Horizon Straddling ISCOs in Spherically Symmetric String Black Holes

Parthapratim Pradhan*  
Department of Physics, Vivekananda Satavarshiki Mahavidyalaya(Affiliated to Vidyasagar University), Manikpara, Jhargram, West Midnapur, West Bengal 721513, India

Received Day Month Year
Revised Day Month Year
Communicated by Managing Editor

The causal geodesics in the equatorial plane of a static extremal charged black holes in heterotic string theory are examined with regard to their geodesic stability, and compared with similar geodesics in the non-extremal situation. Extremization of the effective potential for time-like and null circular geodesics implies that in the extremal limit, the radius of ISCO (Inner-most Stable Circular Orbit) \( r_{ISCO} \), circular photon orbit (CPO) \( r_{ph} \) and marginally bound circular orbit (MBCO) \( r_{mb} \) are coincident with the event horizon \( r_{hor} \) i.e. \( r_{ISCO} = r_{ph} = r_{mb} = r_{hor} = 2M \). Since the proper radial distance on a constant time slice both in Schwarzschild and Painlevé-Gullstrand coordinates become zero, thus these three orbits indeed coincide with the null geodesic generators of the event horizon. This strange behavior is quite different from the static, spherically symmetric extremal Reissner Nordstrom black hole.

Keywords: ISCO, MBCO, CPO, Extremal String BH.

1. Introduction

The motion of geodesics determine important features of the black hole(BH) spacetime. Among the different kinds of geodesics, circular geodesics specially, ISCOs are more interesting. Studies of neutral test particles, both time-like and null, is one way to understand the gravitational field around a BH spacetime. Different theoretical as well as observational predictions i.e. the gravitational red-shift, gravitational bending of light, Shapiro time-delay, Lense-Thirring precession, gravitational leasining and perihelion shift etc. all are physical phenomenon which might be related to the geodesic structure of the BH. Thus it is important to study the proper geodesic structure of the BH spacetime.

It is well known that ISCO [also called marginally stable circular orbit(MSCO)] or LSCO(Least stable circular orbit) plays a crucial role in accretion disk theory. It also useful to estimate the temperature of the accretion disk(ad) via the Eddington
It has been observed that for neutron star’s surface
\[ \frac{GM}{c^2 R} \sim 0.1, \]
for Schwarzschild BH \( \frac{GM}{c^2 R} \sim 16 \) and we find for extremal string BH \( \frac{GM}{c^2 R} \sim 12 \). Here \( R \) is the radius of ISCO. In any cases, for \( M \sim M_\odot, \epsilon \sim 0.5 \), we find the characteristic temperature of the X-ray sources around \( T_{ad} \sim \text{few KeV} \). To compute the binding energy, ISCO also plays an important role.

In Einstein’s general relativity, circular geodesics of arbitrary radii are not possible, there exists a minimum radii below which no circular orbits are possible. The conditions for the existence of ISCO, MBCO and CPO have been considered for the Schwarzschild black hole, Reissner Nordstrom(RN) black-hole, spherically symmetric string BH and charged dilation BH. In the author completely studied the null geodesics of charged BHs in string theory. Gravitational lensing effect of the charged string BHs examined in.

But the study of causal geodesics of charged black holes in string theory, both extreme and non-extremal situations have not been considered in the extant literature. The fact that string theory is a promising candidate for a consistent quantum theory of gravity. Hence, the studies of BH solution of the low energy string theory has a great significance from this perspective. Additionally, the classical equation of motion for string theory has the form of Einstein’s equation plus the Planck scale corrected terms.

Thus in the present work, we wish to investigate in detail the equatorial time-like circular geodesics and null circular geodesics, both extremal and non-extremal cases of four dimensional static, spherically symmetric string BH, which is also called Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) BH and we show that in the extremal limit, the radius of ISCO, CPO and MBCO coalesce to the same point i.e.

\[ r_{ISCO} = r_{ph} = r_{mb} = r_{hor} = 2M. \]

And we also compute that the proper radial distances on a constant time slice, both in Schwarzschild and Painlevé-Gullstrand coordinates, from the horizon to the ISCO becomes zero, therefore these three orbits indeed coincide with the principal null geodesics generators of the event horizon. From the best of my knowledge this result is not reported previously in the literature.

The paper is organized as follows. In section 2, we give the basics of GMGHS space-time. In section 3, we shall analyze in detail the equatorial circular geodesics, both particle orbits and photon orbits for non extreme GMGHS space-time and also compute the ISCO for non-extreme GMGHS BH. In section 4, we present the
particle orbits and photon orbits for extreme GMGHS space-time and compute the ISCO for extreme GMGHS BH. Discussion of the proper radial distance is presented in section 5. Implications of different useful coordinates for GMGHS space-time are described shortly in section 6. Finally, the key conclusions are given in section 7.

2. Preliminaries of the GMGHS Space-time:

The effective action in heterotic string theory\textsuperscript{[13]} in the low energy limit is represented by

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - \frac{1}{12} e^{-4\phi} H_{abc} H^{abc} - 2(\nabla \Phi)^2 - e^{-2\phi} F_{ab} F^{ab} \right]
\]  

(3)

where \( g_{ab} \) is the metric, \( \Phi \) is the dilation field, \( R \) is the scalar curvature and \( F_{ab} = \partial_a A_b - \partial_b A_a \) is the field strength corresponds to the Maxwell field \( A_a \), and \( F_{ab} \) is the Maxwell field associated with a \( U(1) \) subgroup of \( E_8 \times E_8 \) or \( \text{spin}(32) \times_{\mathbb{Z}_2} \), and

\[
H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} - (\Omega_3(A))_{abc}
\]

(4)

where \( B_{ab} \) is the antisymmetric tensor gauge field, and

\[
(\Omega_3(A))_{abc} = \frac{1}{4} (A_a F_{bc} + A_b F_{ca} + A_c F_{ab})
\]

(5)

is the gauge Chern-Simons term. We are interested to explore in this work to the situation when the fields \( H_{abc} \) and \( B_{ab} \) are corresponds to the zero value. Therefore the above action reduces to

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2(\nabla \Phi)^2 - e^{-2\phi} F_{ab} F^{ab} \right]
\]

(6)

and the corresponding field equations are

\[
\nabla_a (e^{-2\phi} F^{ab}) = 0
\]

(7)

\[
\nabla^2 \Phi + \frac{1}{2} e^{-2\phi} F^2 = 0
\]

(8)

\[
R_{ab} = -2\nabla_a \Phi \nabla_b \Phi - 2e^{-2\phi} F_{ac} F_{bc} + \frac{1}{2} g_{ab} e^{-2\phi} F^2
\]

(9)

The static charged BH solutions of the above action was found by Gibbons and Maeda\textsuperscript{[9]} in 1988, and independently by Garfinkle, Horowitz and Strominger\textsuperscript{[10]} in 1991.

The line element of this charged BH is given by

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (r - b) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\]

(10)

where, \( b = \frac{Q^2}{M} e^{-2\phi_0} \) and

\[
e^{-2\phi} = e^{-2\phi_0} \left( 1 - \frac{b}{r} \right), \text{ and } F = Q \sin \theta d\theta \wedge d\phi
\]

(11)
where $\phi_0$ is the asymptotic value of the dilation field, $M$ represents the mass of the BH, $Q$ denote its electric charge and $\phi$ is the scalar field. We have considered throughout the work when the strength of the dilation field is precisely zero i.e. $\phi_0 = 0$. The BH has a regular event horizon at $r_{\text{hor}} = 2M$, which is identical to the Schwarzschild BH. There are some important differences between RN BH and GMGHS BH which are tabulated as below:

| Properties          | RN BH                              | GMGHS BH                        |
|---------------------|------------------------------------|---------------------------------|
| Horizon: $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ | $r_{\text{hor}} = M$               |                                 |
| Extremal limit:     | $M^2 = Q^2$                        | $Q^2 = 2M^2 e^{2\phi_0}$        |                                 |
| Area:               | $A_{\pm} = 4\pi r_{\pm}^2$        | $A = 4\pi r_{\text{hor}}(r_{\text{hor}} - b)$ |                                 |
| Hawking Temp.:      | $T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})}$ | $T_H = \frac{-\phi_0}{8\pi M}$ |                                 |
| Area at extremality:| $A_{\text{ex}} = 4\pi M^2$        | $A_{\text{ex}} = 0$             |                                 |

3. Equatorial Circular Geodesics in GMGHS BH:

Since the space-time has a time-like isometry generated by the time-like Killing vector $\xi \equiv \partial_t$ whose projection along the 4-velocity $u$ of geodesics: $\xi \cdot u = -E$, is conserved along such geodesics. There is also the ‘angular momentum’ $L \equiv \zeta \cdot u$ (where $\zeta \equiv \partial_\phi$) which is similarly conserved. Using these properties together with the normalization of the four velocity, one obtains the radial equation for charged BHs in string theory on the $\theta = \frac{\pi}{2}$ plane:

$$ (u^r)^2 = \dot{r}^2 = E^2 - V_{\text{eff}} = E^2 - \left( \frac{L^2}{r(r-b)} - \epsilon \right) \left( 1 - \frac{2M}{r} \right). $$

(12)

where the standard effective potential for GMGHS space-time is

$$ V_{\text{eff}} = \left( \frac{L^2}{r(r-b)} - \epsilon \right) \left( 1 - \frac{2M}{r} \right). $$

(13)

Here, $\epsilon = -1$ for time-like geodesics, $\epsilon = 0$ for light-like geodesics and $\epsilon = +1$ for space-like geodesics.

3.1. Particle Orbits:

The effective potential for massive particles could be obtained from the above equation by substituting $\epsilon = -1$:

$$ V_{\text{eff}} = \left( 1 + \frac{L^2}{r(r-b)} \right) \left( 1 - \frac{2M}{r} \right). $$

(14)

a) When $L = 0$, we get the radial geodesics and correspondingly the effective potential becomes

$$ V_{\text{eff}} = 1 - \frac{2M}{r}. $$

(15)
If we observe the effective potential for radial time-like geodesics graphically it looks like as in Fig. 1. In the limit $b = 0$ or $Q = 0$, we obtain the effective potential for well known Schwarzschild black hole.

In the diagrams (Fig. 2, Fig. 3), we have described the properties of effective potential as derived in the Eq. (14) for different values of $b$.

To compute the circular geodesic motion of the test particle in the Einstein-Maxwell gravitational field, we must have $r = r_0 = constant$ and from the equation (12), we get

$$V_{eff} = E^2.$$  \hspace{1cm} (16)
Fig. 3. The picture shows the variation of $V_{eff}$ with $r$. Here, $M = 1$. 

and

$$\frac{dV_{eff}}{dr} = 0.$$  \hspace{1cm} (17)$$

Therefore we obtain the energy and angular momentum per unit mass of the test particle along the circular orbits are given by

$$E_0 = \sqrt{\frac{(2r_0 - b)(r_0 - 2M)^2}{r_0[2r_0^2 - (b + 6M)r_0 + 4Mb]}}.$$  \hspace{1cm} (18)$$

and

$$L_0 = \sqrt{\frac{2M(2r_0^2 - (b + 6M)r_0 + 4Mb)}{2}}.$$  \hspace{1cm} (19)$$

We have plotted the energy and angular momentum of the test particle in Fig. 4.

Circular motion of the test particle to be exists when both energy and angular momentum are real and finite, therefore we must have $2r_0^2 - (b + 6M)r_0 + 4Mb > 0$ and $r_0 > b$. Again the angular frequency measured by an asymptotic observers for timelike circular geodesics at $r = r_0$ is given by

$$\Omega_0 = \frac{u^\phi}{u^t} = \sqrt{\frac{2M}{r_0^2(2r_0 - b)}}.$$  \hspace{1cm} (20)$$

In the limit $b \to 0$, we obtain the angular frequency for Schwarzschild BH which is $\Omega_0 = \sqrt{\frac{2M}{r_0^2}}$. In general relativity, circular orbits do not exist for all values of $r$, so the denominator of equations (18,19) real only if $2r_0^2 - (b + 6M)r_0 + 4Mb \geq 0$. The limiting case of equality gives a circular orbit with indefinite energy per unit mass, i.e a photon orbit. This photon orbit is the innermost boundary of the circular
Fig. 4. The figures show the variation of $E_0$ and $L_0$ with $r_0$. Here, $M = 1$.

Fig. 5. The figure displays the variation of $r_{ph}$ with $b/M$. Here, $Q = 1$.

orbits for massive particles. Comparing the above equation of particle orbits with (31) when $r_0 = r_c$, we can see that photon orbits are the limiting case of time-like circular orbit. It occurs at the radius

$$r_{ph} = \frac{1}{4}(b + 6M + \sqrt{b^2 - 20Mb + 36M^2})$$

(21)

It is described in the Fig. 5.

The MBCO \(^1\) can be obtained by setting $E^2 = 1$, then the radius of MBCO is located at

$$r_{mb} = 2M \pm \sqrt{2M(2M - b)}$$

(22)

It has been found in the Fig. 6.
In the limit $b \to 0$, we get $r_{mb} = 4M$, which is the radius of MBCO for Schwarzschild BH. At the extreme limit $b = 2M$, it occurs at the radius $r_{mb} = 2M$.

From the astrophysical point of view the most important class of orbit is innermost stable circular orbit (ISCO), which occurs at the point of inflection of the effective potential $V_{eff}$. Thus at the point of inflection

$$\frac{d^2V_{eff}}{dr^2} = 0$$

with the auxiliary equation $\frac{dV_{eff}}{dr} = 0$. Then the ISCO equation for GMGHS BH is given by

$$r_0^3 - 6Mr_0^2 + 6Mr_0 - 2Mb^2 = 0$$

The real root of the equation gives the radius of ISCO at $r_0 = r_{ISCO}$ which is given by

$$\frac{r_{ISCO}}{M} = 2 + Z + \frac{2(2 - \frac{b}{M})}{Z}$$

$$Z = \left[ 8 - 6\left(\frac{b}{M}\right) + \left(\frac{b}{M}\right)^2 + \sqrt{\left(\frac{b}{M}\right)^4 - 4\left(\frac{b}{M}\right)^3 + 4\left(\frac{b}{M}\right)^2} \right]^\frac{1}{2}$$

It could be found in the Fig. [8]

In the limit $b \to 0$, we obtain $r_{ISCO} = 6M$, which is the radius of ISCO for Schwarzschild BH.

### 3.2. Photon Orbits:

The radial potential that governs the null geodesics can be expressed as

$$U_{eff} = \frac{L^2}{r(r-b)} \left(1 - \frac{2M}{r} \right)$$

Fig. 6. The picture depicts the variation of $\frac{r_{mb}}{M}$ with $\frac{b}{M}$. Here, $b = 1$. 
Fig. 7. The stability threshold is defined by the largest real root of the cubic $y = r_0^3 - 6M r_0^2 + 6Mb r_0 - 2M b^2 = 0$. For $b = 0$ this root has the value $r_0 = 6M$. For $b = 2M$ it has the value $r_0 = 2M$. The figure shows the variation of $y$ with $\frac{M}{M}$.

Fig. 8. The figure shows the variation of $\frac{r_{ISCO}}{M}$ with $\frac{b}{M}$. Here, $b = 1$.

In the limit $b = 0$, we get the effective potential for Schwarzschild case.
In the following Fig. 9 we have drawn the effective potential for photon orbits of GMGHS BH and for various values of \( b \).

For circular null geodesics at \( r = r_c \):

\[
U_{\text{eff}} = E^2 \tag{28}
\]

and

\[
\frac{dU_{\text{eff}}}{dr} = 0 \tag{29}
\]

Thus we obtain the ratio of energy and angular momentum of the test particle evaluated at \( r = r_c \) for CPO:

\[
\frac{E_c}{L_c} = \pm \sqrt{\frac{r_c - 2M}{r_c^2(r_c - b)}} \tag{30}
\]

and

\[
2r_c^2 - (b + 6M)r_c + 4bM = 0. \tag{31}
\]

After introducing the impact parameter \( D_c = \frac{E_c}{L_c} \), the above equation reduces to

\[
\frac{1}{D_c} = \frac{E_c}{L_c} = \sqrt{\frac{r_c - 2M}{r_c^2(r_c - b)}} \tag{32}
\]

Solving equation\(^{(31)}\), one could obtain the radius of the CPO:

\[
(r_c)_\pm = \frac{1}{4} \left( b + 6M \pm \sqrt{b^2 - 20bM + 36M^2} \right) \tag{33}
\]

Here we can easily see that two situation arises first one is that \((r_c)_\pm \geq 2M\) and the second one is \(r_c < 2M\). Also \((r_c)_\pm\) are real when \( b \leq 2M \) or \( b \geq 18M \). Since we...
are interested in this work the case for $b \leq 2M$, therefore the radius of the CPO occurs at

$$r_{ph} = \frac{1}{4} \left( b + 6M + \sqrt{b^2 - 20bM + 36M^2} \right)$$  \hspace{1cm} (34)

Again the angular frequency $\Omega_c$ measured by an asymptotic observers is given by

$$\Omega_c = \frac{u^\phi}{u^t} = \frac{1}{D_c} = \sqrt{\frac{r_c - 2M}{r_c^2(r_c - b)}}$$  \hspace{1cm} (35)

4. Extremal Case:

Now we turn to the extremal cases to see what is happening there.

4.1. Particle Orbits

Proceeding analogously, the corresponding effective potential for extremal GMGHS BH is found to be

$$V_{eff} = \left( 1 + \frac{L^2}{r(r - 2M)} \right) \left( 1 - \frac{2M}{r} \right).$$  \hspace{1cm} (36)

It could be seen in the Fig. 10

Using the condition for circular geodesics of constant $r = r_0$, we obtain the energy and angular momentum per unit mass for the test particle as

$$E_0 = \sqrt{\frac{M}{r_0}}.$$  \hspace{1cm} (37)

and

$$L_0 = \sqrt{Mr_0}.$$  \hspace{1cm} (38)
Fig. 11. The figures display the variation of energy and angular momentum with $r$.

In the Fig. 11, we have drawn the graph for energy and angular momentum. Once again, for Circular geodesic motions of the test particle, both energy and angular momentum are real and finite, therefore we must have $r_0 > M$. Now the most interesting class of circular orbit is the innermost stable circular orbit which occurs at the point of inflection as given by the equation (23). Hence the ISCO equation for the test particle of the extremal GMGHS space-times can be written as

$$ (r_0 - 2M)^3 = 0 $$

Hence the ISCO occurs at the radius $r_0 = r_{ISCO} = 2M$ for extremal GMGHS BH. At the ISCO the values of energy and angular momentum becomes $E_{isco} = \frac{1}{\sqrt{2}}$ and $L_{isco} = \sqrt{2}M$ respectively.

4.2. Photon Orbits:

In this case, the effective potential becomes

$$ U_{eff} = \frac{L^2}{r(r-2M)} \left( 1 - \frac{2M}{r} \right) $$

(40)

For circular geodesics of constant $r = r_c$, the ratio of energy and angular momentum $E_c \over L_c = \pm \frac{1}{r_c}$

(41)

and

$$ (r_c - 2M)^2 = 0. $$

(42)
After introducing the impact parameter \( D_c = \frac{L_c}{E_c} \), the above equation reduces to
\[
D_c = \frac{E_c}{L_c} = \frac{1}{r_c}
\]
(43)

Solving equations(42) one could obtain simply the radius of the circular photon orbit is
\[
r_c = 2M
\]
(44)

Therefore for extremal GMGHS BH, the photon orbit occurs at \( r_{ph} = 2M \).

Therefore from the above analysis, we get for extremal GMGHS BH or extremal string BH
\[
r_{ISCO} = r_{ph} = r_{mb} = 2M.
\]
(45)

These three important geodesic orbits coincide with the event horizon. This is the key point of our investigation.

5. Proper Radial Distance:

Here we shall show that the proper spatial distance on a spatial (constant time) slice, from an exterior point to the horizon gives the zero value for the extremal GMGHS BH. The proper spatial distance on a constant time slice from any point to the event horizon is given by
\[
r_p = \int_{r_{hor}}^{r_{ISCO}} \sqrt{g_{rr}} \, dr = \left( \sqrt{r(r - 2M)} + 2M \ln \left| \sqrt{r} + \sqrt{r - 2M} \right| \right) |_{r_{ISCO}}^{r_{hor}}
\]
(46)

where \( g_{rr} = \frac{r}{r - 2M} \). In the near extremal limit, \( Q = \sqrt{2M(1 - \chi)} \). The event horizon is located at \( r_h = 2M \), the CPO is at \( r_{ph} = M(2 + 2\sqrt{1 - \chi}) \), a MBCO is at \( r_{mb} = 2M(1 + \sqrt{1 - \chi}) \) and the ISCO is at \( r_{ISCO} = 2M(1 + \chi^{1/3} + 2\chi^{2/3}) \) for the equatorial plane. The proper radial distance from photon orbit to event horizon at the extremal limit \( \chi \to 0 \) is \( r_p |_{r_{hor}}^{r_{ISCO}} = 0 \). Again the limiting distance from ISCO to the horizon is \( r_p |_{r_{hor}}^{r_{ISCO}} = 0 \). The distance for marginally bound orbit (\( r_{mb} \)) to the event horizon is \( r_p |_{r_{hor}}^{r_{mb}} = 0 \), which is also vanishing in the extremal limit. Since all the proper radial distances are vanishing at the extremal limit, therefore they must coincides with the null generators of the horizon. It may be noted that the metric components \( g_{rr} \) is independent of \( Q \). Due to this fact, all the proper radial distances from horizon to any exterior point becomes zero.

6. GMGHS space-time in Painlevé-Gullstrand Coordinates

The standard discussion of ISCOs in previous section has given in terms of Schwarzschild coordinates(SC) which are known to be ill-behaved on the event horizon. So in this section, we shall introduce a number of well behaved coordinate systems like ingoing Eddington-Finklestein (EF), Painlevé-Gullstrand (PG)
coordinates which are regular on the event horizon and has some interesting properties. We shall show the effective potential derived in these coordinates are similar to those obtained in SC. Beginning with the EF coordinates, we have the metric as

$$ds^2 = - \left(1 - \frac{2M}{r}\right) d\tau^2 + 2 d\tau dr + r(r-b)(d\theta^2 + \sin^2 \theta d\phi^2)$$ (47)

which is obtained by the following coordinate transformation for Schwarzschild BH

$$v = t + r + 2M \ln \left|\frac{r}{2M} - 1\right|$$ (48)

The above metric is the same as time independent, spherically symmetric geometry with different coordinates and is not singular at $r = 2M$. This type of coordinates are very useful to study the ongoing gravitational collapse. Now if we have given the following transformation

$$dv = dt + \frac{dr}{1 + \sqrt{\frac{2M}{r}}}$$ (49)

then we have found the well known metric of GMGHS BH space-time in Painlevé coordinates:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) d\tau^2 + 2 \sqrt{\frac{2M}{r}} d\tau dr + dr^2 + r(r-b)(d\theta^2 + \sin^2 \theta d\phi^2)$$ (50)

which is unlike Schwarzschild coordinates, are not singular at the horizon. It is manifested that the space-time now well behaved on the horizon. This type of coordinates could be used to calculate the Hawking radiation.

Similar to the Schwarzschild space-time, the GMGHS space-times also have isometries, namely time-like isometries and rotational isometries as we have defined in the section II. Analogously, the radial equation in this coordinate chart is found to be:

$$(1 - f) \dot{r}^2 + \left(1 + \frac{L^2}{r(r-b)}\right) f = \mathcal{E}^2$$ (51)

where $f = 1 - \frac{2M}{r}$.

For circular orbit $\dot{r} = \frac{dr}{d\tau} = 0$ and by substituting the value of $f$ one obtains

$$\mathcal{E}^2 = \mathcal{V}_{eff} = \left(1 + \frac{L^2}{r(r-b)}\right) \left(1 - \frac{2M}{r}\right).$$ (52)

which is exactly similar to the effective potential as we have found in Schwarzschild coordinates.

Let us now study the properties of the radial geodesics with zero angular momentum $L = 0$ and radial free fall of a particle from infinity i.e. $E = 1$ then the Eq.(51) can be written as

$$\frac{dr}{d\tau} = \pm 1.$$ (53)
+ sign for outgoing geodesics and – sign for ingoing geodesics. Now one can compute
the proper time interval which is found to be

$$\tau = \int_{r_{\text{hor}}}^{r_{\text{ISCO}}} dr = r_{\text{ISCO}} - r_{\text{hor}} \ .$$  \hspace{1cm} (54)

This implies that the proper time interval from event horizon to ISCO or event
horizon to MBCO or event horizon to CPO in the extremal limit gives zero value.
This is why, these three orbits coincide with the principal null generators of the
horizon. In schwarzschild coordinates, the above calculation gives identical result
also.

Now if we compare the Schwarzschild time slice and Painlevé time slice for the
GMGHS BH, we have

$$ds_{\text{Sch}}^2 = \frac{r}{r - 2M} dr^2 + r(r - b)d\phi^2$$

$$ds_{\text{Pain}}^2 = dr^2 + r(r - b)d\phi^2 \ .$$  \hspace{1cm} (55)

This implies that in the extremal limit $b = 2M$, on an equatorial constant time slice
the proper radial distance from horizon to any exterior point gives zero value, this
means that in the extremal limit these three orbits namely ISCO, CPO and MBCO
are precisely located on the event horizon, which coincides with the null generators
of the horizon.

7. Discussion

To summarize, we have demonstrated that the geodesic motion of a neutral test
particle for time-like and null circular geodesics in the equatorial plane of both
extreme and non-extreme cases for GMGHS BH. We have compared the ISCO for
non-extremal BH as well as for extremal BH. The interesting findings for extremal
BH is that, ISCO, CPO and MBCO coincides with the event horizon i.e. $r_{\text{ISCO}} =
\ r_{\text{ph}} = r_{\text{mb}} = r_{\text{hor}} = 2M$, which is quite different from extremal RN space-time.
Where the ISCO occurs at $r_{\text{ISCO}} = 4M$ [11], photon sphere [12] is located at $r_{\text{ph}} =
2M$, MBCO is situated at $r_{\text{mb}} = \frac{3+\sqrt{5}}{2}M$ and the horizon is at $r_{\text{hor}} = M$. Thus for
this space-time, the inequality becomes $r_{\text{ISCO}} \neq r_{\text{ph}} \neq r_{\text{mb}} \neq r_{\text{hor}} = M$.

Since the proper radial distance on an equatorial constant time slice, from the
ISCO to event horizon or photon orbit to event horizon or MBCO to event horizon
is exactly zero both in Schwarzschild and Painlevé coordinates, so they are in fact
coalesce with the principal null generators of the horizon. What is interesting in this
space-time is that ISCO lies on the unstable photon sphere. An another interesting
feature of this space-time is that the area $8\pi M(2M - b)$ goes to zero at the extremal
limit, which is also quite different from RN BH.

Acknowledgements

I would like to thank Prof. P. Majumdar of R. M. V. U for helpful discussions. I
also would like to thank the Editor Prof. J. P. S. Lemos and anonymous referee
for their helpful suggestions.

References

1. S. Chandrashekar, *The Mathematical Theory of BHs*, Clarendon Press, Oxford (1983).
2. J. B. Hartle, *Gravity-An Introduction To Einstein’s General Relativity*, Benjamin Cummings, (2003).
3. S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects*, (Wiley, New York, 1983).
4. D. Pugliese, H. Quevedo, R. Ruffini, *Phys. Rev. D* **83**, 024021 (2011).
5. P. Pradhan, P. Majumdar, *Physics Letters A* **375** 474-479 (2011).
6. C. Blaga, P. A. Blaga, *Serb. Astron. Journal* **158** 55-59 (1998).
7. S. Fernando, *Phys. Review D* **85**, 024033 (2012).
8. A. Bhadra, *Phys. Review D* **67** 103009 (2003).
9. G. W. Gibbons, K. Maeda, *Nucl. Phys. B* **298**, 741 (1988).
10. D. Garfinkle, G. T. Horowitz, A. Strominger, *Phys. Review D* **43** (3140) (1991); **45**, (3888) (1992).
11. T. Maki, K. Shiraishi, *Classical Quantum Gravity* **227** (1994).
12. P. Painlevé, *C.R. Acad. Sci.(Paris)* **173** 677 (1921).
13. G. T. Horowitz, “The Dark Side of String Theory: Black Holes and Black Strings”; [arXiv:hep-th/9210119](https://arxiv.org/abs/hep-th/9210119)
14. A. Sen, *Phys. Rev. Lett.* **69**, 1006 (1992).
15. C. M. Claudel, K. S. Virbhadra and G. F. R. Ellis, *Journal of Math Phys* **42**, 818 (2001).