FOURIER COEFFICIENTS FOR THETA REPRESENTATIONS ON
COVERS OF GENERAL LINEAR GROUPS

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Abstract. We show that the theta representations on certain covers of general linear
groups support certain types of unique functionals. The proof involves two types of Fourier
coefficients. The first are semi-Whittaker coefficients, which generalize coefficients intro-
duced by Bump and Ginzburg for the double cover. The covers for which these coefficients
vanish identically (resp. do not vanish for some choice of data) are determined in full.
The second are the Fourier coefficients associated with general unipotent orbits. In par-
ticular, we determine the unipotent orbit attached, in the sense of Ginzburg, to the theta
representations.

1. Introduction

Let $F$ be a number field containing a full set of $n$th roots of unity. Let $\mathbb{A}$ be its adele ring.
Let $\tilde{\text{GL}}_r(\mathbb{A})$ be a metaplectic $n$-fold cover of the general linear group. In their pioneering
work, Kazhdan-Patterson [23] constructed generalized theta representations $\Theta_r$ on $\tilde{\text{GL}}_r(\mathbb{A})$
as multi-residues of Borel Eisenstein series. The local theta representations were also con-
structed as the Langlands quotient of reducible principal series representations. They showed
that (both globally and locally) the generalized theta representations are generic if and only if
$n \geq r$; and uniqueness of Whittaker models holds if and only if $n = r$ or $r+1$ (when $n = r+1$,
the uniqueness property only holds for certain covers). The theta representations and their
unique models have been used to construct Rankin-Selberg integrals for symmetric power
$L$-functions for the general linear groups; see Shimura [31], Gelbart-Jacquet [12], Patterson-
Piatetski-Shapiro [30], Bump-Ginzburg [4], Bump-Ginzburg-Hoffstein [5], and Takeda [32].

Suppose $r > n$. Motivated by the above background, one may ask the following natural
questions:

1. Does $\Theta_r$ support other types of Fourier coefficients?
2. If $\Theta_r$ supports a nonzero Fourier coefficient, when does the uniqueness property hold?
3. If the uniqueness property holds for certain types of Fourier coefficients, can we use it
to construct Rankin-Selberg integrals that represent Euler products?

All the three questions have affirmative answers and this paper mainly addresses the first
two questions. We first introduce a generalization of the Whittaker coefficients, which we
call semi-Whittaker coefficients. Let $\lambda = (r_1 \cdots r_k)$ be a partition of $r$. Let $P_\lambda$ be the
standard parabolic subgroup of $\text{GL}_r$ whose Levi subgroup $M \cong \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}$. Let $U_\lambda$
be its unipotent radical. Let $U$ be the standard unipotent subgroup of $\text{GL}_r$. Fix a nontrivial
additive character $\psi : F\backslash A \rightarrow \mathbb{C}^\times$. Let $\psi_\lambda : U(F)\backslash U(A) \rightarrow \mathbb{C}^\times$ be the character such that it acts as $\psi$ on the simple positive root subgroups contained in $M$, and acts trivially otherwise. The $\lambda$-semi-Whittaker coefficient of $\theta \in \Theta_r$ is defined to be

$$
\int_{U(F)\backslash U(A)} \theta(ug)\psi_\lambda(u) \, du.
$$

When the partition is $\lambda = (r)$, this recovers the usual Whittaker coefficients.

**Theorem 1.1.**

1. If there is an $r_i > n$, then

$$
\int_{U(F)\backslash U(A)} \theta(ug)\psi_\lambda(u) \, du
$$

is zero for all choices of data.

2. If $r_i \leq n$ for all $i$, then

$$
\int_{U(F)\backslash U(A)} \theta(ug)\psi_\lambda(u) \, du
$$

is nonzero for some choice of data.

3. When $r = mn$, i.e. when the rank is a multiple of the degree, and the partition is $\lambda = (nm)$, then global uniqueness of $\lambda$-semi-Whittaker models holds.

We remark that the local version of the above theorem is also established (see Corollary 3.34, 3.36, and Theorem 3.44). Indeed, parts (1) and (3) are proved by using the local results, and part (2) is proved by using a global argument. We also remark that when $n = 2$ and $\lambda = (2^k)$ or $(2^k1)$ (depending on the parity of $r$), such coefficients and their uniqueness properties were already used in Bump and Ginzburg [4] in their work on symmetric square $L$-functions for GL($r$).

The second type of Fourier coefficients we consider is the Fourier coefficients associated with unipotent orbits. The unipotent orbits of GL$_r$ are parameterized by the partitions of $r$ via the Jordan decomposition. Given a unipotent orbit $O$, we can associate a set of Fourier coefficients; see Section 5 below. Roughly speaking, starting with a unipotent orbit $O$, we can define a unipotent subgroup $U_2(O)$. Let $\psi_{U_2(O)} : U_2(O)(F)\backslash U_2(O)(A) \rightarrow \mathbb{C}^\times$ be a character which is in general position. The Fourier coefficient of $\theta \in \Theta_r$ we want to consider is

$$
\int_{U_2(O)(F)\backslash U_2(O)(A)} \theta(ug)\psi_{U_2(O)}(u) \, du.
$$

When the unipotent orbit is $O = (r)$, this also recovers the usual Whittaker coefficients. There is a partial ordering on the set of unipotent orbits. Our goal is to show that there is a unique maximal unipotent orbit that supports nonzero Fourier coefficients of $\Theta_r$ (see Definition 5.1 below). Let $O(\Theta_r)$ be this orbit. The main results for the Fourier coefficients associated with unipotent orbits are summarized as follows (Theorem 6.2, 7.4, and 6.11).

**Theorem 1.2.** (1) Write $r = an + b$ such that $a \in \mathbb{Z}_{\geq 0}$ and $0 \leq b < n$. Then both locally and globally $O(\Theta_r) = (n^ab)$. 

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(2) Let \( v \) be a finite place such that \(|n|_v = 1\) and \( \Theta_{r,v} \) is unramified. If \( r = mn \) and \( \mathcal{O} = (n^m) \), then
\[
\dim \text{Hom}_{U_2(\mathcal{O}(F_v))}(\Theta_{r,v}, \psi U_2(\mathcal{O}), v) = 1.
\]

This unique model is valuable and it already finds applications in Rankin-Selberg integrals for covering groups. In the research announcement by Friedberg, Ginzburg, Kaplan and the author [6], the notion of Whittaker-Speh-Shalika representation was introduced (see Definition 7.5). Such representations are irreducible automorphic representations on \( \tilde{\text{GL}}_r(\mathbb{A}) \) and they possess unique functionals. The Whittaker-Speh-Shalika representations and their uniqueness models are used in the generalization of the doubling methods to covering groups. The theta representations are examples of such representations.

**Theorem 1.3** (Theorem 7.6). When \( r = mn \), \( \Theta_r \) is a Whittaker-Speh-Shalika representation of type \((n, m)\).

This unique functional also plays a role in a new-way integral (Euler products with non-unique models) for covering groups; see Ginzburg [17].

We now describe the ideas of the proofs. The proof of Theorem 1.1 is based on an induction in stages statement. We describe it in the global setup. Such an argument was also used in Bump-Friedberg-Ginzburg [3] where they studied the Fourier coefficients of theta representations on the double covers of odd orthogonal groups. First of all, we can rewrite the \( \lambda \)-semi-Whittaker coefficients as
\[
\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u) \, du = \int_{U\cap M(F)\backslash U\cap M(\mathbb{A})} \int_{U_\lambda(F)\backslash U_\lambda(\mathbb{A})} \theta(vug) \, dv \, \psi_\lambda(u) \, du.
\]
The inner integral is actually a constant term of the theta function. To compute it, we compute the constant term of the Eisenstein series and use the fact that the multi-residue operator and the constant term operator commute. By the standard unfolding argument, the constant term of the Eisenstein series is a sum of Eisenstein series on \( \tilde{M}(\mathbb{A}) \). After applying the multi-residue operator, only one term survives. This implies that the constant term of a theta function is actually a “theta function” on \( \tilde{M}(\mathbb{A}) \). This fact is also called “periodicity” in [23] and [4].

Now we are facing a difficulty which did not appear in [3]. In the double cover of the odd orthogonal case, the constant terms of theta functions give rise to a representation on the cover of the Levi subgroup. In that case, different blocks commute in \( \tilde{M}(\mathbb{A}) \). Thus, one can take theta representations on each block and form the tensor product. It is shown that the tensor product of theta representations on each block is the same as the theta representation on \( \tilde{M}(\mathbb{A}) \). We would like to seek an analogous result for the general linear group. However, in the general linear case, when we restrict the metaplectic cover to \( \tilde{M} \), the blocks never commute (except when \( n = 2 \)). In fact, there is even no natural map between \( \tilde{M} \) and \( \tilde{\text{GL}}_{r_1} \times \cdots \times \tilde{\text{GL}}_{r_k} \). This means that, starting with representations on the \( \text{GL}_{r_i} \), there is no direct way to construct a representation of \( \tilde{M} \).

To overcome this difficulty, a construction called the metaplectic tensor product has been introduced (see Section 3.4 and 4.4). The local version is developed in Mezo [25] and the global version is given in Takeda [33, 34]. Roughly speaking, the construction goes as follows (both locally and globally). Let \( \tilde{\text{GL}}_{r_i}^{(n)} \) be the subgroup of \( \tilde{\text{GL}}_{r_i} \), consisting of those elements...
whose determinants are $n$th powers. Let $\tilde{M}^{(n)}$ be the subgroup of $\tilde{M}$ consisting of those elements such that the determinants of all the blocks are $n$th powers. The $\tilde{GL}_{r_i}$'s commute in $\tilde{M}$, and $\tilde{M}^{(n)}$ is the direct product of $\tilde{GL}_{r_1}^{(n)}, \ldots, \tilde{GL}_{r_k}^{(n)}$ with amalgamated $\mu_n$.

Now start with representations $\pi_i$ on $\tilde{GL}_{r_i}$. We first restrict $\pi_i$ to $\tilde{GL}_{r_i}^{(n)}$, and pick an irreducible constituent $\pi_i^{(n)}$. Then we take the tensor product $\pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)}$. This is a representation of $\tilde{M}^{(n)}$. We then use induction to obtain a representation of $\tilde{M}$. Extra care must be taken in order to establish the well-definedness and irreducibility of such constructions.

**Theorem 1.4 (Rough form).** Both locally and globally,

$$\Theta_{\tilde{M}} \cong \Theta_{r_1} \otimes \cdots \otimes \Theta_{r_k}.$$

The local version is given in Theorem 3.28, and the global version is Theorem 4.3. Once we have the induction in stages statement, Theorem 1.1 can be established by carefully analyzing the restriction and induction process. In the local setup, we give an explicit formula for the dimension of the twisted Jacquet module $J_{U, \psi, \lambda}(\Theta_r)$.

Theorem 1.2 is proved in Sections 6 and 7. The proof consists of two parts. The first part is to show that any unipotent orbit greater than or not comparable to $(n^a b)$ does not support any Fourier coefficients. The second part is to show that $(n^a b)$ actually supports a nonzero Fourier coefficient. The idea is to build a relation between the semi-Whittaker coefficients and the Fourier coefficients associated with unipotent orbits. Once we know enough information about the semi-Whittaker coefficients, the unipotent orbit attached to the representation can be determined.

Two tools play crucial roles in the proof. The first one is called root exchange. This allows us to replace the domain of integration with a slightly different one. The second one is the Fourier expansion. This allows us to enlarge the domain of integration if we know certain coefficients vanish (this is usually related to the vanishing of semi-Whittaker coefficients). When we combine these tools in a systematic way, vanishing and nonvanishing of Fourier coefficients associated with unipotent orbits can be related to the results on the semi-Whittaker coefficients. Furthermore, when $n$ and $b$ have the same parity, we actually establish an identity between these coefficients. In particular, Theorem 1.2 part (2) follows from Theorem 1.1 part (3).

The remainder of this paper is organized as follows. Section 2.1 introduces notations and defines metaplectic covers of general linear groups. Certain issues such as centers and maximal abelian subgroups are also discussed. The local theory of semi-Whittaker functionals is developed in Section 3. We first review the principal series representations and theta representations of $\tilde{GL}_r(F_v)$. In Section 3.2, we give an explicit description of the restriction of these representations to $\tilde{GL}_r^{(n)}(F_v)$. These results are used to provide examples of the metaplectic tensor product in Section 3.5. We then carefully analyze the construction and compute the dimensions of some twisted Jacquet modules in Section 3.6. Section 4 is devoted to the global theory. The nonvanishing part of Theorem 1.1 is proved in Theorem 4.4. In Section 5, we review the association of Fourier coefficients to a unipotent orbit. The unipotent orbit attached to the theta representations is determined in Sections 6 and 7. Section 6 introduces the local argument. The relation between semi-Whittaker coefficients
and Fourier coefficients associated with unipotent orbit is established in a series of lemmas. Section 7 describes the corresponding global picture.

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2. NOTATIONS AND PRELIMINARIES

2.1. Notations. Fix a positive integer $n$ and let

$$\mu_n(F) = \{x \in F : x^n = 1\}$$

be the group of $n$th roots of unity in a field $F$. In this paper we always assume $|\mu_n(F)| = n$. Fix, once and for all, an embedding $\epsilon : \mu_n \to \mathbb{C}^\times$. We always write $\mu_n$ for $\mu_n(F)$, if there is no confusion. We often invoke the convention of omitting $\epsilon$ from the notation.

All representations which we consider are representations where $\mu_n$ acts by scalars by the embedding $\epsilon$. Such representations are called genuine.

If $F$ is a non-Archimedean local field, we denote by $\mathfrak{o}$ the ring of integers of $F$. Let $\text{val}$ be the normalized valuation on $F$. Let $|\cdot|_F$ be the normalized absolute value on $F$. Let $(\cdot, \cdot) = (\cdot, \cdot)_{F,n} : F^\times \times F^\times \to \mu_n(F)$ be the $n$th order Hilbert symbol. It is a bilinear form on $F^\times$ that defines a nondegenerate bilinear form on $F^\times/F^{\times n}$ and satisfies

$$(x, -x) = (x, y)(y, x) = 1, \quad x, y \in F^\times.$$  

When $F$ is a number field, and $v$ is a place of $F$, we denote by $F_v$ the completion of $F$ at $v$. When $v$ is non-Archimedean, we let $\mathfrak{o}_v$ be the ring of integers of $F_v$.

For $\text{GL}_r$, let $B = TU$ be the standard Borel subgroup with unipotent radical $U$ and maximal torus $T$. The set $\Phi = \{(i, j) : 1 \leq i \neq j \leq r\}$ is identified with the set of roots of $\text{GL}_r$ in the usual way. Let $\Phi^+$ denote the set of positive roots with respect to $B$.

For a partition $\lambda = (r_1 \cdots r_k)$ of $r$, let $P_\lambda$ be the standard parabolic subgroup of $\text{GL}_r$ whose Levi part $M_\lambda$ is $\text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}$ embedded diagonally

$$(g_1, \cdots, g_k) \mapsto \text{diag}(g_1, \cdots, g_k), \quad g_i \in \text{GL}_{r_i},$$

and let $U_\lambda$ denote the unipotent radical of $P_\lambda$. We usually write $M$ for $M_\lambda$ when the partition is fixed. We usually write $m \in M$ by $m = \text{diag}(g_1, \cdots, g_k)$ with $g_i \in \text{GL}_{r_i}$. Let $\Phi_\lambda$ and $\Phi^+_\lambda$ denote the set of roots and positive roots contained in $M_\lambda$, respectively. We also write $B_M = B \cap M$ and $U_M = U \cap M$. We sometimes add subscript $M$ to indicate the ambient group. For example, $T_M$ (which is $T$) is viewed as a maximal torus of $M$. We might still use $T$ for $T_M$ when there is no confusion.

Let $W$ be the set of all $r \times r$ permutation matrices. The Weyl group of $\text{GL}_r$ is identified with $W$. We also identify $W$ as the group of permutations of $\{1, 2, \cdots, r\}$ via

$$w = (\delta_{i,w(j)}).$$
The action of $W$ on $\Phi$ is given by $w(i, j) = (w(i), w(j))$. For a Levi subgroup $M_\lambda$, let $W(M_\lambda)$ be the subset of permutation matrices contained in $M_\lambda$. The group $W(M_\lambda)$ is identified with the Weyl group of $M_\lambda$ (as sets). Let

$$w^{M_\lambda} = \begin{pmatrix} & I_{r_2} & \\ & \vdots & \\ I_{r_k} & & \\ & & \\ \end{pmatrix} \in W.$$  

The element $w^{M_\lambda}$ sends $GL_{r_k} \times \cdots \times GL_{r_1}$ to $M_\lambda$.

For any group $G$ and elements $g, h \in G$, we define $gh = ghg^{-1}$. For a subgroup $H \subseteq G$ and a representation $\pi$ of $H$, let $^g\pi$ be the representation of $^gH$ defined by $^g\pi(h') = \pi(g^{-1}h'g)$ for $h' \in ^gH$.

Let $F$ be a local field. Let $\psi$ be a nontrivial additive character on $F$. In this paper we need to consider several characters on various subgroups of $U$. We make the following convention. For a partition $(p_1 \cdots p_k)$ of $r' \leq r$, let $\Delta = \{i : 1 \leq i \leq r'\} \setminus \{p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_k\}$. Let $V$ be a subgroup of $U$ such that $V$ contains all the root subgroups associated to $\alpha = (i, i + 1)$ for $i \in \Delta$. Let $\psi_{(p_1 \cdots p_k)} : V \to \mathbb{C}^\times$ be a character such that $\psi_{(p_1 \cdots p_k)}$ acts as $\psi$ on the root subgroups associated to $\alpha = (i, i + 1)$ for $i \in \Delta$, and acts trivially otherwise. Thus $\psi_{(r)}$ and $\psi_{(1r)}$ are the usual Whittaker character and the trivial character on $U$, respectively. When $F$ is a number field and $\psi$ is a nontrivial additive character of $F \setminus A$, these characters can be defined analogously.

Let $F$ be a non-Archimedean field. Let $\psi_V$ be a character on a unipotent subgroup $V$ of $U$. We use $J_V$ to denote the Jacquet functor with respect to $V$. The functor $J_{V, \psi_V}$ is the twisted Jacquet functor with respect to $(V, \psi_V)$.

Throughout the paper, the induction Ind is not normalized.

2.2. The local metaplectic cover $\widetilde{GL}_r(F)$. Let $F$ be a local field of characteristic 0 that contains all the $n$th roots of unity. Associated to every 2-cocycle $\sigma : GL_r(F) \times GL_r(F) \to \mu_n(F)$, there is a central extension $\widetilde{GL}_r(F)$ of $GL_r(F)$ by $\mu_n$ satisfying an exact sequence

$$1 \to \mu_n \to \widetilde{GL}_r(F) \xrightarrow{p} GL_r(F) \to 1.$$  

We call $\widetilde{GL}_r(F)$ a metaplectic $n$-fold cover of $GL_r(F)$. As a set, we can realize $\widetilde{GL}_r(F)$ as

$$\widetilde{GL}_r(F) = GL_r(F) \times \mu_n = \{(g, \zeta) : g \in GL_r(F), \zeta \in \mu_n\}.$$  

Notice that $\widetilde{GL}_r(F)$ is not the $F$-rational points of an algebraic group, but this notation is standard. We use $\widetilde{GL}_r$ to denote $\widetilde{GL}_r(F)$. This abuse of notation is widely used in this paper, especially in the local setup. The embedding $\iota$ and the projection $p$ are given by

$$\iota(\zeta) = (I_r, \zeta) \text{ and } p(g, \zeta) = g$$

where $I_r$ is the identity element of $GL_r$. The multiplication is defined in terms of $\sigma$ as follows,

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1g_2, \zeta_1\zeta_2\sigma(g_1, g_2)).$$

For any subset $X \subseteq GL_r$, let

$$\widetilde{X} = p^{-1}(X) \subseteq \widetilde{GL}_r.$$
We also fix the section $\mathbf{s} : \text{GL}_r \to \widetilde{\text{GL}}_r$ of $\mathbf{p}$ given by $\mathbf{s}(g) = (g, 1)$. Then for $g_1, g_2 \in \text{GL}_r,
\mathbf{s}(g_1)\mathbf{s}(g_2) = (g_1g_2, \sigma(g_1, g_2)).$

In [23], Kazhdan-Patterson provided 2-cocycles $\sigma^{(c)}$ parameterized by $c \in \mathbb{Z}/n\mathbb{Z}$. They are related by

$$\sigma^{(c)}(g_1, g_2) = \sigma^{(0)}(g_1, g_2)(\det g_1, \det g_2)^c, \quad g_1, g_2 \in \text{GL}_r. \quad (1)$$

In this paper, we use the 2-cocycles constructed in Banks-Levy-Sepanski [1]. The 2-cocycles in [1] satisfy a block compatibility property.

Let $\sigma^{(0)} = \sigma^{(0)}_r$, and $\sigma^{(c)} = \sigma^{(c)}_r$ be related to $\sigma^{(0)}_r$ by Eq. (1). Block compatibility means the following. If $r = r_1 + \cdots + r_k$ and $g_i, g'_i \in \text{GL}_{r_i}$ for $i = 1, \cdots, k$, then

$$\sigma^{(c)}_r(\text{diag}(g_1, \cdots, g_k), \text{diag}(g'_1, \cdots, g'_k)) = \left[ \prod_{i=1}^{k} \sigma^{(c)}_{r_i}(g_i, g'_i) \right] \cdot \left[ \prod_{i<j} (\det g_i, \det g'_i)^{c+1}(\det g_j, \det g'_j)^c \right].$$

Throughout the paper we fix the positive integers $r$ and $n$ and the modulus class $c \in \mathbb{Z}/n\mathbb{Z}$ and let $\sigma = \sigma^{(c)}_r$. Note that the restriction of $\sigma$ to $T$ is given by

$$\sigma(t, t') = \left[ \prod_{i<j} (t_i, t'_j) \right] \cdot \prod_{i,j} (t_i, t'_j)^c$$

for $t = \text{diag}(t_1, \cdots, t_r)$ and $t' = \text{diag}(t'_1, \cdots, t'_r)$.

The group $U$ splits in $\widetilde{\text{GL}}_r$. In fact $\mathbf{s}|_U$ is an embedding of $U$ in $\widetilde{\text{GL}}_r$ ([24] Proposition 2). Let $K = \text{GL}_r(\mathfrak{o})$. When $|n|_F = 1$, $K$ also splits in $\widetilde{\text{GL}}_r$ ([24] Theorem 2). There is a map $\kappa : K \to \mu_n$ such that $g \mapsto \kappa^*(g) = (g, \kappa(g))$ is a group homomorphism from $K$ to $\widetilde{\text{GL}}_r$. We denote its image by $K^\ast$. We shall fix $\kappa$ such that $\kappa^*$ is what Kazhdan-Patterson refer to as the canonical lift of $K$ to $\widetilde{\text{GL}}_r$. It is characterized by the property that

$$\mathbf{s}|_{T \cap K} = \kappa^*|_{T \cap K}, \mathbf{s}|_W = \kappa^*|_W, \text{ and } \mathbf{s}|_{U \cap K} = \kappa^*|_{U \cap K}.$$ ([23] Proposition 0.1.3). The topology of $\widetilde{\text{GL}}_r$ as a locally compact group is determined by this embedding.

2.3. Centers. The following lemma is Takeda [33] Lemma 3.13, which is very useful for us.

**Lemma 2.1.** Let $F$ be a local field. Then for each $g \in \text{GL}_r$ and $a \in F^\times$, $\sigma_r(g, aI_r)\sigma_r(aI_r, g)^{-1} = (\det g, a^{r-1+2cr})$.

**Lemma 2.2.** Let $n_1 = \gcd(n, 2rc+r-1)$, and $n_2 = \frac{n}{n_1}$. Then the center of $\widetilde{\text{GL}}_r$ is

$$Z_{\widetilde{\text{GL}}_r} = \{(zI_r, \zeta) : z^{2rc+r-1} \in F^{\times n}\} = \{(zI_r, \zeta) : z \in F^{\times n_2}\}.$$

The first part is proved in [23] Proposition 0.1.1, and the second part is proved in Chinta-Offen [7] Lemma 1.

The center of $\widetilde{T}$ is also determined in [23]. Let $\widetilde{T}^n = \{(t^n, \zeta) : t \in T\}.$
Lemma 2.3. The center of $\tilde{T}$ is $Z_{\widetilde{\text{GL}}_r} \tilde{T}^n$.

Let $\widetilde{\text{GL}}_r^{(n)} := \{ g \in \widetilde{\text{GL}}_r : \det g \in F^{\times n} \}$. We are interested in this group since it controls the representation theory of $\widetilde{\text{GL}}_r$. Moreover, it plays a role in developing tensor products and parabolic inductions for metaplectic groups; see Section 3.4. Let $\tilde{T}^\square := \widetilde{\text{GL}}_r^{(n)} \cap \tilde{T}$. The centers of $\widetilde{\text{GL}}_r^{(n)}$ and $\tilde{T}^\square$ behave better than the centers of $\text{GL}_r$ and $\tilde{T}$.

Lemma 2.4. The center of $\widetilde{\text{GL}}_r^{(n)}$ is

$$Z_{\widetilde{\text{GL}}_r}^{(n)} = \tilde{Z} \cap \widetilde{\text{GL}}_r^{(n)} = \{(aI_r, \zeta) : a^r \in F^{\times n}\}$$

$$= \{(aI_r, \zeta) : a \in F^{\times \gcd(n,r)}\}.$$

Proof. The first equality is immediate from Lemma 2.1. For the second equality, the proof is exactly the same as in [7] Lemma 1. □

The proof of the following lemma is also straightforward.

Lemma 2.5. The center of $\tilde{T}^\square$ is $Z_{\widetilde{\text{GL}}_r}^{(n)} \tilde{T}^n$.

2.4. Maximal abelian groups. Maximal abelian subgroups of $\tilde{T}$ play an important role in the representation theory of $\tilde{T}$. Let $\tilde{T}_s$ be a maximal abelian subgroup $\tilde{T}$. In Section 3.2-3.6, we assume that $\tilde{T}^\square \cap \tilde{T}_s$ is a maximal abelian subgroup of $\tilde{T}^\square$, unless otherwise specified.

We briefly explain why such a group exists. First we start with a maximal abelian subgroup $\tilde{T}_s^\square$ of $\tilde{T}^\square$. If $\tilde{T}_s^\square$ is not a maximal abelian subgroup of $\tilde{T}$, then we can choose $x \in \tilde{T} - \tilde{T}^\square$ such that the group generated by $\tilde{T}_s^\square$ and $x$ is abelian. Notice $|\tilde{T}/\tilde{T}^\square|$ is finite. Thus we can repeat this process until we obtain a maximal abelian subgroup of $\tilde{T}$.

A concrete construction of maximal abelian groups is given in Section 3.6.2.

2.5. The global metaplectic cover $\widetilde{\text{GL}}_r(\mathbb{A})$. Let $F$ be a number field that contains all the $n$th roots of unity and $\mathbb{A}$ be the ring of adeles. To construct a metaplectic $n$-fold cover of $\text{GL}_r(\mathbb{A})$ of $\text{GL}_r(\mathbb{A})$, we follow [33] Section 2.2. The adelic 2-cocycle $\tau_r$ is defined by

$$\tau_r(g, g') = \prod_v \tau_{r,v}(g_v, g'_v),$$

for $g, g' \in \text{GL}_r(\mathbb{A})$. Here, the local cocycle is obtained from the block-compatible cocycle, multiplied by a suitable coboundary. It can be shown that there is a section $s_r : \text{GL}_r(F) \to \widetilde{\text{GL}}_r(\mathbb{A})$ such that $\text{GL}_r(F)$ splits in $\widetilde{\text{GL}}_r(\mathbb{A})$. The center $Z_{\widetilde{\text{GL}}_r(\mathbb{A})}$ of $\widetilde{\text{GL}}_r(\mathbb{A})$ can be easily found by using the local results. As in the local case, we define

$$\widetilde{\text{GL}}_r^{(n)}(\mathbb{A}) := \{ g \in \widetilde{\text{GL}}_r(\mathbb{A}) : \det g \in \mathbb{A}^{\times n}\}.$$

The group $\widetilde{\text{GL}}_r(\mathbb{A})$ can also be described as a quotient of a restricted direct product of the groups $\widetilde{\text{GL}}_r(F_v)$. First we consider the restricted direct product $\prod_v \widetilde{\text{GL}}_r(F_v)$ with respect to $K_v$ for all $v$ with $v \nmid n$ and $v \nmid \infty$. Denote each element in this restricted direct product by $\prod_v (g_v, \zeta_v)$ so that $g_v \in K_v$ and $\zeta_v = 1$ for almost all $v$. Then

$$\rho : \prod_v \widetilde{\text{GL}}_r(F_v) \to \widetilde{\text{GL}}_r(\mathbb{A}), \quad \prod_v (g_v, \zeta_v) \mapsto (\prod_v g_v, \prod_v \zeta_v)$$

(2)
is surjective group homomorphism. (Notice that $\prod_v \zeta_v$ is a finite product.) We have

$$\prod_v \tilde{\GL}_r(F_v)/\ker \rho \simeq \tilde{\GL}_r(\mathbb{A}).$$

Thus we have the notions of automorphic representations and automorphic forms on $\tilde{\GL}_r(\mathbb{A})$. We now explain how to write an irreducible automorphic representation $\pi$ on $\tilde{\GL}_r(\mathbb{A})$ as the “metaplectic restricted tensor product” $\otimes'_v \pi_v$ in the sense of [34] Section 2. First of all, we view $\pi$ as a representation on the restricted direct product $\prod'_v \tilde{\GL}_r(F_v)$ by pulling it back by $\rho$ in Eq. (2). By the usual tensor product theorem for the restricted tensor product, we obtain $\pi \circ \rho \simeq \otimes'_v \pi_v$, where each $\pi_v$ is genuine. We call $\pi_v$ the irreducible constituent of $\pi$ at $v$. For almost all $v$, $\pi_v$ is $K_v$-spherical. The representation $\otimes'_v \pi_v$ descends to a representation $\tilde{\GL}_r(\mathbb{A})$. Thus we write

$$\pi \simeq \otimes'_v \pi_v.$$ 

Notice that the space of $\otimes'_v \pi_v$ is the same as $\otimes_v \pi_v$.

2.6. Metaplectic cover of Levi subgroups. Let $\lambda = (r_1 \cdots r_k)$ be a partition of $r$. Let $M := M_\lambda$ be the Levi subgroup of $\GL_r$ described in Section 2.1. This section discusses metaplectic covers $\tilde{M}$, both locally and globally. The 2-cocycle $\tau_r$ does not satisfy block-compatibility. To get round it, an equivalent cocycle $\tau_M$ was introduced in [33] Section 3. We use this cocycle to define $\tilde{\GL}_r(\mathbb{A})$.

Let $T_M$ be the maximal torus consisting of diagonal matrices. We write $T_i = T \cap \GL_{r_i}$, where $\GL_{r_i}$ is embedded in $\GL_r$ via

$$g \mapsto \text{diag}(I_{r_1}, \cdots, g, \cdots, I_{r_k}).$$

The torus $T_i$ can be viewed as a maximal torus of $\GL_{r_i}$. Define $\tilde{T}_M = \tilde{T}_M \cap \tilde{M}$. The following results are proved in [33] Section 3. We omit the details.

Lemma 2.6. The center of $\tilde{M}(R)$ is

$$Z_{\tilde{M}(R)} = \left\{ \begin{pmatrix} a_1 I_{r_1} \\ \vdots \\ a_k I_{r_k} \end{pmatrix} : a_i^{r_i+1+2cr} \in R^{x_n} \text{ and } a_1 \equiv \cdots \equiv a_k \mod R^{x_n} \right\}.$$ 

Remark 2.7. Notice that $Z_{\tilde{\GL}_r} \tilde{T}_n = Z_{\tilde{M}} \tilde{T}_n$ and $Z_{\tilde{\GL}_r} \tilde{M}^{(n)} = Z_{\tilde{M}} \tilde{M}^{(n)}$.

Lemma 2.8. The center of $\tilde{M}^{(n)}$ is

$$Z_{\tilde{M}^{(n)}} = \left\{ \begin{pmatrix} a_1 I_{r_1} \\ \vdots \\ a_k I_{r_k} \end{pmatrix}, \zeta \right\} : a_i^{r_i} \in R^{x_n} \right\}.$$ 

Lemma 2.9. The center of $\tilde{T}_M$ is $Z_{\tilde{M}^{(n)}} \tilde{T}_n$. 

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Let $\widetilde{T}_{M,*}$ be a maximal abelian subgroup of $\widetilde{T}_M$. We again assume $\widetilde{T}_{M,*} \cap \widetilde{T}_M^\square$ is a maximal abelian subgroup of $\widetilde{T}_M^\square$.

One can also talk about automorphic forms and representations on $\widetilde{M}(\mathbb{A})$ as well.

3. Local Theory

In this section, $F$ is a non-Archimedean local field. Recall that we use $\widetilde{\text{GL}}_r$ to denote $\text{GL}_r(F)$.

3.1. The principal series representations. The principal series representations of $\widetilde{\text{GL}}_r$ were studied in [23]. For the generalization to metaplectic covers of other reductive groups, see McNamara [24].

We start with the representation theory of $\widetilde{T}$. In general, $\widetilde{T}$ is not abelian, but it is a two-step nilpotent group. The irreducible genuine representations of $\widetilde{T}$ are parameterized in the following way ([24] Theorem 3): start with a genuine character $\chi$ on the center of $\widetilde{T}$ and extend it to a character $\chi'$ on any maximal abelian subgroup $\widetilde{T}_s$, then the induced representation $i(\chi') := \text{Ind}_{\widetilde{T}_s}^{\widetilde{T}} \chi'$ is irreducible (see [24] Theorem 3). This construction is independent of the choice of $\widetilde{T}_s$ and of the extension of characters.

We extend $i(\chi')$ to a representation $i_{\widetilde{B}}(\chi')$ on $\widetilde{B} = \widetilde{T}U$ by letting $U$ act trivially. Let $\delta_B$ be the modular quasicharacter of $B$. Then $\text{Ind}_{\widetilde{B}}^{\widetilde{\text{GL}}_r} i_{\widetilde{B}}(\chi') \delta_B^{1/2}$ is the principal series representation. This representation is denoted by $I(\chi')$, although its isomorphism class only depends on $\chi$.

There is an alternative way to describe the principal series representations. We can extend the character $\chi'$ to $\widetilde{B}_s = \widetilde{T}_sU$, and then induce it to $\widetilde{\text{GL}}_r$. The representation $\text{Ind}_{\widetilde{B}_s}^{\widetilde{\text{GL}}_r} \chi' \delta_B^{1/2}$ is isomorphic to $I(\chi')$.

The representation $I(\chi')$ is irreducible when $\chi$ is in general position. For a positive root $\alpha$, there is an embedding $i_\alpha : \text{SL}_2 \to \text{GL}_r$. Define

$$\chi_\alpha^n(t) = \chi \left( i_\alpha \left( t \begin{pmatrix} 1 & n \\ \cdot & 1 \end{pmatrix} \right. \right).$$

**Theorem 3.1.** Suppose that $\chi_\alpha^n \neq | \cdot |_F^{1/2}$ for all the positive roots $\alpha$. Then $I(\chi')$ is irreducible.

This is proved by the theory of intertwining operators; see [23] Corollary I.2.8.

If $\chi_\alpha^n = | \cdot |_F$ for all the positive simple roots $\alpha$, we call $\chi$ *exceptional*. In this case, $I(\chi')$ is reducible, and we are interested in the unique irreducible subquotient of $I(\chi')$. Recall that the intertwining operator $T_w : I(\chi') \to I(\chi'_w)$ is defined as

$$(T_w f)(g) = \int_{U(w)} f(w^{-1}ug) du.$$

where $U(w)$ is the subgroup of $U$ corresponding to roots $\alpha > 0$ such that $w^{-1} \alpha < 0$. If this converges for all $f \in I(\chi')$ and is non-trivial, then it is a generator of $\text{Hom}_{\widetilde{\text{GL}}_r}(I(\chi'), I(\chi'_w))$. For general $\chi$, the intertwining operator can be defined via analytic continuation.

**Theorem 3.2.** Let $\chi$ be exceptional. Let

$$\Theta(\chi') = \text{Im}(T_{w_0} : I(\chi') \to I(\chi'_w)).$$
where $w_0$ is the longest element of $W$. Then

1. $\Theta(\chi')$ is the unique irreducible subrepresentation of $I(w_0 \chi')$.
2. $\Theta(\chi')$ is the unique irreducible quotient representation of $I(\chi')$.
3. The Jacquet module $J_U(\Theta(\chi')) \cong \text{Ind}_{w_0 \mathcal{T}_*}^{\mathcal{T}_*} (w_0 \chi' \delta_B^{1/2})$.

This is [23] Theorem I.2.9. $\Theta(\chi')$ is called exceptional.

The Whittaker models of exceptional representations are studied in [23] Section I.3. These authors have shown the following results.

**Proposition 3.3.** Suppose that $|n|_F = 1$.

1. The representation $\Theta(\chi')$ has a unique Whittaker model if and only if $n = r$, or otherwise $n = r + 1$, and $2(c + 1) \equiv 0 \pmod{n}$.
2. The representation $\Theta(\chi')$ does not have a Whittaker model if $n \leq r - 1$.
3. The representation $\Theta(\chi')$ has a finite number of independent nonzero Whittaker models if $n \geq r + 1$.

**Remark 3.4.** In the above proposition, parts (1) and (3) are also true when $|n|_F \neq 1$. This is shown in [23] Section II by using global arguments. Part (2) is expected to be true when $|n|_F \neq 1$, but this is known only when $n = 2$; see Kaplan [22] Theorem 2.6 and Flicker-Kazhdan-Savin [9].

**Remark 3.5.** When $r = 1$, we take $\Theta(\chi')$ to be $\text{Ind}_{\mathcal{T}_*}^{\mathcal{T}_*} \chi'$. This fits into the metaplectic tensor product perfectly.

### 3.2. Restrictions

We study the restriction functor $\text{Res}_{\widetilde{\text{GL}}_r}^{\text{GL}_r(n)}$ in this section. We obtain an explicit description of the restriction of the principal series representations and exceptional representations from $\widetilde{\text{GL}}_r$ to $\widetilde{\text{GL}}_r(n)$. This is useful in Section 3.5 where we give explicit examples of the metaplectic tensor product.

Notice that $\widetilde{\text{GL}}_r(n)$ is an open normal subgroup of $\widetilde{\text{GL}}_r$, and $\widetilde{\text{GL}}_r/\widetilde{\text{GL}}_r(n) \cong F^\times/F^\times n$ is finite and abelian. By Gelbart-Knapp [13] Lemma 2.1, if $I(\chi')$ is irreducible, and $\pi$ is an irreducible constituent of $I(\chi')|_{\widetilde{\text{GL}}_r(n)}$, then

$$I(\chi')|_{\widetilde{\text{GL}}_r(n)} = \sum m \pi.$$

The multiplicities $m$ depend only on $I(\chi')$, and the direct sum is over certain elements of $\widetilde{\text{GL}}_r$.

From now on, we assume $\widetilde{T}:= \mathcal{T}_* \cap \widetilde{T}$ is a maximal abelian subgroup of $\widetilde{T}$. Let $\widetilde{B}_* = \widetilde{T}_* U$ and $\widetilde{B}_\square = \widetilde{T}_\square U$.

**Proposition 3.6.**

$$I(\chi')|_{\widetilde{\text{GL}}_r(n)} \cong \bigoplus_{x^{-1} \in \widetilde{T} \cap \widetilde{T}} \text{Ind}_{x \mathcal{B}_\square}^{\widetilde{\text{GL}}_r(n)} (x \chi' \delta_B^{1/2})|_{x \mathcal{B}_\square}. \quad (3)$$

**Proof.** This follows from Bernstein-Zelevinsky [2] Theorem 5.2. We are working with representations of $\widetilde{\text{GL}}_r$. Let us choose triples $\widetilde{B}, \widetilde{T}, U$ with trivial character on $U$ on the induced
functor side, and \( \widetilde{\text{GL}}_r(n), \text{Gl}_r(n), \{1\} \) with trivial character on \( \{1\} \) on the Jacquet functor side. The Jacquet functor in this case is the restriction functor.

The resulting functor is glued by functors indexed by the double coset space \( \widetilde{B}/\text{Gl}_r/\text{Gl}_r(n) \). This double coset space is a singleton since \( \widetilde{T}\text{Gl}_r(n) = \text{Gl}_r(n) \). Therefore, the functor is the composition of the induction functor from \( \widetilde{T} \cap \text{Gl}_r(n) \) to \( \text{Gl}_r(n) \) and the restriction functor \( \text{Res}_{\widetilde{T}/T} \).

By [13] Lemma 2.1, \( \text{Ind}_{\widetilde{T}/T} \chi \mid_{\widetilde{T}/T} \) is a direct sum of irreducible \( \widetilde{T}^{\square} \)-representations. On the other hand, it has a Jordan-Holder series whose composition factors are

\[
\text{Ind}_{\widetilde{T}/T} x' \quad x^{-1} \in \widetilde{T} \setminus \widetilde{T}/\widetilde{T}^{\square}.
\]

Notice that \( \widetilde{T}^{\square} \) is also a Heisenberg group and \( \widetilde{T}^{\square} \cap z\widetilde{T} = z(\widetilde{T}^{\square}) \) is again a maximal abelian subgroup of \( \widetilde{T}^{\square} \). This implies \( \text{Ind}_{\widetilde{T}/T} x' \) is irreducible. Thus,

\[
\langle \text{Ind}_{\widetilde{T}/T} \chi \rangle \mid_{\widetilde{T}/T} = \bigoplus_{x^{-1} \in \widetilde{T} \setminus \widetilde{T}/\widetilde{T}^{\square}} \text{Ind}_{\widetilde{T}/T} x'.
\]

Now the proposition follows. \( \square \)

**Remark 3.7.** Notice that Eq. (3) depends on the choice of maximal abelian subgroup. Indeed, when \( \chi \) is in general position, the condition that \( \widetilde{T} \cap \widetilde{T}^{\square} = \widetilde{T}^{\square} \) implies each component is irreducible. Without this condition, we get a similar decomposition, but the components are reducible.

Next we show that, when \( \chi \) is in general position, the components in Proposition 3.6 are irreducible. Let us write \( V(x\chi') = \text{Ind}_{\widetilde{B}^{\square}/B} (x\chi' \delta_B^{1/2}) \mid_{\widetilde{B}^{\square}} \) for \( x^{-1} \in \widetilde{T} \setminus \widetilde{T}/\widetilde{T}^{\square} \). Thus Proposition 3.6 becomes

\[
I(\chi') \mid_{\text{Gl}_r(n)} \cong \bigoplus_{x^{-1} \in \widetilde{T} \setminus \widetilde{T}/\widetilde{T}^{\square}} V(x\chi').
\]

**Definition 3.8.** A character of \( Z \text{GL}_r(n) \tilde{T}^n \) or \( Z \text{GL}_r(n) \tilde{T}^n \) is called regular if \( w \chi \neq \chi \) for all \( w \in W, w \neq I \).

**Lemma 3.9.**

1. The \( \tilde{T}^{\square} \)-module \( J_U(V(\chi')) \) has a Jordan-Holder series whose composition factors are

\[
\text{Ind}_{\widetilde{T}^{\square}/\widetilde{T}^{\square}}(w \chi' \delta_B^{1/2}) \quad (w \in W).
\]

2. If \( \chi \) is regular, then for any extension \( \chi', \chi' \mid_{Z \text{GL}_r(n) \tilde{T}^n} \) is regular. Moreover,

\[
J_U(V(\chi')) \cong \bigoplus_{w \in W} \text{Ind}_{\widetilde{T}^{\square}/\widetilde{T}^{\square}}(w \chi' \delta_B^{1/2}).
\]

**Proof.** The first part follows from [2] Theorem 5.2. For the second part, we only need to show that \( \chi' \mid_{Z \text{GL}_r(n) \tilde{T}^n} \) is regular. Indeed, if \( \chi \) is regular, then for any \( w \in W \), there exists
Theorem 3.13. Let \( \chi \) be exceptional. Let
\[
V_0(\chi') = \Im(T_{w_0} : V(\chi') \rightarrow V(^{w_0}\chi')),
\]
where \( w_0 \) is the longest elements of \( W \). Then

1. \( V_0(\chi') \) is the unique irreducible subrepresentation of \( V(^{w_0}\chi') \).
2. \( V_0(\chi') \) is the unique irreducible quotient representation of \( V(\chi') \).
3. \( J_U(V_0(\chi')) \cong \Im^{\wedge w_0}T_{\mathbb{Z}}\mathbb{F}_{\wedge} \mathbb{Z}(\delta_B^{1/2}) \).

Proof. Under the assumption, \( T_w : I(\chi') \rightarrow I(^{w}\chi') \) is an isomorphism, and hence its restriction
\[
T_w : V(\chi') \rightarrow V(^{w}\chi')
\]
is again an isomorphism. Arguing as in [23] Corollary I.2.8, we can show that \( V(\chi') \) is irreducible.

Similarly we can deduce results for exceptional representations.

Lemma 3.10. Let \( \chi_1, \chi_2 \) be two quasicharacters of \( \mathbb{Z}_{\mathbb{GL}r}(\mathbb{Z}) \) and let \( \chi_1', \chi_2' \) be extensions to \( \mathbb{Z}_{\mathbb{GL}r}(\mathbb{Z}) \). Suppose \( \chi_1 \) is regular. Then
\[
\dim \Hom_{\mathbb{GL}r}(\mathbb{Z}_{\mathbb{GL}r}(\mathbb{Z})(\chi_1'), \mathbb{Z}_{\mathbb{GL}r}(\mathbb{Z})(\chi_2')) \leq 1.
\]
The equality holds if and only if \( \chi_2 = w\chi_1 \) for some \( w \in W \).

Proof. This is an immediate application of Lemma 3.9, Frobenius reciprocity, and the fact that \( \Im^{\wedge w_0}T_{\mathbb{Z}}\mathbb{F}_{\wedge} \mathbb{Z}(\delta_B^{1/2}) \) is irreducible.

Lemma 3.11. The restriction of the intertwining operator \( T_w : I(\chi') \rightarrow I(^{w}\chi') \) to Eq. (3) gives
\[
T_w : V(^{x}\chi') \rightarrow V(^{wx}\chi').
\]
Proof. Recall that \( I(\chi') \) is the space of smooth functions
\[
I(\chi') = \{ f : \mathbb{GL}_r \rightarrow \mathbb{C} : f \text{ is smooth and } f(bg) = \chi'(b)\delta_B(b)^{1/2}f(g) \text{ for } b \in \mathbb{B}_* \}.
\]
The embedding of \( V(^{x}\chi') \) into \( I(\chi') \) is given as follows. Let \( f \in V(^{x}\chi') \). Define
\[
\tilde{f}(g) = \begin{cases} 
\tilde{f}(xg) & \text{if } x \in \mathbb{GL}_r^{(n)}, \\
0 & \text{otherwise}.
\end{cases}
\]
Then it is straightforward to check that \( \tilde{f} \in I(\chi') \).

Now given \( f \in V(^{x}\chi') \). We can see that \( T_w(\tilde{f}) \) is in the image of \( V(^{wx}\chi') \) in \( I(^{w}\chi') \).

Proposition 3.12. If \( \chi_\alpha^n \neq | \cdot |^{+/1} \) for all positive roots \( \alpha \), then \( V(\chi') \) is irreducible.

Proof. Under the assumption, \( T_w : I(\chi') \rightarrow I(^{w}\chi') \) is an isomorphism, and hence its restriction
\[
T_w : V(\chi') \rightarrow V(^{w}\chi')
\]
is again an isomorphism. Arguing as in [23] Corollary I.2.8, we can show that \( V(\chi') \) is irreducible.

Similarly we can deduce results for exceptional representations.
Proof. The map

\[ T_{w_0} : I(\chi') \to I(\chi') \]

restricts to

\[ T_{w_0} : \bigoplus_{x^{-1} \in T_\circ \backslash \overline{T}} V(x \chi') \to \bigoplus_{x^{-1} \in \overline{T} \cdot \overline{T}} V(x w_0 \chi'). \]

This implies that

\[ \Theta(\chi')|_{\widetilde{GL}_r^{(\pi)}} = \bigoplus_{x^{-1} \in \overline{T} \cdot \overline{T}} V_0(x \chi'). \]

We first show part (3). From the exactness of the Jacquet functor, \( J_U(V_0(\chi')) \) is a subrepresentation of both \( J_U(V(\chi')) \) and \( J_U(\Theta(\chi')) \). Therefore, \( J_U(V(\chi')) \cong \text{Ind}_{w_0 \overline{T} \cdot \overline{T}}^{\overline{G} \widetilde{L}}(w_0 \chi' \delta_B^{1/2}) \).

The representation \( \Theta(\chi')|_{\widetilde{GL}_r^{(\pi)}} \) is a direct sum of irreducible constituents, which are conjugate to each other. Thus \( V_0(\chi') \) is a direct sum of some of these components. This implies that \( J_U(V_0(\chi')) \) is also a direct sum of the corresponding Jacquet modules which are conjugate to each other. Thus \( V_0(\chi') \) is irreducible since \( J_U(V_0(\chi')) \) is irreducible.

If \( \pi \) is another irreducible quotient representation of \( V(\chi') \), then its Jacquet module is a quotient of \( J_U(V(\chi')) \), and hence there is a nonzero homomorphism \( J_U(\pi) \to \text{Ind}_{w \overline{T} \cdot \overline{T}}^{\overline{G} \widetilde{L}}(w \chi' \delta_B^{1/2}) \) for some \( w \in W \). By Frobenius reciprocity, there is a nonzero intertwining map \( \pi \to V(\chi') \).

The composition

\[ V(\chi') \to \pi \to V(\chi') \]

is nonzero and it must be a constant multiple of \( T_w \). Therefore, the composition

\[ V(\chi') \to \pi \to V(\chi') \xrightarrow{T_{w_0}^{-1}} V(w \chi') \]

is \( T_{w_0} \) and its image is \( V_0(\chi') \). We see that \( V_0(\chi') \) is a quotient of \( \pi \), and since \( \pi \) is irreducible, they must be the same. This proves part (2). Part (1) follows from part (2) by duality.

\[ \Box \]

As a corollary, we describe the decomposition of \( \Theta(\chi') \) when restricted to \( \widetilde{GL}_r^{(\pi)} \).

**Corollary 3.14.**

\[ \Theta(\chi')|_{\widetilde{GL}_r^{(\pi)}} \cong \bigoplus_{x^{-1} \in \overline{T} \cdot \overline{T}} V_0(x \chi'). \]

### 3.3. Principal series of Levi subgroups.

Let \( \lambda \) be a partition of \( r \) and write \( \widetilde{M} \) for \( \widetilde{M}_\lambda \). Recall \( B_M = B \cap M \), and \( U_M = U \cap M \). The principal series representations and exceptional representations can be similarly defined on \( \widetilde{M} \). Recall we may identify \( \text{GL}_{r_i} \) as a subgroup of \( M \) via the embedding

\[ g_i \mapsto \text{diag}(I_{r_1}, \ldots, g_i, \ldots, I_{r_k}). \]

Let \( B_i \) be the standard Borel subgroup of \( \text{GL}_{r_i} \) and \( \delta_{B_i} \) be the modular quasicharacter of \( B_i \) in \( \text{GL}_{r_i} \).

Let \( \chi \) be a genuine character of \( Z_{\text{GL}_{r_i}} \overline{T}_M^{n_i} \) and \( \chi' \) be a character of \( \overline{T}_{M,*} \) extending \( \chi \). The genuine representation \( \pi_{\overline{T}_M}(\chi') := \text{Ind}_{\overline{T}_M,*}^{\overline{G} \widetilde{L}}(\chi' \delta_M^{1/2}) \) is irreducible. The principal series representation \( I(\chi') \) is the induced representation \( \text{Ind}_{B_M}^{\overline{G} \widetilde{L}} \pi_{\overline{T}_M}(\chi') \otimes \delta_M^{1/2} \), where \( \delta_M = \delta_{B_1} \otimes \cdots \otimes \delta_{B_k} \).

There is an alternative way to describe it as in the general linear case.
The theory of intertwining operators applies just as the general linear case. Therefore, \(I(\chi')\) is irreducible when \(\chi\) is in general position.

**Theorem 3.15.** Suppose that \(\chi^n_\alpha \neq | \cdot |_F^{\pm 1}\) for all the positive roots \(\alpha\) in \(M\). Then \(I(\chi')\) is irreducible.

If \(\chi^n_\alpha = | \cdot |_F\) for all positive simple roots \(\alpha\) in \(M\), we call it **exceptional**.

**Theorem 3.16.** Let \(\chi\) be exceptional. Let

\[
\Theta(\chi') = \text{Im}(T_{w_{M,0}} : I(\chi') \to I(w_{M,0} \chi')),
\]

where \(w_{M,0}\) is the longest element of \(W(M)\). Then

1. \(\Theta(\chi')\) is the unique irreducible subrepresentation of \(I(w_{M,0} \chi')\).
2. \(\Theta(\chi')\) is the unique irreducible quotient representation of \(I(\chi')\).
3. \(J_{U_M}(\Theta(\chi')) \cong \text{Ind}_{w_{M,0}^1 T_{M}^1}^{\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}} (w_{M,0} \chi^{1/2})\).

We also want to study \(I(\chi')|_{\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n)}\), and \(\Theta(\chi')|_{\mathfrak{g}_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}}}\). The arguments in Section 3.2 apply in this case without essential change. We only state the results here.

**Proposition 3.17.**

\[
I(\chi')|_{\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n)} \cong \bigoplus_{x \in T_{M}^1 : T_{M}^1 \mathfrak{g}_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}} x} \text{Ind}_{x(\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n) \cap B_{\mathfrak{g}, \mathfrak{g}})} (\chi')^{1/2}.
\]

**Proposition 3.18.** If \(\chi^n_\alpha \neq | \cdot |_F^{\pm 1}\) for all positive roots \(\alpha\) in \(M\), then

\[
\text{Ind}_{x(\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n) \cap B_{\mathfrak{g}, \mathfrak{g}})} (\chi')^{1/2}
\]

is irreducible.

As in the general linear case, write \(V(x \chi') = \text{Ind}_{x(\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n) \cap B_{\mathfrak{g}, \mathfrak{g}})} (\chi')^{1/2}\).

**Proposition 3.19.** Let \(\chi\) be exceptional. Let

\[
V_0(\chi') = \text{Im}(T_{w_{M,0}} : V(\chi') \to V(w_{M,0} \chi')),
\]

where \(w_{M,0}\) is the longest elements of \(W(M)\). Then

1. \(V_0(\chi')\) is the unique irreducible subrepresentation of \(V(w_{M,0} \chi')\).
2. \(V_0(\chi')\) is the unique irreducible quotient representation of \(V(\chi')\).
3. \(J_{U_M}(V_0(\chi')) \cong \text{Ind}_{Z_{\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}}}^{\mathfrak{g}_{\tilde{\mathfrak{g}}}} (w_{M,0} \chi^{1/2})\).

**Proposition 3.20.**

\[
\Theta(\chi')|_{\mathfrak{g}_{\tilde{\mathfrak{g}}}, \tilde{\mathfrak{g}}(n)} \cong \bigoplus_{x \in T_{M}^1 : T_{M}^1 \mathfrak{g}_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}} x} V_0(x \chi').
\]

Lastly, let \(\chi'\) be an exceptional character for \(\tilde{\mathfrak{g}}\). Let \(P\) be the parabolic subgroup of \(\mathfrak{g}_{\tilde{\mathfrak{g}}}\) with Levi subgroup \(\mathfrak{g}\), and \(R\) be its unipotent radical. Let \(\delta_P\) be the modular quasicharacter of \(\mathfrak{g}_{\tilde{\mathfrak{g}}}\) with respect to \(P\). Recall we have \(\delta_M \cdot \delta_P = \delta_{\mathfrak{g}_{\tilde{\mathfrak{g}}}}\) and \(w_0 = w_{M,0} w_M\).
Proposition 3.21. The character \( w^M \chi' \cdot \delta_p^{1/2} \) is exceptional for \( M \), and

\[
J_R(\Theta_{\widetilde{GL}_c}(\chi')) \cong \Theta_{\widetilde{M}}(w^M \chi' \cdot \delta_p^{1/2}).
\]

Proof. The Weyl element \( w^M \) permutes blocks of \( M \), and thus the character \( w^M \chi' \cdot \delta_p^{1/2} \) is exceptional for \( \widetilde{M} \). To prove the isomorphism of Jacquet modules, we apply \( J_{UM}(-) \) on both sides. The left-hand side is

\[
J_{UM}(J_R(\Theta_{\widetilde{GL}_c}(\chi'))) = J_U(\Theta_{\widetilde{GL}_c}(\chi')) \cong \text{Ind}_{w_0 T_r}(w_0^0 \chi' \delta_{GL_r});
\]

while the right-hand side is

\[
J_{U\widetilde{M}}(\Theta_{\widetilde{M}}(w^M \chi' \cdot \delta_p^{1/2})) \cong \text{Ind}_{w_{M,0} T_r, w_{M,0}^1} w_0^0 \chi' \cdot \delta_p^{1/2} \delta_{\widetilde{M}} \cong \text{Ind}_{w_0 T_r}(w_0^0 \chi' \delta_{GL_r}).
\]

This implies that \( J_R(\Theta_{\widetilde{GL}_c}(\chi')) \) and \( \Theta_{\widetilde{M}}(w^M \chi' \cdot \delta_p^{1/2}) \) are both irreducible subrepresentations of \( I(w^M \chi' \cdot \delta_p^{1/2}) \). Thus they are isomorphic.

\[\square\]

3.4. The metaplectic tensor product. One of the basic constructions in the representation theory of \( GL_r(F) \) is parabolic induction. Let \( r = r_1 + \cdots + r_k \) be a partition of \( r \), and let \( M = GL_{r_1} \times \cdots \times GL_{r_k} \) be a Levi subgroup. We start with a list of representations, one for each of \( GL_{r_1}, \ldots, GL_{r_k} \), and then form their tensor product to obtain a representation of \( M \). However, since \( \widetilde{M} \) is not simply the amalgamated direct product of the various \( \widetilde{GL}_{r_i} \), this construction cannot be generalized directly to the metaplectic case. Fortunately, we have a replacement, which is defined in Mezo [25]. We review the construction in this section. The two-fold cover case was outlined in Bump and Ginzburg [4], and studied in full detail in Kable [21]. For the global setup and further properties see Takeda [33, 34].

We observe that any element \( m \in \widetilde{M} \) may be written as \( \text{diag}(g_1, \ldots, g_k) \), such that \( p(g_i) \in GL_{r_i} \) for \( 1 \leq i \leq k \). Recall

\[
\widetilde{M}^{(n)} = \{ m \in \widetilde{M} : \det g_1, \ldots, \det g_k \in F^{\times n} \}
\]

and \( \widetilde{GL}^{(n)} = \widetilde{M}^{(n)} \cap \widetilde{GL}_{r_i} \).

Let \( \pi_1, \ldots, \pi_k \) be irreducible genuine representations of \( \widetilde{GL}_{r_1}, \ldots, \widetilde{GL}_{r_k} \), respectively. The construction of the metaplectic tensor product takes several steps.

First of all, for each \( i \), fix an irreducible constituent \( \pi_i^{(n)} \) of the restriction \( \pi_i |_{\widetilde{GL}_{r_i}^{(n)}} \) of \( \pi_i \) to \( \widetilde{GL}_{r_i}^{(n)} \). Then we have

\[
\pi_i |_{\widetilde{GL}_{r_i}^{(n)}} = \sum g m_i^g(\pi_i^{(n)}),
\]

where \( g \) runs through a finite subset of \( \widetilde{GL}_{r_i} \), \( m_i \) is a positive multiplicity and \( g(\pi_i^{(n)}) \) is the representation twisted by \( g \). Then we construct the tensor product representation

\[
\pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)}
\]
of the group $\widetilde{GL}_{r_1}^{(n)} \otimes \cdots \otimes \widetilde{GL}_{r_k}^{(n)}$. Because the representations $\pi_1, \ldots, \pi_k$ are genuine, this tensor product representation descends to a representation of the group $\widetilde{M}^{(n)}$, i.e. the representation factors through the natural surjection

$$\widetilde{GL}_{r_1}^{(n)} \times \cdots \times \widetilde{GL}_{r_k}^{(n)} \twoheadrightarrow \widetilde{M}^{(n)}.$$ 

We denote this representation of $\widetilde{M}^{(n)}$ by

$$\pi^{(n)} := \pi_1^{(n)} \otimes \cdots \otimes \pi_k^{(n)},$$

and call it the metaplectic tensor product of $\pi_1^{(n)}, \ldots, \pi_k^{(n)}$.

Let $\omega$ be a character on the center $Z_{\widetilde{GL}_r}$ such that for all $(aI_r, \zeta) \in Z_{\widetilde{GL}(r)} \cap \widetilde{M}^{(n)}$ where $a \in F^\times$, we have

$$\omega(aI_r, \zeta) = \pi^{(n)}(aI_r, \zeta) = \zeta \pi_1^{(n)}(aI_{r_1}, 1) \cdots \pi_k^{(n)}(aI_{r_k}, 1).$$

Namely, $\omega$ agrees with $\pi^{(n)}$ on the intersection $Z_{\widetilde{GL}_r} \cap \widetilde{M}^{(n)}$. We can extend $\pi^{(n)}$ to the representation

$$\pi^{(n)}_\omega := \omega \pi^{(n)}$$

of $Z_{\widetilde{GL}_r} \widetilde{M}^{(n)}$ by letting $Z_{\widetilde{GL}_r}$ act by $\omega$.

The last step is crucial. If we induce $\pi^{(n)}_\omega$ to $\widetilde{M}$, the resulting representation is usually reducible. To get an irreducible representation, we extend the representation $\pi^{(n)}_\omega$ to a representation $\rho_\omega$ of a subgroup $\widetilde{H}$ of $\widetilde{M}$ so that $\rho_\omega$ satisfies Mackey’s irreducibility criterion and the induced representation

$$\pi_\omega := \text{Ind}_{\widetilde{H}}^{\widetilde{M}} \rho_\omega$$

is irreducible. It is always possible to find such $\widetilde{H}$ and moreover $\widetilde{H}$ can be chosen to be normal. The construction of $\pi_\omega$ is independent of the choices of $\pi_i^{(n)}$, $\widetilde{H}$ and $\rho_\omega$, and it only depends on $\omega$ (see [25] Section 4).

We write

$$\pi_\omega = (\pi_1 \otimes \cdots \otimes \pi_k)_\omega$$

and call it the metaplectic tensor product of $\pi_1, \ldots, \pi_k$ with the character $\omega$.

The metaplectic tensor product $\pi_\omega$ is unique up to twist.

**Proposition 3.22 ([25] Lemma 5.1).** Let

$$\pi_1, \ldots, \pi_k \quad \text{and} \quad \pi_1', \ldots, \pi_k'$$

be genuine representations of $\widetilde{GL}_{r_1}, \ldots, \widetilde{GL}_{r_k}$. They give rise to isomorphic metaplectic tensor products with a character $\omega$, i.e.

$$(\pi_1 \otimes \cdots \otimes \pi_k)_\omega \cong (\pi_1' \otimes \cdots \otimes \pi_k')_\omega$$

if and only for each $i$ there exists a character $\omega_i$ of $\widetilde{GL}_{r_i}$, trivial on $\widetilde{GL}_{r_i}^{(n)}$, such that $\pi_i \cong \omega_i \otimes \pi_i'$.

**Remark 3.23.** Notice that the metaplectic tensor product generally depends on the choice of $\omega$. If the center $Z_{\widetilde{GL}_r}$ is already contained in $\widetilde{M}^{(n)}$, we have $\pi^{(n)}_\omega = \pi^{(n)}$ and hence there is no actual choice for $\omega$ and the metaplectic tensor product is canonical. This is the case, for example, when $n = 2$ or $n \mid r$. 
A representation of \( \widetilde{M} \) is always a metaplectic tensor product ([33], Lemma 4.5). Moreover, we have the following useful lemmas.

**Lemma 3.24** ([33] Lemma 4.6). Let \( \pi \) and \( \pi' \) be irreducible admissible representations of \( \widetilde{M} \). Then \( \pi \) and \( \pi' \) are equivalent if and only if \( \pi \big|_{Z_{GL_r} \widetilde{M}(n)} \) and \( \pi' \big|_{Z_{GL_r} \widetilde{M}(n)} \) have an equivalent constituent.

**Lemma 3.25** ([33] Proposition 4.7). We have
\[
\text{Ind}_{Z_{GL_r} \widetilde{M}(n)}^{\widetilde{M}} \pi_{\omega}^{(n)} = m \pi_{\omega}
\]
for some finite multiplicity \( m \), so every constituent of \( \text{Ind}_{Z_{GL_r} \widetilde{M}(n)}^{\widetilde{M}} \pi_{\omega}^{(n)} = m \pi_{\omega} \) is isomorphic to \( \pi_{\omega} \).

Indeed, we can verify that \( m = [\widetilde{H} : Z_{GL_r} \widetilde{M}(n)] \).

**Lemma 3.26.** We have
\[
\text{Ind}_{\widetilde{M}(n)}^{\widetilde{M}} \pi^{(n)} = m \left( \bigoplus_{\xi} \pi_{\xi} \right)
\]
where \( m = [\widetilde{H} : Z_{GL_r} \widetilde{M}(n)] \) and \( \xi \) is over the finite set of characters of \( Z_{GL_r} \widetilde{M}(n) \) that are trivial on \( \widetilde{M}(n) \).

**Proof.** The proof is the same as in [33] Proposition 4.7; see also [34] Proposition 3.5. \( \square \)

3.5. **Examples.** We give some examples of the metaplectic tensor product in this section. The key ingredient in the proof is Lemma 3.24. This allows us to compare irreducible smooth representations of \( \widetilde{M} \) by restricting to \( Z_{GL_r} \widetilde{M}(n) \).

Let \( \chi \) be a genuine quasicharacter on \( Z_{GL_r} \widetilde{T}^n \), and \( \omega = \chi \big|_{Z_{GL_r}} \) be the central quasicharacter. For each \( i \), let \( \widetilde{T}_{s^i} \) be a maximal abelian subgroup of \( \widetilde{T}_i \). Let \( \widetilde{T}_s \) be the direct product of \( \widetilde{T}_{s^1}, \ldots, \widetilde{T}_{s^k} \) with amalgamated \( \mu_n \). Then \( \widetilde{T}_s \) is a maximal abelian subgroup of \( \widetilde{T} \). Let \( \widetilde{T}_s \) be a maximal abelian subgroup of \( \widetilde{T} \) such that \( \widetilde{T}_s \cap \widetilde{T}_s = \widetilde{T}_s \).

Let \( \chi' \) be an extension of \( \chi \) to \( \widetilde{T}_s \). We may decompose \( \chi' \big|_{\widetilde{T}_s} \) as
\[
\chi_1 \otimes \cdots \otimes \chi_k,
\]
where \( \chi_i \) is a genuine character on \( \widetilde{T}_{s^i} \). Let \( \widetilde{T}_{s^i} \) be a maximal abelian subgroup of \( \widetilde{T}_s \) such that \( \widetilde{T}_{s^i} \cap \widetilde{T}_{s^j} = \widetilde{T}_{s^i} \). We still use \( \chi_i \) to denote an extension of \( \chi_i \) to \( \widetilde{T}_{s^i} \) (this extension is not unique). When \( \chi \) is in general position, so are \( \chi_i \)'s. Therefore the principal series representations \( I(\chi_i) \) on \( GL_{r_i} \) are irreducible.

**Theorem 3.27.** Assume that \( \chi \) is in general position. Then the metaplectic tensor product \( (I(\chi_1) \otimes \cdots \otimes I(\chi_k))_{\omega} \) is independent on the choices of \( \chi_i \). Moreover, as representations of \( \widetilde{M} \),
\[
I(\chi') \cong (I(\chi'_1) \otimes \cdots \otimes I(\chi'_k))_{\omega}
\]
This result shows that, for principal series representations, the metaplectic tensor product can be viewed as an instance of Langlands functoriality on covering groups; see Gan [11].
Proof. Indeed, the choice of the character $\chi_i$ on $\widetilde{T}_{s,i}$ is up to a character of $\widetilde{T}_{s,i}/\widetilde{T}_{s,i}$. Thus the resulting principal series representations differ by a character that is trivial on $\widetilde{\text{GL}}_{r_i}$. By Proposition 3.22, the metaplectic tensor products are still in the same isomorphism class. This proves the well-definedness.

For the second assertion, let us follow the construction of metaplectic tensor product. For $I(\chi'_i)|_{\text{GL}_{r_i}^{(n)}}$, we choose one irreducible constituent $\text{Ind}_{\widetilde{B}_{r_i}}^{\widetilde{\text{GL}}_{r_i}^{(n)}} \chi'_i \delta_{B_i}^{1/2}$. Then as representations of $Z_{\text{GL}_r}^{M^{(n)}}$,

$$\omega(\text{Ind}_{\widetilde{B}_{r_i}}^{\widetilde{\text{GL}}_{r_i}^{(n)}} \chi'_i \delta_{B_i}^{1/2} \otimes \cdots \otimes \text{Ind}_{\widetilde{B}_{r_k}}^{\widetilde{\text{GL}}_{r_k}^{(n)}} \chi'_k \delta_{B_k}^{1/2}) \cong \omega \text{Ind}_{\widetilde{B}_{r_i}}^{\widetilde{M}^{(n)}} \chi'_i \delta_{M}^{1/2}.$$

This is an irreducible constituent of $(I(\chi'_i) \otimes \cdots \otimes I(\chi'_k))|_{Z_{\text{GL}_r}^{M^{(n)}}}$. On the other hand,

$$\omega \text{Ind}_{\widetilde{B}_{r_i}}^{\widetilde{M}^{(n)}} \chi'_i \delta_{M}^{1/2} \cong \text{Ind}_{Z_{\text{GL}_r}^{M^{(n)}}}^{Z_{\text{GL}_r}^{\tilde{M}^{(n)}}} \chi'_i \delta_{M}^{1/2}.$$ 

is also an irreducible constituent of $I(\chi')|_{Z_{\text{GL}_r}^{M^{(n)}}}$. By Lemma 3.24, we are done.

Next, we turn to exceptional representations. We start with an exceptional character $\chi$ on $Z_{\widetilde{\text{GL}}_r}^{\widetilde{\text{T}}}$. and form the exceptional representation $\Theta_{\widetilde{M}}(\chi')$ as the irreducible quotient of $\text{Ind}_{\widetilde{B}_r}^{\widetilde{M}} \chi'_i \delta_{M}^{1/2}$. The characters $\chi'$s are defined as in the previous case.

**Theorem 3.28.** The metaplectic tensor product $(\Theta(\chi'_1) \otimes \cdots \otimes \Theta(\chi'_k))|_{\text{GL}_r}$ is well-defined. As representations of $\widetilde{M}$,

$$\Theta_{\widetilde{M}}(\chi') \cong (\Theta(\chi'_1) \otimes \cdots \otimes \Theta(\chi'_k))|_{\text{GL}_r}.$$

**Proof.** Again, we want to show that both sides have an equivalent irreducible constituent when restricted to $Z_{\text{GL}_r}^{\widetilde{M}^{(n)}}$. For the left-hand side, we choose $V_0(\chi'|_{\widetilde{T}_s})$. This is the unique irreducible subrepresentation of $\text{Ind}_{Z_{\text{GL}_r}^{M^{(n)}}}^{Z_{\text{GL}_r}^{\tilde{M}^{(n)}}} w_{M,0} \chi'_i \delta_{M}^{1/2}$. The Jacquet module of $V_0(\chi'|_{\widetilde{T}_s})$ is

$$J_{U_M}(V_0(\chi'|_{\widetilde{T}_s})) \cong \text{Ind}_{Z_{\text{GL}_r}^{w_{M,0} \tilde{T}_s} w_{M,0}^{-1}}^{Z_{\text{GL}_r}^{M^{(n)}}} w_{M,0} \chi'_i \delta_{M}^{1/2}.$$

On the right-hand side, we choose $\omega(V_0(\chi'_1) \otimes \cdots \otimes V_0(\chi'_k))$, whose Jacquet module is

$$\omega(\text{Ind}_{w_{GL_{r_1},0}(\widetilde{T}_s)}^{\text{GL}_{r_1}} \chi'_i \delta_{B_i}^{1/2} \otimes \cdots \otimes \text{Ind}_{w_{GL_{r_k},0}(\widetilde{T}_s)}^{\text{GL}_{r_k}} \chi'_k \delta_{B_k}^{1/2}) \cong \text{Ind}_{Z_{\text{GL}_r}^{w_{M,0} \tilde{T}_s} w_{M,0}^{-1}}^{Z_{\text{GL}_r}^{M^{(n)}}} w_{M,0} \chi'_i \delta_{M}^{1/2}.$$

Thus $\omega(V_0(\chi'_1) \otimes \cdots \otimes V_0(\chi'_k))$ can be also realized as the unique irreducible subrepresentation of

$$\text{Ind}_{Z_{\text{GL}_r}^{w_{M,0} \tilde{T}_s} w_{M,0}^{-1}}^{Z_{\text{GL}_r}^{M^{(n)}}} w_{M,0} \chi'_i \delta_{M}^{1/2}.$$
Therefore, as representations of $Z_{GL}, \tilde{M}^{(n)}$,

$$V_0(\chi'_1, \tilde{T}, \omega, \omega') \cong \omega(V_0(\chi'_1) \otimes \cdots \otimes V_0(\chi'_k)).$$

By Lemma 3.24, we are done. \qed

**Example 3.29.** Consider the partition $(1^r)$. In this case, $\tilde{M}$ is just $\tilde{T}$ and the metaplectic tensor product is just the representation theory of $\tilde{T}$. The exceptional representation on $\tilde{GL}$ is $\text{Ind}_{A}^{\tilde{GL}} \chi'$, where $A$ is a maximal abelian subgroup of $\tilde{GL}_1$, and $\chi'$ is an extension of $\chi : F^\times \to \mathbb{C}^\times$ to $A$. Notice that $\chi$ is an irreducible constituent of $\left( \text{Ind}_{A}^{\tilde{GL}} \chi' \right)|_{F^\times \cdot \chi}$. Let $\chi_1, \ldots, \chi_r$ be characters of $F^\times \cdot \chi$. Thus the metaplectic tensor product of $\text{Ind}_{A}^{\tilde{GL}} \chi'_1, \ldots, \text{Ind}_{A}^{\tilde{GL}} \chi'_r$ is $\text{Ind}_{\tilde{T}}(\chi_1 \otimes \cdots \otimes \chi_r)'$, where $(\chi_1 \otimes \cdots \otimes \chi_k)'$ is an extension of $\chi_1 \otimes \cdots \otimes \chi_k$ to $\tilde{T}_n$.

### 3.6. Semi-Whittaker functionals

**Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. For a partition $\lambda$ of $r$, let $M = M_\lambda$ be the corresponding Levi subgroup of $GL_r$. We define a character $\psi_\lambda : U_M \to U_M/[U_M, U_M] \to \mathbb{C}^\times$ as follows. Let $\alpha$ is a positive simple root in $U_M$ and $x_\alpha(a)$ be the one-dimensional unipotent subgroup in $U$ corresponding to the root $\alpha$. We define $\psi_\lambda(x_\alpha(a)) = \psi(a)$. We extend this character to $\psi_\lambda : U \to \mathbb{C}^\times$ via the naive projection $U \to U_M$. Notice this character agrees with the character defined in Section 2.1. For a smooth representation $(\pi, V)$ of $GL_r$, a linear functional $L : V \to \mathbb{C}$ is called a $\lambda$-semi-Whittaker functional if $L(\pi(u)v) = \psi_\lambda(u)L(v)$ for all $u \in U, v \in V$. When $\lambda$ is fixed, we simply call it a semi-Whittaker functional.**

**3.6.1. An explicit formula.** We study semi-Whittaker functionals of exceptional representations. First, we have the following observation for Whittaker functionals of exceptional representations on $GL_r$.

Let $\Theta(\chi')$ be an exceptional representation of $\tilde{GL}_r$. Recall $\psi_{(r)} : U \to \mathbb{C}^\times$ is defined as $\psi_{(r)}(u) = \psi(\sum_{i=1}^{r-1} u_{i,i+1})$. Let $d = \dim J_{U,\psi_{(r)}}(\Theta(\chi'))$. If we restrict $\Theta(\chi')$ to $\tilde{GL}_r^{(n)}$, we still have $d = \dim J_{U,\psi_{(r)}}(\Theta(\chi')|_{\tilde{GL}_r^{(n)}})$. By the exactness of Jacquet functor and Corollary 3.14,

$$d = \sum_{x^{-1} \in \tilde{T}_1 \setminus \tilde{T}/\tilde{T}^\circ} \dim J_{U,\psi_{(r)}}(V_0(x\chi'))$$

$$= \sum_{x^{-1} \in \tilde{T}_1 \setminus \tilde{T}/\tilde{T}^\circ} \dim J_{U,\psi_{(r)}}(V_0(\chi')).$$

Therefore,

$$\sum_{x \in \tilde{T}/\tilde{T}^\circ} \dim J_{U,\psi_{(r)}}(V_0(\chi')) = d[\tilde{T}/\tilde{T}^\circ : \tilde{T}^\circ] = d[\tilde{T}_n^* : \tilde{T}_n^*]$$

Now let us return to the setup of the metaplectic tensor product. Let

$$\Theta_{\tilde{M}}(\chi') = (\Theta(\chi'_1) \otimes \cdots \otimes \Theta(\chi'_k)) \omega$$

be an exceptional representation of $\tilde{M}$. Let $d_i = \dim J_{U_{GL_i},\psi_{(r)}}(\Theta(\chi'_i))$. We now choose representatives for $\tilde{T}_i^\circ \setminus \tilde{T}_i$, and combine them together. This gives a set of representatives of
\[ \sum_{x \in \overline{T} \setminus T} \dim J_{U_M, x \psi_\lambda}(V_\theta(x'_1) \otimes \cdots \otimes V_\theta(x'_k)) = \prod_{i=1}^k d_i[\tilde{T}_{*,i} : \tilde{T}_{*,i}^\square]. \]

**Proposition 3.30.**

\[ \dim J_{U_M, \psi_\lambda}(\Theta_{\tilde{M}}(\chi')) = \frac{\prod_{i=1}^k d_i[\tilde{T}_{*,i} : \tilde{T}_{*,i}^\square]}{[\tilde{H} : \tilde{M}^{(n)}]} \cdot \]

**Proof.** Write \( \pi^{(n)} = V_\theta(x'_1) \otimes \cdots \otimes V_\theta(x'_k) \). We have

\[ J_{U_M, \psi_\lambda}(\text{Ind}_{\tilde{M}}^{\tilde{M}(n)} \pi^{(n)}) \cong \bigoplus_{x \in \tilde{M}(n) \setminus \tilde{M}} J_{U_M, x \psi_\lambda}(\pi^{(n)}) = \bigoplus_{x \in \overline{T} \setminus T} J_{U_M, x \psi_\lambda}(\pi^{(n)}). \]

By Lemma 3.26, the dimension of the left-hand side is

\[ [\tilde{H} : \tilde{M}^{(n)}] \dim J_{U_M, \psi_\lambda}(\Theta_{\tilde{M}}(\chi')). \]

The dimension of the right-hand side is \( \prod_{i=1}^k d_i[\tilde{T}_{*,i} : \tilde{T}_{*,i}^\square] \). This proves the result. \( \square \)

Now we proceed to simplify this formula. From now on till the end of this section, we drop the subscript \( \tilde{M} \) when the ambient group is \( \tilde{M} \) to avoid burden on notations. The subscript \( i \) indicates the subgroup considered is in the \( i \)-th block \( \text{GL}_{r_i} \).

Let \( \pi \) be an irreducible constituent of the representation \( \Theta_{\tilde{M}}(\chi') |_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}} \). Recall

\[ \Theta_{\tilde{M}}(\chi') |_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}} = \bigoplus_{x \in \tilde{T} \setminus \overline{T} / Z_{\text{GL}_{r_i}} \overline{T}^\square} x\pi = \bigoplus_{x \in \overline{T} \setminus \overline{T} / \overline{T}^\square} x\pi. \]

Apply the induction functor \( \text{Ind}_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}}^{\tilde{M}} \) and use Lemma 3.25 on the right-hand side. This gives

\[ \text{Ind}_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}}^{\tilde{M}}(\Theta_{\tilde{M}}(\chi')) |_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}} = [\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{H} : Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}] \Theta_{\tilde{M}}(\chi'). \]

Apply the Jacquet functor \( J_{U_M}(-) \). This gives

\[ \text{Ind}_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}}^{\tilde{M}}(\Theta_{\tilde{M}}(\chi')) |_{Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}} = [\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{H} : Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}] J_{U_M}(\Theta_{\tilde{M}}(\chi')). \]

Comparing the dimensions and using \( [\tilde{T} : Z_{\text{GL}_{r_i}} \tilde{M}^{(n)} \cap \tilde{T}] = [\tilde{M} : Z_{\text{GL}_{r_i}} \tilde{M}^{(n)}] \) gives

\[ [\tilde{M} : \tilde{H}] = [\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{T} : \tilde{T} \cap \overline{T}]. \]

Thus

\[ [\tilde{H} : \tilde{M}^{(n)}] = \frac{[\tilde{M} : \tilde{M}^{(n)}]}{[\tilde{T} : \tilde{T} \cap \tilde{T}]} = \frac{[\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{T} : \overline{T} \cap \tilde{T}]}{[\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{T} : \overline{T} \cap \tilde{T}]} = \frac{[\tilde{T} : \overline{T} \cap \tilde{T}][\tilde{T} : \overline{T} \cap \tilde{T}]}{[\tilde{T} : \overline{T} \cap \tilde{T}]}. \]

**Theorem 3.31.**

\[ \dim J_{U_M, \psi_\lambda}(\Theta_{\tilde{M}}(\chi')) = \prod_{i=1}^k \frac{[\tilde{T}_{*,i} : \tilde{T}_{*,i}^\square]}{[\tilde{T}_{*,i} : \tilde{T}_{*,i}^\square]} \prod_{i=1}^k d_i. \]
Remark 3.32. We can see that the same calculation is true for the principal series representations.

Let us mention some immediate corollaries.

**Corollary 3.33.** Suppose $|n|_F = 1$. If $r_i > n$, for some $i$, then

$$J_{U,M,\psi,\lambda}(\Theta_M(\chi')) = 0.$$

*Proof.* This is because when $r_i > n$, $d_i = 0$. $\Box$

**Corollary 3.34.** Suppose $|n|_F = 1$. Let $\Theta_r(\chi')$ be an exceptional representation of $\widetilde{GL}_r$. If $r_i > n$ for some $i$, then $J_{U,\psi,\lambda} (\Theta_r(\chi')) = 0$. In other words, there is no semi-Whittaker functional on $\Theta_r(\chi')$.

*Proof.* In fact,

$$J_{U,\psi,\lambda} (\Theta_r(\chi')) = J_{U,M,\psi,\lambda}(\Theta_T(\chi')) = J_{U,M,\psi,\lambda}(\Theta_M(\chi'_{u_M}, \delta_{P}^{1/2})) = 0.$$

The following corollaries are true without $|n|_F = 1$.

**Corollary 3.35.** When $r_i \leq n$ for all $i$, $J_{U,M,\psi,\lambda}(\Theta_M(\chi')) \neq 0$.

**Corollary 3.36.** When $r_i \leq n$ for all $i$, $J_{U,\psi,\lambda}(\Theta_r(\chi')) \neq 0$.

3.6.2. **Construction of maximal abelian groups.** We now discuss the construction of maximal abelian subgroups. This helps us simplify the formula further.

Given a maximal isotropic subgroup $\Omega$ of the Hilbert symbol, [23] Section 0.3 provides a way to construct maximal abelian subgroups of $T$ under certain assumptions. When $|n|_F = 1$, $F^{\times} \otimes \mathbb{C}$ is a maximal isotropic subgroup of the Hilbert symbol. Let

$$T_0 = \{ \text{diag}(a_1, \cdots, a_r) \in T : \text{val}(a_i) \equiv 0 \mod n \}.$$

Then $Z_{\widetilde{GL}_r} T_0$ is called the standard maximal abelian subgroup of $T$, in the sense of [23] Section I.1. We use $T_0^\ast$ to denote this subgroup.

**Remark 3.37.** Notice that $T_0^\ast \cap T_0^\square$ is usually not a maximal abelian subgroup of $T$, even for $n = 2$. When $n = 2, c = 0$, a “canonical” maximal abelian subgroup was introduced in Bump-Ginzburg [4]. The intersection of their maximal abelian subgroup and $T_0^\square$ is a maximal abelian subgroup of $T_0^\square$.

Let $T_0^\square = T_0 \cap \widetilde{GL}_r^{(n)}$. The following proposition can be proved by imitating the argument in [23] Section 0.3.

**Proposition 3.38.** The group $Z_{\widetilde{GL}_r} T_0^\square$ is a maximal abelian subgroup of $T_0^\square$.

**Remark 3.39.** Our calculation in Section 3.6 relies on the index $[\widetilde{T} : T_0^\ast]$, which is an invariant of $\widetilde{T}$. We can compute it by using the standard maximal abelian subgroup $T_0^\ast$. 

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Remark 3.40. When $|n|_F = 1$, we give an example of maximal abelian subgroup such that its intersection with $\tilde{T}^\square$ is $Z_{\tilde{GL}_n}^* \tilde{T}_o^\square$. Let

$$\tilde{Z}_* = \{(zI_r, \zeta) \in \tilde{Z} : z \in \mathfrak{o}^* F^{\times \gcd(n,r(2r+c+r-1))}\}$$

and

$$\tilde{T}_{o}^{(n')} = \{a \in \tilde{T}_o : \det(a) \in F^{\times \gcd(n,r)}\}.$$

Then $\tilde{Z}_* Z_{\tilde{GL}_n}^* \tilde{T}_o^{(n')} = \tilde{Z}_* \tilde{T}_o^{(n')}$ is a maximal abelian subgroup of $\tilde{T}$ and its intersection with $\tilde{T}^\square$ is $Z_{\tilde{GL}_n}^* \tilde{T}_o^\square$.

Remark 3.41. When $|n|_F \neq 1$, it is usually difficult to construct maximal abelian subgroups of $\tilde{T}$. However, when $n \mid r$, the situation is still nice in the following sense. Let $\Omega$ be an isotopic subgroup of the Hilbert symbol. Then by the construction in [23] Section 0.3,

$$\{(\text{diag}(t_1, \cdots, t_r), \zeta) : t_i \in \Omega, \zeta \in \mu_n\}$$

is a maximal abelian subgroup of $\tilde{T}$.

We now discuss maximal abelian subgroups of $\tilde{T}^\square_M$. For $1 \leq i \leq k$, let $\tilde{T}_{*,i}^\square$ be a maximal abelian subgroup of $\tilde{T}_i^\square$. Let $\tilde{T}_{*}^\square$ be the direct product of $\tilde{T}_{*,1}^\square, \cdots, \tilde{T}_{*,k}^\square$ with amalgamated $\mu_n$. Then $\tilde{T}_*^\square$ is a maximal abelian subgroup of $\tilde{T}^\square$.

Let $\tilde{T}^\square_{M,o} = \tilde{T}_o \cap \tilde{M}(n)$.

Lemma 3.42. The group $Z_{\tilde{M}(n)} \tilde{T}^\square_{M,o}$ is a maximal abelian subgroup of $\tilde{T}^\square$.

The other maximal abelian subgroup we consider is the standard maximal abelian subgroup $\tilde{T}_{M,*}^{st}$.

3.6.3. An explicit formula, continued. We now continue the calculation of our explicit formula. Throughout this section, the ambient group is $\tilde{M}$. We again use the convention after the proof of Proposition 3.30. Thus $\tilde{T}_{*,i}^{st}$ (resp.) is $\tilde{T}_{*,i}^{st}$, $\tilde{T}_{M,o}$, resp.) in the previous section, while $\tilde{T}_{*,i}^{st}$ and $\tilde{T}_{o,i}^{st}$ are the corresponding subgroups in the $i$-th block $\tilde{GL}_{r_i}$.

Theorem 3.43. When $|n|_F = 1$,

$$\dim J_{U_M, \psi_{\lambda}}(\Theta_{\tilde{M}}(\chi')) = \frac{\prod_{i=1}^k [\tilde{T}_{*,i}^{st} : \tilde{T}_{o,i}^{st}]}{[\tilde{T}_{*,i}^{st} : \tilde{T}_{o}^{st}]} \prod_{i=1}^k d_i.$$

Proof. Indeed,

$$\frac{[\tilde{T}_* : \tilde{T}_o]}{[\tilde{T} : \tilde{T}_o]} = \frac{[\tilde{T} : \tilde{T}_*] [\tilde{T}_* : \tilde{T}_o]}{[\tilde{T} : \tilde{T}_*]} = \frac{[\tilde{T}_* : \tilde{T}_o]}{[\tilde{T} : \tilde{T}_*]} \frac{[\tilde{T}_o : \tilde{T}_{*,i}^{st}]}{[\tilde{T}_o : \tilde{T}_{*,i}^{st}]} [\tilde{T}_{*,i}^{st} : \tilde{T}_{o,i}^{st}].$$

Notice that

$$\prod_{i=1}^k \frac{[\tilde{T}_i : \tilde{T}_{i}^{st}]}{[\tilde{T} : \tilde{T}_i]} = \prod_{i=1}^k \frac{[\tilde{T}_{i}^{st} : \tilde{T}_{*,i}^{st}]}{[\tilde{T}_i : \tilde{T}_{*,i}^{st}]} = \prod_{i=1}^k \frac{[\tilde{T}_i : \tilde{T}_{o,i}^{st}]}{[\tilde{T}_i : \tilde{T}_{o}^{st}]} = 1.$$

Combining with Theorem 3.31, we obtain the desired formula. \qed
When $r$ is a multiple of $n$, we have the following uniqueness result. This holds even without the assumption $|n|_F = 1$.

**Theorem 3.44.** If $r = mn$, and $\lambda = (n^m)$, then $J_{U, \psi_\lambda}(\Theta_r(x'))$ is one-dimensional.

**Proof.** When $|n|_F = 1$, this follows from Theorem 3.43. Indeed, in this case we have $\gcd(n, 2rc + r - 1) = 1$. Therefore $Z_{\text{GL}_r} \subset \tilde{T}$, and $[\tilde{T}^* : \tilde{T}_s] = 1$. Similarly, $Z_{\text{GL}_n} \subset \tilde{T}_o$, and $[\tilde{T}^*_{o,i} : \tilde{T}_{o,i}] = 1$. By Proposition 3.3, $d_i = 1$ for all $i$. Therefore $\dim J_{U, \psi_\lambda}(\Theta_r(x')) = 1$.

We now assume $|n|_F \neq 1$. Let $\Omega$ be a maximal isotropic subgroup of the Hilbert symbol. Then

$$\tilde{T}_s := \{ (\text{diag}(t_1, \ldots, t_r), \zeta) : t_i \in \Omega \}$$

is a maximal abelian subgroup of $\tilde{T}$, and $\tilde{T}_s^\square := \tilde{Z}_{\tilde{M}(c)} \cdot (\tilde{T}_s' \cap \tilde{T}_s^\square)$ is a maximal abelian subgroup of $\tilde{T}^\square$. Notice $Z_{\tilde{M}(c)} = \tilde{Z}_{\tilde{M}(c)}$. Moreover, $[\tilde{T}_s : \tilde{T}_s^\square] = \left[ \frac{\tilde{T} : \tilde{T}_s}{\tilde{T} : \tilde{T}_s^\square} \right]$ and

$$\prod_{k=1}^{|\tilde{T} : \tilde{T}_s|} \left[ \frac{\tilde{T} : \tilde{T}_s}{\tilde{T} : \tilde{T}_s^\square} \right] = \prod_{k=1}^{|\tilde{T} : \tilde{T}_s|} \left[ \frac{\tilde{T} : \tilde{T}_s}{\tilde{T} : \tilde{T}_s^\square} \right] = 1.$$

Combining this uniform description with Theorem 3.31, we are done. \qed

**Remark 3.45.** Recall that the metaplectic cover $\text{GL}_r$ depends on an implicit choice of the modulus class $c \in \mathbb{Z}/n\mathbb{Z}$. Our results are true for all $c \in \mathbb{Z}/n\mathbb{Z}$. This is clear for the vanishing result (Corollary 3.34) and nonvanishing result (Corollary 3.36). For the uniqueness result, notice that when $r = mn$, $Z_{\text{GL}_r} = \{ z^n I_r : z \in F^\times \}$. This fact is independent of $c$. Thus the proof of Theorem 3.44 is independent of $c$.

**Remark 3.46.** When $n = 2$, this is \cite{[4]} Proposition 1.3 (i). Indeed, when $r = 2k$ and the partition is $(2^k)$, this follows from Theorem 3.44. When $n = 2$, $r = 2k+1$, and $M$ corresponds to the partition $(2^k1)$. In this case, $d_i = 1$ for all $i$ and $[\tilde{T}^*_{s,i} : \tilde{T}_o] = [F^\times : F^{\times2} \phi^\times]$. Moreover, $[\tilde{T}^*_{s,i} : \tilde{T}_{o,i}] = 1$ if $r_i = 2$; and $[F^\times : F^{\times2} \phi^\times]$ if $r_i = 1$. The twisted Jacquet module of $\Theta_{\tilde{M}}(x')$ is again one-dimensional.

4. **Global Theory**

4.1. **Theta representations.** Let $F$ be a number field containing a full set of $n$th roots of unity $\mu_n$, and let $\mathbb{A}$ denote the adeles of $F$. For $r \geq 2$, let $\tilde{\text{GL}}_r(\mathbb{A})$ denote an $n$-fold metaplectic cover of the general linear group, as in Section 2.5.

We recall the definition of the global theta representations. These representations were constructed in \cite{[23]} using the residues of Eisenstein series as follows. Let $B$ be the standard Borel subgroup of $\text{GL}_r$, and $T \subset B$ denote the maximal torus of $\text{GL}_r$. Let $\omega \in \mathbb{C}$ be a multiplicative character, and define the character $\mu_\omega$ of $T(\mathbb{A})$ by $\mu_\omega(\text{diag}(a_1, \ldots, a_r)) = \prod |a_i|^{\omega}$. Let $Z(\tilde{T}(\mathbb{A}))$ denote the center of $\tilde{T}(\mathbb{A})$. Let $\omega_\omega$ be a genuine character of $Z(\tilde{T}(\mathbb{A}))$ such that $\omega_\omega = \mu_\omega \circ p$ on $\{ (t^n, 1) | t \in T(\mathbb{A}) \}$, where $p$ is the canonical projection from $\tilde{T}(\mathbb{A})$ to $T(\mathbb{A})$. Choose a maximal abelian subgroup $A$ of $\tilde{T}(\mathbb{A})$, extend this character to a character of $A$, and induce it to $\tilde{T}(\mathbb{A})$. Then extend it trivially to $\tilde{B}(\mathbb{A})$ using the canonical projection from $\tilde{B}(\mathbb{A})$ to $\tilde{T}(\mathbb{A})$, and further induce it to the group $\text{GL}_r(\mathbb{A})$. We abuse the notation slightly and
write this induced representation \( \text{Ind}_{B(\mathbb{A})}^{\text{GL}_r(\mathbb{A})} \mu_\Lambda \delta_B^{1/2} \). It follows from [23] that this construction is independent of the choice of \( A \) and of the extension of characters. Let \( E(\mathfrak{g}, g) \) be the Eisenstein series attached to this induced representation. It follows from [23] that when \( n(s_i - s_{i+1}) = 1 \) for \( 1 \leq i \leq r - 1 \), this Eisenstein series has a nonzero residue representation.

Let \( \Lambda \in \mathbb{C}^r \) be such a pole, and we write the residue representation as \( \Theta_{r, \Lambda} \). The poles where we take the residues are usually clear in the context, and thus sometimes we omit it from the notation. The global theta representation \( \Theta_r \) is the metaplectic restricted tensor product of the local exceptional representations \( \Theta_{r,v} \), as explained in Section 2.5. It is shown in [23] that \( \Theta_r \) is generic if and only if \( r \leq n \).

### 4.2. Vanishing results.

**Proposition 4.1.** Let \( \theta \) be in the space of \( \Theta_r \). Let \( \lambda = (r_1 \cdots r_k) \) be a partition of \( r \). If there is an \( r_i > n \) for some \( i \), then

\[
\int_{U(F) \backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u) \, du \equiv 0
\]

for all choices of data.

**Proof.** If

\[
\int_{U(F) \backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u) \, du
\]

is nonzero for some choice of data, then the functional \( l : \Theta_r \to \mathbb{C} \) defined by

\[
\theta \mapsto \int_{U(F) \backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u) \, du
\]

is nonzero. Recall that \( \Theta_r \) is the metaplectic restricted tensor product \( \tilde{\otimes}_v' \Theta_{r,v} \) and the space of \( \tilde{\otimes}_v' \Theta_{r,v} \) is the same as the usual restricted tensor product \( \otimes_v' \Theta_{r,v} \) (see Section 2.5). We can view \( l \) as a functional on \( \otimes_v' \Theta_{r,v} \) as well. We can choose a factorizable vector \( \otimes_v' \theta_0,v \) such that \( l(\otimes_v' \theta_0,v) \neq 0 \).

Let \( w \) be a non-Archimedean place of \( F \) such that \( |n|_w = 1 \) and \( \Theta_r \) is unramified at \( w \). Define a local functional \( l_w : \Theta_{r,w} \to \mathbb{C} \) by

\[
\theta_w \mapsto l(\theta_w \otimes (\otimes_{v \neq w}' \theta_{0,v})).
\]

By our construction, \( l_w \) is nonzero. Now the functional \( l_w \) factors through the twisted Jacquet module of \( \Theta_{r,w} \) for the character \( \psi_\lambda(u) \) on the group \( U(F_w) \). This implies that \( J_{U(F_w),\psi_\lambda}(\Theta_{r,w}) \) is nonzero. This contradicts the local result.

### 4.3. Constant terms I.

Let \( \lambda = (r_1 \cdots r_k) \) be a partition of \( r \). Let \( P_\lambda \) be the standard parabolic subgroup of \( \text{GL}_r \) with Levi subgroup \( M_\lambda \) and unipotent radical \( U_\lambda \).

The goal for this section is to determine the constant term of \( \Theta_r \). We first compute the constant term of the Eisenstein series along \( U_\lambda \). This turns out to be a sum of Eisenstein series on \( \widetilde{M}_\lambda(\mathbb{A}) \), over a subset of the Weyl group \( W \). We exchange the constant term operator and the multi-residue operator, and the constant terms actually span a “theta
representation” on $\tilde{M}_\Lambda(\mathbb{A})$. We then review the construction of the global metaplectic tensor product in Section 4.4, and show that the theta representation on $\tilde{M}_\Lambda(\mathbb{A})$ is actually the global metaplectic tensor product of $\Theta_r$'s.

**Proposition 4.2.** If $\theta \in \Theta_r$, then the constant term

$$m \mapsto \int_{U_\Lambda(F) \setminus U_\Lambda(\mathbb{A})} \theta(um) \, du, \quad m \in \tilde{M}_\Lambda(\mathbb{A})$$

is the residue of an Eisenstein series on $\tilde{M}_\Lambda(\mathbb{A})$.

**Proof.** Let $\theta(g) = \text{Res}_{s=\Lambda} E(\phi, \underline{s}, g)$. We first compute the constant term of the Eisenstein series $E(\phi, \underline{s}, g)$ along $P_\Lambda$. To do this, we introduce the set $W_\Lambda$ which consists of elements $w^{-1}$ such that $w^{-1}(\beta) > 0$ for any $\Phi^+$, and $wT w^{-1} \subset M_\Lambda$. By Mœglin-Waldspurger [28] Proposition 2.1.7(2),

$$E(\phi, \underline{s}, g)_{P_\Lambda} = \sum_{w^{-1} \in W_\Lambda} \sum_{\gamma \in (wBu^{-1} \cap M_\lambda)(F) \setminus M_\Lambda(F)} T(w, \underline{s}) \phi(\underline{s})(\gamma g)$$

$$= \sum_{w^{-1} \in W_\Lambda} E_{M_\lambda} (T(w, \underline{s}) \phi(\underline{s}), w \underline{s}, g)$$

Let $\Lambda$ denote the pole of $E(\phi, \underline{s}, g)$ as in Section 4.1. To compute the constant term of theta function along $P_\Lambda$, we use the fact that the multi-residue operator $\lim_{\underline{s} \to \Lambda} \prod_{i=1}^{r-1} (ns_i - ns_{i+1} - 1)$ and the constant term operator commute. Following an argument as in the proof of Offen-Sayag [29] Lemma 2.4, we deduce that after applying the multi-residue operator, the only term left is the one corresponding to $w^{M_\Lambda}$.

We identify the set of simple roots with $\{(i, i+1) : 1 \leq i \leq r-1\}$. Given $w^{-1} \in W_\Lambda$, let $\Delta^1(w) = \{i : \alpha = (i, i+1), w^{-1}(\alpha) < 0\}$. Notice that by the definition of $W_\Lambda$, $\Delta^1(w)$ is contained in $\{r_1, r_1 + r_2, \ldots \}$. Then the normalized intertwining operator

$$N(w, \underline{s}) = \prod_{i \in \Delta^1(w)} (ns_i - ns_{i+1} - 1) T(w_s, \underline{s})$$

is holomorphic at $\Lambda$. Notice that the action of $w$ on $\underline{s}$ is

$$w(s_1, \ldots, s_r) = (s_{w^{-1}(1)}, \ldots, s_{w^{-1}(r)}).$$

Let

$$\Delta^2(w) = \{i : w^{-1}(i+1) - w^{-1}(i) = 1\} \setminus \{r_1, r_1 + r_2, \ldots \}.$$ 

Then the normalized Eisenstein series

$$\prod_{i \in \Delta^2(w)} (ns_i - ns_{i+1} - 1) E_{M_\lambda}(N(w, \underline{s}) \phi(\underline{s}), w \underline{s}, g)$$

is holomorphic at $\Lambda$. Thus, the terms corresponding to $w^{-1}$ survives after taking multi-residue if and only if

$$\Delta^1(w) \cup \Delta^2(w) = \{1, \ldots, r-1\}.$$ 

This implies that $\Delta^1(w) = \{r_1, r_1 + r_2, \ldots \}$ and $w$ permutes blocks of $M_\Lambda$. The only possibility is $w = w^{M_\Lambda}$. Thus we have shown the following identity

$$\theta(g)_{P_\Lambda} = \text{Res}_{\underline{s}=\Lambda} E_{M_\lambda} (T(w^{M_\Lambda}, \underline{s}) \phi(\underline{s}), w^{M_\Lambda} \underline{s}, g).$$
This finishes the proof.

If we vary \( \theta \in \Theta_r \), then the constant terms of \( \theta \)'s span an irreducible automorphic representation of \( \tilde{M}(\mathbb{A}) \). We denote it by \( \Theta_{\tilde{M}} \). As in the general linear case, \( \Theta_{\tilde{M}} \) is the restricted tensor product of local theta representations of \( \tilde{M}(F_v) \).

4.4. Global metaplectic tensor product. The global metaplectic tensor product was first given in [33] Section 5, and a simplified version is given in [34]. We briefly review the latter construction here.

Assume \( (\pi, V_\pi) \) is an automorphic representation of \( G \), and \( V_\pi \) is a space of functions or maps on the group \( G \), and \( \pi \) is the representation of \( G \) on \( V_\pi \) defined by right translation. Let \( H \subset G \) be a subgroup. Then we define \( \pi|_H \) to be the representation of \( H \) realized in the space

\[
V_\pi|_H := \{ f|_H : f \in V_\pi \}
\]

of restrictions of \( f \in V_\pi \) to \( H \), on which \( H \) acts by right translation.

Let \( \pi_i \) be a genuine irreducible automorphic unitary representation of \( \tilde{G}_r(\mathbb{A}) \). Let \( H_i = \text{GL}_{r_i}(F)\tilde{G}_r(n)(\mathbb{A}) \), and \( \sigma_i = \pi_i|_{H_i} \). Then the restriction \( \pi_i|_{H_i} \) is completely reducible ([34], Proposition 3.9). Hence \( \sigma_i \) is a subrepresentation of \( \pi_i|_{H_i} \).

Note that \( H_i \) is indeed a closed subgroup of \( \tilde{G}_r(n)(\mathbb{A}) \). By the product formula for the Hilbert symbol and block-compatibility of the cocycle, we have the natural surjection

\[
H_1 \times \cdots \times H_k \to M(F)\tilde{M}(n)(\mathbb{A}).
\]

Then consider the space

\[
V_{\sigma_1} \otimes \cdots \otimes V_{\sigma_k}
\]

as functions on the direct product \( H_1 \times \cdots \times H_k \), which gives rise to a representation of \( H_1 \times \cdots \times H_k \). If \( \varphi_i \in V_{\sigma_i} \) for \( i = 1, \cdots, k \), we denote this function by

\[
\varphi_1 \otimes \cdots \otimes \varphi_k,
\]

and denote the space generated by those function by \( V_\sigma \). These functions can be viewed as “automorphic forms” on \( M(F)\tilde{M}(n)(\mathbb{A}) \). The group \( M(F)\tilde{M}(n)(\mathbb{A}) \) acts on \( V_\sigma \) by right translation. Denote this representation by \( \sigma \). This representation is completely reducible ([34] Proposition 3.11).

Fix an irreducible subrepresentation \( \tau \) of \( \sigma \). Then the abelian group

\[
Z_{\tilde{G}_r(\mathbb{A})\cap M(F)\tilde{M}(n)(\mathbb{A})}
\]

acts as a character \( \omega_\tau \) ([34] Lemma 3.17). Choose a “Hecke character” \( \omega \) on \( Z_{\tilde{G}_r(\mathbb{A})} \) by extending \( \omega_\tau \). Then one can extend \( \tau \) to a representation \( \tau_\omega \) on \( Z_{\tilde{G}_r(\mathbb{A})}M(F)\tilde{M}(n)(\mathbb{A}) \). Consider the smooth induced representation

\[
\Pi(\tau_\omega) := \text{Ind}_{Z_{\tilde{G}_r(\mathbb{A})}M(F)\tilde{M}(n)(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau_\omega.
\]

We can view \( \Pi(\tau_\omega) \) as a subrepresentation of \( \mathcal{A}(\tilde{M}) \), which is the space of automorphic forms on \( \tilde{M}(\mathbb{A}) \). Moreover, \( \Pi(\tau_\omega) \) has an irreducible subrepresentation ([34] Proposition 27)
Choose such a representation and denote it by $\pi_\omega$. Then we call $\pi_\omega$ a metaplectic tensor product of $\pi_1, \cdots, \pi_k$ with respect to the character $\omega$ and write

$$\pi_\omega = (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega.$$ 

The representation $\pi_\omega$ has the desired local-global compatibility. Moreover, it is unique up to equivalence, and depends only on $\pi_1, \cdots, \pi_k$ and $\omega$ ([34] Theorem 3.23).

### 4.5. Constant term II

We give the second description of the constant term of the theta function. We show that the theta representation on $\tilde{M}_\lambda(\mathbb{A})$ is in fact the global metaplectic tensor product of theta representations on $\tilde{GL}_{r_i}(\mathbb{A})$.

**Theorem 4.3.** If $\theta \in \Theta_{r_i}$, the constant term

$$m \mapsto \int_{U_{\lambda}(F) \backslash U_{\lambda}(A)} \theta(um) \, du, \quad m \in \tilde{M}_\lambda(\mathbb{A})$$

is in the space $\Theta_{r_1} \tilde{\otimes} \cdots \tilde{\otimes} \Theta_{r_k}$. Indeed,

$$\Theta_{\tilde{M}_\lambda} \simeq \Theta_{r_1} \tilde{\otimes} \cdots \tilde{\otimes} \Theta_{r_k}.$$ 

Here, the global metaplectic tensor product is with respect to the central character $\omega$ of $\Theta_{\tilde{M}_\lambda}$.

**Proof.** Write $\sigma_i = \Theta_{r_i}||H_i$ for $i = 1, \cdots, k$. As explained above, the representation $\sigma_i \tilde{\otimes} \cdots \tilde{\otimes} \sigma_k$ descends to a representation $\sigma$ on $M(F)\tilde{M}^{(n)}(\mathbb{A})$. It suffices to show that

$$\Theta_{\tilde{M}_\lambda} ||_{M(F)\tilde{M}^{(n)}(\mathbb{A})} \hookrightarrow \sigma.$$

Notice the space $\sigma$ contains the metaplectic tensor products with respect to all possible characters $\omega$.

Before we prove this claim we would like to introduce some notations. Let $E(\underline{s}, g)$ be the Eisenstein series on $\tilde{GL}_r(\mathbb{A})$ and let $\Lambda$ be the pole to define the theta function. Let $\Lambda_P \in \mathbb{C}^r$ be the $r$-tuple of complex numbers so that the corresponding $\mu_{\Lambda_P}$ is the modular quasicharacter $\delta_{P_\lambda}$ of $P_\lambda$. Write $\Lambda = (\Lambda_k, \cdots, \Lambda_1)$, where $\Lambda_i \in \mathbb{C}^{r_i}$. Write $\Lambda_P = (\Lambda_{P,1}, \cdots, \Lambda_{P,k})$ such that $\Lambda_{P,i} \in \mathbb{C}^{r_i}$. Notice that all the entries in $\Lambda_{P,i}$ are the same.

Let $f \in \Theta_{\tilde{M}_\lambda} ||_{M(F)\tilde{M}^{(n)}(\mathbb{A})}$. This means that $f$ is the restriction of the residue of an Eisenstein series $E^{\tilde{M}_\lambda}(\underline{s}, g)$ to $M(F)\tilde{M}^{(n)}(\mathbb{A})$. Indeed, if $g \in M(F)\tilde{M}^{(n)}(\mathbb{A})$, then

$$f(g) = \text{Res}_{\underline{s} = w^{M}(\underline{\Lambda}) + \Lambda_P} E(\underline{s}, g)$$

$$= \text{Res}_{\underline{s} = w^{M}(\underline{\Lambda}) + \Lambda_P} \sum_{\gamma \in B_M(F) \backslash M(F)} \phi(\underline{s})(\gamma g)$$

$$= \text{Res}_{\underline{s} = w^{M}(\underline{\Lambda}) + \Lambda_P} \sum_{\gamma \in B_M^{(n)}(F) \backslash M^{(n)}(F)} \phi(\underline{s})(\gamma g).$$

The last equality follows from the following fact: $M^{(n)}(F) \hookrightarrow M(F)$ induces a bijection $B_M^{(n)}(F) \backslash M^{(n)}(F) \leftrightarrow B_M(F) \backslash M(F)$.

By a global analogue of Proposition 3.6, we can view $\text{Ind}_{B_M^{(n)}(\mathbb{A})}^{\tilde{M}_\lambda(\mathbb{A})} \mu_2^{1/2} \delta_M^{1/2}$ as a subspace of $\text{Ind}_{B_M^{(n)}(\mathbb{A})}^{\tilde{M}_\lambda(\mathbb{A})} \mu_2^{1/2} \delta_M^{1/2}$. Without loss of generality, we
may assume that $\phi \in \text{Ind}_{\widetilde{M}^{(n)}(A)}^{\widetilde{\mathcal{G}}^{(n)}(A)} \mu_2 \delta^{1/2}_M \subset \text{Ind}_{\widetilde{B}_i^{(n)}(A)}^{\widetilde{M}^{(n)}(A)} \mu_2 \delta^{1/2}_M$ and furthermore it is decomposable: $\phi = \phi_1 \otimes \cdots \otimes \phi_k$, where $\phi_i \in \text{Ind}_{\widetilde{B}_i^{(n)}(A)}^{\widetilde{G}_i^{(n)}(A)} \mu_2 \delta^{1/2}_{B_i}$. Write $g = \text{diag}(g_1, \ldots, g_k)$ and $\gamma = \text{diag}(\gamma_1, \ldots, \gamma_k)$. Then we can naturally view $f = f_1 \otimes \cdots \otimes f_k$, where $f_i(g_i) = \text{Res}_{\omega = \Lambda_i + \Lambda_{P,i}} \sum_{\gamma_i \in B_i^{(n)}(F)} \phi_{\omega} (\gamma_i g_i)$.

This means that $f_i \in \Theta_{r_i, \Lambda_i + \Lambda_{P,i}}$. We are done.

\[\square\]

4.6. **Global nonvanishing.** Now we prove the global nonvanishing results. Let $\lambda$ be a partition of $r$. Define the Levi subgroup $M = M_\lambda$ as usual. Define the semi-Whittaker functional $\psi_\lambda$ as in the local case. We also write $\psi_\lambda(u) = \psi_1(u_1) \cdots \psi_k(u_k)$ if $u = \text{diag}(u_1, \ldots, u_k) \in U \cap M$.

**Theorem 4.4.** If $r_i \leq n$ for all $i$, then

$$\int_{U(F) \backslash U(A)} \theta(ug) \psi_\lambda(u) \ du$$

is nonzero for some choices of $\theta \in \Theta_r$ and $g \in \widetilde{\text{GL}}_r(A)$.

**Proof.** Notice that

$$\int_{U(F) \backslash U(A)} \theta(ug) \psi_\lambda(u) \ du = \int_{U_M(F) \backslash U_M(A)} \int_{U_{\lambda}(F) \backslash U_{\lambda}(A)} \theta(vug) \ dv \psi_\lambda(u) \ du.$$

By Proposition 4.2, it suffices to show that

$$\int_{U_M(F) \backslash U_M(A)} \theta(ug) \psi_\lambda(u) \ du \neq 0$$

for some choices of $\theta \in \Theta_{\widetilde{M}}$ and $g \in \widetilde{M}(A)$. We now use notations in the proof of Theorem 4.3. Notice that the character $\omega$ in Theorem 4.3 does not contribute anything in this integral. Thus it suffices to show that

$$\int_{U_M(F) \backslash U_M(A)} f(u)(1) \psi_\lambda(u) \ du \neq 0$$

for some $f \in \text{Ind}_{\widetilde{M}^{(n)}(A)}^{\widetilde{M}(A)} \sigma$. Here $f(u)$ is in $\sigma$ and we use $f(u)(1)$ to denote its value at 1. Notice that $U_M(A) \subset M(F) \widetilde{M}^{(n)}(A)$. Thus $f(u)(1) = f(1)(u)$.

Without loss of generality, we can choose $\sigma$ such that $f(1)$ is a simple tensor $f_1 \otimes \cdots \otimes f_k$, where $f_i \in \sigma_i$. Moreover, we can choose $f_i$ such that the Whittaker coefficient of $f_i$ is nonzero, i.e.

$$\int_{U_{\text{GL}_{r_i}(F) \backslash U_{\text{GL}_{r_i}(A)}}} f_i(u_i g_i) \psi_i(u_i) \ du_i \neq 0$$

for all $i$. (This is because when $r_i \leq n$, $\Theta_{r_i}$ is generic.)
Thus,
\[
\int_{U_M(F) \backslash U_M(\mathbb{A})} f(u)(1)\psi_\lambda(u) \, du
= \int_{U_M(F) \backslash U_M(\mathbb{A})} f(1)(u)\psi_\lambda(u) \, du
= \prod_{i=1}^k \int_{U_{GL_{r_i}}(F) \backslash U_{GL_{r_i}}(\mathbb{A})} f_i(u_i)\psi_i(u_i) \, du_i \neq 0.
\]
This proves the theorem. \qed

5. Unipotent Orbits and Fourier Coefficients

For the rest of this paper, we turn to the Fourier coefficients associated with general unipotent orbits. In this section, we explain how to associate a set of Fourier coefficients with a unipotent orbit. A general reference for unipotent orbits is Collingwood-McGovern [8]. (The classification of unipotent orbits for classical groups can be found in [8] Chapter 5.) For the local version of this association see [26, 27]. For global details see Jiang-Liu [20] and Ginzburg [14, 15]. The associated Fourier coefficients are described as integration over certain unipotent subgroups, and the metaplectic cocycles do not contribute to any nontrivial factors. To simplify notations, we only describe this association in the non-metaplectic setup.

We work with the global setup. Let \( F \) be a number field, and \( \mathbb{A} \) be its adele ring. Fix a nontrivial additive character \( \psi: F \backslash \mathbb{A} \to \mathbb{C}^\times \). The unipotent orbits of \( GL_r \) are parameterized by partitions of \( r \). Let \( O = (p_1 \cdots p_k) \) with \( p_1 + \cdots + p_k = r \) be a unipotent orbit. We shall always assume \( p_1 \geq p_2 \geq \cdots \geq p_k > 0 \). To each \( p_i \) we associate the diagonal matrix
\[
\text{diag}(p_i^{-1}, t^{p_i-3}, \ldots, t^{3-p_i}, t^{1-p_i}).
\]
Combining all such diagonal matrices and arranging them in decreasing order of the powers, we obtain a one-dimensional torus \( h_O(t) \). For example, if \( O = (3^21) \), then
\[
h_O(t) = \text{diag}(t^2, t^2, 1, 1, 1, t^{-2}, t^{-2}).
\]

The one-dimensional torus \( h_O(t) \) acts on \( U \) by conjugation. Let \( \alpha \) be a positive root and \( x_\alpha(a) \) be the one-dimensional unipotent subgroup in \( U \) corresponding to the root \( \alpha \). There is a nonnegative integer \( m \) such that
\[
h_O(t)x_\alpha(a)h_O(t)^{-1} = x_\alpha(t^ma). \tag{4}
\]
On the subgroups \( x_\alpha(a) \) which correspond to negative roots \( \alpha \), the torus \( h_O(t) \) acts with non-positive powers.

Given a nonnegative integer \( l \), we denote by \( U_l(O) \) the subgroup of \( U \) generated by all \( x_\alpha(a) \) satisfying the Eq. (4) with \( m \geq l \). We are mainly interested in \( U_l(O) \) where \( l = 2, 3 \).

Let
\[
M(O) = T \cdot \langle x_{\pm\alpha}(a) : h_O(t)x_\alpha(a)h_O(t)^{-1} = x_\alpha(a) \rangle.
\]
The group \( M(O) \) acts by conjugation on the abelian group \( U_2(O)/U_3(O) \). If the ground field is algebraically closed, then under this action of \( M(O) \) on the group \( U_2(O)/U_3(O) \), there is an open orbit. Denote a representative of this orbit by \( u_2 \). It follows from the general theory
that the connected component of the stabilizer of this orbit inside \( M(\mathcal{O}) \) is a reductive group. Denote by \( \text{Stab}_0^\mathcal{O} \) this connected component of the stabilizer of \( u_2 \).

The \( F \)-rational points \( M(\mathcal{O})(F) \) acts on the group of all characters of \( U_2(\mathcal{O})(F) \setminus U_2(\mathcal{O})(\mathbb{A}) \). For each character, its stabilizer is a subgroup of \( M(\mathcal{O})(F) \) as an algebraic group, and hence it is of the form \( C(F) \) for some algebraic group \( C \). If the character is such that the connected component of \( C \) is isomorphic to \( \text{Stab}_0^\mathcal{O} \) over the algebraic closure, it is denoted by \( \psi_{U_2(\mathcal{O})} \).

Notice that the character \( \psi_{U_2(\mathcal{O})} \) is not unique. Given an automorphic function \( \varphi(g) \) on \( \text{GL}_r(\mathbb{A}) \) or its cover, the Fourier coefficients we want to consider are

\[
\int_{U_2(\mathcal{O})(F) \setminus U_2(\mathcal{O})(\mathbb{A})} \varphi(ug)\psi_{U_2(\mathcal{O})}(u) \, du.
\]

In this way, we associate with each unipotent orbit \( \mathcal{O} \) a set of Fourier coefficients. When the partition is \( \mathcal{O} = (r) \), the Fourier coefficients associated to \( \mathcal{O} \) are the Whittaker coefficients.

In order to perform root exchange as in Sections 6.1 and 7.1 below, we also work with a slightly different torus. Let

\[
h'_{\mathcal{O}}(t) = \text{diag}(t^{p_1-1}, t^{p_1-3}, \ldots, t^{1-p_1}, t^{p_2-1}, \ldots, t^{1-p_2}).
\]

Here the first block of size \( p_1 \) is

\[
\text{diag}(t^{p_1-1}, t^{p_1-3}, \ldots, t^{1-p_1}),
\]

while the remaining part \( \text{diag}(t^{p_2-1}, \ldots, t^{1-p_2}) \) is \( h_{(p_2-p_v)}(t) \). For example, if \( \mathcal{O} = (3^21) \), then

\[
h'_{\mathcal{O}}(t) = (t^2, 1, t^{-2}, t^2, 1, 1, t^{-2}).
\]

The tori \( h_{\mathcal{O}}(t) \) and \( h'_{\mathcal{O}}(t) \) are conjugate by an element in the Weyl group of \( \text{GL}_r \). Let \( V_2(\mathcal{O}), \psi_{V_2(\mathcal{O})} \) be the corresponding unipotent subgroup and character, respectively.

Let us recall the partial ordering defined on the set of unipotent orbits. Given \( \mathcal{O}_1 = (p_1 \cdots p_k) \) and \( \mathcal{O}_2 = (q_1 \cdots q_l) \), we say that \( \mathcal{O}_1 \geq \mathcal{O}_2 \) if \( p_1 + \cdots + p_i \geq q_1 + \cdots + q_i \) for all \( 1 \leq i \leq l \). If \( \mathcal{O}_1 \) is not greater than \( \mathcal{O}_2 \) and \( \mathcal{O}_2 \) is not greater than \( \mathcal{O}_1 \), we say that \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are not comparable.

**Definition 5.1.** Let \( \pi \) be an automorphic representation of \( \widetilde{\text{GL}}_r(\mathbb{A}) \). Let \( \mathcal{O}(\pi) \) denote the set of unipotent orbits of \( \text{GL}_r \) defined as follows. A unipotent orbit \( \mathcal{O} \in \mathcal{O}(\pi) \) if \( \pi \) has a nonzero Fourier coefficient which is associated with the unipotent orbit \( \mathcal{O} \), and for all \( \mathcal{O}' > \mathcal{O} \), \( \pi \) has no nonzero Fourier coefficient associated with \( \mathcal{O}' \).

We already describe this association in the global setup. The corresponding local picture could be described analogously. We omit the details.

**Remark 5.2.** It is expected that for any automorphic representation \( \pi \), the set \( \mathcal{O}_G(\pi) \) is a singleton (see [14] Conjecture 5.4). In this paper, the notation \( \mathcal{O}_G(\pi) = \mu \) means that the set \( \mathcal{O}_G(\pi) \) is a singleton, consisting of the orbit \( \mu \) only.

6. Unipotent Orbits: Local Results

We return to the local setup in this section. Fix positive integers \( n, r \) such that \( |n|_F = 1 \). Write \( r = an + b \), where \( a \in \mathbb{Z}_{\geq 0} \) and \( 0 \leq b < n \). Let \( \Theta = \Theta_r \) be an exceptional representation on \( \widetilde{\text{GL}}_r \). The unipotent orbit attached to \( \Theta \) is determined in this section. The key ingredients are the results on the semi-Whittaker functionals. We follow closely the approach given in
Jiang-Liu [20], where they determine the unipotent orbits attached to the residual spectrum of the general linear groups. Here we give a local version with necessary modifications.

**Theorem 6.1.** Let $\mathcal{O} = (p_1 \cdots p_k)$ be a unipotent orbit of $\text{GL}_r$.

(1) If $p_1 > n$, then $J_{U_2(\mathcal{O}), \psi_U(\mathcal{O})}(\Theta) = 0$ (or equivalently, $J_{V_2(\mathcal{O}), \psi_V(\mathcal{O})}(\Theta) = 0$).

(2) If $\mathcal{O} = (n^a b)$, then $J_{U_2(\mathcal{O}), \psi_U(\mathcal{O})}(\Theta) \neq 0$ (or equivalently, $J_{V_2(\mathcal{O}), \psi_V(\mathcal{O})}(\Theta) \neq 0$).

Notice that any unipotent orbit greater than or not comparable with $(n^a b)$ must have $p_1 > a$. Thus we obtain the following result.

**Theorem 6.2.** Let $\Theta$ be an exceptional representation of $\tilde{\text{GL}}_r$. Then $\mathcal{O}(\Theta) = (n^a b)$.

The rest of this section is devoted to proving Theorem 6.1. This theorem is also proved in an unpublished work of Gordan Savin by using the Iwahori-Hecke algebras.

### 6.1. A general lemma.

We start with a general lemma, which is used repeatedly in this section.

Let $G$ be the rational points of a split algebraic group or a cover of such. Let $\mathfrak{u}$ be a maximal nilpotent Lie subalgebra of $\text{Lie}(G)$. Let $\mathfrak{A}, \mathfrak{C}, \mathfrak{X}$ and $\mathfrak{Y}$ be Lie subalgebras of $\mathfrak{u}$, and let $A, C, X, Y$ be the corresponding unipotent subgroups of $G$. Let $\psi_C$ be a nontrivial character of $C$.

We make the following assumptions:

(a) $C, X, Y \subset A$.

(b) $X$ and $Y$ are abelian, normalize $C$ and preserve $\psi_C$.

(c) The commutators $x^{-1}y^{-1}xy$ lie in $C$, for all $x \in X, y \in Y$. In particular, $Y$ normalizes $D = CX$ and $X$ normalizes $B = CY$.

(d) $A = D \rtimes Y = B \rtimes X$.

(e) The set

$$\{ x \mapsto \psi_C(x^{-1}y^{-1}xy) | y \in Y \}$$

is the group of all characters of $X$. Moreover, writing $x = \exp E, y = \exp S$, for $E \in \mathfrak{X}, S \in \mathfrak{Y}$, we have

$$\psi_C(xy^{-1}y^{-1}) = \psi((E, S))$$

where $(, ,)$ is a nondegenerate, bilinear pairing between $\mathfrak{X}$ and $\mathfrak{Y}$.

**Lemma 6.3.** Assume (a)-(e). Let $\pi$ be a smooth representation of $A$. Extend $\psi_C$ trivially to characters $\psi_B$ of $B$ and $\psi_D$ of $D$. Then we have an isomorphism of $C$-modules

$$J_{B, \psi_B}(\pi) \cong J_{D, \psi_D}(\pi).$$

Moreover,

$$J_{C, \psi_C}(\pi) = 0 \iff J_{D, \psi_D}(\pi) = 0 \iff J_{B, \psi_B}(\pi) = 0.$$
Proof. The first isomorphism is proved in Ginzburg-Rallis-Soudry [18] Section 2.2. We now prove the second statement. By symmetry, it suffices to prove
\[ J_{C,\psi_C}(\pi) = 0 \iff J_{D,\psi_D}(\pi) = 0. \]
Clearly if \( J_{C,\psi_C}(\pi) = 0 \), then
\[ J_{D,\psi_D}(\pi) = J_X(J_{C,\psi_C}(\pi)) = 0. \]
Conversely, suppose \( J_{D,\psi_D}(\pi) = 0 \). There is a natural map
\[ T : J_{C,\psi_C}(\pi) \to J_{D,\psi_D}(\pi) = 0 \]
over \( D \). This induces a map of \( A \)-modules
\[ i : J_{C,\psi_C}(\pi) \to \text{Ind}_D^A(J_{D,\psi_D}(\pi)) = 0. \]
It is shown in [18] Section 2.2 that \( i \) is injective. Thus \( J_{C,\psi_C}(\pi) = 0. \) \( \square \)

When \( X \) and \( Y \) are root subgroups, the above lemma is the local version of the root exchange in Friedberg-Ginzburg [10] Section 2.2 and Ginzburg [16] Section 2.2.2. This is always the case in our application. The above assumptions can always be verified by the Steinberg relations.

6.2. Root exchange. Given a unipotent orbit \( O = (p_1 \cdots p_k) \), we define several unipotent subgroups of \( U \). Let \( U_O \) be the subgroup of \( U \) consisting elements of the form
\[ u = \begin{pmatrix} u_1 & n_1 \\ u_2 \end{pmatrix} \]
where \( u_1 \) is a unipotent matrix in \( \text{GL}_{p_1} \), \( n_1 \in \text{Mat}_{p_1 \times (n - p_1)} \) with the last row being zero, and \( u_2 \in U_2((p_2 \cdots p_k)) \subset \text{GL}_{r-p_1} \). We define a character \( \psi_{U_O} : U_O \to \mathbb{C}^\times \) as the product of the Whittaker character on \( u_1 \) and \( \psi_{U_2((p_2 \cdots p_k))} \) on \( u_2 \). We also define a unipotent subgroup \( U'_O \) of \( U_O \) by removing all the root subgroups \( U_\alpha \) in the \( n_1 \) part for all \( \alpha \) such that
\[ h'_O(t)x_\alpha(a)h'_O(t)^{-1} = x_\alpha(ta). \] (5)
Let \( \psi_{U'_O} \) be the restriction of \( \psi_{U_O} \) to \( U'_O \).

Remark 6.4. If \( p_i \)'s have the same parity, then \( U_O = U'_O \).

Lemma 6.5. Let \( \pi \) be a smooth representation of \( \widetilde{\text{GL}}_r \).

(1) \[ J_{V_2(O),\psi_{V_2(O)}}(\pi) \cong J_{U'_O,\psi_{U'_O}}(\pi). \]

(2) \[ J_{V_2(O),\psi_{V_2(O)}}(\pi) = 0 \]
if and only if \[ J_{U_O,\psi_{U_O}}(\pi) = 0. \]

(3) If \( p_i \)'s have the same parity, then \[ J_{V_2(O),\psi_{V_2(O)}}(\pi) \cong J_{U_O,\psi_{U_O}}(\pi). \]
Proof. Part (3) is clear from part (1) and Remark 6.4. We first prove part (1). The strategy is to use the root exchange lemma. Notice that any element of $V_2(\mathcal{O})$ has the following form:

$$u = \begin{pmatrix} u_1 & q \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} I_{p_1} & 0 \\ p & I_{n-p_1} \end{pmatrix},$$

where $u_1 \in \text{GL}_{p_1}$ and $u_2 \in U(p_2 \cdots p_k) \subset \text{GL}_{n-p_1}$ are unipotent matrices, and $p \in \text{Mat}_{p_1 \times (n-p_1)}$ and $q \in \text{Mat}_{(n-p_1) \times p_1}$ are certain matrices to be described later. The character $\psi_{V_2(\mathcal{O})}$ is the product of Whittaker character on $u_1$ and $\psi_{U_2(p_2 \cdots p_k)}$. We use the simple roots in $u_1$ to move root subgroups contained in $p$ to $q$. The desired twisted Jacquet module is obtained after we finish this process.

Let us give more details in the case $\mathcal{O} = (p_1p_2)$. The general case follows by the same argument. There are two cases to consider, depending on the parity of $p_1 - p_2$.

Case 1: $p_1 - p_2$ is even. Notice that in this case part (1) implies part (2) immediately.

We can write $u \in V_2(\mathcal{O})$ as

$$u = \begin{pmatrix} u_1 & n_1 \\ n_2 & u_2 \end{pmatrix}.$$  

Here $u_1 \in \text{GL}_{p_1}, u_2 \in \text{GL}_{p_2}$ are unipotent matrices, and

$$n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \in \text{Mat}_{p_1 \times p_2},$$

where

$$a_1 \in \text{Mat}_{(\frac{p_1+p_2}{2}) \times p_2}, \quad b_1 \in \text{Mat}_{p_2 \times p_2} \text{ is nilpotent}, \quad c_1 = 0 \in \text{Mat}_{(\frac{p_1+p_2}{2}) \times p_2};$$

and

$$n_2 = \begin{pmatrix} 0 & b_2 & c_2 \end{pmatrix} \in \text{Mat}_{p_2 \times p_1},$$

where

$$0 \in \text{Mat}_{p_2 \times (\frac{p_1+p_2}{2}+1)}, \quad b_2 \in \text{Mat}_{p_2 \times p_2} \text{ is upper triangular}, \quad c_2 \in \text{Mat}_{p_2 \times (\frac{p_1+p_2}{2}-1)}.$$

Now we apply the root exchange lemma. For the first column of $b_2$, the only nonzero entry is the root subgroup corresponding to the (negative) root $(p_1 + 1, \frac{p_1+p_2}{2} + 2)$. We now describe the groups $A, B, C, D, X, Y$ in Section 6.1 in our current setting. Let $A = V_2(\mathcal{O})$. Let $C, X$ and $Y$ be the root subgroups corresponding to the roots $(\frac{p_1+p_2}{2} + 1, \frac{p_1+p_2}{2} + 2), (p_1 + 1, \frac{p_1+p_2}{2} + 2)$ and $(\frac{p_1+p_2}{2} + 1, p_1 + 1)$, respectively. This determines the groups $B$ and $D$. Notice that the character $\psi_{V_2(\mathcal{O})}$ is nontrivial on $C$. After applying Lemma 6.3, we replace the root $(p_1 + 1, \frac{p_1+p_2}{2} + 2)$ by the (positive) root $(\frac{p_1+p_2}{2} + 1, p_1 + 1)$ in the twisted Jacquet module.

Similarly, the $i$th column of $b_2$ has $i$ nonzero entries corresponding to the roots

$$\left( j, \frac{p_1 - p_2}{2} + i + 1 \right), \quad j = p_1 + 1, \ldots, p_1 + i.$$

We let $X$ be the group generated by the root subgroups corresponding to these roots. Let $C$ be the root subgroup corresponding to the root $(\frac{p_1+p_2}{2} + i, \frac{p_1+p_2}{2} + i + 1)$. Let $Y$ be the group generated by the root subgroups corresponding to the roots

$$\left( \frac{p_1 - p_2}{2} + i, j \right), \quad j = p_1 + 1, \ldots, p_1 + i.$$
These are exactly the missing entries in the $i$th row in $b_1$. By applying Lemma 6.3, we replace $X$ by $Y$ in the twisted Jacquet module. The $c_2$ part can be handled similarly. Indeed, using the simple roots in $u_1$, the entries in $c_2$ are moved to the first $(p_1 - p_2 - 1)$ rows of $c_1$. Thus, in this case, we have shown that

$$J_{V_2(\mathcal{O})\psi_{V_2(\mathcal{O})}}(\pi) \cong J_{U_0',\psi_{U_0'}}(\pi).$$

Case 2: $p_1 - p_2$ is odd. The proof of part (1) is the same as Case 1, with minor differences. Indeed, $u \in V_2(\mathcal{O})$ can be written as

$$u = \begin{pmatrix} u_1 & n_1 \\ n_2 & u_2 \end{pmatrix}.$$ 

Here $u_1 \in \text{GL}_{p_1}, u_2 \in \text{GL}_{p_2}$ are unipotent matrices, and

$$n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \in \text{Mat}_{p_1 \times p_2},$$

where

$$a_1 \in \text{Mat}_{\left(\frac{p_1 - p_2 - 1}{2}\right) \times p_2}, b_1 \in \text{Mat}_{p_2 \times p_2} \text{ is nilpotent}, c_1 = 0 \in \text{Mat}_{\left(\frac{p_1 - p_2 + 1}{2}\right) \times p_2},$$

and

$$u_2 = \begin{pmatrix} 0 & b_2 & c_2 \end{pmatrix} \in \text{Mat}_{p_2 \times p_1}$$

where

$$0 \in \text{Mat}_{p_2 \times \left(\frac{p_1 - p_2 + 1}{2}\right)}, b_2 \in \text{Mat}_{p_2 \times p_2} \text{ is nilpotent}, c_2 \in \text{Mat}_{p_2 \times \left(\frac{p_1 - p_2 - 1}{2}\right)}.$$

There is no element in the first column of $b_2$, and the first entry of $b_1$ is missing. For the second column of $b_2$, the only nontrivial entry corresponds to the root $(p_1 + 1, \frac{p_1 - p_2 + 1}{2} + 2)$. Let $X$ be the root subgroup corresponding to this root. We now let $C$ and $Y$ be the root subgroups corresponding to the roots $(\frac{p_1 - p_2 + 1}{2} + 1, \frac{p_1 - p_2 + 1}{2} + 2)$ and $(\frac{p_1 - p_2 + 1}{2} + 1, p_1 + 1)$. By applying Lemma 6.3, we can replace $X$ by $Y$ in the twisted Jacquet module. The group $Y$ gives the first entry of the second row of $b_1$. Now we only miss the second entry in the second row of $b_1$.

Similarly, the $(i + 1)$th column of $b_2$ has $i$ entries corresponding to the roots

$$(j, \frac{p_1 - p_2 + 1}{2} + i + 1), \quad j = p_1 + 1, \ldots, p_1 + i.$$ 

Let $X$ be the group generated by the root subgroups corresponding to these roots. Let $C$ be the root subgroup corresponding to the root $(\frac{p_1 - p_2 + 1}{2} + i, \frac{p_1 - p_2 + 1}{2} + i + 1)$. Let $Y$ be the group generated by the root subgroups corresponding to the roots

$$(\frac{p_1 - p_2 + 1}{2} + i, j), \quad j = p_1 + 1, \ldots, p_1 + i.$$ 

By Lemma 6.3, $X$ can be replaced by $Y$ in the twisted Jacquet module. Thus, after this process, we only miss the $(i + 1)$th entry in the $(i + 1)$th row in $b_1$. The $c_2$ part can be handled similarly, and the entries in $c_2$ are moved to the first $(\frac{p_1 - p_2 + 1}{2} - 1)$ rows of $c_1$. The missing entries in $b_1$ are the diagonal entries, which are exactly the root subgroups that are removed in the definition of $U_0'$; see Eq. (5). This finishes the proof of part (1).
For part (2), let \( Y \) be the subgroup of \( V_2(\mathcal{O}) \) such that \( u_1 = I, u_2 = I, a_2, a_1, c_1 = 0, \) and \( b_1 \) is diagonal. Then we can verify that \( Y \) normalizes \( U'_{\mathcal{O}} \) and preserves \( \psi_{U'_{\mathcal{O}}} \). Moreover, \( U'_{\mathcal{O}}Y = U_{\mathcal{O}} \). By Lemma 6.3,

\[
J_{U_{\mathcal{O}}, \psi_{\mathcal{O}}} (\pi) = 0
\]

if and only if

\[
J_{U'_{\mathcal{O}}, \psi'_{\mathcal{O}}} (\pi) = 0
\]

if and only if

\[
J_{V_2(\mathcal{O}), \psi'_{V_2(\mathcal{O})}} (\pi) = 0.
\]

For the general case, we need to proceed inductively. Notice that if we perform root exchange on the \( p_3 \) part using \( p_1 \), what is done in the previous steps is unchanged. Therefore, the lemma is true for a general unipotent orbit \( \mathcal{O} \). \( \square \)

6.3. \textbf{Vanishing results.} Now we prove the vanishing property of the twisted Jacquet modules of \( \Theta \) attached to the unipotent orbits either greater than or not comparable with \((n^{2}b)\).

Let \( V_{1m-1,r-m+1} \) be the unipotent radical of the parabolic subgroup \( P_{1m-1,r-m+1} \) with Levi part \( GL_1 \times (m-1) \times GL_{r-m+1} \). Let

\[
\psi_{m-1}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m}),
\]

and

\[
\tilde{\psi}_{m-1}(v) = \psi(v_{1,2} + \cdots + v_{m-2,m-1})
\]

be two characters of \( V_{1m-1,r-m+1} \). Notice that \( (V_{1m-1,r-m+1}, \psi_{m-1}) \) is the same as \( (U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}}) \) where \( \mathcal{O} = (m^{1-r-m}) \).

We consider slightly more general characters. Let \( m' \geq m \), and

\[
\xi = (\epsilon_m, \epsilon_{m+1}, \cdots, \epsilon_{m'-1}) \in F^{m'-m}.
\]

Let

\[
\psi_{m-1}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m} + \epsilon_m v_{m,m+1} + \cdots + \epsilon_{m'-1} v_{m'-1,m'})
\]

be a character of \( V_{1m'-1,r-m'+1} \).

**Lemma 6.6.** If \( m > n \), then

\[
J_{V_{1m'-1,r-m'+1}, \psi_{m-1,\xi}} (\Theta) = 0.
\]

In particular,

\[
J_{V_{1m-1,r-m+1}, \psi_{m-1}} (\Theta) = 0.
\]

**Proof.** We prove this by induction on \( r - m' \). When \( r = m' \geq m \), the pair \( (V_{1r-1,1}, \psi_{r-1,\xi}) \) can only be \((U, \psi_{\lambda})\) where \( \lambda \) is a partition of the form \((m'' \cdot \cdots \cdot)\) with some \( m'' \geq m \). The result follows Corollary 3.34 since \( m'' \geq m > n \).

Now assume the result is true for \( m' \) and we prove it for \( m' - 1 \) if \( m' - 1 \geq m \). Define \( R_{m'-1} \) to be the subgroup of \( U \) such that any element \( u = (u_{jl}) \in R_{m'-1}, u_{j,j} = 0 \), unless \( j = m' - 1 \). The group \( R_{m'-1} \) acts on \( V_{1m'-2,r-m'+2} \). For any character \( \xi \) of \( R_{m'-1} \),

\[
J_{R_{m'-1}, \xi} (J_{V_{1m'-2,r-m'+2}, \psi_{m'-2,\xi}} (\Theta)) = 0
\]

by induction. This implies

\[
J_{V_{1m'-2,r-m'+2}, \psi_{m'-2,\xi}} (\Theta) = 0.
\]

This finishes the proof. \( \square \)
Lemma 6.7. 

\[ J_{V_{n-1},r-n+1,\psi_{n-1}}(\Theta) \cong J_{V_{n},r-n,\psi_{n}}(\Theta). \]

Proof. The group \( R_n \) acts on \( V_{n-1},r-n+1 \). For any nontrivial character \( \xi \) of \( R_n \),

\[ J_{R_n,\xi}(J_{V_{n-1},r-n+1,\psi_{n-1}}(\Theta)) = 0 \]

by Lemma 6.6. Therefore, the action of \( R_n \) on \( J_{V_{n-1},r-n+1,\psi_{n-1}}(\Theta) \) is trivial, and

\[ J_{V_{n-1},r-n+1,\psi_{n-1}}(\Theta) \cong J_{V_{n},r-n,\psi_{n}}(\Theta). \]

\[ \square \]

Now we are ready to prove Theorem 6.1 part (1). Indeed, since \( p_1 > n \),

\[ J_{U(\mathcal{O}),\psi_{\mathcal{O}}}(\Theta) = J_{\mathcal{O},\psi_{\mathcal{O}}}(\Theta) = 0. \]

Here \( \ast \) is some unipotent subgroup of \( V_{2}(\mathcal{O}) \). By Lemma 6.5, this implies

\[ J_{V_{2}(\mathcal{O}),\psi_{V_{2}(\mathcal{O})}}(\Theta) = 0. \]

6.4. Nonvanishing results. In this subsection, \( \mathcal{O} = (n^{a}b) \). For \( 1 \leq i \leq a \), consider \( V_{1n-1},r-in+1 \) and its characters attached to the partitions \( (ni) \) and \( (n^{i-1}(n+1)) \), respectively. We can prove the following lemma by using the same arguments as in Lemma 6.6 and 6.7.

Lemma 6.8.

1. \( J_{V_{1}(n),r-in,\psi_{(n^{i-1}(n+1))}}(\Theta) = 0. \)

2. \( J_{V_{1}(n),r-in+1,\psi_{(n^{i})}}(\Theta) \cong J_{V_{1}(n),r-in,\psi_{(n^{i})}}(\Theta). \)

Now we prove the following nonvanishing result (Theorem 6.1 part (2)).

Proposition 6.9. \( J_{V_{2}(\mathcal{O}),\psi_{V_{2}(\mathcal{O})}}(\Theta) \neq 0 \).

Proof. It suffices to show that \( J_{U(\mathcal{O}),\psi_{V_{2}(\mathcal{O})}}(\Theta) \neq 0 \). Indeed,

\[ J_{U(\mathcal{O}),\psi_{V_{2}(\mathcal{O})}}(\Theta) = J_{U_{2}((n^{a-1}b)),\psi_{V_{2}((n^{a-1}b))}}(J_{V_{1}(n),r-in+1,\psi_{(n)}}(\Theta)) \]

\[ \cong J_{U_{2}((n^{a-1}b)),\psi_{V_{2}((n^{a-1}b))}}(J_{V_{1}(n),r-in,\psi_{(n)}}(\Theta)). \]

Here, \( U_{2}((n^{a-1}b)) \) is viewed as a subgroup of \( U \) via the embedding \( u \mapsto I_{n},u \). Now we apply root exchange in \( U_{2}((n^{a-1}b)) \). The root exchange does not change anything we did in the previous step. Thus,

\[ J_{U_{2}((n^{a-1}b)),\psi_{V_{2}((n^{a-1}b))}}(J_{V_{1}(n),r-in,\psi_{(n)}}(\Theta)) \neq 0 \]

if and only if

\[ J_{V_{1}(n),r-in,\psi_{(n)}}(\Theta) \neq 0. \]

Here, \( U((n^{a-1}b)) \) is again viewed as a subgroup of \( U \) via the same embedding. By Lemma 6.8,

\[ J_{V_{1}(n),r-in,\psi_{(n)}}(\Theta) = J_{U_{2}((n^{a-2}b)),\psi_{V_{2}((n^{a-2}b))}}(J_{V_{1}(n),r-in+1,\psi_{(n^{2})}}(\Theta)) \]

\[ \cong J_{U_{2}((n^{a-2}b)),\psi_{V_{2}((n^{a-2}b))}}(J_{V_{1}(n),r-in+1,\psi_{(n^{2})}}(\Theta)). \]

Here, \( U_{2}((n^{a-2}b)) \) is viewed as a subgroup of \( U \) via \( u \mapsto I_{2n},u \).
Now we repeat this process inductively. This implies that \( J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \neq 0 \) if and only if

\[
0 \neq J_{U_2(\mathcal{O}),\psi_{U_2(\mathcal{O})}}(J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta)) = J_{U,\psi_{(n^\alpha b)}}(\Theta).
\]

The result follows from the nonvanishing results of the semi-Whittaker functionals (Corollary 3.36).

Now suppose that \( n, b \) have the same parity. By Lemma 6.5 part (3), in all the steps of the above proof, we actually obtain isomorphisms of twisted Jacquet modules, instead of “if and only if” statements. This proves the following result.

**Proposition 6.10.** When \( n \) and \( b \) have the same parity,

\[
J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) \cong J_{U,\psi_\lambda}(\Theta),
\]

where \( \lambda \) is the partition \( (n^\alpha b) \).

When \( r \) is a multiple of \( n \), combining with Theorem 3.44, we obtain the following uniqueness result.

**Theorem 6.11.** When \( r = mn \) and \( \mathcal{O} = (n^m) \),

\[
\dim J_{U_2(\mathcal{O}),\psi_{U_2(\mathcal{O})}}(\Theta) = \dim J_{V_2(\mathcal{O}),\psi_{V_2(\mathcal{O})}}(\Theta) = 1.
\]

**Remark 6.12.** We already proved the local results at good primes. At bad primes, these statements would be valid once we have the corresponding vanishing results of semi-Whittaker functionals.

This new unique functional in Theorem 6.11 is valuable and it already finds applications in Rankin-Selberg integrals for covering groups. The first – doubling constructions for covering groups – will be discussed in Section 7.5. This unique functional also plays a role in a new-way integral (Euler products with non-unique models) for covering groups; see Ginzburg [17].

7. UNIPOTENT ORBITS: GLOBAL RESULTS

We are back to the global situation in this section. Let \( \Theta_r \) be the global theta representation on \( \widehat{\text{GL}_r}(\mathbb{A}) \), defined in Section 4.1. Let \( (n^\alpha b) \) be the unipotent orbit of \( \text{GL}_r \) as in Section 6. We determine \( \mathcal{O}(\Theta_r) \) in this section.

7.1. Root exchange lemma: global version. The following global root exchange lemma is proved in [20] Lemma 5.2; see also Ginzburg-Rallis-Soudry [19] Section 7.1. This is the global version of Lemma 6.3.

Let \( C \) be an \( F \)-subgroup of a maximal unipotent subgroup of \( \text{GL}_r \), and let \( \psi_C \) be a nontrivial character of \( C(F) \setminus C(\mathbb{A}) \). Let \( X, Y \) be two unipotent \( F \)-subgroups satisfying the following conditions:

(a) \( X \) and \( Y \) are abelian and normalize \( C \);
(b) \( X(\mathbb{A}) \) and \( Y(\mathbb{A}) \) preserve \( \psi_C \);
(c) \( X \cap C \) and \( Y \cap C \) are normal in \( X \) and \( Y \), respectively;
(d) \( \psi_C \) is trivial on \( (X \cap C)(\mathbb{A}) \) and \( (Y \cap C)(\mathbb{A}) \);
(e) \([X,Y] \subset C\).
(f) there is a nondegenerate pairing $\langle X \cap C \rangle(\mathbb{A}) \times \langle Y \cap C \rangle(\mathbb{A}) \to \mathbb{C}^\times$, given by $(x,y) \mapsto \psi_C([x,y])$, which is multiplicative in each coordinate, and identifies $(Y \cap C)(F) \setminus Y(F)$ with the dual of $X(F)(\mathbb{A}) \setminus X(\mathbb{A})$, and $(X \cap C)(F) \setminus X(F)$ with the dual of $Y(F)(\mathbb{A}) \setminus Y(\mathbb{A})$.

Let $B = CY$ and $D = CX$, and extend $\psi_C$ trivially to characters of $B(F) \setminus B(\mathbb{A})$ and $D(F) \setminus D(\mathbb{A})$, which are denoted by $\psi_B$ and $\psi_D$, respectively.

**Lemma 7.1.** Assume the quadruple $(C, \psi_C, X, Y)$ satisfies the above conditions. Let $f$ be an automorphic form on $\widetilde{\text{GL}}_r(\mathbb{A})$. Then

$$\int_{C(F) \setminus C(\mathbb{A})} f(cg) \psi_C^{-1}(c) \, dc \equiv 0, \quad \forall g \in \widetilde{\text{GL}}_r(\mathbb{A}),$$

if and only if

$$\int_{B(F) \setminus B(\mathbb{A})} f(ug) \psi_B^{-1}(u) \, du \equiv 0, \quad \forall g \in \widetilde{\text{GL}}_r(\mathbb{A}),$$

if and only if

$$\int_{D(F) \setminus D(\mathbb{A})} f(ug) \psi_D^{-1}(u) \, du \equiv 0, \quad \forall g \in \widetilde{\text{GL}}_r(\mathbb{A}).$$

7.2. Vanishing results.

**Proposition 7.2.** Let $\theta$ be in the space of $\Theta_r$. Let $O$ be a unipotent orbit which is greater than or not comparable to $(n^a b)$. Then the integral

$$\int_{U_2(O)(F) \setminus U_2(O)(\mathbb{A})} \theta(ug) \psi_{U_2(O)}(u) \, du$$

is zero for all choices of data.

**Proof.** As in the case of semi-Whittaker coefficients, the global vanishing result follows from the local vanishing result. If

$$\int_{U_2(O)(F) \setminus U_2(O)(\mathbb{A})} \theta(ug) \psi_{U_2(O)}(u) \, du$$

is nonzero for some choice of data, then the functional $l : \Theta_r \to \mathbb{C}$ defined by

$$\theta \mapsto \int_{U_2(O)(F) \setminus U_2(O)(\mathbb{A})} \theta(ug) \psi_{U_2(O)}(u) \, du$$

is nonzero. As explained in the proof of Proposition 4.1, we can choose a factorizable vector $\otimes_v' \theta_{0,v}$ such that $l(\otimes_v' \theta_{0,v}) \neq 0$.

Let $w$ be a non-Archimedean place of $F$ such that $|n|_w = 1$ and $\Theta_r$ is unramified at $w$. Define a local functional $l_w : \Theta_{r,w} \to \mathbb{C}$ by

$$\theta_w \mapsto l(\theta_w \otimes (\otimes_v' \theta_{0,v})).$$
By our construction, $l_w$ is nonzero. Now the functional $l_w$ factors through the twisted Jacquet module of $\Theta_{r,w}$ for the character $\psi_{U_2(\mathcal{O})}(u)$ on the group $U_2(\mathcal{O})(F_w)$. This implies that $J_{U_2(\mathcal{O})(F_w),\psi_{U_2(\mathcal{O})}}(\Theta_{r,w}) \neq 0$. This contradicts the local result.  

7.3. Nonvanishing results.

**Proposition 7.3.** Let $\theta$ be in the space of $\Theta_r$. Let $\mathcal{O} = (n^a b)$. Then the integral

$$\int_{U_2(\mathcal{O})(F) \backslash U_2(\mathcal{O})(A)} \theta(ug)\psi_{U_2(\mathcal{O})}(u) \, du$$

is nonzero for some choice of data.

**Proof.** The proof is analogous to the local case. Once we have the global root exchange lemma and global vanishing results, the nonvanishing results follow from the corresponding nonvanishing results on the semi-Whittaker coefficients. Notice that the global version of Lemma 6.6, 6.7 and 6.8 can be established by using the corresponding local results. We omit the details. We remark that a similar argument can be found in [20] Section 5.  

7.4. Unipotent orbits attached to Theta representations. Finally we determine the unipotent orbit attached to $\Theta_r$.

**Theorem 7.4.** The unipotent orbit attached to $\Theta_r$ is $(n^a b)$. In other words, $\mathcal{O}(\Theta_r) = (n^a b)$.

**Proof.** By Definition 5.1, this follows from Propositions 7.2 and 7.3.  

7.5. Whittaker-Speh-Shalika representations. In the research announcement [6], the famous doubling method is extended to the covering groups. A family of automorphic representations of $\widetilde{\text{GL}}_r(\mathbb{A})$ are introduced as the induction data of Eisenstein series on covers of suitable split classical groups. We recall the definition here.

**Definition 7.5.** An irreducible genuine automorphic representation $\pi$ of $\widetilde{\text{GL}}_{ab}(\mathbb{A})$ is a Whittaker-Speh-Shalika representation of type $(a, b)$ if:

1. $\mathcal{O}(\pi) = (a^b)$.
2. For a finite place $v$, let $\pi_v$ denote the irreducible constituent of $\pi$ at $v$. Suppose that $\pi_v$ is an unramified representation. Then $\mathcal{O}(\pi_v) = (a^b)$. (That is, the local analogue of part (1) holds.) Moreover,

$$\dim \text{Hom}_{U_2((a^b))(F_v)}(\pi_v, \psi_{U_2((a^b))}) = 1.$$

Thus, when $r$ is a multiple of $n$, we can rephrase Theorem 6.2, 6.11 and 7.4 as follows.

**Theorem 7.6.** When $r = mn$, the representation $\Theta_r$ is a Whittaker-Speh-Shalika representation of type $(n, m)$.

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