SHADOW MARTINGALES – A STOCHASTIC MASS TRANSPORT APPROACH TO THE PEACOCK PROBLEM

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Abstract. Given a family of real probability measures $(\mu_t)_{t \geq 0}$ increasing in convex order (a peacock) we describe a systematic method to create a martingale exactly fitting the marginals at any time. The key object for our approach is the obstructed shadow of a measure in a peacock, a generalization of the (obstructed) shadow introduced in [12, 45]. As input data we take an increasing family of measures $(\nu^{\alpha})_{\alpha \in [0,1]}$ with $\nu^{\alpha}(\mathbb{R}) = \alpha$ that are submeasures of $\mu_0$, called a parametrization of $\mu_0$. Then, for any $\alpha$ we define an evolution $(\eta^{\alpha}_t)_{t \geq 0}$ of the measure $\nu^{\alpha} = \eta^{\alpha}_0$ across our peacock by setting $\eta^{\alpha}_t$ equal to the obstructed shadow of $\nu^{\alpha}$ in $(\mu_s)_{s \in [0,t]}$. We identify conditions on the parametrization $(\nu^{\alpha})_{\alpha \in [0,1]}$ such that this construction leads to a unique martingale measure $\pi$, the shadow martingale, without any assumptions on the peacock. In the case of the left-curtain parametrization $(\nu_{lc}^{\alpha})_{\alpha \in [0,1]}$ we identify the shadow martingale as the unique solution to a continuous-time version of the martingale optimal transport problem.

Furthermore, our method enriches the knowledge on the Predictable Representation Property (PRP) since any shadow martingale comes with a canonical Choquet representation in extremal Markov martingales.

Keywords: peacock problem, peacocks, optimal transport, martingale optimal transport, predictable representation property, Choquet representation

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1. Introduction

Two finite measures $\mu$ and $\mu'$ on $\mathbb{R}$ with finite first moments are said to be in convex order, denoted by $\mu \leq_c \mu'$, if $\int \varphi \, d\mu \leq \int \varphi \, d\mu'$ for all convex $\varphi : \mathbb{R} \to \mathbb{R}$. Peacocks are families $(\mu_t)_{t \geq 0}$ of probability measures on $\mathbb{R}$ with finite first moments that increase in convex order. Given a peacock $(\mu_t)_{t \geq 0}$, the peacock problem is to construct a probability measure $\pi$ such that the canonical process $X = (X_t)_{t \geq 0}$ is a martingale w.r.t. its natural filtration and the marginal distributions coincide with $(\mu_t)_{t \geq 0}$, i.e. $\text{Law}_t(X_t) = \mu_t$ for each $t \geq 0$.

There is a wide range of beautiful solutions to this problem employing different ideas and techniques, e.g. [39, 12, 14, 37, 27, 43, 34, 22, 2, 36, 24]. On the one extreme, there is the fundamental non-constructive result of Kellerer [39] proving the existence of Markov solutions for any given peacock. On the other end of the spectrum, there are very explicit constructions for specific sub classes of peacocks, many of which can be found in the monograph [26] by Hirsch, Protfa, Roynette, and Yor. However, it
is difficult to manage both aspects by constructing an explicit solution for a generic peacock. Only recently there have been contributions in this direction by Lowther [42], Hobson [27], Juillet [36] and Henry-Labordere and Touzi [25].

We propose a new method to systematically construct a martingale associated with a peacock. Thereby, we rely on the rich theory of optimal transport. In optimal transport a coupling of two probability measures is interpreted as a plan to transport one marginal to the other one. More precisely, given a coupling transport a coupling of two probability measures is interpreted as a plan to transport a peacock. Thereby, we rely on the rich theory of optimal transport. In optimal transport, see e.g. [13, 6, 18, 41, 8]. In various striking applications it turned out to be useful to interpret a stochastic process $X \equiv (X_t)_{t \geq 0}$ as a device to transport mass from time 0 to the distribution of $X$ at a (potentially random) time $\tau$. To identify the induced coupling of the distribution of $X$ at time 0 and time $\tau$ it is then necessary to trace the evolution of fixed parts of the initial distribution, e.g. to consider for $A \subset \mathbb{R}$ the evolution in $t$ of

$$\eta_t^A := \mathbb{P}[X_t \in \cdot | X_0 \in A].$$

Observe, that opposed to the classical optimal transport setup knowing the evolution of $\eta_t^A$ for $A \subset \mathbb{R}$ does in general not allow to recover the process $X$. Note that this already fails for two-step processes. For instance, compare the law of $(X_1, X_2, X_3)$ where the three random variables are independent and identically distributed to the law of $(X_1, X_2, X_2)$.

We call a measure $\nu$ a submeasure of $\mu$, if $\nu \leq_+ \mu$, i.e. if $\nu(A) \leq \mu(A)$ for all measurable sets $A$. For instance, the restrictions $(\mu_0)|_A$ of $\mu_0 = \text{Law}(X_0)$ to the measurable sets $A = (-\infty, q]$ are submeasures of $\mu_0$. Using this terminology, the goal of this article is to uniquely define a martingale associated with a peacock $(\mu_t)_{t \geq 0}$ from the following input data only:

- A parametrization of $\mu_0$, i.e. a family of submeasures $(\nu^\alpha)_{\alpha \in [0,1]}$ of $\mu_0$ s.t. $\nu^\alpha(\mathbb{R}) = \alpha$, $\nu^\alpha \leq_+ \nu^\beta$ for $\alpha \leq \beta$, and $\nu^\lambda = \mu_0$.

- For each $\alpha$, the evolution of $\nu^\alpha$ through the marginals $(\mu_t)_{t \geq 0}$, i.e. a family $(\eta^\alpha_t)_{t \geq 0}$ of submeasures of $(\mu_t)_{t \geq 0}$, $\eta^\alpha_t \leq_+ \mu_t$ for all $t \geq 0$, satisfying $\nu^\alpha = \eta^\alpha_0 \leq_+ \eta^\alpha_t$ for $0 \leq s \leq t$. These evolutions also need to be consistent in the sense that $\eta^\alpha_t \leq_+ \eta^\beta_t$ for all $\alpha \leq \beta$ in $[0,1]$ and $t \geq 0$.

It is easy to see that, without further assumptions, this data is not sufficient to uniquely determine the law of a martingale. It turns out that a certain convexity of $(\nu^\alpha)_{\alpha \in [0,1]}$ together with some kind of minimality in the choice of $(\eta^\alpha_t)_{t \geq 0}$ is the key to uniquely define a martingale measure via this procedure.
Figure 1. The shaded area shows the evolution \((\eta^\alpha_t)_{t \geq 0}\) of \(\nu^\alpha\) through \((\mu_t)_{t \geq 0}\) for \(\alpha = \frac{1}{3}\) (left) and \(\alpha = \frac{2}{3}\) (right), respectively. Here \(\mu_t = \text{Unif}_{[-1-t,1+t]}\) and \(I_\alpha = [-\alpha, +\alpha]\). The measures are represented by their density functions w.r.t. the Lebesgue measure. The representation is in 3D-perspective with times evolving transversally to the page.

Before introducing the appropriate notions we would like to present our solution in a special setting which already gives a good idea of the general case (namely Theorem 1.5 in Subsection 1.1):

Corollary 1.1. Let \((\mu_t)_{t \geq 0}\) be a peacock. For any nested family of intervals \((I_\alpha)_{\alpha \in [0,1]}\) for which

(i) \(\mu_0(I_\alpha) = \alpha\) for any \(\alpha \in [0,1]\),
(ii) \(\alpha \mapsto \int_{I_\alpha} y \, d\mu_0(y)\) is a convex function and
(iii) \(\sup I_\alpha < +\infty\) and \(\partial I_\alpha \cap \partial I_\beta = \emptyset\) for all \(\alpha \neq \beta\) in \([0,1]\),

there exists a unique solution \(\pi\) to the peacock problem w.r.t. \((\mu_t)_{t \geq 0}\) such that for any other solution \(\rho\) to the peacock problem w.r.t. \((\mu_t)_{t \geq 0}\) it holds

\[
\text{Law}_\pi(X_t | X_0 \in I_\alpha) \leq c \text{Law}_\rho(X_t | X_0 \in I_\alpha)
\]

for all \(\alpha \in [0,1]\) and \(t \geq 0\). Moreover, \((X_0, X_t)_{t \geq 0}\) is a Markov process under \(\pi\).

Remark 1.2. As the reader should have noticed, a completely rigorous statement of Corollary 1.1 requires to specify on which measurable space the martingale measure \(\pi\) in Corollary 1.1 is defined. Here, as well as in all of the paper, we use \(\mathbb{R}^{[0,\infty)}\) with the Borel \(\sigma\)-algebra induced by the product topology on \(\mathbb{R}^{[0,\infty)}\).

Alternatively, if the map \(t \mapsto \mu_t\) is right-continuous w.r.t. the weak topology (cf. Section 3.1), by standard martingale regularization (see e.g. [46, II §2]) there exists a càdlàg modification of the canonical process on \(\mathbb{R}^{[0,\infty)}\) under \(\pi\) that one can use to define the solution \(\pi\) directly on the Skorokhod space of càdlàg functions from \([0, \infty)\) to \(\mathbb{R}\), see Section 6 for an implementation.

Consider the peacock \((\mu_t)_{t \geq 0}\) consisting of uniform distributions \(\mu_t = \text{Unif}_{[-1-t,1+t]}\) on the intervals \([-1 - t, 1 + t]\) and the interval family \(I_\alpha = [-\alpha, \alpha], \alpha \in [0,1]\). It is not difficult to check that this pair satisfies the conditions (i)-(iii) in Corollary 1.1 and for two choices of \(\alpha\), Figure 1 illustrates the evolution \((\eta^\alpha_t)_{t \geq 0}\) of \(\nu^\alpha = (\mu_0)|_{I_\alpha}\) over time under the solution to the peacock problem constructed in Corollary 1.1.

We would like to highlight a few features of Corollary 1.1 which all appear in the general case, Theorem 1.3 again:

- The parametrization of \(\mu_0\) induced by the intervals \(I_\alpha\), namely \((\mu_0|_{I_\alpha})_{\alpha \in [0,1]}\), is in a certain sense convex, cf. item (iii) in Corollary 1.1.
• The minimality condition (1.1) affects only the conditional one-dimensional marginal distributions under $\pi$. Thus, only the evolution $d\eta_\alpha^t = \alpha \text{Law}_\pi(X_t | X_0 \in I_\alpha)$ of $\nu^\alpha = (\mu_0)|_{I_\alpha}$ is prescribed by the requirement to be minimal in convex order but (a priori) no joint distributions are fixed.

• In particular, (1.1) says that $\text{Law}_\pi(X_t | X_0 \in I_\alpha)$ is minimal in convex order among all solutions to the peacock problem, for every $\alpha$. Explicitly, for every $t$, every $\alpha$, every convex function $\varphi$, and any other solution $\rho$ to the peacock problem w.r.t. $(\mu_t)_{t \geq 0}$ there holds

$$\int \varphi \, d\text{Law}_\pi(X_t | X_0 \in I_\alpha) \leq \int \varphi \, d\text{Law}_\rho(X_t | X_0 \in I_\alpha),$$

so that, in this precise sense, $\text{Law}_\pi(X_t | X_0 \in I_\alpha)$ is as concentrated as possible. Hence, we can think of $\pi$ as a plan to transport $\mu_0|_{I_\alpha}$ through $(\mu_t)_{t \in [0,1]}$ as concentrated as possible subject to the martingale constraint.

• Finally, it will become apparent during the proof of Theorem 1.5 that the Markov property turns out to be a consequence of the fact that $\text{Law}(X|X_0)$ is uniquely determined by its marginal distributions, see Lemma 4.29 Proposition 4.30.

1.1. Main results. Our main results, Theorem 1.5 and 1.6, enlarge the perspective presented in Corollary 1.1 but are of the same nature. They in fact permit further parametrizations of $\mu_0$ and stress the optimal feature of our shadow martingales.

To state Theorem 1.5 we need to introduce the objects that will replace the specific parametrization $(\mu_0|_{I_\alpha})_{\alpha \in [0,1]}$ and property (1.1). We start with the definition of shadows, the concept that will replace (1.1). To not overload the introduction we give a preliminary (but correct) definition and refer to Proposition 4.20 where it is extended to a more general setting.

As before a martingale measure is a probability measure under which the canonical process is a martingale w.r.t. the filtration generated by the process.

Definition 1.3. For all pocks $(\mu_t)_{t \geq 0}$, $t \geq 0$ and $\nu \leq_+ \mu_0$ with $\alpha = \nu(\mathbb{R}) > 0$ the set

$$\left\{ \alpha \text{Law}_\pi(X_t) : \pi \text{ is a martingale measure, } \nu = \alpha \text{Law}_\pi(X_0) \text{ and } \alpha \text{Law}_\pi(X_s) \leq_+ \mu_s \text{ for all } s \in [0,t] \right\}$$

attains a minimum w.r.t. the convex order $\leq_c$. This minimum is called the shadow of $\nu$ in $(\mu_s)_{s \in [0,t]}$ and is denoted by $S^{\mu([0,1])}(\nu)$.

We say that two finite measures $\mu$ and $\mu'$ on $\mathbb{R}$ with finite first moments are in convex-stochastic order, denoted by $\mu \leq_{c,s} \mu'$, if $\int \varphi \, d\mu \leq \int \varphi \, d\mu'$ for all convex and increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$. A parametrization $(\nu^\alpha)_{\alpha \in [0,1]}$ of $\mu_0$ is called $\leq_{c,s}$-convex if for all $\alpha_1 < \alpha_2 < \alpha_3$ in $[0,1]$ it holds

$$\frac{\nu^{\alpha_2} - \nu^{\alpha_1}}{\alpha_2 - \alpha_1} \leq_{c,s} \frac{\nu^{\alpha_3} - \nu^{\alpha_2}}{\alpha_3 - \alpha_2}.$$  

Since property (1.2) can be interpreted as increasing slopes of secant lines for the functions $\alpha \mapsto \int \varphi \, d\mu^\alpha$, $\varphi$ increasing and convex, this property is called $\leq_{c,s}$-convexity. The following three parametrizations, that were introduced in 1.3 for one step processes, are examples of $\leq_{c,s}$-convex parametrizations (cf. Lemma 4.6 for the proof):

• the left-curtain parametrization

$$\nu^\alpha_{lc} = \mu_0|_{(-\infty,F_{\mu_0}^{-1}(\alpha))} + (\alpha - \mu_0(-\infty,F_{\mu_0}^{-1}(\alpha)))\delta_{F_{\mu_0}^{-1}(\alpha)}$$

• the right-curtain parametrization

$$\nu^\alpha_{rc} = \mu_0|_{[F_{\mu_0}^{-1}(\alpha),\infty)}$$

• the midpoint parametrization

$$\nu^\alpha_{mp} = \frac{1}{2} \mu_0|_{(-\infty,F_{\mu_0}^{-1}(\alpha))} + \frac{1}{2} \mu_0|_{[F_{\mu_0}^{-1}(\alpha),\infty)}.$$
where $F_{\mu_0}^{-1}$ is the quantile function of $\mu_0$, i.e., the generalized inverse of the cumulative distribution function $F_{\mu_0}$.

- the sunset parametrization $\nu_{\text{sun}}^\alpha = \alpha \mu_0$ for every $\alpha \in [0, 1]$ and
- the middle-curtain parametrization

$$\nu_{\text{mc}}^\alpha = \mu_0(q_\alpha, q_\alpha') + c_\alpha \delta_{q_\alpha} + \ell_\alpha' \delta_{q_\alpha'}$$

where $q_\alpha \leq q_\alpha'$ and $c_\alpha, \ell_\alpha' \in [0, 1]$ are chosen such that $\nu_{\text{mc}}^\alpha(\mathbb{R}) = \alpha$ and

$$\int y \, d\nu_{\text{mc}}^\alpha(y) = \int y \, d\mu_0(y).$$

We remark that if $\mu_0$ has no atoms the left-monotone and the middle-curtain parametrizations are special cases of the parametrization used in Theorem 1.1. In particular, the parametrization in Figure 3 is the middle-curtain parametrization of the uniform measure on $[-1, 1]$.

The final object that we need to introduce are martingale parametrizations. Their purpose is to allow conditioning on the initial behaviour of a martingale which is not of the form $\{X_t \in I_\alpha\}$ for some Borel set $I_\alpha \subset \mathbb{R}$. Again, as Definition 1.3, Definition 1.4 is a simplified version of the general Definition 1.1. Notice that the notion of submeasure is well-defined for measurable spaces other than $\mathbb{R}$. Moreover, we denote by $\pi^\alpha(X_t \in \cdot)$ the push-forward measure of $\pi^\alpha$ via $X_t$ (it is a measure of mass $\alpha$).

**Definition 1.4.** Let $(\nu^\alpha)_{\alpha \in [0, 1]}$ be a parametrization of $\mu_0$ and $\pi$ the law of a martingale indexed by $[0, \infty)$. A family $(\pi^\alpha)_{\alpha \in [0, 1]}$ of finite measures is called a martingale parametrization of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0, 1]}$ if

1. for every $\alpha \in [0, 1]$ it holds $\pi^\alpha(\mathbb{R}^{[0, \infty)}) = \alpha$,
2. we have $\pi^\alpha \leq \pi^{\alpha'}$ for all $\alpha \leq \alpha'$,
3. for every $\alpha \in (0, 1]$ the measure $\frac{\pi^\alpha}{\alpha}$ is a martingale measure,
4. we have $\pi^1 = \pi$ and
5. it holds $\pi^\alpha(X_0 \in \cdot) = \nu^\alpha$ for all $\alpha \in [0, 1]$.

As we discuss in Subsection 4.2, a martingale parametrization $(\pi^\alpha)_{\alpha \in [0, 1]}$ w.r.t. $(\nu^\alpha)_{\alpha \in [0, 1]}$ is a convenient way of encoding that for each $\alpha \in [0, 1]$ the martingale $\pi$ transports the submeasure $\nu^\alpha$ of $\mu_0$ according to $\pi^\alpha$, i.e., we may interpret $\pi^\alpha$ formally as “$\alpha \text{Law}_{\pi}(X|X_0 \in \nu^\alpha)$”. In particular, any martingale parametrization $(\pi^\alpha)_{\alpha \in [0, 1]}$ w.r.t. $(\nu^\alpha)_{\alpha \in [0, 1]}$ induces a specific evolution of $\nu^\alpha$ for every $\alpha \in [0, 1]$, namely $\eta^\alpha = \pi^\alpha(X_t \in \cdot)$. We would like to stress that there might be several martingale parametrizations of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0, 1]}$ (cf. Example 1.9).

We can now state our first main result:

**Theorem 1.5.** Let $(\mu_t)_{t \geq 0}$ be a peacock and $(\nu^\alpha)_{\alpha \in [0, 1]}$ a $c.s.$ convex parametrization of $\mu_0$. Then, there exists a unique pair $(\pi, (\pi^\alpha)_{\alpha \in [0, 1]})$ where the martingale measure $\pi$ solves the peacock problem w.r.t. $(\mu_t)_{t \in T}$, $(\pi^\alpha)_{\alpha \in [0, 1]}$ is a martingale parametrization of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0, 1]}$ and

$$\pi^\alpha(X_t \in \cdot) = S_{\mu([0,t])}(\nu^\alpha)$$

for all $\alpha \in [0, 1]$ and $t \geq 0$. We call $\pi$ the shadow martingale (measure) w.r.t. $(\mu_t)_{t \geq 0}$ and $(\nu^\alpha)_{\alpha \in [0, 1]}$.

The shadow martingale $\pi$ can be represented as $\pi = \text{Law}(M^U)$ where $U$ is a $[0, 1]$-valued random variable and $(M^\alpha)_{\alpha \in [0, 1]}$ is a family of $\mathbb{R}^{[0, \infty)}$-valued random variables such that
(i) \( \{U\} \cup \{M^a : a \in [0,1]\} \) is a collection of independent random variables,
(ii) the random variable \( U \) is uniformly distributed on \([0,1]\) with \( \text{Law}(M^u|U \leq \alpha) = \frac{1}{\alpha} \nu^\alpha \) for all \( \alpha \in (0,1) \) and
(iii) for each \( a \in [0,1] \), \( (M_t^a)_{t \geq 0} \) is a Markov martingale which is uniquely determined by its (one-dimensional) marginal distributions, i.e. any martingale \( (Y_t)_{t \geq 0} \) with \( \text{Law}(Y_t) = \text{Law}(M_t^a) \) for all \( t \geq 0 \) satisfies \( \text{Law}(Y) = \text{Law}(M^a) \).

Note that the single constraint on \( \pi \) given by (1.3) only involves the evolution \( (\pi^a(X_t \in \cdot))_{t \geq 0} \) of the submeasures \( \nu^a \) and a priori no joint distributions. Hence, the theorem states that taking a \( \leq_{c,e} \)-convex parametrization of the initial marginal \( \mu_0 \) and fixing its evolution to be as concentrated as possible w.r.t. the convex order uniquely characterizes a martingale.

Specializing to the left-curtain parametrization \( (\nu^{\alpha}_c)_{\alpha \in [0,1]} \) and additionally assuming that both \( \mu_0 \) has no atoms and \( t \mapsto \mu_t \) is weakly right-continuous, we can give an alternative characterization of the associated shadow martingale which identifies it as a unique solution to a variant of an optimal transport problem, namely a peacock version of the martingale optimal transport problem:

**Theorem 1.6.** Let \( (\mu_t)_{t \geq 0} \) be a peacock with \( \mu_0(\{x\}) = 0 \) for all \( x \in \mathbb{R} \) and \( c \) a sufficiently integrable and regular cost function with \( \partial_x \partial^2_x c < 0 \) (cf. Theorem 8.4). The shadow martingale \( \pi_c \) w.r.t. \( (\mu_t)_{t \geq 0} \) and \( (\nu^{\alpha}_c)_{\alpha \in [0,1]} \) is the only solution to the peacock problem w.r.t. \( (\mu_t)_{t \geq 0} \) that satisfies
\[
\mathbb{E}_{\pi_c}[c(X_0, X_t)] = \inf \left\{ \mathbb{E}_\rho[c(X_0, X_t)] : \text{\rho sol. to peacock problem w.r.t.}(\mu_t)_{t \geq 0} \right\}
\]
simultaneously for all \( t \geq 0 \).

The necessity of the assumption of no atoms is best seen by looking at the case \( \mu_0 = \delta_0 \). In that case, since the marginals at time \( t \) are given, each solution to the peacock problem w.r.t. \( (\mu_t)_{t \in [0,1]} \) is a solution to the optimization problem (1.4).

When considering only finitely many marginals \( \{\mu_{t_0}, \mu_{t_1}, \ldots, \mu_{t_n}\} \) increasing in convex order, and a corresponding piecewise constant peacock, Theorem 1.6 reduces to a recent theorem by Nutz, Stebegg and Tan [45, Theorem 7.16]. As we will prove in Remark 8.6, in certain situations the martingale measure \( \pi_c \) is the unique limit of these piecewise constant martingales.

**Remark 1.7.** Corollary 7.3, Theorem 7.6 and Theorem 1.6 are well-defined and valid if we replace the index set \([0,\infty)\) by an abstract totally ordered set \((T, \leq)\) with minimal element \( 0 \in T \) (see Corollary 7.3, Theorem 7.6 and Theorem 8.4).

Moreover, the convex-stochastic order is defined via the integrals of increasing convex functions. Of course, one could define an analogue order relation on the set of finite measures via decreasing convex functions. Then the corresponding version of Theorem 7.3 for a “convex-decreasing” parametrization of the initial marginal distribution holds.

### 1.2 Choquet representation and the PRP property.

There is another abstract point of view on our main result Theorem 7.3. Any martingale measure has a Choquet representation, i.e. it can be written as a superposition of martingale measures that are extremal elements of the convex set of all martingale measures. Such a representation is interesting because the extremality in the set of all martingale measures naturally relates to the predictable representation property (PRP). In stochastic analysis a martingale \( M \) is said to satisfy the PRP if and only if any martingale \( X \) adapted to the natural filtration of \( M \) can be represented as a stochastic integral with respect to \( M \).
According to a theorem by Jacod and Yor a martingale satisfies the PRP if and only if its law is extremal in the convex set of all martingale measures (cf. [32, 48, 31]). Hence, any martingale measure is a superposition of martingales with the PRP. To the best of our knowledge, no concrete recipe for the construction of such a representation is known. However, for shadow martingales there exists a natural Choquet representation. In fact, this natural Choquet representation is the driving force behind the proof of Theorem 1.5, especially the uniqueness part.

Given a peacock \( \mu = (\mu_t)_{t \geq 0} \), our construction of the uniquely determined shadow martingale starts with a representation of \( \mu \) as a superposition of peacocks, i.e.

\[
\mu = \int_{[0,1]} \eta^a \, da. \tag{1.5}
\]

This representation is induced by the shadow and the choice of a proper parametrization \((\nu^a)_{a \in [0,1]}\) of \( \mu_0 \) (cf. Lemma 5.4). The peacocks \( \eta^a \) in (1.5) are in general not extremal in the convex set of all peacocks (in the sense that \( 2\eta^a = \eta' + \eta'' \) implies \( \eta' = \eta'' = \eta^a \)) so that (1.5) cannot be called a Choquet representation of \( \mu \). However, they satisfy a very similar property that we call non self-improvable (NSI) (cf. §4.3):

\[
2\eta^a = \eta' + \eta'' \quad \text{and} \quad \eta_0^a = \eta_0^0 = \eta_0^a \quad \text{implies} \quad \eta' = \eta'' = \eta^a. \tag{1.6}
\]

The main consequence of the NSI property is that for every peacock that satisfies this property there exists only one martingale measure that is associated with this peacock. Thus, the unique martingale measures \( \pi^a \) associated with \( \eta^a \) are extremal in the set of martingale measures with fixed initial distribution, i.e.

\[
\begin{cases}
2\pi^a = \rho' + \rho \\
\rho'(X_0 \in \cdot) = \rho''(X_0 \in \cdot) = \pi^a(X_0 \in \cdot)
\end{cases}
\]

implies \( \rho' = \rho'' = \pi^a \). Indeed, if \( \rho' \) and \( \rho'' \) are two martingale measures with \( 2\pi^a = \rho' + \rho'' \) that have the same initial distribution as \( \pi^a \), the marginal distributions of these three objects satisfy (1.6) and thus the marginal distributions of \( \rho' \) and \( \rho'' \) are \( \eta^a \), i.e. they coincide with the marginal distributions of \( \pi^a \). But then the uniqueness of the associated martingale measure given by the NSI property yields \( \pi^a = \rho = \rho' \). The superposition of these special martingale measures

\[
\pi = \int_0^1 \pi^a \, da \tag{1.7}
\]

is exactly the shadow martingale w.r.t. \( \mu \) and \((\nu^a)_{a \in [0,1]}\). More precisely, \( \pi^a \) is the distribution of \( M^a \) in the representation \( \pi = \text{Law}(M^U) \) of the shadow martingale that is described in the second part of Theorem 1.5.

The representation of the shadow martingale in (1.7) is in general not yet a Choquet representation because the martingale measures \( \pi^a \) are only extremal in the set of martingale measures with fixed initial distribution. Nevertheless, we can directly obtain a Choquet representation from (1.7) and then by Jacod and Yor’s theorem we have a rather explicit representation of the shadow martingale as a superposition of martingales that satisfy the PRP. In the case of the left-curtain parametrization, this is particularly easy. The construction of (1.5) is such that for each \( a \in [0,1] \) there holds \( \int_0^a \eta^a_0 \, da = \nu^a_k \) where \((\nu^a_k)_{a \in [0,1]}\) is the left-curtain parametrization. Looking again at the definition of \((\nu^a_k)_{a \in [0,1]}\) in Subsection 1.1 we see that \( \eta^0_0 \) is a Dirac measure for any \( a \in [0,1] \). Hence, the peacocks \( \eta^a \) and the associated martingale measures \( \pi^a \) are automatically extremal in the set of all peacocks and all martingale measures.
Therefore, in the case of the left-curtain parametrization, (1.7) is in fact already a Choquet representation of the shadow martingale $\pi$. More generally, given any $\leq_{c,s}$-convex parametrization $(\nu^a)_{a \in [0,1]}$ of $\mu_0$ we obtain a Choquet representation of the shadow martingale by further disintegrating (1.7) w.r.t. the initial marginal $\mu_0$, i.e. by conditioning on the starting value of $\pi$.

We want to emphasize that this Choquet representation of the shadow martingale is uniquely determined by the representation of the peacock $\mu$ given in (1.5). This representation of $\mu$ is constructed using only the shadow and a $\leq_{c,s}$-convex parametrization of the initial distribution. To show that the peacocks $(\eta^a)_{a \in [0,1]}$ in (1.5) satisfy the NSI property (which is very similar to extremality, cf. (1.6)) is in fact a crucial part of our proof. Moreover, this abstract point of view of our result makes it apparent that the construction of the shadow martingale is purely based on its marginals as an object in the space of peacocks and thus these are intrinsic solutions to the peacock problem.

1.3. Outline. There are several contributions to martingale optimal transport theory and the peacock problem that are related to our results and that we discuss in Section 2. In Section 3 we recall order relations for finite measures and important properties of the peacock problem.

In Section 4 we introduce (martingale) parametrizations, (general obstructed) shadows and non self-improvable peacocks. These concepts are not only essential ingredients of our proof of Theorem 1.5 but are interesting in themselves.

In Section 5 we prove a variant of Theorem 1.5 in the case that the peacock is indexed by a countable set $S \subset [0, \infty)$ that contains $0$ and satisfies $\sup S \in S$. In this setup it is possible to avoid some of the technicalities needed to be able to handle the general case and concentrate on the key steps and ideas of the proof. Let us briefly sketch them in the following paragraphs:

We need to construct a family of measures $(\pi^a)_{a \in [0,1]}$ on $\mathbb{R}^S$ that satisfies both Definition 1.4 (i)-(iii) and property (1.3) and need to show that this family is uniquely determined by these two properties (note that (1.3) already implies that $\pi := \pi^1$ is a solution of the peacock problem w.r.t. $(\mu_t)_{t \in S}$ and that condition (v) of Definition 1.4 is satisfied). To this end, we pursue the following approach:

• STEP 1: Any family of measures $(\pi^a)_{a \in [0,1]}$ on $\mathbb{R}^S$ satisfies Definition 1.4 (i)-(iii) if and only if there exists a family of martingale measures $(\hat{\pi}^a)_{a \in [0,1]}$ such that

$$\pi^a = \int_0^a \hat{\pi}^a \, da$$

for all $a \in [0,1]$. In particular, the family $(\pi^a)_{a \in [0,1]}$ is uniquely determined by any such family $(\hat{\pi}^a)_{a \in [0,1]}$ and for Lebesgue-a.e. $a \in [0,1]$ it holds

$$\hat{\pi}^a = \lim_{h \downarrow 0} \frac{\pi^{a+h} - \pi^a}{h}$$

under an appropriate topology (cf. Subsection 3.1). Thus, $(\pi^a)_{a \in [0,1]}$ satisfies property (1.3) if and only if the following two properties hold: For Lebesgue-a.e. $a \in [0,1]$ and all $t \in S$ the limit

$$\hat{\eta}^a_t = \lim_{h \downarrow 0} \frac{S^\mu_{[0,t] \cap S}(\nu^{a+h}) - S^\mu_{[0,t] \cap S}(\nu^a)}{h}$$

\footnote{Of course with $\mathbb{R}^{[0,\infty)}$ replaced by $\mathbb{R}^S$ and $[0,t]$ replaced by $S \cap [0,t]$.}
exists and the distribution of $X_t$ under $\hat{\pi}^a$ is $\hat{\eta}^a_t$. Step 1 is accomplished in Subsection 5.1.

- **STEP 2**: Step 1 implies that there exists a family $(\pi^a)_{a \in [0,1]}$ with the desired properties, if there exists a family $(\hat{\pi}^a)_{a \in [0,1]}$ of martingale measures on $\mathbb{R}^S$ such that the canonical process under $\hat{\pi}^a$ has marginal distributions $(\hat{\eta}^a_{t})_{t \in S}$ for Lebesgue a.e. $a \in [0,1]$. By Kellerer’s Theorem, for fixed $a \in [0,1]$ such a martingale measure $\hat{\pi}^a$ exists if $(\hat{\eta}^a_{t})_{t \in S}$ is a peacock. Using the calculus rules that we develop for general obstructed shadows, we show in Subsection 5.2 that for Lebesgue-a.e $a \in [0,1]$ the limit in (1.9) exists and that $(\hat{\eta}^a_{t})_{t \in S}$ is a peacock.

- **STEP 3**: Another implication of Step 1 is that the family $(\pi^a)_{a \in [0,1]}$ constructed in Step 2 is uniquely determined if there exists only one martingale measure with marginal distributions $(\hat{\eta}^a_{t})_{t \in S}$ for Lebesgue a.e. $a \in [0,1]$. Unfortunately, just from the defining equation (1.9), the peacock $(\hat{\eta}^a_{t})_{t \in S}$ does not need to satisfy this very restrictive property for all $a \in [0,1]$ where $(\hat{\eta}^a_{t})_{t \in S}$ is defined (cf. Example 9.4).

That being said, it is sufficient for us that $(\hat{\eta}^a_{t})_{t \in S}$ is NSI for Lebesgue-a.e. $a \in [0,1]$ since the NSI property implies the uniqueness of a martingale associated with $(\hat{\eta}^a_{t})_{t \in S}$ (cf. Section 1.2). To show this, we introduce an auxiliary optimization problem and establish a corresponding monotonicity principle (cf. Subsection 5.3). The minimality of the shadow in conjunction with the $\leq_{c,S}$-convexity of $(\nu^a)_{a \in [0,1]}$ implies that $(\hat{\eta}^a_{t})_{a \in [0,1]}$ is a minimizer of this optimization problem which in turn implies that $(\hat{\eta}^a_{t})_{t \in S}$ is NSI for Lebesgue-a.e $a$ as desired (see Subsection 5.4).

If $S$ was finite, we could use the concept of Kellerer dilations as in [13] to show that for all $a \in [0,1]$ where $(\hat{\eta}^a_{t})_{t \in S}$ is well-defined there is only one martingale measure with these one-dimensional marginal distributions (cf. Remark 4.31). However, as shown in Example 9.4 this is not true if $S$ is infinite. This major difference between the case of a finite index set and a countable infinite one, is the reason why we have to develop new tools and techniques and cannot extend methods used in [13] and [45].

In Section 6 we establish Theorem 1.5 in the setting of a continuous time index set $T \subset [0, \infty)$ under the additional assumption that the given peacock is right-continuous. Martingale regularization techniques imply that martingale measures are uniquely determined by their behaviour on a countable index set. We will show in Subsection 6.2 that also the obstructed shadow and the NSI property are determined by the behaviour of the peacock $(\mu_t)_{t \in T}$ restricted to a well chosen countable index set $S \subset T$. This allows us to lift the results from Section 5 to the setting of $T \subset [0, \infty)$ with right-continuous peacock.

In Section 7 we show how we can pass to an abstract totally ordered index set without any assumptions on the peacock. In particular, this completes the proof of Theorem 1.6 (recall that it was stated for the totally ordered space $T = [0, \infty)$). Moreover, we explain how Corollary 1.1 follows from Theorem 1.5.

The proof of Theorem 1.6 is contained in Section 8.

Finally, in Section 9 we discuss counterexamples regarding shadows and NSI peacocks and provide explicit examples of shadow martingales.
2. Related Literature

The theory of optimal transport dates back to Monge (1781) and Kantorovich (1939) and has a huge variety of different facets and applications (see e.g. [47]). Martingale optimal transport is a relatively new part of this theory, that has for instance applications in robust mathematical finance (see e.g. [1] or the book [23]). Given two probability measures $\mu_0$ and $\mu_1$ with $\mu_0 \leq c \mu_1$ and a cost function $c$, the goal is to minimize (or maximize)

$$\pi \mapsto \mathbb{E}_\pi[c(X_0, X_1)]$$

over the set of couplings of $\mu_0$ and $\mu_1$ that additionally satisfy the martingale property.

Among the solutions of the problem (for different cost functions) are the couplings presented by Hobson and Neuberger [29], Hobson and Klimmek [28] and for other related problems the couplings recently introduced in [33] by Jourdain and Margheti and in [13] by Beiglböck and Juillet. Note that martingale optimal transport problems are a special case of a wider class of transport problems as weak optimal transport problems [20, 19] or linear transfers [16].

The left-curtain coupling introduced by Beiglböck and Juillet in [12] is of particular importance for our approach of the peacock problem. Besides being the unique minimizer for a certain class of cost functions (cf. [12]), the left-curtain coupling has several different characterizations, for instance concerning the geometry of its support [35] or in the context of the Skorokhod Embedding problem [10, 13]. Moreover, Hobson and Norgilas show in [30] that it possesses a natural interpretation in Mathematical Finance.

The concept of shadow – that is at the root of this article – is introduced and developed in the same paper [12] as the left-curtain coupling $\pi$. In fact, $\pi$ is introduced by

$$\pi(X_0 \leq a, X_1 \in \cdot) = S^\mu_1(\mu_0|(-\infty, a])$$

for every $a \in \mathbb{R}$. The shadow martingale w.r.t. the left-curtain parametrization is a natural extension of this coupling (and hence also of its discrete time extension by Nutz, Stebegg and Tan in [45]) to the continuous time case. Moreover, shadow martingales extend the concept of shadow couplings introduced in [13].

The name peacock (alias PCOC) which is derived from the French term Processus Croissant pour l’Ordre Convexe and likewise the peacock problem were introduced by Hirsch, Profeta, Roynette and Yor in their monograph [26] and are therefore quite recent. However, the construction of martingales that match given marginal distributions at least goes back to the seminal work of Kellerer [39]. Since then a variety of solutions have been developed before it was subsumed under the name peacock problem. Most of these solutions make more or less restrictive additional assumptions on the peacock, e.g. assuming that the peacock satisfies the (IMRV) property (see [13]) or consists of the marginal distributions of a solution to a certain class of SDE (see [21]). Moreover, the construction of fake Brownian motions (e.g. [2, 22]) can be seen as solutions to a (very specific) peacock problem. The monograph [26] provides a comprehensive overview of solutions to the peacock problem that work with special classes of peacocks.

Recently there were several contributions that face generic peacocks without a rich additional structure. There is the solution of Hobson [27] which is based on the Skorokhod Embedding Problem and the one of Lowther [42] who constructs for continuous peacocks with connected support a solution under which the canonical process is a strong Markov process. The approach closest to our class of solutions is the one parallelly studied by Henry-Labordère, Tan and Touzi [24] and Juillet [36]. The shadow
We denote this topology on $M_T$ topology on probability measures in $T$. If not difficult to see, that, if $(\nu^n)_{n\in\mathbb{N}}$ for all $c$ with $\partial_x \partial_y c < 0$ as $n$ tends to infinity for a suitable chosen sequence of nested finite partitions of $T$ whose mesh tends to zero. In contrast, the solution of $[24, 36]$ — when it exists — is constructed as the limit of the concatenation of the discrete time simultaneous minimizers of

$$\mathbb{E}_\pi[c(X_{t_{k-1}}, X_{t_k})] \quad \forall 1 \leq k \leq n.$$ 

for all $c$ with $\partial_x \partial_y c < 0$. Unsurprisingly, this solution behaves notably differently than the shadow martingale induced by the left-curtain parametrization (see Example 9.10).

Besides this article we are not aware of solutions to the peacock problem that are related to shadow martingales w.r.t. a parametrization which is not the left-curtain one. Similarly, there are no results about uniquely constructing martingales by solely describing how parts of the initial distribution evolve. In fact, the only approach in this direction that we are aware of is [15] but in a non-martingale setup.

3. Preliminaries

In this section we introduce our notation and recall objects and properties that are well known in the context of martingale optimal transport and the peacock problem. Since we want to work at the level of probability distributions (of processes), we sometimes choose a non-standard perspective on standard results.

3.1. Notation. We denote by $\mathcal{M}_0(X)$ (resp. $\mathcal{P}_0(X)$) the set of all finite measures (resp. probability measures) on some measurable space $X$. The underlying space will mostly be the space of functions from $T$ to $\mathbb{R}$, denoted by $\mathbb{R}^T$, for some totally ordered set $(T, \leq)$. In this case, the space $\mathbb{R}^T$ is equipped with the product topology and the corresponding Borel $\sigma$-algebra, the canonical process on $\mathbb{R}^T$ is denoted by $(X_t)_{t \in T}$, i.e. for all $t \in T$

$$X_t : \mathbb{R}^T \ni \omega \mapsto \omega(t) \in \mathbb{R},$$

and $(\mathcal{F}_t)_{t \in T}$ is its natural filtration defined by $\mathcal{F}_t = \sigma(X_s : s \leq t)$.

The set $\mathcal{M}_1(\mathbb{R}^T)$ consists of all $\pi \in \mathcal{M}_0(\mathbb{R}^T)$ for which all one-dimensional marginal distributions have a finite first moment. We equip $\mathcal{M}_1(\mathbb{R}^T)$ with the initial topology generated by the functionals $(I_f)_{f \in G_0 \cup G_1}$, where

$$I_f : \mathcal{M}_1(\mathbb{R}^T) \ni \pi \mapsto \int_{\mathbb{R}^T} f \, d\pi \in \mathbb{R}$$

and

$$G_0 = \{ g \circ (X_{t_1}, \ldots, X_{t_n}) : n \geq 1, \ t_1, \ldots, t_n \in T, \ g \in C_b(\mathbb{R}^n) \}$$
$$G_1 = \{ |X_t| : t \in T \}.$$

We denote this topology on $\mathcal{M}_1(\mathbb{R}^T)$ by $\mathcal{T}_1$. In contrast, we denote by $\mathcal{T}_0$ the initial topology on $\mathcal{M}_0(\mathbb{R}^T)$ that is generated by the functionals $(I_f)_{f \in G_0}$ only. The subspace of probability measures in $\mathcal{M}_1(\mathbb{R}^T)$ is denoted by $\mathcal{P}_1(\mathbb{R}^T)$ and equipped with the inherited topology. If $T$ is finite, the topology on $\mathcal{P}_1(\mathbb{R}^T)$ is induced by the 1-Wasserstein metric $\mathcal{W}_{1, l_1}$ corresponding to the $l_1$-metric on $\mathbb{R}^T$ (see Villani [47, Theorem 6.9]). It is also not difficult to see, that, if $T$ is countable, $\mathcal{M}_1(\mathbb{R}^T)$ is first countable and therefore continuity is equivalent to sequential continuity.
Note now that we can reduce $G_0$ in the definition of $T_0$ to the following set of functions

$$G_0' = \{ \psi \in \mathbb{R}^T \mapsto 1 \} \cup \{ g \circ (X_{t_1}, \ldots, X_{t_n}) : n \geq 1, \; t_1, \ldots, t_n \in T, \; g \in C_c(\mathbb{R}^n) \}$$

To see this, recall that a sequence $(\pi_n)_{n \in \mathbb{N}}$ converges to $\pi$ w.r.t. some initial topology $T$ defined by some set of functions $G$ if and only if $(I_f(\pi_n))_{n \in \mathbb{N}}$ converges to $I_f(\pi)$ in $\mathbb{R}$ for all $f \in G$. If $T$ is finite and $G$ contains $C_c(\mathbb{R}^T)$, the same limit is also satisfied for any continuous function $f$ that can be dominated by a linear combination generated with elements $f_i \in G$, i.e. such that

$$|f| \leq \sum_i \zeta_i f_i.$$ 

This is the reason why (i) the topology generated by $G_0'$ is exactly $T_0$ (also if $T$ is infinite), (ii) functions $g \circ (X_{t_1}, \ldots, X_{t_n})$ where $g$ grows at most linearly at infinity are admissible for $T_1$.

We denote the push-forward of a probability measure $\pi \in \mathcal{M}_1(\mathbb{R}^T)$ under some measurable map $f$ defined on $\mathbb{R}^T$ by $f_\#\pi$. If $\pi$ is a probability distribution, we refer to the push-forward as the law or distribution of $f$ under $\pi$ denoted by $\text{Law}_\pi(f)$. Furthermore, the expression “marginals of a probability measure $\pi$ on $\mathbb{R}^{T'}$” always refers to all the one-dimensional marginal distributions of the canonical process under $\pi$, i.e. to the measures $\text{Law}_\pi(X_t)$ for $t \in T$.

Let $S$ be a subset of $T$ and $\text{proj}^S : \mathbb{R}^T \to \mathbb{R}^S$ the projection on the index set $S$. The induced projection map

$$\mathcal{M}_1(\mathbb{R}^T) \ni \pi \mapsto (\text{proj}^S)_\# \pi \in \mathcal{M}_1(\mathbb{R}^S)$$

is continuous w.r.t. $T_1$ on $\mathcal{M}_1(\mathbb{R}^T)$ and $\mathcal{M}_1(\mathbb{R}^S)$. We denote the measure $\pi$ projected on the coordinates in $S$, i.e. $(\text{proj}^S)_\# \pi$, by the shorter notation $\pi|_S$ as if it were the restriction of a random vector.

Moreover, we denote the cumulative distribution function of a probability measure $\mu$ on $\mathbb{R}$ by $F_\mu$ and its quantile function is

$$F_\mu^{-1} : \alpha \in [0,1] \mapsto \inf \{ x \in \mathbb{R} : F_\mu(x) \geq \alpha \}.$$ 

We also denote by $\lambda$ the Lebesgue measure on $[0,1]$, by $\text{Unif}_I$ the uniform distribution on an interval $I \subset \mathbb{R}$ and by $\delta_x$ the Dirac measure at point $x$.

### 3.2. Order relations and potential functions

We use several partial order relations on $\mathcal{M}_1(\mathbb{R})$. They can all be introduced in a parallel way saying that $\mu \in \mathcal{M}_1(\mathbb{R})$ is smaller than or equal to $\mu' \in \mathcal{M}_1(\mathbb{R})$ if

$$\int_\mathbb{R} \varphi \, d\mu \leq \int_\mathbb{R} \varphi \, d\mu'$$

for every $\varphi$ in a certain positive cone of measurable test functions. These orders and the corresponding cones are:

- The positive order: $\mu \leq_+ \mu'$ if (3.1) holds for all non-negative $\varphi$.
- The convex order: $\mu \leq_c \mu'$ if (3.1) holds for all convex $\varphi$.
- The convex-positive order: $\mu \leq_{c,+} \mu'$ if (3.1) holds for all non-negative convex $\varphi$.
- The convex-stochastic order: $\mu \leq_{c,s} \mu'$ if (3.1) holds for all increasing convex $\varphi$.

Both non-negative and convex functions are bounded from below by an affine function and thus, since the first moments are finite, the integrals in (3.1) are well-defined with values in $(-\infty, \infty]$. Note that the positive order is well-defined for finite measures.
on any measurable space (e.g. $\mathbb{R}^T$). Moreover, recall from the introduction that we call $\pi$ a submeasure of $\pi'$ if $\pi \leq_+ \pi'$.

**Lemma 3.1.** Let $\mu$ and $\mu'$ be in $\mathcal{M}_1(\mathbb{R})$.

(i) If $\mu \leq_c \mu'$, then $\mu(\mathbb{R}) = \mu'(\mathbb{R})$ and $\int_{\mathbb{R}} y \, d\mu(y) = \int_{\mathbb{R}} y \, d\mu'(y)$.

(ii) If $\mu \leq_{c,+} \mu'$ and $\mu(\mathbb{R}) = \mu'(\mathbb{R})$, then $\mu \leq_c \mu'$.

(iii) If $\mu \leq_{c,s} \mu'$ and $\int_{\mathbb{R}} y \, d\mu(y) = \int_{\mathbb{R}} y \, d\mu'(y)$, then $\mu \leq_c \mu'$.

**Proof.** Item (i): The four functions $x \mapsto \pm 1$ and $x \mapsto \pm x$ are convex functions.

Item (ii): If $\mu \leq_{c,+} \mu'$ and $\mu(\mathbb{R}) = \mu'(\mathbb{R})$, equation (3.1) is satisfied for every non-negative convex function and also for $x \mapsto -1$. Thus (3.1) is satisfied for any convex function that is bounded from below. Hence, for a general convex function $\varphi$ with $\int_{\mathbb{R}} \varphi \, d\mu' < +\infty$, (3.1) holds for $\varphi_n = \varphi \vee (-n)$ and the monotone convergence theorem shows that (3.1) holds for $\varphi$ as well.

Item (iii): We use a similar argument adding $x \mapsto -x$ to the set of non-decreasing convex functions. Any convex function is the pointwise increasing limit of a sequence of convex functions with limit slope bounded at $-\infty$.

**Lemma 3.2.** Let $\mu_n, \mu'_n, \mu, \mu' \in \mathcal{M}_1(\mathbb{R})$ for all $n \in \mathbb{N}$.

(i) Suppose $(\mu_n)_{n \in \mathbb{N}}$ and $(\mu'_n)_{n \in \mathbb{N}}$ converge to $\mu$ and $\mu'$ under $\mathcal{T}_0$. If $\mu_n \leq_+ \mu'_n$ for all $n \in \mathbb{N}$, then $\mu \leq_+ \mu'$.

(ii) Suppose $(\mu_n)_{n \in \mathbb{N}}$ and $(\mu'_n)_{n \in \mathbb{N}}$ converge to $\mu$ and $\mu'$ under $\mathcal{T}_1$. For any order relation $\leq_c, \leq_{c,+}$ or $\leq_{c,s}$ represented by $\leq$, the relations $\mu_n \leq_\psi \mu'_n$ for all $n \in \mathbb{N}$ imply $\mu \leq_\psi \mu'$.

**Proof.** Item (i): It is clearly sufficient to test the positive order by indicator functions of closed intervals. For any such function $\varphi$ there exists a sequence $(\varphi_m)_{m \in \mathbb{N}}$ of continuous bounded functions such that $0 \leq \varphi_m \leq \varphi$ for all $m \in \mathbb{N}$ and $\int_{\mathbb{R}} \varphi \, d\mu = \lim_{m \to \infty} \int_{\mathbb{R}} \varphi_m \, d\mu$. Since convergence in $\mathcal{T}_0$ implies that $\int_{\mathbb{R}} \varphi_m \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} \varphi_m \, d\mu_n$ for all $m \in \mathbb{N}$, the claim follows.

Item (ii): For any (non-negative/increasing) convex function $\varphi \in L^1(\mu)$, there exists a sequence $(\varphi_m)_{m \in \mathbb{N}}$ of (non-negative/increasing) convex functions with bounded slope at $\pm \infty$ such that $\varphi_m \leq \varphi$ for all $m \in \mathbb{N}$ and $\int \varphi \, d\mu = \lim_{m \to \infty} \int \varphi_m \, d\mu$. Since the slope of $\varphi_m$ is bounded, there exist $a_m, b_m > 0$ such that $|\varphi_m(x)| \leq a_m|x| + b_m$ for all $x \in \mathbb{R}$ and hence the claim follows because the sequences converge w.r.t. $\mathcal{T}_1$ (cf. Subsection 3.1).

Note that convergence in $\mathcal{T}_0$ does in general not preserve the order relations $\leq_c, \leq_{c,+}$ and $\leq_{c,s}$. However, a sequence that is convergent under $\mathcal{T}_0$ and has a uniform upper bound in $\leq_{c,+}$, is in fact convergent under $\mathcal{T}_1$.

**Lemma 3.3.** Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_1(\mathbb{R})$. If there exists a measure $\theta \in \mathcal{M}_1(\mathbb{R})$ such that $\mu_n \leq_{c,+} \theta$ for all $n \in \mathbb{N}$, then the sequence $(\mu_n)_{n \in \mathbb{N}}$ is uniformly integrable, i.e.

$$\lim_{N \to \infty} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |x| 1_{[-N,N]} \, d\mu_n(x) = 0.$$ 

Hence, $(\mu_n)_{n \in \mathbb{N}}$ converges under $\mathcal{T}_0$ if and only if $(\mu_n)_{n \in \mathbb{N}}$ converges under $\mathcal{T}_1$. Moreover, if $(\mu_n)_{n \in \mathbb{N}}$ converges under $\mathcal{T}_0$ to $\mu \in \mathcal{M}_1(\mathbb{R})$, then it holds $\int f \, d\mu_n \to \int f \, d\mu$ for all continuous $f$ for which $|f|$ is dominated by some convex $\varphi \in L^1(\theta)$. 
Proof. We have \(|x|1_{[-N,N]}(x) \leq (2|x| - N)^+\) for all \(x \in \mathbb{R}\) and \(N \in \mathbb{N}\). Thus, we obtain
\[
\int_{\mathbb{R}} |x|1_{[-N,N]} \, d\mu_n(x) \leq \int_{\mathbb{R}} (2|x| - N)^+ \, d\theta
\]
and this upper bound converges to 0 as \(N\) tends to infinity by dominated convergence. Similarly, for any convex function \(\varphi \in L^1(\theta)\) and continuous \(f\) with \(|f| \leq \varphi\) it holds
\[
\int_{\mathbb{R}} |f|1_{[-N,N]} \, d\mu_n \leq \int_{\mathbb{R}} (\varphi + |x| - N)^+ \, d\theta
\]
for all \(N \in \mathbb{N}\). Hence, standard argumentation via the triangle inequality based on compact continuous functions coinciding with \(f\) on \([-N,N]\) yields that \(\int f \, d\mu_n \to \int f \, d\mu\) as \(n\) tends to infinity for all such \(f\). \(\square\)

Dealing with the convex order, potential functions are a very useful representation of finite measures on \(\mathbb{R}\). They are defined as follows:

**Definition 3.4.** Let \(\mu \in \mathcal{M}_1(\mathbb{R})\). The potential function of \(\mu\) is the function
\[
U(\mu) : \mathbb{R} \ni x \mapsto \int_{\mathbb{R}} |y - x| \, d\mu(y) \in \mathbb{R}^+.
\]

Since elements of \(\mathcal{M}_1(\mathbb{R})\) have finite first moments, the potential function is always well-defined. We collect a few important properties of potential functions below.

**Lemma 3.5** (cf. [12][Proposition 4.1]). Let \(m \in [0, \infty)\) and \(x^* \in \mathbb{R}\). For a function \(u : \mathbb{R} \to \mathbb{R}\) the following statements are equivalent:

(i) There exists a finite measure \(\mu \in \mathcal{M}_1(\mathbb{R})\) with mass \(\mu(\mathbb{R}) = m\) and barycenter \(x^* = \int_{\mathbb{R}} x \, d\mu(x)\) such that \(U(\mu) = u\).

(ii) The function \(u\) is non-negative, convex and satisfies
\[
\lim_{x \to \pm \infty} u(x) - m|x - x^*| = 0.
\]

Moreover, for all \(\mu, \mu' \in \mathcal{M}_1(\mathbb{R})\) we have \(\mu = \mu'\) if and only if \(U(\mu) = U(\mu')\).

Convex ordering and convergence in \(\mathcal{M}_1(\mathbb{R})\) can be well expressed in terms of potential functions.

**Lemma 3.6.** For all \(\mu, \mu'\) and sequences \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}_1(\mathbb{R})\) with \(\mu(\mathbb{R}) = \mu'(\mathbb{R}) = \mu_n(\mathbb{R})\) for all \(n \in \mathbb{N}\), we have the following properties:

(i) It holds \(\mu \leq_L \mu'\) if and only if \(U(\mu) \leq U(\mu')\).

(ii) It holds \(\mu_n \to \mu\) under \(\mathcal{T}_1\) if and only if \(U(\mu_n) \to U(\mu)\) pointwise.

Proof. Since for every \(x \in \mathbb{R}\) the function \(f_x : y \mapsto |y - x|\) is convex the direct implication of (i) is obvious. The reverse implication is part of the folklore (see e.g Exercise 1.7 of [20]). It can be proved as follows: let \(C\) be the cone of real functions \(f\) for which \(\int f \, d\mu \leq \int f \, d\mu'\). It includes the constants and also the functions \(f_x, x \in \mathbb{R}\). Considering both sequences \((f_{\pm n} - n)_{n \in \mathbb{N}}\), by the monotone convergence theorem we obtain \(\pm x \in C\). Hence \(C\) contains any piecewise (we mean with finitely many pieces) affine convex function. By the monotone convergence theorem again we see that every convex function is in \(C\).

Since \(f_x\) is affine close to \(\pm \infty\), the direct implication of (ii) is obvious. For the reverse implication, since all measures have the same finite mass and \(U(\mu_n)(0) \to U(\mu)(0)\) we have \(\int f \, d\mu_n \to_{n \to \infty} \int f \, d\mu\) for \(x \mapsto 1\) and \(x \mapsto |x|\). Therefore it suffices to establish the convergence for every continuous and compactly supported function \(f\). Notice that
the vectorial space spanned by the functions \( f_x \) and the constant functions includes the continuous and piecewise affine functions with compact support. Hence we can conclude by their density in \( C_c(\mathbb{R}) \) for the uniform norm. \( \square \)

Specified to families monotonously increasing in convex-stochastic order, the second part of the previous lemma yields the following result.

**Corollary 3.7.** Let \( T \subset \mathbb{R} \) and \( (\mu_t)_{t \in T} \) be a family in \( \mathcal{M}_1(\mathbb{R}) \) that is increasing in convex-stochastic order, i.e. \( \mu_s \leq_{c.s} \mu_t \) for all \( s \leq t \) in \( T \). There exists a countable set \( S \subset T \) such that \( t \mapsto \mu_t \) is a continuous map from \( T \setminus S \) to \( \mathcal{M}_1(\mathbb{R}) \) under \( T_1 \).

**Proof.** For all \( q \in \mathbb{Q} \) the function

\[ (3.3) \quad t \mapsto U(\mu_t)(q) = \int_\mathbb{R} |y-q| \, d\mu_t(y) = 2 \int_\mathbb{R} (y-q)^+ \, d\mu_t(y) - \int_\mathbb{R} (y-q) \, d\mu_t(y) \]

is continuous except on a countable set \( S_q \) because it is the difference of two functions that are monotonously increasing in \( t \). Set \( S = \bigcup_{q \in \mathbb{Q}} S_q \). Observe that \( \bar{u}_t := \lim_{t \downarrow} U(\mu_s) \) is a well defined convex function as a pointwise limit of convex functions. This limit exists by monotonicity in \( t \) of the integrals in (3.3). Also \( u_t(x) := U(\mu_t)(x) \) is a convex function and we get \( \bar{u}_t = u_t \) on \( \mathbb{Q} \) for all \( t \not\in S \). Since both \( \bar{u}_t \) and \( u_t \) are continuous as convex functions this equality extends to \( \mathbb{R} \). Similarly, it holds \( \lim_{t \uparrow} U(\mu_r)(x) = U(\mu_t)(x) \) for all \( x \in \mathbb{R} \) and \( t \not\in S \). Thus, the map \( t \mapsto U(\mu_t)(x) \) is continuous for every \( x \in \mathbb{R} \) at any time \( t \not\in S \). This transfers to the continuity of \( t \mapsto \mu_t \) outside of \( S \) by Lemma 3.6 (ii). \( \square \)

Despite monotonicity, a family increasing in convex-stochastic order does in general not admit left- and right-limits everywhere under \( T_1 \). For instance, the family \( (\mu_t)_{t \in [0,1]} \) with

\[ \mu_t = \frac{1 - t^2}{2} \delta_{1-t} + \frac{1}{2 - t^2} \delta_{1+t} \quad \mu_1 = \delta_2 \]

does not have a left-limit at 1.

### 3.3. Infimum and supremum in convex order.

**Definition 3.8.** Let \( \mathcal{A} \) be a set of measures in \( \mathcal{M}_1(\mathbb{R}) \). If \( \mathcal{A} \) possesses a smallest upper bound w.r.t. convex order, we call it the convex supremum of \( \mathcal{A} \) and denote it by \( \text{Csup} \mathcal{A} \). It is then the unique measure \( \zeta \) such that

(i) \( \mu \leq_{c} \zeta \) for all \( \mu \in \mathcal{A} \)

(ii) \( \zeta \leq_{c} \zeta' \) for all \( \zeta' \) that satisfy (i).

Similarly, we define \( \text{Cinf} \mathcal{A} \) as the convex infimum, if it exists.

**Proposition 3.9.** Let \( \mathcal{A} \) be a non-empty subset of \( \mathcal{M}_1(\mathbb{R}) \) such that all measures in \( \mathcal{A} \) have the same mass and the same barycenter.

(i) The convex infimum \( \text{Cinf} \mathcal{A} \) exists.

(ii) If there exists some \( \theta \in \mathcal{M}_1(\mathbb{R}) \) such that \( \mu \leq_{c,+} \theta \) holds for all \( \mu \in \mathcal{A} \), then the convex supremum \( \text{Csup} \mathcal{A} \) exists.

Moreover, their potential functions satisfy

\[ U(\text{Cinf} \mathcal{A}) = \text{conv} \left( \inf_{\mu \in \mathcal{A}} U(\mu) \right) \quad \text{and} \quad U(\text{Csup} \mathcal{A}) = \sup_{\mu \in \mathcal{A}} U(\mu). \]

where \( \text{conv}(f) \) denotes the convex hull of a function \( f \), i.e. the largest convex function that is pointwise smaller than \( f \).
Proof. Item (i): Since the measures of $A$ all have the same mass $m$ and barycenter $x$ and $U(m\delta_x)$ is convex, the set $\{g : g \text{ convex, } g \leq \inf_{\mu \in A} U(\mu)\}$ is not empty. Let $u$ be defined by $u(x) = \sup \{g(x) : g \text{ convex, } g \leq \inf_{\mu \in A} U(\mu)\}$. It is convex as the pointwise supremum of convex functions and $U(m\delta_x) \leq u \leq U(\mu)$ for any fixed $\mu \in A$. Hence $u$ possesses the right behavior at $\pm \infty$ in the sense of Lemma 3.5. Therefore $u$ is a potential function and the corresponding measure satisfies the properties of a convex infimum by Lemma 3.6 (i).

Item (ii): According to [12, Lemma 4.5] applied to $m\delta_x$ and $\theta$ the ordering $m\delta_x \leq_{c,+} \theta$ implies that there exists a $\theta' \in \mathcal{M}_1(\mathbb{R})$ with $m\delta_x \leq c \theta'$ such that for all $\eta \in \mathcal{M}_1(\mathbb{R})$ with $m\delta_x \leq \eta \leq_{c,+} \theta$ there holds $\eta \leq c \theta'$ ($\theta'$ is a restriction of $\theta$ in a proper neighborhood of $\pm \infty$, up to atoms).

In particular, we obtain $U(m\delta_x) \leq U(\mu) \leq U(\theta')$ for every $\mu \in A$ and therefore the convex function $u = \sup_{\mu \in A} U(\mu)$ satisfies $U(m\delta_x) \leq u \leq U(\theta')$. Thus, $u$ has the right behavior at $\pm \infty$ in the sense of Lemma 3.5. Hence, by Lemma 3.5 $u$ is a potential function and the corresponding measure satisfies the properties of a convex supremum (see Lemma 3.5 (i)). □

Remark 3.10. The assumption that all measures in $A$ have the same mass and barycenter is equivalent to the assumption that $A$ has a lower bound w.r.t. the convex order. Hence, we don’t need an additional lower bound in (i).

Lemma 3.11. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_1(\mathbb{R})$.

(i) If $\mu_m \leq \mu_n$ for all $n \leq m$ in $\mathbb{N}$, then $(\mu_n)_{n \in \mathbb{N}}$ converges to $\text{CInf}\{\mu_n : n \in \mathbb{N}\}$ under $\mathcal{T}_1$.

(ii) If $\mu_n \leq \mu_m \leq_{c,+} \theta$ for all $n \leq m$ in $\mathbb{N}$ and some $\theta \in \mathcal{M}_1(\mathbb{R})$, then $(\mu_n)_{n \in \mathbb{N}}$ converges to $\text{CSup}\{\mu_n : n \in \mathbb{N}\}$ under $\mathcal{T}_1$.

(iii) If $(\mu'_n)_{n \in \mathbb{N}}$ is another sequence in $\mathcal{M}_1(\mathbb{R})$ and both sequences are increasing in convex order and are uniformly bounded from above in convex-positive order, then

$$\text{CSup}\{\mu_n + \mu'_n : n \in \mathbb{N}\} = \text{CSup}\{\mu_n : n \in \mathbb{N}\} + \text{CSup}\{\mu'_n : n \in \mathbb{N}\}.$$  

Proof. With Proposition 3.9 and Lemma 3.6 we can rewrite this statement in terms of sequences of real functions and then the statement is well known. □

Lemma 3.12. Let $A$ be a non-empty subset of $\mathcal{M}_1(\mathbb{R})$ such that all measures in $A$ have the same mass, the same barycenter and are dominated by some $\theta \in \mathcal{M}_1(\mathbb{R})$ in convex-positive order. If additionally for all $\mu_1, \mu_2 \in A$ there exists some $\mu' \in A$ such that $\mu_1 \leq \mu' \leq \mu_2$, then there exists an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $A$ that converges to $\text{CSup} A$ under $\mathcal{T}_1$.

Proof. The potential function of $\text{CSup} A$ is given by $u = \sup_{\mu \in A} U(\mu)$. For any $q \in \mathbb{Q}$ there exists a sequence $(\nu^q_k)_{k \in \mathbb{N}}$ of measures in $A$ such that for the corresponding potential functions $u_k^q = U(\nu_k^q)$ the sequence $(u_k^q(q))_{k \in \mathbb{N}}$ converges to $u(q)$.

Let $(\eta_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$, set $\mu_1 = \nu_1^{\eta_1}$ and choose a $\mu_n \in A$ that is an upper bound in convex order to the finite set

$$\{\mu_{n-1}\} \cup \{\nu_k^q : 1 \leq k, l \leq n\}$$

which is possible by assumption. Thereby, we get an increasing sequence in $A$ that satisfies $\lim_{n \to \infty} U(\mu_n)(q) = u(q)$ for all $q \in \mathbb{Q}$. Since $(\mu_n)_{n \in \mathbb{N}}$ is increasing in convex order, $\lim_{n \to \infty} U(\mu_n)(x) = \sup_{n \in \mathbb{N}} U(\mu_n)(x)$ for all $x \in \mathbb{R}$. Thus, $\sup_{n \in \mathbb{N}} U(\mu_n)$ and $u$ are convex functions that agree on $\mathbb{Q}$ and, hence, on $\mathbb{R}$. Hence, $\lim_{n \to \infty} U(\mu_n)(x) =$
3.4. Peacocks and Kellerer’s Theorem. In this section we introduce notation regarding peacocks and martingale measures.

We fix a totally ordered index set $(T, \leq)$. As already indicated in Subsection $3.1$, we are not working on the level of processes but with their distributions on the state space $\mathbb{R}^T$. However, we would like to introduce the martingale property and the Markov property that are typically formulated for processes indexed by $T$ and not probability measures on $\mathbb{R}^T$.

**Definition 3.13.** Let $\pi \in \mathcal{P}_1(\mathbb{R}^T)$.

(i) We call $\pi$ a martingale measure if the canonical process $(X_t)_{t \in T}$ is a martingale w.r.t. its natural filtration under $\pi$, i.e. if

$$E_{\pi} \left[ X_t \mid \mathcal{F}_s \right] = X_s \quad \pi\text{-a.e.}$$

for all $s \leq t$ in $T$. The set of all martingale measures on $\mathbb{R}^T$ is denoted by $M_T$.

(ii) The probability measure $\pi$ is said to be Markov if the canonical process $(X_t)_{t \in T}$ is a Markov process under $\pi$, i.e. if

$$E_{\pi} \left[ 1_A(X_t) \mid \mathcal{F}_s \right] = E_{\pi} \left[ 1_A(X_t) \mid X_s \right] \quad \pi\text{-a.e.}$$

for all Borel sets $A \subset \mathbb{R}$ and $s < t$ in $T$.

We equip $M_T$ with the topology inherited from $T_1$ on $\mathcal{P}_1(\mathbb{R}^T)$ and the corresponding Borel $\sigma$-algebra. All subsets of $M_T$ are equipped with the subspace topology and subspace $\sigma$-algebra.

By Jensen’s inequality, the marginal distributions $(\text{Law}_\pi(X_t))_{t \in T}$ of a martingale measure $\pi$ form a family in $\mathcal{P}_1(\mathbb{R})$ that is increasing in convex order. Recall from the introduction that those families are called peacocks:

**Definition 3.14.** We call a family $(\mu_t)_{t \in T}$ in $\mathcal{P}_1(\mathbb{R})$ a peacock, if $\mu_s \leq_c \mu_t$ for all $s \leq t$ and we denote by $P_T$ the set of all peacocks indexed by $T$. Moreover, we say that a martingale measure $\pi$ is associated with a peacock $(\mu_t)_{t \in T}$ if $\text{Law}_\pi(X_t) = \mu_t$ for all $t \in T$.

Since all elements of a family of finite measures increasing in convex order have the same mass (not always $1$) by Lemma $3.1$ (i), they can therefore easily be rescaled to become peacocks. We equip $P_T$ with the inherited product topology on $\mathcal{P}_1(\mathbb{R})^T$, where each factor $\mathcal{P}_1(\mathbb{R})$ is equipped with $T_1$. The corresponding Borel $\sigma$-algebra is the product $\sigma$-algebra.

**Definition 3.15.** Let $S \subset T$ and $(\mu_t)_{t \in S}$ be a family in $\mathcal{P}_1(\mathbb{R})$. By $M_T((\mu_t)_{t \in S})$ we denote the set of all martingale measures $\pi \in M_T$ satisfying $\text{Law}_\pi(X_t) = \mu_t$ for all $t \in S$.

Thanks to the following result we know precisely when $M_T((\mu_t)_{t \in T})$ is not empty.

**Proposition 3.16** (Kellerer Theorem $[39, 40]$). Let $(\mu_t)_{t \in T}$ be a family in $\mathcal{P}_1(\mathbb{R})$. The following are equivalent:

(i) The family $(\mu_t)_{t \in T}$ is a peacock.

---

$^2$Recall that the Borel $\sigma$-algebra of the subspace topology coincides with the subspace $\sigma$-algebra of the Borel $\sigma$-algebra corresponding to the topology on the ambient space. 

---

$u(x)$ for all $x \in \mathbb{R}$ and we can apply Lemma $3.6$ (ii) to conclude that $(\mu_n)_{n \in \mathbb{N}}$ converges to $C_{\text{sup}}A$ under $T_1$. □
(ii) There exists a martingale measure $\pi \in M_T((\mu_t)_{t \in T})$ which can moreover be chosen to be Markov.

The existence of solutions to the peacock problem is also true for martingales on $\mathbb{R}^d$ with $d \geq 2$ (cf. [26]) but it is still an open problem whether in this case the martingale can be chosen to be Markov. An extension to partially ordered sets of indices is possible but only in certain cases (cf. [34]).

4. Parametrizations, shadows, and NSI

The goal of this section is to introduce the three concepts that are crucial on the one hand for the construction of the shadow martingales, namely parametrizations and obstructed shadows, and on the other hand for the uniqueness of the shadow martingale measure, namely the NSI property, cf. Subsection 1.3.

Throughout this section we fix a totally ordered set $(T, \leq)$.

4.1. Parametrizations.

**Definition 4.1.** Let $X$ be a measurable space and $\mu \in \mathcal{P}_0(X)$. A family $(\mu^\alpha)_{\alpha \in [0,1]}$ in $\mathcal{M}_0(X)$ is called a parametrization of $\mu$ if

1. $\mu^\alpha(X) = \alpha$ for all $\alpha \in [0,1]$,
2. $\mu^\alpha \leq + \mu^\alpha'$ for all $\alpha \leq \alpha'$ in $[0,1]$ and
3. $\mu^1 = \mu$.

Each parametrization of a probability measure $\mu$ can be seen as an explicit coupling of $\mu$ with a uniformly distributed random variable on $[0,1]$ that is added to the probability space. Recall, that $\lambda$ denotes the Lebesgue measure on $[0,1]$.

**Remark 4.2.** Let $X$ be a measurable space, $\mu, \nu \in \mathcal{P}_0(X)$ and $(\mu^\alpha)_{\alpha \in [0,1]}$ a family of finite measures on $X$. The following are equivalent:

1. The family $(\mu^\alpha)_{\alpha \in [0,1]}$ is a parametrization of $\mu$.
2. There exists a coupling $\xi$ of $\lambda$ and $\mu$ with $\xi([0,\alpha] \times B) = \mu^\alpha(B)$ for all $\alpha \in [0,1]$ and measurable sets $B \subset E$.

Clearly, the coupling $\xi$ is uniquely determined by $(\mu^\alpha)_{\alpha \in [0,1]}$ and vice versa.

**Lemma 4.3.** Let $X$ be a measurable space, $\mu, \nu \in \mathcal{P}_0(X)$, $(\mu^\alpha)_{\alpha \in [0,1]}$ a parametrization of $\mu$ and $(\nu^\alpha)_{\alpha \in [0,1]}$ a parametrization of $\nu$. If $\mu^\alpha = \nu^\alpha$ for all $\alpha$ in a dense subset $A$ of $[0,1]$, then $\mu^\alpha = \nu^\alpha$ for all $\alpha \in [0,1]$ and, in particular, $\mu = \nu$.

**Proof.** Let $\alpha \in (0,1)$ and $(\alpha_n)_{n \in \mathbb{N}}$ a sequence in $A$ with $\alpha_n \uparrow \alpha$. Remark 4.2 yields that $\mu^\alpha(B) = \lim_{n \to \infty} \mu^{\alpha_n}(B)$ and $\nu^\alpha(B) = \lim_{n \to \infty} \nu^{\alpha_n}(B)$ for all measurable sets $B \subset X$. □

Given a specific peacock, the degree of freedom in our construction (Theorem 1.5) is the choice of a parametrization of the initial marginal. Hence, our primary motivation to consider general (non-interval based) parametrizations of probability measures is to enlarge the set of possible input choices. For instance, an initial distribution that contains atoms cannot satisfy condition (i) in Corollary 1.1. The concept of parametrizations allows us to break these atoms into a continuum of quantiles.
4.1.1. Convex parametrizations.

**Definition 4.4.** Let \( \mu \) be in \( \mathcal{P}_1(\mathbb{R}) \). A parametrization \( (\nu^\alpha)_{\alpha \in [0,1]} \) of \( \mu \) is said to be \( \leq_{c,s} \)-convex if

\[
\frac{\nu^{\alpha_2} - \nu^{\alpha_1}}{\alpha_2 - \alpha_1} \leq_{c,s} \frac{\nu^{\alpha_3} - \nu^{\alpha_2}}{\alpha_3 - \alpha_2}
\]

for all \( \alpha_1 < \alpha_2 < \alpha_3 \) in [0, 1].

Since both sides of inequality (4.1) can be interpreted as the slopes of secant lines of \( \alpha \mapsto \nu^\alpha \) on \([\alpha_1, \alpha_2]\) and \([\alpha_2, \alpha_3]\), the inequality yields that \( \alpha \mapsto \nu^\alpha \) is convex in this sense. Moreover, property (4.1) is equivalent to \( \alpha \mapsto \int \varphi \nu^\alpha \) being convex for all increasing convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \).

**Lemma 4.5.** Let \( \mu \in \mathcal{P}_1(\mathbb{R}) \) and \( (\nu^\alpha)_{\alpha \in [0,1]} \) be a parametrization of \( \mu \). If there exists a sequence of nested intervals \( (I_\alpha)_{\alpha \in [0,1]} \) in \( \mathbb{R} \) such that

(i) \( \sup I_\alpha < +\infty \) and \( \text{supp}(\nu^\alpha) \subset I_\alpha \) for all \( \alpha \in [0, 1] \),

(ii) \( \text{supp}(\nu^{\alpha_2} - \nu^{\alpha_1}) \subset I_{\alpha_1} \) for all \( \alpha_1 < \alpha_2 \) in [0, 1] and

(iii) \( \alpha \mapsto \int y \, d\nu^\alpha(y) \) is convex,

then the parametrization \( (\nu^\alpha)_{\alpha \in [0,1]} \) is \( \leq_{c,s} \)-convex.

**Proof.** For all \( \alpha_1 < \alpha_2 < \alpha_3 \) in [0, 1] the measure
\[
\bar{\nu}_{1,2} := \frac{\nu^{\alpha_2} - \nu^{\alpha_1}}{\alpha_2 - \alpha_1}
\]
is concentrated on \( \overline{I_{\alpha_2}} \) by (i) and
\[
\bar{\nu}_{2,3} := \frac{\nu^{\alpha_3} - \nu^{\alpha_2}}{\alpha_3 - \alpha_2}
\]
is concentrated on the closure of the complement \( \overline{I_{\alpha_2}} \) by (ii). Moreover, both of these measures are probability measures and their barycenters satisfy
\[
\int_{\mathbb{R}} y \, d\bar{\nu}_{1,2}(y) \leq \int_{\mathbb{R}} y \, d\bar{\nu}_{2,3}(y)
\]
because \( \alpha \mapsto \int_{\mathbb{R}} y \, d\nu^\alpha(y) \) is convex by property (iii). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a convex increasing function. Since \( I_{\alpha_2} \) is bounded from above, there exists an increasing affine function \( l(y) = ay + b \) with \( \varphi \leq l \) on \( \overline{I_{\alpha_2}} \) and \( \varphi \geq l \) on \( \overline{I_{\alpha_3}} \). Thus, we obtain
\[
\int_{\mathbb{R}} \varphi \, d\bar{\nu}_{1,2} \leq \int_{\mathbb{R}} l \, d\bar{\nu}_{1,2} = a \int_{\mathbb{R}} y \, d\bar{\nu}_{1,2}(y) + b \leq a \int_{\mathbb{R}} y \, d\bar{\nu}_{2,3}(y) + b = \int_{\mathbb{R}} l \, d\bar{\nu}_{2,3} \leq \int_{\mathbb{R}} \varphi \, d\bar{\nu}_{2,3}
\]
because \( a \geq 0 \).

Note that this Lemma includes the setting of Corollary 4.1. Figure 4.1.1 illustrates the following three \( \leq_{c,s} \)-convex parametrizations:

**Lemma 4.6.** The following parametrizations of \( \mu \in \mathcal{P}_1(\mathbb{R}) \) are \( \leq_{c,s} \)-convex:

(i) The left-curtain parametrization \( (\nu^\alpha)_{\alpha \in [0,1]} \) with
\[
\nu^\alpha = \mu([-\infty, F_\mu^{-1}(\alpha)] + (\alpha - \mu([-\infty, F_\mu^{-1}(\alpha)])) \delta_{F_\mu^{-1}(\alpha)}.
\]
Proof. For item (iii), we have Definition 4.7. of \( \pi \) and \( \alpha \) for all \( \alpha \).

Remark 4.8. It is not difficult to prove that for any \( \alpha \in [0, 1] \), for which there exists a Borel set \( A \subseteq \mathbb{R} \) with \( \nu^\alpha = (\mu_0)_A \), it holds \( \pi^\alpha = \alpha \text{Law}_\pi(X|X_0 \in A) \).

(ii) The middle-curtain parametrization \((\nu^\alpha_{mc})_{\alpha \in [0, 1]}\) with

\[
\nu^\alpha_{mc} = \mu(\{q_\alpha, q_\alpha'\}) + c_\alpha \delta_{q_\alpha} + c'_\alpha \delta_{q_\alpha'}
\]

for \( q_\alpha \leq q_\alpha' \) in \( \mathbb{R} \) and \( c_\alpha, c'_\alpha \in [0, 1] \) such that \( \nu^\alpha_{mc}(\mathbb{R}) = \alpha \) and \( \int y \, d\nu^\alpha_{mc} = \int y \, d\mu \).

(iii) The sunset parametrization \((\nu^\alpha_{sun})_{\alpha \in [0, 1]}\) with \( \nu^1_{sun} = 0 \).

Proof. For item (iii), we have \( \frac{1}{\alpha_2 - \alpha_1}(\nu_{\alpha_2} - \nu_{\alpha_1}) = \mu \) for all \( \alpha_1 < \alpha_2 \) in \([0, 1]\).

For item (i) and item (ii) we can apply Lemma 4.5 because both \( \alpha \mapsto \int_{\mathbb{R}} y \, d\nu^\alpha_{lc}(y) \) and \( \alpha \mapsto \int_{\mathbb{R}} y \, d\nu^\alpha_{mc}(y) \) are convex functions. Indeed, it holds

\[
\frac{1}{\alpha_2 - \alpha_1} \left( \int_{\mathbb{R}} y \, d\nu^\alpha_{lc} - \int_{\mathbb{R}} y \, d\nu^\alpha_{lc} \right) \leq F_{\mu}^{-1}(\alpha_2) \leq \frac{1}{\alpha_3 - \alpha_2} \left( \int_{\mathbb{R}} y \, d\nu^\alpha_{lc} - \int_{\mathbb{R}} y \, d\nu^\alpha_{lc} \right)
\]

for all \( \alpha_1 < \alpha_2 < \alpha_3 \) in \([0, 1]\) and \( \alpha \mapsto \int_{\mathbb{R}} y \, d\nu^\alpha_{mc}(y) = \int_{\mathbb{R}} y \, d\mu \) is constant.

4.1.2. Probability measures on \( \mathbb{R}^T \). To be able to describe the evolution of a submeasure of the initial measure under a measure \( \pi \in \mathcal{M}_T \) we need to consider parametrizations of \( \pi \) as well.

Definition 4.7. Let \( \pi \in \mathcal{P}(\mathbb{R}^T) \).

(i) Let \((\nu^\alpha)_{\alpha \in [0, 1]}\) be a parametrization of the initial marginal Law\( \pi(X_0) \). A family \((\pi^\alpha)_{\alpha \in [0, 1]}\) in \( \mathcal{M}(\mathbb{R}^T) \) is called a parametrization of \( \pi \) w.r.t. \((\nu^\alpha)_{\alpha \in [0, 1]}\) if \((\pi^\alpha)_{\alpha \in [0, 1]}\) is a parametrization of \( \pi \) with \( \pi^\alpha(X_0 = \cdot) = \nu^\alpha \) for all \( \alpha \in [0, 1] \).

(ii) A parametrization \((\pi^\alpha)_{\alpha \in [0, 1]}\) of \( \pi \) is called a martingale parametrization of \( \pi \), if \( \frac{1}{\alpha} \pi^\alpha \in \mathcal{M}_T \) for all \( \alpha \in (0, 1] \).

Remark 4.8. Let \( \pi \) be in \( \mathcal{P}(\mathbb{R}^T) \) and \((\nu^\alpha)_{\alpha \in [0, 1]}\) be a parametrization of the initial marginal Law\( \pi(X_0) \). Moreover, let \((\pi^\alpha)_{\alpha \in [0, 1]}\) be a parametrization of \( \pi \) w.r.t. \((\nu^\alpha)_{\alpha \in [0, 1]}\). It is not difficult to prove that for any \( \alpha \in [0, 1] \), for which there exists a Borel set \( A \subseteq \mathbb{R} \) with \( \nu^\alpha = (\mu_0)_A \), it holds \( \pi^\alpha = \alpha \text{Law}_\pi(X|X_0 \in A) \).
Remark 4.8 suggests that we can interpret $\pi^\alpha$ as the way $\nu^\alpha$ is transported under $\pi$, i.e. we can see $\pi^\alpha(X_t \in \cdot)$ as a formal version of $\alpha\text{Law}_{\pi}(X_t|X_0 \in \nu^\alpha)$. However, one has to be careful with this informal notation because contrarily to $\alpha\text{Law}_{\pi}(X|X_0 \in A)$ that is uniquely defined, there can be several parametrizations $(\pi^\alpha)_{\alpha \in [0,1]}$ of the same measure $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$, each giving another meaning to $\alpha\text{Law}_{\pi}(X_t|X_0 \in \nu^\alpha)$. This is illustrated in Example 4.9 just below. Hence, the correct interpretation of the existence of a parametrization $(\pi^\alpha)_{\alpha \in [0,1]}$ of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ is that $\nu^\alpha = \text{Law}_{\pi^\alpha}(X_0)$ can be transported according to $\pi^\alpha$ as part of the dynamic given by $\pi$.

Example 4.9. Let $(\mu_t)_{t \geq 0}$ be a peacock, $(\nu^\alpha_{\text{sun}})_{\alpha \in [0,1]}$ be the sunset parametrization of $\mu_0$ and let $\pi \in \mathcal{P}_1([0,\infty))$ be associated with $(\mu_t)_{t \geq 0}$. Let $(\pi^\alpha)_{\alpha \in [0,1]}$ be a (martingale) parametrization of $\pi$ w.r.t. $(\nu^\alpha_{\text{sun}})_{\alpha \in [0,1]}$. For $\alpha \in [0,1]$, set $\rho^\alpha = \pi - \pi^{1-\alpha}$. Assume that there is $\bar{\alpha} \in (0,1)$ such that $\pi^{\bar{\alpha}} \neq \rho^{\bar{\alpha}}$. Then, the family $(\rho^\alpha)_{\alpha \in [0,1]}$ is not NSI and different from $(\pi^\alpha)_{\alpha \in [0,1]}$. For a concrete example one can choose $(\mu_t)_{t \geq 0}$, $\pi$ and $(\pi^\alpha)_{\alpha \in [0,1]}$ as in Example 4.11.

Remark 4.10. In the last example the assumption that there is a martingale parametrization satisfying $\pi^{\bar{\alpha}} \neq \rho^{\bar{\alpha}}$ for some $\bar{\alpha} \in (0,1)$ is always satisfied as soon as the peacock is not NSI (see § 4.3. NSI peacocks are extremal elements in the set of peacocks with fixed initial marginal so that they are in a certain sense rare (see Lemma 7.28).

4.2. Shadows. The concept of the shadow of a measure $\nu$ through a family of finite measures is at the center of our construction (cf. Section 1.3). After recalling previous results of Beiglböck and Juillet [12] and Nutz, Stebegg and Tan [45] for simple and finitely obstructed shadows, we establish in Proposition 4.20 the existence of an obstructed shadow in the generality required for our setup.

4.2.1. The simple shadows. We start by recalling the original concept of (simple) shadows developed in [12]. Given two finite measures $\nu$ and $\mu$ on $\mathbb{R}$, the shadow of $\nu$ in $\mu$ is defined as the minimum in convex order among all submeasure of $\mu$ that are in convex order larger than $\nu$. More precisely:

Proposition 4.11 (cf. [12] Lemma 4.6). Let $\nu, \mu \in \mathcal{M}_1(\mathbb{R})$ satisfying $\nu \leq_{c,+} \mu$. There exists a unique finite measure $\eta$ such that

1. $\nu \leq_c \eta$,
2. $\eta \leq_{c,+} \mu$ and
3. for all $\eta' \in \mathcal{M}_1(\mathbb{R})$ with $\nu \leq_c \eta' \leq_{c,+} \mu$ it holds $\eta \leq_c \eta'$.

The measure $\eta$ is denoted by $S^\mu(\nu)$ and called the shadow of $\nu$ in $\mu$.

For a detailed proof we refer to [12]. We only stress that the proof is based on potential functions and the potential function of the shadow has an explicit expression in terms of the potential functions of $\nu$ and $\mu$ stated in the following lemma:

Lemma 4.12. Let $\nu, \mu \in \mathcal{M}_1(\mathbb{R})$ with $\nu \leq_{c,+} \mu$. It holds

$$U(S^\mu(\nu)) = U(\mu) - \text{conv}(U(\mu) - U(\nu))$$

where $\text{conv}(f)$ denotes the convex hull of a function $f$, i.e. the largest convex function that is pointwise smaller than $f$.

Proof. This formula has been brought to our attention by Mathias Beiglböck and can be easily derived from the proof of [12] Lemma 4.6. □
Figure 3. The striped red area on the left represents the shadow of $\nu_1 = \delta_0$ in $\mu_1 = 2 \text{Unif}[-1,1]$ and the shaded area on the right illustrates the shadow of $\nu_2 = \text{Unif}[\frac{-1}{2},\frac{1}{2}]$ in $\mu_2 = \text{Unif}[-2,-1] + \text{Unif}[1,2]$.

Corollary 4.13. For all $\nu, \mu, \mu' \in M_1(\mathbb{R})$ with $\nu \leq_c \mu \leq_c \mu'$ it holds

$$U(S^\mu(\nu)) - U(S^\mu(\nu)) \leq U(\mu') - U(\mu).$$

Proof. This follows from Lemma 4.12 because $\text{conv}(U(\mu') - U(\nu)) \geq \text{conv}(U(\mu) - U(\nu))$ since $\mu \leq_c \mu'$.

The most intuitive example of a shadow is the shadow of an atom.

Example 4.14. Let $\nu \leq_{c,+} \mu$ in $M_1(\mathbb{R})$ such that $\mu$ is atomless. If

(i) $\nu = \alpha \delta_x$ for some $\alpha \geq 0$ and $x \in \mathbb{R}$ or
(ii) there exists an interval $I \subset \mathbb{R}$ with $\text{supp}(\nu) \subset I$ and $\text{supp}(\mu) \subset I^c$,

then there exists an interval $J$ such that $S^\mu(\nu) = \mu_j$.

See [12, Example 4.7] for the proof in case (i). For item (ii) consider the shadow of $\alpha \delta_x$ in $\mu$ as in (i) where $\alpha, x$ are the mass and the barycenter of $\nu$, respectively. Since $\text{supp}(\nu) \subset I$, it holds $\nu \leq_c S^\mu(\alpha \delta_x)$ (see [12, Example 4.2]) and thus $S^\mu(\nu) = S^\mu(\alpha \delta_x) = \mu_j$ for some interval $J$.

For measures $\mu$ that posses atoms these examples can easily be adapted adding to $\mu_j$ one or two atomic masses at the end points of the interval $J$.

The calculation rule (ii) in Proposition 4.15 below in conjunction with Example 4.14 allow to explicitly compute the shadow in many examples: If the measures $\nu$ and $\mu$ are linear combinations of atoms and of finite measures with piecewise linear densities w.r.t. the Lebesgue measure on $\mathbb{R}$, one can iteratively compute the shadow $S^\mu(\nu)$ by solving in each iteration systems of two cubic equations in two variables. Hence, up to numerical approximation, the computation is also possible if the measures have a continuous density w.r.t. the Lebesgue measure.

Proposition 4.15. Let $\nu \leq_{c,+} \mu$ in $M_1(\mathbb{R})$.

(i) For all $\alpha > 0$ it holds $\alpha \nu \leq_{c,+} \alpha \mu$ and $S^{\alpha \mu}(\alpha \nu) = \alpha S^\mu(\nu)$.
(ii) For all $\nu_1 + \nu_2 = \nu$ we have $\nu_2 \leq_{c,+} \mu - S^\mu(\nu_1)$ and $S^\mu(\nu) = S^\mu(\nu_1) + S^{\mu - S^\mu(\nu_1)}(\nu_2)$.

Proof. Item (i) is clear by construction of the shadow and Item (ii) is [12, Theorem 4.8].

Lemma 4.16. Let $\nu, \nu' \leq_{c,+} \mu$. 


(i) If \( \nu \leq_c \nu' \), then \( S^\mu(\nu) \leq_c S^\mu(\nu') \).
(ii) If \( \nu \leq_* \nu' \), then \( S^\mu(\nu) \leq_* S^\mu(\nu') \).
(iii) If \( \nu \leq_{c,s} \nu' \), then \( S^\mu(\nu) \leq_{c,s} S^\mu(\nu') \).

Proof. Item (i) is a direct consequence of the minimality property of \( S^\mu(\nu) \) and (ii) is a direct consequence of Proposition 4.15 (ii). If \( \nu \leq_{c,s} \nu' \), then similar to the shadow, the set \( \{ \eta : \nu \leq_{c,s} \eta \leq_* \mu \} \) has a minimal element w.r.t. the convex-stochastic order that we denote by \( \eta^* \) (cf. [44, Lemma 6.2] for decreasing instead of increasing functions in the definition of \( \leq_{c,s} \)). The minimality implies both \( \eta^* \leq_{c,s} S^\mu(\nu) \) and \( \eta^* \leq_{c,s} S^\mu(\nu') \).

Moreover, we have

\[
\int_{\mathbb{R}} y \, dS^\mu(\nu)(y) = \int_{\mathbb{R}} y \, d\nu(y) \leq \int_{\mathbb{R}} y \, d\eta^*(y).
\]

Hence, \( \eta^* = S^\mu(\nu) \) by Lemma 3.1 (iii) and we conclude.

4.2.2. The obstructed shadow. We now turn to the definition of obstructed shadows. They can conveniently be constructed as a convex supremum over finitely obstructed shadows that were introduced by Nutz, Stebegg, and Tan in [45].

Recall that \((T, \leq)\) is a totally ordered set. Moreover, we fix a family of measures \((\mu_t)_{t \in T} \) in \( M_1(\mathbb{R}) \). To keep the notation compact we will denote

- \((\mu_t)_{t \in S}\) by \( \mu_S \) for all subsets \( S \subseteq T \) and
- use the abbreviation \( T_t = \{ s \in T : s \leq t \} \).

This notation will be used in all following sections.

Definition 4.17. Let \( \nu \in M_1(\mathbb{R}) \) and \( S \subseteq T \). We say \( \nu \leq_{c,+} \mu_S \) if there exists a family \((\eta_t)_{t \in S}\) such that

(i) \( \nu \leq_{c,+} \eta_t \) for all \( t \in S \),
(ii) \( \eta_s \leq \eta_t \) for all \( s \leq t \) in \( S \) and
(iii) \( \eta_t \leq \mu_t \) for all \( t \in S \).

Remark 4.18. If \( T = \{ \ast \} \) is a singleton, then Definition 4.17 coincides with the definition of \( \leq_{c,+} \) in Section 3.2 by choosing \( \eta_\ast = \nu \). Moreover, if \( \nu \leq_{c,+} \mu_T \), then \( \nu \leq_{c,+} \mu_S \) for all \( S \subseteq T \).

In the case that \( T \) is finite, it was observed in [45] that one can recursively define an obstructed shadow through finitely many marginals.

Lemma 4.19 ([45, Lemma 6.7]). Let \( R = \{ r_1 \leq \ldots \leq r_n \} \) be a finite subset of \( T \) and \( \nu \leq_{c,+} \mu_R \). We define inductively the (obstructed) shadow of \( \nu \) through \( \mu_R \) by

\[
S^{\mu_{r_1}, \ldots, \mu_{r_n}}(\nu) = \bigwedge_{r \in R} S^{\mu_{r_1}, \ldots, \mu_{r_{n-1}}}(\nu).
\]

The measure \( S^{\mu_{r_1}, \ldots, \mu_{r_n}}(\nu) \) is the unique minimal element of the set

\[
\{ \eta_{r_n} : (\eta_r)_{r \in R} : \nu \leq_{c} \eta_r \leq_{c} \eta_{r'} \leq_{c} \mu_r \text{ for all } r \leq r' \text{ in } R \}
\]

w.r.t. the convex order \( \leq_{c} \). In particular, \( S^{\mu_{r_1}, \ldots, \mu_{r_n}}(\nu) \leq_{c} \mu_{r_n} \).

Proof. This can be easily shown by induction over \( n \geq 2 \) using Proposition 4.11 and Lemma 4.16. Alternatively, see [45, Lemma 6.7].

By comparing (4.2) with Proposition 4.11 we see that \( S^{\mu_{r_1}, \ldots, \mu_{r_n}}(\nu) \) is the shadow of \( \nu \) in \( \mu_{r_n} \) obstructed by the additional finitely many marginals \( \mu_{r_1}, \ldots, \mu_{r_{n-1}} \). Figure 4 illustrates that an additional obstructing marginal can force the shadow to “spread out” (in convex order).
Taking the convex supremum over all choices of finite subsets $R \subseteq T$ yields the obstructed shadow of $\nu$ through $\mu_T$.

**Proposition 4.20.** Let $\nu \in \mathcal{M}_1(\mathbb{R})$ with $\nu \leq_{c,+} \mu_T$ and suppose there exists $\theta \in \mathcal{M}_1(\mathbb{R})$ such that $\mu_t \leq c, + \theta$ for every $t \in T$. Then, the set
\[
\{ S^{\mu_1, \ldots, \mu_n}(\nu) : \{ r_1 < \ldots < r_n \} \subset T, \ n \geq 1 \}
\]
admits a convex supremum. This is called the (general obstructed) shadow of $\nu$ through $\mu_T$ and is denoted by $S^{\mu_T}(\nu)$. Moreover, there exists a sequence $(R_n)_{n \in \mathbb{N}}$ of nested finite subsets of $T$ such that $(S^{\mu_{R_n}}(\nu))_{n \in \mathbb{N}}$ converges to $S^{\mu_T}(\nu)$ under $T_1$.

**Proof.** By Lemma 4.19, $S^{\mu_1, \ldots, \mu_n}(\nu)$ is increasing in convex order, if we add additional marginals as obstructions. Thus, for all finite $R = \{ r_1, \ldots, r_n \} \subset T$ it holds that $S^{\mu_R}(\nu) = S^{\mu_1, \ldots, \mu_n}(\nu)$, and therefore $S^{\mu_R}(\nu) \leq_{c,+} \mu_{R_n} \leq_{c,+} \theta$. Consequently, the convex supremum exists by Proposition 3.9 (ii).

Again by Lemma 4.19, it holds $S^{\mu_{R_1}}(\nu) \leq_{c,+} S^{\mu_{R_1 \cup R_2}}(\nu)$ and $S^{\mu_{R_2}}(\nu) \leq_{c,+} S^{\mu_{R_1 \cup R_2}}(\nu)$ for all finite $R_1, R_2 \subset T$. Thus, by Lemma 3.12, there exists a sequence of finite sets $(R_n)_{n \in \mathbb{N}}$ such that $(S^{\mu_{R_n}}(\nu))_{n \in \mathbb{N}}$ converges under $T_1$ to the convex supremum. Moreover, one can easily show that $R'_n = \bigcup_{i=1}^n R_i$ is a nested sequence of finite subsets of $T$ for which $(S^{\mu_{R'_n}}(\nu))_{n \in \mathbb{N}}$ converges to $S^{\mu_T}(\nu)$ because $S^{\mu_{R_n}}(\nu) \leq_{c,+} S^{\mu_{R'_n}}(\nu) \leq_{c,+} S^{\mu_T}(\nu)$ for all $n \in \mathbb{N}$ and the convex order is preserved under convergence w.r.t. $T_1$ (see Lemma 3.2). \qed

Proposition 4.20 extends the definitions in Lemma 4.19 and Proposition 4.11 in two ways: Firstly, it allows for infinitely, even uncountably, many obstructions. Secondly, the family $\mu_T$ does not have to be increasing in convex order.

For the remaining part of Section 4.2.2 we will always assume that there exists some $\theta \in \mathcal{M}_1(\mathbb{R})$ with $\mu_t \leq_{c,+} \theta$ for all $t \in T$. If $T$ has a maximal element and $\mu_T$ is a peacock this property is automatically satisfied. The following lemma collects some important consequences of Proposition 4.20.

**Lemma 4.21.** Let $\nu \leq_{c,+} \mu_T$.

![Figure 4](image-url)
Lemma 4.16. Since all of the three order relations are preserved under convergence in $T_1$, the same holds for any sequence $(S_n)_{n \in \mathbb{N}}$ of subsets of $T$ with $S_n \subseteq S'_n$ for all $n \in \mathbb{N}$.

Proof. For the case of finite subsets $S, S', S_n, S'_n$ of $T$, we have already shown items (i) and (ii) in the proof of Proposition 4.20. The items (i)–(v) are simple consequences of these two properties together with Proposition 4.20 and Lemma 3.2.

Proposition 4.20 does not require that $T$ admits a maximal element. However, if such a maximal element exists, then we can recover an analogue of (4.3):

**Proposition 4.22.** Let $\nu \leq c, \mu_T$. For all $u \in T$,

\begin{equation}
S^{\mu_T}(\nu) = \text{Cinf} \{ \eta_t : (\eta_t)_{t \in T_u} \text{ with } \nu \leq \eta_t \leq \mu_t \text{ for all } s \leq t \text{ in } T_u \}
\end{equation}

and the infimum is attained by the family $(\eta_t)_{t \in T_u} = (S^{\mu_T}(\nu))_{t \in T_u}$.

Proof. Set $\eta_t = S^{\mu_T}(\nu)$ for all $t \in T$. We know that $\nu \leq c S^{\mu_T}(\nu)$. Lemma 4.21 (i) shows that $S^{\mu_T}(\nu) \leq S^{\mu_T}(\nu)$ for all $s \leq t$ and we have $S^{\mu_T}(\nu) \leq \mu_t$ by Lemma 4.21 (v). Thus, $S^{\mu_T}(\nu)$ has to be, in convex order, greater than or equal to the convex infimum on the right-hand side of (4.3) for all $u \in T$.

Let $u \in T$ and take a sequence $(R_n)_{n \in \mathbb{N}}$ of finite subsets of $T_u$ given by Proposition 4.20 such that $(S^{\mu_{R_n}}(\nu))_{n \in \mathbb{N}}$ converges to $S^{\mu_T}(\nu)$. Lemma 4.21 (ii) yields that for $R'_n = R_n \cup \{u\}$ the sequence of corresponding shadows converges to $S^{\mu_T}(\nu)$ as well. Any family $(\eta_t')_{t \in T_u}$ with $\nu \leq \eta'_t \leq \mu_t$ for all $s \leq t$ in $T_u$ satisfies

\begin{equation}
S^{\mu_{R'_n}}(\nu) = \text{Cinf} \{ \eta_t' : (\eta_t')_{t \in R'_n} : \nu \leq \eta_t' \leq \mu_t \text{ for all } s \leq t \text{ in } R'_n \}
\end{equation}

where the equality is due to Lemma 4.19. Passing to the limit under $T_1$, shows that $S^{\mu_T}(\nu)$ is smaller in convex order than the right-hand side of (4.3) by Lemma 4.22.

The following lemma generalizes Lemma 4.10 (the simple measure $\mu$ is replaced by a family $\mu_T$).

**Lemma 4.23.** Let $\nu, \nu' \leq c, \mu_T$.

(i) If $\nu \leq c \nu'$, then $S^{\mu_T}(\nu) \leq c S^{\mu_T}(\nu')$.

(ii) If $\nu \leq \nu'$, then $S^{\mu_T}(\nu) \leq S^{\mu_T}(\nu')$.

(iii) If $\nu \leq c, \nu'$, then $S^{\mu_T}(\nu) \leq c S^{\mu_T}(\nu')$.

Moreover, for any peacock $\mu_T$ we have

(iv) if $\mu_T \leq c \mu'_T$ for all $t \in T$, then $\nu \leq c \mu'_T$, and $S^{\mu'_T}(\nu) \leq c S^{\mu_T}(\nu)$.

Proof. Item (i)-(iii): By Proposition 4.20 and Lemma 4.21 (ii), we can find a sequence of nested finite sets $(R_n)_{n \in \mathbb{N}}$ such that both $(S^{\mu_{R_n}}(\nu))_{n \in \mathbb{N}}$ converges weakly to $S^{\mu_T}(\nu)$ and $(S^{\mu_{R_n}}(\nu'))_{n \in \mathbb{N}}$ converges weakly to $S^{\mu_T}(\nu')$. In any of the three cases we get the desired relation between $(S^{\mu_{R_n}}(\nu))_{n \in \mathbb{N}}$ and $(S^{\mu_{R_n}}(\nu'))_{n \in \mathbb{N}}$ by inductively applying Lemma 4.10. Since all of the three order relations are preserved under convergence in $T_1$ by Lemma 3.2, we have shown the claim.
Item (iv) is an immediate consequence of Proposition 4.22 and Lemma 4.21 (iv).

**Lemma 4.24.** Let \( \nu \leq c_1, \mu_T \) and \( \alpha > 0 \). Then \( \alpha \nu \leq c_1, \alpha \mu_T \) and
\[
S^{\alpha \mu_T} (\alpha \nu) = \alpha S^{\mu_T} (\nu),
\]
i.e. the convex supremum is positively 1-homogeneous.

**Proof.** For all \( \alpha > 0 \) and all measures \( \eta, \eta' \in M_1(\mathbb{R}) \) it holds \( \eta \leq c, \eta' \) if and only if \( \alpha \eta \leq c, \alpha \eta' \). \( \square \)

**Proposition 4.25.** Let \( \nu_1, \nu_2 \in M_1(\mathbb{R}) \) with \( \nu = \nu_1 + \nu_2 \leq c_1, \mu_T \). It holds \( \nu_1 \leq c_1, \mu_T, \nu_2 \leq c_1, (\mu_t - S^{\mu_T}(\nu_1))_{t \in T} \) and
\[
S^{\mu_T}(\nu_1 + \nu_2) = S^{\mu_T}(\nu_1) + S^{c}(\mu_t - S^{\mu_T}(\nu_1))_{t \in T} (\nu_2).
\]

**Proof.** First assume that \( T \) is finite. In this case the claim follows from applying Lemma 4.15 (ii) inductively \(|T|\) times.

Now suppose that \( T \) has a maximal element, i.e. \( T = T_u \) for some \( u \in T \). Lemma 4.24 (ii) implies that \( (S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1))_{t \in T_u} \) is a well-defined family in \( M_1(\mathbb{R}) \). We will show that this sequence is increasing in convex order and greater than or equal to \( \nu_2 \). To this end, let \( s, t \in T_u \) with \( s \leq t \) and let \( (R_n)_{n \in N} \) be a sequence of finite sets such that all of the four sequences \( (S^{\mu_T}(R_n)(\nu))_{n \in N}, (S^{\mu_T}(R_n)(\nu_1))_{n \in N}, (S^{\mu_T}(R_n)(\nu))_{n \in N} \) and \( S^{\mu_T}(\nu_1) \) converge to \( S^{\mu_T}(\nu), S^{\mu_T}(\nu_1), S^{\mu_T}(\nu) \) and \( S^{\mu_T}(\nu) \) respectively. Again, this sequence can be constructed by using Proposition 4.22 in conjunction with Lemma 4.21 (i). For all \( n \in N \) we obtain by the initial considerations for finite index sets
\[
S^{\mu_T}(R_n)(\nu) - S^{\mu_T}(R_n)(\nu_1) = S^{(\mu_t - S^{\mu_T}(\nu))_{t \in T_u}}(\nu_2) \leq S^{(\mu_t - S^{\mu_T}(\nu))_{t \in T_u}}(\nu_2) = S^{\mu_T}(\nu_1) - S^{\mu_T}(\nu_1).
\]

Letting \( n \) tend to infinity, this proves that \( (S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1))_{t \in T_u} \) is increasing in convex order and shows \( \nu_2 \leq c_1, (S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1))_{t \in T_u} \). Since additionally \( S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1) \leq c, \mu_t - S^{\mu_T}(\nu_1) \) for all \( t \in T_u \), Proposition 4.22 yields
\[
S^{(\mu_t - S^{\mu_T}(\nu))_{t \in T_u}}(\nu_2) \leq c, S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1).
\]

Similarly, we can apply Proposition 4.22 to see
\[
S^{\mu_T}(\nu) \leq c, S^{\mu_T}(\nu_1) + S^{c}(\mu_t - S^{\mu_T}(\nu_1))_{t \in T_u} (\nu_2).
\]

and therefore both sides are equal.

In the general case, by Lemma 4.21 (iv) it holds
\[
S^{\mu_T}(\nu) = \text{Csup} \left\{ S^{\mu_T}(\nu_1) + S^{c}(\mu_t - S^{\mu_T}(\nu_1))_{t \in T_u} (\nu_2) : u \in T \right\}
\]
\[
= \text{Csup} \left\{ S^{\mu_T}(\nu_1) : u \in T \right\} + \text{Csup} \left\{ S^{c}(\mu_t - S^{\mu_T}(\nu_1))_{t \in T} (\nu_2) : u \in T \right\}
\]
\[
= S^{\mu_T}(\nu_1) + S^{c}(\mu_t - S^{\mu_T}(\nu_1))_{t \in T} (\nu_2)
\]
where the second equality follows from Lemma 3.11 (iii) because both summands are increasing in convex order as \( u \) increases (see Lemma 4.21 (i) and Lemma 4.23 (iv)). \( \square \)

**Remark 4.26.** In the previous proof we have shown that \( (S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1))_{t \in T} \) is increasing in convex order with \( \nu_2 \leq c_1, (S^{\mu_T}(\nu) - S^{\mu_T}(\nu_1))_{t \in T} \) if \( \nu_1 + \nu_2 \leq c_1, \mu_T \) for all \( s \leq t \) in \( T \), since \( S^{\mu_T}(\nu) = \mu_T \), this implies that \( (\mu_t - S^{\mu_T}(\nu_1))_{t \in T} \) is increasing in convex order with \( \nu_2 \leq c_1, (\mu_t - S^{\mu_T}(\nu_1))_{t \in T} \).
4.3. **Non self-improvable peacocks.** After parameterizations and shadows, non self-improvable peacocks are the last conceptual ingredient that we need for the proof of Theorem 4.23. Recall that \((T, \leq)\) is an abstract totally ordered set with minimal element \(0 \in T\) and that we use the notation \(T_r = \{s \in T : s \leq r\} \).

**Definition 4.27.** A peacock \((\mu_t)_{t \in T}\) is called non self-improvable (NSI) if for all peacocks \((\eta_t)_{t \in T}\) with

(i) \(\eta_0 = \mu_0\) and

(ii) \(\eta_t \leq + 2\mu_t\) for all \(t \in T\),

it holds \(\mu_t \leq_c \eta_t\).

The following lemma explains the term “non self-improvable”. Indeed, Item (ii) in Lemma 4.28 shows that an NSI peacock is minimal in convex order for an operation that aims at reducing the peacock \((\mu_t)_{t \in T}\) in convex order at every \(t \in T\) by rearranging the mass constrained to be a submeasure of \(2\mu_t\) at every \(t \in T\). In this sense NSI peacocks cannot be “improved”.

**Lemma 4.28.** Let \((\mu_t)_{t \in T}\) be a peacock. The following are equivalent:

(i) \((\mu_t)_{t \in T}\) is NSI.

(ii) For all \(t \in T\) it holds \(S(2\mu_t)_{t \in T}(\mu_0) = \mu_t\).

(iii) \((\mu_t)_{t \in T}\) is an extreme point of the convex set

\[ K_{\mu_0} = \{ (\eta_t)_{t \in T} \mid (\eta_t)_{t \in T} \text{ is a peacock with } \eta_0 = \mu_0 \}. \]

**Proof.** (i) \(\Rightarrow\) (iii): Assume that \((\mu_t)_{t \in T}\) is NSI and \((\eta_t)_{t \in T}\) and \((\eta'_t)_{t \in T}\) are peacocks with \(\eta_0 = \mu_0 = \eta'_0\) and \(\mu_t = \eta_0 + \frac{1}{2}\eta_t + \frac{1}{2}\eta_t'\) for all \(t \in T\). Then both inequalities \(\eta_t \leq + 2\mu_t\) and \(\eta_t' \leq + 2\mu_t\) hold for all \(t \in T\). Hence, \(\mu_t \leq_c \eta_t\) and \(\mu_t \leq_c \eta_t'\) by the NSI property. Combining these two inequalities with \(\eta_t' = 2\mu_t - \eta_t\), yields \(\mu_t \leq_c \eta_t \leq_c \mu_t\) for all \(t \in T\), i.e. \(\mu_t = \eta_t\). Hence, \((\mu_t)_{t \in T}\) is an extreme point of \(K_{\mu_0}\).

(iii) \(\Rightarrow\) (ii): If \((\mu_t)_{t \in T}\) is an extreme point of \(K_{\mu_0}\), applying first Lemma 4.24 with \(\alpha = 2\) and then Proposition 4.26 with \(\nu_1 = \nu_2 = \mu_0\), we can rewrite \(\mu_t\) as

\[ \mu_t = S(\mu_t)_{t \in T}(\mu_0) = \frac{1}{2}S(2\mu_t)_{t \in T}(\mu_0) = \frac{1}{2}S(2\mu_t)_{t \in T}(\mu_0) + \frac{1}{2}(2\mu_t - S(2\mu_t)_{t \in T}(\mu_0)) \]

for all \(t \in T\). Both \(S(2\mu_t)_{t \in T}(\mu_0)\) and \((2\mu_t - S(2\mu_t)_{t \in T}(\mu_0))_{t \in T}\) are elements of \(K_{\mu_0}\) (see Remark 4.26) and thus extremality yields

\[ S(2\mu_t)_{t \in T}(\mu_0) = 2\mu_t - S(2\mu_t)_{t \in T}(\mu_0) \]

for all \(t \in T\) and hence \((\mu_t)_{t \in T}\) satisfies (ii).

(ii) \(\Rightarrow\) (i): Suppose \((\mu_t)_{t \in T}\) satisfies (ii) and \((\eta_t)_{t \in T}\) is a peacock with \(\eta_0 = \mu_0\) and \(\eta_t \leq + 2\mu_t\) for all \(t \in T\). Then \(\mu_t = S(2\mu_t)_{t \in T}(\mu_0) \leq_c \eta_t\) for all \(t \in T\) by Proposition 4.22 and hence \((\mu_t)_{t \in T}\) is NSI. \(\square\)

The key feature of non self-improvable peacocks is that there is only one martingale measure associated with such peacocks, see Proposition 4.30 below. Even better this martingale measure is necessarily Markov.

**Lemma 4.29.** If \((\mu_t)_{t \in T}\) is a peacock that is NSI, then under any \(\pi \in M_T((\mu_t)_{t \in T})\) the canonical process is a Markov process.

**Proof.** Assume there exists \(\pi \in M_T((\mu_t)_{t \in T})\) for which the canonical process is not a Markov process (in the sense of Definition 3.13 (ii)), i.e. there exist \(r < u\) and a Borel
set $A \subset \mathbb{R}$ such that $\mathbb{E}_\pi[\mathbb{1}_A | \mathcal{F}_r]$ is not $\pi$-a.e. equal to $\mathbb{E}_\pi[\mathbb{1}_A | X_r]$. Since $\mathcal{F}_r$ is the product $\sigma$-algebra generated by the family $(\sigma(X_s))_{s \leq r}$, there exist $n \in \mathbb{N}$, $0 \leq r_1 < \ldots < r_n \leq r < u$ in $T$ such that

$$\pi[X_u \in A | X_{r_1}, \ldots, X_{r_n}, X_r] \neq \pi[X_u \in A | X_r].$$

For $t \geq r$, let $k_t$ be a regular version of $\pi[X_t \in \cdot | X_{r_1}, \ldots, X_{r_n}, X_r]$ and $k'_t$ be a regular version of $\pi[X_t \in \cdot | X_r]$ which exist because $\mathbb{R}^{n+2}$ and $\mathbb{R}^2$ are Polish spaces. The inequality in (4.3) implies that there exists a convex function $\varphi$ s.t. the Borel set

$$\left\{ x \in \mathbb{R}^{r_1, \ldots, r_n, r} : \int_{\mathbb{R}} \varphi(y) \, dk_u(x, dy) \neq \int_{\mathbb{R}} \varphi(y) \, dk'_u(x, dy) \right\}$$

has positive mass under $\pi_{(r_1, \ldots, r_n, r)}$. Suppose there exists $\varepsilon > 0$ such that the Borel set

$$D = \left\{ x \in \mathbb{R}^{r_1, \ldots, r_n, r} : \int_{\mathbb{R}} \varphi(y) \, dk_u(x, dy) \geq \int_{\mathbb{R}} \varphi(y) \, dk'_u(x, dy) + \varepsilon \right\}$$

has positive mass under $\pi_{(r_1, \ldots, r_n, r)}$. We set

$$\eta_t = \begin{cases} \mu_t & t < r \\ \int_{D^c} k_t(x, \cdot) \, d\pi_{(r_1, \ldots, r_n, r)}(x) + \int_D k'_t(x, \cdot) \, d\pi_{(r_1, \ldots, r_n, r)}(x) & t \geq r \end{cases}$$

for all $t \in T$. Then $\eta_0 = \mu_0$, $\eta_t \leq c \eta_{t-1}$ for all $t \in T$ and $\eta_s \leq \varepsilon \eta_t$ for all $s < t \leq r$. For all $r \leq s < t$ and any convex function $\psi$ it holds

$$\int \psi \, d\eta_t = \mathbb{E}_\pi[\mathbb{E}_\pi[\psi(X_t)|X_{r_1}, \ldots, X_{r_n}, X_r] \cdot \mathbb{1}_D + \mathbb{E}_\pi[\psi(X_t)|X_r] \cdot \mathbb{1}_{D^c}]$$

$$= \mathbb{E}_\pi[\mathbb{E}_\pi[\mathbb{E}_\pi[\psi(X_t)|\mathcal{F}_s]|X_{r_1}, \ldots, X_{r_n}, X_r] \cdot \mathbb{1}_D + \mathbb{E}_\pi[\mathbb{E}_\pi[\psi(X_t)|\mathcal{F}_s]|X_r] \cdot \mathbb{1}_{D^c}]$$

$$\geq \mathbb{E}_\pi[\mathbb{E}_\pi[\psi(X_u)|X_{r_1}, \ldots, X_{r_n}, X_r] \cdot \mathbb{1}_D + \mathbb{E}_\pi[\psi(X_u)|X_r] \cdot \mathbb{1}_{D^c}] = \int \psi \, d\eta_s$$

and thus $(\eta_t)_{t \in T}$ is a peacock. Then, since we have for $u > r$

$$\int_{\mathbb{R}} \varphi \, d\mu_u \leq \int_{\mathbb{R}} \varphi \, d\mu_u - \varepsilon \cdot \pi_{(r_1, \ldots, r_n, r)}[D] \leq \int_{\mathbb{R}} \varphi \, d\mu_u,$$

we get a contradiction to $(\mu_t)_{t \in T}$ being NSI. If such an $\varepsilon > 0$ does not exist, then there has to exist an $\varepsilon > 0$ such that the Borel set

$$D' = \left\{ x \in \mathbb{R}^T_r : \int_{\mathbb{R}} \varphi(y) \, dk'_u(x, dy) \geq \int_{\mathbb{R}} \varphi(y) \, dk_u(x, dy) + \varepsilon \right\}$$

has positive mass under $\pi_{(r_1, \ldots, r_n, r)}$. We define $\eta'$ with reversed roles of $k_t$ and $k'_t$ and obtain a contradiction as above.

The Markov property of a martingale measure associated with a NSI peacock allows for a short proof of the crucial uniqueness property.

**Proposition 4.30.** If $(\mu_t)_{t \in T}$ is a peacock that is NSI, then $\mathbb{M}_T(\mu_T)$ consists of only one martingale measure and the canonical process is Markov under this measure.

**Proof.** Let $\pi, \pi' \in \mathbb{M}_T(\mu_T)$ and let $k_{s,t}$ (resp. $k'_{s,t}$) be regular versions of $\pi[X_t \in \cdot | X_s]$ (resp. $\pi'[X_t \in \cdot | X_s]$) for all $0 \leq s < t \leq 1$. These exist because $\mathbb{R}^2$ is a Polish space. Assume that $\pi \neq \pi'$. Lemma 4.29 yields that both $\pi$ and $\pi'$ are Markov processes and thus there must exist $0 < r < u \leq 1$ such that $k_{r,u}(x, \cdot) \neq k'_{r,u}(x, \cdot)$ for all $x$ in a Borel
set with positive \(\mu_r\)-mass. Suppose there exists a convex function \(\varphi\) and \(\varepsilon > 0\) such that the Borel set

\[
D = \left\{ x \in \mathbb{R} : \int_{\mathbb{R}} \varphi \, dk_{r,u}(x,\cdot) \geq \int_{\mathbb{R}} \varphi \, dk'_{r,u}(x,\cdot) + \varepsilon \right\}
\]

has positive mass under \(\mu_r\). We set

\[
\eta_t = \begin{cases} 
\mu_t & t < r \\
\int_{D^c} k_{r,t}(x,\cdot) \, d\mu_r + \int_{D} k'_{r,t}(x,\cdot) \, d\mu_r & t \geq r
\end{cases}
\]

for all \(t \in T\). Then \(\eta_0 = \mu_0, (\eta_t)_{t \in T} \in P_T\) and \(\eta_t \leq +2\mu_t\). Therefore

\[
\int_{\mathbb{R}} \varphi \, d\eta_t \leq \int_{\mathbb{R}} \varphi \, d\mu_u - \varepsilon \cdot \mu_r[D] < \int_{\mathbb{R}} \varphi \, d\mu_u
\]

is a contradiction to \((\mu_t)_{t \in T}\) being NSI. If there exist no convex function and \(\varepsilon > 0\) such that \(D\) has positive mass under \(\mu_r\), there exists a convex function \(\varphi\) and \(\varepsilon > 0\) such that

\[
D' = \left\{ x \in \mathbb{R}^T_r : \int_{\mathbb{R}} \varphi(x) \, dk_{r,u}(x,dy) \geq \int_{\mathbb{R}} \varphi(x) \, dk'_{r,u}(x,dy) + \varepsilon \right\}
\]

has positive mass under \(\mu_r\). We define \(\eta'\) by reversing roles of \(k_{r,t}\) and \(k'_{r,t}\) and obtain a contradiction as above.

\[\square\]

**Remark 4.31.** Suppose that \(T = \{0,1\}\). Then the NSI property is closely related to the concept of Kellerer dilations (cf. [10]) that we explain in the following: For any closed set \(F \subset \mathbb{R}\) the Kellerer dilation is a kernel \(P_F\) defined for every \(x \in \mathbb{R} \cap [\min F, \max F]\) by

\[
P_F(x,\cdot) = \begin{cases} 
\frac{x - x^-}{x + x^-} \delta_{x^+} + \frac{x^+ - x}{x + x^-} \delta_{x^-} & x \notin F \\
\delta_x & x \in F
\end{cases}
\]

where \(x^+ = \min(F \cap [x,\infty))\) and \(x^- = \max(F \cap (-\infty,x])\). Kellerer showed in [10, Satz 25] that for any \(\mu \in P_1(\mathbb{R})\) with \(\text{supp}(\mu) \subseteq [\min F, \max F]\) there is only one martingale measure with marginals \(\mu(dx)\) and \(\mu P_F := \int P_F(x,dy) \mu(dx)\) and it is given by \(\mu(dx)P_F(x,dy)\). In particular, as a consequence of Lemma [4.28] and [13, Lemma 2.8] a peacock \((\mu_0, \mu_1)\) is NSI if and only if \(\mu_1 = \mu_0 \mu_{\text{supp}(\mu_1)}\). Then the law of the unique martingale is \(\mu(dx)P_{\text{supp}(\mu_1)}(x,dy)\).

5. SHADOW MARTINGALES INDEXED BY COUNTABLE SET

In this section we prove Theorem [5.15] that is an analogue of Theorem [1.5] in the case of a countable index set \(T \subset [0,\infty)\) with minimal element 0 and that attains also a maximal element, i.e. \(\sup T \in T\). The proof follows the outline explained in Steps 1–3 in Subsection [1.3]. In Subsection [5.1] we show that martingale parametrizations are almost everywhere differentiable. The existence part of Theorem [5.15] is covered in Subsection [5.2]. In Subsection [5.3] we will introduce an auxiliary optimization problem over peacocks and establish a monotonicity principle for this optimization problem that will allow us in Subsection [5.4] to deduce that any optimizer is necessarily NSI. Finally, in Subsection [5.5] we show that the family of right derivatives of shadow martingales is a solution to our auxiliary optimization problem. This allows us to conclude.
5.1. Right-derivatives of martingale parametrizations. Recall that \( T \) is at most countable. In this subsection we show that in the current setup any martingale parametrization is \( \lambda \)-a.e. right-differentiable.

**Lemma 5.1.** The spaces \( P_T \) and \( M_T \) are Polish.

*Proof.* Since \( T \) is at most countable, by Lemma 3.2 \( P_T \) is a closed subset of the Polish space \( P_1(\mathbb{R})^T \) and thus Polish itself (cf. [38, Exercise 3.3]). In the same way, the closed subspace \( M_T \) is Polish once we have shown that \( P_1(\mathbb{R})^T \) is a Polish space, which can be seen as follows: the Wasserstein metric on the set \( P_1(\mathbb{R})^T \) induced by the metric \( d(x, y) = \sum_{n \in \mathbb{N}} 2^{-n} \min \{|x(f_n) - y(f_n)|, 1\} \) on \( \mathbb{R}^T \) for some surjective \( f : \mathbb{N} \to T \) is a complete separable metric that, moreover, induces the topology on \( P_1(\mathbb{R})^T \) that we defined in Section 3.1. \( \square \)

**Lemma 5.2.** Let \( \pi \in P_1(\mathbb{R})^T \) and \((\pi^\alpha)_{\alpha \in [0,1]} \) a parametrization of \( \pi \). The curve \( \alpha \mapsto \pi^\alpha \) is \( \lambda \)-a.e. right-differentiable, i.e. for \( \lambda \)-a.e. \( \alpha \in [0,1] \) the right-derivative

\[
\hat{\pi}^\alpha = \lim_{h \downarrow 0} \frac{\pi^{\alpha + h} - \pi^\alpha}{h}
\]

exists as a limit in \( P_1(\mathbb{R})^T \) under \( T_1 \).

More precisely, by choosing \( \hat{\pi}^\alpha \) constant on the \( \lambda \)-nullset where the limit in (5.1) does not exist, \( (\hat{\pi}^\alpha)_{\alpha \in [0,1]} \) is a disintegration of the corresponding coupling of \( \lambda \) and \( \pi \) in Remark 4.2 w.r.t. \( \lambda \). In particular, \( \alpha \mapsto \hat{\pi}^\alpha \) is a measurable map from \([0,1]\) to \( P_1(\mathbb{R})^T \).

If \( \pi \in M_T \) and \((\pi^\alpha)_{\alpha \in [0,1]} \) is a martingale parametrization of \( \pi \), then the right derivative \( \hat{\pi}^\alpha \) is an element of \( M_T \) for \( \lambda \)-a.e. \( \alpha \).

*Proof.* Since \((\pi^\alpha)_{\alpha \in [0,1]} \) is a parametrization of \( \pi \), \( \frac{\pi^{\alpha + h} - \pi^\alpha}{h} \) is an element of \( P_1(\mathbb{R})^T \) for all \( \alpha \in [0,1] \) and \( h \in (0,1-\alpha] \). Let \( \xi \) be the coupling of \( \lambda \) and \( \pi \) on \([0,1] \times \mathbb{R}^T \) defined in Remark 4.2 and let \((\xi_x)_{x \in [0,1]} \) be a version of the disintegration of \( \xi \) w.r.t. \( \lambda \) (we can disintegrate the measure \( \xi \) because both \([0,1]\) and \( \mathbb{R}^T \) are Polish spaces, cf. Lemma 5.1).

Recall that the topology \( T_1 \) on \( P_1(\mathbb{R})^T \) is generated by the convergence of the integrals of all functions in \( \mathcal{G}_0 \cup \mathcal{G}_1 \) where

\[
\begin{align*}
\mathcal{G}_0 &= \{ g \circ (X_{t_1}, \ldots, X_{t_n}) : n \geq 1, \ t_1, \ldots, t_n \in T, \ g \in C_b(\mathbb{R}^n) \} \quad \text{and} \\
\mathcal{G}_1 &= \{ |X_t| : t \in T \}.
\end{align*}
\]

For all \( h > 0 \) and \( f \in \mathcal{G}_0 \cup \mathcal{G}_1 \) we have

\[
\int_{\mathbb{R}^T} f \ d \left( \frac{\pi^{\alpha + h} - \pi^\alpha}{h} \right) = \frac{1}{h} \int_{\mathbb{R}^T} f \ d \xi_x \ d\lambda(x)
\]

and since \( x \mapsto \int_{\mathbb{R}^T} f \ d\xi_x \) is measurable and in \( L^1(\lambda) \), the Lebesgue differentiation theorem yields that the integral converges for \( \lambda \)-a.e. \( \alpha \) to \( \int_{\mathbb{R}^T} f \ d\xi_x \) as \( h \to 0 \).

We claim that one can choose this \( \lambda \)-nullset independent of \( f \). Indeed, the set

\[
A_c = \{ g \circ (X_{t_1}, \ldots, X_{t_n}) : n \geq 1, \ t_1, \ldots, t_n \in T, \ g \in C_c(\mathbb{R}^n) \} \subset \mathcal{G}_0
\]

is separable (w.r.t. the supremum norm on \( C(\mathbb{R}^T) \)) because \( T \) is countable and \( C_c(\mathbb{R}^n) \) is separable for all \( n \in \mathbb{N} \). Let \( \mathcal{X} \) be a countable dense subset of \( A_c \). Then there exists a \( \lambda \)-null set \( L \subset [0,1] \) such that (5.2) converges to

\[
\int_{\mathbb{R}^T} f \ d\xi_x
\]
as \( h \to 0 \) for all \( f \in \mathcal{X} \cup \mathcal{G}_1 \) and \( a \notin L \) (\( \mathcal{G}_1 \) is countable). Using the triangle inequality, it follows that this convergence holds for all \( f \in A_{a} \cup \mathcal{G}_1 \) and \( a \notin L \). Moreover, since \( \xi_{a} \) is a probability measure for \( \lambda \)-a.e. \( a \), we conclude that \( (5.2) \) converges as \( h \to 0 \) for all \( f \in \mathcal{G}_0 \cup \mathcal{G}_1 \) and \( \lambda \)-a.e \( a \) because weak and vague convergence coincides for probability measures on a Polish space.

Thus, we have shown that \( \left( \frac{\frac{\pi_{a}^{+} + \eta_{a}^{+}}{h} - \frac{\pi_{a}^{-} + \eta_{a}^{-}}{h}}{h} \right)_{h > 0} \) converges under \( T_{1} \) in \( \mathcal{P}_1(\mathbb{R}^{T}) \) to \( \xi_{a} \) as \( h \downarrow 0 \) for all \( a \) outside the \( \lambda \)-nullset \( L \). Since \( (\xi_{a})_{a \in [0, 1]} \) is a disintegration, the map \( a \mapsto \xi_{a} \) is measurable.

Furthermore, if \( (\pi_{a})_{a \in [0, 1]} \) is a martingale parametrization, notice that for all \( a \in [0, 1] \) and \( h \in (0, 1-a] \) the quotient \( \frac{\pi_{a}^{+} + \eta_{a}^{+}}{h} - \frac{\pi_{a}^{-} + \eta_{a}^{-}}{h} \) is a martingale measure and this property is preserved under convergence w.r.t. \( T_{1} \).

**Corollary 5.3.** Let \( \pi \) and \( \rho \) be in \( \mathcal{P}_1(\mathbb{R}^{T}) \), \( (\pi_{a})_{a \in [0, 1]} \) a parametrization of \( \pi \) and \( (\rho_{a})_{a \in [0, 1]} \) a parametrization of \( \rho \). If the right-derivatives \( (\pi_{a})_{a \in [0, 1]} \) and \( (\rho_{a})_{a \in [0, 1]} \) coincide for \( \lambda \)-a.e. \( a \), then \( \pi_{a} = \rho_{a} \) for all \( a \in [0, 1] \).

**Proof.** This is a direct consequence of Remark 1.2 and Lemma 5.2.

### 5.2. Existence of shadow martingales

In order to show the existence of a martingale parametrization \( (\pi_{a})_{a \in [0, 1]} \) that satisfies

\[
(5.3) \quad \pi_{a}(X_{t} \in \cdot) = S_{T_{t}}(\nu_{a})
\]

for all \( t \in T \) and \( a \in [0, 1] \), we will construct an appropriate family of martingale measures \( (\tilde{\pi}_{a})_{a \in [0, 1]} \) such that \( a \mapsto \tilde{\pi}_{a} \) is a measurable function and

\[
\pi_{a} = \int_{0}^{a} \tilde{\pi}_{a} \, d\alpha
\]

satisfies (5.3) for all \( t \in T \) and \( a \in [0, 1] \), i.e. we construct the right-derivatives of \( \alpha \mapsto \pi_{a} \). To this end, it is necessary and sufficient that the marginal distribution of \( \tilde{\pi}_{a} \) at time \( t \) coincides with the right-derivatives of \( \alpha \mapsto S_{T_{t}}(\nu_{a}) \) for \( \lambda \)-a.e. \( a \).

Recall that \( T \) is a countable subset of \( [0, \infty) \) with \( 0 \in T \) and \( \text{sup} T \in T \).

**Lemma 5.4.** Let \( (\nu_{a})_{a \in [0, 1]} \) be a parametrization of \( \mu_{0} \). There exists a Borel set \( A \subset [0, 1] \) with \( \lambda(A) = 1 \) such that for every \( a \in A \) the following holds:

(i) For all \( t \in T \), the curve \( \alpha \mapsto S_{T_{t}}(\nu_{a}) \) is right-differentiable at \( a \) (in the sense of Lemma 5.2) and we denote this right derivative by \( \tilde{\nu}_{a} \).

(ii) The family \( (\tilde{\nu}_{a})_{t \in T} \) is a peacock with initial value \( \tilde{\nu}_{a}^{0} = \tilde{\nu}_{a} \). Here, \( \tilde{\nu}_{a} \) is the right-derivative of \( \alpha \mapsto \nu_{a} \). \( a \).

Moreover, setting \( \tilde{\nu}_{a}^{0} = \delta_{a} \) for all \( t \in T \) and \( a \notin A \), \( a \mapsto (\tilde{\nu}_{a}^{0})_{t \in T} \) is a measurable map from \( [0, 1] \) to \( \mathcal{P}_{T} \) and, for all \( a \in [0, 1] \) and \( t \in T \), it holds

\[
S_{T_{t}}(\nu_{a}) = \int_{0}^{a} \tilde{\nu}_{a}^{0} \, d\alpha.
\]

**Proof.** It is not difficult to see that \( (S_{T_{t}}(\nu_{a}))_{a \in [0, 1]} \) is a parametrization of \( \mu_{t} \). Hence, for all \( t \in T \) Lemma 5.2 yields that there exists a Borel set \( A_{t} \subset [0, 1] \) with \( \lambda(A_{t}) = 1 \) such that the map \( \alpha \mapsto S_{T_{t}}(\nu_{a}) \) is right-differentiable for all \( a \in A_{t} \). We set \( A = \bigcap_{t \in T} A_{t} \) and denote the right derivatives by \( \tilde{\nu}_{a} \) for all \( a \in A \) and \( t \in T \). Then, item (i) holds.
Moreover, Proposition 4.22 in conjunction with Lemma 4.24 implies that
\[ \hat{\eta}^a_t = \lim_{h \downarrow 0} S^\frac{\mu - S^{\mu}(\nu)}{h} \]
for all \( a \in A \) and \( t \in T \). Clearly, for all \( a \in A \) it holds \( \hat{\eta}^a_0 = \tilde{\nu}^a \) and we obtain
\[ S^\frac{\mu - S^{\mu}(\nu)}{h} \leq c S^\frac{\mu - S^{\mu}(\nu)}{h} \]
for all \( t \leq u \in T \) and \( h > 0 \). Lemma 5.2 shows that the convex order is preserved under convergence in \( T_1 \), Item (ii) follows.

Finally, note that Lemma 5.2 implies that for all \( t \in T \), \( a \mapsto \hat{\eta}^a_t \) is a measurable map from \( [0,1] \) to \( \mathbb{R} \) and \( \hat{\eta}^a_t \) is a disintegration of the coupling \( \xi^t \) between \( \lambda \) and \( \mu_t \) w.r.t. \( \lambda \) that corresponds to the parametrization \( (S^{\mu_t}(\nu^a))_{a \in [0,1]} \) in the sense of Remark 4.2 (setting \( \hat{\eta}^a_t = \delta_0 \) for all \( a \notin A \)). Hence, \( a \mapsto (\hat{\eta}^a_t)_{t \in T} \) is a measurable map from \( [0,1] \) to \( P_T \) and, for all \( a \in [0,1] \) and \( t \in T \), it holds
\[ \int_0^\alpha \hat{\eta}^a_t \, da = \xi^t([0,\alpha]) = S^{\mu_t}(\nu^a). \]

**Lemma 5.5.** There exists a measurable map \( P_T \to M_T \) such that the image of \( (\mu_t)_{t \in T} \) is an element of \( M_T((\mu_t)_{t \in T}) \).

**Proof.** By Lemma 5.1, \( P_T \) and \( M_T \) are Polish spaces. We define \( \Phi : M_T \ni \pi \mapsto (L_{a}(\mu_t))_{t \in T} \in P_T \). It is not difficult to see that \( \Phi \) is continuous (thus measurable) and Proposition 4.10 yields that \( \Phi \) is a surjective map. Since \( T \) is countable with \( \sup T \in T \), the set \( \Phi^{-1}((\mu_t)_{t \in T}) = M_T((\mu_t)_{t \in T}) \) is compact for all \( (\mu_t)_{t \in T} \in P_T \) by [11] Lemma 2.1. The measurable selection theorem of Dellacherie [17] shows that there exists a measurable right-inverse \( \Phi^{-1} \). \( \square \)

With respect to Lemma 5.5 we would like to emphasize an impressive result by Lowther [12]. For continuous peacocks with connected supports the measurable map of Lemma 5.1 can be chosen to be continuous. For these peacocks, Lowther’s map is the unique continuous map.

Now we are able to prove the existence part of Theorem 4.3 for the countable index set \( T \):

**Proposition 5.6.** Let \( (\nu^a)_{a \in [0,1]} \) be a parametrization of \( \mu_0 \). There exists a martingale measure \( \pi \in M_T((\mu_t)_{t \in T}) \) and a martingale parametrization \( (\pi^a)_{a \in [0,1]} \) of \( \pi \) such that
\[ \pi^a(X_t \in \cdot) = S^{\mu_t}(\nu^a) \]
for all \( a \in [0,1] \) and \( t \in T \).

**Proof.** Lemma 4.24 yields that there exists \( A \subset [0,1] \) with \( \lambda(A) = 1 \) and a measurable map \( a \mapsto (\hat{\pi}^a_t)_{t \in T} \) from \( [0,1] \) to \( P_T \) such that \( (\hat{\pi}^a_t)_{t \in T} \) is the right-derivative of \( \alpha \mapsto S^{\mu_t}(\nu^a) \) at \( a \) for all \( a \in A \) and \( t \in T \). Thus, Lemma 5.5 implies that there exists a measurable map \( a \mapsto \hat{\pi}^a \) from \( [0,1] \) to \( M_T((\hat{\pi}^a_t)_{t \in T}) \) for all \( a \in [0,1] \).

We set
\[ \pi^a = \int_0^\alpha \hat{\pi}^a \, da. \]
It is easy to check that \( (\pi^a)_{a \in [0,1]} \) is a well-defined martingale parametrization of the martingale measure \( \pi = \pi^1 \) w.r.t. \( (\nu^a)_{a \in [0,1]} \). Moreover, we have
\[ \pi^a(X_t \in \cdot) = \int_0^\alpha \hat{\pi}^a(X_t \in \cdot) \, da = \int_0^\alpha \hat{\pi}^a \, da = S^{\mu_t}(\nu^a) \]
for all $\alpha \in [0,1]$ and $t \in T$. In particular, (5.4) implies for $\alpha = 1$ that $\pi$ is a solution to the peacock problem w.r.t. $(\mu_t)_{t \in T}$. \hfill $\square$

**Remark 5.7.** Note that the existence of a shadow martingale does not require the parametrization to be $\leq_{c,s}$-convex.

### 5.3. An auxiliary optimization problem.

Recall that we assume $T$ to be at most countable. In this subsection, we will introduce an auxiliary optimization problem over families of peacocks. The main result is a monotonicity principle, i.e. a necessary pointwise optimality condition, similar to $c$-cyclical monotonicity in classical optimal transport, see e.g. [47], or monotonicity principles in stochastic variants of the transport problem, e.g. [13][10]. Our monotonicity principle is similar in spirit to the one recently proved for the weak transport problem [34][39][53].

For a given peacock $(\mu_t)_{t \in T} \in \mathcal{P}_T$ and a family of probability measures $(\tilde{\nu}^a)_{a \in [0,1]}$ such that $a \mapsto \tilde{\nu}^a$ is a measurable map from $[0,1]$ to $\mathcal{P}_1(\mathbb{R})$, we set

$$
\mathcal{A} = \left\{(\theta^a)_{a \in [0,1]} \mid \theta^a \in \mathcal{P}_T, a \mapsto \theta^a \text{ measurable,} \theta^a_0 = \tilde{\nu}^a, \int_0^1 \theta^a_t \, da = \mu_t \right\}.
$$

Let $c : [0,1] \times \mathcal{P}_T \to [0,\infty)$ be a Borel measurable cost function that is linear in the second component. We are interested in properties of solutions to the optimization problem

$$
(5.5) \quad V_A := \inf_{(\theta^a)_{a \in [0,1]} \in \mathcal{A}} \int_0^1 c(a, \theta^a) \, da.
$$

The following monotonicity principle will be essential in the next section.

**Proposition 5.8.** Assume $V_A < \infty$. If $(\theta^a)_{a \in [0,1]}$ is optimal for (5.5), then there exists a measurable set $A \subset [0,1]$ with $\lambda(A) = 1$ such that for all $a < a'$ in $A$ we have

$$
c(a, \theta^a) + c(a', \theta^{a'}) \leq c(a, \theta^a) + c(a', \theta^{a'})
$$

for any two peacocks $(\theta^a_t)_{t \in T}$ and $(\theta^{a'}_t)_{t \in T}$ that satisfy $\theta^a + \theta^{a'} = \theta^a + \theta^{a'}$, $\theta^a_0 = \theta^{a'}_0$ and $\theta^{a'}_0 = \theta^a_0$.

**Proof.** This proof follows closely the proof of [11] Proposition 4.1. Recall that $T$ is at most countable such that $\mathcal{P}_T$ is a Polish space (cf. Lemma 5.1) and thus

$$
\mathcal{B} = \left\{(a,a',\theta^a,\theta^{a'}) \in [0,1]^2 \times \mathcal{P}_T^2 \mid \theta^a + \theta^{a'} = \theta^a + \theta^{a'}, \theta^a_0 = \tilde{\nu}^a, \theta^{a'}_0 = \tilde{\nu}^{a'}, c(a, \theta^a) + c(a', \theta^{a'}) > c(a, \theta^a) + c(a', \theta^{a'}) \right\}
$$

is an analytic set. Likewise the projection of $\mathcal{B}$ onto $[0,1]^2$, denoted by $B$, is analytic. Furthermore Lusin’s theorem [38] Theorem 21.20 states that any analytic set is universally measurable and thus the mass of an analytic set and the integral of an analytically measurable function under any Borel measure are well-defined. We will show that $\xi(B) = 0$ for all couplings $\xi$ of $\lambda$ and $\lambda$. Then [39] Proposition 2.1] (for analytic sets) yields that there exists a $\lambda$-nullset $N$ with $B \subset (N \times [0,1]) \cup ([0,1] \times N)$ and the claim follows by choosing $A = N^c$.

Suppose there exists a coupling $\xi$ of $\lambda$ and $\lambda$ with $\xi(B) > 0$. Then the symmetrized coupling $\xi' = \frac{1}{2} (\xi + s \# \xi)$ where $s : (a, a') \mapsto (a', a)$ is again a coupling of $\lambda$ and $\lambda$ with

\footnote{Recall that a subset of a Polish space is called analytic if it is the image of a continuous function defined on another Polish space. The countable intersection of analytic sets and the preimage of an analytic set under a Borel measurable map are analytic. Since all Borel sets are analytic, we get that $B$ is indeed an analytic set.}
Denote the first (resp. second) component of \( \Phi \) disintegration coincide. For symmetrized and thus the disintegrations w.r.t. the first and the second marginal dis-tribution, \( t \in T \) for all \( a, a' \in B \). We extend \( \tilde{\Phi} \) to an analytically measurable map \( \Phi : [0, 1]^2 \rightarrow \mathbb{P}_T \) by setting

\[
\Phi(a, a') = \begin{cases} 
\tilde{\Phi}(a, a') & (a, a') \in B \\
(\theta^a, \theta^{a'}) & (a, a') \notin B
\end{cases}
\]

Denote the first (resp. second) component of \( \Phi \) by \( \Phi_1 \) (resp. \( \Phi_2 \)). Define

\[
\kappa^a_t = \int_0^1 \Phi_1(a, a') d\xi'(a'), \quad \kappa^a_t = \int_0^1 \Phi_2(a, a') d\xi'(a)
\]

for all \( t \in T \) where \( \xi' \) is the disintegration of \( \xi \) w.r.t. \( \lambda \) (Recall that \( \xi \) is symmetrized and thus the disintegrations w.r.t. the first and the second marginal distribution coincide). For \( a \in [0, 1] \) put \( \kappa^a_0 = \frac{1}{2} (\kappa^a_t + \kappa^a_t) \). Then for all \( a \in [0, 1] \) the family \((\kappa^a_t)_{t \in T} \) is a peacock with \( \kappa^a_0 = \nu^a \). Furthermore, by definition of \( B \),

\[
\int_0^1 \kappa^a_0 da = \int_{[0,1]^2} \left( \frac{\Phi_1(a, a') + \Phi_2(a, a')}{2} \right) d\xi'(a, a')
\]

\[
= \int_{[0,1]^2} \left( \frac{\theta^a + \theta^{a'}}{2} \right) d\xi'(a, a') = \int_0^1 \theta^a_t da = \mu_t
\]

for all \( t \in T \) and thus \((\kappa^a_t)_{t \in [0,1]} \) is an element of \( \mathcal{A} \), i.e. a competitor of \((\theta^a)_{t \in [0,1]} \) in the optimization problem \((5.5)\). Since \( c \) is linear in the second component and \( \xi'(B) > 0 \), it follows that

\[
\int_0^1 c(a, \kappa^a_0) da = \frac{1}{2} \int_0^1 c(a, \kappa^a_0) da + \frac{1}{2} \int_0^1 c(a', \kappa^a_0) da'
\]

\[
= \int_{[0,1]^2} \frac{c(a, \Phi_1(a, a')) + c(a', \Phi_2(a, a'))}{2} d\xi'(a, a')
\]

\[
< \int_{[0,1]^2} \frac{c(a, \theta^a) + c(a', \theta^{a'})}{2} d\xi'(a, a')
\]

\[
= \int_0^1 c(a, \theta^a) da.
\]

This contradicts the fact that \((\theta^a)_{t \in [0,1]} \) is optimal. \[\square\]

**Remark 5.9.** Of course, the same proof works if \( c \) is only convex in the second component instead of being linear.

### 5.4. NSI property for simultaneous optimizers

As a last preparation for the proof of the uniqueness part of Theorem 5.15 in Section 5.5 we will show how the NSI property is closely connected to optimizers of \((5.5)\). For \( t \in T \) we set

\[
c_t(a, \theta) = (1 - a) \int_\mathbb{R} x + \sqrt{1 + x^2} \, d\theta_t.
\]

Clearly, \( c_t \) is an admissible cost function for \((5.5)\) in Subsection 5.3. Throughout this section, we fix some family \((\tilde{\nu}^a)_{t \in [0,1]} \) of probability measures on \( \mathbb{R} \) as input data for the optimization problem \((5.5)\).
The crucial observation, proved in Proposition 5.13, is that, if a family \((\theta^a_{\cdot})_{a \in [0,1]}\) of peacocks minimizes simultaneously the optimization problem (5.5) with cost function \(c_t\) for all \(t \in T\), then \(\theta^a_{\cdot}\) is NSI for \(\lambda\)-almost every \(a\).

**Lemma 5.10.** Let \((\theta_t)_{t \in T}\) and \((\theta'_t)_{t \in T}\) be two peacocks with \(\theta_0 \leq_{c,s} \theta'_0\). There exist two peacocks \((\tilde{\theta}_t)_{t \in T}\) and \((\tilde{\theta}'_t)_{t \in T}\) with

(i) \(\tilde{\theta}_0 = \theta_0, \tilde{\theta}'_0 = \theta'_0\),

(ii) \(\tilde{\theta}_t + \tilde{\theta}'_t = \theta_t + \theta'_t\) for all \(t \in T\) and

(iii) \(\tilde{\theta}_t \leq_{c,s} \theta_t\) and \(\tilde{\theta}_t \leq_{c,s} \theta'_t\) for all \(t \in T\).

**Proof.** Set \(\tilde{\theta}_t = S^{(\theta_t + \theta'_t)}_{c,t}(\theta_0)\) and \(\tilde{\theta}'_t = \theta_t + \theta'_t - \tilde{\theta}_t\) for all \(t \in T\). Both \((\tilde{\theta}_t)_{t \in T}\) and \((\tilde{\theta}'_t)_{t \in T}\) are peacocks by Remark 4.26. They clearly satisfy properties (i) and (ii). Furthermore, for all \(t \in T\) it holds \(\tilde{\theta}_t \leq_{c} S^{\theta_t}_{c,t}(\theta_0) = \theta_t\) by Lemma 4.23 (iv) and

\[\tilde{\theta}_t \leq_{c,s} S^{(\theta_t + \theta'_t)}_{c,t}(\theta_0) \leq_{c} S^{\theta'_t}_{c,t}(\theta_0) = \theta'_t\]

by Lemma 4.23 (iii) and (iv). \(\square\)

Property (ii) and (iii) together imply that we also have \(\tilde{\theta}_t \leq_{c,s} \tilde{\theta}'_t\) and \(\theta'_t \leq_{c,s} \tilde{\theta}'_t\) for all \(t \in T\). Thus, we have sandwiched \(\theta\) and \(\theta'\) between \(\tilde{\theta}\) and \(\tilde{\theta}'\) in convex-stochastic order.

**Lemma 5.11.** If \(x, x', y, y' \in \mathbb{R}\) satisfy \(x + x' = y + y'\) and \(x < y\), then

\[(1 - a)x + (1 - a')x' < (1 - a)y + (1 - a')y'\]

for all \(a < a'\) in \([0,1]\).

**Proof.** The inequality \((1 - a)x + (1 - a')x' < (1 - a)y + (1 - a')y'\) holds if and only if \((a' - a)(y - x) > 0\) because \(y' - x' = x - y\). \(\square\)

**Lemma 5.12.** Let \((\theta^a_{\cdot})_{a \in [0,1]}\) be a minimizer of (5.5) with finite cost \(V_A\) w.r.t. \(c_t\) simultaneously for all \(t \in T\). Then there exists a Borel set \(A \subset [0,1]\) with \(\lambda(A) = 1\) such that for all \(a < a'\) in \(A\) it holds

\[2\theta^a_t - S^{(2\theta^a_{\cdot})_{c,t}}(\theta_0) \leq_{c,s} S^{(2\theta'_{\cdot})_{c,t}}(\theta_0)\]

for all \(t \in T\).

The main idea of the proof is to show that whenever (5.6) is not satisfied for some \(t \in T\), the pair \(((a, \theta^a_{\cdot}), (a', \theta'^a_{\cdot}))\) violates the monotonicity principle in Proposition 5.8 for \(c_t\). However, since the convex-stochastic order is not a total order relation on \(\mathcal{M}_1(\mathbb{R})\), the negation of (5.6) does not imply that the reversed order relation is true but the two measures might just be not comparable in convex-stochastic order. Thus, we use Lemma 5.10 to construct a new pair of competitors that are comparable and bring the essential improvement (cf. (5.8)).

**Proof.** Recall that \(T\) is a countable set. For every \(t \in T\), there exists by Proposition 5.8 a Borel set \(A_t \subset [0,1]\) with \(\lambda(A_t) = 1\) such that for all \(a < a'\) in \(A_t\) we have

\[c_t(a, \theta^a_{\cdot}) + c_t(a', \theta'^a_{\cdot}) \leq c_t(a, \theta'_{\cdot}) + c_t(a', \theta^a_{\cdot})\]

where \((\theta'_t)_{t \in T}\) and \((\theta'^a_t)_{t \in T}\) are any two peacocks with \(\theta' + \theta'^a = a + a'\), \(\theta'_0 = \theta'_0\) and \(\theta'^0_a = \theta'^a_0\).
Put $A = \bigcap_{t \in T} A_t$ and note that $\lambda(A) = 1$. For all $a \in A$ and $t \in T$ we define

$$\theta_t^a := S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a)$$

and $\theta_t^{a+} := 2\theta_t^a - \theta_t^{-}$.

We want to show that $\theta_t^{a+} \leq c,s \theta_t^{a'}$ for all $t \in T$ and all $a < a'$ in $A$. Suppose this is not the case for some $u \in T$ and $a < a'$ in $A$. By Lemma 5.10 there exist two peacocks $\tilde{\theta}$ and $\tilde{\theta}'$ with $\bar{\theta}_0 = \theta_0^{a+} = \tilde{\theta}_0, \bar{\theta}_0' = \theta_0^{a'+} = \tilde{\theta}_0', \tilde{\theta} + \tilde{\theta}' = \theta^{a+} + \theta^{a'-}$, and

$$\tilde{\theta}_t \leq c,s \theta_t^{a+} \quad \text{and} \quad \tilde{\theta}_t \leq c,s \theta_t^{a'}$$

for all $t \in T$. The inequality $\tilde{\theta}_u \leq c,s \theta_u^{a+}$ cannot be an equality because this would imply that $\theta_u^{a+} = \tilde{\theta}_u \leq c,s \theta_u^{a'}$, which we supposed to be false. Hence, it holds

(5.8) $c_u(a, \tilde{\theta}) < c_u(a, \theta^{a+})$

because $x \mapsto x + \sqrt{1 + x^2}$ is strictly increasing and strictly convex. Next, we use $(\tilde{\theta}, \tilde{\theta}')$ to construct a competitor $(\theta', \theta'')$ for $(\theta^a, \theta^{a'})$ in the sense of Proposition 5.8. We set

$$\theta' = \frac{1}{2} \theta^{a+} + \frac{1}{2} \theta^{a'-} \quad \text{and} \quad \theta'' = \frac{1}{2} \theta' + \frac{1}{2} \theta^{a+}.$$ 

The pair $(\theta', \theta'')$ of peacocks is indeed a competitor since $\theta' + \theta'' = \theta^a + \theta^{a'}$, $\theta_0^a = \theta_0^{a'}$ and $\theta_0^{a+} = \theta_0^{a'+}$. Moreover, for $t = u$ it holds

$$c_u(a, \theta') + c_u(a', \theta'') = \frac{1}{2} \left( c_u(a, \theta^{a+}) + c_u(a, \theta^{a'}) + c_u(a', \tilde{\theta}') + c_u(a', \theta^{a'+}) \right)$$

$$< \frac{1}{2} \left( c_u(a, \theta^{a+}) + c_u(a, \theta^{a'}) + c_u(a', \theta^{a+}) + c_u(a', \theta^{a'+}) \right) = c_u(a, \theta^a) + c_u(a', \theta^{a'}$$

by the linearity of $c_u$ in the second component and Lemma 5.11 in conjunction with (5.8). This is a contradiction of (5.7). □

**Proposition 5.13.** Let $(\theta^a)_{a \in [0,1]}$ be a simultaneous minimizer of (5.3) with finite cost $V_A$ w.r.t. $c_t$ for all $t \in T$. Then $(\theta_t^a)_{t \in T}$ is NSI for a.e. $a \in [0,1]$.

**Proof.** By Lemma 5.12 there exists a Borel set $A \subset [0,1]$ with $\lambda(A) = 1$ such that for all $a < a'$ in $A$ and $t \in T$ it holds

(5.9) $2\theta_t^a - S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a) \leq c,s S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a)$

Moreover, by Proposition 4.25 and Lemma 1.23 (ii) it holds

(5.10) $S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a) \leq c,s 2\theta_t^a - S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a)$

for all $a \in [0,1]$ and $t \in T$. Hence, the map

$$a \mapsto S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a)$$

is increasing on $A$ in convex-stochastic order for all $t \in T$. If $\theta^a$ is not NSI for some $a \in A$, there exists at least one $t \in T$ for which (5.10) is not an equality (see Lemma 4.28) and thus the map $a \mapsto S_{(\theta_t^{a'})_{\in T_t}}(\theta_0^a)$ has a discontinuity at $a$ because of (5.9). But since the map is increasing in convex-stochastic order, Corollary 3.7 yields that this can only happen for countably many $a \in A$ for a given $t \in T$. The set $T$ is countable, and hence we obtain that $\theta^a$ is NSI for $\lambda$-a.e. $a$. □

**Remark 5.14.** Referring back to (1.3) the last proposition establishes the decomposition of a peacock into NSI peacocks, cf. (1.15) under the assumption that there is a suitable cost function $c$ such that for a given parametrization $(\nu^a)_{a \in [0,1]}$ there is an optimizer (with finite value) to (5.3). For $\leq c,s$-convex parametrizations this assumption will be established in the next subsection.
5.5. Uniqueness of the shadow martingale.

**Theorem 5.15.** Let $T \subset [0, \infty)$ be a countable index set with $0 \in T$ and $\sup T \in T$ and let $(\nu^\alpha)_{\alpha \in [0,1]}$ be a parametrization of $\mu_0$ that is $\leq_{c,a}$-convex. There exists a unique pair $(\pi^\alpha)_{\alpha \in [0,1]}$ where the martingale measure $\pi \in M_T((\mu_t)_{t \in T})$ solves the peacock problem w.r.t. $(\mu_t)_{t \in T}$, $(\nu^\alpha)_{\alpha \in [0,1]}$ is a martingale parametrization of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ and for all $\alpha$ and $t$ in $T$

\begin{equation}
\pi^\alpha(X_t \in \cdot) = S^\mu_{\pi_t}(\nu^\alpha).
\end{equation}

Moreover, there exists a Borel set $A \subset [0,1]$ with $\lambda(A) = 1$ such that for all $a \in A$ the map $\alpha \mapsto \pi^\alpha$ is right-differentiable at $a$ and the marginals of the right-derivative $\hat{\pi}^a$ at a form a NSI peacock. In particular, $\hat{\pi}^a$ is a Markov martingale measure uniquely defined by its marginal distributions.

**Proof.** We have already proven the existence of a martingale measure $\pi$ and a corresponding martingale parametrization $(\pi^\alpha)_{\alpha \in [0,1]}$ that satisfies (5.11) in Proposition 5.6. Hence, it remains to prove the uniqueness of the family $(\pi^\alpha)_{\alpha \in [0,1]}$.

Let $(\rho^\alpha)_{\alpha \in [0,1]}$ be a martingale parametrization that satisfies (5.11). Lemma 5.2 yields that $\alpha \mapsto \rho^\alpha$ is a.e. right-differentiable and the right-derivatives $(\hat{\rho}^\alpha)_{\alpha \in [0,1]}$ are a family in $\mathcal{M}_T$. These right-derivatives determine $(\rho^\alpha)_{\alpha \in [0,1]}$ uniquely (see Corollary 5.3) and their marginal distributions $(\text{Law}_{\rho^\alpha}(X_t))_{t \in T}$ are determined by (5.11). Hence, the marginal distributions of $\hat{\rho}^\alpha$ coincide with those of $\hat{\pi}^\alpha$ which are denoted by $(\hat{\eta}^\alpha)_{t \in T}$ in $P_T$ (cf. Lemma 5.4). Thus, if $(\hat{\eta}^\alpha)_{t \in T}$ is NSI for $\lambda$-a.e. $a \in [0,1]$, Proposition [1,30] implies that $\hat{\pi}^\alpha = \hat{\rho}^\alpha$ for $\lambda$-a.e. $a \in [0,1]$ and therefore we obtain $\pi^\alpha = \rho^\alpha$ for all $\alpha \in [0,1]$ by Corollary 5.3. Moreover, this would imply that $\hat{\pi}^a$ is Markov for $\lambda$-a.e. $a \in [0,1]$ proving the theorem.

Hence, we need to show that $(\hat{\eta}^\alpha)_{t \in T}$ is a NSI peacock for $\lambda$-a.e. $a$. By Proposition 5.13 it is sufficient to show that $((\hat{\eta}^\alpha)_{t \in T})_{\alpha \in [0,1]}$ is a solution to the optimization problem w.r.t. $c_t$ and $(\hat{\rho}^\alpha)_{\alpha \in [0,1]}$ simultaneously for all $t \in T$. In the current setup we then have

$$
\mathcal{A} = \left\{(\theta^\alpha)_{\alpha \in [0,1]} \mid \theta^\alpha \in P_T, a \mapsto \theta^\alpha \text{ measurable,} \quad \theta_0^\alpha = \hat{\rho}^\alpha, \quad \int_0^\alpha \theta^\alpha_t \, \text{da} = \mu_t \right\}.
$$

It is easy to see that $(\hat{\eta}^\alpha)_{t \in T} \in \mathcal{A}$ and $\int_0^\alpha \hat{\eta}^\alpha_t \, \text{da} = S^\mu_{\pi_t}(\nu^\alpha)$ for all $t \in T$ and $\alpha \in [0,1]$. By the minimality of shadows (cf. Proposition [1,22]), any competitor $(\tilde{\theta}^\alpha)_{\alpha \in [0,1]} \in \mathcal{A}$ satisfies $\int_0^\alpha \tilde{\theta}^\alpha_t \, \text{da} \leq_c \int_0^\alpha \hat{\eta}^\alpha_t \, \text{da}$. Hence, it follows

$$
\int_0^1 c_t(a, \tilde{\theta}^\alpha) \, \text{da} = \int_0^1 \left( \int_a^1 \int_\mathbb{R} \frac{x + \sqrt{1 + x^2}}{2} \, \text{d}\tilde{\theta}^\alpha_t \, \text{da} \right) \, \text{da}
$$

$$
= \int_0^1 \left( \int_\mathbb{R} \frac{x + \sqrt{1 + x^2}}{2} \, \text{d} \left( \int_0^\alpha \tilde{\theta}^\alpha_t \, \text{da} \right) \right) \, \text{da}
$$

$$
\geq \int_0^1 \left( \int_\mathbb{R} \frac{x + \sqrt{1 + x^2}}{2} \, \text{d} \left( \int_0^\alpha \hat{\eta}^\alpha_t \, \text{da} \right) \right) \, \text{da}
$$

$$
= \int_0^1 c_t(a, \hat{\eta}^\alpha) \, \text{da}
$$

for all $t \in T$. This proves the claim. 

\qed
Definition 6.1. (i) We call a peacock $\tilde{\pi}$ a right-continuous map from $T$ to $P_1(\mathbb{R})$ w.r.t. $T_1$ if for all $t \in T$ and $\omega \in \Omega$ we have $\tilde{\pi}_t(\omega) = \tilde{\pi}_u(\omega)$ for all $u \leq t$ in the right-continuous topology.

Remark 5.16. Let $(\Omega_a)_{a \in [0,1]}$ be uncountably many copies of $\mathbb{R}^T$ and set
$$\Omega = [0,1] \times \prod_{a \in [0,1]} \Omega_a.$$ We equip $\Omega$ with the product $\sigma$-algebra and denote by $\mathbb{P}$ the product measure on $\Omega$ generated by $\lambda$ on $[0,1]$ and $\pi^a$ on $\Omega_a$ for all $a \in [0,1]$ (on the Lebesgue-nullset where the right-derivative $\tilde{\pi}^a$ are not defined we choose the Dirac mass of the null-path). It is easy to see that the random variables
$$U(\omega_0, (\omega_a)_{a \in [0,1]}) = \omega_0 \quad \text{and} \quad M^a(\omega_0, (\omega_a)_{a \in [0,1]}) = \omega_a, \ a \in [0,1],$$
satisfy the assertions of Theorem 1.5.

6. Càdlàg shadow martingales indexed by a continuous time set

In this section we show how the results of the previous section can be lifted to the setting of a continuous time index set $T \subset [0, \infty)$ with minimal element 0 $\in T$ under the additional assumption that the given peacock $(\mu_t)_{t \in T}$ is right-continuous, i.e. the map $t \mapsto \mu_t$ is a right-continuous map from $T$ to $P_1(\mathbb{R})$ (under $T_1$).

The key observation is that in the current setup martingale measures $\pi$ (similarly for martingale parametrizations) are uniquely determined by the restriction to a well chosen countable index set $S$, i.e. there exists a unique martingale measure $\pi$ extending $\pi_{|S}$ to the index set $T$ (see Lemma 6.5). This will be established in Section 6.1. In Subsection 6.2 we will show that also the obstructed shadow only depends on $\mu_S$ for some countable family $S$ if $\mu_T$ is a peacock. Consequently, also the NSI property only depends on $\mu_S$ by Lemma 1.28. These results will allow us to provide a proof of Theorem 6.1, a variant of Theorem 1.5 in the case of a continuous time index set $T$ and a right-continuous peacock $\mu_T$ in Subsection 6.3

6.1. Continuous time martingale measures. We fix a subset $T \subset [0, \infty)$ with $0 \in T$ and we equip $T$ with the inherited standard topology.

Recall that a modification of the canonical process $(X_t)_{t \in T}$ under $\pi \in P_1(\mathbb{R}^T)$ is a process $\tilde{X} : \mathbb{R}^T \rightarrow \mathbb{R}$ such that $\pi(\tilde{X}_t = X_t) = 1$ for all $t \in T$. Note that Law$_\pi(\tilde{X}) = \text{Law}_\pi(X)$ and, if $T$ is countable, we get $\tilde{X} = X \pi$-a.e.

Definition 6.1. (i) We call a peacock $(\mu_t)_{t \in T}$ right-continuous, if the map $t \mapsto \mu_t$ is a right-continuous map from $T$ to $P_1(\mathbb{R})$ w.r.t. $T_1$. We denote by $P^T_1 \subset P_T$ the set of all right-continuous peacocks.

(ii) We call a martingale measure $\pi \in M_T$ a càdlàg martingale measure, if there exists a modification $(\tilde{X}_t)_{t \in T}$ of the canonical process under $\pi$ such that $t \mapsto \tilde{X}_t(\omega)$ is a càdlàg function for all $\omega \in \mathbb{R}^T$. We denote the set of all càdlàg martingale measures by $M^T_\pi$.

(iii) We call a martingale parametrization $(\pi^a)_{a \in [0,1]}$ of a càdlàg martingale measure $\pi \in M^T_\pi$ càdlàg, if $\frac{1}{a} \pi^a \in M^T_\pi$ for all $a \in (0,1]$.

Remark 6.2. (i) Lemma 5.5 shows that a peacock is right-continuous (w.r.t. $T_1$) if and only if $t \mapsto \mu_t$ is right-continuous w.r.t. $T_0$ because for all $t \in T$ and $n \downarrow t$ the measures $\mu_t$ and $(\mu_{n_t})_{n \in \mathbb{N}}$ are bounded from above in convex order by $\mu_t$.

(ii) Let $\pi \in M^T_\pi$ and $(\tilde{X}_t)_{t \in T}$ be a modification of the canonical process $(X_t)_{t \in T}$ under $\pi$. Then $(\tilde{X}_t)_{t \in T}$ is a $(\tilde{F}_t)_{t \in T}$-martingale under $\pi$ where $\tilde{F}_t = \sigma(\{\tilde{X}_s : s \in [0,t] \cap T\})$ because $\pi(\tilde{X}_t = X_t) = 1$ for all $t \in T$. 

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We equip both $P_T^F$ and $M_T^F$ with the subspace topology inherited from the product topology on $(P_1(\mathbb{R}))^T$ and $T_1$ on $M_T$. Moreover, we use the notation $M_T^R((\mu_t)_{t \in T})$ for càdlàg martingale measures associated with a peacock $(\mu_t)_{t \in T}$.

Note that the right-continuity of the peacock corresponds to the càdlàg property of the martingale measure:

**Lemma 6.3.** Let $(\mu_t)_{t \in T} \in P_T$ and $\pi \in M_T$ be associated with $(\mu_t)_{t \in T}$.

(i) If $\pi$ is a càdlàg martingale measure, then $(\mu_t)_{t \in [0,1]} \in P_T^R$.

(ii) If $(\mu_t)_{t \in T}$ is a right-continuous peacock, then $\pi \in M_T^F$.

**Proof.** First suppose that $\pi \in M_T^F$. Let $(\tilde{X}_t)_{t \in T}$ be a càdlàg modification of the canonical process under $\pi$ and $\varphi \in C_b(\mathbb{R})$. For any $t \in T$ and all sequences $(t_n)_{n \in \mathbb{N}}$ that converge to $t$ from above, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} \varphi \, d\mu_{t_n} = \lim_{n \to \infty} \mathbb{E}_\pi[\varphi(\tilde{X}_{t_n})] = \mathbb{E}_\pi[\varphi(\tilde{X}_t)] = \int_{\mathbb{R}} \varphi \, d\mu_t$$

by dominated convergence. By Remark 6.2 this convergence also holds under $T_1$.

Conversely, if the marginal distributions are right-continuous, it is a standard result that a martingale has a modification $(\tilde{X}_t)_{t \in T}$ with càdlàg paths (see [46] §2 Theorem 2.8). Remark 6.2 finishes the proof. □

**Lemma 6.4.** There exists a countable set $S \subset T$ that is right-dense in $T$, i.e. for all $t \in T$ there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in $\{s \in S : s \geq t\}$ that converges to $t$. Similarly, there exists a countable left-dense subset of $T$ that is both right- and left-dense.

**Proof.** Let $S_1 = \{t \in T \mid \exists \varepsilon > 0, \ T \cap (t, t + \varepsilon) = \emptyset\}$ be the set of points that are “right-isolated” in $T$ and $S_2$ be a countable dense subset of $T$. Since $S_1$ is countable so is $S := S_1 \cup S_2$. Moreover, it is not difficult to check that any $t \in T$ is the limit of a sequence in $S \cap [t, \infty)$. Thus, $S$ is a countable right-dense subset of $T$. □

The following Lemmas 6.5–6.7 link càdlàg martingale measures and càdlàg martingale parametrizations to their restrictions to a suitable right-dense set:

**Lemma 6.5.** Let $S$ be a countable right-dense subset of $T$.

(i) If $\pi, \rho \in M_T^F$ satisfy $\pi|_S = \rho|_S$, then $\pi = \rho$.

(ii) For all $\pi' \in M_T^R$, there exists a $\pi \in M_T^F$ such that $\pi|_S = \pi'$ (with (i) it is uniquely determined).

**Proof.** Item (i): Let $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ and $\varphi \in C_b(\mathbb{R}^n)$. We can find sequences $(s_k^i)_{k \in \mathbb{N}}$ in $S$, $1 \leq i \leq n$, such that $s_k^i \searrow t_i$ for all $i \in \{1, \ldots, n\}$. Since there exist càdlàg modifications of the canonical process under $\pi$ and $\rho$, we deduce using $\pi|_S = \rho|_S$

$$\int_{\mathbb{R}^n} \varphi(X_{t_1}, \ldots, X_{t_n}) \, d\pi = \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi(X_{s_k^1}, \ldots, X_{s_k^n}) \, d\pi = \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi(X_{s_k^1}, \ldots, X_{s_k^n}) \, d\rho = \int_{\mathbb{R}^T} \varphi(X_{t_1}, \ldots, X_{t_n}) \, d\rho.$$

Item (ii): Since $\pi' \in M_T^R$, there exists a modification $(\tilde{X}_s)_{s \in S}$ of the canonical process $(X_s)_{s \in S}$ on $\mathbb{R}^S$ under $\pi'$ such that any path $s \mapsto X_s(\omega)$ is càdlàg. Furthermore, also
the limits $\lim_{s \uparrow t, s \in S} \hat{X}_s$ and $\lim_{s \uparrow t, s \in S} \hat{X}_s$ for $t \in T \setminus S$ exist (see [11] Proposition 1). We define

$$
(6.1) \quad \mathbb{R}^S \ni \omega \mapsto Y_t(\omega) = \begin{cases} 
\hat{X}_t(\omega) & t \in S \\
\lim_{s \uparrow t} \hat{X}_s(\omega) & t \notin S
\end{cases}
$$

The family $(Y_t)_{t \in T}$ is a well-defined process on the probability space $(\mathbb{R}^S, \mathcal{F}^S, \pi)$ where $\mathcal{F}^S$ denotes the $\sigma$-algebra $\mathcal{F}^S = \bigvee_{s \in S} \mathcal{F}_s$. Here, for definiteness, we denote the canonical filtration on $\mathbb{R}^S$ by $(\mathcal{F}^S_t)_{t \in S}$. Moreover, $Y$ is a martingale w.r.t. the right-continuous filtration $(\mathcal{F}^r_t)_{t \in T}$ on $\mathbb{R}^T$ given by $\mathcal{F}^r_t = \bigcap_{s > t, s \in S} \mathcal{F}_s$ for all $t \in T$ that are not the maximal element and $\mathcal{F}^r = \mathcal{F}^S$ if there exists a maximal element $t^* \in T$. Set $\pi = \text{Law}_{\pi'}(Y) \in \mathcal{P}_1(\mathbb{R}^T)$. Then $\pi$ is a martingale measure because $Y$ is a martingale under $\pi'$. By the càdlàg property of $Y$, the marginal distributions of the canonical process on $\mathbb{R}^T$ under $\pi$ are right-continuous. Hence, $\pi \in \mathcal{M}_T^R$ by Lemma 6.3. By construction, it holds $\pi_{|S} = \pi'$ and item (i) yields that this extension is unique. 

Recall that we fixed an arbitrary index set $T \subset [0, \infty)$ with $0 \in T$.

**Lemma 6.6.** Suppose $T$ admits a maximal element $t^*$ and let $(\pi^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_T^R$.

(i) Let $\pi \in \mathcal{M}_T^R$ and $S \subset T$ be a countable right-dense subset of $T$. If $(\pi^n_{|S})_{n \in \mathbb{N}}$ converges to $\pi_{|S}$ in $\mathcal{M}_S^R$ (under $T_1$) and $(\text{Law}_{\pi^n}(X_t))_{n \in \mathbb{N}}$ converges to $\text{Law}_{\pi}(X_t)$ in $\mathcal{P}_1(\mathbb{R})$ for all $t \in T$, then $(\pi^n)_{n \in \mathbb{N}}$ converges to $\pi$ in $\mathcal{M}_T^R$.

(ii) If the sequence of right-continuous peacocks $((\text{Law}_{\pi^n}(X_t)))_{n \in \mathbb{N}}$ is convergent in $\mathcal{P}_T^R$ with limit $(\mu_t)_{t \in T}$, then there exists a convergent subsequence of $(\pi^n)_{n \in \mathbb{N}}$ in $\mathcal{M}_T^R$ with limit $\pi \in \mathcal{M}_T^R((\mu_t)_{t \in T})$.

**Proof.** Let $S' \subset T$ with $t^* \in S'$ and suppose that $A$ is a compact subset of $\mathcal{P}_1(\mathbb{R})$ under $T_1$. A slight modification of [11] Lemma 1] shows that the set

$$(6.2) \quad \{ \rho \in \mathcal{M}_{S'} \mid \text{Law}_\rho(X_{t^*}) \in A \}$$

is a compact subset of $\mathcal{M}_{S'}$ (under $T_1$).

Item (i): By the definition of the topology $T_1$ on $\mathcal{P}(\mathbb{R}^T)$, the sequence $(\pi^n)_{n \in \mathbb{N}}$ converges to $\pi$ under $T_1$ if and only if $(\pi^n_{|S \cup R})_{n \in \mathbb{N}}$ converges to $\pi_{|S \cup R}$ in $\mathcal{M}_{S \cup R}^R$ under $T_1$ for all finite $R \subset T$. Thus, it is sufficient to show that for any finite subset $R \subset T$, any subsequence of $(\pi^n_{|S \cup R})_{n \in \mathbb{N}}$ has a subsequence that converges to $\pi_{|S \cup R}$ under $T_1$.

Let $R \subset T$ finite and note that $S$ contains $t^*$. Let $(\pi^n_{|S \cup R})_{n \in \mathbb{N}}$ be an arbitrary subsequence of $(\pi^n_{|S \cup R})_{n \in \mathbb{N}}$. Since $(\text{Law}_{\pi^n}(X_{t^*}))_{n \in \mathbb{N}}$ converges to $\text{Law}_{\pi}(X_{t^*})$ under $T_1$, the set $A_1 := \{ \text{Law}_{\pi^n}(X_{t^*}) : n \in \mathbb{N} \} \cup \{ \text{Law}_{\pi}(X_{t^*}) \} \subset \mathcal{P}_1(\mathbb{R})$ is compact w.r.t. $T_1$. Hence, the set in (6.2) with $A := A_1$ and $S' := S \cup R$ is a compact subset of $\mathcal{M}_{S \cup R}$. Consequently, $(\pi^n_{|S \cup R})_{n \in \mathbb{N}}$ has itself a convergent subsequence with limit $\rho^* \in \mathcal{M}_{S \cup R}$. By assumption, the marginal distributions of $\rho^*$ have to coincide with the right-continuous marginal distributions of $\pi_{|S \cup R}$, and thus $\rho^* \in \mathcal{M}_{S \cup R}^R$ by Lemma 6.3 ii). Since $(\pi^n_{|S})_{n \in \mathbb{N}}$ is convergent to $\pi_{|S}$, it holds $\rho^*_{|S} = \pi_{|S}$ and therefore Lemma 6.3 i) yields $\rho^* = \pi_{|S \cup R}$.

Item (ii): Let $S$ be a countable right-dense subset of $T$ that includes $t^*$. Since $(\text{Law}_{\pi^n}(X_{t^*}))_{n \in \mathbb{N}}$ converges to $\mu_{t^*}$ under $T_1$, the set $A_2 := \{ \text{Law}_{\pi^n}(X_{t^*}) : n \in \mathbb{N} \} \cup \{ \mu_{t^*} \} \subset \mathcal{P}_1(\mathbb{R})$ is compact w.r.t. $T_1$. Hence, the set in (6.2) with $A := A_2$ and $S' := S$ is a compact subset of $\mathcal{M}_S$ under $T_1$. Therefore, there exists a convergent subsequence...
Lemma 6.7. Let $T \subset [0, \infty)$ and $S$ be a countable right-dense subset of $T$.

(i) If $(\pi^\alpha)_{\alpha \in [0,1]}$ is a martingale parametrization of the càdlàg martingale measure $\pi$, then $(\pi^\alpha)_{\alpha \in [0,1]}$ is a càdlàg martingale parametrization of $\pi|_S$.

(ii) Let $(\tilde{\pi}^\alpha)_{\alpha \in [0,1]}$ be a càdlàg martingale parametrization of $\tilde{\pi} \in \mathcal{M}^\otimes_T$ and $\pi \in \mathcal{M}^\otimes_T$ the unique extension of $\tilde{\pi}$ given by Lemma 6.3, i.e. $\pi|_S = \tilde{\pi}$. There exists a unique càdlàg martingale parametrization $(\pi^\alpha)_{\alpha \in [0,1]}$ of $\pi$ such that $\pi^\alpha|_S = \tilde{\pi}^\alpha$ for all $\alpha \in [0,1]$.

(iii) Let $(\pi^\alpha)_{\alpha \in [0,1]}$ be a martingale parametrization of $\pi \in \mathcal{M}^\otimes_T$. If $\alpha \mapsto \pi^\alpha|_S$ is right-differentiable at $a \in [0,1]$ (in the sense of Lemma 6.2) and there exists $(\eta_t)_{t \in T} \in \mathcal{P}^\otimes_T$ such that the law of $X_t$ under $\frac{1}{h}(\pi^{a+h} - \pi^a)$ converges under $T_1$ to $\eta_t$ for all $t \in T$, then the map $\alpha \mapsto \pi^\alpha$ is right-differentiable at $a$, i.e.

$$\hat{\pi}^a = \lim_{h \downarrow 0} \frac{\pi^{a+h} - \pi^a}{h}$$

exists as a limit in $\mathcal{P}_1(\mathbb{R}^T)$ under $T_1$. Moreover, the right-derivative of $\alpha \mapsto \pi^\alpha$ at $a$ is an element of $\mathcal{M}^\otimes_T((\eta_t)_{t \in T})$ and its restriction to $S$ is the right-derivative of $\alpha \mapsto \pi^\alpha|_S$ at $a$.

Proof. Item (i): It is straightforward to check that $(\pi^\alpha)_{\alpha \in [0,1]}$ is a (càdlàg) martingale parametrization of $\pi|_S \in \mathcal{M}^\otimes_S$.

Item (ii): For all $\alpha \in [0,1]$ let $\frac{1}{\alpha}\pi^\alpha \in \mathcal{M}^\otimes_T$ be the unique extension of $\frac{1}{\alpha}\tilde{\pi}^\alpha$ given by Lemma 6.3 (ii). Then $\pi^\alpha(\mathbb{R}^T) = \alpha$ and $\pi^1 = \pi$. Finally, $\pi^\alpha \leq \pi^\beta$ follows by considering nonnegative cylinder functions (because they generate the $\sigma$-algebra on $\mathbb{R}^T$).

Item (iii): Recall that the martingale property of a sequence in $\mathcal{P}_1(\mathbb{R}^T)$ is preserved under convergence in $T_1$. The claim follows from Lemma 6.6 (i). \hfill $\Box$

6.2. Shadows obstructed by peacocks and the NSI property. In this section we consider shadows in the special – and for us most important – case in which $(\mu_t)_{t \in T}$ is a peacock. A particular consequence of this assumption, which is not true without the peacock assumption, is that the shadow is uniquely determined by a countable subset of marginal obstructions, i.e. by marginal constraints $(\mu_t)_{t \in S}$ for a countable set $S \subset T$. Combining this with Lemma 5.28 implies that also the NSI property is determined by a well chosen countable subset of obstructions.

Lemma 6.8. Let $\nu \leq c_{\varepsilon,+} \mu_T$ and $s \in T$.

(i) If $t \mapsto \mu_t$ is left-continuous at $s$ w.r.t. $T_1$, then $t \mapsto S^{\mu_T}(\nu)$ is left-continuous at $s$ w.r.t. $T_1$.

(ii) If $t \mapsto \mu_t$ is right-continuous at $s$ w.r.t. $T_1$, then $t \mapsto S^{\mu_T}(\nu)$ is right-continuous at $s$ w.r.t. $T_1$.

Recall that since $\mu_T$ is a peacock, left- and right-continuity of $t \mapsto \mu_t$ is independent from the choice of $T_1$ or $T_0$ (cf. Remark 6.2 (i)).
Proof. Item (i): Let \((t_n)_{n \in \mathbb{N}}\) be a sequence that converges to \(s\) from below. We define the family \((\eta_t)_{t \in T_n}\) in \(\mathcal{M}_1(\mathbb{R})\) by

\[
\eta_t = \begin{cases} 
  S^\mu_{t_t}(\nu) & t < s, \\
  \lim_{t \to s} S^\mu_{t_t}(\nu) & t = s.
\end{cases}
\]

The limit \(\lim_{t \to s} S^\mu_{t_t}(\nu)\) exists by Lemma 3.11 (ii) in conjunction with Lemma 4.21 (i). Clearly, \(\nu \leq c \eta_t \leq c \eta_u\) for all \(t \leq u\) in \(T_s\) and \(\eta_t \leq + \mu_t\) for all \(t < s\) in \(T\). Moreover, since \(\eta_s = \lim_{n \to \infty} \eta_{t_n}\) and \(\eta_{t_n} \leq + \mu_{t_n}\) by Lemma 4.21 (v), we obtain \(\eta_s \leq + \mu_s\) from Lemma 3.2. The claim follows by Proposition 4.22.

Item (ii): Let \((t_n)_{n \in \mathbb{N}}\) be a sequence that converges to \(s\) from above. It holds

\[
S^\mu_{t_n}(\nu) = S^{\mu_{t_n}}_{t_n}(S^{\mu_{t_n}}(\nu)) \text{ for all } n \in \mathbb{N} \text{ where } T_{s,t_n} = \{ t \in T : s < t \leq t_n \}.
\]

Proposition 4.20 states that there exists a sequence of finite sets \((R_k)_{k \in \mathbb{N}}\) such that \((S^\mu_{R_k}(S^{\mu_{R_k}}(\nu)))_{k \in \mathbb{N}}\) converges to \(S^{\mu_{t_n}}_{t_n}(S^{\mu_{t_n}}(\nu)) = S^\mu_{t_n}(\nu)\) and a well-chosen telescopic application of Corollary 4.13 to \((S^\mu_{R_k}(S^{\mu_{R_k}}(\nu)))_{k \in \mathbb{N}}\) in conjunction with Lemma 3.2 implies

\[
U(S^\mu_{R_k}(S^{\mu_{R_k}}(\nu))) - U(S^{\mu_{R_k}}(\nu)) \leq U(\mu_{\max R_k}) - U(\mu_s) \leq U(\mu_{t_n}) - U(\mu_s).
\]

Letting \(k\) tend to infinity yields \(U(S^{\mu_{t_n}}(\nu)) - U(S^{\mu_{t_n}}(\nu)) \leq U(\mu_{t_n}) - U(\mu_s)\) and since \((\mu_t)_{t \in T}\) is right-continuous, by Lemma 3.6 (ii) we obtain

\[
0 \leq \lim_{n \to \infty} U(S^{\mu_{t_n}}(\nu)) - U(S^{\mu_{t_n}}(\nu)) \leq \lim_{n \to \infty} U(\mu_{t_n}) - U(\mu_t) = 0.
\]

Proof. Fix \(t \in T\). The sequence \((S^\mu_{(R_n)_n}(\nu))_{n \in \mathbb{N}}\) is monotonically increasing in convex order as \(n\) tends to infinity and is bounded in convex-positive order by \(\mu_t\). Hence, Lemma 3.11 (ii) yields that this sequence is converging in \(\mathcal{M}_1(\mathbb{R})\) under \(T_1\). We denote the limit by \(\eta_t\) and set \([t]_n := \max(R_n)_t\). Lemma 4.21 (v) implies that \(S^{\mu_{(R_n)_n}}(\nu) \leq + \mu_{[t]_n}\) and since \(S\) is both right- and left-dense in \(T_1\), \(([t]_n)_{n \in \mathbb{N}}\) converges to \(t\) from below.

If \(t \in S\), then \([t]_n = t\) for \(n\) large enough, and, if \(t \notin S\), then \((\mu_{[t]_n})_{n \in \mathbb{N}}\) converges to \(\mu_t\) under \(T_1\). Hence, in both cases Lemma 3.2 implies that \(\eta_t \leq + \mu_t\). By Lemma 3.2 the convex-order is preserved under convergence in \(T_1\) and thus it holds

\[
(6.3) \quad \nu \leq c \eta_t \leq c \eta_t \leq + \mu_t
\]

for all \(s \leq t\) in \(T\). Fix \(u \in T\). Any other family \((\eta_t')_{t \in T}\) which satisfies the relations in (6.3) for all \(s \leq t\) in \(T_u\), satisfies them in particular for all \(s \leq t\) in \((R_n)_u\). Thus, Proposition 4.22 applied to \((R_n)_u\) yields \(S^{\mu_{(R_n)_u}}(\nu) \leq c \eta'_{[t]_n}\), and therefore

\[
\eta_u = \lim_{n \to \infty} S^{\mu_{(R_n)_u}}(\nu) \leq c \sup \{ \eta'_{[t]_n} : n \in \mathbb{N} \} \leq c \eta'_u.
\]
As a consequence of Proposition 4.22 applied to the index set $T_u$ we get for all $u \in T$
\begin{equation}
S^{\mu_{T_u}}(\nu) = \eta_u = \lim_{n \to \infty} S^{\mu_{(R_n)\uparrow}}(\nu) \quad \square
\end{equation}

Corollary 6.10. There exists a countable right- and left-dense set $S \subset T$ such that
\begin{equation}
S^{\mu_S}(\nu) = S^{\mu_T}(\nu)
\end{equation}
for all $t \in T$ and for all $\nu \leq_+ \mu_0$. In fact, $S$ can be any countable right- and left-dense subset of $T$ which contains all discontinuity points of the map $t \mapsto \mu_t$.

Proof. Pick by Corollary 5.17 a countable right- and left-dense subset $S \subset T$ such that $t \mapsto \mu_t$ is continuous on $T \setminus S$. Since $S$ is countable, there exists a nested sequence $(R_n)_{n \in \mathbb{N}}$ such that $\bigcup_{n \in \mathbb{N}} R_n = S$. Let $t$ be in $T$. Lemma 6.9 yields
\begin{equation}
S^{\mu_{R_n}}(\nu) = \lim_{n \to \infty} S^{\mu_{(R_n)\uparrow}}(\nu)
\end{equation}
for all $\nu \leq_+ \mu_0$ and Lemma 4.21 (iii) implies that $(S^{\mu_{(R_n)\uparrow}}(\nu))_{n \in \mathbb{N}}$ converges to $S^{\mu_S}(\nu)$. \quad \square

Recall that the index set $T$ satisfies $T \subset [0, \infty)$ with $0 \in T$.

Corollary 6.11. Let $(\mu_t)_{t \in T}$ be a peacock and $S$ be a countable left- and right-dense subset of $T$ including 0 and all time points where $t \mapsto \mu_t$ is not continuous. Then $(\mu_t)_{t \in T}$ is NSI if and only if $(\mu_t)_{t \in S}$ is NSI.

Proof. This is an easy consequence of Corollary 6.10, Lemma 6.8 and Lemma 4.28 (characterisation of NSI property via generalized obstructed shadows). \quad \square

6.3. Existence and uniqueness of right-continuous shadow martingales. In this subsection, we prove the following right-continuous version of Theorem 1.5 (cf. Remark 5.10).

Theorem 6.12. Let $T \subset [0, \infty)$ with $0 \in T$, $(\mu_t)_{t \in T}$ be a right-continuous peacock and let $(\nu^\alpha)_{\alpha \in [0,1]}$ be a parametrization of $\mu_0$ that is $\leq_{c,s}$-convex. There exists a unique pair $(\pi, (\alpha^\pi)_{\alpha \in [0,1]})$ where the martingale measure $\pi \in \mathcal{M}_{rc}^T((\mu_t)_{t \in T})$ solves the peacock problem w.r.t. $(\mu_t)_{t \in T}$, $(\pi^\alpha)_{\alpha \in [0,1]}$ is a càdlàg martingale parametrization of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ and for all $\alpha$ and $t$ in $T$
\begin{equation}
\pi^\alpha(X_t \in \cdot) = S^{\mu_{[0,t]}}(\nu^\alpha).
\end{equation}

Moreover, there exists a Borel set $A \subset [0,1]$ with $\lambda(A) = 1$ such that for all $\alpha \in A$ the map $\alpha \mapsto \pi^\alpha$ is right-differentiable at $\alpha$ and the marginals of the right-derivative $\dot{\pi}^\alpha$ at a form a NSI peacock. In particular, $\hat{\pi}^\alpha$ is a Markov martingale measure uniquely defined by its marginal distributions.

Proof. For STEPS 1–4 we assume that $T$ admits a maximal element, i.e. $t^* := \sup T \in T$, and we fix a countable left- and right-dense subset $S$ of $T$ that contains both 0 and $t^*$, and all time points where $t \mapsto \mu_t$ is not continuous (cf. Corollary 5.17 and Lemma 6.4).

STEP 1: We show that there exists $\pi \in \mathcal{M}_{rc}^T((\mu_t)_{t \in T})$ and a martingale parametrization $(\pi^\alpha)_{\alpha \in [0,1]}$ of $\pi$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ that satisfies (6.4).

Since $S$ is a countable set and $t^* \in S$, Theorem 5.10 implies that there exists $\tilde{\pi} \in \mathcal{M}_S((\mu_t)_{t \in S})$ and a martingale parametrization $(\tilde{\pi}^\alpha)_{\alpha \in [0,1]}$ of $\tilde{\pi}$ w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ that satisfies (6.5)
\begin{equation}
\tilde{\pi}^\alpha(X_t \in \cdot) = S^{\mu_{S^t}}(\nu^\alpha)
\end{equation}
for all \( t \in S \) and \( \alpha \in [0, 1] \). The peacock \((\mu_t)_{t \in S}\) is right-continuous, thus Lemma 6.8 implies that the map \( t \mapsto S^\mu_{\pi t} (\nu^\alpha) \) is right-continuous for every \( \alpha \in [0, 1] \). Therefore, by Lemma 6.3 (ii) applied to \( \pi \) and \( \frac{1}{\alpha} \pi S \) for every \( \alpha \in (0, 1) \), we have both \( \pi \in \mathcal{M}^r \) and \( (\pi S)_{\alpha \in [0, 1]} \) is a càdlàg martingale parametrization. We can uniquely extend \( \pi \) to \( \pi \in \mathcal{M}^r ((\mu_t)_{t \in T}) \) by Lemma 6.3 (ii) and Lemma 6.7 (ii) shows that we can extend the parametrization \((\pi S)_{\alpha \in [0, 1]} \) to a martingale parametrization \((\pi S)_{\alpha \in [0, 1]} \) of \( \pi \). The set \( S \) contains all discontinuities of the map \( t \mapsto \mu_t \) on \( T \) and hence Corollary 6.10 yields for all \( t \in S \) and \( \alpha \in [0, 1] \) the equality

\[
\pi^\alpha (X_t \in \cdot) = \tilde{\pi}^\alpha (X_t \in \cdot) = S^{\mu S_t} (\nu^\alpha) = S^{\mu [0, t]} (\nu^\alpha). 
\]

Since both \( t \mapsto \pi^\alpha (X_t \in \cdot) \) and \( t \mapsto S^{\mu S_t} (\nu^\alpha) \) are right-continuous functions from \( T \) to \( \mathcal{M}_1 (\mathbb{R}) \) w.r.t. \( T_1 \) (see Lemma 6.3 (i) and Lemma 6.8), we deduce that \( \pi^\alpha \) satisfies (6.4) for all \( t \in T \) and \( \alpha \in [0, 1] \).

STEP 2: We show that \( \pi \) and \((\pi S)_{\alpha \in [0, 1]} \) are uniquely determined.

Let \( \rho \in \mathcal{M}^c ((\mu_t)_{t \in T}) \) and \((\rho^\alpha)_{\alpha \in [0, 1]} \) be a càdlàg martingale parametrization of \( \rho \) w.r.t. \((\nu^\alpha)_{\alpha \in [0, 1]} \) that satisfies

\[
\rho^\alpha (X_t \in \cdot) = S^{\mu S_{\cdot}} (\nu^\alpha).
\]

for all \( t \in T \) and \( \alpha \in [0, 1] \). Lemma 6.7 (i) yields that the restrictions \((\rho^\alpha |_{S})_{\alpha \in [0, 1]} \) are a martingale parametrization of \( \rho_{S} \). Furthermore, we obtain

\[
\rho^\alpha (X_t \in \cdot) = S^{\mu S_{\cdot}} (\nu^\alpha) = S^{\mu S_{t}} (\nu^\alpha)
\]

for all \( t \in S \) and \( \alpha \in [0, 1] \) by Corollary 6.10. The uniqueness part of Theorem 5.15 implies that \( \rho_{S} \) and \( \pi_{S} \) coincide and hence \( \pi = \rho \) by Lemma 6.5 (i).

STEP 3: We show that there exists a Borel set \( A \subset [0, 1] \) with \( \lambda(A) = 1 \) such that \( \alpha \mapsto \pi^\alpha \) is right-differentiable at \( a \) for all \( a \in A \).

Theorem 5.15 yields that there exists a Borel set \( A \subset [0, 1] \) with \( \lambda(A) = 1 \) such that for all \( a \in A \) the curve \( \alpha \mapsto \pi^\alpha |_{S} \) is right-differentiable at \( a \). Hence, by Lemma 6.7 (iii) it suffices to show that the family of marginal distributions of \( \frac{1}{h} (\pi^\alpha + h - \pi^\alpha) \) converges in \( \mathcal{P}^{rc}_{\overline{T}} \) to a right-continuous peacock for all \( a \in A \) to show that \( \alpha \mapsto \pi^\alpha \) is right-differentiable for all \( a \in A \).

To this end, fix \( a \in A \) and note that the distribution of \( X_t \) under \( \frac{1}{h} (\pi^\alpha + h - \pi^\alpha) \) is given by

\[
\zeta^\alpha \not\leq \zeta^a,
\]

due to Corollary 6.10. Since \( \alpha \mapsto \pi^\alpha |_{S} \) is right-differentiable at \( a \), the limit \( \hat{\eta}^a \not\leq \lim_{h \downarrow 0} \zeta^a \not\leq \hat{\eta}^a \) under \( T_1 \) exists for all \( t \in S \). Moreover, Lemma 4.21 (i) implies that \( \zeta^a \) is decreasing in convex-order for \( u \downarrow t \) and thus \( \hat{\eta}^a \not\leq \lim_{h \downarrow 0} \zeta^a \not\leq \hat{\eta}^a \) for all \( t \leq u \) in \( S \) by Lemma 3.2. In particular, Lemma 3.11 (i) shows that for all \( t \in T \) the limit \( \lim_{u \downarrow t, u \in S} \zeta^a \) exists. Since \( \zeta^a \) is decreasing in convex-stochastic order for both \( u \downarrow t \) and \( h \downarrow 0 \) (by Lemma 4.21 (i) and Lemma 4.23 (ii) in conjunction with Lemma 4.23 (iv)), we may interchange the limits and obtain for \( t \in T \) by Lemma 6.8

\[
\lim_{u \downarrow t, u \in S} \tilde{\eta}^a = \lim_{h \downarrow 0} \zeta^a = \lim_{h \downarrow 0} \zeta^a = \lim_{h \downarrow 0} \zeta^a =: \hat{\eta}^a
\]

Thus, the marginal distributions of \((X_t)_{t \in S}\) under \( \frac{1}{h} (\pi^\alpha + h - \pi^\alpha) \) converge in \( \mathcal{P}^{rc}_{\overline{T}} \) to a right-continuous peacock.
STEP 4: We show that for all $a \in A$ the marginal distributions of $(X_t)_{t \in T}$ under the right-derivatives of $\alpha \mapsto \pi^n$ at $a$ are a NSI peacock.

By Lemma 6.7 (iii), the restriction of the right-derivative of $\alpha \mapsto \pi^n$ at $a \in A$ to $S$ is the right-derivative of $\alpha \mapsto \pi^n_S$ at $a \in A$. Since the marginal distributions of the latter are NSI by Theorem 5.15, Corollary 6.11 implies that the marginals of the right-derivative of $\alpha \mapsto \pi^n$ at $a$ are NSI as well.

STEP 5: We remove the assumption that $sup T \in T$.

If $T$ does not admit a maximal element, there exists a sequence $(t^*_n)_{n \in \mathbb{N}}$ in $T$ approaching $sup T \in [0, \infty]$. For every $n \in \mathbb{N}$, we have shown that there exists a unique shadow martingale and a unique càdlàg martingale parametrization w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ corresponding to the right-continuous peacock $(\mu_t)_{t \in T^*_n}$. Since these measures are consistent for different $n$, they define a unique shadow martingale and a unique càdlàg martingale parametrization w.r.t. $(\nu^\alpha)_{\alpha \in [0,1]}$ corresponding to the right-continuous peacock $(\mu_t)_{t \in T}$.

\[ \square \]

7. Shadow martingales indexed by a totally ordered set

In this section, we prove Theorem 7.3 which includes Theorem 1.5 as a special case. The key observation is that the martingale property implies that this abstract setting can be embedded into the right-continuous setup of Section 6 in such a way that the structure and main properties of the obstructed shadows can be lifted from the continuous time case to the abstract setup.

7.1. Embedding into the continuous time setup. Let $(T, \leq)$ be a totally ordered set with minimal element $0 \in T$ and $(\mu_t)_{t \in T}$ a peacock. We define the map $E : T \to [0, \infty)$ as

\[ E(t) = \int_{\mathbb{R}} \sqrt{1+x^2} \, d\mu_t(x) - \int_{\mathbb{R}} \sqrt{1+x^2} \, d\mu_0(x) \]

for all $t \in T$ and set $\tilde{T} = E(T)$. Note that $E(0) = 0$ and hence $0 \in \tilde{T}$. Since $(\mu_t)_{t \in T}$ is a peacock, $\tilde{E}$ is monotonously increasing, $E(t) \leq E(t')$ implies $\mu_t \leq \mu_{t'}$, and $E(t) = E(t')$ is equivalent to $\mu_t = \mu_{t'}$. Moreover, $E(t_n) \rightarrow E(t)$ yields that $(\mu_t)_{t \in \tilde{T}}$ converges to $\mu_t$ under $T_1$ (cf. [11, Section 2.2]). However, $E$ is in general not strictly increasing and thus in general not invertible.

**Lemma 7.1.** We define the family $(\tilde{\mu}_u)_{u \in \tilde{T}}$ by

\[ \tilde{\mu}_u = \mu_t , \text{ where } t \in E^{-1}\{\{u\}\}. \]

For all $u \in \tilde{T}$, the measure $\tilde{\mu}_u$ is well-defined. Moreover, there holds:

(i) The family $(\tilde{\mu}_u)_{u \in \tilde{T}}$ is a peacock with $\tilde{\mu}_{E(t)} = \mu_t$ for all $t \in T$.
(ii) The map $u \mapsto \tilde{\mu}_u$ is continuous from $\tilde{T}$ to $\mathcal{P}_1(\mathbb{R})$ w.r.t. $T_1$.
(iii) For all $t \in T$ and $\nu \in \mathcal{M}_1(\mathbb{R})$ with $\nu \leq \mu_0 = \tilde{\mu}_0$, it holds $S^{\mathcal{PT}_1} (\nu) = S^{\mathcal{PT}_E(t)} (\nu)$.

**Proof.** Item (i): Since $E$ is monotonously increasing, the claim follows immediately.

Item (ii): For $u \in \tilde{T}$ and a sequence $(u_n)_{n \in \mathbb{N}}$ that converges to $u$ we obtain

\[ \lim_{n \to \infty} \int_{\mathbb{R}} \sqrt{1+x^2} \, d\tilde{\mu}_{u_n} = \int_{\mathbb{R}} \sqrt{1+x^2} \, d\mu_0 + \lim_{n \to \infty} u_n = \int_{\mathbb{R}} \sqrt{1+x^2} \, d\mu_0 + u = \int_{\mathbb{R}} \sqrt{1+x^2} \, d\tilde{\mu}_u. \]

Thus, $u \mapsto \tilde{\mu}_u$ is continuous.
Item (iii): Let \((\tilde{\eta}_t)_{u\in \tilde{T}}\) be a family in \(M_1(\mathbb{R})\) with \(\nu \leq_{c} \tilde{\eta}_u \leq_{c} \tilde{\eta}_t \leq_{+} \mu_u\) for all \(u \leq u'\) in \(\tilde{T}\). We set \(\eta_t := \tilde{\eta}_{E(t)}\) for all \(t \in T\). Since \(E\) is increasing, it holds \(\nu \leq_{c} \tilde{\eta}_u \leq_{c} \tilde{\eta}_t \leq_{+} \mu_u\) for all \(t \leq t'\) in \(T\) and furthermore we have \(\eta_t = \tilde{\eta}_{E(t')} \leq_{+} \mu_{E(t')} = \mu_{t'}\) by (ii).

Conversely, let \((\eta_t)_{t\in T}\) be a family with \(\nu \leq_{c} \eta_t \leq_{c} \eta_t' \leq_{+} \mu_t\) for all \(t \leq t'\) in \(T\). We set \(\tilde{\eta}_u := \eta_t\) for any \(u \in E^{-1}(\{t\})\) for all \(u \in T\). Since \(E\) is monotonously increasing, we get \(\nu \leq_{c} \tilde{\eta}_u \leq_{c} \tilde{\eta}_t \leq_{+} \mu_t\) for all \(u \leq u'\) in \(\tilde{T}\). Moreover, there exists \(t' \in T\) with \(E(t') = u'\), and similar as in (ii) it holds as in (i) it holds \(\eta_{E(t')} = \tilde{\eta}_u.\) Thus, \(\tilde{\eta}_u = \tilde{\eta}_{E(t')} = \eta_t \leq_{+} \mu_t = \tilde{\mu}_{E(t')} = \tilde{\mu}_{t'}\).

Then, Proposition \ref{proposition:5.3} implies that \(S^{\mu_{T_t}}(\nu) = S^{\tilde{\mu}_{E(t)}}(\nu)\) for all \(t \in T\).

**Lemma 7.2.** Let \(E^* : M_1(\mathbb{R}^T) \rightarrow M_1(\mathbb{R}^T)\) be the map \(\tilde{\pi} \mapsto E^* (\tilde{\pi})\) where \(E^* (\tilde{\pi})\) is uniquely determined by

\[
E^*(\tilde{\pi})(Y_{t_1}, \ldots, Y_{t_n}, B_n) := \tilde{\pi}(X_{E(t_1)} \in B_1, \ldots, X_{E(t_n)} \in B_n)
\]

for all \(n \in \mathbb{N}, t_1, \ldots, t_n \in T\) and Borel sets \(B_1, \ldots, B_n\) where \(Y\) and \(X\) denote the canonical process on \(\mathbb{R}^T\) and \(\mathbb{R}^T\).

(i) If \((\tilde{\pi}^\alpha)_{\alpha \in [0,1]}\) is a martingale parametrization of \(\tilde{\pi} \in M_T\), then \((E^* (\tilde{\pi}^\alpha))_{\alpha \in [0,1]}\) is a martingale parametrization of \(E^* (\tilde{\pi}) \in M_T\).

(ii) The map \(E^*\) is sequentially continuous, i.e., if \((\tilde{\pi}_n)_{n \in \mathbb{N}}\) converges to \(\tilde{\pi}\) under \(T_1\), then \((E^*(\tilde{\pi}_n))_{n \in \mathbb{N}}\) converges to \(E^*(\tilde{\pi})\) under \(T_1\).

(iii) It holds \(\{\pi^\alpha : \pi^\alpha \leq_{+} \pi, \pi \in M_T((\mu_{t})_{t\in T})\} \subset \text{Im}(E^*)\).

**Proof.** Item (i) and item (ii) are direct consequences of the previous definitions.

Item (iii): Let \(\pi \in M_T((\mu_{t})_{t\in T})\) and \(\pi' \leq_{+} \pi\). For any \(t, t' \in T\) with \(E(t) = E(t')\) it holds \(\mu_t = \mu_{t'}\). Thus, since \(\pi\) is a martingale measure, we have \(\pi(X_t \neq X_{t'}) = 0\) and therefore also \(\pi'(X_t \neq X_{t'}) = 0\). Hence, we can uniquely define the measure \(\tilde{\pi}' \in M_1(\mathbb{R}^T)\) by

\[
\tilde{\pi}'(X_{E(t_1)} \in B_1, \ldots, X_{E(t_n)} \in B_n) := \pi'(Y_{t_1} \in B_1, \ldots, Y_{t_n} \in B_n)
\]

for all \(n \in \mathbb{N}, t_1, \ldots, t_n \in T\) and Borel sets \(B_1, \ldots, B_n\) where \(Y\) and \(X\) denote the canonical processes on \(\mathbb{R}^T\) and \(\mathbb{R}^T\), respectively. Clearly, \(E^*(\tilde{\pi}') = \pi\).

7.2. Existence and uniqueness of shadow couplings.

**Theorem 7.3.** Let \(T\) be a totally ordered set with \(\min T =: 0 \in T\) and let \((\nu^\alpha)_{\alpha \in [0,1]}\) be a parametrization of \(\nu_0\) that is \(\leq_{c}\)-convex. There exists a unique pair \((\pi^\alpha)_{\alpha \in [0,1]}\), where the martingale measure \(\pi \in M_T((\mu_{t})_{t\in T})\) solves the peacock problem w.r.t. \((\mu_{t})_{t\in T}\), \((\pi^\alpha)_{\alpha \in [0,1]}\) is a martingale parametrization of \(\pi\) w.r.t. \((\nu^\alpha)_{\alpha \in [0,1]}\) and for all \(\alpha \in [0,1]\) and \(t\in T\)

\[
\pi^\alpha(X_t \in \cdot) = S^{\nu_{T_t}}(\nu^\alpha).
\]

Moreover, there exists a Borel set \(A \subset [0, \infty)\) with \(\lambda(A) = 1\) such that for all \(a \in A\) the map \(\alpha \mapsto \pi^\alpha\) is right-differentiable at \(a\) and the marginals of the right-derivative \(\tilde{\pi}^\alpha\) at a form a NSI peacock. In particular, \(\tilde{\pi}^\alpha\) is a Markov martingale measure uniquely defined by its marginal distributions.

Observe that Theorem 7.3 encompasses Theorem 1.5 since \([0, \infty)\) is a totally ordered set with minimal element. The stochastic formulation in the last paragraph of Theorem 1.5 follows as in Remark 5.10.
Proof. We will prove the claim by reduction to the right-continuous setting in Theorem 6.12.

STEP 1: We show that there exists a \( \pi \in M_T((\mu_t)_{t \in T}) \) and a martingale parametrization \((\pi^\alpha)_{\alpha \in [0,1]} \) of \( \pi \) w.r.t. \((\nu^\alpha)_{\alpha \in [0,1]} \) such that \((7.1)\) is satisfied.

Let \((\tilde{\mu}_u)_{u \in \tilde{T}} \) be the right-continuous peacock associated with \((\mu_t)_{t \in T} \) in Lemma 7.1. By Theorem 6.12, there exists a unique \( \tilde{\pi} \in M^S_T\big((\tilde{\mu}_u)_{u \in \tilde{T}}\big) \) and a unique càdlàg martingale parametrization \((\tilde{\pi}^\alpha)_{\alpha \in [0,1]} \) of \( \tilde{\pi} \) that satisfies

\[
\tilde{\pi}^\alpha(X_u \in \cdot) = S^\mu\tilde{\pi}(\nu^\alpha)
\]

for all \( u \in \tilde{T} \) and \( \alpha \in [0,1] \). Set \( \pi = E^*(\tilde{\pi}) \) and \( \pi^\alpha = E^*(\tilde{\pi}^\alpha) \) for all \( \alpha \in [0,1] \) where the map \( E^* \) is defined as in Lemma 7.2. Then \( \pi \in M_T((\mu_t)_{t \in T}) \) and \((\pi^\alpha)_{\alpha \in [0,1]} \) is a martingale parametrization of \( \pi \) with

\[
\pi^\alpha(X_t \in \cdot) = \tilde{\pi}^\alpha(X_{E(t)} \in \cdot) = S^{\mu\tilde{\pi}_{E(t)}}(\nu^\alpha) = S^{\mu\pi}(\nu^\alpha)
\]

for all \( t \in T \) and \( \alpha \in [0,1] \) by Lemma 7.1.

STEP 2: We show that \((\pi^\alpha)_{\alpha \in [0,1]} \) (and therefore also \( \pi = \pi^1 \)) is uniquely determined by \((7.1)\).

Let \( \rho \in M_T((\mu_t)_{t \in T}) \) and let \((\rho^\alpha)_{\alpha \in [0,1]} \) be a martingale parametrization of \( \rho \) w.r.t. \((\nu^\alpha)_{\alpha \in [0,1]} \) with

\[
(7.2) \quad \rho^\alpha(X_t \in \cdot) = S^{\mu\pi}(\nu^\alpha)
\]

for all \( \alpha \in [0,1] \) and \( t \geq 0 \). Let \( \tilde{\rho}^\alpha \in (E^*)^{-1}(\rho^\alpha) \) and \( t \in T \) with \( E(t) = u \). There holds

\[
\tilde{\rho}^\alpha(X_u \in \cdot) = \rho^\alpha(X_{E(t)} \in \cdot) = S^{\mu\tilde{\pi}_{E(t)}}(\nu^\alpha) = S^{\mu\pi}(\nu^\alpha)
\]

for all \( \alpha \in [0,1] \) and \( u \in \tilde{T} \) using \((7.2)\) and Lemma 7.1. The uniqueness part of Theorem 6.12 yields that \( \tilde{\pi}^\alpha = \tilde{\rho}^\alpha \) and hence \( \pi^\alpha = \rho^\alpha \) for all \( \alpha \in [0,1] \).

STEP 3: We show that there exists a Borel set \( A \subset [0,1] \) with \( \lambda(A) = 1 \) such that for all \( a \in A \) the map \( \alpha \mapsto \pi^\alpha \) is right-differentiable at \( a \) and the marginals of the right-derivative \( \tilde{\pi}^a \) at \( a \) are a NSI peacock.

By Theorem 6.12 there exists a Borel set \( A \subset [0,1] \) with \( \lambda(A) = 1 \) such that for all \( a \in A \) the map \( \alpha \mapsto \pi^\alpha \) is right-differentiable at \( a \) with right-derivative \( \tilde{\pi}^a \) and the marginals of the right-derivative \( \tilde{\pi}^a \) at \( a \) are a NSI peacock. Note that

\[
\frac{\pi^{\alpha+h} - \pi^{\alpha}}{h} = E^*\left(\frac{\tilde{\pi}^{\alpha+h} - \tilde{\pi}^{\alpha}}{h}\right)
\]

and thus the sequential continuity proven in Lemma 7.2 implies that at all \( a \in A \) the map \( \alpha \mapsto \pi^\alpha \) is right-differentiable with right-derivative \( E^*\left(\tilde{\pi}^a\right) \). Since the marginal distributions of the right-derivative \( \tilde{\pi}^a \) are a NSI peacock, the same is true for \( E^*\left(\tilde{\pi}^a\right) \) by Lemma 7.1 in conjunction with Lemma 4.28.

7.3. Proof of Corollary 1.1. Let \( (T, \leq) \) be a totally ordered set with minimal element \( 0 \in T \) and \((\mu_t)_{t \in T} \) a peacock.

Lemma 7.4. Let \((\nu^\alpha)_{\alpha \in [0,1]} \) a parametrization of \( \mu_0 \) and let \( \pi \) be the shadow martingale w.r.t. \((\mu_t)_{t \in T} \) and \((\nu^\alpha)_{\alpha \in [0,1]} \). Moreover, we denote by \((\pi^\alpha)_{\alpha \in [0,1]} \) the corresponding parametrization of \( \pi \) and by \( \tilde{\pi}^a \) the right-derivative of \( \alpha \mapsto \pi^\alpha \) at \( a \in [0,1] \) (if it exists). If there exists a measurable function \( q : \mathbb{R} \rightarrow [0,1] \) such that

\[
(i) \quad q_{\#}\mu_0 = \lambda \quad \text{and}
\]

...
(ii) \( \hat{\pi}^a(x)(q(X_0) = q(x)) = 1 \) for \( \mu_0 \)-a.e \( x \in \mathbb{R} \),
then for all \( n \in \mathbb{N}, t_1 \leq \ldots \leq t_n \in T, A \in \mathcal{B}(\mathbb{R}^n) \) and \( \sigma\)-algebras \( \mathcal{G} \subset \bigvee_{t \in T} \mathcal{F}_t \) we have \( \pi \)-a.s.
\[
E_\pi \left[ I_A(X_{t_1}, \ldots, X_{t_n}) | X_0, \mathcal{G} \right] = E_{\hat{\pi}^a(x_0)} \left[ I_A(X_{t_1}, \ldots, X_{t_n}) | X_0, \mathcal{G} \right].
\]

**Proof.** Let \( n \in \mathbb{N}, t_1 \leq \ldots \leq t_n \in T, A \in \mathcal{B}(\mathbb{R}^n) \) and fix a \( \sigma\)-algebra \( \mathcal{G} \subset \sigma(\bigcup_{t \in T} \mathcal{F}_t) \).
Moreover, we set \( Y := I_A(X_{t_1}, \ldots, X_{t_n}) \).

It remains to show that \( \text{properties (i)-(iii) of the family } (I_{\alpha})_{\alpha \in [0,1]} \text{ satisfies} \)
\[
E_\pi [I_{\alpha} | X_0, \mathcal{G}] = E_{\hat{\pi}^a(x_0)} [I_{\alpha} | X_0, \mathcal{G}],
\]
for all \( \alpha \in [0,1] \).

**Corollary 7.5.** Let \( (\mu_t)_{t \in T} \) be a peacock and let \( (I_{\alpha})_{\alpha \in [0,1]} \) be a nested family of intervals that satisfies
\begin{enumerate}[label=(\roman*)]
\item \( \mu_0(I_{\alpha}) = \alpha \) for any \( \alpha \in [0,1] \),
\item \( \sup I_{\alpha} < +\infty \) and \( \partial I_{\alpha} \cap \partial I_{\beta} = \emptyset \) for all \( \alpha \neq \beta \) in \( [0,1] \) and for which \( \alpha \mapsto \int_{I_{\alpha}} y \, d\mu_0(y) \) is a convex function.
\end{enumerate}

there exists unique \( \pi \in \mathcal{M}_T((\mu_t)_{t \geq 0}) \) such that for all \( \rho \in \mathcal{M}_T((\mu_t)_{t \geq 0}) \) there holds
\[
\text{Law}_\pi(X_t | X_0 \in I_{\alpha}) \leq_c \text{Law}_\rho(X_t | X_0 \in I_{\alpha})
\]
for all \( \alpha \in [0,1] \) and \( t \geq 0 \). Moreover, \( (X_0, X_t)_{t \geq 0} \) is a Markov process under \( \pi \).

Clearly, this covers the case \( T = [0, \infty) \) as stated in Corollary \ref{cor:(i)}

**Proof.** Lemma \ref{lem:(i)} shows that \( (\nu^\alpha)_{\alpha \in [0,1]} \) with \( \nu^\alpha = \mu_0 I_{\alpha} \) is a \( \leq_c \sigma \)-convex parametrization of \( \mu_0 \). Theorem \ref{thm:4.5} states that there exists a unique shadow martingale \( \pi \) w.r.t. \( (\mu_t)_{t \in T} \) and \( (\nu^\alpha)_{\alpha \in [0,1]} \). The martingale measure \( \pi \) is a solution to the peacock problem w.r.t. \( (\mu_t)_{t \in T} \) and Remark \ref{rem:5.4} yields
\[
\alpha \text{Law}_\pi(X_t | X_0 \in I_{\alpha}) = \pi^\alpha(X_t \in \cdot) = S^\nu_{\mu_t}(\mu_0 I_{\alpha})
\]
for all \( \alpha \in [0,1] \). Thus, \( \pi \) is the unique solution to the peacock problem with
\[
\text{Law}_\pi(X_t | X_0 \in I_{\alpha}) \leq_c \text{Law}_\rho(X_t | X_0 \in I_{\alpha})
\]
for any other \( \rho \in \mathcal{M}_T((\mu_t)_{t \in T}) \).

It remains to show that \( (X_0, X_t)_{t \in T} \) is a Markov process under \( \pi \).

It is not difficult to see that properties (i)-(iii) of the family \( (I_{\alpha})_{\alpha \in [0,1]} \) imply that the pseudo-quantile map \( q : \mathbb{R} \to [0,1] \) defined as
\[
x \mapsto q(x) := \sup \{ \alpha \in [0,1] : x \notin I_{\alpha} \}
\]
meets the requirements of Lemma \ref{lem:7.4} (W.l.o.g. we assume \( I_1 = \mathbb{R} \)). Note that the map \( q \) is Borel measurable because there exists \( x_0 \in \mathbb{R} \) such that \( x_0 \in I_{\alpha} \) for all \( \alpha > 0 \) and \( q \) is monotone on \( (-\infty, x_0) \) and \( [x_0, +\infty) \). Thus, for all Borel sets \( B \subset \mathbb{R}^2 \), we have \( \pi \)-a.e.

\[
(7.3) \quad \begin{cases}
E_\pi [I_B(X_0, X_t) | X_0, \mathcal{F}_s] = E_{\hat{\pi}^a(x_0)} [I_B(X_0, X_t) | X_0, \mathcal{F}_s] \\
E_\pi [I_B(X_0, X_t) | X_0, X_s] = E_{\hat{\pi}^a(x_0)} [I_B(X_0, X_t) | X_0, X_s]
\end{cases}
\]
By Theorem 7.3 there exists $A \subset [0,1]$ with $\lambda(A) = 1$ such that for all $a \in A$ the right derivatives $\dot{\pi}^a$ of $\alpha \mapsto \pi^a$ exist and $(X_t)_{t \in T}$ is a Markov process under $\dot{\pi}^a$. Hence, $(X_0, X_t)$ is a Markov process under $\dot{\pi}^a$ and the claim follows with (7.3).

$\square$

8. The left-curtain shadow martingale

Let $(T, \leq)$ be a totally ordered set with minimal element $0 \in T$ and $(\mu_\alpha)_{\alpha \in T}$ a peacock with $\mu_0(\{x\}) = 0$ for all $x \in \mathbb{R}$. Moreover, let $(\nu^\alpha_\alpha)_{\alpha \in [0,1]}$ be the left-curtain parametrization of $\mu_0$, i.e. $\nu^\alpha_\alpha = \mu_0([\infty, \alpha])$ for all $\alpha \in [0,1]$ because $\mu_0$ is atomless (see Lemma 4.5).

In this section, we give an alternative characterization of the shadow martingale measure w.r.t. $(\mu_\alpha)_{\alpha \in T}$ and the left-curtain parametrization $(\nu^\alpha_\alpha)_{\alpha \in [0,1]}$ of $\mu_0$ in terms of continuous time martingale optimal transport.

More precisely, we prove Theorem 8.4 which is the rigorous version of Theorem 1.6 i.e. we show that the shadow martingale is the unique solution to the continuous time martingale optimal transport problem

\[
\inf \{ \mathbb{E}_{\rho}[c(X_0, X_t)] : \rho \in \mathcal{M}_T(\mu_T) \}
\]

simultaneously for all $t \in T$ and for a specific class of cost functions $c$.

8.1. Optimality. In the following we denote partial derivatives with the indices of the coordinates, e.g. we write $\partial_{122}c$ for $\partial_2 \partial_2 \partial c$.

**Definition 8.1.** A function $c : \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be a martingale Spence-Mirrlees (MSM) cost function if for every $x < x'$ in $\mathbb{R}$ the increment function

\[\Delta_{x,x'}c : y \mapsto c(x',y) - c(x,y)\]

is strictly concave.

We will principally work with MSM cost functions $c$ that are in $C^{1,2}(\mathbb{R}^2)$ and satisfy the sufficient condition $\partial_{122}c < 0$. Typical examples are cost functions of the form $c(x,y) = h(y-x)$ where $h'$ is strictly convex, e.g. $c(x,y) = (y-x)^3$, and cost functions of the form $c(x,y) = \varphi(x)\psi(y)$ where $\varphi$ is strictly decreasing and $\psi$ is strictly convex, e.g. $c(x,y) = -x/y$ for $(x,y) \in \mathbb{R} \times (0,\infty)$. The cost function $c(x,y) = \exp(y-x)$ is in the intersection of the subclasses and $c(x,y) = \sqrt{a(x) + b(x)y^2}$ is a MSM function outside these two subclasses where $a,b$ are non-negative functions for which both $a/\sqrt{b}$ and $b/\sqrt{a}$ are decreasing. The class of MSM cost functions was introduced in [25] by Henry-Labordère and Touzi and is similar to (but should not be mixed up with) the ‘non-twisted’ condition $\partial_{122}c < 0$ in classical optimal transport.

We will show that the left-curtain shadow martingale is the unique simultaneous optimizer of (8.1). Note that the case of finite $T$ has been worked out with different methods by Beiglböck and Juillet [13], Nutz, Stebegg and Tan [45] and Beiglböck, Cox, Huesmann [7]. In fact, the following lemma is very similar to [45] Lemma 7.14.

**Lemma 8.2.** Let $c \in C^{1,2}(\mathbb{R}^2)$. For all $M,N \in \mathbb{N}$ and $(x,y) \in (-\infty, M] \times (-\infty, N]$ there holds

\[
c(x,y) = c(M,y) - (\Delta_{x,M}c)(N) - (y-N)(\Delta_{x,M}c)'(N) + \int_{-\infty}^{M} \int_{-\infty}^{N} \mathbf{1}_{(-\infty,u]}(x)(v-y)^+ \cdot (-\partial_{122}c(u,v)) \, du \, dv.
\]
Proof. Let \( M, N \in \mathbb{N} \) and \( f \in C^2(\mathbb{R}) \). By partial integration, there holds for all \( y \in (-\infty, N] \)

\[
-f(y) = -f(N) + \int_y^N f'(v) \, dv
\]

\[
= -f(N) + [(v - y)f'(v)]_y^N - \int_y^N (v - y)f''(v) \, dv
\]

\[
= -f(N) + (N - y)f'(N) - \int_{-\infty}^y (v - y)^+ f''(v) \, dv.
\]

Applying this to \( f = \Delta_{x,M} \) with \( x \leq M \) and rewriting

\[
(\Delta_{x,M}c)'(v) = \int_{-\infty}^M \mathbb{1}_{[-\infty,u]}(x) \partial_2 c(u,v) \, du
\]

proves the claim. \( \square \)

Lemma 8.2 shows that - up to some boundary terms - the interaction of \( x \) and \( y \) given by \( c \) is basically described by the functions of the form \((x,y) \mapsto \mathbb{1}_{(-\infty,0]}(x) (v - y)^+ \). These, however, are closely connected with the left-curtain shadow martingale.

Lemma 8.3. For all \((u, v) \in \mathbb{R}^2\) we define the function \( c^{u,v} \) as

\[
c^{u,v} : (x,y) \in \mathbb{R}^2 \mapsto \mathbb{1}_{(-\infty,u]}(x) (v - y)^+ \in [0, \infty).
\]

Moreover, let \((\mu_t)_{t \in T}\) be a peacock and \(\pi_{lc}\) the shadow coupling w.r.t. \((\mu_t)_{t \in T}\) and the left-curtain parametrization \((\nu_{lc}^t)_{\alpha \in [0,1]}\) of \(\mu_0\). Let \(\pi \in \mathcal{M}_T((\mu_t)_{t \in T})\).

(i) For every \(t \in T\) and \((u, v) \in \mathbb{R}^2\) there holds

\[
\mathbb{E}_\pi [c^{u,v}(X_0, X_t)] \geq \mathbb{E}_{\pi_{lc}} [c^{u,v}(X_0, X_t)].
\]

(ii) Fix \(t \in T\). If for all \((u, v)\) in a dense set of \(\mathbb{R}^2\) there holds

\[
\mathbb{E}_\pi [c^{u,v}(X_0, X_t)] = \mathbb{E}_{\pi_{lc}} [c^{u,v}(X_0, X_t)],
\]

then \((X_0, X_t)\) have the same law under \(\pi\) and \(\pi_{lc}\).

(iii) Assume \(\mu_0\) is atomless. If for every \(t \in T\) and all \((u, v)\) in a dense set of \(\mathbb{R}^2\) there holds

\[
\mathbb{E}_\pi [c^{u,v}(X_0, X_t)] = \mathbb{E}_{\pi_{lc}} [c^{u,v}(X_0, X_t)],
\]

then \(\pi\) is the shadow martingale \(\pi_{lc}\).

Proof. Item (i): By definition of \(\pi_{lc}\) and Proposition 4.22, there holds for all \(u \in \mathbb{R}\) and \(t \in T\)

\[
\pi_{lc}(X_0 \leq u, X_t \in \cdot) = \mathcal{S}_{\mu_0} ((\mu_0)(-\infty, u]) \leq \pi(X_0 \leq u, X_t \in \cdot).
\]

Since \( y \mapsto (v - y)^+ \) is a convex function for all \(v \in \mathbb{R}\) we get the desired inequality.

Item (ii): Let \(f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}\) be defined as

\[
f_1(u, v) := \mathbb{E}_\pi [c^{u,v}(X_0, X_t)] \quad \text{and} \quad f_2(u, v) := \mathbb{E}_{\pi_{lc}} [c^{u,v}(X_0, X_t)].
\]

Since \(f_1(u, \cdot)\) and \(f_2(u, \cdot)\) are the potential functions of \(\pi_{lc}(X_0 \leq u, X_t \in \cdot)\) and \(\pi(X_0 \leq u, X_t \in \cdot)\) and potential functions are convex, \(u_1\) and \(u_2\) are continuous in \(v\). Moreover, for all \((u, v) \in \mathbb{R}^2\) and \(i \in \{1, 2\}\), dominated convergence yields

\[
\lim_{u_n \downarrow u} f_i(u_n, v) = f_i(u, v)
\]
because \( \mu_t \) has a finite first moment. Hence, since \( f_1 \) and \( f_2 \) coincide on a dense set, they are equal everywhere. In particular, by Lemma 8.7 we obtain for all \( u \in \mathbb{R} \)

\[
\pi_{lc}(X_t \leq u, X_t \in \cdot) = \pi(X_0 \leq u, X_t \in \cdot).
\]

Item (iii): Since \( \mu_0(\{x\}) = 0 \) for all \( x \in \mathbb{R} \), for all \( \alpha \in [0,1] \) there exists \( q_\alpha \in \mathbb{R} \) with

\[
\nu^\alpha = (\mu_0)((-\infty, q_\alpha]].
\]

Hence, \((\pi^\alpha)_{\alpha \in [0,1]}\) with

\[
\pi^\alpha := \alpha \text{Law}_\pi(X | X_0 \leq q_\alpha)
\]

is a parametrization of \( \pi \) w.r.t. \((\nu^\alpha)_{\alpha \in [0,1]}\) and by (ii) (and Lemma 4.3) there holds

\[
\pi^\alpha(X_t \in \cdot) = \alpha \text{Law}_{\pi_{lc}}(X | X_0 \leq q_\alpha) = S^{\mu_t}(\nu^\alpha)
\]

for all \( \alpha \in [0,1] \) and \( t \in T \). By uniqueness of the shadow martingale, \( \pi = \pi_{lc} \).

Theorem 8.4. Let \((\mu_t)_{t \in T}\) be a peacock with \( \mu_0 \) atomless and \( c : \mathbb{R}^2 \to \mathbb{R} \) a MSM cost function. Suppose that

(i) \( c \in C^{1,2}(\mathbb{R}^2) \) with \( \partial_{122}c < 0 \)

(ii) for all \( t \in T \) there exists \( \varphi_1 \in L^1(\mu_0) \) and \( \psi_1 \in L^1(\mu_t) \) such that for all \( (x,y) \in \mathbb{R}^2 \) there holds \( |c(x,y)| \leq \varphi_1(x) + \psi_1(y) \).

Moreover, we assume that at least one of the two following assumptions is satisfied:

(iii-a) For all \( t \in T \) there exists \( \varphi_2 \in L^1(\mu_0) \) and \( \psi_2 \in L^1(\mu_t) \) such that for all \( (x,y) \in \mathbb{R}^2 \) there holds \( |\partial_2 c(x,0)y| \leq \varphi_2(x) + \psi_2(y) \)

(iii-b) there exists \( M \in \mathbb{N} \) such that \( \text{supp}(\mu_0) \subset (-\infty,M] \).

Under these assumptions the shadow martingale \( \pi_{lc} \) w.r.t. \((\mu_t)_{t \in T}\) and the left-curtain parametrization \((\nu^\alpha)_{\alpha \in [0,1]}\) of \( \mu_0 \) is the unique element of \( M_T((\mu_t)_{t \in T}) \) that satisfies

\[
\mathbb{E}_{\pi_{lc}}[c(X_0, X_t)] = \inf \{ \mathbb{E}_\rho[c(X_0, X_t)] : \rho \in M_T((\mu_t)_{t \in T}) \}
\]

simultaneously for all \( t \in T \).

We first concentrate on the central argument of the proof and postpone the proofs of two technical results Lemma 8.7 and Lemma 8.8 to the next subsection.

Proof. Fix \( t \in T \) and for \( \pi_{lc} \) a competitor \( \pi \in M_T((\mu_t)_{t \in T}) \). For all \( m, M, N \in \mathbb{N} \) and \( (x,y) \in \mathbb{R}^2 \) we define

\[
A_{m,M,N} := [-m,M] \times (-\infty,N] \quad \text{and}
\]

\[
R_{M,N} : (x,y) \in \mathbb{R}^2 \mapsto c(M,y) - [\Delta x,MC(N) + (\Delta x,MC)'(N) \cdot (y-N)]
\]

where we recall that \( \Delta x,MC \) denotes the increment function \( y \mapsto c(M,y) - c(x,y) \).

By assumption (i) and Lemma 8.2 we know that for all \( m, M, N \in \mathbb{R} \) and \( (x,y) \in A_{m,M,N} \) there holds

\[
c(x,y) = R_{M,N}(x,y) + \int_{-\infty}^M \int_{-\infty}^N c^{\mu,y}(x,y)(-\partial_{122}c(u,v)) \, du \, dv
\]

where \( c^{\mu,y}(x,y) = 1_{(-\infty,u]}(x)(y-v)_+ \) as in (8.2). Furthermore, in the following we use the notation \( \mathbb{E}_{\pi_{lc}}[g(X)] \) for \( \mathbb{E}_{\pi}[g(X)] - \mathbb{E}_{\pi_{lc}}[g(X)] \).

On the one hand, by assumption (ii) \( c(X_0, X_t) \) is integrable w.r.t. \( \pi \) and \( \pi_{lc} \), and dominated convergence yields

\[
\mathbb{E}_{\pi_{lc}}[c(X_0, X_t)] = \lim_{M \to \infty} \lim_{m \to \infty} \lim_{N \to \infty} \mathbb{E}_{\pi_{lc}}[1_{A_{m,M,N}}(X_0, X_t)c(X_0, X_t)].
\]

On the other hand, for all \( m, M, N \in \mathbb{N} \) the random variable

\[
1_{A_{m,M,N}}(X_0, X_t)R_{M,N}(X_0, X_t)
\]
is integrable w.r.t. $\pi$ and $\pi_{tc}$ because the function $x \mapsto \partial_t c(x, N)$ is continuous on $[-m, M]$ and $\mu_t$ has a finite first moment. We show in Lemma 8.7 that under assumptions (i) and (ii) the following successive limits exist and satisfy

$$\lim_{m \to \infty} \lim_{N \to \infty} \mathbb{E}_{\pi-\pi_{tc}} \left[ 1_{A_{m,M,N}}(X_0, X_t) R_{m,M,N}(X_0, X_t) \right] = \mathbb{E}_{\pi-\pi_{tc}} \left[ 1_{(-\infty,M]}(X_0) c(M, X_t) \right].$$

Since assumption (iii-a) or assumption (iii-b) is satisfied, by Lemma 8.8 there holds

$$\lim_{m \to \infty} \mathbb{E}_{\pi-\pi_{tc}} \left[ 1_{(-\infty,M]}(X_0) c(M, X_t) \right] = 0.$$

Hence, taking the expectation in (8.3), by Fubini’s theorem we have

$$\mathbb{E}_{\pi-\pi_{tc}} [c(X_0, X_t)] = 0 + \lim_{m \to \infty} \lim_{N \to \infty} \int_m^M \int_{-\infty}^\infty \mathbb{E}_{\pi-\pi_{tc}} \left[ c^{u,v}(X_0, X_t) \right] \left(-\partial_{122} c(u, v)\right) du dv.$$

Since $\partial_{122} c < 0$ by assumption (i) and $\mathbb{E}_{\pi-\pi_{tc}} [c^{u,v}(X_0, X_t)] \geq 0$ by Lemma 8.3 (i), by monotone convergence we obtain

$$\mathbb{E}_{\pi-\pi_{tc}} [c(X_0, X_t)] = \int_{-\infty}^\infty \int_{-\infty}^\infty \mathbb{E}_{\pi-\pi_{tc}} \left[ c^{u,v}(X_0, X_t) \right] \left(-\partial_{122} c(u, v)\right) du dv.$$

The claim follows with Lemma 8.3 (i) and Lemma 8.3 (iii). \hfill \Box

**Remark 8.5.** In the following cases the assumptions of Theorem 8.4 are satisfied:

(i) The cost function $c : (x, y) \mapsto \tanh(-x) \sqrt{1 + y^2}$ satisfies (i), (ii) and (iii-a) for every peacock and every $t \in T$.

(ii) Let $c(x, y) := (y - x)^3$ and suppose that $\mu_t$ has a finite third moment for all $t \in T$. Then we can satisfy assumptions (ii) and (iii-a) with the same functions $x \mapsto 4|x|^3$ and $y \mapsto 4|y|^3$.

(iii) Let $\varphi \in C^1(\mathbb{R})$ with $\varphi' > 0$, $\psi \in C^2(\mathbb{R})$ with $\psi'' < 0$, $c(x, y) := \varphi(x) \psi(y)$ and suppose there exist $\frac{1}{p} + \frac{1}{q} = 1$ such that for all $t \in T$ $\varphi \in L^p(\mu_0)$ and $\psi \in L^q(\mu_0)$ and either $\mu_t$ has finite $q$-th moment for all $t \in T$ or there exists $\overline{M} \in \mathbb{N}$ with $\sup(\mu_0) \subset (-\infty, \overline{M})$.

**Remark 8.6.** As mentioned before, Theorem 8.4 for a finite index set $T$ is proven in 8.3 and 8.4. For a general index set $T$, picking a suitable sequence of nested finite subsets $(R_n)_n$ of $T$, it is possible to show the sequence of shadow martingales w.r.t. $(\mu_t)_{t \in R_n}$ and $(\nu^{(\alpha)}_{tc})_{(\alpha \in [0,1]}$ converges to the unique shadow martingale w.r.t. $(\mu_t)_{t \in T}$ and $(\nu^{(\alpha)}_{tc})_{(\alpha \in [0,1]}$. In particular, the unique optimizer of the finite time martingale optimal transport problem provided by 8.3 converge to the unique optimizer of the corresponding continuous-time martingale optimal transport problem.

8.2. Lemma 8.7 and Lemma 8.8. In this subsection we keep the notation from the proof of Theorem 8.4. i.e. for all $m, M, N \in \mathbb{N}$ we set

$$A_{m,M,N} := [-m, M] \times (-\infty, N] \quad \text{and} \quad R_{M,N} : (x, y) \in \mathbb{R}^2 \mapsto c(M, y) - [\Delta_{x,MC} N + (\Delta_{x,MC} N)'(y - N)].$$

and we use the notation $\mathbb{E}_{\pi-\pi_{tc}}[g(X)] := \mathbb{E}_{\pi}[g(X)] - \mathbb{E}_{\pi_{tc}}[g(X)]$. Moreover, given a MSM cost function $c$, for all $x < x'$ in $\mathbb{R}$ and $N \in \mathbb{N}$ we denote by $L^N_{x,x'}$ the tangent of the concave increment function $\Delta_{x,x'} c$ at position $N$, i.e.

$$L^N_{x,x'}(y) := (\Delta_{x,x'} c)(N) + (\Delta_{x,x'} c)'(N)(y - N).$$
for all $y \in \mathbb{R}$.

**Lemma 8.7.** Let $(\mu_t)_{t \in T}$ be a peacock and $c : \mathbb{R}^2 \to \mathbb{R}$ a MSM cost function. Under the assumptions (i) and (ii) of Theorem 8.4, for all $t \in T$ and $M \in \mathbb{N}$ the following successive limit exists and satisfies

$$
\lim_{N \to \infty} \lim_{m \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{A_{m,M,N}}(X_0, X_t) R_{M,N}(X_0, X_t) \right] = \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M)}(X_0) c(M, X_t) \right].
$$

**Proof.** Fix $\pi \in \mathcal{M}_T((\mu_t)_{t \in T})$ and $t \in T$.

**STEP 1:** For all $m, M, N \in \mathbb{N}$ and $x \in [-m, M]$, the tangent function $L_{m,M}^N$ of $c$ is an affine function. Hence, the martingale property yields that

$$
\mathbb{E}_{\pi} \left[ \mathbb{I}_{[-m,M]}(X_0) (c(M, X_t) - R_{M,N}(X_0, X_t)) \right] = \mathbb{E}_{\pi} \left[ \mathbb{I}_{[-m,M]}(X_0) L_{X_0,M}^N(X_t) \right] = \mathbb{E}_{\pi} \left[ \mathbb{I}_{[-m,M]}(X_0) L_{X_0,M}^N(X_0) \right]
$$

is independent of $\pi \in \mathcal{M}_T((\mu_t)_{t \in T})$. Thus,

$$
\mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{[-m,M]}(X_0) R_{M,N}(X_0, X_t) \right] = \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{[-m,M]}(X_0) c(M, X_t) \right].
$$

**STEP 2:** Fix $M \in \mathbb{N}$ and $x \in [-m, M]$. For all $N \in \mathbb{N}$ we define

$$
B_{m,M,N} = [-m, M] \times (N, \infty)
$$
so that $\mathbb{I}_{[-m,M]}(x) = \mathbb{I}_{A_{m,M,N}}(x, y) + \mathbb{I}_{B_{m,M,N}}(x, y)$ for all $y \in \mathbb{R}$. The function

$$
y \mapsto c(M, y) - R_{N,M}(x, y) = L_{x,M}^N(y)
$$

is the tangent of the concave function $\Delta_{x,M}$ at the position $N$. By the properties of concave functions, for every $y \in (N, +\infty)$ (where we recall $N \geq 0$) there holds

$$
\Delta_{x,M}(y) \leq L_{x,M}^N(y) \leq L_{x,M}^0(y).
$$

Hence,

$$
\mathbb{I}_{B_{m,M,N}}(X_0, X_t) | R_{M,N}(X_0, X_t) |
\leq |c(M, X_t)| + |\Delta_{X_0,M}(X_t)| + \mathbb{I}_{[-m,M]}(X_0) |L_{X_0,M}^0(X_t)|.
$$

Note that the r.h.s. is independent of $N$ and integrable w.r.t. $\pi$ and $\pi_{lc}$.

**STEP 3:** With the majorant that we found in Step 2, the dominated convergence theorem yields

$$
\lim_{N \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{B_{m,M,N}}(X_0, X_t) R_{M,N}(X_0, X_t) \right] = 0.
$$

and therefore there holds

$$
\lim_{N \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{A_{m,M,N}}(X_0, X_t) R_{M,N}(X_0, X_t) \right]
= \lim_{N \to \infty} \left( \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{[-m,M]}(X_0) R_{M,N}(X_0, X_t) \right] - \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{B_{m,M,N}}(X_0, X_t) R_{M,N}(X_0, X_t) \right] \right)
= \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{[-m,M]}(X_0) c(M, X_t) \right].
$$

**STEP 4:** For all $M \in \mathbb{N}$ assumption (ii) ensures that $\mathbb{I}_{(-\infty,M]}(X_0) c(M, X_t)$ is integrable w.r.t. $\pi$ and $\pi_{lc}$. Hence, the claim follows with dominated convergence. \qed
Lemma 8.8. Let \((\mu_t)_{t \in T}\) be a peacock and \(c : \mathbb{R}^2 \to \mathbb{R}\) a MSM cost function that satisfies assumptions (i) and (ii) of Theorem 8.4. If additionally assumption (iii-a) or assumption (iii-b) of Theorem 8.4 are satisfied, for all \(t \in T\) there holds

\[
\lim_{M \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M]}(X_0)c(M,X_t) \right] = 0.
\]

Proof. Fix \(\pi \in \mathcal{M}_T(\mu_t)_{t \in T}\) and \(t \in T\).

If assumption (iii-b) is satisfied, \(\mathbb{I}_{(-\infty,M]} = 1\) \(\pi\)-a.e. Hence, for all \(M \geq \overline{M}\) there holds

\[
\mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M]}(X_0)c(M,X_t) \right] = \mathbb{E}_{\pi - \pi_{lc}} \left[ c(M,X_t) \right] = 0
\]

and the claim follows.

Now suppose assumption (iii-a) is satisfied. In this case, w.l.o.g. we can assume that

\[(8.4) \quad c(0,y) = c(x,0) = \partial_x c(x,0) = 0 \quad \text{for all } (x, y) \in \mathbb{R}^2.\]

Otherwise, we replace \(c\) with \(\tilde{c}\) defined as

\[
\tilde{c}(x,y) := \Delta_{0,x}c(y) - (\Delta_{0,x}c)(0) - (\Delta_{0,x}c)'(0)y
\]

\[
= c(x,y) - c(0,y) - (\Delta_{0,x}c)(0) - (\Delta_{0,x}c)'(0)y.
\]

Indeed, \(\tilde{c}\) satisfies assumptions (i), (ii), and (iii-a) or (iii-b) (depending on \(c\)) and \(\tilde{c}(0,y) = \tilde{c}(x,0) = \partial_x \tilde{c}(x,0) = 0\) for all \((x, y) \in \mathbb{R}^2\). Moreover, by the martingale property there holds

\[
\lim_{M \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M]}(X_0)\left(\tilde{c}(M,X_t) - c(M,X_t)\right) \right] = 0
\]

Hence the limit of \(\mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M]}(X_0)c(M,X_t) \right]\) exists and vanishes as \(M\) tends to infinity if and only if the limit of \(\mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(-\infty,M]}(X_0)\tilde{c}(M,X_t) \right]\) exists and vanishes as \(M\) tends to infinity.

For all \(0 \leq x \leq x'\), the MSM property and \((8.4)\) yield that \(\Delta_{x,x'}c\) is a concave function with \((\Delta_{x,x'}c)'(0) = (\Delta_{x,x'}c)'(0) = 0\) and thus \(\Delta_{x,x'}c \leq 0\). Hence, for all \(0 \leq M' \leq M\) and \((x,y) \in \mathbb{R}^2\) we obtain

\[
\mathbb{I}_{(M,\infty)}(x)\Delta_{M,x}c(y) = \mathbb{I}_{(M,M')}\Delta_{M,x}c(y) + \mathbb{I}_{(M',\infty)}(x)\Delta_{M,x}c(y) \leq \mathbb{I}_{(M',\infty)}(x)\Delta_{M,x}c(y) \leq \mathbb{I}_{(M',\infty)}(x)\Delta_{M',x}c(y).
\]

Since for all \((x,y) \in \mathbb{R}^2\) the sequence \(\mathbb{I}_{(M,\infty)}(x)\Delta_{M,x}c(y)\) converges to 0 as \(M\) tends to infinity, monotone convergence applied separately for \(\pi\) and \(\pi_{lc}\) yields

\[
\lim_{M \to \infty} \mathbb{E}_{\pi - \pi_{lc}} \left[ \mathbb{I}_{(M,\infty)}(X_0)\Delta_{M,x}c(X_t) \right] = 0.
\]
Finally, we can conclude

\[
\lim_{M \to \infty} \mathbb{E}_{\pi^{-\nu}} \left[ \mathbb{1}_{(-\infty,M]}(X_0)c(M,X_t) \right]
\]

\[
= \lim_{M \to \infty} \left( \mathbb{E}_{\pi^{-\nu}}[c(M,X_t)] - \mathbb{E}_{\pi^{-\nu}}[\mathbb{1}_{(M,\infty)}(X_0)c(M,X_t)] \right)
\]

\[
= -\lim_{M \to \infty} \mathbb{E}_{\pi^{-\nu}} \left[ \mathbb{1}_{(M,\infty)}(X_0)c(M,X_t) \right]
\]

\[
= \lim_{M \to \infty} \mathbb{E}_{\pi^{-\nu}} \left[ \mathbb{1}_{(M,\infty)}(X_0)\Delta_{M,X_0} c(X_t) \right] - \lim_{M \to \infty} \mathbb{E}_{\pi^{-\nu}} \left[ \mathbb{1}_{(M,\infty)}(X_0)c(X_0,X_t) \right]
\]

\[
= 0
\]

using dominated convergence in the last equality employing assumption (ii). \( \Box \)

9. Examples and counterexamples

The purpose of this section is twofold. On the one hand we show in Section 9.1 that certain continuity properties of the obstructed shadow valid for finite index sets \( T = \{0, \ldots, n\} \) do not hold in general. (In fact this is one of the reasons why we had to develop new tools.) On the other hand, we show explicit examples of shadow martingales. One particular class of examples allowing for very explicit representations is the class of non-obstructed peacocks introduced in Section 9.2. In Section 9.3 we give a couple of concrete examples (non-obstructed and obstructed).

9.1. Continuity of the shadow. In Section 6.2 we showed that the map \( t \mapsto S^\mu(t) (\nu) \) is left- resp. right-continuous whenever the peacock \( \mu_T \) is left- or right-continuous. However, we didn’t discuss any continuity properties of the shadow as a function of \( \nu \) and \( \mu_T \) defined on \( \mathcal{M}_1(\mathbb{R}) \) and \( \mathcal{P}_T \) respectively. For the simple shadow (see Proposition 4.11), Juillet showed in [35, Theorem 2.30] that

\[
\mathcal{W}_1 \left( S^\mu(\nu), S^{\mu'}(\nu') \right) \leq \mathcal{W}_1(\nu, \nu') + 2\mathcal{W}_1(\mu, \mu')
\]

for all \( \nu, \nu' \in \mathcal{M}_1(\mathbb{R}) \), \( \nu \leq_{c,+} \mu \) and \( \nu' \leq_{c,+} \mu' \) where the Wasserstein distance extends to finite non-probability measures through the dual Kantorovich equivalent definition of \( \mathcal{W}_1 \) (cf. [35, Section 1]).

Similarly, an inductive application of (9.1) implies that for finite \( T \) the map \( \mu_T \mapsto S^{\mu_T}(\nu) \) is continuous w.r.t. pointwise convergence of \( \mu_T \) (If \( \mu_T \) is a peacock, then this is exactly the topology on \( \mathcal{P}_T \) ). We want to highlight a few implications of this continuity for the class of NSI peacocks in the case of a finite index set \( T \) (where \( T_t = \{s \leq t : s \in T\} \) as previously):

Lemma 9.1. Let \( T \) be finite with minimal element 0.

(i) The set of NSI peacocks is a closed subset of \( \mathcal{P}_1(\mathbb{R})^T \).

(ii) Given an arbitrary peacock \( (\eta_t)_{t \in T} \), the peacock \( (\mu_t)_{t \in T} \) with

\[
\mu_t := \lim_{n \to \infty} (S(n\eta_t), s \tau^n(\eta_0))_{n \in \mathbb{N}}
\]

for all \( t \in T \) is NSI.

(iii) Let \( (\mu_t)_{t \in T} \) be a peacock, \( (\nu^\alpha)_{\alpha \in [0,1]} \) some parametrization of \( \mu_0 \) and \( (\pi^\alpha)_{\alpha \in [0,1]} \) the martingale parametrization of a shadow martingale \( \pi \) w.r.t. \( (\mu_t)_{t \in T} \) and \( (\nu^\alpha)_{\alpha \in [0,1]} \). If the right-derivative \( \hat{\pi}^\alpha \) exists it is NSI. (If \( (\nu^\alpha)_{\alpha \in [0,1]} \) is the sunset parametrization of \( \mu_0 \), this is an immediate consequence of (ii).)
The proof is rather straightforward given that \( \mu_S \mapsto S^{\mu_S}(\nu) \) is continuous for all \( S \subseteq T \). We omit the details.

Armed with Lemma 9.1, we can give a short proof of the uniqueness of shadow martingales for a finite index set \( T \) independent of Theorem 7.3. Furthermore, appealing to the connection of one step NSI peacocks and Kellerer dilations explained in Remark 4.31, it follows that shadow martingales can be disintegrated into binomial martingales whose law is by definition a concatenation of Kellerer dilations. As a particular case, we recover [13, Theorem 8.3] by Nutz, Stebegg, and Tan.

**Corollary 9.2.** Let \( T = \{0, 1, \ldots, n\} \) be finite, \( (\mu_t)_{t \in T} \) be a peacock, and \( (\nu^\alpha)_{\alpha \in [0, 1]} \) be any parametrization of \( \mu_0 \). There exists a unique shadow martingale \( \pi \) w.r.t. \( (\mu_t)_{t \in T} \) and a unique martingale parametrization \( (\pi^\alpha)_{\alpha \in [0, 1]} \) of \( \pi \) w.r.t. \( (\nu^\alpha)_{\alpha \in [0, 1]} \) such that \( \alpha \mapsto \pi^\alpha \) is right-differentiable at \( \lambda \)-a.e. \( \alpha \in [0, 1] \) and the right-derivative \( \hat{\pi}^\alpha \), whenever it exists, is a binomial martingale measure.

If \( \mu_0(\{x\}) = 0 \) for all \( x \in \mathbb{R} \), then the unique shadow martingale w.r.t. \( (\mu_t)_{t \in T} \) and the left-curtain parametrization \( (\nu^\alpha_t)_{\alpha \in [0, 1]} \) is a binomial martingale.

**Proof.** By Proposition 5.6, there exists a shadow martingale \( \pi \) w.r.t. \( (\mu_t)_{t \in T} \) and \( (\nu^\alpha)_{\alpha \in [0, 1]} \). Denote by \( (\pi^\alpha)_{\alpha \in [0, 1]} \) the corresponding martingale parametrization (recall that Proposition 5.6 does not require that \( (\nu^\alpha)_{\alpha \in [0, 1]} \) be \( \leq c_s \)-convex).

By Lemma 5.7, there exists a Borel set \( A \subset [0, 1] \) with \( \lambda(A) = 1 \) such that \( \alpha \mapsto \pi^\alpha \) is right-differentiable for all \( \alpha \in A \). For each \( \alpha \in A \), we denote by \( (\eta^\alpha_t)_{t \in T} \) the marginal distributions of the right-derivative \( \hat{\pi}^\alpha \). Lemma 9.1 (iii) yields that \( (\eta^\alpha_t)_{t \in T} \) is NSI for all \( \alpha \in A \).

For each \( \alpha \in A \), Lemma 4.28 yields that the one-step peacocks \( (\eta^\alpha_0, \eta^\alpha_1) \) is NSI because \( (\eta^\alpha_t)_{t \in T} \) is NSI. By induction over \( 0 \leq i \leq n \) we obtain by Lemma 4.28 that the one-step peacocks \( (\eta^\alpha_i, \eta^\alpha_{i+1}) \) are NSI. Thus, Remark 4.31 yields that \( \hat{\pi}^\alpha \) is a binomial martingale. In particular, \( \hat{\pi}^\alpha \) is uniquely determined by its marginals and therefore \( \pi \) and \( (\pi^\alpha)_{\alpha \in [0, 1]} \) are uniquely determined by their marginals.

If \( \mu_0(\{x\}) = 0 \) for all \( x \in \mathbb{R} \) and \( (\nu^\alpha)_{\alpha \in [0, 1]} \) is the left-curtain parametrization, we obtain by Lemma 7.4 that

\[
\text{Law}_{\pi}(X|X_0) = \hat{\pi}^q(X_0) \quad \pi\text{-a.e.}
\]

where \( q = F_{\mu_0}^{-1} \) is the quantile function of \( \mu_0 \). The claim follows from the first part. \( \square \)

Let us go back to the case of an infinite index set \( T \). As Example 9.3 below shows, the continuity of \( \mu_S \mapsto S^{\mu_S}(\nu) \) fails in general. Moreover, neither of the items (i)-(iii) of Lemma 9.1 are true any more as Examples 9.3 and 9.4 show.

We would like to stress that the lack of item (ii) (and hence (iii)) of Lemma 9.1 is the main point separating the case of finite index sets from the one of a countably infinite index set \( T \). As a particular consequence, we could not rely on argumentations from [13] and [14] to show uniqueness of shadow martingales but had to develop a new approach.

In Examples 9.3 and 9.4 we consider the index set \( T = [0, 1] \) for notational reasons. The examples can easily be adapted to a countable index set, e.g. by considering \( T = [0, 1] \cap \mathbb{Q} \).
Example 9.3 (Discontinuity of the shadow, NSI is not closed). Let \((\mu^n_t)_{t \in [0,1]}\) be defined by
\[
\mu^n_t = \begin{cases} 
\delta_0 & t \in [0, \frac{n-1}{n}) \\
\delta_1 + \frac{1}{2} (\delta_2 - \delta) & t = 1 \\
\frac{1}{2} \delta_0 + \frac{1}{4} (\delta_2 - \delta) & t \in \left(\frac{n-1}{n}, 1\right]
\end{cases}
\]
for \(t \in [0,1]\) and \(n \in \mathbb{N}\). For all \(t \in [0,1]\) the sequence \((\mu^n_t)_{n \in \mathbb{N}}\) converges in \(\mathcal{M}_1(\mathbb{R})\) to
\[
\mu_t = \begin{cases} 
\delta_0 & t < 1 \\
\frac{1}{2} \delta_0 + \frac{1}{4} (\delta_2 - \delta) & t = 1
\end{cases}
\]
But for \(\nu = \frac{1}{2} \delta_0\), we have
\[
\lim_{n \to \infty} \mathcal{S}^{\mu^n_0,1}(\nu) = \frac{1}{4} \delta_0 + \frac{1}{8} (\delta_2 - \delta) \neq \frac{1}{2} \delta_0 = \mathcal{S}^{\mu_0,1}(\nu).
\]
Thus, the map \(\mu_{[0,1]} \mapsto \mathcal{S}^{\mu_{[0,1]}}(\nu)\) is not (sequentially) continuous. Moreover, any element of the sequence \((\mu^n_t)_{t \in [0,1]}\) is NSI but the limit \((\mu_t)_{t \in [0,1]}\) is not NSI by Lemma 4.28. Hence the subset of NSI peacocks in \(\mathcal{P}_{[0,1]}\) is not closed.

Example 9.4 (Right-derivatives are not NSI). Let \((\mu_t)_{t \in [0,1]}\) be defined by
\[
\mu_t = \begin{cases} 
(1-t) \delta_0 + \frac{1}{2} (\delta_1 + \delta) & t < 1 \\
\frac{1}{3} \delta_0 + \frac{1}{2} (\delta_2 - \delta) & t = 1
\end{cases}
\]
for \(t \in [0,1]\) and let
\[
\nu^\alpha = \alpha \mu_0 = \alpha \delta_0
\]
be the subset parametrization of \(\mu_0\). Moreover, let \(\pi\) be the shadow martingale w.r.t. \((\mu_t)_{t \in [0,1]}\) and \((\nu^\alpha)_{\alpha \in [0,1]}\) and \((\pi^\alpha)_{\alpha \in [0,1]}\) the corresponding martingale parametrization.

For all \(h > 0\) (small enough) and \(\alpha \in [0,1]\), it holds
\[
\frac{\nu^{\alpha+h} - \nu^\alpha}{h} = \mu_0 = \delta_0.
\]
Thus, the right-derivative of \(\alpha \mapsto \nu^\alpha\) exist for all \(\alpha \in [0,1]\) and equals \(\mu_0\). Actually, it is not difficult to show that the right-derivatives \(\pi^\alpha\) of \(\alpha \mapsto \pi^\alpha\) exist for all \(\alpha \in [0,1]\).

However, the marginal distributions of \(\pi^\frac{1}{2}\) are
\[
\pi^\frac{1}{2}(X_t \in \cdot) = \lim_{n \to \infty} \mathcal{S}^{(\mu_0 - \mathcal{S}^{\mu_\frac{1}{2}}(\frac{1}{2} \mu_0)))_{s \in T_1}(\mu_0) = \begin{cases} 
\delta_0 & t < 1 \\
\frac{1}{2} \delta_0 + \frac{1}{4} (\delta_2 - \delta) & t = 1
\end{cases}
\]
Observe that \((\mu_s - \mathcal{S}^{\mu_\frac{1}{2}}(\frac{1}{2} \mu_0)))_{s \geq 0}\) is a peacock by Remark 4.28. Moreover, Lemma 4.28 implies that the family of measures on the right hand side of (9.2) is not an NSI peacock. Hence, items (ii) and (iii) of Lemma 9.7 are not satisfied.

9.2. Non-obstructed shadows. We fix a totally ordered set \((T, \leq)\) with minimal element \(0 \in T\). In this section, we consider the special case that the additional obstructions in the shadow between \(\mu_0\) and \(\mu_t\) imposed by the marginals \((\mu_s)_{s \in T}\) are not binding, i.e. \(\mathcal{S}^{\mu_\frac{1}{2}}(\nu) = \mathcal{S}^{\mu}(\nu)\) for all \(t \in T\). The associated shadow martingales allow for rather explicit representations as will be shown in Proposition 9.8 and illustrated in Examples 9.9, 9.11

Definition 9.5. Let \((\mu_t)_{t \in T}\) be a family in \(\mathcal{P}_1(\mathbb{R})\) and \(\nu \leq_{c,+} \mu_T\). We say that the shadow of \(\nu\) in \((\mu_t)_{t \in T}\) is non-obstructed if for all \(t \in T\)
\[
\mathcal{S}^{\mu_\frac{1}{2}}(\nu) = \mathcal{S}^{\mu}(\nu).
\]
Lemma 9.6. Let \((\mu_t)_{t \in T}\) be a family in \(\mathcal{P}_1(\mathbb{R})\) and \(\nu \leq_{c,+} \mu_T\). The shadow of \(\nu\) in \((\mu_t)_{t \in T}\) is non-obstructed if for all \(s \leq t\) in \(T\) one of the following equivalent conditions is satisfied:

(i) \(S^\mu_s(\nu) \leq_c S^{\mu_t}(\nu)\).
(ii) \(S^\mu_s(\nu) = S^{\mu_t}(\nu)\).

Proof. For fixed \(s \leq t\) in \(T\), the equivalence of (i) and (ii) is straightforward. If (i) is satisfied for all \(s \leq t\), the claim follows with Proposition 4.22 applied to the family \((S^{\mu_s}(\nu))_{s \in T}\).

We are interested in pairs of peacocks \((\mu_t)_{t \in T}\) and \(\leq_{c,s}\)-convex parametrization \((\nu^\alpha)_{\alpha \in [0,1]}\) of \(\mu_0\) for which the shadow of \(\nu^\alpha\) in \((\mu_t)_{t \in T}\) is non-obstructed, for all \(\alpha \in [0,1]\). For an example available in the literature, one can consider the middle-curtain parametrization in combination with peacocks increasing in monotonic order, see [34] and [13] Section 3.1.3. Setting \(x\) the barycenter of \(\mu_0\), this condition can be formulated by \(S^{\mu^\alpha}(\alpha \delta_x) \leq_c S^{\mu_t}(\alpha \delta_x)\) for all \(0 \leq s \leq t\) and \(\alpha \in [0,1]\). One can easily check that it precisely corresponds to peacocks for which the middle-curtain parametrization has non-obstructed shadows. Note that in [34] a non-Markovian generalization of Kellerer’s Theorem 3.16 is given for peacocks in \(\mathcal{P}(\mathbb{R})\) increasing in diatomic convex order that are indexed by a partially ordered set.

The following lemma describes a class of peacocks non-obstructed by parametrizations of “interval type”:

Lemma 9.7. Let \((\mu_t)_{t \in T}\) be a peacock such that there exists a nested family of closed intervals \((I_t)_{t \in T}\) with

(i) \(\text{supp}(\mu_t) \subset I_t\) for all \(t \in T\) and
(ii) \(\mu_t|_{I_s} \leq_+ \mu_s|_{I_t}\) for all \(s \leq t\) in \(T\).

If additionally \(\mu_t(\{x\}) = 0\) for all \(x \in \mathbb{R}\) and \(t \in T\), then, for any closed interval \(I \subset \mathbb{R}\), the shadow of \(\nu = \mu_{0|I}\) in \((\mu_t)_{t \in T}\) is non-obstructed.

Proof. Let \(s \leq t\) in \(T\) and \(I \subset \mathbb{R}\) be an interval. W.l.o.g. we may assume that \(I \subset I_s\). Property (ii) yields that \((\mu_{s|I} - \mu_{t|I}) \in \mathcal{M}_1(\mathbb{R})\) and thus by Proposition 4.15 (ii)

\[(9.3)\]

\[S^\mu_s(\mu_{s|I}) = \mu_{t|I} + S^{\mu_{t|I}}(\mu_{s|I} - \mu_{t|I}) = \mu_{t|I} + \mu_{t|I \cap J} = \mu_{t|J}\]

for some closed interval \(J\) with \(I \subset J \subset I_t\). Such an interval \(J\) exists because \(\text{supp}(\mu_{s|I} - \mu_{t|I})\) is contained in the interval \(I\) and \(\text{supp}(\mu_{t|I\cap J})\) belongs to the closure of the complement of \(I\) (see Lemma 4.1.4 (ii)). Applying \((9.3)\) twice yields both \(S^\mu_s(\nu) = \mu_{t|J}\) and \(S^\mu_s(S^{\mu_s}(\nu)) = \mu_{t|J'}\) for two intervals \(J\) and \(J'\). However, since both measures are in convex order greater than \(\nu\), they have the same mass and barycenter by Lemma 3.1 (i) and hence \(\mu_{t|J} = \mu_{t|J'}\). Thus, we have proven that \(S^\mu_s(\nu) \leq_c S^{\mu_t}(\nu)\) because \(S^\mu_s(\nu) \leq_c S^{\mu_t}(S^{\mu_s}(\nu))\) by default. The claim follows by Lemma 9.6. □

Note that the condition in Lemma 9.7 is conceptually very similar to the Dispersion Assumption introduced by Hobson and Norgilas in [30]. Recall that \(T\) is a totally ordered set with minimal element 0.

Proposition 9.8. Let \((\mu_t)_{t \in T}\) be a peacock and \((\nu^\alpha)_{\alpha \in [0,1]}\) a \(\leq_{c,s}\)-convex parametrization of \(\mu_0\). Let \(\pi \in \mathcal{M}_T((\mu_t)_{t \in T})\) be the corresponding shadow martingale and \((\tilde{\pi}^\alpha)_{\alpha \in [0,1]}\)
the family of right-derivatives. For all \( x \in \mathbb{R} \) and \( a \in [0,1] \), we define the maps \( C_{x,a}^+, C_{x,a}^- : [0,1] \rightarrow \mathbb{R} \) as

\[
C_{x,a}^+(t) = \inf \left( [x, +\infty) \cap \text{supp}(\mu_t - S^\mu(t^a)) \right) \quad \text{and} \\
C_{x,a}^-(t) = \sup \left( (-\infty, x] \cap \text{supp}(\mu_t - S^\mu(t^a)) \right).
\]

If the shadow of \( \nu^\alpha \) in \( (\mu_t)_{t \in T} \) is non-obstructed for all \( \alpha \in [0,1] \), then under \( \hat{\pi}^\alpha \) the process \( (X_t)_{t \in T} \) is a Markov process with

\[
X_t \in \{ C_{X_0,a}^+(t), C_{X_0,a}^-(t) \} \quad \hat{\pi}^\alpha \text{-a.e.}
\]

Moreover, if there exists a measurable function \( q : \mathbb{R} \rightarrow [0,1] \) as in Lemma 7.4, then \( (X_0, X_t)_{t \in T} \) is a Markov process under \( \pi \) that jumps between the two curves

\[
\hat{C}_x^+(t) = \inf \left( [x, +\infty) \cap \text{supp}(\mu_t - S^\mu(t^a)) \right) \quad \text{and} \\
\hat{C}_x^-(t) = \sup \left( (-\infty, x] \cap \text{supp}(\mu_t - S^\mu(t^a)) \right)
\]

depending on the initial value \( X_0 = x \).

**Proof.** Theorem 6.12 yields that there exists a Borel set \( A \subset [0,1] \) with \( \lambda(A) = 1 \) such that for all \( a \in A \) the right-derivative \( \hat{\nu}^\alpha \) of \( \alpha \mapsto \pi^\alpha \) at \( a \) exists and is Markov.

Since the shadow is non-obstructed, the marginal distributions of \( \hat{\pi}^\alpha \) satisfy

\[
\hat{\pi}^\alpha(X_t \in \cdot) = \lim_{h \downarrow 0} S^\mu(\mu_t - S^\mu(t^a)) \left( \frac{\nu^\alpha h - \nu^\alpha}{h} \right)
\]

Note that this is the simple shadow. Since \( \alpha \mapsto \nu^\alpha \) is right-differentiable everywhere with derivative \( \nu^\alpha \), we can apply [13, Lemma 2.8] (in conjunction with Lemma 3.3) for simple shadows, to obtain that

\[
\hat{\pi}^\alpha(X_t \in \cdot) = \nu^\alpha P_{t,a}
\]

where \( P_{t,a} \) denotes the Kellerer dilation onto the set \( \text{supp}(\mu_t - S^\mu(t^a)) \) (see Remark 4.31). There is only one martingale coupling between a measure and its Kellerer projection onto a set \( F \) and it is given by the Kellerer dilation kernel. Hence, (9.4) holds a.e.

In the second case, we know that \( (X_0, X_t)_{t \in T} \) is a Markov process under the unique shadow martingale measure \( \pi \) (cf. Corollary 7.5) and moreover

\[
\pi \left( X_t \in \{ \hat{C}_{X_0}^+, \hat{C}_{X_0}^- \} \right) = \mathbb{E}_{\pi} \left[ \pi \left( X_t \in \{ \hat{C}_{X_0}^+, \hat{C}_{X_0}^- \} \mid X_0 \right) \right]
\]

\[
= \mathbb{E}_{\pi} \left[ \hat{\pi}^{\alpha(X_0)} \left( X_t \in \{ \hat{C}_{X_0}^+, \hat{C}_{X_0}^- \} \mid X_0 \right) \right]
\]

\[
= \mathbb{E}_{\pi} \left[ \hat{\pi}^{\alpha(X_0)} \left( X_t \in \{ C_{X_0,q(X_0)}, C_{X_0,q(X_0)} \} \mid X_0 \right) \right] = 1
\]

by Lemma 7.4. \( \Box \)

### 9.3. Examples of shadow martingales.

In this section we present four examples of shadow martingales. For the first three examples we can apply Proposition 9.8 from the previous subsection. Indeed, using Lemma 9.10 it is easy to check that in these cases the shadow w.r.t. the given peacock and parametrization is non-obstructed. Alternatively, in Example 9.9 and Example 9.10 one could argue via Lemma 9.7 to see that the shadow is non-obstructed. Moreover, in these two examples the second part of Proposition 9.8 is applicable (cf. Proof of Corollary 7.5).
Example 9.9. Let \((\mu_t)_{t \geq 0}\) be the marginal distributions of a standard Brownian motion started at \(t = 1\), i.e. \(\mu_t\) is the Gaussian distribution on \(\mathbb{R}\) with mean 0 and variance \(1 + t\), and let \((\nu_{mc}^\alpha)_{\alpha \in [0,1]}\) be the middle-curtain parametrization of \(\mu_0\).

Let \(\pi_{mc}\) be the shadow martingale w.r.t. \((\mu_t)_{t \geq 0}\) and \((\nu_{mc}^\alpha)_{\alpha \in [0,1]}\). The second part of Proposition 9.8 shows that the canonical process \((X_t)_{t \geq 0}\) is a martingale that jumps between the two curves

\[(9.5) \quad C_{X_0}^- : t \mapsto -|X_0| \cdot \sqrt{1 + t} \quad \text{and} \quad C_{X_0}^+ : t \mapsto |X_0| \cdot \sqrt{1 + t}.
\]

Since \(\text{Law}_{\pi_{mc}}(X_0) = \mu_0\) is fixed, \((9.5)\) characterizes \(\pi_{mc}\) uniquely.

We can describe \(\pi_{mc}\) in purely stochastic terms: If \(N\) is a standard Gaussian distributed random variable, the distribution of the unique càdlàg martingale \((Y_t)_{t \geq 0}\) that satisfies

\[Y_t \in \begin{cases} \{N\} & t = 0 \\ \{-|N| \cdot \sqrt{1 + t}, |N| \cdot \sqrt{1 + t}\} & t > 0 \end{cases} \]

for all \(t \geq 0\) is precisely the shadow martingale measure \(\pi_{mc}\).

Note that in the previous example \((X_t)_{t \geq 0}\) is a Markov process under \(\pi_{lc}\). In general, only the process \((X_0, X_t)_{t \geq 0}\) is a Markov process under the shadow martingale measure by Corollary 1.1 and not the canonical process \((X_t)_{t \geq 0}\) itself. But since in Example 9.9 there holds \(\{C_{X_0}^-(t), C_{X_0}^+(t) : t \geq 0\} \cap \{C_{X_0}^-(t), C_{X_0}^+(t) : t \geq 0\} = \emptyset\) for all \(x \neq y\) (cf. Figure 5), one can reconstruct \(|X_0|\) from \(X_t\) and thus \((X_t)_{t \in [0,1]}\) is Markov.

We face a similar situation in the next example:

Example 9.10. Let \((\mu_t)_{t \geq 0}\) be defined as \(\mu_t = \text{Unif}[-1-t,1+t]\) for all \(t \geq 0\) and let \((\nu_{lc}^\alpha)_{\alpha \in [0,1]}\) be the left-curtain parametrization of \(\mu_0\).

Let \(\pi_{mc}\) be the shadow martingale w.r.t. \((\mu_t)_{t \geq 0}\) and \((\nu_{lc}^\alpha)_{\alpha \in [0,1]}\). The second part of Proposition 9.8 shows that the canonical process \((X_t)_{t \geq 0}\) is a martingale that jumps between the two curves

\[(9.6) \quad C_{X_0}^- : t \mapsto (-1) - \frac{X_0 + 1}{2} \cdot t \quad \text{and} \quad C_{X_0}^+ : t \mapsto X_0 + \frac{X_0 + 1}{2} \cdot t
\]
Figure 6. On the left is a sketch of $C_x^+$ (blue) and $C_x^-$ (red) in Example 9.10. On the right is a sketch of two typical trajectories under the corresponding shadow martingale measure $\pi_{lc}$.

Since $\text{Law}_{\pi_{mc}}(X_0) = \mu_0$ is fixed, (9.10) characterizes $\pi_{lc}$ uniquely.

We can describe $\pi_{lc}$ in purely stochastic terms: If $U$ is a uniformly random variable on $[-1, 1]$, the distribution of the unique càdlàg martingale $(Y_t)_{t \geq 0}$ that satisfies

$$Y_t \in \begin{cases} 
{U} & t = 0 \\
{-1 - \frac{U+1}{2} \cdot t, U + \frac{U+1}{2} \cdot t} & t > 0
\end{cases}$$

for all $t \geq 0$ is precisely the shadow martingale measure $\pi_{lc}$.

As discussed in Section 2, $\pi_{lc}$ is related to the solution of the peacock problem that was recently parallely constructed by Henry-Labordère, Tan and Touzi [24] and Juillet [36]. In the setting of Example 9.10, their solution exists and has the following behavior: Let $(\mu_t)_{t \geq 0}$ be as in Example 9.10. There exists a unique limit-curtain measure $\pi \in M^{lc}_{\mu_0}((\mu_t)_{t \geq 0})$. Under $\pi$ the canonical process consists of trajectories piecewise non-decreasing with jumps down at random times to the bottom $-f(t)$ of the interval. (cf. [36, Theorem B]) This solution to the peacock problem behaves notably differently from the shadow martingale (cf. Figure 6).

For Example 9.9 and Example 9.10 we could apply the second part of Proposition 9.8 that corresponds to Corollary 1.1 because the parametrization of the initial marginal was given by a restriction of $\mu_0$ to intervals of $\mathbb{R}$. Any shadow martingale w.r.t. a sunrise parametrization does not belong to this class. Therefore, in the following example we need to rely on the notion of martingale parametrization to describe the shadow martingale.

Example 9.11. Let $(\mu_t)_{t \geq 0}$ be defined as $\mu_t = \text{Unif}_{[-e^t, e^t]}$ for all $t \geq 0$ and let $(\nu^\alpha_{sun})_{\alpha \in [0,1]}$ be the sunset parametrization of $\mu_0$.

Let $\hat{\pi}_{sun}$ be the shadow martingale w.r.t. $(\mu_t)_{t \geq 0}$ and $(\nu^\alpha_{sun})_{\alpha \in [0,1]}$ and $(\hat{\pi}^a_{sun})_{a \in [0,1]}$ the right-derivatives of the corresponding martingale parametrization. Proposition 9.3 shows that under $\hat{\pi}_{sun}$ the canonical process $(X_t)_{t \geq 0}$ is a martingale that jumps between
Figure 7. This is a sketch of $C^{x,a}_{+}$ (blue) and $C^{x,a}_{-}$ (red) for $a = \frac{1}{2}$ and $x \in \{-1 + \frac{2}{5}i : 0 \leq i \leq 5\}$ (left) and two typical trajectories under the shadow martingale measure $\pi_{\text{sun}}$ (right) in Example 9.11.

The two curves

\begin{align*}
\bar{C}^{X_0,a}_{-} & : t \mapsto \begin{cases} 
X_0 & t \leq -\ln(a) \\
-ae^t & t > -\ln(a)
\end{cases} \quad \text{and} \quad \bar{C}^{X_0,a}_{+} & : t \mapsto \begin{cases} 
X_0 & t \leq -\ln(a) \\
ae^t & t > -\ln(a)
\end{cases}
\end{align*}

Since $\text{Law}_{\hat{\pi}_{\text{sun}}}(X_0) = \nu_{\text{sun}}^{a} = \mu_0$ is fixed, (9.7) characterizes $\hat{\pi}_{\text{sun}}^{a}$ and thereby the shadow martingale $\pi_{\text{sun}}$ uniquely.

We can describe $\pi_{\text{sun}}$ in purely stochastic terms: If $T$ is an exponentially distributed random variable with parameter 1 and $U$ an independent uniformly distributed random variable on $[-1,1]$, the distribution of the unique càdlàg martingale $(Y_t)_{t \geq 0}$ that satisfies

\[ Y_t \in \begin{cases} 
\{U\} & t < T \\
\{-e^{t-T}, e^{t-T}\} & t \geq T
\end{cases} \]

for all $t \geq 0$ is precisely the shadow martingale measure $\pi_{\text{sun}}$.

This example shows that martingale parametrizations and their right-derivatives are essential to describe our solution to the peacock problem. The behavior of the canonical process under every right-derivative $\hat{\pi}_{\text{sun}}^{a}$ is easy to understand and the shadow martingale is a simple mixture of these measures (see Figure 7).

Example 9.11 was more involved than Example 9.9 and Example 9.10 because the parametrization was not given by restrictions to intervals. However, this example still belongs to the special class of non-obstructed peacocks. The last example does not belong to this class.

Example 9.12. Let $(S_n)_{n \in \mathbb{N}}$ be the symmetric simple random walk on $\mathbb{Z}$, i.e. $S_n := \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$ where $(X_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables uniformly distributed on $\{-1,1\}$. Let the family $(\mu_n)_{n \in \mathbb{N}}$ be defined as

\[ \mu_n := \frac{1}{2} \text{Law}(S_n) + \frac{1}{2} \text{Law}(3S_n). \]
for all $n \in \mathbb{N}$ and let $(\nu^\alpha_{\text{sun}})_{\alpha \in [0,1]}$ be the sunset parametrization of $\mu_0 = \delta_0$, i.e. $\nu^\alpha_{\text{sun}} = \alpha \mu_0$.

By Theorem 7.3 there exists a unique shadow martingale $\pi_{\text{sun}}$ w.r.t. $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu^\alpha_{\text{sun}})_{\alpha \in [0,1]}$. Let $(\hat{\pi}^a)_{a \in [0,1]}$ be the family of right-derivatives of the corresponding martingale parametrization. For $a < \frac{1}{2}$, $\hat{\pi}^a$ is the distribution of $(S_n)_{n \in \mathbb{N}}$ and for $a \geq \frac{1}{2}$, $\hat{\pi}^a$ is the distribution of $(3S_n)_{n \in \mathbb{N}}$.

We can describe $\pi_{\text{sun}}$ in purely stochastic terms: If $U$ is a uniformly distributed random variable on $[0,1]$ independent from the simple symmetric random walk $(S_n)_{n \in \mathbb{N}}$, the distribution of the martingale $(Y_n)_{n \in \mathbb{N}}$ defined by

$$Y_n = \begin{cases} S_n & U < \frac{1}{2} \\ 3S_n & U \geq \frac{1}{2} \end{cases}$$

for all $n \in \mathbb{N}$ is precisely the shadow martingale measure $\pi_{\text{sun}}$.

It is easy to see that the shadow of the sunset parametrization in the peacock $(\mu_n)_{n \in \mathbb{N}}$ defined as in Example 9.12 is not non-obstructed similar to Figure 4. Hence, Proposition 9.8 does not apply and in fact under $\hat{\pi}^a$ the canonical process $(X_t)_{t \in \mathbb{N}}$ does not jump between only two curves.

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