Abstract

We prove that it is possible for nonconvex low-rank matrix recovery to contain no spurious local minima when the rank of the unknown ground truth \( r^* < r \) is strictly less than the search rank \( r \), and yet for the claim to be false when \( r^* = r \). Under the restricted isometry property (RIP), we prove, for the general overparameterized regime with \( r^* \leq r \), that an RIP constant of \( \delta < \frac{1}{1 + \sqrt{r^*/r}} \) is sufficient for the inexistence of spurious local minima, and that \( \delta < \frac{1}{1 + 1/\sqrt{r - r^* + 1}} \) is necessary due to existence of counterexamples. Without an explicit control over \( r^* \leq r \), an RIP constant of \( \delta < \frac{1}{2} \) is both necessary and sufficient for the exact recovery of a rank-\( r \) ground truth. But if the ground truth is known a priori to have \( r^* = 1 \), then the sharp RIP threshold for exact recovery is improved to \( \delta < \frac{1}{1 + 1/\sqrt{r}} \).

1 Introduction

The low-rank matrix recovery problem seeks to recover an unknown \( n \times n \) ground truth matrix \( M^* \) that is positive semidefinite (denoted \( M^* \succeq 0 \)) and of low rank \( r \ll n \) from \( m \) known linear measurements

\[
b \overset{\text{def}}{=} \mathcal{A}(M^*) \quad \text{where} \quad \mathcal{A}(M) \overset{\text{def}}{=} \left[ \text{tr}(A_1 M) \ldots \text{tr}(A_m M) \right]^T
\]

and where \( A_1, A_2, \ldots, A_m \in \mathbb{R}^{n \times n} \) are known data matrices. The problem has a well-known convex formulation \([9, 10, 12, 13, 31]\), but it is far more common in practice to solve a nonconvex formulation

\[
\min_{X \in \mathbb{R}^{n \times r}} f_{\mathcal{A}}(X) \quad \text{where} \quad f_{\mathcal{A}}(X) \overset{\text{def}}{=} \| \mathcal{A}(XX^T) - b \|^2 \tag{P}
\]

using a local optimization algorithm, starting from a random initial point. In principle, greedy local optimization can get stuck at a highly polished but fundamentally flawed solution, known as a spurious local minimum. Nevertheless, the nonconvex approach has been
highly effective in practice—both in its speed as well as its consistency in recovering the exact ground truth $M^*$—despite the apparent risk of failure.

In pursuit of a theoretical justification for this empirical success, recent work showed, for certain choices of the linear operator $\mathcal{A}$, that the corresponding nonconvex function $f_\mathcal{A}$ contains no spurious local minima $[3, 5, 20, 21]$:

$$\nabla f_\mathcal{A}(X) = 0, \quad \nabla^2 f_\mathcal{A}(X) \succeq 0 \iff XX^T = M^*.$$ 

Accordingly, any algorithm that converges to a second-order point is guaranteed to solve (P) to global optimality, starting from an arbitrary initial guess. One important line of results assumes that $\mathcal{A}$ satisfies the restricted isometry property (RIP) of Recht, Fazel, and Parrilo $[31]$, which is based in turn on a similar notion of Candes and Tao $[11]$. Intuitively, an RIP operator $\mathcal{A}$ is nearly orthonormal $\|\mathcal{A}(E)\| \approx \|E\|_F$ over low-rank matrices $E$.

**Definition 1.1 (Restricted Isometry Property).** The linear map $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ is said to satisfy $(\delta, 2r)$-RIP if there exists constant $\delta$ with $0 \leq \delta < 1$ and rescaling $\nu > 0$ such that

$$(1 - \delta)\|E\|_F^2 \leq \frac{1}{\nu}\|\mathcal{A}(E)\|^2 \leq (1 + \delta)\|E\|_F^2$$

holds for all $E \in \mathbb{R}^{n \times n}$ satisfying $\text{rank}(E) \leq 2r$.

We write “$\delta$-RIP” to mean “$(\delta, 2r)$-RIP” whenever the rank-$2r$ clause is understood from context. We write “RIP” to mean “$\delta$-RIP with any $\delta < 1$”.

Bhojanapalli, Neyshabur, and Srebro $[3]$ were the first to prove that (P) contains no spurious local minima under $(\delta, 4r)$-RIP with $\delta < 1/5$, and this was soon improved to $(\delta, 2r)$-RIP with $\delta < 1/5$ by Ge, Jin, and Zheng $[21]$. Translated into statements on conditioning, these require the measurement operator $\mathcal{A}$ to have a condition number of $\kappa < 3/2$ when restricted to low-rank matrices.

In practice, the rank of the ground truth $r^* = \text{rank}(M^*)$ is usually unknown. It is common to choose the search rank $r$ conservatively to encompass a range of possible values for $r^*$. Alternatively, the search rank $r$ may be progressively increased until a globally optimal solution is found. Either approaches naturally give rise to the overparameterized regime, in which $r > r^*$, and we have—accidentally or intentionally—allocated more degrees of freedom in the model $M = XX^T$ than exists in the ground truth $M^*$.

Existing “no spurious local minima” guarantees make no distinction between ground truths of different ranks $r^*$, so long as $r^* \leq r$. But in this paper, we show that explicitly controlling for the true rank $r^*$ can improve the RIP threshold. This is due to the fact that, for a fixed search rank $r$, the ground truths with highest ranks $r^*$ are the ones that require the smallest RIP constants $\delta$ to guarantee exact recovery. In fact, ground truths with $r^* = r$ always require $\delta < 1/2$ to guarantee exact recovery; this is the sharpest RIP-based guarantee possible without an explicit control over $r^*$. If higher-rank ground truths can be discounted by hypothesis, then larger RIP constants can be tolerated. For example, if the ground truth is known a priori to have $r^* = 1$, then an RIP constant of $\delta < 1/(1 + 1/\sqrt{r})$ guarantees exact recovery. In terms of conditioning, the prior knowledge that $r^* = 1$ allows the restricted condition number $\kappa$ to be a factor of $\frac{1}{3}(2\sqrt{r} + 1)$ larger.
1.1 Main Results

The main result of this paper is a global recovery guarantee that explicitly accounts for the rank $r^* = \text{rank}(M^*)$ of the ground truth. Our guarantee is sharp for a rank-1 ground truth ($r^* = 1$) and when the search rank coincides with the true rank ($r^* = r$).

**Theorem 1.2 (Overparameterization).** For $r < n$, let $M^* \succeq 0$ satisfy $b = \mathcal{A}(M^*)$ with $r^* \overset{\text{def}}{=} \text{rank}(M^*) \leq r$, and let $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ satisfy $(\delta, r^*)$-RIP.

- If $\delta < 1/(1 + \sqrt{r^*/r})$, then (P) has no spurious local minima:
  \[ \nabla f_{\mathcal{A}}(X) = 0, \quad \nabla^2 f_{\mathcal{A}}(X) \succeq 0 \iff XX^T = M^*. \]

- If $\delta \geq 1/(1 + 1/\sqrt{r-r^*+1})$, then exists a counterexample: a linear map $\mathcal{A}^*$ that satisfies $\delta$-RIP, a ground truth $M^* \succeq 0$ with $\text{rank}(M^*) = r^*$, and an $n \times r$ matrix $X$ with $XX^T \neq M^*$ that satisfies
  \[ f_{\mathcal{A}^*}(X) = \|\mathcal{A}^*(XX^T - M^*)\|^2 > 0, \quad \nabla f_{\mathcal{A}^*}(X) = 0, \quad \nabla^2 f_{\mathcal{A}^*}(X) \succeq 0. \]

Our proof of Theorem 1.2 is built on a recent proof technique of Zhang, Sojoudi, and Lavaei [38], which they use to prove a necessary and sufficient condition for the special $r = r^* = 1$ case. A key difficulty here is the need to solve a certain auxiliary optimization problem to provably global optimality. In the rank-1 case, the auxiliary optimization contains just two variables, and can be solved by hand in closed-form. In the general rank-$r$ case, however, the number of variables grows to $r^2 + 2r - 1$. The rank equality constraint $r^* = \text{rank}(M^*)$ introduces a further dimension of difficulty. Naively repeating their arguments would amount to solving a nonconvex optimization by exhaustive search over a possibly arbitrary number of variables.

This paper presents two major innovations to make the technique work for the general rank-$r$ case, with explicit control over $r^* = \text{rank}(M^*)$. The first is a simple lower-bound based on a relaxation of the auxiliary optimization that is still tight enough to prove global optimality. The second is a set of valid inequalities that enforce $r^* = \text{rank}(M^*)$, and cut the feasible region of the relaxation at the globally optimal solution. These allow us to to provably optimize over an arbitrary number of variables without exhaustive search. In Section 2 we briefly summarize the proof of Zhang et al. [38] and describe its difficulties in the rank-$r$ case. In Section 3 we describe our two innovations in detail, and conclude with a proof Theorem 1.2.

Without an explicit control over the true rank $r^*$, Theorem 1.2 says that $\delta$-RIP with $\delta < 1/2$ is both necessary and sufficient to exactly recover the ground truth $M^*$. The following is the first complete, sharp characterization of global recovery guarantees based on RIP.

**Corollary 1.3 (Sharp threshold).** For $r < n$, let $M^* \succeq 0$ satisfy $b = \mathcal{A}(M^*)$ and $\text{rank}(M^*) \leq r$, and let $\mathcal{A}$ satisfy $(\delta, 2r)$-RIP.

- If $\delta < 1/2$, then (P) has no spurious local minima:
  \[ \nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \iff f(X) = 0 \iff XX^T = M^*. \]
• If \( \delta \geq 1/2 \), then exists a counterexample: a linear map \( \mathcal{A}^* \) that satisfies \( \delta \)-RIP, a ground truth \( M^* \succeq 0 \) with \( \text{rank}(M^*) = r \), and an \( n \times r \) matrix \( X \) with \( XX^T \neq M^* \) that satisfies

\[
 f_{\mathcal{A}^*}(X) \overset{\text{def}}{=} \| \mathcal{A}^*(XX^T - M^*) \|^2 > 0, \quad \nabla f_{\mathcal{A}^*}(X) = 0, \quad \nabla^2 f_{\mathcal{A}^*}(X) \succeq 0.
\]

In fact, increasing the search rank \( r > r^* \) can also eliminate spurious local minima that had previously existed for \( r = r^* \). Figure 1 illustrates this on the 1/2-RIP counterexample used to prove sharpness in Corollary 1.3 for \( r = 1 \). This phenomenon lends further evidence to the popular refrain that overparameterization makes machine learning problems easier to solve. There is, however, an important caveat: the linear operator \( \mathcal{A} \) needs to satisfy a higher-rank \( (\delta', r + r^*) \)-RIP as well as a lower-rank \( (\delta, 2r^*) \)-RIP, and the higher-rank constant \( \delta' \) should not be substantially larger than the lower-rank constant \( \delta \). More work is needed to understand the impact of this qualifier for real-world problems.

Finally, once \( r \geq n \), it is a folklore result implied in the original papers of Burer and Monteiro \( [7, 8] \) that (P) contains no spurious local minima under no further assumptions on \( \mathcal{A} \) and \( b \); see also Journée et al. \( [23, \text{Corollary 8}] \) and Boumal et al. \( [6, \text{Corollary 3.2}] \). For completeness, we give a short proof in Appendix A.

**Proposition 1.4.** For \( \mathcal{A} : \mathbb{R}^{n \times r} \to \mathbb{R}^m \) and \( b \in \mathbb{R}^m \), let \( f(X) = \| \mathcal{A}(XX^T) - b \|^2 \). If \( r \geq n \), then \( \nabla f(X) = 0 \) and \( \nabla^2 f(X) \succeq 0 \iff f(X) \leq f(U) \) for all \( U \in \mathbb{R}^{n \times r} \).

Further imposing RIP ensures exact recovery. However, an \( \mathcal{A} \) that satisfies RIP for \( r \geq n \) must be bijective, and it would be easier to solve a system of linear equations \( \mathcal{A}(M) = b \) to exactly recover \( M^* \).

![Figure 1: Overparameterization eliminates spurious local minima.](image-url)

We consider the 1/2-RIP counterexample that proves the necessity of \( \delta < 1/2 \) for search rank \( r = 1 \) in Corollary 1.3 (see Example 7.1). For \( n = 4 \), the corresponding \( \mathcal{A}^* \) satisfies \( (\delta, r + r^*) \)-RIP with the same optimal RIP constant \( \delta = 1/2 \) for all search rank \( r \in \{1, 2, 3\} \). (Left) With \( r = 1 \), stochastic gradient descent (SGD) results in 11 failures over 100 trials, due to convergence to a spurious local minimum. (Right) Overparameterizing to \( r = 2 \) eliminates all spurious local minima, and SGD now succeeds in all 100 trials. (Our implementation of SGD uses a random Gaussian initialization over \( 10^4 \) steps with learning rate \( 10^{-3} \), classical momentum 0.9, batch-size of 1, sampling with replacement at each step.)
1.2 Related Work

In the literature, the nonconvex approach to low-rank matrix recovery is often called as the Burer–Monteiro method, due to pioneering work by Burer and Monteiro [7,8]. Under the RIP assumption, (P) is frequently known as matrix sensing. The original paper of Bhojanapalli, Neyshabur, and Srebro [3] actually extended their “no spurious local minima” guarantee for noiseless measurements into a polynomial-time guarantee for possibly noisy measurements. The key idea is to establish the strict saddle condition of Ge, Huang, Jin, and Yuan [19], which uses approximate second-order optimality to imply near recovery. Their proof has since been simplified [21], and their results generalized to the nonsymmetric [29, 41] and nonquadratic [26, 30] settings.

All of these recent results mentioned above have been blanket guarantees, in that they allow any second-order algorithm to converge to a global minimum, beginning from any arbitrary initial point, and for any global minimum to recover the ground truth. This is much stronger than previous guarantees based on a sufficiently good initial point [2, 14, 15, 22, 23, 25, 28, 30, 32, 33, 39, 40], the use of a specific algorithm like alternating minimization [22, 23, 25, 28] or gradient descent [14, 34, 36, 40], as well as some notion of local convexity [32, 33], see also Chi, Lu, and Chen [16] for a detailed survey. In fact, a blanket guarantee seems too strong given the nonconvex nature of the problem, but the conditions required also seem too conservative to be broadly applicable to instances of (P) that arise in practice.

The overparameterized regime of matrix sensing has only been recently studied as a separate topic [27, 42]. Most prior “no spurious local minima” guarantees continue to hold in the noiseless setting, although they provide no insight on the benefits or limitations of overparameterization. In the noisy setting, however, most prior results are no longer valid—including global guarantees and algorithm-specific guarantees. A main challenge is the fact that the nonconvex function \( f_A \) is no longer strongly convex within a neighborhood of the ground truth. Recently, Li, Ma, and Zhang [27] studied gradient descent in the overparameterized regime of noisy matrix sensing. Assuming a sufficiently good initialization, they proved that early termination of gradient descent yields a good solution, due to the effects of implicit regularization. Zhuo, Kwon, Ho, and Caramanis [42] improved upon this result, by showing that gradient descent converges unconditionally to a good solution, albeit only at a sublinear rate.

This paper uses exact second-order optimality to imply exact recovery, but our proof technique can—at least in principle—be perturbed in a logical manner to the strict saddle condition [19, 21], which uses approximate second-order optimality to imply approximate recovery. Extending Theorem 1.2 in this manner should then allow us to make polynomial-time guarantees for the noisy case. On the other hand, the strict saddle condition is not expected hold in the overparameterized setting, because the \( r \)-th largest eigenvalue of the ground truth \( \lambda_r(M^*) \) can be close to, or exactly, zero. It remains an important future work to investigate this extension, as a possible path forward in understanding the overparameterization regime within the noisy setting; we give further details in Section 8.

Aside from matrix sensing, there exist two other lines of “no spurious local minima” guarantees for (P). The first line, known as matrix completion, satisfies an RIP-like condition over incoherent matrices. Ge, Lee, and Ma [20] proved the first global recovery guarantee by
augmenting \( \mathcal{P} \) with a regularizer; see also Ge, Jin, and Zheng \[21\] for a unified exposition of matrix sensing and matrix completion. The second line considers the highly overparameterized regime, once \( r \) is large enough to satisfy \( r(r+1)/2 > m \) where \( m \) is the number of measurements. Here, the main result of Boumal, Voroninski, and Bandeira \[5\] implies that \( \mathcal{P} \) can be generically perturbed to contain no spurious local minima. This direction has been further explored by Bhojanapalli et al. \[4\], Boumal et al. \[6\] and in particular by Cifuentes and Moitra \[17\]. Typical values of \( m \) are on the order of \( n \), so these global guarantees generally require \( r \approx \sqrt{n} \).

2 Background: Inexistence of Counterexamples

Our proof of Theorem 1.2 is built on a recent proof technique of Zhang, Sojoudi, and Lavaei \[38\], which they used to prove Theorem 1.2 in the special rank-1 case, where \( r^* = r = 1 \). Repeating this for the general rank-\( r \) case, however, immediately runs into serious difficulty. In this section, we briefly summarize the proof technique, and outline the key challenges that it faces in analyzing the general rank-\( r \) case.

To motivate the argument, suppose that we are given a fixed threshold on the RIP constant \( \lambda \in [0, 1) \) and rank \( r^* \) of the ground truth, and that we wish to prove the following “no spurious local minima” claim as false:

\[
\text{If } \text{rank}(Z^*) = r^* \text{ and } \mathcal{A} \text{ satisfies } \lambda \text{-RIP, then } \nabla f_{\mathcal{A}}(X) = 0, \quad \nabla^2 f_{\mathcal{A}}(X) \succeq 0 \iff XX^T = ZZ^T. \tag{2.1}
\]

The task of finding a counterexample to refute the claim can be posed as a nonconvex feasibility problem:

\[
\text{find } X, Z \in \mathbb{R}^{n \times r}, \mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m \quad \text{s.t. } \nabla f_{\mathcal{A}}(X) = 0, \quad \nabla^2 f_{\mathcal{A}}(X) \succeq 0, \quad XX^T \neq ZZ^T, \quad \text{rank}(Z) = r^*, \quad \mathcal{A} \text{ satisfies } \delta \text{-RIP}. \tag{2.2}
\]

However, if a counterexample does not exist and the problem above is infeasible, then the inexistence of counterexamples serves as proof that the claim in (2.1) is actually true.

The feasibility problem above can be reformulated as an optimization problem. Let \( \delta^* \) be the smallest RIP constant associated with a counterexample, as in

\[
\delta^* \overset{\text{def}}{=} \inf \delta \text{ over } X, Z \in \mathbb{R}^{n \times r}, \mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m
\quad \text{s.t. } \nabla f_{\mathcal{A}}(X) = 0, \quad \nabla^2 f_{\mathcal{A}}(X) \succeq 0, \quad XX^T \neq ZZ^T, \quad \text{rank}(Z) = r^*, \quad \mathcal{A} \text{ satisfies } \delta \text{-RIP}. \tag{2.3}
\]

If \( \lambda > \delta^* \), then the claim in (2.1) is false; there exists a counterexample that satisfies \( \delta^* \text{-RIP} \), and therefore also \( \delta \text{-RIP} \) for any \( \delta \geq \delta^* \), including \( \delta = \lambda \). Conversely, if \( \lambda < \delta^* \), then the claim in (2.1) is true; there can exist no counterexample that satisfy \( \lambda \text{-RIP} \), without contradicting the definition of \( \delta^* \) as the infimum. (The case of \( \lambda = \delta^* \) is determined by whether the infimum is attained.) We conclude, therefore, that \( f_{\mathcal{A}} \) contains no spurious
local minima if $A$ satisfies $\delta$-RIP with $\delta < \delta^\star$, whereas such a claim is necessarily false for $\delta > \delta^\star$ due to the existence of a counterexample.

The optimal value $\delta^\star$ provides a sharp threshold on the RIP constant needed to eliminate spurious local minima. Proving a necessary and sufficient “no spurious local minima” guarantee (like Corollary 1.3) amounts to solving (2.3) to certifiable global optimality. To proceed, Zhang et al. [38] suggest splitting (2.3) into a two-stage minimization

$$\delta^* = \inf_{X,Z \in \mathbb{R}^{n \times r}} \{ \delta(X, Z) : \text{rank}(Z) = r^\star, \ XX^T \neq ZZ^T \}$$

in which $\delta(X, Z)$ is the optimal value for the subproblem over the linear operator $A$ with $X, Z$ fixed.

**Definition 2.1** (Threshold RIP function). Given $X, Z \in \mathbb{R}^{n \times r}$ with $XX^T \neq ZZ^T$, we define

$$\delta(X, Z) \overset{\text{def}}{=} \min_A \left\{ \delta : \begin{array}{l} \nabla f_A(X) = 0, \ \nabla^2 f_A(X) \succeq 0, \\ A \text{ satisfies } \delta\text{-RIP} \end{array} \right\}$$

The point of the splitting—and indeed the main contribution of Zhang et al. [38]—is that $\delta(X, Z)$ can be evaluated exactly as a convex linear matrix inequality (LMI) optimization problem. The exactness here is surprising because the set of $\delta$-RIP operators is nonconvex, and that even verifying whether a given operator $A$ satisfies $\delta$-RIP is already NP-hard (see Weed [35] and the references therein). Problem (2.5) avoids NP-hardness only because it lacks specificity towards a particular $A$.

**Theorem 2.2** (Zhang, Sojoudi, and Lavaei [38, Theorem 8]). For $X \in \mathbb{R}^{n \times r}$ and $Z \in \mathbb{R}^{n \times r^\star}$ such that $XX^T \neq ZZ^T$, define

$$\delta_p(X, Z) \overset{\text{def}}{=} \min_A \left\{ \delta : \begin{array}{l} \nabla f_A(X) = 0, \ \nabla^2 f_A(X) \succeq 0, \\ A \text{ satisfies } (\delta, p)\text{-RIP} \end{array} \right\}.$$ (2.6)

Then, we have for all $p \geq \text{rank}([X, Z]):$

$$\delta_p(X, Z) = \delta_n(X, Z) = \min_A \left\{ \delta : \begin{array}{l} \nabla f_A(X) = 0, \ \nabla^2 f_A(X) \succeq 0, \\ (1 - \delta)I \preceq A^T A \preceq (1 + \delta)I \end{array} \right\}.$$ (2.7)

Moreover, the minimizer $A^*$ for (2.7) is also a minimizer for (2.6).

**Remark 2.3.** Our notation $\delta(X, Z)$ refers to $\delta_{r+r^\star}(X, Z)$. This is without loss of generality because $r + r^\star \geq \text{rank}([X, Z])$ is always true for $X, Z \in \mathbb{R}^{n \times r}$ with $\text{rank}(Z) = r^\star$.

Having established Theorem 2.2 it becomes easy to prove upper-bounds like $\delta^* \leq \delta_{ub}$; these correspond to necessary conditions on the RIP constant (of the form $\delta \leq \delta_{ub}$) that must be satisfied for a “no spurious local minima” guarantee to be possible. We can numerically evaluate $\delta_{ub} = \delta(X, Z)$, by solving the corresponding convex LMI in (2.7) using an interior-point method. The upper-bound can be further refined by numerically optimizing over pairs of $X$ and $Z$.

However, lower-bounds like $\delta^* \geq \delta_{lb}$ are much more valuable, as these correspond directly to sufficient conditions on the RIP constant (of the form $\delta < \delta_{lb}$) for the inexistence of spurious local minima. Put simply, proving a lower-bound on $\delta^*$ is the same thing as
proving a “no spurious local minima” guarantee. Unfortunately, lower-bounds are also much more difficult to prove than upper-bounds; lower-bounds are universal statements over all pairs of $X$ and $Z$, whereas upper-bounds are existential statements that only need hold for a particular pair of $X, Z$. This difficulty is a fundamental limitation of the proof technique; it cannot be addressed by better software, faster computers, nor by collecting more samples.

Ideally, we would derive a closed-form expression for $\delta(X, Z)$, which we then optimize over all $X$ and $Z$. Unfortunately, the LMI in (2.7) is difficult to solve in closed-form, even in the rank-1 case. Instead, Zhang et al. [38] resorted to solving a relaxation of the LMI, in order to obtain a closed-form lower-bound $\delta_{lb}(X, Z) \leq \delta(X, Z)$. There is a trade-off between simplicity and tightness: the relaxation needs to be simple enough to be solvable in closed-form, but still tight enough to give a meaningful bound.

Even if a simple, closed-form expression for $\delta(X, Z)$ were known, it would still need to be minimized over all $X, Z \in \mathbb{R}^{n \times r}$ subject to the constraint $\text{rank}(Z) = r^*$. This nonconvex minimization amounts to an exhaustive search, though the complexity can be reduced by noting the scale- and unitary-invariance:

$$\delta(X, Z) = \delta(\sigma U X V_1, \sigma U Z V_2) \text{ for all } \sigma \neq 0, \quad U, V_1, V_2 \text{ orthonormal.}$$

In the rank-1 case, all $x, z \in \mathbb{R}^n$ with same length ratio $\rho = \|x\|/\|z\|_F$ and incidence angle $\phi$ satisfying $\|x\|\|z\| \cos \phi = x^T z$ have the same value of $\delta(x, z)$. The exhaustive search over $\rho$ and $\phi$ can be performed by hand. In the rank-$r$ case, however, the number of symmetric invariants like $\rho$ and $\phi$ grows as $r + r^* + r r^* - 1$. Even the rank-2 case has up to 7 symmetric invariants. Clearly, exhaustive search is not a viable proof technique for the general rank-$r$ case, particular when the number of invariants is itself arbitrary.

3 Innovations in Our Proof

This paper was motivated by the following heuristic upper-bound $\delta^* \leq \delta(X, Z)$. The corresponding choices of $X, Z$ were found numerically, by applying a zero-order pattern search to $\min_{X,Z} \delta(X, Z)$ while evaluating each $\delta(X, Z)$ using a direct implementation of the LMI within MOSEK [1].

**Proposition 3.1** (Heuristic upper-bound). For $r, r^*, n$ satisfying $1 \leq r^* \leq r < n$, there exist $X, Z \in \mathbb{R}^{n \times r}$ with $\text{rank}(Z) = r^*$ such that $\delta(X, Z) \leq 1/(1 + 1/\sqrt{r - r^* + 1})$.

In Section 7, we state a choice of $X, Z$ for each $r, r^*, n$ whose $\delta(X, Z)$ satisfies Proposition 3.1. Our numerical experiments over tens of thousands of trials did not uncover a smaller $\delta(X, Z)$ than Proposition 3.1. This provides strong evidence that the heuristic upper-bound may in fact be globally optimal.

The main contribution of this paper is a proof of global optimality for the heuristic upper-bound in Proposition 3.1 in the regimes where $r^* \in \{1, r\}$. We introduce two major innovations in our proof to overcome the aforementioned difficulties in proving a lower-bound on $\delta^*$. The first is a simple lower-bound $\delta_{lb}(X, Z) \leq \delta(X, Z)$ that is still tight enough to match Proposition 3.1. The second is a set of valid inequalities that accurately capture

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1 The invariants are the $r$ singular values of $X$, the $r^*$ singular values of $Z$, the $r r^*$ incident angles between their left singular vectors, less one parameter due to scale-invariance.
the rank equality constraint $r^* = \text{rank}(Z)$, without exhaustively searching over $2nr$ input variables or up to $r^2 + 2r - 1$ symmetry invariants. We give more details below, followed by a proof of the main results.

### 3.1 Simple lower-bound $\delta_{lb}(X, Z) \leq \delta(X, Z)$

Our simple lower-bound was motivated by a similar lower-bound used in the proof that $\delta^* = 1/2$ for $r = r^* = 1$ by Zhang, Sojoudi, and Lavaei [35]. Even there, $\delta(x, z)$ was already too difficult to solve in closed-form, so they relaxed the LMI, and reformulated the dual into the following

$$\delta(x, z) \geq \max_{t \geq 0} \frac{\cos \theta(t) - t}{1 + t}, \quad (3.1)$$

where $\cos \theta(t)$ is the optimal value to a certain minimization problem defined in terms of $x$ and $z$ (see Lemma 5.1 below for the exact details). In the rank-1 case, $\cos \theta(t)$ can be solved in closed-form

$$\cos \theta(t) = \psi(\alpha, \beta, t) \overset{\text{def}}{=} \alpha(t/\beta) + \sqrt{1 - \alpha^2} \sqrt{1 - (t/\beta)^2} \quad (3.2)$$

where $\alpha, \beta$ are functions of $x, z$. Substituting (3.2) into (3.1) results in a quasiconvex maximization over $t$ that can also be solved in closed-form

$$\max_{t \geq 0} \frac{\psi(\alpha, \beta, t) - t}{1 + t} = \gamma(\alpha, \beta) \overset{\text{def}}{=} \begin{cases} \sqrt{1 - \alpha^2} \quad \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\ 1 - 2\alpha + \beta^2 \quad \beta < \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}. \end{cases} \quad (3.3)$$

The resulting lower-bound $\delta_{lb}(x, z) \overset{\text{def}}{=} \gamma(\alpha(x, z), \beta(x, z))$ is simple but not always tight. Nevertheless, it is tight enough to satisfy $\delta_{lb}(x, z) \geq 1/2$ for all $x, z \in \mathbb{R}^n$, and this is all that is needed to prove $\delta^* \geq 1/2$ for $r = r^* = 1$.

Repeating the same relaxation for the rank-$r$ case, however, immediately runs into an issue of exploding complexity. The optimization problem that defines $\cos \theta(t)$ is now very complicated, without an obvious closed-form solution. Even if we were to solve $\cos \theta(t)$ in closed-form, we would still have to maximize over $t$ in closed-form to obtain a lower-bound $\delta_{lb}(X, Z)$. And even then, we would still be left with the task of optimizing the resulting lower-bound $\delta_{lb}(X, Z)$ over all $X$ and $Z$ by hand.

Here, our first major innovation is a lower-bound $\cos \theta(t) \geq \psi(\alpha, \beta, t)$ for the rank-$r$ case that takes on the same form as the closed-form solution for the rank-1 case, but with the $\alpha$ and $\beta$ functions suitably generalized to the rank-$r$ case. This way, the simplicity of the rank-1 solution is maintained, but at the cost of further relaxing (3.1), which is itself already a relaxation. The resulting lower-bound $\delta_{lb}(X, Z)$ takes the same form as the rank-1 case. Below, $\sigma_{\text{min}}(\cdot)$ denotes the smallest singular value, and the superscript $\dagger$ denotes the pseudoinverse.

**Theorem 3.2** (Simple lower-bound). For $X, Z \in \mathbb{R}^{n \times r}$ such that $XX^T \neq ZZ^T$, define $\delta_{lb}(X, Z) \overset{\text{def}}{=} \gamma(\alpha(X, Z), \beta(X, Z))$ as in (3.3) where

$$\alpha(X, Z) \overset{\text{def}}{=} \frac{\|Z_\perp Z_\perp^T\|_F}{\|XX^T - ZZ^T\|_F^2}, \quad \beta(X, Z) \overset{\text{def}}{=} \frac{\sigma_{\text{min}}^2(X)}{\|XX^T - ZZ^T\|_F^2} \frac{\text{tr}(Z_\perp Z_\perp^T)}{\|Z_\perp Z_\perp^T\|_F}$$

and $Z_\perp = (I - XX^T)Z$. Then, we have $\delta(X, Z) \geq \delta_{lb}(X, Z)$. 

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Remark 3.3. The definition of $\beta$ becomes ambiguous when $Z_\perp = 0$. This is without loss of precision, because $Z_\perp = 0$ implies $\alpha = 0$, and therefore $\delta_{lb}(X, Z) = 1$ regardless of the value of $\beta$. We can select $\beta(X, Z) = \sigma_{\min}^2(X)/\|XX^T - ZZ^T\|_F$ for $Z_\perp = 0$ to disambiguate, though we will not need to do this throughout the paper.

The proof of Theorem 3.2 is given in Section 5. Remarkably, despite two separate stages of relaxation, the simple lower-bound still remains tight enough to match the heuristic upper-bound in Proposition 3.1 when $r^* \in \{1, r\}$. This is all that is needed to prove the sharp guarantee in Theorem 1.2.

Theorem 3.2 may be of independent interest, because it unifies a number of previous lower-bounds on $\delta(X, Z)$. For $r = 1$, the bound reduces to a bound of Zhang et al. [38, Theorem 12]. For $r > r^* > 1$, however, our numerical results found that $\delta_{lb}(X, Z)$ really does attain its global minimum at the lower-bound $1/(1 + \sqrt{r^*/r})$ used in the proof of Theorem 1.2. This provides evidence that Theorem 1.2 is the best we can do without further improving Theorem 3.2.

3.2 Minimizing $\delta_{lb}(X, Z)$ over all $X, Z$ with $\text{rank}(Z) = r^*$

We are left with the task of minimizing $\delta_{lb}(X, Z)$ in Theorem 3.2 over all possible choices of $X$ and $Z$. In the rank-1 case, Zhang et al. [38] did this by rewriting $\alpha$ and $\beta$ in terms of the two underlying symmetry invariants, the length ratio $\rho = \|x\|/\|z\|$ and the incidence angle $\phi$ satisfying $\|x\|\|z\|\cos\phi = x^Tz$, and then solving the unconstrained optimization over $\rho$ and $\phi$. In the rank-$r$ case, however, the same approach would force us to solve a nonconvex optimization over up to $r^2 + 2r - 1$ symmetry invariants.

Instead, we relax the explicit dependence of $\alpha(X, Z)$ and $\beta(X, Z)$ on their arguments $X, Z$, and minimize $\delta_{lb}(X, Z)$ directly over $\alpha, \beta$ as variables. Adopting a classic strategy from integer programming, we tighten the relaxation by introducing valid inequalities: inequality constraints on $\alpha, \beta$ that would have been valid on $\alpha(X, Z)$ and $\beta(X, Z)$ for all $X$ and $Z$. (Valid inequalities are more commonly known as cutting planes when they are affine.)

To give a pedagogical example, it is easy to see from the definitions that $\alpha(X, Z) \geq 0$ and $\beta(X, Z) \geq 0$, so we may add the constraints $\alpha, \beta \geq 0$ as valid inequalities. However, the resulting relaxation is still far too loose. Optimizing $\delta_{lb}(X, Z)$ over $\alpha, \beta \geq 0$ results in a trivial lower-bound

$$0 = \min_{\alpha, \beta \geq 0} \left\{ \sqrt{1 - \alpha^2} : \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \right\}$$

$$= \inf_{\alpha, \beta \geq 0} \left\{ \frac{1 - 2\alpha\beta + \beta^2}{1 - \beta^2} : \beta < \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \right\}$$

attained in the limit $\alpha = 1$ and $\beta \to 1$.

Our second major innovation is to find simple, nontrivial valid inequalities on $\alpha$ and $\beta$ that fully encapsulate the dependence of $\alpha(X, Z)$ and $\beta(X, Z)$ on their arguments $X, Z$.

Lemma 3.4 (Valid inequalities). For $X, Z \in \mathbb{R}^{n \times r}$, let $\alpha = \alpha(X, Z)$ and $\beta = \beta(X, Z)$ as in Theorem 3.2, and let $r^* = \text{rank}(Z) \leq r$.

1. If $\beta \leq \alpha$, then $\alpha^2 + (r/r^*)\beta^2 \leq 1$. 

2. If \( \beta \geq \alpha \), then \( \alpha \leq 1/\sqrt{1 + r/r^*} \).

At first sight, these valid inequalities appear too simple to be useful. They give up almost all of the information encoded within \( X \) and \( Z \), so the resulting relaxation over \( \alpha \) and \( \beta \) seems unlikely to be tight. Surprisingly, these two valid inequalities are all that is needed to minimize the simple lower-bound \( \delta_{lb}(X, Z) \) to global optimality. The sharpness of the simple lower-bound in turn concludes the proof of our advertised result.

Observe that Lemma 3.4 becomes false if its rank equality constraint \( r^* = \text{rank}(Z) \) were relaxed into an inequality. As such, we cannot prove it by naively parameterizing \( Z \) into an \( n \times r^* \) matrix, since this would only be able to enforce an upper-bound \( \text{rank}(Z) \leq r^* \). The key element in our proof is a generalization of the classic low-rank approximation result of Eckart and Young [18], stated later as Theorem 6.1. The proof of Lemma 3.4 quickly follows from this result; see Section 6.

3.3 Proof of the Main Results

The proof of Theorem 1.2 quickly follows from Proposition 3.1, Theorem 3.2, and Lemma 3.4.

**Theorem 1.2** We have the following sequence of inequalities

\[
\frac{1}{1 + 1/\sqrt{r - r^*} + 1} \geq \delta_* = \inf_{XX^T \neq ZZ^T, \text{rank}(Z) = r^*} \delta(X, Z) \geq \inf_{XX^T \neq ZZ^T, \text{rank}(Z) = r^*} \delta_{lb}(X, Z) \geq \frac{1}{1 + \sqrt{r^*/r}}
\]

over \( X \in \mathbb{R}^{n \times r} \) and \( Z \in \mathbb{R}^{n \times r^*} \). Inequality (a) follows from the heuristic upper-bound in Proposition 3.1. Inequality (b) is due to Theorem 3.2. We will now prove inequality (c) by minimizing the expression in Theorem 3.2, restated here as

\[
\delta_{lb}(X, Z) = \gamma(\alpha, \beta) \overset{\text{def}}{=} \begin{cases} 
\sqrt{1 - \frac{\alpha^2}{1 + \sqrt{1 - \alpha^2}}}, & \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\
\frac{1}{\sqrt{1 - \alpha^2}}, & \beta \leq \frac{1}{1 + \sqrt{1 - \alpha^2}}, 
\end{cases}
\]

over choices of \( \alpha \geq 0 \) and \( \beta \geq 0 \) satisfying Lemma 3.4.

We first resolve the degenerate case of \( r \geq n \). Here, we have either: \( \alpha = 0 \), because \( \text{rank}(X) = n \) and \( Z_{\perp} = (I - XX^T)Z = 0 \); or \( \beta = 0 \), because \( \text{rank}(X) < n \) and \( \sigma_{\min}(X) = 0 \). Either way yields \( \delta_{lb}(X, Z) = 1 \), and hence \( \delta(X, Z) = 1 \).

Otherwise, if \( r < n \), then it is additionally possible for \( \alpha, \beta > 0 \). Let \( t = r^*/r \). We partition the feasible set into three regions:

- If \( \alpha \leq \beta \), then \( \gamma(\alpha, \beta) \geq 1/\sqrt{1 + t} \). Here, we have \( \alpha \leq 1/\sqrt{1 + t^{-1}} \) via Lemma 3.4. We decrease \( \delta_{lb}(X, Z) = \sqrt{1 - \alpha^2} \) by increasing \( \alpha \) until \( \alpha = 1/\sqrt{1 + t^{-1}} \).

- If \( \alpha \geq \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \), then \( \gamma(\alpha, \beta) \geq \gamma(\alpha, \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}) \). Here, we have \( \alpha^2 + t^{-1}\beta^2 \leq 1 \) via Lemma 3.4. We decrease \( \gamma(\alpha, \beta) = \sqrt{1 - \alpha^2} \) by increasing \( \alpha \) until either \( \beta = \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \) or \( \beta = \sqrt{t(1 - \alpha^2)} \). If the latter, we then decrease \( \beta \) along the boundary \( \beta = \sqrt{t(1 - \alpha^2)} \) until \( \beta = \frac{1}{1 + \sqrt{1 - \alpha^2}} \).


• If $\beta \leq \frac{\alpha}{1+\sqrt{1-\alpha^2}}$, then $\gamma(\alpha, \beta) \geq \frac{1}{1 + \sqrt{t}}$. Here, we have $\alpha^2 + t^{-1} \beta^2 \leq 1$ via Lemma 3.4. At the boundary $\beta = \frac{\alpha}{1+\sqrt{1-\alpha^2}} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}$, the two expressions for $\gamma(\beta, \alpha)$ coincide.

Within the interior $\beta < \frac{\alpha}{1+\sqrt{1-\alpha^2}}$, we have $\gamma(\alpha, \beta) = \frac{1-2\alpha\beta + \beta^2}{1-\beta^2}$. The following relaxation yields the desired lower-bound

$$\min_{\alpha, \beta} \left\{ \frac{1-2\alpha\beta + \beta^2}{1-\beta^2} : \alpha^2 + t^{-1} \beta^2 \leq 1 \right\} = \frac{1}{1 + \sqrt{t}}.$$ 

Finally, we verify that $1/\sqrt{T + t} \geq 1/(1 + \sqrt{t})$.

We conclude this section by noting that, in our numerical experiments, the simple lower-bound from Theorem 3.2 really does seem to attain a minimum value of $\min_{X, Z} \delta_{lb}(X, Z) = 1/(1 + \sqrt{r/\tau})$.

In the remainder of this paper, we prove the simple lower-bound of Theorem 3.2 in Section 5, the valid inequalities of Lemma 3.4 in Section 6, and finally, the heuristic upper-bound of Proposition 3.1 in Section 7. We begin with some technical preliminaries, which we use throughout the rest of the paper.

4 Preliminaries

4.1 Notations and basic definitions

Basic linear algebra. Lower-case letters are vectors and upper-case letters are matrices. We use “MATLAB notation” in concatenating vectors and matrices:

$$[a, b] = [a \ b], \quad [a; b] = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \text{diag}(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

and the following short-hand to construct them:

$$[x_i]_{i=1}^n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad [x_{i,j}]_{i=1}^m_{j=1} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}.$$ 

Denote $1 = [1, 1, \ldots, 1]^T$ as the vector-of-ones and $I = \text{diag}(1)$ as the identity matrix; we will infer their dimensions from context. Denote $\langle X, Y \rangle = \text{tr}(X^T Y)$ and $\|X\|_F^2 = \langle X, X \rangle$ as the Frobenius inner product and norm. Denote $\text{nnz}(x)$ as the number of nonzero elements in $x$.

Positive cones and projections. The sets $\mathbb{R}^n \supset \mathbb{R}^n_+$ are the $n$ vectors, and the corresponding positive orthant. Denote $x_+ = \max\{0, x\} = \arg\min\{\|x - y\| : y \in \mathbb{R}^n_+\}$ as
the projection onto the positive orthant. The sets $S^n \supset S^n_+$ are the $n \times n$ real symmetric matrices, and the corresponding positive semidefinite cone. Denote $X_+ = V \max\{0, \Lambda\}V^T = \arg \min\{\|X - Y\|_F : Y \in S_+\}$ as the projection onto the positive semidefinite cone, where $X = V\Lambda V^T$ is the usual eigendecomposition.

**Vectorization and Kronecker product.** Denote $\text{vec}(X)$ as the usual column-stacking vectorizing operator, and $\text{mat}(x)$ as its inverse (the dimensions of the matrix are inferred from context). The Kronecker product $\otimes$ is defined to satisfy the identity $\text{vec}(AXB^T) = (B \otimes A)\text{vec}(X)$.

**Pseudoinverse.** Denote the (Moore–Penrose) pseudoinverse of a rank-$r$ matrix $A$ using its singular value decomposition $A^\dagger \text{def} = VS^{-1}U^T$ where $A = USV^T$, $S \succ 0$, $U^T U = V^T V = I_r$ and $0^\dagger \text{def} = 0$. It is easy to verify from this definition that $A^\dagger b$ minimizes $\|Ax - b\|$ over $x$ while also minimizing $\|x\|$, as in

$$A^\dagger b = \arg \min_{x \in X^*} \|x^*\| \quad \text{where } X^* = \arg \min_x \|Ax - b\|.$$ 

It follows that $AA^\dagger = UU^T$ is the projection onto the column-span of $A$, whereas $A^\dagger A = VV^T$ is the projection onto the row-span of $A$.

**Vector $e$ and matrix $J$.** For a candidate point $X \in \mathbb{R}^{n \times r}$ and a rank-$r$ factorization $Z \in \mathbb{R}^{n \times r}$ of the ground truth $M^* = ZZ^T$, we define the error vector $e \in \mathbb{R}^{n^2}$ and its Jacobian matrix $J \in \mathbb{R}^{n^2 \times nr}$ to satisfy

$$e = \text{vec}(XX^T - ZZ^T), \quad J\text{vec}(Y) = \text{vec}(XY^T + YX^T).$$

We state some basic properties of $e$ and $J$.

**Lemma 4.1.** If $J^\dagger e = 0$, then either $XX^T = ZZ^T$ or $\sigma_r(X) = 0$.

**Lemma 4.2.** We have $I - JJ^\dagger = (I - XX^\dagger) \otimes (I - XX^\dagger)$.

The proofs are straightforward, but we give the details for completeness in Appendix B and Appendix C.

### 4.2 Convex reformulation of $\delta(X, Z)$

Define $A \in \mathbb{R}^{m \times n^2}$ as the matrix representation of the operator $\mathcal{A}$, as in

$$A\text{vec}(M) = \mathcal{A}(M) \quad \text{where } A \text{ def} = [\text{vec}(A_1), \text{vec}(A_2), \ldots, \text{vec}(A_m)]^T,$$

in order to write $f_A$ in quadratic form

$$f_A(X) \text{ def} = \|A(XX^T - ZZ^T)\|^2 = e^T A^T A e.$$

Zhang, Sojoudi, Josz, and Lavaei [37] observed that the gradient $\nabla f_A$ and Hessian $\nabla^2 f_A$ are both linear functions of the kernel matrix $H = A^T A$, as in

$$\nabla f_A(X) = 2J^T He, \quad \nabla^2 f_A(X) = 2I_r \otimes \text{mat}(He) + J^T H J.$$
Accordingly, the following is a convex restriction of the \( \delta(X, Z) \) problem in Definition 2.1:

\[
\delta_{ub}(X, Z) = \min_{\delta, H} \left\{ \delta : J^T H e = 0, \quad 2I_r \otimes \text{mat}(He) + J^T A^T A J \succeq 0, \quad (1 + \delta)I \preceq H \preceq (1 + \delta)I \right\}
\]

(4.1)

This gives an upper-bound on \( \delta(U, Z) \), because every feasible \( H \) in (4.1) factors into an \( A \) that satisfies \( \|A(M)\|^2 / \|M\|^2_F \approx 1 \pm \delta \) for all \( M \in \mathbb{R}^{n \times r} \), but \( \delta \text{-RIP} \) only requires this to hold for \( M \) with \( \text{rank}(M) \leq 2r \). Later, Zhang, Sojoudi, and Lavaei [38] proved the following convex relaxation of the \( \delta(X, Z) \) problem:

\[
\delta(X, Z) \geq \min_{\delta, H} \left\{ \delta : J^T H e = 0, \quad 2I_r \otimes \text{mat}(He) + J^T A^T A J \succeq 0, \quad (1 + \delta)P^T P \preceq P^T H P \preceq (1 + \delta)P^T P \right\}
\]

(4.2)

with \( P = [X, Z] \otimes [X, Z] \) gives a lower-bound on \( \delta(X, Z) \) with the same optimal value as the upper-bound \( \delta_{ub}(X, Z) \) in (4.1), and therefore \( \delta(X, Z) = \delta_{ub}(X, Z) \). (In Section 2 this result was restated as Theorem 2.2.) Optimizing \( \delta(X, Z) \) over choices of \( X, Z \), however, would seem to require solving the convex optimization problem (4.1) in closed-form.

5 Derivation of the Simple Lower-Bound

Even in the rank-1 case, a closed-form expression for \( \delta(x, z) \) is too difficult to obtain. Instead, Zhang et al. [38] solved a relaxation of the corresponding LMI in order to obtain a lower-bound \( \delta_{lb}(x, z) \leq \delta(x, z) \) that is sharp in the sense that \( \delta_{lb}(x, z) \geq 1/2 \) for all \( x, z \). In the general rank-\( r \) case, the same relaxation yields the following lower-bound:

**Lemma 5.1.** For \( X, Z \in \mathbb{R}^{n \times r} \), we have

\[
\delta(X, Z) \geq \max_{t \geq 0} \frac{\cos \theta(t) - t}{1 + t}
\]

(5.1)

where \( \cos \theta(t) \) is defined via a maximization over \( y \in \mathbb{R}^{nr} \) and \( W_{i,j} \in \mathbb{R}^{n \times r} \) for \( i, j \in \{1, 2, \ldots, r\} \) as follows:

\[
\cos \theta(t) \overset{\text{def}}{=} \max_{y,W_{i,j}} \left\{ \frac{e^T [J y - w]}{\|e\|\|J y - w\|} : \frac{J^T J, W}{\|e\|\|J y - w\|} = 2t, \ W \succeq 0 \right\}
\]

(5.2)

and \( W = [W_{i,j}]_{i,j=1}^r \) and \( w = \sum_{i=1}^r \text{vec}(W_{i,i}) \).

**Proof.** The proof follows a step-by-step replication of Zhang et al. [38, Section 7.2]. For completeness, we give the details in Appendix D. \( \square \)

Numerical experiments suggest that the lower-bound in Lemma 5.1 remains sharp for the general rank-\( r \) case, in the sense that, for every choice of \( X, Z \in \mathbb{R}^{n \times r} \), there exists a \( t^* \geq 0 \) such that \( (\cos \theta(t^*) - t^*) / (1 + t^*) \geq 1/2 \). To make this into a rigorous proof, however, will require executing 3 difficult steps: 1) solve \( \cos \theta(t) \) in closed-form; 2) maximize over \( t \geq 0 \) for the best possible lower-bound; and 3) minimize the lower-bound over all possible \( X \) and \( Z \).
In the rank-1 case, the problem in (5.2) has optimal value \( \cos \theta(t) = \psi(\alpha, \beta, t) \), where
\[
\psi(\alpha, \beta, t) \overset{\text{def}}{=} \begin{cases} 
(t/\beta)\alpha + \sqrt{1 - (t/\beta)^2} \sqrt{1 - \alpha^2} & t/\beta \leq \alpha, \\
1 & t/\beta > \alpha,
\end{cases}
\]
and \( \alpha, \beta \) are functions of \( x \) and \( z \). The resulting maximization over \( t \) in (5.1) is quasiconcave, and can be solved in closed-form.

**Lemma 5.2.** Given \( 0 \leq \alpha \leq 1 \) and \( \beta > 0 \), define \( \psi(\alpha, \beta, t) \) as in (5.3). Then,
\[
\max_{t \geq 0} \frac{\psi(\alpha, \beta, t) - t}{1 + t} = \begin{cases} 
\sqrt{1 - \alpha^2} \frac{1 - 2\alpha\beta + \beta^2}{1 - \beta^2} & \beta \geq \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}, \\
\beta < \frac{\alpha}{1 + \sqrt{1 - \alpha^2}}.
\end{cases}
\]

**Proof.** A proof can be found in Zhang et al. [38]. For completeness, we give an alternative, slightly more direct proof in Appendix E. \( \square \)

The rank-\( r \) case encounters immediate difficulty in the first step of solving (5.2) for \( \cos \theta(t) \). While it is technically possible to derive a closed-form solution, the resulting expression would be so complicated as to render the subsequent two steps (maximizing and then minimizing over \( X, Z \)) as intractable. Alternatively, we can lower-bound \( \cos \theta(t) \) by offering a heuristic solution for (5.2); this can be made simple enough to keep the next two steps tractable. However, the resulting lower-bound \( \delta_{lb}(X, Z) \leq \delta(X, Z) \) may no longer be sharp enough to satisfy \( \delta_{lb}(X, Z) \geq 1/2 \) for all \( X, Z \).

Our first major innovation in this paper is the following lemma, which provides a lower-bound \( \cos \theta(t) \geq \psi(\alpha, \beta, t) \) that maintains the *same expression* in (5.3) as the rank-1 case, over different definitions of \( \alpha \) and \( \beta \). Essentially, it tells us how \( \alpha \) and \( \beta \) should be generalized to the rank-\( r \) case.

**Lemma 5.3.** Let \( XX^T \neq ZZ^T \). Define \( \psi(\alpha, \beta, t) \) as in (5.3), and \( \alpha(X, Z) \) and \( \beta(X, Z) \) as in Theorem 3.2. If \( \beta > 0 \), then \( \cos \theta(t) \geq \psi(\alpha(X, Z), \beta(X, Z), t) \).

Once Lemma 5.3 is established, we evoke Lemma 5.2 to solve the maximization in (5.1) over \( t \) in closed-form, to yield the lower-bound \( \delta_{lb}(X, Z) \) stated in Theorem 3.2. Remarkably, this lower-bound remains sharp despite two separate stages of relaxation.

**Proof of Theorem 3.2.** The nondegenerate case of \( e \neq 0 \) and \( \beta > 0 \) follows from the following chain of inequalities
\[
\delta(X, Z) \overset{(a)}{\geq} \max_{t \geq 0} \frac{\cos \theta(t) - t}{1 + t} \overset{(b)}{\geq} \max_{t \geq 0} \frac{\psi(\alpha, \beta, t) - t}{1 + t} \overset{(c)}{=} \delta_{lb}(X, Z),
\]
where \( \cos \theta(t) \) is defined in (5.2) and \( \psi(t) \) in (5.3). Step (a) is due to Lemma 5.1. Step (b) uses the lower-bound \( \cos \theta(t) \geq \psi(\alpha, \beta, t) \) in Lemma 5.3. Step (c) evokes Lemma 5.2.

We now resolve the degenerate case: if \( e \neq 0 \) and \( \beta = 0 \), then \( \delta(X, Z) = \delta_{lb}(X, Z) = 1 \). Clearly \( \delta_{lb}(X, Z) = 1 \) whenever \( \beta = 0 \). To verify \( \delta(X, Z) = 1 \), suppose by contradiction that \( \delta(X, Z) < 1 \), which by its definition means that there exists \( A \) satisfying \( \delta \)-RIP with \( \delta < 1 \) such that \( \nabla f_A(X) = 0 \) and \( \nabla^2 f_A(X) \succeq 0 \). The hypothesis \( \beta = 0 \) ensures that \( X \) is
Given that $\delta < 1$, we must actually have $XX^T = ZZ^T$, but this contradicts our hypothesis that $e \neq 0$. \hfill \square

In the remainder of this section, we will prove Lemma 5.3. We break the bulk of the proof down into two key steps of reformulation, whose proofs we defer to the end of the of this section.

**Lemma 5.4** (Reduction over $y$). Let $e \neq 0$ and $\sigma_r(X) > 0$. We have

$$
\max_y \{ e^T (Jy - w) : \|e\| \|Jy - w\| = 1 \}
$$

$$
= \sqrt{1 - \alpha^2} \sqrt{1 - \|e\|^2} \| (I - JJ^T) w \|^2 - e^T (I - JJ^T) w
$$

**Lemma 5.5** (Reduction over $W_{i,j}$). Let $e \neq 0$ and $\sigma_r(X) > 0$. For $\tau \geq 0$ and $t \geq 0$, we have

$$
\max_{W_{i,j}} \left\{ -e^T (I - JJ^T) w : \|e\| \| (I - JJ^T) w \| = \tau, \ w = \sum_i vec(W_{i,i}) \right\}
$$

$$
= \begin{cases} 
\max_{w \geq 0} \{ d^T w : \|w\| \leq \tau/\|e\|, 1^T w \leq t/s_r \} & \text{if } \tau/\|e\| \leq t/s_r \\
-\infty & \text{if } \tau/\|e\| > t/s_r 
\end{cases}
$$

where $s_r = \lambda_{\min}(X^T X)$ and $d = [d_i]_{i=1}^r$, $d_i = \lambda_i(Z^T Z_{\perp})$ and $Z_{\perp} = (I - XX^T) Z$.

We obtain Lemma 5.3 by lower-bounding the maximization in Lemma 5.5 with a heuristic choice of $w$.

**Proof of Lemma 5.3**. Consider the following reformulation of (5.2)

$$
\max_{y, W_{i,j}} \left\{ e^T (Jy - w) : \|e\| \|Jy - w\| = 1, \ w = \sum_{i=1}^r vec(W_{i,i}) \right\}
$$

over variables $y \in \mathbb{R}^{nr}$ and $W_{i,j} \in \mathbb{R}^{n \times n}$ for $i, j \in \{1, 2, \ldots, r\}$. Clearly, the optimal value in (5.5) must coincide with $\cos \theta(t)$, and the maximizer for (5.5) is also a maximizer for (5.2). Applying Lemma 5.4 to (5.5) yields

$$
\max_{\tau, W_{i,j}} \left\{ \sqrt{1 - \alpha^2} \sqrt{1 - \tau^2} - e^T (I - JJ^T) w : \|e\| \| (I - JJ^T) w \| = \tau, \ w = \sum_{i=1}^r vec(W_{i,i}) \right\}
$$

Applying Lemma 5.5 to (5.6) yields

$$
\cos \theta(t) = \max_{\tau, w} \left\{ \sqrt{1 - \alpha^2} \sqrt{1 - \tau^2} + d^T w : \|w\| \leq \tau/\|e\| \leq t/s_r, \ 1^T w \leq t/s_r \right\}
$$
over variables \( \tau \geq 0 \) and \( w \in \mathbb{R}^r_+ \). We have added the constraint \( \tau/\|e\| \leq t/s_r \) because (5.6) is infeasible for \( \tau/\|e\| > t/s_r \) due to Lemma 5.5. Finally, substituting the heuristic choice of \( w = (\tau d)/(\|d\||e|) \) yields

\[
\cos \theta(t) \geq \max_{\tau \geq 0} \left\{ \sqrt{1 - \alpha^2} \sqrt{1 - \tau^2} + \frac{\|d\|}{\|e\|} \tau : s_r \frac{1^T d}{\|d\|} \tau \leq t \right\}
\]

\[
= \max_{\tau \geq 0} \left\{ \sqrt{1 - \alpha^2} \sqrt{1 - \tau^2} + \alpha \tau : \tau \leq t/\beta \right\}
\]

\[
= \begin{cases} 
\sqrt{1 - \alpha^2} \sqrt{1 - (t/\beta)^2} + \alpha (t/\beta) & \text{if } t/\beta \leq \alpha \\
1 & \text{otherwise}
\end{cases}
\]

as desired. Here, we note that \( \alpha = \|d\|/\|e\| \) and \( \beta = (s_r/\|e\|)(1^T d/\|d\|) \) and that \( 1 = \max_r \{\sqrt{1 - \alpha^2} \sqrt{1 - \tau^2} + \alpha \tau\} \) is attained at \( \tau^* = \alpha \).

At this point, we remark that it is technically possible to reduce (5.7) further, since by Lagrange duality we have

\[
\max_{y \in \mathbb{R}^r} \left\{ d^T w : \|w\| \leq \frac{\tau}{\|e\|}, 1^T w \leq \frac{t}{s_r} \right\} = \min_{\lambda \in \mathbb{R}^r_+} \left\{ \frac{\tau}{\|e\|} \|d - \lambda\| + \frac{t}{s_r} \|\lambda\|_{\infty} \right\}
\]

\[
= \min_{\rho \geq 0} \left\{ \frac{\tau}{\|e\|} \|(d - \rho 1)_{+}\| + \frac{t}{s_r} \rho \right\}.
\]

However, this does not lead to a tractable proof. Even if we resolve the above into a closed-form solution for \( \cos \theta(t) \), we would still need to solve \( \max_{t \geq 0} (\cos \theta(t) - t)/(1 + t) \) in Lemma 5.1 for the lower-bound \( \delta_{lb}(X, Z) \). After this, we must still optimize the lower-bound over all choices of \( X \) and \( Z \).

In the remainder of this section, we will prove the two technical lemmas that we used to prove Lemma 5.3.

### 5.1 Proof of Lemma 5.4

Let \( XX^T \neq ZZ^T \) and \( \sigma_r(X) > 0 \). The following is a trust-region subproblem

\[
\max_y \{ e^T (Jy - w) : \|e\| \|Jy - w\| = 1 \}
\]

whose convex relaxation

\[
\max_y \{ e^T [Jy - w] : \|e\|^2 \|Jy - w\|^2 \leq 1 \}
\]

has optimality conditions that read

\[
-J^T e + \lambda J^T (Jy - w) = 0, \tag{5.8a}
\]

\[
\lambda (\|e\|^2 \|Jy - w\|^2 - 1) = 0. \tag{5.8b}
\]

in which \( \lambda \geq 0 \) is the Lagrange multiplier. The solutions to (5.8a) are of the form \( y = J^T (e/\lambda + w) + h \) where \( h \) satisfies \( Jh = 0 \), but all of these yield the same value of \( Jy \) and
hence

$$Jy - w = \frac{1}{\lambda}JJ^Te - (I - JJ^t)w,$$

(5.9a)

$$\|Jy - w\|^2 = \frac{1}{\lambda^2}\|JJ^Te\|^2 + \|(I - JJ^t)w\|^2.$$  

(5.9b)

Note that we must have $J^te \neq 0$ under the hypotheses of the lemma, because $JJ^te = 0$ would imply either $e = 0$ or $\sigma_r(X) = 0$ via Lemma 4.1. Hence, $\lambda > 0$ and $\|e\|\|Jy - w\| = 1$ via (5.8b). The convex relaxation is tight, and the optimal multiplier $\lambda$ satisfies the following

$$\frac{1}{\lambda^2}\|e\|^2\|JJ^te\|^2 = \|e\|^2\|Jy - w\|^2 - \|e\|^2\|(I - JJ^t)w\|^2$$

$$= 1 - \|e\|^2\|w\|^2,$$

which yields the following optimal value

$$e^T(Jy - w) = \frac{1}{\lambda}e^TJJ^te - e^T(I - JJ^t)w$$

$$= \left(\frac{\|JJ^te\|}{\|e\|}\right) \left(\frac{1}{\lambda}\|e\||\|JJ^te\|\right) - e^Tw = \sqrt{1 - \frac{\alpha^2}{\lambda^2}} = 1 - \|e\|^2\|w\|^2.$$

5.2 Proof of Lemma 5.5

Let $XX^T \neq ZZ^T$ and $\sigma_r(X) > 0$. Let $Q \in \mathbb{R}^{n \times r}$ and $P \in \mathbb{R}^{n \times (n-r)}$ diagonalize the following two matrices while being orthogonal complements of one another

$$XX^T = QSQ^T, \quad (I - XX^\dagger)ZZ^T(I - XX^\dagger) = PGP^T,$$

$$QQ^T + PP^T = [Q, P]^T[Q, P] = I_n, \quad S, G \text{ are diagonal.}$$

Let $S = \text{diag}(s)$ where $s \in \mathbb{R}^r$ and $G = \text{diag}(g)$ where $g \in \mathbb{R}^{n-r}$, and sort $s_1 \geq \cdots \geq s_r \geq 0$.

In this subsection, we prove that the following problem

$$\max_{W_{i,j}} \left\{ -e^T(I - JJ^\dagger)w : \|e\|\|(I - JJ^\dagger)w\| = \tau, \ w = \sum_{i,j} \text{vec}(W_{i,j}) \right\}$$

over $W_{i,j} \in \mathbb{R}^{n \times n}$ for $i, j \in \{1, 2, \ldots, r\}$, has the same optimal value as the following if $\tau/\|e\| \leq t/s_r$

$$\max_{W} \{d^Tw : \|w\| \leq \tau/\|e\|, 1^Tw \leq t/s_r\},$$

(5.10)

and that it is otherwise infeasible. Here, $d \in \mathbb{R}^r$ are the $r$ eigenvalues $d_i = \lambda_i(Z^\dagger Z)_{i} = \lambda_i(Z_{+}Z_{+}^\dagger Z_{+})$ where $Z_{+} = (I - XX^\dagger)Z$. Note that $g_i = \lambda_i(Z_{+}^\dagger Z_{+})$, so $g$ and $d$ coincide in their nonzero elements, despite having a potentially different number of zero elements.

The proof follows from 7 steps of reformulation:

1. We evoke Lemma 4.2 to yield

$$I - JJ^\dagger = (I - XX^\dagger) \otimes (I - XX^\dagger) = PP^T \otimes PP^T,$$

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and this rewrites (5.10) as

$$\max_{W_{i,j} \in \mathbb{R}^{n \times n}} \left\{ \langle G, W \rangle : \|e\| \|W\|_F = \tau, \quad W = \sum_{i=1}^r P^T W_{i,i} P, \right\}.$$  \hspace{1cm} (5.12)

2. Noting that $W \succeq 0$, we write $W = \sum_{j=1}^{nr} \text{vec}(V_j)\text{vec}(V_j)^T$ and observe that

$$\langle J^T J, W \rangle = \sum_{j=1}^{nr} \| XV_j^T + V_j X^T \|_F^2,$$

$$\sum_{i=1}^{nr} W_{i,i} = \sum_{j=1}^p V_j V_j^T.$$  

Splitting each $V_j = Q \hat{V}_j + P \tilde{V}_j$ where $\hat{V}_j = Q^T V_j \in \mathbb{R}^{r \times r}$ and $\tilde{V}_j = P^T V_j \in \mathbb{R}^{(n-r) \times r}$, some simple calculation shows that

$$\| XV_j^T + V_j X^T \|_F^2 = \| S^{\frac{1}{2}} \hat{V}_j + \hat{V}_j S^{\frac{1}{2}} \|_F^2 + 2\| \hat{V}_j S^{\frac{1}{2}} \|_F^2.$$

Viewing $\xi = \sum_j \| S^{\frac{1}{2}} \hat{V}_j + \hat{V}_j S^{\frac{1}{2}} \|_F^2 \geq 0$ as a slack variable yields the following exact reformulation of (5.12):

$$\max_{\hat{V}_j \in \mathbb{R}^{(n-r) \times r}} \left\{ \langle G, \sum_{j=1}^{nr} \hat{V}_j \hat{V}_j^T \rangle : \|e\| \sum_{j=1}^{nr} \hat{V}_j \hat{V}_j^T \|_F = \tau, \quad \sum_{j=1}^{nr} (S, V_j^T \tilde{V}_j) \leq t \right\}.$$  \hspace{1cm} (5.13)

3. Let $\tilde{W} = \sum_{j=1}^{nr} \text{vec}(V_j)\text{vec}(V_j)^T$ and observe that

$$\sum_{j=1}^{nr} \langle S, \hat{V}_j^T \hat{V}_j \rangle = \sum_{j=1}^{nr} \text{vec}(\hat{V}_j^T) (S \otimes I) \text{vec}(\hat{V}_j)$$

$$= \left( S \otimes I, \sum_{j=1}^{nr} \text{vec}(\hat{V}_j)\text{vec}(\hat{V}_j)^T \right) = \langle S \otimes I, \tilde{W} \rangle = \sum_{i=1}^r s_i \text{tr}(\tilde{W}_{i,i}).$$

Therefore, the following is an exact reformulation of (5.13)

$$\max_{\tilde{W}_{i,j} \in \mathbb{R}^{(n-r) \times (n-r)}} \left\{ \sum_{i=1}^r \langle G, \tilde{W}_{i,i} \rangle : \|e\| \sum_{i=1}^r \tilde{W}_{i,i} \|_F = \tau, \quad \sum_{i=1}^r s_i \text{tr}(\tilde{W}_{i,i}) \leq t, \quad \tilde{W} = [\tilde{W}_{i,j}]_{i,j=1}^{r} \succeq 0 \right\}.$$  \hspace{1cm} (5.14)

4. Relax the constraint $\tilde{W} \succeq 0$ into $\tilde{W}_{i,i} \succeq 0$ and optimizing over each $\tilde{W}_{i} \succeq 0$ yields

$$\max_{W_{i,j} \in \mathbb{R}^{(n-r) \times (n-r)}} \left\{ \sum_{i=1}^r \langle G, W_{i,i} \rangle : \|e\| \sum_{i=1}^r W_{i,i} \|_F = \tau, \quad \sum_{i=1}^r s_i \text{tr}(W_{i,i}) \leq t, \quad W_{i,i} \succeq 0 \right\}.$$  \hspace{1cm} (5.15)

The relaxation is exact because the optimal $\tilde{W}_{i}^*$ to the relaxation (5.15) yields a feasible $\tilde{W} = \text{diag}(\tilde{W}_{1}^*, \tilde{W}_{2}^*, \ldots, \tilde{W}_{r}^*) \succeq 0$ for the restriction (5.14) with the same objective value.

5. Restricting $\tilde{W}_{1} = \tilde{W}_{2} = \cdots = \tilde{W}_{r-1} = 0$ and optimizing over $\tilde{W} = \tilde{W}_{r}$ yields

$$\max_{W \in \mathbb{R}^{(n-r) \times (n-r)}} \left\{ \langle G, \tilde{W} \rangle : \|\tilde{W}\|_F = \tau/\|e\|, \quad \text{tr}(\tilde{W}) \leq t/s_r, \quad \tilde{W} \succeq 0 \right\}.$$  \hspace{1cm} (5.16)
The restriction is exact because the optimal $\tilde{W}_i^*$ to the relaxation (5.15) satisfies $s_i \sum_i \tilde{W}_i^* \leq \sum_i s_i W_i^* \leq t$ and therefore yields a feasible $\tilde{W} = \sum_i \tilde{W}_i^*$ for the restriction (5.16) with the same objective value.

6. Here, if $\tau/\|e\| > t/s_r$, then problem (5.16) is infeasible, because $\|\tilde{W}\|_F \leq \text{tr}(\tilde{W})$ for any $\tilde{W} \succeq 0$. Otherwise if $\tau/\|e\| \leq t/s_r$, then let $w = \text{diag}(\tilde{W})$ and optimize

$$\max_{w \in \mathbb{R}^{n-r}} \left\{ g^T w : \|w\| \leq \tau/\|e\|, \ 1^T w \leq t/s_r, \ w \geq 0 \right\}. \quad (5.17)$$

This is an exact relaxation of (5.16). It is a relaxation because $\tilde{W} \succeq 0$ implies $\text{diag}(\tilde{W}) \geq 0$, and $\|\text{diag}(\tilde{W})\|_F \leq \|\tilde{W}\|_F = \tau/\|e\|$, and $\langle D, \tilde{W} \rangle = d^T \text{diag}(\tilde{W})$ and $\text{tr}(\tilde{W}) = 1^T \text{diag}(\tilde{W})$. The relaxation is exact because the optimal $w^*$ for the relaxation (5.17) yields a feasible $W^*$ for the restriction (5.16) with the same objective value:

- If $\|w^*\| = \tau/\|e\|$, then $W^* = \text{diag}(w^*)$ is feasible for (5.16) with $\langle G, W(\gamma) \rangle = g^T w^*$.

- If $\|w^*\| < \tau/\|e\|$, then $1^T w = t/s_r$ because the objective in (5.17) is linear. Here we set $W(\gamma) = \gamma \text{diag}(w^*) + (1 - \gamma) uu^T$ where $u_i = \sqrt{w_i^*}$ so that $\text{tr}(W(\gamma)) = 1^T w^*$ and $\|W(\gamma)\|_F$ is a continuous function satisfying $\|W(0)\|_F = \|w^*\| < \tau/\|e\| \leq t/s_r = 1^T w^* = \|W(1)\|_F$. By the intermediate value theorem, there exists a $0 \leq \gamma^* \leq 1$ such that $\|W(\gamma^*)\| = \tau/\|e\|$, and this yields a feasible $W^* = W(\gamma^*)$ for (5.16) with $\langle G, W(\gamma^*) \rangle = g^T w^*$.

7. Finally, we show that zero elements in $g$ can be added or removed without affecting the optimal value, and hence (5.11) and (5.17) are equivalent. By induction, it suffices to show that augmenting $g$ with a single zero element, as in

$$\max_{[w; \mu] \in \mathbb{R}^{(n-r+1)}} \left\{ g^T w : \|[w; \mu]\| \leq \tau/\|e\|, \ 1^T [w; \mu] \leq t/s_r, \ [w; \mu] \geq 0 \right\}, \quad (5.18)$$

does not affect the optimal value. Indeed, (5.17) is a restriction of (5.18) obtained by enforcing $\mu = 0$, and the restriction is exact because the optimal $[w^*; \mu^*]$ for the relaxation (5.18) yields a feasible $w^*$ for the restriction (5.17) with the same objective value $g^T w^*$.

6 Valid inequalities over $\alpha$ and $\beta$

The second major innovation in our proof is Lemma 3.4 which provided two valid inequalities that allowed us to relax the dependence of the functions $\alpha(X, Z)$ and $\beta(X, Z)$ over $X, Z$, while capturing the rank equality constraint $r^* = \text{rank}(Z)$. In this section, we give a proof of Lemma 3.4. The proof relies crucially on the following generalization of the classic low-rank approximation result of Eckart and Young [18].
Theorem 6.1 (Regularized Eckhart–Young). Given $A \in \mathbb{S}_+^n$ and $B \in \mathbb{S}_+^r$ with $r \leq n$, let $(s_i, u_i)$ and $(d_i, v_i)$ denote the eigenpairs of $A$ and $B$ satisfying

$$A = \sum_{i=1}^{n} s_i u_i u_i^T,$$

$$I_n = \sum_{i=1}^{n} u_i u_i^T,$$

$$s_1 \geq \cdots \geq s_n \geq 0,$$  \hspace{1cm} (6.1a)

$$B = \sum_{i=1}^{r} d_i v_i v_i^T,$$

$$I_r = \sum_{i=1}^{r} v_i v_i^T,$$

$$0 \leq d_1 \leq \cdots \leq d_r.$$  \hspace{1cm} (6.1b)

Then,

$$\min_{Y \in \mathbb{R}^{n \times r}} \left\{ \|A - YY^T\|_F^2 + 2\langle B, Y^TY \rangle \right\} = \sum_{i=1}^{n} s_i^2 - \sum_{i=1}^{r} (s_i - d_i)^+ \right)^2 \right\} \right\}$$  \hspace{1cm} (6.2)

with minimizer $Y^* = \sum_{i=1}^{r} u_i v_i^T \sqrt{(s_i - d_i)^+}$.

Setting $B = 0$ in Theorem 6.1 recovers the original Eckhart–Young Theorem: The best low-rank approximation $YY^T \approx A$ in Frobenius norm is the truncated singular value decomposition $YY^T = \sum_{i=r+1}^{n} s_i u_i u_i^T$, with approximation error $\|A - YY^T\|_F^2 = \sum_{i=r+1}^{n} s_i^2$. Hence, $B \neq 0$ may be viewed as regularizer that prevents $Y^*$ from becoming excessively large.

The proof of Lemma 3.4 quickly follows from Theorem 6.1. In the remainder of this section, we first give a proof of Lemma 3.4 based on Theorem 6.1. Finally, we will give a self-contained proof of Theorem 6.1.

6.1 Proof of Lemma 3.4

We first prove a small technical lemma.

Lemma 6.2. Let $x \in \mathbb{R}_+^n$ satisfy $1^T x \leq \|x\|^2$. Then

$$1^T \left( I - xx^T/\|x\|^2 \right) \ 1 \geq \|(1 - x)^+\|^2.$$

Proof. The right-hand side is the projection distance of $x$ onto the constraint $x \geq 1$, as in

$$\|(1 - x)^+\|^2 = \min_{y \geq 1} \|x - y\|^2.$$

As such, we can increase its value by shrinking $x \leftarrow \alpha x$ towards zero

$$\alpha \leq 1 \implies \|(1 - \alpha x)^+\|^2 \geq \|(1 - x)^+\|^2.$$

We can shrink $x$ this way until the constraint $1^T x \leq \|x\|^2$ becomes active. Define this point as $u = \alpha x$ where $\alpha$ is chosen so that $1^T u = \|u\|^2$. Here, we can verify that

$$1^T \left( I - uu^T/\|u\|^2 \right) 1 = \min_{t} \|1 - tu\|^2 = \|1 - u\|^2$$

because $(1 - u)^T u = 0$ implies $(1 - u) \perp u$. Finally, we observe that

$$\|1 - u\|^2 = \sum_{i=1}^{n} (1 - u_i)^2 \geq \sum_{u_i \leq 1} (1 - u_i)^2 = \|(1 - u)^+\|^2.$$
Combined, this yields
\[ 1^T(I - xx^T/\|x\|^2)1 = 1^T(I - uu^T/\|u\|^2)1 = \|1 - u\|^2 \geq \|(1 - u)_+\|^2 \geq \|(1 - x)_+\|^2 \]
as desired.

Given \(X, Z \in \mathbb{R}^{n \times r}\) with \(\text{rank}(Z) = r^*\), recall that \(\alpha, \beta\) are defined

\[ \alpha(X, Z) = \frac{\|Z_\perp Z_\perp^T\|_F}{\|XX^T - ZZ^T\|_F}, \quad \beta(X, Z) = \frac{\sigma^2_{\min}(X)}{\|XX^T - ZZ^T\|_F} \cdot \frac{\text{tr}(Z_\perp Z_\perp^T)}{\|Z_\perp Z_\perp^T\|_F}. \]

where \(Z_\perp = (I - XX^\top)Z\). Since \(\alpha\) and \(\beta\) are unitarily invariant, we may assume without loss of generality that \(X, Z\) are partitioned as

\[ X = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \quad \text{where } X_1 \in \mathbb{R}^{r \times r}, \ Z_1 \in \mathbb{R}^{r \times r^*}, \ Z_2 \in \mathbb{R}^{(n-r) \times r^*}. \]

(Otherwise, take the QR decomposition \(X = QR\) and note that \(\alpha, \beta\) remain unchanged with \(X \leftarrow Q^TX\) and \(Z \leftarrow Q^TZ\).) Denoting \(s \in \mathbb{R}^r\) as the \(r\) eigenvalues of \(X_1X_1^\top\) and \(d \in \mathbb{R}^{r^*}\) as the \(r^*\) eigenvalues of \(Z_2^\top Z_2\) yields

\[ s_i \overset{\text{def}}{=} \lambda_i(X_1X_1^\top) = \lambda_i(XX^T), \quad s_1 \geq s_2 \geq \cdots \geq s_r \geq 0, \]

\[ d_i \overset{\text{def}}{=} \lambda_i(Z_2^\top Z_2) = \lambda_i(Z_\perp Z_\perp^T), \quad 0 \leq d_1 \leq d_2 \leq \cdots \leq d_{r^*}, \]

allows us to rewrite \(\alpha, \beta\) as

\[ \alpha(X, Z) = \frac{\|d\|}{\|e\|}, \quad \beta(X, Z) = \frac{s_r(1^Td)}{\|e\|\|d\|}, \]

where \(\|e\| = \|XX^T - ZZ^T\|_F\). While this partitioning of \(X\) and \(Z\) appears to relax the rank equality constraint into an inequality \(r^* \leq \text{rank}(Z)\), we clarify after our proof below why \(r^* = \text{rank}(Z)\) always holds true.

**Proof of Lemma 3.4.** We will prove the following lower-bounds on \(\|e\|^2 = \|XX^T - ZZ^T\|_F^2\) in terms of \(s\) and \(d\)

\[ s_r 1^Td \leq \|d\|^2 \implies \|e\|^2 \geq s_r^2 \frac{(1^Td)^2}{\|d\|^2} + \|d\|^2 + (r - r^*)s_r^2, \quad (6.3a) \]

\[ s_r 1^Td \geq \|d\|^2 \implies \|e\|^2 \geq 2\|d\|^2 + (r - r^*)s_r^2. \quad (6.3b) \]

Rewriting the final term

\[ (r - r^*)s_r^2 = (r - r^*) \frac{\|d\|^2}{(1^Td)^2} \beta^2 \|e\|^2 \geq \frac{(r - r^*)}{\text{nnz}(d)} \beta^2 \|e\|^2 \geq \frac{(r - r^*)}{r^*} \beta^2 \|e\|^2, \]
and substituting into (6.3) yields

$$\beta \leq \alpha \quad \implies \quad 1 \geq \beta^2 + \alpha^2 + \frac{(r - r^*)}{r^*} \beta^2 = \alpha^2 + \frac{r}{r^*} \beta^2, \quad (6.4a)$$

$$\beta \geq \alpha \quad \implies \quad 1 \geq 2\alpha^2 + \frac{(r - r^*)}{r^*} \beta^2 \geq \left(1 + \frac{r}{r^*}\right) \alpha^2, \quad (6.4b)$$

as desired. We prove (6.3) by expanding $\|XX^T - ZZ^T\|_F^2$ block-wise and evoking Theorem 6.1 over $Z_1$

$$\|XX^T - ZZ^T\|_F^2 \geq \min_{Z_1 \in \mathbb{R}^{r \times r}} \{\|X_1X_1^T - Z_1Z_1^T\|_F^2 + 2\langle Z_2^TZ_2, Z_1^TZ_1 \rangle\} + \|Z_2Z_2^T\|_F^2$$

$$= \|s\|_2^2 - \|(s' - d)_+\|^2 + \|d\|^2$$

where $s' = [s_i]_{i=1}^r$. To prove (6.3a), fix $1$ to be of length $r^*$. Under $s_r1^Td \leq \|d\|^2$, we have

$$\|s\|_2^2 - \|(s' - d)_+\|^2 \overset{(a)}{\geq} \|s_r1\|^2 - \|(s_r1 - d)_+\|^2 + (r - r^*)s_r^2$$

$$= s_r^2(\|1\|^2 - \|(1 - d/s_r)_+\|^2) + (r - r^*)s_r^2$$

$$\overset{(b)}{\geq} s_r^2([1^T(d/s_r)]^2/\|d/s_r\|^2) + (r - r^*)s_r^2$$

$$= s_r^2(1^Td)^2/\|d\|^2 + (r - r^*)s_r^2.$$

Inequality (a) uses the monotonicity of $s_r^2 - [(s_i - d_i)_+]^2$ with respect to $s_i \geq 0$. Inequality (b) evokes Lemma 6.2, noting that $s_r1^Td \leq \|d\|^2$ is equivalently written $1^T(d/s_r) \leq \|d/s_r\|^2$.

To prove (6.3b), again fix $1$ to be of length $r^*$. Under $s_r1^Td \geq \|d\|^2$ we have

$$\|s\|_2^2 - \|(s' - d)_+\|^2 + \|d\|^2 \overset{(a)}{\geq} \|s_r1\|^2 - \|(s_r1 - d)_+\|^2 + \|d\|^2 + (r - r^*)s_r^2$$

$$\overset{(b)}{\geq} \|s_r1\|^2 - \|s_r1 - d\|^2 + \|d\|^2 + (r - r^*)s_r^2$$

$$= 2s_r1^Td + (r - r^*)s_r^2 \overset{(c)}{\geq} 2\|d\|^2 + (r - r^*)s_r^2.$$

Inequality (a) uses the monotonicity of $s_r^2 - [(s_i - d_i)_+]^2$ with respect to $s_i \geq 0$. Inequality (b) follows from $[(s_i - d_i)_+]^2 \leq (s_i - d_i)^2$. Inequality (c) substitutes the hypothesis that $s_r1^Td \geq \|d\|^2$.

We conclude this section by clarifying how our proof of Lemma 3.4 is able to capture the rank equality constraint $r^* = \text{rank}(Z)$, without relaxing it into an inequality. The key insight is the observation that Theorem 6.1 solves

$$\min_{Z_1 \in \mathbb{R}^{r \times r}} \{\|X_1X_1^T - Z_1Z_1^T\|_F^2 + 2\langle Z_2^TZ_2, Z_1^TZ_1 \rangle\} + \|Z_2Z_2^T\|_F^2$$

with a solution $Z_1$ that satisfies $\lambda_{\min}(Z_2^TZ_2 + Z_1^TZ_1) \geq s_r^2$. If $\text{rank}(X) = r$ and $s_r > 0$, then we have $\text{rank}(Z) = r^*$ as desired. Conversely, if $\text{rank}(X) < r$ and $s_r = 0$, then Lemma 3.4 loses its dependence on $r^*$, so it can remain true even if we allow $\text{rank}(Z) \neq r^*$.  

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6.2 A Regularized Eckhart–Young Theorem

In this subsection, we will solve the following

\[ \phi^* = \min_{X \in \mathbb{R}^{n \times r}} \phi(X) \quad \text{where} \quad \phi(X) \overset{\text{def}}{=} \| S - XX^T \|_F^2 + 2\langle D, X^TX \rangle \quad (6.5) \]

ever diagonal matrices \( S \) and \( D \), with \( S = \text{diag}(s) \) in descending order \( s_1 \geq \cdots \geq s_n \geq 0 \) and \( D = \text{diag}(d) \) in ascending order \( 0 \leq d_1 \leq \cdots \leq d_r \). The solution \( X^* \) to (6.5) then recovers a solution \( Y^* = UX^*V^T \) to (6.2) with the same objective value.

Here, our key insight is the fact that there exists a solution \( X^* \) to (6.5) that is a scaled permutation matrix. Loosely, this is a permutation matrix whose “1” elements have been replaced by real scalars.

**Definition 6.3.** The matrix \( X \in \mathbb{R}^{n \times m} \) is said to be a scaled permutation if it contains at most one nonzero element per column and at most one nonzero element per row.

Problem (6.5) is easy to solve once we restrict \( X \) to be a scaled permutation. The difficulty in the proof comes in the fact that it is not necessary for a solution \( X^* \) to be a scaled permutation if \( S \) or \( D \) have repeated eigenvalues. If \( X^* \) is not a scaled permutation, then we will use the following two linear algebraic lemmas to assert the existence of a scaled permutation \( Y \) with the same objective \( \phi(Y) = \phi(X^*) \).

**Lemma 6.4.** Let \( X \in \mathbb{R}^{n \times r} \) satisfy \((S - XX^T)X = XD\) for real symmetric \( S \) and diagonal \( D \). Then, there exists \( Y \in \mathbb{R}^{m \times n} \) with \( Y^TY \) diagonal that also satisfies \((S - YY^T)Y = YD\) with \( YY^T = XX^T \) and \( \langle D, Y^TY \rangle = \langle D, X^TX \rangle \).

**Proof.** Write \( D = \text{diag}(d_1, \ldots, d_r) \) and assume \( d_1 \leq d_2 \leq \cdots \leq d_r \) without loss of generality. (Otherwise, reorder the columns of \( X \) and \( Y \).) Partition the elements of \( D \) into \( p \) distinct values

\[ D = \text{diag}(\lambda_1 I_{k(1)}, \lambda_2 I_{k(2)}, \ldots, \lambda_p I_{k(p)}) \]

in which the \( j \)-th distinct value \( \lambda_j \) has multiplicity \( k(j) \). Partition the columns of \( X \) into corresponding blocks

\[ X = [ X_1 \quad X_2 \quad \cdots \quad X_p ] \quad , \quad XD = [ \lambda_1 X_1 \quad \lambda_2 X_2 \quad \cdots \quad \lambda_p X_p ] . \]

The \( j \)-th block-column of \((S - XX^T)X = XD\) reads

\[ MX_j = \lambda_j X_j \] where \( M = S - XX^T \).

Hence, the nonzero columns of \( X_j \) must be eigenvectors of \( M \), with corresponding eigenvalue \( \lambda_j \). The matrix \( M \) is real symmetric; it follows from the Spectral Theorem that all of its distinct eigenspaces are orthogonal, and therefore \( X_j^TX_j = 0 \) holds for all \( i \neq j \). The matrix \( X^TX \) is block-diagonal, though the blocks \( X_j^TX_j \) themselves may be dense.

Now, we diagonalize \( X_j^TX_j = V_j \Gamma_j V_j^T \) with diagonal \( \Gamma_j \) and orthonormal \( V_j \). Define \( Y \) with the same column partitions as \( X \) but with its \( j \)-th block-column right-multiplied by \( V_j \), as in

\[ Y = [ Y_1 \quad Y_2 \quad \cdots \quad Y_p ] = [ X_1 V_1 \quad X_2 V_2 \quad \cdots \quad X_p V_p ] . \]
Clearly, \( Y^T Y \) is diagonal because \( Y_j^T Y_j = V_j^T X_j^T X_j V_j = \Gamma_j \) and \( Y_i^T Y_j = V_i^T X_i^T X_j V_j = 0 \). Moreover, observe that \( V = \text{diag}(V_1, \ldots, V_p) \) is an orthonormal matrix that commutes with \( D \), as in

\[
VD = \text{diag}(\lambda_1 V_1, \ldots, \lambda_1 V_p) = DV, \quad VDV^T = V^T DV = D.
\]

Therefore, \( Y = XV \) must satisfy

\[
YY^T = XVV^T X^T = XX^T, \quad (S - YY^T)Y = (S - XX^T) XV = XD V = XV D = YD, \quad \langle D, Y^T Y \rangle = \text{tr}(DV^T X^T XV) = \text{tr}(VDV^T X^T X) = \text{tr}(DX^T X) = \langle D, X^T X \rangle,
\]

as desired. \( \square \)

**Lemma 6.5.** Let \( X \in \mathbb{R}^{n \times r} \) satisfy \( SX = X(D + X^T X) \) with real diagonal \( D, X^T X \) and \( S \). Then, there exists a scaled permutation \( Y \) that also satisfies \( SY = Y(D + Y^T Y) \) with \( Y^T Y = X^T X \) and \( \langle S, YY^T \rangle = \langle S, XX^T \rangle \).

**Proof.** Write \( D + X^T X = G = \text{diag}(g_1, g_2, \ldots, g_n) \) and \( S = \text{diag}(s_1, s_2, \ldots, s_n) \) and assume \( s_1 \geq s_2 \geq \cdots \geq s_n \) without loss of generality. (Otherwise, reorder the rows of \( X \) and \( Y \).) The equation \( SX = XG \) reads at the \( i \)-th row and \( j \)-th column

\[
(s_i - g_j)x_{i,j} = 0 \iff s_i = g_j \text{ or } x_{i,j} = 0
\]

where \( x_{i,j} \) denotes the \((i, j)\)-th element of \( X \). If the elements of \( S \) (resp. \( G \)) are distinct, then \( X \) contains at most a single nonzero element per column (resp. per row). If both \( S \) and \( G \) contain distinct elements, then \( X \) is a scaled permutation, so we set \( Y = X \).

However, \( X \) need not be a scaled permutation if \( S \) or \( G \) contain repeated elements. Instead, we partition the elements \( S \) into \( p \) distinct values

\[
S = \text{diag}(\lambda_1 I_{k(1)}, \lambda_2 I_{k(2)}, \ldots, \lambda_p I_{k(p)}) \quad \text{where } \lambda_1 > \lambda_2 > \cdots > \lambda_p,
\]

in which each distinct \( \lambda_j \) has multiplicity \( k(j) \) in \( S \). Similarly, we reorder the elements of \( G \) by the permutation matrix \( \Pi \) and then partition into the same values

\[
\Pi^T G \Pi = \text{diag}(\lambda_1 I'_{k(1)}, \lambda_2 I'_{k(2)}, \ldots, \lambda_p I'_{k(p)}, \Lambda_{p+1})
\]

with each distinct \( \lambda_j \) has multiplicity \( k'(j) \) in \( G \). Here, we have collected all the remaining, nonmatching elements of \( G \) into \( \Lambda_{p+1} \). The distinctness of \( \lambda_i \) ensures a block-diagonal structure in the reordered equation

\[
SX \Pi = X \Pi (\Pi^T G \Pi) \implies X \Pi = \text{diag}(X_1, X_2, \ldots, X_p).
\]

Here, each \( X_j \) has \( k(j) \) rows and \( k'(j) \) columns, except \( X_p \), which has \( k(j) \) rows, \( k'(j) \) nonzero columns, and \( n - \sum_j k'(j) \) zero columns corresponding to \( \Lambda_{p+1} \). Hence, the matrix \( X \) is a block-scaled permutation, though the blocks \( X_j \) themselves may be dense.
The hypothesis that $X^TX$ is diagonal implies that the columns of $X_j$ are already or-
thogonal. Performing Gram-Schmidt starting from the nonzero columns of $X_j$ yields a QR
factorization satisfying
\[
X_j = \begin{bmatrix} U_{j,1} & U_{j,2} \end{bmatrix} \begin{bmatrix} \Sigma_j \\ 0 \end{bmatrix} = U_j Y_j, \quad U_j \text{ is orthonormal, } \Sigma_j \succeq 0 \text{ is diagonal.}
\]
Now, define $Y$ with the same block partitions and the same column permutation as $X$ but
with its $j$-th block-row left-multiplied by $U_j^T$, as in
\[
Y_j = \text{diag}(Y_1, Y_2, \ldots, Y_p) = \text{diag}(U_1^T X_1, U_2^T X_2, \ldots, U_p^T X_p).
\]
Clearly, the matrix $Y$ is a scaled permutation because each $Y_j$ is diagonal. Moreover, observe
that $U = \text{diag}(U_1, \ldots, U_p)$ is an orthonormal matrix that commutes with $S$, as in
\[
US = \text{diag}(\lambda_1 V_1, \ldots, \lambda_1 V_p) = SU, \quad USU^T = U^T SU = S.
\]
Accordingly, $Y = U^T X$ must satisfy
\[
Y^T Y = X^T U U^T X = X^T X,
\]
\[
Y(D + Y^T Y) = U^T X (D + X^T X) = U^T S X = SU^T X = SY,
\]
\[
\langle S, YY^T \rangle = \text{tr}(SU^T XX^T U) = \text{tr}(USU^T XX^T) = \text{tr}(SXX^T) = \langle D, XX^T \rangle,
\]
as desired.

Proof. Making the change of variable $X = U^T YV$ reformulates (6.2) into (6.5), which we
restate here as
\[
\phi^* = \min_X \phi(X) \text{ where } \phi(X) \overset{\text{def}}{=} \|S - XX^T\|_F^2 + 2 \langle D, XX^T \rangle
\]
with $S, D$ diagonal. Here, $\phi$ is continuous and its sublevel sets are closed and bounded, so
the infimum is attained at a minimizer $X^*$ satisfying $\phi^* = \phi(X)$. This solution $X^*$ then
recovers a solution $Y^* = UX^* V^T$ to (6.2) with the same objective value.

We assert the existence of a scaled permutation $X$ satisfying $\phi^* = \phi(X)$ by evoking
Lemma 6.4 and Lemma 6.5. First, the gradient $\nabla \phi(X)$ exists for all $X$, so any minimizer
$X^*$ must satisfy first order optimality
\[
\nabla \phi(X) = 4XX^T X - 4SX + 4XD = 0,
\]
which can be rewritten as
\[
(S - XX^T) X = XD.
\]
If $(X^*)^T X^*$ is not diagonal, then we use Lemma 6.4 to assert the existence of a $Y$ with
$Y^T Y$ diagonal that satisfies (6.7) and $Y^T Y = X^*(X^*)^T$ and $\langle D, Y^T Y \rangle = \langle D, (X^*)^T X^* \rangle$, and therefore $\nabla \phi(Y) = 0$ and $\phi(Y) = \phi(X^*) = \phi^*$. Taking $(X^*)^T X^*$ to be diagonal, we rewrite first-order optimality (6.6) into the following
\[
SX = X(D + X^T X).
\]
If \( X^\ast \) is not a scaled permutation, then we use Lemma \( \ref{lem:2001} \) to assert the existence of a scaled permutation \( Y \) that satisfies \( \ref{eq:2001} \) and \( Y^TY = (X^\ast)^T X^\ast \) and \( \langle S, YY^T \rangle = \langle S, X^\ast(X^\ast)^T \rangle \), and therefore \( \nabla \phi(Y) = 0 \) and \( \phi(Y) = \phi(X^\ast) = \phi^\ast \). Hence, \( Y \) is a scaled permutation satisfying \( \phi^\ast = \phi(Y) \).

Now, restricting \( X \) to be a scaled permutation, we parametrize \( X^T X \) and \( XX^T \) in terms of the weights \( w \in \mathbb{R}^n \) and the permutation \( \pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \), as in

\[
X^T X = \text{diag}(w_1, w_2, \ldots, w_r)
\]

\[
XX^T = \text{diag}(w_{\pi(1)}, w_{\pi(2)}, \ldots, w_{\pi(n)})
\]

where \( w \geq 0 \) and \( w_i = 0 \) for all \( i > r \). Substituting yields

\[
\phi^\ast = \min_{w \geq 0, \pi} \left\{ \sum_{i=1}^n (w_{\pi(i)} - s_{\pi(i)})^2 + 2 \sum_{i=1}^r d_i w_i : w_i = 0 \text{ for all } i > r \right\}
\]

\[
= \min_{\pi} \sum_{i=1}^r \left[ s_{\pi(i)} - (s_{\pi(i)} - d_i)_+ \right]^2 + \sum_{i=r+1}^n s_{\pi(i)}^2
\]

\[
= \min_{\pi} \sum_{i=1}^r s_i^2 - \sum_{i=1}^r s_{\pi(i)}^2 - \sum_{i=r+1}^n d_i^2
\]

Step (a) keeps the permutation \( \pi \) fixed and optimizes over each individual weight \( w_i \) using the following

\[
\min_{w \geq 0} \{(w - s)^2 + 2dw\} = s^2 - [(s - d)_+]^2
\]

with minimizer \( w = (s - d)_+ \). Step (b) optimizes over the permutation \( \pi \), noting that the summation \( \sum_i \left[ (s_{\pi(i)} - d_i)_+ \right]^2 \) is maximized by sequentially pairing the largest \( s_{\pi(i)} \) with the smallest \( d_i \). Given that \( s \) is already ordered in descending order and \( d \) in ascending order, the optimal \( \pi \) is just the natural ordering and the optimal \( w_i = (s_i - d_i)_+ \) yields \( X^\ast = \text{diag}(\sqrt{(s_i - d_i)_+}) \). □

### 7 Proof of the Heuristic Upper-bound

Finally, we give a constructive proof for the heuristic upper-bound in Proposition \( \ref{prop:2001} \). Concretely, we will prove, for each \( r, r^\ast, n \) in the proposition, that there exists an \( A \) that satisfies \((\delta, r + r^\ast)-\text{RIP}\) with the quoted RIP constant \( \delta \), but whose \( f_A \) admits an \( n \times r \) spurious second-order point \( X \) for the recovery of a rank-\( r^\ast \) ground truth \( ZZ^T \).

**Example 7.1** \( (1/(1 + 1/\sqrt{r - r^\ast + 1})\)-RIP counterexample). For \( r, r^\ast, n \) satisfying \( 1 \leq r^\ast \leq r < n \), let \( u_0, u_1, u_2, \ldots u_{n-1} \) be an orthonormal basis for \( \mathbb{R}^n \), and let \( q = r - r^\ast + 1 \). Define the condition number \( \kappa = 1 + 2\sqrt{q} \) and the measurement operator

\[
A : \mathbb{R}^{n \times n} \to \mathbb{R}^n, \quad M \mapsto \text{vec}([A^{(i,j)}, M]_{i,j=0}^{n-1})
\]
whose $m = n^2$ data matrices satisfy $A^{(i,j)} = \sqrt{r} \cdot u_j u_i^T$ except the following

\[
A^{(0,0)} = \sqrt{\frac{\kappa}{2}} u_0 u_0^T + \sqrt{\frac{\kappa}{2q}} \sum_{i=1}^{q} u_i u_i^T, \quad A^{(1,1)} = \frac{1}{\sqrt{2}} u_0 u_0^T - \frac{1}{\sqrt{2q}} \sum_{i=1}^{q} u_i u_i^T, \\
A^{(i,i)} = \frac{\sqrt{p_i}}{p_i + 1} u_{i-1} u_{i-1}^T - \frac{1}{\sqrt{p_i(p_i + 1)}} \sum_{j=0}^{p_i-1} u_{i+j} u_{i+j}^T \quad \text{for all } i \in \{2, \ldots, q\},
\]

where $p_i = q - i + 1$. The operator $\mathcal{A}$ satisfies $(\delta, r + r^*)$-RIP with optimal constant

\[
\delta = \frac{\kappa - 1}{\kappa + 1} = \frac{1}{1 + 1/\sqrt{q}} = \frac{1}{1 + 1/\sqrt{r - r^* + 1}}.
\]

because $\mathcal{A}$ can be scaled to satisfy

\[
(1 - \delta)\|M\|_{\mathcal{F}}^2 \leq \frac{2}{\kappa + 1} \|\mathcal{A}(M)\|_{\mathcal{F}}^2 \leq (1 + \delta)\|M\|_{\mathcal{F}}^2 \quad \text{for all } M \in \mathbb{R}^{n \times n},
\]

and that the extremal singular vectors $V_1, V_m \neq 0$ both have low-rank

\[
\frac{\|\mathcal{A}(V_1)\|_{\mathcal{F}}^2}{\|V_1\|_{\mathcal{F}}^2} = \kappa, \quad \frac{\|\mathcal{A}(V_m)\|_{\mathcal{F}}^2}{\|V_m\|_{\mathcal{F}}^2} = 1, \quad \text{rank}(V_1), \text{rank}(V_m) \leq r + r^*.
\]

Nevertheless, for the following $n \times r^*$ ground truth $Z$ and $n \times r$ candidate point $X$,

\[
Z = \begin{bmatrix} u_0 & u_{q+1} & u_{q+2} & \cdots & u_r \end{bmatrix}, \quad X = \begin{bmatrix} \xi u_1 & \cdots & \xi u_q & u_{q+1} & u_{q+2} & \cdots & u_r \end{bmatrix},
\]

where $\xi = 1/\sqrt{1 + \sqrt{q}}$, the corresponding $f_\mathcal{A}$ admits $X$ as a second-order critical point

\[
f_\mathcal{A}(X) \overset{\text{def}}{=} \|\mathcal{A}(X X^T - ZZ^T)\|_{\mathcal{F}}^2 = \frac{1 + 2\sqrt{q}}{1 + \sqrt{q}}, \quad \nabla f_\mathcal{A}(X) = 0, \quad \nabla^2 f_\mathcal{A}(X) \succeq 0.
\]

In this section, we will First, recall from Section 4.2 that the RIP threshold function has the following convex reformulation

\[
\delta(X, Z) = \min_{\delta, \mathbf{H}} \left\{ \delta : \begin{bmatrix} J^T \mathbf{e} & 2I_r \otimes \text{mat}(\mathbf{He}) & J^T A^T A J \geq 0 \end{bmatrix} \begin{bmatrix} I_r \end{bmatrix} \right\} \quad (7.1)
\]

where $\mathbf{e} = \text{vec}(XX^T - ZZ^T)$ and its Jacobian $\mathbf{J}$ is defined to satisfy $\mathbf{J}\text{vec}(V) = \text{vec}(XX^T + VV^T)$ for all $V$. We begin by deriving sufficient conditions for $(\delta, \mathbf{H})$ to be feasible for (7.1) with $r^* = 1$.

**Lemma 7.2.** For $r \geq 1$. Let $\kappa = 1 + 2\sqrt{r}$, and let $\mathbf{H}$ be any $(r + 1)^2 \times (r + 1)^2$ matrix that satisfy $I \preceq \mathbf{H} \preceq \kappa I$ and the following $r + 2$ eigenvalue equations

\[
\mathbf{H}\text{vec} \left[ \begin{bmatrix} \sqrt{r} & 0 \\ 0 & I_r \end{bmatrix} \right] = \kappa \cdot \text{vec} \left[ \begin{bmatrix} \sqrt{r} & 0 \\ 0 & I_r \end{bmatrix} \right], \quad \mathbf{H}\text{vec} \left[ \begin{bmatrix} \sqrt{r} & 0 \\ 0 & -I_r \end{bmatrix} \right] = 1 \cdot \text{vec} \left[ \begin{bmatrix} \sqrt{r} & 0 \\ 0 & I_r \end{bmatrix} \right] \\
\mathbf{H}\text{vec} \left[ \begin{bmatrix} 0 & v^T \\ v & 0 \end{bmatrix} \right] = \kappa \cdot \text{vec} \left[ \begin{bmatrix} 0 & v^T \\ v & 0 \end{bmatrix} \right] \quad \text{for all } v \in \mathbb{R}^r.
\]

Then, $(\delta, (1 - \delta)\mathbf{H})$ with $\delta = (1 + \sqrt{r})^{-1}$ is a feasible point for $\delta_{\text{ub}}(X, z)$ in (4.1) with $X = [0; I_r]$ and $z = [\sqrt{1 + \sqrt{r}}; 0]$. 

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Proof. We have \((1 - \delta)I \preceq (1 - \delta)H \preceq (1 + \delta)I\) because \(\kappa = (1 + \delta)/(1 - \delta)\). To verify \(J^T H e = 0\) and \(2I_r \otimes \text{mat}(He) + J^T H J \succeq 0\), we first decompose \(e\) into eigenvectors
\[
\text{mat}(e) = XX^T - ZZ^T = \begin{bmatrix} -(1 + \sqrt{r}) & 0 \\ 0 & I_r \end{bmatrix} = -\frac{1}{2\sqrt{r}} \begin{bmatrix} \sqrt{r} & 0 \\ 0 & I_r \end{bmatrix} = -\frac{1 + 2\sqrt{r}}{2\sqrt{r}} \begin{bmatrix} \sqrt{r} & 0 \\ 0 & -I_r \end{bmatrix}.
\]
Applying \(H\) yields
\[
\text{mat}(He) = -\frac{1 + 2\sqrt{r}}{2\sqrt{r}} \begin{bmatrix} \sqrt{r} & 0 \\ 0 & I_r \end{bmatrix} - \frac{1 + 2\sqrt{r}}{2\sqrt{r}} \begin{bmatrix} \sqrt{r} & 0 \\ 0 & -I_r \end{bmatrix} = \begin{bmatrix} -(1 + 2\sqrt{r}) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\kappa & 0 \\ 0 & 0 \end{bmatrix}.
\]
Now, for any \(v \in \mathbb{R}^{nr}\), write \(V \in \mathbb{R}^{n \times r}\) so that \(v = \text{vec}(V)\), and partition \(V = [v_0^T; V]\) with \(v_0 \in \mathbb{R}^r\) and \(V_1 \in \mathbb{R}^{r \times r}\), so that
\[
J_v = \text{vec}(XV^T + VX^T) = \text{vec} \left( \begin{bmatrix} 0 & v_0^T \\ v_0 & V + V_1^T \end{bmatrix} \right).
\]
We have \(JHe = 0\), since
\[
v^T JHe = \langle XV^T + VX^T, \text{mat}(He) \rangle = \left\langle \begin{bmatrix} 0 & v_0^T \\ v_0 & V + V_1^T \end{bmatrix}, \begin{bmatrix} -\kappa & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = 0,
\]
holds for all \(v \in \mathbb{R}^{nr}\). We also have \(2I_r \otimes \text{mat}(He) + J^T H J \succeq 0\), since
\[
v^T [I_r \otimes \text{mat}(He)] v = \langle \text{mat}(He), VV^T \rangle = \left\langle \begin{bmatrix} -\kappa & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} v_0^T v_0 & v_0^T V_1 \\ V_1 v_0 & V_1 V_1 \end{bmatrix} \right\rangle = -\kappa \|v_0\|^2,
\]
\[
v^T J^T H J v = \left\| H^{1/2} \text{vec} \left( \begin{bmatrix} 0 & v_0^T \\ v_0 & V + V_1^T \end{bmatrix} \right) \right\|^2 \geq \left\| H^{1/2} \text{vec} \left( \begin{bmatrix} 0 & v_0^T \\ v_0 & V + V_1^T \end{bmatrix} \right) \right\|^2 \geq 2\kappa \|v_0\|^2,
\]
and therefore \(2v^T [I_r \otimes \text{mat}(He)] v \geq -v^T J^T H J v\) holds for all \(v \in \mathbb{R}^{nr}\). \(\square\)

We proceed by generalizing Lemma 7.2 to \(r^* > 1\) and also making a change of basis.

Lemma 7.3. For \(r, r^*, n\) satisfying \(1 \leq r^* \leq r < n\), let \(u_0, u_1, u_2, \ldots, u_{n-1}\) be an orthonormal basis for \(\mathbb{R}^n\), and let \(q = r - r^* + 1\). Let \(H\) be any \(n^2 \times n^2\) matrix that satisfy \(I \preceq H \preceq \kappa I\) and the following \(r + 2\) eigenvalue equations
\[
\text{Hvec} \left( \sqrt{q} u_0 u_0^T + \sum_{i=1}^{q} u_i u_i^T \right) = \kappa \cdot \text{vec} \left( \sqrt{q} u_0 u_0^T + \sum_{i=1}^{q} u_i u_i^T \right),
\]
\[
\text{Hvec} \left( \sqrt{q} u_0 u_0^T - \sum_{i=1}^{q} u_i u_i^T \right) = 1 \cdot \text{vec} \left( \sqrt{q} u_0 u_0^T - \sum_{i=1}^{q} u_i u_i^T \right),
\]
\[
\text{Hvec} \left( u_0 u_0^T + u_i u_i^T \right) = \kappa \cdot \text{vec} \left( u_0 u_0^T + u_i u_i^T \right) \quad \text{for all} \ i \in \{1, 2, \ldots, r\},
\]
Then, \((\delta, (1 - \delta)H)\) with \(\delta = (1 + \sqrt{q})^{-1}\) is a feasible point for \(\delta_{ab}(X, Z)\) in (4.1) with
\[
Z = \xi \begin{bmatrix} u_0 & u_{q+1} & u_{q+2} & \cdots & u_r \end{bmatrix}, \quad X = \begin{bmatrix} u_1 & \cdots & u_q & \xi u_{q+1} & \xi u_{q+2} & \cdots & \xi u_r \end{bmatrix}
\]
where \(\xi = \sqrt{1 + \sqrt{q}}\).
Proof. Let \( P = [u_0, u_1, \ldots, u_{n-1}] \). We have \((1 - \delta)I \preceq (1 - \delta)H \preceq (1 + \delta)I\) because \(\kappa = (1 + \delta)/(1 - \delta)\). With the change of basis, we now have \(\text{mat}(H \xi) = -\kappa u_0 u_0^T\). For any \(v = \text{vec}(V) \in \mathbb{R}^{nr}\), we partition \(V = P[v_0^T; V_1; V_2]\) with \(v_0 \in \mathbb{R}^r\) and \(V_1 \in \mathbb{R}^{r \times r}\) and \(V_2 \in \mathbb{R}^{(n-r-1) \times r}\), so that

\[
Jv = \text{vec}(XV^T + VX^T) = \text{vec} \left( P \begin{bmatrix} 0 & v_0^T S & 0 \\ S v_0 & V_1 S + SV_1^T & SV_2^T \\ 0 & V_2 S & 0 \end{bmatrix} P^T \right)
\]

where \(S = \text{diag}(I,q,\xi_{r-q}) \succeq I\). We have \(JH\xi = 0\), since

\[
v^T JH\xi = \left\langle \begin{bmatrix} -\kappa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} v_0^T S & 0 & 0 \\ S v_0 & V_1 S + SV_1^T & SV_2^T \\ 0 & V_2 S & 0 \end{bmatrix} \right\rangle = 0,
\]

holds for all \(v \in \mathbb{R}^{nr}\). We also have \(2I_r \otimes \text{mat}(H\xi) + J^T HJ \succeq 0\), since

\[
v^T[I_r \otimes \text{mat}(H\xi)]v = \left\langle \begin{bmatrix} -\kappa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} v_0^T v_0 & v_0^T V_1 & v_0^T V_2 \\ V_1 v_0 & V_1^T V_1 & V_1^T V_2 \\ V_2 v_0 & V_2^T V_1 & V_2^T V_2 \end{bmatrix} \right\rangle = -\kappa\|v_0\|^2,
\]

\[
v^T J^T HJv = \left\| H^{1/2} \text{vec} \left( P \begin{bmatrix} 0 & v_0^T S & 0 \\ S v_0 & V_1 S + SV_1^T & SV_2^T \\ 0 & V_2 S & 0 \end{bmatrix} P^T \right) \right\|^2 \geq 2\kappa\|S v_0\|^2 \geq 2\kappa\|v_0\|^2,
\]

and therefore \(2v^T[I_r \otimes \text{mat}(H\xi)]v \geq -v^T J^T HJv\) holds for all \(v \in \mathbb{R}^{nr}\).

Now, we verify that Example 7.1 satisfies the conditions in Lemma 7.3. In turn, the correctness of Example 7.1 immediately proves Proposition 3.1.

Proof of correctness for Example 7.1. We construct the matrices \(A^{(i,j)}\) in Example 7.1 by rescaling the induced orthonormal basis \(A^{(i,j)} = \sqrt{\kappa} u_i u_j^T\). Then, we set

\[
A^{(0,0)} = \sqrt{\kappa} \cdot V_0/\|V_0\|_F, \quad A^{(1,1)} = 1 \cdot V_1/\|V_1\|_F
\]

where \(V_0\) and \(V_1\) are the two eigenvectors in Lemma 7.3.

\[
V_0 = \sqrt{q} u_0 u_0^T + \sum_{i=1}^{q} u_i u_i^T, \quad V_1 = \sqrt{q} u_0 u_0^T - \sum_{i=1}^{q} u_i u_i^T.
\]

Finally, we orthogonalize \(A^{(i,i)}\) for \(i \geq 2\) against \(A^{(0,0)}\) and \(A^{(1,1)}\) and against each other using a Gram-Schmidt process, and set \(\|A^{(i,j)}\|_F = 1\) for simplicity. This way, \(\{\text{vec}(A^{(i,j)})\}\) are precisely the orthogonal eigenvectors that diagonalize \(H\), and the fact that \(1 \leq \|A^{(i,j)}\|_F \leq \sqrt{\kappa}\) ensures that \(I \preceq H \preceq \kappa I\) as desired. The fact that \(A^{(0,j)} = \sqrt{\kappa} u_0 u_j^T\) and \(A^{(j,0)} = \sqrt{\kappa} u_j u_0^T\) then ensures that

\[
H \text{vec} \left( u_0 u_j^T + u_i u_0^T \right) = \kappa \cdot \text{vec} \left( u_0 u_i^T + u_i u_0^T \right)
\]

for all \(i\).
The fact that $A^{(0,0)}$ and $A^{(1,1)}$ align with the eigenvectors $V_0$ and $V_1$ ensures that

$$H\text{vec} \left( \sqrt{q}u_0u_0^T + \sum_{i=1}^q u_iu_i^T \right) = \kappa \cdot \text{vec} \left( \sqrt{q}u_0u_0^T + \sum_{i=1}^q u_iu_i^T \right),$$

$$H\text{vec} \left( \sqrt{q}u_0u_0^T - \sum_{i=1}^q u_iu_i^T \right) = 1 \cdot \text{vec} \left( \sqrt{q}u_0u_0^T - \sum_{i=1}^q u_iu_i^T \right).$$

Hence, all the conditions in Lemma 7.3 are satisfied, and $(\delta, (1 - \delta)H)$ is a feasible point for $\delta(X, \hat{Z})$ in (4.1). Finally, the $X, Z$ in Example 7.1 are related to $\hat{X}, \hat{Z}$ via $X = \xi^{-1}\hat{X}$ and $Z = \xi^{-1}\hat{Z}$. This rescales $e = \xi^2\hat{e}$ and $J = \xi J$, but does not affect $J^THe = \xi^3J^T\hat{H}e = 0$ and $2I_r \otimes \text{mat}(He) + J^THJ = \xi^2[2I_r \otimes \text{mat}(\hat{H}e) + J^T\hat{H}J] \succeq 0$ as desired.

8 Concluding Remarks

The main goal of this paper is to establish sufficient conditions for exact recovery with an explicit control over $r^* = \text{rank}(M^*)$. To keep our discussion concise and pedagogical, we focused on exact second-order points for the noiseless variant. But our proof can be largely repeated for approximate critical points satisfying the strict saddle condition of Ge, Huang, Jin, and Yuan [19]:

$$\|\nabla f(X)\| \leq \epsilon_g, \quad \nabla^2 f(X) \succeq -\epsilon_H I \iff \|XX^T - ZZ^T\| \leq \rho,$$

where $\epsilon_g$ and $\epsilon_H$ are thresholds for approximate second-order points, and $\rho$ is a corresponding error estimate. In principle, a precise characterization of the above can be used to derive a polynomial-time guarantee for the noisy case. (See Ge, Jin, and Zheng [21] for an exposition, and also the original proof of Bhojanapalli, Neyshabur, and Srebro [3]) Viewing this extension is an important future work for the overparameterized regime, we conclude by sketching out the details for next steps.

First, recall from Section 2 that our proof works by defining an RIP threshold function (Definition 2.1) and reformulating it exactly as a convex optimization (Theorem 2.2):

$$\delta(X, Z) \overset{\text{def}}{=} \min_{\mathcal{A}} \left\{ \delta : \begin{array}{l} \nabla f_\mathcal{A}(X) = 0, \quad \nabla^2 f_\mathcal{A}(X) \succeq 0, \\ \mathcal{A} \text{ satisfies } \delta\text{-RIP} \end{array} \right\}$$

$$= \min_{\mathcal{A}} \left\{ \delta : \begin{array}{l} \nabla f_\mathcal{A}(X) = 0, \quad \nabla^2 f_\mathcal{A}(X) \succeq 0, \\ (1 - \delta)I \preceq \mathcal{A}^T\mathcal{A} \preceq (1 + \delta)I \end{array} \right\}$$

It is easy to verify (by repeating the proof of Theorem 2.2) that the same convex reformulation remains valid if second-order optimality is only approximately satisfied. This leads to the following definition

$$\delta(X, Z, \epsilon_g, \epsilon_H) \overset{\text{def}}{=} \min_{\mathcal{A}} \left\{ \delta : \begin{array}{l} \|\nabla f_\mathcal{A}(X)\| \leq \epsilon_g, \quad \nabla^2 f_\mathcal{A}(X) \succeq -\epsilon_H I, \\ \mathcal{A} \text{ satisfies } \delta\text{-RIP} \end{array} \right\}$$

$$= \min_{\mathcal{A}} \left\{ \delta : \begin{array}{l} \|\nabla f_\mathcal{A}(X)\| \leq \epsilon_g, \quad \nabla^2 f_\mathcal{A}(X) \succeq -\epsilon_H I, \\ (1 - \delta)I \preceq \mathcal{A}^T\mathcal{A} \preceq (1 + \delta)I \end{array} \right\}$$
where $\delta(X, Z, 0, 0) = \delta(X, Z)$ by construction. Again, we repeat the same definition of the sharp threshold
\[
\delta^*(\epsilon_g, \epsilon_H, \rho) \overset{\text{def}}{=} \inf_{X, Z} \{ \delta(X, Z, \epsilon_g, \epsilon_H) : \|XX^T - ZZ^T\| > \rho, \quad \text{rank}(Z) = r^* \}.
\]
If $\mathcal{A}$ satisfies $\delta$-RIP with $\delta < \delta^*(\epsilon_g, \epsilon_H, \rho)$, then we have the strict saddle condition
\[
\|\nabla f(X)\| \leq \epsilon_g, \quad \nabla^2 f(X) \succeq -\epsilon_H I \iff \|XX^T - ZZ^T\| \leq \rho \quad (8.1)
\]
Conversely, if $\delta \geq \delta^*(\epsilon_g, \epsilon_H, \epsilon_f)$, then nothing can be said due to the existence of a counterexample.

The argument above again requires optimizing an RIP threshold over $X, Z$ with rank$(Z) = r^*$. While $\delta(X, Z, \epsilon_g, \epsilon_H)$ cannot be evaluated in closed-form, it is possible to develop a lower-bound like in Theorem 3.2 by retracing the same steps. First, repeating the proof of Lemma 5.1 for $\delta(X, Z, \epsilon_g, \epsilon_H)$ yields the following lower-bound
\[
\delta(X, Z, \epsilon_g, \epsilon_H) \geq \max_{t, y, W_{i,j}} \frac{e^T[Jy - w] - t}{1 + t} - \epsilon_g\|y\| - \epsilon_H\text{tr}(W) \quad (8.2)
\]
s.t. $\|e\||\|Jy - w\| = 1, \quad \langle J^TW, W \rangle = 2t, \quad W \succeq 0,$
where the variables are $y \in \mathbb{R}^{nr}$ and $W_{i,j} \in \mathbb{R}^{n \times r}$ for $i, j \in \{1, 2, \ldots, r\}$, and $W = [W_{i,j}]_{i,j=1}^{r,r}$ and $w = \sum_{i=1}^r \text{vec}(W_{i,i})$. Previously, in the case of $\epsilon_g = \epsilon_H = 0$, we substituted explicit choices for $y$ and $W$ into (8.2) using Lemma 5.4 and Lemma 5.5 and then optimized over $t$ using Lemma 5.2 to yield the simple lower-bound $\delta_{lb}(X, Z)$ in Theorem 3.2. Now, in the case of $\epsilon_g, \epsilon_H > 0$, we can substitute the same choices of $t, y, W$ into (8.2) to result in lower-bound like
\[
\delta(X, Z, \epsilon_g, \epsilon_H) \geq \delta_{lb}(X, Z, \epsilon_g, \epsilon_H) \overset{\text{def}}{=} \delta_{lb}(X, Z) - \epsilon_g\|y\| - \epsilon_H\text{tr}(W).
\]
Here, the margin $\epsilon_g\|y\| + \epsilon_H\text{tr}(W)$ can be understood as the “loss” associated with an approximately second-order point. Optimizing over $\delta_{lb}(X, Z, \epsilon_g, \epsilon_H)$ over all $\|XX^T - ZZ^T\| \geq \rho$ then yields a similar “loss” associated with the strict saddle condition. We expect this “loss” to reveal the mechanism of the strict saddle condition in the overparameterized case.

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A Proof of Proposition 1.4

We will prove that the function $f(X) = \|A(XX^T) - b\|^2$ has no spurious local minima when the $n \times r$ factor $X$ is sized so that $r \geq n$, as in

$$\nabla f(X) = 0, \quad \nabla^2 f(X) \succeq 0 \iff f(X) = \min_{U \in \mathbb{R}^{n \times r}} f(U).$$

Note that this claim makes no assumptions on $A$ nor $b$. In particular, it does not require RIP, nor the existence of a $Z$ such that $f(Z) = 0$.

To begin, note that a solution to the following convex optimization problem

$$M^\star = \arg \min_{M \succeq 0} \|A(M) - b\|^2$$

must satisfy the following optimality conditions

$$M^\star \succeq 0, \quad A^T(y^\star) \succeq 0, \quad y^\star = A(M^\star) - b, \quad A^T(y^\star)M^\star = 0. \quad (A.1)$$

Now, let $X$ satisfy first-order optimality, as in

$$\nabla f(X) = 2A^T(y)X = 0 \quad \text{where} \quad y = A(XX^T) - b \quad (A.2)$$

and second-order optimality

$$\langle \nabla^2 f(X)[V],V \rangle = 4\langle A^T(y),VV^T \rangle + 2\|A(XX^T + VV^T)\|^2 \geq 0 \quad (A.3)$$

holds for all $V \in \mathbb{R}^{n \times r}$.

We will show that an $X$ that satisfies $[A.2]$ and $[A.3]$ must yield $M^\star = XX^T \succeq 0$ and $y^\star = A(M^\star) - b$ that satisfy $[A.1]$. First, the condition $A^T(y^\star)XX^T = A^T(y^\star)M^\star = 0$ follows from $[A.2]$. If $X$ is rank deficient, as in $\text{rank}(X) < r$, then there exists $u \neq 0$ such that $Xu = 0$. In this case, $A^T(y^\star) \succeq 0$ follows from $[A.3]$, since

$$\langle \nabla^2 f(X)[vu^T],vu^T \rangle = 4\langle A^T(y^\star),vu^Tvu^T \rangle + 2\|A(Xvu^T + vu^TX^T)\|^2$$

$$= 4\|u\|^2v^T[A^T(y^\star)]v \geq 0$$

holds for all $v \in \mathbb{R}^n$. Finally, $X$ is full-rank as in $\text{rank}(X) = r$ only if it is square and invertible. In this case, $A^T(y^\star)X = 0$ from $[A.2]$ implies $A^T(y^\star) = 0$ and therefore $A^T(y^\star) \succeq 0$ is trivially true.
B Proof of Lemma 4.1

We wish to show that $J^T e = 0$ implies either $XX^T = ZZ^T$ or $\sigma_r(X) = 0$. We will use the QR decomposition

$$X = [Q_1 \ Q_2] \begin{bmatrix} X_1 \\ 0 \end{bmatrix} = QR \quad \text{where } QQ^T = I_n \text{ and } X_1 X_1^T > 0$$

to define $Z_1 \overset{\text{def}}{=} Q_1^T Z$ and $Z_2 \overset{\text{def}}{=} Q_2^T Z$.

Note that $J^T e = \arg \min_y \|J y - e\| = 0$ if and only if $e \perp \text{span}(J)$; this latter condition is equivalently written $J^T e = 0$. Rewrite $J^T e = 0$ as the following

$$0 = \langle XY^T + YX^T, XX^T - ZZ^T \rangle = 2\langle (XX^T - ZZ^T)X, Y \rangle \quad \text{for all } Y \in \mathbb{R}^{n \times r}$$

$$\iff (XX^T - ZZ^T)X = 0.$$}

$$\iff \begin{bmatrix} X_1 X_1^T - Z_1 Z_1^T & -Z_1 Z_2^T \\ -Z_2 Z_1^T & -Z_2 Z_2^T \end{bmatrix} \begin{bmatrix} X_1 \\ 0 \end{bmatrix} = 0$$

$$\iff (X_1 X_1^T - Z_1 Z_1^T)X_1 = 0, \quad Z_2 Z_1^T X_1 = 0$$

$$\iff X_1 X_1^T = Z_1 Z_1^T, \quad Z_2 Z_1^T = 0.$$}

If $Z_2 = 0$, then we have $XX^T = ZZ^T$. Otherwise, we have

$$Z_2 Z_1^T = 0 \iff Z_2(Z_1^T Z_1) = 0 \iff \sigma_r(Z_1) = 0.$$}

At the same time

$$X_1 X_1^T = Z_1 Z_1^T \implies \sigma_i(Z_1) = \sigma_i(X_1) = \sigma_i(X) \quad \text{for all } i$$

and so $\sigma_r(Z_1) = 0$ implies that $\sigma_r(X) = 0$.

C Proof of Lemma 4.1

We wish to show that $I - J J^\dagger = (I - XX^\dagger) \otimes (I - XX^\dagger)$. We again use the QR decomposition

$$X = [Q_1 \ Q_2] \begin{bmatrix} X_1 \\ 0 \end{bmatrix} = QR \quad \text{where } QQ^T = I_n \text{ and } X_1 X_1^T > 0$$

to define $Z_1 \overset{\text{def}}{=} Q_1^T Z$ and $Z_2 \overset{\text{def}}{=} Q_2^T Z$.

For an arbitrary $W \in \mathbb{R}^{n \times n}$, the pseudoinverse $J^\dagger \text{vec}(W)$ is solves the following

$$\min_y \|J y - \text{vec}(W)\|_F = \min_y \|XY + YY^T - W\|_F$$

$$= \min_{Y_1, Y_2} \left\| \begin{bmatrix} X_1 Y_1^T + Y_1 X_1^T & X_1 Y_2^T \\ Y_2 X_1^T & 0 \end{bmatrix} - \begin{bmatrix} Q_1^T W Q_1 & Q_1^T W Q_2 \\ Q_2^T W Q_1 & Q_2^T W Q_2 \end{bmatrix} \right\|_F$$

$$= \|Q_2^T W Q_2\|_F = \|Q_2 Q_2^T W Q_2 Q_2^T\|_F.$$}

The fact that there exists $Y_1, Y_2$ such that $X_1 Y_1^T + Y_1 X_1^T = Q_1^T W Q_1$ and $X_1 Y_2^T = Q_1^T W Q_2$ follows from $X_1 X_1^T > 0$. Hence, this implies that

$$(I - J J^\dagger) \text{vec}(W) = \text{vec}(Q_2 Q_2^T W Q_2 Q_2^T)$$

and the claim follows because $Q_2 Q_2^T = I - XX^\dagger$. 37
D Proof of Lemma 5.1

Following Theorem 2.2 and the derivations in Section 4.2, our goal is to solve the following in closed-form
\[
\delta(X, Z) = \min_{\delta, H} \left\{ \delta : \begin{array}{l} J^THe = 0, \quad 2I_r \otimes \text{mat}(He) + J^T AjA^T J \succeq 0, \\ (1 + \delta)I \preceq H \preceq (1 + \delta)I \end{array} \right\}. 
\]

Instead, we solve the following
\[
\eta^* = \max_{\eta, H} \left\{ \eta : J^T He = 0, \quad 2I_r \otimes \text{mat}(He) + J^T J \succeq 0, \quad \eta I \preceq H \preceq I \right\}, \tag{D.1a}
\]
\[
= \min_{y, U, V, W} \left\{ \text{tr}(V) : \begin{array}{l} f(y, W)e^T + ef(y, W) = U - V, \\ \text{tr}(U) = 1, \quad V, U \geq 0, \quad W \geq 0 \end{array} \right\}. \tag{D.1b}
\]

whose Lagrangian dual involves the following function
\[
f(y, W) = Jy + \sum_{i=1}^r \text{vec}(W_{i,i}) \quad \text{and} \quad W_{i,j} \text{ partitions } W = [W_{i,j}]^T_{i,j=1}.
\]

The primal is bounded and the dual is strictly feasible, so strong duality holds, and the two optimal values coincide at \( \eta^* \). This in turn yields \( \delta(X, Z) = (1 - \eta^*)/(1 + \eta^*) \).

The lower-bound in Lemma 5.1 arises from the following relaxation suggested by Zhang et al. [38]:
\[
\eta_{\text{ub}} = \max_{\eta, H} \left\{ \eta : J^T He = 0, \quad 2I_r \otimes \text{mat}(He) + J^T J \succeq 0, \quad \eta I \preceq H \preceq I \right\}, \tag{D.2a}
\]
\[
= \min_{y, U, V, W} \left\{ \text{tr}(V) + \langle J^T J, W \rangle : \begin{array}{l} f(y, W)e^T + ef(y, W) = U - V, \\ \text{tr}(U) = 1, \quad V, U \geq 0, \quad W \geq 0 \end{array} \right\}. \tag{D.2b}
\]

This generates a lower-bound \( \delta(X, Z) \geq (1 - \eta_{\text{ub}})/(1 + \eta_{\text{ub}}) \). Note that (D.2a) is a relaxation of (D.1a) because
\[
H \preceq I, \quad M + J^T J \succeq 0 \quad \implies \quad M + J^T J \succeq 0.
\]

Strong duality again holds because the primal is bounded and the dual is strictly feasible. Note that the dual problem (D.2b) has a closed-form solution over \( U \) and \( V \) [38, Lemma 13]
\[
\text{tr}([M]_-) / \text{tr}([M]_+) = \min_{t, U, V} \left\{ \text{tr}(V) : \begin{array}{l} tM = U - V, \quad t \geq 0, \\ \text{tr}(U) = 1, \quad V, U \geq 0 \end{array} \right\},
\]
and that \( \text{fe}^T + ef^T \) has exactly two nonzero eigenvalues \( e^T f \pm \|e\|\|f\| \) [38, Lemma 14]. Hence, (D.2b) can be rewritten as
\[
\eta_{\text{ub}} = \min_{W, y} \left\{ \frac{(J^T J, W) + (\|e\|\|f\| - e^T f)}{\|e\|\|f\| + e^T f} : W \succeq 0 \right\} = \min_{t \geq 0} \frac{2t + 1 - \cos \theta(t)}{1 + \cos \theta(t)},
\]
where
\[
\cos \theta(t) = \max_{W, y} \left\{ \frac{e^T f}{\|e\|\|f\|} : \frac{(J^T J, W)}{\|e\|\|f\|} = 2t, \quad W \succeq 0 \right\}
\]
coincides with the definition in (5.2). In turn, this yields the following lower-bound on \( \delta(X, Z) \)
\[
\delta_{\text{ub}} = \frac{1 - \eta_{\text{ub}}}{1 + \eta_{\text{ub}}} = \max_{t \geq 0} \frac{\cos \theta(t) - t}{1 + t},
\]
which is the trade-off function claimed in Lemma 5.1.
E Proof of Lemma 5.2

Lemma 5.3 says that
\[ \psi(t) \overset{\text{def}}{=} \begin{cases} (t/\beta) \alpha + \sqrt{1-(t/\beta)^2} \sqrt{1-\alpha^2} & t/\beta \leq \alpha, \\ 1 & t/\beta > \alpha, \end{cases} \]

satisfies \( \cos \theta(t) \geq \psi(t) \). Substituting into Lemma 5.1, we have the following sequence of inequalities
\[
\delta_{lb} \geq \max_{t \geq 0} \left\{ \frac{\psi(t) - t}{1 + t} \right\} = \max_{0 \leq t \leq \alpha} \left\{ \frac{\psi(t) - t}{1 + t} \right\} 
\]
\[
= \max_{0 \leq t \leq \alpha} \left\{ \frac{t \alpha + \sqrt{1 - \alpha^2} \sqrt{1 - t^2} - \beta t}{1 + \beta t} \right\} 
\]
\[
= \max_{0 \leq t \leq 1} \left\{ \frac{t \alpha + \sqrt{1 - \alpha^2} \sqrt{1 - t^2} - \beta t}{1 + \beta t} \right\} 
\]

Observe that the objective value is quasiconcave, since \( \sqrt{1 - t^2} \) is concave with respect to \( t \), so its superlevel sets must be convex
\[
t(\alpha - \beta) + \sqrt{1 - \alpha^2} \sqrt{1 - t^2} \geq \xi(1 + \beta t).
\]
The unconstrained objective has gradient
\[
(1 + \beta t) \left[ (\alpha - \beta) + \sqrt{1 - \alpha^2} \frac{-t}{\sqrt{1 - t^2}} \right] - \beta \left[ t(\alpha - \beta) + \sqrt{1 - \alpha^2} \sqrt{1 - t^2} \right]
\]
\[
= \frac{(\alpha - \beta) - \beta \left[ \sqrt{1 - \alpha^2} \right]}{(1 + \beta t)^2}
\]
The point \( t = 0 \) is optimal if the gradient is negative
\[
(\alpha - \beta) - \beta \left[ \sqrt{1 - \alpha^2} \right] \leq 0 \iff \frac{\alpha}{1 + \sqrt{1 - \alpha^2}} \leq \beta,
\]
and the corresponding optimal value is exactly \( \sqrt{1 - \alpha^2} \). This proves the first condition.

Now, if \( \alpha/(1 + \sqrt{1 - \alpha^2}) > \beta \), then the constraint \( t \geq 0 \) is inactive, so we relax it and solve the following
\[
\max_{|t| \leq 1} \frac{t(\alpha - \beta) + \sqrt{1 - \alpha^2} \sqrt{1 - t^2}}{1 + \beta t} = \max_t \left\{ \frac{1}{1 + \beta t} \left[ \begin{array}{c} t \\ s \end{array} \right]^T \left[ \begin{array}{c} \alpha - \beta \\ \sqrt{1 - \alpha^2} \end{array} \right] \right\} = \max_t \left\{ \left[ \begin{array}{c} t' \\ s' \end{array} \right]^T \left[ \begin{array}{c} \alpha - \beta \\ \sqrt{1 - \alpha^2} \end{array} \right] \right\} = \max_t \left\{ \left[ \begin{array}{c} t' \\ s' \end{array} \right]^T \left[ \begin{array}{c} \alpha - \beta \\ \sqrt{1 - \alpha^2} \end{array} \right] \right\} \leq 1 - \beta t'
\]

In the second to last line, we define \( t' = t/(1 + t) \) and \( s' = s/(1 + t) \) and observe that
\[
\left\| \left[ \begin{array}{c} t' \\ s' \end{array} \right] \right\| = \frac{1}{1 + \beta t} \left\| \left[ \begin{array}{c} t \\ s \end{array} \right] \right\| = \frac{1}{1 + \beta t} = 1 - \beta t = 1 - \beta t'.
\]
The dual problem satisfies strong duality because the primal problem is trivially strictly feasible

$$\min_{\lambda \geq 0} \left\{ \lambda : \left\| \begin{bmatrix} \frac{\alpha}{\sqrt{1-\alpha^2}} \\ \beta \end{bmatrix} - \begin{bmatrix} \beta \\ 0 \end{bmatrix} (1 + \lambda) \right\| \leq \lambda \right\}$$

The objective here is linear, so the constraint is satisfied at optimality. This yields the following quadratic equation, which we solve to yield

$$[\alpha - \beta(1 + \lambda)]^2 + 1 - \alpha^2 = \lambda^2$$
$$\alpha^2 - 2\alpha\beta(1 + \lambda) + \beta^2(1 + \lambda)^2 + 1 - \alpha^2 = (1 + \lambda)^2 - 2\lambda - 1$$
$$-2\alpha\beta(1 + \lambda) + \beta^2(1 + \lambda)^2 = (1 + \lambda)^2 - 2(1 + \lambda)$$
$$-2\alpha\beta + \beta^2(1 + \lambda) = (1 + \lambda) - 2$$
$$\frac{1 - 2\alpha\beta + \beta^2}{1 - \beta^2} = \lambda$$

as desired.