A Generalized Multivariable Adaptive Super-Twisting Control and Observation for Amphibious Robot

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This work was supported in part by the National Key Research and Development Program of China under Grant 2022YFB3706902 and Grant 2022YFB3706901, and in part by the National Natural Science Foundation of China under Grant 51975593.

ABSTRACT This paper aims at the problem of fast stability control of amphibious robot after state switch. A novel method named as generalized multivariable adaptive super-twisting algorithm (GMASTA) is proposed for the finite time attitude control of amphibious robot. In the proposed method, the standard multivariate super-twisting algorithm is improved by introducing a linear term and a higher power term, in order to reduce convergence time and chattering. In addition, the adaptive gain is employed based on a disturbance observer which can avoid a conservative overestimation of the disturbance. For the designed attitude controller, the finite time stability is proved by Lyapunov method. Two numerical examples were carried out. The results from the first example show that GMASTA is better than MSTA in convergence and robustness. The other example is used to illustrate that the disturbance observer based on the GMASTA can realize the accurate observation of the integrated interference. Compared with PID, the proposed method based on GMASTA can converge faster and has better robustness.

INDEX TERMS General finite time control, generalized multivariable adaptive, super-twisting algorithms, adaptive control, sliding mode control.

I. INTRODUCTION As a special kind of service robots, the negative pressure adsorption wall-climbing robot can replace people to work in some dangerous environments, such as the cleaning for concrete-wall or glass-wall, and the maintenance for high-rise buildings [1], [2], [3], structural health detection of viaduct or tunnel structural health monitoring of bridges, nuclear power plants or highway tunnel or railway tunnel [4], [5]. The negative pressure wall-climbing robot can attach to the wall surface due to the negative pressure adsorption. There is a risk of accidental falling off resulted from some faults or uncertainty external disturbances. Therefore, some innovative wall-climbing robots were proposed. One of new wall-climbing robot includes two sub-systems: a wall-climbing part and a flight part based on a quadrotor. It can quickly switch to flight mode when it is in danger of falling off. The main challenge is to design a control system for quickly realizing stabilized flight in very short time during the control mode switch from climbing to flight.

In general, the amphibious robot is a kind of typical multiple input-multiple output, strongly nonlinear, strongly coupled model uncertainties and underactuated system [6]. The flight control system design is a challenging work for the quadrotor. Many control methods for the quadrotor robot have been proposed. Generally, these control methods can be
classified into three categories: the classical linear control, learning-based control, and nonlinear control techniques.

The classical linear control methods mainly include the proportional-integral-derivative (PID) and linear quadratic regulator (LQR). The advantage of PID is that it is unnecessary to establish an accurate robot model when facing the gain adjustment of the actual system [8], [9]. Unlike the PID controller, the LQR adopts a cost function minimizing approach, which can deal with multiple input-multiple output problems. However, it usually occurs steady-state errors during the tracking of quadrotor robot [10], [11].

The learning-based control method is such a control technique which does not require an accurate dynamic model. It just needs some trials and flight data for training the system to control quadrotor robot. The fuzzy control, neural network control and reinforcement learning control are considered as typical learning-based control techniques [11], [12], [13], [14], [15], [16]. However, the control technology based on learning needs enough training data, and cannot always ensure the stability of system. Raza et al. [12] proposed a fuzzy logic based flight controller for an autonomous quadrotor. The quadrotor simulator demonstrated that the fuzzy control is suitable for the nonlinear quadrotor systems with complex uncertainties in the presence of various disturbances, such as sensor noise and high wind conditions. Kayacan et al. [13] studied fuzzy neural networks for the trajectory tracking problem of quadrotor in terms of their tracking accuracy and control efforts. Razmi et al. [14] proposed a neural network-based adaptive sliding mode control to realize the position and attitude tracking control of quadrotor in the presence of parametric uncertainties and external disturbance. Becker-Ehmck et al. [15] trained a thrust-attitude controller for a quadrotor through model-based reinforcement learning. A drone dynamics model is learned from raw sensor observations, without encoding any physics knowledge into the latent state-space.

Some nonlinear methods have also been developed for quadrotor, such as feedback linearization, backstepping, and sliding mode control [17], [18], [19]. The main advantage of feedback linearization is to transform a nonlinear system into a linear model. It will have good performance if there is only a little difference between the linear model and nonlinear system. However, when there is a large model uncertainty, the controller will have poor robustness [17]. The backstep method can deal with the nonlinear part of system and overcome the mismatched disturbance. However, the backstep method exits a problem of excessive parameterization, which makes it so difficult to obtain effective control performance accurately [18]. To stabilize the nonlinear system in finite time, the sliding mode control technology is widely applied. However, the first order sliding mode control is prone to chattering or high-frequency oscillation due to continuous switching [19]. So, a kind of continuous sliding mode controllers named Super-twisting algorithms (STA) has been proposed to reduce the chattering [20], [21], [22], [23], [24], [25]. Tian et al. [26] recently proposed a multivariable super-twisting-like algorithm (STLA), which can drive the attitude tracking errors of quadrotor UAV to zero in finite time. It indicates that the controller based on STA or STLA is suitable for attitude control of quadrotor UAV. In addition, the Generalized Super-Twisting Algorithms (GSTA) was proposed by Moreno [27], in which an extra linear correction terms was added in classical STA. This method provided a faster convergence and an enhanced robustness of the stability. The STA obviously depends on the disturbances and uncertain boundary knowledge, so a multivariable non-decoupled super-twisting algorithm with adaptive gains was proposed to solve the situation where the interference boundary is unknown [28].

In this paper, inspired by [27] and [28], a novel method named as generalized multivariable adaptive super-twisting algorithms (GMSTA) is developed for amphibious robot. In novel method, the standard multivariate super-twisting algorithm is improved by introducing a linear term and a higher power term. These additional degrees of freedom provide faster control convergence rates when the trajectory is far from the origin. In addition, a disturbance observer based on GMSTA is also proposed for avoiding a conservative overestimation of the disturbance. Moreover, a detail proof for the finite time stability was performed using Lyapunov method. Two examples have been provided to validate the effectiveness of the method.

II. PRELIMINARIES

A. NOTATIONS

The following notations will be used.

\[
\left(\|x\|^p \text{sign}(x)\right)' = \|x\|^{p-1} \left[I_m + (p-1) \frac{x x^T}{\|x\|^2}\right] \dot{x},
\]

\[
\text{sign}(x) = \frac{x}{\|x\|}, \quad \text{where} \quad \|x\| = \sqrt{x^T x}, \quad x \in \mathbb{R}^n
\]

where \( p > 1 \)

\( \lambda_{\max}(\cdot) \) and \( \lambda_{\min}(\cdot) \) denotes the largest and smallest eigenvalues a matrix respectively.

B. PRELIMINARIES

Consider that A, B, C, and D are the matrices with compatible dimension. The Kronecker product is a block matrix is defined as follows.

\[
(A \otimes B) = \begin{bmatrix}
    a_{11}B & \cdots & a_{1n}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

The following Kronecker properties hold [28] and [29]

\[
(A \otimes B) (C \otimes D) = AC \otimes BD
\]

\[
A \otimes C + B \otimes C = (A + B) \otimes C
\]

\[
(A \otimes B)^T = A^T \otimes B^T
\]
Consider the following nonlinear dynamical system:

\[ \dot{x} = f(x), \ x \in \mathbb{R}^n, \ t > t_0, x(t_0) = x_0 \]  

where \( f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) may not be continuous. There exists a Filippov solution for system (1). It is assumed that the origin is an equilibrium point of system (1).

**Lemma 1 [23]:** Consider the system (1). There exists a continuously differentiable, positive definite and radially unbounded function \( V(x) \) satisfying

\[ \dot{V}(x) \leq -\beta V(x)^\alpha \]  

with \( \alpha \in (0, 1) \) and \( \beta > 0 \). Then, the value \( x \) in (1) will converge to zero in finite time

\[ T \leq \frac{1}{\beta(1-\alpha)} V(x_0)^{(1-\alpha)} \]  

where \( V(x_0) \) being the initial value of \( V(x) \).

**III. PROBLEM FORMULATION**

The configuration of amphibious robot is based on quadrotor UAV. The robot has four actuators and rotors. It is an underactuated system because it only has the maximum four-rank input matrix for controlling six degrees of freedom, regardless of the number of inputs. The model is derived using the Euler-Lagrange approach. The model [26] of the amphibious robot can be expressed as

\[ \begin{cases} \dot{\Theta} = W \Omega \\ I \dot{\Omega} = -\Omega \times I \Omega + \tau + \Delta(t) \end{cases} \]

where \( I = \text{diag}[I_x, I_y, I_z] \) is a symmetric positive definite constant inertia matrix, \( \Theta = [\phi, \theta, \psi]^T \) is the Euler angle expressed in inertial frame, \( \Omega = [p, q, r] \) is the attitude angular velocity expressed in the body frame, \( \tau = [\tau_1, \tau_2, \tau_3] \) is the control torque vector, \( \Delta(t) \in \mathbb{R}^3 \) represents the bounded external disturbances, and \( \| \Delta(t) \| \leq L \), where \( L \) is a known constant. The matrix \( W \) is calculated by

\[ W = \begin{pmatrix} \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \]

where \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), and \(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\).

In this paper, the generalized multivariable adaptive super-twisting algorithms(GMSTA) is utilized to design the finite time stabilizing attitude controller for amphibious robot in (4), such that the system output \( \Theta = [\phi, \theta, \psi]^T \) track the reference commands \( \Theta_{\text{ref}} = [\phi_{\text{ref}}, \theta_{\text{ref}}, \psi_{\text{ref}}]^T \) in finite time. Define the tracking errors of attitude and attitude angular velocity as \( e_1 = \Theta - \Theta_{\text{ref}} \) and \( e_2 = W \Omega - \dot{\Theta}_{\text{ref}} \), respectively. Taking into account the attitude dynamics of amphibious robot in (4), the error dynamics of attitude are governed by

\[ \begin{align*} 
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= W^{-1} \tau - (W^{-1} \Omega \times I \Omega + \dot{\Theta}_{\text{ref}} - \dot{W} \Omega) + W^{-1} \Delta(t) 
\end{align*} \]

Then the state-space form of the model is given by

\[ \begin{align*} 
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -f(e) + \tau' + \Delta'(t)
\end{align*} \]

where, \( \tau' = WI^{-1} \tau, f(e) = (WI^{-1} \Omega \times I \Omega + \dot{\Theta}_{\text{ref}} - \dot{W} \Omega) \), and \( \Delta'(t) = WI^{-1} \Delta(t) \).

**IV. NEW CONTROL SCHEME**

**A. CONTROL ALGORITHM**

1) **GENERALIZED MULTIVARIABLE SUPER-TWISTING ALGORITHMS (GMSTA)**

Consider the following multivariable super-twisting algorithm

\[ \begin{cases} \dot{s}_1 = -\alpha_0 \phi_1(s_1) + s_2 \\
\dot{s}_2 = -\beta_0 \phi_2(s_1) + d(t) \end{cases} \]

where \( s_1 \in \mathbb{R}^m \) and \( s_2 \in \mathbb{R}^m \) are sliding variables, \( \alpha_0 \) and \( \beta_0 \) are positive constants, \( \phi_2(s_1) \) and \( \phi_1(s_1) \) are given by

\[ \phi_1(s_1) = \mu_1 \|s_1\|^{1/2} \text{sign}(s_1) + \mu_2 \|s_1\|^p \text{sign}(s_1) \]

\[ + \mu_3 s_1, \ \ p > 1 \]

\[ \phi_2(s_1) = \frac{d\phi_1(s_1)}{ds_1} \phi_1(s_1) \]

\[ d(t) \in \mathbb{R}^m \] represents unknown bounded external disturbance, and \( \| d(t) \| \leq L \). Furthermore, \( \frac{d\phi_1(s_1)}{ds_1} \) from (9) satisfies

\[ \frac{d\phi_1(s_1)}{ds_1} = \phi'_1(s_1) = \psi_1 + \psi_2 + \mu_3 \]

where

\[ \psi_1 = \mu_1 \|s_1\|^{1/2} \left[ I_m - \frac{s_1 {s'_1}}{2\|s_1\|^2} \right], \ \text{and} \]

\[ \psi_2 = \mu_2 \|s_1\|^p - 1 \left[ I_m + (p-1) \frac{s_1 {s'_1}}{\|s_1\|^2} \right] \]

And then \( \phi_2(s_1) \) is given by

\[ \phi_2(s_1) = (T_1 + \mu_3 \mu_1 \|s_1\|^{1/2} + \mu_3 \mu_2 \|s_1\|^p) \text{sign}(s_1) + \mu_3 s_1 \]

where

\[ T_1 = \frac{\mu_1^2}{2} + \mu_2 \|s_1\|^{2p-1} + \left( \frac{1}{2} + p \right) \mu_1 \mu_2 \|s_1\|^{p-\frac{1}{2}} \]

In the above GMSTA, a linear term and a higher power term are added in (9) relative to STA.

**Lemma 2:** Consider the system (8), if

\[ \begin{align*} 
L &\geq 1 \\
\mu_1 &< 1 \\
\alpha_0 &> \frac{4}{\beta_0 - 2} \\
\beta_0 &> 2 > 0
\end{align*} \]

Then, the sliding variables \( s_1 = s_2 = 0 \) after finite time. The proof process can be seen in section IV.B.
2) GENERALIZED MULTIVARIABLE ADAPTIVE SUPER-TWISTING ALGORITHMS (GMASTA)

Consider the following generalized multivariable algorithm based on a variation of the adaptive super-twisting algorithm

\[
\begin{align*}
\dot{s}_1 &= -\alpha(t)\phi_1(s_1) + \Gamma(t, \phi_1) + s_2 \\
\dot{s}_2 &= -\beta(t)\phi_2(s_1) + d(t) \\
\Gamma(t, \phi_1) &= -\frac{L}{2L} \left( \frac{d\phi_1}{ds_1} \right)^{-1} \phi_1
\end{align*}
\]

(13)

(14)

where, \( \alpha(t) = \alpha_0 \sqrt{L(t)} \), \( \beta(t) = \beta_0 L(t) \), \( L(t) \in \mathbb{R} \) is an adaptive gain of which the value is given by an adaptation law that will be defined later. When the value of \( L(t) \in \mathbb{R} \) is constant, such that \( L(t) = 0 \), then \( \Gamma(t, \phi_1) = 0 \). Therefore, GMASTA reverts to GMSTA.

Assumption 1: The function \( L(t) \in \mathbb{R} \) is differentiable and satisfies \( L(t) > \max(||d(t)||, l_0) \), \( \forall t \), where \( l_0 > 0 \in \mathbb{R} \), then we consider an adaptation law to guarantee that Assumption 1 is satisfied and also to ensure non-overestimated values for the gains.

GMSTA consists of two parts. In the first part, if Assumption 1 is satisfied. By the values of \( \alpha_0 \) and \( \beta_0 \), in the system (8) during finite time, the sliding variables \( s_1 = \dot{s}_1 = 0 \). In both parts, through the design of dual layer adaptive gain design, Assumption 1 holds in finite time.

Lemma 3: Consider the system (13) with Assumption 1. If

\[
\begin{align*}
\mu_1 &= \mu_2 = \mu_3 = 1 \\
\alpha_0^2 &> \frac{4}{\beta_0 - 2} \\
\beta_0 &> 0
\end{align*}
\]

(15)

Then, after a finite time, the sliding variables \( s_1 = \dot{s}_1 = 0 \).

B. STABILITY PROOF

1) FINITE TIME STABILITY PROOF OF GMASTA

In this section, consider the robust stability analysis and finite time convergence demonstration base on GMASTA.

Proof: choose \( \xi = \left[ \xi_1^T \; \xi_2^T \right]^T \), let \( \dot{\xi}_1 = \sqrt{L(t)}\phi_1(s_1) \),

\[
\dot{\xi}_2 = s_2, \text{ such that}
\]

\[
\begin{align*}
\dot{\xi}_1 &= \phi_1'(s_1) [\dot{s}_1 - \alpha(t)\phi_1(s_1) + \Gamma(t, \phi_1) + s_2] \\
&= \frac{L}{2\sqrt{L}} \phi_1(s_1) + \sqrt{L} \phi_1'(s_1) \dot{s}_1 \\
&= \frac{L}{2\sqrt{L}} \phi_1(s_1) + \sqrt{L} \phi_1'(s_1) \Gamma(t, \phi_1) \\
&\quad + \sqrt{L} \phi_1'(s_1) [\dot{s}_1 - \alpha(t)\phi_1(s_1) + s_2] \\
&= \sqrt{L} \phi_1'(s_1) [\dot{s}_1 - \alpha \xi_1 + \xi_2] \\
&= \sqrt{L} \phi_1'(s_1) [-k_1\phi_1(s_1) + s_2]
\end{align*}
\]

and

\[
\begin{align*}
\dot{\xi}_2 &= -\beta(t)\phi_2(s_1) + s_2 \\
&= -\beta_0 L(t)\phi_1'(s_1)\phi_1(s_1) + d(t)
\end{align*}
\]

\[
\begin{align*}
= \sqrt{L(t)} \left( -\beta_0 \phi_1'(s_1)\sqrt{L(t)}\phi_1(s_1) + \frac{d(t)}{\sqrt{L(t)}} \right) \\
= \sqrt{L(t)} \left( -\beta_0 \phi_1'(s_1)\sqrt{L(t)}\xi_1 + \frac{d(t)}{\sqrt{L(t)}} \right)
\end{align*}
\]

(17)

Furthermore, it satisfies

\[
\begin{align*}
\dot{\xi} &= \begin{bmatrix} -\alpha_0 & 1 \\ -\beta_0 & 0 \end{bmatrix} \otimes \phi_1'(s_1)\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_m \Lambda \\
&= A_0 \xi + B_0 \dot{\Delta} = A \otimes \phi_1'(s_1)\xi + B \otimes I_m \Lambda
\end{align*}
\]

(18)

where

\[
A = \begin{bmatrix} -\alpha_0 & 1 \\ -\beta_0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\quad A_0 = \begin{bmatrix} -\alpha_0 & 1 \\ -\beta_0 & 0 \end{bmatrix}
\]

\[
\otimes \phi_1'(s_1), B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_m, \quad \Lambda = \frac{d(t)}{\sqrt{L(t)}}
\]

Now, consider the following Lyapunov function

\[
V = \xi^T P_n \xi
\]

(19)

where \( P_n = P \otimes I_m \), then \( P \) can be expressed as

\[
P = \begin{bmatrix} p_1 & -p_2 \\ -p_2 & p_3 \end{bmatrix}
\]

The derivative of Lyapunov function can be obtained as

\[
\begin{align*}
\dot{V} &= \dot{\xi}^T P_n \xi + \xi^T P_n \dot{\xi} \\
&= \sqrt{L(t)} \left( \xi^T A_0^T P_n \xi + \xi^T P_n A_0 \xi + 2\xi^T P_n B_0 \Lambda \right) \\
&= \sqrt{L(t)} \left( T_2 + 2\xi^T P_n B_0 \Lambda \right) \\
&= \sqrt{L(t)} \left( T_2 + 2\xi^T P_n B_0 \Lambda \right)
\end{align*}
\]

(20)

where

\[
T_2 = \xi^T A_0^T P_n \xi + \xi^T P_n A_0 \xi \\
+ \xi^T \left( \left( A^T \otimes \phi_1'(s_1) \right) \left( P \otimes I_m \right) \right) \\
+ \xi^T \left( \left( A^T P \otimes \phi_1'(s_1) \right) \left( I_m \otimes \phi_1'(s_1) \right) \right) \\
+ \xi^T \left( \left( A^T P \otimes \phi_1'(s_1) \right) \right) \xi
\]

Suppose \( Q = A^T P + PA \) and \( Q_0 = Q \otimes \psi_1(s_1) \), we have

\[
\dot{V} = \sqrt{L(t)} \left( \xi^T Q_0 \xi + 2\xi^T \left( \xi^T P_n B_0 \Lambda \right) \right)
\]

(21)

Define \( p_1 = -\alpha_0 p_2 + \beta_0 p_3 \), and

\[
Q = \begin{bmatrix} 2 ((\beta_0 + \alpha_0^2) p_2 - \alpha_0 \beta_0 p_3) & 0 \\ 0 & -2p_2 \end{bmatrix}
\]

(22)

where \( Q \) is a negative definite matrix.

It follows from (21), \( \xi^T Q_0 \xi \) satisfies

\[
\begin{align*}
\xi^T Q_0 \xi &= \xi^T \left( Q \otimes \psi_1(s_1) \right) \xi + \xi^T \left( Q \otimes \psi_2(s_1) \right) \\
&\quad + \mu_3 \xi^T \xi \leq T_3 T_4
\end{align*}
\]

(23)
where
\[ T_3 = \mu_1 \frac{1}{2} \| s_1 \|^2 + \mu_2 \| s_1 \|^p - 1 + \mu_3 \]
\[ T_4 = 2 \left( (\beta_0 + \alpha_0^2) p_2 - \alpha_0 \beta_0 p_3 \right) \| \xi_1 \|^2 - 2p_2 \| \xi_2 \|^2 \]
then we can obtain
\[ \lambda_{\min}(\psi_1) = \frac{\mu_1}{2} \| s_1 \|^\frac{1}{2}, \quad \lambda_{\max}(\psi_1) = \frac{\mu_1}{\| s_1 \|^{\frac{1}{2}}} \]  
(24)
\[ \lambda_{\min}(\psi_2) = \mu_2 \| s_1 \|^p - 1, \quad \lambda_{\max}(\psi_2) = \mu_2 p \| s_1 \|^{p - 1} \]  
(25)
The relevant proof is listed in Appendix. Suppose \( \xi = [\| \xi_1 \| \| \xi_2 \|]^T \), hence
\[ \xi^T Q \xi \leq \left( \frac{\mu_1}{2} \| s_1 \|^\frac{1}{2} + \mu_2 \| s_1 \|^{p - 1} + \mu_3 \right) \xi^T Q \xi \]  
(26)
and, \( \xi^T P_{n} B_0 \dot{\Lambda} \) satisfies
\[ 2 \xi^T P_{n} B_0 \Lambda \leq 2p_2 \left| \xi^T \Lambda \right| + 2p_3 \left| \xi^T \Lambda \right| \]
\[ \leq T_5 \left( 2p_2 \| \xi_1 \|^2 + 2p_3 \| \xi_2 \| \| \xi_1 \| \right) \frac{1}{T_5 T_6} \]
\[ \leq T_5 \left( 4 \mu_1^2 p_2 \| \xi_1 \|^2 + 4 \mu_2^2 p_3 \| \xi_2 \| \| \xi_1 \| \right) \]  
(27)
where
\[ T_5 = \mu_1 \frac{1}{2} \| s_1 \|^\frac{1}{2} + \mu_2 \| s_1 \|^{p - 1} + \mu_3 \]
\[ T_6 = \mu_1 \| s_1 \|^\frac{1}{2} + \mu_2 \| s_1 \|^p + \mu_3 \| s_1 \| \]
\[ T_5 T_6 \geq \mu_1 \frac{1}{2} \| s_1 \|^\frac{1}{2} + \mu_2 \| s_1 \|^\frac{1}{2} = \frac{\mu_1^2}{2} \]
Consider (26) and (27), then
\[ \dot{V} \leq -\frac{\sqrt{T(t)}}{(\mu_1 + \mu_2 \| s_1 \|^{p - 1} + \mu_3)} \left( \xi^T M \xi \right) \]  
(28)
where
\[ M = \left[ 2 \left( \alpha_0 \beta_0 p_3 - \left( \beta_0 + \alpha_0^2 + \frac{2}{\mu_1^2} \right) p_2 \right) - \frac{2}{\mu_1^2} p_3 \right] \frac{1}{2p_2} \]
The following inequality (29) is necessary to make \( W \) positive definite when \( p_2 = 1 \), and \( \mu_1 = 1 \).
\[ \left\{ \begin{array}{l}
\alpha_0 \beta_0 p_3 - \left( \beta_0 + \alpha_0^2 + \frac{2}{\mu_1^2} \right) p_2 > 0; \\
\alpha_0 \beta_0 p_3 - \left( \beta_0 + \alpha_0^2 + \frac{2}{\mu_1^2} \right) p_2 - \frac{1}{\mu_1^2} p_3 > 0 \end{array} \right. \]  
(29)
Case 1:
\[ \alpha_0 \beta_0 p_3 - \left( \beta_0 + \alpha_0^2 + 2 \right) - p_3^2 > 0 \]  
(30)
Case 2:
\[ (\alpha_0^2 p_3 - 4 \left( \beta_0 + \alpha_0^2 + 2 \right) > 0 \]  
(31)
Case 3:
\[ \alpha_0^2 > \frac{4}{\beta_0 - 2}; \beta_0 - 2 > 0 \]  
(32)
From Case 2, to be existed \( p_3 \), we have Case 1. From Case 1, it is easy to satisfy (29).
From (28), consider
\[ V = \xi^T P_n \xi \leq \lambda_{\max}(P_n) \| \xi \|^2 \]
\[ = \lambda_{\max}(P_n) \| \xi \|^2 \lambda_{\min}(M) \| \xi \|^2 \leq \xi^T M \xi \]  
(33)
We have
\[ \dot{V} \leq -\frac{\sqrt{T(t)}}{(\mu_1 + \mu_2 \| s_1 \|^{p - 1} + \mu_3)} \lambda_{\min}(M) \lambda_{\max}(P_n) \]  
(34)
From Case 3 and Case 4, it follows that
\[ \dot{V} \leq -\frac{\sqrt{T(t)}}{(\mu_1 + \mu_2 \| s_1 \|^{p - 1} + \mu_3)} \lambda_{\min}(M) \lambda_{\max}(P_n) \]  
(35)
Thus, system (13) converges to zero from the initial trajectory in finite time with \( \xi = 0 \), such that it can be verified that \( s_1 = \dot{s}_1 = 0 \) in finite time.

2) DUAL LAYER ADAPTIVE GAIN DESIGN
The dual layer adaptive gain is designed such that Assumption 1 holds. When the system (13) is in the sliding mode, \( s_1 = \dot{s}_1 = 0 \) and \( s_2 = 0 \), it can be obtained \( d(t) = \beta(t) \theta_2(s_1) \). Consider the formula
\[ u_{eq} = d(t) = \beta(t) \frac{\mu_2^2}{2} \text{sign}(s_1) \]  
(36)
where
\[ h(t) = \beta(t)(T_7 + T_8) \text{sign}(s_1) + \beta(t) \mu_2^2 s_1 \]
\[ T_7 = p_2 \mu_2^2 \| s_1 \|^p - 1 + \left( \frac{1}{2} + p \right) \mu_1 \mu_2 \| s_1 \|^{p - \frac{1}{2}} \]
\[ T_8 = \mu_3 \mu_1 \| s_1 \|^\frac{1}{2} + \mu_3 \mu_2 \| s_1 \|^p \]
In (36), \( u_{eq} \) can be understood as an equivalent control, and a low-pass filter can be used for online estimation
\[ \dot{\sigma}(t) = \frac{1}{\tau} \text{sign}(s_1) - \sigma(t), \quad \dot{u}_{eq} = \beta(t) \frac{\mu_2^2}{2} \sigma(t) + h(t) \]  
(37)
where $0 < \nu \ll 1$ and $\sigma(t)$ is the estimated value of $\text{sign}(s(t))_{eq}$. During the sliding mode approach motion phase, $\tilde{u}_{eq}$ does not represent the equivalent control, but it is still a usable signal employed in the adaptive scheme. The real-time estimate of $d(t)$ can be obtained by (37). The adaptive gain $L(t)$ is designed as follows [31].

$$
\begin{align*}
L(t) &= l(t) + l_0 \\
\dot{l}(t) &= -\gamma(t)\text{sign}(\delta(t)) \\
\gamma(t) &= r(t) + r_0 \\
\dot{r}(t) &= \gamma(\delta(t))
\end{align*}
$$

(38)

where $l_0 > 0$, $r_0 > 0$, $\gamma > 0$. Consider the scalar $\delta(t)$ in (38)

$$
\delta(t) = L(t) - \frac{1}{a\beta_0} \| \tilde{u}_{eq} \| - \epsilon
$$

(39)

and its derivative is given by

$$
\dot{\delta}(t) = - (r(t) + r_0) \text{sign}(\delta(t)) - \frac{1}{a\beta_0} \frac{d}{dt} \| \tilde{u}_{eq} \| (40)
$$

where $a > 0$, $a\beta_0 < 1$, $\epsilon > 0$. The dual layer adaptation scheme (36)-(40) can be adjusted by choosing appropriate $a$ and $\epsilon$.

**Lemma 4:** If the bounded external perturbation $d(t)$ satisfies $\| \tilde{d}(t) \| \leq l$, where the positive constant $l$ is finite and unknown. Then, the designed adaptation scheme (36)-(40) can make assumption 1 is satisfied in finite time.

**Proof:** Consider variables

$$
m(t) = \frac{g}{a\beta_0} - r(t)
$$

(41)

and its derivative is given by

$$
\dot{m}(t) = -\gamma |\delta(t)|
$$

(42)

The value of $g$ must ensure that $\frac{d}{dt} \| \tilde{u}_{eq} \| < g$ is established. Consider the Lyapunov function

$$
V_L = \frac{m^2(t)}{2y} + \frac{\delta^2(t)}{2}
$$

(43)

Combining equations (40), (41), and (42), its derivative can be obtained as

$$
\dot{V}_L \leq -r_0 |\delta(t)|
$$

(44)

So that $m(t)$ and $\delta(t)$ are bounded. Thus, the value of $r(t)$ is limited from (41). From the above analysis, it can be obtained that $\dot{\delta}(t)$ is bounded.

From (43) it can be obtained

$$
r_0 \int_0^t |\delta(t)| \, dt \leq V_L(0)
$$

(45)

Since $\delta(t)$ is consistent and continuous, $|\delta(t)|$ is consistent and continuous. According to Barbalat’s theorem, $|\delta(t)| \to 0$ as $t \to \infty$. Consequently there exists a finite time $t_L$ such that $|\delta(t)| < \frac{\epsilon}{2}$ for all time $t > t_L$. It follows from (39) that

$$
|\delta(t)| \leq \left| L(T) - \frac{1}{a\beta_0} \| \tilde{u}_{eq} \| - \epsilon \right| < \frac{\epsilon}{2}, t > t_L
$$

(46)

From (39) it can be obtained

$$
L(T) > \frac{1}{a\beta_0} \| \tilde{u}_{eq} \| + \frac{\epsilon}{2} > \| \tilde{u}_{eq} \| = d(t), t > t_L
$$

(47)

Hence, Assumption 1 is satisfied.

In the GMASTA algorithm, $L(t) \in R$ is the time-varying adaptive gain parameter, and its value is determined by the double-layer adaptive law. When the value of $L(t)$ remains fixed, then $\dot{L}(t) = 0$, and thus $\Gamma(t, \phi_1) = 0$, GMASTA degenerates into GMSTA. The proof of Lemma 1 is the case where the value of $L(t)$ of GMASTA remains fixed.

**V. SIMULATION**

**A. GENERALIZED MULTIVARIABLE SUPER-TWISTING ALGORITHMS**

In the section, the parameters of robot come from [20], i.e. $I_x = 2.3 \times 10^{-3}$ kg $\cdot m^2$, $I_y = 2.4 \times 10^{-3}$ kg $\cdot m^2$, $I_z = 2.6 \times 10^{-3}$ kg $\cdot m^2$, $L = 3$. The initial conditions are given as

$$
\Theta(0) = [\phi(0), \theta(0), \psi(0)]^T
$$

$$
\phi(0) \in [-0.5, 0.5] rad
$$

$$
\theta(0) \in [-0.4, 0.6] rad
$$

$$
\psi(0) \in [-0.5, 0.5] rad
$$

$$
\Omega(0) = [0 0 0]^T rad/s
$$

(48)

The reference commands of attitude angles are defined by

$$
\Theta_{ref} = [0.1 \sin(t), 0.1 \cos(t)]^T
$$

(49)

The disturbances in the simulation are added as

$$
\Delta = [1 + \sin(5t), 1 + \cos(5t), 0.5(1 + \sin(5t) + \cos(5t))]^T I
$$

(50)

Consider system (4), a sliding manifold has to be designed as

$$
s_1 = e_2 + \int_0^t (\lambda_1 \| e_1 \|^{\sigma_1} \text{sign}(e_1) + \lambda_2 \| e_2 \|^{\sigma_2} \text{sign}(e_2)) \, dt
$$

(51)

Hence, the dynamically equivalent model control is

$$
\tau'_{eq} = WI^{-1} \Omega \times I \Omega + \tilde{\Theta}_{ref}
$$

$$
-WI^{-1} \lambda_1 \| e_1 \|^{\sigma_1} \text{sign}(e_1) - \lambda_2 \| e_2 \|^{\sigma_2} \text{sign}(e_2)
$$

(52)

The multivariable super-twisting algorithm (MSTA) in [26] is designed by

$$
\tau' = \tau'_{eq} - a_0 \|s_1\|^{\frac{1}{2}} \text{sign}(s_1) - 2\beta_0 \int_0^t \|s_1\| \, dt
$$

(53)

where the control parameters are given as

$$
\lambda_1 = 5, \lambda_2 = 4, a_0 = 9, \beta_0 = 18, \sigma_1 = 0.9\sigma_2 = 0.8182
$$

The system can reach the sliding mode surface in finite time.
In this paper, the GMSTA is given by

\[ \tau' = \tau'_{eq} - \alpha \phi_1(s_1) - \beta_0 \int_0^t \phi_2(s_1) dt \]  

(54)

Consider (12), we choose \( \mu_1 = \mu_2 = \mu_3 = \sqrt{3}, \alpha_0 = \frac{9}{\sqrt{3}}, \beta_0 = 18 \). All the initial values of the integral items are selected by 0. The simulation is carried out in the MATLAB/Simulink software with a fixed sampling time 4 ms [26].

The simulation results are shown by Fig.s (1)-(2), from the results, it can be observed that the GMSTA proposed in this paper can converge to zero faster than MSTA [26]. It shows that the system can reach the sliding mode surface faster in the approaching movement stage, and has almost no chattering after stabilization.

The attitude tracking and attitude tracking errors using GMSTA, including roll angle, pitch angle, and yaw angle are presented in Fig.s (3)-(4). It is clear from Fig 3 that the desired attitude commands can be tracked effectively by the proposed control scheme with the bounded external disturbance. Furthermore, from Fig (4), it can be observed that the attitude tracking errors converge to zero. The results validate the fast convergence and robustness of the proposed control methods.

B. DISTURBANCE OBSERVER BASED ON GMASTA

In this section, the parameters are given as follows. \( I_x = 8.93 \times 10^{-2} \text{kg} \cdot \text{m}^2, I_y = 8.93 \times 10^{-2} \text{kg} \cdot \text{m}^2, I_z = 1.628 \times 10^{-4} \text{kg} \cdot \text{m}^2 \). The initial conditions are given as (48). The reference commands of attitude angles are defined by \( \Theta_{ref} = [0.1 \sin(t), 0.1 \cos(t), 0]^T \). The external disturbances are chosen as \( \Delta = [1 + \sin(5t), 1 + \cos(5t), 0.5(1 + \sin(5t) + \cos(5t))]^T I \).

Consider (7) and (13), a disturbance observer based on GMASTA is designed as

\[
\begin{align*}
\dot{k}_1 &= -\alpha(t)\phi_1(w_1) + \Gamma(t, \phi_1) + k_2 + \tau' + f(e) \\
\dot{k}_2 &= -\beta(t)\phi_2(w_1)
\end{align*}
\]

(55)

where, \( k_1 \) is observer value of \( e_2, k_2 \) is observer value of \( \Delta'(t) \), the estimation errors are defined as \( w_1 = k_1 - e_2, w_2 = k_2 - \Delta'(t) \), the estimation error model is designed as

\[
\begin{align*}
\dot{w}_1 &= -\alpha(t)\phi_1(w_1) + \Gamma(t, \phi_1) + w_2 \\
\dot{w}_2 &= -\beta(t)\phi_2(w_1) - \Delta'(t)
\end{align*}
\]

(56)

If \( \| \Delta'(t) \| \) is bounded, it follows from Lemma 3 that there exists a finite time such that the estimation errors converge to zero, \( w_1 = 0, w_2 = 0 \). It is to achieve online estimation of the disturbance. \( w_2 \) is used to realize the equivalent compensation of external interference, so that the attitude system of the amphibious robot can realize the accurate tracking of the given reference command. The controller base on GMSTA is designed as

\[
\tau' = f(e) - \lambda_1 \| e_1 \|^p \text{sign}(e_1) - \lambda_2 \| e_2 \|^q \text{sign}(e_2) - w_2
\]

(57)

where

\[
\begin{align*}
\lambda_1 &= 5, \lambda_2 = 4, \sigma_1 = 0.9, \sigma_2 = 0.8182, \alpha_0 = 9, \beta_0 = 9, \\
\mu_1 &= \mu_2 = \mu_3 = 1, p = 1.5, l_0 = 0.1, r_0 = 0.1, \\
\epsilon &= 0.1, a = 0.1, y = 5
\end{align*}
\]
All the initial values of the integral items are taken as zero. The simulation is carried out in a fixed sampling time 1 ms.

The simulation results are shown by Fig.s (5)-(7). As shown in Fig (5), the observed error $w_1$ and $w_2$ converge to zero in finite time. It shows that $\Delta'(t)$ is accurately estimated $u_{eq}$ in Fig (6). $k_2$ and $\Delta'(t)$ change over time as shown in Fig (7). It can be seen that the interference observer based on the GMASTA design realizes the accurate observation of the integrated interference.

From Fig (8), we observe the expected behavior of the adaptive gain $L(t)$, the adaptive gain can track the upper limit of the disturbance $\Delta'(t)$, the amplitude of the gain changes with the interference, and can suppress the interference with as small as possible control gain, it further improves the control performance, when considering about the advantages of STA’s weakening shake vibration.

The attitude tracking and attitude tracking errors using GMASTA and PID, including roll angle, pitch angle, and yaw angle are presented in Fig.s (9)-(10). It is clear from Fig. (9) that the desired attitude commands can be tracked effectively by the proposed control scheme with the bounded external
VI. CONCLUSION

In this research, a GMASTA is proposed for the finite time attitude control of amphibious robot. The mainly novelty and contribution is that a linear term and a higher power term is introduced based on STA, which effectively reduces convergence time and chattering. Moreover, the adaptive gain is employed based on a disturbance observer. The detail proof for the finite time stability is given based on Lyapunov method. The simulation results show that the proposed GMSTA is better than MSTA and PID in rate of convergence and robustness, which indicates the proposed method can well applied in the amphibious robot. In addition, the method can also employed in the control of other complex nonlinear systems.

APPENDIX

From section 4.2, \( \psi_1 \) and \( \psi_2 \) are given by

\[
\psi_1 = \mu_1 \|s_1\|^{1/2} \left[ I_m - \frac{s_1 s_1^T}{2 \|s_1\|^2} \right] \quad \psi_2 = \mu_2 \|s_1\|^{p-1} \left[ I_m + (p-1) \frac{s_1 s_1^T}{\|s_1\|^2} \right] \tag{58}
\]

From the Cauchy-Schwarz inequality for inner products, one has that \( v^T s_1 \leq \|v\| \|s_1\| \). Therefore, the following property holds for \( n \geq 2 \),

\[
v^T \psi_1 v = \mu_1 \left( \frac{\|v\|^2}{\|s_1\|^2} - \frac{(v^T s_1)^2}{2 \|s_1\|^2} \right) \\
\geq \mu_1 \frac{\|v\|^2}{2 \|s_1\|^2}, \forall v \in \mathbb{R}^n \tag{59}
\]

\[
v^T \psi_2 v = \mu_2 \left( \|s_1\|^{p-1} \|v\|^2 + (p-1) \|s_1\|^{p-1} \frac{(v^T s_1)^2}{\|s_1\|^2} \right) \\
\leq \mu_2 p \|s_1\|^{p-1} \|v\|^2, \forall v \in \mathbb{R}^n \tag{60}
\]
For $\psi_1, s_1 \neq 0$, the minimum value is reached at the maximum value of $v^T s_1$, hence

$$\lambda_{\min} (\psi_1) = \mu_1 \frac{1}{2 \| s_1 \| ^2}$$  \hspace{1cm} (61)

For $\psi_2, s_1 \neq 0$, the maximum value is reached at the maximum value of $v^T s_1$, hence

$$\lambda_{\max} (\psi_2) = \mu_2 P \| s_1 \| ^{p-1}$$  \hspace{1cm} (62)

Therefore, for every $s_1 \neq 0$, $\psi_1$ and $\psi_2$ is a Symmetric and Positive Definite (S.P.D.) matrix. When $v$ is orthogonal to $s_1$, i.e., $v^T s_1 = 0$. Thus, the following inequality is satisfied for every $s_1 \neq 0$.

$$v^T \psi_1 v = \mu_1 \left( \| v \| ^2 \right) - \left( \frac{v^T s_1}{\| s_1 \| ^2} \right) ^2 \leq \mu_1 \| v \| ^2$$  \hspace{1cm} (63)

$$v^T \psi_2 v = \mu_2 \left( \| s_1 \| ^{p-1} \| v \| ^2 + (p - 1) \| s_1 \| ^{p-1} \right) \frac{v^T s_1}{\| s_1 \| ^2} \geq \mu_2 \| s_1 \| ^{p-1} \| v \| ^2$$  \hspace{1cm} (64)

For $\psi_1, s_1 \neq 0$, the maximum value occurs when $v$ is orthogonal to $s_1$, hence

$$\lambda_{\max} (\psi_1) = \mu_1 \frac{1}{\| s_1 \| ^{2}}$$  \hspace{1cm} (65)

For $\psi_2, s_1 \neq 0$, the minimum value occurs when $v$ is orthogonal to $s_1$. Hence

$$\lambda_{\max} (\psi_2) = \mu_2 \| s_1 \| ^{p-1}$$  \hspace{1cm} (66)

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