Noncommutative gerbes and deformation quantization

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Abstract

We define noncommutative gerbes using the language of star products. Quantized twisted Poisson structures are discussed as an explicit realization in the sense of deformation quantization. Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.
1 Introduction

Gerbes [1, 2, 3] are the next step up from a line bundle on the geometric ladder in the following sense: A unitary line bundle is a 1-cocycle in Čech cohomology, i.e., it is a collection of smooth “transition” functions \( g_{\alpha \beta} \) on the intersections \( U_\alpha \cap U_\beta \) of an open cover \( \{ U_\alpha \} \) of a manifold \( M \) satisfying \( g_{\alpha \beta} = - g_{\beta \alpha} \) and \( g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha} = 1 \) on \( U_\alpha \cap U_\beta \cap U_\gamma \). A gerbe is a 2-cocycle in Čech cohomology, i.e., it is a collection \( \lambda = \{ \lambda_{\alpha \beta \gamma} \} \) of maps \( \lambda_{\alpha \beta \gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to U(1) \), valued in the abelian group \( U(1) \), satisfying

\[
\lambda_{\alpha \beta \gamma} = \lambda_{\beta \alpha \gamma} = \lambda_{\gamma \beta \alpha} = 1
\]

and the 2-cocycle condition

\[
\delta \lambda = \lambda_{\beta \gamma \delta} \lambda_{\alpha \gamma \delta} \lambda_{\alpha \beta \delta} \lambda_{\alpha \beta \gamma} = 1
\]

on \( U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \). The collection \( \lambda = \{ \lambda_{\alpha \beta \gamma} \} \) of maps with the stated properties defines a gerbe in the same sense as a collection of transition functions defines a line bundle. In the special case where \( \lambda \) is a Čech 2-coboundary with \( \lambda = \delta h \), i.e., \( \lambda_{\alpha \beta \gamma} = h_{\alpha \beta} h_{\beta \gamma} h_{\gamma \alpha} \), we call the collection \( h = \{ h_{\alpha \beta} \} \) of functions \( h_{\alpha \beta} : U_\alpha \cap U_\beta \to U(1) \) a trivialization of a gerbe. Taking the “difference” of two trivializations \( \{ h_{\alpha \beta} \} \), \( \{ h'_{\alpha \beta} \} \) of a gerbe we step down the geometric ladder again and obtain a line bundle: \( g_{\alpha \beta} \equiv h_{\alpha \beta} / h'_{\alpha \beta} \) satisfies the 1-cocycle condition \( g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha} = 1 \).

A gerbe has a local trivialization for any particular open set \( U_0 \) of the covering: Defining \( h'_{\beta \gamma} \equiv \lambda_{0 \beta \gamma} \) with \( \beta, \gamma \neq 0 \) we find from the 2-cocycle condition of a gerbe that \( \lambda_{\alpha \beta \gamma} = h_{\alpha \beta} h_{\beta \gamma} h_{\gamma \alpha} \). This observation leads to an equivalent definition of a gerbe in terms of line bundles on the double overlaps of the cover. The only difference to the definition of a line bundle from this point of view is that we step up the geometric ladder and use line bundles on \( U_\alpha \cap U_\beta \) rather than transition functions. A gerbe is then a collection of line bundles \( L_{\alpha \beta} \) for each double overlap \( U_\alpha \cap U_\beta \), such that:

G1 There is an isomorphism \( L_{\alpha \beta} \cong L_{\beta \alpha}^{-1} \).

G2 There is a trivialization \( \lambda_{\alpha \beta \gamma} \) of \( L_{\alpha \beta} \otimes L_{\beta \gamma} \otimes L_{\gamma \alpha} \) on \( U_\alpha \cap U_\beta \cap U_\gamma \).

G3 The trivialization \( \lambda_{\alpha \beta \gamma} \) satisfies \( \delta \lambda = 1 \) on \( U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \).

Gerbes are interesting in physics for several reasons: One motivation is the interpretation of D-brane charges in terms of K-theory in the presence of a topologically nontrivial \( B \)-field, when the gauge fields living on D-branes become connections on certain noncommutative algebras rather than on a vector bundle [4]-[12]. Azumaya algebras appear to be a natural choice and give the link to gerbes. Gerbes, rather than line bundles, are the structure that arises in the presence of closed 3-form backgrounds as, e.g., in WZW models and Poisson sigma models with WZW term [9, 13, 14]. Gerbes help illuminate the geometry of mirror symmetry of 3-dimensional Calabi-Yau manifolds [3] and they provide a language to formulate duality transformations with higher order antisymmetric
Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.

The paper is organised as follows: In section 2 we recall the local description of noncommutative line bundles in the framework of deformation quantization. Instead of repeating that construction we shall take the properties that were derived in [16, 17] as a formal definition of a noncommutative line bundle. In the same spirit we define noncommutative gerbes in section 3 using the language of star products and complement this definition with an explicit realization of noncommutative gerbes as quantizations of twisted Poisson structures as introduced in [18] and further discussed in [19].

2 Noncommutative line bundles

Here we collect some facts on noncommutative line bundles [20, 16] that we will need in the sequel. Let $(M, \theta)$ be a general Poisson manifold, and $\ast$ the corresponding Kontsevich’s deformation quantization of the Poisson tensor $\theta$. Further let us consider a good covering $\{U^i\}$ of $M$. For purposes of this paper a noncommutative line bundle $L$ is defined by a collection of local transition functions $G^{ij} \in C^\infty(U^i \cap U^j)[[\hbar]]$, valued in the enveloping algebra of $U(1)$ (see [21]), and a collection of maps $D^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$, formal power series in $\hbar$ starting with identity and with coefficients being differential operators such that

$$G^{ij} \ast G^{jk} = G^{ik}$$

on $U^i \cap U^j \cap U^k$, $G^{ii} = 1$ on $U^i$, and

$$\text{Ad}_s G^{ij} = D^i \circ (D^j)^{-1}$$

on $U^i \cap U^j$ or, equivalently, $D^i(f) \ast G^{ij} = G^{ij} \ast D^j(f)$ for all $f \in C^\infty(U^i \cap U^j)[[\hbar]]$. Obviously, with this definition the local maps $D^i$ can be used to define globally a new star product $\ast'$ (because the inner automorphisms $\text{Ad}_s G^{ij}$ do not affect $\ast'$)

$$D^i(f \ast' g) = D^i f \ast D^i g .$$

We say that two line bundles $L_1 = \{G_1^{ij}, D_1^i, \ast\}$ and $L_2 = \{G_2^{ij}, D_2^i, \ast\}$ are equivalent if there exist a collection of invertible local functions $H^i \in C^\infty(U^i)[[\hbar]]$ such that

$$G_1^{ij} = H^i \ast G_2^{ij} \ast (H^j)^{-1}$$

and

$$D_1^i = \text{Ad}_s H^i \circ D_2^i .$$

\[1\] A noncommutative line bundle is a finite projective module. In the present context it can be understood as a quantization of a line bundle in the sense of deformation quantization. Here we shall take the properties of quantized line bundles as derived in [16, 17] as a formal definition of a noncommutative line bundle.
The tensor product of two line bundles $\mathcal{L}_1 = \{G^{ij}_1, D^i_1, \ast_1\}$ and $\mathcal{L}_2 = \{G^{ij}_2, D^i_2, \ast_2\}$ is well defined if $\ast_2 = \ast'_1$ (or $\ast_1 = \ast'_2$). Then the corresponding tensor product is a line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1 = \mathcal{L}_{21} = \{G^{ij}_{21}, D^i_{12}, \ast_1\}$ defined as

$$G^{ij}_{12} = D^i_1(G^{ij}_2) \ast_1 G^{ij}_1 = G^{ij}_1 \ast_1 D^i_1(G^{ij}_2) \quad (8)$$

and

$$D^i_{12} = D^i_1 \circ D^i_2 \quad (9)$$

The order of indices of $\mathcal{L}_{21}$ indicates the bimodule structure of the corresponding space of sections to be defined later, whereas the first index on the $G_{12}$’s and $D_{12}$’s indicates the star product (here: $\ast_1$) by which the objects multiply.

A section $\Psi = (\Psi^i)$ is a collection of functions $\Psi^i \in C^\infty(U^i)[[h]]$ satisfying consistency relations

$$\Psi^i = G^{ij} \ast \Psi^j \quad (10)$$

on all intersections $U^i \cap U^j$. With this definition the space of sections $\mathcal{E}$ is a right $\mathfrak{A} = (C^\infty(M)[[h]], \ast)$ module. We shall use the notation $\mathcal{E}_\mathfrak{A}$ for it. The right action of the function $f \in \mathfrak{A}$ is the regular one

$$\Psi \cdot f = (\Psi^k \ast f) \quad (11)$$

Using the maps $D^i$ it is easy to turn $\mathcal{E}$ also into a left $\mathfrak{A}' = (C^\infty(M)[[h]], \ast')$ module $\mathfrak{A}' \mathcal{E}$. The left action of $\mathfrak{A}'$ is given by

$$f \cdot \Psi = (D^i(f) \ast \Psi^i) \quad (12)$$

It is easy to check, using (4), that the left action (12) is compatible with (10). From the property (5) of the maps $D^i$ we find

$$f \cdot (g \cdot \Psi) = (f \ast' g) \cdot \Psi \quad (13)$$

Together we have a bimodule structure $\mathfrak{A}' \mathcal{E}_\mathfrak{A}$ on the space of sections. There is an obvious way of tensoring sections. The section

$$\Psi_{12} = D^i_1(\Psi^i_2) \ast_1 \Psi^i_1 \quad (14)$$

is a section of the tensor product line bundle (8), (9). Tensoring of line bundles naturally corresponds to tensoring of bimodules.

Using the Hochschild complex we can introduce a natural differential calculus on the algebra $\mathfrak{A}$. The $p$-cochains, elements of $C^p = \text{Hom}_\mathbb{C}(\mathfrak{A}^{op}, \mathfrak{A})$, play the role of $p$-forms and the derivation $d : C^p \to C^{p+1}$ is given on $C \in C^p$ as

$$dC(f_1, f_2, \ldots, f_{p+1}) = f_1 \ast C(f_2, \ldots, f_{p+1}) - C(f_1 \ast f_2, \ldots, f_{p+1})$$

$$+ C(f_1, f_2 \ast f_3, \ldots, f_{p+1}) - \ldots + (-1)^p C(f_1, f_2, \ldots, f_p \ast f_{p+1})$$

$$+ (-1)^{p+1} C(f_1, f_2, \ldots, f_p) \ast f_{p+1} \quad (15)$$

$^2$Other choices for the differential calculus are of course possible, e.g., the Lie algebra complex.

3
A (contravariant) connection $\nabla : \mathcal{E} \otimes_{\mathcal{A}} C^p \rightarrow \mathcal{E} \otimes_{\mathcal{A}} C^{p+1}$ can now be defined by a formula similar to (15) using the natural extension of the left and right module structure of $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} C^p$. Namely, for a $\Phi \in \mathcal{E} \otimes_{\mathcal{A}} C^p$ we have

$$\nabla \Phi (f_1, f_2, \ldots, f_{p+1}) = f_1 \cdot \Phi (f_2, \ldots, f_{p+1}) - \Phi (f_1 \ast f_2, \ldots, f_{p+1}) + \Phi (f_1, f_2 \ast f_3, \ldots, f_{p+1}) - \ldots + (-1)^p \Phi (f_1, f_2, \ldots, f_p \ast f_{p+1}) + (-1)^{p+1} \Phi (f_1, f_2, \ldots, f_p). f_{p+1}. \tag{16}$$

We also have the cup product

$$(C_1 \cup C_2)(f_1, \ldots, f_{p+q}) = C_1(f_1, \ldots, f_p) \ast C_2(f_{p+1}, \ldots, f_q). \tag{17}$$

The cup product extends to a map from $(\mathcal{E} \otimes_{\mathcal{A}} C^p) \otimes_{\mathcal{A}} C^q$ to $\mathcal{E} \otimes_{\mathcal{A}} C^{p+q}$. The connection $\nabla$ satisfies the graded Leibniz rule with respect to the cup product and thus defines a bona fide connection on the module $\mathcal{E}_{\mathcal{A}}$. On the sections the connection $\nabla$ introduced here is simply the difference between the two actions of $C^\infty(M)[[\hbar]]$ on $\mathcal{E}$:

$$\nabla \Psi (f) = f. \Psi - \Psi. f = (\nabla^i \Psi^i (f)) = (D^i (f) \ast \Psi^i - \Psi^i \ast f). \tag{18}$$

As in [17] we define the gauge potential $\mathcal{A} = (\mathcal{A}^i)$, where the $\mathcal{A}^i : C^\infty(U^i)[[\hbar]] \rightarrow C^\infty(U^i)[[\hbar]]$ are local 1-cochains, by

$$\mathcal{A}^i \equiv D^i - id. \tag{19}$$

Then we have for a section $\Psi = (\Psi^i)$, where the $\Psi^i \in C^\infty_c(U^i)[[\hbar]]$ are local 0-cochains,

$$\nabla^i \Psi^i (f) = d\Psi^i (f) + \mathcal{A}^i (f) \ast \Psi^i, \tag{20}$$

and more generally $\nabla^i \Phi^i = d\Phi^i + \mathcal{A}^i \cup \Phi^i$ with $\Phi = (\Phi^i) \in \mathcal{E} \otimes_{\mathcal{A}} C^p$. In the intersections $U^i \cap U^j$ we have the gauge transformation (cf. (2))

$$\mathcal{A}^i = \text{Ad}_{\mathcal{A}} G^{ij} \circ \mathcal{A}^j + G^{ij} \ast d(G^{ij})^{-1}. \tag{21}$$

The curvature $K_{\nabla} \equiv \nabla^2 : \mathcal{E} \otimes_{\mathcal{A}} C^p \rightarrow \mathcal{E} \otimes_{\mathcal{A}} C^{p+2}$ corresponding to the connection $\nabla$, measures the difference between the two star products $\ast'$ and $\ast$. On a section $\Psi$, it is given by

$$(K_{\nabla} \Psi)(f, g) = (D^i (f \ast' g - f \ast g) \ast \Psi^i). \tag{22}$$

The connection for the tensor product line bundle (2) is given on sections as

$$\nabla_{12} \Psi_{12} = D^i_1 (\nabla_2 \Psi^i_2) \ast_1 \Psi^i_1 + D^i_1 (\Psi^i_2) \ast_1 \nabla_1 \Psi^i_1. \tag{23}$$

Symbolically,

$$\nabla_{12} = \nabla_1 + D_1 (\nabla_2). \tag{24}$$

Let us note that the space of sections $\mathcal{E}$ as a right $\mathcal{A}$-module is projective of finite type. Of course, the same holds if $\mathcal{E}$ is considered as a left $\mathcal{A}'$ module. Also let us note that the two
algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are Morita equivalent. Up to a global isomorphism they must be related by an action of the Picard group $\text{Pic}(M) \cong H^2(M, \mathbb{Z})$ as follows. Let $L \in \text{Pic}(M)$ be a (complex) line bundle on $M$ and $F$ its Chern class. Consider the formal Poisson structure $\theta'$ given by the geometric series

$$\theta' = \theta(1 + \hbar F \theta)^{-1}. \quad (25)$$

In this formula $\theta$ and $F$ are understood as maps $\theta : T^*M \to TM$, $F : TM \to T^*M$ and $\theta'$ is the result of the indicated map compositions. Then $\star'$ must (up to a global isomorphism) be the deformation quantization of $\theta'$ corresponding to some $F \in H^2(M, \mathbb{Z})$. If $F = da$ then the corresponding quantum line bundle is trivial, i.e.,

$$G^{ij} = (H^i)^{-1} \star H^j \quad (26)$$

and the linear map

$$D = \text{Ad}_e H^i \circ D^i \quad (27)$$

defines a global equivalence (a stronger notion than Morita equivalence) of $\star$ and $\star'$.

3 Noncommutative gerbes

Now let us consider any covering $\{U_\alpha\}$ (not necessarily a good one) of a manifold $M$. Here we switch from upper Latin to lower Greek indices to label the local patches. The reason for the different notation will become clear soon. Consider each local patch equipped with its own star product $\star_\alpha$ the deformation quantization of a local Poisson structure $\theta_\alpha$. We assume that on each double intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the local Poisson structures $\theta_\alpha$ and $\theta_\beta$ are related similarly as in the previous section via some integral closed two form $F_{\beta\alpha}$, which is the curvature of a line bundle $L_{\beta\alpha} \in \text{Pic}(U_{\alpha\beta})$

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha} \theta_\beta)^{-1}. \quad (28)$$

Let us now consider a good covering $U^i_{\alpha\beta}$ of each double intersection $U_\alpha \cap U_\beta$ with a noncommutative line bundle $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, D^i_{\alpha\beta}, \star_\alpha\}$, see Figure 2.

$$G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}, \quad G^{ii}_{\alpha\beta} = 1, \quad (29)$$
\[ D_{\alpha\beta}^i(f) \star f G_{\alpha\beta}^{ij} = G_{\alpha\beta}^{ij} \star f D_{\alpha\beta}^j(f) \] (30)

and

\[ D_{\alpha\beta}^i(f \star g) = D_{\alpha\beta}^i(f) \star f D_{\alpha\beta}^i(g). \] (31)

The opposite order of indices labelling the line bundles and the corresponding transition functions and equivalences simply reflects a choice of convention. As in the previous section the order of indices of \( L_{\alpha\beta} \) indicates the bimodule structure of the corresponding space of sections, whereas the order of Greek indices on \( G \)'s and \( D \)'s indicates the star product in which the objects multiply. The product always goes with the first index of the multiplied objects.

A noncommutative gerbe is characterised by the following axioms:

**Axiom 1** \( L_{\alpha\beta} = \{ G_{\beta\alpha}^{ij}, D_{\beta\alpha}^i, \star \beta \} \) and \( L_{\beta\alpha} = \{ G_{\alpha\beta}^{ij}, D_{\alpha\beta}^i, \star \alpha \} \) are related as follows

\[ \{ G_{\beta\alpha}^{ij}, D_{\beta\alpha}^i, \star \beta \} = \{(D_{\alpha\beta}^i)^{-1}, (G_{\alpha\beta}^{ij})^{-1}, \star \beta \} \] (32)

i.e. \( L_{\alpha\beta} = L_{\beta\alpha}^{-1} \). (Notice also that \((D_{\alpha\beta}^{ji})^{-1} = (D_{\alpha\beta}^{ij})^{-1} \).)

**Axiom 2** On the triple intersection \( U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \) the tensor product \( L_{\gamma\beta} \otimes L_{\beta\alpha} \) is equivalent to the line bundle \( L_{\gamma\alpha} \). Explicitly

\[ G_{\alpha\beta}^{ij} \star f D_{\alpha\beta}^j(G_{\beta\gamma}^{ij}) = \Lambda_{\alpha\beta\gamma}^i \star f G_{\alpha\gamma}^{ij} \star f (\Lambda_{\alpha\gamma}^j)^{-1}, \] (33)

\[ D_{\alpha\beta}^i \circ D_{\beta\gamma}^i = \text{Ad}_{\alpha} \Lambda_{\alpha\beta\gamma}^i \circ D_{\alpha\gamma}^i. \] (34)

**Axiom 3** On the quadruple intersection \( U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \cap U_{\delta} \)

\[ \Lambda_{\alpha\beta\gamma}^i \star f \Lambda_{\alpha\gamma\delta}^i = D_{\alpha\beta}^i(\Lambda_{\alpha\beta\delta}^i) \star f \Lambda_{\alpha\gamma\delta}^i, \] (35)

\[ \Lambda_{\alpha\beta\gamma}^i = (\Lambda_{\alpha\beta\gamma}^i)^{-1} \quad \text{and} \quad D_{\alpha\beta}^i(\Lambda_{\beta\gamma\alpha}^i) = \Lambda_{\alpha\beta\gamma}^i. \] (36)

With slight abuse of notation we have used Latin indices \( \{ i, j, \ldots \} \) to label both the good coverings of the intersection of the local patches \( U_{\alpha} \) and the corresponding transition functions of the consistent restrictions of line bundles \( L_{\alpha\beta} \) to these intersections. A short comment on the consistency of Axiom 3 is in order. Let us define

\[ D_{\alpha\beta\gamma}^i = D_{\alpha\beta}^i \circ D_{\beta\gamma}^i \circ D_{\gamma\alpha}^i. \] (37)

Then it is easy to see that

\[ D_{\alpha\beta\gamma}^i \circ D_{\alpha\gamma\delta}^i \circ D_{\alpha\delta\beta}^i = D_{\alpha\beta}^i \circ D_{\beta\gamma\delta}^i \circ D_{\beta\delta\alpha}^i. \] (38)

In view of (34) this implies that

\[ \Lambda_{\alpha\beta\gamma\delta}^i \equiv D_{\alpha\beta}^i(\Lambda_{\beta\gamma\delta}^i) \star f \Lambda_{\alpha\beta\delta}^i \star f \Lambda_{\alpha\delta\gamma}^i \star f \Lambda_{\alpha\gamma\beta}^i. \]
is central. Using this and the associativity of $\star$ together with (33) applied to the triple tensor product $L_{\gamma\delta} \otimes L_{\gamma\beta} \otimes L_{\beta\alpha}$ transition functions

$$G^{ij}_{\alpha\beta\gamma} \equiv G^{ij}_{\alpha\beta} \star \eta^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) \star \eta^j_{\alpha\beta}(D^j_{\beta\gamma}(G^{ij}_{\gamma\delta}))$$

(39)

reveals that $\Lambda^{i}_{\alpha\beta\gamma\delta}$ is independent of $i$. It is therefore consistent to set $\Lambda^{i}_{\alpha\beta\gamma\delta}$ equal to 1. A similar consistency check works also for (36). If we replace all noncommutative line bundles $L_{\alpha\beta}$ in Axioms 1-3 by equivalent ones, we get by definition an equivalent noncommutative gerbe.

There is a natural (contravariant) connection on a quantum gerbe. It is defined using the (contravariant) connections $\nabla_{\alpha\beta} = (\nabla_{\alpha\beta}^i)$ (cf. (16), (18)) on quantum line bundles $L_{\alpha\beta}$. Let us denote by $\nabla_{\alpha\beta\gamma}$ the contravariant connection formed on the triple tensor product $L_{\alpha\beta\gamma} \equiv L_{\alpha\gamma} \otimes L_{\gamma\beta} \otimes L_{\beta\alpha}$ with maps $D^i_{\alpha\beta\gamma}$ and transition functions (39) according to the rule (24). Axiom 2 states that $\Lambda^{i}_{\alpha\beta\gamma}$ is a trivialization of $L_{\alpha\gamma\beta}$ and that

$$\nabla^{i}_{\alpha\beta\gamma} \Lambda^{i}_{\alpha\beta\gamma} = 0 .$$

(40)

Using Axiom 2 one can show that the product bundle

$$L_{\alpha\gamma\delta} = L_{\alpha\gamma} \otimes L_{\gamma\delta} \otimes L_{\alpha\delta} \otimes L_{\beta\gamma} \otimes L_{\beta\delta} \otimes L_{\beta\alpha}$$

(41)

is trivial: it has transition functions $G^{ij}_{\alpha\beta\gamma\delta} = 1$ and maps $D^i_{\alpha\beta\gamma\delta} = \text{id}$. The constant unit section is thus well defined on this bundle. On $L_{\alpha\beta\gamma\delta}$ we also have the section $(\Lambda^{i}_{\alpha\beta\gamma})$. Axiom 3 implies $(\Lambda^{i}_{\alpha\beta\gamma\delta})$ to be the unit section. If two of the indices $\alpha, \beta, \gamma, \delta$ are equal, triviality of the bundle $L_{\alpha\beta\gamma\delta}$ implies (36). Using for example the first relation in (36) one can show that (35) written in the form $D^i_{\alpha\beta}(\Lambda^{i}_{\beta\gamma\delta}) = \text{id}$ is invariant under cyclic permutations of any three of the four factors appearing on the l.h.s..

If we now assume that $F_{\alpha\beta} = da_{\alpha\beta}$ for each $U_{\alpha} \cap U_{\beta}$ then all line bundles $L_{\beta\alpha}$ are trivial

$$G^{ij}_{\alpha\beta\gamma} = (H^{i}_{\alpha\beta})^{-1} \star \eta^{j}_{\alpha\beta} H^{j}_{\alpha\beta}$$

$$D_{\alpha\beta} = \text{Ad}_{\eta^{i}_{\alpha\beta}} H^{i}_{\alpha\beta} \circ D^{i}_{\alpha\beta} .$$

It then easily follows that

$$\Lambda_{\alpha\beta\gamma} \equiv H^{i}_{\alpha\beta} \star \eta^{i}_{\alpha\beta}(H^{i}_{\beta\gamma}) \star \eta^{i}_{\alpha\beta}(D^{i}_{\beta\gamma}(H^{i}_{\gamma\delta})) \star \eta^{i}_{\alpha\beta} \Lambda^{i}_{\alpha\beta\gamma}$$

(42)

defines a global function on the triple intersection $U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$. $\Lambda_{\alpha\beta\gamma}$ is just the quotient of the two sections $(H^{i}_{\alpha\beta} \star \eta^{i}_{\alpha\beta}(H^{i}_{\beta\gamma}) \star \eta^{i}_{\alpha\beta}(D^{i}_{\beta\gamma}(H^{i}_{\gamma\delta}))^{-1})$ and $\Lambda^{i}_{\alpha\beta\gamma}$ of the triple tensor product $L_{\alpha\gamma} \otimes L_{\gamma\beta} \otimes L_{\beta\alpha}$. On the quadruple overlap $U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \cap U_{\delta}$ it satisfies conditions analogous to (35) and (36)

$$\Lambda_{\alpha\beta\gamma} \star \Lambda_{\alpha\gamma\delta} = D_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) \star \eta^{j}_{\alpha\beta} \Lambda^{i}_{\alpha\beta\gamma}$$

(43)

$$\Lambda_{\alpha\beta\gamma} = (\Lambda^{i}_{\alpha\beta\gamma})^{-1} \quad \text{and} \quad D_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) = \Lambda_{\alpha\beta\gamma}$$

(44)
Also
\[ \mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \text{Ad}_* \lambda_{\alpha\beta\gamma}. \] (45)

So we can take formulas (43)-(45) as a definition of a gerbe in the case of a good covering \{U_\alpha\}. The collection of local equivalences \( \mathcal{D}_{\alpha\beta} \) satisfying (45) with \( \lambda_{\alpha\beta\gamma} \) fulfilling (43), (44) defines on \( M \) a stack of algebras [22].

From now on we shall consider only good coverings. A noncommutative gerbe defined by \( \lambda_{\alpha\beta\gamma} \) and \( \mathcal{D}_{\alpha\beta} \) is said to be trivial if there exist a global star product \( * \) on \( M \) and a collection of “twisted” transition functions \( G_{\alpha\beta} \) defined on each overlap \( U_\alpha \cap U_\beta \) and a collection \( \mathcal{D}_\alpha \) of local equivalences between the global product \( * \) and the local products \( *_\alpha \)

\[ \mathcal{D}_\alpha (f * \mathcal{D}_\alpha (g) = \mathcal{D}_\alpha (f *_\alpha g) \]
satisfying the following two conditions:

\[ G_{\alpha\beta} * G_{\beta\gamma} = \mathcal{D}_\alpha (\lambda_{\alpha\beta\gamma}) * G_{\alpha\gamma} \] (46)

and

\[ \text{Ad}_* G_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta}. \] (47)

Locally, every noncommutative gerbe is trivial as is easily seen from (43), (44) and (45) by fixing the index \( \alpha \). Defining as in (19), \( A_\alpha = \mathcal{D}_\alpha - \text{id} \), \( A_{\alpha\beta} = \mathcal{D}_{\alpha\beta} - \text{id} \) we obtain the “twisted” gauge transformations

\[ A_\alpha = \text{Ad}_* G_{\alpha\beta} \circ A_\beta + G_{\alpha\beta} * d(G_{\alpha\beta})^{-1} - \mathcal{D}_\alpha \circ A_{\alpha\beta}. \] (48)

4 Quantization of twisted Poisson structures

Let \( H \in H^3(M, \mathbb{Z}) \) be a closed integral three form on \( M \). Such a form is known to define a gerbe on \( M \). We can find a good covering \{U_\alpha\} and local potentials \( B_\alpha \) with \( H = dB_\alpha \) for \( H \). On \( U_\alpha \cap U_\beta \) the difference of the two local potentials \( B_\alpha - B_\beta \) is closed and hence exact: \( B_\alpha - B_\beta = da_{\alpha\beta} \). On a triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \) we have

\[ a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = -i \lambda_{\alpha\beta\gamma} d\lambda_{\alpha\beta\gamma}^{-1}. \] (49)

The collection of local functions \( \lambda_{\alpha\beta\gamma} \) defines the gerbe.

Let us also assume the existence of a formal antisymmetric bivector field \( \theta = \theta(0) + h\theta(1) + \ldots \) on \( M \) such that

\[ [\theta, \theta] = h \theta^* H, \] (50)

where \([ , ]\) is the Schouten-Nijenhuis bracket and \( \theta^* \) denotes the natural map sending \( n \)-forms to \( n \)-vector fields by “using \( \theta \) to raise indices”. Explicitly, in local coordinates, \( \theta^* H^{ijk} = \theta^m \theta^n \theta^k H_{mno} \). We call \( \theta \) a Poisson structure twisted by \( H \) [18, 8, 13]. On each \( U_\alpha \) we can introduce a local formal Poisson structure \( \theta_\alpha = \theta(1 - hB_\alpha \theta)^{-1} \), \( \{ \theta_\alpha, \theta_\alpha \} = 0 \). The Poisson structures \( \theta_\alpha \) and \( \theta_\beta \) are related on the intersection \( U_\alpha \cap U_\beta \) as in (28)

\[ \theta_\alpha = \theta_\beta (1 + hF_{\alpha\beta} \theta_\beta)^{-1}, \] (51)
with an exact $F_{\beta\alpha} = da_{\beta\alpha}$. Now we can use Formality \[23\] to obtain local star products $\ast_\alpha$ and to construct for each intersection $U_\alpha \cap U_\beta$ the corresponding equivalence maps $D_{\alpha\beta}$. See \[17\], \[16\] for an explicit formula for the equivalence maps. According to our discussion in the previous section these $D_{\alpha\beta}$, supplemented by trivial transition functions, define a collection of trivial line bundles $L_{\beta\alpha}$. On each triple intersection we then have

\[
D_{\alpha\beta} \circ D_{\beta\gamma} \circ D_{\gamma\alpha} = \text{Ad}_{\ast_\alpha} \Lambda_{\alpha\beta\gamma}.
\] (52)

It follows from the discussion after formula (36) that $\Lambda_{\alpha\beta\gamma}$ defines a quantum gerbe (a deformation quantization of the classical gerbe $\lambda_{\alpha\beta\gamma}$) if each of the central functions $\Lambda_{\alpha\beta\gamma\delta}$ introduced there can be chosen to be equal to 1. See \[19\], section 5 and \[24\] that this is really the case.

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