Stochastic Gravity and Self-Organized Critical Cosmology

J. W. Moffat

Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

(October 16, 2018)

Abstract

A stochastic theory of gravity is described in which the metric tensor is a random variable such that the spacetime manifold is a fluctuating physical system at a certain length scale. A general formalism is described for calculating probability densities for gravitational phenomena in a generalization of general relativity (GR), which reduces to classical GR when the magnitude of the metric fluctuations is negligible. Singularities in gravitational collapse and in big-bang cosmology have zero probability of occurring. A model of a self-organized critical universe is described which is independent of its initial conditions.

To be published in the Proceedings of the workshop: Very High Energy Phenomena in the Universe of the XXXIIInd Rencontres de Moriond, Les Arcs, France, January, 1997.
Introduction

Classical general relativity is based on the assumption that the spacetime manifold is $C^2$ smooth down to zero length scales. This assumption is basically the source of the singularity theorems of Hawking and Penrose \[1\], which state that in GR a singularity must be the final state of gravitational collapse and that the big-bang must have occurred as a naked singularity in cosmology, provided the strong and weak energy positivity conditions are satisfied. From our knowledge of physical systems in Nature, through studies of physics, chemistry and biology we know that systems display noise at some length scale in the form of random fluctuations or $1/f^a$ flicker noise or chaotic noise. If we consider spacetime geometry to be a physical system that interacts with matter, then we should expect that it also will have metric fluctuations corresponding to “noise” in the spacetime geometry. Indeed, the assumption that \textit{spacetime geometry is smooth down to zero length scales seems quite unrealistic.}

Wheeler \[2\] suggested some time ago that at the Planck length, $L_P = \sqrt{\hbar G} = 1.6 \times 10^{-33}$ cm spacetime geometry would possess quantum fluctuations with metric fluctuations of order, $\Delta g \sim 10^{-20}$. These are so small that we can ignore them for experimentally accessible physical observations. Quantum gravity describes a strongly coupled phase in which there are no correlations on length scales larger than $L_P$. However, we can consider that spacetime fluctuations occur not involving Planck’s constant $\hbar$, which display correlations at much larger length scales and for self-organized critical gravitational phenomena display correlations at all length scales. Theoretical and experimental studies have shown that non-linear physical systems in “noisy” environments show surprising behavior that does not conform to common intuitive experiences. Thus, we can expect that since GR is a highly non-linear theory the fluctuating spacetime manifold when coupled to gravitational matter exhibits a much richer behavior than is possible in a purely deterministic theory. A gravitational theory based on statistical mechanics can display two kinds of systems: (1) a fine-tuning of parameters that leads to critical points and phase transitions and (2), self-organization which can often describe critical systems with structure spread out over every available scale.
Such properties can be applied to our understanding of the universe at both small and large scales. Indeed, it would be surprising if non-linear gravitational physics is not a rich source of new phenomena determined by non-linear complexity theory.

**Stochastic Gravity**

A probabilistic interpretation of spacetime has been developed based on defining the spacetime metric tensor as a random stochastic variable \[3\]. The probability that the metric \( g \) takes a value between \( g \) and \( dg \) is given by \( P(g)dg \) where \( P(g) \) is a probability distribution defined as a scalar quantity with \( P(g) \geq 0 \) and normalized to unity on its range. A canonical formalism based on a \((3 + 1)\) foliation of spacetime in GR\(^{(3)}\) leads to a Langevin equation for the canonically conjugate momentum variable \( \pi^{\mu\nu} \) treated as a stochastic variable. The gravitational constant \( G \) is defined to be a control parameter with the decomposition:

\[
G_t = G + \sigma \xi_t, 
\]

(1)

where \( G \) is the average value of Newton’s gravitational constant and \( \xi_t \) is Gaussian white noise with \( \langle \xi_t \rangle = 0 \) where \( E \) denotes the expectation value; \( \sigma \) measures the intensity of the geometrical fluctuations of the metric. Thus, \( G_t \) will have a bell-shaped curve distribution, peaked at the average value, \( G \). The stochastic differential equation for the dynamical random variable \( \pi_t^{\mu\nu} \) has the form:

\[
\partial_t \pi_t^{\mu\nu} = f_t^{\mu\nu} + 8\pi GT^{\mu\nu} + 8\pi\sigma \xi_t T^{\mu\nu}, 
\]

(2)

where \( f_t^{\mu\nu} \) is determined by terms involving the \((3 + 1)\) projected curvature tensor, \( \pi_t^{\mu\nu} \) and the lapse and shift functions, \( T^{\mu\nu} \) is the projected stress energy-momentum tensor for matter with the components \( T^\perp \perp = T_{\mu\nu} n^\mu n^\nu \) and \( T^\nu_\perp = h^\nu_\alpha T^{\alpha\beta} n_\beta \); \( h_{\mu\nu} + n_\mu n_\nu \) is the induced spatial metric and \( n^\mu \) is the normal unit vector to the Cauchy spacelike surface, \( \Sigma \). A Fokker-Planck equation can be derived for the probability density, \( p(\pi_t) \):

\[
\partial_t p(y, t|\pi_t, 0) = -\partial_y[F_t(y)p(y, t|\pi_t, 0)] + 32\pi^2 T^2 \partial_{yy} p(y, t|\pi_t, 0), 
\]

(3)

where \( \pi \) and \( T \) denote for convenience the canonically conjugate momentum \( \pi^{\mu\nu} \) and the stress energy-momentum tensor \( T^{\mu\nu} \), respectively. This equation has an exact solution for
a stationary probability density obtained for a gravitational system after a long time has elapsed:

\[
p_S(\pi) = \frac{C}{64\pi^2T^2} \exp\left(\frac{1}{32\pi^2\sigma^2T^2} \int_{\pi} F(u)du\right),
\]

where \(C\) is a normalization constant.

We can formulate a stochastic differential equation for the geodesic motion of a test particle:

\[
du_s^\mu + \Gamma^\mu_{s,\alpha\beta} u_s^\alpha u_s^\beta = -\zeta_s \Gamma^\mu_{s,\alpha\beta} u_s^\alpha u_s^\beta ds,
\]

where \(\zeta_s\) is a Brownian motion process in terms of the proper time \(s\), the Christoffel symbol \(\Gamma^\mu_{s,\alpha\beta}\) is treated as a random variable determined by the stochastic metric \(g_{s,\mu\nu}\), \(u_s^\mu = dx_s^\mu/ds\) denotes the time-like four-velocity, and \(u_s^\mu\) describes this four-velocity as a random variable.

At the length scale for which the fluctuations of spacetime are significant, we can picture a test particle moving in spacetime along a Brownian motion path such that \(u_s^\mu\) does not have a well defined derivative with respect to \(s\) at a point on the world line. For large enough macroscopic length scales for which the spacetime fluctuations can be neglected, the motion of the test particle becomes the same as the deterministic geodesic equation of motion in GR.

A stochastic Raychaudhuri equation can be derived from which we can deduce that caustic singularities in the spacetime manifold can be prevented from occurring if the intensity of fluctuations is big enough for a given length scale, assuming the standard positive energy conditions. In the limit of classical GR, the Hawking-Penrose singularity theorems will continue to hold, for the Brownian motion fluctuations of spacetime are negligible and can be neglected.

Consider the case of inward radial motion in a Schwarzschild geometry. For radial motion we have

\[
\frac{dt}{ds} = \left(1 - \frac{2GM}{r}\right)^{-1}, \quad \frac{dr}{ds} = -\sqrt{\frac{2GM}{r}},
\]
where $M$ denotes the mass of the central particle. The stochastic differential equation for the random variable $r_s$ is given by

$$dr_s = \sqrt{G} f(r_s) ds + \beta(\sqrt{G} + \zeta_s)f(r_s) ds,$$

(7)

where

$$f(r_s) = -\sqrt{\frac{2M}{r_s}},$$

(8)

and $\beta$ is a non-linear function of the Wiener process $\zeta_s$. By taking the white-noise Gaussian limit for short correlation times [3], we get the stationary probability density:

$$p_S(r) = \frac{Cr}{2M} \exp\left(-\frac{2\sqrt{2\sqrt{G}}}{3\sqrt{M}\sigma^2}r^{3/2}\right),$$

(9)

where $C$ is a normalization constant. We see that $r \to \infty$ is a natural boundary: both the drift and the diffusion coefficients vanish as $r \to \infty$. For $r \to 0$ we have $p_S(r) \sim 0$. We therefore arrive at the result that as the particle falls towards the origin there is zero probability for $r(s)$ to have the value zero, and consequently there is zero probability of having a singularity at $r = 0$ in the Schwarzschild solution. The spacetime metric fluctuations smear out the singularity at $r = 0$.

An analysis of the collapse of a spherically symmetric dust cloud in our stochastic gravity theory shows that a similar result follows [3]. As the star collapses there is zero probability for the star having a singularity as the final state of collapse. Moreover, it was also found that in the gravitational collapse of a star there is zero probability of a black hole horizon forming with an infinite red shift. The spacetime fluctuations become extremely intense as $r \to 2GM$ and they quench the infinite red shifts. If we ignore the spacetime fluctuations, then the standard GR results follow: both the singularity and the infinite red shift black hole horizon must occur when a trapped surface forms during collapse. However, this assumes that the spacetime geometry is perfectly smooth in the limit of zero distance scales and infinite frequencies of red shifts- an assumption which seems contrary to all our experiences of physical systems!
An application of stochastic gravity to cosmology yields the result that the probability of a singularity occurring at the big-bang is zero [3]; the spacetime metric fluctuations smear out the singularity at \( t = 0 \).

**Self-organization and Criticality in Cosmology**

A major problem in modern cosmology is this: How could the universe evolve during more than 10 Gyr and become so close to spatial flatness and avoid the horizon problem? Why is the universe so homogeneous and isotropic? How could such a critical state of the universe come about without a severe fine tuning of the parameters?

The usual explanation for these questions is based on the idea of inflation [4]. However, inflation is a type of phenomenon that in statistical mechanics corresponds to the existence of an attractor that requires *fine tuning of the parameters*. Indeed, in all models of inflation introduced to date there exists fine tuning of the parameters of the models. This is the intrinsic weakness of these models.

Recently, it has been proposed that the universe evolves as a self-organized critical system with the expansion of the universe undergoing “punctuated equilibria” [5] with energy being dissipated at all length scales. The idea of a self-organized critical state was introduced by Bak, Tang and Wiesenfeld [6] using a sandpile or a system of coupled damped pendula as models. This remarkable idea has been applied to many different physical systems in condensed matter physics, biology, earthquake studies and economics among others [7].

We postulate that the universe evolves as a self-organized critical system in which the forces of expansion increase the cosmic scale \( R \) in steps with the spacetime fluctuations and the increments of expansion \( R_n \) sliding back in “avalanches” which cannot be detected by the local observer. The universe evolves towards a critical state with the individual spacetime fluctuations and \( R_n \) developing highly cooperative effects. The \( R_n \) (or the density profile \( \Omega_n \)) satisfies a nonlinear discretized diffusion equation with a threshold condition associated with the critical value \( R_c \) or \( \Omega_c \). The dynamics leads to a stable state at some time \( t = t_S \) *completely independent of how the universe began*. We can randomly add expansion, \( R_n \to R_n + 1 \), and induce slides \( R_n \to R_n - 1 \) with a random distribution of
critical expansion differences and a uniformly increasing slope of expansion. The power spectrum for the metric fluctuations $S(\Delta g)$ satisfies a power law

$$S(\Delta g) \sim (\Delta g)^{-\beta}$$

(10)

associated with a self-similar fractal behavior.

We are unable to predict the self-organized critical value for the density profile, $\Omega = \Omega_c$ (observationally $\Omega_c \leq 1$), without further dynamical input from GR. But we can explain with this model how the universe has to evolve to a stable critical value $\Omega = \Omega_c$, which is independent of the initial conditions and without fine tuning of the parameters. The metric fluctuations display $1/f$ flicker noise, correlations of fluctuations occur at all length scales, and the universe evolves at the “edge of chaos”. There is only one possible stable choice (i.e., stable under local perturbations) for the present expanding universe whatever its initial conditions.

According to our assumptions the spacetime geometry fluctuates randomly at some length scale. If we assume that the metric fluctuations are very intense at the beginning of the universe, and that they smear out the light cones locally, then for a given short duration of time $\Delta t$ after the big-bang there will be communication of information “instantaneously” throughout the universe. This will resolve the “horizon” problem and explain the high degree of isotropy and homogeneity of the present universe.

**ACKNOWLEDGMENTS**

I thank the Natural Sciences and Engineering Research Council of Canada for the support of this work.
REFERENCES

[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.

[2] C. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, J. Freeman, San Francisco, 1973.

[3] J. W. Moffat, *Stochastic Gravity*, University of Toronto preprint, UTPT-96-16. gr-qc/9610067, 1996.

[4] A. Guth, Phys. Rev. D 23, 347 (1981); A. Linde, Rep. Prog. Phys. 47, 925 (1984).

[5] J. W. Moffat, *A Self-Organized Critical Universe*, University of Toronto preprint, UTPT-97-03. gr-qc/9702014, 1997.

[6] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A 38, 364 (1988).

[7] P. Bak, *How Nature Works*, Springer-Verlag New York, Inc. 1996.