OPTIMALITY AND DUALITY FOR COMPLEX MULTI-OBJECTIVE PROGRAMMING

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Abstract. We consider a complex multi-objective programming problem (CM-P). In order to establish the optimality conditions of problem (CM-P), we introduce several properties of optimal efficient solutions and scalarization techniques. Furthermore, a certain parametric dual model is discussed, and their duality theorems are proved.

1. Introduction. In complex optimization problems, the earliest study seems to be the paper published by Levinson [16] in 1966 for the case of complex linear programming. Since then, linear, nonlinear, fractional, various types of the complex programming problems are considered [1, 3, 17].

We recall a complex nonlinear programming problem as follows.

\[(P_0) \quad \min \ Re f(z, \z) \]
\[\text{such that } (z, \z) \in X = \{(z, \z) \in \mathbb{C}^{2n} \mid -g(z, \z) \in S\},\]

where \(S\) is a polyhedral cone in \(\mathbb{C}^q\), \(f: \mathbb{C}^{2n} \to \mathbb{C}\) and \(g: \mathbb{C}^{2n} \to \mathbb{C}^q\) are analytic in \(\z = (z, \z) \in Q\), and the set \(Q = \{(z, \z) \in \mathbb{C}^{2n} \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}\) is a linear manifold over the real field.

Throughout this paper, we adopt the following conventions: Given \(z \in \mathbb{C}^p\), the notations \(z^T\), \(z^T\) and \(z^\dagger\) are respectively the conjugate, the transpose and the conjugate transpose of \(z\). Since a nonlinear analytic function \(f(z)\) does not have convex real part, we consider all complex functions defined on a linear manifold of the set \(Q = \{z \in \mathbb{C}^n \mid z \in \mathbb{C}^n\}\) (see Ferrero [10, Proposition 3.1]).

Datta and Bhatia [4] considered a complex minimax programming problem in 1984. Then, many authors are interested in various types of the minimax programming problems in complex spaces. For instance, Lai and Huang [11] formulated a
complex nondifferentiable minimax programming problem as follows.

\[(P_1) \min_\zeta \sup_{\zeta \in \mathbb{X}, \eta \in \mathbb{Y}} \Re \left[ f(\zeta, \eta) + (z^H A z)^{1/2} \right] \]

such that \( X = \{ \zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n} \mid -h(\zeta) \in S \subset \mathbb{C}^p \} \),

where \( Y \) is a specified compact subset in \( \mathbb{C}^{2m} \) and \( A \in \mathbb{C}^{n \times n} \) is a positive semidefinite Hermitian matrix. Lai and Huang also considered a complex nondifferentiable minimax fractional programming problem:

\[(P_2) \min_\zeta \sup_{\zeta \in \mathbb{X}, \eta \in \mathbb{Y}} \frac{\Re \left[ f(\zeta, \eta) + (z^H A z)^{1/2} \right]}{\Re \left[ g(\zeta, \eta) - (z^H B z)^{1/2} \right]} \]

such that \( X = \{ \zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n} \mid -h(\zeta) \in S \subset \mathbb{C}^p \} \),

where \( Y \) is a specified compact subset in \( \mathbb{C}^{2m} \) and \( A, B \in \mathbb{C}^{n \times n} \) are positive semidefinite Hermitian matrices. They established optimality conditions, constructed some duality models and prove the duality theorems under some generalized convexity assumptions. Interested readers can refer to [11, 12, 13, 14, 15].

People are interested in various types of the complex multi-objective programming problems. Duca [5] formulated a certain vectorial optimization problem in a complex space and obtained necessary and sufficient conditions for a point to be the efficient solution of a problem. The same idea was applied in some other papers[6, 7, 8]. Stancu-Minasian et. al. [18], considered a multiple objective linear fractional program in complex spaces of the form:

\[(P_3) \min \left( \begin{array}{c}
\Re(\langle e^1, z \rangle + c_1^0) \\
\ldots \\
\Re(\langle e^p, z \rangle + c_p^0)
\end{array} \right) \]

such that \( X = \{ z \in \mathbb{C}^n \mid Az - b \in S, z \in T \} \),

where \( c_1^0, \ldots, c_p^0 \) are complex numbers, \( e^1, \ldots, e^p \) are vectors, \( S \) is a convex cone in \( \mathbb{C}^m \), \( T \) is a convex cone in \( \mathbb{C}^n \), \( A \in \mathbb{C}^{m \times n} \) is a complex matrix and \( b \in \mathbb{C}^m \) is a vector. They gave several theorems of necessary optimality conditions for problem (P3).

Usually, the objective functions in complex programming problems are the real parts of complex functions. Youssef and Elbrolosy[20, 21] formulated the extended problem, in which the objective function contains its two parts (real and imaginary). The complex extended programming problem is of the form:

\[(P_4) \min f(z, \overline{z}) \]

such that \( (z, \overline{z}) \in X = \{ (z, \overline{z}) \in Q \mid -g(z, \overline{z}) \in S \} \),

where \( S \) is a polyhedral cone in \( \mathbb{C}^m \), \( f : \mathbb{C}^{2n} \to \mathbb{C} \) and \( g : \mathbb{C}^{2n} \to \mathbb{C}^m \) are analytic in \( \zeta = (z, \overline{z}) \in Q \), and the set \( Q = \{ (z, \overline{z}) \mid z \in \mathbb{C}^n \} \subset \mathbb{C}^{2n} \) is a linear manifold over the real field. Necessary optimality conditions of problem (P4) were formulated, and the sufficient optimality conditions had been established under generalized forms of convexity assumptions. Thus, the previous studies can be considered as special cases of real part of complex functions.

Elbrolosy [9] considered the following complex multi-objective programming problem (or called the generalized form of vector optimization problem in complex space),

\[(\text{CMP}) \min f(\zeta) = (f_1(\zeta), \ldots, f_p(\zeta)) \]

such that \( \zeta = (z, \overline{z}) \in X = \{ \zeta \in Q \mid -g(\zeta) \in S \} \),
where $S \subset \mathbb{C}^q$ is a polyhedral cone, and $f : \mathbb{C}^{2n} \to \mathbb{C}^p$, $g : \mathbb{C}^{2n} \to \mathbb{C}^r$ are analytic in $\zeta = (z, \bar{z}) \in Q = \{(z, \bar{z}) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$. Elbrolosy introduced a certain concept of optimal efficient solutions and established optimality conditions of problem (CMP) by using the scalarization techniques. This paper is concerned with the multi-objective complex programming problem (CMP), in which the multi-objective function contains its real and imaginary parts. Based on the necessary optimality conditions of problem (CMP) in Elbrolosy [9] (see Theorem 3.4), we established sufficient optimality conditions of problem (CMP) (see Theorem 3.6), formulated the parametric dual problem (D) and proved their duality theorems under some generalized convexity assumptions (see Section 4).

The paper is organized as follows: In order to investigate the multi-objective programming problem in complex spaces, we introduce the concept of optimal efficient solutions and discuss scalarization techniques in Section 2. In Section 3, we first cite necessary optimality conditions in [9] and then establish sufficient optimality conditions of (CMP) under generalized convexity assumptions. In Section 4, we give the parametric dual problem (D) and establish their duality theorems under suitable generalized convexity assumptions.

2. Notations and preliminary. The contents in this section are similar to those in Elbrolosy [9, 2016]. We first introduce several important definitions in the literature for the multi-objective complex programming, including a certain concept of optimal efficient solution (see Definition 2.4) and the set of minimal efficient elements (see Definition 2.5). Then, Example 2.6, Lemma 2.7, Theorem 2.8, and Theorem 2.9 are respectively similar to Example 1, Proposition 4.2, Theorem 4.4, and Theorem 4.6 in Elbrolosy [9, 2016], respectively. We intend to re-prove the example and these results in this section since they integrate the paper and help the readers get familiar with the definitions given in the beginning of this section.

**Definition 2.1.** (Abrams and Ben-Israel [2]) A set $T$ is a polyhedral cone in $\mathbb{C}^p$ if it is the intersection of finitely many closed half spaces, each containing the origin in its boundary, i.e.,

$$T = \bigcap_{k=1}^r H_{u_k},$$

where $H_{u_k} = \{z \in \mathbb{C}^p \mid \text{Re } z^H u_k \geq 0\}$ for $u_k \in \mathbb{C}^p$ ($k = 1 \ldots r$). Note that $T$ is a polyhedral cone if and only if there is an integer $k$ and matrix $K \in \mathbb{C}^{k \times p}$ such that

$$T = \{z \in \mathbb{C}^p \mid \text{Re}(Kz) \geq 0\} \subset \mathbb{C}^p.$$

**Definition 2.2.** (Abrams and Ben-Israel [2]) The dual cone $T^*$ of the convex cone $T$ is defined by

$$T^* = \{\eta \in \mathbb{C}^p \mid z \in S \Rightarrow \text{Re } \eta^H z \geq 0\}$$

and the interior of $T^*$ is defined by

$$\text{int } (T^*) = \{\eta \in \mathbb{C}^p \mid 0 \neq z \in T \Rightarrow \text{Re } \eta^H z > 0\}.$$ 

**Definition 2.3.** (Abrams and Ben-Israel [2]) Let $T$ be a closed and convex cone in $\mathbb{C}^p$ and $s_0 \in T$. The cone $T$ at $s_0$ is defined by

$$T(s_0) = \{z \in \mathbb{C}^p \mid \text{Re } \mu^H s_0 = 0, \mu \in T^* \Rightarrow \text{Re } \mu^H z \geq 0\}.$$
If $T$ is a polyhedral cone, then $T(s_0)$ is the intersection of those closed half spaces $H_{u_k}$ that includes $s_0$ in their boundaries, i.e.,

$$T(s_0) = \bigcap_{k \in B(s_0)} H_{u_k},$$

where $B(s_0) = \{ k \mid \text{Re } s_0^T(u_k) = 0 \}$. Thus if $s_0 \in \text{int}(T)$, $T(s_0)$ is the whole space $\mathbb{C}^p$.

Let $T \subset \mathbb{C}^p$ be a pointed, closed convex cone. For any $a, b \in \mathbb{C}^p$, the ordered relation notation “$\leq_T$” with respect to cone $T$ is defined by

$$a \leq_T b \text{ if and only of } b - a \in T.$$  

Note that for a nonzero vector $\mu \in T^*$,

$$a \leq_T b \Rightarrow \text{Re}[\mu^T(b - a)] \geq 0.$$  

In real-valued programming problems, we could directly define the optimal points of a real-valued function. Indeed, if $X$ is a nonempty set, $f : X \rightarrow \mathbb{R}$ is a real-valued function and $C \subset X$, then $x_0 \in C$ is a minimum point of $f$ on $C$ if $f(x_0) \leq f(x)$, for all $x \in C$. As for the multi-objective programming problem, we introduce the following concepts of optimal efficient solution (see Definition 2.4). For various properties of optimal efficient solution, one can refer to [9, 19].

**Definition 2.4.** (Optimal efficient solution) (Duca [8, Definition 3.1.1]) Let $X$ be a nonempty subset of $Q = \{ \zeta = (\pi, \tau) \in \mathbb{C}^{2n} \mid z \in \mathbb{C}^n \} \subset \mathbb{C}^{2n}$, $T \subset \mathbb{C}^p$ a pointed and closed convex cone, and $f : X \rightarrow \mathbb{C}^p$.

1. The point $\zeta_0 = (z_0, \pi_0) \in X$ is a minimal efficient (or Pareto-minimal) solution of $f$ with respect to $T$ if there exists no other feasible point $\zeta = (z, \pi) \in X$ such that $f(\zeta_0) - f(\zeta) \in T \setminus \{0\}$.

2. The point $\zeta_0 = (z_0, \pi_0) \in X$ is a maximal efficient (or Pareto-maximal) solution of $f$ with respect to $T$ if there exists no other feasible point $\zeta = (z, \pi) \in X$ such that $f(\zeta_0) - f(\zeta) \in T \setminus \{0\}$.

Note that $\zeta_0 \in X$ is a minimal efficient solution of $f$ with respect to $T$ if $(f(X) - f(\zeta_0)) \cap (-T) = \{0\}$; analogously, $\zeta_0 \in X$ is a maximal efficient solution of $f$ with respect to $T$ if $(f(\zeta_0) - f(X)) \cap (-T) = \{0\}$.

The minimal efficient solution or maximal efficient solution of $f$ with respect to $T$ in multi-objective programming problem is called the optimal efficient solution of $f$ with respect to $T$.

**Definition 2.5.** (Sawaragi, Nakayama and Tanino [19]) Let $T \subset \mathbb{C}^p$ be a pointed, closed and convex cone. The set of minimal efficient elements of a set $Y \subseteq \mathbb{C}^p$ is given by

$$E(Y, T) = \{ \hat{y} \in Y \mid \text{there is no } y \in Y \text{ with } y \neq \hat{y} \text{ such that } \hat{y} \in y + T \}.$$  

We give an example for solving the minimal efficient solution.

**Example 2.1.** (Elbrolosy [9, Example 1]) Consider the following problem

$$(P_{ex}) \quad \min \quad (z\pi, z)$$

subject to

$$z \pi \in X = \{ (z, \pi) \in \mathbb{C}^2 \mid -\frac{\pi}{4} \leq \text{arg}(z) \leq 0, \text{Re}(z) \leq 1 \},$$

with a pointed, closed and convex cone

$$T = T_1 \times T_2 = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2 \mid 0 \leq \text{arg}(z_1) \leq \frac{\pi}{4}, \text{Re}\left[\frac{-1 - 2i}{1 + i}\right]z_2 \geq 0 \right\}.$$
We want to find the minimal efficient solution for problem ($P_{ex}$).

**Solution.** Let $\rho : Q \subseteq \mathbb{C}^2 \to \mathbb{R}^2$ defined by $\rho(z, \bar{z}) = [\text{Re}(z), \text{Im}(z)]^T$ and $\sigma : \mathbb{C}^2 \to \mathbb{R}^4$ defined by $\sigma(z_1, z_2) = [\text{Re}(z_1), \text{Im}(z_1), \text{Re}(z_2), \text{Im}(z_2)]^T$ be two canonical mappings. Then

$$\widehat{X} = \rho(X) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid (z, \bar{z}) = (x + iy, x - iy) \in X \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y \leq 0, \ x + y \geq 0, \ x \leq 1 \right\}$$

and

$$\widehat{T} = \sigma(T) = \left\{ \begin{bmatrix} x_1, y_1, x_2, y_2 \end{bmatrix}^T \in \mathbb{R}^4 \mid (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in T \right\} = \left\{ \begin{bmatrix} x_1, y_1 \end{bmatrix}^T \in \mathbb{R}^2 \mid x_1 \geq y_1, \ y_1 \geq 0 \right\} \times \left\{ \begin{bmatrix} x_2, y_2 \end{bmatrix}^T \in \mathbb{R}^2 \mid \begin{bmatrix} 2y_2 - x_2 \geq 0, \ x_2 - y_2 \geq 0 \end{bmatrix} \right\}$$

The graphs of $\widehat{X}$, $\widehat{T}_1$ and $\widehat{T}_2$ are shown in (a)(b)(c) of Fig. 1, respectively. The objective function $f : \mathbb{C}^2 \to \mathbb{C}^2$ of problem ($P_{ex}$) is given by $f(z, \bar{z}) = (f_1(z, \bar{z}), f_2(z, \bar{z}))$ with $f_1(z, \bar{z}) = z\bar{z} = (x^2 + y^2) + 0i$, $f_2(z, \bar{z}) = z = x + yi$, where $z = x + iy \in \mathbb{C}$ and $x, y \in \mathbb{R}$. We could know that

$$\rho(f_1(X)) = \{ [x_1, y_1]^T \in \mathbb{R}^2 \mid x_1 = x^2 + y^2, \ y_1 = 0 \ \text{and} \ (x, y) \in \widehat{X} \},$$

$$\rho(f_2(X)) = \{ [x_2, y_2]^T \in \mathbb{R}^2 \mid x_2 = x, \ y_2 = y \ \text{and} \ (x, y) \in \widehat{X} \}.$$  

The graphs of $\rho(f_1(X))$ and $\rho(f_2(X))$ are given in (d)(e) of Fig. 1, respectively.

Since the set of minimal efficient elements of $f_1$ with respect to $T_1$ is

$$E(f_1(X), T_1) = \{ \hat{y} \in f_1(X) \mid \text{there is no} \ y \in f_1(X) \ \text{with} \ y \neq \hat{y} \ \text{such that} \ \hat{y} \in y + T_1 \}.$$  

From the graphs of $\widehat{T}_1$ and $\rho(f_1(X))$, we obtain

$$E(f_1(X), T_1) = \{ z \in \mathbb{C} \mid z = 0 + i0 \}.$$  

Similarly, from the graphs of $\widehat{T}_2$ and $\rho(f_2(X))$, the set of minimal efficient elements of $f_2$ with respect to $T_2$ is

$$E(f_2(X), T_2) = \{ z \in \mathbb{C} \mid \arg(z) = -\frac{\pi}{4}, \ \text{Re}(z) \leq 1 \}.$$  

The graphs of $E(f_1(X), T_1)$ and $E(f_2(X), T_2)$ are shown in (f)(g) of Fig.1, respectively. Thus, the set minimal efficient elements of problem ($P_{ex}$) with respect to $T$ is

$$E(f(X), T) = \{ z \mid z = 0 + i0 \} \times \left\{ z \mid \arg(z) = -\frac{\pi}{4}, \ \text{Re}(z) \leq 1 \right\} \subseteq \mathbb{C}^2.$$  

Obviously, if $(z_0, \pi_0)$ is a minimal efficient solution of problem ($P_{ex}$) with respect to $T$ such that

$$f(z_0, \pi_0) = \left( x_0^2 + y_0^2 + 0i, \ x_0 + y_0i \right) \in E(f(X), T),$$  

where $x_0 = \text{Re} \ z_0$ and $y_0 = \text{Im} \ z_0$, then $(z_0, \pi_0) = (0 + 0i, 0 - 0i)$.

**Lemma 2.6.** ([Ekeland, Proposition 4.2]) Let $T \subseteq \mathbb{C}^p$ be a pointed, closed and convex cone and $Y \subseteq \mathbb{C}^p$ be a nonempty set. Then

$$E(Y, T) = E(Y + T, T).$$
Proof.

(1) Prove that $E(Y, T) \supseteq E(Y + T, T)$.
We prove by contradiction. Suppose there exists $\hat{y} \in E(Y + T, T)$ but $\hat{y} \notin E(Y, T)$.

**Case 1.** $\hat{y} \notin Y$:
Since $\hat{y} \in Y + T$ and $\hat{y} \notin Y$, there exist $y' \in Y$ and $0 \neq t \in T$ such that $\hat{y} = y' + t$. In particular, $y' \neq \hat{y}$. Observe that $0 \in T$ implies $Y \subseteq Y + T$.
Therefore, $\hat{y} = y' + t \in Y + T$ with $y' \in Y \subseteq Y + T$ and $y' \neq \hat{y}$, which contradicts to $\hat{y} \in E(Y + T, T)$.

**Case 2.** $\hat{y} \in Y$:
Since $\hat{y} \notin E(Y, T)$ and $\hat{y} \in Y$, there exists $y' \in Y$ with $y' \neq \hat{y}$ such that $\hat{y} = y' + t$ for some $t \in T$. This means that we again have $y' \in Y \subseteq Y + T$ with $y' \neq \hat{y}$ such that $\hat{y} = y' + t \in Y + T$, which contradicts to $\hat{y} \in E(Y + T, T)$.

(2) Prove that $E(Y + T, T) \supseteq E(Y, T)$.
We prove by contradiction again. Suppose there exists $\hat{y} \in E(Y, T)$ but $\hat{y} \notin E(Y + T, T)$, there exists $y' \in Y + T$ with $\hat{y} - y' = t' \in T \setminus \{0\}$. Thus, $y' = y'' + t''$ with $y'' \in Y, t'' \in T$. Since $T$ is a pointed and convex cone, we obtain $\hat{y} = y'' + (t' + t'') \in Y + T$, where $y'' \in Y, (t' + t'') \in T$ and $t' + t'' \neq 0$. Thus, $\hat{y} \notin E(Y, T)$, which leads to a contradiction.
Lemma 2.6, then

The scalarization technique is useful for multi-objective programming problems. We could obtain the existence of minimum efficient solutions of problem (CMP) by scalarized programming problem (SP\_τ).

Given a nonzero vector τ \in \mathbb{C}^p, we consider the following scalarized programming problem (CMP) with respect to τ.

\[
(SP_\tau) \quad \min_{\zeta} \text{Re}[\tau^H f(\zeta)] \\
\text{subject to} \quad \zeta \in X = \{\zeta = (z, \bar{z}) \in Q \mid -g(\zeta) \in S\}.
\]

**Theorem 2.7.** (Elbrolosy [9, Theorem 4.4]) Let T \subset \mathbb{C}^p be a pointed, closed and convex cone and f(X) a convex set. If ζ₀ is a minimal efficient solution of (CMP) with respect to T, then there exists a nonzero vector τ \in T^* such that ζ₀ is an optimal solution of (SP\_τ).

**Proof.** If ζ₀ is a minimal efficient solution of (CMP) with respect to T, and from Lemma 2.6, then f(ζ₀) ∈ E(f(X), T) = E(f(X) + T, T). From the definition of E(f(X) + T, T), we obtain \(f(X) + T - f(ζ₀)) \cap (-T) = \{0\}\). Since both \(f(X) + T\) and T are convex sets, it follows from the separation theorem that there exists a nonzero τ \in \mathbb{C}^p such that

\[
\text{Re} \tau^H[f(ζ) + t - f(ζ₀)] \geq 0 \quad \text{for all } ζ \in X, t \in T, \text{ and} \tag{1}
\]

\[
\text{Re} \tau^H[-k] \leq 0 \quad \text{for all } k \in T. \tag{2}
\]

From inequality (2), the nonzero vector τ lies in T^*. Setting \(t = 0\) in inequality (1), we obtain \(\text{Re} [\tau^H f(ζ)] \geq \text{Re} [\tau^H f(ζ₀)]\) for all \(ζ \in X\), which implies that ζ₀ is an optimal solution of (SP\_τ).

The converse of Theorem 2.7 is described as the following theorem.

**Theorem 2.8.** (Elbrolosy [9, Theorem 4.6]) Let T \subset \mathbb{C}^p be a pointed, closed and convex cone, let τ \in T^* with τ \neq 0. Assume that ζ₀ is an optimal solution of (SP\_τ), and one of the following conditions holds,
(a) nonzero vector τ ∈ int (T^*),
(b) point ζ₀ is the unique optimal solution of (SP\_τ).
Then ζ₀ be the minimal efficient solution of (CMP) with respect to T.

**Proof.** Since ζ₀ solves (SP\_τ), we have

\[
\text{Re} \tau^H[f(ζ) - f(ζ₀)] \geq 0 \quad \text{for each } ζ \in X. \tag{3}
\]

Suppose to the contrary that ζ₀ is not a minimal efficient solution with respect to T, there is a ζ' \in X such that \(f(ζ₀) - f(ζ') \in T \setminus \{0\}\). Since \(0 \neq τ \in T^*\),

\[
\text{Re} \{\tau^H[f(ζ₀) - f(ζ')]\} \geq 0. \tag{4}
\]

If condition (a) holds, the nonzero vector τ \in int (T^*) = \{τ \in \mathbb{C}^p \mid 0 \neq ξ \in T ⇒ \text{Re} \tau^H ξ > 0\}, then inequality (4) becomes

\[
\text{Re} \tau^H[f(ζ₀) - f(ζ')] > 0,
\]

which contradicts to the inequality of (3).

If condition (b) holds, the point ζ₀ is the unique optimal solution of (SP\_τ) for τ \in T^*, then inequality (3) becomes \(\text{Re} \tau^H[f(ζ) - f(ζ₀)] > 0\) for each ζ \in X, which contradicts to the inequality of (4).

Hence, ζ₀ is a minimal efficient solution of (CMP) with respect to T. □
3. Optimality conditions. Given $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$ and analytic function $f: \mathbb{C}^{2n} \to \mathbb{C}^p$ with $f(\zeta) := (f_1(\zeta), \ldots, f_p(\zeta))$ where $f_k: \mathbb{C}^{2n} \to \mathbb{C}$ ($k = 1, \ldots, p$), the gradient expression $\nabla f(\zeta)$ is denoted by

$$\nabla f(\zeta) = \left(\nabla_z f(\zeta), \nabla_{\overline{z}} f(\zeta)\right) \in \mathbb{C}^{p \times 2n}$$

with

$$\nabla_z f(\zeta) = \begin{pmatrix} \frac{\partial}{\partial z_1} f_1(\zeta) & \cdots & \frac{\partial}{\partial z_n} f_1(\zeta) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_1} f_p(\zeta) & \cdots & \frac{\partial}{\partial z_n} f_p(\zeta) \end{pmatrix},$$

$$\nabla_{\overline{z}} f(\zeta) = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} f_1(\zeta) & \cdots & \frac{\partial}{\partial \overline{z}_n} f_1(\zeta) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_1} f_p(\zeta) & \cdots & \frac{\partial}{\partial \overline{z}_n} f_p(\zeta) \end{pmatrix} \in \mathbb{C}^{p \times n}.$$

We express the differential form of a complex function by using the gradient representations as the following lemma.

**Lemma 3.1.** Given $\zeta = (z, \overline{z}), \zeta_0 = (z_0, \overline{z}_0) \in Q \subset \mathbb{C}^{2n}$ and $(v, \overline{v}) = (z - z_0, \overline{z - z_0})$. Suppose that $f(\cdot): \mathbb{C}^{2n} \to \mathbb{C}^p, \tau = (\tau_1, \ldots, \tau_p) \in \mathbb{C}^p$ and $\Phi(\zeta) = (f(\zeta), \tau) = \tau^H f(\zeta)$.

Then

$$\text{Re}[\Phi'(\zeta_0)(\zeta - \zeta_0)] = \text{Re}\left\{ \left.<v, \tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_{\overline{z}} f(\zeta_0)\right> \right\}.$$ 

**Proof.** Since $\langle x, y \rangle = y^H x$ be the standard inner product in complex space,

$$\Phi'(\zeta_0)(\zeta - \zeta_0) = \langle f'(\zeta_0)(\zeta - \zeta_0), \tau \rangle = \langle (\nabla_z f(\zeta_0), \nabla_{\overline{z}} f(\zeta_0)) \left(\begin{array}{c} v \\ \overline{v} \end{array}\right), \tau \rangle = \langle \nabla_z f(\zeta_0)v + \nabla_{\overline{z}} f(\zeta_0)\overline{v}, \tau \rangle = \tau^H \nabla_z f(\zeta_0)v + \tau^H \nabla_{\overline{z}} f(\zeta_0)\overline{v} = \sum_{j=1}^p \sum_{i=1}^n \tau_j \cdot \frac{\partial}{\partial z_i} f_j(\zeta_0) \cdot v_i + \sum_{j=1}^p \sum_{i=1}^n \tau_j \cdot \frac{\partial}{\partial \overline{z}_i} f_j(\zeta_0) \cdot \overline{v}_i$$

$$= \sum_{j=1}^p \sum_{i=1}^n \tau_j \cdot \frac{\partial}{\partial z_i} f_j(\zeta_0) \cdot v_i + \sum_{j=1}^p \sum_{i=1}^n \tau_j \cdot \overline{v}_i \cdot \left(\overline{\tau_j} \cdot \frac{\partial}{\partial z_i} f_j(\zeta_0)\right)$$

$$= \langle v, \tau^T \nabla_z f(\zeta_0) \rangle + \langle \tau^H \nabla_{\overline{z}} f(\zeta_0), v \rangle.$$ 

We obtain

$$\Phi'(\zeta_0)(\zeta - \zeta_0) = \left\langle z - z_0, \tau^T \nabla_z f(\zeta_0) \right\rangle + \left\langle \tau^H \nabla_{\overline{z}} f(\zeta_0), z - z_0 \right\rangle.$$ 

Since $\text{Re} \left[ \langle x, y \rangle \right] = \text{Re} \left[ \langle y, x \rangle \right] = \text{Re} \left[ \langle y, x \rangle \right]$, we have

$$\text{Re} \left[ \Phi'(\zeta_0)(\zeta - \zeta_0) \right] = \text{Re} \left\{ \left\langle z - z_0, \tau^T \nabla_z f(\zeta_0) \right\rangle + \left\langle \tau^H \nabla_{\overline{z}} f(\zeta_0), z - z_0 \right\rangle \right\}$$

$$= \text{Re} \left\{ \left\langle z - z_0, \tau^T \nabla_z f(\zeta_0) \right\rangle \right\} + \text{Re} \left\{ \left\langle z - z_0, \tau^H \nabla_{\overline{z}} f(\zeta_0) \right\rangle \right\}$$

$$= \text{Re} \left\{ \left\langle z - z_0, \tau^T \nabla_z f(\zeta_0) \right\rangle \right\} + \text{Re} \left\{ \left\langle z - z_0, \tau^H \nabla_{\overline{z}} f(\zeta_0) \right\rangle \right\}$$

$$= \text{Re} \left\{ \left\langle z - z_0, \tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_{\overline{z}} f(\zeta_0) \right\rangle \right\}.$$ 

$\square$
Definition 3.2. (Lai and Huang [11, Definition 3]) The problem \((P_0)\) is said to satisfy the **constraint qualification** at a point \(\zeta_0 = (\zeta_0, \mu_0)\), if for any nonzero \(\mu \in S^* \subset \mathbb{C}^q\),

\[
\langle g'_2(\zeta_0)(\zeta - \zeta_0), \mu \rangle \neq 0, \quad \text{for } \zeta \neq \zeta_0.
\]

Under the gradient expression as in Lemma 3.1, the constraint qualification is then expressed by

\[
\mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_\tau g(\zeta_0) \neq 0, \quad \text{for } \mu \neq 0 \text{ in } S^*,
\]

where \(\mu^H = \overline{\mu^T}\).

Abrams and Ben-Israel [3] established the necessary optimality conditions of problem \((P_0)\) which are stated as follows:

**Theorem 3.3.** (Abrams and Ben-Israel [3])

Let \(S\) be a polyhedral cone in \(\mathbb{C}^q\) and \(\zeta_0 = (\zeta_0, \mu_0)\) be an optimal solution of problem \((P_0)\). Suppose that the problem \((P_0)\) possesses the constraint qualification at \(\zeta_0\). Then there exists a nonzero vector \(\mu \in S^* \subset \mathbb{C}^q\) such that

\[
\nabla_z f(\zeta_0) + \nabla_\tau f(\zeta_0) + \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_\tau g(\zeta_0) = 0, \quad (5)
\]

\[
\text{Re} \left( \langle g(\zeta_0), \mu \rangle \right) = 0. \quad (6)
\]

We will obtain the Kuhn-Tucker necessary optimality conditions of problem \((\text{CM-P})\) by using the scalarization technique.

**Theorem 3.4.** (Elbrolosy [9, Theorem 4.9]) Let \(T \subset \mathbb{C}^p\) be a pointed, closed and convex cone, \(S\) a polyhedral cone in \(\mathbb{C}^q\) and \(f(X)\) a convex set. Suppose that the mappings \(f(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p\) and \(g(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^q\) are analytic on \(X \subset Q\), and \(\zeta_0\) is a minimal efficient solution of \((\text{CMP})\) with respect to \(T\). If problem \((\text{CMP})\) possesses the constraint qualification in Definition 3.2 at \(\zeta_0\), there are nonzero vectors \(\tau \in T^* \subset \mathbb{C}^p\) and \(\mu \in S^* \subset \mathbb{C}^q\) satisfying the following conditions.

\[
\tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_\tau f(\zeta_0) + \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_\tau g(\zeta_0) = 0, \quad (7)
\]

\[
\text{Re} \left( \mu^H g(\zeta_0) \right) = 0. \quad (8)
\]

In order to formulate sufficient optimality conditions and duality theorems, we introduce a certain generalized convexity in complex spaces as follows.

**Definition 3.5.** (Lai and Huang [11, Definition 1]) The real part of an analytic function \(f(\cdot)\) from \(\mathbb{C}^{2n}\) to \(\mathbb{R}\) is said to be, respectively,

(i) **(strictly) convex** at \(\zeta_0 \in Q \subset \mathbb{C}^{2n}\) if for all \(\zeta \in Q\)

\[
\text{Re} \left[ f(\zeta) - f(\zeta_0) \right] > \text{Re} \left[ f'(\zeta_0)(\zeta - \zeta_0) \right],
\]

(ii) **(strictly) pseudoconvex** at \(\zeta_0 \in Q\) if for all \(\zeta \in Q\)

\[
\text{Re} \left[ f'(\zeta_0)(\zeta - \zeta_0) \right] \geq 0 \Rightarrow \text{Re} \left[ f(\zeta) - f(\zeta_0) \right] 
\geq 0. \quad (> 0)
\]

(iii) **quasiconvex** at \(\zeta_0 \in Q\) if for all \(\zeta \in Q\)

\[
\text{Re} \left[ f(\zeta) - f(\zeta_0) \right] \leq 0 \Rightarrow \text{Re} \left[ f'(\zeta_0)(\zeta - \zeta_0) \right] \leq 0.
\]

**Theorem 3.6.** (Sufficient optimality conditions) Let \(T \subset \mathbb{C}^p\) be a pointed, closed and convex cone, \(S\) a polyhedral cone in \(\mathbb{C}^q\), and \(f(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p\) and \(g(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^q\) be two analytic mappings on \(X \subset Q\), where \(Q \subset \mathbb{C}^{2n}\). Suppose that \(\zeta_0\) is a feasible solution of \((\text{CM-P})\), and there are nonzero vectors \(\tau \in T^* \subset \mathbb{C}^p\) and \(\mu \in S^* \subset \mathbb{C}^q\) satisfying conditions (7) and (8) in Theorem 3.4. Assume that one of the following conditions (i), (ii) and (iii) holds:
Then, \( \tau \in T^* \) implies that
\[
\text{Re}[\tau^H f(\zeta_0)] \geq \text{Re}[\tau^H f(\zeta)].
\] (9)

Using the feasibility of \( \zeta \) in (CMP), we have \(-g(\zeta) \in S\), and hence for the nonzero vector \( \mu \in S^* \), one has \( \text{Re}[\mu^H g(\zeta)] \leq 0 \). From equality (8), we obtain
\[
\text{Re}[\mu^H g(\zeta_0)] = 0 \geq \text{Re}[\mu^H g(\zeta)].
\] (10)

(a) If hypothesis (i) holds, without loss of generality, assume that \( \text{Re}[\tau^H f(\cdot)] \) is

strictly convex and \( \text{Re}[\mu^H g(\cdot)] \) is

convex at \( \zeta_0 \). Let \( \zeta - \zeta_0 = (z - z_0, \bar{z} - \bar{z}_0) = (v, \overline{\tau}) \). Since \( \text{Re}[\tau^H f(\cdot)] \) is strictly convex at \( \zeta_0 \), and the

inequality (9) holds, we obtained
\[
\text{Re}(v, \tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_{\overline{\tau}} f(\zeta_0)) < 0.
\] (11)

Since \( \text{Re}[\mu^H g(\cdot)] \) is convex at \( \zeta_0 \) and the inequality (10) holds, then
\[
\text{Re}(v, \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_{\overline{\tau}} g(\zeta_0)) \leq 0.
\] (12)

Adding inequalities (11) and (12), we have
\[
\text{Re}(v, \tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_{\overline{\tau}} f(\zeta_0) + \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_{\overline{\tau}} g(\zeta_0)) < 0.
\]

This implies that
\[
\tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_{\overline{\tau}} f(\zeta_0) + \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_{\overline{\tau}} g(\zeta_0) \neq 0.
\] (13)

The inequality (13) contradicts the equality of (7).

(b) If hypothesis (ii) holds, \( \text{Re}[\tau^H f(\cdot)] \) is quasiconvex at \( \zeta_0 \), and from inequality (9), it implies that
\[
\text{Re}(v, \tau^T \nabla_z f(\zeta) + \tau^H \nabla_{\overline{\tau}} f(\zeta)) \leq 0.
\] (14)

Since \( \text{Re}[\mu^H g(\cdot)] \) is strictly pseudoconvex at \( \zeta_0 \) with inequality (10), we obtain
\[
\text{Re}(v, \mu^T \nabla_z g(\zeta) + \mu^H \nabla_{\overline{\tau}} g(\zeta)) < 0.
\] (15)

Adding inequalities (14) and (15), we again obtain inequality (13), which

contradicts to the equality of (7).

(c) If we combine inequalities (9) and (10), then
\[
\text{Re}[\tau^H f(\zeta) + \mu^H g(\zeta)] \leq \text{Re}[\tau^H f(\zeta_0) + \mu^H g(\zeta_0)].
\]

Suppose hypothesis (iii) holds. Since \( \text{Re}[\tau^H f(\cdot) + \mu^H g(\cdot)] \) is strictly

pseudoconvex at \( \zeta_0 \) and the above inequality holds, we get inequality (13), which

contradicts to the equality of (7).

Thus, \( \zeta_0 = (z_0, \bar{z}_0) \) is a minimal efficient solution of (CMP) with respect to \( T \).
4. The parametric duality model. We could constitute the parametric dual problem (D) for problem (CMP) by using the necessary optimality conditions in Theorem 3.4:

\[ (D) \max_{\mathcal{F}_D} K = (K_1, \cdots, K_p), \]

where \( \mathcal{F}_D \) is the set of all feasible solutions \((\tau, \xi, \mu, K)\) subject to

\[
\begin{align*}
\tau^T \nabla_z f(\xi) + \tau^H \nabla_{\tau} f(\xi) + \mu^T \nabla_z g(\xi) + \mu^H \nabla_{\tau} g(\xi) &= 0, \\
\Re \langle f(\xi) - K, \tau \rangle &\geq 0, \\
\Re \langle g(\xi), \mu \rangle &\geq 0, \quad \mu \neq 0 \text{ in } S^*. 
\end{align*}
\]

Here \( K \in \mathbb{C}^p, f(\xi) = (f_1(\xi), \cdots, f_p(\xi)) \) for \( \xi = (\alpha, \overline{\alpha}) \in Q \subset \mathbb{C}^{2n}, \tau \in T^* \subset \mathbb{C}^p \) and \( \mu \in S^* \subset \mathbb{C}^q. \)

The duality theorems of (D) for primary problem (CMP) are established in the following. First, we will prove that the feasible value of (CMP) is not less than the feasible value of (D) under some suitable assumptions.

**Theorem 4.1.** (Weak Duality) Let \( \zeta = (z, \overline{z}) \) be a feasible solution of (CMP), and \((\tau, \xi, \mu, K)\) a feasible solution of (D). Assume that one of the following conditions (i), (ii) and (iii) holds:

(i) one of \( \Re [\tau^H f(\cdot)] \) and \( \Re [\mu^H g(\cdot)] \) is strictly convex and the other is convex at \( \xi \in Q, \) or both are strictly convex at \( \xi \in Q, \)

(ii) \( \Re [\tau^H f(\cdot)] \) is quasiconvex at \( \xi \in Q \) and \( \Re [\mu^H g(\cdot)] \) is strictly pseudoconvex at \( \xi \in Q, \)

(iii) \( \Re [\tau^H f(\cdot)] + \mu^H g(\cdot) \) is strictly pseudoconvex at \( \xi \in Q. \)

Then there is no feasible solution \( \zeta \) for (CMP) such that

\[ K - f(\zeta) \in T \setminus \{0\}. \]

**Proof.** Suppose to the contrary that there exists some feasible solution \( \zeta \) of (CMP) such that

\[ K - f(\zeta) \in T \setminus \{0\}. \]

We know that for the nonzero vector \( \tau \in T^*, \)

\[ \Re \langle K - f(\zeta), \tau \rangle \geq 0. \]

From inequality (17), we obtain \( \Re [\tau^H f(\zeta)] \leq \Re [\tau^H f(\xi)], \) or equivalently,

\[ \Re [\tau^H f(\zeta) - \tau^H f(\xi)] \leq 0. \]

On the other hand, since \( \zeta = (z, \overline{z}) \) is a feasible solution of (CMP), for the nonzero vector \( \mu \in S^* \), we have \( \Re \langle g(\zeta), \mu \rangle \leq 0. \) Thus, for any \( \xi \in Q \) in dual problem (D), inequality (18), thus

\[ \Re [\mu^H g(\zeta) - \mu^H g(\xi)] \leq 0. \]

(a) If hypothesis (i) holds, without loss of generality, assume that \( \Re [\tau^H f(\cdot)] \) is strictly convex and \( \Re [\mu^H g(\cdot)] \) is convex at \( \xi \in Q. \) From inequality (20) and \( \Re [\tau^H f(\cdot)] \) is strictly convex at \( \xi \in Q, \) we obtained

\[ \tau^T \nabla_z f(\xi) + \tau^H \nabla_{\tau} f(\xi) < 0. \]

From inequality (21) and \( \Re [\mu^H g(\cdot)] \) is convex at \( \xi \in Q, \) we obtained

\[ \mu^T \nabla_z g(\xi) + \mu^H \nabla_{\tau} g(\xi) \leq 0. \]
Adding inequalities (22) and (23), we have
\[ \tau^T \nabla_z f(\xi) + \tau^H \nabla_z g(\xi) + \mu^T \nabla_z g(\xi) + \mu^H \nabla_z g(\xi) < 0. \] (24)

The inequality (24) contradicts the equality of (16).

(b) If hypothesis (ii) holds, Re[\tau^H f(\cdot)] is quasiconvex at \(\xi \in Q\) and from inequality (20), it implies that
\[ \tau^T \nabla_z f(\xi) + \tau^H \nabla_z f(\xi) \leq 0. \] (25)

Since Re[\mu^H g(\cdot)] is strictly pseudoconvex at \(\xi \in Q\) and from inequality (21), we obtain
\[ \mu^T \nabla_z g(\xi) + \mu^H \nabla_z g(\xi) < 0. \] (26)

Adding (25) and (26), we again get inequality (24), which contradicts to the equality of (16).

(c) Adding (20) and (21), we get
\[ \text{Re}[\tau^H f(\zeta) + \mu^H g(\zeta)] \leq \text{Re}[\tau^H f(\xi) + \mu^H g(\xi)]. \]

Suppose hypothesis (iii) holds. Since Re[\tau^H f(\cdot) + \mu^H g(\cdot)] is strictly pseudoconvex at \(\xi \in Q\) and the above inequality holds, we get inequality (24), which contradicts to the equality of (16).

Then the proof is complete.

Given an optimal efficient solution of problem (CMP), we can obtain a feasible solution of the dual problem (D), and the following strong duality theorem will be proved.

**Theorem 4.2. (Strong Duality)** Let \(T \subset \mathbb{C}^p\) be a pointed, closed and convex cone. Suppose that \(\zeta_0\) is a minimal efficient solution of (CMP) with respect to \(T\), and problem (CMP) possesses the constraint qualification in Definition 3.2 at \(\zeta_0\). Then there exists a feasible solution \((\tau, \zeta_0, \mu, K)\) of the dual problem (D). If the hypotheses of Theorem 4.1 are fulfilled, then \((\tau, \zeta_0, \mu, K)\) is an optimal efficient solution of (D), and the two problems (CMP) and (D) have the same optimal value.

**Proof.** Let \(\zeta_0 = (z_0, \overline{z_0}) \in Q\) be a minimal efficient solution of problem (CMP) with optimal value \(K\). By Theorem 3.4, there exist \(\tau \in T^* \subset \mathbb{C}^p\) and \(\mu \in S^* \subset \mathbb{C}^q\) such that
\[ \tau^T \nabla_z f(\zeta_0) + \tau^H \nabla_z f(\zeta_0) + \mu^T \nabla_z g(\zeta_0) + \mu^H \nabla_z g(\zeta_0) = 0, \]
\[ \text{Re}[\mu^H g(\zeta_0)] = 0, \]
and therefore conditions (16) and (18) hold. Because \(K\) is the optimal value of problem (CMP), that is \(K = \min f(\zeta) = f(\zeta_0)\). It implies that Re\((f(\zeta_0) - K, \tau) = 0\), condition (17) holds. Hence, \((\tau, \zeta_0, \mu, K)\) is a feasible solution of the dual problem (D).

From Theorem 4.1, the optimality of the feasible solution \((\tau, \zeta_0, \mu, K)\) for (D) reduces to be the optimal value of (D). Indeed, if there exists a feasible solution \((\tau', \zeta', \mu, K')\) of (D) such that \(K' - K \in T \setminus \{0\}\). Since \(K = f(\zeta_0)\) is the optimal value of problem (CMP), we obtain
\[ K' - f(\zeta_0) \in T \setminus \{0\}, \]
which contradicts to Theorem 4.1. 

\[ \square \]
If both optimal efficient solutions of primary problem (CMP) and dual problem (D) exist, then the optimal values of (CMP) and (D) are coincident under some assumptions. We could prove this result as the following theorem.

**Theorem 4.3. (Strictly Converse Duality)**

Let $T \subset \mathbb{C}^p$ be a pointed, closed and convex cone. Suppose that $\tilde{\zeta}$ and $(\tau, \tilde{\xi}, \mu, \tilde{K})$ are optimal efficient solutions of (CMP) and (D) with respect to $T$, respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. In addition, if $\text{Re}[\tau^H f(\cdot)]$ is strictly pseudoconvex at $\hat{\xi} \in Q$ and $\text{Re}[\mu^H g(\cdot)]$ is quasiconvex at $\hat{\xi} \in Q$, then $\hat{\zeta} = \tilde{\zeta}$, and the optimal values of (CMP) and (D) are coincident.

**Proof.** We assume that $\hat{\zeta} \neq \tilde{\zeta}$, and then reach a contradiction. Since $\tilde{\zeta}$ is an optimal efficient solution of (CMP) with optimal value $\tilde{K}$, Theorem 4.2 then implies

$$\tilde{K} = \min f(\tilde{\zeta}) = (f_1(\tilde{\zeta}), \cdots, f_p(\tilde{\zeta})).$$

Hence, $\text{Re}\langle f(\tilde{\zeta}) - \tilde{K}, \tau \rangle = 0$, for nonzero $\tau \in T^*$. That is

$$\text{Re}[\tau^H f(\tilde{\zeta})] = \text{Re}[\tau^H \tilde{K}],$$

for nonzero $\tau \in T^*$. (27)

On other hand, using the feasibility of $\tilde{\zeta}$ of (CMP), $\mu \in S^*$, and inequality (18), we have

$$\text{Re}[\mu^H g(\tilde{\zeta})] \leq 0 \leq \text{Re}[\mu^H \tilde{g}(\tilde{\zeta})].$$

By the hypothesis, $\text{Re}[\mu^H g(\cdot)]$ is quasiconvex at $\hat{\xi} \in Q$. Let $\hat{\zeta} - \tilde{\zeta} = (\check{v}, \tilde{v})$. Then

$$\text{Re}\langle \hat{v}, \mu^T \nabla_z g(\hat{\xi}) + \mu^H \nabla_{\tau^T} g(\tilde{\xi}) \rangle \leq 0.$$

(28)

Using equality (16) and inequality (28), we have

$$\text{Re}\langle \check{v}, \tau^T \nabla_z f(\hat{\xi}) + \tau^H \nabla_{\tau^T} f(\tilde{\xi}) \rangle \geq 0.$$

(29)

Since $\text{R}[\tau^H f(\cdot)]$ is strictly pseudoconvex at $\hat{\xi} \in Q$, inequality (29) implies that

$$\text{Re}[\tau^H f(\hat{\xi})] > \text{Re}[\tau^H f(\tilde{\xi})].$$

Equality (27) further implies

$$\text{Re}[\tau^H \tilde{K}] > \text{Re}[\tau^H f(\tilde{\xi})],$$

which contradicts to the inequality of (17). Therefore, the conclusion is $\hat{\zeta} = \tilde{\zeta}$, and the proof is complete.

5. **Conclusion.** In this paper, we obtain certain sufficient optimality conditions for (CMP), establish parametric dual model (D) for (CMP), and discuss their duality theorems. In the future, we could consider some types of duality model of the following complex fractional multi-objective programming problem.

$$(\text{CFMP}) \quad \min_{\zeta \in X} \left( \frac{f_1(\zeta)}{g_1(\zeta)}, \cdots, \frac{f_p(\zeta)}{g_p(\zeta)} \right)$$

s.t. $X = \{(z, \bar{z}) \in Q \mid -h(\zeta) \in S\},$

where $f_i, g_i$, for $i = 1, \ldots, p$ are analytic functions from $\mathbb{C}^{2n}$ to $\mathbb{C}$, and $h : \mathbb{C}^{2n} \to \mathbb{C}^q$ is an analytic mapping.

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