ASYMPTOTIC RELATIONS BETWEEN INTERPOLATION DIFFERENCES
AND ZETA FUNCTIONS

MICHAEL I. GANZBURG

Abstract. Asymptotic relations between zeta functions (such as, \( \zeta(s) \), \( \beta(s) \), and other Dirichlet \( L \)-functions) and interpolation differences of functions like \( |y|^s \) and their interpolating entire functions of exponential type 1 are discussed. New criteria for zeros of the zeta functions in the critical strip in terms of integrability of the interpolation differences are obtained as well.

1. Introduction

In this paper we find asymptotic relations between zeta functions (such as, \( \zeta(s) \), \( \beta(s) \), and other Dirichlet \( L \)-functions) and interpolation differences of functions like \( |y|^s \) and their interpolating entire functions of exponential type 1. As corollaries, we obtain new criteria for zeros of the zeta functions in the critical strip in terms of integrability of the interpolation differences.

1.1. Notation. Let \( \mathbb{Z} \) denote the set of all integers and let \( \mathbb{Z}_+ \) be the set of all nonnegative integers and \( \mathbb{N} := \mathbb{Z}_+ \setminus \{0\} \). Let \( K(R) := \{w \in \mathbb{C} : |w| = R\} \) be the circle in \( \mathbb{C} = \mathbb{R} + i\mathbb{R} \) centered at the origin of radius \( R > 0 \). In addition, \( |\Omega| \) denotes the Lebesgue measure of a measurable set \( \Omega \subset \mathbb{R} \).

We also use the floor function \( \lfloor a \rfloor, a \geq 0 \), and the gamma function \( \Gamma(s), s \in \mathbb{C} \).

Let \( L^p(\Omega) \) be the space of all measurable complex-valued functions \( F \) on a measurable set \( \Omega \subset \mathbb{R} \) with the finite quasinorm

\[
\|F\|_{L^p(\Omega)} := \begin{cases} 
\left( \int_{\Omega} |F(y)|^p \, dy \right)^{1/p}, & 0 < p < \infty, \\
\text{ess sup}_{y \in \Omega} |F(y)|, & p = \infty.
\end{cases}
\]

We say that an entire function \( g : \mathbb{C} \to \mathbb{C} \) has exponential type \( \sigma \) if for any \( \epsilon > 0 \) there exists a constant \( C = C(\epsilon, g, \sigma) > 0 \) such that for all \( z \in \mathbb{C}, |g(z)| \leq C \exp ((\sigma + \epsilon)|z|) \). The class of all entire functions of exponential type \( \sigma \) is denoted by \( B_\sigma \). Throughout the paper, if no confusion may occur, the same notation is applied to \( g \in B_\sigma \) and its restriction to \( \mathbb{R} \) (e.g., in the form \( g \in B_\sigma \cap L^p(\mathbb{R}) \)). Here, we mostly discuss entire functions of exponential type 1 (EFET1) from \( B_1 \).

2010 Mathematics Subject Classification. Primary 41A05, 11M06, 11M26, Secondary 11M35.

Key words and phrases. Zeta functions, Dirichlet \( L \)-functions, entire functions of exponential type, interpolation difference.
The Fourier transform of a function \( G \in L^1(\mathbb{R}) \) is denoted by the formula

\[ \hat{G}(u) := \int_{\mathbb{R}} G(y)e^{iuy} \, dy. \]

We use the same notation \( \hat{G} \) for the Fourier transform of a tempered distribution \( G \) on \( \mathbb{R} \). By the definition (see, e.g., [15, Sect. 4.1]), \( G \) is a continuous linear functional \( \langle G, \psi \rangle \) on the Schwartz class \( S(\mathbb{R}) \) of all test functions \( \psi \) on \( \mathbb{R} \), and \( \hat{G} \) is defined by the formula \( \langle \hat{G}, \psi \rangle := \langle \psi, \hat{\psi} \rangle, \psi \in S(\mathbb{R}) \).

If a function \( G \) of polynomial growth on \( \mathbb{R} \) is locally integrable, then it generates the tempered distribution \( G \) by the formula \( \langle G, \psi \rangle := \int_{\mathbb{R}} G(y)\psi(y) \, dy, \psi \in S(\mathbb{R}) \).

Throughout the paper \( C, C_1, C_2, \ldots, C_{18} \) denote positive constants independent of essential parameters. Occasionally, we indicate dependence on certain parameters. The same symbol \( C \) does not necessarily denote the same constant in different occurrences, while \( C_j, 1 \leq j \leq 18 \), denotes the same constant in different occurrences.

1.2. Interpolation Differences and Zeta Functions. It was Bernstein who in 1938 initiated the study of polynomial approximation and approximation by EFET1 of the function \( f_s(y) := |y|^s \) by proving the following celebrated result (see [1, Eqn. (36)]) for \( s > 0 \):

\[
\lim_{n \to \infty} n^s \inf_{P_n \in \mathcal{P}_n} \|f_s - P_n\|_{L^\infty([-1,1])} = \inf_{g \in B_1} \|f_s - g\|_{L^\infty(\mathbb{R})} < \infty, \tag{1.1}
\]

where \( \mathcal{P}_n \) is the class of all univariate algebraic polynomials of degree at most \( n \). An essential ingredient of the proof of (1.1) in [1] was the use of the interpolation difference \( \Delta_s(y) := f_s(y) - g_s(y), y \in \mathbb{R} \), where \( g_s \) is the only EFET1 that interpolates \( f_s \) at the nodes \( \{\pi(n + 1/2)\}_{n \in \mathbb{Z}} \) under a certain condition. Bernstein [1, Eqn. (42)] announced without proof the formula for \( g_s \) and the integral representation for \( \Delta_s \). The author [6, Lemma 5 (a)] proved these formulae and obtained similar results for \( |y|^s \text{sgn} y \). Note that \( L^p \)-versions of (1.1) for complex \( s \) with \( \text{Re} \, s > \max\{-1, -1/p\}, p \in (0, \infty) \), or \( \text{Re} \, s = 0, p = \infty \), were recently proved in [9].

The systematic studies of the interpolation difference \( f - g \) and its \( L^1(\mathbb{R}) \)-norm for certain real-valued functions \( f \) (including \( f = f_s, s > 0 \)) were conducted by Vaaler [17], Littmann [13], Carneiro and Vaaler [2], the author [7], and others.

It turns out that the interpolation difference \( \Delta_s \) possesses the following surprising property:

\[
\beta(s) = \frac{\pi}{4 \sin(\pi s/2)} \lim_{|y| \to \infty} \frac{\Delta_s(y)}{\cos y}, \quad s > 0, \quad s \neq 2, 4, \ldots, \tag{1.2}
\]

where \( \beta(s) \) is the Dirichlet beta function (or the Dirichlet \( L \)-function \( L(s, \chi) \) with the character \( \chi \) of modulus 4, see Section 1.3 for definitions). This relation can be extended to a complex \( s \) as well.

In this paper we discuss more general interpolation differences \( \Delta_{k,v}(y) := f_{k,v}(y) - g_{k,v}(y), y \in \mathbb{R} \), related to the function \( \Phi \) (see Section 1.3 for the definition), with a parameter \( v \in (0,1] \) and a
complex \( s, \Re s > 1 \) for \( k = 0 \) and \( \Re s > 0 \) for \( k = 1 \). Here, \( f_{k,s,v} \) is the linear combination of \( |y|^s \) and \( |y|^s \text{sgn} y \), and \( g_{k,s,v} \) is the only EFET1 that interpolates \( f_{k,s,v} \) at the nodes \( \{ \pi(n + k/2) \}_{n=1}^{\infty} \) under certain conditions.

Main results are presented in Section 2. The explicit formulæ for \( f_{k,s,v} \) and \( g_{k,s,v} \) and a general version of (1.2) are given in Theorem 2.1. Special cases associated with the zeta functions \( \zeta, \beta \) and general \( L \)-functions are discussed in Corollary 2.3 and Theorem 2.8 respectively. Different versions of Theorem 2.8 for two special \( L \)-functions are presented in Corollaries 2.11 and 2.12. New criteria for zeros of the zeta functions and general \( L \)-functions in the critical strip in terms of integrability of the interpolation differences are discussed in Corollaries 2.2, 2.5, 2.6, and 2.10.

Preliminaries are discussed below, and the proofs of main results are given in Section 5. The proofs are based on the two lemmas proved in Sections 3 and 4.

1.3. Preliminaries. Here, we discuss certain special functions and their properties.

**Special Functions** \( \Phi, \zeta, \) and \( \beta \). The Lerch transcendent

\[
\Phi(z, s, v) := \sum_{n=0}^{\infty} (v + n)^{-s} n^v z, \quad |z| < 1, \quad v \neq 0, -1, \ldots, \quad s \in \mathbb{C},
\]

of three complex variables can be extended to a different domain by the following integral representation

\[
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1} e^{(1-v)t}}{e^t - z} dt,
\]

where \( \Re v > 0 \) and either \( |z| \leq 1 \), \( z \neq 1 \), \( \Re s > 0 \), or \( z = 1 \), \( \Re s > 1 \) (see [5, Sect. 1.11], [12], [11]).

In this paper we discuss \( \Phi(z, s, v) \) for \( v > 0 \) (mostly for \( v \in (0, 1] \)) and either \( z = 1 \), \( \Re s > 1 \), or \( z = -1 \), \( \Re s > 0 \), because zeta functions can be expressed in terms of this function. The Hurwitz (or generalized) zeta function \( \zeta(s, v) := \sum_{n=0}^{\infty} (v + n)^{-s} \), \( \Re s > 1 \), \( v > 0 \); the Riemann zeta function \( \zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \Re s > 1 \); and the Dirichlet beta function \( \beta(s) := \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-s}, \Re s > 0 \), allow the following representations in terms of \( \Phi(\pm 1, s, v) \) (see [5] Sects. 1.10 and 1.12)):

\[
\zeta(s, v) = \Phi(1, s, v), \quad \Re s > 1, \quad v > 0; \quad \text{(1.4)}
\]

\[
\zeta(s) = \Phi(1, s, 1), \quad \Re s > 1; \quad \text{(1.5)}
\]

\[
\zeta(s) = (2^s - 1)^{-1} \Phi(1, s, 1/2), \quad \Re s > 1; \quad \text{(1.6)}
\]

\[
\zeta(s) = (1 - 2^{-s})^{-1} \Phi(-1, s, 1), \quad \Re s > 0, \quad s \neq 1; \quad \text{(1.7)}
\]

\[
\beta(s) = 2^{-s} \Phi(-1, s, 1/2), \quad \Re s > 0. \quad \text{(1.8)}
\]

Note that formula (1.7) extends \( \zeta(s) \) to \( \Re s > 0, s \neq 1 \).
The Dirichlet Characters and $L$-functions. Let $\chi = \chi(\cdot, q) : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character of modulus $q$ with $q \in \mathbb{N}$, $q > 1$. Then (see, e.g., [14, Sect. 4.2]) $\chi$ is a completely multiplicative and $q$-periodic function on $\mathbb{Z}$ with $|\chi(\cdot)| = 0$ or 1, and $\chi(l) = 0$ if and only if $(l, q) > 1$. In addition,

$$\sum_{l=1}^{q-1} \chi(l) = 0, \quad \chi \neq \chi_0,$$  \hspace{1cm} (1.9)

where

$$\chi_0(l) = \chi_0(l, q) := \begin{cases} 1, & (l, q) = 1, \\ 0, & (l, q) > 1, \end{cases} \quad l \in \mathbb{Z},$$

is the principal character. Two more examples are given below.

$$\chi(l, 3) = \begin{cases} 1, & l \equiv 1 \pmod{3}, \\ -1, & l \equiv 2 \pmod{3}, \\ 0, & l \equiv 0 \pmod{3}, \end{cases} \quad \chi(l, 4) = \begin{cases} (-1)^{(l-1)/2}, & l \text{ odd}, \\ 0, & l \text{ even}, \end{cases} \quad l \in \mathbb{Z}. \quad (1.10)$$

A Dirichlet character $\chi$ of modulus $q$ is called primitive if for every proper divisor $d$ of $q$ (that is, $d < q$), there exists an integer $a \equiv 1 \pmod{d}$, with $(a, q) = 1$ and $\chi(a) \neq 1$. The following properties of primitive characters hold true (see, e.g., [3, Sect. 6.3]):

(i) If $q$ is an odd prime, then every nonprincipal character is primitive.

(ii) If $\chi$ is a primitive character, then

$$\left| \sum_{l=1}^{q-1} \chi(l)e^{2\pi il/q} \right| = \sqrt{q}.$$  \hspace{1cm} (i)

(iii) If $\chi$ is a primitive character, then

$$\sum_{l=1}^{q-1} \chi(l)e^{2\pi il/q} = \bar{\chi}(n) \sum_{l=1}^{q-1} \chi(l)e^{2\pi il/q}, \quad n \in \mathbb{Z},$$

where $\bar{\chi}(n)$ is the complex conjugate of $\chi(n)$. Note that the sum in the right-hand side of this equality is called the Gauss sum.

Let us consider the following equation in $n \in \mathbb{Z}$:

$$\sum_{l=1}^{q-1} \chi(l)e^{2\pi i nl/q} = 0. \quad (1.11)$$

**Proposition 1.1.** Let $\chi \neq \chi_0$ be a primitive character (for example, $\chi \neq \chi_0$ and $q$ is an odd prime by property (i)). Then the following statements hold true:

(a) A number $n \in \mathbb{Z} \setminus \{0\}$ satisfies equation \(1.11\) if and only if $(n, q) > 1$.

(b) The function

$$T_q(2z) := \frac{\sin qz}{\sum_{l=1}^{q-1} \chi(l)e^{i(2l-q)z}}, \quad z \in \mathbb{C}, \quad q > 2,$$  \hspace{1cm} (1.12)
is entire if and only if $q = 3$ or $q = 4$. In addition,

$$T_3(y) = (i/2)(1 + 2 \cos y), \quad T_4(y) = (i/2) \cos y.$$  \hspace{1cm} (1.13)

**Proof.** Statement (a) immediately follows from properties (ii) and (iii) of primitive characters. To prove statement (b), we note first that $T_3$ and $T_4$, given by (1.13), are entire functions. Next, the exponential sum

$$P_q(2z) := (2/i) \sum_{l=1}^{q-1} \chi(l) e^{i(2l-q)z} = (2/i) e^{-qz} \sum_{l=1}^{q-1} \chi(l) e^{2li z}$$  \hspace{1cm} (1.14)

is a trigonometric polynomial of exact degree $q - 2$ (since $\chi(1) \neq 0$), and $|P_q(2\cdot)|$ is a $\pi$-periodic function on $\mathbb{C}$ by (1.14). Then $P_q(2\cdot)$ has exactly $q - 2$ zeroes in the strip $\text{Re } z \in (0, \pi]$. Furthermore, by statement (a), the number of zeroes of $P_{q-2}$ of the form $\pi n/q, n \in \mathbb{N}, n \leq q$, is $q - \varphi(q)$, where $\varphi$ is Euler’s totient function. Therefore, if $T_q(2\cdot)$ is entire, then $\varphi(q) = 2$. This is possible only for $q \in \{3, 4, 6\}$. Finally, $\chi(\cdot, 3)$ and $\chi(\cdot, 4)$ are primitive characters and $\chi(\cdot, 6)$ is an imprimitive character. Thus statement (b) is established. \hfill \Box

Note that all above-mentioned Dirichlet characters have modulus $q > 1$. However, the Dirichlet character of modulus 1 can be defined as well by $\chi = \chi(\cdot, 1) \equiv 1$ on $\mathbb{Z}$.

A Dirichlet $L$-function $L(s, \chi)$ is a meromorphic function on $\mathbb{C}$, which is the holomorphic extension of a Dirichlet $L$-series

$$L(s, \chi) := \sum_{l=1}^{\infty} \frac{\chi(l)}{l^s}, \quad \text{Re } s > 1.$$  \hspace{1cm} (1.15)

In particular (see, e.g., [14, Sect. 4.3]),

$$L(s, \chi(\cdot, 1)) = \zeta(s), \quad L(s, \chi(\cdot, 2)) = L(s, \chi_0(\cdot, 2)), \quad L(s, \chi_0(\cdot, q)) = \zeta(s) \prod_{p \mid q} \left(1 - p^{-s}\right),$$  \hspace{1cm} (1.16)

where $s \in \mathbb{C} \setminus \{1\}, q > 1,$ and $p$ is a prime. Two more examples below follow from (1.10) and (1.15):

$$L(s, \chi(\cdot, 3)) = \sum_{d=0}^{\infty} \left((3d+1)^{-s} - (3d+2)^{-s}\right), \quad L(s, \chi(\cdot, 4)) = \beta(s), \quad \text{Re } s > 0, \quad \chi \neq \chi_0.$$  \hspace{1cm} (1.17)

In general, if $\chi \neq \chi_0$, then $L(s, \chi)$ is an entire function and series (1.15) is convergent for $\text{Re } s > 0$. In addition, the following integral representation for $\text{Re } s > 0$ holds true:

$$L(s, \chi) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^{qx} - 1} \sum_{l=1}^{q-1} \chi(l) e^{(q-l)x} dx, \quad \chi \neq \chi_0, \quad \text{Re } s > 0.$$  \hspace{1cm} (1.18)

This formula easily follows from (1.15) for $\text{Re } s > 1$, and its holomorphic extension to $\text{Re } s > 0$ immediately follows from (1.9). Representation (1.18) can be rewritten in terms of the function $\Phi$.
(or in terms of \( \zeta(s, v) \), see (1.4)) by using (1.3) as

\[
L(s, \chi) = q^{-s} \sum_{l=1}^{q-1} \chi(l) \Phi(1, s, l/q), \quad \text{Re } s > 0. \tag{1.19}
\]

Note again that despite the fact that \( \Phi(1, s, l/q) \) is defined for \( \text{Re } s > 1 \), the \( L \)-function in (1.19) can be holomorphically extended to \( \text{Re } s > 0 \) by (1.18).

2. Main Results

Here, we present limit representations for \( \Phi(\pm 1, s, v) \), \( \zeta(s) \), \( \beta(s) \), and general \( L \)-functions and apply them to new criteria for their zeros.

2.1. Asymptotic Relations for \( \Phi(\pm 1, s, v) \). Throughout the paper we assume that \( k = 0 \) or \( k = 1 \) and

\[
m_k = m_k(s) := \begin{cases} 
\lfloor (\text{Re } s - 1)/2 \rfloor, & k = 0, \\
\lfloor (\text{Re } s)/2 \rfloor, & k = 1. 
\end{cases}
\tag{2.1}
\]

We first discuss the following general results about \( \Phi(-1^k, s, v) \) and its zeros.

**Theorem 2.1.** Let one of the following conditions on \( k, s \), and \( v \) be satisfied:

\[
\begin{align*}
\text{Re } s &\in (1, \infty), \text{ Re } s \neq 3, 5, \ldots, \text{ and } s \notin \mathbb{N}, \text{ if } k = 0 \text{ and } v \in (0, 1); \\
\text{Re } s &\in \bigcup_{d \in \mathbb{Z}_+} (1 + 2d, 2 + 2d), \text{ if } k = 0 \text{ and } v = 1; \\
\text{Re } s &\in (0, \infty), \text{ Re } s \neq 2, 4, \ldots, \text{ and } s \notin \mathbb{N}, \text{ if } k = 1 \text{ and } v \in (0, 1].
\end{align*}
\tag{2.2}
\]

Then the following statements hold true:

(a) The function

\[
g_{k,s,v}(y) := -\sin(y + \pi k/2) \left( \sum_{j=1}^{m_k} (-1)^{j-1} \Gamma(s-2j) \Phi\left((-1)^k, s-2j, v\right) (2y)^{2j} \right.
+ \left. \frac{\pi^{2s} y^{2(m_k+1)}}{\sin(\pi s)} \sum_{n=1-k}^{\infty} \frac{\pi(n+k/2)^{s-2m_k-1} \sin[\pi(n+k/2)v + \pi s/2]}{y^2 - \lfloor \pi(n+k/2) \rfloor^2} \right)
\tag{2.5}
\]

is the only EFET1 that interpolates the function

\[
f_{k,s,v}(y) := -\pi^{2s-2} \left( \frac{|y|^s \sin[(2v-1)y - \pi k/2]}{\sin(\pi s/2)} + \frac{|y|^s \text{sgn } y \cos[(2v-1)y - \pi k/2]}{\cos(\pi s/2)} \right)
\tag{2.6}
\]

at the nodes \( \{\pi(n+k/2)\}_{n \in \mathbb{Z}} \) and satisfies the following conditions: (C1) \( f_{k,s,v} - g_{k,s,v} \in L_\infty(\mathbb{R}) \) and (C2) \( g_{k,s,v}^{(1-k)}(0) = 0 \).
(b) In addition to statement (a), for \( n \in \mathbb{Z} \),

\[
\begin{align*}
  f_{k,s,v}[\pi(n + k/2)] &= g_{k,s,v}[\pi(n + k/2)] \\
  &= -\frac{\pi 2^{s-1}(-1)^{n+k}|\pi n + k/2|^s}{\sin(\pi s)} \sin[\pi(2n + k)v + (\pi s/2)\text{sgn}(n + k/2)].
\end{align*}
\]  

(2.7)

(c) The limit equality

\[
\Phi \left( (-1)^k, s, v \right) = \frac{1}{\Gamma(s)} \lim_{|y| \to \infty} \frac{f_{k,s,v}(y) - g_{k,s,v}(y)}{\sin(y + \pi k/2)}
\]  

(2.8)

is valid.

**Corollary 2.2.** Let \( k, s, \) and \( v \) be as in Theorem 2.1.

(a) If \( \Phi \left( (-1)^k, s, v \right) = 0 \), then for any \( p \in (1/2, \infty) \), \( f_{k,s,v} - g_{k,s,v} \in L_p(\mathbb{R}) \).

(b) If there exists \( p \in (1/2, \infty) \) such that \( f_{k,s,v} - g_{k,s,v} \in L_p(\mathbb{R}) \), then \( \Phi \left( (-1)^k, s, v \right) = 0 \).

2.2. Asymptotic Relations for \( \zeta \) and \( \beta \). Here, we present simplified formulae for \( \zeta(s) \) and \( \beta(s) \) with \( \text{Re} \, s \) belonging to one of the intervals \((1, 2)\), \((1, 3)\), \((0, 2)\). Note that for these values of \( s \) the polynomial in (2.5) is zero. Then the next corollary follows directly from Theorem 2.1 (c) and equalities (1.5) through (1.8).

**Corollary 2.3.** The following statements hold true:

(a) For \( \text{Re} \, s \in (1, 2) \),

\[
\zeta(s) = \frac{1}{\Gamma(s)} \lim_{|y| \to \infty} \frac{f_{0,s,1}(y) - g_{0,s,1}(y)}{\sin y}
\]

\[
= \pi 2^{s-2} \frac{1}{\Gamma(s)} \lim_{|y| \to \infty} \frac{1}{\sin y}
\]

\[
\times \left( -|y|^s \sin y - |y|^s \text{sgn} y \cos y \right) + \frac{2y^2 \sin y}{\cos(\pi s/2)} \sum_{n=1}^{\infty} \left( \frac{1}{y^2 - (\pi n)^2} \right) \, .
\]

(b) For \( \text{Re} \, s \in (1, 3) \),

\[
\zeta(s) = \frac{1}{(2^s - 1) \Gamma(s)} \lim_{|y| \to \infty} \frac{f_{0,s,1/2}(y) - g_{0,s,1/2}(y)}{\sin y}
\]

\[
= \frac{\pi 2^{s-2}}{(2^s - 1) \Gamma(s) \cos(\pi s/2)} \lim_{|y| \to \infty} \frac{1}{\sin y}
\]

\[
\times \left( -|y|^s \text{sgn} y + 2y^2 \sin y \sum_{n=1}^{\infty} \frac{1}{y^2 - (\pi n)^2} \right) \, .
\]
(c) For Re $s \in (0, 2)$,
\[
\begin{align*}
\zeta(s) &= \frac{1}{(1 - 2^{1-s}) \Gamma(s)} \lim_{|y| \to \infty} \frac{f_{1,s,1}(y) - g_{1,s,1}(y)}{\cos y} \\
&= \frac{\pi 2^{s-2}}{(1 - 2^{1-s}) \Gamma(s)} \lim_{|y| \to \infty} \frac{1}{\cos y} \\
&\times \left( |y|^s \cos y - \frac{|y|^s \text{sgn } y \sin y}{\cos(\pi s/2)} - \frac{2y^2 \cos y}{\cos(\pi s/2)} \sum_{n=0}^{\infty} \frac{\pi(n + 1/2)^{s-1}}{y^2 - [\pi(n + 1/2)]^2} \right). 
\end{align*}
\] (2.9)

(d) For Re $s \in (0, 2)$,
\[
\begin{align*}
\beta(s) &= \frac{1}{2^{s-1} \Gamma(s)} \lim_{|y| \to \infty} \frac{f_{1,s,1/2}(y) - g_{1,s,1/2}(y)}{\cos y} \\
&= \frac{\pi}{4 \Gamma(s) \sin(\pi s/2)} \lim_{|y| \to \infty} \frac{1}{\cos y} \\
&\times \left( |y|^s + 2y^2 \cos y \sum_{n=0}^{\infty} \frac{(-1)^n \pi(n + 1/2)^{s-1}}{y^2 - [\pi(n + 1/2)]^2} \right). 
\end{align*}
\] (2.10)

Remark 2.4. The formulae for $f_{k,s,1/2}$ and $g_{k,s,1/2}$ from statements (b) and (d) of Corollary 2.3 for real $s$ were obtained in [6, Lemma 5 (a)]. These results for $k = 1$ were announced in [1, Eqn. (42)].

The specified criteria, presented in the next two corollaries, follow directly from Corollary 2.2 and equalities (1.7) and (1.8). Note that the formulae for the interpolation differences $f_{1,s,1} - g_{1,s,1}$ and $f_{1,s,1/2} - g_{1,s,1/2}$ are given in (2.9) and (2.10), respectively.

**Corollary 2.5.** Let Re $s \in (0, 1)$.

(a) If $\zeta(s) = 0$, then for any $p \in (1/2, \infty)$, $f_{1,s,1} - g_{1,s,1} \in L_p(\mathbb{R})$.

(b) If there exists $p \in (1/2, \infty)$ such that $f_{1,s,1} - g_{1,s,1} \in L_p(\mathbb{R})$, then $\zeta(s) = 0$.

**Corollary 2.6.** Let Re $s \in (0, 1)$.

(a) If $\beta(s) = 0$, then for any $p \in (1/2, \infty)$, $f_{1,s,1/2} - g_{1,s,1/2} \in L_p(\mathbb{R})$.

(b) If there exists $p \in (1/2, \infty)$ such that $f_{1,s,1/2} - g_{1,s,1/2} \in L_p(\mathbb{R})$, then $\beta(s) = 0$.

Remark 2.7. There are dozens of the other criteria (see, e.g., [4]). However, the criteria of Corollaries 2.5 and 2.6 are the first in terms of interpolation of functions like $|y|^s$ by EFET1. Note that certain criteria in terms of interpolation of functions like $|y|^s$ by algebraic polynomials are presented in [8, Sects. 5.3 and 5.4].

2.3. **Asymptotic Relations for Dirichlet $L$-functions.** Recall that $m_1 = m_1(s) = [(\text{Re } s)/2]$ by (2.1). Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character of modulus $q$ with $q > 1$ and $\chi \neq \chi_0$, and let $E(\chi)$ be a symmetric set of all nonzero integers $n$, satisfying equation (1.11). Note that by (1.9),
0 ∈ E(χ). In particular, by Proposition 1.14(a), in case of a primitive character χ, n ∈ E(χ) if and only if (n,q) > 1. In addition, let L(s,χ) be a Dirichlet L-function. Then the following results are valid.

**Theorem 2.8.** Let Re s ∈ (0,∞), Re s ≠ 2, 4, ..., and s ≠ N. Then the following statements hold true:

(a) The function

\[ \gamma_{s,q}(y) := -\sin y \left( q^n \sum_{j=1}^{m_1} (-1)^{j-1} \Gamma(s-2j)L(s-2j,\chi) (2y)^{2j} \right. \]

\[ \left. + \frac{\pi 2^n y^{2(m_1+1)}}{\sin(\pi s)} \sum_{n \in \mathbb{N} \setminus E(\chi)} \frac{(\pi n)^{s-2m_1-1} \sum_{l=1}^{q-1} \chi(l) \sin[2\pi nl/q + \pi s/2]}{y^2 - (\pi n)^2} \right) \quad (2.11) \]

is the only EFET1 that interpolates the function

\[ \varphi_{s,q}(y) := -\pi 2^{s-2} \left( \frac{|y|^s \sum_{l=1}^{q-1} \chi(l) \sin[2\pi l/q - 1)]y^2)}{\sin(\pi s/2)} + \frac{|y|^s \sgn y \sum_{l=1}^{q-1} \chi(l) \cos[2\pi l/q - 1)]y^2)}{\cos(\pi s/2)} \right) \quad (2.12) \]

at the nodes \( \{\pi n\}_{n \in \mathbb{Z}} \) and satisfies the following conditions: (C1*) \( \varphi_{s,q} - \gamma_{s,q} \in L_\infty(\mathbb{R}) \) and (C2*) \( \gamma_{s,q}'(0) = 0 \).

(b) In addition to statement (a),

\[ \varphi_{s,q}(\pi n) = \gamma_{s,q}(\pi n) \]

\[ = \begin{cases} 
0, & n \in \mathbb{N} \setminus E(\chi), \\
-\pi 2^{s-1} (-1)^n |\pi n|^s \sum_{l=1}^{q-1} \chi(l) \sin[2\pi nl/q + (\pi s/2) \sgn n], & n \geq 1, n \in \mathbb{Z} \setminus E(\chi). 
\end{cases} \quad (2.13) \]

(c) The limit equality

\[ L(s,\chi) = \frac{1}{q^n \Gamma(s)} \lim_{|y| \to \infty} \frac{\varphi_{s,q}(y) - \gamma_{s,q}(y)}{\sin y} \quad (2.14) \]

is valid.

**Remark 2.9.** Though Theorem 2.8 holds true for \( q > 1 \) and \( \chi \neq \chi_0 \), certain versions of the theorem are valid for \( q = 1 \) or \( \chi = \chi_0 \). Indeed, (1.3) and the first formula of (1.16) show that if condition (2.3) on s is satisfied, then Theorem 2.1 (see also Corollary 2.3(a) for Re s ∈ (1,2)) can substitute Theorem 2.8 for \( q = 1 \). In addition, it follows from the third formula of (1.16) that the same conclusion can be made for \( q > 1 \) and \( \chi = \chi_0 \) (up to the constant in (1.16)). In particular, statement (b) of Corollary 2.3 follows from relation (2.14) for \( q = 2 \) and \( m_1 = 0 \) by (1.16).

**Corollary 2.10.** Let Re s ∈ (0,1).

(a) If \( L(s,\chi) = 0 \), then for any \( p \in (1/2,\infty) \), \( \varphi_{s,q} - \gamma_{s,q} \in L_p(\mathbb{R}) \).

(b) If there exists \( p \in (1/2,\infty) \) such that \( \varphi_{s,q} - \gamma_{s,q} \in L_p(\mathbb{R}) \), then \( L(s,\chi) = 0 \).
One of the shortcomings of Theorem 2.8 is the same set of nodes \(\{\pi n\}_{|n|=1}\) for all \(q > 1\). Assuming that \(T_q \in B_1\), we can obtain more informative results when we replace in Theorem 2.8 \(\varphi_{s,q}(y)\) with \(\varphi_{s,q}^*(y) := \varphi_{s,q}(qy/2)/P_q(y)\) and \(\gamma_{s,q}(y)\) with \(\gamma_{s,q}^*(y) := \gamma_{s,q}(qy/2)/P_q(y)\), where \(T_q(2\cdot)\) and \(P_q(2\cdot)\) are defined by (1.12) and (1.14), respectively. The advantage of using \(\varphi_{s,q}^*\) and \(\gamma_{s,q}^*\) instead of \(\varphi_{s,q}\) and \(\gamma_{s,q}\) is due to the fact that the EFET1 \(\gamma_{s,q}^*\) interpolates \(\varphi_{s,q}^*\) only at the nodes \(\{ 2\pi n/q \}_{n \in \mathbb{Z} \setminus \{0\}}\), unlike (2.13).

Note that by Proposition 1.1 (b), \(T_q \in B_1\) if and only if \(q = 3\) or \(q = 4\) with \(T_3\) and \(T_4\) given in (1.13) and \(P_3(y) = \sin(y/2), P_4(y) = 2\sin y\). In the following two corollaries we obtain more explicit results when compared with Theorem 2.8 for \(q = 3\) and \(q = 4\). A more explicit version of Theorem 2.8 for \(q = 3\) is presented below.

**Corollary 2.11.** If \(s\) satisfies the conditions of Theorem 2.8, then the following statements hold true:

(a) The function

\[
\gamma_{s,3}^*(y) := (1 + 2\cos y) \left( -3^s \sum_{j=1}^{m_1} (-1)^{j-1} \Gamma(s - 2j) L(s - 2j, \chi(\cdot, 3)) (3y)^{2j} \right.
\]

\[
+ \frac{\pi 3^{s-1/2} y^{2(m_1+1)}}{\sin(\pi s/2)} \left(-\sum_{d=0}^{\infty} \left[ \frac{2\pi(d + 1/3)/2 - 2m_1 - 1}{y^2 - [2\pi(d + 1/3)]^2} \right] + \sum_{d=0}^{\infty} \left[ \frac{2\pi(d + 2/3)/2 - 2m_1 - 1}{y^2 - [2\pi(d + 2/3)]^2} \right] \right) \right)
\]

is the only EFET1 that interpolates the function

\[
\varphi_{s,3}^*(y) := \frac{\pi(3|y|)^s}{2\sin(\pi s/2)}
\]

at the nodes \(\{ 2\pi(d \pm 1/3) \}_{d \in \mathbb{Z}}\) and satisfies the following conditions: \((C1^*)\) \(\varphi_{s,3}^* - \gamma_{s,3}^* \in L_\infty(\mathbb{R})\) and \((C2^*)\) \(\gamma_{s,3}^*(0) = 0\).

(b) In addition to statement (a),

\[
\varphi_{s,3}^*[2\pi(d \pm 1/3)] = \gamma_{s,3}^*[2\pi(d \pm 1/3)] = \frac{\pi 3^s \left[2\pi|d \pm 1/3|\right]^s}{2\sin(\pi s/2)}, \quad d \in \mathbb{Z}.
\]

(c) The limit equality

\[
L(s, \chi(\cdot, 3)) = \frac{1}{3^s \Gamma(s)} \lim_{|y| \to \infty} \frac{\varphi_{s,3}^*(y) - \gamma_{s,3}^*(y)}{1 + 2\cos y}
\]

is valid.

The next corollary shows that for \(q = 4\) Theorem 2.8 can be replaced by more explicit Theorem 2.1 for \(k = 1\) (see also Corollary 2.3 (d) for \(Re s \in (0, 2)\)).

**Corollary 2.12.** If \(s\) satisfies the conditions of Theorem 2.8, then

\[
\varphi_{s,4}^* = 2^s f_{1,s,1/2}, \quad \gamma_{s,4}^* = 2^s g_{1,s,1/2},
\]

\[
\varphi_{s,4} = 2^s f_{1,s,1/2}, \quad \gamma_{s,4} = 2^s g_{1,s,1/2}.
\]
and

\[ L(s, \chi(\cdot, 4)) = \frac{1}{4s \Gamma(s)} \lim_{|y| \to \infty} \frac{\varphi_{s,q}(y) - \gamma_{s,q}(y)}{\cos y}. \]  

(2.20)

Remark 2.13. Versions of Corollary 2.10 for \( q = 3 \) and \( q = 4 \) with \( \varphi_{s,q} \) and \( \gamma_{s,q} \) replaced by \( \varphi_{s,q}^* \) and \( \gamma_{s,q}^* \), respectively, hold true. In case of \( q = 4 \) this version is equivalent to Corollary 2.6.

The proofs of Theorems 2.1 and 2.8 and Corollaries 2.10, 2.11, and 2.12 are presented in Section 5. The proof of Theorem 2.1 is based on the two lemmas that are proved in Sections 3 and 4.

3. Properties of the Integral

Recall that \( \Phi, f_{k,s,v}, \) and \( g_{k,s,v} \) are defined by (1.3), (2.6), and (2.5), respectively.

3.1. Four Major Properties. The proof of Theorem 2.1 is based on properties of the integral

\[ F_{k,s,v}(y) := \int_0^\infty \frac{t^{s-1}e^{(1-v)t}}{(e^t - (-1)^k)(1 + |t/(2y)|^2)} dt, \]

(3.1)

where \( y \in \mathbb{R} \setminus \{0\} \) is a fixed number, \( k = 0 \) or \( k = 1 \), \( v > 0 \), and \( \text{Re} s > 1 \) if \( k = 0 \) and \( \text{Re} s > 0 \) if \( k = 1 \). Note that these conditions guarantee the absolute convergence of the integral and the boundedness of \( F_{k,s,v} \) on \( \mathbb{R} \) (that is, \( F_{k,s,v} \in L_\infty(\mathbb{R}) \)).

Lemma 3.1. If \( y \in \mathbb{R} \setminus \{0\} \), then the following statements hold true.

(a) The following inequality is valid:

\[ |F_{k,s,v}(y) - \Gamma(s)\Phi((-1)^k, s, v)| \leq C_1(k, s, v)y^{-2}. \]  

(3.2)

(b) If \( |y| \leq 1 \), \( \text{Re} s \in (2, \infty) \), and \( v \in (0, 1] \), then

\[ |F_{0,s,v}(y)| \leq C_2(s, v)|y|^{2(\text{Re} s - 1)/\text{Re} s}. \]  

(3.3)

(c) For any \( m \in \mathbb{Z}_+ \) such that \( \text{Re} s - 2m > 1 \) if \( k = 0 \) and \( \text{Re} s - 2m > 0 \) if \( k = 1 \), the function \( F_{k,s,v} \) satisfies the recurrence relation

\[ F_{k,s,v}(y) = \sum_{j=1}^m (-1)^{j-1}\Gamma(s - 2j)\Phi((-1)^k, s - 2j, v)(2y)^{2j} + (-1)^m(2y)^{2m}F_{k,s-2m,v}(y). \]  

(3.4)

(d) Let \( k, s \), and \( v \) satisfy one of conditions (2.2), (2.3), and (2.4). Then the series

\[ \frac{\pi^{2s}}{\sin(\pi s)} \sum_{n=1-k}^\infty \sin(\pi(n+k)/2)^{s-2m_k} \frac{\sin(y + \pi k/2)}{y^2 + (\pi(n+k)/2)^2} \]

in the definition (2.5) of \( g_{k,s,v} \) is convergent for every \( y \in \mathbb{R} \setminus \{0\} \), and the following representation holds true:

\[ \sin(y + \pi k/2)F_{k,s,v}(y) = f_{k,s,v}(y) - g_{k,s,v}(y). \]  

(3.5)
3.2. Proofs of Statements (a), (b), and (c). (a) Inequality (3.2) immediately follows from the equality
\[ \Gamma(s)\Phi\left((-1)^k, s, v\right) - F_{k, s, v}(y) = \int_0^\infty \frac{t^{s+1}e^{(1-v)t}}{(e^t - (-1)^k)(t^2 + 4y^2)} dt. \]

(b) Setting \( \alpha := 2/\text{Re} s \), we split
\[ |F_{0, s, v}(y)| \leq \left( \int_0^{y^\alpha} + \int_{y^\alpha}^{1} + \int_1^{\infty} \right) \frac{t^{\text{Re}s-1}e^{(1-v)t}}{(e^t - 1)(1 + |t/(2y)|^2)} dt = I_1(y) + I_2(y) + I_3(y), \]
where
\[ I_1(y) \leq e^{1-v} \int_0^{y^\alpha} t^{\text{Re}s-2} dt \leq \left( e^{1-v}/(\text{Re} s - 1) \right) y^{\alpha(\text{Re} s - 1)}; \]
\[ I_2(y) \leq 4e^{1-v} \int_{y^\alpha}^{1} \frac{t^{\text{Re}s-3}}{t/y^2} dt \leq 4 \left( e^{1-v}/(\text{Re} s - 2) \right) y^{2-\alpha}; \]
\[ I_3(y) \leq 8y^2 \int_1^{\infty} t^{\text{Re}s-3}e^{-vt} dt \leq 8\Gamma(\text{Re} s - 2)y^2. \]
Combining (3.8), (3.9), and (3.10) with (3.7), we arrive at (3.3).

(c) Using the elementary identity \((h \in \mathbb{R}, u \in \mathbb{R})\)
\[ h^m = (h + u) \sum_{j=1}^{m} (-1)^{j-1}h^{m-j}u^{j-1} + (-1)^m u^m \]
for \( h = t^2 \) and \( u = (2y)^2 \), we obtain
\[ F_{k, s, v}(y) = (2y)^2 \int_0^\infty \frac{t^{s-2m-1}e^{(1-v)t}}{e^t - (-1)^k} \frac{t^{2m}}{t^2 + 4y^2} dt \]
\[ = (2y)^2 \int_0^\infty \frac{t^{s-2m-1}e^{(1-v)t}}{e^t - (-1)^k} \left( \sum_{j=1}^{m} (-1)^{j-1}t^{2(m-j)}(2y)^{2j-2} + (-1)^m \frac{(2y)^{2m}}{t^2 + 4y^2} \right) dt. \]
Hence (3.4) follows from (3.11) and (1.3).

3.3. Properties of \( H \). Recall that \( m_k \) is defined by (2.1). To prove statement (d), we need certain properties of the function
\[ H(w) = H_{k, s, v, y}(w) := \frac{w^{s-1}e^{(1-v)w}}{(e^w - (-1)^k)\left(1 + [w/(2y)]^2\right)}, \quad y \in \mathbb{R} \setminus \{0\}, \quad v \in (0, 1], \]
and its parts. Here, we choose a branch of \( w^{s-1} \) that is holomorphic on \( \mathbb{C} \setminus [0, \infty) \) and satisfies the condition \( \lim_{b \to 0^+} (a + ib)^{s-1} = a^{s-1} \) for \( a > 0 \). We start with two simple properties.

Property 3.2. \( H \) is holomorphic in the complete angle \( 0 < \arg w < 2\pi \) with sides \( L_1 = L_2 = [0, \infty) \), except the points of the set \( E_{k, y} := \{0\} \cup \{2iy, -2iy\} \cup \{\pi i(2n + k) : n \in \mathbb{Z}\} \), which consists of the origin and the simple poles of \( H \).
Property 3.3. For \(|w| > 2\sqrt{2}|y|\),

\[
h_1(w) := \left| w^{s-1} / \left(1 + [w/(2y)]^2 \right) \right| \leq 2y^2|w|^{\operatorname{Re}s-3}, \quad s \in \mathbb{C}.
\]

More relations are discussed in the next three properties.

Property 3.4. For \(v \in (0, 1]\) and \(|w| = \pi(2d + k - 1), \ d \in \mathbb{N}\),

\[
h_2(w) := \left| e^{(1-v)w} \left( e^w - (-1)^k \right)^{-1} \right| \leq \begin{cases} 2e^{-C_3(v)|\operatorname{Re}w|}, & v \in (0, 1), \\ 2, & v = 1, \end{cases}
\]

where \(C_3(v) := \begin{cases} v, & v \in (0, 1/2), \\ 1-v, & v \in [1/2, 1]. \end{cases}\)

Proof. For \(z = a + ib, \ a \in \mathbb{R}, \ b \in \mathbb{R}\), the following identities are valid:

\[
|e^z \pm e^{-z}|^2 = (e^a \pm e^{-a})^2 \cos^2 b + (e^a \mp e^{-a})^2 \sin^2 b = e^{2a} + e^{-2a} \pm 2\cos 2b.
\]  

(3.13)

First, it follows from (3.13) that

\[
|e^z - (-1)^k e^{-z}| \geq (e^{2a} + e^{-2a} - 2)^{1/2} \geq (1/2)e^{|a|}, \quad |a| \geq 1;
\]

\[
|e^z - (-1)^k e^{-z}| \geq e^{|a|} \cos \left( b - \pi(k - 1)/2 \right), \quad |a| < 1.
\]

(3.14)

(3.15)

If \(|z| = \pi(d + (k - 1)/2), \ d \in \mathbb{N}\), then for \(|a| < 1,\)

\[
|\cos \left( b - \pi(k - 1)/2 \right)| = \left| \cos \left( |z| - \sqrt{|z|^2 - a^2} \right) \right| \geq \cos(1/|z|) \geq \cos(2/\pi) > 1/2.
\]  

(3.16)

Next, combining (3.14), (3.15), and (3.16), we obtain

\[
|e^z - (-1)^k e^{-z}| \geq (1/2)e^{\left|\operatorname{Re}z\right|}, \quad |z| = \pi \left( d - (1 - k)/2 \right), \ d \in \mathbb{N}.
\]

(3.17)

Finally, using (3.17) for \(z = w/2, \ |w| = \pi(2d + k - 1), \ d \in \mathbb{N}\), we have

\[
h_2(w) := \frac{e^{(1/2-v)\operatorname{Re}w}}{|e^{w/2} - (-1)^k e^{-w/2}|} \leq \frac{2}{e^{(v-1/2)\operatorname{Re}w + |\operatorname{Re}w|/2}}.
\]

Thus (3.12) follows. \(\square\)

Property 3.5. For \(v \in (0, 1]\) and \(0 < |w| < \min\{2|y|, 2\pi/3\},\)

\[
|H(w)| \leq \frac{C_4(k)|w|^{\operatorname{Re}s+k-2e^{(1-v)|w|}}}{1 - |w/(2y)|^2}.
\]
Proof. It suffices to show that
\[ |e^w - (-1)^k| \geq C_4^{-1}|w|^{1-k}, \quad |w| \leq 2\pi/3. \] (3.18)

For \( w = a + ib, a \in \mathbb{R}, b \in \mathbb{R}, \) and \( |w| \leq 2\pi/3, \) the following relations are valid:
\[ |e^w - (-1)^k| = \left( (e^a - 1)^2 + 4e^a \cos^2 (b/2 - \pi(k - 1)/2) \right)^{1/2} \]
\[ \geq \max \left\{ |e^a - 1|, 2e^{a/2} \cos (b/2 + \pi(1 - k)/2) \right\}. \] (3.19)

It follows from (3.19) that
\[ |e^w - (-1)^k| \geq 1 - e^a \geq 1/2 \geq C_5(k)|w|^{1-k}, \quad a < - \log 2; \] (3.20)
\[ |e^w - (-1)^k| \geq \max \left\{ |a|/2, (1/2)^{1/2}|b|/\pi \right\} \]
\[ \geq C_6(k) \begin{cases} 1, & k = 1, \\ \max \{|a|, |b|\}, & k = 0, \end{cases} \quad C_7(k)|w|^{1-k}, \quad a \geq - \log 2, \quad |b| \leq 2\pi/3. \] (3.21)

Thus (3.18) follows from (3.20) and (3.21) with \( C_4^{-1} := \max\{C_5(k), C_7(k)\}. \) \( \square \)

**Property 3.6.** Let \( \Re s \in (1, 3), \) if \( k = 0 \) and \( v \in (0, 1); \) \( \Re s \in (1, 2) \) if \( k = 0 \) and \( v = 1; \) and \( \Re s \in (0, 2), \) if \( k = 1 \) and \( v \in (0, 1]. \) Then
\[ \lim_{\varepsilon \to 0^+} \int_{K(\varepsilon)} H(w)dw = 0, \] (3.22)
\[ \lim_{d \to \infty} \int_{K(R_d)} H(w)dw = 0, \] (3.23)

where \( R_d := \pi (2d + k - 1), d \in \mathbb{N}. \)

**Proof.** Since the conditions of the property guarantee that \( \Re s + k - 1 > 0, \) we obtain from Property 3.5
\[ \lim_{\varepsilon \to 0^+} \int_{K(\varepsilon)} |H(w)|dw \leq 2\pi C_4(k) \lim_{\varepsilon \to 0^+} \varepsilon^{\Re s + k - 1} = 0. \]

Thus (3.22) is established. Next, for \( |w| = R_d, \) where \( d \in \mathbb{N} \) is large enough, we have from Properties 3.3 and 3.4
\[ |H(w)| = h_1(w)h_2(w) \leq 4y^2|w|^{\Re s - 3} \begin{cases} e^{-C_3(v)|w| \cos(\arg w)}, & v \in (0, 1), \\ 1, & v = 1. \end{cases} \] (3.24)

If \( v = 1 \) and \( \Re s \in (1, 2) \) or \( \Re s \in (0, 2), \) then by (3.24),
\[ \lim_{d \to \infty} \int_{K(R_d)} |H(w)|dw \leq 8\pi y^2 \lim_{d \to \infty} R_d^{\Re s - 2} = 0. \] (3.25)
If \( v \in (0, 1) \) and \( \text{Re } s \in (1, 3) \) or \( \text{Re } s \in (0, 2) \), then using (3.24) and Jordan’s Lemma, we obtain
\[
\lim_{d \to \infty} \int_{K(R_d)} |H(w)|dw \leq 8\pi y^2 \lim_{d \to \infty} R_d^{\text{Re } s - 3} = 0. \tag{3.26}
\]
Thus (3.23) follows from (3.25) and (3.26) in all cases. \( \square \)

3.4. **Proof of Statement (d).** We first prove Lemma 3.1 (d) under the conditions of Property 3.6 when \( m_k = 0 \), that is, we show that the following relations hold true:

\[
\sin(y + \pi k/2)F_{k,s,v}(y) = f_{k,s,v}(y) - g_{k,s,v}(y) = \pi 2^{s-2} \left( -\frac{|y|^s \sin[(2v - 1)y - \pi k/2]}{\sin(\pi s/2)} - \frac{|y|^s \sgn y \cos[(2v - 1)y - \pi k/2]}{\cos(\pi s/2)} \right) + \frac{\pi 2^s |y + \pi k/2|^2}{\sin(\pi s)} \sum_{n=-k}^{\infty} \frac{[\pi(n+k/2)]^{s-1} \sin[\pi(2n+k)v + \pi s/2]}{y^2 - [\pi(n+k/2)]^2}. \tag{3.27}
\]

Indeed, using Properties 3.2 and 3.6 and the Residue Theorem, we obtain
\[
(1 - e^{2\pi is}) \int_0^\infty H(w)dw = \int_{L_1} H(w)dw - e^{2\pi is} \int_{L_2} H(w)dw = 2\pi i \sum_{w \in E_{k,y} \setminus \{0\}} \text{Res}(H, w).
\]
Hence
\[
F_{k,s,v}(y) = \int_0^\infty H(w)dw = \frac{2\pi i}{1 - e^{2\pi is}} \sum_{w \in E_{k,y} \setminus \{0\}} \text{Res}(H, w). \tag{3.28}
\]
(recall that \( s \notin \mathbb{N} \)). Next, by straightforward calculations,
\[
\frac{2\pi i}{1 - e^{2\pi is}} \left( \text{Res}(H, 2yi) + \text{Res}(H, -2yi) \right) = \frac{\pi 2^{s-2} \left( -\frac{|y|^s \sin[(2v - 1)y - \pi k/2]}{\sin(\pi s/2)} - \frac{|y|^s \sgn y \cos[(2v - 1)y - \pi k/2]}{\cos(\pi s/2)} \right) }{\sin(y + \pi k/2)} = f_{k,s,v}(y)/\sin(y + \pi k/2). \tag{3.29}
\]
and
\[
\frac{2\pi i}{1 - e^{2\pi is}} \sum_{n \in \mathbb{Z}, n+2k \neq 0} \text{Res}(H, \pi i(2n+k)) = \frac{\pi 2^s y^2}{1 - e^{2\pi is}} \sum_{n=-k}^{\infty} \frac{[\pi(n+k/2)]^{s-1} \left( e^{-\pi i(2n+k)} + (-1)^{s-1} e^{-\pi i(2n+k)} \right) }{y^2 - [\pi(n+k/2)]^2} = \frac{\pi 2^s y^2}{\sin(\pi s)} \sum_{n=-k}^{\infty} \frac{[\pi(n+k/2)]^{s-1} \sin[\pi(2n+k)v + \pi s/2]}{y^2 - [\pi(n+k/2)]^2} = -g_{k,s,v}(y)/\sin(y + \pi k/2). \tag{3.30}
\]

Then it follows from (3.28), (3.29), and (3.30) that the sum of series (3.5) for \( m_k = 0 \) is equal to
\[
\frac{\sin(\pi s)}{\pi 2^s y^2} \left( \sin(y + \pi k/2) \int_0^\infty H_{k,s,v}(w)dw - f_{k,s,v}(y) \right).
\]
that is, series (3.5) is convergent for \( y \in \mathbb{R} \setminus \{0\} \). Note that the convergence of (3.5) is trivial for \( 0 < \text{Re} \, s < 2, \, s \neq 1 \), but it is not the case for \( 2 < \text{Re} \, s < 3 \).

Combining now (3.28), (3.29), and (3.30), we arrive at (3.27) and (3.6) as well if the conditions of Property 3.6 are satisfied.

Furthermore, we extend (3.6) to the case when \( k, \, s, \) and \( v \) satisfy one of less restrictive conditions (2.2), (2.3), and (2.4) of Theorem 2.1. Since \( k, \, s - 2m_k, \) and \( v \) satisfy the conditions of Property 3.6, we see from (3.27) that

\[
(-1)^m (2y)^{2m} F_{k,s-2m_k,v}(y) = (-1)^m (2y)^{2m} (f_{k,s-2m_k,v}(y) - g_{k,s-2m_k,v}(y)) = f_{k,s,v}(y) - (-1)^m (2y)^{2m} g_{k,s-2m_k,v}(y). \tag{3.31}
\]

Finally, using recurrence relation (3.4) with \( m = m_k \), we arrive at (3.6) from (3.31).

Thus the proof of Lemma 3.1 is completed. \( \square \)

Note that relations like (3.6) for \( v = 1/2 \) and real \( s \) were obtained in [6, Lemma 5 (a)].

4. Properties of \( g_{k,s,v} \).

Recall that \( f_{k,s,v}, \, g_{k,s,v}, \) and \( F_{k,s,v} \) are defined by (2.6), (2.5), and (3.1), respectively. In addition to Lemma 3.1 the proof of Theorem 2.1 is based on certain properties of \( g_{k,s,v} \).

**Lemma 4.1.** Let \( k, \, s, \) and \( v \) satisfy one of conditions (2.2), (2.3), and (2.4) of Theorem 2.1. Then the following statements are valid.

(a) \( g_{k,s,v} \) is an EFET1 that satisfies the conditions (C1) \( f_{k,s,v} - g_{k,s,v} \in L_{\infty}(\mathbb{R}) \) and (C2) \( g_{k,s,v}^{(1-k)}(0) = 0 \).

(b) \( g_{k,s,v} \) interpolates \( f_{k,s,v} \) at the nodes \( \{\pi(n+k/2)\}_{n \in \mathbb{Z}} \). In addition, relations (2.7) hold true.

(c) \( g_{k,s,v} \) is the only EFET1 that interpolates \( f_{k,s,v} \) at the nodes \( \{\pi(n+k/2)\}_{n \in \mathbb{Z}} \) and satisfies (C1) and (C2).

**Proof.** (a) We first prove that \( g_{k,s,v} \in B_1 \). It suffices to show that if \( k, \, s, \) and \( v \) satisfy the conditions of Property 3.6 then \( G(y) := g_{k,s,v}(y)/y^2 \) defined by (3.5) for \( m_k = 0 \) is an EFET1. The traditional technique of proving such a result (see, e.g., [16 Sect. 4.3] and the proof of Corollary 2.11) can be applied here for \( \text{Re} \, s \in (0,2), \, s \neq 1 \), but it is not applicable for \( \text{Re} \, s \in (2,3) \) since series (3.5) for \( m_k = 0 \) is not absolutely convergent. That is why we use a different approach based on the Fourier transform of \( G \) and on the generalized Paley-Wiener theorem.

Note that by (3.6), (2.6), and the boundedness of \( F_{k,s,v} \) on \( \mathbb{R} \),

\[
|g_{k,s,v}(y)| \leq |F_{k,s,v}(y)| + |f_{k,s,v}(y)| \leq C_8(k, s, v)(1 + |y|^{\text{Re} \, s}). \tag{4.1}
\]
Next, if \( \text{Re} \, s \in (0, 2), s \neq 1 \) and \( k = 0 \) or \( k = 1 \), then series (3.5) for \( m_k = 0 \) converges uniformly on \([-1, 1]\). Therefore, by (4.1),

\[
\sup_{y \in \mathbb{R}} |G(y)| < \infty, \quad \text{Re} \, s \in (0, 2), \quad s \neq 1, \quad k = 0 \text{ or } k = 1.
\]  

(4.2)

If \( \text{Re} \, s \in (2, 3) \) and \( k = 0 \), then by (3.3) and (4.1),

\[
|G(y)| \leq C_g(k, s, v) \begin{cases} |y|^{-2/\text{Re} \, s}, & |y| \leq 1, \\ |y|^{\text{Re} \, s - 2}, & |y| > 1, \end{cases} \quad \text{Re} \, s \in (2, 3), \quad k = 0.
\]  

(4.3)

In addition, by Lemma 3.1 (d), the function \( G \) is the limit of continuous functions on \( \mathbb{R} \). Hence \( G \) is a measurable function on \( \mathbb{R} \), and it is locally integrable on \( \mathbb{R} \) by estimates (4.2) and (4.3). Thus \( G \) generates the tempered distribution \( G \) by the formula \( \langle G, \psi \rangle := \int_\mathbb{R} G(y) \psi(y) \, dy \) for every test function \( \psi \) from the Schwartz class \( S(\mathbb{R}) \).

Its distributional Fourier transform is given by the formula

\[
\hat{G}(u) = \frac{\pi 2^s}{\sin(\pi s)} \sum_{n=1-k}^{\infty} [\pi(n + k/2)]^{s-1} \sin[\pi(2n + k)v + \pi s/2] \hat{h}_{k,n}(u),
\]  

(4.4)

where

\[
h_{k,n}(y) := \frac{\sin(y + \pi k/2)}{y^2 - [\pi(n + k/2)]^2}
\]  

(4.5)

and

\[
\hat{h}_{k,n}(u) = \begin{cases} (-1)^{n+k} i^{1-k} \frac{1}{n+k/2} \sin[\pi(n + k/2)u + \pi k/2], & |u| \leq 1, \\ 0, & |u| > 1. \end{cases}
\]  

(4.6)

To prove (4.6), we first note that the following formulae for the Fourier transform:

\[
\int_{\mathbb{R}} \frac{\sin(y + \pi k/2)}{y^2 + b^2} e^{i u y} \, dy = \frac{\pi i^{1-k}}{b} \begin{cases} e^{-b \sinh bu}, & k = 0, \quad |u| \leq 1, \\ e^{-b |u| \text{sgn} u \sinh b}, & k = 0, \quad |u| > 1, \\ e^{-b \cosh bu}, & k = 1, \quad |u| \leq 1, \\ e^{-b |u| \cosh b}, & k = 1, \quad |u| > 1, \end{cases} \]  

(4.7)

immediately follow from the Laplace-type integral

\[
\int_{\mathbb{R}} \cos ay \, dy = \frac{\pi}{b} e^{-|a|b}, \quad \text{Re} \, b > 0, \quad a \in \mathbb{R},
\]  

(see, e.g., [10] Eqn. 3.723.2)). Next, setting \( b = b_{k,n}(\delta) := \delta + i \pi(n + k/2) \) in (4.7), we see that for all \( y \in \mathbb{R}, \delta \in (0, 1), \) and \( |n| \geq 1 - k, \)

\[
\left| \frac{\sin(y + \pi k/2)}{y^2 + b_{k,n}^2(\delta)} \right| \leq \frac{|\sin(y + \pi k/2)|}{\max \{|y^2 - [\pi(n + k/2)]^2 + \delta^2|, 2\pi|n + k/2|\delta\}} \leq \frac{1}{\min \{1/2, \pi|n + k/2|\}} \frac{|\sin(y + \pi k/2)|}{|y^2 - [\pi(n + k/2)]^2|} \leq \frac{4}{\pi}.
\]  

(4.8)
Note that the second inequality of \((4.8)\) is proved by the following two cases:

\[ \delta \in (0, |y^2 - [\pi(n + k/2)]^2|/2] \quad \text{and} \quad \delta \in (|y^2 - [\pi(n + k/2)]^2|/2, 1). \]

Hence by the Dominated Convergence Theorem,

\[ \hat{h}_{k,n}(u) = \lim_{\delta \to 0^+} \int_{\mathbb{R}} \frac{\sin(y + \pi k/2)}{y^2 + b_{k,n}^2(\delta)} e^{iuy} dy, \quad (4.9) \]

and \((4.6)\) follows from \((4.7)\) and \((4.9)\).

Then combining \((4.4)\) and \((4.6)\), we see that the corresponding trigonometric series converges to a tempered distribution on \([-1, 1]\) and the support of the tempered distribution \(\hat{G}\) is a subset of \([-1, 1]\). Finally using the generalized Paley-Wiener theorem (see, e.g., [15, Theorem 7.2.3]), we arrive at \(G \in B_1\) and \(g_{k,s,v} \in B_1\).

Finally, the condition \(g^{(1-k)}_{k,s,v}(0) = 0\) is a consequence of representation \((2.5)\), and it immediately follows from \((3.6)\) that \(f_{k,s,v} - g_{k,s,v} \in L_\infty(\mathbb{R})\).

(b) \(g_{k,s,v}\) interpolates \(f_{k,s,v}\) at the nodes \(\{\pi(n + k/2)\}_{|n|=1-k}\) by \((3.6)\). In addition, \(f_{0,s,v}(0) = g_{0,s,v}(0) = 0\), by \((2.6)\) and \((2.5)\). To verify equalities \((2.7)\), we first find \(f_{k,s,v}[\pi(n + k/2)]\) for \(|n| \geq 1-k\) by a simple calculation and then use \((3.6)\) again. Note that \((2.7)\) is valid for \(n = k = 0\) as well. It is also possible to prove \((2.7)\) without using \((3.6)\) (at least for \(m_k = 0\)) by a straightforward calculation of \(g_{k,s,v}[\pi(n + k/2)]\) that coincides with \(f_{k,s,v}[\pi(n + k/2)]\) for \(n \in \mathbb{Z}\).

(c) Assume that an EFET1 \(g\) interpolates \(f_{k,s,v}\) at the nodes \(\{\pi(n + k/2)\}_{n \in \mathbb{Z}}\), and, in addition, \(f_{k,s,v} - g \in L_\infty(\mathbb{R})\) and \(g^{(1-k)}(0) = 0\). Next, denoting \(\Phi := g_{k,s,v} - g\), we obtain from statements (a) and (b) of Lemma 4.1 that \(\Phi \in B_1 \cap L_\infty(\mathbb{R})\) and \(\Phi(\pi(n + k/2)) = 0\) for \(n \in \mathbb{Z}\). It is well known (see, e.g., [16 Sect. 4.3.1]) that there exists a constant \(C\) such that \(\Phi(y) = C \sin(y + \pi k/2)\). Since \(0 = \Phi^{(1-k)}(0) = C\), we arrive at \(g = g_{k,s,v}\). This completes the proof of Lemma 4.1.

5. Proofs of Main Results

Here, we prove Theorems 2.1 and 2.8 and Corollaries 2.2, 2.10, 2.11, and 2.12.

Proof of Theorem 2.1. Statement (a) of the theorem immediately follows from Lemma 4.1 while statement (c) is the direct consequence of relations \((3.2)\) and \((3.6)\) of Lemma 3.1. Equalities \((2.7)\) of statement (b) are proved in Lemma 4.1 (b). \(\square\)

Proof of Corollary 2.2. (a) If \(\Phi((-1)^k, s, v) = 0\), then by \((3.2)\) and \((3.6)\) of Lemma 3.1

\[ |f_{k,s,v}(y) - g_{k,s,v}(y)| \leq |F_{k,s,v}(y)| \leq C_1(k, s, v)y^{-2}, \quad y \neq 0. \quad (5.1) \]
Thus statement (a) follows from (5.1).

(b) The statement is proved by contradiction. Assume that there exists \( p \in (1/2, \infty) \) such that

\[
\| f_{k,s,v} - g_{k,s,v} \|_{L_p(\mathbb{R})} < \infty, \tag{5.2}
\]

and, in addition, \( \Phi((-1)^k, s, v) \neq 0 \). For \( N > 0 \) let us define a set \( M_N := \{ y \in [-N, N] : |\sin(y + \pi k/2)| \geq 1/2 \} \). Then by statement (c) of Theorem 2.1, there exists \( y_0 > 0 \) such that for all \( N > y_0 \),

\[
\inf_{y \in M_N \setminus [-y_0, y_0]} |f_{k,s,v}(y) - g_{k,s,v}(y)| \geq |\Gamma(s)\Phi((-1)^k, s, v)|/4.
\]

Hence

\[
\| f_{k,s,v} - g_{k,s,v} \|_{L_p(\mathbb{R})} \geq \limsup_{N \to \infty} \| f_{k,s,v} - g_{k,s,v} \|_{L_p(M_N \setminus [-y_0, y_0])} \\
\geq \left( |\Gamma(s)\Phi((-1)^k, s, v)|/4 \right) \lim_{N \to \infty} |M_N \setminus [-y_0, y_0]|^{1/p} = \infty,
\]

which is in contradiction to (5.2).

\[\Box\]

**Proof of Theorem 2.8** The proof is similar to those of Lemmas 3.1 and 4.1. We first assume that \( \text{Re} \, s \in (0, 2), \, s \neq 1 \), i.e., \( m_1 = 0 \). Then it follows from (2.11) that

\[
\gamma_{s,q}(y) = \frac{\pi 2^s y^2}{\sin(\pi s)} \sin y \sum_{n \in \mathbb{N} \setminus E(\chi)} \frac{(\pi n)^{s-1} \sum_{l=1}^{q-1} \chi(l) \sin[2\pi n l/q + \pi s/2]}{y^2 - (\pi n)^2},
\]

since \( \sum_{l=1}^{q-1} \chi(l) \sin[2\pi n l/q + \pi s/2] = 0 \), \( n \in E(\chi) \), by (1.11) and the definition of \( E(\chi) \). Next, the series, representing \( g_{0,s,l/q} \), \( l = 1, \ldots, q - 1 \), in (2.5), as well as the series, representing \( \gamma_{s,q} \) in (5.3), absolutely converge for \( \text{Re} \, s \in (0, 2) \), \( s \neq 1 \). Therefore, the following representations for \( \gamma_{s,q} \) and \( \varphi_{s,q} \) are valid:

\[
\gamma_{s,q}(y) = \sum_{l=1}^{q-1} \chi(l) g_{0,s,l/q}, \quad \varphi_{s,q}(y) = \sum_{l=1}^{q-1} \chi(l) f_{0,s,l/q}, \quad \text{Re} \, s \in (0, 2), \, s \neq 1. \tag{5.4}
\]

Despite equalities (5.4), Theorem 2.8 for \( \text{Re} \, s \in (0, 2) \) does not follow directly from Theorem 2.1 for \( k = 0 \) and \( m_0 = 0 \) because \( \text{Re} \, s \in (1, 3) \) in this case of Theorem 2.1. That is why we briefly discuss below all major steps of Lemmas 3.1 and 4.1 that are used in the proof of Theorem 2.8.

To prove Theorem 2.8 (a), we study properties of the integral

\[
F_{s,q}(y) := \int_0^\infty \sum_{l=1}^{q-1} \chi(l) e^{(1-l/q)t} \frac{t^{s-1}}{(e^t - 1)(1 + |t/(2y)|^2)} dt = \sum_{l=1}^{q-1} \chi(l) F_{0,s,l/q}(y), \tag{5.5}
\]
where $y \in \mathbb{R} \setminus \{0\}$, $\Re s \in (0, 2)$, $s \neq 1$, and $F_{0,s,l/q}$ is defined by (3.1). Then it is easy to see from (5.4) and (1.19) (cf. the proof of Lemma 5.1) (a) that

$$|F_{s,q}^*(y) - \Gamma(s)q^s L(s, \chi)| \leq C_{10}(s,q)y^{-2}. \quad (5.6)$$

Next, we show that the following equality holds true:

$$\sin y F_{s,q}^*(y) = \varphi_{s,q}(y) - \gamma_{s,q}(y). \quad (5.7)$$

The proof is similar to the one of Lemma 3.1 (d). Let us define the function

$$H^*(w) = H_{s,q,w}^*(w) := \frac{w^{s-1} \sum_{l=1}^{q-1} \chi(l)e^{(1-l/q)w}}{(e^w - 1) \left(1 + |w/(2y)|^2\right)}, \quad y \in \mathbb{R} \setminus \{0\},$$

that is holomorphic in the complete angle $0 < \arg w < 2\pi$, except the points of the set $E_y^* := \{0\} \cup \{2iy, -2iy\} \cup \{2\pi in : n \in \mathbb{Z}\}$, which consists of the origin and the simple poles of $H^*$.

We also need the following estimates of $H^*(w)$ for $y \in \mathbb{R} \setminus \{0\}$:

$$|H^*(w)| \leq C_{11}y^{2||w||\Re s - 3e^{-C_{12}|\cos(\arg w)|}}, \quad |w| = \pi(2d - 1), \quad d \in \mathbb{N}; \quad (5.8)$$

$$|H^*(w)| \leq \frac{C_{13}|w|^{\Re s - 1}}{1 - |w/(2y)|^2}, \quad 0 < |w| < \min\{2|y|, 2\pi/3\}, \quad (5.9)$$

where the constants $C_{11}, C_{12}$, and $C_{13}$ are independent of $w$. Inequality (5.8) follows from Properties 3.3 and 3.4 while (5.9) is a consequence of inequality (3.18) for $k = 0$ and elementary estimates

$$\left|\sum_{l=1}^{q-1} \chi(l)e^{(1-l/q)w}\right| \leq \sum_{l=1}^{q-1} \left|e^{(1-l/q)w} - 1\right| \leq (q - 1) e^{2\pi/3|w|}, \quad |w| \leq 2\pi/3.$$

Then using (5.8), (5.9), and Jordan’s Lemma (similarly to the proof of Property 3.6), we arrive at

$$\lim_{\varepsilon \to 0^+} \int_{K(\varepsilon)} H^*(w)dw = \lim_{d \to \infty} \int_{K(\pi(2d - 1))} H^*(w)dw = 0. \quad (5.10)$$

Furthermore, using (5.10) and the Residue Theorem (similarly to the proof of Lemma 3.1 (d)) we obtain

$$F_{s,q}^*(y) = \int_0^\infty H^*(w)dw = \frac{2\pi i}{1 - e^{2\pi is}} \sum_{w \in E_y^* \setminus \{0\}} \text{Res}(H^*, w) = \frac{\varphi_{s,q}(y) - \gamma_{s,q}(y)}{\sin y}.$$

Thus equality (5.7) is established.

Finally, we discuss certain properties of $\gamma_{s,q}$. If $\Re s \in (0, 2)$, then $\gamma_{s,q}$ given by (5.3) belongs to $B_1$. To prove this statement, it suffices to show that $G^*(y) := \gamma_{s,q}(y)/y^2$ is an EFET1. Indeed, we first note that by (5.7), (2.12), and the boundedness of $F_{s,q}^*$ on $\mathbb{R}$,

$$|\gamma_{s,q}(y)| \leq |F_{s,q}^*(y)| + |\varphi_{s,q}(y)| \leq C_{14}(s,q) \left(1 + |y|^{\Re s}\right). \quad (5.11)$$
Next, since \( \Re s \in (0, 2) \), series (5.3) converges uniformly on \([-1, 1]\). Therefore by (5.11),

\[
\sup_{y \in \mathbb{R}} |G^*(y)| < \infty, \quad \Re s \in (0, 2).
\]

(5.12)

In addition, the function \( \gamma_{s,q}(y) \) is the limit of continuous functions on \( \mathbb{R} \). Hence \( G^* \) is a measurable function on \( \mathbb{R} \) and inequality (5.12) holds true. Thus \( G^* \) is locally integrable on \( \mathbb{R} \), and it generates the tempered distribution \( G^* \) by the formula \((G^*, \psi) := \int_{\mathbb{R}} G^*(y)\psi(y)dy\) for every test function \( \psi \) from the Schwartz class \( S(\mathbb{R}) \).

Its distributional Fourier transform is given by the formula

\[
\widehat{G}^*(u) = \frac{\pi^{2s}}{\sin(\pi s)} \sum_{n=1}^{\infty} [\pi n]^{s-1} \sum_{l=1}^{q-1} \chi(l) \sin[2\pi nl/q + \pi s/2] \widehat{h}_{0,n}(u),
\]

(5.13)

where \( h_{0,n} \) and \( \widehat{h}_{0,n} \) are given in (4.5) and (4.6), respectively.

Then combining (5.13) and (4.6), we see that the corresponding trigonometric series converges to a tempered distribution on \([-1, 1]\) and the support of the tempered distribution \( \widehat{G}^* \) is a subset of \([-1, 1]\). Finally, using the generalized Paley-Wiener theorem (see, e.g., [15, Theorem 7.2.3]), we arrive at \( G^* \in B_1 \) and \( \gamma_{s,q} \in B_1 \).

In addition, it is easy to verify equalities (2.13) (see (2.7)) and to show that conditions (C1*) and (C2*) are satisfied. The uniqueness of \( \gamma_{s,q} \) that interpolates \( \varphi_{s,q} \) at the nodes \( \\{\pi n\}_{n \in \mathbb{Z}} \) and satisfies conditions (C1*) and (C2*) can be proved similarly to the proof of Lemma 4.1 (c).

Next, relation (2.14) follows from (5.6) and (5.7). Thus Theorem 2.8 is established for \( \Re s \in (0, 2) \).

Let now \( \Re s \in (0, \infty) \), \( \Re s \neq 2, 4, \ldots \), and \( s \notin \mathbb{N} \). Recall that \( m_1 = \lfloor (\Re s)/2 \rfloor \). Since \( s - 2m_1 \in (0, 2) \), we can replace \( s \) by \( s - 2m_1 \) in (5.7) and obtain the identity

\[
(-1)^{m_1} (2y)^{2m_1} F_{s-2m_1,q}^*(y) = (-1)^{m_1} (2y)^{2m_1} (\varphi_{s-2m_1,q}(y) - \gamma_{s-2m_1,q}(y))
\]

\[
= \varphi_{s,q}(y) - (-1)^{m_1} (2y)^{2m_1} \gamma_{s-2m_1,q}(y).
\]

(5.14)

Finally, using (5.5) and recurrence relation (3.4) with \( m = m_1 \), we arrive at (2.14) from (5.14). Thus the proof of Theorem 2.8 is completed.

Proof of Corollary 2.10 The proof of the corollary is based on inequality (5.6) and statement (c) of Theorem 2.8 similarly to the proof of Corollary 2.2.

Proof of Corollary 2.11 Note that by Proposition 1.1 (a), \( E(\chi(\cdot, 3)) = \{3d \pm 1\}_{d \in \mathbb{Z}} \). Then formulae (2.15), (2.16), and (2.18) immediately follow from the relations

\[
\gamma_{s,3}^*(y) = \gamma_{s,3}(3y/2)/\sin(y/2), \quad \varphi_{s,3}^*(y) = \varphi_{s,3}(3y/2)/\sin(y/2).
\]

(5.15)
and equalities \((2.11), (2.12), \) and \((2.13)\) for \(q = 3\), respectively.

It remains to prove \((2.17)\) and the statement that \(\gamma_{s,3}^*\), given by \((2.15)\), is the only EFET1 that interpolates \(\varphi_{s,3}\) at the nodes \(\{2\pi(d \pm 1/3)\}_{d \in \mathbb{Z}}\) and satisfies conditions \((C1^*)\) and \((C2^*)\).

We first prove that \(\gamma_{s,3}^* \in B_1\). We use the traditional technique (see, e.g., \cite{16} Sect. 4.3) for the proof because we do not know as to whether the Fourier method used in the proof of Lemma 4.1 (a) can be applied in this case.

It follows from \((5.15)\) that \(\gamma_{s,3}^*\) is an entire function since \(\gamma_{s,3}\) is entire by Theorem 2.8 (a) and all zeros of \(\sin(y/2)\) are zeros of \(\gamma_{s,3}(3y/2)\). Therefore, to prove that \(\gamma_{s,3}^* \in B_1\), it suffices to estimate the function series

\[
\psi_r(z) := \sum_{d=0}^{\infty} (d + r/3)^{Re s - 1} \left| \frac{1 + 2 \cos z}{z^2 - [2\pi(d + r/3)]^2} \right|,
\]

where \(r = 1\) or \(r = 2\), \(z = x + iy \in \mathbb{C}\), and \(Re s \in (0, 2)\). Since \(\psi_r\) is an even function, we can assume, without loss of generality, that \(x = \text{Re } z \geq 0\). Next, for every \(x \geq 0\) there exists \(d_{x} \in \mathbb{Z}_{+}\) such that \(|x - 2\pi(d_{x} + r/3)| \leq 4\pi/3\). Then

\[
\psi_r(z) = (d_{x} + r/3)^{Re s - 1} \left| \frac{1 + 2 \cos z}{z^2 - [2\pi(d_{x} + r/3)]^2} \right| + \sum_{d=0, d \neq d_{x}}^{\infty} (d + r/3)^{Re s - 1} \left| \frac{1 + 2 \cos z}{z^2 - [2\pi(d + r/3)]^2} \right| = I_1(z) + I_2(z),
\]

where

\[
I_1(z) \leq 3\pi^{-2} (d_{x} + r/3)^{Re s - 1} e^{|y|} \leq (|z| + 1)^{Re s} e^{|\text{Im } z|}, \quad |y| = |\text{Im } z| \geq \pi.
\]

Furthermore, if \(|y| < \pi\), then setting \(w := z - 2\pi(d_{x} + r/3)\), we see that \(|w| < (5/3)\pi\). Therefore,

\[
\left| \frac{1 + 2 \cos z}{z^2 - 2\pi(d_{x} + r/3)} \right| = 4 \left| \frac{\sin(w/2)}{w} \right| \left| \cos(w/2 + (-1)^{r+1}\pi/6) \right| \leq 4 \sinh(w/2) \cosh(|w|/2 + \pi/6) < C_{15}, \quad |y| \leq \pi,
\]

where \(C_{15} \in (1, 61)\) is an absolute constant. Since \(|z + 2\pi(d_{x} + r/3)|^{-1} \leq [2\pi(d_{x} + r/3)]^{-1}\) for \(x \geq 0\), we obtain from \((5.19)\)

\[
I_1(z) \leq C_{15} (d_{x} + r/3)^{Re s - 2} < C_{15}, \quad |y| \leq \pi.
\]

Combining \((5.18)\) and \((5.20)\), we arrive at the estimate

\[
I_1(z) \leq C_{15} (|z| + 1)^{Re s} e^{|\text{Im } z|}, \quad z \in \mathbb{C}.
\]
Next,
\[
I_2(z) \leq 3e^{|y|} \sum_{d=0, d \neq d_x}^{\infty} (d + r/3)^{\Re s - 1} |x^2 - [2\pi(d + r/3)]^2|^{-1}
\]
\[
\leq 3(2\pi)^{-2}e^{|y|} \sum_{d=0, d \neq d_x}^{\infty} (d + r/3)^{\Re s - 1} (|d - d_x| - 2/3)^{-1} (|d + d_x| + 2(r - 1)/3)^{-1}
\]
\[
\leq 9(2\pi)^{-2}e^{|y|} \sum_{d=0, d \neq d_x}^{\infty} (d + r/3)^{\Re s - 1} |d^2 - d_x^2|^{-1}
\]
\[
\leq C_{16}(s)e^{|y|} \left( \sum_{d=0}^{d_x-1} (d + 1)^{\Re s - 1} (d_x^2 - d^2)^{-1} + \sum_{d=d_x+1}^{\infty} (d + 1)^{\Re s - 1} (d^2 - d_x^2)^{-1} \right)
\]
\[
= C_{16}(s)e^{|y|} (J_{1,d_x} + J_{2,d_x}),
\]
where \(C_{16}(s) := 9(2\pi)^{-2} \max\{1, 3^{1-\Re s}\}\) and \(J_{1,0} = 0\). Then
\[
J_{1,d_x} = \sum_{\nu=1}^{d_x} \frac{(d_x - \nu + 1)^{\Re s - 1}}{\nu (2d_x - \nu)} \leq \begin{cases} 
\frac{d_x^{-1} \sum_{\nu=1}^{d_x} 1/\nu}{\nu^{\Re s - 2} \sum_{\nu=1}^{d_x} 1/\nu}, & \Re s \in (0, 1), \\
\frac{d_x^{\Re s - 2} (\log d_x + 1)}{} & \Re s \in [1, 2), 
\end{cases}
\]
and
\[
J_{2,0} = \sum_{d=1}^{\infty} \frac{(d + 1)^{\Re s - 1}}{d^2}; \quad J_{2,d_x} = \sum_{\nu=1}^{\infty} \frac{(d_x + \nu + 1)^{\Re s - 1}}{\nu (2d_x + \nu)} \leq \sum_{\nu=1}^{\infty} \nu^{\Re s - 3}, \quad d_x \in \mathbb{N}.
\]
Combining (5.22), (5.23), and (5.24), we arrive at the estimate
\[
I_2(z) \leq C_{18}(s)e^{\Im z}, \quad z \in \mathbb{C}.
\]
Finally, taking account of (5.17), (5.21), and (5.25), we obtain the estimate
\[
\psi_z(z) \leq C (|z| + 1)^{\Re s} e^{\Im z}, \quad z \in \mathbb{C},
\]
where \(C\) is independent of \(z\). Hence (5.16) and (2.15) show that \(g_{s,3}^* \in B_1\).

Next, it immediately follows from representation (2.15) that the condition \(\gamma_{s,3}^*(0) = 0\) is satisfied.

In addition, \(\varphi_{s,3}^* - \gamma_{s,3}^* \in L_{\infty}(\mathbb{R})\). Indeed, recall that by (5.14), equality (5.7) is valid for \(\Re s \in (0, \infty)\), \(\Re s \neq 2, 4, \ldots\), and \(s \notin \mathbb{N}\). Then it follows from (5.7) and (5.15) that
\[
(1 + 2 \cos y) (F_{0,s,1/3}(3y/2) - F_{0,s,2/3}(3y/2)) = \varphi_{s,3}^*(y) - \gamma_{s,3}^*(y).
\]
Hence condition (C1*) is satisfied. In addition, (2.17) is a consequence of (2.16) and (5.20). It is also possible to prove (2.17) without using (5.26) (at least for \(m_1 = 0\)) by a straightforward calculation of \(\gamma_{s,3}^*[2\pi(d \pm 1/3)]\) that coincides with \(\varphi_{s,3}^*[2\pi(d \pm 1/3)]\) for \(d \in \mathbb{Z}\).
Finally, let us assume that an EFET $g$ interpolates $\psi_{s,3}^*$ at the nodes $\{2\pi(d \pm 1/3)\}_{d \in \mathbb{Z}}$ and, in addition, $\psi_{s,3}^* - g \in L_\infty(\mathbb{R})$ and $g(0) = 0$. Next, denoting $\mathfrak{G} := \psi_{s,3}^* - g$, we obtain that $\mathfrak{G} \in B_1 \cap L_\infty(\mathbb{R})$ and $\mathfrak{G}(0) = 0$. In addition, $\mathfrak{G}(2\pi(d \pm 1/3)) = 0$, $d \in \mathbb{Z}$. Then the function $\mathfrak{G}_1(y) := \mathfrak{G}(y) \sin(y/2)$ belongs to $B_{3/2} \cap L_\infty(\mathbb{R})$ and $\mathfrak{G}_1\left(\frac{2\pi}{3}\right) = 0$, $n \in \mathbb{Z}$. It is well known (see, e.g., [16, Sect. 4.3.1]) that there exists a constant $C$ such that $\mathfrak{G}_1(y) = C \sin(3y/2)$, that is, $\mathfrak{G}(y) = C(1 + 2 \cos y)$. Since $0 = \mathfrak{G}(0) = 3C$, we arrive at $g = \gamma_{s,3}^*$. This completes the proof of Corollary 2.11.

**Proof of Corollary 2.12.** Formulae (2.19) immediately follow from (2.11) and (2.12), while (2.20) is a consequence of (1.17) and (2.10). In addition, (2.20) follows as well from (2.14) for $q = 4$. □

**References**

[1] S. Bernstein, Sur la meilleure approximation de $|x|^p$ par des polynômes des degrés très élevés, Bull Acad. Sci. USSR Ser. Math. 2 (1938), 181–190.

[2] E. Carneiro, J. D. Vaaler, Some extremal functions in Fourier analysis, III, Constr. Approx., 31 (2010), 259–288.

[3] K. Chandrasekharan, Arithmetical Functions, Springer-Verlag, New York-Berlin, 2012.

[4] B. Conrey, The Riemann hypothesis, Notices Amer. Math. Soc. 50 (2003), 341–353.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.

[6] M. I. Ganzburg, The Bernstein constant and polynomial interpolation at the Chebyshev nodes, J. Approx. Theory 119 (2002), 193–213.

[7] M. I. Ganzburg, $L$-approximation to non-periodic functions, J. Concr. Appl. Math., 8 (2010), no. 2, 208–215.

[8] M. I. Ganzburg, Polynomial interpolation and asymptotic representations for zeta functions. Dissertationes Math. 496 (2013), 117 pp.

[9] M. I. Ganzburg, Asymptotic behaviour of the error of polynomial approximation of functions like $|x|^\alpha+i\beta$, Comput. Methods Funct. Theory 21, (2021), 73–94.

[10] I. S. Gradshteyn, I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, San Diego, 1980.

[11] J. C. Lagarias, W.-C. W. Li, The Lerch zeta function I. Zeta integrals. Forum Math. 24 (2012), 1–48.

[12] A. Laurenčikas, R. Garunkštis, The Lerch Zeta-Function, Kluwer Academic Publishers, Dordrecht, 2002.

[13] F. Littmann, Entire approximations to the truncated powers, Constr. Approx. 22 (2005), 273–295.

[14] K. Prachar, Primzahlverteilung, Springer-Verlag, Berlin-Götingen-Heidelberg, 1957.

[15] R. Strichartz, A Guide to Distribution Theory and Fourier Transforms, World Scientific, River Edge, NJ, 2003.

[16] A. F. Timan, Theory of Approximation of Functions of a Real Variable, Pergamon Press, New York, 1963.

[17] J. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc., 12 (1985), 183–216.

212 WOODBURN DRIVE, HAMPTON, VA 23664, USA

Email address: michael.ganzburg@gmail.com