On the critical exponent $p_c$ of the 3D quasilinear wave equation
\[-\left(1 + (\partial_t \phi)^p\right)\partial_t^2 \phi + \Delta \phi = 0\]
with short pulse initial data. I, global existence

Bingbing Ding$^{1,*}$, Yu Lu$^{1,*}$, Huicheng Yin$^{1,*}$

School of Mathematical Sciences and Mathematical Institute, Nanjing Normal University, Nanjing, 210023, China.

Abstract

For the 3D quasilinear wave equation
\[-\left(1 + (\partial_t \phi)^p\right)\partial_t^2 \phi + \Delta \phi = 0\]
with the short pulse initial data $(\phi, \partial_t \phi)(1, x) = (\delta^{2-\varepsilon_0} \phi_0(\frac{r-1}{\delta}, \omega), \delta^{1-\varepsilon_0} \phi_1(\frac{r-1}{\delta}, \omega))$, where $p \in \mathbb{N}$, $p \geq 2$, $0 < \varepsilon_0 < 1$, $r = |x|$, $\omega = \frac{x}{r} \in S^2$, and $\delta > 0$ is sufficiently small, under the outgoing constraint condition $(\partial_t + \partial_r)^k \phi(1, x) = O(\delta^{2-\varepsilon_0})$ for $k = 1, 2$, we will establish the global existence of smooth large data solution $\phi$ when $p > p_c$ with $p_c = \frac{1}{1-\varepsilon_0}$ being the critical exponent. In the forthcoming paper, when $1 \leq p \leq p_c$, we show the formation of the outgoing shock before the time $t = 2$ under the suitable assumptions of $(\phi_0, \phi_1)$.

Keywords: Short pulse initial data, critical exponent, incoming, outgoing, inverse foliation density, Goursat problem

Mathematical Subject Classification: 35L05, 35L72

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$^*$Ding Bingbing (bbding@njnu.edu.cn, 13851929236@163.com), Lu Yu (15850531017@163.com) and Yin Huicheng (huicheng@nju.edu.cn, 05407@nju.edu.cn) are supported by the NSFC (No.11731007, No.12071223, No.11601236, No.11971237).
In this paper, we are concerned with the following 3D quasilinear wave equation

\[-(1 + (\partial_t \phi)^p) \partial_t^2 \phi + \Delta \phi = 0,\]  \hspace{1cm} (1.1)

where \( p \in \mathbb{N}, p \geq 2, x = (x^1, x^2, x^3) \in \mathbb{R}^3, t \geq 1, \partial = (\partial_t, \partial_x^1, \partial_x^2, \partial_x^3) = (\partial_t, \partial_1, \partial_2, \partial_3) \) and \( \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2. \)

Let (1.1) equip with the small initial data

\[\phi(1, x) = \delta \varphi_0(x), \quad \partial_t \phi(1, x) = \delta \varphi_1(x),\]  \hspace{1cm} (1.2)

where \((\varphi_0(x), \varphi_1(x)) \in C_0^\infty(\mathbb{R}^3), \delta > 0\) is sufficiently small. Then (1.1)-(1.2) has a global smooth solution \( \phi \) (see [9] or Chapter 6 of [8]). On the other hand, if \( p = 1 \) in (1.1), then the smooth solution \( \phi \) of (1.1)-(1.2) will blow up in finite time as long as \((\varphi_0(x), \varphi_1(x)) \not\equiv 0\) (see [1], [2] and [5]).

For \( p = 2 \), let (1.1) equip with the short pulse initial data

\[\phi, \partial_t \phi(1, x) = (\delta^2 \phi_0(\frac{r - 1}{\delta}, \omega), \delta^2 \phi_1(\frac{r - 1}{\delta}, \omega)),\]  \hspace{1cm} (1.3)

where \( r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \omega = (\omega_1, \omega_2, \omega_3) = \frac{x}{r} \in \mathbb{S}^2, \phi_0(s, \omega) \) and \( \phi_1(s, \omega) \in C_0^\infty((-1, 0) \times \mathbb{S}^2) \), moreover, such the incoming constraint condition is posed

\[(\partial_t - \partial_r)^k \phi(1, x) = O(\delta^{\frac{2k}{3}}), \quad k = 1, 2.\]  \hspace{1cm} (1.4)

Then under the suitable assumption of \((\phi_0, \phi_1)\), it is shown in [13] that the smooth solution \( \phi \) of equation

\[-(1 + (\partial_t \phi)^p) \partial_t^2 \phi + \Delta \phi = 0\]

will blow up and further the incoming shock will be formed before the time \( t = 2 \). Here we point out that the “short pulse initial data” (a class of large initial data) are firstly introduced by D. Christodoulou in [4]. For the short pulse data and by the short pulse method, the authors in monumental papers [4] and [10] showed that the black holes can be formed in vacuum spacetime and the blowup mechanism is due to the condensation of the gravitational waves for the 3D Einstein general relativity equations.

Motivated by [13], we now consider (1.1) with the general short pulse initial data

\[\phi, \partial_t \phi(1, x) = (\delta^{2-\varepsilon_0} \phi_0(\frac{r - 1}{\delta}, \omega), \delta^{1-\varepsilon_0} \phi_1(\frac{r - 1}{\delta}, \omega)),\]  \hspace{1cm} (1.5)
where $0 < \varepsilon_0 < 1$, and $(\phi_0(s, \omega), \phi_1(s, \omega)) \in C^\infty_0 \left( (-1, 0) \times S^2 \right)$. Meanwhile, the following outgoing constraint condition is posed

$$(\partial_t + \partial_x)^k \phi(1, x) = O(\delta^{2-\varepsilon_0}), \quad k = 1, 2. \quad (1.6)$$

Note that in order to guarantee the strict hyperbolicity of (1.1), the smallness of $\partial \phi$ should be needed (in particular, when $p$ is odd). However, by the following expression of solution $v$ to the 3D linear wave equation $\Box v = 0$ with $(v, \partial_t v)(0, x) = (v_0, v_1)(x)$,

$$v(t, x) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( \int_{S^2} v_0(x + t\omega) d\omega \right) + \frac{t}{4\pi} \int_{S^2} v_1(x + t\omega) d\omega$$

$$= \frac{1}{4\pi} \int_{S^2} v_0(x + t\omega) d\omega + \frac{t^2}{4\pi} \int_{S^2} \nabla_x v_0(x + t\omega) \cdot \omega d\omega + \frac{t}{4\pi} \int_{S^2} v_1(x + t\omega) d\omega, \quad (1.7)$$

when $\nabla^2 v_0$ or $\nabla v_1$ are large, then $\partial v$ is generally large. Therefore, only in terms of the short pulse initial data (1.5), it is not enough to keep the smallness of solution (1.1) when $\delta > 0$ is small. This means that such an outgoing constraint condition (1.6) is required in order to guarantee the well-posedness of (1.1) with (1.5). On the other hand, the condition (1.6) actually implies the better smallness of $\phi$ along the outgoing directional derivative $\partial_t + \partial_x$ up to orders 2 on the initial time $t = 1$. It is pointed out that when (1.5) is given, $\partial^\beta \phi(1, x) = O(\delta^{2-\varepsilon_0-|\beta|})$ and further $(\partial_t + \partial_x)^k \phi(1, x) = O(\delta^{2-\varepsilon_0-k})$ hold from (1.1), which means the over-determination of (1.6) for arbitrary choice of $(\phi_0, \phi_1)$ in (1.5). Thus the choice of $(\phi_0, \phi_1)$ will be somewhat restricted (see Appendix below).

The main result in the paper is

**Theorem 1.1.** Under the condition (1.6), when $p > p_c = \frac{1}{1-\varepsilon_0}$, for small $\delta > 0$, the equation (1.1) with (1.5) admits a global smooth solution

$$\phi \in C^\infty \left( [1, +\infty) \times \mathbb{R}^3 \right) \text{ with } |\partial \phi| \leq C\delta^{1-\varepsilon_0} t^{-1},$$

where $C > 0$ is a uniform constant independent of $\delta$ and $\varepsilon_0$.

**Remark 1.1.** In our subsequent companion paper [11], when $1 \leq p \leq p_c$, we will show that the smooth solution $\phi$ of (1.1) with (1.5) will blow up and further form the outgoing shock before the time $t = 2$ under the following assumption of $(\phi_0, \phi_1)$: there exists a point $(s_0, \omega_0) \in (-1, 0) \times S^2$ such that

$$\phi_{p-1}^{-1}(s_0, \omega_0) \partial_s \phi_{1}(s_0, \omega_0) > \frac{2}{p} \quad \text{for } 1 \leq p < p_c$$

or

$$\phi_{p-1}^{-1}(s_0, \omega_0) \partial_s \phi_{1}(s_0, \omega_0) > \frac{(p-1)2^p}{p(2^{p-1}-1)} \quad \text{for } p = p_c.$$  

**Remark 1.2.** Due to the special forms of (1.5) and equation (1.1), and the restrictions of $k = 1, 2$, then (1.6) is completely equivalent to

$$(\partial_t + c\partial_x)^k \Omega^\alpha \partial^\beta \phi(1, x) = O(\delta^{2-\varepsilon_0-|\beta|}), \quad 0 \leq k \leq 2, \quad (1.8)$$

where $\Omega \in \{ x^i \partial_j - x^j \partial_i : 1 \leq i < j \leq 3 \}$ stands for the derivatives on $S^2$ and $c = (1 + (\partial_t \phi)^p)^{-\frac{1}{p}}$ is the wave speed.
Remark 1.3. In [12], the authors study the Cauchy problem of the following 3-D semilinear wave equation systems with the short pulse initial data

\[
\begin{aligned}
\square \varphi^I &= \sum_{0 \leq \alpha, \beta \leq 2;\, 1 \leq I, K \leq N} A_{IJK}^{\alpha\beta,I} \partial_\alpha \varphi^J \partial_\beta \varphi^K, \quad I = 1, \ldots, N, \\
(\varphi^I, \partial_t \varphi^I)(1, x) &= (\delta^{\frac{3}{2}} \varphi_0^{(r-1)}(r-1), \omega), \delta^{-\frac{1}{2}} \varphi_1^{(r-1)}(r-1), \omega)),
\end{aligned}
\]

where \(A_{IJK}^{\alpha\beta,I}\) are constants, \(\partial_0 = \partial_v\), \((\varphi_0(s, \omega), \varphi_1(s, \omega)) \in C^\infty_0 \left( (-1, 0) \times S^2 \right)\), the quadratic nonlinear forms satisfy the null conditions

\[
\sum_{0 \leq \alpha, \beta \leq 3} A_{IJK}^{\alpha\beta,I} \xi_\alpha \xi_\beta \equiv 0 \quad \text{for} \quad \xi_0 = -1, \quad \text{any} \quad (\xi_1, \xi_2, \xi_3) \in S^2 \quad \text{and} \quad 1 \leq I, J, K \leq N. \quad \text{Moreover, it is assumed that}
\]

\[
|\varphi_t + \partial_\nu (\varphi^I)^\nu k \Omega^\alpha \partial_\alpha \varphi^I(1, x)| \leq C_{k_o q}^{\nu^1/2-|q|}, \quad k \leq N_0,
\]

where \(C_{k_o q}\) are constants, and \(N_0 \geq 40\) is a sufficiently large integer. Then the global existence of smooth solution \(\varphi = (\varphi^1, \ldots, \varphi^N)\) to (1.9) is established in [12]. Due to the largeness of the integer \(N_0\) in (1.10), it follows from the proof of [12] that (1.9) essentially becomes the small value solution problem inside the cone \(\{ r \leq t - \delta \} \). Recently, by the similar idea of [12], the authors in [15] proved the global existence of the relativistic wave equation

\[
\partial_t \left( \frac{\partial_t \varphi}{\sqrt{1 - (\partial_t \varphi)^2 + |\nabla \varphi|^2}} \right) - \sum_{i=1}^{n} \partial_i \left( \frac{\partial_i \varphi}{\sqrt{1 - (\partial_t \varphi)^2 + |\nabla \varphi|^2}} \right) = 0 \quad (n = 2, 3)
\]

with the short pulse initial data \((\delta^{\frac{3}{2}} \varphi_0^{(r-1)}(r-1), \delta^{\frac{1}{2}} \varphi_1^{(r-1)}(r-1), \omega))\) and the constraint condition \((\partial_t + \partial_\nu)^k \Omega^\alpha \partial_\alpha \varphi(1, x) = O(\delta^{\nu^1/2-|q|})\) for \(k \leq N_0\) \((N_0\) is large enough). By the largeness of \(N_0\) in (1.10) and the methods for treating the semilinear wave equation (one can choose \(u = t - r\) as the optical function), the authors in [12] and [15] can show the global existence of smooth solutions by the energy method. In this paper, we need to treat the large value of \(\phi\) in the whole time-space \(\mathbb{R}_+ \times \mathbb{R}^3\) because of \(k = 1, 2\) in (1.6), and the methods in [12] or [15] seem not to be available for us.

Remark 1.4. For \(k = 3\) or larger number \(k\), (1.6) with the power \(\delta^{2-\varepsilon_0}\) is seriously over-determined and difficult to be realized for the choice of \((\phi_0, \phi_1)\). In Appendix, we will chose \((\phi_0, \phi_1)\) such that (1.6) holds for \(k = 1, 2\).

We now comment on the proof of Theorem 1.1. For the general 4D quasilinear wave equation

\[
\sum_{\alpha, \beta = 0}^{4} g^{\alpha\beta} (\varphi, \partial \varphi) \partial_\alpha \partial_\beta \varphi = 0
\]

and the 2D quasilinear wave equation

\[
\sum_{\alpha, \beta = 0}^{2} g^{\alpha\beta} (\varphi, \partial \varphi) \partial_\alpha \partial_\beta \varphi = 0
\]

with the short pulse initial data \((\delta^{2-\varepsilon_0} \varphi_0^{(r-1)}(r-1), \omega), \delta^{1-\varepsilon_0} \varphi_1^{(r-1)}(r-1), \omega))\), under the corresponding null conditions the authors in [6]-[7] have established the global existence of the smooth solutions \(\varphi\) for the suitable scope of \(\varepsilon_0\) with \(0 < \varepsilon_0 < \varepsilon^0\) and \(\varepsilon^0 < 1\) under the assumptions like (1.6). As in [6]-[7], strongly motivated by the geometric methods of D. Christodoulou [3], we will construct the solution \(\phi\) of (1.1) near the outermost outgoing conic surface \(C_0 = \{(t, x) : t \geq 1 + 2\delta, t = \tau\}\). Introduce the inverse foliation density

\[
\mu = -\frac{1}{(1 + (\partial_t \phi)^p)^r},
\]

where the optical function \(u\) satisfies \(-(1 + (\partial_t \phi)^p)^{\partial_t u \frac{2}{2} + \sum_{i=1}^{3} (\partial_i u)^2 = 0\) with the initial data \(u(1 + 2\delta, x) = 1 + 2\delta - r\). Under the bootstrap assumptions on \(\partial \varphi\) with the suitable time-decay rates and the precise smallness powers of \(\delta\), then \(\mu\) satisfies the equation \(L_\mu = O(\delta^{1-\varepsilon_0})\)
where $L$ is a vectorfield approximating $\partial_t + \partial_r$. By $\mu(1 + 2\delta, x) \sim 1$ and integration along the integral curves of $L$, $\mu \sim 1$ is derived for small $\delta > 0$. The positivity of $\mu$ tells us that the outgoing characteristic conic surfaces never intersect as long as the smooth solution $\phi$ with suitable time-decay rate exists. From this, the global weighted energy estimates of $\phi$ near $C_0$ can be derived and further the bootstrap assumptions are closed. In addition, we can establish that the outgoing characteristic conic surface of equation (1.1) starting from the domain $\{ t = 1 + 2\delta, 1 - 2\delta \leq r \leq 1 + \delta \}$ are almost straight and further contain the surface $\tilde{C}_{2\delta} = \{(t, x) : t \geq 1 + 2\delta, t - r = 2\delta \}$. On the other hand, the crucial estimate $|\partial^\alpha \phi| \lesssim \delta^{2-\varepsilon_0} t^{-1}$ on $\tilde{C}_{2\delta}$ is obtained, which contains the higher order smallness factor $\delta^{2-\varepsilon_0}$ rather than $\delta^{2-\varepsilon_0-|\alpha|}$. Based on such “good” smallness of $\phi$ on $\tilde{C}_{2\delta}$, we can solve the global Goursat problem of (1.1) in the conic domain $B_{2\delta} = \{(t, x) : t \geq 1 + 2\delta, t - r \geq 2\delta \}$. Therefore, the proof of Theorem 1.1 is completed. It is emphasized that compared with [6]-[7], except the differences of space dimensions and time-decay rate of solution $\phi$, one of our main ingredients is to obtain the optimal power estimates of $\delta$ for all the related quantities so that the critical exponent $p_c$ of equation (1.1) can be determined.

Our paper is organized as follows. In Section 2, at first, we state the local existence of the solution $\phi$ to equation (1.1) with (1.5)-(1.6) for $1 \leq t \leq 1 + 2\delta$ and some key estimates of $\phi(1 + 2\delta, x)$. In Subsection 2.2, we give the preliminary knowledge on the Lorentzian geometry, especially, recall the definitions of optical function, inverse foliation density $\mu$, deformation tensor, null frame and some norms of smooth functions. In addition, the equation of $\mu$ is derived, and some basic calculations are given for the covariant derivatives of the null frame and for the deformation tensors. In Section 3, the basic bootstrap assumptions near $C_0$ are given, meanwhile, we also give some estimates on several quantities which will be extensively used in subsequent sections. In Section 4, the global energy estimates for the linearized covariant wave equation $\mu \Box_g \Psi = \Phi$ are derived and some higher order weighted energies and fluxes are defined. In Section 5, under the bootstrap assumptions, the higher order $L^\infty$ and $L^2$ estimates of $\partial \phi$ near $C_0$ are established. In Section 6, the top order $L^2$ estimates for the derivatives of $\chi$ and $\mu$ are established, where $\chi$ is the second fundamental form of the related metric $g$. In Section 7, we first derive the commuted wave equation and then treat the estimates for the resulting error terms. In Section 8, based on all the estimates in the previous sections, the bootstrap arguments are closed and further the global existence of solution $\phi$ to equation (1.1) near $C_0$ is established. On the other hand, the global existence of solution $\phi$ in $B_{2\delta}$ is obtained and then Theorem 1.1 is shown. In Appendix, we prove the existence of $(\phi_0, \phi_1)$ such that the condition (1.6) is satisfied.

Throughout the whole paper, without special mentions, the following notations are used:

- Greek letters $\{\alpha, \beta, \gamma, \cdots\}$ corresponding to the spacetime coordinates are chosen in $\{0,1,2,3\}$;
- Latin letters $\{i, j, k, \cdots\}$ corresponding to the spatial coordinates are chosen in $\{1,2,3\}$; Capital letters $\{A, B, C, \cdots\}$ corresponding to the sphere coordinates are chosen in $\{1,2\}$.
- We use the Einstein summation convention to sum over the repeated upper and lower indices.
- The convention $f \lesssim h$ means that there exists a generic positive constant $C$ independent of the parameter $\delta > 0$ and the variables $(t, x)$ such that $f \leq C h$.
- If $\xi$ is a $(0,2)$-type spacetime tensor, $\Lambda$ is a one-form, $U$ and $V$ are vectorfields, then the contraction of $\xi$ with respect to $U$ and $V$ is defined as $\xi_{UV} = \xi_{\alpha\beta} U^\alpha V^\beta$, and the contraction of $\Lambda$ with respect to $U$ is defined as $\Lambda_{UV} = \Lambda_\alpha U^\alpha$.
- The restriction of quantity $\zeta$ (including the metric $g$, $(m,n)$-type spacetime tensor field) on the sphere is represented by $\zeta$. But if $\zeta$ is already defined on the sphere, it is still represented by $\zeta$.
- $\mathcal{L}_V \xi$ stands for the Lie derivative of $\xi$ with respect to vector $V$, and $\mathcal{L}_V \xi$ is the restriction of $\mathcal{L}_V \xi$ on the sphere.
Finally, such notations are introduced:

\[ C_0 = \{(t, x) : t \geq 1 + 2\delta, t = r\}, \]

\[ B_{2\delta} = \{(t, x) : t \geq 1 + \delta, t - r \geq 2\delta\}, \]

\[ \tilde{C}_{2\delta} = \{(t, x) : t \geq 1 + 2\delta, t - r = 2\delta\}, \]

\[ c = (1 + (\partial_t \phi)^p)^{-\frac{1}{2}}, \]

\[ t_0 = 1 + 2\delta, \]

\[ L = \partial_t + \partial_r, \]

\[ \tilde{L} = \partial_t - \partial_r, \]

\[ \Omega_i = \epsilon_{ij}^k x^j \partial_k, \]

\[ \Omega \in \{\Omega_i : 1 \leq i \leq 3\}, \]

\[ S = t \partial_t + r \partial_r = \frac{t - r}{2} \tilde{L} + \frac{t + r}{2} L, \]

\[ H_i = t \partial_i + x^j \partial_i = \omega^i \left( \frac{r - t}{2} \tilde{L} + \frac{t + r}{2} L \right) + \frac{t \omega^j}{r} \epsilon_{ij}^k \Omega_k, \]

\[ \Sigma_t = \{(t', x) : t' = t, x \in \mathbb{R}^3\}, \]

where \( \epsilon_{ij}^k = \epsilon_{ijk} = -1 \) when \( ijk \) is 123’s odd permutation, and \( \epsilon_{ij}^k = \epsilon_{ijk} = 1 \) when \( ijk \) is 123’s even permutation.

## 2 Some preliminaries

### 2.1 Local existence

In this subsection, for the equation (1.1) with (1.5)-(1.6), we list the local existence of the smooth solution \( \phi \) and some crucial properties for \( 1 \leq t \leq t_0 \). Since the proof is rather analogous to that of Theorem 2.1 in [6] (for the general 4D quasilinear wave equation with the first null condition) or that of Theorem 2.1 in [7] (for the 2D quasilinear wave equations with the first and second null conditions), we omit the details here.

**Theorem 2.1.** For sufficiently small \( \delta > 0 \), the equation (1.1) with (1.5)-(1.6) admits a local smooth solution \( \phi \in C^\infty([1, t_0] \times \mathbb{R}^3) \). Moreover, for \( q \in \mathbb{N}_0^4 \), \( \kappa \in \mathbb{N}_0^3 \), \( k \in \mathbb{N}_0 \) and \( l \in \mathbb{N}_0 \), it holds that

1. \[
|\tilde{L}^k \partial^\kappa \phi(t_0, x)| \lesssim \delta^{2 - |q| - \epsilon_0}, \quad r \in [1 - 2\delta, 1 + 2\delta],
\]

2. \[
|\tilde{L}^l \partial^\kappa \phi(t_0, x)| \lesssim \delta^{2 - |q| - \epsilon_0}, \quad r \in [1 - 3\delta, 1 + \delta].
\]

3. \[
|\partial^\kappa \phi(t_0, x)| \lesssim \begin{cases} 
\delta^{2 - \epsilon_0}, & \text{as } |q| \leq 2,
\delta^4 - |q| - \epsilon_0, & \text{as } |q| > 2, 
\end{cases} \quad r \in [1 - 3\delta, 1 + \delta].
\]

4. \[
|\tilde{L}^{k+l} \tilde{L}^l \partial^\kappa \phi(t_0, x)| \lesssim \delta^{2 - \epsilon_0}, \quad r \in [1 - 2\delta, 1 + \delta].
\]
2.2 The Lorentzian geometry and some related definitions

In this subsection, we give some preliminaries on the related Lorentzian geometry and definitions, which will be utilized as the basic tools later on.

2.2.1. Metric and Christoffel symbols

By the form of (1.1), it is natural to introduce the following inverse spacetime metric

\[ g^{-1} = (g^{\alpha\beta}) = \text{diag}(-\frac{1}{c^2}, 1, 1, 1) \]  

(2.1)

and the corresponding spacetime metric

\[ g = (g_{\alpha\beta}) = \text{diag}(-c^2, 1, 1, 1). \]  

(2.2)

In this case, (1.1) can be rewritten as

\[ -\frac{1}{c^2} \partial_t^2 \phi + \Delta \phi = 0 \]  

(2.3)

or

\[ g^{\alpha\beta} \partial_\alpha \partial_\beta \phi = 0. \]  

(2.4)

In the Cartesian coordinates, the Christoffel symbols of \( g \) are defined by

\[ \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\alpha\gamma}) \]  

(2.5)

\[ \Gamma^\gamma_{\alpha\beta} = g^{\gamma\lambda} \Gamma_{\alpha\lambda\beta}. \]  

(2.6)

Meanwhile, set

\[ \Gamma^\gamma = g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta}. \]  

(2.7)

Definition 2.1. Define \( G \) function as

\[ G^\gamma_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial \varphi_\gamma}, \]

where and below \( \varphi_\gamma = \partial_\gamma \phi \).

Due to \( g_{00} = -c^2 = -(1 + \varphi^p_0)^{-1} \), then the only non-vanishing component of the \( G \) function is

\[ G^0_{00} = pc^4 \varphi^p_0. \]

2.2.2. Optical function, inverse foliation density and null frames

As in [3], one can introduce the following optical function.

Definition 2.2. A \( C^1 \) function \( u(t, x) \) is called the optical function of problem (2.4) if \( u(t, x) \) satisfies the eikonal equation

\[ g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0. \]  

(2.8)

In the paper, we will choose the initial data \( u(t_0, x) = u(1 + 2\delta, x) = 1 + 2\delta - r \) and pose the condition \( \partial_t u > 0 \) for (2.8). According to the definition of optical function, define the inverse foliation density \( \mu \) as in [3],

\[ \mu = -\frac{1}{g^{\alpha\beta} \partial_\alpha u \partial_\beta u} = \frac{1}{c^{-2} \partial_t u}. \]  

(2.9)
By (2.8) and (2.1), then 
\[ -c^{-2}(\partial_t u)^2 + \sum_{i=1}^{3} (\partial_i u)^2 = 0 \]
holds. Since \( \partial_t u = -\partial_r r = -\frac{4}{r} \) on \( t = t_0 \), one then has \( \partial_t u|_{t_0} = c|_{t_0} \) and
\[ \mu|_{t_0} = c|_{t_0}. \]  
(2.10)

Note that the authors in [3], [13] and [14] apply the inverse foliation density to prove the formation of shocks when \( \mu \to 0^+ \) holds with the development of time \( t \). In the paper, on the contrary, we will show \( \mu \geq C > 0 \) as long as the smooth solution \( \phi \) of equation (1.1) exists as in [6]-[?].

Note that 
\[ \dot{L} = -\text{grad} u = -g^{\alpha\beta} \partial_\alpha u \partial_\beta \]  
(2.11)
is a tangent vector field for the outgoing light cone \( \{ u = C \} \). In addition, it is easy to know that \( \dot{L} \) is geodesic and \( \dot{L} t = \mu^{-1} \). Thus, we rescale \( \dot{L} \) as
\[ L = \mu \dot{L}. \]  
(2.12)

To obtain the vector field of the incoming light cone, just as in [13], let
\[ \dot{T} = c^{-1}(\partial_t - L). \]  
(2.13)

Then by the definition of null frames, we set
\[ T = c^{-1} \mu \dot{T}, \]  
(2.14)
\[ \mathcal{L} = c^{-2} \mu L + 2T, \]  
(2.15)
where \( L \) and \( \mathcal{L} \) are two vector fields in the null frame. About the other vector fields \( \{ X_1, X_2 \} \) in the null frame, we take use of the vector field \( L \) to construct them. To this end, one extends the local coordinates \( \{ \theta^1, \theta^2 \} \) on \( S^2 \) as follows
\[ L \theta^A = 0, \ |_{t=1}^{\theta^A} = \theta^A, \]
here and below \( A = 1, 2 \). Subsequently, let
\[ X_1 = \frac{\partial}{\partial \theta^1}, \ \ X_2 = \frac{\partial}{\partial \theta^2}. \]  
(2.16)

A direct computation yields

**Lemma 2.1.** \( \{ L, \mathcal{L}, X_1, X_2 \} \) constitutes a null frame with respect to the metric \( g \) in (2.2), and admits the following identities
\[ g(L, L) = g(\mathcal{L}, \mathcal{L}) = g(L, X_A) = g(\mathcal{L}, X_A) = 0, \]  
\[ g(L, L) = -2\mu. \]

In addition,
\[ g(L, T) = -\mu, \ g(T, T) = c^{-2} \mu^2. \]

And
\[ Lt = 1, \ Lu = 0, \ Tt = 0, \ Tu = 1, \ \mathcal{L} t = c^{-2} \mu, \ \mathcal{L} u = 2. \]

**2.2.3. Domains, coordinates and norms**

As in [14], one can perform the change of coordinates: \( (t, x^1, x^2, x^3) \rightarrow (t, u, \theta^1, \theta^2) \) near \( C_0 \) with
\[
\begin{align*}
  t &= t, \\
  u &= u(t, x), \\
  \theta^1 &= \theta^1(t, x), \\
  \theta^2 &= \theta^2(t, x).
\end{align*}
\]  
(2.17)

For notational convenience, we introduce the following subsets
Definition 2.3.

\[ \Sigma^u_t = \{ (t', u', \vartheta) : t' = t, 0 \leq u' < u \}, u \in [0, 4\delta], \]

\[ C^u_u = \{ (t', u', \vartheta) : t' \geq t_0, u' = u \}, \]

\[ C^u_t = \{ (t', u', \vartheta) : t_0 \leq t' < t, u' = u \}, \]

\[ S_{t,u} = \Sigma_t \cap C_{t,u}, \]

\[ D^{t,u} = \{ (t', u', \vartheta) : t_0 \leq t' < t, 0 \leq u' < u \}. \]

Note that \( \vartheta = (\vartheta^1, \vartheta^2) \) are the coordinates on sphere \( S_{t,u} \). Then under the new coordinate system \((t, u, \vartheta^1, \vartheta^2)\), one has \( L = \frac{\partial}{\partial t}, T = \frac{\partial}{\partial u} - \Xi \) with \( \Xi = \Xi^A X_A \). In addition, it follows from direct computation that

Lemma 2.2. In domain \( D^{t,u} \), the Jacobian determinant of map \((t, u, \vartheta^1, \vartheta^2) \rightarrow (x^0, x^1, x^2, x^3)\) is

\[ \det \frac{\partial (x^0, x^1, x^2, x^3)}{\partial (t, u, \vartheta^1, \vartheta^2)} = c^{-1} \mu \sqrt{\det \varrho}. \]

Remark 2.1. From Lemma 2.2, it is easy to know that if the metric \( \varrho \) are locally regular, that is, \( \det \varrho > 0 \), the transformation of coordinates between \((t, u, \vartheta^1, \vartheta^2)\) and \((x^0, x^1, x^2, x^3)\) then makes sense as long as \( \mu > 0 \).

For the domains with \( \mu > 0 \), we now give some definitions of related integrations and norms, which will be utilized repeatedly in subsequent sections.

Definition 2.4. For any continuous function \( f \), set

\[ \int_{S_{t,u}} f = \int_{S_{t,u}} f(t, u, \vartheta) \sqrt{\det \varrho(t, u, \vartheta)} d\vartheta, \quad \|f\|_{L^2(S_{t,u})}^2 = \int_{S_{t,u}} |f|^2, \]

\[ \int_{C^u_t} f = \int_{t_0}^t \int_{S_{t,u}} f(t, u, \vartheta) \sqrt{\det \varrho(t, u, \vartheta)} d\vartheta dt, \quad \|f\|_{L^2(C^u_t)}^2 = \int_{C^u_t} |f|^2, \]

\[ \int_{\Sigma^u_t} f = \int_{0}^u \int_{S_{t,u}} f(t, u', \vartheta) \sqrt{\det \varrho(t, u', \vartheta)} d\vartheta du', \quad \|f\|_{L^2(S^u_t)}^2 = \int_{\Sigma^u_t} |f|^2, \]

\[ \int_{D^{t,u}} f = \int_{t_0}^t \int_{0}^u \int_{S_{t,u}} f(t, u', \vartheta) \sqrt{\det \varrho(t, u', \vartheta)} d\vartheta du' dt, \quad \|f\|_{L^2(D^{t,u})}^2 = \int_{D^{t,u}} |f|^2. \]

2.2.4. Connection, the second fundamental form and torsion form

Let \( \nabla \) be the Levi-Civita connection of \( g \). Without causing confusion, we use \( \nabla \) to denote the Levi-Civita connection of \( \varrho \).

Under the null frame \( \{ L, \sigma, X_1, X_2 \} \), define the second fundamental forms \( \chi \) and \( \sigma \) as

\[ \chi_{AB} = g(\nabla_A L, X_B), \quad \sigma_{AB} = g(\nabla_A T, X_B). \quad (2.18) \]

And the torsion one forms \( \zeta \) and \( \eta \) are defined by

\[ \zeta_A = g(\nabla_A L, T), \quad \eta_A = -g(\nabla_A T, L). \quad (2.19) \]

Direct computation yields

\[ \sigma_{AB} = -c^{-1} \chi_{AB}, \quad (2.20) \]

\[ \zeta_A = -c^{-1} \mu d_A c, \quad (2.21) \]

\[ \eta_A = -c^{-1} \mu d_A c + d_A \mu. \quad (2.22) \]
Lemma 2.3. For the connection coefficients of the related frames, it holds that
\[ D_L L = \mu^{-1} L \mu L, \quad D_T L = \eta^A X_A - c^{-1} L(c^{-1} \mu) L, \quad D_A L = -\mu^{-1} \zeta_A L + \chi_A^B X_B, \]
\[ D_L T = -\zeta^A X_A - c^{-1} L(c^{-1} \mu) L, \quad D_A T = -\eta_A T - c^{-2} \mu \chi_A^B X_B, \]
\[ D_T T = c^{-3} \mu [T c + L(c^{-1} \mu)] L + \{c^{-1} [T c + L(c^{-1} \mu)] + T \ln(c^{-1} \mu)\} T - c^{-1} \mu \delta^A(c^{-1} \mu) X_A, \]
\[ D_L X_A = D_A L, \quad D_T X_A = D_A T, \quad \nabla_A X_B = \nabla_A^T X_B + \mu^{-1} \chi A B T, \quad D_A L = -\eta_A L - c^{-2} \mu \chi A B X_B, \]
\[ D_A L = -L(c^{-2} \mu) L + 2\eta A X_A, \quad D_T L = -2\zeta^A X_A, \quad D_A T = [\mu^{-1} L + L(c^{-2} \mu)] L - 2\mu \delta^A(c^{-2} \mu) X_A, \]
where \( \zeta^A = g^A B \zeta_B \) and \( \eta^A = g^A B \eta_B \).

2.2.5. Error vectors and rotation vectors

On the initial hypersurface \( \Sigma^R \), one has that \( \tilde{T}^i = -\frac{x^i}{\rho}, \quad L^0 = 1, \quad L^i = \frac{x^i}{\rho} + O(\delta(1-\epsilon_0) \rho) \) and \( \chi A B = \frac{1}{\rho} g A B \). Note that on \( \Sigma^R \), \( r \) is just \( t_0 - u \). For \( t \geq t_0 \), we define the “error vectors” with the components being

Definition 2.5.

\[ \tilde{L}^0 = 0, \]
\[ \tilde{L}^i = L^i - \frac{x^i}{\rho}, \]
\[ \tilde{T}^i = \tilde{T}^i + \frac{x^i}{\rho}, \]
\[ \tilde{\chi} A B = \chi A B - \frac{1}{\rho} g A B, \]

here and below \( \rho = t - u \).

Lemma 2.4. The error vectors satisfy
\[ \tilde{L}^i = -c \tilde{T}^i + (c - 1) \rho^{-1} x^i, \]
\[ \text{tr} \tilde{\chi} = \text{tr} \chi - 2 \rho^{-1}, \]
\[ |\tilde{\chi}|^2 = |\chi|^2 - 2 \rho^{-1} \text{tr} \chi + 2 \rho^{-2}. \]

Let
\[ v_i = g(\Omega_i, \tilde{T}) = \epsilon_{ijk} x^j \tilde{T}^k. \]

Then
\[ R_i = \Omega_i - v_i \tilde{T} \]
are the rotation vector fields of \( S_{t,u} \).

Remark 2.2. The components of \( g \) satisfy
\[ g^{\mu \nu} = g^{\mu \nu} + \frac{1}{2} \mu^{-1} (L^\mu L^\nu + L^\mu L^\nu), \]
\[ g^{\mu \nu} = g A B X_A^\mu X_B^\nu, \]
and for any smooth function \( \Psi \),
\[ g^{\mu \nu} \partial_\mu \Psi \partial_\nu \Psi = |\phi \Psi|^2. \]
2.2.6. Curvature tensor, energy-momentum tensor and deformation tensor

The Riemann curvature tensor $\mathcal{R}$ of $g$ can be defined as follows

$$\mathcal{R}_{WXYZ} = -g(\mathcal{R}_W Y - \mathcal{R}_X Y - \mathcal{R}_{[W,X]} Y, Z).$$  \hspace{1cm} (2.25)$$

Since

$$\mathcal{R}_{\mu\nu\alpha\beta} = \partial_\nu \Gamma_{\mu\beta\alpha} - \partial_\mu \Gamma_{\nu\beta\alpha} + g^{\kappa\lambda} (\Gamma_{\nu\kappa\alpha} \Gamma_{\mu\lambda\beta} - \Gamma_{\nu\kappa\alpha} \Gamma_{\mu\lambda\beta}),$$

by the definitions (2.25) and (2.5), the only non-vanishing components of $\mathcal{R}$ for the metric $g$ in (2.2) are

$$\mathcal{R}_{i0j0} = \mathcal{R}_{0i0j} = c \partial_i \partial_j c,$$  \hspace{1cm} (2.26)$$

and the only non-vanishing component under the frame $\{L, T, X_1, X_2\}$ is

$$\mathcal{R}_{LALB} = -\mu^{-1} c(Tc) \chi_{AB} - \mu^{-1} \rho^{-1} c(Tc) \phi_{AB}$$

$$- \frac{1}{2} pc^4 \rho^{-1} \nabla^2 \chi_{AB} \phi_0 - \frac{3}{2} pc^3 \rho^{-1} \phi_{AB} \phi_0 - \frac{1}{2} p(p - 1) c^4 \rho^{-2} \phi_{AB} \phi_0.$$  \hspace{1cm} (2.27)$$

For the sake of convenience, define

$$\mathcal{R}_{LALB} = -\mu^{-1} c(Tc) \chi_{AB} - \mu^{-1} \rho^{-1} c(Tc) \phi_{AB},$$  \hspace{1cm} (2.28)$$

Here we point out that $\mathcal{R}_{LALB}$ will admit the better time-decay rate and higher smallness orders of $\delta$ than $\mathcal{R}_{LALB}$.

For any smooth function $\Psi$, one can denote its associate energy-momentum tensor by

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\Psi] = (\partial_\alpha \Psi)(\partial_\beta \Psi) - \frac{1}{2} g_{\alpha\beta} g^{\kappa\lambda}(\partial_\kappa \Psi)(\partial_\lambda \Psi).$$  \hspace{1cm} (2.29)$$

The components of energy-momentum tensor in terms of the null frame can be computed as follows

$$Q_{LL} = (L \Psi)^2, \quad Q_{LL} = (L \Psi)^2, \quad Q_{LL} = \mu |\Psi|^2,$$

$$Q_{LA} = L \Psi \phi_A \Psi, \quad Q_{LA} = L \Psi \phi_A \Psi,$$

$$Q_{AB} = \phi_A \Psi \phi_B \Psi - \frac{1}{2} g_{AB} |\Psi|^2 - \mu^{-1} L \Psi \phi_A \Psi.$$  \hspace{1cm} (2.30)$$

For any vector field $V$, denote its associate deformation tensor by

$$(V) \pi_{\alpha\beta} = g(\mathcal{R}_\alpha V, \partial_\beta) + g(\mathcal{R}_\beta V, \partial_\alpha).$$  \hspace{1cm} (2.31)$$

Moreover, for any two vector fields $X, Y$, one has

$$(V) \pi_{XY} = (V) \pi_{\alpha\beta} X^\alpha Y^\beta = g(\mathcal{R}_X V, Y) + g(\mathcal{R}_Y V, X).$$

The components of $(V) \pi$ under the related frames and the metric $g$ in (2.2) can be obtained as follows

(1) When $V = L$,

$$(L) \pi_{LL} = 0, \quad (L) \pi_{LT} = -L \mu, \quad (L) \pi_{TT} = 2c^{-1} \mu L (c^{-1} \mu),$$

$$(L) \phi_{LA} = 0, \quad (L) \phi_{TA} = c^2 \phi_A \phi_0, \quad (L) \phi_{AB} = 2 \chi_{AB},$$

$$(L) \pi_{LL} = -2 L \mu, \quad (L) \pi_{LL} = 4 \mu L (c^{-2} \mu), \quad (L) \phi_{LA} = 2 c^2 \phi_A \phi_0.$$  \hspace{1cm} (2.32)$$
(2) When \( V = T \),

\[
\begin{align*}
(T) \pi_{LL} &= 0, \quad (T) \pi_{LT} = -T \mu, \quad (T) \pi_{TT} = T(e^{-2} \mu^2), \\
(T) \hat{\pi}_{LA} &= -e^2 \hat{\delta}_A\mu^2, \quad (T) \hat{\pi}_{TA} = 0, \quad (T) \hat{\pi}_{AB} = -2e^{-2} \mu \chi_{AB}, \\
(T) \pi_{LL} &= -2T \mu, \quad (T) \pi_{LL} = 4\mu T(e^{-2} \mu^2), \quad (T) \hat{\pi}_{LA} = -\mu \hat{\delta}_A(e^{-2} \mu).
\end{align*}
\]  

(23.3)

(3) When \( V = R_i \),

\[
\begin{align*}
(R_i) \pi_{LL} &= 0, \quad (R_i) \pi_{LT} = -R_i \mu, \quad (R_i) \pi_{TT} = 2e^{-1} \mu R_i(c^{-1} \mu), \\
(R_i) \hat{\pi}_{LA} &= -\chi_{AB} R_i^B + \epsilon_{ijk} \tilde{T}^j A x^k - v_i \hat{\delta}_A, \\
(R_i) \hat{\pi}_{TA} &= c^{-2} \chi_{AB} R_i^B - c^{-2} (c-1) \mu \chi^{-1} \hat{\delta}_{AB} R_i^B + c^{-1} \mu \epsilon_{ijk} \tilde{T}^j A x^k + v_i \hat{\delta}_A(c^{-1} \mu), \\
(R_i) \hat{\pi}_{AB} &= 2e^{-1} v_i \chi_{AB}, \quad (R_i) \pi_{LL} = -2R_i \mu, \quad (R_i) \pi_{LL} = 4\mu R_i(c^{-2} \mu).
\end{align*}
\]  

(23.4)

2.2.7. Lie derivatives and commutators

According to (8.26) in [14], one has

**Lemma 2.5.** For any symmetric 2-tensor \( \xi \) on \( S_{L,w} \),

\[
[[\hat{\nabla} A, \hat{\nabla} Z] \xi]_{BC} = (\hat{\nabla} A \hat{\delta}^Z B) \hat{\nabla} C \xi + (\hat{\nabla} A \hat{\delta}^Z C) \hat{\nabla} B \xi,
\]  

(2.35)

where

\[
\hat{\nabla} A \hat{\delta}^Z B = \frac{1}{2} (\hat{\nabla} A \hat{\delta}^Z B + \hat{\nabla} B \hat{\delta}^Z A - \hat{\nabla} C \hat{\delta}^Z A). \]

For any vector field \( Z \in \{L, \rho L, T, R_1, R_2, R_3\} \) and smooth function \( f \),

\[
[[\hat{\nabla}^2, \hat{\nabla} Z] f]_{AB} = (\hat{\nabla} A \hat{\delta}^Z B) \hat{\nabla} C f
\]  

(2.36)

and

\[
[\Delta, Z] f = (\hat{\nabla} A \hat{\delta}^Z B) \hat{\nabla}^2 B f + (\hat{\nabla} A \hat{\delta}^Z B) \hat{\nabla} B f,
\]

moreover, for any vector field \( Z \in \{L, T, R_1, R_2, R_3\} \),

\[
\hat{\nabla} Z \hat{\delta}_{AB} = (\hat{\nabla} A \hat{\delta}^Z B) \hat{\nabla} B f.
\]

2.2.8. Covariant wave equations and structure equations

We now look for the equation of \( \varphi \) (\( \lambda = 0, 1, 2, 3 \)) under the action of the covariant wave operator \( \Box_g = g^{\alpha\beta} \partial_{\alpha\beta} \) with the help of metric and Christoffel symbol. It follows from direct computation that

\[
\Box_g \varphi \lambda = g^{\alpha\beta} \partial_{\alpha\beta} \varphi \lambda - g^{\alpha\beta} \Gamma_{\alpha\beta} \partial_{\gamma} \varphi \lambda
\]

\[
= (-\frac{1}{c^2} \partial_t^2 \varphi \lambda + \Delta \varphi \lambda) - \Gamma_{\gamma} \partial_{\gamma} \varphi \lambda,
\]  

(2.37)

In addition, taking the derivative on two sides of (1.1) with respect to the variable \( x^\lambda \) derives

\[
-\frac{1}{c^2} \partial_t^2 \varphi \lambda + \Delta \varphi \lambda = p \varphi_0 \partial_t \varphi \lambda.
\]  

(2.38)
By $\partial_t = L + c^2 \mu^{-1}T$ and $\partial_t = c^2 \mu^{-2} T^i T + d^A x^i X_A$, then it follows from (2.37) and (2.38) that

$$\mu \Box_g \varphi_\lambda = F_\lambda,$$

(2.39)

where

$$F_\lambda = \frac{1}{2} p \mu \varphi_0^{p-1} L \varphi_0 L \varphi_\lambda + \frac{1}{2} \mu^2 \varphi_0^{p-1} T \varphi_\lambda + \frac{1}{2} \mu \varphi_0^{p-1} T \varphi_0 L \varphi_\lambda - \frac{1}{2} \mu^2 \varphi_0^{p-1} d \varphi_0 \cdot d \varphi_\lambda.$$  \hspace{1cm} (2.40)

On the other hand, due to

$$\mu \Box_g \varphi_\lambda = \mu g^\alpha \beta \mathcal{D}_\alpha \beta \varphi_\lambda = -(L + \frac{1}{2} \text{tr}_g \chi)L \varphi_\lambda + 2c^{-1} \mu d \varphi_\lambda + \mu \Delta \varphi_\lambda + \frac{1}{2} c^{-2} \mu \text{tr}_g \chi L \varphi_\lambda,$$

this yields

$$(L + \frac{1}{2} \text{tr}_g \chi)L \varphi_\lambda = \mu \Delta \varphi_\lambda = H_\lambda = F_\lambda.$$  \hspace{1cm} (2.41)

where

$$H_\lambda = \frac{1}{2} c^{-2} \mu \text{tr}_g \chi L \varphi_\lambda + 2c^{-1} \mu d \varphi_\lambda.$$  \hspace{1cm} (2.42)

For convenience, define

$$\bar{H}_\lambda = 2c^{-1} \mu d \varphi_\lambda.$$  \hspace{1cm} (2.43)

Next, we give the structure equations of $\mu$, $\chi$, $L^i$ and $\bar{L}^i$.

**Lemma 2.6.** It holds that

$$L \mu = -c T \epsilon + (c^{-1} L \epsilon) \mu,$$

(2.44)

$$L X_{AB} = c^{-1} (L \epsilon) X_{AB} + \chi^C A \chi_{BC} - \mathcal{R}_{LALB},$$

(2.45)

$$\mathcal{L}_T X_{AB} = c^{-1} (L \epsilon) X_{AB} \chi_{AB} + \frac{1}{2} c^{-1} \mu_2 \sigma_{AC} \chi_{BC} + \sigma_{BC} \chi_{AB}$$

$$+ \frac{1}{2} (\mathcal{V}_{A} \epsilon \eta_{B} + \mathcal{V}_{B} \epsilon \eta_{A}) + \frac{1}{2} \mu^{-1} \zeta \eta_{B} \zeta \eta_{A},$$

(2.46)

$$(\text{div}_g \chi)_A = \mathcal{D}_A \text{tr}_g \chi = c^{-1} (d^B \chi_{AB} - c^{-1} (d^A \mu) \text{tr}_g \chi),$$

(2.47)

$$L L^i = c^{-1} (L \epsilon) \bar{L}^i + c^{-1} (L \epsilon) \rho^{-1} x^i - c \mu \varphi_0 \cdot d \varphi_\lambda,$$

(2.48)

$$T L^i = (c^{-1} \mu d \varphi_0 + d \mu \chi^C A \chi_{BC}) d^B x^i - c^{-1} L (c^{-1} \mu) L^i,$$

(2.49)

$$d_A L^i = \tilde{\chi}_{AB} d^B x^i.$$  \hspace{1cm} (2.50)

**Proof.** Since (2.44)-(2.47) are completely similar to (2.32)-(2.36) in [13], we omit the details here.

With the help of Lemma 2.3 and the fact $\partial_t = c^2 \mu^{-2} T^i T + d^A x^i X_A$, one has that by (2.14), Definition 2.5, Lemma 2.3 and (2.44),

$$L L^i = \mathcal{D}_L \bar{L}^i - L^\alpha \mathcal{D}_{\alpha \beta} \Gamma_{\alpha \beta}^i = \mu^{-1} L \mu L^i - c \partial_t \chi$$

$$= c^{-1} (L \epsilon) \bar{L}^i + c^{-1} (L \epsilon) \rho^{-1} x^i - c \mu \varphi_0 \cdot d \varphi_\lambda.$$  \hspace{1cm} (2.51)

Analogously, (2.49) can be obtained by $T L^i = \mathcal{D}_T \bar{L}^i - L^\alpha T^i \Gamma_{\alpha j}^i$ and Lemma 2.3.

Finally, by (4.10c) in [14] and Definition 2.1, we arrive at

$$d_A \bar{L}^i = g^{B C} \tilde{\chi}_{AB} d_C x^i + g^{B C} (-\frac{1}{2} g_{AB} L \varphi_\alpha + \frac{1}{2} g_{CA} d_B \varphi_\alpha - \frac{1}{2} g_{CB} d_A \varphi_\alpha) d_C x^i$$

$$+ (-G_{AB} d_A \varphi_\alpha - \frac{1}{2} g_{AB} d_A \varphi_\alpha) \bar{T}^i$$

$$= \tilde{\chi}_{AB} d^B x^i.$$  \hspace{1cm} (2.52)
3 Bootstrap assumptions and lower order derivative estimates

To show the global existence of solution $\phi$ to equation (1.1) near $C_0$, we will utilize the bootstrap argument. For this purpose, we make the following bootstrap assumptions in $D^{1,u}$

\[
\begin{align*}
\delta^i \|L Z^\alpha \varphi_i\|_{L^\infty(\Sigma^u_t)} &\leq M \delta^{1-\varepsilon_0} t^{-2}, \\
\delta^i \|d Z^\alpha \varphi_i\|_{L^\infty(\Sigma^u_t)} &\leq M \delta^{1-\varepsilon_0} t^{-2}, \\
\delta^i \|T Z^\alpha \varphi_i\|_{L^\infty(\Sigma^u_t)} &\leq M \delta^{1-\varepsilon_0} t^{-3}, \\
\|\nabla \varphi_i\|_{L^\infty(\Sigma^u_t)} &\leq M \delta^{1-\varepsilon_0} t^{-1}, \\
\|\varphi_i\|_{L^\infty(\Sigma^u_t)} &\leq M \delta^{1-\varepsilon_0} t^{-2},
\end{align*}
\] (3.1)

where $|\alpha| \leq N$, $N$ is a large positive integer, $M$ is some positive number which is suitably chosen (at least double bounds of the corresponding quantities on time $t_0$), $Z \in \{\rho L, T, R_1, R_2, R_3\}$, $l$ is the number of $T$ included in $Z^{\alpha}$.

As $L_\mu = \mu \bar{c}^{-1} L c - c T e$ by (2.44), then $\|L_\mu\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-p} + M^p \delta^{(1-\varepsilon_0)p-\mu} t^{-p}$ from (3.1), which deduces that by integrating $L_\mu$ along integral curves of $L$,

\[
\|\mu - 1\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1}.
\] (3.2)

Hence,

\[
\|L_\mu\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-p}.
\] (3.3)

In addition, taking the $i$-th component on both sides of $\partial_i = c^2 \mu^{-2} T^i T + d^A x_i X_A$ yields

\[
1 = c^2 \mu^{-2} |T^i|^2 + d^A x_i d_A x_i = c^2 \mu^{-2} |T^i|^2 + |dx^i|^2,
\]
which immediately gives

\[
|dx^i| \lesssim 1.
\] (3.4)

Similarly to the estimate of $\mu$ in (3.2), $\bar{\chi}$ can also be estimated by integrating $L \bar{\chi}$ along integral curves of $L$.

**Proposition 3.1.** For sufficiently small $\delta > 0$, it holds that

\[
\|\bar{\chi}\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2}.
\] (3.5)

**Proof.** Substituting $X_{AB} = \bar{\chi}_{AB} + \rho^{-1} \bar{\phi}_{AB}$ into (2.45), in view of $L \bar{\phi}_{AB} = 2 \bar{\chi}_{AB}$, one has

\[
L \bar{\chi}_{AB} = c^{-1} (L c) \bar{\chi}_{AB} + \bar{\chi}_A \bar{\chi}_B C + c^{-1} (L c) \rho^{-1} \bar{\phi}_{AB} - \bar{\phi}_{LALB},
\] (3.6)

and then,

\[
L(\rho^4 |\bar{\chi}|^2) = 2 \rho^4 (c^{-1} L c) |\bar{\chi}|^2 - \bar{\chi}_A \bar{\chi}_B C + \rho^{-1} c^{-1} L c t r \bar{\phi} \bar{\chi} - \bar{\chi}^{AB} \bar{\phi}_{LALB}.
\] (3.7)

Note that $\bar{\phi}_{LALB}$’s explicit expression has been given in (2.28) and (2.27). Then by (3.1) and (3.4), we obtain

\[
|L(\rho^2 |\bar{\chi}|)| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-p} + M^p \delta^{(1-\varepsilon_0)p} t^{-2}. \rho^2 |\bar{\chi}| + \rho^2 |\bar{\chi}|^2.
\] (3.8)

Due to

\[
|\bar{\chi}_{AB}|_0 \lesssim |(c - 1) \frac{1}{\rho} \bar{\phi}_{AB}|_0 \lesssim M^p \delta^{(1-\varepsilon_0)p},
\] (3.9)
then integrating (3.8) along integral curves of $L$ from $t_0$ to $t$ yields

$$|\bar{\chi}| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2}.$$ 

With the help of (3.5), we can estimate $d\mu$.

**Proposition 3.2.** For sufficiently small $\delta > 0$, it holds that

$$\|d\mu\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-1}.$$  \hspace{1cm} (3.10)

**Proof.** By (2.44), one has

$$\mathcal{L}_L d\mu = dL\mu = c^{-1} Ld\mu + d(cTc) + \mu d(c^{-1} Lc).$$  \hspace{1cm} (3.11)

Then

$$L(p^2 |d\mu|^2) = 2p^2 \left\{ - \chi^{AB} d_A d_B d\mu + c^{-1} Lc|d\mu|^2 + (d_A(cTc) + \mu d_A(c^{-1} Lc)) d_A d\mu \right\}. $$  \hspace{1cm} (3.12)

Together with (3.4), (3.5) and (3.1), this yields

$$L(p|d\mu|) \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-p} + M^p \delta^{(1-\varepsilon_0)p} t^{-2} \cdot p|d\mu|.$$ 

Thus

$$|d\mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-1}.$$ 

It follows from (2.21) and (2.22) that

**Corollary 3.1.** For sufficiently small $\delta > 0$, one has

$$\|\zeta\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-(p+1)},$$

$$\|\eta\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1} t^{-1}.$$  \hspace{1cm} (3.13)

Based on (3.13), we now estimate $T\mu$.

**Proposition 3.3.** For sufficiently small $\delta > 0$, it holds that

$$\|T\mu\|_{L^\infty(\Sigma^u_t)} \lesssim M^p \delta^{(1-\varepsilon_0)p-2}.$$  \hspace{1cm} (3.14)

**Proof.** It follows from (2.44) and Lemma 2.3 that

$$LT\mu = TL\mu + [L, T]\mu$$

$$= c^{-1} LcT\mu + \left[ - T(cTc) + \mu T(c^{-1} Lc) - (\zeta^A + \eta^A) d_A \mu \right].$$

In addition, by (3.1), (3.2), (3.13) and (3.10), one has

$$|LT\mu| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-(p+1)} |T\mu| + M^p \delta^{(1-\varepsilon_0)p-2} t^{-2}.$$ 

By Gronwall’s inequality, we arrive at

$$|T\mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-2}.$$ 

\hfill $\square$
At the last of this section, we give the lower order derivative $L^\infty$ estimates of some related quantities which will be further utilized in subsequent sections.

**Lemma 3.1.** For sufficiently small $\delta > 0$, it holds that

\[
|\tilde{L}^j| + |\tilde{T}^j| + |R_i\tilde{L}^j| + |R_i\tilde{T}^j| + \rho|L\tilde{L}^j| + \rho|L\tilde{T}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

\[
|T\tilde{L}^j| + |T\tilde{T}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

\[
|v_j| + |R_i v_j| + \rho|Lv_j| \lesssim M^p \delta^{(1-\varepsilon_0)p}, \quad |Tv_j| \lesssim M^p \delta^{(1-\varepsilon_0)p},
\]

\[
|\mathcal{L}_{R_i} \tilde{d} x^j| + |\mathcal{L}_{\rho L} \tilde{d} x^j| \lesssim 1, \quad |\mathcal{L}_T \tilde{d} x^j| \lesssim t^{-1},
\]

\[
|\mathcal{L}_{R_i} R_j| \lesssim t, \quad |\mathcal{L}_{\rho L} R_j| \lesssim M^p \delta^{(1-\varepsilon_0)p}, \quad |\mathcal{L}_T R_j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

\[
|\tilde{\mathcal{L}}_{i2} \tilde{d} x^j| + |\tilde{\mathcal{L}}_{i3} \tilde{d} x^j| \lesssim 1, \quad |\mathcal{L}_T \tilde{d} x^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

\[
|\tilde{T}_{i2} \tilde{d} x^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

(3.15)

**Proof.** We deal with these quantities in (3.15) one by one.

**Part 1. Bounds of $\tilde{L}^j$, $\tilde{T}^j$, $R_i \tilde{L}^j$, $R_i \tilde{T}^j$, $v_j$ and $R_i v_j$**

It is derived from Definition 2.5 and (2.48) that

\[
L(\rho \tilde{L}^i) = c^{-1}(Lc)\rho \tilde{L}^i + c^{-1}(Lc)x^i - \rho c(dA x^i)d^A c,
\]

(3.16)

then

\[
|L(\rho \tilde{L}^i)| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-(p+1)}|\rho \tilde{L}^j| + M^p \delta^{(1-\varepsilon_0)p} t^{-p}.
\]

This, together with Gronwall’s inequality, yields

\[
|\tilde{L}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

which also implies

\[
|\tilde{T}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1}
\]

in view of Lemma 2.4. In addition, it follows from (2.50), (3.5), (2.48) and Lemma 2.4 that

\[
|\tilde{d} \tilde{L}^j| + |\tilde{d} \tilde{T}^j| + |L \tilde{L}^j| + |L \tilde{T}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2}.
\]

Together with $R_i \sim r \tilde{d}$, we have

\[
|R_i \tilde{L}^j| + |R_i \tilde{T}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1}.
\]

On the other hand, since $T \tilde{L}^i = TL^i - \frac{c^{-1}}{\rho} \tilde{\mathcal{L}}_{i2} \tilde{d} x^j + (\frac{c-1}{c\rho})x^i + \frac{c^{-1}}{\rho} \tilde{\mathcal{L}}_{i3} \tilde{d} x^j$, then

\[
|T \tilde{L}^i| + |T \tilde{T}^i| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1}
\]

by Lemma 2.4. The estimates for $v_j$, $R_i v_j$, $Lv_j$ and $Tv_j$ can be obtained directly after using the definition of $v_j$ in (2.23).

**Part 2. Bounds of $\mathcal{L}_{R_i} \tilde{d} x^j$, $\mathcal{L}_{\rho L} \tilde{d} x^j$, $\mathcal{L}_T \tilde{d} x^j$ and $\mathcal{L}_{R_t} R_j$**

Due to $R_i^j = \Omega_i^j - v_i \tilde{T}^j = \epsilon_{ik}^j x^k - v_i \tilde{T}^j$, then

\[
|R_i^j| \lesssim t.$
In view of (2.24) and Definition 2.5, we have $R_k R_i x^j = R_k R_i^j = R_k (\Omega_i^j - v_i T^j)$. Then it follows from the estimates in Part 1 that 

$$|R_k R_i x^j| \lesssim t.$$ 

Thus,

$$|\mathcal{L}_R \delta x^j| = |\delta R_i x^j| \lesssim r^{-1} \sum_{k=1}^3 |R_k R_i x^j| \lesssim 1$$

and

$$|\mathcal{L}_R R_j^k| = |[R_i, R_j]^k| = |R_i R_j x^k - R_j R_i x^k| \lesssim t.$$ 

Similarly, the estimate for $\mathcal{L}_R \delta x^j$ and $\mathcal{L}_T \delta x^j$ could be obtained by the facts $\mathcal{L}_R \delta x^j = \delta (\rho L^j) = \rho \delta L^j + \delta x^j$ and $\mathcal{L}_T \delta x^j = \delta (e^{-1} \mu (T^j - \chi^j)).$

**Part 3. Bounds of $^{(R_i)} \bar{\#}, \ (R_i) \ #_L, \ (R_i) \ #_T, \ (T) \ #, \ \mathcal{L}_R R_j$ and $\mathcal{L}_T R_j$**

It follows from (2.34) and the estimates of $\bar{L}_i$, $v_i$, $\chi$, $\mu$ and $\delta \mu$ that

$$|^{(R_i)} \bar{\#}| + |^{(R_i)} \ #_L| + |^{(R_i)} \ #_T| \lesssim M^p \delta^{(1 - \epsilon_0)} t^{-1}.$$ 

In addition, by (2.33), we have

$$|^{(T)} \ #| = 2c^{-2} |\mu \chi| \lesssim t^{-1}.$$ 

Since $\mathcal{L}_R R_j = (^{(R_i)} \ #_L)$ and $\mathcal{L}_T R_j = (^{(R_i)} \ #_T)$, the estimates of $\mathcal{L}_R R_j$ and $\mathcal{L}_T R_j$ can be obtained directly. \hfill \Box

## 4 Energy estimates for the linearized covariant wave equation

In this section, we focus on the global energy estimates for the smooth function $\Psi$ to the following linearized covariant wave equation

$$\mu \Box_g \Psi = \Phi, \quad (4.1)$$

where $\Psi$ and its derivatives vanish on $\mathcal{C}_0^1$. This procedure is divided into the following four steps.

**Step 1. New divergence form of (4.1)**

By (2.29), one has

$$\Box_g \Psi \cdot \partial_g \Psi = \mathcal{D}^\alpha Q_{\alpha \beta}.$$ 

Then for any vector field $V$, it follows from (2.31) that

$$\Box_g \Psi \cdot V \Psi = \mathcal{D}^\alpha (V) J^\alpha - \frac{1}{2} Q^{\alpha \beta} [\Psi]^{(V)} \pi_{\alpha \beta}, \quad (4.2)$$

where $(V) J^\alpha = Q^\alpha_{\beta} V^\beta$ with $Q^\alpha_{\beta} = g^{\alpha \gamma} Q_{\beta \gamma}$ and $Q^{\alpha \beta} = g^{\alpha \alpha'} g^{\beta \beta'} Q_{\alpha \beta}$.

**Step 2. Integration by parts on domain $D_{1,u}$**

Under the optical coordinate $\{t, u, \theta^1, \theta^2\}$, we have

$$(V) J = (V) J^t \frac{\partial}{\partial t} + (V) J^u \frac{\partial}{\partial u} + (V) J^A \frac{\partial}{\partial \theta^A}, \quad (4.3)$$
Denoting \( N = \partial_t = L + c^2 \mu^{-1} T \) by the normal vector. Then by taking the inner product with \( N \) on both sides of (4.3), one has

\[
(V) J_N = (V) J^t g(L, N) + (V) J^u g(T, N) + (V) J^A g(X_A, N)
\]

\[
= -c^{-2} (V) J^t.
\]

Hence, \( (V) J^t = -c^{-2} (V) J_N \). Similarly, \( (V) J^u = -\mu^{-1} (V) J_L \). Therefore, it follows from \( \sqrt{|\det g|} = \mu \sqrt{|\det g|} \) that

\[
D^\alpha (V) J^\alpha = \frac{1}{\sqrt{|\det g|}} \left[ \frac{\partial}{\partial t} \left( \sqrt{|\det g|} (V) J^t \right) + \frac{\partial}{\partial u} \left( \sqrt{|\det g|} (V) J^u \right) + \frac{\partial}{\partial \beta^A} \left( \sqrt{|\det g|} (V) J^A \right) \right]
\]

\[
= \frac{1}{\sqrt{|\det g|}} \left[ \frac{\partial}{\partial t} (-c^{-2} \mu (V) J_N \sqrt{|\det g|}) + \frac{\partial}{\partial u} (- (V) J_L \sqrt{|\det g|}) + \frac{\partial}{\partial \beta^A} \left( \sqrt{|\det g|} (V) J^A \right) \right].
\]

Integrating over \( D^{t, u} \) to obtain

\[
-\int_{D^{t, u}} \mu D^\alpha (V) J^\alpha = \int_{\Sigma^u} c^{-2} \mu (V) J_N - \int_{\Sigma^u} c^{-2} \mu (V) J_N + \int_{C^{u}} (V) J_L,
\]

where

\[
(V) J_N = (V) J^a N_\alpha = Q^a_\beta V^\beta N_\alpha = Q_{V^N}, (V) J_L = Q_{V^L}.
\]

**Step 3. Energy identity**

Choosing two vector fields \( V_1 = \rho^{2s} L, V_2 = L \) as multipliers with \( 0 < s < \frac{1}{2} \) being any fixed constant. By \( T = \frac{1}{2} (L - c^{-2} \mu L) \) and (2.30), then

\[
Q_{V^1 N} = \frac{1}{2} \rho^{2s} [(L \Psi)^2 + c^2 |d \Psi|^2],
\]

\[
Q_{V^1 L} = \rho^{2s} (L \Psi)^2,
\]

\[
Q_{V^2 N} = \frac{1}{2} \left[ c^2 \mu^{-1} (L \Psi)^2 + \mu |d \Psi|^2 \right],
\]

\[
Q_{V^2 L} = \mu |d \Psi|^2.
\]

By (4.4) and (4.2), we have the following energy identity

\[
E_i[\Psi](t, u) - E_i[\Psi](t_0, u) + F_i[\Psi](t, u) = - \int_{D^{t, u}} \Phi \cdot V_i \Psi - \int_{D^{t, u}} \frac{1}{2} \mu \pi_{\alpha \beta} (V_i)^\alpha (V_i)^\beta, \; i = 1, 2, \quad (4.5)
\]

where the energies \( E_i[\Psi](t, u) \) and fluxes \( F_i[\Psi](t, u) (i = 1, 2) \) are defined as follows

\[
E_1[\Psi](t, u) = \int_{\Sigma^u} \frac{1}{2} \rho^{2s} \left[ c^{-2} \mu (L \Psi)^2 + \mu |d \Psi|^2 \right],
\]

\[
E_2[\Psi](t, u) = \int_{\Sigma^u} \frac{1}{2} \left[ (L \Psi)^2 + c^{-2} \mu^2 |d \Psi|^2 \right],
\]

\[
F_1[\Psi](t, u) = \int_{C^u} \rho^{2s} (L \Psi)^2,
\]

\[
F_2[\Psi](t, u) = \int_{C^u} \mu |d \Psi|^2.
\]

(4.6)
**Step 4.** Error estimates and the energy inequality

Next, we deal with the second integral in the right-hand side of (4.5). By Remark 2.2, one has

\[ -\frac{1}{2}\mu Q^{\alpha\beta}[\Psi](V)\pi_{\alpha\beta} = -\frac{1}{2}\mu g^{\alpha\beta} g^{\lambda\gamma} Q_{\kappa\lambda}[\Psi](V)\pi_{\alpha\beta} \]

\[ = -\frac{1}{2}\mu \left[ -\frac{1}{2}h^{-1}(L^\alpha L^\kappa + L^\alpha L^\kappa) + \theta^{AB} X_\alpha^B X_\beta^B \right] \left[ \frac{1}{2}h^{-1}(L^\beta L^\lambda + L^\beta L^\lambda) + \theta^{CD} X_\beta^C X_\delta^D \right] Q_{\kappa\lambda}(V)\pi_{\alpha\beta} \]

\[ = -\frac{1}{8}h^{-1}(Q_{LL}(V)\pi_{ll} + Q_{LL}(V)\pi_{ll}) - \frac{1}{4}h^{-1}Q_{LL}(V)\pi_{ll} + \frac{1}{2}(Q_L^i(V)\pi_{LA} + Q_L^i(V)\pi_{LA}) - \frac{1}{2}\mu Q^{AB}(V)\pi_{AB}. \]

Then by (2.30), we obtain

\[ -\frac{1}{2}\mu Q^{\alpha\beta}[\Psi](V)[V'_2]\pi_{\alpha\beta} \]

\[ = \frac{1}{2} \left[ L^\mu + \mu L(c^{-2}\mu) \right] |d\Psi|^2 - \mu d_A(c^{-2}\mu) L^\mu d^A \Psi - c^2 d_A(c^{-2}\mu) L^\mu d^A \Psi \]

\[ + c^{-2} \mu d_A(c^{-2}\mu) L^\mu d^A \Psi - \frac{1}{2} c^{-2} \mu^2 \mu tr g\chi|d\Psi|^2 + \frac{1}{2} c^{-2} \mu^2 \mu tr g\chi L^\mu L^\mu \Psi. \]

Applying the results in Section 3 to estimate all the coefficients in (4.7) and (4.8) yields

\[ \int_{D_1} u \cdot \frac{1}{2} \mu Q^{\alpha\beta}[\Psi](V)\pi_{\alpha\beta} \]

\[ \lesssim \int_{D_1} \left\{ \delta^{-1} \rho^{2s-1}(L^\mu)^2 + |d\Psi|^2 + \tau^{-2+2s} \cdot \delta(L^\mu)^2 \right\} \]

\[ \lesssim \delta^{-1} \int_0^u F_1[\Psi](t, u')du' + \int_{t_0}^t \tau^{-2+2s} \cdot \delta E_2[\Psi](\tau, u)d\tau + \delta^{-1} \int_{t_0}^t \cdot \delta F_2[\Psi](t, u')du' \]

and

\[ \delta \int_{D_1} u \cdot \frac{1}{2} \mu Q^{\alpha\beta}[\Psi](V)\pi_{\alpha\beta} \]

\[ \lesssim \int_{D_1} \left\{ M^p \delta^{(1-\varepsilon_0)p-2} \cdot \delta |d\Psi|^2 + \left[ \delta(L^\mu)^2 + \tau^{-2} \cdot \delta(L^\mu)^2 \right] \right\} \]

\[ \lesssim \int_0^u M^p \delta^{(1-\varepsilon_0)p-2} \cdot \delta F_2[\Psi](t, u')du' + \int_0^u \delta F_1[\Psi](t, u')du' + \int_{t_0}^t \tau^{-2} \cdot \delta E_2[\Psi](\tau, u)d\tau. \]

Substituting (4.9) and (4.10) into (4.5) and using the Gronwall’s inequality, we obtain

\[ E_1[\Psi](t, u) + F_1[\Psi](t, u) + \delta E_2[\Psi](t, u) + \delta F_2[\Psi](t, u) \]

\[ \lesssim E_1[\Psi](t_0, u) + \delta E_2[\Psi](t_0, u) + \int_{D_1} u \cdot \rho^{2s}|\Phi| \cdot L^\mu |\Psi| + \delta \int_{D_1} |\Phi| \cdot L^\mu |\Psi|. \]
At the last of this subsection, we define the higher order energy and flux as follows

\[ E_{i,m+1}(t, u) = \sum_{\gamma=0}^{3} \sum_{|\alpha|=m} \delta^{2l} E_{i}[Z^{\alpha} \varphi_{\gamma}](t, u), \quad i = 1, 2, \]

\[ F_{i,m+1}(t, u) = \sum_{\gamma=0}^{3} \sum_{|\alpha|=m} \delta^{2l} F_{i}[Z^{\alpha} \varphi_{\gamma}](t, u), \quad i = 1, 2, \]

\[ E_{i,\leq m+1}(t, u) = \sum_{0 \leq n \leq m} E_{i,n+1}(t, u), \quad \tilde{E}_{i,\leq m+1}(t, u) = \sup_{t_0 \leq \tau \leq t} E_{i,\leq m+1}(\tau, u), \quad i = 1, 2, \]

\[ F_{i,\leq m+1}(t, u) = \sum_{0 \leq n \leq m} F_{i,n+1}(t, u), \quad \tilde{F}_{i,\leq m+1}(t, u) = \sup_{t_0 \leq \tau \leq t} F_{i,m+1}(\tau, u), \quad i = 1, 2, \]

(4.12)

where \( l \) is the number of \( T \) appearing in \( Z^{\alpha} \).

5 Non-top order derivative estimates

5.1 \( L^\infty \) estimates

In Section 3, we have obtained the lower order \( L^\infty \) estimates for some quantities. Since our aim is to close the bootstrap assumptions (3.1), the results obtained in Section 3 are far from enough. For this purpose, we give the higher order estimates.

**Proposition 5.1.** Under the assumptions (3.1), for any vector field \( Z \in \{ \rho L, T, R_1, R_2, R_3 \} \), when \( \delta > 0 \) is small, it holds that for \( |\alpha| \leq N - 1, \)

\[
\delta^l |E_{Z,\bar{X}}^\alpha| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2}, \quad \delta^l |Z^{\alpha+1} \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p - 1},
\]

\[
\delta^l |Z^{\alpha+1} L^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1}, \quad \delta^l |Z^{\alpha+1} v_j| \lesssim M^p \delta^{(1-\varepsilon_0)p},
\]

\[
\delta^l (|L_{Z,\bar{X}}^{\alpha} R_1| + |L_{Z,\bar{X}}^{\alpha} R_2| + |L_{Z,\bar{X}}^{\alpha} R_3|) \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},
\]

(5.1)

where \( l \) is the number of \( T \) appearing in the string of \( Z \).

**Proof.** We will prove this proposition by the induction method with respect to the index \( \alpha \). Note that we have proved (5.1) for the case \( \alpha = 0 \) in Section 3.

We first prove (5.1) which only involves the rotational vector fields, that is, \( Z \in \{ R_1, R_2, R_3 \} \). Assume that (5.1) holds up to the order \( k - 1 \) (\( 1 \leq k \leq N - 1 \)), one needs to show that (5.1) is also true for \( \alpha \) with \( |\alpha| = k \).

1. **Bounds of** \( L_{R_1}^{\alpha} \bar{X} \) and \( L_{R_1}^{\alpha} (R_3) \not\bar{X} \)

Using the expression of \( L_{R_1}^{\alpha} (R_3) \not\bar{X} \) in (2.34) and the induction assumption, we can check that \( |L_{R_1}^{\alpha} (R_3) \not\bar{X}| \lesssim \tau |L_{R_1}^{\alpha} \bar{X}| + M^p \delta^{(1-\varepsilon_0)p} t^{-1} \).

(5.2)

This means that the estimate of \( L_{R_1}^{\alpha} (R_3) \not\bar{X} \) can be obtained once \( L_{R_1}^{\alpha} \bar{X} \) is bounded. Commuting \( L_{R_1}^{\alpha} \) with \( L_{X,AB} \) to derive

\[
L_{X} L_{R_1}^{\alpha} \bar{X} = L_{R_1}^{\alpha} (L_{X,AB}) + \sum_{\beta_1 + \beta_2 = \alpha - 1} L_{R_1}^{\beta_1} L_{[L,R_1]} L_{R_1}^{\beta_2} \bar{X},
\]

(5.3)
Substituting (3.6) and the identity $[L, R_i] = (R_i)^T L C X_C$ into (5.3), and applying Lemma 2.5, (5.2) and induction argument, we arrive at

$$|\mathcal{L}_L^\alpha \mathcal{L}_{R_i}^\alpha \chi| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2} |\mathcal{L}_{R_i}^\alpha \chi| + M^p \delta^{(1-\varepsilon_0)p} t^{-(p+2)}. \tag{5.4}$$

Similarly to (3.7), one has

$$L(\rho^4 |\mathcal{L}_{R_i}^\alpha \chi|^2) = -4 \rho^4 \chi A \cdot \mathcal{L}_{R_i}^\alpha \chi C \cdot \mathcal{L}_{R_i}^\alpha \chi A + 2 \rho^4 \mathcal{L}_{R_i}^\alpha \chi AB \cdot \mathcal{L}_L \mathcal{L}_{R_i}^\alpha \chi AB. \tag{5.5}$$

Then combining (5.4) and (5.5) yields

$$L(\rho^2 |\mathcal{L}_{R_i}^\alpha \chi|) \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2} \cdot \rho^2 |\mathcal{L}_{R_i}^\alpha \chi| + M^p \delta^{(1-\varepsilon_0)p} t^{-p}.$$

By Gronwall’s inequality, then

$$|\mathcal{L}_{R_i}^\alpha \chi| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2}.$$

This, together with (5.2), derives

$$|\mathcal{L}_{R_i}^\alpha (R_j)^T L \chi | \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1}.$$

2. \textbf{Bounds of } $R_i^{\alpha+1} \mu$

Similarly to the estimate $\mathcal{L}_{R_i}^\alpha \chi$, we commute $R_i^{\alpha+1}$ with $L$ to get

$$LR_i^{\alpha+1} \mu = R_i^{\alpha+1} L \mu + \sum_{\beta_1+\beta_2=\alpha} R_i^{\beta_1} [L, R_i] R_i^{\beta_2} \mu. \tag{5.6}$$

By (2.44), the induction assumptions and (3.1), one has

$$|LR_i^{\alpha+1} \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-2} |R_i^{\alpha+1} \mu| + M^p \delta^{(1-\varepsilon_0)p-1} t^{-2}.$$

By Gronwall’s inequality, then

$$|R_i^{\alpha+1} \mu| \lesssim M^p \delta^{(1-\varepsilon_0)p-1}. \tag{5.7}$$

3. \textbf{Bounds of } $R_i^{\alpha+1} \dot{L}_j$, $R_i^{\alpha+1} \dot{v}_j$, $\mathcal{L}_{R_i}^\alpha x^j$, and $\mathcal{L}_{R_i}^\alpha R_j$

It follows (2.50) that $R_i^{\alpha+1} \dot{L}_j = R_i^{\alpha+1} (\dot{\chi}_{AB} \dot{x}^j).$ Then by the induction assumptions, the estimate of $\mathcal{L}_{R_i}^\alpha \dot{\chi}$ and the identity $\dot{L}_j = -\rho (\rho^2 - d) \rho^{-1} x^j$, one has

$$|R_i^{\alpha+1} \dot{L}_j| + |R_i^{\alpha+1} \dot{x}^j| \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1},$$

which gives the estimate of $R_i^{\alpha+1} \dot{v}_j$ directly by (2.23). And hence, $|R_i^{\alpha+2} x^j| \lesssim t$ holds because of $R_i^{\alpha+2} x^j = R_i^{\alpha+1} (\Omega x^j - v_i \dot{T}_i).$ Therefore,

$$|\mathcal{L}_{R_i}^\alpha x^j| \lesssim 1, \text{ and } |\mathcal{L}_{R_i}^\alpha R_j| \lesssim t.$$

4. \textbf{Bounds of } $\mathcal{L}_{R_i}^\alpha (R_j)^T \chi$, $\mathcal{L}_{R_i}^\alpha (R_j)^T \chi_T$, and $\mathcal{L}_{R_i}^\alpha (T)^T \chi$

According the expressions in (2.34) and (2.33), the estimates of $\mathcal{L}_{R_i}^\alpha (R_j)^T \chi$, $\mathcal{L}_{R_i}^\alpha (R_j)^T \chi_T$ and $\mathcal{L}_{R_i}^\alpha (T)^T \chi$ can be obtained immediately with the help of all the results in Parts 1-3 and the induction argument.
It follows from (2.46) and Definition 2.5 that the formula of $\mathcal{L}_T \tilde{\chi}$ is obtained. After taking the Lie derivatives of this formula with respect to the rotational vector fields, one can estimate $\mathcal{L}_{R^1}^{\alpha-1} \mathcal{L}_T \tilde{\chi}$ directly by using the estimates above and (3.1). Therefore, the bound of $L_\Gamma$ can be obtained when $Z \in \{T, R_1, R_2, R_3\}$ and there is only one $T$ in $Z^\alpha$.

Similarly, if $Z \in \{T, R_1, R_2, R_3\}$ and $Z^{\alpha+1}$ only contains one $T$, then $Z^{\alpha+1} \tilde{\ell}^j$, $\mathcal{L}_Z^{\alpha+1} \tilde{\mu}^j$ and $\mathcal{L}_{Z}^{\alpha+1} R_j$ can be estimated since the expressions of $T \tilde{\ell}^j$, $\mathcal{L}_T \tilde{\mu}^j$ and $\mathcal{L}_T R_j$ have appeared in the proof of Lemma 3.1. With the same procedure for deriving (3.14) or (5.17), $Z^{\alpha+1} \mu$ is also estimated. The reminder quantities such as $Z^{\alpha+1} v_j$, $\mathcal{L}_Z^{\alpha}(R_j)$ $\bar{\mathcal{J}}_L$ e.t.c. that contain one derivative of $T$ and other derivatives of the rotational vector fields can be estimated by (2.23), (2.33) and (2.34).

By induction argument with respect to the number of $T$, (5.1) can be proved for $Z \in \{T, R_1, R_2, R_3\}$.

Finally, when the derivatives in $Z^\alpha$ involve the scaling vectorfield $\rho L$, the transport equations (e.g. (2.44), (2.45), (2.48)) can be utilized to derive (5.1).}

\section{$L^2$ estimates}

In this subsection, we shall establish the higher order $L^2$ estimates for some related quantities so that the last two terms of (4.11) can be absorbed by the left hand side, and hence the higher order energy estimates on (2.39) can be completed.

At first, we list two lemmas which are similar to Lemma 7.3 in [13] and Lemma 12.57 in [14].

**Lemma 5.1.** For any $\Psi \in C^1(D^{1,u})$ which vanishes on $C_0$, one has that for small $\delta > 0$,

$$
\int_{\Sigma_t} \Psi^2 \lesssim \delta \int_{\Sigma_t} (\mathcal{L}_Z \Psi)^2 + \mu^2 (L \Psi)^2 \lesssim \delta \{ \rho^{-2} E_1[\Psi](t, u) + E_2[\Psi](t, u) \},
$$

$$
\int_{\Sigma_t} \Psi^2 \lesssim \delta^2 \int_{\Sigma_t} (\mathcal{L}_Z \Psi)^2 + \mu^2 (L \Psi)^2 \lesssim \delta^2 \{ \rho^{-2} E_1[\Psi](t, u) + E_2[\Psi](t, u) \}.
$$

**Lemma 5.2.** For any $f \in C(D^{1,u})$, set

$$
F(t, u, \vartheta) = \int_t^1 f(\tau, u, \vartheta) d\tau.
$$

Under the assumptions (3.1), it holds that for small $\delta > 0$,

$$
\| F \|_{L^2(\Sigma_t^u)} \lesssim \rho(t, u) \int_t^1 \tau^{-1} \| f \|_{L^2(\Sigma_t^u)} d\tau.
$$

Based on the preparations above, we are ready to derive the higher order $L^2$ estimates for some related quantities.

**Proposition 5.2.** Under the assumptions (3.1), when $\delta > 0$ is small, it holds that for $|\alpha| \leq 2N - 6$,

$$
\begin{align*}
&\delta^l \| \mathcal{L}_Z^{\alpha} \chi \|_{L^2(\Sigma_t^u)} + \delta^l \| \mathcal{L}_Z^{\alpha}(R_j) \|_{L^2(\Sigma_t^u)} + \delta^l \| \mathcal{L}_Z^{\alpha}(R_j) \|_{L^2(\Sigma_t^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + \Theta_1^M(t, u),
&\delta^l \| Z^{\alpha+1} \tilde{\ell}^j \|_{L^2(\Sigma_t^u)} + t^{-1} \delta^l \| Z^{\alpha+1} v_j \|_{L^2(\Sigma_t^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + \Theta_1^M(t, u),
&\delta^l \| \mathcal{L}_Z^{\alpha+1} \tilde{\mu}^j \|_{L^2(\Sigma_t^u)} + t^{-1} \delta^l \| \mathcal{L}_Z^{\alpha+1} R_j \|_{L^2(\Sigma_t^u)} \lesssim \delta^2 t + \Theta_1^M(t, u),
&\delta^l \| Z^{\alpha+1} \mu \|_{L^2(\Sigma_t^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + \Theta_1^M(t, u),
&\delta^l \| \mathcal{L}_Z^{\alpha}(R_j) \|_{L^2(\Sigma_t^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + \Theta_1^M(t, u),
&\delta^l \| \mathcal{L}_Z^{\alpha}(T) \|_{L^2(\Sigma_t^u)} \lesssim \delta^2 + \Theta_1^M(t, u),
\end{align*}
$$
where $l$ is the number of $T$ appearing in the string of $Z$, and

\[
\Theta_{M}^{1}(t, u) = M^{p-1} \delta^{(1-\varepsilon_{0})/(p-1)} \left[ \sqrt{E_{1,|\alpha|+2}(t, u)} + \delta \sqrt{E_{2,|\alpha|+2}(t, u)} \right],
\]

\[
\Theta_{M}^{2}(t, u) = M^{p-1} \delta^{(1-\varepsilon_{0})/(p-1)} \left[ \sqrt{E_{1,|\alpha|+2}(t, u)} + \sqrt{E_{2,|\alpha|+2}(t, u)} \right].
\]

**Proof.** We will prove this proposition by the induction method with respect to the index $\alpha$. As in the proof of Proposition 5.1, we first prove the results which only involve the rotational vector fields.

When $\alpha = 0$, in view of Proposition 5.1, the corresponding $L^{2}$ estimates can be directly obtained by the fact $\|f\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim t^{1/2}$ (similar to Corollary 11.30.3 in [14]). For example, $\|\hat{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim \|\hat{\chi}\|_{L^{\infty}(\Sigma_{t}^{\nu})}$.

Assume that Proposition (5.2) holds for the index $\alpha (|\alpha| = k \leq 2N - 7)$ and $Z \in \{R_{1}, R_{2}, R_{3}\}$. For the treatment on the case $|\alpha| = k + 1$, our concrete strategies are to take the $L^{2}$ norms for the factors equipped with the highest order derivatives of the related quantities, meanwhile applying Proposition 5.1 to estimate the corresponding $L^{\infty}$ coefficients in these terms.

1. **Bounds of $\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}$ and $\mathcal{L}_{R_{k}}^{\alpha} (R_{j}) \not\in L^{\infty}$**

With the help of induction assumption, then by (2.34) and Lemma 5.1,

\[
\|\mathcal{L}_{R_{k}}^{\alpha} (R_{j}) \not\in L^{\infty}\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim t \|\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-2} \|\mathcal{L}_{R_{k}}^{\alpha} R_{j}\|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-1} \|R_{k}^{\alpha} \mathcal{L}_{R_{j}}^{\alpha} \mathcal{L}_{R_{k}}^{\alpha} \|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-1} \|R_{k}^{\alpha} \mathcal{L}_{R_{j}}^{\alpha} \mathcal{L}_{R_{k}}^{\alpha} \|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-3} \|R_{k}^{\alpha} \mathcal{L}_{R_{j}}^{\alpha} \mathcal{L}_{R_{k}}^{\alpha} \|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-2} \|R_{k}^{\alpha} \mathcal{L}_{R_{j}}^{\alpha} \mathcal{L}_{R_{k}}^{\alpha} \|_{L^{2}(\Sigma_{t}^{\nu})} + M^{p} \delta^{(1-\varepsilon_{0})} p^{-1} \|R_{k}^{\alpha} \mathcal{L}_{R_{j}}^{\alpha} \mathcal{L}_{R_{k}}^{\alpha} \|_{L^{2}(\Sigma_{t}^{\nu})} \quad (5.8)
\]

Thus, it follows from (5.3) and induction assumption that

\[
\|\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim M^{p} \delta^{(1-\varepsilon_{0})} t^{-2} \|\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} + M^{2p} \delta^{(1-\varepsilon_{0})} t^{-3} + M^{p-1} \delta^{(1-\varepsilon_{0})} \{t^{-1} \sqrt{E_{1,|\alpha|+2}(t, u)} + \delta t^{-1} \sqrt{E_{2,|\alpha|+2}(t, u)} \}. \quad (5.9)
\]

Recalling (5.5), one then has

\[
|L(\rho^{2} |\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}|)| \leq 2|\tilde{\chi}| \cdot \rho^{2} |\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}| + \rho^{2} |L \mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}|. \quad (5.10)
\]

Integrating (5.10) and utilizing Lemma 5.2 yield

\[
\|\rho^{2} \mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim \|\rho^{2} \mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} + \rho \int_{t_{0}}^{t} \tau^{-1} \|L(\rho^{2} |\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}|)\|_{L^{2}(\Sigma_{t}^{\nu})} d\tau.
\]

Hence,

\[
\|\rho \mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} \lesssim M^{p} \delta^{(1-\varepsilon_{0})} t^{-3} + \int_{t_{0}}^{t} \tau \|\mathcal{L}_{R_{k}}^{\alpha} \tilde{\chi}\|_{L^{2}(\Sigma_{t}^{\nu})} d\tau. \quad (5.11)
\]
Substituting (5.9) into (5.11), together with Gronwall’s inequality, we have
\[
\|\rho \mathcal{E}^\alpha_{R_i} \tilde{\chi} \|_{L^2(\Sigma^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \left[ \sqrt{ \tilde{E}_{1,\leq |\alpha|+2}(t,u) } + \delta \sqrt{ \tilde{E}_{2,\leq |\alpha|+2}(t,u) } \right].
\]
Therefore,
\[
\| \mathcal{L}^\alpha_{R_i} \tilde{\chi} \|_{L^2(\Sigma^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} t^{-1} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} t^{-1} \left[ \sqrt{ \tilde{E}_{1,\leq |\alpha|+2}(t,u) } + \delta \sqrt{ \tilde{E}_{2,\leq |\alpha|+2}(t,u) } \right],
\]
which also gives the estimate of \( L^2 \) norm for \( \mathcal{L}^\alpha_{R_i} (R_i) \#_L \) by (5.8).

2. Bounds of \( R_i^{\alpha+1} \tilde{L}^j, R_i^{\alpha+1} v_j, \mathcal{L}^{\alpha+1} x^j, \mathcal{L}^{\alpha+1} R_j, \mathcal{L}^\alpha_{R_i} (R_i) \# \) and \( \mathcal{L}^\alpha_{R_i} (T) \# \)

In the proof of Proposition 5.1, one has known that \( R_i^{\alpha+1} \tilde{L}^j = R_i^\alpha (R_i^A \tilde{\chi}_A B d^j x^j) \). Then by (5.12) and induction argument,
\[
\| R_i^{\alpha+1} \tilde{L}^j \|_{L^2(\Sigma^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} t^{-1} \| \mathcal{E}^\alpha_{R_i} \tilde{\chi} \|_{L^2(\Sigma^u)} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} t^{-1} \| \mathcal{E}^\alpha_{R_i} R_j \|_{L^2(\Sigma^u)} + M^{p} \delta^{(1-\varepsilon_0)p} t^{-2} \| \mathcal{E}^\alpha_{R_i} v_j \|_{L^2(\Sigma^u)} + M^p \delta^{(1-\varepsilon_0)p-1} \| \mathcal{E}^\alpha_{R_i} \tilde{\chi} \|_{L^2(\Sigma^u)} + M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \left[ \sqrt{ \tilde{E}_{1,\leq |\alpha|+2}(t,u) } + \delta \sqrt{ \tilde{E}_{2,\leq |\alpha|+2}(t,u) } \right].
\]
For the other quantities \( R_i^{\alpha+1} v_j, \mathcal{L}^{\alpha+1} x^j, \mathcal{L}^{\alpha+1} R_j, \mathcal{L}^\alpha_{R_i} (R_i) \# \) and \( \mathcal{L}^\alpha_{R_i} (T) \# \), their \( L^2 \) norms over \( \Sigma^u \) can be similarly obtained by the procedure of estimating \( \| R_i^{\alpha+1} \tilde{L}^j \|_{L^2(\Sigma^u)} \).

3. Bounds of \( R_i^{\alpha+1} \mu \) and \( \mathcal{L}^\alpha_{R_i} (R_i) \#_T \)

According to (2.34), by induction argument and Lemma 5.1, we obtain
\[
\| \mathcal{L}^\alpha_{R_i} (R_i) \#_T \|_{L^2(\Sigma^u)} \lesssim M^p \delta^{(1-\varepsilon_0)p} t^{-1} \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} t^{-1} \| \mathcal{E}^\alpha_{R_i} \mu \|_{L^2(\Sigma^u)} + M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} t^{-1} \| R_i^{\alpha+1} \varphi \|_{L^2(\Sigma^u)} \left[ \sqrt{ \tilde{E}_{1,\leq |\alpha|+2}(t,u) } + \delta \sqrt{ \tilde{E}_{2,\leq |\alpha|+2}(t,u) } \right].
\]
This means that once the bound of \( \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} \) is obtained, then the estimate of \( \| \mathcal{L}^\alpha_{R_i} (R_i) \#_T \|_{L^2(\Sigma^u)} \) comes naturally.

Similarly to \( \tilde{\chi} \), by Lemma 5.2, one has
\[
\| R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} \lesssim \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} + \rho \int_{t_0}^t \tau^{-1} \| L R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} d\tau \lesssim M^p \delta^{(1-\varepsilon_0)p+\frac{1}{2}} t + \rho \int_{t_0}^t \tau^{-1} \| L R_i^{\alpha+1} \mu \|_{L^2(\Sigma^u)} d\tau.
\]
Lemma 6.1. Under the assumptions satisfies $satisfies$

$
\| t^{-1} R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)} \lesssim M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} + \int_0^t \tau^{-1} \| LR_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)} d\tau.
$

(5.14)

Next, we estimate $\| LR_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)}$ in (5.14). Recalling (5.6), (2.44) and $[L, R_i] = (R_i)_{T \neq 0} X^A$, then by (5.13), we have

$\| R_i^{\alpha+1} L \mu \|_{L^2(\Sigma_t^\alpha)} \lesssim (M \delta^{1-\varepsilon_0} \tau^{-1})^{p-1} \| TR_i^{\alpha+1} \varphi \|_{L^2(\Sigma_t^\alpha)} + M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} \| L R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)}$

$+ (M \delta^{1-\varepsilon_0} \tau^{-1})^{p-1} \| R_i^{\alpha+1} L \varphi \|_{L^2(\Sigma_t^\alpha)} + (M \delta^{1-\varepsilon_0} \tau^{-1})^{p-1} \| R_i^{\alpha+1} \varphi \|_{L^2(\Sigma_t^\alpha)}$

$+ M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)}$

\[ \lesssim M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)} + M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)}
\]

(5.15)

and

\[ \| \sum_{\beta_1+\beta_2=\alpha} R_i^{\beta_1} [L, R_i] R_i^{\beta_2} \mu \|_{L^2(\Sigma_t^\alpha)} \lesssim M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} \| R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)} + M^p \delta^{2p(1-\varepsilon_0)-\frac{1}{2}} \tau^{-1}
\]

(5.16)

Substituting (5.15) and (5.16) into (5.14), utilizing Gronwall’s inequality, one has

$\| t^{-1} R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)} \lesssim M^p \delta^{(1-\varepsilon_0)p-\frac{1}{2}} + M^p \delta^{2p(1-\varepsilon_0)(p-1)-\frac{1}{2}} \tau^{-1} \left[ \sqrt{E_1, \leq |\alpha|+2(t, u)} + \sqrt{E_2, \leq |\alpha|+2(t, u)} \right].$

Hence, the estimates of $\| R_i^{\alpha+1} \mu \|_{L^2(\Sigma_t^\alpha)}$ and $\| \xi^{\alpha} R_i^{\beta} \|_{L^2(\Sigma_t^\alpha)}$ are obtained with help of (5.13).

If there are $T$ or $\rho L$ in $Z^{\alpha+1}$, we could use the identities in Lemma 2.6, (2.33), (2.34) e.t.c. to get the corresponding $L^2$ bounds. Since the treatments are analogous to those in the end of proof for Proposition 5.1, we omit the details here. \qed

6 Top order $L^2$ estimates for the derivatives of $\nabla \chi$ and $\nabla^2 \mu$

When we try to close the energy estimate in Section 7 below, it is found that the top orders of derivatives of $\varphi$, $\chi$ and $\mu$ for the energy estimates are $2N - 4$, $2N - 5$ and $2N - 4$ respectively. However, as shown in Proposition 5.2, the $L^2$ estimates for the $(2N - 5)^{th}$ order derivatives of $\chi$ and $(2N - 4)^{th}$ order derivatives of $\mu$ can be controlled by the $(2N - 3)^{th}$ order energy of $\varphi$. So there is a mismatch here. To overcome this difficulty, we need to deal with $\chi$ and $\mu$ with the corresponding top order derivatives. Once the estimate of $\Delta \mu$ is established, we can use the elliptic estimate to treat $\nabla^2 \mu$. Before making the elliptic estimates, we shall estimate the Gaussian curvature $G$ of $g$ firstly.

Lemma 6.1. Under the assumptions (3.1), when $\delta > 0$ is small, the Gaussian curvature $G$ of $g$ in $D^{L^u}$ satisfies

$G = (1 + O(M^p \delta(1-\varepsilon_0)p^{-1})) \rho^{-2}.$

(6.1)
Proof. By (2.29) in [13],
\[ G = \frac{1}{2} c^{-2} \left( (\text{tr}_g \chi)^2 - |\chi|^2 \right). \] (6.2)
Together with Lemma 2.4 and (3.5), it follows that
\[ G = \frac{1}{2} c^{-2} \left[ 2\rho^{-2} + (\text{tr}_g \tilde{\chi})^2 + 2\rho^{-1} \text{tr}_g \tilde{\chi} - |\tilde{\chi}|^2 \right] \]
\[ = (1 + O(M^2 (1 - \varepsilon_0) \rho^{-1})) \rho^{-2}. \]
\[ \square \]

With Lemma 6.1, we can get the following two lemmas for the elliptic estimates.

**Lemma 6.2.** For any trace-free symmetric 2-tensor \( \xi \) on \( S_t,u \), it holds that
\[ \int_{S_t,u} (|\nabla \xi|^2 + 2G|\xi|^2) = \int_{S_t,u} 2|\text{div}_g \xi|^2. \] (6.3)
For any function \( f \in C^2(D_t,u) \), it holds that
\[ \int_{S_t,u} (|\nabla^2 f|^2 + G|d f|^2) = \int_{S_t,u} (\Delta f)^2. \] (6.4)

**Proof.** Taking \( \mu = 1 \) in (18.23) and (18.12) of [14], then (6.3) and (6.4) hold immediately. \( \square \)

Next, we deal with \( \nabla \chi \) and \( \nabla^2 \mu \).

### 6.1 Estimates on the derivatives of \( \nabla \chi \)
Recalling (2.28) and (2.27), we now define
\[ \hat{R}_{LL} = g^{AB} \hat{R}_{LALB} = -\frac{1}{2} pc^4 \varphi_0^{-1} \Delta \varphi_0 + R_0, \] (6.5)
where
\[ R_0 = -\frac{3}{2} pc^3 \varphi_0^{p-1} \delta c \cdot d \varphi_0 - \frac{1}{2} p(p-1)c^4 \varphi_0^{p-2} |d \varphi_0|^2. \] (6.6)
By (2.45) and Lemma 2.4, then
\[ L \text{tr}_g \chi = (c^{-1} L c - 2\rho^{-1}) \text{tr}_g \chi + 2\rho^{-2} - |\chi|^2 - \hat{R}_{LL}. \] (6.7)
Note that the highest order derivative of \( \varphi_0 \) in the right hand side of (6.5) is \( \Delta \varphi_0 \). We will replace \( \Delta \varphi_0 \) in \( \hat{R}_{LL} \) with \( L(\partial \varphi) + \text{l.o.t.} \), where and below “l.o.t.” stands for the phrase “lower order terms”. Indeed, by (2.41) and (2.42), we have
\[ \Delta \varphi_0 = \mu^{-1} (L L \varphi_0 + \text{tr}_g \chi T \varphi_0 - \tilde{H}_0 + F_0). \] (6.8)
Hence,
\[ \hat{R}_{LL} = -LE \chi - e \chi - \frac{1}{2} pc^4 \mu^{-1} \varphi_0^{-1} T \varphi_0 \text{tr}_g \chi, \] (6.9)
where
\[ E_\chi = \frac{1}{2} p e^4 \mu^{-1} \phi_0^{-1} L \varphi_0, \]  
\[ e_\chi = -L(\frac{1}{2} p e^4 \mu^{-1} \phi_0^{-1})L \varphi_0 - \frac{1}{2} p e^4 \mu^{-1} \phi_0^{-1}(\tilde{H}_0 - F_0) - R_0. \]  
(6.10)

Substituting (6.9) into (6.7) yields
\[ L(\text{tr}_g \chi - E_\chi) = (\mu^{-1} L \mu - 2 \rho^{-1}) \text{tr}_g \chi + 2 \rho^{-2} - |\chi|^2 + e_\chi. \]  
(6.11)

Let \( E_\chi^\alpha = d \tilde{Z}^\alpha (\text{tr}_g \chi - E_\chi) \) with \( Z \in \{T, R_1, R_2, R_3\} \). Then by the induction argument on (6.11), we have
\[ \mathcal{L}_L E_\chi^\alpha = (\mu^{-1} L \mu - 2 \rho^{-1})E_\chi^\alpha + (\mu^{-1} L \mu - 2 \rho^{-1})d \tilde{Z}^\alpha E_\chi - d \tilde{Z}^\alpha (|\chi|^2) + e_\chi^\alpha, \]  
(6.12)

where
\[ e_\chi^\alpha = \mathcal{L}_Z^\alpha e_\chi^0 + \sum_{\beta_1 + \beta_2 = \alpha, |\beta_1| \geq 1} \tilde{Z}^{\beta_1} (\mu^{-1} L \mu - 2 \rho^{-1})d \tilde{Z}^{\beta_2} \text{tr}_g \chi + \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_Z^{\beta_1} \mathcal{L}_{[L, Z]} E_\chi^{\beta_2}, \]  
(6.13)

\[ e_\chi^0 = d(\mu^{-1} L \mu) \text{tr}_g \chi + d e_\chi. \]

For any 1-form \( \xi \), one has
\[ L(\rho^3 |\xi|^2) = -2 \rho^2 \tilde{X}^{\alpha AB} \xi_A \xi_B + 2 \rho^2 \tilde{g}^{AB} (\mathcal{L}_L \xi_A) \xi_B. \]  
(6.14)

By taking \( \xi = \rho^2 E_\chi^\alpha \) in (6.14) and using (6.12), then
\[ L(\rho^6 |E_\chi^\alpha|^2) = 2 \rho^6 \left[ (\mu^{-1} L \mu) E_\chi^\alpha + (\mu^{-1} L \mu - 2 \rho^{-1})d \tilde{Z}^\alpha E_\chi \right] \]
\[ - d \tilde{Z}^\alpha (|\chi|^2) + e_\chi^\alpha \cdot E_\chi - 2 \rho^6 \tilde{X}^{\alpha AB} E_\chi^\alpha E_\chi_B. \]

Hence,
\[ L(\rho^3 |E_\chi^\alpha|^2) \lesssim \rho^3 \left[ M^p \delta^{(1-\varepsilon_0)p-1} \rho^{-2} |E_\chi^\alpha|^2 + \rho^{-1} |d \tilde{Z}^\alpha E_\chi|^2 + |d \tilde{Z}^\alpha (|\chi|^2)| + |e_\chi^\alpha| \right]. \]  
(6.15)

Using Lemma 5.2 and (6.15) for \( \rho^3 |E_\chi^\alpha| \) to get
\[ \delta^1 \rho^2 E_\chi^\alpha \|_{L^2(\Sigma^\tau_v)} \lesssim M^p \delta^{(1-\varepsilon_0)p-1} \tau^{-2} \rho^2 E_\chi^\alpha \|_{L^2(\Sigma^\tau_v)} \]
\[ + \tau \|d \tilde{Z}^\alpha E_\chi\|_{L^2(\Sigma^\tau_v)} + \tau^2 \|d \tilde{Z}^\alpha (|\chi|^2)\|_{L^2(\Sigma^\tau_v)} + \tau^2 \|e_\chi^\alpha \|_{L^2(\Sigma^\tau_v)} \} d\tau. \]  
(6.16)

Next, we estimate the terms in integrand of (6.16) one by one.

**1a) Estimate of \( d \tilde{Z}^\alpha E_\chi \)**

Due to \( E_\chi = \frac{1}{2} p e^4 \mu^{-1} \phi_0^{-1} L \varphi_0 \), then it follows from (3.1), Proposition 5.2 and Lemma 5.1 that
\[ \delta^1 \|d \tilde{Z}^\alpha E_\chi \|_{L^2(\Sigma^\tau_v)} \]
\[ \lesssim \left( M^{\varepsilon_0} \tau^{-1} \right)^{p-1} \left\{ \tau^{-1} \delta \|T Z^{\leq \alpha+1} \phi_\alpha \|_{L^2(\Sigma^\tau_v)} + \tau^{-1} \delta \|d Z^{\leq \alpha+1} \phi_\alpha \|_{L^2(\Sigma^\tau_v)} \right. \]
\[ + \delta \|Z^{\leq \alpha+1} \phi_\alpha \|_{L^2(\Sigma^\tau_v)} + M \delta^{\varepsilon_0} \tau^{-2} \delta \|Z^{\leq \alpha+1} \mu \|_{L^2(\Sigma^\tau_v)} \]
\[ + M \delta \|Z \mu \|_{L^2(\Sigma^\tau_v)} \}
\[ \lesssim M^{2p} \delta^{(1-\varepsilon_0)p-\frac{3}{2}} \tau^{-3} \rho^p + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \tau^{-p} \left\{ \sqrt{\tilde{E}_{1, \leq |\alpha|+2}^2 (\tau, u)} + \sqrt{\tilde{E}_{2, \leq |\alpha|+2}^2 (\tau, u)} \right\}. \]  
(6.17)
(1-b) Estimate of $d\tilde{Z}^\alpha (|\tilde{\chi}|^2)$

According to (2.47), we have

$$(\text{div}_g \tilde{\chi})_A = d_A \text{tr}_g \chi + I_A, \quad \text{(6.18)}$$

where

$$I_A = c^{-1}(d^B c) \chi_{AB} - c^{-1}(d^A c) \text{tr}_g \chi. \quad \text{(6.19)}$$

By commuting $\mathcal{L}_Z^\alpha$ with $\text{div}_g$, one has from (2.35) that

$$(\text{div}_g \mathcal{L}_Z^\alpha \tilde{\chi})_A = \mathcal{L}_Z^\alpha (\mathcal{L}_Z^{BC} \tilde{\chi} C_{AB}) + \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_Z^{\beta_1} [\mathcal{L}_Z^{BC} (\tilde{\chi}^D_T) \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{BD} + \mathcal{L}_Z^{BC} (\tilde{\chi})^D_T \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{AD} + \mathcal{L}_Z^{BC} \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{AB}].$$

Then (6.18) implies

$$(\text{div}_g \mathcal{L}_Z^\alpha \tilde{\chi})_A = d_A \tilde{Z}^\alpha \text{tr}_g \chi + I_A^{\alpha}, \quad \text{(6.20)}$$

where

$$I_A^{\alpha} = \mathcal{L}_Z^\alpha I_A - \sum_{\beta_1 + \beta_2 = \alpha - 1} \mathcal{L}_Z^{\beta_1} [\mathcal{L}_Z^{BC} (\tilde{\chi}^D_T) \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{BD} + \mathcal{L}_Z^{BC} (\tilde{\chi})^D_T \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{AD} + \mathcal{L}_Z^{BC} \mathcal{L}_Z^{\beta_2} \tilde{\chi}_{AB}].$$

with

$$\delta^\tau ||I^\alpha||_{L^2(\Sigma^\nu)} \lesssim \delta^\tau \delta^\tau \delta^\tau ||\mathcal{L}_Z^{\alpha} \tilde{\chi}||_{L^2(\Sigma^\nu)} + M^p \delta(1 - \varepsilon_0)p \tau^{-3} \{ \delta^\tau ||\mathcal{L}_Z^{\alpha} \tilde{\chi}||_{L^2(\Sigma^\nu)} + \delta^\tau ||\mathcal{L}_Z^{\alpha} \tilde{\chi}||_{L^2(\Sigma^\nu)} \}. \quad \text{(6.21)}$$

It follows from (6.1), (6.3) and (6.20) that

$$\delta^\tau ||\nabla \mathcal{L}_Z^\alpha \tilde{\chi}||_{L^2(\Sigma^\nu)} \lesssim \delta^\tau ||d_A \tilde{Z}^\alpha \text{tr}_g \chi||_{L^2(\Sigma^\nu)} + \delta^\tau ||I^\alpha||_{L^2(\Sigma^\nu)}, \quad \text{(6.22)}$$

and then by (6.23) and (6.22),

$$\delta^\tau ||d\tilde{Z}^\alpha (|\tilde{\chi}|^2)||_{L^2(\Sigma^\nu)} \leq M^p \delta(1 - \varepsilon_0)p \tau^{-2} \delta^\tau ||\nabla \mathcal{L}_Z^\alpha \tilde{\chi}||_{L^2(\Sigma^\nu)} + M^p \delta(1 - \varepsilon_0)p \tau^{-3} \delta^\tau ||\mathcal{L}_Z^\alpha \tilde{\chi}||_{L^2(\Sigma^\nu)}$$

Therefore, using Proposition 5.2 and (6.17) to get

$$\delta^\tau ||d\tilde{Z}^\alpha (|\tilde{\chi}|^2)||_{L^2(\Sigma^\nu)} \leq M^p \delta(1 - \varepsilon_0)p \tau^{-2} ||E\tilde{\chi}||_{L^2(\Sigma^\nu)} + M^p \delta(1 - \varepsilon_0)p \tau^{-4} \delta^\tau \left( \frac{1}{\sqrt{E_1 |\alpha| + 2(\tau, u)} + \sqrt{E_2 |\alpha| + 2(\tau, u)}} \right). \quad \text{(6.24)}$$

(1-c) Estimate of $e^\alpha_{\tilde{\chi}}$
By (6.10), (2.43), (2.40), (2.45) and (6.6), one has
\[
\begin{aligned}
\delta^t \| \mathcal{L}_Z^2 \delta e \chi \|_{L^2(\Sigma^N_t)} \\
\lesssim M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-p+2} \delta^t \| Z^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \\
+ \delta^{-1} \tau^{-1} (M \delta^{1-\varepsilon_0} \tau^{-1}) p^{-1} (\delta^t \| dZ^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} + \tau^{-1} \delta^t \| Z^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)}) \\
\lesssim M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-p+2} \delta^t \| Z^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \\
+ M^{p-1} \delta^{1-\varepsilon_0} (p-1)^{-1} \tau^{-p} \{ \tau^{-s} \sqrt{E_1,\leq|\alpha|+2}(\tau,u) + \delta \tau^{-1} \sqrt{E_2,\leq|\alpha|+2}(\tau,u) \}. 
\end{aligned}
\] (6.25)

For the third term in (6.13), it follows from \((T)\) \(\tilde{\sigma}_L = -c^2 \tilde{d}(e^{-2} \mu)\), (6.12) and the induction argument that
\[
\begin{aligned}
\delta^t \| \sum_{\beta_1+\beta_2=\alpha-1} \mathcal{L}_Z^{\beta_1} \mathcal{L}_E^{\beta_2} \|_{L^2(\Sigma^N_t)} \\
\lesssim \delta^t \| \sum_{\beta_1+\beta_2=\alpha-1} \mathcal{L}_Z^{\beta_1} \left( \mathcal{L}_E^{\beta_2} \nabla \mathcal{E}_X^{\beta_2} + \nabla \mathcal{E}_X^{\beta_2} \mathcal{L}_E^{\beta_2} \right) \|_{L^2(\Sigma^N_t)} \\
\lesssim M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-2} \left\{ \delta^t \| \mathcal{E}_X^{\beta_2} \|_{L^2(\Sigma^N_t)} + \tau^{-1} \delta^t \| dZ^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} + \tau^{-1} \delta^t \| d\tilde{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \right\} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-5} \left\{ \delta^t \| \bar{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} + \delta^t \| \bar{Z}^{\leq \alpha+1} c \|_{L^2(\Sigma^N_t)} \right\} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-4} \delta^t \| \mathcal{L}_Z^{\alpha(R)} \bar{\sigma}_L \|_{L^2(\Sigma^N_t)}. 
\end{aligned}
\] (6.26)

We point out that the estimates of the other terms in (6.13) are easily to be handled. Then after substituting (6.25) and (6.26) into (6.13), one has
\[
\begin{aligned}
\delta^t \| e_\chi^{\alpha} \|_{L^2(\Sigma^N_t)} \\
\lesssim M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-2} \delta^t \| \mathcal{E}_X^{\alpha} \|_{L^2(\Sigma^N_t)} + \tau^{-3} \delta^t \| \bar{L}^{\alpha} \mathcal{E}_X^{\alpha} \|_{L^2(\Sigma^N_t)} + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \tau^{-p} \delta^t \| d\tilde{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \\
+ M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \tau^{-p+1} \{ \tau^{-s} \sqrt{E_1,\leq|\alpha|+2}(\tau,u) + \sqrt{E_2,\leq|\alpha|+2}(\tau,u) \} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-5} \left\{ \delta^t \| \bar{L}^{\alpha(R)} \|_{L^2(\Sigma^N_t)} + \delta^t \| \bar{L}^{\alpha(T)} \|_{L^2(\Sigma^N_t)} + \tau^2 \delta^t \| d\tilde{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \right\} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \tau^{-4} \delta^t \| Z^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} + \delta^t \| \mathcal{L}_Z^{\alpha(R)} \bar{\sigma}_L \|_{L^2(\Sigma^N_t)}. 
\end{aligned}
\]

Therefore, by Proposition 5.2 and (6.17), we arrive at
\[
\begin{aligned}
\int_{t_0}^t \tau^2 \delta^t \| e_\chi^{\alpha} \|_{L^2(\Sigma^N_t)} d\tau \\
\lesssim \int_{t_0}^t M^p \delta^{1-\varepsilon_0} p^{-1} \delta^t \| \mathcal{E}_X^{\alpha} \|_{L^2(\Sigma^N_t)} d\tau + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \tau^{-p} \delta^t \| d\tilde{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \tau \ln t + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \ln t \left\{ \sqrt{E_1,\leq|\alpha|+2}(t,u) + \sqrt{E_2,\leq|\alpha|+2}(t,u) \right\}. 
\end{aligned}
\] (6.27)

Substituting (6.17), (6.24) and (6.27) into (6.16), and applying Gronwall’s inequality, one obtains
\[
\begin{aligned}
\delta^t \| P^2 \mathcal{E}_X^{\alpha} \|_{L^2(\Sigma^N_t)} \lesssim & M^p \delta^{1-\varepsilon_0} p^{-1} \tau \ln t + M^{p-1} \delta^{(1-\varepsilon_0)(p-1)} \tau^{-p} \delta^t \| d\tilde{Z}^{\leq \alpha+1} \|_{L^2(\Sigma^N_t)} \\
+ M^p \delta^{1-\varepsilon_0} p^{-1} \ln t \left\{ \sqrt{E_1,\leq|\alpha|+2}(t,u) + \sqrt{E_2,\leq|\alpha|+2}(t,u) \right\}. 
\end{aligned}
\]
Due to $d\tilde{Z}^\alpha \text{tr}_g \chi = E_\chi + d\tilde{Z}^\alpha E_\chi$, then by (6.17) and (6.23),

$$\delta^l \|d\tilde{Z}^\alpha \text{tr}_g \chi\|_{L^2(\Sigma^t_{t'})} + \delta^l \|\nabla \tilde{Z}^\alpha \chi\|_{L^2(\Sigma^t_{t'})}$$

$$\lesssim M^P \delta^{1-\varepsilon_0}(p-\frac{1}{2})t^{-2} \ln t + M^{p-1}\delta^{(1-\varepsilon_0)(p-1)-1}t^{-\frac{3}{2}-s} \int_0^t \sqrt{\tilde{E}_{1,\leq |\alpha|+2}(t, u')}du'$$

$$+ M^{p-1}\delta^{(1-\varepsilon_0)(p-1)}t^{-2} \ln t \left\{ \sqrt{\tilde{E}_{1,\leq |\alpha|+2}(t, u)} + \sqrt{\tilde{E}_{2,\leq |\alpha|+2}(t, u)} \right\}.$$  

(6.28)

If there exists at least one $\rho L$ in $\tilde{Z}^\alpha$, then by making use of (3.6), the commutators of vector fields and Proposition 5.2, we have

$$\delta^l \|\nabla \tilde{Z}^\alpha \chi\|_{L^2(\Sigma^t_{t'})} \lesssim M^P \delta^{1-\varepsilon_0}(p+\frac{1}{2})t^{-3}$$

$$+ M^{p-1}\delta^{(1-\varepsilon_0)(p-1)} \left\{ t^{-2-s} \sqrt{\tilde{E}_{1,\leq |\alpha|+2}(t, u)} + \delta t^{-3} \sqrt{\tilde{E}_{2,\leq |\alpha|+2}(t, u)} \right\}.$$  

(6.29)

Therefore, for any vector filed $Z \in \{\rho L, T, R_1, R_2, R_3\}$, (6.28) and (6.29) give that

$$\delta^l \|\nabla \tilde{Z}^\alpha \chi\|_{L^2(\Sigma^t_{t'})} + \delta^l \|d\tilde{Z}^\alpha \text{tr}_g \chi\|_{L^2(\Sigma^t_{t'})}$$

$$\lesssim M^P \delta^{1-\varepsilon_0}(p-\frac{1}{2})t^{-2} \ln t + M^{p-1}\delta^{(1-\varepsilon_0)(p-1)-1}t^{-\frac{3}{2}-s} \int_0^t \sqrt{\tilde{E}_{1,\leq |\alpha|+2}(t, u')}du'$$

$$+ M^{p-1}\delta^{(1-\varepsilon_0)(p-1)}t^{-2} \ln t \left\{ \sqrt{\tilde{E}_{1,\leq |\alpha|+2}(t, u)} + \sqrt{\tilde{E}_{2,\leq |\alpha|+2}(t, u)} \right\}.$$  

(6.30)

### 6.2 Estimates on the derivatives of $\nabla^2 \mu$

Similarly to $\text{tr}_g \chi$, we use the transport equation (2.44) to estimate $\Delta \mu$. By Lemma 2.5, one has

$$L\Delta \mu = L\Delta \mu + \left[L, \Delta\right]\mu$$

$$= -\left[cT\Delta c\right] - 2c(d_A(c^{-2}\mu)\chi^{AB} - \frac{1}{2}d^B(c^{-2}\mu)\text{tr}_g \chi)dBc - c^{-1}\mu \chi^{AB}\nabla^2_{AB}c$$

$$- 2\rho^{-1}c^{-1}\mu \Delta c - 2dc \cdot dTc - (\Delta c)Tc + L(\mu \Delta \ln c) - (L\mu)\Delta \ln c$$

$$+ 2\mu \chi^{AB}\nabla^2_{AB} \ln c + 2\rho^{-1}\mu \Delta \ln c + 2d(c^{-1}Lc) \cdot \text{tr}_g \chi - c^{-1}Lc \Delta \mu$$

$$- (d_A \text{tr}_g \chi) d^A \mu - 2\chi^{AB} \nabla^2_{AB} \mu - 2\rho^{-1} \Delta \mu - 2Lc \text{tr}_g \chi$$

(6.31)

where $I_A$ has been given in (6.19). Observe that the term with underline can be removed to the left hand side of (6.31), while the terms with wavy line will be treated by the elliptic estimate and Gronwall’s inequality. The strategy to treat the boxed term which contains the third order derivative of the solution is to transfer it into such a form $L(\partial^{\leq 2} \varphi) + \text{l.o.t.}$. Indeed, by (2.46), we have

$$T \text{tr}_g \chi = T(\varphi^{AB} \chi_{AB}) = \Delta \mu + J$$

(6.32)

with

$$J = c^{-2}\mu \chi^2 - c^{-1}L(c^{-1}\mu)\text{tr}_g \chi - \nabla^2(c^{-1}\mu \text{tr}_g c) + c^{-2}\mu |dc|^2 - c^{-1}(d_A c) d^A \mu.$$  

(6.33)
In addition, (6.8) gives that

\[ \Delta c = \nabla^A \left( -\frac{1}{2} p c^3 \varphi_0^{-1} \mathcal{d}_A \varphi_0 \right) = -\frac{1}{2} p c^3 \mu^{-1} \varphi_0^{-1} (L \varphi_0 \phi_0) - \frac{1}{2} p c^3 \mu^{-1} \varphi_0^{-1} (T \varphi_0) \text{tr}_g \chi \]

(6.34)

Combining (6.34) and (6.32) yields

\[-c T \Delta c = L \left( \frac{1}{2} p c^4 \mu^{-1} \varphi_0^{-1} T L \varphi_0 \right) - c \mu^{-1} (T c) \Delta \mu - L \left( \frac{1}{2} p c^4 \mu^{-1} \varphi_0^{-1} T L \varphi_0 \right) \]

(6.35)

Substituting (6.35) into (6.31), by direct computation then one has

\[ L(\Delta \mu - E_\mu) = (\mu^{-1} L \mu - 2 \rho^{-1}) \Delta \mu - 2 \chi^{AB} \nabla_{AB} \mu - (\mathcal{d}_A \text{tr}_g \chi) \mathcal{d}^A \mu + e_\mu, \]

where

\[ E_\mu = \frac{1}{2} p c^4 \mu^{-1} \varphi_0^{-1} T L \varphi_0 + \mu \Delta \ln c, \]

\[ e_\mu = -L \left( \frac{1}{2} p c^4 \mu^{-1} \varphi_0^{-1} T L \varphi_0 \right) - \frac{1}{2} p c^4 \mu^{-1} \varphi_0^{-1} (T \varphi_0) \right) \]

(6.37)

It is easy to see that \( e_\mu \) is composed of the terms which contain the factors \( \partial \leq 2 \varphi, \partial \leq 1 \mu \) and \( \tilde{\chi} \).

Let \( E_\mu^\alpha = \tilde{Z}^{\alpha} (\Delta \mu - E_\mu) \) with \( Z \in \{ T, R_1, R_2, R_3 \} \). Then by the analogous induction argument on (6.36), we have

\[ LE_\mu^\alpha = (\mu^{-1} L \mu - 2 \rho^{-1}) E_\mu^\alpha + [L, \tilde{Z}] E_\mu^\alpha - 2 \chi^{AB} \nabla_{AB} \mu - (\mathcal{d}_A \text{tr}_g \chi) \mathcal{d}^A \mu + e_\mu^\alpha, \]

where

\[ e_\mu^\alpha = \tilde{Z}^{\alpha} e_\mu + (\mu^{-1} L \mu - 2 \rho^{-1}) \tilde{Z}^{\alpha} E_\mu + \sum_{\beta_1 + \beta_2 = \alpha, |\beta_1| \geq 1} \tilde{Z}^{\beta_1} (\mu^{-1} L \mu - 2 \rho^{-1}) \tilde{Z}^{\beta_2} \Delta \mu + \sum_{\beta_1 + \beta_2 = \alpha, |\beta_1| \geq 1} \tilde{L}_{Z}^{\beta_1} \tilde{X} \cdot \tilde{L}_{Z}^{\beta_2} \nabla_{AB} \mu \]

(6.39)
Since
\[ L(\rho^2 E_{\mu}^\alpha) = \rho^2 \{ \mu^{-1} L_{\mu} \cdot E_{\mu}^\alpha + [L, \bar{Z}] E_{\mu}^\alpha - 2\bar{\chi} A^B \bar{\chi}_2 \bar{\chi} A_{\mu}^B - \mu \cdot \bar{d} Z_{\mu} \bar{\chi} + e_{\mu}^\alpha \}, \]
it follows from Lemma 5.2 for \( \rho^2 E_{\mu}^\alpha \) and Proposition 5.1 that
\[
\delta^l \| \rho E_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} \\
\leq M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \| E_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} + M^p \delta^{1 - \varepsilon_0} p^{-1} \| \bar{d} Z_{\mu} \bar{\chi} \|_{L^2(\Sigma_0^\infty)} + \tau \| e_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} + \tau \| e_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} \int_0^t \| \bar{d} Z_{\mu} \bar{\chi} \|_{L^2(\Sigma_0^\infty)} \, d\tau.
\]

(6.40)

Next, we estimate the terms in the integrand of (6.40) one by one (notice that the \( L^2 \) estimate of \( \bar{d} Z_{\mu} \bar{\chi} \) has been obtained in (6.30)).

(2-a) Estimate of \( \bar{L} Z^2 \bar{\chi} \)

With the help of Lemma 2.5, we have from (6.1) and the elliptic estimate (6.4) that
\[
\delta^l \| \bar{L} Z^2 \bar{\chi} \|_{L^2(\Sigma_0^\infty)} \\
\leq \delta^l \| \bar{L} Z \bar{\chi} \|_{L^2(\Sigma_0^\infty)} + \delta^l \| \bar{L} \bar{\chi} \|_{L^2(\Sigma_0^\infty)} + \delta^l \| \bar{d} Z_{\mu} \bar{\chi} \|_{L^2(\Sigma_0^\infty)} \\
\leq \delta^l \| \bar{L} Z \bar{\chi} \|_{L^2(\Sigma_0^\infty)} + \delta^l \| \bar{d} Z_{\mu} \bar{\chi} \|_{L^2(\Sigma_0^\infty)}.
\]

(6.41)

(2-b) Estimate for \( e_{\mu}^\alpha \)

In view of (6.39) and (6.37), one knows that all terms in (6.39) can be estimated with the help of Propositions 5.2, 5.1 and (3.1), that is,
\[
\delta^l \| e_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} \leq M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-2} \\
+ M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-2} \int_0^t \bar{F}_{1, \leq |\alpha| + 2}(t, u') \, du' \, d\tau.
\]

(6.42)

Substituting (6.41), (6.42) and (6.30) into (6.40), and then applying Gronwall’s inequality to obtain
\[
\delta^l \| \rho E_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} \leq M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \ln t + M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-2} \int_0^t \bar{F}_{1, \leq |\alpha| + 2}(t, u') \, du' \\
+ M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-1} \int_0^t \bar{F}_{1, \leq |\alpha| + 2}(t, u') \, du'.
\]

Hence,
\[
\delta^l \| E_{\mu}^\alpha \|_{L^2(\Sigma_0^\infty)} \leq M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \ln t + M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-2} \int_0^t \bar{F}_{1, \leq |\alpha| + 2}(t, u') \, du' \\
+ M^p \delta^{1 - \varepsilon_0} p^{-1 - \mu} \tau^{-1} \int_0^t \bar{F}_{1, \leq |\alpha| + 2}(t, u') \, du'.
\]
It follows from the definition of $E^\alpha_{\mu}$ and (6.41) that
\[
\delta t \| Z^\alpha \Delta \mu \|_{L^2(\Sigma^t)} + \delta t \| \mathcal{L}_Z^2 \nabla^2 \mu \|_{L^2(\Sigma^t)} \\
\lesssim M^p \delta (1-\varepsilon_0) t^{-1} ln t + M^p \delta (1-\varepsilon_0) (2p-1) t^{-1} \int_0^t \tilde{F}_{1,|\alpha|+2}(t,u) du' + M^p \delta (1-\varepsilon_0) (p-1) t^{-1} \left\{ \sqrt{\tilde{E}_{1,|\alpha|+2}(t,u)} + ln t \sqrt{\tilde{E}_{2,|\alpha|+2}(t,u)} \right\}. (6.43)
\]

As in the estimate of $\nabla \mathcal{L}_Z^2 \tilde{\chi}$, when there exists at least one $\rho L$ in $Z^\alpha$, $\delta t \| \mathcal{L}_Z^2 \nabla^2 \mu \|_{L^2(\Sigma^t)}$ can be estimated with the help of (2.44) and Proposition 5.2. Therefore, together with (6.43), for any vector field $Z \in \{\rho L, T, R_1, R_2, R_3\}$, we have
\[
\delta t \| Z^\alpha \Delta \mu \|_{L^2(\Sigma^t)} + \delta t \| \mathcal{L}_Z^2 \nabla^2 \mu \|_{L^2(\Sigma^t)} \\
\lesssim M^p \delta (1-\varepsilon_0) t^{-1} ln t + M^p \delta (1-\varepsilon_0) (2p-1) t^{-1} \int_0^t \tilde{F}_{1,|\alpha|+2}(t,u) du' + M^p \delta (1-\varepsilon_0) (p-1) t^{-1} \left\{ \sqrt{\tilde{E}_{1,|\alpha|+2}(t,u)} + ln t \sqrt{\tilde{E}_{2,|\alpha|+2}(t,u)} \right\}. (6.44)
\]

7 Commutator estimates

7.1 Commuted covariant wave equation

In order to get the energy estimates for $\varphi_\gamma$ and its derivatives, we now choose $\Psi = \Psi_\gamma^{[|\alpha|+1]} = Z_{|\alpha|+1} \cdots Z_1 \varphi_\gamma$ and $\Phi = \Phi_\gamma^{[|\alpha|+1] = \mu \Box \gamma \Psi_\gamma^{[|\alpha|+1]}$ in (4.11). By commuting vectorfield $Z_k$ ($k = 1, \ldots, |\alpha| + 1$) with (4.1), the induction argument gives
\[
\Phi_\gamma^{[|\alpha|+1]} = \sum_{k=1}^{|\alpha|} \left( Z_{|\alpha|+1} + (Z_{|\alpha|+1}) \lambda \cdots (Z_{|\alpha|+2-k} + (Z_{|\alpha|+2-k}) \lambda) \right) \left( \mu \text{div}_g (Z_{|\alpha|+1-k}) C_\gamma^{[|\alpha|]} \right) \\
+ \mu \text{div}_g (Z_{|\alpha|+1}) C_\gamma^{[|\alpha|]} + (Z_{|\alpha|+1} + (Z_{|\alpha|+1}) \lambda \cdots (Z_{1} + (Z_{1}) \lambda) \Phi_\gamma^{0}, (7.1)
\]

where
\[
\text{div}_g (Z) C_\gamma^j = \partial_\beta \left[ \left( (Z) \pi^{\beta \nu} - \frac{1}{2} g^{\beta \nu} \text{tr}_g (Z) \pi \right) \partial_\nu \Psi_\gamma^j \right],
\]
\[
(Z) \lambda = \frac{1}{2} \text{tr}_g (Z) \pi - \mu^{-1} Z \mu,
\]
\[
\Psi_\gamma^0 = \varphi_\gamma, \Phi_\gamma^0 = \mu \Box \gamma \varphi_\gamma
\]

with $\text{tr}_g (Z) \pi = g^{\alpha \beta} (Z) \pi_{\alpha \beta} = -\frac{1}{2} \mu^{p-1} (Z) \pi_{L^L} + \frac{1}{2} \text{tr}_g (Z) \pi$. Thus,
\[
\langle \rho L \rangle \lambda = \rho \text{tr}_g \tilde{\chi} + 3,
\]
\[
\langle T \rangle \lambda = - c^{-2} \mu \text{tr}_g \chi,
\]
\[
\langle R \rangle \lambda = c^{-1} v \text{tr}_g \chi, (7.2)
\]
Under the frame \( \{ L, L, X_1, X_2 \} \), \( \mu \text{div}_g (Z) C^j_{\gamma} \) could be written as

\[
\mu \text{div}_g (Z) C^j_{\gamma} = -\frac{1}{2} [L + L(c^{-2} \mu) - c^{-2} \mu \text{tr}_g \chi]^j (Z) C^j_{\gamma, L} - \frac{1}{2} (L + \text{tr}_g \chi)^j (Z) C^j_{\gamma, L} + \nabla^A (\mu (Z) Q^j_{\gamma, A}),
\]

(7.3)

where

\[
(Z) C^j_{\gamma, L} = g((Z) C^j_{\gamma}, L) = -\frac{1}{2} \text{tr}_g (Z) L \Psi^j + g((Z) \nabla^A, \Psi^j),
\]

(7.4)

\[
\mu (Z) Q^j_{\gamma, A} = g(\mu (Z) C^j_{\gamma}, X_A) = -\frac{1}{2} (Z) \Psi^j - \frac{1}{2} (Z) L \Psi^j + \frac{1}{2} (Z) \nabla^A \Psi^j + \mu((Z) \nabla^A, \Psi^j) - \frac{1}{2} \text{tr}_g (Z) \nabla^A \Psi^j.
\]

Note that if we substitute (7.4) into (7.3) directly, then a lengthy and tedious equality for \( \mu \text{div}_g (Z) C^j_{\gamma} \) is obtained. To overcome this default and treat these terms more convenient, we will divide \( \mu \text{div}_g (Z) C^j_{\gamma} \) into the following three parts as in [13]:

\[
\mu \text{div}_g (Z) C^j_{\gamma} = (Z) \mathcal{N}^j_{a_1} + (Z) \mathcal{N}^j_{a_2} + (Z) \mathcal{N}^j_{a_3},
\]

where

\[
(Z) \mathcal{N}^j_{a_1} = \left[ \frac{1}{4} L \mu^{-1}(Z) \pi_{LL} + \frac{1}{4} L \text{tr}_g (Z) \Psi^j - \frac{1}{2} \nabla^A (Z) \Psi^j - \frac{1}{2} \nabla^A \Psi^j \right] L \Psi^j - \left[ \frac{1}{2} \nabla^A \Psi^j \right] L \Psi^j + \frac{1}{2} L \text{tr}_g (Z) \Psi^j - \frac{1}{2} \nabla^A \Psi^j,
\]

(7.5)

\[
(Z) \mathcal{N}^j_{a_2} = \frac{1}{2} \text{tr}_g (Z) \Psi^j - \frac{1}{2} \text{tr}_g \chi \frac{1}{2} L \Psi^j - \frac{1}{2} \mu^{-1}(Z) \pi_{LL} L^2 \Psi^j - \frac{1}{2} \nabla^A \Psi^j - \frac{1}{2} \text{tr}_g (Z) \Psi^j - \frac{1}{2} \nabla^A \Psi^j,
\]

(7.6)

\[
(Z) \mathcal{N}^j_{a_3} = \left[ \frac{1}{4} \mu^{-1}(Z) \Psi^j - \frac{1}{4} \mu^{-1}(Z) \text{tr}_g \chi \text{tr}_g (Z) \Psi^j + \frac{1}{2} (Z) \Psi^j - \frac{1}{2} \mu \text{tr}_g (Z) \Psi^j - \frac{1}{2} \mu \text{tr}_g (Z) \Psi^j \right] L \Psi^j + \frac{1}{2} \text{tr}_g (Z) \Psi^j - \frac{1}{2} \text{tr}_g \chi \frac{1}{2} L \Psi^j - \frac{1}{2} \nabla^A \Psi^j - \frac{1}{2} \text{tr}_g \chi \frac{1}{2} L \Psi^j - \frac{1}{2} \nabla^A \Psi^j - \frac{1}{2} \text{tr}_g \chi \frac{1}{2} L \Psi^j - \frac{1}{2} \nabla^A \Psi^j,
\]

(7.7)

One sees that \( (Z) \mathcal{N}^j_{a_1} \) collects the products of the first order derivatives of the deformation tensor and the first order derivatives of \( \Psi^j \), and \( (Z) \mathcal{N}^j_{a_2} \) contains the terms which are the products of the deformation tensor and the second order derivatives of \( \Psi^j \) except the first term which has the better smallness and higher time-decay rate due to (2.41). In addition, \( (Z) \mathcal{N}^j_{a_3} \) is the collections of the products of the deformation tensor and the first order derivatives of \( \Psi^j \) which can be treated more easily.

Through making the preparations for the \( L^2 \) estimates of the quantities in Subsection 5.2 and Section 6, we are ready to handle the error terms \( \int_{\Omega_{1, u}} \rho Z \Phi \cdot L \Psi \) and \( \int_{\Omega_{1, u}} |\Phi \cdot L \Psi| \) in (4.11), and then get the final energy estimates for \( \varphi_\gamma \) and its derivatives.
7.2 Error estimates for the terms originating from $\mathcal{M}_1^{\rho L}$

For the most difficult term $\mathcal{M}_1^{\rho L}$ in $\Phi_1^{\alpha+1}$, the number of the top order derivatives is $|\alpha|$, which means that there will be some terms containing the $(|\alpha| + 1)^{th}$ order derivatives of the deformation tensors. In this case, $\bar{E}_{i, \leq |\alpha| + 3}$ will appear in the right hand side of (4.11) if one only adopts Proposition 5.2. This leads to that it can not be absorbed by the left hand side of (4.11). To overcome such a difficulty, we will carefully examine the expression of $\mathcal{M}_1^{\rho L}$ and apply the estimates in Section 6 to deal with the top order derivatives of $\chi$ and $\mu$.

By substituting the components of the deformation tensor, we can obtain the expression of $\mathcal{M}_1^{\rho L}$.

- **The case of $Z = \rho L$**

By (2.32), one has

$$
\mathcal{M}_1^{\rho L} = \left\{ \frac{\rho L^2}{2} \right\} \left[ -\frac{1}{2} L(\rho \text{tr}_g \chi) - \rho \nabla^A \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right] \right\} L \Psi_{\gamma}^j - \left\{ \mathcal{L}_L \rho \mathcal{L}_c \left( c^{-2} \mu \right) \right\} \nabla^A \chi + \left\{ \frac{1}{2} L(\rho \text{tr}_g \chi) L \Psi_{\gamma}^j \right\}.
$$

For the terms with underline in (7.8), we point out there exists an important cancelation. In fact, it follows from (6.32) that

$$
\frac{1}{2} L(\rho \text{tr}_g \chi) - \rho \nabla^A \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right] = \rho J - \text{tr}_g \chi + \frac{1}{2} c^{-2} \mu L(\rho \text{tr}_g \chi) - \rho \nabla^A \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right],
$$

which implies that the term $\Delta \mu$ has been eliminated.

For the term with wavy line in (7.8), it follows from (2.47) that

$$
\frac{1}{2} L(\rho \text{tr}_g \chi) = -2 \rho \nabla^B \left[ \chi \text{tr}_g \chi g_{AB} \right] - \frac{1}{2} \mu \text{tr}_g \chi g_{AB}.
$$

In addition, we notice that there are some terms in (7.8) whose factors are the derivatives with respect to $L$ and which contain the second order derivative of $\mu$ or the first derivative of $\chi$. In this case, the derivatives of these terms can be treat directly by Proposition 5.2 since $\chi$ and $\mu$ satisfy the transport equations (2.45) and (2.44) respectively.

Therefore, we can arrive at

$$
\mathcal{M}_1^{\rho L} = \rho \mu \text{tr}_g \chi \cdot \nabla^A \Psi_{\gamma}^j + \text{l.o.t.},
$$

- **The case of $Z = T$**

By (2.33), we have

$$
\mathcal{M}_1^{\rho L} = \left\{ LT(c^{-2} \mu) - \frac{1}{2} L(c^{-2} \mu \text{tr}_g \chi) + \frac{1}{2} L \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right] \right\} L \Psi_{\gamma}^j + \left\{ \frac{1}{2} \mathcal{L}_L \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right] \right\} \nabla^A \chi + \left\{ - \frac{1}{2} L(\rho \text{tr}_g \chi) + \frac{1}{2} \nabla^A \left[ c^2 \mathcal{L}_c \left( c^{-2} \mu \right) \right] \right\} L \Psi_{\gamma}^j.
$$
For the terms with underline in (7.11), one has
\[
\frac{1}{2} \mathcal{L}_L [c^2 d_A (c^{-2} \mu)] - d_A T \mu = \frac{1}{2} c^{-2} \mu d_A L \mu + \frac{1}{2} \mathcal{L}_L [c^2 \mu d_A (c^{-2})],
\]
(7.12)
which implies that the term \(d_A T \mu\) has been eliminated.

For the terms with braces in (7.11), we have
\[
- \frac{1}{2} \mathcal{L}(c^{-2} \mu \text{tr} \chi) + \frac{1}{2} \nabla^A \mu [\mu d_A (c^{-2} \mu)]
= - \frac{1}{2} c^{-2} \mu \Delta \mu - c^2 \mu J - \frac{1}{2} c^4 \mu^2 L \text{tr} \chi - \frac{1}{2} L(c^{-2} \mu) \text{tr} \chi + \frac{1}{2} d(c^{-2} \mu) \cdot d \mu + \frac{1}{2} \nabla^A [\mu^2 d_A (c^{-2})]
\]
and
\[
\frac{1}{2} \nabla^A [c^2 d_A (c^{-2} \mu)] = \frac{1}{2} \Delta \mu + \frac{1}{2} \nabla^A [c^2 \mu d_A (c^{-2})].
\]

For the terms with wavy line in (7.11),
\[
- 2 \nabla^B (c^{-2} \mu^2 (\bar{\chi}_{AB} - \frac{1}{2} \text{tr} \chi \bar{\varphi}_{AB}))
= - c^{-2} \mu^2 d_A \text{tr} \chi + 2 c^{-2} \mu \zeta^B \bar{\chi}_{AB} - 2 c^{-2} \mu \zeta_A \text{tr} \chi - 2 c^{-2} \mu \rho^{-1} \zeta_A - 2 \nabla^B (c^{-2} \mu^2) (\bar{\chi}_{AB} - \frac{1}{2} \text{tr} \chi \bar{\varphi}_{AB}).
\]

Therefore, we actually arrive at
\[
(\mathcal{T}) N_i^j = (\Delta \mu) T \Psi_i^j - c^{-2} \mu^2 \text{tr} \chi \cdot d \Psi_i^j + \text{l.o.t.}.
\]

- The case of \(Z = R_i\)

By (2.34), we have
\[
(\mathcal{R}_i) N_i^j = \left\{ \mathcal{L}(c^{-2} \mu) + \frac{1}{2} L(c^{-1} v_i \text{tr} \chi) - \frac{1}{2} \nabla^A (c^{-2} \mu R_i^B \bar{\chi}_{AB}) - \nabla^A (c^{-1} v_i d_A \mu)
- \frac{1}{2} \nabla^A [2 c^{-2} (c - 1) \mu \rho^{-1} \bar{\varphi}_{AB} R_i^B - 3 c^{-2} \mu v_i d_A c + c^{-2} \mu \epsilon_{ijk} \bar{L}^j d_A x^k]
+ 2 c^{-1} \mu \epsilon_{ijk} \bar{L}^j d_A x^k \right\} L \Psi_i^j
+ \left\{ \frac{1}{2} \mathcal{L}_L (R_i^B \bar{\chi}_{AB} - \epsilon_{ijk} \bar{L}^j d_A x^k + v_i d_A c)
- \frac{1}{2} \mathcal{L}_L (R_i^B \bar{\chi}_{AB} - \epsilon_{ijk} \bar{L}^j d_A x^k + v_i d_A c)
+ \left\{ \frac{1}{2} L(c^{-1} v_i \text{tr} \chi) + \frac{1}{2} \nabla^A (R_i^B \bar{\chi}_{AB} - \epsilon_{ijk} \bar{L}^j d_A x^k + v_i d_A c) \right\} L \Psi_i^j.
\]
(7.14)

For the terms with underline in (7.14), one has that by (2.46),
\[
\frac{1}{2} L(c^{-1} v_i \text{tr} \chi) - \nabla^A (c^{-1} v_i d_A \mu)
=c^{-1} v_i J + \frac{1}{2} c^{-3} \mu v_i L \text{tr} \chi + \frac{1}{2} L(c^{-1} v_i) \text{tr} \chi - d(c^{-1} v_i) \cdot d \mu
= \text{l.o.t.}
\]
and
\[
\frac{1}{2} \mathcal{L}_L (R_i^B \chi_{AB}) - d_A R_i \mu = R_i^B \mathcal{L}_T \chi_{AB} - \nabla_A (R_i^B d_B \mu) + \text{l.o.t.} = \text{l.o.t.},
\]
which implies that the terms \(\Delta \mu\) and \(d_A R_i \mu\) have been eliminated.

For the terms with braces in (7.14), we have
\[
- \frac{1}{2} \nabla^A (c^{-2} \mu R_i^B \chi_{AB}) = - \frac{1}{2} c^{-2} \mu \text{tr}_g \chi + \text{l.o.t.}
\]
and
\[
\frac{1}{2} \nabla^A (R_i^B \chi_{AB}) = \frac{1}{2} R_i \text{tr}_g \chi + \text{l.o.t.}.
\]

For the terms with wavy line in (7.14),
\[
- \nabla^B (2c^{-1} \mu \nu_i (\chi_{AB} - \frac{1}{2} \text{tr}_g \chi_{AB})) = - c^{-1} \mu \nu_i d_A \text{tr}_g \chi + \text{l.o.t.}
\]

Therefore, we arrive at
\[
(R_i)_{\mathcal{A}^j} = R_i^A (d_A \text{tr}_g \chi) T \Psi^j - c^{-1} \mu \nu_i d_A \text{tr}_g \chi \cdot d \Psi^j + \text{l.o.t.} \tag{7.15}
\]

By substituting \((Z)_{\mathcal{A}^j}^i\) into (7.1), the resulting terms \((Z^\alpha \Delta \mu) T \Psi^0_\gamma\) from (7.13) and \(R_i^A (d_A Z^\alpha \text{tr}_g \chi) T \Psi^0_\gamma\) from (7.15) need to be treated especially.

Note that \((Z^\alpha \Delta \mu) T \Psi^0_\gamma\) comes from \((Z^\alpha)^{(T)} \mathcal{A}^0_1\) with \(Z^\alpha = Z_{|\alpha|+1} \cdots Z_2\) and \(Z_1 = T\). When the number of \(T\) in \(Z_{|\alpha|+1} \cdots Z_1\) is \(l\), then \(Z^\alpha\) contains \(l - 1\) vector field \(T\)'s. In view of (4.11), one has that by (6.44),
\[
\begin{align*}
\delta^{2l} & \int_{|\alpha|+1} |(Z^\alpha \Delta \mu) T \Psi^0_\gamma| |L \Psi^1_\gamma| \\
& \lesssim M^2 \delta^{3-2\varepsilon_0} \int_{|\alpha|+1} \tau^{-2+2s} \left( \delta^{l+1} \|Z^\alpha \Delta \mu\|_{L^2(\Sigma^\tau)^s} \right)^2 d\tau + \delta^{-1} \int_{|\alpha|+1} \rho^{2s} \left( \delta^{l} |L \Psi^1_\gamma| \right)^2 \\
& \lesssim M^{2p+2} \delta^{(1-\varepsilon_0)(2p+2)-2} + \delta^{-1} \int_{0}^{\tau} \tilde{F}_{1,\leq|\alpha|+2}(t, u') du' \\
& + \delta \int_{0}^{\tau} \tau^{-4+2s} \{ \tilde{E}_{1,\leq|\alpha|+2}(\tau, u) + (\ln^2 \tau) \tilde{E}_{2,\leq|\alpha|+2}(\tau, u) \} d\tau
\end{align*}
\]
and
\[
\begin{align*}
\delta^{2l+1} & \int_{|\alpha|+1} |(Z^\alpha \Delta \mu) T \Psi^0_\gamma| |L \Psi^1_\gamma| \\
& \lesssim M \delta^{2-\varepsilon_0} \int_{|\alpha|+1} \tau^{-1} \left( \delta^{l+1} \|Z^\alpha \Delta \mu\|_{L^2(\Sigma^\tau)^s} \right) \left( \delta^l |L \Psi^0_\gamma| \right) |L \Psi^1_\gamma| d\tau \\
& \lesssim M^{2p+2} \delta^{(1-\varepsilon_0)(2p+2)-2} + \delta \int_{0}^{\tau} \tilde{F}_{1,\leq|\alpha|+2}(t, u') du' \\
& + \delta \int_{0}^{\tau} \tau^{-2} \{ \tilde{E}_{1,\leq|\alpha|+2}(\tau, u) + (\ln \tau) \tilde{E}_{2,\leq|\alpha|+2}(\tau, u) \} d\tau.
\end{align*}
\]
For the term \( R_i^A(d_A Z^n tr_y \gamma) T \Psi^0_\gamma \), we know \( Z_1 = R_i \). If there are \( l \) vector field \( T \)'s in \( Z_{[a]+1} \cdots Z_1 \), then the number of \( T \) in \( Z^n = Z_{[a]+1} \cdots Z_2 \) is just \( l \). Thus, with the help of (6.30), we obtain

\[
\delta^{2l} \int_{D^{t,u}} \rho^{2s} |R_i^A(d_A Z^n tr_y \gamma) T \Psi^0_\gamma | \| L \Psi^{[a]+1}_\gamma | \\
\lesssim M \delta^{1-2\epsilon_0} \int_{t_0}^{t} \rho^{2s}(\delta^n \|dZ^n tr_y \gamma\|_{L^2(\Sigma^n_\gamma)})^2 d\tau + \delta^{-1} \int_{D^{t,u}} \rho^{2s}(\delta^n \|L \Psi^{[a]+1}_\gamma\|)^2 \\
\lesssim M^{2p+2}\delta^{(1-\epsilon_0)(2p+2)-2} + \delta^{-1} \int_{0}^{u} \tilde{F}_{1,\leq[a]+2}(t,u') du' \\
+ \delta \int_{t_0}^{t} \tau^{-4+2s} \ln^2 \tau \{ \tilde{E}_{1,\leq[a]+2}(\tau,u) + \tilde{E}_{2,\leq[a]+2}(\tau,u) \} d\tau \\
\] (7.18)

and

\[
\delta^{2l+1} \int_{D^{t,u}} |R_i^A(d_A Z^n tr_y \gamma) T \Psi^0_\gamma | \| L \Psi^{[a]+1}_\gamma | \\
\lesssim M \delta^{1-\epsilon_0} \int_{t_0}^{t} (\delta^n \|dZ^n tr_y \gamma\|_{L^2(\Sigma^n_\gamma)}) (\delta^n \|\hat{L} \Psi^{[a]+1}_\gamma\|_{L^2(\Sigma^n_\gamma)}) d\tau \\
\lesssim M^{2p+2}\delta^{(1-\epsilon_0)(2p+2)-2} + \delta^{-1} \int_{0}^{u} \tilde{F}_{1,\leq[a]+2}(t,u') du' \\
+ \delta \int_{t_0}^{t} \tau^{-3/2} \{ \tilde{E}_{1,\leq[a]+2}(\tau,u) + \tilde{E}_{2,\leq[a]+2}(\tau,u) \} d\tau. \\
\] (7.19)

For the other terms coming from \( \sum_{k=1}^{[a]} (Z_{[a]+1}+\cdots Z_{[a]+1}) \lambda \cdots (Z_{[a]+2-k}+\cdots Z_{[a]+k}) (Z_{[a]+1-k} A^1_{[a]-k} + Z_{[a]+1} A^1_{[a]+1}) \), they are easier to be handled in (4.11) since (6.30) can be adopted to treat \( d^n Z^n tr_y \gamma \) as in (7.18) and (7.19). On the other hand, Proposition 5.2 can be used to estimate the others in \((Z) A^1_j\).

7.3 Error estimates for the left terms in (7.1)

- **Display the expressions of \((Z) A^2_j\) and \((Z) A^3_j\)**

  Similarly to \((Z) A^1_j\), one has

  \[
  (\rho^L) A^2_j = \rho tr_y \gamma (L + \frac{1}{2} tr_y \gamma)L \Psi^j_\gamma + [\rho (c^{-2} \mu - c^{-2} \mu + 2)] L^2 \Psi^j_\gamma - 2 \rho c^2 (c^{-2} \mu) \cdot dL \Psi^j_\gamma \\
  - (\rho L \mu + \mu) \Delta \Psi^j_\gamma + 2 \rho \mu (\chi^{AB} - \frac{1}{2} tr_y \hat{g}^{AB}) \hat{g}^{AB} \Psi^j_\gamma, \\
  \] (7.20)

  \[
  (T) A^2_j = - c^{-2} \mu tr_y \gamma (L + \frac{1}{2} tr_y \gamma)L \Psi^j_\gamma + T (c^{-2} \mu) L^2 \Psi^j_\gamma + \mu (c^{-2} \mu) \cdot dL \Psi^j_\gamma \\
  + c^2 d_A (c^{-2} \mu) d^A \Psi^j_\gamma - T \mu \Delta \Psi^j_\gamma - 2 c^{-2} \mu^2 (\chi^{AB} - \frac{1}{2} tr_y \hat{g}^{AB}) \hat{g}^{AB} \Psi^j_\gamma, \\
  \] (7.21)

  \[
  (R_i) A^2_j = c^{-1} v_i tr_y \gamma (L + \frac{1}{2} tr_y \gamma)L \Psi^j_\gamma - (c^{-2} \mu (R_i) \hat{f}^{LA} + 2 (R_i) \hat{f}^{T_A} A) d^L \Psi^j_\gamma - R_i \mu \Delta \Psi^j_\gamma \\
  + R_i \hat{f}^{LA} d^L \Psi^j_\gamma + R_i (c^{-2} \mu) L^2 \Psi^j_\gamma + 2 c^{-1} \mu v_i (\chi^{AB} - \frac{1}{2} tr_y \hat{g}^{AB}) \hat{g}^{AB} \Psi^j_\gamma. \\
  \] (7.22)
and
\[
\begin{align*}
\langle \rho L \rangle \mathcal{M}_3^i &= \text{tr}_g \chi \{ \rho L \langle c^{-2} \mu - 2c^{-2} (\mu - 1) + 2(1 - c^{-2}) - \frac{1}{2} \rho c^{-2} \mu \text{tr}_g \chi \} L \Psi_\gamma^i \\
&
+ 2 \rho c^{-2} d^B (c^{-2} \mu) \chi_{AB} d^4 \Psi_\gamma^j, \\
\langle T \rangle \mathcal{M}_3^j &= [T (c^{-2} \mu) \text{tr}_g \chi + \frac{1}{2} c^{-4} \mu^2 (\text{tr}_g \chi)^2 - \frac{1}{2} \rho c^{-2} (\text{tr}_g \chi)] L \Psi_\gamma^j + \frac{1}{2} L (c^{-2} \mu) \text{tr}_g \chi d^A (c^{-2} \mu) \Psi_\gamma^j, \\
\langle R_i \rangle \mathcal{M}_3^j &= [R_i (c^{-2} \mu) \text{tr}_g \chi - \frac{1}{2} c^{-3} \mu \text{tr}_g (c^{-2} \mu)] L \Psi_\gamma^j \\
&+ \left[ c \text{tr}_g (c^{-2} \mu) \text{tr}_g \chi - \frac{1}{2} L (c^{-2} \mu) (R_i) \right] L \Psi_\gamma^j + 2 (R_i) \text{tr}_g \chi d^A (c^{-2} \mu) \Psi_\gamma^j.
\end{align*}
\] (7.23)

(7.24)

(7.25)

Set
\[
\mathcal{M}^{\alpha} = \sum_{k=1}^{\lceil \alpha \rceil} (Z_{\alpha+1} + (Z_{\alpha+1}^*) \hat{\lambda}) \ldots (Z_{\alpha+2-k} + (Z_{\alpha+2-k}^*) \hat{\lambda}) (Z_{\alpha+1-k}^*) \hat{\lambda}^\alpha \mathcal{M}_2^\alpha - (Z_{\alpha+1-k}^*) \hat{\lambda}^\alpha \mathcal{M}_3^\alpha.
\]

Then using (7.20)-(7.25), Proposition 5.1 and 5.2 to derive
\[
\delta^t \| \mathcal{M}^{\alpha} \|_{L^2(\Sigma_{\gamma}^u)} \lesssim M^{p+1} \delta^{1-\varepsilon_0} (p+1)^{-\frac{1}{2}} \tau^{-2} + \tau^{-s} \sqrt{E_{1, \leq 1+2}(\tau, u)} + \delta \tau^{-2} \sqrt{E_{2, \leq 1+2}(\tau, u)}.
\]

Therefore,
\[
\delta^2 \int_{D^1} \| \mathcal{M}^{\alpha} \| \left( \rho^{2s} \left| L \Psi_\gamma^{[\alpha+1]} \right| + \delta \left| L \Psi_\gamma^{[\alpha+1]} \right| \right) \lesssim \delta^2 \int_{D^1} \rho^{2s} \mathcal{M}^{\alpha} \|_{L^2(\Sigma_{\gamma}^u)} d\tau + \delta \int_{D^1} \rho^{2s} d^2 L \Psi_\gamma^{[\alpha+1]} \|_{L^2(\Sigma_{\gamma}^u)} d\tau \lesssim M^{2p+2} \delta^{1-\varepsilon_0} (p+2) + \delta \int_{0}^{\tau} \tau^{-s} \left\{ \hat{E}_{1, \leq 1+2}(\tau, u) + \hat{E}_{2, \leq 1+2}(\tau, u) \right\} d\tau.
\]

- Error estimates on \((Z_{\alpha+1} + (Z_{\alpha+1}^*) \hat{\lambda}) \ldots (Z_{\alpha} + (Z_{\alpha}^*) \hat{\lambda}) \Phi_0^\alpha\)

By (2.39), (2.40) and Proposition 5.2, one has
\[
\delta^t \| (Z_{\alpha+1} + (Z_{\alpha+1}^*) \hat{\lambda}) \ldots (Z_{\alpha} + (Z_{\alpha}^*) \hat{\lambda}) \Phi_0^\alpha \|_{L^2(\Sigma_{\gamma}^u)} \lesssim M^{2p+1} \delta^{1-\varepsilon_0} (p+1)^{-\frac{1}{2}} \tau^{-p} + M P \delta^{1-\varepsilon_0} \tau^{-p} \left\{ \tau^{-s} \sqrt{E_{1, \leq 1+2}(\tau, u)} + \delta \tau^{-1} \sqrt{E_{2, \leq 1+2}(\tau, u)} \right\}.
\]

Then
\[
\delta^2 \int_{D^1} \| (Z_{\alpha+1} + (Z_{\alpha+1}^*) \hat{\lambda}) \ldots (Z_{\alpha} + (Z_{\alpha}^*) \hat{\lambda}) \Phi_0^\alpha \| \left( \rho^{2s} \left| L \Psi_\gamma^{[\alpha+1]} \right| + \delta \left| L \Psi_\gamma^{[\alpha+1]} \right| \right) \lesssim M^{4p+2} \delta^{1-\varepsilon_0} (p+2) + \delta \int_{0}^{\tau} \tau^{-p} \left\{ \hat{E}_{1, \leq 1+2}(\tau, u) + \hat{E}_{2, \leq 1+2}(\tau, u) \right\} d\tau.
\]

(7.27)
8 Global existence

8.1 Global existence of $\phi$ near $C_0$

By substituting the estimates (7.16)-(7.19) and (7.26)-(7.27) into (4.11), then it follows from Gronwall’s inequality that by $p > p_c$,

$$
\tilde{E}_{1,\leq 2N-4}(t,u) + \tilde{F}_{1,\leq 2N-4}(t,u) + \delta \tilde{E}_{2,\leq 2N-4}(t,u) + \delta \tilde{F}_{2,\leq 2N-4}(t,u) \\
\lesssim M^{2p+2}\delta^{1-\varepsilon_0}(2p+2)^{-2} + \delta^{2-2\varepsilon_0}.
$$

(8.1)

Together with the following Sobolev-type embedding formula on $S_{t,u}$

$$
\|f\|_{L^\infty(S_{t,u})} \lesssim \frac{1}{t} \sum_{|\beta| \leq 2} \|R^\beta f\|_{L^2(S_{t,u})},
$$

(8.2)

we have that for $|\alpha| \leq 2N - 7$,

$$
\delta^{|\alpha|} |Z^\alpha \varphi_\gamma| \lesssim \frac{1}{t} \sum_{|\beta| \leq 2} \delta^{|\beta|} \|R^\beta |Z^\alpha \varphi_\gamma|\|_{L^2(S_{t,u})} \\
\lesssim \delta^2 \left( \sqrt{E_{1,\leq 2N-4}} + \sqrt{E_{2,\leq 2N-4}} \right) t^{-1} \\
\lesssim \delta^{2-\varepsilon_0} t^{-1},
$$

(8.3)

where the related bounds are independent of $M$. This improves the bootstrap assumptions (3.1), and hence we have proved the global existence of the solution $\phi$ to the equation (1.1) with the initial data (1.5)-(1.6) in the domain $\tilde{\Omega}^{1,4\delta}$. Moreover, since (8.3) holds, with the same argument as in the proof of Proposition 5.1, the constants in (5.1) can be refined to be independent of $M$ when $|\alpha| \leq 2N - 9$.

When $N$ is large enough, we now claim that on $\tilde{\Gamma}_{2\delta} = \{(t,x) : t \geq 1 + 2\delta, t - r = 2\delta\}$,

$$
|\Gamma^\alpha \phi| \lesssim \delta^{2-\varepsilon_0} t^{-1}, \quad |\alpha| \leq 2N - 8,
$$

(8.4)

here and below $\Gamma \in \{(t+r)\tilde{L}, \tilde{L}, \Omega_1, \Omega_2, \Omega_3\}$.

We next focus on the proof of (8.4).

It follows from $T = c^{-1} \mu \tilde{T}$ and $\tilde{T}^i = \tilde{\varphi}^i + \frac{x^i}{\rho}$ that

$$
\omega_i T^i = \omega_i c^{-1} \mu \tilde{T}^i = c^{-1} \mu (\omega_i \tilde{T}^i - \frac{r}{\rho}).
$$

This, together with $\partial_i = c^2 \mu^{-2} T^i T + d^A x^i X_A$, derives

$$
\partial_r = \sum_{i=1}^3 \omega_i \partial_i = c\mu^{-1} (\omega_i \tilde{T}^i - \frac{r}{\rho}) T + \omega_i d^A x^i X_A.
$$

In addition, by (2.23) and (2.24), one has

$$
\Omega_i = R_i + \epsilon_{ijk} x^j c\mu^{-1} T^k T.
$$
Then by $\partial_t = L + c^2 \mu^{-1} T$, we have

$$\begin{aligned}
\bar{L} &= L + [c^2 \mu^{-1} + c \mu^{-1} (\omega_i \bar{T}^i - \frac{r}{\rho})] T + \omega_i \delta^A x^i X_A, \\
\bar{L} &= L + [c^2 \mu^{-1} - c \mu^{-1} (\omega_i \bar{T}^i - \frac{r}{\rho})] T - \omega_i \delta^A x^i X_A, \\
\Omega_i &= R_i + \epsilon_{ijk} x^j c \mu^{-1} T^k T.
\end{aligned}$$

(8.5)

Due to $g(\bar{T}, \bar{T}) = 1$, then $\frac{r^2}{\rho^2} - 2(\bar{T}^i \omega_i) \frac{\bar{T}^j}{\rho} + g_{ij} \bar{T}^i \bar{T}^j = -1$ and further

$$\frac{\bar{T}^i \omega_i - g_{ij} \bar{T}^i \bar{T}^j}{\sqrt{(\bar{T}^i \omega_i)^2 - g_{ij} \bar{T}^i \bar{T}^j} + 1 + 1 - \bar{T}^i \omega_i}.$$ 

This yields that in $D^{1,4\delta}$,

$$|Z^\gamma \left( \frac{r}{\rho} - 1 \right) | \lesssim \delta^{(1-\varepsilon_0)p-l} t^{-1},$$

(8.6)

where $|\alpha| \leq 2N - 8$ and $l$ is the number of $T$ in $Z^\gamma$. Note that $c^2 \mu^{-1} + c \mu^{-1} (\omega_i \bar{T}^i - \frac{r}{\rho}) = c \mu^{-1} (c-1) + c \mu^{-1} - c \mu^{-1} (1 - \frac{r}{\rho}) + c \mu^{-1} \omega_i \bar{T}^i$. Then by (8.3), (8.6) and (5.1) (neglecting the unimportant constant $M$), we obtain that in $D^{1,4\delta}$,

$$\begin{aligned}
\delta^l |Z^\beta [c^2 \mu^{-1} + c \mu^{-1} (\omega_i \bar{T}^i - \frac{r}{\rho})] | &\lesssim \delta^{(1-\varepsilon_0)p-l} t^{-1}, \\
\delta^l |Z^\beta [c^2 \mu^{-1} - c \mu^{-1} (\omega_i \bar{T}^i - \frac{r}{\rho})] | &\lesssim 1, \\
\delta^l |Z^\beta (\omega_i \delta^A x^i) | &\lesssim 1, \\
\delta^l |Z^\beta (\epsilon_{ijk} x^j c \mu^{-1} T^k) | &\lesssim \delta^{(1-\varepsilon_0)p},
\end{aligned}$$

(8.7)

where $|\beta| \leq 2N - 8$ and $l$ is the number of $T$ in $Z^\gamma$. Therefore, in $D^{1,4\delta}$, combining (8.7), (8.5) with (8.3) derives

$$\delta^l |\Gamma^\alpha \varphi_\gamma | \lesssim \delta^{1-\varepsilon_0} t^{-1}, |\alpha| \leq 2N - 7,$$

(8.8)

where $l$ is the number of $\bar{L}$ in $\Gamma^\alpha$. Recalling that $\phi$ is the solution to (1.1) and $\varphi_\gamma = \partial_\gamma \phi$, then by (8.8) we have

$$\delta^l |\bar{L} \Gamma^\alpha \phi | \lesssim \delta^{1-\varepsilon_0} t^{-1}, |\alpha| \leq 2N - 7.$$

(8.9)

Integrating (8.9) along the integral curves of $\bar{L}$, and choosing the zero boundary value on $C_0$, one has that in $D^{1,4\delta}$,

$$\delta^l |\Gamma^\alpha \phi | \lesssim \delta^{2-\varepsilon_0} t^{-1}, |\alpha| \leq 2N - 7.$$

(8.10)

In addition, it follows from (8.6) that the distance between $C_0$ and $C_{4\delta}$ on the hypersurface $\Sigma_4$ is $4\delta + O(\delta^{1-\varepsilon_0})$ and the characteristic surface $C_u$ ($0 \leq u \leq 4\delta$) is almost straight with the error $O(\delta^{1-\varepsilon_0})$ from the corresponding outgoing light conic surface. Next we improve the estimate (8.10).

Rewriting the equation (1.1) as

$$\bar{L} \bar{L} \phi = \frac{1}{\Theta} \left[ \frac{1}{r} (\bar{L} \phi - \bar{L} \phi) + \frac{1}{r^2} \Delta \phi - \frac{1}{2^{p+2}} \sum_{i=0}^{p} (\bar{L} \phi)^{i} (\bar{L} \phi)^{p-i} (\bar{L} \phi + \bar{L}^2 \phi) \right],$$

(8.11)

where

$$\Theta = 1 + \frac{1}{2^{p+1}} \sum_{i=0}^{p} (\bar{L} \phi)^{i} (\bar{L} \phi)^{p-i}.$$
It follows from (8.11) and direct computation that
\[
\hat{L}(r\tilde{L}\phi) = \tilde{L}\phi + r\tilde{L}\phi = \frac{1}{\Theta}(\Theta - 1)\tilde{L}\phi + \frac{1}{\Theta}[\tilde{L}\phi - \tilde{L}\phi + \frac{1}{r}\Delta\phi + r(\tilde{L}\phi)^p(\tilde{L}^2\phi + \tilde{L}^2\phi)] + \frac{1}{\Theta}G,
\]  
(8.12)

where
\[
G = r\tilde{L}\phi\sum_{i=1}^{p}(\tilde{L}\phi)^{i-1}(\tilde{L}\phi)^{p-i}(\tilde{L}^2\phi + \tilde{L}^2\phi).
\]

This, together with the estimates in (3.1), yields
\[
|\hat{L}(r\tilde{L}\phi)| \lesssim \delta^{1-\varepsilon_0}t^{-p-1}r|\tilde{L}\phi| + \delta^{2-\varepsilon_0}t^{-2}.
\]  
(8.13)

Integrating (8.13) along the integral curves of \(\tilde{L}\) from \(t = t_0\) and applying the estimates in Theorem 2.1, one then has that on \(\tilde{C}_{2\delta}\),
\[
|\tilde{L}\phi| \lesssim \delta^{2-\varepsilon_0}t^{-1}.
\]

Hence, \(|\tilde{L}\phi| \lesssim \delta^{2-\varepsilon_0}t^{-2}\) holds by (8.11). By the induction argument and (8.11), we complete the proof of (8.4).

Finally, we point out that (8.4) will play an important role in solving the global Goursat problem of (1.1) in \(B_{2\delta}\).

### 8.2 Global existence of \(\phi\) in \(B_{2\delta}\)

In this subsection, we establish the global existence of solution \(\phi\) to the equation (1.1) in \(B_{2\delta}\). Set
\[
D_T = \{ (\tau, x) : \tau - \rho \geq 2\delta, t_0 \leq \tau \leq T \} \subset B_{2\delta}.
\]  
(8.14)

**Lemma 8.1.** For any function \(f(t, x) \in C^\infty(R^{1+3}), t \geq 1, (t, x) \in D_T\), we have the following inequalities:

- For \(|x| \leq \frac{1}{4}t\) and any \(s \geq 0\), then
  \[
  |f(t, x)| \lesssim \sum_{i=0}^{2} t^{-\frac{3}{2}}\delta^{(\frac{1}{2}-i)s}\|\tilde{\Gamma}^{-2-i}f(t, \cdot)\|_{L^2(\tilde{r} \leq \frac{1}{2})}.
  \]  
  (8.15)

- For \(|x| \geq \frac{1}{4}t\), then
  \[
  |f(t, x)| \lesssim |f(t, B_{t}^x)| + t^{-1}\|\Omega^{\leq 2}\partial^{\leq 1}f(t, \cdot)\|_{L^2(\frac{4}{3} \leq \tilde{r} \leq t-2\delta)},
  \]  
  (8.16)

where \(\tilde{\Gamma} \in \{ S, H, \Omega \}, (t, B_{t}^x)\) is the intersection point of the boundary \(\tilde{C}_{2\delta}\) and the ray crossing \((t, x)\) which emanates from \((t, 0)\).

**Remark 8.1.** It should be remarked that although this Lemma and its proof are similar to those of Proposition 3.1 in [12], the refined inner estimate, (8.15) is new (the appearance of factor \(\delta^{(\frac{1}{2}-i)s}\)) and is crucial for the treatment of the short pulse initial data as in [7].
Proof. Let \( \chi \) be a non-negative smooth cut off function on \( \mathbb{R}, \) such that

\[
\chi(t) = \begin{cases} 
1, & 0 \leq t \leq \frac{1}{4}, \\
0, & t \geq \frac{1}{2}.
\end{cases}
\]

Set \( f_1(t, x) = \chi\left(\frac{t}{t}\right) f(t, x) \) and \( f_2(t, x) = \left(1 - \chi\left(\frac{t}{t}\right)\right) f(t, x), \) then \( f(t, x) = f_1(t, x) + f_2(t, x) \) and \( \text{supp} f_1 \subset \{(t, x) \mid |x| \leq \frac{1}{4} t\} \), \( \text{supp} f_2 \subset \{(t, x) \mid |x| \geq \frac{1}{2} t\} \).

1. For any point \( (t, x) \) satisfying \( |x| \leq \frac{1}{4} t, f_1(t, x) = f(t, x) \). We perform the change of variable \( x = t\delta y, \) and make use of the Sobolev embedding theorem for \( y \) to have

\[
|f(t, x)| = |f_1(t, t\delta y)| \lesssim \sum_{|\alpha| \leq 2} \left( \int_{\mathbb{R}^3} |\partial_y^\alpha (f(t, t\delta y)\chi(\delta y))|^2 dy \right)^{1/2}
\]

\[
\lesssim \sum_{|\alpha| \leq 2} \left( \int_{|\delta y| \leq 1/2} (t\delta y)^{2|\alpha|} |(\partial_y^\alpha f)(t, t\delta y)|^2 dy \right)^{1/2}
\]

\[
\lesssim \sum_{|\alpha| \leq 2} \left( \int_{|z| \leq \frac{1}{4} t} (t\delta z)^{2|\alpha|-3} |\partial_z^\alpha f(t, z)|^2 dz \right)^{1/2}.
\]

Note that

\[
\partial_t = -\frac{1}{t-r} \left( \frac{x^i}{t+r} S - \frac{t}{t+r} H_i + \frac{x^j}{r/2} \epsilon_{jk}^i \mathcal{O}_k \right)
\]

and \( t \sim t - |z| \) in the domain \( \{(t, z) : |z| \leq \frac{1}{4} t\} \). Then we have

\[
|t\partial_z f(t, z)| \lesssim |t - |z|| \partial_z f(t, z)| \lesssim |\Gamma f(t, z)|,
\]

\[
|t^2 \partial_z^2 f(t, z)| \lesssim \sum_{|\alpha| \leq 2} |\partial_z f(t, z)|.
\]

(8.18)

Therefore, substituting (8.18) into (8.17) yields

\[
|f(t, x)| \lesssim \sum_{i=0}^2 t^{-3/2} \delta^{(i+1)/2} \|\Gamma^2 f(t, \cdot)\|_{L^2(r \leq \frac{1}{4} t)}.
\]

2. When \( (t, x) \) satisfies \( |x| \geq \frac{1}{4} t, \) by Newton-Leibnitz formula and the Sobolev embedding theorem on the spheres \( S^\rho \) with radius \( \rho \) and center at the origin on \( \Sigma_t, \) we have

\[
f^2(t, x) = f^2(t, B^\rho_t) - \frac{t^{-2\delta}}{|x|} \partial_{\rho_t}(f^2(t, \rho \omega)) d\rho
\]

\[
\lesssim f^2(t, B^\rho_t) + \left( \frac{1}{\rho^2} \right)^{1/2} \sum_{|\alpha|, |eta| \leq 2} \|\Omega^\alpha f\|_{L^2(S^\rho_t)} \|\Omega^\beta \partial f\|_{L^2(S^\rho_t)} d\rho
\]

\[
\lesssim f^2(t, B^\rho_t) + \sum_{|\alpha| \leq 2, |eta| \leq 1} t^{-2} \|\Omega^\alpha \partial^\beta f(t, \cdot)\|^2_{L^2(t/4 \leq r \leq t-2\delta)}.
\]

This completes the proof of (8.16).
Define the energy
\[ E_{k,l}(t) = \int_{\Sigma_t \cap D_T} [(\partial_\nu v)^2 + |\nabla v|^2], \quad k + l \leq 6, \] (8.19)
where \( v = \tilde{\Gamma}^k \Omega^l \phi, \tilde{\Gamma} \in \{ \partial, S, H \} \). Thanks to the estimates on \( \Sigma_{t_0} \) in Theorem 2.1, we make the following bootstrap assumptions:

For \( t \geq t_0 \), there exists a uniform constant \( M_0 \) such that
\[ E_{k,l}(t) \leq M_0^2 \delta^{4-2\varepsilon_0}, \quad k = 0, 1, \] (8.20)
\[ E_{k,l}(t) \leq M_0^2 \delta^{7-2k-2\varepsilon_0}, \quad 2 \leq k \leq 6. \]

We next establish the following \( L^\infty \) estimates:

**Proposition 8.1.** Under the assumptions (8.20), for sufficiently small \( \delta > 0 \), it holds that in \( D_T \),
\[ |\partial \Omega^{3,3} \phi| \lesssim M_0 \delta^{\frac{13}{8} - \varepsilon_0} t^{-1}, \]
\[ |\partial \tilde{\Gamma} \Omega^{3,2} \phi| \lesssim M_0 \delta^{-\varepsilon_0} t^{-1}, \] (8.21)
\[ |\partial \tilde{\Gamma}^2 \Omega^{1,1} \phi| \lesssim M_0 \delta^{-\varepsilon_0} t^{-1}, \]
\[ |\partial \tilde{\Gamma}^{3} \phi| \lesssim M_0 \delta^{-1-\varepsilon_0} t^{-1}. \]

**Proof.**

- When \( |x| \leq \frac{t}{4} \), it follows from Lemma 8.1 that
\[
|\partial \Omega^{3,3} \phi| \lesssim t^{-\frac{3}{2}} \left\{ \delta^{-\frac{3}{2}} \| \partial \Omega^{3,3} \phi \|_{L^2(\Sigma_t \cap D_T)} + \delta^{-\frac{1}{2}} \| \Gamma \partial \Omega^{3,3} \phi \|_{L^2(\Sigma_t \cap D_T)} + \delta^{\frac{1}{2}} \| \tilde{\Gamma}^2 \partial \Omega^{3,3} \phi \|_{L^2(\Sigma_t \cap D_T)} \right\}
\lesssim t^{-\frac{3}{2}} \left\{ \delta^{-\frac{3}{2}} \sqrt{E_{0,t}} + \delta^{-\frac{1}{2}} \sqrt{E_{1,t}} + \delta^{\frac{1}{2}} \sqrt{E_{2,t}} \right\}
\lesssim t^{-\frac{3}{2}} M_0 \left\{ \delta^{-\frac{3}{2} + 2 - \varepsilon_0} + \delta^{-\frac{1}{2} + 2 - \varepsilon_0} + \delta^{\frac{1}{2} + \frac{3}{2} - \varepsilon_0} \right\}. \] (8.22)

Choosing \( s = \frac{1}{4} \), then
\[ |\partial \Omega^{3,3} \phi| \lesssim M_0 \delta^{\frac{13}{8} - \varepsilon_0} t^{-\frac{3}{2}}. \]

- When \( \frac{t}{4} \leq |x| \leq t - 2\delta \), by Lemma 8.1 and (8.4), one has
\[
|\partial \Omega^{3,3} \phi| \lesssim |\partial \Omega^{3,3} \phi(t, B_t^2)| + t^{-1} \| \Omega^{3,2} \partial \Omega^{3,3} \phi(t, \cdot) \|_{L^2(\frac{t}{4} \leq r \leq t - 2\delta)}
\lesssim \delta^{2-\varepsilon_0} t^{-1} + t^{-1} \left( \sqrt{E_{0,\leq 6}} + \sqrt{E_{1,\leq 5}} \right)
\lesssim M_0 \delta^{2-\varepsilon_0} t^{-1}. \] (8.23)

Combining (8.22) and (8.23) yields
\[ |\partial \Omega^{3,3} \phi| \lesssim M_0 \delta^{\frac{13}{8} - \varepsilon_0} t^{-1}. \]

For the other cases, the procedures of proof are analogous. Indeed, in Lemma 8.1,

- for \( \partial \tilde{\Gamma} \Omega^{3,2} \phi \), let \( s = \frac{3}{4} \);
- for \( \partial \tilde{\Gamma}^2 \Omega^{1,1} \phi \), let \( s = 1 \);
- for \( \partial \tilde{\Gamma}^{3} \phi \), let \( s = 1 \).

\[ \]
Then Proposition 8.1 is proved.

Next we take the energy estimates of \( \phi \) in \( D_T \). It follows from the integration \( \partial_t v g^{\alpha \beta} \partial_\alpha \partial_\beta v \) over \( D_t \) by parts and direct computation that

\[
\int_{\Sigma_{t_0} \cap D_t} \left[ \frac{1}{c^2} (\partial_t v)^2 + |\nabla v|^2 \right] \lesssim \int_{\Sigma_{t_0} \cap D_t} \left[ \frac{1}{2} (\partial_t v)^2 + |\nabla v|^2 \right] + \int_{\Sigma_{t_0} \cap D_t} \omega^i \partial_i v \partial_t v \left| \partial_t v \right| \quad (8.24)
\]

\[
+ \int_{D_t} |\partial_t v g^{\alpha \beta} \partial_\alpha \partial_\beta v | + \int_{D_t} |(\partial_t \phi)^p - \partial_t^2 \phi | \left| \partial_t v \right|^2 .
\]

On \( \tilde{C}_{2\delta} \), by the estimate (8.4), we have

\[
\int_{\tilde{C}_{2\delta} \cap D_t} \left[ \frac{1}{2} (\partial_t v)^2 + |\nabla v|^2 \right] \lesssim \int_{\tilde{C}_{2\delta} \cap D_t} \left\{ \sum_{i=1}^{3} (\omega^i \partial_i v + \partial_i v)^2 + |\partial_t \phi|^p |(\partial_t v)^2 \right\} \quad (8.25)
\]

\[
\lesssim \delta^{4 - 2\epsilon_0}.
\]

On the initial hypersurface \( \Sigma_{t_0} \cap D_t \), with the help of Theorem 2.1, one has

\[
\int_{\Sigma_{t_0} \cap D_t} \left[ \frac{1}{c^2} (\partial_t v)^2 + |\nabla v|^2 \right] \lesssim \left\{ \begin{array}{ll}
\delta \cdot (\delta^{2 - \epsilon_0})^2 = \delta^{5 - 2\epsilon_0}, & k \leq \min \{1, 6 - l \}, \\
\delta \cdot (\delta^{3 - k - \epsilon_0})^2 = \delta^{7 - 2k - 2\epsilon_0}, & 2 \leq k \leq 6 - l.
\end{array} \right. \quad (8.26)
\]

Therefore, it follows from (8.24)-(8.26) and the Gronwall’s inequality that

\[
E_{k,l}(t) \lesssim \delta^{4 - 2\epsilon_0} + \int_{D_t} |(\partial_t \tilde{\Gamma}^k \Omega^l \phi) (g^{\alpha \beta} \partial_\alpha \partial_\beta \tilde{\Gamma}^k \Omega^l \phi) |, \quad k \leq \min \{1, 6 - l \}, \quad (8.27)
\]

\[
E_{k,l}(t) \lesssim \delta^{7 - 2k - 2\epsilon_0} + \int_{D_t} |(\partial_t \tilde{\Gamma}^k \Omega^l \phi) (g^{\alpha \beta} \partial_\alpha \partial_\beta \tilde{\Gamma}^k \Omega^l \phi) |, \quad 2 \leq k \leq 6 - l. \quad (8.28)
\]

The remaining task is to estimate the term \( \int_{D_t} |(\partial_t \tilde{\Gamma}^k \Omega^l \phi) (g^{\alpha \beta} \partial_\alpha \partial_\beta \tilde{\Gamma}^k \Omega^l \phi) | \) in the right hand sides of (8.27) and (8.28). Acting the operator \( \tilde{\Gamma}^k \Omega^l \) on (1.1) and commuting it with \( g^{\alpha \beta} \partial_\alpha \partial_\beta \) yield

\[
|g^{\alpha \beta} \partial_\alpha \partial_\beta \tilde{\Gamma}^k \Omega^l \phi| \lesssim \sum_{k_1 + \cdots + k_{p+1} \leq k \atop l_1 + \cdots + l_{p+1} \leq l} |\partial \tilde{\Gamma}^{k_1} \Omega^{l_1} \phi| \cdots |\partial \tilde{\Gamma}^{k_p} \Omega^{l_p} \phi| \cdot |\partial^2 \tilde{\Gamma}^{k_{p+1}} \Omega^{l_{p+1}} \phi|. \quad (8.29)
\]

1. The treatment for the cases of \( k = 0 \) and \( l \leq 6 \)

   - If \( l_i \leq l_{p+1} \) holds for all \( 1 \leq i \leq p \), then \( l_i \leq 3 \ (1 \leq i \leq p) \), and it follows from Proposition 8.1
and (8.20) that
\[
\int_{D_t} |\partial \Omega^{\leq 6} \phi| \cdot |\partial \Omega^{l_1} \phi| \cdots |\partial \Omega^{l_p} \phi| \cdot |\partial^2 \Omega^{l_{p+1}} \phi| \\
\lesssim \int_{D_t} |\partial \Omega^{\leq 6} \phi| \cdot |\partial \Omega^{\leq 3} \phi|^p \cdot |\partial^2 \Omega^{\leq 5} \phi| \\
\lesssim \int_{t_0}^t M_0^p \delta(\frac{13}{5} - \epsilon_0)^p \tau^{-p} E_{0,l}(\tau) d\tau + \int_{t_0}^t M_0^p \delta(\frac{13}{8} - \epsilon_0)^p \tau^{-p} E_{1,l}(\tau) d\tau \\
\lesssim \delta^{4 - 2\epsilon_0}.
\]

- If there exists one \( i \) \((1 \leq i \leq p)\) such that \( l_i > l_{p+1} \), then \( l_{p+1} \leq 2 \), and
\[
\int_{D_t} |\partial \Omega^{\leq 6} \phi| \cdot |\partial \Omega^{l_1} \phi| \cdots |\partial \Omega^{l_p} \phi| \cdot |\partial^2 \Omega^{l_{p+1}} \phi| \\
\lesssim \int_{D_t} |\partial \Omega^{\leq 6} \phi| \cdot |\partial \Omega^{\leq 3} \phi|^p \cdot |\partial \Omega^{\leq 6} \phi| \cdot |\partial^2 \Omega^{\leq 2} \phi| \\
\lesssim \int_{t_0}^t M_0^p \delta(\frac{13}{5} - \epsilon_0)(p-1) + \frac{7}{8} - \epsilon_0 \tau^{-p} E_{0,l}(\tau) d\tau \\
\lesssim \delta^{4 - 2\epsilon_0},
\]
where \((\frac{13}{8} - \epsilon_0)(p - 1) + \frac{7}{8} - \epsilon_0 > (1 - \epsilon_0)p - 1 > 0\).

Therefore,
\[
E_{0,l}(t) \lesssim \delta^{4 - 2\epsilon_0}.
\]

**2. The treatment for the cases of \( k = 1 \) and \( l \leq 5 \)**

- If \( k_i + l_i \leq k_{p+1} + l_{p+1} \) for all \( 1 \leq i \leq p \), then \( k_i + l_i \leq 3 \), and
\[
\int_{D_t} |\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \tilde{\Gamma}^{k_1} \Omega^{l_1} \phi| \cdots |\partial \tilde{\Gamma}^{k_p} \Omega^{l_p} \phi| \cdot |\partial^2 \tilde{\Gamma}^{k_{p+1}} \Omega^{l_{p+1}} \phi| \\
\lesssim \int_{D_t} \{|\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \Omega^{\leq 3} \phi|^p \cdot |\partial^2 \tilde{\Gamma} \Omega^{\leq 4} \phi| + |\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \Omega^{\leq 3} \phi|^{p-1} \cdot |\partial \tilde{\Gamma} \Omega^{\leq 2} \phi| \cdot |\partial^2 \Omega^{\leq 5} \phi| \} \\
\lesssim \int_{t_0}^t M_0^p \delta(\frac{13}{5} - \epsilon_0)^p \tau^{-p} E_{1,l}(\tau) d\tau + \int_{t_0}^t M_0^p \delta(\frac{13}{8} - \epsilon_0)^p \tau^{-p} E_{2,l}(\tau) d\tau \\
+ \int_{t_0}^t M_0^p \delta(\frac{13}{8} - \epsilon_0)^p \tau^{-p} E_{1,l}(\tau) d\tau \\
\lesssim \delta^{4 - 2\epsilon_0}.
\]

- If there exists one \( i \) \((1 \leq i \leq p)\) such that \( k_i + l_i > k_{p+1} + l_{p+1} \), then \( k_{p+1} + l_{p+1} \leq 2 \), and
\[
\int_{D_t} |\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \tilde{\Gamma}^{k_1} \Omega^{l_1} \phi| \cdots |\partial \tilde{\Gamma}^{k_p} \Omega^{l_p} \phi| \cdot |\partial^2 \tilde{\Gamma}^{k_{p+1}} \Omega^{l_{p+1}} \phi| \\
\lesssim \int_{D_t} \{|\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \Omega^{\leq 3} \phi|^{p-1} \cdot |\partial \tilde{\Gamma} \Omega^{\leq 5} \phi| \cdot |\partial \Omega^{\leq 3} \phi| \cdot |\partial^2 \Omega^{\leq 2} \phi| \cdot |\partial \tilde{\Gamma} \Omega^{\leq 2} \phi| \cdot |\partial^2 \Omega^{\leq 2} \phi| \} \\
\lesssim \delta^{4 - 2\epsilon_0}.
\]
Therefore,
\[ E_{1,t}(t) \lesssim \delta^{4-2\varepsilon_0}. \]

3. The treatment for the other left cases in (8.29)

For the cases of \( k + l \leq 6 \) \((k \geq 2)\), the treatment procedure is exactly similar to that for the cases of \( k = 0 \) and \( k = 1 \), the details are omitted. Then we can get
\[ E_{k,l}(t) \lesssim \delta^{7-2k-2\varepsilon_0}, \]
which is independent of \( M_0 \).

Note that all the bounded constants above are independent of \( M_0 \). Then the bootstrap assumptions (8.20) can be closed. By combining with the local existence of solution \( \phi \) to (1.1) and the continuous argument, the global existence of \( \phi \) in \( B_{2\delta} \) is established.

Finally, we prove Theorem 1.1.

**Proof.** By Theorem 2.1, we have got the local existence of smooth solution \( \phi \) to equation (1.1) with (1.5)-(1.6). On the other hand, the global existence of \( \phi \) near \( C_0 \) and in \( B_{2\delta} \) has been established in Section 8. Then it follows from the uniqueness of smooth solution to (1.1) that the proof of \( \phi \in C_\infty([1, +\infty) \times \mathbb{R}^3) \) is finished. In addition, \( |\partial\phi| \lesssim \delta^{1-\varepsilon_0}t^{-1} \) comes from Theorem 2.1, (3.1) and the first inequality in (8.21). Thus Theorem 1.1 is proved. \( \Box \)

**Appendix**

A. The existence of short pulse initial data with (1.6)

In this section, we give the existence of short pulse initial data which satisfy (1.6).

Due to \( \partial_t^2 \phi = c^2(\partial_r^2 \phi + \frac{2}{r} \partial_r \phi + \frac{1}{r^2} \Delta \phi) \), then
\[
\tilde{L}^2 \phi = c^2(\partial_r^2 \phi + \frac{2}{r} \partial_r \phi + \frac{1}{r^2} \Delta \phi) + \partial_r^2 \phi + 2\partial_r(\partial_t \phi)
= (c^2 + 1)\partial_r^2 \phi + \frac{2c^2}{r} \partial_r \phi + \frac{c^2}{r^2} \Delta \phi + 2\partial_r(\partial_t \phi).
\]

Assume that \( \phi_0(s, \omega) \in C_\infty^\infty((-1, 0) \times S^2) \) of (1.5) is chosen as the fixed smooth function. By virtue of (1.5), the derivatives of \( \phi \) (up to the second order) can be computed on \( \Sigma_1 \) as follows

\[
\begin{align*}
\partial_t \phi &= \delta^{1-\varepsilon_0} \phi_1, \\
\partial_r \phi &= \delta^{1-\varepsilon_0} \partial_s \phi_0, \\
\partial_r(\partial_t \phi) &= \delta^{-\varepsilon_0} \delta^2 \phi_1, \\
\partial_r^2 \phi &= \delta^{-\varepsilon_0} \delta^2 \phi_0, \\
\Delta \phi &= \delta^{2-\varepsilon_0} \Delta \phi_0.
\end{align*}
\]

Then \( \tilde{L}^2 \phi |_{t=1} \) can be expressed as
\[
\tilde{L}^2 \phi |_{t=1} = (c^2 + 1)\delta^{-\varepsilon_0} \delta^2 \phi_0 + \frac{2c^2}{r} \delta^{1-\varepsilon_0} \partial_s \phi_0 + \frac{c^2}{r^2} \delta^{2-\varepsilon_0} \Delta \phi_0 + 2\delta^{-\varepsilon_0} \partial_s \phi_1.
\]

It is claimed that \( \phi_1 \) in (1.5) can be chosen such that
\[
\tilde{L}^2 \phi |_{t=1} = O(\delta^{2-\varepsilon_0}),
\]
(A.1)
where \( \phi_1(s, \omega) \in C_0^\infty((-1, 0) \times S^2) \). We now make the following assumption for \( \phi_1 \) with

\[
|\partial_s \phi_1| \leq C, \tag{A.2}
\]

where the constant \( C > 0 \) depends only on \( \phi_0 \) but is independent of \( \delta \).

By (A.2), in order to show (A.1), it suffices to prove

\[
(1 + \frac{1}{c^2}) \partial_s^2 \phi_0 + \frac{2}{r} \partial_r \phi_0 + 2c^{-2} \partial_s \phi_1 = O(\delta^2). \tag{A.3}
\]

Since \( (1 + \frac{1}{c^2})^{-1} = \frac{1}{2} - \frac{1}{4} \delta (1-\varepsilon_0)^p \phi_0^p + O(\delta (1-\varepsilon_0)^{2p}) \), it then follows from (A.3) that

\[
\partial_s^2 \phi_0 + \left( \frac{2}{r} \delta \partial_r \phi_0 + 2c^{-2} \partial_s \phi_1 \right) (\frac{1}{2} - \frac{1}{4} \delta (1-\varepsilon_0)^p \phi_0^p + O(\delta (1-\varepsilon_0)^{2p})) = O(\delta^2).
\]

This, together with \( p > p_c = \frac{1}{1-\varepsilon_0} \), yields

\[
\partial_s^2 \phi_0 + \frac{\delta}{r} (1 - \frac{1}{2} \delta (1-\varepsilon_0)^p \phi_0^p) \partial_r \phi_0 + (1 - \frac{1}{2} \delta (1-\varepsilon_0)^p \phi_0^p) c^{-2} \partial_s \phi_1 = O(\delta^2). \tag{A.4}
\]

Due to \( r = 1 + s \delta \) and \( p > p_c \), in order to let (A.4) hold, then one can set

\[
\partial_s^2 \phi_0 + \delta \partial_r \phi_0 + \left( 1 + \frac{1}{2} \delta (1-\varepsilon_0)^p \phi_0^p \right) \partial_s \phi_1 = 0,
\]

which derives \( F(\phi_1, \delta) = \partial_s \phi_0 + \delta \partial_r \phi_0 + \left( \phi_1 + \frac{\delta (1-\varepsilon_0)^p \phi_0^p}{2(1-\varepsilon_0)^p} \phi_1^{p+1} \right) = 0 \). Note that \( F(-\partial_s \phi_0, 0) = 0 \) and \( \partial_{\phi_1} F(\phi_1, \delta) = 1 + \frac{1}{2} \delta (1-\varepsilon_0)^p \phi_0^p > 0 \) for small \( \delta > 0 \). Then it follows from the implicit function theorem and \( F(\phi_1, \delta) = 0 \) that \( \phi_1(s, \omega) \in C_0^\infty((-1, 0) \times S^2) \) and (A.1) are obtained. On the other hand, by \( \partial_t = \frac{1}{2} (\bar{L} - \bar{L}) \) and \( \bar{L} \bar{L} \phi = (c^2 - 1) \Delta \phi \), we have

\[
\partial_t \bar{L} \phi|_{t=1} = \frac{1}{2} [\bar{L}^2 \phi + (1 - c^2) \partial_r^2 \phi - \frac{2c^2}{r} \partial_r \phi - \frac{c^2}{r^2} \Delta \phi]|_{t=1} = O(\delta^{1-\varepsilon_0}).
\]

Then it follows from the integration with respect to \( r \) that

\[
|\bar{L} \phi|_{t=1} \leq \int_{1-\delta}^1 |\partial_r \bar{L} \phi|_{t=1} dr = O(\delta^{2-\varepsilon_0}).
\]

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