We construct a model to explain the muon anomalous magnetic moment, without considering any lepton flavor violations, in the modular $A_4$ symmetry. We have investigated a predictive radiative seesaw model including dark matter candidate at favorable fixed point of $\tau = \omega$ obtained by recent analysis of the stabilized moduli values from the possible configurations of the flux compactifications. In the result, we show our predictions on the Dirac CP and Majorana phases, the neutrino masses, the mass range of dark matter as well as the muon anomalous magnetic moment through the $\chi^2$ analysis.
I. INTRODUCTION

Several flavor puzzles on electron/muon anomalous magnetic moment ($\Delta a_{e/\mu}$), neutrino masses and mixings as well as phases are expected to be explained by theories beyond the standard model (SM). Recently, the electron/muon anomalous magnetic moment ($\Delta a_{e/\mu}$) attracts more attention of particle physicists due to the discrepancy between experiment and the SM predictions. To resolve the $\Delta a_{e/\mu}$, one needs to introduce an extra charged fermion or at least boson that couples to the SM charged-lepton, namely electron/muon. In the other hand, a radiative seesaw model is believed to be one of the elegant solutions to explain neutrino oscillation data as well as $\Delta a_{e/\mu}$ data at low energy scale like the Large Hadron Collider (LHC) that is used to search for the footprint of the model, where the extra fermion or boson has to be added to generate loop diagrams of the active neutrinos mass matrix [1]. This implies that the small Yukawa couplings are not needed even if their extra masses are of the 1 TeV scale. The implementation of the loop diagram is imposed based on a (residual) symmetry, however it depends on the model construction. This also lead us to explain dark matter (DM) candidate as well as $\Delta a_{e/\mu}$. In such scenario, we have to consider constraints of the lepton flavor violations (LFVs) such as $\mu \rightarrow e\gamma$, where it is rather difficult to constraint.

Recently, attractive flavor symmetries are proposed by authors in the papers [2, 3]. They have applied modular motivated by non-Abelian discrete flavor symmetries to quark and lepton sectors. One remarkable advantage of applying this symmetries is the dimensionless couplings of the model can be transformed to non-trivial representations under those symmetries. We then do not need the scalar fields to obtain a predictive mass matrix. Along with this idea, a vast reference has recently appeared in the literature, e.g., $A_4$ [2, 4–28, 30–32], $S_3$ [33–38], $S_4$ [39–45], $A_5$ [44, 46, 47], double covering of $A_5$ [48, 49], larger groups [50], multiple modular symmetries [51], and double covering of $A_4$ [52, 53], $S_4$ [54, 55], and the other types of groups [56] in which masses, mixing, and CP phases for the quark and/or lepton have been predicted. Moreover, a systematic approach to understand the origin of CP transformations has been discussed in Ref. [65], and CP violation in models with modular symmetry was discussed in Refs. [66, 67], and a possible correction from Kähler potential was discussed in Ref. [68]. Furthermore, systematic analysis of the fixed points (stabilizers) has been discussed in Ref. [69]. A very recent paper of Ref. [70] finds a favorable fixed point $\tau = \omega$ among three fixed points, which are the fundamental domain of PSL(2, Z), by systematically analyzing the stabilized moduli values in the possible configurations

---

For interest readers, we provide some literature reviews, which are useful to understand the non-Abelian group and its applications to flavor structure [57–64].
of flux compactifications as well as investigating the probabilities of moduli values.

In this paper, we successfully introduce a term in our construction model in order to explain the muon anomalous magnetic moment without suffering from the LFVs using the modular $A_4$ symmetry. In addition, we construct a predictive neutrino mass model based on a radiative seesaw model. The bosonic or fermionic DM candidate is also considered in the model and its stability is preserved by residual symmetry after the spontaneous symmetry breaking of the modular symmetry. We have performed numerical analysis using the $\chi^2$ fit at 2, 3, and 5 $\sigma$ confidence level (C.L.). We also show the allowed region for each of the C.L. and we focus in the region nearby the most favorable fixed point of $\tau = \omega$ as in the recent theoretical analysis of Ref. [70].

This paper is organized as follows. In Sec. II, we define our construction model where in the formula we consider the muon and electron anomalous magnetic moments, neutral and active neutrino fermion mass matrices, LFVs, and the relic density of the bosonic/fermionic DM candidate. In Sec. III, we present our numerical analysis and our several predictions on CP phases and muon anomalous magnetic moment in the specific region nearby $\tau = \omega$ which satisfy the neutrino oscillation data, LFVs, and DM relic density are shown. In Sec. IV, we devote the conclusions and discussions. In Appendix, we review the modular group and we show how the multiplication rules work in $A_4$ symmetry.

\section{Model Setup}

In this section, we explain our construction model by introducing new fields and assigning the charges under the symmetries of $SU(2)_L \times U(1)_Y \times A_4 \times (-k)$ into the lepton and Higgs sectors. For the fermion sector, we add one singly-charged heavy lepton $E$ and three right-handed Majorana fermions $N_R \equiv [N_{Re}, N_{R\mu}, N_{R\tau}]^T$. The $E$ has $A_4$ singlet with $-1$ charge under the modular weight.

| Field Contents | Charge Assignment |
|----------------|-------------------|
| $SU(2)_L$ $L_L$ | 2 2 2 1 1 1 1 1 |
| $U(1)_Y$ $N_R$ | $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $-1$ $-1$ $-1$ 0 |
| $A_4$ $N_R$ | 1 1 $1''$ 1 $1''$ $1''$ 1 3 |
| $-k$ $N_R$ | 0 $-2$ 0 0 $-2$ 0 $-1$ $-1$ |

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
| Field Contents | Charge Assignment |
|----------------|-------------------|
| $SU(2)_L$ $L_L$ | 2 2 2 1 1 1 1 1 |
| $U(1)_Y$ $N_R$ | $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $-1$ $-1$ $-1$ 0 |
| $A_4$ $N_R$ | 1 1 $1''$ 1 $1''$ $1''$ 1 3 |
| $-k$ $N_R$ | 0 $-2$ 0 0 $-2$ 0 $-1$ $-1$ |
\hline
\end{tabular}
\caption{Field contents of the fermions and their charge assignments under $SU(2)_L \times U(1)_Y \times A_4 \times (-k)$, where the upper index of $N_R$ is the flavor index with $a = 1, 2, 3$, and $SU(3)_C$ singlet for all leptons.}
\end{table}
TABLE II. Field contents of bosons and their charge assignments under $SU(2)_L \times U(1)_Y \times A_4 \times (-k)$, where $SU(3)_C$ singlet for all bosons.

For the boson sector, we introduce an isospin singlet inert boson $\chi$, singly charged one $\chi^+$, and doublet inert one $\eta^*$, where all have zero vacuum expectation values (VEVs). The $\chi$ is denoted by $\chi = (\chi_R + i\chi_I) / \sqrt{2}$ that is $A_4$ singlet with $-1$ modular weight. The $\chi$ plays a role in inducing the muon $g$-2 together with $E$. The $\chi^+$ is $A_4$ singlet with $-1$ modular weight, which plays a role in inducing a valid interaction of the fermionic DM candidate to satisfy the relic density as well as another source of the electron $g$-2. The $\eta^*$ is denoted by $\eta^* = [\eta^-, \eta_R - i\eta_I] / \sqrt{2}$ that is $A_4$ singlet with $-3$ modular weight. The $\eta$ plays a role in generating both the electron $g$-2 and the neutrino mass matrix together with $N_R$. The SM Higgs is defined by $H = [h^+, (v_H + h_0 + iz) / \sqrt{2}]^T$ with charged neutral under $A_4 \times (-k)$, where $h^+$ and $z$ give, respectively, the mass of $W^+$ boson and $Z$ boson in the SM. The SM VEV of $v_H$ is given by 246 GeV. The bosonic field contents and their charged assignments are listed in Table II. Under these symmetries, one writes the valid Lagrangian as follows:

$$-\mathcal{L}_Y = y_e \bar{L}_L e_R + y_\mu Y^{(4)}_{\chi} \bar{L}_L \mu_R + y_\tau \bar{L}_L \tau_R + y_E \bar{E}_L \chi + M_E \bar{E}_L E_R + a_\eta [Y^{(4)}_{3} \otimes \bar{L}_L \otimes N_R] \tilde{\eta}^* + b_\eta [Y^{(6)}_{3} \otimes \bar{L}_L \otimes N_R] \tilde{\eta}^* + c_\eta [Y^{(4)}_{3} \otimes \tilde{L}_L \otimes N_R] \tilde{\eta}^* + m_R [Y^{(2)}_{3} \otimes \bar{N}_R \otimes N_R] + a_\chi [(Y^{(2)}_{3})^* \otimes \bar{e}_R \otimes N_R^C] \chi^- + b_\chi [(Y^{(6)}_{3})^* \otimes \bar{\mu}_R \otimes N_R^C] \chi^- + c_\chi [(Y^{(3)}_{6})^* \otimes \bar{\tau}_R \otimes N_R^C] \chi^- + \text{h.c.},$$

where $y_E$ and $M_E$ respectively include $Y^{(4)}_{\chi}$ and $i/(\tau - \tau^*)$, and $\tilde{\eta}^* \equiv i\sigma_2 \eta^*$ both of which play a role
in making them invariant under these symmetries, \( \sigma_2 \) is the second component of the Pauli matrix. The parentheses of \([\ldots]\) represents singlet under \( A_4 \) by applying its multiplication rules; Yukawa matrices have concrete structures as discussed later. Notice here that the charged-lepton mass matrix is diagonal due to the modular \( A_4 \) symmetry, and \( a, b, c \) are the complex parameters.

The Higgs potential is also given by \([1]\)

\[
\mathcal{V} = \lambda_0 Y_1^{(6)} (\eta^+ H)^2 + \mu_0 Y_1^{(4)} \eta^+ H \chi + V_2^{\text{tri}} + V_4^{\text{tri}} + \text{h.c.},
\]

where we define \( \eta \equiv [\eta^+, (\eta_R + i\eta_I)/\sqrt{2}]^T, \ H \equiv [h^+, (v_H + h_0 + iz_0)/\sqrt{2}]^T, \ \chi \equiv (\chi_R + i\chi_I)/\sqrt{2}, \) and \( V_2^{\text{tri}} \) and \( V_4^{\text{tri}} \) are respectively trivial quadratic and quartic terms of the Higgs potential; \( V_2^{\text{tri}} = \sum_{\phi = H, \chi, \eta} \mu^2_0 |\phi|^2, \ V_4^{\text{tri}} = \sum_{\phi' \leq \phi} \lambda_{\phi'\phi} |\phi\phi'|^2 \). The \( \lambda_0 \) term is important to generate the non-vanishing neutrino mass matrix, which is proportional to the mass difference between \( \eta_R \) and \( \eta_I \); \( m_R^2 - m_I^2 = \lambda_0 Y_1^{(6)} v_H^2 \), where \( m_R, I \) is the mass eigenstate of \( \eta_{R,I} \).

In this work, we take \( \tau = \omega \) (\( \omega \equiv e^{2\pi i/3} \)), which corresponds to one of the fixed points that are favored by systematically analyzing the stabilized moduli values in the possible configurations of the flux compactifications and investigating the probabilities of the moduli values. In Ref. \([70]\), we find that \( Y_1^{(4)} = 0 \). This point is practically very interesting because it is invariant under \( ST \) transformation \( \tau = -1/(1 + \tau) \). Therefore, we have a remnant \( Z_3 \) symmetry where the generators are \( \{ I, ST, (ST)^2 \} \) at this point. In this work, we also take \( \tau \approx \omega \), so we can neglect \( \mu_0 \) in the Higgs potential at the first order approximation. This implies that \( \eta \) and \( \chi \) do not mix in the mass matrix.\(^2\) In our convenience, we write the other representations values at \( \tau = \omega \):

\[
\begin{align*}
Y_1^{(4)} &= \frac{9}{4} Y_0^2 \omega, \\
Y_1^{(6)} &= \frac{27}{8} Y_0^3, \\
Y_3^{(2)} &= Y_0 \{1, \omega, -\frac{1}{2} \omega^2 \} \equiv \{y_1, y_2, y_3\}, \\
Y_3^{(4)} &= \frac{3}{2} Y_0^2 \{1, -\frac{1}{2} \omega, \omega^2 \} \equiv \{y'_1, y'_2, y'_3\}, \\
Y_3^{(6)} &= 0 \equiv \{y''_1, y''_2, y''_3\}, \\
Y_3^{(6)} &= \frac{9}{8} Y_0^3 \{-1, 2\omega, 2\omega^2 \} \equiv \{y'''_1, y'''_2, y'''_3\},
\end{align*}
\]

where \( Y_0 \approx 0.9486 \). More details on the modular symmetry and multiplication rules of the \( A_4 \) symmetry can be found in Appendix.

**A. Muon anomalous magnetic dipole moment**

A muon anomalous magnetic dipole moment (\( \Delta a_\mu \) or muon g-2) has been firstly reported by Brookhaven National Laboratory (BNL). They found that the muon g-2 data has a discrepancy at the \( 3.3\sigma \) level from the SM prediction: \( \Delta a_\mu = (26.1 \pm 8.0) \times 10^{-10} \) \([71]\).\(^2\) If the mixing term is large enough, we might be able to explain electron g-2 sizably. However, this topic is beyond our scope in this paper.
The new contribution to the muon g-2 is explained by the Yukawa Lagrangian proportional of $y_E$ in Eq.(II.1):

$$-\mathcal{L}_{\Delta a_\mu} = y_E \bar{E} L \mu \chi + H.c.,$$

(II.6)

where $y_E$ is real without loss of generality. We notice that this term contributes to the muon g-2 only. The formula for the muon g-2 can be then expressed as follows [72]:

$$\Delta a_\mu = \frac{|y_E|^2}{8\pi^2} \int_0^1 dx \frac{m_\mu^2 x^2 (1-x)}{x(x-1) + M_\mu^2 x + (1-x)m_\chi^2}.$$

(II.7)

B. Neutral fermions

The Yukawa mass matrix coming from $a_\eta, b_\eta, c_\eta$, denoted by $\bar{L} L Y_\eta N R \tilde{\eta}^* \sim \bar{\nu}_L Y_\eta N_R (\eta_R - i\eta_I)/\sqrt{2}$, are written as follows:

$$Y_\eta = \begin{pmatrix} a_\eta & 0 & 0 \\ 0 & b_\eta & 0 \\ 0 & 0 & c_\eta \end{pmatrix} \begin{pmatrix} y_1' & y_3' & y_2' \\ y_1'' + \epsilon_1 y_1''' & y_3'' + \epsilon_3 y_3''' & y_2'' + \epsilon_2 y_2''' \\ y_2' & y_1' & y_3' \end{pmatrix} \approx \frac{3}{2} Y_0^2 \begin{pmatrix} a_\eta & 0 & 0 \\ 0 & b_\eta & 0 \\ 0 & 0 & c_\eta \end{pmatrix} \begin{pmatrix} 1 & \omega^2 & -\frac{1}{2}\omega \\ -\frac{3}{4} Y_0 & \frac{3}{2} Y_0 \omega^2 & \frac{3}{2} Y_0 \omega \\ -\frac{1}{2}\omega & 1 & \omega^2 \end{pmatrix},$$

(II.8)

where $a_\eta, b_\eta, c_\eta$ stand for the real parameters by the phase redefinition of fields, and the $\epsilon_{1,2,3}$ are complex.

The right-hand neutrino mass matrix is given by

$$\mathcal{M}_N = \frac{m_R}{3} \begin{pmatrix} 2y_1 & -y_3 & -y_2 \\ -y_3 & 2y_2 & -y_1 \\ -y_2 & -y_1 & 2y_3 \end{pmatrix} \approx \frac{m_R Y_0}{3} \begin{pmatrix} 2 & \frac{1}{2}\omega^2 & -\omega \\ \frac{1}{2}\omega & 2\omega & -1 \\ -\omega & -1 & -\omega^2 \end{pmatrix},$$

(II.9)

where $m_R$ is taken to be real without loss of generality.

The mass matrix $\mathcal{M}_N$ is then diagonalized by multiplying it with a unitary matrix $V$, it then gives

$$V^T \mathcal{M}_N V = \mathcal{M}_N^{\text{diag}} \equiv \text{diag}(M_1, M_2, M_3).$$

(II.10)

Here, the mass eigenstate $\psi_R$ is defined by $N_R \equiv \sum_{k=1,3} V_{ik} \psi_R$, and its mass eigenvalue is defined by $M_a (a = 1, 2, 3)$. To induce the neutrino mass matrix into the Lagrangian, we rewrite the Lagrangian in terms of the mass eigenstate as

$$-\mathcal{L}_\nu = \frac{1}{\sqrt{2}} \bar{\nu}_L F_{ia} \psi_R (\eta_R - i\eta_I) + H.c.,$$

(II.11)
where $F \equiv Y_\eta V$. The final formula for the mass matrix is given by

$$\begin{align*}
(m_\nu)_{ij} &= \sum_{a=1}^{3} \frac{F_{ia} M_a F_{aj}^T}{2(4\pi)^2} \left[ \frac{m^2_{R_a}}{m^2_{R_a} - M^2_a} \ln \frac{m^2_{R_a}}{M^2_a} - \frac{m^2_i}{m^2_i - M^2_a} \ln \frac{m^2_i}{M^2_a} \right] \\
&\approx \frac{\lambda_0 v^2}{(4\pi)^2} \sum_{a=1}^{3} \frac{F_{ia} M_a F_{aj}^T}{m^2_a - M^2_a} \left[ 1 - \frac{M^2_a}{m^2_0 - M^2_a} \ln \frac{m^2_0}{M^2_a} \right],
\end{align*}$$

(II.12)

where we assume to be $\lambda_0 v^2_H = m^2_R - m^2_1 << m^2_0 \equiv (m^2_R + m^2_1)/2$ in the second line of Eq. (II.12), and $m_\nu$ is diagonalized by using a unitary matrix $U_{PMNS}$ \[73\]; $D_\nu \equiv U_{PMNS}^T m_\nu U_{PMNS}$. Here, we define the dimensionless neutrino mass matrix as $m_\nu \equiv (\lambda_0 v) \tilde{m}_\nu \equiv \kappa \tilde{m}_\nu$, where the $\kappa$ does not depend on the flavor structure. The diagonalization in terms of dimensionless form $\tilde{D}_\nu \equiv U_{PMNS}^T \tilde{m}_\nu U_{PMNS}$ is rewritten. Thus, we fix $\kappa$ by using this formula

$$\begin{align*}
(NH): \quad \kappa^2 &= \frac{|\Delta m^2_{\text{atm}}|}{\tilde{D}^2_{\nu_3} - \tilde{D}^2_{\nu_1}}, \\
(IH): \quad \kappa^2 &= \frac{|\Delta m^2_{\text{atm}}|}{\tilde{D}^2_{\nu_2} - \tilde{D}^2_{\nu_3}},
\end{align*}$$

(II.13)

where $\tilde{m}_\nu$ is diagonalized by $V_\nu^\dagger (\tilde{m}_\nu^0) V_\nu = (\tilde{D}^2_{\nu_1}, \tilde{D}^2_{\nu_2}, \tilde{D}^2_{\nu_3})$ and $\Delta m^2_{\text{atm}}$ is the atmospheric neutrino mass-squared difference. NH and IH stand for the normal and the inverted hierarchies, respectively. Subsequently, the solar neutrino mass-squared difference is described in terms of the $\kappa$ as follows:

$$\Delta m^2_{\text{sol}} = \kappa^2 (\tilde{D}^2_{\nu_2} - \tilde{D}^2_{\nu_1}).$$

(II.14)

This should be within the range of the experimental value. Later, we will adopt NuFit 5.0 \[74\] to our numerical analysis. The neutrinoless double beta decay is also given by

$$\langle m_{ee} \rangle = \kappa |\tilde{D}_{\nu_1} \cos^2 \theta_{12} \cos^2 \theta_{13} + \tilde{D}_{\nu_2} \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_2} + \tilde{D}_{\nu_3} \sin^2 \theta_{13} e^{i(\alpha_3 - 2\delta_{CP})}|,$$

(II.15)

which may be able to observed by future experiment of KamLAND-Zen \[75\].

C. Lepton flavor violations and anomalous magnetic moment

We firstly write the valid Lagrangian to arise LFVs in terms of the mass eigenstate

$$-\mathcal{L}_\nu = \bar{\ell}_L F_{ia} \psi_{R_a} \eta^- + G_{\alpha} \bar{\tau}_R \psi_{R_a}^C \chi^- + G_{\gamma} \bar{\tau}_R \psi_{R_a}^C \chi^- + \text{h.c.},$$

(II.16)
where we define $G^a_e \equiv a^e_\chi [y^a_1 V^a_{1a} + y^a_2 V^a_{2a} + y^a_3 V^a_{3a}]$, $G^a_\mu \equiv c^\mu_\chi [y^a_1 V^a_{1\mu} + y^a_2 V^a_{2\mu} + y^a_3 V^a_{3\mu}]$, and $a$ should be summed up over $1-3$. The corresponding branching ratio is given at one-loop level as follows \cite{76, 77}

$$\text{BR}(\ell_i \rightarrow \ell_j \gamma) = \frac{48\pi^3 a_m G^2_F}{G^2_F} |A_{ij}|^2 \left(1 + \frac{m^2_{ij}}{m^2_i} \right), \quad (\text{II.17})$$

$$A_{ij} = \frac{1}{(4\pi)^2} \sum_{a=1}^3 \left( F^i_{ja} F_{ai} \mathcal{I}(M_a, m_{\eta^-}) + (|\delta_{ia} G^a_i|^2 + |\delta_{ia} G^a_i|^2) \mathcal{I}(M_a, m_{\eta^-}) \right), \quad (\text{II.18})$$

$$\mathcal{I}(m_1, m_2) \simeq \frac{m^2_{m_2} - 6m^4_{m_2}m^2_i + 3m^2_{m_2}m^4_i + 2m^6_i + 6m^2_{m_2}m^4_i \ln \left[ \frac{m^2_{m_2}}{m^2_i} \right]}{12(m^2_{m_2} - m^2_i)^4}, \quad (\text{II.19})$$

where $i, j$ runs over $e, \mu, \tau$, the fine structure constant $\alpha_m \simeq 1/128$, the Fermi constant $G_F \simeq 1.17 \times 10^{-5}$ GeV$^{-2}$, and $(C_{21}, C_{31}, C_{32}) \simeq (1, 0.1784, 0.1736)$. \(\mathcal{I}(m_1, m_2)\) is derived by assuming $m_{i,j} << M_a, m_{\eta^-}$, and notice $\mathcal{I}(m_1, m_2) = \frac{1}{24M_a^2}$ in the limit of $M_a = m_{\eta^-}$. The current experimental upper bounds at 90\% C.L. are \cite{78, 79}

$$\text{BR}(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13}, \quad \text{BR}(\tau \rightarrow e\gamma) < 3.3 \times 10^{-8}, \quad \text{BR}(\tau \rightarrow \mu\gamma) < 4.4 \times 10^{-8}. \quad (\text{II.20})$$

Lepton $g-2$ is obtained via the same interaction with LFVs, which has the form

$$\Delta a^Y_\ell \approx -2m^2_\ell A_{\ell\ell}, \quad (\text{II.21})$$

where $\ell = \mu$ is for the case of the muon $g-2$, while $\ell = e$ for the electron $g-2$. We notice that $\Delta a^Y_\ell$ has a negative sign correction. This means that it is almost impossible to explain the anomaly, since the muon $g-2$ has a positive value. However, the electron $g-2$ has a negative value, so we may use it to explain the anomaly by using this term. The experimental data for the electron $g-2$ is given by \cite{80}

$$\Delta a^Y_e = -(8.8 \pm 3.6) \times 10^{-13}. \quad (\text{II.22})$$

While we impose condition of small value of the muon $g-2$ \cite{71};

$$\Delta a^Y_\mu << (26.1 \pm 8.0) \times 10^{-10}, \quad (\text{II.23})$$

in order to avoid a cancellation of $\Delta a_\mu$ in Eq.(II.7). In our analysis, with the imposing condition value in Eq.(II.23), we then evaluate the value of $\Delta a^Y_e$ in our model and see whether our model value prediction fits with the experimental value of $\Delta a^Y_e$ in Eq.(II.22).

### D. Dark matter

We consider several DM candidates; the lightest neutral boson of $\eta_R$ and $\eta_I$, $\chi$, and the lightest of $\psi_R_i$, $I = 1, 2, 3$.
1. Bosonic DM

As mentioned above, among several DM candidates, we briefly present the DM candidate of the lightest neutral boson of the $\eta_R$ and $\eta_I$, but we firstly concentrate on $\eta_R$. In direct detection searches, one might worry that the DM candidate would be ruled out by the spin independent scattering process via Z-boson portal. However, it can be evaded if the mass difference between $\eta_R$ and $\eta_I$ are bigger than the order 100 keV, since $\eta_R$ couples to Z-boson through $\eta_I$, which is inelastic scattering. If the dominant contribution to explain the correct relic density comes from the kinetic term, the allowed region is very narrow and fixed to be around the pole at the half of the neutral Higgs masses, e.g., $\sim 63$ GeV in case of the SM Higgs, and 534 GeV [81]. On the other hand, when the Yukawa term is dominant to the relic density; $F$ in our case, $F = O(1)$ is required since the cross section is p-wave dominant [82]. In our case, especially, $F$ cannot be so large because of the additional mixing from unitary mixing $V$ that diagonals the neutral heavier fermion $M_N$.

We now move forward to another DM candidate $\chi$. Here, we simply suppose that the relic density would be explained by the Yukawa coupling $y_E$ in Eq.(II.6) by assuming all the other interactions form the Higgs potential is negligibly small. The cross section is then given by [72]

$$\sigma v_{rel} \approx \frac{|y_E|^4}{192 \pi} \frac{M_E^2}{(M_E^2 + m^2_{\chi})^2} v_{rel}^2,$$

(II.24)

where $\chi$ is considered as complex, assuming that $m_\mu << M_E, m_\chi$, and approximation method is used to expand the relative velocity of the DM $v_{rel}^2 \sim 0.3$. The cross section should lie within the range of $[1.78-1.97] \times 10^{-9}$ GeV$^{-2}$ at $2\sigma$ C.L. in order to find the correct relic density. However, we will use more relaxed bound in our numerical analysis; $[1.5 - 2.5] \times 10^{-9}$ GeV$^{-2}$. In DM direct detection, we expect that it would not give us so stringent bounds, since the $\chi$ does not directly couple to the quarks in SM at tree level. The same analysis has been studied in Ref. [83], however it was done at one-loop level.

2. Fermionic DM

Since $F$ is not so large to explain the relic density, the main contribution arises from $G$ term in Eq.(II.16) as follows:

$$- \mathcal{L} = G^1_{\bar{e} e} e_R^{C} X_R^{C} \chi^- + G^1_{\bar{\tau} \tau} \tau_R^{C} X_R^{C} \chi^- + \text{h.c.},$$

(II.25)
where we have defined $X_R \equiv \psi_{R1}$. This implies that DM annihilates into a pair of electron-positron. Thus we assume that $m_e/m_{\chi^\pm} < m_e/m_X < m_{\tau}/m_{\chi^\pm} < m_{\tau}/m_X \sim 0$. The DM thermal-averaged annihilation cross section is then given by

$$\sigma_{v_{\text{rel}}} \approx \frac{(|G_e|^4 + |G_{\tau}|^4) m_X^2 (m_X^2 + m_{\chi^\pm}^2)}{(m_X^2 + m_{\chi^\pm}^2)^4 v_{\text{rel}}^2},$$

(II.26)

where we have assumed to be analog for the bosonic case. In numerical analysis, we only consider the fermionic DM candidate, since it correlates to the neutrino sector via the heavier neutral fermion and modulus $\tau$. In other words, one can easily find the allowed region between the DM mass and the muon anomalous magnetic moment, since the input parameters $y_E$, $M_E$ and $m_\chi$ are independent of the neutrino sector.

### III. NUMERICAL ANALYSIS

In this section, we present our numerical analysis using the following ranges of the input parameters below,

$$\{a_{\chi}, c_{\chi}, \gamma_D\} \in [0.1, 10], \quad \{a_\eta, b_\eta, c_\eta\} \in [0.01, 10], \quad |\epsilon_{1,2,3}| \in [0.01, 100], \quad y_E \in [0.1, \sqrt{\pi}],$$

$$\{m_\chi, m_{\chi^+}, M_E, m_0\} \in [0.1, 1] \text{TeV}, \quad m_R \in [10^{-3}, 1] \text{ TeV}, \quad m_{\eta^+} \in [m_0 \pm 0.005] \text{ TeV},$$

(III.1)

where in our work, we choose the region at nearby $\tau = \omega$. In addition, we also consider the case of the fermionic DM candidate only, because only the fermionic case correlates to the neutrino sector as mention in the last part of the DM subsection.

Under these regions, we randomly scan those input parameters and search for the allowed regions by imposing the constraints of the neutrino oscillation data, LFVs, and the observed relic density replaced by $\sigma v = [1.5 - 2.5] \times 10^{-9}$ GeV$^{-2}$. Next, we will show some plots in terms of the classification of $\chi^2$ square analysis within the range of 1-2$\sigma$ C.L. which is represented by green color, 2-3$\sigma$ C.L. which is represented by yellow color, and 3-5$\sigma$ C.L. which is represented by red color, referring to NuFit 5.0 [74]. In the present work, we adopt the accuracy of $\chi^2$ for five well known dimensionless observables such as $\Delta m_{\text{atm}}^2$, $\Delta m_{\text{sol}}^2$, $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$, and $\sin^2 \theta_{13}$. We find that $BR(\tau \to e\gamma)$ fits with the experimental upper bound, where the maximum values for $BR(\tau \to e\gamma$ and $BR(\tau \to \mu\gamma)$ are respectively $10^{-11}$ and $1.5 \times 10^{-11}$, which are lower than the experimental upper bounds.

In Fig. 1, we show the scatter plots of the real $\tau$ and imaginary $\tau$ at nearby the fixed point $\tau = \omega$. One finds that we definitely confirm the allowed region at nearby the fixed point $\tau = \omega$. The solid line is the fundamental domain boundary at $|\tau| = 1$. 

FIG. 1. The scatter plots for the real $\tau$ and imaginary $\tau$ at nearby the fixed point $\tau = \omega$ in NH. In the $\chi^2$ analysis, the green color represents $1$-$2\sigma$ C.L., yellow color is for $2$-$3\sigma$ C.L., and red color is for $3$-$5\sigma$ C.L.. The black solid line is the boundary of the fundamental domain at $|\tau| = 1$.

FIG. 2. The scatter plots for the DM mass and the muon $g$-$2$. The color representations are the same as in Fig. 1. The black solid lines is the central value of the muon $g - 2$, and the blue and red solid lines are respectively $1\sigma$ and $2\sigma$ bands.

Figure 2 shows the relationship between the DM mass and $\Delta a_\mu$, where the black solid line represents the central value of the muon $g - 2$, and the blue and red solid lines respectively correspond to $1\sigma$, and $2\sigma$ band of $\Delta a_\mu$. We also find that the DM mass $M_X$ lies in the range of $[50-820]$ GeV. However, if the $\Delta a_\mu$ is taken into account, the allowed region is restricted to be $[60-150]$ GeV at $1\sigma$ C.L. and $[50-300]$ GeV at $2\sigma$ C.L..

In the left panel of Fig. 3, we show the scatter plots of the Majorana phases $\alpha_{31}$ and $\alpha_{21}$. The summation of active neutrino mass eigenstates and Dirac CP phase is shown in the middle panel of Fig. 3, and the lightest mass of active neutrinos and neutrinoless double beta decay is shown in the right panel of Fig. 3. In the left panel of Fig. 3, all ranges are satisfied, but they tend to be localized at nearby left-up sector; $\alpha_{31} = [0 - 180]$ deg and $\alpha_{21} = [230 - 350]$ deg. In the middle
FIG. 3. The scatter plots for the Majorana phases $\alpha_{31}$ and $\alpha_{21}$ (Left panel), the summation of active neutrino mass eigenstates and Dirac CP phase (Middle panel), and the lightest mass of active neutrinos and neutrinoless double beta decay, and the right the DM mass and muon g-2 (Right panel). The color representations are the same as in Fig. 1.

Panel of Fig. 3, it is clearly shown that the allowed region is localized at $\sum_i m_i = [0.058 - 0.072]$ eV and $\delta_{CP} = [0 - 80, 260 - 360]$ deg. In the right panel of Fig. 3, we find that $\{m_1, \langle m_{ee} \rangle \} \leq 0.008$ eV.

For the case of IH, we found that the allowed region satisfy the neutrino oscillation data and satisfy the LFVs at 3-5$\sigma$ C.L.. However, the cross section to explain the relic density is the order of $10^{-13}$ GeV$^{-2}$ at the most, and the size of anomalous magnetic moment is the order of $10^{-10}$ at most. Both of them do not reach the favorite values. This implies that we have to rely on the bosonic DM candidate to obtain the correct relic density and sizable anomalous magnetic moment. Thus we do not consider this case furthermore.

IV. CONCLUSIONS AND DISCUSSIONS

We have constructed a model to explain the muon anomalous magnetic moment without considering any LFVs, by using the modular $A_4$ symmetry. We have also constructed a radiative seesaw model including the DM candidate. Specifying the favorite fixed point of $\tau = \omega$ obtaining by the recent analysis of the stabilized moduli values in the possible configurations of the flux compactifications, we have made several predictions in the case of NH such as phases, neutrino masses as shown in Fig. 3 and the range of the DM mass and the muon anomalous magnetic moment as shown in Fig. 2, where we have used the $\chi^2$ analysis. The sum of the neutrino masses lies within the range of $0.12$ eV, which is in favor of the cosmological observation.
ACKNOWLEDGMENTS

The work of P.T.P.H was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) No. 2018R1A5A1025563 and No. 2019R1A2C1005697. The work of J.K. is supported in part by Korean Institute Advanced Studies (KIAS) Individual Grant. No. PG074201. The work of D.K. is supported in part by KIAS Individual Grant. No. PG076201. The work of H.O. was supported by the Junior Research Group (JRG) Program at the Asia-Pacific Center for Theoretical Physics (APCTP) through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government and was supported by the Korean Local Governments-Gyeongsangbuk-do Province and Pohang City. H.O. is sincerely grateful for all the KIAS members.

APPENDIX

In this appendix, we present several properties of the modular $A_4$ symmetry. In general, the modular group $\bar{\Gamma}$ is a group of the linear fractional transformation $\gamma$, acting on the modulus $\tau$ which belongs to the upper-half complex plane and transforms as

$$\tau \to \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1, \quad \text{Im}[\tau] > 0. \quad (IV.1)$$

This is isomorphic to $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I, -I\}$ transformation. Then modular transformation is generated by two transformations $S$ and $T$ defined by:

$$S : \tau \to -\frac{1}{\tau}, \quad T : \tau \to \tau + 1, \quad (IV.2)$$

and they satisfy the following algebraic relations,

$$S^2 = \mathbb{I}, \quad (ST)^3 = \mathbb{I}. \quad (IV.3)$$

More concretely, we fix the basis of $S$ and $T$ as follows:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (IV.4)$$

where $\omega \equiv e^{2\pi i/3}$. 
Thus, we introduce the series of groups $\Gamma(N)$ ($N = 1, 2, 3, \ldots$) that is so-called "principal congruence subgroups of $SL(2, \mathbb{Z})$", which are defined by

$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) , \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N) \right\}$, \hspace{1cm} (IV.5)

and we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\}$ for $N = 2$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$ case, we have $\bar{\Gamma}(N) = \Gamma(N)$, that are infinite normal subgroup of $\bar{\Gamma}$ known as principal congruence subgroups. We thus obtain finite modular groups as the quotient groups defined by $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$. For these finite groups $\Gamma_N$, $T^N = I$ is imposed, and the groups $\Gamma_N$ with $N = 2, 3, 4$ and 5 are isomorphic to $S_3$, $A_4$, $S_4$ and $A_5$, respectively [3].

Modular forms of level $N$ are holomorphic functions $f(\tau)$ which are transformed under the action of $\Gamma(N)$ given by

$$f(\gamma \tau) = (c\tau + d)^k f(\tau) , \hspace{0.5cm} \gamma \in \Gamma(N) ,$$

where $k$ is the so-called as the modular weight.

Under the modular transformation in Eq.(IV.1) in case of $A_4$ ($N = 3$) modular group, a field $\phi^{(I)}$ is also transformed as

$$\phi^{(I)} \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)} ,$$

where $-k_I$ is the modular weight and $\rho^{(I)}(\gamma)$ denotes a unitary representation matrix of $\gamma \in \Gamma(2)$ ($A_4$ representation). Thus Lagrangian such as Yukawa terms can be invariant if sum of modular weight from fields and modular form in corresponding term is zero (also invariant under $A_4$ and gauge symmetry).

The kinetic terms and quadratic terms of scalar fields can be written by

$$\sum_I \frac{\sum_I |\partial_\mu \phi^{(I)}|^2}{(-i\tau + i\bar{\tau})^{k_I}} + \sum_I \frac{|\phi^{(I)}|^2}{(-i\tau + i\bar{\tau})^{k_I}} ,$$

which is invariant under the modular transformation and overall factor is eventually absorbed by a field redefinition consistently. Therefore the Lagrangian associated with these terms should be invariant under the modular symmetry.

The basis of modular forms with weight 2, $Y_3^{(2)} = (y_1, y_2, y_3)$, transforming as a triplet of $A_4$ is
written in terms of Dedekind eta-function \( \eta(\tau) \) and its derivative [2]:

\[
y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right)
\]

\[
\simeq 1 + 12q + 36q^2 + 12q^3 + \cdots ,
\]

(IV.9)

\[
y_2(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right)
\]

\[
\simeq -6q^{1/3}(1 + 7q + 8q^2 + \cdots ),
\]

(IV.10)

\[
y_3(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right)
\]

\[
\simeq -18q^{2/3}(1 + 2q + 5q^2 + \cdots ),
\]

(IV.11)

where \( q = e^{2\pi i \tau} \), and expansion form in terms of \( q \) would sometimes be useful to have numerical analysis.

Then, we can construct the higher order of couplings; e.g., \( Y_1^{(4)}, Y_1^{(6)}, Y_1^{(10)}, Y_3^{(6)}, Y_3^{(6)} \) following the multiplication rules as follows:

\[
Y_1^{(4)} = y_1^2 + 2y_2y_3, \ Y_1^{(6)} = y_1^3 + y_2^3 + y_3 - 3y_1y_2y_3, \ Y_1^{(10)} = Y_1^{(4)}Y_1^{(6)}, \ (IV.12)
\]

\[
Y_3^{(6)} \equiv (y_1', y_2', y_3') = (y_3^3 + 2y_1y_2y_3, y_1'^2y_2 + 2y_2'^2y_3, y_1'^2y_1 + 2y_2'^2y_2), \ (IV.13)
\]

\[
Y_3^{(6)} \equiv (y_1'', y_2'', y_3'') = (y_3'^3 + 2y_1y_2y_3, y_1''^2y_2 + 2y_2''^2y_3, y_1''^2y_1 + 2y_2''^2y_2), \ (IV.14)
\]

where the above relations are constructed by the multiplication rules under \( A_4 \) as shown below:

\[
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}_{3} \otimes \begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix}_{3'} = (a_1b_1 + a_2b_3 + a_3b_2)_{1} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{1'},
\]

\[
\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{1''}
\]

\[
\oplus \left( \frac{1}{3} \begin{pmatrix}
  2a_1b_1 - a_2b_3 - a_3b_2 \\
  2a_3b_3 - a_1b_2 - a_2b_1 \\
  2a_2b_2 - a_1b_3 - a_3b_1
\end{pmatrix}_{3} \oplus \frac{1}{2} \begin{pmatrix}
  a_2b_3 - a_3b_2 \\
  a_1b_2 - a_2b_1 \\
  a_3b_1 - a_1b_3
\end{pmatrix}_{3'} \right),
\]

\[
\begin{align*}
1 \otimes 1 &= 1, & 1' \otimes 1' &= 1'', & 1'' \otimes 1'' &= 1', & 1' \otimes 1'' &= 1 .
\end{align*}
\]

(IV.15)

[1] E. Ma, “Verifiable radiative seesaw mechanism of neutrino mass and dark matter,” Phys. Rev. D 73 (2006) 077301, arXiv:hep-ph/0601225.
[2] F. Feruglio, “Are neutrino masses modular forms?”, pp. 227–266. 2019. arXiv:1706.08749 [hep-ph].

[3] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, “Finite Modular Groups and Lepton Mixing,” Nucl. Phys. B 858 (2012) 437–467, arXiv:1112.1340 [hep-ph].

[4] J. C. Criado and F. Feruglio, “Modular Invariance Faces Precision Neutrino Data,” SciPost Phys. 5 (2018) no. 5, 042, arXiv:1807.01125 [hep-ph].

[5] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, “Modular $A_4$ invariance and neutrino mixing,” JHEP 11 (2018) 196, arXiv:1808.03012 [hep-ph].

[6] H. Okada and M. Tanimoto, “CP violation of quarks in $A_4$ modular invariance,” Phys. Lett. B 791 (2019) 54–61, arXiv:1812.09677 [hep-ph].

[7] T. Nomura and H. Okada, “A modular $A_4$ symmetric model of dark matter and neutrino,” Phys. Lett. B 797 (2019) 134799, arXiv:1904.03937 [hep-ph].

[8] H. Okada and M. Tanimoto, “Towards unification of quark and lepton flavors in $A_4$ modular invariance,” arXiv:1905.13421 [hep-ph].

[9] F. J. de Anda, S. F. King, and E. Perdomo, “$SU(5)$ grand unified theory with $A_4$ modular symmetry,” Phys. Rev. D 101 (2020) no. 1, 015028, arXiv:1812.05620 [hep-ph].

[10] P. Novichkov, S. Petcov, and M. Tanimoto, “Trimaximal Neutrino Mixing from Modular $A_4$ Invariance with Residual Symmetries,” Phys. Lett. B 793 (2019) 247–258, arXiv:1812.11289 [hep-ph].

[11] T. Nomura and H. Okada, “A two loop induced neutrino mass model with modular $A_4$ symmetry,” arXiv:1906.03927 [hep-ph].

[12] H. Okada and Y. Orikasa, “A radiative seesaw model in modular $A_4$ symmetry,” arXiv:1907.13520 [hep-ph].

[13] G.-J. Ding, S. F. King, and X.-G. Liu, “Modular $A_4$ symmetry models of neutrinos and charged leptons,” JHEP 09 (2019) 074, arXiv:1907.11714 [hep-ph].

[14] T. Nomura, H. Okada, and O. Popov, “A modular $A_4$ symmetric scotogenic model,” Phys. Lett. B 803 (2020) 135294, arXiv:1908.07457 [hep-ph].

[15] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, “$A_4$ lepton flavor model and modulus stabilization from $S_4$ modular symmetry,” Phys. Rev. D 100 (2019) no. 11, 115045, arXiv:1909.05139 [hep-ph]. [Erratum: Phys.Rev.D 101, 039904 (2020)].

[16] T. Asaka, Y. Heo, T. H. Tatsuishi, and T. Yoshida, “Modular $A_4$ invariance and leptogenesis,” JHEP 01 (2020) 144, arXiv:1909.06520 [hep-ph].

[17] D. Zhang, “A modular $A_4$ symmetry realization of two-zero textures of the Majorana neutrino mass matrix,” Nucl. Phys. B 952 (2020) 114935, arXiv:1910.07869 [hep-ph].

[18] G.-J. Ding, S. F. King, X.-G. Liu, and J.-N. Lu, “Modular $S_4$ and $A_4$ symmetries and their fixed points: new predictive examples of lepton mixing,” JHEP 12 (2019) 030, arXiv:1910.03460 [hep-ph].

[19] T. Kobayashi, T. Nomura, and T. Shimomura, “Type II seesaw models with modular $A_4$ symmetry,”
[20] T. Nomura, H. Okada, and S. Patra, “An Inverse Seesaw model with $A_4$-modular symmetry,” arXiv:1912.00379 [hep-ph].

[21] X. Wang, “Lepton flavor mixing and CP violation in the minimal type-(I+II) seesaw model with a modular $A_4$ symmetry,” Nucl. Phys. B 957 (2020) 115105, arXiv:1912.13284 [hep-ph].

[22] H. Okada and Y. Shoji, “A radiative seesaw model with three Higgs doublets in modular $A_4$ symmetry,” Nucl. Phys. B 961 (2020) 115216, arXiv:2003.13219 [hep-ph].

[23] H. Okada and M. Tanimoto, “Quark and lepton flavors with common modulus $\tau$ in $A_4$ modular symmetry,” arXiv:2005.00775 [hep-ph].

[24] M. K. Behera, S. Singirala, S. Mishra, and R. Mohanta, “A modular $A_4$ symmetric Scotogenic model for Neutrino mass and Dark Matter,” arXiv:2009.01806 [hep-ph].

[25] M. K. Behera, S. Mishra, S. Singirala, and R. Mohanta, “Implications of $A_4$ modular symmetry on Neutrino mass, Mixing and Leptogenesis with Linear Seesaw,” arXiv:2007.00545 [hep-ph].

[26] T. Nomura and H. Okada, “A linear seesaw model with $A_4$-modular flavor and local $U(1)_{B-L}$ symmetries,” arXiv:2007.04801 [hep-ph].

[27] T. Nomura and H. Okada, “Modular $A_4$ symmetric inverse seesaw model with $SU(2)_L$ multiplet fields,” arXiv:2007.15459 [hep-ph].

[28] T. Asaka, Y. Heo, and T. Yoshida, “Lepton flavor model with modular $A_4$ symmetry in large volume limit,” Phys. Lett. B 811 (2020) 135956, arXiv:2009.12120 [hep-ph].

[29] K. I. Nagao and H. Okada,”Neutrino and dark matter in a gauged $U(1)_R$ symmetry,” arxiv:2008.13686 [hep-ph].

[30] H. Okada and M. Tanimoto, “Modular invariant flavor model of $A_4$ and hierarchical structures at nearby fixed points,” arXiv:2009.14242 [hep-ph].

[31] K. I. Nagao and H. Okada, “Lepton sector in modular $A_4$ and gauged $U(1)_R$ symmetry,” arxiv:2010.03348 [hep-ph].

[32] H. Okada and M. Tanimoto, “Spontaneous CP violation by modulus $\tau$ in $A_4$ model of lepton flavors,” arXiv:2012.01688 [hep-ph].

[33] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, “Neutrino mixing from finite modular groups,” Phys. Rev. D 98 (2018) no. 1, 016004, arXiv:1803.10391 [hep-ph].

[34] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, “Finite modular subgroups for fermion mass matrices and baryon/lepton number violation,” Phys. Lett. B 794 (2020) 114–121, arXiv:1812.11072 [hep-ph].

[35] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, “Modular $S_3$-invariant flavor model in SU(5) grand unified theory,” PTEP 2020 (2020) no. 5, 053B05, arXiv:1906.10341 [hep-ph].

[36] H. Okada and Y. Orikasa, “Modular $S_3$ symmetric radiative seesaw model,” Phys. Rev. D 100 (2019) no. 11, 115037, arXiv:1907.04716 [hep-ph].
[37] S. Mishra, “Neutrino mixing and Leptogenesis with modular $S_3$ symmetry in the framework of type III seesaw,” arXiv:2008.02095 [hep-ph].

[38] X. Du and F. Wang, “SUSY Breaking Constraints on Modular flavor $S_3$ Invariant $SU(5)$ GUT Model,” arXiv:2012.01397 [hep-ph].

[39] J. Penedo and S. Petcov, “Lepton Masses and Mixing from Modular $S_4$ Symmetry,” Nucl. Phys. B 939 (2019) 292–307, arXiv:1806.11040 [hep-ph].

[40] P. Novichkov, J. Penedo, S. Petcov, and A. Titov, “Modular $S_4$ models of lepton masses and mixing,” JHEP 04 (2019) 005, arXiv:1811.04933 [hep-ph].

[41] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, “New $A_4$ lepton flavor model from $S_4$ modular symmetry,” JHEP 02 (2020) 097, arXiv:1907.09141 [hep-ph].

[42] S. F. King and Y.-L. Zhou, “Trimaximal TM$_1$ mixing with two modular $S_4$ groups,” Phys. Rev. D 101 (2020) no. 1, 015001, arXiv:1908.02770 [hep-ph].

[43] H. Okada and Y. Orikasa, “Neutrino mass model with a modular $S_4$ symmetry,” arXiv:1908.08409 [hep-ph].

[44] J. C. Criado, F. Feruglio, and S. J. King, “Modular Invariant Models of Lepton Masses at Levels 4 and 5,” JHEP 02 (2020) 001, arXiv:1908.11867 [hep-ph].

[45] X. Wang and S. Zhou, “The minimal seesaw model with a modular $S_4$ symmetry,” JHEP 05 (2020) 017, arXiv:1910.09473 [hep-ph].

[46] P. Novichkov, J. Penedo, S. Petcov, and A. Titov, “Modular $A_5$ symmetry for flavour model building,” JHEP 04 (2019) 174, arXiv:1812.02158 [hep-ph].

[47] G.-J. Ding, S. F. King, and X.-G. Liu, “Neutrino mass and mixing with $A_5$ modular symmetry,” Phys. Rev. D 100 (2019) no. 11, 115005, arXiv:1903.12588 [hep-ph].

[48] X. Wang, B. Yu, and S. Zhou, “Double Covering of the Modular $A_5$ Group and Lepton Flavor Mixing in the Minimal Seesaw Model,” arXiv:2010.10159 [hep-ph].

[49] C.-Y. Yao, X.-G. Liu, and G.-J. Ding, “Fermion Masses and Mixing from Double Cover and Metaplectic Cover of $A_5$ Modular Group,” arXiv:2011.03501 [hep-ph].

[50] A. Baur, H. P. Nilles, A. Trautner, and P. K. Vaudrevange, “Unification of Flavor, CP, and Modular Symmetries,” Phys. Lett. B 795 (2019) 7–14, arXiv:1901.03251 [hep-th].

[51] I. de Medeiros Varzielas, S. F. King, and Y.-L. Zhou, “Multiple modular symmetries as the origin of flavor,” Phys. Rev. D 101 (2020) no. 5, 055033, arXiv:1906.02208 [hep-ph].

[52] X.-G. Liu and G.-J. Ding, “Neutrino Masses and Mixing from Double Covering of Finite Modular Groups,” JHEP 08 (2019) 134, arXiv:1907.01488 [hep-ph].

[53] P. Chen, G.-J. Ding, J.-N. Lu, and J. W. Valle, “Predictions from warped flavor dynamics based on the $T'$ family group,” Phys. Rev. D 102 (2020) no. 9, 095014, arXiv:2003.02734 [hep-ph].

[54] P. Novichkov, J. Penedo, and S. Petcov, “Double Cover of Modular $S_4$ for Flavour Model Building,” arXiv:2006.03058 [hep-ph].

[55] X.-G. Liu, C.-Y. Yao, and G.-J. Ding, “Modular Invariant Quark and Lepton Models in Double
Covering of $S_4$ Modular Group,” arXiv:2006.10722 [hep-ph].

[56] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, “Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of $\Gamma_N$,” JHEP 11 (2020) 101, arXiv:2007.06188 [hep-th].

[57] G. Altarelli and F. Feruglio, “Discrete Flavor Symmetries and Models of Neutrino Mixing,” Rev. Mod. Phys. 82 (2010) 2701–2729, arXiv:1002.0211 [hep-ph].

[58] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, “Non-Abelian Discrete Symmetries in Particle Physics,” Prog. Theor. Phys. Suppl. 183 (2010) 1–163, arXiv:1003.3552 [hep-th].

[59] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, An introduction to non-Abelian discrete symmetries for particle physicists, vol. 858. 2012.

[60] D. Hernandez and A. Smirnov, “Lepton mixing and discrete symmetries,” Phys. Rev. D 86 (2012) 053014, arXiv:1204.0445 [hep-ph].

[61] S. F. King and C. Luhn, “Neutrino Mass and Mixing with Discrete Symmetry,” Rept. Prog. Phys. 76 (2013) 056201, arXiv:1301.1340 [hep-ph].

[62] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, “Neutrino Mass and Mixing: from Theory to Experiment,” New J. Phys. 16 (2014) 045018, arXiv:1402.4271 [hep-ph].

[63] S. King, “Unified Models of Neutrinos, Flavour and CP Violation,” Prog. Part. Nucl. Phys. 94 (2017) 217–256, arXiv:1701.04413 [hep-ph].

[64] S. Petcov, “Discrete Flavour Symmetries, Neutrino Mixing and Leptonic CP Violation,” Eur. Phys. J. C 78 (2018) no. 9, 709, arXiv:1711.10806 [hep-ph].

[65] A. Baur, H. P. Nilles, A. Trautner, and P. K. Vaudrevange, “A String Theory of Flavor and $\mathcal{CP}$,” Nucl. Phys. B 947 (2019) 114737, arXiv:1908.00805 [hep-th].

[66] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, “CP violation in modular invariant flavor models,” Phys. Rev. D 101 (2020) no. 5, 055046, arXiv:1910.11553 [hep-ph].

[67] P. Novichkov, J. Penedo, S. Petcov, and A. Titov, “Generalised CP Symmetry in Modular-Invariant Models of Flavour,” JHEP 07 (2019) 165, arXiv:1905.11970 [hep-ph].

[68] M.-C. Chen, S. Ramos-Sánchez, and M. Ratz, “A note on the predictions of models with modular flavor symmetries,” Phys. Lett. B 801 (2020) 135153, arXiv:1909.06910 [hep-ph].

[69] I. de Medeiros Varzielas, M. Levy, and Y.-L. Zhou, “Symmetries and stabilisers in modular invariant flavour models,” JHEP 11 (2020) 085, arXiv:2008.05329 [hep-ph].

[70] K. Ishiguro, T. Kobayashi, and H. Otsuka, “Landscape of Modular Symmetric Flavor Models,” arXiv:2011.09154 [hep-ph].

[71] K. Hagiwara, R. Liao, A. D. Martin, D. Nomura, and T. Teubner, “$(g-2)_\mu$ and $\alpha(M_Z^2)$ re-evaluated using new precise data,” J. Phys. G 38 (2011) 085003, arXiv:1105.3149 [hep-ph].

[72] C.-W. Chiang and H. Okada, “A simple model for explaining muon-related anomalies and dark
matters,” Int. J. Mod. Phys. A 34 (2019) no. 20, 1950106, arXiv:1711.07365 [hep-ph].

[73] Z. Maki, M. Nakagawa, and S. Sakata, “Remarks on the unified model of elementary particles,” Prog. Theor. Phys. 28 (1962) 870–880.

[74] I. Esteban, M. Gonzalez-Garcia, M. Maltoni, T. Schwetz, and A. Zhou, “The fate of hints: updated global analysis of three-flavor neutrino oscillations,” JHEP 09 (2020) 178, arXiv:2007.14792 [hep-ph].

[75] KamLAND-Zen Collaboration, A. Gando et al., “Search for Majorana Neutrinos near the Inverted Mass Hierarchy Region with KamLAND-Zen,” Phys. Rev. Lett. 117 (2016) no. 8, 082503, arXiv:1605.02889 [hep-ex]. [Addendum: Phys.Rev.Lett. 117, 109903 (2016)].

[76] M. Lindner, M. Platscher, and F. S. Queiroz, “A Call for New Physics : The Muon Anomalous Magnetic Moment and Lepton Flavor Violation,” Phys. Rept. 731 (2018) 1–82, arXiv:1610.06587 [hep-ph].

[77] S. Baek, T. Nomura, and H. Okada, “An explanation of one-loop induced h → µτ decay,” Phys. Lett. B 759 (2016) 91–98, arXiv:1604.03738 [hep-ph].

[78] MEG Collaboration, A. Baldini et al., “Search for the lepton flavour violating decay µ+ → e+γ with the full dataset of the MEG experiment,” Eur. Phys. J. C 76 (2016) no. 8, 434, arXiv:1605.05081 [hep-ex].

[79] MEG Collaboration, J. Adam et al., “New constraint on the existence of the µ+ → e+γ decay,” Phys. Rev. Lett. 110 (2013) 201801, arXiv:1303.0754 [hep-ex].

[80] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller, “Measurement of the fine-structure constant as a test of the Standard Model,” Science 360 (2018) 191, arXiv:1812.04130 [physics.atom-ph].

[81] T. Hambye, F.-S. Ling, L. Lopez Honorez, and J. Rocher, “Scalar Multiplet Dark Matter,” JHEP 07 (2009) 090, arXiv:0903.4010 [hep-ph]. [Erratum: JHEP 05, 066 (2010)].

[82] C. Boehm and P. Fayet, “Scalar dark matter candidates,” Nucl. Phys. B 683 (2004) 219–263, arXiv:hep-ph/0305261.

[83] D. Schmidt, T. Schwetz, and T. Toma, “Direct Detection of Leptophilic Dark Matter in a Model with Radiative Neutrino Masses,” Phys. Rev. D 85 (2012) 073009, arXiv:1201.0906 [hep-ph].