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Accessibility
An area law for 2D frustration-free spin systems

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Abstract

We prove that the entanglement entropy of the ground state of a locally gapped frustration-free 2D lattice spin system satisfies an area law with respect to a vertical bipartition of the lattice into left and right regions. We first establish that the ground state projector of any locally gapped frustration-free 1D spin system can be approximated to within error $\epsilon$ by a degree $O(\sqrt{n \log(\epsilon^{-1})})$ multivariate polynomial in the interaction terms of the Hamiltonian. This generalizes the optimal bound on the approximate degree of the boolean AND function, which corresponds to the special case of commuting Hamiltonian terms. For 2D spin systems we then construct an approximate ground state projector (AGSP) that employs the optimal 1D approximation in the vicinity of the boundary of the bipartition of interest. This AGSP has sufficiently low entanglement and error to establish the area law using a known technique.

1 Introduction

The information-theoretic view on quantum matter has had widespread impact in physics. For instance, tools from quantum Shannon theory have provided insights into the black-hole paradox [26] and the notion of topological entanglement entropy has been crucial for understanding and classifying phases of matter [28]. This viewpoint has also permeated the computational side of condensed matter physics, and has led to the identification of entropic properties known as the area laws, which are hallmarks of classical simulability in many physically relevant settings. A state of a lattice spin system is said to satisfy an area law if its entanglement entropy with respect to any bipartition scales with the size of its boundary. This restricts the quantum correlations arising in the state, and enables an efficient classical representation of the state for one-dimensional (1D) lattice systems [36]. A breakthrough result of Hastings [24] established an area law for gapped ground states in one dimension. Subsequent improvements were obtained in Refs. [8, 7] using new tools from combinatorics and approximation theory. This led to an efficient classical algorithm for computing ground states [30, 9], providing a rigorous justification for the success of the DMRG algorithm [37]. Recently, area law was also shown for the ground states of 1D long range hamiltonians [29].

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The area law conjecture for two or higher dimensional systems has remained a significant open question, see e.g., Refs. [19, 32, 16, 12]. It can be motivated by the following “locality intuition”:

Locality of correlations in the vicinity of the boundary of a region should imply an area law for the region.

In particular, this suggests that the area law should hold for gapped ground states since they possess a finite correlation length [23]. Whether or not this intuition can be made rigorous remains to be seen [25]. While correlation decay has been shown to imply an area law in 1D [13], the locality intuition has no formal support in higher dimensions.

In this work, we prove that the unique ground state of any frustration-free, locally gapped 2D lattice spin system satisfies an area law scaling of entanglement entropy with respect to a vertical cut that partitions the system into left and right regions, see Fig. 1.

**Theorem 1.1** (Informal). Consider a locally gapped, frustration-free Hamiltonian with geometrically local interactions in 2D and a unique ground state. The ground state entanglement entropy with respect to a vertical bipartition of length \(n\) is at most \(n^{1+O(1/\text{polylog}(n))}\).

A frustration-free quantum spin system has the property that its ground state has minimal energy for each term in the Hamiltonian. Such a system is said to be locally gapped if there exists a positive constant that lower bounds the spectral gap of any subset of the local Hamiltonian terms. We believe that our techniques readily generalized to rectangular bi-partitions on the lattice. This can then be used to prove area laws for other bi-partitions via appropriate tiling, including those featuring gapless edge excitations [10]. Using the techniques introduced in [9] and further developed in [1], Theorem 1.1 readily extends to degenerate ground states. We note that a previous work [17] established the area law for the special case of spin-1/2 frustration-free systems, using an exact characterization of the ground space from Ref. [14]. The area law is known to be false on general graphs [3, 6].

The proof of Theorem 1.1 is obtained via new insights in the approximation theory of quantum ground states. For a Hamiltonian with unique ground state \(|\Omega\rangle\), an \(\epsilon\)-approximate ground state projector (AGSP) is an Hermitian operator \(K\) such that \(K|\Omega\rangle = |\Omega\rangle\) and \(\|K - |\Omega\rangle\langle\Omega|| \leq \epsilon\). In Ref. [8] it has been shown that an AGSP with small error \(\epsilon\) and low entanglement with respect to a given bipartition of the lattice — as measured by its
Schmidt rank SR$(K)^{1}$ — implies an upper bound on the entanglement of the ground state itself across the bipartition. In particular, Ref. [8] shows that if $\epsilon \cdot \text{SR}(K) \leq 1/2$ then the entanglement of the ground state is at most $O(1) \cdot \log(\text{SR}(K))$, see Theorem 4.1 for a precise statement. In this way the study of entanglement in quantum ground states can be reduced to the study of entanglement in low-error AGSPs.

**Approximate degree of quantum ground states** To describe our techniques, consider a collection $\{H_j\}_{j=1}^{n}$ of Hermitian operators satisfying $0 \leq H_j \leq I$ for all $j$. Suppose the Hamiltonian $H \overset{\text{def}}{=} \sum_{j=1}^{n} H_j$ has a unique ground state $|\Omega\rangle$ satisfying $H_j |\Omega\rangle = 0$ for all $j$.

We consider AGSPs which are multivariate polynomial functions

$$K = P(H_1, H_2, \ldots, H_n).$$

(1)

In general, this kind of polynomial is a linear combination of monomials of the form

$$H_{j_1} H_{j_2} \ldots H_{j_m} \quad j_k \in [n].$$

As discussed above, in order to bound the entanglement of the ground state, it suffices to construct an AGSP with sufficiently small error and sufficiently small entanglement. Moreover, the techniques of Ref. [7] suggest that polynomial degree can be taken as a proxy for entanglement in AGSPs of this type. Thus, we ask: what is the minimal polynomial degree $s$ needed to approximate the ground state projector to within a given error $\epsilon$?

Following Ref. [7], it is instructive to consider the special case in which our AGSP (1) can be expressed as $K = p(H)$ where $p$ is a univariate polynomial. This kind of AGSP has the nice feature that it commutes with $H$ and can therefore be diagonalized in the same basis. Using this fact we see that such a polynomial is an $\epsilon$-AGSP iff

$$p(0) = 1 \quad \text{and} \quad \max_{x \in \text{Spec}_+(H)} |p(x)| \leq \epsilon.$$  

(2)

Here Spec$_+(H)$ is the set of nonzero eigenvalues of $H$. Note that since $0 \leq H_j \leq I$ we have $\|H\| \leq n$ and therefore Spec$_+(H) \subseteq [\gamma, n]$ where $\gamma$ is the smallest nonzero eigenvalue or spectral gap of $H$. By choosing $p$ to be a rescaled and shifted Chebyshev polynomial of degree $s$ one obtains an AGSP with [7]

$$\epsilon = e^{-\Omega(s\sqrt{n})}.$$  

(3)

This scaling of error with degree is optimal, a consequence of the extremal property of Chebyshev polynomials [33, Proposition 2.4]. Here we did not use any properties of the Hamiltonian except the fact that Spec$(H) \subseteq [\gamma, n]$. We see that a spectral gap lower bounded by a positive constant ensures a good $\epsilon = O(1)$ approximation by a $O(\sqrt{n})$-degree polynomial. This form of locality in the ground state is somewhat distinct from finite correlation length.

Remarkably, the tradeoff Eq. (3) between polynomial degree and error can be improved in certain important special cases. For example, suppose the Hamiltonian terms are commuting

---

1The Schmidt rank SR$(K)$ of an operator $K$ with respect to a bipartition of the system is the minimal number $R$ of tensor product operators $A_\alpha \otimes B_\alpha$ such that $K = \sum_{\alpha=1}^{R} A_\alpha \otimes B_\alpha$. 

---
projectors, i.e., \([H_i, H_j] = 0\) and \(H_i^2 = H_i\). In that case the problem of approximating the ground state is formally equivalent to the problem of approximating the multivariate AND function of \(n\) binary variables (equivalently, the OR function), see Sec. 3.1 for details\(^2\). The distinguishing feature of the commuting case for our purposes is that, crucially, all eigenvalues of \(H\) are integers between 0 and \(n\) and by exploiting the fact that \(\text{Spec}_+(H) \subseteq \{1, 2, \ldots, n\}\) one can construct a suitable univariate polynomial \(p\) that achieves Eq. (2) with

\[
\epsilon = e^{-\Omega(s^2/n)}.
\]

This improves upon Eq. (3) in the low-error regime \(s \gg \sqrt{n}\) and is known to be optimal in the commuting case [27].

More generally, for a collection of possibly non-commuting operators \(\{H_j\}_{j=1}^n\) let us call a degree-\(s\) multivariate polynomial AGSP (1) **optimal** if the approximation error matches Eq. (4). Our first result establishes that one-dimensional frustration-free locally-gapped ground states can be optimally approximated in this sense.

**Theorem 1.2** (Optimal approximation of 1D ground states, informal). For any constant \(\delta \in (0, 1/2)\) and \(s \in (\sqrt{n}, n^{1-\delta})\), there is a degree \(O(s)\) polynomial which approximates the ground state projector of a locally-gapped 1D frustration-free quantum spin system to within error Eq. (4).

We emphasize that the AGSP in the above theorem is a multivariate polynomial of the form (1), and as far as we know cannot be expressed as a univariate polynomial function of \(H\). This is because we may have \([H_i, H_{i+1}] \neq 0\), and — in contrast with the commuting case — the spectrum \(\text{Spec}_+(H)\) does not appear to have a nice characterization that allows us to improve upon Eq. (3) by a suitable choice of univariate polynomial \(p\). We construct our AGSP via a recursive application of the robust polynomial method of Ref. [4] (where a subvolume law for the same class of systems was shown). The resulting polynomial, which is detailed in Sec. 3.2, has a structure which is reminiscent of a renormalization group flow.

Although it concerns 1D systems, Theorem 1.2 turns out to be just what we need to establish the area law in two dimensions. The key insight is captured by the following modified locality intuition that we propose, which asserts a direct link between linear-degree optimal polynomial approximations and area laws:

**A linear-degree optimal polynomial approximation for the ground state in the vicinity of the boundary of a region should imply an area law for the region.**

Here we mean linear in \(n\), the number of inputs of the multivariate polynomial (cf. Eq. (1)). To understand where this comes from, suppose we can construct an optimal linear-degree polynomial \(P\) that approximates the ground state projector and is localized in a width \(\sim w\) neighborhood of the boundary of the bipartition of interest (here we are intentionally vague about the meaning of localized, see Sec. 4 for details). Thus, the degree of \(P\) is \(\sim w \cdot \text{area}\) and its error is \(\epsilon \leq e^{-\Omega(w\cdot\text{area})}\), where `area` is the size of the boundary. Now consider an AGSP \(K = P^q\) for some positive integer \(q\). The total polynomial degree

\(^2\)The \(\epsilon\)-approximate degree of \(\text{AND}\) has the remarkable low-error behaviour \(\deg_{\epsilon}(\text{AND}) = O(\sqrt{n \log(\epsilon^{-1})})\) [15, 18]. The log under the square root reflects the fact that the error probability of Grover’s search algorithm can be reduced using a better strategy than the naive parallel amplification.
of $K$ is thus $D = qw \cdot \text{area}$, and its error is $\epsilon' = \epsilon^q \leq e^{-\Omega(qw\cdot\text{area})}$. Now we shall assume that the polynomial $K^q$ is nicely behaved in a certain sense first identified in Ref. [8]. In particular, we assume that its Schmidt rank is amortized over the width $w$ neighborhood of the boundary, in that it scales as $\text{SR} \sim e^{O(D/w+w\cdot\text{area})}$. Choosing $q = \Omega(w)$ (say) and letting $w$ be a large constant, we can ensure $\epsilon' \cdot \text{SR} \leq 1/2$, with $\log(\text{SR}) = O(\text{area})$. Thus, applying the aforementioned method from Ref. [8] we would obtain the desired upper bound $O(\text{area})$ on the ground state entanglement entropy.

Since the boundary of a region on a 2D lattice is one-dimensional, the above argument suggests that the 2D area law should follow from optimal linear-degree polynomial approximations in 1D. To make this work, in Sec. 4 we show how our 1D approximation can be used “in the vicinity of the boundary of the region” and we relate the entanglement of the resulting AGSP to the polynomial degree. The area law is then established using the aforementioned method from Ref. [8]. The astute reader may note that Theorem 1.2 does not quite provide a linear-degree optimal polynomial as the degree must be $n^{1-\delta}$ for some $\delta \in (0, 1/2)$; a careful treatment of the $\delta \to 0$ limit leads to the slight deviation $n^{1+o(1)}$ from area law behaviour in Theorem 1.1, see Sec. 3 for details.

**Discussion** There are at least three significant questions left open by our work. Firstly, one may ask if the assumption of a local spectral gap can be removed or replaced with one concerning the global spectral gap of the Hamiltonian. We believe that this could lead to a generalization of our techniques to frustrated systems. To make progress here may require a deeper understanding of the interplay between the local spectral gap and the gap of the full hamiltonian. Secondly, it is natural to ask if ground states of locally gapped frustration-free systems can be approximated by efficiently representable tensor networks such as PEPS of small bond dimension [35]. While it is known that a 2D area law does not imply such a representation [21], a more detailed study of the optimal polynomial approximations considered here may provide insight into this question. Finally, a natural open question is to extend our results to local hamiltonian systems on higher dimensional lattices. As mentioned earlier, this is closely related to the question of approximating ground states by linear-degree optimal polynomials.

The rest of the paper is organized as follows. In Sec. 2 we review some basic approximation tools, the Chebyshev and robust polynomials. In Sec. 3 we deploy them to construct optimal polynomial AGSPs for a family of quantum systems that includes 1D quantum spin systems and we establish Theorem 1.2. Finally, in Sec. 4 we adapt our methods to the 2D setting and prove the area law Theorem 1.1.

## 2 Polynomial approximation toolkit

In this section we describe methods for approximating multivariate functions by polynomials. We first describe polynomial approximations with real-valued variable inputs. Then we generalize these methods to the local Hamiltonian setting by allowing operator-valued inputs.
2.1 Approximation of functions

Following Ref. [4] we shall build polynomial approximations by combining two well-known ingredients: the univariate Chebyshev polynomials and a robust polynomial [34].

We will use a rescaled and shifted Chebyshev polynomial defined as follows. For every \( s \in \mathbb{R}_{\geq 0} \) and \( \eta \in (0, 1) \) we define a polynomial \( T_{\eta,s} : [0, 1] \rightarrow \mathbb{R} \) of degree \( \lceil s \rceil \) by

\[
T_{\eta,s}(x) \overset{\text{def}}{=} \frac{T_{\lceil s \rceil}\left(\frac{2(1-x)}{1-\eta} - 1 \right)}{T_{\lceil s \rceil}\left(\frac{2}{1-\eta} - 1 \right)},
\]

where \( T_j \) is the Chebyshev polynomial of the first kind. To ease notation later on, the parameter \( s \) which determines the degree is not required to be an integer. The polynomial Eq. (5) has the following property which is a direct consequence of Lemma 4.1 of Ref. [7].

**Lemma 2.1** ([7]). For every \( s \in \mathbb{R}_{\geq 0} \) and \( \eta \in (0, 1) \) we have \( T_{\eta,s}(0) = 1 \) and

\[
|T_{\eta,s}(x)| \leq 2e^{-2s\sqrt{\eta}} \quad \eta \leq x \leq 1.
\]

Next, we describe the robust polynomial. Our starting point is the function \( B : [0, 1] \rightarrow \{0, 1\} \) defined by

\[
B(x) \overset{\text{def}}{=} \begin{cases} 
1, & x = 1 \\
0, & 0 \leq x < 1.
\end{cases}
\]

This function rounds \( x \) to a bit in a one-sided fashion. Using (6) we define a (one-sided) “robust product” that takes real inputs \( x_1, x_2, \ldots, x_m \in [0, 1] \) and outputs 1 if and only if they are all equal to 1:

\[
\operatorname{Rob}(x_1, x_2, \ldots, x_m) \overset{\text{def}}{=} B(x_1)B(x_2)\ldots B(x_m).
\]

We note that since \( B(x_j)^2 = B(x_j) \) we may also express Eq. (7) as

\[
\operatorname{Rob}(x_1, x_2, \ldots, x_m) = (B(x_m)B(x_{m-1})\ldots B(x_1))(B(x_1)B(x_2)\ldots B(x_m)).
\]

The left-right symmetric expression will be useful to us momentarily when we extend the definition of the function to allow operator-valued inputs.

The robust polynomial of interest is an approximation to Eq. (7). To this end, let

\[
B_i(x) \overset{\text{def}}{=} \begin{cases} 
x, & i = 1 \\
x^{i-1}(x - 1), & 2 \leq i \leq \infty.
\end{cases}
\]

Note that for any \( x \in [0, 1] \) we have \( B(x) = \lim_{i \rightarrow \infty} \sum_{j=1}^{i} B_j(x) \). Define

\[
\widetilde{\operatorname{Rob}}(x_1, x_2, \ldots, x_m) \overset{\text{def}}{=} \sum_{(i_1+i'_1)+\ldots+(i_m+i'_m)\leq3m} (B_{i_m}(x_m)\ldots B_{i'_1}(x_1))(B_{i_1}(x_1)\ldots B_{i_m}(x_m)).
\]
The above expression is obtained by starting with Eq. (8), substituting \( B \leftarrow \sum_{j=1}^{\infty} B_j \), and then truncating the summation so that the total degree of the polynomial is at most 3\( m \) (this is somewhat arbitrary). This polynomial is a good approximation to \( \text{Rob} \) in the following sense.

**Lemma 2.2** (Special case of Lemma 2.4). Suppose that \( x_1, x_2, \ldots, x_m \in [0, \varepsilon] \cup \{1\} \) for some \( \varepsilon \leq 1/10 \). Then \( |\text{Rob}(x_1, \ldots, x_m) - \text{Rob}(x_1, \ldots, x_m)| \leq (10\varepsilon)^m \).

### 2.2 Approximation of operators

Let us now extend our definitions from the previous section to allow operator-valued inputs. Suppose \( O \) is a Hermitian operator with all eigenvalues in the interval \([0, 1]\). The operator-valued Chebyshev polynomial \( T_{\eta,s}(O) \) is defined in the usual way by substituting \( x \leftarrow O \) in Eq. (5). By applying Lemma 2.1 to each eigenvalue of \( O \), we obtain the following.

**Lemma 2.3.** Let \( s \in \mathbb{R}_{\geq 0} \) and \( \eta \in (0, 1) \). Suppose that \( O \) is an Hermitian operator with eigenvalues in the interval \([0, \eta, 1]\) and let \( \Pi \) be the projector onto the nullspace of \( O \). Then \( T_{\eta,s}(O)\Pi = \Pi \) and \( \|T_{\eta,s}(O) - \Pi\| \leq 2e^{-2s\sqrt{\eta}} \).

For the robust polynomial, we start by defining \( B(O) \) to be the projector onto the eigenspace of \( O \) with eigenvalue 1. For Hermitian operators \( O_1, O_2, \ldots, O_m \) such that each \( O_i \) has eigenvalues in the interval \([0, 1]\), we define a Hermitian robust product which generalizes Eq. (8):

\[
\text{Rob}(O_1, O_2, \ldots, O_m) \overset{\text{def}}{=} C^\dagger C \quad \text{where} \quad C \overset{\text{def}}{=} B(O_1)B(O_2)\ldots B(O_m).
\]

Note that due to the possible non-commutativity of the \( \{O_i\} \) operators, \( \text{Rob}(O_1, O_2, \ldots, O_m) \) is generally not the projector onto the intersection of the \(+1\) eigenspaces of these operators.

We also define the operator-valued polynomial \( B_i(O) \) for positive integers \( i \) by substituting \( x \leftarrow O \) in Eq. (9). The robust polynomial is defined in parallel with (10), i.e.,

\[
\tilde{\text{Rob}}(O_1, O_2, \ldots, O_m) \overset{\text{def}}{=} \sum_{(i_1+i_1')+\ldots+(i_m+i_m') \leq 3m} (B_{i_1}(O_m) \ldots B_{i_1'}(O_1)) (B_{i_2}(O_1) \ldots B_{i_2'}(O_m)) \cdot (11)
\]

One can easily check that the operator in Eq. (11) is Hermitian. Let us now establish the following error bound.

**Lemma 2.4.** Suppose that the eigenvalues of all operators \( \{O_i\}_{i=1}^{m} \) lie in the range \([0, \varepsilon] \cup \{1\}\) for some \( \varepsilon \leq 1/10 \). Then

\[
\|\tilde{\text{Rob}}(O_1, \ldots, O_m) - \text{Rob}(O_1, \ldots, O_m)\| \leq (10\varepsilon)^m.
\]

**Proof.** To ease notation in this proof, we use the shorthand \( \tilde{i} \overset{\text{def}}{=} (i_1, i_1', i_2, i_2', \ldots, i_m, i_m') \) for a tuple of \( 2m \) positive integers, \( \text{sum}(\tilde{i}) \overset{\text{def}}{=} i_1 + i_1' + i_2 + i_2' + \ldots + i_m + i_m' \) for their sum, and

\[
M(\tilde{i}) \overset{\text{def}}{=} B_{i_1}(O_m) \ldots B_{i_1'}(O_1)B_{i_2}(O_1) \ldots B_{i_m}(O_m)
\]
for the product that appears in Eq. (11). Using Eq. (9) we see that for any Hermitian operator $O$ with eigenvalues in $[0, \varepsilon] \cup \{1\}$ we have $\|B_i(O)\| \leq \varepsilon^i$ for all $i \geq 1$. Therefore $\|M(\vec{i})\| \leq \varepsilon^{|\text{sum}(\vec{i})|-2m}$ and, for any integer $q \geq 2m$,

$$
\left\| \sum_{\text{sum}(\vec{i})=q} M(\vec{i}) \right\| \leq \sum_{\text{sum}(\vec{i})=q} \varepsilon^{2m} = \left( \frac{q-1}{2m-1} \right) \varepsilon^{2m} \leq 2 \varepsilon \varepsilon^{2m}.
$$

Here we used the fact that each component of $\vec{i}$ is a positive integer and so $\vec{i}$ is a composition of $q$ with exactly $2m$ parts; the number of compositions of an integer $n$ with $k$ parts is $\binom{n-1}{k-1}$. In the last inequality in Eq. (12) we used the upper bound $\binom{a}{b} \leq 2^b$. Now let $J > 3m$ be a positive integer to be fixed later. We have

$$
\sum_{i_1, i_1', \ldots, i_m, i_m' = 1}^{J} M(\vec{i}) - \tilde{\text{Rob}}(O_1, \ldots, O_m) = \sum_{q=3m+1}^{2mJ} \sum_{\text{sum}(\vec{i})=q} M(\vec{i})
$$

and therefore

$$
\left\| \sum_{i_1, i_1', \ldots, i_m, i_m' = 1}^{J} M(\vec{i}) - \tilde{\text{Rob}}(O_1, O_2, \ldots, O_m) \right\| \leq \varepsilon^{2m} \sum_{q=3m+1}^{2mJ} (2\varepsilon)^q \leq (8\varepsilon)^m \left( \frac{2\varepsilon}{1-2\varepsilon} \right) \leq (8\varepsilon)^m,
$$

where in the last step we used $\varepsilon \leq 1/10$. Using Eq. (9) gives $\| \sum_{1 \leq i \leq J} B_i(O) - B(O) \| \leq \varepsilon^J$ for any Hermitian $O$ with eigenvalues in $[0, \varepsilon] \cup \{1\}$. Applying this bound $2m$ times and using the triangle inequality gives

$$
\left\| \sum_{i_1, i_1', \ldots, i_m, i_m' = 1}^{J} M(\vec{i}) - \text{Rob}(O_1, \ldots, O_m) \right\| \leq 2m \varepsilon^J.
$$

Let us choose $J$ to be large enough so that the right-hand-side of Eq. (14) is at most $(10\varepsilon)^m - (8\varepsilon)^m$. Combining Eqs. (13, 14) using the triangle inequality completes the proof. 

We also use the following claims which summarize simple properties of $\tilde{\text{Rob}}$.

**Claim 2.5.** Let $O_1, O_2, \ldots, O_m$ be Hermitian, with eigenvalues in the range $[0, 1]$. Suppose there exists a projector $\Pi$ such that $O_j \Pi = \Pi$ for all $j \in [m]$. Then $\tilde{\text{Rob}}(O_1, \ldots, O_m) \Pi = \Pi$.

**Proof.** Note that for all $j \in [m]$, $B_1(O_j) \Pi = O_j \Pi = \Pi$ and for all $i \geq 2$, $B_i(O_j) \Pi = 0$. Thus, $\tilde{\text{Rob}}(O_1, \ldots, O_m) \Pi = B_1(O_m) \ldots B_1(O_1) B_1(0) \ldots B_1(O_m) \Pi = \Pi$.

**Claim 2.6.** The polynomial $\tilde{\text{Rob}}(O_1, \ldots, O_m)$ can be expressed as a linear combination of at most $2^{5m}$ monomials of the form

$$
O_{i_1}^{a_1} O_{i_2}^{a_2} \ldots O_{i_{2m}}^{a_{2m}},
$$

where $i_1, i_2, \ldots, i_{2m} \in [m]$ and $\{a_i\}$ are positive integers satisfying $\sum_{j=1}^{2m} a_j \leq 3m$. 


Proof. The definition Eq. (11) expresses \( \widetilde{\text{Rob}}(O_1, \ldots, O_m) \) as a sum of \( \binom{3m-1}{2m-1} \leq 2^{3m-1} \leq 8^m \) operators

\[
B_{i_m}(O_m) \cdots B_{i_1}(O_1) B_{i_1}(O_1) \cdots B_{i_m}(O_m) \quad (i_1 + i_1') + \ldots + (i_m + i_m') \leq 3m.
\] (16)

Now observe from Eq. (9) that \( B_i(O) \in \{O, O^j - O^{i-1}\} \). Therefore each term Eq. (16) can be expanded as a sum of at most \( 2^{2m} \) monomials of the form Eq. (15). Therefore \( \widetilde{\text{Rob}}(O_1, \ldots, O_m) \) expands into at most \( 8^m \cdot 2^{2m} = 2^{5m} \) terms of the form Eq. (15).

\[\square\]

### 3 Optimal ground state approximations

Throughout this section we consider the following scenario. We are given a set of Hermitian operators \( \{H_j\}_{j=1}^n \) such that

\[
0 \leq H_j \leq I \quad \text{for all } j \in [n],
\] (17)

which act on some finite-dimensional Hilbert space \( \mathcal{H} \). We are interested in the nullspace of the operator

\[
H = \sum_{j=1}^n H_j.
\]

Let us write \( \Pi \) for the projector onto the nullspace of \( H \). In other words, \( \Pi \) projects onto the intersection of the nullspaces of all operators \( H_j \) (we are interested in the case where \( \Pi \) is nonzero). Our goal is to approximate \( \Pi \) by a low-degree polynomial in the operators \( \{H_j\} \).

In Sec. 3.1 and Sec. 3.2 we work in a general setting and in particular we do not assume a tensor product structure of the Hilbert space \( \mathcal{H} \) or geometric locality of the operators \( \{H_j\} \). In Sec. 3.1 we consider the simplest case in which all operators \( H_j \) are mutually commuting and we describe the known optimal tradeoff between approximation degree and error. Then, in Sec. 3.2 we show that optimal approximations can be obtained more generally for noncommuting operators which satisfy certain gap and merge properties. These properties themselves assert a kind of one-dimensional structure with respect to the given ordering \( 1 \leq j \leq n \) of the operators. In Sec. 3.3 we describe how a direct application of these results provides optimal ground state approximations for one-dimensional, locally-gapped, frustration-free qudit Hamiltonians. Later we will see how the results of Sec. 3.2 can provide low-entanglement approximations of ground states in the 2D setup.

#### 3.1 Commuting projectors

We begin with the easy case in which all \( \{H_i\} \) are commuting projectors:

\[
H_i^2 = H_i \quad \text{and} \quad [H_i, H_j] = 0 \quad \text{for all } i, j \in [n].
\] (18)

In this case \( (I - H_i) \) is the projector onto the nullspace of \( H_i \), and due to the commutativity Eq. (18) we may express \( \Pi \) exactly as the degree-\( n \) polynomial \( \Pi = \prod_{i=1}^n (I - H_i) \). Our
goal is to construct a lower degree polynomial $P$ that approximates $\Pi$. Since all operators \{${H}_i$\} commute and have \{0,1\} eigenvalues, we may work in a basis in which they are simultaneously diagonal and the problem reduces to that of approximating the product of binary variables $x \in \{0,1\}^n$ which label the eigenvalues of \{${I} - {H}_i$\}$_i$. (Note that here we do not require any properties of the basis which simultaneously diagonalizes these operators, only that it exists). In other words, the problem of approximating the ground space projector for a Hamiltonian which is a sum of commuting projectors, reduces to the problem of approximating the boolean AND function

$$\text{AND}(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } x_1 = x_2 = \ldots = x_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$  

We are faced with the task of constructing a multilinear polynomial $p$ which $\epsilon$-approximates AND in the sense that $|p(x) - \text{AND}(x)| \leq \epsilon$ for each $x \in \{0,1\}^n$. Remarkably, it is possible to achieve an arbitrarily small constant error $\epsilon = O(1)$ using a polynomial of degree $O(\sqrt{n})$ [31]. For example, one can use the Chebyshev polynomial $T_{n/s}(\frac{1}{n} \sum_{i=1}^n x_i)$ of degree $\lceil s \rceil$ which achieves an approximation error $\epsilon = e^{-\Omega(\sqrt{n})}$ as can be seen from Lemma 2.1. Similarly, the acceptance probability of the standard Grover search algorithm [22], viewed as a function of the input bit string $x$ provided as an oracle, constructs such an approximating polynomial [11]. However, neither of these polynomials has optimal degree in the low-error regime where $\epsilon$ decreases with $n$. In that regime an optimal polynomial can be constructed via a low-error refinement of Grover search [15, 18] (see also Ref. [27]).

Here we provide a different family of polynomials that give an optimal approximation to the AND function. These polynomials are obtained in a simple way by combining the Chebyshev polynomial $T_{n/s}$ and the robust polynomial Rob from Sec. 2.1. Soon we will see how this construction can be extended to the non-commuting case. It is unclear to us whether one can alternatively extend the known optimal polynomials constructed in Refs. [27, 15, 18].

**Theorem 3.1 (Optimal approximation of AND).** Let $n$ be a positive integer. For every real number $s \in (\sqrt{n}, n)$, there exists a polynomial $P(x)$ of degree $O(s)$ such that

$$|P(x) - \text{AND}(x)| = e^{-\Omega(s^2)} \quad \text{for all } x \in \{0,1\}^n.$$  

**Proof.** Define the positive integer $t = \lceil \frac{n^2}{s} \rceil$ and note that $1 \leq t \leq n$ due to the specified bounds on $s$. Let $p(y) \overset{\text{def}}{=} T_{\frac{1}{t}2\sqrt{t}}(y)$. From Lemma 2.1 we see that

$$p(0) = 1 \quad \text{and} \quad |p(y)| \leq 2 \cdot e^{-\frac{1}{t}} \leq \frac{1}{20} \quad \text{for all } \frac{1}{t} \leq y \leq 1. \quad (19)$$  

Since $t \leq n$, we may construct a partition $[n] = I_1 \cup I_2 \cup \ldots \cup I_\xi$ where $\xi \overset{\text{def}}{=} \lceil n/t \rceil$ and $|I_k| \leq t$ for all $1 \leq k \leq \xi$. Our polynomial approximation to AND is defined as

$$73A72EA3P(x) = \widetilde{\text{Rob}} \left( p \left( 1 - \frac{1}{|I_1|} \sum_{j \in I_1} x_j \right), p \left( 1 - \frac{1}{|I_2|} \sum_{j \in I_2} x_j \right), \ldots, p \left( 1 - \frac{1}{|I_\xi|} \sum_{j \in I_\xi} x_j \right) \right).$$  

10
Figure 2: An interval of length \( n \) is decomposed into smaller intervals of length \( n/r \) each. The overlap between consecutive intervals is exactly \( n^2/r \). The number of intervals is \( \xi = 2r - 1 \).

Now we observe that the \( k \)th input to the \( \widetilde{\text{Rob}} \) function on the RHS approximates the AND of all bits in the set \( I_k \). To see this, note that \( 1 - \frac{1}{|I_k|} \sum_{j \in I_k} x_j = 0 \) when \( x_j = 1 \) for all \( j \in I_k \), and \( 1 - \frac{1}{|I_k|} \sum_{j \in I_k} x_j \geq 1/t \) if one or more \( x_j = 0 \). Using this fact and Eq. (19), we see that for each \( 1 \leq k \leq \xi \) we have

\[
\left| p \left( 1 - \frac{1}{|I_k|} \sum_{j \in I_k} x_j \right) - \prod_{j \in I_k} x_j \right| \leq \frac{1}{20}. \tag{20}
\]

Now applying Lemma 2.2 with \( \varepsilon = 1/20 \), and noting that \( \text{Rob}(x) = \text{AND}(x) \) we see that, for each \( x \in \{0, 1\}^n \),

\[
|P(x) - \text{AND}(x)| \leq 2^{-\xi} \leq 2^{-n/t} \leq 2^{-s^2/n}.
\]

The degree of the polynomial is at most \( 3\xi \cdot 2 \sqrt{t} = O(s) \). \( \square \)

### 3.2 Operators with gap and merge properties

We now consider a more general case in which the operators \( \{H_j\}_{j=1}^n \) still satisfy (17), but may not be projectors and are not assumed to commute. For any subset \( S \subseteq [n] \) of the operators, we define the corresponding Hamiltonian

\[
H_S \overset{\text{def}}{=} \sum_{j \in S} H_j
\]

and the projector \( \Pi_S \) onto its nullspace. Similarly, we define \( \text{gap}(H_S) \) to be the smallest nonzero eigenvalue of \( H_S \). A crucial difference between our setting here and the commuting setting considered previously, is that a product \( \Pi_S \Pi_T \) is not in general equal to \( \Pi_{S \cup T} \).

We require our operators to satisfy two properties which are defined with respect to the given ordering \( 1 \leq j \leq n \). To describe these properties it will be convenient to define an interval as a contiguous subset \( \{j, j+1, \ldots, k-1, k\} \subseteq [n] \). The gap property states a lower bound \( \Delta \) on the smallest nonzero eigenvalue of any interval Hamiltonian \( H_S \). The merge property asserts that \( \Pi_S \Pi_T \cong \Pi_{S \cup T} \) for overlapping intervals \( S, T \), with error decreasing exponentially in the size of the overlap region. We now state these properties more precisely.

**Definition 3.2.** Operators \( \{H_j\}_{j=1}^n \) satisfy the gap and merge properties if, for some \( \Delta \in (0, 1] \), the following conditions hold for all intervals \( S \subseteq [n] \) and any partition \( S = ABC \) into

\( \footnote{We use the convention that \( \text{gap}(h) = 1 \) if \( h = 0 \).} \)
three consecutive intervals:

\[ \text{gap}(H_S) \geq \Delta \quad \text{[Gap property]} \quad (21) \]
\[ \|\Pi_{AB}\Pi_{BC} - \Pi_S\| \leq 2e^{-|B|\sqrt{\Delta}} \quad \text{[Merge property].} \quad (22) \]

Note that the parameter \( \Delta \) in this definition appears in both the gap and merge properties. One could alternatively consider a more general definition where each of these properties has its own parameter, though we will not need to.

In the following we show that the optimal scaling \( e^{-n(O(s^2)} \) of error with degree \( s \) can be recovered in this noncommutative setting, by a recursive use of the robust polynomial, with one use of the Chebyshev polynomial and gap property in the base level of the recursion. The analysis uses the merge property to bound the error in the recursion. The following theorem describes our results for the case where the approximation degree scales less than linearly in \( n \).

**Theorem 3.3 (Less than linear degree).** Suppose \( \{H_j\}_{j=1}^n \) satisfy Eqs. (17, 21, 22) for some \( \Delta \in (0, 1] \). Let \( \delta \in (0, 1/4) \) be fixed and let \( s \) be a real number satisfying

\[ 2\sqrt{n}\Delta^{-1/2} \leq s \leq (1/4)n^{1-\delta}\Delta^{-1/4}. \quad (23) \]

There is a degree \( O(s) \) Hermitian multivariate polynomial \( P \) in the operators \( \{H_j\}_{j=1}^n \) such that

\[ P\Pi = \Pi \quad \text{and} \quad \|P - \Pi\| = e^{-\frac{s^2}{4n}}. \quad (24) \]

In the above, the big-O notation hides a constant which depends only on \( \delta \). We shall also be interested in a case where \( \delta \) is taken very close to 0 and the degree is close to linear. This almost-linear degree approximation will be used to establish the area law for two-dimensional spin systems. For that application it will be useful to describe the structure of the polynomial \( P \) in more detail. To this end, we first define certain families \( P(\alpha, \beta) \) of elementary polynomials as follows.

**Definition 3.4.** For \( \alpha, \beta > 0 \), let \( P(\alpha, \beta) \) denote the set of polynomials of the form

\[(H_{S_1})^{j_1}(H_{S_2})^{j_2}\ldots(H_{S_k})^{j_k} \quad j_1 + j_2 + \ldots + j_k \leq \alpha \quad \text{and} \quad k \leq \beta.\]

where \( j_1, j_2, \ldots, j_k \) are positive integers and each set \( S_1, S_2, \ldots, S_k \subseteq [n] \) is an interval.

**Theorem 3.5 (Near-linear degree).** Suppose \( \{H_j\}_{j=1}^n \) satisfy Eqs. (17, 21, 22) for some \( \Delta \in (0, 1] \) and that \( n \geq C\Delta^{-1} \), where \( C > 0 \) is some absolute constant. There exist real numbers

\[ \alpha \leq n\Delta^{-1/4} \quad \text{and} \quad \beta = \Delta^{1/2}n^{1-O((\log n)^{-1/4})} \quad (25) \]

such that the following holds. There exists a Hermitian multivariate polynomial \( P \) in the operators \( \{H_i\}_{i=1}^n \) of degree at most \( \alpha \) such that

\[ P\Pi = \Pi \quad \text{and} \quad \|P - \Pi\| \leq \exp\left(-\beta e^{\sqrt{\log(n)}}\right), \quad (26) \]

and such that \( P \) can be expressed as a linear combination of at most \( (2\alpha)^\beta \) elements of \( P(\alpha, \beta) \).
Theorems 3.3 and 3.5 are obtained as consequences of the following Lemma, which treats the special case where \( n \) has the form \( tr^{b-1} \) for suitably chosen positive integers \( t, r, b \). It constructs an approximating polynomial \( P \) recursively, with \( b \) levels of recursion.

**Lemma 3.6.** Suppose \( n = tr^{b-1} \) for positive integers \( t, r, b \) such that \( t \) is even and

\[
\frac{16r}{t\sqrt{\Delta}} \leq 1/\Gamma \leq 1
\]

for some real positive number \( \Gamma \). There is a Hermitian polynomial \( P \) of the operators \( \{H_i\}_{i=1}^n \) of degree at most \( s \), where

\[
s \overset{\text{def}}{=} \frac{4n \cdot 6^{b-1}\Gamma}{\sqrt{t\Delta}} \quad \text{and} \quad P\Pi = \Pi \quad \text{and} \quad \|P - \Pi\| \leq \frac{1}{200} \exp\left(-\Gamma r^{b-1}\right)
\]

Moreover, \( P \) can be expressed as a linear combination of at most \( (2s)^{(6r)^{b-1}} \) polynomials from the set \( \mathcal{P}(s,(6r)^{b-1}) \).

**Proof of Lemma 3.6.** Let us fix \( t \) and \( r \) satisfying Eq. (27). We show the claim by induction on \( b \).

First consider the base case \( b = 1 \). In this case we have \( n = t \) and we take

\[
P \overset{\text{def}}{=} T_{\Delta/s} \left( \frac{1}{t} \sum_{j=1}^{t} H_j \right), \quad s = 4\Gamma \sqrt{t/\Delta}
\]

which is a polynomial of degree at most \( s = 4\Gamma \sqrt{t/\Delta} = 4n\Gamma/(\sqrt{t\Delta}) \) as claimed. By construction, \( P \) is Hermitian, and applying Lemma 2.3 and using the gap property we get \( P\Pi = \Pi \) and \( \|P - \Pi\| \leq 2e^{-s\Gamma} \leq 1/200e^{-\Gamma} \) as required. Finally, note that \( P \) is a univariate polynomial of degree at most \( s \) in \( H_S \), where \( S = \{1, 2, \ldots, t\} \). Thus \( P \) is a linear combination of at most \( s + 1 \leq 2s \) elements of \( \mathcal{P}(s,1) \).

Next let \( b \geq 2 \) and suppose the claim is true for \( b-1 \). Let us subdivide our operator labels \([n]\) into \( \xi = 2r - 1 \) overlapping intervals of length \( n/r \) as depicted in Fig. 2. Consecutive intervals overlap in \( n/(2r) \) places (this is an integer as \( t \) is even). Let us write these intervals as \([n] = I_1 \cup I_2 \cup \ldots I_\xi \). We apply the inductive hypothesis to obtain an approximation \( P^{(\xi)} \)
to the ground state projector $\Pi^{(j)}$ of each interval $I_j$. The inductive hypothesis states that $P^{(j)}$ is a Hermitian polynomial of degree at most

$$\frac{4(n/r) \cdot 6^{b-2} \Gamma}{\sqrt{t \Delta}}$$

(29)

and satisfies

$$P^{(j)} \Pi^{(j)} = \Pi^{(j)} \quad \text{and} \quad \|P^{(j)} - \Pi^{(j)}\| \leq \frac{1}{200} e^{-\Gamma r^{b-2}}.$$

We then define our polynomial approximation to the ground space of the whole chain as

$$P = \widetilde{\text{Rob}}(P^{(1)}, P^{(2)}, \ldots, P^{(\xi)}).$$

(30)

Since $\widetilde{\text{Rob}}$ is a polynomial of degree at most $3 \xi \leq 6r$, and each input is Hermitian and has degree at most Eq. (29), $P$ is Hermitian and has degree upper bounded as in Eq. (28). Applying Claim 2.5 we see that $P \Pi = \Pi$. Applying Lemma 2.4 we get

$$\|P - \text{Rob}(P^{(1)}, P^{(2)}, \ldots, P^{(\xi)})\| \leq \left(\frac{1}{20} e^{-\Gamma r^{b-2}}\right)^{\xi}.$$

(31)

Using the merge property (22) and the triangle inequality we get

$$\|\Pi^{(1)} \Pi^{(2)} \ldots \Pi^{(\xi)} - \Pi\| \leq 2 \xi \exp\left(-\frac{\sqrt{\Delta n}}{2r}\right).$$

(32)

Now recall that $\text{Rob}(P^{(1)}, P^{(2)}, \ldots, P^{(\xi)}) = C^\dagger C$ where $C = \Pi^{(1)} \Pi^{(2)} \ldots \Pi^{(\xi)}$. Substituting in Eq. (31), applying the triangle inequality, and using Eq. (32), we arrive at

$$\|P - \Pi\| \leq \|P - C^\dagger C\| + \|C^\dagger C - \Pi\| \leq \left(\frac{1}{20} e^{-\Gamma r^{b-2}}\right)^{\xi} + 4 \xi \exp\left(-\frac{\sqrt{\Delta n}}{2r}\right).$$

To complete the proof we show that each of terms on the RHS is at most $1/400 \exp (-\Gamma r^{b-1})$. The first term is bounded in this way since $\xi \geq r$ and $r \geq 2^4$. For the second term, we write

$$4 \xi \exp\left(-\frac{\sqrt{\Delta n}}{2r}\right) = 4 \xi \exp\left(-\frac{tr^{b-2} \sqrt{\Delta}}{2}\right) \leq 8r \exp (-8\Gamma r^{b-1}) \leq 1/400 \exp (-\Gamma r^{b-1}).$$

where we used Eq. (27) and in the last inequality we used the fact that $r, b \geq 2$ and $\Gamma \geq 1$.

Finally, let us show that $P$ has the claimed structure. The inductive hypothesis states that each $P^{(j)}$ is a sum of at most $(2s/6r)^{(6r)^{b-2}}$ polynomials from the set $\mathcal{P}(s/6r, (6r)^{b-2})$. Using this fact and Claim 2.6 with $m \leftarrow \xi$ we see that Eq. (30) is a sum of at most

$$2^{5\xi} \left((2s/6r)^{(6r)^{b-2}}\right)^{3\xi}$$

(33)

polynomials from the set $\mathcal{P}(3\xi s/6r, (6r)^{b-2}3\xi)$. Using the upper bound $\xi \leq 2r$ we see that each of the latter elementary polynomials is in the set $\mathcal{P}(s, (6r)^{b-1})$, and that the number of them Eq. (33) is at most $2^{10r}(2s/6r)^{(6r)^{b-1}} \leq (2s)^{(6r)^{b-1}}$, where we used $2^{10r}/(6r)^{6r} \leq 1$.

\footnote{If $r = 1$ then $b > 1$ is the same as $b = 1$ which is handled above.}
Let us now see how to obtain Theorems 3.3, 3.5 from Lemma 3.6. The proofs are along
the same lines so we handle them both below. As noted above, the key ideas are all contained
in Lemma 3.6 and all that is left is to choose the parameters \( t, r, b \) in a suitable manner. The
analysis is somewhat tedious as several parameters are required to be integers.

\textbf{Proof of Theorem 3.3 and Theorem 3.5.} Suppose \( \{H_j\}_{j=1}^n \) satisfy Eqs. (17, 21, 22), \( \delta \in (0, 1/4) \),
and that \( s \) is a real number satisfying Eq. (23). We shall specify an integer \( n' \geq n \) for which
a suitable polynomial approximation \( P' \), of a certain degree \( s' \), can be constructed using
Lemma 3.6. By a simple padding argument, this implies a polynomial approximation \( P \)
with degree \( s' \) and the same approximation error for the original system of \( n \) operators\(^5\). In
particular, we choose \( n' = tr^{b-1} \) where

\[
b = 1 + \lceil (2\delta)^{-1} \rceil \quad t = 2\lceil \frac{n^2}{\Delta s^2} \rceil \quad r = \lceil (n/t)^{1/(b-1)} \rceil.
\]

(34)

Clearly \( t, b, r \) are positive integers and \( t \) is even. Note that the lower bound from Eq. (23)
implies \( t \leq n \) and therefore \( r \leq 2(n/t)^{\frac{1}{b-1}} \). Using this fact we see that

\[
n' = tr^{b-1} \leq 2^{b-1}n.
\]

(35)

We now show that the condition Eq. (27) is satisfied as long as \( \Gamma \leq n^{4\delta^2} \).

\[
\frac{16r \Gamma}{t \sqrt{\Delta}} \leq \frac{32 \Gamma}{n \sqrt{\Delta}} \frac{(n/t)^{1+\frac{1}{2(b-1)}}}{1+2\delta} \leq \frac{32 \Gamma}{n \sqrt{\Delta}} \frac{(\Delta s^2/2n)^{1+2\delta}}{2^{1+24\delta}} \leq \frac{32}{2^{1+24\delta}} \Delta \Gamma \leq 1
\]

(36)

where in the second-to-last inequality we upper bounded \( s \) using Eq. (23) and in the last
inequality we used the fact that \( \Delta \leq 1 \) and \( \Gamma \leq n^{4\delta^2} \).

Let us now bound the degree \( s' \) and approximation error of the polynomial \( P' \) obtained
from Lemma 3.6 with the choices Eq. (34). Using the fact that \( t \geq 2n^2/(s^2 \Delta) \) we get

\[
s' = \frac{4n' \cdot 6^{b-1} \Gamma}{\sqrt{t \Delta}} \leq 2\sqrt{2} s \cdot (n'/n) \cdot 6^{b-1} \Gamma \leq 2\sqrt{2} \cdot (12)^{b-1} s \Gamma
\]

(37)

where we used Eq. (35). The approximation error satisfies

\[
\|P' - \Pi'\| \leq (1/200)e^{-\Gamma r^{b-1}} \leq e^{-n \Gamma/t} \leq e^{-\Delta s^2 \Gamma/4n},
\]

(38)

where we used \( n \leq n' \) and in the last inequality we used the fact that \( t \leq \frac{4n^2}{\Delta s^2} \). Theorems
3.3 and 3.5 are obtained as special cases of the above.

Theorem 3.3 is obtained in the special case that \( \delta \) is a fixed constant and with the choice
\( \Gamma = 1 \). In this case we have \( b = O(1) \) and using Eq. (37) we see that our polynomial has
degree \( s' = O(s) \). The approximation error Eq. (38) has the desired form since \( \Gamma = 1 \).

Now let us prove Theorem 3.5. This is obtained by specializing to the case

\[
s = (1/4) \Delta^{-1/4} n^{1-\delta} \quad \delta = (\log n)^{-1/4} \quad \Gamma = n^{4\delta^2} = e^{4\sqrt{\log n}}.
\]

(39)

\(^5\)It suffices to set \( H_j = 0 \) for all \( n+1 \leq j \leq n' \). This does not change the nullspace projector \( \Pi \). Moreover,
the new system also satisfies the gap and merge properties for the same \( \Delta \).
Note that with these choices we have \( b - 1 = \lceil (\log n)^{1/4} \rceil \). Here we have chosen \( s \) at the upper limit of Eq. (23). We also need to verify that the lower bound in Eq. (23) holds (that is, the range of allowed degrees Eq. (23) is nonempty). We see that this constraint is satisfied as long as \( n^{2-4d} \Delta \geq (64)^2 \). This follows from our assumption \( n \Delta \geq C \) for some sufficiently large absolute constant \( C \).

Lemma 3.6 then states that our polynomial \( P' \) is a sum of at most \( (2\alpha)^\beta \) elements of \( P(\alpha, \beta) \), where \( \alpha \overset{\text{def}}{=} s' \) and \( \beta \overset{\text{def}}{=} (6r)^{b-1} \). The approximation error, using the first upper bound in Eq. (38), is at most

\[
e^{-\beta \Gamma/6^{b-1}} = \exp \left( -\beta e^{4\sqrt{\log n}6^{-(b-1)}} \right) \leq \exp \left( -\beta e^{\sqrt{\log(n)}} \right)
\]

where we used the fact that \( e^{3\sqrt{\log(n)}} \geq 6^{\lceil (\log n)^{1/4} \rceil} \) for \( n \geq 2 \).

It remains to establish Eq. (25). Using Eq. (37) and plugging in our choices from Eq. (39) we get

\[
\alpha \leq \frac{1}{\sqrt{2}} \Delta^{-1/4} n^{1-(\log n)^{-1/4}+4(\log n)^{-1/2}} (12)^{\lceil (\log n)^{1/4} \rceil}.
\]

Since \( (12)^{\lceil (\log n)^{1/4} \rceil} = n^{O((\log n)^{-3/4})} \), we see from the above that for \( n \) larger than some absolute constant we have \( \alpha \leq n \Delta^{-1/4} \) as claimed (we ensure this by choosing \( C \) sufficiently large).

To bound \( \beta \), we use the facts that \( 1 \leq (n'/n) \leq n^{O((\log n)^{-3/4})} \) (cf. Eq. (35)) and \( t = \Theta(\Delta^{-1/2} n^{(\log n)^{-1/4}}) \) which follows from our choices Eqs. (39,34). Combining these bounds we get

\[
\beta = 6^{b-1} n'/t = \Delta^{1/2} n^{1-O((\log n)^{-1/4})}
\]

as claimed. \( \Box \)

### 3.3 Application to 1D quantum spin systems

As a prototypical application of the results of the previous section, here we specialize to the case of frustration-free one-dimensional quantum spin systems with a local gap.

Consider a 1D system of \( n + 1 \) qudits of local dimension \( d \geq 2 \). The Hilbert space is \((\mathbb{C}^d)^{\otimes n+1}\) and the Hamiltonian is \( H = \sum_{j=1}^n H_j \), where each operator \( H_j \) satisfies \( 0 \leq H_j \leq I \) and acts nontrivially only on qudits \( j \) and \( j + 1 \) (and as the identity on all other qudits). The *local gap* \( \gamma \) is defined as the minimum spectral gap of a subset of Hamiltonian terms

\[
\gamma \overset{\text{def}}{=} \min_{S \subseteq [n]} \text{gap}(\sum_{j \in S} H_j).
\]

By definition, operators \( \{H_j\}_{j=1}^n \) satisfy the gap property Eq. (21) with \( \Delta = \gamma \). Below we show that the merge property is satisfied with \( \Delta = \gamma/80 \) (a consequence of the “detectability lemma” [2, 5]). Therefore we may substitute \( \Delta = \gamma/80 \) in Theorems 3.3 and 3.5 to obtain optimal approximations to the ground state projector \( \Pi \), as claimed in Theorem 1.2.

**Lemma 3.7** ([5]). Suppose \( H = \sum_{j=1}^n H_j \) is a 1D frustration-free qudit Hamiltonian with local gap \( \gamma \). Then \( \{H_j\}_{j=1}^n \) satisfy the merge property Eq. (22) with \( \Delta = \gamma/80 \).
Proof sketch. Let $S \subseteq [n]$ be an interval partitioned as $S = ABC$. Define the “detectability lemma” operator of the interval $S$ by:

$$DL_S \overset{\text{def}}{=} \prod_{j \in S \cap \{1,3,5,\ldots\}} \Pi_j \prod_{j \in S \cap \{2,4,6,\ldots\}} \Pi_j$$

where $\Pi_j$ is the projector onto the nullspace of $H_j$. Clearly the nullspace of $(I - DL_S^\dagger DL_S)$ is equal to that of $\sum_{j \in S} H_j$. Moreover, the detectability lemma, as summarized in Theorem A.1 (setting $g = 2$) implies

$$\text{gap}(I - DL_S^\dagger DL_S) \geq \gamma/5.$$ 

One can show (cf. Claim 6 of Ref. [5])

$$\Pi_{AB}(DL_S^\dagger DL_S)^q \Pi_{BC} = \Pi_{AB} \Pi_{BC} \quad \text{for all integers } 0 \leq q \leq \frac{|B|}{8}. \quad (40)$$

The above implies $\Pi_{AB} f(I - DL_S^\dagger DL_S) \Pi_{BC} = \Pi_{AB} \Pi_{BC}$ for any polynomial $f$ of degree at most $|B|/8$, such that $f(0) = 1$. Using this and $\Pi_{AB} \Pi_{ABC} \Pi_{BC} = \Pi_{ABC}$ gives

$$\|\Pi_{AB} \Pi_{BC} - \Pi_{ABC}\| = \|\Pi_{AB} (T_{\gamma/5,|B|/8} (I - DL_S^\dagger DL_S) - \Pi_{ABC}) \Pi_{BC}\| \leq 2e^{-\frac{|B|}{4\sqrt{\gamma/5}}},$$

where in the last inequality we used Lemma 2.3.

\[\square\]

4 2D Area law

Here we consider a 2D locally gapped, frustration-free quantum spin system along with a bipartition of the qubits into two regions. We use the results of Sec. 3.2 to construct a polynomial approximate ground state projector (AGSP) which has a kind of 1D structure along the boundary of the bipartition. We show that this AGSP has low enough error as a function of its Schmidt rank across the bipartition, to establish the area law as stated in Theorem 1.1 using the method from Refs. [2, 8, 7].

Consider a system of qudits of local dimension $d$ arranged at the vertices of an $L \times (n+1)$ grid with $n + 1$ rows and $L$ columns, as shown in Fig. 1. The Hilbert space is $(\mathbb{C}^d)^{\otimes L(n+1)}$, and we index qudits by their (column, row) coordinates $(i, j) \in [L] \times [n + 1]$. We consider a Hamiltonian which acts as a sum of local projectors

$$H_0 = \sum_{i=1}^{L-1} \sum_{j=1}^{n} h_{ij} \quad h_{ij}^2 = h_{ij}$$

where $h_{ij}$ acts nontrivially only on the qudits in the set $\{i, i+1\} \times \{j, j+1\}$. We assume that $H_0$ has a unique ground state $|\Omega\rangle$ such that $H_0 |\Omega\rangle = 0$. Since $h_{ij} \geq 0$, the latter condition

\[\text{This is without loss of generality. Consider a frustration-free hamiltonian } H' = \sum c_I h'_{i,j}, \text{ where } c_I \geq h'_{i,j} \geq 0 \text{ are not projectors. Let } h_{i,j} \text{ be the projector onto the span of } h'_{i,j}, \text{ so that } c_I h_{i,j} \geq h'_{i,j}. \text{ The local spectral gap of } H_0 \text{ is at least } \frac{1}{\varepsilon} \text{ times the local spectral gap of } H' \text{ and they have the same ground space.}\]
is equivalent to the frustration-free property \( h_{ij} |\Omega\rangle = 0 \) for all \( i, j \). Our results depend on the local gap of \( H_0 \):

\[
\gamma \overset{\text{def}}{=} \min_{S \subseteq [L-1] \times [n]} \text{gap} \left( \sum_{(i,j) \in S} h_{ij} \right). \tag{41} \text{[Local gap]}
\]

(recall \( \text{gap}(M) \) denotes the smallest nonzero eigenvalue of a positive semidefinite operator \( M \).) We note that for our purposes it would in fact be sufficient to consider a local gap in which the minimization is restricted to rectangular regions.

We consider a bipartition of the lattice into left and right regions, corresponding to a “vertical cut” between a given column \( c \) and \( c + 1 \), as depicted in Fig. 1. In the following we write \( \text{SR}(M) \) for the Schmidt rank of an operator with respect to the cut.

To bound the entanglement entropy of \(|\Omega\rangle\), we use the powerful method of approximate ground state projectors (AGSP) developed in Refs. [24, 2, 8, 7]. The following theorem is obtained by specializing Corollary III.4 of Ref. [8] to the case of Hermitian \( K \).

**Theorem 4.1** (Entanglement entropy from AGSP [8]). Suppose \( K \) is a Hermitian operator satisfying \( K |\Omega\rangle = |\Omega\rangle \) and

\[
\| K - |\Omega\rangle \langle\Omega| \cdot \text{SR}(K) \leq \frac{1}{2}.
\]

Then the entanglement entropy of \(|\Omega\rangle\) across the cut is upper bounded by \( O(1) \cdot \log(\text{SR}(K)) \).

We use the results of Sec. 3.2 to construct a suitable AGSP \( K \). To this end we first construct a system of operators \( \{ H_j \}_{j=1}^n \) which has the gap and merge properties Eqs. (21, 22).

We are going to focus our attention on a band of \( w \) columns of qudits centered around the cut \( (c, c + 1) \); see Figure 3(a). Here \( w \) is an integer that we will choose later to depend only on the cut length \( n \). For now, suppose WLOG that the cut is not too close to the left or right boundary of the lattice, i.e., \( w \leq c \leq L - w \) (otherwise we can ensure this by a padding argument). We reorganize the indices of qudits by changing \((i, j) \rightarrow (i-(c-w/2), j)\) such that the cut is between indices \((w/2, w/2 + 1) \times [n + 1]\), and the region of width \( w \) is \((1, w) \times [n + 1]\). Let \( \Pi_{L,j} \) and \( \Pi_{R,j} \) project onto the ground space of all local terms \( h_{ij} \) with \( i < 1 \) and \( i \geq w \) respectively. For each row \( 1 \leq j \leq n \) define

\[
H_j \overset{\text{def}}{=} \frac{1}{2w} \left( \sum_{1 \leq i < w} h_{ij} + (I - \Pi_{L,j}) + (I - \Pi_{R,j}) \right), \tag{42}
\]

see Figure 3(a). The norm of this operator is bounded as \( \| H_j \| \leq \frac{1}{2w} (w + 2) \leq 1 \). The following lemma is proved in Appendix A using the detectability lemma machinery [2, 5].

**Lemma 4.2.** The operators \( \{ H_j \}_{j=1}^n \) satisfy the gap and merge properties Eqs. (21, 22) with \( \Delta = \Theta(\gamma/w) \).

Therefore the operators \( \{ H_j \}_{j=1}^n \) satisfy the requirements of Theorem 3.5. The approximating polynomial from the latter theorem is a sum of elementary polynomials from the set \( \mathcal{P}(a, b) \) from Definition 3.4. We shall use the following lemma to bound the Schmidt rank of
each of them. A key feature of the bound given below is that the Schmidt rank is amortized across the $w$ columns in the sense that its exponential scaling with degree is $(\text{total degree})/w$ instead of simply (total degree). This feature (in a slightly different setting) was also key to the 1D area-law bound from Ref. [7].

**Lemma 4.3** (Schmidt rank amortization). $\text{SR}(Q) \leq (16\alpha^4 d^4 n)^{\frac{w}{w} + b + w n}$ for all $Q \in \mathcal{P}(a, b)$.

The proof of Lemma 4.3 is provided in Appendix B. We are now in a position to prove our main result:

**Theorem 4.4.** Suppose $\gamma = \Omega(1)$ and $d = O(1)$. The entanglement entropy of $|\Omega\rangle$ across the cut is at most $n^{1 + O((\log n)^{-1/5})}$.

**Proof.** As a first step we apply Theorem 3.5 with the system of operators $\{H_j\}_{j=1}^n$ defined in Eq. (42). Lemma 4.2 states that we may set $\Delta = \Theta(\gamma/w) = \Theta(1/w)$. Let $P$ be the polynomial provided by the theorem. Our AGSP is defined as $K \overset{\text{def}}{=} P w^2$, which is Hermitian (since $P$ is) and satisfies $K|\Omega\rangle = |\Omega\rangle$, since $\sum_{1 \leq j \leq n} H_j$ has unique ground state $|\Omega\rangle$. The error bound from the theorem gives

$$\|K - |\Omega\rangle\langle\Omega|\| \leq e^{-\beta w^2 e^{\sqrt{\log n}}}. \quad (43)$$

The theorem also implies that $K$ can be expressed as a linear combination of $(2\alpha)^w \beta$ polynomials from $\mathcal{P}(w^2 \alpha, w^2 \beta)$, where

$$\alpha = O(n w^{1/4}) \quad \text{and} \quad \beta = w^{-1/2} n^{1 - O((\log n)^{-1/4})}.$$ 

Thus, applying Lemma 4.3

$$\text{SR}(K) \leq (2\alpha)^w \beta \max_{Q \in \mathcal{P}(w^2 \alpha, w^2 \beta)} \text{SR}(Q) \leq (16\alpha^4 w^8 nd^4)^{w\alpha + 2w^2 \beta + wn}. \quad (44)$$

As a first simplification, let us focus on the term in parentheses above. Our choice of $w$ below (Eq. (48)) satisfies $w \leq O(n)$ and therefore $16\alpha^4 w^8 nd^4 = O(\text{poly}(n))$, where we used $d = O(1)$ and $\gamma = \Omega(1)$. Using this in Eq. (44) and combining with Eq. (43) gives

$$\|K - |\Omega\rangle\langle\Omega|\| \cdot \text{SR}(K) \leq \exp \left( - \beta w^2 e^{\sqrt{\log n}} + (w\alpha + \beta w^2 + wn) \cdot O(\log n) \right). \quad (45)$$
Taking $n$ large enough that $e^{\sqrt{\log n} - O(\log n)} \geq 1$, and substituting the values $\alpha, \beta$ gives
\[
\|K - |\Omega\rangle\langle \Omega|\cdot SR(K) \leq \exp \left( - \beta w^2 + (w \alpha + wn) \cdot O(\log n) \right) \leq \exp \left( - n^{1-O(\log n)^{-1/4}} w^{3/2} + O(w^{5/4} n \log n) \right)
\]
Now let us choose
\[
w = \Theta(n^{(\log n)^{-1/5}})
\]
(here $1/5$ is somewhat arbitrary). With this choice, the RHS of Eq. (47) can be made less than $1/2$ for $n$ sufficiently large. Applying Theorem 4.1 we get that the entanglement entropy of $|\Omega\rangle$ across the cut is at most $O(1) \cdot \log(SR(K)) \leq n^{1+O((\log n)^{-1/5})}$.

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A Gap and merge properties in 2D

Here we prove Lemma 4.2, showing that the operators \( \{ H_j \}_{j=1}^n \) satisfy the gap and merge properties Eqs. (21, 22) with \( \Delta = \Theta(\gamma/w) \). Crucial to our analysis is the following theorem which encapsulates the detectability lemma [2, 5] and its converse [20].

**Theorem A.1** (Corollary 1 and Lemma 4 in [5], [20]). Let \( Q_1, Q_2, \ldots, Q_m \) be a collection of Hermitian projectors such that each projector commutes with all but at most \( g \) others. Define \( DL := \prod_{i=1}^m (I - Q_i) \), where we fix some (arbitrary) ordering of terms in the product. Then

\[
4 \cdot \text{gap} \left( \sum_{i=1}^m Q_i \right) \geq \text{gap} (I - DL \dagger DL) \geq \frac{1}{g^2 + 1} \cdot \text{gap} \left( \sum_{i=1}^m Q_i \right).
\]

**Merge property**: The proof is very similar to that of Lemma 3.7. Fix an interval \( S \subseteq [n] \) and let \( \Pi_S \) be the projector onto the ground space of \( H_S \). The projectors \( \{ h_{ij} \}_{i \in [L-1], j \in S} \) can be partitioned into four subsets \( \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4 \) such that projectors within each subset mutually commute. Let us define \( DL_S \overset{\text{def}}{=} \prod_{k \in [4]} \left( \prod_{h_{ij} \in \mathcal{G}_k} (I - h_{i,j}) \right) \). Fix a partition \( S = ABC \) (Figure 3(b)). Analogous to Eq. (40), we have

\[
\Pi_{AB}(DL_S \dagger DL_S)^q \Pi_{BC} = \Pi_{AB} \Pi_{BC} \quad \text{for all integers} \quad 0 \leq q \leq \left\lfloor \frac{|B|}{8} \right\rfloor.
\]

**Gap property**: Next we show \( \text{gap}(H_S) = \Theta(\gamma/w) \), where

\[
H_S = \sum_{j \in S} H_j = \frac{1}{2w} \left( \sum_{1 \leq i < w, j \in S} h_{ij} + \sum_{j \in S} (I - \Pi_{L,j}) + \sum_{j \in S} (I - \Pi_{R,j}) \right).
\]
To this end, define
\[
DL' \overset{\text{def}}{=} \left( \prod_{j \in S, j=\text{even}} \Pi_{L,j} \Pi_{R,j} \right) \left( \prod_{1 \leq i < w, j \in S} (I - h_{ij}) \right) \left( \prod_{j \in S, j=\text{odd}} \Pi_{L,j} \Pi_{R,j} \right). \tag{49}
\]

The nullspace \( \Pi_S \) of \( H_S \) is the nullspace of \( I - (DL')^\dagger DL' \). Using Theorem A.1, we find that gap\((2wH_S) = \Theta \left( \text{gap} \left( I - (DL')^\dagger DL' \right) \right) \). To establish the gap property, it suffices to show that gap\((I - (DL')^\dagger DL') = \Theta(\gamma) \). To show this we will use the following identities, which are a consequence of \( h_{ij} \Pi_{L,j} = 0 \) (\( h_{ij} \Pi_{R,j} = 0 \)) for \( i < 1 \) (\( i \geq w \)):

\[
\prod_{j \in S, j=\text{even}} \Pi_{L,j} \Pi_{R,j} = \prod_{j \in S, j=\text{even}} \Pi_{L,j} \left( \prod_{i < 1 \text{ or } i \geq w, j \in S, j=\text{even}} (I - h_{ij}) \right),
\]

\[
\prod_{j \in S, j=\text{odd}} \Pi_{L,j} \Pi_{R,j} = \left( \prod_{i < 1 \text{ or } i \geq w, j \in S, j=\text{odd}} (I - h_{ij}) \right) \prod_{j \in S, j=\text{odd}} \Pi_{L,j} \Pi_{R,j}.
\]

Substituting in Eq. (49), we find

\[
DL' = ( \prod_{j \in S, j=\text{even}} \Pi_{L,j} \Pi_{R,j} ) DL'' ( \prod_{j \in S, j=\text{odd}} \Pi_{L,j} \Pi_{R,j} ), \quad \text{where}
\]

\[
DL'' \overset{\text{def}}{=} \left( \prod_{i < 1 \text{ or } i \geq w, j \in S, j=\text{even}} (I - h_{ij}) \right) \left( \prod_{1 \leq i < w, j \in S} (I - h_{ij}) \right) \left( \prod_{i < 1 \text{ or } i \geq w, j \in S, j=\text{odd}} (I - h_{ij}) \right).
\]

Now \( DL'' \) is a product of all the projectors \( I - h_{ij} \) with \( j \in S \). Applying Theorem A.1 we see that gap\((I - (DL'')^\dagger DL'') = \Theta \left( \text{gap} \left( \sum_{i,j \in S} h_{ij} \right) \right) = \Theta(\gamma) \). Furthermore, \( I - (DL')^\dagger DL' \) and \( I - (DL'')^\dagger DL'' \) have the same nullspace \( \Pi_S \). Thus, gap\((I - (DL')^\dagger DL') \geq \text{gap} \left( I - (DL')^\dagger DL'' \right) = \Theta(\gamma) \). This completes the proof.

### B Schmidt rank amortization

Here we prove Lemma 4.3. Let \( Q \in \mathcal{P}(a, b) \) be given. By definition, \( Q = (H_{S_1})^{j_1}(H_{S_2})^{j_2} \ldots (H_{S_k})^{j_k} \), where \( k \leq b \) and \( \sum_{p=1}^k j_p \leq a \). Without loss of generality we assume \( k = b \) and \( \sum_{p=1}^k j_p = a \) (the claimed upper bound in Lemma 4.3 is an increasing function of \( a, b \)).

Following Ref. [7], we introduce complex variables \( Z \overset{\text{def}}{=} \{ Z_i \}_{i \in [w]} \in \mathbb{C}^w \) and replace \( h_{ij} \leftarrow Z_i h_{ij} \) for all \( i \in [w] \). That is, for each \( j \in [n] \), we define

\[
H_j(Z) \overset{\text{def}}{=} \frac{1}{2w} (I - \Pi_{L,j} + I - \Pi_{R,j} + \sum_{i \in [w]} h_{ij} Z_i).
\]

and for \( S \subseteq [n] \) let \( H_S(Z) = \sum_{j \in S} H_j(Z) \). Similarly, define

\[
Q(Z) \overset{\text{def}}{=} (H_{S_1}(Z))^{j_1} \cdot (H_{S_2}(Z))^{j_2} \cdot \ldots \cdot (H_{S_k}(Z))^{j_k} \tag{50}
\]

We view \( Q(Z) \) as a multivariate polynomial in the components of \( Z \in \mathbb{C}^w \) with operator-valued coefficients. We are interested in the entanglement of \( Q(Z) \) for \( Z = (1, 1, \ldots, 1) \) across
the given vertical cut \((c, c+1) \times [n+1]\). Below we use the notation \(\text{SR}_r(M)\) to denote the Schmidt rank of an operator \(M\) across a vertical cut \((r, r+1) \times [n+1]\), omitting the subscript if \(r = c\), i.e., \(\text{SR}(M) \equiv \text{SR}_c(M)\). We prove the following generalization of Lemma 4.3.

Lemma B.1 (Generalization of Lemma 4.3). For each \(Z \in \mathbb{C}^w\) we have

\[
\text{SR}(Q(Z)) \leq (16a^4 d^4 n)^{\frac{a}{2} + b + wn}. \tag{51}
\]

To prove this lemma, we use the following simple fact about polynomials. For a polynomial \(g(x) = \sum_{j=0}^{p} c_j x^j\) we will use the notation \([x^\ell] g(x) \equiv c_\ell\) to denote the coefficient of \(x^\ell\). This notation extends to multivariate polynomials, e.g., \([x^2](2x^2y - x^2) = 2y - 1\), or \([xy^2](x + y)^3 = 3\).

Claim B.2. Let \(g(x)\) be a degree-\(p\) polynomial. There exist \(x_0, x_1, \ldots, x_p \in \mathbb{C}\) such that each coefficient \([x^c] g(x)\) can be expressed as a linear combination of \(g(x_0), \ldots, g(x_p)\) with complex coefficients.

Proof. Take \(x_j = e^{2\pi ij/(p+1)}\) for \(0 \leq j \leq p\). Then using the inverse discrete Fourier transform, we get \([x^\ell] g(x) = (p+1)^{-1} \sum_{j=0}^{p} e^{-2\pi ijc/(p+1)} g(x_j)\). \(\square\)

We first bound the entanglement of a coefficient of a term in the product (50).

Lemma B.3. For any \(r \in [w]\), interval \(S \subseteq [n]\) and integers \(\ell \leq k \leq a\) we have

\[
\text{SR}_r([Z_r^\ell](H_S(Z))^k) \leq (4a^2 d^4 n)^{\ell+1}. \tag{53}
\]

Proof. Let \(B = \frac{1}{2w} \sum_{j \in S} h_{rj}\) and write

\[
H_S(Z) = A + BZ_r \tag{52}
\]

with \(A = A_{\text{left}} + A_{\text{right}}\), where

\[
A_{\text{left}} = \frac{1}{2w} \sum_{j \in S} (I - \Pi_{L,j} + \sum_{i=1}^{r-1} h_{ij} Z_i) \quad A_{\text{right}} = \frac{1}{2w} \sum_{j \in S} (I - \Pi_{R,j} + \sum_{i=r+1}^{w} h_{ij} Z_i). \]

Using \([A_{\text{left}}, A_{\text{right}}] = 0\) gives \(A^p = \sum_{q=0}^{p} \binom{p}{q} A_{\text{left}}^q A_{\text{right}}^{p-q}\). Since \(\text{SR}_r(A_{\text{left,right}}) = 1\) we get

\[
\text{SR}_r(A^p) \leq p + 1. \tag{53}
\]

Since \(h_{rj}\) acts on two qudits on each side of the cut, we have \(\text{SR}_r(h_{rj}) \leq d^4\) and therefore

\[
\text{SR}_r(B) \leq d^4 |S| \leq d^4 n. \tag{54}
\]

Using Eq. (52) we can expand

\[
[Z_r^\ell](H_S(Z))^k = \sum_{x_0 + \ldots + x_\ell = k-\ell} A^{x_0} B A^{x_1} B \ldots B A^{x_\ell} \tag{55}
\]
Using the bounds Eqs. (53, 54) and the fact that \( x_i \leq k \) for each \( i \), we see that
\[
\text{SR}_r(A^x BA x_1 B \ldots BA x_r) \leq (k + 1)^{(r+1)}(d^4n)^r.
\]
The sum in Eq. (55) contains at most \((k + 1)^{(r+1)}\) terms (since \( 0 \leq x_i \leq k \)). Hence
\[
\text{SR}_r(\{Z_r^\ell\} (H_S(Z))^k) \leq (k + 1)^{2r+2} \cdot (d^4n)^r \leq (4a^2d^4n)^r,
\]
where in the last inequality we used the fact that \( k \leq a \) and \( a \geq 1 \), hence \((k+1)^2 \leq 4a^2\). \qed

We are now in position to prove Lemma B.1.

**Proof of Lemma B.1.** Let \( \mathbb{N} \overset{\text{def}}{=} \{0, 1, 2, \ldots\} \) and \( \mathcal{K} \overset{\text{def}}{=} \{\alpha \in \mathbb{N}^w : \sum_{i=1}^w \alpha_i = a\} \). We have
\[
Q(Z) = \sum_{\alpha \in \mathcal{K}} Q_{\alpha} \prod_{j=1}^w Z_{\alpha j}^{\alpha_j} \quad Q_{\alpha} \overset{\text{def}}{=} [Z_1^{\alpha_1} Z_2^{\alpha_2} \ldots Z_w^{\alpha_w}]Q(Z).
\]
Since \( 0 \leq \alpha_i \leq a \) for each \( i \), we have \( |\mathcal{K}| \leq (a + 1)^w \leq (2a)^w \), and 73A72EA3
\[
\text{SR}(Q(Z)) \leq (2a)^w \max_{\alpha \in \mathcal{K}} \text{SR}(Q_{\alpha}) \tag{56}
\]
To upper bound the RHS, let \( \alpha \in \mathcal{K} \) be given. Let \( M \overset{\text{def}}{=} \min_{i \in [w]} \alpha_i \) and let \( r \in [w] \) be such that \( \alpha_r = M \). Since \( \sum_{i=1}^w \alpha_i = a \), we have \( M \leq a/w \). Below we show
\[
\text{SR}_r(Q_{\alpha}) \leq (2a)^{w+b}(4a^2d^4n)^{M+b} \tag{57}
\]
Since \(|r - c| \leq w/2\) and each column of the lattice contains \( n \) qudits, Eq. (57) implies
\[
\text{SR}(Q_{\alpha}) \leq d^{wn}\text{SR}_r(Q_{\alpha}) \leq d^{wn}(2a)^{w+b}(4a^2d^4n)^{a/w+b} \leq (2a)^{w+b}(4a^2d^4n)^{a/w+b+w/n}
\]
where we used \( M \leq a/w \). Eq. (51) follows by plugging into Eq. (56) and bounding \((4a^2)^w (2a)^b \leq (4a^2)^{a/w+b+w/n}\).

To establish Eq. (57), define \( G(Z) \overset{\text{def}}{=} [Z_r^M]Q(Z) \) (a function of \( Z_i \) with \( i \neq r \)) and write
\[
Q_{\alpha} = [Z_1^{\alpha_1} Z_2^{\alpha_2} \ldots Z_w^{\alpha_w}]Q(Z) = [Z_1^{\alpha_1} \ldots Z_{r-1}^{\alpha_{r-1}} Z_r^{\alpha_r+1} \ldots Z_w^{\alpha_w}]G(Z). \tag{58}
\]
Applying Claim B.2 \( w - 1 \) times inductively and using \( \alpha_i \leq a \), we see that \( Q_{\alpha} \) is a linear combination of at most \((a + 1)^{w-1}\) operators \( G(X) \) with \( X \in \mathbb{C}^{w-1} \). Therefore
\[
\text{SR}_r(Q_{\alpha}) \leq (a + 1)^{w-1} \max_{X \in \mathbb{C}^{w-1}} \text{SR}_r(G(X)). \tag{59}
\]
Finally, using Eq. (50) we obtain
\[
G(Z) = [Z_r^M]Q(Z) = \sum_{\ell_1 + \ell_2 + \ldots \ell_b = M} [Z_r^\ell_1](H_S(Z)^{j_1})[Z_r^\ell_2](H_S(Z)^{j_2})\ldots[Z_r^\ell_b](H_S(Z)^{j_b}).
\]
Since \( 0 \leq \ell_i \leq M \), the number of terms in the above sum is at most \((M+1)^b \leq (2a)^b\). Using this fact and Lemma B.3 we get
\[
\text{SR}_r(G(Z)) \leq \sum_{\ell_1 + \ell_2 + \ldots \ell_b = M} \prod_{p=1}^b \text{SR}_r([Z_r^{\ell_p}](H_S(Z)^{j_p})) \leq (2a)^b(4a^2d^4n)^{M+b}.
\]
Substituting in Eq. (59) and bounding \((a + 1)^w \leq (2a)^w\) completes the proof of Eq. (57). \qed