PERMUTATIONS OF THE INTEGERS INDUCE ONLY THE TRIVIAL AUTOMORPHISM OF THE TURING DEGREES

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Abstract. Is there a nontrivial automorphism of the Turing degrees? It is a major open problem of computability theory. Past results have limited how nontrivial automorphisms could possibly be. Here we consider instead how an automorphism might be induced by a function on reals, or even by a function on integers. We show that a permutation of \( \omega \) cannot induce any nontrivial automorphism of the Turing degrees of members of \( 2^\omega \), and in fact any permutation that induces the trivial automorphism must be computable.

A main idea of the proof is to consider the members of \( 2^\omega \) to be probabilities, and use statistics: from random outcomes from a distribution we can compute that distribution, but not much more.

§1. Introduction. Let \( \mathcal{D}_T \) denote the set of Turing degrees and let \( \leq \) denote its ordering. This article gives a partial answer to the following famous question.

QUESTION 1.1. Does there exist a nontrivial automorphism of \( \mathcal{D}_T \)?

DEFINITION 1.2. A bijection \( \pi : \mathcal{D}_T \to \mathcal{D}_T \) is an automorphism of \( \mathcal{D}_T \) if for all \( x, y \in \mathcal{D}_T \), \( x \leq y \) iff \( \pi(x) \leq \pi(y) \). If moreover there exists an \( x \) with \( \pi(x) \neq x \) then \( \pi \) is nontrivial.

Question 1.1 has a long history. Already in 1977, Jockusch and Solovay [11] showed that each jump-preserving automorphism of the Turing degrees is the identity above \( 0^{(4)} \). Nerode and Shore 1980 [18] showed that each automorphism (not necessarily jump-preserving) is equal to the identity on some cone \( \{ a : a \geq b \} \). Slaman and Woodin [25] showed that each automorphism is equal to the identity on the cone above \( 0''' \). An obstacle to reducing the base of the cone to \( 0' \) and ultimately \( 0 \) is that Turing reducibility is \( \Sigma^0_2 \), but not \( \Pi^0_2 \) or \( \Sigma^0_2 \) in the sense of descriptive set theory.

In the other direction, S. Barry Cooper [5] claimed to construct a nontrivial automorphism, induced by a continuous function on \( 2^\omega \). That claim could ultimately not be independently verified, however. In contrast to Cooper’s 85-page effort, here we will attack the problem by ruling out a certain simple but natural possibility: automorphisms induced by permutations of finite objects.

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Table 1. Current status of the (non)rigidity of various degree structures.

| Structure | $D_m$ | $D_{tt}$ | $D_T$ | $D_e$ | $D_h$ |
|-----------|------|--------|------|------|------|
| Status    | not rigid | ? | ? | ? | rigid |
| Citations | [7,19,20] | [1,16] | [25] | [4] | Slaman, Woodin 1990 [23] |

The thought that a permutation of $\omega$ might induce an automorphism of $D_T$ might seem frivolous. But Haught and Slaman [9] did use permutations of the integers to obtain automorphisms of the polynomial-time Turing degrees in an ideal (below a fixed set).

**Theorem 1.3** (Haught and Slaman [9]). There is a permutation of $2^{<\omega}$ (equivalently, of $\omega$) that induces a nontrivial automorphism of $(\text{PTIME}^A, \leq_{PT})$

for some $A$.

Our proof below shows, informally speaking, that any ideal, for any reducibility, where the degrees are sufficiently closed under iterated exponential time reductions, will have no nontrivial automorphism induced by a permutation belonging to the ideal. This includes T (Turing), wtt (weak truth-table), tt (truth-table), and reducibilities based on the complexity classes EXPTIME and ELEMENTARY. But the argument only works when the permutation belongs to the ideal. In Theorem 1.3 it is not asserted that the permutation itself belongs to the ideal $\text{PTIME}^A$.

Our result can be seen as a contrast to the following work of Kent.

**Definition 1.4.** $A \subset \omega$ is cohesive if for each recursively enumerable set $W_e$, either $A \cap W_e$ is finite or $A \cap (\omega \setminus W_e)$ is finite.

**Theorem 1.5** (Kent [21, Theorem 12.3.IX], [12,13]). There exists a permutation $f$ such that

(i) for all recursively enumerable $B$, $f(B)$, and $f^{-1}(B)$ are recursively enumerable (and hence for all recursive $A$, $f(A)$, and $f^{-1}(A)$ are recursive);

(ii) $f$ is not recursive.

**Proof.** Kent’s permutation is just any permutation of a cohesive set (and the identity off the cohesive set). ⊣

For an overview of the situation for rigidity in degree structures see Table 1. The situation for the many-one degrees $D_m$ was long known and at the other extreme the same is true for the hyperdegrees $D_h$. Next to be settled may be the enumeration degrees $D_e$, see [4]. For the truth-table degrees $D_{tt}$ some restrictions on possible automorphisms are known.

§2. **Inducing automorphisms.** Let us now make precise what it means that a permutation of the integers $\omega$ induces a map on equivalence classes of the reals $2^\omega$. 
Definition 2.1. The pullback of a function \( f : \omega \to \omega \) is the function \( f^* : \omega^\omega \to \omega^\omega \) given by
\[
f^*(A)(n) = A(f(n)).
\]
We often write \( F = f^* \). Given a set \( S \subseteq \omega \), let \( \mathcal{D}_S = S^\omega / \equiv_T \). Thus the elements of \( \mathcal{D}_S \) are of the form
\[
[g]_S = \{ h \in S^\omega \mid h \equiv_T g \}, \quad g \in S^\omega.
\]
Given \( F : S^\omega \to S^\omega \) for which
\[
A \equiv_T B \implies F(A) \equiv_T F(B),
\]
we may define \( F_S : \mathcal{D}_S \to \mathcal{D}_S \) by
\[
F_S([A]_S) = [F(A)]_S.
\]
If \( F = f^*_S \) then we say that \( F_S \) and \( F \) are both induced by \( f \).

In Definition 2.1 we are mostly interested in the case where \( f \) is a bijection, but the definition does not require it.

Lemma 2.2. For each \( f : \omega \to \omega \) and each \( S \subseteq \omega \), the pullback \( f^* \) maps \( S^\omega \) into \( S^\omega \).

Proof. \( A \in S^\omega, n \in \omega \implies f^*(A)(n) = A(f(n)) \in S. \)

In light of Lemma 2.2, we can define:

Definition 2.3. \( f^*_S : \mathcal{D}_S \to \mathcal{D}_S \) is the map given by
\[
f^*_S([g]_S) = [f^*(g)]_S.
\]

Our main result concerns \( \mathcal{D}_S \) with \( S = 2 = \{0,1\} \). The easier corresponding result for \( S = \omega \) is Theorem 2.4.

Theorem 2.4. Let \( f : \omega \to \omega \) be a bijection and let \( f^* \) be its pullback. If \( f^*_S \) is an automorphism of \( \mathcal{D}_S \) for some infinite computable set \( S \), then \( f \) is computable.

Proof. Let \( \eta : \omega \to S \) be a computable bijection between \( \omega \) and \( S \). Then for all \( x \in \omega \),
\[
f^*(\eta \circ f^{-1})(x) = (\eta \circ f^{-1})(f(x)) = \eta(f^{-1}(f(x))) = \eta(x).
\]
Since \( \eta \in S^\omega \) is computable and \( f^*_S \) is an automorphism, \( \eta \circ f^{-1} \in S^\omega \) must be computable. Hence \( f \) is computable.

§3. Permutations preserve randomness. Perhaps surprisingly, and certainly going against the idea of purity of methods, we shall make use of Bernoulli measures to obtain our main result.

Definition 3.1. Let \( 0 \leq p \leq 1 \). The Bernoulli measure \( \mu_p \) is the product measure on \( 2^\omega \) of the measure on \( 2 = \{0,1\} \) given by a Bernoulli random variable with parameter \( p \):
\[
\mu(\{1\}) = p = 1 - \mu(\{0\}).
\]
A set $A$ is variously called
- $\mu_p$-random,
- $\mu_p$-ML-random, or
- $\mu_p$-Martin-Löf random
if it does not belong to any set $\bigcap_{n \in \omega} U_n$ where the sets $U_n$ are uniformly $\Sigma_0^0(p)$ and $\mu_p U_n \leq 2^{-n}$ for each $n$. If this holds when the $U_n$ are uniformly $\Sigma_0^1$ (without needing access to $p$) then $A$ is called Hippocratic $\mu_p$-random. For an oracle $f$, $A$ is $f$-$\mu_p$-random if the above statement holds with $\Sigma_0^0(p)$ replaced by $\Sigma_0^1(f \oplus p)$.

Note that for each $n \in \omega$,
\[ \mu_p(\{ X \in 2^\omega : X(n) = 1 \}) = p, \]
where $X(0), X(1), X(2), \ldots$ are mutually independent random variables.

Hippocratic randomness has also been called blind randomness. It is not essential to our current aim but it lets us avoid excessive oracle-relativization of statements, using the following result.

**Theorem 3.2 (Kjos-Hanssen [14]).** Hippocratic $\mu_p$-randomness is the same as $\mu_p$-randomness.

Generalizations of the following Theorem 3.3 have been studied under the heading of randomness conservation (sometimes called randomness preservation). See [3, Theorem 3.2] and [10, Proposition 5] which is explained in further detail in [2]. The Theorem expresses the simple idea that if $B$ is random for some measure, then $F(B)$ should be random for the corresponding image measure, assuming $F$ does not have special knowledge of $B$.

**Theorem 3.3.** If $B$ is $f$-$\mu_p$-random, $F = f^*$ and $A = F(B)$ or $A = F^{-1}(B)$, then $A$ is $f$-$\mu_p$-random.

**Proof.** First note that $f^{-1}$-$\mu_p$-randomness is the same as $f$-$\mu_p$-randomness since $f \equiv_T f^{-1}$. Thus the result for $A = F^{-1}(B)$ follows from the result for $A = F(B)$. So suppose $A = F(B)$ and $A$ is not $f$-$\mu_p$-random. So by Theorem 3.2, $A \in \bigcap_n U_n$ where $\{U_n\}_n$ is a Hippocratic test for $f$-$\mu_p$-ML-randomness. That is, $U_n$ is uniformly $\Sigma_0^0(f)$ and $\mu_p(U_n) \leq 2^{-n}$. Then
\[ B \in \{ X \mid F(X) \in \bigcap_n U_n \} = \bigcap_n V_n, \]
where
\[ V_n = \{ X \mid F(X) \in U_n \} = F^{-1}(U_n). \]
We claim that $V_n$ is $\Sigma_0^0(f)$ (uniformly in $n$) and $\mu_p(V_n) = \mu_p(U_n)$. Write $U_n = \bigcup_k [\sigma_k]$ where the strings $\sigma_k$ are all incomparable. Then
\[ V_n = \bigcup_k F^{-1}([\sigma_k]) \]
and
\[ \mu_p[\sigma_k] = \mu_p F^{-1}([\sigma_k]) \]
and the $F^{-1}([\sigma_k]), k \in \omega$ are still disjoint and clopen. (If we think of $\sigma \in 2^{<\omega}$ as a partial function from $\omega$ to 2 then
\[
F^{-1}([\sigma]) = \{ X \mid F(X) \in [\sigma]\}
\]

\[
= \{ X \mid X(f(n)) = \sigma(n), n < |\sigma| \} = \{(f(n), \sigma(n)) \mid n < |\sigma|\}.
\]

Thus \(\{V_n\}_n\) is another Hippocratic \(f\)-\(\mu_p\)-ML test, and so \(B\) is not \(f\)-\(\mu_p\)-random, which completes the proof.

**Theorem 3.4.** \(\mu_p(\{A : A \geq_T p\}) = 1\), in fact if \(A\) is \(\mu_p\)-ML-random then \(A\) computes \(p\).

**Proof.** Kjos-Hanssen [14] showed that each Hippocratic \(\mu_p\)-random set computes \(p\). In particular, each \(\mu_p\)-random set computes \(p\).

The result that each Hippocratic \(\mu_p\)-random set computes \(p\) is actually an expression of the Law of Large Numbers. The basic idea of our main result will be the following:

From a random sequence of outcomes with respect to a distribution we can compute that distribution, but not much more. Thus a cone \([b, \infty)\) in the Turing degrees has positive \(\mu_p\)-measure iff \(b \leq p\). If our automorphism is \(\mu_p\)-measure-preserving for each \(p\), then it must consequently be the identity.

Theorem 3.4 gives one direction, that if \(p \geq b\) then the cone above \(b\) has measure 1. The other direction will follow from Theorem 4.9 below.

**§4. Other cones have small measure.** Ben Miller proved the following extension (Theorem 4.5) of the Lebesgue Density Theorem to Bernoulli measures and beyond [17, Proposition 2.10].

**Definition 4.1.** An ultrametric space is a metric space with metric \(d\) satisfying the strong triangle inequality

\[
d(x, y) \leq \max\{d(x, z), d(z, y)\}.
\]

**Definition 4.2.** A Polish space is a separable completely metrizable topological space.

**Definition 4.3.** In a metric space \((X, d)\), we let \(B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}\).

**Definition 4.4.** For any measure \(\mu\) define the conditional measure by

\[
\mu(A \mid B) = \frac{\mu(A \cap B)}{\mu(B)}.
\]

**Theorem 4.5 ([17, Proposition 2.10]).** Suppose that \(X\) is a Polish ultrametric space, \(\mu\) is a probability measure on \(X\), and \(A \subseteq X\) is Borel. Then

\[
\lim_{\varepsilon \to 0} \mu(A \mid B(x, \varepsilon)) = 1
\]

for \(\mu\)-almost every \(x \in A\).

A measurable subset \(A\) of Cantor space has density \(d\) at \(x\), with respect to a measure \(\mu\), if

\[
\lim_n \mu(A \mid [x \upharpoonright n]) = d.
\]

where \([\sigma]\) denotes the set of all reals that have \(\sigma\) as a prefix.
Theorem 4.6. Let $A$ be a Borel subset of Cantor space. Then
\[
\lim_{n \to \infty} \mu_p(A \mid [x \upharpoonright n]) = 1
\]
for $\mu_p$-almost every $x \in A$.

Proof. The ultrametric $d(x, y) = 2^{-\min\{n : x(n) \neq y(n)\}}$ induces the standard topology on $2^\omega$. Hence we apply Theorem 4.5. \hfill \Box

Remark 4.7. Lest the reader think that Lebesgue’s Density Theorem for $\mu_p$ is a recent result, we mention that it follows from results of Federer [8, sections 2.8–2.9] on directionally limited metric spaces, see Simmons [24, Theorem 9.1]. There it is shown that Lebesgue’s differentiation theorem, and consequently Lebesgue’s density theorem, holds for any finite Borel measure on a separable, locally compact ultrametric space.

The next important ingredient in our development will be the result of Sacks [22] and independently de Leeuw, Moore, Shannon, and Shapiro [6] that each nontrivial cone in the Turing degrees has measure zero. Here we use Theorem 4.6 to extend this to $\mu_p$. If a set $A$ is computably enumerable using an oracle for $p$ then we write that $A$ is c.e. in $p$; if the complement of $A$ is c.e. in $p$ then $A$ is said to be co-c.e. in $p$.

Theorem 4.8. If $\mu_p(\{x \in 2^\omega : W^x_e = A\}) > 0$ then $A$ is c.e. in $p$.

Proof. Suppose $\mu_p(S) > 0$ where $S = \{x : W^x_e = A\}$. By Theorem 4.6, almost all elements of $S$ are such that $S$ has density 1 at $X$. In particular, there is at least one such $x$. Then, there is an $n$ such that $\mu_p(S \mid [x \upharpoonright n]) > \frac{1}{2}$. Let $\sigma = x \upharpoonright n$. We can now enumerate $A$ using $p$ by taking a “vote” among the sets extending $\sigma$. More precisely, $n \in A$ iff
\[
\mu_p(\{y : \sigma < y \land n \in W^y_c\}) > \frac{1}{2},
\]
and the set of $n$ for which this holds is clearly c.e. in $p$. \hfill \Box

Theorem 4.9. Let $p$ and $q$ be real numbers in $[0, 1]$. Then
\[
\mu_p(\{A : q \leq_T A\}) = 1 \quad \implies \quad q \leq_T p.
\]

Proof. If $A$ can compute $q$ then $A$ can enumerate both $q$ and the complement of $q$. Hence by Theorem 4.8, $q$ is both c.e. in $p$ and co-c.e. in $p$; hence $q \leq_T p$. \hfill \Box

§5. Main result. We are now ready to prove our main result Theorem 5.1 that no nontrivial automorphism of the Turing degrees is induced by a permutation of $\omega$.

Theorem 5.1. If $\pi$ is an automorphism of $\mathcal{D}_2$ which is induced by a permutation of $\omega$ then $\pi(p) = p$ for each $p \in \mathcal{D}_2$.

Proof. Fix a permutation $f : \omega \to \omega$ and let $F = f^* \upharpoonright 2^\omega$. Let $B$ be $f^{-\mu_p}$-random and $A = F^{-1}(B)$. By Theorem 3.3, $A$ is $f^{-\mu_p}$-random. Thus

\footnote{We are grateful to the editor, Laurent Bienvenu, for this reference.}
by Theorem 3.4, \( p \leq_T A \). Hence, by the homomorphism property of \( F \), \( F(p) \leq_T F(A) = B \).

Thus we have shown that every \( f \)-\( \mu_\rho \)-random set computes \( F(p) \). In particular, \( \mu_\rho \)-almost every real computes \( F(p) \). By Theorem 4.9, it follows that \( F(p) \leq_T p \).

The rest is standard: considering the inverse \( f^{-1} \) we also obtain \( F^{-1}(p) \leq_T p \), and hence \( p \leq_T F(p) \). So \( F(p) \equiv_T p \) and \( F \) induces the identity automorphism.

Actually Theorem 5.1 can be strengthened as shown in the next two results Theorems 5.2 and 5.3.

**Theorem 5.2.** Let \( f : \omega \to \omega \) be a permutation. Let \( F = f^* \) be its pullback (Definition 2.1) to \( 2^\omega \). If for positive Lebesgue measure many \( G, F(G) \leq_T G \), then \( f \) is recursive.

**Proof.** Let us write \( p_n(A) = A + n = A \cup \{n\} \) and \( m_n(A) = A - n = A \setminus \{n\} \). Note that \( F(A - n) \) and \( F(A + n) \) differ in exactly one bit, namely \( f^{-1}(n) \), for all \( A \):

\[
F(A - n)(b) \neq F(A + n)(b) \iff (A - n)(f(b)) \neq (A + n)(f(b))
\]

\[
\iff n = f(b) \iff b = f^{-1}(n),
\]

that is,

\[
\{A : \text{for all } b, \ F(A + n)(b) \neq F(A - n)(b) \iff b = f^{-1}(n)\} = 2^\omega.
\]

Thus, to determine that \( b = f^{-1}(n) \), we need to find such an \( A \); this will be done by a more elaborate version of the voting method from Theorem 4.8.

Namely, by the Lebesgue Density Theorem we can get a \( \Phi \) and a \( \sigma \) such that, if \( \mu_\sigma \) denotes conditional probability on \( \sigma \) and \( E = \{A : F(A) = \Phi^A\} \), then

\[\mu_\sigma(E) \geq 0.95.\]

As usual write \( p_n^{-1}E = \{A : p_n(A) \in E\} \). Note that

\[E \subseteq p_n^{-1}(E) \cup m_n^{-1}(E)\]

and \( E^c \subseteq p_n^{-1}(E^c) \cup m_n^{-1}(E^c) \), whence

\[\mu_\sigma(E) \leq \mu_\sigma(p_n^{-1}(E) \cup m_n^{-1}(E)) \leq \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E)).\]

We now have

\[\mu_\sigma\{A : F(A + n) = \Phi^{A+n}\} \geq 0.9 \quad \text{and} \quad \mu_\sigma\{A : F(A - n) = \Phi^{A-n}\} \geq 0.9.\]

Indeed, the events \( A \in m_n^{-1}(E), A \in p_n^{-1}(E) \) are each independent of the event \( n \in A \), so for \( n > |\sigma| \),

\[0.95 \leq \mu_\sigma(E)\]

\[= \mu_\sigma(\{A : A \in E \text{ and } (n \in A \text{ or } n \notin A)\})\]

\[= \mu_\sigma(\{A : A \in E \text{ and } n \in A\}) + \mu_\sigma(\{A : A \in E \text{ and } n \notin A\})\]

\[= \mu_\sigma(E \mid n \in A) \mu_\sigma(n \in A) + \mu_\sigma(E \mid n \notin A) \mu_\sigma(n \notin A)\]

\[= \mu_\sigma(p_n^{-1}(E) \mid n \in A) \mu_\sigma(n \in A) + \mu_\sigma(m_n^{-1}(E) \mid n \notin A) \mu_\sigma(n \notin A)\]
which gives

\[ 1.9 \leq \mu_\sigma(p_n^{-1}(E)) + \mu_\sigma(m_n^{-1}(E)) \leq 1 + \min\{\mu_\sigma(p_n^{-1}(E)), \mu_\sigma(m_n^{-1}(E))\} \]

Let \( D_{n,b} = \{A : \Phi^{A+n}(b) \downarrow \Phi^{A-n}(b) \downarrow\}. \) For \( n > |\sigma|, \)

\[ \mu_\sigma(D_{n,f^{-1}(n)} \setminus \bigcup\{D_{n,b} : b \neq f^{-1}(n)\}) = \mu_\sigma\{A : (\forall b)(A \in D_{n,b} \iff b = f^{-1}(n))\} \geq 0.8 \]

since

\[ \mu_\sigma\{A : \neg(\forall b)(A \in D_{n,b} \iff b = f^{-1}(n))\} \leq \mu_\sigma(\neg p_n^{-1}(E)) + \mu_\sigma(\neg m_n^{-1}(E)) \leq 0.1 + 0.1 = 0.2. \]

Therefore, given any \( n, \) we can compute \( f^{-1}(n); \) enumerate computations until we have found some bit \( b \) such that

\[ \mu_\sigma D_{n,b} \geq 0.8. \]

Then \( b = f^{-1}(n). \) Thus, \( f^{-1} \) is computable, and hence so is \( f. \)

**Theorem 5.3.** If \( \pi \) is an automorphism of \( \mathcal{D}_2 \) which is induced by a permutation \( f \) of \( \omega \) then \( f \) is recursive.

**Proof.** By Theorem 5.1, \( f^*(G) \equiv_T G \) for each \( G \in 2^\omega. \) By Theorem 5.2, \( f \) is recursive.

**Remark 5.4.** Theorem 5.3 is a strengthening of Theorem 5.1. Namely, if \( \pi \) is an automorphism then applying Theorem 5.3 to \( \pi \) and \( \pi^{-1}, \) we have \( \pi(p) \leq p \) and \( \pi^{-1}(p) \leq p \) for each \( p. \) This gives \( \pi(p) = p. \)

§6. **Measure-preserving homeomorphisms of the Cantor set.** Our results can be generalized somewhat, with the following three results.

**Proposition 6.1.** A permutation of \( \omega \) induces a homeomorphism of \( 2^\omega \) that is \( \mu_p \)-preserving for each \( p. \)

**Proposition 6.2.** There exist homeomorphisms of \( 2^\omega \) that are \( \mu_p \)-preserving for each \( p, \) but are not induced by a permutation.

**Theorem 6.3.** Suppose \( \phi \) is a homeomorphism of \( 2^\omega \) which is \( \mu_p \)-preserving for all \( p \) (it suffices to require this for infinitely many \( p, \) or for a single transcendental \( p). \) Suppose \( \phi \) induces an automorphism \( \pi \) of the Turing degrees. Then \( \pi = id. \)

The proof of Proposition 6.2 is simple: for \( x \in 2^\omega \) and denoting concatenation of finite and infinite strings by juxtaposition, we let \( f(100x) = 001x \) and \( f(001x) = f(100x), \) and let \( f \) be the identity on inputs of the form \( 000x, 010x, 011x, 101x, 110x, \) and \( 111x. \) Note that three bits or more are necessary here: the simpler map \( f(01x) = 10x, f(10x) = 01x \) is induced by a permutation.

We omit the proof of Theorem 6.3, which is along similar lines to that of Theorem 5.1. By Proposition 6.1, Theorem 6.3 is (like Theorem 5.3) a strengthening of Theorem 5.1.
There are other kinds of functions that one may wonder whether induce nontrivial automorphisms of the Turing degrees. We close with an easy example.

**Theorem 6.4.** A polynomial cannot restrict to a homeomorphism of $[0,1]$ inducing a nontrivial automorphism of $\mathcal{D}_T$.

**Proof.** A polynomial that maps all computable points to computable points must be computable. This follows from the effectivity in the unsolvability theorem, in which the relevant matrix is the Vandermonde matrix.

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