Left–right symmetry of finite finitistic dimension

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Abstract
We show that the finitistic dimension conjecture for finite-dimensional algebras is equivalent to the left–right symmetry of finite finitistic dimension for finite-dimensional algebras. We also prove the equivalent statement for injective generation.

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1 INTRODUCTION

The homological conjectures are a collection of longstanding open questions about the representation theory of finite-dimensional algebras. A summary can be found in [10]. One of the strongest conjectures is the finitistic dimension conjecture, which concerns the finiteness of a numerical invariant associated to an algebra. If it is true, then several other homological conjectures also hold, including the Nunke condition [16] and the Nakayama conjecture [15]. In 1991, Happel noted that it is unclear if an algebra satisfying the finitistic dimension conjecture implies that its opposite algebra also satisfies the conjecture [10]. We show that, heuristically, it is unclear for good reason; this property is equivalent to the finitistic dimension conjecture.
Numerical invariants have long been used to measure the complexity of mathematical structures. Some examples in representation theory are homological dimensions, such as the global dimension [5], the finitistic dimension [2] and the dominant dimension [15]. These three dimensions have been studied extensively, particularly with a view to classify when they are finite. To have finite global dimension is quite a restrictive property for a ring; even straightforward algebras like \( k[\varepsilon]/\langle \varepsilon^2 \rangle \) do not satisfy this property. Whereas, the other two dimensions are conjectured to be finite for quite large classes of rings. Indeed, the Nakayama conjecture asserts that the dominant dimension of a finite-dimensional algebra is infinite if and only if the algebra is self-injective [15], and the finitistic dimension conjecture asserts that the finitistic dimension of a finite-dimensional algebra is finite. These conjectures remain open in general, but have been verified for various classes of algebras including commutative algebras [2], radical cube zero algebras [8], monomial algebras [9] and representation dimension three algebras [12]. A survey on the finitistic dimension conjecture can be found in [11].

Homological dimensions can be defined in terms of either left or right modules. Sometimes, the value of a dimension is unaffected by this choice, that is, the dimension is left–right symmetric. For a finite-dimensional algebra, this is true for both the global dimension [3, Theorem 4] and the dominant dimension [14, Theorem 4]. However, it is not true for the finitistic dimension; not even when restricted to quiver algebras [13, Example 2.2]. In fact, the left and right finitistic dimensions of a quiver algebra can be arbitrarily different [9, Example 1.2].

For a ring \( \Lambda \), a left \( \Lambda \)-module is simply a right \( \Lambda^{\text{op}} \)-module. As the dominant dimension of a finite-dimensional algebra is left–right symmetric, it follows that an algebra satisfies the Nakayama conjecture if and only if its opposite algebra does. The main result of this paper shows that this property is satisfied by the finitistic dimension conjecture if and only if the finitistic dimension conjecture itself holds.

**Theorem A** (Theorem 3.4). The following three statements are equivalent.

(i) The finitistic dimension conjecture holds for all finite-dimensional algebras.

(ii) For all finite-dimensional algebras \( \Lambda \), the finitistic dimension of \( \Lambda \) being finite implies that the finitistic dimension of \( \Lambda^{\text{op}} \) is finite.

(iii) The finitistic dimension conjecture holds for all finite-dimensional algebras \( \Lambda \) such that the finitistic dimension of \( \Lambda^{\text{op}} \) is zero.

To prove Theorem A, we define a ring construction that takes a basic finite-dimensional algebra \( A \) and produces a related algebra \( \tilde{A} \). The finitistic dimension of \( \tilde{A} \) bounds the finitistic dimension of \( A \) from above. Moreover, the algebra \( \tilde{A} \) is constructed in such a way that several of the homological properties of its opposite algebra are straightforward to compute. In fact, the finitistic dimension of \( \tilde{A}^{\text{op}} \) is always zero, regardless of the choice of the starting algebra \( A \).

One natural setting for many homological properties of rings and their modules is the derived category. Indeed, a finite-dimensional algebra has finite global dimension if and only if its bounded derived category is generated, as a triangulated category, by the injective modules. Rickard proved that, for a finite-dimensional algebra, finite finitistic dimension can also be seen as a property of the derived category [17, Theorem 4.4]. More generally, if the unbounded derived category of a finite-dimensional algebra is generated, as a triangulated category with set indexed coproducts, by the injective modules, then the finitistic dimension conjecture holds for that algebra [17, Theorem 4.3]. In Section 4, we use the construction \( \tilde{A} \) to prove a similar statement to Theorem A for this generation property.
Outline

In Section 2, we take a basic finite-dimensional algebra \( A \) over an algebraically closed field and define a related algebra \( \tilde{A} \) whose homological properties are connected to those of \( A \). In Section 3, we focus on the finitistic dimensions of \( A, \tilde{A} \) and \( A^{\text{op}} \) and prove Theorem A. Section 4 has the same structure as Section 3, except we focus on injective generation rather than the finitistic dimension.

Notation

All rings and ring homomorphisms are unital, and modules are right modules unless otherwise stated. Let \( \Lambda \) be a finite-dimensional algebra over a field \( k \) with \( M_{\Lambda} \) a right \( \Lambda \)-module.

- \( \text{Mod-}\Lambda \) (mod-\( \Lambda \)) is the category of (finitely generated) right \( \Lambda \)-modules.
- \( \text{pd}_{\Lambda}(M) \) is the projective dimension of \( M \).
- Let \( e \in \Lambda \) be an idempotent.
  - \( D(\Lambda e) \) is the injective right \( \Lambda \)-module \( \text{Hom}_{k}(\text{Hom}_{\Lambda}(e\Lambda,\Lambda),k) \).
  - \( S_{\Lambda}(e) \) is the semi-simple right \( \Lambda \)-module \( e\Lambda/\text{rad}(e\Lambda) \). When the ring \( \Lambda \) is clear from context, we omit the subscript \( \Lambda \).
- \( D(\Lambda) \) is the unbounded derived category of cochain complexes of right \( \Lambda \)-modules.
- \( \text{Inj-}\Lambda \) is the category of injective right \( \Lambda \)-modules.

2 | THE CONSTRUCTION OF \( \tilde{A} \)

In this section, we take a finite-dimensional algebra \( A \) and construct a new, related algebra \( \tilde{A} \). As the finitistic dimension is invariant under field extensions [13, Theorem 2.5] and Morita equivalence, we define \( \tilde{A} \) for a basic finite-dimensional algebra \( A \) over an algebraically closed field \( k \). To each simple \( A \)-module, we adjoin a copy of the two-dimensional algebra \( k[\tilde{\ell}] / \langle \tilde{\ell}^2 \rangle \). We do this in such a way that \( \tilde{A} \) can be realised as a triangular matrix ring where \( A \) is one of the diagonal entries.

Construction 2.1. Let \( A \) be a basic finite-dimensional algebra over an algebraically closed field \( k \). Let \( \{e_i : 1 \leq i \leq n\} \subset A \) be a set of primitive idempotents such that \( \{S_A(e_i) : 1 \leq i \leq n\} \) is an irredundant set of isomorphism classes of the simple right \( A \)-modules. Let \( M \) be the \( n \)-dimensional free \( k \)-module with basis \( \{\tilde{m}_i : 1 \leq i \leq n\} \). Define a right \( k \)-linear \( A \)-module structure on \( M \) by

\[
\tilde{m}_i e_i = \tilde{m}_i \quad \text{and} \quad \tilde{m}_i a = 0 \quad \text{for} \quad a \in \text{rad}(A).
\]

For each \( 1 \leq i \leq n \), let \( B_i \) denote the local ring \( k[\hat{\ell}_i] / \langle \hat{\ell}_i^2 \rangle \) and let \( \hat{e}_i \) denote its multiplicative identity. Let \( B \) denote the product ring \( \prod_{i=1}^n B_i \). Define a left \( k \)-linear \( B \)-module structure on \( M \) by

\[
\hat{e}_i \tilde{m}_i = \tilde{m}_i \quad \text{and} \quad \hat{b} \tilde{m}_i = 0 \quad \text{for} \quad \hat{b} \in \text{rad}(B).
\]
Then $\tilde{A}$ is the triangular matrix ring

$$\begin{pmatrix} A & 0 \\ \tilde{M}_A & B \end{pmatrix}.$$

When $A$ is a quiver algebra, Construction 2.1 can be realised via simple graph operations on the underlying quiver.

**Example 2.2.** Let $A$ be the quiver algebra $kQ/I$ where $k$ is an algebraically closed field, $Q$ is the quiver

![Quiver Diagram](image)

and $I$ is the ideal $\langle \alpha\beta, \beta\gamma, \gamma\alpha, \delta\beta, \zeta\epsilon, \epsilon\alpha^2, \epsilon\gamma\beta, \delta\alpha^2, \zeta\delta - \delta\alpha \rangle$. Then the indecomposable projective $A$-modules can be represented by the diagrams below. In these diagrams, the vertices represent the composition factors of the module and the arrows represent the action of the algebra on these composition factors. A systematic study of these diagrams can be found in [1].

The injective $A$-modules can be represented by the diagrams below.

The simple $A$-modules correspond to the vertices of the quiver $Q$. Therefore, in this case, $M$ is a 3-dimensional $k$-vector space with basis $\{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$, where $\tilde{e}_i\tilde{m}_i = \tilde{m}_i$ for each $1 \leq i \leq 3$. Moreover, $B$ is a product of three copies of $k[\tilde{e}]/(\tilde{e}^2)$ and $\text{rad}(k[\tilde{e}]/(\tilde{e}^2))$ is equal to $\text{span}_k\{\tilde{e}\}$. So,
we can realise $\tilde{A}$ as the quiver algebra $k\tilde{Q}/\tilde{I}$ where $\tilde{Q}$ is the quiver

$$
\begin{array}{c}
\gamma \\
\downarrow \ \\
\delta \\
\downarrow \ \\
\alpha \\
\downarrow \\
1
\end{array}$$

and $\tilde{I}$ is the ideal

$$\langle r, \tilde{m}_1a, \tilde{e}_i\tilde{m}_1, \tilde{e}_i^2 : r \in I, a \in \text{rad}(A), 1 \leq i \leq 3 \rangle.$$

The projective $\tilde{A}$-modules can be represented by the diagrams below.

$$
\begin{array}{cccccccc}
e_1\tilde{A} & e_2\tilde{A} & e_3\tilde{A} & \tilde{e}_1\tilde{A} & \tilde{e}_2\tilde{A} & \tilde{e}_3\tilde{A} \\
1 & 2 & 3 & \tilde{m}_1 & \tilde{m}_2 & \tilde{m}_3 \\
1 & 2 & 1 & 3 & 1 & 2 \\
1 & 2 & 1 & 1 & 1 \\
\end{array}
$$

The injective $\tilde{A}$-modules can be represented by the diagrams below.

$$
\begin{array}{cccccccc}
& & & & & & & \\
\tilde{m}_1 & 2 & 1 & 3 & \tilde{m}_2 & 1 & 3 & \tilde{m}_3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & 3 & 1 & 3 & 2 & 3 & 3 & \\
\end{array}
$$

\[ D(\tilde{A}e_1) \quad D(\tilde{A}e_2) \quad D(\tilde{A}e_3) \quad D(\tilde{A}\tilde{e}_1) \quad D(\tilde{A}\tilde{e}_2) \quad D(\tilde{A}\tilde{e}_3) \]

**Lemma 2.3.** Let $A$ be a basic finite-dimensional algebra over an algebraically closed field $k$, with $\tilde{A}$ defined as in Construction 2.1. Then each simple right $\tilde{A}^{\text{op}}$-module is isomorphic to $S_{\tilde{A}^{\text{op}}}(f)$ for some $f \in \{e_i, \tilde{e}_i : 1 \leq i \leq n\}$. Moreover, for each $1 \leq i \leq n$, there exists a short exact sequence of right $\tilde{A}^{\text{op}}$-modules

$$
0 \longrightarrow S_{\tilde{A}^{\text{op}}}(\tilde{e}_i) \longrightarrow D(\tilde{A}^{\text{op}}\tilde{e}_i) \longrightarrow S_{\tilde{A}^{\text{op}}}(e_i) \oplus S_{\tilde{A}^{\text{op}}}(\tilde{e}_i) \longrightarrow 0.
$$
Proof. The primitive idempotents of a triangular matrix ring correspond to the primitive idempotents of its diagonal components. Therefore, the primitive idempotents of $\tilde{A}^{op}$ are $e_i$ and $\tilde{e}_i$ for $1 \leq i \leq n$. Consequently, the simple right $\tilde{A}^{op}$-modules are of the form $S_{\tilde{A}^{op}}(f)$ where $f$ is either $e_i$ or $\tilde{e}_i$ for some $1 \leq i \leq n$.

For each $1 \leq i \leq n$, the radical of $B_i$ is $\text{span}_k \{ \tilde{e}_i \}$. So, it follows that the radical of $\tilde{A}$ is

$$\text{span}_k \{ a, m_i, \tilde{e}_i : a \in \text{rad}(A), 1 \leq i \leq n \}.$$

Fix $1 \leq i \leq n$. Then $\text{rad}(\tilde{e}_i \tilde{A})$ is equal to $\text{span}_k \{ m_i, \tilde{e}_i \}$ as an $\tilde{A}$-module. Moreover, $\text{rad}^2(\tilde{e}_i \tilde{A})$ is trivial, so $\text{rad}(\tilde{e}_i \tilde{A})$ is semi-simple. As the primitive idempotents of $\tilde{A}$ are $e_j$ and $\tilde{e}_j$ for $1 \leq j \leq n$ and

$$m_i e_i = m_i \text{ and } \tilde{e}_i e_i = \tilde{e}_i,$$

it follows that $\text{rad}(\tilde{e}_i \tilde{A})$ decomposes as a right $\tilde{A}$-module to $S_{\tilde{A}}(e_i) \oplus S_{\tilde{A}}(\tilde{e}_i)$. So, there exists a short exact sequence of right $\tilde{A}$-modules

$$0 \rightarrow S_{\tilde{A}}(e_i) \oplus S_{\tilde{A}}(\tilde{e}_i) \rightarrow \tilde{e}_i \tilde{A} \rightarrow S_{\tilde{A}}(\tilde{e}_i) \rightarrow 0.$$

Note that $\tilde{e}_i \tilde{A}$ is equal to $\tilde{A}^{op} \tilde{e}_i$ and $\tilde{A}$ is a finite-dimensional algebra over $k$. So, the application of the duality functor

$$D = \text{Hom}_k(-, k) : \text{mod-} \tilde{A} \rightarrow \text{mod-} \tilde{A}^{op}$$

to this sequence yields the required short exact sequence of right $\tilde{A}^{op}$-modules. \hfill \Box

3 | THE FINITISTIC DIMENSION

In this section, we focus on the properties of the finitistic dimensions of the finite-dimensional algebras $A$, $\tilde{A}$ and $\tilde{A}^{op}$. We start with the definition of the finitistic dimension.

**Definition 3.1** (The (little/big) finitistic dimension). Let $\Lambda$ be a finite-dimensional algebra. Then the little finitistic dimension of $\Lambda$ is

$$\text{findim}(\Lambda) = \sup \{ \text{pd}_\Lambda(M) : M \in \text{mod-} \Lambda \text{ and } \text{pd}_\Lambda(M) < \infty \}.$$

The big finitistic dimension of $\Lambda$ is

$$\text{FinDim}(\Lambda) = \sup \{ \text{pd}_\Lambda(M) : M \in \text{Mod-} \Lambda \text{ and } \text{pd}_\Lambda(M) < \infty \}.$$

The *little finitistic dimension conjecture* is the assertion that $\text{findim}(\Lambda) < \infty$ for all finite-dimensional algebras $\Lambda$. The analogous statement for the big finitistic dimension is the *big finitistic dimension conjecture*. 
The main result of this section, namely Theorem 3.4, concerns the relationship between the finitistic dimension of a finite-dimensional algebra and its opposite algebra. The proof of the result is the combination of two propositions that we state and prove now. In particular, we show that \( \text{findim}(A) \leq \text{findim}(\tilde{A}) \) in Proposition 3.2 and that \( \text{findim}(\tilde{A}^{\text{op}}) = 0 \) in Proposition 3.3.

**Proposition 3.2.** Let \( A \) be a basic finite-dimensional algebra over an algebraically closed field, with \( \tilde{A} \) defined as in Construction 2.1. Then \( \text{findim}(A) \leq \text{findim}(\tilde{A}) \) and \( \text{FinDim}(A) \leq \text{FinDim}(\tilde{A}) \).

*Proof.* As \( \tilde{A} \) is a triangular matrix ring, the statements follow by [7, Corollary 4.21].

**Proposition 3.3.** Let \( A \) be a basic finite-dimensional algebra over an algebraically closed field, with \( \tilde{A} \) defined as in Construction 2.1. Then \( \text{findim}(\tilde{A}^{\text{op}}) = 0 \) and \( \text{FinDim}(\tilde{A}^{\text{op}}) = 0 \).

*Proof.* As \( \text{findim}(\tilde{A}^{\text{op}}) \leq \text{FinDim}(\tilde{A}^{\text{op}}) \), it suffices to prove that \( \text{FinDim}(\tilde{A}^{\text{op}}) = 0 \). Suppose that \( \Lambda \) is a finite-dimensional algebra over a field. Then \( \text{FinDim}(\Lambda) = 0 \) if and only if there is a non-zero homomorphism from the dual module \( D(\Lambda) \) to every simple \( \Lambda \)-module [4, Lemma 6.2]. Therefore, by Lemma 2.3, it follows that \( \text{FinDim}(\tilde{A}^{\text{op}}) = 0 \).

**Theorem 3.4.** The following three statements are equivalent.

(i) The little finitistic dimension conjecture holds for all finite-dimensional algebras.

(ii) For all finite-dimensional algebras \( \Lambda \), \( \text{findim}(\Lambda) < \infty \) implies that \( \text{findim}(\Lambda^{\text{op}}) < \infty \).

(iii) The little finitistic dimension conjecture holds for all finite-dimensional algebras \( \Lambda \) such that \( \text{findim}(\Lambda^{\text{op}}) = 0 \).

*Proof.* The first statement implies both the second and the third statement trivially. For the implication from the second statement to the third statement, suppose there exists a finite-dimensional algebra \( \Lambda \) such that \( \text{findim}(\Lambda^{\text{op}}) = 0 \) but \( \text{findim}(\Lambda) = \infty \). Then the finiteness of the little finitistic dimension of \( \Lambda \) is not left–right symmetric. So, all that remains to be shown is that the third statement implies the first statement.

Let \( C \) be a finite-dimensional algebra over a field \( k \). Then \( \tilde{C} = C \otimes_k \tilde{k} \) is a finite-dimensional algebra over the algebraically closed field \( \tilde{k} \), and \( \text{findim}(C) \leq \text{findim}(\tilde{C}) \) by [13, Proposition 2.1]. Every finite-dimensional algebra is Morita equivalent to a basic finite-dimensional algebra and the finitistic dimension is invariant under Morita equivalence. So, there exists a basic finite-dimensional algebra \( A \) over the field \( \tilde{k} \) such that \( \text{findim}(C) \leq \text{findim}(A) \).

Let \( \tilde{A} \) be defined as in Construction 2.1. Then \( \text{findim}(\tilde{A}^{\text{op}}) = 0 \), by Proposition 3.3 and so, \( \tilde{A} \) satisfies the finitistic dimension conjecture. Hence, by Proposition 3.2, we have that

\[
\text{findim}(C) \leq \text{findim}(A) \leq \text{findim}(\tilde{A}) < \infty,
\]

and the finitistic dimension conjecture holds for \( C \).

**Theorem 3.5.** The following three statements are equivalent.

(i) The big finitistic dimension conjecture holds for all finite-dimensional algebras.

(ii) For all finite-dimensional algebras \( \Lambda \), \( \text{FinDim}(\Lambda) < \infty \) implies that \( \text{FinDim}(\Lambda^{\text{op}}) < \infty \).

(iii) The big finitistic dimension conjecture holds for all finite-dimensional algebras \( \Lambda \) such that \( \text{FinDim}(\Lambda^{\text{op}}) = 0 \).
Proof. This is the same as the proof of Theorem 3.4, using [13, Proposition 2.1] and Propositions 3.2 and 3.3.

4 INJECTIVE GENERATION

This section has the same structure as Section 3 except we focus on generation properties of the derived categories of the finite-dimensional algebras $A, \tilde{A}$ and $\tilde{A}^{\text{op}}$. We start with the definition of a localising subcategory.

**Definition 4.1 (Localising subcategory).** Let $\Lambda$ be a finite-dimensional algebra over a field. A triangulated subcategory of $D(\Lambda)$ is a localising subcategory if it is closed under set indexed coproducts. Let $S$ be a class of complexes of $\Lambda$-modules. Denote the smallest localising subcategory that contains $S$ by $\text{Loc}_{D(\Lambda)}(S)$.

It is well-known that the derived category $D(\Lambda)$ is compactly generated or, equivalently, that the smallest localising subcategory that contains the regular module $\Lambda$ is the entire derived category. We ask when this property holds for the injective modules.

**Definition 4.2 (Injectives generate).** Let $\Lambda$ be a finite-dimensional algebra over a field. Then injectives generate for $\Lambda$ if $\text{Loc}_\Lambda(\text{Inj-}\Lambda) = D(\Lambda)$.

This property is strongly connected to the homological conjectures. In particular, if injectives generate for a finite-dimensional algebra over a field, then the big finitistic dimension conjecture holds for that algebra [17, Theorem 4.3]. As such, it is perhaps unsurprising that an analogous set of results to those in Section 3 hold in the more general setting of injective generation.

**Proposition 4.3.** Let $A$ be a basic finite-dimensional algebra over an algebraically closed field, with $\tilde{A}$ defined as in Construction 2.1. If injectives generate for $\tilde{A}$, then injectives generate for $A$.

Proof. As $\tilde{A}$ is a triangular matrix ring, the statement holds by [6, Example 6.11].

**Proposition 4.4.** Let $A$ be a basic finite-dimensional algebra over an algebraically closed field, with $\tilde{A}$ defined as in Construction 2.1. Then injectives generate for $\tilde{A}^{\text{op}}$.

Proof. As $\tilde{A}^{\text{op}}$ is a finite-dimensional algebra over a field, the smallest localising subcategory of $D(\tilde{A}^{\text{op}})$ that contains the simple $\tilde{A}^{\text{op}}$-modules is equal to $D(\tilde{A}^{\text{op}})$ by [17, Lemma 6.1]. So, it suffices to show that the simple $\tilde{A}^{\text{op}}$-modules lie in $\text{Loc}_{\tilde{A}^{\text{op}}}(\text{Inj-}\tilde{A}^{\text{op}})$.

Fix $1 \leq i \leq n$. By Lemma 2.3, there exists a short exact sequence of $\tilde{A}^{\text{op}}$-modules

$$0 \longrightarrow S(\tilde{e}_i) \longrightarrow D(\tilde{A}^{\text{op}} \tilde{e}_i) \xrightarrow{(\tilde{m}_i \tilde{\epsilon}_i)} (S(e_i) \oplus S(\tilde{e}_i)) \longrightarrow 0.$$ 

Both the kernel and image of $(\tilde{m}_i \tilde{\epsilon}_i)$ contain a copy of $S(\tilde{e}_i)$ as a direct summand. So, the short exact sequence can be spliced with itself to build a bounded above complex

$$X_i = \cdots \rightarrow D(\tilde{A}^{\text{op}} \tilde{e}_i) \rightarrow D(\tilde{A}^{\text{op}} \tilde{e}_i) \rightarrow D(\tilde{A}^{\text{op}} \tilde{e}_i) \rightarrow 0 \rightarrow 0 \rightarrow \cdots.$$
whose cohomology modules are given by

\[ H^n(X_i) \cong \begin{cases} S(e_i) \oplus S(\tilde{e}_i), & \text{if } n = 0, \\ S(e_i), & \text{if } n < 0, \\ 0, & \text{otherwise.} \end{cases} \]

As \( X_i \) is a bounded above complex of injective \( \tilde{A}^{\text{op}} \)-modules, it lies in \( \text{Loc}_{\tilde{A}^{\text{op}}} (\text{Inj-} \tilde{A}^{\text{op}}) \) by \cite{17, Proposition 2.1 h}. Moreover, \( H^n(X_i) \) is a quotient of \( X_i^n \) for each \( n \in \mathbb{Z} \), so the canonical projection morphism

\[ X_i \to \bigoplus_{n \in \mathbb{Z}} H^n(X_i)[-n] \]

is a quasi-isomorphism. Localising subcategories are closed under quasi-isomorphisms and direct summands, so both \( S(e_i) \) and \( S(\tilde{e}_i) \) lie in \( \text{Loc}_{\tilde{A}^{\text{op}}} (\text{Inj-} \tilde{A}^{\text{op}}) \). Consequently, all the simple \( \tilde{A}^{\text{op}} \)-modules lie in \( \text{Loc}_{\tilde{A}^{\text{op}}} (\text{Inj-} \tilde{A}^{\text{op}}) \) by Lemma 2.3, and so, injectives generate for \( \tilde{A}^{\text{op}} \).

**Remark 4.5.** If \( A \) is a quiver algebra, then the complex \( X_i \) used in the proof of Proposition 4.4 can be illustrated via diagrams of modules. In particular, it is the complex

\[ X_i = \cdots \to m_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} m_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} m_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} m_i \xrightarrow{i} \tilde{e}_i \xrightarrow{i} \tilde{e}_i \to 0 \to \cdots. \]

**Theorem 4.6.** The following two statements are equivalent.

(i) Injectives generate for all finite-dimensional algebras.
(ii) For all finite-dimensional algebras \( \Lambda \), injectives generate for \( \Lambda \) implies that injectives generate for \( \Lambda^{\text{op}} \).

**Proof.** Similarly to the proof of Theorem 3.4, it suffices to consider basic finite-dimensional algebras over algebraically closed fields. In particular, let \( C \) be a finite-dimensional algebra over a field \( k \). Then \( \bar{C} = C \otimes_k \bar{k} \) is a finite-dimensional algebra over the algebraically closed field \( \bar{k} \). Moreover, if injectives generate for \( \bar{C} \), then injectives generate for \( C \) by the application of \cite[Lemma 5.2]{6} to the ring homomorphism \( C \hookrightarrow \bar{C} \). Every finite-dimensional algebra is Morita equivalent to a basic finite-dimensional algebra and injective generation is invariant under Morita equivalence because it is invariant under derived equivalence \cite[Theorem 3.4]{17}. So, there exists a basic finite-dimensional algebra \( A \) over \( \bar{k} \) such that if injectives generate for \( A \), then injectives generate for \( C \).

Thus, the statement follows similarly to the proof of Theorem 3.4, using Propositions 4.4 and 4.3.

**Remark 4.7.** There is a dual concept to injective generation called projective cogeneration that concerns the smallest triangulated subcategory of the derived category that contains the projective modules and is closed under set indexed products. See \cite[section 5]{17}. The analogous statement to Theorem 4.6 for projective cogeneration also holds. The proof of the analogous theorem has the same structure except we replace Propositions 4.3 and 4.4 by their dual statements.
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