ON THE RANGE OF THE ATTENUATED MAGNETIC RAY TRANSFORM FOR CONNECTIONS AND HIGGS FIELDS

GARETH AINSWORTH AND YERNAT M. ASSYLBEKOV

Abstract. For a two-dimensional simple magnetic system, we study the attenuated magnetic ray transform $I_{A,\Phi}$, with attenuation given by a unitary connection $A$ and a skew-Hermitian Higgs field $\Phi$. We give a description for the range of $I_{A,\Phi}$ acting on $\mathbb{C}^n$-valued tensor fields.

1. Introduction

1.1. Magnetic flows. Consider a compact oriented Riemannian manifold $(M, g)$ with boundary. Let $\pi : TM \rightarrow M$ denote the canonical projection, $\pi : (x, v) \mapsto x$, $x \in M$, $v \in T_x M$. Denote by $\omega_0$ the canonical symplectic form on $TM$, which is obtained by pulling back the canonical symplectic form of $T^*M$ via the Riemannian metric. Let $H : TM \rightarrow \mathbb{R}$ be defined by

$$H(x, v) = \frac{1}{2}|v|^2_{g(x)}, \quad (x, v) \in TM.$$ 

The Hamiltonian flow of $H$ with respect to $\omega_0$ gives rise to the geodesic flow of $(M, g)$. Let $\Omega$ be a closed 2-form on $M$ and consider the new symplectic form $\omega$ defined as

$$\omega = \omega_0 + \pi^*\Omega.$$ 

The Hamiltonian flow of $H$ with respect to $\omega$ gives rise to the magnetic geodesic flow $\phi_t : TM \rightarrow TM$. This flow models the motion of a unit charge of unit mass in a magnetic field whose Lorentz force $Y : TM \rightarrow TM$ is the bundle map uniquely determined by

$$\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta \rangle$$

for all $x \in M$ and $\xi, \eta \in T_x M$. Every trajectory of the magnetic flow is a curve on $M$ called a magnetic geodesic. Magnetic geodesics satisfy Newton’s law of motion

$$\nabla_{\dot{\gamma}}\dot{\gamma} = Y_\gamma(\dot{\gamma}).$$

(1)

Here $\nabla$ is the Levi-Civita connection of $g$. The triple $(M, g, \Omega)$ is said to be a magnetic system. In the absence of a magnetic field, that is $\Omega = 0$, we recover the ordinary geodesic flow and ordinary geodesics. Note that magnetic geodesics are not time reversible, unless $\Omega = 0$. Note also that magnetic geodesics have constant speed, and hence are restricted to a specific energy level - we will consider the magnetic flow on the unit sphere bundle $SM = \{(x, v) \in TM : |v| = 1\}$. This
is not a restriction at all from a dynamical point of view, since other energy levels may be understood by simply changing $\Omega$ to $c\Omega$, where $c \in \mathbb{R}$.

Magnetic flows were first considered in [2, 3] and it was shown in [4, 11, 15, 16, 17, 19] that they are related to dynamical systems, symplectic geometry, classical mechanics and mathematical mechanics. Inverse problems related to magnetic flows were studied in [1, 5, 6].

Let $\Lambda$ stand for the second fundamental form of $\partial M$ and $\nu(x)$ for the inward unit normal to $\partial M$ at $x$. We say that $\partial M$ is strictly magnetic convex if
\[
\Lambda(x, \xi) > \langle Y_x(\xi), \nu(x) \rangle
\]
for all $(x, \xi) \in S(\partial M)$. Note that if we replace $\xi$ by $-\xi$, we can put an absolute value in the right-hand side of (2). In particular, magnetic convexity is stronger than the Riemannian analogue.

For $x \in M$, we define the magnetic exponential map at $x$ to be the partial map $\exp^\mu_x : T_x M \to M$ given by
\[
\exp^\mu_x(t\xi) = \pi \circ \phi^t_x(\xi), \quad t \geq 0, \quad \xi \in S_x M.
\]
In [5] it is shown that, for every $x \in M$, $\exp^\mu_x$ is a $C^1$-smooth partial map on $T_x M$ which is $C^\infty$-smooth on $T_x M \setminus \{0\}$.

**Definition 1.1.** We say that a magnetic system $(M, g, \alpha)$ is a simple magnetic system if $\partial M$ is strictly magnetic convex and the magnetic exponential map $\exp^\mu_x : (\exp^\mu_x)^{-1}(M) \to M$ is a diffeomorphism for every $x \in M$.

In this case $M$ is diffeomorphic to the unit ball of Euclidean space, and therefore $\Omega = d\alpha$ for some 1-form $\alpha$ on $M$. This definition is a generalization of the notion of a simple Riemannian manifold. The latter naturally arose in the context of the boundary rigidity problem [14]. Throughout this paper we only consider simple magnetic systems.

### 1.2. Attenuated magnetic ray transform for a unitary connection and Higgs field.

We consider a unitary connection and a skew-Hermitian Higgs field on the trivial bundle $M \times \mathbb{C}^n$. We define a unitary connection as a matrix-valued smooth map $A : TM \to \mathfrak{u}(n)$ which for fixed $x \in M$ is linear in $v \in T_x M$, and define a skew-Hermitian Higgs field as a matrix-valued smooth map $\Phi : M \to \mathfrak{u}(n)$.

The connection $A$ induces a covariant derivative which acts on sections of $M \times \mathbb{C}^n$ by $d_A := d + A$. Saying $A$ is unitary means the following holds for the inner product of sections $s_1, s_2$ of $M \times \mathbb{C}^n$
\[
d(s_1, s_2) = (d_A s_1, s_2) + (s_1, d_A s_2).
\]
Pairs of unitary connections and skew-Hermitian Higgs fields $(A, \Phi)$ are very important in the Yang-Mills-Higgs theories, since they correspond to the most popular structure groups $U(n)$ or $SU(n)$, see [7, 8, 12, 13].

On the boundary of $M$, we consider the set of inward and outward unit vectors defined as
\[
\partial_+ SM = \{(x, v) \in SM : x \in \partial M, \langle v, \nu(x) \rangle \geq 0\},
\]
\[
\partial_- SM = \{(x, v) \in SM : x \in \partial M, \langle v, \nu(x) \rangle \leq 0\},
\]
where $\nu$ is the unit inner normal to $\partial M$. The magnetic geodesics entering $M$ can be parametrized by $\partial_+ SM$. We say that a magnetic system $(M, g, \Omega)$ is non-trapping if for any $(x, v) \in SM$ the time $\tau_+(x, v)$ when the magnetic geodesic $\gamma_{x,v}$, with $x = \gamma_{x,v}(0), \ v = \dot{\gamma}_{x,v}(0)$, exits $M$ is finite. In particular, simple magnetic systems are non-trapping [5].

Let $G_\mu$ denote the generating vector field of the magnetic flow $\phi_t$. Given $f \in C^\infty(SM, \mathbb{C}^n)$, consider the following transport equation for $u : SM \to \mathbb{C}^n$

$$G_\mu u + Au + \Phi u = -f \text{ in } SM, \quad u|_{\partial_- SM} = 0.$$ 

Here $A$ and $\Phi$ act on functions on $SM$ by matrix multiplication. This equation has a unique solution $u^f$, since on any fixed magnetic geodesic the transport equation is a linear system of ODEs with zero initial condition.

**Definition 1.2.** The **attenuated magnetic ray transform** of $f \in C^\infty(SM, \mathbb{C}^n)$, with attenuation determined by a unitary connection $\Lambda : TM \to u(n)$ and a skew-Hermitian Higgs field $\Phi : M \to \mathbb{C}^n$, is given by

$$I_{A,\Phi} f := u^f|_{\partial_+ SM}.$$ 

It is clear that a general function $f \in C^\infty(SM, \mathbb{C}^n)$ cannot be determined by its attenuated magnetic ray transform, since $f$ depends on more variables than $I_{A,\Phi} f$. Moreover, one can easily see that the functions of the following type are always in the kernel of $I_{A,\Phi}$

$$(G_\mu + A + \Phi)u, \quad u \in C^\infty(SM, \mathbb{C}^n), \quad u|_{\partial(SM)} = 0. \quad (3)$$

However, in applications one often needs to invert the transform $I_{A,\Phi}$ acting on functions on $SM$ arising from symmetric tensor fields. Further, we will consider this particular case.

Let $f = f_{i_1 \ldots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ be a $\mathbb{C}^n$-valued, smooth symmetric $m$-tensor field on $M$. Then a tensor field induces a smooth function $f_m \in C^\infty(SM, \mathbb{C}^n)$ by

$$f_m(x, v) = f_{i_1 \ldots i_m}(x) v^{i_1} \cdots v^{i_m}.$$ 

We denote by $C^\infty(S_m(M), \mathbb{C}^n)$ the bundle of smooth $\mathbb{C}^n$-valued, (covariant) symmetric $m$-tensor fields on $M$. When $m = 1$, we also use the notation $C^\infty(A^1(M), \mathbb{C}^n)$.

By $I_{A,\Phi}^m$ we denote the following operator

$$I_{A,\Phi}^m f := I_{A,\Phi} f_m, \quad f \in C^\infty(S_m(M), \mathbb{C}^n).$$

We will frequently identify the tensor field $f \in C^\infty(S_m(M), \mathbb{C}^n)$ with the corresponding function $f_m \in C^\infty(SM, \mathbb{C}^n)$.

The magnetic field and Higgs field couple tensors of degrees $m$ and $m - 1$, therefore, we have to consider $m$-tensors and $(m - 1)$-tensors simultaneously. This observation implies that we have to study $\mathcal{I}_{A,\Phi}^m$ which acts on the product space $C^\infty(S_m(M), \mathbb{C}^n) \times C^\infty(S_{m-1}(M), \mathbb{C}^n)$ and is defined as

$$\mathcal{I}_{A,\Phi}^m[f, h] := I_{A,\Phi}^m f + I_{A,\Phi}^{m-1} h, \quad [f, h] \in C^\infty(S_m(M), \mathbb{C}^n) \times C^\infty(S_{m-1}(M), \mathbb{C}^n).$$
1.3. Range description. In this paper we give a characterization of the range of $\mathcal{I}_{A,\Phi}$. More precisely, we give a description for the functions in $C^\infty(\partial_+ SM, \mathbb{C}^n)$ which are in the range of $\mathcal{I}_{A,\Phi}$. For the description we use the following boundary data: the scattering relation (see Section 2.1), the scattering data of the pair $(A, \Phi)$ (see Section 2.2) and the fibrewise Hilbert transform at the boundary (see Section 2.5). Given $w \in C^\infty(\partial_+ SM, \mathbb{C}^n)$ we define $w^\sharp$ to be the unique solution to transport equation

$$G_\mu w^\sharp + A w^\sharp + \Phi w^\sharp = 0, \quad w^\sharp|_{\partial_+ SM} = w.$$  

Moreover, we define $S^\infty_{A,\Phi}(\partial_+ SM, \mathbb{C}^n)$ to be the set of all $w \in C^\infty(\partial_+ SM, \mathbb{C}^n)$ such that $w^\sharp$ is smooth.

We introduce the operator $B_{A,\Phi} : C(\partial(SM), \mathbb{C}^n) \to C(\partial_+ SM, \mathbb{C}^n)$ defined by

$$B_{A,\Phi}a := [(C_A^{-1}a) \circ S - a]|_{\partial_+ SM}.$$  

Let $a$ be a smooth function and $f = (G_\mu + A + \Phi)a$. Suppose that $u^f$ solves

$$G_\mu u + (A + \Phi)u = -f$$ 

with $u|_{\partial_- SM} = 0$. Then clearly $G_\mu (u^f + a) + (A + \Phi)(u^f + a) = 0$. Since $(u^f + a)|_{\partial_+ SM} = a|_{\partial_+ SM}$ we deduce that

$$\mathcal{I}_{A,\Phi}((G_\mu + A + \Phi)a) = u^f|_{\partial_+ SM} = [(C_A^{-1}a) \circ S - a]|_{\partial_+ SM} = B_{A,\Phi}(a)|_{\partial(SM)}.$$  

Next we introduce the operator $P : S^\infty_{A,\Phi}(\partial_+ SM, \mathbb{C}^n) \to C^\infty(\partial_+ SM, \mathbb{C}^n)$ defined by $P_{A,\Phi} := B_{A,\Phi}^* Q_{A,\Phi}$. Clearly the operator $P$ is completely determined by the scattering relation $S$ and scattering data $C_{A,\Phi}$. 

Recall a connection $A$ induces an operator $d_A$, acting on $\mathbb{C}^n$-valued differential forms on $M$ by the formula $d_A\alpha = d\alpha + A \wedge \alpha$. By $d_A^*$ we denote the dual of $d_A$ with respect to $L^2$-norm on the space of forms. Then it is not difficult to check that $d^*_A = - \ast d_A \ast$. We use the notation $\mathfrak{H}_A$ for the space of all 1-forms $\eta$ with $d_A \eta = d_A^* \eta = 0$ and $j^* \eta = 0$ where $j : \partial M \to M$ is the inclusion map. The elements of this space are called $A$-harmonic forms. Note that $\mathfrak{H}_A$ is a finite dimensional space, since the equations defining $\mathfrak{H}_A$ are an elliptic system with regular boundary condition, see [27, Section 5.11]. Since $M$ is a disk, we have $\mathfrak{H}_A = 0$ whenever $A = 0$.

We can now state our main result.

**Theorem 1.3.** Let $(M, g, \alpha)$ be a two-dimensional simple magnetic system, $A$ a unitary connection and $\Phi$ a skew-Hermitian Higgs field. Then a function $u \in C^\infty(\partial_+ SM, \mathbb{C}^n)$ belongs to the range of $\mathcal{I}_{A,\Phi}$ if and only if

$$u = P_{A,\Phi}w + I_{A,\Phi}\eta$$ 

for some $w \in S^\infty_{A,\Phi}(\partial_+ SM, \mathbb{C}^n)$ and for some $\eta \in \mathfrak{H}_A$.

Theorem 1.3 was proved by Paternain, Salo and Uhlmann [21] in the case of the geodesic flow. This was used to give a description of the range of unattenuated geodesic ray transform acting on tensors. For the characterization of the range of
the ray transform on higher order tensors our main result is stated in Section 3: Theorem 3.2.

The crucial difficulty when one deals with a magnetic field or Higgs field is the fact that the concomitant transport equation couples different Fourier components. Even if one restricts oneself to the geodesic case, by adding a Higgs field this difficulty already presents an obstacle to the approach of [21]. The key idea that is utilized to overcome this, and which represents the major contribution of this paper, is a result on the “simultaneous” surjectivity of the adjoints of the ray transform \( I_{A,\Phi}^0 \) and \( I_{A,\Phi}^1 \) - this is made precise in Theorem 4.2. This in turn relies on the ellipticity of the operator \((I_{A,\Phi}^{0,1})^* I_{A,\Phi}^{0,1}\) and that the kernel of \( I_{A,\Phi}^{0,1} \) is trivial [1] - here the critical concept is that \( I_{A,\Phi}^{0,1} \) is defined such that it has domain \( \Omega_0 \oplus \Omega_1 \), not \( \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1 \). If the latter domain is chosen, then the kernel of the ray transform has a natural obstruction, and in particular, is not identically 0.

1.4. Structure of the paper. The structure of the paper is as follows. In Section 2 we recall some facts and definitions from [1, 23] that will be used in our paper. The proof of Theorem 3.2 is given in Section 3. In Section 4 we discuss the surjectivity properties of the adjoint of the attenuated ray transform. Finally, in Section 5 we give the proof of Theorem 1.3.

2. Preliminaries

2.1. Scattering relation. Let \((M, g, \alpha)\) be a simple magnetic system. For \((x, \xi) \in \mathcal{S}M\), we denote by \(\gamma_{x, \xi}\) the magnetic geodesic on \(M\) such that \(\gamma_{x, \xi}(0) = x, \dot{\gamma}_{x, \xi}(0) = \xi\). By \(\tau_+(x, \xi)\) and \(\tau_-(x, \xi)\) we denote the nonnegative and nonpositive times, respectively, when \(\gamma_{x, \xi}\) exits \(\mathcal{M}\). Simplicity of \((M, g, \alpha)\) implies that \(\tau_+\) and \(\tau_-\) are continuous on \(\mathcal{S}M\) and smooth on \(\mathcal{S}M \setminus \mathcal{S}(\partial\mathcal{M})\). Furthermore, from [5, Lemma 2.3] we also have that \(\tau_+|_{\partial_+\mathcal{S}M}\) and \(\tau_-|_{\partial_-\mathcal{S}M}\) are smooth.

The scattering relation \(\mathcal{S} : \partial_+\mathcal{S}M \to \partial_-\mathcal{S}M\) is defined as

\[
\mathcal{S}(x, \xi) := (\phi_{\tau_+(x, \xi)}(x, \xi)) = (\gamma_{x, \xi}(\tau_+(x, \xi)), \dot{\gamma}_{x, \xi}(\tau_+(x, \xi))).
\]

From the above comments on \(\tau_+\), we conclude that the scattering relation \(\mathcal{S}\) is a smooth map.

For a given \(w \in C^\infty(\partial_+\mathcal{S}M, \mathbb{C}^n)\), the transport equation

\[
G_\mu u = 0 \quad \text{in} \quad \mathcal{S}M, \quad u|_{\partial_+\mathcal{S}M} = w
\]

has the solution \(u = w_\psi := w \circ \mathcal{S}^{-1} \circ \psi\) where \(\psi : \mathcal{S}M \to \partial_-\mathcal{S}M\) is the end point map \(\psi(x, \xi) := \phi_{\tau_+(x, \xi)}(x, \xi)\).

2.2. Scattering data of a unitary connection and skew-Hermitian Higgs field. Let \(U_{A,\Phi} : \mathcal{S}M \to U(n)\) be the unique solution of the transport equation

\[
G_\mu U_{A,\Phi} + (A + \Phi)U_{A,\Phi} = 0 \quad \text{in} \quad \mathcal{S}M, \quad U_{A,\Phi}|_{\partial_+\mathcal{S}M} = \text{Id}.
\]

The map \(C_{A,\Phi} : \partial_-\mathcal{S}M \to U(n)\) defined by \(C_{A,\Phi} := U_{A,\Phi}|_{\partial_-\mathcal{S}M}\) is called the scattering data of the pair \((A, \Phi)\). If \(G : \mathcal{S}M \to U(n)\) is a smooth map such that \(G|_{\partial\mathcal{M}} = \text{Id}\), then it is not difficult to check that the pair \((G^{-1}dG + G^{-1}AG, G^{-1}\Phi G)\)
has the same scattering data. It was proved in [1] that \((A, \Phi)\) can be determined by the scattering data \(C_{A, \Phi}\) up to such a gauge equivalence.

Now, for a given \(w \in C^\infty(\partial_+ SM, \mathbb{C}^n)\) consider the unique solution \(w^\# : SM \to \mathbb{C}^n\) to the transport equation
\[
G_\mu w^\# + Aw^\# + \Phi w^\# = 0 \quad \text{in} \quad SM, \quad w^\#|_{\partial_+ SM} = w.
\]

Observe that \(w^\#(x, v) = U_{A, \Phi}(x, v)w_\psi(x, v)\). Using the scattering relation \(S\) and the scattering data \(C_{A, \Phi}\), we introduce the operator
\[
Q_{A, \Phi} : C(\partial_+ SM, \mathbb{C}^n) \to C(\partial SM, \mathbb{C}^n)
\]
defined by
\[
Q_{A, \Phi}w(x, v) := \begin{cases} w(x, v) & (x, v) \in \partial_+ SM, \\ C_{A, \Phi}(x, v) (w \circ S^{-1})(x, v) & (x, v) \in \partial_- SM. \end{cases}
\]

Then clearly \(w^\#|_{\partial(SM)} = Q_{A, \Phi}w(x, v)\). The space of those \(w\) for which \(w^\#\) is smooth in \(SM\) is denoted by
\[
\mathcal{S}_{A, \Phi}^\infty(\partial_+ SM, \mathbb{C}^n) = \{ w \in C^\infty(\partial_+ SM, \mathbb{C}^n) : w^\# \in C^\infty(SM, \mathbb{C}^n) \}.
\]

This space was characterized in [1, Lemma 4.2] in terms of the operator \(Q_{A, \Phi}\) as follows:
\[
\mathcal{S}_{A, \Phi}^\infty(\partial_+ SM, \mathbb{C}^n) = \{ w \in C^\infty(\partial_+ SM, \mathbb{C}^n) : Q_{A, \Phi}w \in C^\infty(SM, \mathbb{C}^n) \}.
\]

Using the fundamental solution \(U_{A, \Phi}\), we can also give an integral expression for the ray transform. Recall \(I_{A, \Phi}f := u^f|_{\partial_+ SM}\) where \(u^f\) is the unique solution to
\[
G_\mu u + Au + \Phi u = -f \quad \text{in} \quad SM, \quad u|_{\partial_- SM} = 0.
\]

Note that \(U_{A, \Phi}^{-1}\) solves \(G_\mu U_{A, \Phi}^{-1} - U_{A, \Phi}^{-1}(A + \Phi) = 0\). Therefore, \(G_\mu(U_{A, \Phi}^{-1}u^f) = -U_{A, \Phi}^{-1}f\). Integrating from 0 to \(\tau_+(x, v)\) for \((x, v) \in \partial_+ SM\) we obtain the following expression
\[
u^f(x, v) = \int_0^{\tau_+(x,v)} U_{A, \Phi}^{-1}(\phi_t(x, v)) f(\phi_t(x, v)) \, dt.
\]

### 2.3. Geometry and Fourier analysis on \(SM\)

Since \(M\) is assumed to be oriented, there is a circle action on the fibres of \(SM\) with infinitesimal generator \(V\) called the vertical vector field. Let \(X\) denote the generator of the geodesic flow of \(g\). We complete \(X, V\) to a global frame of \(T(SM)\) by defining the vector field \(X_\perp := [V, X]\), where \([\cdot, \cdot]\) is the Lie bracket for vector fields. It is easy to see that the generator of magnetic flow \(\phi_t\) can be expressed in terms of the global frame \((X, X_\perp, V)\) in the following form
\[
G_\mu = X + \lambda V,
\]
where \(\lambda\) is the unique function satisfying \(\Omega = \lambda d\text{Vol}_g\) with \(d\text{Vol}_g\) being the area form of \(M\).

For any two functions \(u, v : SM \to \mathbb{C}^n\) define an inner product:
\[
\langle u, v \rangle = \int_{SM} (u, v)_{\mathbb{C}^n} \, d\Sigma^3,
\]
where \(d\Sigma^3\) is the Liouville measure of \(g\) on \(SM\). The space \(L^2(SM, \mathbb{C}^n)\) decomposes orthogonally as a direct sum
\[
L^2(SM, \mathbb{C}^n) = \bigoplus_{k \in \mathbb{Z}} H_k
\]
where \(H_k\) is the eigenspace of \(-iV\) corresponding to the eigenvalue \(k\). Any function \(u \in C^\infty(SM, \mathbb{C}^n)\) has a Fourier series expansion
\[
u = \sum_{k=-\infty}^{\infty} u_k,
\]
where \(u_k \in \Omega_k := C^\infty(SM, \mathbb{C}^n) \cap H_k\).

2.4. Some elliptic operators of Guillemin and Kazhdan. Now we introduce the following first order elliptic operators due to Guillemin and Kazhdan [9]
\[
\eta_+, \eta_- : C^\infty(SM, \mathbb{C}^n) \rightarrow C^\infty(SM, \mathbb{C}^n)
\]
defined by
\[
\eta_+ := \frac{1}{2}(X + iX_\perp), \quad \eta_- := \frac{1}{2}(X - iX_\perp).
\]
By the commutation relations \([-iV, \eta_+] = \eta_+\) and \([-iV, \eta_-] = -\eta_-\) we see that
\[
\eta_+ : \Omega_k \rightarrow \Omega_{k+1}, \quad \eta_- : \Omega_k \rightarrow \Omega_{k-1}.
\]
We will use these operators in the last two sections.

2.5. Fibrewise Hilbert transform. An important tool in our approach is the fibrewise Hilbert transform \(H : C^\infty(SM, \mathbb{C}) \rightarrow C^\infty(SM, \mathbb{C})\), which we define in terms of Fourier coefficients as
\[
\mathcal{H}(u_k) = -\text{sgn}(k)iu_k,
\]
where we use the convention \(\text{sgn}(0) = 0\). Moreover, \(H(u) = \sum_k \mathcal{H}(u_k)\). Note that
\[
(\text{Id} + i\mathcal{H})u = u_0 + 2 \sum_{k=1}^{\infty} u_k, \quad (\text{Id} - i\mathcal{H})u = u_0 + 2 \sum_{k=-\infty}^{-1} u_k.
\]

The following commutator formula, which was derived by Pestov and Uhlmann in [24] and generalized in [1, 23], will play an important role.
\[
[\mathcal{H}, \mathbf{G}_\mu + A + \Phi]u = (X_\perp + *A)u_0 + ((X_\perp + *A)u)_0, \quad u \in C^\infty(SM, \mathbb{C}). \quad (4)
\]
This formula has been frequently used in recent works on inverse problems, see [21, 22, 23, 24, 25, 26].
3. Range characterizations for higher order tensors

Let \( \kappa \) denote the canonical line bundle of \( M \), whose complex structure is that induced by its metric \( g \). For \( k \in \mathbb{N} \) we denote by \( \Gamma(M, \kappa^\otimes k) \) the set of \( k \)-th tensor power of canonical line bundle. It was explained in [20, Section 2] the set \( \Gamma(M, \kappa^\otimes k) \) can be identified with \( \Omega_k \). Roughly speaking, for a given \( \xi \in \Gamma(M, \kappa^\otimes k) \) we obtain a corresponding function on \( \Omega_k \) via the one-to-one map \( SM \ni (x, v) \mapsto \xi_x(v^\otimes k) \).

Since \( M \) is a disk, there is a nonvanishing \( \xi \in \Gamma(M, \kappa) \). Define a function \( h : SM \to S^1 \) by setting \( h(x, v) := \xi_x(v)/|\xi_x(v)| \), and hence \( h \in \Omega_1 \). For the description of the range of \( \mathcal{I}^m_A \), we use the following unitary connection \( A_h := -h^{-1} X h \text{Id} \) and skew-Hermitian Higgs field \( \Phi_\lambda := -i \lambda \text{Id} \). Then it is easy to see that \( A_h \) and \( \Phi_\lambda \) satisfy

\[-h^{-1} G_\mu h \text{Id} = A_h + \Phi_\lambda.\]

We start with characterizing the range of attenuated magnetic ray transform \( I_{A, \Phi} \) restricted to \( \Omega_{m-1} \oplus \Omega_m \oplus \Omega_{m+1} \):

\[
I_{m, A, \Phi}^\pm := I_{A, \Phi}[\Omega_{m-1} \oplus \Omega_m \oplus \Omega_{m+1}] : \Omega_{m-1} \oplus \Omega_m \oplus \Omega_{m+1} \to C^\infty(\partial_+ SM, \mathbb{C}).
\]

We describe the range of \( I_{m, A, \Phi}^\pm \) and then use it in the description of the range of \( \mathcal{I}^m_A \). Consider \( f \in \Omega_{m-1} \oplus \Omega_m \oplus \Omega_{m+1} \) and \( h^m \in \Omega_m \), we have \( h^{-m} f \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1 \).

Therefore

\[
\mathcal{I}^1_{A-mA_h, \Phi-m\Phi_\lambda}(h^{-m} f) = (h^{-m}|_{\partial_+ SM})I_{m, A, \Phi}^\pm f.
\]

The range of the left hand side of (5) was described in Theorem 1.3. Thus we directly conclude the following result.

**Theorem 3.1.** Let \( (M, g, \alpha) \) be a simple two-dimensional magnetic system, \( A \) a unitary connection and \( \Phi \) a skew-Hermitian Higgs field. Then a function \( u \in C^\infty(\partial_+ SM, \mathbb{C}) \) belongs to the range of \( I_{m, A, \Phi}^\pm \) if and only if

\[
u = (h^m|_{\partial_+ SM}) \left(P_{A-mA_h, \Phi-m\Phi_\lambda}(w) + I^1_{A-mA_h, \Phi-m\Phi_\lambda}(\eta)\right)
\]

for some \( w \in \mathcal{S}_A^{\infty}(A, \Phi-m\Phi_\lambda)(\partial_+ SM, \mathbb{C}) \) and \( \eta \in \mathcal{S}_{A-mA_h} \).

Recall that, according to [22, Section 2], there is a one-to-one correspondence between \( C^\infty(S_m(M), \mathbb{C}^n) \) and a subspace of the set of functions on \( SM \) of the form \( f = \sum_{k=-m}^m f_k \), \( f_k \in \Omega_k \). Let \( F \) be the smooth function on \( SM \) corresponding to the pair \( f, h \in C^\infty(S_m(M), \mathbb{C}^n) \times C^\infty(S_{m-1}(M), \mathbb{C}^n) \):

\[F(x, v) := f_{i_1} \cdots f_{i_m}(x) v^{i_1} \cdots v^{i_m} + h_{i_1} \cdots i_{m-1}(x) v^{i_1} \cdots v^{i_{m-1}}.\]

Then we can write \( F = \sum_{k=-m}^m F_k \) for suitable \( F_k \in \Omega_k \). Therefore, we have

\[
\mathcal{I}^m_{A, \Phi}[f, h] = \sum_{k=-[m-1]/3}^{[(m+1)/3]} I^\pm_{3k, A, \Phi}(F_{3k-1} + F_{3k} + F_{3k+1}).
\]

Using this and Theorem 3.1, we obtain our second main result of the current paper.
Let \((M, g, \alpha)\) be a simple two-dimensional magnetic system, \(A\) a unitary connection and \(\Phi\) a skew-Hermitian Higgs field. Then a function \(u \in C^\infty(\partial_+ SM, \mathbb{C}^n)\) belongs to the range of \(I_{A, \Phi}^1\) if and only if there are \(w_{3k} \in S_{A-3kA_0}^\infty(\partial_+ SM, \mathbb{C})\) and \(\eta_{3k} \in \Delta_{A-3kA_0}\) for all \(k = -[(m+1)/3], \ldots, [(m+1)/3]\) such that
\[
\begin{aligned}
\sum_{k = -[(m+1)/3]}^{[(m+1)/3]} (h_{3k}^\dagger |\partial_+ SM\rangle (P_{A-3kA_0} \Phi - 3kA_0 \partial_+ SM)(w_{3k}^\dagger) + I_{A-3kA_0}^1 \Phi - 3kA_0 \eta_{3k}^\dagger).
\end{aligned}
\]

4. Surjectivity properties of \(I_{A, \Phi}^1\)

Let \(d\Sigma^2\) be the volume form on \(\partial(SM)\). In the space of \(\mathbb{C}^n\)-valued functions on \(\partial_+ SM\) define the inner product
\[
\langle h, h' \rangle_\mu = \int_{\partial_+ SM} (h, h')_{\mathbb{C}^n} d\mu(x, v)
\]
where \(d\mu(x, v) = \langle v, \nu(x) \rangle d\Sigma^2(x, v)\). Denote the corresponding Hilbert space by \(L^2_\mu(\partial_+ SM, \mathbb{C}^n)\). As in [21], using the integral representation for \(I_{A, \Phi}\) and Santaló formula [5, Lemma A.8], one can show that \(I_{A, \Phi}\) can be extended to a bounded operator \(I_{A, \Phi} : L^2(SM, \mathbb{C}^n) \to L^2(\partial_+ SM, \mathbb{C}^n)\).

4.1. Adjoint of \(I_{A, \Phi}^1\).

In this section we give an expression for the adjoint of \(I_{A, \Phi}^1\)
\[
(I_{A, \Phi}^1)^* : L^2_\mu(\partial_+ SM, \mathbb{C}^n) \to L^2(\Lambda^1(M, \mathbb{C}^n) \times L^2(M, \mathbb{C}^n)).
\]

Let \(f\) be a smooth \(\mathbb{C}^n\)-valued function on \(M\), \(\omega\) be a smooth \(\mathbb{C}^n\)-valued 1-form on \(M\) and \(h \in L^2_\mu(\partial_+ SM, \mathbb{C}^n)\), then using the Santaló formula and the integral representation for \(I_{A, \Phi}^1\), we have
\[
\langle I_{A, \Phi}^1[f], h \rangle_\mu = \int_{\partial_+ SM} (I_{A, \Phi}^1[f](x, \zeta), h(x, \zeta))_{\mathbb{C}^n} d\mu(x, \zeta)
\]
\[
= \int_{SM} (U_{A, \Phi}^{-1} \omega(x, \zeta) \circ f(x), h_\psi(x, \zeta))_{\mathbb{C}^n} d\Sigma^3(x, \zeta).
\]

Let \(d\text{Vol}_g(x)\) be a measure on \(M\) and \(d\sigma_x\) be a measure on \(S_xM\). Then
\[
\langle I_{A, \Phi}^1[f], h \rangle_\mu = \int_M \left( \omega_i(x), \int_{S_xM} \zeta^i (U_{A, \Phi}^{-1})^*(x, \zeta) h_\psi(x, \zeta) d\sigma_x(\zeta) \right)_{\mathbb{C}^n} d\text{Vol}_g(x)
\]
\[
+ \int_M \left( f(x), \int_{S_xM} (U_{A, \Phi}^{-1})^*(x, \zeta) h_\psi(x, \zeta) d\sigma_x(\zeta) \right)_{\mathbb{C}^n} d\text{Vol}_g(x).
\]

Therefore, if \(A\) is unitary and \(\Phi\) is skew-Hermitian, we have
\[
(I_{A, \Phi}^1)^*(h) = \left[ \int_{S_xM} \zeta^i h^\dagger(x, \zeta) d\sigma_x(\zeta), \int_{S_xM} h^\dagger(x, \zeta) d\sigma_x(\zeta) \right].
\]

Moreover, from [21, Remark 5.2] and (6) we see that the function on \(SM\) which corresponds to \((I_{A, \Phi}^1)^*(h)\), and is denoted again by \((I_{A, \Phi}^1)^*(h)\), has the following form
\[
(I_{A, \Phi}^1)^*(h)(x, v) = \pi h^\dagger_{-1}(x, v) + 2\pi h^\dagger_0(x, v) + \pi h^\dagger_3(x, v), \quad (x, v) \in SM.
\]
4.2. Normal operators. Let \( I_{A,\Phi}^{0,1} \) denote the restriction of \( I_{A,\Phi} \) to \( \Omega_0 \oplus \Omega_1 \). Consider the corresponding normal operator defined as
\[
N_{A,\Phi}^{0,1} := (I_{A,\Phi}^{0,1})^* I_{A,\Phi}^{0,1} : \Omega_0 \oplus \Omega_1 \to \Omega_0 \oplus \Omega_1.
\]

**Lemma 4.1.** Let \( (M, g, \alpha) \) be a two-dimensional simple magnetic system, \( A \) a unitary connection and \( \Phi \) a skew-Hermitian Higgs field. Then \( N_{A,\Phi} \) is elliptic pseudodifferential operator in \( M^{\text{int}} \) of order \(-1\).

**Proof.** First, we consider the normal operator corresponding to \( I_{A,\Phi}^1 \):
\[
N_{A,\Phi} := (I_{A,\Phi}^1)^* I_{A,\Phi}^1 : L^2(\Lambda^1(M), \mathbb{C}^n) \times L^2(M, \mathbb{C}^n) \to L^2(\Lambda^1(M), \mathbb{C}^n) \times L^2(M, \mathbb{C}^n).
\]

From the integral representation for \( I_{A,\Phi} \) and expression (6), we have
\[
N_{A,\Phi}[\omega, f](x) = [N_{A,\Phi}^{10} f + N_{A,\Phi}^{11} \omega, N_{A,\Phi}^{00} f + N_{A,\Phi}^{01} \omega]
\]
with
\[
\begin{align*}
(N_{A,\Phi}^{00} f)(x) &= \int_{S_x M} U_{A,\Phi}(x, \zeta) \int_{\tau_+(x, \zeta)} U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) f(\gamma_{x,\zeta}(t)) \, dt \, d\sigma_x(\zeta), \\
(N_{A,\Phi}^{01} \omega)(x) &= \int_{S_x M} U_{A,\Phi}(x, \zeta) \int_{\tau_+(x, \zeta)} U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) \omega_j(\gamma_{x,\zeta}(t)) \xi_j x,\zeta(t) \, dt \, d\sigma_x(\zeta), \\
(N_{A,\Phi}^{10} f)(x) &= \int_{S_x M} \xi_i U_{A,\Phi}(x, \zeta) \int_{\tau_-(x, \zeta)} U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) f(\gamma_{x,\zeta}(t)) \, dt \, d\sigma_x(\zeta), \\
(N_{A,\Phi}^{11} \omega)(x) &= \int_{S_x M} \xi_i U_{A,\Phi}(x, \zeta) \int_{\tau_-(x, \zeta)} U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) \omega_j(\gamma_{x,\zeta}(t)) \xi_j x,\zeta(t) \, dt \, d\sigma_x(\zeta).
\end{align*}
\]

Repeating the same arguments as in \[5, \text{Section } 4.2\], the use of \[5, \text{Lemma B.1}\] gives that
\[
\sigma_p(N_{A,\Phi}^{00})(x, \xi) = c_2 |\xi|^{-1}, \quad \sigma_p(N_{A,\Phi}^{11})(x, \xi) = c_2 |\xi|^{-1} (\delta_i^j - \xi_i \xi_j/|\xi|^2), \\
\sigma_p(N_{A,\Phi}^{01})(x, \xi) = 0, \quad \sigma_p(N_{A,\Phi}^{10})(x, \xi) = 0,
\]
where \(|\xi|^2 = g_{ij} \xi^i \xi^j\) and \(\xi_i = g_{ij} \xi^j\).

Let \( \beta = \beta_z \, dx^1 + \beta_z \, dx^2 \) be a smooth 1-form on \( M \) and consider be the corresponding function on \( SM \):
\[
\beta(x, v) = \beta_z(v^1 + \beta_z(x)v^2).
\]

Then \( V\beta(x, v) = \beta_z(x)v^1 - \beta_z(x)v^2 \). Compute the terms of the Fourier expansion \( \beta = \beta_{-1} + \beta_1 \):
\[
\begin{align*}
\beta_{-1}(x, v) &= \frac{1}{2} (\beta(x, v) + i V\beta(x, v)) = \frac{1}{2} (\beta_z(x) + i \beta_z(x))(v^1 - iv^2), \\
\beta_1(x, v) &= \frac{1}{2} (\beta(x, v) - i V\beta(x, v)) = \frac{1}{2} (\beta_z(x) - i \beta_z(x))(v^1 + iv^2). \quad (8)
\end{align*}
\]

Therefore, we can conclude that \( \beta \in \Omega_1 \) if and only if
\[
\beta_z = i \beta_z. \quad (9)
\]
Now, consider the normal operator $N_{A,\Phi}$ corresponding to the restricted ray transform $I_{A,\Phi}^{0,1}$. Let $f \in \Omega_0$ and $\omega \in \Omega_1$. As in the case of $N_{A,\Phi}$, we have

$$N_{A,\Phi}[\omega, f](x) = [N_{A,\Phi}^{10} f + N_{A,\Phi}^{11} \omega, N_{A,\Phi}^{00} f + N_{A,\Phi}^{01} \omega]$$

where, thanks to (9), we have

$$(N_{A,\Phi}^{00} f)(x) = \int_{S_x M} U_{A,\Phi}(x, \zeta) \int_{\tau}(x, \zeta) \ U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) f(\gamma_{x,\zeta}(t)) \ dt \ d\sigma_x(\zeta),$$

$$(N_{A,\Phi}^{01} \omega)(x) = \int_{S_x M} U_{A,\Phi}(x, \zeta) \times \int_{\tau}(x, \zeta) U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) \omega_x(\gamma_{x,\zeta}(t)) \left(\gamma_{x,\zeta}^1(t) + i \gamma_{x,\zeta}^1(t)\right) \ dt \ d\sigma_x(\zeta),$$

$$(N_{A,\Phi}^{10} f)(x) = \int_{S_x M} \zeta_i U_{A,\Phi}(x, \xi) \int_{\tau}(x, \zeta) \ U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) f(\gamma_{x,\zeta}(t)) \ dt \ d\sigma_x(\zeta),$$

$$(N_{A,\Phi}^{11} \omega)(x) = \int_{S_x M} \zeta_i U_{A,\Phi}(x, \xi) \times \int_{\tau}(x, \zeta) U_{A,\Phi}^{-1}(\phi_t(x, \zeta)) \omega_x(\gamma_{x,\zeta}(t)) \left(\gamma_{x,\zeta}^1(t) + i \gamma_{x,\zeta}^1(t)\right) \ dt \ d\sigma_x(\zeta).$$

Using [5, Lemma B.1] we see that the principal symbols of the above operators can be written as linear combinations of the principal symbols of $N_{A,\Phi}$, hence using the expressions for the latter from [5, Proposition 7.2] we can conclude the following

$$\sigma_p(N_{A,\Phi}^{00})(x, \xi) = \sigma_p(N_{A,\Phi}^{10})(x, \xi) = c_2 |\xi|^{-1},$$

$$\sigma_p(N_{A,\Phi}^{01})(x, \xi) = \sigma_p(N_{A,\Phi}^{11})(x, \xi) = i\sigma_p(N_{A,\Phi}^{01})(x, \xi) = 0.$$
4.3. Surjectivity of adjoints. The aim of this section is to prove the following analogue of [21, Theorem 5.4] and [21, Theorem 5.5].

**Theorem 4.2.** Let \((M, g, \alpha)\) be a two-dimensional simple magnetic system, \(A\) a unitary connection and \(\Phi\) a skew-Hermitian Higgs field. Given \([\omega, f] \in C^\infty(\Lambda^1(M), \mathbb{C}^n) \times C^\infty(M, \mathbb{C}^n)\) with \(d^*_A \omega = 2\Phi f\), there exists \(w \in \mathcal{S}_{Xg}^\infty(\partial_+ SM, \mathbb{C}^n)\) such that \((T^*_{A, \Phi})^*(w) = [\omega, f] \) for \(w \in \Omega_m\). For the proof we need the following:

**Proposition 4.3.** For given \(f \in \Omega_m\) and \(w \in \Omega_{m+1}\), there is \(u \in C^\infty(SM, \mathbb{C}^n)\) such that \((G_\mu + A + \Phi)u = 0\) and \(u_m = f\), \(u_{m+1} = w\).

**Proof.** First, we deal with the case \(m = 0\). The statement is equivalent to the surjectivity of \((f_{A, \Phi}^{0,1})^*\). The proof follows the same ideas as [25, Theorem 1.4]. We only mention the main tools: The first main ingredient is Lemma 4.1, which says that \(N_{A, \Phi}^{0,1}\) is an elliptic pseudo-differential operator of order \(-1\). This ensures closed range of a suitable extension. Second key result is [1, Theorem 1.2], which in our case says that if \(f \in \Omega_0\) and \(w \in \Omega_1\) with \(f_{A, \Phi}^{0,1}(w + f) = 0\), then \(w \equiv 0\). This gives the desired surjectivity for \((f_{A, \Phi}^{0,1})^*\).

Now, we prove the statement for any \(m \in \mathbb{Z}\). Fix a non-vanishing \(h \in \Omega_1\). As before, consider the unitary connection \(A_h = -h^{-1}Xh\) \(\text{Id} \) and skew-Hermitian Higgs field \(\Phi_\lambda = -i\lambda \text{Id} \) satisfying

\[-h^{-1}G_\mu h \text{Id} = A_h + \Phi_\lambda.\]

Note that for any \(a \in C^\infty(SM, \mathbb{C}^n)\) the following holds

\[(G_\mu + A + \Phi - mA_h - m\Phi_\lambda)a = -h^{-m}((G_\mu + A + \Phi)(hm a)).\]  

By what we have already proved above for the case \(m = 0\), there exists \(a \in C^\infty(SM, \mathbb{C}^n)\) such that

\[(G_\mu + A + \Phi - mA_h - m\Phi_\lambda)a = 0\]

and \(a_0 = h^{-m} f\), \(a_1 = h^{-m} w\). Set \(u_0 := hm a\), then clearly \(u_m = f\) and \(u_{m+1} = w\). By equality (10), we have \((G_\mu + A + \Phi)u = 0\), which finishes the proof. 

**Proof of Theorem 4.2.** Let \(\omega\) be a smooth \(\mathbb{C}^n\)-valued 1-form on \(M\) and let \(f \in C^\infty(M, \mathbb{C}^n)\). By Proposition 4.3, there exist \(u, u' \in C^\infty(SM, \mathbb{C}^n)\) such that

\[
\begin{cases}
(G_\mu + A + \Phi)u = 0, \\
u_{-1} = \omega_{-1}, \quad u_0 = f,
\end{cases}
\quad \text{and} \quad \begin{cases}
(G_\mu + A + \Phi)u' = 0, \\
u'_0 = f, \quad u'_1 = \omega_1.
\end{cases}
\]

Consider \(w \in C^\infty(SM, \mathbb{C}^n)\) defined as

\[w := \sum_{k=-\infty}^{-1} u_k + \sum_{k=1}^{\infty} u'_k + f.\]

It is clear that \(w_0 = f\) and \(w_{-1} + w_1 = \omega_{-1} + \omega_1 = \omega\). Introduce the operators \(\mu_\pm = \eta_\pm + A_{\pm 1}\). Then \(X + A = \mu_- + \mu_+\). It is easy to check that

\[(G_\mu + A + \Phi)w = 0.\]
In particular, we have
\[ \mu_- w_{-1} + \mu_+ w_1 + \Phi w_0 = 0. \]

By [21, Lemma 6.2], this is equivalent to our assumption \( d_*^1 \omega = 2\Phi f \). Solutions of the transport equation are unique once boundary data is specified, therefore, \( w|_{\partial I_{SM}} \) satisfies the requirement of our theorem. \( \square \)

5. Proof of Theorem 1.3

Let \( w^\tau \) be any smooth solution of the transport equation \( G_{\mu}w^\tau + (A + \Phi)w^\tau = 0 \). Applying (4) to \( w^\tau \) we get
\[ -(G_{\mu} + A + \Phi)Hw^\tau = (X_\perp + \ast A)w^\tau_0 + (X_\perp w^\tau + \ast Aw^\tau)_0. \]
Since \( X_\perp f = \ast df \) for \( f \in \Omega_0 \) we have
\[ (X_\perp + \ast A)w^\tau_0 = \ast d_A w^\tau_0. \]
Since \( X_\perp = i(\eta_- - \eta_+) \) and \( \ast(A_{-1} + A_1) = i(A_{-1} - A_1) \) we obtain
\[ (X_\perp w^\tau_0 + \ast Aw^\tau_0 = i(\eta_- w^\tau_0 - \eta_+ w^\tau_0) + i(A_{-1} w^\tau_0 - A_1 w^\tau_0) = i(\mu_- w^\tau_0 - \mu_+ w^\tau_0) \]
where \( \mu_\pm = \eta_\pm + A_\pm \). According to [21, Lemma 6.2] the following holds
\[ \ast d_A \alpha = 2i(\mu_- \alpha - \mu_+ \alpha_0). \]
Collecting everything together and using (7) we have
\[ -2\pi (G_{\mu} + A + \Phi)Hw^\tau = 2\pi \ast d_A w^\tau_0 + \pi \ast d_A (w^\tau_0 + w^\tau_1) = \ast d_A (I_{1,A,\Phi})^* \ast w. \]
Applying \( I_{1,A,\Phi} \) to the above equality we obtain
\[ -2\pi P_{A,\Phi} = I_{1,A,\Phi} \ast d_A (I_{1,A,\Phi})^*. \] (11)

We also need the following result whose proof is postponed until after the proof of Theorem 1.3.

Lemma 5.1. Let \((M, g)\) be a Riemannian disk, \( A \) a unitary connection and \( \Phi \) a skew-Hermitian Higgs field.

(a) Let \( \alpha \) be a smooth \( C^n \)-valued 1-form. Then there exist functions \( a, p \in C^\infty(M, C^n) \) and \( \eta \in H_A \) such that \( p|_{\partial M} = 0 \) and \( d_A p + \ast d_A a + \eta = \alpha \).

(b) Given \( f, a \in C^\infty(M, C^n) \) there is a smooth \( C^n \)-valued 1-form \( \beta \) with \( \ast d_A \beta = f \) and \( d_A^* \beta = \Phi a \).

Proof of Theorem 1.3. Suppose that \( u = P_{A,\Phi} w + I_{1,A,\Phi}^1 \eta \) for \( \eta \in H_A \), then (11) shows that \( u \) belongs to the range of \( I_{1,A,\Phi} \). Conversely, if \( u \) belongs to the range of \( I_{1,A,\Phi} \), then \( u = I_{1,A,\Phi}[\omega, f] \) for some smooth \( C^n \)-valued function \( f \) and 1-form \( \omega \) on \( M \). By item (a) of Lemma 5.1 we can find \( a, p \in C^\infty(M, C^n) \) with \( p|_{\partial M} = 0 \) and \( \eta \in H_A \) such that
\[ d_A p + \ast d_A a + \eta = \omega. \]
Since \( I_{1,A,\Phi}[d_A p, \Phi p] = 0 \), we have \( I_{1,A,\Phi}[\omega, f] = I_{1,A,\Phi} \eta + I_{1,A,\Phi}[\ast d_A a, f - \Phi p] \). By item (b) of Lemma 5.1 we can find \( C^n \)-valued 1-form \( \beta \) such that \( \ast d_A \beta = f - \Phi p \) and \( d_A^* \beta = 2\Phi a \). Therefore we obtain \( I_{1,A,\Phi}[\omega, f] = I_{1,A,\Phi} \eta + I_{1,A,\Phi}[\ast d_A a, \ast d_A \beta] \). By
There is \( w \in S^\infty_{\infty}(\partial SM, \mathbb{C}^n) \) such that \((I_{A,\Phi})^*(w) = [\beta, a] \). Using (11) we conclude that \( u = I_{A,\Phi}[\omega, f] = I_{A,\Phi}\eta - 2\pi P_{A,\Phi}(w) \).

Item (a) of Lemma 5.1 was proved in [21, Lemma 6.1-(1)]. To prove item (b) of Lemma 5.1, consider the Laplacian corresponding to \( dA - \Delta A = d\ast A dA + dA d\ast A \).

This operator acts on \( \mathbb{C}^n \)-valued graded forms and maps \( k \)-forms to \( k \)-forms. The following result directly implies item (b) of Lemma 5.1.

**Lemma 5.2.** Given \( f, a \in C^\infty(M, \mathbb{C}^n) \) there is a smooth \( \mathbb{C}^n \)-valued 1-form \( \beta \) with \( d\ast A \beta = f \) and \( dA \beta = \Phi a \ dVol_g \).

**Proof.** The proof is essentially identical to the proof of [21, Lemma 6.6]. Look for \( \beta \) of the form \( \beta = dA u^0 + d\ast A u^2 \) where \( u^0, u^2 \) are smooth forms. Then we need \( u^0, u^2 \) to satisfy

\[
d\ast A dA u^0 + (F_A)^* u^2 = f, \quad (F_A)u^0 + dA d\ast A u^2 = \Phi a \ dVol_g,
\]

where \( F_A = dA \circ dA = dA + A \wedge A \) is the curvature of \( dA \). Writing \( u = u^0 + u^2 \), these equations are equivalent to

\[
(-\Delta A + R)u = f + \Phi a \ dVol_g,
\]

for some operator \( R \) of order 0. Then [21, Lemma 6.5] implies the existence of a smooth solution \( u \), hence the existence of the desired \( \beta \).

**Acknowledgements.** The first author wishes to thank his advisor, Gabriel Paternain, for all his encouragement and support. The second author would like to express his acknowledgements to Professor Gunther Uhlmann, for constant assistance and support. The work of the second author was partially supported by NSF.

**References**

[1] G. Ainsworth, *The attenuated magnetic ray transform on surfaces*, Inverse Problems and Imaging, 7 no. 1 (2013), 27–46.

[2] D. V. Anosov, Y. G. Sinai, *Certain smooth ergodic systems* [Russian], Uspekhi Mat. Nauk, 22 (1967), 107–172.

[3] V. I. Arnold, *Some remarks on flows of line elements and frames*, Sov. Math. Dokl., 2 (1961), 562–564.

[4] V. I. Arnold, A. B. Givental, “Symplectic Geometry”, Dynamical Systems IV, Encyclopaedia of Mathematical Sciences, Springer Verlag, Berlin, 1990.

[5] N. S. Dairbekov, G. P. Paternain, P. Stefanov, G. Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*, Adv. Math., 216 (2007), 535–609.

[6] N. Dairbekov, G. Uhlmann, *Reconstructing the Metric and Magnetic Field from the Scattering Relation*, Inverse Problems and Imaging, 4 (2010), 397–409.

[7] M. Dunajski, *Solitons, instantons, and twistors*. Oxford Graduate Texts in Mathematics, 19. Oxford University Press, Oxford, 2010.

[8] N. J. Hitchin, G. B. Segal, R. S. Ward, *Integrable systems. Twistors, loop groups, and Riemann surfaces*, Oxford Graduate Texts in Mathematics, 4. The Clarendon Press, Oxford University Press, New York, 1999.

[9] V. Guillemin, D. Kazhdan, *Some inverse spectral results for negatively curved 2-manifolds*, Topology 19 (1980), 301–312.
[10] P. Juhlin, 1992, Principles of Doppler tomography, Tekniska Hogskolan i Lund, Matematiska Institutionen, LUTFD2/TFMA-92/7002.
[11] V. V. Kozlov, *Calculus of variations in the large and classical mechanics*, Russian Math. Surveys, 40 (2011), no. 2, 37–71.
[12] N. Manton, P. Sutcliffe, Topological solitons. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2004.
[13] L.J. Mason, N. M. J. Woodhouse, *Integrability, self-duality, and twistor theory*, London Mathematical Society Monographs. New Series, 15. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.
[14] R. Michel, *Sur la rigidité imposée par la longueur des géodésiques*, Invent. Math., 65 (1981), 71–83.
[15] S. P. Novikov, *Variational methods and periodic solutions of equations of Kirchhoff type. II*, J. Functional Anal. Appl., 15 (1981), 263–274.
[16] S. P. Novikov, *Hamiltonian formalism and a multivalued analogue of Morse theory*, Russian Math. Surveys, 37 (1982), no. 5, 1–56.
[17] S. P. Novikov, I. Shmel’tser, *Periodic solutions of the Kirchhoff equations for the free motion of a rigid body in a liquid, and the extended Lyusternik-Schnirelmann-Morse theory. I.*, J. Functional Anal. Appl., 15 (1981), 197–207.
[18] G. P. Paternain, *Transparent connections over negatively curved surfaces*, J. Mod. Dyn. 3 (2009), 311–333.
[19] G. P. Paternain, M. Paternain, *Anosov geodesic flows and twisted symplectic structures*, in International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañe), F. Ledrappier, J. Lewowicz, S. Newhouse eds, Pitman Research Notes in Math. 362 (1996), 132–145.
[20] G. P. Paternain, M. Salo, G. Uhlmann, *Spectral rigidity and invariant distributions on Anosov surfaces*, preprint, arXiv: 1208.4943.
[21] G. P. Paternain, M. Salo, G. Uhlmann, *On the range of the attenuated ray transform for unitary connections*, to appear, International Math. Research Notices (IMRN).
[22] G. P. Paternain, M. Salo, G. Uhlmann, *Tensor tomography on surfaces*, Inventiones Math., 193 (2013), 229–247.
[23] G. P. Paternain, M. Salo, G. Uhlmann, *The attenuated ray transform for connections and Higgs fields*, Geom. Funct. Anal. 22 (2012), 1460–1489.
[24] L. Pestov, G. Uhlmann, *On the characterization of the range and inversion formulas for the geodesic X-ray transform*, International Math. Research Notices, 80 (2004), 4331–4347.
[25] L. Pestov, G. Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math. 161, no. 2 (2005), 1089-1106.
[26] M. Salo, G. Uhlmann, *The attenuated ray transform on simple surfaces*, J. Diff. Geom. 88 (2011), no. 1, 161–187.
[27] M. E. Taylor, *Partial Differential Equations I. Basic Theory*. Second edition. Applied Mathematical Sciences, 115. Springer, New York, 2011.

Trinity College, Cambridge, CB2 1TQ, United Kingdom
E-mail address: ga296@cam.ac.uk

University of Washington, Department of Mathematics, Seattle, WA 98195-4350, USA
E-mail address: yassylbekov@yahoo.com