On a characterization of the complex hyperbolic space

Ovidiu Munteanu

February 1, 2008

Abstract
Consider a compact Kähler manifold $M^m$ with Ricci curvature lower bound $\text{Ric}_M \geq -2(m+1)$. Assume that its universal cover $\tilde{M}$ has maximal bottom of spectrum $\lambda_1(\tilde{M}) = m^2$. Then we prove that $\tilde{M}$ is isometric to the complex hyperbolic space $\mathbb{C}\mathbb{H}^m$.

1 Introduction

Complete Riemannian manifolds with Ricci curvature lower bound have been the object of study of many authors and there are very interesting results about such manifolds. An important approach is to see how the spectrum of the Laplacian interacts with the geometry of the manifold. A famous result that we recall here is S.Y. Cheng’s comparison theorem [C]. If the Ricci curvature of a complete noncompact Riemannian manifold $N^n$ of dimension $n$ is bounded from below by $\text{Ric}_N \geq -(n-1)$, then Cheng’s theorem asserts that the bottom of the spectrum of the Laplacian has an upper bound $\lambda_1(N) \leq \frac{(n-1)^2}{4} = \lambda_1(\mathbb{H}^n)$. This result is sharp, but we should point out that there are in fact many manifolds with maximal $\lambda_1$, more examples can be found by considering hyperbolic manifolds $N = \mathbb{H}^n/\Gamma$ obtained by the quotient of $\mathbb{H}^n$ by a Kleinian group $\Gamma$ ([S]). While in general we cannot determine the class of manifolds with $\lambda_1$ achieving its maximal value, recently there has been important progress in some directions.

Research partially supported by NSF grant No. DMS-0503735
P. Li and J. Wang have studied the structure at infinity of a complete noncompact Riemannian manifold that has $Ric_N \geq -(n-1)$ and maximal bottom of spectrum $\lambda_1(N) = \frac{(n-1)^2}{4}$. They proved that either the manifold is connected at infinity (i.e. it has one end) or it has two ends. In case it has two ends then it must split as a warped product of a compact manifold with the real line $[L-W2]$. Their result has since been extended in many other situations, e.g. Kähler manifolds, quaternionic Kähler manifolds or locally symmetric spaces.

Recently X. Wang $[W]$ has obtained an interesting result in a different setting. Suppose $N^n$ is a compact Riemannian manifold with $Ric_N \geq -(n-1)$. Consider $\pi: \tilde{N} \to N$ its universal cover and assume that $\lambda_1(\tilde{N}) = \frac{(n-1)^2}{4}$. Then $\tilde{N}$ is isometric to the hyperbolic space $\mathbb{H}^n$.

Wang proved this theorem using the notion of Kaimanovich entropy $\beta$,

$$\beta = -\lim_{t \to \infty} \frac{1}{t} \int_{\tilde{N}} p(t, x, y) \log p(t, x, y) \, dy,$$

where $p$ denotes the heat kernel on $\tilde{N}$, which also plays an important role in our discussion.

It should be pointed out that in Wang’s theorem if the manifold $N$ is assumed to have negative curvature (and removing the lower bound on Ricci curvature assumption) then stronger results are already known from the work of Ledrappier, Foulon, Labourie, Besson, Courtois, Gallot $[L, F-L, B-C-G]$. In this case it can be proved that if $h$ denotes the volume entropy of $N$ defined by

$$h = \lim_{R \to \infty} \log \frac{Vol(B_p(R))}{R}$$

and $\lambda_1(\tilde{N}) = \frac{1}{4}h^2$ then $N$ is locally symmetric.

However, Wang’s theorem is quite powerful because it does not assume negative curvature.

It is a natural question to investigate these issues on Kähler manifolds. A first question that one should ask is if Cheng’s estimate can be improved in this case. The model space that we work with is now the complex hyperbolic space $\mathbb{CH}^m$. Recently Li-Wang have proved $[L-W1]$ that for a complete noncompact Kähler manifold $M^m$ of complex dimension $m$ if the bisectional curvature is bounded from below by $BK_M \geq -1$, then $\lambda_1(M) \leq m^2 = \lambda_1(\mathbb{CH}^m)$. They proved that in fact if the bottom of spectrum $\lambda_1(M)$ achieves its maximal value, then the manifold is either connected.
at infinity or it has two ends and in this latter case it is diffeomorphic to the product of a compact manifold with the real line and the Kähler metric on $M$ has a specialized form.

We recently improved (see [M]) Li-Wang’s results for complete Kähler manifolds that have a Ricci curvature lower bound, $\text{Ric}_M \geq -2(m+1)$, which is a weaker assumption than bisectional curvature lower bound. To prove the estimate for $\lambda_1(M)$ and the structure at infinity for manifolds with maximum $\lambda_1$ we used a new argument, a sharp integral estimate for the gradient of a certain class of harmonic functions. In this paper we will use our argument to estimate the Kaimanovich entropy from above, which will imply the following result.

**Theorem 1** Let $M^m$ be a compact Kähler manifold of complex dimension $m$ and with Ricci curvature bounded from below by $\text{Ric}_M \geq -2(m+1)$. Assume its universal cover $\pi: \tilde{M} \rightarrow M$ has maximal bottom of spectrum, $\lambda_1(\tilde{M}) = m^2$. Then $\tilde{M}$ is isometric to the complex hyperbolic space $\mathbb{CH}^m$.

We want to comment now about the particular case when $M$ has negative curvature.

For Kähler manifolds with bisectional curvature lower bound $\text{BK}_M \geq -1$ it follows from Li-Wang [L-W1] that volume entropy verifies the sharp estimate $h \leq 2m$. So maximal bottom of spectrum in this case implies $\lambda_1 = \frac{1}{4}h^2$.

However, for only Ricci curvature lower bound $\text{Ric}_M \geq -2(m+1)$ it is not known if $h \leq 2m$, so it is not clear how to apply the Besson-Courtois-Gallot theorem in the negative curvature case.

**Acknowledgement.** The author would like to express his deep gratitude to his advisor, Professor Peter Li, for continuous help, encouragement and many valuable discussions.
2 Proof of the Theorem

First, let us set the notation. We use the notations in \([L-W, M]\). If 
\(ds^2 = h_{\alpha \bar{\beta}} dz^\alpha d\bar{z}^{\beta}\) is the Kähler metric on 
\(\tilde{M}\), then \(Re(ds^2)\) defines a Riemannian metric on 
\(\tilde{M}\).

Note that if \(\{e_1, e_2, \ldots, e_{2m}\}\) with 
\(e_{2k} = Je_{2k-1}\) for \(k \in \{1, 2, \ldots, m\}\) is 
an orthonormal frame with respect to the Riemannian metric on 
\(\tilde{M}\) then 
\(\{v_1, \ldots, v_m\}\) is a unitary frame of 
\(T_{x,0}^1 \tilde{M}\), where

\[ v_k = \frac{1}{2} \left( e_{2k-1} - \sqrt{-1} e_{2k} \right) . \]

In this notation the following formulas hold

\[
\nabla f \cdot \nabla g = 2 \left( f_{\alpha} f_{\bar{\alpha}} + g_{\alpha} g_{\bar{\alpha}} \right) \\
\Delta f = 4 f_{\alpha \bar{\alpha}} .
\]

In the statement of the theorem, the Ricci curvature lower bound refers to 
the Riemannian metric and it is equivalent to saying 
\(Ric_{\alpha \bar{\beta}} \geq -(m + 1) \delta_{\alpha \bar{\beta}}\) 
with respect to any unitary frame.

To prove the theorem we follow the argument in \([W]\) and use the results 
in \([M]\).

We first need to recall some facts about the Kaimanovich entropy.

There are a few equivalent formulations of this entropy. First, it can be 
defined as a limit of the heat kernel:

\[
\beta = \lim_{t \to \infty} \left( -\frac{1}{t} \int_{\tilde{M}} p(t, x, y) \log p(t, x, y) \, dy \right) ,
\]

where \(p\) is the heat kernel on \(\tilde{M}\). This definition is useful because it can be 
showed that (a result of Ledrappier \([L]\))

\[
\beta \geq 4 \lambda_1 \left( \tilde{M} \right) .
\]

There is another very useful formula for \(\beta\), using the minimal Martin boundary 
of \(\tilde{M}\). Let us quickly recall some known facts (see e.g. \([A]\)).

Let \(H \left( \tilde{M} \right)\) denote the space of harmonic functions on 
\(\tilde{M}\), with the topology of uniform convergence on compact sets. Observe that 
\(K_O = \{ u \in H \left( \tilde{M} \right) : u(O) = 1, \ u > 0 \}\) is a compact and convex subset of 
\(H \left( \tilde{M} \right)\)
so denote with $\partial^* \widetilde{M}$ the set of extremal points of $K_O$, i.e. points in $K_O$ that do not lie in any open line segment in $K_O$. Note that a point of $K_O$ is extremal iff it is a minimal harmonic function normalized at $O$, therefore $\partial^* \widetilde{M}$ is the minimal Martin boundary of $\widetilde{M}$. Since $K_O$ is a metric space and it is compact and convex, by a theorem of Choquet it results that for any positive harmonic function $h$ there is a unique Borel measure $\mu^h$ on the set of extremal points of $K_O$ so that

$$h(x) = \int_{\partial^* \widetilde{M}} \xi(x) \, d\mu^h(\xi).$$

In particular, for $h = 1$ there exists a unique measure $\nu$ on $\partial^* \widetilde{M}$ so that for any $x \in \widetilde{M}$,

$$\int_{\partial^* \widetilde{M}} \xi(x) \, d\nu(\xi) = 1.$$  

Let $\Gamma$ denote the group of deck transformations on $\widetilde{M}$, then there is a natural action of $\Gamma$ on $\partial^* \widetilde{M}$, defined by

$$(\gamma \xi)(x) = \frac{\xi(\gamma^{-1}x)}{\xi(\gamma^{-1}O)},$$

for any $\xi \in \partial^* \widetilde{M}$ and for any $\gamma \in \Gamma$.

It is important to know how the measure $\nu$ is changed by the action of $\Gamma$ on $\partial^* \widetilde{M}$, it can be easily seen that if $\eta = \gamma \xi$, then

$$\frac{d\nu(\eta)}{d\nu(\xi)} = \xi(\gamma^{-1}O).$$

For $x \in \widetilde{M}$ define

$$\omega(x) = \int_{\partial^* \widetilde{M}} \xi^{-1}(x) |\nabla \xi|^2(x) \, d\nu(\xi),$$

and notice that $\omega$ descends on $M$. Indeed, for any $\gamma \in \Gamma$ we have that

$$|\nabla \xi|^2(\gamma x) = |\nabla (\gamma^* \xi)|^2(x),$$

where $\gamma^* \xi$ is the pull back of $\xi$, i.e. $\gamma^* \xi = \xi \circ \gamma$. Then it is easy to check using the Radon-Nikodym derivative that for $\eta = \gamma^{-1} \xi$ we have

$$\xi^{-1}(\gamma x) |\nabla \xi|^2(\gamma x) \, d\nu(\xi) = \eta^{-1}(x) |\nabla \eta|^2(x) \, d\nu(\eta).$$
Then it clearly follows that
\[
\omega(\gamma x) = \int_{\partial^* \tilde{M}} \xi^{-1}(\gamma x) |\nabla \xi|^2(\gamma x) \, d\nu(\xi)
= \int_{\partial^* \tilde{M}} \eta^{-1}(x) |\nabla \eta|^2(x) \, d\nu(\eta)
= \omega(x).
\]

We have showed that in fact \(\omega\) is a well defined function on \(M\). This function can be used now to give another formula for the Kaimanovich entropy. Everywhere in this paper we will denote by \(dv\) the normalized Riemannian volume form i.e.
\[
dv = \frac{1}{\int_M \sqrt{g} \, dx} \left( \sqrt{g} \, dx \right).
\]

By a formula of Kaimanovich (\cite{K}, also see \cite{L, W}) the entropy can also be expressed as
\[
\beta = \int_M \omega \, dv
= \int_M \left( \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\nabla \xi|^2(x) \, d\nu(\xi) \right) \, dv.
\]
So we have the following
\[
4\lambda_1 \left( \tilde{M} \right) \leq \int_M \left( \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\nabla \xi|^2(x) \, d\nu(\xi) \right) \, dv. \quad (1)
\]

For Riemannian manifolds X, Wang has used this inequality together with the sharp Yau’s gradient estimate (\cite{L-W2}) to prove his result in the Riemannian setting.

For our problem, a sharp pointwise gradient estimate for Kähler manifolds is not known to be true, but we know a way to obtain a sharp integral estimate for the gradient of harmonic functions. So the goal is to show that
\[
\int_M \left( \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\nabla \xi|^2(x) \, d\nu(\xi) \right) \, dv \leq 4m^2.
\]
To show this, we use the argument in \cite{M}. The main technical point now is to justify integration by parts (and in what sense) that was used in \cite{M}.
Let $u = \log \xi$, then a simple computation shows that

$$u_{\alpha \overline{\beta}} = \xi^{-1} \xi_{\alpha \overline{\beta}} - \xi^{-2} \xi_{\alpha} \xi_{\overline{\beta}}.$$  

For a fixed $x \in \widetilde{M}$ consider

$$\int_{\partial^* \widetilde{M}} \xi(x) |u_{\alpha \overline{\beta}}|^2 (x) \, d\nu (\xi).$$

We first claim that this integral is a finite number (depending on $x$). Indeed, since $\partial^* \widetilde{M}$ is compact and $d\nu$ is a finite measure, it suffices to show the integrand is bounded. But this is true because for fixed $x$ we can bound $|\xi_{\alpha \overline{\beta}}(x)| \leq C (x) \xi(O) = C (x)$. This can be seen as follows. Consider $B_O(R)$ a geodesic ball of radius $R$ big enough so that $x \in B_O(R)$. Note that there exists a constant $A > 0$ so that $\Delta |\xi_{\alpha \overline{\beta}}| \geq -A |\xi_{\alpha \overline{\beta}}|$ on $B_O(R)$. Such a constant $A$ can be chosen to depend on the lower bound of the bisectional curvature on $B_O(R)$, using the Bochner formula. Using now the mean value inequality we get that there exists a constant $C_1$ depending on $R$ and $A$ so that

$$|\xi_{\alpha \overline{\beta}}|^2 (x) \leq C_1 \int_{B_O(R)} |\xi_{\alpha \overline{\beta}}|^2.$$  

It is known that by using integration by parts and suitable cut-off functions that there exists a constant $C_2$ so that

$$\int_{B_O(R)} |\xi_{\alpha \overline{\beta}}|^2 \leq C_2 \int_{B_O(2R)} \xi^2.$$  

The right side of this inequality can now be bounded by $C_3 \xi^2 (O)$, using the Harnack inequality. Obviously, these constants will depend on $R$, nevertheless it follows that for $x$ fixed $|\xi_{\alpha \overline{\beta}}(x)|$ will be bounded uniformly for all $\xi$, which was our claim.

The second claim is that the function thus obtained actually descends on $M$. This claim can be showed as above, now using the fact that since $M$ is Kähler, the deck transformations are holomorphic, therefore for $\gamma \in \Gamma$ and $\gamma^* \xi$ the pull back of $\xi$ we have

$$\left| \left( \log \xi \right)_{\alpha \overline{\beta}} \right|^2 (\gamma x) = \left| \left( \log (\gamma^* \xi) \right)_{\alpha \overline{\beta}} \right|^2 (x).$$  

The rest of the proof follows the same line as for the gradient of $\xi$ (see above).
Therefore it makes sense to consider the following quantity:

\[
\int_M \int_{\partial^* \tilde{M}} \xi(x) |u_{\alpha\beta}(x)|^2(x) \, d\nu(\xi) \, dv = \int_M \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\xi_{\alpha\beta}|^2(x) \, d\nu(\xi) \, dv \\
-2 \int_M \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_{\alpha\beta}\xi_{\alpha\beta}) (x) \, d\nu(\xi) \, dv \\
+ \frac{1}{16} \int_M \int_{\partial^* \tilde{M}} \xi^{-3}(x) |\nabla \xi|^4(x) \, d\nu(\xi) \, dv,
\]

where each of the integrals in the right side are also well defined by a similar discussion.

We now want to justify integration by parts to show that

\[
\int_M \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\xi_{\alpha\beta}|^2(x) \, d\nu(\xi) \, dv = \int_M \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_{\alpha\beta}\xi_{\alpha\beta}) (x) \, d\nu(\xi) \, dv
\]

Consider \((U_i)\) a covering of \(M\) with small open sets and let \(\rho_i\) be a partition of unity subordinated to this covering. We can choose \((U_i)\) so that each \(U_i\) is diffeomorphic to an open set \(\tilde{U}_i \subset \tilde{M}\) via \(\pi\). We then have

\[
\int_M \int_{\partial^* \tilde{M}} \xi^{-1}(x) |\xi_{\alpha\beta}|^2(x) \, d\nu(\xi) \, dv \\
= \int_M \int_{\partial^* \tilde{M}} \xi^{-1}(x) \xi_{\alpha\beta}(x) \left(\xi_{\alpha}(x) \sum_i \rho_i(\pi(x)) \right)_\beta \, d\nu(\xi) \, dv \\
= \sum_i \int_M \int_{\partial^* \tilde{M}} \xi^{-1}(x) \xi_{\alpha\beta}(x) \left(\xi_{\alpha}(x) \rho_i(\pi(x)) \right)_\beta \, d\nu(\xi) \, dv \\
= \sum_i \int_{\tilde{U}_i} \int_{\partial^* \tilde{M}} \xi^{-1}(x) \xi_{\alpha\beta}(x) \left(\xi_{\alpha}(x) \rho_i(\pi(x)) \right)_\beta \, d\nu(\xi) \, dv \\
= -\sum_i \int_{\tilde{U}_i} \int_{\partial^* \tilde{M}} (\xi^{-1}(x) \xi_{\alpha\beta}(x))_\beta \left(\xi_{\alpha}(x) \rho_i(\pi(x)) \right) \, d\nu(\xi) \, dv \\
= \sum_i \int_{\tilde{U}_i} \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_{\alpha\beta}\xi_{\alpha\beta}) (x) (\xi_{\alpha}(x) \rho_i(\pi(x))) \, d\nu(\xi) \, dv
\]
\[\sum_i \int_{U_i} \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_\alpha \xi_\beta (x) (\xi_\alpha (x) \rho_i (\pi (x))) d\nu (\xi) dv \]
\[= \sum_i \int_{M} \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_\alpha \xi_\beta (x) (\xi_\alpha (x) \rho_i (\pi (x))) d\nu (\xi) dv \]
\[= \int_{M} \int_{\partial^* \tilde{M}} \xi^{-2}(x) (\xi_\alpha \xi_\beta (x) (\xi_\alpha (x) \rho_i (\pi (x))) d\nu (\xi) dv.\]

Let us mark out that everywhere in this formulas (and in the paper) a priori the integrals on the minimal Martin boundary are taken for any (parameter) \(x \in \tilde{M}\). Then, it can be justified that in fact these integrals on \(\partial^* \tilde{M}\) are invariant by the group of deck transformations, so they are well defined functions on \(M\). With this in mind, in the third line above one should also justify that for each \(i\) the functions on \(\tilde{M}\) defined by \(x \rightarrow \int_{\partial^* \tilde{M}} \xi^{-1}(x) \xi_\alpha (x) \rho_i (\pi (x))) d\nu (\xi)\) descend on \(M\). This can be done by the same argument, and using that \(\gamma^* (\rho_i \circ \pi) = \rho_i \circ \pi\), for any \(\gamma \in \Gamma\). It is also important that the function in \(\xi\) which is integrated on the minimal Martin boundary (for example \(\xi \rightarrow \xi^{-1} \xi_\alpha (\xi_\alpha (x) \rho_i (\pi (x))) \beta\)) be homogeneous of degree 1 in \(\xi\). Thus we want to remark that not quite any integration by parts is allowed by this procedure of lifting the integrals on the universal covering.

This argument will be applied below every time we integrate by parts, it is easy to check that the argument works in each case.

To simplify the writing, we will henceforth omit to write the argument \(x\) and the measure \(d\nu\), but we always assume the integrals on \(\partial^* \tilde{M}\) are taken with respect to \(d\nu\) and that all the functions integrated on \(\partial^* \tilde{M}\) depend on \(x \in \tilde{M}\). For each of these integrals on the minimal Martin boundary it can be justified that it is invariant by the group of deck transformations so it legitimately defines a function on \(M\).

We have thus proved that

\[\int_M \int_{\partial^* \tilde{M}} \xi |u_{\alpha\beta}|^2 = - \int_M \int_{\partial^* \tilde{M}} \xi^{-2}(\xi_\alpha \xi_\beta (x) (\xi_\alpha (x) \rho_i (\pi (x))) + \frac{1}{16} \int_M \int_{\partial^* \tilde{M}} \xi^{-3} |\nabla \xi|^4.\]
Let us use again integration by parts to see that
\[- \int_M \int_{\partial^* \widetilde M} \xi^{-2} (\xi_{\alpha \beta} \xi_{\alpha \beta}) = \int_M \int_{\partial^* \widetilde M} \xi_\alpha (\xi^{-2} \xi_{\alpha \beta}) \xi_\beta \]
\[= - \frac{1}{8} \int_M \int_{\partial^* \widetilde M} \xi^{-3} |\nabla \xi|^4 + \int_M \int_{\partial^* \widetilde M} \xi^{-2} \xi_{\alpha \beta} \xi_{\alpha \beta} \cdot \]

Note that the following inequality holds on \( \widetilde M \):

\[|\xi_{\alpha \beta} \xi_{\alpha \beta}| \leq \frac{1}{4} |\xi_{\alpha \beta}| |\nabla \xi|^2 \]
so that we get

\[2 \int_M \int_{\partial^* \widetilde M} \xi^{-2} \xi_{\alpha \beta} \xi_{\alpha \beta} \leq \int_M \int_{\partial^* \widetilde M} 2 (\xi^{-1/2} |\xi_{\alpha \beta}|) \left( \frac{1}{4} \xi^{-3/2} |\nabla \xi|^2 \right) \]
\[\leq \frac{m}{m+1} \int_M \int_{\partial^* \widetilde M} \xi^{-1} |\xi_{\alpha \beta}|^2 + \frac{1}{16} \frac{m+1}{m} \int_M \int_{\partial^* \widetilde M} \xi^{-3} |\nabla \xi|^4 . \] (3)

Moreover, again integrating by parts we have

\[\int_M \int_{\partial^* \widetilde M} \xi^{-1} |\xi_{\alpha \beta}|^2 = \int_M \int_{\partial^* \widetilde M} \xi^{-2} \xi_{\alpha \beta} \xi_{\alpha \beta} - \int_M \int_{\partial^* \widetilde M} \xi^{-1} \xi_{\alpha \beta} \xi_{\alpha \beta} \]
\[\leq \int_M \int_{\partial^* \widetilde M} \xi^{-2} \xi_{\alpha \beta} \xi_{\alpha \beta} + \frac{m+1}{4} \int_M \int_{\partial^* \widetilde M} \xi^{-1} |\nabla \xi|^2 , \]
using that \( \xi \) is harmonic, the Ricci identities and the lower bound of the Ricci curvature:

\[-\xi_{\alpha \beta} \xi_{\alpha \beta} = -\xi_{\alpha \beta} \xi_{\alpha \beta} \]
\[= -\xi_{\alpha \beta} \xi_{\alpha \beta} - Ric_{\alpha \beta \alpha} \xi_{\alpha \beta} \]
\[= -Ric_{\alpha \beta \alpha} \xi_{\alpha \beta} \]
\[\leq (m+1) \xi_{\alpha \beta} \xi_{\alpha \beta} \]
\[= \frac{m+1}{4} |\nabla \xi|^2 . \]

Plug this inequality into (3) and it follows

\[\frac{m+2}{m+1} \int_M \int_{\partial^* \widetilde M} \xi^{-2} \xi_{\alpha \beta} \xi_{\alpha \beta} \leq \frac{m}{4} \int_M \int_{\partial^* \widetilde M} \xi^{-1} |\nabla \xi|^2 \]
\[+ \frac{1}{16} \frac{m+1}{m} \int_M \int_{\partial^* \widetilde M} \xi^{-3} |\nabla \xi|^4 . \]
Getting back to (2) we obtain
\[-\int_M \int_{\partial^* \tilde{M}} \xi^{-2} \xi_{\alpha\beta} \xi_\alpha \xi_\beta \leq \left( -\frac{1}{8} + \frac{1}{16} \frac{(m+1)^2}{m(m+2)} \right) \int_M \int_{\partial^* \tilde{M}} |\nabla \xi|^4 + \frac{m(m+1)}{4(m+2)} \int_M \int_{\partial^* \tilde{M}} \xi^{-1} |\nabla \xi|^2 \]

We have thus proved that
\[\int_M \int_{\partial^* \tilde{M}} |u_{\alpha\beta}|^2 \leq \frac{1}{16} \frac{m}{m+2} \int_M \int_{\partial^* \tilde{M}} \xi^{-3} |\nabla \xi|^4 + \frac{m(m+1)}{4(m+2)} \int_M \int_{\partial^* \tilde{M}} \xi^{-1} |\nabla \xi|^2.\]

The estimate from below is straightforward:
\[|u_{\alpha\beta}|^2 \geq \sum_\alpha |u_{\alpha\bar{\alpha}}|^2 \geq \frac{1}{m} \left| \sum_\alpha u_{\alpha\bar{\alpha}} \right|^2 = \frac{1}{16m} \xi^{-4} |\nabla \xi|^4.\]

Hence, this shows that
\[\int_M \int_{\partial^* \tilde{M}} \xi^{-3} |\nabla \xi|^4 \leq 4m \int_M \int_{\partial^* \tilde{M}} \xi^{-1} |\nabla \xi|^2. \tag{4}\]

Finally, using the Schwarz inequality and the fact that \(\int_{\partial^* \tilde{M}} \xi = 1\) we get
\[\int_M \int_{\partial^* \tilde{M}} \xi^{-1} |\nabla \xi|^2 \leq \left( \int_M \int_{\partial^* \tilde{M}} \xi^{-3} |\nabla \xi|^4 \right)^{\frac{1}{2}} \left( \int_M \int_{\partial^* \tilde{M}} \xi \right)^{\frac{1}{2}} = \left( \int_M \int_{\partial^* \tilde{M}} \xi^{-3} |\nabla \xi|^4 \right)^{\frac{1}{2}}.\]

Combined with (4) and (11) this gives indeed that
\[4\lambda_1 \left( \tilde{M} \right) \leq \int_M \left( \int_{\partial^* \tilde{M}} \xi^{-1} (x) |\nabla \xi|^2 (x) d\nu (\xi) \right) dv \leq 4m^2,
\]
as claimed.

Since we know \(\lambda_1 \left( \tilde{M} \right) = m^2\), it follows that all inequalities used in this proof will be (pointwise) equalities on \(\tilde{M}\) for almost all \(\xi \in \partial^* \tilde{M}\). Indeed,
this is true because everywhere in our proof the inequalities were proved by integrating on $\partial^* \tilde{M}$ some inequalities at $x \in \tilde{M}$ that hold for each $\xi \in \partial^* \tilde{M}$.

Tracing back our argument, in $[M]$ we proved that for $B = \frac{1}{2m} \log \xi$ we have

\[
|\nabla B| = 1
\]

\[
Hess_B (X, Y) = -g(X, Y) + g(\nabla B, X)g(\nabla B, Y) - g(J \nabla B, X)g(J \nabla B, Y)
\]

where $Hess_B$ denotes the real Hessian of $B$.

From the work of Li-Wang [L-W] we know that in this case, if the manifold has bounded curvature then it is isometric to $\mathbb{C} \mathbb{H}^m$. This is always the case for our setting, since $\tilde{M}$ covers a compact manifold, so its curvature is bounded. Q.E.D.

References

[A] A. Ancona, *Theorie du potentiel sur les graphes et les varietes*, École d’été de Probabilités de Saint-Flour XVIII-1988, 1-112, Lecture Notes in Math. 1427, Springer, Berlin, 1990.

[B-C-G] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courboure strictement négative*, Geom. Funct. Anal. 5 (1995), no. 5, 731-799.

[C] S.Y. Cheng, Eigenvalue comparison theorems and its geometric applications, Math. Z. 143 (1975), 289-297.

[F-L] P. Foulon and F. Labourie, *Sur les variétés compactes asymptotique-ment harmoniques*, Invent. Math. 109 (1992), no.1, 97-111.

[K] V. Kaimanovich, *Brownian motion and harmonic functions on covering manifolds. An entropy approach*, Soviet Math. Dokl. 33 (1986) 812-816.

[L] F. Ledrappier, *Harmonic measures and Bowen-Margulis measures*, Israel J. Math. 71 (1990) 275-287.

[L1] F. Ledrappier, *Profil d’entropie dans le cas continu*, Hommage a P.A. Meyer et J. Neveu, Asterisque No 236 (1996), 189-198.
[L-W] P. Li and J. Wang, *Connectedness at infinity of complete Kähler manifolds*, preprint.

[L-W1] P. Li and J. Wang, *Comparison theorem for Kähler manifolds and the positivity of spectrum*, J. Diff. Geom., 69 (2005) 43-74.

[L-W2] P. Li and J. Wang, *Complete manifolds with positive spectrum II*, J. Diff. Geom., 62 (2002), no.1, 143-162.

[M] O. Munteanu, *A sharp estimate for the bottom of the spectrum of the Laplacian on Kähler manifolds*, math.DG/0703098.

[S] D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Diff. Geom. 25 (1987), 327-351.

[W] X. Wang, *Harmonic functions, entropy, and a characterisation of the hyperbolic space*, to appear in Journal of Geometric Analysis.