U(1) × U(1) QUATERNIONIC METRICS FROM HARMONIC SUPERSPACE

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Abstract

We construct, using harmonic superspace and the quaternionic quotient approach, a quaternionic-Kähler extension of the most general two centres hyper-Kähler metric. It possesses $U(1) \times U(1)$ isometry, contains as special cases the quaternionic-Kähler extensions of the Taub-NUT and Eguchi-Hanson metrics and exhibits an extra one-parameter freedom which disappears in the hyper-Kähler limit. Some emphasis is put on the relation between this class of quaternionic-Kähler metrics and self-dual Weyl solutions of the coupled Einstein-Maxwell equations. The relation between our explicit results and the recent general ansatz of Calderbank and Pedersen for quaternionic-Kähler metrics with $U(1) \times U(1)$ isometries is traced in detail.

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1 Introduction

Recently, there was a surge of interest in the explicit construction of metrics for various classes of the hyper-Kähler (HK) and quaternionic-Kähler (QK) manifolds, caused by the important role these manifolds play in string theory (see, e.g., [1]-[5]). At present, there exist a few approaches to tackling this difficult problem [6]-[25]. One of them proceeds from the generic actions of bosonic nonlinear sigma models with the HK and QK target manifolds [8]-[13], [19]-[23].

Such generic actions, respectively for the HK and QK sigma models, were constructed in [8, 12, 13] and [19, 20, 21, 23] within the harmonic superspace (HSS) method [26, 27], based on the renowned one-to-one correspondence [28, 29] between the HK and QK manifolds on the one hand, and global and local $N = 2, d = 4$ supersymmetries on the other. It was proved in [28, 29] that the most general self-coupling of $N = 2$ matter supermultiplets (hypermultiplets) in the rigid or local $N = 2$ supersymmetry, necessarily implies, respectively, the HK or QK target geometry for the hypermultiplet physical bosonic fields. Conversely, any HK or QK bosonic sigma model can be lifted to a rigidly or locally $N = 2$ supersymmetric nonlinear sigma model. Most general off-shell actions for such $N = 2$ sigma models were constructed in [13, 19] in the framework of $N = 2$ harmonic superspace (HSS) [26] as the only one to offer such an opportunity. As was proved in [13, 21] starting from the general definition of HK or QK geometries as the properly constrained Riemannian ones, the corresponding analytic superfield Lagrangians of interaction have a nice geometric interpretation as the HK or QK potentials. These are the fundamental objects of the HK and QK geometries (like the Kähler potential in Kähler geometry). They encode the entire information about the local properties of the relevant bosonic metric, in particular, about its isometries. Then, based on the one-to-one correspondence mentioned above, the generic HK and QK sigma model bosonic actions can be obtained simply by discarding the fermionic fields in the general harmonic superspace sigma model actions. For the QK case such a generic bosonic action was constructed in [23]. The actions of physical bosons containing the explicit HK or QK metric associated with the given harmonic potential appear in general as the result of elimination of infinite sets of auxiliary fields contained in the off-shell hypermultiplet harmonic analytic superfields. This procedure amounts to solving some differential equations on the internal sphere $S^2$ parametrized by the $SU(2)$ harmonic variables. It is a difficult problem in general to solve such equations. However, as was shown in [8, 23], in the cases with isometries the computations can be radically simplified by using the harmonic superspace version of the HK [1, 6] or QK [13-18] quotient constructions. One of the attractive features of the HSS quotient is that it allows one, at all steps of computation, to keep manifest the corresponding isometries of the metric which come out as internal symmetries of the HSS sigma model Lagrangian with a transparent origin. It is especially interesting and tempting to apply this method for the explicit calculation of new inhomogeneous QK metrics. Indeed, whereas a lot of the HK metrics of this sort was explicitly constructed (both in 4- and higher-dimensional cases, see, e.g., [30]-[33], [14]), not too many analogous QK metrics are known to date.

In [23], using the HSS quotient techniques, we constructed QK extensions of the well-known [22] Taub-NUT and Eguchi-Hanson 4-dimensional HK metrics and discussed some their distinguished geometric features. In one or another (though rather implicit) form
these QK metrics already appeared in the literature (see, e.g., [17, 22, 34]) and our detailed treatment of them was a preparatory step to reveal capacities of the HSS approach for working out more interesting and less known examples.

In [11], the double Taub-NUT HK metric was derived from the HSS approach by directly solving the corresponding harmonic differential equations. It turns out that the HSS quotient approach allows one to reproduce the same answer much easier, and it nicely works as well in the QK case, where solving similar harmonic equations would bear a much more involved problem. In [35] we constructed a QK extension of the double Taub-NUT metric using the HSS quotient approach.

The present paper is intended, on the one hand, to give the detailed proof of some statements made in the letter [35] and to perform a further comparison with the available ansatzes for QK metrics. On the other hand, we demonstrate here that the HSS quotient approach suggests a further extension of the class of explicit QK metrics presented in [35]. All of them possess $U(1) \times U(1)$ isometry and are characterized by two additional free parameters. In the HK limit they go over into a generalization of the standard double Taub-NUT metric with two unequal “masses”, one of the new parameters being just the ratio of these “masses”. Another parameter does not show up in the HK limit, but it proves essential at the non-vanishing contraction parameter (Einstein constant). Thus we observe the existence of a one-parameter class of non-equivalent QK metrics having the same HK limit.

In section 2 we remind the basic facts about the HSS action of generic QK sigma model, as it was derived in [23]. In section 3 we construct the HSS quotient for the considered case of the QK double-Taub-NUT sigma model: proceed from a sum of the HSS “free” actions of three $Q^+$ hypermultiplets (having the hyperbolic $\mathbb{H}H^3$ manifold as the target space) and then gauge two common commuting one-parameter symmetries of these actions by two non-propagating $N = 2$ vector multiplets. The freedom in embedding these two symmetries in the variety of symmetries of the “free” action is characterized by two arbitrary constants which specify the most general QK extension of the double Taub-NUT metric. The intermediate steps leading to the final 4-dimensional metrics are described in section 4. The metric is read off after fixing the appropriate gauges and solving two sets of algebraic constraints appearing as the equations of motion for the auxiliary fields of the gauge multiplets. In section 5 we bring the metrics into the final form. Using the Przanowski-Tod ansatz [34, 36], we make an independent check that the metrics are indeed self-dual Einstein. Several limiting cases are also discussed. In section 6 we examine our metrics in the context of the literature related to self-dual Einstein geometries [37]-[43], including Flaherty’s equivalence to the (self-dual Weyl) solutions of the coupled Einstein-Maxwell equations [10].

Just after publication of our letter [35] reporting the construction of a QK extension of the double Taub-NUT metric in the HSS approach, Calderbank and Pedersen [43] have obtained the exact linearization of any four-dimensional QK metric with two commuting Killing vectors. After a short review of their results in section 6.5, we give the precise relation between their coordinates and ours.

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1 The QK metric presented in [35] corresponds to the minimal case, when both extra parameters are equal to zero.
2 The generic HSS action of QK sigma models

In [23] the generic action of QK sigma models with $4n$ dimensional target manifold of physical bosons was obtained as a pure bosonic part of the general off-shell HSS action of $n$ self-interacting matter hypermultiplets coupled to the so-called principal version of $N = 2$ Einstein supergravity [19]. The gauge multiplet of the latter, in the language of $N = 2$ conformal SG, consists of the $N = 2$ Weyl multiplet ($24 + 24$ off-shell components), the compensating vector multiplet ($8 + 8$ off-shell components) and the compensating hypermultiplet ($\infty + \infty$ off-shell components). It is the only version which admits the most general hypermultiplet matter self-couplings and thus, in accord with the theorem of [29], the most general QK metric in the sector of physical bosons. The matter and compensating hypermultiplets are described by the superfields $Q^+_r(\zeta)$ and $q^+_a(\zeta)$, $r = 1, \ldots, 2n$, $a = 1, 2$, given on the harmonic analytic $N = 2$ superspace

$$ (\zeta) = (x^m, \theta^{+\mu}, \bar{\theta}^{+\bar{\mu}}, u^{+i}, u^{-k}) , \quad (2.1) $$

where the coordinates $u^{+i}, u^{-k}$, $u^{+i}u^{-i} = 1$, $i, k = 1, 2$, are the $SU(2)/U(1)$ harmonic variables. These superfields obey the pseudo-reality conditions

$$ (a) \quad Q^{+r} \equiv (\tilde{Q}^+_r) = \Omega^{rs}Q^+_s \quad , \quad (b) \quad q^{+a} \equiv (\tilde{q}^+_a) = \epsilon^{ab}q^+_b \quad , \quad (2.2) $$

where $\Omega^{rs}$ and $\epsilon^{ab}$ ($\epsilon^{12} = -\epsilon_{12} = -1$) are the skew-symmetric constant $Sp(n)$ and $Sp(1) \sim SU(2)$ tensors. The generalized conjugation $\sim$ is the product of the ordinary complex conjugation and a Weyl reflection of the sphere $S^2 \sim SU(2)/U(1)$ parametrized by $u^{\pm i}$. The superspace $(2.1)$ is real with respect to this generalized conjugation which acts in the following way on the superspace coordinates:

$$ \tilde{x}^m = x^m \ , \ \tilde{\theta}^{+\mu} = \bar{\theta}^{+\bar{\mu}} \ , \ \tilde{\bar{\theta}}^{+\bar{\mu}} = -\theta^{+\mu} \ , \ \tilde{u}^{\pm i} = u^{\pm i} \ , \ \tilde{u}^{\pm i} = -u^{\pm i} . $$

In the QH sigma model action to be given below we shall need to know only the bosonic components in the $\theta$-expansion of the above superfields:

$$ q^{+a}(\zeta) = f^{+a}(x, u) + i(\theta^+\sigma^m\bar{\theta}^+ A_{m}^{-a}(x, u) + (\theta^+)^2(\bar{\theta}^+)^2 g^{-3a}(x, u) $$

$$ Q^{+r}(\zeta) = F^{+r}(x, u) + i(\theta^+\sigma^m\bar{\theta}^+ B_{m}^{-r}(x, u) + (\theta^+)^2(\bar{\theta}^+)^2 G^{-3r}(x, u) \quad (2.3) $$

(possible terms $\sim (\theta^+)^2$ or $\sim (\bar{\theta}^+)^2$ can be shown to fully drop out from the final action and so can be discarded from the very beginning). The component fields still have general harmonic expansions off shell. The physical bosonic components $F^{ri}(x), f^{ai}(x)$ are defined as the lowest components in the harmonic expansions of $F^{+r}(x, u), f^{+a}(x, u)$

$$ F^{+r}(x, u) = F^{ri}(x)u^+_i + \cdots \ , \quad f^{+a}(x, u) = f^{ai}(x)u^+_i + \cdots \ , \quad (F^{ri}(x)) = \Omega_{rs}\epsilon_{ik}F^{sk}(x) \ , \quad (f^{ai}(x)) = \epsilon_{ab}\epsilon_{ik}f^{bk}(x) . \quad (2.4) $$

Further details can be found in [23] and [20].

The bosonic QK sigma model action derived in [23] consists of the two parts

$$ S_{QK} = \frac{1}{2} \int d\zeta(-4) \left\{ -q_{a}^{+}D^{++}q^{+a} + \frac{\kappa^2}{\gamma^2}(u^{-i}q^{+i})^2 \left[ Q_{r}^{+}D^{++}Q^{+r} + L^{++}Q^{+} + L^{++}Q^{+} + L^{++}Q^{+} \right] \right\} $$

$$ -\frac{1}{2\kappa^2} \int d^4x \left[ D(x) + V^{mij}(x)\nu_{mij}(x) \right] \equiv S_{QQ} + S_{SG} . \quad (2.5) $$
Here, $d\zeta^{(-4)} = d^4x d^2 \theta^i d^2 \bar{\theta}^j du$ is the measure of integration over \([2.1]\), the covariant harmonic derivative $\mathcal{D}^{++}$ is defined by

$$
\mathcal{D}^{++} = D^{++} + (\theta^+)^2(\bar{\theta}^+)^2 \{ \partial(x) \partial^{-} + 6 \mathcal{V}^{m(ij)}(x)u^i_a u^j_a \partial_m \},
$$

(2.6)

with $D^{++} = \partial^{++} - 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m$, the non-propagating fields $D$, $\mathcal{V}^{ij}_m = \mathcal{V}^{ij}_{m}$ are inherited from the $N = 2$ Weyl multiplet, $\kappa^2 ([\kappa] = -1)$ is the Einstein constant (or, from the geometric standpoint, the parameter of contraction to the HK case), $\gamma ([\gamma] = -1)$ is the sigma model constant (chosen equal to 1 from now on), and the “target” harmonic variable $v^a$ is defined by

$$
v^a = \frac{q^a}{u^a} = u^a - \frac{u^+ q^a}{u^a} u^{-a}, \quad v^a u^a = 1.
$$

(2.7)

The function $L^{q+}(Q^+, v^+, u_-)$ is the analytic QK potential, the object which encodes the full information about the relevant QK metric.

The action \([2.3]\) possesses a local $SU(2)$ invariance, the remnant of the $N = 2$ supergravity gauge group, with $\mathcal{V}^{ij}_{m}(x)$ as the gauge field. The precise form of the $SU(2)_{\text{loc}}$ transformations leaving the $S_{q,Q}$ part of \([2.3]\) invariant can be inferred from the realization of the group of $N = 2$ conformal SG as restricted diffeomorphisms of the analytic superspace \([2.1]\), \([4]\). This can be achieved by fixing a WZ gauge for the Weyl multiplet and neglecting all its field components besides $D(x)$, $\mathcal{V}^{ik}_{m}(x)$, $\epsilon^a_m(x) \to \delta^a_m$ and all the residual gauge invariance parameters besides the $SU(2)_{\text{loc}}$ one $\lambda^{ik}(x) = \lambda^{ki}(x)$. These transformations read \footnote{They were not explicitly given in \([23]\) and earlier papers on the subject.}

$$
\delta u^+_i = \Lambda^{++} u^+_i , \quad \delta u^-_i = 0,
$$

(2.8)

$$
\delta \theta^+ = \lambda^+ \theta^+ - i(\theta^+)^2(\bar{\theta}^+)^2(\sigma^m \bar{\theta}^+ \partial_m \lambda^- - \partial^{++} \partial^{--} - \lambda^{--})
$$

$$
\delta \bar{\theta}^+= \lambda^- \theta^- - i(\bar{\theta}^+)^2(\sigma^m \theta^+ \partial_m \lambda^- - \partial^{++} \partial^{--} - \lambda^{--})
$$

$$
\delta x^m = -2i\theta^+ \sigma^m \bar{\theta}^+ \lambda^- + 6(\bar{\theta}^+)^2(\bar{\theta}^+)^2 \mathcal{V}^{--} \lambda^- = \lambda^m(\bar{\theta}^+)^2
$$

(2.9)

$$
\delta \mathcal{D}^{++} = -\Lambda^{++} D^0 , \quad D^0 = u^+ i \frac{\partial}{\partial u^+} - u^- i \frac{\partial}{\partial u^-} +\theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+}
$$

$$
\delta q^{a}(\bar{\lambda}) \simeq q^{a'}(\bar{\lambda}) - q^{a}(\bar{\lambda}) = -\frac{1}{2} \Lambda(\bar{\lambda}) q^{a}(\bar{\lambda}),
$$

(2.10)

$$
\delta Q^{- \gamma}(\bar{\lambda}) \simeq Q^{- \gamma'}(\bar{\lambda}) - Q^{- \gamma}(\bar{\lambda}) = 0,
$$

(2.11)

$$
\Lambda(\bar{\lambda}) = \partial_m \lambda^m + \partial^- \lambda^{++} - \partial_+ \lambda^m - \partial_+ \bar{\theta}^+.
$$

(2.12)

Here

$$
\lambda^{\pm \pm} = \lambda^{ik}(x) u^+_i u^+_k, \quad \lambda^{+-} = \lambda^{ik}(x) u^-_i u^-_k, \quad \mathcal{V}^{--} = \mathcal{V}^{ij}_{m}(x) u^-_i u^-_j \lambda^{--} = \mathcal{V}^{ij}_{m}(x) u^-_i u^-_j.
$$

To these transformations one should add the transformation laws of the fields $D(x)$ and $\mathcal{V}^{ik}_{m}(x)$

$$
\delta^s D(x) = 2 \partial_m \lambda^{ik}(x) \mathcal{V}^{m(ij)}_{ik}(x), \quad \delta^s \mathcal{V}^{ik}_{m}(x) = -\partial_m \lambda^{ik}(x) + 2 \lambda^{(i(k)}_{m) j(} (x),
$$

(2.13)
which uniquely follow from the transformation law (2.9). It is easy to see that the $S_{SG}$ part of (2.4) is invariant under (2.13), implying the $SU(2)_{loc}$ invariance of the full action (2.5). Note that the QK potential $L^{++}(Q^{+},v^{+},u^{-})$ in (2.3) is $SU(2)_{loc}$ invariant because its arguments $Q^{+}, v^{+}$ and $u^{-}$ behave as scalars under the above transformations. The transformations (2.8)-(2.12) entail the following simple $SU(2)_{loc}$ transformation rules for the lowest components $f^{+a}(x,u), F^{++r}(x,u)$ in the $\theta$-expansion (2.3)

$$\delta^* f^{+a} = \lambda^+ f^{+a} - \lambda^+ \partial^- f^{+a}, \quad \delta^* F^{++r} = -\lambda^+ \partial^- F^{++r}. \quad (2.14)$$

The procedure of obtaining the QK metric from the action (2.5) goes through a few steps. First one integrates over $\theta$s in $S_{Q,K}$, then varies with respect to the non-propagating fields $g^{-3a}(x,u), G^{-3r}(x,u), A_{-a}(x,u), B_{m}^{-r}(x,u), D(x)$ and $V_{ij}^{m}(x)$, solve the resulting non-dynamical equations and substitute the solution back into (2.5), thus expressing everything in terms of the physical components $f^{ai}(x)$ and $F^{ri}(x)$. Varying with respect to $D(x)$ and $V_{ik}^{m}(x)$ yields the important constraint relating $f^{\alpha a}$ and $F^{++r}$:

$$\int du \left[ f^{\alpha a} \partial^- f_a^{+} - \kappa^2 (u^{-} f^{+})^2 F^{++r} \partial^- F^{++r} \right] = \frac{1}{\kappa^2} \quad (2.15)$$

and the general expression for $V_{ik}^{m}(x)$ in terms of the hypermultiplet fields

$$V_{ik}^{m}(x) = 3\kappa^2 \int du u^{-k} \left[ f^{\alpha a} \partial_m f_a^{+} - \kappa^2 (u^{-} f^{+})^2 F^{++r} \partial_m F^{++r} \right]. \quad (2.16)$$

As the next step, one fixes a gauge with respect to the $SU(2)_{loc}$ transformations defined above. Most convenient is the gauge leaving only the singlet part in $f^{ai}(x)$

$$f_{i}^{a}(x) = \delta_{a}^{i} \omega(x) \quad (2.17)$$

(in what follows, we shall permanently use just this gauge). Finally, using the constraint (2.15), one expresses $\omega$ in terms of $F^{ri}(x)$, substitutes this expression into the action and reads off the QK metric on the $4n$ dimensional target space parametrized by $F^{ri}(x)$.

An essential assumption is that $\omega$ is a constant in the flat (hyper-Kähler) limit which is achieved by putting altogether

$$|\kappa| \omega = 1, \quad (2.18)$$

and then setting

$$\kappa = 0. \quad (2.19)$$

Note that in order to approach the HK limit in (2.3) in the unambiguous way, one should firstly eliminate the non-propagating field $V_{ik}^{m}(x)$ by its algebraic equation of motion and also perform varying with respect to the auxiliary field $F(x)$. Taking into account that the composite field $V_{ik}^{m}(x) \sim O(\kappa^2)$ and $q^{+a} \rightarrow u^{+a}|\kappa|^{-1}$ in the HK limit, one observes that any dependence on $q^{+a}, D$ and $V_{ik}^{m}$ disappears in this limit, and (2.3) goes into the HSS action of generic HK sigma model of $n$ hypermultiplets $Q^{+r} (r = 1, ..., 2n) \quad [12, 13].$ The constraint (2.15) becomes just the identity $1 = 1$. Another possibility is to remove the fields $D(x), V_{ik}^{m}(x)$ from (2.5) by equating them to zero. In this case one reproduces

$^{3}$Though looking rather involved, the transformations (2.8)-(2.13) can be straightforwardly checked to be closed, with the Lie bracket parameter $\lambda_{br}^{ik} = \lambda_{2}^{k} \lambda_{1}^{i} - \lambda_{1}^{k} \lambda_{2}^{i}$. 

5
the HSS action of the most general conformally-invariant HK sigma model with $n + 1$
hypermultiplets [27, 23, 45] (the former compensator $q^+(\zeta)$ enters it on equal footing
with other hypermultiplets). One can reverse the argument, i.e. start from such HK
sigma model action and reproduce the QK sigma model one (2.3) by coupling the HK
action to the non-propagating fields $D(x)$ and $V^m(x)$ in order to restore the local $SU(2)$
symmetry and to be able to remove the remaining (non-gauge) bosonic degree of freedom
in $f^+$ by the constraint (2.15). This is the content of the so-called “$N = 2$ superconformal
quotient” approach to the construction of 4$n$-dimensional QK manifolds from the 4$(n + 1)$-
dimensional HK ones [15, 46, 47, 24, 25]. In what follows we shall not need to resort to
such an interpretation and shall proceed from the general QK sigma model action (2.7).

3 QK extensions of the “double Taub-NUT” sigma model from HSS quotient

As already mentioned, on the road to the explicit QK metrics one needs to solve the
differential equations on $S^2$ for $f^+(x, u)$, $F^+(x, u)$ which follow by varying the QK
sigma model action with respect to the non-propagating fields $g^{-3a}(x, u)$ and $G^{-3r}(x, u)$.
No regular methods of solving such nonlinear equation are known so far, and this can (and
does) bear some troubles in general. However, in a number of interesting examples there is
a way around this difficulty, the HSS quotient method (it should not be confused with the
“superconformal quotient” mentioned in the end of the previous section). It can be applied
both in the HK [11] and QK [20, 23] cases. In it, one proceeds from a system of several “free”
hypermultiplets (with $L^+ = 0$ in (2.5), which corresponds to a $\mathbb{H}H^n \sim Sp(1, n)/Sp(1) \times
Sp(n)$ sigma model) and gauges some symmetries of this system in the analytic superspace
by non-propagating $N = 2$ vector multiplets represented by the gauge superfields $V^{++}(\zeta)$
(once again, only bosonic components of these superfields are of relevance). In one of
possible gauges these superfields can be fully integrated out, producing a non-trivial QK
(or HK) potential $L^+$ with the necessity to solve nonlinear harmonic equations. But in
another gauge (Wess-Zumino gauge) the harmonic equations remarkably become linear
and can be easily solved. All the nonlinearity in this gauge proves to be concentrated in
nonlinear algebraic constraints on the hypermultiplet physical fields. These constraints
are enforced by the auxiliary fields of vector multiplets as Lagrange multipliers. They are
much easier to solve as compared to the differential equations on $S^2$. This allows one to
get the explicit form of the QK (or HK) metric at cost of a comparatively little effort.

In [23], we exemplified the HSS quotient approach by QK extensions of the Taub-NUT
and Eguchi-Hanson (EH) metrics. Here we elaborate on a more interesting and non-trivial
case of the QK nonlinear sigma model generalizing the HK model with the “double Taub-
NUT” target manifold. The HSS action of the latter model was proposed in [11], and the
relevant HK metric was directly computed in [11] (it belongs to the class of two-center
ALF metrics, with the triholomorphic $U(1) \times U(1)$ isometry). Here we construct, using
the HSS QK quotient method, the QK sigma model action going into that of [9, 11] in the
HK limit. We find an interesting degeneracy suggested by the QK quotient: there is a one-
parameter family of the QK metrics, all having $U(1) \times U(1)$ isometry and reproducing
the double Taub NUT metric in the HK limit. More general QK action contains one
more parameter which survives in the HK limit and corresponds to a generalization of
the double Taub NUT metric by non-equal “masses” in its two-centre potential.

3.1 Minimal QK double-Taub-NUT HSS action

The actions we wish to construct have as their “parent action” the QK action including three hypermultiplet superfields of the type \( Q^{++} \) with the vanishing \( L^{++} \). So it corresponds to the “flat” QK manifold \( \mathbb{H}H^3 \sim Sp(1,3)/Sp(1) \times Sp(3) \). For our specific purposes we relabel this superfield triade as

\[
Q_{A}^{++}, \ g^{++}, \quad a = 1,2; \ r = 1,2; \ A = 1,2. \tag{3.1}
\]

The indices \( a \) and \( r \) are the doublet indices of two (initially independent) Pauli-Gürsey type \( SU(2) \) groups realized on \( Q^{+} \) and \( g^{+} \), the index \( A \) is an extra \( SO(2) \) index. Each of these three superfields satisfies the pseudo-reality condition (2.2).

We wish to end up with a 4-dimensional quaternionic metric. So, following the general strategy of the quotient method, we need to gauge two commuting one-parameter \( (U(1)) \) symmetries of this action. In this case the total number of algebraic constraints and residual gauge invariances in the WZ gauge is expected to be just 8, which is needed for reducing the original 12-dimensional physical bosons target space to the 4-dimensional one. These \( U(1) \) symmetries should be commuting, otherwise their gauging would entail gauging the symmetries appearing in their commutator. This would result in further constraints trivializing the theory.

The selection of two commuting symmetries to be gauged and the form of the final gauge-invariant HSS action are to a great extent specified by the natural requirement that the resulting action has two different limits corresponding to the earlier considered HSS quotient actions of the QK extensions of Taub-NUT and Eguchi-Hanson metrics [23].

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\[
S_{dTN} = \frac{1}{2} \int d\zeta^{(-4)} \mathcal{L}_{dTN}^{+4} - \frac{1}{2\kappa^2} \int d^4x \left[ D(x) + V^{mij}(x)\nu_{mij}(x) \right], \tag{3.2}
\]

where

\[
\mathcal{L}_{dTN}^{+4} = -q_{a}^{+}D^{++}q^{++} + \kappa^{2}(u^{+}\cdot q^{+})^{2} \left[ Q_{rA}^{++}D^{++}Q_{A}^{++} + g_{r}^{+}D^{++}g^{++} \right. \\
+ \left. W^{++} \left( Q_{A}^{++}Q_{AB}^{+} - \kappa^{2}c^{(ij)}g_{i}^{++}g_{j}^{++} + c^{(ij)}v_{i}^{++}v_{j}^{++} \right) \right. \\
+ \left. V^{++} \left( 2(u^{+}\cdot g^{+}) - a^{(r)}Q_{A}^{++}Q_{A}^{+} \right) \right] \tag{3.3}
\]

and the second term \( S_{SG} \) is common for all QK sigma model actions. In (3.3), \( V^{++}(\zeta) \) and \( W^{++}(\zeta) \) are two analytic gauge abelian superfields, \( c^{(ij)} \) and \( a^{(rm)} \) are two sets of independent \( SU(2) \) breaking parameters satisfying the pseudo-reality conditions

\[
(c^{(ij)}) = \epsilon_{ik}\epsilon_{jk}(kl), \quad (a^{(rm)}) = \epsilon_{rn}\epsilon_{ms}(ms). \tag{3.4}
\]

The Lagrangian can be checked to be invariant under the following two commuting gauge \( U(1) \) transformations, with the parameters \( \varepsilon(\zeta) \) and \( \varphi(\zeta) \)

\[
\delta\varepsilon Q_{A}^{++} = \varepsilon \left[ e_{AB}Q_{B}^{++} - \kappa^{2}c^{(ij)}v_{i}^{++}u_{j}^{++}Q_{A}^{++} \right], \quad \delta\varepsilon g^{++} = \varepsilon \kappa^{2} \left[ c^{(rm)}g_{n}^{++} - c^{(ij)}v_{i}^{++}u_{j}^{++}g^{++} \right], \\
\delta\varepsilon q^{++} = \varepsilon\kappa^{2}c^{(ab)}q_{+}^{++}, \quad \delta\varepsilon W^{++} = D^{++}, \tag{3.5}
\]

\[\text{To avoid a possible confusion, let us recall that the original general QK sigma model action (2.3).}\]
\[ \delta_\varphi Q_{A}^{+r} = \varphi a^{(rb)} Q_{bA}^{+} - \varphi \kappa^{2} (u^{-} \cdot g^{+}) Q_{A}^{+r} , \quad \delta_\varphi g^{+r} = \varphi \left[ u^{+r} - \kappa^{2} (u^{-} \cdot g^{+}) g^{+r} \right] , \\
\delta_\varphi q^{+a} = \varphi \kappa^{2} (u^{-} \cdot q^{+}) g^{+a} , \quad \delta_\varphi V^{++} = D^{++} \varphi . \]  

(3.6)

This gauge freedom will be fully fixed at the end. The only surviving global symmetries of the action will be two commuting \( U(1) \). One of them comes from the Pauli-Gürsey \( SU(2) \) acting on \( Q_{A}^{+a} \) and broken by the constant triplet \( a^{(bc)} \). Another \( U(1) \) is the result of breaking of the \( SU(2) \) which uniformly rotates the doublet indices of harmonics and those of \( q^{+a} \) and \( g^{+r} \). It does not commute with supersymmetry (in the full \( N = 2 \) supersymmetric version of (3.3)) and forms the diagonal subgroup in the product of three independent \( SU(2) \)'s realized on these quantities in the “free” case; this product gets broken down to the diagonal \( SU(2) \), and further to \( U(1) \), due to the presence of explicit harmonics and constants \( c^{(ik)} \) in the interaction terms in (3.3). These two \( U(1) \) symmetries are going to be isometries of the final QK metric, the first one becoming triholomorphic in the HK limit. The fields \( D(x) \) and \( \nu^{(ik)}(x) \) are inert under any isometry (modulo some rotations in the indices \( i,j \) after fixing the gauge (2.17)), and so are \( D^{++} \) and the \( S_{SG} \) part of (3.2). The harmonics \( v^{+a} \), as follows from their definition (2.7), undergo some appropriate transformations induced by those of \( q^{+a} \) in (3.3) and (3.4). Note that the presence of the \( g \)-field term in the supercurrent (Killing potential) to which \( W^{++} \) couples in (3.3), in parallel with the \( v^{+}_{i} \) term (becoming the Fayet-Iliopoulos term in the HK limit), is required for ensuring the invariance of this supercurrent under the \( \varphi \) gauge transformations. This in turn implies the non-trivial transformation property of \( g^{+r} \) under the \( \epsilon \) gauge group in (3.3). In the HK limit the \( g \)-field term drops out and \( g^{+r} \) becomes inert under the \( \epsilon \) transformations.

By fixing the appropriate broken \( SU(2) \) symmetries in (3.3), we can leave only one real component in each of the \( SU(2) \) breaking vectors \( a^{rf} \) and \( c^{ik} \). Thus the relevant QK metric is characterized by three real parameters: two \( SU(2) \) breaking ones and the Einstein constant \( \kappa^{2} \). The \( SU(2) \) breaking parameters survive in the HK limit.

**The QK EH and Taub-NUT sigma model limits**

It is easy to see that the action (3.2), (3.3) is indeed a generalization of the HSS quotient actions describing QK extensions of the EH and Taub-NUT sigma models.

Putting \( g^{+r} = a^{(rm)} = 0 \) yields the QK EH action as it was given in [20, 23]:

\[ L_{dTN}^{+4} \Rightarrow L_{EH}^{+4} = -q_{a}^{+} D^{++} q^{+a} + \kappa^{2} (u^{-} \cdot q^{+})^{2} \left[ Q_{A}^{+} D^{++} Q_{A}^{+r} + W^{++} (Q_{A}^{+a} Q_{aB}^{+} + c^{(ij)} v^{+}_{i} v^{+}_{j}) \right] . \]  

(3.7)

Putting \( Q_{2}^{+a} = c^{(ik)} = 0 \) yields the QK Taub-NUT action [23]

\[ L_{dTN}^{+4} \Rightarrow L_{TN}^{+4} = -q_{a}^{+} D^{++} q^{+a} + \kappa^{2} (u^{-} \cdot q^{+})^{2} \left[ g_{r}^{+} D^{++} g^{+r} + Q_{1r}^{+} D^{++} Q_{1r}^{+} + V^{++} (2 (v^{+} \cdot g^{+} - a^{(rf)} Q_{r}^{+} Q_{f}^{+}) \right] . \]  

(3.8)

contains a dimensionful sigma model constant \( \gamma, [\gamma] = -1 \), which we have put equal to 1 for convenience. Actually, it is present in an implicit form in the appropriate places of eq. (3.3) and subsequent formulae, thus removing an apparent discrepancy in the dimensions of various involved quantities. From now on, we assign the following dimensions to the basic involved objects and the gauge transformation parameters (in mass units): \([q] = [Q] = 1, [W^{++}] = 0, [V^{++}] = 1, [c] = 2, [a] = -1, [\epsilon] = 0, [\varphi] = 1\). With this choice, \( \gamma \) nowhere re-appears on its own right.
The HSS action with $g^{+r}$ eliminated

Representing $g^{+r}$ as

$$g^{+r} = (u^- \cdot g^+)v^{+r} - (v^+ \cdot g^+)u^{-r},$$

fixing the gauge with respect to the $\varphi$ transformations by the condition

$$(u^- \cdot g^+) = 0,$$

varying with respect to the non-propagating superfield $V^{++}$ and eliminating altogether $(v^+ \cdot g^+)$ by the resulting algebraic equation,

$$(v^+ \cdot g^+) \equiv L^{++} = \frac{1}{2} u^{rI} q^{+I} Q^{+A},$$

we arrive at the following equivalent form of (3.3), with only two matter hypermultiplets $Q^{+a}_A$ being involved

$$L^{+4}_{dTN} = -q^+_a D^{++} q^{+a} + \kappa^2 (u^- \cdot q^+)^2 [q^+_a D^{++} q^{+r}_A + L^{++} L^{++} + W^{++} (Q^{+a}_A \epsilon_{AB} - \kappa^2 c^{(ij)} u_i u^- L^{++} L^{++} + c^{(ij)} v_i v_j^+)] \tag{3.9}.$$  

In the HK limit $\kappa^2 \to 0$ $(q^{+a} \to |\kappa|^{-1} u^{+a}, |\kappa|(u^- \cdot q^+) \to 1)$ the corresponding action goes into the HSS action describing the double Taub-NUT manifold [11]. Thus (3.2), (3.3) is the natural QK generalization of the action of [11] and therefore the relevant metric is expected to be a QK generalization of the double Taub-NUT HK metric. We shall calculate it and its some generalizations in the next sections by choosing another, Wess-Zumino gauge in the relevant gauged QK sigma model actions.

### 3.2 Generalizations

In order to better understand the symmetry structure of the action (3.3) and to construct its generalizations, let us make the field redefinition

$$\hat{Q}^{+a}_A = |\kappa|(u^- \cdot q^+) Q^{+a}_A, \quad \hat{g}^{+r} = |\kappa|(u^- \cdot q^+) g^{+r}_A.$$

In terms of the redefined superfields, eqs. (3.3), (3.5) and (3.6) are simplified to

$$L^{+4}_{dTN} = -q^+_a D^{++} q^{+a} + \hat{Q}^{+_A} D^{++} \hat{Q}^{+r}_A + \hat{g}^{+r} D^{++} \hat{g}^{+r} + W^{++} [\hat{Q}^{+_A} \hat{Q}^{+r}_A \epsilon_{AB} - \kappa^2 c^{(ij)}(\hat{g}^{+i}_j + \hat{q}^{+i}_j)]$$

$$+ V^{++} [2|\kappa|(q^+ \cdot \hat{g}^+) - a^{(rf)} \hat{Q}^{+_A} \hat{Q}^{+r}_A], \tag{3.11}.$$  

$$\delta_{\varepsilon} \hat{Q}^{+r}_A = \varepsilon \epsilon_{AB} \hat{Q}^{+r}_B, \quad \delta_{\varepsilon} \hat{g}^{+r} = \varepsilon \epsilon^{(nr)} \hat{g}^{+r}_n, \quad \delta_{\varepsilon} q^{+a} = \varepsilon \kappa^2 c^{(ab)} q^{+a}_b, \tag{3.12}$$

$$\delta_{\varphi} \hat{Q}^{+r}_A = \varphi a^{(rb)} \hat{Q}^{+r}_B, \quad \delta_{\varphi} \hat{g}^{+r} = \varphi |\kappa| q^{+r}, \quad \delta_{\varphi} q^{+a} = \varphi |\kappa| g^{+a} \tag{3.13}.$$  

\footnote{For the precise correspondence one should choose $a^{12} = i a, a^{11} = a^{22} = 0$ by appropriately fixing the frame with respect to the broken Pauli-Gürsey $SU(2)$ symmetry of $Q^{+r}$.}
(the gauge superfields $W^{++}$, $V^{++}$ have the same transformation laws as before).

This form of gauge transformations clearly shows that the corresponding rigid transformations are linear combinations of four independent mutually commuting one-parameter symmetries which are enjoyed by the free part of the Lagrangian (3.11): (a) $SO(2)$ symmetry realized on the capital index of $\hat{Q}_A^{\pm}$; (b) a diagonal $U(1)$ subgroup in the product of two commuting $SU(2)_{PG}$ groups realized on $q^{a}$ and $\hat{g}^{r}$, with $\epsilon_{ijk}$ as the $U(1)$ generator; (c) $U(1)$ subgroup of the $SU(2)_{PG}$ group acting on $\hat{Q}_A^{tr}$, with $a^{rs}$ as the $U(1)$ generator; (d) a hyperbolic rotation of $q^{a}$ and $\hat{g}^{r}$,

$$\delta \hat{g}^{+r} = \varphi |\kappa| q^{+a} + \delta q^{+a} = \varphi |\kappa| \hat{g}^{+a}.$$  \hspace{1cm} (3.14)

Note that the bilinear form invariant under (3.14) is just $c^{(ij)}(\hat{g}^{i}_r \hat{g}^{j}_r - q^{i}_r q^{j}_r)$. This explains the presence of this expression in the $\varepsilon$-Killing potential (first square brackets in (3.11)): the $q^{+}$ term which is needed for making one of two basic constraints of the theory meaningful and solvable (see below) should be accompanied by the proper $\hat{g}^{+}$ term in order to comply with the symmetry (3.14). One is led to $\varepsilon$-gauge the diagonal $U(1)$ subgroup in the product of two independent $SU(2)_{PG}$ groups realized on $q^{a}$ and $\hat{g}^{r}$ just in order to gain this expression in the relevant Killing potential. In the HK limit $|\kappa| q^{+a} \rightarrow u^{+a}$, $\kappa \rightarrow 0$ the symmetry (3.14) becomes gauging of the familiar shift symmetry of the free hypermultiplet action:

$$\delta \hat{g}^{+r} = \varphi u^{+r}, \delta u^{+a} = 0.$$  \hspace{1cm} (3.15)

Thus we come to the conclusion that our original Lagrangian (3.13) is the simplest and natural choice yielding the double Taub-NUT HK action in the $\kappa \rightarrow 0$ limit, but it is by no means the unique one. Indeed, one could gauge two most general independent combinations of the four commuting $U(1)$ symmetries just mentioned. The corresponding generalization of (3.11) which still has a smooth $\kappa \rightarrow 0$ limit is as follows

$$\mathcal{L}_{dTN}^{++} = -q^{+}_a D^{++} q^{+a} + \hat{Q}_A^{++} D^{++} \hat{Q}_A^{++} + \hat{g}^{+r} D^{++} \hat{g}^{+r}$$

$$+ W^{++} \left[ \hat{Q}_A^{++} \hat{Q}_{AB}^{++} \varepsilon_{AB} - \kappa^2 c^{(ij)} \left( \hat{g}^{i}_r \hat{g}^{j}_r - q^{i}_r q^{j}_r \right) - \beta_0 a^{(rf)} \hat{Q}_{rA}^{++} \hat{Q}_{fA}^{++} \right]$$

$$+ V^{++} \left[ 2|\kappa| \left( q^{+} \cdot \hat{g}^{+} \right) - a^{(rf)} \hat{Q}_{rA}^{+} \hat{Q}_{fA}^{+} - \alpha_0 \kappa^2 c^{(ij)} \left( \hat{g}^{i}_r \hat{g}^{j}_r - q^{i}_r q^{j}_r \right) \right],$$

(3.16)

with $\alpha_0$ and $\beta_0 \left( [\alpha_0] = -1, [\beta_0] = 1 \right)$ being two new real independent parameters. It is straightforward to find the precise modification of the gauge transformation rules (3.12), (3.13):

$$\tilde{\delta}_e \hat{Q}_A^{++} = \delta_e \hat{Q}_A^{++} + \varepsilon \beta_0 a^{(rb)} \hat{Q}_{bA}^{++}, \quad \tilde{\delta}_\varphi \hat{g}^{+r} = \delta_\varphi \hat{g}^{+r} + \varphi \alpha_0 \kappa^2 c^{(rn)} \hat{g}^{+n},$$

$$\tilde{\delta}_e q^{+a} = \delta_e q^{+a} + \varphi \alpha_0 \kappa^2 c^{(ab)} q^{+b},$$

(3.17)

(the rest of transformations remains unchanged).

**Limits and truncations**

In the HK limit the generalized Lagrangian is reduced to

$$\mathcal{L}_{dTN}^{++} (\kappa \rightarrow 0) = \hat{Q}_r^{+} D^{++} \hat{Q}_r^{+} + \hat{g}_r^{+} D^{++} \hat{g}_r^{+}$$

$$+ W^{++} \left[ \hat{Q}_r^{+} \hat{Q}_{AB}^{+} \varepsilon_{AB} - \beta_0 a^{(rf)} \hat{Q}_{rA}^{+} \hat{Q}_{fA}^{+} + c^{(ij)} u^{+}_i u^{+}_j \right]$$

$$+ V^{++} \left[ 2 (u^{+} \cdot \hat{g}^{+}) - a^{(rf)} \hat{Q}_{rA}^{+} \hat{Q}_{fA}^{+} + \alpha_0 c^{(ij)} u^{+}_i u^{+}_j \right].$$

(3.18)
It is easy to see that the $\alpha_0$ term in the second bracket in (3.18) can be removed by the redefinition
\[ \hat{g}^{+r} \Rightarrow \hat{g}^{+r} - \frac{1}{2} \alpha_0 \epsilon_r^i u_i^+ , \] (3.19)
which does not affect the kinetic term of $\hat{g}^{+r}$. At the same time, no such a redefinition is possible in the QK Lagrangian (3.16), so $\alpha_0$ is the essentially new parameter of the corresponding QK metric. This $\alpha_0$-freedom disappears in the HK limit.

Thus the associated class of QK metrics includes two extra free parameters $\alpha_0$ and $\beta_0$ besides the $SU(2)$ breaking parameters and Einstein constant which characterize the minimal case treated before. But only one of them, $\beta_0$, is retained in the HK limit. Here we encounter a new (to the best of our knowledge) phenomenon of violation of the one-to-one correspondence between the HK manifolds and their QK counterparts.

It remains to understand the meaning of the parameter $\beta_0$. At $\beta_0 = 0$, we have the $\alpha_0$-modified QK double Taub-NUT action. To see what happens at non-zero $\beta_0$, it is instructive to take a modified EH limit in (3.18). Let us redefine
\[ a^{ik} = \frac{1}{\beta_0} \tilde{a}^{ik} \]
and then put $\hat{g}^{+r} = 0$, $\beta_0 \to \infty$ with keeping $\tilde{a}^{ik}$ finite and non-vanishing. Then (3.18) goes into
\[ L^+_{EH} (\kappa \to 0) = \hat{Q}^+_{rA} D^{++} \hat{Q}^+_{A} + W^{++} \left[ \hat{Q}^+_{rA} Q^+_{AB} \hat{Q}^+_{A} - \tilde{a}^{(r)} \hat{Q}^+_{rA} \hat{Q}^+_{A} + \epsilon^{(ij)} u_i^+ u_j^+ \right] . \] (3.20)
It is shown in section 5.5 that this HSS action produces a generalization of the standard two-centre Eguchi-Hanson metric by bringing in two unequal “masses” $1 - a$ and $1 + a$ in the numerators of poles in the relevant two-centre potential, with $a = \sqrt{\frac{1}{2} \tilde{a}^{ik} \tilde{a}_{ik}}$ ($c^{ik}$ specifies the centres like in the standard EH case [3]). Then it is clear that the action (3.18) describes a similar non-equal masses modification of the double Taub-NUT metric as a non-trivial “hybrid” of the Taub-NUT and unequal masses EH metrics, with $\beta_0$ measuring the ratio of the masses.

The general Lagrangian (3.16) has still two commuting rigid $U(1)$ symmetries which constitute the $U(1) \times U(1)$ isometry of the related QK metric. As distinct from the QK Taub-NUT and EH truncations (3.8) and (3.7) of (3.2), in which the isometries are enhanced to $U(2)$ [20, 23], the same truncations made in the Lagrangian (3.16) lead to generalized QK Taub-NUT and EH metrics having only $U(1) \times U(1)$ isometries. In the QK Taub-NUT truncation which is performed by putting $\hat{Q}^+_{2} = \beta_0 = 0$, $c^{ik} = 0$, $\alpha_0 c^{ik} \equiv \tilde{c}^{ik} \neq 0$ in (3.16),
\[ L^+_{4, TN} \Rightarrow L^+_{4, TN} = - q^+ + \hat{Q}^+_{r} D^{++} q^+ + \hat{Q}^+_{r} D^{++} \hat{Q}^+_{r} + \hat{g}^+ D^{++} \hat{g}^+ + V^{++} \left[ 2|\kappa| (q^+ \cdot \hat{g}^+) - a^{(r)} \hat{Q}^+_{r} \hat{Q}^+_{r} - \kappa^2 \epsilon^{(ij)} (\hat{g}^+ \hat{g}^+ + q^+ q^+) \right] , \] (3.21)
this isometry is again enhanced to $U(2)$ after taking the HK limit, because any dependence on the breaking parameter $\tilde{c}^{ik}$ disappears in this limit (after the redefinition like (3.19)). At the same time, in the QK EH truncation ($\hat{g}^{+r} = \alpha_0 = a^{r} = 0$, $\beta_0 a^{r} \equiv \tilde{a}^{r} \neq 0$ in (3.16)) the $U(1) \times U(1)$ isometry is retained in the HK limit, as clearly seen from the
form of the limiting HK Lagrangian (3.20) (parameters \(\tilde{a}^{ik}\) break \(SU(2)_{PG}\) and \(c^{ik}\) break the \(SU(2)\) which rotates harmonics).

**Alternative HSS quotient**

Finally, we wish to point out that the QK sigma model actions we considered up to now give rise to the QK metrics which are one or another generalization of the HK double Taub-NUT metric. This is closely related to the property that one of the symmetries of the free QK action of \((q^{+a}, \hat{Q}_{A}^{+r}, \hat{g}^{+r})\) which we gauge always includes as a part the hyperbolic \(\hat{g}^{+r}\), \(q^{+a}\) rotation (3.14) becoming a pure shift (3.15) of \(g^{+r}\) in the HK limit. This ensures the existence of the QK Taub-NUT truncation for the considered class of QK metrics. An essentially different class of QK metrics can be constructed by gauging two independent combinations of those mutually commuting \(U(1)\) symmetries of the free action which are realized as the homogeneous phase transformations of the involved superfields. The most general gauged QK sigma model of this kind is specified by the following superfield Lagrangian

\[
L_{dEH}^{++} = -a^{+}D^{++}q^{+a} + \hat{Q}_{rA}^{+}D^{++}Q_{A}^{+} + \hat{g}_{r}^{+}D^{++}\hat{g}^{+r} + W^{++} \left[ \hat{Q}_{A}^{+}Q_{aB}^{+} + \gamma_0 d^{(ik)}\hat{g}_{i}^{+}\hat{g}_{k}^{+} + \beta_0 a^{(r)}\hat{Q}_{rA}^{+}Q_{A}^{+} + c^{(ik)}u^{+}_{i}u^{+}_{j} \right] + V^{++} \left[ d^{(ik)}\hat{g}_{i}^{+}\hat{g}_{k}^{+} - a^{(r)}\hat{Q}_{rA}^{+}\hat{Q}_{A}^{+} + \alpha_0 c^{(ij)}u^{+}_{i}u^{+}_{j} \right],
\]

(3.22)

where the involved constants are different from those in (3.14), despite being denoted by the same letters. To see to which kind of the 4-dimensional HK sigma model the QK Lagrangian (3.22) corresponds, let us examine its HK limit

\[
L_{dEH}^{++}(\kappa \to 0) = \hat{Q}_{rA}^{+}D^{++}Q_{A}^{+} + \hat{g}_{r}^{+}D^{++}\hat{g}^{+r} + W^{++} \left[ \hat{Q}_{A}^{+}Q_{aB}^{+} + \gamma_0 d^{(ik)}\hat{g}_{i}^{+}\hat{g}_{k}^{+} + \beta_0 a^{(r)}\hat{Q}_{rA}^{+}Q_{A}^{+} + c^{(ik)}u^{+}_{i}u^{+}_{j} \right] + V^{++} \left[ d^{(ik)}\hat{g}_{i}^{+}\hat{g}_{k}^{+} - a^{(r)}\hat{Q}_{rA}^{+}\hat{Q}_{A}^{+} + \alpha_0 c^{(ij)}u^{+}_{i}u^{+}_{j} \right].
\]

(3.23)

Under the truncation \(\hat{g}^{+r} = 0, \alpha_0 = 0, \beta_0 a^{rf} \equiv \bar{a}^{rf} \neq 0, a^{rf} = 0\) it goes into the Lagrangian (3.20) which corresponds to the EH model with unequal masses, while under the truncation \(Q_{2}^{+a} = 0, Q_{1}^{+a} \equiv Q^{+a}, \gamma_0 = \beta_0 = 0, \alpha_0 c^{ik} \equiv \bar{c}^{ik} \neq 0, c^{ik} = 0\) it is reduced to the following expression

\[
L_{EH}^{++} = \hat{Q}_{rA}^{+}D^{++}\hat{Q}_{A}^{+} + \hat{g}_{r}^{+}D^{++}\hat{g}^{+r} + V^{++} \left[ d^{(ik)}\hat{g}_{i}^{+}\hat{g}_{k}^{+} - a^{(r)}\hat{Q}_{rA}^{+}\hat{Q}_{A}^{+} + \bar{c}^{(ij)}u^{+}_{i}u^{+}_{j} \right].
\]

(3.24)

This HSS Lagrangian can be shown to yield again a EH sigma model with unequal masses. The parameters of this model are different from those pertinent to the first truncation. Thus (3.23) defines a “hybrid” between two different EH sigma models, and the associated QK sigma model (3.22) could be called the “QK double EH sigma model”.

As the final remark, we note that in the case of three hypermultiplets in the HK case one can define mutually-commuting independent shifting symmetries of the form (3.13).

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\(^{6}\)We expect that the related QK metrics fall into the class of QK metrics described by the Plebanski-Demianski Ansatz (3.3); this is not the case for the QK double Taub-NUT metrics, see section 6.4.
separately for each hypermultiplet. Accordingly, one can use them to define different HSS quotient actions (actually, all such actions, with at least two independent shifting symmetries (3.13) gauged, yield the Taub-NUT sigma model, while those where all three such symmetries are gauged yield a trivial free 4-dimensional HK sigma model). No such an option exists in the QK case: any other hyperbolic rotation like (3.14) (in the planes \((Q_1^\alpha, q^{i\alpha})\) or \((\hat{Q}_2^{\alpha a}, q^{i\alpha})\)) does not commute with (3.14) and the third one of the same kind. For this reason, we are allowed to use only one such hyperbolic symmetry in the gauged combinations of independent \(U(1)\) symmetries in the course of constructing the relevant HSS quotient actions. Of course, this is related to the fact that the full symmetry of the “flat” QK action of \((q^{+a}, \hat{Q}_A^{+a}, \hat{g}^{+r})\) is \(Sp(1,3)\), while the analogous symmetry of the relevant limiting HK action is a contraction of \(Sp(1,3)\), with a bigger number of the mutually commuting abelian subgroups.

4 From the HSS actions to QK metrics

4.1 Preparatory steps

As already mentioned, the basic advantage of the HSS quotient as compared to the approach based on solving nonlinear harmonic equations is the opportunity to choose the WZ gauge for \(W^{++}\) and \(V^{++}\) by using the \(\varepsilon\) and \(\varphi\) gauge freedom (see (3.3), (3.6)). In this gauge the harmonic differential equations for the lowest components \(f^{+a}(x, u, \hat{F}_A^{+r}(x, u)\) and \(\hat{g}^{+r}(x, u)\) of the superfields \(q^{+a}(\zeta), \hat{Q}_A^{+r}(\zeta)\) and \(\hat{g}^{+r}(\zeta)\) become linear and can be straightforwardly solved.

In the WZ gauge the gauge superfields has the following short expansion

\[
W^{++}(\zeta) = i\theta^+ \sigma^m \bar{\theta}^+ W_m(x) + (\theta^+)^2(\bar{\theta}^+)^2 P^{(ik)}(x) \bar{u}_i^- u_k^- ,
\]

\[
V^{++}(\zeta) = i\theta^+ \sigma^m \bar{\theta}^+ V_m(x) + (\theta^+)^2(\bar{\theta}^+)^2 T^{(ik)}(x) \bar{u}_i^- u_k^- \tag{4.1}
\]

(like in (2.3), we omitted possible terms proportional to the monomials \((\theta^+)^2\) and \((\bar{\theta}^+)^2\) because the equations of motion for the corresponding fields are irrelevant to our problem of computing the final target QK metrics). At the intermediate steps it is convenient to deal with the hypermultiplet superfields \(\hat{Q}_A^{+a}, \hat{g}^{+r}\) related to the original superfields by (3.10). They have the same \(\theta\) expansions (2.3), with “hat” above all the component fields. Due to the structure of the WZ-gauge (4.1), the highest components in the \(\theta\) expansions of the superfields \(\hat{Q}_A^{+a}, \hat{g}^{+r}\) and \(q^{+a}\) \((\hat{G}_A^{+3a}(x, u), \hat{g}^{-3r}(x, u)\) and \(f^{-3a}(x, u)\)) appear only in the kinetic part of (3.3). This results in the linear harmonic equations for \(f^{+a}(x, u, \hat{F}_A^{+r}(x, u)\) and \(\hat{g}^{+r}(x, u)\):

\[
\partial^{++} f^{+a} = 0 \Rightarrow f^{+a} = f_a^{+i}(x)u_i^+ , \quad \partial^{++} \hat{F}^{+r} = 0 \Rightarrow \hat{F}^{+b} = \hat{F}_A^{bi}(x)u_i^+ ,
\]

\[
\partial^{++} \hat{g}^{+r} = 0 \Rightarrow \hat{g}^{+r} = \hat{g}^{ri}(x)u_i^+ . \tag{4.2}
\]

It is easy to check that these equations are covariant under the \(SU(2)_{\text{loc}}\) transformations (2.14) which act on \(f^{+a}, \hat{F}_A^{+r}\) and \(g^{+r}\) as follows:

\[
\delta^* f^{+a} = \lambda^+ f^{+a} - \lambda^- \partial^- f^{+a} , \quad \delta^* \hat{F}_A^{+r} = \lambda^+ \hat{F}_A^{+r} - \lambda^- \partial^- \hat{F}_A^{+r} , \quad \delta^* \hat{g}^{+r} = \lambda^+ \hat{g}^{+r} - \lambda^- \partial^- \hat{g}^{+r} . \tag{4.3}
\]
(in checking this, one must use the properties \( \partial^{++}\lambda^{++} = 0 \), \( \partial^{++}\lambda^{+-} = \lambda^{++} \), \([\partial^{++},\partial^{--}] = \partial^{0} = u^{+}\partial/\partial u^{+} - u^{-}\partial/\partial u^{-} \) and \( \partial^{0}(f^{+a}, \hat{F}^{+r}, \hat{g}^{+r}) = (f^{+a}, \hat{F}^{+r}, \hat{g}^{+r}) \)). These transformations entail the following ones for the bosonic fields of physical dimension

\[
\delta f^{ai}(x) = \lambda^{i}_{k}(x) f^{ak}(x) , \quad \delta \hat{F}^{ri}_{A}(x) = \lambda^{i}_{k}(x) \hat{F}^{rk}_{A}(x) , \quad \delta \hat{g}^{ri}(x) = \lambda^{i}_{k}(x) \hat{g}^{rk}(x). \quad (4.4)
\]

This step is common for all QK sigma model actions considered in the previous section. The next common step is to vary with respect to the SG fields \( D(x) \) and \( V^{ik}_{m}(x) \) in order to obtain the appropriate particular forms of the constraint \((2.13)\) and the expression \((2.16)\). Bearing in mind the harmonic “shortness” \((4.2)\), we find

\[
\frac{k^2}{2} f^2 = 1 + \frac{k^2}{2} (\hat{F}^2 + \hat{g}^2) , \quad (4.5)
\]

\[
V^{ik}_{m} = k^2 \left( f^{a(i} \partial_{m} f^{j)} - \hat{F}^{ri}_{A} \partial_{m} \hat{F}^{rj}_{A} - \hat{g}^{(i} \partial_{m} \hat{g}^{j)} \right) , \quad (4.6)
\]

where

\[
f^2 = f^{ai} f_{ai} , \quad \hat{F}^2 = \hat{F}^{ri}_{A} \hat{F}^{rj}_{A} , \quad \hat{g}^{ri} \hat{g}^{ri}. \]

Taking into account the constraint \((4.7)\), it is easy to check that the \( SU(2)_{loc} \) transformation laws \((4.3)\) imply just the transformation law \((2.13)\) for the composite gauge field \((4.6)\).

One more common step is enforcing the gauge \((2.17)\)

\[
f^{ai}(x) = \delta^{i}_{\omega}(x) \Rightarrow f^{ai} f_{ai} \Rightarrow 2\omega^2. \quad (4.7)
\]

For what follows it will be useful to give how the residual gauge symmetries of the WZ gauge \((1.1)\) with the parameters \( \varepsilon(x) = \varepsilon(\zeta) \) and \( \varphi(x) = \varphi(\zeta) \) are realized in the gauge \((1.7)\) (in the general case of gauge symmetries \((3.17)\), \((3.12)\), \((3.13)\))

\[
\delta^{i}_{\varepsilon} \hat{F}^{ri}_{A} = \varepsilon_{AB} \hat{F}^{ri}_{B} + \varepsilon_{AB} \alpha^{rs} \hat{F}^{ri}_{sA} - \lambda^{i}_{k} \hat{F}^{rk}_{kA} , \quad \delta^{i}_{\varphi} \hat{F}^{ri}_{A} = \varphi^{rs} \hat{F}^{ri}_{sA} - \lambda^{i}_{k} \hat{F}^{rk}_{kA} , \quad \delta^{i}_{\varphi} \hat{g}^{ri} = \varphi \left| \kappa \right| \hat{g}^{ri} \omega + \varphi \alpha^{0} \kappa^2 c^{rs} \hat{g}^{s} - \lambda^{i}_{k} \hat{g}^{rk} ,
\]

\[
\delta^{i}_{\omega} = \frac{1}{2} \varphi \left| \kappa \right| (\epsilon_{ai} \hat{g}^{ai}) , \quad (4.9)
\]

where \( \lambda^{i}_{k} \), \( \lambda^{i}_{k} \) are the parameters of two different induced \( SU(2)_{loc} \) transformations needed to preserve the gauge \((4.7)\)

\[
\lambda^{i}_{k} = -\varepsilon \kappa^2 c^{ik} , \quad \lambda^{i}_{k} = -\varphi \left( \frac{\left| \kappa \right|}{\omega} \hat{g}^{ri} + \alpha^{0} \kappa^2 c^{ri} \right) . \quad (4.10)
\]

From now on, we fully fix the residual \( \varphi(x) \) gauge symmetry by gauging away the singlet part of \( g^{ri}(x) \):

\[
\epsilon_{ir} g^{ri}(x) = 0 \Rightarrow g^{ri}(x) = g^{(ri)}(x). \quad (4.11)
\]

The residual \( SO(2) \) gauge freedom, with the parameter \( \varepsilon(x) \), will be kept for the moment.

We shall explain further steps on the example of the simplest QK double Taub-NUT action \((3.2)\), \((3.3)\) and then indicate the modifications which should be made in the resulting physical bosons action in order to encompass the general case \((3.16)\).
These steps are technical (though sometimes amounting to rather lengthy computations) and quite similar to those expounded in [23] on the examples of the QK extensions of the Taub-NUT and EH metrics. So here we shall describe them rather schematically.

Firstly one substitutes the solution (4.2) back into the action (3.2), (3.3) (with the θ-integration performed) and varies with respect to the remaining non-propagating (vector) fields of the hypermultiplet superfields \( (A_m^a(x, u), \hat{B}_{4m}(x, u) \) and \( \hat{b}_{m}^{-r}(x, u) \) in the θ-expansions of \( q^{+a}, \hat{Q}^{+a}_{A} \) and \( \hat{g}^{+r} \), respectively). Then one substitutes the resulting expressions for these fields into the action (together with those for \( \omega(x) \) and \( \mathcal{V}_m^{ij}(x) \), eqs. (1.5), (2.13)) and performs the \( u \)-integration. At this stage it is convenient to redefine the remaining fields as follows

\[
F_A^{ai} = \frac{1}{\kappa \omega} \hat{F}_A^{ai}, \quad g^{ri} = \frac{2}{\kappa \omega} \hat{g}^{ri}.
\]  

(4.12)

In terms of the redefined fields and with taking account of the gauges (4.7), (4.11), the composite fields \( \omega \) and \( \mathcal{V}_m^{ij} \) are given by the following expressions:

\[
\kappa \omega = \frac{1}{\sqrt{1 - \lambda^2 g^2 - 2\lambda F^2}}, \quad \mathcal{V}_m^{ij} = -16\lambda^2 \omega^2 \left[ F_A^{ai} \partial_m F_j^{aj} + \frac{1}{4} g^{ri} \partial_m g^{jr} \right],
\]  

(4.13)

where

\[
F^2 \equiv F_A^{ai} F_{aiA}, \quad g^2 \equiv g^{ri} g_{ri}, \quad \lambda \equiv \frac{\mu^2}{4}.
\]  

(4.14)

After substituting everything back into the action we get the following intermediate expression for the \( x \)-space Lagrangian density \( \mathcal{L}_{dTN}(x) \):

\[
\mathcal{L}_{dTN}(x) = \mathcal{L}_0(x) + \mathcal{L}_{vec}(x),
\]  

(4.15)

where

\[
\mathcal{L}_0(x) = \frac{1}{D^2} \left\{ \mathcal{D} \left( X + \frac{Y}{4} \right) + \lambda \left( g^2 \cdot \frac{Y}{8} + 2T \right) \right\}
\]  

(4.16)

with

\[
\mathcal{D} = 1 - \frac{\lambda}{2} g^2 - 2\lambda F^2, \quad X = \frac{1}{2} \partial_m F_{aiA} \partial_m F_A^{ai}, \quad Y = \frac{1}{2} \partial_m g_{ij} \partial_m g^{ij},
\]

\[
T = F_a^i B \partial_m F_B^{aj} \left( F_{aiA} \partial_m F_{jA}^{ai} + \frac{1}{2} g_{ir} \partial_m g^{jr} \right)
\]  

(4.17)

and

\[
\mathcal{L}_{vec}(x) = \frac{1}{D} \left[ \alpha(W^m W_m) + \beta(V^m V_m) + \gamma(W^m V_m) + W^m K_m + V^m J_m \right],
\]  

(4.18)

with

\[
J_m = \frac{1}{2} a_{ab} F_a^i F_{aiB}, \quad K_m = -\frac{1}{2} \epsilon_{AB} F_A^{ai} \partial_m F_{aiB} - \frac{\lambda}{2} c_{ij} g^i_s \partial_m g^{sj},
\]

\[
\alpha = \frac{1}{2} \left( \frac{F^2}{4} - \lambda \hat{c}^2 + \frac{\lambda^2}{2} \hat{c}^2 g^2 \right), \quad \beta = \frac{1}{4} \left( 1 + \frac{\hat{a}^2}{4} F^2 - \frac{\lambda}{2} g^2 \right),
\]

\[
\gamma = \frac{1}{4} a_{ab} F_a^i F_{biB} \epsilon_{AB} - \lambda (c \cdot g), \quad \hat{c}^2 \equiv c_{ik} c_{ik}, \quad \hat{a}^2 \equiv a_{ab} a_{ab}.
\]  

(4.19)

(4.20)

---

7This relation was misprinted in [35].
After integrating out the non-propagating gauge fields $W^m(x)$ and $V^m(x)$, the part $\mathcal{L}_{\text{vec}}$ acquires the typical nonlinear sigma model form
\[
\mathcal{L}_{\text{vec}} \Rightarrow \frac{1}{D} Z , \quad Z = \frac{1}{4 \alpha \beta - \gamma^2} \left\{ \gamma (J \cdot K) - \alpha (J \cdot J) - \beta (K \cdot K) \right\} . \tag{4.21}
\]

The resulting sigma model action should be supplemented by two algebraic constraints on the involved fields
\[
F_a^{(i)} F_{aB}^{(j)} \epsilon_{AB} - \lambda g^{(ti)} g^{(rj)} c_{(tr)} + c^{(ij)} = 0 , \tag{4.22}
\]
\[
g^{ij} - a^{ab} F_{aB}^i F_{bB}^j = 0 , \tag{4.23}
\]
which follow from varying the action with respect to the auxiliary fields $P^{(ik)}(x)$ and $T^{(ik)}(x)$ in the WZ gauge (4.1). Keeping in mind these 6 constraints and one residual gauge ($SO(2)$) invariance, one is left just with four independent bosonic target coordinates as compared with eleven such coordinates explicitly present in (4.16), (4.21). The problem now is to solve eqs. (4.22), (4.23), and thus to obtain the final sigma model action with 4-dimensional QK target manifold. This will be the subject of our further presentation.

Here, as the convenient starting point for the geometrical treatment in section 5, it is worth to give how the full distance looks before solving the constraints (4.22), (4.23)
\[
g = \frac{1}{D^2} \left\{ D \left( X' + Z' + \frac{Y'}{4} \right) + \lambda \left( g^{2} \cdot \frac{Y'}{8} + 2 T' \right) \right\} . \tag{4.24}
\]
The quantities with “prime” are obtained from those defined above by replacing altogether “$\partial_m$” by “$d$”, thus passing to the distance in the target space. For instance,
\[
X' = \frac{1}{2} dF_{ai} A dF_{ai}, \quad Y' = \frac{1}{2} dg_{ij} dg^{ij} . \tag{4.25}
\]
Note that this metric includes three free parameters. These are the Einstein constant related to $\lambda (\lambda \equiv \frac{\kappa^2}{4})$, and two $SU(2)$ breaking parameters: the triplet $c^{(ij)}$, which breaks the $SU(2)_{\text{SUSY}}$ to $U(1)$, and the triplet $a^{(ab)}$, which breaks the Pauli-Gürsey $SU(2)$ to $U(1)$. The final isometry group is therefore $U(1) \times U(1)$. Constraints (4.22), (4.23) are manifestly covariant under these isometries. For convenience, from now on we choose the following frame with respect to the broken $SU(2)$ groups
\[
c^{12} = ic, \quad c^{11} = c^{22} = 0, \quad a^{12} = ia, \quad a^{11} = a^{22} = 0 , \tag{4.26}
\]
with real parameters $a$ and $c$. In this frame, the squares (4.20) become
\[
\hat{c}^2 = 2c^2 , \quad \hat{a}^2 = 2a^2 .
\]

Let us now discuss which modifications the distance (4.24) undergoes if one starts from the general QK double Taub-NUT action corresponding to the Lagrangian (3.16). Since the difference between (3.12) and (3.16) is solely in the structure of supercurrents (Killing potentials) to which gauge superfields $W^{++}$ and $V^{++}$ couple, the only modifications entailed by passing to (3.16) are the appropriate changes in the $Z'$-part of (4.24) and in the
constraints (4.22), (4.23). Namely, one should make the following replacements in $Z'$:

$$
\alpha \Rightarrow \hat{\alpha} = \alpha + \frac{1}{16} \beta_0 \hat{a}^2 F^2 + \frac{1}{4} \beta_0 a^i F_{rA}^i F_{fjB} \epsilon_{AB},
$$

$$
\beta \Rightarrow \hat{\beta} = \beta - \lambda \alpha_0 (g \cdot c) - \frac{1}{2} \alpha_0^2 \hat{c}^2 \left(1 - \frac{\lambda}{2} g^2 \right),
$$

$$
\gamma \Rightarrow \hat{\gamma} = \gamma + \frac{1}{8} \beta_0 \hat{a}^2 F^2 - \lambda \alpha_0 \hat{c}^2 \left(1 - \frac{\lambda}{2} g^2 \right),
$$

$$
K_m \Rightarrow \hat{K}_m = K_m + \frac{1}{2} \beta_0 a^f F_{rA}^i dF_{fjA},
$$

$$
J_m \Rightarrow \hat{J}_m = J_m - \frac{1}{2} \lambda \alpha_0 c^{tr} g^{r_l} d g_{kr},
$$

and pass to the following modification of the constraints (4.22), (4.23):

$$
F_A^{ai} F_{aB}^{ij} \epsilon_{AB} - \lambda g^{(ti)} g^{(rj)} c_{(tr)} - \beta_0 a^{ab} F_{aB}^i F_{bB}^j + c^{(ij)} = 0 ,
$$

$$
g^{ij} - a^{ab} F_{aB}^i F_{bB}^j + \alpha_0 \left[c^{ij} - \lambda g^{(ti)} g^{(rj)} c_{(tr)}\right] = 0 .
$$

### 4.2 Solving the constraints

In order to find the final form of the QK target metric corresponding to the HSS Lagrangian (3.3) or its generalization (3.16), we should solve the constraints (4.22), (4.23) or their generalization (4.28), (4.29). It is a non-trivial step to find the true coordinates to solve these constraints. Indeed, a direct substitution of $g_{ij}$ from (4.23) into (4.22) gives a quartic constraint for $F_{aiA}$ which is very difficult to solve as compared to the HK case [9, 11] where the analogous constraint is merely quadratic. In the general case (4.28), (4.29) the situation is even worse.

In view of these difficulties, it proves more fruitful to take as independent coordinates just the components of the triplet $g^{(ri)}$, 

$$
g^{12} = g^{21} \equiv iah, \quad \overline{h} = h, \quad g^{11} \equiv g, \quad g^{22} = \overline{g},
$$

and one angular variable from $F_A^{ai}$. Then the above 6 constraints and one residual gauge invariance (the $\epsilon(x)$ one) allow us to eliminate the remaining 7 components of $F_A^{ai}$ in terms of 4 independent coordinates thus defined. Following the same strategy as in the previous subsection, we shall first explain how to solve eqs. (4.22) and (4.23) in this way and then indicate the modifications giving rise to the solution of the general two-parameter set of constraints (4.28), (4.29). We relabel the components of $F_A^{ai}$ as follows

$$
\begin{cases}
F_{A=1}^{a1} i=2 = \frac{1}{2} (\mathcal{F} + \mathcal{K}), & F_{A=1}^{a1} i=1 = \frac{1}{2} (\mathcal{P} + \mathcal{V}), \\
F_{A=2}^{a2} i=2 = \frac{1}{2t} (\mathcal{F} - \mathcal{K}), & F_{A=2}^{a2} i=1 = \frac{1}{2t} (\mathcal{P} - \mathcal{V}), \\
F_{A=2}^{a1} i=1 = -F_{A=1}^{a1} i=2, & F_{A=2}^{a2} i=2 = F_{A=1}^{a1} i=1,
\end{cases}
$$
and substitute this into (4.22), (4.23). After some simple algebra, the constraints can be equivalently rewritten in the following form

(a) \( P \bar{F} = -\frac{i}{2a} A_+ \), (and c.c.) ;
(b) \( \mathcal{V} \bar{K} = -\frac{i}{2a} A_+ \), (and c.c.) ;
(c) \( F \bar{F} - P \bar{P} = B_+ \); (d) \( \mathcal{V} \bar{V} - \mathcal{K} \bar{K} = B_- \).

(4.30)

Here

\( A_\pm = 1 \pm 2\lambda a^2 c h \), \( B_\pm = c(1 + \lambda a^2 r^2) \pm h A_\pm \), \( r^2 = h^2 + t^2 \), \( g \bar{g} = a^2 t^2 \).

Next, one expresses \( \bar{P} \) and \( \bar{K} \) from (4.30) and substitutes them into (4.31), which gives two quadratic equations for \( F \bar{F} \equiv X \) and \( \mathcal{V} \bar{V} \equiv Y \),

\[
X^2 - X B_+ - \frac{1}{4} t^2 A_+^2 = 0, \quad Y^2 - Y B_- - \frac{1}{4} t^2 A_-^2 = 0.
\]

(4.32)

Solving these equations, selecting the solution which is regular in the limit \( g = \bar{g} = h = 0 \) and properly fixing the phases of \( F, P, V \) and \( K \) in terms of the phase of \( g \) with taking account of the residual \( \varepsilon(x) \) gauge freedom, we find the general solution of (4.22), (4.23) in the following concise form

\[
P = -i M e^{i(\phi + \alpha/\rho - \mu \rho)} , \quad F = R e^{i(\phi - \mu \rho)} , \quad K = i S e^{i(\phi - \alpha/\rho + \mu \rho)} , \quad V = L e^{i(\phi + \mu \rho)} ,
\]

(4.33)

and

\[
g = at e^{i(\alpha/\rho - 8\mu \lambda c)} . \tag{4.34}
\]

The various functions involved are

\[
L = \sqrt{\frac{1}{2}(\sqrt{\Delta_+} + B_-)} , \quad R = \sqrt{\frac{1}{2}(\sqrt{\Delta_+} + B_+)} ,
\]

\[
M = \sqrt{\frac{1}{2}(\sqrt{\Delta_-} - B_+)} , \quad S = \sqrt{\frac{1}{2}(\sqrt{\Delta_-} - B_-)} ,
\]

where

\[
\Delta_\pm = B_\pm^2 + t^2 A_\pm^2 .
\]

The true coordinates are \((\phi, \alpha, h, t)\). An extra angle \(\mu\) parametrizes the residual local \(SO(2)\) transformations which act as shifts of \(\mu\) by the parameter \(\varepsilon(x)\), \(\mu \rightarrow \mu + \varepsilon\). To see this, one must rewrite the \(\varepsilon\)-transformation law of \(F_A^r\) following from that of \(\tilde{F}_A^r\), eq. (4.8) (at \(\beta_0 = \alpha_0 = 0\)),

\[
\delta_\varepsilon F_A^r = \varepsilon \epsilon_{AB} F_B^r + \varepsilon \kappa^2 e^{ik} F_k^r ,
\]

in terms of the newly defined variables and in the \(SU(2)\) frame (4.26)

\[
\delta_\varepsilon F = -i \varepsilon \rho_- F , \quad \delta_\varepsilon V = i \varepsilon \rho_- V , \quad \delta_\varepsilon P = -i \varepsilon \rho_+ P , \quad \delta_\varepsilon K = i \varepsilon \rho_+ K , \quad \delta_\varepsilon h = 0 , \quad \delta_\varepsilon g = -8i \varepsilon \lambda c g . \tag{4.35}
\]

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As a consequence of gauge invariance of (4.16), the final form of the metric should not depend on $\mu$ and we can choose the latter at will. For instance, we can change the precise dependence of phases in (4.33), (4.34) on $\phi$ and $\alpha$. In what follows we shall stick just to the above parametrization. Explicitly keeping $\mu$ at the intermediate steps of calculations is a good self-consistency check: this gauge parameter should fully drop out from the correct final expression for the metric.

Finally, let us indicate the modifications which should be made in the above solution to adapt it to the general set of constraints (4.28), (4.29). It is convenient to represent the latter in the following equivalent form

$$F_{aB}(iA_{ij}F_{kB}) - \beta_0 g^{ij} + (1 - \alpha_0 \beta_0) [e^{ij} - \lambda g^{(ij)} g^{(rj)} c_{(r)}] = 0, \quad (4.36)$$

$$g^{ij} - a_{ab} F_{aB} F_{bB} + \alpha_0 [e^{ij} - \lambda g^{(ij)} g^{(rj)} c_{(r)}] = 0. \quad (4.37)$$

Then, following the same line as in the case of $\beta_0 = \alpha_0 = 0$, one gets the general solution in the form

$$P = -i\bar{M} e^{i(\phi + a/\rho - \mu_\rho)} e^{-i\mu_0 a}, \quad F = \tilde{R} e^{i(\phi - \mu_\rho)} e^{-i\mu_0 a},$$
$$\mathcal{K} = i\tilde{S} e^{i(\phi - a/\rho + \mu_\rho)} e^{-i\mu_0 a}, \quad \mathcal{V} = \tilde{L} e^{i(\phi + \mu_\rho)} e^{-i\mu_0 a}, \quad (4.38)$$

where the functions with “tilde” are related to those defined earlier by the following replacements

$$A_{\pm} \Rightarrow \tilde{A}_{\pm} = (1 \pm a \beta_0) (1 - 2\alpha_0 \lambda a c h) \pm 2\lambda a^2 c h,$$
$$B_{\pm} \Rightarrow \tilde{B}_{\pm} = \left[1 \pm \frac{\alpha_0}{a} (1 \mp a \beta_0)\right] B_{\pm} - a \left[\beta_0 + \frac{\alpha_0}{a^2} (1 \mp a \beta_0)\right] h. \quad (4.39)$$

The appearance of an additional phase factor in (4.34) is due to the fact that in the general case the $\varepsilon$ transformations (4.35) acquire the common extra piece proportional to $\beta_0$:

$$\delta_\varepsilon F \Rightarrow \delta_\varepsilon F - i\varepsilon \beta_0 a F,$$

etc. The QK Taub-NUT and QK EH truncations of the general solution correspond to imposing the following conditions:

**QK Taub-NUT**: $\beta_0 = 0$, $c = 0$, $\alpha_0 c \equiv \tilde{\alpha}_0 \neq 0$, \quad (4.40)

**QK EH**: $\alpha_0 = 0$, $\beta_0 a \equiv \tilde{\beta}_0 \neq 0$, $a \Rightarrow 0$. \quad (4.41)

 Respectively, in these two limits we have

**QK Taub-NUT**:

$$\tilde{A}_{\pm} = (1 - 2\lambda a \tilde{\alpha}_0 h) \equiv \tilde{\tilde{A}}, \quad \tilde{B}_{\pm} = \pm \left[\tilde{h} + \frac{\tilde{\alpha}_0}{a} (1 + \lambda a^2 r^2)\right] \equiv \pm \tilde{\tilde{B}},$$
$$\tilde{\Delta}_+ = \tilde{\Delta}_- = \tilde{\tilde{B}}^2 + t^2 \tilde{\tilde{A}}^2 \Rightarrow \tilde{\tilde{L}} = \tilde{M}, \quad \tilde{\tilde{R}} = \tilde{S}, \quad (4.42)$$

**QK EH**:

$$\tilde{\tilde{A}}_{\pm} = (1 \pm \tilde{\beta}_0), \quad \tilde{\tilde{B}}_{\pm} = c \pm (1 \mp \tilde{\beta}_0) h,$$
$$\tilde{\Delta}_{\pm} = \left[\pm (1 \mp \tilde{\beta}_0)\right]^2 + (1 \mp \tilde{\beta}_0)^2 t^2. \quad (4.43)$$
Note that in the Taub-NUT case we can obviously choose, up to a gauge freedom,
\[ \mathcal{P} = \mathcal{V} , \quad \mathcal{F} = \mathcal{K} \Rightarrow e^{2i\mu} = -ie^{i\alpha} , \]
which, according to the above definition of \( \mathcal{P} , \mathcal{V} , \mathcal{F} , \mathcal{V} \), corresponds just to the truncation \( Q_{A=2}^+ = 0 \) at the level of the general HSS Lagrangian (3.16). Also note that for taking the QK EH limit in the original form of constraints (4.36), (4.37) in the unambiguous way, one should firstly rescale \( g^{ik} \to a g^{(ik)} \). The corresponding limiting QK metrics can be obtained by taking the limits (4.40), (4.41) in the QK metric associated with the general choice of \( \alpha_0 \neq 0 , \beta_0 \neq 0 \).

5 The structure of general metric

5.1 First set of coordinates

To obtain the metric, we substitute the explicit form (4.38) of the coordinates into the distance (4.24) and compute it. The algebraic manipulations to be done in order to cast the resulting expression in a readable form are rather involved, and Mathematica was intensively used while doing this job. To simplify matters, we make the change of coordinates
\[ T = \frac{2t}{1 - a^2 \lambda r^2} , \quad H = \frac{2h}{1 - a^2 \lambda r^2} , \quad \rho = \sqrt{T^2 + H^2} \tag{5.1} \]
and use the notations
\[ \beta = \frac{a \beta_0}{1 - 4c \lambda} , \quad c_\pm = \left( \frac{1}{1 \mp a \beta_0} \pm \frac{\alpha_0}{a} \right) c , \]
\[ \delta_\pm = \frac{4 \Delta_\pm}{(1 - a^2 \lambda r^2)^2} = (1 + 4a^2 \lambda c_\pm^2) T^2 + (H \pm 2c_\pm)^2 . \]
The final result for the metric \( g \) can be presented in terms of 4 functions \( D , A , P , Q \)
\[ 4D^2 g = \frac{P}{A} \left( d\phi + \frac{Q}{4P} d\alpha \right)^2 + A \left( g_0 + \frac{1 + a^2 \lambda \rho^2}{P} T^2 d\alpha^2 \right) \tag{5.2} \]
where
\[ g_0 = \frac{dH^2 + dT^2 + a^2 \lambda (T dH - H dT)^2}{1 + a^2 \lambda \rho^2} \tag{5.3} \]
is the metric on the two-sphere \( (a^2 \lambda < 0) \), on flat space \( (a^2 \lambda = 0) \), or on the hyperbolic plane \( (a^2 \lambda > 0) \).

The various involved functions are as follows
\[ D = \frac{D}{1 - a^2 \lambda r^2} = 1 - \lambda \left( (1 + a \beta_0) \sqrt{\delta_-} + (1 - a \beta_0) \sqrt{\delta_+} \right) , \]
\[ A = \frac{a^2}{4} + \frac{1}{4} \left( (1 + a \beta_0) \frac{1 - 4a^2 \lambda c_-^2}{\sqrt{\delta_-}} + (1 - a \beta_0) \frac{1 - 4a^2 \lambda c_+^2}{\sqrt{\delta_+}} \right) \]
\[ + a^2 \lambda H \left( \frac{(1 + a \beta_0) c_-}{\sqrt{\delta_-}} - \frac{(1 - a \beta_0) c_+}{\sqrt{\delta_+}} \right) - \frac{4 c^2 \lambda}{1 - a^2 \beta_0^2} \frac{1 + a^2 \lambda H^2}{\sqrt{\delta_-} \sqrt{\delta_+}}, \]

\[ P = (1 + a^2 \lambda \rho^2) \left( 1 - 2 c \lambda \frac{H + 2 c_+}{\sqrt{\delta_+}} + 2 c \lambda \frac{H - 2 c_-}{\sqrt{\delta_-}} \right)^2 \]

\[ + 4 c^2 \lambda^2 T^2 \left(-a \alpha_0 - \frac{1 + 2 a^2 \lambda c_- H}{\sqrt{\delta_-}} + \frac{1 - 2 a^2 \lambda c_+ H}{\sqrt{\delta_+}} \right)^2, \]

\[ Q = - \left( 1 + a^2 \lambda \rho^2 \right) \left( 2 \beta (1 + 2 c \lambda) + (1 + \beta (1 + 4 c \lambda)) \frac{H - 2 c_-}{\sqrt{\delta_-}} \right. \]

\[ + (1 - \beta (1 + 4 c \lambda)) \frac{H + 2 c_+}{\sqrt{\delta_+}} - 4 c \beta \lambda \frac{(H - 2 c_-) (H + 2 c_+)}{\sqrt{\delta_-} \sqrt{\delta_+}} \]

\[ - 2 c a \lambda T^2 \left(a^2 \alpha_0 - 2 (1 - 2 c a \alpha_0^2) \beta a \lambda \right. \]

\[ + (a + \alpha_0 + \alpha_0 \beta (1 + 4 c \lambda)) \frac{1 + 2 c_- a^2 \lambda H}{\sqrt{\delta_-}} \]

\[ - (a - \alpha_0 + \alpha_0 \beta (1 + 4 c \lambda)) \frac{1 - 2 c_+ a^2 \lambda H}{\sqrt{\delta_+}} \]

\[ - \frac{2 \beta (1 + 2 c_- a^2 \lambda H) (1 - 2 c_+ a^2 \lambda H)}{a} \left( \sqrt{\delta_-} \sqrt{\delta_+} \right). \quad (5.4) \]

The isometry group \( U(1) \times U(1) \) acts by translations on \( \phi \) and \( \alpha \).

### 5.2 Second set of coordinates

In order to verify that \( g \) is self-dual Einstein (see section 5.4), it is more convenient to use coordinates \( s \) and \( x \) defined by

\[ T = s \sqrt{1 - x^2}, \quad H = s x. \quad (5.5) \]

We then get for the metric the expression

\[ 4D^2 g = \frac{P}{A} \left( d\phi + \frac{Q}{4 P} d\alpha \right)^2 \]

\[ + A \left( \frac{ds^2}{1 + a^2 \lambda s^2} + \frac{s^2 dx^2}{1 - x^2} + \frac{s^2 (1 + a^2 \lambda s^2) (1 - x^2)}{P} d\alpha^2 \right). \quad (5.6) \]

The functions \( A, P, Q \) and \( D \) are still the same as in (5.4), up to the substitution (5.5), and the functions \( \delta_\pm \) can be written as

\[ \delta_\pm = \frac{1}{a^2 \lambda} \left[ (1 + 4 a^2 \lambda c_\pm^2) (1 + a^2 \lambda s^2) - (1 \mp 2 a^2 \lambda c_\pm s x)^2 \right]. \]
5.3 Third set of coordinates ($\alpha_0 = \beta_0 = 0$ case)

In the limit $\alpha_0 \to 0$ and $\beta_0 \to 0$, the metric $g$ reduces to the quaternionic extension of the double Taub-NUT metric given in [35]. For this particular case one can get rid of the square roots by switching to the spheroidal coordinates $(u, \theta)$,

$$T = \frac{\sqrt{u^2 - 4c^2}}{\sqrt{1 + 4a^2\lambda c^2}} \sin \theta, \quad H = u \cos \theta. \quad (5.7)$$

In these coordinates:

$$\sqrt{\delta_{\pm}} = u \pm 2c \cos \theta.$$

It is convenient to scale the angles $\phi$ and $\alpha$ according to

$$\hat{\phi} = \frac{\phi}{1 + 4a^2\lambda c^2}, \quad \hat{\alpha} = \frac{\alpha}{1 + 4a^2\lambda c^2}.$$

Then the metric at $\alpha_0 = \beta_0 = 0$ becomes

$$4D^2 g = (1 + a^2\lambda u^2) \left( \frac{\hat{P}}{A} \left( d\hat{\phi} + \frac{\hat{Q}}{4\hat{P}} d\hat{\alpha} \right)^2 + \hat{A} \left( \hat{g}_0 + \frac{(u^2 - 4c^2)(1 + 4a^2\lambda c^2 \cos^2 \theta)}{\hat{P}} \sin^2 \theta d\hat{\alpha}^2 \right) \right), \quad (5.8)$$

where

$$\hat{g}_0 = (u^2 - 4c^2 \cos^2 \theta) g_0 = \frac{du^2}{(u^2 - 4c^2)(1 + a^2\lambda u^2)} + \frac{d\theta^2}{1 + 4a^2\lambda c^2 \cos^2 \theta}$$

and

$$4 \hat{A} = 4(u^2 - 4c^2 \cos^2 \theta) A = (2 + a^2u)(u - 8c^2\lambda) - 4a^2c^2D^2 \cos^2 \theta,$$

$$\hat{P} = \frac{(1 + 4a^2\lambda c^2)(u^2 - 4c^2 \cos^2 \theta)}{1 + a^2\lambda u^2} P = 4c^2 \sin^2 \theta (1 + 4a^2\lambda c^2 \cos^2 \theta)D^2 + (u^2 - 4c^2)(1 + 4a^2\lambda c^2 \cos^2 \theta - 16\lambda c^2 \sin^2 \theta),$$

$$\hat{Q} = \frac{(1 + 4a^2\lambda c^2)(u^2 - 4c^2 \cos^2 \theta)}{1 + a^2\lambda u^2} Q = -2(u^2 - 4c^2)(1 + 4a^2\lambda c^2) \cos \theta,$$

$$D = 1 - 2\lambda u. \quad (5.9)$$

5.4 Einstein and self-dual Weyl properties of the metric

A four-dimensional QK metric is nothing but an Einstein metric with self-dual Weyl tensor. This property should be inherent to the metric $g$ given by (5.2), since we started
from the generic HSS action for QK sigma models. However, checking these properties explicitly is a good test of the correctness of our computations.

We first consider the particular case $\alpha_0 = \beta_0 = 0$ because the use of the spheroidal-like coordinates (5.7) greatly simplifies the metric as can be seen from relations (5.8) and (5.9). Despite these simplifications, intensive use of Mathematica was needed to compute the spin connection, the anti-self-dual curvature $R_i^-$ and to check the crucial relation (see the Appendix for the notation):

$$R_i^- = -16\lambda \Xi_i^- .$$

It simultaneously establishes that the metric is indeed self-dual Einstein, with

$$Ric (g) = \Lambda g , \quad \frac{\Lambda}{3} = -16\lambda , \quad W_i^- = 0 .$$

For non-vanishing $\alpha_0$ or $\beta_0$, such a check is no longer feasible because of the strong increase in complexity of various functions appearing in the metric. Moreover, in this case we failed to find any proper generalization of the spheroidal-like coordinates (5.7) which would allow us to get rid of the square roots $\sqrt{\delta^+}$ and $\sqrt{\delta^-}$.

In order to by-pass these difficulties we have used an approach due to Przanowski [36] and Tod [34], which reduces the verification of the self-dual Einstein property to simpler checks. We shall begin with a description of their construction.

One starts from an Einstein metric $g$ (more precisely, $Ric (g) = \Lambda g$). Furthermore it will be supposed that this metric has (at least) one Killing vector with the associated 1-form $K = K_\mu dx^\mu$. Differentiating $K$ gives

$$dK = dK_i^+ \Xi_i^+ + dK_i^- \Xi_i^- , \quad \Xi_i^\pm = e_0 \wedge e_i \pm \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k ,$$

for some vierbein of the metric $g$. We can extract, from $dK$, an integrable complex structure $I$ and a coordinate $w$ according to

$$I = \frac{dK_i^-}{\sqrt{\sum_i (dK_i^-)^2}} \Xi_i^- , \quad w = -\frac{\Lambda}{3 \sqrt{\sum_i (dK_i^-)^2}} . \quad (5.10)$$

Using these elements one can formulate

**Proposition 1 ([36],[34])** There exist real coordinates $w$, $\nu$ and $\mu$ such that any Einstein metric $g$ with self-dual Weyl tensor and a Killing vector $\partial_\phi$ can be written as

$$g = \frac{1}{w^2} \left[ \frac{1}{W} (d\phi + \Theta)^2 + W (e^\nu (dv^2 + d\mu^2) + dw^2) \right] . \quad (5.11)$$

This metric will be self-dual Einstein iff

$$\begin{align*} 
(a) & \quad -2 \frac{\Lambda}{3} W = 2 - w \partial_w v , \\
(b) & \quad (\partial^2_\nu + \partial^2_\mu) v + \partial^2_w (e^\nu) = 0 , \\
(c) & \quad -d\Theta = \partial_\nu W d\mu \wedge dw + \partial_\mu W dw \wedge dv + \partial_w (W e^\nu) dv \wedge d\mu . 
\end{align*} \quad (5.12)$$
The following remarks are in order:

1. The relation (5.12b) is the celebrated continuous Toda equation.

2. Except for this Toda equation, the checks of the self-dual Einstein property are reduced to solving first order partial differential equations.

3. Relation (5.11) shows that any self-dual Einstein metric with at least one Killing is conformal to a subclass of Kähler scalar-flat metrics (see section 6.1 for the proof).

Let us now use this approach to analyze our metric (5.6) in the $(s, x)$ coordinates and to check whether it obeys the conditions (5.12).

We take for vierbein

\[
e_0 = \frac{1}{\sqrt{W}} \left( d\phi + \Theta \right) , \quad e_1 = \frac{\sqrt{A}}{2D} \frac{ds}{\sqrt{1 + a^2 \lambda s^2}} , \quad W = \frac{4D^2A}{P} ,
\]

\[
e_2 = \frac{\sqrt{A}}{2D} \frac{dx}{\sqrt{1 - s^2}} , \quad e_3 = \frac{W}{4D^2} s \sqrt{1 + a^2 \lambda s^2} \sqrt{1 - x^2} \, d\alpha ,
\]

and consider the Killing $\partial_{\phi}$, with the 1-form

\[
K = \frac{1}{W} (d\phi + \Theta) = \frac{1}{\sqrt{W}} e_0 .
\]

The computation of $\sum_i (dK_i^-)^2$ eventually leads to the identification

\[
w = -\frac{\Lambda}{3} \frac{D}{4 \lambda \sqrt{\delta(\hat{c})}} , \quad (5.13)
\]

where

\[
\delta(\hat{c}) = \frac{1}{a^2 \lambda} \left[ (1 + 4a^2 \lambda \hat{c}\hat{c}) (1 + a^2 \lambda s^2) - (1 - 2a^2 \lambda \hat{c} s) \right] ,
\]

\[
2 \hat{c} = (1 - a \beta_0) c_+ - (1 + a \beta_0) c_- = 2c_0 \frac{\alpha_0}{a} . \quad (5.14)
\]

Then, comparing the metric $g$ in the form (5.6) with (5.11), we express the quantities $W$, $\mu$ and $e^v$ entering (5.11) in terms of ours

\[
W = \frac{W}{w^2} , \quad \mu = \alpha , \quad e^v = \frac{s^2 (1 + a^2 \lambda s^2)}{16 D^4} (1 - x^2) w^4 . \quad (5.15)
\]

Simultaneously, we obtain the expressions for the partial derivatives of $\nu$

\[
\partial_x \nu = -\frac{4D^2}{(1 - x^2) w^2} \partial_s w , \quad \partial_s \nu = \frac{4D^2 \partial_x w}{s^2 (1 + a^2 \lambda s^2)} w^2 . \quad (5.16)
\]

Two expressions for the mixed derivative $\partial_s \partial_x \nu$ coincide as a consequence of the relation:

\[
s^2 (1 + a^2 \lambda s^2) \partial_s^2 D + (1 - x^2) \partial_x^2 D = 0 . \quad (5.17)
\]

Checking the relation (5.12b) suggests the identification

\[
\frac{\Lambda}{3} = -16 \lambda . \quad (5.18)
\]

Then the remaining equations (b) and (c) in (5.12) have been explicitly checked using Mathematica, and shown to be valid. This proves that our general metric (5.6) is self-dual Einstein.
5.5 Limiting cases

The hyper-Kähler limit

Using the coordinates $H$ and $T$ (defined in 5.1), in the limit $\lambda \to 0$, the metric (5.2) can be written as the multicentre structure

$$4 \, g(\lambda \to 0) = \frac{1}{V} \left( d\Phi + A \right)^2 + V \, g_0(\lambda \to 0) ,$$

with the flat 3-metric and the angle $\Phi$ defined by

$$g_0(\lambda \to 0) = dH^2 + dT^2 + T^2 d\alpha^2 , \quad \Phi = \phi - \frac{a \beta_0}{2} \alpha .$$

The potential $V$ and the connection $A$ are, respectively,

$$V = \frac{1}{4} \left( a^2 + \frac{1 + a \beta_0}{\sqrt{\delta_-}} + \frac{1 - a \beta_0}{\sqrt{\delta_+}} \right) , \quad (5.19)$$

$$A = - \frac{1}{4} \left( (1 + a \beta_0) \frac{H - 2 c_-}{\sqrt{\delta_-}} + (1 - a \beta_0) \frac{H + 2 c_+}{\sqrt{\delta_+}} \right) d\alpha , \quad (5.20)$$

with

$$\delta_\pm = (H \pm 2c_\pm)^2 + T^2 , \quad c_\pm = \frac{c}{1 \mp a \beta_0} .$$

Since $\alpha_0$ is an irrelevant parameter in the limit $\lambda \to 0$ (it can be removed from the metric by a shift of $H$), we put it equal to zero from the very beginning.

The potential shows two centres at $T = 0 , \ H = \mp 2 c_\pm$ with different masses $\frac{1 \mp a \beta_0}{4}$ and $V(\infty) = \frac{a^2}{4}$. An easy computation gives the fundamental multicentre relation

$$dV = - \star dA .$$

For $a \neq 0, \beta_0 = 0$, we have the double Taub-NUT metric; for $a \neq 0, c = 0$ and $a \neq 0, a \beta_0 = \pm 1$, $c_\pm$ finite, we have the Taub-NUT metric; for $a = 0$, we have the Eguchi-Hanson metric.

The quaternionic Taub-NUT limit

In order to show that in the limit $c \to 0$ we recover the quaternionic Taub-NUT metric, we switch to new coordinates $(\hat{s}, \hat{\theta})$ defined by

$$H = \frac{2 \hat{s} \cos \hat{\theta}}{a \, 1 - \lambda \hat{s}^2} , \quad T = \frac{2 \hat{s} \sin \hat{\theta}}{a \, 1 - \lambda \hat{s}^2} .$$

The metric $g$ coincides, up to a constant factor $\frac{1}{2 a}$, with the metric given by relation (5.4) in [22]:

$$2 \, a \, g(c \to 0) = \frac{1}{2} \left( \frac{\dot{B} \dot{C}}{\hat{s} C^2} \, ds^2 + \frac{\dot{s} \dot{B}}{C^2} (\sigma_1^2 + \sigma_2^2) + \frac{\dot{s} \dot{A}^2}{B C^2} \sigma_3^2 \right) ,$$

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where
\[
\hat{A} = 1 - \hat{R} \hat{\lambda}^2 \hat{s}^2, \quad \hat{B} = 1 + \hat{\lambda}^2 \hat{s} (4 + \hat{R} \hat{s}), \quad \hat{C} = 1 + \hat{R} \hat{s} + \hat{R} \hat{\lambda}^2 \hat{s}^2,
\]
and
\[
\begin{align*}
\sigma_1^2 + \sigma_2^2 &= d\hat{\theta}^2 + \sin^2 \hat{\theta} \, d\alpha^2, \\
\sigma_3 &= (-2 d\phi + a \beta_0 d\alpha) + \cos \hat{\theta} \, d\alpha, \\
\hat{R} &= -\frac{4}{a}, \quad \hat{\lambda}^2 = \frac{a}{4}.
\end{align*}
\]
(5.21)

Various limits of the quaternionic Taub-NUT metric can be found in [22]. Let us just remark here that in the limit \( \hat{R} \to 0 \) we once again recover the standard Taub-NUT metric.

**The quaternionic Eguchi-Hanson limit**

In the limit \( a \to 0 \) with \( a \beta_0 = \tilde{\beta}_0 \neq 0 \), it is more convenient to study the metric in coordinates in which the square roots disappear. Thus, we define the coordinates \( \hat{s} \) and \( \hat{\theta} \) by
\[
T = \frac{2}{1 - \beta_0^2} \sqrt{\hat{s}^2 - c^2 \sin \hat{\theta}} , \quad H = \frac{2}{1 - \beta_0^2} \hat{s} \cos \hat{\theta} + c_+ - c_+ ,
\]
so that
\[
\sqrt{\delta_{\pm}} = \frac{2}{1 - \beta_0^2} (\hat{s} \pm c \cos \hat{\theta}) .
\]
The metric can now be expressed as
\[
4 (1 - \beta_0^2) \hat{C}^2 g(a \to 0, \beta_0) = \frac{\hat{s}^2 - c^2}{\hat{s} B} \mathcal{G}^2 + \hat{s} B \left( \frac{d\hat{s}^2}{\hat{s}^2 - c^2} + d\hat{\theta}^2 + \sin^2 \hat{\theta} \mathcal{H}^2 \right) ,
\]
with
\[
\begin{align*}
\hat{C} &= 1 - \frac{\kappa^2}{1 - \beta_0^2} \left( \hat{s} - c \tilde{\beta}_0 \cos \hat{\theta} \right) , \quad \mathcal{G} = - \left( 1 + \beta \, \cos \hat{\theta} \right) d\alpha + 2 \cos \hat{\theta} \, d\phi , \\
\hat{s} \hat{B} &= \hat{s} - \kappa^2 c^2 + c \tilde{\beta}_0 \cos \hat{\theta} , \quad \mathcal{H} = \frac{1}{\hat{s} B} \left[ - (\hat{s} - c) \, \beta \, d\alpha + 2 (\hat{s} - \kappa^2 c^2) \, d\phi \right]
\end{align*}
\]
where
\[
\kappa^2 = 4 \lambda , \quad \beta = \frac{\tilde{\beta}_0}{1 - 4 c \lambda} .
\]
One can see that in the limit \( a \to 0 \), the parameter \( \alpha_0 \) fully drops out from the metric.

If we now take the limit \( \beta_0 = a \beta_0 \to 0 \), we reproduce the quaternionic Eguchi-Hanson metric derived in [23] (see equation (4.7) of this reference) :
\[
4 \hat{C}^2 g(a \to 0) = \frac{\hat{s}^2 - c^2}{\hat{s} B} \hat{\sigma}_3^2 + \hat{s} \hat{B} \left( \frac{d\hat{s}^2}{\hat{s}^2 - c^2} + \hat{\sigma}_1^2 + \hat{\sigma}_2^2 \right) ,
\]
with
\[
\begin{align*}
\hat{C} &= 1 - \kappa^2 \hat{s} , \quad \hat{\sigma}_3 = (-d\alpha) + \cos \hat{\theta} \, (2 \, d\phi) , \\
\hat{s} \hat{B} &= \hat{s} - \kappa^2 c^2 , \quad \hat{\sigma}_1^2 + \hat{\sigma}_2^2 = d\hat{\theta} + \sin^2 \hat{\theta} \, (2 \, d\phi)^2 .
\end{align*}
\]
In conclusion, let us point out that, whereas the parameters $a$, $c$ and $\beta_0$ have a counterpart in the HK limit, this is not the case for the parameter $\alpha_0$. This distinguished parameter is specific just for the QK metrics.

6 Connection with the literature

Metrics with self-dual Weyl tensor may appear as:

1. Kähler scalar-flat metrics,
2. Self-dual Einstein metrics (considered in this work),
3. Metrics in the system of coupled Einstein-Maxwell fields.

In order to exhibit the relationships between these classes and to find out how our metrics correlate with them, let us begin with the description, due to LeBrun, of the Kähler scalar-flat metrics with one Killing vector.

6.1 Kähler scalar-flat metrics in LeBrun setting

These metrics, with self-dual Weyl tensor, have received attention in [11]. There, it was proved that any such metric, with at least one Killing vector $K = \partial_\nu$, can be written as

$$g = \frac{1}{W}(dt + \tilde{\Theta})^2 + W[ dw^2 + e^\nu (d\nu^2 + d\mu^2)] = \sum_{A=0}^{3} e_A^2 , \quad (6.1)$$

where the functions $v$ and $W$ must be solutions of the following equations

$$(\partial^2_\nu + \partial^2_\mu) v + \partial_\mu (e^\nu) = 0 , \quad (\partial^2_\nu + \partial^2_\mu) W + \partial_\nu (W e^\nu) = 0 . \quad (6.2)$$

The connection one-form $\tilde{\Theta}$ is then obtained from

$$d\tilde{\Theta} = \partial_\nu (W) d\mu \wedge dw + \partial_\mu (W) dw \wedge d\nu + \partial_\nu (W e^\nu) d\nu \wedge d\mu . \quad (6.3)$$

The vierbein, defined in relation (5.1), is taken to be

$$e_0 = \frac{dt + \tilde{\Theta}}{\sqrt{W}} , \quad e_1 = \sqrt{We^\nu} d\nu , \quad e_2 = \sqrt{We^\nu} d\mu , \quad e_3 = \sqrt{W} dw .$$

In terms of the self-dual two-forms $\Xi^+ i = e_0 \wedge e_i \pm \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k$ the Kähler form is anti-self-dual,

$$\Omega = dw \wedge (dt + \tilde{\Theta}) + W e^\nu d\nu \wedge d\mu = -\Xi^- 3 , \quad (6.4)$$

while the Ricci form is self-dual,

$$2\hat{\rho} = \frac{1}{\sqrt{e^\nu}} \partial_\nu \left( \frac{\partial_\mu v}{W} \right) \cdot \Xi^+_1 + \frac{1}{\sqrt{e^\mu}} \partial_\mu \left( \frac{\partial_\nu v}{W} \right) \cdot \Xi^+_2 + \partial_\nu \left( \frac{\partial_\mu v}{W} \right) \cdot \Xi^+_3 . \quad (6.5)$$
Now, comparing (5.11) and (6.1), we observe that any self-dual Einstein metric with at least one Killing, in particular the metric (5.2), is conformally related to a subclass of Kähler scalar-flat metrics, with the identifications:

\[ \tilde{\Theta} = -\Theta , \quad dt = -d\phi , \quad d\mu = d\alpha , \quad g = w^2 g . \]

In [41], a large class of explicit solutions of (5.2) was obtained. Taking

\[ q = \sqrt{2w} , \quad e^v = q^2 , \quad V = q^2 W , \quad \gamma = \frac{dv^2 + d\mu^2 + dq^2}{q^2} , \]

where \( \gamma \) is the hyperbolic 3-space, these metrics have the form

\[ q^2 \left[ \frac{1}{V} (dt + \Theta)^2 + V \gamma \right] , \tag{6.6} \]

where \( V \) is some real harmonic function on \( \gamma \).

LeBrun obtained the potential \( V \) as a sum of monopoles in this hyperbolic space. In the limit where the hyperbolic space becomes flat, one recovers the multicentre metrics. However, the possibility that these metrics could be conformally Einstein has been ruled out by Pedersen and Tod in [42]. Therefore the metrics (6.6) bear no relation to our metric (5.2).

### 6.2 Flaherty’s equivalence

Let us now examine Flaherty’s equivalence relating Kähler scalar-flat metrics and self-dual metrics solving the coupled Einstein-Maxwell field equations.

In [40] Flaherty has proved:

**Proposition 2** The following two classes of metrics are equivalent:

1. Any Kähler scalar-flat metric.
2. Any metric which is a solution of the coupled Einstein-Maxwell equations

\[
\begin{align*}
\text{Ric}_{\mu\nu} &= \frac{1}{2} \left( F_{\mu\rho} g^{\rho\sigma} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \\
\text{d}F^- &= 0 , \\
\text{d}F^+ &= 0 ,
\end{align*}
\tag{6.7}
\]

with self-dual Weyl tensor \( (W^- = 0) \).

In this equivalence the self-dual parts of the Maxwell field strength are given by

\[ F^- \propto \Omega , \quad F^+ \propto \hat{\rho} , \]

where \( \Omega \) denotes the Kähler form and \( \hat{\rho} \) the Ricci form of the Kähler metric.

In the euclidean case, this equivalence can be easily checked for metrics with at least one Killing vector, using the LeBrun framework. One can check eqs. (6.7) and find the self-dual parts of the field strength two-forms:

\[ F^- = -\frac{m}{2} \Omega , \quad F^+ = \frac{2}{m} \hat{\rho} , \tag{6.8} \]
where \( m \) is an arbitrary real parameter.

This equivalence and the property that any self-dual Einstein metric with one Killing is conformal to some Kähler scalar-flat metric suggest that the Weyl-self-dual metrics which solve the Einstein-Maxwell system may hide, up to some conformal factor, a self-dual Einstein metric. Let us now examine two known classes of the metrics giving solution of the Einstein-Maxwell system (in general, they are not Weyl-self-dual) in order to see whether the metric \((5.2)\) is conformally related to any of them. We shall find that the answer is negative in both cases. This means that \((5.2)\) determines a new explicit solution of the Einstein-Maxwell system, with the conformal factor \( w \) given in \((5.13)\).

6.3 The metrics of Perjès-Israel-Wilson

These metrics are solutions of the Einstein-Maxwell field equations. They were derived independently, for the minkowskian signature, by Perjès [37] and Israel and Wilson [38]. Their continuation to the euclidean signature was given by Yuille [48] and Whitt [49] who discussed their global properties and their possible applications in the path integral approach to quantum gravity.

These metrics have at least one Killing vector \( \partial_t \). Their local form is given by

\[
g = \frac{1}{V} (dt + A)^2 + V \gamma_0 , \quad V = U \tilde{U} , \quad \gamma_0 = d\vec{x} \cdot d\vec{x} .
\]

(6.9)

The real functions \( U \) and \( \tilde{U} \) must be harmonic

\[
\Delta U = \Delta \tilde{U} = 0 ,
\]

(6.10)

and the connection one-form \( A \) is constrained by

\[
\star_{\gamma_0} dA = \tilde{U} dU - U d\tilde{U} .
\]

(6.11)

The star and laplacian are taken with respect to the three dimensional flat space with cartesian coordinates \( \vec{x} \). Clearly, when \( U \) or \( \tilde{U} \) are constant we come back to the multicentre metrics.

In order to check the previous assertions, let us define the vierbein \( e_A \) by

\[
e_0 = \frac{1}{\sqrt{V}} (dt + A) , \quad e_i = \sqrt{V} dx_i , \quad i = 1, 2, 3 .
\]

It is an easy task to compute the matrices \( A, B \) and \( C \) giving the curvature (see the Appendix for the definitions and notation). One finds, upon using the relations \((6.10)\), \((6.11)\), the simple expressions

\[
A_{ij} = \frac{1}{V} \left[ \frac{\partial^2 U}{U} - 3 \frac{\partial_i U \partial_j U}{U^2} + \delta_{ij} \frac{(\partial_i U)^2}{U^2} \right] ,
\]

(6.12)

\[
B_{ij} = -\frac{1}{V^2} \partial_i U \partial_j \tilde{U} ,
\]

\[
C_{ij} = \frac{1}{V} \left[ \frac{\partial^2 \tilde{U}}{\tilde{U}} - 3 \frac{\partial_i \tilde{U} \partial_j \tilde{U}}{\tilde{U}^2} + \delta_{ij} \frac{(\partial_i \tilde{U})^2}{\tilde{U}^2} \right] ,
\]

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where the derivatives are taken with respect to the cartesian coordinates $\vec{x}$. The scalar curvature $R = 4(TrA)$ vanishes as it should.

The first equation in (6.7) gives for the field strength

$$ F = F^+ + F^- = \partial_i \left[ U^{-1} \right] \cdot \vec{\Xi}_i - \partial_i \left[ \tilde{U}^{-1} \right] \cdot \vec{\Xi}_i^+ . $$

Using (6.10), (6.11) one can check that these field strengths indeed obey the Maxwell equations:

$$ dF^+ = dF^- = 0 . $$

Let us prove the following:

**Proposition 3** The Perjès-Israel-Wilson metrics are self-dual Weyl only in the two cases:

1. When $\tilde{U}$ is a constant: they are homothetic to the multicentre metrics.
2. When $\tilde{U} = m/|\vec{x} - \vec{x}_0|$: they are conformal to the multicentre metrics.

**Proof:** Let us impose, for instance, the condition that the Weyl tensor is self-dual (i.e., $W^- = 0$). Using (6.12) and (6.10), the corresponding constraints can be written as

$$ \partial_i \partial_j \left( \frac{1}{U^2} \right) - \frac{1}{3} \delta_{ij} \Delta \left( \frac{1}{U^2} \right) = 0 . \quad (6.13) $$

Acting on the left hand side by $\partial_k$ gives

$$ \partial_j \Delta \left( \frac{1}{U^2} \right) = 0 \quad \implies \quad \Delta \left( \frac{1}{U^2} \right) = \text{const} \equiv 6B . $$

Then one can integrate relation (6.13) to

$$ \left( \frac{1}{U^2} \right) = A + \vec{f} \cdot \vec{x} + Br^2 , \quad r^2 = \vec{x} \cdot \vec{x} , $$

where $A$ and $\vec{f}$ are integration constants. The requirement that $\tilde{U}$ is harmonic (eq. (6.10)) amounts to the relation

$$ \vec{f} \cdot \vec{f} = 4AB . $$

If $B$ vanishes, the harmonic function $\tilde{U}$ is evidently reduced to a constant which can be scaled to 1. Then the relations (6.10), (6.11) imply that the metric is homothetic to some multicentre one.

If $B$ does not vanish, we can write $\tilde{U} = m/|\vec{x} - \vec{x}_0|$ which can be simplified to $1/r$ by rescaling and translation of $\vec{x}$. The metric (6.9) becomes

$$ g = r^2 \hat{g} , $$

with

$$ \hat{g} = \left[ \frac{1}{V} (d\tau + A)^2 + \hat{V} \hat{\gamma} \right] , \quad \hat{\gamma} = \frac{1}{r^4} \hat{\gamma}_0 , \quad \hat{V} = rU . $$
Using spherical coordinates we have

\[ \hat{\gamma}_0 = dr^2 + r^2 \, d\Omega^2 \quad \implies \quad \hat{\gamma} = d\rho^2 + \rho^2 \, d\Omega^2 , \quad \rho = 1/r , \]

thus establishing that \( \hat{\gamma} \) is flat. Then relation (6.11) becomes

\[ -\hat{\gamma} dA = d\hat{V} , \]

showing that \( \hat{\gamma} \) is some multicentre. This completes the proof.

Proposition 3 tells us that the metrics of Perjé-Ireland-Wilson, when they have self-dual Weyl tensor, are never conformal to Einstein metrics (with non-vanishing cosmological constant). This implies that our metric (5.2) can never be transformed to the Perjé-Ireland-Wilson form.

6.4 The Plebanski-Demianski metric

In [39] Plebanski and Demianski have derived a minkowskian solution of the coupled Einstein-Maxwell field equations. Its euclidean version, obtained by complex changes of coordinates and parameters, can be written in the form \( g_{PD} = \sum_{\alpha=0}^{3} e_{\alpha}^2 \), with the vierbein

\[
\begin{align*}
\frac{1}{1+pq} \sqrt{\frac{p^2 - q^2}{X(p)}} dp , & \quad e_1 = \frac{1}{1+pq} \sqrt{\frac{p^2 - q^2}{Y(q)}} dq , \\
\frac{1}{1+pq} \sqrt{\frac{X(p)}{p^2 - q^2}} (d\tau + q^2 d\sigma) , & \quad e_2 = \frac{1}{1+pq} \sqrt{\frac{Y(q)}{p^2 - q^2}} (d\tau + p^2 d\sigma) .
\end{align*}
\]

where

\[
\begin{align*}
X(p) &= \left( g_0^2 - \gamma + \frac{\lambda}{6} \right) - 2l p + \epsilon p^2 - 2m p^3 - \left( e^2 + \gamma + \frac{\lambda}{6} \right) p^4 , \\
Y(q) &= \left( e^2 + \gamma - \frac{\lambda}{6} \right) - 2m q - \epsilon q^2 - 2l q^3 - \left( g_0^2 - \gamma - \frac{\lambda}{6} \right) q^4 .
\end{align*}
\]

It displays 6 real parameters besides the cosmological constant \( \lambda \) and possesses \( U(1) \times U(1) \) isometry realized by shifts of \( \tau \) and \( \sigma \).

The meaning of the parameters \( e, g_0, l \) and \( m \) follows from:

**Proposition 4** The Plebanski-Demianski metrics are

- **Einstein** for \( e = g_0 = 0 \).
- **Einstein with self-dual Weyl tensor** \( (W^- = 0) \) for \( e = g_0 = 0 \) and \( l = m \).
- **Einstein with anti-self-dual Weyl tensor** \( (W^+ = 0) \) for \( e = g_0 = 0 \) and \( l = -m \).
**Proof**: The proposition follows from the computation of the curvature matrices $A, B$ and $C$ defined in Appendix.

We are going to show that our metric (5.2) lies outside the above ansatz. To this end, we shall work with an anti-self-dual Weyl tensor ($W^- = 0$) and analyze the $\lambda \to 0$ limit of $g_{PD}$. We switch to the tri-holomorphic Killing vector $\partial_\phi$ by making the change of coordinates $d\phi = d\tau$, $d\alpha = d\sigma + d\tau$.

It leads to the limiting metric

$$g_{PD}(\lambda \to 0) = \frac{1}{V}(d\phi + A)^2 + V \gamma_0,$$

with the potential

$$V = \frac{(1 + pq)^2(p^2 - q^2)}{D}, \quad D = (1 - q^2)^2X(p) + (1 - p^2)^2Y(q),$$

the gauge field one-form

$$A = \frac{q^2(1 - q^2)X(p) + p^2(1 - p^2)Y(q)}{D} d\alpha,$$

and the three dimensional metric

$$\gamma_0 = \frac{D}{(1 + pq)^4} \left( \frac{dp^2}{X(p)} + \frac{dq^2}{Y(q)} \right) + \frac{X(p)Y(q)}{(1 + pq)^4} d\alpha^2.$$

One can explicitly check the relation

$$\star_{\gamma_0} dA = \pm dV.$$

To prove that (6.14) is indeed multicentre, we define cartesian coordinates $\vec{x}$ by

$$\begin{cases} x = A \sin \left[ \sqrt{m^2 + \gamma(2\gamma + \epsilon)} \alpha \right], \\ y = A \cos \left[ \sqrt{m^2 + \gamma(2\gamma + \epsilon)} \alpha \right], \\ z = B, \end{cases}$$

with

$$A = \frac{1}{(1 + pq)^2} \sqrt{\frac{X(p)Y(q)}{m^2 + \gamma(2\gamma + \epsilon)}}, \quad B = -\frac{m(p - q)(1 - pq) + \gamma(p^2 + q^2) + \epsilon pq}{\sqrt{m^2 + \gamma(2\gamma + \epsilon)(1 + pq)^2}}.$$

One can check that these coordinates make manifest the flatness of the metric (6.17)

$$\gamma_0 = (dx)^2 + (dy)^2 + (dz)^2.$$

For comparing (6.14) with the HK limit of our metric we need to express the potential $V$ in terms of the coordinates (6.18).
For $m = 0$, as observed in the original paper [39], the metric (6.14) is flat: this is a special case which needs a separate analysis. We have

$$V = \frac{1}{2\sqrt{\gamma(\epsilon - 2\gamma)}} \frac{1}{\sqrt{x^2 + y^2 + Z^2}}, \quad Z = z + \frac{\epsilon}{2\sqrt{\gamma(2\gamma + \epsilon)}},$$

provided that the expressions within square roots are positive. This potential corresponds to a mass at the origin, and is known to yield a flat four-dimensional metric [32].

For $m \neq 0$, we define new parameters by

$$\cosh \phi = \frac{\epsilon - 2\gamma}{4m}, \quad \phi \geq 0, \quad c = \frac{\sqrt{(\epsilon - 2\gamma)^2 - 16m^2}}{4\sqrt{m^2 + \gamma(2\gamma + \epsilon)}},$$

and

$$Z = z + \frac{2\gamma + \epsilon}{4\sqrt{m^2 + \gamma(2\gamma + \epsilon)}}, \quad d_{\pm} = x^2 + y^2 + (Z \pm c)^2.$$

In this notation, the potential (6.15) becomes

$$V = \mu \left( \frac{\eta}{\sqrt{d_-}} + \frac{1/\eta}{\sqrt{d_+}} \right), \quad (6.19)$$

with

$$\eta^2 = \frac{e^{-\phi}}{\sqrt{c^2 + 1 + c}}, \quad \mu = \frac{\sqrt{c}}{4m(\sinh \phi)^{3/2}}.$$

The HK limit of the Plebanski-Demianski metric therefore gives an ALE generalization of the Eguchi-Hanson metric (recovered for $\eta = 1$) with two different masses. It is reduced to the flat metric, up to rescaling, in the limits $\eta \to 0$ and $\eta \to \infty$.

The potential (6.19) is a particular case $a = 0$, $a\beta_0 \neq 0$ of our limiting HK potential (5.19):

$$V = \frac{1}{4} \left( a^2 + \frac{1 + a\beta_0}{\sqrt{\delta_-}} + \frac{1 - a\beta_0}{\sqrt{\delta_+}} \right).$$

The conclusion is that our general metric (5.2) cannot be embedded into the Plebanski-Demianski class of self-dual Einstein metrics because their HK limits are different.

Summarizing the discussion in sections 6.3 and 6.4, we observe that our metric (5.2) cannot be reduced to either known class of metrics solving the Einstein-Maxwell equations. Hence, by Flaherty’s equivalence, it provides (up to conformal factor (5.13)) a new family of explicit solutions of this system. For the minimal case $\alpha_0 = \beta_0 = 0$ this fact was pointed out in [35].

### 6.5 The linearization by Calderbank and Pedersen

Quite recently, while we were typing this article, Calderbank and Pedersen [43] have exhibited a class of self-dual Einstein metrics with two commuting (and hypersurface generating) Killing vectors. To describe their metrics, two main ingredients are needed:
1. A function $F(\rho, \eta)$ which is a solution of the linear differential equation

$$\rho^2(F_{\rho\rho} + F_{\eta\eta}) = \frac{3}{4}F.$$  \hfill (6.20)

It is an eigenfunction of the laplacian in the hyperbolic plane $\mathcal{H}^2$ with metric

$$g_0(\mathcal{H}^2) = \frac{d\rho^2 + d\eta^2}{\rho^2}, \quad \rho > 0.$$  \hfill (6.21)

2. The set of one-forms

$$\alpha = \sqrt{\rho} \, d\alpha, \quad \beta = \frac{d\phi + \eta \, d\alpha}{\sqrt{\rho}}.$$ 

In terms of these, the full metric is

$$g = \frac{F^2 - 4\rho^2(F_{\rho\rho}^2 + F_{\eta\eta}^2)}{4F^2} \cdot g_0(\mathcal{H}^2) + \left[\frac{(F - 2\rho F_{\rho}) \alpha - 2\rho F_{\eta} \beta}{F^2} + \frac{(2\rho F_{\rho} \alpha - (F + 2\rho F_{\rho}) \beta)^2}{F^2(4F^2 - 4\rho^2(F_{\rho}^2 + F_{\eta}^2))}\right].$$  \hfill (6.22)

The main result of [43] is a theorem which states that these metrics with two commuting Killings are self-dual Einstein with non-vanishing scalar curvature and that any such metric has locally the structure given by the expression (6.22).

In order to get a deeper insight into the construction of [43], it is convenient to pass to a function $G$ according to $F = G/\sqrt{\rho}$. The metric $g$ becomes

$$G^2 \, g = \frac{1}{W} (d\phi + \Theta)^2 + W \gamma,$$

with

$$\Theta = \left(\frac{G G_{\eta}}{G_{\rho}^2 + G_{\eta}^2} - \eta\right) d\alpha,$$

$$\gamma = \rho^2 \, d\alpha^2 + (G_{\rho}^2 + G_{\eta}^2) \left( d\rho^2 + d\eta^2 \right).$$  \hfill (6.23)

Following Tod, we can now compute the anti-self-dual part of $dK$, where $K$ is the 1-form associated with the Killing $\partial_\phi$. After some algebra, using (6.20), we obtain

$$K = \frac{1}{G^2 W} (d\phi + \Theta), \quad dK^- = -\frac{1}{G \sqrt{G_{\rho}^2 + G_{\eta}^2}} \left( G_{\rho} \, \Xi_1 + G_{\eta} \, \Xi_2 \right),$$

from which we conclude that in fact $G$ is proportional to Tod’s coordinate $w$, defined in (5.10). Taking $G = w$, relation (6.20) becomes

$$w_{\rho\rho} + w_{\eta\eta} = \frac{1}{\rho} \, w.$$  \hfill (6.24)

Using relations (3.4) and (3.5), and switching from Tod’s coordinates $(w, \nu)$ to the $(\rho, \eta)$ coordinates, we can obtain the Kähler form $\Omega$ and the Ricci form $\hat{\rho}$ in this setting:

$$\Omega = -dw \wedge d\phi + (\eta \, w_\rho - \rho \, w_\eta) \, d\rho \wedge d\alpha + (\rho \, w_\rho + \eta \, w_\eta - w) \, d\eta \wedge d\alpha,$$

$$\hat{\rho} = -d \left[ \frac{1}{w \, W} (d\phi + \Theta) + \frac{1}{w} \left( d\phi - \eta \, d\alpha \right) \right].$$  \hfill (6.25)
The Kähler form $\Omega$ is closed as a consequence of \((6.24)\). One can check that $\Omega$ and $\hat{\rho}$ possess opposite self-dualities. In view of Flaherty’s equivalence, the metrics described by the Calderbank-Pedersen ansatz are conformally related to a subclass of metrics solving the coupled Einstein-Maxwell equations. Then the two-forms \((6.25)\) specify the field strengths of the corresponding Maxwell field \((6.8)\).

We are now in a position to establish the precise connection between our coordinates $s$ and $x$ and the coordinates $\rho$ and $\eta$ in the hyperbolic plane $H^2$. To this end, we have to identify the pieces which are independent of the choice of basis for the Killing vectors, i.e. the pieces involving $\gamma$. One gets the correspondence:

$$
\rho = \frac{4}{\delta(\hat{c})} s \sqrt{1 - x^2} \sqrt{1 + a^2 s^2 \lambda},
$$

$$
\eta = \frac{2}{\hat{c}} \left( \frac{s (s + 2 \hat{c} x)}{\delta(\hat{c})} - 1 \right),
$$

(6.26)

where $\delta(\hat{c})$ was defined in \((5.14)\) and $\hat{c} = \alpha_0 c/a$. Let us notice that the coordinate $\eta$ is defined up to an additive constant that can always be re-absorbed through a redefinition of the Killing $\partial_\phi$. The check of equation \((5.12a)\) gives $\Lambda = 3 \leftrightarrow \lambda = -1/16$. One can then invert relations \((6.26)\):

$$
x = \frac{|2 - c \alpha_0|}{2 \sqrt{(c \alpha_0 \eta + 2 a)^2 + c^2 \alpha_0^2 \rho^2}} \left( \eta - \frac{2 a}{2 - c \alpha_0} \right)^2 + \rho^2
$$

$$
- \frac{1}{2} \sqrt{\left( \frac{c \alpha_0}{a} \eta + 2 \right)^2 + \left( \frac{c \alpha_0}{a} \right)^2 \rho^2}
$$

$$
s = \frac{8}{|2 - c \alpha_0|} \left( \eta - \frac{2 a}{2 - c \alpha_0} \right)^2 + \rho^2 + \frac{1}{4} |2 + c \alpha_0| \sqrt{\left( \frac{2 a}{2 + c \alpha_0} \right)^2 + \rho^2}
$$

$$
+ \frac{|c|}{8} \sqrt{\left( \eta - \frac{2}{c} (1 - a \beta_0) \right)^2 + \rho^2}.
$$

As discussed in section 5.5, in the analysis of the QK Eguchi-Hanson limit, for $a \to 0$ the parameter $\alpha_0$ becomes irrelevant since it disappears from the metric. The above coordinate $s$ is well defined in the limit $a \to 0$ only if we first put $\alpha_0 = 0$.

Having the explicit expressions for $s, x$, it is then possible to compute $w(\rho, \eta)$ which was given in \((5.13)\) as a function of $s, x$:

$$
w = \frac{1}{4} |2 - c \alpha_0| \sqrt{\left( \eta - \frac{2 a}{2 - c \alpha_0} \right)^2 + \rho^2} + \frac{1}{4} |2 + c \alpha_0| \sqrt{\left( \eta + \frac{2 a}{2 + c \alpha_0} \right)^2 + \rho^2}
$$

$$
+ \frac{|c|}{8} \sqrt{\left( \eta - \frac{2}{c} (1 - a \beta_0) \right)^2 + \rho^2}.
$$

It is easy to check that $w(\rho, \eta)$ satisfies eq. \((5.24)\)
Let us finally give \( w(\rho, \eta) \) in the two interesting cases \( a \to 0 \) (QK-EH) and \( c \to 0 \) (QK-TN):

\[
w_{\text{QK-EH}}(\rho, \eta) = \sqrt{\eta^2 + \rho^2 + \frac{|c|}{8} \sqrt{(\eta - \frac{2}{c})^2 + \rho^2 + \frac{|c|}{8} \sqrt{(\eta + \frac{2}{c})^2 + \rho^2}}}
\]

\[
w_{\text{QK-TN}}(\rho, \eta) = \frac{1}{2} + \frac{1}{2} \sqrt{(\eta - a)^2 + \rho^2 + \frac{1}{2} \sqrt{(\eta + a)^2 + \rho^2}}.
\]

Using these relations we can, e.g., compute the forms \( \Omega \) and \( \hat{\rho} \) (6.25) for our metrics and, via the correspondence (6.8), to find the relevant Maxwell field strengths.

7 Conclusions

In this paper, proceeding from the general HSS formulation of QK sigma models, we have constructed a wide class of \( U(1) \times U(1) \) 4-dimensional QK metrics extending most general two-centre HK metrics. These QK metrics supply, via Flaherty’s equivalence [40], a new family of explicit solutions of the coupled Einstein-Maxwell equations. We have given the precise embedding of our metrics in the framework of general \( U(1) \times U(1) \) ansatz of Calderbank and Pedersen [43].

The HSS approach gives QK metrics in the form which admits a transparent interpretation of the involved parameters as the symmetry breaking ones and possesses a clear hyper-Kähler limit, with the Einstein constant as a contraction parameter. However, despite these attractive features, it does not immediately provide the natural coordinates best suited to display the final linearization of the self-dual Einstein equations along the line of ref. [13]. It would be interesting to explore what the choice of such coordinates means in the language of the original hypermultiplet superfields parametrizing the general HSS action of QK sigma models. One more obvious direction of further study could consist in applying our HSS methods for explicit construction of higher-dimensional QK metrics generalizing, e.g., the HK metrics constructed in [14].

One of possible physical applications of the QK metrics presented here is to utilize them in the context of gauged five-dimensional supergravities. The latter seemingly provide an appropriate framework for supersymmetric extensions of the famous Randall-Sundrum scenario (for a recent review, see [50]). The presence of matter hypermultiplets seems necessary for the existence of such (smooth) extensions (see, e.g., [51]). To analyse various possibilities, it is important to know the structure of the scalar potential which is obtained by gauging isometries of the QK manifold parametrized by the hypermultiplets. Until now, in the actual computations (e.g., in [52, 53]) there was mainly used the so-called universal hypermultiplet [4] corresponding to the homogeneous QK manifold \( SU(2,1)/U(2) \). It would be tempting to study models with non-homogeneous QK manifolds possessing isometries and, in particular, with those considered here. It is straightforward to gauge the \( U(1) \times U(1) \) isometries of our HSS actions following the general recipe of ref. [20] (in order to generate scalar potentials, the relevant gauge supermultiplets should be propagating, in contrast to the non-propagating ones employed in the HSS quotient). The \( SU(2,1)/U(2) \) QK manifold is a special case [23] of the QK Eguchi-Hanson limit of our \( U(1) \times U(1) \) class of QK manifolds, so the scalar potentials associated with our metrics and inheriting
all free parameters of the latter may offer new possibilities as compared to the case of
universal hypermultiplet.

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Appendix. Definitions and notation

For a given metric \( g \), the vierbein \( e_a, \ a = 0, 1, 2, 3 \), is such that \( g = \sum_a e_a^2 \). The spin
connection \( \omega_{ab} \) is defined by

\[ de_a + \omega_{ab} \wedge e_b = 0, \quad \omega_{ab} = -\omega_{ba}, \]

with self-dual components

\[ \omega_i^\pm = \omega_0 i \pm \frac{1}{2} \epsilon_{ijk} \omega_{jk}, \]

and similarly for the curvature

\[ R_{ab} = d\omega_{ab} + \omega_{as} \wedge \omega_{sb} = \frac{1}{2} R_{ab, st} e_s \wedge e_t, \quad \rightarrow \quad R_i^\pm = R_0 i \pm \frac{1}{2} \epsilon_{ijk} R_{jk}. \]

We take for the Ricci tensor and scalar curvature

\[ Ric_{ab} = R_{as, ba}, \quad R = Ric_{ss}. \]

It is useful to define the two-forms of definite self-duality by

\[ \Xi_i^\pm = e_0 \wedge e_i \pm \frac{1}{2} \epsilon_{ijk} e_j \wedge e_k. \]

Using this basis, the curvature and Ricci tensor are encoded in the three matrices \( A, B \)
and \( C \) such that

\[ R_i^+ = A_{ij} \Xi_j^+ + B_{ij} \Xi_j^- , \quad R_i^- = B_{ij}^t \Xi_j^+ + C_{ij} \Xi_j^- , \]

where the matrices \( A \) and \( C \) are symmetric.

The Ricci components in the vierbein basis are

\[ Ric_{00} = Tr(A + B), \quad Ric_{0i} = -\frac{1}{2} \epsilon_{ijk} (B_{jk} - B_{jk}^t), \quad Ric_{ij} = Tr(A - B) \delta_{ij} + B_{ij} + B_{ij}^t, \]
and the scalar curvature is

\[ R = 4 \left( Tr A \right) = 4 \left( Tr C \right). \]

The Einstein condition \( \text{Ric}_{ab} = \Lambda \delta_{ab} \) is seen to be equivalent to the vanishing of the matrix \( B \) and we have \( Tr C = Tr A = \Lambda \).

One further defines the Weyl tensor

\[ W_{ab,cd} = R_{ab,cd} + \frac{R}{6} \left( \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} \right) - \frac{1}{2} \left( \delta_{ac} \text{Ric}_{bd} - \delta_{ad} \text{Ric}_{bc} + \delta_{bd} \text{Ric}_{ac} - \delta_{bc} \text{Ric}_{ad} \right). \]

The corresponding two-forms

\[ W_{ab} = \frac{1}{2} W_{ab,cd} \epsilon^c \wedge \epsilon^d, \]

and their self-dual parts are given by

\[
\begin{cases}
W_i^+ &\equiv W_{0i} + \frac{1}{2} \epsilon_{ijk} W_{jk} = W_{ij}^+ \Xi^+ \quad , \quad W_{ij}^+ = A_{ij} - \frac{1}{3} \left( Tr A \right) \delta_{ij} , \\
W_i^- &\equiv W_{0i} - \frac{1}{2} \epsilon_{ijk} W_{jk} = W_{ij}^- \Xi^- \quad , \quad W_{ij}^- = C_{ij} - \frac{1}{3} \left( Tr C \right) \delta_{ij}.
\end{cases}
\]

We conclude that for an Einstein space with self-dual Weyl tensor (i.e., \( W_i^- = 0 \)) we should have

\[ C_{ij} = \frac{\Lambda}{3} \delta_{ij} \quad \iff \quad R_i^- = \frac{\Lambda}{3} \Xi_i^- . \]

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