \) and a nodoid \( (b = 1/\mu) \).

We modify Proposition 5.7 to a more convenient statement by replacing \( (bu, bv) \) by \( (u, v) \) and \( a/b \) by \( \mu \).

**Proposition 6.1.** Let \( (\mu, b) \in \mathfrak{M} = \{ (\mu, b) \mid 0 \leq \mu \leq 1, \ 1 \leq b \leq 1/\mu \} \) and set
\begin{align*}
c_1 &= \sqrt{1 - \mu^2b^2}(b^2 - 1), \\
n_0(u) &= \left( \frac{1 - \mu^2b^2\sn^2}{-\mu\sn} \right)(u, \mu), \\
x_0(u) &= \left( \frac{\mu(b\cn\dn - ic_1\sn)}{\sqrt{1 - \mu^2b^2}\sn^2} \right)(u, \mu), \\
g(u) &= \frac{c_1}{b} \Pi(\am(u, \mu), \mu^2b^2, \mu).
\end{align*}
Then, for a non-zero constant $H$,

$$-2Hx = \begin{pmatrix} e^{iv/b} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{ig(u)} & 0 \\ 0 & 1 \end{pmatrix} \{n_0(u) + \check{x}_0(u)\} + \begin{pmatrix} 0 \\ c_1v/b \end{pmatrix} \quad (6.2)$$

gives a cmc-$H$ helicoid $x: U \to \mathbb{E}^3$.

Conversely, any cmc-$H$ helicoid can be parametrized in this manner.

### 6.2. Boundedness by circular cylinders

In this paper, we say that a helicoidal surface $H$ is bounded (or bounded outward) if there exists a circular cylinder $C$ such that $H$ is included inside of $C$. Similarly, $H$ is said to be bounded inward if there exists a circular cylinder $C$ such that $H$ is included outside of $C$.

Thanks to the formula (6.2), we can easily estimate the boundedness of cmc-$H$ helicoids.

**Proposition 6.2.** Any helicoidal surface of positive constant Gaussian curvature is bounded. It is also bounded inward except for the parallel surface of an unduloid.

**Proof.** We may suppose a helicoidal surface of positive constant Gaussian curvature is given by the cgc-companion $\check{x}$ of (6.2). We have only to prove the assertion for a special value of $H$, hence we assume $H = -1/2$ here.

For $\check{x} = \check{t}(x,y,z)$, set $p(\check{x}) = t(x,y,0)$. Then, by straightforward computation, we have

$$|p(\check{x})|^2 = \left| \frac{\mu b \cn \dn - ic_1 \sn}{\sqrt{1 - \mu^2 b^2 \sn^2}}(u, \mu) \right|^2 = \mu^2 b^2 (b^2 - \sn^2(u, \mu)).$$

Since $0 \leq \sn^2(u, \mu) \leq 1$, we have

$$\mu b \sqrt{b^2 - 1} \leq |p(\check{x})| \leq \mu b^2. \quad (6.3)$$

We may exclude the case $\mu = 0$ where the image of $\check{x}$ degenerates to a line. Thus, unless $\mu = 0$, the image of $\check{x}$ is inside a circular cylinder of radius $\mu b^2$, and outside a circular cylinder of radius $\mu b \sqrt{b^2 - 1}$ if $b \neq 1$.

If $b = 1$, then the inner radius equals 0, indeed, $x$ is an unduloid. \hfill \box

For a cgc-companion with a general value $H$ ($\neq 0$), the inequality (6.3) is

$$\mu b \sqrt{b^2 - 1}/|2H| \leq |p(\check{x})| \leq \mu b^2/|2H|,$$

that is,

$$\mu b \sqrt{b^2 - 1}/\sqrt{K} \leq |p(\check{x})| \leq \mu b^2/\sqrt{K}. \quad (6.4)$$

In (6.4), the equality can happen. We call $\mu b \sqrt{b^2 - 1}/\sqrt{K}$, $\mu b^2/\sqrt{K}$ the inner radius, outer radius of $\check{x}$, respectively. The outer radius is simply called the radius as long as there is no confusion.

**Proposition 6.3** (cf. Remark 4.15 of [7]). Any cmc-$H$ helicoid is bounded.

**Proof.** We may assume that a cmc-$H$ helicoid $x$ is given by (6.2). We have only to prove the assertion for a special value of $H$, hence we assume $H = -1/2$ here.
For \( x = \ell(x, y, z) \), set \( p(x) = \ell(x, y, 0) \). Then, by straightforward computation, we have

\[
|p(x)|^2 = \left| \sqrt{1 - \mu^2 b^2 \text{sn}^2(u, \mu) + \frac{\mu b (b \text{dn} - ic_1 \text{sn})}{\sqrt{1 - \mu^2 b^2 \text{sn}^2}} - (u, \mu) \right|^2
\]

\[
= 1 + \mu^2 b^2 - 2\mu^2 b^2 \text{sn}^2(u, \mu) - 2\mu b^2 \text{cn}(u, \mu) \text{dn}(u, \mu)
\]

\[
= 1 + \mu^2 b^2 - 2\mu^2 b^2 + 2\mu b^2 (\mu b^2 + t \sqrt{(1 - \mu^2) + \mu^2 t^2})
\]

where we set \( t = \text{cn}(u, \mu) \). On the other hand, one can verify

\[
\mu - 1 \leq \mu^2 + t \sqrt{(1 - \mu^2) + \mu^2 t^2} \leq \mu + 1 \quad \text{if} \quad 0 \leq \mu \leq 1, -1 \leq t \leq 1.
\]

Therefore,

\[
1 + \mu^2 b^2 - 2\mu^2 b^2 + 2\mu b^2 (\mu - 1) \leq |p(x)|^2 \leq 1 + \mu^2 b^2 - 2\mu^2 b^2 + 2\mu b^2 (\mu + 1),
\]

Hence

\[
(1 - \mu b^2)^2 \leq |p(x)|^2 \leq (1 + \mu b^2)^2,
\]

i.e.,

\[
|1 - \mu b^2| \leq |p(x)| \leq 1 + \mu b^2.
\]

Thus, \( x \) is inside a circular cylinder of radius \( 1 + \mu b^2 \) and outside a circular cylinder of radius \( |1 - \mu b^2| \).

For a \( \text{cmc-H} \) helicoid with a general value \( H \neq 0 \), the inequality (6.5) is

\[
|1 - \mu b^2|/|2H| \leq |p(x)| \leq (1 + \mu b^2)/|2H|.
\]

Note that the equalities in (6.6) can happen. We call \(|1 - \mu b^2|/|2H|, 1 + \mu b^2/|2H|\) the inner radius, outer radius of \( x \), respectively. The outer radius is simply called the radius as long as there is no confusion.

In [7], the inequality (6.6) was already pointed out, however it was not fully discussed there whether \( x \) can have zero inner radius (see Remark 4.15 in [7]). The inequality (6.6) tells us that a \( \text{cmc-H} \) helicoid \( x \) has inner radius 0 if and only if \( \mu b^2 = 1 \). In other words, a \( \text{cmc-H} \) helicoid \( x \) is bounded inward if and only if \( \mu b^2 = 1 \). Thus, a \( \text{cmc-H} \) helicoid \( x \) is bounded both inward and outward, unless \( \mu b^2 = 1 \). We also note that the case where \( \mu b^2 = 1 \) does actually occur. In fact, we can assert the following:

**Proposition 6.4.** Except for the circular cylinders, each associated family includes a \( \text{cmc-H} \) helicoid with zero inner radius.

**Proof.** Let \( x \) be a \( \text{cmc-H} \) helicoid determined by \( (\mu, b) \in \mathcal{M} \), and assume \( x \) is not a circular cylinder, i.e., \( \mu \neq 0 \). If we fix \( \mu \), then there exists a unique \( b \) such that \( (\mu, b) \in \mathcal{M} \) and \( \mu b^2 = 1 \).

If a \( \text{cmc-H} \) helicoid \( x \) has zero inner radius, then there exists a point on \( x(M) \cap z \)-axis. The trajectory of such a point by a helicoidal motion forms a line coincident with \( z \)-axis. Thus we can assert as follows:

**Corollary 6.5.** Each associated family includes a \( \text{cmc-H} \) helicoid on which a straight line lies.

In Figures 5 and 6, \( \text{cmc-H} \) helicoids on which a line (the axis of helicoidal motions) lies are shown. The \( \text{cmc-H} \) helicoid in Figure 6 has a period (periods will be explained in the next subsection).
6.3. Cmc helicoidal cylinders. It can happen that a helicoidal surface $x$ has a period, that is, it reduces to a map from $S^1 \times \mathbb{R}$ to $\mathbb{E}^3$. A helicoidal surface having a period is called a helicoidal cylinder. We show that both cmc-$H$ helicoids and their cgc-companions can have periods in some cases.

In this section, let $H = -1/2$, and let

$$K := K(\mu), \quad E := E(\mu), \quad \Pi := \Pi(\mu^2 b^2, \mu)$$

denote the complete elliptic integrals of the first, second and third kind, respectively.

**Lemma 6.6.** For any fixed $(\mu, b) \in \mathfrak{M}$, the following equations of quasi-periodicity hold:

- $g(u + 2K) = g(u) + \frac{2c_1}{b} \Pi$,
- $n_0(u + 2K) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_0(u)$,
- $\tilde{x}_0(u + 2K) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{x}_0(u) + \left(2(b - \frac{1}{b})K - 2bE\right)$.

A proof of Lemma 6.6 is straightforward by the definition of $g, n_0, \tilde{x}_0$ (cf. Proposition 6.1) and the (quasi-)periodicity of elliptic integrals and Jacobi’s elliptic functions. So the proof is omitted here and is left to the reader.

**Proposition 6.7.** Let $\tilde{x}$ be a cgc-companion of a cmc-$H$ helicoid determined by $(\mu, b) \in \mathfrak{M}$. Then $\tilde{x}$ is periodic if and only if

$$\Phi(\mu, b) := \frac{c_1^2 \Pi + b^2 E + (1 - b^2)K}{\pi bc_1} \in \mathbb{Q}, \quad (6.8)$$

where $c_1 = \sqrt{(1-\mu^2 b^2)/(b^2-1)}$ and $K, E, \Pi$ are complete elliptic integrals (6.7).
Proof. It follows from Proposition 6.1 and Lemma 6.6 that
\[
\tilde{x}(u + 2mK, v + h) = \begin{pmatrix} e^{i(2mc_1\Pi + h)/b + \text{mix}} 0 \\ 0 1 \end{pmatrix} \tilde{x}(u, v) + \begin{pmatrix} 0 \\ c_1 h/b + 2m \{ (b - \frac{1}{b})K - bE \} \end{pmatrix}
\]
for \(m \in \mathbb{Z}\) and \(h \in \mathbb{R}\). Hence, if we were able to choose \(m, h\) so that
\[
\frac{1}{b} (2mc_1\Pi + h) + m\pi \in 2\pi \mathbb{Z}
\]
\[
2m \{ (b - \frac{1}{b})K - bE \} + c_1 h = 0 \quad (6.9)
\]
then
\[
\tilde{x}(u + 2mK, v + h) = \tilde{x}(u, v)
\]
holds.

Assume first that there exist \(m, h\) satisfying (6.9). Then
\[
h = -\frac{2m}{c_1} \{ (b^2 - 1)K - b^2E \} = \frac{2m}{c_1} \{ b^2E + (1 - b^2)K \}. \quad (6.10)
\]
Substituting (6.10) into the first equation of (6.9), we have
\[
m(c_1 \Pi + \frac{\pi}{2}) + \frac{m}{bc_1} \{ b^2E + (1 - b^2)K \} \in \pi \mathbb{Z},
\]
that is,
\[
m \pi \Phi(\mu, b) + \frac{m}{2} \pi \in \pi \mathbb{Z}.
\]
Therefore the condition (6.8) holds.

Conversely we suppose (6.8). Then \(\Phi(\mu, b) + \frac{1}{2} \in \mathbb{Q}\), that is, there exist mutually prime integers \(p, q\) such that
\[
\Phi(\mu, b) + \frac{1}{2} = \frac{q}{p},
\]
namely,
\[
\frac{c_1}{b} \Pi + \frac{\pi}{2} + \frac{1}{bc_1} \{ b^2E + (1 - b^2)K \} = \frac{q}{p} \pi.
\]
Hence,
\[
p(c_1 \Pi + \frac{\pi}{2}) + \frac{p}{bc_1} \{ b^2E + (1 - b^2)K \} \in \pi \mathbb{Z}.
\]
Therefore,
\[
m := p, \quad h := \frac{2p}{c_1} \{ b^2E + (1 - b^2)K \} \quad (6.11)
\]
satisfy (6.9). \(\square\)

Corollary 6.8. There exist infinitely many non-congruent helicoidal cylinders with constant positive Gaussian curvature, i.e., periodic helicoidal wave fronts with constant positive Gaussian curvature.

Proof. The function \(\Phi(\mu, b)\) is non-constant and real-analytic in \((\mu, b)\). Hence, there exist infinitely many \((\mu, b)\) such that \(\Phi(\mu, b) \in \mathbb{Q}\). \(\square\)
From now on, by a **helicoidal cgc-cylinder**, we mean a helicoidal cylinder with positive constant Gaussian curvature.

When (6.8) is satisfied, there exist \( m \in \mathbb{Z} \) and \( h \in \mathbb{R} \) such that

\[
\vec{x}(u + 2mK, v + h) = \vec{x}(u, v)
\]

(6.12)

holds for any \( u, v \). We always choose \( m, h \) so that \( m \) is the minimum positive integer. Then \( m \) equals the number of cuspidal edges of \( \vec{x} \) (cf. Remark 5.12). Thus \( \vec{x} \) is (non-)co-orientable if \( m \) is even (odd). We can verify it precisely as follows: At the same time to (6.12),

\[
n(u + 2mK, v + h) = (-1)^m n(u, v).
\]

This implies that if \( m \) is even, then \( n \) has the same period as \( \vec{x} \), but if \( m \) is odd, then \( n \) reverse its direction. Namely, if \( m \) is chosen to be an odd number then \( \vec{x} \) is non-co-orientable. (A wave front is said to be co-orientable if it posses a global unit normal field. For details, see [15], etc.) Hence we can assert that there are infinitely many non-co-orientable helicoidal cgc-cylinders.

**Example 6.9.** Let \( \mu = 1/2 \). The graph of the function \( b \mapsto \Phi(1/2, b) + 1/2 \) is as in Figure 7.

![Figure 7](image-url)

First, we consider an equation \( \Phi(1/2, b) + 1/2 = 2(= 2/1) \). This equation has two distinct solutions \( b = b_1, b_2 \). They are approximately \( b_1 \approx 1.07213, b_2 \approx 1.99434 \).

For \( b_1 \), the formula (6.11) indicates \( m = 1, h = h_1 \approx 8.7932 \) which satisfy (6.9). For \( b_2 \), the formula (6.11) indicates \( m = 1, h = h_2 \approx 12.6016 \) which satisfy (6.9). Since \( m = 1 \), they are non-co-orientable helicoidal cgc-cylinders having a single cuspidal edge. Indeed, graphics of helicoidal cgc-cylinders of \( (\mu, b) = (1/2, b_1), (1/2, b_2) \) are as in Figures 8, 9.

In the next place, we consider an equation \( \Phi(1/2, b) + 1/2 = 3/2 \). This equation has two distinct solutions \( b = b'_1, b'_2 \). They are approximately \( b'_1 \approx 1.19174, b'_2 \approx 1.97619 \).

For \( b'_1 \), the formula (6.11) indicates \( m = 2, h = h'_1 \approx 10.57012 \) which satisfy (6.9). For \( b'_2 \), the formula (6.11) indicates \( m = 2, h = h'_2 \approx 12.70952 \) which satisfy (6.9). Since \( m = 2 \), they are co-orientable helicoidal cgc-cylinders having two cuspidal edges. Indeed, graphics of helicoidal cgc-cylinders of \( (\mu, b) = (1/2, b'_1), (1/2, b'_2) \) are as in Figures 10, 11.

Finally, we note that the condition (6.8) is also the period condition for cmc-\( H \) helicoids (not only for cgc-companion). Indeed, there exist \( m \in \mathbb{Z} \) and \( h \in \mathbb{R} \) such
that
\[ \mathbf{x}(u + 2mK, v + h) = \mathbf{x}(u, v) + (-1)^m \mathbf{n}(u, v) \]
when (6.8) is satisfied. Therefore,
\[ \mathbf{x}(u + 2mK, v + h) = \mathbf{x}(u, v) + \mathbf{n}(u, v) = \mathbf{x}(u, v) \text{ if } m \text{ is even}, \]
\[ \mathbf{x}(u + 4mK, v + h) = \mathbf{x}(u, v) \text{ if } m \text{ is odd}. \]
Thus \( \mathbf{x} \) is periodic, i.e., \( \mathbf{x} \) is a helicoidal cylinder with constant mean curvature. For any fixed \( \mu \), there are infinitely many values of \( b \) such that (6.8) is satisfied. Thus we have given another proof of the following theorem due to Burstall-Kilian.

**Theorem 6.10** (Theorem 7.6 in [5]). *In each associated family of a Delaunay surface, there are infinitely many non-congruent cylinders with screw-motion symmetry.*
Graphics in Figures 12, 13 are cmc-$H$ helicoidal cylinders determined by $\mu = 1/2$ and $\Phi(1/2, b) + 1/2 = 3/2$, that is, parallel cmc-$H$ surfaces of those in Figures 10, 11.

Figure 12. $(\mu, b) = (1/2, b'_1)$ Figure 13. $(\mu, b) = (1/2, b'_2)$

6.4. Geometric interpretation. Without loss of generality, we fix $H = -1/2$ in this subsection.

We have seen that a cmc-$H$ helicoid (or equivalently a cgc-companion) is determined by two parameters $\mu, b$. We can restate this so that a cmc-$H$ helicoid (or equivalently a cgc-companion) is determined by the pitch and radius. In fact, for a cmc-$H$ helicoid $x$ determined by $(\mu, b) \in \mathcal{M}$, the pitch $\lambda$ and the outer radius $R$ are given by

$$\lambda = c_1 = \sqrt{(1 - \mu^2b^2)(b^2 - 1)}, \quad R = 1 + \mu b^2. \quad (6.13)$$

(6.13) gives a bijective correspondence between $\mathcal{M}$ and

$$\mathcal{L} = \{ (\lambda, R) \mid 0 \leq \lambda < \infty, \ 1 \leq R < \infty \},$$

whose inverse is

$$\mu = \frac{\sqrt{\lambda^2 + R^2} - \sqrt{\lambda^2 + (R - 2)^2}}{\sqrt{\lambda^2 + R^2} + \sqrt{\lambda^2 + (R - 2)^2}}, \quad b = \frac{1}{2} \left\{ \sqrt{\lambda^2 + R^2} + \sqrt{\lambda^2 + (R - 2)^2} \right\}. \quad (6.14)$$

Moreover we have the following assertion.

**Theorem 6.11.** It is determined by the pitch and radius whether a cmc-$H$ helicoid has a period or not. More precisely, a cmc-$H$ helicoid of the pitch $\lambda$ and the radius $R$ has a period if and only if the value of $\Phi = \Phi(\mu, b)$ substituted with (6.14) is rational.
Let $\rho$ denote the inner radius of $x$. Then $\rho = |1 - \mu b^2| = |2 - R|$. Thus the first equation of (6.14) is written as

$$\mu = \frac{\sqrt{\lambda^2 + R^2} - \sqrt{\lambda^2 + \rho^2}}{\sqrt{\lambda^2 + R^2} + \sqrt{\lambda^2 + \rho^2}}$$

(6.15)

An application of (6.15) is the following.

**Theorem 6.12.** Two cmc-$H$ helicoids belong to the same associated family if and only if the values of

$$\frac{\lambda^2 + \rho^2}{\lambda^2 + R^2} = \frac{\lambda^2 + (R - 2)^2}{\lambda^2 + R^2}$$

are coincident.

**Proof.** Two cmc-$H$ helicoids belong to the same associated family if and only if the value of $\mu \in [0, 1]$ are coincident. On the other hand, (6.15) can be rewritten as

$$\frac{\sqrt{\lambda^2 + \rho^2}}{\sqrt{\lambda^2 + R^2}} = \frac{1 - \mu}{1 + \mu}.$$

□

**Corollary 6.13.** An unduloid and a nodoid are associated (i.e., locally isometric) if and only if the ratios

$$\frac{\rho}{R}$$

of the inner radius $\rho$ and the outer radius $R$ are coincident.

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