Simulation of vortex laser beams propagation in parabolic index media based on fractional Fourier transform

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Abstract. We use radial Fractional Fourier transform to model vortex laser beams propagation in optical waveguides with parabolic dependence of the refractive index. To overcome calculation difficulties at distances proportional to a quarter of the period we use varied calculation step. Numerical results for vortex modes superposition propagation in a parabolic optical fiber show that the transverse beam structure can be changed significantly during the propagation. To provide stable transverse distribution input scale modes should be in accordance with fiber parameters.

1. Introduction
Beams with helical phase singularity are very perspective in telecommunication applications as new information channels as mode multiplexing [1, 2]. They can be used for nano-imaging [3]. We use radial Fractional Fourier transform (FrFT) to model vortex laser beams propagation in optical waveguides with parabolic dependence of the refractive index [4]. The canonical FrFT was considered [5] as the Fourier transform of \(p\)-order, where \(p\) is the real value. Currently the FrFT is actively used in optical image processing [6] and can be optically implemented by means of several spherical and/or cylindrical lenses [7-9]. Note, cylindrical lenses are used for astigmatic transformation in order to form vortex beams [9-14]. For real fibers a bounded FrFT should be used [15, 16] and problems of beams propagation are solved numerically. Application of FrFT for laser beam propagation in gradient index media has calculation problems at distances proportional to a quarter of the period [17, 18]. We have overcome these difficulties using the varied calculation step. Several examples of vortex beams propagation are demonstrated.

2. Theoretical background
In a gradient parabolic fiber the refractive index is given by \(n^2(r) = n_0^2(1 - \alpha^2 r^2)\), where \(n_0\) is the refractive index on the fiber’s optical axis and \(\alpha\) is a constant that defines the curvature of the refractive index profile. Based on the paraxial approximation for the parabolic medium \(\alpha r \ll 1\) we can get the integral propagation operator:
\begin{equation}
E(r, \varphi, z) = -\frac{ik\alpha}{2\pi \sin(\alpha z)} \exp(ikz) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} r^2 \right] \times \int_0^{2\pi} \int_0^\infty E_n(\rho, \theta) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} \rho^2 \right] \exp \left[ -\frac{ik\alpha}{\sin(\alpha z)} \rho r \cos(\varphi - \theta) \right] \rho d\rho d\theta,
\end{equation}

which is FrFT in polar system with order \( p = \alpha z \); \( k = 2\pi/\lambda \) is the wavenumber, \( \lambda \) is the wavelength of an illuminating light.

Consider a laser beam with a singular component:

\begin{equation}
E_0(\rho, \theta) = E_m(\rho) \exp(i m \theta),
\end{equation}

where \( E_m(\rho) \) is the radial component of the beam, \( \exp(i m \theta) \) is the optical vortex of order \( m \).

Substituting the expression (2) into the formula (1), we obtain:

\begin{equation}
E(r, \varphi, z) = -\frac{ik\alpha}{2\pi \sin(\alpha z)} \exp(ikz) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} r^2 \right] \exp(i m \varphi) \times \int_0^{2\pi} \int_0^\infty E_n(\rho, \theta) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} \rho^2 \right] \exp \left[ -\frac{ik\alpha}{\sin(\alpha z)} \rho r \cos(\varphi - \theta) \right] \rho d\rho d\theta.
\end{equation}

In order to calculate the integral over \( \theta \), we apply this relation:

\begin{equation}
J_m(x) = \frac{i^m}{2\pi} \int_0^{2\pi} \exp(i m t) \exp(-ix \cos t) dt,
\end{equation}

where \( J_m(x) \) is Bessel function of order \( m \).

Using the expression (4) instead of equation (3), we obtain:

\begin{equation}
E_n(r, \varphi, z) = \text{FrFT} \left[ E_n(\rho) \exp(i m \theta) \right] = -\frac{ik\alpha}{\sin(\alpha z)} \exp(ikz) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} r^2 \right] \times i^{-m} \exp(i m \varphi) \int_0^{2\pi} \int_0^\infty E_n(\rho) \exp \left[ \frac{ik\alpha}{2 \tan(\alpha z)} \rho^2 \right] J_m \left[ \frac{kr \rho}{\sin(\alpha z)} \right] \rho d\rho.
\end{equation}

We can make use formula (5) as the sum to modulate the propagation of the superposition of vortex functions. For example, let the superposition is situated at the entrance:

\begin{equation}
E_n(\rho, \theta) = \sum_q C_{nq} E_n(\rho) \exp(i m_q \theta),
\end{equation}

where \( C_{nq} \) are the superposition coefficients of functions as in equation (2). Then the result of the transformation is easily obtained from the relation:

\begin{equation}
E(r, \varphi, z) = \sum_q C_{nq} \text{FrFT} \left[ E_n(\rho) \exp(i m_q \theta) \right].
\end{equation}
3. Numerical simulations

The following parameters were used in the calculations: radius of the parabolic fiber is $R=10\lambda$, parameter $\alpha = 0.0628\lambda^{-1}$. Laguerre-Gaussian modes (LG) [19] were considered in the calculations:

$$GL_m(n,\phi,z) = \frac{\sigma_0}{\sigma(z)} \left[ \frac{2n!}{\pi (n+|m|)!} \right] \exp\left[ i(2n+m+1)\eta(z) \right] \exp\left[ ikz + ikr^2 \frac{2}{2R(z)} \right] \times$$

$$\times \exp\left[ -r^2 \frac{\sigma_0}{\sigma(z)} \left[ \sqrt{2} r \frac{\sigma_0}{\sigma(z)} \right] \frac{n}{|m|} \frac{2r^2}{\sigma(z)^2} \right] \exp(im\phi),$$

where $L_n^m(x) = \frac{1}{n!} \frac{d^n}{dx^n} \left( e^{-x^2} x^m \right)$ is the generalized Laguerre polynomial [20], $k=2\pi/\lambda$ is the light wavenumber for the wavelength $\lambda$, $\sigma(z) = \sigma_0 \left[ 1 + (z/z_0)^2 \right]^{1/2}$ is the beam effective radius, $\sigma_0$ is the waist radius, $z_0 = k\sigma_0^2/2$ is the confocal parameter. $R(z) = z \left[ 1 + (z/z_0)^2 \right]$ is the radius of curvature of the parabolic front of light field, $\eta(z) = \arctan(z/z_0)$ is Gouy phase, $\phi = \arctan(y/x)$.

By using the diffraction optical elements [21-22], we can effectively generate separate LG modes (8) and their superposition (6) with given complex coefficients.

The propagation of LG modes with indexes $(n,m)=(3,2)$ and $(n,m)=(0,5)$, respectively, in a parabolic fiber at the distance of $z \in [0,50\lambda]$ is shown in figures 1 and 2. The modes sizes were matched with the fiber parameters to avoid large-scale changes during the propagation. For this purpose the waist radius (8) was chosen as $\sigma = 2\lambda$. If we increase or decrease the waist radius, the transverse distribution will change its scale during propagation as shown in figure 3.

Although the idea of mode multiplexing in the fiber communication channels has been known for a long time [23-24], practical interest in this subject has been renewed lately due to vortex laser beams [1-3], possessing orbital angular momentum. «Single-ring» LG modes with a zero radial index $n=0$ and different azimuthal indexes $m$ (figures 2, 3) are of particular interest.

**Figure 1.** The propagation of LG modes with indexes $(n,m)=(3,2)$ in a parabolic fiber, $\sigma = 2\lambda$ : the distribution at the entrance (a) the intensity, (b) the phase and (c) the longitudinal intensity distribution $z \in [0,50\lambda]$.

**Figure 2.** The propagation of LG modes with indexes $(n,m)=(0,5)$ in a parabolic fiber, $\sigma = 2\lambda$ : the distribution at the entrance (a) the intensity, (b) the phase and (c) the longitudinal intensity distribution $z \in [0,50\lambda]$.
Figure 3. The propagation of LG modes with indexes \((n,m)=(0,5)\) is the longitudinal intensity distribution where (a) \(\sigma = 1.5\lambda\) and (b) \(\sigma = 2.5\lambda\).

The propagation of the «single-ring» LG modes superposition with different vortex of order \(m\) and the waist radius \(\sigma\) in a single physical carrier (an optical fiber) is shown in Tables 1-3. In particular, the propagation of the two LG modes superposition with parameters \(m_1 = 1, m_2 = 7, \sigma_1 = \sigma_2 = 2\lambda\) was shown in Table 1.

Table 1. The propagation of the two LG modes superposition with parameters \(m_1 = 1, m_2 = 7, \sigma_1 = \sigma_2 = 2\lambda\).

| \(z\) | \(z=0\) | \(z=12.5\lambda\) | \(z=25\lambda\) | \(z=37.5\lambda\) | \(z=50\lambda\) |
|-------|--------|------------------|----------------|----------------|----------------|
| Transverse distributions on a period (intensity and phase) | | | | | |
| The longitudinal intensity distribution, \(z \in [0,50\lambda]\) | | | | | |

Table 2. The propagation of the two LG modes superposition with parameters \(m_1 = 2, m_2 = -5, \sigma_1 = \sigma_2 = 2\lambda\).

| \(z\) | \(z=0\) | \(z=12.5\lambda\) | \(z=25\lambda\) | \(z=37.5\lambda\) | \(z=50\lambda\) |
|-------|--------|------------------|----------------|----------------|----------------|
| Transverse distributions on a period (intensity and phase) | | | | | |
| The longitudinal intensity distribution, \(z \in [0,50\lambda]\) | | | | | |
As can be seen from Tables 1 and 2, the transverse pattern in an ideal fiber does not change by using of the harmonized scales of the two modes superpositions, and it only rotates [26]. When there are mismatched scales and when a larger number of modes are used in the superposition (6), more complex patterns appears (Tables 3 and 4).

Table 3. The propagation of the two LG modes superposition with parameters $m_1 = 2$, $m_2 = -5$, $\sigma_1 = 2\lambda$, $\sigma_2 = 3\lambda$.

| $z$         | The transverse distributions on a period (intensity and phase) |
|-------------|-------------------------------------------------------------|
| $z=0$       | ![Image](https://via.placeholder.com/150)                   |
| $z=12.5\lambda$ | ![Image](https://via.placeholder.com/150)                   |
| $z=25\lambda$   | ![Image](https://via.placeholder.com/150)                   |
| $z=37.5\lambda$  | ![Image](https://via.placeholder.com/150)                   |
| $z=50\lambda$     | ![Image](https://via.placeholder.com/150)                   |

The longitudinal intensity distribution, $z \in [0,50\lambda]$.

Table 4. The propagation of the three LG modes superposition with parameters $m_1 = 2$, $m_2 = 5$, $m_3 = 7$, $\sigma_1 = 3\lambda$, $\sigma_2 = 2\lambda$, $\sigma_3 = \lambda$.

| $z$         | The transverse distributions on a period (intensity and phase) |
|-------------|-------------------------------------------------------------|
| $z=0$       | ![Image](https://via.placeholder.com/150)                   |
| $z=12.5\lambda$ | ![Image](https://via.placeholder.com/150)                   |
| $z=25\lambda$   | ![Image](https://via.placeholder.com/150)                   |
| $z=37.5\lambda$  | ![Image](https://via.placeholder.com/150)                   |
| $z=50\lambda$     | ![Image](https://via.placeholder.com/150)                   |

The longitudinal intensity distribution, $z \in [0,50\lambda]$.

4. Summary and conclusions
To sum up, might be said that the propagation of the vortex modes superpositions in an optical fiber with a parabolic refractive index based on the FrFT was considered. It was presented that the transverse beam structure can be changed significantly during the propagation, especially when there is the inconsistency of the scales included in the superposition of modes with fiber parameters.

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