MEAN CURVATURE FLOW OF AREA DECREASING MAPS IN CODIMENSION TWO

RENAN ASSIMOS, ANDREAS SAVAS-HALILAJ, AND KNUT SMOCZYK

Abstract. We consider the graphical mean curvature flow of strictly area decreasing maps $f : M \to N$ between a compact Riemannian manifold $M$ of dimension $m > 1$ and a complete Riemannian surface $N$ of bounded geometry. We prove long-time existence of the flow and that the strictly area decreasing property is preserved, when the Ricci curvature $\text{Ric}_M$ on $M$ satisfies the main condition

$$\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,$$

where $\sigma_M, \sigma_N$ denote the sectional curvatures on $M$ and $N$, respectively. When $M$ has dimension 2 or 3, this condition is equivalent to $\text{Scal}_M \geq \text{Scal}_N$, where $\text{Scal}_M, \text{Scal}_N$ denote the scalar curvatures on $M$ and $N$, respectively. In addition, if $\text{Ric}_M \geq \sup_N \sigma_N$, then the mean curvature stays uniformly bounded and uniform $C^1$-bounds for the maps imply uniform $C^k$-bounds, for any $k \geq 1$. Under the assumption that the main curvature condition holds, we obtain such $C^1$-bounds if $\text{Ric}_M \geq 0$ and we obtain smooth convergence to a minimal map if $\text{Ric}_M \geq \sup\{0, \sup_N \sigma_N\}$. These results significantly improve known results on the graphical mean curvature flow in codimension 2.

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1. Introduction and summary

Suppose that \( f : M \to N \) is a smooth map between the Riemannian manifolds \( M \) and \( N \) and let

\[
\Gamma_f := \{(x, f(x)) \in M \times N : x \in M \}
\]
denote the graph of \( f \). We deform \( \Gamma_f \) by the mean curvature flow. Some general questions are whether the flow stays graphical, it exists for all times, and it converges to a minimal graphical submanifold \( \Gamma_\infty \) generated by a smooth map \( f_\infty : M \to N \). In this case, \( f_\infty \) is called a minimal map and can be regarded as a canonical representative of the homotopy class of \( f \).

The first result concerning the evolution of graphs by its mean curvature was obtained by Ecker and Huisken [10]. They proved long-time existence of the mean curvature flow of entire graphical hypersurfaces in the euclidean space. Moreover, they proved convergence to flat subspaces under the assumption that the graph is straight at infinity. For general maps between arbitrary Riemannian manifolds the situation is much more complicated. However, under suitable conditions on the differential of \( f \) and on the curvatures of the Riemannian manifolds \( M \) and \( N \), it is possible to establish long-time existence and convergence of the graphical mean curvature flow; for example see [6, 7, 13–17, 20, 27–29, 31, 32, 34, 36, 37].

A map \( f : M \to N \) between Riemannian manifolds is called strictly area decreasing, if

\[
|df(v) \wedge df(w)| < |v \wedge w|, \text{ for all } v, w \in TM.
\]

One of the first results for the graphical mean curvature flow in higher codimension was obtained by Tsui and Wang [36], where they proved that each initial strictly area decreasing map \( f : S^m \to S^n \) between unit spheres of dimensions \( m, n \geq 2 \) smoothly converges to a constant map under the flow. This first result has been generalized much further by other authors to the case of strictly area decreasing maps between Riemannian manifolds of non-constant curvature; see for instance [15,17,27,29]. For example, in [27] we proved that the mean curvature flow smoothly deforms a strictly area decreasing map \( f : M \to N \) into a constant one, if the manifolds \( M \) and \( N \) are compact, the Ricci curvature \( \text{Ric}_M \) of \( M \) and the sectional curvatures \( \sigma_M \) and \( \sigma_N \) of the Riemannian manifolds \( M \) and \( N \), respectively, satisfy

\[
\sigma_M > -\sigma \quad \text{and} \quad \text{Ric}_M \geq (m-1)\sigma \geq (m-1)\sigma_N
\]

for some positive constant \( \sigma > 0 \), where \( m \) is the dimension of \( M \). Optimal results were obtained in [29] in the case of weakly area decreasing maps between Riemannian surfaces.
In general, long-time existence of the mean curvature flow holds if the norm of the second fundamental form does not blow up in finite time. In some situations, estimates of the second fundamental form can be derived for graphical mean curvature flows. For example, if  \( f : M \to N \) is a strictly area decreasing map between compact Riemannian surfaces whose sectional curvatures satisfy

\[
\sigma_M \geq \max_N \sigma_N,
\]

then the norm of the second fundamental form of the evolving graphs stays uniformly bounded; see [29, Theorem B] for more details. In general, such estimates are hard to prove for mean curvature flows of graphs generated by maps between Riemannian manifolds of arbitrary dimensions. However, in some situations it turns out that an estimate for the mean curvature of the evolving graphs implies long-time existence of the mean curvature flow and such estimates are easier to obtain; for example in [28] this was shown when the initial map \( f \) is a strictly length decreasing map between Riemannian manifolds satisfying the curvature assumptions in [O].

In this paper, we focus on the case of codimension two. More precisely, we consider strictly area decreasing maps \( f : M \to N \) between a compact Riemannian manifold \( M \) of dimension at least two and a complete Riemannian surface \( N \) with bounded geometry, that is the curvature of \( N \) and its derivatives of all orders are uniformly bounded, and the injectivity radius is positive. Before we state one of our main results, we introduce some relevant curvature conditions.

**Definition 1.1.** Let \((M, g_M)\) be a Riemannian manifold of dimension \( m > 1 \) and let \((N, g_N)\) be a Riemannian surface.

(A) We say that the curvature condition \((A)\) holds, if the Ricci curvature \( \text{Ric}_M \) of \( M \) satisfies

\[
\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,
\]

where \( \sigma_M, \sigma_N \) denote the sectional curvatures of \( M, N \) and \( v, w \) are any orthonormal tangent vectors.

(B) We say that the curvature condition \((B)\) holds, if the Ricci curvature of \( M \) satisfies

\[
\text{Ric}_M(v, v) \geq 0,
\]

that is, if the Ricci curvature on \( M \) is non-negative.

(C) We say that the curvature condition \((C)\) holds, if the Ricci curvature of \( M \) satisfies

\[
\text{Ric}_M(v, v) \geq \sup_N \sigma_N,
\]

where \( v \) is any unit tangent vector on \( M \).

Note that \((C)\) implies \((B)\) if \( \sup_N \sigma_N \geq 0 \) and that \((B)\) implies \((C)\) if \( \sup_N \sigma_N \leq 0 \). In particular, \((B) \Leftrightarrow (C)\) if \( \sup_N \sigma_N = 0 \). We will discuss these curvature conditions in more detail in Remark 2.2.

The main results of the paper concerning the mean curvature flow are contained in Theorem A and its corollaries which are presented in Section 2. Roughly speaking, we obtain long-time existence of the mean curvature flow of strictly area decreasing maps under the curvature condition \((A)\) and smooth convergence to minimal maps under the conditions \((A), (B), \) and \((C)\). These minimal maps will be completely classified in Theorem F in Section 3.
Let us describe now the general idea and the main ingredients used in the proof. We would like to emphasise that, for the long-time existence proof and for the convergence to a minimal map, an estimate on the mean curvature plays a very crucial role. The main idea to exclude the formation of finite time singularities is to properly rescale near a singular point that might occur during the evolution. Then the singularity is modelled by an ancient mean curvature flow that is not totally geodesic. Since we show that under the curvature assumption (A) the mean curvature stays uniformly bounded on finite time intervals, this ancient solution is given by a non-flat complete minimal submanifold of codimension two in an euclidean space. According to the Bernstein type results in [1] and [38], such minimal graphs are totally geodesic, leading to a contradiction. Consequently, finite time singularities cannot occur. The non-negativity of the Ricci curvature of $M$ is then used to show that the maps stay uniformly strictly area decreasing and uniformly bounded in $C^1(M,N)$. In this case the aforementioned method can be carried over to prove that the second fundamental form stays uniformly bounded in time for the long-time solution. It is well-known that a uniform $C^2$-bound for the mean curvature flow in a complete Riemannian manifold of bounded geometry implies uniform $C^k$-bounds for all $k \geq 2$. Then the remaining statements of Theorem A follow.

The structure of the paper is as follows: In the next section, we state Theorem A. Moreover, we prove some corollaries including results concerning the homotopy type of maps with values in surfaces. In Section 3 we classify strictly area decreasing minimal maps under the curvature conditions (A) and (B). In Section 4 we recall the basic geometric quantities that are needed in the proofs of our results. The crucial estimate on the mean curvature vector along the evolving graphs is obtained in Section 5. In Section 6, in terms of $m$-convex functions on the target manifold, we derive a generalization of a well-known barrier principle for hypersurfaces to higher codimensions. Finally, the main results are proved in Section 7.

2. Long-time existence and convergence of the mean curvature flow

Our main results concerning the mean curvature flow of graphs are stated in the next theorem.

**Theorem A.** Let $(M,g_M)$ be a compact Riemannian manifold of dimension $m > 1$ and let $(N,g_N)$ be a complete Riemannian surface of bounded geometry. Suppose $f_0 : M \to N$ is strictly area decreasing.

(a) If the curvature condition (A) holds, that is if

$$\text{Ric}_M(v,v) + \text{Ric}_M(w,w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,$$

then the induced graphical mean curvature flow exists for $t \in [0, \infty)$, and the evolving maps $f_t : M \to N$ remain strictly area decreasing for all $t$.

(b) If the curvature conditions (A) and (B) hold, that is if

$$\text{Ric}_M(v,v) + \text{Ric}_M(w,w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,$$

$$\text{Ric}_M(v,v) \geq 0,$$

then $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^1(M,N)$ and remains uniformly strictly area decreasing.
(c) If the curvature conditions \((A)\) and \((C)\) hold, that is if
\[
\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,
\]
\[
\text{Ric}_M(v, v) \geq \sup_N \sigma_N,
\]
then the mean curvature stays uniformly bounded. If \(\{f_t\}_{t \in [0, \infty)}\) is uniformly bounded in \(C^1(M, N)\), then \(\{f_t\}_{t \in [0, \infty)}\) is uniformly bounded in \(C^k(M, N)\), for all \(k \geq 1\).

(d) Suppose that the curvature conditions \((A)\), \((B)\) and \((C)\) hold, that is we have
\[
\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,
\]
\[
\text{Ric}_M(v, v) \geq \max \{0, \sup_N \sigma_N\},
\]
for orthonormal tangent vectors \(v, w\). Then we get the following results:

1. \(\{f_t\}_{t \in [0, \infty)}\) is uniformly bounded in \(C^k(M, N)\), for all \(k \geq 1\).
2. In the following cases \(\{f_t\}_{t \in [0, \infty)}\) is uniformly bounded in \(C^0(M, N)\):
   (i) \(\text{Ric}_M > 0\).
   (ii) \(N\) satisfies one of the listed conditions:
      (ii.1) \(N\) is compact.
      (ii.2) \(\sup_N \sigma_N \leq 0\) and \(N\) is simply connected, that is diffeomorphic to \(\mathbb{R}^2\).
      (ii.3) \(\sup_N \sigma_N \leq 0\) and \(N\) contains a totally convex subset \(\mathcal{C}\), that is \(\mathcal{C}\) contains any geodesic in \(N\) with endpoints in \(\mathcal{C}\).
      (ii.4) There exists a constant \(c \in \mathbb{R}\) and a smooth function \(\psi : N \to \mathbb{R}\) such that \(\psi\) is convex on \(N^c := \{y \in N : \psi(y) < c\}\), \(N^c\) is compact and \(f_0(M) \subset N^c\).
3. Under the assumption that \(\{f_t\}_{t \in [0, \infty)}\) is uniformly bounded in \(C^k(M, N)\), for all \(k \geq 0\), the following holds:
   (i) There exists a subsequence \(\{f_{t_n}\}_{n \in \mathbb{N}}\), \(\lim_{n \to \infty} t_n = \infty\), that smoothly converges to one of the minimal maps classified in Theorem \((A)\).
   (ii) If there exists a subsequence \(\{f_{t_n}\}_{n \in \mathbb{N}}\) that converges in \(C^0(M, N)\) to a constant map, then the whole flow \(\{f_t\}_{t \in [0, \infty)}\) smoothly converges to this constant map.
   (iii) If there exists a point \(x \in M\) such that \(\text{Ric}_M(x) > 0\), then the flow \(\{f_t\}_{t \in [0, \infty)}\) smoothly converges to a constant map.
   (iv) If \((M, g_M)\) and \((N, g_N)\) are real analytic, then the flow smoothly converges to one of the minimal maps classified in Theorem \((A)\).

The proof of this result will be given in Section \((A)\). Let us discuss some of its interesting corollaries. For example, with a simple scaling argument we get the following.

**Corollary B.** Let \(M\) be a compact manifold with non-vanishing Euler characteristic \(\chi(M)\), and let \(N\) be a compact Riemann surface of genus bigger than one.

(a) If \(M\) is Kähler with vanishing first Chern class \(c_1(M)\), then any smooth map \(f : M \to N\) is smoothly null-homotopic.

(b) More generally, the same result holds if \(M\) admits an analytic metric of non-negative Ricci curvature.
Proof. Let us start by proving part (b). Let \((M, g_M)\) be real analytic with an analytic metric of non-negative Ricci curvature. Since \(N\) has genus bigger than one, we can endow \(N\) with a complete analytic Riemannian metric \(\hat{g}_N\) of constant negative curvature \(\hat{\sigma}_N\). Let \(\hat{\lambda}, \hat{\mu}\) denote the singular values of a smooth map \(f : M \to N\) with respect to the metrics \(g_M\) and \(\hat{g}_N\).

For a constant \(r > 0\) define the new metric \(g_N := r^2 \hat{g}_N\). Then the sectional curvature \(\sigma_N\) of \(g_N\), and the singular values \(\lambda\) and \(\mu\) of \(f\) with respect to \(g_M\) and \(g_N\) are given by

\[
\lambda = r \hat{\lambda}, \quad \mu = r \hat{\mu}, \quad \sigma_N = r^{-2} \hat{\sigma}_N.
\]

(2.1)

If we choose \(r\) sufficiently small, then \(f\) will become strictly area decreasing and \(\sigma_N\) will be so small that all curvature conditions in (A), (B) and (C) are satisfied. In particular, all assumptions in Theorem A are fulfilled. Applying the mean curvature flow to the graph of \(f\) in \((M, g_M) \times (N, g_N)\), Theorem A implies that \(f\) is homotopic to a constant map if \(M\) does not have vanishing Euler characteristic. This proves part (b). If \(M\) is a Kähler manifold with vanishing first Chern class, then by a famous theorem of Yau [39], \(M\) admits a Ricci flat Kähler metric and Kähler manifolds are analytic. This proves the corollary. □

Remark 2.1. Most of the Calabi-Yau manifolds have non-vanishing Euler characteristic. For example, the Euler number of K3-surfaces is 24. The statement in (b) cannot be extended to the case where \(N\) is \(S^2\) or \(T^2\). Neither the Hopf fibration \(f : S^3 \to S^2\) nor the projections \(\pi_{S^2} : S^1 \times S^2 \to S^2, \pi_{T^2} : S^1 \times T^2 \to T^2\) are homotopic to a constant map or to a geodesic.

Remark 2.2. It is easy to construct examples of long-time existence but no convergence. For example, take \(M = S^1 \times S^2\) with the standard product metric and for \(N\) choose the cylinder \(N = S^1 \times \mathbb{R}\) with a rotationally symmetric metric of non-positive curvature that is of funnel type like the surface depicted in Figure 1(a). Then \(\text{Ric}_M \geq 0 \geq \sigma_N\) and condition (A) is satisfied since \(M\) has non-negative sectional curvatures. For a fixed number \(z_0 \in \mathbb{R}\), let \(c_0 : S^1 \to N\) be the circle \(c_0(s) = (s, z_0)\) and define \(f_0(s, p) := c_0(s)\) for all \((s, p) \in S^1 \times S^2\). Then \(f_0\) admits just one non-zero singular value \(\mu\) and \(f_0\) is strictly area decreasing. The solution \(f_t\) to the mean curvature flow will be of the form \(f_t(s, p) = (s, z(t))\), where \(z : [0, \infty) \to \mathbb{R}\) is a smooth function that becomes unbounded when \(t \to \infty\). Note that all conditions in Theorem A(d) are satisfied, except those conditions in (2) guaranteeing \(C^0\)-bounds. However, the solution is uniformly bounded in \(C^k(M, N)\) for all \(k \geq 1\) and the mean curvature tends to zero when \(t \to \infty\). Nevertheless, there exist rotationally symmetric hyperbolic metrics on the cylinder for which we can apply Theorem A(d). For example, the closed geodesic \(C\) on the one-sheet hyperboloid depicted in Figure 1(b) is a totally convex subset.

Remark 2.3. We add some remarks concerning the curvature conditions and the results of Theorem A

(a) In dimension \(m = 2\), the curvature condition (A) is equivalent to

\[
\sigma_M \geq \sup_N \sigma_N.
\]

Therefore, we recover the main curvature condition and the results obtained in [29].
Figure 1. (a) Long-time existence but no convergence can occur in the case where the target manifold $N$ admits a funnel type metric of non-positive curvature without closed geodesics. The images $f_t(M)$ then might slide off to infinity, while at the same time the mean curvature and all $C^k$-norms of $f_t$, $k \geq 2$, tend to zero and the $C^1$-norm of $f_t$ remains uniformly bounded. (b) The images of $f_t$ stay in a compact region of $N$, if $N$ has non-positive curvature and contains a totally convex subset $\mathcal{C}$.

(b) Suppose that $m = 3$. Then, for any orthonormal frame $\{\alpha_1, \alpha_2, \alpha_3\}$ with respect to the Riemannian metric $g_M$, we have

$$\text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) = \sigma_M(\alpha_1 \wedge \alpha_2) + \sigma_M(\alpha_1 \wedge \alpha_3) + \sigma_M(\alpha_2 \wedge \alpha_3) - \sigma_M(\alpha_1 \wedge \alpha_2) = \text{Scal}_M/2.$$ 

Hence, the curvature condition (A) is equivalent to

$$\frac{1}{2} \text{Scal}_M \geq \sup_N \sigma_N. \quad (2.2)$$

Let us mention that, due to a result of Hamilton [12], a compact 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form, that is to a quotient $\mathbb{S}^3/G$ of the sphere $\mathbb{S}^3$ by a finite subgroup $G \subset O(4)$. According to the seminal work of Perelman [22–24], an oriented compact 3-manifold of positive scalar curvature is diffeomorphic to a connected sum of spherical space forms and finitely many copies of $\mathbb{S}^1 \times \mathbb{S}^2$; see also [21] and [4, Theorem 1.1]. It follows directly from the strong parabolic maximum principle and the evolution equation of the Ricci curvature under Ricci flow [12] that, a metric of non-negative Ricci curvature on a compact 3-manifold that has strictly positive Ricci curvature somewhere will immediately become Ricci positive under the Ricci flow. Moreover, Aubin [2, page 397] proved that a compact manifold of dimension $m$ that admits a metric of non-negative Ricci curvature which is strictly positive at some point, admits a metric of strictly positive Ricci curvature.
(c) Suppose now that $m = 4$ and consider an orthonormal frame \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} with respect to the Riemannian metric $g_M$. Then,

\[
\text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) \\
= \sigma_M(\alpha_1 \wedge \alpha_2) + \sigma_M(\alpha_1 \wedge \alpha_3) + \sigma_M(\alpha_1 \wedge \alpha_4) \\
+ \sigma_M(\alpha_1 \wedge \alpha_2) + \sigma_M(\alpha_2 \wedge \alpha_3) + \sigma_M(\alpha_2 \wedge \alpha_4) - \sigma_M(\alpha_1 \wedge \alpha_2) \\
= \text{Scal}_M/2 - \sigma_M(\alpha_3 \wedge \alpha_4).
\]

Hence, the curvature condition (A) is equivalent to

\[
\frac{1}{2} \text{Scal}_M - \sigma_M(v \wedge w) \geq \sup_N \sigma_N, \quad \text{for all } v, w \in TM.
\]

(d) If $(M, g_M)$ and $(N, g_N)$ satisfy (A), then at each $x \in M$ the scalar curvature of $M$ can be estimated by

\[
\text{Scal}_M \geq \frac{m(m-1)}{2m-3} \sup_N \sigma_N, \quad \text{(2.3)}
\]

and, for $m \neq 3$, equality occurs if and only if

\[
\sigma_M(v \wedge w) = \frac{1}{2m-3} \sup_N \sigma_N
\]

for all sectional curvatures at $x$.

**Proof.** Consider an orthonormal frame \{\alpha_1, \ldots, \alpha_m\} with respect to $g_M$. Then, for any $i \geq 2$, we get from (A) that

\[
\text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_i, \alpha_i) - \sigma_M(\alpha_1 \wedge \alpha_i) \geq \sup_N \sigma_N.
\]

Taking a trace for $i \geq 2$, we obtain

\[
(m-3) \text{Ric}_M(v, v) + \text{Scal}_M \geq (m-1) \sup_N \sigma_N, \quad \text{(2.4)}
\]

for any unit tangent vector $v$ of $M$. Taking another trace, we deduce (2.3). Now suppose that at $x \in M$ we have

\[
\text{Scal}_M = \frac{m(m-1)}{2m-3} \sup_N \sigma_N.
\]

Then (2.4) implies that at $x$ all Ricci curvatures satisfy

\[
(m-3) \text{Ric}_M(\alpha_i, \alpha_i) + \frac{m(m-1)}{2m-3} \sup_N \sigma_N \geq (m-1) \sup_N \sigma_N,
\]

that is

\[
(m-3) \left( \text{Ric}_M(\alpha_i, \alpha_i) - \frac{m-1}{2m-3} \sup_N \sigma_N \right) \geq 0.
\]

Since the trace on the left hand side vanishes and $m \neq 3$ the tensor inside the brackets vanishes and all Ricci curvatures are constant, that is

\[
\text{Ric}_M(\alpha_i, \alpha_i) = \frac{m-1}{2m-3} \sup_N \sigma_N.
\]

Inserting this into (A), we obtain

\[
\frac{1}{2m-3} \sup_N \sigma_N - \sigma_M(\alpha_i \wedge \alpha_j) \geq 0.
\]
Since the trace over $j \neq i$ gives again 0, we may argue as before to conclude that equality holds. Hence all sectional curvatures satisfy $\sigma_M(\alpha_i \wedge \alpha_j) = (2m - 3)^{-1} \sup_N \sigma_N$. \hfill \textcircled{2}

Note, when $m$ equals 2 or 3, inequality (2.3) is equivalent to (A). In addition, if $m \neq 3$ and at each $x \in M$ there exist at least two distinct sectional curvatures, then (A) can only be satisfied as a strict inequality.

(e) As we already mentioned above, the convergence results in [27] and [28] were obtained under the curvature assumption (O). It turns out that (O) implies the validity of (A) and, in particular, in this case (A) becomes even strict when $m > 2$. Indeed, if $m = 2$ the conclusion follows from $\text{Ric}_M(v, v) = \text{Ric}_M(w, w)$ for any $v, w \in TM$. In case $m > 2$, it suffices to check this for an orthonormal frame $\{\alpha_1, \ldots, \alpha_m\}$ for which the Ricci tensor becomes diagonal. Then, for any $i \neq j$, we get

$$
\text{Ric}_M(\alpha_i, \alpha_i) + \text{Ric}_M(\alpha_j, \alpha_j) - \sigma_M(\alpha_i \wedge \alpha_j) = \text{Ric}_M(\alpha_i, \alpha_i) + \sum_{k \neq j} \sigma_M(\alpha_k \wedge \alpha_j) - \sigma_M(\alpha_i \wedge \alpha_j) = (m - 1)\sigma - (m - 2)\sigma = \sigma 

\geq \sup_N \sigma_N.
$$

However, (A) does not imply (O), hence the curvature condition (A) is more general than (O). To obtain a better picture, let us assume that the sectional curvatures of $(M, g_M)$ are all constant to $\sigma_M$ and that the curvature of $N$ is given by a constant $\sigma_N$. The curvature condition (O) of [27] is then equivalent to

$$
\sigma_M \geq \sigma_N \quad \text{and} \quad \sigma_M > 0.
$$

When the sectional curvatures are constant, (A), (B) and (C) are equivalent to (in this order):

$$
(2m - 3)\sigma_M \geq \sigma_N, \\
\sigma_M \geq 0, \\
(m - 1)\sigma_M \geq \sigma_N.
$$

Therefore, the long-time existence and convergence results in Theorem A are stronger than those in [27].

The description of the homotopy groups $\pi_m(S^n)$ for $m > n$ is one of the central problems in Algebraic Topology. Although in many cases they have been computed explicitly, in general, the problem is still open. An interesting question is to determine when a map between spheres is homotopically trivial. In this direction, we obtain the following result.

**Corollary C.** For the standard unit spheres $(S^m, g_{S^m})$ and $(S^2, g_{S^2})$ let us define

$$
\mathcal{A}_{m-1} := \{ f \in C^\infty(S^m, S^2) : \lambda \mu < m - 1 \},
$$

where $\lambda$ and $\mu$ denote the singular values of $f$. Then for $m > 1$ and for any $f_0 \in \mathcal{A}_{m-1}$ there exists a smooth homotopy $\{ f_t \}_{t \in [0, \infty)} \subset \mathcal{A}_{m-1}$ deforming $f_0$ into a constant map. This homotopy can be given by the mean curvature flow of $f_0$ as a map between $(S^m, g_{S^m})$ and the scaled 2-sphere $(S^2, (m - 1)^{-1} g_{S^2})$. In particular, $\mathcal{A}_{m-1}$ is smoothly contractible.
**Proof.** Maps in $\mathcal{A}_{m-1}$ are strictly area decreasing maps from $(\mathbb{S}^m, g_{\mathbb{S}^m})$ to $(\mathbb{S}^2, (m-1)^{-1} g_{\mathbb{S}^2})$. The sectional curvature of

$$g_N := (m-1)^{-1} g_{\mathbb{S}^2}$$

is $m - 1$ and the result follows from Remark 2.3(e), because in this case the curvature conditions in Theorem A are equivalent to $m - 1 \geq \sigma_N$. □

**Remark 2.4.** It is well-known that the groups $\pi_m(\mathbb{S}^2)$ are non-trivial for $m \geq 2$ and are finite for $m \geq 4$; see [3, 9, 11]. Consequently, in Corollary C we cannot increase the upper bound for $\lambda \mu$ arbitrarily without losing the contractibility of the corresponding set

$$\mathcal{A}_{m,c} := \{ f \in C^{\infty}(\mathbb{S}^m, \mathbb{S}^2) : \lambda \mu < c \}.$$

A natural problem arises; to determine the number

$$c_m := \sup\{ c > 0 : \mathcal{A}_{m,c} \text{ is smoothly contractible} \}.$$

The Hopf fibration $f : \mathbb{S}^3 \to \mathbb{S}^2$ has constant singular values $\lambda = \mu = 2$. Moreover, it is minimal, but not totally geodesic, and not homotopic to a constant map; see [19, Remark 1]. Hence, from Corollary C we see that

$$2 \leq c_3 \leq 4 \quad \text{and} \quad m - 1 \leq c_m < \infty,$$

for $m > 2$. Since the identity map $\text{Id} : \mathbb{S}^2 \to \mathbb{S}^2$ is not homotopic to the constant map, we have that $c_2 = 1$.

We would like to emphasise that in Theorem A also the long-time existence of the graphical mean curvature flow is obtained under much weaker assumptions on the curvatures than those in [27]. For example, in dimension $m = 3$, the condition $\text{Scal}_M \geq \text{Scal}_N$ suffices. With respect to convergence, we cover the case where $N$ is a complete non-positively curved Riemannian surface and the Riemannian manifold $M$ has either non-negative sectional curvature or $M$ is 3-dimensional with non-negative Ricci curvature. We can summarise this in the following corollary.

**Corollary D.** Let $(M, g_M)$ be a compact 3-manifold and let $(N, g_N)$ be a complete surface of bounded geometry that satisfy the curvature condition

$$\text{Ric}_M(v, v) \geq \max\{0, \sup_N \sigma_N\},$$

for all unit tangent vectors $v$. Then, the curvature conditions $[A]$, $[B]$ and $[C]$ in Theorem A are satisfied and for any strictly area decreasing initial map $f_0 : M \to N$ the results in Theorem A(d) apply.

**Proof.** Conditions $[B]$ and $[C]$ hold by assumption. Since $m = 3$, the curvature condition in $[A]$ is equivalent to

$$\text{Scal}_M \geq 2 \sup_N \sigma_N.$$

We distinguish two cases.

(i) $\sup_N \sigma_N \geq 0$. In this case $[C] \Rightarrow \text{Scal}_M \geq 3 \sup_N \sigma_N \geq 2 \sup_N \sigma_N$.

(ii) $\sup_N \sigma_N \leq 0$. In this case $[B] \Rightarrow \text{Scal}_M \geq 0 \geq \sup_N \sigma_N \geq 2 \sup_N \sigma_N$.

Therefore, the curvature condition $[A]$ holds and Theorem A applies. □
The next corollary follows easily from the Künneth formula and the fact that compact manifolds with positive Ricci curvature do not admit non-trivial harmonic 1-forms. Here we can give an alternative proof of this result using the mean curvature flow.

**Corollary E.** Let \( M = L \times N \) be the product of a compact manifold \( L \) and a compact surface \( N \) of genus bigger than one. Then \( M \) does not admit any Riemannian metric of positive Ricci curvature.

**Proof.** The projection \( \pi_N : L \times N \to N \) is not homotopic to a constant map. If \( L \times N \) admits a metric of positive Ricci curvature, then we can equip \( N \) with a metric of sufficiently negative constant curvature such that \( \pi_N \) becomes strictly area decreasing and such that the curvature conditions (A), (B) and (C) hold. Theorem A implies that \( \pi_N \) can be deformed into a constant map by mean curvature flow. This is a contradiction. \( \square \)

The case \( N = T^2 \) cannot be covered in Corollary E by our method since the proof uses a scaling argument that fails in this case.

### 3. Classification of strictly area decreasing minimal maps

In this section we classify all strictly area decreasing minimal maps that satisfy the curvature conditions (A) and (B). In particular, this classifies all possible limits under the mean curvature flow in Theorem A. If \( f : M \to N \) is smooth, and \( \lambda_1 \geq \cdots \geq \lambda_m \) are the singular values of \( df \) (for a precise definition see Section 4), then \( \dim N = 2 \) implies that at each point \( x \in M \) we can have at most two non-trivial singular values, namely \( \mu := \lambda_1 \) and \( \lambda := \lambda_2 \).

**Theorem F.** Let \( (M, g_M) \) be a compact Riemannian manifold of dimension \( m > 1 \) and let \( (N, g_N) \) be a complete Riemannian surface such that (A) and (B) hold, that is we have

\[
\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,
\]

for orthonormal tangent vectors \( v, w \). Suppose \( f : M \to N \) is a strictly area decreasing minimal map. Then \( f \) is totally geodesic, the rank of \( df \) and the singular values \( \lambda \) and \( \mu \) of \( f \) are constant.

(a) If \( \text{rank}(df) = 0 \), then \( f \) is constant and \( \lambda = \mu = 0 \).

(b) If \( \text{rank}(df) > 0 \), then \( f : M \to f(M) \) is a submersion. For \( y \in f(M) \) the fibers \( K_y \) are compact embedded and totally geodesic submanifolds of codimension \( \text{rank}(df) \) in \( M \). The fiber \( K_y \) is isometric to a compact Riemannian manifold \( (K, g_K) \) of non-negative Ricci curvature that does not depend on \( y \). The horizontal integral submanifolds are complete totally geodesic submanifolds in \( M \) that intersect the fibers orthogonally. The manifold \( (M, g_M) \) is locally isometric to a product \( (L \times K, g_L \times g_K) \). The Euler characteristic \( \chi(M) \) of \( M \) vanishes, and, at each point \( x \in M \), the kernel of the Ricci operator is non-trivial. More precisely, depending on the rank of \( df \), we distinguish two cases:

(i) \( \text{rank}(df) = 1 \). In this case \( \lambda = 0 \) and \( \mu > 0 \). Moreover, \( \gamma := f(M) \) is a closed geodesic in \( N \). The horizontal leaves are geodesics orthogonal to the fibers, and the map \( f : (M, g_M) \to (\gamma, \mu^{-2} g_\gamma) \) is a Riemannian submersion, where \( g_\gamma \) denotes the metric on \( \gamma \) as a submanifold in \( (N, g_N) \).
(ii) \( \text{rank}(df) = 2 \). In this case \( \lambda > 0 \) and \( \mu > 0 \). The image \( f(M) \) coincides with \( N \) and \( N \) is diffeomorphic to a torus \( T^2 \) or to a Klein bottle \( T^2/\mathbb{Z}_2 \). The metric \( g_N \) and the metrics on the horizontal leaves are flat. Additionally, \( f : (M, g_M) \rightarrow (N, \mu^{-2}g_N) \) is a Riemannian submersion, if \( \lambda = \mu \).

We will postpone the proof of this theorem to Section 7.

**Corollary G.** If, in addition to the assumptions made in Theorem F, there exists a point \( x \in M \) with \( \text{Ric}_M(x) > 0 \), then strictly area decreasing minimal maps \( f : M \rightarrow N \) are constant.

**Proof.** If \( \text{Ric}_M(x) > 0 \) at some point \( x \in M \), then part (b) in Theorem F is impossible since this part requires the kernel of the Ricci operator to be non-trivial at each point. □

4. **Geometry of graphs**

In this section, we follow the notations of our previous papers [27–29] and recall some basic terminology and facts related to the geometry of graphical submanifolds.

4.1. **First fundamental form and connections.**

The product \( M \times N \) will be regarded as a Riemannian manifold equipped with the metric \( g_{M \times N} = \langle \cdot, \cdot \rangle := g_M \times g_N \).

The **graph** of a map \( f : M \rightarrow N \) is defined to be the submanifold \( \Gamma_f := \{(x, f(x)) \in M \times N : x \in M\} \) of the Riemannian product \( M \times N \). The graph \( \Gamma_f \) can be parametrized via the embedding \( F : M \rightarrow M \times N \), given by \( F := \text{Id}_M \times f \), where \( \text{Id}_M \) is the identity map of \( M \). The Riemannian metric on \( M \) induced by \( F \) will be denoted by \( g := F^*g_{M \times N} \) and will be called the **graphical metric**. The Levi-Civita connection on \( M \) with respect to the induced metric \( g \) is denoted by \( \nabla \), the curvature tensor by \( R \) and the Ricci curvature by \( \text{Ric} \).

The two natural projections \( \pi_M : M \times N \rightarrow M \) and \( \pi_N : M \times N \rightarrow N \) are submersions. The tangent bundle of the product manifold \( M \times N \) splits into the direct sum \( T(M \times N) = TM \oplus TN \).

The four metric tensors \( g_M, g_N, g_{M \times N} \) and \( g \) are related by the equations
\[
g_{M \times N} := \pi_M^*g_M + \pi_N^*g_N \quad \text{and} \quad g := F^*g_{M \times N} = g_M + f^*g_N.
\]

As in [27–29], let us define the symmetric 2-tensors
\[
s_{M \times N} := \pi_M^*s_M - \pi_N^*s_N \quad \text{and} \quad S := F^*s_{M \times N} = g_M - f^*g_N.
\]

The Levi-Civita connection \( \nabla^{g_{M \times N}} \) associated with the metric \( g_{M \times N} \) is related to the Levi-Civita connections \( \nabla^g_M \) on \( (M, g_M) \) and \( \nabla^g_N \) on \( (N, g_N) \) by the equation
\[
\nabla^{g_{M \times N}} = \pi_M^*\nabla^g_M \oplus \pi_N^*\nabla^g_N,
\]
and the corresponding curvature operator $R_{M \times N}$ on $M \times N$ with respect to the metric $g_{M \times N}$ is related to the curvature operators $R_M$ on $(M, g_M)$ and $R_N$ on $(N, g_N)$ by equation
\[ R_{M \times N} = \pi^*_M R_M \oplus \pi^*_N R_N. \]

### 4.2. The second fundamental form.

The differential $dF$ of $F$ can be regarded as a section of the bundle $F^* T(M \times N) \otimes T^* M$. In the sequel, we will denote all full connections on bundles over $M$ which are induced by the Levi-Civita connection of $g_{M \times N}$ via $F : M \to M \times N$ by the same letter $\nabla$. The second fundamental form of $F$ is given by
\[ A := \nabla dF, \]
that is
\[ A(v, w) := (\nabla dF)(v, w), \]
for any tangent vectors $v, w \in T M$. In terms of the connections $\nabla^F$ and $\nabla^f$ on the pull-back bundles $F^* T(M \times N)$ and $f^* T N$, respectively, we can express $A$ by the equations
\begin{align*}
A(X, Y) &= \nabla^F_X (dF(Y)) - dF(\nabla_X Y) \\
&= \left( \nabla^M_X Y, \nabla^f_X (df(Y)) \right) - \left( \nabla_X Y, df(\nabla_X Y) \right) \\
&= \left( \nabla^M_X Y - \nabla_X Y, \nabla^f_X (df(Y)) - df(\nabla_X Y) \right),
\end{align*}
where $X, Y \in \mathfrak{X}(M)$ are arbitrary smooth vector fields on $M$. If $\xi$ is a normal vector of the graph, then the symmetric bilinear form $A^\xi$, given by
\[ A^\xi(v, w) := \langle A(v, w), \xi \rangle, \]
will be called the second fundamental form with respect to the normal $\xi$. The mean curvature vector field of the graph $\Gamma_f$ is the trace of $A$ with respect to the graphical metric $g$,
\[ H := \text{trace}_g A \]
and $H$ is a section in the normal bundle $T^\perp M$. The graph $\Gamma_f$, and likewise the map $f$, are called minimal if $H$ vanishes identically.

The relation between the curvature tensors $R$ and $R_{M \times N}$ is expressed by Gauß' equation
\begin{equation}
(R - F^* R_{M \times N})(v_1, w_1, v_2, w_2) = \langle A(v_1, v_2), A(w_1, w_2) \rangle - \langle A(v_1, w_2), A(w_1, v_2) \rangle,
\end{equation}
and the second fundamental form satisfies the Codazzi equation
\begin{equation}
(\nabla_u A)(v, w) - (\nabla_v A)(u, w) = R_{M \times N}(dF(u), dF(v))dF(w) - dF(R(u, v)w),
\end{equation}
for any $u, v, w, v_1, w_1, v_2, w_2 \in T M$.

Throughout this paper, we will use latin indices to indicate components of tensors with respect to frames in the tangent bundle that are orthonormal with respect to $g$. For example, if $\{e_1, \ldots, e_m\}$ is a local orthonormal frame of the tangent bundle and $\xi$ is a local vector field in the normal bundle of $M$, then
\[ A_{ij} = A(e_i, e_j) \quad \text{and} \quad A^\xi_{ij} = \langle A(e_i, e_j), \xi \rangle. \]
4.3. Singular value decomposition of maps in codimension two.

Fix a point \( x \in M \) and let \( \lambda_1^2 \geq \ldots \geq \lambda_m^2 \) denote the eigenvalues of \( f^*g_N \) at \( x \) with respect to \( g_M \). The corresponding values \( \lambda_i \geq 0, \ i \in \{1, \ldots, m\} \), are called singular values of the differential \( df \) of \( f \) at the point \( x \). The singular values are Lipschitz continuous functions on the manifold \( M \).

Suppose that \( M \) has dimension at least two and that \( N \) is a complete Riemannian surface. In this case, we have \( \text{rank}(df) \leq 2 \) and consequently there exist at most two non-vanishing singular values, which we denote for simplicity by \( \lambda := \lambda_1 \) and \( \mu := \lambda_2 \). At each fixed point \( x \in M \), one may consider an orthonormal basis \( \{\alpha_1, \ldots, \alpha_m\} \) of \( T_xM \) with respect to \( g_M \) that diagonalizes \( f^*g_N \). Therefore, at \( x \) we have

\[
(f^*g_N(\alpha_i, \alpha_j))_{i,j} = \text{diag}(\lambda^2, \mu^2, 0, \ldots, 0).
\]

In addition, at \( f(x) \in N \) we may consider an orthonormal basis \( \{\beta_1, \beta_2\} \) of \( T_{f(x)}N \) with respect to \( g_N \) such that

\[
df(\alpha_1) = \lambda\beta_1, \quad df(\alpha_2) = \mu\beta_2 \quad \text{and} \quad df(\alpha_i) = 0, \ \text{for} \ i \geq 3.
\]

We then define another basis \( \{e_1, \ldots, e_m\} \) of \( T_xM \) and a basis \( \{\xi, \eta\} \) of \( T^\perp xM \) in terms of the singular values, namely

\[
e_1 := \frac{\alpha_1}{\sqrt{1 + \lambda^2}}, \quad e_2 := \frac{\alpha_2}{\sqrt{1 + \mu^2}}, \quad e_i := \alpha_i, \ \text{for} \ i \geq 3,
\]

and

\[
\xi := -\lambda\alpha_1 \oplus \beta_1, \quad \eta := -\mu\alpha_2 \oplus \beta_2.
\]

One can easily check that \( \{e_1, \ldots, e_m\} \) defines an orthonormal basis of \( T_xM \) with respect to the induced graphical metric \( g \), and that \( \{\xi, \eta\} \) form an orthonormal basis of the normal space \( T^\perp xM \) of the graph \( \Gamma_f \) at \( F(x) \). With respect to the orthonormal basis

\[
\{df(e_1), \ldots, df(e_m); \xi, \eta\},
\]

the 2-tensor \( s_{M \times N} \) has the following matrix representation

\[
s_{M \times N} = \begin{pmatrix}
\frac{1 - \lambda^2}{1 + \lambda^2} & 0 & 0 & \ldots & 0 & -\frac{2\lambda}{1 + \lambda^2} & 0 \\
0 & \frac{1 - \mu^2}{1 + \mu^2} & 0 & \ldots & 0 & -\frac{2\mu}{1 + \mu^2} & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
-\frac{2\lambda}{1 + \lambda^2} & 0 & 0 & \ldots & -\frac{1 - \lambda^2}{1 + \lambda^2} & 0 & 0 \\
0 & -\frac{2\mu}{1 + \mu^2} & 0 & \ldots & -\frac{1 - \mu^2}{1 + \mu^2} & 0 & 0
\end{pmatrix}
\]

Consequently, the pull-back \( S = F^*s_{M \times N} \) to the tangent bundle of \( M \) satisfies

\[
(S(e_i, e_j))_{i,j} = \text{diag}\left(\frac{1 - \lambda^2}{1 + \lambda^2}, \frac{1 - \mu^2}{1 + \mu^2}, 1, \ldots, 1\right).
\]

(4.4)
Moreover, the restriction $S^\perp$ of $s_{M \times N}$ to the normal bundle of $\Gamma_f$ satisfies the identities

\[ S^\perp(\xi, \xi) = \frac{1 - \lambda^2}{1 + \lambda^2}, \quad S^\perp(\eta, \eta) = \frac{1 - \mu^2}{1 + \mu^2}, \quad \text{and} \quad S^\perp(\xi, \eta) = 0. \]  

(4.5)

Let us introduce also the quantities $T_{11}$ and $T_{22}$ given by

\[ T_{11} := s_{M \times N}(dF(e_1), \xi) = -\frac{2\lambda}{1 + \lambda^2} \quad \text{and} \quad T_{22} := s_{M \times N}(dF(e_2), \eta) = -\frac{2\mu}{1 + \mu^2}, \]  

(4.6)

which represents the mixed terms of $s_{M \times N}$.

The differential $df$ of the map $f$ induces the natural map $\Lambda^2 df : \Lambda^2 TM \rightarrow \Lambda^2 TN$ given by

\[ \Lambda^2 df(v, w) := df(v) \wedge df(w) \]

for any $v, w \in TM$. The supremum norm

\[ |\Lambda^2 df| := \sup_{v, w \neq 0} \frac{|df(v) \wedge df(w)|}{|v \wedge w|} \]

of $\Lambda^2 df$ measures how much the differential of $f$ stretches the 2-dimensional areas of $M$. A smooth map $f$ is called weakly area decreasing if $|\Lambda^2 df| \leq 1$ and strictly area decreasing if $|\Lambda^2 df| < 1$. The above conditions can be expressed in terms of the singular values of $f$ by the inequalities $\lambda \cdot \mu \leq 1$ and $\lambda \cdot \mu < 1$, respectively. Consider now the smooth function $p : M \rightarrow \mathbb{R}$ given by

\[ p := \text{tr}_g S + 2 - m = S_{11} + S_{22} = \frac{2(1 - \lambda^2 \mu^2)}{(1 + \lambda^2)(1 + \mu^2)}. \]

Note that, in codimension two, the map $f$ is weakly area decreasing if and only if $p$ is non-negative and $f$ is strictly area decreasing if and only if $p$ is positive.

5. Estimates for the graphical mean curvature flow

Let $f : M \rightarrow N$ be a smooth map between Riemannian manifolds $M$ and $N$ and let

\[ F_0 := \text{Id}_M \times f : M \rightarrow \Gamma_f \subset M \times N. \]

We deform the graph $\Gamma_f$ of $f$ by the mean curvature flow in $M \times N$, that is we consider the family of immersions $F : M \times [0, T) \rightarrow M \times N$ satisfying the evolution equation

\[ \frac{dF}{dt}(x, t) = H(x, t), \quad F(x, 0) = F_0(x). \]  

(MCF)

where $(x, t) \in M \times [0, T)$, $H(x, t)$ is the mean curvature vector field at $x \in M$ given by the immersion $F_t : M \rightarrow M \times N$, $F_t(\cdot) := F(\cdot, t)$, and where $T$ denotes the maximal time of existence of a smooth solution of equation (MCF).

5.1. First order estimates for area decreasing maps.

In order to investigate under which curvature conditions the area decreasing property is preserved under the flow, we need to compute the evolution equation of the function

\[ p := \text{tr}_g S + 2 - m. \]

We have the following general result for graphs in codimension two.
Lemma 5.1. The function $p$ satisfies the evolution equation

$$(\nabla_{\partial_t} - \Delta) p = 2 |A|^2 + 2 \sum_{k=1,i=3}^m |A_k^i|^2 (1 - S_{11}) + 2 \sum_{k=1,i=3}^m |A_k^i|^2 (1 - S_{22})$$

$$+ \frac{1}{2} p \left( 4 \sum_{k=1}^m |A_k^i T_{22} + A_{2k}^i T_{11}|^2 - |\nabla p|^2 \right) + Q,$$  \hspace{1cm} (5.1)

where $Q$ is the first order quantity given by

$$Q := \frac{2 \lambda^2 \mu^2 (2 + p)}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right)$$

$$+ \frac{2 \lambda^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_1, \alpha_1) + \frac{2 \mu^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_2, \alpha_2),$$  \hspace{1cm} (5.2)

and $\{\alpha_1, \ldots, \alpha_m\}$, $\{e_1, \ldots, e_m\}$, $\{\xi, \eta\}$ are the special bases arising from the singular value decomposition defined in subsection 4.3.

Proof. To derive the evolution equation of $p$, we use the evolution equations of $g$ and $S$ that were derived in [27]. Recall that

$$(\nabla_{\partial_t} g)(v, w) = -2 A^H(v, w),$$  \hspace{1cm} (5.3)

and

$$(\nabla_{\partial_t} S - \Delta S)(v, w) = -S(\text{Ric} \, v, w) - S(\text{Ric} \, w, v) - 2 \sum_{k=1}^m S^\perp(\text{A}(e_k, v), \text{A}(e_k, w))$$

$$+ 2 \sum_{k=1}^m \left( \text{R}_M - f^* \text{R}_N \right)(e_k, v, e_k, w),$$  \hspace{1cm} (5.4)

for any $v, w \in TM$. Combining (5.3) with the trace of the Gauß equation (4.2), that is with

$$\text{Ric}(v, w) = \sum_{k=1}^m (F^* \text{R}_{M \times N})(e_k, v, e_k, w)$$

$$+ \sum_{k=1}^m \langle \text{A}(e_k, e_k), \text{A}(v, w) \rangle - \sum_{k=1}^m \langle \text{A}(v, e_k), \text{A}(w, e_k) \rangle$$

$$= \sum_{k=1}^m \left( \text{R}_M + f^* \text{R}_N \right)(e_k, v, e_k, w)$$

$$+ \sum_{k=1}^m \langle \text{A}(e_k, e_k), \text{A}(v, w) \rangle - \sum_{k=1}^m \langle \text{A}(v, e_k), \text{A}(w, e_k) \rangle$$
we obtain

$$\left( \nabla \partial_t - \Delta \right) p = 2 \left( \sum_{k,l=1}^{m} \langle A_{kl}, A_{kl} \rangle S_{ll} - \sum_{k,l=1}^{m} S^{\perp}(A_{kl}, A_{kl}) \right)$$

$$= A$$

$$+ 2 \sum_{k,l=1}^{m} \left( R_M - f R_N \right)_{klkl} - 2 \sum_{k,l=1}^{m} \left( R_M + f R_N \right)_{klkl} S_{ll} \, . \quad (5.5)$$

In codimension two we are able to simplify this equation further. We start with the terms on the right hand side of the first line in (5.5). Since

$$S_{ll} = 1, \quad \text{for } l \geq 3, \quad S^{\perp}(x, x) = - S(e_1, e_1), \quad S^{\perp}(y, y) = - S(e_2, e_2) \quad \text{and} \quad S^{\perp}(x, y) = 0,$$

we get

$$A = \sum_{k=1}^{m} \langle A_{kl}, A_{kl} \rangle S_{ll} - \sum_{k,l=1}^{m} \langle A_{kl}, A_{kl} \rangle$$

$$= \sum_{k=1}^{m} |A_{k1}|^2 S_{11} + \sum_{k=1}^{m} |A_{k2}|^2 S_{22} + \sum_{k,l=1,i=3}^{m} |A_{kl}|^2 + |A^\xi|^2 S_{11} + |A^\eta|^2 S_{22}$$

$$= p |A|^2 + \sum_{k=1,i=3}^{m} \left( |A^\xi_{k1}|^2 + |A^\eta_{k1}|^2 \right)$$

$$+ \left( \sum_{k=1}^{m} |A_{k1}|^2 - |A^\eta|^2 \right) S_{11} + \left( \sum_{k=1}^{m} |A_{k2}|^2 - |A^\xi|^2 \right) S_{22} \, .$$

On the other hand

$$\sum_{k=1}^{m} |A_{k1}|^2 - |A^\eta|^2 = \sum_{k=1}^{m} \left( |A^\xi_{k1}|^2 - |A^\eta_{k2}|^2 \right) - \sum_{k=1,i=3}^{m} |A^\eta_{ki}|^2$$

and

$$\sum_{k=1}^{m} |A_{k2}|^2 - |A^\xi|^2 = \sum_{k=1}^{m} \left( |A^\eta_{k2}|^2 - |A^\xi_{k1}|^2 \right) - \sum_{k=1,i=3}^{m} |A^\xi_{ki}|^2 \, .$$

Consequently,

$$A = p |A|^2 + \sum_{k=1,i=3}^{m} |A^\xi_{k1}|^2 (1 - S_{11}) + \sum_{k=1,i=3}^{m} |A^\eta_{ki}|^2 (1 - S_{22})$$

$$- \sum_{k=1}^{m} \left( |A^\xi_{k1}|^2 - |A^\eta_{k2}|^2 \right) (S_{22} - S_{11}) \, . \quad (5.6)$$

We want to express $B$ in terms of $|\nabla p|^2$, therefore we need a different expression for $|\nabla p|^2$. Namely, since

$$(\nabla_{e_k} S)(v, v) = 2 s_{M \times N}(A(e_k, v), dF(v)) \quad (5.7)$$
we get
\[ \nabla_{e_k} p = 2 \sum_{i=1}^{m} s_{M \times N}(A(e_k, e_i), dF(e_i)) \]
\[ = 2 \sum_{i=1}^{m} s_{M \times N}(\xi, dF(e_i))A_{ik}^{\xi} + 2 \sum_{i=1}^{m} s_{M \times N}(\eta, dF(e_i))A_{ik}^{\eta}, \]
from where we deduce that
\[ \nabla_{e_k} p = 2 A_{1k}^{\xi} T_{11} + 2 A_{2k}^{\eta} T_{22}. \]
Recalling that \( S_{ll}^2 + T_{ll}^2 = 1 \), for \( l \in \{1, 2\} \), we obtain the following equation for \( |\nabla p|^2 \).
\[ 4 p B = 4 p \sum_{k=1}^{m} (|A_{1k}^{\xi}|^2 - |A_{2k}^{\eta}|^2)(S_{22} - S_{11}) \]
\[ = 4 \sum_{k=1}^{m} (|A_{1k}^{\xi}|^2 - |A_{2k}^{\eta}|^2)(S_{22} - S_{11}) \]
\[ = |\nabla p|^2 - 4 \sum_{k=1}^{m} |A_{1k}^{\xi} T_{22} + A_{2k}^{\eta} T_{11}|^2. \] (5.8)
Combining (5.5)–(5.8), we derive that at points where \( p > 0 \) it holds
\[ (\nabla \partial_t - \Delta) p = 2 p |A|^2 + 2 \sum_{k=1,i=3}^{m} |A_{ki}^{\xi}|^2(1 - S_{11}) + 2 \sum_{k=1,i=3}^{m} |A_{ki}^{\eta}|^2(1 - S_{22}) \]
\[ + \frac{1}{2} p \left( 4 \sum_{k=1}^{m} |A_{1k}^{\xi} T_{22} + A_{2k}^{\eta} T_{11}|^2 - |\nabla p|^2 \right) \]
\[ + 2 \sum_{k,l=1}^{m} (R_M)_{kkl}(1 - S_{ll}) - 2 \sum_{k,l=1}^{m} (f_{l}^{R}R_{N})_{kkl}(1 + S_{ll}). \] (5.9)
Since \( S_{ll} = 1 \) and \( \lambda_l = 0 \) for \( l \geq 3 \), the first term \( C_1 \) in the last line of (5.9) simplifies to
\[ C_1 = 2 \sum_{k,l=1}^{m} (R_M)_{kkl}(1 - S_{ll}) = 2 \sum_{k,l=1}^{m} R_M(e_k, e_l, e_k, e_l)(1 - S(e_l, e_l)) \]
\[ = \frac{4\lambda^2}{1 + \lambda^2} \sum_{k=1}^{m} R_M(e_k, e_1, e_k, e_1) + \frac{4\mu^2}{1 + \mu^2} \sum_{k=1}^{m} R_M(e_k, e_2, e_k, e_2) \]
\[ = \frac{2\lambda^2}{(1 + \lambda^2)^2} \sum_{k=1}^{m} \left( 1 + \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \right) \sigma_M(\alpha_1 \land \alpha_k) \]
\[ + \frac{2\mu^2}{(1 + \mu^2)^2} \sum_{k=1}^{m} \left( 1 + \frac{1 - \lambda_k^2}{1 + \lambda_k^2} \right) \sigma_M(\alpha_2 \land \alpha_k). \]
Analysing the curvature terms involving $R_M$, we get that
\[
C_1 = \frac{2\lambda^2}{(1 + \lambda^2)^2} \text{Ric}_M(\alpha_1, \alpha_1) + \frac{2\mu^2}{(1 + \mu^2)^2} \text{Ric}_M(\alpha_2, \alpha_2)
+ \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \sigma_M(\alpha_1 \land \alpha_2)
+ \frac{2\lambda^2}{(1 + \lambda^2)^2} \sum_{k \geq 3} \sigma_M(\alpha_1 \land \alpha_k) + \frac{2\mu^2}{(1 + \mu^2)^2} \sum_{k \geq 3} \sigma_M(\alpha_2 \land \alpha_k).
\]

Hence,
\[
C_1 = \frac{2\lambda^2}{(1 + \lambda^2)^2} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \sum_{k \geq 3} \sigma_M(\alpha_1 \land \alpha_k) \right)
+ \frac{2\mu^2}{(1 + \mu^2)^2} \left( \text{Ric}_M(\alpha_2, \alpha_2) + \sum_{k \geq 3} \sigma_M(\alpha_2 \land \alpha_k) \right)
+ \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \sigma_M(\alpha_1 \land \alpha_2).
\]

Applying the formula $2ac + 2bd = (a + b)(c + d) + (a - b)(c - d)$ to the first two lines in the last equality, we derive
\[
C_1 = \left( \frac{\lambda^2}{(1 + \lambda^2)^2} + \frac{\mu^2}{(1 + \mu^2)^2} \right)
\times \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) + \sum_{k \geq 3} \left( \sigma_M(\alpha_1 \land \alpha_k) + \sigma_M(\alpha_2 \land \alpha_k) \right) \right)
+ \left( \frac{\lambda^2}{(1 + \lambda^2)^2} - \frac{\mu^2}{(1 + \mu^2)^2} \right)
\times \left( \text{Ric}_M(\alpha_1, \alpha_1) - \text{Ric}_M(\alpha_2, \alpha_2) + \sum_{k \geq 3} \left( \sigma_M(\alpha_1 \land \alpha_k) - \sigma_M(\alpha_2 \land \alpha_k) \right) \right)
+ \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \sigma_M(\alpha_1 \land \alpha_2).
\]
Observe now that
\[
\frac{\lambda^2}{(1 + \lambda^2)^2} - \frac{\mu^2}{(1 + \mu^2)^2} = \frac{(\lambda^2 - \mu^2)(1 - \lambda^2 \mu^2)}{(1 + \lambda^2)^2(1 + \mu^2)^2} = \frac{\lambda^2 - \mu^2}{(1 + \lambda^2)(1 + \mu^2)} \cdot \frac{p}{2}.
\]
Using the identities
\[
\sum_{k \geq 3} \sigma_M(\alpha_1 \land \alpha_k) = \text{Ric}_M(\alpha_1, \alpha_1) - \sigma_M(\alpha_1 \land \alpha_2)
\]
and
\[
\sum_{k \geq 3} \sigma_M(\alpha_2 \land \alpha_k) = \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \land \alpha_2)
\]
we obtain that
\[ C_1 = 2 \left( \frac{\lambda^2}{(1 + \lambda^2)^2} + \frac{\mu^2}{(1 + \mu^2)^2} \right) \]
\[ \times \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) \right) \]
\[ + \frac{(\lambda^2 - \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) - \text{Ric}_M(\alpha_2, \alpha_2) \right) \]
\[ + \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \sigma_M(\alpha_1 \wedge \alpha_2). \]

By our choice of the local frames, the last term \( C_2 \) in (5.9) is given by
\[ C_2 = 2 \sum_{k,l=1}^{m} (f^* R_N)_{kkl}(1 + S_{kl}) \]
\[ = 2(2 + p)R_N(df(e_1), df(e_2), df(e_1), df(e_2)). \]

Hence,
\[ C_2 = \frac{(4 + 2 p) \lambda^2 \mu^2 \sigma_N}{(1 + \lambda^2)(1 + \mu^2)} \]
\[ = 2 \sigma_N \left( \frac{\lambda^2}{(1 + \lambda^2)^2} + \frac{\mu^2}{(1 + \mu^2)^2} \right) - \frac{\lambda^2 + \mu^2}{(1 + \lambda^2)(1 + \mu^2)} p. \]

where \( \sigma_N \) is the sectional curvature of \( N \). Hence, the crucial quantity \( Q \) in (5.2) can be written in the simplified form
\[ Q = 2 \left( \frac{\lambda^2}{(1 + \lambda^2)^2} + \frac{\mu^2}{(1 + \mu^2)^2} \right) \]
\[ \times \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right) \]
\[ + \frac{(\lambda^2 - \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) - \text{Ric}_M(\alpha_2, \alpha_2) \right) \]
\[ + \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \left( \sigma_M(\alpha_1 \wedge \alpha_2) + \sigma_N \right) \]

or, equivalently,
\[ Q = \left( \frac{2\lambda^2}{(1 + \lambda^2)^2} + \frac{2\mu^2}{(1 + \mu^2)^2} - \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \right) \]
\[ \times \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right) \]
\[ + \frac{(\lambda^2 - \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) - \text{Ric}_M(\alpha_2, \alpha_2) \right) \]
\[ + \frac{(\lambda^2 + \mu^2) p}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) \right). \]
From the last identity, we deduce that

\[
Q = \frac{4\lambda^2\mu^2(2 + \lambda^2 + \mu^2)}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right)
+ \frac{2\lambda^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_1, \alpha_1) + \frac{2\mu^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_2, \alpha_2)
+ \alpha \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right)
+ \frac{2\lambda^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_1, \alpha_1) + \frac{2\mu^2 p}{(1 + \lambda^2)(1 + \mu^2)} \text{Ric}_M(\alpha_2, \alpha_2).
\] (5.10)

Combining (5.9) and (5.10), we obtain the desired evolution equation (5.1) for \( p \).

**Lemma 5.2.** Let \( M \) be a compact Riemannian manifold of dimension \( m > 1 \) and let \( N \) be a complete Riemannian surface of bounded geometry. Suppose they satisfy the main curvature condition (A). Let \([0, T)\) denote the maximal time interval on which the smooth solution of the mean curvature flow \( \{F_t\}_{t \in [0, T]} : M \to M \times N \) exists, with the initial condition given by \( F_0 = \text{Id}_M \times f_0 \), and where \( f_0 : M \to N \) is a strictly area decreasing map. Then the following hold:

(a) The flow remains graphical for all \( t \in [0, T) \).

(b) There exist constants \( c_0, c_1 > 0 \) depending on \( f_0 \) such that

\[
p \geq \frac{2c_0 e^{\varepsilon_0 t}}{\sqrt{1 + c_0^2 e^{2\varepsilon_0 t}}} \quad \text{and} \quad |df_t|_{g_M}^2 \leq c_1 e^{-\varepsilon_0 t},
\] (5.11)

where \( f_t : M \to N \) are the smooth maps induced by \( F_t \) and where the constant \( \varepsilon_0 \) is defined by

\[
\varepsilon_0 := \begin{cases} 
\frac{1}{4} \min_M \text{Ric}_M, & \text{if } \min_M \text{Ric}_M \geq 0, \\
\frac{1}{2} \min_M \text{Ric}_M, & \text{if } \min_M \text{Ric}_M < 0.
\end{cases}
\]

In particular, if \( \text{Ric}_M \geq 0 \) or if \( T < \infty \), then the smooth family \( \{f_t\}_{t \in [0, \infty)} \) remains uniformly strictly area decreasing and uniformly bounded in \( C^1(M, N) \) for all \( t \in [0, T) \).

**Proof.** From the compactness of \( M \), it follows that the evolving submanifolds will stay graphical at least on some time interval \([0, T_g)\) with \( 0 < T_g \leq T \). More precisely, there exist smooth families of diffeomorphisms \( \{\varphi_t\}_{t \in [0, T_g)} \subset \text{Diff}(M) \) and maps \( \{f_t\}_{t \in [0, T_g)} : M \to N \) such that \( F_t \circ \varphi_t = \text{Id}_M \times f_t \), for any \( t \in [0, T_g) \). They are given by \( \varphi_t = \pi_M \circ F_t \) and \( f_t = \pi_N \circ F_t \).

The function \( \varphi : M \times [0, T_g) \to \mathbb{R} \), given by \( \varphi(t) := \min \{ p(x, t) : x \in M \} \), is continuous. Since \( f_0 \) is strictly area decreasing and \( M \) is compact, we have \( \varphi_0 := \varphi(0) > 0 \). Let \( T_a \leq T_g \) denote the maximal time such that \( \varphi(t) > 0 \) for all \( t \in [0, T_a) \).

The inequality

\[
1 - \frac{p^2}{4} \leq \frac{2(\lambda^2 + \mu^2)}{(1 + \lambda^2)(1 + \mu^2)} \leq 2 \left( 1 - \frac{p^2}{4} \right)
\] (5.12)
is elementary and, together with the curvature assumption (A), it implies that the quantity $Q$ in equation (5.2) can be estimated by

$$Q \geq \varepsilon_0 p(4 - p^2),$$

where

$$\varepsilon_0 := \begin{cases} \frac{1}{4} \min_M \text{Ric}_M, & \text{if } \min_M \text{Ric}_M \geq 0, \\ \frac{1}{2} \min_M \text{Ric}_M, & \text{if } \min_M \text{Ric}_M < 0. \end{cases}$$

From the evolution equation (5.11) for $p$, we derive the following estimate

$$(\nabla \partial_t - \Delta) p \geq \varepsilon_0 p(4 - p^2) - \frac{1}{2} |\nabla p|^2, \quad \text{on } M \times [0, T_0).$$

Therefore the parabolic maximum principle shows that on $M \times [0, T_0)$ we get the first estimate in (5.11), namely

$$p \geq \frac{2c_0 e^{\varepsilon_0 t}}{\sqrt{1 + c_0^2 e^{2\varepsilon_0 t}}},$$

where $c_0$ is the positive constant determined by $2c_0/\sqrt{1 + c_0^2} = \varrho_0$. Therefore $p$ cannot become zero in finite time and in particular $T_0 = T_0$. Moreover, we have

$$\frac{1 - \mu^2}{1 + \mu^2} = p - 1 - \lambda^2 \geq p - 1 \geq \frac{2c_0 e^{\varepsilon_0 t}}{1 + c_0^2 e^{2\varepsilon_0 t}} - 1,$$

and since $\mu$ denotes the largest singular value, we get

$$|d t|^2_{g_M} = \lambda^2 + \mu^2 \leq 2 \mu^2 \leq 2 \frac{\sqrt{1 + c_0^2 e^{2\varepsilon_0 t}} - c_0 e^{\varepsilon_0 t}}{c_0 e^{\varepsilon_0 t}} = 2 \sqrt{1 + \frac{1}{c_0^2 e^{2\varepsilon_0 t}} - 2} \leq \frac{2}{c_0} e^{-\varepsilon_0 t},$$

from which we obtain the second estimate in (5.11), now with $c_1 := 2/c_0$. It is well-known that the mean curvature flow stays graphical as long as the maps $f_t$ stay bounded in $C^1$. Therefore our estimate implies $T_0 = T$. \hfill \Box

### 5.2. Estimates for the mean curvature

To obtain long-time existence of the flow one needs $C^2$-estimates. To derive such estimates we first prove an estimate on the mean curvature.

**Lemma 5.3.** At points where the mean curvature $H$ is non-zero, we have

$$(\nabla \partial_t - \Delta) |H|^2 \leq -2 |\nabla |H|^2 |^2 + 2 |A|^2 |H|^2 + \mathcal{R},$$

(5.13)

where $\mathcal{R}$ is the quantity given by

$$\mathcal{R} = \frac{2\lambda^2 H^2 |H|^2}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right)$$

$$+ 2 \text{Ric}_M(v, v) - \frac{2\lambda^2 H^2 |H|^2}{(1 + \lambda^2)(1 + \mu^2)} \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) \right)$$

$$+ 2\sigma_N |w|^2. \quad (5.14)$$

Here, the vectors $v, w$ are given by

$$v := \frac{\lambda H^\xi}{\sqrt{1 + \lambda^2}} \alpha_1 + \frac{\mu H^n}{\sqrt{1 + \mu^2}} \alpha_2 \quad \text{and} \quad w := -\frac{\lambda H^n}{\sqrt{1 + \lambda^2}} \alpha_1 + \frac{\mu H^\xi}{\sqrt{1 + \mu^2}} \alpha_2, \quad (5.15)$$

where $\xi = \frac{1}{2} \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2)$ and $\mu$ is the smallest mean curvature.
where \( \{\alpha_1, \ldots, \alpha_m\}, \{\xi, \eta\} \) are the special bases arising from the singular value decomposition defined in subsection 4.3.

**Proof.** Recall that [33, Corollary 3.8] the squared norm of the mean curvature vector evolves in time according to the equation

\[
(\nabla_{\partial_t} - \Delta)|H|^2 = -2|\nabla \perp H|^2 + 2|A^H|^2 + 2 \sum_{k=1}^{m} R_{M \times N}(dF(e_k), H, dF(e_k), H). \tag{5.16}
\]

From the Cauchy-Schwarz inequality we have

\[
|A^H|^2 \leq |A|^2 |H|^2.
\]

Moreover, at points where \( H \neq 0 \), we have

\[
|\nabla |H|^2 |^2 = 4|H|^2 |\nabla |H||^2,
\]

and

\[
|\nabla \perp H|^2 = |2(\nabla \perp H, H)|^2 \leq 4|H|^2 |\nabla \perp H|^2.
\]

Hence \( |\nabla \perp H|^2 \geq |\nabla |H||^2 \) and we conclude that

\[
(\nabla_{\partial_t} - \Delta)|H|^2 \leq -2|\nabla |H|^2 |^2 + 2|A|^2 |H|^2 + 2 \sum_{k=1}^{m} R_{M \times N}(dF(e_k), H, dF(e_k), H). \tag{5.17}
\]

Now let us compute the last curvature term in (5.17), which we call \( \mathcal{R} \) in the sequel,

\[
\mathcal{R} := 2 \sum_{k=1}^{m} R_{M \times N}(dF(e_k), H, dF(e_k), H).
\]

Recall that

\[
\xi = -\lambda \alpha_1 \oplus \beta_1 \quad \text{and} \quad \eta = -\mu \alpha_2 \oplus \beta_2.
\]

With these definitions, we compute

\[
\mathcal{R} = 2 \sum_{k=1}^{m} R_{M \times N} \left( \frac{\alpha_k + \lambda \beta_k}{\sqrt{1 + \lambda^2}} H^\xi \xi + H^\eta \eta, \frac{\alpha_k + \lambda \beta_k}{\sqrt{1 + \lambda^2}} H^\xi \xi + H^\eta \eta \right)
\]

\[
= \sum_{k=1}^{m} \frac{2}{1 + \lambda_k^2} R_M \left( \frac{\alpha_k}{\sqrt{1 + \lambda_k^2}} \frac{\lambda H^\xi \xi}{\sqrt{1 + \lambda^2}} \alpha_k + \frac{\mu H^\eta \eta}{\sqrt{1 + \mu^2}} \alpha_k, \frac{\lambda H^\xi \xi}{\sqrt{1 + \lambda^2}} \alpha_k + \frac{\mu H^\eta \eta}{\sqrt{1 + \mu^2}} \alpha_k \right)
\]

\[
= \mathcal{D}_1
\]

\[
+ \sum_{k=1}^{2} \frac{2 \lambda_k^2}{1 + \lambda_k^2} R_N \left( \frac{\beta_k}{\sqrt{1 + \lambda_k^2}} \frac{H^\xi \xi}{\sqrt{1 + \lambda^2}} \beta_k + \frac{H^\eta \eta}{\sqrt{1 + \mu^2}} \beta_k, \frac{H^\xi \xi}{\sqrt{1 + \lambda^2}} \beta_k + \frac{H^\eta \eta}{\sqrt{1 + \mu^2}} \beta_k \right)
\]

\[
= \mathcal{D}_2
\]

For \( \mathcal{D}_2 \) we get

\[
\mathcal{D}_2 = \frac{2 \lambda^2 |H^\eta|^2 + 2 \mu^2 |H^\xi|^2}{(1 + \lambda^2)(1 + \mu^2)} \sigma_N.
\]
In the next step we compute \( D_1 \), and use \( v \) defined as in [5.15] to obtain
\[
D_1 = 2\mu^2|H|^2 + \lambda^2|H|^{\xi^2} \sigma_M (\alpha_1 \wedge \alpha_2) + 2|v|^2 \sum_{k \geq 3} R_M \left( \frac{v}{|v|}, \alpha_k, \frac{v}{|v|} \right)
\]
\[
= 2\mu^2|H|^2 + \lambda^2|H|^{\xi^2} \sigma_M (\alpha_1 \wedge \alpha_2) + 2|v|^2 \left\{ \frac{\text{Ric}_M \left( \frac{v}{|v|}, \frac{v}{|v|} \right) - \sigma_M (\alpha_1 \wedge \alpha_2)}{1 + \lambda^2 (1 + \mu^2)} \right\}
\]
\[
= 2 \text{Ric}_M (v, v) + 2\sigma_M (\alpha_1 \wedge \alpha_2) \left\{ \frac{\mu^2|H|^2 + \lambda^2|H|^{\xi^2}}{1 + \lambda^2 (1 + \mu^2)} - \frac{\lambda^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} + \mu^2|H|^2 \right\}
\]
\[
= 2 \text{Ric}_M (v, v) - \frac{2\lambda^2 \mu^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} \sigma_M (\alpha_1 \wedge \alpha_2) + \frac{2\lambda^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} \sigma_N
\]

Thus
\[
R = 2 \text{Ric}_M (v, v) - \frac{2\lambda^2 \mu^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} \sigma_M (\alpha_1, \alpha_2) + \frac{2\lambda^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} \sigma_N
\]
\[
+ 2 \text{Ric}_M (v, v) - \frac{2\lambda^2 \mu^2|H|^2}{1 + \lambda^2 (1 + \mu^2)} \left( \text{Ric}_M (\alpha_1, \alpha_1) + \text{Ric}_M (\alpha_2, \alpha_2) - \sigma_M (\alpha_1 \wedge \alpha_2) - \sigma_N \right)
\]
\[
+ 2\sigma_N \frac{\lambda^2|H|^2 + \mu^2|H|^{\xi^2} + \lambda^2 \mu^2|H|^2}{1 + \lambda^2 (1 + \mu^2)}
\]

This proves the lemma. \( \square \)

**Lemma 5.4.** Let us assume the main curvature condition [A]. At points where the mean curvature \( H \) is non-zero, the function \( \Theta := p^{-1} \langle H \rangle^2 \) satisfies
\[
(\nabla_{\partial_t} - \Delta) \Theta \leq p^{-1} \langle \nabla \Theta, \nabla p \rangle - 2p^{-1} \left( \text{Ric}_M (w, w) - \sigma_N |w|^2 \right), \tag{5.18}
\]
where \( w \) is defined as in [5.15].

**Proof.** Note that, from the area decreasing property, the evolution equation for \( p \), and from [A] we get
\[
(\nabla_{\partial_t} - \Delta) p \geq -\frac{1}{2p} |\nabla p|^2 + 2|A|^2 p
\]
\[
+ \frac{2\lambda^2 \mu^2 p}{1 + \lambda^2 (1 + \mu^2)} \left( \text{Ric}_M (\alpha_1, \alpha_1) + \text{Ric}_M (\alpha_2, \alpha_2) - \sigma_M (\alpha_1 \wedge \alpha_2) - \sigma_N \right)
\]
\[
+ \frac{2p}{1 + \lambda^2 (1 + \mu^2)} \left( \lambda^2 \text{Ric}_M (\alpha_1, \alpha_1) + \mu^2 \text{Ric}_M (\alpha_2, \alpha_2) \right). \tag{5.19}
\]
Then [5.19], [5.13] and the formula
\[
(\nabla_{\partial_t} - \Delta) \Theta - 2p^{-1} \langle \nabla p, \nabla \Theta \rangle = p^{-1} (\nabla_{\partial_t} - \Delta) |H|^2 - p^{-2} |H|^2 (\nabla_{\partial_t} - \Delta) p
\]
imply, after some cancellations, that at points where $H \neq 0$, it holds

\[
(\nabla_{\partial_t} - \Delta) \Theta - 2 p^{-1} \langle \nabla \Theta, \nabla p \rangle \leq -2 p^{-1} |\nabla H|^2 + \frac{1}{2} p^{-3} |H|^2 |\nabla p|^2 + 2 \Theta \left( \frac{\text{Ric}_M(v,v) + \text{Ric}_N(w)^2}{|H|^2} - \frac{\lambda^2}{1 + \lambda^2} \text{Ric}_M(\alpha_1, \alpha_1) - \frac{\mu^2}{1 + \mu^2} \text{Ric}_M(\alpha_2, \alpha_2) \right). \tag{5.20}
\]

Note that, the term $\mathcal{E}$ is of the form $\mathcal{E} = 2 \Theta \mathcal{F}$ and that $\mathcal{F}$ might vanish at some points, for example, if $\lambda = \mu = 0$ or if $\lambda = |H| = 0$. This shows that we cannot expect the estimate $\mathcal{F} < 0$ to hold in general. Since we assume $H \neq 0$, the two gradient terms in the first line of (5.20) can be combined and this gives

\[
-2 p^{-1} |\nabla H|^2 + \frac{1}{2} p^{-3} |H|^2 |\nabla p|^2 = -\frac{1}{2} \Theta^{-1} |\nabla \Theta|^2 - p^{-1} \langle \nabla \Theta, \nabla p \rangle. \tag{5.21}
\]

From the definition of $v, w$ in (5.15), we get

\[
\text{Ric}_M(v,v) + \text{Ric}_M(w,w)
= \frac{\lambda^2 |H|^2}{1 + \lambda^2} \text{Ric}_M(\alpha_1, \alpha_1) + 2 \frac{\lambda \mu |H|^2 H^\xi}{1 + \lambda^2} \frac{\mu^2 |H|^2}{1 + \mu^2} \text{Ric}_M(\alpha_1, \alpha_1) + 2 \frac{\lambda \mu |H|^2 H^\eta}{1 + \lambda^2} \frac{\mu^2 |H|^2}{1 + \mu^2} \text{Ric}_M(\alpha_1, \alpha_1)
+ \lambda^2 |H|^2 \frac{\mu^2 |H|^2}{1 + \mu^2} \text{Ric}_M(\alpha_2, \alpha_2)
+ \lambda^2 |H|^2 \frac{\mu^2 |H|^2}{1 + \mu^2} \text{Ric}_M(\alpha_2, \alpha_2)

\]

Therefore, together with (5.21), we can simplify (5.20) and finally obtain the desired inequality for $\Theta$. \hfill \Box

Now observe that

\[
|w|^2 = \frac{\lambda^2}{1 + \lambda^2} |H|^2 + \frac{\mu^2}{1 + \mu^2} |H|^2
= \frac{\mu^2}{1 + \mu^2} |H|^2 + \left( \frac{\lambda^2}{1 + \lambda^2} - \frac{\mu^2}{1 + \mu^2} \right) |H|^2
= \frac{\lambda^2 - \mu^2}{(1 + \lambda^2)(1 + \mu^2)} |H|^2
\leq \frac{\mu^2}{1 + \mu^2} |H|^2 \leq |H|^2.
\]

Let

\[
\varepsilon_1 := \sup_N \sigma_N - \min_{|w|=1} \langle \text{Ric}_M(u,u) \rangle.
\tag{5.22}
\]

Then, at points where $H \neq 0$, inequality (5.18) implies the estimate

\[
(\nabla_{\partial_t} - \Delta) \Theta \leq p^{-1} \langle \nabla \Theta, \nabla p \rangle + 2 \max \{0, \varepsilon_1\} \Theta. \tag{5.23}
\]

Applying the maximum principle to (5.23), taking into account Lemma 5.2 and the fact that $p \leq 2$, we immediately obtain the following estimate for the mean curvature.
Lemma 5.5. Let $M$ be a compact Riemannian manifold $M$ of dimension $m > 1$ and let $N$ be a complete Riemannian surface of bounded geometry. Suppose they satisfy the main curvature condition $[\mathcal{A}]$. Let $[0, T)$ denote the maximal time interval on which the smooth solution of the mean curvature flow $\{F_t\}_{t \in [0, T)} : M \to M \times N$ exists, with the initial condition given by $F_0 = \text{Id}_M \times f_0$, and where $f_0 : M \to N$ is a strictly area decreasing map. Then the following hold:

(a) The function $\Theta := |H|^2 / p$ is well-defined for all $t \in [0, T)$ and it satisfies the estimate

$$\Theta \leq \max_{t=0} \Theta \cdot e^{2 \max \{0, \varepsilon_1\} t}, \text{ for all } t \in [0, T),$$

where $\varepsilon_1$ is the constant defined in (5.22).

(b) There exists a constant $a_0 > 0$, depending only on $f_0$, such that

$$|H|^2 \leq a_0 \cdot e^{2 \max \{0, \varepsilon_1\} t}, \text{ for all } t \in [0, T).$$

In particular, if $\text{Ric}_M \geq \sup_N \sigma_N$, then $|H|^2 \leq a_0$ for all $t \in [0, T)$.

Inequality (5.25) will be exploited in the next section to obtain long-time existence of the flow. Once long-time existence is known, the uniform estimate on $H$ in the case $\text{Ric}_M \geq \sup_N \sigma_N$ will be used to derive uniform $C^k$-bounds on $f$ for all $k \geq 1$ and $t \in [0, \infty)$.

6. The barrier theorem and an entire graph lemma

In the proof of the main theorem, we will need the following barrier theorem that generalizes the well-known barrier theorem for mean curvature flow of hypersurfaces to any codimension. Before we state and prove it, we recall the definition of $m$-convexity.

**Definition 6.1.** A smooth function $\phi : P \to \mathbb{R}$ on a Riemannian manifold $(P, g_P)$ of dimension $p \geq m$ is called $m$-convex at $y \in P$, if the Hessian $D^2 \phi$ of $\phi$ at $y$ satisfies

$$\sum_{k=1}^m D^2 \phi(e_k, e_k) \geq 0$$

for any choice of $m$ orthonormal vectors $\{e_1, \ldots, e_m\} \in T_y P$.

**Theorem H** (Barrier theorem for the mean curvature flow).

Let $F_t : M \to (P, g_P)$, $t \in [0, T)$ be a mean curvature flow of a compact manifold $M$ of dimension $m$ into a complete Riemannian manifold $(P, g_P)$ of dimension $p$. Suppose that $\phi : P \to \mathbb{R}$ is a smooth function and $c \in \mathbb{R}$ a constant such that $\phi$ is $m$-convex on the sub-level set

$$P^c := \{ y \in P : \phi(y) < c \}.$$

If the initial image $F_0(M)$ is contained in $P^c$, then $F_t(M) \subset P^c$ for all $t \in [0, T)$.

**Proof.** We define the function $\omega : M \times [0, T) \to \mathbb{R}$, given by

$$\omega(x, t) := \phi(F_t(x)).$$
Since $\partial_t F_t = H_t$ we get

$$\partial_t \omega = D\phi(H_t)$$

and moreover

$$\Delta \omega = \text{trace}_{g_t}(F_t^*D^2\phi) + D\phi(H_t),$$

where $\Delta$ denotes the Laplace-Beltrami operator on $M$ with respect to the induced metric $g_t = F_t^*g_P$. Thus

$$\partial_t \omega = \Delta \omega - \text{trace}_{g_t}(F_t^*D^2\phi).$$

Since $\phi$ is $m$-convex on $P^c$ and $F_t$ is an immersion, we see that

$$\text{trace}_{g_t}(F_t^*D^2\phi) \geq 0$$

as long as $F_t(M) \subset P^c$. Now, since $M$ is compact and $P^c$ is open, we observe that $F_t(M) \subset P^c$ will hold on some maximal time interval $[0, t_0) \subset [0, T)$. It remains to show that $t_0 = T$. Assume $t_0 < T$. By continuity, we have the estimate

$$\partial_t \omega \leq \Delta \omega$$

on the interval $[0, t_0]$. Then the strong parabolic maximum principle implies that $\omega < c$ on all of $[0, t_0]$ which gives $F_{t_0}(M) \subset P^c$. This contradicts the maximality of $t_0$. Thus $t_0 = T$ and $F_t(M) \subset P^c$ for all $t \in [0, T)$.

□

Example 6.2. For $m > 1$ a convex function is $m$-convex. However, the converse does not hold. For example, the function $\phi(x, y, z, w) = x^2 + y^2 + z^2 - w^2$ is 2-convex on $\mathbb{R}^4$ but not convex. Therefore, any compact surface $M^2 \subset \mathbb{R}^4$ immersed into the interior domain bounded by the one-sheet 3-dimensional hyperboloid $H = \{(x, y, z, w) : x^2 + y^2 + z^2 - w^2 = c\}$, $c > 0$, will stay in that interior domain under mean curvature flow.

Remark 6.3. As we pointed out in Remark 2.2, the long-time existence of the mean curvature flow does not ensure smooth convergence, since for example the evolving submanifolds might slide off to infinity. However, in some situations, the geometry of the ambient space forces the submanifolds to stay in a compact region. For instance, if the ambient space possesses a compact totally convex set $\mathcal{C}$, then we can use Theorem 14 with $\phi$ chosen as the squared distance function to $\mathcal{C}$ to show that the flow stays in a compact region. Recently, Tsai and Wang [35] introduced the notion of strongly stable minimal submanifolds. They proved that if $\Sigma$ is an $m$-dimensional compact strongly stable minimal submanifold of a Riemannian manifold $P$, then the squared distance function to $\Sigma$ is $m$-convex in a tubular neighbourhood of $\Sigma$. Moreover, if $\Gamma$ is a compact $m$-dimensional submanifold that is $C^1$-close to $\Sigma$, then the mean curvature flow $\Gamma_t$ with $\Gamma_0 = \Gamma$ exists for all time, and $\Gamma_t$ smoothly converges to $\Sigma$ as $t \to \infty$. We refer also to the work of Lotay and Schulze [15] for further generalizations and applications of the stability result in [35].

The next lemma turns out to be very useful and it is a direct consequence of the preceding barrier theorem.

Lemma 6.4. Let $(M, g_M)$ be a compact and $(N, g_N)$ a complete Riemannian manifold. Suppose $\{f_t\}_{t \in [0, \infty)}$ is uniformly bounded in $C^k(M, N)$, for all $k \geq 1$, and their graphs evolve by mean curvature flow. If there exists a sequence of times $\{t_n\}_{n \in \mathbb{N}}$, with $\lim_{n \to \infty} t_n = \infty$, such that the sequence $\{f_{t_n}\}_{n \in \mathbb{N}}$ converges in $C^0(M, N)$ to a constant map $f_\infty : M \to N$, then the whole flow $\{f_t\}_{t \in (0, \infty)}$ smoothly converges to $f_\infty$. 
Proof. Let $F_t : M \to M \times N$ denote the mean curvature flow of $F_0 := \text{Id}_M \times f_0$. Then $f_t = \pi_N \circ F_t$, where $\pi_N : M \times N \to N$ is the projection onto the second factor. By assumption, there exist a point $y \in N$ and a sequence $\{t_n\}_{n \in \mathbb{N}}$, with $\lim_{n \to \infty} t_n = \infty$, such that
\[
\lim_{n \to \infty} \text{dist}_N(y, f_{t_n}(x)) = 0 \text{ for all } x \in M,
\]
where $\text{dist}_N$ denotes the distance function on $N$. Let $\mathcal{B}(y, r)$ be the geodesic ball of $N$ with radius $r$ centered at the point $y \in N$, and denote by $\varrho_y : \mathcal{B}(y, r) \to \mathbb{R}$ the function given by
\[
\varrho_y(z) := \text{dist}_N(y, z).
\]
For sufficiently small $r > 0$, $\varrho_y$ is smooth and strictly convex on $\mathcal{B}(y, r)$. Since $M$ is compact, the sets $f_{t_n}(M)$ uniformly tend to $\{y\}$ as $n \to \infty$. Therefore, for any $j \in \mathbb{N}$ there exists a sufficiently large time $t_{n_j}$ such that the image $f_{t_{n_j}}(M)$ is contained in the geodesic ball $\mathcal{B}(y, r/j)$; see Figure 2. For a fixed $j$, define the compact set $C_j \subset N$ by
\[
C_j := \mathcal{B}(y, r/j).
\]
Then the function
\[
\phi := \varrho_y \circ \pi_N : M \times N \to \mathbb{R}, \quad \phi(x, z) = \varrho_y(z)
\]
is smooth and convex on its sub-level set
\[
P^{r/j} := M \times C_j = \{(x, z) \in P := M \times N : \phi(x, z) \leq r/j\}.
\]
Applying the barrier theorem to $\phi$, we see that $F_t(M) \subset P^{r/j}$ for all $t \geq t_{n_j}$, which is equivalent to $f_t(M) \subset C_j$ for all $t \geq t_{n_j}$. This proves
\[
\lim_{t \to \infty} \text{dist}_N(y, f_t(x)) = 0 \text{ for all } x \in M,
\]
that is, $\{f_t\}_{t \in [0, \infty)}$ converges uniformly in $C^0(M, N)$ to the constant map $f_\infty : M \to N$, $f_\infty \equiv y$. Thus $\{f_t\}_{t \in [0, \infty)}$ is uniformly bounded in $C^k(M, N)$, for all $k \geq 0$. We claim that this implies
\[
\lim_{t \to \infty} \|f_t\|_{C^k(M, N)} = 0, \text{ for all } k \geq 1.
\]
Indeed, if this does not hold, then there exist $k \geq 1$, $\varepsilon > 0$ and a sequence $\{t_n\}_{n \in \mathbb{N}}$ with $\lim_{t \to \infty} t_n = \infty$ such that
\[
\|f_{t_n}\|_{C^k(M, N)} \geq \varepsilon, \text{ for all } n \in \mathbb{N}.
\]
Since $\{f_{t_n}\}_{n \in \mathbb{N}}$ is uniformly bounded in $C^k(M, N)$, for all $k \geq 0$, the Arzelà-Ascoli Theorem implies that there exists a subsequence $\{f_{t_{n_j}}\}_{j \in \mathbb{N}}$ smoothly converging to a limit map
\[
\hat{f} : M \to N.
\]
But the same subsequence already converges in $C^0(M, N)$ to $f_\infty$, so the map $\hat{f}$ must coincide with $f_\infty$. Therefore
\[
\varepsilon \leq \lim_{j \to \infty} \|f_{t_{n_j}}\|_{C^k(M, N)} = \|f_\infty\|_{C^k(M, N)} = 0,
\]
because $Df_\infty = 0$ and $k \geq 1$. This contradicts the choices of $k, \varepsilon$ and $\{t_n\}_{n \in \mathbb{N}}$. 

We will also need the following elementary lemma.

Lemma 6.5 (Entire graph lemma). Let $f : \Omega \to \mathbb{R}^n$ be a smooth map on an open domain $\Omega \subset \mathbb{R}^m$ and $C^1$-bounded. Then the graph $\Gamma_f$ is complete if and only if $f$ is entire, that is $\Omega = \mathbb{R}^m$. 


Proof. Let us consider the two metric spaces \((\Omega, d_{\text{euc}})\) and \((\Gamma_f, d_g)\), where \(d_{\text{euc}}\) denotes the usual euclidean distance function, and \(d_g\) is the distance function on the graph \(\Gamma_f\), induced by its Riemannian metric \(g\). By the Hopf-Rinow theorem, the metric space \((\Gamma_f, d_g)\) is complete if and only if \((\Gamma_f, g)\) is a complete Riemannian manifold. Moreover, since \(\Omega\) is an open domain, and because complete sets are closed, the metric space \((\Omega, d_{\text{euc}})\) is complete if and only if \(\Omega = \mathbb{R}^m\). To conclude the proof we need to show that the metric space \((\Omega, d_{\text{euc}})\) is complete if and only if \((\Gamma_f, d_g)\) is complete. On one hand, the map

\[
F := \text{Id}_\Omega \times f : \Omega \to \Gamma_f
\]

provides a homeomorphism between these metric spaces, with its inverse being the projection \(\pi : \Gamma_f \to \Omega\). Since \(\pi\) is a homeomorphism, we deduce that the metric space \((\Omega, d_g)\) is complete if and only if \((\Gamma_f, g)\) is complete. Hence, \((\Omega, d_{\text{euc}})\) is complete if and only if \((\Gamma_f, d_g)\) is complete.

Remark. There exist complete graphs \(\Gamma_f\) that are not entire if one drops the condition on the \(C^1\)-boundedness of \(f\).

7. Proof of main results

In this section, we will prove the main results stated in Theorems [1] and [2]. To this end, we need to recall the blow-up analysis of singularities in the mean curvature flow.

7.1. Blow-up analysis.

The following blow-up procedure is well-known; for example see [8].

**Proposition 7.1** (Blow-up limit). Let \(M\) be a \(m\)-dimensional compact manifold and suppose that \(F : M \times [0, T) \to P\) is a solution of the mean curvature flow, where \((P, g_P)\) is a Riemannian manifold of dimension \(p\) with bounded geometry, and \(T \leq \infty\) its maximal time of existence. Suppose that there exists a point \(x_\infty \in M\), and a sequence \(\{(x_j, t_j)\}_{j \in \mathbb{N}}\) of points in \(M \times [0, T)\) with \(\lim x_j = x_\infty\), \(\lim t_j = T\), such that

\[
|A(x_j, t_j)| = \max_{(x, t) \in M \times [0, t_j]} |A(x, t)| =: a_j \to \infty.
\]
Consider the family of maps $F_j : M \times [L_j, R_j] \to (P, a_j^2 g_P)$, $j \in \mathbb{N}$, given by

$$F_j(x, s) := F_{j,s}(x) := F(x, s/a_j^2 + t_j),$$

where

$$L_j := -a_j^2 t_j \quad \text{and} \quad R_j := \begin{cases} a_j^2(T - t_j) & \text{if } T < \infty \\ \infty & \text{if } T = \infty. \end{cases}$$

Then the following hold:

(a) For each $j \in \mathbb{N}$, the family of maps $\{F_{j,s}\}_{s \in [L_j, R_j]}$ evolves by mean curvature flow in time $s$. The second fundamental forms $A_j$ of $F_j$ satisfy the identities

$$A_j(x, s) = A(x, s/a_j^2 + t_j) \quad \text{and} \quad |A_j(x, s)| = a_j^{-1} |A(x, s/a_j^2 + t_j)|,$$

Moreover, for any $s \leq 0$, we have $|A_j| \leq 1$ and $|A_j(x, 0)| = 1$, for any $j \in \mathbb{N}$.\(^1\)

(b) For any $s \leq 0$, the sequence of pointed Riemannian manifolds $\{(M_j, F_{j,s}(a_j^2 g_P), x_j)\}_{j \in \mathbb{N}}$ smoothly subconverges in the Cheeger-Gromov sense to a connected complete pointed Riemannian manifold $(M_\infty, g_\infty(s), x_\infty)$, where the manifold $M_\infty$ does not depend on the choice of $s$. Moreover, the sequence of pointed manifolds $\{(P, a_j^2 g_P, F_j(x, s))\}_{j \in \mathbb{N}}$ smoothly subconverges to the standard euclidean space $(\mathbb{R}^p, g_{\text{eucl}}, 0)$.

(c) There is an ancient solution $F_\infty : M_\infty \times (-\infty, 0] \to \mathbb{R}^p$ of the mean curvature flow such that, for each fixed time $s \leq 0$, the sequence $\{F_{j,s}\}_{j \in \mathbb{N}}$ smoothly subconverges in the Cheeger-Gromov sense to $F_{\infty,s}$. This convergence is uniform with respect to the parameter $s$. Additionally, $|A_{F_\infty}| \leq 1$ and $|A_{F_\infty}(x_\infty, 0)| = 1$.

(d) If $T = \infty$, then $R_j = \infty$. If $T < \infty$ and the singularity is of type-II, then $R_j \to \infty$. In both cases, the limiting mean curvature flow $F_\infty$ can be constructed on the whole time axis $(-\infty, \infty)$, hence it gives an eternal solution of the mean curvature flow.

7.2. Proofs of Theorem A and Theorem F

We are now ready to prove our main results starting with the proof of Theorem F.

Proof of Theorem F.

Let $(M, g_M)$ be a compact Riemannian manifold of dimension $m \geq 1$ and let $(N, g_N)$ be a complete Riemannian surface such that (A) and (B) hold, that is we have

$$\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,$$

$$\text{Ric}_M(v, v) \geq 0,$$

for orthonormal tangent vectors $v, w$. Suppose $f : M \to N$ is a strictly area decreasing minimal map. Then $f$ is totally geodesic, the rank of $df$ and the singular values $\lambda$ and $\mu$ of $f$ are constant.

\(^1\)All the norms are regarded with respect to the Riemannian metrics induced by the corresponding immersions.
Suppose $f_{\min}: M \to N$ is a smooth and strictly area decreasing minimal map. Since $p > 0$, $H = 0$, and $\partial_t p = 0$ we may use the evolution equation (5.1) of $p$ in Lemma 5.1 to conclude
\begin{equation}
\Delta p - \frac{1}{2p} |\nabla p|^2 + 2p |A|^2 + Q \leq 0 \tag{7.1}
\end{equation}

From the curvature conditions (A) and (B), it follows that the term $Q$ in (7.1) is non-negative. Hence
\begin{equation}
\Delta \sqrt{p} + \sqrt{p} |A|^2 = \frac{1}{2\sqrt{p}} \left( \Delta p - \frac{1}{2p} |\nabla p|^2 + 2p |A|^2 \right) \leq 0.
\end{equation}

Integration gives $|A|^2 = 0$ and therefore $f_{\min}$ must be totally geodesic. Once we know that $f_{\min}$ is totally geodesic, equation (5.7) shows that $\nabla S = 0$ and hence the singular values $\lambda, \mu$ must be constant functions on $M$, in particular $p$ is constant. This proves the first part of the theorem. It remains to describe the possible cases.

(a) If $\text{rank}(df) = 0$, then $f$ is constant and $\lambda = \mu = 0$.

(b) If $\text{rank}(df) > 0$, then $f: M \to f(M)$ is a submersion. For $y \in f(M)$ the fibers $K_y$ are compact embedded and totally geodesic submanifolds of codimension $\text{rank}(df)$ in $M$. The fiber $K_y$ is isometric to a compact Riemannian manifold $(K, g_K)$ of non-negative Ricci curvature that does not depend on $y$. The horizontal integral submanifolds are complete totally geodesic submanifolds in $M$ that intersect the fibers orthogonally. The manifold $(M, g_M)$ is locally isometric to a product $(L \times K, g_L \times g_K)$. The Euler characteristic $\chi(M)$ of $M$ vanishes, and at each point $x \in M$ the kernel of the Ricci operator is non-trivial. More precisely, depending on the rank of $df$, we distinguish two cases:

(i) $\text{rank}(df) = 1$. In this case $\lambda = 0$ and $\mu > 0$. Moreover, $\gamma := f(M)$ is a closed geodesic in $N$. The horizontal leaves are geodesics orthogonal to the fibers, and the map $f: (M, g_M) \to (\gamma, \mu^{-2} g_\gamma)$ is a Riemannian submersion, where $g_\gamma$ denotes the metric on $\gamma$ as a submanifold in $(N, g_N)$.

(ii) $\text{rank}(df) = 2$. In this case $\lambda > 0$, $\mu > 0$. The image $f(M)$ coincides with $N$ and $N$ is diffeomorphic to a torus $\mathbb{T}^2$ or to a Klein bottle $\mathbb{T}^2/\mathbb{Z}_2$. The metric $g_N$ and the metrics on the horizontal leaves are flat. Moreover, $f: (M, g_M) \to (N, \mu^{-2} g_N)$ is a Riemannian submersion, if $\lambda = \mu$.

(a) $\text{rank}(df) = 0$. Clearly, $f$ must be constant and $\lambda = \mu = 0$.

(b) $\text{rank}(df) > 0$. Once we know that $f$ is totally geodesic, equation (7.1) implies $Q = 0$. Therefore, from (A), (B), $p > 0$, and from the definition of $Q$ we obtain the following equations:

\begin{align*}
0 &= \lambda^2 \mu^2 \left( \text{Ric}_M(\alpha_1, \alpha_1) + \text{Ric}_M(\alpha_2, \alpha_2) - \sigma_M(\alpha_1 \wedge \alpha_2) - \sigma_N \right), \tag{7.2} \\
0 &= \lambda^2 \text{Ric}_M(\alpha_1, \alpha_1), \tag{7.3} \\
0 &= \mu^2 \text{Ric}_M(\alpha_2, \alpha_2). \tag{7.4}
\end{align*}
We claim that $\mathcal{V} := \ker df$ and $\mathcal{H} := (\ker df) \perp$ are parallel distributions on $M$, where $\mathcal{H}$ is the horizontal distribution given by the orthogonal complement of $\mathcal{V}$ with respect to the graphical metric $g$ on $M$. The distributions are certainly smooth since at each point $x \in M$, the fiber $\mathcal{V}_x$ is the kernel of the smooth bilinear form $S - g$ and the nullity of $S - g$ is fixed, because the eigenvalues of $S$ are constant. Since the second fundamental form $A$ vanishes, equation (4.1) shows that the Levi-Civita connections of $g_M$ and $g$ coincide, that is
\[
\nabla_X Y = \nabla_X^{g_M} Y, \text{ for all } X, Y \in \mathfrak{X}(M).
\] (7.5)
In particular, the geodesics on $M$ with respect to these metrics coincide. Moreover, again by equation (4.1), we get
\[
\nabla_X Y = \nabla_X^{g_M} Y \in \Gamma(\mathcal{V}), \text{ for all } X \in \mathfrak{X}(M) \text{ and all smooth sections } Y \in \Gamma(\mathcal{V}).
\] (7.6)
Using the fact that the connections are metric with respect to $g$ and $g_M$, and that the two distributions are orthogonal to each other with respect to $g$, we see that in addition
\[
\nabla_X Y = \nabla_X^{g_M} Y \in \Gamma(\mathcal{H}), \text{ for all } X \in \mathfrak{X}(M) \text{ and all } Y \in \Gamma(\mathcal{H}).
\] (7.7)
Equations (7.6) and (7.7) imply that both distributions are parallel. In particular, they are involutive. Therefore, we may apply Frobenius’ Theorem, and conclude that for each point $x \in M$ there exist uniquely determined integral leaves $V_x$ of $\mathcal{V}$ and $H_x$ of $\mathcal{H}$. Since the distributions are parallel and orthogonal to each other, the integral leaves $V_x$ and $H_x$ are complete and totally geodesic submanifolds of $M$ that intersect orthogonally in $x$. Since $M$ is compact and the integral leaves $V_x$ are the pre-images $K_y$ of points $y \in f(M)$, $V_x$ must be closed and embedded. Thus, $(M, g_M)$ is locally isometric to the Riemannian product of two manifolds $(L, g_L)$ and $(K, g_K)$ of non-negative Ricci curvature, and $f : M \to f(M)$ is a submersion. The set $f(M)$ is compact, because $M$ is compact and $f$ continuous. Therefore, if rank$(df) = 1$, then $\gamma$ must be a closed 1-dimensional submanifold of $N$, and because $f$ is totally geodesic, this curve must be a geodesic. If rank$(df) = 2$, then $f(M)$ must coincide with $N$, because submersions are open maps and $N$ is connected.

Claim: If rank$(df) = 2$, then the horizontal leaves and $(N, g_N)$ are flat and the surface $N$ is diffeomorphic to a torus $\mathbb{T}^2$ or to a Klein bottle $\mathbb{T}^2/\mathbb{Z}_2$.

Proof of the claim. Since $(M, g_M)$ is locally a product manifold, the tangent vectors $\alpha_1, \alpha_2$ in the singular value decomposition are given by horizontal vectors. From (7.2)-(7.4) we get
\[
\text{Ric}_M(\alpha_1, \alpha_1) = \text{Ric}_M(\alpha_2, \alpha_2) = \sigma_M(\alpha_1 \wedge \alpha_2) = 0.
\]
Since $\mathcal{H}$ is 2-dimensional and totally geodesic, it is flat. To see that $(N, g_N)$ is flat, we use equation (7.2) again, and get $\sigma_N \circ f = 0$. As we have already seen, $f(M) = N$. Therefore $\sigma_N \equiv 0$. This proves the claim. Since $g_N$ is flat and $N$ is compact, we conclude that $N$ is diffeomorphic to a torus $\mathbb{T}^2$ or to a Klein bottle $\mathbb{T}^2/\mathbb{Z}_2$.

In particular, this proves that the kernel of the Ricci operator is non-trivial, because $M$ splits locally into the Riemannian product of the fibers and the horizontal leaves.

If rank$(df) = 1$, and $\alpha$ is a horizontal vector of unit length, then by definition of the singular values $df(\alpha) = \mu \cdot \beta$, for a unit tangent vector $\beta$ to the curve $\gamma$. Therefore, if we equip $\gamma$ with the metric $\mu^{-2} g_\gamma$, then $df$ becomes an isometry. This proves that

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2In this article we assume manifolds are connected.
\( f(M, g_M) \rightarrow (\gamma, \mu^{-2} g_{\gamma}) \) is a Riemannian submersion. In the same way we see that \( f(M, g_M) \rightarrow (N, \mu^{-2} g_N) \) is a Riemannian submersion, if \( \lambda = \mu \).

It remains to show that the Euler characteristic of \( M \) vanishes. Any vector field \( W \in H(f(M)) \) can be lifted in a unique way to a smooth horizontal vector field \( \alpha \in \Gamma(H) \) on \( M \), that is \( df(\alpha) = W \circ f \). In particular, if \( W \) is a non-vanishing vector field, then \( \alpha \) is non-vanishing since \( f \) is a submersion and \( \alpha \in \Gamma(H) \). The image \( f(M) \) is diffeomorphic to either of \( S^1 \), \( T^2 \) or \( T^2/Z_2 \), and there exist non-vanishing vector fields on these target manifolds. Thus, we obtain non-vanishing horizontal vector fields on \( M \).

By the Poincaré-Hopf Theorem this shows that the Euler characteristic \( \chi(M) \) vanishes.

This finishes the proof of Theorem \( \Box \)

Proof of Theorem \( \Box \)

Let \( (M, g_M) \) be a compact Riemannian manifold of dimension \( m > 1 \) and let \( (N, g_N) \) be a complete Riemannian surface of bounded geometry. Suppose \( f_0 : M \rightarrow N \) is strictly area decreasing.

(a) If the curvature condition \( \Box \) holds, that is if

\[
\text{Ric}_M(v, v) + \text{Ric}_M(w, w) - \sigma_M(v \wedge w) \geq \sup_N \sigma_N,
\]

then the induced graphical mean curvature flow exists for \( t \in [0, \infty) \), and the evolving maps \( f_t : M \rightarrow N \) remain strictly area decreasing for all \( t \).

We already know from Lemma \( \ref{lem:area-decreasing-flow} \) that the flow remains graphical as long as it exists and that all maps \( f_t, t \in [0, T) \), stay strictly area decreasing. Thus, it remains to be proven that the maximal time of existence \( T \) is infinite. Suppose by contradiction that \( T < \infty \). Thus there exists a sequence \( \{(x_j, t_j)\}_{j \in \mathbb{N}} \) in \( M \times [0, T) \) such that

\[
\lim t_j = T, \quad a_j = \max_{(x, t) \in M \times [0, t_j]} |A|(x, t) = |A(x_j, t_j)| \quad \text{and} \quad \lim a_j = \infty.
\]

Let \( F_j : M \times [-a_j^2 t_j, 0] \rightarrow (M \times N, a_j^2 (g_M \times g_N)), \quad j \in \mathbb{N} \), be the family of graphs of the maps

\[
f_{s/a_j^2 + t_j} : M \rightarrow N, \quad s \in [-a_j^2 t_j, 0].
\]

The singular values of \( f_{s/a_j^2 + t_j} \), considered as a map between the Riemannian manifolds \( (M, a_j^2 g_M) \) and \( (N, a_j^2 g_N) \), coincide with the singular values of the same map, considered as a map between the Riemannian manifolds \( (M, g_M) \) and \( (N, g_N) \), for any \( j \in \mathbb{N} \) and any \( s \in [-a_j^2 t_j, 0] \).

Moreover, the mean curvature vector \( H_j \) of \( F_j \) is related to the mean curvature vector \( H \) of \( F \) by

\[
H_j(x, s) = a_j^{-2} H(x, s/a_j^2 + t_j),
\]

for any \( (x, s) \in M \times [-a_j^2 t_j, 0] \). Since we assume \( T < \infty \), the estimate in \( \ref{lem:mean-curvature-flow} \) of Lemma \( \ref{lem:mean-curvature-flow} \) implies that the norm of the mean curvature vector \( |H| \) is uniformly bounded in time and since the convergence in Proposition \( \ref{prop:area-decreasing-flow} \) is smooth, it follows that the ancient solution \( F_\infty : M_\infty \rightarrow \mathbb{R}^m \times \mathbb{R}^2 \) given in Proposition \( \ref{prop:area-decreasing-flow} \) is a non-totally geodesic complete minimal immersion. From Lemma \( \ref{lem:area-decreasing-flow} \) it follows that the singular values of \( f_t \) remain uniformly bounded as \( t \rightarrow T \). Then Lemma \( \ref{lem:mean-curvature-flow} \) implies that \( M_\infty = \mathbb{R}^m \). Therefore, \( F_\infty : \mathbb{R}^m \rightarrow \mathbb{R}^{m+2} \)
is an entire minimal strictly area decreasing graph in $\mathbb{R}^{m+2}$ that is uniformly bounded in $C^1$. Due to the Bernstein type result in [1, Theorem 1.1] we obtain that the immersion $F_\infty : \mathbb{R}^m \to \mathbb{R}^{m+2}$ is totally geodesic; see also [37, Theorem 1.1]. This contradicts Theorem 7.1(c). Consequently, the maximal time $T$ of existence of the flow must be infinite. This proves part (a) of Theorem A. 

(b) If the curvature conditions (A) and (B) hold, that is if
\[
\text{Ric}_M(v,v) + \text{Ric}_M(w,w) - \sigma_M(v \wedge w) \geq \sup N \sigma_N, \\
\text{Ric}_M(v,v) \geq 0,
\]
then $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^1(M,N)$ and remains uniformly strictly area decreasing.

Since $\text{Ric}_M \geq 0$, the constant $\varepsilon_0$ in inequality (5.11) is non-negative and therefore $\{f_t\}_{t \in [0,\infty)}$ remains uniformly strictly area decreasing and uniformly bounded in $C^1(M,N)$. This proves part (b) of Theorem A.

(c) If the curvature conditions (A) and (C) hold, that is if
\[
\text{Ric}_M(v,v) + \text{Ric}_M(w,w) - \sigma_M(v \wedge w) \geq \sup N \sigma_N, \\
\text{Ric}_M(v,v) \geq \max\{0, \sup N \sigma_N\},
\]
then the mean curvature stays uniformly bounded. If $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^1(M,N)$, then $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^k(M,N)$, for all $k \geq 1$.

The uniform bound on the mean curvature follows directly from inequality (5.25) in Lemma 5.5. It is well-known that a uniform $C^2$-bound in the mean curvature flow implies uniform $C^k$-bounds for all $k \geq 2$, if the ambient manifold $(N, g_N)$ is complete with bounded geometry. To obtain a uniform $C^2$-bound we need to show that the norm $|A|$ of the second fundamental form stays uniformly bounded in time. We may then argue in the same way as in part (a) of the proof to derive a contradiction, if $\limsup_{t \to \infty} |A| = \infty$. Note that we need the uniform $C^1$-bound to apply the entire graph lemma and the Bernstein theorem in [1]. This proves part (c).

(d) Suppose that the curvature conditions (A), (B) and (C) hold, that is we have
\[
\text{Ric}_M(v,v) + \text{Ric}_M(w,w) - \sigma_M(v \wedge w) \geq \sup N \sigma_N, \\
\text{Ric}_M(v,v) \geq \max\{0, \sup N \sigma_N\}
\]
for orthonormal tangent vectors $v, w$. Then we get the following results:
(1) $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^k(M,N)$, for all $k \geq 1$.
(2) In the following cases $\{f_t\}_{t \in [0,\infty)}$ is uniformly bounded in $C^0(M,N)$:
(i) $\text{Ric}_M > 0$.

(ii) $N$ satisfies one of the listed conditions:

(ii.1) $N$ is compact.

(ii.2) $\sup_N \sigma_N \leq 0$ and the surface $N$ is simply connected, that is diffeomorphic to $\mathbb{R}^2$.

(ii.3) $\sup_N \sigma_N \leq 0$ and $N$ contains a totally convex subset $\mathcal{C}$, that is $\mathcal{C}$ contains any geodesic in $N$ with endpoints in $\mathcal{C}$.

(ii.4) There exists a constant $c \in \mathbb{R}$ and a smooth function $\psi : N \to \mathbb{R}$ such that $\psi$ is convex on $N^c := \{ y \in N : \psi(y) < c \}$, $N^c$ is compact and $f_0(M) \subset N^c$.

(3) Under the assumption that $\{ f_t \}_{t \in [0, \infty)}$ is uniformly bounded in $C^k(M, N)$, for all $k \geq 0$, the following holds:

(i) There exists a subsequence $\{ f_{t_n} \}_{n \in \mathbb{N}}$, $\lim_{n \to \infty} t_n = \infty$, that smoothly converges to one of the minimal maps classified in Theorem F.

(ii) If there exists a subsequence $\{ f_{t_n} \}_{n \in \mathbb{N}}$ that converges in $C^0(M, N)$ to a constant map, then the whole flow $\{ f_t \}_{t \in [0, \infty)}$ smoothly converges to this constant map.

(iii) If there exists a point $x \in M$ such that $\text{Ric}_M(x) > 0$, then the flow $\{ f_t \}_{t \in [0, \infty)}$ smoothly converges to a constant map.

(iv) If $(M, g_M)$ and $(N, g_N)$ are real analytic, then the flow smoothly converges to one of the minimal maps classified in Theorem F.

(1) This follows from combining (b) and (c).

(2) We show that for all cases listed in (2) there exists a compact subset $C \subset N$ such that $f_t(M) \subset C$ for all $t$.

(i) $\text{Ric}_M > 0$. From estimate (5.11) in Lemma 5.2 it follows that there exist positive constants $c_1, \varepsilon_0$ such that

$$|df_t|_{g_M}^2 \leq c_1 e^{-\varepsilon_0 t}, \text{ for all } t \in [0, \infty).$$

Clearly $|df_t|_{g_M}$ tends to zero as time goes to infinity. Fix a time $t$, take a geodesic curve $\gamma : [0, 1] \to (M, g_M)$ that connects two arbitrary fixed points $x, y \in M$, and define the smooth curve $\varphi : [0, 1] \to (N, g_N)$ given by $\varphi = f_t \circ \gamma$. Therefore,

$$\text{dist}_N (f_t(x), f_t(y)) \leq \int_0^1 |\varphi'(s)| ds = \int_0^1 |(f_t \circ \gamma)'(s)| ds$$

$$\leq \int_0^1 |(df_t)_{\varphi(s)}|_{g_M} |\gamma'(s)| ds$$

$$= L(\gamma) \int_0^1 |(df_t)_{\varphi(s)}|_{g_M} ds$$

$$\leq L(\gamma) \sqrt{c_1 e^{-\varepsilon_0 t}/2},$$
where \( L(\gamma) \) denotes the length of \( \gamma \). Since this holds for any geodesic curve \( \gamma \) connecting \( x, y \), we obtain
\[
\text{dist}_N (f_t(x), f_t(y)) \leq c_2 e^{-\varepsilon_0 t/2},
\]
where \( c_2 > 0 \) is a constant depending on \( c_1 \), the diameter of \((M, g_M)\), and where \( \text{dist}_N \) is the distance on \( N \) induced by \( g_N \). Thus, the diameter \( \text{diam}(f_t(M)) \) of \( f_t(M) \) tends exponentially to zero as \( t \to \infty \).

Let \( B(q, r) \) be the geodesic ball of \( N \) with radius \( r \) centered at a point \( q \in N \) and denote by \( \varrho_q \) the function given by
\[
\varrho_q(y) := \text{dist}_N(q, y).
\]
Because the surface \( N \) has bounded geometry, due to a theorem of Whitehead (see for example [25, Theorem 29, page 177]), there exists a positive constant \( r_0 < \text{inj}_{g_N} (N) \) depending only on the geometry of \((N, g_N)\), such that \( \varrho_q \) is smooth and strictly convex on \( B(q, r_0) \) for all \( q \in N \).

Since the diameters of the sets \( f_t(M) \) shrink to zero, there exists a sufficiently large time \( t_0 \) such that the image \( f_{t_0}(M) \) is contained in a geodesic ball \( B(p, r_0) \); see Figure 2. We may now proceed exactly as in the proof of Lemma 6.4 to show that \( f_t(M) \subset C \) for all \( t \geq t_0 \).

**Figure 2.** In the case \( \text{Ric}_M > 0 \), the images \( f_t(M) \) will be contained in a small geodesic ball \( B(p, r) \) for all \( t > t_0 \) and a sufficiently large \( t_0 \).

(ii.1) \( N \) is compact. Choose \( C := N \).

(ii.2) \( N \) is diffeomorphic to \( \mathbb{R}^2 \). Since the curvature of \( N \) is non-positive, the distance function \( \varrho_p : N \to \mathbb{R} \) to any fixed point \( p \in N \) is globally smooth and convex (see [5, Theorem 4.1]). Similarly as in (i), we choose \( \phi := \varrho_p \circ \pi_N : M \times N \to \mathbb{R} \) as a globally defined convex function on \( P \), and apply Theorem 11 to \( \phi \) and the set \( C := B(p, r) \), where \( r > 0 \) is chosen so large that \( f_0(M) \subset C \). This yields that \( f_t(M) \subset C \) for all \( t \).
(ii.3) $N$ is complete and contains a totally convex subset $\mathcal{C}$. In this case, the distance function $\varrho_{\mathcal{C}}(q):=d_N(\mathcal{C},q)$ is globally convex (see [3, Remarks 4.3(1)]). We can proceed as in (ii.2) with $\phi:=\varrho_{\mathcal{C}} \circ \pi_N$ and the compact set $C \subset N$ chosen as the closure of a sub-level set of $\varrho_{\mathcal{C}}$ that contains $f_0(M)$, yielding $f_t(M) \subset C$ for all $t$.

(ii.4) We proceed as in (ii.3) with $C:=N^{\complement}$ and $\phi:=\psi \circ \pi_N$.

This proves part (2) of (d).

(3) (i) The volume measure $d\mu$ on $\Gamma_f$ evolves by
\[ \partial_t d\mu = -|H|^2 d\mu. \]  
(7.8)
Since $M$ is compact and the flow exists for all time, integrating (7.8) gives
\[ \int_0^\infty \left( \int_M |H|^2 d\mu \right) dt < \infty. \]
Hence, there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, $\lim_{n \to \infty} t_n = \infty$, such that
\[ \lim_{n \to \infty} \int_M |H|^2 d\mu \bigg|_{t=t_n} = 0. \]  
(7.9)
Because $\{f_{t_n}\}_{n \in \mathbb{N}}$ is uniformly bounded in $C^k(M, N)$ for any $k \geq 0$, there exists a subsequence (denoted again by $\{f_{t_n}\}_{n \in \mathbb{N}}$) that smoothly converges to a limit map $f_\infty$. By (7.9), this limit map must be minimal.

(ii) This follows from Lemma 6.4.

(iii) This follows from (i) and Corollary 11.

(iv) Let us assume that $(M, g_M)$ and $(N, g_N)$ are both real analytic. Since we have already shown that $\{f_t\}_{t \in [0, \infty)}$ contains a subsequence $\{f_{t_n}\}_{n \in \mathbb{N}}$ that smoothly converges to a totally geodesic map $f_\infty$, and since $(M, g_M)$ and $(N, g_N)$ are analytic, a deep result of Leon Simon [30] shows that the family $\{f_t\}_{t \in [0, \infty)}$ converges smoothly and uniformly to $f_\infty$.

This completes the proof of part (3)(iv) and of Theorem A. \qed

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Renan Assimos  
Leibniz University Hannover  
Institute of Differential Geometry  
Welfengarten 1  
30167 Hannover, Germany  

Email address: renan.assimos@math.uni-hannover.de

Andreas Savas-Halilaj  
University of Ioannina  
Department of Mathematics  
Section of Algebra and Geometry  
45110 Ioannina, Greece  

Email address: ansavas@uoi.gr

Knut Smoczyk  
Leibniz University Hannover  
Institute of Differential Geometry  
and Riemann Center for Geometry and Physics  
Welfengarten 1  
30167 Hannover, Germany  

Email address: smoczyk@math.uni-hannover.de