On $a-F$ dimensional interpolation

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A dimensional interpolation between the free energy and conformal anomaly on spheres is derived for free scalar and spinor fields on the basis of standard field theory. The regularisation used is an extension of one by Candelas and Weinberg. It yields a (known) simple integral which is shown to be identical to the interpolations introduced by Giombi and Klebanov using earlier AdS/CFT results. The extension to GJMS–type higher derivatives is made with a hint of a possible, kinematic resolution of the non–minimal Type-B mismatch being presented.

Another form of the sphere conformal anomaly is given.

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1. Introduction

Spectral zeta functions on spheres, and other symmetrical spaces, have been analysed for many years, probably beginning with [1]. In this paper, in order to effect the necessary analytic continuation, Minakshisundaram used a Bessel transform to rewrite the $\zeta$–function.

Expressed in more general terms, this gives the $\zeta$–function representation,

$$\sum_{\lambda} \lambda^{-2 - \alpha^2} = \sqrt{\frac{\pi}{\Gamma(s)}} \int_0^\infty d\tau \frac{K^{1/2}(\tau)}{(2\alpha)\tau^{s-1/2}} \mathcal{I}_{s-1/2}(\alpha \tau)$$

where $K^{1/2}(\tau)$ is the ‘cylinder’ heat-kernel

$$K^{1/2}(\tau) = \sum_{\lambda} e^{-\lambda \tau}$$

$\mathcal{I}_\nu(z)$ is the first modified Bessel function.

The terminology reflects the fact that the squares, $\lambda^2$, are usually the eigenvalues of a second order Laplace-type operator so that $K^{1/2}(\tau)$ is the heat-kernel of a square-root (pseudo)-operator, sometimes referred to as the wave operator.

The sum is over the, not necessarily distinct, eigenvalues. If the distinct eigenlevels, $\lambda_n$, are linear functions of the integer label, $n$, then $K^{1/2}(\tau)$ is the degeneracy generating function.

The representation (1) applied to spheres, $S^d$, (for scalars and spinors) was employed by Candelas and Weinberg, [2], to compute the effective action. For this one needs the derivative at zero, $Z'(0, \alpha)$. Their continuation to $s = 0$ involved a complex contour manoeuvre which worked only in odd dimensions, $d$. (See also Chodos and Myers, [3]). The method has been used to evaluate Rényi entropies, [4], the effective action for GJMS operators, [5], [6], and the boundary $F$–theorem, [7], [8]. As a very particular result, it yields the effective action for free fields as an easily computable integral. Of course, there are many other ways of finding this well known quantity.

On the other hand, motivated by AdS/CFT, and dimensional regularisation, an interpolation between the effective action (sometimes referred to as the free energy, ‘$F$’) for odd dimensions and the conformal anomaly ‘$a$’ for even has been suggested, [9]. A more extensive analysis of extending AdS/CFT to fractional dimensions,

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2 This has more recently appeared as the character.
and thence an interpolation, has recently appeared, [10], relating combinations of higher spins in the bulk to a CFT on the (spherical) boundary. The Bessel function transform is employed on the AdS side, and a contour method used that works in any dimension.\footnote{The detailed application of this transform places the eigenvalue in the Bessel argument and the mass-like parameter in the exponential. For the CFT, I do the opposite. This has the advantage that, for the restricted aim of finding an interpolation between \( F \) and \( a, s \) can be set to zero and the Bessel disappears. Any sum over the parameter \( \alpha \) can still be taken, as in the GJMS case.} In the light of this, it is interesting to reconsider the original method of Candelas and Weinberg and whether it can be rescued, in some form, for even dimensions. I do this for standard free scalar fields on the full sphere and quickly extend to spinors.

In the next section, I review the Candelas and Weinberg approach, and rewrite the scalar result to hold for both odd and even dimensions. This leads to an interpolation in the dimension. I next show that this agrees with the interpolation introduced by Giombi and Klebanov, [9]. This process is then repeated for spinors.

Further sections generalise these notions to higher derivative, GJMS-like operators. Their possible relevance to the non–minimal Type–B free energy mismatch found in AdS/CFT is discussed in section 6.

2. The formulae

I can quote the relevant \( \zeta \)-function expression taken from [4] equn. (9) adjusted for the full sphere and conformal coupling (\( \alpha = 1/2 \)),

\[
\zeta_d(s) = \frac{2^{2-d} \sqrt{\pi}}{\Gamma(s)} \int_0^\infty dz \cos z \sinh \frac{d}{2} z (2z)^{s-1/2} I_{s-1/2}(z) = \frac{2^{2-d} \sqrt{\pi}}{\Gamma(s)} \int_0^\infty dz f(z) 
\]

\[\equiv \frac{I(s)}{\Gamma(s)}, \tag{2}\]

The integral converges for \( s \) sufficiently large, but a continuation to around \( s = 0 \) is required. The Candelas and Weinberg method extends the integration variable, \( z \), into the complex plane and employs the parity behaviour of the integrand,

\[f(e^{\pm \pi i} z) = e^{\pm \pi i \sigma} f(z), \quad \sigma = 2s - d - 1,\]
to give the continuations,

\[ I_\pm(s) = \frac{2^{2-d}\sqrt{\pi}}{1 + e^{\pm\pi i\sigma}} \int_{C_\pm} dz \frac{\cosh z}{\sinh^{d/2} z} (2z)^{s-1/2} \Gamma_{s-1/2}(z), \tag{3} \]

where \( z = x + iy \), and the contours, \( C_\pm \), run from \(-\infty \pm iy_0 \) to \( \infty \pm iy_0 \) with \( 0 < y_0 < \pi \) \( i.e. \) above or below the real axis so avoiding the branch point at \( z = 0 \) and any other singularities. The cut (from 0 to \(-\infty \)) disappears when \( \sigma \) is integral which happens at the ‘physical’ values \( s = 0 \) and \( d \in \mathbb{Z} \).

For even dimensions, the prefactor in (3) diverges at \( s = 0 \) which has precluded its direct use in this case. The question is whether information can still be extracted.

I seek, in some ways, to reverse the introduction of the contour by reverting to a single sided integral. I keep \( \sigma \) general for as long as possible and rewrite the integral as follows, (using \(-x \pm iy_0 = e^{\pm i\pi}(x \pm iy_0)^* \mid_{C_\pm} \)),

\[
\int_{C_\pm} dz f(z) = \int_{-\infty}^{\infty} dx f(z) = \int_{0}^{\infty} dx f(z) + \int_{-\infty}^{0} dx f(z) \\
= \int_{0}^{\infty} dx f(z) + \int_{0}^{\infty} dx f(-x \pm iy_0) = \int_{0}^{\infty} dx f(z) + \int_{0}^{\infty} dx f(e^{\pm i\pi} z^*) \\
= \int_{0}^{\infty} dx (f(z) + e^{\pm i\pi \sigma} f(z^*)) \\
= \int_{0}^{\infty} dx ((1 + e^{\pm i\pi \sigma}) \text{Re} f(z) + i(1 - e^{\pm i\pi \sigma}) \text{Im} f(z)).
\]

Hence

\[ I_\pm(s) = 2^{2-d}\sqrt{\pi} \int_{0}^{\infty} dx (\text{Re} f(z) \pm \tan(\pi \sigma/2) \text{Im} f(z)). \tag{4} \]

To see the consequences of this result I look at the effective action, and use the \( \zeta \)-function form given in [11], as being the most convenient.

Generally, the free energy, \( F \), is determined by,

\[
-2F = \lim_{s \to 0} \frac{\zeta_d(s)}{s} = \zeta'_d(0) + \frac{\zeta_d(0)}{s} \bigg|_{s \to 0}, \tag{5} \]

which displays the log det part and the conformal anomaly, which is the pole residue.

In the particular case here,

\[ F_\pm = -\frac{1}{2} \lim_{s \to 0} \frac{I_\pm(s)}{\Gamma(s+1)}. \tag{6} \]
My objective is an interpolation between odd and even dimensions, so I firstly consider these special values of \( d \) at which \( F_+ = F_- \).

The odd \( d \) case is given in [4] with the result that the conformal anomaly is zero, and the log det is

\[
F_\pm = -\frac{1}{2} \zeta_d'(0) = -\frac{1}{2} I_\pm (0) \quad \text{odd } d .
\] (7)

For even \( d \), I am interested only in the conformal anomaly as being the universal quantity in \( F \). The second term in (4) contains the factor \( \tan(\pi \sigma/2) = -\cot(\pi s) \) giving the required pole with associated residue,

\[
\zeta(0) = \mp \frac{2^{2-d}}{\pi} \int_0^\infty dx \Im \frac{\cosh^2 z}{z \sinh^d z} , \quad \text{even } d.
\] (8)

I will now introduce a dimensional regularisation, and hence an interpolation, by assuming, initially, that \( d \) is such that \( F \) remains finite as \( s \to 0 \) and examining the resulting function of \( d \).

In accordance with this, from (6),

\[
F_\pm (d) = \frac{1}{2} \lim_{s \to 0} \frac{I_\pm (s)}{\Gamma(s + 1)} = -\frac{1}{2} I_\pm (0)
\]

\[
= -2^{1-d} \int_0^\infty dx \left( \Re \pm \cot(\frac{\pi d}{2}) \Im \right) \frac{\cosh^2 z}{z \sinh^d z} ,
\]

which suggests the introduction of the function

\[
\tilde{F}_\pm (d) = -\sin(\pi d/2) F_\pm (d) = 2^{1-d} \int_0^\infty dx \left( \sin(\frac{\pi d}{2}) \Re \pm \cos(\frac{\pi d}{2}) \Im \right) \frac{\cosh^2 z}{z \sinh^d z} ,
\] (9)

which interpolates between \((-1)^{(d-1)/2} \zeta'(0)/2\), (7), for odd \( d \) and \((-1)^{d/2} \pi \zeta(0)/2\), (8), for even.

As in [4], simplification occurs by choosing the contours \( z = x \pm i\pi/2 \). Then

\[
\frac{\cosh^2 z}{z \sinh^d z} = (x \mp i\pi/2) \left( \cos(\frac{\pi d}{2}) \mp i \sin(\frac{\pi d}{2}) \right) \frac{\sech^d x - \sech^{d-2} x}{x^2 + \pi^2/4} .
\]

The real and imaginary parts of the product of the first two brackets are, (the arguments are \( \pi d/2 \)),

\[
\Re = x \cos -\frac{\pi}{2} \sin , \quad \Im = \mp \frac{\pi}{2} \cos \mp x \sin
\]

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and hence the combination in (9) is,

\[(x \cos \frac{\pi}{2} \sin) \sin \pm (\mp \frac{\pi}{2} \cos x \sin) \cos = -\frac{\pi}{2},\]

and \(\tilde{F}\) reads (‘b’ for boson),

\[\tilde{F}_b(k) = -2^{-d} \pi \int_0^\infty dx \frac{\text{sech}^d x - \text{sech}^{d-2} x}{x^2 + \pi^2/4} = -2^{1-d} (J(d) - J(d-2)),\]

(10)

where

\[J(d) \equiv \int_0^\infty dx \frac{1}{(x^2 + 1) \cosh^d \pi x/2}.\]

(11)

I note that the influence of the cut has disappeared, \(F_+ = F_-.\)

For odd dimensions, the \(J\) integral has been introduced in [4], further studied in [5] as the quantity \(f_d = J(d)/\pi\) and there calculated for odd and even \(d\). For even \(d\), a simple residue calculation gives,

\[J(d) = (-1)^{d/2} \frac{\pi}{2^d d!} D_d^{(d)}, \quad \text{even } d,\]

where \(D_{2\nu}^{(m)}\) are Nörlund \(D\)-numbers, tabulated and easily calculated. This gives yet another expression for the scalar conformal anomaly on the even \(d\)-sphere,

\[\zeta_d(0) = \frac{2^{1-d}}{d!} (D_d^{(d)} - d(d-1) D_d^{(d-2)}).\]

(12)

For odd \(d\), \(J(d)\) is a finite sum of Dirichlet \(\eta\) functions and \(\log 2 = \eta(1)\). For the purposes here, I do not need it.

The integral \(J(d)\) can also be calculated by expanding the powers of sech in derivatives and then integrating by parts. For odd \(d\) this was done in [4] where Jensen’s integral form of the Riemann \(\zeta\)-function was employed. For even \(d\), the same route leads to another of Jensen’s forms.

### 3. Connections

The interpolation, for standard (real) free scalar fields, suggested by Giombi and Klebanov, [9], using the results of an AdS/CFT construction, equals \(\tilde{F}_{\text{GK}}(1)\) where,

\[\tilde{F}_{\text{GK}}^b(k) = \frac{1}{\Gamma(d+1)} \int_0^k du \, u \sin \pi u \Gamma\left(\frac{d}{2} + u\right) \Gamma\left(\frac{d}{2} - u\right).\]

(13)
In AdS/CFT, \( k \) is related to the field scaling dimension, *cf* [12,13].

Because \( \tilde{F}_{GK}(1) \) coincides with \( \tilde{F}(d) \), (10), on the integers, the two interpolations must be identical by Carlson’s theorem. However it is amusing to show this directly.

For this purpose the standard integral,

\[
\int_0^\infty dy \frac{\cosh u \pi y}{\cosh^d(\pi y/2)} = 2^{d-1} \frac{\Gamma(d/2 + u) \Gamma(d/2 - u)}{\pi \Gamma(d)},
\]

(14)

is useful.

Hence,

\[
\tilde{F}_{GK}^b(1) = \frac{2^{1-d}}{\pi d} \int_0^\infty dy \frac{\Gamma(d/2 + u) \Gamma(d/2 - u)}{\cosh^d(\pi y/2)} \int_0^\pi du \, u \sin u \, \cosh uy.
\]

(15)

Using,

\[
\int_0^\pi du \, u \sin u \, \cosh uy = \pi \frac{\cosh \pi y}{y^2 + 1} + \sinh \pi y \, \partial_y \frac{1}{y^2 + 1},
\]

we therefore need (the hyperbolic arguments are all \( \pi y/2 \)),

\[
\int_0^\infty dy \left( \frac{\pi(-\text{sech}^d + 2\text{sech}^{d-2})}{y^2 + 1} + 2\partial_y \left( \frac{1}{y^2 + 1} \right) \sinh \text{sech}^{d-1} \right)
\]

\[
= \int_0^\infty dy \left( \frac{\pi(-\text{sech}^d + 2\text{sech}^{d-2})}{y^2 + 1} + \frac{1}{\pi (y^2 + 1)(d-2)} \partial_y^2 \text{sech}^{d-2} \right)
\]

\[
= \pi d \int_0^\infty dy \frac{\pi(-\text{sech}^d + \text{sech}^{d-2})}{y^2 + 1},
\]

after integrating by parts and employing,

\[
\frac{4}{\pi^2} \partial_y^2 \text{sech}^{d-2}(\pi y/2) = (d - 2)^2 \text{sech}^{d-2}(\pi y/2) - (d - 2)(d - 1) \text{sech}^d(\pi y/2).
\]

Agreement with (10) is thus reached.
4. Spinors

The spin-1/2 expressions can likewise be shown to be coincident. The calculation is algebraically somewhat easier. The only difference is in the form of the function \( f(z) \) in (2). The necessary mode information is summarised in [6] which also employs the Candelas and Weinberg approach (for odd dimensions). The formulae in [4] show firstly that the \( \cosh z \) factor is replaced by unity and also the parameter \( \alpha \) is zero. These differences ultimately mean that the \( \cosh^2 z \) factor in (9) does not appear and the answer for the interpolation then follows in the same manner as before yielding,

\[
\tilde{F}_f(d) = 2^{1-d} J(d). \tag{16}
\]

Spin degeneracy factors have been removed.

This should be compared with the corresponding AdS/CFT interpolation, \( \tilde{F}^{f}_{GK}(1/2) \), with, [9],

\[
\tilde{F}^{f}_{GK}(k) = \frac{2}{\Gamma(d+1)} \int_0^k du \cos \pi u \Gamma\left(\frac{d+1}{2} + u\right) \Gamma\left(\frac{d+1}{2} - u\right). \tag{17}
\]

The equivalence of these two expressions again follows from the integral (14) using

\[
\int_0^{k\pi} du \cos u \cosh uy = \frac{y \cos \pi k \sinh k\pi y + \sin \pi k \cosh k\pi y}{1 + y^2}
\]

From (16) and (10) one obtains the boson–fermion relation

\[
\tilde{F}_b(d) = -\tilde{F}_f(d) + \frac{1}{4} \tilde{F}_f(d - 2),
\]

for interpolations, which is not so apparent from the forms (13) and (17).

On the integers, this implies relations between the free energies and between the conformal anomalies. For example, reinstating the spin factors, the usual conformal anomalies, \( C^{b,f}_{d/2} \), are connected by,

\[
C^{b}_{d/2} = -2^{-d/2-1}(2C^f_{d/2} + C^f_{d/2-1}),
\]

which can be checked from existing numbers.

These relations can be obtained from other representations.
5. Higher derivatives

I point out that although there are intimate linkages, my discussion is entirely boundary field–theoretic and makes no specific use of any CFT techniques or notions.

Motivated by conformal geometry questions, the functional determinants of operators of the form,

$$\Omega_k(d) = \frac{\Gamma(B + k + 1/2)}{\Gamma(B - k + 1/2)}, \quad (18)$$

arise. Here $k$ is a real parameter and, for scalars, $B = B_b = \sqrt{Y_d + 1/4}$ where $Y_d$ is the conformally covariant Penrose–Yamabe Laplacian. For Dirac spinors $B = B_f = (\nabla^2)^{1/2} = |\nabla|$ with an overall sign factor of $\nabla/|\nabla|$ being understood, [14].

For integral or half–integral $k$, $\Omega_k(d)$ factorises to a differential operator of order $2k$. In the integral case, for scalars, this product is Branson’s spherical realisation of the more general GJMS operators whose theory I do not need. I call such scalars, ‘regular’. The lowest example is just $\Omega_1(d) = Y_d$. The next one, $\Omega_2$ is the Paneitz operator. For half–integral $k$, scalars, termed ‘irregular’, propagate non–locally by a pseudo–operator, the lowest example being $\Omega_{1/2} = \sqrt{Y_d + 1/4}$.

The opposite prevails for spinors with $\Omega_{1/2} = \nabla$ being regular (Dirac) and $\Omega_1 = (\nabla/|\nabla|)(\nabla^2 - 1/4)$ irregular.

More explicitly, the product is, for integer $k$,

$$\Omega_k^{(E)}(d) = \prod_{j=0}^{k-1} (B^2 - (j + 1/2)^2) \quad (19)$$

and, for half integer $k = l + 1/2$,

$$\Omega_k^{(O)}(d) = B \prod_{h=1}^{l} (B^2 - h^2). \quad (20)$$

In fact, it does not much matter which factorised form one uses. For example, if one elects to use (19) for spinors (the irregular choice) a continuation, if possible, of the answer to $k$ half integral gives the (higher derivative) regular Dirac values, cf [6]. In this regard, the equality,

$$\log \det \Omega_k^{(E)} = \log \det \Omega_l^{(O)}, \quad k = l + 1/2,$$

was already shown analytically for all $d$ in [6].
It is now possible to apply standard field theory techniques essentially to each factor in the product. The sums of the logdets coming from the specific GJMS products can be given in closed form as functions of $k$. The expressions are given in [5] for scalars and [6] for spinors. Furthermore, the analysis of section 2 can be extended to yield interpolating quantities. I simply give the answers, which are most neatly expressed as contours,

$$\tilde{F}_b(d, k) = -\frac{e^{\mp \pi i (d+1)/2}}{2^{d+1}} \int_{C_{\pm}} dz \frac{\cosh z \sinh 2kz}{z \sinh^{d+1} z}$$

$$\tilde{F}_f(d, k) = \frac{e^{\mp \pi i (d+1)/2}}{2^d} \int_{C_{\pm}} dz \frac{\sinh 2kz}{z \sinh^{d+1} z}.$$  

These are useful formally as they stand. For example one has the simple relation,

$$\tilde{F}_f(d, 1) = -4\tilde{F}_b(d, 1/2)$$  

between the lowest irregular fields.

Trigonometry also enables the value at any $k$, if this is integral or half-integral, to be found from the values with $k = 1$ or $1/2$ as a sum over varying dimension. The algebraic details are exposed in [5,6]. Evaluated at odd and even dimension, these formulae connect free energies and conformal anomalies respectively.

As a typical consequence, the scalar GJMS conformal anomalies, $C_{d/2}(k) = k \zeta(0)$, can be expressed as linear combinations of the standard (regular) $(k = 1)$ anomalies at differing dimensions. For example,

$$C_{d/2}(4) = 4C_{d/2}(1) + 10C_{d/2-1}(1) + 6C_{d/2-2}(1) + C_{d/2-3}(1),$$

although this is not much use.

In order to have a calculable interpolating formula for any (real) $d$ and $k$, it is best to find a real form similar to (10) (which applies only for $k = 1$). Setting $z = x + i\pi/2$, algebra turns the above contour integrals into,

$$\tilde{F}_f(d, k) = \frac{1}{2^{d-1}} \int_0^\infty dy \frac{y \sinh(k\pi y) \cos(k\pi) + \cosh(k\pi y) \sin(k\pi)}{(y^2 + 1) \cosh^{d+1}(\pi y/2)}$$

for spinors while the scalar formula is,

$$\tilde{F}_b(d, k) = -\frac{1}{2^d} \int_0^\infty dy \sinh(\pi y/2) \frac{\sinh(k\pi y) \cos(k\pi) - y \cosh(k\pi y) \sin(k\pi)}{(y^2 + 1) \cosh^{d+1}(\pi y/2)}. $$

\footnote{Some numbers are given by Diaz, [13].}

4 Some numbers are given by Diaz, [13].
Use of the integral (14) shows, after a while, that these expressions are equal to the Giombi and Klebanov interpolations, \( \tilde{F}^f_{GK}(k) \) and \( \tilde{F}^b_{GK}(k) \) of (17) and (13). They are easily evaluated numerically subject to being convergent which places the restrictions \( |k| \leq d/2 \) for bosons and \( |k| \leq (d + 1)/2 \) for fermions. These could be termed GJMS existence conditions. The limits are when the integrands in (13) and (17) first acquire poles and also when negative modes appear for the operator \( \Omega_k \), (18). I do not consider the effects of this nor the remedy at this time.

The expressions (21) can be calculated explicitly by, e.g., lifting the contour, \( C \), to \( i\infty \), leaving a finite sum of \( \eta \)-functions or of \( \zeta \)-functions for the ‘physical’ situation (i.e. integral \( d \) and \( 2k \) even or odd). I do not need them but the case of even \( d \) is easy to deal with.

Averaging the (equal) \( C_+ \) and \( C_- \) expressions, the signs are such as to allow the two contours to be combined into a loop around the origin and evaluated as a single residue there. This is the source of the simple expression, (12), for the conformal anomaly. As another example, I enlarge on the conformal anomaly of irregular fields, say that of a \( k = 1/2 \) boson (equivalent to a \( k = 1 \) fermion to a factor of 4). The result is

\[
C_{d/2}(1/2) = \frac{2^{-d}}{(d-1)d!} D^{(d-1)}_d,
\]

and a few values are

\[
\frac{1}{24}, \frac{17}{5760}, \frac{367}{967680}, -\frac{27859}{464486400}, -\frac{1295803}{122624409600}.
\]

for \( d = 2, 4, 6, 8 \).

6. The non–minimal Type B mismatch

A clue to a possible resolution of the Type-B theory mismatch between bulk and boundary thus presents itself in the following way.

The AdS/CFT calculation of sums of free energies for bulk higher spin fields by Günaydin, Skvorstov and Tran, [15], (see also [16]), in the case of the Type–B spectrum were found by explicit evaluation, dimension by dimension, to arrange themselves according to (I use their notation),

\[
-\frac{1}{2} \zeta'_{HS}(0) = \frac{1}{4} \delta \tilde{F}^{\psi}_{\Delta=d/2+1} = -\delta \tilde{F}^{\phi}_{\Delta=d/2+1/2}, \tag{23}
\]

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where $\delta \tilde{F}_\Delta = \tilde{F}_G^f(k)$ and $\delta \tilde{F}_\Delta^b = \tilde{F}_b^G(k)$ with $k = \Delta - d/2$.

On the other hand, the (odd $d$) boundary field theoretic free energies found here equal $(−1)^{(d+1)/2} \tilde{F}_{b,f}^G(d, k)$ and, because of the equality of the $\tilde{F}_{G}^{f,b}(k)$ and $\tilde{F}_{b,f}(d, k)$ indicated previously, one sees that the AdS free energies are, up to alternating signs, one quarter of the free energy of an irregular fermion ($k = 1$) (per component) or, equivalently, the free energy of an irregular boson ($k = 1/2$).

It is not clear whether these irregular fields could be classified as sensible dual field theories in the light of their non–locality i.e. of their integral operator propagation. Even if they can be, there is still the problem of finding appropriate shifts of the bulk coupling constant needed for consistency with AdS/CFT, [16], and also of showing that the associated higher spin spectrum in the bulk is effectively unchanged. For these reasons the calculation here should be thought of simply as a pointer to a resolution or, possibly, merely as a relocation of the puzzle.

It is worth noting that the difference in free energies on the even $(d + 1)$–hemisphere of two regular ($k = 1$) scalars, one obeying Dirichlet and the other Neumann conditions on the rim, equals one quarter of the irregular ($k = 1$) spinor free energy on the odd $d$–sphere, [8].5

7. Conclusion and discussion

It has been demonstrated, for free fields, that an interpolation between odd and even dimensions obtained on AdS/CFT methods by dimensional regularisation can be obtained field theoretically on the boundary by extending an old technique which produces a different, but equivalent, form.

Numerically, there is little to choose between the expressions (10) and (13). The former is over an infinite range, but converges rapidly. The latter is over a finite interval but has to call the gamma function.

The extension to higher derivatives has been made and a minimal resolution of the type-B mismatch suggested consisting of taking the dual field theory to be a free irregular (non–local) one, but questions remain.

The bulk higher–spin combinations dual to the other higher derivative theories remain to be uncovered, cf [17].

5 This generalises to any $k$. 

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Acknowledgments

I thank Evgeny Skvortsov for very helpful instruction.

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