A new class of symmetric distributions including the elliptically symmetric logistic

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ABSTRACT
We introduce a new broad and flexible class of multivariate elliptically symmetric distributions including the elliptically symmetric logistic and multivariate normal. Various probabilistic properties of the new distribution are studied, including the distribution of linear transformations, marginal distributions, conditional distributions, moments, stochastic representations and characteristic function. We also consider estimation issues associated with the mean vector and the dispersion matrix. An analysis of a real life data set is presented for illustrative purposes.

1. Introduction
The elliptically symmetric logistic distribution with density
\[
f(x) = \frac{\Gamma\left(\frac{n}{2}\right)|\Sigma|^{-\frac{1}{2}}}{\pi^{\frac{n}{2}} \prod_{i=1}^{n} \int_{0}^{\infty} u^{2-1} \frac{\exp(-u)}{1 + \exp(-u)} du} \exp\left(-\frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{2}\right), \quad x \in \mathbb{R}^n,
\]
has been originally introduced by Jensen (1985) (as the generalization of the multivariate normal distribution) and studied by Fang, Kotz and Ng (1990), Balakrishnan (1992), Kano (1994), Volodin (1999) and Gómez-Sánchez-Manzano, Gómez-Villegas, and Marín (2006). Several applications of multivariate symmetric logistic distribution in risk management, quantitative finance and actuarial science can be found in many literatures such as Valdez and Chernih (2003), Landsman (2004), Landsman and Valdez (2003) and Landsman, Makov, and Shushi (2016, 2018), Xiao and Valdez (2015). Note that several authors (cf. Gumbel 1961; Malik and Abraham 1973; Ali, Mikhail, and Haq 1978; Fang and Xu 1989; Kotz, Balakrishnan, and Johnson 2000; Yeh 2010; Ghosh and Alzaatreh 2018; Hu and Lin 2018) have studied the multivariate logistic distribution using different definitions.

The elliptically symmetric logistic distribution belongs to the elliptically contoured distributions family (also called an elliptically symmetric distributions family) Elln(\mu, \Sigma, g) with the location parameter \(\mu\), the scale parameter \(\Sigma\) and the density generator
\[
g(u) = \frac{\exp(-u)}{(1 + \exp(-u))^2}.
\]
However, research work on the multivariate symmetric logistic distribution in recent decade is rather scarce compared to much research has focused on other elliptical distributions such as multivariate normal, multivariate Student $t$, multivariate Cauchy, multivariate power exponential distribution, Kotz-type distribution and multivariate skew-normal distributions; see the books and papers of Fang, Kotz, and Ng (1990), Johnson, Kotz, and Balakrishnan (1995) and Kotz, Balakrishnan, and Johnson (2000), Azzalini and Regoli (2012), Nadarajah (2003), Arellano-Valle and Azzalini (2013), Battey and Linton (2014), Arashi and Nadarajah (2017), Arellano-Valle, Ferreira, and Genton (2018) and the references therein. The relative few literatures on the properties of elliptically symmetric logistic distribution drive the authors to further study the properties of this distribution. Moreover, we will define a new family of multivariate distributions including the elliptically symmetric logistic and multivariate normal and study the probabilistic and statistics properties of the distributions included by this family.

In some literature, the joint pdf of $n$-dimensional elliptically symmetric logistic distribution is defined as

$$f(x) = c_n |\Sigma|^{-\frac{1}{2}} \frac{\exp\left(-\frac{1}{2} q(x)\right)}{(1 + \exp\left(-\frac{1}{2} q(x)\right))^2}, x \in \mathbb{R}^n,$$

where $q(x) = (x - \mu)^T \Sigma^{-1} (x - \mu)$. The density generator is

$$g_n(u) = \frac{\exp(-u)}{(1 + \exp(-u))^2},$$

and the normalizing constant $c_n$ is given by

$$c_n = \frac{\Gamma\left(\frac{1}{2}\right)}{(2\pi)^{\frac{n}{2}}} \left[ \int_0^\infty x^{\frac{n}{2} - 1} g_n(x) dx \right]^{-1}.$$  \hspace{1cm} (1.1)

Interested readers may refer to Landsman and Valdez (2003) and Landsman, Makov, and Shushi (2016, 2018), for more details on the elliptically symmetric logistic distribution and its applications. As pointed out by Landsman and Valdez (2003) this normalizing constant has been mistakenly pointed in Fang, Kotz, and Ng (1990), Gupta, Varga, and Bodnar (2013), Xiao and Valdez (2015). Further simplification of the normalizing constant $c_n$ suggests by Landsman and Valdez (2003):

$$c_n = (2\pi)^{-\frac{n}{2}} \left[ \sum_{j=1}^\infty (-1)^{j-1} j^{1-\frac{n}{2}} \right]^{-1}.$$ \hspace{1cm} (1.2)

by using the expansion

$$\frac{e^{-x}}{(1 + e^{-x})^2} = \sum_{j=1}^\infty (-1)^{j-1} j e^{-jx}, x > 0.$$ \hspace{1cm} (1.3)

We observe that the formula (1.2) has no meaning when $n = 1$ and 2, since the series

$$\sum_{j=1}^\infty (-1)^{j-1} \sqrt{j} \text{ and } \sum_{j=1}^\infty (-1)^{j-1}$$

are divergent.
We now give the definition of a generalized elliptically symmetric logistic (GESL) distribution including the elliptically symmetric logistic distribution and multivariate normal.

**Definition 1.1** The \( n \)-dimensional random vector \( X \) is said to have a GESL distribution with location parameter \( \mu \) (\( n \)-dimensional vector) and dispersion matrix \( \Sigma \) (\( n \times n \) matrix with \( \Sigma > 0 \)) if its pdf has the form

\[
f(x) = d_n |\Sigma|^{-\frac{1}{2}} g((x - \mu)^T \Sigma^{-1} (x - \mu)), x \in \mathbb{R}^n,
\]

where \( d_n \) is the normalizing constant and will be determined in the next section, \( a, b, r > 0 \) are constants and

\[
g(u) = \frac{\exp(-bu)}{(1 + \exp(-au))^r},
\]

is its density generator. If \( X \) belongs to the GESL distribution, we shall write \( X \sim GML_n(\mu, \Sigma, g) \).

The GESL distribution is a particular case of an elliptically distribution, so \( X \) admits the stochastic representation

\[
X = \mu + \sqrt{\Sigma} A' U^{(n)},
\]

where \( A \) is a square matrix such that \( A' A = \Sigma, U^{(n)} \) is uniformly distributed on the unit sphere surface in \( \mathbb{R}^n \), and \( R \geq 0 \) is independent of \( U^{(n)} \) and has the pdf given by

\[
f_R(v) = \frac{1}{\sqrt{2\pi}^n} v^{\frac{n-1}{2}} g(v), v \geq 0.
\]

For more details see Cambanis, Huang, and Simons (1981). Note that the \( n \)-dimensional elliptically symmetric logistic distribution can be deduced as a special case of (1.4) by setting \( a = b = 1 \) and \( r = 2 \); For \( b = \frac{1}{2} \) and \( r = 0 \), we obtain the multivariate normal distribution; The density generator (1.5) covers all the density generators of the generalized logistic types I–IV distributions in Arashi and Nadarajah (2017). Figures 1–4 illustrate...
Figure 2. $n = 2, a = b = 1, r = 1$.

Figure 3. $n = 2, a = b = 1, r = 2$.

Figure 4. $n = 2, a = b = 1, r = 5$. 
the density functions of (1.4) with \( n = 2, a = b = 1 \) and \( r = 0.5, 1, 2, 5 \). Figures 5–8 illustrate the density functions of (1.4) with \( n = 1 \) and various choices of \( a, b \) and \( r \).

The rest of the article is organized as follows. In Section 2, we discuss the expression of the normalizing constant in (1.1) and give the correct values for \( c_1 \) and \( c_2 \), and provide a equivalent expression for \( c_n, n \geq 3 \). In Sections 3–6, we will investigate some of the properties of this new class of the multivariate distributions. More specifically, we find the distributions of linear transformations, the marginal and conditional distributions, the moments and the characteristic functions. The estimations associated with the mean vector and the dispersion matrix are given in Section 7. A real-life data set is used to illustrate the application of generalized elliptically symmetric logistic distribution in Section 8.

### 2. Evaluation of the normalizing constants

We first collect some facts of the Riemann zeta function and the generalized Hurwitz–Lerch zeta function which will be used in the sequel. The Riemann zeta function \( \zeta \) is defined as

\[
\zeta(s) = \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, & \text{if } \Re(s) > 1, \\
\frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, & \text{if } \Re(s) > 0, s \neq 1,
\end{cases}
\]

which can, except for a simple pole at \( s = 1 \) with its residue 1, be continued meromorphically to the whole complex \( s \)-plane; see, for details Srivastava (2003), Choi, Cho, and Srivastava (2004). Recall that (see Cvijović and Klinowski 2002; Arakawa, Ibukiyami, and Kanek 2014)
\[
\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}, \quad n \in \mathbb{N}_0,
\]  
(2.1)

and

\[
\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(u) \cot(\pi u) du, \quad n \in \mathbb{N},
\]  
(2.2)

where \(B_n = B_n(0)\) are the \(n\)th Bernoulli numbers and \(B_n(x)\) are Bernoulli polynomials defined by the generating function
The Bernoulli numbers are well-tabulated (see e.g., Srivastava 2003):

\[ B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{2n+1} = 0 (n = 1, 2, \ldots), \ldots \]

The functions \( \zeta(s) \) have the following integral representations (cf. Srivastava and Choi 2012, 169, 172)

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \, dt, \quad \mathcal{R}(s) > 1,
\]

and

\[
\zeta(s) = \frac{(1 - 2^{-1-s})^{-1}}{\Gamma(s+1)} \int_0^\infty \frac{t^{s-1}e^t}{(e^t + 1)^2} \, dt, \quad \mathcal{R}(s) > 0.
\]

Note that there is an extra 2 in (51) of Srivastava and Choi (2012, 172).

The generalized Hurwitz–Lerch zeta function is defined by (cf. Lin, Srivastava, and Wang 2006)

\[
\Psi_{\mu}(z, s, a) = \frac{1}{\Gamma(\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(\mu + n)}{n!} \frac{z^n}{(n + a)^s},
\]

which has an integral representation

\[
\Psi_{\mu}(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-at}}{(1 - ze^{-t})^\mu} \, dt, \quad \text{where } \mathcal{R}(a) > 0; \mathcal{R}(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \mathcal{R}(s) > 1 \text{ when } z = 1.
\]
Theorem 2.1. Consider the normalizing constant $d_n$ defined in (1.4). Then

$$d_n = \left(\frac{a}{\pi}\right)^{\frac{n}{2}} \left[\Psi_r^{\ast}(-1, \frac{n}{2}, \frac{b}{a})\right]^{-1},$$

(2.4)

where $\Psi_r^{\ast}$ is the generalized Hurwitz–Lerch zeta function.

Proof. By using the formula (1.37) in Denuit et al. (2005), we have

$$d_n = \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi)^{\frac{n}{2}}} \left[\int_0^\infty x^{\frac{n}{2}-1} g(x) dx\right]^{-1}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi)^{\frac{n}{2}}} \left[\int_0^\infty x^{\frac{n}{2}-1} \frac{\exp(-bx)}{(1 + \exp(-ax))^r} dx\right]^{-1}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{(\pi)^{\frac{n}{2}}} \left[\int_0^\infty x^{\frac{n}{2}-1} \frac{\exp(-\frac{b}{a}x)}{(1 + \exp(-x))^r} dx\right]^{-1}$$

$$= \left(\frac{a}{\pi}\right)^{\frac{n}{2}} \left[\Psi_r^{\ast}(-1, \frac{n}{2}, \frac{b}{a})\right]^{-1},$$

as desired.

Corollary 2.1. Consider the normalizing constant $c_n$ defined in (1.1).

(i) If $n = 1$, then

$$c_1 = \frac{1}{\sqrt{2\pi}} \left(\Psi_2^{\ast}(-1, \frac{1}{2}, 1)\right)^{-1},$$

where $\Psi_2^{\ast}$ is the generalized Hurwitz–Lerch zeta function.

(ii) If $n = 2$, then $c_2 = \pi^{-1}$.

(iii) If $n = 4$, then $c_4 = \frac{1}{4\pi^2 \ln 2}$.

(iv) If $n \geq 3, n \neq 4$, then

$$c_n = \pi^{-2} \left[(2^{\frac{n}{2}} - 4)\zeta\left(\frac{n}{2} - 1\right)\right]^{-1},$$

where $\zeta$ is the Riemann zeta function.

Proof. (i) The result follows by letting $n = 1, a = b = 1, r = 2$ in (2.4) and some algebras.
(ii) If \( n = 2 \), by (1.1) we have

\[
c_n = \frac{\Gamma\left(\frac{3}{2}\right)}{(2\pi)^{\frac{3}{2}}} \left[ \int_0^\infty x^{\frac{1}{2}} - 1 \frac{\exp(-x)}{(1 + \exp(-x))^2} \, dx \right]^{-1} = \frac{1}{\pi},
\]

where we have used the fact that

\[
h(x) = \frac{2 \exp(-x)}{(1 + \exp(-x))^2}, \quad x > 0,
\]

is the pdf of the half-logistic distribution.

(iii) By (1.2),

\[
c_4 = (2\pi)^{-2} \left[ \sum_{j=1}^\infty (-1)^{j-1} \frac{1}{j} \right]^{-1} = \frac{1}{4\pi^2 \ln 2},
\]

where we have used the well-known fact

\[
\sum_{n=1}^\infty (-1)^{n-1} \frac{1}{n} = \ln 2.
\]

(iv) If \( n \geq 3, n \neq 4 \), by making use of

\[
\zeta(s) = \frac{1}{1 - 2^{1-s}}, \quad \Psi_2(-1, s + 1, 1), \quad R(s) > 0,
\]

we have

\[
c_n = (2\pi)^{-\frac{3}{2}} \left[ \Psi_2(-1, \frac{n}{2}, 1) \right]^{-1} = \pi^{-\frac{3}{2}} \left[ (2^{\frac{n}{2}} - 4) \zeta\left(\frac{n}{2} - 1\right) \right]^{-1},
\]

where \( \zeta \) is the Riemann zeta function.

**Remark 2.1.** When \( \frac{n}{2} - 1 \) are even or odd positive integers, we apply formulae (2.1) and (2.2) to (2.4): For \( n = 4m + 2, m = 1, 2, \ldots \), the following formula holds,

\[
c_n = \frac{(2m)!}{(-1)^{m+1} 2^{2m-1} \pi^{2m} (2^\frac{n}{2} - 4) B_{2m}};
\]

For \( n = 4m + 4, m = 1, 2, \ldots \), the following formula holds,

\[
c_n = \frac{(2m + 1)!}{2^{2m-1} (2^\frac{n}{2} - 4) (-1)^{m+1} (2\pi)^{2m+1} \int_0^1 B_{2m+1}(u) \cot(\pi u) \, du}.
\]

**Example 2.1.** Let us compute \( c_n \) for small even \( n \). By using the fact that (cf. Srivastava and Choi 2012, 167)

\[
\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450},
\]

we get
\[ c_6 = \frac{3}{2} \pi^{-5}, c_{10} = \frac{45}{14} \pi^{-9}, c_{14} = \frac{945}{124} \pi^{-13}, c_{18} = \frac{4725}{254} \pi^{-17}. \]

**Remark 2.2.** Using the relationship of \( c_n \) and \( d_n \) above we find the following formula

\[ \Psi'_2(-1, 1, 1) = \frac{1}{2}, \Psi'_2(-1, 2, 1) = \ln 2, \]

and

\[ \Psi'_2\left(-1, \frac{n}{2}, 1\right) = (2^{\frac{n}{2}} - 4)\zeta\left(\frac{n}{2} - 1\right), n \geq 3, n \neq 4. \]

### 3. Linear transformations

Consider the affine transformations of the form \( Y = B X + b \) of a random vector \( X \sim GML_n(\mu, \Sigma, g) \). If \( B \) is a non-singular \( n \times n \) matrix it can be easily verified by definition or by the characteristic function in Section 6 that \( Y \sim GML_n(B\mu + b, B\Sigma B', g) \). However, when \( B \) is a \( m \times n \) matrix with \( m < n \) and \( \text{rank}(B) = m \), the following theorem shows that the density generator of \( Y \) is not necessarily \( g \), it may dependent on \( n \) and \( m \).

**Theorem 3.1.** Let \( X \sim GML_n(\mu, \Sigma, g) \), with stochastic representation \( X = \mu + \sqrt{A'U}^{(n)} \). Let \( Y = BX + b \), where \( B \) is a \( m \times n \) matrix with \( m < n \) and \( \text{rank}(B) = m \) and \( b \in \mathbb{R}^m \). Then \( Y \sim Ell_n(B\mu + b, B\Sigma B', g_Y) \) with

\[ g_Y(t) = e^{-bt} a^{-\frac{n-m}{2}} \Gamma\left(\frac{n-m}{2}\right) \Psi'_r\left(-e^{-at}, \frac{n-m}{2}, \frac{b}{a}\right), \]

where \( \Psi'_r \) is the generalized Hurwitz–Lerch zeta function. Moreover, \( Y \) admits the stochastic representation

\[ Y = \mu_Y + \sqrt{R_Y A_Y'U}^{(m)}, \]

where \( A'_Y A_Y = B\Sigma B' \) and \( R_Y = R_X B \). Here, \( B \) is the non negative random variable independent of \( R \) with distribution Beta\(\left(\frac{m}{2}, \frac{n-m}{2}\right)\).

**Proof.** The result follows from Theorems 2.15 and 2.16 in Fang, Kotz, and Ng (1990). In the sequel, we provided an alternative proof. In doing so we consider the transformation

\[ Y' = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} X + \begin{pmatrix} b \\ 0_{n-m} \end{pmatrix}, \]

where \( C \) is any given matrix such that \( \begin{pmatrix} B \\ C \end{pmatrix} \) is non-singular. So that

\[ Y' \sim GML_n\left(\begin{pmatrix} B\mu + b \\ C\mu \end{pmatrix}, \begin{pmatrix} B\Sigma B' & B\Sigma C' \\ C\Sigma B' & C\Sigma C' \end{pmatrix}, g\right), \]
The result follows, since
\[
\int_{\mathbb{R}^{n-m}} g \left( \sum_{i=1}^{n} x_i^2 \right) \prod_{j=m+1}^{n} dx_j = 2^{n-m} \int_{\mathbb{R}^{n-m}} g \left( \sum_{i=1}^{n} x_i^2 \right) dx_{m+1} \cdots dx_n
\]
\[
= \int_{\mathbb{R}^{n-m}} g \left( \sum_{i=1}^{m} x_i^2 + \sum_{j=m+1}^{n} u_j \right) \prod_{j=m+1}^{n} u_j^{-\frac{1}{2}} du_{m+1} \cdots du_n
\]
\[
= \int_{D_1} g \left( \sum_{i=1}^{m} x_i^2 + y_n \right) \prod_{j=m+1}^{n-1} y_j^{-\frac{1}{2}} \left( y_n - \sum_{j=m+1}^{n-1} y_j \right)^{-\frac{1}{2}} dy_{m+1} \cdots dy_n
\]
\[
= \int_{D_2} \left( 1 - \sum_{j=m+1}^{n-1} v_j \right) \prod_{j=m+1}^{n-1} \left( v_j^{-\frac{1}{2}} dv_j \right) \int_0^{\infty} g \left( \sum_{i=1}^{m} x_i^2 + y \right) y^{\frac{n-m}{2}} dy
\]
\[
= \frac{\pi^{\frac{n-m}{2}}}{\Gamma\left(\frac{n-m}{2}\right)} \int_0^{\infty} g \left( \sum_{i=1}^{m} x_i^2 + y \right) y^{\frac{n-m}{2}-1} dy,
\]
where
\[
D_1 = \left\{ (y_{m+1}, \ldots, y_n) \mid y_i \geq 0, i = m+1, \ldots, n, \sum_{j=m+1}^{n-1} y_j \leq y_n \right\},
\]
and
\[
D_2 = \left\{ (v_{m+1}, \ldots, v_{n-1}) \mid v_i \geq 0, i = m+1, \ldots, n-1, \sum_{j=m+1}^{n-1} v_j \leq 1 \right\}.
\]

This ends the proof.

Taking \( B = (\mathbf{x}_1, \ldots, \mathbf{x}_n) := \mathbf{x}' \) in Theorem 3.1 leads to
\[
\mathbf{x}'X \sim \text{Ell}_1(\mathbf{x}' \mu, \mathbf{x}' \Sigma x, g_{1,n}).
\]
In particular,
\[
X_k \sim \text{Ell}_1(\mu_k, \sigma_k^2, g_{1,n}), k = 1, 2, \ldots, n,
\]
and
\[
\sum_{k=1}^{n} X_k \sim \text{Ell}_1 \left( \sum_{k=1}^{n} \mu_k, \sum_{k=1}^{n} \sum_{l=1}^{n} \sigma_{kl}, g_{1,n} \right),
\]
where
\[
g_{1,n}(t) = a^{-\frac{n-1}{2}} \Gamma \left( \frac{n-1}{2} \right) \Psi_r \left( -e^{-at}, \frac{n-1}{2}, \frac{b}{a} \right).
\]
4. Marginal and conditional distributions

For fixed \( m < n \), consider the partitions of \( \mathbf{X}, \mu, \Sigma \) given below

\[
\mathbf{X} = \begin{pmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},
\]

where \( \mathbf{X}^{(1)}, \mu^{(1)} \in \mathbb{R}^{m} (m < n), \mathbf{X}^{(2)}, \mu^{(2)} \in \mathbb{R}^{n-m}, \Sigma_{11} \) is \( m \times m \) matrix, \( \Sigma_{12} \) is \( m \times (n-m) \) matrix, \( \Sigma_{21} \) is \( (n-m) \times m \) matrix, \( \Sigma_{22} \) is \( (n-m) \times (n-m) \) matrix.

The following theorem gives the result on the marginal distributions of \( \mathbf{X}^{(1)} \) and \( \mathbf{X}^{(2)} \).

**Theorem 4.1.** Let \( \mathbf{X} \sim \text{GML}_{n}(\mu, \Sigma, g) \), where \( g \) is defined as (1.6), then

1. \( \mathbf{X}^{(1)} \sim \text{Ell}_{m}(\mu^{(1)}, \Sigma_{11}, g_{(m)}) \),
2. \( \mathbf{X}^{(2)} \sim \text{Ell}_{n-m}(\mu^{(2)}, \Sigma_{22}, g_{(n-m)}) \),

where \( g_{(k)} \) is the function given by

\[
g_{(k)}(t) = e^{-bt} \cdot \frac{a^{-\frac{a}{2}r}}{\Gamma\left(\frac{n-k}{2}\right)} \Psi_{r}^{+}\left(-e^{-at}, \frac{n-k}{2}, \frac{b}{a}\right),
\]

where \( \Psi_{r}^{+} \) is the generalized Hurwitz–Leh r zeta function.

In particular, if \( m = n - 2 \) and \( a = b \), then

\[
g_{(n-2)}(t) = \begin{cases} \frac{1}{a(r-1)} \left(1 - \frac{1}{(1 + e^{-at})^{r-1}}\right), & \text{if } r \neq 1, \\
1 - \ln (1 + e^{-at}), & \text{if } r = 1.
\end{cases}
\]

**Proof.** Taking \( B = (I_{m}0_{m \times (n-m)}) \) in Theorem 3.1, where \( I_{m} \) is the \( m \times m \) identity matrix and \( 0_{m \times (n-m)} \) is the \( m \times (n-m) \) null matrix, we get \( \mathbf{X}^{(1)} = B \mathbf{X} \). Applying Theorem 3.1 the result (i) follows. The proof of (ii) is similarly.

An interesting question is to know whether the marginal pdfs of (1.4) have (1.5) as their density generator. From Theorem 4.1 we see that the marginal pdfs of (1.4) depending on the dimension \( n \) and have not (1.5) as their density generator. Conversely, if an \( n \)-dimensional multivariate elliptically symmetric distribution with \( p \)-dimensional marginal generalized elliptically symmetric logistic, then, the density generator \( g \) is determined by

\[
\frac{\exp \left(-bu\right)}{1 + \exp \left(-au\right)} = \int_{t}^{\infty} (w - t)^{p-1} g(w) dw, \quad (4.1)
\]

For example, if \( a = b = 1, n = 2, r = 2, p = 1 \), then, (4.1) becomes

\[
\frac{\exp \left(-t\right)}{(1 + \exp \left(-t\right))^{2}} = \int_{0}^{\infty} w^{-\frac{1}{2}} g(t + w) dw, \quad (4.2)
\]

from which we get

\[
g(t) = \sum_{k=1}^{\infty} (-1)^{k-1} k e^{-kt}, t > 0. \quad (4.3)
\]
The 2-dimensional multivariate elliptically symmetric distribution with the density generator (4.3) is not a generalized elliptically symmetric logistic distribution. If \( a = b = 1, n = 3, r = 2, p = 1 \), then (4.1) becomes
\[
\frac{\exp(-t)}{(1 + \exp(-t))^2} = \int_t^\infty g(w)dw.
\] (4.4)
from which we get
\[
g(t) = \frac{e^{-t} - e^{-2t}}{(1 + e^{-2t})^3}, t > 0.
\] (4.5)
The 3-dimensional multivariate elliptically symmetric distribution with the density generator (4.5) is not a generalized elliptically symmetric logistic distribution.

Therefore, the distribution (1.4) is not dimensionally coherent or consistent. A spherical distribution with density generator \( f \) is said to have the consistency property if
\[
\int_{-\infty}^{\infty} f\left(\sum_{i=1}^{n+1} x_i^2\right) dx_{n+1} = f\left(\sum_{i=1}^{n} x_i^2\right)
\] (4.6)
for any integer \( n \) and almost all \( x \in \mathbb{R}^n \). This consistency property ensures that any marginal distribution of \( X \) also belongs to the same spherical family. Kano (1994) gave several necessary and sufficient conditions for a spherical distribution to satisfy (4.6) and list some examples. Dimensionally coherent elliptically distributions are multivariate Normal, multivariate Student \( t \), multivariate Cauchy and symmetric stable. Distributions that do not have this property include the multivariate Logistic \( \exp(-bt)/(1 + \exp(-at))^r \), Pearson type II \( \{(1-u)^{-r}\} \) with \( v > 1 \), Pearson type VII \( \{(1+u)^{-r}\} \) with \( v > \frac{n}{2} \), Kotz type \( \{u^{n-1}e^{-nu^2}\} \) with \( r, s > 0, 2N + n > 2 \) and multivariate Bessel distributions.

Voldin (1999) provided exact formulae for pdf of spherically symmetric distribution with logistic marginals. Applying formulae (1) and (2) in Voldin (1999) to elliptically symmetric logistic distribution with density generator (1.5) yields the density generators of pdf of spherically symmetric distribution with generalized elliptically symmetric logistic marginals. That is for \( n = 2m + 1, m \in \mathbb{N}^+ \),
\[
g_n(t) = \frac{(-1)^m \partial^m}{\pi^m} \frac{\exp(-bt)}{(1 + \exp(-at))^r}, t \geq 0,
\] (4.7)
and for \( n = 2m, m \in \mathbb{N}^+ \)
\[
g_n(t) = \frac{(-1)^m \partial^m}{\pi^m} \int_0^\infty \frac{\exp(-b(t + z^2))}{(1 + \exp(-a(t + z^2)))^r} dz, t \geq 0.
\] (4.8)
The following theorem gives the conditional distribution of \( X^{(2)} \) given \( X^{(1)} \).

**Theorem 4.2.** Let \( X \sim GML_n(\mu, \Sigma, g) \), where \( g \) is defined as (1.5). Conditionally on \( X^{(1)} = x^{(1)} \), we have the conditional distribution of \( X^{(2)} \) is the elliptical distribution \( Ell_{n-m}(\mu_{2,1}, \Sigma_{22,1}, g_{(2,1)}) \), where
\[ \mu_{2,1} = \mu^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(x^{(1)} - \mu^{(1)}), \]

and

\[ g_{(2,1)}(t) = \frac{e^{-b[t+(x^{(1)}-\mu^{(1)})\Sigma_{11}^{-1}(x^{(1)}-\mu^{(1))}]}(1 + e^{-a[t+(x^{(1)}-\mu^{(1)})\Sigma_{11}^{-1}(x^{(1)}-\mu^{(1))}]})^{r}}{t \geq 0}. \]

**Proof.** The conditional density of \( X^{(2)} \) given \( X^{(1)} = x^{(1)} \) is given by

\[ f(x^{(2)}|x^{(1)}) = \frac{f(x)}{P(X^{(1)} = x^{(1)})}, \]

the result follows from (1.4) and Theorem 4.1(i), since

\[ |\Sigma| = |\Sigma_{11}| : |\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}|, \]

and for each \( x \in \mathbb{R}^n \),

\[ x'\Sigma^{-1}x = x^{(1)'}\Sigma_{11}^{-1}x^{(1)} + x_{2,1}'\Sigma_{22,1}^{-1}x_{2,1}. \]

Here,

\[ x_{2,1} = x^{(2)} - \Sigma_{21}\Sigma_{11}^{-1}x^{(1)}. \]

**Remark 4.1.** Note that this conditional pdf is not a \((n - m)\)-variate multivariate elliptically symmetric logistic distribution unless \( x^{(1)} = \mu^{(1)} \).

### 5. Moments

In this section, we derive the moments of \( X \). From (1.7), we get, for real number \( l > 0 \),

\[ E(R^{l}) = \frac{1}{\int_{0}^{\infty} t^{\frac{n}{2}-1}g(t)dt} \int_{0}^{\infty} z^{\frac{n}{2}+l-1}g(z)dz \]

\[ = \frac{1}{\int_{0}^{\infty} t^{\frac{n}{2}-1}g(t)dt} \left( \frac{1}{a} \right) \int_{0}^{\infty} z^{\frac{n}{2}+l-1} \frac{e^{-bz}}{(1 + e^{-az})^{r}} dz \]

\[ = \frac{1}{\int_{0}^{\infty} t^{\frac{n}{2}-1}g(t)dt} \left( \frac{1}{a} \right) \Gamma\left( \frac{n}{2} + l \right) \Psi_{r}^{*}\left( -1, \frac{n}{2} + l, \frac{b}{a} \right) \]

\[ = \frac{1}{\Gamma\left( \frac{n}{2} \right) \Psi_{r}^{*}\left( -1, \frac{n}{2}, \frac{b}{a} \right)}, \]

where \( \Psi_{r}^{*} \) is the generalized Hurwitz–Lerch zeta function.

**Theorem 5.1.** Let \( X \sim GML_{n}(\mu, \Sigma, g) \), where \( g \) is defined as (1.5).
The expectation and the covariance are:

\[ E(X) = \mu, \; \text{Cov}(X) = \frac{\Psi_r^*(\left(-1, \frac{n}{2} + 1, \frac{b}{a}\right))}{2a\Psi_r^*(\left(-1, \frac{n}{2}, \frac{b}{a}\right))} \Sigma; \] (5.1)

For any integers \( m_1, \ldots, m_n \), with \( m = \sum_{i=1}^n m_i \), the product moments of \( Y := \Sigma^{-\frac{1}{2}}(X - \mu) \) are

\[ E \left( \prod_{i=1}^n Y_i^{2m_i} \right) = \frac{\left( \frac{1}{a} \right)^m \Gamma \left( \frac{n}{2} + m \right) \Psi_r^* \left(-1, \frac{n}{2} + m, \frac{b}{a}\right)}{\left( \frac{a}{2} \right)^m \Gamma \left( \frac{n}{2} \right) \Psi_r^* \left(-1, \frac{n}{2}, \frac{b}{a}\right)} \prod_{i=1}^n (2m_i)! \]

where \( a^{[k]} = a(a+1) \cdots (a+k-1) \), \( \Psi^*_r \) is the generalized Hurwitz–Lerch zeta function.

**Proof.**

(i) By using (1.6) we have \( E(X) = \mu \) since \( E(U^{(n)}) = 0 \), and

\[ \text{Cov}(X) = \frac{1}{n} E(R) \Sigma = \frac{\Psi_r^*(\left(-1, \frac{n}{2} + 1, \frac{b}{a}\right))}{2a\Psi_r^*(\left(-1, \frac{n}{2}, \frac{b}{a}\right))} \Sigma. \]

(ii) By Equation (2.18) in Fang, Kotz, and Ng (1990), the product moments of \( Y \) are:

\[ E \left( \prod_{i=1}^n U_i^{2m_i} \right) = E(R^n) E \left( \prod_{i=1}^n U_i^{2m_i} \right). \]

The result follows since (cf. Fang, Kotz, and Ng 1990)

\[ E \left( \prod_{i=1}^n U_i^{2m_i} \right) = \frac{1}{(\frac{a}{2})^n} \prod_{i=1}^n (2m_i)! \]

6. Characteristic function

**Theorem 6.1.** Let \( X \sim GML_n(\mu, \Sigma, g) \), where \( g \) is defined as (1.5).

The characteristic function of \( X \) can be expressed in the following form:

(i) If \( n = 1 \), assume that

\[ f(x) = \frac{d_1}{s} g \left( \frac{(x - \mu)^2}{\sigma^2} \right), \quad -\infty < x < \infty, \]

where \( \mu, \sigma > 0 \) are real number. Then

\[ \psi_X(t) = d_1 e^{it\mu} \int_0^\infty \cos (t\sigma \sqrt{y}) \frac{e^{-by}}{\sqrt{y}(1 + e^{-ay})} dy, \; t \in (-\infty, \infty). \]

(ii) If \( n \geq 2 \), then

\[ \psi_X(t) = \frac{1}{B(\frac{n-1}{2}, \frac{1}{2})} \Psi_r^* \left(-1, \frac{n}{2} + \frac{b}{a}\right) \int_0^\pi \Psi_r^* \left(-1, \frac{n}{2}, \frac{b}{a} - \frac{i}{a} t^* \cos \theta \right) e^{-i0} d\theta, \]

where \( \Psi^*_r \) is the generalized Hurwitz–Lerch zeta function.
Proof.

(i) By definition, we have

\[ \psi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{d}{dx} g \left( \frac{(x - \mu)^2}{\sigma^2} \right) dx \]

\[ = 2d_1 e^{it\mu} \int_0^\infty \cos (t\sigma x) g(x^2) dx \]

\[ = d_1 e^{it\mu} \int_0^\infty \cos (t\sigma \sqrt{y}) y^{-1} g(y) dy \]

\[ = d_1 e^{it\mu} \int_0^\infty \cos (t\sigma \sqrt{y}) \frac{e^{-by}}{\sqrt{y}(1 + e^{-ay})} dy. \]

(ii) Using (1.6) and note that the independence of \( R \) and \( U(n) \),

\[ \psi_X(t) = E(e^{itX}) = e^{itE} \left( e^{-\frac{1}{2}R^2} U(n) \right) \]

\[ = e^{itE} \left[ E(e^{-\frac{1}{2}R^2} U(n) | R) \right] \]

\[ = e^{itE} \left[ \Omega_n(Rt' \Sigma t) \right] \]

\[ = e^{itE} \int_0^\infty \Omega_n(vt' \Sigma t) P(R < v) dv \]

\[ = \frac{1}{B(n - 1, \frac{1}{2})} \int_0^\infty \Omega_n(vt' \Sigma t) \frac{t^{-1} \exp (-bv)}{\left(1 + \exp (-av)\right)^t} dv \]

\[ = \frac{1}{B(n - 1, \frac{1}{2})} \int_0^\infty \Omega_n(vt' \Sigma t) \frac{t^{-1} \exp (-bv)}{\left(1 + \exp (-av)\right)^t} dv \]

\[ = \frac{1}{B(n - 1, \frac{1}{2})} \Phi_r \left( -1, \frac{n}{2}, \frac{b}{a} \right) e^{it\mu} \int_0^\infty \Omega_n(vt' \Sigma t) \sin^{-2} t \, d\theta, \]

where \( \Omega_n(t'|t) \) is the characteristic function of \( U(n) \) (see Fang, Kotz, and Ng 1990, 3.1):

\[ \Omega_n(||t||^2) = \frac{1}{B(n - 1, \frac{1}{2})} \int_0^\pi \exp (i||t|| \cos \theta) \sin^{n-2} \theta d\theta. \]

Remark 6.1. Using the following equivalent forms of \( \Omega_n(||t||^2) \) (see Fang, Kotz, and Ng 1990, 3.2, 3.3)

\[ \Omega_n(||t||^2) = \frac{\Gamma(n/2)}{\sqrt{\pi} \Gamma((n-1)/2)} \sum_{k=0}^{\infty} \frac{(-1)^k ||t||^{2k} \Gamma(n-1) \Gamma(2k+1)}{(2k)! \Gamma((n+2k)/2)}. \]
and
\[ \Omega_n(||t||^2) = {}_0F_1\left(\frac{n}{2}; -\frac{1}{4} ||t||^2 \right), \]
we get the following equivalent forms of the characteristic function \( \psi_X(t) \):
\[
\psi_X(t) = \frac{e^{it\mu}}{\sqrt{\pi} \Psi^*_{r\left(-1, \frac{n}{2}, \frac{b}{a}\right)}} \sum_{k=0}^{\infty} \frac{\left((-1)^k \Gamma\left(\frac{k+1}{2}\right)\right)}{(2k)! \Gamma\left(\frac{n+2k}{2}\right)} \Psi_{r\left(-1, \frac{n}{2} + k, \frac{b}{a}\right)},
\]
and
\[
\psi_X(t) = \frac{e^{it\mu}}{\Psi^*_{r\left(-1, \frac{n}{2}, \frac{b}{a}\right)}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} t^\Sigma t\right)^k}{(\frac{n}{2})^k k!} \Psi_{r\left(-1, \frac{n}{2} + k, \frac{b}{a}\right)},
\]
where \( {}_0F_1(\cdot; \cdot) \) is the generalized hypergeometric function, \( \Psi^*_{r} \) is the generalized Hurwitz-Lerch zeta function and \( a^{[k]} \) are the ascending factorials, i.e., \( a^{[k]} = a(a+1) \cdots (a+k-1) \).

7. Estimation of parameters

In this section, we consider estimation problems associated with the mean vector \( \mu \) and the dispersion matrix \( \Sigma \).

7.1. Moments estimation

Assume that \( X_1, ..., X_N \) are \( N \) random sample from \( GML_n(\mu, \Sigma, g) \). The sample mean \( \bar{X} \) and covariance matrix \( S \) are defined as
\[
\bar{X} = \frac{1}{N} \sum_{k=1}^{N} X_k, \quad S = \frac{1}{N-1} \sum_{k=1}^{N} (X_k - \bar{X})(X_k - \bar{X})'.
\]
Equating the theoretical expectation and covariance of \( X_i \) given by (5.1) and the corresponding sample mean and covariance matrix, we obtain
\[
\hat{\mu} = \frac{1}{N} \sum_{k=1}^{N} X_k, \quad \hat{\Sigma} = \frac{2a\Psi^*_{r\left(-1, \frac{n}{2}, \frac{b}{a}\right)}}{\Psi_{r\left(-1, \frac{n}{2} + 1, \frac{b}{a}\right)}} S.
\]
By using the identity
\[
S = \frac{1}{N-1} \sum_{k=1}^{N} (X_k - \mu)(X_k - \mu)' - \frac{N}{N-1} (\bar{X} - \mu)(\bar{X} - \mu)',
\]
and
\[ E(\bar{X} - \mu)(\bar{X} - \mu)' = \frac{1}{N} \Sigma, \]
we have
\[
E(\mu) = \mu, E(\Sigma) = \frac{2a\Psi_r(-1, \frac{n}{2}, \frac{b}{a})}{\Psi_r(-1, \frac{n}{2} + 1, \frac{b}{a})}\Sigma.
\]

Note that when \(r = 0\) and \(a, b > 0\),
\[
\Psi_r(-1, \frac{n}{2}, \frac{b}{a}) = \Psi_r(-1, \frac{n}{2} + 1, \frac{b}{a}) = \left(\frac{b}{a}\right)^{\frac{n}{2} + 1},
\]
and hence
\[
\frac{\Psi_r(-1, \frac{n}{2}, \frac{b}{a})}{\Psi_r(-1, \frac{n}{2} + 1, \frac{b}{a})} = \frac{b}{a};
\]

Thus \(\hat{\Sigma} = 2b\hat{S}\) and \(E(\Sigma) = 2b\Sigma\). In particular, when \(r = 0, a > 0\) and \(b = \frac{1}{2}\), \(\text{GML}_n(\mu, \Sigma, g)\) reduces to the multivariate normal distribution \(N_n(\mu, \Sigma)\) and in this case, \(\hat{\Sigma} = S\) and \(E(\Sigma) = \Sigma\).

Table 1 lists some moments estimation of \(\sigma^2\) for various \(a, b\) and \(r\) in the case of \(n = 1\). Setting
\[
S_N^2 = \frac{1}{N - 1} \sum_{k=1}^{N} (X_k - \bar{X})^2
\]

### 7.2. Maximum likelihood estimation

Suppose \(X_1, ..., X_N\) are \(N\) random sample from \(\text{GML}_n(\mu, \Sigma, g)\) and \(x_1, ..., x_N\) are their observations. Then, the log-likelihood function is given by
\[
\log L(\mu, \Sigma) = N\log(d_n) - \frac{N}{2}\log|\Sigma| + \sum_{i=1}^{N} \log g(x_i - \mu)^\top \Sigma^{-1} (x_i - \mu).
\]

The MLEs of \(\mu\) and \(\Sigma\) are obtained by maximizing simultaneously with respective to \(\mu\) and \(\Sigma\). Differentiating with respect to \(\mu\) and \(\Sigma\) yields the system of estimating equations
\[
\hat{\mu} = \text{ave} \left[ \rho'(s_i)(X_i) \right] / \text{ave}(\rho'(s_i)), \tag{7.1}
\]
\[
\hat{\Sigma} = \text{ave} \left[ \rho'(s_i)(X_i - \hat{\mu})(X_i - \hat{\mu})^\top \right], \tag{7.2}
\]

where \(\rho(u) = 2bu + 2r \log(1 + e^{-au})\) and \(s_i = (X_i - \hat{\mu})^\top \hat{\Sigma}^{-1} (X_i - \hat{\mu})\). Here “ave” stands for the arithmetic average over \(i = 1, ..., N\). Note that \(\rho\) is differentiable with derivatives
\[
\rho'(u) = 2b - \frac{2rae^{-au}}{1 + e^{-au}},
\]
and
have of normal distribution. For more details see Maronna et al. (2019, 204). Start with arbitrary initial values. Tyler (1991) implies that (7.1) and (7.2) have a unique solution. For particular, when $a = b = 1$, $\rho'(u) = 2b - ar \geq 0$, and so that

$$\left(\sqrt{u} \rho'(u)\right)' = \frac{1}{2} u^{-\frac{3}{2}} \rho'(u) + u^2 \rho''(u) \geq 0, u \geq 0,$$

which means that $\sqrt{u} \rho'(u)$ is an increasing function of $u > 0$. The result in Kent and Tyler (1991) implies that (7.1) and (7.2) have a unique solution. For $r = 0$ or $a = 0$, we have $\rho'(u) = 2b$, and thus

$$\hat{\mu} = \bar{X}, \quad \hat{\Sigma} = \frac{2b}{N} \sum_{i=1}^{N} (X_i - \hat{\mu})(X_i - \hat{\mu})' , E(\hat{\Sigma}) = \frac{2b(N - 1)}{N} \Sigma$$

In particular, when $r = 0$ or $a = 0$ and $b = \frac{1}{2}$, we have $\rho'(u) = 1$ which reduces the case of normal distribution. For more details see Maronna et al. (2019, 204). Start with arbitrary initial values $\mu^{(0)}$ and $\Sigma^{(0)}$, the following iterative equations converge to the MLEs of $\mu$ and $\Sigma$:

$$\mu^{(k+1)} = \text{ave}(\rho'(s_i^{(k)}))X_i)/\text{ave}(\rho'(s_i^{(k)})), k = 0, 1, 2 \cdots,$$

$$\Sigma^{(k+1)} = \text{ave}[\rho'(s_i^{(k)})(X_i - \mu^{(k)})(X_i - \mu^{(k)})'], k = 0, 1, 2 \cdots$$

where

$$s_i^{(k)} = (X_i - \mu^{(k)})'\Sigma^{(k)-1}(X_i - \mu^{(k)}).$$

### Table 1. Moments estimation of $\sigma^2$.

| Distribution | $\frac{b-1}{2}$, $r = 2$ | $\frac{b-1}{2}$, $r = 2$ | $\frac{b-1}{2}$, $r = 2$ | $\frac{b-1}{2}$, $r = 2$ | $\frac{b-1}{2}$, $r = 2$ | $\frac{b-1}{2}$, $r = 2$ |
|--------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\sigma^2$   | 0.62$\sigma^2_{y}$ | 1.26$\sigma^2_{y}$ | 2.81$\sigma^2_{y}$ | 0.89$\sigma^2_{y}$ | 1.80$\sigma^2_{y}$ | 3.72$\sigma^2_{y}$ |

### Table 2. The strength in GPA for single-carbon fibers data.

| GPA | Parameter estimates | Log-likelihood | AIC |
|-----|---------------------|----------------|-----|
| 1.901 | $a = 1, b = \frac{1}{2}$ | $\hat{\mu} = 3.0593$ | $\hat{\sigma}^2 = 0.3794$ | $-58.8664$ | 121.7329 |
| 2.454 | $a = b = 1$ | $\hat{\mu} = 3.1858$ | $\hat{\sigma}^2 = 0.7964$ | $-64.0761$ | 132.1522 |
| 2.624 | $a = 2b = 1$ | $\hat{\mu} = 3.1223$ | $\hat{\sigma}^2 = 0.6324$ | $-61.6561$ | 127.3121 |
| 2.977 | $a = 2b = \frac{1}{2}$, $r = 0$ | $\hat{\mu} = 3.0593$ | $\hat{\sigma}^2 = 0.1897$ | $-58.8664$ | 121.7329 |
| 3.264 | $a = 2b = \frac{1}{2}$, $r = \frac{1}{2}$ | $\hat{\mu} = 3.0593$ | $\hat{\sigma}^2 = 1.5176$ | $-58.8664$ | 121.7329 |

### Table 3. Parameter estimates of the generalized logistic distribution fitted to GPA data.

| Parameter estimates | GPA | Log-likelihood | AIC |
|---------------------|-----|----------------|-----|
| $\mu$ | $\hat{\mu} = 3.0593$ | $\hat{\sigma}^2 = 0.3794$ | $-58.8664$ | 121.7329 |
| $\sigma^2$ | $\hat{\sigma}^2 = 0.7964$ | $\hat{\sigma}^2 = 0.6324$ | $-61.6561$ | 127.3121 |
| $\mu$ | $\hat{\mu} = 3.1858$ | $\hat{\sigma}^2 = 0.1897$ | $-58.8664$ | 121.7329 |
| $\sigma^2$ | $\hat{\sigma}^2 = 0.1897$ | $\hat{\sigma}^2 = 1.5176$ | $-58.8664$ | 121.7329 |
8. Application

To illustrate the applicability of the GESL distribution, a data set is applied to the GESL distribution, and the result is compared with normal distribution and symmetric logistic distribution. The data set in Table 2 was used by Gupta and Kundu (2010) which represents the strength measured in GPA for single carbon fiber and impregnated 1000-carbon fiber tows. The sample mean and variance are 3.0593 and 0.3794, respectively. The maximum likelihood estimates, the log-likelihood value and the Akaike information criterion (AIC) are listed in Table 3. Note that when \( a = b = 1 \) and \( r = 2 \) corresponding to the symmetric logistic distribution, and when \( b = \frac{1}{2} \) and \( r = 0 \) corresponding to the normal case. The results show that the GESL distribution provides the best fit for certain particular parameters. For the case \( r = 0 \) and \( n = 1 \), one has

\[
AIC = -2 \log L(\hat{\mu}, \hat{\sigma}^2) + 4 = N \log (2\pi S_N^2) + N + 4,
\]

where

\[
S_N^2 = \frac{1}{N} \sum_{k=1}^{N} (X_k - \bar{X})^2.
\]

Therefore, the values AIC and Log-likelihood do not depend on \( a \) and \( b \).

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References

Ali, M. M., N. N. Mikhail, and M. S. Haq. 1978. A class of bivariate distributions including the bivariate logistic. *Journal of Multivariate Analysis* 8 (3):405–12. doi:10.1016/0047-259X(78)90063-5.

Arakawa, T., T. Ibukiyama, and M. Kaneko. 2014. Bernoulli numbers and zeta functions, with an appendix by Don Zagier. Tokyo: Springer Monographs in Math, Springer.

Arashi, M., and S. Nadarajah. 2017. Generalized elliptical distributions. *Communications in Statistics - Theory and Methods* 46 (13):6412–32. doi:10.1080/03610926.2015.1129415.

Arellano-Valle, R. B., and A. Azzalini. 2013. The centred parameterization and related quantities of the skew-\( t \) distribution. *Journal of Multivariate Analysis* 113:73–90. doi:10.1016/j.jmva.2011.05.016.

Arellano-Valle, R. B., C. S. Ferreira, and M. G. Genton. 2018. Scale and shape mixtures of multivariate skew-normal distributions. *Journal of Multivariate Analysis* 166:98–110. doi:10.1016/j.jmva.2018.02.007.

Azzalini, A., and G. Regoli. 2012. Some properties of skew-symmetric distributions. *Annals of the Institute of Statistical Mathematics* 64 (4):857–79. doi:10.1007/s10463-011-0338-5.

Balakrishnan, N. 1992. *Handbook of the logistic distribution*. New York: Marcel Dekker.

Battey, H., and O. Linton. 2014. Nonparametric estimation of multivariate elliptic densities via finite mixture sieves. *Journal of Multivariate Analysis* 123:43–67. doi:10.1016/j.jmva.2013.08.013.
Cambanis, S., S. Huang, and G. Simons. 1981. On the theory of elliptically contoured distributions. *Journal of Multivariate Analysis* 11:365–85.

Choi, J., Y. J. Cho, and H. M. Srivastava. 2004. Series involving the zeta function and multiple Gamma functions. *Applied Mathematics and Computation* 159 (2):509–37. doi:10.1016/j.amc.2003.08.134.

Cvijović, D., and J. Klinowski. 2002. Integral representations of the Riemann zeta function for odd-integer arguments. The *Journal of Computational and Applied Mathematics*. 142 (2): 435–9. doi:10.1016/S0377-0427(02)00358-8.

Denuit, M., J. Dhaene, M. Goovaerts, and R. Kaas. 2005. *Actuarial theory for dependent risks: Measures, orders and models*. Chichester: Wiley.

Fang, K. T., and J. L. Xu. 1989. A class of multivariate distributions including the multivariate logistic. *Journal of Mathematical Research and Exposition* 9:91–100.

Fang, K. T., S. Kotz, and K. W. Ng. 1990. *Symmetric multivariate and related distributions*. London: Chapman and Hall.

Ghosh, I., and A. Alzaatreh. 2018. A new class of generalized logistic distribution. *Communications in Statistics - Theory and Methods* 47 (9):2043–55. doi:10.1080/03610926.2013.835420.

Gómez-Sánchez-Manzano, E., M. A. Gómez-Villegas, and J. M. Marín. 2006. Sequences of elliptical distributions and mixtures of normal distributions. *Journal of Multivariate Analysis* 97 (2):295–310. doi:10.1016/j.jmva.2005.03.008.

Gumbel, E. J. 1961. Bivariate logistic distribution. *Journal of the American Statistical Association* 56 (294):335–49. doi:10.1080/01621459.1961.10482117.

Gupta, A. K., T. Varga, and T. Bodnar. 2013. *Elliptically contoured models in statistics and portfolio theory*. 2nd ed. New York: Springer Science + Business Media.

Gupta, R. D., and D. Kundu. 2010. Generalized logistic distributions. *Journal of Applied Statistical Science* 18 (1):1–23.

Hu, C.-Y., and G. D. Lin. 2018. Characterizations of the logistic and related distributions. *Journal of Mathematical Analysis and Applications* 463 (1):79–92. doi:10.1016/j.jmaa.2018.03.003.

Jensen, D. R. 1985. Multivariate distributions. In *Encyclopedia of statistical sciences*, ed. S. Kotz, N. L. Johnson, and C. B. Read, vol. 6, 43–55. Wiley.

Johnson, N. L., S. Kotz, and N. Balakrishnan. 1995. *Continuous univariate distributions*. 2nd ed., vol. 2, 113–63. Wiley.

Kano, Y. 1994. Consistency property of elliptic probability density functions. *Journal of Multivariate Analysis* 51 (1):139–47. doi:10.1006/jmva.1994.1054.

Kent, J. T., and D. E. Tyler. 1991. Redescending M-estimates of multivariate location and scatter. *The Annals of Statistics* 19 (4):2102–19. doi:10.1214/aos/1176348388.

Kotz, S., N. Balakrishnan, and N. L. Johnson. 2000. Continuous multivariate distributions. In *Models and Applications*, vol. 1. Wiley.

Landsman, Z. 2004. On the generalization of Esscher and variance premiums modified for the elliptical family of distributions. *Insurance: Mathematics and Economics* 35 (3):563–79. doi:10.1016/j.insmatheco.2004.07.006.

Landsman, Z., U. Makov, and T. Shushi. 2016. Tail conditional moments for elliptical and log-elliptical distributions. *Insurance: Mathematics and Economics* 71:179–88. doi:10.1016/j.insmatheco.2016.09.001.

Landsman, Z., U. Makov, and T. Shushi. 2018. A multivariate tail covariance measure for elliptical distributions. *Insurance: Mathematics and Economics* 81:27–35. doi:10.1016/j.insmatheco.2018.04.002.

Landsman, Z. M., and E. A. Valdez. 2003. Tail conditional expectations for elliptical distributions. *North American Actuarial Journal* 7 (4):55–71. doi:10.1090/10820277.2003.10596118.

Lin, S. D., H. M. Srivastava, and P. Y. Wang. 2006. Some expansion formulas for a class of generalized Hurwitz-Lerch zeta functions. *Integral Transforms and Special Functions* 17 (11):817–27. doi:10.1080/10652460600926923.

Malik, H. J., and B. Abraham. 1973. Multivariate logistic distribution. *The Annals of Statistics* 1 (3):588–90. doi:10.1214/aos/1176342430.

Maronna, R. A., R. D. Martin, V. J. Yohai, and M. Salibin-Barrera. 2019. *Robust Statistics: Theory and Methods (with R)*. 2nd ed. Wiley.
Nadarajah, S. 2003. The Kotz-type distribution with applications. *Statistics* 37 (4):341–58. doi:10.1080/0233188031000078060.

Srivastava, H. M. 2003. Certain classes of series associated with the zeta and related functions. *Applied Mathematics and Computation* 141 (1):13–49. doi:10.1016/S0096-3003(02)00318-1.

Srivastava, H. M., and J. Choi. 2012. *Zeta and q-zeta functions and associated series and integrals*. Amsterdam: Elsevier.

Valdez, E. A., and A. Chernih. 2003. Wang’s capital allocation formula for elliptically contoured distributions. *Insurance: Mathematics and Economics* 33 (3):517–32. doi:10.1016/j.insmatheco.2003.07.003.

Volodin, A. 1999. Spherically symmetric logistic distribution. *Journal of Multivariate Analysis* 70 (2):202–6. doi:10.1006/jmva.1999.1826.

Xiao, Y., and E. A. Valdez. 2015. A Black-Litterman asset allocation model under elliptical distributions. *Quantitative Finance* 15 (3):509–19. doi:10.1080/14697688.2013.836283.

Yeh, H. C. 2010. Multivariate semi-logistic distributions. *Journal of Multivariate Analysis* 101 (4):893–908. doi:10.1016/j.jmva.2009.09.002.