PARTIAL SUMS OF THE GIBONACCI SEQUENCE

PANKAJ JYOTI MAHANTA

ABSTRACT. Recently, Chu studied some properties of the partial sums of the sequence $P^k(F_n)$, where $P(F_n) = \left(\sum_{i=1}^{n} F_i\right)_{n \geq 1}$ and $(F_n)_{n \geq 1}$ is the Fibonacci sequence, and gave its combinatorial interpretation. We generalize those results, introduce colored Schreier sets, and give another equivalent combinatorial interpretation by means of lattice path.

1. Introduction

The Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$, with initial terms $F_1 = 1$ and $F_2 = 1$. One of its generalizations is the Gibonacci sequence, which is defined by $G_n = G_{n-1} + G_{n-2}$, with initial terms $G_1$ and $G_2$, where the initial two terms can be any positive integer. Both the sequences are related by the following identity

$$G_n = G_1 F_{n-2} + G_2 F_{n-1}$$

for all $n > 2$.

We refer the reader to Benjamin and Quinn’s book [BQ03], and the author’s joint work with Saikia [MS21].

In a recent paper [Chu21], Chu defined a function $P$ such that

$$P(F_n) := \left(\sum_{i=1}^{n} F_i\right)_{n \geq 1}.$$ 

Chu also defined that

$$P^k(F_n) := P(P^{k-1}(F_n))$$

for all $k \geq 2$, and denoted the $n$th term of the sequence $P^k(F_n)$ by $a_k(n)$. We generalize these to Gibonacci sequence, that is

$$P(G_n) := \left(\sum_{i=1}^{n} G_i\right)_{n \geq 1}$$

and $P^k(G_n) := P(P^{k-1}(G_n))$ for all $k \geq 2$.

We denote the $n$th term of the sequence $P^k(G_n)$ by $a_k^{(G_1,G_2)}(n)$. So, $a_k^{(F_1,F_2)}(n) = a_k(n)$. For simplicity, sometimes we write $a'_k(n)$ instead of $a_k^{(G_1,G_2)}(n)$. So, $a'_k(n) = a_k(n-1) + a'_{k-1}(n)$.

2. Generalization of $a_k(n)$

We start with the following interesting binomial coefficient identity.

**Proposition 2.1.** For all non-negative integers $k$ and $n$,

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}.$$ 

2020 Mathematics Subject Classification. 11B39, 05A19.

Key words and phrases. Gibonacci sequence, Fibonacci sequence, partial sums, colored Schreier Set, lattice path.
It can be easily proved by mathematical induction on $n$ using the Pascal’s identity \( \binom{n}{k} + \binom{n+1}{k+1} \). Note that \( \binom{0}{0} = 1 \), and if $n \geq 0$ and $\ell < 0$ then \( \binom{n}{\ell} = 0 \).

**Theorem 2.2.** For all integers $n, k \geq 1$, we have

\[
a_k^{\{G_1, G_2\}}(n) = \sum_{i=0}^{n-1} \binom{k-1+i}{k-1} G_{n-i}.\]

The theorem can be easily proved by mathematical induction using Proposition 2.1.

**Corollary 2.3** (Generalization of Lemma 2.1, [Chu21]). For $k \geq 0$, we have

\[
a'_k(3) = (k+1) G_2 + \left( \binom{k+1}{k-1} + 1 \right) G_1.\]

**Corollary 2.4** (Generalization of Theorem 1.1, [Chu21]). For all $n, k \geq 1$, we have

\[
a'_k(n) = a'_{k-1}(n+2) - \left( \binom{n+k-1}{k-1} \right) G_2 - \left( \binom{n+k-1}{k-2} \right) G_1.
\]

When $k = 1$, it gives us the well-known identity $\sum_{i=1}^{n} G_i = G_{n+2} - G_2$, and then $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

**Proof of Corollary 2.4.** We have,

\[
a'_{k-1}(n+2) = \sum_{i=0}^{n+1} \binom{k-2+i}{k-2} G_{n+2-i}
= G_{n+2} + (k-1) G_{n+1} + \sum_{i=0}^{n-1} \binom{k+i}{k-2} G_{n-i}
= k G_2 + \left( 1 + \binom{k}{k-2} \right) G_n + \sum_{i=1}^{n-1} \left( k + \binom{k+i}{k-2} \right) G_{n-i}.
\]

(Since we get $G_{n+2} = \sum_{i=0}^{n} G_{n-i} + G_2$ and $G_{n+1} = \sum_{i=1}^{n-1} G_{n-i} + G_2$.)

Therefore, $a'_{k-1}(n+2) - a'_k(n)$ is equal to
kG_2 + \binom{k}{k-2} G_n + \sum_{i=1}^{n-1} \left( k + \binom{k+i}{k-2} - \binom{k-1+i}{k-1} \right) G_{n-i}

= \left( k + \binom{k}{k-2} \right) G_2 + \left( k + \binom{k+1}{k-2} - \binom{k}{k-1} \right) G_{n-1} + \sum_{i=2}^{n-1} \left( k + \binom{k+i}{k-2} + \binom{k+i}{k-2} - \binom{k-1+i}{k-1} \right) G_{n-i}

(Since G_n = \sum_{i=2}^{n-1} G_{n-i} + G_2.)

= \left( \binom{k+1}{k-1} \right) G_2 + \left( \binom{k+1}{k-1} \right) G_{n-1} + \sum_{i=2}^{n-1} \left( \binom{k+1}{k-1} + \binom{k+i}{k-2} - \binom{k-1+i}{k-1} \right) G_{n-i}

= \left( \binom{k+2}{k-1} \right) G_2 + \left( \binom{k+2}{k-2} \right) G_{n-2} + \sum_{i=3}^{n-1} \left( \binom{k+2}{k-1} + \binom{k+i}{k-2} - \binom{k-1+i}{k-1} \right) G_{n-i}

(Since G_{n-1} = \sum_{i=2}^{n-1} G_{n-i} + G_2.)

= \left( \binom{k+3}{k-1} \right) G_2 + \left( \binom{k+3}{k-2} \right) G_{n-3} + \sum_{i=4}^{n-1} \left( \binom{k+3}{k-1} + \binom{k+i}{k-2} - \binom{k-1+i}{k-1} \right) G_{n-i}

(Since G_{n-2} = \sum_{i=3}^{n-1} G_{n-i} + G_2.)

Proceeding in this way upto \( (n-2) \) steps we get that the difference is equal to

\left( \binom{k+n-2}{k-1} \right) G_2 + \left( \binom{k+n-2}{k-2} \right) G_{n-2} + \left( \binom{k+n-2}{k-1} + \binom{k+n-1}{k-2} - \binom{k-1+n-1}{k-1} \right) G_1,

which is equal to

\left( \binom{n+k-1}{k-1} \right) G_2 + \left( \binom{n+k-1}{k-2} \right) G_1.

\square

3. A COMBINATORIAL INTERPRETATION OF \( a^{(G_1,G_2)}_k(n) \)

A finite subset \( S \) of natural numbers is called a Schreier set if \( \min S \geq |S| \). By counting some Schreier sets Chu gave a combinatorial interpretation of \( a_k(n) \). We generalize it for \( a^{(G_1,G_2)}_k(n) \).

**Definition 3.1** (Chu). For any integers \( n \geq 1 \), and \( k \geq 0 \),

\[ s_k(n) := \# \{ S \subset \{ 1, 2, 3, \ldots, n \} : |S| \geq k, \text{ and } \min S \geq |S| \} \]

It is easy to prove the following proposition.

**Proposition 3.2.** For any natural number \( n \),

\[ \# \{ S \subset \{ 1, 2, 3, \ldots, n \} : |S| = \ell, \text{ and } \min S > \ell \} = \binom{n-\ell}{\ell}, \]
and

$$\# \{ S \subseteq \{1, 2, 3, \ldots, n\} : |S| = \ell, \text{ and } \min S = \ell \} = \binom{n-\ell}{\ell-1}. $$

This implies that

$$s_k(n) = \sum_{\ell \geq k} \left( \binom{n-\ell}{\ell} + \binom{n-\ell}{\ell-1} \right).$$

Now we count some Schreier sets where in each set, one specific element is of different type. We call these sets colored Schreier sets. Here, for the sets in

$$\{ S \subseteq \{1, 2, 3, \ldots, n\} : |S| = \ell, \text{ and } \min S = \ell \},$$

the element $\ell$ occurs in $G_1$ different colors. And for the sets in

$$\{ S \subseteq \{1, 2, 3, \ldots, n\} : |S| = \ell, \text{ and } \min S > \ell \},$$

the element which is equal to $|S|$ occurs in $G_2$ different colors. For example, for $G_1 = 3$ and $G_2 = 2$, let $R$, $B$, and $G$ be three different colors, and then all the colored Schreier sets corresponding to $\{ S \subseteq \{1, 2, 3, 4, 5, 6\} : |S| \geq 2, \text{ and } \min S \geq |S| \}$ are

$$\{ 2^R, 3 \}, \{ 2^R, 4 \}, \{ 2^R, 5 \}, \{ 2^R, 6 \}, \{ 3^R, 4 \}, \{ 3^R, 5 \}, \{ 3^R, 6 \}, \{ 4^R, 5 \}, \{ 4^R, 6 \}, \{ 5^R, 6 \},$$
$$\{ 2^B, 3 \}, \{ 2^B, 4 \}, \{ 2^B, 5 \}, \{ 2^B, 6 \}, \{ 3^B, 4 \}, \{ 3^B, 5 \}, \{ 3^B, 6 \}, \{ 4^B, 5 \}, \{ 4^B, 6 \}, \{ 5^B, 6 \},$$
$$\{ 2^G, 3 \}, \{ 2^G, 4 \}, \{ 2^G, 5 \}, \{ 2^G, 6 \},$$
$$\{ 3^R, 4 \}, \{ 3^R, 5 \}, \{ 3^R, 6 \}, \{ 4^R, 5 \}, \{ 4^R, 6 \}, \{ 5^R, 5 \}, \{ 5^R, 6 \},$$
$$\{ 3^B, 4 \}, \{ 3^B, 5 \}, \{ 3^B, 6 \}, \{ 4^B, 5 \}, \{ 4^B, 6 \}, \{ 5^B, 6 \},$$
$$\{ 3^G, 4 \}, \{ 3^G, 5 \}, \{ 3^G, 6 \}. $$

The total number of colored Schreier sets for particular $n$ and $k$ is given by

$$s_k^{\{G_1,G_2\}}(n) := \sum_{\ell \geq k} \left( \binom{n-\ell}{\ell} G_2 + \binom{n-\ell}{\ell-1} G_1 \right).$$

For simplicity, sometimes we denote it by $s_k(n)$.

**Proposition 3.3** (Generalization of Corollary 2.3, [Chu21]). For $k \geq 0$ and $n \geq 1$, we get

$$s_k^{\{G_1,G_2\}}(n) = s_k^{\{G_1,G_2\}}(n) - \binom{n-k}{k} G_2 - \binom{n-k}{k-1} G_1. $$

**Theorem 3.4** (Generalization of Theorem 1.3, [Chu21]). For $k \geq 0$ and $n \geq 1$, we get

$$s_k^{\{G_1,G_2\}}(n) = a_k^{\{G_1,G_2\}}(n - 2(k-1)).$$

**Proof.** First we show that $s'_0(n) = a'_0(n + 2) = G_{n+2}$. We have,

$$s'_0(n) = \sum_{\ell \geq 0} \left( \binom{n-\ell}{\ell} G_2 + \binom{n-\ell}{\ell-1} G_1 \right) = F_{n+1} G_2 + F_n G_1 \quad (\text{Since for } n \geq 1, F_n = \sum_{i=0}^{[n-1]} \binom{n-i-1}{i}, \text{ which is a well-known identity.})$$
$$= G_{n+2}. $$

The remaining part of the proof is similar to that of the Theorem 1.3 of [Chu21].
4. Combinatorial interpretation of $a_{k}^{(G_1,G_2)}(n)$ by means of lattice paths

Let us define a set of lattice paths in a $k \times n$ grid, which start from $(\ell, 0)$, where $\ell \geq k$, and end on the line joining $(0, k)$ and $(n, k)$, and which consist only of steps in the upward or rightward directions, such that

- the first step is always in upward direction,
- only one step can be taken at once in the upward direction,
- one or more steps can be taken at once in the rightward direction.

The square located in the $i$th column and the $j$th row from the lower left corner of a grid is called the $(i, j)$-cell of the grid. If $\ell = k$ then we give $G_1$ colors to the $(k, 1)$-cell, if $\ell > k$ then we give $G_2$ colors to the $(\ell, 1)$-cell, and when $k = 0$ then we consider that only one lattice path is there along with $G_2$ colors. We observe that, the total lattice paths for $0 \leq k \leq \left\lfloor \frac{n + 1}{2} \right\rfloor$ is equal to $s_{k}^{(G_1,G_2)}(n) = a_{k}^{(G_1,G_2)}(n - 2(k - 1))$. For example, Figure 1 shows all lattice paths for $s_{3}^{'}(6)$.

![Lattice paths for $s_{3}^{'}(6)$](image)

Figure 1. Lattice paths for $s_{3}^{'}(6) = a_{3}^{'}(2)$.

If a lattice path starts from $(0, 0)$ and ends at $(n, k)$ in a $k \times n$ grid, and it consists only steps in the upward or rightward directions, then the total number of such lattice paths is equal to $\binom{n + k}{k}$. We can construct all such lattice paths for $0 \leq k \leq \left\lfloor \frac{n + 1}{2} \right\rfloor$. By Lemma 2.2 of [Chu21] we get, if $G_1 = F_1$ and $G_2 = F_2$, then the set of the above lattice paths is in one-to-one correspondence with the set of these lattice paths in a $k \times (n - 2k + 1)$ grid for $0 \leq k \leq \left\lfloor \frac{n + 1}{2} \right\rfloor$.

For example, the total lattice paths of this type in a $3 \times 1$ grid is 4, and from Figure 1 we get $s_{3}^{(F_1,F_2)}(6) = 4$.

Acknowledgements

The author would like to thank Manjil P. Saikia for his helpful comments.

References

[BQ03] Arthur T Benjamin and Jennifer J Quinn. Proofs that really count: the art of combinatorial proof, volume 27. American Mathematical Soc., 2003.
[Chu21] Hung Viet Chu. Partial sums of the Fibonacci sequence. Fibonacci Quart., 59(2):132–135, 2021.
[MS21] Pankaj Jyoti Mahanta and Manjil P. Saikia. Some new and old Gibonacci identities. Rocky Mountain Journal of Mathematics, accepted, 2021.

GONIT SORA, DHALPUR, ASSAM 784165, INDIA
Email address: pankaj@gonitsora.com