Deformation Quantization on Singular Coadjoint Orbits.

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ABSTRACT. Invariant star products are constructed on minimal coadjoint orbits of all the simple Lie algebras. Explicit expressions are given for the generators of the Joseph ideals and the associated infinitesimal characters.

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1. Introduction.

Let \( \mathfrak{g} \) be a Lie algebra over a field \( K \), \( \mathfrak{g}' \) its K-vector space dual and \( G \) its adjoint Lie group. The group acts smoothly on \( \mathfrak{g}' \) and each orbit is a variety of the type \( K[x_1, \ldots, x_N]/(R) \), where \( R \) is a set of polynomial relations. And each variety in this very rich collection comes fitted with a natural symplectic structure. As shown by Kirillov, there are interesting relationships between these coadjoint orbits and representation theory; Kostant [K1] and Souriau [S] brought in ideas from classical and quantum mechanics and especially from quantization. Meanwhile Gerstenhaber had developed his deformation theory [G1], and eventually it was understood that quantization on a symplectic space is a deformation of the algebra of functions. One can go further, to regard every deformation as quantization.

It is remarkable, we think, that the greater part of recent work on this subject has tended to downplay the special features that coadjoint orbits inherit from the Lie algebra. This paper is a contribution to the study of invariant star products, a type of deformation-quantization that makes more intimate contact with Lie structure. We shall also borrow from the analogy with quantization of mechanical and field theoretic systems. It will turn out that the view of quantization that was introduced by Weyl, and that is intimately tied to Bohr’s correspondence principle, is a most efficient one; it supplements cohomological methods precisely where those become inefficient.

History

A first correspondence between quantum mechanical and classical observables was established by Weyl [Wl] (classical-to-quantum) and Wigner [Wr] (the other way). The formula for pulling back the non-commutative commutator of operators to classical observables is due to Moyal [M]. The resulting deformation of the Poisson bracket, and of the product of functions on phase space, was subjected to mathematical analysis by Vey [Vy], following the study of one-differentiable deformations in [FLS].

The papers [BFFLS] generalized this notion and proposed a new axiomatic approach to quantization, interpreted as a deformation and formulated in terms of a general type of associative star product. Following Moyal and Vey one considers the space of formal series, in a parameter \( \hbar \), of \( C^\infty \) functions on a symplectic space \( W \), with
an associative product of the form

\[ f \ast g = fg + \frac{i\hbar}{2} \{ f, g \} + \sum_{n=2}^{\infty} \hbar^n C_n(f, g), \quad f, g \in C^\infty W, \]

where \( fg \) is the ordinary product of functions, \( \{ ., . \} \) is the Poisson bracket, and the cochains \( C_n \) are often (but not in this paper) taken to be bi-differential operators.

There were at least four different developments.

1. The problem of classification of star products on symplectic spaces, up to a natural but very weak form of equivalence, was investigated by Gutt and others [Gt]. The existence of star products on an arbitrary symplectic manifold was established by de Wilde and Lecomte [dL] and by Fedosov [F], culminating with the results of Kontsevich who demonstrated the existence of star products on an arbitrary Poisson manifold [Kh], [T]. The importance of these results is that they are global statements about (smooth) manifolds.

2. A generalization of Weyl’s original correspondence is required in field theories and is referred to as ‘the ordering problem’. There have been applications to mechanical problems as well [AW]. Attention is called to the power of this method in the local algebraic context. From this point of view the existence of associative star products is not surprising, although a concise expression for \( f \ast g \) may be difficult to obtain.

3. The concept of invariant star products on coadjoint orbits of Lie algebras was proposed in [BFFLS]. The existence of invariant star products on any ‘regular’ coadjoint orbit of a semisimple Lie algebra was demonstrated in the same paper. This result was obtained by setting up an explicit, invariant type of generalized Weyl correspondence. The original Weyl correspondence yields, in particular, an invariant star product for the linear symplectic algebra of the manifold. Invariant star products are implicit in recent studies of nilpotent orbits, especially those that deal with the Joseph ideal, e.g. [BJ]. There was an important parallell development in the work of Berezin [Bn].

4. Star quantization was used as a tool in representation theory, to generalize the method of geometric quantization of Kostant [K1] and Souriau [S]. See for example [F1], [Gt], [W]. This idea has not yet realized its full potential.
Cohomology

Since the first paper by Vey it has been clear that the classification of deformations of the algebra of functions of a smooth manifold is intimately related to the Hochschild cohomology of the manifold. Less well known is the role played by the BGS decomposition of this complex. As an example we recall that the nonexistence of abelian deformations on a smooth manifold is related to the vanishing of the Harrison component of Hochchild cohomology. More general algebraic manifolds offer more room for deformations. The lifting of a Poisson structure to a star product is governed by components of cohomology that are purely local, being associated with the singularities. This strongly suggests that global Poisson structures on varieties more general than smooth manifolds lift to global star products. But the results reported here are local.

Results

This paper was intended as a preliminary study of the deformations of the coordinate algebras of some algebraic varieties with singularities, a context in which the BGS decomposition could be expected to have some interesting applications. The coadjoint orbits of simple Lie groups offers an especially rich and interesting family of examples.

A familiar reduction paradigm was used to reduce the cohomology to a complex of closed, linear chains. It had been expected that this would lead to an easy classification of essential deformations. It turned out, however, to be difficult to obtain enough information about the homology of the reduced complex. Before us experts [BJ], applying much heavier machinery ([Be], [BGS]) to the same problem, had the same experience and were forced to fall back on heuristic (?) arguments. See below, and in Section 6.3.

Returning to the point of view that sees a star product as a correspondence between ordinary polynomials and star polynomials, we were able to complete the calculations. The principal results of this paper are as follows.

(a) The Hochschild cohomology of the coordinate algebra of an algebraic variety defined by quadratic relations is isomorphic to that of its restriction to linear, closed chains. Obstructions to extending a first order star product to a formal, associative product, to all orders in the deformation parameter, can therefore be reduced to a study of the star products $x \ast x$ and $x \ast x \ast x$ for $x$ of degree 1.
(b) The minimal, coadjoint orbit of a complex, simple Lie algebra $\neq \mathfrak{sl}(n)$ admits a one-parameter family of invariant star products/deformations. In the case of $\mathfrak{sl}(n)$ we determine a 2-parameter family of deformations, including an interesting 1-parameter abelian subfamily. In all the other cases there is a unique, invariant star product such that, $\forall u, v \in \mathfrak{g}'$,

$$u \ast v - v \ast u = \hbar \{u, v\}, \quad u \ast u = uu + k,$$

for some $k \in K$; see Section 6. The value of $k$ is determined by an examination of the next case,

$$u \ast u \ast u = uuu + \phi(u),$$

where $\phi$ is a polynomial of degree one. Both $k$ and $\phi$ are uniquely determined by the relations that define the orbit. A uniform calculation covers the five exceptional algebras. For $\mathfrak{sl}(n)$ and the other classical simple Lie algebras we calculate the generators of the Joseph ideals and determine the associated highest weight modules. The uniqueness of the Joseph ideal is a corollary, see [WS].

Outline.

Section 2 contains a short introduction to formal star products and the BGS decomposition of Hochshild cohomology. Section 3 is a study of the Hochschild cohomology of algebraic varieties defined by a set of quadratic relations. A principal tool used here is a reduction of the Hochschild complex to a subcomplex of closed, linear chains; Theorem 3.1.2. The method is effective when the underlying algebra is finitely generated and thus graded, with only positive degrees. The BGS decomposition is used throughout.

Section 4 is a brief introduction to invariant star products. Section 4.3 gives an example of the appearance of finite representations within the program of star quantization. Included here is the first example (known to me) of harmonic polynomials in the enveloping algebra of a simple Lie algebra. We show an ‘instance of a deformation’ (not a formal deformation) in which the deformed variety (the spectrum of the deformation of the ring of coordinate functions) is a finite union of disconnected varieties.

Section 5 is an informal study of singular, nilpotent orbits. It serves to introduce this subject to nonexperts, and to build some support for our own intuition. Associated to these orbits, and to the Joseph ideals, are certain very special, unitary representations that play a conspicuous role in physics.
Section 6 examines invariant star products on the most interesting coadjoint orbits, those of minimal dimensions, with their Joseph ideals. Attempting to calculate the cohomology we encounter a difficulty that had already been met by Braverman and Joseph [BJ], and fail to obtain a sufficiently detailed description of the space \( \text{Hoch}_3 \) of the coordinate algebra. For the solution of this problem we offer only conjecture 6.1.1, but we circumvent the difficulty by an independent, direct calculation. It is done by reformulating the search for an invariant star product as a correspondence principle, in the spirit of Weyl’s symmetric ordering. Detailed knowledge of the cohomology of the restricted complex is not needed. Generators of the Joseph ideals are determined.

The Lie algebras \( \mathfrak{sl}(n) \) and \( \mathfrak{so}(n) \) are treated separately and all the calculations are included, with proofs relegated to an Appendix. The case of \( \mathfrak{sp}(2n) \) is too well known to warrant much attention. The five exceptional simple Lie algebras are handled uniformly together, all the calculations are in the main text, they are both short and easy.

Within the family of generally noncommutative star products there may be a subfamily of non trivial abelian ones. According to Braverman and Joseph, this would contradict the fact that the minimal orbits - excepting the case of \( \mathfrak{sl}(n) \) - are rigid. Granted that a deformation of the ring of coordinate functions implies a deformation of its spectrum; but is it known that the deformed spectrum of an abelian deformation is always an algebraic variety, or that every equivariant deformation of a coadjoint orbit is a coadjoint orbit? In any case we confirm that abelian deformations of the coordinate algebra exists only in the case of \( \mathfrak{sl}(n) \), and that in that case the spectrum can be identified with a neighbouring orbit of the same dimension.

2. Associative star products and cohomology.

2.1 Formal \( \ast \)-products.

A formal, abelian \( \ast \)-product on a commutative algebra \( A \) is a commutative, associative product on the space of formal power series in a formal parameter \( \hbar \) with coefficients in \( A \), given by a formal series

\[
f \ast g = fg + \sum_{n>0} \hbar^n C_n(f,g).
\] (2.1)
Associativity is the condition that $f \ast (g \ast h) = (f \ast g) \ast h$, or

$$\sum_{m,n=0}^{k} \hbar^{m+n} \left( C_m(f, C_n(g, h)) - C_m(C_n(f, g), h) \right) = 0, \quad (2.2)$$

where $C_0(f, g) = fg$. This must be interpreted as an identity in $\hbar$; thus

$$\sum_{m,n=0}^{k} \delta_{m+n,k} \left( C_m(f, C_n(g, h)) - C_m(C_n(f, g), h) \right) = 0, \quad k = 1, 2, \cdots. \quad (2.3)$$

The formal $\ast$-product (2.1) is associative to order $p$ if Eq.(2.3) holds for $k = 1, \cdots p$.

A first order $\ast$-product is a product

$$f \ast g = fg + \hbar C_1(f, g), \quad (2.4)$$

associative to first order in $\hbar$, which makes $C_1$ be a closed Hochschild cochain, namely

$$\partial C_1(f, g, h) := fC_1(g, h) - C_1(fg, h) + C_1(f, gh) - C_1(f, g)h = 0.$$

If $C_1$ is exact; that is, if there is a 1-cochain such that

$$C_1(f, g) = \partial E(f, g),$$

then to first order in $\hbar$ Eq.(2.4) can be written

$$(f - \hbar E(f)) \ast (g - \hbar E(g)) = fg - \hbar E(fg);$$

essential first order deformations are classified by Hoch$^2$. [G1]

Suppose that a formal $\ast$-product is associative to order $p \geq 1$; this statement involves $C_1, \cdots, C_p$ only, and we suppose these cochains fixed. Then the condition that must be satisfied by $C_{p+1}$, in order that the $\ast$-product be associative to order $p + 1$, is

$$\sum_{m,n=0}^{p} \left( C_m(C_n(f, g), h) - C_m(f, C_n(g, h)) \right) = \partial C_{p+1}(f, g, h). \quad (2.5)$$

The left hand side is closed [G1],[BFFLS]; this minor miracle is responsible for the fact that commutative algebras are not isolated in the family of associative algebras. The right hand side of Eq. (2.5) is a Hochschild coboundary. An obstruction to the existence of a two-cochain $C_{p+1}$ that would solve Eq. (2.5) is thus an element of Hoch$^3$. This statement will be sharpened below.
2.2. The BGS decomposition of Hochschild (co-)homology.

The $p$-chains of the Hochschild homology complex of a commutative algebra $A$ are the $p$-tuples $a = \sum a_1 \otimes \cdots \otimes a_p \in A^\otimes p$, and the differential is defined by
\[ da = a_1 a_2 \otimes a_3 \cdots \otimes a_p - a_1 \otimes a_2 a_3 \otimes a_4 \cdots \otimes a_p + \cdots + (-)^p a_1 \otimes \cdots a_{p-2} \otimes a_{p-1} a_p. \]

The $p$-cochains are maps $A^\otimes p \to A$, and the differential is
\[ \partial C(a_1, \cdots, a_{p+1}) = a_1 C(a_2, \cdots, a_{p-1}) - C(da) + (-)^{p+1} C(a_1, \cdots, a_p) a_{p+1}. \]

The Hochschild cochain complex splits into a finite or infinite sum of direct summands. (If the algebra is generated by $N$ generators then there are only $N$ nonzero summands.) After the pioneering work of Harrison [H] and Barr [B],[G2], the complete decomposition of the Hochschild cohomology of a commutative algebra was found by Gerstenhaber and Schack [GS]. The decomposition is based on the action of $S_n$ on $n$-cochains, and on the existence of $n$ idempotents $e_n(k)$, $k = 1, \cdots, n$, in $\mathbb{C}S_n$, $\sum_k e_n(k) = 1$, with the property that
\[ \partial \circ e_n(k) = e_{n+1}(k) \circ \partial. \]

There is thus a decomposition $C^n = \sum_{k=1}^{\infty} C_{n,k}$ of the space of $n$-cochains, and $\text{Hoch}^n = \sum_{k=1}^{\infty} \mathcal{H}^{n,k}$ with $\mathcal{H}^{n,1} =: \text{Harr}^n$.

A generating function was found by Garsia [G],
\[ \sum_{k=1}^{n} x^k e_n(k) = \frac{1}{n!} \sum_{\sigma \in S_n} (x - d_\sigma)(x - d_\sigma + 1) \cdots (x - d_\sigma + n - 1) \text{sgn}(\sigma) \sigma, \]
where $d_\sigma$ is the number of descents, $\sigma(i) > \sigma(i+1)$, in $\sigma(1 \cdots n)$. (Example: $\sigma(1234) = 3142$ has one descent, from 2 to 3.) The simplest idempotents are
\[ e_2(1)12 = \frac{1}{2}(12 + 21), \quad e_2(2)12 = \frac{1}{2}(12 - 21), \]
\[ e_3(1)123 = \frac{1}{6}(2(123 - 321) + 132 - 231 + 213 - 312), \]
\[ e_3(2)123 = \frac{1}{2}(123 + 321) \]
\[ e_n(n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma. \]
The space of Hochschild $n$-chains decomposes in the same way, $C_n = \sum_{k=1}^{n} C_{n,k}$ with

$$d \circ e_n(k) = e_{n-1}(k) \circ d$$

so that $\text{Hoch}_n = \sum_{k=1}^{n} H_{n,k}$ with $H_{n,1} = \text{Harr}_n$.

### 2.3. Star products and the BGS decomposition.

Every 2-cochain $C$ has a decomposition

$$C = C^+ + C^-, \quad C^+ \in C^{2,1}, \ C^- \in C^{2,2}.$$ 

Associativity of the star product to order $\hbar$ is the requirement that both 2-cochains be closed, $\partial C^+ = \partial C^- = 0$. A first order deformation of the algebra is inessential if both forms are exact. In the case of a smooth manifold Harr$^2$ is empty and choosing $C_1^+ = 0$ entails no essential loss. Returning now to Eq.(2.5) we have mappings

$$
\begin{array}{ccc}
\mathcal{Z}^{2,1} \times \mathcal{Z}^{2,1} & \rightarrow & \mathcal{Z}^{3,1} \\
\mathcal{Z}^{2,2} \times \mathcal{Z}^{2,2} & \rightarrow & \mathcal{Z}^{3,3} \\
\mathcal{Z}^{2,1} \times \mathcal{Z}^{2,2} & \rightarrow & \mathcal{Z}^{3,2} \\
\end{array}
$$

(2.6)

The first column of arrows represents the construction on the left side of Eq. (2.5). The second column of arrows is the mapping by the differential. Because $\mathcal{B}^{3,3}$ is empty; the obstruction in $\mathcal{Z}^{3,3} = \mathcal{H}^{3,3}$ demands that the antisymmetric part of the left side of (2.5) vanish, this is the Jacobi identity, satisfied if $C_1^-$ is a Poisson bracket. The first line shows that the obstruction to abelian deformations is the Harrison component $\text{Harr}^3 = \mathcal{H}^{3,1} \subset \text{Hoch}^3$. In the case of smooth manifolds $\text{Hoch}^n = \mathcal{H}^{n,n} = 0$ and abelian deformations are inessential. The familiar deformations with $C_1 = C_1^-$ encounter no additional obstructions to order $\hbar^2$. We now turn to a preliminary investigation of algebraic varieties with singularities. We shall find varieties for which $\mathcal{Z}^{3,1}$, $\mathcal{Z}^{3,2}$ and $\mathcal{Z}^{3,1}$ are all non empty.
3. Some varieties with singularities.

3.1. Conic varieties defined by quadratic relations.

These are algebraic varieties of the type \( \mathcal{O}^N / R \), where \( R = \{ g_\alpha \}_{\alpha=1,2,...} \) is a set of homogeneous, quadratic forms,

\[
g_\alpha = \sum_{i,j=1}^{N} g^{ij}_\alpha x_i x_j, \quad \alpha = 1, 2, ... .
\] (3.1)

Let \( A \) be the graded coordinate algebra \( A = \mathcal{O}[x_1, ..., x_N] / (R) \), and \( A_+ \) the subalgebra of positive degrees; the restriction to positive degrees is essential. Cochains on \( A_+ \) extend naturally to \( A \), but the homology of \( A_+ \) is richer than that of \( A \). (The generators of \( A_+ \) are not exact.)

3.1.1. Definition. A chain \( a = a_1 \otimes a_2 \otimes ... \otimes a_p \) will be said to be ‘linear’ if each \( a_k, k = 1, ..., p \) is of degree 1. The ‘restricted complex’ is the restriction of the Hochschild complex of \( A_+ \) to closed, linear chains.

In the case of a smooth manifold, or more generally in the case of a regular commutative algebra, the chains of the restricted complex are the antisymmetric ones; the next result reduces in that case to a famous theorem of Hochschild, Kostant and Rosenfeld [HKR]. For other generalizations see [FG], [FK] and Sect. 3.3.

Let \( A_+ \) be as above, the subalgebra of \( A = \mathcal{O}[x_1, ..., x_N] / (R) \) obtained by restriction to positive degrees.

3.1.2. Theorem. The Hochschild complex of \( A_+ \) is quasi-isomorphic to the restricted complex; that is, their (co-)homologies are isomorphic.

Proof. The restriction of a closed/exact form is closed/exact. Conversely, every closed/exact restricted cochain is the restriction of a closed/exact Hochschild cochain. It is enough to consider homogeneous chains; that is, \( a = a_1 \otimes ... \otimes a_n \) such that each factor \( a_k, k = 1, ..., n \) is of well defined degree. (The only grading that we use is the total polynomial degree.) To show that every restricted (= closed, linear) \( n \)-cochain extends to a closed, Hochschild \( n \)-cochain we consider the formula

\[
\partial C(a_1, ..., a_{n+1}) = a_1 C(a^1) - C(da) + (-)^{n+1} C(a^{n+1})a_{n+1},
\] (3.2)
where \( a^1 = a_2 \otimes \ldots \otimes a_{n+1} \) and \( a^{n+1} = a_1 \otimes \ldots \otimes a_n \). Evidently the degree of \( a \) is higher than the degrees of \( a^1 \) and \( a^{n+1} \). This formula can therefore be used to try to extend closedness, recursively to higher degrees. The obstruction is \( da = 0 \), but it is easy to verify that the remaining terms in (3.2) vanish when \( a \) is exact and \( C \) is closed on lower degrees. The obstruction comes from homology; if the restricted \( n \)-cochain \( C \) has the property that \( \partial C(h) = 0 \) for a representative \( h \) of every homology class of \( n + 1 \)-chains, then it extends to a closed, Hochschild \( n \)-cochain.

3.2. The simple cone; one quadratic relation.

Retain all the definitions but suppose that \( R = g \) is just one quadratic form. In this case the following holds.

3.2.1. Proposition. Every closed chain is homologous to a linear chain and no linear chain is exact. The space \( Z_{2k+1} \) of closed, linear \((2k+1)\)-chains is spanned by the following \( n = 2k + l \)-chains, \( n = 1, 2, \ldots \), with \( Z_{2k+l,k+l} \in Z_{2k+1,k+l} \),

\[
(Z_{2k+l,k+l})_{m_1 \ldots m_l} = g^{i_1j_1 \ldots i_kj_k} \sum_\sigma (-)^\sigma x_{i_1} \otimes \ldots \otimes x_{j_k} \otimes x_{m_1} \otimes \ldots \otimes x_{m_l},
\]

where the sum is over all permutations of \( i_1 \ldots j_k m_1 \ldots m_l \) that preserve the internal order of each pair \((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\).

Examples,

\[
(Z_{1,1})_i = x_i, \quad i = 1, \ldots, N,
\]

\[
(Z_{2,2})_{ij} = x_i \wedge x_j, \quad Z_{2,1} = g^{ij} x_i \otimes x_j,
\]

\[
(Z_{3,3})_{ijk} = x_i \wedge x_j \wedge x_k, \quad (Z_{3,2})_k = g^{ij} (x_i \otimes x_j \otimes x_k - x_i \otimes x_k \otimes x_j + x_k \otimes x_i \otimes x_j),
\]

\[
(Z_{4,4})_{ijkl} = x_i \wedge x_j \wedge x_k \wedge x_l,
\]

\[
(Z_{4,3})_{kl} = g^{ij} (x_i \otimes x_j \otimes x_k \otimes x_l + x_k \otimes x_i \otimes x_j \otimes x_l + x_i \otimes x_l \otimes x_j \otimes x_k
\]

\[
+ x_k \otimes x_l \otimes x_i \otimes x_j + x_i \otimes x_k \otimes x_l \otimes x_j + x_l \otimes x_i \otimes x_k \otimes x_j - k, l),
\]

\[
(Z_{4,2}) = g^{ij} g^{kl} (x_i \otimes x_k \otimes x_l \otimes x_j - x_i \otimes x_k \otimes x_j \otimes x_l + x_i \otimes x_j \otimes x_k \otimes x_l).
\]

From this point onwards, the notation \( Z_{k,l}, B_{k,l}, Z^{k,l}, B^{k,l} \) stands for spaces defined with reference to the restricted complex. The dimension of \( Z_{2k+l,k+l} \subset H_{2k+l,k+l} \) is
or 0. We note that $Z_{3,1}$ is empty. To lowest order, a star product is determined by the 2-cochain $C_1$, and up to equivalence by the restricted 2-cochains; that is, by their values $C^-(x_i \wedge x_j)$ and $g^{ij}C^+(x_i, x_j)$ on the homological basis. The differential $\partial C_1^+$ is in $B^{3,1}$, and since $Z_{3,1}$ is empty every restricted, symmetric 2-cochain is closed. The differential $\partial C_1^-$ is in $B^{3,2}$; $C_1^-$ is closed iff $\partial C_1^-$ vanishes on $Z_{3,2}$,

$$\partial C_1^-(Z_{3,2}) = 4g^{ij}x_iC_1^-(x_j, x_k) = 0.$$ 

This will be interpreted as the statement that the ‘Hamiltonian’ vector fields $C_1^-(x_k, \cdot)$ be tangential to the constraint surface.

The addition of an exact form $\partial E$ to $C_1$ (we like to think of it as a ‘gauge transformation’) does not affect $C_1^-$ but it adds $2g^{ij}x_iE(x_j)$ to $C_1^+(Z_{2,1}) = g^{ij}C_1^+(x_i, x_j)$.

### 3.2.2. Example

Suppose that the 2-form $g$ is nondegenerate; then $C_1^+$ is fixed up to equivalence by its value $c = g^{ij}C_1^+(x_i, x_j)|_{x=0}$ at $x_1 = x_2 = \ldots = 0$. That is; $\mathcal{H}^{2,2}$ is the space of tangential vector fields on the cone and $\mathcal{H}^{2,1} = \mathcal{C}$. (See 3.2.8 for equivariant cohomology.)

We examine the obstructions to extending the star product to all orders in $\hbar$, referring to Eq. s (2.5) and (2.6).

### 3.2.3. The emptiness of $B^{3,3}$ is an obstruction that must be circumvented by imposing the Jacobi identity on $C_1^-$. Because $C_1^-$ is closed this entails that it extends to a unique Poisson bracket on $A$. Recall that, if $(a, b) \mapsto \{a, b\}$ is a Poisson bracket on $A$ then $\forall f \in A$ the mapping $f^\# : A \to A$ defined by $a \mapsto \{f, a\}$ is a derivation. The projection of (2.5) on $Z_{2,3}$ is now solved by taking $C_n^-(x_i, x_j) = 0, \ i, j = 1, \ldots, N, \ n > 1$. This choice is implicit in the context of differentiable deformations, and it is one of the axioms of invariant star products. We do not investigate alternatives.

Summary: The projection of Eq. (2.6) on $Z_{3,3}$ leads to

1. $C_1^-(x_i, x_j) = \{x_i, x_j\}$ extends to a Poisson bracket on $A$,
2. $\{x_i, g(x, x)\} = 0$ is the condition that $C_1$ be closed,
3. $C_n^-(x_i, x_j) = 0, \ n > 1, \ i, j = 1, \ldots, N$.  

(3.3)
3.2.4. The projection of Eq. (2.5) on $Z_{3,2}$ takes the form

\[
\sum_{m+n=p+1} \left[ -4g^{ij}C_m^+(C_n^-(x_i,x_k),x_j) + 2g^{ij}C_m^-(C_n^+(x_i,x_j),x_k) \right]
\]

\[= \partial C_{p+1}^- (Z_{3,2}) \kappa = 4g^{ij}x_i C_{p+1}^-(x_j,x_k), \quad k = 1, \ldots, N. \]

(3.4)

If $g$ is nondegenerate then we can restrict the value of $C_1 ^+(x_i, x_j)$ to $\mathbb{C}$; see Example 3.2.2. The obstruction is then the value of the left side at $x = 0$. In view of (3.3), (3.4) simplifies to

\[
2g^{ij}C_p ^+(\{x_i, x_k\}, x_j) - g^{ij}\{C_p ^+(x_i, x_j), x_k\} = 0.
\]

(3.5)

3.2.5. Example. If $\{x_i, x_k\} = \epsilon_{ik}^m$, the coefficients $(\epsilon_{ik}^m)$ the structure constants of a simple Lie algebra $\mathfrak{g}$, then (3.5) is satisfied when $C_p ^+$ is the Killing form of $\mathfrak{g}$.

3.2.6. Example. In the case of $sl(n)$ there is an equivariant 3-tensor $f : A \otimes A \to A$ and $C_p ^+ = f$ also solves (3.5).

3.2.7. Example. Choose coordinates such that $g(x, x) = x_i ^2 - \rho(x)$, where $\rho(x)$ is a polynomial in $x_1, \ldots, x_N$, at most linear in $x_N$. A regular function on the cone, being the restriction of a polynomial in $x_1, \ldots, x_N$, has a unique decomposition of the form

$f = f_1 + x_N f_2$, where $f_1, f_2$ are polynomials in $x_1, \ldots, x_{N-1}$. Define ([F2])

\[
f \ast g = fg + h f_2 g_2 = fg |_{x_N ^2 = \rho(x) + h}.
\]

(3.6)

This star product is associative to all orders; it can be interpreted as the ordinary product of functions on the hyperboloid $g = h$. This deformation, in which $C_1$ is symmetric, can be followed by another deformation in which $C_1$ is antisymmetric, leading to a Poisson bracket such that the vector fields $\{x_i, \cdot\}$ are tangent to the hyperboloid $g(x, x) = h$; no Harrison cohomology intervenes in either stage. When both deformations are combined we note that, with $C_1 ^+$ as we have defined it in Eq.(3.6), $C_1 ^+ (f, g) = f_2 g_2$, the only contribution to the left side of Eq.(3.5) at $x = 0$ comes from the linear term in $\{x_i, x_k\}$. Closure of $C_m^-$ implies that the vector field $\{\cdot, x_k\}$ is tangent to $g(x, x) = h$.

3.2.8. Equivariant cohomology. In the context of Lie algebras and invariant star products all maps will be equivariant. This affects the question of exactness, as in 3.2.2, and will be taken into account as the occasion arises. Theorem 3.1.2 is not affected.
3.3. Case of one polynomial relation.

Let \( A = \mathcal{O}[x_1, \ldots, x_N]/(g) \) where \( g \) is a polynomial without constant or linear terms.

3.3.1. Grading. We can no longer restrict our attention to homogeneous chains. By an appropriate linear transformation of variables we can bring the polynomial \( g \) to the form \( x_1^t + h(x_1, \ldots, x_N) \), where \( h \) is a polynomial of degree less than \( t \) in \( x_1 \). A ‘normalized’ polynomial is one that is of degree less than \( t \) in \( x_1 \); to each regular function on \( \mathcal{R}/g \) there is just one normalized polynomial and the degree of a regular function is defined to be the degree of this normalized polynomial.

Let \( g \) be a normalized polynomial in \( x_1, \ldots, x_n \), without a constant term and without linear terms. Choose a presentation

\[
g = \sum_{a, b=1}^{K} g^{ab} y_a y_b, \quad g^{ab} = g^{ba} \in \mathcal{O}, \quad a, b = 1, \ldots, K.
\]

where each of the polynomial factors \( y_a \), \( a = 1, \ldots, k \), has no constant term, and let \( A \) be the filtered algebra \( A = \mathbb{C}[x_1, \ldots, x_N]/(g) \) with \( A_+ \) the sum of the positive degrees of \( A \). To each element of \( A \) is associated a unique, normalized polynomial.

Let \( Z_{2k+l} \) denote the space of closed \( p \)-chains, spanned by the following \( p \)-chains, \( p = 2k + l = 1, 2, \ldots \) and \( m_1, \ldots, m_l = 1, \ldots, N \),

\[
(Z_{2k+l,k+l})_{m_1 \ldots m_l} = g^{a_1 b_1} \ldots g^{a_k b_k} \sum_{\sigma} (-1)^{\sigma} \sigma(y_{a_1} \otimes \ldots \otimes y_{b_k} \otimes x_{m_1} \otimes \ldots \otimes x_{m_l}),
\]

where the sum is over all permutations that preserve the internal order of each pair \((a_1, b_1), \ldots, (a_k, b_k)\).

3.3.2. Theorem. The restriction of the Hochschild complex to such chains is a quasi homomorphism.

The proof of the theorem is as the proof of 3.1.2, but needs the following lemma. Let \( A_+ \) be the filtered algebra without unit as above. The degree of an \( A_+ \)-chain \( a_1 \otimes \ldots \otimes a_p \) is the sum of the degrees of its factors.

3.3.3. Lemma. If an \( A_+ \) \( p \)-chain is exact, then there is a \( p + 1 \)-chain \( b \), with the same degree as \( a \), such that \( a = db \). [FK]
4. Invariant star products on coadjoint orbits.

4.1. Background.

The origin of this problem is as follows [AM]. Let \( W \) be a symplectic space with Poisson tensor \( \Lambda \) and let there be given an action by \( \text{so}(3) \), generated by hamiltonian vector fields \( \Lambda(dL_i) \), where \( L_1, L_2, L_3 \in C^\infty W \) satisfy the following Poisson bracket relations,

\[
\{L_i, L_j\} = \epsilon_{ijk} L_k.
\]

The Casimir element

\[
Q = \sum_i (L_i)^2
\]

is an invariant; that is, \( \{L_i, Q\} = 0 \). The hamiltonian vector fields leave invariant each surface \( Q = \text{constant} \), each such surface is a symplectic leaf with an induced Poisson structure. The problem: to invent an equivariant ordering (an invariant star product) such that the star polynomial \( W(Q) := \sum L_i \ast L_i \) is invariant, \( L_i \ast W(Q) = W(Q) \ast L_i \), and fixed, \( \sum L_i \ast L_i \ast f = qf \) for some \( q \in \mathcal{O}, \forall f \).

4.2. Invariant star products.

Let \( G \) be a Lie group, \( \mathfrak{g} \) the Lie algebra of \( G \), \( \mathfrak{g}' \) the real vector space dual, and \( W \) an orbit of the coadjoint action of the connected component of \( G \) in \( \mathfrak{g}' \). This defines a homomorphism from the symmetric algebra \( S(\mathfrak{g}) \) into \( C^\infty W \). There is a natural Poisson structure on \( W \), such that \( h\{a,b\} = [a,b] \) for \( a, b \in \mathfrak{g} \). (We identify \( \mathfrak{g} \) with \( \mathfrak{g}' \).)

4.2.1. Definition. A star product on a coadjoint orbit \( W \) is \( \mathfrak{g} \)-invariant if, for all \( k \in \mathbb{C}; \ a, b \in \mathfrak{g}; \ f, g \in C^\infty W \),

\[
\begin{align*}
 k \ast a &= a \ast k = ka, \\
 a \ast b - b \ast a &= h\{a,b\}, \\
\{a, f \ast g\} &= \{a, f\} \ast g + f \ast \{a, g\}.
\end{align*}
\]

4.2.2. Remark. Coadjoint orbits provide a plethora of symplectic spaces, but to invoke the assistance of a Lie group for that purpose alone is somewhat odd. It seems more natural, in this context, to investigate star products that incorporate additional elements of group theoretical structure.
Given an associative star product on \( W \), a linear map \( \mathcal{W} \) from the symmetric algebra into \( C^\infty W \) is defined as follows,

\[
\mathcal{W} : a^n \mapsto \mathcal{W}(a^n) = a \ast a \ast \ldots \in C^\infty W, \quad a \in \mathfrak{g},
\]

Conversely, any invertible linear map \( \mathcal{W} \) that associates a \( C^\infty \) function to each formal star monomial defines an associative star product on \( C^\infty W \). For any polynomial \( P(a) \) we write \( P(a, \ast) \) for \( \mathcal{W}(P(a)) \).

The original Moyal product is the unique associative, invariant product for \( W = \mathbb{R}^{2N} \) with the standard Poisson bracket such that \( \mathcal{W}(a^n) = a^n \) for every \( a \) that is linear in the natural coordinates. It is invariant under the Lie algebra of affine symplectic transformations. The domain includes the space of regular functions (the space of polynomials in the generators).

A recipe for the construction of all invariant star products for any compact, semisimple Lie algebra, on any regular coadjoint orbit, was formulated almost 30 years ago. (For non-regular orbits of a compact group see [Ll].)

**4.2.3. Definition.** A star product on \( W \) is nondegenerate if the space of star polynomials (actually, the image by \( \mathcal{W} \)) is dense in the space of \( C^\infty \) functions on \( W \).

**4.2.4. Theorem.** An associative, nondegenerate, invariant star product on a coadjoint orbit \( W \) of a compact, semisimple Lie algebra \( \mathfrak{g} \) is given by an infinitesimal character \( Z(\mathfrak{g}) \mapsto \mathcal{C} \) and the formulas

\[
P_n(a, \ast) := C_n P_n(a), \quad C_n \in \mathcal{C} - \{0\}, \quad C_0 = C_1 = 1,
\]

\[
a \ast b - b \ast a = \hbar [a, b], \quad a, b \in \mathfrak{g}, \quad n = 0, 1,
\]

where \( P_0 = 1, \ P_1(a) = a, \ \{P_2, P_3, \ldots\} \) is a complete set of irreducible, harmonic elements of \( S(\mathfrak{g}' ) \) and \( P_n(a, \ast) \) are the corresponding star polynomials. \[\text{[BFFLS]}\]

In the case when \( \mathfrak{g} = \text{so}(3) \), \( P_n(a) \) is a solid Legendre polynomial and the polynomials \( P_n(a, \ast) \) can be obtained from the recursion relation (found and solved in [BF])

\[
(n + 1) P_{n+1}(a, \ast) = (2n + 1) a \ast P_n(a, \ast) - n(q + \frac{1 - n^2}{4}\hbar^2)[a]^2 P_{n-1}(a, \ast),
\]

16
with \( P_0(a, \ast) = 1 \). The parameter \( q \) is the value of the Casimir operator \( \sum L_i \ast L_i \).

The statement of the theorem remains valid for noncompact, semisimple Lie algebras and regular orbits. However, the restriction to polynomials would not be appropriate.

### 4.3. Finite dimensional representations.

An invariant star product gives an action of \( g \) on the star algebra, and on \( C^\infty(W) \), by the homomorphisms \( \pi_l : a \mapsto a \ast \) and similarly by \( \pi_r : a \mapsto \ast(-a) \),

\[
\pi_l(a) f = a \ast f, \quad \pi_l(a) f = -f \ast a, \quad f \in C^\infty W.
\]

In the case of compact Lie algebras we expect to find finite dimensional representations. How this actually comes about can be seen explicitly in the case \( g = so(3) \).

#### 4.3.1. Example. Proposition. Let \( g = so(3) \); an invariant, associative star product is of one of two types, both defined as in Eq.(4.2).

(1) All \( C_n \neq 0 \); the action \( \pi_l \) or \( \pi_r \) generated by \( a \ast \) or \( \ast(-a) \), \( a \in g \) is not semisimple. In this case the choice \( C_n = 1 + o(h) \) for \( n > 1 \) provides an equivariant deformation for every value \( q \) of the Casimir \( \sum L_i \ast L_i \).

(2) If the infinitesimal character takes the Casimir to the value \( q = l(l + 1) \), \( 2l \in \{0, 1, 2, \ldots\} \) then the algebra of star polynomials contains an ideal generated by \( P_{2l+1}(a, \ast) \). The quotient is a finite dimensional \( \ast \)-algebra and is spanned by the projection of \( \{P_n(a, \ast)\}_{n=0,1,\ldots,2l} \). The action of \( a \ast \) and \( \ast(-a) \) is equivalent to the direct product of two copies of the irreducible representation of \( su(2) \), each with dimension \( 2l + 1 \). The polynomial \( P_{2l+1}(a, \ast) \) reduces in this case to

\[
P_{2l+1}(a, \ast) \propto \prod_{m=-l}^{l} (a \ast -m |a|), \quad a = \sum a_i L_i, \quad |a| := \sqrt{\sum a_i^2},
\]

and every \( P_n(a, \ast) \) with \( n > 2l + 1 \) contains \( P_{2l+1}(a, \ast) \) as a factor.

The spectrum, the space of maximal ideals, has a finite number of disconnected components.
5. Introduction to singular orbits.

5.1. The coadjoint orbits of so(2,1).

Let $L_1, L_2, L_3$ be the standard basis for the real Lie algebra $\mathfrak{so}(2,1)$, with relations $[L_i, L_j] = \epsilon_{ijk} L_k$ and Killing form $g^{ij} L_i L_j = -(L_i)^2 - (L_2)^2 + (L_3)^2$. The moment map interprets $L_1, L_2, L_3$ as coordinates for the coadjoint space, and in this role we denote them by the symbols $x = (x_1, x_2, x_3)$. The regular orbits are the loci of $g(x) = g^{ij} x_i x_j = \text{const} \neq 0$ and the nilpotent orbit is the algebraic variety

$$M = \mathbb{R}^3 / g(x).$$

A regular function on $M$ is the restriction of a polynomial in $x_1, x_2, x_3$. This variety is a simple cone in the sense of Section 3.2. The first order star product $x_i \ast x_j = x_i x_j + \hbar C^1(x_i, x_j)$ is equivariant if $C^{-1}_i(x_i, x_j) = (1/2) \epsilon_{ijk} x_k$ and $C^+_i(x_i, x_j) \propto g_{ij}$. Since $g$ is nondegenerate there is a two parameter family of essential, first order equivariant deformations, indexed by $\hbar$ and $C^+_i|_{x=0}$. Every first order, equivariant deformation can be extended to an invariant star product (to all orders in $\hbar$); for example, by the method outlined above. Questions of domains have not yet been adequately discussed, to our knowledge.

5.2. Minimal orbits of $\mathfrak{sl}(n), n > 2$.

If the matrix $U \in \mathfrak{sl}(n)$ lies on the minimal orbit, then $U^2 = 0$, but this relation is not enough to define a minimal orbit. The minimal orbit is defined by the relations

$$U_a^b U_c^d - U_a^d U_c^b = 0, \quad a, b, c, d = 1, ..., n. \quad (5.1)$$

These are solved by the factorization

$$U_a^b = p_a q^b, \quad \text{with} \quad q \cdot p := q^a p_a = 0, \quad (5.2)$$

which defines an imbedding of $\mathfrak{g}$ into the space of second order polynomials on $P^{2n-2} = \mathbb{R}^{2n}/(p \cdot q, \approx)$, where $\approx$ is the equivalence relation $\forall \lambda > 0 : (\lambda p, \lambda^{-1} q) \approx (p, q)$. A star product can be defined on this space by introducing the Poisson bracket defined by
\{q^b, p_a\} = \delta^b_a, \ {q^b, q^d} = \{p_a, p_c\} = 0 \text{ and quantizing this in the manner of Weyl, see Sections 5.4 and 6.3.}

### 5.3 Minimal orbits of $\text{sp}(2n)$, $\text{so}(n)$ and the others.

$\text{Sp}(2n)$ is the algebra of traceless matrices that leaves invariant an anti symmetric, nondegenerate 2-form $\eta$: $U^b_a \eta_{bc} = L_{ac} = L_{ca}$. The imbedding $L_{ac} = \xi_a \xi_c$, with $\xi_1, ..., \xi_{2n} = q^1, ..., q^n, p_1, ..., p_n$ incorporates all the relations that define the minimal orbit, namely

\[ L_{ab} L_{cd} = L_{ad} L_{cb}. \]

The Lie algebra $\text{so}(n)$ is as $\text{sp}(2n)$ except that the form $\eta$ is symmetric. Analogy with $\text{sp}(2n)$ suggests using Grassmann variables, replacing the commutative affine algebra by a supercommutative super Lie algebra, as is done in field theories with fermions. An alternative is the imbedding $L_{ac} = q_a p_c - q^a p_a, (\text{with } q_a = q^b \eta_{ba})$. The relations that define the minimal orbit are

\[ \frac{\sum_{cyc(abc)}}{} L_{ab} L_{cd} = 0, \quad \eta^{bc} L_{ab} L_{cd} = 0. \tag{5.3} \]

The first is implied by the embedding and the second can be incorporated by restriction to $\eta^{ab} p_a p_b = \eta^{ab} q_a q_b = q^a p_a = 0$ and projecting on a quotient.

The Lie algebra $G_2$ is a subalgebra of $\text{so}(7)$. The minimal orbit can be parameterized as that of $\text{so}(7)$ with the additional condition $p \times q = 0$.

### 5.4. Associated representations and Joseph ideals.

#### 5.4.1. Background.

The ideals in the enveloping algebra of a compact, simple Lie algebra are fixed by a central character; the noncompact case is more interesting. An example is well known to physicists. Consider the Lie algebra $\text{so}(4,2)$, with the usual basis \{L_{ab}\}_{a,b=1,\ldots,6} and relations

\[ [L_{ab}, L_{cd}] = \eta_{bc} L_{ad} - a, b - c, d, \tag{5.4} \]

where $\eta$ is the pseudo Euclidean metric. The tensor $g^{bc}(L_{ab} L_{cd} + a, d)$ in the enveloping algebra reduces, in a certain irreducible and unitarizable representation, to fixed numerical values, so that relations of the type

\[ \eta^{bc}(L_{ab} L_{cd} + a, d) = -h^2 \eta_{ad} \tag{5.5} \]
hold in the representation. This particular representation is well known from its appearance in the analysis of the Schroedinger theory of the hydrogen atom and in conformal field theory. The algebra also enters the description of Keplerian orbits on a 6-dimensional phase space.

This orbit is of interest, inter alia, because Kostant’s method of geometric quantization encounters a difficulty, the non existence of an invariant quantization \([K1]\). (It was shown by Joseph that this is true of all minimal orbits except the case of \(sl(n)\) \([J]\).) Although the corresponding quantum theory is known, an invariant Wigner-Weyl correspondence is not. The existence of an invariant star product associated with such a correspondence is strongly expected to exist, but it has not been constructed. Nevertheless, the relation (5.5) suggests that there is an invariant star product such that

\[
\eta^{bc}(L_{ab} \ast L_{cd} + a, d) = -\hbar^2 \eta_{ad}.
\]

(5.6)

A question that motivated this work is whether such a deformation exists, and if it is a deformation in the direction of the Poisson bracket. The undoubted presence of interesting homology on this highly singular orbit was expected to play a role in invariant quantization. A preliminary exploration of the associated representations will show us what to expect.

### 5.4.2. Associated representations of \(sl(n)\).

Let \(V_N\) denote a space of functions on \(\mathbb{C}^n\), spanned by a set of functions \(x_1^{r_1}, \ldots, x_n^{r_n}\) with \(N = \sum_i r_i\) fixed, and let \(\{\tilde{U}^b_a\}_{a,b=1,\ldots,n}\) be the family of operators given by \(\tilde{U}^b_a = \hbar(x_a \partial/\partial x_b - (N/n)\delta_a^b)\) in \(V_N\). Then \([\tilde{U}^b_a, \tilde{U}^d_c] = \hbar(\delta_b^d \tilde{U}^a_c - \delta_a^d \tilde{U}^b_c)\), which are the relations of \(sl(n)\) with the association that identifies \(\tilde{U}^b_a / \hbar\) with the unit matrix \(E_{ab}\) when \(a \neq b\) and with \(E_{aa} - 1/N\) when \(a = b\).

To be more precise, consider the real form \(su(n-1, 1)\), with the compact subalgebra \(su(n-1)\) generated by \(\{U^b_a\}_{a,b=1,\ldots,n-1}\). Taking \(r_1, \ldots, r_{n-1}\) to run over the natural numbers one obtains, for a range of values of \(N\), a unitarizable, highest weight representation, with the highest weight reducing to zero on \(su(n-1)\). This representation is finite dimensional if \(N \in \{0, 1, \ldots\}\). The relations that define the minimal orbit are \(U^b_a U^d_c - b, d = 0\), while

\[
\tilde{U}^b_a \tilde{U}^d_c - b, d = \hbar \delta_a^c \tilde{U}^d_a - \hbar N \frac{N}{n} (\delta_a^b \tilde{U}^d_c + \delta_c^d \tilde{U}^b_a) - \hbar^2 N \frac{N}{n} (\frac{N}{n} + 1) \delta_a^b \delta_c^d - b, d.
\]

(5.7)
An invariant deformation in the direction of the Poisson bracket would have

\[ U_a^b U_c^d = U_a^b U_c^d + (\hbar/2)\{U_a^b, U_c^d\} + \hbar C^+ (U_a^b, U_c^d). \] (5.8)

In this setting, because of the very strong relations that characterize the orbit, the most general equivariant, symmetric 2-form \( C^+ \) takes the form

\[ \hbar C^+ (U_a^b, U_c^d) = (k/2)(\delta_c^b U_a^d + \delta_a^d U_c^b - \frac{1}{n}(\delta_a^d U_c^b + \delta_c^b U_a^d)) + k'(\delta_a^d \delta_c^b - \frac{1}{n} \delta_a^b \delta_c^d). \] (5.9)

This yields relations just like (5.7) if the parameters \( k, k' \) are appropriately related to the degree \( N \) of the homogeneous functions in the vector space \( V_N \), namely if

\[ k(1 + \frac{2}{n}) = -\hbar(1 + 2N/n), \quad k'(1 + \frac{1}{n}) = \hbar^2 \frac{N}{n}(\frac{N}{n} + 1). \] (5.10)

It turns out that such a deformation exists if and only if the parameters \( k \) and \( k' \) are related to each other, precisely as implied by (5.10).

5.4.3. Associated representation of \( so(n) \) and \( sp(2n) \). Let \( V_N \) be as in 5.4.2, and \( \eta \) a symmetric, nondegenerate 2-form on \( \mathbb{C}^n \). Let \( \partial_a = \eta_{ab} \partial / \partial x_b \) and \( \tilde{L}_{ab} = \hbar(x_a \partial_b - x_b \partial_a) \), which gives a formal representation of \( so(n, \mathbb{C}) \) and the formula

\[ \tilde{L}_{ab} \tilde{L}_{cd} = \hbar^2(x_a x_c \partial_b \partial_d + \eta_{be} x_a \partial_d - a, b - c, d). \]

We want to simplify this representation as much as possible and therefore restrict the variables to the cone \( \eta(x, x) = 0 \) and the space \( V_N \) to the subspace of harmonic functions. (One verifies that the choice \( N = 2 - n/2 \) of the degree of homogeneity allows the Laplace operator a well defined action on functions defined on the cone.) The first term on the right now satisfies the constraints. This is is not yet a model for a star product since the first order operators on the right do not combine to \( \tilde{L} \)'s. But another way to write the last relation is

\[ \tilde{L}_{ab} \tilde{L}_{cd} = \hbar^2(x_a x_c \partial_b \partial_d + \frac{1}{2}(\eta_{be} x_a \partial_d + \eta_{ad} x_d \partial_a) - a, b - c, d) \]

\[ + \frac{\hbar}{2} \eta_{be}(L_{ad} - a, b - c, d). \]

The first line satisfies the constraints and if we take this as a model for the classical part then we are led to look for a star product of the form

\[ L_{ab} * L_{cd} = L_{ab} L_{cd} + \frac{\hbar}{2}(\eta_{be} L_{ad} - a, b - c, d) + o(\hbar^2). \]
It turns out that such an invariant star product exist iff the last term is precisely $\hbar^2(2 - n/2)(1 - n)$ times the Killing form. See [BZ] for a complete discussion of singular representations of $\mathfrak{so}(p,q)$.

In the case $n = 6$, $N = -1$, restricting to the real form $\mathfrak{so}(4,2)$ and taking $r_1, ..., r_4 \in \mathbb{N}$, $r_5/r_6 \in \mathbb{N}$, one recovers the unitarizable representation that is realized on the space of solutions of a massless scalar field in 4 dimensions, in the form discovered by Dirac. The same representation appears in the theory of the hydrogen atom, where it is realized by self adjoint operators in $L_2(\mathbb{R}, d^3x/r)$.

Let $V_{2n}$ be the space of polynomials on a complex vector space of dimension $2n$ endowed with an antisymmetric, nondegenerate 2-form $\eta$ and coordinates $\xi_1, ..., \xi_{2n}$. A Poisson bracket is defined by $\{\xi_a, \xi_b\} = \eta_{ab}$. Let $\partial_a := \eta_{ab}\partial/\partial x_b$ and $\tilde{L}_{ab} = \hbar(\xi_a\partial_b + \xi_b\partial_a)$; then a very similar analysis leads to the idea of a star product such that

$$L_{ab} \ast L_{cd} = L_{ab} \ast L_{cd} + \frac{\hbar}{2}(\eta_{bc}\tilde{L}_{ad} + a, b + c, d) + \frac{\hbar^2}{2}\eta_{ab}\eta_{cd}.$$ 

This is of course the Moyal star product, restricted to $\mathfrak{sp}(2n)$.

6. Invariant star product on the minimal orbits.

6.1. Computation of the homology.

A minimal, coadjoint orbit of $\mathfrak{g}$ is an orbit through a highest weight vector of the coadjoint action, equal to a highest root of $\mathfrak{g}$. By a theorem of Kostant [K2] one knows that the orbit is a variety

$$M = \mathfrak{g}[x_1, ..., x_N]/(R),$$

where $N$ is the order of $\mathfrak{g}$ and $(R)$ is the ideal generated by a set $R = \{g_1, ..., g_K\}$ of quadratic relations. We choose the quadratic polynomials so that they are linearly independent and express each $g_\alpha$ as $g_\alpha = g^{ij}_\alpha x_i x_j$ with $g^{ij}_\alpha$ symmetric.

We need a generalization of the results in Section 3.2 to this case of multiple relations, up to the level of 3-chains and 3-cochains. It is clear that $\text{Hoch}_2$ is the space spanned by the following chains,

$$(Z_{2,1})_\alpha = g^{ij}_\alpha x_i \otimes x_j, \quad \alpha = 1, 2, ..., \text{ and } (Z_{2,2})_{ij} = x_i \wedge x_j, \quad i, j = 1, ..., N.$$
Every closed 3-chain is homologous to a linear one,

\[ a = x_i \otimes x_j \otimes x_k A^{ijk} \]

with \( A^{ijk} \in \mathcal{G} \), and this chain is closed iff

\[ x_i x_j A^{ijk} = 0 = x_j x_k A^{ijk} = 0. \]

Hence

\[ A^{ijk} + A^{jik} = g^{ij} c^k, \quad A^{ijk} + A^{ikj} = c'_i g^{jk}, \]

with complex coefficients \( c_\alpha, c'_\alpha \). This can be solved iff

\[ \sum_{\text{cyc}} g^{ij} c^k = \sum_{\text{cyc}} c'_i g^{jk}, \]

where the sums are over cyclic permutations, which implies that \( c'^i_\alpha = c^i_\alpha + \rho^i_\alpha \), with the coefficients \( \rho^i_\alpha \) subject to \( \sum_{\text{cyc}} \rho^i_\alpha g^{jk} = 0 \), as follows,

\[ 6A^{ijk} = \text{Alt} A^{ijk} + 3(g^{ij} c^k - g^{ik} c^j + g^{jk} c^i) + 2(\rho^i_\alpha g^{jk} - i, j), \quad \sum_{\text{cyc}} \rho^i_\alpha g^{jk} = 0. \]

The three terms lie in \( \mathbb{Z}_3 = \mathcal{H}_3 \), \( \mathbb{Z}_2 = \mathcal{H}_2 \) and \( \mathbb{Z}_1 = \mathcal{H}_1 \), respectively. The first two are of the form listed in Proposition 3.2.1. The third space, which is empty in the case that there is only one relation, has not been determined.

6.1.1. Conjecture. The space \( \mathbb{Z}_3 = \mathcal{H}_3 \) is spanned by chains of the form

\[ e_3(1)g^{ij}_\alpha g^{kl}_\beta (x_i \otimes \{x_j, x_k\} \otimes x_l), \]

where \( \{g_\alpha\} \) is the full set of binary relations and \( e_3(1) \) is the BGS idempotent.

6.1.2. Example. For \( \text{sl}(n) \) the two relations \( U^b a^d U^c - a, c = 0 \), \( U^f d^h U^g - f, h = 0 \) generate in this manner the closed chain

\[ U^b \otimes U^c \otimes U^h - a, c - f, h. \]

The difficulty will be overcome with the help of the correspondence principle, an adaptation of Weyl’s symmetric ordering.
6.2. Invariant star products and correspondence principle.

We take a fresh point of departure. Suppose that an invariant star product has the following property
\[ S(x_{i_1} \ast \ldots \ast x_{i_p}) = x_{i_1} \ldots x_{i_p} + \phi_{i_1 \ldots i_p}, \quad (6.1) \]
where \( S \) stands for symmetrization and the function \( \phi \) is a formal series in \( \bar{h} \).

6.2.1. Remark. We shall postulate that, to each order in \( \bar{h} \), \( \phi \) is a polynomial of order less than \( p \). In this way we guarantee an important property of the deformation: the Poincaré-Witt basis is preserved. Actually, in the present context this is a weak limitation, since equivariant 2-cochains of higher order are scarce by reason of the constraints.

Invariance of the star product imposes the requirement that the map \( \phi : A \rightarrow A \) defined by \( \phi : x_{i_1} \ldots x_{i_p} \mapsto \phi_{i_1 \ldots i_p} \) be equivariant for the adjoint action. The only other requirement is the obvious one that the correspondence must be consistent with the relations that define the variety. Applying these constraints to both sides of Eq.\((6.1)\) results in conditions on the map \( \phi \). We shall calculate these conditions explicitly for monomials of order 2 and 3. We shall show that these conditions are precisely the same as those implied by associativity, confirming the rather obvious fact that associativity is not a separate concern. So long as the correspondence is consistent with the constraints (now the only issue), detailed knowledge of the restricted homology spaces is not required. Cohomology was crucial for the demonstration of an extension to higher orders (Theorem 2.1.3), but it is not the best tool for establishing the basis at low orders.

Let the symbol \( S \) stand for symmetrization in the order of factors and set
\[ S(x_i \ast x_j) = x_i x_j + \psi_{ij}, \quad S(x_i \ast x_j \ast x_k) = x_i x_j x_k + \phi_{ijk}, \quad (6.2) \]
and recall that \( x_i \ast x_j - x_j \ast x_i = \bar{h}\{x_i, x_j\} = \bar{h}\epsilon_{ij}^m x_m \). The polynomials \( \phi_{ijk} \) and \( \psi_{ij} \) are assumed to define equivariant maps as explained above.

6.2.1. Proposition. Assume that the equivariant polynomials \( \psi \) and \( \phi \) have been chosen so that relations \((6.2)\) are consistent with the constraints. Then there is an invariant, associative star product such that \((6.2)\) holds.
Proof. It follows from (6.2) that

\[(x_i x_j) * x_k = x_i * x_j * x_k - \frac{\hbar}{2} \varepsilon_{ij} m x_m * x_k - \psi_{ij} * x_k.\]

The symmetrized star product is

\[S(x_i * x_j * x_k) = \frac{1}{6} \sum_{\sigma \in S_3} (x_i * x_j * x_k) \]
\[= \frac{1}{2} x_i * x_j * x_k + \frac{\hbar}{3} x_i * \{x_k, x_j\} + \frac{\hbar}{6} \{x_k, x_i\} * x_j + i, j,\]

and we deduce that

\[(x_i x_j) * x_k = x_i x_j x_k + \phi_{ijk} - \psi_{ij} * x_k - \frac{\hbar}{2} \psi(\{x_k, x_i\}, x_j + i, j)\]
\[= \frac{\hbar}{2} \{x_k, x_i\} x_j + i, j\] - \frac{\hbar^2}{12} \{x_i, \{x_k, x_j\}\} + i, j. \]

Taking \(\psi\) to be equivariant leads to some cancellations,

\[(x_i x_j) * x_k = x_i x_j x_k + \phi_{ijk} - \psi_{ij} x_k - \psi(\psi_{ij}, x_k)\]
\[- \frac{\hbar}{2} \{x_k, x_i\} x_j + i, j\] - \frac{\hbar^2}{12} \{x_i, \{x_k, x_j\}\} + i, j.\] \hspace{1cm} (6.3)

Similarly

\[x_i * (x_j x_k) = x_i x_j x_k + \phi_{ijk} - x_i x_k - \psi(x_i, \psi_{jk})\]
\[- \frac{\hbar}{2} \{x_k, x_i\} x_j + j, k\] - \frac{\hbar^2}{12} \{\{x_j, x_i\}, x_k\} + j, k.\]

These equations yield explicit expressions for the the values of the two-forms \(C_1\) and \(C_2\) defined by

\[(x_i x_j) * x_k = x_i x_j x_k + \hbar C_1(x_i x_j, x_k) + \hbar^2 C_2(x_i x_j, x_k),\]
\[x_i * (x_j x_k) = x_i x_j x_k + \hbar C_1(x_i, x_j x_k) + \hbar^2 C_2(x_i, x_j x_k),\] \hspace{1cm} (6.4)

These values solve the condition for associativity of the star product on linear chains. Theorem 2.3.1 then assures us that they can be satisfied in general. The proposition is proved.
In the present approach associativity is satisfied trivially. What is far from trivial is the existence of a function $\phi_{ijk}$ that solves (6.3). The obstructions are the constraints. Application of $g_{ij}^\alpha$ to the first and $g_{jk}^\alpha$ to the second, gives

$$g_{ij}^\alpha \phi_{ijk} = (g_{ij}^\alpha \psi_{ij}) * x_k + \hbar g_{ij}^\alpha \epsilon_{ki}^m \psi_{mj} + \frac{\hbar^2}{6} g_{ij}^\alpha \epsilon_{ki}^m \epsilon_{mj}^p x_p, \quad (6.5)$$

$$g_{ik}^\alpha \phi_{ijk} = x_i * (g_{ik}^\alpha \psi_{jk}) + \hbar g_{ik}^\alpha \epsilon_{ki}^m \psi_{mj} + \frac{\hbar^2}{6} g_{ik}^\alpha \epsilon_{ki}^m \epsilon_{jm}^p x_p. \quad (6.6)$$

Since $\phi$ is symmetric both right hand expressions must agree,

$$(g_{ij}^\alpha \psi_{ij}) * x_k - x_k * (g_{ij}^\alpha \psi_{ij}) = 2\hbar g_{ij}^\alpha \epsilon_{ik}^m \psi_{mj}.$$  

The only equivariant tensors available to use for the 2-chain $\psi$ are the Killing form and, in the case of $sl(n)$, a term linear in the generators, as above. Both satisfy this last condition, so of the two equations (6.5-6) it is enough to examine the first. The problem of consistency of (6.1) is reduced to the existence of $\phi_{ijk}$ that solves Eq.s (6.5).

A complete determination of the restricted cohomology is not required; it is enough to know the relations that define the orbit.

6.3. Calculations for $sl(n)$.

6.3.1. Solving the constraints. Applied to $sl(n)$, $x_i \rightarrow U_a^b$, with the notation and the commutation relations as in Sections 5.2-4, the above result (6.3) take the form

$$\frac{1}{2}(U_a^b * U_c^d + U_c^d * U_a^b) = U_a^b U_c^d + \psi_{bd}$$

and

$$(U_a^b U_c^d) * U_e^f = U_a^b U_c^d U_e^f + \phi_{ace}^{bdf} - \psi_{ac}^{bdf} U_e^f - \psi(\psi_{ac}^{bdf}, U_e^f)$$

$$- \frac{\hbar}{2} \left(\{U_e^f, U_a^b\} U_c^d + \{U_e^f, U_c^d\} U_a^b\right)$$

$$- \frac{\hbar^2}{12} \left(\{U_a^b, \{U_e^f, U_c^d\}\} + \{U_c^d, \{U_e^f, U_a^b\}\}\right). \quad (6.7)$$

A better notation is to express $u \in \mathfrak{g}'$ as $u = (AU) = A_b^a U_a^b$. The coefficients $A_b^a$ are coordinates for $\mathfrak{g}$; that is, $A$ ranges over the matrices of the adjoint representation of $\mathfrak{g}$. The first relation becomes

$$u * u = u^2 + \psi(u, u).$$
Equivariance restricts $\psi$: 

$$
\psi(u, u) = k(AAU) + k'(AA), \quad k, k' \in \mathcal{C},
$$

(6.8)

where $(AAU)$ and $(AA)$ indicate traces of product of matrices. Similarly,

$$
\psi(u, u) = u^3 + \phi(u, u, u),
$$

(6.8.8)

where $(AAU)$ and $(AA)$ indicate traces of product of matrices. Similarly,

$$
u * u = u^3 + \phi(u, u, u),
$$

with

$$
\phi(u, u, u) = \phi_1(AAU)u + \phi_2(AAAU) + \phi_3(AA)u + \phi_4(AAA).
$$

(6.9)

(There are other invariants but their inclusion here is not allowed by the relations.) Let $\psi$ be given in the form (6.8) and look at Eq.(6.7) as an equation to determine $\phi$. Recall that the minimal orbit is defined by the relations $U_{ab}^c U_{cd}^e = U_{ad}^c U_{bd}^e$, $a, ..., d = 1, ..., n$.

6.3.2. Proposition. **Eq.(6.7) is consistent with the relations $U_{ab}^c U_{cd}^e = U_{ad}^c U_{bd}^e$ if and only if the parameters $k$ and $k'$ are related to each other as follows,**

$$
4k'(1 + \frac{1}{n}) = k^2(1 + \frac{2}{n})^2 - h^2.
$$

(6.10)

The proof is in the Appendix.

6.3.3. Generators of the ideal. Proposition. **The Joseph ideal for $sl(n)$ is generated by the relations**

$$
U_{ab}^c U_{cd}^e - U_{ad}^c U_{bd}^e = (h/2)(\delta_{a}^{d}U_{b}^{c} - \delta_{a}^{c}U_{b}^{d} - \delta_{d}^{c}U_{a}^{b} + \delta_{d}^{b}U_{a}^{c})
$$

$$+
(k/2)(1 + \frac{2}{n})(\delta_{a}^{d}U_{b}^{c} + \delta_{d}^{b}U_{a}^{c} - \delta_{b}^{d}U_{a}^{c} - \delta_{d}^{c}U_{a}^{b})
$$

$$+
k'(1 + \frac{1}{n})(\delta_{a}^{d}\delta_{b}^{c} - \delta_{a}^{c}\delta_{b}^{d}).
$$

(6.11)

**Proof.** We find the relations of the deformed algebra by eliminating the original product from

$$
U_{ab}^c U_{cd}^e = U_{ad}^c U_{bd}^e + (h/2)(\delta_{a}^{d}U_{b}^{c} - \delta_{a}^{c}U_{b}^{d})
$$

$$+
(k/2)(\delta_{a}^{d}U_{b}^{c} + \delta_{d}^{b}U_{a}^{c} - \frac{2}{n}(\delta_{c}^{d}U_{a}^{b} + \delta_{a}^{b}U_{c}^{d})
$$

$$+
k'(\delta_{a}^{d}\delta_{b}^{c} - \frac{1}{n}\delta_{a}^{c}\delta_{b}^{d}).
$$

(6.12)
We have seen that this relation is equivalent (given the Poisson bracket) to the simple relation \( u \ast u = u^2 + k(AA_U) + k'(AA) \).

A highest weight module over the deformed algebra is a module generated by a vector \( v \) such that

\[
U^b_a \ast v = 0, \quad a < b = 2, \ldots, n, \quad U^a_a \ast v = \lambda_a v, \quad a = 1, \ldots, n, \quad (6.13)
\]

with \( \lambda \in \mathfrak{g}^n, \sum_1^n \lambda_a = 0 \). (The Cartan subalgebra consists of diagonal matrices.)

**6.3.4. Highest weight module. Proposition.** A highest weight module of the deformed algebra exists if and only if the relation \((6.10)\) holds. In that case the highest weight is one of the following, \(1 \leq m \leq n\),

\[
\lambda_1 = \lambda_2 = \ldots = \lambda_{m-1} = -\frac{h}{2} - \gamma, \quad \lambda_{m+1} = \ldots = \lambda_n = \frac{h}{2} - \gamma. \quad (6.14)
\]

with \( \lambda_m \) determined by the fact that \( \sum \lambda_i = 0 \) and \( \gamma = (k/2)(1 + \frac{2}{n}) \).

**Proof.** The only relations that are changed by the deformation are those where \( a, c \) intersects \( bd \), so we may limit ourselves to the case that \( b = a \). Also, if \( a \) is equal to \( c \) or to \( d \) the relations are just the usual commutation relations; so take \( c, d \neq a \). Then

\[
U^a_c \ast U^d_c - U^d_c \ast U^a_c = \frac{h}{2}(-\delta^d_c U^a_c + U^d_c) - \frac{k}{2}(1 + \frac{2}{n})(U^d_c + \delta^d_c U^a_c) - k'(1 + \frac{1}{n})\delta^d_c.
\]

In particular, for \( a \neq c \),

\[
U^a_c \ast U^a_c - U^c_a \ast U^a_a = \frac{h}{2}(U^a_c - U^a_a) - (k/2)(1 + \frac{2}{n})(U^a_c + U^a_a) - k'(1 + \frac{1}{n}).
\]

Applying this to the highest weight vector we obtain, for \( a > c \),

\[
\lambda_a \lambda_c = (h/2)(\lambda_c - \lambda_a) - \gamma(\lambda_a + \lambda_c) - \gamma', \quad \gamma := (k/2)(1 + \frac{2}{n}), \quad \gamma' := k'(1 + \frac{1}{n}).
\]

It follows that,

\[
\lambda_a = \frac{(\frac{h}{2} - \gamma)\lambda_c - \gamma'}{\lambda_c + \frac{h}{2} + \gamma}, \quad a > c.
\]

Suppose \( n > 2 \). Assume that \( \gamma' \neq \gamma^2 - (\frac{h}{2})^2 \), then taking \( c = 1 \) we get \( \lambda_2 = \ldots = \lambda_n \) and taking \( a = n \) we find that \( \lambda_1 = \ldots = \lambda_{n-1} \). Since \( \sum \lambda_i = 0 \) this is not interesting. We conclude that

\[
\gamma' = \gamma^2 - (\frac{h}{2})^2, \quad (6.15)
\]

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which is the same as (6.10). Then there is an integer \( m \), \( 1 \leq m \leq n \) such that \( \lambda_m \) differs from its neighbours, and the statement of the proposition follows immediately.

**6.3.5. Abelian deformation.** When \( \bar{h} = 0 \) the deformed algebra is commutative, so a representation is just a character, or a maximal ideal. The group acts on the maximal ideals and among the maximal ideals there are upper triangular ones. We have seen, Eq.(6.10) that this is possible only if either \( n = 2 \) or if (6.10) holds

\[
4k'(1 + \frac{1}{n}) = k^2(1 + \frac{2}{n})^2.
\]

In this case the deformed variety is the space of traceless matrices with all but one of the eigenvalues equal to \(-\frac{k}{2}(1 + \frac{2}{n})\).

**6.4. Calculations for so(n).**

**6.4.1. Solving the constraints.** Applied to \( \text{sl}(n) \), \( x_i \to L_{ab} \), with the notation and the commutation relations as in Section 5.4.3, we have

\[
\frac{1}{2}(L_{ab} * L_{cd} + L_{cd} * L_{ab}) = L_{ab}L_{cd} + \psi_{ab,cd}
\]

and the above result (6.3) takes the form

\[
(L_{ab}L_{cd})*L_{ef} = (L_{ab}L_{cd})L_{ef} + \phi_{ab,cd,ef} - \psi_{ab,cd}L_{ef}
\]

\[
- \frac{\bar{h}}{2} \left( \{L_{ef}, L_{ab}\}L_{cd} + \{L_{ef}, L_{cd}\}L_{ab} \right)
\]

\[
- \frac{\bar{h}^2}{12} \left( \{L_{ab}, \{L_{ef}, L_{cd}\}\} + \{L_{cd}, \{L_{ef}, L_{ab}\}\} \right)
\]

(6.16)

A much better notation is to express \( u \in \mathfrak{g}' \) as \( u = (AL) = A^{ab}L_{ab} \), the coefficients \( A^{ab} = -A^{ba} \) coordinates for \( \mathfrak{g} \); then these equations become

\[
u * u = u^2 + \psi(u,u).
\]

(6.17)

and

\[
u^2 * v = u^2v + \phi(A, A, B) - \psi(A, A)v - \bar{h}[v, u]u - \frac{\bar{h}^2}{6}\{u, \{v, u\}\}.
\]

Equivariance restricts \( \psi \) and \( \phi \):

\[
\psi_{A,A} = k(AA), \quad \phi(AAA) = \phi_1(AAAL) + \phi_2(AA)u,
\]
where \((AAU)\) and \((AA)\) indicate traces of product of matrices. (In this context we take the form \(\eta\) that defines \(so(n)\) to be \(\eta_{ab} = 0, a \neq b, \eta_{aa} = 1, a = 1, ..., n, \) so that two-forms become matrices without any fuss.) We fix the parameter \(k\) and look at Eq.(6.16) as an equation to determine the coefficients \(\phi_1, \phi_2\). Then we have:

6.4.2. Proposition. Eq.(6.16) is consistent with the relations that define the minimal orbit if and only if the parameter \(k\) takes the value \(\bar{h}^2(n - 4)/(n - 1)\). In this case the parameters \(\phi_1\) and \(\phi_2\) are fixed by (6.16).

The proof is in the Appendix.

6.4.3. Generators of the ideal. Proposition. The Joseph ideal for \(so(n)\) is generated by (the commutation relations and) the relations

\[
S(\eta^{bc}L_{ab} \ast L_{cd}) + \frac{\bar{h}^2}{2}(n - 4)\eta_{ad} = 0, \quad \sum_{\text{cyc}(abc)} (L_{ab} \ast L_{cd} - \bar{h}\eta_{ad}L_{bc}) = 0. \tag{6.18}
\]

These relations were derived by Binegar and Zierau [BZ], who also determined the highest weight module and the associated unitary representations of \(SO(p, q)\). We are interested, nevertheless, in deriving these results with the help of the star product. Eq.(6.18) is a direct consequence of (6.17), that read in full

\[
L_{ab} \ast L_{cd} = L_{ab}L_{cd} + \frac{\bar{h}}{2}[L_{ab}, L_{cd}] + \frac{k}{2}(\eta_{ac}\delta_{bd} - a, b).
\]

6.5. Highest weight module for \(so(2\ell + 1)\).

Let \(\{E_{ab}\}, a, b = 1, ...,\) denote the unit \(n\)-by-\(n\) matrices as earlier, and take

\[
\eta = \sum_{a=1}^{n} \delta_{a,a'}, \quad a' := n + 1 - a.
\]

A basis for \(so(\eta)\) in the natural representation is the set \(\{L_{ab}\}, a < b \in \{i, ..., n\}\) of matrices

\[
L_{ab} = (E_{ab} - E_{ba})\eta = E_{ab'} - E_{ba'}.
\]

The Cartan subalgebra of choice has the basis

\[
H_a = L_{aa'} - L_{a'a}, \quad a = 1, ..., \ell.
\]
The set of positive root vectors is the collection \( \{ L_{ab} \} \), \( a+b \leq n \), \( a < b \). Our calculations are insensitive to the parity of \( n \); nevertheless we note the following facts. When \( n = 2\ell + 1 \) the simple roots are \( L_{1,n-1}, L_{2,n-2}, \ldots, L_{\ell-1,\ell}, L_{\ell-1,\ell+1} \) and the associated roots are \( \alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \), \( i = 1, \ldots, \ell - 1 \) with 1 in the \( i \)'th place and \( \alpha_\ell = (0, \ldots, 0, 1) \). When \( n = 2\ell \) the simple roots are \( L_{1,n-1}, L_{2,n-2}, \ldots, L_{\ell-1,\ell}, L_{\ell-1,\ell+1} \) and the associated roots are \( \alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \), \( i = 1, \ldots, \ell - 1 \) with 1 in the \( i \)'th place and \( \alpha_\ell = (0, \ldots, 0, 1, 1) \). All this as in Bourbaki.

Having fixed the root system and the Cartan subalgebra we define a highest weight module as one generated by a character \( H_a \mapsto \lambda_a \in \mathcal{G} \) and a (highest weight) vector \( v_0 \) with the property

\[
L_{ab}v_0 = 0, \quad a + b \leq n, \\
H_a v_0 = \hbar \lambda_a v_0, \quad a = 1, \ldots, \ell.
\]

**Proposition 6.5.1.** A highest weight module of the deformed algebra exists if and only if the parameter \( k \) takes the value \( \hbar^2 (n-4)/(n-1) \). In that case the highest weight is one of the following, for \( k \in \{1, \ldots, \ell\} \),

\[
\lambda_1 = \ldots = \lambda_{k-1} = -1, \quad \lambda_k = k + 1 - \frac{n}{2}, \quad \lambda_{k+1} = \ldots = \lambda_\ell = 0.
\]

The proof is in the Appendix.

Joseph’s choice is \( k = \ell - 1 \). When \( n = 2\ell + 1 \), \( \mathrm{so}(n) = \mathrm{so}(2\ell + 1) = B_\ell \),

\[
\lambda + \rho = \omega_1 + \ldots + \omega_{\ell-3} + \frac{1}{2}(\omega_{\ell-2} + \omega_{\ell-1}) + \omega_{\ell} \\
= (\ell - \frac{3}{2}, \ell - 1, \frac{5}{2}, 1, 1, \frac{1}{2}), \quad \lambda = (-1, \ldots, -1, -\frac{1}{2}, 0).
\]

When \( n = 5 \) this corresponds to the Bose singleton. Another choice is \( \lambda = (-1, \ldots, -1, \frac{1}{2}) \); in the case \( n = 5 \) it is the Fermi singleton. Both are unitarizable (after taking a quotient) representations of \( \mathrm{so}(3,2) \). Binegar and Zierau take \( k = 1 \).

When \( n = 2\ell \), \( \mathrm{so}(n) = \mathrm{so}(2\ell) = D_\ell \), Joseph again takes \( k = \ell - 1 \) and

\[
\lambda + \rho = \omega_1 + \ldots + \omega_{\ell-3} + \omega_{\ell-1} + \omega_{\ell} \\
= (\ell - 3, \ell - 2, 1, 1, 0), \quad \lambda = (-1, -1, \ldots -1, 0, 0).
\]
When \( n = 6 \) this is the highest weight of the representation of the conformal group by a scalar massless field in 4 dimensions, the same representation that appears in Schrödinger’s hydrogen atom. Binegar and Zierau take \( k = 1 \).

### 6.6. Calculations for the exceptional simple Lie algebras.

#### 6.6.1. The relations.

Let \( K \) denote the Killing form,

\[
K(x_i, x_j) = K_{ij} = -\text{tr}(x_i x_j) = -\epsilon_{m}^{i} \epsilon_{n}^{j} m
\]

and \( K^{ij} \) the matrix elements of the inverse matrix. The reduction of the symmetric part of the adjoint representation is governed by the operator \( \text{Lin } \mathfrak{g} \otimes \mathfrak{g} \) defined by

\[
L : x_i \otimes x_j \mapsto K^{mn} \epsilon_{m}^{i} s \epsilon_{n}^{j} t x_s \otimes x_t = K^{mn}[x_i, x_m] \otimes [x_j, x_n]. \tag{6.19}
\]

The symmetric product of the adjoint representation by itself decomposes into a sum of three terms, \( \text{Id} \oplus V_2 \oplus V_3 \). The last contains the extremal weights and is the minimal orbit in \( \mathfrak{g} \). The relations that define the minimal orbit are the projections on the first two. The one-dimensional component expresses the condition that

\[
K^{ij} x_i x_j = 0,
\]

jointly they fix the eigenspace of \( L \) in \( \mathfrak{g} \otimes \mathfrak{g} \). The eigenvalues \( l_i = l_1, l_2, l_3 \) associated with the eigenspaces \( \text{Id}, V_2, V_3 \) are as follows. In all cases, \( l_1 = 1 \), and \( l_2 + l_3 = -1/6 \).

The values of \( l_3 \) are taken from [MP],

| \( \mathfrak{g} \) | \( G_2 \) | \( F_4 \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
|-------|-------|-------|-------|-------|-------|
| \( l_3 \) | 1/4   | 1/9   | 1/12  | 1/18  | 1/30  |
| \( D \)  | 27    | 52    | 78    | 133   | 248   |

The number \( D \) is the dimension of \( \mathfrak{g} \).

We turn to Eq.(6.3). Actually, the exceptional simple Lie algebras are the easiest to deal with, and all five can be done uniformly. The immediate reason for this is the non-existence of an irreducible, invariant fourth order polynomial. The only equivariant, symmetric tensors are (see 6.2.1)

\[
\psi_{ij} = k K_{ij}, \quad \phi_{ijk} = \frac{k'}{3} \sum_{cyc} K_{ij} x_k, \tag{6.20}
\]

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with $k$ and $k'$ to be determined by the relations. So that Eq. (6.3) simplifies:

\[
(x_i x_j) * x_k - x_i x_j x_k + \frac{\hbar}{2} \left( \{x_k, x_i\} x_j + i, j \right) = \phi_{ijk} - \psi_{ij} x_k - \frac{\hbar^2}{12} \left( \{x_i, \{x_k, x_j\}\} + i, j \right).
\]

(6.21)

6.6.2. Theorem. Let $g$ be one of the five exceptional simple Lie algebras, and $K$ the Killing form. There exists a unique, invariant star product on the minimal nilpotent orbit, such that

\[
S(x_i * x_j) = x_i x_j + k K_{ij}, \quad k \in \mathfrak{c}',
\]

for one and only one choice of $k, k' \in \mathfrak{c}'$, namely $k = \frac{l_2 - 1}{D} \frac{\hbar^2}{6} + l_2 \frac{\hbar^2}{12}$, $k' = \frac{\hbar^2}{4}$.

Proof of the Theorem. It has already been pointed out that the only possible forms of the deformed products $x_i * x_j = x_i x_j + o(\hbar)$ and $S(x_i * x_j * x_k) = x_i x_j x_k + o(\hbar)$ are

\[
S(x_i * x_j) = x_i x_j + \psi_{ij} x_k, \quad S(x_i * x_j * x_k) = x_i x_j + \phi_{ijk},
\]

with the equivariant maps $\psi, \phi$ as in (6.20). The left side of (6.21) satisfies the constraints; the question is whether the right side does. Fix the index $k$ and define vectors $X, Y$ in $g \otimes g$ with components

\[
X_{ij} = K_{ik} x_j + i, j, \quad Y_{ij} = K_{ij} x_k.
\]

Let $L_s$ denote the projection of the operator $L$ on the symmetric part of $g \otimes g$; then the right hand side of (6.21) is the vector

\[
\frac{k'}{3} (X + Y) - k Y - \frac{\hbar^2}{12} L_s X.
\]

(6.22)

The operator $L_s$ satisfies the characteristic equation $(L_s - 1)(L_s - l_2)(L_s - l_3) = 1$, and since $L_s Y = Y$ there is a constant $c$ such that

\[
(L_s - l_2)(L_s - l_3) X = c Y.
\]

To determine the constant we contract this equation with $K$ and find that

\[
c = \frac{2}{D} (1 - l_2)(1 - l_3).
\]
With this, the vector (6.22) reduces to

\[ Z := c^{-1}(\frac{k'}{3} - k)(L_s - l_2)(L_s - l_3)X + \frac{k'}{3}X - \frac{\hbar^2}{12}L_sX \]

Projecting on the trivial representation we get

\[ 0 = \frac{k'}{3}(D + 2) - kD - \frac{\hbar^2}{6} = 0, \quad D = \dim g. \tag{6.23} \]

All the constraints are expressed by \((L_s - l_3)Z = 0\), which yields 3 conditions

\[
(1 - l_3)c^{-1}(\frac{k'}{3} - k) - \frac{\hbar^2}{12} = 0,
\]

\[
(1 - l_3)c^{-1}(\frac{k'}{3} - k)(l_2 + l_3) + \frac{k'}{3} = l_3 \frac{\hbar^2}{12}, \tag{6.24}
\]

\[
(1 - l_3)c^{-1}(\frac{k'}{3} - k)l_2l_3 = l_3 \frac{k'}{3}.
\]

All four equations (6.23-4) agree on unique values of \(k\) and \(k'\),

\[ k' = l_2 \frac{\hbar^2}{4}, \quad k - \frac{k'}{3} = \frac{l_2 - 1 \hbar^2}{D}. \]

The proposition is proved.

6.6.3. The Josef ideal. It is of course generated by the relations

\[ x_i * x_j - x_j * x_i = \hbar \{x_i, x_j\}, \]

and

\[ (L_s - l_3)(x_i * x_j - kK_{ij}) = 0. \]

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Appendix

Proof of Proposition 6.3.2. The lack of symmetry in (6.7) forces the introduction of a second variable, \( v = (BU) \); then the equation can be written as

\[
u^2 * v = u^2 v + \phi(v, u, v) - \psi(u, u, v) - \hbar \{ v, u \} u - \frac{\hbar^2}{6} \{ u, \{ v, u \} \},\]

with \( \phi \) as in (6.9),

\[
\phi(v, u, v) = + \frac{\phi_1}{3} \left( (AAU) v + 2(ABU) u \right) + \frac{\phi_2}{3} \left( (AABU) + (ABAU) + (BAAU) \right) + \frac{\phi_3}{3} \left( (AA) v + 2(AB) u \right) + \phi_4(AAB).
\]

Exchange of the upper indices on the two \( A \)’s is a type of Fierz transformation. The effect is as follows.

\[
(ABAU) \mapsto (AB) u - \frac{1}{n} (AABU) - \frac{1}{n} (AAUB) + \frac{1}{n^2} (AA) v
\]
\[
(AABU) \mapsto - \frac{2}{n} (AABU) + \frac{1}{n^2} (AA) v
\]
\[
(ABU) u \mapsto (ABU) u - \frac{1}{n} (AAU) v
\]
\[
(BAU) u \mapsto (BAU) u - \frac{1}{n} (AAU) v,
\]
\[
(AAU) \mapsto - \frac{2}{n} (AAU), \quad (AAB) \mapsto - \frac{2}{n} (AAB),
\]
\[
(AB) u \mapsto (AUAB), \quad (AA) \mapsto - \frac{1}{n} (AA).
\]

and

\[
\phi(v, u, v) \mapsto + \frac{\phi_1}{3} \left( - \frac{2}{n} (AAU) v + 2(ABU) u - \frac{2}{n} (AAU) v \right) + \frac{\phi_2}{3} \left( - \frac{3}{n} (AABU + AAUB) + \frac{3}{n^2} (AA) v + (AB) u \right) + \frac{\phi_3}{3} \left( - \frac{1}{n} (AA) v + 2(ABAU) \right) + \phi_4 \frac{2}{n} (AAB).
\]
Furthermore,

\[
-\psi(u, u)v = -k(AAU)v - k'(AA)v \mapsto \frac{2k}{n}(AAU)v + \frac{k'}{n}(AA)v \\
-\psi(\psi(u, u), v) = -\frac{k^2}{2}(AABU + AAUB) - \frac{2k}{n}(AA)v - kk'(AAB) \\
\mapsto \frac{k^2}{n}(AABU + AAUB) - \frac{2k}{n}(AA)v \\
- \frac{k^2}{n}(AA)v + \frac{k'}{n}(AAB) \\
- \frac{h^2}{6}(u, \{v, u\}) = \frac{h^2}{3}(ABAU) + \frac{h^2}{6}(AABU + AAUB) \\
\mapsto \frac{h^2}{3}(- (AB)u + \frac{1}{n}(AABU + AAUB) - \frac{1}{n^2}(AA)v) \\
- \frac{h^2}{6}(- \frac{2}{n}(AABU + AAUB) + \frac{2}{n^2}(AA)v.)
\]

Applying the constraint \(U^b_a U^d_c = U^d_a U^b_c\) to (6.7) we get

\[
\frac{1}{3} \phi_1(1 + \frac{2}{n}) - k(1 + \frac{2}{n}) = 0, \\
2\phi_2(1 + \frac{3}{n}) - 3k'^2(1 + \frac{2}{n}) + \frac{h^2}{n} = 0, \\
\phi_2 - 2\phi_3 - \frac{h^2}{n} = 0, \\
\phi_3(1 + \frac{1}{n}) - 3k'(1 + \frac{1}{n}) - \frac{3}{n^2} \phi_2 + \frac{3}{n}k^2(1 + \frac{2}{n}) = 0, \\
\phi_4(1 + \frac{2}{n}) - kk'(1 + \frac{2}{n}) = 0.
\]

Eliminating the parameters \(\phi_i\), we find the relation

\[
4k'(1 + \frac{1}{n}) = k^2(1 + \frac{2}{n})^2 - \frac{h^2}{n}.
\]

The proposition is proved.

**Proof of Proposition 6.4.2.** Recall that the relations are

\[
\sum_{cyc(abc)} L_{ab} L_{cd} = 0, \quad \eta^{bc}_{bc} L_{ab} L_{cd} = 0.
\]

Applying the first relation, contraction on \(b, c\), to Eq. (6.16) one gets

\[
0 = \eta^{bc}_{bc} \phi_{ab, cd, ef} - \frac{k}{2}(n - 1)\eta_{ad} L_{ef} - \frac{h^2}{12}((n - 4)(\eta_{fd} L_{ae} + a, d - e, f) - 4\eta_{ad} L_{ef}),
\]

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and
\[ \eta^{bc} \phi_{ab,cd,ef} = \left( \frac{\phi_1}{24}(n-1) - \frac{\phi_2}{6} \right) (\eta_{fd} L_{ae} + a, d - e, f) + \left( - \frac{\phi_1}{6} + \frac{\phi_2}{6}(n-1) \right) \eta_{ad} L_{ef}. \]

The second relation (a kind of Fierz transformation) gives
\[ 0 = \sum_{\text{cyc}} \phi_{ab,cd,ef} + \frac{\hbar^2}{3} \eta_{fa}(\eta_{de} L_{bc} + \eta_{ec} L_{db}) - e, f \]
and
\[ \sum_{\text{cyc}} \phi_{ab,cd,ef} = \left( \frac{\phi_1}{12} + \frac{\phi_2}{6} \right) \sum_{\text{cyc}} \eta_{fa}(\eta_{de} L_{cb} + \eta_{ec} L_{bd}) - e, f. \]
Thus
\[ \frac{\phi_1}{24}(n-1) - \frac{\phi_2}{6} - \frac{\hbar^2}{12}(n-4) = 0, \]
\[ - \frac{\phi_1}{6} + \frac{\phi_2}{6}(n-1) - \frac{k}{2}(n-1) + \frac{\hbar^2}{3} = 0, \]
\[ \frac{\phi_1}{12} + \frac{\phi_2}{6} + \frac{\hbar^2}{3} = 0. \]
Finally,
\[ \phi_1(n+1) = 2\hbar^2(n-8), \quad \phi_2(1+n) = -3\hbar^2(n-2), \]
\[ k(n-1) = \hbar^2(n-4). \]
The proposition is proved.

**Proof of Proposition 6.5.1.** All Cartan subalgebras are isomorphic and any two systems of simple roots are related by a transformation of the Weyl group. The ideal determines only the infinitesimal character \( \chi(\lambda) := \rho + \lambda \), where \( \rho \) is half the sum of the positive roots and \( \lambda \) is the highest weight, up to a Weyl reflection. It does not distinguish between weights that are related by a Weyl reflection of the infinitesimal character. In the case of \( B_l = \mathfrak{so}(2\ell+1) \) the formula is \( \rho = (\ell - \frac{1}{2}, \ell - \frac{3}{2}, ..., \frac{1}{2}) \).

The problem is to determine the possible values of the infinitesimal character.

We begin with the relation
\[ \sum_b L_{db} \ast L_{b' a} - \frac{\hbar}{2}(n-2)L_{da} + \frac{k}{2}(n-1)\eta_{da} = 0, \]
in the case $a + d = n + 1, a < d$. Applied to $v_0$ it gives $(L_{da} \to -\lambda_a)$

$$\left( \sum_{b \leq a} L_{db} \ast L_{b'} a + \frac{\hbar^2}{2} (n - 2) \lambda_a + \frac{k}{2} (n - 1) \right) v_0 = 0.$$ 

Why? Well if $b > a$ then $b' < a'$ and $a + b' < n + 1$ and $L_{b'} a v_0 = 0$. Thus

$$\left( \sum_{b < a} \{ L_{db}, L_{b'} a \} + \lambda_a^2 + \frac{\hbar^2}{2} (n - 2) \lambda_a + \frac{k}{2} (n - 1) \right) v_0 = 0,$$

The bracket is $\hbar^2 (\lambda_b - \lambda_a)$, $b = 1, ..., a - 1$, so finally

$$(1 - a) \lambda_a + (\lambda_1 + ... + \lambda_{a-1}) + \lambda_a^2 + \frac{1}{2} (n - 2) \lambda_a + \frac{1}{2} (n - 4) = 0,$$

In particular

$$\lambda_1^2 + \frac{1}{2} \lambda_1 (n - 2) + \frac{1}{2} (n - 4) = 0,$$

or

$$(\lambda_1 + 1) \left( \lambda_1 + \frac{n - 4}{2} \right) = 0.$$ 

Thus

$$(1 - a) \lambda_a + (\lambda_1 + ... + \lambda_{a-1}) + \lambda_a^2 - \lambda_1^2 + \frac{1}{2} (n - 2) (\lambda_a - \lambda_1) = 0,$$

and

$$(\lambda_a - \lambda_{a-1}) \left( \lambda_a + \lambda_{a-1} + \frac{1}{2} (n - a) \right) = 0.$$ 

We return to

$$\sum_b L_{db} \ast L_{b'} a - \frac{\hbar}{2} (n - 2) L_{da} + \frac{k}{2} (n - 1) \eta_{da} = 0,$$

now in the case that $a + d = n + 2, 2 \leq a < d$. Applying to the highest weight

$$\left( \sum_{b \leq a} L_{db} \ast L_{b'} a - \frac{\hbar}{2} (n - 2) L_{da} \right) v_0 = 0,$$

or

$$\left( \sum_{b \leq a-2} L_{db} \ast L_{b'} a + L_{da} \ast L_{a' a} + L_{d'd'} L_{da} - \frac{\hbar}{2} (n - 2) L_{da} \right) v_0 = 0.$$ 

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The first term can be replaced by the bracket \( \sum_{b \leq a-2} [L_{ab}, L_{b'a}] = \hbar \sum_{b \leq a-2} L_{da} \), so for \( a = 2, \ldots, \ell \),

\[
(a - 1)L_{da} - \lambda_a L_{da} - \lambda_a - 1 L_{da} - \frac{1}{2}(n - 2)L_{da})v_0 = 0.
\]

Hence

\[
(\lambda_a + \lambda_a - 1 + \frac{1}{2}(n - 2a))L_{da}v_0 = 0, \quad a = 2, \ldots, \ell.
\]

Note that \( \{L_{da}, L_{a'd'}\}v_0 = \hbar^2(\lambda_a - \lambda_{a-1})v_0 \). The information contained in this last result is therefore precisely the same as in (6.*), for when \( \lambda_2 = \ldots = \lambda_n = 0 \), then \( L_{ab}v_0 = 0, \ a, b > 1 \).

Next, the other relation,

\[
\sum_{cycbcd} (L_{ab} - \hbar \eta_{ab}) \ast L_{cd} = 0,
\]

applied to \( v_0 \) in the case \( a + b = c + d = n + 1, \ a < b, c < d, a \neq c \),

\[
((\lambda_a - 1)\lambda_c + L_{ac} \ast L_{db} + L_{ad} \ast L_{bc})v_0 = 0.
\]

Evaluating this in two cases, \( a > b, a < c \) we find that \( \lambda_\prec \) is -1 or \( \lambda_\succ \) is 0. That completes the proof.

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