A CHARACTER FORMULA FOR REPRESENTATIONS OF LOOP GROUPS BASED ON NON-SIMPLY CONNECTED LIE GROUPS

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1. Introduction

In this paper we compute characters of certain irreducible representations of loop groups based on non simply connected Lie groups. Apart from a general representation theoretic interest, our motivation to study these characters comes from the fact that they appear naturally in the theory of moduli spaces of semistable principal bundles over elliptic curves.

The characters of highest weight representations of loop groups based on simply connected Lie groups are well understood due to the Kac-Weyl character formula: Let $G$ be a simply connected complex Lie group whose Lie algebra is simple. We denote by $L(G)$ the group of holomorphic maps from $\mathbb{C}^*$ to $G$. This group possesses a universal central extension $\hat{L}(G)$ which, viewed as a manifold, is a non-trivial $\mathbb{C}^*$-bundle over $L(G)$. The natural multiplicative action of $\mathbb{C}^*$ on $L(G)$ lifts uniquely to an action of $\mathbb{C}^*$ on $\hat{L}(G)$ by group automorphisms. Let $V$ be an irreducible highest weight representation of $\hat{L}(G)$. Such a representation extends to a representation of the semidirect product $\hat{L}(G) \rtimes \mathbb{C}^*$. One can show that for any $q$ with $|q| < 1$, the element $(g, q) \in \hat{L}(G) \rtimes \mathbb{C}^*$ viewed as an operator on $V$ extends to a trace class operator on the Hilbert space completion of $V$. So one can define the character $\chi_V$ of $V$ at a point $(g, q)$ with $|q| < 1$ as the trace of the operator $(g, q)$. This defines a holomorphic and conjugacy-invariant function on the space $\hat{L}(G) \times D^*$, where $D^*$ denotes the punctured unit disk in $\mathbb{C}$. The Kac-Weyl character formula gives an explicit formula for the character $\chi_V$ restricted to a certain family of tori in $\hat{L}(G) \times D^*$ in terms of theta functions. This is enough to describe the character completely since almost every conjugacy class in $\hat{L}(G) \times D^*$ intersects this family of tori.

If the group $G$ is not simply connected, the loop group $L(G)$ consists of several connected components which are labeled by the fundamental group of $G$. In this case, central extensions of $L(G)$ have been constructed in [1]. We will review this construction in section 2. Let $\hat{L}(G)$ denote such a central extension. The natural action of $\mathbb{C}^*$ on $L(G)$ does not lift to $\hat{L}(G)$. Instead, a finite covering $\hat{C}^*$ of
\(\mathbb{C}^*\) acts on \(\hat{L}(G)\) covering the natural \(\mathbb{C}^*\)-action on \(L(G)\). So similar to the simply connected case, we can consider the semidirect product \(\hat{L}(G) \rtimes \mathbb{C}^*\). We are interested in representations of \(\hat{L}(G) \rtimes \mathbb{C}^*\) which, restricted to the connected component of \(\hat{L}(G) \rtimes \mathbb{C}^*\) containing the identity, decompose into a direct sum of irreducible highest weight representations. These representations have been classified in [1]. Let \(V\) be such a representation. For \(\tilde{q} \in \mathbb{C}^*\), let \(q\) denote the image under the natural projection \(\mathbb{C}^* \to \mathbb{C}^*\). As in the simply connected case, one shows that any \((g, \tilde{q}) \in \hat{L}(G) \rtimes \mathbb{C}^*\) with \(|q| < 1\) extends to a trace class operator on the Hilbert space completion of \(V\). Thus, one can define the character \(\chi_{V}\) of the representation \(V\) exactly as in the simply connected case.

The main goal of this paper is to give an explicit formula for the character \(\chi_{V}\) restricted to the connected components of \(\hat{L}(G) \rtimes \mathbb{C}^*\) which do not contain the identity element. This gives a generalization of the Kac-Weyl character formula. While the usual approach to the Kac-Weyl formula is rather algebraic, we work in a completely geometric setting. In particular, we identify the characters with sections in certain line bundles over an Abelian variety. To do this, we have to realize the set of semisimple conjugacy classes in a connected component of \(\hat{L}(G) \rtimes \mathbb{C}^*\) as the total space of a line bundle over a family of Abelian varieties over \(D^*\). Then we show that the characters have to satisfy a certain differential equation.

In the simply connected case, the differential equation has been derived in [EK]. Our main step is a generalization of this equation to the non-simply connected case. Finally, we use the differential equation to obtain an explicit formula for the character \(\chi_{V}\). In the simply connected case, this gives an easy proof of the Kac-Weyl character formula which is similar to Weyl’s original proof of his character formula for compact Lie groups. In the non-simply connected case, we obtain a formula for the characters which very much resembles the Kac-Weyl character formula (Theorem 5.5). The main difference is that the character restricted to a connected component of \(\hat{L}(G) \rtimes \mathbb{C}^*\) not containing the identity is not governed by the root system \(\Delta\) of the Lie algebra of \(\hat{L}(G) \rtimes \mathbb{C}^*\) but by a new root system \(\tilde{\Delta}_c\), which can be obtained from \(\Delta\) by a “folding” process. It is interesting to note that the Lie algebra corresponding to root system \(\tilde{\Delta}_c\) can, in general, not be realized as a subalgebra of the affine Lie algebra corresponding to \(\tilde{\Delta}\). In this way, the situation resembles the case of characters of irreducible representations of non-connected compact Lie groups [W]. Also, Fuchs et al. [FRS, FSS] have obtained similar results calculating the characters of representations of Kac-Moody algebras twisted by outer automorphisms. These so called ”twining characters” appear in a conjecture concerning Verlinde formulas for non-simply connected Lie groups [FS].

Our main motivation for the study of characters of irreducible representations of loop groups based on non simply connected Lie groups comes from the theory of moduli spaces of semistable \(G\)-bundles on elliptic curves. For a given group \(G\) and an elliptic curve \(E_q = \mathbb{C}^*/q\mathbb{Z}\) with \(q \in D^*\), the moduli space of semistable \(G\)-bundles over \(E_q\) consists of several connected components which are labeled by the elements of the fundamental group of \(G\). The knowledge of the characters of \(\hat{L}(G) \rtimes \mathbb{C}^*\) allows to construct an analogue of a Steinberg cross section in each connected component of \(\hat{L}(G) \rtimes \{q\}\) for any \(\tilde{q} \in \mathbb{C}^*\) such that \(q \in D^*\) (see e.g. [B]) for the construction of a Steinberg cross section in loop groups based on simply connected Lie groups and [M] for the case of non-connected semisimple algebraic groups). It turns out that there is a natural action of \(\mathbb{C}^*\) on this cross section, and that the space of
orbits of this action is isomorphic to the connected component of the moduli space of semistable $G$–bundles on $E_q$ which corresponds to a connected component of $\tilde{L}(G) \times \{\tilde{q}\}$. On the other hand, by construction, the cross section can be identified with an affine space $\mathbb{C}^*$ and the $\mathbb{C}^*$–action becomes linear in this identification. So the approach outlined above gives a new proof of a result of Friedman and Morgan \cite{FM2} which states that each component of the moduli space is isomorphic to a weighted projective space. These ideas will be published elsewhere.

The organization of this paper is as follows. In section \ref{2} we review the construction of central extensions $\tilde{L}(G)$ of loop groups $L(G)$ based on non simply connected Lie groups $G$ and describe their representation theory. In section \ref{3} we study certain conjugacy classes in these groups. In section \ref{4} we identify the characters of the representations of $\tilde{L}(G)$ introduced in section \ref{2} with sections of a line bundle over a family of Abelian varieties and deduce a differential equation for these sections. Finally, in section \ref{5} we put everything together and deduce an explicit formula for the characters. In particular, in \ref{5} we describe how the "folded" root system $\tilde{\Delta}_{σ}$ appears for these characters. In the appendix we list the root systems $\tilde{\Delta}_{σ}$, together with some other data corresponding to non simply connected Lie groups.

2. Affine Lie groups and algebras

2.1. Affine Lie algebras. We begin by recalling some facts from the theory of affine Lie algebras. Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. We denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by $\Delta$ and let $\mathfrak{h}^\ast \subset \mathfrak{g}$ be the real vector space spanned by the co-roots of $\mathfrak{g}$.

The loop algebra $L(\mathfrak{g})$ of $\mathfrak{g}$ is the Lie algebra of holomorphic maps from $\mathbb{C}^*$ to $\mathfrak{g}$. The (untwisted) affine Lie algebra corresponding to $\mathfrak{g}$ is a certain extension of $L(\mathfrak{g})$. Let us fix some $k \in \mathbb{C}$ and consider the Lie algebra $\tilde{L}^k(\mathfrak{g}) = L\mathfrak{g} \oplus \mathbb{C}C \oplus \mathbb{C}D$, where the Lie bracket on $\tilde{L}^k(\mathfrak{g})$ is given by

\[ [C, x(z)] = [C, D] = 0, \quad [D, x(z)] = z \frac{d}{dz} x(z), \]

and

\[ [x(z), y(z)] = [x, y](z) + \frac{k}{2\pi i} \int_{|z| = 1} \langle \frac{d}{dz} x(z), y(z) \rangle dz \cdot C \]

Here $[x, y](z)$ denotes the pointwise commutator of $x$ and $y$, and $\langle \cdot, \cdot \rangle$ is the normalized invariant bilinear form on $\mathfrak{g}$ (i.e. the Killing form on $\mathfrak{g}$ normalized in such a way that $\langle \alpha, \alpha \rangle = 2$ for the long roots $\alpha$ of $\mathfrak{g}$). Note that the Lie algebras $\tilde{L}^k(\mathfrak{g})$ are isomorphic for all $k \neq 0$. However, for different $k$ they define non-equivalent central extensions of $L(\mathfrak{g}) \oplus \mathbb{C}D$. For $k = 1$, we usually omit the $k$ and denote the corresponding Lie algebra simply by $\tilde{L}(\mathfrak{g})$.

If $\mathfrak{g}$ is simple, the subalgebra $\tilde{L}(\mathfrak{g})_{\text{pol}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}C \oplus \mathbb{C}D \subset \tilde{L}(\mathfrak{g})$ of polynomial loops is an untwisted affine Lie algebra in the sense of \cite{K}, and $\tilde{L}(\mathfrak{g})_{\text{pol}}$ can be viewed as a certain completion of it (see \cite{GW}). The Lie algebra $\tilde{L}(\mathfrak{g})_{\text{pol}}$ has a root space decomposition in the following sense: Set $\bar{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}C \oplus \mathbb{C}D$ and choose an element $\delta \in (\mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}C \oplus \mathbb{R}D)^\ast$ dual to $D$. Then the root system $\bar{\Delta}$ of $\tilde{L}(\mathfrak{g})_{\text{pol}}$ is given by

\[ \bar{\Delta} = \{ \alpha + n\delta \mid \alpha \in \Delta, \ n \in \mathbb{Z} \} \cup \{ n\delta \mid n \in \mathbb{Z} \setminus \{0\} \} \]
and we can write
\[ \tilde{L}(g)_{\text{pol}} = \tilde{h} \oplus \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} \tilde{L}(g)_{\tilde{\alpha}} \]
with \( \tilde{L}(g)_{\tilde{\alpha}} = g_{\tilde{\alpha}} \otimes \mathbb{Z}^n \) if \( \tilde{\alpha} = \alpha + n\delta \), and \( g_{\tilde{\alpha}} = \mathfrak{h} \otimes \mathbb{Z}^n \) if \( \alpha = n\delta \).

The set \( \tilde{\Delta} \) is an affine root system. Let \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) be a basis of \( \Delta \) and let \( \theta \) denote the highest root of \( \Delta \) with respect to this basis. Then we can define the set \( \tilde{\Delta} = \{\alpha_0 = \delta - \theta, \alpha_1, \ldots, \alpha_l\} \), which is a basis of \( \tilde{\Delta} \). The Dynkin diagram of \( \tilde{\Delta} \) is defined in the usual sense and it turns out that it is the extended Dynkin diagram corresponding to \( \Delta \) (see [K]). The affine root system \( \tilde{\Delta} \) decomposes into
\[ \tilde{\Delta} = \tilde{\Delta}_+ \cup \tilde{\Delta}_- , \]
where the set \( \tilde{\Delta}_+ \) of positive roots is given by
\[ \tilde{\Delta}_+ = \Delta_+ \cup \{\alpha + n\delta \mid \alpha \in \Delta \cup \{0\}, \ n > 0\} , \]
and \( \tilde{\Delta}_- = -\tilde{\Delta}_+ \).

By definition, the set of real roots of \( \tilde{L}(g) \) is the set
\[ \tilde{\Delta}^{re} = \{\alpha + n\delta \mid \alpha \in \Delta\} \subset \tilde{\Delta} , \]
and the set of positive real roots is given by \( \tilde{\Delta}^{re}_+ = \tilde{\Delta}^{re} \cap \tilde{\Delta}_+ \).

Sometimes we will need to consider twisted affine Lie algebras. If the finite dimensional Lie algebra \( g \) admits an outer automorphism \( \sigma \) of finite order \( \text{ord}(\sigma) = r \), one can define the twisted loop algebra
\[ L(g, \sigma) = \{X \in L(g) \mid \sigma(X(z)) = X(e^{i\pi/r}z)\} . \]
The corresponding affine Lie algebra \( \tilde{L}(g, \sigma) \) is constructed in a similar manner as the untwisted algebra. It has a root space decomposition which is only slightly more complicated than in the untwisted case (see e.g. [K]).

2.2. Loop groups and affine Lie groups. Let \( G \) be a complex simply connected semisimple Lie group with Lie algebra \( g \) and suppose that \( g \) is simple. The loop group \( L(G) \) of \( G \) is the group of holomorphic maps from \( \mathbb{C}^* \) to \( G \) with pointwise multiplication. This is a Lie group with Lie algebra \( L(g) \). Let \( \tilde{L}(G) \) denote the universal central extension of \( L(G) \). The central extension \( \tilde{L}(G) \) can be defined via the embedding of \( L(G) \) into the “differentiable loop” group studied by Pressley and Segal [PS]. Topologically, \( \tilde{L}(G) \) is a non-trivial holomorphic principal \( \mathbb{C}^* \)-bundle over \( L(G) \). In fact, there exists a central extension \( \tilde{L}^k(G) \) of \( L(G) \) for each \( k \in \mathbb{N} \). The group \( \tilde{L}^k(G) \) is called the level \( k \) central extension of \( L(G) \). Its Lie algebra is given by \( \tilde{L}^k(g) = \tilde{L}^k(g)/CD \). The universal central extension is just the level 1 extension of \( L(G) \). The group \( \mathbb{C}^* \) acts naturally on \( L(G) \) by \( (q \circ g)(z) = g(q^{-1}z) \) and we can consider the semidirect product \( L(G) \ltimes \mathbb{C}^* \). There is a \( \mathbb{C}^* \)-action on \( \tilde{L}^k(G) \) which covers the \( \mathbb{C}^* \)-action on \( L(G) \), and we denote the semidirect product \( \tilde{L}^k(G) \ltimes \mathbb{C}^* \) by \( \tilde{L}^k(G) \). Its Lie algebra is the affine Lie algebra \( \tilde{L}^k(g) \) described in the last section.

Now assume that \( G \) is of the form \( G = \tilde{G}/Z \), where \( \tilde{G} \) is simply connected and simple, and \( Z \subset \tilde{G} \) is a subgroup of the center of \( G \). Since the group \( Z \) may be identified with the fundamental group of \( G \), the loop group \( L(G) \) consists of \( |Z| \) connected components. In particular, the connected component of \( L(G) \) containing the identity element is isomorphic to \( L(\tilde{G})/Z \). We shall now indicate, following
Toledano Laredo, how to construct certain central extensions of \( L(G) \). We will first consider the group \( L_Z(\tilde{G}) \) of holomorphic maps \( g : \mathbb{C} \to \tilde{G} \) such that \( g(t)g(t + 1)^{-1} \in Z \). Identifying the variable \( z \) with \( e^{2\pi \text{i}t} \), we see that the group \( L(G) \) is isomorphic to \( L_Z(\tilde{G})/Z \). Furthermore, the connected component of \( L_Z(\tilde{G}) \) containing the identity element is isomorphic to \( L(\tilde{G}) \).

The goal is to construct all central extensions of \( L_Z(\tilde{G}) \) and then see which of these extensions are pullbacks of central extensions of \( L(G) \). To this end, let \( \tilde{T} \subset \tilde{G} \) and \( T = \tilde{T}/Z \subset G \) denote maximal tori of \( \tilde{G} \) and \( G \), and let \( \Lambda(\tilde{T}) = \text{Hom}_{\text{alg grp}}(\mathbb{C}^*, \tilde{T}) \) and \( \Lambda(T) = \text{Hom}_{\text{alg grp}}(\mathbb{C}^*, T) \) denote the respective co-character lattices. Then \( \Lambda(T)/\Lambda(\tilde{T}) \cong Z \). The lattice \( \Lambda(T) \) can be identified with a subgroup of \( L_Z(\tilde{G}) \) by viewing it as a lattice in \( h_\mathfrak{g} \subset \mathfrak{h} \) and identifying an element \( \beta \in \Lambda(T) \) with the "open loop" \( t \mapsto \exp(2\pi \text{i}t\beta) \). We can define a subgroup \( N \subset L(\tilde{G}) \rtimes \Lambda(T) \) via

\[
N = \{ (\lambda, \lambda^{-1}) \mid \lambda \in \Lambda(\tilde{T}) \}.
\]

Then we have

\[
L_Z(\tilde{G}) \cong \left( L(\tilde{G}) \rtimes \Lambda(T) \right)/N.
\]

Choose a central extension \( \hat{\Lambda}(T) \) of the lattice \( \Lambda(T) \) by \( \mathbb{C}^* \). Any such central extension is uniquely determined by a skew-symmetric \( \mathbb{Z} \)-bilinear form (the commutator map) \( \omega \) on \( \Lambda(T) \) which is defined by

\[
\omega(\lambda, \mu) = \hat{\lambda}\hat{\mu}\lambda^{-1}\mu^{-1}.
\]

Here, \( \hat{\lambda} \) and \( \hat{\mu} \) are arbitrary lifts of \( \lambda, \mu \in \Lambda(T) \) to \( \hat{\Lambda}(T) \). Let \( \hat{L}^k(\tilde{G}) \) be the central extension of \( L(\tilde{G}) \) of level \( k \). Suppose that \( \hat{\Lambda}(T) \) is a central extension of \( \Lambda(T) \) such that its commutator map satisfies

\[
\omega(\lambda, \mu) = (-1)^{k(\lambda, \mu)} \text{ for all } \lambda \in \Lambda(\tilde{T}) \text{ and } \mu \in \Lambda(T).
\]

Then one can construct a central extension of \( L_Z(\tilde{G}) \) as follows: The group \( \Lambda(T) \subset L_Z(\tilde{G}) \) acts on \( L(\tilde{G}) \) by conjugation. This action uniquely lifts to an action of \( \Lambda(T) \) on the central extension \( \hat{L}^k(\tilde{G}) \). We can consider the semidirect product \( \hat{L}^k(\tilde{G}) \rtimes \hat{\Lambda}(T) \), where the action of \( \hat{\Lambda}(T) \) on \( \hat{L}^k(\tilde{G}) \) factors through the action of \( \Lambda(T) \). Now, the lattice \( \Lambda(\tilde{T}) \) is a subgroup of \( L(\tilde{G}) \) so that the restriction of the central extension of \( L(\tilde{G}) \) to this lattice yields a central extension \( \hat{\Lambda}(\tilde{T}) \) of \( \Lambda(\tilde{T}) \).

On the other hand, we can restrict the central extension \( \hat{\Lambda}(T) \) of \( \Lambda(T) \) to the sublattice \( \Lambda(\tilde{T}) \). The compatibility condition of equation (1) implies in particular that \( \omega(\lambda, \mu) = (-1)^{k(\lambda, \mu)} \) for all \( \lambda, \mu \in \Lambda(\tilde{T}) \). This implies that the two extensions \( \hat{\Lambda}(T) \) are equivalent ([PS], Proposition 4.8.1). We may therefore consider the subgroup

\[
\tilde{N} = \{ (\hat{\lambda}, \lambda^{-1}) \mid \hat{\lambda} \in \hat{\Lambda}(\tilde{T}) \} \subset \hat{L}^k(\tilde{G}) \rtimes \hat{\Lambda}(T).
\]

Now, using the full compatibility condition ([1], Proposition 3.3.1) that \( \tilde{N} \) is a normal subgroup in \( \hat{L}^k(\tilde{G}) \rtimes \hat{\Lambda}(T) \). Therefore, the quotient

\[
\hat{L}^k_Z(\tilde{G}) = \left( \hat{L}^k(\tilde{G}) \rtimes \hat{\Lambda}(T) \right)/\tilde{N}
\]

is a central extension of \( L_Z(\tilde{G}) \).

We then have the following theorem ([1], Theorem 3.2.1 and Proposition 3.3.1).
Theorem 2.1. Every central extension of \( L_Z(\tilde{G}) \) is uniquely determined by the level \( k \) of the corresponding central extension of \( L(\tilde{G}) \) and by a commutator map \( \omega \) defining a central extension of \( \Lambda(T) \) which satisfies the compatibility condition of equation (1). The corresponding central extension of \( L_Z(\tilde{G}) \) is the one described in equation (2).

Remark 2.2. Note that Theorem 2.1 restricts the possible levels at which central extensions of \( L_Z(\tilde{G}) \) can exist. For example, \( L_{(-id)}(SL(2, \mathbb{C})) \) does not possess any central extensions of odd level. Indeed, we have \( \Lambda(\tilde{T}) = \alpha\mathbb{Z} \) and \( \Lambda(T) = \frac{\pi}{2}\mathbb{Z} \) with \( \langle \alpha, \alpha \rangle = 2 \). Now, for odd level, the compatibility requirement of equation (1) requires \( \omega(\alpha, \frac{\pi}{2}) = -1 \) which is in contradiction to bilinearity and skew-symmetry of \( \omega \).

Definition 2.3. Let \( k_f \) be the smallest level at which a central extension on \( L_Z(\tilde{G}) \) exists. This \( k_f \) is called the fundamental level of \( G \). Let \( k_b \) be the smallest positive integer such that the restriction of \( k_b \langle \cdot, \cdot \rangle \) to \( \Lambda(T) \) is integral. \( k_b \) is called the fundamental level of \( G \).

Obviously, for the fundamental level one has \( k_f \in \{1, 2\} \). One can show (11) that the basic level of \( G \) is always a multiple of the fundamental level of \( G \). The fundamental and basic levels of the simple Lie groups are computed in [T]. We list the basic levels in the appendix of this paper. Finally, we have (11, Proposition 3.5.1):

Proposition 2.4. A central extension of \( L_Z(\tilde{G}) \) is a pull-back of a central extension of \( L(G) \) only if its level \( k \) is a multiple of the basic level \( k_b \) of \( G \). Conversely, if \( k_b | k \), the subgroup \( Z \subset \tilde{L}_Z^k(\tilde{G}) \) corresponding to the canonical embedding \( \tilde{G} \hookrightarrow \tilde{L}_Z^k(\tilde{G}) \) is central and we have

\[
\tilde{L}_Z^k(\tilde{G}) \cong \pi^*(\tilde{L}_Z^k(\tilde{G})/Z).
\]

Let us fix a commutator map \( \omega \) satisfying the compatibility requirement from equation (1) for the rest of this paper.

From now on let us assume that \( Z = \langle c \rangle \) is a cyclic group. The group \( \mathbb{C} \) acts naturally on \( L_Z(\tilde{G}) \) by translations. This action factors through an action of \( \mathbb{C}/\text{ord}(c)\mathbb{Z} \). We view \( \mathbb{C}/\text{ord}(c)\mathbb{Z} \) as an \( \text{ord}(c) \)-fold covering of \( \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \), which we denote by \( \tilde{\mathbb{C}}^* \). Thus, we can define the semidirect product \( L_Z(\tilde{G}) \times \mathbb{C}^* \). The action of \( \mathbb{C}^* \) on \( L_Z(\tilde{G}) \) described above lifts to any central extension of \( \tilde{L}_Z^k(\tilde{G}) \) of \( L_Z(\tilde{G}) \) (11, Proposition 3.4.1). So we can define the group \( \tilde{L}_Z^k(\tilde{G}) = \tilde{L}_Z^k(\tilde{G}) \times \tilde{\mathbb{C}}^* \). Furthermore, if the level \( k \) of the central extension \( \tilde{L}_Z^k(\tilde{G}) \) is a multiple of the basic level \( k_b \) of \( G \), the group \( Z = \langle c \rangle \) is a central subgroup of \( \tilde{L}_Z(\tilde{G}) \) and we can define

\[
\tilde{L}^k(G) = \tilde{L}_Z^k(\tilde{G})/Z.
\]

Finally, note that \( \mathbb{C}^* \) acts naturally on the loop group \( L(G) \). However, contrary to the simply connected case, this action does not necessarily lift to all central extensions of \( L(G) \). In fact, we have (11, 3.5.10)

Proposition 2.5. The rotation action of \( \mathbb{C}^* \) on \( L(G) \) lifts to a central extension of \( L(G) \) of level \( k \) if and only if \( k\langle \lambda, \lambda \rangle \in 2\mathbb{Z} \) for all \( \lambda \in \Lambda(T) \), i.e. if \( \Lambda(T) \) endowed with \( k\langle \cdot, \cdot \rangle \) is an even lattice.
2.3. The adjoint action of $\tilde{L}^k(G)$ on its Lie algebra. Suppose as before that $G = \tilde{G}/\tilde{Z}$ where $\tilde{G}$ is simply connected, and $Z = \langle c \rangle$ is a cyclic subgroup of the center of $\tilde{G}$. Consider the centrally extended loop group $\tilde{L}^k(\tilde{G}) = \tilde{L}^k(\tilde{G}) \rtimes \tilde{C}^*$ introduced in the last section. Since the center of any Lie group acts trivially in the adjoint representation, the adjoint action of $\tilde{L}^k(\tilde{G})$ on its Lie algebra $\tilde{L}^k(\mathfrak{g})$ factors through $L_Z(\tilde{G}) \rtimes \tilde{C}^*$. The $\tilde{C}^*$-part acts by translations, so the only interesting part is the action of $L_Z(\tilde{G})$. Let $\zeta$ be an element of $L_Z(\tilde{G})$. Then the adjoint action of $\zeta$ on $\tilde{L}^k(\mathfrak{g})$ is given by (\cite{[T]} Corollary 3.4.2, \cite{PS})

(3) $\text{Ad}(\zeta) : X + aC + bD \mapsto \zeta X^{-1} - \frac{b}{2\pi i} \zeta^{-1} +$

$$\left(a + \frac{k}{2\pi i} \int_0^1 \langle X(t), \zeta^{-1}(t)\dot{\zeta}(t) \rangle dt \right. - \frac{kb}{8\pi^2 z^2} \int_0^1 \langle \zeta^{-1}(t)\dot{\zeta}(t), \zeta^{-1}(t)\dot{\zeta}(t) \rangle dt \left. \right) C + bD .$$

Here, $X$ is an element of the loop algebra $L(\mathfrak{g})$, and $\dot{\zeta}$ denotes the derivative of $\zeta$ with respect to $t$. Finally, as before, we have identified $\tilde{C}^*$ with $\mathbb{C}/\mathbb{Z}$ by identifying the coordinate $z$ with $e^{2\pi it}$.

We are interested in the action of a specific element of $\sigma_c \in L_Z(\tilde{G})$ which is the product $c\cdot \zeta_c w_c$ of an “open loop” $\zeta_c$ and an element $w_c \in G$ which are defined as follows. As before, let $\theta$ denote the highest root of $\mathfrak{g}$. The set of elements $\alpha_i \in \Pi$ which have coefficient $m_j = 1$ in the expansion $\theta = \sum_{j=1}^l m_j \alpha_j$ can be identified with the non-trivial elements of the center of $\tilde{G}$. Indeed, let $\{\lambda_j\} \subset \mathfrak{h}$ be the dual basis corresponding to $\Pi \subset \mathfrak{h}^*$. Then the condition $m_j = 1$ implies that $\exp(2\pi i \lambda_j)$ is an element of the center of $\tilde{G}$.

Let $\alpha_c$ denote the root $\alpha$ which is identified with the generator $c \in Z$ in this identification and let $\lambda_c \in \mathfrak{h}$ denote the corresponding fundamental weight of $\mathfrak{g}$. There exists a unique element $w_c \in W$ which permutes the set $\Pi \cup \{-\theta\}$ and maps $-\theta$ to $\alpha_c$ (see \cite{[T]}, Proposition 4.1.2). Furthermore, let $\{e_{\alpha} \mid \alpha \in \Delta_+\}$ be a Chevalley basis of the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Then we can choose a representative $\tilde{w}_c$ of $w_c$ in $N(T)$ such that $\tilde{w}_c(e_{\alpha}) = e_{w_c(\alpha)}$ for all $\alpha \in \Pi$. From now on, we will denote both the Weyl group element $w_c$ as well as its representative $\tilde{w}_c \in N_G(T)$ simply by $w_c$. Finally, we can define an “open loop” $\zeta_c$ in $G$ via $\zeta_c(t) = \exp(2\pi it \lambda_c)$. Now, the element $\sigma_c$ is defined as

$$\sigma_c = \zeta_c^{-1} w_c .$$

The action of $\sigma_c$ on $\tilde{L}^k(\mathfrak{g})$ can be described explicitly in terms of the root space decomposition: To each root $\alpha$ of $\mathfrak{g}$ choose a co-root $h_\alpha \in \mathfrak{h}$. Set $h_{\alpha_0} = h_{-\theta} + kC$. Since $\langle \lambda_c, \alpha_c \rangle = 1$ and $\langle \lambda_c, \alpha \rangle = 0$ for all $\alpha \in \Pi$ with $\alpha \neq \alpha_c$, we find that the action of $\sigma_c$ on $\mathfrak{h}$ is given by

$$D \mapsto D + \lambda_c - \frac{k}{2} ||\lambda_c||^2 C ,$$

$$h_{\alpha_0} \mapsto h_{\alpha_0} , \quad h_{w_c^{-1}(\theta)} \mapsto h_{\alpha_0} , \quad \text{and}$$

$$h_{\alpha_i} \mapsto h_{w_c(\alpha_i)} \quad \text{for all other } i .$$

A set of generators for $\tilde{L}(\mathfrak{g})_{red}$ is given by the set $\{e_{\alpha}, f_{\bar{\alpha}} \mid \alpha \in \bar{\Pi}\}$ with $e_{\bar{\alpha}} = e_{\alpha} \otimes 1$ for $1 \leq i \leq l$ and $e_{\bar{\alpha}} = e_{-\theta} \otimes z$, and accordingly $f_{\bar{\alpha}} = f_{\alpha} \otimes 1$ for $1 \leq i \leq l$ and $f_{\bar{\alpha}} = f_{-\theta} \otimes z^{-1}$. It is straightforward to check that $\sigma_c$ permutes the $e_{\bar{\alpha}}$, according to its action on $\bar{\Pi}$ and similarly for the $f_{\bar{\alpha}}$. 
2.4. Integrable representations and characters. As before, let $\theta$ denote the highest root of $g$ and fix some non-negative integer $k \in \mathbb{Z}_{\geq 0}$. Let $P_+$ be the set of dominant weights of $g$ with respect to $\Pi$, and let $P^k_+$ denote the set of $\lambda \in P_+$ such that $\langle \lambda, \theta \rangle \leq k$. To each pair $(\lambda, k)$ with $\lambda \in P^k_+$, we can associate an irreducible highest weight module $V_{\lambda,k}$ of $\widehat{L}(g)_{\text{pol}}$ such that the center of $\hat{g}$ acts as the scalar $k$ (see [K]). Letting $D$ act on the highest weight vector of $V_{\lambda,k}$ as an arbitrary scalar uniquely determines an irreducible highest weight representation of $\widehat{L}(g)_{\text{pol}}$. We will denote by $V_{\lambda,k}$ the highest weight representation of $\widehat{L}(g)_{\text{pol}}$ such that $D$ acts trivially on the highest weight vector.

It was shown by H. Garland [G] that $V_{\lambda,k}$ admits a positive definite Hermitian form $(.,.)$ which is contravariant with respect to the anti-linear Cartan involution on $\widehat{L}(g)_{\text{pol}}$. Let us denote by $V^{ss}_{\lambda,k}$ the $L^2$-completion of $V_{\lambda,k}$ with respect to this norm defined by the Hermitian form. That is, if $\{v_{\lambda,\mu,i}\} \in I(\lambda)$ is an orthonormal basis of the weight subspace $V_{\lambda,k}[\mu]$ of $V_{\lambda,k}$, then

$$V^{ss}_{\lambda,k} = \left\{ \sum_{\mu,i} a_{\mu,i} v_{\lambda,\mu,i} \mid \sum_{\mu,i} |a_{\mu,i}|^2 < \infty \right\}.$$ 

Analogously, we define the analytic completion of $V_{\lambda,k}$ to be the space

$$V^{an}_{\lambda,k} = \left\{ \sum_{\mu,i} a_{\mu,i} v_{\lambda,\mu,i} \mid \text{there exists a } 0 < q < 1 \text{ s.t. } a_{\mu,i} = O \left(q^{-D(\mu)}\right), \mu \to \infty \right\},$$

where $D(\mu)$ denotes the (non-positive) degree of the weight $\mu$ in the homogeneous grading. By definition, $V^{ss}_{\lambda,k}$ is a Hilbert space and $V^{an}_{\lambda,k}$ is a dense subspace in it. It is known ([GW], [EFK]), that the action of $\widehat{L}(g)_{\text{pol}}$ on $V_{\lambda,k}$ extends by continuity to an action of $\widehat{L}(g)$ on $V^{an}_{\lambda,k}$, but not to an action on $V^{ss}_{\lambda,k}$.

We now turn to the representation theory of the affine Lie groups. We first consider the case that the corresponding finite dimensional Lie group is simply connected. In this case, the following result is known (see e.g. [EFK], Theorem 2.2 and Lemma 2.3).

**Theorem 2.6.**

(i) The action of the Lie algebra $\widehat{L}(g)$ on $V^{an}_{\lambda,k}$ uniquely integrates to an action of $\widehat{L}(G)$.

(ii) For any $q \in \mathbb{C}^*$ with $|q| < 1$ and any $g \in \widehat{L}(G)$, the operator $gq^{-D} : V^{an}_{\lambda,k} \to V^{an}_{\lambda,k}$ uniquely extends to a trace class operator on $V^{ss}_{\lambda,k}$.

Let $q \in \mathbb{C}^*$ and denote by $\widehat{L}(G)_q$ the subset of $\widehat{L}(G)$ of elements of the form $(g, q)$ with $g \in \widehat{L}(G)$. This subset is invariant under conjugation in $\widehat{L}(G)$. Furthermore, let us introduce the semigroup $\widehat{L}(G)_{<1} = \bigcup_{|q|<1} \widehat{L}(G)_q$. As a manifold, $\widehat{L}(G)_{<1}$ is isomorphic to $\widehat{L}(G) \times D^*$, where $D^*$ denotes the punctured unit disk in $\mathbb{C}$. Since any element in $\widehat{L}(G)_{<1}$ extends to a trace class operator on $V^{ss}_{\lambda,k}$, we can introduce functions $\chi_{\lambda,k} : \widehat{L}(G)_{<1} \to \mathbb{C}$ via

$$\chi_{\lambda,k}(g, q) = \text{Tr}_{V^{ss}_{\lambda,k}}(gq^{-D}).$$

The function $\chi_{\lambda,k}$ is the character of the module $V_{\lambda,k}$.

**Proposition 2.7** ([EFK], Lemma 2.4, Proposition 2.5). The functions $\chi_{\lambda,k}$ are holomorphic and conjugacy invariant.
By definition, the central element $C$ of $\tilde{L}(g)$ acts on $V_{\lambda,k}^{an}$ by scalar multiplication with $k$. Therefore, if $\iota : \mathbb{C}^* \to \tilde{L}(G)$ denotes the identification of $\mathbb{C}^*$ with the center of $\tilde{L}(G)$, we have $\chi_{\lambda,k}(\iota(u)g) = u^k \chi_{\lambda,k}(g)$ for all $u \in \mathbb{C}^*$.

Now let $G = \tilde{G}/\mathbb{Z}$ be non simply connected and let us fix once and for all a central extension $\tilde{L}_Z(\tilde{G})$ of $L_Z(\tilde{G})$ of level $k_f$. Suppose that $V$ is an irreducible representation of $\tilde{L}_Z(\tilde{G})$ of level $k$. Then $k$ is necessarily a multiple of $k_f$. Restricting this representation to the connected component of $\tilde{L}_Z(\tilde{G})$ containing the identity element yields a level $k$ representation of $\tilde{L}(\tilde{G})$. We shall always assume that we can decompose $V$ as

$$V \cong \bigoplus_{\lambda \in L} V_{\lambda,k}^{an}$$

as a representation of $\tilde{L}(\tilde{G})$. Such representations are called negative energy representations of $\tilde{L}_Z(\tilde{G})$.

Remember the automorphism $\sigma_c$ introduced in the last section. As we have seen, $\sigma_c$ acts as an automorphism of the Dynkin diagram of $\tilde{L}(g)$, so $\sigma_c$ permutes the set $P^k_+$ of level $k$-representations of $\tilde{L}(g)$, pol. The following Theorem is essentially Theorem 6.1 and Corollary 7.3. of [T] (note that [T] deals with projective representations).

**Theorem 2.8.** Let $k$ be a multiple of the fundamental level of $G$. To each $\sigma_c$-orbit $I \subset P_+^k$ there exist $\text{ord}(c)/|I|$ different irreducible negative energy representations of $\tilde{L}_Z(\tilde{G})$ which, restricted to a representation of $\tilde{L}(\tilde{G})$ decompose according to equation (4).

A representation $V_{I,k}$ of $\tilde{L}_Z(\tilde{G})$ factors through a representation of $\tilde{L}_k(G)$ if and only if the level $k$ is a multiple of the basic level $k_b$ of $G$ and $Z$ acts trivially on $V_{I,k}$.

Suppose that the $\sigma_c$-orbit $I$ consists of a single element $\lambda$. Then $\sigma_c$ acts on the highest weight subspace $V_{\lambda}$. Since $V_{\lambda}$ is one-dimensional, $\sigma_c$ acts by a scalar. From now on, we shall assume that $\sigma_c$ acts as the identity on $V_{\lambda}$. This determines the $\tilde{L}_Z(\tilde{G})$–module $V_{I,k}$ corresponding to $I$ and $k$ uniquely.

Finally, letting $D$ act trivially on all highest weight vectors $v_{\lambda,k} \in V_{I,k}$, we can extend the representation $V_{I,k}$ of $\tilde{L}_Z(\tilde{G})$ to a representation of $\tilde{L}_Z(\tilde{G}) = \tilde{L}_Z(\tilde{G}) \times \mathbb{C}^*$ which factors through a representation of $\tilde{L}(G)$ if $Z$ acts trivially on $V_{I,k}$ and $k_b | k$.

(Recall that $\mathbb{C}^*$ denotes the ord($c$)-fold covering of $\mathbb{C}^*$ which acts on $\tilde{L}_Z(\tilde{G})$ covering the natural $\mathbb{C}^*$-action on $L_Z(G)$). For any $\tilde{q} \in \tilde{\mathbb{C}}^*$, let us denote by $q$ its image under the natural projection $\tilde{\mathbb{C}}^* \to \mathbb{C}^*$.

It is easy to see that $\sigma_c$ acts as a unitary operator in $V_{I,k}$. So we have

**Corollary 2.9.** Let $k$ be a multiple of $k_f$ (resp. $k_b$). For any $\tilde{q} \in \tilde{\mathbb{C}}^*$ with $|q| < 1$ and any $g \in \tilde{L}_Z(\tilde{G})$ (resp. $g \in \tilde{L}(G)$), the operator $g\tilde{q}^{-D} : V_{I,k}^{an} \to V_{I,k}^{an}$ uniquely extends to a trace class operator on $V_{I,k}^{ss} = \bigoplus_{\lambda \in I} V_{\lambda,k}^{ss}$.

As before, we can compute the traces of the trace class operators. Abusing terminology slightly, we define the functions

$$\chi_{I,k}(g, \tilde{q}^{-D}) = Tr|_{V_{I,k}^{an}}(g\tilde{q}^{-D})$$
on $\tilde{L}Z(\bar{G})_{<1}$ resp. $\tilde{L}(G)_{<1}$ not distinguishing the different spaces they are defined on. (The semi groups $\tilde{L}Z(\bar{G})_{<1}$ and $\tilde{L}(G)_{<1}$ are defined analogously to the simply connected case). Finally, again using the fact that $\sigma_c$ acts as a unitary operator on $V_{I,k}^{ss}$, we get

**Corollary 2.10.** The functions $\chi_{I,k}$ are holomorphic and conjugacy invariant.

The main goal of this paper is to compute the functions $\chi_{I,k}$ explicitly which will yield a generalization of the Kac-Weyl character formula. This will be done in section 5.3. Here, we state a trivial observation: The character $\chi_{I,k}$ will yield a generalization of the Kac-Weyl character formula. This will be done in section 5.3. Here, we state a trivial observation: The character $\chi_{I,k}$ restricted to the component of $LZ(\bar{G})_{<1}$ containing the element $(e,q)$ is just the sum of the characters $\chi_{I,k} = \sum_{\lambda \in I} \chi_{\lambda,k}$ of the characters $\chi_{\lambda,k}$ of $\tilde{L}(G)$ considered above. On the other hand, $\sigma_c$ permutes the highest weight vectors of the representations $V_{\lambda,k}$ with $\lambda \in I$. So if the $\sigma_c$-orbit $I$ consists of more that one element, the function $\chi_{I,k}$ restricted to the connected component of $LZ(\bar{G})_{<1}$ containing the element $(\sigma_c,q)$ vanishes.

3. Conjugacy classes

3.1. **Conjugacy classes and principal bundles.** Since the characters $\chi_{I,k}$ are conjugacy invariant functions on $\tilde{L}^k(G)_q$, it is necessary to have a good understanding of the conjugacy classes in $\tilde{L}^k(G)_q$ in order to understand the characters. The fundamental result in this direction is an observation due to E. Looijenga which gives a one-to-one correspondence between the $L(G)$-conjugacy classes in $L(G) \times \{q\} \subset L(G) \rtimes \mathbb{C}^*$ and the isomorphism classes of holomorphic principal $G$-bundles over the elliptic curve $E_q = \mathbb{C}^*/q\mathbb{Z}$. To be more precise, let $G = \bar{G}/Z$, as above. Then, up to $C^\infty$-isomorphism, every principal $G$-bundle over $E_q$ is determined by its topological class, which is an element in $\pi_1(G) \cong Z$. We can classify holomorphic principal $G$-bundles of a fixed topological class $c$ as follows. Consider the connected component $L(G)_c$ of the loop group $L(G)$ which corresponds to $c$. The group $L(G) \rtimes \mathbb{C}^*$ acts on the set $L(G)_c \times \{q\} \subset L(G) \rtimes \mathbb{C}^*$ by conjugation. Looijenga’s observation gives a one-to-one correspondence between the set of holomorphic $G$-bundles on $E_q$ of topological type $c$ and the set of $L(G) \times \{1\}$-orbits in $L(G)_c \times \{q\}$. This correspondence comes about as follows. For any element $(g,q) \in L(G)_c \times \{q\}$ consider the $G$-bundle $B_q$ over $E_q$ which is defined as follows. View $E_q$ as the annulus $|q| \leq |z| \leq 1$ in the complex plane with the boundaries identified via $z \mapsto qz$. Then take the trivial bundle over the annulus and define the bundle $B_q$ over $E_q$ by describing the gluing map which identifies the fibers over the points identified under $z \mapsto qz$. This is given by $f(qz) = g(z)f(z)$, where $f$ takes values in $G$. Obviously, this construction gives a holomorphic $G$-bundle of topological type $c$ and the following theorem which is due to E. Looijenga is not hard to prove (see e.g. [FN]).

**Theorem 3.1.**

(i) Two elements $(g_1,q), (g_2,q) \in L(G)_c \times \{q\}$ are conjugate under $L(G) \rtimes \{1\}$ if and only if the corresponding holomorphic $G$-bundles $B_{g_1}$ and $B_{g_2}$ are isomorphic.

(ii) For any holomorphic $G$-bundle over $E_q$ of topological type $c$, there exists an element $g \in L(G)_c$ such that $B \cong B_g$. 
Following [EFN], we call an $L(G) \times \{1\}$-orbit in $L(G)_c \times \{q\}$ semisimple if the corresponding principal $G$-bundle over $E_q$ comes from a representation of the fundamental group of $E_q$ inside a maximal compact subgroup of $G$. We call an element $(g, q) \in L(G) \times \{q\}$ semisimple if the corresponding $L(G) \times \{1\}$-orbit is semisimple. It is known that almost every conjugacy class is semisimple. To be more precise, if $\{B_t\}_{t \in T}$ is a holomorphic family of holomorphic principal bundles on $E_q$ parametrized by a complex space $T$, then the set subset $\tilde{T}_0$ of bundles which are flat and unitary is nonempty and Zariski open in $T$ (see [R]).

3.2. The simply connected case. The set of semisimple $L(G) \times \{1\}$-orbits in $L(G)_c \times \{q\}$ can be described more explicitly. We start with the case that $G$ is simply connected (the non-simply connected case will be considered in section 3.3). The affine Weyl group $\tilde{W} = W \ltimes \Lambda(T)$ can be identified with the group $N_{L(G) \times C^\ast}(T \times C^\ast)/(T \times C^\ast)$ where $N_{L(G) \times C^\ast}(T \times C^\ast)$ denotes the normalizer of $T \times C^\ast$ in $L(G) \times C^\ast$ (see [PS]). In this sense, $\tilde{W}$ is the Weyl group of $\tilde{L}(G)$. It acts on the torus $T \times C^\ast$ via $(w, \beta) : (\xi, q) \mapsto (w\xi q^{-\beta}, q)$. The following proposition follows from the definition of semisimplicity and the classification of stable and unitary $G$-bundles over $E_q$ ([EF], [FHM]).

**Proposition 3.2.** Let $G$ be simply connected. Every semisimple element in $L(G) \times \{q\}$ is $L(G) \times \{1\}$-conjugate to an element of the form $(\xi, q)$ with $\xi \in T$. Two elements $(\xi_1, q)$ and $(\xi_2, q)$ with $\xi_1, \xi_2 \in T$ are conjugate if and only if they are in the same orbit under the action of the affine Weyl group $\tilde{W}$ on $T \times C^\ast$.

**Remark 3.3.** We have $T = (C^\ast)^r$ for some $r \in \mathbb{N}$. The set $T / q^{\Lambda(T)}$ is an Abelian variety isomorphic to the product $E_q \otimes_{\mathbb{Z}} \Lambda(T)$ and there is a natural action of $W$ on $E_q \otimes_{\mathbb{Z}} \Lambda(T)$. In this way, the set of semisimple $L(G) \times \{1\}$-conjugacy classes in $L(G) \times \{q\}$ can be identified with the set $E_q \otimes_{\mathbb{Z}} \Lambda(T) / W$.

Conjugacy classes in the centrally extended group $\bar{L}^k(G)$ can be described as follows. We are interested in the semisimple conjugacy classes in $\bar{L}^k(G)_q$, i.e. conjugacy classes which project to semisimple conjugacy classes under the natural projection $\bar{L}^k(G)_q \to L(G) \times \{q\}$. The set $\bar{L}^k(G)_q$ is the total space of a fiber-bundle over $L(G) \times \{q\}$ with fiber $C^\ast$ and conjugation with an element of $L(G) \times \{1\}$ induces an automorphism of this bundle. So the set of semisimple $L(G) \times \{1\}$-orbits in $\bar{L}^k(G)_q$ will be the total space of a $C^\ast$-bundle over the set of semisimple $L(G) \times \{1\}$-orbits in $L(G)_q$. Using equation (3), one can describe the bundle explicitly.

Let $\tilde{T}_q$ be the set of all elements in $\bar{L}^k(G)_q$ which project to an element of the form $(\xi, q)$ with $\xi \in T$. As a complex manifold this set is isomorphic to $T \times C^\ast \times \{q\}$.

**Proposition 3.4.** The set of semisimple conjugacy classes in $\bar{L}^k(G)_q$ is given by the quotient $\tilde{T}_q / \tilde{W}$, where $(w, \beta) \in \tilde{W} = W \ltimes \Lambda(T)$ acts on $\tilde{T}_q$ as follows:

$$(w, 1)(\xi, u, q) = (w(\xi), u, q),$$

$$(1, \beta)(\xi, u, q) = (\xi q^{-\beta}, u q^{-k/2}(\beta, \beta) \beta(\xi^k), q).$$

Here, $\beta(\cdot)$ denotes the value of $\beta$ as a character of $T$ and $h$ and $h^*$ are identified via the normalized invariant bilinear form on $\mathfrak{g}$.

**Remark 3.5.** The quotient $\tilde{T}_q / \Lambda(T)$ described in Proposition 3.4 is the set of non-zero vectors in a holomorphic line bundle $L^k$ over the Abelian variety $E_q \otimes_{\mathbb{Z}} \Lambda(T)$. 

The action of the finite Weyl group $W$ on this variety induces an action of $W$ on $L^k$.

3.3. The non-simply connected case. If $G$ is not simply connected, the set of semisimple conjugacy classes in the connected component $L(G)_c \times \{q\}$ of $L(G) \times \mathbb{C}^*$ can be described similarly to the simply connected case although the corresponding analysis is slightly more involved. Throughout this section let us assume that $G$ is of the form $G = \tilde{G}/Z$, where $\tilde{G}$ is simply connected and $Z = \langle c \rangle$ is a cyclic subgroup of the center of $\tilde{G}$.

Recall the definition of the element $\sigma_c = \zeta_c^{-1}w_c$ from section 2.3. The element $(\sigma_c, 1) \in L(G) \rtimes \mathbb{C}^*$ acts on the torus $T \times \mathbb{C}^* \subset L(G) \rtimes \mathbb{C}^*$ by conjugation. We denote by $(T \times \mathbb{C}^*)_0^{\sigma_c}$ the connected component of the fixed point set $(T \times \mathbb{C}^*)^{\sigma_c}$ which contains the identity element.

Denote by $L(G)_c \times \{q\}$ the connected component of $L(G) \times \{q\}$ containing the element $(\sigma_c, q)$. The following lemma follows from the definition of semisimplicity and the classification of flat and unitary $G$-bundles of topological type $c$ on the elliptic curve $E_q$ ([S] Equation 2.5, [FM1]).

**Lemma 3.6.** Every semisimple element in $L(G)_c \times \{q\}$ is conjugate under $L(G)$ to an element of the form $(\sigma_c, \xi, q) \in (\sigma_c, 1)(T \times \mathbb{C}^*)_0^{\sigma_c}$.

It remains to check, which elements of $(\sigma_c, 1)(T \times \mathbb{C}^*)_0^{\sigma_c}$ are conjugate in $L(G) \times \mathbb{C}^*$. Obviously, it is enough, to consider conjugation with elements of $L(G)_0 \rtimes \mathbb{C}^*$, where $L(G)_0$ denotes the connected component of $L(G)$ containing the identity. In particular, $L(G)_0$ consists of loops which are contractible. So we have to study the “twisted Weyl group”

$$\tilde{W}_{\sigma_c} = N_{L(G)_0 \times \mathbb{C}^*}((\sigma_c, 1)(T \times \mathbb{C}^*)_0^{\sigma_c})/(T \times \mathbb{C}^*)_0^{\sigma_c}$$

and its action on the set $(\sigma_c, 1)(T \times \mathbb{C}^*)_0^{\sigma_c}$. First, note that if some $(g, q)$ normalizes $(\sigma_c, 1)(T \times \mathbb{C}^*)_0^{\sigma_c}$, then it normalizes $(T \times \mathbb{C}^*)_0^{\sigma_c}$.

**Lemma 3.7.** If some element $(g, q) \in L(G) \rtimes \mathbb{C}^*$ normalizes the torus $(T \times \mathbb{C}^*)_0^{\sigma_c}$, then it has to normalize the torus $T \times \mathbb{C}^*$ as well.

**Proof.** Conjugation by an element $(g, q) \in L(G) \rtimes \mathbb{C}^*$ induces an automorphism of the Lie algebra $L(g) \oplus CD$ of $L(G) \rtimes \mathbb{C}^*$ which we will denote by $q_g$. Consider the root space decomposition

$$L(g)_{\text{pol}} \oplus CD = (\mathfrak{h} \oplus CD) \oplus \bigoplus_{\alpha \in \Delta} L(g)_\alpha.$$

The automorphism $\sigma_c$ acts on $L(g) \oplus CD$ leaving $\mathfrak{h} \oplus CD$ invariant. We can choose an element $X \in (\mathfrak{h} \oplus CD)^{\sigma_c}$ such that $\alpha(X) \neq 0$ for all $\alpha \in \Delta$. This choice of $X$ insures that the condition $[X, Y] = 0$ already implies $Y \in \mathfrak{h} \oplus CD$. But for $Y \in \mathfrak{h} \oplus CD$ and $g \in N_{L(G) \times \mathbb{C}^*}(T \times \mathbb{C}^*)_0^{\sigma_c}$ we have $[X, g_q(Y)] = 0$ so that $g_q$ indeed leaves $\mathfrak{h} \oplus CD$ invariant. Finally, we can use the exponential map $\exp : L(g) \oplus CD \to L(G) \rtimes \mathbb{C}^*$ to go back to the group level. \hfill \Box

Let us denote by $\tilde{W}$ the group

$$\tilde{W} = N_{L(G)_0 \times \mathbb{C}^*}(T \times \mathbb{C}^*)/(T \times \mathbb{C}^*).$$

Then we have the following Lemma.
Lemma 3.8. The twisted Weyl group $\tilde{W}_{\sigma_c}$ is isomorphic to the semidirect product
$$\tilde{W}_{\sigma_c} \cong \tilde{W}_{\sigma_c} \times \left( (T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right).$$
of the $\sigma_c$-invariant part $\tilde{W}_{\sigma_c}$ of $\tilde{W}$ and the finite group $((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c})$.

**Proof.** Lemma 3.7 allows to define a map
$$\varphi : \tilde{W}_{\sigma_c} \rightarrow \tilde{W}$$
via
$$(g, q)(T \times \mathbb{C}^*)_{0}^{\sigma_c} \mapsto (g, q)(T \times \mathbb{C}^*).$$

It is easy to check that
$$(\sigma_c, 1)\varphi ((g, q)(T \times \mathbb{C}^*)_{0}^{\sigma_c}) (\sigma_c, 1)^{-1} = \varphi ((g, q)(T \times \mathbb{C}^*)_{0}^{\sigma_c})$$
so that $\varphi$ defines a map $\tilde{W}_{\sigma_c} \rightarrow \tilde{W}_{\sigma_c}$. The kernel of $\varphi$ is given by
$$\ker(\varphi) = (\left((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right).$$

Finally, one can show that that the map $\varphi : \tilde{W}_{\sigma_c} \rightarrow \tilde{W}_{\sigma_c}$ is surjective and that the exact sequence
$$\{1\} \rightarrow (\left((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right) \rightarrow \tilde{W}_{\sigma_c} \xrightarrow{\varphi} \tilde{W}_{\sigma_c} \rightarrow \{1\}$$
splits (see e.g. [W] or [M] for the case of finite Weyl groups). □

It remains to describe the action of the groups $\tilde{W}_{\sigma_c}$ and $((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}$
on $\sigma_c, 1)\left((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right)$. Note that conjugation by $(\sigma_c, 1)$ maps an element $(\xi, q) \in T \times \mathbb{C}^*$ to $(w_c(\xi)q^{-\lambda_c}, q)$. Let us choose some $h_0 \in \mathfrak{h}_R$ such that $w_c(h_0) = h_0 + \lambda_c$.

Then we can define a bijective map
$$T_0^{w_c} \times \mathbb{C} \rightarrow (T \times \mathbb{C}^*)_{0}^{\sigma_c}$$
via
$$(\xi, q) \mapsto (\xi q^{h_0}, q).$$

This shows that $(\left((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right)$ is in fact isomorphic to $(T/T_0^{w_c})^{w_c}$. Let $T_0^{w_c} = T/(id - w_c)T$ denote the torus of co-invariants under $w_c$. There is a natural projection $T_0^{w_c} \rightarrow T_{w_c}$ whose kernel is isomorphic to $(T/T_0^{w_c})^{w_c}$. This projection allows us to embed the co-character lattice $\Lambda(T_{w_c})$ of $T_{w_c}$ into $\mathfrak{h}^{w_c}$ such that $\Lambda(T_0^{w_c}) \subset \Lambda(T_{w_c})$. The group $(T/T_0^{w_c})^{w_c}$ acts on $(\sigma_c, 1)\left((T \times \mathbb{C}^*)/\left((T \times \mathbb{C}^*)_{0}\right)^{\sigma_c}\right)$ via translations.

**Lemma 3.9.** Suppose that $\sigma_c$ is not the order $n$ automorphism of the extended Dynkin diagram corresponding to $A_{n-1}$. Then the group $\tilde{W}_{\sigma_c}$ is isomorphic to a semidirect product
$$\tilde{W}_{\sigma_c} \cong \tilde{W}_0 \rtimes \Lambda(T_{w_c}).$$

Here, $\tilde{W}_0$ is a finite Weyl group acting irreducibly on $\mathfrak{h}^{w_c}$. The action of $\tilde{W}_{\sigma_c}$ on
$$(\sigma_c, 1)(T \times \mathbb{C}^*)_{0}^{\sigma_c}$$
is given by
$$(w, 1) (\left((\sigma_c, 1)(\xi q^{h_0}, q)\right) = (\sigma_c, 1)(w(\xi)q^{h_0}, q)$$
and
$$(1, \beta) (\left((\sigma_c, 1)(\xi q^{h_0}, q)\right) = (\sigma_c, 1)(\xi q^{-\beta}q^{h_0}, q).$$
Proof. Fix some $q \neq 1$ in $\mathbb{C}^*$ and view the tangent space of $T \times \{q\}$ as an affine subspace of the tangent space $\mathfrak{h} \oplus \mathbb{C}D$ of $T \times \mathbb{C}^*$. We can identify this space with $\mathfrak{h}$. Since the Weyl group $W$ maps $T \times \{q\}$ to itself, it acts on $\mathfrak{h}$ by affine transformations. It is a standard fact that $\tilde{W}$ acts on $\mathfrak{h}$ by affine reflections, and its lattice of translations is given by $\Lambda(T)$. Let $a$ be a fundamental domain for the induced action of $\tilde{W}$ on $\mathfrak{h}_R$. Then $\tilde{W}$ is generated by the reflections in the walls of $a$. Now, $\sigma_c$ induces an affine map on $\mathfrak{h}_R$ which maps the set of reflection hyperplanes of $\tilde{W}$ to itself. Furthermore, we can choose a fundamental domain $a_0$ which is mapped to itself by $\sigma_c$. So $a_0 \cap \mathfrak{h}_R^e \neq \emptyset$. We claim that the action of $\tilde{W}^\sigma_c$ on $\mathfrak{h}_R^e$ is generated by the set of reflections in the walls of $a_0 \cap \mathfrak{h}_R^e$. Indeed, let $h_c$ be a wall of $a_0 \cap \mathfrak{h}_R^e$ which is the intersection of the walls $h_1, \ldots, h_r$ of $a_0$. Since each $x \in h_1 \cap \ldots \cap h_r$ is fixed under $\sigma_c$, the map $\sigma_c$ permutes the hyperplanes $h_1, \ldots, h_r$. Furthermore, since $h_1 \cap \ldots \cap h_r$ is supposed to be a wall of $a_0 \cap \mathfrak{h}_R^e$ (i.e., an affine subspace of $\mathfrak{h}^e_\sigma$ of codimension 1), we can assume the $h_c$ to consist of a single $\sigma_c$-orbit. Let $s_c : \mathfrak{h}_R \to \mathfrak{h}_R$ denote the reflection in the affine hyperplane $h_c$. Since $\sigma_c$ is not the order $n$-automorphism of the extended Dynkin diagram corresponding to $A_{n-1}$, we have that either for all simple roots $\alpha_i \in \Pi$, the simple roots $\alpha_i$ and $\sigma_c(\alpha_i)$ are non connected in the Dynkin diagram of $\tilde{A}$ in which case $s_i$ and $s_{\sigma_c(i)}$ commute. Or the $\sigma_c$-orbit through $\alpha_c$ consists of exactly two elements $s_1$ and $s_2$ say, and $(s_1s_2)^3 = 1$. In the first case, $s_1s_2\cdots s_r$ commutes with $\sigma_c$. Furthermore, the restriction of $s_1s_2\cdots s_r$ leaves an affine hyperplane of $\mathfrak{h}^e_\sigma$ invariant so that it acts as an affine reflection on $\mathfrak{h}^e_\sigma$. In the second case, we have to consider the element $s_1s_2s_1 \in \tilde{W}$ which commutes with $\sigma_c$ and acts as an affine reflection on $\mathfrak{h}^e_\sigma$. Since $a_0 \cap \mathfrak{h}_R^e$ is a fundamental domain for the action of $\tilde{W}^\sigma_c$ acting on $\mathfrak{h}^e_\sigma$, the subgroup of $\tilde{W}^\sigma_c$ generated by the reflections in the walls of $a_0 \cap \mathfrak{h}_R^e$ generate the whole of $\tilde{W}^\sigma_c$.

Finally, it is straight forward to check that the lattice of translations of the action of $\tilde{W}^\sigma_c$ on $\mathfrak{h}^e_\sigma$ is given by the lattice
\[
\Lambda(T_{w_c}) = \left\{ \frac{1}{\text{ord}(w_c)} \sum_{i=1}^{\text{ord}(w_c)} w_c^i(\beta) \mid \beta \in \Lambda(T) \right\}.
\]

Putting everything together, we get the analogue of Proposition 3.2 in the non-simply connected case.

Proposition 3.10. Let $G$ be of the form $G = \tilde{G}/\mathbb{Z}$. Every semisimple element in $L(G)c \times \{q\}$ is conjugate to an element of the form $(\sigma_c, \xi q^{h_0}, q)$, where $\xi \in T_0^{w_c}$ and $h_0 \in \mathfrak{h}_R$ is chosen such that $w_c(h_0) = h_0 + \lambda_c$.

Two elements $(\sigma_c, \xi_1 q^{h_0}, q)$ and $(\sigma_c, \xi_2 q^{h_0}, q)$ are conjugate if and only if they are in the same orbit under the action of the group $W_{\sigma_c}$ on $(\sigma_c, 1) (T \times \mathbb{C}^*)^{w_c}$.

Remark 3.11. Remember that we have identified $(T/T_0^{w_c})^{w_c}$ with a finite subgroup of $T_0^{w_c}$. The quotient $T_0^{w_c}/(T/T_0^{w_c})^{w_c}$ is isomorphic to the torus $T_{w_c}$ of invariants. So analogously to Remark 3.3, the set of semisimple conjugacy classes in $L(G)c \times \{q\}$ can be identified with the quotient $E_q \otimes \mathbb{Z} \Lambda(T_{w_c})/W_0$.

Finally, let us consider semisimple conjugacy classes in a central extension $\tilde{L}^k(G)$. Remember that $\tilde{L}^k(G)$ is given as a semidirect product $\tilde{L}^k(G) \cong \hat{L}^k(G) \rtimes \hat{\mathbb{C}}^*$ where
\( \hat{L}^k(G) \) denotes a level \( k \) central extension of \( L(G) \) corresponding to some commutator map \( \omega \), and \( \hat{\mathbb{C}}^* \) is the ord(c)-fold covering of \( \mathbb{C}^* \). Let \( L(G)_c \) denote the connected component of the loop group \( L(G) \) corresponding to the element \( c \in \mathbb{Z} \) and denote by \( \hat{L}^k(G)_c \) the set of elements in \( \hat{L}^k(G) \) which project to an element in \( L(G)_c \). Furthermore, fix some \( \tilde{q} \in \hat{\mathbb{C}}^* \). Since conjugation in \( \hat{L}^k(G) \) leaves the sets \( \hat{L}^k(G) \times \{ \tilde{q} \} \) invariant, we can consider the connected component \( \hat{L}^k(G)_c \times \{ \tilde{q} \} \) which is also invariant under conjugation. The set of semisimple conjugacy classes in this component will be the total space of a \( \mathbb{C}^* \)-bundle over the space of semisimple conjugacy classes in \( L(G)_c \times \{ \tilde{q} \} \). For any element \( \tilde{q} \in \hat{\mathbb{C}}^* \), let \( q \in \mathbb{C}^* \) denote the image of \( \tilde{q} \) under the natural projection \( \hat{\mathbb{C}}^* \to \mathbb{C}^* \). Since \( \tilde{q} \) acts on \( L(G) \) via rotation by \( q \), the set of semisimple conjugacy classes in \( L(G)_c \times \{ \tilde{q} \} \) is isomorphic to the set of semisimple conjugacy classes in \( L(G)_c \times \{ q \} \).

Recall from Lemma 3.9 and Remark 3.11 that \( \hat{W}_{\sigma} = (W_0 \rtimes \Lambda(T_{w_0})) \rtimes (T/T_0^{w_{c'}}) \), where \( \Lambda(T_{w_0}) \) is identified with a sublattice of \( \mathfrak{h}^c_{\mathbb{C}} \).

Let us denote by \( \hat{T}_q^{w_{c'}} \) the set of elements in \( \hat{L}^k(G)_c \times \{ \tilde{q} \} \) which project to an element of the form \( (\sigma, \xi q^{b_0}, \tilde{q}) \) with \( \xi \in T_0^{w_{c'}} \). As a complex manifold, this set is isomorphic to \( T_0^{w_{c'}} \times \mathbb{C}^* \). Using Lemma 3.9 Remark 3.11 and formula 3.4, we find the analogue of Proposition 3.3 in the non-simply connected case.

**Proposition 3.12.** The set of semisimple \( L(G)_c \)-conjugacy classes in the connected component of \( \hat{L}^k(G) \times \{ \tilde{q} \} \) corresponding to \( c \in \mathbb{Z} \) is given by the quotient \( \hat{T}_q^{w_{c'}} / \hat{W}_{\sigma} \), where \( (w, \xi_0, \beta) \in \hat{W}_{\sigma} = (W_0 \rtimes \Lambda(T_{w_0})) \rtimes (T/T_0^{w_{c'}}) \) acts as follows:

\[
(w, 1, 1)(\xi, u, \tilde{q}) = (w \xi, u, \tilde{q}),
\]

\[
(1, \beta, 1)(\xi, u, \tilde{q}) = \left( \xi q^{-\beta}, uq^{-(k/2)(\beta, \beta)} \beta(\xi^k), \tilde{q} \right) \quad \text{for } \beta \in \Lambda(T_0^{w_{c'}}).
\]

and

\[
(1, 1, \xi_0)(\xi, u, \tilde{q}) = (\xi_0 u, u, \tilde{q}),
\]

Here, \( \beta(\cdot) \) denotes the value of \( \beta \) as a character of \( T_{w_0} \) and \( \mathfrak{h} \) and \( \mathfrak{h}^c \) are identified via the normalized invariant bilinear form on \( \mathfrak{g} \).

**Remark 3.13.** The quotient \( \hat{T}_q^{w_{c'}} / \Lambda(T_{w_0}) \) described in Proposition 3.12 is the set of non-zero vectors in a line bundle \( L_{w_0}^k \) over the Abelian variety \( E_q \otimes \mathbb{Z} \Lambda(T_{w_0}) \) introduced in Remark 3.20. The action of the finite Weyl group \( W_0 \) on this variety induces an action of \( W_0 \) on \( L_{w_0}^k \).

4. **The differential equation for affine characters**

4.1. **The characters as sections of a line bundle.** Throughout this section, we shall assume that \( G \) is of the form \( G = \hat{G}/\mathbb{Z} \), where \( Z = \langle c \rangle \) is a cyclic subgroup of the center of \( \hat{G} \). The simply connected case follows from the calculations below by setting \( c = id \). As before, let \( k \) be a multiple of the basic level of \( G \), and let \( \chi_{I,k} \) denote the character of the \( \hat{L}^k(G) \)-module \( V_{I,k} \). Here, \( I \) is a \( \sigma_c \)-orbit in \( P_+^k \). So the representation \( V_{I,k} \) restricted to the connected component \( \hat{L}^k(G) \) containing the identity decomposes into a direct sum \( V_{I,k} = \bigoplus_{\lambda \in I} V_{\lambda,k} \) of irreducible highest weight representations of \( \hat{L}^k(G) \). We have seen in the end of section 2.4.4 that if \( I \) consist of more than one element, the character \( \chi_{I,k} \) restricted to the connected component of \( \hat{L}^k(G) \) corresponding to the element \( c \) vanishes. So the only interesting characters on this component are the ones coming from
representations \(V_{I,k}\), where \(I\) consists of a single element \(\lambda\) which is necessarily invariant under \(\sigma_c\). Let us denote the corresponding character by \(\chi_{\lambda,k}\).

As before, let \(L^k(G)_c\) denote the connected component of \(L(G)\) containing the element \(\sigma_c\), and let \(\hat{L}^k(G)_c\) be the connected component of \(\hat{L}^k(G)\) which consists of elements which project to some \(g \in L(G)_c\). Fix some \(\bar{q} \in \hat{C}^*\) and let \(q\) denote the image of \(\bar{q}\) under the natural projection \(\hat{C}^* \to C^*\). Let us assume that \(|q| < 1\).

Then the character \(\chi_{\lambda,k}\) defines a holomorphic function on \(\hat{L}^k(G)_c \times \{\bar{q}\}\). Since almost every element in \(\hat{L}^k(G)_c \times \{\bar{q}\}\) is semisimple, the function \(\chi_{\lambda,k}\) is uniquely determined by its values on \(\hat{T}_{\bar{q}} \times \{\bar{q}\}\). As before, fix some \(h_0 \in \mathfrak{h}\) such that \(w_c(h_0) = h_0 + \lambda_c\). Remember that \(\hat{T}_{\bar{q}}^{\mathfrak{w}_c} \times \{\bar{q}\}\) was defined as the set of elements of \(\hat{L}^k(G)_c \times \{\bar{q}\}\) which project to an element of the form \((\sigma_c \xi \Omega_{h_0}, \bar{q})\) with \(\xi \in T_0^{\mathfrak{w}_c}\) under the natural projection \(\hat{L}^k(G)_c \times \{\bar{q}\} \to L(G) \times \{\bar{q}\}\).

Finally, recall the identification \(\iota : C^* \to \hat{L}^k(G)_c\) of \(C^*\) with the center of \(\hat{L}^k(G)\).

If \(g = (\sigma_c \xi \Omega_{h_0}, u, \bar{q}) \in \hat{T}_{\bar{q}}^{\mathfrak{w}_c}\), we can use the identity

\[
\chi_{\lambda,k}(\sigma_c \xi \Omega_{h_0}, u, \bar{q}) = u^k \chi_{\lambda,k}(\sigma_c \xi \Omega_{h_0}, 1, \bar{q})
\]

to get rid of the central variable. Thus, for fixed \(\bar{q}\), we can view the character \(\chi_{\lambda,k}\) as a section of the line bundle \(L_{\mathfrak{w}_c}\) introduced in Remark 3.13. We shall abuse notation slightly and denote this section by \(\chi_{\lambda,k}\) as well. So locally, we can view \(\chi_{\lambda,k}\) as a function on \((T \times C^*)_0\). We will derive a differential equation for this function in section 4.3 by varying the variable \(\bar{q}\).

Let us change our notation slightly. For \(q \in C^*\) with \(|q| < 1\) fix \(\tau \in C\) with \(Im(\tau) > 0\). Let \(L \subset C\) be the lattice generated by 1 and \(\tau\). Then the elliptic curve \(E_\tau = C/L\) is isomorphic to the curve \(E_q\) considered in the last paragraph via the map \(x \mapsto e^{2\pi i x\tau}\). This allows to identify the Abelian variety \(E_q \otimes \Lambda(T_{\mathfrak{w}_c}) / \Lambda(T_{\mathfrak{w}_c}) \otimes \tau \Lambda(T_{\mathfrak{w}_c})\). In this identification, we can view the character \(\chi_{\lambda,k}\) at some fixed \(\bar{q}\) as a function \(\chi_{\lambda,k}^\sigma (\cdot ; \bar{q})\) on \(\mathfrak{h}^{\mathfrak{w}_c}\) which has to satisfy the following identity:

\[
\chi_{\lambda,k}^\sigma (h + \beta + \tau \beta' ; \bar{q}) = \exp(-2\pi i k \langle \beta', v \rangle - \pi i k \tau \langle \beta', \beta' \rangle) \chi_{\lambda,k}^\sigma (h ; \bar{q})
\]

for any \(\beta, \beta' \in \Lambda(T_{\mathfrak{w}_c})\). From now on, we shall switch between the different viewpoints for the characters freely.

4.2. The action of \(\sigma_c\) on the derivation \(D\). Throughout this section, let \(k\) be a multiple of the basic level of \(G\) and let \(\lambda \in P_+^k\) be a highest weight of \(\mathfrak{g}\) which is invariant under \(\sigma_c\). So the \(\sigma_c\)-orbit \(\mathfrak{f}\) through \(\lambda\) consists of a single element.

Let \(h_1, \ldots, h_l\) be an orthonormal basis of \(\mathfrak{h}\) and choose \(e_\alpha \in \mathfrak{g}_\alpha\) and \(f_\alpha \in \mathfrak{g}_{-\alpha}\) for each \(\alpha \in \Delta_+\) such that the set \(\{e_\alpha, f_{\alpha}, h_j \mid \alpha \in \Delta_+ \text{ and } 1 \leq j \leq l\}\) is an orthonormal basis of \(\mathfrak{g}\). We can make this choice in such a way that \(w_c(e_\alpha) = \pm e_{w_c(\alpha)}\) for all \(\alpha \in \Delta\) and similarly for \(f_\alpha\) (see section 2.3).

For \(\tilde{\alpha} = \alpha + n\delta \in \tilde{\Delta}^c\) let us define

\[
e_{\tilde{\alpha}} = \begin{cases} e_\alpha \otimes z^n & \text{if } \alpha \in \Delta_+, \ n \geq 0, \\ f_\alpha \otimes z^n & \text{if } \alpha \in \Delta_-, \ n > 0. \end{cases}
\]

Similarly for \(\tilde{\alpha} = \alpha + n\delta \in \tilde{\Delta}^c_+\), we set

\[
f_{\tilde{\alpha}} = \begin{cases} f_\alpha \otimes z^{-n} & \text{if } \alpha \in \Delta_+, \ n \geq 0, \\ e_\alpha \otimes z^{-n} & \text{if } \alpha \in \Delta_-, \ n > 0. \end{cases}
\]
We use the usual notation $X \otimes z^n = X^{(n)}$ for $X \in \mathfrak{g}$. Then, using the explicit expression of the Kac-Casimir operator \cite[12.8]{K}, on $V_{\lambda,k}$, we can express $D$ as an operator on the highest weight representation $V_{\lambda,k}$ of $\tilde{L}(\mathfrak{g})$ via

\begin{equation}
2(k + h^\vee)D = c(\lambda)Id - 2 \sum_{\alpha \in \Delta^\vee} f_{\alpha} e_\alpha - 2 \sum_{n=1}^{\infty} \sum_{j=1}^{l} h_j^{(-n)} h_j^{(n)} - \sum_{j=1}^{l} (h_j)^2 - \sum_{\alpha \in \Delta^+} h_\alpha,
\end{equation}

where $c(\lambda) = \|\lambda + \rho\|^2 - \|\rho\|^2$ and $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$.

In this section, we determine how the expression (5) for the derivation $D$ behaves under the automorphism $\sigma_c$ of $\tilde{L}(\mathfrak{g})$. Let $(\Delta^r^+)^{\sigma_c}$ be the set of $\sigma_c$-orbits in $\Delta^r^+$. Let $m_{\tilde{\alpha}} = ||[\tilde{\alpha}]||$ denote the cardinality of the $\sigma_c$-orbit through $\tilde{\alpha}$. Let $e_\tilde{\alpha}$ be a $m_{\tilde{\alpha}}$-th root of unity. Then we set

$$e^{\tilde{\alpha}_s}_\tilde{\alpha} = \sum_{j=1}^{m_{\tilde{\alpha}}} e^j_{\tilde{\alpha}} \sigma^j_c(e_\tilde{\alpha}),$$

and accordingly

$$f^{\tilde{\alpha}_s}_\tilde{\alpha} = \sum_{j=1}^{m_{\tilde{\alpha}}} e^j_{\tilde{\alpha}} \sigma^j_c(f_\tilde{\alpha}).$$

Obviously, $e^{\tilde{\alpha}_s}_\tilde{\alpha}$ and $f^{\tilde{\alpha}_s}_\tilde{\alpha}$ lie in the $e_\tilde{\alpha}^{-1}$-eigenspace of $\sigma_c$. Similarly, recall that $\sigma_c$ acts linearly on the space $\mathfrak{h} \oplus \mathbb{C}$. For $h \in \mathfrak{h} \oplus \mathbb{C}$ and any ord($\sigma_c$)-th root of unity $\epsilon$ we define

$$h^\epsilon = \sum_{j=1}^{\text{ord}(\sigma_c)} \epsilon^j \sigma^j_c(h).$$

Finally, we can decompose $\mathfrak{h} = \mathfrak{h}_0 \oplus \ldots \oplus \mathfrak{h}_{\text{ord}(w_c)-1}$ where $\mathfrak{h}_j$ denotes the $e^{2\pi i \text{ord}(\sigma_c) j}$-eigenspace of the action of the Weyl group element $w_c$ acting on $\mathfrak{h}$. Furthermore, we can choose the basis $h_1, \ldots, h_l$ of $\mathfrak{h}$ such that it is adapted to this decomposition of $\mathfrak{h}$. That is, let $h_{j_1}, \ldots, h_{j_l}$ be an orthonormal basis of the space $\mathfrak{h}_j$. Then the set \{ $h_{j_s} \mid 0 \leq j \leq \text{ord}(w_c) - 1, 1 \leq s \leq l_j$ \} is an orthonormal basis of $\mathfrak{h}$.

Let us fix a representative $\tilde{\alpha}$ for each $\sigma_c$-orbit $[\tilde{\alpha}]$ in $\Delta^r^+$. Furthermore, from now on let us fix the roots of unity $e_\tilde{\alpha} = e^{2\pi i / m_{\tilde{\alpha}}}$ and $\epsilon = e^{2\pi i / \text{ord}(\sigma_c)}$. Using the expression from equation (5) for the derivation $D$ and the fact that $\sigma_c$ acts as an automorphism on the universal enveloping algebra of $\tilde{L}(\mathfrak{g})$, we can write

\begin{equation}
\frac{2(k + h^\vee)}{\text{ord}(\sigma_c)} \sum_{j=1}^{\text{ord}(\sigma_c)} \sigma^j_c(D) =
\end{equation}

\begin{align}
&c(\lambda) - 2 \sum_{[\tilde{\alpha}] \in (\Delta^\vee)^{\sigma_c}} \frac{1}{m_{\tilde{\alpha}}} \sum_{j=1}^{m_{\tilde{\alpha}}} f^{\tilde{\alpha}_s}_\tilde{\alpha} e^{\tilde{\alpha}_s}_\tilde{\alpha} - \frac{2}{\text{ord}(\sigma_c)} \sum_{n=1}^{\text{ord}(w_c)} \sum_{j=1}^{l_j} \sum_{s=1}^{2r_j} h^{(-n)}_{j,s} h^{(n)}_{j,s} \\
&- \frac{1}{\text{ord}(\sigma_c)^2} \sum_{j=1}^{\text{ord}(\sigma_c)} \sum_{s=1}^{l_j} \sum_{r=1}^{\text{ord}(\sigma_c)} h^{r}_{j,s} h^{-r}_{j,s} - \frac{1}{\text{ord}(\sigma_c)} \sum_{\alpha \in \Delta^+} \sigma^j_\alpha(h_\alpha)
\end{align}
4.3. The differential equation. We now derive a differential equation for the character $\chi_{\lambda,k}$.

Recall that $\sigma_c$ acts on $\mathfrak{h}$. Fix some $h \in \mathfrak{h}^w$ and some $\tau \in \mathbb{C}$. Then the element

$$\tilde{H}_0 = h - \tau \frac{1}{\text{ord}(\sigma_c)} \sum_{j=1}^{\text{ord}(\sigma_c)} \sigma_c^j(D)$$

is invariant under $\sigma_c$. But $\tilde{H}_0$ contains a non-zero central term which comes from $\sum_{j=1}^{\text{ord}(\sigma_c)} \sigma_c^j(D)$. Let $C(D)$ denote this central term. Then

$$H_0 = \tilde{H}_0 - C(D) \in \mathfrak{h}^w \oplus \mathbb{C}D,$$

and $H_0$ is invariant under the residual action of $\sigma_c$ on $\mathfrak{h} \oplus \mathbb{C}D \cong \mathfrak{h}/CC$. Therefore we have

$$e^{2\pi i H_0} = e^{2\pi i h} \left( \frac{1}{\text{ord}(\sigma_c)} \sum_{j=1}^{\text{ord}(\sigma_c)} \sigma_c^j(D) \right) e^{2\pi i C(D)} \in (T \times \mathbb{C}^*)^{\sigma_c}$$

where we have set $q = e^{2\pi i \tau}$ as usual. If $\text{Im}(\tau) > 0$, we have $|q| < 1$ so that the character $\chi_{\lambda,k}(\sigma_c e^{2\pi i H_0})$ converges. (Remember that we have identified the character $\chi_{\lambda,k}$ with a section of the family of line bundle $L_{w_c}^\alpha$ which we view locally as a function on $(T \times \mathbb{C}^*)^{\sigma_c}$.)

We now want to calculate the derivative

$$\frac{2(k + h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} \chi_{\lambda,k}(\sigma_c e^{2\pi i H_0}).$$

For simplicity, we will calculate $\frac{2(k + h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0})$ and disregard all central terms which come from the action of $\sigma_c$ on the derivation $D$. Using equation (6), we can compute

$$\frac{2(k + h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) = -Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0}) 2\frac{k + h^\vee}{\text{ord}(\sigma_c)} \sum_{j=1}^{\text{ord}(\sigma_c)} \sigma_c^j(D))$$

$$= -c(\lambda) \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) + 2 \sum_{[\tilde{\alpha}] \in (\Delta^+_{\mathbb{C}})^{\sigma_c}} \frac{1}{m_{\tilde{\alpha}}} A_{[\tilde{\alpha}]}
$$

$$+ \frac{2}{\text{ord}(\sigma_c)} \sum_{n=1}^{\infty} \sum_{j=1}^{\text{ord}(w_n)} B_{n,j}
$$

$$+ \frac{1}{\text{ord}(\sigma_c)^2} \sum_{j=1}^{\text{ord}(\sigma_c)} C_j + \frac{1}{\text{ord}(\sigma_c)} \sum_{\alpha \in \Delta^+_+} \sum_{j=1}^{\text{ord}(\sigma_c)} E_{\alpha,j},$$

where we have set

$$A_{[\tilde{\alpha}]} = \sum_{j=1}^{m_{\tilde{\alpha}}} A_{\tilde{\alpha},j}$$

with

$$A_{\tilde{\alpha},j} = Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} f_{\tilde{\alpha}}^j \epsilon_{\tilde{\alpha}}^{-j})$$

for some fixed $\tilde{\alpha} \in [\tilde{\alpha}]$. It is clear that $A_{[\tilde{\alpha}]}$ does not depend on the choice of the representative $\tilde{\alpha}$. Similarly,

$$B_{n,j} = \sum_{s=1}^{\dim(h)} \sum_{r=1}^{\text{ord}(w_s)} e^{2\pi i r} B_{n,j,s}$$
with
\[ B_{n,j,s} = Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} h_{s,j}^{(-n)} h_{s,j}^{(n)}) , \]
and
\[ C_j = \sum_{s=1}^{l_1} \sum_{r=1}^{\text{ord}(\sigma_c)} C_{j,r,s} \]
with
\[ C_{j,s,r} = Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} h_{j,s}^{e^{-r}}) . \]
Finally,
\[ E_{\alpha,j} = Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} \sigma_c^j(h_\alpha)) . \]

We can compute the summands of equation (7) more explicitly. Let us first note that since \( \tilde{H}_0 \) is invariant under \( \sigma_c \), we have \( \sigma_j^j(\tilde{\alpha})(\tilde{H}_0) = \tilde{\alpha}(\tilde{H}_0) \) for all \( j \in \mathbb{N} \), so that for any root \( \tilde{\alpha} \in \Delta \), the value of \( \tilde{H}_0 \) on the \( \sigma_c \)-orbit through \( \tilde{\alpha} \) is constant. Furthermore, for any real root \( \tilde{\alpha} \in \Delta^\vee \), we have chosen the \( e_\tilde{\alpha} \) and \( f_\tilde{\alpha} \) such that \( \sigma_c(e_\tilde{\alpha}) = \pm e_\tilde{\alpha} \) and similarly for \( f_\tilde{\alpha} \). Let us define
\[ s(\tilde{\alpha}) = \begin{cases} -1 & \text{if } \sigma_c(\tilde{\alpha}) = \tilde{\alpha} \text{ and } \sigma_c(e_\tilde{\alpha}) = -e_\tilde{\alpha} \\ 1 & \text{in all other cases} \end{cases} \]
If \( s(\tilde{\alpha}) = 1 \), we can calculate
\[ A_{\tilde{\alpha},j} = Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} f_\tilde{\alpha}^{e^{-j}} e_\tilde{\alpha}^{e^{-j}}) \]
\[ = e^{-j} Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} f_\tilde{\alpha}^{e^{-j}}) \]
\[ = e^{-j} e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)} Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0} e^{-j} f_\tilde{\alpha}^{e^{-j}}) \]
\[ = e^{-j} e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)} Tr_{V_{\lambda,k}}(\sigma_c e^{2\pi i \tilde{H}_0}(f_\tilde{\alpha}^{e^{-j}} f_\tilde{\alpha}^{e^{-j}} + [e_\tilde{\alpha}^{e^{-j}}, f_\tilde{\alpha}^{e^{-j}}])). \]
In the first step, we have used the cyclic property of the trace. In the second step we have used the fact that \( e_\tilde{\alpha}^{e^{-j}} \) lies in the \( e_\tilde{\alpha}^{e^{-j}} \)-eigenspace of the action of \( \sigma_c \) on \( \tilde{L}(\mathfrak{g}) \). The remaining steps are the standard commutation relations in \( \tilde{L}(\mathfrak{g}) \). The commutator in the last step can be computed as follows:
\[ [e_\tilde{\alpha}^{e^{-j}}, f_\tilde{\alpha}^{e^{-j}}] = \sum_{s=1}^{m_2} \sum_{r=1}^{m_2} e_\tilde{\alpha}^{e^{-j}} e_\tilde{\alpha}^{e^{-j}} [\sigma_s^j(e_\tilde{\alpha}), \sigma_s^j(f_\tilde{\alpha})] \]
\[ = \sum_{s=1}^{m_2} \sigma_s^j([e_\tilde{\alpha}, f_\tilde{\alpha}]) , \]
since we have \( [\sigma_s^j(e_\tilde{\alpha}), \sigma_s^j(f_\tilde{\alpha})] = 0 \) whenever \( s \neq r \). Indeed, by assumption, we have \( \sigma_s^j(\tilde{\alpha}) \neq \sigma_s^j(\tilde{\alpha}) \) for \( s \neq r \). Since \( \sigma_c \) acts as a diagram automorphism on the Dynkin diagram of the affine root system \( \Delta \), there exists a root basis \( \Pi' \) of \( \Delta \) which contains both \( \sigma_s^j(\tilde{\alpha}) \) and \( \sigma_s^j(\tilde{\alpha}) \). Now, we can find some element \( w \) in the Weyl group of \( \Delta \) such that \( w(\Pi') = \Pi \) (\cite{K}, Proposition 5.9). So we have \( [\sigma_s^j(e_\tilde{\alpha}), \sigma_s^j(f_\tilde{\alpha})] = [e_\tilde{\alpha}, f_\tilde{\alpha}] \) for two simple roots \( \tilde{\alpha}_i \neq \tilde{\alpha}_j \in \Pi' \). The last commutator has to vanish due to the Serre relations for \( \tilde{L}(\mathfrak{g})_{\text{pol}} \) (see \cite{K}, 1.2).

If \( \tilde{\alpha} = \alpha + n\delta \) with \( \alpha \in \Delta_+ \), we have
\[ [e_\tilde{\alpha}, f_\tilde{\alpha}] = h_\alpha + nC , \]
where \( h_\alpha \in \mathfrak{h} \) is the co-root of \( \mathfrak{g} \) corresponding to the root \( \alpha \in \Delta \). Similarly, if \( \tilde{\alpha} = \alpha + n\delta \) with \( \alpha \in \Delta_+ \), one has

\[
[e_{\tilde{\alpha}}, f_{\tilde{\alpha}}] = -h_\alpha + nC.
\]

Let us write \( n(\tilde{\alpha}) = n \) for \( \tilde{\alpha} = \alpha + n\delta \) and \( h_\tilde{\alpha} = h_\alpha \) if \( \alpha \in \Delta_+ \) and \( h_\tilde{\alpha} = -h_\alpha \) if \( \alpha \in \Delta_- \). Then the calculations above give us

\[
A_{\tilde{\alpha},j} = e_{\tilde{\alpha}}^j e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)} \left( A_{\alpha,j} + \sum_{s=1}^{m_{\tilde{\alpha}}} e_s^*(h_{\tilde{\alpha}}) + \bar{\chi}_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) \right)
\]

or equivalently

\[
A_{\tilde{\alpha},j} = e_{\tilde{\alpha}}^j e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)} \left( \sum_{s=1}^{m_{\tilde{\alpha}}} e_s^*(h_{\tilde{\alpha}}) + \bar{\chi}_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) \right).
\]

Summing over the \( j \)'s, we can write

\[
A_{[\tilde{\alpha}]} = e_{\tilde{\alpha}} e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)} \left( \sum_{s=1}^{m_{\tilde{\alpha}}} w_s^*(h_{\tilde{\alpha}}) \right) + \bar{\chi}_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) + Y_{[\tilde{\alpha}]} C
\]

for some \( y_{[\tilde{\alpha}]} \in \mathbb{C} \), we can finally write

\[
A_{[\tilde{\alpha}]} = \frac{e_{\tilde{\alpha}} e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}}{1 - e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}} \left( \sum_{s=1}^{m_{\tilde{\alpha}}} \frac{1}{2\pi i} \frac{\partial}{\partial w_s^*(h_{\tilde{\alpha}})} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) \right) + \frac{k m_{\tilde{\alpha}}^2 n(\tilde{\alpha}) e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}}{1 - e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) + Y_{[\tilde{\alpha}]} C
\]

for some \( Y_{[\tilde{\alpha}]} \in \mathbb{C} \) which comes from the action of \( \sigma_c \) on the derivation \( D \).

If \( s(\tilde{\alpha}) = -1 \), we necessarily have \( m_{\tilde{\alpha}} = 1 \). A similar calculation as above gives

\[
A_{[\tilde{\alpha}]} = \frac{-e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}}{1 + e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}} \left( \frac{1}{2\pi i} \frac{\partial}{\partial w_s^*(h_{\tilde{\alpha}})} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) \right) + \frac{k n(\tilde{\alpha}) e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}}{1 + e^{-2\pi i \tilde{\alpha}(\tilde{H}_0)}} \chi_{\lambda,k}(\sigma_c e^{2\pi i \tilde{H}_0}) + Y_{[\tilde{\alpha}]} C
\]

A similar calculation for the \( B_{n,j,s} \) gives

\[
B_{n,j,s} = \text{Tr}_{\lambda,k} \left( \sigma_c e^{2\pi i \tilde{H}_0} h_{j,s}(-n) h_{j,s}(n) \right)
\]

\[
= e^{-j e^{-2\pi i \delta(\tilde{H}_0)}} \text{Tr}_{\lambda,k} \left( \sigma_c e^{2\pi i \tilde{H}_0} \left( h_{j,s}(-n) h_{j,s}(n) + [h_{j,s}(n), h_{j,s}(-n)] \right) \right).
\]
Using the fact that \( \delta(\tilde{H}_0) = -\tau \) and \([h_{j,s}^{(n)}, h_{j,s}^{(-n)}] = nC\), we get

\[
B_{n,j,s} = \frac{\epsilon^{-j}q^n nk\chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0})}{1 - \epsilon^{-j}q^n}
\]

(11)

This shows that \( B_{n,j,s} \) in fact does not depend on \( s \). So we get

\[
B_{n,j} = \dim(\mathfrak{h}_j) \frac{\epsilon^{-j}q^n nk\chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0})}{1 - \epsilon^{-j}q^n}.
\]

(12)

This sum vanishes if \( \text{ord}(\sigma_c) \nmid 2j \), otherwise we get

\[
B_{n,j} = \text{ord}(\sigma_c) \dim(\mathfrak{h}_j) \frac{\epsilon^{-j}q^n nk\chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0})}{1 - \epsilon^{-j}q^n}.
\]

(13)

Finally, since \( h_{j,s}^{\epsilon^r} \) is in the \( \epsilon^{-r} \)-eigenspace of the action of \( \sigma_c \) on \( \mathfrak{h} \oplus \mathbb{C}C \), a similar calculation gives

\[
C_{j,s,r} = Tr_{\chi_{\lambda,k}}(\sigma_c e^{2\pi i\tilde{H}_0} h_{j,s}^{\epsilon^r}) = \epsilon^{-r} C_{j,s,r}.
\]

This shows that \( C_{j,s,r} = 0 \) unless \( \epsilon^r = 1 \). Now, if \( j = 0 \), we have \( h_{j,s}^{1} = \text{ord}(\sigma_c) h_{j,s}, \) so that we get

\[
C_0 = \sum_{s=1}^{l_0} C_{0,s,0} = \frac{\text{ord}(\sigma_c)^2}{(2\pi i)^2} \Delta_{h_0} \chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0}),
\]

(14)

where \( \Delta_{h_0} \) denotes the Laplace operator on \( h_0 \). On the other hand, for \( j \neq 0 \), we have \( h_{j,s}^{1} = a_{s,j} C \) for some \( a_{s,j} \in \mathbb{C} \) so that we can drop these terms since they come from the action of \( \sigma_c \) on \( D \).

Since all roots \( \tilde{\alpha} \) vanish on the center \( C \) of \( \tilde{L}(\mathfrak{g}) \), we have \( \tilde{\alpha}(\tilde{H}_0) = \tilde{\alpha}(H_0) \) for all \( \tilde{\alpha} \in \tilde{\Delta} \). So putting the calculations above together, reordering the terms slightly, and keeping in mind that we have to disregard all central terms coming from the action of \( \sigma_c \) on the derivation \( D \), we get

\[
\frac{2(k + h^{\epsilon^r})}{2\pi i} \frac{\partial}{\partial r}(\chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0})) = (c(\lambda) + A_1 + A_2 + B + E - \frac{1}{4\pi i} \Delta_{h_0}) \chi_{\lambda,k}(\sigma_c e^{2\pi i\tilde{H}_0})
\]

(15)

where we have set

\[
A_1 = \sum_{[\tilde{\alpha}] \in (\Delta_{+}^{\infty})^{\epsilon r}} \frac{s(\tilde{\alpha})2e^{-2\pi i m_{\tilde{\alpha}}(\tilde{H}_0)}}{1 - s(\tilde{\alpha})e^{-2\pi i m_{\tilde{\alpha}}(\tilde{H}_0)}} \sum_{s=1}^{m_{\tilde{\alpha}}} \frac{1}{2\pi i} \frac{\partial}{\partial w_s^\epsilon(\tilde{h}_\alpha)},
\]

\[
A_2 = \sum_{[\tilde{\alpha}] \in (\Delta_{+}^{\infty})^{\epsilon r}} \frac{s(\tilde{\alpha})2km_{\tilde{\alpha}}(\tilde{\alpha})e^{-2\pi i m_{\tilde{\alpha}}(\tilde{H}_0)}}{1 - s(\tilde{\alpha})e^{-2\pi i m_{\tilde{\alpha}}(\tilde{H}_0)}},
\]

\[
B = \sum_{n=1}^{\text{ord}(\sigma_c)} \sum_{j=1}^{2} 2 \dim(\mathfrak{h}_j) \frac{\epsilon^{-j}q^n nk}{1 - \epsilon^{-j}q^n},
\]

\[
E = \frac{1}{\text{ord}(\sigma_c)} \sum_{[\alpha] \in \Delta_{+}} \sum_{j=1}^{\text{ord}(\sigma_c)} \frac{1}{2\pi i} \frac{\partial}{\partial w_s^\epsilon(\tilde{h}_\alpha)}.
\]
Equation (15) is the desired differential equation for the characters. We will now simplify the equation. To do this, let us make some definitions. First, recall the element \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \) and set

\[
\rho_{w_c} = \frac{1}{ord(w_c)} \sum_{j=1}^{ord(w_c)} w_c(\rho).
\]

Then we can define the function

\[
F_{w_c}(H_0) = e^{2\pi i \rho_{w_c}(H_0)} \prod_{[\tilde{\alpha}] \in (\Delta_+^\vee)^{\rho_c}} (1 - s(\tilde{\alpha})e^{-2\pi i m_\tilde{\alpha}(H_0)}) \prod_{n=1}^{\infty} \prod_{j=1}^{ord(w_c)} (1 - e^{-q^n}) \dim(b_n).
\]

By the definition of \( H_0 \) and invariance of \( \delta \) under \( \sigma_c \), we have \( \tilde{\alpha}(H_0) = \alpha(H_0) - \tau n \) for \( \tilde{\alpha} = \alpha + n\delta \). Hence we get

\[
\frac{1}{2\pi i} \frac{\partial}{\partial \tau} F_{w_c}(H_0) = -\left( \frac{1}{2k} A_2 + \frac{1}{2k} B \right) F_{w_c}(H_0).
\]

Now, \( F_{w_c} \) is a section of the line bundle \( L_{w_c}^\vee \) from remark 3.13 so \( \chi_{\lambda,k} F_{w_c} \) is a section of \( L_{w_c}^{k+h^\vee} \). Therefore we get

\[
(16) \quad \frac{1}{2\pi i} \frac{2(k + h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} (\chi_{\lambda,k} F_{w_c}) = \frac{2(k + h^\vee)}{2\pi i} \left( \frac{\partial \chi_{\lambda,k}}{\partial \tau} + \chi_{\lambda,k} \frac{\partial F_{w_c}}{\partial \tau} \right)
\]

\[
= \left( -c(\lambda) + A_1 + E - \frac{1}{4\pi i} \Delta_{h_{w_c}} \right) \chi_{\lambda,k}
\]

Next, we take care of the Laplacian. We use the formula

\[
\Delta_{h_{w_c}}(\chi_{\lambda,k} F_{w_c}) = (\Delta_{h_{w_c}} \chi_{\lambda,k}) F_{w_c} + 2\langle \nabla \chi_{\lambda,k}, \nabla F_{w_c} \rangle + \chi_{\lambda,k} \langle \Delta_{h_{w_c}} F_{w_c} \rangle.
\]

But we have

\[
(17) \quad \frac{1}{2\pi i} \nabla F_{w_c} = \sum_{s=1}^{\dim(b_0)} \rho_{w_c}(h_0,s) F_{w_c,h_0,s} + \sum_{s=1}^{\dim(b_0)} \sum_{[\tilde{\alpha}] \in (\Delta_+^\vee)^{\rho_c}} m_{\tilde{\alpha}} \tilde{\alpha}(h_0,s) \frac{s(\tilde{\alpha})e^{-2\pi i m_\tilde{\alpha}(H_0)}}{1 - s(\tilde{\alpha})e^{-2\pi i m_\tilde{\alpha}(H_0)}} F_{w_c,h_0,s}
\]

\[
(18) \quad = F_{w_c} h_{w_c} + \sum_{[\tilde{\alpha}] \in (\Delta_+^\vee)^{\rho_c}} m_{\tilde{\alpha}} \frac{s(\tilde{\alpha})e^{-2\pi i m_\tilde{\alpha}(H_0)}}{1 - s(\tilde{\alpha})e^{-2\pi i m_\tilde{\alpha}(H_0)}} F_{w_c,h_{\alpha}}
\]

where we have used the notation from before for an orthogonal basis \( h_{0,s} \) of \( h_0 = h_{w_c} \). Since \( h_{0,s} \in h \), we have \( \tilde{\alpha}(h_{0,s}) = \alpha(h_{0,s}) \) for \( \tilde{\alpha} = \alpha + n\delta \). Now, since we have

\[
\langle \nabla \chi_{\lambda,k}, h \rangle = \frac{\partial}{\partial h} \chi_{\lambda,k}
\]

and

\[
h_{w_c} = \frac{1}{\sqrt{2\dim(\sigma_c)}} \sum_{\alpha \in \Delta_+} \sum_{j=1}^{ord(\sigma_c)} w_c^j(h_{\alpha}),
\]

\[\text{with } \sum_{j=1}^{ord(\sigma_c)} w_c^j(h_{\alpha}) \text{ the } \sigma_c \text{-eigenspace of } h_{w_c} \text{ corresponding to } \alpha \text{ and } \sum_{j=1}^{ord(\sigma_c)} w_c^j(h_{\alpha}) \text{ the } \sigma_c \text{-eigenspace of } h_{w_c} \text{ corresponding to } \alpha.\]
we get
\[
\langle \frac{1}{2\pi i} \nabla \lambda, \frac{1}{2\pi i} \nabla \mathcal{F}_{w_c} \rangle = \frac{1}{2} (E \lambda, k) \mathcal{F}_{w_c} + \frac{1}{2} (A_1 \lambda, k) \mathcal{F}_{w_c}
\]

So putting everything together, we get
\[
\frac{1}{F_{w_c}} \left( \frac{2(k + \h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} (\lambda, k \mathcal{F}_{w_c}) + \frac{1}{4\pi^2} \Delta_{h_0} (\lambda, k \mathcal{F}_{w_c}) - \lambda, k \frac{1}{4\pi^2} \Delta_{h_0} \mathcal{F}_{w_c} \right) = -c(\lambda) \chi_{\lambda, k}.
\]

A direct calculation gives
\[
\frac{1}{4\pi^2} \Delta_{h_0} \mathcal{F}_{w_c} = -|\rho_{w_c}|^2 \mathcal{F}_{w_c}
\]
so that we arrive at the final equation:

**Theorem 4.1.** Let \( k \) be a multiple of the basic level of \( G \), and let \( \lambda \in P^*_k \) be a highest weight of \( G \) which is invariant under the action of \( \sigma_c \). Then the character \( \chi_{\lambda, k} \) of the representation \( V_{\lambda, k} \) restricted to the torus \( (\sigma_c, 1)(T \times \mathbb{C}^*)_0 \) has to satisfy the following differential equation:

\[
\frac{1}{F_{w_c}} \left( \frac{2(k + \h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} (\lambda, k \mathcal{F}_{w_c}) + \frac{1}{(2\pi i)^2} \Delta_{h_0} - |\rho_{w_c}|^2 \right) \chi_{\lambda, k} \mathcal{F}_{w_c} = c(\lambda) \chi_{\lambda, k}.
\]

**Remark 4.2.** If we set \( \sigma_c = id \) theorem 4.1 gives a differential equation for the characters of highest weight representations of loop groups based on simply connected Lie groups. This case is treated in more generality in [EK] (see also [EFK]), where a differential equation for traces of intertwining maps between representations of loop groups is derived. Introducing the automorphism \( \sigma_c \), one can find a differential equation for traces of intertwining maps between representations of loop groups based on non-simply connected Lie groups.

**Remark 4.3.** We have only treated the case of automorphisms \( \sigma_c \) associated to the center of the simply connected Lie group \( \widetilde{G} \). The same calculations work if one replaces the automorphism \( \sigma_c \) by an arbitrary finite order automorphism \( \sigma \) of the Lie algebra \( \widetilde{L}(\mathfrak{g}) \) which comes from an automorphism of the Dynkin diagram of \( \widetilde{L}(\mathfrak{g}) \).

## 5. Character formulas

### 5.1. Theta functions

The line bundle \( \mathcal{L}_k \) introduced in Remark 3.3 has been studied in [Lo]. Here we give a brief account of the main results. As in the end of section 4.1 fix some \( \tau \in \mathbb{C} \) with \( \text{Im}(\tau) > 0 \) and let \( q = e^{2\pi i \tau} \). Let \( L \subset \mathbb{C} \) be the lattice generated by 1 and \( \tau \). Then the elliptic curve \( E_\tau = \mathbb{C}/L \) is isomorphic to the curve \( E_q \) considered in the last paragraph via the map \( x \mapsto e^{2\pi i x} \). Denote by \( \Gamma(L^k) \) the space of holomorphic sections of \( L^k \). Let \( \Gamma(L^k)^W \), resp. \( \Gamma(L^k)^{-W} \) denote the spaces of holomorphic \( W \)-invariant, resp. \( W \)-anti invariant sections of \( L^k \). Looijenga’s theorem describes the set \( \Gamma(L^k)^{-W} \) explicitly in terms of theta functions. Since we also want to consider sections in the line bundles \( L_{w_c}^k \) from Remark 3.3 we start with some general statements.

Let \( V \) be a finite dimensional Euclidean vector space and denote the bilinear form in \( V \) by \( \langle , \rangle \). In what follows, we will freely identify \( V \) and \( V^* \) via the bilinear form \( \langle , \rangle \). Let \( M \) be some integer lattice in \( V \). Let us denote by \( M^* \subset V \) the dual
lattice corresponding to $M$. That is, $M^* = \{ \lambda \in V \mid \langle \alpha, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in M \}$. For $\mu \in M^*$ and $k \in \mathbb{N}$, define the theta function $\Theta_{\mu,k}$ on $V \otimes \mathbb{C}$ via

$$\Theta_{\mu,k}(v) = \exp(-\frac{1}{k} \pi i \tau \langle \mu, \mu \rangle) \sum_{\gamma \in \frac{1}{k} \mu + M} \exp(2\pi ik \tau \langle \gamma, \gamma \rangle).$$

This function converges absolutely on compact sets and satisfies the identity

$$\Theta_{\mu,k}(v + \beta + \tau \beta') = \exp(-2\pi ik \langle \beta', v \rangle - \pi i k \tau \langle \beta', \beta' \rangle) \Theta_{\mu,k}(v)$$

for any $\beta, \beta' \in \Lambda(T)$. To emphasize the dependence of $\Theta_{\mu,k}$ on $\tau$, we will write $\Theta_{\mu,k}(v; \tau)$.

Now suppose that $G$ is a simply connected Lie group and take $V_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}}$ endowed with the normalized Killing form $\langle \cdot, \cdot \rangle$. Since $\Lambda(T)$ is an integer lattice with respect to $\langle \cdot, \cdot \rangle$, we can take $M$ to be $\Lambda(T)$. It is clear from equation (22) that in this situation, the functions $\Theta_{\mu,k}(\cdot; \tau)$ define holomorphic sections of the line bundle $L^k$.

We can define the anti-invariant theta functions

$$A \Theta_{\mu,k} = \sum_{w \in W} (-1)^{l(w)} \Theta_{w\mu},$$

where $W$ the Weyl group of $G$, and $l(w)$ denotes the length of $w$ in $W$.

Let $a \subset \mathfrak{h}_{\mathbb{R}}$ be a fundamental alcove for the action of the affine Weyl group $\tilde{W} = W \ltimes \Lambda(T)$. Then the following result is due to [Lo].

**Proposition 5.1.** Let $k$ be a positive integer. The anti-invariant theta functions $A \Theta_{\mu,k}$ with $\mu \in \Lambda(T)^* \cap ka$ form a basis in $\Gamma(L^k)^{-W}$.

Let $h^\vee$ denote the dual Coxeter number of the root system $\Delta$ of $\mathfrak{g}$. As usual, we set $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$, where $\Delta_+$ denotes the set of positive roots of $\Delta$ with respect to the Weyl chamber containing the fundamental alcove $a$. For $\sigma_c = id$ and $h \in \mathfrak{h}$, the function $\mathcal{F} = \mathcal{F}_{id}$ which was defined in section 5.3 can be written as

$$\mathcal{F}_{id}(h - \tau D) = \exp(2\pi i \langle \rho, h \rangle) \prod_{n=1}^{\infty} (1 - q^n) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i \langle \alpha, h \rangle)) \prod_{\alpha \in \Delta} \prod_{n=1}^{\infty} (1 - q^n \exp(-2\pi i \langle \alpha, z \rangle)),$$

where, as before, $l$ denotes the rank of $\mathfrak{g}$. It is well known that $\mathcal{F}$ is a holomorphic, $W$ anti-invariant section of $L^{h^\vee}$ (see [Lo]).

If $G$ is of the form $G = \tilde{G}/\mathbb{Z}$, where $Z = \langle c \rangle$ is a cyclic subgroup of the center of the simply connected group $\tilde{G}$, we have to consider the lattice $\Lambda(T_{w_c}) \subset \mathfrak{h}^w$. Let $\langle \cdot, \cdot \rangle$ denote the normalized Killing form restricted to $\mathfrak{h}^w$. As before, let $k_b$ denote the basic level of $G$. A case by case check shows that $\Lambda(T_{w_c})$ is an integral lattice with respect to the bilinear form $k_b \langle \cdot, \cdot \rangle$. Denote by $\Lambda(T_{w_c})_0^* \subset \mathfrak{h}^w_{\mathbb{R}}$ the dual lattice of $\Lambda(T_{w_c})$ with respect to the bilinear form $k_b \langle \cdot, \cdot \rangle$. Then, for $\mu \in \Lambda(T_{w_c})_0^*$ and $k$ a multiple of $k_b$, the theta function $\Theta_{\mu,k}$ is a section of the line bundle $L^k_{w_c}$ which was defined in Remark 5.13. As before, we define the anti-invariant theta functions

$$A_0 \Theta_{\mu,k} = \sum_{w \in W_0} (-1)^{l(w)} \Theta_{w\mu},$$
where $W_0$ is the finite Weyl group introduced in Proposition 5.4 and $l(w)$ denotes the length of $w$ in $W_0$. Let $a_{w_0} \in h^\mathbb{R}_{>0}$ be a fundamental alcove for the action of the affine Weyl group $\tilde{W} = W_0 \ltimes \Lambda(T_{w_0})$. Then we have (see [K], Chapter 13):

**Proposition 5.2.** Let $k$ be a multiple of the basic level $k_0$ of $G$. The anti-invariant theta functions $A_0 \Theta_{\mu,k}$ with $\mu \in \Lambda(T_{w_0})^* \cap k a_{w_0}$ form a basis in $\Gamma(L^k_{w_0})^{-W_0}$.

Finally, the following proposition is straightforward to check.

**Proposition 5.3.** The function $F_{w_0}$ defines a $W_0$-anti-invariant holomorphic section of the line bundle $L^h_{w_0}$.

5.2. The Kac-Weyl character formula. Throughout this section let $G$ be simply connected. Let $\chi_{\lambda,k}$ denote the character of the $L(G)$-module $V_{\lambda,k}$ of highest weight $\lambda$ and level $k$. The goal of this section is to find an explicit formula for the character $\chi_{\lambda,k}$ viewed as a section of the family of line bundles $L^k$ as described in Section 5.1.

Since $F = F_{id}$ is a $W$-anti-invariant section of $L^h$, the product $\chi_{\lambda,k} F$ defines a $W$-anti-invariant section of the bundle $L^{(k + h)}$. By Proposition 5.1, we can write this product uniquely as

$$\chi_{\lambda,k}(h; \tau) F(h; \tau) = \sum_{\mu \in \Lambda(T)^* \cap (k + h)^* \mathfrak{a}} f_\mu(\tau) A_0 \Theta_{\mu,k + h}(h; \tau).$$

We can naturally identify the lattice $\Lambda(T)^*$, with the weight lattice $P$ of $g$. So the sum in equation (23) ranges over $\mu \in (k + h)^* \mathfrak{a}$.

Now we let $\tau$ vary in the upper half plane. In the situation at hand, the differential equation from theorem 5.1 reads

$$\left( -\frac{2(2k + h^\vee)}{2\pi i} \frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2} \Delta_h - \langle \rho, \rho \rangle \right) (\chi_{\lambda,k}(h, \tau) F(h, \tau)) = \langle \lambda, \lambda + 2\rho \rangle \chi_{\lambda,k}(h, \tau) F(h, \tau),$$

where $\Delta_h$ is the Laplacian on $h$.

Substituting equation (23) into differential equation (24) and keeping in mind that the $A_0 \Theta_{\mu,k}$ are linearly independent, we find

$$f_\mu(\tau) = a_\mu \exp \left( \frac{2\pi i \tau}{2(2k + h^\vee)} \langle \lambda + \rho, \lambda + \rho \rangle - \langle \mu, \mu \rangle \right),$$

where $a_\mu$ is a constant depending only on $\mu$. So we get

$$\chi_{\lambda,k}(h, \tau) F(h, \tau) = \sum_{\mu \in P \cap (k + h)^* \mathfrak{a}} a_\mu \exp \left( \frac{2\pi i \tau}{2(2k + h^\vee)} \langle \lambda + \rho, \lambda + \rho \rangle - \langle \mu, \mu \rangle \right) A_0 \Theta_{\mu,k + h}(h; \tau).$$

By definition of $\chi_{\lambda,k}$, we can write

$$\chi_{\lambda,k}(h, \tau) = \sum_{\tilde{\mu} \in P(\lambda,k)} \dim V_{\lambda,k}[\tilde{\mu}] \cdot q^{-D(\tilde{\mu})} \exp(2\pi i \langle \tilde{\mu}, h \rangle),$$

where $P(\lambda,k) \subset (h \oplus \mathbb{C}C \oplus \mathbb{C}D)^*$ denotes the set of weights of $V_{\lambda,k}$, and for $\tilde{\mu} \in P(\lambda,k)$, the space $V_{\lambda,k}[\tilde{\mu}]$ denotes the corresponding weight space. Since $V_{\lambda,k}$ is a highest weight module of highest weight $(\lambda,k,0)$, we know that for all weights $\tilde{\mu} \in P(\lambda,k)$, the difference $(\lambda,k,0) - \tilde{\mu}$ has to be a sum of positive roots of $\tilde{L}(g)$.
For any $\mu u \in P$, let us set

$$\tilde{\mu} = \left(\mu, k + h^\vee, \frac{\|\lambda + \rho\|^2 - \|\mu\|^2}{2(k + h^\vee)}\right)$$

It follows from equation (27) that, whenever $a_\mu \neq 0$ in equation (26), then $(\lambda + \rho, k + h^\vee, 0) - \tilde{\mu}$ has to be a sum of positive roots of $\hat{L}(g)$. Furthermore, we have $\|\tilde{\mu}\|^2 = \|(\lambda + \rho, k + h^\vee, 0)\|^2$ and $\langle \tilde{\alpha}, \tilde{\mu} \rangle \geq 0$ for all $\tilde{\alpha} \in \tilde{\Pi}$. Together, these observations imply that $\tilde{\mu} = (\lambda + \rho, k + h^\vee, 0)$ (see e.g. [PS], Lemma 14.4.7).

Putting everything together, we have proved the following theorem.

**Theorem 5.4 (Kac-Weyl character formula).** The character $\chi_{\lambda,k}$ of the integrable highest weight module $V_{\lambda,k}$ of highest weight $\lambda$ and level $k$ at the point $(h, \tau) \in \mathfrak{h} \times \mathbb{H}$, where $\mathbb{H}$ denotes the upper half plane in $\mathbb{C}$, is given by

$$\chi_{\lambda,k}(h; \tau) = \frac{\mathcal{A}_h(\lambda + \rho, k + h^\vee (h, \tau))}{\mathcal{F}(h, \tau)}.$$

### 5.3. Characters for non-connected loop groups

In this section we want to derive an analogue of the Kac-Weyl character formula in the case that $G$ not simply connected. As always, let $G = G/Z$ where $Z = \langle c \rangle$ is a cyclic subgroup of the center of the simply connected Lie group $\tilde{G}$. Let $k$ be a multiple of the basic level of $G$, and let $\lambda \in P_+^k$ be invariant under the action of $\sigma_c$ on $P_+^k$. Let $\chi_{\lambda,k}$ denote character of the $\hat{L}(G)$–module $V_{\lambda,k}$. One checks directly, that under the action of $\sigma_c$ on $V_{\lambda,k}$, the weight space $V_{\lambda,k}[\tilde{\mu}]$ is mapped to $V_{\lambda,k}[\sigma_c(\tilde{\mu})]$. Recall the definition of $H_0$ from the beginning of section 4.3. By definition of the character $\chi_{\lambda,k}^{\sigma_c}$ (see section 4.1), we can write

$$\chi_{\lambda,k}^{\sigma_c}(H_0) = \sum_{\tilde{\mu} \in \mathcal{P}(\lambda,k)} \text{Tr}(\sigma_c)|_{V_{\lambda,k}[\tilde{\mu}]} e^{2\pi i \langle \tilde{\mu}, H_0 \rangle}.$$  

On the other hand, $W_0$–anti-invariance of $\chi_{\lambda,k}^{\sigma_c}$ and the differential equation from Theorem 4.1 together with Proposition 5.2 imply that we can write

$$\chi_{\lambda,k}^{\sigma_c}(H_0)\mathcal{F}_{w_c}(H_0) = \sum_{\mu \in \Lambda(T_{w_c})} a_\mu f_\mu \mathcal{A}_0(\theta_{\mu, k + h^\vee})$$

where $a_\mu$ is a constant depending only on $\mu$, and

$$f_\mu = \exp\left(\frac{2\pi i \tau}{2(k + h^\vee)}\langle \lambda + \rho_{w_c}, \lambda + \rho_{w_c} \rangle - \langle \mu, \mu \rangle\right).$$

Let us set

$$\tilde{\mu} = \left(\mu, k + h^\vee, \frac{\|\lambda + \rho_{w_c}\|^2 - \|\mu\|^2}{2(k + h^\vee)}\right)$$

Similar to the simply connected case, it follows from equations (28) and (29) that whenever $a_\mu \neq 0$ in equation (29), then $(\lambda + \rho_{w_c}, k + h^\vee, 0) - \tilde{\mu}$ has to be a $\sigma_c$-invariant sum of positive roots of $\hat{L}(g)$. Furthermore, we have

$$\|(\lambda + \rho_{w_c}, k + h^\vee, 0)\|^2 = \|\tilde{\mu}\|^2$$

and $\langle \tilde{\alpha}, \tilde{\mu} \rangle \geq 0$ for all $\tilde{\alpha} \in \tilde{\Pi}$. As in the simply connected case, these observations imply $\lambda + \rho_{w_c} = \mu$. So we have proved the following theorem.
Theorem 5.5. The character $\chi_{\lambda,k}$ of the integrable highest weight module $V_{\lambda,k}$ of highest weight $\lambda$ and level $k$ at the point $(\sigma_cH_0)$ is given by

$$\chi_{\lambda,k}^{\sigma_c}(H_0) = \frac{A_0\Theta_{\lambda+\rho_c,k+h^*(H_0)}}{F_{w_c}(H_0)}.$$ 

5.4. The root system $\tilde{\Delta}_{\sigma_c}$. Finally, let us take a closer look at the denominator $F_{w_c}(H_0)$ appearing in Theorem 5.5. We have to distinguish two cases. First, let us suppose that for any simple root $\tilde{\alpha} \in \tilde{\Pi}$, the roots $\tilde{\alpha}$ and $\sigma_c(\tilde{\alpha})$ are not connected in the Dynkin diagram of $\Delta$. In this case, one can easily show that $\sigma_c(e_{\tilde{\alpha}}) = e_{\sigma_c(\tilde{\alpha})}$, so that $s(\tilde{\alpha}) = 1$ for all real roots $\alpha \in \tilde{\Delta}^c$. For any root $\tilde{\alpha} \in \Delta$ let us denote by $\tilde{\alpha}_{\sigma_c}$ its restriction to the subspace $\tilde{\mathfrak{h}}^{\sigma_c} \subset \tilde{\mathfrak{h}}$. Then the set

$$\{m\tilde{\alpha}_{\sigma_c} | \tilde{\alpha} \in \tilde{\Delta}^c\}$$

is the set of real roots of an affine root system which we will denote by $\tilde{\Delta}_{\sigma_c}$.

Now suppose that there exists some $\tilde{\alpha} \in \Delta$ such that $\tilde{\alpha} \neq \sigma_c(\tilde{\alpha})$ are not orthogonal. Since we have excluded the case that $\sigma_c$ is the order $n + 1$ automorphism of the extended Dynkin diagram of $A_n$, we can assume that $\sigma_c^2(\tilde{\alpha}) = \tilde{\alpha}$. In this case, one can show that

$$\sigma_c(e_{\tilde{\alpha}}) = -1^{\text{ht}(\tilde{\alpha})+1}e_{\sigma_c(\tilde{\alpha})},$$

where $\text{ht}(\tilde{\alpha})$ denotes the height of the root $\tilde{\alpha}$ with respect to the basis $\tilde{\Pi}$ (see e.g. [K], 7.10.1 for the case of finite root systems). Looking at the explicit form of the automorphism $\sigma_c$, we see that if there exists a simple root $\tilde{\alpha}$ such that $\tilde{\alpha} \neq \sigma_c(\tilde{\alpha})$ are not orthogonal, then we have $\tilde{\alpha} \neq \sigma_c(\tilde{\alpha})$ for all simple roots of $\Delta$. Therefore, if $\sigma_c(\tilde{\alpha}) = \tilde{\alpha}$, then there exists some $\tilde{\beta} \in \Delta$ with $\sigma_c(\tilde{\beta}) \neq \tilde{\beta}$ and $\tilde{\beta} + \sigma_c(\tilde{\beta}) = \tilde{\alpha}$. Hence, in this case $\text{ht}(\tilde{\alpha})$ is necessarily even so that $s(\tilde{\alpha}) = -1$ whenever $\sigma_c(\tilde{\alpha}) = \tilde{\alpha}$. As above, the set

$$\{m\tilde{\alpha}_{\sigma_c} | \tilde{\alpha} \in \tilde{\Delta}, \sigma_c(\tilde{\alpha}) \neq \tilde{\alpha}, \text{ and } \tilde{\alpha} \text{ and } \sigma_c(\tilde{\alpha}) \text{ are orthogonal}\} \cup$$

$$\{2m\tilde{\alpha}_{\sigma_c} | \tilde{\alpha} \in \tilde{\Delta}, \sigma_c(\tilde{\alpha}) \neq \tilde{\alpha}, \text{ and } \tilde{\alpha} \text{ and } \sigma_c(\tilde{\alpha}) \text{ are not orthogonal}\}$$

is the set of real roots of an affine root system which we also denote by $\tilde{\Delta}_{\sigma_c}$. In both cases, the smallest positive imaginary root of $\tilde{\Delta}_{\sigma_c}$ is given by $\delta_{\sigma_c} = \text{ord}(\sigma_c)\delta$, where $\delta$ denotes the smallest positive imaginary root of $\Delta$. We will list the types of $\tilde{\Delta}_{\sigma_c}$ in the end of this paper.

Now, using $(1+x)(1-x) = (1-x^2)$, we can write

$$\prod_{[\tilde{\alpha}] \in (\tilde{\Delta}^c^*^c)_{\sigma_c}} (1 - s(\tilde{\alpha})e^{-2\pi im_{\tilde{\alpha}}^{\sigma_c}(H_0)}) = \prod_{\tilde{\alpha} \in \tilde{\Delta}^c_{\sigma_c}} (1 - e^{-2\pi i\tilde{\alpha}(H_0)})$$

From this, we see that up to the factor

$$f(q) = \frac{\prod_{n=1}^{\infty} \prod_{j=1}^{\text{ord}(w_c)} (1 - e^{-j\pi}(\pi)^{\dim(\mathfrak{h}_j)})}{\prod_{n=1}^{\infty} (1 - q^{\text{ord}(\sigma_c)n}\text{mult}(n\delta_{\sigma_c}))},$$

the function $F_{w_c}$ can be identified with the Kac-Weyl denominator corresponding to the affine root system $\tilde{\Delta}_{\sigma_c}$. Similarly, the group $\tilde{W}_{\sigma_c} = W_0 \times \Lambda(T_{w_c})$ is isomorphic to the Weyl group of the root system $\tilde{\Delta}_{\sigma_c}$. So the character $\chi_{\lambda,k}$ closely resembles a character of an irreducible highest weight module of an affine Lie algebra corresponding to the root system $\tilde{\Delta}_{\sigma_c}$.
Remark 5.6. Since we are mainly interested in the case of loop groups based on connected but not necessarily simply connected Lie groups $G$, we restricted our attention to automorphisms $\sigma_c$ of the extended Dynkin diagram of $\Delta$ which are associated to elements of the center of the universal cover $\tilde{G}$ of $G$. The arguments in this section can be extended to the case of characters of loop groups based on non-connected Lie groups. In this case, one has to consider the full automorphism group of the extended Dynkin diagram of $\Delta$.

Remark 5.7. If $\tilde{\Delta}$ is the root system of an affine Lie algebra $\tilde{L}(g)$, and $\sigma$ is an automorphism of the Dynkin diagram of $\tilde{\Delta}$ then the affine Lie algebra corresponding to the root system $\tilde{\Delta}_\sigma$ was also realized in [FRS] [FSS], where for an outer automorphism $\sigma$ of a (generalized) Kac-Moody Lie algebra $g$, the $\sigma$-twisted characters of highest weight representations are calculated. It turns out that these twisted characters can be identified with untwisted characters of the orbit Lie algebra corresponding to $g$ and $\sigma$.

6. Appendix: Some data on affine root systems and their automorphisms

The following table lists some data corresponding to the non-simply connected Lie groups. The basic levels of non-simply connected Lie groups have been calculated in [T]. See also [FSS] for a list of the root systems $\tilde{\Delta}_\sigma$ for general automorphisms $\sigma$ of the Dynkin diagram of $\tilde{\Delta}$. The notation for affine root systems in the table below is the same as in [K].

| $G$       | $\langle c \rangle$ | $G = \tilde{G}/\langle c \rangle$ | $k_b$ | $\Delta$       | $\Delta_{\sigma_c}$ |
|-----------|----------------------|-----------------------------------|-------|----------------|----------------------|
| $\text{SL}_n$ $n \geq 2$ | $\mathbb{Z}_r$ | smallest $k$ with $\frac{n(n-1)}{r}k \in \mathbb{Z}$ | $A_{n-1}^{(1)}$ | $A_{n/r-1}^{(1)}$ if $r \neq n$, $\emptyset$ if $r = n$ |
| $\text{Spin}_{2n+1}$ $n \geq 2$ | $\mathbb{Z}_2$ | $SO_{2n+1}$ | $1$ | $B_n^{(1)}$ | $A_{2n-1}^{(2)}$ |
| $\text{Spin}_{4n}$ $n \geq 1$ | $\mathbb{Z}_2$ | $B_n^{(1)}$ | $A_{2n}^{(2)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 1$ | $\mathbb{Z}_2$ | $SO_{2n+1}$ | $2$ | $C_{2n+1}^{(1)}$ | $C_n^{(1)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 2$ | $\mathbb{Z}_2$ | $SO_{4n}$ | $1$ | $D_{2n}^{(1)}$ | $C_{2n-2}^{(1)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 2$ | $\mathbb{Z}_{2}^{2}$ | $\mathbb{Z}_{2}^{2}$ | $1$ if $n$ even, $2$ if $n$ odd | $D_{2n}^{(1)}$ | $B_n^{(1)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 2$ | $\mathbb{Z}_{2}^{4}$ | $\mathbb{Z}_{4}$ | $4$ | $D_{2n+1}^{(1)}$ | $C_{2n-1}^{(1)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 2$ | $\mathbb{Z}_{4}$ | $PSO_{4n+2}$ | $3$ | $E_{6}^{(1)}$ | $C_n^{(1)}$ |
| $\text{Spin}_{4n+2}$ $n \geq 2$ | $\mathbb{Z}_{2}^{2}$ | $2$ | $E_{7}^{(1)}$ | $F_4^{(1)}$ |

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