Analysis of Flexible Bars and Frames with Large Displacements of Nodes By Finite Element Method in the Form of Classical Mixed Method

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Abstract. This article is the continuation of the work made bt the authors on the development of the algorithms that implement the finite element method in the form of a classical mixed method for the analysis of geometrically nonlinear bar systems [1-3]. The paper describes an improved algorithm of the formation of the nonlinear governing equations system for flexible plane frames and bars with large displacements of nodes based on the finite element method in a mixed classical form and the use of the procedure of step-by-step loading. An example of the analysis is given.

1. Introduction
The issue of numerical solution of geometrically nonlinear problems belonging to the analysis of various types of structures is a subject of many works [4,5]. Almost in all the works, the solutions are based on the use of the finite element method in displacements.

One of the methods of the reliable verification of these solutions is the FEM in the form of classical mixed method, which is developed by the authors.

2. Construction of the system of governing equations
The algorithm of the analysis through the given method is demonstrated by the example of a frame of polygonal shape (figure 1,a). The initial (original) geometry for the \(i\)\(^{th}\) and \((i+1)^{th}\) finite elements of the frame are shown in figure 1,b.

![Frame of polygonal shape](image1.png)

**Figure 1.** (a) frame of polygonal shape; (b) \(i^{th}\) and \((i+1)^{th}\) finite elements of the frame.
Figure 2 shows the basic system of the mixed method for these finite elements for the nonlinear statement analysis at the first step of loading $\Delta P^{(1)}_i = P_i/k$, where $k$ - is the number of stages of additional loading set by the conditions of the linear work of a finite element at each step of additional loading. In this case, the load can be either in the form of concentrated forces $P_{ix}$, $P_{iy}$, or as the form of concentrated moment $M_i = P_{ix,i}$.

![Diagram](image)

**Figure 2.** Basic system of the mixed method for $i^{th}$ and $(i+1)^{th}$ finite elements.

At the first step of additional loading we get the following system of governing equations in the linear statement for the load $P^{(1)}_i = \Delta P_i$:

1. \[ R_{ix,j} = q_{ix,j} \cos \alpha_j + q_{ib,j} \sin \alpha_j - q_{iy,j+1} \cos \alpha_{j+1} - q_{ib,j+1} \sin \alpha_{j+1} - \Delta P_{ix,j} = 0, \]
2. \[ R_{iy,j} = q_{iy,j} \cos \alpha_j + q_{ib,j} \sin \alpha_j - q_{ix,j+1} \cos \alpha_{j+1} + q_{ib,j+1} + \Delta P_{iy,j} = 0, \]
3. \[ R_{ij,j} = -q_{iy,j} \sin \alpha_j + q_{ix,j+1} \cos \alpha_{j+1} - q_{ib,j} - q_{ib,j+1} + \Delta P_{ij,j} = 0 \]

- the equilibrium equations for the node $i$;

4. \[ \Delta q_{ix,i} = (q_{ix,i} - q_{ix,i+1}) \cos \alpha_i + (q_{ix,i+1} - q_{ny,i}) \frac{l_i}{2} + (q_{ib,i} - q_{ib,i+1}) \sin \alpha_i + q_{ix,i} \frac{l_i}{12EI_i} + \delta_{p,i} = 0, \]
5. \[ \Delta q_{iy,i} = -q_{iy,i+1} + q_{ny,i} \frac{l_i}{EI_i} + \delta_{p,i} = 0, \]
6. \[ \Delta q_{ij,i} = (q_{ix,i} - q_{ix,i+1}) \sin \alpha_i + (q_{ix,i+1} - q_{ny,i}) \cos \alpha_i + q_{ix,i} \frac{l_i}{EF_i} + \delta_{p,i} = 0 \]

- the strain compatibility equations at the cross-section of $i^{th}$ finite element.

As a result of the solution of the linear equations system, the values $q_{ix,i}^{(1)}, q_{ix,i+1}^{(1)}, ..., q_{nx,i}^{(1)}$ after the first step of additional loading will be found, and the geometry of the stained structure will be determined on their basis. This state of strain (figure 3) is the initial one for the second step of additional loading.

To form the governing equations at this step of additional loading, it is necessary to know the coordinates of the point $C$ and the angle of the cross-section of the FE at this point (figure 4).
In the linear statement, we find:

$$\varphi_{x,i}^{\text{new}} = q_{x,i}^{(1)} - \frac{l_i^2}{8EI} + q_{k,i}^{(1)} \frac{l_i}{2EI}, \quad \varphi_{y,i}^{\text{new}} = q_{y,i}^{(1)} - \frac{l_i^2}{8EI} + q_{k,i}^{(1)} \frac{l_i}{2EI}. \tag{3}$$

In the state of strain (figure 3)

$$\varphi_{x,i}^{\text{new}} = \alpha_i + q_{x,i}^{(1)}, \quad \varphi_{y,i}^{\text{new}} = \alpha_i + q_{y,i}^{(1)}, \quad x_{i,x}(i+1) = q_{x,i}^{(1)} + \Delta x_{i,x}(i+1), \quad x_{i,y}(i+1) = q_{y,i}^{(1)} + \Delta x_{i,y}(i+1). \tag{3a}$$

For this state of strain, let us write the refined nonlinear governing equations of equilibrium and strain compatibility:

1) $\Delta P_{x,i}^{(1)} = \left(q_{x,i}^{(1)} + \Delta q_{x,i}^{(1)}\right) \cdot \cos \left[\varphi_{x,i}^{\text{new}} + \varphi_{x,i}^{\text{old}} + \Delta \varphi_{x,i}^{\text{new}}\right] - \left(q_{x,i+1}^{(1)} + \Delta q_{x,i+1}^{(1)}\right) \cdot \cos \left[\varphi_{x,i+1}^{\text{new}} + \varphi_{x,i+1}^{\text{old}} + \Delta \varphi_{x,i+1}^{\text{new}}\right] + \left(q_{x,i}^{(1)} + \Delta q_{x,i}^{(1)}\right) \cdot \sin \left[\varphi_{x,i}^{\text{new}} + \varphi_{x,i}^{\text{old}} + \Delta \varphi_{x,i}^{\text{new}}\right] - \left(q_{x,i+1}^{(1)} + \Delta q_{x,i+1}^{(1)}\right) \cdot \sin \left[\varphi_{x,i+1}^{\text{new}} + \varphi_{x,i+1}^{\text{old}} + \Delta \varphi_{x,i+1}^{\text{new}}\right] = -\Delta P_{x,i}^{(1)} = 0,$$

2) $\Delta P_{y,i}^{(1)} = \left(q_{y,i}^{(1)} + \Delta q_{y,i}^{(1)}\right) \cdot \cos \left[\varphi_{y,i}^{\text{new}} + \varphi_{y,i}^{\text{old}} + \Delta \varphi_{y,i}^{\text{new}}\right] - \left(q_{y,i+1}^{(1)} + \Delta q_{y,i+1}^{(1)}\right) \cdot \cos \left[\varphi_{y,i+1}^{\text{new}} + \varphi_{y,i+1}^{\text{old}} + \Delta \varphi_{y,i+1}^{\text{new}}\right] + \left(q_{y,i}^{(1)} + \Delta q_{y,i}^{(1)}\right) \cdot \sin \left[\varphi_{y,i}^{\text{new}} + \varphi_{y,i}^{\text{old}} + \Delta \varphi_{y,i}^{\text{new}}\right] - \left(q_{y,i+1}^{(1)} + \Delta q_{y,i+1}^{(1)}\right) \cdot \sin \left[\varphi_{y,i+1}^{\text{new}} + \varphi_{y,i+1}^{\text{old}} + \Delta \varphi_{y,i+1}^{\text{new}}\right] = -\Delta P_{y,i}^{(1)} = 0,$$

3) $\Delta P_{x,i}^{(1)} = \left(q_{x,i}^{(1)} + \Delta q_{x,i}^{(1)}\right) \cdot \cos \left[\varphi_{x,i}^{\text{new}} + \varphi_{x,i}^{\text{old}} + \Delta \varphi_{x,i}^{\text{new}}\right] - \left(q_{x,i+1}^{(1)} + \Delta q_{x,i+1}^{(1)}\right) \cdot \cos \left[\varphi_{x,i+1}^{\text{new}} + \varphi_{x,i+1}^{\text{old}} + \Delta \varphi_{x,i+1}^{\text{new}}\right] + \left(q_{x,i}^{(1)} + \Delta q_{x,i}^{(1)}\right) \cdot \sin \left[\varphi_{x,i}^{\text{new}} + \varphi_{x,i}^{\text{old}} + \Delta \varphi_{x,i}^{\text{new}}\right] - \left(q_{x,i+1}^{(1)} + \Delta q_{x,i+1}^{(1)}\right) \cdot \sin \left[\varphi_{x,i+1}^{\text{new}} + \varphi_{x,i+1}^{\text{old}} + \Delta \varphi_{x,i+1}^{\text{new}}\right] = -\Delta P_{x,i}^{(1)} = 0,$$

- the equilibrium equations of the node $i$;
4) \( \Delta q^{(i)}_{2,j} = \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \cos \left[ \varphi_{i}^{(p)}(1) + \varphi_{r,j}^{(p)} + \Delta \varphi_{r,j}^{(p)} \right] - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \cos \left[ \varphi_{i}^{(p)}(1) + \varphi_{r,j}^{(p)} + \Delta \varphi_{r,j}^{(p)} \right] + \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{2} \cdot \cos \varphi_{r,j}^{(p)} - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \cos \varphi_{r,j}^{(p)} - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{12 E_I} + \delta_{s,j}^{(i)} = 0, \)

5) \( \Delta q^{(i)}_{3,j} = -\left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \varphi_{r,j}^{(p)} - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{4 E_I} + \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{E_I} + \delta_{s,j}^{(i)} = 0, \)

6) \( \Delta q^{(i)}_{4,j} = \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \sin \left[ \varphi_{i}^{(p)}(1) + \varphi_{r,j}^{(p)} + \Delta \varphi_{r,j}^{(p)} \right] - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \sin \left[ \varphi_{i}^{(p)}(1) + \varphi_{r,j}^{(p)} + \Delta \varphi_{r,j}^{(p)} \right] - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{2} \left[ 1 - \cos \left( \varphi_{i}^{(p)}(1) \right) \right] - \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{2} \left[ 1 - \cos \left( \varphi_{i}^{(p)}(1) \right) \right] + \left( q^{(i)}_{2,j} + \Delta q^{(i)}_{2,j} \right) \cdot \frac{l}{E_I} + \delta_{s,j}^{(i)} = 0 \)

- the strain compatibility equations at the cross-section of \( i \) th finite element.

In this system of equations, the unknowns are the incremental values \( \Delta q^{(i)}_i \) for the \( q_i \) found through the linear analysis.

Let us assign the system of equations (1) and (2) as the linear operator

\[
L_1 \left( l_i, E, E_F, n, q_i^{(1)}, \Delta \varphi_i^{(1)} \right) = 0,
\]

and the system of equations (4) and (5) as the nonlinear operator

\[
L_2 \left( l_i, E, E_F, n, q_i^{(1)} + \Delta q_i^{(1)}, \Delta \varphi_i^{(1)} \right) = 0.
\]

The difference of the operators \( L_2 - L_1 = L_3 \left( \Delta q_i^{(1)} \right) \) gives the operator \( L_3 \) in the form of a system of nonlinear equations for the increments \( \Delta q_i^{(1)} \).

Solving the nonlinear system of equations (4,5) is possible only through iteration in the linearized form at each iteration. For the linearization, the most convenient way is the expansion of trigonometric functions of \( \sin (\varphi + \Delta \varphi) \), \( \cos (\varphi + \Delta \varphi) \) in a Taylor series [6]:

\[
\sin (\varphi + \Delta \varphi) = \Delta \varphi - \frac{\Delta \varphi^3}{3!} + \varphi \cdot \left( 1 - \frac{\Delta \varphi^2}{2!} + \frac{\varphi^3}{3!} \right) - \frac{\varphi^3}{3!} \left( 1 - \frac{\varphi^2}{2!} \right) + ..., \]

\[
\cos (\varphi + \Delta \varphi) = 1 - \frac{\Delta \varphi^2}{2!} - \varphi \cdot \left( \Delta \varphi - \frac{\Delta \varphi^3}{3!} - \frac{\varphi^3}{3!} \left( 1 - \frac{\varphi^2}{2!} \right) + \frac{\varphi^3}{3!} \right) + ... .
\]

Saving only the quantities of the first order of smallness in these expressions, we get:

\[
\sin (\varphi + \Delta \varphi) \approx \frac{\varphi^3}{3!} + \Delta \varphi \cdot \left( 1 + \frac{\varphi^3}{2!} \right), \quad \cos (\varphi + \Delta \varphi) \approx \frac{\varphi^3}{3!} - \Delta \varphi \cdot \left( \frac{\varphi^3}{2!} \right) .
\]

The results of the solution of the equations system (4,5) linearized in this way are the incremental values of the parameters of the stress-strain state of the structure, i.e. \( \Delta q_i^{(1)} \) at the first iteration, as well as the refined values

\[
q_i^{(2)} = q_i^{(1)} + \Delta q_i^{(1)} .
\]

The substitution of these values into the nonlinear operator (7) gives residuals \( R_i^{(1)} \).

The further iterative process is described by the authors in [1].
In accordance with it, refining corrections are inserted into the operator $L^{(2)}$:

$$L^{(2)}(l, EI, EF, n, \{q_{i}^{(2)} + \Delta q_{i}^{(2)}\}, \Delta P^{(i)}) = 0.$$ (10)

The linearization of this operator is carried out in the same way as at the first iteration. The results of the solution give new corrections $\Delta q_{i}^{(2)}$ and new values

$$q_{i}^{(3)} = q_{i}^{(2)} + \Delta q_{i}^{(2)}.$$

The iterations stop when the required degree of accuracy of the analysis is achieved.

After that, one should proceed to the second step of loading with the same nodal load $P^{(II)} = \Delta P^{I}$ since the geometry of the structure after the first step of loading is original and the structure in this condition before loading $P^{(II)}$ is non-loaded.

The formulation of the systems of the governing equations (1,2) and (4,5) is performed through the substitution of $q_{i}^{I}$ as the known quantities found after the first step of loading into them. The iterative process at this step of loading is performed by the same algorithm as at the first step.

Summing up the results of the first and the second steps of loading, we will get the original geometry for the third step of loading, etc.

3. Test example

To illustrate the algorithm, let us consider the analysis of a flexible cantilever beam loaded by the concentrated force $P$ at the end (figure 5).

![Flexible cantilever beam loaded by the concentrated force $P$ at the end.](image)

Figure 5. Flexible cantilever beam loaded by the concentrated force $P$ at the end.

Beam parameter: $L=10\, m$, $EI=100\, t\cdot m^2$, $EF=10^8\, t$, $P=4t$.

The solution to this problem is performed by finite element method in the form of classical mixed method for the above described algorithm. The solution results and compare them with analytical [7] are given in table 1. In fig.6 shows a deformed state of the beam

| Parameter | Classical mixed FEM analysis | Analytical solution | Accuracy, % |
|-----------|-----------------------------|---------------------|-------------|
| $\Delta x_{(i-1)} \cdot m$ | -3.25 | -3.29 | 1.22 |
| $\Delta y_{(i-1)} \cdot m$ | -6.77 | -6.70 | 1.04 |
| $M_{(i-0)} \cdot t \cdot m$ | 26.8 | 26.8 | 0 |
4. Conclusions
Finite element method in the form of classical mixed method is an alternative to the traditional FEM in displacements. It allows you to perform verifizierung results obtained by other methods.

In the considered nonlinear problem it has the advantage of simultaneously obtaining both displacements and efforts in the nodal points of the element mesh with the same degree of accuracy as in the traditional FEM.

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References
[1] Ignatyev V A, Ignatyev A V, Galishnikova V V and Onishchenko E V 2014 Nonlinear structural mechanics of bar systems. Theoretical fundamentals. Analysis cases (Volgograd: VolgGASU Publ.)
[2] Ignatyev V A, Ignatyev A V and Zhidelev A V 2006 Mixed Form of Finite Element Method in Problems of Structural Mechanics (Volgograd: VolgGASU Publ.)
[3] Ignatyev V A, Ignatyev A V and Onishchenko E V 2016 Analysis of Systems with Unilateral Constraints through the Finite Element Method in the Form of a Classical Mixed Method Procedia Engineering vol 150 pp 1754–59
[4] 2015 Verification report on the LIRA-SAPR software package (Moscow) pp 161–4
[5] Hmyrov A F, Kokorev G V, Sukum S E, Simakov S N and Kasatkina L I 1984 Discrete method of analysis of the forms of equilibrium of a flexible cantilever beam under the action of arbitrary loads Research on construction mechanics of structures. Collection of scientific papers (Voronezh)
[6] Ignatiev V A 1973 Calculation regular of rod systems (Saratov) pp 84–92
[7] Landau L D and Lifshitz E M 1987 Theory of elasticity (Moscow: Nauka) p 106
[8] Petrov V V 1975 Method of successive loadings in nonlinear theory of plates and shells (Saratov: SSTU Publ.)
[9] Scott Michel H and Filippou Filip C 2007 Response Gradients for Nonlinear Beam-Column Elements under Large Displacements Journal of structural Engineering pp 155–165
[10] Petrov V V 2014 Incremental non-linear structural mechanics (Moscow: Infra - Engineering)