BETHE SUBALGEBRAS OF $U_q(\widehat{\mathfrak{g}_n})$ VIA SHUFFLE ALGEBRAS

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Abstract. In this article we construct certain commutative subalgebras of the big shuffle algebra of type $A_n^{(1)}$. This can be considered as a generalization of the similar construction for the small shuffle algebra, obtained in [PHHSY]. We present a functional realization of these subalgebras. The latter identifies them with the Bethe subalgebras of $U_q(\widehat{\mathfrak{g}_n})$.

Introduction

Elliptic shuffle algebras were first introduced and studied by the first author and A. Odesskii, see [FO1, FO2, FO3]. In the loc.cit., they were associated with an elliptic curve $E$ endowed with two automorphisms $\tau_1, \tau_2$. Degenerating $E$ into $\mathbb{CP}^1$ with a double point, one obtains small shuffle algebras, depending on two parameters (alternatively $q_1, q_2, q_3$ with $q_1 q_2 q_3 = 1$). These algebras became of interest in the recent years, due to their geometric interpretations and different algebraic incarnations (see [FHHSY, FT, N1, SV] for the related results).

In this paper we study another kind of degenerations, obtained by degenerating the elliptic curve $E$ into a chain of intersecting $\mathbb{CP}^1$’s with the incidence matrix being of $A_n^{(1)}$-type. The degenerations of $\tau_1, \tau_2$ are given by two discrete parameters $a, b \in \mathbb{Z}_n$ and $2n$ continuous parameters $u_1, \ldots, u_{2n}$. We call the corresponding algebras the big shuffle algebras. We consider only the simplest case: \{a, b, −a − b\} = {−1, 0, 1}, $u_i = 1$. This algebra was also recently considered in [N2], where it was identified with the positive half of the quantum toroidal $\hat{U}_{q,d}(\mathfrak{sl}_n)$.

The aim of this paper is to study particular large commutative subalgebras of the big shuffle algebra $S$, similar to the one from [PHHSY]. We also establish a functional realization of these subalgebras (which seems to be new even for the small shuffle algebras). It allows to identify those with the standard Bethe subalgebras of the quantum affine algebra $U_q(\widehat{\mathfrak{g}_n})$, which is horizontally embedded into the quantum toroidal algebra $\hat{U}_{q,d}(\mathfrak{sl}_n)$. This construction admits a one-parameter deformation, leading to the commutative subalgebras $A(s_1, \ldots, s_n; t) \subset S^2 \cdot \wedge^2$. These algebras are closely related to the study of nonlocal integrals of motion for the deformed $W$-algebras $W_{q,t}(\mathfrak{sl}_n)$ from [FKSW], as well as provide a framework for the generalization of the recent results from [EJMM2] to $\hat{U}_{q,d}(\mathfrak{sl}_n)$. This will be elaborated elsewhere.

This paper is organized as follows:

• In Section 1, we recall the definition and key results about the quantum toroidal algebra $\hat{U}_{q,d}(\mathfrak{sl}_n)$, $n > 2$. We also recall the notion of the small shuffle algebra $S^m$ and its commutative subalgebra $A^m$, and introduce a higher rank generalization, the big shuffle algebra $S$.

• In Section 2, we introduce a family of subspaces $A(s_1, \ldots, s_n) \subset S$ depending on $n$ parameters and generalizing the construction of $A^m \subset S^m$. We prove that $A(s_1, \ldots, s_n; \frac{1}{s_1 \cdots s_{n-1}})$ is a polynomial algebra on explicitly given generators if $s_1, \ldots, s_{n-1}$ are generic, see Theorem 2.3.

• In Section 3, we use the universal $R$-matrix and vertex-type representations to establish an alternative viewpoint towards $A(s_1, \ldots, s_n)$. This allows to identify them with the well-known Bethe subalgebras of the quantum affine $U_q(\widehat{\mathfrak{g}_n})$, horizontally embedded into $\hat{U}_{q,d}(\mathfrak{sl}_n)$.

• In Section 4, we discuss the generalizations of previous results to the cases $n = 1, 2$. 


1. Basic definitions and constructions

1.1. The quantum toroidal of $\mathfrak{sl}_n$.

Let $q, d \in \mathbb{C}^*$ be two parameters. We set $[n] := \{0, 1, \ldots, n-1\}$, $[n]^\times := [n] \setminus \{0\}$, the former viewed as a set of mod $n$ residues. Let $q_m(z) := \frac{q^m - q^{-m}}{z - q^{-1}}$. Define $\{a_{i,j}, m_{i,j} | i, j \in [n]\}$ by

$$a_{i,i} = 2, a_{i,i+1} = -1, m_{i,i+1} = 1, a_{i,j} = m_{i,j} = 0$$

otherwise.

The quantum toroidal algebra of $\mathfrak{sl}_n$, denoted $U_{q,d}(\mathfrak{sl}_n)$, is the unital associative algebra, generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma_{\pm 1/2}, q^\pm d_1, q^\pm d_2\}_{k \in [n]}$ with the following defining relations:

(T0.1) \[
\left[\psi^\pm_i(z), \psi^\pm_j(w)\right] = 0, \quad \gamma^{\pm 1/2} \text{ central},
\]

(T0.2) \[
\psi^\pm_{i,0} \cdot \psi^\mp_{i,0} = \gamma^{1/2} \cdot \gamma^{1/2} = q^d_1 \cdot q^d_1 = q^d_2 \cdot q^d_2 = 1,
\]

(T0.3) \[
q^d_1 e_i(z) q^{-d_1} = e_i(qz), \quad q^d_1 f_i(z) q^{-d_1} = f_i(qz), \quad q^d_i \psi^\pm_i(z) q^{-d_1} = \psi^\pm_i(qz),
\]

(T0.4) \[
q^d_1 e_i(z) q^{-d_2} = q e_i(z), \quad q^d_1 f_i(z) q^{-d_2} = q^{-1} f_i(z), \quad q^d_2 \psi^\pm_i(z) q^{-d_2} = \psi^\pm_i(z),
\]

(T1) \[
g_{a_{i,j}}(\gamma^{-d_{m_{i,j}}^0} z/w) \psi^+_i(z) \psi^-_j(w) = g_{a_{i,j}}(\gamma^{d_{m_{i,j}}^0} z/w) \psi^-_j(w) \psi^+_i(z),
\]

(T2) \[
e_i(z) e_j(w) = g_{a_{i,j}}(d_{m_{i,j}}^0 z/w) e_j(w) e_i(z),
\]

(T3) \[
f_i(z) f_j(w) = g_{a_{i,j}}(d_{m_{i,j}}^0 z/w)^{-1} f_j(w) f_i(z),
\]

(T4) \[
(q - q^{-1})[e_i(z), f_j(w)] = \delta_{i,j} \left(\delta(\gamma w/z) \psi^+_i(\gamma^{1/2} w) - \delta(\gamma z/w) \psi^-_i(\gamma^{1/2} z)\right),
\]

(T5) \[
\psi^+_i(z) e_j(w) = g_{a_{i,j}}(\gamma^{1/2} d_{m_{i,j}}^0 z/w) e_j(w) \psi^+_i(z),
\]

(T6) \[
\psi^-_i(z) f_j(w) = g_{a_{i,j}}(\gamma^{1/2} d_{m_{i,j}}^0 z/w)^{-1} f_j(w) \psi^+_i(z),
\]

(T7.1) \[
\text{Sym}_{\otimes 2} [e_i(z_1), e_i(z_2), e_{i,\pm 1}(w)] \big|_{q \rightarrow 1} = 0,
\]

(T7.2) \[
\text{Sym}_{\otimes 2} [f_i(z_1), f_i(z_2), f_{i,\pm 1}(w)] \big|_{q \rightarrow 1} = 0,
\]

where the generating series are defined as follows:

$$e_i(z) := \sum_{k = -\infty}^{\infty} e_{i,k} z^{-k}, \quad f_i(z) := \sum_{k = -\infty}^{\infty} f_{i,k} z^{-k}, \quad \psi^\pm_i(z) := \psi^\pm_{i,0} + \sum_{i,j > 0} \psi_{i,j} z^{i+j}, \quad \delta(z) := \sum_{k = -\infty}^{\infty} z^k.$$

It will be convenient to use the generators $\{h_{i,k}\}$ instead of $\{\psi_{i,j}\}$, defined by

$$\exp \left(\pm(q - q^{-1}) \sum_{k > 0} h_{i,k} z^{i+k} \right) = \psi^\pm_i(z) := \psi^\pm_{i,0} \psi^\pm_i(z), \quad h_{i,k} \in \mathbb{C}[\psi^\pm_{i,0}, \psi_{i,\pm 1}, \psi_{i,\pm 2}, \ldots].$$

Then the relations (T5,T6) are equivalent to the following (we use $[m] := (q^m - q^{-m})/(q - q^{-1})$):

(T5') \[
\psi_{i,0} e_{j,l} = q^{a_{i,j} c_{j,l}} e_{i,l} \psi_{i,0}, \quad [h_{i,\pm k}, e_{j,l}] = d^{-km_{i,j}} \gamma^{-k/2} \frac{[ka_{i,j}]}{k} e_{j,l} \pm k (k > 0),
\]

(T6') \[
\psi_{i,0} f_{j,l} = q^{-a_{i,j} c_{j,l}} f_{i,l} \psi_{i,0}, \quad [h_{i,\pm k}, f_{j,l}] = -d^{-km_{i,j}} \gamma^{-k/2} \frac{[ka_{i,j}]}{k} f_{j,l} \pm k (k > 0).
\]

We also introduce $h_{i,0}, c, c'$ via $\psi_{i,0} = q^{h_{i,0}} \gamma^{1/2} = q^c, c' = \sum_i h_{i,0}$, so that $c, c'$ are central.

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1 Our notation are consistent with those of [LY], but following [S] we add the elements $q^d_1, q^d_2$ satisfying (T0.3), (T0.3). This update is essential for our discussion of the Drinfeld double and the universal $R$-matrix.
Let $\hat{U}^-$ and $\hat{U}^+$ be the subalgebras of $U_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,j}\}$ and $\{f_{i,j}\}$ respectively, while $\hat{U}^0$ is generated by $\{\psi_{i,j}, \psi^{-1}_{i,0}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}$.

**Proposition 1.1.** (Triangular decomposition) The multiplication map $m : \hat{U}^- \otimes \hat{U}^0 \otimes \hat{U}^+ \to U_{q,d}(\mathfrak{sl}_n)$ is an isomorphism of vector spaces.

We equip the algebra $U_{q,d}(\mathfrak{sl}_n)$ with the $\mathbb{Z}^n \times \mathbb{Z}$-grading by assigning

\[
\deg(e_{i,k}) := (1; k), \quad \deg(f_{i,k}) := (-1; k), \quad \deg(\psi_{i,k}) := (0; k), \quad \deg(x) := (0; 0) \text{ for } x = \psi_{i,0}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}.
\]

**1.2.** Horizontal and vertical $U_q(\mathfrak{sl}_n)$

Following [VV], we introduce the vertical and horizontal copies of the quantum affine algebra of $\mathfrak{sl}_n$, denoted $\hat{U}_q(\mathfrak{sl}_n)$, inside $U_{q,d}(\mathfrak{sl}_n)$. Consider a subalgebra $\hat{U}^\psi(\mathfrak{sl}_n)$ of $U_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi^{-1}_{i,0}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2} | i \in [n]^\times\}$. This algebra is isomorphic to $U_q(\mathfrak{sl}_n)$, realized via the “new Drinfeld presentation”. Let $\hat{U}^h(\mathfrak{sl}_n)$ be the subalgebra of $U_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,0}, f_{i,0}, \psi^{-1}_{i,0}, q^{\pm d_2} | i \in [n]\}$. This algebra is also isomorphic to $U_q(\mathfrak{sl}_n)$, realized via the classical Drinfeld-Jimbo presentation.

Following [FJMM1], we recall a slight upgrade of this construction, which provides two copies of the quantum affine algebra of $\mathfrak{sl}_n$, rather than $\mathfrak{sl}_n$, inside $U_{q,d}(\mathfrak{sl}_n)$. For every $r \neq 0$, choose $\{a_{i,r} | i \in [n]\} = \{c_{i,r} | i \in [n]\}$ to be a nontrivial solution of the following system of linear equations\(^2\)

\[
\sum_{i=0}^{n-1} c_{i,r} \frac{[r a_{i,j}]}{[r]} = 0, \quad j \in [n]^\times.
\]

Let $\mathfrak{h}^\psi$ be the subspace of $U_{q,d}(\mathfrak{sl}_n)$ spanned by

\[
\mathfrak{h}^\psi_r = \begin{cases} 
\sum_{i \in [n]} c_{i,r} h_{i,r} & \text{if } r \neq 0 \\
\gamma^{1/2} & \text{if } r = 0
\end{cases}.
\]

Note that $\mathfrak{h}^\psi$ is well-defined and commutes with $\hat{U}^\psi(\mathfrak{sl}_n)$, due to (T5’, T6’). Moreover, $\mathfrak{h}^\psi$ is isomorphic to the Heisenberg Lie algebra. Let $\hat{U}^\psi(\mathfrak{gl}_n)$ be the subalgebra of $U_{q,d}(\mathfrak{sl}_n)$, generated by $\hat{U}^\psi(\mathfrak{sl}_n)$ and $\mathfrak{h}^\psi$. The above discussions imply that $\hat{U}^\psi(\mathfrak{gl}_n) \simeq U_q(\mathfrak{gl}_n)$, the quantum affine algebra of $\mathfrak{gl}_n$. We let $\hat{U}^\psi(\mathfrak{gl}_1) \subset \hat{U}^\psi(\mathfrak{gl}_n)$ be the subalgebra generated by $\mathfrak{h}^\psi$.

Our next goal is to provide a horizontal copy of $\hat{U}_q(\mathfrak{gl}_n)$, containing $\hat{U}^h(\mathfrak{sl}_n)$. One way to do this was proposed in [FJMM1] and it is based on the following beautiful result:

**Theorem 1.2.** [M] There exists an automorphism $\pi$ of $\hat{U}_{q,d}(\mathfrak{sl}_n)$ such that

\[
\pi(\hat{U}^\psi(\mathfrak{sl}_n)) = \hat{U}^h(\mathfrak{sl}_n), \quad \pi(\hat{U}^h(\mathfrak{sl}_n)) = \hat{U}^\psi(\mathfrak{sl}_n),
\]

\[
\pi(q^{d_1}) = q^{d_2}, \quad \pi(q^{d_2}) = q^{-d_1}, \quad \pi(q^\gamma) = q^\psi, \quad \pi(q^\psi) = q^{-\gamma}.
\]

Let us define $\mathfrak{h} := \pi(\mathfrak{h}^\psi)$ and let $\hat{U}^h(\mathfrak{gl}_n)$ be the subalgebra of $\hat{U}_{q,d}(\mathfrak{sl}_n)$, generated by $\hat{U}^h(\mathfrak{sl}_n)$ and $\mathfrak{h}$. Then $\hat{U}^h(\mathfrak{gl}_n) = \pi(\hat{U}^\psi(\mathfrak{gl}_n))$ and it is isomorphic to $U_q(\mathfrak{gl}_n)$. We also define $\hat{U}^h(\mathfrak{gl}_1) \subset \hat{U}^h(\mathfrak{gl}_n)$ as the subalgebra generated by $\mathfrak{h}$.

However, this construction is not very enlightening, as the images $\pi(\mathfrak{h}^\psi)$ are hardly computable. An alternative approach, based on the RTT-realization of $U_q(\mathfrak{gl}_n)$, was proposed in [N2]. We will discuss the related results in Section 2.2.
1.3. Hopf pairing, Drinfeld double and the universal $R$-matrix.

We recall the general notion of Hopf pairing, following [KRT, Chapter 3]. Given two Hopf algebras $A$ and $B$ with invertible antipodes $S_A$ and $S_B$, the bilinear map

$$\varphi : A \times B \to \mathbb{C}$$

is called the Hopf pairing if it satisfies the following properties:

$$\varphi(a, bb') = \varphi(a_1, b) \varphi(a_2, b') \quad \forall \ a \in A, \ b, b' \in B,$$
$$\varphi(aa', b) = \varphi(a, b_2) \varphi(a', b_1) \quad \forall \ a, a' \in A, \ b \in B,$$
$$\varphi(1_A, b) = \epsilon_B(b) \quad \forall \ a \in A, \ b \in B,$$
$$\varphi(S_A(a), b) = \varphi(a, S_B^{-1}(b)) \quad \forall \ a \in A, \ b \in B,$$

where we use the Sweedler notation for the coproduct:

$$\Delta(x) = x_1 \otimes x_2.$$

For such a data, one can define the generalized Drinfeld double $D_\varphi(A, B)$ as follows:

**Theorem 1.3.** [KRT, Theorem 3.2] There is a unique Hopf algebra $D_\varphi(A, B)$ satisfying the following properties:

(i) As coalgebras $D_\varphi(A, B) \simeq A \otimes B$.  
(ii) Under the natural inclusions

$$A \hookrightarrow D_\varphi(A, B)$$

given by $a \mapsto a \otimes 1$,

$$B \hookrightarrow D_\varphi(A, B)$$

given by $b \mapsto 1 \otimes b$,

$A$ and $B$ are Hopf subalgebras of $D_\varphi(A, B)$.  
(iii) For any $a \in A, b \in B$, we have

$$(a \otimes 1) \cdot (1 \otimes b) = a \otimes b$$

and

$$(1 \otimes b) \cdot (a \otimes 1) = \varphi(S_A^{-1}(a_1), b_1) \varphi(a_3, b_3) a_2 \otimes b_2.$$  

**Remark 1.4.** The notion of the Drinfeld double is reserved for the case $B = A^{* \cop}$ with $\varphi$ being the natural pairing.

A Hopf algebra $A$ is quasitriangular (formally quasitriangular) if there is an invertible element $R \in A \otimes A$ (or $R \in \hat{A} \otimes \hat{A}$) satisfying the following properties:

$$R \Delta(x) = \Delta^{op}(x)R \quad \forall \ x \in A,$$
$$\Delta \otimes \text{Id}(R) = R^{13} R^{23},$$
$$\text{Id} \otimes \Delta(R) = R^{13} R^{12}.$$  

Such an element $R$ is called the universal $R$-matrix of $A$.

The fundamental property of Drinfeld doubles is their quasitriangularity:

**Theorem 1.5.** [KRT, Theorem 3.2] For a nondegenerate Hopf pairing $\varphi : A \times B \to \mathbb{C}$, the generalized Drinfeld double $D_\varphi(A, B)$ is formally quasitriangular with the universal $R$-matrix

$$R = \sum_i e_i \otimes e_i^*,$$

where $\{e_i\}$ is a basis of $A$ and $\{e_i^*\}$ is the dual basis of $B$ (with respect to $\varphi$).
1.4. The algebra $\bar{U}_{q,d}(\mathfrak{sl}_n)$ as a Drinfeld double.

In order to apply the constructions of the previous section to the quantum toroidal algebra $\bar{U}_{q,d}(\mathfrak{sl}_n)$ and its subalgebras, we need to endow the former with a Hopf algebra structure. This was first done (in a more general setup) in [D1, Theorem 2.1]:

**Theorem 1.6.** The formulas (H1-H9) endow $\bar{U}_{q,d}(\mathfrak{sl}_n)$ with a topological Hopf algebra structure:

(H1) \[ \Delta(e_i(z)) = e_i(z) \otimes 1 + \psi_i^- (\gamma_{(1)}^{1/2} z) \otimes e_i(\gamma_{(1)} z), \]

(H2) \[ \Delta(f_i(z)) = 1 \otimes f_i(z) + f_i(\gamma_{(2)} z) \otimes \psi_i^+ (\gamma_{(2)}^{1/2} z), \]

(H3) \[ \Delta(\psi_i^\pm(z)) = \psi_i^\pm(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi_i^\pm(\gamma_{(1)}^{\mp 1/2} z), \]

(H4) \[ \Delta(x) = x \otimes x \text{ for } x = \gamma_{(1)}^{1/2}, q^{d_1}, q^{d_2}, \]

(H5) \[ \epsilon(e_i(z)) = \epsilon(f_i(z)) = 0, \quad \epsilon(\psi_i^\pm(z)) = 1, \]

(H6) \[ \epsilon(x) = 1 \text{ for } x = \gamma_{(1)}^{1/2}, q^{d_1}, q^{d_2}, \]

(H7) \[ S(e_i(z)) = -\psi_i^-(\gamma_{(1)}^{-1/2} z)^{-1} e_i(\gamma_{(1)}^{-1} z), \]

(H8) \[ S(f_i(z)) = -f_i(\gamma_{(1)}^{-1} z) \psi_i^+(\gamma_{(1)}^{-1/2} z) ^{-1}, \]

(H9) \[ S(x) = x^{-1} \text{ for } x = \gamma_{(1)}^{1/2}, q^{d_1}, q^{d_2}, \psi_i^\pm(z), \]

where $\gamma_{(1)}^{1/2} := \gamma^{1/2} \otimes 1$ and $\gamma_{(2)}^{1/2} := 1 \otimes \gamma^{1/2}$.

Let $\bar{U}^\geq$ be the subalgebra of $\bar{U}_{q,d}(\mathfrak{sl}_n)$ generated by \{\(e_{i,k}, \psi_{i,j}, \psi_{i,0}\) | $\gamma^{1/2}, q^{d_1}, q^{d_2} \in \mathbb{N}$\}, and let $\bar{U}^\leq$ be the subalgebra of $\bar{U}_{q,d}(\mathfrak{sl}_n)$ generated by \{\(f_{i,k}, \psi_{i,l}, \psi_{i,0}\) | $\gamma^{1/2}, q^{d_1}, q^{d_2} \in \mathbb{N}$\}.

Now we are ready to state the main result of this section (the proof is straightforward):

**Theorem 1.7.** (a) There exists a unique Hopf algebra pairing $\varphi : \bar{U}^\geq \times \bar{U}^\leq \to C$ satisfying

(P1) \[ \varphi(e_i(z), f_j(w)) = \frac{\delta_{i,j}}{q - q^{-1}} \cdot \delta \left( \frac{z}{w} \right), \quad \varphi(\psi_i^-(z), \psi_i^+(w)) = g_{a_{i,j}}(d_{m+1} z/w), \]

(P2) \[ \varphi(e_i(z), x^-) = \varphi(x^+, f_i(z)) = 0 \text{ for } x^\pm = \psi_j^\mp(w), \psi_{j,0}^{\pm 1}, \gamma_{(1)}^{1/2}, q^{d_1}, q^{d_2}, \]

(P3) \[ \varphi(\gamma_{(1)}^{1/2}, q^{d_1}) = \varphi(q^{d_1}, \gamma_{(1)}^{1/2}) = q^{-1/2}, \quad \varphi(\psi_i^-(z), q^{d_2}) = q^{-1}, \quad \varphi(q^{d_2}, \psi_i^+(z)) = q, \]

(P4) \[ \varphi(\psi_i^-(z), x) = \varphi(x, \psi_i^+(z)) = 1 \text{ for } x = \gamma_{(1)}^{1/2}, q^{d_1}, \]

(P5) \[ \varphi(\gamma_{(1)}^{1/2}, q^{d_2}) = \varphi(q^{d_2}, \gamma_{(1)}^{1/2}) = \varphi(\gamma_{(1)}^{1/2}, q^{d_1}) = \varphi(\gamma_{(1)}^{1/2}, q^{d_1}) = \varphi(\gamma_{(1)}^{1/2}, q^{d_1}) = 1 \text{ for } 1 \leq k, l \leq 2. \]

(b) The natural Hopf algebra homomorphism $D_{\varphi}(\bar{U}^\geq, \bar{U}^\leq) \to \bar{U}_{q,d}(\mathfrak{sl}_n)$ induces the isomorphism

$$
\Xi : D_{\varphi}(\bar{U}^\geq, \bar{U}^\leq)/I \cong \bar{U}_{q,d}(\mathfrak{sl}_n) \text{ with } I := (x \otimes 1 - 1 \otimes x | x = \psi_{i,0}^{\pm 1}, \gamma_{(1)}^{1/2}, q^{d_1}, q^{d_2}).
$$

(c) Consider a slight modification $\tilde{\bar{U}}_{q,d}(\mathfrak{sl}_n)$, obtained from $\bar{U}_{q,d}(\mathfrak{sl}_n)$ by “throwing away” the generator $q^{d_1}$ and taking the quotient by the central element $c'$. As in (b), this algebra admits the double Drinfeld realization $D_{\varphi'}(\tilde{\bar{U}}^\geq, \tilde{\bar{U}}^\leq)$, where $\tilde{\bar{U}}^\leq$ and $\tilde{\bar{U}}^\geq$ are obtained from $\bar{U}^\leq$ and $\bar{U}^\geq$ by “throwing away” $q^{d_1}$ and taking the quotient by $c'$, while $\varphi'$ is induced from $\varphi$.

(d) The pairings $\varphi$ and $\varphi'$ are nondegenerate iff $q, qd, qd^{-1}$ are not roots of unity.
1.5. Bethe subalgebras.

Let us recall the standard way of constructing large commutative subalgebras of a quasitriangular Hopf algebra $A$. Fix a group-like element $x \in A$ (or in appropriate completion $A^{\wedge}$). For an $A$-representation $V$, we consider the transfer matrix

$$T_V(x) := (1 \otimes \text{tr}_V)((1 \otimes x)R)$$

if the latter is well-defined. The properties of the $R$-matrix imply

$$T_{V_1 \otimes V_2}(x) = T_{V_1}(x) + T_{V_2}(x),$$

$$T_{V_1 \otimes V_2}(x) = T_{V_2}(x) \cdot T_{V_1}(x).$$

In particular, we see that $T_{V_1}(x) \cdot T_{V_2}(x) = T_{V_2}(x) \cdot T_{V_1}(x)$. To summarize, $\bullet \mapsto T_\bullet(x)$ is a ring homomorphism from the Grothendieck group of the suitable subcategory of $A$-modules to the suitable subcategory of $A^{\wedge}$, with the image being a commutative subalgebra of that completion.

The commutative subalgebras constructed in this way are sometimes called the Bethe subalgebras. In Section 3, we will apply this setup to $A = \tilde{U}_{q,d}(\mathfrak{sl}_n)$ and a generic “Cartan” group-like element $x = q^{h_1 h_1 + \cdots + h_n h_n} d_1$.

1.6. Small shuffle algebra.

As a motivating point for the current paper, we briefly recall the notion of the small shuffle algebra and its particular commutative subalgebra. Consider an $\mathbb{N}$-graded $\mathbb{C}$-vector space $S^{sm} = \bigoplus_{n \geq 0} S^{sm}_n$, where $S^{sm}_n$ consists of rational functions $f(x_1, \ldots, x_n)$ with $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^G$ and $\Delta(x_1, \ldots, x_n) := \prod_{i \neq j} (x_i - x_j)$. Define the star-product $\star: S^{sm}_k \times S^{sm}_l \to S^{sm}_{k+l}$ by

$$(F \star G)(x_1, \ldots, x_{k+l}) := \text{Sym}_{s_k+s_l} \left(F(x_1, \ldots, x_k)G(x_{k+1}, \ldots, x_{k+l}) \prod_{i < j} \lambda(x_i/x_j)\right)$$

with

$$\lambda(x) := \frac{(q_1 x - 1)(q_2 x - 1)(q_3 x - 1)}{(x - 1)^3}, \text{ where } q_i \in \mathbb{C}^* \text{ and } q_1 q_2 q_3 = 1.$$ 

This endows $S^{sm}$ with a structure of an associative unital $\mathbb{C}$-algebra.

We say that an element $f(x_1, \ldots, x_n) \in S^{sm}$ satisfies the wheel conditions iff $f(x_1, \ldots, x_n) = 0$ for any $\{x_1, \ldots, x_n\} \subset \mathbb{C}$ such that $x_i/x_j = q_1, x_2/x_3 = q_j, i \neq j$. Let $S^{sm}_n \subset S^{sm}$ be an $\mathbb{N}$-graded subspace, consisting of all such elements. The subspace $S^{sm}$ is $\star$-closed.

**Definition 1.8.** The algebra $(S^{sm}, \star)$ is called the small shuffle algebra.

Following [FHHSY], we introduce an important $\mathbb{N}$-graded subspace $A^{sm} = \bigoplus_n A^{sm}_n$ of $S^{sm}$. Its degree $n$ component is defined by $A^{sm}_n := \{F \in S^{sm}_n | \partial^{(0,k)} F = 0 \forall 0 \leq k \leq n\}$, where

$$\partial^{(0,k)} F := \lim_{\xi \to 0} F(x_1, \ldots, \xi x_{n-k+1}, \ldots, \xi x_n), \partial^{(\infty,k)} F := \lim_{\xi \to \infty} F(x_1, \ldots, \xi x_{n-k+1}, \ldots, \xi x_n).$$

This subspace satisfies the following properties:

**Theorem 1.9.** [FHHSY Section 2] We have:

(a) Suppose $F \in S^{sm}_n$ and $\partial^{(\infty,k)} F \neq 0$ for all $1 \leq k \leq n$, then $F \in A^{sm}_n$.

(b) The subspace $A^{sm} \subset S^{sm}$ is $\star$-commutative.

(c) $A^{sm}$ is $\star$-closed and it is a polynomial algebra in $\{K_j\}_{j \geq 1}$ with $K_j \in S^{sm}_j$ defined by:

$$K_1(x_1) = x_1^0, \ K_2(x_1, x_2) = \frac{(x_1 - q_1 x_2)(x_2 - q_1 x_1)}{(x_1 - x_2)^2}, \ K_m(x_1, \ldots, x_m) = \prod_{1 \leq i < j \leq m} K_2(x_i, x_j).$$
1.7. Big shuffle algebra.
Consider an $\mathbb{N}^n$-graded $\mathbb{C}$-vector space
\[ S = \bigoplus_{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n} S_{\mathbf{k}}, \]
where $S_{k_1, \ldots, k_n}$ consists of $\prod S_{k_i}$-symmetric rational functions in the variables $\{x_{i,j}\}_{1 \leq i \leq k_i}$. We also fix an $n \times n$ matrix of rational functions $\Omega = (\omega_{i,j}(z))_{i,j=1}^n \in \text{Mat}_{n \times n}(\mathbb{C}(z))$ by setting
\[ \omega_{i,i}(z) = \frac{z-q^{-2}}{z-1}, \quad \omega_{i,i+1}(z) = \frac{d^{-1}z-q}{z-1}, \quad \omega_{i,i-1}(z) = \frac{z-q d^{-1}}{z-1}, \quad \text{and} \quad \omega_{i,j}(z) = 1 \text{ otherwise}. \]

Let us now introduce the bilinear $\star$-product on $S$: given $f \in S_{\mathbf{k}}, g \in S_{\mathbf{t}}$ define $f \star g \in S_{\mathbf{k+t}}$ by
\[ (f \star g)(x_{1,1}, \ldots, x_{1,k_1+1}; \ldots; x_{n,1}, \ldots, x_{n,k_n+l}) := \text{Sym} \prod_{i} S_{\mathbf{k}_i}, \]
\[ (f(x_{1,1}, \ldots, x_{1,k_1+1}; \ldots; x_{n,1}, \ldots, x_{n,k_n+l}) g(x_{1,k_1+1}, \ldots, x_{1,k_1+l}; \ldots, x_{n,k_n+l}) \times \prod_{i,j=1}^n \prod_{j' < k_i} \omega_{i,i'}(x_{i,j}/x_{i',j'}). \]

This endows $S$ with a structure of an associative unital algebra with the unit $1 \in S_{0,\ldots,0}$. We will be interested only in a certain subspace of $S$, defined by the pole and wheel conditions:

- We say that $F \in S_{\mathbf{k}}$ satisfies the pole conditions iff
\[ F = \frac{f(x_{1,1}, \ldots, x_{n,k_n})}{\prod_{i=1}^n \prod_{j' < k_i}^j (x_{i,j} - x_{i+1,j'})}, \quad \text{where} \quad f \in (\mathbb{C}[x_{i,j,1}^{\pm 1}]_{1 \leq i \leq n}) \prod S_{\mathbf{k}_i}. \]
- We say that $F \in S_{\mathbf{k}}$ satisfies the wheel conditions iff $F(x_{1,1}, \ldots, x_{n,k_n}) = 0$ for any collection $\{x_{i,j}\} \subset \mathbb{C}$ such that $x_{i,j}/x_{i+1,l} = q d^{\pm 1}, x_{i+1,j}/x_{i,j+1} = q d^{-1}, 1 \leq i \leq n, j_1, j_2 \leq k_i, l \leq k_{i+1}$.

Let $S_{\mathbf{k}} = S_{\mathbf{k}}^* \subseteq S_{\mathbf{k}}$ be the subspace of all elements $F$ satisfying the above two conditions and set
\[ S := \bigoplus_{\mathbf{k} \in \mathbb{N}^n} S_{\mathbf{k}}. \]

Further $S_{\mathbf{k}} = \oplus_{d \in \mathbb{Z}} S_{\mathbf{k},d}$ with $S_{\mathbf{k},d} := \{ F \in S_{\mathbf{k}} \mid \text{tot.deg}(F) = d \}$. The following is straightforward:

**Lemma 1.10.** The subspace $S \subseteq S$ is $\star$-closed.

Now we are ready to introduce the main algebra of this paper:

**Definition 1.11.** The algebra $(S, \star)$ is called the big shuffle algebra (of $A^{(1)}_{n-1}$-type).

1.8. Relation between $S$ and $\hat{U}^+$. 
Recall the subalgebra $\hat{U}^+$ of $\hat{U}_{q,d}(\mathfrak{g}_n^\mathbb{C})$ from Section 1.1. By standard results $\hat{U}^+$ is generated by $\{e_{i,j}\}_{i \in [n]}$ with the defining relations (T2, T7.1). The following theorem is straightforward:

**Proposition 1.12.** The map $e_{i,j} \mapsto x_{i+1,j+1} \in S$ extends to a homomorphism $\Psi : \hat{U}^+ \to S$.

As a consequence: $\text{Im}(\Psi) \subset S$. The following beautiful result was recently proved by Negut:

**Theorem 1.13.** [N2] Theorem 1.1] The homomorphism $\Psi : \hat{U} \to S$ is an isomorphism of $\mathbb{N}^n \times \mathbb{Z}$-graded algebras.

**Remark 1.14.** In the loc. cit. $d = 1$, but the proof can be easily modified for any $d$. Note that the algebra $A^+$ from [N2] is isomorphic to our $S$ with $d = 1$ via the map $S_{1,d} \to A^+$ given by
\[ F(x_{1,1}, \ldots, x_{n,k_n}) \mapsto q \sum_{i,j}^n \frac{k_i(k_i-1)}{2} F(z_{i,1}, \ldots, z_{i,k_n}) \prod_{i=1}^n \prod_{j \neq j'} \frac{z_{i,j} - z_{i,j'}}{q^{-1} z_{i,j} - q z_{i,j'}} \prod_{i=1}^n \prod_{j \neq j'} \frac{z_{i,j} - z_{i+1,j'}}{z_{i,j} - q z_{i+1,j'}}. \]
2. Subalgebras $A(s_1, \ldots, s_n)$

2.1. Key constructions.

In this section we introduce the key objects of our paper, the commutative subalgebras of $S$, analogous to $A^{sm} \subset S^{sm}$ from Section 1.6. The new feature of our setup (in comparison to the small shuffle algebras) is that we get an $(n - 1)$-dimensional family of those.

For any $0 \leq \mathbf{t} \leq \mathbf{f} \in \mathbb{N}^n$, $\xi \in \mathbb{C}^*$ and $F \in S^\mathbf{t}$, we define $F^\mathbf{t}_\xi \in \mathbb{C}(x_{1,1}, \ldots, x_{n,k_n})$ by

$$F^\mathbf{t}_\xi := F(\xi \cdot x_{1,1}, \ldots, \xi \cdot x_{1,l_1}, x_{1,l_1+1}, \ldots, x_{1,k_1}; \ldots; \xi \cdot x_{n,1}, \ldots, \xi \cdot x_{n,l_n}, x_{n,l_n+1}, \ldots, x_{n,k_n})$$

For any integer numbers $a \leq b$, define the degree vector $\mathbf{t} := [a; b] \in \mathbb{N}^n$ by

$$\mathbf{t} = (l_1, \ldots, l_n) \text{ with } l_i = \#\{c \in \mathbb{Z} | a \leq c \leq b \text{ and } c \equiv i \text{ (mod } n\}.$$ 

For such a choice of $\mathbf{t}$, we will denote $F^\mathbf{t}_\xi$ simply by $F^{(a,b)}_\xi$.

**Definition 2.1.** For any $\mathbf{s} = (s_1, \ldots, s_n) \in (\mathbb{C}^*)^n$, consider an $\mathbb{N}^n$-graded subspace $A(\mathbf{s}) \subset S$ whose degree $\mathbf{t} = (k_1, \ldots, k_n)$ is defined by

$$A(\mathbf{s}) := \left\{ F \in S_{\mathbf{t}, 0} \mid \partial^{(\infty; a, b)} F = \prod_{i=a}^{b} s_i \cdot \partial^{(0; a, b)} F \quad \forall [a; b] \leq \mathbf{t} \right\},$$

where $\partial^{(\infty; a, b)} F := \lim_{\xi \to \infty} F^{(a,b)}_\xi$, $\partial^{(0; a, b)} F := \lim_{\xi \to 0} F^{(a,b)}_\xi$, and $s_{nr+c} := s_c$ for $1 \leq c \leq n, r \in \mathbb{Z}$.

A certain class of such elements is provided by the following result:

**Lemma 2.2.** For any $k \in \mathbb{N}$, $\mu \in \mathbb{C}$, and $\mathbf{s} \in (\mathbb{C}^*)^n$, define $F^{(\mu)}_k(\mathbf{s}) \in S_{k, \ldots, k}$ by

$$F_k^{(\mu)}(\mathbf{s}) := \prod_{i=1}^{n} \prod_{1 \leq j \neq j' \leq k} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i=1}^{n} \prod_{1 \leq j \leq k} (x_{i,j} - \mu \prod_{j=1}^{k} x_{i,j+1}) .$$

If $s_1 \cdots s_n = 1$, then $F^{(\mu)}_k(\mathbf{s}) \in A(\mathbf{s})$.

**Proof.**

Without loss of generality we can assume $\mu \neq 0$, $a = 1$, $b = nr + c$, $0 \leq r \leq k - 1, 1 \leq c \leq n$. Then $l_1 = \ldots = l_c = r + 1$ and $l_{c+1} = \ldots = l_n = r$. As $\xi \to \infty$, the function $F^{(\mu)}_k(\mathbf{s})^{(a,b)}$ grows at the speed $\xi^{\sum_{i=1}^{n} l_i(l_i+1)-1} \sum_{i=1}^{n} \max(l_i+1, l_i)$, while as $\xi \to 0$, the function $F^{(\mu)}_k(\mathbf{s})^{(a)}$ grows at the speed $\xi^{\sum_{i=1}^{n} l_i(l_i+1)+1} \sum_{i=1}^{n} \min(l_i, l_i+1)$. For the above values of $l_i$, both powers of $\xi$ are zero and hence both limits $\partial^{(\infty; a, b)} F^{(\mu)}_k(\mathbf{s})$ and $\partial^{(0; a, b)} F^{(\mu)}_k(\mathbf{s})$ exist. Moreover, for $\alpha$ being 0 or $\infty$, we have $\partial^{(0; a, b)} F^{(\mu)}_k(\mathbf{s}) = (-1)^{\sum_{i=1}^{n} l_i(l_i+1)} q^{-2 \sum_{i=1}^{n} l_i} \cdot G \cdot \prod_{i=1}^{n} G_{\alpha, i}$, where

$$G = \frac{\prod_{i=1}^{n} \prod_{1 \leq j \neq j' \leq k} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i=1}^{n} \prod_{1 \leq j \leq k} (x_{i,j} - \mu \prod_{j=1}^{k} x_{i,j+1})}{\prod_{i=1}^{n} \prod_{1 \leq j \leq l_i} (x_{i,j} - x_{i,j+1}) \cdot \prod_{i=1}^{n} \prod_{l_i+1 \leq j \leq k} (x_{i,j} - x_{i,j+1})} .$$

$$G^{(\infty)\alpha,i} = \prod_{j=1}^{k-i+1} x_{i,j}^{l_i+1-l_i-i-2l_i}, \quad \begin{cases} s_1 \cdots s_i \prod_{j=1}^{k} x_{i,j} - \mu \prod_{j=1}^{k} x_{i,j+1} \text{ if } l_i = l_i+1 \\
 s_1 \cdots s_i \prod_{j=1}^{k} x_{i,j} \text{ if } l_i > l_i+1 \\
 -\mu \prod_{j=1}^{k} x_{i,j+1} \text{ if } l_i < l_i+1 
\end{cases}$$

$$G^{(0)\alpha,i} = \prod_{j=1}^{k-i+1} x_{i,j}^{-l_i+1-l_i-i+2l_i}, \quad \begin{cases} s_1 \cdots s_i \prod_{j=1}^{k} x_{i,j} - \mu \prod_{j=1}^{k} x_{i,j+1} \text{ if } l_i = l_i+1 \\
 -\mu \prod_{j=1}^{k} x_{i,j+1} \text{ if } l_i > l_i+1 \\
 s_1 \cdots s_i \prod_{j=1}^{k} x_{i,j} \text{ if } l_i < l_i+1 
\end{cases}$$

The equality $\partial^{(\infty; a, b)} F^{(\mu)}_k(\mathbf{s}) = \prod_{i=1}^{n} s_i \cdot \partial^{(0; a, b)} F^{(\mu)}_k(\mathbf{s})$ follows, while tot. deg $F^{(\mu)}_k(\mathbf{s}) = 0$. $\square$
2.2. Main result.

A collection \( \{s_1, \ldots, s_n\} \subset \mathbb{C}^* \) satisfying \( s_1 \cdots s_n = 1 \) is called generic iff
\[
s_1^\alpha_1 \cdots s_n^\alpha_n \in q^\mathbb{Z} \cdot d^2 \Rightarrow \alpha_1 = \ldots = \alpha_n.
\]
The main result of this section describes \( \mathcal{A}(s_1, \ldots, s_n) \) for such generic \( n \)-tuples \( \{s_1, \ldots, s_n\} \).

**Theorem 2.3.** For a generic \( \sigma = \{s_1, \ldots, s_n\} \) satisfying \( s_1 \cdots s_n = 1 \), the space \( \mathcal{A}(\sigma) \) is shuffle-generated by \( \{F^\mu_k(\sigma) | k \in \mathbb{N}_+, \mu \in \mathbb{C}\} \). Moreover, \( \mathcal{A}(\sigma) \) is a polynomial algebra in free generators \( \{F^\mu_k(\sigma) | k \in \mathbb{N}_+, 1 \leq l \leq n\} \) for arbitrary pair-wise distinct \( \mu_1, \ldots, \mu_n \in \mathbb{C} \).

The proof of this theorem will proceed in several steps. First, we will use the analogue of the Gordon filtration from [FHHSY], further generalized in [N2] to prove Theorem 1.13 in order to obtain the upper bound on dimensions of \( \mathcal{A}(\sigma) \). Next, we will show that the subalgebra \( \mathcal{A}'(\sigma) \subset S \), shuffle generated by all \( F^\mu_k(\sigma) \), belongs to \( \mathcal{A}(\sigma) \). We will use another filtration to argue that the dimension of \( \mathcal{A}'(\sigma) \) is at least as big as the upper bound for the dimension of \( \mathcal{A}(\sigma) \), implying \( \mathcal{A}'(\sigma) = \mathcal{A}(\sigma) \). Similar arguments will also imply the commutativity of \( \mathcal{A}(\sigma) \).

**Lemma 2.4.** Consider the polynomial algebra \( \mathcal{R} = \mathbb{C}[T_{i,m}]_{1 \leq i \leq n} \) with \( \deg(T_{i,m}) = m \). Then:
(a) For \( \tau = k\delta := (k, \ldots, k) \), we have \( \dim \mathcal{A}(\tau) \leq \dim \mathcal{R} \).
(b) For \( \tau \notin \{0, \delta, 2\delta, \ldots\} \), we have \( \dim \mathcal{A}(\tau) = 0 \).

**Proof.**

An unordered set \( L \) of integer intervals \( \{[a_1, b_1], \ldots, [a_r, b_r]\} \) is called a partition of \( \tau \) (denoted \( L \vdash \tau \)) if \( \tau = [a_1; b_1] + \cdots + [a_r; b_r] \). We order the elements of \( L \) so that \( b_1 - a_1 \geq b_2 - a_2 \geq \cdots \geq b_r - a_r \). The two sets \( L \) and \( L' \) as above are said to be equivalent iff \( |L| = |L'| \) and we can order their elements so that \( b'_i - a'_i = a_i - a_i = c_i \) for all \( i \) and some \( c_i \in \mathbb{Z} \). Note that the collection of \( L \cup L' \) up to the above equivalence, is finite for any \( \tau \in \mathbb{N}^n \). Finally, we say \( L' > L \) if there exists \( s \), such that \( b'_s - a'_s > b_s - a_s \) and \( b'_t - a'_t = b_t - a_t \) for \( 1 \leq t \leq s - 1 \).

Any \( L \vdash \tau \) defines a linear map \( \phi_L : \mathcal{A}(\tau) \to \mathbb{C}[y_1^\pm, \ldots, y_n^\pm] \) as follows. Split the variables \( \{x_{i,j}\} \) in \( r \) groups, each group corresponding to one of the \( L \)-intervals. Specialize the variables corresponding to the interval \( [a_t, b_t] \) to \( (qd)^{-a_t} \cdot y_t, \ldots, (qd)^{-b_t} \cdot y_t \) in the natural order. For
\[
F = \frac{f(x_1, \ldots, x_n, b)}{\prod_{i=1}^n \prod_{1 \leq j \leq k_i} (x_{i,j} - x_{i,j+1}')} \in \mathcal{A}(\tau),
\]
define \( \phi_L(F) \) as the corresponding specialization of \( f \). The result is independent of our splitting of variables since \( f \) is symmetric. Finally, we define the filtration on \( \mathcal{A}(\tau) \) by
\[
\mathcal{A}(\tau)_L^L := \bigcap_{L' > L} \text{Ker}(\phi_{L'}).\n\]

Let us now consider the images \( \phi_L(\mathcal{A}(\tau)_L^L) \) for any \( L \vdash \tau \). For \( F \in \mathcal{A}(\tau)_L^L \), we have:
- The total degree \( \text{tot.deg}(\phi_L(F)) = \sum_{i=1}^n k_i k_{i+1} \), since \( \text{tot.deg}(F) = 0 \).
- For each \( 1 \leq t \leq r \), the degree of \( \phi_L(F) \) with respect to \( y_t \) is bounded by
\[
\deg_{y_t}(\phi_L(F)) \leq \sum_{i=1}^n (l'_t(k_i - 1 + k_{i+1}) + l'_t l_{i+1})
\]
due to the existence of the limit \( \partial^{(\infty; a_t, b_t)}_F \) (here \( \vec{t} := [a_t; b_t] \) for \( 1 \leq t \leq r \)).

On the other hand, the wheel conditions for \( F \) guarantee that \( \phi_L(F)(y_1, \ldots, y_r) \) becomes zero under the following specializations:

(i) \( (qd)^{-x} \cdot y_u = (q/d)(qd)^{-x} \cdot y_u \) for any \( 1 \leq u < v \leq r \), \( a_u \leq x < b_u, a_v \leq x' \leq b_v, x' \equiv x + 1 \),
(ii) \( (qd)^{-x} \cdot y_v = (d/q)(qd)^{-x} \cdot y_u \) for any \( 1 \leq u < v \leq r \), \( a_u \leq x < b_u, a_v \leq x' \leq b_v, x' \equiv x - 1 \).
Finally, the conditions $\phi_L(F) = 0$ for any $L' > L$ guarantee that $\phi_L(F)(y_1, \ldots, y_r)$ becomes zero under the following specializations:

(iii) $(qd)\cdot x' y_v = (qd)\cdot b_{u-1} y_u$ for any $1 \leq u < v \leq r$, $a_u \leq x' \leq b_v$, $x' \equiv b_u + 1$.

(iv) $(qd)\cdot x' y_v = qd \cdot y_u$ for any $1 \leq u < v \leq r$, $a_v \leq x' \leq b_u$, $x' \equiv a_u - 1$.

In particular, we see that $\phi_L(F)$ is divisible by $Q_L \in \mathbb{C}[y_1, \ldots, y_r]$ defined as the product of linear terms in $y_i$, coming from (i)-(iv) (if some of these coincide, we still count them with the correct multiplicity). Note that

$$\text{tot.deg}(Q_L) = \sum_{1 \leq u < v \leq r} \sum_{i=1}^{n} (l_i^{u} t_{i+1}^{v} + l_i^{v} t_{i-1}^{u}) = \sum_{i=1}^{n} k_i k_{i+1} - \sum_{i=1}^{n} l_i^{t_i^{v}} t_{i+1}^{v},$$

while the degree with respect to each variable $y_t$ ($1 \leq t \leq r$) is given by

$$\deg_{y_t}(Q_L) = \sum_{i=1}^{n} (l_i^{t_i}(k_i + k_{i+1}) - 2l_i^{t_i}).$$

Define $r_L := \phi_L(F)/Q_L \in \mathbb{C}[y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$. Then:

$$\text{tot.deg}(r_L) = \sum_{i=1}^{n} \sum_{i \neq 0} l_i^{t_i} l_{i+1}^{t_{i+1}} \quad \text{and} \quad \deg_{y_t}(r_L) \leq \sum_{i=1}^{n} l_i^{t_i^{v}} t_{i+1}^{v}.$$

Thus $r_L = \nu \cdot \prod_{t=1}^{r} \frac{\sum_{i=1}^{n} l_i^{t_i^{v}}}{y_t^{t_i^{v}}} + \prod_{1 \leq u < v \leq r} \prod_{a_u \leq x' \leq b_v} ((qd)\cdot x' y_u - (qd)\cdot x' y_v)$ for some $\nu' \in \mathbb{C}^*$. The condition $F \in A(\{\xi\})$ implies

$$\lim_{\xi \to \infty} \left( \frac{\phi_L(F)}{Q} \right)_{|\nu_t = \xi y_t} = s_{a_1} \cdots s_{b_n}, \quad \lim_{\xi \to 0} \left( \frac{\phi_L(F)}{Q} \right)_{|\nu_t = \xi y_t} \quad \forall \ 1 \leq t \leq r.$$

For $\nu \neq 0$, this equality ensures $s_{a_1} \cdots s_{b_n} \in \mathbb{Q}^2 \cdot d^2$. Due to our condition on $\{s_i\}$, we get $b_i - a_i + 1 \geq c_i$ for every $1 \leq t \leq r$ and some $c_i \in \mathbb{N}$. The claim (ii) of the lemma is now obvious, while part (i) of the lemma follows from the inequality $\dim A(\{\xi\}) \leq \sum_{L \in \mathcal{F}} \dim \phi_L(A(\{\xi\}))$.  

**Lemma 2.5.** Let $A(\{\xi\})$ be the subalgebra of $S$ generated by $\{F_{k_i}^\mu(\{\xi\})\}_{k_i \geq 1}$. Then $A'(\{\xi\}) \subset A(\{\xi\})$.

**Proof.**

It suffices to show $F_{k_1, \ldots, k_r}^\mu(\{\xi\}) := F_{k_1}^\mu(\{\xi\}) \ast \cdots \ast F_{k_r}^\mu(\{\xi\}) \in A(\{\xi\})$ for any $r$, $k_i \geq 1$, and $\mu_i \in \mathbb{C}^*$. The case of $r = 1$ has been already treated in Lemma 2.2. The arguments for general $r$ are similar. Choose any $a \leq b$, such that $[a; b] \leq k_0$, where $k := k_1 + \cdots + k_r$. We can further assume $a = 1$. Let us consider any summand from the definition of $F_{k_1, \ldots, k_r}^\mu(\{\xi\})$ with $\vec{t} := [a; b]$ variables being multiplied by $\xi$. We will check that as $\xi$ tends to $\infty$ or 0, both limits exist and differ by the constant $s_{a_1} \cdots s_{b_n}$.

For a fixed summand as above, define $\{\vec{t}''\}_{t'' \in \mathbb{N}}$ satisfying $\vec{t}'' = \vec{t}'' + \cdots + \vec{t}''$ by considering those variables $x_{i,j}$ which are multiplied by $\xi$ and get substituted into $F_{k_i}^\mu(\{\xi\})$. Following the proof of Lemma 2.2, the function $F_{k_i}^\mu(\{\xi\})^\vec{t}''$ grows at the speed $\xi^{\sum_{i=1}^{n} l_i^{t_i^{v}} t_{i+1}^{v} - \sum_{i=1}^{n} \max(t_i^{v}, t_{i+1}^{v})}$.  

### References

1. Boris Feigin and Alexander Tsymbaliuk
2. [Additional references as needed]
as $\xi \to \infty$, and at the speed $\xi^{\sum_{i=1}^{n} l_i t_i (-l_{i+1} + l_i - 1) + \sum_{i=1}^{n} \min(l_i, l_{i+1})}$ as $\xi \to 0$. To estimate these powers, we note that $(a - b)(a - b - 1) \geq 0$ for any $a, b \in \mathbb{N}$, implying

$$\min(a, b) + \frac{a^2 + b^2 - a - b}{2} \geq ab$$

with equality holding iff $a - b \in \{-1, 0, 1\}$. Therefore,

$$\sum_{i=1}^{n} l_i (l_{i+1} - l_i - 1) + \sum_{i=1}^{n} \max\{l_i, l_{i+1}\} \leq 0 \text{ and } \sum_{i=1}^{n} l_i (l_{i+1} - l_i - 1) + \sum_{i=1}^{n} \min(l_i, l_{i+1}) \geq 0,$n

with equalities holding iff $l_i - l_{i+1} \in \{\pm 1, 0\}$ for any $1 \leq i \leq n$. Since the limits of

$$\omega_{i,j}(\xi \cdot x, y), \omega_{i,j}(x, \xi \cdot y), \omega_{i,j}(\xi \cdot x, \xi \cdot y) \text{ as } \xi \to 0, \infty \text{ exist } \forall i, j,$$

the limits of the corresponding summands in the symmetrization are well-defined as either $\xi \to 0, \infty$. Moreover, they are both zero if $|l_i - l_{i+1}| > 1$ for some $1 \leq t \leq r, 1 \leq i \leq n$.

Assuming finally that $|l_i - l_{i+1}| \leq 1$ for any $t, i$, the formulas from the proof of Lemma 2.2 imply that the ratio of the limits as $\xi$ goes to $\infty$ and 0 equals to

$$\prod_{i=1}^{n} \prod_{j=1}^{n} \left(\frac{s_1 \cdots s_i}{-\mu t}\right)^{l_i - l_{i+1}} = \prod_{i=1}^{n} (s_1 \cdots s_i)^{l_i - l_{i+1}} = s_a \cdots s_b.$$

The result follows. \hfill \Box

**Lemma 2.6.** For any $k \in \mathbb{N}$, we have $\dim \mathcal{A}'(\overline{\pi})_{k()} \geq \dim \mathcal{R}_k$.

**Proof.**

Choose any pairwise distinct $\mu_1, \ldots, \mu_n \in \mathbb{C}$ and consider a subspace $\mathcal{A}'(\overline{\pi})$ of $\mathcal{A}'(\overline{\pi})$ spanned by $F_{k_1}(\overline{\pi}) \cdots F_{k_r}(\overline{\pi})$ with $r \geq 0, k_1 \geq k_2 \geq \cdots \geq k_r > 0$, and $1 \leq i_1, \ldots, i_r \leq n$. It suffices to show

$$\dim \mathcal{A}'(\overline{\pi})_{k()} \geq \dim \mathcal{R}_k.$$

For a Young diagram $\lambda$ we introduce the specialization map

$$\varphi_\lambda : S_{|\lambda|} \rightarrow \mathbb{C}^{\{y_{i,j}^{1 \leq j \leq l(\lambda)}\}}$$

by specializing the variables $x_{i,j}$ as follows

$$x_{i_1, i_2 + \cdots + i_{\lambda_1 + j} \rightarrow q^{2j} y_{i,t} \text{ for any } 1 \leq t \leq l(\lambda), 1 \leq j \leq \lambda_t.}$$

It is clear that for any $\overline{k} = (k_1, \ldots, k_r)$ with $\sum k_i = k = |\lambda|$ and $\overline{k} > \lambda'$ (here $>$ denotes the lexicographic order on Young diagrams and $\lambda'$ denotes the transposed to $\lambda$ Young diagram), we have $\varphi_\lambda(F_{\overline{k}}(\overline{\pi})) = 0$ for all $\overline{\pi} \in \mathcal{C}^r$. Therefore, it remains to prove

$$\sum_{\overline{k} \vdash k} \dim \varphi_\lambda \left(\text{span} \left\{F_{\overline{k}}(\overline{\pi}) | \overline{\pi} \in \mathcal{C}^r\right\}\right) \geq \dim \mathcal{R}_k.$$

Let us first consider the case $k_1 = \ldots = k_r \Rightarrow k = rk_1$. Then

$$\varphi_\lambda(F_{\overline{k}}(\overline{\pi})) = Z \cdot \prod_{i=1}^{n} \prod_{j=1}^{n} (s_1 \cdots s_i \prod_{j=1}^{k_i} y_{i,j} - \mu t \prod_{j=1}^{k_{i+1}} y_{i+1,j})$$

for a certain nonzero common factor $Z$. Define $Y_{i} := y_{i,1} \cdots y_{i,k_i}$. Since the polynomials

$$f_t(Y_1, \ldots, Y_n) := \prod_{i=1}^{n} (s_1 \cdots s_i Y_i - \mu_t Y_{i+1}), \ 1 \leq t \leq n,$$

are algebraically independent, we immediately get the required dimension estimate for this particular $\overline{k}$. The general case follows immediately. \hfill \Box
By Lemmas 2.3, 2.4, the subspace $A(\mathfrak{F})$ is generated by $F^i_k(\mathfrak{F})$ and has the prescribed dimensions of each $\mathbb{N}^n$-graded component.

**Lemma 2.7.** The algebra $A(\mathfrak{F})$ is commutative. Moreover, for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ with $\mu_i \neq \mu_j$ for $i \neq j$, there is an isomorphism $\mathfrak{F} \xrightarrow{\sim} A(\mathfrak{F})$ given by $T_{i,k} \mapsto F^i_k(\mathfrak{F})$.

**Proof.**

It suffices to prove $F^i_{m_1}(\mathfrak{F}) \cdot F^j_{m_2}(\mathfrak{F}) = F^j_{m_2}(\mathfrak{F}) \cdot F^i_{m_1}(\mathfrak{F})$ for any $m_1, m_2 \in \mathbb{N}$ and $\nu_1, \nu_2 \in \mathbb{C}$. Define $F := F^i_{m_1}\nu_1(\mathfrak{F}) - F^j_{m_2}\nu_2(\mathfrak{F})$. Due to previous lemmas, $F$ can be written as a certain linear combination of $F^m_k(\mathfrak{F})$ with $k = (k_1 \geq k_2 \geq \cdots)$.

We claim that $\varphi_{(2, 1, m_1 + m_2 - 2)}(F) = 0$. Together with the properties of $\varphi_\lambda$ discussed before, this equality implies $F = \sum_{i=1}^n \pi_i \cdot F^\nu_i_{m_1 + m_2}(\mathfrak{F})$ for some $\pi_i \in \mathbb{C}$. Let us multiply both sides of this equality by $\prod_{i=1}^n \prod_{1 \leq j < j' \leq m_1 + m_2} (x_{i,j} - x_{i,j'})$ and consider a specialization $x_{i,j} \mapsto y_i \forall i, j$. The left hand side will clearly specialize to 0, while the right hand side will specialize to

$$\prod_{i=1}^n ((1 - q^{-2})y_i)^{m_1 + m_2 - 1} \cdot \sum_{r=1}^n \pi_r \cdot \prod_{i=1}^n (s_1 \cdot s_i(y_i - y_{r+1})),$$

This expression vanishes if $\pi_1 = \ldots = \pi_n = 0$, and so $F = 0$ as required.

Finally, let us prove the equality $\varphi_{(2, 1, m_1 + m_2 - 2)}(F) = 0$. The statement is obvious when either $m_1$ or $m_2$ is zero. To prove for general $m_1, m_2 > 0$, we can assume by induction that

$$F^i_{m'_1}(\mathfrak{F}) \cdot F^j_{m'_2}(\mathfrak{F}) = F^j_{m'_2}(\mathfrak{F}) \cdot F^i_{m'_1}(\mathfrak{F})$$

for any $m'_1 < m_1, m'_2 < m_2$, $\nu'_1, \nu'_2 \in \mathbb{C}$, and $\prod_i s'_i = 1$ (though $s'_i$ are not necessarily generic).

By straightforward computations, we find

$$\varphi_{(2, 1, m_1 + m_2 - 2)}(F^i_{m'_1}F^j_{m'_2}(\mathfrak{F})) = \text{Sym}(A_1 \cdot B_1),$$

where the symmetrization is taken with respect to all permutations of $\{y_{ij}\}_{1 \leq i < j \leq n}$ preserving index $i$, $A_1 \in \mathbb{C}((y_{ij}))$ is symmetric, while $B_1$ is given by the following explicit formula

$$B_1 = \prod_{i=1}^n \prod_{1 \leq j < j' < m_1 + m_2} \prod_{1 \leq 1 \leq i' < m_1 + m_2} \omega_{i,i'}(y_{i,j}/y_{i',j'}) \times
$$

$$\prod_{i=1}^n \prod_{1 \leq j < j' < m_1 + m_2} \prod_{1 \leq 1 \leq i' < m_1 + m_2} \omega_{i,i'}(y_{i,j}/y_{i',j'}) \times
$$

where $n_i := n_i \cdot \nu_i y_{i,1} / y_{i,1}$, $s'_{i,1} := s_i^{1/2}$, $\kappa := \prod_{i=1}^n y_i^{1/2} y_{i,1}^{1/2}$.

Permuting $m_1 \leftrightarrow m_2$, $\nu_1 \leftrightarrow \nu_2$, we get

$$\varphi_{(2, 1, m_1 + m_2 - 2)}(F^i_{m'_1}F^j_{m'_2}(\mathfrak{F})) = \kappa \cdot A_1 \cdot (F^j_{m'_2}F^i_{m'_1}(\mathfrak{F}))(y_{1,2}, \ldots, y_{n-1,n-m_1-m_2+1}, \ldots, y_{n,n-m_1-m_2-1}).$$

Applying the induction assumption, we find

$$\varphi_{(2, 1, m_1 + m_2 - 2)}(F) = \kappa \cdot A_1 \cdot [F^i_{m'_1}(\mathfrak{F}), F^j_{m'_2}(\mathfrak{F})](y_{1,2}, \ldots, y_{n,n-m_1-m_2-1}).$$

This proves the inductive step and, hence, completes the proof of the claim.

The results of Theorem 2.3 follow immediately by combining the above four lemmas.

**Remark 2.8.** The proof of Lemma 2.3 implies $A(\mathfrak{F}) = \mathbb{C}$ for any $s_1, \ldots, s_n \in \mathbb{C}^*$ such that $\prod_i s_i^{\alpha_i} \notin q^2 \cdot d^\alpha$ unless $\alpha_1 = \ldots = \alpha_n = 0$. 


2.3. Shuffle realization of $\mathring{U}^h(\mathfrak{gl}_n)^+$ and $\mathring{U}^h(\mathfrak{gl}_1)^+$.

In [N2], the author introduced the notion of the slope filtration on $S$. For a zero slope, the corresponding subspace $A^0 \subset S$ is $\mathbb{N}^n$-graded with the graded component $A^k_\mathbb{F}$ given by

$$F \in A^k_\mathbb{F} \iff F \in S_{k,0} \text{ and } \lim_{\xi \to \infty} F^\xi_\xi 0 \leq \ell \leq \mathfrak{k}.$$  

While proving Theorem 1.13, the author obtained the following description of $A^0$.

**Proposition 2.9.** [N2 Lemma 4.4]
(a) The isomorphism $\Psi : \mathring{U}^{+} \mathring{\rightarrow} S$ identifies $\mathring{U}^h(\mathfrak{gl}_n)^+$ with $A^0$.
(b) Under the isomorphism $\Psi^h : \mathring{U}^h(\mathfrak{gl}_1)^+ \mathring{\rightarrow} A^0$ from (a), the image $X_k := \Psi^h(h^k_\mathfrak{l})$ of the $k$-th generator $h^k_\mathfrak{l} \in \mathring{U}^h(\mathfrak{gl}_1)^+ \subset \mathring{U}^h(\mathfrak{gl}_n)^+$ is uniquely (up to a constant) characterized by

$$X_k \in S_{k,0} \text{ and } \lim_{\xi \to \infty} (X_k)^{\ell}_\xi = 0 \quad 0 < \ell < k\delta.$$  

This proposition provides a shuffle characterization of both $\mathring{U}^h(\mathfrak{gl}_n)^+$ and $\mathring{U}^h(\mathfrak{gl}_1)^+$. In particular, we immediately obtain the following result:

**Theorem 2.10.** We have $A(\mathfrak{t}) \subset \mathring{U}^h(\mathfrak{gl}_n)^+$ for generic $\{s_i\}$ such that $s_1 \cdots s_n = 1$.

**Proof.**
Combining the proof of Lemma 2.3 together with Theorem 2.3, we see that for any $0 \leq \ell \leq \mathfrak{k}$ and $F \in A(\mathfrak{t})_\mathbb{F}$, the limit $\lim_{\xi \to \infty} F^\xi_\xi$ exists. Now the result follows from Proposition 2.9(b).

We complete this section by providing explicit formulas for the elements $X_k = \Psi^h(h^k_\mathfrak{l}) \in S$ (this answers one of the questions raised in [N2 Section 5.6]). Consider the elements

$$F_k := \prod_{i=1}^{n} \prod_{1 \leq j \neq k}^{n} (q^{-1} x_{i,j} - qx_{i,j'}) \cdot \prod_{i=1}^{n} \prod_{j \neq k}^{n} (x_{i+1,j} - x_{i,j}) \in S_{k,0}.$$  

Note that $F_k = \frac{(-q^{-1})^{nk}}{s_1 s_2 \cdots s_n} : F^k_\mathfrak{p}(\mathfrak{t}) \in A(\mathfrak{t})$ for any $\{s_i\}$ such that $s_1 \cdots s_n = 1$. We also define

$$L_k \in S_{k,0} \text{ via } \exp \left( \sum_{k=1}^{\infty} L_k t^k \right) = \sum_{k=0}^{\infty} F_k t^k.$$  

The relevant properties of these elements are formulated in our next theorem:

**Theorem 2.11.** (a) For $\ell \notin \{0, \delta, 2\delta, \ldots, k\delta\}$, we have $\lim_{\xi \to \infty} (F_k)^\ell_\xi = 0$.
(b) For any $0 \leq l \leq k$, we have $\lim_{\xi \to \infty} (F_k)^l_\xi = F_l \cdot F_{k-l}$.
(c) For any $0 < \ell < k\delta$, we have $\lim_{\xi \to \infty} (L_k)^\ell_\xi = 0$.

**Proof.**
(a) For any $0 \leq \ell \leq k\delta$, the function $(F_k)^\ell_\xi$ grows at the speed $\xi^{-\sum_{i=1}^{n} l_i^2 + \sum_{i=1}^{n} l_i l_{i+1}}$ as $\xi \to \infty$.

Note that $-\sum_{i=1}^{n} l_i^2 + \sum_{i=1}^{n} l_i l_{i+1} = -\frac{1}{2} \sum_{i=1}^{n} (l_i - l_{i+1})^2 \leq 0$. Moreover, the equality holds iff $l_1 = \ldots = l_n \iff \ell \in \{0, \delta, 2\delta, \ldots\}$. Part (a) follows.
(b) Straightforward.
(c) Standard (it is actually equivalent to the general exponential relation between group-like elements and primitive elements; see [N2 Section 4.3] for the related coproduct). $$\square$$

**Corollary 2.12.** Combining this result with Proposition 2.9(b), we see that $L_k$ and $X_k$ coincide up to a nonzero constant, and the isomorphism $\Psi^h$ identifies $\mathring{U}^h(\mathfrak{gl}_1)^+$ with $\mathbb{C}[F_1, F_2, \ldots]$. 

3. Functional realization of $A(\mathfrak{s})$

We provide an alternative viewpoint on the subspaces $A(\mathfrak{s})$ for generic $\{s_i\}$ with $s_1 \cdots s_n = 1$.

3.1. Vertex representations $W(p)_n$.

We start by recalling the construction of vertex $\check{U}_{q,d}(\mathfrak{s}^+_n)$-representations from [S], which generalize the classical Frenkel-Kac construction. Let $S_n$ be the "generalized" Heisenberg algebra, generated by $\{H_{i,k}|i \in [n], k \in \mathbb{Z}\setminus\{0\}\}$ and a central element $H_0$ with the defining relations

$$[H_{i,k}, H_{j,l}] = d^{-km_{i,j}} \frac{[k]_1 [ka_{i,j}]}{k} \delta_{k,-l} \cdot H_0.$$  

Let $S^+_{n}$ be the Lie subalgebra generated by $\{H_{i,k}|i \in [n], k > 0\} \sqcup \{H_0\}$, and let $\mathbb{C} v_0$ be the $S^+_{n}$-representation with $H_{i,k}$ acting trivially and $H_0$ acting via the identity operator. The induced representation $F_n := \text{Ind}_{S^+_{n}}^{S_n} \mathbb{C} v_0$ is called the Fock representation of $S_n$.

We denote by $\{\tilde{\alpha}_i\}_{i=1}^{n-1}$ the simple roots of $\mathfrak{s}_n$, by $\{\tilde{\Lambda}_i\}_{i=1}^{n-1}$ the fundamental weights of $\mathfrak{s}_n$, by $\{\tilde{h}_i\}_{i=1}^{n-1}$ the simple coroots of $\mathfrak{s}_n$. Let $Q := \bigoplus_{i=1}^{n-1} \mathbb{Z} \tilde{h}_i$ be the root lattice of $\mathfrak{s}_n$, $P := \bigoplus_{i=1}^{n-1} \mathbb{Z} \tilde{\alpha}_i \oplus \mathbb{Z} \tilde{\Lambda}_n = \mathbb{Z} \tilde{\alpha}_n \oplus \mathbb{Z} \tilde{\Lambda}_n$ the weight lattice of $\mathfrak{s}_n$. Let $Q(P)$ be the $\mathbb{Q}$-algebra.

For every $0 \leq p \leq n-1$, define the space

$$W(p)_n := F_n \otimes Q(P) e^{\tilde{\lambda}_p}.$$ 

Consider the operators $H_{i,k}, e^{\tilde{\alpha}}, \partial_{\tilde{\alpha}_i}, z^{H_{i,k}}$ acting on $W(p)_n$, which assign to every element

$$v \otimes e^{\tilde{\beta}} = H_{i_1,-k_1} \cdots H_{i_N,-k_N} (v_0) \otimes e^{n-1_{i=1} m_j \tilde{\alpha}_j + \tilde{\lambda}_p} \in W(p)_n$$

the following values:

$$H_{i,k} (v \otimes e^{\tilde{\beta}}) := (H_{i,k} v) \otimes e^{\tilde{\beta}}, e^{\tilde{\alpha}} (v \otimes e^{\tilde{\beta}}) := v \otimes e^{\tilde{\beta} e^{\tilde{\alpha}}}, \partial_{\tilde{\alpha}_i} (v \otimes e^{\tilde{\beta}}) := (\tilde{h}_i, \tilde{\beta}) v \otimes e^{\tilde{\beta}},$$

$$z^{H_{i,k}} (v \otimes e^{\tilde{\beta}}) := z^{(\tilde{h}_i, \tilde{\beta}) (\frac{1}{2} \sum_{j=1}^{n-1} (m_j \tilde{\alpha}_j))} v \otimes e^{\tilde{\beta}}, d (v \otimes e^{\tilde{\beta}}) := - (\sum (\tilde{h}_i + ((\tilde{\beta}, \tilde{\alpha}) - (\tilde{\lambda}_p, \tilde{\lambda}_p))/2) v \otimes e^{\tilde{\beta}}.$$ 

The following result provides a natural structure of an $\check{U}_{q,d}(\mathfrak{s}_n)$-module on $W(p)_n$.

**Proposition 3.1.** [S Proposition 3.2.2] For any $\tilde{c} = (c_0, \ldots, c_{n-1}) \in (\mathbb{C}^\ast)^n$ and $0 \leq p \leq n-1$, the following formulas define an action of $\check{U}_{q,d}(\mathfrak{s}_n)$ on $W(p)_n$:

$$\rho_{p, \tilde{c}}(e_i(z)) = c_i \cdot \exp \left( \sum_{k > 0} \frac{q^{-k/2}}{[k]} H_{i,-k} z^k \right) \cdot \left( - \sum_{k > 0} \frac{q^{-k/2}}{[k]} H_{i,k} z^{-k} \right),$$

$$\rho_{p, \tilde{c}}(f_i(z)) = c_i^{-1} \cdot \exp \left( - \sum_{k > 0} \frac{q^{-k/2}}{[k]} H_{i,-k} z^k \right) \cdot \left( \sum_{k > 0} \frac{q^{k/2}}{[k]} H_{i,k} z^{-k} \right),$$

$$\rho_{p, \tilde{c}}(\psi_i^{\pm}(z)) = \exp \left( \pm (q^{-1} - q^{-1}) \sum_{k > 0} H_{i, \pm k} z^{\pm k} \right), q^{\pm \partial_{\tilde{\alpha}_i}},$$

$$\rho_{p, \tilde{c}}(\gamma^{\pm 1/2}) = q^{\pm 1/2}, \quad \rho_{p, \tilde{c}}(q^{d}) = q^{d}.$$
3.2. Functionals $\phi_{p,\bar{c}}^0, \phi_{p,\bar{c}}^\bar{u}, \phi_{p,\bar{c}}^{\bar{u},t}$ on $\hat{U}^\leq$.

In this subsection we introduce and “explicitly compute” three functionals on $\hat{U}^\leq$.

- **Top matrix coefficient.**

  Consider the functional
  \[
  \phi_{p,\bar{c}}^0 : \hat{U}^\leq \to \mathbb{C}
  \]
  defined by $\phi_{p,\bar{c}}^0(A) := \langle v_0 \otimes e^{\bar{\lambda}_p} | \rho_{p,\bar{c}}(A) | v_0 \otimes e^{\bar{\lambda}_p} \rangle$.

  Since $h_{i,j}(v_0 \otimes e^{\bar{\lambda}_p}) = 0$ for $j > 0$, it remains to compute the values of $\phi_{p,\bar{c}}^0$ evaluated at
  \[
  f_{i_1,j_1} f_{i_2,j_2} \cdots f_{i_m,j_m} \psi_{r_0,0}^{i_0} \cdots \psi_{r_{n-1},0}^{i_{n-1}} \cdot (\gamma/2)^{a} (q^{d_i})^{b}
  \]
  with $a, b \in \mathbb{Z}$, $\bar{r} := (r_0, \ldots, r_{n-1}) \in \mathbb{Z}^n$ and $\sum_{s=1}^{m} a_{i_s} = 0 \in \bar{Q}$. The latter condition means that the multiset $\{i_1, \ldots, i_m\}$ contains an equal number of each of the indices $\{0, \ldots, n - 1\}$. Due to the defining quadratic relation (T3) of $\hat{U}_{q,d}(\mathfrak{sl}_n)$, it suffices to compute the series
  \[
  \phi_{p,\bar{c},N,\bar{r},a,b}(z_0,1,\ldots,z_{n-1},N) := \phi_{p,\bar{c}}^0 \left( \prod_{j=1}^{N} (f_0(z_0,j) \cdots f_{n-1}(z_{n-1},j)) \cdot \prod_{i=0}^{N-1} \psi_{r_i,0}^{i} \cdot \gamma^{a/2} q^{bd_i} \right).
  \]

  In this expression we order the $z$-variables as follows:
  \[
  z_0,1,\ldots,z_{n-1},1, z_0,2,\ldots,z_{n-1},2,\ldots,z_0,N,\ldots,z_{n-1},N.
  \]

  Normally ordering the product $\prod_{j=1}^{N} (f_0(z_0,j) \cdots f_{n-1}(z_{n-1},j))$, we get the following result:

**Proposition 3.2.** For $n > 2$, we have:

\[
\phi_{p,\bar{c},N,\bar{r},a,b}(z_0,1,\ldots,z_{n-1},N) = (c_0 \cdots c_{n-1})^{-N} q^{a/2+r_p-r_d} \frac{d^{N-2} \cdot \prod_{j=1}^{N} z_{j,p}}{\prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N}(z_{i,j} - z_{i,j'})(z_{i,j} - q^{2}z_{i,j'}) \cdot \prod_{j=1}^{N} z_{i,j}}
\]

- **Top level graded trace.**

  Recall the operator $d$ acting diagonally in the natural basis of $W(p)_n$. Clearly all its eigenvalues are in $-\mathbb{N}$. Let $M(p)_n \subset W(p)_n$ be the zero $d$-eigenspace. The following is obvious:

**Lemma 3.3.** (a) The subspace $M(p)_n$ is $U_q(\mathfrak{sl}_n)$-invariant and is isomorphic to the irreducible highest weight $U_q(\mathfrak{sl}_n)$-module $L_\alpha(\mathbb{A}_p)$.

(b) For any $\bar{\sigma} = \{1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq n\}$, let $\bar{\lambda}_p^\bar{\sigma}$ be the $\mathfrak{sl}_n$-weight having entries $1 - \frac{\bar{\sigma}}{n}$ at the places $\{\sigma_i\}_{i=1}^{n}$ and $-\frac{\bar{\sigma}}{n}$ elsewhere. Then $\{v_0 \otimes e^{\bar{\lambda}_p^\bar{\sigma}}\}_{\bar{\sigma}}$ form a basis of $M(p)_n$.

Define the degree operators $d_1, \ldots, d_{n-1}$ acting on $W(p)_n$ by

\[
d_r(v \otimes e^{\sum_{j=1}^{n-1} m_j \alpha_j + \bar{\lambda}_p}) = -m_r \cdot v \otimes e^{\sum_{j=1}^{n-1} m_j \alpha_j + \bar{\lambda}_p} \ \forall \ v \in F_n.
\]

For any $\bar{u} = (u_1, \ldots, u_{n-1}) \in (\mathbb{C}^*)^n$, consider the functional

\[
\phi_{p,\bar{c}}^{\bar{u}} : \hat{U}^\leq \to \mathbb{C}
\]

defined by $\phi_{p,\bar{c}}^{\bar{u}}(A) := \sum_{\sigma} \langle v_0 \otimes e^{\bar{\lambda}_p^\sigma} | \rho_{p,\bar{c}}(A) u_1^{d_1} \cdots u_{n-1}^{d_{n-1}} | v_0 \otimes e^{\bar{\lambda}_p^\sigma} \rangle$,

computing the $\bar{Q}$-graded trace of the $A$-action on the subspace $M(p)_n$ (here $u_i^{d_i}$ makes sense as $d_i$ acts with integer eigenvalues). Since $h_{i,j}(v_0 \otimes e^{\bar{\lambda}_p^\sigma}) = 0$ for $j > 0$, it suffices to compute the generating series

\[
\phi_{p,\bar{c},N,\bar{r},a,b}(z_0,1,\ldots,z_{n-1},N) := \phi_{p,\bar{c}}^{\bar{u}} \left( \prod_{j=1}^{N} (f_0(z_0,j) \cdots f_{n-1}(z_{n-1},j)) \cdot \prod_{i=0}^{N-1} \psi_{r_i,0}^{i} \cdot \gamma^{a/2} q^{bd_i} \right).
\]
Normally ordering the product $\prod_{j=1}^{N} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j}))$, we get the following result:

**Proposition 3.4.** For $n > 2$, we have:

$$
\phi^0_{p,N} f_0(z_0, \ldots, z_{n-1,N}) = (c_0 \cdots c_{n-1})^{-N} q^{a/2} d^{N(n-2)/2} \times \prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N} (z_{i,j} - q^{2} z_{i,j'}) \times \prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N} (z_{i,j} - q^{d-1} z_{i,j'}) \times (-1)^p \prod_{j=1}^{p} \prod_{u_1, \ldots, u_d} \cdot [\mu^p] \left\{ \prod_{j=1}^{N} \left( \prod_{i=0}^{N} z_{i+1,j} - \mu u_1 \cdots u_d q^{i+1-r_i} \prod_{j=1}^{N} z_{i,j} \right) \right\},
$$

where $[\mu^p] \ldots$ denotes the coefficient of $\mu^p$ in $\ldots$.

- **Full graded trace.**

Finally we introduce the most general functional

$$
\phi^0_{p,N} : U^\leq \longrightarrow \mathbb{C}[[t]]
$$

defined by $\phi^0_{p,N} (A) := \text{tr}_{W(p)} (\rho_p \bar{c}(A) u_{d^1} \cdots u_{d^{n-1}-1} t^{-d})$,

computing the $\bar{Q} \times \mathbb{N}$-graded trace of the $A$-action on the representation $W(p)_n$. Due to the quadratic relations and the $Q$-grading, it suffices to compute the following generating series:

$$
\phi^0_{p,N} f_0(z_0, \ldots, z_{n-1,N}) = (c_0 \cdots c_{n-1})^{-N} q^{a/2} d^{N(n-2)/2} \times \prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N} (z_{i,j} - z_{i,j'})(z_{i,j} - q^{2} z_{i,j'}) \times \prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N} (z_{i,j} - q^{d-1} z_{i,j'}) \times \prod_{i=0}^{n-1} \prod_{1 \leq j < j' \leq N} (z_{i,j} - q^{2} z_{i,j'}) \times q^{d-1} \cdot \prod_{j=1}^{N} z_{0,j} \cdot \prod_{i=0}^{N} \prod_{j=1}^{N} z_{i,j} \cdot \theta(\bar{y}, \Omega) \times
$$

$$
\frac{1}{(T; T)^{n}} \prod_{i=0}^{n-1} \prod_{a=1}^{k_i} \frac{(T q^{2} z_{a,b}; T)^{\infty} \cdot (T q^{-1} d^{-1} z_{a,b}; T)^{\infty}}{(T q^{2} z_{a,b}; T)^{\infty} \cdot (T q^{-1} d^{-1} z_{a,b}; T)^{\infty}} \times
$$

$$
\prod_{i=0}^{n-1} \prod_{a=1}^{k_i} \prod_{b=1}^{w_{i,b}} (T q^{2} z_{a,b}; T)^{\infty} \cdot (T q^{-1} d^{-1} z_{a,b}; T)^{\infty} \cdot (T q^{-1} d^{-1} z_{a,b}; T)^{\infty},
$$

where $T := \frac{1}{q}$ and $\theta(\bar{y}, \Omega) := \sum_{\bar{u} \in \mathbb{Z}^{n}} \exp(2\pi i \vec{y} \cdot \vec{u} \Omega + i\vec{u} \bar{y})$ is the classical Riemann theta function with $\Omega = \frac{1}{2\pi i} \sqrt{\alpha} \ln(T) \cdot \prod_{i,j=1}^{N} \frac{z_{i,j}^{2} z_{j+1,i}}{z_{i,j}^{2} z_{j+1,i}}
$$

We start with the following two auxiliary results:

**Lemma 3.6.** The matrix $\left( \frac{[k_i]}{k_i} d_{k_i,j} \right)_{i,j=0}^{n-1}$ is nondegenerate iff $q^{2k}, (dq)^k, (d^{-1} q)^k \neq 1$. 

Therefore if $q^2, dq, d^{-1}q$ are not roots of unity, we can choose a new basis $\{\tilde{H}_{i,-k}\}_{i=0}^{n-1}$ of the space spanned by $\{H_{0,-k}, \ldots, H_{n-1,-k}\}$, such that $[H_{i,k}, \tilde{H}_{j,-l}] = \delta_{i,j} \delta_{k,l} H_0$ for any $i, j$ and $k, l > 0$. In particular, the elements $\{H_{i,k}, \tilde{H}_{i,-k}, H_0\}_{k=0}^{n-1}$ form a Heisenberg Lie algebra $\mathfrak{h}_i$ for any $i$, and $\mathfrak{h}_i$ commutes with $\mathfrak{h}_j$ for any $0 \leq i \neq j \leq n - 1$.

**Lemma 3.7.** Let $a$ be a Heisenberg Lie algebra with the basis $\{a_k\}_{k \in \mathbb{Z}}$ and the commutator relation $[a_k, a_l] = \delta_{k,-l} \lambda_k a_0$. Consider the Fock $a$-representation $F := \text{Ind}^a_C$ with the central charge $a_0 = 1$ and the degree operator $d$ on $F$ with $[d, a_k] = k a_k$. Then for any $x_j, y_j \in C$:

$$\text{tr}_F \left\{ \exp \left( \sum_{j=1}^{\infty} x_j a_{-j} \right) \cdot \exp \left( \sum_{j=1}^{\infty} y_j a_j \right) \cdot t^{-d} \right\} = \frac{1}{(t; t)_\infty} \exp \left( \sum_{j=1}^{\infty} x_j y_j \lambda_j^t \frac{1}{1 - t^j} \right).$$

**Proof.** Applying the formula $(a_{l-j} v_0 | a_{k} a_{k}^t | a_{l-j} v_0) = (l - 1) \cdots (l - k + 1) \lambda^k$ for $k \leq 0$, we get

$$\text{tr}_F \left\{ \exp \left( \sum_{j=1}^{\infty} x_j a_{-j} \right) \cdot \exp \left( \sum_{j=1}^{\infty} y_j a_j \right) \cdot t^{-d} \right\} = \sum_{k_1, k_2, \ldots \geq 0} \text{tr}_F \left( \prod_{j=1}^{\infty} \frac{(x_j y_j \lambda_j^k)}{(k_j^l)^{1/2}} \cdot t^{-d} \right) =$$

$$\prod_{j=1}^{\infty} \left\{ \sum_{k_j = 0}^{\infty} \frac{(x_j y_j \lambda_j^k)}{(k_j^l)^{1/2}} \cdot t^{1/j} \right\} = \prod_{j=1}^{\infty} \left\{ \sum_{k_j = 0}^{\infty} \frac{(x_j y_j \lambda_j^k)}{(k_j^l)^{1/2}} \cdot \frac{1}{(1 - t^j)^{k_j + 1}} \right\}.$$

The result follows. $\square$

**Proof of Theorem 3.6.**

Reordering the factors of $\prod_{j=1}^{n} (f_0(z_{0,j}) \cdots f_{n-1}(z_{n-1,j})) \cdot \prod_{i=0}^{n-1} \prod_{j=1}^{k_i} \psi_i^+ (w_{i,j}) \cdot \prod_{i=0}^{n-1} \bar{\psi}^i_0$ in the normal order, we gain the product of factors from the first two lines of $(\phi)$. The $Q \times \mathbb{Z}$-graded trace of the normally ordered product splits as $\text{tr}_1 \cdot \text{tr}_2$, where

$$\text{tr}_1 = \text{tr}_{\mathcal{Q}(Q)e^{A'}} \left( q \prod_{i=0}^{n} \prod_{j=1}^{k_i} \tilde{H}_{i,-k} \right) \cdot \prod_{i=0}^{n-1} \prod_{j=1}^{k_i} \frac{u_{i,j}}{t^{d^{(1)}}} \left( q^{-1} \right) =$$

$$\text{tr}_2 = \text{tr}_{F_n} \left( \exp \left( \sum_{i=0}^{n-1} \sum_{k>0} u_{i,k} \tilde{H}_{i,-k} \right) \cdot \exp \left( \sum_{i=0}^{n-1} \sum_{k>0} \left( u_{i,k}^{(1)} \tilde{H}_{i,k} + u_{i,k}^{(2)} \right) \right) \cdot \left( t/q^b \right)^{d^{(1)}} \right),$$

with

$$u_{i,k} := -q^{k/2} \sum_{j=1}^{N} z_{i,j}^{k}, \quad v_{i,k}^{(1)} := q^{k/2} \sum_{j=1}^{N} z_{i,j}^{k}, \quad v_{i,k}^{(2)} := q^{-k} \sum_{j=1}^{N} w_{i,j}^{k}$$

and the operators $d^{(1)} \in \text{End}(F_n), d^{(2)} \in \text{End}(\mathcal{Q}(Q)e^{A'})$ defined by

$$d^{(1)}(H_{i_1,-k_1} \cdots H_{i_l,-k_l} v_0) = \sum_{i=1}^{l} k_i \cdot H_{i,-k_i} \cdots H_{i_l,-k_l} v_0, \quad d^{(2)}(e^\tilde{\beta}) = \left( \tilde{\beta}, \tilde{\beta} \right) - \left( \Lambda_p, \Lambda_p \right) / 2 \cdot e^\tilde{\beta}.$$

The computation of $\text{tr}_1$ is straightforward and we get exactly the expression from the third line of $(\phi)$. To evaluate $\text{tr}_2$, we rewrite $\sum_{i=0}^{n-1} \sum_{k>0} u_{i,k} \tilde{H}_{i,-k} = \sum_{i=0}^{n-1} \sum_{k>0} \tilde{u}_{i,k} \tilde{H}_{i,-k}$ with $\tilde{H}_{i,-k}$ defined right after Lemma 3.8 and $\tilde{u}_{i,k} = \sum_{i=0}^{n-1} d^{-km_{i,j}} \sum_{i=0}^{k} \tilde{u}_{i,k}$. The commutativity of $\mathfrak{h}_i$ and $\mathfrak{h}_j$ for $i \neq j$ allows us to rewrite $\text{tr}_2$ as a product of the corresponding traces over the $\mathfrak{h}_i$-Fock modules. Applying Lemma 3.7, we see (after routine computations) that $\text{tr}_2$ is equal to the product of the factors from the last two lines in $(\phi)$. $\square$
3.3. Functionals via pairing.

Recall the Hopf algebra pairing \( \varphi' : \hat{U}^\ge \times \hat{U}^\le \to \mathbb{C} \) from Theorem \([7, 17]\). As \( \varphi' \) is non-degenerate, there exist unique elements \( X^0_{p, \varepsilon}, X^u_{p, \varepsilon} \in \hat{U}^\ge \wedge \) and \( X^{u, t}_{p, \varepsilon} \in \hat{U}^\ge \wedge [t] \) such that

\[
\varphi^0_{p, \varepsilon}(X) = \varphi'(X^0_{p, \varepsilon}, X), \quad \varphi^u_{p, \varepsilon}(X) = \varphi'(X^u_{p, \varepsilon}, X), \quad \varphi^{u, t}_{p, \varepsilon}(X) = \varphi'(X^{u, t}_{p, \varepsilon}, X) \quad \forall X \in \hat{U}^\le .
\]

The goal of this section is to find these elements explicitly.

We will actually compute these elements in the shuffle presentation. In order to do this, we first extend the isomorphism \( \Psi \) of Theorem \([13, 19]\) to the isomorphism

\[
\Psi^2 : \hat{U}^\ge \simeq S_{N}^\ge.
\]

Here \( S_{N}^\ge \) is generated by \( S \) and the formal generators \( \psi_{i,k}(k < 0), \psi_{i,1}, \gamma_{\pm 1/2}, q^{d_1} \) with the defining relations compatible with those for \( \hat{U}^\ge \). In particular, for \( F \in S_{N,d}^\ge \) we have

\[
q^{d_1} F q^{-d_1} = q^d \cdot F.
\]

We define \( \Gamma^0_{p, \varepsilon}, \Gamma^u_{p, \varepsilon}, \Gamma^{u, t}_{p, \varepsilon} \) as the images of \( X^0_{p, \varepsilon}, X^u_{p, \varepsilon}, X^{u, t}_{p, \varepsilon} \) under isomorphism \( \Psi^2 \), respectively. Now we are ready to state the main result of this section:

**Theorem 3.8.** We have the following formulas:

(a) \( \Gamma^0_{p, \varepsilon} = \sum_{N=0}^{\infty} (c_0 \cdots c_{n-1})^{-N} \cdot \Gamma^0_{p, N} \cdot q^{\frac{\lambda}{p} - d_1} \) with \( \Gamma^0_{p, N} \in S_{N,d}^\ge \) given by

\[
\Gamma^0_{p, N} = (1 - q^{-2})^n N \cdot \prod_{j=1}^{N} x_{i,j} \cdot \prod_{j=1}^{n} \prod_{i,j \neq j'} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i=1}^{n} \prod_{j=1}^{N} x_{i,j}.
\]

(b) \( \Gamma^u_{p, \varepsilon} = \sum_{N \geq 0} (c_0 \cdots c_{n-1})^{-N} \Gamma^u_{p, N} \cdot q^{-d_1} \) with \( \Gamma^u_{p, N} \in S_{N,d}^\ge \) given by

\[
\Gamma^u_{p, N} = (1 - q^{-2})^n N \cdot \prod_{j=1}^{n} \prod_{i,j \neq j'} (x_{i,j} - q^{-2} x_{i,j'}) \cdot (1)^p [\mu] \cdot \prod_{i=1}^{n} \prod_{j=1}^{N} x_{i,j}.
\]

where in the last product we take all the \( x_{i,j} \) to the left and all \( q^{\delta_i} \) to the right.

(c) \( \Gamma^{u, t}_{p, \varepsilon} = \sum_{N \geq 0} (c_0 \cdots c_{n-1})^{-N} \Gamma^{u, t}_{p, N} \cdot q^{\frac{\lambda}{p} - d_1} \) with \( \Gamma^{u, t}_{p, N} \in S_{N,d}^\ge \) given by

\[
\Gamma^{u, t}_{p, N} = (1 - q^{-2})^n N \cdot \prod_{j=1}^{n} \prod_{i,j \neq j'} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \theta(x, \bar{x}) \times
\]

\[
\prod_{i=1}^{n} \prod_{j=1}^{N} \left( \frac{(t x_{i,j})}{x_{i,j}} \right) \cdot \prod_{k=1}^{n} \prod_{a=1}^{N} \left( \frac{(t q d x_{i,j})}{x_{i,j}} \right) \cdot \prod_{a=1}^{n} \prod_{b=1}^{N} \psi_{a,b}(k \cdot q^{1/2} x_{i,a})
\]

where \( \bar{x} = (x_1, \ldots, x_{n-1}) \) with \( x_i = \frac{1}{2} \ln \left( \frac{u_i q^{d_1} \psi_{i,0} \prod_{j=1}^{N} x_{i,j} x_{i+1,j}^{2-j}}{x_{i+1,j}^{2-j}} \right) \)

and in the above products we take all \( x_{i,j} \) to the left and all \( \psi_{i,j} \) to the right.
The proof of this theorem follows by combining Proposition 3.2, Proposition 3.4 and Theorem 3.5 with the following technical lemma:

**Lemma 3.9.** (a) For any elements $a \in \tilde{U}^{+}, a' \in \tilde{U}^{\tilde{z}} \cap \tilde{U}^{0}, b \in \tilde{U}^{-}, b' \in \tilde{U}^{\tilde{z}} \cap \tilde{U}^{0}$, we have

$$\varphi(aa', bb') = \varphi(a, b) \cdot \varphi(a', b').$$

(b) For any $k_i, l_i \in \mathbb{N}$ and $A, B, C, A', B', C', a_i, b_i \in \mathbb{Z}$, we have

$$\varphi\left(\prod_{i=0}^{n-1} \tilde{\varphi}_{i}^{a_i} (\tilde{z}_{i,a}) \cdot \prod_{i=0}^{n-1} \psi_{i,0}^{0} \cdot \gamma^{A/2} q^{B_d} q^{C_d} \prod_{j=0}^{n-1} \tilde{\varphi}_{j}^{b_j} (w_{j,b}) \cdot \prod_{i=0}^{n-1} \psi_{i,0}^{0} \cdot \gamma^{A'/2} q^{B'_d} q^{C'_d}\right) =$$

$$q^{-\frac{d}{4} A'B - \frac{d}{4} AB' + C'} \sum_{a_i + C \sum a' + \sum_{i,j} a_i a'_{i,j}} \prod_{i,j=0}^{n-1} \prod_{a=b=1}^{k_i, k_j} \frac{w_{j,b} - q^{a_{i,j}} d^{m_{i,j}} z_{i,a}}{w_{j,b} - q^{-a_{i,j}} d^{m_{i,j}} z_{i,a}}.$$

(c) For $\tilde{r} = (r_0, \ldots, r_{n-1}), \tilde{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{N}^n$ and elements $X \in \tilde{U}^{+}, Y \in \tilde{U}^{-}$ of the form

$$X = E_{0,a_0} \cdots E_{0,a_{n-1}}, E_{n-1,a_{n-1}}, \cdots, E_{n-1,a_{n-1}}, Y = F_{0,b_0} \cdots F_{0,b_{n-1}}, F_{n-1,b_{n-1}}, \cdots, F_{n-1,b_{n-1}},$$

the pairing $\varphi(X, Y)$ is expressed by an integral formula similar to [N2 Proposition 3.10]:

$$\varphi(X, Y) = \frac{\delta_{\tilde{r}, \tilde{s}}}{(q - q^{-1})^{\sum r_i}} \int \prod_{i,j<r_i, s_i} b_{0,n_1}^{(i,j)} \cdot (u_{n_1} - s_{n_1}) \cdot \Psi(X)(u_0, \ldots, u_{n_1}, r_{n_1}) \prod_{i=0}^{n-1} \prod_{j=0}^{r_i} \prod_{j'=r_i}^{s_i} \omega_{i,j}(u_{i,j}/u_{i,j'}) \cdot (u_{i,j}/u_{i,j'}) d u_{i,j} \cdot 2 \pi \sqrt{1 - u_{i,j}}.$$

3.4. Bethe incarnation of $A(\tilde{\tau})$.

Recalling the notion of transfer matrix from Section 1.3, it is easy to see that

$$X_{p,c}^{\tilde{u},\tilde{l}} = T_{W(p), \infty}(u_1^{-\tilde{l}_1} \cdots u_{n-1}^{-\tilde{l}_{n-1}} - d_1) \cdot \prod_{j=0}^{n-1} u_j^{\tilde{l}_j},$$

which provides a more elegant definition of $X_{p,c}^{\tilde{u},\tilde{l}}$. Moreover, the elements $X_{p,c}^{\tilde{u},\tilde{l}}$ can be thought of as certain truncations of $X_{p,c}^{\tilde{u},\tilde{l}}$ obtained by setting $t \to 0$, while $X_{p,c}^{0,0}$ are obtained by setting further $u_1, \ldots, u_{n-1} \to 0$.

The commutativity of the Bethe subalgebras implies the commutativity of $\{\Gamma_{p,c}^{0,0}\}_{p,c} \in \mathbb{P}$, hence of $\{\Gamma_{p,c}^{0,0}\}_{p,c} \in \mathbb{P}_{\mathbb{C}^n}$. As a result, we get the commutativity of the families $\{\Gamma_{p,c}^{0,0}\}_{p,c} \in \mathbb{P}_{\mathbb{C}^n}$ and $\{\Gamma_{p,c}^{0,0}\}_{p,c} \in \mathbb{P}_{\mathbb{C}^{n-1}}$. Due to Theorem 3.5(b), the elements $\Gamma_{p,c}^{0,0}$ have the same form as the generators of the subalgebra $A(s_1, \ldots, s_n)$ from Section 2 with $s_i \in \mathbb{C}^* e^{C \otimes \tilde{P}}$. Since $e^h (h \in \tilde{P})$ commute with $\otimes h S_{k}$, we see that those $s_i$ can be treated as formal parameters with $s_1 \cdots s_n = 1$ and $\{s_i\}$ still being generic (note that $\{s_i\}$ are generic for any choice of $\{u_i\}$).

Finally, let us notice that while $\tilde{U}_{q,d}(\mathfrak{sl}_n)$ contained the horizontal copy of $U_q(\mathfrak{gl}_n)$, the algebra $\tilde{U}_{q,d}(\mathfrak{sl}_n)$ contains a horizontal copy of $U_q(\mathfrak{gl}_n)$ (that is no $q^{\pm d}$ and with trivial central charge $c' = 0$). The subspace $M(p)_n$ is $U_q(\mathfrak{gl}_n)$-invariant and is just the $p$-th fundamental representation. By standard results $U_q(\mathfrak{gl}_n)$ admits a double construction similar to the one for $U_{q,d}(\mathfrak{sl}_n)$. Combining all the previous discussions with the construction of the universal $R$-matrices for $\tilde{U}_{q,d}(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_n)$, we get the following result:

**Theorem 3.10.** The Bethe subalgebra of $U_q(\mathfrak{gl}_n)$ corresponding to the group-like element $x = u_1^{-\tilde{l}_1} \cdots u_{n-1}^{-\tilde{l}_{n-1}}$ can be identified with $A(s_1, \ldots, s_n)$ (here $s_i \in \mathbb{C}^* e^{C \otimes \tilde{P}}$ as above).
4. Generalizations to $n=1,2$ case

It turns out that all the previous results proved for $n>2$ can be actually generalized to the $n=1,2$ cases. The goal of this last section is to explain the required slight modifications.

4.1. $n=1$ case.

The quantum toroidal algebra $\hat{U}_{q,d}(\mathfrak{gl}_1)$ has been extensively studied in the last few years. Roughly speaking, one just needs to modify the quadratic relations from Section 1.1 by replacing

$$g_{a_{1,i}}(t) \sim \frac{(q^2 t - 1)(q^{-1} d t - 1)(q^{-1} d^{-1} t - 1)}{(t - q^2)(t - q^{-1} d)(t - q^{-1} d^{-1})},$$

and by replacing the Serre relations (T7.1, T7.2) by

$$\text{Sym}_{\mathfrak{sl}_1} z_2 z^{-1}_3 \cdot [e(z_1), [e(z_2), e(z_3)]] = 0 = \text{Sym}_{\mathfrak{sl}_1} z_2 z^{-1}_3 \cdot [f(z_1), [f(z_2), f(z_3)]]].$$

Analogously to the $n>2$ case, the map $e_i \mapsto x^i$ extends to the isomorphism $\hat{U}(\mathfrak{gl}_1)^+ \xrightarrow{\sim} \mathcal{A}_{\text{sm}}$.

The results of Section 2 recover the same commutative algebra $\mathcal{A}_{\text{sm}}$ we started from. On the other hand, we can apply the constructions of Section 3 to the Fock $\mathcal{F}$-representation $\{F_c\}_{c \in \mathbb{C}^*}$. As a result we will get:

- The elements $\Gamma^0_c$ (corresponding to the top matrix coefficient functional $\phi^0_c$) are given by
  $$\Gamma^0_c = \sum_{N=0}^{\infty} c^{-N} q^{-N(N-1)/2} H_N(x_1, \ldots, x_N) \cdot q^{d1},$$

- The elements $\Gamma^1_c$ (corresponding to the full graded trace functional $\phi^1_c$) are given by
  $$\Gamma^1_c = \sum_{N=0}^{\infty} c^{-N} q^{-N(N-1)/2} \prod_{a,b=1}^N \frac{(\bar{t} \bar{t} q \frac{q}{t}; q)_\infty (\bar{t} \bar{t} q \frac{q}{t}; q)_\infty}{(t q \frac{q}{t}; q)_\infty (t q \frac{q}{t}; q)_\infty} \cdot \prod_{k=0}^N \frac{1}{\bar{t}^k q^{1/2} x_a} \cdot q^{d1}.$$

4.2. $n=2$ case.

For $n=2$ we need first to redefine both the quantum toroidal and the shuffle algebras.

- **Quantum toroidal algebra of $\mathfrak{sl}_2$.**
  One needs to slightly modify the defining relations (T0.1-T7.2) of $\hat{U}_{q,d}(\mathfrak{sl}_2)$ (see [FJMM1]).
  The function $g_{a_{1,i}}(t)$ from the relations (T1,T2,T3,T5,T6) should be changed as follows:
  $$g_{a_{1,i}}(t) \sim \frac{q^2 t - 1}{t - q^2}, \quad g_{a_{1,i+1}}(t) \sim \frac{(d t - q)(d^{-1} t - q)}{(q t - d)(q t - d^{-1})},$$

  while the cubic Serre relations (T7.1, T7.2) should be replaced with quartic Serre relations
  $$\text{Sym}_{\mathfrak{sl}_2} [e_i(z_1), [e_i(z_2), e_i(z_3), e_{i+1}(w)]_{q^2}]_{q^{-2}} = 0 = \text{Sym}_{\mathfrak{sl}_2} [f_i(z_1), [f_i(z_2), f_i(z_3), f_{i+1}(w)]_{q^2}]_{q^{-2}}.$$

- **Big shuffle algebra of type $A_1^{(1)}$.**
  One needs to modify the matrix $\Omega$ used to define the $\ast$-product as follows:
  $$\omega_{i,i}(t) = \frac{t - q^{-2}}{t - 1}, \quad \omega_{i,i+1}(t) = \frac{(t - q d)(t - q d^{-1})}{(t - 1)^2}.$$

- **Vertex representations $W(p)_2$.**
  Finally, we need to slightly modify the formulas of $\rho_{p,\bar{c}}$ from Theorem 3.1
  (i) we redefine the commutator relations of the Heisenberg algebra $\mathcal{S}_n$ as follows:
  $$[H_{1,k}, H_{1,l}] = \frac{[k] \cdot [2k]}{k} \cdot \delta_{k,-l} H_0, \quad [H_{1,k}, H_{1,l+1}] = -(d^k + d^{-k}) \frac{[k] \cdot [k]}{k} \cdot \delta_{k,-l} H_0.$$

(ii) we also redefine the operator $z^{H_{1,o}}$ via
  $$z^{H_{1,o}}(v \otimes \bar{c}) := z^{(H_{1,o})} v \otimes \bar{c}.$$

Once the above modifications are made, all the results from Sections 2-3 still hold.
Acknowledgments. We are grateful to A. Negut and J. Shiraishi for stimulating discussions. A. T. is thankful to P. Etingof and H. Nakajima for their interest and support. A. T. thanks the Research Institute for Mathematical Sciences (Kyoto) and the Japan Society for the Promotion of Science for support during the last stage of this project. B. F. acknowledges the financial support of a subsidy granted to the Higher School of Economics by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

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