STRONG PSEUDO-AMENABILITY OF SOME BANACH ALGEBRAS

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Abstract. In this paper we introduce a new notion of strong pseudo-amenability for Banach algebras. We study strong pseudo-amenability of some Matrix algebras. Using this tool, we characterize strong pseudo-amenability of \( \ell^1(S) \), provided that \( S \) is a uniformly locally finite semigroup. As an application we show that for a Brandt semigroup \( S = M^0(G, I) \), \( \ell^1(S) \) is strong pseudo-amenable if and only if \( G \) is amenable and \( I \) is finite. We give some examples to show the differences of strong pseudo-amenability and other classical notions of amenability.

1. Introduction and Preliminaries

Johnson introduced the class of amenable Banach algebras. A Banach algebra \( A \) is called amenable, if there exists a bounded net \((m_\alpha)\) in \( A \otimes_p A \) such that \( a \cdot m_\alpha - m_\alpha \cdot a \to 0 \) and \( \pi_A(m_\alpha)a \to a \) for every \( a \in A \). For further information about the history of amenability see [19].

By removing the boundedness condition in the definition of amenability, Ghahramani and Zhang in [14] introduced and studied two generalized notions of amenability, named pseudo-amenability and pseudo-contractibility. A Banach algebra \( A \) is called pseudo-amenable(pseudo-contractible) if there exists a not necessarily bounded net \((m_\alpha)\) in \( A \otimes_p A \) such that \( a \cdot m_\alpha - m_\alpha \cdot a \to 0 \) and \( \pi_A(m_\alpha)a \to a \) for every \( a \in A \), respectively. Recently pseudo-amenability and pseudo-contractibility of the archimedean semigroup algebras and the uniformly locally finite semigroup algebras have investigated in [10], [9] and [24]. In fact the main results of [10] and [9] are about characterizing pseudo-amenability and pseudo-contractibility of \( \ell^1(S) \), where \( S = M^0(G, I) \) is the Brandt semigroup over an index set \( I \). They showed that \( \ell^1(S) \) is pseudo-amenable (pseudo-contractible) if and only if \( G \) is amenable (\( G \) is a finite group and \( I \) is a finite index set), respectively. For further information about pseudo-amenability and pseudo-contractibility of general Banach algebras the readers refer to [5].

Motivated by these considerations this question raised "Is there a notion of amenability which stands between pseudo-amenability and pseudo-contractibility for the Brandt semigroup algebras?", that is, under which notion of amenability for the Brandt semigroup algebra \( \ell^1(M^0(G, I)) \), \( G \) becomes amenable and \( I \) becomes finite. In order to answer this question author defines a new notion of amenability, named strong pseudo-amenability. Here we give the definition of our new notion.

Definition 1.1. A Banach algebra \( A \) is called \textit{strong pseudo-amenable}, if there exists a (not necessarily bounded) net \((m_\alpha)\) in \( (A \otimes_p A)^{**} \) such that

\[
a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad a \pi_A^{**}(m_\alpha) = \pi_A^{**}(m_\alpha)a \to a \quad (a \in A).
\]
In this paper, we study the basic properties of strong pseudo-amenable Banach algebras. We show that strong pseudo-amenability is weaker than pseudo-contractibility but it is stronger than pseudo-amenability. We investigate strong pseudo-amenability of matrix algebras. Using this tool we characterize strong pseudo-amenability of \( \ell^1(S) \), whenever \( S \) is a uniformly locally finite semigroup. In particular, we show that \( \ell^1(S) \) is strong pseudo amenable if and only if \( I \) is a finite index set and \( G \) is amenable, where \( S = M^0(G, I) \) is the Brandt semigroup. Finally we give some examples that shows the differences between strong pseudo-amenability and other classical concepts of amenability.

We present some standard notations and definitions that we shall need in this paper. Let \( A \) be a Banach algebra. If \( X \) is a Banach \( A \)-bimodule, then \( X^* \) is also a Banach \( A \)-bimodule via the following actions
\[
(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).
\]
Let \( A \) and \( B \) be Banach algebras. The projective tensor product \( A \otimes_p B \) with the following multiplication is a Banach algebra
\[
(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2 \quad (a_1, a_2 \in A, b_1b_2 \in B).
\]
Also \( A \otimes_p A \) with the following action becomes a Banach \( A \)-bimodule:
\[
a_1 \cdot a_2 \otimes a_3 = a_1a_2 \otimes a_3, \quad a_2 \otimes a_3 \cdot a_1 = a_2 \otimes a_3a_1, \quad (a_1, a_2, a_3 \in A).
\]
The product morphism \( \pi_A : A \otimes_p A \to A \) is specified by \( \pi_A(a \otimes b) = ab \) for every \( a, b \in A \).

2. Basic properties of strong pseudo-amenability

**Proposition 2.1.** Let \( A \) be a strong pseudo-amenable Banach algebra. Then \( A \) is pseudo-amenable.

**Proof.** Since \( A \) is strong pseudo-amenable, there exists a net \( (m_n) \) in \( (A \otimes_p A)^* \) such that \( a \cdot m_n - m_n \cdot a \to 0 \) and \( \pi_A^*(m_n)a = a\pi_A^*(m_n) \to a \), for every \( a \in A \). Take \( \epsilon > 0 \) and arbitrary finite subsets \( F \subseteq A, \Lambda \subseteq (A \otimes_p A)^* \) and \( L \subseteq A^* \). It follows that
\[
||a \cdot m_n - m_n \cdot a|| < \epsilon, \quad ||\pi_A^*(m_n)a - a|| < \epsilon,
\]
for every \( a \in F \). It is well-known that for each \( \alpha \), there exists a net \( (n_{\beta})_\alpha \) in \( A \otimes_p A \) such that \( n_{\beta} \overset{w^*}{\to} m_\alpha \).

Thus using \( w^*\)-continuity of \( \pi_A^* \), we have
\[
\pi_A(n_{\beta}(\gamma)) = \pi_A^*(n_{\beta}(\gamma)) \overset{w^*}{\to} \pi_A^*(m_\alpha).
\]
Hence there exists \( \beta = \beta(\epsilon, F, \Lambda, L) \) such that
\[
||a \cdot n_{\beta}(\gamma) - a \cdot m_\alpha|| < \frac{\epsilon}{K}, \quad ||n_{\beta}(\gamma) \cdot a - m_\alpha \cdot a|| < \frac{\epsilon}{K}
\]
and
\[
||\pi_A(n_{\beta}(\gamma))a - \pi_A^*(m_\alpha)a|| < \frac{\epsilon}{2L_0}, \quad ||\pi_A^*(m_\alpha)a - a|| < \frac{\epsilon}{2L_0}
\]
for every \( a \in F, f \in \Lambda \) and \( g \in L \), where \( K = \sup\{||f|| : f \in \Lambda\} \) and \( L_0 = \sup\{||f|| : f \in L\} \). So for a \( c \in \mathbb{R}^+ \) we have
\[
||a \cdot n_{\beta}(\gamma) - n_{\beta}(\gamma) \cdot a|| < \frac{\epsilon}{K}
\]
and
\[
||a(\gamma) - \pi_A(n_{\beta}(\gamma)a(\gamma)|| < \frac{\epsilon}{L_0}\]
for every \( a \in F, f \in \Lambda \) and \( g \in L \). It follows that there exists a net \((n_{\beta(\epsilon,F,\Lambda,L)}^a)_{(a,\epsilon,F,\Lambda,L)}\) in \( A \otimes_p A \) which satisfies
\[
a \cdot n_{\beta(\epsilon,F,\Lambda,L)}^a - n_{\beta(\epsilon,F,\Lambda,L)}^a \cdot a \xrightarrow{w} 0, \quad \pi_A(n_{\beta(\epsilon,F,\Lambda,L)}^a)a - a \xrightarrow{w} 0, \quad (a \in A).
\]

Using Mazur Lemma we can assume that
\[
a \cdot n_{\beta(\epsilon,F,\Lambda,L)}^a - n_{\beta(\epsilon,F,\Lambda,L)}^a \cdot a \xrightarrow{||\cdot||} 0, \quad \pi_A(n_{\beta(\epsilon,F,\Lambda,L)}^a)a - a \xrightarrow{||\cdot||} 0, \quad (a \in A).
\]

Therefore \( A \) is pseudo-amenable.

Let \( A \) be a Banach algebra and \( \phi \in \Delta(A) \). A Banach algebra \( A \) is called approximately left \( \phi \)-amenable, if there exists a (not necessarily bounded) net \((n_\alpha)\) in \( A \) such that
\[
am_\alpha - \phi(a)m_\alpha \to 0, \quad \phi(m_\alpha) \to 1, \quad (a \in A).
\]
For further information see [1].

**Corollary 2.2.** Let \( A \) be a Banach algebra and \( \phi \in \Delta(A) \). If \( A \) is strong pseudo-amenable, then \( A \) is approximately left \( \phi \)-amenable.

**Proof.** By Proposition 2.1 strong pseudo-amenability of \( A \) implies that \( A \) is pseudo-amenable. So there exists a net \((m_\alpha)\) in \( A \otimes_p A \) such that \( a \cdot m_\alpha - m_\alpha \cdot a \to 0 \) and \( \pi_A(m_\alpha)a \to a \), for every \( a \in A \). Define \( T : A \otimes_p A \to A \) by \( T(a \otimes b) = \phi(b)a \) for every \( a, b \in A \). Clearly \( T \) is a bounded linear map. Set \( n_\alpha = T(m_\alpha) \). One can easily see that
\[
am_\alpha - \phi(a)n_\alpha \to 0, \quad \phi(n_\alpha) = \phi(T(m_\alpha)) = \phi(\pi_A(m_\alpha)) \to 1, \quad (a \in A).
\]
Then \( A \) is approximately left \( \phi \)-amenable.

A Banach algebra \( A \) is called pseudo-contractible if there exists a net \((m_\alpha)\) in \( A \otimes_p A \) such that \( a \cdot m_\alpha = m_\alpha \cdot a \) and \( \pi_A(m_\alpha)a \to a \) for each \( a \in A \), see [14].

**Lemma 2.3.** Let \( A \) be a pseudo-contractible Banach algebra. Then \( A \) is strong pseudo-amenable.

**Proof.** Clear.

**Lemma 2.4.** Let \( A \) be a commutative pseudo-amenable Banach algebra. Then \( A \) is strong pseudo-amenable.

**Proof.** Clear.

A Banach algebra \( A \) is called biflat if there exists a bounded \( A \)-bimodule morphism \( \rho : A \to (A \otimes_p A)^{**} \) such that \( \pi_A^* \circ \rho(a) = a \) for each \( a \in A \). See [19].

**Lemma 2.5.** Let \( A \) be a biflat Banach algebra with a central approximate identity. Then \( A \) is strong pseudo-amenable.

**Proof.** Since \( A \) is biflat, there exists a bounded \( A \)-bimodule morphism \( \rho : A \to (A \otimes_p A)^{**} \) such that \( \pi_A^* \circ \rho(a) = a \) for each \( a \in A \). Let \((e_\alpha)\) be a central approximate identity for \( A \). Define \( m_\alpha = \rho(e_\alpha) \).

Since \( \rho \) is a bounded \( A \)-bimodule morphism, we have \( a \cdot m_\alpha = m_\alpha \cdot a \) and \( \pi_A^*(m_\alpha)a = a\pi_A^*(m_\alpha) \to a \), for every \( a \in A \).
Remark 2.6. In the previous lemma we can replace the biflatness with the existence of a (not necessarily bounded net) of $A$-bimodule morphism $\rho_\alpha : A \to (A \otimes_p A)^{**}$ which satisfies $\pi_A \circ \rho_\alpha (a) \xrightarrow{|| \cdot ||} a$. Now using the similar argument as in the proof of previous and iterated limit theorem [17, p. 69], we can see that $A$ is strong pseudo-amenable.

Proposition 2.7. Suppose that $A$ and $B$ are Banach algebras. Let $A$ be strong pseudo-amenable. If $T : A \to B$ is a continuous epimorphism, then $B$ is strong pseudo-amenable.

Proof. Since $A$ is strong pseudo-amenable, there exists a net $(m_\alpha)$ in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A^*(m_\alpha)a = a\pi_A^*(m_\alpha) \to a$, for every $a \in A$. Define $T \otimes T : A \otimes_p A \to B \otimes_p B$ by $T \otimes T(a \otimes b) = T(a) \otimes T(b)$ for every $a, b \in A$. Clearly $T \otimes T$ is a bounded linear map. So we have

$$T(a) \cdot (T \otimes T)^*(m_\alpha) - (T \otimes T)^*(m_\alpha) \cdot T(a) = (T \otimes T)^*(a \cdot m_\alpha - m_\alpha \cdot a) \to 0, \quad (a \in A).$$

and

$$\pi_B^* \circ (T \otimes T)^*(m_\alpha)\pi_T(a) - T(a)\pi_B^* \circ (T \otimes T)^*(m_\alpha)$$

$$= (\pi_B \circ (T \otimes T))^*(m_\alpha \cdot a) - (\pi_B \circ (T \otimes T))^*(a \cdot m_\alpha)$$

$$= T^{**} \circ \pi_A^{**}(m_\alpha \cdot a) - T^{**} \circ \pi_A^{**}(a \cdot m_\alpha)$$

$$= T^{**}(\pi_A^{**}(m_\alpha)a - a\pi_A^{**}(m_\alpha)) = T^{**}(0) = 0$$

Also

$$\pi_B^* \circ (T \otimes T)^*(m_\alpha)\pi_T(a) - T(a) = (\pi_B \circ (T \otimes T))^*(m_\alpha \cdot a) - T(a) = T^{**}(\pi_A^{**}(m_\alpha)a - a) \to 0,$$

for every $a \in A$. Then $B$ is strong pseudo-amenable. □

Corollary 2.8. Let $A$ be a Banach algebra and $I$ be a closed ideal of $A$. If $A$ is strong pseudo-amenable, then $A/I$ is strong pseudo-amenable.

Proof. The quotient map is a bounded epimorphism from $A$ onto $A/I$, now apply previous proposition. □

Lemma 2.9. Let $A$ and $B$ be Banach algebras. Suppose that $B$ has a non-zero idempotent. If $A \otimes_p B$ is strong pseudo-amenable, then $A$ is strong pseudo-amenable.

Proof. It deduces from a small modification of the argument of [19, Proposition 3.5]. In fact suppose that $b_0$ is a non-zero idempotent of $B$. Using Hahn-Banach theorem there exists a bounded linear map $f \in B^*$ such that $f(bb_0) = f(b_0b)$ and $f(b_0) = 1$, for every $b \in B$. Define $T_{b_0} : A \otimes_p B \to A \otimes_p A$ by $T_{b_0}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = f(b_0b_1b_2)a_1 \otimes a_2$ for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since $A \otimes_p B$ is strong pseudo-amenable, there exists a net $(m_\alpha)$ in $(A \otimes_p B)^{**}$ such that

$$x \cdot m_\alpha - m_\alpha \cdot x \to 0, \quad \pi_A^{**}B(m_\alpha)x = x\pi_A^{**}B(m_\alpha) \to x, \quad (x \in A \otimes_p B).$$

Now one can readily see that

$$a \cdot T_{b_0}^{**}(m_\alpha) - T_{b_0}^{**}(m_\alpha)a \to 0, \quad a\pi_A^{**} \circ T_{b_0}^{**}(m_\alpha) = \pi_A^{**} \circ T_{b_0}^{**}(m_\alpha)a \to a,$$

for each $a \in A$. It follows that $A$ is strong pseudo-amenable. □
3. Strong pseudo-amenability of matrix algebras

Let $A$ be a Banach algebra and $I$ be a totally ordered set. The set of $I \times I$ upper triangular matrices, with entries from $A$ and the usual matrix operations and also finite $\ell^1$-norm, is a Banach algebra and it denotes with $UP(I, A)$.

**Theorem 3.1.** Let $I$ be a totally ordered set with smallest element and let $A$ be a Banach algebra with $\phi \in \Delta(A)$. Then $UP(I, A)$ is strong pseudo-amenability if and only if $A$ is strong pseudo-amenability and $|I| = 1$.

**Proof.** Let $i_0$ be the smallest element and $\phi \in \Delta(A)$. Suppose that $UP(I, A)$ is strong pseudo-amenability. Suppose conversely that $|I| > 1$. Define $\psi : UP(I, A) \to \mathbb{C}$ by $\psi((a_{i,j})_{i,j}) = \phi(a_{i,i})$ for every $(a_{i,j})_{i,j} \in UP(I, A)$. Clearly $\psi$ is a character on $UP(I, A)$. Since $UP(I, A)$ is strong pseudo-amenability, by Corollary 2.2 $UP(I, A)$ is approximately left $\psi$-amenability. So by $[1]$ there exists a net $(n_a)_{i,j}$ in $UP(I, A)$ such that $\alpha_n - \psi(a)n_a \to 0$ and $\psi(n_a) \to 1$ for every $a \in UP(I, A)$. Set

$$J = \{(a_{i,j}) \in UP(I, A) | a_{i,j} = 0, \ i \neq i_0 \}.$$ 

It is easy to see that $J$ is a closed ideal of $UP(I, A)$ and $\psi|J \neq 0$. So there exists a $j$ in $J$ such that $\psi(j) = 1$. Replacing $(n_a)$ with $(n_a, j)$ we can assume that $(n_a)$ is a net in $J$ such that $\alpha_n - \psi(a)n_a \to 0$ and $\psi(n_a) \to 1$ for every $a \in J$. Suppose that $n_a$ in $J$ has a form

$$\begin{pmatrix} a_{00} & a_{01} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

for some nets $(a_{00}, n_a)$ and $(a_{01}, n_a)$ in $A$. Note that since $|I| > 1$, the matrix $n_a$ must have at least two columns. Also $\psi(n_a) \to 1$ implies that $\phi(a_{00}) \to 1$. Let $x$ be an element of $A$ such that $\phi(x) = 1$. Set $a = \begin{pmatrix} 0 & x & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \in J$. Clearly $a \in \ker \psi$. Put $a$ in the following fact $\alpha_n - \psi(a)n_a \to 0$. It follows that $a_{00}n_a x \to 0$. Application of $\phi$ on $a_{00}n_a x \to 0$ implies that $\phi(a_{00}n_a x) = \phi(a_{00}) \phi(x) = \phi(a_{00}) \to 0$, which is impossible. So $|I| = 1$ and $UP(I, A) = A$ which implies that $A$ is strong pseudo-amenability.

Converse is clear. \hfill $\square$

Suppose that $A$ is a Banach algebra and $I$ is a non-empty set. We denote $M_I(A)$ for the Banach algebra of $I \times I$-matrices over $A$, with the finite $\ell^1$-norm and the matrix multiplication. This class of Banach algebras belongs to $\ell^1$-Munn algebras, see $[8]$. We also denote $\varepsilon_{i,j}$ for a matrix belongs to $M_I(\mathbb{C})$ which $(i, j)$-entry is 1 and 0 elsewhere. The map $\theta : M_I(A) \to A \otimes_\varepsilon M_I(\mathbb{C})$ defined by $\theta((a_{i,j})) = \sum_{i,j} a_{i,j} \otimes \varepsilon_{i,j}$ is an isometric algebra isomorphism.

**Theorem 3.2.** Let $I$ be a non-empty set. Then $M_I(\mathbb{C})^{**}$ is strong pseudo-amenability if and only if $I$ is finite.

**Proof.** Let $A = M_I(\mathbb{C})$. Suppose that $A^{**}$ is strong pseudo-amenability. There exists a net $(m_a)$ in $(A^{**} \otimes_\varepsilon A^{**})^{**}$ such that $a \cdot m_a - m_a \cdot a \to 0$ and $\pi_{A^{**}}(m_a)a = a\pi_{A^{**}}(m_a) \to a$, for every $a \in A^{**}$. So there exists a net $\pi_{A^{**}}(m_a) \in A^{****}$ such that $\pi_{A^{**}}(m_a) \to a$, for every $a \in A$. Thus for each $a$ we have
a net \((m^\alpha_\beta)_\beta\) in \(A^{**}\) such that \(m^\alpha_\beta \xrightarrow{w^*} n_\alpha\) and \(\|m^\alpha_\beta\| \leq \|n_\alpha\|\). Then we have
\[
am^\alpha_\beta - m^\alpha_\beta a \xrightarrow{w} 0, \quad m^\alpha_\beta(f) \to n_\alpha(f),
\]
where \(f \in A^{**}\) such that \(n_\alpha(f) \neq 0\). Take \(\epsilon > 0\) and \(F = \{a_1, a_2, a_3, \ldots, a_r\}\) an arbitrary subset of \(A\).
Define
\[
V_\alpha = \{(a_1n - na_1, a_2n - na_2, \ldots, a_rn - na_r, n(f) - n_\alpha(f)) | n \in A^{**}, \|n\| \leq \|n_\alpha\|\},
\]
clearly \(V_\alpha\) is a convex subset of \((\prod_{\alpha=1}^r A^{**}) \otimes \mathbb{C}\). It is easy to see that \((0,0,\ldots,0)\) belongs to \(\overline{V_\alpha}^w\).
Since the norm topology and the weak topology on the convex sets are the same, we can assume that \((0,0,\ldots,0)\) belongs to \(\overline{V_\alpha}^w\).
So there exists an element \(m_{(F,\epsilon)}\) in \(A^{**}\) which
\[
\|a_i m_{(F,\epsilon)} - m_{(F,\epsilon)} a_i\| < \epsilon, \quad \|m_{(F,\epsilon)}(f) - n_\alpha(f)\| < \epsilon, \quad \|m_{(F,\epsilon)}\| \leq \|n_\alpha\|
\]
for every \(i \in \{1,2,\ldots,r\}\). It follows that the net \((m_{(F,\epsilon)}(a))_{a \in A}\) in \(A^{**}\) satisfies
\[
am_{(F,\epsilon)}(a) \xrightarrow{w^*} 0, \quad m_{(F,\epsilon)}(f) \xrightarrow{w} n_\alpha(f), \quad \|m_{(F,\epsilon)}\| \leq \|n_\alpha\|
\]
for every \(a \in A\). Since \(n_\alpha(f) \neq 0\) we may assume that \(m_{(F,\epsilon)}(f)\) stays away from 0. On the other hand there exists a net \((m^v_{(F,\epsilon)})\) in \(A\) such that \(m^v_{(F,\epsilon)} \xrightarrow{w^*} m_{(F,\epsilon)}\) and \(\|m^v_{(F,\epsilon)}\| \leq \|m_{(F,\epsilon)}\| \leq \|n_\alpha\|\). So
\[
am^v_{(F,\epsilon)} \xrightarrow{w^*} am_{(F,\epsilon)} = m^v_{(F,\epsilon)} a \xrightarrow{w^*} m_{(F,\epsilon)} a, \quad (a \in A).
\]
Since \(am_{(F,\epsilon)} = m_{(F,\epsilon)} a\) \(\xrightarrow{w^*} 0\), we may assume that \(am_{(F,\epsilon)} = m_{(F,\epsilon)} a \xrightarrow{w^*} 0\). It follows that
\[
w^* - \lim_{(F,\epsilon)} am_{(F,\epsilon)} = w^* - \lim_{v} (am^v_{(F,\epsilon)} - m^v_{(F,\epsilon)} a)
\]
\[
= w^* - \lim_{v} am^v_{(F,\epsilon)} - am^v_{(F,\epsilon)} a + am^v_{(F,\epsilon)} a + m^v_{(F,\epsilon)} a - m_{(F,\epsilon)} a = 0
\]
and
\[
w^* - \lim_{v} am^v_{(F,\epsilon)} = n_\alpha.
\]
Now using iterated limit theorem [17, p. 69], we can find a net \((m_{(v,F,\epsilon)})\) in \(A\) such that
\[
w^* - \lim_{(v,F,\epsilon)} am_{(v,F,\epsilon)} = m_{(v,F,\epsilon)} a \xrightarrow{w} 0.
\]
Since \((m_{(v,F,\epsilon)})\) is a net in \(A\), we have \(am_{(v,F,\epsilon)} = m_{(v,F,\epsilon)} a \xrightarrow{w^*} 0\). Now we follow the similar arguments as in [21] Example 4.1(iii) to show that \(I\) is finite.
Let \(m_{(v,F,\epsilon)} = (y^{ij}_{(v,F,\epsilon)})\), where \(y^{ij}_{(v,F,\epsilon)} \in \mathbb{C}\) for every \(i, j \in I\). Since the product of the weak topology on \(\mathbb{C}\) coincides with the weak topology on \(A\) [23, Theorem 4.3], for a fixed \(i_0 \in \Lambda\), we have \(\varepsilon_{i_0,j} m_{(v,F,\epsilon)} = m_{(v,F,\epsilon)}^{i_0,j} \xrightarrow{w^*} 0\). Thus \(y^{ij}_{(v,F,\epsilon)} - y^{i_0,j}_{(v,F,\epsilon)} \xrightarrow{w^*} 0\) and \(y^{ij}_{(v,F,\epsilon)} \xrightarrow{w^*} 0\), whenever \(i \neq j\). The boundedness of \((m_{(v,F,\epsilon)})\), implies that \((y^{i_0,j}_{(v,F,\epsilon)})\) is a bounded net in \(\mathbb{C}\). Thus \((y^{i_0,j}_{(v,F,\epsilon)})\) has a convergence subnet, denote it again with \((y^{i,j}_{u,v,F,\epsilon})\). Suppose that \((y^{i_0,j}_{u,v,F,\epsilon})\) converges to \(l\) with respect to \(\|\cdot\|\). On the other hand \(y^{ij}_{(v,F,\epsilon)} - y^{i_0,j}_{(v,F,\epsilon)} \xrightarrow{w^*} 0\), implies that \(y^{ij}_{(v,F,\epsilon)} - y^{i,j}_{u,v,F,\epsilon} \xrightarrow{w} 0\) (because \(\mathbb{C}\) is a Hilbert space). Thus \(y^{ij}_{(v,F,\epsilon)} \xrightarrow{w} l\) for every \(j \in I\). We claim that \(l \neq 0\). On the contrary suppose that \(l = 0\). Then by [23, Theorem 4.3] we have \(m_{(v,F,\epsilon)} \xrightarrow{w} 0\). Then \(f(m_{(v,F,\epsilon)}) \to 0\). Also we have \(f(m_{(v,F,\epsilon)}) = m_{(v,F,\epsilon)}(f) \to n_\alpha(f) \neq 0\), which reveals a contradiction. Hence \(l\) must be a non-zero number. Therefore the facts \(y^{ij}_{(v,F,\epsilon)} - y^{i,j}_{u,v,F,\epsilon} \xrightarrow{w} 0\) and \(y^{ij}_{(v,F,\epsilon)} \xrightarrow{w} 0\) in conjunction with [23, Theorem 4.3], give that \(y^{i,j}_{(v,F,\epsilon)} \xrightarrow{w} y_0\), where \(y_0\) is a matrix with \(l\) in the diagonal position and 0 elsewhere. So we have \(y_0 \in \text{Conv}(y^{i,j}_{(v,F,\epsilon)})^{w^*} = \text{Conv}(y^{i,j}_{u,v,F,\epsilon})^{w^*}\). It
implies that \( y_0 \in A \). But \( \infty = \sum_{j \in I} |l| = \sum_{j \in I} |y_0^j| = ||y_0|| < \infty \), provided that \( I \) is infinite which is a contradiction. So \( I \) must be finite.

For converse, let \( I \) be finite. Then \( M_I(\mathbb{C})^{**} = M_I(\mathbb{C}^{**}) = M_I(\mathbb{C}) \). Using [18, Proposition 2.7] we know that \( M_I(\mathbb{C}) \) is biflat with an identity. So Lemma 2.3 implies that \( M_I(\mathbb{C}) \) is strong pseudo-amenable. \( \square \)

We can use the similar arguments as in the previous theorem and shows the following result.

**Theorem 3.3.** Let \( I \) be a non-empty set. Then \( M_I(\mathbb{C}) \) is strong pseudo-amenable if and only if \( I \) is finite.

**Remark 3.4.** We give a pseudo-amenable Banach algebra which is not strong pseudo-amenable.

Let \( I \) be an infinite set. Using [18, Proposition 2.7], \( M_I(\mathbb{C}) \) is biflat. By [10, Proposition 3.6], \( M_I(\mathbb{C}) \) has an approximate identity. Then [10, Proposition 3.5] implies that \( M_I(\mathbb{C}) \) is pseudo-amenable. But by previous theorem \( M_I(\mathbb{C}) \) is not strong pseudo-amenable.

4. SOME APPLICATIONS FOR BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

In this section we study strong pseudo-amenability of the measure algebras, the group algebras and some semigroup algebras related to locally compact groups.

**Proposition 4.1.** Let \( G \) be a locally compact group. Then \( L^1(G) \) is strong pseudo-amenable if and only if \( G \) is amenable.

**Proof.** Suppose that \( L^1(G) \) is strong pseudo-amenable. Then by Proposition 2.1, \( L^1(G) \) is pseudo-amenable. So by [14, Proposition 4.1], \( G \) is amenable.

For converse, let \( G \) be amenable. By Johnson theorem \( L^1(G) \) is amenable. Therefore there exists \( M \in (L^1(G) \otimes_p L^1(G))^{**} \) such that \( a \cdot M = M \cdot a \) and \( \pi^{**}_{L^1(G)}(M)a = a\pi^{**}_{L^1(G)}(M) = a \) for every \( a \in L^1(G) \). Then \( L^1(G) \) is strong pseudo-amenable. \( \square \)

**Remark 4.2.** In fact in the proof of the previous proposition we showed that, if a Banach algebra \( A \) is amenable, then \( A \) is strong pseudo-amenable.

**Proposition 4.3.** Let \( G \) be a locally compact group. Then \( M(G) \) is strong pseudo-amenable if and only if \( G \) is discrete and amenable.

**Proof.** Suppose that \( M(G) \) is strong pseudo-amenable. Then by Proposition 2.1, \( M(G) \) is pseudo-amenable. So by [14, Proposition 4.2], \( G \) is discrete and amenable.

For converse, let \( G \) discrete and amenable. Then by the main result of [1], \( M(G) \) is amenable. Applying Remark 4.2 implies that \( M(G) \) is strong pseudo-amenable. \( \square \)

**Proposition 4.4.** Let \( G \) be a locally compact group. Then \( L^1(G)^{**} \) is strong pseudo-amenable if and only if \( G \) is finite.

**Proof.** Suppose that \( L^1(G)^{**} \) is strong pseudo-amenable. Then by Proposition 2.1, \( L^1(G)^{**} \) is pseudo-amenable. So by [14, Proposition 4.2], \( G \) is finite.

For converse, let \( G \) finite. Clearly \( L^1(G)^{**} \) is amenable. Applying Remark 4.2 implies that \( M(G) \) is strong pseudo-amenable. \( \square \)
We present some notions of semigroup theory. Our standard reference of semigroup theory is [15]. Let $S$ be a semigroup and let $E(S)$ be the set of its idempotents. There exists a partial order on $E(S)$ which is defined by

$$s \leq t \iff s = st = ts \quad (s, t \in E(S)).$$

A semigroup $S$ is called inverse semigroup, if for every $s \in S$ there exists $s^* \in S$ such that $ss^* = s$ and $s^*ss^* = s$. If $S$ is an inverse semigroup, then there exists a partial order on $S$ which coincides with the partial order on $E(S)$. Indeed

$$s \leq t \iff s = ss^*t \quad (s, t \in S).$$

For every $x \in S$, we denote $[x] = \{y \in S | y \leq x\}$. $S$ is called locally finite (uniformly locally finite) if for each $x \in S$, $|[x]| < \infty$ (sup$\{|[x]| : x \in S\} < \infty$), respectively.

Suppose that $S$ is an inverse semigroup. Then the maximal subgroup of $S$ at $p \in E(S)$ is denoted by $G_p = \{s \in S | ss^* = s^*s = p\}$.

Let $S$ be an inverse semigroup. There exists an equivalence relation $D$ on $S$ such that $sDt$ if and only if there exists $x \in S$ such that $ss^* = xx^*$ and $x^*t = x^*x$. We denote $\{D_\lambda : \lambda \in \Lambda\}$ for the collection of $D$-classes and $E(D_\lambda) = E(S) \cap D_\lambda$.

**Theorem 4.5.** Let $S$ be an inverse semigroup such that $E(S)$ is uniformly locally finite. Then the following are equivalent:

(i) $E(S)$ is strong pseudo-amenable;

(ii) Each maximal subgroup of $S$ is amenable and each $D$-class has finitely many idempotent elements.

**Proof.** Let $E(S)$ be strong pseudo-amenable. Since $S$ is a uniformly locally finite inverse semigroup, using [18] Theorem 2.18 we have

$$E(S) \cong E - \bigoplus\{M_{E(D_\lambda)}(E(G_p))\}.$$ 

Thus $M_{E(D_\lambda)}(E(G_p))$ is a homomorphic image of $E(S)$. Then by Proposition 2.7 strong pseudo-amenability of $E(S)$ implies that $M_{E(D_\lambda)}(E(G_p))$ is strong pseudo-amenable. It is well-known that $E(G_p)$ has an identity (then has an idempotent element). Hence by Lemma 2.9, $M_{E(D_\lambda)}$ is strong pseudo-amenable. Now by Theorem 4.3 $E(D_\lambda)$ is finite. Also since $M_{E(D_\lambda)}$ has an idempotent, again by Lemma 2.4 $E(G_p)$ is strong pseudo-amenable. Applying Proposition 4.4 $G_p$ is amenable.

For converse, suppose that $E(D_\lambda)$ is finite and $G_p$ is amenable for every $\lambda$. Johnson theorem implies that $E(G_p)$ is 1-amenable (so it is 1-biflat). By [18] Proposition 2.7 it follows that $M_{E(D_\lambda)}(E(G_p))$ is 1-biflat. So [18] Proposition 2.5] gives that $M_{E(D_\lambda)}(E(G_p))$ is 1-biflat. Using

$$E(S) \cong E - \bigoplus\{M_{E(D_\lambda)}(E(G_p))\},$$

and [18] Proposition 2.3], we have $E(S)$ is 1-biflat. The finiteness of $E(D_\lambda)$ deduces that $E(G_p)$ has an identity. Therefore It is easy to see that $E(S)$ has a central approximate identity. Now by Lemma 2.5 biflatness of $E(S)$ gives that $E(S)$ is strong pseudo-amenable.

We recall that a Banach algebra $A$ is approximately amenable, if for each Banach $A$-bimodule $X$ and each bounded derivation $D : A \to X^*$ there exists a net $(x_\alpha)$ in $X^*$ such that

$$D(a) = \lim \alpha a \cdot x_\alpha - x_\alpha \cdot a \quad (a \in A),$$
for more details see [12] and [13].

For a locally compact group $G$ and a non-empty set $I$, set

$$M^0(G, I) = \{(g)_{i,j} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{i,j}$ denotes the $I \times I$ matrix with $g$ in $(i,j)$-position and zero elsewhere. With the following multiplication $M^0(G, I)$ becomes a semigroup

$$(g)_{i,j} \ast (h)_{k,l} = \begin{cases} (gh)_{il} & j = k \\ 0 & j \neq k, \end{cases}$$

It is well known that $M^0(G, I)$ is an inverse semigroup with $(g)^{-1}_{i,j} = (g^{-1})_{j,i}$. This semigroup is called Brandt semigroup over $G$ with index set $I$, which by the arguments as in [10, Corollary 3.8], $M^0(G, I)$ becomes a uniformly locally finite inverse semigroup.

**Theorem 4.6.** Let $S = M^0(G, I)$ be a Brandt semigroup. Then the following are equivalent:

(i) $\ell^1(S)$ is strong pseudo-amenable;

(ii) $G$ is amenable and $I$ is finite;

(iii) $\ell^1(S)$ is approximately amenable.

**Proof.** (i)⇒(ii) Using [7] Remark, p 315, we know that $\ell^1(S)$ is isometrically isomorphic with $[M_I(\mathbb{C}) \otimes_p \ell^1(G)] \oplus_1 \mathbb{C}$. Applying Proposition 2.7, $M_I(\mathbb{C}) \otimes_p \ell^1(G)$ is strong pseudo-amenable. Since $\ell^1(G)$ has an identity, by Lemma 2.9, $M_I(\mathbb{C})$ is strong pseudo-amenable. Hence by Theorem 3.3, $I$ must be finite. On the other hand the finiteness of $I$ implies that $M_I(\mathbb{C})$ has a unit. So Lemma 2.9 implies that $\ell^1(G)$ is strong amenable. Now by Proposition 4.1, $G$ is amenable.

(ii)⇔(iii) By the main result of [20], it is clear. \qed

**Remark 4.7.** There exists a pseudo-amenable semigroup algebra which is not strong pseudo-amenable.

To see this, let $G$ be an amenable locally compact group. Suppose that $I$ is an infinite set. By [10 Corollary 3.8] $\ell^1(S)$ is pseudo-amenable but using previous theorem $\ell^1(S)$ is not strong pseudo-amenable, whenever $S = M^0(G, I)$ is a Brandt semigroup.

Also there exists a strong pseudo-amenable semigroup algebra which is not pseudo-contractible.

To see this, let $G$ be an infinite amenable group. Suppose that $I$ is a finite set. By previous theorem $\ell^1(S)$ is strong pseudo-amenable but [9 Corollary 2.5] implies that $\ell^1(S)$ is not pseudo-contractible, whenever $S = M^0(G, I)$ is a Brandt semigroup.

5. Examples

**Example 5.1.** We present some strong pseudo-amenable Banach algebras which is not amenable.

(i) Suppose that $G$ is the integer Heisenberg group. We know that $G$ is discrete and amenable, see [19]. Therefore by the main result of [11] the Fourier algebra $A(G)$ is not amenable. But by the Leptin theorem (see [19]), the amenability of $G$ implies that $A(G)$ has a bounded approximate identity. Since $A(G)$ is a commutative Banach algebra, $A(G)$ has a central approximate identity.
Hence [22] Theorem 4.2] follows that \( A(G) \) is pseudo-contractible. Now by Lemma 2.3 \( A(G) \) is strong pseudo-amenable.

(ii) Let \( S = \mathbb{N} \). Equip \( S \) with \( \max \) as its product. Then the semigroup algebra \( \ell^1(S) \) is not amenable. To see this on contrary suppose that \( \ell^1(S) \) is amenable. Then by [7], \( E(S) \) must be finite which is impossible. We claim that \( \ell^1(S) \) is strong pseudo-amenable. By [5, p. 113] \( \ell^1(S) \) is approximate amenable. Since \( \ell^1(S) \) has an identity, \( \ell^1(S) \) is pseudo-amenable. So commutativity of \( \ell^1(S) \) follows that \( \ell^1(S) \) is strong pseudo-amenable.

(iii) Let \( S = \mathbb{N} \cup \{0\} \). With the following action

\[
m \ast n = \begin{cases} m & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}
\]

\( S \) becomes a semigroup. Clearly \( S \) is commutative and \( E(S) = \mathbb{N} \cup \{0\} \). So by [7], \( \ell^1(S) \) is not amenable. On the other hand since \( S \) is a uniformly locally finite semilattice, [3, Corollary 2.7] implies that \( \ell^1(S) \) is pseudo-amenable. Thus by Lemma 2.3 \( \ell^1(S) \) is strong pseudo-amenable.

Example 5.2. We give a strong pseudo-amenable Banach algebra which is not approximately amenable

A Banach algebra \( A \) is approximately biprojective if there exists a (not necessarily bounded ) net \( (\rho_a) \) of bounded linear \( A \)-bimodule morphisms from \( A \) into \( A \otimes_p A \) such that \( \pi_A \circ \rho_a(a) - a \to 0 \), for every \( a \in A \), see [25]. Suppose that \( A = \ell^2(\mathbb{N}) \). With the pointwise multiplication, \( A \) becomes a Banach algebra. By the main result of [6], \( A \) is not approximately amenable. But by [25, Example p-3239], \( A \) is an approximately biprojective Banach algebra with a central approximate identity. Then by [14, Proposition 3.8], \( A \) is pseudo-contractible. It follows that \( A \) is strong pseudo-amenable.

Example 5.3. We give a biflat semigroup algebra which is not strong pseudo-amenable. So we can not remove the hypothesis " the existence of central approximate identity" from Lemma 2.3

Let \( S \) be the right-zero semigroup with \( |S| > 1 \), that is, \( st = t \) for every \( s, t \in S \). We denote \( \phi_S \) for the augmentation character on \( \ell^1(S) \). It is easy to show that \( fg = \phi_S(fg) \). Pick \( f_0 \in \ell^1(S) \) such that \( \phi_S(f_0) = 1 \). Define \( \rho : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S) \) by \( \rho(f) = f_0 \otimes f \). It is easy to see that \( \pi_{\ell^1(S)} \circ \rho(f) = f \) and \( \rho \) is a bounded \( \ell^1(S) \)-bimodule morphism. So \( \ell^1(S) \) is biflat. Suppose conversely that \( \ell^1(S) \) is strong pseudo-amenable. So by Proposition 2.1 \( \ell^1(S) \) is pseudo-amenable. Hence \( \ell^1(S) \) has an approximate identity, say \( (e_n) \). It leads that

\[
f_0 = \lim f_0 e_n = \lim \phi_S(f_0)e_n = \lim e_n.
\]

Suppose that \( s_1, s_2 \) be two arbitrary elements in \( S \). Thus \( \delta_{s_1} = \lim \delta_{s_1} e_n = \delta_{s_1} f_0 = f_0 \) and \( \delta_{s_2} = \lim \delta_{s_2} e_n = \delta_{s_2} f_0 = f_0 \) which implies that \( \delta_{s_1} = \delta_{s_2} \). Then \( s_1 = s_2 \). Therefore \( |S| = 1 \), which is a contradiction.

Remark 5.4. A Banach algebra \( A \) is called Johnson pseudo-contractible, if there exists a net \( (m_\alpha) \) in \( (A \otimes_p A)^{**} \) such that

\[
a \cdot m_\alpha = m_\alpha \cdot a, \quad \pi^*_A(m_\alpha)a \to a, \quad (a \in A).
\]
For more information about Johnson pseudo-contractibility, see [21]. By Example 5.1[ii] we know that $\ell^1(\mathbb{N}_{\text{max}})$ is strong pseudo-amenable. But by [2] Example 2.6 $\ell^1(\mathbb{N}_{\text{max}})$ is not Johnson pseudo-contractible.

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