FORMULAS GENERALIZING PAPPUS AND DESARGUES

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Abstract. The Theorems of Pappus and Desargues are generalized by two special formulas that hold in the three-dimensional vector space over a field.

1. Introduction

The Theorems of Pappus and Desargues hold in the projective plane over an arbitrary field \( \mathbb{F} \). This projective plane can be constructed in the three-dimensional vector space \( \mathbb{F}^3 \) by taking the points and lines to be the one-dimensional and two-dimensional subspaces of \( \mathbb{F}^3 \). Two special formulas involving cross products and determinants in \( \mathbb{F}^3 \) have the Theorems of Pappus and Desargues as corollaries. These formulas were suspected and found in the spring semester of 1997 as a result of teaching the second semester of college geometry from [2]. The material for this paper was extracted from the lecture notes for Math 436: Geometry [1]. The lecture notes were based on the textbook [2].

2. The vector space \( \mathbb{F}^3 \)

Let \( \mathbb{F} \) be an arbitrary field. The elements of \( \mathbb{F}^3 \) are written as column vectors.

Definition 1. For all \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{F}^3 \), \( \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{F}^3 \), and \( \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{F}^3 \),

- the determinant of \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) is
  \[ \det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1y_2z_3 + x_3y_1z_2 + x_2y_3z_1 - x_1y_3z_2 - x_3y_2z_1 - x_2y_1z_3 \in \mathbb{F}, \]
- the inner product of \( \mathbf{x} \) and \( \mathbf{y} \) is
  \[ \langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 \in \mathbb{F}, \]
- the cross product of \( \mathbf{x} \) and \( \mathbf{y} \) is
  \[ \mathbf{x} \times \mathbf{y} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \in \mathbb{F}^3. \]

For arbitrary sets of vectors \( X, Y \subseteq \mathbb{F}^3 \), \( X + Y \) is their complex sum:
\[ X + Y = \{ \mathbf{x} + \mathbf{y} : \mathbf{x} \in X, \mathbf{y} \in Y \}. \]

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To prove the Pappus and Desargues Theorems we use the following pair of specially designed formulas involving cross products and determinants.

**Theorem 1.** For any field $\mathbb{F}$ and any $u, v, w, x, y, z \in \mathbb{F}^3$,

\[
\begin{align*}
(P) & \quad \det[(u \times y) \times (x \times v), (w \times x) \times (z \times u), (v \times z) \times (y \times w)] \\
& = \det[v, u, x] \det[u, w, z] \det[v, y, x] \det[y, z, w] \det[u, w, v], \\
(D) & \quad \det[(w \times u) \times (z \times x), (u \times v) \times (x \times y), (v \times w) \times (y \times z)] \\
& = \det[x, y, z] \det[u \times x, v \times y, w \times z] \det[u, v, w].
\end{align*}
\]

The proof of Theorem 1 could be left to the reader since it is merely computational: for each formula, expand both sides in 18 variables (3 for each of 6 vectors) and see that the expansions are the same. Nevertheless, a proof of Theorem 1 using simpler properties of cross products and determinants is included as an appendix.

### 3. The projective plane over $\mathbb{F}$.

The projective plane over $\mathbb{F}$ has points, lines, and an incidence relation between them: a point may be on a line.

**Definition 2.** The points $X, Y, \ldots$ of the projective plane over $\mathbb{F}$ are the one-dimensional subspaces of the three-dimensional vector space $\mathbb{F}^3$ over $\mathbb{F}$, and its lines $\lambda, \mu, \ldots$ are the two-dimensional subspaces of $\mathbb{F}^3$. A point is on a line if it is contained in that line:

$$X \text{ is on } \lambda \text{ iff } X \subseteq \lambda.$$  

Every non-zero vector in $\mathbb{F}^3$ determines both a point and a line. Suppose $0 \neq x, y \in \mathbb{F}^3$. Let

$$X = \{rx : r \in \mathbb{F}\} \text{ and } \lambda = \{z : \langle z, y \rangle = 0\}.$$  

Then $X$ is the 1-dimensional subspace generated by $x$ and $\lambda$ is the 2-dimensional subspace of vectors perpendicular to $y$, so $X$ is a point and $\lambda$ is a line in the projective plane over $\mathbb{F}$. The vectors $x$ and $y$ are called (sets of) homogeneous coordinates for $X$ and $\lambda$, respectively. The incidence relation between a point and a line holds if the inner product of their homogeneous coordinates is 0 (their homogeneous coordinates are perpendicular):

$$X \text{ is on } \lambda \text{ iff } \langle x, y \rangle = 0.$$  

Suppose $X$ and $Y$ are distinct points. Then there are non-zero vectors $x, y \in \mathbb{F}^3$ such that

$$X = \{rx : r \in \mathbb{F}\} \text{ and } Y = \{sy : s \in \mathbb{F}\}.$$  

The unique line containing both $X$ and $Y$ is the 2-dimensional subspace generated by $x$ and $y$:

$$X + Y = \{rx + sy : r, s \in \mathbb{F}\}.$$  

The cross product $x \times y$ is perpendicular to both $x$ and $y$, so any vector perpendicular to $x \times y$ will be in the unique line $X + Y$ that contains $X$ and $Y$:

$$X + Y = \{z : \langle z, x \times y \rangle = 0\}.$$  

Thus points with homogeneous coordinates $x$ and $y$ determine a line with homogeneous coordinates $x \times y$. Suppose two lines, $\lambda$ and $\mu$, have homogeneous coordinates $x, y \in \mathbb{F}$, respectively:

$$\lambda = \{z : \langle z, x \rangle = 0\}, \quad \mu = \{z : \langle z, y \rangle = 0\}.$$
Note that \( x \) and \( y \) are linearly independent, since otherwise \( \lambda = \mu \) (and there is only one line, not two, as postulated). Then \( \lambda \cap \mu \) is a 1-dimensional subspace of \( \mathbb{F}^3 \) (a point) generated by any non-zero vector perpendicular to both \( x \) and \( y \), such as \( x \times y \):

\[
\lambda \cap \mu = \{ t(x \times y) : t \in \mathbb{F} \}.
\]

Thus, two lines with homogeneous coordinates \( x \) and \( y \) meet at a point with homogeneous coordinates \( x \times y \).

Suppose \( 0 \neq x, y, z \in \mathbb{F}^3 \). Let \( X, Y, \) and \( Z \) be the points, and \( \lambda, \mu, \) and \( \nu \) the lines, with homogeneous coordinates \( x, y \) and \( z \), respectively:

\[
X = \{ rx : r \in \mathbb{F} \}, \quad Y = \{ ry : r \in \mathbb{F} \}, \quad Z = \{ rz : r \in \mathbb{F} \},
\]

\[
\lambda = \{ u : \langle u, x \rangle = 0 \}, \quad \mu = \{ u : \langle u, y \rangle = 0 \}, \quad \nu = \{ u : \langle u, z \rangle = 0 \}.
\]

By definition, \( X, Y, Z \) are collinear iff \( X + Y = X + Z = Y + Z \), and \( \lambda, \mu, \nu \) are concurrent iff \( \lambda \cap \mu \cap \nu \) is a point. Concurrency and collinearity are equivalent to the homogeneous coordinates having determinant zero:

\[
X, Y, Z \text{ are collinear \iff } \det[x, y, z] = 0 \quad \text{iff} \quad \lambda, \mu, \nu \text{ are concurrent.}
\]

4. The Theorems of Pappus and Desargues

It is convenient to state the Theorems of Pappus and Desargues together because between them they deal with all fifteen lines passing through six points (fifteen is six taken two at a time). Twelve of these lines are intersected in pairs to produce six more points, three for Pappus and the other three for Desargues, while the remaining three lines are involved in the conclusion of Desargues.

**Theorem 2** (Pappus-Desargues). Let \( U, V, W, X, Y, Z \) be six points in the projective plane over \( \mathbb{F} \). Define six more points

\[
O = (V + Z) \cap (Y + W), \quad R = (V + W) \cap (Y + Z),
\]

\[
P = (W + X) \cap (Z + U), \quad S = (W + U) \cap (Z + X),
\]

\[
Q = (U + Y) \cap (X + V), \quad T = (U + V) \cap (X + Y).
\]

**Pappus** If \( U, V, W \) are collinear and \( X, Y, Z \) are collinear then \( O, P, Q \) are collinear.

**Desargues** If \( U, V, W \) are not collinear and \( X, Y, Z \) are not collinear then the lines \( U + X, V + Y, W + Z \) are concurrent iff the points \( R, S, T \) are collinear.

**Proof.** Choose non-zero vectors \( u, v, w, x, y, z \in \mathbb{F}^3 \) that are homogeneous coordinates for the points \( U, V, W, X, Y, Z \), respectively: \( U = \{ ru : r \in \mathbb{F} \} \), etc. From the definitions of the additional six points we obtain homogeneous coordinates for them as cross products of cross products:

\[
O = \{ r ((v \times z) \times (y \times w)) : r \in \mathbb{F} \},
\]

\[
P = \{ r ((v \times x) \times (z \times u)) : r \in \mathbb{F} \},
\]

\[
Q = \{ r ((u \times y) \times (x \times v)) : r \in \mathbb{F} \},
\]

\[
R = \{ r ((v \times w) \times (y \times z)) : r \in \mathbb{F} \},
\]

\[
S = \{ r ((w \times u) \times (z \times x)) : r \in \mathbb{F} \},
\]

\[
T = \{ r ((u \times v) \times (x \times y)) : r \in \mathbb{F} \}.
\]

For Pappus’s Theorem, assume \( U, V, W \) are collinear and \( X, Y, Z \) are collinear. Then

\[
\det[y, x, z] = 0 = \det[u, w, v],
\]
so by formula (P) we get
\[
\det([u \times y] \times (x \times v), (w \times x) \times (z \times u), (v \times z) \times (y \times w)] \\
= \det[v, u, x] \det[u, w, z] \det[w, v, y] \det[y, x, z] \\
+ \det[x, y, v] \det[z, x, u] \det[y, z, w] \det[u, w, v] = 0,
\]
hence \(O, P, Q\) are collinear. For Desargues’s Theorem, assume that \(U, V, W\) are not collinear and \(X, Y, Z\) are not collinear. Hence
\[
\det[u, v, w] \neq 0 \neq \det[x, y, z].
\]
The conclusion of Desargues’s Theorem is that the following two statements are equivalent (the first one says \(U + X, V + Y, W + Z\) are concurrent, the second says \(R, S, T\) are collinear):

(1) \[0 = \det[u \times x, v \times y, w \times z],\]
(2) \[0 = \det[(w \times u) \times (z \times x), (u \times v) \times (x \times y), (v \times w) \times (y \times z)].\]

This equivalence follows immediately from formula (D), written here with the non-collinearity assumptions included:
\[
\det[(w \times u) \times (z \times x), (u \times v) \times (x \times y), (v \times w) \times (y \times z)] \\
= \det[x, y, z] \det[u \times x, v \times y, w \times z] \det[u, v, w].
\]

Notice that (1) implies (2) without the non-collinearity assumptions but to get (1) from (2) requires knowing that if a product of three numbers is zero and two of them are not zero then the third one must be zero. Here the non-collinearity assumptions are needed. For example, if \(X, Y, Z\) are collinear then \(R, S, T\) are also collinear because they lie on the same line as \(X, Y, Z\), but the lines \(U + X, V + Y, W + Z\) are free to not concur.

With or without any of the assumptions of Pappus or Desargues, the formulas (P) and (D) express explicit numerical relationships holding among the six points and fifteen lines connecting them. The Theorems of Desargues and Pappus just deal with cases in which some of the determinants in the formulas are zero. But the formulas (P) and (D) hold all the time. They are in this sense strict generalizations of the Theorems of Pappus and Desargues for the projective planes arising from fields. For example, given six arbitrary points \(U, V, W, X, Y, Z\) in the projective plane over \(F\), we can create four triples of points, one of which is \(O, P, Q\) from Theorem (P) and the other three are
\[
(Z + V) \cap (U + W), \quad (Y + U) \cap (X + Z), \quad (W + X) \cap (V + Y), \\
(Y + U) \cap (W + V), \quad (X + W) \cap (Z + Y), \quad (V + Z) \cap (U + X), \\
(Z + W) \cap (V + X), \quad (U + V) \cap (Y + Z), \quad (X + Y) \cap (W + U),
\]
with the property that homogeneous coordinates for these triples all have the same determinant, by (P). Therefore either all four triples are collinear or none of them are collinear. Pappus’s Theorem says only that if \(U, V, W\) and \(X, Y, Z\) are collinear then \(O, P, Q\) and the other three triples are collinear.
Appendix: Proof of Theorem \[1\]

Here are the properties of determinants and cross products used in the proofs of formulas \[\text{(P)}\] and \[\text{(D)}\].

Cross products are perpendicular to their factors:

\[(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{x} = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 0\]

The scalar triple product (box product) formula:

\[(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) \mathbf{y} - (\mathbf{y} \cdot \mathbf{z}) \mathbf{x}\]

Determinants with proportional inputs are zero:

\[\det[\mathbf{x}, \mathbf{y}, r\mathbf{z}] = 0\]

Determinants are invariant under cyclic permutations:

\[\det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \det[\mathbf{y}, \mathbf{z}, \mathbf{x}] = \det[\mathbf{z}, \mathbf{x}, \mathbf{y}]\]

Switching determinant inputs changes sign:

\[\det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = -\det[\mathbf{y}, \mathbf{z}, \mathbf{x}] = -\det[\mathbf{z}, \mathbf{x}, \mathbf{y}] = -\det[\mathbf{y}, \mathbf{x}, \mathbf{z}]\]

Determinants are unchanged by transposition:

\[\det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \det[\mathbf{x}, \mathbf{y}, \mathbf{z}]\]

Scalars move in and out of determinants:

\[r\det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \det[r\mathbf{x}, \mathbf{y}, \mathbf{z}] = \det[\mathbf{x}, r\mathbf{y}, \mathbf{z}] = \det[\mathbf{x}, \mathbf{y}, rz]\]

Determinants distribute over vector addition:

\[\det[\mathbf{w} + \mathbf{x}, \mathbf{y}, \mathbf{z}] = \det[\mathbf{w}, \mathbf{y}, \mathbf{z}] + \det[\mathbf{x}, \mathbf{y}, \mathbf{z}]\]

A product of determinants is the determinant of inner products:

\[\det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \det[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \begin{vmatrix} \langle \mathbf{u}, \mathbf{x} \rangle & \langle \mathbf{u}, \mathbf{y} \rangle & \langle \mathbf{u}, \mathbf{z} \rangle \\ \langle \mathbf{v}, \mathbf{x} \rangle & \langle \mathbf{v}, \mathbf{y} \rangle & \langle \mathbf{v}, \mathbf{z} \rangle \\ \langle \mathbf{w}, \mathbf{x} \rangle & \langle \mathbf{w}, \mathbf{y} \rangle & \langle \mathbf{w}, \mathbf{z} \rangle \end{vmatrix}\]

The vector triple product formula:

\[(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) \mathbf{y} - (\mathbf{y} \cdot \mathbf{z}) \mathbf{x}\]

The vector quadruple product formula:

\[(\mathbf{w} \times \mathbf{x}) \times (\mathbf{y} \times \mathbf{z}) = \det[\mathbf{z}, \mathbf{w}, \mathbf{y}] \mathbf{x} + \det[\mathbf{y}, \mathbf{x}, \mathbf{z}] \mathbf{w}\]

Proof of the vector quadruple product formula:

\[(\mathbf{w} \times \mathbf{x}) \times (\mathbf{y} \times \mathbf{z}) = \langle \mathbf{w}, \mathbf{y} \times \mathbf{z} \rangle \mathbf{x} - \langle \mathbf{w}, \mathbf{y} \times \mathbf{z} \rangle \mathbf{y} \times \mathbf{z} \]

by \[\text{(12)}\]

\[= \det[\mathbf{w}, \mathbf{y}, \mathbf{z}] \mathbf{x} - \det[\mathbf{x}, \mathbf{y}, \mathbf{z}] \mathbf{w} \]

by \[\text{(8)}\]

\[= \det[\mathbf{z}, \mathbf{w}, \mathbf{y}] \mathbf{x} + \det[\mathbf{y}, \mathbf{x}, \mathbf{z}] \mathbf{w} \]

by \[\text{(9)}\], \[\text{(7)}\]
Proof of formula (D). Define three vectors and use the vector quadruple product formula (13) to rewrite them.

\[ p = (u \times y) \times (x \times v) = \det[v, u, x]y + \det[x, y, v]u, \]
\[ q = (w \times x) \times (z \times u) = \det[u, w, z]x + \det[z, x, u]w, \]
\[ r = (v \times x) \times (y \times w) = \det[w, v, y]z + \det[y, z, w]v. \]

Then \( \det[p, q, r] \), the left side of (P), is the sum of eight determinants by the distributive law (10). Move all the scalars (the determinants) out by (9), obtaining another sum of eight terms. Six of these cancel with others as indicated by the notations to the right of the affected terms. What remains are the two terms on the right side of (P).

\[ \det[p, q, r] = \det[\det[v, u, x]y + \det[x, y, v]u, \]
\[ \quad \det[u, w, z]x + \det[z, x, u]w], \]
\[ = \det[\det[v, u, x]y, \det[u, w, z]x, \det[w, v, y]z] + \]
\[ \quad + \det[\det[v, u, x]y, \det[u, w, z]x, \det[w, v, y]z] + \]
\[ \quad + \det[\det[x, y, v]u, \det[u, w, z]x, \det[w, v, y]z] + \]
\[ \quad + \det[\det[x, y, v]u, \det[z, x, u]w, \det[w, v, y]z] + \]
\[ \quad + \det[\det[x, y, v]u, \det[z, x, u]w, \det[y, z, w]v] + \]
\[ \quad + \det[\det[v, u, x]y, \det[z, x, u]w, \det[y, z, w]v] + \]
\[ \quad + \det[\det[v, u, x]y, \det[z, x, u]w, \det[y, z, w]v] = \]
\[ r_1 = -r_5 \text{ by (6), (7)}\]
\[ r_2 = -r_3 \text{ by (6), (7)}\]
\[ r_3 \]
\[ r_4 = -r_6 \text{ by (7)}\]
\[ r_5 \]
\[ r_6 \]
\[ = \det[v, u, x] \det[u, w, z] \det[w, v, y] \det[y, x, z] + \]
\[ r_5 \]
\[ r_6 \]

Proof of formula (D). Define three vectors and use the vector quadruple product formula (13).

\[ p = (w \times u) \times (z \times x) = \det[x, w, z]u + \det[z, u, x]w, \]
\[ q = (u \times v) \times (x \times y) = \det[y, u, x]v + \det[x, v, y]u, \]
\[ r = (v \times w) \times (y \times z) = \det[z, v, y]w + \det[y, w, z]v. \]
Then \( \det[p, q, r] \), the left side of (11), is the sum of eight determinants by (10), six of which are zero by (5) because two of the argument vectors are proportional.

\[
\begin{align*}
\det[p, q, r] &= \det[y, u, x]v + \det[x, v, y]u, \\
\det[z, v, y]w + \det[y, w, z]v \\
= & \det[\det[x, w, z]u + \det[z, u, x]w, \\
& \det[y, u, x]v + \det[x, v, y]u, \\
& \det[z, v, y]w + \det[y, w, z]v \\
= & 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
& 0 \text{ by (5)} \\
= & \det[x, w, z]\det[y, u, x]\det[z, v, y]\det[u, v, w] \\
& + \det[z, u, x]\det[x, v, y]\det[y, w, z]\det[u, v, w] \\
& \text{by (9)} \\
= & (\det[x, w, z]\det[y, u, x]\det[z, v, y] \det[u, v, w] \\
& + \det[z, u, x]\det[x, v, y]\det[y, w, z] \det[u, v, w] \\
& \text{by (5)} \\
= & \begin{vmatrix}
0 & \det[x, v, y] & \det[x, w, z] \\
\det[y, u, x] & 0 & \det[y, w, z] \\
\det[z, u, x] & \det[z, v, y] & 0
\end{vmatrix} \det[u, v, w] \\
& \text{by definition of } \det \\
= & \begin{vmatrix}
x, u \times x & x, v \times y & x, w \times z \\
y, u \times x & y, v \times y & y, w \times z \\
z, u \times x & z, v \times y & z, w \times z
\end{vmatrix} \\
& \det[u, v, w] \\
& \text{by (1) and (3)} \\
= & \begin{vmatrix}
x \\
y \\
z
\end{vmatrix} \det[u \times x, v \times y, w \times z] \det[u, v, w] \\
& \text{by (11)} \\
= & \det[x, y, z] \det[u \times x, v \times y, w \times z] \det[u, v, w] \\
& \text{by (5)}
\end{align*}
\]

References

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