Implications of the Generalized Entropy Formalisms on the Newtonian Gravity and Dynamics

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Employing the Verlinde’s hypothesis, and considering two well-known generalized entropy formalisms, two modifications to the Newtonian gravity are derived. In addition, it has been shown that the generalized entropy measures may also provide theoretical basis for the Modified Newtonian Dynamics (MOND) theory and generate its modified forms. Since these entropy measures are also successful in describing the current accelerated universe, our results indicate that the origin of dark sectors of cosmos may be unified to meeting the generalized entropy measures instead of the Boltzmann-Gibbs entropy by the gravitational systems due to the long-range nature of gravity.

I. INTRODUCTION

The correspondence between the first law of thermodynamics on the boundary of spacetime and the field equations of gravity, describing the system evolution in bulk [1–4], suggests a profound connection between the laws of thermodynamics and gravity which further supports the holography proposal [5–9]. Based on the Verlinde’s hypothesis [7], the tendency of systems to increase their entropy leads to the emergence of gravity between the holographic screens. It is important to note that the entropy expression plays a crucial role in this theory. In fact, diverse corrections to the entropy-area relation presents various modifications to the gravitational theories and their corresponding cosmology [10–28]. It has also been shown [22–27] that if the Unruh temperature [29] is attributed to the holographic screen, then the quantum statistics of surface degrees of freedom may help us in obtaining a theoretical basis for the Modified Newtonian Dynamics (MOND) theory [30–32]. It is also worthy to note here that the quantum aspects of surface degrees of freedom imply that $a_0$, appeared in MOND theory, depends on the energy content of system [14, 23–27].

One property of gravity is its long-range nature which motivates physicists to use generalized entropy formalisms [33–39] in order to study various gravitational and cosmological phenomena [21, 22, 40–57]. In the generalized entropy formalisms, systems are described by the power-law distributions of probabilities ($P_i^q$) instead of the ordinary linear distribution ($P_i$) [33, 39], and the additional new free parameters, such as $q$, can be evaluated by fitting with data [33, 39]. The Tsallis generalized entropy [33, 34], an one-free parameter measure, can be combined with the Verlinde’s hypothesis [7] to obtain a MOND theory [14]. Additionally, using the Verlinde approach, it has also been shown that the power-law and logarithmic corrections to the Bekenstein entropy lead to modified versions for the MOND theory [14]. These attempts motivate us to study relations between those generalized entropies which are successful in describing cosmological and gravitational phenomena [21, 22, 40–57] and various MOND theories.

In the present Letter, by taking the various generalized entropy formalisms as well as the entropic origin of gravity into account, we are going to derive some MOND theories and the implications of these generalized entropy measures on the Newtonian gravity. We are focusing on those the generalized entropy measures successful in describing the current accelerated universe [52–57]. The paper is organized as follows. Some generalized entropy measures and the general relation between the system entropy and the gravitational force in the Verlinde approach have been shown in the next section. In sec. III, we address some MOND theories and corrected Newtonian gravities based on the generalized entropies. The last section is devoted to a summary. For the sake of simplicity, we set $G = h = c = k_B = 1$, where $k_B$ is the Boltzmann constant, throughout the article.

II. GENERALIZED ENTROPY FORMALISM, VERLINDE HYPOTHESIS AND THE GRAVITATIONAL FORCE

For a system with $W$ discrete states, where each state has probability $P_i$, Tsallis and Rényi entropies are defined as

$$S_T = \frac{1}{1-q} \sum_{i=1}^{W} (P_i^q - P_i),$$

(1)
and

$$S = \frac{1}{1-q} \ln \sum_{i=1}^{W} P_i^q, \quad (2)$$

respectively, where $q$ is a free parameter. Using the fact that $\sum_i P_i = 1$, we can combine these equations to arrive at

$$(1-q)S + 1 = e^{(1-q)S}, \quad (3)$$

which finally leads to

$$S = \frac{1}{\delta} \ln(1 + \delta S_T), \quad (4)$$

where we have used the $\delta = 1 - q$ expression. There is also a two-parametric generalized entropy which is called the Sharma-Mittal entropy and is written as

$$S_{SM} = \frac{1}{\alpha} ((1 + \delta S_T)^\alpha - 1), \quad (5)$$

where $\alpha = 1 - r$, and $r$ is a new free parameter. Some cosmic applications of $S$ and $S_{SM}$ can be found in Refs. [41, 53]. In general, the free parameters $r$ and $q$ should be evaluated by comparing the theory with the observations [39], meaning that the free parameters are not the same for all systems. This is in full agreement with gravitational and cosmological studies [22, 50, 51].

In addition, we assume that the system has a boundary with area $A$, and consists $N$ degrees of freedom which satisfy the energy equipartition law [17, 52]

$$E = M = \frac{NT}{2}. \quad (6)$$

Here, $T$ is the boundary temperature and $M$ denotes the mass content of the system. Moreover, in our unit, $A$ and $N$ are in a mutual relation as

$$A = N, \quad (7)$$

claiming that the area change per unit change of information is one (or equally $\Delta A = \Delta N = 1$) [7, 12]. It was argued that the Bekenstein’s entropy expression ($S_B = A/4$), the entropy of a system with boundary $A$ [48], is a proper candidate for $S_T$ [41, 42, 53], a result confirmed by using the Tsallis formalism to evaluate the black hole entropy in loop quantum gravity [49]. Thus, replacing $S_B$ with $S_T$ in the above generalized entropy measures, one can easily find the entropy-area relation in generalized entropy formalism [52, 54].

Based on the entropic force scenario, the absolute value of gravitational force applied from a source $M$ to the test particle $m$ located at the distance $\Delta x = \eta \lambda_m$ from the holographic screen of radius $R$ covering $M$, is evaluated as [7, 12]

$$F = T \frac{\Delta S}{\Delta x}. \quad (8)$$

Now, since $\frac{\Delta S}{\Delta x} = \frac{\Delta S}{\Delta A} \frac{\Delta A}{\Delta x} = \frac{1}{\eta \lambda_m} \frac{dS}{dA} [7, 13]$, using Eqs. [6] and [7], this equation can be written as

$$F = \left( \frac{1}{2\pi \eta} \right) \frac{Mm}{R^2} \frac{dS}{dA}, \quad (9)$$

where $A = 4\pi R^2$. We also used the Compton wavelength expression $\lambda_m = 1/m$, as well as Eq. [7] to obtain Eq. [9].

III. POSSIBLE MOND THEORIES

As we mentioned, the Rényi and Sharma-Mittal entropies can provide suitable description for the current accelerated universe and thus dark energy [53, 54]. Here, using the introduced generalized entropy formalisms, Eq. [3], the $S_B = S_T$ relation [41], and Eq. [9], we are going to get the various MOND theories allowed by employing the Rényi and Sharma-Mittal entropies to the system.

A. Rényi Entropy

Now, inserting Eq. [4] into equation [4], one can easily obtain

$$F = \left( \frac{1}{8\pi \eta} \right) \frac{Mm}{R^2} \frac{1}{\delta A/4 + 1}. \quad (10)$$

Since we set $G = 1$, we should have $F = \frac{Mm}{R^2}$ at the $\delta = 0$ limit leading to $8\pi \eta = 1$ in full agreement with Eq. [12]. Finally, defining $A_0 \equiv \delta \pi \lambda$ and $a \equiv M/R^2$, one can rewrite Eq. [10] as

$$F = \left( \frac{1}{1 + A_0/a} \right) ma = f(a)ma, \quad (11)$$

which recovers the Newtonian gravity at the appropriate limit of $\delta \to 0$ (or equally $A_0 \to 0$). In addition, for $a \gg A_0$ ($a \ll A_0$), we have $F \simeq ma = Mm/R^2$ ($F \simeq m^2 a^2 = Mm/\pi \delta R^4$) meaning that the Modified Newtonian force obtained in Eq. [11] has similarities with a MOND theory with simple interpolating function $\mu \left( \frac{a}{a_0} \right) = 1/(1 + A_0/a)$.

From the phenomenological point of view, following [59, 60] and working with the accelerations, one can introduce vectors and divides Eq. [11] as
\[ \vec{a} = \vec{a}_n + \vec{a}_{nn}, \]  

(12)

where the total acceleration \( \vec{a} \) is given by the ordinary Newtonian acceleration \( \vec{a}_n \) plus the acceleration \( \vec{a}_{nn} \) which is due to the non-Newtonian force. Now, taking the square of Eq. (12) one gets [59]:

\[ \vec{a}_{nn} \cdot \vec{a}_n = \frac{1}{2} (a^2 - a_n^2 - a_{nn}^2). \]  

(13)

In this equation the dot represents the three-dimensional scalar product. Equation (13) is a general relation which expresses the unknown vector \( \vec{a}_{nn} \) in terms of the total acceleration \( \vec{a} \), of the acceleration due to the non-Newtonian force \( \vec{a}_{nn} \) and of the magnitudes \( a^2, a_n^2 \) and \( a_{nn}^2 \). From Eq. (13), one obtains the acceleration \( \vec{a}_n \) as being [59]

\[ \vec{a}_n = \frac{1}{2} (a^2 - a_n^2 - a_{nn}^2) \frac{\vec{d}}{a_{nn} a} + \vec{b} \times \vec{a}_{nn}, \]  

(14)

where \( \vec{b} \) is is an arbitrary vector perpendicular to the acceleration \( \vec{a}_{nn} \). For the sake of simplicity, one assumes \( \vec{b} = 0 \) [59]. As one wants the mathematical consistency of Eq. (14), one needs \( a_{nn} \cdot a \neq 0 \). In other words, the accelerations \( a_{nn} \) and \( a \) cannot be orthogonal to each other. We will assume that both of them are parallel. Thus, Eq. (14) becomes

\[ \vec{a}_n = \frac{1}{2} (a^2 - a_n^2 - a_{nn}^2) \frac{\vec{d}}{a_{nn} a}, \]  

(15)

Assuming that \( \vec{a}_{ng} \) dominates, which means \( a_n \ll a \), one gets

\[ a_n \approx \frac{a \vec{d}}{2a_{nn}} (1 - \frac{a_{nn}^2}{a^2}). \]  

(16)

If one defines [59, 60]

\[ a_e^{-1} \equiv \frac{1}{2a_{nn}} (1 - \frac{a_{nn}^2}{a^2}), \]  

(17)

then Eq. (16) becomes

\[ \vec{a}_n \approx \frac{a}{a_e} \vec{d}, \]  

(18)

which can be combined with Eq. (13) to obtain

\[ a \approx (a_e a_n)^{\frac{1}{2}}. \]  

(19)

We now recall that the standard Newtonian acceleration is

\[ \vec{a}_n = \frac{M}{r^2} \hat{a}_r, \]  

(20)

helping us in writing the total acceleration as

\[ \vec{a} = (a_e M)^{\frac{1}{2}} \hat{a}_r = \frac{v_r^2}{r} \hat{a}_r, \]  

(21)

where

\[ v_r = (a_e M)^{\frac{1}{2}}, \]  

(22)

is the rotation velocity of a test mass under the influence of the non-Newtonian force. By applying our analysis to the galaxies rotation curves, one can identify in a natural way \( a_e \) with \( a_0 \approx 10^{-10} m/s^2 \), which is analogous to the Milgrom’s MOND acceleration [30–32].

It has been argued that since the holographic screen, assumed in previous section, can be considered as the system boundary [7, 23, 24], one can attribute Unruh temperature [20]

\[ T = \frac{a}{2 \pi}, \]  

(23)

to this boundary [7, 10, 11, 23, 24, 47]. Combining this equation with Eqs. (6) and (7), we get

\[ A = \frac{4 \pi E}{a}, \]  

(24)

as a relation between the surface area of holographic screen and the acceleration of test particle. This relation indicates that \( A \) is decreased by increasing \( a \). It is a true result because attracting the test particle by the source, their mutual interval and thus the holographic screen are shrinking.

Now, bearing the \( \frac{dS}{dT} = \frac{1}{\eta_0} \frac{dS}{dA} \) and \( \lambda_m = \frac{1}{\eta_0} \) relations in mind, by inserting Eqs. (24) and (24) into Eq. (8) and using Eq. (14), one can find

\[ F = \frac{1}{8 \pi \eta} \frac{ma}{1 + \frac{m}{a}}, \]  

(25)

in which

\[ a_0 = \delta \pi M. \]  

(26)

It is obvious that the Newtonian dynamics is obtainable at the appropriate limit \( \delta = 0 \) (\( a_0 = 0 \)), whenever \( 8 \pi \eta = 1 \), a result in full accordance with what is obtained in Eq. (10). Therefore, in order to obtain a MOND theory with simple interpolating function \( \mu \left( \frac{m}{a} \right) = 1/(1 + a_0/a) \), we should have \( \eta = 1/8 \pi \) which
leads to \( a_0 = \delta \pi M \). Thus \( a_0 = A_0 \), and the amount of \( a_0 \) depends on both of \( \delta \) and the mass content of source.

It is worth to mention here that the linear dependency of \( a_0 \) to the mass (energy) content of system, obtained in Eq. (20), is in full agreement with the results of applying quantum statistics to the surface degrees of freedom [14, 20, 27]. This result also says that \( \delta \) should be evaluated by comparing the observation with Eq. (26), and its value is not necessarily the same for all the galaxies. For example, if \( a_0 \approx 10^{-1} m/s^2 \) [24, 30, 32], then we can use Eq. (26) in order to find \( \delta \) (or equally \( q \)) for a system with mass \( M \). It is in agreement with the spirit of generalized entropy formalism in which the values of free parameters should be evaluated by fitting the theory with the observations [33, 34].

B. Sharma-Mitall Entropy

Employing the \( S_B = S_T = A \) relation [49] and Eq. (7), one can rewrite Eq. (5) as

\[
S_{SM} = \frac{1}{\alpha} \left( (1 + \frac{\delta A}{4})^\frac{1}{\alpha} - 1 \right). \tag{27}
\]

It is the generalization of both the Tsallis and Rényi entropy measures [33] which reduces to the Bekenstein entropy when \( \alpha = \delta \) [55]. Now, following the approaches led to Eqs. (10) and (25), we reach

\[
F = \left( \frac{Mm}{R^2} \right) \left[ \frac{1}{\frac{\delta A}{4} + 1} \right]^{1 - \frac{1}{\alpha}}, \tag{28}
\]

and

\[
F = ma \left[ \frac{1}{\frac{\delta a}{4} + 1} \right]^{1 - \frac{1}{\alpha}}, \tag{29}
\]

respectively, where \( a_0 \) meets Eq. (20). In fact, since the Newtonian gravity and dynamics are obtainable whenever the Bekenstein entropy is considered [14], our results should reduce to those of the Newton at the \( \alpha = \delta \) limit. This expectation is obeyed whenever \( \alpha = \delta \) leading to \( S_{SM} = S_B \) [53] and \( 8\pi\eta = 1 \) used to obtain the above equations.

Let us focus on the last equation indeed a MOND-like theory with the interpolating function \( \xi \left( \frac{a_0}{a} \right) = \left\{ \begin{array}{ll} 1, & a \gg a_0 \text{ (independent of the values of } \delta \text{ and } \alpha), \\ \left( \frac{a_0}{a} \right)^{\frac{1}{\alpha}} & \text{for } \frac{a_0}{a} < 1, \\ \left( \frac{a_0}{a} \right)^{\frac{1}{\alpha} - 1} & \text{for } \frac{a_0}{a} > 1 \end{array} \right. \tag{30} \)

which finally leads to

\[
F = \left\{ \begin{array}{ll} ma, & a \gg a_0 \text{ (independent of the values of } \delta \text{ and } \alpha), \\ ma \left( \frac{a_0}{a} \right)^{\frac{1}{\alpha}} & \text{for } \frac{a_0}{a} < 1, \\ ma \left( \frac{a_0}{a} \right)^{\frac{1}{\alpha} - 1} & \text{for } \frac{a_0}{a} > 1 \end{array} \right. \tag{31}
\]

for the force felt by a particle with mass \( m \) and acceleration \( a \). Therefore, the Sharma-Mitall entropy modifies the MOND theory as Eq. (31). Similar modifications to the MOND theory have also been obtained by using the logarithmic and power-law corrections of entropy [14].

IV. SUMMARY

Bearing the entropic force scenario in mind, we studied some consequences of applying two generalized entropy formalisms, successful in describing the current accelerated universe [53–57], to the gravitational systems. It was shown that the Rényi entropy modifies the Newtonian gravity. Moreover, we also found out that this entropy measure can provide a theoretical basis for the MOND theory with the simple interpolating function \( \mu \left( \frac{a_0}{a} \right) = \frac{1}{1 + a_0/a} \). The Sharma-Mitall entropy has also been studied showing that this entropy can modify both the Newtonian gravity and the MOND theory. Our results express that, the same as the dark energy origin [53–57], the nature of MOND theory may be attributed to the long-range aspect of gravity which may force the gravitational systems to obey the generalized entropy formalisms instead of the ordinary Boltzmann-Gibbs entropy.

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