Deformations from a given Kähler metric to a twisted cscK metric

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1 Introduction

In 1950’s, E. Calabi(cf. [2], [3]) has raised the famous Calabi’s conjecture stating that there exists a unique Kähler metric in any given Kähler class whose Ricci form is any given 2-form representing the first chern class. This conjecture was later proved in 1970’s by the celebrated works of S. T. Yau([25]), E. Calabi([1]) and T. Aubin([11]) using continuity method to solve the complex Monge-Ampère equation. In particular, when the first chern class is zero or negative, their works imply the existence of Kähler-Einstein metrics. When the first chern class is positive, Tian has made contributions towards understanding precisely when a solution exists([22]).

In 1980’s, E. Calabi(cf. [5], [6]) proposed a broader program aiming to find the extremal metrics as the generalization of Kähler-Einstein metrics in an arbitrary Kähler class. As a special case of extremal metrics, the existence problem of the constant scalar curvature Kähler(cscK) metrics fits into a general picture of symplectic geometry as described by S. K. Donaldson([13]). It was well known by now that the existence of Kähler-Einstein metrics or cscK metrics was equivalent to some notion of "stability" in algebraic geometry (c.f. Yau-Tian-Donaldson conjecture([24], [22] and [12])). Recently, this conjecture was settled in the Fano case by the crucial contributions of Chen-Donaldson-Sun (cf. [8], [9] and [10]).

In a series of remarkable work ([14], [15], [16] and [17]), S. K. Donaldson proved the existence of cscK metric on a K-stable toric surface. Very little was known for a general Kähler class in higher dimensions. Recently, X. Chen initiates a new program attacking the existence problem of cscK metrics via a new continuity path in [7], which connects the usual cscK metric equation with a second order elliptic equation. As in [7], for a positive closed (1,1)-form $\chi$, we define the Kähler metric $\omega_\varphi$ satisfying

$$t(R_\varphi - R) - (1 - t)(\text{tr}_\varphi \chi - \chi) = 0$$

(1)

the twisted cscK metric, where $R_\varphi$ denote the scalar curvature of $\omega_\varphi$, $R = \frac{\text{c}_1(M)[\omega][n-1]}{[\omega]^{[n]}}$ and $\chi = \frac{|\chi|\omega^{[n-1]}}{[\omega]^{[n]}}$. In his same paper, X. Chen also showed the openness of this path when $0 < t < 1$. And in a subsequent paper [11], X.Chen, M. Păun and Y. Zeng used the openness result at $t = 1$ to give a new proof of the uniqueness theorem of extremal metrics. Similar
notions of twisted cscK metrics could also be found in earlier papers of J. Fine \cite{18}, J. Stoppa \cite{21} and Lejmi-Székelyhidi \cite{20}.

In this paper, we’ll prove the openness of the new continuity path introduced in \cite{7} at $t = 0$. And it adds further evidence that the path is the right one to work on. Following from a simple observation, we could choose $\chi = \omega$ in (1) such that (1) always has trivial solution at $t = 0$. Our purpose of this paper is to prove the following main theorem:

**Theorem 1.1.** Suppose $(M, \omega)$ is a closed Kähler manifold. Then, for any $t > 0$ sufficiently small, there exist a unique smooth Kähler metric $\omega_{\varphi_t}$ such that

$$t(R_{\varphi_t} - \bar{R}) - (1 - t)(\text{tr}_{\varphi_t} \omega - n) = 0. \tag{2}$$

Notice here, when $t > 0$, (2) is a 4th order nonlinear elliptic equation while at $t = 0$, we get a second order equation. Thus, it’s not clear which function space we should choose if we want to apply the inverse function theorem. Fortunately, if we denote $\varphi_0 = 0$, then

$$t(R_{\varphi_0} - \bar{R}) - (1 - t)(\text{tr}_{\varphi_0} \omega - n) = t(R_{\varphi_0} - \bar{R}) \to 0, \text{ as } t \to 0$$

in any $C^k(M)$ norm. It suggests that $\varphi_0$ is very close to a twisted cscK metric when $t > 0$ sufficiently small. Thus, if we take $\varphi_0$ as base point and apply the inverse function theorem at $\varphi = \varphi_0$, there’s a slight chance that it contains "0" in its neighborhood of image of $\varphi_0$ where every element has a pre-image. However, later we find out that the radius of neighbourhood of $t(R_{\varphi_0} - \bar{R})$ which has pre-images decreases faster than $t^2$ while "0" lies in only the radius $t$ neighborhood.

To overcome this difficulty, we’ll first introduce basic notions in Section 2 and reduce (2) from a 4th order equation to a second order equation

$$r \theta_{\varphi} + \varphi = 0. \tag{3}$$

Then in Section 3 we could expand the above equation in power series of $r$ and collect the same order terms of $r$ to see possible ways of cancelations. Then we could choose $\varphi_1$ closer to the critical point than $\varphi_0$ as shown in Lemma 3.1. Namely, we choose $\varphi_1$ such that "0" is in the $r^4$ neighborhood of $r \theta_{\varphi_1} + \varphi_1$ in $C^\alpha$ space. And in Section 4, by intense calculations, we show that the radius of neighborhood of $r \theta_{\varphi_1} + \varphi_1$ which has pre-images is greater than $r^{3+\epsilon}$ for some $\epsilon > 0$. Eventually "0" will fall into the $r^{3+\epsilon}$ neighborhood of image of $\varphi_1$. Thus, it has a pre-image.

Without further notice, the "C" in each estimate means a constant depending on the complex dimension $n$, the background metric $\omega$, the topological constant $\bar{R}$ and $0 < \alpha < 1$ unless specified.

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2 Preliminary

Suppose \((M, \omega)\) is a closed Kähler manifold. Denote the space of normalized smooth Kähler potentials as

\[
\mathcal{H}_\omega = \{ \phi \in C^\infty(M) | \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0, \int_M \phi \omega^n = 0 \}.
\]  

For \(\phi \in \mathcal{H}_\omega\), we denote \(R_\phi\) the scalar curvature of \(\omega_\phi\) and \(R = \frac{[c_1(M)] [\omega^{n-1}]}{[\omega][n]}\).

In [7], Chen has introduced a continuity path in \(\mathcal{H}_\omega\) for a closed positive \((1,1)\)-form \(\chi\) as

\[
t(R_\phi - R) - (1 - t)(\text{tr}_\phi \chi - \chi) = 0,
\]

where \(\chi = \frac{[\chi][\omega^{n-1}]}{[\omega][n]}\).

In particular, as described in the introduction, we could simply choose the closed positive \((1,1)\)-form to be \(\omega\). Thus, (5) always has a trivial solution at \(t = 0\). As in defining the Futaki invariant in [23], we could solve the Laplacian equation for any \(\phi \in \mathcal{H}_\omega\)

\[
\Delta_\phi f = R_\phi - R.
\]

We denote the solution of (6) as \(\theta_\phi\) with the normalization \(\int_M \theta_\phi \omega^n = 0\). Since

\[
R_\phi - R = \text{tr}_\phi(\text{Ric}_\phi - \text{Ric}(\omega)) + \text{Ric}(\omega) - R \omega - R \sqrt{-1} \partial \bar{\partial} \phi,
\]

we have

\[
\theta_\phi = -\log \frac{\omega^n_\phi}{\omega^n} + R_\phi + \int_M \log \frac{\omega^n_\phi}{\omega^n} \omega^n + P_\phi,
\]

where \(P_\phi\) is determined by

\[
\Delta_\phi P_\phi = \text{tr}_\phi(\text{Ric}(\omega) - R_\omega), \int_M P_\phi \omega^n = 0.
\]

Given the notion of \(\theta_\phi\) above, we could reduce the continuity path equation (5) with \(\chi = \omega\) from a 4th order PDE to a Monge-Ampère type of equation as

\[
t \theta_\phi + (1 - t) \phi = 0.
\]

Following the discussion above, Theorem 1 will be an easy corollary of the following theorem:

**Theorem 2.1.** Suppose \((M, \omega)\) is a closed Kähler manifold. Then, for any \(r > 0\) sufficiently small, there exists a unique \(\phi_r \in \mathcal{H}_\omega\) such that

\[
r \theta_{\phi_r} + \phi_r = 0.
\]
In our paper, we’ll repeatedly use schauder estimate of Laplacian equation. Thus, let’s introduce it here as the following Lemma:

**Lemma 2.2.** If $\varphi \in C^{2,\alpha}(M)$ with $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$ and $u \in C^{2,\alpha}(M)$ satisfies
\[
\Delta \varphi u = f, \int_M u\omega^n = 0
\] (13)
for some $f \in C^{\alpha}(M)$. Then
\[
\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^{\alpha}(M)}.
\] (14)

**Proof.** Proof of Lemma 2.2. By Schauder estimate [19], we can get
\[
\|u\|_{C^{2,\alpha}(M)} \leq C\left(\|u\|_{L^\infty(M)} + \|f\|_{C^{\alpha}(M)}\right).
\] (15)
To bound $\|u\|_{L^\infty(M)}$, we first multiply $u$ on both hand sides of (13), integrate against $\omega^\varphi$, and we get that
\[
\int_M |\nabla u|^2\omega^\varphi = \int_M -fu\omega^\varphi \leq C\|f\|_{L^\infty(M)}\|u\|_{L^2(M,\omega)}.
\] (16)
On the other hand,
\[
\int_M |\nabla u|^2\omega^\varphi \geq \frac{1}{C}\int_M |\nabla u|^2\omega^n \geq \frac{1}{C}\|u\|_{L^2(M,\omega)}^2.
\] (17)
Thus, combining the above two inequalities, we can get
\[
\|u\|_{L^2(M,\omega)} \leq C\|f\|_{L^\infty(M)}.
\] (18)
Then, by Moser iteration [19], we can get that
\[
\|u\|_{L^\infty(M)} \leq C\left(\|u\|_{L^2(M)} + \|f\|_{L^\infty(M)}\right) \leq C\|f\|_{C^{\alpha}(M)}.
\] (19)
This ends the proof.

### 3 Choose the base point

Let’s first introduce the space we’re going to work on. Define for $0 < \alpha < 1$ and $k \in \mathbb{N}$
\[
\mathcal{H}_\omega^{2,\alpha} = \{\varphi \in C^{2,\alpha}(M) | \omega^\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \int_M \varphi\omega^n = 0\},
\] (20)
\[
C_{\omega}^{k,\alpha}(M) = \{f \in C^{k,\alpha}(M) | \int_M f\omega^n = 0\}.
\] (21)
More generally, $\theta_{\varphi}$ could be defined on the space $\mathcal{H}_\omega^{2,\alpha}$ if we took the definition as in (9). Therefore, we define, still denoted by $\theta_{\varphi}$,
\[
\theta : \mathcal{H}_\omega^{2,\alpha} \rightarrow C_{\omega}^{\alpha}(M)
\]
\[
\varphi \mapsto \theta_{\varphi} = -\log \frac{\omega^\varphi}{\omega^n} - R\varphi + \int_M \log \frac{\omega^\varphi}{\omega^n}\omega^n + P_{\varphi},
\]
\[4\]
where $P_\varphi \in C_2^\alpha(M) \subset C_\omega^\alpha(M)$ is determined by
\[ \Delta_\varphi P_\varphi = \text{tr}_\varphi (\text{Ric}(\omega) - \mathcal{R}_\omega), \int_M P_\varphi \omega^n = 0. \] (22)

Define
\[ F_\varphi : \mathcal{H}^2_\omega \to C_\omega^\alpha(M), \]
\[ \varphi \mapsto r \theta_\varphi + \varphi. \]

Denote $\varphi_0 = 0 \in \mathcal{H}^2_\omega$. As we described in the introduction, $\varphi_0$ is not enough for our purpose. We need to find "better" base point to apply inverse function theorem.

Let
\[ \varphi_1 = \varphi_0 + ru_1 + \frac{r^2}{2} u_2 + \frac{r^3}{6} u_3, \] (23)
where $u_1$'s are smooth functions on $M$ with $\int_M u_i \omega^n = 0$ that we'll specify later. First we'll expand $F_\varphi(\varphi_1)$ in terms of $r$ at $r = 0$. Denote $u_r = ru_1 + \frac{r^2}{2} u_2 + \frac{r^3}{6} u_3$. Compute
\[ \frac{\partial \theta_\varphi}{\partial r} = -\Delta_\varphi \hat{u}_r - \mathcal{R} \hat{u}_r + \int_M \Delta_\varphi \hat{u}_r \omega^n + \mathcal{D}P|_{\varphi_1}(\hat{u}_r), \] (24)
where $\mathcal{D}P|_{\varphi_1} : C_2^\alpha(M) \to C_\omega^\alpha(M)$ is the linearization of $P_{\varphi}$ at $\varphi = \varphi_1$ and it satisfies
\[ \Delta_{\varphi_1}(\mathcal{D}P|_{\varphi_1}(u)) = \langle \partial \partial u, \partial \partial P_{\varphi_1} - (\text{Ric}(\omega) - \mathcal{R}_\omega) \rangle_{\varphi_1}, \int_M (\mathcal{D}P|_{\varphi_1}(u)) \omega^n = 0. \] (25)

Take one more derivative of $\theta_\varphi$, we get
\[ \frac{\partial^2 \theta_\varphi}{\partial^2 r} = -(\Delta_\varphi \hat{u}_r - |\partial \partial \hat{u}_r|_{\varphi_1}^2) - \mathcal{R} \hat{u}_r + \int_M (\Delta_\varphi \hat{u}_r - |\partial \partial \hat{u}_r|_{\varphi_1}^2) \omega^n + \mathcal{D}P|_{\varphi_1}(\hat{u}_r) + \left( \frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi_1}(\hat{u}_r, \hat{u}_r) \right) \] (26)
where the last term is given by the unique solution of the following elliptic equation
\[ \Delta_\varphi f = 2(\partial \partial u_r, \partial \partial (\mathcal{D}P|_{\varphi_1}(u_r)))_{\varphi_1} - \hat{u}_r \partial \partial \hat{u}_r \partial \partial (P_{\varphi_1,j}\hat{u}_r, (\text{Ric}(\omega) - \mathcal{R}_\omega)_{jj}) \]
\[ - \hat{u}_r \partial \partial \hat{u}_r \partial \partial (P_{\varphi_1,p} - (\text{Ric}(\omega) - \mathcal{R}_\omega)_{pj}) \]
with $\int_M f \omega^n = 0$. Thus, we get the expansion of $F_\varphi(\varphi_1)$ of $r$ at $r = 0$,
\[ F_\varphi(\varphi_1) = r \theta_\varphi + \varphi_1 \] (27)
\[ = \varphi_0 + ru_1 + \theta_\varphi + \frac{r^2}{2} u_2 + 2 \frac{\partial \theta_\varphi}{\partial r}|_{r=0} + \frac{r^3}{6} u_3 + 3 \frac{\partial^2 \theta_\varphi}{\partial^2 r}|_{r=0} + O(r^4). \] (28)

It suggests that we should define
\[ u_1 = -\theta_\varphi, \]
\[ u_2 = -2 \frac{\partial \theta_\varphi}{\partial r}|_{r=0} = -2 \left( -\Delta_\varphi u_1 - \mathcal{R} u_1 + \mathcal{D}P|_{\varphi_0}(u_1) \right) \]
\[ u_3 = -3 \frac{\partial^2 \theta_\varphi}{\partial^2 r}|_{r=0} = -3 \left( -\Delta_\varphi u_2 - \mathcal{R} u_2 + \mathcal{D}P|_{\varphi_0}(u_2) + |\partial \partial u_2|_{\varphi_0}^2 - \int_M |\partial \partial u_1|_{\varphi_0}^2 \omega^n \right. \]
\[ \left. + \left( \frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi_0}(u_1, u_1) \right) \right) \]
It’s clear from definitions that $u_s'$s are fixed smooth functions with $C^k$ norm bounds only depend on $\varphi_0$. Therefore, we could choose $r > 0$ sufficiently small such that $\varphi_1 \in H^{2,\alpha}_{\omega}$ with $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{\ell}$ and we expect that $F_r(\varphi_1)$ is $r^4$ close to "0" in appropriate norms. This observation can be made more precise as the following lemma:

**Lemma 3.1.** Notations as described above, for $r > 0$ sufficiently small, we have

$$
\|F_r(\varphi_1)\|_{C^{0}(M)} \leq Cr^4.
$$

**Proof.** Proof of Lemma 3.1 It suffices to show that

$$
\|\theta_{\varphi_1} - (\theta_{\varphi_0} + r \frac{\partial \theta_{\varphi_1}}{\partial r}|_{r=0} + \frac{r^2}{2} \frac{\partial^2 \theta_{\varphi_1}}{\partial^2 r}|_{r=0})\|_{C^{0}(M)} \leq Cr^3. \tag{29}
$$

By Taylor expansion theorem, we could write the remaining error of the function $\theta_{\varphi_1}$ and its second order Taylor expansion as an integral,

$$
R(x) = \frac{1}{2!} \int_0^r (r-s)^2 \left( \frac{\partial^2 \theta_{\varphi_1}}{\partial^2 s}|_{r=s}(x) \right) ds. \tag{30}
$$

So it suffices to show that for any $s \in [0, r]$ with $r > 0$ sufficiently small

$$
\|\frac{\partial^3 \theta_{\varphi_1}}{\partial^3 r}|_{r=s}\|_{C^{0}(M)} \leq C. \tag{31}
$$

Denote $\varphi_s = \varphi_0 + su_1 + \frac{s^2}{2} u_2 + \frac{s^3}{6} u_3$ and $u_s = su_1 + \frac{s^2}{2} u_2 + \frac{s^3}{6} u_3$. Compute

$$
\frac{\partial^3 \theta_{\varphi_1}}{\partial^3 s}|_{r=s} = -(\Delta \varphi_s u_s^{(3)} - 3(\partial \bar{\partial} u_s^{(1)}, \partial \bar{\partial} u_s^{(2)})\varphi_s + 2(\partial \bar{\partial} u_s^{(1)})^3)
$$

$$
- \int_M (\Delta \varphi_s u_s^{(3)} - 3(\partial \bar{\partial} u_s^{(1)}, \partial \bar{\partial} u_s^{(2)})\varphi_s + 2(\partial \bar{\partial} u_s^{(1)})^3) \omega^n - Ru_s^{(3)}
$$

$$
+ DP|_{\varphi_s}(u_s^{(3)}) + 2(\frac{\partial}{\partial \varphi} DP|_{\varphi_s})|_{\varphi=\varphi_s}(u_s^{(2)}, u_s^{(1)}) + (\frac{\partial}{\partial \varphi} DP|_{\varphi_s})|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(2)})
$$

$$
+ (\frac{\partial^2}{\partial^2 \varphi} DP|_{\varphi_s})|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(1)}, u_s^{(1)}).
$$

It’s obvious that the first two lines has uniform $C^\alpha$ norm as we expected. Therefore, we need to estimate the last four terms of the above equation. Let’s first consider $P_{\varphi_s}$. It satisfies the Laplacian equation as described in Lemma 2.2 so we get that

$$
\|P_{\varphi_s}\|_{C^{2,\alpha}(M)} \leq C. \tag{32}
$$

Then we can estimate $DP|_{\varphi_s}(u)$ using (32) and Lemma 2.2 since it satisfies the similar Laplacian equation with right hand side depending on second order derivatives of $P_{\varphi_s}$, we can conclude that

$$
\|DP|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}. \tag{33}
$$
Thus, we could further estimate the term using the same argument in Lemma 2.

\[ \|\left(\frac{\partial}{\partial \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(u, v)\|_{C^{2,\alpha}(M)} \leq C \|u\|_{C^{2,\alpha}(M)} \|v\|_{C^{2,\alpha}(M)}. \] (34)

Finally, we could estimate the term \( \left(\frac{\partial^2}{\partial^2 \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(u, v, w) \) which satisfies the equation

\[
\Delta_{\varphi} f = \langle \partial \bar{\partial} w, \partial \bar{\partial} \left(\left(\frac{\partial}{\partial \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(u, v)\right)_{\varphi} + \partial \bar{\partial} u, \partial \bar{\partial} \left(\left(\frac{\partial}{\partial \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(v, w)\right)_{\varphi} + \partial \bar{\partial} v, \partial \bar{\partial} \left(\left(\frac{\partial}{\partial \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(u)\right)_{\varphi} + \partial \bar{\partial} u \ast \partial \bar{\partial} w \ast \partial \bar{\partial} \left(\mathcal{D}P_{\varphi}\right)(v) + \partial \bar{\partial} v \ast \partial \bar{\partial} u \ast \partial \bar{\partial} \left(\mathcal{D}P_{\varphi}\right)(w) + \partial \bar{\partial} u \ast \partial \bar{\partial} v \ast \partial \bar{\partial} w \ast \left(\partial \bar{\partial} P_{\varphi} - \operatorname{Ric}(\omega) - \bar{R}\omega\right), \int_{M} f \omega^{n} = 0. \] (35)

Thus by the Lemma 2, we can conclude that

\[ \|\left(\frac{\partial^2}{\partial^2 \varphi} \mathcal{D}P_{\varphi}\right)_{\varphi}(u, v, w)\| \leq C \|u\|_{C^{2,\alpha}(M)} \|v\|_{C^{2,\alpha}(M)} \|w\|_{C^{2,\alpha}(M)}. \] (36)

Since \( \|u_s^{(i)}\|_{C^{2,\alpha}(M)} \leq C \) for \( 1 \leq i \leq 3 \),

\[ \left\| \frac{\partial^3 \theta_{\varphi_1}}{\partial^3 \tau} \right\|_{C^0(M)} \leq C. \] (37)

Thus it ends the proof of the lemma.

\[ \square \]

4 Proof of Theorem 1.1

In last section, we have shown that for \( r > 0 \) sufficiently small, \( \|F_{r}(\varphi_1)\|_{C^0(M)} \leq Cr^4 \). Next we’ll construct a contract map defined on a \( r^{1+\epsilon} \) neighborhood of \( \varphi_1 \) in \( C^{2,\alpha} \) space, which is similar to the proof of inverse function theorem. Since \( F_{r}(\varphi_1) \) is \( r^4 \) small in \( C^0 \) norm, we could start the iterating process from \( \varphi_1 \) and keep every following term stay within the prescribed \( r^{1+\epsilon} \) neighborhood of \( \varphi_1 \).

First, we have to understand the linearization of \( F_{r} : H^2_\omega \to C^0_\omega(M) \) at \( \varphi = \varphi_1 \). Compute

\[
\mathcal{D}F_{r}|_{\varphi_1} : C^{2,\alpha}(M) \to C^0_\omega(M)
\]

\[
u \mapsto -r \Delta_{\varphi_1} u + (1 - r R) u + r \left( \int_M (\Delta_{\varphi_1} u) \omega^n + \mathcal{D}P_{\varphi_1}(u) \right),
\]

where \( \mathcal{D}P_{\varphi_1}(u) \) satisfies

\[
\Delta_{\varphi_1} \left( \mathcal{D}P_{\varphi_1}(u) \right) = \langle \partial \bar{\partial} u, \left( \partial \bar{\partial} P_{\varphi_1} - (\operatorname{Ric}(\omega) - \bar{R}\omega) \right) \rangle_{\varphi_1}, \int_M \left( \mathcal{D}P_{\varphi_1}(u) \right) \omega^n = 0. \] (38)

We summarize the properties of \( \mathcal{D}F_{r}|_{\varphi_1} \) as the following lemma:
Lemma 4.1. Suppose $0 < \alpha < 1$. Then, for $r > 0$ sufficiently small, the linearizaiton of $F_r : \mathcal{H}^2_\omega \rightarrow C^\alpha_\omega(M)$ at $\varphi = \varphi_1$, $DF_r|_{\varphi_1} : C^2_\omega(M) \rightarrow C^\alpha_\omega(M)$, is injective and also surjective. Moreover, the operator norm of the inverse of $(DF_r|_{\varphi_1})$ has the upper bound

$$\| (DF_r|_{\varphi_1})^{-1} \| \leq C r^{-\frac{2-\alpha}{1-\alpha}}.$$ 

Before proving Lemma 4.1, we'll need the estimate of $\mathcal{D}P|_{\varphi}(u)$ for $\|\varphi\|_{C^2_\omega(M)} \leq \frac{1}{2}$. We summarize it as the following lemma:

Lemma 4.2. Suppose $\|\varphi\|_{C^2_\omega(M)} \leq \frac{1}{2}$, then we have the estimate for any $1 < p < \infty$,

$$\| (\mathcal{D}P|_{\varphi}(u)) \|_{L^p(M)} \leq C_p \|u\|_{L^p(M)}. \quad (42)$$

Remark. Since $\omega$ and $\omega_\varphi$ are equivalent metrics if $\|\varphi\|_{C^2_\omega(M)} \leq \frac{1}{2}$, we make no efforts to distinguish between $L^p$ spaces with respect to the two metrics hereafter.

Proof. We first introduce the Green function $G_\varphi(x, y)$ of the metric $\omega_\varphi$. Then we define

$$T(u)(x) = \int_M G_\varphi(x, y) \left( u(P_{\varphi,ij} - (\text{Ric}(\omega) - R_{\omega})_{ij}) \right)_y \omega_\varphi^n \quad (43)$$

$$= \int_M (G_\varphi(x, y))_{ij} \left( u(P_{\varphi,ij} - (\text{Ric}(\omega) - R_{\omega})_{ij}) \right)_y \omega_\varphi^n. \quad (44)$$

Since

$$\Delta_\varphi \big( \mathcal{D}P|_{\varphi}(u) \big) = \left( u(P_{\varphi,ij} - (\text{Ric}(\omega) - R_{\omega})_{ij}) \right)_{ij} \int_M (\mathcal{D}P|_{\varphi}(u)) \omega_\varphi^n = 0, \quad (45)$$

we have

$$\mathcal{D}P|_{\varphi}(u) = T(u) - \int_M T(u) \omega_\varphi^n \quad (46)$$

For $i, j \in \mathbb{N}$, we define the operator

$$T_{\bar{i}j}f = \int_M (G_\varphi(x, y))_{\bar{i}j} f(y) \omega_\varphi^n.$$

$T_{\bar{i}j}$ is a Calderon-Zygmund([19]) operator which maps $L^p$ functions to $L^p$ functions for any $1 < p < \infty$. Moreover we can show $T_{\bar{i}j}$ has uniform norms. To see this, we consider the Laplacian equation

$$\Delta_\varphi u = f, \int_M u \omega_\varphi^n = 0. \quad (47)$$

Thus, we see that the solution satisfies

$$\frac{\partial^2}{\partial z_i \partial z_j} u(x) = \left( T_{\bar{i}j} f \right)(x). \quad (48)$$
So it suffices to show the uniform $W^{2,2}$ estimates of (17), which follows from the fact that
\[ \| \varphi \|_{C^{2,\alpha}(M)} \leq \frac{1}{2} \] and the standard $L^p$ theory of elliptic equations (19). We have the estimate for any $p \in (1, +\infty)$
\[ \| T^{ij}f \|_{L^p(M)} \leq C_p \| f \|_{L^p(M)}. \] (49)
Thus, taking advantages of the above estimate, we can get
\[ \| T(u) \|_{L^p(M)} \leq \sum_{k,l} C_p \| u \parallel \big( g^{ij} P_{\varphi,ij} - (\text{Ric}(\omega) - \mathcal{R} \omega)_{ij} \big) \|_{L^p(M)} \] (50)
\[ \leq C_p \| u \|_{L^p(M)}. \] (51)
Thus, we have for any $1 < p < \infty$
\[ \| (\mathcal{D}P|_{\varphi}(u)) \|_{L^p(M)} \leq C_p \| u \|_{L^p(M)}. \] (52)
This ends the proof of Lemma 4.2

Now we can prove Lemma 4.1

Proof. Proof of Lemma 4.1. First we show that $\mathcal{D}F_r|_{\varphi_1}$ is injective. Suppose there exists $u \in C^{2,\alpha}_{\omega_1}(M)$ such that
\[ -r \Delta_{\varphi_1}u + (1 - r \mathcal{R})u + r \left( \int_M (\Delta_{\varphi_1}u) \omega_1^n + \mathcal{D}P|_{\varphi_1}(u) \right) = 0. \] (53)
It suffices to show that $u = 0$. Multiply $u$ on both hand sides of (53) and integrate against $\omega_{\varphi_1}^n$.
\[ 0 = r \int_M |\nabla u|_{\varphi_1}^2 \omega_{\varphi_1}^n + (1 - r \mathcal{R}) \int_M u^2 \omega_{\varphi_1}^n + r \left( \int_M (\Delta_{\varphi_1}u) \omega_1^n \right) \left( \int_M u \omega_{\varphi_1}^n \right) + r \int_M (\mathcal{D}P|_{\varphi_1}(u)) u \omega_{\varphi_1}^n \] (54)
\[ \geq (1 - r \mathcal{R}) \int_M u^2 \omega_{\varphi_1}^n + r \left\{ \left( \int_M (\Delta_{\varphi_1}u) \omega_1^n \right) \left( \int_M u \omega_{\varphi_1}^n \right) + \int_M (\mathcal{D}P|_{\varphi_1}(u)) u \omega_{\varphi_1}^n \right\}. \] (55)
We focus on estimates of the later two terms in (55). Consider
\[ \int_M (\Delta_{\varphi_1}u) \omega_1^n = \int_M (\Delta_{\varphi_1}u) \left( \frac{\omega_1^n}{\omega_{\varphi_1}^n} \right) \omega_{\varphi_1}^n = \int_M u (\Delta_{\varphi_1} \left( \frac{\omega_1^n}{\omega_{\varphi_1}^n} \right) \omega_{\varphi_1}^n) \] (56)
\[ = \int_M u \omega_{\varphi_1}^n g^{ij}_{\varphi_1} \left( - g^{k_l}_{\varphi_1} \Gamma_{\varphi_1,kl} + g^{kp}_{\varphi_1} g^{dl}_{\varphi_1} \Gamma_{\varphi_1,pl} \Gamma_{\varphi_1,q} \Gamma_{\varphi_1,k} + g^{pq}_{\varphi_1} g^{kl}_{\varphi_1} \right) \omega_{\varphi_1}^n \] where the derivatives are covariant derivatives of $\omega$. Thus, we have that
\[ r \left( \int_M (\Delta_{\varphi_1}u) \omega_1^n \right) \left( \int_M u \omega_{\varphi_1}^n \right) \geq -Cr^2 \int_M u^2 \omega_{\varphi_1}^n \] (55)
To estimate the last term of (55), we need the following estimate of $(\mathcal{D}P|_{\varphi_1})$. 

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9
Choosing $r > 0$ sufficiently small, we can get $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$. Using Lemma 4, we get

$$\|\left(DP|_{\varphi_1}(u)\right)\|_{L^2(M)} \leq C\|u\|_{L^2(M)}. \tag{57}$$

Thus for the last term in (55) we have the estimate

$$\int_M \left(DP|_{\varphi_1}(u)\right) u\omega^n_\varphi \geq -C \int_M u^2\omega^n_\varphi. \tag{58}$$

Therefore, combining the estimates above, we have that

$$0 \geq (1 - C r) \int_M u^2\omega^n_\varphi. \tag{59}$$

It implies that when $r > 0$ sufficiently small, we have that $u = 0$. So we have proved the injectivity of $\left(DF_r|_{\varphi_1}\right)$.

Next, we show the surjectivity of $\left(DF_r|_{\varphi_1}\right)$ and the upper bound of $\|\left(DF_r|_{\varphi_1}\right)^{-1}\|$ together. For $f \in C^\alpha(M)$, we'll use continuity method to solve the equation

$$DF_r|_{\varphi_1}(u) = f. \tag{60}$$

Define for $s \in [0,1]$,

$$L_s : C^{2,\alpha}(M) \to C^\alpha(M) \tag{61}$$

$$u \mapsto -r \Delta_{\varphi_1} u + (1 - rR) u + sr \left(\int_M (\Delta_{\varphi_1} u) \omega^n + DP|_{\varphi_1}(u)\right). \tag{62}$$

First, we show that for any $s \in [0,1]$,

$$\|u\|_{C^{2,\alpha}(M)} \leq C_r \|L_s u\|_{C^\alpha(M)}. \tag{63}$$

From the definition of $L_s$, we get that,

$$\Delta_{\varphi_1} u = -\frac{1}{r} L_s u + \frac{1 - rR}{r} u + s \left(\int_M (\Delta_{\varphi_1} u) \omega^n + DP|_{\varphi_1}(u)\right). \tag{64}$$

Since we choose $r > 0$ sufficiently small s.t. $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$, we can get from Schauder estimate,

$$\|u\|_{C^{2,\alpha}(M)} \leq C \left(\|\Delta_{\varphi_1} u\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right)$$

$$\leq C \left(\frac{1}{r} \|L_s u\|_{C^\alpha(M)} + \frac{1}{r} \|u\|_{C^\alpha(M)} + \int_M (\Delta_{\varphi_1} u) \omega^n + \|DP|_{\varphi_1}(u)\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right)$$

$$\leq C_0 \left(\frac{1}{r} \|L_s u\|_{C^\alpha(M)} + \frac{1}{r} \|u\|_{C^\alpha(M)} + \|DP|_{\varphi_1}(u)\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right).$$

By interpolations [19], we have

$$\|u\|_{C^\alpha(M)} \leq \frac{r}{4C_0} \|u\|_{C^{2,\alpha}(M)} + C r^{-\frac{\alpha}{1-\alpha}} \|u\|_{L^\infty(M)}. \tag{65}$$
Also, for term \(\|D^P \varphi_1(u)\|_{C^\infty(M)}\), since it satisfies equation (61), we have estimate
\[
\|D^P \varphi_1(u)\|_{C^\infty(M)} \leq C (\|\partial \bar{D} u, (\partial \bar{D} \varphi_1 - (\text{Ric}(\omega) - \bar{R} \omega))\|_{\varphi_1} \|L^\infty(M) (66)
\]
\[
\leq C \|\partial \bar{D} u\|_{L^\infty(M)} \leq \frac{1}{4 C_0} \|u\|_{C^2,0(M)} + C \|u\|_{L^\infty(M)}. (67)
\]
Combining estimates of (65) and (66), we have that
\[
\|u\|_{C^2,0(M)} \leq C \left( \frac{1}{r} \|L_s u\|_{C^0(M)} + r^{-\frac{1}{p}} \|u\|_{L^\infty(M)} \right). (68)
\]
Now we focus on estimates of \(\|u\|_{L^\infty(M)}\). For \(p > 1\), we could first multiply \(|u|^p\) on both hand sides of (64) and integrate against \(\omega^{\varphi_1}\) on the region \(\{u > 0\}\). Then we’ll get by a similar argument which we use to prove the injectivity,
\[
\frac{1}{r^r} \int_{u > 0} (L_s u) u^p \omega^{\varphi_1} \geq \int_{u > 0} pu^{p-1}|\nabla u|^2 \omega^{\varphi_1} + \frac{1}{r} \int_{u > 0} u^{p+1} \omega^{\varphi_1} \leq C \left( \int_{u > 0} |u|^p \omega^{\varphi_1} \right) \left( \int_{u > 0} u \omega^{\varphi_1} \right)
\]
\[
\geq 1 - \frac{r R}{p} \int_{u > 0} u^{p+1} \omega^{\varphi_1} - C_p \int_{u > 0} u^{p+1} \omega^{\varphi_1}.
\]
Multiply \(|u|^p\) on both hand sides of (64) and integrate against \(\omega^{\varphi_1}\) on the region \(\{u < 0\}\). Similarly we get
\[
\frac{1}{r^r} \int_{u < 0} (L_s u) |u|^p \omega^{\varphi_1} \geq \int_{u < 0} pu^{p-1}|\nabla u|^2 \omega^{\varphi_1} + \frac{1}{r} \int_{u < 0} u^{p+1} \omega^{\varphi_1} \leq C \left( \int_{u < 0} |u|^p \omega^{\varphi_1} \right) \left( \int_{u < 0} u \omega^{\varphi_1} \right)
\]
\[
\geq 1 - \frac{r R}{p} \int_{u < 0} u^{p+1} \omega^{\varphi_1} - C_p \int_{u < 0} u^{p+1} \omega^{\varphi_1}.
\]
Thus, we get for \(p < p_0 < \infty\), we could choose our \(r > 0\) small such that
\[
\frac{1}{r} \int_{u > 0} |u|^{p+1} \omega^{\varphi_1} \leq \frac{1}{r} \|u\|_{L^{p+1}(M)} \|L_s u\|_{L^{p+1}(M)}. (70)
\]
And then for \(p < p_0 + 1\)
\[
\|u\|_{L^p(M)} \leq C \|L_s u\|_{L^p(M)} \leq C \|L_s u\|_{L^\infty(M)}. (71)
\]
By \(L^p\) theory of elliptic equation for (64), we get for \(p < p_0 + 1\)
\[
\|u\|_{W^{2,p}(M)} \leq C \left( \frac{1}{r} \|L_s u\|_{L^p(M)} + \frac{1}{r} \|u\|_{L^p(M)} \right) (72)
\]
\[
\leq C \|L_s u\|_{L^\infty(M)}. (73)
\]
By sobolev embedding, we can get that for \(p > n\)
\[
\|u\|_{L^\infty(M)} \leq C \|u\|_{W^{2,p}(M)} \leq \frac{C}{r} \|L_s u\|_{L^\infty(M)}. (74)
\]
Therefore, we conclude that
\[
\|u\|_{C^2,0(M)} \leq C r^{-\frac{2n}{p}} \|L_s u\|_{C^0(M)}. (75)
\]
Since the norm is independent of $s \in [0, 1]$ and obviously $L_0 : C^{2,\alpha}(M) \to C^\alpha(M)$ is onto, thus by continuity method in [10], we conclude that $L_1 : C^{2,\alpha}(M) \to C^\alpha(M)$ is also onto. Thus we have shown that $\mathcal{D}F_{r|\varphi_1} = L_1$ is surjective. And

$$\|\left(\mathcal{D}F_{r|\varphi_1}\right)^{-1}(f)\|_{C^{2,\alpha}(M)} \leq C r^{-\frac{2-\alpha}{1-\alpha}} \|f\|_{C^\alpha(M)}. \quad (76)$$

This ends the proof of Lemma 4.3.

Define functional $\Psi$ in a $C^{2,\alpha}$-neighborhood of $\varphi_1$ as

$$\Psi : \mathcal{H}^{2,\alpha}_\omega \to C^{2,\alpha}_\omega(M)$$

$$\varphi \mapsto \varphi + \left(\mathcal{D}F_{r|\varphi_1}\right)^{-1}\left(- F_r(\varphi)\right)$$

Our goal is to find $\varphi \in \mathcal{H}^{2,\alpha}_\omega$ such that $F_r(\varphi) = 0$. Given the definition of $\Psi$, our problem comes down to find the fixed point of $\Psi$. So we need to show that $\Psi$ is a contraction in a small neighborhood of $\varphi_1 \in \mathcal{H}^{2,\alpha}_\omega$.

**Lemma 4.3.** There exists some $\delta > 0$, such that if $\varphi, \tilde{\varphi} \in \mathcal{H}^{2,\alpha}_\omega$ with $\|\varphi - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}} \delta$ and $\|\tilde{\varphi} - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}} \delta$, then

$$\|\Psi(\varphi) - \Psi(\tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}.$$  \quad (77)

**Proof.** Denote $\varphi_s = s \varphi + (1 - s) \tilde{\varphi}$. Suppose $\|\varphi - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}} \delta$ and $\|\tilde{\varphi} - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}} \delta$. We’ll specify $\delta > 0$ later.

We have

$$\Psi(\varphi) - \Psi(\tilde{\varphi}) = \int_0^1 \frac{\partial}{\partial s} \Psi(\varphi_s) ds$$

$$= (\varphi - \tilde{\varphi}) - \int_0^1 \left(\mathcal{D}F_{r|\varphi_1}\right)^{-1}\left(\mathcal{D}F_{r|\varphi_s}(\varphi - \tilde{\varphi})\right) ds$$

$$= - \int_0^1 \left(\mathcal{D}F_{r|\varphi_1}\right)^{-1}\left\{\left(\mathcal{D}F_{r|\varphi_s} - \mathcal{D}F_{r|\varphi_1}\right)(\varphi - \tilde{\varphi})\right\} ds.$$ 

We consider the term

$$\left(\mathcal{D}F_{r|\varphi_s} - \mathcal{D}F_{r|\varphi_1}\right)(\varphi - \tilde{\varphi}) = r\left(- (\Delta_{\varphi_s} - \Delta_{\varphi_1})(\varphi - \tilde{\varphi}) + \int_M \left((\Delta_{\varphi_s} - \Delta_{\varphi_1})(\varphi - \tilde{\varphi})\right) \omega^n\right.$$

$$+ \left((\mathcal{D}P_{r|\varphi_s} - \mathcal{D}P_{r|\varphi_1})(\varphi - \tilde{\varphi})\right).$$

Thus, we know that

$$\left(\mathcal{D}F_{r|\varphi_s} - \mathcal{D}F_{r|\varphi_1}\right)(\varphi - \tilde{\varphi})\|_{C^\alpha(M)}$$

$$\leq C r^{\frac{1}{1-\alpha}} \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)} + \|\left((\mathcal{D}P_{r|\varphi_s} - \mathcal{D}P_{r|\varphi_1})(\varphi - \tilde{\varphi})\right)\|_{C^\alpha(M)).}$$  \quad (79)$$

By definitions of $\mathcal{D}P_{r|\varphi}$ in [11], we have

$$\Delta_{\varphi_1}\left(\mathcal{D}P_{r|\varphi_1}(u)\right) = \left(\partial\partial u, \left(\partial\partial P_{r|\varphi_1} - (\text{Ric}(\omega) - R\omega)\right)\right)_{\varphi_1}, \int_M \left(\mathcal{D}P_{r|\varphi_1}(u)\right) \omega^n = 0$$

$$\Delta_{\varphi_s}\left(\mathcal{D}P_{r|\varphi_s}(u)\right) = \left(\partial\partial u, \left(\partial\partial P_{r|\varphi_s} - (\text{Ric}(\omega) - R\omega)\right)\right)_{\varphi_s}, \int_M \left(\mathcal{D}P_{r|\varphi_s}(u)\right) \omega^n = 0.$$  \quad (81)
So
\[
\Delta_{\varphi_1}(\mathcal{D}P|_{\varphi_1}(u) - \mathcal{D}P|_{\varphi_s}(u)) = \langle \partial \bar{\partial}u, (\partial \bar{\partial}P_{\varphi_1} - (\text{Ric}(\omega) - R\omega)) \rangle_{\varphi_1} - \langle \partial \bar{\partial}u, (\partial \bar{\partial}P_{\varphi_s} - (\text{Ric}(\omega) - R\omega)) \rangle_{\varphi_s}
+ (\Delta_{\varphi_s} - \Delta_{\varphi_1})(\mathcal{D}P|_{\varphi_s}(u))
\]
\[
= u_{i\bar{j}}(g^{i\bar{j}}_{\varphi_1}g^{k\bar{k}}_{\varphi_1} - g^{i\bar{j}}_{\varphi_s}g^{k\bar{k}}_{\varphi_s})(P_{\varphi_1,\bar{k}\bar{l}} + u_{i\bar{j}}g^{i\bar{j}}_{\varphi_s}g^{k\bar{k}}_{\varphi_s}(P_{\varphi_1} - P_{\varphi_s})
- u_{i\bar{j}}(g^{i\bar{j}}_{\varphi_1}g^{k\bar{k}}_{\varphi_1} - g^{i\bar{j}}_{\varphi_s}g^{k\bar{k}}_{\varphi_s})(\text{Ric}(\omega) - R(\omega))_{\bar{k}\bar{l}} + (g^{k\bar{k}}_{\varphi_1} - g^{k\bar{k}}_{\varphi_s})(\mathcal{D}P|_{\varphi_s}(u))_{\bar{k}\bar{l}}
\]
Thus, by Schauder estimate and previous estimate about $P_{\varphi}$ and $\mathcal{D}P|_{\varphi}(u)$ in Section 3,
\[
\|\mathcal{D}P|_{\varphi_1}(u) - \mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)} \leq C r^\frac{\alpha}{r} \delta \|u\|_{C^{2,\alpha}(M)} + \|\mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)}
+ C\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)}\|u\|_{C^{2,\alpha}(M)}
\leq C r^\frac{\alpha}{r} \delta \|u\|_{C^{2,\alpha}(M)} + C\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)}\|u\|_{C^{2,\alpha}(M)}.
\]
Since we have
\[
\Delta_{\varphi_1}(P_{\varphi_1} - P_{\varphi_s}) = (g^{k\bar{k}}_{\varphi_1} - g^{k\bar{k}}_{\varphi_s})P_{\varphi_s,k\bar{l}} + (g^{k\bar{k}}_{\varphi_1} - g^{k\bar{k}}_{\varphi_s})(\text{Ric}(\omega) - R\omega)_{k\bar{l}},
\]
then
\[
\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)} \leq C r^\frac{\alpha}{r} \delta.
\]
Thus, we have
\[
\|\mathcal{D}P|_{\varphi_1}(\varphi - \tilde{\varphi})\|_{C^{\alpha}(M)} \leq \|\mathcal{D}P|_{\varphi_1} - \mathcal{D}P|_{\varphi_s}(\varphi - \tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq C r^\frac{\alpha}{r} \delta\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}.
\]
By Lemma 4.3, we have that
\[
\|\mathcal{D}F_{r}|_{\varphi_1}^{-1}\{\mathcal{D}F_{r}|_{\varphi_s} - \mathcal{D}F_{r}|_{\varphi_1}(\varphi - \tilde{\varphi})\}\|_{C^{2,\alpha}(M)} \leq C r^{-\frac{\alpha}{rr}} \delta\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}
\leq C\delta\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}.
\]
And then
\[
\|\Psi(\varphi) - \Psi(\tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq C\delta\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}.
\]
We could choose $\delta > 0$ sufficiently small such that $C\delta < \frac{1}{2}$, and thus it ends the proof of Lemma 4.3. \qed

Now we’re ready to prove the Theorem 2.1.

Proof. Denote the constant $\delta > 0$ in Lemma 4.3 as $\delta_0$. Define for $k \in \mathbb{Z}$
\[
\varphi_k = \Psi^{k-1}(\varphi_1).
\]
Ultimately, we want to show that $\varphi_k \to \varphi_\infty$ in $C^{2,\alpha}(M)$ norm for some $\varphi_\infty \in H_{\omega}^{2,\alpha}$ as $k \to \infty$. We choose the start point to be $\varphi_1$, thus we need to show that $\varphi_2$ stays in the neighborhood of $\varphi_1$ for $\Psi$ to be contraction. Compute

$$
\|\varphi_2 - \varphi_1\|_{C^{2,\alpha}(M)} = \|\left(DF_r|_{\varphi_1}\right)^{-1}\left(1 - Fr(\varphi_1)\right)\|_{C^{2,\alpha}(M)}
\leq C r^{-\frac{2}{1-\alpha}}\|Fr(\varphi_1)\|_{C^{\alpha}(M)}
\leq (C r^{-\frac{2}{1-\alpha}}) r^{\frac{1}{1-\alpha}}.
$$

where we use Lemma 3.1 and Lemma 4.1. It’s obvious we could choose $\alpha = \frac{1}{4}$ and $r > 0$ sufficiently small such that

$$
\|\varphi_2 - \varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2} r^{\frac{1}{1-\alpha}} \delta_0. \tag{88}
$$

By induction, we could get that for any $k \in \mathbb{Z}$

$$
\|\varphi_k - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{k}{2(1-\alpha)}} \delta_0, \tag{89}
$$

and

$$
\|\varphi_{k+1} - \varphi_k\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\|\varphi_k - \varphi_{k-1}\|_{C^{2,\alpha}(M)} \leq \left(\frac{1}{2}\right)^k r^{\frac{k}{2(1-\alpha)}} \delta_0. \tag{90}
$$

Thus, we conclude that there exists some $\varphi_\infty \in H_{\omega}^{2,\alpha}$ such that $\varphi_k \to \varphi_\infty$ in $C^{2,\alpha}(M)$ as $k \to \infty$. Thus, we get

$$
Fr(\varphi_\infty) = 0. \tag{91}
$$

From the regularity of elliptic equation, we could immediately see that $\varphi_\infty \in C^{\infty}(M)$. Also it’s clear that

$$
\|\varphi_\infty\|_{C^{2,\alpha}(M)} \leq \|\varphi_1\|_{C^{2,\alpha}(M)} + r^{\frac{1}{1-\alpha}} \delta_0 \leq C r \to 0, \text{ as } r \to 0. \tag{92}
$$

Then we finish the proof of Theorem 2.1.

\[\Box\]

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