Semi-classical string solutions for $\mathcal{N}=1$ SYM

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Abstract

We study semi-classically the dynamics of string solitons in the Maldacena-Nuñez background, dual in the infra-red to $\mathcal{N}=1$, $d=4$ SYM. For closed string configurations rotating in the $S^2 \times \mathbb{R}$ space wrapped by the stack of $N$ D-branes we find a behaviour that indicates the decoupling of the stringy Kaluza-Klein modes with sufficiently large R-symmetry charge. We show that the spectrum of a pulsating string configuration in $S^2$ coincides with that of a $\mathcal{N}=2$ super Sine-Gordon model. Closed string configurations spinning in the transversal $S^3$ give a relation of the energy and the conserved angular momentum identical to that obtained for configurations spinning in the $S^5$ of the $AdS_5 \times S^5$, dual to $\mathcal{N}=4$ SYM. In order to obtain non-trivial relations between the energy and the spin, we also consider conical-like configurations stretching along a radial variable in the unwrapped directions of the system of D-branes and simultaneously along the transversal direction. We find that in this precise case, these configurations are unstable –contrary to other backgrounds, where we show that they are stable. We point out that in the Poincaré-like coordinates used for the Maldacena-Nuñez background it seems that it is not possible to reproduce the well-known field theory relation between the energy and the angular momentum. We reach a similar conclusion for the Klebanov-Strassler background, by showing that the conical-like configurations are also unstable.

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1 Motivation

The AdS/CFT correspondence constitutes a framework where string theory and gauge field theories have married successfully. The conjecture \[1\]-\[3\] involves the equivalence between string theory on the bulk of a curved background, $\text{AdS}_5 \times S^5$, and a gauge theory defined on the boundary, $\mathcal{N}=4 \text{ SU}(N)$ super Yang-Mills theory (SYM). The duality has been tested in the supergravity approximation, where the curvature $\mathcal{R}$ is small and $\alpha'\mathcal{R}$ can be neglected. However, any direct relation between string theory and gauge field theory is extremely difficult to obtain. In fact, achieving the weak coupling regime in the string sigma-model becomes a strong coupling regime in the field theory side, where nowadays our knowledge is restricted to lattice calculations.

One way to overcome the restriction due to the supergravity approximation, is to find a suitable limit \[4\] where string theory becomes fully solvable \[5\] on NS or RR backgrounds \[6\]. Once this limit is achieved in the string theory side one can indeed apply the AdS/CFT arguments and look for the corresponding dual field theory states. This comparison is a overwhelming task, that in the field theory side is equivalent to solving large-$N$ SYM. It has become clear only recently that for certain classes of operators the comparison is still possible without summing all the $1/N$ series. These correspond to states having large quantum numbers such as R-charge or spin \[7\]. In addition, in string theory these states are stationary and semi-classical \[8\]. Moreover, they do not encode directly any reference to the strength of the interaction and therefore they can be used all the way up to the perturbative regime in the field theory side. In particular, for large rotating strings in $\text{AdS}_5$ one finds the dispersion relation

$$E - S = \frac{\sqrt{\lambda_{\text{AdS}_5}}}{\pi} \ln \left( \frac{S}{\sqrt{\lambda_{\text{AdS}_5}}} \right), \quad \lambda_{\text{AdS}_5} = g_{\text{YM}}^2 N, \quad (1.1)$$

for the energy and the spin. Due to conformal invariance $E = \Delta$, with $\Delta$ being the anomalous dimension on $\mathbb{R}^4$. The resemblance with the perturbative result for the anomalous dimension of twist-two operators in the conformal gauge theory is striking \[9\]

$$\Delta - S = f(\lambda) \ln S. \quad (1.2)$$

The same semi-classical analysis as presented in \[8\] has been applied to more general cases such as non-supersymmetric or non-conformal theories \[10\]-\[16\] in the hope to gain some understanding on these theories. However, it is not clear how to properly generalise \(1.1\) because, due to mixing, there is no one-to-one map between states and operators in non-conformal theories. There has been recent proposals suggesting as solutions of \(1.2\) non-periodic solitons in time-dependent backgrounds \[17\]. It is our aim to generalise in the same spirit as in \[8\] one of the most relevant $\mathcal{N}=1$ SYM model \[18\]. We shall bear in mind \(1.1\) and search for a similar relation. It has been checked that, in the superconformal case $\mathcal{N}=4$, \(1.1\) is protected under renormalisation, in the sense that quantum corrections to \(1.1\) proportional to $\ln^2(\sqrt{\lambda})$ cancel out, (see \[19\] for the string calculation) at least at one-loop level (see also \[20\] for a field theory analogue with R-charge instead of spin).
We shall explore different settings for classical closed string configurations, stable in the background of [18], and study relationships between the conserved quantities that arise, like energy, spin or other quantum numbers. There are still many open points that deserve clarification, the main one being the exact string/gauge field duality realisation. Much work must be done before the fogginess of the present landscape fully dissipates. Nevertheless, and despite the fact that the dictionary is not yet ready, we think it is worth to explore one side of the correspondence –the stringy side– with the view that our results must bear some hints on the behaviour of the dual field theory, yet to be determined.

The paper is organised as follows. In section 2 we briefly review the $\mathcal{N}=1$ background of interest, stressing its main properties and mainly the role of the Kaluza-Klein (KK) states. We shall closely follow [18] in the slightly adapted notation of [21]. In the next sections we explore all the relevant closed strings configurations: In section 3 we consider strings spinning in the $S^2$ and stretching along the transversal direction, whereas in section 4 we study oscillating configurations along the equator of the $S^2$. In section 5 the strings are spinning in the transversal $S^3$. Conical configurations stretching along a radial variable in $\mathbb{R}^3$ and also along the transversal direction are considered in section 6. We devote section 7 to summarising our results. Comments on the application of the variational principle for the Nambu-Goto action describing folded strings are gathered in the appendix A.

## 2 $\mathcal{N}=1$ Background

We shall consider mainly the semi-classical analysis in the background produced by a stack of $N$ D5-branes located at the origin of the transversal coordinate $\rho$, [22], and partially wrapping a supersymmetric cycle inside a Calabi-Yau three-fold. The unwrapped sector of the world-volume contains the field theory in four dimensions, in the infrared $\mathcal{N}=1$ SYM. A certain amount of supersymmetry, namely four supercharges, is kept after the partial twist. It is also assumed that in this procedure some world-volume fields become massive enough to decouple.

In the string frame the relevant fields are given by the metric

$$ds_{10}^2 = e^{\Phi_D} \left[ dx_{0,3}^2 + N \alpha' g_s \left( d\rho^2 + e^{2g(\rho)} d\Omega_2^2 + \frac{1}{4} (\omega^a - A^a)^2 \right) \right],$$

the dilaton field

$$e^{2\Phi_D} = \frac{\sinh 2\rho}{2 e^{g(\rho)}},$$

and the Ramond-Ramond three-form

$$F[3] = dC[2] = \frac{Ng_s}{4} \left[ -(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \sum_{a=1}^3 F^a \wedge (w^a - A^a) \right].$$

We have introduced the $SU(2)_{L}$ gauge field

$$A = \frac{1}{2} \left( \sigma^1 a(\rho) d\theta + \sigma^2 a(\rho) \sin \theta d\varphi + \sigma^3 \cos \theta d\varphi \right),$$

For sake of clarity, we shall work in string units and only restore factors eventually.
together with its field strength $F^a$, and the left-invariant one-form parameterising the tree-sphere

$$\omega^1 + i\omega^2 = e^{-i\psi} \left( d\tilde{\theta} + i\sin \tilde{\theta} d\phi \right),$$
$$\omega^2 = d\psi + \cos \tilde{\theta} d\phi. \quad (2.7)$$

In addition we have also made use of two functions of the radial variable $\rho$, which are given by

$$e^{2\varrho(\rho)} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \quad a(\rho) = \frac{2\rho}{\sinh 2\rho}. \quad (2.8)$$

A good description of the four dimensional field theory is obtained from the D5-brane in a regime of energies where higher stringy modes as well as the Kaluza-Klein states on the $S^2$ decouple. It is expected that the back-reaction of the D-brane deforms the initial Calabi-Yau space [23] essentially by shrinking the $S^2$ and blowing-up the $S^3$. It is then mandatory to trace back the role played by the point-like Kaluza-Klein modes in (2.3). The validity of the supergravity approximation to string theory relies on three conditions:

i) The smallness of the scalar field (2.4). This restricts the reliability of (2.3) to the infrared region. Surprisingly enough, if one pushes the theory beyond this regime and calculates the $\beta$-function, one obtains agreement with the perturbative expression and can even predict non-perturbative contributions due to fractional instantons [21, 24].

ii) The smallness of the curvature. This implies that its maximum value (attained at the origin) must be bounded

$$|R| \leq \frac{32}{3} \frac{1}{N g_s} \ll 1. \quad (2.9)$$

iii) The effective four-dimensional field theory should not be sensible to the massive Kaluza-Klein modes and therefore their masses should be heavy enough in order to decouple. This implies the condition

$$N g_s \ll 1. \quad (2.10)$$

Conditions (2.9), (2.10) are incompatible and one is forced to give up one of them. In section 3 we shall see how this claim can be modified once we consider solitonic string configurations in the supergravity background.

The main expectations for the dual field theory, as predicted from the supergravity side, (2.3)–(2.6), are confinement and the correct $\beta$-function while the most relevant failure is the contamination with Kaluza-Klein modes which can not decouple in the deep infrared.

We can also S-dualise the gravity solution and switch to a NS5-brane description. The field theory side is now replaced by a little string theory. The metric S-dual to that in (2.3) is [18]

$$ds^2_{10} = dx^2_0 + N\alpha' g_s \left( d\rho^2 + e^{2\varrho(\rho)} d\Omega^2_2 + \frac{1}{4} (\omega^a - A^a)^2 \right), \quad (2.11)$$

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2It is worth noting that the non-commutative version of this model [25] has a region of parameters where the decoupling of the massive KK modes is compatible with the small curvature condition.
where the dilaton behaves now as

\[ e^{2\Phi} = \frac{2e^{g(\rho)}}{\sinh 2\rho}, \quad (2.12) \]

and the field strength \( F_3 \) is unchanged.

### 3 String rotating in \( S^2 \times \mathbb{R} \)

We shall consider in this section a spinning string in \( S^2 \times \mathbb{R} \), where \( \mathbb{R} \) stands for the \( \rho \) variable. From the point of view of the field content this corresponds to the Kaluza-Klein modes associated with the \( S^2 \), namely the stringy Kaluza-Klein states in contradistinction to the supergravity Kaluza-Klein modes (or point-like configurations, see section 2). In performing a semi-classical analysis the hope is to gain some insight in their effects and substantiate their possible decoupling beyond the supergravity approximation.

We shall place a string with fixed coordinates in the equator of the \( S^3 \) and in the flat space-time, which is isotropic. With the remaining variables we consider the classical string configurations

\[ t = e^\tau, \quad \varphi = e\omega\tau, \quad \theta(\sigma), \quad \rho(\sigma), \quad (3.1) \]

with the string rotating around its centre of mass located at \( \rho = 0 \). Notice that due to the twisting the \( \text{SO}(3) \) isometry in (2.3) is broken down to \( \text{U}(1) \), thus allowing to interpret the global charge, \( J \), defined by the rotating configuration (3.1) as the one corresponding to the R-symmetry of \( \mathcal{N}=1 \) SYM. More specifically we are interested in the relation \( E = E(J) \), where \( E \) is just the system energy. Obviously, instead of (3.1) one can also impose interpolating configurations involving rotation on both the \( S^2 \) and the spatial directions of the unwrapped part of the worldvolume, \( \mathbb{R}^3 \) [26]. Besides the R-symmetry, this will involve the spin. But this will have little to add to the present discussion on the Kaluza-Klein modes.

In the Nambu-Goto action we shall choose a gauge where \( e = 1 \). The simplest configuration, with \( \theta \) constant, is only stable for \( \theta = \frac{\pi}{2} \), and the remaining gauge freedom is absorbed in the variable \( \rho \). The Lagrangian is thus given by

\[ L = -4\sqrt{\lambda} \int_0^{\rho_0} d\rho e^{\Phi(\rho)} \sqrt{1 - \lambda \beta(\rho)(\dot{\varphi})^2}, \quad \lambda = N g_s, \quad (3.2) \]

where

\[ \beta(\rho) \equiv e^{2g(\rho)} + \frac{1}{4} a^2(\rho), \quad (3.3) \]

and \( \dot{\varphi} \equiv d\varphi/d\tau \). The factor of 4 in (3.2) arises because of the folding of the string. The string turning point, \( \rho_0 \), can be found by applying the variational principle to the Lagrangian (3.2) (see appendix 4), and gives the extreme solution for the positivity in the square root

\[ 1 - \lambda \omega^2 \beta(\rho_0) = 0, \quad (3.4) \]
which in turn will guarantee a real energy and R-charge. Notice in particular that at $\rho = \rho_0$, the turning point of the folded string, the derivative of $\rho$ with respect to $\sigma$ should vanish. The energy and the R-charge are directly derived from (3.2),

\[
E = \frac{4\sqrt{\lambda}}{2\pi} \int_{0}^{\rho_0} d\rho \frac{e^{\Phi D}}{\sqrt{1 - \lambda \omega^2 \beta(\rho)}},
\]

\[
\omega J = \frac{4\sqrt{\lambda}}{2\pi} \int_{0}^{\rho_0} d\rho \frac{\lambda \omega^2 \beta(\rho) e^{\Phi D}}{\sqrt{1 - \lambda \omega^2 \beta(\rho)}}.
\] (3.5)

It is worth re-discussing the above expressions in the conformal gauge. First of all because it clarifies whether a particular ansatz fulfils all the requirements imposed by the equations of motion. This can certainly be traced back in the Nambu-Goto action but is somewhat cumbersome. Secondly because it will be a good check of the restriction imposed in the integration range of the $\rho$ variable. The world-sheet action is in this case

\[
S = -\frac{1}{4\pi} \int d\tau d\sigma G_{ij} \partial_\alpha X^i \partial^\alpha X^j,
\] (3.6)

that must be supplemented with the conditions

\[
T_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^j G_{ij} - \frac{1}{2} \eta_{\alpha\beta} (\eta^{\gamma\delta} \partial_\gamma X^i \partial_\delta X^j G_{ij}) = 0.
\] (3.7)

Notice that the components of the energy-momentum tensor $T_{\alpha\beta}$ are constants of motion for the conformal gauge action (3.6) \(^3\). It is also worth mentioning that the choice $\theta = \pi/2$ we picked out in the Nambu-Goto action simply arises in the conformal gauge from the equation of motion for $\theta$. For the string configuration depicted in (3.1) and $\theta = \pi/2$, the contents of these constraints becomes the single relationship

\[
\lambda(\rho')^2 = e^2 (1 - \lambda \omega^2 \beta(\rho)) ,
\] (3.8)

that reduces to (3.4) once we set $\rho'|_{\rho = \rho_0} = 0$. Notice that this identifies the turning point obtained above within the Nambu-Goto approach. Hereafter primes stand for derivatives with respect to $\sigma$. The constant $e$ is adjusted in order to get a period of $2\pi$ in the function $\rho(\sigma)$

\[
d\sigma = d\rho \frac{\sqrt{\lambda}}{e \sqrt{1 - \lambda \omega^2 \beta(\rho)}}.
\] (3.9)

Using (3.6) and (3.1) one obtains the space-time energy and spin in terms of $\sigma$

\[
E = \frac{1}{2\pi} e \int d\sigma e^{\Phi D},
\]

\[
\omega J = \frac{\lambda}{2\pi} e \omega^2 \int d\sigma e^{\Phi D} \beta(\rho),
\] (3.10)

\(^3\)They must vanish because they were constraints for the Polyakov action before implementing any gauge fixing (the Lagrangian in the conformal gauge (3.6) is regular and therefore has no constraints of its own).
that with the use of (3.8) reduce to the previous expressions (3.5). 4

To get some elementary information we analyse in particular the limiting cases of long and short strings.

**Long strings:** We shall first discuss long closed strings [8]. We expect that the string feels the $S^2$ curvature and hence a change in the $E(J)$ relation with respect to the flat case would be induced. In the ultraviolet (3.3) can be approximated by $\beta(\rho) \approx \rho$, and the dilaton term by $e^{\Phi_D} \approx \rho^{1/2} e^{\rho}$. This would suffice in the present situation to display non-analytical terms. The situation corresponds, from (3.8), to approach $\omega \to 0^+$, in contrast to the AdS$_5$ case, $\omega \to 1^-$. Hence $\rho_0$ becomes large. Within this approximation (3.10) can be expressed in terms of hypergeometric functions and reduces to

$$E - \omega J = \frac{\sqrt{\lambda} \sqrt{\pi} \Gamma(\frac{3}{4})}{\pi} \rho_0^{3/4} \text{$_1F_1$}(\frac{3}{4}, \frac{9}{4}; \rho_0) \rho_0^{1/2} e^{\rho_0},$$

$$\omega J = \frac{\sqrt{\lambda} \sqrt{\pi} \Gamma(\frac{7}{4})}{\pi} \rho_0^{3/4} \text{$_1F_1$}(\frac{7}{4}, \frac{9}{4}; \rho_0) \rho_0^{1/4} e^{\rho_0},$$

(3.11)

where in the last step we have just kept the leading contribution in the $\text{$_1F_1$}(a, b; \rho_0)$ function as $\rho_0 \to \infty$. Both operators, $E$ and $J$ diverge with $\rho$ while, similar to the AdS$_5$ case, the ratio $E/\omega J$ remains finite. In this approximation we may use, in string units,

$$\omega^2 = \frac{1}{\lambda \rho_0} = \frac{1}{R^2 \rho_0}, \quad R^2 = \lambda \alpha' = N \alpha' g_s,$$

and we have the leading terms for $E$, $\omega J$ and $E - \omega J$,

$$E \approx \omega J = \frac{R}{\sqrt{\pi}} \rho_0^{1/4} e^{\rho_0}, \quad E - \omega J = \frac{1}{2} \frac{R}{\sqrt{\pi}} \rho_0^{-3/4} e^{\rho_0}.$$

Keeping the next-to-leading order in $\rho_0$ and since $\rho_0 \approx \log J$ for large $\rho_0$, we can write this expression up to the subleading term as

$$E = \frac{J}{R \log^{1/2} J} + \frac{1}{2} \frac{J}{R \log^{3/2} J} + \ldots.$$

(3.12)

where the ellipsis stands for higher terms in $R$. The dependence on the R-charge $J$ in this last expression shows that, keeping ourselves in the supergravity approximation, as long as $J$ is sufficiently large whilst $R$ is keep fixed, the energy of the KK stringy modes increases and their effects decouple. The content of (3.12), found in the string theory side, has no direct reference to $g_s$ (only to the product $N g_s$) and its validity is beyond the weak coupling regime. Furthermore it still holds at strong coupling and therefore, even if to the best of our knowledge it is not known, it should have an analogous expression in the perturbative gauge field context, as (1.1) has its equivalent (1.2) [8].

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4Since the parameter $e$ is usually different for unity in the conformal gauge, to get an interpretation of (3.10) as space-time quantities, one must take the Lagrangian such that the action is expressed as $S = \int dt L$, with $t$ instead of $\tau$. 
It is distressing that a rescaling of $R$ yielding (for fixed $g_s$) $N \to \infty$ can change drastically the aforementioned pattern concerning the KK decoupling. The $R \to \infty$ limit can be achieved, for instance, by transforming the (2.3) geometry into a parallel plane-wave space-time [27].

Let us inspect which are the consequences as far as the KK decoupling is concerning by going to the parameter corner

$$R \to \infty, J \to \infty, \quad (3.13)$$

with $\frac{J}{R}$ fixed but otherwise free, either large or small. By virtue of (3.13), and bearing in mind that we have just argued that (3.12) holds at any $g_s$, this implies $N \to \infty$. The value of $\lambda_{t'\text{Hooft}} = g_{YM}^2 N$ can be read directly from the behaviour of the $\beta$-function [28, 29]

$$\frac{1}{\lambda_{t'\text{Hooft}}} \sim \frac{g_s}{4\pi^2\rho^2} \text{ for } \rho \to 0, \quad \frac{1}{\lambda_{t'\text{Hooft}}} \sim \frac{g_s}{4\pi^2\rho} \text{ for } \rho \to \infty, \quad (3.14)$$

diverging in the infrared and vanishing in the ultraviolet. In the last case $g_{YM}^2$ tends to zero sufficiently fast, whilst $N$ is taken large, ensuring the applicability of perturbation theory in the field theory dual. While the former implies a slow decreasing of $g_{YM}^2$ when approaching the infra-red region and the field theory dual becomes strongly coupled. Thus, after the double scaling (3.13) we have neither recovered after all the t’Hooft limit, i.e. $N \to \infty, g_{YM}^2 \to 0$ with $\lambda_{t'\text{Hooft}} = g_{YM}^2 N$ fixed nor the superconformal case equivalent of (3.13), BMN limit, where [7]

$$R \to \infty, J \to \infty, \quad \text{with } J^2 \frac{1}{R_{AdS_5 \times S^5}} \text{ and } g_{YM}^2 \text{ fixed } \quad (3.15)$$

with the only restriction $g_{YM}^2 \ll 1$ but otherwise not vanishing. It is evident that this last large-$N$ limit is different from the usual t’Hooft limit. One can argue that even if we have previously found a decoupling of the KK modes, it is not so obvious that this holds true in all the parameter space $(R, J)$, see the scaling (3.13). However, one can not rule out a priori that a scaling of the type $J > R$ will not still make plausible a decoupling of the Kaluza-Klein modes even in this corner, $R \to \infty$.

One can argue that the description of the D5 brane background in the ultraviolet region is not suitable for physical purposes because the blowing up of the dilaton. To elucidate further the decoupling of the Kaluza-Klein states we repeat the calculations in the S-dualised metric (2.11). Since our configuration does not couple to the NS-NS background field, the action to consider is again Nambu-Goto. It is convenient to switch to the following set of variables

\footnote{Contrary to (3.13), in (3.15) $g_{YM}$ can remain fixed, for a given $g_s$, because in the superconformal case $g_{YM}^2 = 4\pi g_s$ without involving any factor of $N$.}
that would allow a non-trivial combination of the operators $E$ and $J$. In the present case, and because the function $\Phi_D$ is not present in the metric, we should retain the full expression for $\beta(\rho)$ in obtaining the turning point, which for large values of $\rho$ is given by

$$\lambda \omega^2 = \frac{1}{\rho_0 \coth(2\rho_0)}. \quad (3.17)$$

Expanding (3.16) around $\rho_0$, and subtracting the appropriate combination on $P_+, P_-$ in order to cancel the leading term one finds

$$P_+ - 5P_- = 4\frac{\sqrt{\lambda}}{2\pi} \frac{\rho_0^2 \cosh(\rho_0) \sech(\rho_0)}{\sqrt{1 - \rho_0 \cosh(\rho_0) \sech(\rho_0)}} \to 8\frac{\sqrt{\lambda}}{\pi} \rho_0^2 e^{-2\rho_0}. \quad (3.18)$$

Using the leading contribution to $P_+$ or $P_-$

$$P_+ = 5P_- \approx 4\frac{\sqrt{\lambda}}{2\pi} \frac{10}{3} \rho_0, \quad (3.19)$$

we arrive finally at the closed expression

$$E = \left( \frac{3}{\sqrt{\pi}} \right)^{2/3} \left( \frac{J}{\sqrt{R}} \right)^{2/3} + \ldots. \quad (3.20)$$

We see again from this expression, similarly to (3.12), that the energy increases with $J$, signalling the decoupling of the KK stringy states with sufficiently large values of the R-charge.

In addition we can inspect the behaviour of the KK states once the parallel plane-wave limit for (2.11) is taken. As previously the condition $R \to \infty$ arises. A similar argument as has been presented previously indicates that the same kind of scaling, $J \sim \sqrt{R}$ still holds here and would probably suffice for the decoupling.

Summarising, we can conclude that, contrary to the Kaluza-Klein point-like modes, the KK stringy modes on the $S^2$ decouple for states with sufficient large quantum-number $J$. This must be the case from the very beginning not only for the stringy KK modes but also for higher realisations: as the field theory dual does not posses an explicit $U(1)$ symmetry the states corresponding to this charge should decouple. This means that although there is the presence of KK modes that prevents the supergravity model from being the exact dual of $\mathcal{N} = 1$ SYM for large $N$, the extent of this KK contamination is lighter than it could have been expected in view of the incompatibility between (2.9) and (2.10). However, the
relation for $E(J)$ that we find for the $S^2$ stringy modes is not of the (1.1) type, neither we think it is forced to be, because it has nothing special to do the field theory side in the noncompact dimensions. Surprisingly enough this is not the case for other backgrounds [31] where from a similar treatment it seems possible to obtain (1.1), although is not clear to us its interpretation. If in addition we also perform the parallel plane-wave limit, we are forced to consider a double scaling limit ($R, J \to \infty$) that neither ends in the classical ’t Hooft limit nor in the BMN one. Under certain circumstances this limit can lead also to a possible decoupling. However, in this last case, it is less clear to us the role played by non-planar diagrams in the dual field theory side. Probably and by analogy with the AdS case, the choice $J \sim R$ for the D5-brane system is the critical situation where non-planar diagrams in the field theory side are neither dominant nor subdominant with respect to the planar ones [32].

**Short strings:** The situation for small $\rho_0$ values implies large $\omega$. We expect a similar behaviour to that in flat Minkowski space-time. Furthermore the role of the dilaton is higher order in $\rho$ and thus both, D5-brane and NS5-brane, backgrounds should give the same results. As before we take a spinning string with the configuration (3.1). Expanding (3.16) around $\rho_0 \approx 0$ we get the leading behaviours

$$E = \omega J = \sqrt{\frac{3}{2}} \omega, \quad (3.21)$$

so that

$$E^2 = \sqrt{\frac{3}{2}} J \quad (3.22)$$

Expression (3.22) should be compared with the flat case $E^2 = 2J$. Between both relations there is a mismatch of a factor $\sqrt{\frac{3}{2}}$, thus we are not getting exactly flat space. This factor is easily understood from the infra-red asymptotics of (2.3) [cf. (2.11)]. At small $\rho$, $\beta(\rho) \to \frac{1}{4}$, whereas the analogous expression in flat space tends to zero. To be more definite, even if the prefactor $e^{2g(\rho)}$ of the $S^2$ vanishes in this limit, the contribution of the $a(\rho)$ field remains due to the twisting preventing the $S^2$ from eventually shrinking. Probably this can be amended by using the $S^2$ parametrisation of [29].

4 **String oscillating in $S^2 \times \mathbb{R}$**

It has become more or less clear that all the metrics which possess a conformal invariance behave similarly when oscillating strings are considered. The model in [18] lacks of the presence of this symmetry and hence to get a more elementary information on the role played by conformal invariance, or its absence thereof, we consider a multiwrapped closed string which oscillates around the centre of the $S^2 \times \mathbb{R}$.

The motion of the string we are describing is a pulsation that reaches a maximum for $\rho$ at some $\rho_{\text{max}}$, then the string shrinks, collapses for $\rho = 0$ and expands again until $\rho_{\text{max}}$. The
explicit configuration of this *pulsating* string is

\[ t = \tau, \quad \varphi(\sigma), \quad \rho(\tau), \quad \theta = \frac{\pi}{2}, \quad (4.1) \]

where \( \varphi(\sigma + 2\pi) = \varphi(\sigma) + 2\pi m \), denoting \( m \) the number of times the string wraps the \( \varphi \) direction. Note that we could in principle have taken a seemingly more general configuration, \( \varphi(\tau, \sigma) \), but the geometrical meaning of the Nambu-Goto action for the string makes it indistinguishable from the one derived from (4.1).

The Nambu-Goto action restricted to the configuration (4.1) is

\[ S = -m \int d\tau e^{\Phi_D} \sqrt{(1 - \lambda \dot{\rho}^2)\lambda \beta(\rho)}, \quad (4.2) \]

which defines a one-dimensional classical system to which we can apply the usual canonical analysis. This immediately leads to the momenta

\[ \Pi_\rho = e^{\Phi_D} \frac{\lambda^2 m \dot{\rho} \beta(\rho)}{\sqrt{(1 - \lambda \dot{\rho}^2)\lambda \beta(\rho)}}, \quad (4.3) \]

and the -squared- Hamiltonian

\[ \mathcal{H}^2 = e^{2\Phi_D} \frac{\lambda m^2 \beta(\rho)}{1 - \lambda \dot{\rho}^2}. \quad (4.4) \]

From (4.4) and (4.3) one obtains

\[ \mathcal{H} = \sqrt{\frac{\Pi_\rho^2}{\lambda} + e^{2\Phi_D} \lambda m^2 \beta(\rho)}, \quad (4.5) \]

which defines a one-dimensional potential

\[ V(\rho) = e^{2\Phi_D} \lambda m^2 \beta(\rho). \quad (4.6) \]

In what follows we shall find the energy eigenvalues using the WKB approximation and compare with [15]. These authors suggest for non-conformal AdS backgrounds in the high-energy limit a *universal* behaviour

\[ E \sim n + f(\lambda)\sqrt{mn}, \quad (4.7) \]

where \( n \) is the principal quantum number and \( f \) refers to any function of the coupling constant.

The WKB expression gives

\[ I \equiv \int_0^{\rho_{\text{max}}} d\rho \sqrt{\lambda E^2 - e^{2\Phi_D} \lambda^2 m^2 \beta(\rho)} = \left(n + \frac{1}{2}\right) \pi. \quad (4.8) \]
Notice that the potential \( I(\rho) \) grows exponentially as \( \rho \) increases, thus \( E \) large implies inherently \( \rho \) large. Under this approximation

\[
I \approx \sqrt{\lambda} E \ln \left( \frac{4E^2}{\lambda m^2} \right) + \mathcal{O} \left( \frac{1}{E} \right),
\]

expression that matches with the energy levels (with \( \sqrt{E} \) instead of \( E \)) of a N=2 super Sine-Gordon model \[33]. At this point we can not tell whether this matching is coincidental or relies on more fundamental grounds. If the latter were the case, our results could even suggest that in the deep ultraviolet \( \rho \) defines an integrable system. By passing notice that \( \mathcal{O} \) is not like as \( \mathcal{O} \), therefore \( \mathcal{O} \) does not belong to the same equivalence class of the AdS models at finite temperature.

In addition we have explored a second limit, the deep IR, where we demand \( \sqrt{\lambda} E \) to be small. Then the string is restricted to be close to the origin of the radial variable \( \rho \) and therefore, in the short string case we discussed previously, in principle we expect it not to feel the curvature. The potential near the origin can be obtained by expanding \( V(\rho) \) as a power series and looks like that of a shifted harmonic oscillator

\[
V(\rho) = \lambda^2 m^2 \left( 1 + \frac{20}{9} \rho^2 \right).
\]

Then the squared-energy turns out to be proportional to the excitation level of the oscillator, given by the product \( mn \),

\[
E_n^2 - E_0^2 \approx \sqrt{\lambda} mn,
\]

that, as expected, matches with the spectrum of the pulsating string in flat space \( E_n^2 - E_0^2 \approx 4mn \).

5 String rotating in \( S^3 \)

Instead of decoupling the \( S^3 \) by just freezing its coordinates as has been done in the previous section one can similarly place a spinning string with the centre of mass located at the north pole of the \( S^3 \) and at the origin of the transversal coordinate \( 6 \) and study its states. These should correspond to highly excited string states \[8\]. It should also reinforce the idea that the holographic coordinate \( \rho \) is an essential ingredient to obtain a non-trivial behaviour in the \( E(J) \) relation. Let us write the \( S^3 \) metric coming from \[23\] as

\[
d\Omega_3^2 = \frac{1}{4} \lambda \sum_a (w^a)^2 = \frac{1}{4} \lambda \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 + \cos \tilde{\theta}^2 d\tilde{\phi}^2 \right),
\]

where we have changed to a different set of coordinates for the \( S^3 \) (see the appendix in \[21\]). The notation for the angular coordinates is exclusive for this section. We shall adopt the ansatz

\[
t = e^\tau, \quad \tilde{\theta}(\sigma), \quad \tilde{\phi} = e^{\omega \tau}, \quad \tilde{\phi} = \text{constant}.
\]

\(^6\)The configuration is only stable at \( \rho = 0 \) because this is the only point for which \( \frac{d\rho}{d\rho} = 0 \).
A simple inspection of (5.1) under the restriction (5.2) reveals the similitude with the case of a spinning string in the $S^5$ discussed in [8] where the $\text{AdS}_5$ radius is fixed to $R^2 = \frac{1}{4}\lambda$. Therefore we recall the findings in [8] stressing some of their features.

Working out in the conformal gauge it can be checked that (5.2) is consistent with the equations of motion and the constraint (3.7). The latter comes to be

$$R^2 \left(\tilde{\theta}^\prime\right)^2 = e^2 \left(1 - R^2\omega^2 \sin^2 \tilde{\theta}\right), \quad R^2 = \frac{1}{4}\lambda,$$

and gives rise to the bound $R\omega \geq 1$. The exact value of the constant $e$ is obtained by demanding a period $2\pi$ in the variable $\sigma$. It is rather straightforward with the help of (5.3) to obtain the expressions for the space-time energy and the $\text{SO}(4)$ quantum number in terms of the turning point $\tilde{\theta}_0$ (the string is now stretched along the $\theta$ direction)

$$E = \frac{1}{2\pi} e \int_0^{2\pi} d\sigma = 4\frac{R}{2\pi} F(\tilde{\theta}_0|R^2\omega^2),$$

$$J = \frac{R^2}{2\pi} \omega \int_0^{2\pi} d\sigma \sin^2 \tilde{\theta} = 4\frac{R}{2\pi\omega} \left\{ F(\tilde{\theta}_0|R^2\omega^2) - E(\tilde{\theta}_0|R^2\omega^2) \right\},$$

with

$$\sin \tilde{\theta}_0 = \frac{1}{R\omega},$$

and where we have made use of the elliptic integrals of 1st and 2nd kind, $F$ and $E$ respectively. In the case at hand the regime of interest, large $J$, is obtained in the limit $R\omega \to 1^+, \tilde{\theta}_0 \sim \pi/2$. Then both quantities are large while the difference $E - \omega J$ remains finite

$$E - \omega J \to 4\frac{R^2}{2\pi} = \frac{\lambda}{2\pi}. \quad (5.6)$$

6 String rotating in $\mathbb{R}^3$

So far we have dealt entirely with the transverse coordinates to $\mathbb{R}^3$. It is natural to ask whether it is possible to obtain a relation similar to (1.1) by using the flat and the holographic coordinates. Naively, one would think that is quite unlikely, because in the string frame the flat part, $\mathbb{R}^4$, of (2.3) is trivial in the sense that the transverse coordinate $\rho$ enters as a global common factor to all the coordinates. In some sense to obtain a non-trivial relation similar to (1.1) involves different weighting in some of these coordinates.

6.1 Flat $\mathbb{R}^3$

It is worth, before proceeding with the analysis of (2.3), to clean up the field in the simplest case: plain flat space-metric. Due to the special way in which the transverse coordinate $\rho$ comes in (2.3) we shall learn in this manner almost all the needed features.
Figure 1: Conical configuration for a closed folded string in flat space. Keeping its centre of mass at the vertex of the cone, the string stretches, at a given time, along the generator of the cone, rotating in such a way that its extremes –the points where the folding takes place at the boundary of the cone– move at the speed of light. Configurations a) (complete) and b) (incomplete) are distinguished because the stretching along the $x_3$ direction is different.

As we showed above it is equivalent to use the Nambu-Goto action or the conformal gauge formalism. The analysis will be performed in the former, but the same results can be attained in the latter.

Let us consider a closed string with, for stability reasons, its centre of mass static and fixed at some arbitrary point that in the remainder should be taken as the origin of coordinates. We take cylindrical coordinates $r$, $\varphi$, $x_3$ in $\mathbb{R}^3$. The string, folded, spins over the surface of a cone as depicted in fig. 1 with the configuration

$$t = \tau, \quad r(\sigma), \quad \varphi = \omega \tau, \quad x_3(\sigma). \quad (6.1)$$

The Lagrangian density is given by ($L = \int d\sigma \mathcal{L}$)

$$\mathcal{L} = -\frac{1}{2\pi} \sqrt{\left(1 - r^2 \omega^2\right) \left(\left(r'\right)^2 + \left(x_3'\right)^2\right)}, \quad (6.2)$$

where we have already substituted $\omega$ for $\dot{\varphi}$, which is a trivial solution of the equations of motion for the time evolution. The equations for the variables $r$ and $x_3$, depending on $\sigma$, yield two constants of motion,

$$X[r, x_3](r')^2 + r^2 \omega^2 = c_1, \quad X[r, x_3](x_3')^2 = c_2. \quad (6.3)$$

where

$$X[r, x_3] = \frac{1 - r^2 \omega^2}{\left(r'\right)^2 + \left(x_3'\right)^2}. \quad (6.4)$$
A similar argument to the one developed in the appendix (A) can be used now to show that the turning points for the folded string must be located at $r$ satisfying $1 - r^2\omega^2 = 0$. Combining the two constants of motion one gets, after some manipulations,

$$(1 - c_1 - c_2)(1 - r^2\omega^2)(r')^2 = 0,$$  (6.5)

and since we expect a configuration where the factors $(1 - r^2\omega^2)$ and $(r')^2$ can be simultaneously different from zero, we conclude that $c_1 + c_2 = 1$. On the other hand notice that whereas the freedom of $\tau$ reparameterisations has been already fixed with $t = \tau$, the freedom of $\sigma$ reparameterisations is still with us. We use it by fixing $X = 1$. Then the second constant of motion reads $(x_3')^2 = c_2$, but since the configuration must be periodical in $\sigma$, we end up with $c_2 = 0$ and $x_3 = x_{30}$, where $x_{30}$ is an arbitrary constant. As for the $r$ variable, we have a standard oscillatory dependence on $\sigma$, $r(\sigma) = \sin(\omega\sigma)$. Therefore the conical configuration is not stable and we end up with a spinning string in flat two dimensional space.

One can indeed use a more general decomposition in (6.1), for instance

$$t = \tau, \quad r(\sigma), \quad \varphi = \omega\tau, \quad x_3(\tau, \sigma)$$  (6.6)

The result turns out to be identical to the previous one, but with the plane of motion (coordinates $r, \varphi$) boosted along the $x_3$ direction. Let us mention also that considering configurations like $t = \tau + \tilde{t}(\sigma)$ does not change our conclusions as regards to their stability.

To summarise, we have learnt that in flat space-time there can not exist stable string configurations involving the transverse coordinate, and the strings are confined to rotate in a plane. Is this behaviour changed by the flatness of (2.3)? Essentially the main difference between the flat case and the $\mathbb{R}^3$ part depicted in (2.3) is the presence of the global factor $e^{\Phi_D}$ in the metric. Then it becomes clear that in the deep infrared this is not the case because the effect of this factor disappears and we recover flat Minkowski space-time. We shall turn now to the full analysis of (2.3).

### 6.2 $\mathbb{R}^3 \times \mathbb{R}$ in $\mathcal{N}=1$ SYM

Now $\mathbb{R}$ stands again for the $\rho$ variable. We shall focus the analysis in (2.3), and contrary to the previous section we shall employ the conformal gauge formalism, but once more the Nambu-Goto action leads to the same conclusions, although it is more messy to use in the equations of motion.

The equations of motion for the $-consistent-$ configuration

$$t = e\tau, \quad \varphi = e\omega\tau, \quad r(\sigma), \quad \rho(\sigma),$$  (6.7)

are

$$\partial_\sigma \left( e^{\Phi_D} r' \right) + e^{\Phi_D} r e^2 \omega^2 = 0,$$  (6.8)

$$2\partial_\sigma \left( N g_s e^{\Phi_D} \rho' \right) - \frac{\partial e^{\Phi_D}}{\partial \rho} \left( e^2 - r^2 e^2 \omega^2 + (r')^2 + N g_s (\rho')^2 \right) = 0,$$  (6.9)
while the rest of the equations of motion are satisfied trivially. In addition the equation for
the constraint leads to
\[(r')^2 + Ng_s(\rho')^2 = e^2 \left(1 - r^2 \omega^2\right) .\] (6.10)

It is quite straightforward to reduce (6.9) to (6.8) with the help of (6.10) and therefore in
the remainder we shall only deal with two equations.

To begin with let us see that \(r(\sigma)\) and \(\rho(\sigma)\) can not be related via a continuous transfor-
mation. Suppose \(\rho = H(r)\) then (6.8) and (6.10) gives
\[\left(1 - r^2 \omega^2\right) \frac{\dot{H}}{1 + \dot{H}^2} \frac{d\Phi_D}{d\rho} \left[\frac{1}{2} \frac{\partial}{\partial r} \left(1 - r^2 \omega^2\right) \frac{\dot{H}}{1 + \dot{H}^2}\right] = 0 .\] (6.11)

Although to the best of our knowledge (6.11) can not be solved analytically we have checked
numerically that there is no possible solution if in addition we implement the boundary
conditions \(\rho'(r_0) = 0, \rho(r_0) = \text{constant}\). The first one is equivalent to demanding that the \(\rho\)
edge be at \(r = r_0\). This is a quite plausible condition in view of (6.10). The statement of
non-existence of solution is independent on the second condition.

The picture we have achieved is even clearer by examining directly (6.8) and (6.9), that
is, by turning to the most general case. The equations of motion can be written as
\[r'' + r e^2 \omega^2 + \frac{d\Phi_D}{d\rho} \rho' \rho' = 0 ,\] (6.12)
\[N \rho'' - \frac{d\Phi_D}{d\rho} (r')^2 = 0 .\] (6.13)

The variables \(r\) and \(\rho\) are radial in the sense that they are limited to positive values, but for
the true content of these equations it is best to consider them extended to negative values as
well (in fact, this has been already done without previous warning in the preceding section
for the variable \(r\) in the flat case). Now the only addition that must be made is to take the
function \(\Phi_D\) as an even function of the variable \(\rho\) which indeed corresponds to its analytical
continuation. It is then easy to realize that, since \(\frac{d\Phi_D}{d\rho} \geq 0\) for \(\rho \geq 0\), equation (6.13) implies
that \(\rho(\sigma)\) can never be a periodical function except for the trivial configur ation with
\(\rho(0) = 0\). In some sense it is as we had the wrong sign for \(\Phi_D\). As will be
explained below, if the function \(\Phi_D\) in (2.3) had been defined with the opposite sign, we
would have found stable conical-like configurations.

Thus, for \(\rho = 0\) we are back to the solution of a string rotating in a plane in flat space. It
is worth commenting a little more on the general solution for the equations (6.12) and (6.13).
The second equation is highly non-trivial once (2.4) is used but in its actual form can be seen
to represent a forced oscillator. The last term in the first equation (6.12) acts as a damping
factor with a non-constant coefficient for the term linear in the velocity \(r'\), \(b(\sigma) = \frac{d\Phi_D}{d\rho} \rho'(\sigma)\).
Notice that, with the initial conditions in \(\sigma\) described below, from (6.13) we can conclude
that \(\rho'(\sigma)\) (and \(\rho(\sigma)\)) is a monotonically increasing function. Combining this fact together
with the positivity property \(\frac{d\Phi_D}{d\rho} \geq 0\) we can infer that the \(b(\sigma)\) coefficient is positively
defined, and as a consequence the amplitude in \(r(\sigma)\) decreases as we increase \(\sigma\). One obtains
Figure 2: Behaviour of $r(\sigma)$ (full line) and $\rho(\sigma)$ (dashed curve) for a standard solution of (6.12) and (6.13). The variable $r$ describes a damped oscillator (the damping is induced by the last term in (6.12)) whereas the variable $\rho$ tends to a configuration with constant slope $\rho'$, never periodical.

a critical or over-critical oscillator depending on the exact boundary conditions. Moreover, as we search for periodic solutions in $r$, $r(\sigma) = r(\sigma + 2\pi)$, we only shall consider the former case. Together with (6.12) we have implemented the following set of initial conditions: at some $\sigma_0$ corresponding to $r = r_0$, both coordinates, $r$ and $\rho$ acquire an extremal value, a fact that is supported by (6.10). At the same point the values of the functions are constants

$$r(\sigma_0) = \text{constant}_1, \quad \rho(\sigma_0) = \text{constant}_2, \quad r'(\sigma_0) = 0, \quad \rho'(\sigma_0) = 0.$$  \hspace{1cm} (6.14)

We are able to find a solution for (6.12) and (6.13) under the conditions (6.14). We choose the initial value of the constants in order to get an over-critical behaviour, but otherwise the solution is rather general. The solution fails to be periodic in $\rho(\sigma)$ as fig. 2 shows and the function $r(\sigma)$ behaves as expected.

As a conclusion we can say that in the case of $\mathcal{N}=1$ SYM the background of D5-brane, (2.3), fails to give any stable configuration in $\mathbb{R}^4 \times \mathbb{R}$. Just by direct inspection one can reach the same conclusion for configurations in the background of the S-dual metric, (2.11) since after a simple rescaling of the $\rho$ coordinate, this background –for the part that is relevant here– reduces to the flat space considered in section 6. The fact that our configurations are not stable suggest that what are we really seeing are decaying solutions in the gravity side. If we knew better the dictionary of the correspondence in this $\mathcal{N} = 1$ case, we could identify the corresponding operators in the dual gauge theory side. Then the unstability of our configurations would signal that in strong coupling these operators quickly branch into other operators. Since this result is beyond the perturbative control of the gauge theory, we can conclude that the unstability discussed above in the gravity side may encode the information of these non-perturbative aspects of the behaviour of certain operators in the
gauge theory.

### 6.3 Poincaré vs. Global coordinates

In the previous section we have showed the impossibility of stabilising the string in the \( \mathbb{R}^4 \times \mathbb{R} \) space, and thus, there is no hope to obtain a relation between the energy and the angular momentum (or spin) close to (1.1). In view of the field theory findings [35] one might wonder about the failure of the procedure leading to (1.1).

In what follows we shall comment on the reason of this drawback. Let's start rewriting (2.3) as

\[
\frac{ds^2}{f(\rho)} = \left( dx^2_{0,3} + d\rho^2 + \ldots \right),
\]

where ellipsis stands for compactified coordinates transverse to the \( N \) D-branes. If we compare with the \( AdS_5 \times S^5 \) metric in Poincaré coordinates

\[
\frac{ds^2}{R^2} = \frac{d\rho^2}{\rho^2} \left( dx^2_{0,3} + dx_pdx_p \right), \quad \rho^2 = x_p^2 \quad (p = 4, \ldots, 9),
\]

we conclude that even if (6.15) refers to a global patch (in the sense that is not singular) its form is like written in Poincaré coordinates as (6.16). As is known in the \( AdS_5 \times S^5 \) case, the relation (1.1) can only be obtained in global coordinates and not in Poincaré [17]. In the latter case is also impossible to obtain conical-like configurations as those depicted in fig. 1, even though in this case it is less clear how to properly deal with the variable \( \rho \) at the origin.

One can conclude erroneously from the above discussion that it might be impossible at all to find a gravity description for a gauge theory in a non-compact \( \mathbb{R}^4 \) of the form (6.15) admitting stable configurations as those in (6.7). Astonishingly enough, the trial \( f(\rho) = e^{-\Phi_D} \) gives a permitted solution –depicted in fig. 3. It represents a string singly folded in the variable \( \rho \) and doubly folded in \( r \), even though the periodicity may depend on the exact details of the initial conditions. This “fine tuning” of the initial conditions in \( \sigma \) is explained by the fact that we are looking for solutions, see (6.7), that represent strings rotating homogeneously, that is, at constant velocity. If the initial parameters are not correctly adjusted, then the configuration develops oscillations in time along the \( r \) and \( \rho \) directions, that imply that the rotation will no longer be homogeneous and the periodicity of the variables \( r \) and \( \rho \) will be spoiled.

Also, using for \( f(\rho) \) several trial functions, we are able to find stable configurations of the same kind, or even with different folding ratios between the \( r \) and \( \rho \) variables. We think that this type of configurations, simultaneously stretched along a parallel direction in the unwrapped part of the D-branes and a transversal direction, may play a role in backgrounds that allow for their stability, of which the one considered here, with the “wrong sign” for the function \( \Phi_D \), has been just a toy example.

### 6.4 More on \( \mathcal{N}=1 \) SYM

It is interesting to stress that although we have found conical-like stable configurations in the preceding section, they do not correspond to an specific background solution of string theory.
Figure 3: Using the “wrong sign” for $\Phi_D$, we get stable conical-like configurations for the string. Here we represent the solutions $r(\sigma)$ (full line) and $\rho(\sigma)$ (dashed curve), with some specific initial conditions.

One may ask whether this can be the case for other $\mathcal{N}=1$ SYM models as the one presented in [36]. The models [18] and [36] describe essentially in the same manner the infrared region but they differ substantially in the ultraviolet. This is the reason we find [36] worthy of considering. The construction of the supergravity dual is based on an $\text{SU}(M + N) \times \text{SU}(N)$ gauge group that can be obtained by locating $N$ D3-branes and $M$ wrapped D5-branes at the tip of a conifold. In the infra-red the theory cascades down to $\text{SU}(M)$ if $N$ is multiple of $M$ and the conifold is replaced by a deformed conifold [37], [38]. Then the 10 dimensional metric is a warped product of $\mathbb{R}^{3,1}$ and the deformed conifold

$$
\begin{align*}
\bar{ds}^2_{10} &= h^{-1/2}(\rho)dx_{0,3}^2 + h^{1/2}(\rho)\frac{\epsilon^{4/3}}{6K(\rho)^2}d\rho^2 + \ldots,
\end{align*}
$$

(6.17)

where the ellipsis stand for certain tensor products of differentials of angular variables of the Calabi-Yau metric of the deformed conifold. We assume that via the equations of motion we can fix the values of the angular variables in order to place the string in a stable configuration. The functions $K(\rho)$ and $h(\rho)$ are given by

$$
\begin{align*}
K(\rho) &= \frac{(\sinh(2\rho) - 2\rho)^{1/3}}{2^{1/3} \sinh(\rho)}, \quad h(\rho) = M^2 2^{2/3} \epsilon^{-8/3} \int_{\rho}^{\infty} dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}.
\end{align*}
$$

(6.18)
In the sequel we shall set \( \epsilon = 12^{1/4} \) [36]. As in section 6 we consider configurations (6.7) but in the background (6.17). The equations of motion for the radial variables are

\[
\begin{align*}
  r'' + r\omega^2 e^2 + \frac{1}{A(\rho)} \frac{\partial A}{\partial \rho} \rho' r' &= 0, \\
  \rho'' + \frac{1}{2B(\rho)} \frac{\partial B}{\partial \rho} (\rho')^2 - \frac{1}{2B(\rho)} \frac{\partial A}{\partial \rho} \left( e^2 + (r')^2 - r^2 \omega^2 e^2 \right) &= 0,
\end{align*}
\]

where we have defined

\[
A(\rho) \equiv h^{-1/2}(\rho), \quad B(\rho) \equiv \frac{\epsilon^{4/3} h^{1/2}(\rho)}{6 K^2(\rho)},
\]

In addition we also have the constraint

\[
B(\rho) (\rho')^2 + A(\rho) \left( -e^2 + (r')^2 + r^2 \omega^2 \right) = 0.
\]

The solution of the system (6.19)-(6.20) is by far more involved that (6.12)-(6.13). There is anyhow a simple argument to guess the solution of (6.19)-(6.20) with the boundary conditions (6.14): as one infer from the behaviour of the function \( A(\rho) \) and \( B(\rho) \) in the deep ultraviolet

\[
\begin{align*}
  \frac{1}{A(\rho)} \frac{\partial A}{\partial \rho} (\rho \text{ large}) &\rightarrow \text{constant}_1, \\
  \frac{1}{B(\rho)} \frac{\partial A}{\partial \rho} (\rho \text{ large}) &\rightarrow 0, \\
  \frac{1}{B(\rho)} \frac{\partial B}{\partial \rho} (\rho \text{ large}) &\rightarrow \text{constant}_2,
\end{align*}
\]

the system (6.19)-(6.20) partially decouples and the function \( \rho(\sigma) \), as in the previous background, becomes again non-periodic. We have checked numerically these expectations. In view of the results obtained in [39], where from a similar model [40] a relation like (1.1) was obtained, our conclusions are not clear apriori. The key difference is that while the model presented in [40] is based on AdS space [36] is not asymptotic to AdS.

### 7 Summary

We have discussed several closed folded string configurations with stable motion in the Maldacena-Nuñez supergravity background. For strings rotating in the \( S^2 \), which is the cycle wrapped by the D-branes, we find a relationship, (3.12), between the energy and the constant \( \lambda' \equiv J/R \) that is naturally interpreted as a relationship for the energy levels of KK stringy modes on the \( S^2 \). There is fairly good indication of the decoupling of these modes in the field theory side for operators with large values of the R-charge. Furthermore, in the parallel plane-wave limit we are forced to consider a double scaling limit in order to properly decouple the KK states. A scaling \( J \sim R \) is sufficient for this purpose. When we study the limit of these configurations for small values of the transversal variable \( \rho \) we find that, due to the twisting that has been introduced in order to the Maldacena-Nuñez background to preserve some supersymmetry, the relation between the energy and the R-charge, (3.22), is...
not exactly that of flat space, resulting in this case in a change of the slope for the leading Regge trajectory.

We show also stable configurations for strings in the $S^2$ and oscillating along the transversal direction, given by the variable $\rho$. In this case the expression for the energy levels exactly matches (with $\sqrt{E}$ instead of $E$) that of a N=2 super Sine-Gordon model. This could suggest that in the deep ultraviolet the model could define an integrable system. We also note that our results do not belong to the same equivalence class of the AdS models at finite temperature.

Strings rotating in the transversal $S^3$ exhibit a behaviour identical to that found in \cite{8} for strings rotating in the $S^5$ part of the $AdS_5 \times S^5$.

Looking for a non-trivial relation between the energy and the spin, we have considered configurations stretching simultaneously on a radial variable in the non-compact directions of the D-branes and in the transversal variable $\rho$. We find that such configurations are unstable in the specific background of \cite{18} and we only get trivial flat space rotating strings located at $\rho = 0$. We also verify that the same results apply for the Klebanov-Strassler background. We discuss nevertheless the possibility of having stable configurations of this kind for other backgrounds and, surprisingly enough, we find that a background with a change of sign of the function $\Phi_D$ present in the metric \cite{23} allows for this type of configurations, but unfortunately so far we can not interpreted from the point of view of the supergravity solutions. We have not neither pursue an exhaustive analysis involving compact direction in the metric as in \cite{31}. Perhaps one can find a relation similar to \cite{11}, but is less clear how to interpret it form a physical point of view.

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A Constraints in the Nambu-Goto action

A standard analysis of the Nambu-Goto (NG) action yields the conclusion that there are no Lagrangian constraints. The reason is as follows. Consider the NG action for the bosonic string in an arbitrary target space background

$$\mathcal{L}_{NG} = \sqrt{|g_{\alpha \beta}|}$$  \hspace{1cm} (A.1)

with $g_{\alpha \beta} = G_{ij}(X)\partial_\alpha X^i \partial_\beta X^j$. Under general diffeomorphism invariance,

$$\delta X^i = \epsilon^\alpha \partial_\alpha X^i,$$

for arbitrary infinitesimal diffeomorphism $\epsilon^\alpha(\tau, \sigma)$, $\mathcal{L}_{NG}$ behaves as a scalar density, that is,

$$\delta \mathcal{L}_{NG} = \partial_\alpha(\epsilon^\alpha \mathcal{L}_{NG}),$$ \hspace{1cm} (A.2)
and considering that
\[ \delta L_{\text{NG}} = [L_{\text{NG}}]_i \delta X^i + \partial_\alpha (\frac{\partial L_{\text{NG}}}{\partial (\partial_\alpha X^i)} \delta X^i), \] (A.3)
(where \([L_{\text{NG}}]_i\) stands for the Euler-Lagrange derivative) we end up with the Noether relation
\[ [L_{\text{NG}}]_i \delta X^i + \partial_\alpha (\frac{\partial L_{\text{NG}}}{\partial (\partial_\alpha X^i)} \delta X^i - \epsilon^\alpha L_{\text{NG}}) = 0 \] (A.4)
identically. Out of this relation, and taking into account the arbitrariness of \(\epsilon^\alpha\), we obtain the purported Lagrangian constraints
\[ \frac{\partial L_{\text{NG}}}{\partial (\partial_\alpha X^i)} \partial_\beta X^i - \delta_\beta L_{\text{NG}} = 0, \] (A.5)
which are nothing but the components of the string worldvolume energy-momentum tensor. But it turns out that these constraints are void, as one can check that they are mere identities 7.

Nevertheless, as we shall see below, special configurations for the string may introduce constraints which are not standard in the Dirac sense. Here we are interested in configurations describing rotating folded closed strings, and the effect of the appearance of a non-Dirac constraint, which is caused by the folding, is already present in the simplest of cases, the folded closed string rotating in flat space.

The relevant part of the metric is just
\[ ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2, \] (A.6)
and we consider a configuration
\[ t = \tau, \quad \varphi = \omega \tau, \quad r(\sigma). \]
Note that the first equality fixes the gauge for \(\tau\) reparameterisation. The NG action becomes
\[ L_{\text{NG}} = |r'| \sqrt{1 - r^2 \omega^2}. \] (A.7)
Our configuration is intended to describe a closed string rotating around its centre of mass, located at \(r = 0\), folded, and stretching along the radial direction, with symmetric (with respect to \(r = 0\)) turning points. A partial use of the remaining gauge freedom allows us to consider four pieces composing the string, the first piece being for \(\sigma \in [0, \pi/2]\) with \(r'(\sigma) \geq 0\), and similar expressions for the rest. The action then becomes
\[ S = 4 \int d\tau \int_0^{\pi/2} d\sigma r'(\sigma) \sqrt{1 - r^2 \omega^2}. \] (A.8)

7Another way to look at this fact (no Lagrangian constraints for the NG action) is through the canonical formalism. One can show that there are only two primary Hamiltonian constraints and that they are first class. In such case, it is easy to prove that there can no be Lagrangian constraints.
The only variable left, \( r(\sigma) \), makes the Lagrangian to be a total derivative -with respect to the \( \sigma \) coordinate. Therefore, at first sight, it would seem as if the function \( r(\sigma) \) remains completely arbitrary, for there will be no e.o.m. for it, except for the only natural requirement of the positivity of the term in the square root, that is, \( r \leq 1/\omega \). But this analysis is incomplete, as we shall show now, because there is a boundary effect, caused by the folding of the string, that has been overlooked.

For the first quarter of the string, with parameterisation \( \sigma \in [0, \pi/2] \), the string will stretch from \( r = 0 \) to some \( r = r_0 \leq 1/\omega \), the action can then be written

\[
S = 4 \int d\tau \int_0^{r_0} dr \sqrt{1 - r^2\omega^2}.
\]

(A.9)

Now the action for this configuration of the string is determined by a single parameter \( r_0 \). The correct application of the variational principle requires the action to be extremised with respect to this parameter, that is,

\[
\frac{\partial}{\partial r_0} \int_0^{r_0} dr \sqrt{1 - r^2\omega^2},
\]

(A.10)

giving

\[
\sqrt{1 - r_0^2\omega^2} = 0,
\]

(A.11)

implying,

\[
r_0 = \frac{1}{\omega}.
\]

(A.12)

This is the constraint that restricts the setting of initial conditions for the string. It is not of the usual Dirac type, for it only restricts the positions of two points -the turning points- of the whole string. It is a constraint that tells how far in the radial direction the string stretches. It turns out that it stretches all that it can: until the extremes move at the speed of light.

This result is obtainable in a straightforward way in the conformal gauge, but it is easy to be missed using the NG approach (in fact, as we have observed, since the Lagrangian \( [A.7] \) is locally a total derivative -in \( \sigma \)- one could naively expect that there will no be e.o.m. for \( r \), and so the speed of light condition will be overlooked). What we have shown, therefore, is the equivalence between the NG and the conformal gauge approaches for the description of this type of configurations. This is something that is formally guaranteed by the equivalence of gauges, but that does not spares us of the subtleties involved. Observe in addition that when one works in the conformal gauge, the conformal factor becomes singular at the turning points.

It is worth noting that this type of configurations of folded closed strings has the same dynamics as open strings with Neumann boundary conditions.
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