Counterterm method in dilaton gravity and the critical behavior of dilaton black holes with power-Maxwell field

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We investigate the critical behavior of an \((n+1)\)-dimensional topological dilaton black holes, in an extended phase space in both canonical and grand-canonical ensembles, when the gauge field is in the form of power-Maxwell field. In order to do this we introduce for the first time the counterterms that remove the divergences of the action in dilaton gravity for the solutions with curved boundary. Using the counterterm method, we calculate the conserved quantities and the action and therefore Gibbs free energy in both the canonical and grand-canonical ensembles. We treat the cosmological constant as a thermodynamic pressure, and its conjugate quantity as a thermodynamic volume. In the presence of power-Maxwell field, we find an analogy between the topological dilaton black holes with van der Walls liquid-gas system in all dimensions provided the dilaton coupling constant \(\alpha\) and the power parameter \(p\) are chosen properly. Interestingly enough, we observe that the power-Maxwell dilaton black holes admit the phase transition in both canonical and grand-canonical ensembles. This is in contrast to RN-AdS, Einstein-Maxwell-dilaton and Born-Infeld-dilaton black holes, which only admit the phase transition in the canonical ensemble. Besides, we calculate the critical quantities and show that they depend on \(\alpha\), \(n\) and \(p\). Finally, we obtain the critical exponents in two ensembles and show that they are independent of the model parameters and have the same values as mean field theory.

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I. INTRODUCTION

It has been shown that one can extend the thermodynamic phase space of a Reissner-Nordstrom (RN) black holes in an anti-de Sitter (AdS) space, by considering the cosmological constant as a thermodynamic pressure, \(P = -\Lambda / 8\pi\) and its conjugate quantity as a thermodynamic volume \([1,2]\). The studies on the critical behavior of black hole spacetimes, in a wide range of gravity theories, have got a lot of enthusiasm. Let us review some works in this direction. For example, \(P-V\) criticality of charged AdS black holes has been investigated in \([3]\) and it was shown that indeed there is a complete analogy for RN-AdS black holes with the van der Walls liquid-gas system. In particular, it was found that the critical exponents of this system coincide with those of the van der Waals system \([7]\). When the gauge field is the Born-Infeld nonlinear electrodynamics, extended phase space thermodynamics of charged-AdS black holes have been investigated in \([8]\). In this case, one needs to introduce a new thermodynamic quantity conjugate to the Born-Infeld parameter which is required for consistency of both the first law of thermodynamics and the corresponding Smarr relation \([8]\). The studies were also extended to the rotating black holes. In this regards, phase transition, critical behavior, and critical exponents of Myers-Perry black holes have been explored in \([9]\). Besides, it was shown that charged and rotating black holes in three spacetime dimensions do not exhibit critical phenomena \([8]\). Other studies on the critical behavior of black hole spacetimes in an extended phase space have been carried out in \([10,16]\).

It is also of great interest to generalize the study to dilaton gravity \([17]\). Critical behavior of black holes in Einstein-Maxwell-dilaton (EMd) gravity in the presence of Liouville-type potentials, which is regarded as the generalization of the cosmological constant, has been explored in \([18]\). Although, the asymptotic behavior of these solutions \([18]\) are neither flat nor AdS, it was observed that the critical exponents have the universal mean field values and do not depend on the details of the system, although the thermodynamic quantities depend on the dilaton coupling constant, \(\alpha\) \([18]\). The studies were also extended to the nonlinear electrodynamics in dilaton gravity. In the context of Einstein-Born-Infeld-dilaton (EBId) gravity, critical behavior of \((n+1)\)-dimensional topological black holes in an extended phase space was explored in \([19]\). By interpreting the constant \(\Lambda\) and the BI parameter \(\beta\) as thermodynamic quantities, it was shown that the the phase space can be enlarged. It was also argued that although thermodynamic quantities depend on the dilaton coupling constant, BI parameter and the dimension of the spacetime, they are universal and are independent of metric parameters \([19]\).
One may also interested in studying the critical behavior of the dilaton black holes when the gauge field is in the form of the power-Maxwell field. There are some motivations for studying the critical behavior of the nonlinear power-Maxwell Lagrangian, instead of the usual linear Maxwell case. The first reason comes from the fact that, while the Maxwell Lagrangian is only conformally invariant in four dimensions, the power-Maxwell field is conformally invariant in \((n + 1)\)-dimensional spacetime for \(p = (n + 1)/4\), where \(p\) is the power parameter of the Power-Maxwell Lagrangian. Besides, it is worthwhile to investigate the effects of exponent \(p\) on the critical behavior of the black holes and see whether it can change the \(P-V\) criticality and the critical exponents of the system or not. The investigations on the black object solutions coupled to a conformally invariant Maxwell field have got a lot of attention from different perspective \([20, 30]\). Thermodynamics and thermal stability, in canonical and grand-canonical ensembles, of higher dimensional topological dilaton black holes with power-Maxwell field have been studied in \([31]\). It was shown that the solutions exist provided one assumes three Liouville-type potentials \([31]\). Such a procedure causes the resulting physical quantities to depend on the choice of reference background. Especially, in the case of dilaton gravity, our calculations show that the action calculated by use of the subtraction method is not correct. Another way of removing the divergences in the action and conserved quantities is through the use of counterterm method. In this method one may remove the divergences in the action for \((\Lambda)dS\) solutions by adding counterterms which are functional of the boundary curvature invariants \([35, 37]\). Indeed, by using this method one can calculate the action and conserved quantities intrinsically without reliance on any reference spacetime \([35, 40]\). Due to this fact, the counterterm method has been applied to many cases such as black holes with rotation, NUT charge and various topologies \([41, 43]\). Here we want to generalize the counterterm method to the case of Einstein gravity in the presence of a dilaton field for the solutions with curved boundary. In Einstein gravity, although there may exist a large number of possible boundary curvature invariants, only a finite number of them are non-vanishing on a boundary at infinity. But in the case of dilaton field coupled to gravity, the asymptotic behavior of the solutions may be neither \((\Lambda)dS\) nor flat and therefore there are an infinite number of non-vanishing boundary curvature invariant terms at infinity for the case of curved horizon. Only for black holes with flat horizon, the curvature of the boundary vanishes and there exist only one term proportional to square root of the determinant of the boundary metric \([43]\). In the case of solutions with curved boundary and non-\(\Lambda dS\) asymptotic, due to the fact that there exist an infinite number of counterterms, it is difficult to use the counterterm method in order to calculate the finite action. Due to this fact, this method has not been used till now. In this paper, for the first time, we introduce the counterterm method for the calculation of finite action and physical quantities in dilaton black holes with curved horizon. By using the counterterm method we calculate the action and Gibbs free energy in both canonical and grand canonical ensembles to study the phase transition of the system and compare them with van der Waals fluid.

This paper is outlined as follows. In Sec. \(\text{II}\) we introduce counterterms in dilaton gravity to get a finite value for mass and action in both canonical and grand canonical ensembles. In Sec. \(\text{III}\) we present basic field equations and a class of \((n + 1)\)-dimensional topological dilaton black hole solutions coupled to a nonlinear power-Maxwell field and review their thermodynamic properties. In Sec. \(\text{IV}\) we study the phase structure of the solutions and present the generalized Smarr relation in the presence of the dilaton field. In Sec. \(\text{V}\) we investigate the analogy of the obtained dilaton black holes with van der Waals liquid-gas system in the canonical ensemble by fixing charge at infinity. In this ensemble, we obtain the equation of state, the critical behavior and critical exponents and study Gibbs free energy of the solutions. In Sec. \(\text{VI}\) we consider the possibility of the phase transition in the grand-canonical ensemble by fixing the electric potential at infinity and find that in contrast to the RN-\(\Lambda dS\), EMd and EBld black holes, the phase transition occurs for \(p \neq 1\). Then, we calculate the critical exponents and find that they match to mean field value (the same as the van der Waals liquid). Also, we obtain the expression for Gibbs free energy in this ensemble and study its behavior. The last section is devoted to summary and conclusions.

II. COUNTERTERM METHOD IN DILATON GRAVITY

In this section we want to introduce the counterterms for Einstein-dilaton gravity. The action of \((n+1)\)-dimensional \((n \geq 3)\) coupled to the dilaton field can be written as

\[
I_{\text{bulk}} = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left\{ R - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) \right\},
\]

(1)
where $R$ is the Ricci scalar, $\Phi$ is the dilaton field and $V(\Phi)$ is a potential for $\Phi$. The bulk action of Einstein gravity which is coupled to the dilaton field does not have a well-defined variational principle. In order to make it an action with a well-defined variational principle, one should add the following Gibbons-Hawking surface term to the bulk action

$$I_{GH} = -\frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-\gamma} K,$$

where $\gamma_{ij}$ and $K$ denote the induced metric and extrinsic curvature of the boundary $\partial M$, respectively. It is clear that the action $I_{bulk} + I_{GH}$ is not finite. As in the case of Einstein-Hilbert action, one needs to add counterterms to the action to get a finite value. Since the counterterms should be reduced to those of Einstein gravity in the absence of dilaton, these counterterms may be written from the curvature invariants of the boundary metric. The coefficients of the boundary terms should be chosen such that the divergences in the bulk are canceled for all possible boundary topologies permitted by the equations of motion. For asymptotic AdS solutions, we have only a finite number of counterterms that do not vanish at infinity. Here, because of considering a dilaton field coupled to gravity, the asymptotic behaviour of the solutions is not AdS and therefore we may have an infinite number of counterterms which do not vanish at infinity. First we consider the case in four dimensions.

### A. Counterterms in four dimensions

Defining $l \equiv [(\alpha^2 - 3)/\Lambda]^{1/2}$, the counterterms in 4 dimensions may be written as:

$$I_{ct} = \frac{1}{8\pi} \int d^3x \sqrt{-\gamma} \left\{ \frac{2}{l e^{-\alpha \Phi}} + \frac{le^{-\alpha \Phi}}{2(1-\alpha^4)} R - \sum_{s=2}^{\infty} \frac{A_s}{2(\alpha^4 - 1)^s} \left( \frac{le^{-\alpha \Phi}}{2} \right)^{2s-1} \right\},$$

(2)

where

$$A_2 = 1, \quad A_3 = 2, \quad A_s = A_{s-2} + 2A_{s-1}; \quad s \geq 4.$$

It is worth to mention that in the absence of dilaton ($\alpha = 0$) each term in the summation is zero and therefore the counterterm reduces to that of AdS solutions. But, in the presence of dilaton each term in the summation of Eq. (2) may not vanish at infinity. As $\alpha$ increases, the number of non-vanishing terms in the series will increase. Of course as Eqs. (3) and (4) show $\alpha$ is less than 1 in four dimensions. For example for $\alpha < 1/3$, all the terms in the summation vanish and therefore the counterterms of Einstein gravity remove the divergences in the action while for $1/3 \leq \alpha < 3/5$ only the first does not vanish. Indeed, the number of non-vanishing terms in the summation for $\alpha < (2N-1)/(2N+1)$ is equal to $N$. Fortunately at infinity, the summation of all the divergent terms in the total action ($I_{bulk} + I_{GH} + I_{ct}$), including the summation term of Eq. (2), goes to zero as $s$ goes to infinity. Thus, in order to calculate the finite action, one only needs to calculate the finite terms in the first two terms of Eq. (2) in four dimensions. That is, one need to consider only the generalization of the counterterms of Einstein gravity in the presence of dilaton:

$$I_{total} = \text{Finite terms of } \left\{ I_{bulk} + I_{GH} + \frac{1}{8\pi} \int d^3x \sqrt{-\gamma} \left( \frac{2}{l e^{-\alpha \Phi}} + \frac{le^{-\alpha \Phi}}{2(1-\alpha^4)} R \right) \right\}.$$

Having the total finite action, one can use the Brown and York definition to construct a divergence free stress-energy tensor as

$$T^{ab} = \frac{1}{8\pi} \left\{ (K^{ab} - K \gamma^{ab}) + \frac{2}{l e^{-\alpha \Phi}} \gamma^{ab} - \frac{le^{-\alpha \Phi}}{(1-\alpha^4)} (R^{ab} - \frac{1}{2} R \gamma^{ab}) \right\}$$

$$+ \sum_{s=2}^{\infty} \frac{sA_s}{(\alpha^4 - 1)^s} \left( \frac{l e^{-\alpha \Phi}}{2} \right)^{2s-1} \left[ R^{s-1}(R^{ab} - \frac{1}{2s} R \gamma^{ab}) - (\nabla^a \nabla^b - g^{ab} \nabla^2) R^{s-1} \right].$$

(3)

To compute the conserved charges of the spacetime, we choose a spacelike surface $\Sigma$ in $\partial M$ with metric $\sigma_{ij}$, and write the boundary metric in ADM form:

$$\gamma_{ab}dx^a dx^b = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right),$$

(4)
where the coordinates $\varphi^i$ are the angular variables parameterizing the hypersurface of constant $r$ around the origin, and $N$ and $\Psi^i$ are the lapse and shift functions respectively. The conserved charges associated to a Killing vector $\xi^a$ is

$$Q(\xi) = \int_{\Sigma} d^2x \sqrt{\sigma} u^a T_{ab} \xi^b,$$

where $\sigma$ is the determinant of the metric $\sigma_{ij}$ and $u^a$ is the normal to the quasilocal boundary hypersurface $\Sigma$. For boundaries with timelike Killing vector ($\xi = \partial_t$) one obtains the conserved mass of the system enclosed by the boundary $\Sigma$. Again the summation of all the divergent terms in the above equation at infinity is zero and therefore one should consider only the finite terms in the first three terms of Eq. (9) in order to calculate the conserved quantities from Eq. (5).

B. Counterterms in $(n + 1)$ dimensions

Now, it is easy to generalize the counterterms of previous subsection to the case of $(n + 1)$ dimensions. Defining

$$l = \left(\frac{(\alpha^2 - n)(n - 1)}{2\Lambda}\right)^{1/2},$$

the counterterm may be written as

$$I_{ct} = \frac{1}{8\pi} \int d^3x \sqrt{-\gamma} \left\{ \frac{n - 1}{l e^{-\alpha\Phi}} + \frac{4e^{-\alpha\Phi}}{(1 - \alpha^2)(n - 2 + \alpha^2)} R + \frac{4l e^{-3\alpha\Phi}}{2(n - 4)(1 - \alpha^2)(n - 2 + \alpha^2)^2} [R_{ab} R^{ab} - \frac{n}{4(n - 1)} R^2] + ... \right\}$$

where $[(n + 1)/2]$ denotes the integer part of $(n + 1)/2$. Again although each term in the summation in Eq. (7) may not be zero, the summation of all the divergent terms in the total action is zero at infinity provided one chooses $B_s$ correctly. Thus in order to calculate the finite action, one only needs to consider the finite terms in the counterterms which are the generalization of counterterms of Einstein gravity. That is, the finite action is

$$I_{total} = \text{Finite terms of} \left\{ I_{bulk} + I_{GH} + \frac{1}{8\pi} \int d^3x \sqrt{-\gamma} \left[ \frac{2}{l e^{-\alpha\Phi}} + \frac{8e^{-\alpha\Phi}}{(1 - \alpha^2)} R \right] + \frac{l e^{-3\alpha\Phi}}{2(n - 4)(1 - \alpha^2)(n - 2 + \alpha^2)^2} \left( R_{ab} R^{ab} - \frac{n}{4(n - 1)} R^2 \right) + ... \right\}.$$ 

The finite stress energy tensor and conserved charges are

$$T^{ab} = \frac{1}{8\pi} \left\{ (K^{ab} - K\gamma^{ab}) + \frac{n - 1}{l e^{-\alpha\Phi}} \gamma^{ab} - \frac{4e^{-\alpha\Phi}}{(1 - \alpha^2)(n - 2 + \alpha^2)} (R_{ab} - \frac{1}{2} R \gamma^{ab}) \right\}$$

$$- \frac{l e^{-3\alpha\Phi}}{(1 - \alpha^2)(n - 4)(n - 2 + \alpha^2)^2} \left[ \frac{1}{2} \gamma^{ab} (R_{cd} R_{cd} - \frac{n}{4(n - 1)} R^2) - \frac{n}{2(n - 1)} R R_{ab} \right] + 2R_{cd} R^{abcd} - \frac{n - 2}{2(n - 1)} \nabla^a \nabla^b R + \nabla^2 R_{ab} - \frac{1}{2(n - 1)} \gamma^{ab} \nabla^2 R \right\} + ...$$

$$+ \sum_{s = [(n+1)/2]} B_s \left( \frac{l e^{-\alpha\Phi}}{2} \right)^{2s-1} \left[ R^{s-1}(R_{ab} - \frac{1}{2s} R \gamma^{ab}) - (\nabla^a \nabla^b - g^{ab} \nabla^2) R^{s-1} \right] \right\},$$

where

$$Q(\xi) = \int_{\Sigma} d^{n-1}x \sqrt{\sigma} u^a T_{ab} \xi^b,$$

respectively. Again, the summation of all the divergent terms in Eq. (10) is zero at infinity and one should consider only the finite terms of Eq. (10). In other words, since all the terms in the summation of Eq. (10) are zero at infinity, one should consider only the counterterms which are the generalization of counterterms of Einstein gravity in the presence of dilaton without the summation term.
III. REVIEW OF TOPOLOGICAL DILATON BLACK HOLES WITH POWER-MAXWELL FIELD

We consider the action of \((n + 1)\)-dimensional \((n \geq 3)\) Einstein gravity with power-Maxwell Lagrangian which is coupled to the dilaton field \([31]\)

\[
I = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left\{ -\frac{n}{n-1} \left( V(\Phi) + \Phi \right) - \frac{4}{n-1} \left( \nabla^2 \Phi \right)^2 - V(\Phi) + \left( -e^{-4\alpha\Phi/(n-1)} F \right)^p \right\},
\]

where \(R\) is the Ricci scalar, \(\Phi\) is the dilaton field, \(V(\Phi)\) is a potential for \(\Phi\), and \(p\) and \(\alpha\) are two constants determining the nonlinearity of the electromagnetic field and the strength of coupling of the scalar and electromagnetic field, respectively. \(F = F_{\lambda\mu} F^{\lambda\mu}\), where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor and \(A_\mu\) is the electromagnetic potential. In order to obtain the field equations, we vary the action (11) with respect to the gravitational field, the dilaton field \(\Phi\) and the gauge field \(A_\mu\). We find \([31]\)

\[
\mathcal{R}_{\mu\nu} = \left\{ \frac{1}{n-1} V(\Phi) + \frac{(2p-1)}{n-1} \left( -F e^{-4\alpha\Phi/(n-1)} \right)^p \right\} g_{\mu\nu}
\]

\[
+ \frac{4}{n-1} \partial_\mu \Phi \partial_\nu \Phi + 2 p e^{-4\alpha\Phi/(n-1)} \left( -F \right)^{p-1} F_{\mu\lambda} F^{\lambda\nu},
\]

\[
\nabla^2 \Phi - \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{p\alpha}{2} e^{-4\alpha\Phi/(n-1)} \left( -F \right)^p = 0,
\]

\[
\partial_\mu \left( \sqrt{-g} e^{-4\alpha\Phi/(n-1)} \left( -F \right)^{p-1} F^{\mu\nu} \right) = 0.
\]

In order to construct static topological black hole solutions of the above field equations, we take the line elements of spacetime in the form

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) h_{ij} dx^i dx^j,
\]

where \(f(r)\) and \(R(r)\) are functions of \(r\) which should be determined, and \(h_{ij}\) is a function of coordinates \(x^i\) which spanned an \((n-1)\)-dimensional hypersurface with constant scalar curvature \((n-1)(n-2)k\). The constant \(k = 0, -1,\) and \(+1\) for flat, hyperbolic and spherical hypersurfaces. The solution of Eq. (13) is given

\[
F_{\nu\tau} = \frac{q e^{(n-1)\Phi(x)}}{(rR)^{n-1}},
\]

where \(q\) is an integration constant related to charge of the black hole. Substituting (15) and (16) in the field equations (12) and (13), we arrive at

\[
f'' + \frac{(n-1)f'}{r} + \frac{(n-1)f'R'}{r^2} + \frac{2V}{n-1} - \frac{2[1+(n-3)]p}{n-1} \left( 2q^2 (rR)^{-2(n-1)/[2(n-1)]} e^{4\alpha\Phi/(n-1)} \right)^p = 0,
\]

\[
f'' + \frac{(n-1)f'}{r} + \frac{(n-1)f'R'}{r^2} + \frac{2V}{n-1} + \frac{4(n-1)f'R'}{rR} + \frac{2(n-1)f'R''}{rR} + \frac{8f\Phi^2}{n-1} - \frac{2[1+(n-3)]p}{n-1} \left( 2q^2 (rR)^{-2(n-1)/[2(n-1)]} e^{4\alpha\Phi/(n-1)} \right)^p = 0,
\]

\[
\frac{f'}{r} + \frac{f'R'}{r^2} + \frac{2V}{R^2} + \frac{2(n-1)f'R'}{rR} + \frac{(n-2)R^2 f}{R^2} + \frac{fR''}{R} - \frac{k(n-2)}{n-1} + \frac{V}{R^2} + \frac{2n-1}{n-1} \left( 2q^2 (rR)^{-2(n-1)/[2(n-1)]} e^{4\alpha\Phi/(n-1)} \right)^p = 0,
\]

\[
f \Phi'' + \Phi' f' + \frac{(n-1)f'\Phi'}{r} + \frac{(n-1)f'R'\Phi'}{R} - \frac{n-1}{8} \frac{dV}{d\Phi} - \frac{p\alpha}{2} \left( 2q^2 (rR)^{-2(n-1)/[2(n-1)]} e^{4\alpha\Phi/(n-1)} \right)^p = 0,
\]
where the prime stands for the derivative with respect to $r$. It was argued in Ref. [31] that in order to have exact topological solutions with an arbitrary dilaton coupling parameter $\alpha$, the dilaton potential should be chosen with the combination of three Liouville-type,

$$V(\Phi) = 2\Lambda_1 e^{2\zeta_1 \Phi} + 2\Lambda_2 e^{2\zeta_2 \Phi} + 2\Lambda_3 e^{2\zeta_3 \Phi}, \quad (21)$$

where $\Lambda_1$, $\Lambda_2$, $\Lambda$, $\zeta_1$, $\zeta_2$ and $\zeta_3$ are constants. Note that the topological black holes in EMd theory, can be constructed with two Liouville terms in the dilaton potential [43, 46].

In order to solve the system of equations (17)-(20) for three unknown functions $f(r)$, $R(r)$ and $\Phi(r)$, we make the ansatz

$$R(r) = e^{2\alpha \Phi(r)/(n-1)}. \quad (22)$$

Subtracting (17) from (18), after using (22), we find the following equation for the scalar field,

$$\Phi'' + \frac{2(\alpha^2 + 1)\Phi'^2}{\alpha(n-1)} + \frac{2\Phi'}{r} = 0, \quad (23)$$

which admits the following solution

$$\Phi(r) = \frac{(n-1)\alpha}{2(\alpha^2 + 1)} \ln \left( \frac{b}{r} \right). \quad (24)$$

Substituting (22) and (24) in Eqs. (18)-(20), one can easily show that these equation have a unique consistent solution of the form [31]

$$f(r) = \frac{k(n-2)(1+\alpha^2)r^{2\gamma}}{(1-\alpha^2)(\alpha^2+n-2)b^{2\gamma}} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2p p(1+\alpha)^2(2p-1)^2b^{-2(n-2)(2p-1)^2}q^{2p}}{\Pi(n+\alpha^2-2p)r^{-2(p(n-1)p+1)-2p(n-2)^2}} - \frac{2\Lambda b^{2\gamma}(1+\alpha^2)^2r^{2(1-\gamma)}}{(n-1)(n-\alpha^2)}, \quad (25)$$

where $b$ is an arbitrary non-zero positive constant, $\gamma = \alpha^2/(\alpha^2 + 1)$, $\Pi = \alpha^2 + (n-1-\alpha^2)p$, and the constants should be fixed as

$$\zeta_1 = \frac{2}{(n-1)\alpha}, \quad \zeta_2 = \frac{2p(n-1+\alpha^2)}{(n-1)(2p-1)\alpha}, \quad \zeta_3 = \frac{2\alpha}{n-1}, \quad \Lambda_1 = \frac{k(n-1)(n-2)\alpha^2}{2b^2(\alpha^2-1)}, \quad \Lambda_2 = \frac{2p-1(2p-1)(p-1)\alpha^2q^{2p}}{b^{2(n-1)p}}, \quad (26)$$

It is worth noting that in the linear Maxwell case where $p = 1$, we have $\Lambda_2 = 0$ and hence the potential has two terms. Indeed, the term $2\Lambda_2 e^{2\zeta_2 \Phi}$ in the Liouville potential is necessary in order to have solution (25) for the field equations of power-law Maxwell field in dilaton gravity. Note that $\Lambda$ remains as a free parameter which plays the role of the cosmological constant and we assume to be negative and take it in the form $\Lambda = -(n-\alpha^2)(n-1)/2l^2$. The parameter $m$ in Eq. (25) is the integration constant which is known as the geometrical mass and can be written in term of horizon radius as

$$m(r_+) = \frac{k(n-2)b^{-2\gamma}r_+^{\alpha^2+n-2}}{(2\gamma-1)(\gamma-1)(\alpha^2+n-2)} + \frac{2p p (2p-1)^2 b^{-2(n-2)(2p-1)^2}q^{2p} r_+^{-2(n-2)(\alpha^2+n)}}{(\gamma-1)^2(\alpha^2-2p+n)\Pi} + \frac{b^{2\gamma} r_+^{\alpha^2+n-2}}{l^2(\gamma-1)^2(n-\alpha^2)}, \quad (27)$$

where $r_+$ is the positive real root of $f(r_+) = 0$. In the limiting case where $p = 1$, solution (25) reduces to the topological dilaton black holes of EMd gravity presented in Ref. [43, 46]. In case of linear Maxwell theory ($p = 1$)
and in the absence of dilaton field \((\alpha = \gamma = 0)\), solution \((25)\) reduces to asymptotically AdS topological black hole (see for example \([47]\))

\[
f(r) = k - \frac{m}{r^{n-2}} + \frac{2q^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2\Lambda}{n(n-1)}r^2.
\]  (28)

The gauge potential \(A_t\) corresponding to the electromagnetic field \((16)\) is given by

\[
A_t = \frac{qb^{2p+1-n}}{\Upsilon r_+^4},
\]  (29)

where \(\Upsilon = \frac{(n - 2p + \alpha^2)}{[(2p - 1)(1 + \alpha^2)]}\). Imposing two conditions, namely (i) the electric potential \(A_t\) should have a finite value at infinity and (ii) the term including \(m\) in spacial infinity should vanish, lead to the following restrictions on the parameters \(p\) and \(\alpha\) \([31]\),

For \(\frac{1}{2} < p < \frac{n}{2}\), \(0 \leq \alpha^2 < n - 2\), \(\frac{30}{31}\)

For \(\frac{n}{2} < p < n - 1\), \(2p - n < \alpha^2 < n - 2\). \(\frac{31}{31}\)

The Hawking temperature can be written as

\[
T_+ = \frac{(1 + \alpha^2)}{4\pi} \left( \frac{k(n-2)}{b^{2\gamma}(1-\alpha^2)r_+^{1-2\gamma}} - \frac{2\Lambda b^{2\gamma}r_+^{1-2\gamma}}{n-1} - \frac{2\rho (2p-1) b^{-2\gamma}b^{2p-2(1-\gamma)+1}}{\Pi r_+^{2p-1}} \right).
\]  (32)

One can calculate thermodynamic quantities such as entropy, charge and electric potential of the black hole per unit volume \(\omega_{n-1}\) as \([31]\)

\[
S = \frac{b(n-1)\gamma b^{(n-1)(1-\gamma)}}{4},
\]  (33)

\[
Q = \frac{\tilde{q}}{4\pi}, \quad \tilde{q} = 2^{p-1}q^{2p-1}.
\]  (34)

\[
U = \frac{Cqb^{(2p-n+1)\gamma}}{\Upsilon r_+^{(2p-1)}}.
\]  (35)

where \(C = (n-1)p^2/\Pi\). For \(p = 1\) we have \(\tilde{q} = q\), as expected.

### IV. EXTENDED PHASE SPACE AND SMARR FORMULA

In this section, we would like to investigate the thermodynamics of topological dilaton black holes with power-Maxwell field. First, we calculate the mass through the use of counterterm method. Using Eqs. \([3]\) and \([10]\), one obtains

\[
M = \frac{b(n-1)\gamma (n-1)}{16\pi(\alpha^2 + 1)} m.
\]  (36)

We shall construct a Smarr relation in an extended phase space in which the cosmological constant is treated as thermodynamic variable. The conjugate quantity of the cosmological constant, which is proportional to the pressure, is the volume. As we know the entropy of black hole is a quarter of the area of the horizon, so the thermodynamic volume \(V\) is obtained as

\[
V = \int 4Sdr_+ = \frac{\alpha^2 + 1}{\alpha^2 + n} b^{(n-1)\gamma} r_+^{2\gamma + 2} \omega_{n-1}.
\]  (37)
In an extended phase space, $M$ can be a function of thermodynamic quantities entropy, pressure and charge. Hence, the first law takes the form

$$dM = TdS + UdQ + PdV.$$  \hspace{1cm} (38)

It is easy to show that the conjugate quantity of the thermodynamic volume is [18]

$$P = -\frac{\Lambda n - \gamma(n-1)}{8\pi n - \gamma(n+1)} \left( \frac{b}{r_+} \right)^{2\gamma} = -\frac{(n + \alpha^2)}{8\pi(n - \alpha^2)} \left( \frac{b}{r_+} \right)^{2\gamma} \Lambda,$$  \hspace{1cm} (39)

which is proportional to the cosmological constant $\Lambda$. In the absence of dilaton ($\gamma = 0 = \alpha$) we can see that the above expression for $P$ reduces to the pressure of the RN-AdS black holes [7]. One may note that the above expression for the pressure is the same as that of EMd [18] and EBId black holes [19]. Also, it is clear that the pressure is positive provided $\alpha < \sqrt{n}$ exactly similar to [18] and [19]. This is consistent with the argument given in [31], which states that the topological dilaton black hole solutions have reasonable behavior provided $\alpha < \sqrt{n} - 2$. The Smarr relation may be obtained from the first law of black hole thermodynamics and a scaling dimensional argument [2]. One obtains

$$M = \frac{n - 1}{n - 2 + \alpha^2} TS + \frac{p(n - 3 + \alpha^2) + 1}{p(n - 2 + \alpha^2)} UQ + \frac{2(\alpha^2 - 1)}{n - 2 + \alpha^2} VP.$$  \hspace{1cm} (40)

One may note that the above generalized Smarr formula reduces to those of Refs. [18] in the limit $p = 1$.

In what follows, we study the phase transition of the power-Maxwell dilaton black holes in an extended phase space in both canonical and grand-canonical ensembles, separately.

\section{V. PHASE TRANSITION IN CANONICAL ENSEMBLE}

In order to study the phase transition of system, we first consider the canonical ensemble. In this ensemble the charge $Q$ of the black hole is regarded as a fixed extensive parameter.

A. Equation of state

Using Eqs. (32) and (39) for fixed charge, one may write

$$P = \frac{\Gamma T}{r_+} - \frac{k(n - 2)(1 + \alpha^2)\Gamma}{4\pi(1 - \alpha^2)b^{2\gamma}r_+^{2\gamma}} + \frac{2^p p(1 + \alpha^2)(2p - 1)b^{-2(n-2)p}}{4\pi \tilde{r}_+^{2(n-2)p-1}} \Gamma q^{2p},$$  \hspace{1cm} (41)

where

$$\Gamma = \frac{(n - 1)(n + \alpha^2)}{4(n - \alpha^2)(\alpha^2 + 1)}.$$  \hspace{1cm} (42)

From Eq. (37), we see that $r_+$ is a function of the thermodynamic volume $V$, so the above equation can be regarded as the equation of state $P(V,T)$. Before proceeding further, we translate the ‘geometric’ equation of state (41) to a physical case by performing a dimensional analysis. We identify the following relations between geometric quantities and physical pressure and temperature

$$P = \frac{hc}{l_p^2} P, \quad T = \frac{hc}{\kappa} T,$$  \hspace{1cm} (43)

where the Planck length is $l_p = \sqrt{\hbar G/c^3}$ and $\kappa$ is the Boltzmann constant. In terms of these new definitions, Eq. (41) can be written as

$$P = \frac{\Gamma \kappa T}{l_p^2 r_+} - \frac{khc(n - 2)(1 + \alpha^2)\Gamma}{4\pi l_p^2(1 - \alpha^2)b^{2\gamma}r_+^{2\gamma}} + \frac{2^p p(1 + \alpha^2)(2p - 1)b^{-2(n-2)p}}{4\pi \tilde{r}_+^{2(n-2)p-1}} \Gamma q^{2p} \frac{hc}{l_p^2 r_+^{2(n-2)p-1}}.$$  \hspace{1cm} (43)

Now, comparing the above physical equation of state with the van der Walls equation [7]

$$P = \frac{T}{v} + ...,$$
we understand that the specific volume $v$ of the fluid in terms of the horizon radius should be written as,

$$v = \frac{l^2 p r^+}{\Gamma},$$  \hspace{1cm} (44)

Returning to the geometrical units ($G = \hbar = c = 1 \Rightarrow l^2_p = 1$), the equation of state (41) can be written

$$P = \frac{T}{v} - \frac{k(n - 2)(1 + \alpha^2)\Gamma}{4\pi(1 - \alpha^2)b^2\gamma(\Gamma v)^2 - 2\gamma} + \frac{2^p(1 + \alpha^2)(2p - 1)b^2}{4\pi(\Gamma v)^{2p-1} - 2\gamma} \frac{(2^{p-1} - \Gamma(2p-1))}{2^{p-1}} \Gamma q^{2p}. $$  \hspace{1cm} (45)

To compare the critical behavior of the system with van der Waals fluid, we should plot isotherm diagrams. The corresponding $P - v$ diagrams are displayed in Figs. 1-3. The behavior of the isotherms diagrams depends on how deep we are in nonlinear regime and the value of $n$. We observe that in order to have critical behavior, the dimension

FIG. 1: $P - v$ diagram for power-Maxwell dilaton black holes. Here we have taken $b = 1$, $q = 1$, $n = 3$, $k = 1$, $p = 0.8$ and $\alpha = 0.3$.

FIG. 2: $P - v$ diagram for power-Maxwell dilaton black holes. Here we have taken $b = 1$, $q = 1$, $n = 3$, $k = 1$, $p = 1.2$ and $\alpha = 0.3$.

FIG. 3: $P - v$ diagram for power-Maxwell dilaton black holes. Here we have taken $b = 1$, $q = 1$, $n = 5$, $k = 1$, $p = 3$ and $\alpha = 0.3$. 
FIG. 4: The behavior of $P_c$ versus $p$ for $b = 1$, $q = 1$, $n = 3$, $k = 1$ and $\alpha = 0.3$.

of spacetime, $n$, should increase with increasing the power parameter $p$. This can be easily understand as follows. The second term in (45) is independent of $p$ while the third term depends on $p$ sensitively. To see critical behavior, the power of $v$ in the denominator of the third term should be larger than this power in the second term. From the other point of view, to satisfy (30), $n$ should increase as $p$ increases. The critical point can be obtained by solving the following equations

$$\frac{\partial P}{\partial v} \bigg|_{T_c} = 0, \quad \frac{\partial^2 P}{\partial v^2} \bigg|_{T_c} = 0,$$

which leads to

$$v_c = -\frac{1}{\Gamma X} b^{-\frac{\alpha^2 + 1}{2}} X^{\frac{(2p-1)(\alpha^2 + 1)}{2}},$$

$$P_c = \frac{k(n-2)\Gamma \Delta}{4\pi p(n + \alpha^2 - 1)} X^{-\frac{(2p-1)}{2}} b^{-\frac{2p\alpha^2}{2}},$$

$$T_c = \frac{k\Delta(n-2)}{\pi(1 - \alpha^2)(2pn + \alpha^2 + 1 - 4p)} X^{\frac{2p-1 - \alpha^2}{4}} b^{\frac{-\alpha^2 + 2n - 2p + 1}{4}},$$

where

$$X = \frac{(-4p + \alpha^2 + 1 + 2pn)(n + \alpha^2 - 1)q^{2p}2^{2p}}{2p - 1)k\Pi(n-2)},$$

$$\Delta = pn + p\alpha^2 - 3p + 1.$$  

Using the above critical values, $\rho_c$ is obtained as

$$\rho_c = \frac{(-4p + \alpha^2 + 1 + 2pn)(\alpha^2 - 1)}{4p(n + \alpha^2 - 1)},$$

As one expects, the above $\rho_c$ reduces to that of Ref. [18] for $p = 1$,

$$\rho_c = \frac{P_cv_c}{T_c} = \frac{(1 - \alpha^2)(2n - 3 + \alpha^2)}{(4n - 4 + 4\alpha^2)},$$

and in the absence of the dilaton field ($\alpha = 0 = \gamma$) in four dimensions ($n = 3$), it reduces to $3/8$ which is the characteristic of van der Waals fluid [7]. It is easy to see that $\rho_c$ is positive provided $\alpha < 1$. It is also notable that $P_c$ and $T_c$ decrease as $p$ increases (see Figs. 4, 5).

B. Critical exponents

The behavior of physical quantities in the vicinity of critical point can be characterized by the critical exponents. So, following the approach of [8], one can calculate the critical exponents $\alpha'$, $\beta'$, $\gamma'$ and $\delta'$ for the phase transition
of an \((n + 1)\)-dimensional charged dilaton black hole in the presence of power-Maxwell field. To obtain the critical exponents, we define the reduced thermodynamic variables as

\[
p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}.
\]

Therefore, equation of state (45) translate into the law of corresponding state,

\[
p = \frac{1}{\rho_c} \frac{\tau}{\nu} - \frac{k(n - 2)(1 + \alpha^2)}{4\pi P_c (1 - \alpha^2) b^2 \gamma (\Gamma \nu v_c)^{2-2\gamma}} + \frac{2\rho (1 + \alpha^2) (2\rho - 1) b^{-2(n-2)\nu(\Gamma \nu v_c)}}{4\pi \Pi P_c (\Gamma \nu v_c)}.
\]

Although this law depends on parameter \(\gamma\) and \(p\) but as we will see this doesn’t affect the behavior of the critical exponents. To calculate the critical exponent \(\alpha'\), we consider the entropy \(S\) as a function of \(T\) and \(V\). Using (37) we have

\[
S = S(T, V) = \frac{b^{(n-1)\gamma} \omega_{n-1}}{4} \left[ V \left( (n-1)(1 - \gamma) + 1 \right) / (b^{(n-1)\gamma} \omega_{n-1}) \right]^{\frac{(n-1)(1-\gamma)}{(n-1)(1-\gamma)+1}}.
\]
Obviously, this is independent of \( T \) and therefore the specific heat vanishes, \( C_V = T \left( \partial S / \partial T \right)_V = 0 \). Since the exponent \( \alpha' \) governs the behavior of the specific heat at constant volume \( C_V \propto |\tau - 1|^{\alpha'} \), hence we have \( \alpha' = 0 \).

Expanding Eq. (52) near the critical point

\[
\tau = t + 1, \quad \nu = (\omega + 1)^{\frac{1}{3}},
\]

where \( \varepsilon \) is a positive parameter defined as \( \varepsilon = n - \gamma (n - 1) = (n + \alpha^2) / (1 + \alpha^2) \) and following the method of Ref. [8], we obtain

\[
p = 1 + At - Bt\omega - C\omega^3 + O(t\omega^2, \omega^4),
\]

with

\[
A = \frac{1}{\rho_c}, \quad B = \frac{1}{\varepsilon \rho_c},
\]

\[
C = \frac{2(n - 1 + \alpha^2)}{3(2p - 1)(1 + \alpha^2)^2 \varepsilon^3}.
\]

Differentiating Eq. (55) at a fixed \( t < 0 \) with respect to \( \omega \), we get

\[
dP = -P_c (Bt + 3C\omega^2) \, d\omega.
\]

Now, we apply the Maxwell’s equal area law [7]. Denoting the volume of small and large black holes with \( \omega_s \) and \( \omega_l \), respectively, we obtain

\[
p = 1 + At - Bt\omega_l - C\omega_l^3 = 1 + At - Bt\omega_s - C\omega_s^3,
\]

\[
0 = \int_{\omega_s}^{\omega_l} \omega dP.
\]

Eq. (59) leads to the unique non-trivial solution

\[
\omega_l = -\omega_s = \sqrt{-\frac{Bt}{C}}.
\]

which gives the order parameter \( \eta = V_c (\omega_l - \omega_s) \) as

\[
\eta = 2V_c \omega_l = 2\sqrt{-\frac{B}{C}} t^{1/2}.
\]

Thus, the exponent \( \beta' \) which describes the behavior of the order parameter \( \eta \) near the critical point is \( \beta' = 1/2 \).

To calculate the exponent \( \gamma' \), we may determine the behavior of the isothermal compressibility near the critical point.

Differentiating Eq. (55) with respect to \( V \), one obtains

\[
\left. \frac{\partial V}{\partial P} \right|_T = -\frac{V_c}{BP_c} \frac{1}{t} + O(\omega).
\]

Hence, the isothermal compressibility near the critical point may be written as

\[
\kappa_T = -\frac{1}{V} \left. \frac{\partial V}{\partial P} \right|_T \propto -\frac{V_c}{BP_c} \frac{1}{t} \quad \Rightarrow \quad \gamma' = 1.
\]

Finally, the shape of the critical isotherm \( t = 0 \) is given by (55)

\[
p - 1 = -C\omega^3 \quad \Rightarrow \quad \delta' = 3.
\]

Thus, we have shown that for power-Maxwell-dilaton black holes in \((n + 1)\) dimensions, the critical exponents have the same values as in case of the RN-AdS black holes [7], EMd black holes [18] and EBId black holes [19].
C. Gibbs free energy in canonical ensemble

In the canonical ensemble with fixed charge, the potential, which is the free energy of the system presents the thermodynamic behavior of a system in a standard approach. In order to calculate the free energy of a gravitational system one may evaluate the Euclidean on-shell action. Moreover, to make the action well-defined and finite, one should add the Gibbons-Hawking boundary term and counterterms to the bulk action. Also as we are working in canonical ensemble we should consider a boundary term for electromagnetic field namely

\[ I_s = -\frac{P}{4\pi} \int \sqrt{-\gamma} e^{-2\rho \Phi(r)} (-F)^{\mu \nu} A_{\nu}. \]  

(65)

\[ I = I_{\text{bulk}} + I_{\text{GH}} + I_{\text{ct}} + I_s. \]  

(66)

\[ G = \frac{r_+}{4} + \frac{2\pi P(\alpha^2 - 1)(\alpha^2 + 1) b^2 \gamma^2 r_+^{\frac{2\alpha + 2}{\alpha + 1}}}{(3 + \alpha^2)} + \frac{(\alpha^2 + 1)(2p - 1)(2p + \alpha^2 + 1)2\rho q^2 p b^{-\gamma(2-2p)}}{4(2p + \alpha^2 + p\alpha^2)(\alpha^2 + 3 - 2p)} r_+^{-\frac{\alpha^2 + 3 - 2p}{(2p - 1)(\alpha^2 + 1)}}. \]  

(67)

where \( r_+ \) is understood as a function of pressure and temperature via equation of state [44]. Moreover since we are considering an extended phase space, we may associate Gibbs free energy with the \( G = M - TS \) [1]. In this way the Gibbs free energy can be obtained as

\[ G = \frac{\omega n_1(\alpha^2 + 1)}{16\pi} \left( \frac{(n - 2)k}{\alpha^2 + n - 2} b^{\gamma(n-3)} r_+^{\frac{2\alpha + 2}{\alpha + 1}} + \frac{16\pi P(\alpha^2 - 1)}{(n - 1)(\alpha^2 + b) r_+^{-\gamma(n-1)}} \right) \]

\[ + \frac{\omega n_1(\alpha^2 + 1)}{16\pi} \left( \frac{(2p - 1)(2p - 4p + \alpha^2 + 1)2p q^{2p} b^{-\gamma(n-2p)}}{\Pi(\alpha^2 + n - 2p)} r_+^{-\frac{\alpha^2 + 3 - 2p}{(2p - 1)(\alpha^2 + 1)}} \right), \]  

(68)

which reduces to the (67) for \( n = 3 \) and also can reduce to the result obtained for black holes in EMD gravity as \( p = 1 \) [18]. The behavior of the Gibbs free energy is shown in Figs. [50]. From these figures we see that there is a swallowtail behavior. It means we have first order phase transition in the system.

VI. PHASE TRANSITION IN GRAND-CANONICAL ENSEMBLE

In this section we investigate the phase transition in grand-canonical ensemble by fixing the electric potential at infinity. It is worthwhile to note that, as one expects for linear Maxwell field \( (p = 1) \), we cannot see criticality in the grand canonical ensemble while as we shall see below the system may encounter a critical behavior in case of the power-Maxwell field with \( p \neq 1 \).
FIG. 9: Gibbs free energy versus $T$ for $b = 1$, $n = 5$, $q = 1$, $k = 1$, $p = 2$ and $\alpha = 0.2$.

FIG. 10: $P - v$ diagram for $b = 1$, $n = 3$, $U = 1$, $k = 1$, $p = 1.2$ and $\alpha = 0.3$.

A. Equation of state

To study the critical behavior in grand canonical ensemble, we put

$$q = \frac{U \gamma r_+}{C b^{2(2p-1)}}.$$

Using (41) with $r_+ = v\Gamma$ one may can rewrite equation of state in the following form

$$P = \frac{T}{v} - \frac{k(n-2)(1+\alpha^2)\Gamma}{4\pi(1-\alpha^2)b^2\gamma(\Gamma v)^{2-\gamma}} + \frac{p(2p-1)(1+\alpha^2)\Gamma}{4\pi\Pi} \frac{\sqrt{2U(n-2p+\alpha^2)}}{(2p-1)(1+\alpha^2)b^\gamma c(\Gamma v)^{1-\gamma}}^{2p}.$$

In order to compare the critical behavior of the system with van der Waals gas, we should plot isotherm diagrams. The corresponding $P - v$ diagrams are displayed in Figs. 10-12. The critical point can be obtained by solving the

FIG. 11: $P - v$ for diagram for $b = 1$, $n = 3$, $U = 1$, $k = 1$, $p = 2$ and $\alpha = 0.3$. 
where

\[
Z = \frac{2^p U^{2p} p^2 (2p - 1)(2p - \alpha^2 - 1)(n - 2p + \alpha^2)^{2p}}{k(n - 2)\Pi \{c(2p - 1)(\alpha^2 + 1)\}^{2p}}.
\]

One can see the behavior of \(P_c\) and \(T_c\) in the Figs. 13-16. Using the above critical values, \(\rho_c\) is obtained as

\[
\rho_c = \frac{(\alpha^2 + 1 - 2p)(\alpha^2 - 1)}{4p},
\]

As one expects, the above \(\rho_c\) reduces to that of Ref. 13 as \(\alpha\) goes to zero. In the absence of the dilaton field \((\alpha = 0 = \gamma)\) for \(p = 2\), it reduces to \(3/8\) which is the characteristic of van der Waals fluid. It is clear that \(\rho_c\) is independent of \(n\).

### B. Critical exponents

Following the method we adopted in the canonical ensemble, we can obtain

\[
p = 1 + At - Bt^2 - C\omega^3 + O(t\omega^2, \omega^4),
\]
FIG. 14: The behavior of \( T_c \) versus \( p \) for \( b = 1, q = 1, n = 3, k = 1 \) and \( \alpha = 0.3 \).

FIG. 15: \( P_c - \alpha \) diagram for \( b = 1, q = 1, n = 3, q = 1 \) and \( k = 1 \).

with

\[
A = \frac{1}{\rho_c}, \quad B = \frac{1}{\varepsilon \rho_c}, \tag{75}
\]

\[
C = \frac{2p}{3(1 + \alpha^2)\varepsilon^3}. \tag{76}
\]

Thus, we obtain the same critical exponent as we found already in the canonical ensemble

\[
\alpha' = 0, \beta' = 1/2, \gamma' = 1, \delta' = 3. \tag{77}
\]

C. Gibbs free energy in grand canonical ensemble

As we have done for the canonical ensemble, in the grand-canonical ensemble we can calculate Gibbs free energy by calculating the on-shell action. Since we are working in the grand canonical ensemble, we should fix electric potential on the boundary. This implies that we must ignore the surface term (65). So the total action becomes

\[
I = I_{\text{bulk}} + I_{\text{GH}} + I_{\text{ct}}. \tag{78}
\]

\[
G = \frac{\tau_+}{4} + \frac{2\pi P(\alpha^2 - 1)(\alpha^2 + 1)}{(3 + \alpha^2)} b^2 \gamma^2 \tau_+^{\frac{3\alpha^2}{(\alpha^2+1)}} + \frac{2\pi U^{2p} p(3 + \alpha^2 - 2p)^2 p^{-1}(1 - \alpha^2)}{4c^{2p} p(\alpha^2 + 1)^{2p-1}} \left(\frac{2p - 1}{2p - 2}\right)^{\frac{3-2p+\alpha^2}{(\alpha^2+1)}} b^{-\gamma(2p-2)}. \tag{79}
\]
FIG. 16: (Color online) $T_c - \alpha$ diagram for $b = 1$, $q = 1$, $n = 4$, $q = 1$ and $k = 1$.

FIG. 17: Gibbs free energy versus $T$ for $b = 1$, $n = 3$, $U = 1$, $k = 1$, $p = 6/5$ and $\alpha = 0$.

It is a matter of calculation to show the above Gibbs energy which is obtained from action is same as

$$G = M - TS - \frac{\Pi}{2p}QU,$$  \hspace{1cm} (80)

in the extended phase space for $n = 3$. The behavior of the Gibbs free energy is shown in Figs. 17-18.

VII. SUMMERY AND CONCLUSIONS

In this paper, we have investigated the critical behavior of $(n + 1)$-dimensional dilaton black holes in the presence of power Maxwell field. For this purpose, we have calculated the finite action in both canonical and grand canonical ensembles. We have introduced, for the first time, the counterterm method for spacetimes with curved boundary in dilaton gravity. Due to the fact that the asymptotic behavior of the solutions is not (A)dS, the number of non-vanishing counterterms constructed by the curvature invariant of the boundary at infinity may be infinite. We found out that although as $\alpha$ approaches to $\sqrt{n - 2}$, the number of counterterm goes to infinity, the summation of all the divergent terms in the total action is zero at infinity and therefore one only needs to calculate the finite terms of the total action \([8]\). In other words the number of counterterms which should be calculated in Einstein-dilaton gravity is
the same as that of Einstein gravity and one only needs to calculate the finite terms of them.

In order to have solutions in Einstein-dilaton gravity in the presence of power-Maxwell field, one needs three Liouville type potentials [31]. One of the Liouville type potential contains a constant $\Lambda$, which plays the role of cosmological constant and the others guarantee the existence of the solution. We extended the phase space by considering the constant $\Lambda$ to be treated as thermodynamic pressure, and its conjugate quantity as a thermodynamic volume. By calculating the thermodynamic quantities, we obtained the Smarr relation, which reduces to the Smarr relation in the absence of dilaton field given in [18] and in the limit $p = 1$ it reduces to those of [18]. After constructing the Smarr relation, we used the pressure and Hawking temperature to build the equation of state in the canonical and grand canonical ensembles. Then, we plotted $P-v$ isotherm diagrams. These figures show the analogy between our system and the van der Walls fluid, with the same phase transition in both ensembles. Interstingly enough, we found that in contrast to the RN-AdS, EMd and EBId black holes which has the phase transition only in canonical ensemble, dilaton black holes with power-Maxwell gauge field admit the critical behaviour in both canonical and grand canonical ensembles. We also found that the critical behavior can be occurred only for black holes with spherical horizon ($k = 1$). Then, we obtained the critical pressure, volume and temperature both for the canonical and grand canonical ensembles and by using them we calculate the action and Gibbs free energy. We have considered the behavior of the Gibbs free energy and found that there is a swallowtail behavior for Gibbs free energy as a function of temperature in both ensembles which shows there is a first order small-large black holes phase transition in the system. Finally, we calculated the critical exponents and found that while the critical quantities are different in two ensembles, these exponents are the same and they are the same as van der Waals system. This, implies that the inclusion of nonlinear electrodynamics, dilaton field or extra dimensions do not change the critical exponents.

Finally, we would like to mention that in this work we considered the power-Maxwell field as the gauge field. It is worth investigating the effects of dilaton on the critical behavior of black holes in the presence of other nonlinear gauge field such as logarithmic and exponential electrodynamics.

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