On the family of trajectories of an analytic gradient
flow converging to a critical point

by Zbigniew Szafraniec

Abstract. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic function. There are presented sufficient conditions for existence of an infinite family of trajectories of the gradient flow $\dot{x} = \nabla f(x)$ which converge to a critical point of $f$.

1 Introduction.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic function. According to Łojasiewicz [12], the limit set of an integral curve of the dynamical system $\dot{x} = \nabla f(x)$ is either empty or contain a single critical point of $f$. So the family of trajectories which converge to a critical point is a natural object of study in the theory of gradient dynamical systems.

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function defined in a neighborhood of the origin, having a critical point at 0. We shall write $T(f)$ for the set of non-trivial trajectories of the gradient flow $\dot{x} = \nabla f(x)$ which converge to the origin. There is a natural problem: is $T(f)$ infinite? (In the planar case this is equivalent to the problem whether the stable set of the origin has a non-empty interior?)

In some cases the answer is rather obvious. Let $S_r = S_{n-1}^n \cap \{ f < 0 \}$, where $S_{n-1}^n = \{ x \in \mathbb{R}^n \mid |x| = r \}$, $0 < r \ll 1$. By [13], [16], if $T(f)$ is finite then each cohomology group $H^i(S_r)$ is trivial for $i \geq 1$. Hence, if there exists $i \geq 1$ with $H^i(S_r) \neq 0$, then $T(f)$ is infinite.

In [2] (see also Section 3) there is presented an intrinsic filtration of $T(f)$ given in terms of characteristic exponents and asymptotic critical values of $f$. Unfortunately, these numbers are difficult to compute. This is why in this paper we present methods which are more easy to apply.

Let $\omega : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be the homogeneous initial form associated with $f$. In some cases investigating properties of this form may provide simple conditions which guarantee that $T(f)$ is infinite. Put $\Omega = S_{n-1}^n \cap \{ \omega < 0 \}$. We shall show that $T(f)$ is infinite if at least one cohomology group $H^i(\Omega)$, where $i \geq 1$, is non-trivial. Applying the Moussu results [14] one may also show that the same holds true if there exists at least one non-degenerate critical point of $\omega|\Omega$ which is not a local minimum.
However, if \( n = 2 \) and \( S_r \neq S_1^r \) then none of the above assumptions would hold, but \( T(f) \) may be infinite. (See Example 4.3.)

The main result of this paper says that \( T(f) \) is infinite if \( \text{rank} \, H^0(S_r) < \text{rank} \, H^0(\Omega) \). In particular, if \( n = 2 \) and \( S_r \neq S_1^r \) then it is enough to verify that \( S_r \) has less connected components than \( \Omega \).

As a corollary we shall show that the inequality \( \chi(S_r) < \chi(\Omega) \) implies that \( T(f) \) is infinite. (It is proper to add that there exist efficient methods of computing these Euler-Poincaré characteristics (see [11], [17]).)

It is worth pointing out that according to Moussu [14, Theorem 3] the family \( T(f) \) always contain trajectories which are represented by real analytic curves converging to the origin. In some cases a family of those analytic curves can be infinite.

The paper is organized as follows. In Section 2 we prove preliminary results about the homotopy type of some semi-analytic sets. In Section 3 we present properties of important geometric invariants associated with trajectories of the gradient flow. In Section 4 we prove the main result of this paper (Theorem 4.1), and we show how to apply it. References [1], [4], [10], [16] present significant related results and applications.

## 2 Preliminaries.

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be an analytic function defined in an open neighbourhood of the origin. Let \( \mathbb{Q}^+ \) denote the set of positive rationals. For \( \ell \in \mathbb{Q}^+, a < 0, y < 0 \) and \( r > 0 \) we shall write

\[
B_r^n = \{ x \in \mathbb{R}^n \mid |x| \leq r \}, \quad S_r^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = r \},
\]

\[
V_{\ell,a} = \{ x \in \mathbb{R}^n \setminus \{0\} \mid f(x) \leq a|x|^{\ell} \},
\]

\[
S_{r,a}^{\ell,a} = S_r^{n-1} \cap V_{\ell,a}, \quad B_{\ell,a}^n = B_r^n \cap V_{\ell,a}, \quad F_{\ell,a}(y) = f^{-1}(y) \cap V_{\ell,a},
\]

\[
D_{\ell,a}(y) = f^{-1}([y,0)) \cap V_{\ell,a} = \{ x \in V_{\ell,a} \mid y \leq f(x) < 0 \}.
\]

**Lemma 2.1.** Assume that \( \ell \in \mathbb{Q}^+ \) and \( a < 0 \). If \( 0 < -y \ll r \ll 1 \) then the sets \( S_{r,a}^{\ell,a} \) and \( F_{\ell,a}(y) \) are homotopy equivalent. In particular, the cohomology groups \( H^*(S_{r,a}^{\ell,a}) \) and \( H^*(F_{\ell,a}(y)) \) are isomorphic.

**Proof.** For \( x \in V_{\ell,a} \cup \{0\} \) lying sufficiently close to the origin we have \( |x|^{1/2} \geq |f(x)| \geq |a| \cdot |x|^\ell \), so that in particular functions \( f(x), |x|^2 \) restricted to this
set are proper. According to the local triviality of analytic mappings (see [5]), they are locally trivial. So there is $r_0 > 0$ such that $|x| : B^r_{r_0} \to (0, r_0]$ is a trivial fibration. Hence the inclusion $S^{r_0}_r \subset B^r_{r_0}$ is a homotopy equivalence for each $0 < r \leq r_0$.

By similar arguments, there is $y_0 < 0$ such that $D^{r, a}(y_0) \subset B^r_{r_0}$ and $f : D^{r, a}(y_0) \to [y_0, 0)$ is a trivial fibration, so that the inclusion $F^{r, a}(y) \subset D^{r, a}(y)$ is a homotopy equivalence for each $y \leq y_0 < 0$.

So, if $0 < -y \ll r \ll 1$ then we may assume that $r \leq r_0$, $y_0 \leq y$, and

$$D^{r, a}(y) \subset B^r_{r_0} \subset D^{r, a}(y_0) \subset B^r_{r_0}.$$ 

As inclusions $D^{r, a}(y) \subset D^{r, a}(y_0)$ and $B^{r, a}_r \subset B^{r, a}_{r_0}$ are homotopy equivalencies, then $D^{r, a}(y) \subset B^{r, a}_r$ is a homotopy equivalence too. Then $F^{r, a}(y)$ is homotopy equivalent to $S^{r, a}$.

For $0 < -y \ll r \ll 1$ we shall write

$$F_r(y) = B^r_r \cap f^{-1}(y), \quad S_r = \{x \in S^{r-1}_r \mid f(x) < 0\}.$$ 

We call the set $F_r(y)$ the real Milnor fibre. According to [13], it is either an $(n-1)$-dimensional compact manifold with boundary or an empty set. Moreover, the sets $F_r(y)$ and $S_r$ are homotopy equivalent.

**Corollary 2.2.** If $0 < -y \ll r \ll 1$ then the cohomology groups $H^*(S_r)$ and $H^*(F_r(y))$ are isomorphic.

Let $\omega$ be the initial form associated with $f$ and let $g = f - \omega$, so that $f = \omega + g$. Denote by $d$ the degree of $\omega$. Hence $g = O(|x|^{d+1})$.

**Lemma 2.3.** If $0 < r \ll -a \ll 1$ then sets $S^{d, a}_r = S^{n-1}_r \cap \{f \leq ar^d\}$, $S^{n-1}_r \cap \{\omega \leq ar^d\}$ and $\Omega = S^{n-1}_r \cap \{\omega < 0\}$ have the same homotopy type.

**Proof.** For $r \in \mathbb{R}$ sufficiently close to zero and $x \in S^{n-1}$ we have

$$f(rx) = \omega(rx) + g(rx) = r^d\omega(x) + r^{d+1}G(x, r),$$

where $G(x, r)$ is an analytic function defined in an open neighbourhood of $S^{n-1}_r \times \{0\}$. Put $H(x, r) = \omega(x) + rG(x, r)$, and $H_r = H(\cdot, r) : S^{n-1}_r \to \mathbb{R}$.

By [13 Corollary 2.8], there exists $a_0 < 0$ such that any $a_0 < a < 0$ is a regular value of $\omega|S^{n-1}_r$. Hence there exists $r_0 > 0$ such that $a$ is a regular value of every $H_r$, where $-r_0 < r < r_0$. Then

$$\{(x, r) \in S^{n-1}_r \times (-r_0, r_0) \mid H(x, r) \leq a\}$$
is an $n$-dimensional manifold with boundary $S^{n-1} \times (-r_0, r_0) \cap H^{-1}(a)$. By the implicit function theorem, the mapping $(x, r) \mapsto r$ restricted to both above manifolds is a proper submersion. By Ehresmann’s theorem, it is a locally trivial fibration. Hence if $r$ is sufficiently close to zero then the manifolds $S^{n-1} \cap \{ \omega \leq a \} = \{ x \in S^{n-1} \mid H(x, 0) \leq a \}$ and $S^{n-1} \cap \{ H_r \leq a \} = \{ x \in S^{n-1} \mid H(x, r) \leq a \}$ are homeomorphic.

The set $S^{n-1} \cap \{ \omega \leq a \}$ is a deformation retract of $\Omega = S^{n-1} \cap \{ \omega < 0 \}$, so that these sets have the same homotopy type.

We have $f(rx) = r^d H_r(x)$. Hence $x \in S^{n-1} \cap \{ H_r \leq a \}$ if and only if $rx \in S^{n-1}_r \cap \{ f \leq ar^d \}$, and the proof is complete.

### 3 Geometric invariants of the function.

In the beginning of this section we present some results obtained by Kurdyka et al. [8], [9] in the course of proving Thom’s gradient conjecture. In exposition and notation we follow closely these papers.

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function defined in a neighborhood of the origin, having a critical point at $0$. The gradient $\nabla f(x)$ splits into its radial component $\frac{\partial f}{\partial r}(x) \frac{x}{|x|}$ and the spherical one $\nabla' f(x) = \nabla f(x) - \frac{\partial f}{\partial r}(x) \frac{x}{|x|}$.

We shall denote $\frac{\partial f}{\partial r}$ by $\partial_r f$.

For $\epsilon > 0$ define $W^\epsilon = \{ x \mid f(x) \neq 0, \epsilon |\nabla' f| \leq |\partial_r f| \}$. There exists a finite subset of positive rationals $L(f) \subset \mathbb{Q}^+$ such that for any $\epsilon > 0$ and any sequence $W^\epsilon \ni x \to 0$ there is a subsequence $W^\epsilon \ni x' \to 0$ and $\ell \in L(f)$ such that

$$\frac{|x'| \partial_r f(x')}{f(x')} \to \ell.$$

Elements of $L(f)$ are called characteristic exponents.

Fix $\ell > 0$, not necessarily in $L(f)$, and consider $F = f/|x|^{\ell}$ defined in the complement of the origin. We say that $a \in \mathbb{R}$ is an asymptotic critical value of $F$ at the origin if there exists a sequence $x \to 0, x \neq 0$, such that

$$|x| \cdot |\nabla F(x)| \to 0, \quad F(x) = \frac{f(x)}{|x|^{\ell}} \to a.$$

The set of asymptotic critical values of $F$ is finite.

The real number $a \neq 0$ is an asymptotic critical value if and only if there
exists a sequence \( x \to 0, x \neq 0 \), such that
\[
\frac{\nabla' f(x)}{\partial_x f(x)} \to 0 , \quad \frac{f(x)}{|x|^{\ell}} \to a.
\]

Hence the set
\[
L'(f) = \{ (\ell, a) \mid \ell \in L(f), a < 0 \text{ is an asymptotic critical value of } f/|x|^{\ell} \}
\]
is a finite subset of \( \mathbb{Q}^+ \times \mathbb{R}_- \), where \( \mathbb{R}_- \) is the set of negative real numbers.

We shall write \( T(f) \) for the set of non-trivial trajectories of the gradient flow \( \dot{x} = \nabla f(x) \) converging to the origin. By Section 6 of [8], for every such a trajectory \( x(t) \), with \( x(t) \to 0 \), there exists a unique pair \( (\ell', a') \in L'(f) \) such that \( f(x(t))/|x(t)|^{\ell'} \to a' \). There is a natural partition of \( T(f) \) associated with \( L'(f) \). Namely for \( (\ell', a') \in L'(f) \),
\[
T^{\ell', a'}(f) = \{ x(t) \in T(f) \mid f(x(t))/|x(t)|^{\ell'} \to a' \text{ as } x(t) \to 0 \}.
\]

In the set \( \mathbb{Q}^+ \times \mathbb{R}_- \) we may introduce the lexicographic order
\[
(\ell', a') \leq (\ell, a) \text{ if } \ell' < \ell, \text{ or } \ell' = \ell \text{ and } a' \leq a.
\]

Take \( (\ell, a) \in \mathbb{Q}^+ \times \mathbb{R}_- \setminus L'(f) \). We shall write
\[
\tilde{T}^{\ell, a}(f) = \bigcup T^{\ell', a'}(f), \text{ where } (\ell', a') < (\ell, a) \text{ and } (\ell', a') \in L'(f).
\]

According to [15], there are \( 0 < -y \ll r \ll 1 \) such that each trajectory \( x(t) \in T(f) \) intersects \( F_r(y) \) transversally at exactly one point. Let \( \Gamma(f) \subset F_r(y) \) be the union of all those points. So there is a natural one-to-one correspondence between trajectories in \( T(f) \) and points in \( \Gamma(f) \). The same way one may define the set \( \Gamma^{\ell', a'}(f) \subset F_r(y) \) (resp. \( \tilde{T}^{\ell, a}(f) \subset F_r(y) \)) whose points are in one-to-one correspondence with trajectories from \( T^{\ell', a'}(f) \) (resp. \( \tilde{T}^{\ell, a}(f) \)). In particular, for \( (\ell, a) \in \mathbb{Q}^+ \times \mathbb{R}_- \setminus L'(f) \) the set
\[
\tilde{\Gamma}^{\ell, a}(f) = \bigcup \Gamma^{\ell', a'}(f), \text{ where } (\ell', a') < (\ell, a) \text{ and } (\ell', a') \in L'(f),
\]
is a subset of \( \Gamma(f) \).

By [15 Theorem 12], [2 Theorem 6] we have
Theorem 3.1. If $0 < -y \ll r \ll 1$ then the inclusion $\Gamma(f) \subset F_r(y)$ induces an isomorphism
\[
\tilde{H}^*(\Gamma(f)) \cong H^*(F_r(y)),
\]
where $\tilde{H}^*(\cdot)$ is the Čech-Alexander cohomology group. In particular $\Gamma(f)$ has the same (finite) number of connected components as $F_r(y)$.

Moreover, for every $(\ell, a) \in \mathbb{Q}^+ \times \mathbb{R}_- \setminus L'(f)$ the set $\tilde{\Gamma}^{\ell,a}(f)$ is a closed subset of $F^{\ell,a}(y)$. The inclusion induces an isomorphism
\[
\tilde{H}^*(\tilde{\Gamma}^{\ell,a}(f)) \cong H^*(F^{\ell,a}(y)).
\]

Remark 3.2. If $\Gamma(f)$ is infinite then it contains at least one compact and infinite connected component, which is obviously not a zero-dimensional space. If that is the case then the Menger-Urysohn dimension as well as the Čech-Lebesgue covering dimension of this component is at least one (see [3]).

By Lemma 2.1 and Corollary 2.2 we get

Corollary 3.3. There is an isomorphism $\tilde{H}^*(\Gamma(f)) \cong H^*(S_r)$. In particular $\Gamma(f)$ has the same (finite) number of connected components as $S_r$.

Moreover, for every $(\ell, a) \in \mathbb{Q}^+ \times \mathbb{R}_- \setminus L'(f)$, if $0 < r \ll 1$ then
\[
\tilde{H}^*(\tilde{\Gamma}^{\ell,a}(f)) \cong H^*(S_r^{\ell,a}).
\]

In particular, if there exists $i \geq 1$ such that $H^i(S_r) \neq 0$ then $T(f)$ is infinite. So, if $S_r \neq \emptyset$ and the Euler-Poincaré characteristic $\chi(S_r) \leq 0$, then $T(f)$ is infinite.

Example 3.4. The polynomial $f(x, y, z) = x^3 + x^2z - y^2$ is weighted homogeneous. Of course $S_r \neq \emptyset$. By [17, p.245], the Euler-Poincaré characteristic $\chi(S_r^2 \cap \{f \geq 0\}) = 2$. By the Alexander duality theorem we have $\chi(S_r) = 0$. Hence the set $T(f)$ is infinite.

Proposition 3.5. If $H^i(\Omega) \neq 0$ for some $i \geq 1$ then $T(f)$ is infinite.

Proof. As $L'(f)$ is finite, there are $0 < -a \ll r \ll 1$ such that $(d, a) \not\in L'(f)$. By Corollary 3.3 and Lemma 2.3 we have
\[
\tilde{H}^i(\tilde{\Gamma}^{d,a}(f)) \cong H^i(S_r^{d,a}) \cong H^i(\Omega) \neq 0.
\]

In particular $\tilde{\Gamma}^{d,a}(f)$ is infinite. Hence $\tilde{T}^{d,a}(f)$, as well as $T(f)$, is infinite. \qed
Corollary 3.6. If $\omega$ is a quadratic form which may be reduced to the diagonal form $-x_1^2 - \cdots - x_{i+1}^2 + x_{i+2}^2 + \cdots + x_j^2$, where $i \geq 1$, then

$$\bar{H}^i(\tilde{\Gamma}^2, a(f)) \simeq H^i(\Omega) \simeq H^i(S^i) \neq 0.$$ 

Hence $T(f)$ is infinite.

Example 3.7. Let $f(x, y, z) = z(x^2 + y^2) + x^2 y^2 z - z^4$. It is easy to see that $S_r = S^2_r \cap \{f < 0\}$ is homeomorphic to a union of two disjoint 2-discs, so that $H^i(S_r) = 0$ for $i \geq 1$. As $\omega = z(x^2 + y^2)$, then $\Omega$ is homeomorphic to $S^1 \times (0, 1)$, and so $H^1(\Omega) \neq 0$. Hence $T(f)$ is infinite.

Corollary 3.8. If $\Omega \neq \emptyset$ and the Euler-Poincaré characteristic $\chi(\Omega) \leq 0$, then $T(f)$ is infinite.

Investigating the gradient flow in polar coordinates and applying arguments presented by Moussu in [14, p.449] the reader may also prove the next proposition. (As its proof would require to introduce other techniques, so we omit it here.)

Proposition 3.9. Suppose that there exists a non-degenerate critical point of $\omega|\Omega$ which is not a local minimum. Then $T(f)$ is infinite.

In particular, if there exists a non-degenerate local maximum of $\omega|\Omega$ then the interior of the stable set of the origin is non-empty.

Example 3.10. Let $f(x, y) = x^3 + 3xy^2 + x^2 y^2$, so that $\omega = x^3 + 3xy^2$. It is easy to see that $\omega|S^1$ has a non-degenerate local maximum at $(-1, 0) \in \Omega$. Then the interior of the stable set of the origin is non-empty. In particular $T(f)$ is infinite.

4 Main results

Theorem 4.1. Suppose that $f : \mathbb{R}^n, 0 \to \mathbb{R}$ is an analytic function having a critical point at the origin.

If $\text{rank } H^0(S_r) < \text{rank } H^0(\Omega)$, i.e. the number of connected components of $S_r$ is smaller than the number of connected components of $\Omega$, then the set of trajectories of the gradient flow $\dot{x} = \nabla f(x)$ converging to the origin is infinite.
Proof. Suppose, contrary to our claim, that \( T(f) \) is finite. Then \( \Gamma(f) \) is finite, and for any \((\ell, a) \in \mathbb{Q}^+ \times \mathbb{R} \setminus L'(f)\) the set \( \tilde{\Gamma}^{\ell, a}(f) \) is finite too. Hence \( \text{rank} \ H^0(\tilde{\Gamma}^{\ell, a}(f)) \) equals the number of elements in \( \tilde{\Gamma}^{\ell, a}(f) \).

By Lemma 2.3, there exist \( 0 < r \ll -a \ll 1 \) such that \( \Omega \) and \( S^r_{d,a} \) have the same homotopy type. By Corollary 3.3, the group \( H^i(S_r) \) is isomorphic to \( H^i(\Gamma(f)) \). Hence \( \text{rank} \ H^0(S_r) = \text{rank} \ H^0(\Gamma(f)) \) equals the number of elements in \( \Gamma(f) \). Moreover, \( \text{rank} \ H^0(\Omega) = \text{rank} \ H^0(S^r_{d,a}) = \text{rank} \ H^0(\tilde{\Gamma}^{d,a}(f)) \) equals the number of elements in \( \tilde{\Gamma}^{d,a}(f) \).

As \( \tilde{\Gamma}^{d,a}(f) \subset \Gamma(f) \), then \( \text{rank} \ H^0(\Omega) \leq \text{rank} \ H^0(S^r_{d,a}) \), which contradicts the assumption. \( \square \)

**Theorem 4.2.** If \( \chi(S_r) < \chi(\Omega) \) then \( T(f) \) is infinite.

**Proof.** By Corollary 3.3 and Proposition 3.5, it is enough to consider the case where all cohomology groups \( H^i(S_r), H^i(\Omega) \), where \( i \geq 1 \), are trivial.

If that is the case then \( \text{rank} \ H^0(S_r) = \chi(S_r) < \chi(\Omega) = \text{rank} \ H^0(\Omega) \). By Theorem 4.1, the set \( T(f) \) is infinite. \( \square \)

**Example 4.3.** Let \( f(x, y) = x^3 - y^2 \), so that \( \omega = -y^2 \). Then \( \Omega = \{(x, y) \in S^1 \mid -y^2 < 0\} = S^1 \setminus \{(\pm 1, 0)\} \). Obviously \( \Omega \) has two connected components and \( H^i(\Omega) = 0 \) for any \( i \geq 1 \). The function \( \omega|\Omega \) has exactly two critical (minimum) points at \((0, \pm 1)\). As \( S_r \) is homeomorphic to an interval, then by Theorem 4.1 the set \( T(f) \) is infinite. \( \square \)

**Example 4.4.** Let \( f(x, y, z) = xyz - z^4 \), so that \( \omega = xyz \). It is easy to see that \( \Omega \) is homeomorphic to a disjoint union of four discs, and \( S_r \) is homeomorphic to a disjoint union of two discs. By Theorem 4.1 the set \( T(f) \) is infinite.

**Example 4.5.** Let \( f(x, y, z) = xyz + x^4y - 2y^4z + 3xz^4 \), so that \( f \) has an isolated critical point at the origin and \( \omega = xyz \). Applying Andrzej Łęcki computer program (see [11]) we have verified that the local topological degree of the mapping

\[ \mathbb{R}^3, 0 \ni (x, y, z) \mapsto -\nabla f(x, y, z) \in \mathbb{R}^3, 0 \]

equals zero. By [6], [7], the Euler-Poincaré characteristic \( \chi(S^2_r \cap \{f \geq 0\}) = 1 - 0 = 1 \). By the Alexander duality theorem \( \chi(S_r) = 1 \). By Theorem 4.2 the set \( T(f) \) is infinite. \( \square \)
References

[1] C. Böhm, R. Lafuente, M. Simon, Optimal curvature estimates for homogeneous Ricci flows, Int. Math. Res. Not. IMRN 14 (2019) 4431-4468.

[2] A. Dzedzej, Z. Szafraniec, On families of trajectories of an analytic gradient vector field, Ann. Polon. Math. 87 (2005) 99-109.

[3] R. Engelking, Outline of general topology. Translated from the Polish by K. Sieklucki North-Holland Publishing Co., Amsterdam; PWN-Polish Scientific Publishers, Warsaw; Interscience Publishers Division John Wiley & Sons, Inc., New York, 1968.

[4] P. Goldstein, Gradient flow of a harmonic function in $\mathbb{R}^3$, J. Differential Equations 247 (9) (2009) 2517-2557.

[5] R. Hardt, Stratification of real analytic mappings and images, Invent. Math. 28 (1975) 193-208.

[6] G. M. Khimshiashvili, On the local degree of a smooth mapping, Comm. Acad. Sci. Georgian SSR. 85 (1977) 309-311 (in Russian).

[7] G. M. Khimshiashvili, On the local degree of a smooth mapping, Trudy Tbilisi Math. Inst. 64 (1980) 105-124.

[8] K. Kurdyka, T. Mostowski, A. Parusiński, Proof of the gradient conjecture of R. Thom, Ann. of Math. 152 (2000) 763-792.

[9] K. Kurdyka, A. Parusiński, Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture, Advanced Studies in Pure Mathematics 43 (2006), Singularity Theory and Its Applications, pp. 137-177.

[10] C. Lageman, Convergence of gradient-like dynamical systems and optimization algorithms, Ph.D. Thesis, University of Würzburg, 2007.

[11] A. Łęcki, Z. Szafraniec, Applications of the Eisenbud & Levine’s theorem to real algebraic geometry, Computational Algebraic Geometry, Progr. Math. 109, Birkhäuser 1993, 177-184.
[12] S. Łojasiewicz, Sur les trajectoires du gradient d’une fonction analytique, Seminari di Geometria 1982-1983, Università di Bologna, Istituto di Geometria, Dipartamento di Matematica (1984) 115-117.

[13] J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies 61, Princeton Univ. Press, Princeton, NJ, 1968.

[14] R. Moussu, Sur la dynamique des gradients. Existence de varietes invariantes, Math. Ann. 307 (1997) 445-460.

[15] A. Nowel, Z. Szafraniec, On trajectories of analytic gradient vector fields, J. Differential Equations 184 (2002) 215-223.

[16] A. Nowel, Z. Szafraniec, On trajectories of analytic gradient vector fields on analytic manifolds, Topol. Methods Nonlinear Anal. 25 (2005) 167-182.

[17] Z. Szafraniec, Topological invariants of weighted homogeneous polynomials, Glasgow Math. J. 33 (1991) 241-245.

Zbigniew SZAFRANIEC
Institute of Mathematics, University of Gdańsk
80-952 Gdańsk, Wita Stwosza 57, Poland
Zbigniew.Szafraniec@mat.ug.edu.pl