Ordered multiplicity inverse eigenvalue problem for graphs on six vertices

John Ahn* Christine Alar† Beth Bjorkman‡ Steve Butler‡ Joshua Carlson† Audrey Goodnight§ Haley Knox¶ Casandra Monroe∥ Michael C. Wigal∗∗

Abstract

For a graph $G$, we associate a family of real symmetric matrices, $S(G)$, where for any $M \in S(G)$, the location of the nonzero off-diagonal entries of $M$ are governed by the adjacency structure of $G$. The ordered multiplicity Inverse Eigenvalue Problem of a Graph (IEPG) is concerned with finding all attainable ordered lists of eigenvalue multiplicities for matrices in $S(G)$.

For connected graphs of order six, we offer significant progress on the IEPG, as well as a complete solution to the ordered multiplicity IEPG. We also show that while $K_{m,n}$ with $\min(m,n) \geq 3$ attains a particular ordered multiplicity list, it cannot do so with arbitrary spectrum.

1 Introduction

A graph $G$ consists of a vertex set $V(G)$ and an edge set $E(G)$. Given $G$ with vertices $v_1, \ldots, v_n$, a real symmetric matrix $M$ is in $S(G)$ if for all $i \neq j$, $M_{i,j} = 0$ if and only if $v_i v_j \notin E(G)$; there are no restrictions on the diagonal entries.

The spectrum of a matrix $M$ is the set of eigenvalues of $M$. Let $\lambda_1 < \cdots < \lambda_k$ be the distinct eigenvalues of $M$ in increasing order, and let $\gamma_i$ be the multiplicity of $\lambda_i$ as an eigenvalue of $M$. Then the ordered multiplicity list of $M$ is $(\gamma_1, \ldots, \gamma_k)$. (With this convention, the spectrum of $M$ is \{ $\lambda_1^{(\gamma_1)}, \lambda_2^{(\gamma_2)}, \ldots, \lambda_k^{(\gamma_k)}$ \}.)

The Inverse Eigenvalue Problem of a Graph (IEPG) is stated as follows: given $G$ and a set of numbers $L = \{\ell_1, \ldots, \ell_n\}$, does there exist a matrix $M \in S(G)$ with spectrum $L$?

This problem has been completely resolved through graphs on five vertices (see [5]). More information about the IEPG can be found in the survey of Hogben [12]. Because

---

* Bowdoin College, Brunswick, ME 04011, USA jtahn@bowdoin.edu
† San Francisco State University, San Francisco, CA 94132, USA dalar@mail.sfsu.edu
‡ Iowa State University, Ames, IA 50011, USA {bjorkman,butler,jmsdg7}@iastate.edu
§ Agnes Scott College, Decatur, GA 30030, USA audrey.goodnight@gmail.com
¶ Eastern Connecticut State University, Willimantic, CT 06286, USA knoxh@my.easternct.edu
∥ Princeton University, Princeton, NJ 08544, USA cdm5@princeton.edu
∗∗ West Virginia University, Morgantown, WV 26505, USA mcwigal@mix.wvu.edu
of the difficulty of the IEPG, many relaxations have been considered; previous works have examined inverse inertia (see [7]), minimum rank and maximum nullity (see [9]), and the minimum number of distinct eigenvalues (see [8]).

We consider the ordered multiplicity inverse eigenvalue problem for graphs, a slight relaxation of the IEPG: given a graph $G$ and an ordered list of integers $\Gamma = (\gamma_1, \ldots, \gamma_k)$, does there exist a matrix $M \in \mathcal{S}(G)$ that attains $\Gamma$ as its ordered multiplicity list?

We label graphs using the Atlas of Graphs [13]; these graphs are also reproduced in the appendix for reference. In this paper, we solve the ordered multiplicity IEPG for all connected graphs of order six. The result is summarized in Figure 1: the graphs are in 26 different equivalence classes based on what ordered multiplicity lists are attainable. To determine what lists are attainable, locate the equivalence class it belongs in and then read off all ordered multiplicity lists (and reversals) on the edges of a directed path from that equivalence class to $\emptyset$. This diagram also gives all possible relationships between equivalence classes, namely the equivalence class containing $G$ attains all ordered multiplicity lists as the equivalence class containing $H$ if and only if there is a directed path from the equivalence class containing $G$ to the one containing $H$.

We say a graph $G$ is spectrally arbitrary for an ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ if for any $\lambda_1 < \cdots < \lambda_k$, there is a matrix $M \in \mathcal{S}(G)$ with spectrum $\{\lambda_1^{(\gamma_1)}, \ldots, \lambda_k^{(\gamma_k)}\}$. Many of the techniques we use show that a graph is spectrally arbitrary for an ordered multiplicity list. In the appendix, for each ordered multiplicity list, we give all graphs which can attain that list, indicating those which are not known to be attainable with arbitrary spectrum.

We proceed as follows. In Section 2 we will review what is known for the IEPG for graphs on five or fewer vertices which we will build on for the case of six vertices. In Section 3 we will introduce a technique we call cloning and how it connects with ordered multiplicity lists of eigenvalues. In Section 4 we justify what ordered multiplicity lists are unattainable, while in Section 5 we justify what ordered multiplicity lists are attainable for graphs on six vertices. In Section 6 we show that $K_{m,n}$ with $\min(m, n) \geq 3$ is a graph for which the ordered multiplicity IEPG differs from the IEPG. Finally, in Section 7 we give concluding remarks.

Because we are working on graphs with six or fewer vertices, it will be unambiguous to write the ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ as $\gamma_1 \ldots \gamma_k$. We will say a graph $G$ attains $\gamma_1 \ldots \gamma_k$ if there is some $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$; similarly, $G$ does not attain $\gamma_1 \ldots \gamma_k$ if there is no $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$. We note that a graph attains $\gamma_1 \ldots \gamma_k$ if and only if it attains $\gamma_k \ldots \gamma_1$, which follows by noting if $M \in \mathcal{S}(G)$ then so also is $-M$.

## 2 IEPG for graphs on five or fewer vertices

The IEPG for all graphs of order at most five was recently solved by Barrett et al. [5]. They showed that for graphs with five or fewer vertices, the IEPG is equivalent to the ordered multiplicity IEPG. Thus, a graph on five or fewer vertices attains a given spectrum if and only if the corresponding multiplicity list is attainable. Two of the main tools used to solve this problem were the Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP), introduced in an earlier paper (see [6]).

**Definition 1.** An $n \times n$ symmetric matrix $A$ has the **Strong Spectral Property (SSP)** if the
Figure 1: There are 26 equivalence classes; those in blue have one graph while those in yellow have their full membership given in the boxes. To determine attainable ordered multiplicity lists for a graph, find its equivalence class in the diagram and take any path to ∅; the multiplicity lists (and reversals) that occur on the edges of the path are the only ones attainable. The graph $G$ attains all multiplicity lists that $H$ attains if and only if there is a directed path from the class containing $G$ to the class containing $H$; the difference in what is attainable are the multiplicity lists which occur on any directed path between them.
only symmetric matrix \( X \) satisfying \( A \circ X = I \circ X = AX -XA = O \) is \( X = O \) (where “\( \circ \)” indicates the Hadamard, or entry-wise, product of matrices).

**Definition 2.** An \( n \times n \) symmetric matrix \( A \) satisfies the **Strong Multiplicity Property (SMP)** if the only symmetric matrix \( X \) satisfying \( A \circ X = I \circ X = AX -XA = O \) and \( \text{tr}(A^iX) = 0 \) for \( i = 2, \ldots, n-1 \) is \( X = O \).

These properties are important for the IEPG because SSP (SMP) allows us to determine the attainability of certain spectra (ordered multiplicity lists) for many graphs simultaneously. It should be noted that testing if a matrix has SSP (SMP) reduces to showing if a large linear system has full rank. This has been implemented and is available online (see [11]); any matrix which is claimed to have SSP follows either from Barrett et al. [5] or by using the online implementation.

**Theorem 1** (Barrett et al. [6] Theorems 2.10 and 2.20). If \( A \in S(G) \) has SSP (SMP), then every supergraph of \( G \) with the same vertex set has an SSP (SMP) realization with the same spectrum (ordered multiplicity list).

**Theorem 2** (Barrett et al. [6], Theorem 3.8). If \( A \in S(G) \) and \( B \in S(H) \) both have SSP (SMP) and \( \text{spec}(A) \cap \text{spec}(B) = \emptyset \), then \( A \oplus B \in S(G \cup H) \) has SSP (SMP).

Using the above theorems and several constructions, Barrett et al. [5] determined that the multiplicity lists given in Table 1 are attainable with SSP, and those in Table 2 are attainable but without SMP or SSP. Moreover, they established that the graphs can attain any spectrum compatible with the ordered multiplicity lists it attains. Thus, the IEPG and the ordered multiplicity IEPG are equivalent for graphs on five or fewer vertices.

## 3 Cloning vertices and ordered multiplicity lists

In this section, we introduce **cloning**, a graph operation that, given \( G \) and \( v \in V(G) \), constructs a new graph \( H \) by adding a new vertex \( v' \) which is a twin of \( v \). This operation is sometimes referred to as duplicating or blow-ups.

**Definition 3.** Two vertices \( u \) and \( w \) are **twins** in \( H \) if \( N_H(u) \setminus \{w\} = N_H(w) \setminus \{u\} \), where \( N_H(v) \) is the set of neighbors of \( v \) (i.e., vertices which share an edge with \( v \)).

Twins do not need to be adjacent. This leads to two variants of cloning: cloning \( v \) with an edge requires that \( v \sim v' \) (i.e., \( v \) and \( v' \) are adjacent), while cloning without an edge requires \( v \not\sim v' \) (i.e., \( v \) and \( v' \) are not adjacent).

**Theorem 3.** Let \( G \) be a graph with \( M \in S(G) \) having multiplicity list \( (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_k) \) where the eigenvalue 0 has multiplicity \( \gamma_i \). Then the following two cases are possible:

1. If the diagonal entry of \( M \) corresponding to \( v_j \) is zero, then the graph \( H \) attained from \( G \) by cloning \( v_j \) without an edge has a matrix \( N \in S(H) \) that attains the multiplicity list \( (\gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_k) \).
| Graphs | Attainable Ordered multiplicity lists |
|--------|-------------------------------------|
| G1     | 1                                   |
| G3     | 11                                  |
| G6     | 111                                 |
| G7     | 111, 12, 21                         |
| G14    | 1111                                |
| G13    | 1111, 121                           |
| G15    | 1111, 121, 112, 211                 |
| G16, G17 | 1111, 121, 112, 211, 22       |
| G18    | 1111, 121, 112, 211, 22, 13, 31    |
| G31    | 11111                               |
| G29, G30 | 11111, 1121, 1211                 |
| G35, G36 | 11111, 1121, 1211, 1112, 2111   |
| G34, G38 | 11111, 1121, 1211, 1112, 2111, 122, 221 |
| G37, G40, G41 | 11111, 1121, 1112, 2111, 122, 221, 212 |
| G42, G43, G47 | 11111, 1121, 1112, 2111, 122, 221, 212 |
| G44, G46 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131 |
| G45    | 11111, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311 |
| G48, G49 | 11111, 1121, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32 |
| G50, G51 | 11111, 1121, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41 |
| G52    | 11111, 1121, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41 |

Table 1: Realizable ordered multiplicity lists for connected graphs with five or fewer vertices and are attainable with SSP.

| Graphs | Attainable Ordered multiplicity lists |
|--------|-------------------------------------|
| G29    | 131                                  |
| G42    | 113, 131, 311                         |

Table 2: Realizable ordered multiplicity lists for connected graphs with five or fewer vertices which cannot have SMP or SSP.

2. If the diagonal entry of $M$ corresponding to $v_j$ is nonzero, then the graph $H$ attained from $G$ by cloning $v_j$ with an edge has a matrix $N \in S(H)$ that attains the multiplicity list $(\gamma_1, \ldots, \gamma_i + 1, \ldots \gamma_k)$.

Proof. Let $-\lambda_1 \leq \cdots \leq \lambda_{n-\gamma_i}$ be the nonzero eigenvalues of $M$ where $a$ is the last index such that $-\lambda_a < 0$. Let $x_1, \ldots, x_{n-\gamma_i}$ be the corresponding orthonormal eigenvectors. Then

$$M = \sum_{k=1}^{n-\gamma_i} \lambda_k x_k x_k^T$$

$$= \begin{pmatrix} x_1 & \cdots & x_{n-\gamma_i} \end{pmatrix} \begin{pmatrix} -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_i} \end{pmatrix} \begin{pmatrix} x_1^T \\ \vdots \\ x_{n-\gamma_i}^T \end{pmatrix}$$

5
\[ (\sqrt{\lambda_1}x_1 \cdots \sqrt{\lambda_{n-\gamma}x_{n-\gamma}}) \begin{pmatrix} -I_a & O \\ O & I_{n-\gamma_1-a} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1}x_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma}x_{n-\gamma}^T} \end{pmatrix}. \]

The columns of
\[ Y = \begin{pmatrix} \sqrt{\lambda_1}x_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma}x_{n-\gamma}^T} \end{pmatrix} \]
are an orthogonal representation of \( G \) with respect to the indefinite inner product \( S \). That is, if \( y_k \) denotes the \( k \)-th column then \( y_k^T S y_\ell = 0 \) if and only if \( v_k \not\sim v_\ell \) in \( G \).

Let \( Z \) be a \((n - \gamma_i) \times (n + 1)\) matrix with columns as follows,
- \( z_k = y_k \) for \( 1 \leq k < j \);
- \( z_j = z_{j+1} = \frac{1}{\sqrt{2}}y_j \);
- \( z_k = y_{k-1} \) for \( j + 1 < k \leq n + 1 \).

Now consider the matrix \( N = Z^T S Z \). Since \( N \) is real symmetric, there exists some graph \( H \) such that \( N \in S(H) \). The following two observations will now conclude the proof.

First, use the columns of \( Z \) as an orthogonal representation for \( H \) with respect to \( S \). This corresponds to the graph \( G \) with the vertex \( v_j \) cloned (i.e., columns still have the same orthogonality relationships as given by \( Y \)). This will have cloned with an edge if and only if \( y_j^T S y_j \neq 0 \). The latter holds if and only if the diagonal entry of \( M \) corresponding to \( v_j \) is nonzero.

Second, the inner product of any two rows of \( Z \) agree with the inner product of the corresponding rows of \( Y \). So the nonzero eigenvalues of \( N \) are
\[
SZZ^T = \begin{pmatrix} -I_a & O \\ O & I_{n-\gamma_1-a} \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_1} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_1} \end{pmatrix},
\]
which are the same as those of \( M \). Hence \( N \) has the same spectrum of \( M \) with the addition of a single eigenvalue of 0, giving us the desired ordered multiplicity list. \( \square \)

**Corollary 4.** Let \( G \) be a graph without isolated vertices, and let \( M \in S(G) \) with multiplicity list \( (m_1, \ldots, m_k) \). If a graph \( H \) is attained by cloning \( v \in V(G) \) with an edge, then \( H \) attains the multiplicity list \( (m_1 + 1, \ldots, m_k) \).

**Proof.** By translation we can assume \( M \) is positive semi-definite with nullity \( m_1 \). If any diagonal entry were 0, this would force a row and column of zeroes. However, this implies that \( G \) contains an isolated vertex, a contradiction. Thus, the entries of the diagonal are nonzero, and so we apply the previous theorem by cloning with an edge. \( \square \)

### 4 Unattainable multiplicity lists for graphs

In this section we will determine which ordered multiplicity lists are unattainable for connected graphs on six vertices.
4.1 Using known graph parameters

Since we can assign any particular eigenvalue to 0 by translation, we have the following observations.

**Observation 1.** If $M(G)$ denotes the maximum nullity of a matrix in $S(G)$, then all entries of a multiplicity list of a matrix in $S(G)$ are bounded above by $M(G)$.

**Observation 2.** If $M_+(G)$ denotes the maximum nullity of a positive semidefinite matrix in $S(G)$, then the first (and by reversal from negation, the last) entry of a multiplicity list of a matrix in $S(G)$ are bounded above by $M_+(G)$.

In general, the computation of $M(G)$ and $M_+(G)$ is an open problem. However, for graphs on seven or fewer vertices, known techniques can find these values (in particular, the inertia tables—see [4]). It suffices to provide an upper bound for these parameters, which can be done through the combinatorial parameters $Z(G)$ and $Z_+(G)$, respectively known as the zero-forcing number and semidefinite zero-forcing number of a graph. Because of their combinatorial nature, $Z(G)$ and $Z_+(G)$ can be easily computed for small graphs through exhaustive analysis. The definition of these parameters, as well as related extensions and results, can be found in the survey of Fallat and Hogben [9]. For our purposes, we will use the following result.

**Lemma 5** (AIM [1], Prop. 2.4). For any graph $G$, we have $M(G) \leq Z(G)$.

**Lemma 6** (Barioli et al. [2], Theorem 3.5). For any graph $G$, we have $M_+(G) \leq Z_+(G)$.

Another useful parameter is $q(G)$, the minimum number of distinct eigenvalues.

**Observation 3.** The length of any ordered multiplicity list for $M \in S(G)$ is at least $q(G)$.

This is a harder parameter to compute; for connected graphs of order at most six, $q(G)$ was recently determined (see [8]). For all connected graphs of order six, the parameters $Z(G)$, $Z_+(G)$, and $q(G)$ are given in Table 3 and rule out many ordered multiplicity lists.

4.2 Previous results to rule out ordered multiplicity lists

The following two results, both from [5], rule out several cases.

**Lemma 7** (Barrett et al. [5], Lemma 3.3). If $G$ is a connected unicyclic graph with odd girth, then at least one of the first or last eigenvalues has multiplicity one.

This rules out 2112 and 222 for $G_{92}$, $G_{93}$, $G_{94}$, $G_{95}$, $G_{100}$, and $G_{104}$.

**Lemma 8** (Barrett et al. [5], Lemmas 2.3 and 5.2). A generalized star or a generalized 3-sun does not allow an ordered multiplicity list with consecutive multiple eigenvalues.

This rules out 1122, 1221, and 2211 for $G_{77}$ and $G_{78}$ (generalized stars) and $G_{94}$ (3-sun). The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca [10], and in particular it follows that 1212 and 2121 are not attainable for the graph $C_6$ ($G_{105}$).
| \( G \) | \( Z \) | \( Z_+ \) | \( q \) | \( G \) | \( Z \) | \( Z_+ \) | \( q \) | \( G \) | \( Z \) | \( Z_+ \) | \( q \) | \( G \) | \( Z \) | \( Z_+ \) | \( q \) |
|-------|---|---|---|-------|---|---|---|-------|---|---|---|-------|---|---|---|
| \( G77 \) | 4 | 1 | 3 | \( G121 \) | 3 | 2 | 3 | \( G151 \) | 3 | 3 | 3 | \( G181 \) | 3 | 3 | 2 |
| \( G78 \) | 3 | 1 | 4 | \( G122 \) | 2 | 2 | 4 | \( G152 \) | 2 | 2 | 3 | \( G182 \) | 3 | 3 | 3 |
| \( G79 \) | 2 | 1 | 4 | \( G123 \) | 2 | 2 | 4 | \( G153 \) | 3 | 3 | 3 | \( G183 \) | 3 | 3 | 3 |
| \( G80 \) | 2 | 1 | 5 | \( G124 \) | 2 | 2 | 4 | \( G154 \) | 3 | 3 | 2 | \( G184 \) | 3 | 3 | 3 |
| \( G81 \) | 2 | 1 | 5 | \( G125 \) | 3 | 2 | 3 | \( G156 \) | 3 | 3 | 3 | \( G185 \) | 3 | 3 | 3 |
| \( G83 \) | 1 | 1 | 6 | \( G126 \) | 3 | 3 | 3 | \( G157 \) | 3 | 3 | 3 | \( G186 \) | 3 | 3 | 3 |
| \( G92 \) | 3 | 2 | 3 | \( G127 \) | 2 | 2 | 3 | \( G158 \) | 3 | 3 | 3 | \( G187 \) | 3 | 3 | 3 |
| \( G93 \) | 2 | 2 | 4 | \( G128 \) | 2 | 2 | 3 | \( G159 \) | 3 | 3 | 3 | \( G188 \) | 3 | 3 | 2 |
| \( G94 \) | 2 | 2 | 4 | \( G129 \) | 3 | 3 | 4 | \( G160 \) | 3 | 3 | 3 | \( G189 \) | 4 | 3 | 3 |
| \( G95 \) | 2 | 2 | 4 | \( G130 \) | 3 | 3 | 4 | \( G161 \) | 4 | 2 | 3 | \( G190 \) | 4 | 4 | 2 |
| \( G96 \) | 2 | 2 | 3 | \( G133 \) | 3 | 3 | 3 | \( G162 \) | 3 | 2 | 3 | \( G191 \) | 4 | 4 | 3 |
| \( G97 \) | 2 | 2 | 5 | \( G134 \) | 3 | 3 | 4 | \( G163 \) | 3 | 3 | 3 | \( G192 \) | 3 | 3 | 2 |
| \( G98 \) | 2 | 2 | 4 | \( G135 \) | 3 | 2 | 3 | \( G164 \) | 2 | 2 | 3 | \( G193 \) | 3 | 3 | 3 |
| \( G99 \) | 2 | 2 | 3 | \( G136 \) | 2 | 2 | 3 | \( G165 \) | 4 | 4 | 3 | \( G194 \) | 4 | 4 | 2 |
| \( G100 \) | 3 | 2 | 4 | \( G137 \) | 2 | 2 | 3 | \( G166 \) | 3 | 3 | 3 | \( G195 \) | 4 | 4 | 2 |
| \( G102 \) | 2 | 2 | 5 | \( G138 \) | 3 | 2 | 3 | \( G167 \) | 2 | 2 | 3 | \( G196 \) | 3 | 3 | 2 |
| \( G103 \) | 2 | 2 | 4 | \( G139 \) | 2 | 2 | 4 | \( G168 \) | 3 | 3 | 2 | \( G197 \) | 4 | 3 | 2 |
| \( G104 \) | 2 | 2 | 4 | \( G140 \) | 3 | 3 | 3 | \( G169 \) | 3 | 3 | 3 | \( G198 \) | 3 | 3 | 2 |
| \( G105 \) | 2 | 2 | 3 | \( G141 \) | 3 | 3 | 3 | \( G170 \) | 3 | 3 | 3 | \( G199 \) | 4 | 4 | 2 |
| \( G111 \) | 2 | 2 | 3 | \( G142 \) | 3 | 3 | 4 | \( G171 \) | 3 | 3 | 3 | \( G200 \) | 4 | 4 | 2 |
| \( G112 \) | 2 | 2 | 4 | \( G143 \) | 3 | 3 | 3 | \( G172 \) | 3 | 3 | 3 | \( G201 \) | 4 | 3 | 2 |
| \( G113 \) | 2 | 2 | 4 | \( G144 \) | 3 | 3 | 3 | \( G173 \) | 3 | 3 | 3 | \( G202 \) | 3 | 3 | 2 |
| \( G114 \) | 3 | 2 | 3 | \( G145 \) | 3 | 2 | 3 | \( G174 \) | 3 | 3 | 2 | \( G203 \) | 4 | 4 | 2 |
| \( G115 \) | 2 | 2 | 3 | \( G146 \) | 4 | 2 | 3 | \( G175 \) | 4 | 3 | 2 | \( G204 \) | 4 | 4 | 2 |
| \( G117 \) | 3 | 3 | 3 | \( G147 \) | 2 | 2 | 3 | \( G177 \) | 3 | 3 | 3 | \( G205 \) | 4 | 4 | 2 |
| \( G118 \) | 2 | 2 | 3 | \( G148 \) | 2 | 2 | 3 | \( G178 \) | 3 | 3 | 3 | \( G206 \) | 4 | 4 | 2 |
| \( G119 \) | 3 | 3 | 4 | \( G149 \) | 3 | 2 | 3 | \( G179 \) | 3 | 3 | 3 | \( G207 \) | 4 | 4 | 2 |
| \( G120 \) | 2 | 2 | 4 | \( G150 \) | 3 | 3 | 3 | \( G180 \) | 3 | 3 | 3 | \( G208 \) | 5 | 5 | 2 |

Table 3: The values for \( Z(G) \), \( Z_+(G) \), and \( q(G) \) for connected graphs on six vertices.

### 4.3 Remaining Cases

After applying the graph parameters, Lemma 7, and Lemma 8, the remaining unattainable cases are shown in Figure 2.

Proving that these cases are unattainable will be done by contradiction; namely, by assuming the existence of a matrix that achieves the specified ordered multiplicity list on a graph. An examination of the corresponding orthogonal representation allows us to argue that two of the vectors are scalar multiples. This pair of vectors allows us to “declone” the original graph to a smaller graph with a corresponding matrix that has an impossible ordered multiplicity list.

Recall that cloning works by taking a single vertex and creating a pair of vertices where the vector corresponding with each vertex is a scalar multiple of the original. Decloning does
the opposite: if we can show that the vectors corresponding to a pair of vertices must be scalar multiples, then the two vertices must be twins. We can then delete one of the twins and decrease an entry in the ordered multiplicity list.

**Lemma 9 (Decloning).** Let $G$ be a graph with $M = Q^T SQ \in S(G)$ having multiplicity list $(\gamma_1, \ldots, \gamma_k)$ where the eigenvalue 0 has multiplicity $\gamma_i$. Further assume that $S$ is symmetric and has dimension $n - \gamma_i$, where $n$ is the order of $G$. If the columns of $Q$ corresponding to vertices $a$ and $b$ are scalar multiples, then there exists $N \in S(H)$ where $H$ is attained from $G$ by deleting vertex $a$ and $N$ has ordered multiplicity list $(\gamma_1, \ldots, \gamma_i - 1, \ldots, \gamma_k)$.

**Proof.** Let $Q = (x_1 x_2 \cdots x_n)$, and without loss of generality assume that $x_1$ and $x_2$ are scalar multiples, i.e., $x_2 = \alpha x_1$. Let $\hat{Q} = (\sqrt{1 + \alpha^2} x_2 \cdots x_n)$; we claim $N = \hat{Q}^T S \hat{Q}$.

Note that two vertices $u$ and $v$ in the graph are not adjacent if and only if $x_u^T S x_v = 0$. Since scaling by a nonzero value does not change this, we have that the adjacencies are preserved and $N \in S(H)$. It remains to check the spectrum of $N$, but for this we note that $QQ^T = \hat{Q}\hat{Q}^T$ since the dot products of any pair of corresponding rows are identical. Since the nonzero portion of the spectrum comes from $SQQ^T = S\hat{Q}\hat{Q}^T$, the result follows.

**Proposition 10.** If a symmetric matrix $M$ has nullity three and ordered multiplicity 312, then $M = Q^T I Q$ where $I$ is the $3 \times 3$ identity matrix.

**Proof.** The matrix $M$ has spectrum $\{0^{(3)}, \lambda^{(1)}, \mu^{(2)}\}$. Let $x$ be a unit eigenvector for $\lambda$ and $y, z$ be orthonormal eigenvectors for $\mu$. We have that

$$M = (x \ y \ z) \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} x^T \\ y^T \\ z^T \end{pmatrix} = (\sqrt{\lambda} x \ \sqrt{\mu} y \ \sqrt{\mu} z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda} x^T \\ \sqrt{\mu} y^T \\ \sqrt{\mu} z^T \end{pmatrix}.$$

**Theorem 11.** The graphs $G_{117}$ and $G_{133}$ cannot attain ordered multiplicity lists 312 or 213.

**Proof.** Label the vertices of $G_{117}$ as shown in Figure 3. Suppose $M \in S(G_{117})$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 10, we can write $M = Q^T I Q$, where $Q$ is a matrix which forms an orthogonal representation for $G_{117}$. 

![Figure 2: The remaining unattainable ordered multiplicity lists.](image-url)
Since $v_1$ and $v_3$ are not adjacent, the corresponding vectors $v_1$ and $v_3$ are orthogonal and thus form a plane in $\mathbb{R}^3$. Both $v_5$ and $v_6$ are not adjacent to $v_1$ and $v_3$, so $v_5$ and $v_6$ are orthogonal to this plane and must be scalar multiples.

The decloning lemma implies that the graph with vertex $e$ deleted has a matrix that attains 212. However, this graph is odd unicyclic and thus cannot attain 212 by Lemma 7.

A similar argument establishes the result for $G_{133}$.

**Theorem 12.** The graph $G_{153}$ cannot attain ordered multiplicity lists 213 or 312.

**Proof.** Label the vertices of $G_{153}$ as shown in Figure 4. Suppose that $M \in S(G_{153})$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 10 we can write $M = Q^T S Q$, where $Q$ forms an orthogonal representation for $G_{153}$.

If $v_1$ and $v_2$ are scalar multiples, the decloning lemma implies a matrix for $C_5$ with multiplicity list 212, which is impossible by Lemma 7.

If $v_1$ and $v_2$ are not scalar multiples, then $v_1$ and $v_2$ form a plane in $\mathbb{R}^3$. Since vertices $v_4$ and $v_5$ are not adjacent to $v_1$ and $v_2$, the vectors $v_4$ and $v_5$ are orthogonal to the aforementioned plane. Thus, $v_4$ and $v_5$ are scalar multiples, an impossibility given that $v_4$ and $v_5$ have distinct neighbors.

In either case, we get a contradiction; thus, $G_{153}$ cannot attain 312.

For the remaining cases, we first establish an analogous result to Proposition 10.

**Proposition 13.** If a symmetric matrix $M$ has nullity three and ordered multiplicity 132, then $M = Q^T S Q$ where $S = \text{diag}(-1, 1, 1)$.

**Proof.** The matrix $M$ has spectrum $\{-\lambda^{(1)}, 0^{(3)}, \mu^{(2)}\}$. Let $x$ be a unit eigenvector for $-\lambda$ and $y, z$ be orthonormal eigenvectors for $\mu$. We have

$$M = \begin{pmatrix} x^T \ y^T \ z^T \end{pmatrix} \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} x^T \ y^T \ z^T \end{pmatrix} = \begin{pmatrix} \sqrt{-\lambda} x \ \
\sqrt{\mu} y \ \
\sqrt{\mu} z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{-\lambda} x^T \\ \sqrt{\mu} y^T \\ \sqrt{\mu} z^T \end{pmatrix} = Q.$$


Theorem 14. The graph G_{117} cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of G_{117} as shown in Figure 5. Suppose that \( M \in S(G_{117}) \) attains ordered multiplicity list 132 with nullity three. Applying Proposition 13, \( M = Q^T S Q \) where \( S = \text{diag}(-1, 1, 1) \).

Since \( v_1, v_3, \) and \( v_5 \) have distinct sets of neighbors, no two of \( v_1, v_3 \) and \( v_5 \) are scalar multiples of each other.

Because \( S \) is invertible, \( R = (v_1^T S, v_3^T S) \) has rank two and nullity one. Since \( v_1, v_3 \notin N(v_5) \), we have \( Rv_5 = (0, 0) \); similarly, \( v_1, v_3 \notin N(v_6) \), so \( Rv_6 = (0, 0) \). Thus, \( v_5 \) and \( v_6 \) are scalar multiples. A similar argument allows us to conclude that \( v_3 \) and \( v_4 \) are scalar multiples.

Now we can apply the decloning lemma twice (i.e., once for each scalar multiple pair) to produce a matrix for the graph \( K_{1,3} \) which attains 112. But \( Z_+(K_{1,3}) = 1 \), so the end terms of any multiplicity list of \( K_{1,3} \) must be 1, a contradiction. \( \square \)

In the following proposition, we introduce a tool that shows two vectors are scalar multiples, a technique similar to the one used in the previous results.

Proposition 15. If \( S \) is a 3\times3 symmetric invertible matrix and \( x, y \) are vectors that satisfy \( x^T S x = y^T S y = x^T S y = 0 \), then \( x \) and \( y \) are scalar multiples of each other.

Proof. Assume \( x \) and \( y \) are not scalar multiples; then \( Sx \) and \( Sy \) are also not scalar multiples. This implies the matrix

\[
R = \begin{pmatrix} x^T S \\ y^T S \end{pmatrix}
\]

has rank two and nullity one. On the other hand, by our hypothesis we have

\[
Rx = \begin{pmatrix} x^T Sx \\ y^T Sx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x^T Sy \\ y^T Sy \end{pmatrix} = Ry
\]

which shows that \( R \) has nullity at least two, a contradiction. \( \square \)

Theorem 16. The graph G_{121} cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of G_{121} as shown in Figure 5(a). Suppose that \( M \in S(G_{121}) \) has an ordered multiplicity list 132 with nullity three.

We claim that \( M_{3,3} = M_{5,5} = M_{6,6} = 0 \). To see this, suppose that \( M_{6,6} \neq 0 \); cloning \( v_6 \) with an edge produces the graph shown in Figure 5(b) which attains an ordered multiplicity list 142. The set marked in Figure 5(b) is a zero forcing set of order three. Thus, the cloned graph cannot attain 142, a contradiction. Hence, \( M_{6,6} = 0 \), and \( M_{3,3} = M_{5,5} = 0 \) by symmetry.

We apply Proposition 13 to write \( M = Q^T S Q \) where \( S = \text{diag}(-1, 1, 1) \). Since \( v_3, v_5, \) and \( v_6 \) form an independent set, we have \( v_3^T S v_6 = v_5^T S v_5 = v_6^T S v_6 = 0 \). Moreover, because \( M_{3,3} = M_{5,5} = M_{6,6} = 0 \), we have \( v_3^T S v_3 = v_5^T S v_5 = v_6^T S v_6 = 0 \). From Proposition 15 it follows that \( v_3, v_5, \) and \( v_6 \) are pairwise scalar multiples.

Applying the decloning lemma twice produces a matrix for \( P_4 \) which attains 112. However \( P_4 \) can only attain 1111, a contradiction. \( \square \)
Theorem 17. The graphs $G_{125}$ and $G_{138}$ cannot attain ordered multiplicity lists 132 or 231.

Proof. Label the vertices of $G_{125}$ as shown in Figure 5(a). Suppose $M \in \mathcal{S}(G_{125})$ has spectrum $\{-\lambda, 0^{(3)}, 2^{(2)}\}$ with $\lambda > 0$ (by scale and shift, this holds without loss of generality).

We claim that $M_{5,5} = M_{6,6} = 0$. To see this, suppose that $M_{5,5} \neq 0$; cloning $v_5$ with an edge produces the graph shown in Figure 6(b) which attains an ordered multiplicity list 142. The set marked in Figure 6(b) is a zero forcing set of order three. Thus, the cloned graph cannot attain 142, a contradiction. Hence, $M_{5,5} = 0$, and $M_{6,6} = 0$ by symmetry.

Applying Proposition 13 gives $M = Q^T S Q$, where $S = \text{diag}(-1, 1, 1)$. Note $v_5 \approx v_6$, so $v_5^T S v_6 = 0$. Moreover, $M_{5,5} = M_{6,6} = 0$, so $v_5^T S v_5 = v_6^T S v_6 = 0$. From Proposition 15 it follows that $v_5$ and $v_6$ are scalar multiples.

Applying the decloning lemma, we attain a matrix $N$ for the banner graph as labeled in Figure 7, where $N$ has eigenvalues $\{-\lambda, 0^{(2)}, 2^{(1)}\}$ and $N_{5,5} = 0$; particularly, $N$ has the following form, where $c_i$ are nonzero and $d_i$ are arbitrary:
The matrix $N$ has rank three; moreover, the first three rows are linearly independent. Therefore the fifth row is a linear combination of the first three rows, which implies that $c_4 = \alpha c_2$ and $c_5 = \alpha c_3$ for some $\alpha \neq 0$.

Let $R = (N - I)^2$ which has spectrum $\{1^{(4)}, (1 + \lambda)^2\}$. The only graphs that attain the ordered multiplicity list 41 are unions of complete graphs with isolated vertices. Since $R_{1,3} = c_1 c_2 \neq 0$ and $R_{1,5} = 0$, we must have that $v_1$ is in the clique and $v_5$ is isolated. In particular

$$0 = R_{2,5} = c_2 c_4 + c_3 c_5 = \alpha (c_2^2 + c_3^2) \neq 0,$$

which is impossible.

A similar argument establishes the result for $G_{138}$. \hfill $\square$

## 5 Attainable multiplicity lists for graphs

In this section we establish which ordered multiplicity lists are attainable for connected graphs on six vertices. We first utilize techniques which will also achieve arbitrary spectrum, and then describe those that fail to do so.

Before we begin, we note that a few special cases have already been done in the literature; we refer the reader elsewhere for details on the following cases.

- The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca.

**Theorem 18** (Fernandes and Fonseca [10], Theorem 3.3). The numbers $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ are the spectrum for some matrix $M \in S(C_n)$ if and only if

$$\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 < \cdots$$

or

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 < \lambda_6 < \cdots.$$
(In particular \(G_{105}\) attains 11112, 21111, 11211, 12111, 11211, 1122, 2211, 1221, and 2112 spectrally arbitrary.)

- All trees on six vertices are generalized stars or double stars for which the IEPG has been solved (see [3]). (In particular \(G_{77}\) attains 1131 and \(G_{79}\) attains 1221 with arbitrary spectra.)

- The graph \(G_{129}\) attains 222; a construction can be found in [8].

5.1 Using SSP for connected graphs of order at most five

Let \(G\) be a disconnected graph on six vertices. From each of its connected components, select an attainable ordered multiplicity list (see Table 1). Let \(\gamma_1 \ldots \gamma_k\) be the ordered multiplicity list built by interlacing in some way these lists. By Theorem 2 if each attains its multiplicity list with SSP, \(G\) attains \(\gamma_1 \ldots \gamma_k\) with SSP. By Theorem 1 any supergraph of \(G\)—namely, any connected supergraph \(H\) of order six—attains \(\gamma_1 \ldots \gamma_k\) (with SSP as well).

Since the components are graphs on at most five vertices, all of the attainable multiplicity lists are spectrally arbitrary. Since SSP preserves spectrum, any ordered multiplicity list constructed in this fashion must be spectrally arbitrary as well. Exhaustively performing the process detailed above gives the results listed in Table 4.

5.2 Using cloning for connected graphs of order five

Let \(G\) be a connected graph on five vertices, and let \(H\) be the graph constructed by cloning \(v \in \nu(G)\) with an edge. If \(G\) attains the ordered multiplicity list \((\gamma_1, \ldots, \gamma_k)\), by Corollary 4 it follows that \(H\) attains both \((\gamma_1 + 1, \ldots, \gamma_k)\) and \((\gamma_1, \ldots, \gamma_k + 1)\).

The method above produces the results listed in Table 5 (note we exclude information that follows from previous results).

The full application of Theorem 3 generally requires a constructed matrix; then, confirming the value of the appropriate diagonal entry allows for the manipulation of the interior entries of the corresponding ordered multiplicity list. However, an explicit construction is not always necessary. Given a prescribed multiplicity list, for some graphs \(G\) we can sometimes guarantee that a particular diagonal entry of any \(M \in \mathcal{S}(G)\) must be zero (or nonzero), as we now demonstrate.

**Proposition 19.** Let \(M \in \mathcal{S}(G_{40})\) have nullity two, then the diagonal entry of \(M\) corresponding to the leaf is nonzero. In particular, for any ordered multiplicity list of \(G_{40}\) with a 2 we can clone the leaf vertex with an edge to get \(G_{144}\) and change the 2 to a 3.

**Proof.** In Figure 8 we have \(G_{40}\) and the two possible graphs that result from cloning without an edge (\(G_{111}\)) and with an edge (\(G_{144}\)). In addition we have marked minimal zero forcing sets for the two clones.

The graph \(G_{111}\) has a zero forcing number of two, which would imply that the maximum nullity (and hence also maximum multiplicity of an eigenvalue) is at most two. Therefore no matrix associated with \(G_{111}\) can have an ordered multiplicity list with an entry which is \(\geq 3\). On the other hand that is possible for \(G_{144}\).
| Graph(s) | Supergraphs | Multiplicity lists |
|----------|-------------|--------------------|
| G54 ∪ G1 | G191, G200, G205, G207, G208 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111, 1123, 132, 213, 231, 312, 321, 114, 141, 411 |
| G48 ∪ G1 | G140, G141, G143, G156, G157, G158, G159, G160, G166, G168, G170, G172, G173, G177, G178, G179, G180, G181, G182, G183, G184, G185, G186, G188, G189, G190, G192, G193, G194, G195, G196, G197, G198, G199, G201, G202, G203, G204, G206 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111, 1123, 132, 213, 231, 312, 321 |
| G18 ∪ 2G1 | G133, G134, G142, G165, G169 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111 |
| G44 ∪ G1 | G121, G125, G135, G138, G146, G149, G154, G161, G162, G175 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1131, 1311 |
| G7 ∪ G7 | G96, G98, G99, G103, G111, G112, G113, G114, G115, G117, G118, G119, G120, G122, G123, G124, G126, G128, G129, G130, G136, G137, G139, G144, G145, G147, G148, G150, G151, G152, G153, G163, G164, G167, G171, G174, G187 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211 |
| G34 ∪ G1 | G92, G93, G95, G104, G127 | 111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2122, 2211 |
| G38 ∪ G1 | G94, G97, G100, G102 | 111111, 11112, 11121, 11211, 12111, 21111 |
| G7 ∪ 3G1 | G94, G97, G100, G102 | 111111, 11112, 11121, 11211, 12111, 21111 |
| G13 ∪ 2G1 | G77, G78, G79, G80, G81 | 111111, 11121, 11211, 12111 |
| 6G1 | G83, G105 | 111111 |

Table 4: Using SSP properties for graphs of order at most five to attain (spectrally arbitrary) ordered multiplicity lists for graphs of order six.

We can run cloning with the matrix \( M \) which has nullity two and produce a matrix that has nullity three. This can only be possible if we cloned to \( G_{144} \) and not \( G_{111} \) which means that the diagonal entry corresponding to the leaf vertex is nonzero.

Finally, since we can always translate any particular eigenvalue to 0 then for any matrix in \( \mathcal{S}(G_{40}) \) with an entry of 2 first we translate so that the entry corresponds to an eigenvalue of 0, apply the preceding argument, and then translate back.

A similar argument works for several other cases which are listed in Table 6.

Because we know that graphs on five vertices attain their ordered multiplicity lists spec-
### Table 5: Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices.

| Graph | Cloned graph(s) | Multiplicity lists |
|-------|----------------|--------------------|
| G29   | G92, G161      | 132, 231           |
| G30   | G100           | 1122, 1212, 2121, 2211 |
| G34   | G117, G133, G179 | 1113, 123, 222, 3111, 321 |
| G35   | G119           | 1113, 3111         |
| G36   | G130, G150     | 1113, 3111         |
| G37   | G126, G141, G143, G168 | 1113, 123, 213, 222, 3111, 312, 321 |
| G38   | G153           | 1113, 123, 222, 3111, 321 |
| G40   | G144, G156, G177, G192 | 1113, 123, 213, 222, 3111, 312, 321 |
| G41   | G150, G160, G178, G181 | 1113, 123, 213, 222, 3111, 312, 321 |
| G42   | G165, G195     | 114, 123, 132, 213, 222, 231, 312, 321, 411 |
| G43   | G169, G172, G185 | 123, 213, 222, 312, 321 |
| G44   | G170, G189     | 222                |
| G45   | G165, G191, G200 | 114, 123, 132, 213, 222, 231, 312, 321, 411 |
| G46   | G179, G201     | 222                |
| G47   | G183, G193, G202 | 222                |
| G48   | G190, G194, G199 | 114, 222, 24, 33, 411, 42 |
| G49   | G195, G200, G205 | 114, 222, 24, 33, 411, 42 |
| G50   | G203, G206     | 222, 24, 33, 42   |
| G51   | G205, G207     | 15, 222, 24, 33, 42, 51 |
| G52   | G208           | 15, 222, 24, 33, 42, 51 |

Table 5: Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices.

Figure 8: G40 and its two clones via the leaf. Minimal zero forcing sets have been marked for the clones.

Trally arbitrarily and cloning preserves the eigenvalues, we can conclude that these lists found by cloning are also spectrally arbitrary.

### 5.3 Constructions of graphs which are spectrally arbitrary

Several cases were handled by finding matrices, usually through the aid of orthogonal representations. Constructions with SSP allowed multiple cases to be handled simultaneously.

**Proposition 20.** For G96 we can attain 222 spectrally arbitrary using SSP matrices.
Table 6: Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices where a middle entry is increased.

| Graph | Cloned graph | Multiplicity lists |
|-------|--------------|--------------------|
| G29   | G77          | 141                |
| G30   | G78          | 1131, 1311         |
| G30   | G100         | 1131, 1311         |
| G30   | G114         | 1131, 1311         |
| G34   | G133         | 132, 231           |
| G35   | G119         | 1131, 1311         |
| G37   | G126         | 1131, 1311, 132, 231|
| G40   | G144         | 1131, 1311, 132, 231|
| G42   | G165         | 141                |
| G44   | G146         | 141                |
| G46   | G161         | 141                |
| G48   | G190         | 141                |
| G48   | G194         | 141                |
| G49   | G195         | 141                |

Proof. For \( b > 0 \), the following matrix for \( G_96 \) attains \( \{0^{(2)}, 1^{(2)}, (1 + 4b^2)^{(2)}\} \) with SSP.

\[
\begin{pmatrix}
4b^2 & \sqrt{2}b & -\sqrt{2}b & 0 & 0 & 0 \\
\sqrt{2}b & 1 & 0 & b & 0 & 0 \\
-\sqrt{2}b & 0 & 1 & b & 0 & 0 \\
0 & b & b & 4b^2 & b & b \\
0 & 0 & 0 & b & 1 & 0 \\
0 & 0 & 0 & b & 0 & 1 \\
\end{pmatrix}
\]

By choosing \( b \), we can make the ratio of the two gaps between the eigenvalues arbitrarily large. By scaling and shifting, this establishes attainability with arbitrary spectrum.

By Theorem \[1\] we have that all supergraphs which contain \( G_96 \) can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs \( G_{111}, G_{114}, G_{118}, G_{121}, G_{135}, G_{136}, G_{137}, G_{140}, G_{145}, G_{146}, G_{147}, G_{148}, G_{149}, G_{157}, G_{158}, G_{159}, G_{161}, G_{162}, G_{163}, G_{164}, G_{166}, G_{167}, G_{171}, G_{173}, G_{180}, G_{182}, G_{184}, G_{186}, G_{187}, G_{188}, G_{196}, G_{197}, G_{198}, \) and \( G_{204} \) are spectrally arbitrary for 222.

Proposition 21. For \( G_{105} \) we can attain 222 spectrally arbitrary using SSP matrices.

Proof. The matrix

\[
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & a \\
-1 & 0 & a & 0 & 0 & 0 \\
0 & a & a^2 & a & 0 & 0 \\
0 & 0 & a & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & a \\
a & 0 & 0 & 0 & a & a^2 \\
\end{pmatrix}
\]

\( \in S(G_{105}) \)
has SSP and has spectrum
\[{\{\frac{1}{2}a^2 - \frac{1}{2}\sqrt{a^4 + 10a^2 + 5} - \frac{1}{2}, 0^{(2)}, (\frac{1}{2}a^2 + \frac{1}{2}\sqrt{a^4 + 10a^2 + 5} - \frac{1}{2})^{(2)}}\}}.

Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval \([1, \infty)\):

\[\frac{a^2 + \sqrt{a^4 + 10a^2 + 5} - 1}{-a^2 + \sqrt{a^4 + 10a^2 + 5} + 1}.
\]

This is continuous for \(a \geq 1\) and if \(a = 1\) we get 1. Furthermore, the numerator has growth \(a^2\) and the denominator approaches 6 so the ratio is unbounded.

By Theorem 1, we have that all supergraphs which contain \(G_{105}\) can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs \(G_{127}, G_{128}, G_{151}, G_{152}, G_{154}, G_{174},\) and \(G_{175}\) are spectrally arbitrary for 222.

**Proposition 22.** For \(G_{99}\) we can attain 222 spectrally arbitrary using non-SSP matrices.

*Proof.** For \(a > 0\), the following matrix for \(G_{99}\) attains \(\{0^{(2)}, 1^{(2)}, (1 + 3a^2)^{(2)}\}\).

\[
\begin{pmatrix}
1 & a & 0 & 0 & 0 & 0 \\
a & 3a^2 & a & a & 0 & 0 \\
0 & a & 1 & 0 & -a & 0 \\
0 & a & 0 & 1 & a & 0 \\
0 & 0 & -a & a & 3a^2 & a \\
0 & 0 & 0 & 0 & a & 1
\end{pmatrix}
\]

By choosing \(a\) we can make the ratios of the two gaps arbitrarily big. By scaling and shifting, this establishes attainability with arbitrary spectrum.

It is impossible for any matrix to have 222 and SSP for this graph as \(G_{112}\) is a supergraph which cannot attain 222 because \(q(G_{112}) = 4\). □

**Proposition 23.** For \(G_{189}\) we can attain 141 spectrally arbitrary using SSP matrices.

*Proof.** The matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & a \\
0 & 0 & 0 & 1 & 1 & a \\
0 & 0 & 0 & 1 & 1 & a \\
1 & 1 & 1 & -1 & -1 & 0 \\
1 & 1 & 1 & -1 & -1 & 0 \\
a & a & a & 0 & 0 & a^2
\end{pmatrix}
\]

has SSP and has characteristic polynomial \(p(x) = x^4(x^2 - (a^2 - 2)x - (5a^2 + 6))\). The spectrum is

\[{\{\frac{1}{2}((a^2 - 2) - \sqrt{a^4 + 16a^2 + 28}), 0^{(4)}, \frac{1}{2}((a^2 - 2) + \sqrt{a^4 + 16a^2 + 28})\}}.
\]
Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval \([1, \infty)\):

\[
\frac{(a^2 - 2) + \sqrt{a^4 + 16a^2 + 28}}{-(a^2 - 2) + \sqrt{a^4 + 16a^2 + 28}}
\]

This is continuous for \(a \geq \sqrt{2}\) and if \(a = \sqrt{2}\) we get 1. Furthermore, the numerator has growth \(a^2\) and the denominator approaches 10 so the ratio is unbounded.

By Theorem 1 all supergraphs which contain \(G_{189}\) can attain the ordered multiplicity list \(141\) with arbitrary spectrum. Thus, the graphs \(G_{197}, G_{199}, G_{201}, G_{203},\) and \(G_{206}\) are spectrally arbitrary for \(141\).

The remaining case for \(141\) is \(G_{204}\). Consider

\[
\begin{pmatrix}
  a^2 & 0 & a & a & a & a \\
  0 & -1 & -1 & -1 & 1 & 1 \\
  a & -1 & 0 & 0 & 2 & 2 \\
  a & -1 & 0 & 0 & 2 & 2 \\
  a & 1 & 2 & 2 & 0 & 0 \\
  a & 1 & 2 & 2 & 0 & 0
\end{pmatrix}
\in S(G_{204}),
\]

which has eigenvalues \([-5, 0, 4 + a^2]\). Arbitrary spectrums are attained through appropriate choices of \(a \geq 1\), scaling, and shifting.

**Proposition 24.** For \(G_{151}\) we can attain \(213, 312, 1113, \) and \(3111\) spectrally arbitrary using SSP matrices.

**Proof.** For \(a, b > 0\), the following matrix for \(G_{151}\) has spectrum \(\{0^{(3)}, 2a^2, 2a^2 + 2, a^2 + b^2 + 2\}\) and satisfies SSP:

\[
\begin{pmatrix}
  a^2 & a & 0 & 0 & a^2 & 0 \\
  a & a^2 + 2 & a & 0 & 0 & ab \\
  0 & a & a^2 & a^2 & 0 & 0 \\
  0 & 0 & a^2 & a^2 + 1 & 1 & -b \\
  a^2 & 0 & 0 & 1 & a^2 + 1 & -b \\
  0 & ab & 0 & -b & -b & b^2
\end{pmatrix}.
\]

Setting \(a = b\) gives 312 with a fixed gap between the last two eigenvalues and an arbitrary gap between the first two; the attainability of an arbitrary spectrum follows.

For \(3111\), by scale and shift we can assume a non-negative spectrum such that 0 is the eigenvalue of multiplicity 3, and the gap between the first and second positive eigenvalues is 2. Set \(2a^2\) and \(a^2 + b^2 + 2\) to the first and third positive eigenvalues, respectively, and solve for \(a\) and \(b\). Thus, an arbitrary spectrum is attainable.

By Theorem 1 we have that all supergraphs of \(G_{151}\) can attain the ordered multiplicity lists \(213, 312, 1113, \) and \(3111\) with arbitrary spectrum. Thus, the graphs \(G_{171}\) and \(G_{187}\) are spectrally arbitrary for \(213, 312, 1113,\) and \(3111\).

**Proposition 25.** For \(G_{163}\) we can attain \(213, 312, 1113,\) and \(3111\) spectrally arbitrary using SSP matrices.
Proof. For $a, b > 0$, the following matrix for $G_{163}$ has spectrum $\{0^{(3)}, 1 + 3a^2, 1 + a^2 + 2b^2\}$ and satisfies SSP:

$$
\begin{pmatrix}
    a^2 + b^2 & b & 0 & a^2 & a \\
    b^2 & a^2 + b^2 & b & a & -a^2 & 0 \\
    b & b & 1 & 0 & 0 & 0 \\
    0 & a & 0 & 1 & -a & 0 \\
    a^2 & -a^2 & 0 & -a & 2a^2 & a \\
    a & 0 & 0 & 0 & a & 1
\end{pmatrix}.
$$

By appropriate choices of $a, b$, scaling, and translating, this attains $312, 213, 3111$, and $1113$ spectrally arbitrary.

5.4 Two distinct eigenvalues

For the graphs $G_{154}, G_{168}, G_{174}, G_{175}, G_{181}, G_{186}, G_{188}, G_{192}, G_{196}, G_{197}, G_{198}, G_{201}$, and $G_{202}$, $q(G) = 2$ and $Z_+(G) = 3$ (see Table 3), which implies that all these graphs attain $33$. (For more on matrix realizations for these graphs, see [8].) For the graph $G_{204}$, this attains $33$ by the matrix below on the left; and $42$ ($24$) by the matrix below on the right.

$$
\begin{pmatrix}
    0 & 0 & 1 & 1 & 1 & 1 \\
    0 & 0 & 1 & 1 & -1 & -1 \\
    1 & 1 & 0 & 0 & 1 & -1 \\
    1 & 1 & 0 & 0 & -1 & 1 \\
    1 & -1 & 1 & -1 & 0 & 0 \\
    1 & -1 & -1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 1 & -1 & 2 & -1 \\
    0 & 1 & 1 & 1 & 1 & 2 \\
    -1 & 1 & 0 & 2 & -1 & 3 \\
    2 & 1 & 3 & -1 & 5 & 0 \\
    -1 & 2 & 1 & 3 & 0 & 5
\end{pmatrix}
$$

When a matrix for a graph has two distinct eigenvalues, we can modify the matrix to get additional attainable ordered multiplicity lists. The following will suffice for our purposes (generalizations are possible for graphs with larger order).

Lemma 26. Let $G \not= K_6$ be a connected graph on six vertices. If $G$ attains ordered multiplicity list $33$, then with arbitrary spectrum it attains multiplicity lists $1113, 123, 213, 312, 321$, and $3111$. Similarly, if $G$ attains $42$ or $24$, then with arbitrary spectrum it attains $411$ and $114$.

Proof. Since $G$ is not the complete graph, there are two non-adjacent vertices, which we assume to be $v_1$ and $v_2$.

Let $M \in S(G)$ attain the ordered multiplicity list $33$ with spectrum $\{0^{(3)}, 1^{(3)}\}$. We can write $M = Q^TQ$, where $Q$ is a $3 \times 6$ matrix whose rows are any orthonormal basis of the eigenspace of 1 (in particular, $M$ is the projection matrix onto the eigenspace associated with eigenvalue 1).

We claim we can choose our orthonormal basis so that for some $x, y \neq 0$,

$$
Q = \begin{pmatrix}
    x & 0 & * & * & * & * \\
    0 & y & * & * & * & * \\
    0 & 0 & * & * & * & *
\end{pmatrix}.
$$

To see this, first consider the vectors $a, b, c$, which form a basis for our eigenspace.
• Note that no fixed entry can be 0 for all three of $a, b, c$; this would imply that the corresponding vertex is isolated, a contradiction given that our graph is connected. Thus, at least one vector has a nonzero first entry. By taking linear combinations, we can assume that the first entry is 0 for $a$ and $b$ and nonzero for $c$.

• Run Gram-Schmidt on $a, b, c$ (in this order) to get an orthonormal set $a', b', c'$, where the first entries of $a', b'$ are 0 and the first entry of $c'$ is nonzero. Note the second entry of $c'$ must be zero; otherwise, using this set as an orthonormal basis for $Q$ would force $M_{1,2} \neq 0$, a contradiction given that $v_1 \sim v_2$.

• Now repeat the argument for $a', b'$ by taking a linear combination so that the second entry of $a'$ is zero. Run Gram-Schmidt again to produce $a'', b''$.

• Thus, the rows of $Q$ are (from top to bottom) $c', b'', a''$.

Note we can now introduce parameters $\lambda, \mu > 0$ to give

$$
\tilde{Q} = \begin{pmatrix}
\lambda x & 0 & * & * & * \\
0 & \mu y & * & * & * \\
0 & 0 & * & * & *
\end{pmatrix}.
$$

Since the orthogonality of the columns of $\tilde{Q}$ agree with the orthogonality of the columns of $Q$, the matrix $\tilde{M} = \tilde{Q}^T \tilde{Q} \in S(G)$. Because the nonzero eigenvalues of $\tilde{M}$ are the norms of the rows, $\tilde{M}$ has spectrum $\{0^{(3)}, 1, 1 + (\lambda^2 - 1)x^2, 1 + (\mu^2 - 1)y^2\}$. Appropriate choices of $\lambda$ and $\mu$, combined with scaling and translation, arbitrarily attain any spectrum that starts or ends with an eigenvalue of multiplicity 3.

A similar argument handles the 42 case.

This lemma establishes that 1113, 123, 213, 312, 321, 3111 are all attainable with arbitrary spectrum for graphs $G_{154}$, $G_{174}$, and $G_{175}$; similarly, 411 and 114 are attainable with arbitrary spectrum for $G_{204}$.

### 5.5 Graph minor results

We now turn to results for graphs which attain certain ordered multiplicity lists, but which are not enough to prove we do so arbitrarily. We start with the following result which connects SSP and graph minors.

**Theorem 27** (Barrett et al. [5], Theorem 6.12). Let $G$ be attained from $H$ by contraction of a single edge, and let $M \in S(G)$ have SSP and ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$. Then there is $N \in S(H)$ with ordered multiplicity list $(\gamma_1, \ldots, \gamma_k, 1)$.

Applying Theorem 27 by looking for minors on graphs of order six gives the results listed in Table 7 (we exclude information that follows from previous results).

Note Theorem 27 does not guarantee spectrally arbitrary results; the newly appended one on the ordered multiplicity list might need to be large (see [5] for more information).
| Graph(s)                          | Minor | Multiplicity lists |
|----------------------------------|-------|--------------------|
| G129, G145, G151, G153, G171, G174, G187 | G44   | 1311, 1131         |
| G151, G153, G154, G169, G171, G174, G175, G187 | G48   | 321, 123, 231, 132 |
| G100                            | G34   | 1221               |

Table 7: Using graph minors to attain ordered multiplicity lists for graphs of order six.

### 5.6 Graphs with SSP/SMP

Table 8 lists matrices that attain the corresponding ordered multiplicity list for that graph. All these matrices have either SSP or SMP, which gives the results listed in Table 7.

| Graph | Multiplicity list(s) | Matrix |
|-------|----------------------|--------|
| G105  | 2112                 | \[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & -2 \\
1 & 0 & 2 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 \\
-2 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\] |
| G125  | 222                  | \[
\begin{pmatrix}
-1 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & 1 & 1 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\] |
| G129  | 132, 231             | \[
\begin{pmatrix}
2 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 2 & 2 \\
0 & -1 & 0 & 0 & 2 & 1
\end{pmatrix}
\] |

Table 8: Matrices with SSP or SMP

- Because G127 is a supergraph of G105, G127 attains 2112.
- Because G138 is a supergraph of G125, G138 attains 222.
- Because G145, G149, and G162 are supergraphs of G129, they attain 132 and 231.

### 5.7 Remaining cases

Table 9 gives the remaining attainable cases. The construction of these matrices included exhaustive searches for matrices with simple entries (i.e., 0, ±1), as well as the use of orthogonal representations with respect to some (possibly indefinite) inner product.

22
| \((1 1 1 1 0 0 0)\) | \(\in S(92)\) | \((1 1 1 1 1 0 0)\) | \(\in S(94)\) |
|---|---|---|---|
| \(0 0 \sqrt{2} -1 0 1 1\) | \(\in S(114)\) | \(0 1 0 1 1 0 1\) | \(\in S(115)\) |
| \(0 0 \sqrt{2} -1 0 1 1\) | \(\in S(117)\) | \(0 1 0 1 1 0 1\) | \(\in S(130)\) |
| \(\sqrt{3} \sqrt{3} 0 0 2 2\) | \(\in S(135)\) | \(0 0 0 0 1 1\) | \(\in S(146)\) |
| \(\sqrt{3} \sqrt{3} 0 0 2 2\) | \(\in S(150)\) | \(\gamma^2 \gamma \gamma 0 0\) | \(\gamma^2 = \sqrt{10} - 2\) |
| \(5 \sqrt{5} 0 0 0 0\) | \(\in S(163)\) | \(1 0 1 0 1 0\) | \(\in S(163)\) |

Table 9: Remaining cases.
6 Ordered Multiplicity IEPG differs from IEPG

Through five vertices, the ordered multiplicity IEPG and the IEPG are equivalent: a graph attains an ordered multiplicity list if and only if it attains that multiplicity list with arbitrary spectrum. However, for graphs of order at least six, this relationship no longer holds.

**Theorem 28.** The complete bipartite graph $K_{m,n}$ where $\min(m,n) \geq 3$ attains the ordered multiplicity list $(1, m + n - 2, 1)$, but not spectrally arbitrary.

**Proof.** The spectrum of the adjacency matrix of $K_{m,n}$ is $\{-\sqrt{mn}, 0^{(m+n-2)}, \sqrt{mn}\}$. Thus, $K_{m,n}$ attains $(1, m + n - 2, 1)$. Note the gaps between consecutive eigenvalues are equal. We claim that any $M \in S(K_{m,n})$ that attains $(1, m + n - 2, 1)$ preserves this relationship. Thus, $K_{m,n}$ cannot attain $(1, m + n - 2, 1)$ with arbitrary spectrum.

Let $M \in S(K_{m,n})$ attain multiplicity list $(1, m + n - 2, 1)$, where after translation we may assume 0 is the eigenvalue of multiplicity $m + n - 2$. Let $-\lambda_1 < \lambda_2$ be the nonzero eigenvalues of $M$ with the corresponding orthogonal eigenvectors $x_1$ and $x_2$. Note,

$$M = -\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T = (x_1 \quad x_2) \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1^T \\ x_2^T \end{pmatrix} = (\sqrt{-\lambda_1} x_1 \quad \sqrt{\lambda_2} x_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{-\lambda_1} x_1^T \\ \sqrt{\lambda_2} x_2^T \end{pmatrix}.$$

Let $y_i$ be the $i^{th}$ column of $Y$; thus, $y_i \in \mathbb{R}^2$. Note the association between $y_i$ and $v_i$ of $K_{m,n}$: two vertices $v_i \sim v_j$ if and only if $y_i^T S y_j = 0$. Moreover, $K_{m,n}$ is connected, so $y_i \neq 0$.

Let $a, b \in K_{m,n}$ be nonadjacent. Since $\min(m,n) \geq 3$, there exists $c$ such that $a, b,$ and $c$ are pairwise nonadjacent. If the corresponding vectors $a$ and $b$ are not scalar multiples, then the matrix

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix}$$

has rank two. However, the adjacency structure of $K_{m,n}$ requires that

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows the matrix lacks full rank, a contradiction. Thus, $a$ and $b$ are scalar multiples.

By symmetry, the vectors associated with pairwise nonadjacent vertices must be scalar multiples. These vectors must also satisfy $a^T S a = 0$, and thus must have the form $(\pm x)$. Hence, $Y$ is of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m & \beta_1 & \beta_2 & \cdots & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m & -\beta_1 & -\beta_2 & \cdots & -\beta_n \end{pmatrix},$$

for appropriate choice of $\alpha_i$ and $\beta_j$, so $M$ has spectrum $\{0^{(m+n-2)}, \pm 2\sqrt{(\sum \alpha_i^2)(\sum \beta_j^2)}\}$. Therefore $K_{m,n}$ cannot attain $(1, m + n - 2, 1)$ with arbitrary spectrum.

**Corollary 29.** The graph $K_{3,3}$ (G175) attains 141, but not spectrally arbitrary.
7 Conclusion

We have given a complete solution for the ordered multiplicity IEPG for connected graphs on six vertices. Moreover, many of the techniques used for attainability also allow for an arbitrary spectrum, allowing for significant progress on the IEPG for connected graphs on six vertices. In particular, there are 1326 cases of attainability. Among these, 1285 are known to be done with arbitrary spectrum, one does so without, and 40 remain undetermined (see Table 10). Finishing these cases, and thus solving the IEPG for graphs on six vertices, is an open problem.

| Multiplicity list(s) | Remaining cases |
|----------------------|-----------------|
| 1212, 2121           | G94             |
| 1221                 | G100            |
| 2112                 | G127            |
| 222                  | G115, G125, G129, G138 |
| 1131, 1311           | G92, G117, G129, G130, G145, G150, G151, G153, G163, G171, G174, G187 |
| 123, 321             | G151, G163, G171, G187 |
| 132, 231             | G114, G129, G135, G145, G146, G149, G150, G151, G153, G154, G162, G163, G169, G171, G174, G175, G187 |

Table 10: The remaining cases for the IEPG for connected graphs on six vertices.

We also showed that $K_{m,n}$ with $\min(m, n) \geq 3$ has at least one attainable multiplicity list which cannot be attained spectrally arbitrary. This shows that the ordered multiplicity IEPG and the IEPG differ for graphs on six or more vertices.

A natural next problem to consider is the ordered multiplicity IEPG for connected graphs on seven or more vertices. While many of the techniques implemented in this work can be utilized in that setting (and indeed solves “most” of the cases), the IEPG becomes dramatically harder. Thus, new tools will likely be needed to continue to make progress.

The difficulty lies in part with the number of cases involved, both in terms of the number of graphs and the number of potential ordered multiplicity lists. In addition, one of the most useful tools we had was SMP and SSP which allowed for simultaneous handling of many cases by establishing a result for a graph and all its supergraphs. This does have some limitations.

**Observation 4.** If $M \in S(G)$ attains $\gamma_1 \ldots \gamma_k$ and $H$ is a supergraph which does not attain $\gamma_1 \ldots \gamma_k$, then $M$ does not have SSP or SMP.

This can be used to explain why the cases given in Table 2 do not have SSP, and for graphs on six vertices can be used to show that there are over forty occurrences where a graph attains an ordered multiplicity list but does so without SSP or SMP. These cases required either cloning or finding some appropriate orthogonal constructions. As the number of vertices increases the number of cases needing individual attention (and hence difficulty) will rise as well.
Acknowledgments

The research was conducted at the 2017 REU program held at Iowa State University, which was supported by NSF DMS 1457443. Steve Butler was partially supported by a grant from the Simons Foundation (#427264).

References

[1] AIM Minimum Rank – Special Graphs Work Group, Zero forcing sets and the minimum rank of graphs, *Linear Algebra and its Applications*, **428** (2008), 1628–1648.

[2] Francesco Barioli, Wayne Barrett, Shaun Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, P. van der Driessche, Hein van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra and its Applications* **433** (2010) 401-411.

[3] Francesco Barioli, Shaun Fallat, On the eigenvalues of generalized and double generalized stars, *Linear and Multilinear Algebra* **53** (2005), 269–291.

[4] Wayne Barrett, Steve Butler, H. Tracy Hall, John Sinkovic, Wasin So, Colin Starr, Amy Yielding, Computing inertia sets using atoms, *Linear Algebra and its Applications* **436** (2012), 4489–4502.

[5] Wayne Barrett, Steve Butler, Shaun Fallat, H. Tracy Hall, Jephian Lin, Leslie Hogben, Bryan Shader, Michael Young, The inverse eigenvalue problem of a graph: multiplicities and minors, [arXiv:1708.00064](https://arxiv.org/abs/1708.00064).

[6] Wayne Barrett, Shaun Fallat, H. Tracy Hall, Leslie Hogben, Jephian Lin, Bryan Shader, Generalizations of the strong Arnold property and the minimum number of distinct eigenvalues of a graph, Electronic Journal of Combinatorics, **24** (2017), P2.40, 28 pp.

[7] Wayne Barrett, H. Tracy Hall, Raphael Loewy, The inverse inertia problem for graphs: Cut vertices, trees, and a counterexample, *Linear Algebra and its Applications* **431** (2009), 1147–1191.

[8] Beth Bjorkman, Leslie Hogben, Scarlette Ponce, Carolyn Reinhart, Theodore Tranel, Applications of analysis to the determination of the minimum number of distinct eigenvalues of a graph, [arXiv:1708.01821](https://arxiv.org/abs/1708.01821).

[9] Shaun Fallat, Leslie Hogben, Minimum rank, maximum nullity, and zero forcing number of graphs, in *Handbook of Linear Algebra, 2nd edition*, L. Hogben, ed., CRC Press, Boca Raton, 36 pp.

[10] Rosário Fernandes, C.M. da Fonseca, The inverse eigenvalue problem for Hermitian matrices whose graphs are cycles, *Linear and Multilinear Algebra* **57** (2009), 673–682.

[11] H. Tracy Hall, Code in Sage for determining if a matrix satisfies Strong Arnold Property (SAP), Strong Spectral Property (SSP), or Strong Multiplicity Property (SMP). Sage worksheet available [https://sage.math.iastate.edu/home/pub/79/](https://sage.math.iastate.edu/home/pub/79/).
[12] Leslie Hogben, Spectral graph theory and the inverse eigenvalue problem of a graph, *Electronic Journal of Linear Algebra* **14** (2005), 12–31.

[13] Ronald C. Read, Robin J. Wilson, *An Atlas of Graphs*, Oxford University Press, New York, 1998, xii+454 pp.
A  Connected graphs on six vertices realizing given ordered multiplicities

For the following ordered multiplicities we list all connected graphs which can achieve a given multiplicity list. Any graph which is underlined is a graph which has not yet been determined to be spectrally arbitrary for that ordered multiplicity list. Any graph which is boxed is a graph which has been shown to not be spectrally arbitrary for that multiplicity list.

| 111111 | G77  | G78  | G79  | G80  | G81  | G83  | G92  | G93  | G94  | G95  | G96  | G97  |
|---------|------|------|------|------|------|------|------|------|------|------|------|------|
|         | G98  | G99  | G100 | G102 | G103 | G104 | G105 | G111 | G112 | G113 | G114 | G115 |
|         | G117 | G118 | G119 | G120 | G121 | G122 | G123 | G124 | G125 | G126 | G127 | G128 |
|         | G129 | G130 | G133 | G134 | G135 | G136 | G137 | G138 | G139 | G140 | G141 | G142 |
|         | G143 | G144 | G145 | G146 | G147 | G148 | G149 | G150 | G151 | G152 | G153 | G154 |
|         | G156 | G157 | G158 | G159 | G160 | G161 | G162 | G163 | G164 | G165 | G166 | G167 |
|         | G168 | G169 | G170 | G171 | G172 | G173 | G174 | G175 | G177 | G178 | G179 | G180 |
|         | G181 | G182 | G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 | G191 | G192 |
|         | G193 | G194 | G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 | G203 | G204 |
|         | G205 | G206 | G207 | G208 |      |      |      |      |      |      |      |      |

| 11112 and 21111 | G92  | G93  | G94  | G95  | G96  | G97  | G98  | G99  | G100 | G102 | G103 | G104 |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | G105 | G111 | G112 | G113 | G114 | G115 | G117 | G118 | G119 | G120 | G121 | G122 |
|                 | G123 | G124 | G125 | G126 | G127 | G128 | G129 | G130 | G133 | G134 | G135 | G136 |
|                 | G137 | G138 | G139 | G140 | G141 | G142 | G143 | G144 | G145 | G146 | G147 | G148 |
|                 | G149 | G150 | G151 | G152 | G153 | G154 | G156 | G157 | G158 | G159 | G160 | G161 |
|                 | G162 | G163 | G164 | G165 | G166 | G167 | G168 | G169 | G170 | G171 | G172 | G173 |
|                 | G174 | G175 | G177 | G178 | G179 | G180 | G181 | G182 | G183 | G184 | G185 | G186 |
|                 | G187 | G188 | G189 | G190 | G191 | G192 | G193 | G194 | G195 | G196 | G197 | G198 |
|                 | G199 | G200 | G201 | G202 | G203 | G204 | G205 | G206 | G207 | G208 |      |      |

| 11121 and 12111 | G77  | G78  | G79  | G80  | G81  | G92  | G93  | G94  | G95  | G96  | G97  | G98  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | G99  | G100 | G102 | G103 | G104 | G105 | G111 | G112 | G113 | G114 | G115 | G117 |
|                 | G118 | G119 | G120 | G121 | G122 | G123 | G124 | G125 | G126 | G127 | G128 | G129 |
|                 | G130 | G133 | G134 | G135 | G136 | G137 | G138 | G139 | G140 | G141 | G142 | G143 |
|                 | G144 | G145 | G146 | G147 | G148 | G149 | G150 | G151 | G152 | G153 | G154 | G156 |
|                 | G157 | G158 | G159 | G160 | G161 | G162 | G163 | G164 | G165 | G166 | G167 | G168 |
|                 | G169 | G170 | G171 | G172 | G173 | G174 | G175 | G177 | G178 | G179 | G180 | G181 |
|                 | G182 | G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 | G191 | G192 | G193 |
|                 | G194 | G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 | G203 | G204 | G205 |
|                 | G206 | G207 | G208 |      |      |      |      |      |      |      |      |      |
### 11211

| G77 | G78 | G79 | G80 | G81 | G92 | G93 | G94 | G95 | G96 | G97 | G98 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| G99 | G100| G102| G103| G104| G105| G111| G112| G113| G114| G115| G117|
| G118| G119| G120| G121| G122| G123| G124| G125| G126| G127| G128| G129|
| G130| G133| G134| G135| G136| G137| G138| G139| G140| G141| G142| G143|
| G144| G145| G146| G147| G148| G149| G150| G151| G152| G153| G154| G156|
| G157| G158| G159| G160| G161| G162| G163| G164| G165| G166| G167| G168|
| G169| G170| G171| G172| G173| G174| G175| G177| G178| G179| G180| G181|
| G182| G183| G184| G185| G186| G187| G188| G189| G190| G191| G192| G193|
| G194| G195| G196| G197| G198| G199| G200| G201| G202| G203| G204| G205|
| G206| G207| G208|

### 1122 and 2211

| G92 | G93 | G95 | G96 | G98 | G99 | G100| G103| G104| G105| G111| G112|
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| G113| G114| G115| G117| G118| G119| G120| G121| G122| G123| G124| G125|
| G126| G127| G128| G129| G130| G133| G134| G135| G136| G137| G138| G139|
| G140| G141| G142| G143| G144| G145| G146| G147| G148| G149| G150| G151|
| G152| G153| G154| G156| G157| G158| G159| G160| G161| G162| G163| G164|
| G165| G166| G167| G168| G169| G170| G171| G172| G173| G174| G175| G177|
| G178| G179| G180| G181| G182| G183| G184| G185| G186| G187| G188| G189|
| G190| G191| G192| G193| G194| G195| G196| G197| G198| G199| G200| G201|
| G202| G203| G204| G205| G206| G207| G208|

### 1221

| G79 | G92 | G93 | G95 | G96 | G98 | G99 | G100| G103| G104| G105| G111|
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| G112| G113| G114| G115| G117| G118| G119| G120| G121| G122| G123| G124|
| G125| G126| G127| G128| G129| G130| G133| G134| G135| G136| G137| G138|
| G139| G140| G141| G142| G143| G144| G145| G146| G147| G148| G149| G150|
| G151| G152| G153| G154| G156| G157| G158| G159| G160| G161| G162| G163|
| G164| G165| G166| G167| G168| G169| G170| G171| G172| G173| G174| G175|
| G177| G178| G179| G180| G181| G182| G183| G184| G185| G186| G187| G188|
| G189| G190| G191| G192| G193| G194| G195| G196| G197| G198| G199| G200|
| G201| G202| G203| G204| G205| G206| G207| G208|

### 1212 and 2121

| G92 | G93 | G94 | G95 | G96 | G98 | G99 | G100| G103| G104| G111| G112|
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| G113| G114| G115| G117| G118| G119| G120| G121| G122| G123| G124| G125|
| G126| G127| G128| G129| G130| G133| G134| G135| G136| G137| G138| G139|
| G140| G141| G142| G143| G144| G145| G146| G147| G148| G149| G150| G151|
| G152| G153| G154| G156| G157| G158| G159| G160| G161| G162| G163| G164|
| G165| G166| G167| G168| G169| G170| G171| G172| G173| G174| G175| G177|
| G178| G179| G180| G181| G182| G183| G184| G185| G186| G187| G188| G189|
| G190| G191| G192| G193| G194| G195| G196| G197| G198| G199| G200| G201|
| G202| G203| G204| G205| G206| G207| G208|

29
| 2112 |
|------------------|
| G96 | G98 | G99 | G103 | G105 | G111 | G112 | G113 | G114 | G115 | G117 | G118 |
| G119 | G120 | G121 | G122 | G123 | G124 | G125 | G126 | G127 | G128 | G129 | G130 |
| G133 | G134 | G135 | G136 | G137 | G138 | G139 | G140 | G141 | G142 | G143 | G144 |
| G145 | G146 | G147 | G148 | G149 | G150 | G151 | G152 | G153 | G154 | G156 | G157 |
| G158 | G159 | G160 | G161 | G162 | G163 | G164 | G165 | G166 | G167 | G168 | G169 |
| G170 | G171 | G172 | G173 | G174 | G175 | G176 | G177 | G178 | G179 | G180 | G181 | G182 |
| G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 | G191 | G192 | G193 | G194 |
| G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 | G203 | G204 | G205 | G206 |
| G207 | G208 |

| 222 |
|------------------|
| G96 | G99 | G105 | G111 | G114 | G115 | G117 | G118 | G121 | G125 | G126 | G127 |
| G128 | G129 | G133 | G135 | G136 | G137 | G138 | G140 | G141 | G143 | G144 | G145 |
| G146 | G147 | G148 | G149 | G150 | G151 | G152 | G153 | G154 | G156 | G157 | G158 |
| G159 | G160 | G161 | G162 | G163 | G164 | G165 | G166 | G167 | G168 | G169 | G170 |
| G171 | G172 | G173 | G174 | G175 | G176 | G177 | G178 | G179 | G180 | G181 | G182 |
| G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 | G191 | G192 | G193 | G194 |
| G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 | G203 | G204 | G205 | G206 |
| G207 | G208 |

| 1113 and 3111 |
|------------------|
| G117 | G119 | G126 | G130 | G133 | G134 | G140 | G141 | G142 | G143 | G144 | G150 |
| G151 | G153 | G154 | G156 | G157 | G158 | G159 | G160 | G163 | G165 | G166 | G168 |
| G169 | G170 | G171 | G172 | G173 | G174 | G175 | G177 | G178 | G179 | G180 | G181 |
| G182 | G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 | G191 | G192 | G193 |
| G194 | G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 | G203 | G204 | G205 |
| G206 | G207 | G208 |

| 1131 and 1311 |
|------------------|
| G77 | G78 | G92 | G100 | G114 | G117 | G119 | G121 | G125 | G126 | G129 | G130 |
| G133 | G134 | G135 | G138 | G140 | G141 | G142 | G143 | G144 | G145 | G146 | G149 |
| G150 | G151 | G153 | G154 | G156 | G157 | G158 | G159 | G160 | G161 | G162 | G163 |
| G165 | G166 | G168 | G169 | G170 | G171 | G172 | G173 | G174 | G175 | G177 | G178 |
| G179 | G180 | G181 | G182 | G183 | G184 | G185 | G186 | G187 | G188 | G189 | G190 |
| G191 | G192 | G193 | G194 | G195 | G196 | G197 | G198 | G199 | G200 | G201 | G202 |
| G203 | G204 | G205 | G206 | G207 | G208 |

| 123 and 321 |
|------------------|
| G117 | G126 | G133 | G140 | G141 | G143 | G144 | G150 | G151 | G153 | G154 | G156 |
| G157 | G158 | G159 | G160 | G163 | G165 | G166 | G168 | G169 | G170 | G171 | G172 |
| G173 | G174 | G175 | G177 | G178 | G179 | G180 | G181 | G182 | G183 | G184 | G185 |
| G186 | G187 | G188 | G189 | G190 | G191 | G192 | G193 | G194 | G195 | G196 | G197 |
| G198 | G199 | G200 | G201 | G202 | G203 | G204 | G205 | G206 | G207 | G208 |
### 132 and 231

| 132 | 231 |
|-----|-----|
| G92 | G114 |
| G126 | G129 |
| G133 | G135 |
| G140 | G141 |
| G143 | G144 |
| G145 | G146 |
| G149 | G150 |
| G151 | G153 |
| G154 | G156 |
| G157 | G158 |
| G159 | G160 |
| G161 | G162 |
| G163 | G165 |
| G166 | G168 |
| G169 | G170 |
| G171 | G172 |
| G173 | G174 |
| G175 | G176 |
| G178 | G179 |
| G180 | G181 |
| G182 | G183 |
| G184 | G185 |
| G186 | G187 |
| G188 | G189 |
| G190 | G191 |
| G192 | G193 |
| G194 | G195 |
| G196 | G197 |
| G198 | G199 |
| G200 | G201 |
| G202 | G203 |
| G204 | G205 |
| G206 | G207 |
| G208 |   |

### 213 and 312

| 213 | 312 |
|-----|-----|
| G126 | G140 |
| G141 | G143 |
| G144 | G150 |
| G151 | G154 |
| G156 | G157 |
| G158 | G159 |
| G160 | G163 |
| G165 | G166 |
| G168 | G169 |
| G170 | G171 |
| G172 | G173 |
| G174 | G175 |
| G176 | G177 |
| G178 | G179 |
| G180 | G181 |
| G182 | G183 |
| G184 | G185 |
| G186 | G187 |
| G188 | G189 |
| G190 | G191 |
| G192 | G193 |
| G194 | G195 |
| G196 | G197 |
| G198 | G199 |
| G200 | G201 |
| G202 | G203 |
| G204 | G205 |
| G206 | G207 |
| G208 |   |

### 33

|    |    |
|----|----|
| G154 | G168 |
| G174 | G175 |
| G181 | G186 |
| G188 | G190 |
| G192 | G194 |
| G195 | G196 |
| G197 | G199 |
| G200 | G201 |
| G202 | G203 |
| G204 | G205 |
| G206 | G207 |
| G208 |   |

### 114 and 411

|    |    |
|----|----|
| G165 | G190 |
| G191 | G194 |
| G195 | G199 |
| G200 | G203 |
| G204 | G205 |
| G206 | G207 |
| G208 |   |

### 141

|    |    |
|----|----|
| G77 | G146 |
| G161 | G165 |
| [G175] | G189 |
| G190 | G191 |
| G194 | G195 |
| G197 | G199 |
| G200 | G201 |
| G203 | G204 |
| G205 | G206 |
| G207 | G208 |

### 24 and 42

|    |    |
|----|----|
| G190 | G194 |
| G195 | G199 |
| G200 | G203 |
| G204 | G205 |
| G206 | G207 |
| G208 |   |

### 15 and 51

|    |    |
|----|----|
| G208 |   |
## B Attainable ordered multiplicity lists

For each connected graph on six or fewer vertices we give the Atlas of Graph numbering, draw the graph, and list all attainable ordered multiplicity lists.

| G1   |   | 1   |
|------|---|-----|
| G3   |   | 11  |
| G6   |   | 111 |
| G7   |   | 111, 12, 21 |
| G13  |   | 1111, 121 |
| G14  |   | 1111 |
| G15  |   | 1111, 121, 112, 211 |
| G16  |   | 1111, 121, 112, 211, 22 |
| G17  |   | 1111, 121, 112, 211, 22 |
| G18  |   | 1111, 121, 112, 211, 22, 13, 31 |
| G29  |   | 11111, 1121, 1211, 131 |
| G30 | 11111, 1121, 1211 |
|-----|------------------|
| G31 | 11111            |
| G34 | 11111, 1121, 1211, 1112, 2111, 122, 221 |
| G35 | 11111, 1121, 1211, 1112, 2111 |
| G36 | 11111, 1121, 1211, 1112, 2111 |
| G37 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212 |
| G38 | 11111, 1121, 1211, 1112, 2111, 122, 221 |
| G40 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212 |
| G41 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212 |
| G42 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212, 113, 131, 311 |
| G43 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212 |
| G44 | 11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131 |
|   |   |   |
|---|---|---|
| G45 | ![Diagram](G45) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311 |
| G46 | ![Diagram](G46) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131 |
| G47 | ![Diagram](G47) | 11111, 1121, 1211, 1112, 2111, 122, 212 |
| G48 | ![Diagram](G48) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311, 23, 32 |
| G49 | ![Diagram](G49) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311, 23, 32 |
| G50 | ![Diagram](G50) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311, 23, 32 |
| G51 | ![Diagram](G51) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311, 23, 32 |
| G52 | ![Diagram](G52) | 11111, 1121, 1211, 1112, 2111, 122, 212, 131, 113, 311, 23, 32, 14, 41 |
| G77 | ![Diagram](G77) | 111111, 11121, 12111, 11211, 1131, 1311, 141 |
| G78 | ![Diagram](G78) | 111111, 11121, 12111, 11211, 1131, 1311 |
| G79 | ![Diagram](G79) | 111111, 11121, 12111, 11211, 1221 |
| G80 | ![Diagram](G80) | 111111, 11121, 12111, 11211 |
| G81 | ![Diagram](image1) | 11111, 11121, 12111, 11211 |
|-----|-------------------|----------------------------|
| G83 | ![Diagram](image2) | 111111 |  
| G92 | ![Diagram](image3) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 1131, 1311, 132, 231 |
| G93 | ![Diagram](image4) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221 |
| G94 | ![Diagram](image5) | 11111, 11112, 21111, 11121, 12111, 11211, 1212, 2121 |
| G95 | ![Diagram](image6) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221 |
| G96 | ![Diagram](image7) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G97 | ![Diagram](image8) | 11111, 11112, 21111, 11121, 12111, 11211, 1122 |
| G98 | ![Diagram](image9) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G99 | ![Diagram](image10) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G100 | ![Diagram](image11) | 11111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 1131, 1311 |
| G102 | ![Diagram](image12) | 11111, 11112, 21111, 11121, 12111, 11211 |
| G103 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G104 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221 |
| G105 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 222 |
| G111 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G112 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G113 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G114 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231 |
| G115 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G117 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321 |
| G118 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G119 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311 |
| G120 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G121 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311) |
| G122 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212) |
| G123 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212) |
| G124 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212) |
| G125 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311) |
| G126 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311, 123, 321, 132, 231, 213, 312) |
| G127 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222) |
| G128 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222) |
| G129 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311, 132, 231) |
| G130 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311) |
| G133 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311, 1311) |
| G134 | ![Diagram](111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 2212, 222, 1131, 1311) |
| G135     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231 |
| G136     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G137     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G138     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311 |
| G139     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112 |
| G140     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 213, 312 |
| G141     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 213, 312 |
| G142     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311 |
| G143     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 213, 312 |
| G144     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 213, 312 |
| G145     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231 |
| G146     | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 141 |
| G147 | ![Diagram](G147) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G148 | ![Diagram](G148) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G149 | ![Diagram](G149) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231 |
| G150 | ![Diagram](G150) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G151 | ![Diagram](G151) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G152 | ![Diagram](G152) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G153 | ![Diagram](G153) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G154 | ![Diagram](G154) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G156 | ![Diagram](G156) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G157 | ![Diagram](G157) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G158 | ![Diagram](G158) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G159 | ![Diagram](G159) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G160 | ![Diagram](image1) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G161 | ![Diagram](image2) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 141 |
| G162 | ![Diagram](image3) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 213, 312 |
| G163 | ![Diagram](image4) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G164 | ![Diagram](image5) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G165 | ![Diagram](image6) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 114, 411, 141 |
| G166 | ![Diagram](image7) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1311, 123, 321, 132, 231, 213, 312 |
| G167 | ![Diagram](image8) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222 |
| G168 | ![Diagram](image9) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G169 | ![Diagram](image10) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1131, 123, 321, 132, 231, 213, 312 |
| G170 | ![Diagram](image11) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1311, 123, 321, 132, 231, 213, 312 |
| G171 | ![Diagram](image12) | 111111, 111112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G172 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G173 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G174 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312, 33 |
| G175 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312, 33, 141 |
| G177 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G178 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G179 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G180 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G181 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312, 33 |
| G182 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G183 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G184 | ![Graph](image) | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 1312, 231, 213, 312 |
| G185  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G186  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G187  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G188  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G189  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 141 |
| G190  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141 |
| G191  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 141 |
| G192  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G193  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312 |
| G194  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141 |
| G195  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141 |
| G196  | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G197 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141 |
| G198 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G199 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G200 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G201 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G202 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33 |
| G203 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G204 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G205 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G206 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G207 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42 |
| G208 | 111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 14, 42, 15, 51 |