STOCHASTIC DECOMPOSITION OF THE $M/M/1$ QUEUE WITH ENVIRONMENT DEPENDENT WORKING VACATION

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Abstract. We consider an M/M/1 queue with n types of working vacation. After a non zero busy period if the server finds the system empty, it opts for one of the n types of working vacation depending on the environment. On vacation completion epoch finding an empty system, it remains in the respective vacation. We demonstrate the stochastic decomposition structure of the queue length and waiting time of this M/M/1 queue and obtain the distribution of the additional queue length and additional delay.

Keywords: working vacation; environment; stochastic decomposition.

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1. INTRODUCTION

If a queue is empty the server remains idle. The idle time of the server can be utilized for supplementary jobs. This gives rise to extensive research work in the field of vacation queueing models. The details regarding the research on queueing models can be found in the survey of Doshi [1], the monograph of Takagi [2] and Tian and Zhang [3]. If the number of customers in the queue is less, the functioning of the server at a slow rate will reduce the operating cost, energy consumption, and start-up cost. These advantages are pointing towards

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working vacation. A working vacation is an extension of regular vacation. In working vacation, instead of completely stopping the service, the server provides service at a slow rate. Working vacation reduces the chance of reneging of the customers compared to normal vacation. In this era of high demand for commodities and services which are available in a short spell, the concept of working vacation is very useful. This may be the main reason for the extensive research work going on in working vacation queueing models.

Servi and Finn [4] introduced the idea of working vacation in queueing models and applied the results to the performance analysis of gateway routers in fiber communication networks. Baba [5] studied GI/M/1 queue with working vacation.[6] gives a comprehensive overview of the research results and analysis methods of vacation queue, including its applications in the communication networks. Stochastic decomposition results for the number of customers in the system in the case of an exhaustive service were first obtained by Fuhrmann [7] and then confirmed by Doshi [1]. Fuhrmann and Cooper [8] and Shanthikumar [9] established stochastic decomposition structures for a classical M/G/1 queue with general vacations. [10] demonstrates stochastic decomposition in an M/M/1 queue with working vacation. In [11] the authors discuss an M/M/1 queue with $n$ types of vacations where the server opts out one among these $n$ vacations depending on the environment.

In this model, we consider a single server queueing system with working vacation. On completion of service, if the server finds the system empty, he goes for a working vacation. There are $n$ types of working vacations. After a busy period, depending on the environment, the server opts for $i^{th}$ type of vacation with probability $p_i, 1 \leq i \leq n$. During vacation, if customers arrive, the server provides service at a lower rate. On completion of service during vacation, if there is no customer in the system the server continues to be on vacation. Otherwise, the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service at the normal rate. On completion of vacation, if the server finds the system empty, he remains on the corresponding vacation. We demonstrate stochastic decomposition of the queue length and waiting time using the method of induction and Little’s formula[12].
The rest of the paper is organized as follows. In section 2 the model is described in detail. Section 3 discusses the stochastic decomposition structures of the number of customers in the system and waiting time and obtains the distributions of additional queue length and additional delay.

2. Model Description

Consider a single server queueing system with working vacation in which arrival occurs according to a Poisson process with parameter $\lambda$. The service time is exponentially distributed with parameter $\mu$. On completion of a service if the server finds the system empty it goes for a working vacation. There are $n$ types of working vacations. Depending on the environment, after a busy period, the server goes for $i^{th}$ type of vacation with probability $p_i$, $1 \leq i \leq n$. The duration of $i^{th}$ type of vacation is exponentially distributed with parameter $\gamma_i$, $1 \leq i \leq n$. During vacation, if customers arrive, the server provides service at a lower rate $\mu_i$, provided the server is in $i^{th}$ type of vacation, $1 \leq i \leq n$. On completion of service during vacation, if there is no customer in the system the server continues to stay on vacation. Otherwise, the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service at the normal rate $\mu$. On completion of vacation, if the server finds the system empty, it remains on the corresponding vacations. Figure 1 is a diagrammatic representation of the model.
3. **Mathematical Description**

We establish the stochastic decomposition of the state space by induction on the number of environmental factors.

**Case.1** First we consider the case of $n = 2$.

Let $N(t)$ be the number of customers in the system and $S(t)$ be the status of the server at time $t$:

$$S(t) = \begin{cases} 
0, & \text{if the server is serving in normal mode;} \\
1, & \text{if server is in the type I working vacation;} \\
2, & \text{if server is in the type II working vacation;}
\end{cases}$$

Then $X = \{X(t), t \geq 0\}$ where $X(t) = (N(t), S(t))$ is a continuous time Markov chain with state space $\{0, 1\} \cup \{0, 2\} \cup \{(j, k), \ j = 1, 2, \ldots; k = 0, 1, 2\}$. The infinitesimal generator associated with the Markov chain is

$$Q_1 = \begin{bmatrix}
B_0 & B_1 & A_0 \\
B_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots
\end{bmatrix}$$

where $-B_0 = A_0 = \lambda I$,

$$B_1 = \begin{bmatrix}
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
\mu p_1 & \mu p_2 \\
\mu_1 & 0 \\
0 & \mu_2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
\mu & 0 & 0 \\
\mu_1 & 0 & 0 \\
\mu_2 & 0 & 0
\end{bmatrix},$$

$$A_1 = \begin{bmatrix}
-\lambda - \mu & 0 & 0 \\
\theta_1 & -\lambda - \mu_1 - \theta_1 & 0 \\
\theta_2 & 0 & -\lambda - \mu_2 - \theta_2
\end{bmatrix}$$

**Stability analysis.** We have $A = A_0 + A_1 + A_2 = \begin{bmatrix}
0 & 0 & 0 \\
\theta_1 + \mu_1 & -\mu_1 - \theta_1 & 0 \\
\theta_2 + \mu_2 & 0 & -\mu_2 - \theta_2
\end{bmatrix}$

Then $A$ is the infinitesimal generator of a Markov chain with state space $\{0, 1, 2\}$ which represents the status of the server. Let $y = (y_0, y_1, y_2)$ be the invariant probability vector of $A$. Then
yA = 0 and ye = 1. The left drift rate of the original Markov chain is yA2e and that for right drift is yA0e. Left drift indicates a service completion and right drift represents arrival of customer. Thus the system is stable if and only if yA0e < yA2e. Here yA0e = λ and yA2e = µ.

Hence we have

**Theorem:** The system is stable if and only if \( \lambda < \mu \).

### 3.1. Steady State Analysis

For the analysis of the model it is necessary to solve for the minimal non-negative solution \( R_1 \) of the matrix quadratic equation

\[
R_1^2 A_2 + R_1 A_1 + A_0 = 0.
\]

Since the Matrices \( A_2, A_1, A_0 \) are lower triangular \( R_1 \) is also lower triangular. Solving (1) we obtain \( R_1 \) as

\[
R_1 = \begin{bmatrix}
  r_0 & 0 & 0 \\
  r_1 & r_1 & 0 \\
  r_2 & 0 & r_2
\end{bmatrix}
\]

where \( r_0 = \rho, \ r_1 = \frac{\rho(\lambda + \theta_1)}{(\lambda + \mu_1 + \theta_1)}, \ r_1 = \frac{\lambda}{(\lambda + \mu_1 + \theta_1)} \)

and \( r_2 = \frac{\lambda}{(\lambda + \mu_2 + \theta_2)} \).

Let \( x = (x_0, x_1, x_2, \ldots) \) be the steady state probability vector associated with the Markov process \( X \). Here \( x_0 = (x_{01}, x_{02}) \) and \( x_i = (x_{i0}, x_{i1}, x_{i2}), i = 1, 2, \ldots, \infty \). Assume that \( x_i = x_1 R_i^{-1} \), \( i = 2, 3, \ldots \), then \( x \) can be obtained by solving \( xQ = 0 \) using the boundary condition

\[
x_0 e + x_1 (I - R_1)^{-1} e = 1.
\]

From \( xQ = 0 \) we get

\[
x_0 B_0 + x_1 B_2 = 0.
\]

(3)

\[
x_0 B_1 + x_1 (A_1 + R_1 A_2) = 0.
\]

(4)

From (3) and (4) we will get

\[
\mu p_1 x_{10} + \mu_1 x_{11} = (\lambda)x_{01}.
\]

(5)

\[
\mu p_2 x_{10} + \mu_2 x_{12} = (\lambda)x_{02}.
\]

(6)

\[
\mu x_{10} = (\lambda + \theta_1)x_{11} + (\lambda + \theta_2)x_{12}.
\]

(7)
\[ \lambda x_{01} = (\lambda + \mu_1 + \theta_1)x_{11}. \]

(9) \[ \lambda x_{02} = (\lambda + \mu_2 + \theta_2)x_{12}. \]

Assume \( x_{01} = k_1 \) and \( x_{02} = k_2 \), then from (8) and (9), \( x_{11} = r_1k_1, \ x_{12} = r_2k_2 \). Substituting the values of \( x_{11} \) and \( x_{01} \) in (5) we will get \( x_{10} = \frac{k_1r_1}{p_1} \). Also

\[ k_2 = \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} k_1 \]

To find the value of \( k_1 \) we use the normalizing condition

\[ x_0 e + x_1 (I - R_1)^{-1} e = 1. \]

Let \( r_0' = 1 - r_0, r_1' = 1 - r_1, r_2' = 1 - r_2 \); then \( (I - R_1)^{-1} = \begin{bmatrix} 1/r_0' & 0 & 0 \\ -r_1/r_0' r_1' & 1/r_1' & 0 \\ -r_2/r_0' r_2' & 0 & 1/r_2' \end{bmatrix} \)

Using (2)

\[ k_1 \left[ 1 + \frac{r_1}{p_1 r_0} + \frac{r_1 r_1'}{r_1} - \frac{r_1 r_1'}{r_0 r_1} \right] + k_2 \left[ 1 + \frac{r_2}{r_2} - \frac{r_2 r_2'}{r_0 r_2} \right] = 1. \]

Substituting \( k_2 \) in (10)

\[ k_1 \left[ 1 + \frac{r_1}{p_1 r_0} + \frac{r_1 r_1'}{r_1} - \frac{r_1 r_1'}{r_0 r_1} + \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} \left[ 1 + \frac{r_2}{r_2} - \frac{r_2 r_2'}{r_0 r_2} \right] \right] = 1. \]

From (11) \( k_1 = \frac{1}{\left[ 1 + \frac{r_1}{p_1 r_0} + \frac{r_1 r_1'}{r_1} - \frac{r_1 r_1'}{r_0 r_1} + \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} \left[ 1 + \frac{r_2}{r_2} - \frac{r_2 r_2'}{r_0 r_2} \right] \right]} \).

Now \( R_1^{k-1} = \begin{bmatrix} r_0^{(k-1)} & 0 & 0 \\ r_1^{(k-1)} & 0 & 0 \\ r_2^{(k-1)} & 0 & 0 \end{bmatrix} \)

and

\[ x_k e = x_10 r_0^{k-1} + x_{11} \left[ r_1^{(k-1)} + r_1 \frac{r_0^{(k-1)} - r_1^{(k-1)}}{r_0 r_1 - r_1^2} \right] + x_{12} \left[ r_2^{(k-1)} + r_2 \frac{r_0^{(k-1)} - r_2^{(k-1)}}{r_0 r_2 - r_2^2} \right] \] for \( k > 1 \).

Let \( Q_v(z) \) be the PGF associated with the number of customers in the system. Then

\[ Q_v(z) = \sum_{n=0}^{\infty} x_n e z^n \]

\[ = x_0 + x_1 + \frac{x_10}{1 - r_0 z} + \frac{x_{11} r_0}{1 - r_0 z - r_1 z} + \frac{x_{12} r_0}{1 - r_0 z - r_2 z} + \frac{x_{11} r_0}{1 - r_0 z - r_1 z} \left[ \frac{1}{1 - r_0 z} - \frac{1}{1 - r_1 z} \right] + \frac{x_{12} z}{r_0 r_2 - r_2 z} \left[ \frac{1}{1 - r_0 z} - \frac{1}{1 - r_2 z} \right] \]

\[ = \frac{1 - r_0}{1 - r_0 z} \left[ x_0 \left( \frac{1 - r_0 z}{1 - r_0} \right) + x_1 \left( \frac{1 - r_0 z}{1 - r_0} \right) + x_{11} \left( \frac{1 - r_0 z}{1 - r_1 z} \right) + x_{12} z \left( \frac{1 - r_0 z}{1 - r_2 z} \right) \right] \]

\[ + \frac{x_{11} r_0}{r_0 r_1 - r_1 z} \left[ \frac{1}{1 - r_0 z} - \frac{1}{1 - r_1 z} \right] + \frac{x_{12} z}{r_0 r_2 - r_2 z} \left[ \frac{1}{1 - r_0 z} - \frac{1}{1 - r_2 z} \right] \]

\[ Q_v'(z) = \frac{r_0}{1 - r_0 z} \left[ x_0 \left( \frac{1 - r_0 z}{1 - r_0} \right) + x_1 \left( \frac{1 - r_0 z}{1 - r_0} \right) + x_{10} \left( \frac{1 - r_0 z}{1 - r_1 z} \right) + x_{12} \left( \frac{1 - r_0 z}{1 - r_2 z} \right) \right] \]
The state space of generator associated with the Markov chain is
\[ Q_s = r \]

\[ x_{11} r_1 (1-r_0) \left( \frac{1}{r_0} - \frac{1}{1-r_1} \right) + x_{12} r_2 (1-r_0) \left( \frac{1}{r_0} - \frac{1}{1-r_2} \right) + x_{11} r_1 (1-r_0) \left( \frac{1}{r_0} - \frac{1}{1-r_1} \right) + \left( \frac{1}{r_0} - \frac{1}{1-r_0} \right) \left( \frac{1}{r_0} \right) \]

\[ -r_0 x_0 - r_0 x_2 + x_10 + x_{11} r_1 (r_0 - r_1) + x_{12} r_2 (r_0 - r_2) + x_{11} r_1 (r_0 - r_1) \left( \frac{1}{r_0} - \frac{1}{1-r_1} \right) \]

\[ x_{12} \left( r_0 - r_2 - r_2 \right) \left( \frac{1}{1-r_2^2} \right) \]

Expected queue length \( E(\bar{X}) = Q_v(1) = \frac{r_0}{1-r_0} + \left( \frac{1}{1-r_0} \right) \left[ -r_0 x_0 - r_0 x_2 + x_10 + x_{11} r_1 (r_0 - r_1) + x_{12} r_2 (r_0 - r_2) + x_{11} r_1 (r_0 - r_1) \left( \frac{1}{r_0} - \frac{1}{1-r_1} \right) \right] \]

\[ \frac{1}{1-r_0} + \left( \frac{1}{1-r_0} \right) \left[ -r_0 k_1 - r_0 k_2 + \frac{k_1 r_1}{r_1} + \frac{r_0 k_1 r_1}{r_1} \left( r_0 - r_2 \right) \left( \frac{1}{1-r_1} \right) \right] + k_2 r_2 \left( r_0 - r_2 - r_2 \right) \left( \frac{1}{1-r_2^2} \right) \]

\[ -r_0 + \frac{r_2 r_2}{r_0 - r_2} + r_2 \left( r_0 - r_2 - r_2 \right) \left( \frac{1}{1-r_2^2} \right) \]

\[ \text{Case.2} \] Now consider the case of \( n = 3 \). Then \( S(t) \) has four states. \n\[ S(t) = \begin{cases} 
0, & \text{if the server is serving in normal mode;} \\
1, & \text{if server is in the type I working vacation;} \\
2, & \text{if server is in the type II working vacation;} \\
3, & \text{if server is in the type III working vacation;}
\end{cases} \]

The state space of \( X \) is \( \{(0,k) | k = 1, 2, 3 \} \cup \{(j,k) | j = 1, 2, \ldots ; k = 0, 1, 2, 3 \} \). The infinitesimal generator associated with the Markov chain is \( Q_2 = \)

\[
\begin{pmatrix}
B_0 & B_1 \\
B_2 & A_0
\end{pmatrix}
= \begin{pmatrix}
A_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \ddots & \ddots
\end{pmatrix}
\]

where \(-B_0 = A_0 = \lambda I_3\), \(B_1 = \)

\[
\begin{pmatrix}
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
\]

\(B_2 = \)

\[
\begin{pmatrix}
\mu p_1 & \mu p_2 & \mu p_3 \\
\mu_1 & 0 & 0 \\
0 & \mu_2 & 0 \\
0 & 0 & \mu_3
\end{pmatrix}
\]
Using the normalizing condition

\[ x_0 e + x_1 (I - R_2)^{-1} e = 1, \]

we get
Now we consider the case where there are $n \geq 4$ distinct type of vacations. Then

$$S(t) \text{ has } n + 1 \text{ distinct values.}$$

$$S(t) = \begin{cases} 
0, & \text{if the server is serving in normal mode;} \\
i, & \text{if server is in the } i^{th} \text{ type working vacation, } 1 \leq i \leq n;
\end{cases}$$

The state space of $X$ is $\{(0,k)/k = 1,2,\ldots,n\} \cup \{(j,k)/j = 0,1,2,\ldots; k = 1,2,\ldots,n\}$ The infinitesimal generator associated with the Markov chain is

$$Q_n = \begin{bmatrix}
B_0 & B_1 \\
B_2 & A_1 & A_0 \\
& A_2 & A_1 & A_0 \\
& & & \ddots & \ddots & \ddots
\end{bmatrix} \quad \text{where } B_1 = \begin{bmatrix}
0 & \lambda \\
& & \lambda \\
& & & \ddots & \ddots
\end{bmatrix}_{n \times (n+1)}.$$
\[
B_2 = \begin{bmatrix}
\mu \, p_1 & \mu \, p_2 & \ldots & \mu \, p_n \\
\mu_1 & & & \\
& \mu_2 & & \\
& & \ddots & \\
& & & \mu_n
\end{bmatrix},
A_2 = \begin{bmatrix}
\mu \\
\mu_1 \\
\vdots \\
\mu_n
\end{bmatrix},
\]

\[-B_0 = A_0 = \lambda I_n\]

\[
A_1 = \begin{bmatrix}
-\lambda - \mu & \theta_1 & & & \\
\theta_1 & -\lambda - \mu - \theta_1 & & & \\
& \theta_2 & -\lambda - \mu - \theta_2 & & \\
& & \ddots & \ddots & \\
& & \theta_n & & -\lambda - \mu - \theta_n
\end{bmatrix}
\]

As in the earlier sections

\[
A_0 + A_1 + A_2 = \begin{bmatrix}
-\lambda - \mu & \theta_1 + \mu_1 & -\mu_1 - \theta_1 \\
\theta_2 + \mu_2 & -\mu_2 - \theta_2 & \\
\vdots & \ddots & \ddots \\
\theta_n + \mu_n & & -\mu_n - \theta_n
\end{bmatrix}
\]

Let \( y = (y_0, y_1, y_2, \ldots, y_n) \) be the invariant probability vector of \( A \) satisfying \( yA = 0 \) and \( ye = 1 \). The system is stable if and only if \( yA_0 e < yA_2 e \). Here \( yA_0 e = \lambda \) and \( yA_2 e = \mu \).

Theorem: The system is stable if and only if \( \lambda < \mu \)

\[
R_n = \begin{bmatrix}
r_0 \\
r_1 & \bar{r}_1 \\
0 & r_2 & \bar{r}_2 \\
& \ddots & \ddots & \ddots \\
r_n & & 0 & \bar{r}_n
\end{bmatrix}
\]

where \( r_0 = \rho, r_i = \frac{\rho(\lambda + \theta_i)}{(\lambda + \mu_i + \theta_i)} \),

\[
\bar{r}_i = \frac{\lambda}{(\lambda + \mu_i + \theta_i)}.
\]

Let \( x = (x_0, x_1, x_2, \ldots) \) be the steady state probability vector associated with the Markov
chain \( X \). Here \( x_0 = (x_{01}, x_{02}, \ldots, x_{0n}) \) and \( x_i = (x_{i0}, x_{i1}, x_{i2}, \ldots, x_{in}), i = 1, 2, \ldots \) Then assuming
\( x_{0j} = k_j, 1 \leq i \leq n, \) we get \( x_{1j} = r_j k_j, x_{10} = \frac{kr_1}{p_1}. \)
Also \( k_j = \frac{\mu_{pj} \bar{r}_j}{p_1 (\lambda - \mu_{pj})} r_1. \) Let \( \bar{r}_i = 1 - \bar{r}_i, 1 \leq i \leq n, r_0' = 1 - r_0, \)
\( \ell_i = 1/r_i', 0 \leq i \leq n, \chi_i = \frac{1}{r_i'} / (r_i', r_0'), 1 \leq i \leq n \) then,
\[
(I - R_n)^{-1} = \begin{bmatrix}
\ell_0 \\
\chi_1 & \ell_2 \\
\chi_2 & \ell_2 \\
\vdots & \ddots \end{bmatrix}
\]
\[
k_1 = \left[ 1 + \frac{r_1}{\bar{r}_1} - \frac{r_1}{\bar{r}_0} + \frac{\mu \mu_{pj} r_1}{p_1} \left( 1 + \frac{\bar{r}_j - r_j \bar{r}_j}{\bar{r}_j - r_0} \right) \right]^{-1}
\]
Now \( R_n^{k-1} = \)
\[
\begin{bmatrix}
\frac{r_0}{r_1} & r_0 \bar{r}_1 \\
\frac{r_1}{r_2} - \frac{r_1}{r_2} & \frac{r_2}{r_1} \\
\vdots & \ddots \\
\frac{r_n}{r_{n-1}} & \frac{r_n}{r_n} - \frac{r_0}{r_n}
\end{bmatrix}
\]
\( x_i e = x_{10} r_0^{k-1} + \sum_{j=1}^{n} x_{1j} \left( \bar{r}_1^{(k-1)} + r_j \frac{1}{r_0 - \bar{r}_j} \right) \) for \( k > 1. \)

Then \( Q_1(z) = \sum_{n=0}^{\infty} x_n z^n \)
\[
x_j = \sum_{j=1}^{n} x_{1j} z + \sum_{j=1}^{n} x_{1j} r_j z + \sum_{j=1}^{n} x_{1j} r_j z \left[ \frac{1}{1 - r_0 z} \right] - \frac{x_{10} z}{1 - r_0 z}
\]
Expected queue length \( E(\bar{L}) = Q_1'(1) \)
\[
= r_0 \frac{1}{1 - r_0} + \sum_{j=1}^{n} \left( k_j - r_j \bar{r}_j \right) \left[ r_0 + r_j \frac{1}{p_1} + \bar{r}_j \left( \frac{r_0 - r_j}{r_0 - \bar{r}_j} \right) \left( \frac{1 - r_0 + r_0 \bar{r}_j}{1 - r_0 \bar{r}_j} \right) \right]
\]
The above discussions lead to

**Theorem (Stochastic decomposition):** The expected queue length \( E(\bar{L}) \) can be decomposed into the sum of the expectations of \( n + 1 \) independent random variables as:
\[ E(\bar{L}) = E(L) + \sum_{i=1}^{n} E(L_{V_i}) \] where \( E(L) \) is the queue length of classical \( M/M/1 \) queue and \( \sum_{i=1}^{n} E(L_{V_i}) \) is the additional queue length due to \( n \) types of vacations.

### 3.2. Stationary waiting time.

Using Little’s formula the expected waiting time \( E(\bar{W}) = \frac{E(L)}{\lambda} \).

(12)

\[ E(\bar{W}) = \left( \frac{1}{\mu - \lambda} + \frac{1}{\lambda} \sum_{i=1}^{n} E(L_{V_i}) \right) \]

From (12) it is clear that the expected waiting time can be decomposed into the sum of \( n + 1 \) independent random variables: \( E(\bar{W}) = E(W) + \sum_{i=1}^{n} E(W_{V_i}) \). where \( E(W) \) is the expected waiting time of a customer in the \( M/M/1 \) queue and \( \sum_{i=1}^{n} E(L_{V_i}) \) is the additional waiting time due to \( n \) types of vacations.

### Conflict of Interests

The author(s) declare that there is no conflict of interests.

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