Mathematical Model of Magnetic Tornado in Solar Plasma

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Abstract. The mathematical model is proposed of the magnetic tornado in magnetohydrodynamic plasma. This model is based on the equations of mechanics of continua and the Maxwell electrodynamics. Two specific examples are considered to study the cross-effect of the plasma and magnetic field dynamics, which may result in the occurrence of the magnetic tornado. The plasma-magnetic field interaction is of particular interest in the case of the ideal plasma motion along the magnetic field lines. In this case, the studies of the magnetic tornado can be reduced to the purely hydrodynamic consideration of the tornado occurrence in atmospheric air. The relevant numerical results obtained theoretically confirm the existence of vortex formations identified with the magnetic super-tornados observed in the chromosphere of the solar plasma.

1. Introduction

A magnetic tornado is generally understood as the vortex high-speed flow of the electrically conducting gas in the electromagnetic field. Such flows have been repeatedly observed in the solar plasma [1–4]. There are reasons to believe [5] that magnetic tornados originating in the solar chromosphere form a channel for transporting energy from the lower layers of the Sun into the solar corona and cause its anomalous heating, the formation of sunspots and the generation of solar wind.

When constructing a mathematical theory describing the magnetic tornados, it is natural to use the experience obtained during the mathematical modelling of tornados in the atmosphere of the Earth. At present, it is believed that tornado occurs as a result of the vortex motion of air in the mother cloud. According to observations [6, 7], the vortex generating tornado has a complex three-dimensional structure consisting of the simultaneous air circulation in the poloidal and azimuthal directions and forming the curved torus (vortex ring).

The causes of the vortex ring occurrence, as well as the mechanism for tornado generation by this ring, are poorly understood. The tornado formation model should take into account the three-dimensionality of its motion, the turbulence of air currents, the dependence of the dynamic viscosity of air on the presence of water and sand dust, etc. [8–11]. There occurs a number of serious problems when performing the direct mathematical modelling of the tornado formation based on the numerical solving the equations of gas dynamics as applied to the Earth’s atmosphere. The main of these problems is the proper choice the initial and boundary conditions.

Setting these conditions on the basis of the observational data inevitably results in the large errors of the calculated tornado parameters. In this regard, we proposed [12, 13] a simple mathematical model that makes it possible to study the cause-and-effect relation between the vortex air motion in the
mother cloud and the appearance of the twisted ascending air flow. Moreover, this model can be used
to obtain theoretical solution to the problem of predicting and preventing tornados.

In the present work, the ideas of [12, 13] are generalized in order to study the magnetic tornados on
the Sun. It is assumed that the magnetic tornado arises is the plasma, the dynamics of which obeys the
equations of the classical MHD theory developed by H. Alfvén [14]. In further consideration,
a magnetic tornado is understood as the stationary flow of the solar plasma. Its parameters are axially
symmetric with respect to some straight line intersecting the center of the Sun, and its axial velocity
has a finite positive limit as the distance from the solar center goes to infinity. Such flows can arise,
for example, in the case of the axially symmetric pinching of the solar plasma around a certain straight
line intersecting the center of the Sun, due to the mechanism for the anomalous acceleration of plasma
particles in the z-pinches [15]. Another possible mechanism for the magnetic tornado generation is
associated with the presence of the plane plasma vortex in the chromosphere (arising, say, due to the
plasma electric drift in the crossed electromagnetic fields), the vortex plane being orthogonal to some
straight line intersecting the Sun center. In this case, the vortex center lies on the mentioned straight
line, and the plasma pressure at the periphery is higher than that in the central region [12, 13].

The goals of this study are to theoretically confirm the existence of magnetic tornados observed in
the solar plasma and to indicate some ways of the mathematical investigation of this phenomenon. The
first problem is solved by means of studying the special classes of exact solutions to the classical
MHD equations, among which there are the solutions describing the flows of the magnetic tornado
type. The second problem is reduced to constructing and studying the equations describing the
magnetic tornado, the numerical and analytical solution of which makes it possible to search for the
magnetic tornados in practice.

2. Magnetic tornado equations

We consider the dynamics of the plasma obeying the classical MHD equations for incompressible
viscous plasma of finite electric conductivity in the constant gravitational field [16]:

$$\text{div}\mathbf{U} = 0, \quad \rho = \text{const}, \quad \rho \frac{\partial \mathbf{U}}{\partial t} + \text{Div}\left(\rho \mathbf{U} \cdot \mathbf{U} + \left(p + \frac{H^2}{8\pi}I_3\right)\mathbf{H} \cdot \mathbf{H} - \frac{4\pi}{4\pi} + 2\mu \text{Div}\mathbf{U} + \rho \mathbf{g}\right)$$

$$\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \text{rot}\mathbf{E} = 0, \quad \text{div}\mathbf{H} = 0, \quad \mathbf{E} = \frac{1}{\sigma} \mathbf{U}, \quad \mathbf{j} = \frac{c}{4\pi} \text{rot}\mathbf{H},$$

where $\mu = \text{const}$ is the plasma hydrodynamic viscosity, $\sigma = \text{const}$ is its conductivity, $\mathbf{g} = \text{const}$ is the
acceleration of gravity, $I_3$ is the three-dimensional identity tensor, and $\text{def}\mathbf{U}$ is the strain tensor of the
vector field $\mathbf{U}$.

In the axisymmetric case, $\partial / \partial \varphi = 0$, and the particular solutions to set of Eqs. (1) (in cylindrical
coordinates) can be tried in the following form:

$$U_r = rA(t,z), \quad U_\varphi = rB(t,z), \quad U_z = C(t,z), \quad p = r^2Q(t,z) + \Phi(t,z),$$

$$H_r = rP(t,z), \quad H_\varphi = rS(t,z), \quad H_z = G(t,z),$$

where the functions $A, B, C, P, S, Q, \Phi$ are to be found. Substituting expressions (2) into set of Eqs.
(1), we easily obtain the following result.

**Theorem 1.** In the axially symmetric case, functions (2) give a solution to set of Eqs. (1) if-and-
only-if the complex-valued functions $u = A + iB$ and $w = P + iS$ and the real-valued functions $C$ and
$G$ satisfy the following set of equations:
\[ \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - G \frac{\partial w}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 u}{\partial z^2} + u^2 - \frac{w^2}{4\pi} + \frac{2\gamma_0(t)}{\rho} = 0, \quad \frac{\partial C}{\partial z} = -2 \text{Re} u, \quad (3) \]

\[ \frac{\partial w}{\partial t} + C \frac{\partial w}{\partial z} - G \frac{\partial u}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial G}{\partial z} = -2 \text{Re} w, \quad 0 \leq z < +\infty, \quad t \geq 0, \]

and there is \( z_0 \) such that the condition

\[ \left( 1 \frac{\partial G}{\partial t} + \nu_m \frac{\partial P}{\partial z} - CP + AG \right)_{t=z_0} = 0, \quad P = \text{Re} w, \quad A = \text{Re} u, \quad (4) \]

holds for all \( t \), where \( \gamma_0(t) \) is an arbitrary real-valued function, \( \nu_m = c^2 / (4\pi\sigma) \) is the magnetic viscosity, and the symbol \( \text{Re} \) designates the real part of a complex number. For the given solution to set of Eqs. (3), the function \( Q \) can be obtained in the following form:

\[ Q = \gamma_0(t) - \left| w \right|^2 / (8\pi) = \gamma_0(t) - (P^2 + S^2) / (8\pi), \quad (5) \]

and the function \( \Phi \) can be found as the unique solution (to within an additive constant) to the equation

\[ \frac{\partial C}{\partial t} + \frac{\partial}{\partial z} \left( \frac{C^2}{2} + \frac{2\mu}{\rho} \text{Re} u + \frac{\Phi}{\rho} + g_z \right) = 0. \quad (6) \]

We call set of Eqs. (3) the set of equations of the magnetic tornado. It is easy to obtain that if condition (4) is satisfied for set of Eqs. (3) at some point \( z_0 \), then it will hold for all \( z_0 \). In what follows, we assume that \( z_0 \) is the boundary point of the \( z \)-domain, in which the solution to set of Eqs. (3) is tried, \( z_0 \leq z < +\infty \). Therefore, we say that condition (4) is the boundary condition. For simplicity, we assume that \( z_0 = 0 \). In the special case of \( w = 0 \) and \( G = 0 \) (no magnetic field), the boundary condition (4) is satisfied and set of Eqs. (3) can be reduced to

\[ \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{\mu}{\rho} \frac{\partial^2 u}{\partial z^2} + u^2 + \frac{2\gamma_0}{\rho} = 0, \quad \frac{\partial C}{\partial z} = -2 \text{Re} u. \quad (7) \]

Set of Eqs. (7) is called the set of equations of the tornado in atmospheric air. It was studied in [12, 13], the results of which are essentially used in the present research.

3. Magnetic tornado equations in dimensionless form

In the dimensionless form, set of Eqs. (3) and boundary conditions (4) can be written as follows:

\[ \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - M^2_G \frac{\partial w}{\partial z} - \frac{1}{R_m} \frac{\partial^2 u}{\partial z^2} + u^2 - M^2_A w^2 + \Gamma = 0, \quad \frac{\partial C}{\partial z} = -2 \text{Re} u, \quad (8) \]

\[ \frac{\partial w}{\partial t} + C \frac{\partial w}{\partial z} - G \frac{\partial u}{\partial z} - \frac{1}{R_m} \frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial G}{\partial z} = -2 \text{Re} w, \quad 0 \leq z < +\infty, \quad t \geq 0, \quad (9) \]

where \( R = \rho L_0 U_0 / \mu \), \( R_m = L_0 U_0 / \nu_m \) are the hydrodynamic and magnetic Reynolds numbers, respectively, \( M_A = v_A / U_0 \) is the Alfvén Mach number, \( v_A = H_0(4\pi\rho)^{-1/2} \) is the Alfvén velocity,
\[ \Gamma = 2 \gamma_0(t) \gamma_0 / \rho, \quad t_0 = L_0 / U_0; \] and \( L_0, \) \( H_0, \) and \( U_0, \) are the characteristic length, magnetic field strength, and velocity, respectively.

We supplement set of Eqs. (8) with the initial and boundary conditions. For the stationary flows, the initial conditions are unnecessary, while the boundary conditions for the \( u \) and \( w \) functions are usually set at points \( z = 0 \) and \( z = \infty, \) and for the \( C \) and \( G \) functions, they are set at \( z = 0. \) It is difficult to deliberately choose the boundary conditions on the basis of the solar plasma observations. Of course, it is necessary to use boundary condition (9), but it is not enough. The additional ideas can be picked up from the experimental study [17] of the twisted flows in mercury plasma generated by a rotating disk, which generalizes the results obtained by T. von Karman [18] to the case of the plasma flows [19, 20].

Next, we consider some special solutions to set of Eqs. (8).

**Definitions.**
1. A stationary solution \((u, w, C, G)\) to set of Eqs. (8), (9) is called the magnetic tornado (or antitornado), if there exists the finite limit \( \lim_{z \to \infty} C(z) > 0 \) (or \( \lim_{z \to \infty} C(z) < 0 \)).
2. A stationary solution to set of Eqs. (8), (9) is called the magnetic cyclone (or anticyclone), if it satisfies the asymptotic relation \( C(z) \sim \alpha z^n \) (i.e. \( C(z) / (az^n) \to 1, \) as \( z \to +\infty \)) at some values of \( \alpha > 0 \) (or \( \alpha < 0 \)) and \( n \geq 0 \) (depending on this solution). Additionally, if we have \( \Gamma > 0 \) (or \( \Gamma < 0 \)), then this stationary solution is called the magnetic cyclone (or anticyclone) of the hydrodynamic type. The numbers \( \alpha \) and \( n \) are called the indices of the cyclone (or anticyclone).

The \( \alpha \) and \( n \) indices characterize the strength of the cyclone (anticyclone): the greater are \( |\alpha| \) and \( n, \) the higher is the energy of these flows. In further consideration, it is usually assumed that \( n = 0 \) or \( n = 1. \) Any magnetic tornado (or antitornado) is the magnetic cyclone (or anticyclone) with the index \( n = 0 \). Another example is given by the exact stationary solutions to set of Eqs. (8), (9), which have the form \( u_0(z) = A_0 + iB_0, \quad w_0(z) = P_0 + iS_0, \quad C_0(z) = -2A_0z, \) and \( G_0(z) = -2P_0z, \) where the constant complex numbers \( A_0 + iB_0 \) and \( P_0 + iS_0 \) satisfy the condition \( (A_0 + iB_0)^2 + \Gamma = M_z^0(P_0 + iS_0)^2. \) Then, for \( A_0 > 0 \ (A_0 < 0), \) they describe the anticyclone (cyclone) with the index \( n = 1. \)

4. **Stationary solutions to the tornado equations (summary of results)**
Tornado equations (7) can be written in the dimensionless form

\[ \frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{1}{R} \frac{\partial^2 u}{\partial z^2} + u^2 + \Gamma = 0, \quad \frac{\partial C}{\partial z} = -2\text{Re}u, \quad 0 \leq z < +\infty, \quad t \geq 0. \] (10)

Set of Eqs. (10) is nonlinear, and we have no mathematical theorem that can help us to specify the boundary conditions ensuring the existence and the uniqueness of the solution to the corresponding boundary-value problem for set of Eqs. (10). Physical considerations suggest the following boundary conditions [13]:

\[ u(0) = u^0 \in \mathcal{C}, \quad C(0) = 0, \quad u(\infty) = i\sqrt{\Gamma} \] (11)

The existence and the uniqueness of a stationary solution to problem (10), (11) can be established numerically using the stabilization method described in [13]. The numerical analysis allows concluding that if we substitute the last condition in Eq. (11) for \( u(\infty) \neq i\sqrt{\Gamma}, \) then for any \( u^0 \in \mathcal{C}, \) there will be no stationary solutions to the corresponding problem (10), (11). Otherwise, the existence and the uniqueness of a stationary solution to problem (10), (11) and the type of this solution depend on the boundary value \( u^0 \) at the origin. The character of this dependence is not quite clear and requires further investigation. We restrict ourselves to the following special case of boundary condition (11):

\[ u(0) = i\omega, \quad \omega > 0, \quad C(0) = 0, \quad u(\infty) = i\sqrt{\Gamma}. \] (12)
The solutions to problem (10), (12) depend on three real parameters: \( \omega > 0 \), \( R > 0 \), and \( \Gamma \). With the help of the similarity conversion, this dependence can be reduced to the dependence on a single parameter.

**Theorem 2.** Let \( \omega > 0 \) and

\[
 u = \omega \tilde{u}, \quad C = \omega^{1/2} R^{-1/2} \tilde{C}, \quad z = x(\omega R)^{-1/2}, \quad t = \tau \omega^{-1}. \tag{13}
\]

Then the \( u \), \( C \) functions of \((t, z)\) arguments are the solution to set of Eqs. (10) if-and-only-if the \( \tilde{u} \), \( \tilde{C} \) functions of \((\tau, x)\) argument, are the solution to the following set of equations:

\[
 \frac{\partial \tilde{u}}{\partial \tau} + \frac{\partial \tilde{C}}{\partial x} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\Gamma}{\omega^2} \tilde{u} + \frac{\partial \tilde{C}}{\partial x} = -2 \text{Re} \tilde{u}. \tag{14}
\]

Boundary-value problem (10), (12) rewritten in the \( \tilde{u}(\tau, x) \), \( \tilde{C}(\tau, x) \) variables is equivalent to problem (14), (15), where

\[
 \tilde{u}(0) = i, \quad \tilde{C}(0) = 0, \quad \tilde{u}(\infty) = i \sqrt{\Gamma / \omega^2}, \tag{15}
\]

the solutions to which obviously depend on the single parameter \( \Gamma / \omega^2 \) and make it possible to recover the solutions to problem (10), (12) with the help of similarity conversion (13).

We consider the following boundary conditions:

\[
 u(0) = i, \quad C(0) = 0, \quad u(\infty) = i \sqrt{\Gamma}, \tag{16}
\]

where for \( \Gamma \geq 0 \), the nonnegative root value is chosen, and for \( \Gamma < 0 \), it is assumed that \( i \sqrt{\Gamma} = \sqrt{|\Gamma|} \).

The numerical analysis based on the stabilization method [12, 13] shows that problem (10), (16) has the following solutions:

1. For \( \Gamma > 1 \), it has a stationary solution of the tornado type;
2. For \( 1 > \Gamma \geq 0 \), it has a stationary solution of the antitornado type;
3. For \( \Gamma = 1 \), it has the exact solution \( u(z) \equiv i \), \( C(z) \equiv 0 \);
4. For \( \Gamma < 0 \), it has a stationary solution of the anticyclone type with the indices \( \alpha = -2 \sqrt{|\Gamma|} \), and \( n = 1 \), in particular, \( C(z) \sim -2 \sqrt{|\Gamma|} z \), \( z \to +\infty \).

The graphical illustrations to the results (1)–(4) are given in [12, 26]. For different initial conditions, the numerical analysis based on the stabilization method has not revealed the possibility of being non-unique for the stationary solutions to the initial boundary-value problems considered above. The similarity conversion (13) allows making the following important conclusion concerning the boundary condition (12) (with the nonnegative root value for \( \Gamma \geq 0 \), and \( i \sqrt{\Gamma} = \sqrt{|\Gamma|} \) for \( \Gamma < 0 \)):

**there is the critical value \( \Gamma_{\alpha} = \omega^2 \) such that for \( \Gamma > \Gamma_{\alpha} \), problem (10), (12) has a stationary solution of the tornado type, and for \( 0 \leq \Gamma < \Gamma_{\alpha} \), there is a stationary solution of the antitornado type; for \( \Gamma = \Gamma_{\alpha} \), the stationary solution has the form \( u(z) \equiv i \omega \), \( C(z) \equiv 0 \); and finally, for \( \Gamma < 0 \), problem (10), (12) has a stationary solution of the anticyclone type with the indices \( \alpha = -2 \sqrt{|\Gamma|} \), \( n = 1 \).**

We note that the numerical analysis performed shows that if \( \Gamma < 0 \), then for the boundary condition (11) with \( i \sqrt{\Gamma} = -\sqrt{|\Gamma|} \) and \( u^0 \neq -\sqrt{|\Gamma|} \), problem (10), (11) has no stationary solutions, and for \( u^0 = -\sqrt{|\Gamma|} \), there is an unstable physically unrealizable stationary solution \( u(z) \equiv -\sqrt{|\Gamma|} \), \( C(z) = 2 \sqrt{|\Gamma|} z \).
Next, we indicate a method for obtaining the important approximate formulas by means of performing the similarity conversion.

**Theorem 3.** Let \( \Gamma > 0 \) and

\[
  u = I^{-1/2} \tilde{u}, \quad C = I^{-1/4} R^{-1/2} \tilde{C}, \quad z = x I^{-1/4} R^{-1/2}, \quad t = \tau I^{-1/2}.
\]

Then the \( u, C \) functions of \((t,z)\) arguments are the solution to set of Eqs. (10) if-and-only-if the \( \tilde{u}, \tilde{C} \) of \((\tau,x)\) functions are the solutions to the following set of equations:

\[
  \frac{\partial \tilde{u}}{\partial \tau} + C \frac{\partial \tilde{u}}{\partial x} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u}^2 + 1 = 0, \quad \frac{\partial \tilde{C}}{\partial x} = -2 \text{Re} \tilde{u}, \quad \tag{18}
\]

If \( u(z), C(z) \) is the stationary solution to problem (10), (11) with \( \Gamma > 0 \) (and the positive root is taken in conditions (11)), then Theorem 3 ensures that \( \tilde{u}(x) = \Gamma^{-1/2} u(x \Gamma^{-1/4} R^{-1/2}) \), \( \tilde{C}(x) = \Gamma^{-1/4} R^{1/2} C(x \Gamma^{-1/4} R^{-1/2}) \) is the stationary solution to set of Eqs. (18) with the boundary conditions \( \tilde{u}(0) = u^0 \Gamma^{-1/2}, \tilde{C}(0) = 0, \tilde{u}(\infty) = i \). If \( |u^0| \Gamma^{-1/2} \ll 1 \), then, assuming the analytical dependence of the stationary solution to set of Eqs. (18) with the above boundary conditions on the boundary value of the \( \tilde{u} \) function at the origin \( \tilde{u}(0) \), one can construct the perturbation theory with respect to the parameter \( \tilde{u}(0) \) [26] and calculate (with any desired accuracy) the \( \tilde{u}(x), \tilde{C}(x) \) functions and, accordingly, the \( u(z), C(z) \) functions using the similarity conversion (17). Here, the \( \tilde{u}(x), \tilde{C}(x) \) functions should be regarded as the perturbations (with respect to the parameter \( \tilde{u}(0) \)) of the \( u_0(x), C_0(x) \) functions, which are the stationary solution to set of Eqs. (18) with the unperturbed boundary conditions: \( \tilde{u}(0) = 0, \tilde{C}(0) = 0, \tilde{u}(\infty) = i \). The graphs of these functions obtained numerically and represented in Figure 1 show that this solution is of the tornado type.

![Figure 1. Graphs of \( C_0(x) \), \( \text{Re} u_0(x) \), and \( \text{Im} u_0(x) \) functions with boundary conditions \( u(0) = 0 \), \( u(\infty) = i \), \( C(0) = 0 \).](image)

We limit ourselves to the zero-order approximation (the first, second and higher-order approximations are considered in [26]), assuming that \( \tilde{u}(x) \approx u_0(x), \tilde{C}(x) \approx C_0(x) \). Thus, we can obtain the approximate formulas \( u(z) \approx \Gamma^{1/2} u_0(z \Gamma^{1/4} R^{1/2}), C(z) \approx \Gamma^{1/4} R^{-1/2} C_0(z \Gamma^{1/4} R^{1/2}) \). We note that these formulas do not depend on the boundary value \( u(0) = u^0 \), and their accuracy increases as \( |u^0| \Gamma^{-1/2} \rightarrow 0 \). The second formula makes it possible to calculate the vertical velocity at infinity: \( C(\infty) \approx \Gamma^{1/4} R^{-1/2} C_0(\infty) \). The last relation can be refined [26], if we prove the validity of the following asymptotic formula: \( C(\infty) \sim \Gamma^{1/2} R^{-1/2} C_0(\infty) \) at \( |u^0| \Gamma^{-1/2} \rightarrow 0 \). The graphs in Figure 1 imply that \( C_0(\infty) \approx 1.35 \), and thus, we finally obtain the following important asymptotic formula for the vertical velocity at infinity (that will be used in further consideration):
\[ C(\infty) = 1.35 \Gamma^{1/4} R^{-1/2}, \quad u^0 \Gamma^{-1/2} \to 0. \] (19)

In particular, the \( C(\infty) \) velocity increases proportionally to \( \Gamma^{1/4} \), as \( \Gamma \to +\infty \) and the \( u^0 \) value is fixed, which is confirmed by the numerical analysis.

### 5. Tornado in azimuthal magnetic field

We consider the solutions to set of Eqs. (8) with \( G = 0 \) and the purely imaginary constant function \( w \). Then, \( w = iS, \ S = \text{const} \), the boundary conditions (9) are satisfied, and set of Eqs. (8), (9) can be reduced to the following set of equations:

\[
\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{1}{R} \frac{\partial^2 u}{\partial z^2} + M_A^2 S^2 + \Gamma + u^2 = 0, \quad \frac{\partial C}{\partial z} = -2 \text{Re} u, \quad 0 \leq z < +\infty, \quad t \geq 0, \tag{20}
\]

which differs from the tornado equations (10) only by the following substitution: \( \Gamma \to M_A^2 S^2 + \Gamma \).

The case of \( S = \text{const}, w = iS \), and \( G = 0 \) corresponds to the purely azimuthal magnetic field, \( H_r = H_z = 0, \ H_\phi = rS \), which has no singularities on the \( z \)-axis and is generated by the homogeneous axial current (\( j_r = j_\phi = 0, \ j_z = \text{const} \)), and the \( j_z \) current density is \( j_z = 2S \). Set of Eqs. (20) makes it possible to study the effect of the said purely azimuthal magnetic field on the axisymmetric plasma flows of the form (2), in particular, the tornado flows; in this case, the moving plasma has no effect on the magnetic field. We consider the solution to set of Eqs. (20) with the following boundary conditions:

\[
u(0) = i\omega, \quad \omega \geq 0, \quad C(0) = 0, \quad u(\infty) = \begin{cases} i(\Gamma + M_A^2 S^2)^{1/2}, & \Gamma + M_A^2 S^2 \geq 0, \\ \left|\Gamma + M_A^2 S^2\right|^{1/2}, & \Gamma + M_A^2 S^2 < 0, \end{cases} \tag{21}
\]

The results (1)–(4) obtained in Section 4 imply that the stationary solutions to boundary-value problem (20), (21) exist and are either of the tornado, or antitornado, or anticyclone type, depending on the values of \( \Gamma \) and \( S \). This is illustrated in Figure 2, in which the \( \alpha \) and \( n \) indices correspond to the anticyclone and the boundary parabolas \( \Gamma = -M_A^2 S^2 \) and \( \Gamma = \omega^2 - M_A^2 S^2 \) separate the domains of existence of different types of flows. The parabola \( \Gamma = -M_A^2 S^2 \) is included into the antitornado region, while the parabola \( \Gamma = \omega^2 - M_A^2 S^2 \) consists of the points that correspond to the plasma uniform rotation around the vertical axis with the constant angular velocity \( \omega \) in the azimuthal magnetic field \( H_\phi = rS \).

**Figure 2.** Flow types corresponding to different \( \Gamma \) and \( S \).

The results obtained make it possible to draw three important conclusions:
(1) the azimuthal magnetic field $H_\phi = rS$ displaces down the boundaries separating three types of flows: the tornado ($\Gamma > \Gamma_\omega = \omega^2 - M_k^2 S^2$), antitornado ($\Gamma_\omega > \Gamma \geq -M_k^2 S^2$), and anticyclone ($\Gamma < -M_k^2 S^2$);

(2) in the strong azimuthal magnetic field $H_\phi = rS$, $|S| > \omega / M_k$, the tornado-type flow can appear also for $\Gamma < 0$, i.e., when, according to expression (5), the total (magnetic and hydrodynamic) pressure at the tornado center is higher than that at its periphery;

(3) for $\omega(\Gamma + M_k^2)^{1/2} \ll 1$, the stationary solution to problem (20), (21) is of the tornado-type. According to formula (19), its vertical velocity at infinity is approximately equal to $C(\infty) \cong 1.35(\Gamma + M_k^2 S^2)^{1/4} R^{1/2}$. In particular, the constant uniform axial current with the density $j_z = 2S$ increases the tornado vertical velocity at infinity, and we obtain the following asymptotic formula:

$$C(\infty) \cong 1.35 M_k^2 \left[ j_z \right]^{1/2} R^{1/2} / \sqrt{2}, \quad \left| j_z \right| \rightarrow +\infty, \quad \omega = \text{const.}$$

An additional discussion of this topic can be found in [27].

6. Tornado along magnetic field lines
We consider the plasma flows of the form (2) along the magnetic field lines. It is known [21] that in the case of the stationary plasma flows, for which the vectors of the magnetic field strength and hydrodynamic velocity are parallel to each other, the integration of the classical MHD equations can be reduced to the integration of the ordinary hydrodynamic equations. In particular, this is true for the stationary solutions to set of Eqs. (8) with $R_m = \infty$ (ideally conducting plasma), for which $\omega = ku$ and $G = kC$, where $k$ is an arbitrary given real constant. For such solutions, boundary condition (9) is satisfied and set of Eqs. (8) can be reduced to the following set of equations:

$$\left(1 - k^2 M_k^2 \right) C \frac{\partial u}{\partial z} - \frac{1}{R} \frac{\partial^2 u}{\partial z^2} + \left(1 - k^2 M_k^2 \right) u^2 + \Gamma = 0, \quad \frac{\partial C}{\partial z} = -2 \text{Re } u. \quad (22)$$

In what follows, the product $k^2 M_k^2$ is referred to as the magnetization parameter, which characterizes the contribution of the magnetic field to the plasma motion. It is easy to obtain that the $k^2 M_k^2$ product is equal to the ratio of the magnetic field energy to the kinetic plasma energy, both divided by the unit volume, $k^2 M_k^2 = H^2 / (8\pi) : \rho U^2 / 2$. For $k^2 M_k^2 \rightarrow 0$, the effect of the magnetic field on the plasma is infinitesimal. For $k^2 M_k^2 \rightarrow +\infty$ and also for $k^2 M_k^2 \rightarrow 1$ (which is the most interesting), this effect sharply increases and $k^2 M_k^2 = 1$ is the bifurcation value that separates different types of the stationary flows. For $k^2 M_k^2 = 1$, set of Eqs. (22) becomes linear and can be easily integrated:

$$u(z) = \frac{1}{2} R \Gamma z^2 + K_o z + K_1, \quad K_o, K_1 \in \mathbb{C},$$

$$C(z) = -\frac{1}{3} R \Gamma z^3 - (\text{Re } K_o) z^2 - 2(\text{Re } K_1) z + K_2, \quad K_2 \in \mathbb{R}, \quad (23)$$

where $K_o, K_1, K_2$ are arbitrary constants. Formulas (23) define the real five-dimensional vector space that for $\Gamma \neq 0$, consists of cyclones and anticyclones with the indices $\alpha = -R \Gamma / 3$, $n = 3$, which are not, obviously, the hydrodynamic-type formations.

For $k^2 M_k^2 \neq 1$, set of Eqs. (22) can be reduced to tornado equations (10). Indeed, for $k^2 M_k^2 < 1$, the change of variables
\[ C = \tilde{C}(1 - k^2 M_A^2)^{-1/2}, \quad z = x(1 - k^2 M_A^2)^{-1/2}, \quad \tilde{\Gamma} = \Gamma(1 - k^2 M_A^2)^{-1} \]  

(24)

reduces set of Eqs. (22) to the equations

\[ \frac{\tilde{C}}{\partial x} - \frac{1}{R \partial x^2} + u^2 + \tilde{\Gamma} = 0, \quad \frac{\partial \tilde{C}}{\partial x} = -2 \text{Re} u, \quad 0 \leq x < +\infty, \]  

(25)

that coincide with tornado equations (10) for \( \partial / \partial t = 0 \).

For \( k^2 M_A^2 > 1 \), the change of variables

\[ C = -\tilde{C}(k^2 M_A^2 - 1)^{-1/2}, \quad z = x(k^2 M_A^2 - 1)^{-1/2}, \quad u = -\tilde{v}, \quad \tilde{\Gamma} = \Gamma(k^2 M_A^2 - 1)^{-1}, \]  

(26)

(the bar on top denotes the complex conjugation) reduces set of Eqs. (22) to the following form:

\[ \frac{\tilde{C}}{\partial x} - \frac{1}{R \partial x^2} + \tilde{v}^2 - \tilde{\Gamma} = 0, \quad \frac{\partial \tilde{C}}{\partial x} = -2 \text{Re} \tilde{v}, \quad 0 \leq x < +\infty, \]  

(27)

that coincides with set of Eqs (10) for \( \partial / \partial t = 0 \) and \( \Gamma \) substituted for \( -\Gamma \).

Thus, the studies of the stationary axisymmetric ideal plasma flows of type (2) along the magnetic field lines can be completely reduced to the studies of the stationary solutions to tornado equations (10).

According to the results (1)–(4) obtained in Section 4, set of Eqs. (25) has solutions with the following boundary values:

\[ u(0) = i\omega, \quad \omega > 0, \quad \tilde{C}(0) = 0, \quad u(\infty) = \begin{cases} i\tilde{\Gamma}^{1/2}, & \tilde{\Gamma} \geq 0, \\ i|\tilde{\Gamma}|^{1/2}, & \tilde{\Gamma} < 0, \end{cases} \]  

and set of Eqs. (27) has solutions with the following boundary values:

\[ \tilde{v}(0) = i\omega, \quad \omega > 0, \quad \tilde{C}(0) = 0, \quad \tilde{v}(\infty) = \begin{cases} \tilde{\Gamma}^{1/2}, & \tilde{\Gamma} > 0, \\ i|\tilde{\Gamma}|^{1/2}, & \tilde{\Gamma} \leq 0, \end{cases} \]  

and these solutions are either the tornados, or antitornados, or anticyclones. Using changes of variables (24), (26), we conclude that for \( k^2 M_A^2 \neq 1 \), set of Eqs (22) has solutions satisfying the following boundary conditions

\[ u(0) = i\omega, \quad \omega > 0, \quad C(0) = 0, \quad u(\infty) = \begin{cases} \left( \frac{\Gamma}{1 - k^2 M_A^2} \right)^{1/2}, & \frac{\Gamma}{1 - k^2 M_A^2} \geq 0, \\ \text{sgn}(1 - k^2 M_A^2) \left( \frac{\Gamma}{1 - k^2 M_A^2} \right)^{1/2}, & \frac{\Gamma}{1 - k^2 M_A^2} < 0, \end{cases} \]  

(28)

and these solutions are of the four types: tornado, antitornado, cyclone, and anticyclone, depending on the values of \( \Gamma \) and \( k \), as shown in Figure 3.
The interval \((-M_A^{-1}, M_A^{-1})\) on the \(k\)-axis corresponds to the antitornado region, and the infinite intervals \((-\infty, -M_A^{-1})\) and \((M_A^{-1}, +\infty)\) correspond to the tornado regions; the points of the parabola \(\Gamma = \omega^2 (1 - k^2 M_A^2)\), which are not on the \(k\)-axis, correspond to the solutions to set of Eqs. (22) describing plasma rotation around the \(z\)-axis with the constant angular velocity \(\omega\). Figure 3 also presents the \(\alpha\) and \(n\) indices of the cyclone and anticyclone solutions, as well as the \(u(\infty)\) value at infinity. For \(\omega|1 - k^2 M_A^2|^{1/2} \ll 1\) the flows of the tornado or antitornado types described by the solutions to problem (22), (28) have the following vertical velocities at infinity (according to expressions (19), (24), and (26))

\[
C(\infty) = 1.35 R^{1/2} \left\{ \begin{array}{ll}
\Gamma^{1/4} (1 - k^2 M_A^2)^{-3/4} & \text{for tornado}, \\
-\Gamma^{1/4} (k^2 M_A^2 - 1)^{-3/4} & \text{for antitornado}.
\end{array} \right.
\]

Let us perform some calculations. In view of boundary conditions (28), for the tornado and the antitornado solutions, we obtain:

\[
\frac{H_z(\infty)}{H_z(0)} = \frac{u(\infty)}{u(0)} = \left( \frac{\Gamma}{1 - k^2 M_A^2} \right)^{1/2} = \frac{1}{\omega} \left( \frac{\Gamma_{\text{cr}}}{\Gamma} \right)^{1/2}, \quad \Gamma_{\text{cr}} = \omega^2 \left(1 - k^2 M_A^2\right).
\]

where \(\Gamma_{\text{cr}}\) is the boundary point on the straight line \(\Gamma\) separating the tornado and antitornado regions (Figure 3). Let the \(u(z)\), \(C(z)\) functions be the stationary solution to problem (10), (12) with \(\Gamma \geq 0\); let the \(u_0(z)\), \(C_0(z)\) functions be the corresponding antitornado solution for \(\Gamma = 0\); and let the \(u_{\text{Kar}}(z)\), \(C_{\text{Kar}}(z)\) functions be the solution to problem (10), (12) for \(\omega = 1\), \(R = 1\), and \(\Gamma = 0\) obtained by T. von Karman [18]. In accordance with [24], we obtain \(C_{\text{Kar}}(\infty) = -0.886\). As shown in [12], the \(C(\infty)\) function is the monotonically increasing continuous function of \(\Gamma\) such that \(\lim_{\Gamma \to 0} C(\infty) = C_0(\infty)\). Theorem 2 implies that \(u_0(z) = \omega u_{\text{Kar}}(\sqrt{\omega R} z)\), \(C_0(z) = \sqrt{\omega / R} C_{\text{Kar}}(\sqrt{\omega R} z)\), and therefore, \(C_0(\infty) = \sqrt{\omega / R} C_{\text{Kar}}(\infty) = -0.886 \sqrt{\omega / R}\). Let us calculate the \(H_z(\infty)\) values for \(|k M_A| \to 1 \pm 0\). For the tornado solution at \(|k M_A| \to 1 - 0\), we obtain the following asymptotic formula:

\[
H_z(\infty) = k C(\infty) = \frac{k \dot{C}(\infty)}{(1 - k^2 M_A^2)^{1/2}} = \frac{1.35 k \Gamma^{1/4} R^{-1/2}}{(1 - k^2 M_A^2)^{3/4}} \to \pm \infty,
\]
where \( \tilde{C}(x) \) is the solution to set of Eqs. (25), \( \pm \) coincides with \( \text{sgn} \, k \), and we have used change of variables (24) and asymptotic formula (19). For the antitornado solution at \( |kM_\lambda| \to 1+0 \), we obtain similar expression:

\[
H_y(\infty) = kC(\infty) = -\frac{k\tilde{C}(\infty)}{(k^2M_\lambda^2-1)^{1/2}} \sim -\frac{1.35k|\Gamma|^{1/4}R^{-1/2}}{(k^2M_\lambda^2-1)^{3/4}} \to \pm \infty,
\]

where \( \tilde{C}(x) \) is the solution to set of Eqs. (27). Finally, for the tornado solution at \( |k| \to +\infty \), we obtain

\[
H_y(\infty) = kC(\infty) = -\frac{k\tilde{C}(\infty)}{(k^2M_\lambda^2-1)^{1/2}} \to -\frac{kC_y(\infty)}{|k|M_\lambda} = \text{sgn} \, k \cdot \frac{0.886 \sqrt{\omega}}{M_\lambda} \sqrt{k},
\]

where \( \tilde{C}(x) \) is the solution to set of Eqs. (27). This implies that \( \lim_{|k| \to +\infty} C(\infty) = 0 \), and boundary conditions (28) imply that \( \lim_{|k| \to +\infty} u(\infty) = 0 \).

The above results allow us making the following conclusions.

(i) There is the critical (bifurcation) value of the magnetization parameter equal to 1 that separates different types of plasma flows: tornados from cyclones and antitornados from anticyclones. As the point \( (k, \Gamma) \) moves towards the bifurcation lines \( k = \pm M_\lambda^{-1} \), the distinctions between the said types of flows increase. This is manifested, firstly, in an unlimited increase in different plasma velocities (see formulas (28) and (29)): on the one hand, for the tornados and antitornados, the vertical velocity and the angular rotation velocity increase, and, on the other hand, for the cyclones and anticyclones, the radial velocity and the index \( \alpha \) increase. Secondly, the arbitrarily strong magnetic field is generated (see formulas (31) and (32)). In particular, the super-high-power magnetic (super)tornados can occur, in which the plasma flows along the magnetic field and has the finite magnetization.

(ii) In the strong magnetic field with the magnetization parameter higher than 1, there can occur the tornado-type plasma flows along the magnetic field lines, in which the total (hydrodynamic and magnetic) pressure at the tornado centre is higher than that at its periphery.

(iii) For the fixed \( \Gamma \) and the magnetization parameter tending to infinity, the plasma velocity in the cyclones and tornados tends to zero (see formulas (28) and (29)), and the azimuthal magnetic field component has the finite limit given by formula (33); thus, the strong magnetic field hampers the occurrence of the magnetic tornados and cyclones.

(iv) If the magnetic field is weak (\( |kM_\lambda| < 1 \)), then, according to formula (30), the azimuthal magnetic field component at infinity \( H_y(\infty) \) will become \( (\Gamma / \Gamma_\alpha)^{1/2} \) times higher (for the tornado) or lower (for the antitornado), as compared to the azimuthal magnetic field at \( z = 0 \). On the contrary, in the strong magnetic field (\( |kM_\lambda| > 1 \)), the azimuthal magnetic field component at infinity becomes \( (\Gamma / \Gamma_\alpha)^{1/2} \) times lower (for the tornado) or higher (for the antitornado), as compared to \( H_y(0) \).

7. Conclusions
In this article, we proposed the mathematical model of the magnetic tornado phenomenon based on finding the exact solutions to the classical MHD equations of a special form and obtained the equations of the magnetic tornado. The complexification of equations and the use of similarity conversion make it possible to reduce the magnetic tornado equations to a set of two nonlinear heat conduction equations for the complex “temperatures” and reduce the dimension of the problem.

We have considered two classes of solutions to the magnetic tornado equations. These classes describe the plasma flows of the tornado type, which can be studied on the basis of the equations of the tornado in atmospheric air. The analysis performed provides the theoretical justification for the
existence of magnetic tornados in plasma and also reveals a number of important regularities in the interaction of the magnetic field with plasma in the tornado.

8. References

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