On the moduli space of isometric surfaces with the same mean curvature in 4-dimensional space forms

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Abstract

We study the moduli space of congruence classes of isometric surfaces with the same mean curvature in 4-dimensional space forms. Having the same mean curvature means that there exists a parallel vector bundle isometry between the normal bundles that preserves the mean curvature vector fields. We prove that if both Gauss lifts of a compact surface to the twistor bundle are not vertically harmonic, then there exist at most three nontrivial congruence classes. We show that surfaces with a vertically harmonic Gauss lift, allow locally a one-parameter family of isometric deformations with the same mean curvature. This family is trivial only if the surface is superconformal. For such compact surfaces with non-parallel mean curvature, we prove that the moduli space is the disjoint union of two sets, each one being either finite, or a circle. In particular, for surfaces in $\mathbb{R}^4$ we prove that the moduli space is a finite set, under a condition on the Euler numbers of the tangent and normal bundles.

1 Introduction

A basic problem in surface theory is to investigate which data are sufficient to determine a surface (up to congruence) in a complete simply-connected 3-dimensional space form $Q^3_c$ of curvature $c$. Bonnet’s fundamental theorem implies that two isometric surfaces in $Q^3_c$ are congruent if and only if their second fundamental forms coincide. Bonnet [5] raised the problem to what extent a surface is determined by the metric and the mean curvature. Generically, a surface in $Q^3_c$ is uniquely determined by these data. The exceptions are the Bonnet surfaces that include the constant mean curvature (CMC) surfaces.

There has been a lot of interest in the following natural problem: given an isometric immersion $f: M \to Q^3_c$ of a 2-dimensional Riemannian manifold $M$, how many noncongruent isometric immersions of $M$ into $Q^3_c$ can exist with the same mean curvature with

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This problem has been studied locally or globally by Bonnet [5], Cartan [6], Lawson [32], Hoffman [26], Tribuzy [38], Chern [12], Roussos-Hernandez [35] and Kenmotsu [30] among others. Lawson [32] proved that if $M$ is simply-connected and $f$ is a CMC surface in $\mathbb{Q}^3_c$, then the space of isometric immersions with the same mean curvature is the circle $S^1$, unless $f$ is totally-umbilical. If $M$ is not simply-connected, Smyth and Tinaglia [37] showed that this space is either $S^1$, or a finite set. In particular, it is a finite set if $M$ is compact (cf. [2, 3]). On the other hand, Chern [12] provided the classification of all simply-connected, umbilic-free surfaces in $\mathbb{R}^3$ with non-constant mean curvature, for which the space of isometric immersions with the same mean curvature is infinite. Lawson and Tribuzy [33], proved that a compact oriented 2-dimensional Riemannian manifold admits at most two noncongruent isometric immersions in $\mathbb{Q}^3_c$, with the same non-constant mean curvature. Interesting results have been obtained for surfaces in Riemannian homogeneous 3-manifolds with a 4-dimensional isometry group (cf. [1, 16, 21]).

For surfaces in 4-dimensional space forms, the fundamental theorem of submanifolds asserts that two isometric immersions $f, \tilde{f}: M \to \mathbb{Q}^4_c$ are congruent if they have the same second fundamental form. By the latter, we mean that there exists a parallel vector bundle isometry between their normal bundles that preserves the second fundamental forms. Inspired by Bonnet’s question for surfaces in $\mathbb{Q}^3_c$, we are interested in the following problem: given an isometric immersion $f: M \to \mathbb{Q}^4_c$ of a 2-dimensional Riemannian manifold $M$, how many noncongruent isometric immersions of $M$ into $\mathbb{Q}^4_c$ can exist with the same mean curvature with $f$? Two isometric immersions $f, \tilde{f}: M \to \mathbb{Q}^4_c$ are said to have the same mean curvature if there exists a parallel vector bundle isometry between their normal bundles that preserves the mean curvature vector fields.

The aim of the paper is to study the moduli space $M(f)$ of congruence classes of isometric immersions that have the same mean curvature with $f$. Our results are mostly global in nature. We show that the structure of the moduli space of a surface in $\mathbb{R}^4$ is controlled by the behavior of its Gauss map. Recall that the Grassmannian $Gr(2,4)$ of oriented 2-planes in $\mathbb{R}^4$, can be identified with the product $S^2_+ \times S^2_-$ of two spheres. Accordingly, given an isometric immersion $f: M \to \mathbb{R}^4$, its Gauss map $g: M \to Gr(2,4)$ decomposes into a pair of maps as $g = (g_+, g_-): M \to S^2_+ \times S^2_-$. Our first result is a Lawson-Tribuzy type theorem [33] for compact surfaces in $\mathbb{R}^4$.

**Theorem 1.** Let $f: M \to \mathbb{R}^4$ be an isometric immersion of a compact, oriented 2-dimensional Riemannian manifold $M$. If both components $g_+$ and $g_-$ of the Gauss map of $f$ are not harmonic, then there exist at most three nontrivial congruence classes of isometric immersions of $M$ into $\mathbb{R}^4$, that have the same mean curvature with $f$. In particular, there exists at most one nontrivial class, if $M$ is homeomorphic to $S^3$.

For surfaces in nonflat space forms, the structure of the moduli space is controlled by the behavior of the Gauss lifts $G_+: M \to \mathcal{Z}_+$ and $G_-: M \to \mathcal{Z}_-$. Here $\mathcal{Z}_+$ and $\mathcal{Z}_-$ stand for the two connected components of the twistor bundle $\mathcal{Z}$ of $\mathbb{Q}^4_c$, which is a Riemannian manifold.
Theorem 2. Let \( f : M \to \mathbb{Q}_c^4 \) be an isometric immersion of a compact, oriented 2-dimensional Riemannian manifold \( M \). If both Gauss lifts \( G_+ \) and \( G_- \) of \( f \) are not vertically harmonic, then there exist at most three nontrivial congruence classes of isometric immersions of \( M \) into \( \mathbb{Q}_c^4 \), that have the same mean curvature with \( f \). In particular, there exists at most one nontrivial class, if \( M \) is homeomorphic to \( \mathbb{S}^2 \).

In particular, compact surfaces in \( \mathbb{Q}_c^4 \) whose both Gauss lifts are not vertically harmonic, do not allow nontrivial global isometric deformations that preserve the mean curvature.

It is now interesting to study the case where at least one of the Gauss lifts is vertically harmonic. We recall that Ruh and Vilms [36], and later Jensen and Rigoli [28], proved that both Gauss lifts are vertically harmonic if and only if the mean curvature vector field is parallel in the normal connection. Such surfaces are either minimal, or they lie as CMC surfaces in a totally geodesic or totally umbilical hypersurface of the ambient space \( \mathbb{Q}_c^4 \) (cf. [10, 41]). Dajczer and Gromoll [13] proved that any simply-connected minimal surface admits a one-parameter associated family of isometric deformations through minimal surfaces. Furthermore, this family is trivial if and only if \( f \) is superconformal. In the next theorem, we extend this result to non-minimal surfaces in \( \mathbb{Q}_c^4 \) with a vertically harmonic Gauss lift. Indeed, we prove that such surfaces allow a one-parameter family of isometric deformations that preserve the mean curvature. It is worth noticing that this applies to certain classes of Lagrangian surfaces, studied by Castro and Urbano [8, 9] (we refer to Section 3.3 for details).

Theorem 3. Let \( f : M \to \mathbb{Q}_c^4 \) be a non-minimal isometric immersion of a 2-dimensional oriented and simply-connected Riemannian manifold \( M \). If the Gauss lift \( G_\pm \) of \( f \) is vertically harmonic, then:

(i) There exists a one-parameter family of isometric immersions \( f_\theta^\pm : M \to \mathbb{Q}_c^4, \theta \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}, \) which have the same mean curvature with \( f_0^\pm = f \).

(ii) If \( f \) is superconformal, then \( f_\theta^\pm \) is congruent to \( f \) for any \( \theta \).

(iii) If there exist \( \theta \neq \tilde{\theta} \in S^1 \) such that \( f_\theta^\pm \) is congruent to \( f_{\tilde{\theta}}^\pm \), then \( f \) is superconformal.

The above theorem indicates that surfaces with a vertically harmonic Gauss lift inherit some of the properties of CMC surfaces in 3-dimensional space forms. For instance, we prove in Section 4 that such surfaces satisfy Ricci-like conditions that extend the Ricci condition for CMC surfaces (cf. [32]). Furthermore, the above family can be viewed as an extension of the associated family of CMC surfaces.

The following result determines the structure of the moduli space of compact surfaces with a vertically harmonic Gauss lift. It was inspired by the recent work of Smyth and Tinaglia [37], and extends results in [2, 3].

Theorem 4. Let \( f : M \to \mathbb{Q}_c^4 \) be an isometric immersion of a compact, oriented 2-dimensional Riemannian manifold \( M \) with vertically harmonic Gauss lift \( G_\pm \).
(i) If the mean curvature vector field of \( f \) is non-parallel, then the moduli space \( \mathcal{M}(f) \) is the disjoint union of two sets, each one being either finite, or the circle \( S^1 \).

(ii) If \( c = 0 \) and the Euler numbers \( \chi \) and \( \chi_N \) of the tangent and normal bundles satisfy \( \chi \neq \mp \chi_N \), then \( \mathcal{M}(f) \) is a finite set.

The paper is organized as follows: In Section 2, we fix the notation and give some preliminaries. In Section 3, we discuss surfaces with the same mean curvature and assign to a pair of such surfaces a holomorphic differential which is called the distortion differential. The moduli space splits into disjoint components. We provide information for these components for compact surfaces and then give the proofs of Theorems 1 and 2. As an application of our results, we provide a short proof of Lawson-Tribuzy theorem and a recent result [24] for Lagrangian surfaces in \( \mathbb{R}^4 \). Moreover, we show that for compact superconformal surfaces there exists at most one nontrivial congruence class with the same mean curvature. Section 4 is devoted to surfaces with a vertically harmonic Gauss lift. We prove that such surfaces satisfy Ricci-like conditions and give the proof of Theorem 3. As a consequence of Theorem 3, we show that the moduli space of simply-connected surfaces with non-vanishing parallel mean curvature vector field is the torus \( S^1 \times S^1 \), and the two-parameter associated family coincides with the one given by Eschenburg-Tribuzy [19]. Finally, we give the proof of Theorem 4.

2 Preliminaries

Throughout the paper, \( M \) is a connected, oriented 2-dimensional Riemannian manifold. Let \( f: M \to \mathbb{Q}_c^4 \) be a surface, i.e., an isometric immersion into the complete simply-connected 4-dimensional space form of curvature \( c \). Denote by \( N_f M \) the normal bundle of \( f \) and by \( \nabla_\perp, R_\perp \) the normal connection and its curvature tensor, respectively. Let \( \alpha: TM \times TM \to N_f M \) be the second fundamental form of \( f \) and \( A_\xi \) the symmetric endomorphism of \( TM \) defined by \( \langle A_\xi X, Y \rangle = \langle \alpha(X, Y), \xi \rangle \), where \( \xi \in N_f M \) and \( \langle \cdot, \cdot \rangle \) stands for the Riemannian metric of \( \mathbb{Q}_c^4 \). The Gauss, Codazzi and Ricci equations for \( f \) are respectively

\[
(K - c)((X \wedge Y)Z, W) = \langle \alpha(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, Z), \alpha(Y, W) \rangle,
\]

\[
(\nabla_X^\perp \alpha)(Y, Z) = (\nabla_Y^\perp \alpha)(X, Z),
\]

\[
R_\perp(X, Y)\xi = \alpha(X, A_\xi Y) - \alpha(A_\xi X, Y),
\]

where \( K \) is the Gaussian curvature, \( X, Y, Z, W \in TM \) and \( (X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y \).

The orientations of \( M \) and \( \mathbb{Q}_c^4 \) induce an orientation on the normal bundle. The normal curvature \( K_N \) of \( f \) is given by

\[
K_N = \langle R_\perp(e_1, e_2)e_4, e_3 \rangle,
\]

where \( \{e_1, e_2\} \) and \( \{e_3, e_4\} \) are positively oriented orthonormal frame fields of \( TM \) and \( N_f M \), respectively. Notice that if \( \tau \) is an orientation-reversing isometry of \( \mathbb{Q}_c^4 \), then \( f \) and
\( \tau \circ f \) have opposite normal curvatures. The Gauss and the normal curvatures satisfy the equations

\[
d\omega_{12} = -K\omega_1 \wedge \omega_2, \quad d\omega_{34} = -K_N\omega_1 \wedge \omega_2,
\]

where \( \{\omega_j\} \) is the dual frame field of \( \{e_j\} \), \( 1 \leq j \leq 4 \), and the connection forms \( \omega_{kl} \), \( 1 \leq k,l \leq 4 \), are given by

\[
d\omega_k = \sum_{m=1}^{4} \omega_{km} \wedge \omega_m, \quad 1 \leq k \leq 4.
\]

If \( M \) is compact, the Euler-Poincaré characteristics \( \chi, \chi_N \) of \( TM \) and \( N_fM \), are given respectively, by

\[
2\pi\chi = \int_M K, \quad 2\pi\chi_N = \int_M K_N.
\]

For a symmetric section \( \beta \in \Gamma(\text{Hom}(TM \times TM, N_fM)) \), the ellipse associated to \( \beta \) at each \( p \in M \) is defined by

\[
E_\beta(p) = \{\beta(X,X) : X \in T_pM, \|X\| = 1\}.
\]

It is indeed an ellipse on \( N_fM(p) \) centered at \( \text{trace}\beta(p)/2 \), which may degenerate into a line segment or a point. In particular, the ellipse associated to the second fundamental form is denoted by \( E_f \), is centered at the mean curvature vector \( H \) and is called the curvature ellipse of \( f \). It is parametrized by

\[
\alpha(X_\theta, X_\theta) = H(p) + \cos 2\theta \frac{(\alpha_{11} - \alpha_{22})}{2} + \sin 2\theta \alpha_{12},
\]

where \( X_\theta = \cos \theta e_1 + \sin \theta e_2 \), \( \alpha_{ij} = \alpha(e_i, e_j) \), \( i,j = 1,2 \), and \( \{e_1, e_2\} \) is an orthonormal basis of \( T_pM \). The Ricci equation is written equivalently at \( p \) as

\[
R^\perp(e_1, e_2) = (\alpha_{11} - \alpha_{22}) \wedge \alpha_{12}.
\]

Clearly, the ellipse degenerates into a line segment or a point if and only if the vectors \( (\alpha_{11} - \alpha_{22})/2 \) and \( \alpha_{12} \) are linearly dependent, or equivalently, if \( R^\perp = 0 \) at \( p \). At a point where the curvature ellipse is nondegenerate, \( K_N \) is positive if and only if the orientation induced on the ellipse as \( X_\theta \) traverses positively the unit tangent circle, coincides with the orientation of the normal plane (cf. \[22\]). Let \( \lambda_1, \lambda_2 \) be the length of the semiaxes of \( E_f \). Using the Gauss equation and (5), we have that (cf. \[34\])

\[
\lambda_1^2 + \lambda_2^2 = \|H\|^2 - (K - c), \quad \lambda_1 \lambda_2 = \frac{1}{\pi} A(E_f) = \frac{1}{2} |K_N|
\]

at any point, where \( A(E_f) \) is the area of the curvature ellipse. Therefore,

\[
\|H\|^2 - (K - c) \geq |K_N|.
\]
A point $p \in M$ is called pseudo-umbilic if the curvature ellipse is a circle at $p$. A pseudo-umbilic point is called umbilic if the circle degenerates into a point. From (6) it follows that the set $M_0(f)$ of pseudo-umbilic points of $f$ is characterized as

$$M_0(f) = \{ p \in M : \|H\|^2 - (K - c) = |K_N| \}.$$  

A surface for which any point is pseudo-umbilic is called superconformal. By setting

$$M_0^+(f) = \{ p \in M_0(f) : \pm K_N \geq 0 \};$$

it is clear that $M_0(f) = M_0^+(f) \cup M_0^-(f)$ and the set $M_1(f)$ of umbilic points is

$$M_1(f) = M_0^+(f) \cap M_0^-(f) = \{ p \in M : \|H\|^2 = (K - c) \}.$$ 

For later use we need the following elementary fact.

**Lemma 5.** Let $f : M \to \mathbb{Q}^4_+$ be a surface and $\gamma \in \Gamma(\text{Hom}(TM \times TM, N_f M))$ a symmetric section. Assume that the ellipse $E_{\gamma}$ associated to $\gamma$ is not a circle at a point $p \in M$. Then, there exist positively oriented orthonormal frame fields $\{e_1, e_2\}$ of $TM$, $\{e_3, e_4\}$ of $N_f M$, on a neighbourhood $U$ of $p$, and $\kappa, \mu \in C^\infty(U)$ with $\kappa > |\mu|$, such that $\gamma_{11} - \gamma_{22} = 2\kappa e_3$ and $\gamma_{12} = \mu e_4$, where $\gamma_{ij} = \gamma(e_i, e_j)$, $i, j = 1, 2$.

**Proof:** Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a positively oriented orthonormal tangent frame field around $p$ and set $X_t = \cos te_1 + \sin te_2$, $t \in \mathbb{R}$. The ellipse $E_{\gamma}(q)$ is parametrized by

$$\gamma(X_t(q), X_t(q)) = \text{trace}\gamma(q)/2 + \cos 2tu(q) + \sin 2tv(q),$$

where $u = (\tilde{\gamma}_{11} - \tilde{\gamma}_{22})/2$, $v = \tilde{\gamma}_{12}$ and $\gamma_{ij} = \gamma(\tilde{e}_i, \tilde{e}_j)$, $i, j = 1, 2$. Our assumption implies that at least one of the quantities $\|u\| - \|v\|$, $\langle u, v \rangle$ is non-zero at $p$. By continuity, we have that either $\|u\| \neq \|v\|$, or $\langle u, v \rangle \neq 0$ everywhere on a neighbourhood $U$ of $p$. Let $q \in U$. The function $r'(t) = \|\tilde{\gamma}(X_t(q), X_t(q))\|^2$, where $\tilde{\gamma}$ is the traceless part of $\gamma$, attains its maximum at $t_0$. Clearly, $\tilde{\gamma}(X_{t_0}(q), X_{t_0}(q))$ is a major semiaxis of $E_{\gamma}(q)$ and $\tilde{\gamma}(X_{t_0}(q), X_{t_0+\pi/2}(q))$ is a minor semiaxis. From $r'(t_0) = 0$ and $r''(t_0) \leq 0$, we obtain that

$$\sin 4t_0 \left( \|u\|^2 - \|v\|^2 \right)(q) = 2 \cos 4t_0 \langle u, v \rangle(q)$$

and

$$\cos 4t_0 \left( \|u\|^2 - \|v\|^2 \right)(q) + 2 \sin 4t_0 \langle u, v \rangle(q) \geq 0.$$ 

Define the function $\omega \in C^\infty(U)$ by

$$\omega = \frac{1}{4} \arctan \left( \frac{2 \langle u, v \rangle}{\|u\|^2 - \|v\|^2} \right) \text{ modulo } 2\pi,$$
if \( \|u\| \neq \|v\| \) on \( U \), where the branch of arctan is such that \( \cos 4\omega (\|u\|^2 - \|v\|^2) \geq 0 \). If \( \langle u, v \rangle \neq 0 \) on \( U \), then \( \omega \) is defined by

\[
\omega = \frac{1}{4} \text{arccot} \left( \frac{\|u\|^2 - \|v\|^2}{2 \langle u, v \rangle} \right) \text{ modulo } 2\pi,
\]

where the branch of arccot is such that \( \sin 4\omega \langle u, v \rangle \geq 0 \). We consider the frame field \( e_1 = \cos \omega \bar{e}_1 + \sin \omega \bar{e}_2, \ e_2 = -\sin \omega \bar{e}_1 + \cos \omega \bar{e}_2 \) and the positively oriented orthonormal frame field \( \{e_3, e_4\} \) in the normal bundle such that \( \hat{\gamma}(e_1, e_1) = \|\hat{\gamma}(e_1, e_1)\| e_3 \). By the choice of \( \omega \), we have that \( \hat{\gamma}(e_1, e_1) \) is a major semiaxis of \( E_\gamma \). Then, the proof follows with \( \kappa = \|\hat{\gamma}(e_1, e_1)\| \) and \( \mu = \langle \hat{\gamma}(e_1, e_2), e_4 \rangle \).

### 2.1 Complexification and associated differentials

The complexified tangent bundle \( TM \otimes \mathbb{C} \) of a 2-dimensional oriented Riemannian manifold \( M \), decomposes into the eigenspaces of the complex structure \( J \), denoted by \( T^{(1,0)}M \) and \( T^{(0,1)}M \), corresponding to the eigenvalues \( i \) and \( -i \), respectively. Let \( (U, z = x + iy) \) be a local complex coordinate on \( M \). The Wirtinger operators are defined on \( U \) by \( \overline{\partial} = \partial_x - i\partial_y \) and \( \partial = \partial_x + i\partial_y \), where \( \partial_x = \partial/\partial x \) and \( \partial_y = \partial/\partial y \).

Let \( E \) be a complex vector bundle over \( M \) equipped with a connection \( \nabla^E \). An \( E \)-valued differential \( \Psi \) of \( r \)-order is an \( E \)-valued \( r \)-covariant tensor field on \( M \) of holomorphic type \((r, 0)\). The \( r \)-differential \( \Psi \) is called holomorphic (cf. [4]) if its covariant derivative \( \nabla^E \Psi \) has holomorphic type \((r + 1, 0)\). On \( U \) with complex coordinate \( z \), \( \Psi \) has the form \( \Psi = \psi dz^r \), where \( \psi : U \to E \) is given by \( \psi = \Psi(\partial, \ldots, \partial) \). Then \( \Psi \) is holomorphic if and only if

\[
\nabla^E_{\overline{\partial}} \psi = 0,
\]

i.e., \( \psi \) is a holomorphic section. The following result, proved in [4, 11], will be used in the sequel.

**Lemma 6.** Assume that the \( E \)-valued differential \( \Psi \) is holomorphic and let \( p \in M \) be such that \( \Psi(p) = 0 \). Let \( (U, z) \) be a local complex coordinate with \( z(p) = 0 \). Then either \( \Psi \equiv 0 \) on \( U \); or \( \Psi = z^m \Psi^r \), where \( m \) is a positive integer and \( \Psi^r(p) \neq 0 \).
where
\[ \alpha(\partial, \bar{\partial}) = \frac{\lambda^2}{2} \left( \frac{\alpha_{11} - \alpha_{22}}{2} - i\alpha_{12} \right), \quad \alpha_{ij} = \alpha(e_i, e_j), \quad i, j = 1, 2, \quad \text{and} \quad \alpha(\partial, \bar{\partial}) = \frac{\lambda^2}{2} H. \] (7)

The Codazzi equation is equivalent to
\[ \nabla_\perp \bar{\partial} \alpha(\partial, \bar{\partial}) = \frac{\lambda^2}{2} \nabla_\perp \partial H. \] (8)

The Hopf differential of \( f \) is the quadratic \( N_f M \otimes \mathbb{C} \)-valued differential \( \Phi = \alpha^{(2,0)} \) with local expression \( \Phi = \alpha(\partial, \bar{\partial}) dz^2 \). It follows from (8) that \( \Phi \) is holomorphic if and only if the mean curvature vector field \( H \) is parallel.

Let \( J^\perp \) be the complex structure of \( N_f M \) defined by the metric and the orientation. The complexified normal bundle decomposes as
\[ N_f M \otimes \mathbb{C} = N_f^- M \oplus N_f^+ M \]
into the eigenspaces \( N_f^- M \) and \( N_f^+ M \) of \( J^\perp \), corresponding to the eigenvalues \( i \) and \( -i \), respectively. Any section \( \xi \in N_f M \otimes \mathbb{C} \) is decomposed as \( \xi = \xi^- + \xi^+ \), where
\[ \xi^\pm = \pi^\pm(\xi) = \frac{1}{2}(\xi \pm iJ^\perp \xi). \]

A section \( \xi \) of \( N_f M \otimes \mathbb{C} \) is called isotropic if at any point of \( M \), either \( \xi = \xi^- \), or \( \xi = \xi^+ \). This is equivalent to \( \langle \xi, \xi \rangle = 0 \), where \( \langle \cdot, \cdot \rangle \) is the \( \mathbb{C} \)-bilinear extension of the metric. Notice that \( \langle \zeta, \eta \rangle = 0 \) for \( \zeta \in N_f^- M \) and \( \eta \in N_f^+ M \), implies that \( \zeta = 0 \) or \( \eta = 0 \). According to the above decomposition, the Hopf differential splits as
\[ \Phi = \Phi^- + \Phi^+, \quad \text{where} \quad \Phi^\pm = \pi^\pm \circ \Phi. \]

**Lemma 7.**

(i) The zero-sets of \( \Phi^\pm \) and \( \Phi \), are \( M_0^\pm(f) \) and \( M_1(f) \), respectively.

(ii) The surface \( f \) is superconformal with normal curvature \( \pm K_N \geq 0 \) if and only if \( \Phi^\pm \equiv 0 \). In particular, if \( f \) is superconformal, then \( K_N \) vanishes precisely on \( M_1(f) \).

**Proof:** In terms of a local complex coordinate \( z \) around a point \( p \), it is clear that \( \Phi^\pm(p) = 0 \) if and only if \( J^\perp \alpha(\partial, \bar{\partial}) = \pm i\alpha(\partial, \bar{\partial}) \) at \( p \). It follows from (7) that \( \mathcal{E}_f(p) \) is a circle if and only if \( \alpha(\partial, \bar{\partial}) \) is isotropic. Bearing in mind (4) and (5), we conclude that \( \Phi^\pm(p) = 0 \) if and only if \( \mathcal{E}_f(p) \) is a circle and \( \pm K_N(p) \geq 0 \). Obviously, \( \Phi \) vanishes precisely at the points where both \( \Phi^\pm \) vanish, i.e., the umbilic points. This proves part (i), and the first assertion of part (ii) follows immediately. If \( f \) is superconformal, then the second equation in (6) implies that the normal curvature vanishes precisely at the umbilic points. \( \blacksquare \)
2.2 Twistor spaces and Gauss lifts

We recall some known facts about the twistor theory of 4-dimensional space forms. The reader may consult [17, 20], although the paper of Jensen and Rigoli [28] is closer to our approach. Let \( O(Q^4_e) \) be the principal \( O(4) \)-bundle of orthonormal frames in \( Q^4_c \), which has two connected components denoted by \( O_+(Q^4_c) \) and \( O_-(Q^4_c) \), corresponding to the two connected components of \( O(4) \). The twistor bundle \( Z \) of \( Q^4_c \) is defined as the set of all pairs \( (p, \tilde{J}) \), where \( p \in Q^4_c \) and \( \tilde{J} \) is an orthogonal complex structure on \( T_p Q^4_c \). The twistor projection \( g: Z \to Q^4_c \) is defined by \( g(p, \tilde{J}) = p \), and \( Z \) is an \( O(4)/U(2) \)-fiber bundle over \( Q^4_c \), which is associated to \( O(Q^4_c) \). Indeed, at a point \( p \in Q^4_c \) and for any orthonormal frame \( e = (e_1, e_2, e_3, e_4) \) of \( T_p Q^4_c \), define an orthogonal complex structure \( \tilde{J}_e \) by

\[
\tilde{J}_e e_1 = e_2, \quad \tilde{J}_e e_3 = e_4, \quad \tilde{J}_e^2 = -I.
\]

Any orthogonal complex structure on \( T_p Q^4_c \) is equal to \( \tilde{J}_e \) for some orthonormal frame \( e \) of \( T_p Q^4_c \) and \( \tilde{J}_e = \tilde{J}_\tilde{e} \) if and only if \( \tilde{e} = eA \) for some \( A \in U(2) \). Thus, the set of all orthogonal complex structures on \( T_p Q^4_c \) is \( O(4)/U(2) \) and has two connected components isomorphic to \( SO(4)/U(2) = \{ \tilde{J}_e : e \text{ is a } \pm \text{ oriented frame of } T_p Q^4_c \} \). Hence, the twistor bundle is

\[
Z = O(Q^4_c) \times_{O(4)} O(4)/U(2) = O(Q^4_c)/U(2)
\]

and its two connected components are denoted by \( Z_+ \) and \( Z_- \). Each projection \( \varrho_\pm: Z_+ \to \mathbb{Q}_c^4 \) is a \( P^1(\mathbb{C}) \simeq S^2 \)-fiber bundle over \( \mathbb{Q}_c^4 \).

A one-parameter family of Riemannian metrics \( g_t, \ t > 0 \), is defined on \( Z \) in a natural way, making \( \varrho_+ \) and \( \varrho_- \) Riemannian submersions. With respect to the (common) decomposition of the tangent bundle of \( Z_\pm \) induced by the Levi-Civit\'a connection of \( g_t \)

\[
TZ_\pm = T^h Z_\pm \oplus T^v Z_\pm
\]

into horizontal and vertical subbundles, the metric \( g_t \) is given by the pull-back of the metric of \( \mathbb{Q}_c^4 \) to the horizontal subspaces and by adding the \( t^2 \)-fold of the metric of the fibers.

Denote by \( Gr_2(TQ^4_c) \) the Grassmann bundle of oriented 2-planes tangent to \( \mathbb{Q}_c^4 \). There are projections

\[
\Pi_+: Gr_2(TQ^4_c) \to Z_+ \quad \text{and} \quad \Pi_-: Gr_2(TQ^4_c) \to Z_-
\]

defined as follows; if \( \zeta \subset T_p \mathbb{Q}_c^4 \) is an oriented 2-plane, then \( \Pi_\pm(p, \zeta) \) is the complex structure on \( T_p \mathbb{Q}_c^4 \) corresponding to the rotation by \( +\pi/2 \) on \( \zeta \) and the rotation by \( \pm\pi/2 \) on \( \zeta^\perp \). The Gauss lift \( G_f: M \to Gr_2(TQ^4_c) \), of an oriented surface \( f: M \to \mathbb{Q}_c^4 \) is defined by \( G_f(p) = (f(p), f_*T_pM) \). The Gaussian lifts of \( f \) to the twistor bundle are the maps

\[
G_+: M \to Z_+ \quad \text{and} \quad G_-: M \to Z_-, \quad \text{where} \ G_\pm = \Pi_\pm \circ G_f.
\]
At any point \( p \in M \), we obviously have \( G_\pm(p) = (f(p), \tilde{J}_\pm(f(p))) \), where
\[
\tilde{J}_\pm(f(p)) = \begin{cases} 
  f_* \circ J(p), & \text{on } f_* T_p M, \\
  \pm J^\perp(p), & \text{on } N_f M(p).
\end{cases}
\]

The Gauss lift \( G_\pm : M \to (\mathcal{Z}_\pm, g_t) \) is called \textit{vertically harmonic} if its tension field has vanishing vertical component with respect to the decomposition \( T \mathcal{Z}_\pm = T^h \mathcal{Z}_\pm \oplus T^v \mathcal{Z}_\pm \).

Let \( \{e_j\}, 1 \leq j \leq 4 \), be a \( \pm \) oriented, local adapted orthonormal frame field of \( \mathcal{Q}_{c}^4 \), where \( \{e_1, e_2\} \) is in the orientation of \( TM \). Denote by \( \{\omega_j\}, 1 \leq j \leq 4 \), the corresponding coframe and by \( \omega_{kl}, 1 \leq k, l \leq 4 \), the connection forms given by (3). The pull-back of \( g_t \) on \( M \) under \( G_\pm \), is related to the metric \( ds^2 \) of \( M \) as follows
\[
G_\pm^*(g_t) = ds^2 + t^2 \left( (\omega_{13} - \omega_{24})^2 + (\omega_{14} - \omega_{23})^2 \right).
\]

The covariant differential of the mean curvature vector field \( H = H^3 e_3 + H^4 e_4 \) is given by
\[
\nabla^\perp H = \sum_{a=3}^{4} (dH^a + \sum_{b=3}^{4} H^b \omega_{ba}) \otimes e_a = \sum_{a=3}^{4} \sum_{j=1}^{4} H^a_j \omega_j \otimes e_a.
\] (9)

The following proposition relates the vertical harmonicity of the Gauss lift \( G_\pm \) with the holomorphicity of the differential \( \Phi^\pm \) and the holomorphicity of the section \( H^\pm \). The equivalence of (i) and (iv) below, is a slight modification of Theorem 8.1. in [28] for space forms, where the scalar curvature of \( \mathcal{Q}_{c}^4 \) is normalized to be equal to \( c \). It was also proved by Hasegawa [23] who studied surfaces with a vertically harmonic Gauss lift.

**Proposition 8.** Let \( f : M \to \mathcal{Q}_{c}^4 \) be a surface with mean curvature vector field \( H \). The following are equivalent:

(i) The Gauss lift \( G_\pm : M \to (\mathcal{Z}_\pm, g_t) \) of \( f \) is vertically harmonic.

(ii) The differential \( \Phi^\pm \) is holomorphic.

(iii) The section \( H^\pm \) is anti-holomorphic.

(iv) \( \nabla^\perp_{JX} H = \pm J^\perp \nabla^\perp_X H \), for any \( X \in TM \).

**Proof:** The equivalence of (ii), (iii) and (iv) is an immediate consequence of the Codazzi equation (8). We prove that (i) is equivalent to (iv). The tension field of \( G_\pm \), in terms of an appropriate frame field \( \{E_k^\pm, 1 \leq k \leq 6\} \) of \( \mathcal{Z}_\pm \), is given by (cf. [28])
\[
\tau(G_\pm) = \sum_{k=1}^{6} B_k^\pm \, E_k^\pm,
\]
where
\[
\begin{align*}
B_j^\pm &= 0 \text{ for } j = 1, 2; \quad B_3^\pm = 2H^a(1 - ct^2) \text{ for } a = 3, 4, \\
B_5^\pm &= 2t(H_2^4 - H_1^3), \quad B_6^\pm = -2t(H_1^4 + H_2^3).
\end{align*}
\]
Its vertical component is given by
\[(\tau(G_{\pm}))^v = B_5^\pm E_5^\pm + B_6^\pm E_6^\pm\]
and therefore, \(G_{\pm}\) is vertically harmonic if and only if \(H_2^3 = H_1^3\) and \(H_1^4 = -H_2^3\). Using (9) and taking into account the orientation of the frame field of \(Q_4\), the result follows after a straightforward computation.

From the above proof it follows that if \(t^2 = 1/c\), then \(G_{\pm}\) is vertically harmonic if and only if it is harmonic.

Proposition 8 and Lemma 7(ii) imply that any superconformal surface \(f: M \rightarrow Q_4_c\) with \(\pm K_N \geq 0\) has vertically harmonic Gauss lift \(G_{\pm}\). The Gauss lift \(G_{\pm}\) of such surfaces is holomorphic with respect to a complex structure \(\mathcal{J}\) on \(\mathcal{Z}\), which endows \((\mathcal{Z}, g_0)\) with the structure of a Hermitian manifold (cf. [17, 28]).

It is clear from Proposition 8 that both Gauss lifts are vertically harmonic if and only if the surface has parallel mean curvature vector field in the normal connection.

3 Surfaces with the same mean curvature

3.1 Bonnet pairs and the distortion differential

Let \(M\) be a 2-dimensional oriented Riemannian manifold and \(f, \tilde{f}: M \rightarrow Q_4\) be isometric immersions with second fundamental forms \(\alpha, \tilde{\alpha}\) and mean curvature vector fields \(H, \tilde{H}\), respectively. The surfaces \(f, \tilde{f}\) are said to have the same mean curvature, if there exists a parallel vector bundle isometry \(T: N_f M \rightarrow N_{\tilde{f}} M\) such that \(TH = \tilde{H}\).

The case of minimal surfaces has been studied in [14] and [40]. In this section, we assume that all surfaces under consideration are non-minimal.

Suppose that \(f, \tilde{f}: M \rightarrow Q_4\) have the same mean curvature and let \(T: N_f M \rightarrow N_{\tilde{f}} M\) be a parallel vector bundle isometry satisfying \(TH = \tilde{H}\). After an eventual composition of one of the surfaces with an orientation-reversing isometry of \(Q_4\), we may hereafter suppose that \(T\) is orientation-preserving. To such a pair \((f, \tilde{f})\) we assign a holomorphic differential which is going to play a fundamental role in the sequel. The section of \(\text{Hom}(TM \times TM, N_f M)\) given by
\[D^T_{f,\tilde{f}} = \alpha - T^{-1} \circ \tilde{\alpha}\]
measures how far the surfaces deviate from being congruent. Since \(D^T_{f,\tilde{f}}\) is traceless, its \(\mathbb{C}\)-bilinear extension decomposes into its \((k, l)\)-components, \(k + l = 2\), as
\[D^T_{f,\tilde{f}} = (D^T_{f,\tilde{f}})^{(2,0)} + (D^T_{f,\tilde{f}})^{(0,2)},\]
where \((D^T_{f,\tilde{f}})^{(2,0)} = (\overline{(D^T_{f,\tilde{f}})^{(2,0)}})\).

We are interested into the \((2,0)\)-part which is given by
\[Q^T_{f,\tilde{f}} = (D^T_{f,\tilde{f}})^{(2,0)} = \Phi - T^{-1} \circ \tilde{\Phi},\]
where \(\Phi, \tilde{\Phi}\) stand for the Hopf differentials of \(f, \tilde{f}\), respectively.
Lemma 9. Let \( f, \tilde{f} : M \to \mathbb{Q}^4 \) be surfaces and \( T : N_f M \to N_{\tilde{f}} M \) an orientation-preserving, parallel vector bundle isometry satisfying \( TH = \tilde{H} \). Then:

(i) The quadratic differential \( Q^T_{f, \tilde{f}} \) is holomorphic and independent of \( T \).

(ii) The normal curvatures of the surfaces are equal and the curvature ellipses \( E_f, E_{\tilde{f}} \) are congruent at any point of \( M \). In particular, \( M^\pm_0(f) = M^\pm_0(\tilde{f}) \).

Proof: (i) From our assumption it follows that the section \( T^{-1} \circ \tilde{\alpha} \) of \( \text{Hom}(TM \times TM, N_f M) \) satisfies the Codazzi equation for the data on \( N_f M \) and thus, \( Q^T_{f, \tilde{f}} \) is holomorphic by (8).

Suppose that there exists another orientation-preserving parallel vector bundle isometry \( S : N_f M \to N_{\tilde{f}} M \) with \( SH = \tilde{H} \). We argue that \( Q^T_{f, \tilde{f}} \equiv Q^S_{f, \tilde{f}} \). Set \( L = T^{-1} \circ S \) and \( U = \{ p \in M : H(p) \neq 0 \} \). On \( N_f U \), \( L \) preserves both of \( H \) and \( J^\perp H \) and thus, \( T = S \) on \( N_f U \). Therefore, the holomorphic differential \( Q^T_{f, \tilde{f}} - Q^S_{f, \tilde{f}} \) vanishes identically on the open subset \( U \) of \( M \). Then by Lemma 6, we obtain that \( Q^T_{f, \tilde{f}} \equiv Q^S_{f, \tilde{f}} \) on \( M \).

(ii) The vector bundle isometry \( T \) preserves the normal curvature tensors. Since it is orientation-preserving, (1) implies that the normal curvatures of \( f, \tilde{f} \) are equal. The fact that the curvature ellipses are congruent, now follows from (6) and this completes the proof.

Lemma 9(i) allows us to assign to each pair of surfaces \((f, \tilde{f})\) with the same mean curvature, a holomorphic differential denoted by \( Q_{f, \tilde{f}} \), which is called the distortion differential of the pair and is given by

\[
Q_{f, \tilde{f}} = \Phi - T^{-1} \circ \tilde{\Phi}.
\]

Obviously, \( Q_{f, \tilde{f}} \equiv 0 \) if and only if \( f \) and \( \tilde{f} \) are congruent. To simplify the notation, we denote the distortion differential associated to the pair \((f, \tilde{f})\) by \( Q \), whenever there is no danger of confusion. A pair \((f, \tilde{f})\) of noncongruent surfaces with the same mean curvature is called a Bonnet pair. In this case, the zero-set of \( Q \) is denoted by \( Z \) and according to Lemmas 6 and 9(i), consists of isolated points only.

With respect to the decomposition \( N_f M \otimes \mathbb{C} = N^{-}_f M \oplus N^{+}_f M \), the distortion differential \( Q \) splits as

\[
Q = Q^- + Q^+,
\] where \( Q^\pm = \pi^\pm \circ Q \).

It follows from Lemma 9(i) that the differentials

\[
Q^\pm = \Phi^\pm - T^{-1} \circ \tilde{\Phi}^\pm
\]

are both holomorphic. According to Lemma 6, either \( Q^\pm \equiv 0 \), or its zero-set \( Z^\pm \) consists of isolated points only.
3.2 The decomposition of the moduli space

Let \( f : M \to \mathbb{Q}^4_\epsilon \) be a non-minimal oriented surface. We denote by \( \mathcal{M}(f) \) the moduli space of congruence classes of all isometric immersions of \( M \) into \( \mathbb{Q}^4_\epsilon \), that have the same mean curvature with \( f \). Since the distortion differential of a Bonnet pair does not vanish identically, the moduli space can be written as

\[
\mathcal{M}(f) = N^-(f) \cup N^+(f) \cup \{f\},
\]

where

\[
N^\pm(f) = \{ \tilde{f} : Q^{\pm}_{f,\tilde{f}} \neq 0 \}/\text{Isom}^+(\mathbb{Q}^4_\epsilon),
\]

\( \{f\} \) is the trivial congruence class and \( \text{Isom}^+(\mathbb{Q}^4_\epsilon) \) is the group of orientation-preserving isometries of \( \mathbb{Q}^4_\epsilon \). Furthermore, \( \mathcal{M}(f) \) decomposes into disjoint components as

\[
\mathcal{M}(f) = \mathcal{M}^*(f) \cup \mathcal{M}^-(f) \cup \mathcal{M}^+(f) \cup \{f\},
\]

where

\[
\mathcal{M}^\pm(f) = N^\pm(f) \setminus N^\mp(f) = \{ \tilde{f} : Q^{\pm}_{f,\tilde{f}} \equiv Q^{\mp}_{f,\tilde{f}} \}/\text{Isom}^+(\mathbb{Q}^4_\epsilon),
\]

and

\[
\mathcal{M}^*(f) = N^-(f) \cap N^+(f) = \{ \tilde{f} : Q^-_{f,\tilde{f}} \neq 0 \text{ and } Q^+_{f,\tilde{f}} \neq 0 \}/\text{Isom}^+(\mathbb{Q}^4_\epsilon).
\]

Hereafter, whenever we refer to a surface in the moduli space we mean its congruence class.

The following theorem provides information about the structure of the components of the decomposition of the moduli space \( \mathcal{M}(f) \) of a compact surface and is fundamental for the proofs of our results.

**Theorem 10.** Let \( f : M \to \mathbb{Q}^4_\epsilon \) be an isometric immersion of a compact, oriented 2-dimensional Riemannian manifold.

(i) If the Gauss lift \( G^\pm \) of \( f \) is not vertically harmonic, then \( \mathcal{M}^\pm(f) \) contains at most one congruence class. Moreover, if \( \tilde{f} \in \mathcal{M}^*(f) \) then \( \mathcal{M}^*(f) \cup \mathcal{M}^\pm(f) = \{ \tilde{f} \} \cup \mathcal{M}^\pm(\tilde{f}) \).

(ii) If both Gauss lifts of \( f \) are not vertically harmonic, then \( \mathcal{M}^*(f) \) contains at most one congruence class. In particular, \( \mathcal{M}^*(f) = \emptyset \) if \( M \) is homeomorphic to \( S^2 \).

For the proof of the above theorem we need a series of auxiliary results. In view of Lemma 9(ii), we denote by \( M_0 = M^+_0 \cup M^-_0 \) and \( M_1 \) the set of pseudo-umbilic and umbilic points of a Bonnet pair, respectively.

**Lemma 11.** For any Bonnet pair \( (f, \tilde{f}) \) we have that \( M_1 \) is isolated and \( M^\pm_0 \subset Z^\pm \). In particular, \( M^\pm_0 \) is isolated if \( \tilde{f} \in N^\pm(f) \).

**Proof:** The fact that \( M^\pm_0 \subset Z^\pm \) follows immediately from Lemma 7(i) and (10). Since \( M_1 \subset Z \) and \( Z \) consists of isolated points, the umbilic points are isolated. If \( \tilde{f} \in N^\pm(f) \), then \( Z^\pm \) is isolated and this completes the proof.
Proposition 12. If \( \tilde{f} \in \mathcal{N}^\pm(f) \), then there exists \( \theta^\pm \in \mathcal{C}^\infty(M \setminus Z^\pm) \) with values in \((0, 2\pi)\), such that the distortion differential of the pair \((f, \tilde{f})\) satisfies

\[
Q^\pm = (1 - e^{\mp i\theta^\pm})\Phi^\pm \text{ on } M \setminus Z^\pm.
\]  

(11)

Proof: We set \( \beta = T^{-1} \circ \alpha \), where \( T: N_f M \to N_{\tilde{f}}M \) is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry.

If \( \text{int}(M_0) \neq \emptyset \), then from Lemma 11 we obtain that \( \text{int}(M_0) \subset M_0^\pm \). Lemma 7(ii) implies that \( \pm K_N < 0, \Phi^\pm \equiv 0 \) and \( \Phi^\pm \neq 0 \) on \( \text{int}(M_0 \setminus Z^\pm) \). Let \( z \) be a local complex coordinate defined on a simply-connected neighbourhood \( V \subset \text{int}(M_0 \setminus Z^\pm) \). From Lemma 9(ii), it follows that the isotropic sections \( \alpha(\partial, \partial) \) and \( \beta(\partial, \partial) \) have the same length. Hence, there exists \( \tau \in \mathcal{C}^\infty(V) \) with values in \((0, 2\pi)\), such that

\[
\beta(\partial, \partial) = J^\perp_\tau \alpha(\partial, \partial),
\]

where the rotation \( J^\perp_\tau = \cos \tau I + \sin \tau J^\perp \) satisfies \( J^\perp_\tau = e^{\mp i\tau} I \) on \( N_f^\pm M \). Since \( \Phi^\pm \neq 0 \) on \( \text{int}(M_0 \setminus Z^\pm) \), the function \( \tau \) is well-defined modulo \( 2\pi \) on \( \text{int}(M_0 \setminus Z^\pm) \). Moreover, it is non-vanishing modulo \( 2\pi \) on \( \text{int}(M_0 \setminus Z^\pm) \) and thus, there exists a branch in \( \mathcal{C}^\infty(\text{int}(M_0 \setminus Z^\pm)) \) with values in \((0, 2\pi)\). By setting \( \theta^\pm = \tau \), we have that (11) holds on \( \text{int}(M_0 \setminus Z^\pm) \). In particular, the assertion is obvious if \( M = M_0 \).

Assume that \( M \neq M_0 \) and let \( p \in M \setminus M_0 \). According to Lemma 5, there exist smooth frame fields \( \{e_1, e_2, e_3, e_4\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \) on a neighbourhood \( U \subset M \setminus M_0 \) of \( p \), such that

\[
\alpha_{11} - \alpha_{22} = 2\kappa e_3, \quad \alpha_{12} = \mu e_4,
\]

where \( \alpha_{ij} = \alpha(e_i, e_j), j = 1, 2 \), and

\[
\beta_{11} - \beta_{22} = 2\bar{\kappa}\tilde{e}_3, \quad \beta_{12} = \bar{\mu}\tilde{e}_4,
\]

where \( \beta_{ij} = \beta(\tilde{e}_i, \tilde{e}_j), j = 1, 2 \).

Lemma 9(ii) yields that the ellipses \( E_f(q) \) and \( E_\beta(q) \) are congruent at any point \( q \in U \) and consequently, \( \kappa = \bar{\kappa} \). Using (1) and (5), we obtain that \( K_N = 2\kappa \mu \) and \( \tilde{K}_N = 2\bar{\kappa}\bar{\mu} \). Then, Lemma 9(ii) implies that \( \mu = \bar{\mu} \). Setting \( \tilde{e}_3 - i\tilde{e}_4 = e^{i\theta}(e_3 - ie_4) \) for some \( \theta \in \mathcal{C}^\infty(U) \), we have that

\[
J^\perp_\theta (\alpha_{11} - \alpha_{22}) = \beta_{11} - \beta_{22} \quad \text{and} \quad J^\perp_\theta \alpha_{12} = \beta_{12} \quad \text{on } U,
\]

where \( J^\perp_\theta = \cos \theta I + \sin \theta J^\perp \). This gives

\[
\beta(\tilde{e}_1 - i\tilde{e}_2, \tilde{e}_1 - i\tilde{e}_2) = J^\perp_\theta (\alpha(e_1 - ie_2, e_1 - ie_2)).
\]

Setting \( \tilde{e}_1 - i\tilde{e}_2 = e^{i\sigma}(e_1 - ie_2) \) for some \( \sigma \in \mathcal{C}^\infty(U) \), the above is written equivalently as

\[
T^{-1} \circ \Phi = e^{i\theta^-} \Phi^- + e^{-i\theta^+} \Phi^+, \quad \text{where} \quad \theta^\pm = \theta \pm 2\sigma.
\]

Since \( \Phi^- \) and \( \Phi^+ \) are everywhere non-vanishing on \( M \setminus M_0 \), the functions \( \theta^- \) and \( \theta^+ \) are well-defined modulo \( 2\pi \) on \( M \setminus M_0 \). From the assumption \( \tilde{f} \in \mathcal{N}^\pm(f) \), it follows that
\( \theta^\pm \) is non-vanishing modulo \( 2\pi \) on \( M \setminus (M_0 \cup Z^\pm) \) and thus, there exists a branch in \( \mathcal{C}^\infty(M \setminus (M_0 \cup Z^\pm)) \) with values in \( (0, 2\pi) \). Obviously, (11) holds on \( M \setminus (M_0 \cup Z^\pm) \).

Lemma 9(ii) implies that for a point \( q \in M_0 \setminus (\text{int}(M_0) \cup Z^\pm) \), there exists a unique number \( l(q) \in (0, 2\pi) \) such that

\[
T^{-1} \circ \tilde{\Phi}(q) = J^\perp_{l(q)} \Phi(q),
\]

where the rotation is given by \( J^\perp_{l(q)} = e^{\mp il(q)}I \), since \( q \in M_0 \setminus 0 \). We extend \( \theta^\pm \) on \( M \setminus Z^\pm \) by setting \( \theta^\pm(q) = l(q) \). Then, (11) holds on \( M \setminus Z^\pm \). Since \( Q^\pm \) and \( \Phi^\pm \) are everywhere non-vanishing on \( M \setminus Z^\pm \), from (11) it follows that \( \theta^\pm \) is smooth. \( \blacksquare \)

Let \( f: M \to \mathbb{Q}_c^4 \) be a surface with Hopf differential \( \Phi \). In terms of a complex chart \( (U, z) \), \( \Phi^\pm \) is written as

\[
\Phi^\pm = \phi^\pm dz^2.
\]

Moreover, there exist smooth complex functions \( h^+ \) and \( h^- \) such that

\[
\nabla^\perp_{\bar{\partial}} \phi^\pm = h^\pm \phi^\pm \quad \text{on} \quad U \setminus M_0^\pm.
\]

The following lemma is essential for the proof of Theorem 10.

**Lemma 13.** Let \( \tilde{f} \in \mathcal{N}^\pm(f) \). In terms of a complex chart \( (U, z) \), the function \( \theta^\pm \) in Proposition 12 satisfies on \( U \setminus Z^\pm \) the equations

\[
A^\pm e^{\mp 2i\theta^\pm} - 2i(\text{Im } A^\pm)e^{\mp i\theta^\pm} - \overline{A^\pm} = 0, \quad (14)
\]

\[
\theta^\pm_{z\bar{z}} = \mp A^\pm(1 - e^{\mp i\theta^\pm}), \quad (15)
\]

where \( A^\pm = i (h^\pm_z - |h^\pm|^2) \).

**Proof:** By differentiating (11) with respect to \( \bar{\partial} \) in the normal connection, and using (13) and the holomorphicity of \( Q^\pm \), we obtain

\[
\left( h^\pm(1 - e^{\mp i\theta^\pm}) \pm ie^{\mp i\theta^\pm} \theta^\pm_z \right) \phi^\pm = 0.
\]

Since \( \phi^\pm \neq 0 \) on \( U \setminus Z^\pm \), we have

\[
\theta^\pm_z = \mp ih^\pm(1 - e^{\mp i\theta^\pm}) \quad \text{and} \quad \theta^\pm_{\bar{z}} = \pm ih^\pm(1 - e^{\mp i\theta^\pm}).
\]

Differentiating the above, we find

\[
\theta^\pm_{z\bar{z}} = \mp A^\pm(1 - e^{\mp i\theta^\pm}) \quad \text{and} \quad \theta^\pm_{z\bar{z}} = \mp \overline{A^\pm}(1 - e^{\mp i\theta^\pm}),
\]

from which (14) and (15) follow immediately. \( \blacksquare \)

**Lemma 14.** (i) If \( f_1 \in \mathcal{M}^-(f_3) \) and \( f_2 \in \mathcal{M}^+(f_3) \), then \( f_1 \in \mathcal{M}^+(f_2) \).

(ii) If \( f_1, f_2 \in \mathcal{M}^\pm(f_3) \), then \( f_1 \in \mathcal{M}^\pm(f_2) \).
Proof: Let $T_{jk}: N_{f_j}M \to N_{f_k}M$, $1 \leq j, k \leq 3$, $j \neq k$, be orientation and mean curvature vector field-preserving, parallel vector bundle isometries. Denote by $Q_{jk}$ and $\Phi_j$ the distortion differential of the pair $(f_j, f_k)$ and the Hopf differential of $f_j$, respectively. From Lemma 9(i), we know that $Q_{jk}$ is independent of $T_{jk}$. Hence,

$$Q_{12} = \Phi_1 - T_{12}^{-1} \circ \Phi_2 = \Phi_1 - (T_{31} \circ T_{32}^{-1}) \circ \Phi_2,$$

or equivalently,

$$T_{31}^{-1} \circ Q_{12} = T_{31}^{-1} \circ \Phi_1 - T_{32}^{-1} \circ \Phi_2 = Q_{31} - Q_{32}.$$

Therefore,

$$Q_{ij}^\pm = T_{31} \circ (Q_{31}^\pm - Q_{32}^\pm)$$

and the results follow immediately.

Proof of Theorem 10: We claim that if there exist $f_1, f_2 \in N^\pm(f)$ with $f_1 \in N^\pm(f_2)$, then the Gauss lift $G_\pm$ of $f$ is vertically harmonic. To unify the notation, set $f_3 = f$ and denote by $Q_{jk}$ the distortion differential of the pair $(f_j, f_k)$. Let $\hat{Z}$ be the set containing the zeros of the holomorphic differentials $Q^\pm_{jk}$, $1 \leq j, k \leq 3$, $j \neq k$. According to Lemma 6, $\hat{Z}$ consists of isolated points only. Appealing to Proposition 12, we know that there exists $\theta_1^\pm \in C^\infty(M \setminus \hat{Z})$, $j = 1, 2$, with values in $(0, 2\pi)$ such that

$$Q_{3j}^\pm = (1 - e^{\pm i\theta_1^\pm})\Phi^\pm,$$

where $\Phi$ is the Hopf differential of $f$. Then we have

$$Q_{31}^\pm - Q_{32}^\pm = (e^{\pm i\theta_2^\pm} - e^{\pm i\theta_1^\pm})\Phi^\pm.$$

Since the zeros of $Q_{12}^\pm$ are contained in $\hat{Z}$, the above and (16) imply that $\theta_1^\pm \neq \theta_2^\pm$ at every point in $M \setminus \hat{Z}$. Then (14), viewed as a polynomial equation has three distinct roots, namely $e^{\pm i\theta_1^\pm}, e^{\pm i\theta_2^\pm}, 1$. Therefore, $A^\pm = 0$. From (15) it follows that $\theta_1^\pm$ is harmonic on $M \setminus \hat{Z}$. Since it is bounded and $\hat{Z}$ consists of isolated points, it can be extended to a bounded harmonic function on $M$, which has to be constant by the maximum principle. Then (17) shows that $\Phi^\pm$ is holomorphic. Proposition 8 implies that the Gauss lift $G_\pm$ is vertically harmonic and this proves the claim.

(i) Suppose to the contrary that there exist noncongruent $f_1, f_2 \in \mathcal{M}^\pm(f) \subset N^\pm(f)$. From Lemma 14(ii), we have that $f_1 \in \mathcal{M}^\pm(f_2) \subset N^\pm(f_2)$. Therefore, the Gauss lift $G^\pm$ is vertically harmonic, a contradiction.

For the second assertion, assume that there exists $f_1 \in \mathcal{M}^\pm(f)$. If $f_1 \in \mathcal{M}^\pm(f)$, then Lemma 14(ii) implies that $f \in \mathcal{M}^\pm(f)$, which is a contradiction. Therefore, $f_1 \not\in \mathcal{M}^\pm(f)$ and thus, $\{\tilde{f}\} \cup \mathcal{M}^\pm(f) \subset N^\pm(f)$, which obviously holds if $\mathcal{M}^\pm(f) = \emptyset$. The converse inclusion is obvious if $N^\pm(f) = \{\tilde{f}\}$. Assume that there exists $f_1 \in N^\pm(f) \setminus \{\tilde{f}\}$. From the claim proved above, it follows that $f_1 \in \mathcal{M}^\pm(f)$ and thus, $N^\pm(f) \subset \{\tilde{f}\} \cup \mathcal{M}^\pm(f)$. 

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(ii) Suppose to the contrary that there exist noncongruent \( f_1, f_2 \in \mathcal{M}^*(f) \). Since both \( G_+ \) and \( G_- \) are not vertically harmonic, from the above claim we obtain that \( f_1 \not\in N^+(f_2) \cup N^-(f_2) \), which is a contradiction since \( (f_1, f_2) \) is a Bonnet pair.

If \( M \) is homeomorphic to the sphere, then for any \( f \in \mathcal{M}(f) \setminus \{ f \} \), the fourth-order differential \( \langle Q^-, Q^+ \rangle \) is holomorphic with zero-set \( Z^- \cup Z^+ \), where \( Q \) is the distortion differential of the pair \( (f, \bar{f}) \). From the Riemann-Roch theorem we have that \( \langle Q^-, Q^+ \rangle \equiv 0 \). Hence, either \( Q^- \equiv 0 \) or \( Q^+ \equiv 0 \) and consequently \( \mathcal{M}^*(f) = \emptyset \).

**Proof of Theorem 2:** Theorem 10 implies that \( \mathcal{M}(f) \setminus \{ f \} \) contains at most three congruence classes. Assume that \( M \) is homeomorphic to \( S^2 \). Theorem 10 shows that \( \mathcal{M}^*(f) = \emptyset \) and each one of \( \mathcal{M}^+(f) \) and \( \mathcal{M}^-(f) \) contains at most one congruence class. Suppose that there exist \( f_1 \in \mathcal{M}^+(f) \) and \( f_2 \in \mathcal{M}^-(f) \). From Lemma 14(i) it follows that \( f_1 \in \mathcal{M}^*(f_2) \), which contradicts Theorem 10(ii). Therefore, \( \mathcal{M}(f) \setminus \{ f \} \) contains at most one congruence class.

**Proof of Theorem 1:** In the case of \( \mathbb{R}^4 \), \( (Z_\pm, g_t) \) is isometric to the product \( \mathbb{R}^4 \times S^2(t) \). The Grassmann bundle is trivial \( Gr_2(\mathbb{R}^4) \simeq \mathbb{R}^4 \times Gr(2, 4) \) and the Gauss lift of \( f \) to the Grassmann bundle is given by \( G_f = (f, g) \), where \( g = (g_+, g_-) \): \( M \to S^2_+ \times S^2_- \) is the Gauss map of \( f \) and the radius of \( S^2_\pm \) is \( 1/\sqrt{2} \). The Gauss lift \( G_\pm \) of \( f \) to the twistor bundle is then given by \( G_\pm = (f, \sqrt{2t}g_\pm) \) and it is vertically harmonic if and only if \( g_\pm \) is harmonic. Now, the proof follows immediately from Theorem 2.

**Remark 15.** In the proof of Theorem 10, compactness is only required for the use of the maximum principle. This theorem and also Theorems 1, 2 and the results of the next subsection, still hold true if \( M \) is parabolic. In particular, this includes the case where \( M \) is complete with non-negative Gaussian curvature.

### 3.3 Applications to certain classes of surfaces

The following result due to Lawson-Tribuzy [33], is an easy application of Theorem 10.

**Theorem 16.** Let \( M \) be a compact oriented 2-dimensional Riemannian manifold and \( h \in C^\infty(M) \). If \( h \) is not constant, then there exist at most two congruence classes of isometric immersions of \( M \) into \( \mathbb{Q}_c^2 \) with mean curvature \( h \). In particular, there exists at most one congruence class if \( M \) is homeomorphic to \( S^2 \).

**Proof:** Suppose that there exist noncongruent isometric immersions \( f_1, f_2 : M \to \mathbb{Q}_c^2 \) with mean curvature function \( h \), and let \( \xi_1, \xi_2 \) be their unit normal vector fields. Consider a totally geodesic inclusion \( j : \mathbb{Q}_c^2 \to \mathbb{Q}_c^4 \). The isometric immersion \( \hat{f}_k = j \circ f_k : M \to \mathbb{Q}_c^4 \), \( k = 1, 2 \), has non-parallel mean curvature vector field \( h_j, \xi_k \). Proposition 8 implies that it has non vertically harmonic Gauss lifts. The parallel vector bundle isometry \( T : N f_1 M \to N f_2 M \) given by \( T j, \xi_1 = j, \xi_2, T(J^1 j, \xi_1) = J^2 j, \xi_2 \), preserves the mean curvature vector fields, where \( J^k \) is the complex structure of the normal bundle of \( f_k \), \( k = 1, 2 \). Since the
image of the second fundamental form of \( \hat{f}_k \) is contained in the line bundle spanned by \( j_*\xi_k, k = 1, 2 \), it follows that \( Z = Z^- = Z^+ \). Hence, \( \hat{f}_2 \in \mathcal{M}^*(\hat{f}_1) \) and the proof follows from Theorem 10(ii).

Theorem 17. Let \( f: M \to \mathbb{Q}^4_c \) be a superconformal surface. If \( M \) is compact and oriented, then there exists at most one nontrivial congruence class of isometric immersions of \( M \) into \( \mathbb{Q}^4_c \), with the same mean curvature with \( f \).

Proof: Assume that \( f \) is non-minimal and let \( (f, \hat{f}) \) be a Bonnet pair. From Lemmas 7(ii) and 11 it follows that the normal curvature is everywhere non-vanishing on \( M \setminus Z \). Therefore, \( \pm K_N \geq 0 \) on \( M \). Lemma 7(ii) implies that \( \Phi^\pm \equiv 0 \) and thus, \( \hat{f} \in \mathcal{M}^\mp(f) \). Proposition 8 yields that \( G_\mp \) is vertically harmonic. If \( G_\mp \) is not vertically harmonic, then by Theorem 10(i) we obtain that \( \mathcal{M}^\mp(f) = \{ \hat{f} \} \) and consequently, \( \mathcal{M}(f) = \{ f, \hat{f} \} \).

If \( G_\mp \) is vertically harmonic, then \( f \) has parallel mean curvature vector field. Therefore, \( K_N = 0 \) on \( M \) and Lemma 7(ii) implies that \( f \) is totally umbilical. In the case where \( f \) is minimal, the result follows from [29] or [39].

We give an application to Lagrangian surfaces in \( \mathbb{R}^4 \). Let \( \tilde{J} \) be a canonical complex structure on \( \mathbb{R}^4 \) which is compatible with the orientation, i.e., for orthonormal vectors \( e_1, e_2 \in \mathbb{R}^4 \), the oriented orthonormal basis \( \{ e_1, e_2, \tilde{J}e_1, \tilde{J}e_2 \} \) is in the orientation of \( \mathbb{R}^4 \). Denote by \( \Omega(\cdot, \cdot) = \langle \cdot, \tilde{J} \cdot \rangle \) the associated Kähler form. A surface \( f: M \to \mathbb{R}^4 \) is Lagrangian if \( f^*\tilde{J} \hat{f} \). The Gauss map of a Lagrangian surface is given by \( \Theta = \hat{f}_1A_{\hat{f}_1}X, X \in TM \). Thus, the trilinear map \( C_f \) on \( TM \) given by

\[
C_f(X, Y, Z) = \Omega(\alpha(X, Y), f_*Z)
\]

is symmetric. Associated to \( f \) are its mean curvature form \( \Upsilon_f \) and the cubic differential \( \Theta_f \), given by

\[
\Upsilon_f = \Omega(H, f_*\partial)dz, \quad \Theta_f = \Omega(\alpha(\partial, \partial), f_*\partial)dz^3,
\]

in terms of a local complex coordinate \( z \), where \( \Omega \) and \( \hat{J}_f \) have been extended \( \mathbb{C} \)-linearly. Since \( \tilde{J} \) is compatible with the orientation, \( \hat{J}_f: TM \otimes \mathbb{C} \to N_fM \otimes \mathbb{C} \) satisfies

\[
\hat{J}_fT^{(1,0)}M = N_f^- M \quad \text{and} \quad \hat{J}_fT^{(0,1)}M = N_f^+ M.
\]

The Maslov form \( \varpi_f \) of \( f \), is the 1-form on \( M \) defined by \( \varpi_f(X) = (1/\pi)\Omega(f_*X, H) \). The Gauss map of a Lagrangian surface is \( g = (g_+, g_-): M \to S^2_+ \times S^2_- \), i.e., its second component lies in a great circle of \( S^2_- \). Lagrangian surfaces with conformal (respectively, harmonic) Maslov form provide examples of surfaces in \( \mathbb{R}^4 \) with harmonic \( g_+ \) (respectively, \( g_- \)). Indeed, the following was proved in [8].

Proposition 18. Let \( f: M \to (\mathbb{R}^4, \tilde{J}) \) be a Lagrangian surface. The following are equivalent:
(i) The Maslov form $\omega_f$ is conformal (respectively, harmonic).
(ii) The differential $\Theta_f$ (respectively, $\Upsilon_f$) is holomorphic.
(iii) The component $g_+$ (respectively, $g_-$) is harmonic.

Using Theorem 10, we are able to give a short proof of the following result due to He, Ma and Wang [24].

Theorem 19. Let $f: M \to (\mathbb{R}^4, \tilde{J})$ be a compact, oriented Lagrangian surface with mean curvature form $\Upsilon$. If its Maslov form is not conformal, then there exists at most one nontrivial congruence class of Lagrangian isometric immersions of $M$ into $(\mathbb{R}^4, \tilde{J})$, with mean curvature form $\Upsilon$.

Proof: Suppose that $f, \tilde{f}: M \to (\mathbb{R}^4, \tilde{J})$ are noncongruent Lagrangian surfaces with mean curvature forms $\Upsilon = \tilde{\Upsilon}$. It follows that $T = \tilde{J}_f \circ \tilde{J}_f^{-1}: N_f M \to N_{\tilde{f}} M$ is an orientation and mean curvature vector field-preserving, parallel vector bundle isometry. Let $(U, z)$ be a complex chart. From our assumption, we have that $C_f(\partial, \partial, \bar{\partial}) = C_{\tilde{f}}(\partial, \partial, \bar{\partial})$. Hence,

$$\langle \phi_{\tilde{f}} - T^{-1} \circ \phi_f^{-}, \tilde{J}_f \bar{\partial} \rangle \equiv 0 \text{ on } U,$$

where $\phi_{\tilde{f}}$ and $\phi_f^{-}$ are given by (12). Since $\phi_{\tilde{f}} - T^{-1} \circ \phi_f^{-} \in N_f^{-} U$ and $\tilde{J}_f \bar{\partial} \in N_{\tilde{f}} U$, it follows that $\phi_{\tilde{f}} - T^{-1} \circ \phi_f^{-} \equiv 0$ on $U$. Therefore, $\tilde{f} \in \mathcal{M}^+(f)$ and the proof follows from Theorem 10(i) and Proposition 18.

In [8] it was proved that if $f: M \to \mathbb{R}^4$ is a Lagrangian isometric immersion of a compact, oriented Riemannian manifold with conformal (respectively, harmonic) Maslov form, then genus($M$) ≤ 1 (respectively, genus($M$) ≥ 1). The classification of compact, oriented Lagrangian surfaces in $\mathbb{R}^4$ with conformal Maslov form, was given in [8]. It turns out that there exist Lagrangian tori in $\mathbb{R}^4$ with non-parallel mean curvature vector field and conformal Maslov form. Lagrangian surfaces with harmonic Maslov form are Hamiltonian minimal. Examples of Hamiltonian minimal Lagrangian tori in $\mathbb{R}^4$, with non-parallel mean curvature vector field, were constructed in [9] and the complete classification was given in [25]. Furthermore, it was proved in [7] that the only compact, orientable superconformal Lagrangian surface in $\mathbb{R}^4$ is the Whitney sphere. Therefore, there exist compact, oriented non-superconformal surfaces in $\mathbb{R}^4$, whose only one of the components $g_+, g_-$ of their Gauss map is harmonic.

4 Surfaces with a vertically harmonic Gauss lift

We need some facts about absolute value type functions (cf. [18] or [19]). A smooth complex function $t$ on $M$ is called of holomorphic type if locally it is expressed as $t = t_0 t_1$, where $t_0$ is holomorphic and $t_1$ is smooth without zeros. A non-negative function $u$ on $M$ is called of absolute value type, if there exists a function $t$ on $M$ of holomorphic type such
that } u = |t|. If an absolute value type function } } u does not vanish identically, then its zeros are isolated and they have well-defined multiplicities. Furthermore, the Laplacian } \Delta \log u \text{ is still defined and smooth at the zeros. If } M \text{ is compact and } u \text{ is an absolute value type function on } M, \text{ then}

\[ \int_M \Delta \log u = -2\pi N(u), \]

where } N(u) \text{ is the number of zeros of } u, \text{ counted with multiplicities.}

In the next proposition, we show that surfaces with a vertically harmonic Gauss lift satisfy Ricci-like conditions that extend the well-known Ricci condition (cf. [32]) for CMC surfaces in 3-dimensional space forms.

**Proposition 20.** Let } f : M \to Q_4^c \text{ be a non-minimal surface with mean curvature vector field } H \text{ and vertically harmonic Gauss lift } G_\pm. \text{ Then:}

(i) The quadratic differential } \Psi^\pm = \langle \Phi^\pm, H^\mp \rangle \text{ is holomorphic with zero-set } Z(\Psi^\pm) = M_0^\pm(f) \cup \{p \in M : H(p) = 0\}.

(ii) The functions } \|H\|^2 \text{ and } \|H\|^2 - (K - c) \mp K_N \text{ are of absolute value type with even multiplicities.}

(iii) We have that

\[ \Delta \log \|H\|^2 = \mp 2K_N, \]  \hfill (18)

\[ \Delta \log (\|H\|^2 - (K - c) \mp K_N) = 2(2K \pm K_N) \text{ if } \Psi^\pm \neq 0. \]  \hfill (19)

**Proof:** (i) The holomorphicity of } \Psi^\pm \text{ follows from Proposition 8. The zeros of } \Psi^\pm \text{ are precisely the points where } \langle \Phi^\pm, H \rangle = 0, \text{ which is equivalent to } \Phi^\pm = 0 \text{ at points where } H \neq 0.

(ii) Let } (U, z) \text{ be a complex chart. From Proposition 8(iv) we have}

\[ \nabla_{\bar{\partial}} H = \pm iJ^\perp \nabla_{\bar{\partial}} H. \]

This is equivalently written as

\[ (H^3 \pm iH^4) z = \mp i\omega_{34}(\bar{\partial})(H^3 \pm iH^4), \]

where } H = H^3e_3 + H^4e_4, \text{ and } \{e_3, e_4 = J^\perp e_3\} \text{ is a local orthonormal frame field of } N_f M. \text{ From [18, Lemma 9.1.] it follows that the function } H^3 \pm iH^4 \text{ is of holomorphic type and this proves our claim for } \|H\|^2.

From part (i), the function } \langle \phi^\pm, H^\mp \rangle \text{ is holomorphic, where } \phi^\pm \text{ is given by (12). Moreover,}

\[ |\langle \phi^\pm, H^\mp \rangle|^2 = \frac{\lambda^4 \|H\|^2}{16} (\|H\|^2 - (K - c) \mp K_N), \]  \hfill (20)

where } \lambda \text{ is the conformal factor. Clearly, the function}

\[ t = \frac{4\langle \phi^\pm, H^\mp \rangle}{\lambda^2 (H^3 \pm iH^4)} \]
can be smoothly extended to the zeros of $H$ as a holomorphic type function. Since $|t^2| = \|H\|^2 - (K - c) \mp K_N$, this completes the proof.

(iii) Away from the zeros of $H$, we consider the local orthonormal frame field \( \{e_3 = H/\|H\|, e_4 = J^\perp e_3\} \) of the normal bundle. Using Proposition 8(iv), we find that the normal connection form is given by

\[ \omega_{34} = \pm \star d \log \|H\|. \]

Then (18) follows from (2) and the above.

We choose a complex chart with coordinate $z$, away from the zeros of $\Psi^\pm$. From the holomorphicity of $\Psi^\pm$ we have that

\[ \Delta \log \|\phi^\pm, H^\mp\|^2 = 0. \]

Equation (19) follows from (20) and the fact that $\Delta \log \lambda = -K$.

**Proposition 21.** Let $f : M \rightarrow \mathbb{Q}^4_c$ be a compact surface with mean curvature vector field $H$ and vertically harmonic Gauss lift $G^\pm$.

(i) If $M$ is homeomorphic to $\mathbb{S}^2$, then $f$ is superconformal.

(ii) If $f$ is non-minimal, then

\[ 2\chi_N = \pm N(\|H\|^2). \]

(iii) If $f$ is neither minimal nor superconformal, then

\[ 2(2\chi \pm \chi_N) = -N\left(\|H\|^2 - (K - c) \mp K_N\right). \]

**Proof:** (i) From the assumption and Proposition 20(i) we obtain that $\Psi^\pm \equiv 0$. That $f$ is superconformal follows from Proposition 20(ii) if $f$ is non-minimal. If $f$ is minimal, then (8) implies that $\langle \Phi^-, \Phi^+ \rangle$ is a holomorphic fourth-order differential, whose zero-set is precisely $M_0$. Clearly, $f$ is superconformal.

The proofs of (ii) and (iii) follow immediately from Proposition 20(ii), by integrating (18) and (19), respectively.

**Proof of Theorem 3:** (i) For any $\theta \in \mathbb{R}$ define the symmetric section $\beta^\pm_\theta \in \Gamma(\text{Hom}(TM \times TM, N_f M))$ by

\[ \beta^\pm_\theta(X,Y) = J^\perp_{\theta/2} \left( \alpha(J_{\mp\theta/4} X, J_{\mp\theta/4} Y) - \langle X, Y \rangle H \right) + \langle X, Y \rangle H, \]

where $X, Y \in TM$, $J^\perp_\theta = \cos \theta I + \sin \theta J^\perp$ and $J_\theta = \cos \theta I + \sin \theta J$. We argue that $\beta^\pm_\theta$ satisfies the Gauss, Codazzi and Ricci equations. Clearly, we have that

\[ (\beta^+_\theta)^{(2,0)} = \Phi^- + e^{-i\theta} \Phi^+, \quad (\beta^-_\theta)^{(2,0)} = e^{i\theta} \Phi^- + \Phi^+ \]

and

\[ (\beta^\pm_\theta)^{(1,1)} = \alpha^{(1,1)}. \]
In terms of a local complex coordinate \( z = x + iy \), the Gauss equation for \( f \) is written as

\[(K - c)\lambda^4/4 = \|\alpha(\partial, \bar{\partial})\|^2 - \|\alpha(\partial, \bar{\partial})\|^2,\]

where \( \lambda \) is the conformal factor. Using that \( \|\alpha(\partial, \bar{\partial})\| = \|\beta^\pm_0(\partial, \bar{\partial})\| \), we deduce that \( \beta^\pm_0 \) satisfies the Gauss equation. Setting \( e_1 = \partial_x/\lambda, e_2 = \partial_y/\lambda \) and using (21), we have that

\[(\beta^\pm_1 - \beta^\pm_2) \wedge \beta^\pm_2 = (\alpha_{11} - \alpha_{22}) \wedge \alpha_{12},\]

where \( \alpha_{ij} = \alpha(e_i, e_j) \) and \( \beta^\pm_{ij} = \beta^\pm_0(e_i, e_j) \), \( i, j = 1, 2 \). Therefore, \( \beta^\pm_0 \) satisfies (5). Using (8) and Proposition 8, (21) gives that

\[\nabla^\perp_\theta \beta^\pm_0(\partial, \bar{\partial}) = \frac{\lambda^2}{2} \nabla^\perp_\theta H\]

and thus, \( \beta^\pm_0 \) satisfies the Codazzi equation. By the fundamental theorem of submanifolds, for every \( \theta \in \mathbb{R} \) there exists an isometric immersion \( f^\pm_\theta : M \to \mathbb{Q}^4_\pm \) and an orientation-preserving parallel vector bundle isometry \( T_\theta : N_f \to N_{f^\pm_\theta} M \) such that \( \alpha_{f^\pm_\theta} = T_\theta \circ \beta^\pm_0 \). Clearly, \( T_\theta H \) is the mean curvature vector field of \( f^\pm_\theta \), for any \( \theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi \mathbb{Z} \) and \( f^\pm_0 = f \).

(ii) Suppose that \( f \) is superconformal. We claim that \( \Phi^\pm \equiv 0 \). From Proposition 8, we know that \( \Phi^\pm \) is holomorphic. Hence, either its zeros are isolated, or \( \Phi^\pm \equiv 0 \). Assuming that \( \Phi^\pm \not\equiv 0 \), Lemma 7(ii) implies that \( \Phi^\pm \equiv 0 \) and consequently \( \Phi \) is holomorphic. Then, the mean curvature vector field is parallel and thus, \( f \) is totally umbilical, a contradiction. Hence, \( \Phi^\pm \equiv 0 \) and (21) yields that \( (\beta^\pm_0)^{(2,0)} = \Phi \) for any \( \theta \in \mathbb{S}^1 \). This implies that each \( T_\theta \) preserves the Hopf differential and the mean curvature vector field and consequently, it preserves the second fundamental form as well. This shows that the family is trivial.

(iii) Without loss of generality, we may assume that \( \hat{\theta} = 0 \). The distortion differential of the pair \((f, f^\pm_\theta)\) vanishes identically. Lemma 9(i) implies that any orientation and mean curvature vector field-preserving parallel vector bundle isometry \( T : N_f M \to N_{f^\pm_\theta} M \) preserves the Hopf differential, and consequently the second fundamental form as well. Hence, \( \beta^\pm_0 = T_\theta^{-1} \circ \alpha_{f_\theta} = \alpha \) and (21) implies that \( (1 - e^{\pm i\theta})\Phi^\pm \equiv 0 \). Since \( \theta \neq 0 \), the last relation yields \( \Phi^\pm \equiv 0 \) and thus, \( f \) is superconformal. \( \blacksquare \)

**Proposition 22.** Let \( f : M \to \mathbb{Q}^4_\pm \) be a simply-connected surface with vertically harmonic Gauss lift \( G_\pm \). If \( f \) is neither minimal nor superconformal, then \( \{f\} \cup M^\pm(f) = \mathbb{S}^1 \).

**Proof:** Let \( \hat{f} \in M^\pm(f) \). Appealing to Proposition 12, we have that the distortion differential of the pair \((f, \hat{f})\) is written as

\[Q_{f, \hat{f}} = (1 - e^{\pm i\theta})\Phi^\pm \text{ on } M \setminus Z(Q_{f, \hat{f}}),\]

where \( \theta^\pm \in C^\infty(M \setminus Z(Q_{f, \hat{f}})) \). From Lemma 9(i) and Proposition 8, we know that \( Q_{f, \hat{f}} \) and \( \Phi^\pm \) are holomorphic. Therefore, \( \theta^\pm \) is constant. Using (21), we obtain that \( Q_{f, f^\pm_\theta} \equiv Q_{f, \hat{f}} \) and thus, \( Q_{f, f^\pm_\theta} \equiv 0 \). Consequently, \( \hat{f} \) is congruent to \( f^\pm_\theta \) and this completes the proof. \( \blacksquare \)
The following proposition determines the moduli space of simply-connected surfaces with parallel mean curvature vector field. The two-parameter family given here, coincides up to a parameter transformation, with the one given by Eschenburg-Tribuzy [19].

**Proposition 23.** Let \( f: M \rightarrow \mathbb{Q}^4_c \) be a simply-connected surface with parallel mean curvature vector field \( H \neq 0 \). Then:

(i) There exists a two-parameter family of isometric immersions \( f_{\theta, \varphi}: M \rightarrow \mathbb{Q}^4_c, (\theta, \varphi) \in \mathbb{S}^1 \times \mathbb{S}^1 \), which have the same mean curvature with \( f_{0,0} = f \).

(ii) The family is trivial if and only if \( f \) is totally umbilical.

(iii) If \( f \) is not totally umbilical, then \( \mathcal{M}(f) = \mathbb{S}^1 \times \mathbb{S}^1 \).

**Proof:**

(i) Since both Gauss lifts are vertically harmonic, from Theorem 3 we may consider the two-parameter family \( f_{\theta, \varphi} = (f_{\theta})^+_{\varphi}, \theta, \varphi \in \mathbb{S}^1 \). Clearly, \( f_{\theta, \varphi} \) has the same mean curvature with \( f \).

(ii) From Theorem 3, it is clear that \( f_{\theta, \varphi} \) is congruent to \( f_{\tilde{\theta}, \tilde{\varphi}} \) for \( (\theta, \varphi) \neq (\tilde{\theta}, \tilde{\varphi}) \in \mathbb{S}^1 \times \mathbb{S}^1 \) if and only if \( f \) is superconformal. Since \( H \) is parallel, this can only occur if \( f \) is totally umbilical.

(iii) Let \( \hat{f} \in \mathcal{M}(f) \). Since \( M_0 = M_1 \subset Z(Q_{f,j}) \), from Proposition 12 it follows that

\[
Q_{f,j} = (1 - e^{i\theta})\Phi^- + (1 - e^{-i\theta})\Phi^+ \quad \text{on} \quad M \setminus Z(Q_{f,j}),
\]

for \( \theta^+ \in C^\infty(M \setminus Z(Q_{f,j})) \), with \( \theta^\pm = 0 \) if \( \hat{f} \in \mathcal{M}^+(f) \). From Lemma 9(i) and Proposition 8, we know that \( Q_{f,j}, \Phi^- \) and \( \Phi^+ \) are holomorphic. The above relation yields that the functions \( \theta^- \) and \( \theta^+ \) are constant. Using (21), one easily checks that the distortion differential \( Q \) of the pair \( (f, f_{\theta, \varphi}) \) is given by

\[
Q = (1 - e^{i\theta})\Phi^- + (1 - e^{-i\varphi})\Phi^+.
\]  

From (22) it follows that the distortion differentials of the pairs \( (f, \hat{f}) \) and \( (f, f_{\theta^-}) \) are equal and thus, the distortion differential of the pair \( (\hat{f}, f_{\theta^-, \theta^+}) \) vanishes identically. Therefore, \( \hat{f} \) is congruent to \( f_{\theta^-} \) and this completes the proof.

**Remark 24.**

(i) It is clear that if \( f \) is not totally umbilical, then \( \{f\} \cup \mathcal{M}^-(f) = \mathbb{S}^1 \times \{0\} \) and \( \{f\} \cup \mathcal{M}^+(f) = \{0\} \times \mathbb{S}^1 \).

(ii) We recall (cf. [10, 41]) that any surface with parallel mean curvature vector field \( H \neq 0 \) splits as \( f = j \circ f' \), where \( j: \mathbb{Q}^3_c \rightarrow \mathbb{Q}^4_c, c' \geq c \), is a totally umbilical inclusion and \( f': M \rightarrow \mathbb{Q}^3_c \), is a CMC-\( h' \) surface with \( h' = \pm(\|H\|^2 - (c' - c))^{1/2} \). It is known that there exists locally a bijective correspondence (the so-called Lawson correspondence [32, Theorem 8]) between CMC surfaces in 3-dimensional space forms. Since \( f_{\theta, \varphi} = j \circ f_{\theta, \varphi} \) and \( \|H_{f_{\theta, \varphi}}\| = \|H\| \), the surfaces \( f' \) and \( \hat{f}_{\theta, \varphi} \) are in Lawson correspondence for any \( \theta, \varphi \in \mathbb{S}^1 \). In particular, \( f_{\theta, 2\pi - \theta} \) is congruent to \( j \circ f_{\theta} \), where \( f', \theta \in \mathbb{S}^1 \), is the associated family of \( f' \) in \( \mathbb{Q}^3_{c'} \) as a CMC surface.
4.1 The moduli space of non-simply-connected surfaces

Let $M$ be a 2-dimensional oriented Riemannian manifold with nontrivial fundamental group and $f: M \to \mathbb{Q}_c^4$ a non-minimal isometric immersion. Consider the universal cover $(\tilde{M}, \tilde{\pi})$ of $M$, equipped with metric and orientation that make the covering map $\tilde{\pi}: \tilde{M} \to M$ an orientation-preserving local isometry. Then, $\tilde{f} = f \circ \tilde{\pi}: \tilde{M} \to \mathbb{Q}_c^4$ is an isometric immersion. It is clear that the Gauss lift $G_\pm$ of $\tilde{f}$ is vertically harmonic if and only if the Gauss lift $G_\pm$ of $f$ is vertically harmonic.

If $(f = f_1, f_2)$ is a Bonnet pair, then $(\tilde{f}_1, \tilde{f}_2)$ is also a Bonnet pair, where $\tilde{f}_j = f_j \circ \tilde{\pi}, j = 1, 2$. Moreover, $f_2 \in \mathcal{N}_{\pm}(f_1)$, if and only if $\tilde{f}_2 \in \mathcal{N}_{\pm}(\tilde{f}_1)$. If $G_\pm$ is vertically harmonic and $f_2 \in \mathcal{M}_{\pm}(f_1)$, then from Proposition 22 it follows that $\tilde{f}_2$ is congruent to some $\tilde{f}_{\theta}^\pm$ in the associated family of $\tilde{f}_1$. Therefore $\{f\} \cup \mathcal{M}_{\pm}(f)$ can be parametrized by the set

$$\left\{ \theta \in \mathbb{S}^1: \text{there exists } f_{\theta}: M \to \mathbb{Q}_c^4 \text{ such that } \tilde{f}_{\theta}^\pm = f_{\theta} \circ \tilde{\pi} \right\}.$$ 

In particular, if $H$ is parallel, then by Proposition 23, the moduli space $\mathcal{M}(f)$ can be parametrized by the set

$$\left\{ (\theta, \varphi) \in \mathbb{S}^1 \times \mathbb{S}^1: \text{there exists } f_{\theta, \varphi}: M \to \mathbb{Q}_c^4 \text{ such that } \tilde{f}_{\theta, \varphi} = f_{\theta, \varphi} \circ \tilde{\pi} \right\}.$$ 

The following is essential for the proof of Theorem 4. For its proof we adopt techniques used in [15, 37, 40].

**Proposition 25.** Let $f: M \to \mathbb{Q}_c^4$ be a non-minimal surface with mean curvature vector field $H$.

(i) If the Gauss lift $G_\pm$ of $f$ is vertically harmonic, then $\{f\} \cup \mathcal{M}_{\pm}(f)$ is either a finite set, or the circle $\mathbb{S}^1$.

(ii) If $H$ is parallel, then either $\mathcal{M}(f) = \mathbb{S}^1 \times \mathbb{S}^1$, or it locally decomposes as $\mathcal{M}(f) = \mathcal{V}_0 \cup \mathcal{V}_1$, where each $\mathcal{V}_d, d = 0, 1$, is either empty, or a disjoint finite union of $d$-dimensional real-analytic varieties.

**Proof:** (i) We claim that for any $\sigma \in \mathcal{D}$ in the group of deck transformations of the universal cover $\tilde{\pi}: \tilde{M} \to M$, the surfaces $\tilde{f}_{\theta}^\pm: \tilde{M} \to \mathbb{Q}_c^4$ in the associated family of $\tilde{f} = f \circ \tilde{\pi}$ and $\tilde{f}_{\theta}^\pm \circ \sigma$ are congruent for any $\theta \in \mathbb{S}^1$. It is sufficient to show the existence of a parallel vector bundle isometry between the normal bundles of $\tilde{f}_{\theta}^\pm$ and $\tilde{f}_{\theta}^\pm \circ \sigma$ that preserves the second fundamental forms. Let $T_\theta$ be the parallel vector bundle isometry between the normal bundles of $\tilde{f}$ and $\tilde{f}_{\theta}^\pm$ such that

$$\alpha_{\tilde{f}_{\theta}^\pm}(X, Y) = T_\theta \left( \tilde{J}_{\theta/2}(\alpha_{\tilde{f}}(\tilde{J}_{\pm \theta/4}X, \tilde{J}_{\pm \theta/4}Y) - \langle X, Y \rangle H_{\tilde{f}}) + \langle X, Y \rangle H_{\tilde{f}} \right)$$

for any $X, Y \in T\tilde{M}$, where $\tilde{J}_{\theta/2} = \cos \theta \tilde{I} + \sin \theta \tilde{J}^\perp$, $\tilde{J}_0 = \cos \theta \tilde{I} + \sin \theta \tilde{J}$ and $\tilde{J}^\perp, \tilde{J}$ stand for the complex structures of $N_{\tilde{f}}\tilde{M}$ and $T\tilde{M}$, respectively. Since $\sigma$ is a deck transformation, we have that $\tilde{f} \circ \sigma = \tilde{f}$ and thus, the normal spaces satisfy $N_{\tilde{f}}\tilde{M}(p) = N_{\tilde{f}}\tilde{M}(\sigma(p))$ at
any \( p \in \tilde{M} \). We define the vector bundle isometry \( \Sigma_\theta : N_{\tilde{f}_\theta^\pm} \tilde{M} \to N_{\tilde{f}_\theta^\pm} \tilde{M} \) which is given pointwise by

\[
\Sigma_\theta|_p(\xi) = T_\theta|_{\sigma(p)} \circ (T_\theta|_p)^{-1}(\xi), \quad \xi \in N_{\tilde{f}_\theta^\pm} \tilde{M}(p).
\]

The second fundamental forms of \( \tilde{f}_\theta^\pm \) and \( \tilde{f}_\theta^\pm \circ \sigma \) are related at \( p \in \tilde{M} \) by

\[
\alpha_{\tilde{f}_\theta^\pm \circ \sigma}|_p(X, Y) = \alpha_{\tilde{f}_\theta^\pm}|_{\sigma(p)}(\sigma_* X, \sigma_* Y)
\]

\[
= T_\theta|_{\sigma(p)} \left( \tilde{J}_{\theta/2}^\perp (\alpha_{\tilde{f}_\theta^\pm}|_{\sigma(p)}(\tilde{J}_{\theta/4} \sigma_* X, \tilde{J}_{\theta/4} \sigma_* Y) - \langle X, Y \rangle \tilde{H}_{\tilde{f}}|_{\sigma(p)}) + \langle X, Y \rangle \tilde{H}_{\tilde{f}}|_p \right)
\]

\[
= T_\theta|_{\sigma(p)} \left( \tilde{J}_{\theta/2}^\perp (\alpha_{\tilde{f}_\theta^\pm}|_{\sigma(p)}(\tilde{J}_{\theta/4} \sigma_* X, \tilde{J}_{\theta/4} \sigma_* Y) - \langle X, Y \rangle \tilde{H}_{\tilde{f}}|_{\sigma(p)}) + \langle X, Y \rangle \tilde{H}_{\tilde{f}}|_p \right)
\]

\[
= \Sigma_\theta|_p \circ \alpha_{\tilde{f}_\theta^\pm}|_p(X, Y)
\]

for any \( X, Y \in \tilde{T}_M \) and thus, \( \Sigma_\theta \) preserves the second fundamental forms. For any section \( \xi \) of \( N_{\tilde{f}_\theta^\pm} \tilde{M} \) we have \( \Sigma_\theta \xi = T_\theta(\eta \circ \sigma^{-1}) \circ \sigma \), where \( \xi = T_\theta \eta \) for a section \( \eta \) of \( N_{\tilde{f}} \tilde{M} \). Using the fact that for any section \( \delta \) of \( N_{\tilde{f}} \tilde{M} \) and any deck transformation \( \sigma \) we have that \( \nabla_X^\perp (\delta \circ \sigma) = \nabla_{\sigma_* X}^\perp \delta \circ \sigma \), we obtain

\[
(\nabla_X^\perp \Sigma_\theta) \xi = \nabla_X^\perp (T_\theta(\eta \circ \sigma^{-1}) \circ \sigma) - T_\theta (\nabla_X^\perp \eta \circ \sigma^{-1}) \circ \sigma
\]

\[
= (\nabla_{\sigma_* X} T_\theta(\eta \circ \sigma^{-1})) \circ \sigma - T_\theta (\nabla_X^\perp \eta \circ \sigma^{-1}) \circ \sigma
\]

\[
= T_\theta (\nabla_{\sigma_* X}^\perp (\eta \circ \sigma^{-1}) - \nabla_X^\perp \eta \circ \sigma^{-1}) \circ \sigma,
\]

where, by abuse of notation, \( \nabla^\perp \) stands for the normal connection of \( \tilde{f}, \tilde{f}_\theta^\pm \) and \( \tilde{f}_\theta^\pm \circ \sigma \).

Observe that

\[
\nabla_{\sigma_* X}^\perp (\eta \circ \sigma^{-1}) = \nabla_X^\perp \eta \circ \sigma^{-1},
\]

and thus \( \Sigma_\theta \) is parallel and the claim has been proved.

This allows us to define a homomorphism \( S_\theta : \mathcal{D} \to \text{Isom}(\mathbb{Q}_{c}^\times) \) for each \( \theta \in [0, 2\pi] \), such that

\[
\tilde{f}_\theta^\pm \circ \sigma = S_\theta \circ \tilde{f}_\theta^\pm, \quad \sigma \in \mathcal{D}.
\]

Thus, \( \theta \in \{ f \} \cup \mathcal{M}_c^\pm(f) \) if and only if \( S_\theta(\mathcal{D}) = \{ I \} \). Assume that \( \{ f \} \cup \mathcal{M}_c^\pm(f) \) is infinite and let \( \{ \theta_m \} \) be a sequence in \( \{ f \} \cup \mathcal{M}_c^\pm(f) \) which converges to some \( \theta_0 \in [0, 2\pi] \). From \( S_{\theta_m}(\mathcal{D}) = \{ I \} \) for all \( m \in \mathbb{N} \) we obtain that \( S_{\theta_0}(\mathcal{D}) = \{ I \} \). Let \( \sigma \in \mathcal{D} \). By the mean value theorem applied to each entry \( (S_\theta(\sigma))_{jk} \) of the corresponding matrix, we have

\[
\frac{d}{d\theta}(S_\theta(\sigma))_{jk}(\theta_m) = 0
\]  

(23)
for some \( \hat{\theta}_m \) which lies between \( \theta_0 \) and \( \theta_m \). By continuity it follows that

\[
\frac{d}{d\hat{\theta}} (S_{\hat{\theta}}(\sigma))_{jk}(\theta_0) = 0.
\]

Consider the sequence \( \{\hat{\theta}_m\} \) that converges to \( \theta_0 \) and observe that in view of (23), a similar argument gives

\[
\frac{d^2}{d\theta^2} (S_{\theta}(\sigma))_{jk}(\theta_0) = 0.
\]

Repeating the argument yields

\[
\frac{d^n}{d\theta^n} (S_{\theta}(\sigma))_{jk}(\theta_0) = 0
\]

for any integer \( n \geq 1 \). From the definition of the associated family, it is clear that \( f_{\theta}^\pm \) depends on the parameter \( \theta \) in a real-analytic way. Since \( S_{\theta}(\sigma) \) is an analytic curve in \( \text{Isom}(\mathbb{Q}_4^1) \), we conclude that \( S_{\theta}(\sigma) = I \) for each \( \sigma \in \mathcal{D} \), and thus \( \{f\} \cup M^\pm(f) = S^1 \).

(ii) We claim that for any \( \sigma \in \mathcal{D} \), the surfaces \( \tilde{f}_{\theta,\varphi} : \tilde{M} \to \mathbb{Q}_4^1 \) and \( \tilde{f}_{\theta,\varphi} \circ \sigma \) in \( \mathcal{M}(\tilde{f}) \) are congruent for any \( (\theta, \varphi) \in S^1 \times S^1 \). Let \( T_{\theta,\varphi} \) be the parallel vector bundle isometry between the normal bundles of \( \tilde{f} \) and \( \tilde{f}_{\theta,\varphi} \) such that

\[
\alpha_{\tilde{f}_{\theta,\varphi}}(X,Y) = T_{\theta,\varphi} \left( \frac{\tilde{J}_{(\theta+\varphi)/2} \left( \alpha_{\tilde{f}}(\tilde{J}_{(\theta-\varphi)/4} X, \tilde{J}_{(\theta-\varphi)/4} Y) - \langle X,Y \rangle H_{\tilde{f}} \right)}{\langle X,Y \rangle H_{\tilde{f}}} \right)
\]

for any \( X,Y \in TM \). We define the vector bundle isometry \( \Sigma_{\theta,\varphi} : N_{\tilde{f}_{\theta,\varphi}} \tilde{M} \to N_{\tilde{f}_{\theta,\varphi} \circ \sigma} \tilde{M} \) which is given pointwise by

\[
\Sigma_{\theta,\varphi}|_p(\xi) = T_{\theta,\varphi}|_{\sigma(p)} \circ (T_{\theta,\varphi}|_p)^{-1}(\xi), \quad \xi \in N_{\tilde{f}_{\theta,\varphi}} \tilde{M}(p).
\]

As in the proof of part (i) above, it can be shown that \( \Sigma_{\theta,\varphi} \) is parallel and preserves the second fundamental forms, and the claim follows. This allows us to define a homomorphism \( S_{\theta,\varphi} : \mathcal{D} \to \text{Isom}(\mathbb{Q}_4^1) \) for each \( \theta, \varphi \in [0, 2\pi] \), such that

\[
\tilde{f}_{\theta,\varphi} \circ \sigma = S_{\theta,\varphi} \circ \tilde{f}_{\theta,\varphi}, \quad \sigma \in \mathcal{D}.
\]

Clearly, \( (\theta, \varphi) \in \mathcal{M}(f) \) if and only if \( S_{\theta,\varphi}(\mathcal{D}) = \{I\} \). Since \( \tilde{f}_{\theta,\varphi} \) is real-analytic with respect to \( (\theta, \varphi) \), it follows that \( \mathcal{M}(f) \) is a real-analytic set. According to Lojacewicz’s structure theorem [31, Theorem 6.3.3.], \( \mathcal{M}(f) \) locally decomposes as

\[
\mathcal{M}(f) = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2,
\]

where each \( \mathcal{V}_d, 0 \leq d \leq 2 \), is either empty, or a disjoint finite union of \( d \)-dimensional real-analytic subvarieties. If \( \mathcal{M}(f) \neq S^1 \times S^1 \), then \( \mathcal{V}_2 = \emptyset \) and this completes the proof. \( \blacksquare \)
4.2 Surfaces in $\mathbb{R}^4$

In the sequel, we deal with surfaces in $\mathbb{R}^4$ whose one component of the Gauss map is harmonic. We regard the Grassmannian $Gr(2, 4)$ of oriented 2-planes in $\mathbb{R}^4$ as a submanifold in $\Lambda^2\mathbb{R}^4$ via the Plücker embedding. The inner product of two simple 2-vectors in $\Lambda^2\mathbb{R}^4$ is given by

$$\langle\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle\rangle = \det(\langle v_j, w_k \rangle).$$

Then, $\Lambda^2\mathbb{R}^4$ splits orthogonally into the eigenspaces of the Hodge star operator $\star$, denoted by $\Lambda^2_+\mathbb{R}^4$ and $\Lambda^2_-\mathbb{R}^4$, corresponding to the eigenvalues 1 and $-1$, respectively. An element $a \wedge b$ of $Gr(2, 4)$, where $a, b$ are orthonormal vectors in $\mathbb{R}^4$, decomposes as

$$a \wedge b = (a \wedge b)_+ + (a \wedge b)_-,$$

where $(a \wedge b)_\pm = \frac{1}{2}(a \wedge b \pm \star(a \wedge b)).$

Therefore, $Gr(2, 4)$ can be identified with the product $S^2_+ \times S^2_-$, where $S^2_\pm$ is the sphere of radius $1/\sqrt{2}$ in $\Lambda^2_\pm\mathbb{R}^4$, centered at the origin.

Let $f: M \rightarrow \mathbb{R}^4$ be a non-minimal surface, with mean curvature vector field $H$ and Gauss map $g = (g_+, g_-): M \rightarrow S^2_+ \times S^2_-$. In terms of a local complex coordinate $z$ away from the zeros of $H$, the components of the Gauss map are given by

$$g_\pm = -\frac{i}{\lambda^2} f_\ast \partial \wedge f_\ast \bar{\partial} \mp \frac{i}{\|H\|^2} H^- \wedge H^+,$$

where $\lambda$ is the conformal factor. The differential $\Psi_\pm$ is written as

$$\Psi_\pm = \psi_\pm dz^2,$$

and $\phi_\pm$ is given by (12). The Gauss and Weingarten formulas become respectively,

$$\nabla_{\partial} f_\ast \partial = (\log \lambda^2) z f_\ast \partial + \frac{2\psi^-}{\|H\|^2} H^- + \frac{2\psi^+}{\|H\|^2} H^+,$$

$$\nabla_{\partial} f_\ast \bar{\partial} = \frac{\lambda^2}{2}(H^- + H^+),$$

$$\nabla_{\partial} H^\pm = -\frac{\|H\|^2}{2} f_\ast \partial - \frac{2\psi^\pm}{\lambda^2} f_\ast \bar{\partial} + \frac{2\langle \nabla_{\partial} H^\pm, H^\mp \rangle}{\|H\|^2} H^\pm,$$

where $\nabla$ is the induced connection on the induced bundle $f^\ast T\mathbb{R}^4$.

**Lemma 26.** Let $f: M \rightarrow \mathbb{R}^4$ be a non-minimal surface. If the component $g_\pm$ of the Gauss map of $f$ is harmonic, then its height functions in $\Lambda^2_\pm\mathbb{R}^4$ are eigenfunctions of the elliptic operator $\Delta + 2(2\|H\|^2 - K \mp K_N)$, corresponding to the zero eigenvalue.

**Proof:** Let $z$ be a local complex coordinate away from the isolated zeros of $H$ (see Proposition 20(ii)). By using (26)-(28), equation (24) yields

$$(g_\pm)_z = \frac{4i\psi^\pm}{\lambda^2\|H\|^2} f_\ast \partial \wedge H^\mp - if_\ast \partial \wedge H^\mp.$$
Differentiating (29) with respect to $\bar{z}$, we obtain that the normal component of $(g_{\pm})_{z\bar{z}}$ with respect to $S^2_{\pm}$ is given by

$$( (g_{\pm})_{z\bar{z}} )^\perp = -\frac{\lambda^2}{2} (2\|H\|^2 - K \mp K_N) g_{\pm}.$$  

For an arbitrary vector $v_{\pm} \in \Lambda^2_{\pm}\mathbb{R}^4$ we have

$$\Delta \langle \langle g_{\pm}, v_{\pm} \rangle \rangle = \langle \langle \tau(g_{\pm}) + \frac{4}{\lambda^2} ((g_{\pm})_{z\bar{z}})^\perp, v_{\pm} \rangle \rangle,$$

where $\tau(g_{\pm})$ is the tension field of $g_{\pm}$. The result follows from the above and the harmonicity of $g_{\pm}$. $\blacksquare$

**Lemma 27.** Let $f : M \to \mathbb{R}^4$ be a surface, which is neither minimal nor superconformal. Assume that $g_{\pm}$ is harmonic and that there exist surfaces $f_j \in \mathcal{M}^\pm(f)$ with $\bar{f}_{\pm} = f_j \circ \bar{\pi}$, and vectors $v_{\pm}^j \in \Lambda^2_{\pm}\mathbb{R}^4 \setminus \{0\}$, $j = 1, \ldots, n$, such that the Gauss maps $g^j = (g_{\pm}^j, g_{\pm}^j)$ of $f_j$ satisfy

$$\sum_{j=1}^n \langle \langle g_{\pm}^j, v_{\pm}^j \rangle \rangle = 0. \quad (30)$$

Then:

(i) The differential $U^\pm = u^\pm dz$ is holomorphic, where

$$u^\pm = \sum_{j=1}^n \langle \langle f_j \partial \wedge H^\pm_j, v_{\pm}^j \rangle \rangle \quad (31)$$

and $H_j$ is the mean curvature vector field of $f_j$.

(ii) If $U^\pm \equiv 0$, then

$$\sum_{j=1}^n e^{i\theta_j} \langle \langle g_{\pm}^j, v_{\pm}^j \rangle \rangle = 0. \quad (32)$$

**Proof:** From (21) and since by definition $\Psi_{f_j}^\pm = \langle \Phi_{f_j}^\pm, H^\pm_j \rangle$, we have that

$$\Psi_{f_j}^\pm = e^{\mp i\theta_j} \Psi^\pm, \quad j = 1, \ldots, n.$$  

Let $(U, z)$ be a complex chart. On $U \setminus Z(\Psi^\pm)$, (29) yields

$$(g_{\pm}^j)_z = e^{\mp i\theta_j} \frac{4i\psi_{\pm}}{\lambda^2\|H\|^2} f_j \bar{\partial} \wedge H_j^\mp - if_j \partial \wedge H_j^\pm, \quad j = 1, \ldots, n. \quad (33)$$

Differentiating (30) with respect to $z$ and using (33), we find that

$$u^\pm = \frac{4\psi_{\pm}}{\lambda^2\|H\|^2} \sum_{j=1}^n e^{\mp i\theta_j} \langle \langle f_j \bar{\partial} \wedge H^\pm_j, v_{\pm}^j \rangle \rangle \text{ on } U \setminus Z(\Psi^\pm). \quad (34)$$

28
From Proposition 8 it follows that $H^\pm_j$ is an anti-holomorphic section. Hence, 
\[ \nabla_\partial H^\pm_j = 0 \quad \text{and} \quad (\|H\|^2)_z = 2(\nabla_\partial H^\pm_j, H^\pm_j), \quad j = 1, \ldots, n. \] (35)
Differentiating (34) with respect to $z$, and using (27), (28), (35), (24) and (34), we obtain that 
\[ u^\pm_z = u^\pm_0 \left( \log \frac{\psi^\pm}{\lambda^2\|H\|^2} \right) + 2i\psi^\pm \sum_{j=1}^n e^{\pm i\theta_j} \langle \langle g^j_\pm, v^j_\pm \rangle \rangle \text{ on } U \setminus Z(\Psi^\pm). \] (36)
On the other hand, differentiating (31) with respect to $z$, and using (26), (28), (35), (24) and (31), we find that 
\[ u^\pm_z = u^\pm_0 \left( \log \left( \frac{\lambda^2\|H\|^2}{2i} \right) \right) - 2i\psi^\pm \sum_{j=1}^n e^{\pm i\theta_j} \langle \langle g^j_\pm, v^j_\pm \rangle \rangle \text{ on } U \setminus Z(\Psi^\pm). \] (37)

(i) From (27), (28), (35) and (24), we have that 
\[ u^\pm_z = \frac{\lambda^2\|H\|^2}{2i} \sum_{j=1}^n \langle \langle g^j_\pm, v^j_\pm \rangle \rangle \text{ on } U \] (38)
and the claim follows from (30).

(ii) Using (36) and (37), we obtain that 
\[ \sum_{j=1}^n e^{\pm i\theta_j} \langle \langle g^j_\pm, v^j_\pm \rangle \rangle = \frac{i u^\pm_0}{4\psi^\pm} \left( \log \frac{\psi^\pm}{\lambda^4\|H\|^4} \right)_z \text{ on } U \setminus Z(\Psi^\pm) \] (39)
and (32) follows from (39).

Theorem 28. Let $f: M \to \mathbb{R}^4$ be a non-superconformal isometric immersion of a compact, oriented 2-dimensional Riemannian manifold, with mean curvature vector field $H$ and Gauss map $g = (g_+, g_-): M \to S^2_+ \times S^2_-$. 

(i) If $g^\pm_\pm$ is harmonic and $\chi \neq \mp \chi_N$, then $\mathcal{M}^\pm(f)$ is a finite set.

(ii) If $H$ is parallel and $\chi \neq 0$, then $\mathcal{M}(f)$ is a finite set.

Proof: (i) Suppose that $\mathcal{M}^\pm(f)$ is infinite and consider surfaces $f_j \in \mathcal{M}^\pm(f)$ such that $f^\pm_{\theta_j} = f_j \circ \tilde{\pi}$, $j = 1, \ldots, n$, with $0 < \theta_1 < \cdots < \theta_n < \pi$ or $\pi < \theta_1 < \cdots < \theta_n < 2\pi$. We prove that the height functions of the $\Lambda^2_\pm \mathbb{R}^4$-component of the Gauss maps of $f_j$ are linearly independent. Suppose to the contrary that (30) holds for vectors $v^j_\pm \in \Lambda^2_\pm \mathbb{R}^4 \setminus \{0\}$, $j = 1, \ldots, n$.

We claim that $\mathcal{U}^\pm \equiv 0$. Arguing indirectly, assume that $\mathcal{U}^\pm \neq 0$. From Lemmas 6 and 27(i), it follows that its zero-set $Z(\mathcal{U}^\pm)$ is isolated. Let $z$ be a complex coordinate in a connected neighbourhood $U \subset M \setminus (Z(\Psi^\pm) \cup Z(\mathcal{U}^\pm))$. From (36) and (37), we obtain 
\[ \left( \log \frac{\psi^\pm}{(u^\pm)^2} \right)_z = 0. \]
Using Proposition 20(i) and Lemma 27(i), we have

\[
\left( \log \frac{\psi^\pm}{(u^\pm)^2} \right)_z = 0.
\]

Therefore,

\[
\psi^\pm = c(u^\pm)^2 \tag{40}
\]

on \( U \), for a non-zero constant \( c \in \mathbb{C} \). It is easy to see that \( c \) is independent of the complex coordinate and thus, \( \psi^\pm = c U^\pm \otimes U^\pm \) on \( M \). We argue that \( Z(\psi^\pm) = Z(U^\pm) \neq \emptyset \). Indeed, if \( Z(\psi^\pm) = \emptyset \), then the holomorphic differential \( \psi^\pm \) is everywhere nonvanishing and by the Riemann-Roch theorem we obtain that \( \chi = 0 \). On the other hand, Proposition 20(i) implies that \( H \) is everywhere nonvanishing and Proposition 21(ii) gives \( \chi_N = 0 \). This contradicts our assumption. Let \( Z(\psi^\pm) = \{p_1, \ldots, p_k\} \) and consider a complex chart \((U, z)\) around \( p_r, r = 1, \ldots, k \), with \( z(p_r) = 0 \). Since \( U^\pm = u^\pm dz \) is holomorphic, there exists a positive integer \( m_r \) such that around \( p_r \) we have

\[
u^\pm = z^{m_r} \hat{u}, \quad \text{where } \hat{u} \text{ is holomorphic with } \hat{u}(0) \neq 0. \tag{41}
\]

Hence, from (40) we have that \(|\psi^\pm|^2 = |z|^{4m_r} |c|^2 |\hat{u}|^4\), or equivalently, bearing in mind (20) and (25)

\[
\|H\|^2(\|H\|^2 - K \mp K_N) = |z|^{4m_r} u_1, \quad \text{where } u_1 \text{ is smooth and positive.}
\]

Proposition 20(ii) implies that there exist non-negative integers \( l_r, s_r \) such that

\[
\|H\|^2 = |z|^{2l_r} u_2 \quad \text{and} \quad \|H\|^2 - K \mp K_N = |z|^{2s_r} u_3,
\]

where \( u_2, u_3 \) are smooth and positive. It is clear that \( s_r = 2m_r - l_r \). From (39), by using (40), (41) and the above, on \( U \setminus Z(\psi^\pm) \) we have that

\[
\sum_{j=1}^n e^{\pi i \theta_j} \left\langle \langle g_{\pm}^j, v_{\pm}^j \rangle \right\rangle = \frac{i \lambda^2 \|H\|^2}{2c(u^\pm)^2} \left( \frac{u^\pm}{\lambda^2 \|H\|^2} \right)_z = \frac{i \lambda^2 z^{l_r} \hat{u}}{2cz^{2m_r} u_2^2} \left( \frac{z^{m_r} \hat{u}}{\lambda^2 z^{l_r} \hat{u}} \right)_z,
\]

or equivalently

\[
\sum_{j=1}^n e^{\pi i \theta_j} \left\langle \langle g_{\pm}^j, v_{\pm}^j \rangle \right\rangle = \frac{1}{z^{m_r + 1}} \frac{i \lambda^2 u_2}{2c\hat{u}^2} \left( (m_r - l_r) \frac{\hat{u}}{\lambda^2 u_2} + z \left( \frac{\hat{u}}{\lambda^2 u_2} \right)_z \right).
\]

If \( m_r \neq l_r \) for some \( r = 1, \ldots, k \), then the right-hand side of the above has a pole at \( z = 0 \), whereas the left-hand side is bounded. Hence, \( m_r = l_r = s_r \) for any \( r = 1, \ldots, k \). Then, Proposition 21 implies that \( \chi = \mp \chi_N \), which is a contradiction. Therefore, \( U^\pm \equiv 0 \) and this proves the claim.
According to Lemma 27(ii), (32) is valid, or equivalently
\[ \sum_{j=1}^{n} \cos \theta_j \langle g^j_{\pm}, v^j_{\pm} \rangle = 0 \quad \text{and} \quad \sum_{j=1}^{n} \sin \theta_j \langle g^j_{\pm}, v^j_{\pm} \rangle = 0. \]

Eliminating \( \langle g^n_{\pm}, v^n_{\pm} \rangle \), we obtain
\[ \sum_{j=1}^{n-1} \langle g^j_{\pm}, w^j_{\pm} \rangle = 0, \]
where \( w^j_{\pm} = \sin(\theta_n - \theta_j) v^j_{\pm} \neq 0, j = 1, \ldots, n-1 \). By induction, we finally find that
\( \langle g^n_{\pm}, w^n_{\pm} \rangle = 0 \) for some non-zero vector \( w_{\pm} \in \Lambda^2_{\pm} \mathbb{R}^4 \). Therefore, \( g^n_{\pm} \) takes values in a great circle of \( S^2_{\pm} \) and thus, its Jacobian \( J_{g^n_{\pm}} \) vanishes. On the other hand, we know that (cf. [27, Proposition 4.5])
\[ K = J_{g^n_{\pm}} + J_{g^n_{\mp}} \quad \text{and} \quad K_N = J_{g^n_{\pm}} - J_{g^n_{\mp}}. \]
Hence, we conclude that \( K = \mp K_N \), which contradicts our topological assumption. Therefore, we have proved that the height functions of the \( \Lambda^2_{\pm} \mathbb{R}^4 \)-component of the Gauss maps of \( f_j \) are linearly independent. This contradicts Lemma 26, since the eigenspaces of an elliptic operator are finite dimensional. Hence, \( M^\pm(f) \) is a finite set.

(ii) Assume that \( M(f) \) is infinite. Then there exists a sequence \( f_k \in M(f) \) such that \( f_{\theta_k, \varphi_k} = f_k \circ \tilde{\pi} \), for which \( (\theta_l, \varphi_l) \neq (\theta_m, \varphi_m) \) for \( l \neq m \). Without loss of generality, we may assume that either \( 0 < \theta_l < \theta_m < \pi \), or \( \pi < \theta_l < \theta_m < 2\pi \), for \( l, m \in \mathbb{N} \) with \( l < m \). We prove that the height functions of the \( \Lambda^2_{\pm} \mathbb{R}^4 \)-component of the Gauss maps of \( f_j, j = 1, \ldots, n \), are linearly independent. Suppose to the contrary that (30) holds for vectors \( v^j_{\pm} \in \Lambda^2 \mathbb{R}^4 \setminus \{0\}, j = 1, \ldots, n \). From (22) it follows that \( \Psi^{-1}_f = e^{i\theta^f} \Psi^{-f} \). Consequently, the relations (33)-(39) are valid and thus, the conclusion of Lemma 27 also holds. Taking into account that \( K_N = 0 \), we can prove as in the proof of part (i) that our topological assumption implies \( U^- \equiv 0 \). The remaining of the proof is the same with the one of part (i).

**Proof of Theorem 4**: (i) From Proposition 25(i) we know that \( \{f\} \cup M^\pm(f) \) is either finite, or the circle \( S^1 \). We show that the same holds true for the set \( M^*(f) \cup M^\mp(f) \).

Suppose that \( M^*(f) \cup M^\mp(f) \) is infinite. Since \( G_{\mp} \) is not vertically harmonic, from Theorem 10(i) it follows that \( M^\mp(f) \) contains at most one congruence class and thus, \( M^*(f) \) is infinite. For \( \tilde{f} \in M^*(f) \), Theorem 10(i) implies that \( M^*(f) \cup M^\mp(f) = \{f\} \cup M^\mp(f) \) and the proof follows from Proposition 25(i) applied to the surface \( f \).

(ii) The case where \( H \) is parallel has been proved in Theorem 28(ii). Assume that \( H \) is non-parallel and suppose to the contrary that \( M(f) \) is infinite. From Theorem 28(ii) it follows that \( M^\pm(f) \) is finite. Since \( g_{\mp} \) is not harmonic, Theorem 10(i) implies that \( M^\mp(f) \) contains at most one congruence class and therefore, \( M^*(f) \) is infinite. Theorem 10(i) yields that \( M^*(f) \cup M^\mp(f) = \{\tilde{f}\} \cup M^\mp(\tilde{f}) \) for any \( \tilde{f} \in M^*(f) \), which contradicts Theorem 28(i) for \( \tilde{f} \).
References

[1] U. Abresch and H. Rosenberg, A Hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$, Acta Math. 193 (2004), no. 2, 141–174, DOI 10.1007/BF02392562. MR2134864

[2] R. Aiyama, K. Akutagawa, R. Miyaoka, and M. Umehara, A global correspondence between CMC-surfaces in $S^3$ and pairs of non-conformal harmonic maps into $S^2$, Proc. Amer. Math. Soc. 128 (2000), no. 3, 939–941, DOI 10.1090/S0002-9939-99-05580-X. MR1707134

[3] A.I. Bobenko and M. Umehara, Monodromy of isometric deformations of CMC surfaces, Hiroshima Math. J. 31 (2001), no. 2, 291–297. MR1849192

[4] J. Bolton, T.J. Willmore, and L.M. Woodward, Immersions of surfaces into space forms, Global differential geometry and global analysis 1984 (Berlin, 1984), Lecture Notes in Math., vol. 1156, Springer, Berlin, 1985, pp. 46–58, DOI 10.1007/BFb0075085, (to appear in print). MR824061

[5] O. Bonnet, Mémoire sur la théorie des surfaces applicables, J. Éc. Polyt. 42 (1867), 72–92.

[6] É. Cartan, Sur les couples de surfaces applicables avec conservation des courbures principales, Bull. Sci. Math. (2) 66 (1942), 55–72, 74–85 (French). MR0009876

[7] I. Castro, Lagrangian surfaces with circular ellipse of curvature in complex space forms, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 1, 239–245, DOI 10.1017/S0305004103007126. MR2034586

[8] I. Castro and F. Urbano, Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form, Tohoku Math. J. (2) 45 (1993), no. 4, 565–582, DOI 10.2748/tmj/1178225850. MR1245723

[9] ______, Examples of unstable Hamiltonian-minimal Lagrangian tori in $C^2$, Compositio Math. 111 (1998), no. 1, 1–14, DOI 10.1023/A:1000332524827. MR1611051

[10] B.-Y. Chen, On the surface with parallel mean curvature vector, Indiana Univ. Math. J. 22 (1972/73), 655–666. MR0315606

[11] S.S. Chern, On the minimal immersions of the two-sphere in a space of constant curvature, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, N.J., 1970, pp. 27–40. MR0362151

[12] ______, Deformation of surfaces preserving principal curvatures, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 155–163. MR780041

[13] M. Dajczer and D. Gromoll, Real Kaehler submanifolds and uniqueness of the Gauss map, J. Differential Geom. 22 (1985), no. 1, 13–28. MR826424

[14] ______, Euclidean hypersurfaces with isometric Gauss maps, Math. Z. 191 (1986), no. 2, 201–205, DOI 10.1007/BF01164024. MR818664

[15] M. Dajczer and Th. Vlachos, Isometric deformations of isotropic surfaces, Arch. Math. (Basel) 106 (2016), no. 2, 189–200, DOI 10.1007/s00013-015-0857-z. MR3453994

[16] B. Daniel, Isometric immersions into 3-dimensional homogeneous manifolds, Comment. Math. Helv. 82 (2007), no. 1, 87–131, DOI 10.4171/CMH/86. MR2296059

[17] J. Eells and S. Salamon, Twistorial construction of harmonic maps of surfaces into four-manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), no. 4, 589–640 (1986). MR848842

[18] J.H. Eschenburg, I.V. Guadalupe, and R. Tribuzy, The fundamental equations of minimal surfaces in $CP^2$, Math. Ann. 270 (1985), no. 4, 571–598, DOI 10.1007/BF01455305. MR776173

[19] J.H. Eschenburg and R. Tribuzy, Constant mean curvature surfaces in 4-space forms, Rend. Sem. Mat. Univ. Padova 79 (1988), 185–202. MR964030

32
[20] T. Friedrich, *On surfaces in four-spaces*, Ann. Global Anal. Geom. 2 (1984), no. 3, 257–287, DOI 10.1007/BF01876417. MR777909

[21] J.A. Gálvez, A. Martínez, and P. Mira, *The Bonnet problem for surfaces in homogeneous 3-manifolds*, Comm. Anal. Geom. 16 (2008), no. 5, 907–935, DOI 10.4310/CAG.2008.v16.n5.a1. MR2471362

[22] I.V. Guadalupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. 106 (1983), no. 1, 95–103. MR694674

[23] K. Hasegawa, *On surfaces whose twistor lifts are harmonic sections*, J. Geom. Phys. 57 (2007), no. 7, 1549–1566, DOI 10.1016/j.geomphys.2007.01.004. MR2310605

[24] H. He, H. Ma, and E. Wang, *Lagrangian Bonnet Pairs in Complex Space Forms*, ArXiv e-prints (2015), available at http://arxiv.org/abs/1503.08566.

[25] F. Hélein and P. Romon, *Weierstrass representation of Lagrangian surfaces in four-dimensional space using spinors and quaternions*, Comment. Math. Helv. 75 (2000), no. 4, 668–680, DOI 10.1007/s000140050144. MR1789181

[26] D.A. Hoffman, *Surfaces of constant mean curvature in manifolds of constant curvature*, J. Differential Geometry 8 (1973), 161–176. MR0390973

[27] D.A. Hoffman and R. Osserman, *The Gauss map of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$*, Proc. London Math. Soc. (3) 50 (1985), no. 1, 27–56, DOI 10.1112/plms/s3-50.1.27. MR765367

[28] G.R. Jensen and M. Rigoli, *Twistor and Gauss lifts of surfaces in four-manifolds*, Recent developments in geometry (Los Angeles, CA, 1987), Contemp. Math., vol. 101, Amer. Math. Soc., Providence, RI, 1989, pp. 197–232, DOI 10.1090/conm/101/1034983, (to appear in print). MR1034983

[29] G.D. Johnson, *An intrinsic characterization of a class of minimal surfaces in constant curvature manifolds*, Pacific J. Math. 149 (1991), no. 1, 113–125. MR1099786

[30] K. Kenmotsu, *An intrinsic characterization of $H$-deformable surfaces*, J. London Math. Soc. (2) 49 (1994), no. 3, 555–568, DOI 10.1112/jlms/49.3.555. MR1271550

[31] S.G. Krantz and H.R. Parks, *A primer of real analytic functions*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2002. MR1916029

[32] H.B. Lawson, *Complete minimal surfaces in $S^3$*, Ann. of Math. (2) 92 (1970), 335–374. MR0270280

[33] H.B. Lawson and R. Tribuzy, *On the mean curvature function for compact surfaces*, J. Differential Geom. 16 (1981), no. 2, 179–183. MR638784

[34] J.A. Little, *On singularities of submanifolds of higher dimensional Euclidean spaces*, Ann. Mat. Pura Appl. (4) 83 (1969), 261–335. MR0271970

[35] I.M. Roussos and G.E. Hernández, *On the number of distinct isometric immersions of a Riemannian surface into $\mathbb{R}^3$ with given mean curvature*, Amer. J. Math. 112 (1990), no. 1, 71–85, DOI 10.2307/2374853. MR1037603

[36] E.A. Ruh and J. Vilms, *The tension field of the Gauss map*, Trans. Amer. Math. Soc. 149 (1970), 569–573. MR0259768

[37] B. Smyth and G. Tinaglia, *The number of constant mean curvature isometric immersions of a surface*, Comment. Math. Helv. 88 (2013), no. 1, 163–183, DOI 10.4171/CMH/281. MR3008916

[38] R. Tribuzy, *A characterization of tori with constant mean curvature in space form*, Bol. Soc. Brasil. Mat. 11 (1980), no. 2, 259–274, DOI 10.1007/BF02584641. MR671469
[39] Th. Vlachos, *Congruence of minimal surfaces and higher fundamental forms*, Manuscripta Math. **110** (2003), no. 1, 77–91, DOI 10.1007/s00229-002-0310-z. MR1951801

[40] ______, *Isometric deformations of minimal surfaces in $S^4$*, Illinois J. Math. **58** (2014), no. 2, 369–380. MR3367653

[41] S.T. Yau, *Submanifolds with constant mean curvature. I, II*, Amer. J. Math. **96** (1974), 346–366; ibid. 97 (1975), 76–100. MR0370443

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