1. Introduction

It is shown in [EO], Proposition 2.17, that a finite dimensional quasi-Hopf algebra with radical of codimension 1 is semisimple and 1-dimensional. On the other hand, there exist quasi-Hopf (in fact, Hopf) algebras, whose radical has codimension 2. Namely, it is known [N] that these are exactly the Nichols Hopf algebras $H_{2^n}$, of dimension $2^n$, $n \geq 1$ (one for each value of $n$).

The main result of this paper is that if $H$ is a finite dimensional quasi-Hopf algebra over $\mathbb{C}$ with radical of codimension 2, then $H$ is twist equivalent to a Nichols Hopf algebra $H_{2^n}$, $n \geq 1$, or to a lifting of one of the four special quasi-Hopf algebras $H(2)$, $H_+(8)$, $H_-(8)$, $H(32)$ of dimensions 2, 8, 8, and 32, defined in Section 3. As a corollary we obtain that any finite tensor category which has two invertible objects and no other simple object is equivalent to $\text{Rep}(H_{2^n})$ for a unique $n \geq 1$, or to a deformation of the representation category of $H(2)$, $H_+(8)$, $H_-(8)$, or $H(32)$. In the case of $H(2)$ such a lifting (deformation) must clearly be trivial; for the other three cases we plan to study possible liftings in a later publication.

As another corollary we prove that any nonsemisimple quasi-Hopf algebra of dimension 4 is twist equivalent to $H_4$.

Thus, this paper should be regarded as a beginning of the structure theory of finite dimensional basic quasi-Hopf algebras. It is our expectation that this theory will be a nontrivial generalization of the deep and beautiful theory of finite dimensional basic (or, equivalently, pointed) Hopf algebras (see [AS] and references therein).

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2. Preliminaries

All constructions in this paper are done over the field of complex numbers $\mathbb{C}$.

2.1. Recall that a finite rigid tensor category $\mathcal{C}$ over $\mathbb{C}$ is a rigid tensor category over $\mathbb{C}$ with finitely many simple objects and enough projectives such that the neutral object $1$ is simple (see [EO]). Let $\text{Irr}(\mathcal{C})$ denote the (finite) set of isomorphism classes of simple objects of $\mathcal{C}$. Then to each object $X \in \mathcal{C}$ there is attached a
positive number $\text{FPdim}(X)$, called the Frobenius-Perron (FP) dimension of $X$, and the Frobenius-Perron (FP) dimension of $C$ is defined by

$$\text{FPdim}(C) = \sum_{X \in \text{Irr}(C)} \text{FPdim}(X) \text{FPdim}(P_X),$$

where $P_X$ denotes the projective cover of $X$. By Proposition 2.6 in [LO], $C$ is equivalent to $\text{Rep}(H)$, $H$ a finite dimensional quasi-Hopf algebra, if and only if the FP-dimensions of its objects are integers. We refer the reader to [D] for the definition of a quasi-Hopf algebra and the notion of twist (= gauge) equivalence between quasi-Hopf algebras.

2.2. Recall that Nichols’ Hopf algebra $H_{2n}$ ($n \geq 1$) is the $2^n$-dimensional Hopf algebra generated by $g, x_1, \ldots, x_{n-1}$ with relations $g^2 = 1$, $x_i^2 = 0$ ($1 \leq i \leq n-1$), $x_ix_j = -x_jx_i$ ($1 \leq i \neq j \leq n-1$) and $gx_i = -x_ig$, where $g$ is a grouplike element and $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$. The Hopf algebra $H_4$ is known as the Sweedler’s Hopf algebra [5]. Nichols’ Hopf algebras are the unique (up to isomorphism) Hopf algebras with radical of codimension 2. It is well known (and easy to check) that $H_{2n}$ is self dual.

2.3. The following is the simplest (and well known) example of a quasi-Hopf algebra not twist equivalent to a Hopf algebra. The 2-dimensional quasi-Hopf algebra $H(2)$ is generated by a grouplike element $g$ such that $g^2 = 1$, with associator $\Phi = 1 - 2p_+ \otimes p_- \otimes p_-$ (where $p_+ := (1 + g)/2, p_- := (1 - g)/2$), distinguished elements $\alpha = g, \beta = 1$, and antipode $S(g) = g$. It is not twist equivalent to a Hopf algebra, and any 2-dimensional quasi-Hopf algebra is well known to be twist equivalent either to $\mathbb{C}[\mathbb{Z}_2]$ or to $H(2)$.

2.4. Let $H$ be a finite dimensional algebra, and let $I := \text{Rad}(H)$ be the Jacobson radical of $H$. Then we have a filtration on $H$ by powers of $I$. So one can consider the associated graded algebra $\text{gr}(H) = \bigoplus_{k \geq 0} H_k$, $H_k := I^k/I^{k+1}$ ($I_0 := H$).

The following lemma is standard.

Lemma 2.1. (i) Let $L$ be any linear complement of $I^2$ in $H$. Then $L$ generates $H$ as an algebra.

(ii) $\text{gr}(H)$ is generated by $H_0$ and $H_1$.

Proof. (i) Let $B \subseteq H$ be the subalgebra generated by $L$. We will show by induction in $k$ that $B + I^k = H$ for all $k$. This implies the statement, since $I^N = 0$ for some $N$. The base of induction, $k = 2$, is clear. So assume that $k > 2$. By the induction assumption, all we need to show is that $I^{k-1} \subseteq B + I^k$. So let us take $a \in I^{k-1}$, and show that $a \in B + I^k$. We may assume that $a = i_1 \cdots i_{k-1}$, where $i_m \in I$ for all $m$. Let $\tilde{i}_m \in L \cap I$ be the unique elements such that $\tilde{i}_m - i_m \in I^2$. Then $\tilde{a} := \tilde{i}_1 \cdots \tilde{i}_{k-1} \in B$, and $a - \tilde{a} \in I^k$. We are done.

(ii) Clearly, $\text{gr}(I)$ is the radical of $\text{gr}(H)$. Thus we only need to show that $\text{gr}(I^2) = \text{gr}(I)^2$ (then everything follows from (i)). The inclusion $\text{gr}(I^2) \supseteq \text{gr}(I)^2$ is clear, so let us prove the opposite inclusion. Since both spaces are graded, it suffices to pick a homogeneous element $a \in \text{gr}(I^2)$ and show that it lies in $\text{gr}(I)^2$. Suppose $a$ has degree $k$, i.e. $a \in I^k/I^{k+1}$. Let $\tilde{a}$ be a lifting of $a$ to $I^k$. Then there exist elements $i_{j1}, \ldots, i_{jk} \in I, j = 1, \ldots, m$, such that $\tilde{a} = \sum_j i_{j1} \cdots i_{jk}$. Let $i_{j1}, \ldots, i_{jk}$ be the projections of $i_{j1}, \ldots, i_{jk}$ to $I/I^2 \subseteq \text{gr}(I)$. Then $a = \sum_j \tilde{i}_{j1} \cdots \tilde{i}_{jk}$. Since $k \geq 2$, we are done. \qed
2.5. Let $H$ be a basic quasi-Hopf algebra (i.e., all irreducible $H$-modules are 1-dimensional) and let $I := \text{Rad}(H)$ be the radical of $H$. Then $I$ is a quasi-Hopf ideal (i.e. $\Delta(I) \subset H \otimes I + I \otimes H$, $S(I) = I$ and $\varepsilon(I) = 0$), so we have a quasi-Hopf algebra filtration on $H$ by powers of $I$. The associated graded algebra $\text{gr}(H) = \bigoplus_k H_k$ is thus also a quasi-Hopf algebra. We have $H_0 = \text{Fun}(G)$, where $G$ is a finite group (of characters of $H$), $H_1 = \bigoplus_{a,b \in G} \text{Ext}^1(a, b)^*$. Note that by definition, the associator of $\text{gr}(H)$ lives in $H_0^{\otimes 3}$, so it corresponds to an element of $H^3(G, \mathbb{C}^*)$.

3. The main results

Our first main theorem is the following.

**Theorem 3.1.** Let $H$ be a finite dimensional quasi-Hopf algebra over $\mathbb{C}$ with radical of codimension 2. Suppose that the associator of $H$ is trivial on 1-dimensional $H$-modules (i.e., the subcategory of $\text{Rep}(H)$ generated by 1-dimensional objects is equivalent to $\text{Rep}(\mathbb{Z}_2)$ with the trivial associator). Then $H$ is twist equivalent to a Nichols Hopf algebra $H_{2^n}$, $n \geq 1$.

The proof of the theorem is given in Section 4.

The assumption on the triviality of the associator of $H$ on 1-dimensional modules is clearly essential already for $\dim(H) = 2$, because of the existence of the quasi-Hopf algebra $H(2)$. However, this assumption is also essential in higher dimensions, as the following propositions show.

**Proposition 3.2.** There exist two 8-dimensional quasi-Hopf algebras $H_{\pm}(8)$ (permutated by the action of the Galois group), which have the following structure. As algebras $H_{\pm}(8)$ are generated by $g, x$ with the relations $gx = -xg$, $g^2 = 1$, $x^4 = 0$. The element $g$ is grouplike, while the coproduct of $x$ is given by the formula

$$\Delta(x) = x \otimes (p_+ \pm ip_-) + 1 \otimes p_+ x + g \otimes p_- x,$$

where $p_+ := (1 + g)/2, p_- := (1 - g)/2$. The associator is $\Phi = 1 - 2p_\pm \otimes p_- \otimes p_-$, the distinguished elements are $\alpha = g$, $\beta = 1$, and the antipode is $S(g) = g, S(x) = -x(p_+ \pm ip_-)$.

**Proposition 3.3.** There exists a 32-dimensional quasi-Hopf algebra $H(32)$, which has the following structure. As an algebra $H(32)$ is generated by $g, x, y$ with the relations $gx = -xg$, $gy = -yg$, $g^2 = 1$, $x^4 = 0$, $y^4 = 0$, $xy + iyx = 0$. The element $g$ is grouplike, while the coproducts of $x, y$ are given by the formulas

$$\Delta(x) = x \otimes (p_+ + ip_-) + 1 \otimes p_+ x + g \otimes p_- x,$$

$$\Delta(y) = y \otimes (p_+ - ip_-) + 1 \otimes p_+ y + g \otimes p_- y,$$

where $p_+ := (1 + g)/2, p_- := (1 - g)/2$. The associator is $\Phi = 1 - 2p_\pm \otimes p_- \otimes p_-$, the distinguished elements are $\alpha = g$, $\beta = 1$, and the antipode is $S(g) = g, S(x) = -x(p_+ + ip_-), S(y) = y(p_+ - ip_-)$. Thus, $H(32)$ is generated by its quasi-Hopf subalgebras $H_{\pm}(8)$ and $H_{-}(8)$ generated by $g, x$ and $g, y$, respectively.

Propositions 3.2 and 3.3 are proved in Section 5.

Nonetheless, it turns out that outside of dimensions 2, 8 and 32 the assumption of the triviality of the associator of $H$ on 1-dimensional modules is satisfied automatically. This striking fact follows from our second main theorem, which is the following.
Theorem 3.4. Let \( H \) be a finite dimensional quasi-Hopf algebra over \( \mathbb{C} \) with radical of codimension 2. Suppose that the associator of \( H \) is nontrivial on 1-dimensional \( H \)-modules (i.e., the subcategory of \( \text{Rep}(H) \) generated by 1-dimensional objects is equivalent to \( \text{Rep}(H(2)) \)). Then \( \text{gr}(H) \) is twist equivalent to \( H(2), H_+(8), H_-(8), \) or \( H(32) \); in particular, the dimension of \( H \) is 2, 8 or 32. The quasi-Hopf algebras \( H_+(8) \) and \( H_-(8) \) are not twist equivalent.

Theorem 3.4 is proved in Section 6.

Corollary 3.5. Any finite tensor category which has two invertible objects and no other simple object is tensor equivalent to \( \text{Rep}(H_{2n}) \) for a unique \( n \geq 1 \), or to a deformation of the representation category of \( H(2), H_+(8), H_-(8), \) or \( H(32) \).

Proof. The FP-dimension of any object in \( C \) is an integer. Therefore, by Proposition 2.6 in [EO], \( C \) is equivalent to a representation category of a finite dimensional quasi-Hopf algebra with radical of codimension 2. The result follows now from Theorems 3.1 and 3.4.

Corollary 3.6. Any nonsemisimple quasi-Hopf algebra of dimension 4 is twist equivalent to \( H_4 \).

Proof. By dimension counting it follows that \( H \) is basic; that is, all irreducible representations of \( H \) are 1-dimensional. Moreover, by Theorem 2.17 in [EO], \( H \) has more than one irreducible representation. Let \( \chi \) be a non-trivial 1-dimensional representation of \( H \), and let \( P_\chi \) denote the projective cover of \( \chi \). Then \( P_\chi = \chi \otimes P_\varepsilon \), where \( P_\varepsilon \) is the projective cover of the trivial representation \( \varepsilon \). Therefore 4 equals \( \dim(P_\varepsilon) \) times the number of simple objects in \( \text{Rep}(H) \). Since both numbers are greater than 1, we have that they are equal to 2. So \( H \) has 2 irreducible representations \( \varepsilon \) and \( \chi \), with \( \chi^2 = \varepsilon \). The result follows now from Theorems 3.1 and 3.4.

Corollary 3.7. A nonsemisimple finite tensor category \( C \) of FP-dimension 4 is tensor equivalent to \( \text{Rep}(H_4) \).

Proof. It suffices to show that \( C \) has integer FP-dimensions of objects; in this case by Proposition 2.6 in [EO], \( C \) is a representation category of a 4-dimensional quasi-Hopf algebra, and Corollary 3.6 applies.

Let \( X \) be a simple object of \( C \) of FP-dimension \( d \). If \( X \otimes X^* \neq \mathbf{1} \) then \( X \otimes X^* \) contains as constituents \( \mathbf{1} \) and another object. So if \( d \neq 1 \) then \( d \geq \sqrt{2} \). On the other hand, the projective cover \( P_1 \) of the neutral object has to involve other objects, so it is at least 2-dimensional. This shows that the only chance for \( C \) to have dimension 4 is when all simple objects are 1-dimensional. We are done.

Remark 3.8. We note that semisimple finite tensor categories of FP-dimension 4 are easy to classify. Such a category is either the category of modules over the group algebra of a group of order 4 with associativity defined by a 3-cocycle on this group, or a Tambara-Yamagami category [TY] corresponding to the Ising model (see [ENO], Proposition 8.32).

4. Proof of Theorem 3.1

By assumption, the associator of \( \text{gr}(H) \) is trivial. Therefore \( \text{gr}(H) \) is twist equivalent to a Hopf algebra, and by the result of Nichols [N], \( \text{gr}(H) \) is twist equivalent to a Nichols Hopf algebra \( A := H_{2^\infty} \).
From this point we will identify \( \text{gr}(H) \) and \( A \) as quasi-Hopf algebras.

We will now show that \( H \) itself is twist equivalent to a Nichols Hopf algebra, completing the proof of the theorem.

Let \( I_r \) be the radical of \( H^\otimes r \). Then of course \( I_r = \bigoplus_{k=1}^r H^\otimes k-1 \otimes I \otimes H^\otimes r-k \).

Let \( \Phi \) be the associator of \( H \). Then \( \Phi = 1 + \phi \), where \( \phi \in I_3 \). Assume that \( \phi \in I_3 \), but \( \phi \notin I_3^{m+1} \). We will show that by twisting we can change \( \phi \) so that it will belong to \( I_3^{m+1} \). Then by a chain of twists we can make sure that \( \phi = 0 \), and by Nichols’ result we are done.

Let \( \phi' \) be the projection of \( \phi \) to \( I_3^m/I_3^{m+1} = A^\otimes 3 [m] \). Obviously, \( \phi' \) is a 3-cocycle of \( A^* \) with trivial coefficients. But \( A^* = A = \mathbb{C}[\mathbb{Z}_2] \ltimes \Lambda V \), and

\[
H^3(A, \mathbb{C}) = H^3(\mathbb{C}[\mathbb{Z}_2] \ltimes \Lambda V, \mathbb{C}) = H^3(\Lambda V, \mathbb{C}) = 2^3 V^* = 0,
\]

since \( \mathbb{Z}_2 \) acts on \( V \) by sign. Thus there exists an element \( j' \in A^\otimes 2 [m] \) such that \( dj' = \phi' \). Let \( j \) be a lifting of \( j' \) to \( I_3^m \). Let \( J := 1 + j \). Let \( \Phi' \) be the result of twisting \( \Phi \) by \( J \), and \( \Phi' = 1 + \phi J \). Then \( \phi J \in I_3^{m+1} \), as desired.

5. Proof of Propositions

5.1. Proof of Proposition 3.2 Let us first show that there exist quasi-Hopf algebras \( H_\pm(\infty) \) defined in the same way as \( H_\pm(8) \) but without the relation \( x^4 = 0 \).

To show this, it is sufficient to check that \( \Phi(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x)\Phi \) (the rest of the relations are straightforward). This relation is checked by a simple direct computation.

Now we must show that the ideal generated by \( x^4 \) in the quasi-Hopf algebras \( H_\pm(\infty) \) is a quasi-Hopf ideal. To show this, we compute:

\[
\Delta(x^4) = x^2 \otimes g + [(1+i)(p_+ \otimes p_+ + p_- \otimes p_+) + (1-i)(p_+ \otimes p_- - p_- \otimes p_+)](x \otimes x) + g \otimes x^2,
\]

and hence

\[
\Delta(x^4) = x^4 \otimes 1 + 1 \otimes x^4.
\]

So \( x^4 \) generates a quasi-Hopf ideal, and the quotients \( H_\pm(8) := H_\pm(\infty)/\langle x^4 \rangle \) (obviously of dimension 8) are quasi-Hopf algebras. We are done.

5.2. Proof of Proposition 3.3 Let \( H_{\pm}(\infty) \) be the amalgamated product \( H_+ \otimes_{H(2)} H_- \) (i.e. the algebra defined as \( H(32) \) but without the relation \( xy + i’yx = 0 \)). This is obviously a quasi-Hopf algebra (of infinite dimension). We must show that the principal ideal in \( H_{\pm}(\infty) \) generated by \( z := xy + i’yx \) is a quasi-Hopf ideal. This follows from the easily established relation \( \Delta(z) = z \otimes 1 + 1 \otimes z \).

Thus, \( H(32) := H_{\pm}(\infty)/\langle z \rangle \) is a quasi-Hopf algebra. It is easy to show that the elements \( g^j x^k y^l \), \( j, k, l = 0, 1, 2, 3 \), form a basis in \( H(32) \). Thus, \( H(32) \) is 32-dimensional.

6. Proof of Theorem

Let \( p_+, p_- \) be the primitive idempotents of \( H_0 \), and let \( g := p_+ - p_- \). Let \( \Phi_0 \) be the associator of \( \text{gr}(H) \). By using a twist, we may assume that \( \Phi_0 = 1 - 2p_- \otimes p_- \otimes p_- \) (the only, up to equivalence, nontrivial associator for \( \mathbb{C}[\mathbb{Z}_2] \)).

Let \( x \) be an element of \( H_1 \). By Theorem 2.17 in [CO], \( \text{Ext}_1^1(e, e) = \text{Ext}_1^1(\chi, \chi) = 0 \), thus by subsection 2.5,

\[
g x g^{-1} = -x.
\]
Let $\chi$ be the nontrivial character of $H_0$, and for $z \in H$ define
\[ \xi(z) := (\chi \otimes \text{id})(\Delta(z)) \text{ and } \eta(z) := (\text{id} \otimes \chi)(\Delta(z)). \]
Then $\xi(g) = \eta(g) = -g$.

For $x \in H_1$ we have
\[ \Delta(x) = x \otimes p_+ + p_+ \otimes x + \eta(x) \otimes p_- + p_- \otimes \xi(x). \]

By the quasi-coassociativity axiom, we have $\Phi_0(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x)\Phi_0$.

**Lemma 6.1.** The equation $\Phi_0(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x)\Phi_0$ is equivalent to the relations
\[ \xi^2(x) = -x, \eta^2(x) = -x, \text{ and } \xi\eta(x) = -\eta\xi(x). \]

**Proof.** For degree reasons the coassociativity equation is equivalent to the system of three equations obtained by applying $\chi \otimes \chi \otimes \text{id}$, $\chi \otimes \text{id} \otimes \chi$, and $\text{id} \otimes \chi \otimes \chi$. Since
\[ (\chi \otimes \chi \otimes \text{id})(\Phi_0) = (\chi \otimes \text{id} \otimes \chi)(\Phi_0) = (\text{id} \otimes \chi \otimes \chi)(\Phi_0) = g, \]
application of $\chi \otimes \text{id} \otimes \chi$ gives
\[
\begin{align*}
\xi\eta(x) &= \xi(((\chi \otimes \chi)(\Delta(x)))) \\
&= (\chi \otimes \text{id} \otimes \chi)((\text{id} \otimes \Delta)\Delta(x)) \\
&= (\chi \otimes \text{id} \otimes \chi)(\Phi_0(\Delta \otimes \text{id})\Delta(x)\Phi_0^{-1}) \\
&= g(\chi \otimes \text{id} \otimes \chi)(\Delta \otimes \text{id})\Delta(x)g^{-1} \\
&= g\eta\xi(x)g^{-1},
\end{align*}
\]
which is equivalent to $\xi\eta(x) = -\eta\xi(x)$.

Also, application of $\chi \otimes \chi \otimes \text{id}$ yields
\[
\begin{align*}
\xi^2(x) &= \xi(((\chi \otimes \chi)(\Delta(x)))) \\
&= (\chi \otimes \chi \otimes \text{id})(\text{id} \otimes \Delta)\Delta(x) \\
&= (\chi \otimes \chi \otimes \text{id})(\Phi_0(\Delta \otimes \text{id})\Delta(x)\Phi_0^{-1}) = g\xi g^{-1} = -x,
\end{align*}
\]
and similarly application of $\text{id} \otimes \chi \otimes \chi$ yields $\eta^2(x) = -x$. \hfill $\square$

Let $L_g$ be the operator of left multiplication by $g$ in $H_1$. Then $L_g^2 = \text{id}$, and $L_g\xi = -\xi L_g$, $L_g\eta = -\eta L_g$. Thus, using Lemma 6.1, we see that the operators $\xi, \eta, L_g$ define an action on $H_1$ of the Clifford algebra $Cl_3$ of a 3-dimensional inner product space.

The following lemma is standard.

**Lemma 6.2.** The algebra $Cl_3$ is semisimple and has two irreducible representations $W_\pm$, which are both 2-dimensional. They are spanned by elements $x$ and $gx$ with $\eta(x) = ix$, $\xi(x) = \pm gx$.

(Here we abuse the notation by writing $g$ instead of $L_g$.)

Lemma 6.2 implies that as a $Cl_3$-module, $H_1 = nW_+ \oplus mW_-$. Let $x \in W_\pm \subseteq H_1$ be an eigenvector of $\eta$ with eigenvalue $i$. Then we have
\[ \Delta(x) = x \otimes (p_+ \pm ip_-) + p_+ \otimes x + p_- \otimes gx \\
= x \otimes (p_+ \pm ip_-) + 1 \otimes p_+ x + g \otimes p_- x. \]
Together with Lemma 6.1 (ii), this implies the following proposition:
Proposition 6.3. The quasi-Hopf algebra \( gr(H) \) is generated by elements \( g \) of degree 0, and \( x_1, \ldots, x_n, y_1, \ldots, y_m \) of degree 1, which satisfy the relations \( gx_j = -x_j g, gy_j = -y_j g, g^2 = 1 \) (and possibly other relations). The coproduct of \( gr(H) \) is defined by the formulas
\[
\Delta(g) = g \otimes g, \\
\Delta(x_j) = x_j \otimes (p_+ + ip_-) + 1 \otimes p_+ x_j + g \otimes p_- x_j, \\
\Delta(y_j) = y_j \otimes (p_+ - ip_-) + 1 \otimes p_+ y_j + g \otimes p_- y_j.
\]

Lemma 6.4. In \( gr(H) \), we have \( x_j^4 = y_j^4 = 0 \) for all \( j, l \).

Proof. It suffices to show that \( x_j^4 = 0 \), the case of \( y_l \) is obtained by changing \( i \) to \(-i\). Let \( x = x_j \). Using Proposition 6.3 we find that
\[
\Delta(x^2) = x^2 \otimes g + [(1 + i)(p_+ \otimes p_+ + p_- \otimes p_-) + (1 - i)(p_+ \otimes p_- - p_- \otimes p_+)](x \otimes x) + g \otimes x^2,
\]
and hence
\[
\Delta(x^4) = x^4 \otimes 1 + 1 \otimes x^4.
\]
But a finite dimensional quasi-Hopf algebra cannot contain nonzero primitive elements (this is proved as in the Hopf case). Thus, \( x^4 = 0 \).

Lemma 6.5. The numbers \( m \) and \( n \) are either 0 or 1.

Proof. Clearly, it suffices to show that \( n = 0 \) or 1. Assume the contrary, i.e. that \( n \geq 2 \).

Introduce the element \( z := g(x_2 x_1 - ix_1 x_2) \). Using Proposition 6.3 it is checked directly that
\[
\Delta(z) = z \otimes 1 + 1 \otimes z + 2(gx_2 \otimes p_+ x_1 + ix_2 \otimes p_- x_1).
\]
Let \( N \) be the smallest integer such that \( z^N = 0 \). It exists since \( gr(H) \) is finite dimensional. Taking the coproduct of this equation and looking at the terms of bidegree \((2N - 1, 1)\), we find
\[
\sum_{k=1}^{N} z^{k-1} x_2 z^{N-k} = 0.
\]
Let us now apply the coproduct to this equality, and look at the terms of bidegree \((2N - 2, 1)\) which have a factor \( x_2 \) in the second component. This yields \( N z^{N-1} = 0 \), which is a contradiction with the minimality of \( N \).

Lemma 6.6. (i) If \( m = n = 0 \), \( gr(H) = H(2) \).
(ii) If \( n = 1, m = 0 \) then \( gr(H) = H_+(8) \).
(iii) If \( n = 0, m = 1 \) then \( gr(H) = H_-(8) \).
(iv) If \( n = 1, m = 0 \) then \( gr(H) = H(32) \).

Proof. Statements (i)-(iii) of the lemma follow from the arguments above, since it is easy to check that the algebras \( H_\pm(8) \) do not have nontrivial graded quasi-Hopf ideals which do not intersect with \( H_1 \). It remains to prove statement (iv).

Assume \( m = n = 1 \), and set \( x := x_1, y := y_1 \). Consider the element \( z := xy + iyx \). A direct computation shows that
\[
\Delta(z) = z \otimes 1 + g \otimes z.
\]
Since \( gz = zg \) and \( gr(H) \) is finite dimensional, we must have \( z = 0 \). Thus, \( gr(H) \) is a quotient of \( H(32) \). But it is easy to check directly that \( H(32) \) does not have
Finally, let us show that $H_+(8)$ is not twist equivalent to $H_-(8)$. Suppose they are. Then there exists a twist $I$ for $H_-(8)$, and an isomorphism of algebras $f : H_+(8) \to H_-(8)^I$. Such an isomorphism obviously preserves filtration by powers of the radical, so we can take its degree zero part. Thus we can assume, without loss of generality, that $f$ preserves the grading and $I$ is of degree zero. Then $f$ is the identity on the degree zero part, and $\Phi^f = \Phi$. So $I$ is a twist of $\mathbb{C}[\mathbb{Z}_2]$, hence a coboundary, and thus we may assume that $I = 1 \otimes 1$ (by changing $f$). But then $f$ cannot exist, since $S^2(a) = ia$ on $H_+(8)_1$ and $S^2(a) = -ia$ on $H_-(8)_1$.

This completes the proof of Theorem 3.4.

7. Relation to pointed Hopf algebras

Let $H = \bigoplus_{k \geq 0} H_k$ be a graded quasi-Hopf algebra with radical of codimension 2 and nontrivial associator on 1-dimensional representations.

The main result of this section is that it is actually possible to embed $H$ into a twice bigger quasi-Hopf algebra $\tilde{H}$, which is twist equivalent to a basic Hopf algebra $H'$ with $H'/\text{Rad}(H') = \mathbb{C}[\mathbb{Z}_4]$. This fact should have generalizations to the case of general graded basic quasi-Hopf algebras, which may facilitate applications to the quasi-Hopf case of known deep results about pointed Hopf algebras.

To construct $\tilde{H}$, let us adjoin a new element $a$ to $H$ which is grouplike, $a^2 = g$, and $a x a^{-1} = i^k z$ for $z \in H_k$. Then the algebra $\tilde{H}$ generated by $a$ and $H$ (of twice bigger dimension than that of $H$) is a quasi-Hopf algebra graded by nonnegative integers, with $\tilde{H}_0 = \mathbb{C}[\mathbb{Z}_4]$.

**Proposition 7.1.** $\tilde{H}$ is twist equivalent to a Hopf algebra.

**Proof.** We claim that the image of the associator $\Phi_0$ in $\tilde{H}_0^3$ is homologically trivial. This is because the natural map $f' : H^3(\mathbb{Z}_2, \mathbb{C}^*) \to H^3(\mathbb{Z}_4, \mathbb{C}^*)$ induced by the projection $f : \mathbb{Z}_4 \to \mathbb{Z}_2$ is zero. Indeed, for a cyclic group $G$ one has $H^3(G, \mathbb{C}^*) = G^* \otimes G^*$ (functorially in $G$). Thus for any morphism of cyclic groups $f : G_1 \to G_2$, the induced map of third cohomology groups is $f' = f^* \otimes f^*$. In our case $f : \mathbb{Z}_4 \to \mathbb{Z}_2$, so $f^* : \mathbb{Z}_2 \to \mathbb{Z}_2$ is given by $f^*(1) = 2$. Hence, $(f^* \otimes f^*)(1 \otimes 1) = 2 \otimes 2 = 1 \otimes 4 = 0$, as desired.

Thus, by a twist in $\tilde{H}_0^3$, we can kill $\Phi_0$. So $\tilde{H}$ is twist equivalent to a graded pointed Hopf algebra $A$ generated in degree 1 with $A/\text{Rad}(A) = \mathbb{Z}_4$. $\square$

**Example 7.2.** Let $A_\pm(16)$ (for each choice of sign) be the Hopf algebra of dimension 16 generated by $a, x$ with relations $a^4 = 1, axa^{-1} = ix, x^4 = 0$, such that $a$ is grouplike and $\Delta(x) = 1 \otimes x + x \otimes a^{-1}$.

Let $A(64)$ be the Hopf algebra of dimension 64 generated by $a, x, y$ with relations $a^4 = 1, axa^{-1} = ix, aya^{-1} = iy, x^4 = 0, y^4 = 0, xy + iyx = 0$, such that $a$ is grouplike and $\Delta(x) = 1 \otimes x + x \otimes a^{-1}, \Delta(y) = 1 + y \otimes a$.

In other words, $A_\pm(16) = U_q(\mathfrak{b}_+)$, where $\mathfrak{b}_+$ is the Borel subalgebra in $\mathfrak{sl}_2$, and $q = \exp(\pm i/4)$, while $A(64) = \text{gr}(U_q(\mathfrak{sl}_2^*))$, for the same $q$.

**Proposition 7.3.** (i) $\tilde{A}_\pm(16)$ is twist equivalent to $A_\pm(16)$.

(ii) $H(32)$ is twist equivalent to $A(64)$. 
Proof. (i) Consider the Hopf algebra $A$ obtained by twisting away the associator in $\tilde{H}_+(8)$. Then $A = \bigoplus_{k \geq 0} A_k$ is a 16-dimensional graded basic (and pointed) Hopf algebra generated in degree 1. Grouplike elements of this Hopf algebra form a group $\mathbb{Z}_4$, and $A_1$ is a free rank 1 module over $A_0$ under left multiplication. So $A = A_+(16)$ or $A = A_-(16)$. To decide the sign, assume that $J$ is a pseudotwist for $A_s(16)$, and $f : \tilde{H}_+(8) \to A_s(16)$ is an isomorphism (where $s$ is a choice of sign). Since $f$ preserves the filtration by powers of the radical, we may assume (by taking the degree 0 part) that $f$ preserves the grading and $J$ has degree 0. Then $\Phi = (1 \otimes 1 \otimes 1)^J$. This implies that if $Q := \sum d_j S(e_j)$, where $J^{-1} = \sum d_j \otimes e_j$, then $QS(Q)^{-1} = a^2$. Hence the squares of the antipodes in $\tilde{H}_+(8)$ and $A_s(16)$ differ by conjugation by $a^2$, i.e. by a sign on the degree 1 component. For $\tilde{H}_+(8)$, the eigenvalue of $S^2$ in degree 1 is $i$. Thus $s = +$.

(ii) Consider the Hopf algebra $A$ obtained by twisting away the associator in $H(32)$. This is a 64-dimensional graded basic (and pointed) Hopf algebra, which is a quotient of the amalgamated product of $A_+(16)$ (generated by $a, x$) and $A_-(16)$ (generated by $a, y$). It is easy to see that $xy + iyx$ is a primitive element, so it must be zero. Thus, $A = A(64)$, which is the quotient of this amalgamated product by the relation $xy + iyx = 0$. 

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