EXISTENCE OF MULTI-PEAK SOLUTIONS TO THE
SCHNACKENBERG MODEL WITH HETEROGENEITY ON
METRIC GRAPHS

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ABSTRACT. In this paper, we study the existence of spiky stationary solutions of the Schnakenberg model with heterogeneity on compact metric graphs. These solutions are constructed by using the Liapunov–Schmidt reduction method and taking the same strategy as that in [14, 11]. First, we give the abstract theorem on the existence of multi-peak solutions for general compact metric graphs under several assumptions for the associated Green’s function. In particular, we reveal that how locations of concentration points and amplitudes of spiky solutions are determined by the interaction of the heterogeneity with the geometry of the compact metric graph, represented by Green’s function. Second, we apply our abstract theorem to the \(Y\)-shaped metric graph and the \(H\)-shaped metric graph in non-heterogeneity case. In particular, we show the precise effect of the geometry of those compact graphs to the locations of concentration points for these concrete graphs, respectively.

1. Introduction and main results. In 1979, Schnakenberg proposed the model which describes an autocatalytic chemical reaction [20]. This model is called the Schnakenberg model and is well-known as a model in pattern formation. We first explain the following one-dimensional Schnakenberg model with heterogeneity:

\[
\begin{align*}
  u_t - \varepsilon^2 u_{xx} &= -u + g(x)u^2v, \\
  \varepsilon v_t - Dv_{xx} &= \frac{1}{L} - \frac{c}{\varepsilon}g(x)u^2v,
\end{align*}
\]

in \((-1,1) \times (0,\infty)\) with Neumann boundary condition. \(u(x,t)\) and \(v(x,t)\) represent the density of two chemical substances at \((x,t) \in (-1,1) \times (0,\infty)\). Here, \(c\) is a positive constant, \(L := |(-1,1)| = 2\) is the length of the interval \((-1,1)\), and \(\varepsilon^2 > 0\) and \(D > 0\) are diffusion coefficients of \(u\) and \(v\), respectively. Moreover, \(g(x)\) is a positive function on the interval \((-1,1)\). In particular, \(g(x)\) represents the reaction rate depending on \(x \in (-1,1)\), for example by the effect of temperature. Although the standard Schnakenberg model [20] is the case \(g(x) = 1\), the model with heterogeneity is also important to understand more realistic phenomenon.

The Schnakenberg model have been actively studied in many papers (see e.g. [24] and the reference therein.) Recently, there are the study of the existence and
stability of spike patterns for the Schnakenberg model on a one-dimensional interval with heterogeneity [14, 10, 11, 1]. In these works, the effect of heterogeneity on the existence of spiky solutions and their stability were investigated. The standard Schnakenberg model without heterogeneity has been studied in [9, 22].

Next, to explain our setting in details, we describe a metric graph \( \mathcal{G} = (V, E) \) with a set of vertices \( V \) and a set of edges \( E \). In this paper, we treat a metric graph satisfying the following conditions:

- (H1) \( \mathcal{G} \) is connected and compact.
- (H2) \( \mathcal{G} \) has a finite number of edges.

Each edge \( e \in E \) has two vertices \( v_1, v_2 \in V \) and we fix the orientation \( \overrightarrow{v_1v_2} \) for the edge \( e \). We denote by \( 0 < l_e < +\infty \) the length of \( e \) and by \( L := \sum_{e \in E} l_e \) the total length of the graph \( \mathcal{G} \). We also denote by \( e^o := e \setminus \partial e \). To treat a function on the graph, we furthermore identify the edge \( e \) with the interval \( I^e := [0, l_e] \). For the edge \( e \in E \), we use a local coordinate \( x_e \in I^e \) in such a way that \( v_1 \) corresponding to \( x_0 = 0 \) and \( v_2 \) corresponding to \( x_e = l_e \) according to the orientation \( \overrightarrow{v_1v_2} \).

There are many works on the study of solutions to several partial differential equations on compact or non-compact metric graphs, which appear in biological networks and quantum physics (see e.g. [3, 2, 19, 18]). Moreover, partial differential equations on metric graphs also appears as limiting problems for the model on higher dimensional thin domains (see e.g. [17, 7]). Especially, concerning with scalar reaction-diffusion equations, we mention the work of Yanagida [21], which studied the stability of stationary solution of scalar reaction-diffusion equation on a compact metric graph and the work of Jinbo and Morita [15], which studied the asymptotic behavior of entire solutions of reaction-diffusion equation on non-compact metric graphs. We also mention the work of [5], which studied chemotaxis system on a compact metric graph, and the work of Du et al. [6], Shi et al. [16], which studied the dynamics of solutions of reaction-diffusion-advection equation.

In this paper, motivated by these works, we study the effect of the geometry of the compact metric graph \( \mathcal{G} \) and the heterogeneity on the existence of one-peak and two-peak solutions of the steady-state problem for the following Schnakenberg model on a compact metric graph \( \mathcal{G} = (V, E) \):

\[
\begin{align*}
-\varepsilon^2 u'' &= -u + g(x)u^2v, \quad -Dv'' = \frac{1}{L} - \frac{c}{\varepsilon}g(x)u^2v, \quad x \in \mathcal{G}, \\
\sum_{e \ni v} \frac{du_e}{dx_e}(v) &= \sum_{e \ni v} \frac{dv_e}{dx_e}(v) = 0, \quad v \in V,
\end{align*}
\]  

(1.2)

where \( u \) and \( v \) are continuous functions on the graph \( \mathcal{G} \) and are \( C^2 \) functions on each edge \( e \in E \). Moreover, assume that \( g(x) \) is continuous function on \( \mathcal{G} \) and \( C^3 \) function on each \( e \in E \). Here, \( e \ni v \) means that the edge \( e \in E \) is incident at the vertex \( v \), and \( d\eta_e/dx_e(v) \) represents the outer normal to the boundary \( \partial x_e = 0 \) or \( x_e = l_e \) at \( v \), respectively. The last condition is called the Kirchhoff condition. In particular, if a vertex \( v \in V \) is an endpoint, then the Kirchhoff condition becomes the Neumann boundary condition.

In this section, we state our main results on the existence of one-peak or two-peak solutions in an abstract setting and later, we present more concrete results for the \( Y \)-shaped graph and \( H \)-shaped graph, respectively, which reveal the effect of the geometry of metric graphs to the location of concentration points of spiky solutions.
We need several preliminaries to state our main result in details. Let us consider the equation \( w'' - w + w^2 = 0 \) in \( \mathbb{R} \), where \( w > 0 \), \( w(0) = \max_{y \in \mathbb{R}} w \), and \( \lim_{y \to \infty} w(y) = 0 \). Then, we have the unique solution \( w(y) = 3/(2(\cosh(y/2))^2) \) and \( \int_{\mathbb{R}} w^2 dy = 6 \). We introduce a cut-off function \( \chi \) as follows:

\[
\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi(x) = 1 \ (|x| < 1/4), \quad \chi(x) = 0 \ (|x| > 1/2).
\]  

(1.3)

Next, we introduce the following two function spaces frequently used in this paper.

\[
H^p(\mathcal{G}) := \{ u \in C(\mathcal{G}) \mid u_e \in H^p(e) \text{ for each } e \in E \}, \quad p = 1, 2,
\]

\[
H^2_N(\mathcal{G}) := \{ u \in H^2(\mathcal{G}) \mid \sum_{e \in V} \frac{du_e}{dx_e}(v) = 0 \text{ for each } v \in V \}.
\]

Since we will use the rescaling throughout this paper, we introduce the following notations. Stretching the graph \( \mathcal{G} \) with the scale \( \varepsilon^{-1} \), put \( \mathcal{G}_\varepsilon = (V_\varepsilon, E_\varepsilon) := \varepsilon^{-1} \mathcal{G} = (\varepsilon^{-1} V, \varepsilon^{-1} E) \) and \( I^*_\varepsilon := [0, \varepsilon^{-1} l] \) \( (e \in E) \). For a function \( \eta : \mathcal{G} \to \mathbb{R} \), we define \( \eta(y) := \eta(\varepsilon y) \ (y \in \mathcal{G}_\varepsilon) \) and \( \eta(y) := \eta_\varepsilon(\varepsilon y) \ (y \in I^*_\varepsilon) \). Let \( e, e_1, e_2 \in E \), arbitrarily.

First, we define \( r_1(t) := \min\{t, l_e - t\} \) for \( t \in e^o \). Next, let \( t = (t_1, t_2) \in e_1^e \times e_2^e \). In particular, when \( e_1 = e_2 \), we set \( t_1 < t_2 \). Then, we define \( r_2(t) \) as follows:

\[
r_2(t) := \begin{cases} \min\{t_1, t_2, l_{e_1} - t_1, l_{e_2} - t_2\}, & \text{when } e_1 \neq e_2, \\ \min\{t_1, l_{e_1} - t_2, 2^{-1}t_1 - t_2\}, & \text{when } e_1 = e_2. \end{cases}
\]

(1.4)

For simplicity, we often denote \( r_1(t) \) and \( r_2(t) \) by \( r(t) \). Finally, for \( t^0 \in e \) and \( t^0 \in e_1 \times e_2 \), we define \( B(\varepsilon^{3/4}, t^0) := \{ t \in e \mid |t - t^0| \leq \varepsilon^{3/4} \} \) and \( B(\varepsilon^{3/4}, t^0) := \{ t \in e_1 \times e_2 \mid |t - t^0| \leq \varepsilon^{3/4} \} \), respectively.

Now, assume that \( f \in L^2(\mathcal{G}) \) and \( \int_{\mathcal{G}} f(x) dx = 0 \). Let \( \eta(x) \in H^2_N(\mathcal{G}) \) be a solution of the following equation:

\[
D\eta''(x) = f(x), \quad x \in \mathcal{G}.
\]

(1.5)

Then, \( \eta(x) \) can be written as

\[
\eta(x) = \frac{1}{L} \int_{\mathcal{G}} \eta(s) ds = \int_{\mathcal{G}} G(x, s) f(s) ds.
\]

(1.6)

where \( G(x, s) \) is the associated Green’s function and satisfies \( \int_{\mathcal{G}} G(x, s) ds = 0 \). In Appendix A, we will explain this fact in details. In this paper, for Green’s function above, we assume the following five conditions:

(G1) There exists a constant \( C \) such that \( |G(x, s)| \leq C \) for \( x, s \in \mathcal{G} \).

(G2) For any \( e \in E \), there exists a constant \( C \) such that

\[
|G(x, s) - G(x', s)| \leq C|x - x'| \tag{1.7}
\]

for any \( x, x' \in e \) and \( s \in G \).

(G3) For any \( e \in E \), there exists a constant \( C \) such that

\[
|G(x, s) - G(x, s')| \leq C|s - s'| \tag{1.8}
\]

for any \( s, s' \in e \) and \( x \in G \).

Take any edges \( e_i, e_j \in E \) with \( i, j \in \{1, 2\} \). Furthermore, we assume the following condition for the edges \( e_i \) and \( e_j \).

(G4) For \( i, j \in \{1, 2\} \) and \( t_i \in e_i, t_j \in e_j \), let us define

\[
G^i(y, z + t_j) := G(y + t_i, z + t_j) - G(t_i, z + t_j)
\]

(1.9)

for \( y, z \in ( -r(t), r(t)) \). Then, \( G^i(y, z + t_j) \) can be written as

\[
G^i(y, z + t_j) = m_{ij}(t) y + K_{ij}(y, z), \tag{1.10}
\]
where \( m_{ij}(t) \) is some constant and \( K_{ij}(y, z) \) is some function satisfying the following two conditions: (i) \( K_{ij}(y, z) = O(|y|) \). (ii) \( \int_{-r(t)}^{r(t)} K_{ij}(y, z) w(z) dz \) is an even function with respect to \( y \) on the interval \((-r(t), r(t))\).

Finally, we assume that following condition.

\((G5)\) It holds that 
\[ G(t_1, t_2) - G(t_2, t_2) > G(t_1, t_1) - G(t_2, t_1), \quad t_1, t_2 \in \mathcal{G}, \quad t_1 \neq t_2. \]  

Remark 1. To calculate Green’s function above, it suffices to find the representation formula of the solution \( \eta \) with \( \int_\mathcal{G} \eta(x) dx = 0 \) as follows:

\[ \eta(x) = \int_\mathcal{G} G(x, s) f(s) ds. \]

Note that we should also require the condition \( \int_\mathcal{G} G(x, s) ds = 0 \). In Lemma 2.1 and Lemma 3.1, we will calculate Green’s function above for the Y-shaped graph and the H-shaped graph, respectively.

First, for the existence of a one-peak solution, we assume the following condition:

**The assumption for a one-peak solution:** For a given edge \( e \in E \) and \( t \in e \), we define the notation \( F(t) \) by

\[ F(t) := m(t) + 6cg(t) g(t)^{-2}, \]

where \( m(t) = m_{11}(t, t) \), and we assume the following condition:

\((A0)\). There exists a point \( t^0 \in e^o \) for some \( e \in E \) such that

\[ F(t^0) = 0, \quad F'(t^0) \neq 0. \]

Also, to state the existence theorem of a one-peak solution, we define the notation \( \xi(t) \) as follows:

\[ \frac{6c}{g(t)} \xi(t) = 1. \]

We state the existence of a one-peak solution.

**Theorem 1.1.** Assume \((G1)\), \((G2)\), and \((G3)\). Moreover, we assume \((G4)_{e,e} \) and \((A0)_{e} \) for some edge \( e \in E \). Then, for \( \varepsilon > 0 \) sufficiently small, \((1.2)\) has a one-peak solution \((u_{\varepsilon,1}(x), v_{\varepsilon,1}(x)) \in H^2_N(\mathcal{G}) \) \( \times H^2_N(\mathcal{G}) \). Moreover, the solution satisfies that:

(1) \( u_{\varepsilon,1}(x) \) concentrates near some point \( x = t^\varepsilon \in B(\varepsilon^{3/4}, t^0) \subset e \). The asymptotic form of \( u_{\varepsilon,1}(x) \) is given by \( \phi_{\varepsilon, t^\varepsilon}(x) + \phi_{e, t^\varepsilon}(x) \), where

\[ w_{\varepsilon, t^\varepsilon}(x) := \frac{1}{g(t^\varepsilon) \xi(t^\varepsilon)} w\left( \frac{x - t^\varepsilon}{\varepsilon} \right) \left( \frac{x - t^\varepsilon}{r_0[1]} \right), \quad r_0[1] := 10^{-1} r_1(t^0), \]

and \( \phi_{e, t^\varepsilon} \in H^2_N(\mathcal{G}) \) with \( \| \phi_{e, t^\varepsilon} \|_{H^2(\mathcal{G})} \leq \varepsilon \) for some constant \( C_0 > 0 \) independent of \( \varepsilon > 0 \).

(2) \( v_{\varepsilon,1}(x) \) satisfies \( v_{\varepsilon,1}(t^\varepsilon) = \xi(t^\varepsilon) + O(\varepsilon) \) as \( \varepsilon \to 0 \).

Remark 2. Throughout this paper, by using \((1.15)\), we will simply represent the solutions \( u_{\varepsilon,1}(x) \) and \( v_{\varepsilon,1}(t^\varepsilon) \) above, and the concentration point \( t^\varepsilon \) above by \( u_{\varepsilon,1}(x) \sim (6c)^{-1} w(\varepsilon^{-1} (x - t^0)) \), \( u_{\varepsilon,1}(t^\varepsilon) \sim 6cg(t^0)^{-1} \), and \( t^\varepsilon \sim t^0 \), respectively.

Next, for the existence of a two-peak solution, we assume the following condition:

**The assumptions for a two-peak solution:** Take any edges \( e_i, e_j \in E \) with \( i, j \in \{1, 2\} \). Let us define

\[ \sigma_j(t) := G(t_1, t_j) - G(t_2, t_j), \quad t_j \in e_j, \quad j = 1, 2. \]
Moreover, when $e_1 = e_2$, we assume $t_1^0 < t_2^0$.

(A2) The following matrix $A(t^0, \xi^0)$ is a regular matrix:
$$A(t^0, \xi^0) := \begin{pmatrix} 1 + \sigma_1(t^0) a^0_1 & -1 + \sigma_2(t^0) a^0_2 \\ a^0_1 & a^0_2 \end{pmatrix}, \quad a^0_j := \frac{6c}{g(t^0_j)(\xi^0_j)^2}. \quad (1.19)$$

By Lemma 6.2, which will be proven in Appendix C, for each point $t := (t_1, t_2)$ near the point $t^0$, there exists a unique solution $(\xi_1(t), \xi_2(t))$ such that
$$\xi_1(t) - \xi_2(t) = 6c \sum_{k=1}^{2} \frac{\sigma_k(t)}{g(t_k)\xi_k(t)}, \quad \sum_{k=1}^{2} \frac{6c}{g(t_k)\xi_k(t)} = 1, \quad (1.20)$$
and $\xi_k(t^0) = \xi^0_k$. By using $\xi_j(t)$, we denote
$$F_j(t) := 6c \sum_{k=1}^{2} \frac{m_{jk}(t)}{g(t_k)\xi_k(t)} + \frac{g'(t_j)\xi_j(t)}{g(t_j)}, \quad j = 1, 2, \quad (1.21)$$
and
$$A(t) := \begin{pmatrix} \partial_{t_1}F_1(t) & \partial_{t_1}F_2(t) \\ \partial_{t_2}F_1(t) & \partial_{t_2}F_2(t) \end{pmatrix}. \quad (1.22)$$

Then, we assume the following condition:
(A3) $\det A(t^0) \neq 0$.

We state the existence of a two-peak solution.

**Theorem 1.2.** Take any edges $e_i, e_j \in E$ with $i, j \in \{1, 2\}$. For Green's function $G(x, s)$, assume five conditions (G1), (G2), (G3), (G4), (G5), (G6). Moreover, for the amplitudes $\{c_j^0\}_{j=1}^{2}$ and concentration points $\{t_j^0\}_{j=1}^{2}$, assume three conditions (A1), (A2), and (A3). Then, for $\varepsilon > 0$ sufficiently small, (1.2) has a two-peak solution $(u_{\varepsilon, 2}(x), v_{\varepsilon, 2}(x)) \in H^2(\mathcal{G}) \times H^2(\mathcal{G})$. Moreover, the solution satisfies that:
(1) $u_{\varepsilon, 2}(x)$ concentrates near some point $t^\varepsilon := (t_1^\varepsilon, t_2^\varepsilon) \in B(\varepsilon^{3/4}, t^0)$. The asymptotic form of $u_{\varepsilon, 2}(x)$ is given by $u_{\varepsilon, 2}(x) = \sum_{j=1}^{2} w_{\varepsilon, t_j^\varepsilon}(x) + \phi_{\varepsilon, t^\varepsilon}(x)$, where
$$w_{\varepsilon, t_j^\varepsilon}(x) := \frac{1}{g(t_j^\varepsilon)} w\left(\frac{x - t_j^\varepsilon}{\varepsilon}\right) \chi\left(\frac{x - t_j^\varepsilon}{r_0[2]}\right), \quad r_0[2] := 10^{-1}r_2(t^0), \quad (1.23)$$
and $\phi_{\varepsilon, t^\varepsilon} \in H^2(\mathcal{G})$ with $\|\phi_{\varepsilon, t^\varepsilon}\|_{H^2(\mathcal{G})} \leq C_0\varepsilon$ for some constant $C_0 > 0$ independent of $\varepsilon > 0$.
(2) For $j = 1, 2$, it holds that $v_{\varepsilon, 2}(t_j^\varepsilon) = \xi_j(t_j^\varepsilon) + O(\varepsilon)$ as $\varepsilon \to 0$. 

We assume the following three conditions:
(A1) Then, there exist $\xi^0 := (\xi^0_1, \xi^0_2) \in (0, \infty) \times (0, \infty)$ and $t^0 := (t^0_1, t^0_2) \in e_1 \times e_2$ such that
$$\begin{cases}
F_1^0 := 6c \sum_{k=1}^{2} \frac{m_{1k}(t^0)}{g(t^0_k)\xi^0_k} + \frac{g'(t^0_1)\xi^0_1}{g(t^0_1)} = 0, \\
F_2^0 := 6c \sum_{k=1}^{2} \frac{m_{2k}(t^0)}{g(t^0_k)\xi^0_k} + \frac{g'(t^0_2)\xi^0_2}{g(t^0_2)} = 0,
\end{cases} \quad (1.18)$$
Moreover, when $e_1 = e_2$, we assume $t^0_1 < t^0_2$.

(A2) Then, there exist $\xi^0_j \in (0, \infty)$ such that
$$\xi^0_1 - \xi^0_2 = 6c \sum_{k=1}^{2} \frac{\sigma_k(t^0)}{g(t^0_k)\xi^0_k},$$
and
$$\sum_{k=1}^{2} \frac{6c}{g(t^0_k)\xi^0_k} = 1.$$
Remark 3. Throughout this paper, we will simply represent the solutions $u_{\varepsilon,2}(x)$ and $v_{\varepsilon,2}(t_j^\varepsilon)$ above, and the concentration point $t_j^\varepsilon$ above by

$$u_{\varepsilon,2}(x) \sim \sum_{j=1}^{2} \frac{1}{g(t_j^0)\varepsilon^0} \psi\left(\frac{x-t_j^0}{\varepsilon}\right), \quad v_{\varepsilon,2}(t_j^\varepsilon) \sim \xi_j^0,$$

and $t_j^\varepsilon \sim t_j^0$, respectively. We note that, by a formal calculation, the assumption (A1) is necessary in order to $t_j^0$ are the limit of concentration points and $\xi_j^0$ is a limit of the amplitude of $v_\varepsilon(t_j^\varepsilon)$.

Remark 4. For the stability of the solutions constructed in the abstract theorem of this paper, the first author has been established the stability result under the additional assumption for Green’s function in details (see [12].) Also, in [12], the first author has given the precise stability results of spike solutions with $g(x) = 1$ constructed in Section 2 and 3 for the Y-shaped graph and the H-shaped graph, respectively. In Section 2, we will only present the stability results for Y-shaped graph.

Remark 5. Now, we refer to the Gierer-Meinhardt (GM) model [8], which has common solution structures as in the Schnakenberg model. For the existence and stability of multi-peak solutions of the standard GM model on the one-dimensional interval, see e.g. [24]. For the GM model with heterogeneity, which is called precursors, Wei and Winter studied the existence and stability of spiky solutions in [23, 25]. In [13], the first author also initiated the study of the GM model on metric graphs and established the existence of one-peak solutions for the GM model with heterogeneity different from precursors on the Y-shaped metric graph under the assumption for the associated Green’s function.

This paper is organized as follows: In Section 2, we apply our abstract theorem to the Y-shaped graph $G$ with $g(x) = 1$ (see Theorem 2.2 and Theorem 2.3.) Moreover, by using the typical heterogeneity function, we show the effect of heterogeneity on the location of the concentration point of a one-peak solution. In Section 3, we apply our abstract theorem to the H-shaped graph $G$ with $g(x) = 1$ (see Theorem 3.2 and Theorem 3.3.) In Section 4, we prepare basic estimates for the proof of our abstract theorem. In Section 5, we give the proof of our abstract theorem by using the Liapunov–Schmidt reduction method. In Section 6, we give the proof of the representation formula of Green’s functions for the Y-shaped graph and the H-shaped graph, respectively. Moreover, we also give the proof of some technical lemma.

Throughout this paper, we use the notation $C$ to denote a positive constant which is independent of $\varepsilon$ and are different from line to line.

2. Application of the existence theorem to Y-shaped metric graph. In this section, we consider the Y-shaped metric graph $G = \bigcup_{j=1}^{3} e_j$ with one junction at the common vertex, and the edge $e_j$ has two vertices $O$ and $P_j$, respectively. We denote by $l_j := |e_j|$ the length of $e_j$ and by $L := \sum_{j=1}^{3} l_j$ the total length of the graph of $G$. Now, we identify $e_j$ with the interval $[0,l_j]$. A coordinate $x_{e_j} \in [0,l_j]$ is chosen, in such a way that $O$ corresponds to $x_{e_j} = 0$ and $P_j$ to $x_{e_j} = l_j$. Then, the Kirchhoff condition for (1.2) becomes

$$u_1'(0) + u_2'(0) + u_3'(0) = 0, \quad v_1'(0) + v_2'(0) + v_3'(0) = 0,$$

(2.1)
where $u_j(x) := u|_{e_j}(x)$ and $v_j(x) := v|_{e_j}(x)$ ($j = 1, 2, 3$). Green’s function on the Y-shaped metric graph is given by the following lemma:

**Lemma 2.1.** Green’s function $G(x, s)$ on Y-shaped metric graph is given by

$$DG(x, s) = \frac{1}{2} \left[ |x - s| - (x + s) \right] \chi_{e_j}(s) - \frac{1}{2L} (x - l_j)^2 + \frac{l_j^2}{2L}$$

$$- \frac{1}{2L} \sum_{k=1}^{3} (s - l_k)^2 \chi_{e_k}(s) + \frac{1}{2L} \sum_{k=1}^{3} l_k^2 \chi_{e_k}(s) - \frac{1}{3L} \sum_{k=1}^{3} l_k^3$$

(2.2)

for $x \in e_j$ ($j = 1, 2, 3$) and $s \in G$. Moreover, $G(x, s)$ satisfies the conditions (G1)–(G5).

We give the proof of Lemma 2.1 in details in Appendix B for reader’s convenience.

![Figure 1. Y-shaped metric graph.](image)

2.1. **The case of a one-peak solution.** In this subsection, we consider the existence of a one-peak solution on the Y-shaped metric graph in non-heterogeneity case and heterogeneity case, respectively.

2.1.1. **Non-heterogeneity case.** We first describe the case of $g(x) = 1$.

**Theorem 2.2 (One-peak solution on the Y-shaped metric graph).** Assume that $l_1 > L/2$. Then, we can construct a one-peak solution $(u_{\varepsilon,1}(x), v_{\varepsilon,1}(x))$ which concentrates near the point $t^{\varepsilon} \in e_1$ near $t^0 = l_1 - L/2 \in e_1$. Moreover, the solution satisfies

$$u_{\varepsilon,1}(x) \sim \frac{1}{6c} w \left( \frac{x - t^0}{\varepsilon} \right), \quad v_{\varepsilon,1}(t^{\varepsilon}) \sim 6c.$$  

(2.3)

**Remark 6.** In [12], it is shown that $u_{\varepsilon,1}(x)$ is stable for any $D < +\infty$. Note that, the case of $l_1 = l_2 = l_3$ does not satisfy the assumption $l_1 > L/2$.

**Proof of Theorem 2.2.** By Theorem 1.1 and Lemma 2.1, it suffices to check the assumption (A0)\textsubscript{e1}. Using Lemma 2.1, we have

$$F(t) = -\frac{1}{D} \left( \frac{1}{2} + \frac{t - l_1}{L} \right), \quad F'(t) = -\frac{1}{DL} \neq 0$$

(2.4)

for $0 < t < l_1$. $F(t^0) = 0$ implies

$$t^0 = l_1 - L/2.$$  

(2.5)
Hence, to obtain a one-peak solution which concentrates near a point \( t^0 \in e_1 \), we need to assume that
\[
 l_1 > L/2. 
\]  
(2.6)
Thus we finish the proof of Theorem 2.2.

2.1.2. Heterogeneity case. Next, to study the effect of heterogeneity, we consider the following case:
\[
 g(x) = x + 1 \ (x \in e_1), \quad g(x) = 1 \ (x \in e_1 \cup e_2), \quad l_1 = l_2 = l_3 = 2.
\]
Then, we have
\[
 F(t) = -\frac{t + 1}{6D} + \frac{6c}{(t + 1)^2}, \quad F'(t) = -\frac{1}{6D} - \frac{12c}{(t + 1)^3} < 0.
\]
From \( F(t^0) = 0 \), we obtain the concentration point
\[
 t^0 = (36cD)^{1/3} - 1 \in e_1, \quad \frac{1}{36c} < D < \frac{3}{4c}.
\]
Hence, by Theorem 1.1, we can construct the one-peak solution concentrating at near the point \( t^0 \) above. Moreover, by [12, Theorem 2.3], we see that this solution is stable. Compared with Theorem 2.2, the construction in the case of \( l_1 = l_2 = l_3 \) is made possible by taking the suitable heterogeneity.

2.2. The case of a two-peak solution. We next give the existence theorem of a two-peak solution with \( g(x) = 1 \) on the Y-shaped metric graph. In particular, we consider a two-peak solution with the same heights of the spikes.

**Theorem 2.3 (Two-peak solution on the Y-shaped metric graph).** Let
\[
 S := \{24cD, 24cD + L/8\}. \quad \text{We have the following two results:}
\]

**Case A:** there are two concentration points on different edges, respectively. Assume that
\[
 l_1 = l_2 =: l, \quad l > \frac{L}{4}, \quad \frac{L}{4} \notin S. \quad \text{(2.7)}
\]
Then, we can construct a two-peak solution \((u_{\varepsilon,2}(x), v_{\varepsilon,2}(x)) = (u_{\varepsilon,2}^A(x), v_{\varepsilon,2}^A(x))\) which concentrates near the point \((t_{1,0}^0, t_{2,0}^0) \in e_1 \times e_2\) near
\[
 (t_{1,0}^0, t_{2,0}^0) = \left(l - \frac{L}{4}, l - \frac{L}{4}\right) \in e_1 \times e_2. \quad \text{(2.8)}
\]
Moreover, the solution satisfies
\[
 u_{\varepsilon,2}(x) \sim \sum_{k=1}^2 \frac{1}{12c} w\left(\frac{x - t_{k,0}^0}{\varepsilon}\right), \quad v_{\varepsilon,2}(t_{j,0}^0) \sim 12c, \quad j = 1, 2. \quad \text{(2.9)}
\]

**Case B:** there are two concentration points on same edge. Assume that
\[
 l_1 > \frac{3}{4}L, \quad \frac{L}{4} \notin S. \quad \text{(2.10)}
\]
Then, we can construct a two-peak solution \((u_{\varepsilon,2}(x), v_{\varepsilon,2}(x)) = (u_{\varepsilon,2}^B(x), v_{\varepsilon,2}^B(x))\) which concentrates near the point \((t_{1,0}^0, t_{2,0}^0) \in e_1 \times e_1\) such that
\[
 t_{1,0}^0 \sim t_{1,0}^0 = l_1 - \frac{3}{4}L \in e_1, \quad t_{2,0}^0 \sim t_{2,0}^0 = l_1 - \frac{L}{4} \in e_1.
\]
Moreover, the solution satisfies the asymptotic formula similar to (2.9).
Remark 7. The stability of the solutions constructed Theorem 2.3 is shown in [12]. For comparison, we note that a two-peak symmetric solution in the interval \((-1, 1)\) has the stability threshold \(D_2 := \frac{L}{192c}\) and is stable (resp. unstable) for \(D < D_2\) (resp. \(D > D_2\)). Here, \(L := 2\) is the length of the interval. In [12], it is shown that \((u_{x,2}^B, v_{x,2}^B)\) has the stability threshold

\[
D_2^B := \frac{L}{192c}.
\]

On the other hand, if \(l_3 < L/4\), then \((u_{x,2}^A, v_{x,2}^A)\) has the different stability threshold

\[
D_2^B := \frac{L}{192c} - \frac{l_3}{48c}
\]

from \((u_{x,2}^B, v_{x,2}^B)\) (and the one-dimensional interval case.) Moreover, if \(l_3 > L/4\), then \((u_{x,2}^A, v_{x,2}^A)\) is unstable for any \(D < +\infty\).

Proof of Theorem 2.3. By Theorem 1.2 and Lemma 2.1, it suffices to check the assumptions (A1)–(A3). Let \(t_1^0\) and \(t_2^0\) be two concentration points of a two-peak solution. Since we consider the case of the same heights of the spikes, we get

\[
\xi_1^0 = \xi_2^0 = 12c,
\]

where we used \(\sum_{k=1}^{2} 6c(\xi_k^0)^{-1} = 1\) (see (1.18)). Also, for \(j = 1, 2\), it holds that

\[
a_j^0 = 6c(\xi_j^0)^{-2} = (24c)^{-1} =: a.
\]

Let us demonstrate in the Case A and Case B, respectively.

Case A. Let \(t_1^0 \in e_1\) and \(t_2^0 \in e_2\). By Lemma 2.1, we obtain

\[
D(\sigma_j(t^0)) = (-1)^j r_j^0 - \frac{1}{2L} (t_j^0 - l_1)^2 + \frac{1}{2L} (t_j^0 - l_2)^2 + \frac{1}{2L} (t_j^0 - l_2)^2.
\]

Then it holds that

\[
D(\sigma_1(t^0) + \sigma_2(t^0)) = -t_1^0 + t_2^0 - \frac{1}{L} (t_1^0 - l_1)^2 + \frac{1}{L} (t_2^0 - l_2)^2 + \frac{1}{L} (t_2^0 - l_2)^2
\]

and

\[
D(\sigma_1(t^0) - \sigma_2(t^0)) = -t_1^0 - t_2^0.
\]

We also have

\[
\begin{pmatrix}
    m_{11}(t^0) & m_{12}(t^0) \\
    m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix}
= \frac{1}{D}
\begin{pmatrix}
    -\frac{1}{2} - \frac{t_j^0 - l_1}{L} & -\frac{t_j^0 - l_1}{L} \\
    -\frac{t_j^0 - l_2}{L} & -\frac{1}{2} - \frac{t_j^0 - l_2}{L}
\end{pmatrix},
\]

We first check the assumption (A1). By (1.18) and (2.11), we have

\[
\sum_{k=1}^{2} m_{1k}(t^0) = 0, \quad \sum_{k=1}^{2} m_{2k}(t^0) = 0, \quad \sum_{k=1}^{2} \sigma_k(t^0) = 0,
\]

where \(\sigma_k(t^0) = G(t_1^0, t_k^0) - G(t_2^0, t_k^0)\) for \(j = 1, 2\). By solving \(\sum_{k=1}^{2} m_{jk}(t^0) = 0\), we obtain

\[
t_j^0 = l_j - \frac{L}{4}, \quad j = 1, 2.
\]

By (2.14) and (2.18), we have \(D(\sigma_1(t^0) + \sigma_2(t^0)) = -l_1 + l_2 + L^{-1}(t_2^0 - l_2)^2\). Since \(\sigma_1(t^0) + \sigma_2(t^0) = 0\), we have

\[
l_1 = l_2 =: l.
\]
Thus we also need the condition
\[ l > \frac{L}{4}. \tag{2.20} \]
Hence, we finish the check of (A1). Next, we check the assumption (A2). We have
\[ \det \mathcal{A}(t^0, \xi^0) = 2a + a^2(\sigma_1(t^0) - \sigma_2(t^0)) = 2a - a^2D^{-1}(t_1^0 + t_2^0), \tag{2.21} \]
where \( a = (2c)^{-1} \) (see (2.12)). Then, \( \det \mathcal{A}(t^0, \xi^0) \neq 0 \) implies
\[ l - \frac{L}{4} \neq 24cD \tag{2.22} \]
and hence the check of (A2) is finished. Finally, we check the assumption (A3). From (2.14) and (2.18), it follows that
\[
\begin{cases}
D\partial_{\xi_1}(\sigma_1 + \sigma_2)(t^0) = -1 - \frac{2}{L}(t_1^0 - l_1) = -\frac{1}{2}, \\
D\partial_{\xi_2}(\sigma_1 + \sigma_2)(t^0) = 1 + \frac{2}{L}(t_2^0 - l_2) = \frac{1}{2}.
\end{cases} \tag{2.23}
\]
By substituting \( t_j^0 = l - 4^{-1}L \) into (2.16), it holds that
\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} = \frac{1}{4D} \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}. \tag{2.24}
\]
Also, (2.16) implies \( \mathcal{M}_1(t^0) = -2(DL)^{-1}I \), where \( \mathcal{M}_1(t^0) \) is defined by (6.63). Therefore, combining Lemma 6.3, (2.23), and the two matrices above, we obtain
\[
\mathcal{M}(t^0) = -\frac{1}{DL}I - \frac{a^2}{16D^2 \det \mathcal{A}(t^0, \xi^0)} \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}^2 \\
= -\mu_1 I - \mu_2 \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}, \tag{2.25}
\]
where
\[
\mu_1 := \frac{1}{DL}, \quad \mu_2 := \frac{a^2}{8D^2 \det \mathcal{A}(t^0, \xi^0)}. \tag{2.26}
\]
Since \( \det \mathcal{M}(t^0) = -\mu_1(\mu_1 + 2\mu_2) \neq 0 \), we deduce
\[ l - \frac{L}{4} \neq 24cD + \frac{L}{8}. \tag{2.27} \]
Thus we complete the check of (A3).

**Case B.** Let \( t_1^0, t_2^0 \in e_1 \). Then, we have
\[
D\sigma_1(t^0) = -\frac{1}{2L}(t_1^0 - l_1)^2 + \frac{1}{2L}(t_2^0 - l_2)^2, \tag{2.28}
\]
\[
D\sigma_2(t^0) = -t_1^0 + t_2^0 - \frac{1}{2L}(t_1^0 - l_1)^2 + \frac{1}{2L}(t_2^0 - l_2)^2, \tag{2.29}
\]
and
\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} = \frac{1}{D} \begin{pmatrix}
-\frac{1}{2} & -\frac{t_1^0 - l_1}{L} & -1 - \frac{t_2^0 - l_2}{L} \\
-t_1^0 - l_1 & -\frac{t_2^0 - l_2}{L} & -1 - \frac{t_2^0 - l_2}{L}
\end{pmatrix}. \tag{2.30}
\]
We first check (A1). Solving \( \sum_{k=1}^2 m_{jk}(t^0) = 0 \), we have
\[ t_1^0 = l_1 - \frac{3}{4}L, \quad t_2^0 = l_1 - \frac{L}{4}. \tag{2.31} \]
Then, we need the condition
\[ l_1 > \frac{3}{4}L. \tag{2.32} \]
Moreover, by (2.31), we can calculate
\[ D(\sigma_1(t^0) + \sigma_2(t^0)) = -t_0^0 + t_0^2 - \frac{(t_0^0 - l_1)^2}{L} + \frac{(t_0^2 - l_1)^2}{L} = \frac{3}{4}L - \frac{1}{4}L + L - \frac{9}{16}L = 0. \]
Thus we finish the check of (A1). We next check (A2). By (2.28) and (2.29), it holds that
\[ \det \mathcal{A}(t^0, \xi^0) = 2a - a^2D^{-1}(t_2^0 - t_1^0). \tag{2.33} \]
Since \( \det \mathcal{A}(t^0, \xi^0) \neq 0 \), we obtain
\[ \frac{L}{4} \neq 24cD \tag{2.34} \]
and hence we finish the check of (A2). Finally, we check (A3). By (2.28), (2.29), and (2.31), it holds that
\[ D\partial_{t_1} \sigma_1 + \sigma_2)(t^0) = \frac{1}{2}, \quad D\partial_{t_2} \sigma_1 + \sigma_2)(t^0) = \frac{1}{2} \tag{2.35} \]
By substituting (2.31) into (2.30), it holds that
\[ \begin{pmatrix} m_{11}(t^0) \\ m_{21}(t^0) \\ m_{12}(t^0) \\ m_{22}(t^0) \end{pmatrix} = \frac{1}{4D} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{2.36} \]
Moreover, (2.30) implies \( \mathcal{M}(t^0) = -2(DL)^{-1}I \). Hence, combining Lemma 6.3, (2.35) and the two matrices above, we obtain
\[ \mathcal{M}(t^0) = -\mu_1 I - \mu_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{2.37} \]
where
\[ \mu_1 := \frac{1}{DL}, \quad \mu_2 := \frac{a^2}{8D^2 \det \mathcal{A}(t^0, \xi^0)}. \tag{2.38} \]
Since \( \det \mathcal{M}(t^0) = -\mu_1(\mu_1 + 2\mu_2) \neq 0 \), we see that
\[ \frac{L}{4} \neq 24cD + \frac{L}{8}. \tag{2.39} \]
Thus we complete the check of (A3).

2.3. Discussion. In this subsection, we discuss the Y-shaped graph case and the one-dimensional interval case. Now, we recall the results of multi-peak symmetric solution on the interval \((-1, 1)\). In the work of Wei et al. [9], it is known that a one-peak symmetric solution concentrates near \( x = 0 \) and a two-peak symmetric solution concentrates near \( x = \pm 1/2 \) (see Figure 2). By Theorem 2.2, a one-peak solution on the Y-shaped graph (see Figure 3) has a spike at a distant of \( L/2 \) from a boundary on \( e_1 \). On the other hand, we need \( A + l_2 + l_3 = L/2 \) instead of \( A = L/2 \), where \( A \) is the length between a concentration point \( t_0 \) and a junction \( O \).

Next, we consider a two-peak solution on the Y-shaped graph (see Figure 4). In Case A of Theorem 2.3, the solution has two spikes at a distant of \( L/4 \) from a boundary on \( e_1 \) and \( e_2 \), respectively. Also, the length condition \( l_1 = l_2 \) is required and hence we have an additional condition \( A_1 = A_2 \), where \( A_j \) is the length between a concentration point \( t^0_j \) and a junction \( O \). This is a new phenomenon. Moreover, we need \( A_1 + A_2 + l_3 = L/2 \). In Case B, we see that the distance between a concentration point \( t^0_j \) a boundary on \( e_1 \) is \( L/4 \) and \( A_1 + l_2 + l_3 = L/4 \) is required, where \( A \) is the
Figure 2. A one-peak solution and a two-peak solution on the interval $(-1, 1)$. \( L \) is a length of $(-1, 1)$.

Figure 3. A concentration point \( t_0 \) of a one-peak solution on the \( Y \)-shaped graph. By Theorem 2.2, we have \( A + l_2 + l_3 = L/2 \).

Figure 4. Concentration points \( t_0^1, t_0^2 \) of a two-peak solution on the \( Y \)-shaped graph. Case A: By Theorem 2.3, it holds that \( l_1 = l_2 \) and \( A_1 + A_2 + l_3 = L/2 \). Moreover, we also have \( A_1 = A_2 \). Case B: A distance between \( t_0^1 \) and \( t_0^2 \) is \( L/2 \) and \( A + l_2 + l_3 = L/4 \) holds.

length between a concentration point \( t_0^j \) and a junction \( O \). Furthermore, we have \( t_0^1 - t_0^2 = L/2 \), in other words, a distance between two spikes is \( L/2 \).

3. Application of existence theorem to \( H \)-shaped metric graph. In this section, we consider the \( H \)-shaped metric graph \( G = \bigcup_{j=1}^{5} e_j \) with two junction. When \( j = 1, 2, 3 \), the edge \( e_j \) join two vertices \( O \) and \( P_j \). Moreover, when \( j = 4, 5 \), the edge \( e_j \) join two vertices \( P_3 \) and \( P_j \). We write \( l_j := |e_j| \) and \( L := \sum_{j=1}^{5} l_j \). Now, we identify \( e_j \) with the interval \([0, l_j] \). When \( j = 1, 2, 3 \), a coordinate \( x_{e_j} \in [0, l_j] \) is chosen, in such a way that \( O \) corresponds to \( x_{e_j} = 0 \) and \( P_j \) to \( x_{e_j} = l_j \). When \( j = 4, 5 \), a coordinate \( x_{e_j} \in [0, l_j] \), in such a way that \( P_3 \) corresponds to \( x_{e_j} = 0 \).
and $P_j$ to $x_{e_j} = l_j$. Thus the Kirchhoff condition for (1.2) becomes
\[
\begin{align*}
    u_1'(0) + u_2'(0) + u_3'(0) &= 0, \\
    v_1'(0) + v_2'(0) + v_3'(0) &= 0, \\
    -u_4'(l_4) + u_4'(0) + u_5'(0) &= 0, \\
    -v_4'(l_3) + v_4'(0) + v_5'(0) &= 0,
\end{align*}
\]

where $u_j(x) := u_{|e_j}(x)$ and $v_j(x) := v_{|e_j}(x)$ ($j = 1, 2, \ldots, 5$).

**Figure 5. H-shaped metric graph.**

Green’s function on the $H$-shaped metric graph is given by the following lemma:

**Lemma 3.1.** Let $x_j \in e_j$ ($j = 1, 2, \ldots, 5$) and $s \in \mathcal{G}$. We define $d_j(s)$ as follows:
\[
d_j(s) = \begin{cases} 
    \frac{l_4 + l_5}{L} s \chi_{e_3}(s) + \frac{l_2^2}{2L} - \frac{l_3}{L} (l_4 + l_5) + l_3 (\chi_{e_4}(s) + \chi_{e_5}(s)) & (j = 1, 2), \\
    \left(\frac{l_4 + l_5}{L} - 1\right) s \chi_{e_3}(s) + \frac{l_2^2}{2L} + \frac{l_3^2}{2L} & (j = 4, 5).
\end{cases}
\]

Then, Green’s function $G(x,s)$ on $H$-shaped metric graph is given by
\[
DG(x_3, s) = \frac{1}{2} \left[ |x_3 - s| - (x_3 + s) \right] \chi_{e_3}(s) \\
- \frac{1}{2L} (x_3 - l_3)^2 + \frac{l_4 + l_5}{L} \left[ \frac{l_4}{L} (l_1 + l_2) + (x_3 - l_3) \right] \\
- \left[ \frac{l_4}{L} (l_1 + l_2 + l_3) \right] (\chi_{e_4}(s) + \chi_{e_5}(s)) + \frac{l_4 + l_5}{L} s \chi_{e_3}(s) + \frac{l_3^2}{2L} \\
- \frac{1}{2L} \sum_{k=1}^{5} (s - l_k)^2 \chi_{e_k}(s) + \frac{1}{2L} \sum_{k=1}^{5} l_k^2 \chi_{e_k}(s) - \frac{1}{3L^2} \sum_{k=1}^{5} l_k^3
\]

and
\[
DG(x_j, s) = \frac{1}{2} \left[ |x_j - s| - (x_j + s) \right] \chi_{e_j}(s) - \frac{1}{2L} (x_j - l_j)^2 + d_j(s) \\
- \frac{l_4}{L} (l_1 + l_2 + l_3) (\chi_{e_4}(s) + \chi_{e_5}(s)) - \frac{1}{2L} \sum_{k=1}^{5} (s - l_k)^2 \chi_{e_k}(s) \\
+ \frac{l_3}{L^2} (l_1 + l_2) (l_4 + l_5) + \frac{1}{2L} \sum_{k=1}^{5} l_k^2 \chi_{e_k}(s) - \frac{1}{3L^2} \sum_{k=1}^{5} l_k^3
\]

for $j \neq 3$. Moreover, $G(x,s)$ satisfies the conditions (G1)–(G5).

We give the proof of Lemma 3.1 in details in Appendix B for reader’s convenience.
3.1. The case of a one-peak solution. In this subsection, we give the existence theorem of a one-peak solution with \( g(x) = 1 \) on the \( H \)-shaped metric graph.

**Theorem 3.2 (One-peak solution on the \( H \)-shaped metric graph).** Fix \( j \in \{1, 2, \ldots, 5\} \) and assume that \( l_j > l_j + (l_4 + l_5)\delta_3 - L/2 > 0 \). Then, we can construct a one-peak solution \((u_{\varepsilon,1}(x), v_{\varepsilon,1}(x))\) which concentrates near the point \( t^\varepsilon \in e_j \) such that

\[
t^\varepsilon \sim t^0 = l_j + (l_4 + l_5)\delta_3 - \frac{L}{2} \in e_j.
\]

Moreover, the solution satisfies

\[
u_{\varepsilon,1}(x) \sim \frac{1}{6c} w \left( \frac{x - t^0}{\varepsilon} \right), \quad v_{\varepsilon,1}(t^\varepsilon) \sim 6c.
\]

**Proof.** By Theorem 1.1 and Lemma 3.1, it suffices to check the assumption \((A0)_{e_j}\).

Let \( t^0 \in e_j \) be a concentration point of a one-peak solution. From Lemma 3.1, it follows that

\[
F(t) = -\frac{1}{DL} \left( \frac{1}{2} + \frac{t - l_j}{L} - \frac{l_4 + l_5}{L}\delta_3 \right), \quad F'(t) = -\frac{1}{DL} \neq 0,
\]

for \( 0 < t < l_j \). Since \( F(t^0) = 0 \), we have

\[
t^0 = l_j + (l_4 + l_5)\delta_3 - \frac{L}{2}
\]

and hence we need the following condition for the concentration point:

\[
l_j > l_j + (l_4 + l_5)\delta_3 - \frac{L}{2} > 0.
\]

Thus we complete the proof of Theorem 3.2. \( \square \)

3.2. The case of a two-peak solution. We next investigate the existence of a two-peak solution with \( g(x) = 1 \). We consider the case of the same heights of the spikes in the same argument as Section 2.

**Theorem 3.3 (Two-peak solution on the \( H \)-shaped metric graph).** Let \( S := \{24cD, 24cD + L/8\} \). We have the following four results:

1. The case there are two concentration points on different edges, respectively.

   **Case A (two concentration points on \( e_1 \) and \( e_2 \)).** Assume that

   \[
l_1 = l_2 = : l, \quad l > \frac{L}{4}, \quad l - \frac{L}{4} \notin S.
\]

   Then, we can construct a two-peak solution \((u_{\varepsilon,2}(x), v_{\varepsilon,2}(x))\) which concentrates near the point \((t^1, t^2) \in e_1 \times e_2 \) such that

   \[
t^1 \sim t^0_1 = l - \frac{L}{4} \in e_1, \quad t^2 \sim t^0_2 = l - \frac{L}{4} \in e_2.
\]

   Moreover, the solution satisfies

   \[
u_{\varepsilon,2}(x) \sim \sum_{k=1}^{2} \frac{1}{12c} w \left( \frac{x - t^0_k}{\varepsilon} \right), \quad v_{\varepsilon,2}(t^\varepsilon_j) \sim 12c, \quad j = 1, 2.
\]

   **Case B (two concentration points on \( e_1 \) and \( e_4 \)).** Assume that

   \[
l_1 + c_2l_3 = l_4 + c_1l_3 =: l, \quad l_1, l_4 > \frac{L}{4}, \quad l - \frac{L}{4} \notin S,
\]

   \[
   \text{for } 0 < t < l_j. \quad \text{Since } F(t^0) = 0, \quad \text{we have}
   \]

   \[
   t^0 = l_j + (l_4 + l_5)\delta_3 - \frac{L}{2}
   \]

   and hence we need the following condition for the concentration point:

   \[
l_j > l_j + (l_4 + l_5)\delta_3 - \frac{L}{2} > 0.
\]

Thus we complete the proof of Theorem 3.3.
Then, we can construct a two-peak solution \((u_{\epsilon,2}(x), v_{\epsilon,2}(x))\) which concentrates near the point \((t^*_1, t^*_2) \in e_1 \times e_2\) such that
\[
t^*_1 \sim t^*_1 = l_1 - \frac{L}{4} \in e_1, \quad t^*_2 \sim t^*_2 = l_4 - \frac{L}{4} \in e_4.
\]
Moreover, the solution satisfies the asymptotic formula similar to (3.11).

**Case C (two concentration points on \(e_1\) and \(e_3\)).** Assume that
\[
l_1 = l_3 + l_4 + l_5 =: l, \quad l > \frac{L}{4}, \quad l - \frac{L}{4} \notin S. \tag{3.14}
\]
Then, we can construct a two-peak solution \((u_{\epsilon,2}(x), v_{\epsilon,2}(x))\) which concentrates near the point \((t^*_1, t^*_2) \in e_1 \times e_3\) such that
\[
t^*_1 \sim t^*_1 = l_1 - \frac{L}{4} \in e_1, \quad t^*_2 \sim t^*_2 = l_3 + l_5 - \frac{L}{4} \in e_3.
\]
Moreover, the solution satisfies the asymptotic formula similar to (3.11).

**(2) The case there are two concentration points on same edge.**

**Case D.** Fix \(j \in \{1, 2, \ldots, 5\}\) and assume that
\[
\hat{t}_j := l_j + (l_4 + l_5)\delta_{3j} > \frac{3}{4}L, \quad l - \frac{L}{4} \notin S. \tag{3.15}
\]
Then, we can construct a two-peak solution \((u_{\epsilon,2}(x), v_{\epsilon,2}(x))\) which concentrates near the point \((t^*_1, t^*_2) \in e_j \times e_j\) such that
\[
t^*_1 \sim t^*_1 = \hat{t}_j - \frac{3}{4}L \in e_j, \quad t^*_2 \sim t^*_2 = \hat{t}_j - \frac{L}{4} \in e_j.
\]
Moreover, the solution satisfies the asymptotic formula similar to (3.11).

**Proof.** By Theorem 1.2 and Lemma 3.1, it suffices to check the assumptions (A1)–(A3). Let \(t^*_1\) and \(t^*_2\) be two concentration points of a two-peak solution. Since we consider the case of the same heights of the spikes, we obtain
\[
\xi^*_1 = \xi^*_2 = 12c, \tag{3.16}
\]
and
\[
a^*_j = 6c(\xi^*_1)^{-2} = (24c)^{-1} =: a \tag{3.17}
\]
for \(j = 1, 2\). Let us demonstrate **Case A**–**Case D**.

**Case A.** Let \(t^*_1 \in e_1\) and \(t^*_2 \in e_2\). From Lemma 3.1, we have
\[
D\sigma_j(t^0) = (-1)^j t^0_j - \frac{1}{2L} (t^0_j - l_1)^2 + \frac{1}{2L} (t^0_j - l_2)^2 + \frac{1}{2L} (l_1 - l_2^2) \tag{3.18}
\]
and
\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} = \frac{1}{D} \begin{pmatrix}
-\frac{1}{2} - \frac{t^0_j - l_1}{L} & -\frac{t^0_j - l_1}{L} \\
-\frac{t^0_j - l_2}{L} & -\frac{1}{2} - \frac{t^0_j - l_2}{L}
\end{pmatrix}. \tag{3.19}
\]
By the same argument as that in the proof of Case A of Theorem 2.3, we see that the following condition satisfies (A1):
\[
t^*_1 = l_1 - \frac{L}{4}, \quad t^*_2 = l_2 - \frac{L}{4}, \quad l_1 = l_2 =: l, \quad l > \frac{L}{4}. \tag{3.20}
\]
Moreover, the following condition satisfies (A2) and (A3):

\[ l - \frac{L}{4} \notin S. \]  

(3.21)

**Case B.** Let \( t_1^0 \in e_1 \) and \( t_2^0 \in e_4 \). From Lemma 3.1, we have

\[
D\sigma_j(t^0) = (-1)^j t_j^0 - \frac{1}{2L}(t_1^0 - l_1)^2 + \frac{1}{2L}(t_2^0 - l_2)^2
+ \frac{1}{2L}(l_1^2 - l_4^2 - l_3^2) - \frac{l_3}{L}(l_4 + l_5 - \delta_{j2}L)
\]  

(3.22)

and

\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} = \frac{1}{D} \begin{pmatrix}
-\frac{1}{2} - \frac{t_0^0 - l_1}{L} & -\frac{l_0^0 - l_1}{L} \\
-\frac{t_0^0 - l_4}{L} & -\frac{1}{2} - \frac{t_0^0 - l_4}{L}
\end{pmatrix}.
\]  

(3.23)

We first check (A1). From \( \sum_{k=1}^2 m_{jk}(t^0) = 0 \), we obtain the following condition:

\[ t_1^0 = l_1 - \frac{L}{4}, \quad t_2^0 = l_4 - \frac{L}{4}, \quad l_1, l_4 > \frac{L}{4}. \]  

(3.24)

Also, by (3.22) and (3.24), we can calculate

\[
D(\sigma_1(t^0) + \sigma_2(t^0)) = -\frac{t_0^0 + t_0^0}{L} + \frac{1}{L}(l_1^2 - l_4^2 - l_3^2) + \frac{l_3}{L}(l_4 - 2l_4 - 2l_5)
\]

\[ = (l_1 - l_4)(-1 + \frac{l_1 + l_4}{L} + \frac{l_3}{L})(l_4 - l_4 - l_5)
\]

\[ = -L^{-1}(l_1 - l_4)(l_4 + l_5) + L^{-1}l_3(l_4 - l_5).
\]  

(3.25)

Since \( \sigma_1(t^0) + \sigma_2(t^0) = 0 \), we have

\[ l_1 + c_2 l_3 = l_4 + c_1 l_3 := l,
\]  

(3.26)

where

\[ c_1 := \frac{l_2}{l_2 + l_5}, \quad c_2 := \frac{l_5}{l_2 + l_5}.
\]  

(3.27)

In particular, from (3.26) and (3.24), we see that

\[ t_1^0 + c_2 l_3 = t_2^0 + c_1 l_3, \quad c_1 + c_2 = 1.
\]  

(3.28)

Hence, we finish the check of (A1). Now, by using (3.26) and (3.28), it holds that

\[ \det A(t^0, \xi^0) = 2a - a^2D^{-1}(t_1^0 + t_2^0 + l_3) = 2a - 2a^2D^{-1}(l - 4^{-1}L).
\]  

(3.29)

Also, we have

\[ D\partial_l(\sigma_1 + \sigma_2)(t^0) = -\frac{1}{2}, \quad D\partial_l(\sigma_1 + \sigma_2)(t^0) = \frac{1}{2},
\]  

(3.30)

and

\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} = \frac{1}{4D} \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}.
\]  

(3.31)

Hence, by the same argument as that in the proof of Case A of Theorem 2.3, we deduce that the following condition satisfies (A2) and (A3):

\[ l - \frac{L}{4} \notin S.
\]  

(3.32)
Case C. Let \( t^0_1 \in e_1 \) and \( t^0_2 \in e_3 \). From Lemma 3.1, we see that
\[
D\sigma_j(t^0) = (-1)^j t^0_j - \frac{1}{2L} (t^0_1 - l_1)^2 + \frac{1}{2L} (t^0_2 - l_3)^2 - \frac{t_4 + l_5}{L} t^0_2 + \frac{t^2_1}{2L} - \frac{t^2_3}{2L} \tag{3.33}
\]
and
\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix}
= \frac{1}{D} \begin{pmatrix}
-\frac{1}{2} - \frac{t^0_1 - l_1}{L} & -\frac{t^0_1 - l_1}{L} \\
-\frac{t^0_2 - l_3}{L} + \frac{l_4 + l_5}{L} & -\frac{1}{2} - \frac{t^0_2 - l_3}{L} + \frac{l_4 + l_5}{L}
\end{pmatrix} , \tag{3.34}
\]
First, by solving \( \sum_{k=1}^2 m_{jk}(t^0) = 0 \), we have
\[
t^0_1 = l_1 - \frac{L}{4}, \quad t^0_2 = l_3 + l_4 + l_5 - \frac{L}{4} . \tag{3.35}
\]
Thus the following conditions are required:
\[
l_1 > \frac{L}{4}, \quad l_3 + l_4 + l_5 > \frac{L}{4} . \tag{3.36}
\]
Hence, we complete the check of (A1). Next, by (3.33) and (3.35), we can calculate
\[
(D(\sigma_1(t^0) + \sigma_2(t^0))
= -t^0_1 + t^0_2 \frac{1}{2L} (t^0_1 - l_1)^2 + \frac{1}{L} (t^0_2 - l_3)^2 - 2L t^0_3 l_4 + l_5 \frac{1}{2L (t^0_2 - l_3)}
= (l_3 + l_4 + l_5 - l_1) \left( 1 - \frac{l_1 + l_3 + l_4 + l_5}{L} \right) . \tag{3.37}
\]
Since \( \sigma_1(t^0) + \sigma_2(t^0) = 0 \), we have
\[
l_1 = l_3 + l_4 + l_5 =: l . \tag{3.38}
\]
In particular, from the condition above and (3.35), we see that
\[
t^0_1 = t^0_2 . \tag{3.39}
\]
Moreover, by the same argument as that in the proof of Case A of Theorem 2.3, we obtain the following condition, which satisfies (A2) and (A3):
\[
l - \frac{L}{4} \notin S . \tag{3.40}
\]
Case D. Let \( t^0_1, t^0_2 \in e_j \). From Lemma 3.1, we see that
\[
D\sigma_1(t^0) = -\frac{1}{2L} \left[ t^0_1 - l_1 - (l_4 + l_5) \delta_{3j} \right]^2 + \frac{1}{2L} \left[ t^0_2 - l_j - (l_4 + l_5) \delta_{3j} \right]^2, \tag{3.41}
\]
\[
D\sigma_2(t^0) = -t^0_1 + t^0_2 - \frac{1}{2L} \left[ t^0_1 - l_1 - (l_4 + l_5) \delta_{3j} \right]^2 + \frac{1}{2L} \left[ t^0_2 - l_j - (l_4 + l_5) \delta_{3j} \right]^2, \tag{3.42}
\]
and
\[
\begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix}
= \frac{1}{D} \begin{pmatrix}
-\frac{1}{2} - \frac{t^0_1 - l_3}{L} + \frac{l_4 + l_5}{L} \delta_{3j} & -1 - \frac{t^0_1 - l_3}{L} + \frac{l_4 + l_5}{L} \delta_{3j} \\
-\frac{t^0_2 - l_3}{L} + \frac{l_4 + l_5}{L} \delta_{3j} & -1 - \frac{t^0_2 - l_3}{L} + \frac{l_4 + l_5}{L} \delta_{3j}
\end{pmatrix} . \tag{3.43}
\]
Thus, by the same calculation as that in Case B of Theorem 2.3, we obtain the following conditions, which satisfy (A1)–(A3):

\[
\begin{align*}
t_1^0 &= \hat{t}_j - \frac{3}{4}L, & t_2^0 &= \hat{t}_j - \frac{L}{4}, & \hat{t}_j := l_j + (l_4 + l_5)\delta_{3j} > \frac{3}{4}L, & \frac{L}{4} &\notin S. \\
\end{align*}
\]

(3.44)

3.3. Discussion. In this subsection, we discuss the \( H \)-shaped metric graph. First, we consider a one-peak solution. By Theorem 3.2, when there a spike on a edge except \( e_3 \), we have the same result as that in the \( Y \)-shaped graph case. When there a spike on the edge \( e_3 \), we require two length conditions \( A_1 + l_1 + l_2 = \frac{L}{2} \) and \( A_2 + l_4 + l_5 = \frac{L}{2} \), where \( A_1 \) and \( A_2 \) are the length between a concentration point \( t_0^1 \) and two junctions \( O \) and \( P_3 \), respectively.

Next, we consider a two-peak solution. In Case A, Case C, and Case D (see Figure 7), we have the same results as that in the \( Y \)-shaped graph case. In Case C, we see that distant between a concentration point \( t^0_1 \) a boundary on \( e_1 \) is \( L/4 \) and \( A + l_4 + l_5 = \frac{L}{4} \), where \( A \) is the length between \( t_2^0 \) and a junction \( P_3 \). Also, we need \( t_1^0 + t_2^0 + l_2 = \frac{L}{2} \). Furthermore, since \( l_1 = l_3 + l_4 + l_5 \) is required, we have \( A_1 = A_2 \).

In Case B (see Figure 8), the solution has two spikes at a distant of \( \frac{L}{4} \) from a boundary on \( e_1 \) and \( e_4 \), respectively. Moreover, we have the length condition \( A_1 + A_2 + l_2 + l_3 + l_5 = \frac{L}{2} \), where \( A_1 \) is the length between \( t_2^0 \) and the junction \( O \) and \( A_2 \) is the length between \( t_2^0 \) and the junction \( P_3 \). On the other hand, we have not \( A_1 = A_2 \) but \( A_1 + B = A_2 + (l_4 - B) \), where \( B \) is the point which divide \( e_3 = [0, l_3] \) internally in the ratio \( l_5 : l_2 \). This is a new phenomenon, compared with Case A, Case C, and Case D.

From results of Section 2 and Section 3, we conclude that the geometry of the whole metric graph has effect on the location of concentration points of spiky solutions. The location of the concentration point of one-peak solutions is determined by the total distance from the concentration point to the boundaries. The location of the concentration point of two-peak solutions is basically determined by the relations between the total distance from the concentration points to the boundaries and the junctions, but is determined by a complex relation as in the Case B for the \( H \)-shaped metric graph. Now, let \( e \) be a long segment and \( \hat{G} \) be an arbitrary metric graph satisfying the conditions (H1) and (H2). We consider the metric graph \( \hat{G} := \hat{G} \cup \{e\} \) with one junction at the common vertex \( O \). From the calculation on \( Y \)-shaped graph and \( H \)-shaped graph, we have a conjecture (see Figure 9) that a
Figure 7. Concentration points of a two-peak solution on the \(H\)-shaped graph (Case A, Case C, and Case D). Case A: Using Theorem 3.3, we obtain \(l_1 = l_2\). Case C: \(A_1 + A_2 + l_2 = L/2\) and \(l_1 = l_3 + l_4 + l_5\) is required. Then, we also have \(A_1 = A_2\). Case D: A distance between \(t_0^1\) and \(t_0^2\) is \(L/2\) and \(A_1 + l_1 + l_2 = A_2 + l_4 + l_5 = L/4\) holds.

Figure 8. Concentration points of a two-peak solution on the \(H\)-shaped graph (Case B). The point \(B \in [0, l_3]\) divides \([0, l_3]\) internally in the ratio \(l_5 : l_2\). We have \(A_1 + B = A_2 + (l_3 - B)\).

One-peak solution concentrates near \(t_0 = l_e - L/2 \in e\) and satisfies \(A + \hat{L} = L/2\), where \(A\) is the distant between \(t_0\) and a vertex \(O\), and \(L\) and \(\hat{L}\) are total lengths of \(G\) and \(\hat{G}\), respectively. By Theorem 1.1, we need Green’s function on this metric graph \(G\) satisfying (G1)–(G4)\(\in\cdot\in\). However, it is difficult to calculate this Green’s function so far.

4. Preliminaries and basic estimates for the proof of existence theorem.

4.1. Several notations. In this subsection, we describe several notations frequently used in this paper. Let \(e, e_1, e_2 \in G, t \in e\), and \(t = (t_1, t_2) \in e_1 \times e_2\). For the construction of a one-peak solution, for \(e \in E\), we define

\[
P(r_0[1]) := \{ t \in e \mid r(t) > 2r_0[1] \},
\]

(4.1)

where \(r_0[1]\) is defined by (1.16). For a function \(\eta : I^e \to \mathbb{R}\), we define the following rescaled function around \(t \in P(r_0[1])\):

\[
\tilde{\eta}(y) := \eta(\varepsilon y + t), \quad y \in L_{e, r_0[1]} := (-r_0[1]\varepsilon^{-1}, r_0[1]\varepsilon^{-1}).
\]

(4.2)
Remark 8. For the construction of a two-peak solution, let us put
\[ \int_{I} \eta(y) \, dy \, \text{where the constant} \, C \, \text{is independent of} \, \varepsilon, \text{depending on the situation throughout this paper.} \]
For the convenience of the notation, we may denote \( \tilde{\eta}(y) \) by \( \eta^{-}(y) \) or \( (\eta)^{-}(y) \). For \( t \in P(r_{0}[1]) \), we define \( w_{\varepsilon,t}(x) \) as follows:
\[
 w_{\varepsilon,t}(x) := \frac{1}{g(t)\xi(t)} w \left( \frac{x - t}{\varepsilon} \right) \chi \left( \frac{x - t}{r_{0}[1]} \right). \tag{4.3}
\]
For the construction of a two-peak solution, for \( (e_{1}, e_{2}) \in E \times E \), we define
\[
P(r_{0}[2]) := \begin{cases} 
\{ t \in e_{1} \times e_{2} \mid \delta_{2}(t) > 2r_{0}[2]\} & \text{when} \ e_{1} \neq e_{2}, \\
\{ t \in e_{1} \times e_{1} \mid t_{1} < t_{2}, \ \delta_{2}(t) > 2r_{0}[2]\} & \text{when} \ e_{1} = e_{2},
\end{cases} \tag{4.4}
\]
where \( r_{0}[2] \) is defined by (1.23). For a function \( \eta: I_{j} \to \mathbb{R} \) \( (j = 1, 2) \), we define the following rescaled function around \( t \in P(r_{0}[2]) \):
\[
\tilde{\eta}^{j}(y) := \eta(\varepsilon y + t_{j}), \quad y \in I_{\varepsilon,r_{0}[2]} := (-r_{0}[2]\varepsilon^{-1}, r_{0}[2]\varepsilon^{-1}). \tag{4.5}
\]
For the convenience of the notation, we may denote \( \tilde{\eta}^{j}(y) \) by \( \eta^{-j}(y) \) or \( (\eta)^{-j}(y) \). Also, we may drop \( j \) of \( \tilde{\eta}^{j}(y) \) and write \( \tilde{\eta}(y) \) simply.
For \( t \in P(r_{0}[2]) \), we define \( w_{\varepsilon,t}(x) \) as follows:
\[
 w_{\varepsilon,t}(x) := \frac{1}{g(t)\xi(t)} w \left( \frac{x - t}{\varepsilon} \right) \chi \left( \frac{x - t}{r_{0}[2]} \right), \tag{4.6}
\]
\[
 w_{\varepsilon,t}(x) := \frac{1}{g(t)\xi(t)} w \left( \frac{x - t}{\varepsilon} \right) \chi \left( \frac{x - t}{r_{0}[2]} \right), \tag{4.6}
\]
\[
 \text{Remark 8. Unless otherwise noted, we use the notation} \ r_{0} \ \text{to denote} \ r_{0}[1] \ \text{and} \ r_{0}[2] \ \text{depending on the situation throughout this paper.}
\]
4.2. Basic estimates. In this subsection, we describe several estimates frequently used in this paper. Let us put
\[
\mathcal{B}(C_{0}) := \{ \tilde{\eta} \in H^{2}_{N}(G_{e}) \mid \| \tilde{\eta} \|_{H^{2}(G_{e})} \leq C_{0}\varepsilon \}, \tag{4.7}
\]
where the constant \( C_{0} \) is independent of \( \varepsilon > 0 \) and elements of \( P(r_{0}) \), which will be chosen suitably in Subsection 5.3. For \( g \) and \( w_{\varepsilon,t} \), it is easy to check that, for \( j = 1, 2, \)
\[
 \tilde{g}^{j}(y) = g(t_{j}) + \varepsilon y g^{j}(t_{j}) + O(\varepsilon^{2}y^{2}), \quad y \in I_{\varepsilon,r_{0}}, \tag{4.8}
\]
\[
 \int_{I_{\varepsilon,r_{0}}} (\tilde{w}_{\varepsilon,t}^{j})^{2}dy = \frac{6}{g(t_{j})^{2}\xi(t_{j})^{2}} + O(e^{-q/\varepsilon}), \tag{4.9}
\]
and
\[
 \int_{I_{\varepsilon,r_{0}}} y(\tilde{w}_{\varepsilon,t}^{j})^{2}(\tilde{w}_{\varepsilon,t}^{j})^{\prime}dy = -\frac{1}{3g(t_{j})^{2}\xi(t_{j})^{2}} \int_{\mathbb{R}} w^{3}dy + O(e^{-q/\varepsilon}), \tag{4.10}
\]
where the constant \( q > 0 \) is independent of \( \varepsilon \). We also have similar estimates for \( w_{\varepsilon,t} \).

**Lemma 4.1.** Let \( \overline{\phi} \in B(C_0) \) and \( e, e_1, e_2 \in \mathcal{G} \) with \( e_1 \neq e_2 \). Then we have the following results:

1. For \( t \in P(r_0[1]) \), it holds that
   \[
   c \int_{I_{x,t}^e} \frac{\overline{\phi}(w_{\varepsilon,t} + \overline{\phi})^2}{g(t)\xi(t)} dy = \frac{6c}{g(t)\xi(t)^2} + O(\varepsilon). \tag{4.11}
   \]

2. For \( t \in (e \times e) \cap P(r_0[2]) \), it holds that
   \[
   c \int_{I_{x,t}^e} \overline{\phi}(w_{\varepsilon,t} + \overline{\phi})^2 dy = a_1 + a_2 + O(\varepsilon), \tag{4.12}
   \]
   where
   \[
   a_j = a_j(t) := \frac{6c}{g(t_j)\xi_j(t)^2}. \tag{4.13}
   \]

3. For \( t \in (e_1 \times e_2) \cap P(r_0[2]) \), it holds that
   \[
   c \int_{\Omega_{x,t}^e} \overline{\phi}(w_{\varepsilon,t} + \overline{\phi})^2 dy = a_j + O(\varepsilon), \quad j = 1, 2. \tag{4.14}
   \]

**Remark 9.** In particular, since \( a_j^0 \) is defined by \( a_j^0 := 6c\overline{g}(r_j^0)^{-1}(\xi_j^0)^{-2} \) (see (1.19)), \( a_j(t) \) satisfies \( a_j(t^0) = a_j^0 \).

**Proof.** We first show (2) and (3). Since \( \| \overline{\phi} \|_{L^2(I_{x,t}^e)} \leq C\varepsilon \) for each \( e \in E \), we obtain
\[
\int_{I_{x,t}^e} \overline{\phi}(w_{\varepsilon,t} + \overline{\phi})^2 dy = \int_{I_{x,t}^e} \overline{\phi} w_{\varepsilon,t}^2 + 2\overline{\phi} w_{\varepsilon,t} + \overline{\phi}^2 dy = \int_{I_{x,t}^e} \overline{\phi} w_{\varepsilon,t}^2 dy + O(\varepsilon). \tag{4.15}
\]
Now, we note that \( w_{\varepsilon,t}(y) = 0 \) for \( |\varepsilon y - t_j| \geq r_0/2 \). In addition, if \( e_1 = e_2 =: e \), then \( w_{\varepsilon,t_1}(y)w_{\varepsilon,t_2}(y) = 0 \) for \( y \in I_{x,t}^e \). Thus, if \( e_1 = e_2 =: e \), then by using the two facts above and (4.9), it holds that
\[
\int_{I_{x,t}^e} \overline{\phi} w_{\varepsilon,t}^2 dy \tag{4.16}
\]
\[
= c\int_{I_{x,t}^e} \overline{\phi}(w_{\varepsilon,t_{1}}^2 + w_{\varepsilon,t_{2}}^2) dy = c \int_{I_{x,r_0}} \overline{\phi}(\overline{w}_{\varepsilon,t_{1}}^2 + \overline{w}_{\varepsilon,t_{2}}^2) dy = a_1 + a_2 + O(e^{-q/\varepsilon}).
\]
Also, if \( e_1 \neq e_2 \), then we see that
\[
\int_{I_{x,t}^e} \overline{\phi} w_{\varepsilon,t}^2 dy = c \int_{I_{x,t}^e} \overline{\phi} w_{\varepsilon,t}^2 dy = c \int_{I_{x,r_0}} \overline{\phi} w_{\varepsilon,t}^2 dy = a_j + O(e^{-q/\varepsilon}). \tag{4.17}
\]
Thus (2) and (3) are verified, respectively. Also, by the argument above, (1) can be shown in a similar way. Hence, we complete the proof of Lemma 4.1. \( \square \)

Now, let \( \overline{\phi} \in B(C_0) \) and \( t \in P(r_0) \) be arbitrarily. We study the estimates of the solution of the following equation:
\[
\begin{cases}
-D\eta'' + \frac{c}{\varepsilon} g(w_{\varepsilon,t} + \phi)^2 \eta = \frac{h}{\varepsilon}, & x \in \mathcal{G}, \\
n \sum_{\varepsilon \cdot v} \frac{d\eta_{v}}{dx_{v}}(v) = 0, & v \in V,
\end{cases} \tag{4.18}
\]
where \( h(x) \) is a given function on \( L^2(\mathcal{G}) \). We first obtain the following key lemma:
Lemma 4.2. We have the following estimates:
\[ \|\eta\|_{L^\infty(G_\varepsilon)} \leq C\|\eta\|_{L^1(G_\varepsilon)}, \quad \|\eta\|_{L^2(G_\varepsilon)} \leq C\sqrt{\varepsilon}\|\eta\|_{L^1(G_\varepsilon)}. \]  
(4.19)

Once we have the estimates in Lemma 4.2, by the same argument as in the proof of Lemma 2.3 in [11], the following lemma holds:

Lemma 4.3 ([11, Lemma 2.3]). We have the following estimates:

1. \( \overline{\eta}^j(y) = \eta(t_j) + O(\varepsilon|y|\|\eta\|_{L^1(G_\varepsilon)}) \) for \( y \in I_{\varepsilon, r_0} \).
2. \( \overline{g}^j(y)\overline{\eta}^j(y) = g(t_j)\eta(t_j) + O(\varepsilon|y|\|\eta\|_{L^1(G_\varepsilon)}) \) for \( y \in I_{\varepsilon, r_0} \).

For Lemma 4.2, and lemmas and propositions in Section 5, we will mainly give their proofs of the case of a two-peak solution.

Proof of Lemma 4.2. Letting \( x = \varepsilon y \) for (4.18), we have
\[ -\frac{D}{\varepsilon^2} \eta'' + \frac{c}{\varepsilon} \overline{g}(\overline{w}_{\varepsilon,t} + \overline{\phi})^2 \eta = \frac{\overline{h}}{\varepsilon}, \quad y \in G_\varepsilon. \]
(4.20)
Multiplying both side of the equation above by \( \eta \) and integrating over \( G_\varepsilon \), we obtain
\[ \int_{G_\varepsilon} \frac{D}{\varepsilon^2} (\eta')^2 + \frac{c}{\varepsilon} \overline{g}(\overline{w}_{\varepsilon,t} + \overline{\phi})^2 \eta^2 dy = \int_{G_\varepsilon} \frac{\overline{h}}{\varepsilon} \eta dy \leq \frac{1}{\varepsilon} \|\overline{h}\|_{L^1(G_\varepsilon)} \|\eta\|_{L^\infty(G_\varepsilon)} \]
(4.21)
and hence
\[ \|\eta\|_{L^2(G_\varepsilon)} \leq \sqrt{\frac{\varepsilon}{D}} \|\overline{h}\|_{L^1(G_\varepsilon)} \|\eta\|_{L^\infty(G_\varepsilon)}. \]
(4.22)
On the other hand, integrating (4.20) over \( I_{\varepsilon}^y \), we see that
\[ -D(\overline{\eta}(\varepsilon^{-1}l_{e_1}) - \overline{\eta}(0)) + c \varepsilon \int_{I_{\varepsilon}^y} \overline{g}(\overline{w}_{\varepsilon,t} + \overline{\phi})^2 \eta dy = \varepsilon \int_{I_{\varepsilon}^y} \overline{h} dy. \]
(4.23)
Also, since \( \overline{w}_{\varepsilon,t}(y) = 0 \) for \( y \in I_{\varepsilon}^y (e \neq e_1, e_2) \), by integrating (4.20) over \( [0, y] \) \( (y \in I_{\varepsilon}^y, e \neq e_1, e_2) \), we obtain
\[ -D(\overline{\eta}(y) - \overline{\eta}(0)) + c \varepsilon \int_0^y \overline{g} \overline{\phi}^2 \eta dz = \varepsilon \int_0^y \overline{h} dz. \]
(4.24)
Moreover, by integrating the equation above over \( I_{\varepsilon}^y \), it holds that
\[ -D \int_{I_{\varepsilon}^y} \overline{\eta}(y) dy + D\varepsilon^{-1}l_{e_1} \overline{\eta}(0) \]
\[ + c \varepsilon \int_{I_{\varepsilon}^y} \left( \int_0^y \overline{g} \overline{\phi}^2 \eta dz \right) dy = \varepsilon \int_{I_{\varepsilon}^y} \left( \int_0^y \overline{h} dz \right) dy. \]
(4.25)
Here, by Hölder’s inequality, we estimate \( |\int_{I_{\varepsilon}^y} \overline{\eta}(y) dy| \leq C\varepsilon^{-1/2}\|\overline{\eta}\|_{L^2(G_\varepsilon)} \). Furthermore, we get \( |\int_{I_{\varepsilon}^y} (\int_0^y \overline{h} dz) dy| \leq C\varepsilon^{-1}\|\overline{h}\|_{L^1(G_\varepsilon)} \) and
\[ \left| \int_{I_{\varepsilon}^y} \left( \int_0^y \overline{g} \overline{\phi}^2 \eta dz \right) dy \right| \leq C\varepsilon^{-1}\|\overline{\eta}\|_{L^\infty(G_\varepsilon)} \|\overline{\phi}\|_{L^2(G_\varepsilon)}^2 \leq C\varepsilon\|\overline{\eta}\|_{L^\infty(G_\varepsilon)} \]
(4.26)
Thus, combining (4.25) and the three estimates above, we have
\[ \varepsilon^{-1} \overline{\eta}(0) = O(\|\overline{h}\|_{L^1(G_\varepsilon)}) + O(\varepsilon^{-1/2}\|\overline{\eta}\|_{L^2(G_\varepsilon)}) + O(\varepsilon^2\|\overline{\eta}\|_{L^\infty(G_\varepsilon)}). \]
(4.27)
Similarly, we see that
\[ \varepsilon^{-1} \overline{\eta}(\varepsilon^{-1}l_{e_1}) = O(\|\overline{h}\|_{L^1(G_\varepsilon)}) + O(\varepsilon^{-1/2}\|\overline{\eta}\|_{L^2(G_\varepsilon)}) + O(\varepsilon^2\|\overline{\eta}\|_{L^\infty(G_\varepsilon)}). \]
(4.28)
Now, we show the following claim:
Claim 1. It holds that

\[
c \int_{I^*_y} \psi (\overline{w}_{\overline{r}, \overline{t}} + \overline{\phi})^2 \eta dy = K_j(0) + O(\varepsilon \| \eta \|_{L^\infty(G)}) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) \tag{4.29}
\]

and

\[
c \int_{I^*_y} \psi (\overline{w}_{\overline{r}, \overline{t}} + \overline{\phi})^2 \eta dy = K_j(\varepsilon^{-1} \lambda_j) + O(\varepsilon \| \eta \|_{L^\infty(G)}) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}). \tag{4.30}
\]

where

\[
K_j(y) := \begin{cases} (a_1 + a_2) \eta(y), & \text{when } e_1 = e_2, \\ a_j \eta_y(y), & \text{when } e_1 \neq e_2, \end{cases} \tag{4.31}
\]

Also, \( a_j \) is defined in Lemma 4.1.

Proof of Claim 1. Note that

\[
\eta \in y - \eta(0) = \int_0^y \eta'(z) dz = O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) , \quad e \in E. \tag{4.32}
\]

Thus, by Lemma 4.1, we can calculate

\[
c \int_{I^*_y} \psi (\overline{w}_{\overline{r}, \overline{t}} + \overline{\phi})^2 \eta dy
\]

\[
eq c \eta(0) \int_{I^*_y} \psi (\overline{w}_{\overline{r}, \overline{t}} + \overline{\phi})^2 dy + c \int_{I^*_y} \psi (\overline{w}_{\overline{r}, \overline{t}} + \overline{\phi})^2 O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) dy
\]

\[
eq K_j(0) + c \eta_j(0) O(\varepsilon) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)})
\]

\[
eq K_j(0) + O(\varepsilon \| \eta \|_{L^\infty(G)}) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}). \tag{4.33}
\]

Also, \( \overline{\eta}(\varepsilon^{-1} \lambda_j) - \overline{\eta}(0) = \int_0^{\varepsilon^{-1} \lambda_j} \overline{\eta}'(z) dz = O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) \) and Lemma 4.1, we obtain (4.30). Hence, Claim 1 is verified.

Applying Claim 1 to (4.23), for \( y = 0, \varepsilon^{-1} \lambda_j \), we have

\[
\varepsilon^{-1}(\overline{\eta}(0) - \overline{\eta}(\varepsilon^{-1} \lambda_j)) + D^{-1} K_j(y)
\]

\[
= O(\| \overline{\eta} \|_{L^1(G)} ) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) + O(\varepsilon \| \eta \|_{L^\infty(G)}). \tag{4.34}
\]

Next, we give the following claim:

Claim 2. It holds that

\[
|\overline{\eta}(y)| \leq C(\varepsilon + \frac{1}{4m}) \| \eta \|_{L^\infty(G)}, \tag{4.35}
\]

where \( m \) is the number of edges of \( G \).

Proof of Claim 2. We divide the proof into three cases:

Case 1. Let us assume \( e_1 = e_2 = e \), and let \( v_1 \) and \( v_2 \) be vertices of \( e \). Employing (4.27), (4.28), and (4.34), we deduce

\[
\varepsilon^{-1} \sum_{e^* \in \varepsilon e^* \varepsilon v_1} \frac{d \overline{\eta}}{d y_e^*}(\frac{v_1}{\varepsilon}) + \varepsilon^{-1} \sum_{e^* \in \varepsilon e^* \varepsilon v_2} \frac{d \overline{\eta}}{d y_e^*}(\frac{v_2}{\varepsilon}) + (a_1 + a_2) \overline{\eta}(\varepsilon^{-1} v)
\]

\[
= O(\| \overline{\eta} \|_{L^1(G)} ) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) + O(\varepsilon \| \eta \|_{L^\infty(G)}), \tag{4.36}
\]

for \( v = v_1, v_2 \). By the Kirchhoff condition, the estimate above implies

\[
\overline{\eta}(\varepsilon^{-1} v) = O(\| \overline{\eta} \|_{L^1(G)} ) + O(\varepsilon^{-1/2} \| \eta \|_{L^2(G)}) + O(\varepsilon \| \eta \|_{L^\infty(G)}). \tag{4.37}
\]
Hence, using (4.22), (4.32), and the estimate above, we can estimate
\[
|\overline{\eta}(y)| \leq |\overline{\eta}(y) - \overline{\eta}(0)| + |\overline{\eta}(0)|
\]
\[
\leq C|\overline{\eta}|_{L^1(G_0)} + C\varepsilon^{-1/2}|\overline{\eta'}|_{L^2(G_0)} + C\varepsilon |\overline{\eta}|_{L^\infty(G_0)}
\]
\[
\leq C|\overline{\eta}|_{L^1(G_0)} + C|\overline{\eta}|_{L^1(G_0)}^{1/2} |\overline{\eta}|_{L^\infty(G_0)}^{1/2} + C\varepsilon |\overline{\eta}|_{L^\infty(G_0)}
\]
\[
\leq C|\overline{\eta}|_{L^1(G_0)} + \left( C\varepsilon + \frac{1}{4m} \right) |\overline{\eta}|_{L^\infty(G_0)}.
\] (4.38)

**Case 2.** Let us assume that \(e_1 \neq e_2\), and \(e_1\) and \(e_2\) have a common vertex. The edge \(e_j\) join two vertices \(v\) and \(v_j\). From (4.27), (4.28), and (4.34), it follows that
\[
\varepsilon^{-1} \sum_{\varepsilon^{-1}e \succ \varepsilon^{-1}v_1} \frac{d\overline{\eta}_{e\varepsilon}}{dy_{e\varepsilon}} \left( \frac{v_1}{\varepsilon} \right) + \varepsilon^{-1} \sum_{\varepsilon^{-1}e \succ \varepsilon^{-1}v_2} \frac{d\overline{\eta}_{e\varepsilon}}{dy_{e\varepsilon}} \left( \frac{v_2}{\varepsilon} \right)
\]
\[
+ \varepsilon^{-1} \sum_{\varepsilon^{-1}e \succ \varepsilon^{-1}v} \frac{d\overline{\eta}_{e\varepsilon}}{dy_{e\varepsilon}} \left( \frac{v}{\varepsilon} \right) + a_1 \overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v) + a_2 \overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v)
\]
\[
= O(|\overline{\eta}|_{L^1(G_0)}) + O(\varepsilon^{-1/2} |\overline{\eta'}|_{L^2(G_0)}) + O(\varepsilon |\overline{\eta}|_{L^\infty(G_0)}).
\] (4.39)

Since \(v\) is a common vertex, we have \(\overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v) = \overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v)\). Thus, by the Kirchhoff condition, we obtain
\[
\overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v) = O(|\overline{\eta}|_{L^1(G_0)}) + O(\varepsilon^{-1/2} |\overline{\eta'}|_{L^2(G_0)}) + O(\varepsilon |\overline{\eta}|_{L^\infty(G_0)}).
\] (4.40)

Hence, by the same calculation as that in Case 1, we conclude (4.35).

**Case 3.** Let us assume that \(e_1 \neq e_2\), and \(e_1\) and \(e_2\) don’t have a common vertex. The edge \(e_j\) join two vertices \(v_j^1\) and \(v_j^2\). From (4.27), (4.28), and (4.34), it follows that
\[
\varepsilon^{-1} \sum_{\varepsilon^{-1}e \succ \varepsilon^{-1}v_j^1} \frac{d\overline{\eta}_{e\varepsilon}}{dy_{e\varepsilon}} \left( \frac{v_j^1}{\varepsilon} \right) + \varepsilon^{-1} \sum_{\varepsilon^{-1}e \succ \varepsilon^{-1}v_j^2} \frac{d\overline{\eta}_{e\varepsilon}}{dy_{e\varepsilon}} \left( \frac{v_j^2}{\varepsilon} \right) + a_j \overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v)
\]
\[
= O(|\overline{\eta}|_{L^1(G_0)}) + O(\varepsilon^{-1/2} |\overline{\eta'}|_{L^2(G_0)}) + O(\varepsilon |\overline{\eta}|_{L^\infty(G_0)}).
\] (4.41)

for \(v = v_j^1, v_j^2\). Hence, by the same calculation as that in Case 1, we conclude (4.35). Thus we complete the proof of Claim 2.

Now, let us assume that two edges \(e_j\) and \(e \neq e_j\) have a common vertex \(v\). Since \(\overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v) = \overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v)\), by using Claim 2, we see that
\[
|\overline{\eta}(\varepsilon^{-1}v)| = |\overline{\eta}_{e\varepsilon}(\varepsilon^{-1}v)| \leq C|\overline{\eta}|_{L^1(G_0)} + \left( C\varepsilon + \frac{1}{4m} \right) |\overline{\eta}|_{L^\infty(G_0)}.
\] (4.42)

Also, from (4.32) and (4.22), we notice that
\[
|\overline{\eta}(y) - \overline{\eta}(\varepsilon^{-1}v)| \leq C\varepsilon^{-1/2} |\overline{\eta'}|_{L^2(G_0)}
\]
\[
\leq C|\overline{\eta}|_{L^1(G_0)}^{1/2} |\overline{\eta}|_{L^\infty(G_0)}^{1/2} \leq \left( C\varepsilon + \frac{1}{4m} \right) |\overline{\eta}|_{L^\infty(G_0)}.
\] (4.43)

Hence, combining (4.42) and (4.43), we obtain
\[
|\overline{\eta}(y)| \leq |\overline{\eta}(y) - \overline{\eta}(\varepsilon^{-1}v)| + |\overline{\eta}(\varepsilon^{-1}v)|
\]
\[
\leq C|\overline{\eta}|_{L^1(G_0)} + \left( C\varepsilon + \frac{1}{4m} + \frac{1}{4m} \right) |\overline{\eta}|_{L^\infty(G_0)}.
\] (4.44)
Applying the same argument as that in the above into each edge $e \in E$, we deduce
\[
|\eta(y)| \leq |\eta(x) - \eta(x) + |\eta(y)| | + |\eta(y)| | \leq C\|\eta\|_{L^1(G_e)} + \left( C\|\sum_{k=1}^{m} \frac{1}{4k} \|L\|_{L^1(G_e)} \right) \|\eta\|_{L^\infty(G_e)} \tag{4.45}
\]
and hence
\[
\|\eta\|_{L^\infty(G_e)} \leq C\|\eta\|_{L^1(G_e)} + \left( C\|\sum_{k=1}^{m} \frac{1}{4k} \|L\|_{L^1(G_e)} \right) \|\eta\|_{L^\infty(G_e)}. \tag{4.46}
\]
Therefore, taking $\epsilon > 0$ sufficiently small, we conclude
\[
\|\eta\|_{L^\infty(G_e)} \leq C\|\eta\|_{L^1(G_e)}. \tag{4.47}
\]
Moreover, substituting the estimate above into (4.22), we have
\[
\|\eta\|_{L^2(G_e)} \leq C\sqrt{\|\eta\|_{L^1(G_e)}}. \tag{4.48}
\]
Thus we complete the proof of Lemma 4.2. \qed

5. Existence of multi-peak solution on metric graphs.

5.1. Strategy and outline of the proof of existence theorem. For the existence of a one-peak solution (Theorem 1.1), we can use the same argument as that in the proof of the existence of a one-peak solution on the interval $(−1, 1)$ (see [11, Theorem 1.1].) We can also apply its argument to the existence of a two-peak solution (Theorem 1.2). Thus we mainly explain the strategy and the outline of the proof of Theorem 1.2. For each $t \in P(r_0[2])$, we take $u(x) = w_{x,t}(x) + \phi$ with $\phi \in \mathcal{B}(C_0)$. Then, for sufficiently small $\epsilon > 0$, we can find a unique solution $v = T[u] = T[w_{x,t} + \phi]$ of the second equation of (1.2). For the definition of $T$ and the fundamental properties in details, see Subsection 5.2. To show Theorem 1.2, we find a pair of $\phi \in H^2_N(G)$ and $t \in P(r_0[2])$ satisfying the first equation of (1.2). To this end, we need the Liapunov–Schmidt reduction method and Brouwer’s fixed point theorem.

Now we introduce several operators and function spaces. We define the operator $S_e[\eta]$ as follows:
\[
S_e[\eta] := \epsilon^2 u'' - u + gT[w(u)]u^2, \quad u \in H^2_N(G). \tag{5.1}
\]
For $\phi \in H^2(G)$, we denote by $R_{e,t}[\phi]$ a unique solution of
\[
\begin{cases}
-DR_{e,t}[\phi]'' + \frac{c}{\epsilon} gw_{c,t} R_{c,t}[\phi] = -\frac{2c}{\epsilon} gT[w_{c,t}] w_{c,t} \phi, & x \in G, \\
\sum_{e \in V} \frac{dR_{e,t}[\phi]}{dx} = 0, & v \in V. \tag{5.2}
\end{cases}
\]
Then, it holds that $R_{e,t}[\phi] = \langle T'[w_{e,t}], \phi \rangle$, where $\langle T'[w_{e,t}], \phi \rangle$ is the Fréchet derivative of $T$ at $w_{e,t}$ (see [14, Proposition 3.1].) Moreover, we obtain
\[
\begin{cases}
T[w_{e,t} + \phi] = T[w_{e,t}] + R_{e,t}[\phi] + N_t[\phi], \\
N_t[\phi] = o(||\phi||_{H^2(G)}) \text{ as } ||\phi||_{H^2(G)} \to 0. \tag{5.3}
\end{cases}
\]
We define the linear operator $L_{e,t}[\phi]$ as follows:
\[
L_{e,t}[\phi] := \phi'' - \phi + 2\tilde{g} T[w_{e,t}] \tilde{w}_{e,t} \phi + \tilde{g} R_{e,t}[\phi] \tilde{w}_{e,t}^2, \quad \phi \in H^2_N(G_e). \tag{5.4}
\]
Hence, by (5.3) and (5.4), we have
\[
S_e[w_{e,t} + \phi] = S_e[w_{e,t}] + L_{e,t}[\phi] + N_{e,t}[\phi], \tag{5.5}
\]
To this end, we will carry out the following two steps:

Step 1. Let $t \in P(r_0[2])$. We will find $\phi \in H^2_N(G)$ which solves the problem
\begin{equation}
\pi_{e,t}^+[N_1,t(\phi)] = 0. \tag{5.13}
\end{equation}
By using Lemma 5.5 which will be proven in Subsection 5.3, we see that $L_{e,t}$ is invertible for sufficiently small $e > 0$. Thus, the problem above is equivalent to
\begin{equation}
\bar{\phi} = -(L_{e,t}^+)^{-1} \circ \pi_{e,t}^+ [N_1,t(\phi)] - (L_{e,t}^+)^{-1} \circ \pi_{e,t}^+ [N_1,t(\phi)] =: M_{e,t}[\phi], \tag{5.14}
\end{equation}
where we used (5.5). By using the contraction mapping principle, we will show that, for each $t \in P(r_0[2])$, there exists a unique $\bar{\phi}_{e,t} \in B(C_0) \cap C_{e,t}$ such that $M_{e,t}[\bar{\phi}_{e,t}] = \bar{\phi}_{e,t}$. Then, $\bar{\phi}_{e,t}$ is continuous with respect to $t$.

Step 2. From Step 1, we have
\begin{equation}
\bar{S}_e[w_{e,t} + \phi_{e,t}] \in C_{e,t} \tag{5.15}
\end{equation}
for any $t \in P(r_0[2])$. By using Brouwer's fixed point theorem, we will find a point $\bar{t}^e \in B(0^3/4, t^0)$ such that $\bar{S}_e[w_{e,t^e} + \phi_{e,t^e}] \perp C_{e,t^e}$, namely
\begin{equation}
\int_{G_e} \bar{S}_e[w_{e,t^e} + \phi_{e,t^e}] w_{e,t^e} dy = 0. \tag{5.16}
\end{equation}
Thus we conclude $S_e[w_{e,t^e} + \phi_{e,t^e}] = 0$. 

We introduce the approximate kernel and co-kernel of $L_{e,t}$, respectively, as follows:
\begin{align}
K_{e,t} & := \text{span}\{\xi_j(t)w_{e,t_j} | j = 1, 2\} \subset H^2_N(G_e), \tag{5.7} \\
C_{e,t} & := \text{span}\{\xi_j(t)w_{e,t_j} | j = 1, 2\} \subset L^2(G_e). \tag{5.8}
\end{align}
Moreover, we put
\begin{align}
K_{e,t}^+ & := \left\{ f \in H^2_N(G_e) | \int_{G_e} f w_{e,t_j} dy = 0, j = 1, 2 \right\} \tag{5.9} \\
C_{e,t}^+ & := \left\{ f \in L^2(G_e) | \int_{G_e} f w_{e,t_j} dy = 0, j = 1, 2 \right\}. \tag{5.10}
\end{align}
We also define the projection $\pi_{e,t}^+: L^2(G_e) \to C_{e,t}^+$ as follows:
\begin{equation}
\pi_{e,t}^+[f] := f - \frac{2}{\|w_{e,t_j}\|_{L^2(G_e)}} w_{e,t_j}, \quad f \in L^2(G_e). \tag{5.11}
\end{equation}
Finally, we define the operator $L_{e,t}^+: K_{e,t}^+ \to C_{e,t}^+$ by
\begin{equation}
L_{e,t}^+ := \pi_{e,t}^+ \circ L_{e,t}. \tag{5.12}
\end{equation}
To construct the solution of (1.2), it suffices to solve the equation $S_e[w_{e,t} + \phi] = 0$. 

where
\begin{equation}
N_{1,t}(\phi) := \gamma T[w_{e,t}] \phi^2 + \gamma(2w_{e,t} \phi + \phi^2)R_{e,t}(\phi) + \gamma(w_{e,t} + \phi)^2N_t(\phi). \tag{5.6}
\end{equation}
5.2. Study of the non-local operator \( T \) and its Fréchet derivative. In this subsection, we establish the several estimates for non-local operator \( T \) and the \( R_{\varepsilon,t} \), which is its Fréchet derivative at \( w_{\varepsilon,t} \). Let \( u = w_{\varepsilon,t} + \phi \in H^1_0(\mathcal{G}) \) with \( \phi \in \mathcal{B}(C_0) \) and let \( T[u] \) be the unique solution of the following solution (see [14, Subsection 3.1.]):

\[
\begin{cases}
-DT[u]^\varepsilon + \frac{c}{\varepsilon} gu^2 T[u] = \frac{1}{L}, & x \in \mathcal{G}, \\
\sum_{\varepsilon < v} dT[u]_v (v) = 0, & v \in V.
\end{cases}
\]

(5.17)

First, for the non-local operator \( T \), we have the two following lemmas.

**Lemma 5.1.** We have the following estimates:

1. \( ||T[w_{\varepsilon,t} + \phi]||_{L^\infty(\mathcal{G}_t)} \leq C \).
2. \( T[w_{\varepsilon,t} + \phi](t_j) = \xi_j(t) + O(\varepsilon) \).
3. \( (T[w_{\varepsilon,t} + \phi])^{-\varepsilon}(y) = \xi_j(t) + O(\varepsilon|y|) + O(\varepsilon) \) for \( y \in I_{\varepsilon,r_0} \).
4. \( \varrho_j^{-\varepsilon}(y)(T[w_{\varepsilon,t} + \phi])^{-\varepsilon}(y) = g(t_j) \xi_j(t) + O(\varepsilon|y|) + O(\varepsilon) \) for \( y \in I_{\varepsilon,r_0} \).

**Proof.** Applying Lemma 4.2 to \( \eta := T[w_{\varepsilon,t} + \phi] \) and \( h := \varepsilon/L \), by \( ||T||_{L^\infty(\mathcal{G}_t)} \) we have \( ||T[w_{\varepsilon,t} + \phi]||_{L^\infty(\mathcal{G}_t)} \leq C \). Moreover, by Lemma 4.3 and \( \int_{\mathcal{G}_t} \tilde{h}dy = 1 \), it holds that

\[
(T[w_{\varepsilon,t} + \phi])^{-\varepsilon}(y) = T[w_{\varepsilon,t} + \phi](t_j) + O(\varepsilon|y|)
\]

(5.18)

and

\[
\varrho_j^{-\varepsilon}(y)(T[w_{\varepsilon,t} + \phi])^{-\varepsilon}(y) = g(t_j) T[w_{\varepsilon,t} + \phi](t_j) + O(\varepsilon|y|)
\]

(5.19)

for \( y \in I_{\varepsilon,r_0} \). If (2) is shown, by combining (2) and the two formulas above, the proof of Lemma 5.1 is finished. Thus we concentrate on the proof of (2). For simplicity, we put

\[
T(x) := T[w_{\varepsilon,t} + \phi](x), \quad \tau_j := T[w_{\varepsilon,t} + \phi](t_j).
\]

(5.20)

First, integrating (5.17) over \( \mathcal{G} \) and letting \( x = \varepsilon y \), we have \( c \int_{\mathcal{G}_t} \varrho(\tilde{w}_{\varepsilon,t} + \tilde{\varphi})^2 Tdy = 1 \). By using (5.19), it holds that

\[
c \int_{\mathcal{G}_t} \varrho(\tilde{w}_{\varepsilon,t} + \tilde{\varphi})^2 Tdy = c \sum_{j=1}^{2} \int_{I_{\varepsilon,r_0}} \varrho_j^2(\tilde{w}_{\varepsilon,t})^j (T)^{-\varepsilon} dy + O(\varepsilon) = \sum_{j=1}^{2} a_j \tau_j + O(\varepsilon).
\]

Thus we obtain

\[
\sum_{j=1}^{2} a_j \tau_j + O(\varepsilon) = 1.
\]

(5.21)

In the case of a one-peak solution, the formula above becomes

\[
\frac{6c}{g(t)|\xi(t)|^2} T[w_{\varepsilon,t} + \phi](t) + O(\varepsilon) = 1.
\]

(5.22)

Then, since \( 6cg(t)^{-1} \xi(t)^{-1} = 1 \), we conclude \( T[w_{\varepsilon,t} + \phi](t) = \xi(t) + O(\varepsilon) \) immediately.

Thus we consider the case of a two-peak solution below. Since \( T(x) \) satisfies (5.17), by using (1.6), we can represent \( T \) as follows:

\[
T(x) - \frac{1}{L} \int_{\mathcal{G}} T(s)ds = c \int_{\mathcal{G}} G(x,s) g(s)(w_{\varepsilon,t}(s) + \phi(s))^2 T(s)ds.
\]

(5.23)

Since \( ||T||_{L^\infty(\mathcal{G}_t)} \leq C \) and \( ||\varrho||_{L^2(\mathcal{G}_t)} \leq C\varepsilon \), we have

\[
\tau_j - \frac{1}{L} \int_{\mathcal{G}} Tds = c \int_{\mathcal{G}_t} G(t_j, \varepsilon \varphi) \varrho w_{\varepsilon,t}^2 Tdz + O(\varepsilon).
\]

(5.24)
Here, note that the condition (G3) implies \( G(t_j, \varepsilon z + t_k) = G(t_j, t_k) + (\varepsilon |z|) \) for \( z \in I_{\varepsilon, r_0} \). Hence, using this estimate, (4.9), and (5.19), we can calculate

\[
c \int_{G} G(t_j, \varepsilon z) \overline{g} \overline{w}^{-2} T dz = \frac{2 \sum_{j=1}^{2} G(t_j, \varepsilon z + t_j) \overline{g} \overline{w}^{-2} (T)^{-j} dz}{G(t_j, t_k) g(t_k) \tau_k \int_{I_{\varepsilon, r_0}} (w^{-j})^2 dz + O(\varepsilon)} = 6c \sum_{k=1}^{2} \frac{G(t_j, t_k)}{g(t_k) \xi_k(t)} \tau_k + O(\varepsilon).
\]

Thus, combining (5.24) and (5.25), we deduce

\[
\tau_1 - \tau_2 = \sum_{j=1}^{2} a_j \sigma_j(t) \tau_j + O(\varepsilon),
\]

where \( a_j := 6c g(t_j)^{-1} \xi_j(t)^{-2} \) and \( \sigma_j(t) := G(t_1, t_j) - G(t_2, t_j) \). Recall that \( \xi_1(t) \) and \( \xi_2(t) \) satisfy

\[
\xi_1(t) - \xi_2(t) = \frac{6c}{2} \sum_{k=1}^{2} \frac{\sigma_k(t)}{g(t_k) \xi_k(t)} \sum_{k=1}^{2} \frac{6c}{g(t_k) \xi_k(t)} = 1
\]

(see (1.20).) Combining (5.26), (5.21) and the two equations above, we have

\[
\hat{\tau}_1 - \hat{\tau}_2 = \sum_{j=1}^{2} a_j \sigma_j(t) \hat{\tau}_j + O(\varepsilon), \quad \sum_{j=1}^{2} a_j \hat{\tau}_j = O(\varepsilon), \quad \hat{\tau}_j := \tau_j - \xi_j(t)
\]

and hence

\[
A(t)(\hat{\tau}_1, \hat{\tau}_2)^T = O(\varepsilon),
\]

where the matrix \( A(t) \) is defined by

\[
A(t) := \begin{pmatrix} 1 - \sigma_1(t) a_1 & -1 - \sigma_2(t) a_2 \\ a_1 & a_2 \end{pmatrix}.
\]

The condition (G5) implies

\[
\det A(t) = a_1 + a_2 + a_1 a_2 (\sigma_2(t) - \sigma_1(t)) > 0
\]

for \( t \in P(r_0[2]) \). Thus \( A(t) \) is a regular matrix and hence we conclude \( \hat{\tau}_j = \tau_j - \xi_j(t) = O(\varepsilon) \). Hence, we finish the proof of (2). \( \square \)

**Lemma 5.2** (e.g. [11, Lemma 3.2]). We have the two following formulas:

1. In the case of a one-peak solution, it holds that

\[
(T[w_{\varepsilon, t}]) \sim (y) - (T[w_{\varepsilon, t}]) \sim (0) = \varepsilon y m(t) + \kappa(y) + O(\varepsilon^2 |y|),
\]

where \( m(t) = m_{11}(t, t) \), and

\[
\kappa(y) := c g(t) \xi(t) \int_{I_{\varepsilon, r_0}} K_{11}(\varepsilon y, \varepsilon z) (w^{-2}) dz.
\]

is an even function on \( I_{\varepsilon, r_0} \) and satisfies \( \kappa(y) = O(\varepsilon |y|) \). (2) In the case of a two-peak solution, it holds that

\[
(T[w_{\varepsilon, t}]) \sim (y) - (T[w_{\varepsilon, t}]) \sim (0) = 6c \varepsilon y \sum_{k=1}^{2} \frac{m_{jk}(t)}{g(t_k) \xi_k(t)} + \kappa_j(y) + O(\varepsilon^2 |y|),
\]

where \( m_{jk}(t) \) is a symmetric matrix and \( \kappa_j(y) \) is an even function on \( I_{\varepsilon, r_0} \) and satisfies \( \kappa_j(y) = O(\varepsilon |y|) \).
where
\[
\kappa_j(y) := c \sum_{k=1}^{2} g(t_k) \xi_k(t) \int_{I_{\varepsilon, r_0}} K_{jk}(\varepsilon y, \varepsilon z) (\tilde{w}_{\varepsilon, t_k})^2 dz.
\] (5.35)
is an even function on $I_{\varepsilon, r_0}$ and satisfies $\kappa_j(y) = O(\varepsilon |y|)$.

**Remark 10.** The property of $\kappa(y)$ is derived from $(G4)_{\varepsilon, \varepsilon}$. Moreover, for each $j$, the property of $\kappa_j(y)$ is derived from $(G4)_{\varepsilon_j, \varepsilon_k}$ ($k = 1, 2$).

Next, for the Fréchet derivative $R_{\varepsilon, t}$, we have the two following lemmas.

**Lemma 5.3.** Let us take $\phi \in H^2(\mathcal{G})$. Then, we have the following estimates:

1. $||R_{\varepsilon, t}[\phi]||_{L^\infty(\mathcal{G})} \leq C ||\tilde{\phi}||_{L^2(\mathcal{G})}$.

2. $(R_{\varepsilon, t}[\phi])^{-1}(y) = R_{\varepsilon, t}[\phi](t_j) + O(\varepsilon |y| ||\tilde{\phi}||_{L^2(\mathcal{G})})$ for $y \in I_{\varepsilon, r_0}$.

3. $(R_{\varepsilon, t}[\phi])^{-1}(y) \tilde{\phi}(y) = R_{\varepsilon, t}[\phi](t_j) g(t_j) + O(\varepsilon |y| ||\tilde{\phi}||_{L^2(\mathcal{G})})$ for $y \in I_{\varepsilon, r_0}$.

**Proof.** Let $h := -2c g \int w_{\varepsilon, t} \tilde{w}_{\varepsilon, t} \tilde{\phi}$. By $||\tilde{h}||_{L^\infty(\mathcal{G})} \leq C$ and Hölder’s inequality, we get $||h||_{L^2(\mathcal{G})} \leq C ||\tilde{\phi}||_{L^2(\mathcal{G})}$. Thus, by applying Lemma 4.2 and Lemma 4.3 to the equation (5.2), we obtain (1)–(3).

Here, let us define the matrix $P(t)$ as follows:
\[
P(t) := -\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} A(t)^{-1} \begin{pmatrix} \sigma_1(t) & \sigma_2(t) \\ -1 & -1 \end{pmatrix},
\] (5.36)
where the matrix $A(t)$ is defined by (5.30).

**Lemma 5.4.** We have the following results:

1. *In the case of a one-peak solution, the asymptotic behavior of $R_{\varepsilon, t}[\phi](t)$ is given by*
\[
R_{\varepsilon, t}[\phi](t) = -2cg(t) \xi(t)^2 \int_{I_{\varepsilon, r_0}} \tilde{w}_{\varepsilon, t}(y) \tilde{\phi}^{-1}(y) dy + O(\varepsilon ||\tilde{\phi}||_{L^2(\mathcal{G})}).
\] (5.37)

2. *In the case of a two-peak solution, the asymptotic behavior of $R_{\varepsilon, t}[\phi](t_j)$ is given by*
\[
\frac{1}{6c} \begin{pmatrix} a_1 R_{\varepsilon, t}[\phi](t_1) \\ a_2 R_{\varepsilon, t}[\phi](t_2) \end{pmatrix} = -2 \int_{I_{\varepsilon, r_0}} P(t) \Phi_\varepsilon(y) dy \int \tilde{w}^2 dy + O(\varepsilon ||\tilde{\phi}||_{L^2(\mathcal{G})}),
\] (5.38)
where
\[
\Phi_\varepsilon(y) := \begin{pmatrix} g(t_1) \xi(t_1) \tilde{w}_{\varepsilon, t_1}^{-1}(y) \tilde{\phi}^{-1}(y) \\ g(t_2) \xi(t_2) \tilde{w}_{\varepsilon, t_2}^{-2}(y) \tilde{\phi}^{-2}(y) \end{pmatrix}.
\] (5.39)

3. *The matrix $P(t)$ can be diagonalized. Moreover, it holds that det$(I - 2P(t^0)) \neq 0$.*

**Proof.** Since we can use the same argument as that in the proof of Lemma 3.3 in [11] for the case of a one-peak solution, we omit the proof of (1).

Next, let us show (2). For simplicity, we write
\[
R_\varepsilon(x) := R_{\varepsilon, t}[\phi](x), \quad \tau_j := R_{\varepsilon, t}[\phi](t_j).
\] (5.40)

Since $R_\varepsilon(x)$ satisfies (5.2), by using (1.6), we have
\[
R_\varepsilon(x) - \frac{1}{L} \int_{\mathcal{G}} R_\varepsilon(s) ds
= \int_{\mathcal{G}} G(x, s) \left( \frac{c}{\varepsilon} g(s) w_{\varepsilon, t}(s)^2 R_\varepsilon(s) + \frac{2c}{\varepsilon} g(s) w_{\varepsilon, t}(s) T[w_{\varepsilon, t}](s) \phi(s) \right) ds
\]
Moreover, combining Lemma 5.3 and By using Lemma 5.1, it holds that

\[
K_1 = c \sum_{j=1}^{2} \int_{I_{\varepsilon,\bar{r}_0}} G(x, \varepsilon z + t_j) \tilde{g}_j^i (\tilde{w}_{\varepsilon, t_j})^2 R_{\varepsilon}^{-j} \, dz
\]

\[
= c \sum_{j=1}^{2} g(t_j) \tau_j \sum_{I_{\varepsilon,\bar{r}_0}} G(x, \varepsilon z + t_j) (\tilde{w}_{\varepsilon, t_j})^2 \, dz + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)})
\]

\[
= \sum_{j=1}^{2} a_j \tau_j G(x, t_j) + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)}). \quad (5.42)
\]

Moreover, by Lemma 5.3, we have

\[
K_2 = 2c \sum_{j=1}^{2} \int_{I_{\varepsilon,\bar{r}_0}} \tilde{g}_j^i (T[w_{\varepsilon, t_j}])^{-j} \tilde{w}_{\varepsilon, t_j}^j \phi^{-j} G(x, \varepsilon z + t_j) \, dz
\]

\[
= 2c \sum_{j=1}^{2} g(t_j) \xi_j(t) G(x, t_j) \int_{I_{\varepsilon,\bar{r}_0}} \tilde{w}_{\varepsilon, t_j}^j \phi^{-j} \, dz + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)}). \quad (5.43)
\]

Substituting (5.42) and (5.43) into (5.41), we obtain

\[
R_{\varepsilon}(x) = \frac{1}{L} \int_{\mathcal{G}} R_{\varepsilon}(s) \, ds
\]

\[
= \sum_{j=1}^{2} a_j \tau_j G(x, t_j) + 2c \sum_{j=1}^{2} g(t_j) \xi_j(t) G(x, t_j) \int_{I_{\varepsilon,\bar{r}_0}} \tilde{w}_{\varepsilon, t_j}^j \phi^{-j} \, dz + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)})
\]

and hence

\[
\tau_1 - \tau_2 = R_{\varepsilon}(t_1) - R_{\varepsilon}(t_2)
\]

\[
= \sum_{j=1}^{2} a_j \tau_j \sigma_j(t) + 2c \sum_{j=1}^{2} g(t_j) \xi_j(t) \sigma_j(t) \int_{I_{\varepsilon,\bar{r}_0}} \tilde{w}_{\varepsilon, t_j}^j \phi^{-j} \, dz + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)}). \quad (5.44)
\]

On the other hand, integrating (5.2) over \( \mathcal{G} \) and letting \( x = \varepsilon y + t_j \), we deduce

\[
c \sum_{j=1}^{2} \int_{I_{\varepsilon, \bar{r}_0}} \tilde{g}_j^i (\tilde{w}_{\varepsilon, t_j})^2 R_{\varepsilon}^{-j} \, dy = -2c \sum_{j=1}^{2} \int_{I_{\varepsilon, \bar{r}_0}} \tilde{g}_j^i \tilde{w}_{\varepsilon, t_j}^j (T[w_{\varepsilon, t_j}])^{-j} \phi^{-j} \, dy. \quad (5.45)
\]

By using Lemma 5.1, it holds that

\[
\text{(RHS of (5.45))} = -2c \sum_{j=1}^{2} g(t_j) \xi_j(t) \int_{I_{\varepsilon, \bar{r}_0}} \tilde{w}_{\varepsilon, t_j}^j \phi^{-j} \, dz + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)}). \quad (5.46)
\]

Moreover, combining Lemma 5.3 and \( \alpha_j := 6c g(t_j)^{-1} \xi_j(t)^{-2} \), we obtain

\[
\text{(LHS of (5.45))} = \sum_{j=1}^{2} a_j \tau_j + O(\varepsilon \| \tilde{\varphi} \|_{L^2(G_1)}). \quad (5.47)
\]
Hence, by the two formulas above, we see that
\[
\sum_{j=1}^{2} a_j \tau_j = -2c^2 \sum_{j=1}^{2} g(t_j) \xi_j(t) \int_{t_{x,t_j}^j} \omega_{x,t_j}^j \phi_j^j dz + O(\|\vec{\phi}\|_{L^2(G_x)}). \tag{5.48}
\]
Therefore, combining (5.44) and (5.48), we have
\[
R_{||\vec{\phi}\|_{L^2(G_x)}} = (\tau_1 \tau_2) = 2c \begin{pmatrix} \sigma_1(t) & \sigma_2(t) \\ -1 & -1 \end{pmatrix} \int_{t_{x,r_0}} \Phi_\varepsilon(y) dy + O(\varepsilon^2)_{L^2(G_x)}, \tag{5.49}
\]
where
\[
\Phi_\varepsilon(y) := \begin{pmatrix} g(t_1) \xi_1(t) \omega_{x,t_1}^{1-1}(y) \phi_1^{1}(y) \\ g(t_2) \xi_2(t) \omega_{x,t_2}^{1}(y) \phi_2^{1}(y) \end{pmatrix}
\]
To both sides of the formula above, multiplying the matrix
\[
\frac{1}{6c} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} A(t)^{-1}
\]
and using \( \int_{E} \omega^2 dy = 6 \), we conclude (5.38).

We finally show (3). First, it is easy to see that \( P(t) \) has two eigenvalues 1 and \( 1 - (a_1 + a_2) / \det A(t) \). Since \( (a_1 + a_2) / \det A(t) > 0 \), we have \( 1 - (a_1 + a_2) / \det A(t) \neq 1 \) and hence \( P(t) \) can be diagonalized. Next, note that
\[
I - 2P(t) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} A(t)^{-1} X(t), \tag{5.50}
\]
where
\[
X(t) := A(t) \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{pmatrix} + 2 \begin{pmatrix} \sigma_1(t) & \sigma_2(t) \\ -1 & -1 \end{pmatrix}. \tag{5.51}
\]
Here, \( X(t) \) becomes
\[
X(t) = \begin{pmatrix} a_1^{-1} + \sigma_1(t) & -a_2^{-1} + \sigma_2(t) \\ -1 & -1 \end{pmatrix}. \tag{5.52}
\]
Hence, we can calculate
\[
\det(I - 2P(t^0)) = (\det A(t^0))^{-1} \det \left[ X(t^0) \begin{pmatrix} a_1^0 & 0 \\ 0 & a_2^0 \end{pmatrix} \right] = - (\det A(t^0))^{-1} \det A(t^0, \xi^0) \neq 0, \tag{5.53}
\]
where we used the assumption (A2). Thus we finish the proof of (3). \( \square \)

5.3. The Liapunov–Schmidt reduction method. In this subsection, by using the Liapunov–Schmidt reduction method, we will solve the problem (5.13). We first show the invertibility of \( L_{x,t}^\perp \).

**Lemma 5.5.** There exist \( \varepsilon_0 > 0 \) and \( \lambda > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0) \) and any \( t \in P(t_0) \), the following inequality holds:
\[
||L_{x,t}^\perp||_{L^2(G_x)} \geq \lambda ||\vec{\phi}||_{H^2(G_x)}, \quad \vec{\phi} \in K_{x,t}^\perp. \tag{5.54}
\]
Furthermore, \( \text{Ran}(L_{x,t}^\perp) = C_{x,t}^\perp \) holds for \( \varepsilon \in (0, \varepsilon_0) \), and hence, \( L_{x,t}^\perp : K_{x,t} \to C_{x,t}^\perp \) has a bounded inverse \( (L_{x,t}^\perp)^{-1} \). Here, we note that \( \varepsilon_0 \) and \( \lambda \) depend only on \( w(x) \) and \( g(x) \).
By using Lemma 5.4, we will show Lemma 5.5. We give its proof in details in Appendix D for reader’s convenience.

By Lemma 4.2, we obtain the following estimates for the operators $T$, $N_t$, and $N_{1,t}$. Here, $N_t$ and $N_{1,t}$ are defined (5.3) and (5.6), respectively.

**Lemma 5.6 ([14, Lemma 3.4]).** For $\phi_1, \phi_2 \in B(C_0)$ and $t \in P(r_0)$, the following estimates hold:

- $(1) \|T[w_{\varepsilon,t} + \phi_1] - T[w_{\varepsilon,t} + \phi_2]\|_{L^\infty(G_\varepsilon)} \leq C\|\phi_1 - \phi_2\|_{L^2(G_\varepsilon)}$.
- $(2) \|N_t[\phi_1] - N_t[\phi_2]\|_{L^\infty(G_\varepsilon)} \leq C(\|\phi_1\|_{L^2(G_\varepsilon)} + \|\phi_2\|_{L^2(G_\varepsilon)}) \|\phi_1 - \phi_2\|_{L^2(G_\varepsilon)}$.
- $(3) \|N_{1,t}[\phi_1] - N_{1,t}[\phi_2]\|_{L^2(G_\varepsilon)} \leq C(\|\phi_1\|_{L^2(G_\varepsilon)} + \|\phi_2\|_{L^2(G_\varepsilon)}) \|\phi_1 - \phi_2\|_{L^2(G_\varepsilon)}$.

The proof can be done by the same argument as that in the proof of Lemma 3.4 in [14]. Hence, we omit the proof of Lemma 5.6.

To solve (5.13), we find a fixed point of $M_{\varepsilon,t}$ which is defined by (5.14). We first show that $M_{\varepsilon,t}$ is a contraction mapping on $B(C_0)$.

**Proposition 5.7.** There exists $\varepsilon_\ast > 0$ such that $M_{\varepsilon,t} : B(C_0) \cap K_{\varepsilon,t} \to B(C_0) \cap K_{\varepsilon,t}$ is a contraction mapping for $\varepsilon \in (0, \varepsilon_\ast)$ and $t \in P(r_0)$.

**Proof.** We first determine the constant $C_0$. Now, from $w'' - w + w^2 = 0$ and Lemma 5.1 (4), it follows that

$$\|S_{\varepsilon}[w_{\varepsilon,t}]\|_{L^2(G_\varepsilon)} \leq C\varepsilon. \quad (5.55)$$

Let $\bar{\phi} \in B(C_0)$. By Lemma 5.5, we can estimate

$$\|M_{\varepsilon,t}[\bar{\phi}]\|_{H^2(G_\varepsilon)} \leq \lambda^{-1}(\|\pi_{\varepsilon,t}^+[S_{\varepsilon}[w_{\varepsilon,t}]]\|_{L^2(G_\varepsilon)} + \|\pi_{\varepsilon,t}^+[N_{1,t}][\bar{\phi}]\|_{L^2(G_\varepsilon)}) \leq \lambda^{-1}(C_3\varepsilon + C_4C_3^2\varepsilon^2), \quad (5.56)$$

where the constants $C_3$ and $C_4$ are positive and depend only on $g$ and $w$. Therefore, letting

$$\frac{C_0}{2} > \frac{C_3}{\lambda} \quad (5.57)$$

and $\varepsilon_1 > 0$ be small so that $\varepsilon_1 < \lambda/(2C_0C_4)$, we see that $\|M_{\varepsilon,t}[\bar{\phi}]\|_{H^2(G_\varepsilon)} \leq C_0\varepsilon$ for $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. Thus $M_{\varepsilon,t}$ is a mapping from $B(C_0)$ into itself for $0 < \varepsilon < \min(\varepsilon_0, \varepsilon_1)$. Let us show that $M_{\varepsilon,t}$ is a contraction mapping on $B(C_0) \cap K_{\varepsilon,t}$. For $\bar{\phi}_1, \bar{\phi}_2 \in B(C_0)$, by Lemma 5.5 and Lemma 5.6 (3), we estimate

$$\|M_{\varepsilon,t}[^1\bar{\phi}_1] - M_{\varepsilon,t}[^2\bar{\phi}_2]\|_{H^2(G_\varepsilon)} \leq \lambda^{-1}(\|\pi_{\varepsilon,t}^+[N_{1,t}[^1\bar{\phi}_1]]\|_{L^2(G_\varepsilon)} + \|\pi_{\varepsilon,t}^+[N_{1,t}[^2\bar{\phi}_2]]\|_{L^2(G_\varepsilon)}) \leq \lambda^{-1}(\|N_{1,t}[^1\bar{\phi}_1] - N_{1,t}[^2\bar{\phi}_2]\|_{L^2(G_\varepsilon)}) \leq 2CC_0\lambda^{-1}\varepsilon\|\bar{\phi}_1 - \bar{\phi}_2\|_{L^2(G_\varepsilon)}.$$ 

Hence, moreover letting $\varepsilon_2 > 0$ be small so that $2CC_0\lambda^{-1}\varepsilon_2 < 1$, we see that $M_{\varepsilon,t}$ is a contraction mapping on $B(C_0) \cap K_{\varepsilon,t}$ for all $0 < \varepsilon < \varepsilon_* := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. \end{proof}

**Proposition 5.8.** For $\varepsilon \in (0, \varepsilon_\ast)$ and $t \in P(r_0)$, there exists a unique $\bar{\phi}_{\varepsilon,t} \in B(C_0) \cap K_{\varepsilon,t}$ such that $\pi_{\varepsilon,t}^+[S_{\varepsilon}[w_{\varepsilon,t} + \phi_{\varepsilon,t}]] = 0$. Furthermore, $\phi_{\varepsilon,t}$ is continuous with respect to $t$. \end{proof}
Proof. By using Proposition 5.7 and the contraction mapping principle, there exists a unique \( \phi_{\epsilon,t} \in B(C(0) \cap C^1) \) such that \( M_{\epsilon,t}[\phi_{\epsilon,t}] = \phi_{\epsilon,t} \) for \( \epsilon \in (0, \epsilon_*) \) and \( t \in P(r_0) \).
Thus we have \( \pi_{\epsilon,t}^\perp[S_{\epsilon}[w_{\epsilon,t}^* + \phi_{\epsilon,t}]] = 0 \). By the same argument as that in the proof of Proposition 3.7 in [11], we also see that \( \phi_{\epsilon,t} \) is continuous with respect to \( t \). Hence, we omit its proof. Thus we complete the proof of Proposition 5.8. \( \square \)

5.4. Reduced problem. In this subsection, we will prove the existence theorems (Theorem 1.1 and Theorem 1.2). To this end, we carry out Step 2, explained in Subsection 5.1.

Let us define the function \( W_{\epsilon,j}(t) \) for \( t \in P(r_0[2]) \) as follows:

\[
W_{\epsilon,j}(t) := \frac{1}{\epsilon} \int_{G_\epsilon} S_{\epsilon}[w_{\epsilon,t}^* + \phi_{\epsilon,t}] \xi_j(t) w_{\epsilon,t} \, dy.
\]  
(5.58)

By using \( W_{\epsilon,j}(t) \), we seek the location of the concentration point of a two-peak solution. In the case of a one-peak solution, we use the following function for \( t \in P(r_0[1]) \):

\[
W_{\epsilon}(t) := \frac{1}{\epsilon} \int_{G_\epsilon} S_{\epsilon}[w_{\epsilon,t}^* + \phi_{\epsilon,t}] \xi(t) w_{\epsilon,t} \, dy.
\]  
(5.59)

Now, we give the following two asymptotic formulas of \( W_{\epsilon,j}(t) \) and \( W_{\epsilon}(t) \), respectively:

**Proposition 5.9.** Let \( t \in P(r_0[1]) \) and \( t = (t_1, t_2) \in P(r_0[2]) \).

1. In the case of a one-peak solution, we have

\[
W_{\epsilon}(t) = \hat{c} \hat{F}(t) + O(\epsilon),
\]  
(5.60)

where \( \hat{c} := -(3g(t)^3\xi(t)^2)^{-1} \int_R w^3 \, dy < 0 \).

2. In the case of a two-peak solution, we have

\[
W_{\epsilon,j}(t) = \hat{c}_j \hat{F}_j(t) + O(\epsilon),
\]  
(5.61)

where \( \hat{c}_j := -(3g(t_j)^3\xi_j(t_j)^2)^{-1} \int_R w^3 \, dy < 0 \).

**Proof.** We can use the same argument as that in the proof of Proposition 3.8 in [11] for the case of a one-peak solution. Thus we show only (2). We write \( \phi_{\epsilon,t} = \phi \) and \( \xi_j(t) = \xi_j \) simply. By using (5.5), it holds that

\[
W_{\epsilon,j}(t) = \epsilon^{-1} \xi_j \int \frac{1}{G_\epsilon} S_{\epsilon}[w_{\epsilon,t}^*] w_{\epsilon,t} \, dy + L_{\epsilon,t} \langle \hat{\phi} \rangle w_{\epsilon,t} \, dy + N_{1,t} \langle \hat{\phi} \rangle w_{\epsilon,t} \, dy =: K_1 + K_2 + K_3.
\]  
(5.62)

First, we study the asymptotic behavior of \( K_1 \). Note that \( w_{\epsilon,t}^{-j} = w_{\epsilon,t}^{-j} \) is an even function on \( I_{\epsilon,r_0} \). Using (4.8), we have

\[
K_1 = \epsilon^{-1} \xi_j \int_{I_{\epsilon,r_0}} \tilde{g}^j T[w_{\epsilon,t}]^{-j}(w_{\epsilon,t_j}^{-j})^2(w_{\epsilon,t_j}^{-j})' \, dy = K_{1,1} + K_{1,2} + O(\epsilon),
\]  
(5.63)

where

\[
K_{1,1} := \epsilon^{-1} g(t_j) \xi_j \int_{I_{\epsilon,r_0}} T[w_{\epsilon,t}]^{-j}(w_{\epsilon,t_j}^{-j})^2(w_{\epsilon,t_j}^{-j})' \, dy
\]  
(5.64)

and

\[
K_{1,2} := g'(t_j) \xi_j \int_{I_{\epsilon,r_0}} y T[w_{\epsilon,t}]^{-j}(w_{\epsilon,t_j}^{-j})^2(w_{\epsilon,t_j}^{-j})' \, dy.
\]  
(5.65)
Applying Lemma 5.2 and (4.10) to $K_{1,1}$, we obtain
\[
K_{1,1} = \varepsilon^{-1} g(t_j) \xi_j \int_{I_{\varepsilon, r_0}} \left[ T[w_{\varepsilon, t}]^{-j}(y) - T[w_{\varepsilon, t}]^{-j}(0) \right] (w_{\varepsilon, t_j})'^2 (w_{\varepsilon, t_j})' dy
\]
\[
= g(t_j) \xi_j \left( 6 c \sum_{k=1}^2 \frac{m_{jk}(t)}{g(t_k)\xi_k} \right) \int_{I_{\varepsilon, r_0}} g(w_{\varepsilon, t_j})'^2 (w_{\varepsilon, t_j})' dy
\]
\[
+ \varepsilon^{-1} g(t_j) \xi_j \int_{I_{\varepsilon, r_0}} \kappa_j(y)(w_{\varepsilon, t_j})'^2 (w_{\varepsilon, t_j})' dy + O(\varepsilon)
\]  
(5.66)

Since $(w_{\varepsilon, t_j})'^2 (w_{\varepsilon, t_j})'$ is an odd function on $I_{\varepsilon, r_0}$ and $\kappa_j(y)$ is an even function on $I_{\varepsilon, r_0}$, we see that
\[
\int_{I_{\varepsilon, r_0}} \kappa_j(y)(w_{\varepsilon, t_j})'^2 (w_{\varepsilon, t_j})' dy = 0
\]  
(5.67)

Thus we have
\[
K_{1,1} = - \int \frac{w^3 dy}{3 g(t)^{2\xi_j^2}} \left( 6 c \sum_{k=1}^2 \frac{m_{jk}(t)}{g(t_k)\xi_k} \right) + O(\varepsilon),
\]  
(5.68)

where we used (4.10). Moreover, by Lemma 5.1 (3) and (4.10), we have
\[
K_{1,2} = g'(t_j) \xi_j^2 \int_{I_{\varepsilon, r_0}} g(w_{\varepsilon, t_j})^3 (w_{\varepsilon, t_j})' dy + O(\varepsilon) = - g'(t_j) \int \frac{w^3 dy}{3 g(t)^3 \xi_j} + O(\varepsilon)
\]  
(5.69)

Thus, combining (5.68) and (5.69), we obtain
\[
K_1 = - \int \frac{w^3 dy}{3 g(t)^{2\xi_j^2}} \left( 6 c \sum_{k=1}^2 \frac{m_{jk}(t)}{g(t_k)\xi_k} + \frac{g'(t_j) \xi_j}{g(t_j)} \right) + O(\varepsilon) = \hat{c}_j F_j(t) + O(\varepsilon).
\]  
(5.70)

Next, we study the asymptotic behaviors of $K_2$ and $K_3$, respectively. Since $\varphi \in B(C_0) \cap K_{\varepsilon, t}$, we can apply Claim 2 in the proof of Lemma 5.5 to $K_2$. Thus we see that
\[
K_2 = \varepsilon^{-1} \xi_j O(\varepsilon\|\varphi\|_{L^2(G)}) = O(\varepsilon).
\]  
(5.71)

Also, by Lemma 5.6, we can estimate
\[
|K_3| \leq \varepsilon^{-1} \xi_j N_{1, t} [\varphi] \|\varphi\|_{L^2(G)} \|w_{\varepsilon, t_j}'\|_{L^2(G)} \leq C \varepsilon^{-1} \|\varphi\|_{L^2(G)}^2 \leq C \varepsilon
\]  
(5.72)

Hence, substituting (5.70), $K_2 = O(\varepsilon)$, and $K_3 = O(\varepsilon)$ into (5.62), we complete the proof of (2).

We give the proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 and Theorem 1.2. We first show Theorem 1.2. Taking $\varepsilon > 0$ sufficiently small if necessary, we may assume $B(\varepsilon^{3/4}, t^0) \subset P(r_0)$. By Proposition 5.8, for $t \in B(\varepsilon^{3/4}, t^0)$, there exists a unique $\varphi_t \in K_{\varepsilon, t} \cap B(C_0)$ such that $\pi_{\varepsilon, t}[S_{\varepsilon}[w_{\varepsilon, t} + \varphi_{\varepsilon, t}]] = 0$, which implies $S_{\varepsilon}[w_{\varepsilon, t} + \varphi_{\varepsilon, t}] \in C_{\varepsilon, t}$. Thus it suffices to seek a point $t \in B(\varepsilon^{3/4}, t^0)$ such that $S_{\varepsilon}[w_{\varepsilon, t} + \varphi_{\varepsilon, t}] \in C_{\varepsilon, t}$, i.e.,
\[
W_{\varepsilon}(t) := (\hat{c}^{-1}_1 W_{\varepsilon, 1}(t), \hat{c}_2^{-1} W_{\varepsilon, 2}(t))^T = 0.
\]  
(5.73)

From (A1), (A2), and Proposition 5.9 (2), we see that
\[
W_{\varepsilon}(t) = (F_1(t), F_2(t))^T + O(\varepsilon) = M(t^0)(t - t^0) + O(\varepsilon).
\]  
(5.74)

Let $f(t) := t - M(t^0)^{-1} W_{\varepsilon}(t)$. Then, by (5.74), we have
\[
|f(t) - t^0| = |t - t^0 - M(t^0)^{-1} W_{\varepsilon}(t)| = O(\varepsilon)
\]  
(5.75)
Lemma 6.1. For any $t \in B(\varepsilon^{3/4}, t^0)$. Thus $f(t) \in B(\varepsilon^{3/4}, t^0)$ for $t \in B(\varepsilon^{3/4}, t^0)$. By applying Brouwer’s fixed point theorem to $f(t)$, there exists a point $t^* \in B(\varepsilon^{3/4}, t^0)$ such that $W_{\varepsilon}(t^*) = 0$. Therefore, by the argument in Subsection 5.1, $(u_{\varepsilon,2}(x), v_{\varepsilon,2}(x)) = (u(x), v(x)) = (w_{e, t^*}(x) + \phi_{e, t^*}(x), T[w_{e, t^*} + \phi_{e, t^*}](x))$ is a solution of the system (1.2). Moreover, by Lemma 5.1 (2), we obtain $v_{e,2}(t_j^*) = T[w_{e, t^*} + \phi_{e, t^*}](t_j^*) = \xi_j(t^*) + O(\varepsilon)$. Thus we complete the proof of Theorem 1.2.

We can also apply the same argument above to the proof of Theorem 1.1. In particular, by the intermediate value theorem, we can find the point $t^* \in B(\varepsilon^{3/4}, t^0)$ such that $W_{\varepsilon}(t^*) = 0$ (see the proof of Theorem 1.1 in Subsection 3.4 in [11].) Thus we obtain Theorem 1.1.

\[
\text{Appendix.}
\]

6. Appendix A. In this subsection, we explain the representation by Green's function for general compact and connected metric graphs $G$. We use the following function spaces:

\[
X := \left\{ \eta \in H^1(G) \mid \int_G \eta(x)dx = 0 \right\}, \quad Y := \left\{ \eta \in L^2(G) \mid \int_G \eta(x)dx = 0 \right\}.
\] (6.1)

For any $x, y \in G$, let $K$ be a union of subintervals and edges connecting two points $x$ and $y$ in the graph $G$. Then, combining the well-known fundamental formula of calculus on each intervals and edges, we have $\eta(x) - \eta(y) = \int_K \eta'(x)dx$ ($\eta \in H^1(G)$). Thus, integrating over $y \in G$, we obtain the following Poincaré's inequality

\[
\int_G |\eta(x) - \frac{1}{|G|} \int_G \eta(y)dy|^2 dx \leq C \int_G |\eta'(x)|^2 dx, \quad \eta \in H^1(G)
\] (6.2)

for some constant $C$. Moreover, there exists a constant $C$ such that the following inequality

\[
|\eta(x)| \leq C\|\eta\|_{H^1(G)}
\] (6.3)

holds for any $\eta \in H^1(G)$ and any $x \in G$. Therefore, $X$ and $Y$ are Hilbert spaces with the inner products

\[
(\eta, \psi)_X := (\eta', \psi')_{L^2(G)} = \int_G \eta'(x)\psi'(x)dx = \sum_{e \in E} \int_e \eta'(x)\psi'(x)dx,
\] (6.4)

\[
(f, g)_Y := \int_G f(x)g(x)dx = \sum_{e \in E} \int_e f(x)g(x)dx.
\] (6.5)

Define

\[
\|\eta\|_X := \sqrt{(\eta, \eta)_X} = \left( \int_G |\eta'(x)|^2 dx \right)^{\frac{1}{2}}, \quad \|f\|_Y := \sqrt{(f, f)_Y} = \left( \int_G |f(x)|^2 dx \right)^{\frac{1}{2}}.
\]

Now, applying the Riesz’s representation theorem, we obtain the following lemma.

**Lemma 6.1.** For any $f \in Y$, there exists a unique solution $\eta \in X$ which satisfies

\[
-D \int_G \eta'(x)\varphi'(x)dx = \int_G f(x)\varphi(x)dx
\] (6.6)

for any $\varphi \in X$. Moreover, there exists a constant $C$ such that

\[
\|\eta\|_{H^1(G)} \leq C\|f\|_{L^2(G)}.
\] (6.7)
Remark 11. For $f \in Y$, it is easy to see that the solution $\eta \in X$ obtained in Lemma 6.1 above belongs to $H^2(\mathcal{G})$ and satisfies

\[
\begin{cases}
D\eta''(x) = f(x), & x \in \mathcal{G}, \\
\sum_{x \in \mathcal{V}} \frac{d\eta}{dx}(v) = 0, & v \in V.
\end{cases}
\] (6.8)

Furthermore, by the estimate (6.3), we have

\[
|\eta(x)| \leq C\|f\|_Y \quad (x \in \mathcal{G}).
\] (6.9)

Now, fix $x \in \mathcal{G}$ and consider the map $\Phi : f \in Y \rightarrow \eta(x)$, where $\eta \in X$ is the unique solution obtained by Lemma 6.1. Then, by the estimate (6.9), we have $\Phi \in Y^*$. Hence, from the Riesz’s representation theorem, there exists a unique function $g_x \in Y$ such that

\[
\Phi(f) = \eta(x) = (g_x, f)_Y = \int_{\mathcal{G}} g_x(y)f(y)dy \quad (f \in Y).
\] (6.10)

We denote $G(x, y) := g_x(y)$. Thus, we obtain the representation formula

\[
\eta(x) = \int_{\mathcal{G}} G(x, y)f(y)dy
\] (6.11)

for the unique solution $\eta \in X$. Since $G(x, y) \in Y$ as a function in $y$ variable, $\int_{\mathcal{G}} G(x, y)dy = 0$ holds. Finally, if $\eta \in H^2_N(\mathcal{G})$ satisfies (6.8), then $\tilde{\eta}(x) := \eta(x) - \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} \eta(y)dy$ belongs to $X$ and the unique solution to the problem (6.6). Hence, $\eta(x)$ satisfies satisfies

\[
\eta(x) - \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} \eta(y)dy = \int_{\mathcal{G}} G(x, y)f(y)dy.
\] (6.12)

6.2. Appendix B. In this subsection, we give the proof of Lemma 2.1 and Lemma 3.1.

Proof of Lemma 2.1. We first calculate Green’s function $G(x, s)$. In (1.5), we can assume that $\int_{\mathcal{G}} \eta(x)dx = 0$. Also, it suffices to calculate in the case $D = 1$. We have

\[
-\eta''_1(x_1) = \eta'_1(l_1) - \eta'_1(x_1) = \int_{x_1}^{l_1} \eta''_1(s)ds = \int_{x_1}^{l_1} f_1(s)ds
\] (6.13)

for $x_1 \in e_1 = [0, l_1]$. Thus we deduce

\[
\eta_1(l_1) - \eta_1(x_1) = \int_{x_1}^{l_1} \eta'_1(s)ds = -\int_{x_1}^{l_1} \left( \int_{s}^{l_1} f_1(t)dt \right)ds = \int_{x_1}^{l_1} (x_1 - s)f_1(s)ds.
\]

Hence, in the same way, we obtain

\[
\eta_i(x_i) = C_i - \int_{x_i}^{l_i} (x_i - s)f_i(s)ds, \quad C_i := \eta_i(l_i), \quad i = 1, 2, 3,
\] (6.14)

for $x_i \in e_i = [0, l_i]$. Since $\eta_1(0) = \eta_2(0) = \eta_3(0)$, we see that

\[
C_1 + \int_{0}^{l_1} s f_1(s)ds = C_2 + \int_{0}^{l_2} s f_2(s)ds = C_3 + \int_{0}^{l_3} s f_3(s)ds.
\] (6.15)

The formulas above imply

\[
C_j = C_1 - \int_{0}^{l_1} s f_3(s)ds + \int_{0}^{l_1} s f_1(s)ds, \quad j = 2, 3.
\] (6.16)
On the other hand, from \( \int_{G} \eta(x)dx = 0 \) and (6.14), it follows that

\[
0 = \int_{G} \eta(x)dx = \sum_{i=1}^{3} C_{i}d_{i} - \sum_{i=1}^{3} \int_{x_i}^{l_{i}} \left( \int_{x_i}^{x} (x_{i} - s)f_{i}(s)ds \right)dx_{i}. \tag{6.17}
\]

Now, by changing the order of integration, it holds that

\[
\int_{0}^{l_{i}} \left( \int_{x_i}^{x} (x_{i} - s)f_{i}(s)ds \right)dx_{i} = \int_{0}^{l_{i}} \left( \int_{0}^{x} (x_{i} - s)dx_{i} \right)f_{i}(s)ds
= -\frac{1}{2} \int_{0}^{l_{i}} s^{2}f_{i}(s)ds. \tag{6.18}
\]

Thus

\[
\sum_{i=1}^{3} C_{i}d_{i} = -\frac{1}{2} \sum_{i=1}^{3} \int_{0}^{l_{i}} s^{2}f_{i}(s)ds. \tag{6.19}
\]

Substituting (6.16) into (6.19), we obtain

\[
C_{1}(l_{1} + l_{2} + l_{3}) = -(l_{1} + l_{2} + l_{3}) \int_{0}^{l_{1}} s f_{1}(s)ds
+ \sum_{i=1}^{3} l_{i} \int_{0}^{l_{i}} s f_{i}(s)ds - \frac{1}{2} \sum_{i=1}^{3} \int_{0}^{l_{i}} s^{2}f_{i}(s)ds. \tag{6.20}
\]

Put \( L := l_{1} + l_{2} + l_{3} \). Hence, we have

\[
C_{1} = -\int_{0}^{l_{1}} s f_{1}ds + \frac{3}{2L} \sum_{i=1}^{3} l_{i} \int_{0}^{l_{i}} s f_{i}ds - \frac{3}{2L} \sum_{i=1}^{3} \int_{0}^{l_{i}} s^{2}f_{i}ds
= -\int_{0}^{l_{1}} s f_{1}ds - \frac{1}{2L} \sum_{i=1}^{3} \int_{0}^{l_{i}} (s - l_{i})^{2}f_{i}ds + \frac{1}{2L} \sum_{i=1}^{3} l_{i}^{2} \int_{0}^{l_{i}} f_{i}ds. \tag{6.21}
\]

Thus (6.14) and (6.21) yield

\[
\eta_{1}(x_{1}) = -\int_{0}^{x_{1}} s f_{1}ds - \int_{x_{1}}^{l_{1}} x_{1} f_{1}ds - \frac{1}{2L} \sum_{i=1}^{3} \int_{0}^{l_{i}} (s - l_{i})^{2}f_{i}ds + \frac{1}{2L} \sum_{i=1}^{3} l_{i}^{2} \int_{0}^{l_{i}} f_{i}ds
= -\int_{0}^{l_{1}} \min\{s, x_{1}\} f_{1}ds - \frac{1}{2L} \sum_{i=1}^{3} \int_{0}^{l_{i}} (s - l_{i})^{2}f_{i}ds + \frac{1}{2L} \sum_{i=1}^{3} l_{i}^{2} \int_{0}^{l_{i}} f_{i}ds. \tag{6.22}
\]

Here, note that \( \min\{a, b\} = 2^{-1}(a + b - |a - b|) \). Thus we arrive at

\[
\eta_{1}(x_{1}) = \int_{G} \hat{G}(x_{1}, s)f(s)ds, \quad x_{1} \in e_{1}, \tag{6.23}
\]

where

\[
\hat{G}(x_{1}, s) = \frac{1}{2} \left[ |s - x_{1} - (s + x_{1})| \right] \chi_{e_{1}}(s) - \frac{1}{2L} \sum_{i=1}^{3} (s - l_{i})^{2} \chi_{e_{i}}(s) + \frac{1}{2L} \sum_{i=1}^{3} l_{i}^{2} \chi_{e_{i}}(s). \tag{6.24}
\]

Therefore, since \( \int_{G} G(x_{1}, s)ds = 0 \), we have

\[
G(x_{1}, s) = \hat{G}(x_{1}, s) - \frac{1}{2L} (x_{1} - l_{1})^{2} + \frac{l_{1}^{2}}{2L} - \frac{1}{3L^{2}} \sum_{i=1}^{3} l_{i}^{3}. \tag{6.25}
\]

Similarly, we have the formula \( G(x_{i}, s) \) for \( x_{i} \in e_{i} (i = 2, 3) \).
Next, we check that Green’s function $G(x,s)$ above satisfies the conditions \((G1)-(G5)\). It is easy to show \((G1)\). By using $|x-z| - |x'-z| \leq |x-x'|$ for $x,x',z \in \mathbb{R}$, we can show \((A2)\) and \((G3)\) easily. Thus we omit the proofs of \((G1)-(G3)\). Let $(t_1,t_2, l_1,l_2) \in \varepsilon_1 \times \varepsilon_2$. We show \((G5)\). Recall that $\sigma_j(t) := G(t_1,t_j) - G(t_2,t_j)$ for $j = 1,2$. When $e_1 \neq e_2$, it holds that

$$D\sigma_j(t) = (-1)^j t_j - \frac{1}{2L}(t_1 - l_1)^2 + \frac{1}{2L}(t_2 - l_2)^2 + \frac{1}{2L}(l_1^2 - l_2^2). \quad (6.26)$$

Thus we have $D(\sigma_2(t) - \sigma_1(t)) = t_1 + t_2 > 0$. Also, when $e_1 = e_2$ and $t_2 > t_1$, it holds that

$$D\sigma_1(t) = -\frac{1}{2L}(t_1 - l_1)^2 + \frac{1}{2L}(t_2 - l_2)^2, \quad (6.27)$$

and

$$D\sigma_2(t) = -t_1 + t_2 - \frac{1}{2L}(t_1 - l_1)^2 + \frac{1}{2L}(t_2 - l_2)^2. \quad (6.28)$$

Thus we have $D(\sigma_2(t) - \sigma_1(t)) = t_2 - t_1 > 0$. Hence, \((G5)\) is verified. Finally, we show \((G4)_{e_i,e_j}\). Let us take $y,z \in (-r(t), r(t))$. If $i = j$, we have

$$DG^i(y,z + t_i) = \left[ -\frac{1}{2} - \frac{t_i - l_i}{L} \right] y + \frac{1}{2}(|y - z| - |z|) - \frac{y^2}{2L}. \quad (6.29)$$

Thus we can take

$$m_{ii}(t) = -\frac{1}{2D} - \frac{t_i - l_i}{DL}, \quad K_{ii}(y,z) = \frac{1}{2D}(|y-z|-|z|) - \frac{y^2}{2DL}. \quad (6.30)$$

We consider the case of $i \neq j$. If $e_i \neq e_j$, then we obtain

$$DG^i(y,z + t_j) = -\frac{t_i - l_i}{L}y - \frac{y^2}{2L}. \quad (6.31)$$

and hence we can take

$$m_{ij}(t) = -\frac{t_i - l_i}{DL}, \quad K_{ij}(y,z) = -\frac{y^2}{2DL}. \quad (6.32)$$

On the other hand, if $e_1 = e_2$ and $t_2 > t_1$, then we obtain

$$DG^i(y,z + t_j) = \frac{1}{2} \left[ |y + t_i - t_j| - |t_i - t_j| - y \right] - \frac{t_i - l_i}{L}y - \frac{y^2}{2L}. \quad (6.33)$$

and hence

$$m_{12}(t) = -\frac{1}{D} - \frac{t_1 - l_1}{DL}, \quad m_{21}(t) = -\frac{t_2 - l_2}{DL}, \quad K_{ij}(y,z) = -\frac{y^2}{2DL}. \quad (6.34)$$

Here, note that $l_1 = l_2$. Moreover, we see that all $K_{ij}(y,z)$ above satisfy the following two conditions: (i) $K_{ij}(y,z) = O(|y|)$, (ii) $\int_{-r(t)}^{r(t)} K_{ij}(y,z)(w(\varepsilon^{-1}z)\chi(z))^2 dz$ is an even function with respect to $y$ on the interval $(-r(t), r(t))$. Here we used that $w(\varepsilon^{-1}z)\chi(z)$ is an even function on the interval $(-r(t), r(t))$. Thus \((G4)_{e_i,e_j}\) is verified.

\textbf{Proof of Lemma 3.1.} We first calculate Green’s function $G(x,s)$. In \((1.5)\), we can assume that $\int_{\mathbb{R}}^\eta(x)dx = 0$. Also, it suffices to calculate in the case $D = 1$. Let $x_i \in \varepsilon_i = [0,l_i]$ ($i = 1,2,\ldots,5$). In the same way as the proof of Lemma 2.1, we get

$$\eta_i(x_i) = C_i - \int_{x_i}^{x_i} (x_i - s)f_i(s)ds, \quad C_i := \eta_i(l_i), \quad i = 1,2,4,5. \quad (6.35)$$
Also, we have
\[ \eta_3(x_3) = C_3 + (x_3 - l_3)\eta'_3(l_3) - \int_{x_3}^{l_3} (x_3 - s)f_3(s)ds, \quad C_3 := \eta_3(l_3). \quad (6.36) \]

(6.35) and \( \eta_1(0) = \eta_2(0) = \eta_3(0) \) yield
\[ C_1 + \int_0^{l_1} s f_1(s)ds = C_2 + \int_0^{l_2} s f_2(s)ds = C_3 - l_3\eta'_3(l_3) + \int_0^{l_3} s f_3(s)ds. \quad (6.37) \]

Similarly, (6.36) and \( \eta_3(l_3) = \eta_4(0) = \eta_5(0) \) yield
\[ C_3 = C_4 + \int_0^{l_4} s f_4(s)ds = C_5 + \int_0^{l_5} s f_5(s)ds. \quad (6.38) \]

On the other hand, combining \( \int_G \eta(x)dx = 0 \), (6.35), and (6.36), we can calculate
\[ 0 = \int_G \eta(x)dx = \sum_{i=1}^5 C_i l_i - \sum_{i=1}^5 \int_{x_i}^{l_i} \left( \int_{x_i}^{l_i} (x_i - s)f_i(s)ds \right)dx_i - \frac{l_3^3}{2}\eta'_3(l_3) \]
\[ = \sum_{i=1}^5 C_i l_i + \frac{5}{2} \sum_{i=1}^5 \int_{x_i}^{l_i} s^2 f_i(s)ds - \frac{l_3^2}{2}\eta'_3(l_3). \quad (6.39) \]

Thus (6.37) and (6.38) imply
\[ \sum_{i=1}^5 C_i l_i = C_3L + (l_1 + l_2 + l_3) \int_0^{l_3} s f_3ds - (l_1 + l_2)l_3\eta'_3(l_3) - l_4 \sum_{i=1}^5 \int_{0}^{l_i} s f_i dt \quad (6.40) \]

Thus, combining the equation above and (6.39), we have
\[ C_3 = -L^{-1}(l_1 + l_2 + l_3) \int_0^{l_3} s f_3ds + \frac{1}{L} \sum_{i=1}^5 l_i \int_0^{l_i} s f_i dt \]
\[ + L^{-1}(l_1 + l_2)l_3\eta'_3(l_3) + \frac{l_3^2}{2L}\eta'_3(l_3) - \frac{1}{2L} \sum_{i=1}^5 \int_0^{l_i} s^2 f_i(s)ds \]
\[ = -\left( 1 - \frac{l_1 + l_2}{L} \right) \int_0^{l_3} s f_3ds + \frac{l_3}{L} \left( l_1 + l_2 + \frac{l_3}{2} \right)\eta'_3(l_3) \]
\[ - \frac{1}{2L} \sum_{i=1}^5 \int_0^{l_i} (s - l_i)^2 f_i ds + \frac{1}{2L} \sum_{i=1}^5 l_i \int_0^{l_i} f_i ds. \quad (6.41) \]

where we used \( L = \sum_{i=1}^5 l_i \). Here, from the Kirchhoff condition, we notice that
\[ \eta'_3(l_3) = \eta'_4(0) + \eta'_5(0) = -\int_0^{l_4} f_4 ds - \int_0^{l_5} f_5 ds. \quad (6.42) \]

Hence, we obtain
\[ \eta_3(x_3) = (x_3 - l_3)\eta'_3(l_3) - \int_{x_3}^{l_3} (x_3 - s)f_3(s)ds \]
\[- \left( 1 - \frac{l_1 + l_2}{L} \right) \int_0^{l_3} s f_3ds + \frac{l_3}{L} \left( l_1 + l_2 + \frac{l_3}{2} \right)\eta'_3(l_3) \]
\[ - \frac{1}{2L} \sum_{i=1}^5 \int_0^{l_i} (s - l_i)^2 f_i ds + \frac{1}{2L} \sum_{i=1}^5 l_i \int_0^{l_i} f_i ds. \]
Finally, since

Thus we deduce


\[
\eta_3(x_3) = \int_G \hat{G}(x_3, s) f(s) ds, \quad x_3 \in e_3, \tag{6.44}
\]

where

\[
\hat{G}(x_3, s) = \frac{1}{2} \left[ |s - x_3| - (s + x_3) \right] \chi_{e_3}(s) + \frac{l_4 + l_5}{L} s \chi_{e_3}(s) - \left[ \frac{l_3}{L} \left( l_1 + l_2 + \frac{l_3}{2} \right) + (x_3 - l_3) \right] \left( \chi_{e_4}(s) + \chi_{e_5}(s) \right) - \frac{1}{2L} \sum_{i=1}^{5} (s - l_i)^2 \chi_{e_i}(s), \tag{6.45}
\]

Finally, since \( \int_G G(x_3, s) ds = 0 \), we have

\[
G(x_3, s) = \hat{G}(x_3, s) + \frac{l_4 + l_5}{L} \left[ \frac{l_3}{L} \left( l_1 + l_2 \right) + (x_3 - l_3) \right] - \frac{1}{2L} (x_3 - l_3)^2 + \frac{l_3^2}{2L} - \frac{1}{3L^2} \sum_{i=1}^{5} l_i^3. \tag{6.46}
\]

Next, we calculate \( \eta_1(x_1) \). Then, by using (6.35), (6.37), and (6.41), we see that

\[
\eta_1(x_1) = C_3 - l_3 \eta_2^0(l_3) + \int_0^{l_3} s f_3(s) ds - \int_0^{l_1} s f_1(s) ds - \int_{x_1}^{l_1} (x_1 - s) f_1(s) ds.
\]

\[
= - \int_0^{l_1} \min\{s, x_1\} f_1(s) ds + \frac{l_4 + l_5}{L} \int_0^{l_3} s f_3(s) ds - \frac{l_3}{L} \left( l_1 + l_2 + \frac{l_3}{2} - L \right) \left( \int_0^{l_4} f_4 ds + \int_0^{l_5} f_5 ds \right) - \frac{1}{2L} \sum_{i=1}^{5} (s - l_i)^2 f_i ds + \frac{1}{2L} \sum_{i=1}^{5} l_i^2 f_i ds. \tag{6.47}
\]

Thus we deduce

\[
\eta_1(x_1) = \int_G \hat{G}(x_1, s) f(s) ds, \quad x_1 \in e_1, \tag{6.48}
\]

where

\[
\hat{G}(x_1, s) = \frac{1}{2} \left[ |s - x_1| - (s + x_1) \right] \chi_{e_1}(s) + \frac{l_4 + l_5}{L} s \chi_{e_3}(s) - \frac{l_3}{L} \left( l_1 + l_2 + \frac{l_3}{2} - L \right) \left( \chi_{e_4}(s) + \chi_{e_5}(s) \right) - \frac{1}{2L} \sum_{i=1}^{5} (s - l_i)^2 \chi_{e_i}(s). \tag{6.49}
\]
Similarly, we have the formula \( G(x_1, s) \).

Thus we deduce

\[
G(x_1, s) = \frac{l_3}{L^2}(l_1 + l_2 - L)(l_4 + l_5) - \frac{1}{2L}(x_1 - l_1)^2 + \frac{l_1^2}{2L} - \frac{1}{3L^2}\sum_{i=1}^{5} l_i^3. \quad (6.50)
\]

Similarly, we have the formula \( G(x_2, s) \).

Next, we calculate \( \eta_4(x_4) \). Then, by using (6.35), (6.38), and (6.41), we see that

\[
\eta_4(x_4) = C_3 - \int_{0}^{l_4} s f_4(s) ds - \int_{x_4}^{l_4} (x_4 - s)f_4(s) ds.
\]

Thus we deduce

\[
\eta_4(x_4) = \int_{0}^{l_4} \tilde{G}(x_4, s)f(s) ds, \quad x_4 \in e_4, \quad (6.52)
\]

where

\[
\tilde{G}(x_4, s)\frac{1}{2}[s - x_4] - (s + x_4)]\chi_{e_4}(s) - \left(1 - \frac{l_4 + l_5}{L}\right)s\chi_{e_3}(s)
\]

\[
- \frac{l_3}{L}\left(l_1 + l_2 + \frac{l_3}{2}\right)(\chi_{e_4}(s) + \chi_{e_3}(s))
\]

\[
- \frac{1}{2L}\sum_{i=1}^{5}(s - l_i)^2\chi_{e_i}(s) + \frac{1}{2L}\sum_{i=1}^{5} l_i^2\chi_{e_i}(s). \quad (6.53)
\]

Thus we conclude

\[
G(x_1, s) = \tilde{G}(x_1, s) + \frac{l_3}{L^2}(l_1 + l_2)(l_4 + l_5) + \frac{l_3^2}{2L} - \frac{1}{2L}(x_1 - l_4)^2 + \frac{l_4^2}{2L} - \frac{1}{3L^2}\sum_{i=1}^{5} l_i^3. \quad (6.54)
\]

Similarly, we have the formula \( G(x_5, s) \).

Next, we check that Green’s function \( G(x, s) \) above satisfies the conditions (G1)–(G5). Their conditions can be checked by the same argument as that in the proof of Lemma 2.1. Thus we consider only the case of \( (t_1, t_2) \in e_1 \times e_3 \). It is easy to show (G1)–(G3). We show (G5). Since

\[
D\sigma_j(t) = (-1)^j t_j - \frac{1}{2L}(t_1 - l_1)^2 + \frac{1}{2L}(t_2 - l_3)^2 - \frac{l_4 + l_5}{L}t_2 + \frac{l_1^2}{2L} - \frac{l_3^2}{2L}, \quad (6.55)
\]

we have \( D(\sigma_2(t) - \sigma_1(t)) = t_1 + t_2 > 0 \). Thus (G5) is verified. Next, we show (G4)_{e_i, e_j}. Let us take \( y, z \in (-r(t), r(t)) \). If \( i = j \), by using (6.46) and (6.50), the we can take

\[
m_{11}(t) = -\frac{1}{2D} - \frac{t_1 - l_1}{DL}, \quad m_{22}(t) = -\frac{1}{2D} - \frac{t_2 - l_3}{DL} + \frac{l_4 + l_5}{DL}, \quad (6.56)
\]

and

\[
K_{ii}(y, z) = \frac{1}{2D}(|y - z| - |z|) - \frac{y^2}{2DL}. \quad (6.57)
\]
On the other hand, if \( i \neq j \), then we can take
\[
m_{12}(t) = -\frac{t_1 - l_1}{DL}, \quad m_{21}(t) = -\frac{t_2 - l_3}{DL} + \frac{l_4 + l_5}{DL},
\]
and
\[
K_{ij}(y, z) = -\frac{y^2}{2DL}.
\]
Moreover, all \( K_{ij}(y, z) \) above satisfy the two conditions (i) and (ii) in (G4\(e_i, e_j\)). Thus (G4\(e_i, e_j\)) is verified.

6.3. Appendix C. In this section, we deal with the notation \( \xi_j \), which is introduced in (1.20), and the matrix \( \mathcal{M}(t^0) \), respectively. We first give a lemma of the existence of \( \xi_j \).

**Lemma 6.2.** For each point \( t := (t_1, t_2) \) near the point \( t^0 \) satisfying (A1) and (A2), there exists a unique solution \( (\xi_1(t), \xi_2(t)) \) such that
\[
\xi_1(t) - \xi_2(t) = 6c \sum_{k=1}^2 \frac{\sigma_k(t)}{g(t_k)\xi_k(t)}, \quad \sum_{k=1}^2 \frac{6c}{g(t_k)\xi_k(t)} = 1, \quad \xi_k(t) = \xi_k^0.
\]

**Proof.** We define two functions \( f_1(t, \xi) \) and \( f_2(t, \xi) \) as follows:
\[
f_1(t, \xi) := \xi_1(t) - \xi_2(t) - 6c \sum_{k=1}^2 \frac{\sigma_k(t)}{g(t_k)\xi_k(t)}, \quad f_2(t, \xi) := 1 - \sum_{k=1}^2 \frac{6c}{g(t_k)\xi_k(t)},
\]
where \( \xi := (\xi_1(t), \xi_2(t)) \). Let us take \( t = t^0 \) and \( \xi = \xi^0 = (\xi_1^0, \xi_2^0) \). Then (A1) implies \( f_1(t^0, \xi^0) = 0 \). Here, differentiating \( f_1 \) and \( f_2 \) with respect to \( \xi_1 \) and \( \xi_2 \), respectively, we have
\[
\begin{pmatrix}
\partial_{\xi_1} f_1(t^0, \xi^0) & \partial_{\xi_2} f_1(t^0, \xi^0) \\
\partial_{\xi_1} f_2(t^0, \xi^0) & \partial_{\xi_2} f_2(t^0, \xi^0)
\end{pmatrix} = \begin{pmatrix}
1 + \sigma_1(t^0)a_1^0 & -1 + \sigma_2(t^0)a_2^0 \\
a_1^0 & a_2^0
\end{pmatrix},
\]
where \( a_j^0 := 6cg(t_j)^{-1}(\xi_j^0)^{-2} \). Hence, by using (A2), the matrix above is regular. Thus, by the implicit function theorem, we obtain a unique solution \( (\xi_1(t), \xi_2(t)) \) satisfying (6.60).

Now, we consider the case \( g(x) = 1 \) and the two-peak solution with the same heights of the spikes. Then, we have \( \xi_1^0 = \xi_2^0 = 12c \) and hence \( a_1^0 = a_2^0 = (24c)^{-1} := a \), where \( a_j^0 \) is defined by (1.19). Then, we have the following lemma for the matrix \( \mathcal{M}(t^0) \):

**Lemma 6.3.** It holds that
\[
\mathcal{M}(t^0) = \frac{1}{2} \mathcal{M}_1(t^0) - \frac{a^2}{2 \det \mathcal{A}(t^0, \xi^0)} \begin{pmatrix}
m_{11}(t^0) & m_{12}(t^0) \\
m_{21}(t^0) & m_{22}(t^0)
\end{pmatrix} \mathcal{M}_2(t^0),
\]
where
\[
\mathcal{M}_1(t^0) := \begin{pmatrix}
\partial_{t_1}(m_{11} + m_{12})(t^0) & \partial_{t_2}(m_{11} + m_{12})(t^0) \\
\partial_{t_1}(m_{21} + m_{22})(t^0) & \partial_{t_2}(m_{21} + m_{22})(t^0)
\end{pmatrix},
\]
and
\[
\mathcal{M}_2(t^0) := \begin{pmatrix}
\partial_{t_1}(\sigma_1 + \sigma_2)(t^0) & \partial_{t_2}(\sigma_1 + \sigma_2)(t^0) \\
-\partial_{t_1}(\sigma_1 + \sigma_2)(t^0) & -\partial_{t_2}(\sigma_1 + \sigma_2)(t^0)
\end{pmatrix}.
\]
Proof. Recall that \( F_j(t) := 6c \sum_{k=1}^{2} m_{jk}(t) \xi_k(t)^{-1} \) for \( j = 1, 2 \). First, differentiating \( F_j(t) \) with respect to \( t_i \), we calculate

\[
\partial_{t_i} F_j(t^0) = 6c \sum_{k=1}^{2} \frac{\partial_{t_i} m_{jk}(t^0)}{\xi_k(t^0)} - 6c \sum_{k=1}^{2} \frac{m_{jk}(t^0)}{\xi_k(t^0)^2} \partial_{t_i} \xi_k(t^0)
\]

\[
= -\frac{1}{2} \sum_{k=1}^{2} \partial_{t_i} m_{jk}(t^0) - a \sum_{k=1}^{2} m_{jk}(t^0) \partial_{t_i} \xi_k(t^0), \quad (6.65)
\]

where we used \( \xi_k(t^0) = \theta_k^0 = 12c \) and \( 6c \xi_k(t^0)^{-2} = 6c(\xi_k^0)^{-2} = a \). Thus we have

\[
\mathcal{M}(t^0) = \frac{1}{2} \mathcal{M}_1(t^0) - a \left( \begin{array}{cc} m_{11}(t^0) & m_{12}(t^0) \\ m_{21}(t^0) & m_{22}(t^0) \end{array} \right) \nabla \xi^0,
\]

where

\[
\nabla \xi^0 := \left( \begin{array}{c} \partial_{t_1} \xi_1(t^0) \\ \partial_{t_2} \xi_1(t^0) \\ \partial_{t_1} \xi_2(t^0) \\ \partial_{t_2} \xi_2(t^0) \end{array} \right).
\]

Next, differentiating (1.20) with respect to \( t_i \), we get

\[
\partial_{t_i} \xi_1(t) - \partial_{t_i} \xi_2(t) = 6c \sum_{k=1}^{2} \frac{\partial_{t_i} \sigma_k(t)}{\xi_k(t)} - 6c \sum_{k=1}^{2} \frac{\sigma_k(t)}{\xi_k(t)^2} \partial_{t_i} \xi_k(t),
\]

and

\[
\sum_{k=1}^{2} \frac{6c}{\xi_k(t)^2} \partial_{t_i} \xi_k(t) = 0.
\]

Hence, the two equations above yield

\[
\mathcal{A}(t^0, \xi^0) \nabla \xi^0 = \left( \begin{array}{cc} 6c(\xi_1^0)^{-1} & 6c(\xi_2^0)^{-1} \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \partial_{t_1} \sigma_1(t^0) & \partial_{t_2} \sigma_1(t^0) \\ \partial_{t_1} \sigma_2(t^0) & \partial_{t_2} \sigma_2(t^0) \end{array} \right),
\]

where \( \mathcal{A}(t^0, \xi^0) \) is defined by (1.19). Since \( 6c(\xi_j^0)^{-1} = 2^{-1} \) and

\[
\mathcal{A}(t^0, \xi^0)^{-1} = \frac{1}{\det \mathcal{A}(t^0, \xi^0)} \left( \begin{array}{cc} a & 1 - \sigma_2(t^0)a \\ -a & 1 + \sigma_1(t^0)a \end{array} \right),
\]

we have

\[
\mathcal{A}(t^0, \xi^0)^{-1} \left( \begin{array}{cc} 6c(\xi_1^0)^{-1} & 6c(\xi_2^0)^{-1} \\ 0 & 0 \end{array} \right) = \frac{a}{2 \det \mathcal{A}(t^0, \xi^0)} \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right).
\]

Hence, we arrive at

\[
\nabla \xi^0 = \frac{a}{2 \det \mathcal{A}(t^0, \xi^0)} \left( \begin{array}{cc} \partial_{t_1} (\sigma_1 + \sigma_2)(t^0) & \partial_{t_2} (\sigma_1 + \sigma_2)(t^0) \\ -\partial_{t_1} (\sigma_1 + \sigma_2)(t^0) & -\partial_{t_2} (\sigma_1 + \sigma_2)(t^0) \end{array} \right).
\]

Combining (6.66) and (6.73), we conclude (6.62). Thus we finish the proof. \( \square \)

6.4. Appendix D. In this subsection, we give the proof of Lemma 5.5.

Proof of Lemma 5.5. In the case of a one-peak solution, the proof can be done by an argument similar to that in the proof of [11, Lemma 3.4]. Thus we concentrate on the case of a two-peak solution. We show (5.54). Suppose (5.54) is false. Then there exist \( \{\varepsilon_n\}_{n \geq 1}, \{t^n\}_{n \geq 1} \subset P(r_0) \), and \( \{\phi_n\}_{n \geq 1} \subset K^+_{\varepsilon_n, t^n} \) such that

\[
\begin{aligned}
\varepsilon_n &\to 0, \quad \|L_{\varepsilon_n, t^n}^{+} \bar{\phi}_n\|_{L^2(\varepsilon_n)} \to 0, \quad (n \to \infty) \\
\|\phi_n\|_{H^2(\varepsilon_n)} &\equiv 1, \quad n = 1, 2, \ldots.
\end{aligned}
\]

(6.74)
Put \( \bar{\phi}_{n,i}(y) =: \bar{\phi}_{n,i}(y) \). By using a suitable extension, we can extend \( \bar{\phi}_{n,i} \) from \( I_{\varepsilon_n}^i \) to \( \mathbb{R} \) in \( H^2 \) (see e.g. \([4, 14]\)). Thus we have \( \|\bar{\phi}_{n,i}\|_{H^2(\mathbb{R})} \leq M \) for some constant \( M \) independent of \( n \), and hence we have \( \|\bar{\phi}_{n,i}^\sim\|_{H^2(\mathbb{R})} \leq M \). By using Rellich’s compactness theorem, there exist a subsequence \( \{\phi_{n,k,i}^\sim\}_{n,k \geq 1} \subset \{\phi_{n,i}^\sim\}_{n \geq 1} \) and \( \phi_i \) such that

\[
\begin{cases}
\phi_{n,k,i}^\sim \rightharpoonup \phi_i & \text{weakly in } H^2(\mathbb{R}), \\
\phi_{n,k,i}^\sim \to \phi_i & \text{in } C^1_{\text{loc}}(\mathbb{R}).
\end{cases}
\] (6.75)

For simplicity, we denote \( \varepsilon_n \) and \( n_k \) by \( \varepsilon \) and \( n \), respectively. Now, we introduce the following two linear operator:

\[
l_{0,i}^{\varepsilon} := (\phi_{n,i}^\sim)'' - \phi_{n,i}^\sim + 2\varepsilon y T[ w_{\varepsilon,t^n} ] y w_{\varepsilon,t^n} i \phi_{n,i}^\sim + \varepsilon R_{\varepsilon,t^n} \phi_{n,i}^\sim (w_{\varepsilon,t^n})^2,
\]

where \( \varepsilon^\sim_0 := 6c(t^n)^{-1}(t^n_0)^{-2} \) and \( \gamma_i := \lim_{n \to \infty} R_{\varepsilon,t^n} \phi_{n,i}(t^n_0) \). Now, we have the following claim:

**Claim 1.** For any \( \zeta \in C_0^\infty(\mathbb{R}) \), it holds that \( (l_{\varepsilon,t^n}[\phi_{n,i}^\sim], \zeta)_{L^2(I_{\varepsilon,r_0})} \to (l_{0}[\phi_i], \zeta)_{L^2(\mathbb{R})} \).

**Proof of Claim 1.** Let us take \( \zeta \in C_0^\infty(\mathbb{R}) \) arbitrary and choose a sufficiently large \( n \) such that \( \varepsilon := \sup(\zeta) \subset I_{\varepsilon,r_0} \). Now, we note that \( |yw(y)| \leq C \) for \( y \in \mathbb{R} \). Using Lemma 5.1 (4), we obtain

\[
\tilde{\gamma}(y)T[w_{\varepsilon,t^n}]^\sim(y)w_{\varepsilon,t^n} i \phi_{n,i}^\sim(y) + O(\varepsilon) = w(y) + O(\varepsilon).
\] (6.76)

for \( y \in K \). Thus it holds that

\[
\tilde{\gamma}(y)T[w_{\varepsilon,t^n}]^\sim(y)w_{\varepsilon,t^n} i \phi_{n,i}^\sim(y)\zeta(y) \to w(y)\phi_i(y)\zeta(y).
\] (6.77)

Also, by Lemma 5.3 (3), we see that

\[
\tilde{\gamma}(y)R_{\varepsilon,t^n} \phi_{n,i}^\sim(y)w_{\varepsilon,t^n} i \zeta(y) \to (6c)^{-1}a_0\gamma_i w(y)^2 \zeta(y).
\] (6.78)

Therefore, using Lebesgue’s convergence theorem, we deduce

\[
(l_{\varepsilon,t^n}[\phi_{n,i}^\sim], \zeta)_{L^2(I_{\varepsilon,r_0})} \to (l_{0}[\phi_i], \zeta)_{L^2(\mathbb{R})} = (l_{0}[\phi_i], \zeta)_{L^2(\mathbb{R})}.
\]

Hence, Claim 1 is verified.

On the other hand, letting \( \varepsilon = \varepsilon + t_0^n \), we estimate

\[
\int_{I_{\varepsilon,r_0}} l_{\varepsilon,t^n}[\phi_{n,i}^\sim]^2 dy \\
\leq \int_{\beta_\varepsilon} L_{\varepsilon,t^n} [\phi_{n,i}^\sim]^2 dz = \left| L_{\varepsilon,t^n} [\phi_{n,i}^\sim] + \sum_{j=1}^2 \frac{(L_{\varepsilon,t^n} [\phi_{n,i}^\sim], w_{\varepsilon,t_j^n})_{L^2(\beta_\varepsilon)}^2}{\|w_{\varepsilon,t_j^n}\|_{L^2(\beta_\varepsilon)}^2} \right|^2.
\]

Thus, by Hölder’s inequality and the inequality above, we estimate

\[
|l_{\varepsilon,t^n}[\phi_{n,i}^\sim], L^2(I_{\varepsilon,r_0})| \leq C \|L_{\varepsilon,t^n} [\phi_{n,i}^\sim]\|_{L^2(\beta_\varepsilon)} + C \sum_{j=1}^2 |(L_{\varepsilon,t^n} [\phi_{n,i}^\sim], w_{\varepsilon,t_j^n})_{L^2(\beta_\varepsilon)}|.
\] (6.79)

Here, we need the following claim:

**Claim 2.** It holds that \( (L_{\varepsilon,t^n} [\phi_{n,i}^\sim], w_{\varepsilon,t_j^n})_{L^2(\beta_\varepsilon)} = O(\varepsilon \|\phi_{n,i}\|_{L^2(\beta_\varepsilon)}) \). In particular, if \( \|\phi_{n,i}\|_{L^2(\beta_\varepsilon)} \leq C \), then \( (L_{\varepsilon,t^n} [\phi_{n,i}^\sim], w_{\varepsilon,t_j^n})_{L^2(\beta_\varepsilon)} \to 0 \) as \( n \to \infty \) (\( \varepsilon = \varepsilon_n \to 0 \).
Moreover, by Lemma 5.3, we have
\[ w \text{ where } K_1 := \int_{I_{\varepsilon,r}} (\tilde{w}^{\varepsilon,t_0'''} - \tilde{w}^{\varepsilon,t_0''} + 2\tilde{g}^i T[w_{\varepsilon,t}]\tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''} \phi^{\varepsilon,i} dy), \]  
(6.81)
and
\[ K_2 := \int_{I_{\varepsilon,r}} \tilde{g}^i R_{\varepsilon,t} [\phi_0] \tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''} \phi^{\varepsilon,i} dy. \]  
(6.82)

Here, \( w''' - w' + 2w'w = 0 \) implies \( \tilde{w}^{\varepsilon,t_0'''} - \tilde{w}^{\varepsilon,t_0''} + 2g(t_0)^i \tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''} = O(\varepsilon^{q/2}) \), where the constant \( q > 0 \) is independent of \( \varepsilon \). Thus, applying Lemma 5.1 (4) and the estimate above to \( K_1 \), we can calculate
\[ K_1 = \int_{I_{\varepsilon,r}} (\tilde{w}^{\varepsilon,t_0'''} - \tilde{w}^{\varepsilon,t_0''} + 2g(t_0)^i \tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''} \phi^{\varepsilon,i} dy) + O(\|\phi_0\|_{L^2(\varepsilon,r)}), \]
(6.83)

Moreover, by Lemma 5.3, we have
\[ K_2 = g(t_0)^i R_{\varepsilon,t} [\phi_0] [t_0^i] \int_{I_{\varepsilon,r}} (\tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''} \phi^{\varepsilon,i} dy) + O(\|\phi_0\|_{L^2(\varepsilon,r)}), \]
where we used that \( (\tilde{w}^{\varepsilon,t_0''} \tilde{w}^{\varepsilon,t_0''})' \) is an odd function on \( I_{\varepsilon,r} \). Thus Claim 2 is verified. \( \square \)

Hence, combining (6.74), (6.79), and Claim 2, we deduce \( |(y_{\varepsilon}, t_0^i)[\phi_0^\varepsilon, \zeta]_{L^2(I_{\varepsilon,r})}| \to 0 \) for any \( \zeta \in C_0^\infty(\varepsilon) \). Thus this and Claim 1 yield \( (l_0[\phi_0], \zeta)_{C(\varepsilon,r)} = 0 \) for any \( \zeta \in C_0^\infty(\varepsilon) \), which implies \( l_0[\phi_0] = 0 \). Now, Lemma 5.4 (2) implies
\[ \frac{1}{6c} \left( a_{11}^2 \gamma_1 \right) = -2 \int_{\varepsilon} w(y) P(t) \Phi_0(y) dy, \]  
(6.84)
where \( \gamma_1 \) and \( \Phi_0(y) \) are defined by \( \gamma_i := \lim_{n \to \infty} R_{\varepsilon,t} [\phi_0](t_n) \) and \( \Phi_0(y) := (\phi_1(y), \phi_2(y))^T \), respectively, and \( P(t) \) is defined by (5.36). Therefore, we obtain
\[ L_0 \Phi_0 := \Phi_0' - \Phi_0 + 2w \Phi_0 - 2 \int_{\varepsilon} w(y) P(t) \Phi_0(y) dy \int_{\varepsilon} w^2 dy = 0. \]
(6.85)

Moreover, from Lemma 5.4 (3), we see that 1/2 \( \not\in \sigma(P(t)) \) and the matrix \( P(t) \) is diagonalized. To study the operator \( L_0 \), we need the following lemma:

**Lemma 6.4** ([9]). Let \( a_1 \) and \( a_2 \) be the eigenvalues of \( B \) and \( X_0 = \text{span}\{w'(y)\} \). Also, let us define the operator \( L : (H^2(\varepsilon))^2 \to (L^2(\varepsilon))^2 \) as follows:
\[ L \Phi := \Phi'' - \Phi + 2w \Phi - 2 \int_{\varepsilon} w B \Phi dy \int_{\varepsilon} w^2 dy w^2 \]
(6.86)
Assume that $2b_{ij} \neq 1$ and the matrix $\mathcal{B}$ can be diagonalized. Then, we have $\ker(L) = (X_0)^2$ and $\ker(L^*) = (X_0)^2$.

Thus, by Lemma 6.4, we have $\ker(L^t_0) = (X_0)^2$. On the other hand, $\phi_n \in K_{\varepsilon,t}^\perp$ implies $0 = g(t^n_m)\xi(t^n_j) \int_{I_{x,0}} \overline{w_{\varepsilon,t^n_j}} \phi_n \phi_n^{\ast} \text{d}y \rightarrow \int_{\mathbb{R}} w' \phi_i \text{d}y$. Therefore, we conclude $\Phi = 0$.

(5.4) implies

$$-\phi_n^{\ast} + \phi_n = -L_{\varepsilon,t^n}^{\varepsilon} \phi_n + 2\varepsilon T[w_{\varepsilon,t^n}] \overline{\phi_n} + 2\varepsilon R_{\varepsilon,t^n}^{\varepsilon} \phi_n \overline{w_{\varepsilon,t^n}} =: f_n$$

with $\sum_{\varepsilon^{-1} < \varepsilon^{-1} \epsilon \phi_{n,t^n}/d\epsilon \varepsilon} = 0$ for $v \in V$. Applying a priori elliptic estimate to the equality above, we estimate $\|\phi_n\|_{H^2(G_\varepsilon)} \leq C\|f_n\|_{L^2(G_\varepsilon)} \leq C(K_1 + K_2 + K_3)$. Then, the following claim holds:

**Claim 3.** $\|\phi_n\|_{H^2(G_\varepsilon)} \to 0$ as $n \to \infty$.

If Claim 3 is correct, then this contradicts $\|\phi_n\|_{H^2(G_\varepsilon)} = 1$, and hence, we complete the proof of (5.4).

**Proof of Claim 3.** It suffices to show $K_j \to 0$. First, we can estimate

$$\|L_{\varepsilon,t^n}^{\varepsilon} \phi_n\|_{L^2(G_\varepsilon)} \leq C\|L_{\varepsilon,t^n}^{\varepsilon} \phi_n\|_{L^2(G_\varepsilon)} + C\sum_{j=1}^2 \|w_{\varepsilon,t^n}\|_{L^2(\varepsilon,t^n)}\|. \|L_{\varepsilon,t^n}^{\varepsilon} \phi_n\|_{L^2(\varepsilon,t^n)}\|.$$

Hence, by (6.74) and Claim 2, we have $K_1 \to 0$. Next, by Lemma 5.1 (4), it holds that

$$K_2 \leq C\sum_{j=1}^2 \|g(t^n_j) \tau[w_{\varepsilon,t^n}] (t^n_j) \overline{\phi_n^{\ast}} + \phi_n^{\ast} \|_{L^2(\varepsilon,t^n)} + O(\varepsilon).$$

Thus, by Lebesgue’s convergence theorem and $\Phi = 0$, we have

$$\|g(t^n_j) \tau[w_{\varepsilon,t^n}] (t^n_j) \overline{w_{\varepsilon,t^n}} \phi_n^{\ast} \|_{L^2(\varepsilon,t^n)} \to \|w \phi_j \|_{L^2(\varepsilon,t^n)} = 0$$

and hence $K_2 \to 0$. Finally, we study $K_3$. By (6.83) and $\Phi = 0$, we obtain

$$\text{lim}_{n \to \infty} R_{\varepsilon,t^n}^{\varepsilon} \phi_n (t^n_j) = \gamma_j = 0.$$

Hence, by Lemma 5.3, we see that

$$K_3 \leq C\sum_{j=1}^2 \|g(t^n_j) R_{\varepsilon,t^n}^{\varepsilon} \phi_n (t^n_j) \overline{w_{\varepsilon,t^n}} \phi_n^{\ast} \|_{L^2(\varepsilon,t^n)} + O(\varepsilon) \to 0.$$

Thus we finish the proof of Claim 3.

We denote the conjugate operator of $L_{\varepsilon,t}^\perp$ by $(L_{\varepsilon,t}^\perp)^*$. To complete the proof of $\text{Ran}(L_{\varepsilon,t}) = C_{\varepsilon,t}^\perp$, i.e., Lemma 5.5, it suffices to show that $(L_{\varepsilon,t}^\perp)^*$ is injective from $K_{\varepsilon,t}^\perp$ to $C_{\varepsilon,t}^\perp$. By the same argument as was used in the proof of (5.4), we can show $(L_{\varepsilon,t}^\perp)^*$ is injective for $t \in P(r_0)$ and sufficiently small $\varepsilon$. Hence, we omit the details (see e.g. the proof of [14, Lemma 3.2].)

\[\square\]
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