Field configurations with half-integer angular momentum in purely bosonic theories without topological charge

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It is shown that purely bosonic field theories can have configurations with half-integer angular momentum even when the topological magnetic charge of the configuration vanishes. This result is applicable whenever there is a non-Abelian gauge theory with particles that transform in the fundamental representation of the non-Abelian symmetry group.

The occurrence of half-integer angular momentum configurations in purely bosonic theories has been known and studied for over two decades in the context of magnetic monopoles \[1,2\]. The basic idea is described quite simply as follows. Let us suppose that a theory has magnetic monopoles having asymptotic magnetic field

\[
B = \frac{\mathbf{M} \cdot \hat{e}_i}{g \, r^2},
\]

where \(\mathbf{M}\) is the matrix-valued magnetic charge and \(g\) is a coupling constant. Consider the monopole in the presence of some electric charges in the theory. Then the angular momentum in the gauge fields is

\[
|\mathbf{J}| = \left| \int d^3 x \text{Tr} [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})] \right| = \left| - \sum_i m_i e_i \right|,
\]

where, \(m_i, e_i\) are the magnetic and electric charges and the different possible types of charges are labelled by \(i\). The Dirac quantization condition permits the product of magnetic and electric charge to be half-integral and hence the angular momentum can be half-integral even with purely bosonic particles present in the original theory.

The best known example where half-integral spin is obtained is in the case of the ‘t Hooft-Polyakov monopole \[3,4\] with an additional scalar field transforming in the fundamental representation of \(SU(2)\) \[5\]. There the monopole only has a single type of charge \((i = 1)\) and

\[
J = -me = \frac{1}{2}.
\]

This phenomenon is described by saying that the \(SU(2)\) isospin degree of freedom has led to the (half-) spin of the monopole. The spin-statistics relation is valid for the monopole \[5\] and so the monopoles are fermionic objects in a theory of bosons.

Spin from isospin has been considered for the more complicated \(SU(5)\) monopoles by Lykken and Strominger \[6\] and we shall now discuss and generalize their analysis. The idea is to calculate \(\sum m_i e_i\) for various monopoles and electric charges. For the monopoles formed in

\[
SU(5) \rightarrow [SU(3) \times SU(2) \times U(1)]/\mathbb{Z}_6 = K \quad (1)
\]

there are four magnetic charges \((i = 1, \ldots, 4)\) corresponding to the four diagonal generators of \(K\). These are \(\lambda_3\) and \(\lambda_8\) of \(SU(3)\), \(\tau_3\) of \(SU(2)\), and, \(Y\) of \(U(1)\). The magnetic charges can be chosen to be:

\[
m_1 = 0, m_2 = \frac{n_8}{\sqrt{3}g}, m_3 = \frac{n_3}{2g}, m_4 = \frac{-1}{2g} \sqrt{\frac{5}{3}} n_1,
\]

where, if \(n\) is the winding of the monopole,

\[
n_{8,\min} = n(\text{mod } 3), \quad n_{3,\min} = n(\text{mod } 2).
\]

A scalar field \(H\) transforming in the fundamental representation of \(SU(5)\) has five components (labelled by \(h = 1, 2, \ldots, 5\)) which have the following electric charges:

\[
e_1^h = \frac{g}{2} \left( \begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array} \right), \quad e_2^h = \frac{g}{2\sqrt{3}} \left( \begin{array}{c} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{array} \right),
\]

\[
e_3^h = \frac{g}{2} \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{array} \right), \quad e_4^h = \frac{g}{\sqrt{15}} \left( \begin{array}{c} 1 \\ 1 \\ -3/2 \\ -3/2 \end{array} \right),
\]

Then, we find

\[
J^h = -\sum_i m_i e_i^h = \left( \begin{array}{c} (-n_8 + n_1)/6 \\ (-n_8 + n_1)/6 \\ (+2n_8 + n_1)/6 \\ (-n_3 - n_1)/4 \\ (+n_3 - n_1)/4 \end{array} \right).
\]
By inserting appropriate values of \( n_8, n_3 \) and \( n_1 \) for each winding \( n \), one can determine whether the monopole can have half-integral spin. For \( \pm n = 1, 2, 3, 4 \), and, 6, the monopoles are stable \( \Box \), and for minimal values of \( n_8 \) and \( n_3 \), half-integral spin is always possible.

This shows that one can get spin from isospin for \( SU(5) \) monopoles. But we are interested in spin from isospin in the absence of magnetic charge. For this, consider the unit winding monopole already analysed in \( \Box \). This has \( n_1 = 1, n_3 = 1, n_8 = 1 \) and so

\[
J_{+1}^h = \begin{pmatrix} 0 \\ 0 \\ +1/2 \\ -1/2 \\ 0 \end{pmatrix}
\]

Also consider the winding \(-1\) monopole with \( n_8 = -1, n_3 = +1, n_1 = -1 \). Then,

\[
J_{-1}^h = \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ +1/2 \end{pmatrix}
\]

Next consider the situation where a winding \( +1 \) and winding \( -1 \) are both present. Then,

\[
J_{+1-1}^h = - \sum_i (m_i + m_i') e_i = J_{+1}^h + J_{-1}^h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/2 \\ +1/2 \end{pmatrix}
\]

So the spin can indeed be half - even though the net topological magnetic charge of the system vanishes. The underlying reason for this is that the \( SU(3) \) and \( SU(2) \) gauge fields contribute to the angular momentum but the topological charge only depends on the \( U(1) \) charge.

The monopole and antimonopole pair with spin 1/2 also form a bound state. This can be seen by considering the interaction potential between an \((n_1, n_3, n_8)\) and an \((n'_1, n'_3, n'_8)\) monopole \( \Box \): \( V(r) = \frac{1}{4\alpha r} [n_1 n'_1 \text{Tr}(Y^2)(1 - e^{-\mu_0 r}) + n_3 n'_3 \text{Tr}(\tau_3 \tau'_3)(1 - e^{-\mu_3 r}) + n_8 n'_8 \text{Tr}(\lambda_8 \lambda'_8)(1 - e^{-\mu_8 r})] \) \( \Box \)

where \( r \) is the monopole-antimonopole separation, \( \alpha = g^2/4\pi \), the primes on \( \tau_3' \) and \( \lambda'_8 \) denote that these could be in orientations that are different from \( \tau_3 \) and \( \lambda_8 \), and the parameters \( \mu_0, \mu_3 \) and \( \mu_8 = \mu_3/2 \) are masses of adjoint scalar fields in the model in \( \Box \). We are interested in the case \( \mu_0 << \mu_8 \). Furthermore, the monopoles we are considering have \( \lambda'_8 = \lambda_8, \tau'_3 = \tau_3, n'_1 = n_1 = 1, n'_3 = +n_3 = 1 \) and \( n'_8 = -n_8 = 1 \). The normalization used in \( \Box \) is: \( \text{Tr}(Y^2) = 5/6, \text{Tr}(\tau_3^2) = 1/2, \text{and} \text{Tr}(\lambda_8^2) = 2/3 \). This leads to:

\[
V(r) = \frac{1}{4\alpha r} [-\frac{5}{6}(1 - e^{-\mu_0 r}) + \frac{1}{2}(1 - e^{-2\mu_3 r}) - \frac{2}{3}(1 - e^{-\mu_8 r})] .
\]

For \( r \) near zero, a Taylor expansion gives:

\[
V \rightarrow \frac{1}{3} \left( \frac{\mu_8 - 5}{2} \mu_0 \right) - \frac{1}{3} \left( \frac{\mu_8^2 - 5}{4} \mu_0^2 \right) r + ...
\]

Therefore \( V \) is positive and decreasing near the origin. As \( r \rightarrow \infty \), \( V \) approaches zero from below since the exponentials can be neglected at large \( r \) leading to \( V \approx -1/r \). This explains the schematic shape of the potential shown in Fig. \( \Box \). The presence of a minimum in the potential shows that the monopole and antimonopole can form a bound state. (This is similar to the monopole-antimonopole bound state found by Taubes \( \Box \) in an \( SO(3) \) model.) In the presence of a suitable electric charge, this bound state carries half-integral angular momentum.

FIG. 1. A schematic plot of the interaction potential of the monopole with \((n_1, n_3, n_8) = (1, 1, 1)\) and the antimonopole with \((n_1, n_3, n_8) = (-1, +1, -1)\).
(and similarly $l$) by 1, we can change the spin of certain components by units of 1/2. In other words, purely gluonic excitations of the monopole, having nothing to do with topological charge, can change the spin by half an integer.

It is also possible to write down localized field configurations (not solutions) in an $SU(2)$ model which carry half-integral angular momentum. Consider the $SU(2)$ gauge model with a scalar field $\Phi$ that transforms in the fundamental representation of $SU(2)$. (Note that there is no adjoint scalar field present.) Then the gauge field configuration of the Bogomolnyi-Prasad-Sommerfield (BPS) monopole [10,11]:

$$A_i^a = \frac{\epsilon_{aij}}{gr} \left[ 1 - \frac{Cr}{\sinh(Cr)} \right] \hat{x}_j,$$

where $C$ is any constant, together with a quanta of the $\Phi$ field, has angular momentum 1/2 for every value of $C$. This follows because we know that the gauge fields of the BPS monopole and a quanta of $\Phi$ carry half-integer angular momentum [10,11]. However, the unbroken $SU(2)$ model has no monopoles and the gauge field (3) has no topological charge. Another way of stating this result is that non-Abelian theories may have embedded monopole configurations [12] which do not carry any topological charge but can still yield half-integral angular momentum in the presence of a suitable electric charge.

A potential application of these results may be to QCD where we have $SU(3)_c$ gauge fields and hence can consider configurations of the form given in eq. (3) (within an $SU(2)$ subgroup of $SU(3)_c$) together with the quarks that are known to occur in fundamental representations of $SU(3)_c$. In such a situation, anomalous values of the total angular momentum may be possible when the spin of a quark is combined with the angular momentum in the color gauge fields \(i.e.\) gluonic degrees of freedom). Another possible application may be to the bosonic sector of the unbroken standard electroweak model.

The possibility of obtaining half-integral angular momentum without magnetic charge is particularly relevant to the construction of a dual standard model [8,13,14] in which every known quark and lepton corresponds to a (dualized) magnetic monopole. Now, the neutrino is a spin half particle with zero electric charge and so its magnetic counterpart would have to be a spin half object with zero magnetic charge. What we have seen here is that such a state might be possible, although it is still not clear if the classical configuration can be (third) quantized as a massless spin-half particle.

Finally we should point out the difference between the present half-integer spin configurations and objects such as Skyrmions which can also be quantized as fermions. In the present case, the angular momentum is calculated classically and the half-integral value crucially depends on the non-Abelian nature of the gauge fields. In the case of the Skyrmion, the classical object does not carry half-integral values of angular momentum. Only when the Skyrmion is quantized together with a Wess-Zumino-Witten term, does one get the possibility of half-integral angular momentum. A similarity, however, is that the Skyrmion needs to be stabilized against collapse by including some higher derivative terms in the model. In our case, the gauge configurations (such as (3)) are unstable to spreading out. If desired, they can be stabilized by the inclusion of gravitational forces.

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