Details for "Least-Squares Prices of Games"

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Abstract

This paper is intended to help the readers to understand the article:
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Remark to Theorem 1.1. To understand Theorem 1.1, it is useful to run the following Mathematica program. For any positive values of $a$, $b$, and $r$, the theoretical growth rate $E^r$ and the simulated growth rate (“geometric mean”) $x^{(1/\text{Repeat})}$ are almost equal.

```mathematica
a=19; b=1; r=0.05; Print["theoretical growth rate = ",E^r];
EA=(a+b)/2;k=(1-Sqrt[1-1/E^(2r)])/2;
If[EA>Sqrt[a*b]*E^r,uA=k*a+(1-k)*b;t=uA(E^r-uA)/((a-uA)(uA-b))];
x=1;Repeat=100000;
Do[If[Random[]<0.5,x=x*t*a/uA+x*(1-t),x=x*t*b/uA+x*(1-t)],{n,1,Repeat}];
Print["simulated growth rate = ",x^(1/Repeat)];
```

Proof of Theorem 1.1. From Remark 3.1, in the case where

$$
\frac{1}{e^r} \exp(\int \log a(x) dF(x)) = \frac{1}{e^r} \exp\left(\frac{\log a + \log b}{2}\right) = \frac{\sqrt{ab}}{e^r}
$$

that is, in the case where $(a + b)/(2\sqrt{ab}) = E/\sqrt{ab} \leq e^r$, the price is given by $\exp(\int \log a(x) dF(x))/e^r = \sqrt{ab}/e^r$, and the optimal proportion of investment is 1. Otherwise, the price $u > 0$ and the optimal proportion of investment $t_u > 0$ are determined by the simultaneous equations

$$
\begin{aligned}
\exp(\int \log(\frac{a(x)u}{a(x)t_u-t_u+u}) dF(x)) &= \frac{\sqrt{(at_u-ut_u+u)(bt_u-ut_u+u)}}{u} = e^r, \\
\int \frac{a(x)u}{a(x)t_u-t_u+u} dF(x) &= \frac{a-u}{2(at_u-ut_u+u)} + \frac{b-u}{2(bt_u-ut_u+u)} = 0.
\end{aligned}
$$

It is not difficult to verify that the solutions are given by $u = \kappa a + (1 - \kappa)b$ and $t_u = u(E-u)/(a-u(u-b))$, where $\kappa := (1 - \sqrt{1-1/e^{2r}})/2$.

Lemma D.1. $u_r^A = ku_r^A$ for $k > 0$.

Proof. From Remark 3.1, in the case where $\exp(\int \log a(x) dF(x))/e^r \leq 1/\int 1/a(x) dF(x)$, we have $\exp(\int \log(ka(x)) dF(x))/e^r \leq 1/\int 1/(ka(x)) dF(x)$ and $u_r^A = \exp(\int \log(ka(x)) dF(x))/e^r = k \exp(\int \log a(x) dF(x))/e^r = ku_r^A$. In the other case, as $a(x)/u = ka(x)/(ku)$...
Lemma D.2. \( T \) is convex.

Proof. By definition, for each \((t_i)\) and \((t'_i)\) in \( T \) we have
\[
\frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + t_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} \leq 1 \quad \text{and} \quad \frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + t'_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} \leq 1
\]
for each \((p_i)\) \in \( Q \). Thus,
\[
\frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + (qt_i + (1-q)t'_i)(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} \leq 1 \quad (0 \leq q \leq 1),
\]
which implies the conclusion. \qed

Lemma D.3. \( T \) is closed.

Proof. Put
\[
f_{(p_i)}((t_i)) := \frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + t_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})},
\]
then, as \( \sum_{i=1}^{n} p_i u_{r_i}^{A_i} \geq \min_{1 \leq i \leq n} u_{r_i}^{A_i} > 0 \), \( f_{(p_i)}((t_i)) \) is continuous with respect to \((t_i) \in S\). Therefore, \( \{(t_i) \in S : f_{(p_i)}((t_i)) \leq 1\} \) is closed in \( S \) for each \((p_i) \in Q \). Thus, \( \cap_{(p_i) \in Q} \{(t_i) \in S : f_{(p_i)}((t_i)) \leq 1\} = \{(t_i) \in S : L((t_i)) \leq 1\} = T \) is closed. \qed

Remark to Definition 2.1. In Definition 2.1, we can write
\[
L((t_i)) := \max_{(p_i) \in Q} \frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + t_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} \quad ((t_i) \in S),
\]
because \( \sum_{i=1}^{n} p_i u_{r_i}^{A_i} \) is continuous with respect to \((p_i) \in Q \) (see Theorem D.19). Moreover, by Berge’s maximum theorem [8, Theorem 2.1], \( L((t_i)) \) is continuous with respect to \((t_i) \in S\).

Lemma D.4. \( L((x_i)) = 1 \) for \( u_{r_i}^{A_i} : \Omega = u_{r_i}^{A_i} + x_i(E_{A_i}/e^r - u_{r_i}^{A_i}) \) \( (0 \leq i \leq 1) \).

Proof. From the continuity of \( \sum_{i=1}^{n} p_i u_{r_i}^{A_i} \), \( f_{(p_i)}((t_i)) = \frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + t_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} \) is uniformly continuous with respect to \((((t_i),(p_i)))\) on the compact set \( S \times Q \). Assume \( L((x_i)) < 1 \) and choose \((q_i) \in Q\) such that
\[
L((x_i)) = \max_{(p_i) \in Q} \frac{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i} + x_i(E_{A_i}/e^r - u_{r_i}^{A_i}))}{\sum_{i=1}^{n} p_i (u_{r_i}^{A_i})} = \frac{\sum_{i=1}^{n} q_i u_{r_i}^{A_i}}{\sum_{i=1}^{n} q_i u_{r_i}^{A_i}} < 1.
\]
If \( x_j > 0 \) exists, then there is a \( 0 < \varepsilon < 1 \) such that \( L((x'_j)) < 1 \), where \( x'_j = \varepsilon x_j \), \( x'_i = x_i \) \((i \neq j)\) and \( \sum_{i=1}^{n} (x'_i)^2 < \sum_{i=1}^{n} x_i^2 \), which is a contradiction. On the other hand, if \( x_i = 0 \) for each \( 0 \leq i \leq 1 \), then \( L((0)) < 1 \), which is also a contradiction. \( \square \)

Notation D.5. For two games \( A = (a(x), dF(x)) \) and \( B = (b(x), dF(x)) \), we use the following notation:
\[
f(p) := \exp(\int \log(pa(x) + (1-p)b(x))dF(x))/e^r \quad (0 \leq p \leq 1).
\]
Lemma D.6. $f(p) \leq g(p)$, if $g(p)$ exists.

Proof. As the function $\exp(\int \log((pa(x) + (1 - p)b(x))t/u - t + 1)dF(x))$ is concave with respect to $t$ (see [3, Lemma 4.7]), it reaches its maximum at $t = t_u$. Therefore, we have

$$e^r = \exp(\int \log(\frac{pa(x) + (1 - p)b(x)}{g(p)})t_{g(p)} - t_{g(p)} + 1)dF(x)) \geq \frac{\exp(\int \log(pa(x) + (1 - p)b(x))dF(x))}{g(p)}.$$ 

On the other hand, we have $e^r = \exp(\int \log(pa(x) + (1 - p)b(x))dF(x))/f(p)$. Therefore, $1/f(p) \geq 1/g(p)$, which implies the conclusion. \hfill \Box

Lemma D.7. The following four properties are equivalent at a point $p \in [0, 1]$.

1. $f(p) = g(p) = h(p)$.
2. $f(p) = g(p)$.
3. $f(p) = h(p)$.
4. $g(p) = h(p)$.

Proof. $3 \implies 2$. We write $c(x) := pa(x) + (1 - p)b(x)$. From $f(p) = h(p)$, we have

$$\frac{\exp(\int \log(c(x))dF(x))}{e^r} = \frac{1}{\int c(x)dF(x)}.$$ 

Write $u$ for this value and put $t_u := 1$. Then, we obtain

$$\begin{cases} 
\exp(\int \log(\frac{c(x)}{u}t_u - t_u + 1)dF(x)) = \exp(\int \log \frac{c(x)}{u}dF(x)) = e^r, \\
\int \frac{c(x) - u}{c(x)}dF(x) = 1 - u \int \frac{1}{c(x)}dF(x) = 0.
\end{cases}$$

Therefore, by the uniqueness of the solutions (see [3, Section 6]), we have $u = g(p)$.

$(4) \implies (2)$. Put $u := g(p)$ and $H := 1/h(p)$. Then, $u = h(p)$ implies $u = 1/H$. From [3, Lemmas 4.12, 4.16, and 4.21], we obtain $e^r = H \exp(\int \log c(x)dF(x))$, which implies $h(p) = f(p)$.

The other cases can be obtained in a similar fashion. \hfill \Box

Lemma D.8. $f(p)$ is concave on $[0, 1]$. 

$g(p)$ is defined by $u$ of the simultaneous equations:

$$\begin{cases} 
\exp(\int \log(\frac{pa(x) + (1 - p)b(x)}{u})t_u - t_u + 1)dF(x)) = e^r, \\
\int \frac{pa(x) + (1 - p)b(x)}{u} - u + u)dF(x) = 0 \quad (0 \leq p \leq 1).
\end{cases}$$

$h(p) := 1/(\int 1/(pa(x) + (1 - p)b(x))dF(x)) \quad (0 \leq p \leq 1)$.

$$u(p) := u_{pA + (1 - p)B} = \begin{cases} 
f(p) \text{ if } f(p) \leq h(p), \\
g(p) \text{ if } f(p) > h(p) \quad (0 \leq p \leq 1).
\end{cases}$$
Proof. Let \( \{p, q, \lambda\} \subset [0, 1] \). By the fact that \( \lambda \exp(\int \log a(x)dF(x)) = \exp(\int \log(\lambda a(x))dF(x)) \) and using [2, Theorem 185], we obtain
\[
\lambda f(p) + (1 - \lambda) f(q) = \left( \exp(\int \log(\lambda p a(x) + (1 - p)b(x))dF(x)) + \exp(\int \log((1 - \lambda) q a(x) + (1 - \lambda)(1 - q)b(x))dF(x)) \right) e^r \leq \frac{\exp(\int \log((\lambda p + (1 - \lambda)q)a(x) + (1 - (\lambda p + (1 - \lambda)q)b(x))dF(x))}{e^r} = f(\lambda p + (1 - \lambda)q).
\]
Thus, we have the conclusion. \(\square\)

Lemma D.9. \( g(p) \) is concave on \([0, 1]\) if \( g(p) \) exists for each \( p \in [0, 1] \).

Proof. Let \( \{p, q, \lambda\} \subset [0, 1], C := (c(x), dF(x)) \), and \( D := (d(x), dF(x)) \), where \( c(x) := pa(x) + (1 - p)b(x) \) and \( d(x) := qa(x) + (1 - q)b(x) \). Notice that
\[
e^r = \exp(\int \log \left( \frac{c(x)}{u^C} t_C - t_C + 1 \right)dF(x)),
\]
\[
e^r = \exp(\int \log \left( \frac{d(x)}{u^D} t_D - t_D + 1 \right)dF(x)),
\]
where \( u^C := g(p) \), \( u^D := g(q) \), \( t_C := t_{u^C} \), and \( t_D := t_{u^D} \). Be careful that \( u^C \in \{f(p), g(p)\} \) is not necessarily equal to \( u^C \). Put \( \mu := \lambda/(\lambda + (1 - \lambda)t_C u^D/(t_D u^C)) \) and \( \hat{t} := \mu t_C + (1 - \mu) t_D \). Then, using [2, Theorem 185], we have
\[
e^r = \mu \exp(\int \log \left( \frac{c(x)}{u^C} t_C - t_C + 1 \right)dF(x))
\]
\[
+ (1 - \mu) \exp(\int \log \left( \frac{d(x)}{u^D} t_D - t_D + 1 \right)dF(x))
\]
\[
= \exp(\int \log \left( \frac{\mu c(x)}{u^C} t_C - \mu t_C + \mu \right)dF(x))
\]
\[
+ \exp(\int \log \left( \frac{(1 - \mu)d(x)}{u^D} t_D - (1 - \mu)t_D + (1 - \mu) \right)dF(x))
\]
\[
\leq \exp(\int \log \left( \frac{\mu c(x)}{u^C} t_C + (1 - \mu)d(x) \right)t_D - (\mu t_C + (1 - \mu)t_D + 1) \right)dF(x))
\]
\[
= \exp(\int \log \left( \frac{\lambda c(x) + (1 - \lambda)d(x)}{\lambda u^C + (1 - \lambda) u^D} \right) \hat{t} - \hat{t} + 1 \right)dF(x)).
\]

On the other hand, we have
\[
e^r = \exp(\int \log \left( \frac{\lambda c(x) + (1 - \lambda)d(x)}{u^{\lambda C + (1 - \lambda)D}} \right) t_{u^{\lambda C + (1 - \lambda)D}} - t_{u^{\lambda C + (1 - \lambda)D}} + 1) \right)dF(x))
\]
\[
\geq \exp(\int \log \left( \frac{\lambda c(x) + (1 - \lambda)d(x)}{u^{\lambda C + (1 - \lambda)D}} \right) \hat{t} - \hat{t} + 1 \right)dF(x)).
\]

Therefore,
\[
\exp(\int \log \left( \frac{\lambda c(x) + (1 - \lambda)d(x)}{\lambda u^C + (1 - \lambda) u^D} \right) \hat{t} - \hat{t} + 1 \right)dF(x))
\]
\[
\leq \exp(\int \log \left( \frac{\lambda c(x) + (1 - \lambda)d(x)}{\lambda u^C + (1 - \lambda) u^D} \right) \hat{t} - \hat{t} + 1 \right)dF(x)),
\]
Proof. As the function \((1 - \lambda)u + \lambda g\), which implies \(g(\lambda p + (1 - \lambda)q) \geq \lambda g(p) + (1 - \lambda)g(q)\). \(\square\)

**Lemma D.10.** \(h(p)\) is concave on \([0, 1]\).

*Proof.* Let \(\{p, q, \lambda\} \in [0, 1]\). Then, using [2, Theorem 214], we obtain

\[
\lambda h(p) + (1 - \lambda)h(q) = \frac{\lambda}{\int_{po(x)+(1-p)b(x)} dF(x)} + \frac{1 - \lambda}{\int_{qa(x)+(1-q)b(x)} dF(x)}
\]

\[
= \frac{1}{\int_{\lambda po(x)+(1-p)b(x)} dF(x)} + \frac{1}{\int_{(1-\lambda)qa(x)+(1-q)b(x)} dF(x)}
\]

\[
\leq \frac{1}{\int_{\lambda po(x)+(1-p)b(x)} dF(x)} + \frac{1}{\int_{(1-\lambda)qa(x)+(1-q)b(x)} dF(x)} = h(\lambda p + (1 - \lambda)q).
\]

\(\square\)

**Lemma D.11.** \(f(p)\) is continuous on \([0, 1]\).

*Proof.* As \(f(p)\) is concave, it is continuous with respect to \(0 < p < 1\) (See [7, Theorem 10.3]). It is sufficient to prove the assertion in the case where \(p \to 1^-\). As the function \((1 - p)/p\) is strictly decreasing with respect to \(p \in (0, 1)\), using Lebesgue’s theorem, we obtain

\[
\lim_{p \to 1^-} f(p) = \frac{1}{e^r} \lim_{p \to 1^-} p \exp\left(\int \log(a(x) + \frac{1 - p}{p} b(x)) dF(x)\right) = \frac{1}{e^r} \exp\left(\int \log a(x) dF(x)\right) = f(1),
\]

where \(b(x) \geq 0\). \(\square\)

**Lemma D.12.** \(h(p)\) is continuous on \([0, 1]\).

*Proof.* It is not difficult to verify that \(0 \leq b(p) < pE^A + (1 - p)E^B < \infty\). Similar to case of Lemma D.11, we obtain the conclusion. \(\square\)

**Lemma D.13.** Let \(\alpha(x, y)\) be continuous with respect to \((x, y) \in (-\delta, \delta)^n \times (0, \delta)\) for some positive number \(\delta > 0\), and nondecreasing with respect to \(y \in (0, \delta)\) for each \(x \in (-\delta, \delta)^n\). Let \(\beta(x)\) be continuous with respect to \(x \in (-\delta, \delta)^n\), satisfying \(\lim_{y \to 0^+} \alpha(x, y) = \beta(x)\) for each \(x \in (-\delta, \delta)^n\). Then, \(\alpha(x, y)\) has a unique continuous extension on \((-\delta, \delta)^n \times (0, \delta)\).

*Proof.* Define \(\alpha(x, 0) := \beta(x)\ (x \in (-\delta, \delta)^n)\). It is sufficient to prove that \(\alpha(x, y)\) is continuous at \((0, 0)\). Choose \(\varepsilon > 0\). (1) As \(\beta(x)\) is continuous at \(x = (0, 0)\), \(0 < \delta_1 < \delta\) exists such that \(|\beta(x) - \beta((0))| < \varepsilon\) if \(x \in (-\delta_1, \delta_1)^n\). (2) As \(\lim_{y \to 0^+} \alpha((0), y) = \beta((0))\), \(0 < \delta_2 < \delta_1\) exists such that \(|\alpha((0), y) - \beta((0))| < \varepsilon\) if \(0 < y < \delta_2\). (3) As \(\alpha(x, y)\) is continuous at \((0, \delta_2/2)\), \(0 < \delta_3 < \delta_2/2\) exists such that \(|\alpha(x, y) - \alpha((0), \delta_2/2)| < \varepsilon\) if \(x \in (-\delta_3, \delta_3)^n\) and \(|y - \delta_2/2| < \delta_3\). Therefore, for each \((x', y')\) such that \(x' \in (-\delta_3, \delta_3)^n\) and \(0 \leq y' < \delta_2/2\), we have \(\beta(x') \leq \alpha(x', y') \leq \alpha(x', \delta_2/2)\). Thus, \(\alpha(x', y') - \alpha((0), 0) \geq \beta(x') - \beta((0)) > -\varepsilon\). Moreover, \(\alpha(x', y') - \alpha((0), 0) \leq \alpha(x', \delta_2/2) - \beta((0)) \leq \alpha(x', \delta_2/2) - \alpha((0), \delta_2/2) + \alpha((0), \delta_2/2) - \beta((0)) < 2\varepsilon\). Hence, we have the conclusion. \(\square\)

**Remark to Lemma D.13.** Lemma D.13 is valid if the condition \((-\delta, \delta)^n \times (0, \delta)\) is replaced by \([0, \delta)^n \times (0, \delta)\) and/or the term “nondecreasing” is replaced by “nondecreasing or nonincreasing.”
Lemma D.14. Assume \( f(p) \geq h(p) \) for each \( p \in [0, 1] \) and choose \( L := \sup_{0<p<1} g(p) + 1 \). Then the function
\[
V_t(p, t, u) := \int \frac{pa(x) + (1-p)b(x) - u}{(pa(x) + (1-p)b(x)) t - ut + u} dF(x)
\]
is upper and lower semicontinuous on \( D = \{(p, t, u) : 0 \leq p \leq 1, 0 \leq t \leq 1, \text{ and } f(p) \leq u \leq L\} \). Moreover, \( V_t(p, t, u) = -\infty \) if and only if \( t = 1 \) and \( h(p) = 0 \).

Proof. As \( f(p) \geq h(p) \), \( g(p) \) exists such that \( g(p) \leq pE^A + (1-p)E^B \). Put \( c(x) := pa(x) + (1-p)b(x) \), then \( V_t(p, t, u) = \int (c(x) - u)/(c(x)t - ut + u) dF(x) \).

From Hartogs’ theorem, \( V_t(p, t, u) \) is analytic in \( D := \{(p, t, u) : 0 < p < 1, 0 < t < 1, \text{ and } f(p) < u < L\} \) (see [3, Lemma 3.1]).

First, assume that \( h(p) > 0 \) for each \( p \in [0, 1] \). Put \( U(x, p, t, u) := (c(x) - u)/(c(x)t - ut + u) \), then \( V_t(p, t, u) = \int U(x, p, t, u) dF(x) \).

(1) From
\[
\frac{\partial U}{\partial t}(x, p, t, u) = -\left(\frac{c(x) - u}{c(x)t - ut + u}\right)^2 \leq 0,
\]
we can use Lebesgue’s monotone theorem to obtain
\[
\lim_{t \to 0^+} V_t(p, t, u) = V_t(p, 0, u) = \frac{pE^A + (1-p)E^B}{u} - 1,
\]
\[
\lim_{t \to 1^-} V_t(p, t, u) = V_t(p, 1, u) = 1 - \frac{u}{h(p)}.
\]
Notice that \( V_t(p, 0, u) \) and \( V_t(p, 1, u) \) are analytic in \( \{(p, u) : 0 < p < 1 \text{ and } f(p) < u < L\} \) (see [3, Lemma 3.1]). Moreover, as \( f(p) > 0 \) and \( h(p) > 0 \), we have the following properties:
\[
\lim_{p \to 0^+} V_t(p, 0, u) = V_t(0, 0, u), \quad \lim_{p \to 0^+} V_t(p, 1, u) = V_t(0, 1, u),
\]
\[
\lim_{p \to 1^-} V_t(p, 0, u) = V_t(1, 0, u), \quad \lim_{p \to 1^-} V_t(p, 1, u) = V_t(1, 1, u).
\]
To obtain these equalities, we have used the inequalities \( \partial^2 U(x, p, t, u)/\partial p^2 \leq 0 \) and \( \partial U(x, p, t, u)/\partial u \leq 0 \), which will be shown in (2) and (3).

(2) From
\[
\frac{\partial^2 U}{\partial p^2}(x, p, t, u) = -\frac{2ut (a(x) - b(x))^2}{(c(x)t - ut + u)^3} \leq 0,
\]
\( U(x, p, t, u) \) is concave with respect to \( p \in (0, 1) \). Therefore, \( U(x, p, t, u) \) is nondecreasing or nonincreasing on \( (0, \varepsilon) \) for some positive \( \varepsilon \). Therefore, using Lebesgue’s monotone theorem, we obtain
\[
\lim_{p \to 0^+} V_t(p, t, u) = V_t(0, t, u) = \int \frac{b(x) - u}{b(x)t - ut + u} dF(x),
\]
\[
\lim_{p \to 1^-} V_t(p, t, u) = V_t(1, t, u) = \int \frac{a(x) - u}{a(x)t - ut + u} dF(x).
\]
Notice that \( V_t(0, t, u) \) and \( V_t(1, t, u) \) are analytic in \( \{(t, u) : 0 < t < 1 \text{ and } f(p) < u < L\} \) (see [3, Lemma 3.1]). Moreover, as \( f(p) > 0 \) and \( h(p) > 0 \), we have the
following properties:

\[
\begin{align*}
\lim_{t \to 0^+} V_t(0, t, u) &= V_t(0, 0, u), & \lim_{t \to 0^+} V_t(1, t, u) &= V_t(1, 0, u), \\
\lim_{t \to 1^-} V_t(0, t, u) &= V_t(0, 1, u), & \lim_{t \to 1^-} V_t(1, t, u) &= V_t(1, 1, u), \\
\lim_{u \to f(0)^-} V_t(0, t, u) &= V_t(0, t, f(0)), & \lim_{u \to f(p)^-} V_t(1, t, u) &= V_t(1, t, f(1)), \\
\lim_{u \to L^+} V_t(0, t, u) &= V_t(0, t, L), & \lim_{u \to L^+} V_t(1, t, u) &= V_t(1, t, L).
\end{align*}
\]

To obtain these equalities, we have used the inequality \( \partial U(x, p, t, u)/\partial u \leq 0 \), which will be shown in (3).

(3) By the inequality

\[
\frac{\partial U}{\partial u}(x, p, t, u) = -\frac{c(x)}{(c(x)t - ut + u)^2} \leq 0,
\]

we can use Lebesgue’s monotone theorem to obtain

\[
\begin{align*}
\lim_{u \to f(p)^+} V_t(p, t, u) &= V_t(p, t, f(p)), \\
\lim_{u \to L^-} V_t(p, t, u) &= V_t(p, t, L).
\end{align*}
\]

As \( f(p) \) is analytic in \( (0, 1) \), \( V_t(p, t, f(p)) \) and \( V_t(p, t, L) \) are analytic in \( \{(p, t) : 0 < p < 1 \text{ and } 0 < t < 1\} \) (see [3, Lemma 3.1]). Moreover, as \( f(p) > 0 \) and \( h(p) > 0 \), we have the following properties:

\[
\begin{align*}
\lim_{t \to 0^-} V_t(p, t, f(p)) &= V_t(p, 0, f(p)) = \frac{pE_A + (1 - p)E_B}{f(p)} - 1, \\
\lim_{t \to 0^+} V_t(p, t, f(p)) &= V_t(p, 1, f(p)) = 1 - \frac{f(p)}{h(p)}, \\
\lim_{p \to 0^+} V_t(p, t, L) &= V_t(0, t, L), \\
\lim_{p \to 1^-} V_t(p, t, L) &= V_t(1, t, L), \\
\lim_{t \to 0^-} V_t(p, t, L) &= V_t(p, 0, L) = \frac{pE_A + (1 - p)E_B}{L} - 1, \\
\lim_{t \to 1^+} V_t(p, t, L) &= V_t(p, 1, L) = 1 - \frac{L}{h(p)}.
\end{align*}
\]

By the relations

\[
V_t(p, t, f(p)) = \frac{1}{t} - \frac{1}{t} \int \frac{1}{\frac{c(x)}{f(p)t} + 1 - t} dF(x),
\]

and Lebesgue’s dominated theorem, we have

\[
\begin{align*}
\lim_{p \to 0^+} V_t(p, t, f(p)) &= V_t(0, t, f(0)), \\
\lim_{p \to 1^-} V_t(p, t, f(p)) &= V_t(1, t, f(1)).
\end{align*}
\]

However, these convergences are not necessarily monotonic. The continuity of \( V_t(p, t, f(p)) \) near the boundaries \( \{(p, t, u) : p = 0, 0 \leq t \leq 1, \text{ and } u = f(0)\} \) and \( \{(p, t, u) \)
Therefore, for each real number 0, thus, for each real number in $V$.

Consider the case where $\tilde{h}(p) = 0$ (0 ≤ $p$ ≤ 1), and put $L' := \min(f(0), f(1))$. As $\partial U(x, p, t, u)/\partial u$ ≤ 0, we obtain

$$V_i(p, t, u) = \frac{c(x) - u}{c(x)t - ut + u}dF(x) \leq \frac{c(x) - L'}{c(x)t - L't + L'}dF(x).$$

Put $W(p, t) := \int (c(x) - L')/ (c(x)t - L't + L')dF(x)$, then, as $\partial U(x, p, t, L')/\partial t$ ≤ 0, $W(p, t)$ is nonincreasing with respect to $t \in (0, 1)$ and $\lim_{t \rightarrow 1} W(p, t) = -\infty$. Thus, for each real number $m$ and $p \in [0, 1]$, $t_p > 0$ exists such that $1 - t_p < t' < 1$ implies $W(p, t') < m - 2$. As $W(p, t)$ is continuous at $(p, 1 - t_p/2)$, $0 < \delta_p < t_p/2$ exists such that the conditions $p - \delta_p < p' < p + \delta_p$, $0 \leq t \leq 1$, and $1 - t_p/2 - \delta_p < t' < 1 - t_p/2 + \delta_p$ imply $W(p, 1 - t_p/2 - 1 < W(p', t') < W(p, 1 - t_p/2) + 1$. It should be noted that the set of open intervals $\{(p - \delta_p, p + \delta_p) \cap [0, 1]\}_{0 \leq p \leq 1}$ is an open covering of the compact set $[0, 1]$. Therefore, a finite subcovering $\{(p_i - \delta_{p_i}, p_i + \delta_{p_i}) \cap [0, 1]\}_{i=1,2,...,m}$ exists. Put $\delta := \min_{i=1,2,...,m} \delta_{p_i}$, then for each $0 \leq p \leq 1$ and $1 - \delta < t'' < 1$, we have $W(p', t'') < m - 1$. It is not difficult to see that $V_i(p, t, u)$ is continuous on the compact set $K := \{(p, t, u) : 0 \leq p \leq 1, 0 \leq t \leq 1 - \delta, and f(p) \leq u \leq L\}$. Therefore, $\{(p, t, u) \in \overline{D} : V_i(p, t, u) \geq m\} \subset K \subset D$. This implies that $\{(p, t, u) : V_i(p, t, u) < m\}$ is open in $D$.

Consider the case where $h(p_0) > 0$ for some $p_0 \in [0, 1]$. As $h(p)$ is concave, the compact set $\{p : h(p) = 0\}$ is $\{1\}$, or $\{0, 1\}$. In each case, the upper semicontinuity can be proved in a similar fashion as above.

[3, Lemma 3.1] The function $w_{\beta}(z) := \int (a(x) - \beta)/(a(x)z - \beta + \beta)dF(x)$ is analytic with respect to two complex variables $z := t + si$ and $\beta := u + hi$ such that

(a) $\max(\varepsilon, \xi) < u < L$,
(b) $|h| < \varepsilon^6/(32(L + 1)R^2)$,
(c) $|z| < R$, and $z \notin \{z : |s| \leq \varepsilon\} \cap \{z : t \leq \varepsilon or t \geq u/ (u - \xi) - \varepsilon\}$,

where $0 < \varepsilon < \min\{1/2, 1/(2(u - \xi))\}$, $\max(\varepsilon, \xi) < L < +\infty$, $\max(2, u/(u - \xi)) < R < +\infty$, $i := \sqrt{-1}$, $\xi := \inf_{x} a(x)$, $\arg(z) = s$, and $\arg(\beta) = h$.

Lemma D.15. Under the assumption of Lemma D.14,

$$V(p, t, u) := \int \log\left(\frac{pa(x) + (1 - p)b(x)}{u}\right)\frac{1}{t - t + 1)dF(x),$$

is continuous on $\overline{D} = \{(p, t, u) : 0 \leq p \leq 1, 0 \leq t \leq 1, and f(p) \leq u \leq L\}$.

Proof. As $\partial V(p, t, u)/\partial t = V_i(p, t, u)$ on $D$ and $V(p, t, u) < 0$ on $\overline{D}$, using Lemmas D.13 and D.14, we have the conclusion.

Lemma D.16. $g(p)$ is continuous on $[0, 1]$ if $f(p) \geq h(p)$ for each $p \in [0, 1]$. 


Proof. From \( f(p) \geq h(p) \), \( g(p) \) exists such that \( g(p) \geq f(p) \geq h(p) \) (see Remark 3.1 and Lemma D.6). As \( g(p) \) is concave, it is continuous with respect to \( 0 < p < 1 \) (See [7, Theorem 10.3]), and so \( c := \lim_{p \to 1^-} g(p) \geq g(1) \) exists. It is sufficient to show that \( c = g(1) \).

By Lemmas D.14 and D.15, the set \( K := \{(p,t,u) \in \overline{D} : V(p,t,u) = r \text{ and } V_t(p,t,u) = 0\} \) is compact. As \( c = \lim_{p \to 1^-} g(p) \), a strictly increasing sequence \( \{p_n\} \) exists such that \( \lim_{n \to \infty} p_n = 1 \) and \( \lim_{n \to \infty} g(p_n) = c \). As the sequence \( \{(p_n, t_g(p_n), g(p_n))\} \) is in the compact set \( K \), a subsequence \( \{p'_n\} \subset \{p_n\} \) exists such that \( t_* := \lim_{n \to \infty} t_g(p'_n) \) and \( (1, t_*, c) \in K \). By the uniqueness of the solutions of the simultaneous equations (see Remark 3.1 and [3, Section 6]), we obtain \( c = g(1) \). \( \square \)

**Lemma D.17.** \( u(p) \) is concave on \([0, 1] \).

**Proof.** Let \( \{p, q, \lambda\} \subset [0, 1], p \leq q, \) and \( r := \lambda p + (1 - \lambda)q \). We will show that \( \lambda u(p) + (1 - \lambda)u(q) \leq u(r) \). From Lemmas D.6 and D.9, we have \( \lambda u(p) + (1 - \lambda)u(q) \leq \lambda g(p) + (1 - \lambda)g(q) \leq g(r) \). Therefore, if \( u(r) = g(r) \), then the assertion is proved. Henceforth, we assume that \( u(r) = f(r) < g(r) \).

In the case where \( u(p) = f(p) \) and \( u(q) = f(q) \), the assertion follows from Lemma D.8. In the case where \( u(p) = g(p) > f(p) \) and \( u(q) = g(q) > f(q) \), there are \( p < z < r \) and \( r < w < q \) such that \( u(z) = f(z) = g(z) = h(z) \) and \( u(w) = f(w) = g(w) = h(w) \). As \( u(p) \) is concave, from \( p < z < r < w < q \), we have

\[
\lambda g(p) + (1 - \lambda)g(q) \leq \frac{w - r}{w - z} g(z) + \frac{r - z}{w - z} g(w)
= \frac{w - r}{w - z} f(z) + \frac{r - z}{w - z} f(w) \leq f(r),
\]

which implies that \( \lambda u(p) + (1 - \lambda)u(q) \leq u(r) \).

The other cases can be obtained in a similar fashion. \( \square \)

**Lemma D.18.** \( u(p) \) is continuous with respect to \( p \in [0, 1] \).

**Proof.** As \( u(p) \) is concave, it is continuous with respect to \( 0 < p < 1 \) (See [7, Theorem 10.3]). We will only show that \( \lim_{p \to 1^-} u(p) = u(1) \). In the case where \( h(1) > f(1) \), \( u(p) = f(p) \) in a neighborhood of 1. Therefore, the continuity of \( f(p) \) deduces \( \lim_{p \to 1^-} u(p) = \lim_{p \to 1^-} f(p) = f(1) = u(1) \). Similarly, in the case where \( h(1) < f(1) \), we have \( \lim_{p \to 1^-} u(p) = \lim_{p \to 1^-} g(p) = g(1) = u(1) \). In the case where \( h(1) = f(1) \), using Lemma D.7, we have \( h(1) = \liminf_{p \to 1^-} min(f(p), g(p)) \leq \liminf_{p \to 1^-} u(p) \leq \limsup_{p \to 1^-} u(p) \leq \limsup_{p \to 1^-} \max(f(p), g(p)) = h(1) \), which implies the conclusion. \( \square \)

**Theorem D.19.** \( \sum_{i=1}^{n} p_i A_i \) is continuous with respect to \( (p_i) \in Q \).

**Proof.** As \( u^A \) is finite and concave on \( Q \) (Lemma D.17), \( u^A \) is continuous on the relative interior of \( Q \) (see [7, Theorem 10.1]) and has a unique continuous extension on \( Q \) (see [7, Theorems 10.3 and 20.5]). Therefore, we need to show the relation \( \lim_{p \to 1^-} u^A + (1 - p) B = u^A \), where \( A \) or \( B \) is a relative boundary point or a relative interior point of \( Q \), respectively. In this instance, Lemma D.18 leads to the conclusion. \( \square \)
[3, Example 6.6] The European put option is given by
\[ a(x) = \max(K - Se^{rT}e^x, 0), \quad dF(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x+\frac{\sigma^2}{2}+\frac{\sigma^2}{2})^2}{2\sigma^2 T}} dx. \]

We assume that the stock price \( Y = Se^{rT}e^X \) is lognormally distributed with volatility \( \sigma \sqrt{T} \), where \( S \) is the current stock price, \( r \) is the continuously compounded interest rate, \( K \) is the exercise price of the put option, and \( T \) is the exercise period. The expectation \( E \) is given by
\[
E = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\log K} (K - Se^{rT}e^x) e^{-\frac{(x+\frac{\sigma^2}{2})^2}{2\sigma^2 T}} dx
= KN\left(-\frac{\log S}{\sigma \sqrt{T}} + \frac{r - \frac{\sigma^2}{2}}{\sigma \sqrt{T}}\right) - Se^{rT}N\left(-\frac{\log S}{\sigma \sqrt{T}} + \frac{r + \frac{\sigma^2}{2}}{\sigma \sqrt{T}}\right),
\]
where \( N(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} d\sqrt{2\pi} \) is the cumulative standard normal distribution function.

When \( S = 90, K = 120, T = 2, \sigma = 0.1, \) and \( r = 0.04, \) we have \( \xi = 0, E \approx 22.9848, \) and \( H := \int 1/a(x)dF(x) = +\infty. \) Therefore, from Theorems 4.1 and 5.1, \( G_u(t_u) (u \in (0, E)) \) strictly decreases from +\( \infty \) to 1. The equations \( w_u(t_u) = 0 \) and \( G_u(t_u) = e^{0.08} \) yield the price \( u \approx 17.8157. \) With this price, if investors continue to invest \( t_u \approx 0.5434 \) of their current capital, they can maximize the limit expectation of growth rate to \( e^{0.08} \approx 1.0833. \)

In general, the equation \( E/u = e^{rT} \) yields the price
\[
u = Ke^{-rT}N\left(-\frac{\log S}{\sigma \sqrt{T}} + \frac{r - \frac{\sigma^2}{2}}{\sigma \sqrt{T}}\right) - SN\left(-\frac{\log S}{\sigma \sqrt{T}} + \frac{r + \frac{\sigma^2}{2}}{\sigma \sqrt{T}}\right),
\]
which is the Black-Scholes formula for the European put option. Substituting the above mentioned values for this formula, we obtain the (higher) price \( u \approx 21.2176 \) (> 17.8157). With this price, if the investors continue to invest \( t_u \approx 0.2278 \) of their current capital, they can maximize the limit expectation of growth rate to \( 1.0096 \) (< 1.0833).

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