We consider massive heterogeneous datasets with intrinsic network structure, i.e., big data over networks. These datasets can be modelled by graph signals, which are defined over large-scale irregular graphs representing complex networks. We show that (semi-supervised) learning of the entire underlying graph signal based on incomplete information provided by few initial labels can be reduced to a compressed sensing recovery problem within the cosparse analysis model. This reduction provides two things: First, it allows to apply highly developed compressed sensing methods to the learning problem. In particular, by implementing a recent primal-dual method for convex optimization, we obtain a sparse label propagation algorithm. Moreover, by casting the learning problem within compressed sensing, we are able to derive sufficient conditions on the graph structure and available label information, such that sparse label propagation is accurate.

Index Terms— compressed sensing, cosparse analysis model, big data, semi-supervised learning, complex networks, total variation, nullspace property

I. INTRODUCTION

We live in the era of data deluge, where modern technological systems generate huge amounts of heterogeneous datasets, i.e., big data [8], [22]. These datasets are partially labeled or unlabelled and contain mixtures of different content forms (audio, video, text). Moreover, in many applications, e.g., social networks, proteomics, communication networks, or sensor networks, the observed datasets carry an intrinsic network structure. In order to perform (semi-)supervised learning (regression or classification) it is convenient to model such big data over networks by signals which are defined over a graph representing the underlying network structure. The central smoothness hypothesis of supervised learning [2], [5], requires graph signal values of close-by nodes to be similar. Several approaches have been proposed for exploiting this smoothness hypothesis using various measures of signal smoothness. One line of work is based on the graph Laplacian quadratic form as smoothness measure. This measure is appealing as it results in linear learning problems which can be solved efficiently, e.g., by variants of Gauss-Seidel or multi-grid methods for solving large linear systems of equations [10]. A second line of work considers the total variation semi-norm for measuring signal smoothness [11], [13], [14]. The resulting recovery problems are non-linear but still amount to solving a convex optimization problems, for which efficient algorithms are available [19].

Prior Art. While there have been proposed several efficient learning methods for smooth graph signals, the characterisation of the resulting learning accuracy is still in its infancy. The few existing approaches are mainly based on the spectral properties of the graph Laplacian matrix which is infeasible for irregular large-scale graphs representing complex networks. The closest to our work is [20], [21], which provides sufficient conditions such that a variant of the LASSO method accurately learns a clustered graph signal. By contrast to [20], [21], we assume graph signal values available only on a small subset of nodes.

Contribution. We cast the problem of learning a smooth graph signal within the cosparse analysis model of compressed sensing. This approach naturally suggests to use efficient convex optimization methods for analysis $\ell_1$ minimization as learning algorithms for massive network structured datasets. Moreover, by applying the analysis null space property to smooth graph signals defined over complex networks, we obtain sufficient conditions on the network topology and sampling strategy such that accurate learning is possible.

Notation. Vectors and matrices are denoted by boldface lower-case and upper-case letters, respectively. The vector with all entries equal to one (zero) is denoted $\mathbf{1}$ ($\mathbf{0}$). For a matrix $\mathbf{A}$, we denote its transpose and nullspace by $\mathbf{A}^T$ and $\mathcal{K}(\mathbf{A}) := \{ \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0} \}$. The $\ell_1$, $\ell_2$ and $\ell_\infty$ norm of a vector $\mathbf{u} = (u_1, \ldots, u_N)^T$ is denoted $\|\mathbf{u}\|_1 = \sum_i |u_i|$, $\|\mathbf{u}\|_2 = \sqrt{\sum_i |u_i|^2}$ and $\|\mathbf{u}\|_\infty = \max_i |u_i|$, respectively. For a vector $\mathbf{x}$ and index set $\mathcal{B}$, we denote by $\mathbf{x}_\mathcal{B}$ the particular vector obtained by zeroing all entries $\mathbf{x}$ whose indices are not in $\mathcal{B}$. We denote the support of a vector $\mathbf{x}$ by $\text{supp}(\mathbf{x}) := \{i : x_i \neq 0\}$. The spectral norm of a matrix $\mathbf{D}$ is denoted $\|\mathbf{D}\|_2 := \max_{\|\mathbf{x}\|_2=1} \|\mathbf{D}\mathbf{x}\|_2$.

A graph is denoted $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nodes $\mathcal{V} = \{1, \ldots, N\}$ which are connected by undirected edges $\{i, j\} \in \mathcal{E}$.

Given a subset $\mathcal{C} \subseteq \mathcal{V}$ of nodes, its cutset $\partial \mathcal{C} \subseteq \mathcal{E}$ is made up of all edges $\{i, j\} \in \mathcal{E}$ with $i \in \mathcal{C}$ and $j \in \mathcal{V} \setminus \mathcal{C}$. The nodes of $\mathcal{C}$ which are incident to the edges in the cutset $\partial \mathcal{C}$
II. PROBLEM FORMULATION

We consider (semi-)supervised machine learning from a dataset, composed of \( N \) datapoints, which is represented by a graph \( G = (\mathcal{V}, \mathcal{E}) \) (cf. Fig. 1). A particular node \( i \in \mathcal{V} \) represents a single data point, which might be, e.g., an image, an audio recording or a bag-of-word histogram of a document. The nodes are connected by undirected edges \( \{i, j\} \in \mathcal{E} \), which encode some notion of similarity between the data points. In what follows, we only consider simple graphs, i.e., without self loops.

We define the degree of a node \( i \in \mathcal{V} \) as the number of its neighbors \( \mathcal{N}(i) := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\} \), i.e.,
\[
d_i := |\mathcal{N}(i)|.
\] (1)

The maximum node degree is denoted \( d_{\text{max}} := \max_{i \in \mathcal{V}} d_i \). For a graph \( G \), we define its Laplacian matrix \( L \) element-wise via
\[
L_{i,j} = \begin{cases} 
-1 & \text{if } \{i, j\} \in \mathcal{E} \\
 d_i & \text{if } i = j \\
 0 & \text{else.} 
\end{cases}
\] (2)

The spectral norm of the graph Laplacian is related to the maximum node degree \( d_{\text{max}} \) via [7, Lemma 1.7],
\[
\|L\|_2 \leq 2d_{\text{max}}.
\] (3)

II-A. Graph and Edge Signals

To each data point represented by some node \( i \in \mathcal{V} \), we assign a label \( x[i] \), which will be assumed real-valued for simplicity. Using a graph representation of datasets, we can interpret these labels as the signal values of a graph signal. More formally, a graph signal \( x \) over a graph \( G \) is a mapping \( \mathcal{V} \to \mathbb{R} \), which assigns (labels) every node \( i \in \mathcal{V} \) with the signal value \( x[i] \in \mathbb{R} \). The space of all graph signals, which is also known as the vertex space (cf. [9]), is denoted by \( \mathbb{R}^{\mathcal{V}} \). For a subset \( \mathcal{C} \subseteq \mathcal{V} \) of nodes, we define the indicator graph signal \( t_{\mathcal{C}} \in \mathbb{R}^{\mathcal{V}} \) node-wise by
\[
t_{\mathcal{C}}[i] = \begin{cases} 
1 & \text{if } i \in \mathcal{C} \\
0 & \text{else.} 
\end{cases}
\] (4)

In what follows, it will be useful to associate with the graph \( G \) an arbitrary but fixed orientation, i.e., we declare for each edge \( e = \{i, j\} \) one node as the head \( e^+ \) and the other node as the tail \( e^- \). For a particular orientation, we can associate a graph \( G \) with its incidence matrix \( \mathbf{D} \in \{-1, 0, 1\}^{\mathcal{E} \times |\mathcal{V}|} \). This matrix is defined element-wise as [20]
\[
D_{e,i} = \begin{cases} 
1 & \text{if } i = e^+ \text{ for some } j \in \mathcal{V} \\
-1 & \text{if } i = e^- \text{ for some } j \in \mathcal{V} \\
0 & \text{else.}
\end{cases}
\] (5)

As can be verified easily, the null space \( \mathcal{K}(\mathbf{D}) \) of the incidence matrix \( \mathbf{D} \) is given by all graph signals being constant over the connected components of \( G \). Moreover, the incidence matrix \( \mathbf{D} \) of a graph is closely related to its Laplacian matrix \( \mathbf{L} \). Indeed, we have [1, Chap. 4]
\[
\mathbf{L} = \mathbf{D} \mathbf{D}^T,
\] (6)
which implies
\[
\|\mathbf{D}\|_2 \leq 2d_{\text{max}}.
\] (7)

We consider the problem of learning a smooth graph signal \( x \) representing a labels of all datapoints. The learning is based solely on few initial labels \( x[i] \) which are provided for the nodes \( i \) belonging to a (small) sampling set
\[
\mathcal{M} := \{i_1, \ldots, i_M\} \subseteq \mathcal{V}.
\] (8)

The number of samples \( M = |\mathcal{M}| \) is typically much smaller than the graph size \( N \).

II-B. Smoothness and Sparsity

Our approach to graph signal recovery is based on the central smoothness assumption behind virtually any supervised machine learning method: data points in high-density regions have similar labels [5]. For graph signals representing datasets, this assumption translates to requiring signal values \( x[i], x[j] \) of nodes \( i, j \in \mathcal{V} \) belonging to a well-connected subset of nodes (cluster) to be similar. In particular, this
smoothness assumption reduces to the model of low-pass signals in digital signal processing where time samples at adjacent time instants are strongly correlated (cf. Fig. 1(a)) and adjacent pixels of natural images tend to have similar properties (cf. Fig. 1(b)).

We will quantify the smoothness of a graph signal \( x \in \mathbb{R}^{|V|} \) via the total variation (TV)

\[
\|x\|_{TV} := \sum_{\{i, j\} \in E} |x[j] - x|i].
\] (9)

Using the incidence matrix \( D \) (cf. (5)), we can represent the TV as

\[
\|x\|_{TV} = \|Dx\|_1.
\] (10)

A natural strategy to learn the entire labelling \( x \) from the observed initial labels \( \{x[i]\}_{i \in M} \) is via the optimization problem

\[
\hat{x} \in \arg \min_{\hat{x} \in \mathbb{R}^{|V|}} \|\hat{x}\|_{TV} \text{ s.t. } \hat{x}_M = x_M.
\] (9)

\[
\begin{align*}
\hat{x} &\in \arg \min_{\hat{x} \in \mathbb{R}^{|V|}} \|D\hat{x}\|_1 \text{ s.t. } \hat{x}_M = x_M. 
\end{align*}
\] (11)

As the notation suggests, there might be multiple solutions for (11). However, any solution \( \hat{x} \) of (11) is characterized by (i) it is consistent with the observed labels and (ii) it has minimum TV among all graph signals which are consistent.

The semi-supervised learning problem (11) is a convex optimization problem (8), since the objective function \( f(\hat{x}) = \|D\hat{x}\|_1 \) is a composition of the linear transform represented by \( D \) with the convex norm \( \|u\|_1 \) (8). We highlight that the problem (11) belongs to a particular subclass of convex problems, i.e., those which are equivalent to a linear program (LP) (3). Indeed, (11) is equivalent to

\[
\hat{x} \in \arg \min_{\hat{x}} 1^\top c \\
\text{s.t. } D\hat{x} \leq c \\
-\hat{x} \leq c \\
x_M = x_M.
\] (12)

The learning problem (11) allows for efficient algorithms. In particular, using the equivalent LP formulation (12), we can apply highly developed LP solvers (8) to (11). Moreover, there exist scalable convex optimization methods to solve directly (11) (cf. (23) and the references therein). In particular, the efficient first order primal-dual method by Pock and Chambolle (4) applied to (11) yields Algorithm 1 which we coin sparse label propagation.

Algorithm 1 Sparse Label Propagation

**Input:** incidence matrix \( D \in \mathbb{R}^{|E| \times |V|} \), maximum node degree \( d_{max} \), sampling set \( M \), initial labels \( \{x[i]\}_{i \in M} \).

**Initialize:** \( k := 0 \), \( z^{(0)} := 0 \), \( x^{(0)} := 0 \), \( \hat{x}^{(0)} := 0 \), \( y^{(0)} := 0 \).

**repeat**

1. \( y^{(k+1)} := T(y^{(k)} + (1/2\sqrt{d_{max}})Dz^{(k)}) \)
2. \( z^{(k+1)} := 2x^{(k+1)} - x^{(k)} \)
3. \( x^{(k+1)} := \begin{cases} x[i] & \text{for } i \in M \\ r[i] & \text{else.} \end{cases} \)
4. \( k := k + 1 \)
5. **until** stopping criterion is satisfied

**Output:** \( \hat{x}^{(k)} := (1/k)\hat{x}^{(k)} \)

with some absolute constant \( c_1 \). This convergence rate, i.e., the optimality gap \( \|\hat{x}^{(k)}\|_{TV} - \|\hat{x}\|_{TV} \) being inversely proportional to the iteration number \( k \), is essentially optimal among all first order (sub-)gradient methods for solving non-smooth optimization problems of the type (11) (cf. (12)).

### III. WHEN IS LEARNING ACCURATE?

We have formulated the problem of recovering a smooth graph signal \( x \in \mathbb{R}^{|V|} \) from the observation of its values on the sampling set \( M \subseteq V \) as a convex optimization problem (11). Applying state-of-the-art optimization methods to (11) results in a variety of efficient algorithms for learning smooth graph signals \( x \), i.e., having low TV \( \|x\|_{TV} \), from the observation of the graph signal values \( x_M \) on a small sampling set \( M \subseteq V \).

However, for these algorithms to be accurate we have to ensure that the solutions \( \hat{x} \) of (11) are close to the true underlying graph signal \( \hat{x} \in \mathbb{R}^{|V|} \). We now present conditions which guarantee that any solution \( \hat{x} \) is close to \( x \). In this regard, we will make the following definition.

**Definition 1.** Consider a partition \( \mathcal{F} = \{C_1, C_2, \ldots, C_{|E|}\} \) of pairwise disjoint subsets of nodes (cluster) \( C_l \subseteq V \) and a set of sampled nodes \( M \subseteq \mathcal{V} \). We call \( \mathcal{F} \) to be resolved by \( M \) if there is no edge set \( \mathcal{D} \subseteq \mathcal{E}(C_l) \) of cardinality \( |\mathcal{D}| < 2|\partial C_l| \), which separates the shore \( C_l^\uparrow \) from \( M \cap C_l \).

In Fig. 2 we have depicted a single cluster \( C \subseteq \mathcal{F} \) out of a partition \( \mathcal{F} \) which is resolved by the sampling set \( M \subseteq \mathcal{V} \).
Our first main result applies to graph signals $x$ which, in addition to having a small $\|x\|_{TV}$, are clustered, i.e.,

$$
x = \sum_{C \in F} a_C t_C \quad \text{(cf. (4))} \quad (14)
$$

with a partition $F = \{C_1, \ldots, C_{|F|}\}$ of disjoint clusters $C_i \subseteq V$. The signal model (14), which also has been used in [20], [21], is closely related to the stochastic block model (SBM) [17]. Indeed, the SBM is obtained from (14) by choosing the coefficients $a_C$ uniquely for each cluster, i.e., $a_C \in \{1, \ldots, |F|\}$. Moreover, the SBM provides a generative (stochastic) model for the edges within and between the clusters $C_i$.

**Theorem 2.** For a partition $F$, consider a graph signal $x \in \mathbb{R}^{|V|}$ of the form (14) whose values are observed only at nodes in the sampling set $M$. If the sampling set $M$ resolves the partition $F$, the solution of (11) is unique and coincides with $x$.

Thus, for a graph signal $x \in \mathbb{R}^{|V|}$ of the form (11), with a partition $F$ of clusters which is resolved by the sampling set $M$, any learning algorithm based on solving (11) can be expected to be accurate.

The scope of Theorem 2 is somewhat limited as it applies only to graph signals of the form (14). We now state a more general result that applies to any graph signal $x \in \mathbb{R}^{|V|}$.

**Theorem 3.** Consider a partition $F$ and an arbitrary graph signal $x \in \mathbb{R}^{|V|}$ which is observed only at nodes in the sampling set $M$. If the $M$ resolves $F$, then any solution $\hat{x}$ of (11) satisfies

$$
\|D(\hat{x} - x)\|_1 \leq 6 \min_a \|x - \sum_{C \in F} a_C t_C\|_1. \quad (15)
$$

Thus, as long as the underlying graph signal $x$ can be well approximated by a clustered signal of the form $\sum_{C \in F} a_C t_C$, with suitable coefficients $a_C$, the solution to the learning problem (11) is an accurate proxy for $x$.

**IV. PROOFS**

Let us begin with a high-level outline of the proofs for Theorem 2 and Theorem 3. The optimization problem (11) is exactly the “analysis $\ell_1$-minimization” used for recovery within the cosparse analysis model of compressed sensing [18]. A necessary and sufficient condition for analysis $\ell_1$-minimization (11) delivering the correct solution, i.e., the true underlying graph signal $x$, is the analysis nullspace property [16], [18]. Our approach is then to reformulate this analysis nullspace property in terms of connectivity properties of the graph $G$.

**IV-A. Network Nullspace Property**

As an intermediate step towards proving our main results Theorem 2 and 3 we now reformulate the stable analysis nullspace property [16], [18] in graph signal terminology.

**Definition 4.** Consider a graph $G$ containing the sampling set $\mathcal{M} \subseteq V$. Let us define the kernel of the sampling set $\mathcal{M}$ as

$$
\mathcal{K}(\mathcal{M}) := \{\hat{x} \in \mathbb{R}^{|V|} : \hat{x}_{\mathcal{M}} = 0\}. \quad (16)
$$

We say that the sampling set $\mathcal{M}$ satisfies the network null space property (NNSP-$\mathcal{S}$) w.r.t. an edge set $\mathcal{S} \subseteq \mathcal{E}$ if

$$
\|D u|_{\mathcal{E}\setminus\mathcal{S}}\|_1 \geq 2\|D u|_{\mathcal{S}}\|_1 \text{ for any } u \in \mathcal{K}(\mathcal{M})\setminus\{0\}. \quad (17)
$$

Let us now consider a graph signal $x$, defined over $G$, with associated edge signal $\tilde{x} = Dx$ with support $S_x = \text{supp}(\tilde{x})$. If the graph $G$ and the sampling set $\mathcal{M}$ is such that the NNSP-$S_x$ is satisfied, then the solution of (11) is unique and given by $x$.

**Lemma 5.** Consider a graph signal $x \in \mathbb{R}^{|V|}$ with edge support $S_x = \text{supp}(Dx)$ which is observed only at the nodes in the sampling set $\mathcal{M} \subseteq V$. If NNSP-$S_x$ holds, the solution of (11) is unique and coincides with $x$.

**Proof:** Let us consider a graph signal $x \in \mathbb{R}^{|V|}$ defined over $G$ and observed over sampling set $\mathcal{M}$ such that condition (17) is satisfied. Assume there exists another graph signal $\tilde{x} (\neq x)$ being feasible for (11), i.e., $\tilde{x}_{\mathcal{M}} = x_{\mathcal{M}}$, such that the difference vector $u := \tilde{x} - x$ belongs to the kernel $\mathcal{K}(\mathcal{M})$ (cf. (16)). Note that, since $(Dx)_{S_x} = Dx$,

$$
(D\tilde{x})_{S_x} = (Dx)_{S_x}. \quad (18)
$$

Moreover, by the triangle inequality we have

$$
\|(D\tilde{x})_{S_x}\|_1 \geq \|(Dx)_{S_x}\|_1 - \|(Du)_{S_x}\|_1 = \|Dx\|_1 - \|(Du)_{S_x}\|_1. \quad (19)
$$
However, since \( \|D\hat{x}\|_1 = \|(D\hat{x})_{S}\|_1 + \|(D\hat{x})_{E\setminus S}\|_1 \),
\[
\|D\hat{x}\|_1 = \|(D\hat{x})_{S}\|_1 + \|(D\hat{x})_{E\setminus S}\|_1, \tag{13}
\]
\[
\overset{(13)}{=} \|(D\hat{x})_{S}\|_1 + \|(Du)_{E\setminus S}\|_1, \tag{19}
\]
\[
\overset{(19)}{=} \|Dx\|_1 - \|(Du)_{S}\|_1 + \|(Du)_{E\setminus S}\|_1, \tag{17}
\]
\[
\overset{(17)}{=} \|Dx\|_1. \tag{20}
\]

We will also need another more practical result which applies to graph signals \( x \in \mathbb{R}^{|V|} \) whose associated edge signal \( Dx \in \mathbb{R}^{|E|} \) is not strictly sparse but which is well concentrated on a small subset \( S' \subseteq E \) of edges.

**Lemma 6.** Consider a graph signal \( x \in \mathbb{R}^{|V|} \) which is observed at the nodes in the sampling set \( M \subseteq V \). If the condition NNSP-\( S' \) is valid for the edge set \( S' \subseteq E \), then any solution \( \hat{x} \) of \((11)\) satisfies
\[
\|D(x - \hat{x})\|_1 \leq 6\|(Dx)_{E\setminus S'}\|_1. \tag{21}
\]

**Proof:** The argument closely follows that in the proof of [15, Theorem 8]. First, observe that for any solution \( \hat{x} \) of \((11)\), we have
\[
\|D\hat{x}\|_1 \leq \|Dx\|_1, \tag{22}
\]
since \( x \) is trivially feasible for \((11)\). From \((22)\), we obtain further
\[
\|(Dx)_{S'}\|_1 + \|(D\hat{x})_{E\setminus S'}\|_1 \leq \|(D\hat{x})_{S'}\|_1 + \|(Dx)_{E\setminus S'}\|_1. \tag{23}
\]
Since \( \hat{x} \) is feasible for \((11)\), i.e., \( \hat{x}_M = x_M \), the difference signal \( v := \hat{x} - x \) belongs to the kernel \( K(M) \) (cf. [16]). Applying triangle inequality to \((23)\), we get
\[
\|((Dv)_{E\setminus S'})_1 + \|Dv\|_{E\setminus S'}_1 \leq \|Dv\|_{E\setminus S'}_1 + \|Dv\|_{E\setminus S'}_1 \leq 2\|(Dv)_{E\setminus S'}\|_1. \tag{24}
\]
Combining this inequality with NNSP-\( S' \) (cf. [17]), we get
\[
\|Dv\|_{E\setminus S'}_1 \leq 2\|(Dv)_{E\setminus S'}\|_1. \tag{25}
\]
Using NNSP-\( S' \) again,
\[
\|D(x - \hat{x})\|_1 = \|Dv\|_1 = \|(Dv)_{S}\|_1 + \|(Dv)_{E\setminus S}\|_1 \overset{(24)}{=} (3/2)\|(Dv)_{E\setminus S}\|_1 \overset{(23)}{=} 6\|(Dx)_{E\setminus S'}\|_1. \tag{28}
\]

Let us now render Lemma 5 and Lemma 6 for graph signals \( x \) of the form \((14)\). In particular, we will give now a sufficient condition on the graph topology via the clusters \( C_i \in F \) and the sampling set \( M \) such that NNSP-\( M \) (cf. Definition 4) is satisfied. To this end, let us define the cutset \( \partial C \) of a given subset of nodes \( C \subseteq V \) as
\[
\partial C := \{ e = (i, j) \in E : i \in C, \text{ and } j \in V \setminus C \}. \tag{29}
\]

**Lemma 7.** Consider a partition \( F = \{ C_1, \ldots, C_m \} \) of node subsets (clusters) \( C_i \). For \( C_i \in F \), let any edge \( e \) in the cutset \( \partial C_i \) be connected by two edge-disjoint paths \( P_{1}^{e} \) and \( P_{2}^{e} \) within \( C_i \) to sampled nodes \( i \in \mathcal{M} \cap C_i \). If all the paths \( \{ P_{1}^{e}, P_{2}^{e} \}_{e \in \partial C_i} \) can be chosen to be edge-disjoint, the condition NNSP-\( S \) is satisfied for \( S = \bigcup_{C_i} \partial C_i \).

**Proof:** Consider a graph signal \( x \in \mathbb{R}^{|V|} \) of the form \((14)\). The support \( S = supp(Dx) \) then satisfies
\[
S \subseteq \bigcup_{C_i \in F} \partial C_i. \tag{30}
\]
This graph signal induces an edge signal \( v = Du \). We now verify that \( v_{E \setminus S} \geq 2v_{S} \). According to \((30)\), any non-zero entry of \( v_{E \setminus S} \) corresponds to a particular edge \( e \) which is in the cutsets \( \partial C_i \), \( \partial C_j \) of two neighboring clusters \( C_i, C_j \in F \), i.e., we have \( e \in \partial C_i \cap \partial C_j \). Then, we can at least find two edge-disjoint paths \( P_{1}^{e}, P_{2}^{e} \) containing \( e \) and which connects two sampled nodes \( i_1 \in \mathcal{M} \cap C_i \) and \( i_2 \in \mathcal{M} \cap C_j \). Thus,
\[
0 \leq u_{[i_1]} - u_{[i_2]} = \sum_{(i,j) \in P_{1}^{e}} v_{[i,j]} - \sum_{(i,j) \in P_{2}^{e}} v_{[i,j]}, \tag{31}
\]
which implies, in turn,
\[
|v_{[e]}| \leq \sum_{(i,j) \in P_{1}^{e}} |v_{[i,j]}| + \sum_{(i,j) \in P_{2}^{e}} |v_{[i,j]}|. \tag{32}
\]
Thus, we have
\[
\|v_{E \setminus S}\|_1 \overset{(a)}{=} \sum_{e \in \bigcup_{i} \partial C_i} |v_{[e]}| \overset{(b)}{=} \sum_{e \in \bigcup_{i} \partial C_i} (|v_{[i,j]}| + \sum_{(i,j) \in P_{2}^{e}} |v_{[i,j]}|) \overset{(c)}{=} 2 \sum_{e \in \bigcup_{i} \partial C_i} |v_{[e]}| \tag{34}
\]
where step (a) holds since the paths \( \{ P_{1}^{e}, P_{2}^{e} \}_{e \in \partial C_i} \) are edge-disjoint. ■
In order to apply Lemma 7, we need to verify that for each cluster \( C_l \in \mathcal{F} \), there exist edge-disjoint paths such that any edge in the cutset \( \partial C_l \) is connected by at least two different paths \( P_1, P_2 \), to some node in the sampling set \( M \). To this end, we will tailor a fundamental result of network flow theory, to our setting in the following lemma.

**Lemma 8.** Consider a subset of nodes (cluster) \( C \subseteq V \) which contains the sampled nodes \( M \cap C \). If there is no edge set \( D \subseteq E(C) \) of cardinality \( |D| < 2|\partial C| \) which separates \( C \) from \( M \cap C \), there exist edge-disjoint paths \( \{P_{1/2}\} \) such that each edge \( e \in \partial C \) is connected by two paths \( P_1, P_2 \) to some sampled node \( i \in M \cap C \).

**Proof:** see [9, Cor. 3.3.4].

**IV-B. Proof of Theorem 2**

Combine Definition 1 with Lemma 7, Lemma 8 and Lemma 5.

**IV-C. Proof of Theorem 3**

Combine Definition 1 with Lemma 7, Lemma 8 and Lemma 6 (for the choice \( S' = \bigcup_{C_l \in \mathcal{F}} \partial C_l \)).

**V. CONCLUSIONS**

We have considered the problem of learning sparse graph signals defined over complex networks from observing the signal at a small subset of sampled nodes only. By casting this (semi-supervised) learning problem within the cosparse analysis model of compressed sensing, we derived a sufficient condition on the graph topology and sampling set such that a convex learning method is accurate. This condition is based on the connectivity properties of the underlying network. Roughly speaking it requires to sample more densely in clusters with a large cut-size.

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