Light Cone Condition for a Thermalized QED Vacuum

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1. INTRODUCTION

As an application of the QED effective action approach at finite temperature [1]-[5], we investigate the aberration of low-frequency photons from the usual vacuum light cone in the presence of a heat bath. The propagation of photons can deviate from the classically free vacuum behavior when external perturbations exert their influence on the vacuum polarization of the photon. In a perturbative language, virtual processes, e.g., the electron-positron loop, confer the properties of the participant particles to the photon. By this means, the photon can therefore acquire, for instance, a “size” (of the order of the Compton wavelength of the participant particle) and a “charge distribution”. Finally, external perturbations, e.g., electromagnetic or gravitational fields, cavity-like boundary conditions, or a heat bath, can interact with these induced photon properties. The resulting modified photon propagation can be expressed in terms of a light cone condition for light propagating in a thermalized QED vacuum. In an intermediate-temperature domain, the velocity shift equals minus the shift of the refractive index associated with the modified vacuum.

Concentrating on the deviation of the speed of light from the unmodified-vacuum velocity $c$ (=1 in our units) by thermal effects, the velocity shift\(^1\) in the low-temperature region, $T \ll m$, where $m$ denotes the electron mass, yields $\delta v^2 \sim -T^4/m^4$. This important result can be obtained from the two-loop polarization tensor, in which the radiative photon within the fermion loop is considered to be thermalized [7], [8]. The latter approach was also put forward in [8].

Since the effective action provides us with a convenient interface between the full quantum theory and classical field theory, it is the appropriate tool with which to describe electromagnetic waves of low frequency ($\omega \ll m$), which are the low-energy degrees of freedom of QED. The modified, e.g., thermalized, vacuum is viewed as a medium with optical properties which exert an influence on the propagation of a plane wave. In the present work, we rigorously extend the effective action approach to the light cone condition as applied in [3] to the finite-temperature case. For this, the dependence of the effective action on the complete set of gauge and Lorentz invariants of the given situation has to be considered; contrary to the zero-temperature case involving the two standard invariants $F$ and $G$, we have to take two further invariants into account which arise from an additional four-vector associated with the heat bath. This is compulsory for an exact treatment of the problem in the effective action approach, and has not been considered in earlier works. The only input for deducing the light cone condition for low-frequency waves will be a general Lagrangian whose dependence on electromagnetic field and temperature is solely dictated by Lorentz covariance and

\(^1\)The velocity shift equals minus the shift of the refractive index associated with the modified vacuum.

\(^2\)In this reference, the correct numerical result was found for the first time; furthermore, the similarities between the present problem and the Scharnhorst effect [10], i.e., light propagation in a Casimir vacuum, have been pointed out.
gauge invariance. Therefore, the derivation of the light cone condition does not rely on perturbation theory or a perturbative expansion of the Lagrangian.

As an application for our formalism, we fall back on perturbation theory and employ the thermal one-loop effective action of QED which allows for an investigation of light propagation in a low- and intermediate-temperature domain, $T \sim m$. It is a priori clear that the low-temperature velocity shift $\sim T^2$ will not be found to this order of calculation, since it is a two-loop effect in an exact field theoretic framework. Unfortunately, a full two-loop calculation of the thermal QED effective action has not yet been performed. At intermediate temperature, $T \sim m$, the one-loop contribution is expected to be the dominant one. Apart from soft logarithmic corrections, we find a maximum velocity shift which reduces the light velocity by $\delta v^2_{\text{max}} = -\frac{\delta}{2}$. Our results do not apply to values of temperature at which the physics of the $e^+e^-$-gas is dominated by plasma effects, $T \gg m$. Then, signal propagation will completely be determined by plasma modes which are not considered in our approach. Especially, waves of low-frequency cease to propagate, since the appearance of a plasma mass serves as a cutoff for low frequencies.

II. EFFECTIVE ACTION AT FINITE TEMPERATURE

The low-energy effective action of QED for slowly varying fields at finite temperature and zero density can only depend on a small set of invariants. In particular, a dependence of the effective action on the (infinitely many) invariants involving derivatives of the fields can be neglected if the typical scale associated with the variation of the fields in spacetime is much larger than the Compton wavelength of the electron ($\lambda_c = 1/m$). Concerning light propagation, this is a reasonable assumption as long as the frequency of the propagating wave is much smaller than the electron mass ($\omega \ll m$) and possible additional background fields are almost homogeneous.

In the sense of relativistic equilibrium thermodynamics, the Lorentz covariant and gauge invariant building blocks of these invariants are given by the field strength tensor and its dual, $F^{\mu\nu}$, $*F^{\mu\nu}$, and by the heat-bath vector $n^\mu$. The latter is on the one hand characterized by the value of its invariant scalar product $n^\mu n_\mu = -T^2$, where $T$ denotes the heat-bath temperature, and on the other hand related to the heat-bath 4-velocity $u^\mu$ via the invariant parameter $T$: $n^\mu = T u^\mu$.

From these objects, we can construct the following set of invariants, which represents a conventional choice:

$$F = \frac{1}{4} F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2),$$
$$G = \frac{1}{4} F^{\mu\nu}F_{\mu\nu} = -E \cdot B,$$
$$E := \frac{1}{T^2} \left(n_\alpha F^{\alpha\mu}\right) \left(n_\beta F^{\beta\mu}\right) \equiv \left(u_\alpha F^{\alpha\mu}\right) \left(u_\beta F^{\beta\mu}\right),$$
$$T = \sqrt{-n^\mu n_\mu}.$$  

Incidentally, parity invariance requires the action to be an even function of $G$ and prohibits the generation of a “Chern-Simons”-like term $\sim u_\mu F^{\mu\nu} A_\nu$. Without loss of generality, we assume $G > 0$ for reasons of uniqueness. The convenience of the choice of $E$ becomes clear in the heat-bath rest frame, where we find $E = E^2$. It is easy to check that the maximum number of classical invariants obeying these symmetry requirements is four (see, e.g. Sec. III of Ref. [5]); the set of Eq. (2) is thus complete.

From these symmetry considerations, we conclude that the low-energy effective Lagrangian of QED for slowly varying fields at finite temperature can generally be written as

$$\mathcal{L} = \mathcal{L}(F, G, E, T).$$

In other words, we are dealing with the local part (no derivatives of the fields) of the complete non-local effective action (involving infinitely many derivative terms of the fields). Of course, even the local part of the exact effective action incorporating the contributions of all loops is presently out of reach. Hence in practical applications, we will rely on a perturbative expansion of Eq. (2); in fact, only the lowest order has been identified up to now. The derivation of this one-loop QED effective action for arbitrary constant electromagnetic fields at finite temperature has required more effort than the purely magnetic case [4]. The first comprehensive study has been elaborated by Elmfors and Skagerstam [3] employing the real-time formalism. For our purposes, we make

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3 In [4], the loop counting refers to the order of the Heisenberg-Euler Lagrangian employed during the approximation process for thermalizing the fermions and therefore differs from the present one.

4 We employ the metric $g = \text{diag}(-1,1,1,1)$. 

5 Apart from the implicit (and soft) spacetime dependence of the fields, the system under consideration is homogeneous in space and time (thermal equilibrium). Therefore, the effective Lagrangian cannot depend explicitly on the coordinates $x^\mu$. Although the finite-temperature case closely resembles the case of a Casimir vacuum, the latter is not invariant under translations orthogonal to the Casimir plates; hence, the effective Lagrangian for the Casimir vacuum can and, in fact, does depend explicitly on the spacetime coordinates. In order to perform the considerations of the present work for the Casimir vacuum, we thus had to include another invariant of the form $x^\mu n_\mu$, where $n^\mu$ denotes the normal vector associated with the Casimir plates.
use of the representation of the effective action as given in [3]. Therein, the field and temperature dependence is expressed in terms of the complete set of invariants as given in Eq. (1).

Furthermore, introducing the abbreviations

\[ a := \left( \sqrt{F^2 + G^2 + F} \right)^{1/2}, \]
\[ b := \left( \sqrt{F^2 + G^2 - F} \right)^{1/2}, \]  
(3)

the thermal contribution to the one-loop effective Lagrangian at zero density reads [3]

\[ \mathcal{L}^{1T} = \frac{1}{4 \pi^2} \int_0^\infty ds \frac{e^{-im^2s}}{s^3} eas \cot(eas) ebs \coth(ebs) \]
\[ \times \sum_{n=1}^\infty (-1)^n e^{ib(s)} \frac{s^2}{n^2}, \]

whereby the exponent is given by

\[ h(s) = \frac{b^2 - \mathcal{E}}{a^2 + b^2} e a \cot eas + \frac{a^2 + \mathcal{E}}{a^2 + b^2} e b \cot ebs. \]  
(5)

The complete effective Lagrangian then consists of \( \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}^1 + \mathcal{L}^{1T} + \mathcal{L}_\gamma \), where \( \mathcal{L}_{cl} = -F \) denotes the classical Lagrangian and \( \mathcal{L}^1 \) represents the well-known one-loop part [12] at zero temperature, i.e., the Heisenberg-Euler-Schwinger Lagrangian. For later use, we also incorporated the free photonic contribution \( \mathcal{L}_\gamma \), which is equal to minus the free energy density of a free photon gas (Stefan-Boltzmann law), \( \mathcal{L}_\gamma(T) = \frac{e^2}{16\pi} T^4 \). Incidentally, only the zero-temperature one-loop part \( \mathcal{L}^1 \) is affected by renormalization: for fixing the counter-terms, we require the complete Lagrangian to meet with the pure classical part in the weak-field limit (at \( T = 0 \)).

### III. LIGHT CONE CONDITION

The system under consideration is a propagating plane wave field in the presence of a vacuum that is modified by an external constant electromagnetic field at finite temperature and zero density. Following [3], we assume the propagating wave to be of low frequency and neglect any vacuum modifications caused by the propagating light itself. In other words, we neglect dispersive effects and exclude non-linear self-interactions of the wave. Thus, we can linearize the field equations with respect to this plane wave field.

Within the framework of these assumptions, the field equations are obtained from the Euler-Lagrange equations of motion of the effective thermal QED Lagrangian:

\[ 0 = 2 \partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu \nu}} = 2 \partial_\mu \left[ \partial_F \mathcal{L} \frac{\partial F}{\partial F_{\mu \nu}} + \partial_G \mathcal{L} \frac{\partial G}{\partial F_{\mu \nu}} + \partial_F \mathcal{L} \frac{\partial F}{\partial F_{\mu \nu}} \right]. \]  
(6)

The Lagrangian \( \mathcal{L} \) that we are going to insert into Eq. (6) contains the classical part as well as the zero-temperature and thermal one-loop contributions: \( \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}^1 + \mathcal{L}^{1T} \); however, the following derivation of the light cone condition does not rely on any perturbative approximation of \( \mathcal{L} \). It is only necessary that the field dependence of the Lagrangian be completely contained in a set of three linearly independent invariants for which we take the standard invariants \( F, G \) and the invariant \( E \) as defined in Eq. (1). Note that the differentiation with respect to \( F_{\mu \nu} \) has to be performed with regard to its anti-symmetry properties.

After moving the space-time derivative to the right employing the Bianchi-identity \( \partial_\mu F_{\mu \nu} = 0 \), we may write Eq. (6) in the form

\[ 0 = \partial_F \mathcal{L} \partial_\mu F^{\mu \nu} + \partial_\nu \mathcal{L} F^{\mu \nu} \left( \frac{1}{2} M_{\alpha \beta}^{\mu \nu} \right), \]  
(7)

where \( M_{\alpha \beta}^{\mu \nu} \) denotes the tensor,

\[ \frac{1}{2} M_{\alpha \beta}^{\mu \nu} = 4 \partial_\xi \mathcal{L} u_{[\mu} \delta_{[\beta]}^\nu] \]
\[ + F_{\mu \nu} \left[ \frac{\partial^2 \mathcal{L}}{2} F_{\alpha \beta} + \frac{\partial_\xi \mathcal{L}}{2} F_{\alpha \beta} + 2 \partial_\xi \mathcal{L} u_{[\alpha}(uF)_{\beta]} \right] \]
\[ + *F_{\mu \nu} \left[ \frac{\partial_\xi \mathcal{L}}{2} F_{\alpha \beta} + \frac{\partial^2 \mathcal{L}}{2} F_{\alpha \beta} + 2 \partial_\xi \mathcal{L} u_{[\alpha}(uF)_{\beta]} \right] \]
\[ + 4 u^{[\mu}(uF)^{\nu]} \left[ \frac{\partial_\xi \mathcal{L}}{2} F_{\alpha \beta} + \partial_\xi \mathcal{L} F_{\alpha \beta} \right] *F_{\alpha \beta} + 2 \partial_\xi \mathcal{L} u_{[\alpha}(uF)_{\beta]} \right], \]  
(8)

where we employed matrix notation, \( (Fk)^{\nu} \equiv F^\nu_{\alpha} k^\alpha \), or \( (ku) = k^\nu u_\nu \), and a prescription for anti-symmetrizing tensor indices: \( A^{[\mu]} = \frac{1}{2} (A^{\nu} - A^{\nu}) \). Note that the tensor \( M_{\alpha \beta}^{\mu \nu} \) is anti-symmetric with respect to its upper as well as its lower indices, and contains first and second derivatives of the Lagrangian with respect to the three invariants. As mentioned above, there is no explicit dependence of \( \mathcal{L} \) on the coordinates because of homogeneity in spacetime. Equation (8) represents the effective field equations for slowly varying fields in a thermalized QED vacuum. It replaces the linear Maxwell equations in vacuum \( \partial_\mu F^{\mu \nu} = 0 \), defining a new theory of “classical” non-linear electrodynamics.

Following the techniques of [3], we adapt these equations to the case of a propagating plane wave: first, we decompose the electromagnetic field strength into the field strength of the plane wave field \( f^{\mu \nu} \) and an additional homogeneous background field \( F^{\mu \nu} \): \( F^{\mu \nu} \rightarrow f^{\mu \nu} + F^{\mu \nu} \). Now derivatives of the field act only on the plane wave part: \( \partial_\mu (f^{\alpha \beta} + f^{\alpha \beta}) = \partial_\mu f^{\alpha \beta} \). Transition to Fourier space leads us to the replacement \( -i \partial_\mu f^{\alpha \beta} \rightarrow k_\mu f^{\alpha \beta} \), where \( k_\mu \) denotes the wave vector of the plane wave field \( f^{\alpha \beta} \). The latter is proportional to \( f^{\alpha \beta} \sim k^\alpha e^\beta - k^\beta e^\alpha \), whereby \( e^\alpha \) represents a polarization vector of the plane wave. Furthermore, we adopt the Lorentz gauge condition for the plane wave field and sum over polarization.
states; the latter step is, on the one hand, convenient, and, on the other hand, reasonable for a purely thermalized vacuum which is naturally isotropic. In the case of an additional background field, we lose information about effects of birefringence by summing over polarization states (cf. later). Finally, we arrive at the desired light cone condition:

$$0 = 2 \partial \mathcal{F} \mathcal{L} k^2 + 2 \left( \frac{1}{2} M^{\mu \nu}_{\alpha \beta} \right) k_{\mu} k_{\nu}.$$  \hspace{1cm} (9)

Since we have linearized with respect to the propagating plane wave field, the field quantities appearing in Eq. (9) now describe an externally applied background field only. Following summation over polarization states, the field strength tensor of the plane wave field has completely dropped out of the formula. This corresponds to our approximation of neglecting vacuum modifications caused by the plane wave field itself. In this sense, the plane wave field resembles a small test charge of classical electrodynamics.

Evaluating the contractions of $M^{\mu \nu}_{\alpha \beta}$ with $k_{\mu} k_{\nu}$, we find the light cone condition of a soft plane wave propagating in an external field at finite temperature in its final form,

$$\begin{align*}
0 &= (\partial \mathcal{F} \mathcal{L} + G \partial \mathcal{G} \mathcal{L} - \mathcal{F} \mathcal{L}(\partial \mathcal{G} \mathcal{L})) k^2 + \frac{1}{2} (\partial^2 + \partial_\mathcal{L}) \mathcal{L} (F_k)^\mu (F_k)^\nu - \partial \mathcal{F} \mathcal{L} k^2 + 2 \left[ \partial \mathcal{G} \mathcal{L} + E \partial \mathcal{L} \right] (k u)^2 \\
&+ 2 \partial \mathcal{F} \mathcal{L} (k u) (F_k)^\mu (F_k)^\nu + 2 \partial \mathcal{G} \mathcal{L} (k u) (\mathcal{F}^\mu (F_k)^\nu + \mathcal{F}^\nu (F_k)^\mu) + 2 \left( \partial \mathcal{F} \mathcal{L} - \partial \mathcal{G} \mathcal{L} \right) (u F_k)^2 + 2 \partial \mathcal{G} \mathcal{L} (u F_k)(u^* F_k).
\end{align*}$$  \hspace{1cm} (10)

Let us stress once more that the field invariants $\mathcal{F}, \mathcal{G}, \mathcal{L}$ in this equation characterize an externally applied background field only. The first line in Eq. (10) corresponds to the purely field-modified light cone condition as derived in \[\cite{6}\], which can conveniently be expressed in terms of the energy-momentum tensor (or its vacuum expectation value) of the electromagnetic field.

In the following, we simply confine ourselves to the case of vanishing external field: $F^{\mu \nu}, \mathcal{F}, \mathcal{G}, \mathcal{L} \rightarrow 0$. In this limit, the light cone condition becomes extremely simplified, and, except for the term linear in $\mathcal{F}$ which is filtered out by $\partial \mathcal{F} \mathcal{L}, \mathcal{F}, \mathcal{G}, \mathcal{L} = 0$, only the first term in square brackets in Eq. (10) survives:

$$0 = (\partial \mathcal{F} \mathcal{L} - \partial \mathcal{G} \mathcal{L}) k^2 + 2 \partial \mathcal{G} \mathcal{L} (k u)^2, \quad \text{for } \mathcal{F}, \mathcal{G}, \mathcal{L} \rightarrow 0.$$  \hspace{1cm} (11)

Introducing the phase velocity $v = k^0 / (k \mathcal{L})$, which is identical to the group velocity in the low-frequency limit, we may rewrite the light cone condition in the heat-bath rest frame ((ku)^2 = (k^0)^2) in terms of the squared velocity, $v^2 = 1 + \frac{2 \partial \mathcal{G} \mathcal{L}}{(-\partial \mathcal{F} \mathcal{L} + \partial \mathcal{G} \mathcal{L})}.  \hspace{1cm} (12)$

In the present case of light propagation in a thermalized vacuum, it is physically reasonable to expect that $v \leq 1$ holds rigorously, since the propagating photons do not only virtually interact with the usual vacuum modes (zero-point fluctuations) but also with the thermally excited modes. This may be interpreted as a kind of “resistance” implying a decrease of the propagation velocity.

In order to maintain $v \leq 1$, the fraction in the denominator of Eq. (12) should always be positive. Note that this statement is independent of any loop approximation.

Now we turn to perturbation theory, where $\mathcal{L}$ can be assumed to consist of the classical term $\mathcal{L}_{\text{cl}} = -\mathcal{F}$ plus quantum correction terms such as those discussed in Sec. II. The latter are considered to be small compared to the classical one; hence, Eq. (12) to lowest order in the perturbation simplifies to:

$$v^2 \simeq 1 - 2 \partial \mathcal{G} \mathcal{L}.$$  \hspace{1cm} (13)

As will be demonstrated below, this formula describes sufficiently the thermally induced velocity shift to one loop for reasonable values of temperature.

For the remainder of the section, we will transcribe the light cone condition Eq. (11) into a form which has proved useful for gaining intuitive insight into the problem of light propagation in modified vacua $\cite{6}$. For this, we introduce the vacuum expectation value of the energy-momentum tensor,

$$\langle T^{\mu \nu} \rangle = \mathcal{L} g^{\mu \nu} + T \partial \mathcal{L} u^\mu u^\nu,$$  \hspace{1cm} (14)

which can be obtained by varying the effective action with respect to the metric $g_{\mu \nu}$, and treating the variable $T$ as $T = \sqrt{-g_{\mu \nu} \langle T^{\mu \nu} \rangle}$. Equation (14) is the finite-temperature analogue of Eq. (19) of $\cite{6}$ where the vacuum expectation value of $T^{\mu \nu}$ for an electromagnetic background has been considered. Solving Eq. (14) for $u^\mu u^\nu$, we may substitute this into Eq. (11) and arrive at:

$$k^2 = Q_T \langle T^{\mu \nu} \rangle k_{\mu} k_{\nu},$$  \hspace{1cm} (15)

where the so-called effective action charge $Q_T$ for the present system is given by:

$$Q_T = \frac{2 \partial \mathcal{G} \mathcal{L}}{-\partial \mathcal{F} \mathcal{L} + \partial \mathcal{G} \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial \mathcal{G} \mathcal{L}} \mathcal{L}}.  \hspace{1cm} (16)$$

From Eq. (15), we can read off that the deformation of the light cone is determined by the (renormalized) energy density of the modified vacuum. The proportionality factor $Q_T$ depends on the parameters of the system via the

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\footnote{In $\cite{6}$, the name “effective action charge” derives from the resemblance between the corresponding equation to Eq. (10) for electromagnetic fields and the Poisson equation, $2Q_T = (\partial \mathcal{F} + \partial_\mathcal{L}) \mathcal{L}$. This accidental feature does not hold in the present case; however, we stick to this nomenclature, since essential properties as well as the underlying physical picture also hold here.}
effective Lagrangian. Under the reasonable assumption that the deformation of the light cone is bounded even in the high-temperature domain, one can conclude that the factor \( Q \) has to decrease sufficiently fast for increasing temperature. Therefore, we expect \( Q \) to be locally (charge-like) distributed in the parameter space of temperature. This justifies the nomenclature effective action change. In fact, the one-loop results will confirm this picture.

IV. ONE-LOOP RESULTS

As an application, we insert the one-loop effective action into the light cone condition. Let us first consider a thermalized vacuum with vanishing background fields. Therefore, the velocity shift stems from the thermal contribution \( \mathcal{L}^{1T} \) only, and we need \( \partial_\mathcal{E}\mathcal{L} = \partial_\mathcal{E}\mathcal{L}^{1T} \) in the zero-field limit:

\[
\partial_\mathcal{E}\mathcal{L}^{1T} \equiv -\frac{\alpha}{3\pi} \sum_{n=1}^{\infty} (-1)^n \left( \frac{m}{T} \right)^n K_1(\varphi n).
\]

(17)

Regarding Eq. (14), we also need \( \partial_\mathcal{F}\mathcal{L} \); for this, note that the zero-temperature one-loop part is renormalized in such a way that the term linear in \( \mathcal{E} \) vanishes in order to recover the pure Maxwell theory in the weak-field limit. Hence, it is only the thermal contribution \( \mathcal{L}^{1T} \) which provides for an additional term linear in \( \mathcal{F} \):

\[
\partial_\mathcal{F}\mathcal{L} = -1 + \partial_\mathcal{F}\mathcal{L}^{1T},
\]

where the \(-1\) stems from the Maxwell part \( \mathcal{L}^{cl} \). In the zero-field limit, we get:

\[
\partial_\mathcal{F}\mathcal{L}^{1T} = -\frac{4\alpha}{3\pi} \sum_{n=1}^{\infty} (-1)^n K_0(\varphi n).
\]

(18)

For Eqs. (17) and (18), we employed the fact that summation and proper-time integration in Eq. (9) are interchangeable in the zero-field limit. The proper-time integrals can then be identified as a representation of the modified Bessel function \( K_1 \) and \( K_0 \) [13].

In the limiting cases of low, intermediate and high temperature, Eqs. (17) and (18) can be expanded in terms of the parameter \( \frac{T}{m} \). Let us first consider Eq. (18); at low temperature, only the first term of the sum must be taken into account, which leads to

\[
\partial_\mathcal{F}\mathcal{L}^{1T}(T \ll m) \simeq \frac{2\alpha}{3\pi} \sqrt{\frac{2\pi T}{m}} e^{-\varphi}.
\]

(19)

For an intermediate- or high-temperature expansion, the infinite sum must be completely taken into account; this can be achieved employing the techniques proposed in the appendix B of [3]. The result for Eq. (18) at \( T \sim m \) is

\[
\partial_\mathcal{F}\mathcal{L}^{1T} = \frac{2\alpha}{3\pi} \left[ (0.666 \ldots) + (0.814 \ldots) \ln \frac{T}{m} \right],
\]

(20)

where the numbers stem from pure integrals over analytic functions. Incidentally, the expansion for \( T \gg m \) is formally identical to Eq. (20) with the factor of 0.814 replaced by 1 as found in [14].

For our purpose of investigating low- and intermediate-temperature domains, we observe that \( \partial_\mathcal{F}\mathcal{L}^{1T}(T) = \mathcal{O}(\alpha/\pi) \ll 1 \). For the calculation of the velocity shifts to order \( \alpha/\pi \), it is thus sufficient to employ Eq. (14), i.e., neglecting \( \partial_\mathcal{F}\mathcal{L}^{1T} \) compared to \( \partial_\mathcal{F}\mathcal{L}^{cl} = -1 \).

Turning to the low-temperature expansion of Eq. (17), we have to take into account only the first term of the sum:

\[
\partial_\mathcal{E}\mathcal{L}^{1T}(T \ll m) \simeq \frac{\alpha}{3\pi} \sqrt{\frac{2\pi}{T}} \sqrt{\frac{m}{T}} e^{-\varphi}.
\]

(21)

Inserting this into Eq. (13), we find for the squared velocity at low temperature an exponentially decreasing modification:

\[
v^2(T \ll m) \simeq 1 - \frac{\alpha}{3\pi} \sqrt{\frac{2\pi}{T}} \sqrt{\frac{m}{T}} e^{-\varphi}.
\]

(22)

The high-temperature expansion of Eq. (15) can be worked out along the lines of App. B of [3], leading to

\[
\partial_\mathcal{E}\mathcal{L}^{1T}(T \gg m) \simeq \frac{\alpha}{6\pi} \left( 1 - k_2 \frac{m^2}{T^2} + \mathcal{O}\left( \frac{m^3}{T^4} \right) \right),
\]

(23)

where \( k_2 \) denotes the number \( k_2 = 0.23139 \ldots \) and arises from a parameter-independent integral over analytic functions during the expansion. The light cone condition then yields for the squared velocities at high temperature:

\[
v^2(T \gg m) \simeq 1 - \frac{\alpha}{3\pi} \left( 1 + k_2 \frac{m^2}{T^2} \right) + \ldots,
\]

(24)

which implies a maximum velocity shift of \( \delta v_{\text{max}}^2 = \frac{\alpha}{m^2} \approx -\frac{1}{12\pi} \).

It should, however, be noted that an expansion for \( T/m \gg 1 \) is a formal trick to extract analytical results. For identifying the values of temperature to which the light cone condition Eq. (14) is applicable, we notice that the frequency of the plane wave should, on the one hand, be smaller than the electron mass in order to justify the assumption of slowly varying fields, and on the other hand, be larger than the plasma frequency in order to ensure the existence of such a propagating mode:

\[
\omega_p \ll \omega \ll m.
\]

(25)

For the plasma frequency corresponding to the Debye screening mass, we employ the representation found in [3]:

\[
\omega_p^2 = -\frac{\alpha}{\pi} m^2 \sum_{n=1}^{\infty} (-1)^n K_2(\varphi n).
\]

(26)
The maximum assumption of temperature up to which the low-frequency assumption can formally be justified is determined by \( \omega_0(T_{\text{max}}) = m \). Numerically, one finds \( T_{\text{max}}/m \approx 5.74 \ldots \). Of course, this is just a formal value; in order to obtain reasonable results, i.e., in order to satisfy relation (25), the actual temperature should be kept smaller than this maximum value.

Nevertheless, the numerical results given below confirm that the formal expansion for \( T/m \gg 1 \) is already appropriate for \( T \sim m \), which justifies extracting physical conclusions from this analytical procedure.

V. DISCUSSION

A. Low Temperature

At temperatures well below the fermion mass, the one-loop modification of the velocity of light as described by Eq. (23) vanishes exponentially. It is the mass of the fermions in the loop that is responsible for the attenuation of thermal effects, since it counteracts thermal excitations of higher virtual modes. As a consequence, the usual hierarchy of loop calculations is inverted: the dominant thermal contribution to the low-temperature velocity shift stems from the two-loop graph in which the internal photon line is thermalized. Since the photon is massless, thermal excitations produce a low-temperature velocity shift of \( \delta v = -\frac{4k\pi}{m^2T} \alpha^2 \). Incidentally, the three-loop diagram involving two radiative thermalized photons of course contributes subdominantly \( \sim \alpha^3 T^4/m^4 \), as can be deduced from the findings of [13].

However, we would like to point out that the dominance of the two-loop contribution does not seem to hold over the complete range where \( T < m \), as can be discovered numerically. The increasing influence of the one-loop term is due to the factor of \( T^{-1/2} \) and, of course, the proportionality to \( \alpha \) (and not \( \alpha^2 \)) in Eq. (23). The one-loop contribution becomes dominant for comparably low temperatures of \( T/m \approx 0.058 \) (Fig. 1).

Of course, this statement should be handled with care because here we are comparing a one-loop result with...
thermalized fermions to a two-loop results without thermalized fermions. Nevertheless, an increasing two-loop contribution from thermalized fermions will always be suppressed by a factor of $\alpha$, which could only be compensated for by an unexpected conspiracy of numerical pre-factors.

$$\begin{array}{c}
0.01 & 0.06 & 0.11 & 0.16 \\
10^{-14} & 10^{-13} & 10^{-12} & 10^{-11} & 10^{-10} & 10^{-9} & 10^{-8} & 10^{-7} & 10^{-6}
\end{array}$$

FIG. 1. Low-temperature velocity shift $\delta v (\hat{=} \frac{1}{2} \delta v^2)$ in units of the vacuum velocity $c = 1$ versus the dimensionless temperature scale $T/m$; the one-loop contribution as given in Eq. (22) exceeds the well-known two-loop result $\sim T^4/m^4$ for comparably low values of temperature.

B. High Temperature

As discussed at the end of Sec. IV, the results of the formal high-temperature expansion should be applied to values of temperature of the order of the electron mass, $T \sim m$, where plasma effects [17] do not yet dominate the physical properties of the modified vacuum. At even higher temperatures, $T \gg m$, the magnitude of the plasma frequency $\omega_p$ serves as a cutoff for the low-frequency waves which belong to the main ingredients of our formalism. Then our calculations become meaningless, because the low-frequency modes simply do not propagate.

Nevertheless, already in the temperature region where $T \sim m$, our results indicate that the thermal excitations are no longer seriously damped by the mass of the fermion in the loop. The one-loop contribution obviously dominates the velocity shift; therefore, the usual loop hierarchy is completely restored.

It is remarkable that a maximally possible velocity shift to this order of calculation exists which is simply given by $\delta v^2_{\text{max}} = -\frac{\alpha}{6\pi}$ (Fig. 2). Of course, this maximal velocity shift is only reached asymptotically, and therefore, strictly speaking, lies beyond the scope of the present formalism; nevertheless, the actual velocity shift already comes close to the maximum value in the intermediate-temperature domain where $T \sim m$ (Fig. 2). To complete the discussion, it should be mentioned that the inclusion of the contributions from Eq. (13) increase the negative velocity shift proportional to $\frac{\alpha}{6\pi} \ln \frac{T}{m}$ for $T \geq m$. For reasonable values of temperature, this contribution is in fact completely negligible.

C. Low Temperature and Weak Magnetic Field

In order to justify the low-temperature expansion leading to Eq. (29), we restrict the following discussion of Eq. (30) to maximal values of temperature of $T_{\text{max}}/m \approx 1/3$.

Let us stress once more that the present results derive from a light cone condition which represents a sum-rule for different polarization modes. In fact, the QED vacuum under the influence of an oriented background magnetic field is reminiscent of an anisotropic medium with different refractive indices, i.e, velocity shifts, for each polarization mode. In this sense, the velocity shifts found in Eq. (30) represent the arithmetic averages of those for the single polarization modes. A detailed discussion of magnetically induced birefringence at finite-temperature for which the eigenvalue problem of the tensor $M_{\alpha\beta}^{\mu\nu}$ (Eq. 8) has to be solved is out of the scope of the present work.

Concentrating on those terms in Eq. (30) which depend on the magnetic field, we encounter the well-known zero-temperature velocity shift (last term of the first line)
with its typical dependence on the angle \( \Theta = \delta(\hat{k}, \hat{B}) \); furthermore, there is also one thermal correction with this \( \Theta \)-dependence (second line), and another which is independent of the direction of the magnetic field (third line). Although both thermal corrections vanish exponentially in the zero-temperature limit, the factors of \( T \) in the numerator provide for a strong increase for \( T/m > 0.1 \).

Numerical analysis shows that the second line of Eq. (30) contributes with an opposite sign compared to the zero-temperature part. Since both exhibit the same dependence on the magnetic field and the angle \( \Theta \), we may add them and find that, on the one hand, thermal effects diminish the zero-temperature velocity shift in a magnetic background by an amount of, e.g., \( \sim 20\% \) at \( T/m \sim 0.25 \). On the other hand, the third line of Eq. (30) contributes with the same sign as the zero-temperature velocity shift, and becomes comparable to the latter (for \( \Theta = \pi/2 \)), e.g., \( \sim 80\% \) of the zero-temperature velocity shift at \( T/m \sim 0.25 \). Therefore, at these values of temperature, the velocity shift loses its strong dependence on the direction of the magnetic field and the propagation direction and is partly dependent on the energy density of the magnetic field.

D. Effective Action Charge

In Eq. (15), we were able to formulate the thermally induced deformation of the light cone in terms of the vacuum expectation value of the energy-momentum tensor times a factor \( Q_T \) called effective action charge \( \Lambda \). The charge concept has been introduced as a useful picture which provides for an intuitive understanding of this proportionality factor. Since any perturbed QED vacuum can be expected to control the magnitude of the velocity shifts even for vacuum modifications of high energy density, this factor \( Q_T \) has to decrease sufficiently fast for increasing energy scale parameters (in this case: temperature). Therefore, the factor \( Q_T \) should exhibit a localized distribution in this parameter space.

For calculating the effective action charge \( Q_T \) according to Eq. (16) for a thermalized QED vacuum, we do not only consider the one-loop contribution from the thermalized fermions \( \mathcal{L}^{1T} \), but also take the free photonic part \( \mathcal{L}_\gamma = \frac{\pi}{4} T^4 \) into account. Although it does not exert an influence on the velocity shift and drops out of Eq. (15), we have to include it in the present considerations in order to work with the complete vacuum expectation value of the energy-momentum tensor for the thermalized QED vacuum.

To lowest order in \( \alpha/\pi \), the formula for the effective action charge Eq. (16) reduces to:

\[
Q_T \simeq \frac{2 \partial_E \mathcal{L}}{T \partial_T \mathcal{L}} = \frac{2 \partial_E \mathcal{L}}{T \partial_T \mathcal{L}_\gamma + T \partial_T \mathcal{L}^{1T}},
\]

where \( \partial_E \mathcal{L}^{1T} \) is given in Eq. (17), and the terms in the denominator can be written as:

\[
T \partial_T \mathcal{L}_\gamma(T) = \frac{4\pi^2}{45} T^4, \\
T \partial_T \mathcal{L}^{1T}(T) = -\frac{2}{\pi^2} m^4 T \sum_{n=1}^\infty \frac{(-1)^n}{n} K_3(\frac{\pi}{2} n).
\]

Note that the appearance of \( \partial_T \mathcal{L}_\gamma \) and \( \partial_T \mathcal{L}^{1T} \) in the denominator of Eq. (32) corresponds to the appearance of a photonic and a fermionic part in the energy-momentum tensor. One usually expects the fermionic part to become important only for high temperature, \( T \gg m \), where the fermions become ultra-relativistic.

![FIG. 3. One-loop contribution to the effective action charge \( Q_T \) in units of \( \frac{1}{4\pi} \); for high temperature, \( Q_T \) decreases proportional to \( \frac{1}{T} \). The solid line corresponds to \( Q_T \) as given in Eq. (15); for the dashed line, the fermionic contributions \( \sim T \partial_T \mathcal{L}^{1T} \) have been omitted. In the low-temperature limit, the effective action charge to one loop vanishes due to the influence of a finite electron mass.](image)

On the one hand, we indeed find the expected localized behavior, as can be seen in Fig. 3: the effective action charge vanishes for high as well as low temperatures; in between, it develops a maximum at \( T/m \approx 0.22 \). On the other hand, it is interesting to note that the inclusion of the fermionic contributions \( \sim T \partial_T \mathcal{L}^{1T} \) (solid line) in the denominator of Eq. (15) becomes important for values of temperature close to the maximum of \( Q_T \). This again indicates that the thermalization of the fermions becomes important already for comparably low values of temperature.

Contrary to the cases discussed in \( \mathcal{L} \), the one-loop effective action charge is not centered at the origin. This is because the electron mass damps the fermionic thermal fluctuations exponentially for small values of temperature.
VI. CONCLUDING REMARKS

First, we would like to stress that the one-loop contributions to the velocity shift as calculated in the present work do not fit into the scheme of the “unified formula” proposed in [6]; the latter connects the velocity shift with a shift of the vacuum energy density caused by external influences with a universal constant coefficient as proportionality factor. In the language of Eq. (15), this coefficient has to be identified with the effective action charge $Q_T$ which in the present case is not constant at all but almost carries the complete physical information of the problem. This misfit indicates that the “unified formula” is an artefact of an approximation scheme rather than a fundamental principle.

Furthermore, it should be pointed out that the maximum velocity shift, $-\delta v_{\text{max}}^2 = \frac{\alpha}{3\pi}$, cannot be viewed as an experimentally significant limiting value, since light propagation will successively be dominated by plasma effects for increasing temperature at $T \sim m$. Nevertheless, the existence and amount of such a maximum shift are at least interesting from a theoretical viewpoint, since they characterize the classically forbidden interaction between the modified vacuum and a photon which is exposed to the effects of vacuum polarization. Supposing there were no collective excitations constituting a plasmon for high temperature, then QED would not allow for an arbitrarily strong influence of a heat bath on the propagation of light for reasonable values of temperature. Of course, for extremely high temperature, the logarithmic corrections from $\partial_T L$ in Eq. (12) would also significantly increase the negative velocity shift, slowing down the speed of light; e.g., another shift of $\alpha/3\pi$ would be reached at $T/m \simeq 6.4 \cdot 10^7$.

The problem of calculating the intermediate- / high-temperature velocity shift was also approached [9] employing a method reminiscent of the Born-Opennheimer approximation: the coupling of the external field to the fermion loop was considered as an adiabatic interaction, while the photonic fluctuations were treated as the “fast” degrees of freedom. This calculation can be rated as a two-loop calculation, since the thermalization of the fermions has been approximately taken into account by using a thermal one-loop Lagrangian; photonic fluctuations were introduced in the same way as in the low-temperature case, i.e., by taking thermal vacuum expectation values of the field quantities. Similarly to the present work, a maximum velocity shift has been found which is about three orders of magnitude smaller than the one-loop result given above. Indeed, this fits into the usual hierarchy of perturbation theory.

To overcome the approximation scheme of [6], which treats the thermalization of fermions and photons on a different footing, the present paper offers an appropriate tool by working with the complete set of invariants of the thermalized QED vacuum. However, for a direct verification of the results of [6] for high temperature, a first-principles two-loop calculation of the effective action should be carried out in the same sense as suggested in the present work.

Beyond the clarification of these technical questions, a two-loop calculation of the thermal effective action will be useful for studying further phenomenological aspects of the physics of strong fields, such as thermally induced pair production or photon splitting. In view of the present results, one might speculate that, for these effects, two-loop contributions will be dominant at low temperature, since thermal one-loop corrections have proved to be vanishing (for pair production [13]) or of minor importance (for photon splitting [14]).

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