WEAK DIFFERENTIABILITY OF SOLUTIONS TO SDES WITH SEMI-MONOTONE DRIFTS

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Abstract. In this work we prove Malliavin differentiability for the solution to an SDE with locally Lipschitz and semi-monotone drift. To this end we construct a sequence of SDEs with globally Lipschitz drifts. We show that the solutions of these SDEs converge to the solution of the original SDE and the $p$-moments of their Malliavin derivatives are uniformly bounded.

1. Introduction

In recent years, there were attempts to generalize existence, uniqueness, and smoothness results to SDEs with non-globally Lipschitz coefficients, which have many applications in Financial Mathematics [9, 3, 1, 14]. In [6, 16] the authors studied the existence of a global stochastic flow for SDEs with unbounded and Hölder continuous drifts and nondegenerate diffusion coefficients. Zhang considered the flow of stochastic transport equations which could have irregular coefficients [17].

The SDE we consider has both non-globally Lipschitz and semi-monotone drift. Such equations come mostly from finance, biology, and dynamical systems and are more challenging when considered on infinite dimensional spaces. (see e.g. [2, 18, 7])

In this paper, we consider an SDE with locally Lipschitz and monotone drift and globally Lipschitz diffusion. We prove the existence of a unique infinitely Malliavin differentiable strong solution to this SDE.

Since the drift of the SDE we consider is not globally Lipschitz, we will
construct a sequence of SDEs with globally Lipschitz drifts whose solutions are Malliavin differentiable of all order. In this way we can apply the classical Malliavin calculus to these solutions. Then we can find a uniform bound for the moments of all the Malliavin derivatives of solutions. We will prove that the solutions to the constructed sequence of SDEs converge to the solution of the desired SDE. Then by the uniform boundedness of the moments of the mentioned solutions and the convergence result we are able to prove infinite Malliavin differentiability of the solution to the original SDE.

The organization of the paper is as follows. In section 2, we recall some basic results from Malliavin calculus that will be used in the paper, the prerequisites could be found in Nualart’s book [15], in this section we state also our assumptions and main results. Section 3 involves the construction of our approximating SDEs with globally Lipschitz coefficients, and the proof of convergence of their solutions to the unique solution of the original SDE (2.1). In section 4, we will prove uniform boundedness of the Malliavin derivatives associated to the approximating processes, which results to the infinitely weak differentiability of the solution to SDE (2.1).

2. Some basic results from Malliavin calculus

Let $\Omega$ denote the Wiener space $C_0([0,T]; R^d)$. We furnish $\Omega$ with the $\| , \|_{\infty}$-norm making it a (separable) Banach space. Consider $(\Omega, \mathcal{F}, P)$ a complete probability space, in which $\mathcal{F}$ is generated by the open sets of the Banach space, $W_t$ is a d-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by $W_t$. Consider the Hilbert space $H := L^2([0,T]; R^d)$. Let $\{W(h), h \in H\}$ denote a Gaussian process associated to the Hilbert space $H$ and $W(h) = \int_0^\infty h(t) dW_t$. We denote by $C^\infty_p(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : R^n \rightarrow R$ such that $f$ and all of its partial derivatives have polynomial growth. Let $S$ denote the class of all smooth random variables $F : \Omega \rightarrow R$ such that $F = f(W(h_1), ..., W(h_n))$, for some $f$ belonging to $C^\infty_p(\mathbb{R}^n)$ and $h_1, ..., h_n \in H$ for some $n \geq 1$.

The derivative of the smooth random variable $F \in S$ is an $H$-valued random variable given by

$$D_tF = \Sigma_{i=1}^n \partial_i f(W(h_1), ..., W(h_n)) h_i(t).$$
The operator $D$ from $L^p(\Omega)$ to $L^p(\Omega, H)$ is closable. For every $p \geq 1$, we denote its domain by $\mathbb{D}^{1,p}$ which is exactly the closure of $\mathcal{S}$ with respect to $\| \cdot \|_{1,p}$ where

$$\| F \|_{1,p} = \left( E|F|^p + \| DF \|_{L^p(\Omega; H)}^p \right)^{\frac{1}{p}}.$$ 

(see [15]). One can also define the $k$-th order derivative of $F$ as a random vector in $[0, T]^k \times \Omega$. We denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm

$$\| F \|_{k,p} = \left( E|F|^p + \| D^{i_1, \ldots, i_k} F \|_{L^p(\Omega; H \otimes k)}^p \right)^{\frac{1}{p}},$$

and define $\mathbb{D}^\infty := \bigcap_{k,p} \mathbb{D}^{k,p}$.

Consider the following stochastic differential equation

$$dX_t = [b(X_t) + f(X_t)]dt + \sigma(X_t)dW_t, \quad X_0 = x_0.$$ 

where $b, f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are measurable functions and $\sigma : \mathbb{R}^d \rightarrow \mathcal{M}_{d \times d}(\mathbb{R})$ is a measurable $C^\infty$ function. We denote by $\mathcal{L}$ the second-order differential operator associated to SDE (2.1):

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)^i_j(x) \partial_i \partial_j + \sum_{i=1}^d [b^i(x) + f^i(x)] \partial_i$$

where $*$ denotes transpose. We use the upper index to show a specified row, and the subindex to show a specified column of a matrix.

Kusuoka and Stroock has shown the following result [11, Theorem 1.9.].

**Result 2.1.** Assume that the coefficients $b$, $\sigma$ and $f$ in (2.1) are globally Lipschitz and all of their derivatives have polynomial growth, then (2.1) has a strong solution in $\mathbb{D}^\infty$ whose Malliavin derivative satisfies the following linear equation. For every $r \leq t$

$$D_r X_i^j = \sigma^i_j(X_r) + \int_r^t (\nabla b^i(X_s) + \nabla f^i(X_s)) D_s X_s ds$$

$$+ \int_r^t \nabla \sigma^i_j(Y_s) D_s X_s dW_s^j,$$

where for $r > t$, $D_r X_t = 0$. Also it holds

$$\sup_{0 \leq r \leq T} \mathbb{E} \left[ \sup_{r \leq s \leq T} \| D_s^i X_s^j \| \right] < \infty.$$
Throughout the paper we assume that $b$, $f$ and $\sigma$ satisfy the following Hypothesis.

**Hypothesis 2.2.**

1. The function $b$ is an $C^\infty$ uniformly monotone function, i.e., there exists a constant $K > 0$ such that for every $x, y \in \mathbb{R}^d$,

$$< b(y) - b(x), y - x > \leq -K|y - x|^2.$$  

where $(.,.)$ denotes the scalar product in $\mathbb{R}^d$. Furthermore, $b$ is locally Lipschitz and all of its derivatives have polynomial growth. i.e., for each $x \in \mathbb{R}^d$ and each multiindex $\alpha$ with $|\alpha| = m$, there exist positive constants $\gamma_m$ and $q_m$ such that

$$|\partial_\alpha b(x)|^2 \leq \gamma_m(1 + |x|^{q_m})$$

Also, set $\xi := \max_{m \geq 1} q_m < \infty$.

2. The functions $f$ and $\sigma$ are $C^\infty$, globally Lipschitz with Lipschitz constant $k_1$ and all of their derivatives are bounded. Furthermore $f$ has linear growth, i.e. for every $x \in \mathbb{R}^d$,

$$|f(x)| \leq k_1(1 + |x|).$$

Hypothesis (2.2) yields to the following useful inequalities

$$\langle b(a) + f(a), a \rangle \vee |\sigma(a)|^2 \leq \alpha + \beta|a|^2 \quad \forall a \in \mathbb{R}^d,$$

where

$$\alpha := \frac{1}{2}|b(0)|^2 + k_1^2 \vee 2|\sigma(0)|^2, \quad \text{and} \quad \beta := (-K + 1 + k_1^2) \vee 2k_1^2,$$

and

$$\langle \nabla b(x)y, y \rangle \leq -K|y|^2 \quad \forall x, y \in \mathbb{R}^d.$$
3. Approximation of the solution

For each \( n \geq 1 \), define the stopping time \( \tau_n \) via

\[
\tau_n := \inf \{ t \mid |X_t| \geq n \xi \}.
\]

**Lemma 3.1.** For each \( t \in [0, T] \) and \( p > 1 \), the unique solution \( X_t \) of (2.1) belongs to \( L^p(\Omega) \) and does not explode in finite time.

**Proof.** To proceed, first we use Fatou’s lemma to show that \( X_t \in L^p(\Omega) \) and does not explode. Then, we prove the uniqueness of the solution to SDE (2.1).

By the definition of \( L \) and (2.5), we have

\[
\mathcal{L}|X_t|^p = p|X_t|^{p-2}(X_t, b(X_t) + f(X_t)) + \frac{p}{2}|X_t|^{p-2}||\sigma(X_t)||^2
+ \frac{p(p-2)}{2}|X_t|^{p-4}|\langle X_t, \sigma(X_t) \rangle|^2
\]

\[
\leq p|X_t|^{p-2}(X_t, b(X_t) + f(X_t)) + \frac{p(p-1)}{2}|X_t|^{p-2}||\sigma(X_t)||^2
\]

\[
\leq p\left( \beta + (p-1)k_1^2 \right)|X_t|^p + p\left( \alpha + (p-1)k_1^2 \right)|X_t|^{p-2}
\]

(3.1)

\[
\therefore \beta_p|X_t|^p + \alpha_p|X_t|^{p-2}.
\]

Applying Itô’s formula and using (3.1),

(3.2)

\[
\frac{d}{dt} \mathbb{E}[|X_{t\wedge \tau_n}|^p] = \mathbb{E}[\mathcal{L}|X_{t\wedge \tau_n}|^p] \leq \beta_p \mathbb{E}[|X_{t\wedge \tau_n}|^p] + \alpha_p \mathbb{E}[|X_{t\wedge \tau_n}|^{p-2}].
\]

Setting \( p = 2 \) and using Gronwall’s inequality, we have

(3.3)

\[
\mathbb{E}[|X_{t\wedge \tau_n}|^2] \leq |x_0|^2 \alpha_2 \exp\{\beta_2 T\}.
\]

From (3.3) we can deduce the following inequality

\[
\left( \frac{n}{2} - 1 \right) \frac{1}{\sigma^2} P\left( t \geq \tau_n \right) \leq |x_0|^2 \alpha_2 \exp\{\beta_2 T\}.
\]

Letting \( n \) tend to \( \infty \), then \( \lim_{n \to \infty} \tau_n = \infty \) almost surely, which implies that \( X_t \) does not explode in any finite time interval \([0, T]\). Also, let \( n \) tend to infinity in (3.3) and use Fatou’s lemma, then

\[
\mathbb{E}(|X_t|^2) \leq \mathbb{E}\left( \liminf_{n \to \infty} |X_{t\wedge \tau_n}|^2 \right) \leq \liminf_{n \to \infty} \mathbb{E}\left( |X_{t\wedge \tau_n}|^2 \right) \leq |x_0|^2 \alpha_2 \exp\{\beta_2 T\}.
\]
Finally by (3.2) and induction on \( p \) we conclude that \( X_t \in L^p(\Omega) \).

To prove uniqueness, we assume that the SDE (2.1) has two strong solutions \( X_t \) and \( Y_t \). Since \( X_t, Y_t \in L^2(\Omega) \), applying Itô’s formula we have

\[
\frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2] = 2 \mathbb{E}[\langle X_t - Y_t, b(X_t) - b(Y_t) \rangle] + 2 \mathbb{E}[\langle X_t - Y_t, f(X_t) - f(Y_t) \rangle] + \mathbb{E}[|\sigma(X_t) - \sigma(Y_t)|^2]
\]

From which by (2.2) and the Lipschitz property of \( \sigma \) and \( f \) we derive

\[
\frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2] \leq (-2K + 2k_1) \mathbb{E}[|X_t - Y_t|^2].
\]

By Gronwall’s inequality which is proved in [8, Lemma 1.1] we conclude that \( \mathbb{E}[|X_t - Y_t|^2] = 0 \). So that

\[
P\left(|X_t - Y_t| = 0 \text{ for all } t \in Q \cap [0, T]\right) = 0,
\]

where \( Q \) denotes the set of rational numbers. Since \( t \rightarrow |X_t - Y_t| \) is continuous, then

\[
P\left(|X_t - Y_t| = 0 \text{ for all } t \in [0, T]\right) = 0,
\]

and the uniqueness is proved. \( \Box \)

For every integer \( n > 0 \) let us choose some smooth functions \( \phi_n : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \phi_n = 1 \) on \( A_n := \{ x \in \mathbb{R} ; |x| \leq n^k \} \) and \( \phi_n = 0 \) outside \( A_{2n^k} \) \( (\xi \text{ defined in Hypothesis 2.2 part (1)}) \) and for each multiindex \( L \) with \( |L| = l \geq 1 \),

\[
\sup_{n,x} \left( |\partial_L \phi_n| + |\langle b, \partial_L \phi_n \rangle| \right) \leq M_l
\]

for some \( M_l > 0 \). (See Appendix and the proof of Lemma 2.1.1 in [15]).

Now, set

\[
b_n(x) := \phi_n(x)b(x)
\]

for every \( x \in \mathbb{R}^d \) and \( n > 0 \). Then \( b_n \) would be globally Lipschitz and continuously differentiable. By (2.3) for each \( x \in \mathbb{R}^d \) and each multiindex \( L \) with \( |L| = l \), there exist positive constants \( \Gamma_l \) and \( p_l \) such that

\[
|\partial_L b_n(x)|^2 \leq \Gamma_l(1 + |x|^{p_l}).
\]
Now by Result 2.1, the SDE (3.6) has a strong solution in $D^\infty$, that is, there exists $X_t^n$ in $D^\infty$ which satisfies

\begin{equation}
X_t^n = x_0 + \int_0^t [b_n(X_s^n) + f(X_s^n)]ds + \int_0^t \sigma(X_s^n)dW_s
\end{equation}

We denote by $L_n$ the infinitesimal operator associated to SDE (3.6):

\begin{equation}
L_n = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)^i_j(x) \partial_i \partial_j + \sum_{i=1}^d [b^i_n(x) + f^i(x)] \partial_i.
\end{equation}

We will show that the sequence $X_t^n$ converges to the unique strong solution $X_t$ to the SDE (2.1) and that the moments of $DX_t^n$ are uniformly bounded with respect to $n$ and $t$. This way we can use Lemma 1.2.3 in [15] to derive the Malliavin differentiability of $X_t$ and show that $X_t \in D^\infty$.

**Lemma 3.2.** For each $t \in [0,T]$ and $p > 1$, the sequence $X_t^n$ converges to $X_t$ in $L^p(\Omega)$.

**Proof.** To proceed, we prove the almost sure convergence of $X_t^n$ to $X_t$. Then by showing the uniform integrability of $X_t^n$ we will conclude. Let $X^n_{\tau_n}$ denote $X$ stopped at $\tau_n$. By the choice of $\phi_n(\cdot)$, it follows that $X^n_{\tau_n} = X^n_t$ for all $t \leq \tau_n$. So, for fixed $t \in [0,T]$, letting $n$ tend to $\infty$, we have $\lim_{n \to \infty} X^n_t = X^n_{\tau_n} = X_t^\infty = X_t$ a.s.

Now, we are going to prove the uniform integrability of the sequence $X_t^n$. We will show that for every $p > 1$, there exists $c_p > 0$ such that

\begin{equation}
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|X_t^n|^p] \leq c_p.
\end{equation}

By the definition of $L_n$, we have
Using Itô’s formula, we have

\[
\frac{d}{dt} \mathbb{E} \left[ |X^n_t - x_0|^p \right] = \mathbb{E} \left[ \mathcal{L}_n(|X^n_t - x_0|^p) \right]
\]

\[
\leq \alpha_p \mathbb{E} \left[ |X^n_t - x_0|^p \right] + \beta_p \mathbb{E} \left[ |X^n_t - x_0|^{p-2} \right].
\]

Setting \( p = 2 \) and applying Gronwall’s inequality, (3.7) will be proved for \( p = 2 \). By induction on \( p \) and by the following inequality
\[
\frac{d}{dt} \mathbb{E} \left[ |X^n_t - x_0|^p \right] = \mathbb{E} \left[ \mathcal{L}_n(|X^n_t - x_0|^p) \right] \\
\leq \alpha_p \mathbb{E} \left[ |X^n_t - x_0|^p \right] + \beta_p \left( \mathbb{E} \left[ |X^n_t - x_0|^{p-1} \right] \right)^{1-\frac{1}{p-1}},
\]
(3.7) will be proved for every \( p \geq 2 \).

Now by almost sure convergence of \( X^n_t \) to \( X_t \) and by inequality (3.7) the proof of Lemma is completed. \( \square \)

4. Weak differentiability in the Wiener space

In this section, first we use Lemma 1.2.3 in [15] to derive Malliavin differentiability of the solution to (2.1). Then we show that \( X_t \in \mathbb{D}^\infty \).

**Lemma 4.1.** Assume that Hypothesis 2.2 holds, then the unique strong solution of SDE (2.1) is in \( \mathbb{D}^{1,p} \) for every \( p > 1 \). Moreover, for \( r \leq t \)

\[
D_r X^i_t = \sigma^i(X_r) + \int_r^t [\nabla b^i(X_s) + \nabla f^i(X_s)]D_r X_s ds \\
+ \int_r^t \nabla \sigma^i(X_s).D_r X_sdW^i_s,
\]
and for \( r > t \), \( D_r X^i_t = 0 \), where \( \sigma_l(X_s) \) is the \( l \)-th column of \( \sigma(X_s) \) and \( u.C \) denotes the product \( C^*u \) of matrix \( C^* \) and vector \( u \).

**Proof.** By Result 2.1 we know that for every \( r \leq t \) and \( 1 \leq i \leq d \)

\[
D_r(X^n_t)^i = \sigma^i(X^n_r) + \int_r^t [\nabla b^n_i(X^n_s) + \nabla f^n_i(X^n_s)]D_r X^n_s ds \\
+ \int_r^t \nabla \sigma^n_i(X^n_s).D_r X^n_sdW^i_s,
\]
and for every \( r > t \), \( D_r(X^n_t)^i = 0 \).

Now by Lemma 1.2.3 in [15], it is sufficient to show that

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E} \left[ \|DX^n_t\|_H^p \right] \leq c_p.
\]
(4.1)

To this end, note that for every \( 1 \leq i \leq d \) by Itô’s formula

\[
\mathbb{E} \left[ |D_r(X^n_t)^i|^p \right] = \mathbb{E} \left[ |\sigma^i(X^n_r)|^p \right] + \mathbb{E} \left[ \int_r^t G_n \left( |D_r(X^n_s)^i|^p \right) ds \right] + \mathbb{E} \left[ M^n_t \right],
\]

(4.2)
where

\[
\mathcal{G}_n \left( |D_r(X^n_s)^i|^p \right) = p|D_r(X^n_s)^i|^{p-2} S_{i,s} \\
\quad + p|D_r(X^n_s)^i|^{p-2} \langle D_r(X^n_s)^i, \nabla f^i(X^n_s) \rangle \\
\quad + \frac{p}{2} |D_r(X^n_s)^i|^{p-2} |\nabla \sigma^i(X^n_s) \cdot D_r X^n_s|^2 \\
\quad + \frac{p(p-2)}{2} |D_r(X^n_s)^i|^{p-4} \langle D_r(X^n_s)^i, \nabla \sigma^i(X^n_s) \cdot D_r X^n_s \rangle^2,
\]

\[
S_{i,s} := \langle D_r(X^n_s)^i, \nabla b^i_n(X^n_s) \cdot D_r X^n_s \rangle,
\]
and

\[
M^n_t := \int_t^t p|D_r(X^n_s)^i|^{p-2} \langle D_r(X^n_s)^i, \nabla \sigma^i(X^n_s) \cdot D_r X^n_s dW^i_s \rangle.
\]

Notice that by Result 2.1, \(M^n_t\) is a local martingale and thus \(\mathbb{E}[M^n_t] = 0\).

Since \(\sigma\) and \(f\) have bounded derivatives, there exists some \(\gamma > 0\) such that

\[
\frac{p}{2} |D_r(X^n_s)^i|^{p-2} |\nabla \sigma^i(X^n_s) \cdot D_r X^n_s|^2 \\
\quad + \frac{p(p-2)}{2} |D_r(X^n_s)^i|^{p-4} \langle D_r(X^n_s)^i, \nabla \sigma^i(X^n_s) \cdot D_r X^n_s \rangle^2 \\
\leq \gamma \frac{p(p-1)}{2} |D_r(X^n_s)^i|^{p-2} |D_r X^n_s|^2,
\]

and

\[
p|D_r(X^n_s)^i|^{p-2} \langle D_r(X^n_s)^i, \nabla f^i(X^n_s) \rangle \\
\leq \frac{p}{2} |D_r(X^n_s)^i|^p + \gamma \frac{p}{2} |D_r(X^n_s)^i|^{p-2} |D_r X^n_s|^2.
\]

Using (2.7) and (3.4), for \(0 \leq s \leq T\) we have
\[
\sum_{i=1}^{d} S_{i,t,s} = \sum_{j=1}^{d} \langle \nabla b_n(X^n_s) D^j_t X^n_s, D^j_t X^n_s \rangle \\
= \sum_{j=1}^{d} \phi_n(X^n_s) \langle \nabla b(X^n_s) D^j_t X^n_s, D^j_t X^n_s \rangle \\
+ \sum_{j=1}^{d} \langle \langle b(X^n_s), \nabla \phi_n(X^n_s) \rangle D^j_t X^n_s, D^j_t X^n_s \rangle \\
\leq (-K\phi_n(X^n_s) + M_1) \sum_{j=1}^{d} |D^j_t X^n_s|^2 \leq M_1 \sum_{j=1}^{d} |D^j_t X^n_s|^2
\]

where \( D^j_t X^n_s \) is the \( j \)-th column of \( DX^n_s \). For every \( Y = (Y^1, \cdots, Y^d) \in \mathbb{R}^d \) and \( 1 \leq i \leq d \)

\[
|Y^i|^p \leq |Y|^p,
\]

and

\[
|Y|^p \leq 2^{\frac{p}{p-1}} \sum_{i} |Y^i|^p.
\]

Thus substituting (4.5), (4.3) and (4.4) in (4.2) and taking summation on \( i \) we derive:
\[ \mathbb{E}\left[ |D_r X_t^n|^p \right] \leq 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \mathbb{E}\left[ |D_r (X^n)^i|^p \right] \]

\[ \leq 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \mathbb{E}\left[ |\sigma^i(X^n)|^p \right] \]

\[ + 2^{\frac{p}{p-1}} pdM_1 \sum_{i=1}^{d} \int_r^t \mathbb{E}\left[ |D_r (X^n)^i|^{p-2} |D_r X^n_s|^2 \right] ds \]

\[ + 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \int_r^t \mathbb{E}\left[ \frac{p}{2} |D_r (X^n)^i|^p \right] ds \]

\[ + 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \gamma \frac{p}{2} \int_r^t \mathbb{E}\left[ |D_r (X^n)^i|^{p-2} |D_r X^n_s|^2 \right] ds \]

\[ + 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \gamma \frac{p(p-1)}{2} \int_r^t \mathbb{E}\left[ |D_r (X^n)^i|^{p-2} |D_r X^n_s|^2 \right] ds. \]

Now we can find a constant \( \alpha_p' > 0 \) such that

\[ \mathbb{E}\left[ |D_r X_t^n|^p \right] \leq 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \mathbb{E}\left[ |\sigma^i(X^n)|^p \right] + \alpha_p' \int_r^t \mathbb{E}\left[ |D_r X^n_s|^p \right] ds. \]

Using Gronwall’s inequality, we have

\[ \mathbb{E}\left[ |D_r X_t^n|^p \right] \leq 2^{\frac{p}{p-1}} \sum_{i=1}^{d} \mathbb{E}\left[ |\sigma^i(X^n)|^p \right] \exp\{\alpha_p' T\}. \]

From which by the Lipschitz property of \( \sigma \) and inequality (3.7) the result follows. \( \square \)

Here we are going to prove higher order differentiability of \( X_t \). For simplicity, we will only show the second order differentiability. For every real-valued function \( f \) and random variables \( F \) and \( G \), set **\( \nabla f(x)FG := \sum i, j \partial_i \partial_j f(x) F^i G^j ** \) and \( D^{i,k}_{r,\tau} F = D^i_r D^k_\tau F \).

**Lemma 4.2.** Assuming Hypothesis 2.2, the unique strong solution of SDE (2.1) is in \( \mathbb{D}^{2,p} \), for every \( p > 1, \) and
\[ D_{r,r}^{i,k} X_t^i = A_{r,r}^{ij} \]
\[ + \int_{\tau \land r}^t \left( \langle \nabla \sigma^i (X_s), D_{r,r}^{i,k} X_s \rangle + \Delta \sigma^i (X_s) D_r^k X_s D_r^j X_s \right) dW_s^l \]
\[ + \int_{\tau \land r}^t \langle \nabla b^i (X_s) + \nabla f^i (X_s), D_{r,r}^{i,k} X_s \rangle ds \]
\[ + \int_{\tau \land r}^t \left( \Delta b^i (X_s) + \Delta f^i (X_s) \right) D_r^k X_s D_r^j X_s ds, \]
where
\[ A_{r,r}^{ij} = \langle \nabla \sigma^j (X_r), D_r^k X_r \rangle + \sum_{l=1}^d \langle \nabla \sigma^l (X_r), D_r^j X_r \rangle, \]
and \( D_r X_r = 0 \) for \( \tau > r \), and \( D_r X_r = 0 \) for \( \tau < r \).

**Proof.** Since \( X^n_t \in \mathbb{D}^\infty \), by Result 2.1 for \( \tau_0 := \tau \land r \) we have
\[ D_{r,r}^{i,k} (X^n_t)^i = A_{r,r}^{ij} \]
\[ + \int_{\tau_0}^t \left( \langle \nabla \sigma^i (X^n_s), D_{r,r}^{i,k} X^n_s \rangle + \Delta \sigma^i (X^n_s) D_r^k X^n_s D_r^j X^n_s \right) dW_s^l \]
\[ + \int_{\tau_0}^t \langle \nabla b^i (X^n_s) + \nabla f^i (X^n_s), D_{r,r}^{i,k} X^n_s \rangle ds \]
\[ + \int_{\tau_0}^t \left( \Delta b^i (X^n_s) + \Delta f^i (X^n_s) \right) D_r^k X^n_s D_r^j X^n_s ds, \]
where
\[ A_{n,r}^{ij} = \langle \nabla \sigma^j (X^n_r), D_r^k X^n_r \rangle + \sum_{l=1}^d \langle \nabla \sigma^l (X^n_r), D_r^j X^n_r \rangle, \]
and \( D_r X^n_r = 0 \) for \( \tau > r \). Similarly we have \( D_r X^n_r = 0 \) for \( \tau < r \). By Lemma 1.2.3 in [15], it remains only to find some \( c_2 > 0 \) such that
\[ \sup_n E \left[ \| D_{r,r}^{i,k} X_t^i \|_{H \otimes H}^p \right] < c_2. \]

By Itô’s formula, for every \( 1 \leq i \leq d \) we have
\[ \mathbb{E} \left[ |D_{r,r}^{i,k} (X^n_t)^i|^p \right] = \mathbb{E} \left[ |A_{n,r}^{ij}|^p \right] + \mathbb{E} \left[ \int_{\tau}^t G_{n}^{ij} \left( |D_{r,r}^{i,k} (X^n_s)|^p \right) ds \right] + \mathbb{E} \left[ M_{n}^{ij} (t) \right], \]
where
\[
G^{ij}_n \left( |D^{j,k}_{r,\tau}(X^n_s)i|^p \right) = p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}I_1 + \frac{p}{2}|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2} \sum_{l=1}^d I_2(l) + \frac{p(p-2)}{2}|D^{j,k}_{r,\tau}(X^n_s)i|^{p-4}I_3,
\]
in which
\[
I_1 := D^{j,k}_{r,\tau}(X^n_s)i \left( \langle \nabla b^i_n(X^n_s) + \nabla f^i(X^n_s), D^{j,k}_{r,\tau}(X^n_s) \rangle ight) + \left[ \Delta b^i_n(X^n_s) + \Delta f^i(X^n_s) \right] D^{i,j}_{r,\tau} X^n_s D^{j,k}_{r,\tau} X^n_s,
\]
\[
I_2(l) := \left[ \left| \Delta \sigma^l_i(X^n_s)D^{i,j}_{r,\tau} X^n_s D^{j,k}_{r,\tau} X^n_s \right| + \left| \langle \nabla \sigma^l_i(X^n_s), D^{j,k}_{r,\tau}(X^n_s) \rangle \right| \right]^2,
\]
\[
I_3 := |D^{j,k}_{r,\tau}(X^n_s)i| \left( \Delta \sigma^l_i(X^n_s)D^{i,j}_{r,\tau} X^n_s D^{j,k}_{r,\tau} X^n_s + \langle \nabla \sigma^l_i(X^n_s), D^{j,k}_{r,\tau}(X^n_s) \rangle \right)^2,
\]
and
\[
M^{ij}_n(t) := \int_0^t p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2} (D^{j,k}_{r,\tau}(X^n_s)i, I_2(l))dW^l_s.
\]
Notice that by Result 2.1, \(M^{ij}_n(t) \) is a local martingale and thus \(E[M^{ij}_n(t)] = 0\).
Now, we are going to find appropriate upper bounds for \(I_1, I_2(l)\) and \(I_3\).
As \(\sigma\) has bounded derivatives, we can find some \(\gamma'_1 > 0\) such that
\[
\frac{p}{2}|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2} \sum_{l=1}^d I_2(l) + \frac{p(p-2)}{2}|D^{j,k}_{r,\tau}(X^n_s)i|^{p-4}I_3 \leq \frac{p(p-1)}{2} \left( |D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}|D^{j,k}_{r,\tau}(X^n_s)i|^2 + |D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}|D^{j,k}_{r,\tau}(X^n_s)i|^2 |D^{j,k}_{r,\tau}(X^n_s)i|^2 \right).
\]
(4.10)
Also by the boundedness of \(f\) and the derivatives of \(\sigma\), the polynomial growth of the derivatives of \(b\) and (3.5), there exist some \(\gamma'_2 > 0\) and \(q > 0\) such that
\[
p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}I_1 = p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}J_1 + p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}J_2 + p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}D^{j,k}_{r,\tau}(X^n_s)i \langle \nabla f^i(X^n_s), D^{j,k}_{r,\tau}(X^n_s) \rangle \\
\leq p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}J_1 + \gamma'_2 p|D^{j,k}_{r,\tau}(X^n_s)i|^{p-2}D^{j,k}_{r,\tau}(X^n_s)i |D^{j,k}_{r,\tau}(X^n_s)i|^2 |D^{j,k}_{r,\tau}(X^n_s)i|^2 (1 + |X^n_s|^{p_2})^2 \\
+ p\gamma'_2 |D^{j,k}_{r,\tau}(X^n_s)i|^p + p\gamma'_2 |D^{j,k}_{r,\tau}(X^n_s)i|^{p-2} |D^{j,k}_{r,\tau}(X^n_s)i|^2,
\]
(4.11)
where
\[ J_1 := D^{j,k}_{r,s}(X^n_s)^i(\nabla b^n_s(X^n_s), D^{j,k}_{r,s}X^n_s), \]
and
\[ J_2 := D^{j,k}_{r,s}(X^n_s)^i\left(\left[\Delta b^n_s(X^n_s) + \Delta f^n_s(X^n_s)\right]D^k_sX^n_s D^j_sX^n_s\right) \]
By (2.7) and (3.4), for every \(0 \leq s \leq T\) we have
\[
\sum_{i=1}^{d} J_1 = \langle \nabla b^n_s(X^n_s) D^{j,k}_{r,s}X^n_s, D^{j,k}_{r,s}X^n_s \rangle
\]
\[
+ \langle (b^n_s(X^n_s), \nabla \phi^n_s(X^n_s)) D^{j,k}_{r,s}X^n_s, D^{j,k}_{r,s}X^n_s \rangle
\]
\[
\leq (-K\phi^n_s(X^n_s) + M_1)|D^{j,k}_{r,s}X^n_s|^2 \leq M_1|D^{j,k}_{r,s}X^n_s|^2.
\]
Now, substitute (4.11) and (4.10) in (4.9), sum up on \(i\) and then use (4.12) and (4.6) to derive
\[
\sum_{i=1}^{d} E\left[|D^{j,k}_{r,s}(X^n_t)^i|^p\right] =
\sum_{i=1}^{d} E\left[|A^{j,i}_{n,r,s}|^p\right] + p(M_1 + 2d\gamma_2 + d\gamma_1\frac{p(p-1)}{2}) \int_{\tau_0}^{t} E\left[|D^{j,k}_{r,s}X^n_s|^p\right] ds
\]
\[
+ \sum_{i=1}^{d} \gamma_2 p \int_{\tau_0}^{t} E\left[|D^{j,k}_{r,s}(X^n_s)^i|^p|D^k_sX^n_s|^2|D^j_sX^n_s|^2(1 + |X^n_s|^{p_2})^2\right] ds
\]
\[
+ \sum_{i=1}^{d} \gamma_1 \frac{p(p-1)}{2} \int_{\tau_0}^{t} E\left[|D^{j,k}_{r,s}(X^n_s)^i|^p|D^k_sX^n_s|^2|D^j_sX^n_s|^2\right] ds.
\]
To bound the terms in the right hand side of the above inequality, we need the following version of the Young's inequality. For \(p \geq 2\) and for all \(a, c\) and \(\triangle_1 > 0\) we have:
\[
a^{p-2}c^2 \leq \triangle_1^2 \frac{p-2}{p} a^p + \frac{2}{p\triangle_1^{p-2}} c^p.
\]
Using (4.14) with \(\triangle_1 = 1\) and \(a = |D^{j,k}_{r,s}(X^n_s)^i|\) we find some bounds for the last four terms in (4.13) which depend only on \(\int_{\tau_0}^{t} E\left[|D^{j,k}_{r,s}X^n_s|^p\right] ds\).
and some terms which could be bounded by a constant. So for the last term in (4.13) we have

\[
\sum_{i=1}^{d} \gamma_1^{p(p-1)/2} \int_{\tau_0}^{t} \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p - |D^{i,k}_r X^n_s|^2 |D^{i,k}_r X^n_s|^2 \right] ds \leq \\
\sum_{i=1}^{d} \gamma_2^p \int_{\tau_0}^{t} \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p - |D^{i,k}_r X^n_s|^2 |D^{i,k}_r X^n_s|^2 (1 + |X^n_s|^{p_2})^2 \right] ds.
\]

and for the third summand in (4.13) we have

\[
\sum_{i=1}^{d} \gamma_2^p \int_{\tau_0}^{t} \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p - |D^{i,k}_r X^n_s|^2 |D^{i,k}_r X^n_s|^2 (1 + |X^n_s|^{p_2})^2 \right] ds \leq \\
\sum_{i=1}^{d} \gamma_2^p \int_{\tau_0}^{t} \left( (p-2) \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p \right] + 2 \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p |D^{i,k}_r X^n_s|^p (1 + |X^n_s|^{p_2})^2 \right] \right) ds.
\]

Substituting these bounds in the right hand side of (4.9) and using (3.7), (4.1) and (4.7), we can find some positive constants \(c_1(p)\) and \(c_2(p)\) such that

\[
\mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p \right] \leq 2^{p-1} \sum_{i=1}^{d} \mathbb{E} \left[ |A_{ij,n}^i,\tau,\rho|^p \right] + c_2(p) + c_1(p) \int_{\tau_0}^{t} \mathbb{E} \left[ |D^{j,k}_r(X^n_s)|^p \right] ds.
\]

Now, from (4.1), (3.7) and the definition of \(A_{ij,n}^i,\tau,\rho\) (in which we have used the boundedness of the derivatives of \(\sigma\)), Gronwall’s inequality gives us (4.8).

In the same way, one can easily show that for every multiindex \(\alpha\)

\[
(4.15) \quad \sup_n \mathbb{E}(\|D^\alpha X^n_t\|^p_{H^\alpha}) < \infty
\]

and then by Lemma 1.2.3 in [15] deduce the following theorem.

**Theorem 4.3.** The SDE (2.1) has a unique strong solution in \(D^\infty\).

**Appendix A. Constructing the approximating functions for the drift**

Here we construct the functions \(b_n\) mentioned in section 2. This construction is motivated by Berhanu in [4, Theorem 2.9.]. Assume that \(U \subset V\) are two open sets in \(\mathbb{R}^d\) with distance \(a > 0\). For \(0 \leq \epsilon \leq a\), define \(U_\epsilon = \{x; d(x, U) < \epsilon\}\). Then \(U_\epsilon = \bigcup_{x \in U} B_\epsilon(x)\) and \(U \subseteq U_\epsilon \subseteq V\). Fix \(\epsilon\) such that \(0 < 2\epsilon \leq a\) and let \(h^\epsilon(x)\) be the characteristic function
of $U_\epsilon$. For $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp}\psi \subseteq B_1(0)$ and $\int \psi(x)dx = 1$, set $\psi_\epsilon(x) = \frac{1}{\epsilon^d} \psi(\frac{x}{\epsilon})$. Now consider the construction function

$$
\psi_\epsilon \star h_\epsilon(x) = \int_{\mathbb{R}^d} \psi_\epsilon(y) h_\epsilon(x - y)dy
$$

for $0 < 2\epsilon < d$. Since $\text{supp}\psi_\epsilon \subseteq B_\epsilon(0)$, $\psi_\epsilon \star h_\epsilon = 1$ on $U$ and $\psi_\epsilon \star h_\epsilon = 0$ outside $U_{2\epsilon}$. Note that for each multiindex $\alpha$,

$$
\partial_\alpha (\psi_\epsilon \star h_\epsilon)(x) = \int \partial_\alpha (\psi_\epsilon(y)) h_\epsilon(x - y)dy = \frac{1}{\epsilon^{d+|\alpha|}} \int (\partial_\alpha \psi)(\frac{y}{\epsilon}) h_\epsilon(x - y)dy
$$

(A.1)

$$
= \frac{1}{\epsilon^{|\alpha|}} \int (\partial_\alpha \psi)(z) h_\epsilon(x - \epsilon z)dz \leq \|\psi\|_\infty \frac{1}{\epsilon^{|\alpha|}}
$$

Now, let $n \geq 1$ and set $U = B_{n\epsilon}(0)$, $V = B_{2n\epsilon}(0)$ and $\epsilon = n^\xi$. Then the functions $\phi_n(x) := \psi_\epsilon \star h_\epsilon$ satisfy $\phi_n(x) = 1$ on $U$ and $\phi_n(x) = 0$ outside $V$. Since $\text{supp}\phi_n(x) \subseteq B_{2n\epsilon}(0)$, by (A.1) and (2.3) for each multiindex $\alpha$ with $|\alpha| = c \geq 1$, we have

$$
|b(x)\partial_\alpha \phi_n(x)| \leq |b(x)\chi_{|x|\leq 2n\epsilon}| \|\psi\|_\infty \frac{1}{n^{\xi|\alpha|}}
$$

$$
\leq \gamma_c (1 + 2^\xi n^\xi) \|\psi\|_\infty \frac{1}{n^{\xi|\alpha|}} \leq 2^{\xi+1} \gamma_c \|\psi\|_\infty,
$$

and

$$
|\partial_\alpha \phi_n(x)| \leq \|\psi\|_\infty.
$$

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References

[1] ALÒS, E., AND EWALD, C. O. Malliavin differentiability of the Heston volatility and applications to option pricing, Adv. in Appl. Probab. 40(1): (2008) 14–162.
[2] Bahlali, K. Flows of homeomorphisms of stochastic differential equations with measurable drifts, Stochastics and Stochastics Reports 67: (1999) 53–82.
[3] Bavouzet, M. P., AND Messaoud, M. Computation of Greeks using Malliavin’s calculus in jump type market models, Electronic Journal of Probability 11(10): (2006) 276–300.
[4] Berhanu, S. Approximation by smooth functions and distributions, http://www.math.temple.edu/berhanu, (2001).
[5] Bichteler, K., Gravereaux, J-B., and Jacod, J. Malliavin Calculus for Processes with Jumps, Vol 2, Gordon and Breach Science Publishers, Amesterdam, 1987.
[6] Flandoli, F., Gubinelli, M., and Priola, E. Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift, Bulletin des sciences mathematiques 134: (2010) 405–422.
[7] Gyöngy, I., and Millet, A. Rate of convergence of implicit approximations for stochastic evolution equations, Stochastic Differential Equations: Theory and Applications, A volume in honor of professor Boris L. Rosovskii, Interdisciplinary Mathematical Sciences, 2, World Scientific (2007) 281–310.
[8] Khas’minskii, R.Z. Stochastic Stability of Differential Equations, second ed., Springer-Verlag, Berlin, 2012.
[9] A. Kohatsu-Higa, A., and Montero, M. Malliavin Calculus in Finance, Handbook of computational and numerical methods in finance, Birkhäuser Boston, Boston, MA, (2004) 111–174.
[10] Kusuoka, S., and Stroock, D. Applications of the Malliavin calculus, Part I, Stochastic Analysis, (Kyoto/Katata), North-Holland Math. Library 32: (1984) 271–306.
[11] Kusuoka, S., and Stroock, D. Applications of Malliavin calculus, Part II, J. Fac. Sci. Uni. Tokyo, Sect. IA. MATH. 32: (1985) 1–76.
[12] Mao, X. 1997, Stochastic Differential Equations and Their Applications, Horwood Publishing Limited, England.
[13] Mao, X., and Szpruch, L. Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients, Submitted on 9 Apr 2012, arxiv.org/abs/1204.1874v1.
[14] Marco, S. D. On probability distributions of diffusions and financial models with nonglobally smooth coefficients, PhD. Thesis, http://cermics.enpc.fr/de-marcs/home.html.
[15] Nualart, D. The Malliavin Calculus and Related Topics, second ed., Springer Verlag, Berlin, 2006.
[16] Zhang, X., Stochastic flows and Bismut formula for stochastic Hamiltonian systems, Stochastic Processes and their Applications, 120: (2010) 1929–1949.
[17] Zhang, X. Stochastic flows of SDEs with irregular coefficients and stochastic transport equations, Bull. Sci. math. 134: (2010) 340–378.
[18] Zangeneh, B. Z. Semilinear stochastic evolution equations with monotone nonlinearities, Stochastics and Stochastics Reports 3: (1995)129–174.

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