Reaction-diffusion systems with constant diffusivities: conditional symmetries and form-preserving transformations

Roman Cherniha†‡ and Vasyl’ Davydovych†

† Institute of Mathematics, Ukrainian National Academy of Sciences, 3 Tereshchenkivs’ka Street, Kyiv 01601, Ukraine
‡ Department of Mathematics, National University ‘Kyiv-Mohyla Academy’, 2 Skovoroda Street, Kyiv 04070, Ukraine

E-mail: cherniha@imath.kiev.ua and davydovych@imath.kiev.ua

Abstract

Q-conditional symmetries (nonclassical symmetries) for a general class of two-component reaction-diffusion systems with constant diffusivities are studied. Using the recently introduced notion of Q-conditional symmetries of the first type (R. Cherniha J. Phys. A: Math. Theor., 2010. vol. 43., 405207), an exhaustive list of reaction-diffusion systems admitting such symmetry is derived. The form-preserving transformations for this class of systems are constructed and it is shown that this list contains only non-equivalent systems. The obtained symmetries permit to reduce the reaction-diffusion systems under study to two-dimensional systems of ordinary differential equations and to find exact solutions. As a non-trivial example, multiparameter families of exact solutions are explicitly constructed for two nonlinear reaction-diffusion systems. A possible interpretation to a biologically motivated model is presented.

1 Introduction

The paper is devoted to the investigation of the two-component reaction-diffusion (RD) systems of the form

\[
\begin{align*}
    u_t &= d_1 u_{xx} + F(u, v), \\
    v_t &= d_2 v_{xx} + G(u, v).
\end{align*}
\]

(1)

where \( u = u(t, x) \) and \( v = v(t, x) \) are two unknown functions representing the densities of populations (cells), the concentrations of chemicals, the pressures in thin films, etc. \( F \) and \( G \) are the given smooth functions describing interaction between them and environment, \( d_1 \) and \( d_2 \) are diffusivities assumed to be positive constants. The subscripts \( t \) and \( x \) denote differentiation with respect to these variables. The class of RD systems (1) generalizes many well-known nonlinear second-order models and is used to describe various processes in physics, biology, chemistry and ecology (see, e.g., the well-known books [1, 2, 3, 4, 5]).

Nevertheless the search for Lie symmetries of the class of RD systems (1) was initiated about 30 years ago [6], this problem was completely solved only during the last decade because of its complexity. Now one can claim that all possible Lie symmetries of (1) were completely described in [7, 8, 9].
The time is therefore ripe for a complete description of non-Lie symmetries for the class of the RD systems \([1]\). However, it seems to be extremely difficult task because, firstly, several definitions of non-Lie symmetries have been introduced (non-classical symmetry \([1, 10]\), conditional symmetry \([11, 12]\), generalized conditional symmetry \([13, 14]\) etc.), secondly, the complete description of non-Lie symmetries needs to solve the corresponding system of determining equations, which is non-linear and can fully be solved only in exceptional cases.

Hereafter we use the most common notion among non-Lie symmetries, non-classical symmetry, which we continuously call the \(Q\)-conditional symmetry following the well-known book \([11]\) and our previous papers \([15, 16]\). It is well-known that the notion of \(Q\)-conditional symmetry plays an important role in investigation of the nonlinear evolution equations because, having such symmetries in the explicit form, one may construct new exact solutions, which are not obtainable by the classical Lie machinery. However, for a complete description of such symmetries, one needs to solve the corresponding non-linear system of determining equations that usually is very difficult task. Thus, to solve the \(Q\)-conditional symmetry classification problem for the class of RD systems \([1]\), one should look for new constructive approaches helping to solve the relevant nonlinear system of determining equations. A possible approach was recently proposed in \([17]\) and is used in this paper.

It can be noted that the diffusion coefficient \(d_1\) in system \([1]\) can be omitted without losing of generality because the simple substitution

\[
t \to t/d_1, F \to -d_1C^1, G \to -d_2C^2
\]

reduces the system to the form

\[
\begin{align*}
  u_{xx} &= u_t + C^1(u, v), \\
  v_{xx} &= dv_t + C^2(u, v),
\end{align*}
\]

where \(d = \frac{d_1}{d_2}\). Thus, we consider system \([2]\) in what follows.

The paper is organized as follows. In section 2, two different definitions of \(Q\)-conditional invariance for the class of RD systems \([2]\) are presented and the system of determining equations is derived. The theorem giving the complete description of \(Q\)-conditional symmetries of the first type is proved. In section 3, the form-preserving transformations for the class of RD systems \([2]\) are constructed and applied to the RD systems derived in section 2. In section 4, the \(Q\)-conditional symmetry obtained for reducing of the RD systems to the ODE systems are applied. Examples of finding exact solutions are presented together with a possible interpretation for population dynamics. Finally, we summarize and discuss the results obtained in the last section.

## 2 Conditional symmetries of the RD systems

Here we use the definition of \(Q\)-conditional symmetry of the first type for the RD systems (see \([17]\) for details). It is well-known that to find Lie invariance operators, one needs to consider system \([2]\) as the manifold \(M = \{S_1 = 0, S_2 = 0\}\) where

\[
\begin{align*}
  S_1 &\equiv u_{xx} - u_t - C^1(u, v), \\
  S_2 &\equiv v_{xx} - dv_t - C^2(u, v),
\end{align*}
\]
in the prolonged space of the variables: \( t, x, u, v, u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}, u_{tt}, v_{tt} \). According to the definition, system (2) is invariant under the transformations generated by the infinitesimal operator

\[
Q = \xi^0(t, x, u, v)\partial_t + \xi^1(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v,
\]

if the following invariance conditions are satisfied:

\[
\frac{Q S_1}{\mathcal{M}} = 0,
\]

\[
\frac{Q S_2}{\mathcal{M}} = 0.
\]

The operator \( Q_2 \) is the second prolongation of the operator \( Q \), i.e.

\[
Q_2 = Q + \rho^1_t \frac{\partial}{\partial u_t} + \rho^2_t \frac{\partial}{\partial v_t} + \rho^1_x \frac{\partial}{\partial u_x} + \rho^2_x \frac{\partial}{\partial v_x} + \sigma^1_{xx} \frac{\partial}{\partial u_{xx}} + \sigma^2_{xx} \frac{\partial}{\partial v_{xx}},
\]

where the coefficients \( \rho \) and \( \sigma \) with relevant subscripts are expressed via the functions \( \xi^0, \xi^1, \eta^1 \) and \( \eta^2 \) by well-known formulae (see, e.g., [11, 18, 19]).

Hereafter the listed above differential operators act on functions and differential expressions in a natural way, particularly \( Q(u) = \xi^0 u_t + \xi^1 u_x - \eta^1 \) and \( Q(v) = \xi^0 v_t + \xi^1 v_x - \eta^2 \).

**Definition 1.** [17] Operator (3) is called the \( Q \)-conditional symmetry of the first type for the RD system (2) if the following invariance conditions are satisfied:

\[
\frac{Q S_1}{\mathcal{M}_1} = 0,
\]

\[
\frac{Q S_2}{\mathcal{M}_1} = 0,
\]

where the manifold \( \mathcal{M}_1 \) is either \( \{ S_1 = 0, S_2 = 0, Q(u) = 0 \} \) or \( \{ S_1 = 0, S_2 = 0, Q(v) = 0 \} \).

**Definition 2.** Operator (3) is called the \( Q \)-conditional symmetry of the second type, i.e., the standard non-classical symmetry for the RD system (2) if the following invariance conditions are satisfied:

\[
\frac{Q S_1}{\mathcal{M}_2} = 0,
\]

\[
\frac{Q S_2}{\mathcal{M}_2} = 0,
\]

where the manifold \( \mathcal{M}_2 = \{ S_1 = 0, S_2 = 0, Q(u) = 0, Q(v) = 0 \} \).

**Remark 1.** It is easily seen that \( \mathcal{M}_2 \subset \mathcal{M}_1 \subset \mathcal{M} \), hence, each Lie symmetry is automatically the \( Q \)-conditional symmetry of the first and second type, while each \( Q \)-conditional symmetry of the first type is one of the second type (non-classical symmetry).

**Remark 2.** To the best of our knowledge, there are not many paper devoted to search of \( Q \)-conditional symmetries for the systems of PDEs [20, 21, 22, 23, 24]. One may easily check that Definition 2 was only used in all these papers.
Statement. Let us assume that

\[ X = (h_1(t, x) v + h_0(t, x)) \partial_v, \]  

(hereafter \( h_1(t, x) \) and \( h_0(t, x) \) are the given functions) is the Lie symmetry operator of the RD system \([2]\) while \( Q_1 \) is the known \( Q \)-conditional symmetry of the first type, which was found using the manifold \( \mathcal{M}_1 = \{ S_1 = 0, S_2 = 0, Q(u) = 0 \} \). Then any linear combination \( C_1 Q_1 + C_2 X \) (\( C_1 \) and \( C_2 \neq 0 \) are arbitrary constants) produces new \( Q \)-conditional symmetry of the first type.

Application of definition 2 for finding \( Q \)-conditional symmetry (non-classical symmetry) operators of the RD system \([2]\) leads to a complicated system of determining equations (DEs) (see system \((19)\) in \([17]\)), which seems to be extremely difficult for exact solving.

It turns out that application of definition 1 leads to essentially simpler system of DEs, which can be fully integrated. Here we present the result under the restrictions \( \xi^0 \neq 0 \) and \( d \neq 1 \) (the cases \( \xi^0 = 0 \) and \( d = 1 \) must be investigated separately). Thus, the system of DEs corresponding to the manifold \( \mathcal{M}_1 = \{ S_1 = 0, S_2 = 0, Q(u) = 0 \} \) takes the form

\[ \begin{align*}
\xi_x^0 &= \xi_u^0 = \xi_v^0 = \xi_u^1 = \xi_v^1 = 0 , \\
\eta_v^1 &= \eta_u^1 = \eta_{uu}^2 = \eta_{uv}^2 = \eta_{vv}^2 = 0 , \\
2\xi^0 \eta_{xx}^2 + (d - 1) \xi^1 \eta_u^2 &= 0, \\
2\eta_{xx}^1 + \xi^1 &= 0, \\
2\eta_{xx}^2 + d \xi^1_t &= 0, \\
2\xi^1_x - \xi^0 &= 0, \\
\eta^1 C_u^1 + \eta^2 C_v^1 + (2\xi^1_x - \eta_u^1) C^1 &= \eta_{xx}^1 - \eta^1_t, \\
\eta^1 C_u^2 + \eta^2 C_v^2 + (2\xi^1_x - \eta_v^2) C^2 &= \eta_{xx}^2 C^1 + (1 - d) \frac{\eta^1}{\xi^0} \eta_u^2 + \eta_{xx}^2 - d \eta_v^2. 
\end{align*} \]

Note that there is no any need to solve the similar system of DEs corresponding to the manifold \( \mathcal{M}_1^* = \{ S_1 = 0, S_2 = 0, Q(v) = 0 \} \) because the discrete transformations \( u \to v, v \to u \) transform each symmetry found using \( \mathcal{M}_1 \) to one corresponding to the manifold \( \mathcal{M}_1^* \).

It should be also noted that we find purely conditional symmetry operators, i.e., exclude all such operators, which are equivalent to Lie symmetry operators described in \([7, 8]\). Having this aim, we use the system DEs for search Lie symmetry operators (see \([10]\) for details):

\[ \begin{align*}
\xi_x^0 &= \xi_u^0 = \xi_v^0 = \xi_u^1 = \xi_v^1 = 0 , \\
\eta_v^1 &= \eta_u^1 = \eta_{uu}^2 = \eta_{uv}^2 = 0 , \\
2\xi^1_x - \xi^0 &= 0, \\
2\eta_{xx}^1 + \xi^1 &= 0, \\
2\eta_{xx}^2 + d \xi^1_t &= 0, \\
\eta^1 C_u^1 + \eta^2 C_v^1 + (2\xi^1_x - \eta_u^1) C^1 &= \eta_{xx}^1 - \eta^1_t, \\
\eta^1 C_u^2 + \eta^2 C_v^2 + (2\xi^1_x - \eta_v^2) C^2 &= \eta_{xx}^2 C^1 + (1 - d) \frac{\eta^1}{\xi^0} \eta_u^2 + \eta_{xx}^2 - d \eta_v^2.
\end{align*} \]
Comparing DEs (5)–(12) with (13)–(19) one concludes that \( \eta^2 \neq 0 \) is the necessary and sufficient condition, which guarantees this property.

Now we need to solve the nonlinear system (5)–(12). Obviously equations (5) and (6) can be easily integrated:

\[
\xi^0 = \xi^0(t), \quad \xi^1 = \xi^1(t, x), \\
\eta^1 = r^1(t, x)u + p^1(t, x), \quad \eta^2 = q(t, x)u + r^2(t, x)v + p^2(t, x),
\]

where \( \xi^0(t), \xi^1(t, x), q(t, x), r^k(t, x), p^k(t, x) \) \( (k = 1, 2) \) are to-be-determined functions. Thus, substituting (20) into (7)–(12), one obtains the nonlinear system of PDEs:

\[
\begin{align*}
2\xi^0 q_x + \xi^1 (d - 1) q &= 0, \\
2r_x^1 + \xi^1_t &= 0, \\
2r_x^2 + d \xi^1_t &= 0, \\
2\xi^2_t - \xi^1_t &= 0, \\
(r^1 u + p^1) C_u^1 + (q u + r^2 v + p^2) C_v^1 + (2\xi^1_t - r^1) C^1 &= (r^1_{xx} - r^1_t) u + p^1_{xx} - p^1_t, \\
(r^1 u + p^1) C_u^2 + (q u + r^2 v + p^2) C_v^2 + (2\xi^1_t - r^2) C^2 &= q C^1 + \frac{r^1 u + p^1}{\xi^0} q (1 - d) + (22)
\end{align*}
\]

and

\[
\begin{align*}
(r^2_{xx} - dr^1_t) v + (q_{xx} - dq_t) u + p^2_{xx} - dp^1_t,
\end{align*}
\]

to find the functions \( \xi^0(t), \xi^1(t, x), q(t, x) \neq 0, r^k(t, x), p^k(t, x) \). In other words, all possible \( Q \)-conditional symmetries of the first type are easily constructed provided the general solution of system (21)–(26) is known.

**Theorem 1** The nonlinear RD system (2) with \( d \neq 1 \) is invariant under the \( Q \)-conditional operator of the first type (3) if and only if one and the corresponding operator have the forms listed in Table 1. Any other RD system admitting such kind of \( Q \)-conditional operator is reduced to one of those from Table 1 by the local transformations

\[
\begin{align*}
t &\rightarrow C_1 t + C_2, \\
x &\rightarrow C_3 x + C_4, \\
u &\rightarrow C_5 e^{C_7 t} u + C_7 t + C_8, \\
v &\rightarrow C_9 e^{C_13 t} v + C_{11} t^2 + C_{12} t + C_{13},
\end{align*}
\]

(27)

with correctly-specified constants \( C_l, l = 1, \ldots, 13 \) and/or the discrete transformations

\[
\begin{align*}
u &\rightarrow v, \ v \rightarrow u.
\end{align*}
\]

Simultaneously the relevant operator is reduced by possible adding a Lie symmetry operator of the form \((h_1(t, x)v + h_0(t, x)) \partial_v\) to those from Table 1.

**Sketch of proof.** To prove the theorem one needs to solve the nonlinear PDE system (21)–(26) with restriction \( q(t, x) \neq 0 \). We remind the reader that \( C^1 \) and \( C^2 \) should be treated as unknown functions. As follows from the preliminary analysis (see equations (25) and (26) involving the functions \( C^1 \) and \( C^2 \)), we should examine 6 cases:
$r^1 = r^2 = p^1 = 0$,
$2^1 = r^2 = 0$, $p^1 \neq 0$,
$3^1 = p^1 = 0$, $r^2 \neq 0$,
$4^1 = r^1 = 0$, $p^1 \neq 0$, $r^2 \neq 0$,
$5^2 = r^1 = 0$, $r^1 \neq 0$,
$6^2 = r^1 \neq 0$, $r^2 \neq 0$.

Solving system (21)–(26) in each case one obtains the list of $Q$-conditional symmetries of the first type together with the correctly-specified functions $C^1$ and $C^2$. Note that the symmetry operators have the different structures depending on the case.

Let us consider case (1) in details. Equations (25) and (26) take the form
\[
\begin{align*}
&(qu + p^2)C^1_v + 2\xi^1_vC^1 = 0, \\
&(qu + p^2)C^2_v + 2\xi^1_vC^2 = qC^1 + (q_{xx} - dq_t)u + p^2_{xx} - dp_t^2.
\end{align*}
\]

(29)

Differentiating the first equation of (29) with respect to $x$, one arrives at the equation $(q_xu + p^2_x)C^1_v = 0$, which lead to the requirement $C^1_v = 0$. In fact, if $q_{xx} \neq 0$ then immediately $C^1_v = 0$. If $q_{xx} = 0$ then equation (21) produces $\xi^1 = 0$, hence, $C^1_v = 0$. Thus, the first equation of system (29) takes the form $\xi^1_vC^1 = 0$ and two subcases $\xi^1_v \neq 0$ and $\xi^1_v = 0$ should be examined.

The general solution of (29) with $\xi^1_v \neq 0$ is
\[
C^1 = 0, \quad C^2 = \exp \left(-\frac{2\xi^1_x}{qu + p^2}v\right)g(u) + \frac{q_{xx} - dq_t}{2\xi^1_x}u + \frac{p^2_{xx} - dp_t^2}{2\xi^1_x},
\]
\[
(30)
\]

where $g(u)$ is an arbitrary (at the moment) function. Because the function $C^2$ doesn’t depend on $t$ and $x$, equation (30) with $g(u) \neq 0$ immediately produces the restrictions $q = \alpha_1\xi^1_x$, $p^2 = \alpha_2\xi^1_x$, where $\alpha_1$ and $\alpha_2$ are arbitrary constants. Differentiating equation (24) with respect to $x$, one obtains $\xi^1_{xx} = 0$. So $q_x \equiv \alpha_1\xi^1_{xx} = 0$, however, this contradicts to the assumption $\xi^1_v \neq 0$. The remaining possibility $g(u) = 0$ leads to the linear RD system (2).

Now we examine the subcase $\xi^1_v = 0$, i.e., $\xi^1 = \alpha_1 = const$. The general solution of (29) takes the form
\[
C^1 = f(u), \quad C^2 = \frac{qf(u) + \left(q_{xx} - dq_t\right)u + p^2_{xx} - dp_t^2}{qu + p^2}v + g(u),
\]
\[
(31)
\]

where $f(u)$ and $g(u)$ are arbitrary (at the moment) functions.

If $f(u)$ is an arbitrary function then we obtain $p^2 = \beta q$ ($\beta = const$) hence $C^2 = \frac{f(u)}{u + \beta}v + \alpha v + g(u)$, where $\alpha = \frac{q_{xx} - dq_t}{q}$. Having this, we use renaming $\frac{f(u)}{u + \beta} \rightarrow f(u)$ and solve the overdetermined system
\[
\begin{align*}
q_{xx} - dq_t &= 0, \\
2q_x + \lambda_1(d - 1)q &= 0.
\end{align*}
\]

Thus, the system of DEs (21)–(26) is completely solved (under above listed restrictions !) and we obtain the conditional symmetry operator
\[
Q = \partial_t + \lambda_1\partial_x + \lambda_2 \exp \left(\frac{\lambda_1(1 - d)}{2}x + \frac{\lambda_2^2(1 - d)^2 - 4\alpha}{4d}t\right)(u + \beta)\partial_v,
\]
where $\lambda_1$ and $\lambda_2 \neq 0$ are arbitrary constants, of the RD system
\begin{align}
  u_{xx} &= u_t + (u + \beta)f(u), \\
  v_{xx} &= dv_t + f(u)v + \alpha v + g(u).
\end{align}
(32)

Finally, using the simple transformation
\[ u \to u - \beta, \]
one sees that it is exactly case 6 of Table 1.

To complete the examination of case (1) we look for the correctly-specified function $f(u)$, which satisfies (31) without the restriction $p^2 = \beta q$. Indeed, if one finds the differential consequences of the second order (see equation for $C^2$) then $C^2_{ux} = 0$, $C^2_{vt} = 0$ and two algebraic equation to find the function $f(u)$ are obtained:

\begin{align}
  (q_t p^2 - q p_t^2) f &= ((q_{xx} - dq_t)u + p_{xx}^2 - dp_t^2)(q_t u + p_t^2) - \\
  ((q_{xx} - dq_t)u + (p_{xx}^2 - dp_t^2)_t)(qu + p^2), \\
  (q_x p^2 - q q_x^2) f &= ((q_{xx} - dq_t)u + p_{xx}^2 - dp_t^2)(q_x u + p_x^2) - \\
  ((q_{xx} - dq_t)u + (p_{xx}^2 - dp_t^2)_x)(qu + p^2),
\end{align}

Thus, $f(u) = \alpha_1 + \alpha_2 u + \alpha_3 u^2$ provided $p^2 \neq \beta q$. Substituting this expression into (31) and making the standard routine, one arrives at case 8 of Table 1 if $\alpha_3 \neq 0$ and case 9 if $\alpha_3 = 0$.

Cases (2)–(6) were treated in the similar way and the results are listed in Table 1. It should be noted that several local transformations ( (33) is the simplest example) were used to reduce the number of cases and simplify structures of the relevant RD systems. These transformations can be presented in the general form (27).

The sketch of proof is now completed. ■

**Table 1.** Q-conditional symmetry operators of the RD system (2) with $d \neq 1.$

| \(C^1(u, v)\) | \(C^2(u, v)\) | \(Q\) |
|-----------------|-----------------|-----|
| 1. \(uf(\omega)\) | \(u^k g(\omega) + u f(\omega) + \alpha(1 - d)\) \(\omega = u^{-k}(v - u)\) | \(\partial_t + \alpha u \partial_u + \alpha(1 - k)u + kv\partial_v,\) \(\alpha \neq 0, k \neq 1\) |
| 2. \(uf(\omega)\) | \(u g(\omega) + \alpha(1 - d) \ln u + f(\omega) \ln u, \) \(\omega = u \exp(-\frac{u}{k})\) | \(\partial_t + \alpha u \partial_u + \alpha(u + v)\partial_v,\) \(\alpha \neq 0\) |
| 3. \(uf(\omega)\) | \(g(\omega) + uf(\omega) + \alpha(1 - d)\) \(\omega = u \exp(u - v)\) | \(\partial_t + \alpha u \partial_u + \alpha(u + 1)\partial_v,\) \(\alpha \neq 0\) |
| 4. \(f(\omega)\) | \(e^u g(\omega) - f(\omega) - \alpha(1 - d)\) \(\omega = e^{-u}(u + v)\) | \(\partial_t + \alpha \partial_u + \alpha(u + v - 1)\partial_v,\) \(\alpha \neq 0\) |
| 5. \(f(\omega)\) | \(uf(\omega) + g(\omega) + (1 - d)u\) \(\omega = u^2 - 2v\) | \(\partial_t + \partial_u + u\partial_v\) |
|   |  $uf(u)$ | $vf(u) + g(u) + \alpha v$ | $\partial_t + \frac{2\lambda_1}{1-d} \partial_x + qu \partial_v$, $\lambda_2 \neq 0$ |
|---|---------|--------------------------|----------------------------------------------------------------------------------|
|   | $f(u)$  | $(u + v)(g(u) + \alpha \ln(u + v)) - f(u)$ | $\partial_t + \lambda \exp(-\frac{q^2}{2} t)(u + v) \partial_v$, $\lambda \neq 0$ |
| 8. | $\alpha_1 + \alpha_2 u \frac{u}{u^2}$ | $g(u) + uv$ | $\partial_t + \frac{2\lambda_1}{1-d} \partial_x + (qu + p^2) \partial_v$, $q = \varphi_1(t) \exp(\lambda_1 x)$, $\varphi \neq 0$, $p^2 = \left(\lambda_1^2 + \alpha_2\right)\varphi_1 - d\varphi_1''(\lambda_1 x)$ |
| 9. | $\alpha_1 + \alpha_2 u$ | $g(u) + \alpha_3 v$ | $\partial_t + \frac{2\lambda_1}{1-d} \partial_x + (qu + p^2) \partial_v$, $q = \lambda_2 \exp\left(\lambda_1 x + \frac{\lambda_1^2 + \alpha_2 - \alpha_1 t}{d}\right)$, $p^\alpha = dp^2 + \alpha_3 p^2 - 1$, $\lambda_2 \neq 0$ |
| 10. | $\alpha_1 + \alpha_2 u + \frac{\alpha_4 \ln(u + v)}{\alpha_4}$ | $\alpha_3 v + (\alpha_3 - \alpha_2)u - \frac{\alpha_4 \ln(u + v)}{\alpha_4}$ | $\partial_t + \left(\psi(x) \exp(\frac{\alpha_4}{\alpha_1} x) - \frac{\alpha_4}{\alpha_1}\right)(\partial_u - \partial_v)$, $\alpha_1 \alpha_2 \alpha_4 \neq 0$ |
| 11. | $\alpha_2 u + v$ | $\alpha_3 v + \frac{1}{2}(1 + d)\frac{v^2}{u} + \alpha_1 u \ln u + \alpha_4 u$ | $\partial_t + \varphi_2(t) \partial_v(u \partial_u + v \partial_v) - \partial_\varphi_2(t) \partial_v$, $\varphi \neq 0$ |
| 12. | $\alpha_2 u$ | $\alpha_3 v + \alpha_4 u + u^k$ | $\partial_t + 2\lambda_1 u \partial_u + (\alpha_3 u)u + \lambda_2 k v \partial_v$, $\alpha_3 \alpha_4 \alpha_2 \neq 0$, $k \neq 1$ |
| 13. | $\alpha_1 u \ln u$ | $\alpha_3 v + \alpha_1 v u + \alpha_2 u^\frac{1}{2}$ | $\partial_t + \frac{2\lambda_1}{1-d} \partial_x + \lambda_2 e^{-\alpha_1 t} u \partial_u + (qu + \frac{\lambda_2}{d} e^{-\alpha_1 t}) \partial_v$, $q = \lambda_3 \exp\left(\lambda_1 x + \frac{\lambda_1^2 - \alpha_1 t}{d}\right)$, $\alpha_1 \neq \lambda_2 \neq 0$ |
| 14. | $\alpha_1 u \ln u$ | $\alpha_3 v + \alpha_1 v \ln u + \alpha_4 u$ | $\partial_t + \lambda_2 e^{-\alpha_1 t} u \partial_u + (\varphi_3(t)u + (\frac{\lambda_2}{d} e^{-\alpha_1 t} + \alpha_3 v) \partial_v$, $\alpha_1 \lambda_2 \varphi_3 \neq 0$ |
| 15. | $\alpha_1 u \ln u$ | $\alpha_3 v + \alpha_1 v u + \alpha_4 u + \alpha_2 u^\frac{1}{2}$ | $\partial_t + \lambda_2 e^{-\alpha_1 t} u \partial_u + (\varphi_5(t)u + \frac{\lambda_2}{d} e^{-\alpha_1 t} v) \partial_v$, $\alpha_1 \alpha_2 \alpha_4 \lambda_2 \varphi_5 \neq 0$ |
| 16. | $\alpha_1 u \ln u$ | $\alpha_3 v + \alpha_1 d v \ln u + \alpha_1 (1 - d) u \ln u - \alpha_3 u$ | $\partial_t + \lambda_2 e^{-\alpha_1 t} u \partial_u + (\alpha_1 u + \lambda_2 e^{-\alpha_1 t} \alpha_1 v) \partial_v$, $\alpha_1 \neq \lambda_2 \neq 0$ |
| 17. | $0$ | $\alpha_3 v + \ln u$ | $\partial_t + \frac{2\lambda_1}{1-d} \partial_x + \lambda_2 u \partial_u + (qu + p^2) \partial_v$, $q = \lambda_3 \exp\left(\lambda_1 x + \frac{\lambda_1^2 - \alpha_1 - \lambda_3 (1-d) t}{d}\right)$, $p^\alpha = dp^2 + \alpha_3 p^2 + \lambda_2$, $\lambda_2 \lambda_3 \neq 0$ |
In Table 1, the functions $\psi(x), \varphi_1(t), \varphi_2(t), \varphi_4(t), \varphi_5(t), \varphi_8(t)$, and $\varphi_9(t)$ are the general solutions of the equations

\[
\psi'' - \left( \frac{\alpha_1 \alpha_2}{\alpha_4 (1 - d)} + \alpha_2 \right) \psi = 0,
\]

\[
d\dot{\varphi}_1 - (2\lambda_1^2 + \alpha_1) \dot{\varphi}_1 + \frac{\lambda_1^4 + \alpha_2 \lambda_1^2 + \alpha_1}{d} \varphi_1 = 0,
\]

\[
d\dot{\varphi}_2 - (\alpha_2 - \alpha_3 + (1 - d) \varphi_2) \dot{\varphi}_2 - \alpha_1 \varphi_2 = 0,
\]

\[
d\dot{\varphi}_4 + (\alpha_3 + \lambda_2 (d - 1) e^{-\alpha_1 t}) \varphi_4 - \alpha_4 (\lambda_3 + \lambda_2 \frac{1 - d}{d} e^{-\alpha_1 t}) = 0,
\]

\[
d\dot{\varphi}_5 + (\alpha_3 + \lambda_2 (d - 1) e^{-\alpha_1 t}) \varphi_5 - \alpha_4 \lambda_2 \frac{1 - d}{d} e^{-\alpha_1 t} = 0,
\]
\[ d\ddot{\varphi}_8 - \alpha_2 \dot{\varphi}_8 + \frac{\alpha_1}{d} \varphi_8 + \frac{\alpha_3}{d} = 0, \]

and

\[ d\ddot{\varphi}_9 - (\lambda_1^2(1 + d) + \alpha_2(1 - d) - \alpha_3) \dot{\varphi}_9 + (\lambda_1^4 - \alpha_3 \lambda_1^2 - \alpha_2(\alpha_2 - \alpha_3)) \varphi_9 = 0, \]

respectively. The functions

\[ \varphi_3(t) = \begin{cases} 
\lambda_3 \exp\left(\frac{\alpha_2 - \alpha_3 + \lambda_2(1-d)}{d} t\right) + \frac{\alpha_4 \lambda_2(1-k)}{\alpha_2 - \alpha_3 + \lambda_2(1-d)}, & \text{if } \alpha_2 \neq \alpha_3 - \lambda_2(1-d), \\
-\frac{\alpha_4 \lambda_2(1-k)}{d} t + \lambda_3, & \text{if } \alpha_2 = \alpha_3 - \lambda_2(1-d); 
\end{cases} \]

\[ \varphi_{10}(t) = \begin{cases} 
\lambda_3 \exp\left(-\frac{\alpha_3}{d} t\right) + \frac{\alpha_4 \lambda_2}{\alpha_3}, & \text{if } \alpha_3 \neq 0, \\
\frac{\alpha_4 \lambda_2}{d} t, & \text{if } \alpha_3 = 0. 
\end{cases} \]

Finally, the function \( \varphi_6(t) = \varphi_3(t) \) at \( k = 0 \), while \( \varphi_7(t) = \varphi_3(t) \) at \( k = 0 \) and \( \alpha_4 = 1 \). Hereafter the upper dot index denotes differentiation with respect to the variable \( t \).

### 3 Form-preserving transformations of the RD systems

A natural question is: Can we claim that 26 systems listed in Table 1 are inequivalent up to any local substitutions (not only of the form (27))? It turns out that the answer is positive. To present the rigorous proof of this, we used the notion of the set of form-preserving point transformations introduced in [25] and now extensively used for Lie symmetry classification problems (see, e.g., [26, 27]). Note that finding these transformations for systems of PDEs is a difficult problem because of technical problems occurring in computations and there is no many results for systems (paper [28] is one of the first presenting an explicit result for a class of systems).

The form-preserving transformations present the most general and correctly-specified form of local substitutions, which can map some equations from a given class to other those belonging to the same class. They contain as particular cases the well-known equivalence transformations and discrete transformations, which maps each equation from the class to another one from this class, used in the well-known Ovsiannikov method of Lie symmetry classification. Here we construct such transformations with the aim to show that Table 1 cannot be shortened.

**Theorem 2** An arbitrary RD system of the form (2) with \( d \neq 1 \) can be reduced to another system of the same form

\[ \begin{align*}
    w_{yy} &= w_r + F^1(w, z), \\
    z_{yy} &= \lambda z_r + F^2(w, z),
\end{align*} \tag{34} \]

by the non-degenerate local transformation

\[ \begin{align*}
    \tau &= a(t, x, u, v), & y &= b(t, x, u, v), \tag{35} \\
    w &= \varphi(t, x, u, v), & z &= \psi(t, x, u, v). \tag{36}
\end{align*} \]
if and only if the smooth functions $a$, $b$, $\varphi$ and $\psi$ satisfy one of two sets conditions listed below.

I. $a = \alpha(t)$, $b = \beta(t)x + \gamma(t)$, $\alpha \beta \neq 0$, 
\[\varphi = f(t) \exp\left(-\frac{1}{16}(\beta x^2 + 2\dot{\gamma}x)\right)u + P(t,x), \quad f \neq 0,\]
\[\psi = g(t) \exp\left(-\frac{1}{16}(\beta x^2 + 2\dot{\gamma}x)\right)v + Q(t,x), \quad g \neq 0,\]

where the functions $\alpha(t), \beta(t), f(t), g(t), \gamma(t), P(t,x)$, and $Q(t,x)$ are such that the equalities
\[\dot{\alpha} = \beta^2, \quad \lambda = d,\]
\[\beta^2 F_1(\varphi, \psi) = \varphi_u C_1(u,v) + \varphi_{xx} - \varphi_t - 2\frac{\varphi_x}{\varphi_u} \varphi_{xu},\]
\[\beta^2 F_2(\varphi, \psi) = \psi_v C_2(u,v) + \psi_{xx} - \psi_t - 2\frac{\psi_x}{\psi_v} \psi_{xv}\]
take place;

II. $a = \alpha(t)$, $b = \beta(t)x + \gamma(t)$, $\alpha \beta \neq 0$, 
\[\varphi = f(t) \exp\left(-d\frac{1}{16}(\beta x^2 + 2\dot{\gamma}x)\right)v + P(t,x), \quad f \neq 0,\]
\[\psi = g(t) \exp\left(-d\frac{1}{16}(\beta x^2 + 2\dot{\gamma}x)\right)u + Q(t,x), \quad g \neq 0,\]

where the functions $\alpha(t), \beta(t), \gamma(t), f(t), g(t), P(t,x)$ and $Q(t,x)$ are such that the equalities
\[\dot{\alpha} = \lambda \beta^2, \quad \lambda = \frac{1}{d},\]
\[\beta^2 F_1(\varphi, \psi) = \varphi_v C^2(u,v) + \varphi_{xx} - d\varphi_t - 2\frac{\varphi_x}{\varphi_v} \varphi_{xv},\]
\[\beta^2 F_2(\varphi, \psi) = \psi_u C_1(u,v) + \psi_{xx} - \psi_t - 2\frac{\psi_x}{\psi_u} \psi_{xu}\]
take place.

**Proof.** First of all we note that each non-degenerate transformation (35) – (36) should satisfy the condition

\[\Delta_1 = \begin{vmatrix}
a_x & a_t & a_u & a_v \\
b_x & b_t & b_u & b_v \\
\varphi_x & \varphi_t & \varphi_u & \varphi_v \\
\psi_x & \psi_t & \psi_u & \psi_v
\end{vmatrix} \neq 0,\]

which is used to prove the theorem.

Let us choose an arbitrary RD system of the form (2). The main idea of the proof is based on substituting the expressions for $u_{xx}$, $v_{xx}$, $u_t$, $v_t$ using the formulae (35) and (36) into this
system and on analysis conditions when the system obtained is equivalent to system (34). The expressions for the first-order derivatives have the form

\[
\begin{align*}
    u_x &= \frac{\varphi_x - a_x w_x - b_x w_y}{a_u w_x + b_u w_y - \varphi_v} a_v w_x + b_v w_y - \varphi_v, \\
    \psi_x &= \frac{\psi_x - a_x \varphi_x - b_x \varphi_y}{a_u z_x + b_u z_y - \psi_v} a_v \varphi_x + b_v \varphi_y - \psi_v.
\end{align*}
\]

The expressions for the second-order derivatives are very cumbersome, however, it can be noted that they contain the derivative \( w_{xy} \) and \( w_{xy} \). Because \( \tau \) is a new time-variable we conclude that the coefficient next to \( w_{xy} \) and \( w_{xy} \) must vanish otherwise system (34) are not obtainable. These coefficients vanish if and only if the equalities take place:

\[
a_x = a_u = a_v = b_u = b_v = 0 \Rightarrow a = \alpha(t), \ b = b(t, x).
\]  

Moreover, taking into account (45), the restrictions

\[
\hat{\alpha} \neq 0, \quad \Delta_2 = \begin{vmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{vmatrix} \neq 0
\]

are also obtained.

Having the set of equalities (46), the expressions for \( u_{xx} \) and \( u_t \) can be essentially simplified, namely:

\[
\begin{align*}
    u_{xx} &= \frac{\psi_v b_x^2}{\Delta_2} w_{xy} - \frac{\psi_u b_x^2}{\Delta_2} z_{yy} + \frac{(\psi_v b_x)(\Delta_{xy} - \Delta_{yy})}{\Delta_2^3} w_{xy} - \frac{(\psi_u b_x)(\Delta_{xz} - \Delta_{zx})}{\Delta_2^3} w_{xy}, \\
    u_t &= \frac{1}{\Delta_2^2} (\psi_v (\hat{\alpha} w_x + b_t w_y - \varphi_t) - \varphi_v (\hat{\alpha} z_x + b_t z_y - \psi_t)).
\end{align*}
\]  

Substituting (47) into the first equation of (2). Omitting the full expression of the equation obtained, we note that one contains the terms

\[
(i) \quad \frac{\varphi_v b_x^2}{\Delta_2} (z_{yy} - \frac{\hat{\alpha}}{b_x^2} z_{xy}), \quad (ii) \quad \frac{\psi_u b_x^2}{\Delta_2} (w_{yy} - \frac{\hat{\alpha}}{b_x^2} w_{xy}),
\]

while other terms don’t depend on \( z_{yy}, w_{yy}, z_{xy}, \) and \( w_{xy} \).

Now there is two possibilities. If the first equation of (2) is transformed into the first one of (34) then we immediately obtain

\[
\hat{\alpha} = b_x^2, \quad \varphi_v = 0.
\]  

\[12\]
If the first equation of (2) is transformed into the second one the conditions
\[ \dot{\alpha} = \lambda b^2_x, \quad \psi_v = 0 \] (49)
must be satisfied.

Let us consider conditions (48). Taking into account (45) and \( \phi_v = 0 \), the restriction \( \phi_u \psi_v \neq 0 \) springs up.

On the other hand, \( b(t, x) = \beta(t)x + \gamma(t) \), follows from (48) where \( \beta \) and \( \gamma \) are arbitrary smooth functions. Thus, the first equation from (38) is derived.

Substituting (48) into expressions for \( u_{xx} \) and \( u_t \) (see formulae (47)), one obtains
\[ \begin{align*}
  u_{xx} &= \frac{\beta^2}{\phi_u} w_{yy} - \frac{2\beta}{\phi_u} w_y + \frac{2}{\phi_u} \left( \frac{\phi_u}{\phi_v} \right) \beta w_y - \left( \beta - \frac{\phi_u}{\phi_v} \right)^2, \\
  u_t &= \frac{1}{\phi_u} (\alpha w + (\beta x + \dot{\beta}) w_y - \phi_t).
\end{align*} \] (50)

Since the first equation of system (34) doesn’t contain the terms \( w_y \) and \( w^2_y \), we should vanish the relevant coefficient, namely:
\[ \frac{2\beta}{\phi_u} + \frac{\dot{\beta} + \dot{\gamma}}{\phi_v} = 0, \]
\[ \phi_{uu} = 0. \]

The general solution of this system can be easily constructed so that obtains
\[ \varphi = f(t) \exp \left( -\frac{1}{4\beta} (\beta x^2 + 2\dot{\gamma} x) \right) u + P(t, x), \] (51)
where \( f(t) \neq 0 \) and \( P(t, x) \) are arbitrary functions at the moment. Thus, the first, second and third equations from (37) are derived. Moreover, substituting (50) and (51) into the first equation of system (2), we arrive at the equation
\[ w_{yy} = w_\tau + \frac{\varphi_u}{\beta^2} \left( C^1(u, v) - \varphi_t + \frac{2\varphi_v \varphi_{ux} - \varphi_u \varphi_{xx}}{\phi_u} \right). \] (52)

Now one realizes that (52) coincides with the first equation of system (34) iff condition (39) takes place.

The analogous routine involving the second equation of system (2) leads to the condition \( \dot{\alpha} = \frac{1}{\beta} \beta^2 \Rightarrow \lambda = d \) (see (48)), the function \( \psi \) of the form (37) and equation (40).

Analogous examination of conditions (49) leads to transformations (41) and to equalities (42)–(44).

The proof is now completed. ■

**Consequence 1.** The set of transformations (27) arising in theorem 1 is a subset of form-preserving transformations (37).

**Consequence 2.** If the nonlinear RD system of the form (11) is transformed to another one from this class, say, to the system
\[ \begin{align*}
  u_{t*} &= d_1 u_{x*xx} + F^*(u^*, v^*), \\
  v_{t*} &= d_2 v_{x*xx} + G^*(u^*, v^*)
\end{align*} \]
by a local substitution then they have the proportional diffusivities. Moreover, there are two linear combinations for the reaction terms $F$ and $F^*$, and for $G$ and $G^*$ resulting $\alpha_1 u + \alpha_2$ and $\alpha_3 v + \alpha_4$, respectively (here $\alpha_k, k = 1, \ldots, 4$ are correctly-specified constants).

Roughly speaking, consequence 2 says that the locally-equivalent RD systems have the same structure up to additive terms $\alpha_1 u + \alpha_2$ and $\alpha_3 v + \alpha_4$. At the first sight, there are some systems in Table 1 satisfying this consequence, for example in cases 17 and 18. However, according to consequence 2, the term $\alpha_4 u$ arising in the second equation of the RD system (see case 18) cannot be removed by any local substitution. We have carefully checked all cases listed in Table 1 and concluded that there are no any locally-equivalent systems therein.

Thus, we have shown that the list of RD systems presented in Table 1 cannot be reduced (shortened) by any local substitution.

4 New exact solutions and their possible interpretation

It is well-known that using the known $Q$-conditional symmetry (non-classical symmetry), one reduces the given system of PDEs to a system of ODEs via the same procedure as for classical Lie symmetries. Since each $Q$-conditional symmetry of the first type is automatically one of the second type, i.e., non-classical symmetry, we apply this procedure for finding exact solutions. Thus, to construct an ansatz corresponding to the given operator $Q$, the system of the linear first-order PDEs

$$Q(u) = 0, \quad Q(v) = 0$$

(53)

should be solved. Substituting the ansatz obtained into the RD system with correctly-specified coefficients, one obtains the reduced system of ODEs.

Let us construct exact solutions of the non-linear RD system listed in the case 1 of Table 1, when the system and the corresponding symmetry operator have the form

$$u_t = u_{xx} - uf(\omega),$$
$$dv_t = v_{xx} - u^k g(\omega) - u(f(\omega) + \alpha(1 - d)), \omega = u^{-k}(v - u)$$

(54)

and

$$Q = \partial_t + \alpha u \partial_u + \alpha((1 - k)u + kv)\partial_v.$$  

(55)

In this case system (53) takes the form

$$u_t = \alpha u,$$
$$v_t = \alpha(1 - k)u + \alpha kv$$

(56)

and its general solution produces the ansatz (the functions $u$ and $v$ depend on two variables $t$ and $x$):

$$u = \varphi(x)e^{\alpha t},$$
$$v = \psi(x)e^{\alpha t} + \varphi(x)e^{\alpha t},$$

(57)
where \( \varphi(x) \) and \( \psi(x) \) are new unknown functions. Substituting ansatz (57) into (54), one obtains so called reduced system of ODEs

\[
\begin{align*}
\varphi'' &= \varphi(\alpha + f(\omega)), \\
\psi'' &= \varphi^k g(\omega) + \alpha k d \psi, \quad \omega = \psi^{-k}. 
\end{align*}
\] (58)

Because system (58) is non-linear (excepting, of course, some special cases) it can be integrated only for the correctly-specified functions \( f \) and \( g \). We specify \( f \) and \( g \) in such a way, when the RD system in question will be still non-linear (otherwise the result will be rather trivial). Thus, setting

\[
\begin{align*}
f(\omega) &= \gamma \omega^\frac{1}{k} - \alpha, \\
g(\omega) &= \beta \omega, \\
\end{align*}
\] (59)

\( \beta \) and \( \gamma \) are arbitrary non-zero constants, the RD system takes the form

\[
\begin{align*}
\frac{du}{dt} &= u_{xx} - \gamma (v - u)^\frac{1}{k} + \alpha u, \\
\frac{dv}{dt} &= v_{xx} - \gamma (v - u)^\frac{1}{k} - \beta v + (\beta + \alpha d) u, 
\end{align*}
\] (59)

while the corresponding reduced system is

\[
\begin{align*}
\varphi'' &= \gamma \psi^\frac{1}{k}, \\
\psi'' &= (\beta + \alpha k d) \psi. 
\end{align*}
\] (60)

The general solution of (60) can be easily constructed:

\[
\begin{align*}
\varphi(x) &= \gamma \int \left( \int \psi^\frac{1}{k}(x) dx \right) dx + c_3 x + c_4, \\
\psi(x) &= \begin{cases} 
  c_1 \exp(\mu x) + c_2 \exp(-\mu x), & \text{if } \mu^2 = \beta + \alpha k d > 0, \\
  c_1 \cos(\nu x) + c_2 \sin(\nu x), & \text{if } \nu^2 = -(\beta + \alpha k d) > 0, \\
  c_1 x + c_2, & \text{if } \beta + \alpha k d = 0.
\end{cases}
\end{align*}
\] (62)

Thus, substituting (61) and (62) into (57), the 4-parameter family of solutions for the non-linear RD system (59) is constructed.

Hereafter we highlight the solutions satisfying the zero Neumann boundary conditions, which widely arise in biologically motivated boundary-value problems. Hence, setting \( c_3 = c_4 = 0, \ k = \frac{1}{n}, \ \psi = c_1 \cos(\nu x) \), one obtains the solution

\[
\begin{align*}
u &= -\gamma \frac{c_1^2}{9\nu^2} (\cos^2(\nu x) + 6) \cos(\nu x) e^{\alpha t}, \\
v &= c_1 \cos(\nu x) e^{\frac{-1}{9}\alpha t} + u.
\end{align*}
\]

It can be noted that this solution satisfies the zero Neumann boundary conditions

\[
\begin{align*}
u_x|_{x=0} &= 0, \ v_x|_{x=0} = 0, \ u_x|_{x=j\frac{\pi}{k}} = 0, \ v_x|_{x=j\frac{\pi}{k}} = 0
\end{align*}
\]

on the interval \([0, j\frac{\pi}{k}]\), where \( j \in \mathbb{N} \).

Let us set \( f(\omega) = -(a_1 + b\omega), \ g(\omega) = (\alpha(1 - d) - a_1)\omega \) (hereafter \( \alpha, a_1 \) and \( b \) are arbitrary non-zero constants) in (54), hence, it takes the form

\[
\begin{align*}
u_t &= u_{xx} + a_1 u - bu^{2-k} + bu^{1-k}, \\
v_t &= v_{xx} - a_2 u - bu^{2-k} + bv^{1-k}, 
\end{align*}
\] (63)
where $a_2 = \alpha(1 - d) - a_1$. The corresponding reduced system of ODEs is

\[
\begin{align*}
\varphi'' + b\varphi \varphi^{1-k} + (a_1 - \alpha)\varphi &= 0, \\
\psi'' &= (\alpha kd + a_2)\psi.
\end{align*}
\]

Nevertheless we have not constructed the general solution of system (64), its particular solution was found by setting $\psi = -\delta$, $\delta \neq 0$. In this case, the first-order ODE

\[
\varphi' = \pm \sqrt{(\alpha - a_1)\varphi^2 + \frac{2b\delta}{2-k}\varphi^{2-k} + c_1}, \quad \alpha = \frac{a_1}{d(k-1) + 1}, \quad k \neq 2
\]

for the function $\varphi$ is obtained (the value $k = 2$ is special and leads to ODE $\varphi' = \pm \sqrt{(\alpha - a_1)\varphi^2 + 2b\delta \ln \varphi + c_1}$).

If $c_1 \neq 0$ then the general solution of (65) can be expressed via hypergeometric functions. Here we present the solution for (65) with $c_1 = 0$, $k \neq 0$:

\[
\varphi(x) = \begin{cases} 
\left(\frac{\beta \left(\tan^2 \left(\frac{\sqrt{\alpha - a_1}}{2}(x \pm c_2)\right) + 1\right)}{2-k}\right)^{\frac{1}{k}}, & a_1 > \alpha, \\
\left(-\beta \left(\tanh^2 \left(\frac{\sqrt{\alpha - a_1}}{2}(x \pm c_2)\right) - 1\right)\right)^{\frac{1}{k}}, & a_1 < \alpha,
\end{cases}
\]

($\beta = \frac{(a_1 - \alpha)(2-k)}{2b\delta}$), which seems to be the most interesting. Note that another arbitrary constant can be removed by the trivial substitution $x \pm c_2 \to x$.

Now we rewrite system (63) setting $v \to -v$ with the aim to obtain a biologically motivated model. So the system takes form

\[
\begin{align*}
u_t &= u_{xx} + u(a_1 - bu^{1-k}) - buv^{1-k}, \\
v_t &= v_{xx} + v(-a_2 + bu^{1-k}) + bu^{2-k},
\end{align*}
\]

Figure 1: Exact solution (68) with $k = 0.5$, $\delta = 6$, $d = 4$, $a_1 = 5$, $b = 3$. 
where all coefficients (excepting $k$) should be positive. (67) can be treated as a prey-predator model for the population dynamics. In fact, the species $u$ is prey and described by the first equation. Its population decreases proportionally to the predator density $v$. The natural birth-death rule for the prey is $u(a_1 - bu^{1-k})$ and can be treated as a generalization of the standard logistic rule $u(a_1 - bu)$ (see, e.g., [2]). The similar arguments are also valid for the second equation. The model should involve also the zero Neumann boundary conditions (zero-flux on the boundaries), which indicate that both species cannot widespread over the globe but occupy a bounded domain.

Using (57) with $v \rightarrow -v$, $\psi = -\delta$ and (66) with $a_1 > \alpha$ we construct the exact solution

$$
\begin{align*}
    u &= \left( \beta (\tan^2 (\frac{k\sqrt{a_1 - \alpha}}{2} x) + 1) \right)^{-\frac{1}{k}} e^{\alpha t}, \\
    v &= \delta e^{\alpha kt} - \left( \beta (\tan^2 (\frac{k\sqrt{a_1 - \alpha}}{2} x) + 1) \right)^{-\frac{1}{k}} e^{\alpha t}
\end{align*}
$$

(68)

of (67). It turns out that solution can describe interaction between prey and predator on the space interval $[0, l]$, (here $l = \frac{2\pi j}{k\sqrt{a_1 - \alpha}}$, $j \in \mathbb{N}$) provided

$$
0 < k < 1 - \frac{1}{d}, \quad 0 < \delta \leq \left( \frac{(2 - k)(a_1 - \alpha)}{2b} \right)^{\frac{1}{k}}, \quad \alpha = \frac{a_1}{d(k - 1) + 1} < 1.
$$

(69)

One easily checks that solution (68) is non-negative, bounded in the domain $\Omega = \{(t, x) \in (0, +\infty) \times (0, l)\}$ and satisfy the given zero Neumann boundary conditions, i.e.

$$
\begin{align*}
    u_x|_{x=0} &= 0, \quad v_x|_{x=0} = 0, \quad u_x|_{x=l} = 0, \quad v_x|_{x=l} = 0.
\end{align*}
$$

As example we present this solution (68) with the parameters satisfying the restrictions (69) in Fig. 1. This solution can describe such type of the interaction between the species $u$ and $v$ when both of them eventually die, i.e. $(u, v) \rightarrow (0, 0)$ if $t \rightarrow +\infty$.

5 Conclusions

In this paper, $Q$-conditional symmetries for the class of RD systems (2) (that is equivalent to the class of systems (1)) and their application for finding exact solutions are studied. Following the recent paper [17], the notion of $Q$-conditional symmetry of the first type was used for these purposes. The main result is presented in theorem 1 giving the exhaustive list of RD systems of the form (1) with $d_1 \neq d_2$ (the case $d_1 = d_2$ should be analyzed separately), which admit such symmetry. It turns out that there are exactly 26 locally-inequivalent RD systems admitting the $Q$-conditional symmetry operators of the first type of the form (3) with $\xi^0 \neq 0$ (the case $\xi^0 = 0$ should be analyzed separately). To show local non-equivalence of the systems listed in Table 1, we proved theorem 2 describing the set of form-preserving point transformations for the class of RD systems (2). Note that all the operators found are inequivalent to the Lie symmetry operators presented in [7, 8] because the necessary and sufficient condition, which guarantees this property, was used.
The $Q$-conditional operator listed in case 1 of Table 1 was used to construct the non-Lie ansatz and to reduce two nonlinear RD systems to the corresponding ODE systems. Solving these ODE systems, the two-parameter families of exact solutions were explicitly constructed for the RD systems in question. Moreover, application of the exact solutions for solving the prey-predator system (67) was presented. It turns out that the relevant boundary value problem with the zero Neumann conditions can be exactly solved and the solution can describe the densities of two interacting species.

The work is in progress to construct conditional symmetries for multicomponent RD systems. In particular case, a wide list of the $Q$-conditional symmetries of the first type for the three-component diffusive Lotka-Volterra system is presented in [29].

Finally, we point out that this paper is a natural continuation of the recent paper [16], where RD systems with non-constant diffusivities were examined.

6 Acknowledgment

R.Ch. thanks the Organizing Committee of the 7th Workshop ‘Algebra, Geometry, and Mathematical Physics’ (Mulhouse, 24-26 October 2011) for the financial support.

References

[1] Ames, W.F.: Nonlinear Partial Differential Equations in Engineering. Academic, New York (1972)

[2] Murray, J.D.: Mathematical Biology. Springer, Berlin (1989)

[3] Murray, J.D.: Mathematical Biology II: Spatial Models and Biomedical Applications. Springer, Berlin (2003)

[4] Aris, R.: The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts. Clarendon, Oxford (1975)

[5] Okubo, A., Levin, S.A.: Diffusion and Ecological Problems. Modern Perspectives, 2nd edn. Springer, Berlin (2001)

[6] Zulehner, W., Ames, W.F.: Group analysis of a semilinear vector diffusion equation. Nonlinear Analysis 7, 945–69 (1983)

[7] Cherniha, R., King, J.R.: Lie Symmetries of Nonlinear Multidimensional Reaction-Diffusion Systems: I. J. Phys. A: Math. Gen. 33, 267–82, 7839–41 (2000)

[8] Cherniha, R., King, J.R.: Lie Symmetries of Nonlinear Multidimensional Reaction-Diffusion Systems: II. J. Phys. A: Math. Gen. 36, 405–25 (2003)

[9] Nikitin, A.G.: Group classification of systems of non-linear reaction-diffusion equations. Ukrainian Math. Bull. 2, 153-204 (2005)

[10] Bluman, G.W., Cole, J.D.: The general similarity solution of the heat equation J. Math. Mech. 18, 1025-42 (1969)
[11] Fushchych, W.I., Shtelen, W.M., Serov, M.I.: Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics. Kluwer, Dordrecht (1993)

[12] Cherniha, R., Henkel, M.: On nonlinear partial differential equations with an infinite-dimensional conditional symmetry. J. Math. Anal. Appl. 298, 487–500 (2004)

[13] Fokas, A.S., Liu, Q.M.: Nonlinear interaction of traveling waves of nonintegrable equations. Phys. Rev. Lett. 72, 3293–3296 (1994)

[14] Liu, Q.M., Fokas, A.S.: Exact interaction of solitary waves for certain nonintegrable equations. J. Math. Phys. 37, 324–345 (1996)

[15] Cherniha, R., Davydovych, V.: Conditional symmetries and exact solutions of the diffusive Lotka-Volterra system. Math. Comput. Modelling. 54, 1238–1251 (2011)

[16] Cherniha, R., Davydovych, V.: Conditional symmetries and exact solutions of nonlinear reaction-diffusion systems with non-constant diffusivities. Commun. Nonlinear. Sci. Numer. Simulat. 17, 3177–3188 (2012)

[17] Cherniha, R.: Conditional symmetries for systems of PDEs: new definition and its application for reaction-diffusion systems. J. Phys. A: Math. and Theor. 43, 405207 (13pp) (2010)

[18] Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, Berlin (1986)

[19] Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, Berlin (1989)

[20] Cherniha, R., Pliukhin, O.: New conditional symmetries and exact solutions of reaction-diffusion systems with power diffusivities. J. Phys. A: Math. Theor. 41, 185208 (15pp) (2008)

[21] Barannyk, T.: Symmetry and Exact Solutions for Systems of Nonlinear Reaction-Diffusion Equations. Proceedings of Institute of Mathematics of NAS of Ukraine. 43, 80–85 (2002)

[22] Murata, S.: Non-classical symmetry and Riemann invariants. Int. J. Non-Lin. Mech. 41, 242-246 (2006)

[23] Arrigo, D.J., Ekrut, D.A., Fliss, J.R., Long, Le: Nonclassical symmetries of a class of Burgers’ systems. J. Math. Anal. Appl. 371, 813–820 (2010)

[24] Cherniha, R., Serov, M.: Nonlinear Systems of the Burgers-type Equations: Lie and Q-conditional Symmetries, Ansätze and Solutions. J.Math.Anal.Appl. 282, 305–328 (2003)

[25] Kingston, J.G.: On point transformations of evolution equations. J. Phys. A. 24, 769–774 (1991)

[26] Cherniha, R., Serov, M., Rassokha, I.: Lie Symmetries and Form–preserving Transformations of Reaction–Diffusion–Convection Equations. J. Math. Anal. Appl. 342, 1363–1379 (2008)

[27] Popovych, R., Sophocleous, C., Vaneeva, O.: Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source. Acta Appl. Math. 106, 1–46 (2009)

[28] Cherniha, R., Myroniuk, L.: Lie Symmetries and Exact Solutions of a Class of Thin Film Equations. J. Phys. Math. 2, 1–19 (2010)

[29] Cherniha, R., Davydovych, V.: Lie and Conditional symmetries of the 3-D diffusive Lotka-Volterra system. J. Phys. A: Math. Theor. – 2013. – Vol. 46. – 185204 (14 pp).