Characterization of non-perturbative qubit channel induced by a quantum field

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In this work we provide some characterization of the quantum channel induced by non-perturbative interaction between a single qubit with a quantized massless scalar field in arbitrary globally hyperbolic curved spacetimes. The qubit interacts with the field via Unruh-DeWitt detector model and we consider two non-perturbative regimes: (i) when the interaction is very rapid, effectively at a single instant in time (\textit{delta-coupled detector}); and (ii) when the qubit has degenerate energy level (\textit{gapless detector}). We organize the results in terms of quantum channels and Weyl algebras of observables in the algebraic quantum field theory (AQFT). We collect various quantum information-theoretic results pertaining to these channels, such as entropy production of the field and the qubit, recoverability of the qubit channels, and causal propagation of noise due to the interactions. We show that by treating the displacement and squeezing operations as elements of the Weyl algebra, we can generalize existing non-perturbative calculations involving the qubit channels to non-quasifree Gaussian states in curved spacetimes with little extra effort and provide transparent interpretation of these unitaries in real space. We also generalize the existing result about cohering and decohering power of a quantum channel induced by the quantum field to curved spacetimes in a very compact manner.

I. INTRODUCTION

In standard quantum information theory, the role of relativity is passive, in the sense that one accepts principles such as impossibility of superluminal signalling and derive consequences from them. Some of the no-go results such as Bell inequality and no-cloning theorem are intimately tied to the impossibility to send information faster than the speed of light. At the same time, it is well known a relativistic quantum theory requires us to work with quantum fields. In the absence of quantum theory of gravity, our best understanding of the interplay between quantum theory and gravity is given by the framework of quantum field theory (QFT) in curved spacetimes. Much of what is now known as relativistic quantum information (RQI) seeks to understand various features and of QFT in curved spacetimes using the toolbox from quantum information theory.

One of the most common and useful approaches in RQI involves the use of Unruh-DeWitt (UDW) particle detector model \textsuperscript{[1][2]}, where one couples locally a qubit (which acts as a localized quantum-mechanical ‘detector’) to a quantum field living on top of a generic curved spacetime. It is a simplified model of light-matter interaction representing a monopole-scalar model of atomic dipole-electromagnetic interaction in quantum optics. This model has been refined to admit fully covariant description that allows for arbitrary trajectories and finite-size effect \textsuperscript{3} \textsuperscript{4}, as well as quantized centre of mass degrees of freedom \textsuperscript{5}, higher multipoles and spins. The UDW model is also useful for studying fundamental physics associated to relativistic trajectories or genuine quantum effects in curved spacetimes, such as the Unruh and Hawking effects.

The more important advantages of the UDW model is that it is versatile and strong enough to provide answers to fundamental questions that cannot be directly settled within quantum field theory in curved spacetimes. For example, it allows us to define local measurement theory \textsuperscript{6} for quantum fields even though projective measurements in quantum field theory violate relativistic causality \textsuperscript{7}. Furthermore, since the UDW model is easily generalized to include multiple detectors, it is straightforward to apply it to study relativistic quantum communication (RQC) between two localized parties in curved spacetimes \textsuperscript{8} \textsuperscript{15}. There are numerous other applications of the UDW model in other contexts (see, e.g., \textsuperscript{16} \textsuperscript{20} and references therein).

Our work is largely motivated by the observation that there has been relatively few works on characterization of relativistic quantum channels built using the UDW detector model. There has been some rather general and remarkable results, such as showing the entanglement-breaking nature of certain relativistic communication channels \textsuperscript{10} \textsuperscript{11} or proving “no-go theorems” on entanglement extraction from the quantum field \textsuperscript{27}. However, these results are often restricted to Minkowski spacetime (which is nonetheless important) and relativistic quantum channels are much less understood compared to the non-relativistic counterparts. More recent work \textsuperscript{13} \textsuperscript{15} has exploited the possibility of using non-perturbative methods to obtain very general results regarding certain class of relativistic quantum communication channels in arbitrary curved spacetimes, making use of the full power of algebraic approach to quantum field theory (AQFT).

In this work we aim to fill this gap in the literature by providing some characterization and general results for single-qubit quantum channel within UDW framework. More specifically, we will obtain and collect various properties of quantum channel acting on the qubit detector induced by the interaction with a
quantum field, as well as the corresponding channel associated to the quantum field itself. We will work with non-perturbative regimes: (i) when the interaction is very rapid, effectively at a single instant in time (delta-coupled detector); and (ii) when the qubit has degenerate energy level (gapless detector). We will show that these channels (both qubit and field channels) can be very naturally understood in terms of the algebra of observables of the quantum field theory in the AQFT framework.

It should be stressed that by “non-perturbative regime” we mean that the unitary induced by the detector-field interaction for both gapless and delta-coupling approaches can be worked out without performing any truncation in the sense of Dyson series expansion. This relies on the fact that we can handle the time-ordering operation directly in these settings, so the unitary can be written as a finite linear combination of tensor products of bounded operators. We do not mean that this calculation is non-perturbative in the sense that we solved exactly for the full interacting theory of the detector-field system as a dynamical system. We will thus refer to the latter has having the detector-field system as a dynamical system. Some readers may also prefer to interpret gapless and delta-coupled regimes as the regime where effectively we are performing “resummation” of the series expansion of the unitary evolution.

In more detail, we collect various quantum information-theoretic results pertaining to these channels. On the qubit side, we will explicitly construct the Kraus representations, analyse the entropy production, recoverability, entanglement-breaking property. On the field side, we will show causal behaviour of noise generated by the interaction and generalizations to non-quasifree Gaussian states via coherent and squeezing operations. The generalization to Gaussian states above make use of the observation that displacement and squeezing can be thought of as elements of the Weyl algebra, which allows for very nice interpretation of the behaviour of the correlation functions for these states. Altogether these analyses provide us with two free bonuses: (1) we can generalize the results on cohering power in [28] for free with new insights, and (2) we can show explicitly that in terms of Rényi entropy, both the field and detector’s entropy productions are equal.

As part of the goal to make the use of algebraic approach more accessible, we have provided a more condensed version of the review of AQFT for scalar fields described in [15, 29–30]. We also provide a very brief introduction to von Neumann algebras for the full algebra of observables, which are relevant to make sense of density matrices in AQFT and the computation of the field channel. For example, this helps us understand the computations of Rényi entropy in terms of type I von Neumann algebra of the algebra of observables, in contrast to the more well-known fact about local algebras.

Our paper is organized as follows. In Section III we introduce the bare minimum of AQFT approach and also von Neumann algebras needed in this work. In Section IV we introduce the Unruh-DeWitt detector model and the two non-perturbative methods available for us. In Section V we study some properties of the qubit channels, namely mixed unitary property, recoverability, and entanglement-breaking (or lack thereof) property. In Section VI we study the channel associated to the field state and generalize the results to non-quasifree Gaussian states obtained via squeezing and displacement operations. In Section VII we study two free bonus outputs of this paper, namely cohering power of the qubit channel, and explicit computation of Rényi entropy of the field. We adopt the units $c = \hbar = 1$ and we use mostly-plus signature for the metric.

II. LIGHTNING REVIEW OF ALGEBRAIC QFT AND VON NEUMANN ALGEBRA

In this section we review some aspects of algebraic framework of QFT and von Neumann algebra. We will only cover the very bare minimum to understand or perform calculations outlined in this work to keep it self-contained while not unnecessarily burdening the discussion with background information. For AQFT part, this will be a more condensed version of the summary given in [29, 30], which in turn are based on [31–35]. For von Neumann algebras, we have very much benefitted from [35–37] (more details can be found in [38–40]). Readers can also skip to Section III onwards or Section IV if they are more interested in the main content of this work, referring to this section only when certain details need to be consulted.

A. Algebra of observables

We consider a real scalar field $\phi$ in $(3+1)$-dimensional globally hyperbolic Lorentzian spacetime $(M, g_{ab})$. The field obeys the Klein-Gordon equation

$$P_\phi = 0, \quad P = \nabla_a \nabla^a - m^2 - \xi R,$$

where $\xi \geq 0$, $R$ is the Ricci scalar and $\nabla$ is the Levi-Civita connection with respect to $g_{ab}$. Global hyperbolicity means that $M \cong \mathbb{R} \times \Sigma$ where $\Sigma$ is a Cauchy surface: in such spacetimes, the Klein-Gordon equation admits well-posed initial value problem throughout and we also have a good notion of “constant-time slices”. For example, in flat space we have natural global coordinates $(t, \mathbf{x})$, with Cauchy surface $\Sigma \cong \mathbb{R}^3$ and any constant-$t$ surfaces serve as a good Cauchy surface.

In the large scheme of things, quantization in algebraic framework makes a great deal of use of ingredients in classical field theory. The idea is that we need to construct algebra of observables $\mathcal{A}(M)$ for the field theory as well as quantum states on which $\mathcal{A}(M)$ acts.
We will see that the building blocks of the QFT come from constructing solutions of the wave equation \(^1\). These solutions can be built using appropriate choice of Green’s functions, and we need to provide a “symplectic structure” to realize the dynamical content of the theory, including the implementation of canonical commutation relations (CCR). Finally, we need to construct quantum states without reference to any Hilbert space structure, due to the well-known existence of many unitary inequivalent Hilbert space representations. We will see that there are \textit{a priori} too many options, and the consensus is to pick a subclass of Hadamard states which encode the notion that all states should look “the same” locally and as close to flat space QFT as possible.

Let \( f \in C^\infty_0(M) \) be a smooth compactly supported test function on \( M \). The \textit{retarded and advanced propagators} \( E^\pm = E^\pm(x,y) \) associated to the Klein-Gordon operator \( P \) are Green’s functions obeying

\[
E^\pm f \equiv (E^\pm f)(x) := \int dV' E^\pm(x,x') f(x'),
\]

where here \( dV' = d^4x'\sqrt{-g} \) is the invariant volume element. These solve the inhomogeneous wave equation \( P(E^\pm f) = f \). The \textit{causal propagator} is defined to be the advanced-minus-retarded propagator \( E = E^- - E^+ \). The relevant fact for us is the following: if \( O \) is an open neighbourhood of some Cauchy surface \( \Sigma \) and \( \varphi \in \text{Sol}_R(M) \) is any real solution to Eq. \((1)\) with compact Cauchy data, then there exists \( f \in C^\infty_0(M) \) with \( \text{supp}(f) \subset O \) such that \( \varphi = Ef \). \(^3\)

In AQFT, the quantization of the real scalar field theory \( \hat{\varphi} \) is to be viewed as an \( \mathbb{R} \)-linear mapping from the space of smooth compactly supported test functions to a unital \(*\)-algebra \( \mathcal{A}(M) \) given by

\[
\hat{\varphi} : C^\infty_0(M) \to \mathcal{A}(M), \quad f \mapsto \hat{\varphi}(f),
\]

which satisfy the following properties:

(a) (Hermiticity) \( \hat{\varphi}(f)\dagger = \hat{\varphi}(f) \) for all \( f \in C^\infty_0(M) \);

(b) (Klein-Gordon) \( \hat{\varphi}(Pf) = 0 \) for all \( f \in C^\infty_0(M) \);

(c) (Canonical commutation relations (CCR)) \( \{\hat{\varphi}(f),\hat{\varphi}(g)\} = iE(f,g)\mathbb{1} \) for all \( f,g \in C^\infty_0(M) \), where \( E(f,g) \) is the smeared causal propagator

\[
E(f,g) := \int dV f(x)(Eg)(x).
\]

(d) (Time slice axiom) \( \mathcal{A}(M) \) is generated by the unit element \( \mathbb{1} \) and the smeared field operators \( \hat{\varphi}(f) \) for all \( f \in C^\infty_0(M) \) with \( \text{supp}(f) \subset O \), where \( O \) a fixed open neighbourhood of some Cauchy slice \( \Sigma \).

We say that \(*\)-algebra \( \mathcal{A}(M) \) is the \textit{algebra of observables} of the field. The \textit{smeared} field operator reads

\[
\hat{\varphi}(f) = \int dV \hat{\varphi}(x)f(x).
\]

The (unsmearred) field operator \( \hat{\varphi}(x) \) commonly used in canonical quantization should be thought of as an operator-valued distribution.

The dynamical content of the field theory is reflected by the symplectic structure as follows. The vector space of solutions \( \text{Sol}_R(M) \) can be equipped with a symplectic form \( \sigma : \text{Sol}_R(M) \times \text{Sol}_R(M) \to \mathbb{R} \), defined as

\[
\sigma(\phi_1, \phi_2) := \int_{\Sigma_t} d\Sigma^a \left[ \phi_1 \nabla_a \phi_2 - \phi_2 \nabla_a \phi_1 \right],
\]

where \( d\Sigma^a = -t^a d\Sigma, \) \(-t^a\) is the inward-directed unit normal to the Cauchy surface \( \Sigma_t \), and \( d\Sigma = \sqrt{\varepsilon} d^3x \) is the induced volume form on \( \Sigma_t \). \(^1\) The field operator \( \hat{\phi}(f) \) can be expressed as \textit{symplectically smeared field operator} \(^3\)

\[
\hat{\phi}(f) \equiv \sigma(Ef, \hat{\phi}),
\]

and the CCR algebra can be written as

\[
\sigma(Ef, \hat{\phi}), \sigma(Eg, \hat{\phi}) = i\sigma(Ef, Eg)\mathbb{1} = iE(f,g)\mathbb{1},
\]

where \( \sigma(Ef, Eg) = E(f,g) \) in the second equality follows from Eq. \((5)\) and \((7)\). While in our case it is not directly necessary to construct \( \mathcal{A}(M) \) with explicit reference to \( \sigma \), the symplectic form \((6)\) will be essential when we want to make connection to standard canonical quantization. In particular, we will need to define Klein-Gordon inner product for the one-particle Hilbert space associated to “positive-frequency solutions”.

Since \( \hat{\phi}(f) \in \mathcal{A}(M) \) are unbounded operators, for free fields it is more convenient technically to work with its “exponentiated version” which forms a \textit{Weyl algebra} \( W(M) \), whose elements are bounded operators. The Weyl algebra \( W(M) \) is a unital \(*\)-algebra generated by elements that formally take the form

\[
W(Ef) \equiv e^{i\hat{\phi}(f)}, \quad f \in C^\infty_0(M).
\]

These elements satisfy \textit{Weyl relations}:

\[
W(Ef)\dagger = W(-Ef), \quad W(E(Pf)) = \mathbb{1},
\]

\[
W(Ef)W(Eg) = e^{-\frac{i}{\hbar}E(f,g)}W(E(f+g))
\]

where \( f,g \in C^\infty_0(M) \). Note that relativistic causality (or microcausality) is given by the third Weyl relations. For the rest of this work we try to stick mostly with \( W(M) \).

### B. Algebraic states and quasifree states

In AQFT the state is called an \textit{algebraic state}, defined by a \( \mathbb{C} \)-linear functional \( \omega : W(M) \to \mathbb{C} \) (similarly for \( \mathcal{A}(M) \)) such that

\[
\omega(\mathbb{1}) = 1, \quad \omega(A^\dagger A) \geq 0 \quad \forall A \in W(M).
\]

\(^1\) As is well-known, this definition is independent of the choice of Cauchy surface.
The state $\omega$ is pure if it cannot be written as $\omega = \alpha \omega_1 + (1-\alpha)\omega_2$ for any $\alpha \in (0,1) \text{ and any two algebraic states } \omega_1,\omega_2$; otherwise we say that the state is mixed.

The relationship with standard canonical quantization comes from the Gelfand-Naimark-Segal (GNS) reconstruction theorem [31, 34, 35]: we have a GNS triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$, where $\pi_\omega : \mathbb{W}(\mathcal{M}) \to \mathcal{B}(\mathcal{H}_\omega)$ is a Hilbert space representation with respect to state $\omega$. In its GNS representation, any algebraic state $\omega$ is realized as a vector state $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ and $A \in \mathbb{W}(\mathcal{M})$ are represented as bounded operators $\hat{A} := \pi_\omega(A) \in \mathcal{B}(\mathcal{H}_\omega)$. We can thus write $\omega(A) = \langle \Omega_\omega | \hat{A} | \Omega_\omega \rangle$. Since QFT in curved spacetimes has infinitely many unitarily inequivalent representations of the CCR algebra, the algebraic framework allows us not to pick any one of them until the very last step and work with all representations at once.

One of the most basic objects in QFT is the n-point correlation function defined by

$$\mathbb{W}(f_1, \ldots, f_n) := \omega(\hat{\phi}(f_1) \cdots \hat{\phi}(f_n))$$

where $f_j \in C^\infty_c(\mathcal{M})$ and for a fixed algebraic state $\omega$. It is to be understood that the RHS is computed within some GNS representation of $\mathbb{A}(\mathcal{M})$. The GNS representation of the Weyl algebra $\mathbb{W}(\mathcal{M})$ allows us to calculate Eq. (12) by differentiation: for example, the smeared Wightman function reads

$$\mathbb{W}(f,g) = -\frac{\partial^2}{\partial s \partial t} \bigg|_{s,t=0} \omega(e^{i\hat{\phi}(sf)} e^{i\hat{\phi}(tg)})$$

where the RHS is calculated in the GNS representation of $\mathbb{W}(\mathcal{M})$ (since there is no good notion of derivatives directly on the Weyl algebra [32]). As an example, in flat spacetime the vacuum GNS representation associated to vacuum state $\omega_0$ gives us the Minkowski vacuum $|\Omega_\omega_0\rangle = |\Omega_\mathcal{M}\rangle$.

The general agreement among AQFT practitioners is that physically reasonable states should be Hadamard states [43, 44]. Very roughly speaking, these states respect local flatness and finite expectation values of all observables appropriately [43]. A particularly nice subclass of Hadamard states are quasifree states: for these states, all odd-point functions in the sense of vanish and all higher even-point functions can be written as in terms of just two-point function.

Wightman two-point functions need not vanish and higher-point functions only depend on one- and two-point functions.

specified once we know the Wightman two-point functions associated to the quasifree state $\omega$: we have

$$\omega(W(E_f)) = e^{-\frac{i}{2}\mathbb{W}(f,f)}.$$

At this point, we can simply take Eq. (14) as the definition of quasifree states (see, e.g., [29, 30, 41, 42] for more details). This is very useful because for most practical computations, we do know how to calculate the smeared Wightman function especially if one is familiar with canonical quantization (many examples of the calculations can be found in standard texts such as [45]).

The most important one is the vacuum state $\omega_0$, where we can write the (unsmeared) vacuum Wightman function as

$$\mathbb{W}_0(f,g) = \int d^4k \ u_k(x) u_k^*(y),$$

where $u_k(x)$ are “positive-frequency” modes of Klein-Gordon operator $P$ normalized with respect to Klein-Gordon inner product $\langle \phi_1, \phi_2 \rangle_{\omega_0} := \sigma(\phi_1^* \phi_2)$, where $\phi_j \in \mathbb{W}(\mathcal{M})$ are complexified solutions to Eq. (1) (compare this with canonical quantization discussed in [45]). In situations where $\{u_k\}$ are known explicitly, we can often calculate the symmetrically smeared two-point function (sometimes exactly):

$$\mathbb{W}_0(f,f) = \int dV dV' f(x)f(y)\mathbb{W}_0(x,y).$$

In principle, we can compute any Wightman n-point functions for any algebraic state in their GNS representation. However, it is often most convenient to obtain the expression in relation to the vacuum representation, so that they take the form

$$\mathbb{W}(f,g) = \mathbb{W}_0(f,g) + \Delta \mathbb{W}(f,g),$$

where $\Delta \mathbb{W}(f,g)$ accounts for deviations from vacuum Wightman function [31, 38, 49]. Some explicit calculations of $\Delta \mathbb{W}(x,y)$ in flat spacetime for Fock states, thermal states, coherent and squeezed states, can be found in [47, 48], among many others.

### C. von Neumann algebra for algebra of observables

The GNS reconstruction theorem actually says something more general: given an algebraic state $\omega$ on a $C^*$ algebra, we obtain a $*$-representation $\pi_\omega$ so that we can view the the original algebra as a $C^*$-subalgebra of bounded operators $\mathcal{B}(\mathcal{H}_\omega)$. In our case, $\mathbb{W}(\mathcal{M})$ is a unital $C^*$-algebra that is also closed under what is known as weak $*$-topology, thus $\pi_\omega(\mathbb{W}(\mathcal{M}))$ is a very special class of $C^*$-subalgebra of $\mathcal{B}(\mathcal{H}_\omega)$ called von Neumann algebra.

In what follows we will write $\mathfrak{M} = \pi_\omega(\mathbb{W}(\mathcal{M}))$.

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2 This is also known as Wightman n-point functions to distinguish it from other correlation functions.

3 The term Gaussian states refers to generalization when the one-point functions need not vanish and higher-point functions only depend on one- and two-point functions.

4 Note that there are $C^*$ algebras that cannot be viewed as von Neumann algebra: one (natural) example is the so-called approximately finite-dimensional (AF) $C^*$ algebra [35, 40].
Given a $*$-representation $\pi_\omega : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\omega)$, a density operator $\rho \in \mathcal{D}(\mathcal{H}_\omega)$ is a positive semi-definite operator with unit trace (hence also trace-class operators) that naturally gives rise to algebraic state $\omega_\rho$, defined by

$$\omega_\rho(A) = \text{tr}(\rho \pi_\omega(A)). \quad (18)$$

The set of all such states $S_{\pi_\omega}(\mathfrak{A}) = \{\omega_\rho : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\omega)\}$ is called a folium of $\pi_\omega$. The way this is phrased suggests, for good reasons, that there are several subtleties to be aware of.

1. Not all states $\omega$ arise in this way: the best example is given by KMS thermal states, which in quantum field theory cannot be written as density matrix (because $\text{tr}e^{-\beta H}$ is unbounded) in the vacuum GNS representation associated to vacuum state $\omega_0$. Of course in its own GNS representation, any algebraic state $\omega$ is representation is given by a vector state so its density matrix is $\rho_\omega = |\Omega_\omega\rangle\langle\Omega_\omega|$.

2. A very important point that is often not emphasized enough [59, 51] is that if $\rho = |\Psi\rangle\langle\Psi|$, it is not true that $\omega_\rho$ is always pure! This has to do with the fact that $\pi_\omega(\mathfrak{A})$ can be a proper subalgebra of $\mathcal{B}(\mathcal{H}_\omega)$, so the restriction to $\pi_\omega(\mathfrak{A})$ will be in general mixed. In other words, this only holds if and only if $\pi_\omega$ is irreducible representation. Again, thermal state $\omega_\beta$ give an example: it is a mixed quasi-free state and it can be written as $\rho_\beta = |\Omega_\omega\rangle\langle\Omega_\omega|$ in its own GNS representation, but it is a reducible representation (it is a “double copy” of the vacuum representation).

3. Similarly, if $\omega_\rho = \omega_\rho'$ then it is not true that $\rho = \rho'$. The problem is that $\omega_\rho, \omega_\rho'$ may not generate the same GNS representations, so a priori the Hilbert spaces may not be the same.

These caveats are closely related to the existence of many possibly inequivalent representations of the $C^*$-algebra, which is certainly prevalent in QFT. For finite-dimensional quantum systems, these caveats are unnecessary since all finite-dimensional representations of the $C^*$-algebra of complex matrices $M_n(\mathbb{C})$ with the same dimension are unitarily equivalent.

We can now define what a von Neumann algebra is, following [32]. A von Neumann algebra is a $C^*$-algebra $\mathfrak{A}$ with a distinguished folium of normal states — which spans a linear space $S(\mathfrak{A})$ of all linear functionals on $\mathfrak{A}$ — with the property that

1. If $A, B \in \mathfrak{A}$ such that $\omega(A) = \omega(B)$ for all $\omega \in S(\mathfrak{A})$, then $A = B$.

2. If $f : S(\mathfrak{A}) \to \mathbb{C}$ is a bounded linear functional, then there exists $A \in \mathfrak{A}$ such that $f(\omega) = \omega(A)$ for all $\omega \in S(\mathfrak{A})$. The space $S(\mathfrak{A})$ is also written as $\mathfrak{A}_*$, the predual of $\mathfrak{A}$. In words, any state $\omega$ in this distinguished folium generates representation in the same Hilbert space $\mathcal{H}_\omega$, so that different states $\omega, \omega' \in S_n(\mathfrak{A})$ are represented as different density operators on $\mathcal{H}_\omega$ (their GNS representations are unitarily equivalent).

In the context of this work, all we need is the fact that for (quasifree) Hadamard states, the von Neumann algebra $\pi_\omega(\mathcal{W}(\mathcal{M}))$ that arises from the GNS representation is a type I$_\infty$ von Neumann algebra [59]. The property of being type I$_\infty$ von Neumann algebra is what enables us to calculate the Rényi entropy of the field after interaction with a qubit detector, as we will show in Section 11. The type I property follows directly from the fact that the Hilbert space $\mathcal{H}_\omega$ is infinite-dimensional and there exists rank-1 projectors with no proper subprojectors (namely, the GNS vector state $|\Omega_\omega\rangle\langle\Omega_\omega|$).

Furthermore, recall that all normal states $\omega$ associated to the vacuum representation $\pi_\omega(\mathcal{W}(\mathcal{M}))$ can be represented as some density matrix $\rho \in \mathcal{D}(\mathcal{H}_\omega)$ in this representation. We say that the states $\omega_\rho$ are normal to $\omega_0$, and we write

$$\omega_\rho(A) = \text{tr}(\rho \pi_\omega(A)). \quad (19)$$

Physically, this means that normal states of $\pi_\omega(\mathcal{W}(\mathcal{M}))$ (or states normal to $\omega_0$) contain “finite number of excitations” in the sense of number operator over all modes.

We conclude this section with a brief mention of one of the most exciting (and also most difficult) research line involving local algebras in QFT. One of the reasons why entanglement theory in field theory deserves much more attention is because the local algebras associated to subregions $\pi_\omega(\mathcal{W}(\mathcal{O}))$, where $\mathcal{O} \subset \mathcal{M}$ is a bounded region, is a Type III$_1$ von Neumann algebra (see [33, 57, 50, 51] for more details). From practical standpoint, this means that $\pi_\omega(\mathcal{W}(\mathcal{O}))$ has no pure state in the distinguished folium of normal states of $\pi_\omega(\mathcal{W}(\mathcal{M}))$. Hence it makes no sense to think of states on $\pi_\omega(\mathcal{W}(\mathcal{O}))$ as “statistical mixture of pure states” and we cannot write it in terms of density matrix in the GNS representation of the full algebra. This problem does not exist in finite dimensions (whose associated von Neumann algebra is always type I). Very few computations outside conformal field theory have been achieved, though there has been some progress with the help of Tomita-Takesaki modular theory (see, e.g., [32, 53, 58]).

5 In practice, a normal state $\omega$ means it can be represented as density matrices in its GNS representation. This is not possible, for instance, if $\mathcal{H}_\omega$ is not separable. We say that KMS state $\omega_\beta$ is not normal in the vacuum GNS representation but is normal with respect to its own GNS representation: normal states are always defined with respect to some von Neumann algebra.

6 See [59] for details on classification of von Neumann algebras.

7 This is one way to see why the KMS representation associated to $\omega_\beta$ is not unitarily equivalent to the vacuum representation and their states do not lie in the same folium: this manifests in our (well-known) inability to express thermal states as trace-class density matrix in $\mathcal{H}_{\omega_0}$ [51].

8 This is important because a $C^*$ algebra it does have pure states, but on a different (non-separable) Hilbert space [52].
III. COVARIANT UDW DETECTOR MODEL

Let us first review the covariant generalization of the Unruh-DeWitt (UDW) detector model that was developed in \[3, 4\]. The detector is taken to be a two-level system with free Hamiltonian given by

\[ h_0 = \frac{\Omega}{2} (\hat{\sigma}^z + 1), \tag{20} \]

where \( \hat{\sigma}^z \) is the usual Pauli-Z operator, whose ground and excited states \( |g\rangle, |e\rangle \) have energy \( 0, \Omega \) respectively. Let \( \tau \) be the proper time of the detector whose centre of mass travels along the worldline \( x(\tau) \). The covariant generalization of the Unruh-DeWitt model can be defined via the unitary time evolution of the form (in interaction picture) \[3, 4\]

\[ U = \mathcal{T}_\tau \exp \left[ -i \int dV f(x) \hat{\mu}(\tau(x)) \otimes \hat{\phi}(x) \right], \tag{21} \]

where \( dV = \sqrt{-g} \, dx \) is the invariant volume element, \( f \in C_0^\infty(\mathcal{M}) \) denotes interaction region between the detector and the scalar field. The monopole moment operator \( \hat{\mu} \) is taken to be the Pauli-X operator \( \hat{\sigma}^x \), which in interaction picture reads

\[ \hat{\mu}(\tau) = \hat{\sigma}^x(\tau) = \hat{\sigma}^+ e^{i0x} + \hat{\sigma}^- e^{-i0x}, \tag{22} \]

where \( \hat{\sigma}^\pm \) are the \( su(2) \) ladder operators with \( \hat{\sigma}^+ |g\rangle = |e\rangle \) and \( \hat{\sigma}^- |e\rangle = |g\rangle \). We can take the time ordering to be with respect to the global time function \( t \). The unsmeared field operator \( \hat{\phi}(x) \) is to be understood as operator-valued distribution.

At this point, depending on the problem at hand one may proceed to evaluate the time evolution perturbatively or non-perturbatively. There is a great deal of flexibility when one chooses to work within perturbative regime, but there is mild causality violation and “broken covariance” whose origin can be traced to the combination of time-ordering and non-relativistic nature of the detector model \[3, 4\]. In this work, we would like to obtain more robust results regarding the qubit interaction with a quantum field, so we will work with non-perturbative regimes. These correspond to (i) delta-coupled detector regime, where the interaction is very rapid, effectively at a single instant in time; and (ii) gapless detector regime, where the detector’s energy levels are degenerate. Below we show how these enable us to obtain non-perturbative results, relying on the fact that in both cases the time-ordering operator can be removed nicely.

A. Delta coupling

The delta-coupled detector is the regime where the interaction timescale is assumed to be much faster than all the relevant timescales of the problem, so that the interaction can be taken to occur at a single instant in time (with respect to some time function, typically the detector’s proper time or the global time function). This is often suitable to model one-shot fast processes instead of long-time processes such as thermalization and relaxation. If we assume that the detector interacts with the field only at \( \tau = \tau_0 \) in its own centre-of-mass rest frame, the spacetime smearing is given by

\[ f(x) : = \lambda \eta \delta(\tau - \tau_0) F(x), \tag{23} \]

where \( F(x) \) gives the spatial profile of the detector, \( \lambda \) is coupling strength and \( \eta \) is a constant with dimension of [Length]. The fractur notation \( f \) is to stress the fact that the spacetime smearing is not arbitrary compactly supported smooth functions on \( \mathcal{M} \).

The unitary time evolution in the delta-coupling model simplifies greatly: we get

\[ U = \exp \left[ -i \hat{\mu}(\tau_0) \otimes \hat{\phi}(f) \right]. \tag{24} \]

This unitary can be reorganized into a finite sum of bounded operators

\[ U = 1 \otimes \cos \hat{\phi}(f) - i \hat{\mu}(\tau_0) \otimes \sin \hat{\phi}(f). \tag{25} \]

Alternatively, we can use the spectral decomposition of the monopole operator \( \hat{\mu}(\tau_0) \) and write

\[ U = \sum_j P_j(\tau_0) \otimes e^{-i\eta_j \hat{\phi}(f)}, \tag{26} \]

where \( P_j(\tau_0) \) are eigenprojectors of \( \hat{\mu}(\tau_0) \) with eigenvalues \( \eta_j \). Note that different \( \tau_0 \) simply labels different families of “rotated” monopole operators, thus different \( \tau_0 \) gives different projectors.\(^9\) The spectral decomposition (26) allows us to generalize the delta coupling to arbitrary representations of the SU(2) for the detector by replacing \( \hat{\mu}(\tau) = J^z(\tau) \), with UDW model corresponding to \( j = 1/2 \) case (see, e.g., \[59\], for an application of higher spin systems).

At this point, it is useful to pause to comment on the delta smearing function. We have used the fractur notation \( f \) for the delta smearing in Eq. (23) because mathematically \( f \) is not in \( C_0^\infty(\mathcal{M}) \). Therefore, strictly speaking the results involving delta-coupled detector is not at the level of rigour of pure mathematics as we are “abusing” the assumptions that go into the construction of Weyl algebra \( \mathcal{W}(\mathcal{M}) \). However, one can perform brute-force calculation analogous to that in Appendix A of \[60\] and show that the same result can be obtained as if \( e^{\hat{\phi}(f)} \in \mathcal{W}(\mathcal{M}) \), i.e., by regarding \( f \in C_0^\infty(\mathcal{M}) \). Therefore we will use the properties of Weyl algebra in Section \[11\] for computations liberally in spite of this technical complication. For gapless detector model we are fine since \( f \in C_0^\infty(\mathcal{M}) \): the simplicity really comes from the time-independence of the monopole operator of the qubit rather than the choice of particular spacetime smearing, as we will now show.

\(^9\) That is, once we fix the monopole operator at \( \tau_0 \), the monopole at other time parameter is fixed by the free Hamiltonian \( h_0 \).
B. Gapless detector

The gapless detector regime is obtained by setting the energy gap in the free Hamiltonian $\Omega = 0$. This is the regime where the internal dynamics of the detector is assumed to be much slower than all the relevant timescales of the problem, hence its internal dynamics are frozen. Under this assumption, the expression simplifies greatly as the monopole operator is constant in time: $\hat{\mu}(\tau) = \hat{\mu}(0) \equiv \hat{\mu}$ for all $\tau$. The unitary operator then reduces to \cite{13}

$$U = T_t \exp \left[ -i \hat{\mu} \otimes \phi(f) \right], \quad (27)$$

where \cite{10}

$$\hat{\phi}(f) := \int dV f(x) \hat{\phi}(x), \quad f \in C^\infty(M). \quad (28)$$

In particular, in gapless detector model we can study the situation where the detector is switched on smoothly for long interaction times. The non-perturbative calculation is made possible by the fact that the unitary can be rewritten in terms of Magnus expansion:

$$U = \exp \sum_{j=1}^\infty \Xi_j, \quad (29)$$

$$\Xi_1 = -i \hat{\mu} \otimes \hat{\phi}(f), \quad \Xi_2 = -(1 \otimes 1)\Delta, \quad (30)$$

and $\Xi_{j \geq 3} = 0$. Here we defined $\Delta$ to be

$$\Delta = \frac{1}{2} \int dt dt' \Theta(t - t') i \Delta(t, t'), \quad (31a)$$

$$\Delta(t, t') := \int \Sigma_t \int \Sigma_{t'} d^3x \int d^3x' f(t, x) E(x, x') f(t', x'). \quad (31b)$$

Consequently, the unitary reduces to

$$U = e^{-i\Delta} e^{-i\hat{\phi}(f)}. \quad (32)$$

Here the time parameter can be chosen to be global coordinate time or the proper time of the detector. There is a sense in which the gapless model is simpler than the delta coupling in that the monopole moment operator is completely time-independent, hence the simplification does not rely on the use of Fermi normal coordinates.

Comparison with the delta-coupling model reveals that both regimes are very similar, since for a single detector they differ only by a global phase and the choice of spacetime smearing $f$. This is to be expected since both regimes correspond to the situations where the detector’s internal dynamics is much slower than the rest.

In the delta coupling regime, both the free dynamics of the field and the detector are slower than the interaction timescale, while for gapless regime the detector’s internal dynamics is taken to be much slower than the interaction timescale and the field’s free dynamics. For short, finite-time interactions, they would give similar results (see, e.g., \cite{15}). When the interaction time is taken to be large, the gapless model will show some differences compared to the delta coupling counterpart, as we will see in the next section.

IV. QUBIT CHANNEL INDUCED BY NON-PERTURBATIVE INTERACTION WITH QUANTUM FIELD

In this section we construct the qubit channel induced by the non-perturbative interactions with the field and its complementary channel as we define below. We restrict our attention to quasifree state of the field. In what follows we set $\mathcal{H}_\lambda, \mathcal{H}_b \cong \mathbb{C}^2$ be qubit Hilbert spaces and $\mathcal{H}_\phi$ be the Hilbert space associated to the field obtained from GNS representation of an algebraic state $\omega$. Let $\rho^0_\lambda \in \mathcal{D}(\mathcal{H}_\lambda)$ be initial density matrix of qubit in $\mathcal{H}_\lambda$, $\rho^0_\phi \in \mathcal{D}(\mathcal{H}_\phi)$ be the initial density matrix of the field (in the distinguished folium of its GNS representation).

A quantum channel $\Phi : \mathcal{D}(\mathcal{H}_\lambda) \to \mathcal{D}(\mathcal{H}_b)$ can always be written in the Stinespring representation by embedding the system into a larger Hilbert space with an auxiliary environment, such that

$$\Phi(\rho^0_\lambda) = \text{tr}_E (V \rho^0_\lambda V^\dagger) \quad (33)$$

where $\mathcal{H}_E$ is the Hilbert space of an (auxiliary) environment and $V : \mathcal{H}_\lambda \to \mathcal{H}_b \otimes \mathcal{H}_E$ is a linear isometry. This is closely related (but different!) to the statement in Stinespring dilation theorem \cite{61}, which says that for any channel $\Phi$ there exists some pure initial state $|e_0\rangle\langle e_0| \in \mathcal{D}(\mathcal{H}_E)$ and unitary $U : \mathcal{H}_\lambda \otimes \mathcal{H}_E \to \mathcal{H}_b \otimes \mathcal{H}_E$ such that\cite{11}

$$\Phi(\rho^0_\lambda) = \text{tr}_E (U(\rho^0_\lambda \otimes |e_0\rangle\langle e_0|)U^\dagger). \quad (34)$$

We have the relation $V = U(1 \otimes |e_0\rangle)$. Let us consider the detector and the field to be prepared in the initially uncorrelated state

$$\hat{\rho}_0 = \rho^0_\lambda \otimes \rho^0_\phi. \quad (35)$$

We can then define the qubit channel $\Phi$ in terms of the Stinespring-like representations:

$$\Phi(\rho^0_\lambda) = \text{tr}_\phi (U(\rho^0_\lambda \otimes \rho^0_\phi)U^\dagger). \quad (36a)$$

\footnote{We cannot drop the requirement that the initial state of the environment is pure even in finite dimensions. To see this, replace $|e_0\rangle\langle e_0|$ with maximally mixed state $1/d$ and we see that only unital channels (channels $\Phi$ such that $\Phi(1) = 1$) can be represented this way.}
Let us write $C_t = \cos \hat{\phi}(f)$ and $S_t = \sin \hat{\phi}(f)$, where $f = f$ is the spacetime smear function for delta-coupled detector and $f = f$ for gapless detector. Using the fact that we can write (c.f. Section II C)
\[
\omega(A) = \text{tr}_\phi(\hat{\rho}_0^0 A) \quad (37)
\]
the full unitary evolution gives the total joint state
\[
\hat{\rho}_{0\phi} = \hat{\rho}_0^0 \otimes C_t \hat{\rho}_0^0 C_t + \mu_0 \hat{\rho}_0^0 \hat{\mu}_0 \otimes S_t \hat{\rho}_0^0 S_t
\]
\[
- i \mu_0 \hat{\rho}_0^0 \otimes S_t \hat{\rho}_0^0 C_t + i \hat{\mu}_0 \hat{\rho}_0^0 \otimes C_t \hat{\rho}_0^0 S_t . \quad (38)
\]
Here $\hat{\mu}_0 = \hat{\mu}(\tau_0)$ for delta-coupled detector and $\hat{\mu}_0 = \mu$ for gapless detector.

For the remainder of this section we will show that for single delta-coupled detector or gapless detector, the channel is essentially that of a “rotated” Pauli channel that implements generalized bit-flip. The difference with the conventional Pauli channel is that the probability of doing “rotated” bit flip is completely governed by the interaction profile (spacetime smearing) with the field, and hence by the field fluctuations itself. The qubit channel has the intuitive property that large quantum field fluctuation leads to essentially random flip. In what follows we will restrict our attention to the case when the field state is quasifree (c.f. Section II).

A. Delta coupling regime

Let us construct the qubit channel for delta-coupling case more explicitly. Tracing out the field’s degrees of freedom and using Eq. (37), the channel $\Phi$ is given by
\[
\Phi(\hat{\rho}_0^0) = \omega(C_t^2) \hat{\rho}_0^0 + \omega(S_t^2) \hat{\mu}(\tau_0) \hat{\rho}_0^0 \hat{\mu}(\tau_0)
\]
\[
+ i [\hat{\rho}_0^0, \hat{\mu}(\tau_0)] \omega(S_t C_t), \quad (39)
\]
where we used the fact that $\omega(S_t C_t) = \omega(C_t S_t)$.

The expression in Eq. (39) is good enough for most purposes, but it will be convenient later to rewrite it as follows. From Eq. (26), the eigenprojectors of $\hat{\mu}(\tau_0)$ are given by $P_{\pm}(\tau_0) = \frac{1}{2}(1 \pm \hat{\mu}(\tau_0))$ associated to eigenvalues $p_{\pm} = \pm 1$. Hence we can rewrite the unitary (25) into
\[
U = \frac{1 - \hat{\mu}(\tau_0)}{2} \otimes e^{i\hat{\phi}(f)} + \frac{1 + \hat{\mu}(\tau_0)}{2} \otimes e^{-i\hat{\phi}(f)} . \quad (40)
\]
Following the convention in [13 [15 [29 [62, let us define
\[
\nu_f := \omega(e^{2i\hat{\phi}(f)}) \in \mathbb{C} . \quad (41)
\]
For quasifree states gives $\nu_f = e^{-2\omega(f)} \in [0,1]$, where $\omega(f)$ is the symmetrically smeared Wightman two-point function associated to the algebraic state $\omega$. We can now rewrite the channel in Kraus representation
\[
\Phi(\hat{\rho}_0^0) = \sum_{j=0}^{2} K_j \hat{\rho}_0^0 K_j^\dagger , \quad (42)
\]
where the Kraus operators are
\[
K_0 = \sqrt{\frac{1 - |\nu_f|}{2}} \mathbb{1} , \quad K_1 = \sqrt{\frac{1 - |\nu_f|}{2}} \hat{\mu}(\tau_0) , \quad (43a)
\]
\[
K_2 = \sqrt{\frac{|\nu_f| + Re \nu_f}{2} - i \sqrt{\frac{|\nu_f| - Re \nu_f}{2}}} \hat{\mu}(\tau_0) . \quad (43b)
\]
We will use this expression from time to time.

If we go back to Eq. (39), for quasifree states we have $\omega(C_t S_t) = \frac{1}{2} \omega(S_t) = 0$ (e.g., using Lemma 1 in [29]), so that the channel can be readily written in the Kraus representation:
\[
\Phi(\hat{\rho}_0^0) = A_0 \hat{\rho}_0^0 A_0^\dagger + A_1 \hat{\rho}_0^0 A_1^\dagger , \quad (44a)
\]
\[
A_0 = \sqrt{\omega(C_t^2) \mathbb{1}} , \quad A_1 = \sqrt{\omega(S_t^2)} \hat{\mu}(\tau_0) , \quad (44b)
\]
where the square root is possible because $C_t^2$ and $S_t^2$ are positive-semidefinite. We can do better: due to quasifree property, we know that $\omega(S_t) = 0$, hence
\[
\omega(C_t) = \omega(C_t + i S_t) = \omega(W(f)) = e^{-\frac{1}{2} \omega(f)} . \quad (45)
\]
Consequently, we get
\[
\omega(C_t^2) = \frac{1 + \omega(2C_t)}{2} = \frac{1 + e^{-2\omega(f)}}{2} , \quad (46a)
\]
\[
\omega(S_t^2) = \frac{1 - \omega(2C_t)}{2} = \frac{1 - e^{-2\omega(f)}}{2} . \quad (46b)
\]

Therefore, the channel can now be recast into a “rotated” Pauli channel
\[
\Phi(\hat{\rho}_0^0) = q \hat{\rho}_0^0 + (1 - q) \hat{\mu}(\tau_0) \hat{\rho}_0^0 \hat{\mu}(\tau_0) , \quad (47)
\]
with a “rotated bit-flip” probability $1 - p$ and $q = \omega(C_t^2) = \frac{1 + e^{-2\omega(f)}}{2} . \quad (48)$

If $\hat{\mu}(\tau_0) = \hat{\sigma}^+ \ldots$ then it is exactly the usual bit flip channel in the eigenbasis of $\hat{\sigma}^z$, so it is somewhat a “rotated” bit flip channel.

The Pauli channel Eq. (39) has very nice interpretation in terms of field’s vacuum fluctuations. For very sharply peaked spacetime smearing $f$ which corresponds to finer resolution, the value of $W(f, f)$ increases, and in fact it is UV-divergent in the pointlike limit $f \rightarrow \delta^4(x)$. In other words, trying to resolve smaller spacetime regions (or equivalently, probing the UV structure of the field) introduces a lot of noise, and in the pointlike limit the channel becomes random (rotated) bit-flip channel. In the opposite extreme, the larger the interaction region, the fluctuations of the field are smoothed

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12 We can also use the Kraus representation in Eq. (42) to obtain this simpler Kraus representation using the fact that for quasifree states $\text{Re} \nu_f = \nu_f$. 

out and this reduces the bit-flip probability. For delta-coupling, the size of \( \text{supp}(f) \) is effectively fixed by the size of the qubit detector, so one cannot suppress the vacuum fluctuations freely. To turn it another way, this says that for rapid interaction, high fluctuations imply poor statistics— the smaller the detector used to resolve smaller region, the worse our statistical inference of the output state since it is hard to distinguish from uniform random flip.

### B. Gapless regime

For gapless detectors, the story is quite different because one can suppress this by interacting with the fields smoothly and adiabatically for long enough times. The qubit channel for the gapless regime is structurally the same, which is given by

\[
\Psi(\hat{\rho}_f^0) = \omega(C_f^2)\hat{\rho}_f^0 + \omega(S_f^2)\hat{\mu}\hat{\rho}_f^0\hat{\mu} + i[\hat{\rho}_f^0, \hat{\mu}]\omega(S_f C_f). \tag{49}
\]

This is very similar to the delta-coupling except that the monopole operator \( \hat{\mu} \equiv \hat{\mu}(0) \) is now fixed in time (since the free Hamiltonian vanishes): for instance, we can set it to be any Pauli operators \( \hat{\sigma}^\pm \). Consequently, the full Kraus representation takes the form

\[
\Psi(\hat{\rho}_f^0) = \sum_{j=0}^{2} K_j \hat{\rho}_f^0 K_j, \tag{50}
\]

where the Kraus operators are

\[
K_0 = \sqrt{1 - |\nu_f|^2} \mathbb{I}, \quad K_1 = \sqrt{1 - |\nu_f|^2} \hat{\mu}^\dagger, \tag{51a}
\]
\[
K_2 = \frac{|\nu_f|}{\sqrt{2}} \mathbb{I} - i\sqrt{\frac{1 - |\nu_f|^2}{2}} \hat{\mu}. \tag{51b}
\]

As before, for quasifree states we can trace out the field’s degrees of freedom using Eq. (37) and the channel \( \Psi \) reduces to a simpler Kraus representation

\[
\Psi(\hat{\rho}_f^0) = B_0 \hat{\rho}_f^0 B_0^\dagger + B_1 \hat{\rho}_f^0 B_1^\dagger, \tag{52a}
\]
\[
B_0 = \sqrt{\omega(C_f^2)} \mathbb{I}, \quad B_1 = \sqrt{\omega(S_f^2)}\hat{\mu}. \tag{52b}
\]

Just like in the delta-coupling case, the channel can now be recast into a Pauli channel of the form

\[
\Phi(\hat{\rho}_f^0) = \hat{\rho}_f^0 + (1 - p)\hat{\mu}\hat{\rho}_f^0\hat{\mu}, \tag{53}
\]

which corresponds to actual bit flip in the energy eigenbasis of \( \hat{\sigma}^z \) if we choose \( \hat{\mu} = \hat{\sigma}^z \). In this case, the bit-flip probability is \( 1 - p \) where

\[
p = \omega(C_f^2) = \frac{1 + e^{-2W(f,f)}}{2}. \tag{54}
\]

By inspecting Eq. (39) and (52) it is clear that the two channels \( \Phi, \Psi \) share the same functional form—we can even choose \( \mu(\tau_0) \) in delta coupling to match the gapless monopole operator \( \hat{\mu} \). However, there are important physical differences for both channels because the spacetime smearing for gapless model can be manipulated arbitrarily \( f \), unlike the more restricted smearing \( f \) for delta-coupled detector. Crucially, we can suppress the vacuum fluctuations appearing in the Pauli channel (52) to arbitrarily small value by having the interaction last for very long times in a smooth and carefully controlled manner. To illustrate this point, we can consider an adiabatic smooth switching such as Gaussian in the Fermi normal coordinates\(^\text{13}\)

\[
f(x) \sim e^{-x^2/T^2} F(x). \tag{55}
\]

By taking the limit \( T \to \infty \), we have \( W(f, f) \to 0 \), thus \( p \to 1 \) and

\[
\Psi(\hat{\rho}_f^0) \to \hat{\rho}_f^0. \tag{56}
\]

In other words, the channel approaches the identity (noiseless) channel, even if the detector’s spatial extent can be chosen to be arbitrarily small. Indeed, we know an example of this well-known phenomenon when an inertial qubit detector in its ground state interacts with a quantum field vacuum: if turned on long enough and carefully, the detector registers no response and is unlikely to get excited. In contrast, the probability \( q \) for delta-coupled detector is completely fixed by its spatial profile (effectively by the shape of the detector) and the coupling strength, so there is not much freedom to suppress \( q \); for gapless detector, it has one extra freedom of using the switching the detector on and off “carefully”.

We should also note that in the literature gapless detector model has been used to study scenarios when the detector is uniformly accelerating (see, e.g., \(^\text{14} \text{ [64]}\)). The idea is to take the limit of spacetime smearing \( f \) such that it is only compactly supported along an accelerating trajectory (i.e., pointlike limit). In this case, the smeared Wightman function along this trajectory reduces to that of thermal state with Unruh temperature \( T_U = \alpha/(2\pi) \), in effect we are replacing

\[
W(f, f) \to W_\beta(\chi_T, \chi_T), \tag{57}
\]

where \( W_\beta \) is the Kubo-Martin-Schwinger (KMS) thermal state with inverse Unruh temperature \( \beta = T_U^{-1} \) and \( \chi_T \equiv \chi(\tau/T) \) is the switching function along the detector’s accelerating trajectory with characteristic width \( T \) (such as Gaussian \( e^{-\tau^2/T^2} \)). Since thermal states are quasifree states, the qubit channel for gapless detector

\(^\text{13}\) Borrowing the terminology from Achim Kempf \text{[63]}.

\(^\text{14}\) One can always choose a compactly supported smooth function which is closely approximated by the Gaussian switching with similar effective width and shape, but for simplicity Gaussian is easier to demonstrate what we want.
is still given by Eq. (62), but with the replacement on the flip probability $p$. Interestingly, it is now clear how the detector cannot thermalize. First, we have the flip probability given by

$$p(\beta) = \frac{1}{2} (1 + e^{-2e_\beta(x, x')}) \to \frac{1}{2}$$

as $T \to \infty$, i.e., long interaction times. One can check this by direct computation (see [14])

$$W_\beta(\chi_T, \chi_T) \propto \int_0^\infty d\omega \omega \coth(\beta\omega/2)|\chi_T(\omega)|^2.$$  (59)

Hence in the long time limit the diagonal components approaches 1/2. However, proper thermalization requires that the off-diagonal components completely decohere for arbitrary initial states after long enough times. This is not possible for arbitrary states because, for instance, the channel $\Psi$ has a one-parameter family of fixed points which generally has nonzero coherence in the energy eigenbasis of $\hat{H}$: any state $|\rho_0\rangle$ with $[\hat{p}_0, \hat{\mu}_0] = 0$ will not be thermalized. In the case when $\mu_0 = \hat{\sigma}^x$, all states with Bloch vector $\vec{r} = (r_x, 0, 0)$ will be fixed points of $\Psi$ and hence they never decohere under the action of the channel. Since maximally mixed state is not the unique stationary state of this channel, we can view this as a situation where the open system thermalizing dynamics has constants of motion [63–67].

C. State dependence and field fluctuations

It is worth noting that the channels $\Phi, \Psi$ are very general—they are valid in arbitrary globally hyperbolic spacetime $M$, generic choice of interaction region governed by the spacetime smearings, or even any choice of quasifree states of the field. Everything is completely controlled by the symmetrically smeared Wightman two-point functions (i.e., the variance) of the field and by the interaction region.

To illustrate this point, consider the Kubo-Martinschwenk (KMS) thermal state, which is also a quasifree state. All we need to know is that the Wightman two-point function is given by

$$W_\beta(x, x') = W_0(x, x') + \Delta W_\beta(x, x'),$$  (60)

where $W_0(x, x')$ is the vacuum Wightman two-point function and $\Delta W_\beta(x, x') = \int d^3k \frac{u_k(x)u_k(x') + u_k^*(x)u_k^*(x')}{e^{\beta\omega_k} - 1}.$  (61)

The expression for $\Delta W_\beta(x, x')$ can be found from canonical quantization following procedures such as in [27], or more algebraically following [14]. It follows that

$$W_\beta(f, f) = W_0(f, f) + \Delta W_\beta(f, f) \geq W_0(f, f),$$  (62)

since $\Delta W_\beta(f, f) \geq 0$. This is just the statement that at finite temperature, the field is “noisier” than the vacuum, with thermal excitation increasing the bit flip probability. This is true for any KMS state in any curved spacetimes, and we hardly needed to do much more calculations. For many non-vacuum states of interest, we have

$$W(f, f) \geq W_0(f, f) \geq 0,$$  (63)

so field excitations tend to add more noise to the channel. This additional noise is “above the vacuum state”, in the sense that even smooth adiabatic switching for gapless detector cannot make general $W(f, f) \to 0$, hence it truly reflects finite-energy field excitations.

V. SOME PROPERTIES OF THE UDW QUBIT CHANNELS

Below we consider some basic properties of this channel, using the standard toolbox of quantum information theory. Since the channels $\Phi, \Psi$ are structurally very similar, we will use the gapless detector $\Psi$ as a reference once for a single detector, the channel has more freedom in the choice of spacetime smearing than the delta-coupled detector. In what follows the claims for $\Psi$ will also apply for $\Phi$ unless otherwise stated.

We will focus on three lines of analyses: (1) mixed unitarity, (2) recoverability, and (3) entanglement-breaking property. These are chosen based on the rarity of their uses in relativistic quantum information (while they are commonly used in standard quantum information theory). Some of the explicit calculations may be useful for future references in similar contexts involving UDW detectors.

A. Mixed unitary property and entropy

The first observation we can make is that the qubit channel $\Psi$ is unital ($\Psi(1) = 1$), hence it is also mixed unitary i.e., they can be written as (at most) $d^2$ convex combinations of unitaries where $d = \dim \mathcal{H}_0$:

$$\Psi(\rho_0) = \sum_{j=0}^{d^2-1} p_j U_j \rho_0 U_j^\dagger, \quad \sum_j p_j = 1,$$  (64)

15 Arguably, gapless detectors lack a scale to even speak of thermalization. Here we will think of gapless detector’s “thermalization” as analogous to being at “high-temperature limit” where $\beta\Omega \to 0$. We do not commit to calling gapless detector dynamics undergoing constant accelerated motion a thermalization process in the sense of Unruh effect.

16 It is clear from Eq. (64) that all mixed unitary channels are unital, but in higher dimensional Hilbert spaces there exists unital channels that are not mixed unitary. It is closely related to the concept of $n$-noisy operations.
with \( p_j \geq 0 \) and the sum involves at most 4 terms for a single qubit. In fact, the Kraus representation \([50]\) gave us the explicit unitaries and the probabilities:

\[
\begin{align}
p_0 &= \frac{1 - |\nu_j|^2}{2}, \quad U_0 = \frac{1}{\sqrt{p_0}} K_0, \quad (65a) \\
p_1 &= \frac{1 - |\nu_j|^2}{2}, \quad U_1 = \frac{1}{\sqrt{p_1}} K_1, \quad (65b) \\
p_2 &= |\nu_j|^2, \quad U_2 = \frac{1}{\sqrt{p_2}} K_2, \quad (65c)
\end{align}
\]

with \( p_3 = 0 \). For delta-coupling we just replace \( f \rightarrow \hat{f} \) and the Kraus operators \( K_j \rightarrow K_j \) as given in Eq. \([64]\).

The mixed unitary property is related to the concept of majorization. We say that a \( d \)-dimensional density operator \( \sigma \) majorizes another \( d \)-dimensional density operator \( \rho \), denoted \( \rho \prec \sigma \), if all their eigenvalues (arranged in decreasing order) obey:

\[
\begin{align}
\sum_{j=1}^{k} \lambda(\sigma) \geq \sum_{j=1}^{k} \lambda(\rho), & \quad \forall 1 \leq k \leq d - 1, \quad (66a) \\
\sum_{j=1}^{d} \lambda(\sigma) = \sum_{j=1}^{d} \lambda(\rho). & \quad (66b)
\end{align}
\]

Uhlmann’s theorem states that \( \sigma \) majorizes \( \rho \) if and only if it follows that the input state of the qubit channel \( \Psi \) majorizes the output state, i.e., \( \Psi(\hat{\rho}_0) \prec \hat{\rho}_0 \).

One immediate implication of majorization property is based on a proposition that can be found in \([71, 72]\).

**Theorem 1.** The state \( \sigma \) majorizes \( \rho \) if and only if for all continuous convex functions \( f : [\lambda_1(\rho), \lambda_1(\sigma)] \rightarrow \mathbb{R} \) the following inequality holds:

\[
\sum_{j=1}^{d} f(\lambda(\sigma)) \geq \sum_{j=1}^{d} f(\lambda(\rho)). \quad (67)
\]

The function \( f \) above is called Schur-convex; it is said to be Schur-concave if the opposite inequality holds. Since von Neumann entropy

\[
S(\rho) := - \text{tr} \rho \log \rho \quad (68)
\]

is Schur-concave, the majorization property implies that the qubit channel \( \Phi \) must increase entropy, with equality (constant entropy) if and only if \( \Psi(\hat{\rho}_0) = \hat{\rho}_0 \). Another quantity called \( \alpha \)-Rényi entropy, defined as \([73, 74]\)

\[
S_\alpha(\rho) = - \frac{1}{\alpha - 1} \text{tr}(\rho^\alpha), \quad \alpha \in (0, 1) \cup (1, \infty) \quad (69)
\]

is also Schur-concave. This shows that for the channel \( \Psi \) (without requiring quasiisometry assumption) the detector will be entangled with the field unless we pick the initial state of the detector to commute with the Kraus operators of \( \Psi \), i.e., \( \rho_0^0, \mu_0 \). This occurs, for instance, when \( \hat{\rho}_0 \) is an eigenstate of \( \rho_0 \). Interestingly, for gapless detectors it also means that switching the detector carefully for longer times decreases entropy production, since as shown in Section \([IV]\) in the limit of very long times we have \( \Psi(\rho_0^0) \rightarrow \rho_0^0 \).

### B. Recoverability of the qubit channel

Given a channel \( \mathcal{E} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}) \), we say that the channel is reversible if there exists another channel \( \mathcal{E}^* \) such that \( (\mathcal{E}^* \circ \mathcal{E})(\rho) = \rho \) for all \( \rho \in \mathcal{D}(\mathcal{H}) \). This definition is however too strong, as the only channel that has this property is the unitary channel \( \mathcal{E}(\rho) = U \rho U^\dagger \). Following \([75]\), one can relax this requirement in two ways. First, we say \( \mathcal{E} \) is invertible if we only require that \( \mathcal{E}^* \) is just a linear superoperator (not necessarily a channel). This relaxation works well enough to capture open system dynamics, but some maps with certain fixed points will not be covered by this definition. This motivates the notion of recoverability of a channel.

A channel \( \mathcal{E} \) is said to be recoverable (or reversible/sufficient \([76]\)) if there is another channel \( \mathcal{R} \) and a subset of density matrices \( \mathcal{S}(\mathcal{H}) \subseteq \mathcal{D}(\mathcal{H}) \) such that \( (\mathcal{R} \circ \mathcal{E})(\rho) = \rho \) for all \( \rho \in \mathcal{S}(\mathcal{H}) \). The map \( \mathcal{R} \) is then called the recovery map for the subset \( \mathcal{S}(\mathcal{H}) \). The most famous recovery map is the Petz recovery map \([77, 78]\), usually denoted by \( \mathcal{R}_{\sigma, \mathcal{E}}^p \) and defined by

\[
\mathcal{R}_{\sigma, \mathcal{E}}^p(\rho) = \sigma^{\frac{1}{2}} \mathcal{E}^\dagger \left( (\mathcal{E}(\sigma)^{-\frac{1}{2}} \rho \mathcal{E}(\sigma)^{-\frac{1}{2}}) \sigma^{\frac{1}{2}} \right). \quad (70)
\]

Here the operator exponents are defined on their respective supports, \( \sigma \in \mathcal{D}(\mathcal{H}) \) is a reference state, and \( \mathcal{E}^\dagger \) is the adjoint channel. Petz’s theorem states that if the channel does not decrease relative entropy, i.e.,

\[
D(\rho||\eta) = D(\mathcal{E}(\rho)||\mathcal{E}(\eta)) \quad \forall \rho, \eta \in \mathcal{S}(\mathcal{H}), \quad (71)
\]

where \( \mathcal{S}(\mathcal{H}) \) and \( D(\rho||\sigma) = \text{tr} \rho (\log \rho - \log \sigma) \), then there exists \( \sigma \in \mathcal{D}(\mathcal{H}) \) such that

\[
(\mathcal{R}_{\sigma, \mathcal{E}}^p \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in \mathcal{S}(\mathcal{H}). \quad (72)
\]

Note that relative entropy \( D(\rho||\eta) \) is finite if and only if \( \text{supp}(\rho) \subseteq \text{supp}(\eta) \). Also, by construction we have \( \mathcal{R}_{\sigma, \mathcal{E}}^p(\sigma) = \sigma \), i.e., Petz map always recovers its reference state perfectly. We can thus view Petz theorem as obstructions of extending \( \mathcal{S}(\mathcal{H}) \) to \( \mathcal{D}(\mathcal{H}) \).

For our qubit channels in Eq. \([62]\) (also applies to \( \Phi \) in Eq. \([39]\)), the Petz recovery map is actually useless because with the exceptions of fixed points of \( \Psi \), the relative entropy does decrease. Let us suppose that the reference state is maximally mixed \( \sigma = 1/2 \) which is a fixed point of \( \Psi \). Then

\[
D(\hat{\rho}_0^0||\sigma) = 1 - S(\hat{\rho}_0^0), \quad (73a)
\]

\[
D(\Psi(\hat{\rho}_0^0)||\Psi(\sigma)) = 1 - S(\Psi(\hat{\rho}_0^0)). \quad (73b)
\]

Therefore the two relative entropies are equal if and only if \( \hat{\rho}_0 \) is a fixed point of \( \Psi \), which are the states we do

\[17\]

In terms of Kraus representations, if \( \mathcal{E}(\rho) = \sum_j A_j \rho A_j^\dagger \) then the adjoint channel has the Kraus representation given by \( \mathcal{E}^\dagger(\rho) = \sum_j A_j^\dagger \rho A_j \). See \([69, 74]\) for more details.
not need to bother correcting. It is therefore useful to abandon perfect recoverability by making appropriate tradeoffs.

In surprisingly recent work [75], one proposed strategy is to simply give up perfect recoverability and aim for finding the optimal reference state \( \sigma^* \) that achieves maximum *average fidelity* over all input states. Specifically, given fidelity function \( F(\rho, \sigma) = ||\sqrt{\rho\sigma}\sqrt{\rho\sigma}||^2_1 \) where \( ||A||_1 = \text{tr}|A^\dagger A| \) is the trace norm, the optimal reference state is \( \sigma^* = \text{argmax}_\sigma F_R(\sigma) \), where

\[
F_R(\sigma) := \int d\mu_\rho F(\rho, (R \circ E)(\rho)) ,
\]

where \( d\mu_\rho \) is a uniform measure over all input states [75]. In order to construct the recovery channel (which in our case will be Petz map) according to this strategy, we need to calculate the adjoint channel \( \Psi^\dagger \): in terms of Kraus representation, we have

\[
\Psi^\dagger(\hat{\rho}_0^0) = A_0^\dagger \hat{\rho}_0^0 A_0 + A_1^\dagger \hat{\rho}_0^0 A_1 = \Psi(\hat{\rho}_0^0) .
\] (74)

That is, the channel is its own adjoint because the Kraus operators are Hermitian (the channel is said to be self-dual). Maximally mixed state is expected to be the optimal reference state for unital channels, hence the recovery map is simply given by \( R^P_{\sigma,\Psi} = \Psi \), the channel itself. This is exactly the same situation for dephasing channel considered in [75], where the authors showed that it is a better strategy to simply not apply the recovery channel for arbitrary input states.

Another strategy we could use is to use the notion of approximate recoverability [79] and universal recoverability [70]. The idea of approximate recoverability is that we would like to relax the requirement in Petz’s theorem to allow for “small” decrease in relative entropy, where “small” can be made precise as follows. Let \( \rho, \sigma \) be two density operators such that \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \) and consider a quantum channel \( E \). Given the monotonicity of relative entropy (also known as data-processing inequality):

\[
D(\rho||\sigma) \geq D(E(\rho)||E(\sigma)) ,
\] (75)

the approximate recoverability theorem provides the following refinement [79]:

**Theorem 2.** Let \( \rho, \sigma, E \) be given as above. Then there exists a recovery channel \( R^P_{\sigma,E} \) such that

\[
D(\rho||\sigma) - D(E(\rho)||E(\sigma)) \geq -\log \sup_{t \in \mathbb{R}} F(\rho, (R^{P,t}_{\sigma,E} \circ E)(\rho)) ,
\] (76)

where \( 0 \leq F(\cdot, \cdot) \leq 1 \) is the fidelity between two states and \( (R^{P,t}_{\sigma,E} \circ E)(\sigma) = \sigma \) is the “rotated” Petz recovery map

\[
R^{P,t}_{\sigma,E} = U_{\eta^t} \circ R^{P}_{\sigma,E} \circ U_{\eta^{-t}} ,
\] (77)

with \( U_{\eta^t} \) partial isometry map \( U(\cdot) := \eta^t(\cdot)\eta^{-t} \). In general the value of \( t \) that achieves the supremum depends on \( \rho \) for a given fixed reference state \( \sigma \).

This theorem provides the obstruction to recovering \( \rho \) perfectly, and is a refinement of monotonicity of relative entropy (since the RHS of (76) is always non-negative). Furthermore, if \( \sigma \) is chosen to be a fixed point of \( E \) then \( R^{P,t}_{\sigma,E} = R^{P}_{\sigma,E} \), which is desirable because the value of \( t \) in the rotated Petz map generally depends on the input state \( \rho \).

Let us calculate this for two explicit examples, since such computation is hard to come by in the literature. First, let us choose the initial state to be in the ground state \( \rho^0_0 = |g\rangle\langle g| \), \( \mu_0 = \hat{\sigma}^\dagger \), and we pick reference state \( \sigma = \mathbb{1}/2 \), then

\[
F(\rho^0_0, (R^{P}_{\sigma,E} \circ \Psi)(\rho^0_0)) = (1 - p)^2 .
\] (78)

It follows from the definition of Shannon entropy \( H(p) = -p \log_2 p - (1 - p) \log_2(1 - p) \) that for initially ground state we have

\[
S(\Psi(\rho^0_0)) = H(\rho) \geq -\log_2 F(\rho^0_0, (R^{P}_{\sigma,E} \circ \Psi)(\rho^0_0)) ,
\] (79)

with equality if and only if \( p = 1/2 \), as shown in Figure 1.

The recovery map given above still has the undesirable property that we need different channels for different input states in general, especially when the reference state is not required to be maximally mixed. This motivates a more universal recovery version that is independent of the initial state: this is known as the “twirled” Petz recovery map, and the result is given in [70].

**Theorem 3.** Let \( \rho, \sigma, E \) be given as per Theorem 2. Then for any input state whose support is contained in the support of \( \sigma \), there exists a recovery channel \( R^{P,t}_{\sigma,E} \) such that

\[
D(\rho||\sigma) - D(E(\rho)||E(\sigma)) \geq -\int_{\mathbb{R}} dt p(t) \log F(\rho, (R^{P,t/2}_{\sigma,E} \circ E)(\rho)) ,
\] (80)
where \( p(t) = \frac{1}{2}(\cosh(\pi t) + 1)^{-1} \) is a probability density function on \( \mathbb{R} \).

This removes the time-dependent supremum for each input state \( \rho \), and hence is particularly suitable in the context of quantum error correction where one would like to have “oblivious” recovery channel that can correct generic input states.

We now show that this theorem actually buys us nothing new for the channel \( \Psi \), especially if we wish for \( \text{R}^{P}_{\sigma,\Psi} \) to be tractable. To do this, we first choose the reference state \( \sigma \) to be any fixed point of \( \Psi \) so that \( \Psi(\sigma) = \sigma \) (for which maximally mixed state is one of them). We will always pick \( \sigma \) to be max-rank so that the relative entropy is finite. It follows that \( \sigma \) is also the fixed point of \( \text{R}^{P}_{\sigma,\Psi} \), and this immediately gives us the time-transformation property \[80]:
\[
\Psi(t) = \sigma^t \Psi(\sigma^{-it}(\cdot)^t) \sigma^{it},
\]
which also holds for the adjoint map. Therefore, the rotated map reduces to the usual Petz map \( \text{R}^{P}_{\sigma,\Psi} = \text{R}^\circ_{\sigma,\Psi} \), and Theorem 3 reduces to
\[
D(\rho||\sigma) - D(\Psi(\rho)||\Psi(\sigma)) \geq - \log F(\rho, (\text{R}^P_{\sigma,\Psi} \circ \Psi)(\rho)). \tag{82}
\]

However, by self-duality we have \( \text{R}^P_{\sigma,\Psi} = \Psi \), hence the universal recovery reduces to the standard recovery result. It is worth noting that the bound \[82] is not tight in general \[81].

It is useful to summarize by noting that Figure 1 quantifies how well the approximate recoverability is in terms of entropy production: the fidelity of the approximate recovery is very close to the entropy of the output state in the low noise regime \( p \to 1 \). In the other extreme, “high noise” regime \( p \to 1/2 \) tells us that the output state is nothing but maximally mixed state, so the recovery channel cannot do anything (since \( \Psi \) unit: the entropy production is so large (maximal) that nothing is recoverable from the initial state. The intermediate regime \( p \in (1/2, 1) \) constitutes a “gap” between how much entropy is produced and how much information can be recovered from the state. If instead we consider state recovery by asking what is the closest state in fidelity/trace distance by optimize the reference state to find optimal recovery as per \[73] , then we see that the recovery channel is redundant in this case because it is self-dual (\( \Psi = \Psi \)) — performing recovery channel only adds more noise.

Interestingly, the story can be quite different if we consider thermal states of the detector, defined in terms of Gibbs state
\[
\rho_\beta = \frac{e^{-\beta \mathcal{H}}}{\text{tr} e^{-\beta \mathcal{H}}} = \left( \begin{array}{cc} 1 + e^{-\beta \mathcal{H}} & 0 \\ 0 & \frac{1}{1 + e^{-\beta \mathcal{H}}} \end{array} \right)
\]
where \( \mathcal{H} = \frac{\Omega}{2}(\hat{\sigma}^2 + \mathbb{1}) \) is the free Hamiltonian and \( \beta = T^{-1} \) is the inverse temperature. Clearly for gapless detector this reduces to maximally mixed state so we already know that \( \mathbb{1}/2 \) is a fixed point of \( \Psi \) and no recoverability is needed or possible. However, for delta-coupled detector thermal state makes sense and we can ask what happens after the instantaneous interaction via channel \( \Phi \). Using maximally mixed state \( \sigma = \mathbb{1}/2 \) as a reference, the relative entropy difference is given by
\[
D(\rho\beta||\sigma) - D(\Phi(\rho\beta)||\sigma) = \frac{\beta e^{\beta \mathcal{H}} + ((p-1)e^{\beta \mathcal{H}} - p) \log (p - (p-1)e^{\beta \mathcal{H}})}{e^{\beta \mathcal{H}} + 1} + \frac{(p-1-pe^{\beta \mathcal{H}}) \log (p(e^{\beta \mathcal{H}} - 1) + 1)}{e^{\beta \mathcal{H}} + 1}. \tag{84}
\]
The fidelity that enters into the lower bound reads
\[
F(\rho\beta, (\text{R}^P_{\sigma,\Psi} \circ \Phi)(\rho\beta)) = \left( \frac{e^{\beta \mathcal{H}} + \sqrt{2(p-1)pe^{\beta \mathcal{H}} - 2(p-1)p}}{e^{\beta \mathcal{H}} + 1} \right)^2. \tag{85}
\]
The results are shown in Figure 2. Observe that in the high-temperature regime (small \( \beta \mathcal{H} \)), there is always a significant gap between the Petz map lower bound and the relative entropy difference (0 < \( \beta \mathcal{H} < 1 \), so that the recovery channel cannot “keep up” with the entropy generation and the inequality cannot be saturated even if \( p \to 1/2 \). In contrast, in the low-temperature regime, the inequality can be saturated in both extremes \( p \to 1 \) and \( p \to 1/2 \), although for \( p \to 1/2 \) it is saturated “trivially” because the recovery channel cannot do any recovery. Note that \( p \) can only approach 1 in delta-coupling case in the weak-coupling limit, since the detector size completely determines the spacetime smearing, unlike gapless case where careful switching can be used to suppress field fluctuations.

C. Entanglement-breaking property

Next, let us show that the qubit channel \( \Psi \) is not generically an entanglement-breaking channel, since a related question was addressed in \[10] \[11]. A channel \( \Psi \) is said to be an entanglement-breaking channel if any joint state with an arbitrary environment \( \mathcal{R} \) given by
\[
\rho_{\text{in}} := (\mathbb{1} \otimes \Phi)(\rho_{\text{in},0}) \tag{86}
\]
is separable. This can be straightforwardly proven using the fact that its Choi matrix
\[
\mathcal{J}(\Psi) = \frac{1}{2}(\mathbb{1} \otimes \Psi)(\mathbb{1} \otimes \mathbb{1}) \tag{87}
\]
is separable \[74\], Prop. 3.31. Here \( |\mathbb{1}\rangle = \sum |j\rangle |j\rangle \) is an unnormalized Bell state obtained by applying vectorization map to the identity operator (the factor 1/2 is used to normalize the Bell state so the Choi matrix is a valid density matrix).
It suffices to consider the case when the monopole operator is $\hat{\sigma}_0 = \hat{\sigma}^z$, since we only need a counterexample. Using a faithful entanglement measure called negativity, defined by [82]

$$N[\rho_{ab}] = \frac{||\rho^a_{ab}||_1 - 1}{2},$$  \hspace{1cm} (88)

where $\rho^a_{ab}$ is the partial transpose of $\rho_{ab}$ and setting $\rho_{ab} = \mathcal{J}(\Phi)$, we see that

$$N[\mathcal{J}(\Phi)] = \frac{1}{2}|1 - 2p|,$$  \hspace{1cm} (89)

so that negativity is zero if and only if $p = 1/2$. This also follows from a nice characterization by Ruska [83] that a qubit channel is entanglement-breaking if and only if $\mathcal{J}(\Psi) \leq \mathbb{I}/2$ (that is, its singular values are less than $1/2$). Since $p = \frac{1}{2}(1 + e^{-2\omega(f,f)})$, it follows that the qubit channel is only very close to being entanglement-breaking channel when the field fluctuation is very large ($p \approx 1/2$). It is worth noting that this result is not the same as what was found in [10, 11], since there the channel they are considering is $\Xi : \mathcal{D}(\mathcal{H}_\phi) \rightarrow \mathcal{D}(\mathcal{H}_\phi)$ defined as

$$\Xi(\rho^0_\phi) = tr_B(U(\rho^0_B \otimes \rho^0_\phi)^\dagger) ,$$  \hspace{1cm} (90)

for delta coupling in Minkowski spacetime. Notably, their result proves that the \textit{complementary channel} of $\Phi$, denoted $\Phi^c \equiv \Xi$, is entanglement-breaking. We will say more about complementary channels in Section VI.

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\footnote{Also, it cannot be entanglement-breaking because otherwise there exists Kraus representations with Kraus operators all of which are rank-$1$, and clearly $K_0$ is max-rank and unitaries relating different Kraus representations cannot change the rank.}

FIG. 2. Comparison of the exact difference between relative entropies $\Delta(\rho_\beta||\sigma) := D(\rho_\beta||\sigma) - D(\Phi(\rho_\beta)||\Phi(\sigma))$ where $\sigma = \mathbb{I}/2$ (dotted line) for thermal state $\rho_\beta$. The lower bound in terms of fidelity from Theorem 2 is given by the solid line.

VI. FIELD AND COMPLEMENTARY CHANNEL IN AQFT FRAMEWORK

The analysis in Section V tells us that there is a lot we can understand and say for the qubit channel in the non-perturbative regime of the UDW model. This has to do with the simplicity of the channel, which is the quasifree case reduces to a very transparent (rotated) Pauli channel [59] and [52]: these are essentially the analog of bit-flip channels. The important input from relativity is that the “flipping” probability is governed by the interaction with a relativistic field in curved spacetimes, i.e., it depends on the states of motion of the detectors, local curvature of spacetimes, and also the choice of states of the quantum field (except that we demanded quasifree states for simplicity, which includes physically relevant vacuum and thermal states). We have organized them in such a way that the main results apply generally regardless of these choices.

Arguably, what can be more interesting in this work concerns the kind of quantum channel that outputs the field state after interaction with a UDW detector. This can be given in terms of the \textit{complementary channel} $\Psi^c$ (resp. $\Phi^c$) of the qubit channel in Section V or directly in terms of the channel acting on the field state itself. It has been known in flat space that this induces some sort of “cat states”, i.e., superposition of two coherent states $|\pm\alpha\rangle$ for some $\alpha$ [11, 27]. We will generalize this in curved spacetime and streamline the results to match AQFT framework. One of the main purposes of this section is to demonstrate some calculations involving quantum field state in AQFT framework which are often not explicitly done in the literature.

Given the qubit channel $\Phi$, its \textit{complementary channel} $\Psi^c : \mathcal{D}(\mathcal{H}_\phi) \rightarrow \mathcal{D}(\mathcal{H}_\phi)$ sends the output to the environment instead, i.e.,

$$\Psi(\rho^0_\phi) = tr_B(V\rho^0_B V^\dagger), \quad \Psi^c(\rho^0_\phi) = tr_B(\rho^0_\phi V V^\dagger).$$  \hspace{1cm} (91)

While complementary channel is often hard to understand in isolation, certain classes of channels called \textit{degradable channels} can be understood via its complementary channel [74]. Also note that the complementary channel is different from the channel acting on the environment state itself, since in this case we have $\tilde{\Psi} : \mathcal{D}(\mathcal{H}_B) \rightarrow \mathcal{D}(\mathcal{H}_B)$ with

$$\tilde{\Psi}(\rho_{b,0}) = tr_B(U(\rho^0_B \otimes \rho_{b,0})U^\dagger).$$  \hspace{1cm} (92)

Notably, the input Hilbert spaces are different. The complementary channel $\Psi^c$ of $\Psi$, and the channel for the field $\Psi$ are written in terms of the Stinespring-like representations as

$$\Psi^c(\rho^0_\phi) = tr_B(U(\rho^0_B \otimes \rho^0_\phi)^\dagger),$$  \hspace{1cm} (93a)

$$\tilde{\Psi}(\rho_{b,0}) = tr_B(U(\rho^0_B \otimes \rho_{b,0}))U^\dagger.$$  \hspace{1cm} (93b)

In what follows we are interested in the output of the channel, so $\tilde{\Psi}$ and $\Psi^c$ have the same output. However,
in general these define different channels and they can have distinct properties. For example, $\Psi$ is not expected to be entanglement-breaking, while $\Psi'$ was proven to be so \[10, 11\], essentially because for $\Psi'$ the input state only appears as expectation values in the coefficient $\alpha$ of the channel.

A. Basic properties of the output state of the complementary channel

We are interested in studying the output of the field channel $\Phi$ or complementary channel $\Psi'$, and we will use them interchangeably unless their differences matter. For the complementary channel $\Psi'$ in Eq. (93a), we see from Eq. (38) the output is identical to the output of $\Phi$, which reads

$$\Psi'(\rho^0) = \tilde{\Psi}(\rho^0) = C_f \rho^0 C_f + S_f \rho^0 S_f + \alpha(C_f \rho^0 S_f - S_f \rho^0 C_f),$$

where $\alpha = \text{tr}(\tilde{\rho} \rho^0)$. The way to understand the complementary channel in the algebraic framework is by considering $\Psi'(\rho^0)$ as being associated to a new state $\omega'$, such that for any $A \in \mathcal{W}(\mathcal{M})$ we have (via Eq. (97))

$$\omega'(A) = \omega(C_f AC_f) + \omega(S_f AS_f) + \alpha(\omega(S_f AC_f) - \omega(C_f AS_f)).$$

Each of these terms can be evaluated purely using the Weyl relations of $\mathcal{W}(\mathcal{M})$.

First, for convenience we use the shorthand $[f] = W(Ef)$, so that

$$C_f = \frac{1}{2}([f] + [-f]), \quad S_f = \frac{1}{2i}([f] - [-f]).$$

Since generators of $\mathcal{W}(\mathcal{M})$ are of the form $A = W(Eg)$ for some compactly supported $g \in C^\infty_0(\mathcal{M})$, it suffices to consider only $A = W(Eg)$. We get

$$4\omega(C_f AC_f) = \omega([2f + g]) + \omega([g - 2f]) + 2\omega([g]) \cos E(f, g),$$
$$-4\omega(S_f AS_f) = \omega([2f + g]) + \omega([g - 2f]) - 2\omega([g]) \cos E(f, g),$$
$$4i\omega(C_f AS_f) = \omega([2f + g]) - \omega([-2f + g]) - 2\omega([g]) \sin E(f, g),$$
$$4i\omega(S_f AC_f) = \omega([2f + g]) - \omega([-2f + g]) + 2\omega([g]) \sin E(f, g).$$

Together, the new algebraic state is related to the old one by the relation

$$\omega'(A) = \omega(A)(\cos E(f, g) + i \alpha \sin E(f, g)).$$

We see that starting with an algebraic state $\omega$, we can obtain the evolved state $\omega'$ by first calculating the complementary channel and evaluate the action on the Weyl generators.

It is instructive at this point to check that indeed it is a state, since its appearance in Eq. (98) makes this far from obvious (to us anyway). First, we check $\omega'$ is normalized. The unit element of the Weyl algebra corresponds to $1 = W(E(0))$ by setting $g = 0$, hence we have

$$\omega'(1) = (\cos E(f, 0) + i \alpha \sin E(f, 0)) = 1.$$

We can also use Eq. (95) and we get

$$\omega'(1) = \omega(C_f^2 + S_f^2) = 1,$$

where we used the fact that $\omega(S_f C_f) = \omega(C_f S_f)$ and $C_f^2 + S_f^2 = 1$

Second, we need to show that $\omega'(A^1 A) \geq 0$. This is perhaps obvious for Weyl algebra $\mathcal{W}(\mathcal{M})$ if we only work with the Weyl generators, since then $A^1 A = 1$. Furthermore, elements of the Weyl algebra consists of exponentiation of finite products of smeared field operators, so this covers all of it. What we would like to do, however, is to show that quantities in the algebra of observables $\mathcal{A}(\mathcal{M})$ like $\omega'(\hat{\phi}(g)\hat{\phi}(g))$ should have non-negative values for any $g \in C^\infty_0(\mathcal{M})$. This quantity is the (symmetrically smeared) Wightman two-point function $W'(g, g)$ at some region where supp$g$ away from the interaction region supp$g$. We can show that indeed the operator $\hat{\phi}(g)\hat{\phi}(g) \geq 0$ gives non-negative expectation value.\[19\] The way to do this is to use the Weyl relations, by setting formally

$$A = e^{i(s + t)\hat{\phi}(g)}, \quad s, t \in \mathbb{R}$$

and then take the derivatives with respect to $s$ and $t$: we will get the following nice expression

$$W'(g, g) = \omega'(\hat{\phi}(g)^2) = W(g, g) + E(f, g)^2,$$

This result kills three birds with one stone: it says that (1) indeed the new state gives non-negative expectation value for positive-semidefinite (smeared) operators; (2) the only place the (symmetric) field fluctuations change (i.e., $W'(g, g) \neq W(g, g)$) after interaction with the detector is determined by the causal propagator; (3) a single detector-field interaction for both gapless and delta-coupled case always introduces more noise where the causal influence propagates, since $E(f, g)^2 \geq 0$. Eq. (102) makes the relativistic property of the field manifest, since the fluctuations only impact regions causally connected to $f$ via the field.

Next, we show that the state $\omega'$ is mixed despite its appearance of being a simple modulation by $\cos E(f, g)$ and $\sin E(f, g)$. One simple way to see this is to consider

\[19\] Note that since $g$ is the same for both field operators, we have no issue with the domain despite multiplying two unbounded operators.
the special case when \( \alpha = 0 \) (by choosing suitable state of the detector). Since \( \alpha = 0 \), we get
\[
\omega'(A) = \omega(C_f A C_f) + \omega(S_f A S_f). \tag{103}
\]
Now we define two algebraic states \( \omega_1 \) and \( \omega_2 \):
\[
\omega_1(A) = \frac{\omega(C_f A C_f)}{\omega(C_f^2)}, \quad \omega_2(A) = \frac{\omega(S_f A S_f)}{\omega(S_f^2)}, \tag{104}
\]
we can rewrite \( \omega' \) as a convex combination
\[
\omega' = \omega(C_f^2) \omega_1 + \omega(S_f^2) \omega_2, \tag{105}
\]
with \( \omega(C_f^2) + \omega(S_f^2) = 1 \). Since a state is pure if and only if it cannot be written as a strict convex combination of two algebraic states [34], it follows that \( \omega' \) is a state. Furthermore, this means that the detector and the field are necessarily entangled after the interaction. For \( \alpha \neq 0 \), one can make similar argument, essentially by finding the associated Kraus operators and multiplied by some “modulation” by \( LHS \) of Eq. (98) must be expressible as linear combination.

Note that while Eq. (105) does not allow us to conclude easily from this expression whether the state is mixed or not, it has another advantage: it shows that the interaction is Gaussian, i.e., it maps Gaussian states to Gaussian states. To see this, suppose that \( \omega \) is Gaussian so that \( \omega(\phi(f)) \) and \( \omega(\phi(g)) \) fully characterize the state for all \( f, g \in C_0^\infty(M) \). Then clearly the LHS of Eq. (105) must be expressible as linear combinations of products of one-point and two-point functions and multiplied by some “modulation” by \( \cos E(f, g) \) and \( \sin E(f, g) \) which cannot change the Gaussian character of \( \omega' \).

B. Displacement and squeezing as adjoint channel on observables

In UDW settings, the common choice for the field’s initial state is the vacuum state, which we denote here by \( \omega_0 \). This is only one of the many classes of Gaussian states (fully characterized by one-point and two-point functions) in quantum field theory. There are at least three types of non-vacuum states that are of great interest: thermal states, squeezed vacuum states, and coherent states. Thermal and squeezed vacuum are quasifree states (vanishing one-point functions), while coherent states are Gaussian states that are not quasifree. We can construct more Gaussian states by series of coherent displacement and squeezing operations on any Gaussian states. Below we will show that we can rephrase coherent and squeezing operations as elements of the Weyl algebra — this will allow us to very straightforwardly generalize the qubit channel calculations to these states with little effort.

A generic coherent state is given by the displacement operator acting on the vacuum state:
\[
|\alpha\rangle := D(\alpha) |0\rangle, \tag{106}
\]
where \( \alpha \) is the coherent amplitude (which is typically multimode, see [57]) and \( D(\alpha) \) is the displacement operator. In the Fock representation, it reads
\[
D(\alpha) = e^{\int d^n k (\alpha(k) a_k - \alpha(k)^* a_k^\dagger)}. \tag{107}
\]
Now we make the following observation. Let \( g \in C_0^\infty(M) \) and define
\[
\alpha(k) := i \int dV g(x) u_k(x). \tag{108}
\]
By construction \( \alpha \in L^2(\mathbb{R}^n) \), i.e., \( \int d^n k |\alpha(k)|^2 < \infty \) since \( g(x) \) is compactly supported smooth function although \( \alpha(k) \) will not be compactly supported. Eq. (108) implies that we can view
\[
D(\alpha) = e^{i\phi(\alpha)} \in \mathcal{W}(\mathcal{M}) \tag{109}
\]
for some \( g \in C_0^\infty(M) \); we write \( g := \alpha \) to make suggestive analogy to Fourier transform (which is indeed the Fourier transform when \( \mathcal{M} \) is Minkowski space up to a prefactor i).

We can also define squeezed vacuum state by the action [47],
\[
|S(\zeta)\rangle := S(\zeta) |0\rangle, \tag{110}
\]
where in the Fock representation it is given by
\[
S(\zeta) = e^{\frac{i}{2} \int d^n k d^n k' (\zeta(k, k') a_k a_{k'}^\dagger - H.c.)}. \tag{111}
\]
Now let \( \zeta \in C_0^\infty(M) \), using suggestive notation as before, and define
\[
\zeta(k, k') := 2i \int dV dV' \zeta(x) \zeta(x') u_k(x) u_{k'}(x'). \tag{112}
\]
We assume in this case that the squeezing amplitudes \( \zeta(k, k') \) can be written in this way, hence as before we can view the squeezing operator as an element of the Weyl algebra, namely
\[
S(\zeta) = e^{i\hat{\phi}(\zeta)^2} \in \mathcal{W}(\mathcal{M}) \tag{113}
\]
In other words, it is the exponentiation of bi-local smeared operator \( \hat{\phi}(\zeta)^2 \in \mathcal{A}(\mathcal{M}) \).

\[\text{[20] We are not the first to regard coherent states and squeezed states in QFT this way (see e.g. [55, 56]), though usually this is framed “backwards”: they define coherent and squeezed states directly via } e^{i\phi(\alpha)} \text{ and } e^{i\phi(\alpha)^2} \text{ respectively instead of a more optically-motivated definition via ladder operators } a_k, a_k^\dagger.\]

\[\text{[21] A priori this is not covering all possible squeezing operations, since it includes possibly momentum-entangling squeezing where } \zeta(k, k') \neq \zeta_1(k) \zeta_2(k') \text{ for some functions } \zeta_j. \text{ We restrict our attention to this subclass for simplicity.}\]
The first main takeaway is that the reinterpretation of the coherent displacement and squeezing operations as Weyl elements essentially allows us to view the action of the algebraic coherent state on the observables in “Heisenberg picture”. First let us use coherent state $|\alpha\rangle$ as an example. At the level of GNS representation the vacuum state $\omega_0$, we have the coherent state density matrix $\hat{\rho}_{\phi,\alpha}^0 := D(\alpha)\hat{\rho}_0^0 D(\alpha)^\dagger$ where $\hat{\rho}_0^0 = |0\rangle\langle 0|$. More generally, we can treat displacement operation as a unitary channel $U_\alpha : \mathcal{D}(H_\phi) \rightarrow \mathcal{D}(H_\phi)$ given by

$$U_\alpha(\hat{\rho}_0^0) = D(\alpha)\hat{\rho}_0^0 D(\alpha)^\dagger.$$  

(114)

It follows then

$$\omega_\alpha(A) := \text{tr}\left(\hat{\rho}_{\phi,\alpha}^0 \hat{A}\right) = \text{tr}\left(\hat{\rho}_0^0 D(\alpha)^\dagger \hat{A} D(\alpha)\right) = \omega_0(U_\alpha^\dagger(A)),$$  

(115)

where $U_\alpha^\dagger(\cdot) := D(\alpha)^\dagger(\cdot)D(\alpha)$ is the adjoint channel of $U_\alpha$ and $\omega_0$ is the vacuum state. The adjoint channel acting on the observable is clearly reminiscent of Heisenberg picture unitary rotation of the observable. The case for squeezed vacuum state works similarly: we have the unitary squeezing channel

$$V_\zeta(\hat{\rho}_0^0) = S(\zeta)\hat{\rho}_0^0 S(\zeta)^\dagger.$$  

(116)

The squeezed vacuum state is $\hat{\rho}_{\phi,\zeta}^0 := S(\zeta)\hat{\rho}_0^0 S(\zeta)^\dagger$ with $\hat{\rho}_0^0 = |0\rangle\langle 0|$ and it follows that

$$\omega_{\zeta}(A) = \text{tr}\left(\hat{\rho}_{\phi,\zeta}^0 S(\zeta)^\dagger \hat{A} S(\zeta)\right) = \omega_0(V_\zeta^\dagger(A)),$$  

(117)

where $V_{\zeta}^\dagger(\cdot) := S(\zeta)^\dagger(\cdot)S(\zeta)$ is the corresponding adjoint channel. These conversions seem to be a very trivial move, however we will see in the next subsection that it is precisely this step that allows us to generalize various calculations to coherent and squeezed states despite not being quasifree. This is because we do know how to take expectations with respect to $\omega_\zeta$ based on the definition of quasifree state [14]. In principle we could replace $\omega_0$ with any quasifree state, but we focus on the vacuum state for simplicity.

The other takeaway is that since $D(\alpha)$ and $S(\zeta)$ are both unitary elements of the Weyl algebra, the result from the previous subsections carry over straightforwardly for these states even though they are not quasifree: for example, we have

$$W_\alpha(f, f) = \omega_\alpha(\hat{\phi}(f)\hat{\phi}(f)) = \omega_0(D(\alpha)^\dagger \hat{\phi}(f)\hat{\phi}(f)D(\alpha)).$$  

(118)

However, from Baker-Campbell-Hausdorff (BCH) formula and CCR we have

$$D(\alpha)^\dagger \hat{\phi}(f)D(\alpha) = \hat{\phi}(f) - i[\hat{\phi}(\alpha), \hat{\phi}(f)] = \hat{\phi}(f) + E(\alpha, f)\mathds{1}. $$  

(119)

Therefore, we get

$$W_\alpha(f, f) = W_0(f, f) + E(\alpha, f)^2 \geq W_0(f, f).$$  

(120)

We immediately get the result that the coherent state has larger noise contribution than the vacuum state and in particular this occurs when $\text{supp}(\alpha) \cap \text{supp}(f) \neq \emptyset$. This can be understood as follows: since coherent states are not invariant under the full spacetime isometry group (in flat space it is the Poincaré group), the coherent amplitude’s “Fourier transform” must be localized somewhere in spacetime with support given by that of $\alpha$, and it is here that the coherent excitations add to the field fluctuations.

Similarly, for squeezed state we get

$$S(\zeta)^\dagger \hat{\phi}(f)S(\zeta) = \hat{\phi}(f) - i[\hat{\phi}(\zeta)^2, \hat{\phi}(f)] = \hat{\phi}(f) + 2E(\zeta, f)\hat{\phi}(\zeta) \equiv \hat{\phi}(h_\zeta).$$  

(121)

This has the nice interpretation that the squeezing operation “squeezes” the smearing profile $f$ into $h_\zeta := f + 2E(\zeta, f)$. Therefore, we get

$$W_{\zeta}(f, f) = W_0(f, f) + 2E(\zeta, f)\omega_0(\{\hat{\phi}_\zeta(\hat{\phi}_\zeta(f))\}) + 4E(\zeta, f)^2W_0(\zeta, \zeta).$$  

(122)

Unlike the case for coherent state, it is no longer the case that any choice of squeezing amplitude leads to larger noise than the vacuum since the second term is not positive semidefinite. What remains true, however, is that again the impact of squeezing on the field fluctuations is not uniform in spacetime, since it is controlled by the “Fourier transform” $\zeta$. Observe that in flat space where $\zeta$ is indeed the Fourier transform of $\zeta$, we can choose $L^2$-integrable function $\zeta$ with $\zeta$ compactly supported, so that the squeezing only impacts the region $\text{supp}(\zeta)$ causally connected to interaction region $f$. A version of this spatial dependence of squeezing on detector dynamics in flat spacetime was analyzed in [17].

This calculation generalizes to multiple displacement and squeezing without having to solve any momentum integrals: for example, if we consider

$$|\beta + \alpha\rangle := D(\beta)D(\alpha)|0\rangle,$$  

(123)

we see that Eq. [119]

$$D(\alpha)^\dagger D(\beta)^\dagger \hat{\phi}(f)D(\beta)D(\alpha) = D(\alpha)^\dagger \hat{\phi}(f)D(\alpha) + E(\beta, f)\mathds{1} = \hat{\phi}(f) + E(\beta, f) + E(\beta, f)\mathds{1} = D(\alpha + \beta)^\dagger \hat{\phi}(f)D(\alpha + \beta).$$  

(124)

In the second equality we have used the linearity of the causal propagator. Thus we see that arbitrary sequence of displacement operators does not pose any extra effort. Note that a sequence of squeezing operations is straightforward because the state remains quasifree: we have

$$V_{\eta}^\dagger \circ V_{\zeta}^\dagger(\hat{\phi}(f)) = V_{\eta}(\hat{\phi}(h_\zeta)) = \hat{\phi}(h_{\eta, \zeta}).$$  

(125a)

$$h_{\eta, \zeta} := h_\zeta + 2E(\eta, h_\zeta),$$  

(125b)
with $h_{\zeta} = f + 2E(\zeta, f)$.

As a more non-trivial example, consider a squeezed coherent state $|\zeta; \alpha\rangle := S(\zeta)D(\alpha)|0\rangle$. Using Eq. (119) and Eq. (121) we get

$$D(\alpha)^\dagger S(\zeta)^\dagger \hat{\phi}(f)S(\zeta)D(\alpha)$$

$$= D(\alpha)^\dagger \hat{\phi}(f)D(\alpha) + D(\alpha)^\dagger (2E(\zeta, f) \hat{\phi}(\zeta))D(\alpha)$$

$$= \hat{\phi}(f) + E(\bar{\alpha}, f)1 + 2E(\zeta, f)\hat{\phi}(\zeta) + 2E(\zeta, f)E(\bar{\alpha}, f)1$$

$$= \hat{\phi}(h) + E(\bar{\alpha}, h)1,$$  \hspace{1cm} (126)

where $h_{\zeta} := f + 2E(\zeta, f)\zeta$. Eq. (126) suggests that the action of adjoint coherent and squeezing channels on "deformed smearing" $\hat{\phi}(h_{\zeta})$, and hence, the two unitary adjoint channels commute up to a phase:

$$(U_{\alpha, f}^\dagger \circ V_{\alpha, f}^\dagger - V_{\alpha, f}^\dagger \circ U_{\alpha, f}^\dagger)(\hat{\phi}(f)) = E(\zeta, f)E(\bar{\alpha}, \zeta)1.$$  \hspace{1cm} (127)

Compared to the usual momentum-space calculations involving ladder operators, this computation is manifestly simpler. Furthermore, by definition of adjoint we also get the same result for the state, i.e., displacement and squeezing acting on the state commutes up to a phase. These calculations show that we can study much more general class of Gaussian states by using the nice properties of displacement and squeezing operations by reframing them as elements of the Weyl algebra.

C. Qubit channel revisited: coherent and squeezed states

The expressions we just obtained give us a very straightforward generalization of the qubit channel in Section 11 to two important classes of non-quasifree Gaussian states, namely for algebraic states associated to coherent and squeezed states. This works because we can treat the coherent and squeezing as adjoint channel acting on the observable elements, and then take expectation values with respect to a reference quasifree state (such as the vacuum). Since expectation values associated to quasifree states are given directly in terms of the Wightman two-point functions, the change into Heisenberg picture gives us a way to algebraically generalize the qubit channel calculations into more general non-quasifree (but Gaussian) settings.

Let us first consider coherent field state and we take the gapless detector as an example. For coherent state with coherent amplitude $\alpha$, we have$^{22}$

$$\Psi_\alpha(\rho^0_h) = \omega_\alpha(C_f^2)\rho^0_h + \omega_\alpha(S_f^2)\mu^0_h + \frac{i}{2}[\rho^0_h, \mu]\omega_\alpha(S_fC_f).$$  \hspace{1cm} (128)

For quasifree states, the coefficients $\omega(C^2_f), \omega(S^2_f)$ and $\omega(S_fC_f)$ are determined directly by definition \[14\] but this will not be automatic for $\omega_\alpha$. However, using Eq. (115), we have

$$\omega_\alpha(C^2_f) = \omega_0(C^2 \bar{C}^2 - \frac{1}{2}) + \omega_0(\bar{C}^2 - \frac{1}{2}),$$

$$\omega_\alpha(S^2_f) = \omega_0(C^2 - \frac{1}{2}) + \omega_0(S^2 - \frac{1}{2}),$$

$$\omega_\alpha(S_fC_f) = -i\omega_0(C \bar{C} - \frac{1}{2}) + i\omega_0(C - \frac{1}{2})S_fC_f + \omega_0(C^2 - \frac{1}{2})S_fC_f,$$

where $\nu_f = e^{-2\omega_0(f, f)}$. As expected, unlike the quasifree case $\omega_\alpha(S_fC_f)$ is no longer zero. These can be straightforwardly computed by direct computation using Weyl relations, or more neatly using "trigonometric lemma" in \[29\] Lemma 1.

Notice that the coefficients in Eq. (128) are computed exactly the same as what goes into the calculations involving two-qubit communication settings (see, e.g., Eq. (44)-(49) of \[29\]). The resulting expression is simply a modulation of $\nu_f$ that appears in Eq. (62) by the sine and cosine of the causal propagator. We reproduce the vacuum result when $\alpha \rightarrow 0$. This result also shows that coherent state of the field modify expectation values of the field state non-uniformly in spacetime: in particular, if $\bar{\alpha}$ is compactly supported in $\mathcal{M}$, then $\Psi_\alpha \neq \Psi$ only when $\bar{\alpha}$ is causally connected to $f$ — that is, when the coherent excitations are causally connected to the interaction region. This is to be expected since physically meaningful coherent states must have excitations that are sufficiently localized in spacetime.

For squeezed states, we can perform analogous calculation but we will have to work out the coefficients more directly as follows. From the BCH formula we have

$$S(\zeta)^\dagger e^{i\hat{\phi}(f)}S(\zeta) = e^{i(\hat{\phi}(f) - 2E(f, \zeta)\hat{\phi}(\zeta))}.$$  \hspace{1cm} (130)

Therefore, we have for instance

$$\omega_\zeta(C^2_f) = \frac{1}{2} + \frac{1}{2}\omega_\zeta(e^{2i\hat{\phi}(f)} + e^{-2i\hat{\phi}(f)})$$

$$= \frac{1}{2} + \frac{1}{2}(\omega_0(e^{2i\hat{\phi}(h_+)}) + \omega_0(e^{-2i\hat{\phi}(h_-)))),$$  \hspace{1cm} (131)

where $h_{\pm} = \pm f \mp 2E(f, \zeta)\zeta$. It follows that

$$\omega_0(e^{i\hat{\phi}(h_{\pm})}) = e^{-2W_0(h_{\pm}, h_{\pm})},$$

$$W_0(h_{\pm}, h_{\pm}) = W_0(f, f) + 4E(f, \zeta)W_0(\zeta, \zeta) - 2E(f, \zeta)Re(W_0(f, \zeta))$$

$$\equiv W_\zeta(f, f).$$  \hspace{1cm} (133)

This agrees with the computation in Eq. (122). The same approach can be used for the other coefficients involving $\omega_\zeta(S_fC_f)$ and $\omega_\zeta(S^2_f)$. Note that now the question of whether squeezing can reduce noise from field

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$^{22}$ Alternatively, we could use the full Kraus representations directly, and in that case we will be mostly manipulating expectation values of exponentials instead of sines and cosines. The sine and cosine version is slightly more cumbersome but has the advantage of not having to track the imaginary unit $i$. 
fluctuations is equivalent to the question of whether $h_{\pm}$ satisfies

$$W_0(h_{\pm}, h_{\pm}) \leq W_0(f, f),$$

which can be checked by direct computation.

In principle, we can generalize this to arbitrary Gaussian states obtainable from the vacuum state via a sequence of displacement and squeezing operations, by making use of the sort of computations done in Eqs. (124) and (126), hence our preceding calculations are very general.

**VII. FURTHER CHARACTERIZATIONS**

To close this work, we briefly discuss some further characterizations we can do with the qubit channel and its corresponding field/complementary channels. We focus on some calculations that are not often discussed in the literature: (1) cohering and decohering power of a detector interacting with a quantum field; (2) computation of Rényi entropy associated to the UDW interaction.

### A. Cohering and decohering power

In [28], the ability of a massive scalar field in flat space in generating or destroying coherence of the UDW detector coupled to it was investigated using the delta-coupled model. A nice feature of the calculations done in [28] is the recognition that the “(de)cohering power” of the field is naturally interpreted in terms of the qubit channel acting on the qubit detector. They also studied parametric dependence of the detector (coupling strengths, energy gap, etc.), dependence on field mass as well as choice of states (coherent and thermal states). It is worth noting that coherence is a basis-dependent concept that nonetheless has an interpretation as a physical resource [84, 85]. One natural choice would be to express the qubit states in terms of its energy eigenbasis, then coherence is defined with respect to this basis. In this case, a state $\rho$ has zero coherence in the energy eigenbasis if and only if it is a fixed point of the completely dephasing channel with respect to energy eigenbasis:

$$\Delta(\rho) = \sum_j \rho_{jj} |j\rangle\langle j|.)$$

In order to have a notion of coherence as a physical resource, we need to make sure that free operations (namely, maximally incoherent operations) cannot generate coherence for free states (those with no coherence). The cohering power is then defined to be the measure of a channel in generating coherence. Any coherence measure $C$ must be nonzero on states with nonzero coherence, non-increasing under maximally incoherent operations, and is a convex function in the sense that $C(\sum_j p_j \rho_j) \leq \sum_j p_j C(\rho_j)$. Among many options, one commonly used measure is the so-called $\ell_1$-norm of coherence, defined by [86, 89]

$$C_{\ell_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|,$$

which for qubits reduce to $C_{\ell_1}(\rho) = 2|\rho_{12}| = 2|\rho_{21}|$, where $\rho_{12}$ (or $\rho_{21} = \rho_{12}^*$) is the off-diagonal component.

Performing analogous computation as done in [28], we see that the $\ell_1$-cohering power is

$$C_{\ell_1}(\mathcal{E}) = |\omega(S_{2f})|,$$

where $\mathcal{E} = \Phi, \Psi$ and $f = \bar{f}, f$ for delta-coupled and gapless detector models respectively.

Observe that we can make many general claims just from Eq. (137) without doing any integrals (although the brute-force computation in [28] is reassuring, including the fact that the same result essentially follows also for gapless detectors simply because the channels $\Phi, \Psi$ are identical up to the definition of monopole operator $\mu_0$). For example, any quasifree states has vanishing cohering power which follows directly from the fact that $\omega(S_f) = 0$ for any quasifree states. Hence we do not need to compute $C_{\ell_1}$ for (squeezed) vacuum states and KMS thermal states, regardless of the details of interaction profile, field mass, trajectories, or spacetime curvature. Similarly, for coherent states considered in [28], we can generalize this by noting from our previous calculation in Section VI that for coherent state $|\alpha\rangle$ we have

$$\omega_{\alpha}(S_{2f}) = 2\omega_{\alpha}(S_C f) = \nu_f \sin(2E(\bar{\alpha}, f)),$$

where we have used Eq. (129c) in the second equality and we recall that $E$ is the causal propagator and $\bar{\alpha} = \int d^3k \, \alpha(k) u_k(x)$ with $u_k(x)$ mode functions associated to particular GNS representation (canonical quantization). In particular, this expression has a very transparent interpretation: to maximize cohering power, we need to have $E(\bar{\alpha}, f) \approx \pi/4$ for any fixed spacetime smearing $f$, so it tells us that one should really optimize based on the “Fourier transform” of $\bar{\alpha}$ instead. It is also clear now that for delta-coupled detector, smaller detectors are more difficult to gain coherence simply because $\nu_f \leq 1$ and it decreases exponentially with smaller size and there is only this much help that optimization over coherent states can provide (being part of sine function).

For completeness, we also include briefly the notion of $\ell_1$-decohering power [28, 86, 88]: it is a measure of the ability of the field to reduce coherence. We consider the definition in [28], which for qubits in this setting reduces to a very simple expression:

$$D_{\ell_1}(\mathcal{E}) = 1 - |\omega(C_{2f})|.$$
The term $\omega(C_{2\mathcal{F}})$ no longer vanishes for quasifree states, but we can use the same old trick exploiting $\omega(S_{2\mathcal{F}}) = 0$ for these states:

$$D_{\ell_1}(\mathcal{E}) = 1 - |\omega(e^{2\psi(t)})| = 1 - e^{-2W(t,f)} .$$  \hspace{1cm} (140)

This expression is completely controlled by the smeared Wightman two-point functions, and comparison with Eq. (62) of [28] shows that the function $I(\beta)$ defined there is related simply by $\lambda^2 I(\beta) = 2W_{\beta}(t,f)$, where $W_{\beta}$ is the KMS state Wightman function. In particular, our construction would work for gapless detectors and arbitrary trajectories even if spacetime is not flat with no further computation (and for simple cases like conformally flat spacetimes there is hardly any extra work needed). Generalizations for non-quasifree Gaussian states can be straightforwardly done following the computations for coherent and squeezing operations in Section [VI]. The reader is referred to Section V of [28] for explicit calculations on the numerical dependence on variation of masses and energy gap for delta-coupled detectors.

B. Rényi entropy of the quantum field

As mentioned in Section [IVC] the algebra of observables $\mathcal{W}(\mathcal{M})$ associated to the full spacetime is known to be a type $I_{\infty}$ von Neumann algebra (subscript due to infinite-dimensional Hilbert space representation), since the ground state $|0\rangle$ associated to the vacuum representation of algebraic state $\omega_0$ is a rank-1 projector (even if the vacuum is not unique). However, the full algebra is typically less interesting for the QFT community because what is perhaps more important (and deserves much more effort) is the computation of entropic quantities associated to subregions of spacetimes. It is known that a typical bounded region $\mathcal{O} \subset \mathcal{M}$ (such as a bounded double-cone) is associated to subalgebra $\mathcal{W}(\mathcal{M})$ which is type III. For type III von Neumann algebras, the main difficulties come from the fact that there is no density matrix representation for physically reasonable algebraic state $\omega$, which makes computation of any entropic quantities intractable in most cases.

Note that even though Type $I_{\infty}$ von Neumann algebras are “easier”, being infinite-dimensional means that many computations are confined to formal equations. For example, after interaction with the detector we know that the field state becomes mixed (which we denoted by $\omega'$ in Section [VI]). Thus we know that the von Neumann entropy, which is well-defined for Type I algebras, can be calculated. If $\omega$ is the initial state, then $\rho_{\omega'}$ is the density matrix in the GNS representation of $\omega$ and we have

$$S(\rho_{\omega'}) = -\int d\mu_{\rho_{\omega'}}(\lambda) \lambda \log_2 \lambda ,$$  \hspace{1cm} (141)

where $\mu_{\rho_{\omega'}}$ defines a projective-valued measure associated to $\rho_{\omega'}$. This expression is essentially the infinite-dimensional generalization of spectral decomposition evaluated on the spectrum of $\rho_{\omega'}$ (see [91] for more details). Most of the time we do not have good control over how to evaluate such integral. The situation is worse in Type III algebra, where von Neumann entropy simply does not exist and one has to work with UV-safe quantities such as relative entropy (see, e.g., [30]).

Here our goal is to show one instance where the situation improves and still is interesting enough for us to calculate. In particular, if we work instead with Rényi entropy [73], we can actually do some computations. The definition of $\alpha$-Rényi entropy is

$$S_\alpha(\rho) = -\frac{1}{\alpha - 1} \log_2 \text{tr}(\rho^\alpha) ,$$  \hspace{1cm} (142)

where $\alpha \in [0,1) \cup (1,\infty)$. The limit $\alpha \to 1$ is the von Neumann entropy. Usually Rényi entropy is motivated and defined in terms of $\alpha$-Rényi divergence, which generalizes the concept of relative entropy between two states. There are countless important applications of Rényi divergence, with several generalizations and operational implications (see, e.g., [74] and refs therein).

The special case where $\alpha = 2$ is naturally identified with purity of a state, and this gives us a candidate of a measure of mixedness of the field state [27] and also a measure of entanglement between the detector and the field (if the joint state is initially pure). Let us show how this can be done nicely for initially quasifree state $\omega$ (with GNS vector state $\rho_0^\beta$). For simplicity we restrict our attention to the case when the detector state obeys $\text{tr}(\rho_0^\beta S_f) = 0$ so that the final state of the field is as given in Eq. (103). In terms of density matrix in the GNS representation of $\omega$, we have

$$\rho_{\omega'} = C_f |\Omega_\omega\rangle\langle \Omega_\omega| C_f + S_f |\Omega_\omega\rangle\langle \Omega_\omega| S_f .$$  \hspace{1cm} (143)

For vacuum representation we simply have $|\Omega_\omega\rangle = |0\rangle$. It follows that

$$\text{tr}(\rho_{\omega'}) = \text{tr}(\rho_0 C_f^2) + \text{tr}(\rho_0 S_f^2) + 2 \text{tr}(\rho_0 C_f S_f)^2 = \omega(C_f^2) + \omega(S_f^2) + 2 \omega(C_f S_f)^2 = \frac{1}{2} (1 + \nu_f^2) \leq 1 .$$  \hspace{1cm} (144)

Hence the Rényi 2-entropy after interaction can be written in terms of smeared Wightman function

$$S_2(\omega') = S_2(\rho_{\omega'}) = 1 - \log_2 (1 + e^{-4W(t,f)}) .$$  \hspace{1cm} (145)

The generalization to non-quasifree Gaussian states and other values of $\alpha \neq 1$ are straightforward.

The fact that the state is generally mixed is in itself not questionable nor surprising. The point we would like to bring forward here is that the degree of mixedness can be computed quantitatively in this natural setting.

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24 Rényi 2-entropy is also distinguished by the fact that it is measurable experimentally without state tomography, i.e., computable without directly knowing the state of the system [91].
First, entropy vanishes in the limit of vanishing interaction $f \to 0$, so any finite-region interactions lead to mixed states of the field as expected. However, for gapless detector interacting with the field’s vacuum state, we know that we can suppress $W_0(f,f) \to 0$ by suitably turning the detector on and off carefully. Hence we can make the field state arbitrarily close to vacuum state (which is pure). In contrast, for thermal states which is a mixed quasifree state, we know that $W_0(f,f) = W_0(f,f) + \Delta W_0(f,f) > 0$ so long as $f \neq 0$. Hence the Rényi entropy is never pure no matter how carefully the detector interaction is manipulated.

For inertial gapless detector in flat space, we can even find out the quantitative behaviour of Rényi entropy exactly: we get

$$W_\beta(f,f) = \lambda^2 \int \frac{d^3k}{2(2\pi)^3 \omega_k} |\tilde{\chi}(k)|^2 |\tilde{F}(k)|^2 \coth(\beta \omega_k),$$

(146)

where $f(x) = \lambda \chi(\tau) F(x)$, $\chi$ and $F$ are switching and spatial profiles in the detector’s rest frame, $\tilde{\chi}$ and $\tilde{F}$ are their respective Fourier transforms, and $\lambda$ is the coupling constant. For pointlike gapless detector centred at the origin with Gaussian switching, we have $F(x) = \delta^3(x)$ and $\chi(\tau) = e^{-r^2/T^2}$, where $T$ is the switching width. We can then plot the Rényi entropy $S_2(\omega')$ as a function of temperature $\beta^{-1}$, as shown in Figure 3.

Notice that it has the expected low and high-temperature limits: (1) it maximizes Rényi 2-entropy for large temperature (or very short interactions), hence it gives a well-defined measure of how close it is to “maximally mixed state” for field theory even though $\mathbb{I}$ is not trace-class in infinite dimensions; (2) it minimizes Rényi entropy at low temperatures (or very long and carefully switched on interactions), which is to be expected since the zero-temperature limit is precisely the vacuum state $|0\rangle$ (which has maximum purity). If the field has nonzero mass, it only adds to the entropy production, since the purity for massless field is an upper bound for the purity when the mass $m > 0$. Interestingly, this result is related to the discussion of accelerating gapless detector in Section IV, we already know that for the accelerating detector (with proper acceleration $a$) to thermalize with respect to some vacuum state of the field, we need to have long interaction $T$ (accelerating for very long times). Since the smeared Wightman function is the same as inertial detector interacting with a thermal state at the same temperature $\beta^{-1} = a/(2\pi)$, we see that thermalization of detectors is intimately connected with entropy production of the quantum field: in fact Figure 3 sets the scale for how large the acceleration needs to be relative to interaction duration to produce certain amount of entropy for the field.

Finally, it is worth noting that we even have the equality of entropy production for both detector and the field: it is straightforward to check, using the definition of the gapless channel $\Psi$ in (52), that for quasifree states we have (subject to $\text{tr}(\rho_0 A) = 0$ for simplicity)

$$S_2(\Psi(\rho_0^d)) = S_2(\omega'),$$

(147)

and in particular this is the case for thermal states — hence Figure 3 might as well be about entropy production of an accelerating detector interacting with field vacuum. Note that the RHS is a calculation for infinite-dimensional system, while the LHS is for a single two-dimensional qubit. Practitioners who are experts in (infinite-dimensional) quantum thermodynamics may or may not find this surprising, but we find this remarkable that such computation can be done straightforwardly and independently (instead of assuming that initially pure joint detector-field state remains pure when computing von Neumann entropy). Furthermore, we have enough control to actually write Rényi entropy in closed form (145), completely determined in terms of smeared Wightman two-point functions.

**VIII. CONCLUSION AND OUTLOOK**

In this work provided some characterization and general results for single-qubit quantum channel within UDW framework associated to detector-field interaction in two distinct non-perturbative regimes: (i) when the interaction is very rapid, effectively at a single instant in time (delta-coupled detector); and (ii) when the qubit has degenerate energy level (gapless detector). We showed that these channels are very similar, and they

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25 This is not to be confused with temperature, which we have always denoted by $T = \beta^{-1}$ in this work.

**FIG. 3.** Rényi entropy of the field state after interaction with a gapless detector as a function of temperature $\beta^{-1}$ (in units of switching duration $T$) for various value of field mass. The detector is taken to be pointlike and the switching scale is set by $T$. The low temperature limit leads to vanishing Rényi entropy and the high temperature limit approaches maximum entropy of 1. If we including finite size effect of the detector, the result is qualitatively similar to adding mass of the field.
can be very naturally understood in terms of the algebra of observables of the quantum field theory in the AQFT framework.

We collected several quantum information-theoretic results for these channels, some of which are rarely computed explicitly in the literature. For the qubit side, we showed that the channel has a straightforward Kraus representation and we can understand many of its properties very explicitly. For the field theory side, we demonstrated the causal behaviour of noise generated by the interaction and we showed that it is possible to generalize to non-quasifree Gaussian states via coherent and squeezing operations. In particular, the cohering and squeezing operations can be thought of as elements of the Weyl algebra, which allows for very nice interpretation of the behaviour of the correlation functions for these states. Finally, we showed that as a straightforward consequence of these computations, we could straightforwardly generalize the results on cohering power in \[25\] with new insights, as well as showing explicitly that both the field and detector’s Rényi entropy productions are equal.

We find it (pleasantly) surprising that we can do quite a lot in non-perturbative regimes with very minimal effort and mostly ‘algebraically’, with nearly no integrals involved except when actual numbers are needed (i.e., the computation of smeared Wightman function \(W(f, f')\)) even when the fields are initially in non-quasifree Gaussian states. For this reason there is good hope that the spirit of the analysis here can carry forward to more qubits (e.g., in communication settings, such as \[13, 14\]) without too much difficulty, although it may need some better reorganization.

There are several further extensions that we can consider following this work and we will briefly mention three of them that appear more immediately relevant. First, there are situations where one would like to think of multiple rapid-repeated interactions in delta-coupling model as being analogous to collision models \[92–95\] and indeed this connection was studied for non-relativistic bosonic bath in \[96\]. The authors were able to frame the analysis in terms of Weyl relations of the canonical commutation relations, so we expect that relativistic generalization is straightforward and it is interesting to see if relativistic considerations have anything to say regarding CP-divisibility of the induced qubit channel.

Second, the tractability of our calculations in this work suggests that at the very least, characterization for two-qubits and three-qubit non-perturbative interactions may not be too difficult. For two-qubit systems, in particular, where two-party communication is most naturally set in, there has been quite a few known results in the non-perturbative regimes in flat spacetimes (see, e.g., the thorough work in \[10, 11, 27\]), though this has changed recently to include curved backgrounds exploiting the sort of generalities we consider here \[13, 15, 29, 62\]. The three-qubit system calculation has been only confined to entanglement and mutual information harvesting in flat space \[20, 97, 98\], and there is also an example on sabotaging of correlations where they consider arbitrary number of detectors were considered in flat space \[25\]. It is actually not difficult to show that there are ways to organize these calculations in the same spirit as this work in curved spacetimes. Multi-party characterization will also enable us to consider more complicated quantum communication protocols: investigation for a three-party example is currently in progress.

Last but not least, some of these results will change for higher spin systems, e.g., when we take the detector to be the spin-\(j\) representation of \(SU(2)\) instead of spin-1/2 system. To illustrate one such difference, note that for qubits (\(j = 1/2\)) the \(SU(2)\)-covariant Wigner negativity is known to be completely specified by its purity and hence the Bloch length, while for \(j \geq 1\) this is no longer the case \[99, 100\]. Hence, even for classification of non-classicality, higher-spin systems no longer share the same property. Characterization of single spin-\(j\) system is therefore of independent interest, and this is currently pursued in \[59\].

As a final closing remark, we are currently investigating a remote possibility that there is a way to formulate a toy model of scattering theory that captures some essential aspects of infrared-finite problem in quantum electrodynamics (QED) \[101\]. Notably, if two classical electromagnetic field configurations lead to the same electromagnetic memory for a fixed large gauge transformations, then the Fock representations are unitarily equivalent, but the vacuum state of one Fock representation associated to one field configuration appears as coherent state in the other Fock representation. Our methodologies are particularly suited for coherent states (see Section VI), so perhaps this can very well be one way to construct an exactly solvable toy model that we need as a first step in formulating the “superscattering operator” suggested in \[101\]. If it works, it would be one neat example where RQI provides some input for high energy physics calculation. Notice that even for scalar theory, formulating scattering theory in this way is not too difficult, since the bulk-to-boundary correspondence is well-known (see \[80\] for recent work that comes closest to what we do here).

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