Non-trivial $t$-intersecting families for vector spaces

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Abstract

Let $V$ be an $n$-dimensional vector space over a finite field $\mathbb{F}_q$. In this paper we describe the structure of maximal non-trivial $t$-intersecting families of $k$-dimensional subspaces of $V$ with large size. We also determine the non-trivial $t$-intersecting families with maximum size. In the special case when $t = 1$ our result gives rise to the well-known Hilton-Milner Theorem for vector spaces.

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1 Introduction

The study of intersecting families has long been an important area of research in combinatorics [6, 14] ever since the birth of the celebrated Erdős-Ko-Rado Theorem [7]. In this paper we give a description of the structure of maximal non-trivial $t$-intersecting families of $k$-subspaces of an $n$-dimensional vector space over a finite field whose size is a bit smaller than the bound in the Erdős-Ko-Rado Theorem for vector spaces. In particular, we extend the Hilton-Milner Theorem for vector spaces [4] by describing the structure of non-trivial $t$-intersecting families of vector spaces with maximum size.

Let $n$ and $k$ be integers with $1 \leq k \leq n$. Write $[n] = \{1, 2, \ldots, n\}$ and denote by $\binom{[n]}{k}$ the family of all $k$-subsets of $[n]$. For any positive integer $t$, a family $\mathcal{F} \subseteq \binom{[n]}{k}$ is said to be $t$-intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{F}$. A family is called intersecting if it is 1-intersecting. A $t$-intersecting family is called trivial if all its members contain a common specified $t$-subset of $[n]$, and non-trivial otherwise.

The Erdős-Ko-Rado Theorem gives the maximum size of a $t$-intersecting family and shows further that any $t$-intersecting family with maximum size is a trivial family consisting
of all \( k \)-subsets that contain a fixed \( t \)-subset of \([n]\) for \( n > (t + 1)(k - t + 1) \) \([7, 8, 23]\). In \([2, 10]\), the structure of such extremal families for any positive integers \( t, k \) and \( n \) was described. Determining the structure of non-trivial \( t \)-intersecting families of \( k \)-subsets of \([n]\) with maximum size was a long-standing problem. The first such result is the Hilton-Milner Theorem \([16]\) which describes the structure of such families for \( t = 1 \). A complete solution to this problem for any \( t \) was obtained by Ahlsvede and Khachatrian \([1]\). Recently, other maximal non-trivial intersecting families with large size have been studied. For example, Kostochka and Mubayi \([19]\) described the structure of intersecting families of \( k \)-subsets of \([n]\) whose size is quite a bit smaller than the bound \( \binom{n}{k} - 1 \) given by the Erdős-Ko-Rado Theorem. In \([15]\), Han and Kohayakawa determined the maximum size of an intersecting family which is not a subfamily of any largest or second largest maximal intersecting family, and characterized all families achieving that extremal value.

The Erdős-Ko-Rado Theorem and the Hilton-Milner Theorem for finite sets have natural extensions to vector spaces. Let \( n \) and \( k \) be integers with \( 1 \leq k \leq n \), and \( V \) an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_q \), where \( q \) is necessarily a prime power. We use \( \binom{V}{k} \) to denote the family of all \( k \)-dimensional subspaces of \( V \). In the sequel we will abbreviate “\( k \)-dimensional subspace” to “\( k \)-subspace”. Recall that for any positive integers \( a \) and \( b \) the Gaussian binomial coefficient is defined by

\[
\binom{a}{b} = \prod_{0 \leq i < b} q^{a - i} - 1.
\]

In addition, we set \( \binom{a}{0} = 1 \) and \( \binom{a}{-c} = 0 \) if \( c \) is a negative integer. It is well known that the size of \( \binom{V}{k} \) is equal to \( \binom{n}{k} \).

For any positive integer \( t \), a family \( \mathcal{F} \subseteq \binom{V}{k} \) is called \( t \)-intersecting if \( \dim(A \cap B) \geq t \) for all \( A, B \in \mathcal{F} \). A family is called intersecting if it is 1-intersecting. A \( t \)-intersecting family \( \mathcal{F} \subseteq \binom{V}{k} \) is called trivial if all its members contain a common specified \( t \)-subspace of \( V \) and non-trivial otherwise. In general, the triviality of an intersecting family is determined by the following parameter introduced in \([4]\): For any \( \mathcal{F} \subseteq \binom{V}{k} \), the covering number \( \tau(\mathcal{F}) \) of \( \mathcal{F} \) is the minimum dimension of a subspace \( T \) of \( V \) such that \( \dim(T \cap F) \geq 1 \) for every \( F \in \mathcal{F} \). It is clear that an intersecting family \( \mathcal{F} \) is trivial if and only if \( \tau(\mathcal{F}) = 1 \).

Let \( n, k \) and \( t \) be positive integers with \( n \geq 2k \geq 2t \), and \( \mathcal{F} \subseteq \binom{V}{k} \) a \( t \)-intersecting family with maximum size. The Erdős-Ko-Rado Theorem for vector spaces shows that \( \mathcal{F} \) must be a trivial family consisting of all \( k \)-subspaces of \( V \) which contain a fixed \( t \)-subspace of \( V \), or \( n = 2k \) and \( \mathcal{F} \) consists of all \( k \)-subspaces of a fixed \((n-t)\)-subspace of \( V \) \([6, 12, 17, 20]\). Using the covering number, Blokhuis et al. \([4]\) obtained a vector space version of the Hilton-Milner Theorem, which described the structure of any non-trivial intersecting family with maximum size.

In this paper we study maximal non-trivial \( t \)-intersecting families of \( k \)-subspaces of \( V \) for any positive integer \( t \). By \([3, \text{Remark (ii) in Section 9.3}]\) any maximal non-trivial \((k - 1)\)-intersecting family of \( k \)-subspaces of \( V \) is the collection of all \( k \)-subspaces contained in a fixed \((k+1)\)-subspace of \( V \). Henceforth we will only consider the case when \( 6 \leq 2k \leq n \) and \( 1 \leq t \leq k - 2 \).

To present our results let us first introduce the following three constructions of \( t \)-intersecting families of \( k \)-subspaces of \( V \).

**Family 1.** Let \( X \) and \( M \) be subspaces of \( V \) such that \( X \subseteq M \), \( \dim(X) = t \) and \( \dim(M) =
Define
\[ H_1(X, M) = \left\{ F \in \binom{V}{k} \mid X \subseteq F, \dim(F \cap M) \geq t + 1 \right\} \cup \binom{M}{k}. \]  

**Family II.** Let \( X, M \) and \( C \) be subspaces of \( V \) such that \( X \subseteq M \subseteq C, \) \( \dim(X) = t, \) \( \dim(M) = k \) and \( \dim(C) = c, \) where \( c \in \{k + 1, k + 2, \ldots, 2k - t, n\} \). Define
\[ H_2(X, M, C) = A(X, M) \cup B(X, M, C) \cup C(X, M, C), \]
where
\[
A(X, M) = \left\{ F \in \binom{V}{k} \mid X \subseteq F, \dim(F \cap M) \geq t + 1 \right\},
\]
\[
B(X, M, C) = \left\{ F \in \binom{V}{k} \mid F \cap M = X, \dim(F \cap C) = c - k + t \right\},
\]
\[
C(X, M, C) = \left\{ F \in \binom{C}{k} \mid \dim(F \cap X) = t - 1, \dim(F \cap M) = k - 1 \right\}.
\]

**Family III.** Let \( Z \) be a \((t + 2)\)-subspace of \( V \). Define
\[ H_3(Z) = \left\{ F \in \binom{V}{k} \mid \dim(F \cap Z) \geq t + 1 \right\}. \]

It is straightforward to verify that \( H_1(X, M), H_2(X, M, C), H_2(X, M, V) \) and \( H_3(Z) \) are all non-trivial \( t \)-intersecting families of \( k \)-subspaces of \( V \).

**Remark 1** In Family II, if \( C \) satisfies \( \dim(C) = k + 1, \) then \( H_2(X, M, C) = H_1(X, C); \) if \( t \) and \( k \) satisfy \( t = k - 2, \) then \( H_2(X, M, V) = H_3(M). \)

Our first main result describes the structure of all maximal non-trivial \( t \)-intersecting families of \( k \)-subspaces of \( V \) with large size.

**Theorem 1.1** Let \( n, k \) and \( t \) be positive integers with \( t \leq k - 2 \) and \( 2k + t + \min\{4, 2t\} \leq n. \) If \( \mathcal{F} \subseteq \binom{V}{k} \) is a maximal non-trivial \( t \)-intersecting family and
\[
|\mathcal{F}| \geq \binom{k - t}{1} \binom{n - t - 1}{k - t - 1} - q \binom{k - t}{2} \binom{n - t - 2}{k - t - 2},
\]
then one of the following holds:

(i) \( \mathcal{F} = H_2(X, M, C) \) for some \( t \)-subspace \( X, \) \( k \)-subspace \( M \) and \( c \)-subspace \( C \) of \( V \) with \( X \subseteq M \subseteq C \) and \( c \in \{k + 1, k + 2, \ldots, 2k - t, n\}; \)

(ii) \( \mathcal{F} = H_3(Z) \) for some \((t + 2)\)-subspace \( Z \) of \( V, \) and \( \frac{k}{2} - 1 \leq t \leq k - 2. \)

By comparing the size of the families given in Theorem 1.1, we can describe the structure of the non-trivial \( t \)-intersecting families with maximum size. Our second main result is as follows.
Theorem 1.2 Let $n, k$ and $t$ be positive integers with $t \leq k-2$ and $2k+t+\min\{4,2t\} \leq n$. Then, for any non-trivial $t$-intersecting family $\mathcal{F} \subseteq \binom{V}{k}$, the following hold:

(i) if $1 \leq t \leq \frac{k}{2} - 1$, then

$$|\mathcal{F}| \leq \binom{n-t}{k-t} - q^{(t+1-t)(k-t)} \binom{n-k-1}{k-t} + q^{k+1-t} \binom{t}{1},$$

and equality holds if and only if $\mathcal{F} = \mathcal{H}_1(X,M)$ for some $t$-subspace $X$ and $(k+1)$-subspace $M$ of $V$ with $X \subset M$;

(ii) if $\frac{k}{2} - 1 < t \leq k-2$, then

$$|\mathcal{F}| \leq \binom{t + 2}{1} \binom{n-t-1}{k-t-1} - q \binom{t + 1}{1} \binom{n-t-2}{k-t-2},$$

and equality holds if and only if $\mathcal{F} = \mathcal{H}_3(Z)$ for some $(t+2)$-subspace $Z$ of $V$, or $(t,k) = (1,3)$ and $\mathcal{F} = \mathcal{H}_1(X,M)$ for some $1$-subspace $X$ and $3$-subspace $M$ of $V$ with $X \subset M$.

In the special case when $t = 1$, Theorem 1.2 gives rise to the Hilton-Milner Theorem for vector spaces with $n \geq 2k+3$ ([4]).

The rest of this paper is organized as follows. In the next section we will prove a number of inequalities for the sizes of the intersecting families in Families I, II and III. In §3 we will prove some upper bounds for the sizes of non-trivial $t$-intersecting families of subspaces of $V$ using a key notion—$t$-covering number, which is a generalization of the covering number. After these preparations we will prove Theorems 1.1 and 1.2 in §4.

2 Inequalities for the sizes of the constructed families

2.1 Equalities and formulas involving the Gaussian binomial coefficients

This subsection is a preparation for §2.2 and §3. The following lemma can be easily proved.

Lemma 2.1 Let $m$ and $i$ be positive integers with $i \leq m$. Then the following hold:

(i) $\binom{m}{i} = \binom{m-1}{i-1} + q^{i} \binom{m-1}{i}$ and $\binom{m}{i} = \frac{q^m-1}{q-1} \cdot \frac{m-1}{i-1}$;

(ii) $q^{m-i} < \frac{q^m-1}{q-1} < q^{m-i+1}$ and $q^{i-m-1} < \frac{q^m-1}{q-1} < q^{i-m}$ if $i < m$;

(iii) $q^{i(m-i)} \leq \binom{m}{i} < q^{i(m-i+1)}$, and $q^{i(m-i)} < \binom{m}{i}$ if $i < m$;

(iv) $\frac{q^m-1}{q-1} < 2q^{m-i}$.

Set

$$g_1(t,n) = \binom{t+2}{1} \binom{n-t-1}{t+1} - q \binom{t+1}{1} \binom{n-t-2}{t},$$

$$g_2(t,n) = \binom{n-t}{t+2} - q^{(t+2)^2} \binom{n-2t-2}{t+2}.$$
Lemma 2.2 We have
\[ g_1(t, n) - g_2(t, n) = \sum_{j=1}^{t} q^{j(t+2)+1} \left( \begin{array}{c} t+1-j \\ t \end{array} \right) \left( \begin{array}{c} n-t-2-j \\ t \end{array} \right). \]

Proof. By Lemma 2.1(i), we have
\[ g_1(t, n) = \left[ \begin{array}{c} n-t-1 \\ t+1 \end{array} \right] + q \left[ \begin{array}{c} t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-1 \\ t+1 \end{array} \right] - q \left[ \begin{array}{c} t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ t \end{array} \right] \]
\[ = \left[ \begin{array}{c} n-t-1 \\ t+1 \end{array} \right] + q^{t+1} \left[ \begin{array}{c} t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ t+1 \end{array} \right]. \]

Using Lemma 2.1(i) repeatedly, we can show that
\[ g_2(t, n) = \sum_{i=1}^{t+2} q^{(t+2)(i-1)} \left[ \begin{array}{c} n-t-i \\ t+1 \end{array} \right]. \]

Set
\[ f(a) = \sum_{j=1}^{a} q^{j(t+2)+1} \left[ \begin{array}{c} t+1-j \\ t \end{array} \right] \left[ \begin{array}{c} n-t-2-j \\ t \end{array} \right] + q^{(a+1)(t+2)} \left[ \begin{array}{c} t+1-a \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2-a \\ t+1 \end{array} \right] \]
\[ - \sum_{i=a+2}^{t+2} q^{(t+2)(i-1)} \left[ \begin{array}{c} n-t-i \\ t+1 \end{array} \right] \]
for \( a \in \{0, 1, \ldots, t\}. \) Then
\[ g_1(t, n) - g_2(t, n) = q^{t+2} \left[ \begin{array}{c} t+1 \\ t \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ t+1 \end{array} \right] - \sum_{i=2}^{t+2} q^{(t+2)(i-1)} \left[ \begin{array}{c} n-t-i \\ t+1 \end{array} \right] = f(0). \]

On the other hand, by Lemma 2.1(i), we have
\[ f(a+1) - f(a) = q^{(a+1)(t+2)+1} \left[ \begin{array}{c} t-a \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-3-a \\ t \end{array} \right] + q^{(a+2)(t+2)} \left[ \begin{array}{c} t-a \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-3-a \\ t+1 \end{array} \right] \]
\[ - q^{(a+1)(t+2)} \left[ \begin{array}{c} t+1-a \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2-a \\ t+1 \end{array} \right] + q^{(t+2)(a+1)} \left[ \begin{array}{c} n-t-a-2 \\ t+1 \end{array} \right] \]
\[ = 0. \]

Since this holds for each \( a, \) we obtain \( f(0) = f(1) = \cdots = f(t). \) Therefore,
\[ g_1(t, n) - g_2(t, n) = f(0) = f(t) = \sum_{j=1}^{t} q^{j(t+2)+1} \left[ \begin{array}{c} t+1-j \\ t \end{array} \right] \left[ \begin{array}{c} n-t-2-j \\ t \end{array} \right] \]
as required. \( \square \)

Let \( W \) be an \((e+l)\)-dimensional vector space over \( \mathbb{F}_q, \) where \( l, e \geq 1, \) and let \( L \) be a fixed \( l \)-subspace of \( W. \) We say that an \( m \)-subspace \( U \) is of type \((m, h)\) if \( \dim(U \cap L) = h. \) Define \( \mathcal{M}(m, h; e+l, e) \) to be the set of all subspaces of \( W \) with type \((m, h). \)
Lemma 2.3 ([21, Lemma 2.1]) \( \mathcal{M}(m, k; e + l, e) \) is non-empty if and only if \( 0 \leq h \leq l \) and \( 0 \leq m - h \leq e \). Moreover, if \( \mathcal{M}(m, h; e + l, e) \) is non-empty, then

\[
|\mathcal{M}(m, h; e + l, e)| = q^{(m-h)(l-h)} \left[ \begin{array}{c} e \\ m-h \\ h \end{array} \right].
\]

Define

\[
N'(m_1, h_1; m, h; e + l, e)
\]
to be the number of subspaces of \( W \) with type \((m, h)\) containing a given subspace with type \((m_1, h_1)\). Observe that \(|\mathcal{M}(m, h; e + l, e)| = N'(0,0; m, h; e + l, e)\).

Lemma 2.4 ([22]) \( N'(m_1, h_1; m, h; e + l, e) \neq 0 \) if and only if \( 0 \leq h_1 \leq h \leq l \) and \( 0 \leq m_1 - h_1 \leq m - h \leq e \). Moreover, if \( N'(m_1, h_1; m, h; e + l, e) \neq 0 \), then

\[
N'(m_1, h_1; m, h; e + l, e) = q^{(l-h)(m-h-m_1+h_1)} \left[ \begin{array}{c} e - (m_1 - h_1) \\ (m-h)-(m_1-h_1) \\ h-h_1 \end{array} \right].
\]

Let

\[
h_1(t, k + 1) = |\mathcal{H}_1(X, M)|,
\]

\[
h_2(t, k, c) = |\mathcal{H}_2(X, M, C)|, \text{ for } c \in \{k + 1, k + 2, \ldots, 2k - t, n\},
\]

and

\[
h_3(t + 2) = |\mathcal{H}_3(Z)|.
\]

The following lemma gives the sizes of Families I, II and III.

Lemma 2.5 Suppose \( c \in \{k + 1, k + 2, \ldots, 2k - t, n\} \). Then the following hold:

\[
h_1(t, k + 1) = \left[ \begin{array}{c} n-t \\ k-t \end{array} \right] - q^{(k+1-t)(k-t)} \left[ \begin{array}{c} n-k-1 \\ k-t \end{array} \right] + q^{k+1-t} \left[ \begin{array}{c} t \\ 1 \end{array} \right]; \quad (4)
\]

\[
h_2(t, k, c) = \left[ \begin{array}{c} n-t \\ k-t \end{array} \right] - q^{(k-t)^2} \left[ \begin{array}{c} n-k \\ k-t \end{array} \right] + q^{(k-t)^2} \left[ \begin{array}{c} n-c \\ 2k-c-t \end{array} \right] + q^{k-t+1} \left[ \begin{array}{c} c-k \\ 1 \end{array} \right] \left[ \begin{array}{c} t \\ 1 \end{array} \right]; \quad (5)
\]

\[
h_3(t + 2) = \left[ \begin{array}{c} t + 2 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-1 \\ k-t-1 \end{array} \right] - q^{t+1} \left[ \begin{array}{c} t + 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right]. \quad (6)
\]

Proof. Suppose that \( X, M \) and \( C \) are subspaces of \( V \) with \( X \subseteq M \subseteq C \) such that \( \dim(X) = t \), \( \dim(M) = k \) and \( \dim(C) = c \). Then

\[
\mathcal{A}(X, M) = \left\{ F \in \left[ \begin{array}{c} V \\ k \end{array} \right] \mid X \subseteq F \right\} \setminus \left\{ F \in \left[ \begin{array}{c} V \\ k \end{array} \right] \mid F \cap M = X \right\},
\]

which implies that \(|\mathcal{A}(X, M)| = \left[ \begin{array}{c} n-t \\ k-t \end{array} \right] - q^{(k-t)^2} \left[ \begin{array}{c} n-k \\ k-t \end{array} \right]\) by Lemmas 2.3 and 2.4. From Corollary 2.3 and Lemma 2.1 in [13], we have

\[
|\mathcal{B}(X, M, C)| = q^{(k-t)^2} \left[ \begin{array}{c} n-c \\ 2k-c-t \end{array} \right] \text{ and } |\mathcal{C}(X, M, C)| = q^{k-t+1} \left[ \begin{array}{c} c-k \\ 1 \end{array} \right] \left[ \begin{array}{c} t \\ 1 \end{array} \right].
\]

Since \( h_2(t, k, c) = |\mathcal{A}(X, M)| + |\mathcal{B}(X, M, C)| + |\mathcal{C}(X, M, C)| \), we obtain (5) immediately. By Remark 1 and Lemma 2.1(i), we obtain (4).

Consider the family \( \mathcal{H}_3(Z) \), where \( Z \) is a \((t + 2)\)-subspace of \( V \). By Lemma 2.3, the number of \( k \)-subspaces \( F \) of \( V \) satisfying \( \dim(F \cap Z) = t + 1 \) is \( q^{k-t-1} \left[ \begin{array}{c} n-t-2 \\ k-t-1 \end{array} \right] \left[ \begin{array}{c} t+2 \\ 1 \end{array} \right] \), and the number of \( k \)-subspaces \( F \) of \( V \) satisfying \( \dim(F \cap Z) = t + 2 \) is \( \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right] \). Combining these with Lemma 2.2(i), we obtain (6) immediately. \( \square \)
2.2 Inequalities for \( h_1(t, k + 1), h_2(t, k, c) \) and \( h_3(t + 2) \)

**Lemma 2.6** Let \( n, k \) and \( t \) be positive integers with \( 6 \leq 2k \leq n \) and \( 1 \leq t \leq k - 2 \).

(i) We have

\[ h_1(t, k + 1) = h_2(t, k, k + 1), \]

and

\[ h_2(t, k, c) > h_2(t, k, c + 1) \]

for \( c \in \{k + 1, k + 2, \ldots, 2k - t - 1\} \).

(ii) Assume that \( 1 \leq t \leq k - 3 \). If \( 2k \leq n \leq (k - t)^2 - 1 \), or \( n = (k - t)^2 \) and \( q \geq 3 \), or \( (n, q, t) = ((k - t)^2, 2, 1) \), then

\[ h_2(t, k, 2k - t) > h_2(t, k, n). \]

If \( n \geq (k - t)^2 + 1 \), or \( (n, q) = ((k - t)^2, 2) \) and \( t \geq 2 \), then

\[ h_2(t, k, k + 1) > h_2(t, k, n) > h_2(t, k, 2k - t). \]

(iii) Assume that \( t = k - 2 \). If \( t = 1 \), then

\[ h_2(t, k, n) = h_2(t, k, k + 1); \]

and if \( t \geq 2 \), then

\[ h_2(t, t + 2, n) > h_2(t, t + 2, t + 3). \]

**Proof.** (i) As seen in Remark 1 we have \( h_1(t, k + 1) = h_2(t, k, k + 1) \) for \( 1 \leq t \leq k - 2 \). For \( c \in \{k + 1, k + 2, \ldots, 2k - t - 1\} \), by Lemma 2.1(i), we have

\[ h_2(t, k, c) - h_2(t, k, c + 1) = q^{(k-t)^2+2k-c-t} \left[ \frac{n-c-1}{2k-c-t} \right] - q^{c-t+1} \left[ t \right]. \]

Since \( \left[ \frac{n-c-1}{2k-c-t} \right] \geq q^{(2k-c-t)(n-2k+t-1)} \) and \( \left[ t \right] < q^t \) by Lemma 2.1(iii), and since

\[
\begin{align*}
(k-t)^2 + 2k - c - t + (2k - c - t)(n - 2k + t - 1) - c - 1 \\
= (k-t-1)^2 + (2k - c - t - 1)(n - 2k + t + 1) + (n - 2k) - 1 \\
\geq 0,
\end{align*}
\]

we obtain \( h_2(t, k, c) - h_2(t, k, c + 1) > 0 \) for \( c \in \{k + 1, k + 2, \ldots, 2k - t - 1\} \).

(ii) Note that

\[ h_2(t, k, n) - h_2(t, k, c) = q^{c-t+1} \left[ t \right] \left[ \frac{n-c}{1} \right] - q^{(k-t)^2} \left[ \frac{n-c}{2k-c-t} \right] \] (7)
for any $c \in \{k + 1, k + 2, \ldots, 2k - t\}$. When $c = k + 1$, by (7) and Lemma 2.1(iii), and noting that $t \leq k - 3$ and $2k \leq n$, we obtain
\[
h_2(t, k, n) - h_2(t, k, k + 1) = q^{k-t+2} \left[ t \left[ \begin{array}{c} n-k-1 \\ 1 \end{array} \right] - q^{(k-t)^2} \left[ n-k-1 \right] _{k-t-1} \right] < q^{n+1} - q^{(n-k+1)(k-t-1)+1} < 0.
\]

When $c = 2k - t$, by (7) again, we have
\[
h_2(t, k, n) - h_2(t, k, 2k - t) = q^{2k-2t+1} \left[ t \left[ \begin{array}{c} n-2k+t \\ 1 \end{array} \right] - q^{(k-t)^2} \right].
\]
By Lemma 2.1(iii), if $2k \leq n \leq (k-t)^2 - 1$, then
\[
h_2(t, k, n) - h_2(t, k, 2k - t) < q^{n+1} - q^{(k-t)^2} \leq 0;
\]
and if $n \geq (k-t)^2 + 1$, then
\[
h_2(t, k, n) - h_2(t, k, 2k - t) > q^{n-1} - q^{(k-t)^2} \geq 0.
\]

Now assume that $n = (k-t)^2$. Then
\[
h_2(t, k, n) - h_2(t, k, 2k - t) = \frac{1}{(q-1)^2} \left( (q^t - 1)(q^{n-t+1} - q^{2k-2t+1}) - q^{(k-t)^2}(q-1)^2 \right) = \frac{1}{(q-1)^2} \left( q^{(k-t)^2}(-q^2 + 3q - 1 - q^{-t+1}) - q^{2k-2t+1}(q^t - 1) \right).
\]
If $q \geq 3$, then $-q^2 + 3q \leq 0$, and $h_2(t, k, n) - h_2(t, k, 2k - t) < 0$. If $q = 2$ and $t = 1$, then $h_2(t, k, n) - h_2(t, k, 2k - t) = -2^{2k-1} < 0$. If $q = 2$ and $t \geq 2$, then
\[
h_2(t, k, n) - h_2(t, k, 2k - t) = 2^{(k-t)^2} (1 - 2^{-t+1}) - 2^{2k-2t+1}(2^t - 1) \geq 2^{(k-t)^2-1} (2^{2k-2t+1} + 2^{2k-2t+1}) > 0
\]
as $(k-t)^2 = n \geq 2k$.

(iii) If $t = k - 2$, then by (7) we have
\[
h_2(t, k, n) - h_2(t, k, k + 1) = q^{k-2t} \left[ n-k-1 \right] _{k-t-1} - q^{(n-k-1)(n-t-2)}.
\]
It is clear that $h_2(t, k, n) - h_2(t, k, k + 1) = 0$ if $t = 1$ and $h_2(t, k, n) - h_2(t, k, k + 1) > 0$ if $t \geq 2$.

Define
\[
f(n, k, t) = \left[ n-t-1 \right] _{k-t-1} - q \left[ n-t-2 \right] _{k-t-2}.
\]

**Lemma 2.7** Let $n, k$ and $t$ be positive integers with $6 \leq 2k \leq n$ and $1 \leq t \leq k - 2$.

(i) $\min\{h_2(t, k, 2k - t), h_2(t, k, n)\} \geq f(n, k, t)$. 

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If $1 \leq t \leq k - 3$, then
\[
h_1(t, k + 1) \leq \left[ \begin{array}{c} k - t + 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ k - t - 1 \end{array} \right]. \tag{9}
\]

If $1 \leq t \leq k - 4$, then
\[
h_1(t, k + 1) \leq \left[ \begin{array}{c} k - t + 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ k - t - 1 \end{array} \right] - q^{(k-t-1)(k-t-2)+1} \left[ \begin{array}{c} n - k - 1 \\ k - t - 2 \end{array} \right] \left[ \begin{array}{c} k + 1 - t \\ 2 \end{array} \right]. \tag{10}
\]

**Proof.** Let $X$ and $M$ be subspaces of $V$ with $\dim(X) = t$ and $X \subseteq M$. For each $i \in \{t, t + 1, \ldots, k\}$, set
\[
A_i(X, M) = \left\{ F \in \left[ V \atop k \right] \mid X \subseteq F, \dim(F \cap M) = i \right\}
\]
and
\[
\mathcal{L}_i(X, M) = \left\{ (I, F) \in \left[ V \atop i \right] \times \left[ V \atop k \right] \mid X \subseteq I \subseteq M, I \subseteq F \right\}.
\]
Using Lemma 2.1 and double counting $|\mathcal{L}_i(X, M)|$, we obtain
\[
|\mathcal{L}_i(X, M)| = \sum_{j=i}^{k} |A_j(X, M)| \cdot \left[ \begin{array}{c} j - t \\ i - t \end{array} \right] = \left[ \begin{array}{c} \dim(M) - t \\ i - t \end{array} \right] \left[ \begin{array}{c} n - i \\ k - i \end{array} \right]. \tag{11}
\]
In particular, we have
\[
|\mathcal{L}_{t+1}(X, M)| = \sum_{j=t+1}^{k} |A_j(X, M)| + \sum_{j=t+2}^{k} |A_j(X, M)| \cdot \left( \left[ \begin{array}{c} j - t \\ 1 \end{array} \right] - 1 \right). \tag{12}
\]

(i) Let $M$ be a $k$-subspace of $V$ and $A(X, M)$ the family constructed in Family II. Observe that $A(X, M) = \bigcup_{j=t+1}^{k} A_j(X, M)$. By (11) and (12), we obtain
\[
|\mathcal{L}_{t+1}(X, M)| = \left[ \begin{array}{c} k - t \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ k - t - 1 \end{array} \right]
\leq |A(X, M)| + \sum_{j=t+2}^{k} |A_j(X, M)| \cdot q^{\left[ \begin{array}{c} j - t \\ 2 \end{array} \right]}
= |A(X, M)| + q |\mathcal{L}_{t+2}(X, M)|
= |A(X, M)| + q \left[ \begin{array}{c} k - t \\ 2 \end{array} \right] \left[ \begin{array}{c} n - t - 2 \\ k - t - 2 \end{array} \right].
\]
That is,
\[
|A(X, M)| \geq \left[ \begin{array}{c} k - t \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ k - t - 1 \end{array} \right] - q \left[ \begin{array}{c} k - t \\ 2 \end{array} \right] \left[ \begin{array}{c} n - t - 2 \\ k - t - 2 \end{array} \right].
\]
We then obtain (i) by the definitions of $\mathcal{H}_2(X, M, C)$ and $\mathcal{H}_2(X, M, V)$ and the proof of Lemma 2.5.
(ii) Let $M$ be a $(k + 1)$-subspace of $V$ and $\mathcal{A}'(X, M) = \cup_{j=t+1}^{k} \mathcal{A}_j(X, M)$. By (11) and (12) again, we have

$$|L_{t+1}(X, M)| = \left[ k - t + 1 \right] \left[ n - t - 1 \right] \left[ 1 \right]$$

$$= |A'(X, M)| + \sum_{j=t+2}^{k} |A_j(X, M)| \left( \left[ j - t \right] - 1 \right).$$

Since $1 \leq t \leq k - 3$ and $|A_{k-1}(X, M)| = N'(t; t; k, k-1; n, n-k-1)$, by Lemma 2.1(iii) and the assumption $n \geq 2k$, we have

$$|A_{k-1}(X, M)| \left( \left[ k - 1 - t \right] - 1 \right) = q^2 \left[ n - k - 1 \right] \left[ k + 1 - t \right] \left[ 1 \right] \cdot q \left[ k - t - 2 \right]$$

$$> q^{n+2k-3t-4} > q^{k+1} \geq q^{k+1-t} \left[ t \right].$$

Observe that, for any $F \in [M]_k$, $F \in A'(X, M)$ if and only if $X \subseteq F$. Thus, by Lemma 2.3, we have

$$\left| [M]_k \setminus A'(X, M) \right| = \left[ k + 1 \right] \left[ k - t \right] - \left[ k + 1 - t \right] = q^{k+1-t} \left[ t \right].$$

So by the construction of $H_1(X, M)$ we then obtain

$$h_1(t, k+1) = |A'(X, M)| + q^{k+1-t} \left[ t \right]$$

$$\leq |L_{t+1}(X, M)|$$

$$= \left[ k - t + 1 \right] \left[ n - t - 1 \right] \left[ 1 \right]$$

as required.

(iii) Continuing our discussion in (ii), if $t \leq k - 4$, then $t + 2 \neq k - 1$ and hence

$$h_1(t, k+1) = |A'(X, M)| + q^{k+1-t} \left[ t \right]$$

$$\leq |L_{t+1}(X, M)| - |A_{t+2}(X, M)| \left( \left[ 2 \right] - 1 \right).$$

This together with $|A_{t+2}(X, M)| = N'(t; t; k, t+2; n, n-k-1)$ and Lemma 2.4 yields (10).

\[ \square \]

**Lemma 2.8** Let $n, k$ and $t$ be positive integers with $6 \leq 2k \leq n$ and $1 \leq t \leq k - 2$. Let $f(n, k, t)$ be the function defined in (8).

(i) If $1 \leq t < \frac{k}{2} - 1$, then $h_3(t + 2) < f(n, k, t)$.

(ii) If $\frac{k}{2} - 1 \leq t \leq k - 2$, then $h_3(t + 2) \geq f(n, k, t)$. 
Proof. Let
\[ f_1(n, k, t) = \frac{f(n, k, t) - h_3(t + 2)}{[\frac{n-t-2}{k-t-2}]} . \]

By (6), we have
\[ f_1(n, k, t) = \frac{(q^{k-t} - q^{t+2})(q^{n-t-1} - 1)}{(q-1)(q^{k-t-1} - 1)} + q \left[ \frac{t + 1}{1} \right] - q \left[ \frac{k-t}{2} \right] . \]

(i) Suppose that \( 1 \leq t < \frac{k}{2} - 1 \). Since \( k > 2t + 2 \) and \( n \geq 2k \), by Lemma 2.1(ii)(iii), we have \( f_1(n, k, t) > q^{n-t-1} + q^{t+1} - q^{2k-2t-1} > 0 \), which implies \( f(n, k, t) > h_3(t + 2) \) as required.

(ii) If \( t = \frac{k}{2} - 1 \), then
\[ f_1(n, k, t) = q \left[ \frac{t + 1}{1} \right] - q \left[ \frac{k-t}{2} \right] < 0 . \]

If \( \frac{k}{2} - 1 < t \leq k - 2 \), then by Lemma 2.1(ii)(iii) and the assumption \( n \geq 2k \), we have \( f_1(n, k, t) < -q^{n-k+t+1} + q^{t+2} - q^{2k-2t-3} < 0 \). In either case we obtain \( f(n, k, t) < h_3(t + 2) \) as required.

Combining Lemmas 2.6(i), 2.7(i) and 2.8(i), we obtain that if \( 1 \leq t < \frac{k}{2} - 1 \) then \( \min\{h_2(t, k, c), h_2(t, k, n)\} > h_3(t + 2) \) for any \( c \in \{k + 1, k + 2, \ldots, 2k - t\} \). The next lemma gives several inequalities involving \( h_2(t, k, c) \), \( h_2(t, k, n) \) and \( h_3(t + 2) \) in the case when \( \frac{k}{2} - 1 \leq t \leq k - 2 \).

**Lemma 2.9** Let \( n, k \) and \( t \) be positive integers with \( 6 \leq 2k \leq n \) and \( \frac{k}{2} - 1 \leq t \leq k - 2 \).

(i) Assume that \( t = \frac{k}{2} - 1 \). If \( t = 1 \) and \( 8 \leq n \leq 9 \), then \( h_2(t, k, 2k - t) > h_3(t + 2) \); if \( t = 1 \) and \( n \geq 10 \), or \( t \geq 2 \), then \( h_2(t, k, k + 1) > h_3(t + 2) > h_2(t, k, 2k - t) \); if \( t = 1 \), then \( h_2(t, k, n) = h_3(t + 2) \); if \( t \geq 2 \), then \( h_2(t, k, k + 1) > h_3(t + 2) > h_2(t, k, n) \).

(ii) Assume that \( \frac{k}{2} - \frac{1}{2} \leq t \leq k - 3 \). Then \( h_3(t + 2) > h_2(t, k, k + 1) \).

(iii) Assume that \( t = k - 2 \). If \( t = 1 \), then \( h_2(t, k, n) = h_3(t + 2) = h_2(t, k, k + 1) \); if \( t \geq 2 \), then \( h_2(t, k, n) = h_3(t + 2) > h_2(t, k, k + 1) \).

**Proof.** (i) Assume that \( t = \frac{k}{2} - 1 \). By (5), (6) and Lemma 2.2, we have
\[ h_3(t + 2) - h_2(t, k, 2k - t) \]
\[ = \sum_{j=1}^{t} q^{j(t+2)+1} \left[ \left[ t + 1 - j \right] \left[ n - t - 2 - j \right] \right] - q^{(t+2)} - q^{t+2} \left[ \left[ t + 2 \right] \left[ 1 \right] \right] . \]
It is routine to verify that \( h_2(t, k, 2k - t) > h_3(t + 2) \) when \( t = 1 \) and \( 8 \leq n \leq 9 \) and \( h_2(t, k, 2k - t) < h_3(t + 2) \) when \( t = 1 \) and \( n = 10 \). If \( t = 1 \) and \( n \geq 11 \), or \( t \geq 2 \), then
\[ h_3(t + 2) - h_2(t, k, 2k - t) > q^{(t+2)+1} \left[ \left[ n - 2t - 2 \right] \left[ t \right] \right] - q^{(t+2)} - q^{t+2} \left[ \left[ t + 2 \right] \left[ 1 \right] \right] \]
\[ > q^{(n-2)+1} - q^{(t+2)} - q^{3t+5} \]
\[ > 0 . \]
By (5), (6) and Lemma 2.2 again, we have
\[ h_3(t + 2) - h_2(t, k, n) = \sum_{j=1}^{t} q^{j(t+2)+1}\left[\frac{t + 1 - j}{t}\right]^{n - t - 2 - j} - q^{t+3}\left[\frac{n - 2t - 2}{t}\right]^{1}. \]

It is straightforward to verify that \( h_3(t, k, n) = h_4(t + 2) \) when \( t = 1 \). If \( t \geq 2 \), then
\[
\begin{align*}
    h_3(t + 2) - h_2(t, k, n) &> q^{j(t+2)+1}\left[\frac{n - 2t - 2}{t}\right]^{1} - q^{t+3}\left[\frac{n - 2t - 2}{t}\right]^{1} \\
    &> q^{t(n-2t)+1} - q^{n+1} \\
    &> 0.
\end{align*}
\]

By (5), (6) and Lemma 2.2 again, we have
\[ h_2(t, k, k+1) - h_4(t+2) = q^{(t+2)^2}\left[\frac{n - 2t - 3}{t+1}\right] - \sum_{j=1}^{t} q^{j(t+2)+1}\left[\frac{t + 1 - j}{t}\right]^{n - t - 2 - j} + q^{t+3}\left[\frac{1}{1}\right]. \]

Since by Lemma 2.1(ii),
\[ q^{j(t+2)+1}\left[\frac{t + 1 - j}{t}\right]^{n - t - 2 - j} < q^{nt-2t^2+j+2} \]
for \( j \in \{1, 2, \ldots, t\} \), we have
\[ \sum_{j=1}^{t} q^{j(t+2)+1}\left[\frac{t + 1 - j}{t}\right]^{n - t - 2 - j} < q^{nt-2t^2+2t} \sum_{j=1}^{t} q^{j} < 2q^{nt-2t^2+2t}. \]

Since \( q \geq 2 \), by Lemma 2.1(iii), we have
\[ q^{(t+2)^2}\left[\frac{n - 2t - 3}{t+1}\right] > 2 \cdot q^{2t+3} \cdot q^{(t+1)(n-3t-4)} = 2q^{nt-2t^2+n-3t-1}. \]

Thus \( h_2(t, k, k+1) > h_3(t+2) \) as \( n \geq 2k = 4t + 4 \).

(ii) Assume that \( \frac{k}{2} < t \leq k - 3 \). By (6) and (9), we have
\[ h_3(t + 2) = q^{n-t-1} - 1\left[\frac{t + 1}{1}\right] - q^{t+1}. \]

and
\[ h_1(t, k + 1) = q^{n-t-1} - 1\left[\frac{k - t + 1}{1}\right]. \]

Since by Lemma 2.1(ii),
\[
\begin{align*}
    &\frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} \cdot \left[\frac{t + 2}{1}\right] - q^{t+1} - \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} \cdot \left[\frac{k - t + 1}{1}\right] \\
    &> \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} \cdot \left(\frac{q^{2t-k+1} - 1}{q^{t-1}} - q^{q^t - 1}\right) \\
    &> q^{n-k-t+1} - q^{t+1} \\
    &> 0,
\end{align*}
\]
we obtain $h_3(t + 2) > h_1(t, k + 1)$.

Assume that $t = \frac{k}{2} - \frac{1}{2} \leq k - 3$ and $k \geq 7$. Then $t \leq k - 4$. By (6) and (10), we have

$$h_3(t + 2) = \left[ \begin{array}{c} t + 2 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ t \end{array} \right] - q \left[ \begin{array}{c} t + 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 2 \\ t - 1 \end{array} \right]$$

and

$$h_1(t, k + 1) \leq \left[ \begin{array}{c} t + 2 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 1 \\ t \end{array} \right] - q^{t(t-1)+1} \left[ \begin{array}{c} n - 2t - 2 \\ t - 1 \end{array} \right] \left[ \begin{array}{c} t + 2 \\ 2 \end{array} \right].$$

So by Lemma 2.1 (iii) we have

$$h_3(t + 2) - h_1(t, k + 1) \geq q^{t(t-1)+1} \left[ \begin{array}{c} n - 2t - 2 \\ t - 1 \end{array} \right] \left[ \begin{array}{c} t + 2 \\ 2 \end{array} \right] - q \left[ \begin{array}{c} t + 1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - t - 2 \\ t - 1 \end{array} \right]$$

$$\geq q^{t(t-1)(n-2t+1)} - q^{t(t-1)(n-2)+t+2} = 0.$$

Finally, assume that $t = \frac{k}{2} - \frac{1}{2} \leq k - 3$ and $k = 5$. By (6) and (4), we have

$$\left[ \begin{array}{c} n - 3 \\ 2 \end{array} \right] = \left[ \begin{array}{c} n - 4 \\ 1 \end{array} \right] + q^{2} \left[ \begin{array}{c} n - 4 \\ 2 \end{array} \right]$$

and

$$\left[ \begin{array}{c} n - 2 \\ 3 \end{array} \right] = \sum_{i=1}^{4} q^{3(i-1)} \left[ \begin{array}{c} n - 2 - i \\ 2 \end{array} \right] + q^{12} \left[ \begin{array}{c} n - 6 \\ 3 \end{array} \right].$$

It follows that

$$h_3(t + 2) - h_1(t, k + 1) = q^{3} \left[ \begin{array}{c} n - 4 \\ 2 \end{array} \right] - \sum_{i=2}^{4} q^{3(i-1)} \left[ \begin{array}{c} n - 2 - i \\ 2 \end{array} \right] - q^{4} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]$$

$$= q^{4} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - 4 \\ 2 \end{array} \right] - q^{6} \left[ \begin{array}{c} n - 5 \\ 2 \end{array} \right] - q^{8} \left[ \begin{array}{c} n - 6 \\ 2 \end{array} \right] - q^{4} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]$$

$$= q^{4} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] \left[ \begin{array}{c} n - 5 \\ 2 \end{array} \right] + q^{7} \left[ \begin{array}{c} n - 6 \\ 1 \end{array} \right] - q^{4} \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]$$

$$> 0.$$

(iii) This follows from the definitions of $\mathcal{H}_2(X, M, V)$ and $\mathcal{H}_3(Z)$ and the assumption that $\dim(M) = \dim(Z) = k = t + 2$. \hfill \Box

3 Upper bounds for non-trivial $t$-intersecting families

In this section we prove a number of upper bounds on the size of a maximal non-trivial $t$-intersecting family of $k$-subspaces of $V$. For any family $\mathcal{F} \subseteq \binom{V}{k}$ and any subspace $S$ of $V$, define

$$\mathcal{F}_S = \{ F \in \mathcal{F} \mid S \subseteq F \}.$$

Lemma 3.1 Let $\mathcal{F} \subseteq \binom{V}{k}$ be a $t$-intersecting family and $S$ an $s$-subspace of $V$, where $t - 1 \leq s \leq k - 1$. If there exists $F' \in \mathcal{F}$ such that $\dim(S \cap F') = r < t$, then for each $i \in \{1, 2, \ldots, t-r\}$ there exists an $(s+i)$-subspace $T_i$ with $S \subseteq T_i$ such that $|\mathcal{F}_S| \leq \binom{k-r}{i} |\mathcal{F}_{T_i}|$. 

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Proof. For each \( i \in \{1, 2, \ldots, t - r\} \), let
\[
\mathcal{H}_i = \left\{ H \in \left[ S + F' \atop s + i \right] \mid S \subseteq H \right\}.
\]
Observe that \( |\mathcal{H}_i| = \left[ \binom{k - i}{r} \right] \) by Lemma 2.3. Since \( \mathcal{F} \) is a \( t \)-intersecting family, for any \( F \in \mathcal{F}_S \), we have \( \dim(F \cap F') \geq t \), which implies that \( \dim(F \cap (F' + S)) \geq s + t - r \) and there exists \( H \in \mathcal{H}_i \) such that \( H \subseteq F \). Therefore, \( \mathcal{F}_S = \bigcup_{H \in \mathcal{H}_i} \mathcal{F}_H \). Let \( T_i \) be a subspace in \( \mathcal{H}_i \) such that \( |\mathcal{F}_H| \leq |\mathcal{F}_{T_i}| \) for all \( H \in \mathcal{H}_i \). Then \( |\mathcal{F}_S| \leq \left[ \binom{k - i}{r} \right] |\mathcal{F}_{T_i}| \) as desired. \( \Box \)

Lemma 2.3 implies that \( |\mathcal{F}_T| \leq \left[ \frac{n - \dim(T)}{k - \dim(T)} \right] \) for any subspace \( T \) of \( V \). So we have the following lemma.

**Lemma 3.2** Let \( \mathcal{F} \subseteq \binom{V}{k} \) be a \( t \)-intersecting family and \( S \) an \( s \)-subspace of \( V \), where \( t - 1 \leq s \leq k - 1 \). If there exists \( F' \in \mathcal{F} \) such that \( \dim(S \cap F') = r < t \), then \( |\mathcal{F}_S| \leq \left[ \binom{k - r}{t - r} \right] \left[ \frac{n - s - t + r}{k - s - t + r} \right] \).

For a \( t \)-intersecting family \( \mathcal{F} \subseteq \binom{V}{k} \), we define the \( t \)-covering number \( \tau_t(\mathcal{F}) \) of \( \mathcal{F} \) to be the minimum dimension of a subspace \( T \) of \( V \) such that \( \dim(T \cap F) \geq t \) for any \( F \in \mathcal{F} \). Note that for any non-trivial \( t \)-intersecting family \( \mathcal{F} \subseteq \binom{V}{k} \) we have \( t + 1 \leq \tau_t(\mathcal{F}) \leq k \).

**Remark 2** In [5], Cao also respectively described the structure of maximal non-trivial \( t \)-intersecting families with large size for finite sets and distance-regular graphs of bilinear forms by defining their \( t \)-covering number. It is shown that \( t \)-covering number is a useful notion to describe the structure of maximal non-trivial \( t \)-intersecting families.

### 3.1 The case \( \tau_t(\mathcal{F}) = t + 1 \)

**Assumption 1** Let \( n, k \) and \( t \) be positive integers with \( 6 \leq 2k \leq n \) and \( 1 \leq t \leq k - 2 \). Let \( \mathcal{F} \subseteq \binom{V}{k} \) be a maximal non-trivial \( t \)-intersecting family with \( \tau_t(\mathcal{F}) = t + 1 \). Define
\[
\mathcal{T} = \left\{ T \in \left[ \begin{array}{c} V \\ t+1 \end{array} \right] \mid \dim(T \cap F) \geq t \quad \text{for any} \quad F \in \mathcal{F} \right\}.
\]

**Lemma 3.3** Let \( n, k, t, \mathcal{F} \) and \( \mathcal{T} \) be as in Assumption 1. Then \( \mathcal{T} \) is a \( t \)-intersecting family with \( t \leq \tau_t(\mathcal{T}) \leq t + 1 \). Moreover, the following hold:

(i) if \( \tau_t(\mathcal{T}) = t \), then there exist a \( t \)-subspace \( X \) and an \( l \)-subspace \( M \) of \( V \) with \( X \subseteq M \) and \( t + 1 \leq l \leq k + 1 \) such that
\[
\mathcal{T} = \left\{ T \in \left[ \begin{array}{c} M \\ t+1 \end{array} \right] \mid X \subseteq T \right\}; \quad (13)
\]

(ii) if \( \tau_t(\mathcal{T}) = t + 1 \), then there exists a \( (t + 2) \)-subspace \( Z \) of \( V \) such that \( \mathcal{T} = \left[ \begin{array}{c} Z \\ t+1 \end{array} \right] \).

**Proof.** The maximality of \( \mathcal{F} \) implies that, for any \( T \in \mathcal{T} \), \( \mathcal{F} \) contains all \( k \)-subspaces of \( V \) containing \( T \). Since \( 2k \leq n \), for any \( T_1, T_2 \in \mathcal{T} \), if \( \dim(T_1 \cap T_2) < t \), then there must exist \( F_1, F_2 \in \mathcal{F} \) such that \( T_1 \subseteq F_1, T_2 \subseteq F_2 \) and \( \dim(F_1 \cap F_2) < t \). However, this is impossible as
$\mathcal{F}$ is maximal $t$-intersecting. Hence $\dim(T_1 \cap T_2) \geq t$ and $\mathcal{T} \subseteq \left[ \frac{V}{t+1} \right]$ is a $t$-intersecting family with $t \leq \tau_t(\mathcal{T}) \leq t + 1$.

(i) Suppose that $\tau_t(\mathcal{T}) = t$. Then there exists a $t$-subspace $X$ of $V$ such that $X$ is contained in every $(t + 1)$-subspace in $\mathcal{T}$. Assume that $M = \sum_{T \in \mathcal{T}} T$ and $\dim(M) = l$. It suffices to prove (13) and $t + 1 \leq \dim(M) \leq k + 1$. Since $\tau_t(\mathcal{F}) = t + 1$, we have $\mathcal{F} \setminus \mathcal{F}_X \neq \emptyset$.

Let $F'$ be any member of $\mathcal{F} \setminus \mathcal{F}_X$. Observe that $\dim(X \cap F') \leq t - 1$. For any $T \in \mathcal{T}$, since $X \subseteq T$ and $\dim(T \cap F') \geq t$, we have $\dim(X \cap F') = t - 1$ and $\dim(T \cap (X + F')) \geq t + 1$, which together imply that $\dim(X + F') = k + 1$ and $T \subseteq X + F'$. Hence $M = \sum_{T \in \mathcal{T}} T \subseteq X + F'$ and $t + 1 \leq l \leq k + 1$. It is clear that $\mathcal{T} \subseteq \left\{ T \in \left[ \frac{M}{l+1} \right] \mid X \subseteq T \right\}$. Let $T'$ be any $(t + 1)$-subspace of $M$ with $X \subseteq T'$. For any $F \in \mathcal{F}$, if $X \subseteq F$, then $\dim(T' \cap F) \geq t$; if $X \nsubseteq F$, then $T' \subseteq X + F$ from the above discussion, which implies $\dim(T' \cap F) \geq t$ by $\dim(X + F) = k + 1$. Hence $T' \in \mathcal{T}$ and (13) is proved.

(ii) Suppose that $\tau_t(\mathcal{T}) = t + 1$. Let $A, B, C \in \mathcal{T}$ be distinct subspaces such that $A \cap B$, $A \cap C$ and $B \cap C$ are pairwise distinct. Since $\mathcal{T}$ is $t$-intersecting, we have $\dim(A \cap B) = \dim(A \cap C) = \dim(B \cap C) = t$, which together with $\dim(C) = t + 1$ implies that $C = (A \cap C) + (B \cap C) \subseteq A + B$. Hence, $A + C \subseteq A + B$ and $B + C \subseteq A + B$, which imply that $A + B = A + C = B + C$.

Since $\tau_t(\mathcal{T}) = t + 1$, there exist three distinct subspaces $T_1, T_2, T_3 \in \mathcal{T}$ such that $T_1 \cap T_2$, $T_1 \cap T_3$ and $T_2 \cap T_3$ are pairwise distinct. For any $T \in \mathcal{T} \setminus \{T_1, T_2, T_3\}$, if $T \cap T_1 = T \cap T_2 = T \cap T_3$, then $\dim(T \cap T_1) = t_1$, $T \cap T_1 \subseteq T_2$ and $T \cap T_1 \subseteq T_3$, which imply that $T \cap T_1 \cap T_2 = T_1 \cap T_3$, a contradiction. Hence there exist $T_1, T_3 \in \{T_1, T_2, T_3\}$ such that $T \cap T_1 \neq T \cap T_3$ and

$$T = (T \cap T_1) + (T \cap T_3) \subseteq T_1 + T_2 = T_1 + T_3 + T_2 = T_3 + T_3.$$ 

Let $Z = T_1 + T_2$. Then $\mathcal{T} \subseteq \left[ \frac{Z}{t+1} \right]$. We now prove that $\left[ \frac{Z}{t+1} \right] \subseteq \mathcal{T}$. In fact, for any $F \in \mathcal{F}$, if $F \cap T_1 = F \cap T_2 = F \cap T_3$, then $F \cap T_1 \subseteq T_1$ for each $i \in \{1, 2, 3\}$. But this is impossible because $T_1 \cap T_2, T_1 \cap T_3$ and $T_2 \cap T_3$ are pairwise distinct and $\dim(F \cap T_1) \geq t$. Hence there exist $T_i, T_j \in \{T_1, T_2, T_3\}$ such that $F \cap T_i \neq F \cap T_j$, which implies $\dim(F \cap Z) \geq t + 1$. So for any $F \in \mathcal{F}$ and $F' \in \left[ \frac{Z}{t+1} \right]$ we have $\dim(F \cap F') \geq t$. Therefore, $\mathcal{T} = \left[ \frac{Z}{t+1} \right]$ as desired.

**Lemma 3.4** Let $n, k, t, \mathcal{F}$ and $\mathcal{T}$ be as in Assumption 1, and set $M = \sum_{T \in \mathcal{T}} T$. Suppose that $\tau_t(\mathcal{T}) = t$, $\dim(M) = k + 1$ and $X$ is a $t$-subspace of $V$ which is contained in each $T \in \mathcal{T}$. Then

$$\mathcal{F} = \left\{ F \in \left[ \frac{V}{k} \right] \mid X \subseteq F, \dim(F \cap M) \geq t + 1 \right\} \cup \left[ \frac{M}{k} \right].$$

**Proof.** It follows from the proof of Lemma 3.3 that, for any $F \in \mathcal{F} \setminus \mathcal{F}_X$, we have $M = F + X$ and hence $F \in \left[ \frac{M}{k} \right]$. Let $\mathcal{A}' = \left\{ F \in \left[ \frac{V}{k} \right] \mid X \subseteq F, \dim(F \cap M) \geq t + 1 \right\}$ and $\mathcal{F}'$ be a fixed member of $\mathcal{F} \setminus \mathcal{F}_X$. For any $F \in \mathcal{F}_X$, we have $\dim(F \cap F') \geq t$, $\dim(F' \cap X) \leq t - 1$ and $M = F' + X$. Thus $\dim(F \cap M) \geq t + 1$ and so $\mathcal{F}_X \subseteq \mathcal{A}'$. Note that $\mathcal{A}' \cup \left[ \frac{M}{k} \right]$ is a $t$-intersecting family. Therefore, $\mathcal{F} = \mathcal{A}' \cup \left[ \frac{M}{k} \right]$ by the maximality of $\mathcal{F}$. 

**Lemma 3.5** Let $n, k, t, \mathcal{F}$ and $\mathcal{T}$ be as in Assumption 1, and set $M = \sum_{T \in \mathcal{T}} T$. Suppose that $\tau_t(\mathcal{T}) = t$, $\dim(M) = k$ and $X$ is a $t$-subspace of $V$ which is contained in each $T \in \mathcal{T}$. Set
\[ C = M + \sum_{F \in F \setminus F_X} F \] and \( c = \dim(C) \). Then either \( k + 2 \leq c \leq 2k - t \) or \( c = n \). Moreover, the following hold:

(i) if \( k + 2 \leq c \leq 2k - t \), then \( F = H_2(X, M, C) \); and

(ii) if \( c = n \), then \( t \neq k - 2 \) and \( F = H_2(X, M, V) \).

Proof. It follows from the proof of Lemma 3.3 that, for any \( F \in F \setminus F_X \), we have \( \dim(F \cap X) = t - 1 \) and \( M \subseteq X + F \). Since \( X \subseteq M \), we then have \( \dim(F \cap M) = k - 1 \). Note that \( c \geq k + 1 \) by the definition of \( c \).

Choose \( F_1 \in F \setminus F_X \). Then \( \dim(F_1 + M) = k + 1 \). If \( c > k + 1 \), then there exists \( F_2 \in F \setminus F_X \) such that \( F_2 \nsubseteq F_1 + M \), which implies \( F_2 \cap (F_1 + M) = F_2 \cap M \). Similarly, if \( c > k + 2 \), then there exists \( F_3 \in F \setminus F_X \) such that \( F_3 \nsubseteq F_1 + F_2 + M \), which implies \( F_3 \cap (F_1 + F_2 + M) = F_3 \cap M \). Continuing, by mathematical induction we can prove that there exist \( F_1, F_2, \ldots, F_{c-k} \in F \setminus F_X \) such that

\[ F_i \cap \left( M + \sum_{j=1}^{i-1} F_j \right) = F_i \cap M \tag{14} \]

for \( i \in \{1, 2, \ldots, c-k\} \). If there exists \( F' \in F \) such that \( F' \cap M = X \), then for any \( i \in \{1, 2, \ldots, c-k\} \), there exists \( y_i \in F_i \setminus M \) such that \( y_i \in F' \) as \( \dim(F' \cap F_i) \geq t \) and \( \dim(F' \cap F_i \cap M) = t - 1 \). Let \( x_1, x_2, \ldots, x_t \) be a basis of \( X \). By (14) and the choice of \( F_1, F_2, \ldots, F_{c-k} \), one can easily show that \( x_1, x_2, y_1, \ldots, y_{c-k} \) are linearly independent in \( F' \).

Suppose that \( c \geq 2k - t + 1 \). If there exists \( F' \in F \) such that \( F' \cap M = X \), then by the above discussion we can obtain \( c - k + t \) vectors in \( F' \) which are linearly independent, but this is impossible. Thus \( \dim(A_1 \cap M) \geq t + 1 \) for any \( A_1 \in F_X \). By the maximality of \( F \), it is readily seen that any \( k \)-subspace \( A_2 \) of \( V \) satisfying \( \dim(A_2 \cap X) = t - 1 \) and \( \dim(A_2 \cap M) = k - 1 \) must be in \( F \). Hence \( C = V \) and \( c = n \).

On the other hand, we have \( c \geq k + 2 \), for otherwise we would have \( c = k + 1 \) and \( \dim(T \cap F) \geq t \) for any \( T \in \binom{C}{t+1} \) with \( X \subseteq T \) and any \( F \in F \), which imply \( T \subseteq M \), a contradiction.

So far we have proved that either \( k + 2 \leq c \leq 2k - t \) or \( c = n \). It remains to prove (i) and (ii). Denote \( A = A(X, M), B = B(X, M, C) \) and \( C = C(X, M, C) \).

(i) Suppose that \( k + 2 \leq c \leq 2k - t \). Since \( \dim(F \cap X) = t - 1 \) and \( \dim(F \cap M) = k - 1 \) for any \( F \in F \setminus F_X \), we have \( F \setminus F_X \subseteq C \). For any \( F' \in F_X \), if \( \dim(F' \cap M) \geq t + 1 \), then \( F' \in A \); if \( F' \cap M = X \), then \( \dim(F' \cap C) = c-k+t \) and so \( F' \in B \) by the discussion above. Thus \( F \subseteq A \cup B \cup C \). It is routine to verify that \( A \cup B \cup C \) is a \( t \)-intersecting family. Thus, by the maximality of \( F \), we obtain \( F = A \cup B \cup C \).

(ii) Suppose that \( c = n \). Then \( F = A \cup C \) by the discussion in (i) and the maximality of \( F \). If \( t = k - 2 \), then \( F = H_2(X, M, V) = H_3(M) \), which implies that \( \tau_1(F) = k \), a contradiction. \( \square \)

**Lemma 3.6** Let \( n, k, t, F \) and \( T \) be as in Assumption 1. Suppose that \( \tau_1(T) = t + 1 \) and \( T = \binom{Z}{t+1} \) for some \((t+2)\)-subspace \( Z \) of \( V \). Then \( F = H_3(Z) \).
Proof. Since $\mathcal{T} = \left[ \begin{array}{c} Z \\ {t+1} \end{array} \right]$, we have $\dim(F \cap Z) \geq t$ for any $F \in \mathcal{F}$. If there exists $F' \in \mathcal{F}$ such that $\dim(F' \cap Z) = t$, then there exists $T' \in \mathcal{T}$ such that $\dim(F' \cap T') = t-1$, a contradiction. Hence $\mathcal{F} \subseteq \mathcal{H}_3(Z)$. Since $\mathcal{H}_3(Z)$ is t-intersecting and $\mathcal{F}$ is maximal t-intersecting, we obtain $\mathcal{F} = \mathcal{H}_3(Z)$ as desired. \hfill $\square$

Lemma 3.7 Let $n$, $k$, $t$, $\mathcal{F}$ and $\mathcal{T}$ be as in Assumption 1.

(i) If $|\mathcal{T}| = 1$, then

$$|\mathcal{F}| \leq \left[ \begin{array}{c} n-t-1 \\ k-t-1 \end{array} \right] + q \left[ \begin{array}{c} t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right].$$

(ii) Suppose that $|\mathcal{T}| \geq 2$ and for some $t$-subspace $X$ and $l$-subspace $M$ of $V$ with $X \subseteq M$, $\mathcal{T}$ is a collection of $(t+1)$-subspaces of $V$ containing $X$ and contained in $M$. Then

$$|\mathcal{F}| \leq \left[ \begin{array}{c} l-t \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-1 \\ k-t-1 \end{array} \right] + q^{l-t} \left[ \begin{array}{c} k-l+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right]$$

$$+ q^{k+1-t} \left[ \begin{array}{c} t \\ 1 \end{array} \right] \left[ \begin{array}{c} n-l \\ k-l+1 \end{array} \right].$$

(15)

Moreover, if $l = t+2$, then

$$|\mathcal{F}| \leq \left[ \begin{array}{c} l-t \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-1 \\ k-t-1 \end{array} \right] + q^{l-t} \left[ \begin{array}{c} k-l+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right]$$

$$+ q^{t} \left[ \begin{array}{c} t \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} n-t-2 \\ k-t-2 \end{array} \right].$$

(16)

(iii) If $|\mathcal{T}| \geq 2$ and $\mathcal{T} = \left[ \begin{array}{c} Z \\ {t+1} \end{array} \right]$ for some $(t+2)$-subspace $Z$ of $V$, then $|\mathcal{F}| = h_3(t+2)$.

Proof. (i) Let $T$ be the unique $(t+1)$-subspace of $V$ in $\mathcal{T}$. Since $\dim(T \cap F) \geq t$ for any $F \in \mathcal{F}$, we have

$$\mathcal{F} = \mathcal{F}_T \cup \left( \bigcup_{S \in \left[ \begin{array}{c} T \\ {t+1} \end{array} \right]} (\mathcal{F}_S \setminus \mathcal{F}_T) \right).$$

(17)

We now give an upper bound on $|\mathcal{F}_S \setminus \mathcal{F}_T|$ for any fixed $S \in \left[ \begin{array}{c} T \\ {t+1} \end{array} \right]$. Since $\tau_1(\mathcal{F}) = t+1$, there exists $F' \in \mathcal{F} \setminus \mathcal{F}_S$ such that $\dim(S \cap F') = t-1$ as $\dim(F' \cap T) \geq t$. So $T = (F' \cap T) + S$ and $T \subseteq F' + S$. For any $F \in \mathcal{F}_S \setminus \mathcal{F}_T$, we have $(F \cap F') + S \subseteq F \cap (F' + S)$. Since $\dim(F \cap F') \geq t$ and $\dim(F \cap F' \cap S) \leq t-1$, we have $\dim(F \cap (F' + S)) \geq t + 1$. Hence there exists a $(t+1)$-subspace $H$ such that $H \neq T$, $S \subseteq H \subseteq S + F'$ and $H \subseteq F$. Therefore,

$$\mathcal{F}_S \setminus \mathcal{F}_T = \bigcup_{S \in \left[ \begin{array}{c} T \\ {t+1} \end{array} \right], H \neq T, \dim H = t+1} \mathcal{F}_H.$$ 

(18)

Consider an arbitrary $(t+1)$-subspace $H$ of $V$ satisfying $H \neq T$ and $S \subseteq H \subseteq S + F'$. Since $T$ is the unique $(t+1)$-subspace of $V$ such that $\dim(T \cap F) \geq t$ for $F \in \mathcal{F}$, there exists $A \in \mathcal{F}$ such that $\dim(H \cap A) < t$. Hence $\dim(H \cap A) = t-1$ as $\dim(H \cap T) = \dim(S) = t$.
and \( \dim(T \cap A) \geq t \). By Lemma 3.2, we have \( |\mathcal{F}_H| \leq \left[ \frac{k-t+1}{1} \right]_{[n-t]}^{[n-t-2]} \). By Lemma 2.3, we obtain \( |\mathcal{F}| \leq \left[ \frac{n-t-1}{k-t-1} \right] \) and
\[
\left\{ H \in \left[ \frac{S+F'}{t+1} \right] | S \subseteq H, H \neq T \right\} = \left[ \frac{k-t+1}{1} \right] - 1 = q^{\frac{k-t}{1}}.
\]
It follows from (17) and (18) that
\[
|\mathcal{F}| \leq \left[ \frac{n-t-1}{k-t-1} \right] + \left[ \frac{t+1}{1} \right] \cdot q^{\frac{k-t}{1}} \left[ \frac{k-t+1}{1} \right] \left[ \frac{n-t-1}{k-t-1} \right].
\]

(ii) We will prove the desired upper bound on \( |\mathcal{F}| \) by establishing upper bounds on \( |\mathcal{F}_X| \) and \( |\mathcal{F} \setminus \mathcal{F}_X| \). Since \( \tau_i(\mathcal{F}) = t+1 \), we have \( \dim(F \cap X) \geq t+1 \) for any \( F \in \mathcal{F} \), and there exists \( F' \in \mathcal{F} \) such that \( \dim(X \cap F') = t+1 \). It follows from the proof of Lemma 3.3 that \( X \subseteq M \subseteq X + F' \).

For any \( F \in \mathcal{F}_X \), we have \( \dim(F \cap (X + F')) \geq t+1 \) as \( X \subseteq F \) and \( \dim(F \cap F') \geq t \). So
\[
\mathcal{F}_X = \bigcup_{X \subseteq H_1, H_1 \in [\frac{M}{t+1}]} \mathcal{F}_H_1 \bigcup_{X \subseteq H_2, H_2 \in [\frac{M}{t+1}]} \mathcal{F}_{H_2}.
\]
Since by Lemma 2.3, \( |\mathcal{F}_{H_1}| \leq \left[ \frac{n-(t+1)}{k-(t+1)} \right] \) for any \( H_1 \in [\frac{M}{t+1}] \), we have \( \bigcup_{X \subseteq H_1, H_1 \in [\frac{M}{t+1}]} \mathcal{F}_H_1 \leq \left[ \frac{n-(t+1)}{k-(t+1)} \right] \). For any \( H_2 \in [\frac{X+F'}{t+1}] \) with \( X \subseteq H_2 \), we have \( H_2 \notin \mathcal{T} \) and so there exists \( A \in \mathcal{F} \) such that \( \dim(H_2 \cap A) < t \). Hence \( \dim(H_2 \cap A) = t-1 \) as \( \dim(A \cap X) \geq t-1 \). It follows from Lemma 3.2 that \( |\mathcal{F}_H_2| \leq \left[ \frac{k-l-1}{1} \right]_{[n]}^{[n-(t+1)-1]} \). Note from Lemma 2.3 that
\[
\left\{ H_2 \in \left[ \frac{X+F'}{t+1} \right] \mid X \subseteq H_2 \right\} = \left[ \frac{k+1-t}{1} \right] - \left[ \frac{k-t}{1} \right] = q^{k-t} \left[ \frac{k-l+1}{1} \right].
\]
Therefore,
\[
|\mathcal{F}_X| \leq \left[ \frac{l-t}{1} \right] \left[ \frac{n-t-1}{k-t-1} \right] + q^{k-t} \left[ \frac{k-l+1}{1} \right] \left[ \frac{k-t+1}{1} \right] \left[ \frac{n-t-2}{k-t-2} \right].
\]

For any \( F \in \mathcal{F} \setminus \mathcal{F}_X \) and any \( T \in \mathcal{T} \), since \( \dim(F \cap X) = t-1 \) and \( X \not\subseteq F \cap T \), we have \( T = (F \cap T) + X \subseteq F + X \). Thus, for any \( F \in \mathcal{F} \setminus \mathcal{F}_X \), we have \( M = \sum_{T \subseteq T} T \subseteq F + X \), which implies \( \dim(M \cap F) = l-1 \). Hence \( \mathcal{F} \setminus \mathcal{F}_X \subseteq \left\{ F \in [\frac{M}{k}] \mid \dim(F \cap M) = l-1, X \not\subseteq F \right\} \). Observe from Lemma 2.3 that the number of \( k \)-subspaces \( F \) of \( V \) satisfying \( \dim(F \cap M) = l-1 \) is \( q^{k-l+1} \left[ \frac{n-t}{k+l+1} \right]_{[l]} \), and the number of \( k \)-subspaces \( F \) of \( V \) satisfying \( \dim(F \cap M) = l-1 \) and \( X \subseteq F \) is \( N'(t; t; k, l-1; n, n-l) \). By Lemma 2.4, we then have
\[
|\mathcal{F} \setminus \mathcal{F}_X| \leq q^{k-l+1} \left[ \frac{n-l}{k-l+1} \right]_{[l]} - N'(t; t; k, l-1; n, n-l)
\]
\[
= q^{k-l+1} \left[ \frac{n-l}{k-l+1} \right]_{[l]} - q^{k-l+1} \left[ \frac{n-l}{k-l+1} \right]_{[l-t]} - q^{k-t+1} \left[ \frac{n-l}{k-l+1} \right]_{[1]}
\]
\[
= q^{k-l+1} \left[ \frac{l}{k-l+1} \right]_{[n-l]}
\]
Combining (20) and (21), we obtain (15).
Now let us consider the case when \( l = t + 2 \). From the discussion above, we have \( \dim(M \cap F) = l - 1 = t + 1 \) for any \( F \in \mathcal{F} \setminus \mathcal{F}_X \), which implies

\[
\mathcal{F} \setminus \mathcal{F}_X \subseteq \bigcup_{X \not\subseteq L, \, L \in [t+1]^{M}} \mathcal{F}_L.
\]

For any \( L \in [t+1]^{M} \) with \( X \not\subseteq L \), since \( L \not\in \mathcal{T} \) and \( \dim(F \cap M) \geq t \) for any \( F \in \mathcal{F} \), there exists \( F' \in \mathcal{F} \) such that \( \dim(F' \cap L) = t \). So \( |\mathcal{F}_L| \leq \binom{k-t+1}{1} \binom{n-t-2}{k-t-2} \) by Lemma 3.2. Since by Lemma 2.3 the number of \((t+1)\)-subspaces \( L \) of \( M \) with \( X \not\subseteq L \) is equal to \( \binom{t+2}{t+1} - \binom{t}{t+1} \), we have

\[
|\mathcal{F} \setminus \mathcal{F}_X| \leq q^{\binom{t}{1}} \binom{k-t+1}{1} \binom{n-t-2}{k-t-2}.
\]

Combining (20) and (22), we obtain (15).

(iii) The desired equality follows from Lemma 3.6 and (6).

\[\square\]

### 3.2 The case \( \tau_t(\mathcal{F}) \geq t + 2 \)

In [4], Blokhuis et al. proved the following upper bound for \( |\mathcal{F}| \) in the case when \( t = 1 \).

**Lemma 3.8** ([4]) Let \( n \) and \( k \) be positive integers with \( 6 \leq 2k \leq n \), and let \( \mathcal{F} \subseteq \binom{V}{k} \) be a maximal intersecting family with \( 3 \leq \tau_1(\mathcal{F}) = m \leq k \). Let \( \mathcal{T} \) be the set of all \( m \)-subspaces \( T \) of \( V \) which satisfy \( \dim(T \cap F) \geq 1 \) for any \( F \in \mathcal{F} \). Then the following hold:

(i) if \( m = k \), then \( |\mathcal{F}| \leq \binom{k}{1}^k \);

(ii) if \( m < k \) and \( |\mathcal{T}| \geq 2 \), then

\[
|\mathcal{F}| \leq \binom{m-1}{1} \binom{k}{1}^{m-1} \binom{n-m}{k-m} + q^{2(m-1)} \binom{k-1}{1}^{m-2} \binom{n-m}{k-m};
\]

(iii) if \( m < k \) and \( |\mathcal{T}| = 1 \), then

\[
|\mathcal{F}| \leq \binom{m-1}{1} \binom{k-1}{1}^{m-2} \binom{n-m}{k-m} + q^{m-1} \binom{k-m+1}{1} \binom{m}{1} \binom{k-1}{1}^{m-1} \binom{n-m-1}{k-m-1}.
\]

Using this lemma, we now prove the following bound for \( \mathcal{F} \) with \( \tau_1(\mathcal{F}) < k \).

**Lemma 3.9** Let \( n \) and \( k \) be positive integers with \( 9 \leq 2k + 3 \leq n \), and let \( \mathcal{F} \subseteq \binom{V}{k} \) be a maximal intersecting family with \( 3 \leq \tau_1(\mathcal{F}) < k \). Then

\[
|\mathcal{F}| \leq (q+1) \binom{k}{1}^2 \binom{n-3}{k-3} + q^4 \binom{k}{1} \binom{n-3}{k-3}.
\]
Proof. Let $u_1(n, k, m)$ and $u_2(n, k, m)$ be the upper bounds in (23) and (24), respectively. By Lemma 2.1(iii) and the assumption $n \geq 2k + 3$, for $3 \leq m < k$, we have

$$\frac{u_1(n, k, m) - u_2(n, k, m)}{\binom{n}{k}^{m-2}} = \frac{q^{n-m} - 1}{q^{k-m} - 1} \left( q^m \binom{m-1}{1} \binom{k-m}{1} + q^{2(m-1)} \right) - q^{m-1} \binom{k-m+1}{1} \binom{m}{1} \binom{k}{1} > q^{n-k} \binom{q^{m+k-3} + q^{2(m-1)}} - q^{2(k+m)} > 0.$$ 

Thus, for $m \in \{3, 4, \ldots, k - 1\}$, we have

$$u_1(n, k, m) > u_2(n, k, m).$$ \hspace{1cm} (26)

For any $m \in \{3, 4, \ldots, k - 2\}$, by Lemma 2.1(iii), (23) and $n \geq 2k + 3$, we have

$$\frac{u_1(n, k, m) - u_1(n, k, m + 1)}{\binom{n}{k}^{m-2}} = \frac{q^{n-m} - 1}{q^{k-m} - 1} \left( q^m \binom{m-1}{1} \binom{k-m}{1} + q^{2(m-1)} \right) - q^{m-1} \binom{k-m+1}{1} \binom{m}{1} \binom{k}{1} > q^{n-k} \binom{q^{m+k-3} + q^{2(m-1)}} - q^{2(k+m)} > 0.$$ 

So $u_1(n, k, m)$ is decreasing as $m \in \{3, 4, \ldots, k - 1\}$ increases. Combining this with (26) and Lemma 3.8, we obtain $|\mathcal{F}| \leq u_1(n, 3, m)$ for $m = \tau_1(\mathcal{F})$, which yields (25) as $u_1(n, 3, m)$ is exactly the right-hand side of (25). \hfill \Box

Lemma 3.10 Let $n, k$ and $t$ be positive integers with $8 \leq 2k \leq n$ and $2 \leq t \leq k - 2$, and let $\mathcal{F} \subseteq \binom{[V]}{k}$ be a maximal $t$-intersecting family with $t + 2 \leq \tau_1(\mathcal{F}) = m \leq k$. Then

$$|\mathcal{F}| \leq \binom{m}{t} \binom{k}{1}^{m-t-2} \binom{k-t+1}{1} \binom{n-m}{k-m}.$$ 

Moreover, if $n \geq 2k + t + 1$, then

$$|\mathcal{F}| \leq \binom{t+2}{2} \binom{k-t+1}{1} \binom{n-t-2}{k-t-2}.$$ 

Proof. Let $T$ be an $m$-subspace of $V$ which satisfies $\dim(T \cap F) \geq t$ for any $F \in \mathcal{F}$. Then $\mathcal{F} = \bigcup_{H \subseteq [V]} \mathcal{F}_H$ and hence there exists $H_1 \subseteq [V]$ such that $|\mathcal{F}| \leq \binom{m}{t} |\mathcal{F}_{H_1}|$. If $m \geq t + 3$, using Lemma 3.1 repeatedly, then there exist $H_2 \subseteq [V_{t+1}], H_3 \subseteq [V_{t+2}], \ldots, H_{m-t-1} \subseteq [V_{m-2}]$ such that $H_i \subseteq H_{i-1}$ and $|\mathcal{F}_{H_i}| \leq \binom{k}{1} |\mathcal{F}_{H_{i+1}}|$ for each $i \in \{1, 2, \ldots, m - t - 2\}$. Thus there exists $H' \subseteq [V_{m-2}]$ such that

$$|\mathcal{F}| \leq \binom{m}{t} \binom{k}{1}^{m-t-2} |\mathcal{F}_{H'}|.$$ 

Since $\tau_t(\mathcal{F}) > m - 2$, we have $\mathcal{F} \setminus \mathcal{F}_{H'} \neq \emptyset$ and $\dim(F \cap H') \leq t - 1$ for any $F \in \mathcal{F} \setminus \mathcal{F}_{H'}$. 

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Case 1. \( \dim(F \cap H') \leq t - 2 \) for all \( F \in \mathcal{F} \setminus \mathcal{F}_{H'} \).

Let \( F_1 \) be a fixed \( k \)-subspace in \( \mathcal{F} \setminus \mathcal{F}_{H'} \). Let \( s_1 = \dim(F_1 \cap H') \) so that \( 0 \leq s_1 \leq t - 2 \). By Lemma 3.2, we have

\[
|\mathcal{F}_{H'}| \leq \binom{k - s_1}{t - s_1} \binom{n - m + 2 - t + s_1}{k - m + 2 - t + s_1},
\]

which implies that

\[
|\mathcal{F}| \leq \binom{m}{t} \binom{k}{1} \binom{m - t - 2}{1} \binom{k - s_1}{t - s_1} \binom{n - m + 2 - t + s_1}{k - m + 2 - t + s_1}.
\] (27)

Let

\[
g(s) = \binom{k - s}{t - s} \binom{n - m + 2 - t + s}{k - m + 2 - t + s}
\]

for \( s \in \{0, 1, \ldots, t - 2\} \). By \( n \geq 2k \) and Lemma 2.1(ii), we have

\[
\frac{g(s + 1)}{g(s)} = \frac{\left(q^{t-s} - 1\right)(q^{n-m+3-t+s} - 1)}{(q^{k-s} - 1)(q^{k-m+3-t+s} - 1)} > q^{n-2k+t-1} > 1
\]

for \( s \in \{0, 1, \ldots, t - 3\} \). That is, the function \( g(s) \) is increasing as \( s \in \{0, 1, \ldots, t - 2\} \) increases. This together with (27) yields

\[
|\mathcal{F}| \leq \binom{m}{t} g(s_1) \leq \binom{m}{t} g(t - 2) = \binom{m}{t} \binom{k}{1} \binom{m - t - 2}{2} \binom{k - t + 2}{k - t - 2}.
\] (28)

Case 2. There exists \( F_2 \in \mathcal{F} \setminus \mathcal{F}_{H'} \) such that \( \dim(F_2 \cap H') = t - 1 \).

By Lemma 3.1, there exists an \( (m-1) \)-subspace \( H'' \) such that \( |\mathcal{F}_{H''}| \leq \binom{k-t+1}{1} |\mathcal{F}_{H'''}| \).

Hence \( |\mathcal{F}| \leq \binom{m}{t} \binom{k}{1} \binom{m - t - 2}{1} \binom{k - t + 1}{1} |\mathcal{F}_{H'''}| \). Since \( \tau_1(\mathcal{F}) > m - 1 \), there exists \( F_3 \in \mathcal{F} \) such that \( \dim(F_3 \cap H'') \leq t - 1 \).

If \( \dim(F_3 \cap H'') = t - 1 \), then there exists an \( m \)-subspace \( H''' \) with \( H'' \subseteq H''' \) such that \( |\mathcal{F}_{H'''}| \leq \binom{k-t+1}{1} |\mathcal{F}_{H'''}| \). Since \( |\mathcal{F}_{H'''}| \leq \binom{n-m}{k-m} \) by Lemma 2.3, we have

\[
|\mathcal{F}| \leq \binom{m}{t} \binom{k}{1} \binom{m - t - 2}{1} \binom{k - t + 1}{1} \binom{n - m}{k - m}.
\] (29)

Suppose that \( \dim(F_3 \cap H'') = s_2 \leq t - 2 \). By Lemma 3.2, we have

\[
|\mathcal{F}_{H''}| \leq \binom{k - s_2}{t - s_2} \binom{n - m + 1 - t + s_2}{k - m + 1 - t + s_2}.
\]

Similar to Case 1, it is straightforward to verify that the function \( \binom{k-s}{t-s} \binom{n-m+1-t+s}{k-m+1-t+s} \) is increasing as \( s \in \{0, 1, \ldots, t - 2\} \) increases. Hence

\[
|\mathcal{F}| \leq \binom{m}{t} \binom{k}{1} \binom{m - t - 2}{1} \binom{k - t + 1}{1} \binom{n - m - 1}{k - m - 1}.
\] (30)

By Lemma 2.1 (ii) and \( n \geq 2k \), it is straightforward to verify that

\[
\binom{k - t + 1}{1}^2 \binom{n - m}{k - m} \geq \max \left\{ \binom{k - t + 2}{1} \binom{n - m}{k - m}, \binom{k - t + 1}{1} \binom{k - t + 2}{1} \binom{n - m - 1}{k - m - 1} \right\}.
\]
This together with (28), (29) and (30) yields
\[ |\mathcal{F}| \leq \binom{m}{t} \binom{k}{1}^{m-t-2} \binom{k-t+1}{1}^{2} \binom{n-m}{k-m}. \]  
(31)

Let
\[ p(m') = \binom{m'}{t} \binom{k}{1}^{m'-t-2} \binom{n-m'}{k-m'} \]
for \( m' \in \{t + 2, t + 3, \ldots, k\} \). By \( n \geq 2k \) and Lemma 2.1 (ii), we have
\[ \frac{p(m')}{p(m' + 1)} = \frac{(q^{m'-t+1} - 1)(q - 1)(q^{n-m'} - 1)}{(q^{m'+1} - 1)(q^k - 1)(q^{k-m'} - 1)} > q^{n-2k-t-1} \geq 1 \]
for \( m' \in \{t + 2, t + 3, \ldots, k - 1\} \). That is, the function \( q(m') \) is decreasing as \( m' \in \{t + 2, t + 3, \ldots, k\} \) increases. This together with (31) yields
\[ |\mathcal{F}| \leq \binom{k-t+1}{1}^2 p(t+2) \leq \binom{t+2}{2} \binom{k-t+1}{1}^2 \binom{n-t-2}{k-t-2}. \]
Therefore, the desired upper bounds follow.

\[ \square \]

4 Proofs of Theorems 1.1 and 1.2

\textit{Proof of Theorem 1.1.} Let \( n, k \) and \( t \) be positive integers with \( k \geq 3 \). Suppose that \( n \geq 2k + t + \min\{4, 2t\} \). That is, if \( t = 1 \), then \( n \geq 2k + 3 \), and if \( t \geq 2 \), then \( n \geq 2k + t + 4 \). Let \( \mathcal{F} \subseteq \binom{V}{t} \) be a maximal non-trivial \( t \)-intersecting family which is not any of the exceptional families in (i) and (ii) of Theorem 1.1. Set
\[ f_2(n, k, t) = \frac{f(n, k, t) - |\mathcal{F}|}{\binom{n-t-2}{k-t-2}}, \]
where the function \( f \) is as defined in (8). It suffices to prove \( f(n, k, t) > |\mathcal{F}| \) or equivalently \( f_2(n, k, t) > 0 \).

Let \( \mathcal{T} \) be the set of all \( \tau_t(\mathcal{F}) \)-subspaces \( T \) of \( V \) which satisfy \( \dim(T \cap F) \geq t \) for any \( F \in \mathcal{F} \).

Case 1. \( \tau_t(\mathcal{F}) = t + 1 \).

Case 1.1. \( |\mathcal{T}| = 1 \).

In this case, by Lemma 3.7(i), we have
\[ q^{-1} f_2(n, k, t) \geq \binom{n-t-1}{1} - \binom{k-t}{2} - \binom{t+1}{1} \binom{k-t}{1} \binom{k-t+1}{1}. \]

If \( (t, q) = (1, 2) \), then \( n \geq 2k + 3 \) and
\begin{align*}
\frac{3}{2} f_2(n, k, t) & \geq 3 \cdot (2^{n-2} - 1) - (2^{k-1} - 1)(2^{k-2} - 1) - 9 \cdot (2^{k-1} - 1)(2^k - 1) \\
& = 3 \cdot 2^{n-2} - 37 \cdot 2^{k-3} + 9 \cdot 2^k + 10 \cdot 2^{k-1} + 2^{k-2} - 13 \\
& > 0
\end{align*}
as desired. Suppose that \( n \geq 2k + 3 \) and \( q \geq 3 \) if \( t = 1 \), and that \( n \geq 2k + t + 4 \) if \( t \geq 2 \). By Lemma 2.1(iii)(iv), we have

\[
q^{-1} f_2(n, k, t) > q^{n-t-2} - q^{2(k-t-1)} - 8q^{2k-t-1} > 0
\]
as desired.

**Case 1.2.** \( |T| \geq 2 \) and \( \tau_t(T) = t \).

Let \( M = \sum_{T \in \mathcal{T}} T \) and \( l = \dim(M) \). Since \( \mathcal{F} \) is a maximal non-trivial \( t \)-intersecting family other than any of the exceptional families in Theorem 1.1, we have \( l \leq k - 1 \) by Lemmas 3.4 and 3.5.

Let us first consider the case when \( l = t + 2 \). Then \( k \geq 4 \) as \( l \leq k - 1 \). By (16) and Lemma 2.1(iii), we have

\[
q^{-2} f_2(n, k, t) \geq q^{n-t-1} \left[ \begin{array}{c} k-t-1 \\ 1 \end{array} \right] - q^{-1} \left[ \begin{array}{c} k-t \\ 2 \end{array} \right] - q^{n-t} \left[ \begin{array}{c} k-t-1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] - \left[ \begin{array}{c} t \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right]
\]

Thus, if \( t = 1 \), then

\[
q^{-2} f_2(n, k, t) > q^{k+1} \left( n-k-5 - q^{k-6} - q^{k-3} - 1 \right) > 0
\]
as \( n \geq 2k + 3 \). If \( t \geq 2 \), then

\[
n - t - 3 \geq \max\{2k - 2t, k + 1\} + 2
\]
as \( n \geq 2k + t + 4 \), and hence the inequality above implies \( q^{-2} f_2(n, k, t) > 0 \). In either case we have \( f_2(n, k, t) > 0 \) as desired.

Now consider the case when \( t + 3 \leq l \leq k - 1 \). Then

\[
\left[ \begin{array}{c} n-l \\ k-l+1 \end{array} \right] \leq \left[ \begin{array}{c} n-t-3 \\ k-t-2 \end{array} \right].
\]

Since \( n - t - 1 \geq \max\{2k - 2t + 2, t + 3\} + 2 \) and \( l \leq k - 2 \), by (15) we have

\[
f_2(n, k, t) \geq \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} \left[ \begin{array}{c} k-t \\ 2 \end{array} \right] - q^{n-t} \left[ \begin{array}{c} k-l+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t \\ 1 \end{array} \right] - q^{k+1-t} \left[ \begin{array}{c} t \\ 1 \end{array} \right] \cdot \frac{n-l}{k-l+1} \frac{k+l+1}{k-t-2} - q^{n-t-1} \left[ \begin{array}{c} k-t-1 \\ 2 \end{array} \right] - q^{n-t} \left[ \begin{array}{c} k-l+1 \\ 1 \end{array} \right] \left[ \begin{array}{c} k-t+1 \\ 1 \end{array} \right] - q^{k+1-t} \left[ \begin{array}{c} t \\ 1 \end{array} \right] \cdot \frac{q^n - 1}{q^{n-t-2} - 1}
\]

as desired.

**Case 1.3.** \( |T| \geq 2 \) and \( \tau_t(T) = t + 1 \).

In this case, by Lemma 3.7(i) and Lemma 2.8(i), we have \( f(n, k, t) > |F| \) if \( 1 \leq t \leq \frac{k}{2} - \frac{3}{2} \).

**Case 2.** \( t + 2 \leq \tau_t(F) \leq k \).
Case 2.1. \( t = 1 \).

Since \( t = 1 \), we have \( 3 \leq \tau_1(\mathcal{F}) \leq k \). Consider the case \( \tau_1(\mathcal{F}) = k \) first. By Lemma 3.8(i), in this case we have

\[
f(n, k, t) - |\mathcal{F}| \geq \left[ \frac{n-1}{k-1} \right] \left[ \frac{n-3}{k-2} \right] - q \left[ \frac{k-1}{2} \right] \left[ \frac{n-3}{k-3} \right] - \left[ \frac{k^k}{1} \right].
\]

(32)

If \((n, k, q) = (9, 3, 2)\), then \( f(n, k, 1) - |\mathcal{F}| \geq 36 > 0 \). If \((n, k) = (9, 3)\) and \( q \geq 3 \), then \( f(n, k, 1) - |\mathcal{F}| \geq q^7 + q^6 - q^5 - 4q^4 - 5q^3 - 4q^2 - 2q > 0 \). Since when \( k = 3 \) the right-hand side of (32) is increasing with \( n \), we have \( f(n, 3, 1) - |\mathcal{F}| > 0 \) for \( n \geq 10 \). If \((n, k, q) = (11, 4, 2)\), then \( f(n, k, 1) - |\mathcal{F}| \geq 249850 > 0 \). If \((n, k) = (11, 4)\) and \( q \geq 3 \), or \( n = 2k + 3 \) and \( k \geq 5 \), or \( n \geq 2k + 4 \) and \( k \geq 4 \), then by Lemma 2.1(iii)(iv),

\[
f(n, k, 1) - |\mathcal{F}| \geq \left[ \frac{n-3}{k-3} \right] (q^{n-2} - q^{2k-3}) - 2k \cdot q^{k(k-1)}
\]

\[
> q^{(k-3)(n-k)+n-3} - 2k \cdot q^{k(k-1)}
\]

\[
= 0.
\]

Now let us consider the case when \( 3 \leq \tau_1(\mathcal{F}) < k \). In this case, by Lemma 3.9, we have

\[
f_2(n, k, 1) \geq \frac{q^{n-2} - 1}{q^{k-2} - 1} \left[ \frac{k-1}{1} \right] - q \left[ \frac{k-1}{2} \right] - (q + 1) \left[ \frac{k^2}{1} \right] - q^4 \left[ \frac{k}{1} \right].
\]

If \( q \geq 3 \), then by \( n \geq 2k + 3 \), \( k \geq 4 \) and Lemma 2.1(iii)(iv), we obtain

\[
f_2(n, k, 1) > q^{n-2} - q^{2k-3} - 4(q + 1)q^{2k-2} - q^{k+4}
\]

\[
\geq q^{2k-3} (q^4 - 1 - 4q^2 - 4q - q^{2k-4})
\]

\[
> 0.
\]

If \( q = 2 \), then by \( n \geq 2k + 3 \), \( k \geq 4 \) and \((2n-2 - 1)(2^{k-1} - 1)/(2^{k-2} - 1) > 2^{2k+2} \) we obtain

\[
f_2(n, k, 1) \geq \frac{(2n-2 - 1)(2^{k-1} - 1)}{2^{k-2} - 1} - \frac{2}{3} (2^{k-1} - 1)(2^{k-2} - 1) - 3(2^{k-1})^2 - 2^4(2^k - 1)
\]

\[
> \frac{11}{3} \cdot 2^{2k-2} - 19 \cdot 2^{k-1} + \frac{37}{3}
\]

\[
> 0.
\]

Case 2.2. \( t \geq 2 \).

By Lemma 3.10, we have

\[
f_2(n, k, t) \geq \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} \left[ \frac{k - t}{1} \right] - q \left[ \frac{k - t}{2} \right] - \left[ \frac{t + 2}{2} \right] \left[ \frac{k - t + 1}{1} \right]^2.
\]

Assume that \( n = 2k + t + 4 \) and \( q = 2 \) first. We have

\[
f_2(n, k, t)
\]

\[
= \frac{(2^{k+3} - 1)(2^{k-t} - 1)}{2^{k-t-1} - 1} - \frac{2}{3} (2^{k-t} - 1)(2^{k-t-1} - 1) - \frac{1}{3} (2^{t+2} - 1)(2^{t+1} - 1)(2^{k-t+1} - 1)^2
\]

\[
> 2^{2k+4} - \frac{1}{3} \cdot 2^{2k-2t} - \frac{1}{3} \cdot 2^{2k+5}
\]

\[
> 0.
\]

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Now assume that \( n = 2k + t + 4 \) and \( q \geq 3 \), or \( n \geq 2k + t + 5 \). Since \( q^{\left\lfloor \frac{k-t}{2} \right\rfloor} < \left\lfloor \frac{k-t+1}{2} \right\rfloor^2 \) and \( t^2 + 1 \leq 4q^t \), by Lemma 2.1 (ii)(iv), we have

\[
f_2(n, k, t) > \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} - 4q^t \left\lfloor \frac{k-t+1}{2} \right\rfloor^2
= \left\lfloor \frac{k-t}{2} \right\rfloor \left( \frac{q^{n-t-1} - 1}{q^{k-t-1} - 1} - 4q^t (q^{k-t+1} - 1)^2 \right)
> \left\lfloor \frac{k-t}{2} \right\rfloor \left( q^{n-k} - 16q^{k+t+1} \right)
\geq 0.
\]

This completes the proof. 

\[\blacksquare\]

**Proof of Theorem 1.2.** The result follows from Theorem 1.1, Remark 1 and Lemmas 2.5, 2.6, 2.8 and 2.9. 

\[\blacksquare\]

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