Laplace copulas of multifactor gamma distributions are new generalized Farlie-Gumbel-Morgenstern copulas

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Abstract

This paper provides bifactor gamma distribution, trivariate gamma distribution and two copula families on $[0, 1]^n$ obtained from the Laplace transforms of the multivariate gamma distribution and the multi-factor gamma distribution given by $[P(\theta)]^{-\lambda}$ and $[P(\theta)]^{-\lambda} \prod_{i=1}^n (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}$ respectively, where $P$ is an affine polynomial with respect to the $n$ variables $\theta_1, \ldots, \theta_n$.

These copulas are new generalized Farlie-Gumbel-Morgenstern copulas and allow in particular to obtain multivariate gamma distributions for which the cumulative distribution functions and the probability distribution functions are known.

KEY WORDS: Archimedean copula, cumulative distribution function, copula, exponential families, infinitely divisible distribution, generalized Farlie-Gumbel-Morgenstern copulas, generalized hypergeometric function, generalized Lauricella functions, Horn function, Kendall’s tau, Laplace copula, Laplace transform, multi-factor gamma distribution, multivariate gamma distribution, Spearman’s rho.
1 Introduction

This paper is motivated by the opportunity to produce explicit cumulative distribution functions (cdf) and probability distribution functions (pdf) of multivariate gamma distributions and multi-factor gamma distributions. In this way, we present two new classes of multivariate copulas generalizing the Farlie-Gumbel-Morgenstern Copulas. From Sklar (Sklar, 1959) who states that the cdf $F$ of a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with continuous marginal cdfs can be uniquely written in the form

$$F(x_1, \ldots, x_n) = C[F_1(x_1), \ldots, F_n(x_n)], \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $n$ is the dimension of the random vector $\mathbf{X}$, where $C : [0, 1]^n \to [0, 1]$ is a copula and where $F_1, \ldots, F_n$ are the marginal cdfs of $\mathbf{X}$.

If $f$ is the pdf of the random vector $\mathbf{X}$ and $f_1, \ldots, f_n$ the marginal pdfs of $\mathbf{X}$, then

$$c(u_1, \ldots, u_n) = \frac{\partial^n}{\partial u_1 \cdots \partial u_n} C(u_1, \ldots, u_n),$$

and we have the following equality

$$f(x_1, \ldots, x_n) = c[F_1(x_1), \ldots, F_n(x_n)] f_1(x_1) \cdots f_n(x_n).$$

From Equality (1) it is possible to express the cdf $F$ of $\mathbf{X}$. This expression cannot simply explicitly give the copula $C$ for the multivariate gamma distribution associated with $(P, \lambda)$ and the multi-factor gamma distribution associated with $(P, \Lambda)$. But with Joe (Joe, 1997), we can give copulas deduced of the Laplace transform of the multivariate gamma distribution associated with $(P, \lambda)$ and the Laplace transform of the multi-factor gamma distribution associated with $(P, \Lambda)$. By applying the formulas (1) and (2) we can give a new multivariate gamma distribution associated with $(P, \lambda)$ and a new multi-factor gamma distribution associated with $(P, \Lambda)$ for which we have an explicit formula for its cdf and its pdf.

The paper is organized as follows. Section 2 gives definitions of multivariate gamma distributions and multi-factor gamma distributions for which Laplace transform is given, and considers the bidimensional and tridimensional case. Section 3 defines the Laplace copula. Section 4 states the two main results. Section 5 applies the main cases to bidimensional and tridimensional cases. In particular, the Joe’s family BB10 is generalized. For ease of the fluent exposition of the paper, proofs are collected in the Appendix.
2 Multivariate gamma distributions and multi-factor gamma distributions

In the literature, the multivariate gamma distributions on \( \mathbb{R}^n \) have several non-equivalent definitions. Many authors require only that the marginal distributions are ordinary gamma distributions (Balakrishnan et al., 1997). In the present paper, we use the extension of the classical one-dimensional definition to \( \mathbb{R}^n \) obtained as follows: we consider an affine polynomial \( P(\theta) \) in \( \theta = (\theta_1, \ldots, \theta_n) \) where ‘affine’ means that, for \( j = 1, \ldots, n, \partial^2 P/\partial \theta_j^2 = 0 \). We also assume that \( P(0) = 1 \). For instance, for \( n = 2 \), we have \( P(\theta_1, \theta_2) = 1 + p_1 \theta_1 + p_2 \theta_2 + p_{(1,2)} \theta_1 \theta_2 \).

We denote by \( \mathfrak{P}_n = \mathfrak{P}([n]) \) the family of all subsets of \( [n] \) and \( \mathfrak{P}_n^* \) the family of non-empty subsets of \( [n] = \{1, \ldots, n\} \). For simplicity, if \( n \) is fixed and if there is no ambiguity, we denote these families by \( \mathfrak{P} \) and \( \mathfrak{P}^* \), respectively.

We denote by \( \mathbb{N} \) the set of non-negative integers. If \( z = (z_1, \ldots, z_n) \in \mathbb{R}^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), then \( \alpha! = \alpha_1! \ldots \alpha_n! \), \( |\alpha| = \alpha_1 + \ldots + \alpha_n \), \( a_\alpha = a_{\alpha_1, \ldots, \alpha_n} \) and

\[
\mathcal{Z}^\alpha = \prod_{i=1}^n z_i^{\alpha_i} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.
\]

For \( T \) in \( \mathfrak{P}_n \), we simplify the above notation by writing \( \mathcal{Z}^T = \prod_{t \in T} z_t \) instead of \( \mathcal{Z}^{1_T} \) where

\[
1_T = (\alpha_1, \ldots, \alpha_n) \text{ with } \alpha_i = 1 \text{ if } i \in T \text{ and } \alpha_i = 0 \text{ if } i \notin T.
\]

We also write \( \mathcal{Z}^{-T} = \prod_{t \in T} 1/z_t \). For a mapping \( a : \mathfrak{P} \rightarrow \mathbb{R} \), we shall use the notation \( a : \mathfrak{P} \rightarrow \mathbb{R}, T \mapsto a_T \). In this notation, an affine polynomial with constant term equal to 1 is \( P(\theta) = \sum_{T \in \mathfrak{P}} p_T \theta^T \), with \( p_{\emptyset} = 1 \). For simplicity, if \( T = \{t_1, \ldots, t_k\} \), we denote \( a_{\{t_1, \ldots, t_k\}} = a_{t_1 \ldots t_k} \). The indicator function of a set \( S \) is denoted by \( \mathbb{1}_S \), that is, \( \mathbb{1}_S (x) = 1 \) for \( x \in S \) and 0 for \( x \notin S \).

We fix \( \lambda > 0 \). If a random vector \( \mathbf{X} = (X_1, \ldots, X_n) \) on \( \mathbb{R}^n \) with probability distribution (pd) \( \mu_X \) is such that its Laplace transform is

\[
\mathbb{E} \{ \exp \left[ - (\theta_1 X_1 + \cdots + \theta_n X_n) \right] \} = [P(\theta)]^{-\lambda}, \tag{3}
\]

where \( \mathbb{E} \) denotes the expectation, for a set of \( \theta \) with non-empty interior, then we denote \( \mu_X = \gamma(\lambda), \) and \( \gamma(\lambda) \) will be called the multivariate gamma distribution associated with \( (P, \lambda) \). These multivariate gamma distributions occur naturally in the classification of natural exponential families in \( \mathbb{R}^n \) (Bar-Lev et al., 1994).

The marginal distributions of the multivariate gamma distribution associated with \( (P, \lambda) \) are ordinary gamma distributions of parameters \( (p_i, \lambda) \) for \( i = 1, \ldots, n \), with Laplace transform \( (1 + p_i \theta_i)^{-\lambda} \), and pd \( \gamma(p_i, \lambda) (dx) = x^{\lambda - 1} p_i^{-\lambda} / \Gamma (\lambda) \exp (-x/p_i) 1_{(0,\infty)} (x)dx \). We extend the first definition to the multi-factor
generalized hypergeometric function (Slater, 1966) defined by
\[ \gamma_{n} = \frac{\Gamma(X)}{\Gamma(Y)}. \]
where \( \gamma_{n} \) is the gamma distribution associated with \( X \) (except for the case \( n = 0 \)).

Let \( \Lambda = (\lambda, \lambda_1, \ldots, \lambda_n) \) and \( \lambda_i \geq 0 \) for all \( i = 1, \ldots, n \).

by its Laplace transform
\[ E \{ \exp [-(\theta_1 X_1 + \cdots + \theta_n X_n)] \} = [P(\theta)]^{-\lambda} \prod_{i=1}^{n} (1 + p_i \theta_i)^{-((\lambda_i - \lambda)).} \tag{4} \]
The marginal distributions of the multi-factor gamma distribution associated with \( (P, \Lambda) \) are ordinary gamma distributions of parameters \( (p_i, \lambda_i) \) for \( i = 1, \ldots, n \).

We state first a proposition, whose proof is obvious.

**Proposition 1** A random vector \( X \) with distribution \( \gamma_{P, \Lambda} \) can be obtained in the following way:
Let \( Y \) be a vector with distribution \( \gamma_{P, \lambda} \). Let \( Z = (Z_1, \ldots, Z_n) \) be a random vector constituted of independent margins for which its pds are \( \gamma_{(p_i, \lambda_i)} \), and such that \( Z \) and \( Y \) are independent. Then the vector \( X = Y + Z \) has Laplace transform \( L_X(\theta_1, \theta_2) \), and consequently is a multi-factor gamma distribution associated to \( (P, \Lambda) \).

For the bidimensional case, Dussauchoy and Berland, (Dussauchoy and Berland, 1972) consider the random vector \( X = (X_1, X_2) \) with Laplace transform
\[ L_X(\theta_1, \theta_2) = (1 + p_1 \theta_1)^{-\lambda_1} (1 + p_2 \theta_2)^{-\lambda_2} \left[ 1 - \frac{p_1 \theta_1}{(1 + p_1 \theta_1) (1 + p_2 \theta_2)} \right]^{-\lambda} = (1 + p_1 \theta_1)^{-((\lambda_1 + \lambda_2))} \left( 1 + p_2 \theta_2 \right)^{-((\lambda_2 - \lambda))} (1 + p_1 \theta_1 + p_2 \theta_2 + p_12 \theta_1)^{-\lambda}, \tag{5} \]
where \( r_{12} = 1 - p_12 / (p_1 p_2) > 0 \) and \( p_1, p_2, r_{12} > 0 \).

Unfortunately the Laplace transform of these pds is simple, but its pdfs and cdfs are unknown, except for the case \( n = 2 \) for the multivariate gamma distribution associated with \( (P, \lambda) \). Let \( F^p_m \) be the generalized hypergeometric function (Slater, 1966) defined by
\[ F^p_m(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_m; z) = \sum_{k=0}^{\infty} \left( \frac{\alpha_1}{\beta_1 \cdots \beta_m} \right)_k z^k, \tag{6} \]
where \( (a)_k = \Gamma(a + k)/\Gamma(a) \) for \( a > 0 \) and \( k \in \mathbb{N} \), or more generally by \( (a)_0 = 1 \), \( (a)_{n+1} = (a + n)(a) \), \( \forall n \in \mathbb{N} \), \( \forall a \in \mathbb{R} \), is the Pochhammer’s symbol. For simplification, we denote \( F^0_m \) by \( F_m \). Bernardoff (Bernardoff, 2006) gives the

**Proposition 2** Let \( P(\theta_1, \theta_2) = 1 + p_1 \theta_1 + p_2 \theta_2 + p_12 \theta_1 \) be an affine polynomial where \( p_1, p_2 > 0 \) and \( p_12 > 0 \). Let \( \mu = \gamma_{(P, \lambda)} \) be the gamma distribution associated to \( (P, \lambda) \). The measure \( \mu \) exists if and only if \( c = (p_1 p_2 - p_12) / p_12 = p_1 p_2 / p_12^2 > 0 \). Then we have
\[ \gamma_{(P, \lambda)}(dx_1, dx_2) = \frac{p_12^{\lambda}}{\Gamma(\lambda)} e^{-p_12 x_1} \frac{p_12^{\lambda}}{p_12} (x_1 x_2)^{\lambda-1} F_1(\lambda; c x_1 x_2)^{1/\lambda} (x_1 x_2)^{1/\lambda} \chi_{(0, \infty)^2}(x) d(x). \tag{7} \]
For the case \( \Lambda = (\lambda, \lambda, \lambda) \), the \textit{multi-factor gamma distribution} associated with \((P, \Lambda)\) is named by Chatelain et al. (Chatelain et al., 2008) the \textit{multisensor gamma distribution} associated with \((P, \lambda, \lambda)\) and they have proved that its pd is given by the equality

\[
\gamma_{(P, \Lambda)}(dx_1, dx_2) = \frac{p_1 \lambda^{-\lambda} - (\lambda - \lambda) p_2}{\Gamma(\lambda) \Gamma(\lambda)} x_1^{\lambda_1-1} x_2^{\lambda_2-1} e^{-\frac{p_1 x_1}{p_2} + \frac{p_2 x_2}{12 P, \mu x_1 x_2} \Phi_3(\lambda - \lambda; \lambda; \lambda)} \times \mathbb{I}_{(0, \infty)^2}(x_1, x_2) \, dx_1 \, dx_2,
\]

where

\[
\Phi_3(a; b; x, y) = \sum_{m,n \geq 0} \frac{(a)_m (b)_n}{(m+n)!} \frac{y^n}{m! n!}
\]

is the Horn function.

For the bidimensional general case, we have the following Theorem. Let \( F_I \) be the function defined by

\[
F_I(a, b, c, x) = \sum_{m_1, m_2, m_3 = 0}^{\infty} \frac{(a)_{m_1} (b)_{m_2} (c)_{m_3}}{(a+c)_{m_1+m_3} (b+c)_{m_2+m_3} m_1! m_2! m_3!} \, x^{m_1} e^{z_{12} x^{m_2} x^{m_3}}.
\]

it is a particular generalized Lauricella function defined, by example, in (Panda, 1973).

**Theorem 3** The pd of \( \gamma_{(P, (\lambda, \lambda, \lambda))} \) is given by the equality

\[
\gamma_{(P, (\lambda, \lambda, \lambda))}(dx_1, dx_2) = \frac{p_2 \lambda^{-\lambda} - (\lambda - \lambda) p_2}{\Gamma(\lambda) \Gamma(\lambda)} x_1^{\lambda_1-1} x_2^{\lambda_2-1} e^{-\frac{p_1 x_1}{p_2} + \frac{p_2 x_2}{12 P, \mu x_1 x_2} \Phi_3(\lambda - \lambda; \lambda; \lambda)} \times F_I \left( \lambda_1 - \lambda, \lambda_2 - \lambda, \lambda, \frac{p_1 x_1}{p_2 x_2}, \frac{p_2 x_2}{x_2}, \frac{c_1 x_1}{x_2} \right) \mathbb{I}_{(0, \infty)^2}(x_1, x_2) \, dx_1 \, dx_2,
\]

If we get \( \lambda_1 = \lambda \) in the equality \( \boxed{11} \), we obtain Chatelain and Tourneret’s result \( \boxed{3} \) because

\[
F_I(0, \lambda_2 - \lambda, \lambda, z_1, z_2, z_3) = \sum_{m_2, m_3 = 0}^{\infty} \frac{(b)_{m_2} z_{m_3}}{(b+c)_{m_2+m_3} m_2! m_3!} = \Phi_3(b; b+c; z_2, z_3).
\]

Bernardoff (Bernardoff, 2006) gives the following Proposition:

**Proposition 4** Let \( \mu \) be a multivariate gamma distribution on \( \mathbb{R}^n \) associated to \((P, \lambda)\). Assume that \( \mu \) is not concentrated on a linear subspace of \( \mathbb{R}^n \) of the form \( \{x \in \mathbb{R}^n; x_k = 0\} \) for some \( k \in [n] = \{1, \ldots, n\} \).

Then:

(i) For all \( i \in [n] \), \( p_i \neq 0 \).

(ii) If \( p_1, \ldots, p_k < 0 \) and \( p_{k+1}, \ldots, p_n > 0 \), then \( \text{Supp}(\mu) \subset (\infty, 0]^k \times [0, \infty)^{n-k} \).

(iii) If \( p_1, \ldots, p_n > 0 \) then \( p_n \geq 0 \).
Bernardoff (Bernardoff, 2006) gives a necessary and sufficient condition for infinite divisibility of the multivariate gamma distribution associated with \((P, \lambda)\), in the sense that the Laplace transform of \(\gamma_{(P, \lambda)}\) power \(t\) for all positive \(t\) is still the Laplace transform of a positive measure, by the following theorem:

**Theorem 5** Let \(\mu = \gamma_{P, \lambda}\) be a gamma distribution associated with \((P, \lambda)\), where \(\lambda > 0\) and \(P(\theta) = \sum_{T \in \Psi_n} p_T \theta^T\) is such that \(p_i > 0, \text{ for all } i \in [n], \text{ and } p_{[n]} > 0\). Let \(\tilde{P}(\theta) = \sum_{T \in \Psi_n} \tilde{p}_T \theta^T\) be the affine polynomial such that \(\tilde{p}_T = -p_T/p_{[n]}\) for all \(T \in \Psi_n\), where \(\mathcal{T} = [n] \setminus T\). Let

\[
\tilde{b}_S = b_S(\tilde{P}) = \sum_{k=1}^{[S]} (k-1)! \sum_{T \in \Pi_k^S} \prod_{T \in T} \tilde{p}_T,
\]

where \([S]\) denotes the cardinality of the set \(S\), and \(\Pi_k^S\) denotes the set of all partitions of \(S\) into \(k\) non-empty subsets of \(S\). Then the measure \(\mu\) is infinitely divisible if and only if

\[
\tilde{p}_i < 0 \text{ for all } i \in [n],
\]

and

\[
\tilde{b}_S > 0 \text{ for all } S \in \Psi_n^\ast \text{ such that } |S| > 2.
\]

**Corollary 6** By the properties of infinite divisible distributions we conclude that the necessary and sufficient conditions for infinite divisibility of a multivariate gamma distribution associated to \((P, \lambda)\) of theorem 5, are also necessary and sufficient conditions for infinite divisibility of multivariate multifactor gamma distribution associated to \((P, \lambda)\).

To illustrate the difficulty to calculate the multivariate gamma distribution associated with \((P, \lambda)\) we give, for the tridimensional case, the following theorem. Let \(F_{II}\) be the function defined by

\[
F_{II}(\lambda_1, \lambda_2, z_1, z_2, z_3, z_4) = \sum_{m_1, \ldots, m_4 = 0}^{\infty} \frac{1}{(\lambda_1)_1 m_1 + (\lambda_2)_2 m_2 + m_3} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4}}{m_1! m_2! m_3! m_4!},
\]

it is still a particular generalized Lauricella function.

**Theorem 7** In the case \(n = 3\), \(p_i > 0\) for \(i \in [3]\), \(p_{ij} > 0\) for \((i, j) \in [3]^2\), \(\tilde{b}_{ij} = -\frac{b_{ij}}{p_{123}} + \frac{p_{13} p_{23} p_{1} p_{2}}{p_{123}^2} \geq 0\) for \(i \neq j\) and \(\{i, j, k\} = [3]\), \(p_{123} > 0\), and

\[
\tilde{b}_{123} = -\frac{1}{p_{123}} + \frac{p_{12} p_{1}}{p_{123}^2} + \frac{p_{13} p_{2}}{p_{123}^2} + \frac{p_{23} p_{1}}{p_{123}^2} + \frac{2 p_{12} p_{13} p_{23}}{p_{123}^3} \geq 0,
\]

the infinitely divisible multivariate gamma distribution \(\gamma_{(\lambda, P)}\) associated with \((P, \lambda)\), is given by the formula

\[
\gamma_{(\lambda, P)}(dx) = \frac{p_{123}^{\lambda - 1}}{[\Gamma(\lambda)]^3} \exp(\tilde{p}_1 x_1 + \tilde{p}_2 x_2 + \tilde{p}_3 x_3) (x_1 x_2 x_3)^{\lambda - 1}
\times F_{II}(\lambda, \lambda, \tilde{b}_{13} x_1 x_3, \tilde{b}_{23} x_2 x_3, \tilde{b}_{12} x_1 x_2, \tilde{b}_{13} x_1 x_3, \tilde{b}_{23} x_2 x_3) \mathbb{1}_{(0, \infty)^3}(x) \, dx.
\]
Remark 8 The case $p_{123} = 0$ is solved by Letac and Wesowski (Letac and Wesowski, 2008).

Remark 9 If $\tilde{b}_{12} = \tilde{b}_{13} = \tilde{b}_{23} = 0$, Theorem 4 gives

$$\gamma(\lambda, \mu) (dx) = \frac{p_{123}}{\Gamma(\lambda)} \exp(\tilde{p}_1 x_1 + \tilde{p}_2 x_2 + \tilde{p}_3 x_3) (x_1 x_2 x_3)^{\lambda - 1} F_2 \left( \lambda, \lambda; \tilde{b}_{123} x_1 x_2 x_3 \right) I_{(0, \infty)^3} (x) \, dx,$$

and if we put $\lambda = 1$ in this last equality, we obtain the Kibble and Moran distribution given in (Balakrishnan et al., 2000).

3 Laplace copula

We recall the following theorem (Marshall and Olkin, 1988), Theorem 2.1, p. 835, and his Corollary

Theorem 10 Let $H_1, \ldots, H_n$, be univariate cdfs, and let $G$ be an $n$-variate cdf such that $G(0, \ldots, 0) = 1$, with univariate margins $G_i$ ($i = 1, \ldots, n$). Denote the Laplace transforms of $G$ and $G_i$, respectively, by $\phi$ and $\phi_i$ ($i = 1, \ldots, n$). Let $K$ be a copula. If $F_i(x) = \exp \left\{ -\phi_i^{-1} [H_i(x)] \right\}$ ($i = 1, \ldots, n$), then

$$H(x_1, \ldots, x_n) = \int \cdots \int K \left\{ [F_1(x_1)]^{\theta_1}, \ldots, [F_n(x_n)]^{\theta_n} \right\} \, dG(\theta_1, \ldots, \theta_n) \quad (16)$$

is an $n$-variate cdf with marginals $H_1, \ldots, H_n$.

Corollary 11 Under the conditions of the Theorem 10 with

$$K(x_1, \ldots, x_n) = \prod_{i=1}^n x_i, (0 \leq x_i \leq 1; \ i = 1, \ldots, n),$$

$$H(x_1, \ldots, x_n) = \phi \left\{ \phi_1^{-1} [H_1(x_1)], \ldots, \phi_n^{-1} [H_n(x_n)] \right\} \quad (17)$$

defines an $n$-variate cdf with marginals $H_1, \ldots, H_n$.

Therefore copula for $H$ is associated to $G$ and is given by the formula

$$C(u_1, \ldots, u_n) = \phi \left[ \phi_1^{-1} (u_1), \ldots, \phi_n^{-1} (u_n) \right]. \quad (18)$$

Remark 12 If we know $\phi$ and $\phi_i$, for $i = 1, \ldots, n$, we can find the copula $C$, that is the case for the multivariate gamma distributions and for the multivariate multi-factor gamma distributions. For the case $H_i = G_i$, we can write:

$$H(x_1, \ldots, x_n) = \phi \left\{ \phi_1^{-1} [G_1(x_1)], \ldots, \phi_n^{-1} [G_n(x_n)] \right\}, \quad (19)$$

and this last formula defines $n$-variate cdf gamma $H$ with margin distributions $G_1, \ldots, G_n$.

Then the copula associated to $H$ is

$$C(u_1, \ldots, u_n) = \phi \left[ \phi_1^{-1} (u_1), \ldots, \phi_n^{-1} (u_n) \right].$$
Now, from the Formula 18 we can give the following Definition

**Definition 13** Let \(X = (X_1, X_2, \ldots, X_n)\) be a random vector in \([0, +\infty[^n\) with Laplace transform \(\varphi_X\) defined by \(L_X(\theta) = L_X(\theta_1, \ldots, \theta_n) = \varphi_X(\theta_1, \ldots, \theta_n) = \mathbb{E}\{\exp[-(\theta_1 X_1 + \cdots + \theta_n X_n)]\}\), and let \(\varphi_X\) be the Laplace transform of the random variable \(X_i\), defined by \(L_{X_i}(\theta_i) = \varphi_{X_i}(\theta_i) = \mathbb{E}\{\exp[-\theta_i X_i]\}\). The function \(\varphi_{X_i}\) is a one and onto, and is decreasing of \([0, +\infty[\) onto \([0, 1]\). Its inverse function is denoted by \(\varphi_{X_i}^{-1}\). Then the function \(C_{L_X}\) defined by

\[
C_{L_X}(u_1, \ldots, u_n) = \varphi_X[\varphi_{X_1}^{-1}(u_1), \ldots, \varphi_{X_n}^{-1}(u_n)]
\]

is a copula. We call this copula \(C_{L_X}\) the Laplace copula associated to the random vector \(X\). If \(X\) has pd \(\mu_X(dx)\), then we denote still \(C_{L_X}(u_1, \ldots, u_n) = C_{\mu_X}(u_1, \ldots, u_n)\).

**Proposition 14** Let \(F_1, \ldots, F_n\), be \(n\) univariate cdfs on \([0, +\infty[. The relation

\[
F(x_1, \ldots, x_n) = \varphi_X\left\{\varphi_{X_1}^{-1}[F_1(x_1)], \ldots, \varphi_{X_n}^{-1}[F_n(x_n)]\right\}
\]

defines a cdf \(F\) with marginal cdfs \(F_1, \ldots, F_n\). Then the copula associated to \(F\) is \(C_{L_X}\).

If we choose, for \(i = 1, 2, \ldots, n\), \(F_i = F_{X_i}\), the cdf of \(X_i\), then \(F\) is the cdf of a random vector \(X\) for which the marginal cdfs are the cdfs of \(X_i\), and we have \(F \neq F_X\) where \(F_X\) calls the cdf of \(X\). In the case of multivariate gamma distributions and multi-factor gamma distributions, we get other multivariate gamma distributions and multi-factor gamma distributions with given copula.

Laplace copula can be seen as a generalization of Archimedean copula (Joe, 1997, 2014).

### 4 Main results

Now, we are in the capacity to give the two main results of this paper

**Theorem 15** Let \(P\) an affine polynomial in the \(n\) variables \(\theta_i\), \(i = 1, \ldots, n\), with \(P(0) = 1\), then the Laplace copula of the multivariate gamma distribution \(\gamma_{(P, \lambda)}\) associated to \((P, \lambda)\) is

\[
C_{L\gamma_{(P, \lambda)}}(v) = v^n[1 + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P}1_T) \prod_{i \in T} (1 - \frac{1}{v_i})]^\lambda,
\]

where \(v = (v_1, \ldots, v_n)\), \(|T|\) is the cardinality of \(T\) and the vector \(\frac{1}{P}1_T\) is defined by \(\left(\frac{1}{P}1_T\right)_i = \frac{1}{P_i}\) if \(i \in T\), \(\left(\frac{1}{P}1_T\right)_i = 0\) if \(i \notin T\), for \(i \in \{1, 2, \ldots, n\}\).

This family is given, by example, for the simpler case \(P(\theta) = \prod_{i=1}^{n}(1+p_i \theta_i) - \beta p^{(n)} \theta^{(n)}\), with \(0 \leq \beta < 1\), corresponding to the family gived by (Fang et al., 2000), namely \(C(v) = v^n[1 - \beta \prod_{i=1}^{n}(1 - \frac{1}{v_i})]^\lambda\).
Theorem 16 Let $P$ an affine polynomial in the $n$ variables $\theta_i$, $i = 1, \ldots, n$, with $P(0) = 1$, let $\Lambda = (\lambda, \lambda_1, \ldots, \lambda_n)$ with $\lambda_1 > \lambda > 0$, for $i \in \{1, 2, \ldots, n\}$, then the Laplace copula of the multivariate multi-factor gamma distribution $\gamma_{(P, \Lambda)}$ associated to $(P, \Lambda)$ is

$$C_{L_{\gamma_{(P, \Lambda)}}}(v) = v^{[n]}[1 + \sum_{T \subseteq [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P} 1_T) \prod_{t \in T}(1 - v^\Lambda_t)]^{-\lambda}. \quad (21)$$

This copula family is a new generalization of Farlie-Gumbel-Morgenstern Copulas.

Bekrizadeh et al., 2012 propose a similar formula only for $\lambda$ being a negative integer.

We note that, if the conditions of Theorem 15 are checked, formulas (20) and (21) define copulas.

For these copulas, if we inject univariate gamma distributions $\gamma_{(p, \lambda)}$ in the first case or $\gamma_{(p, \lambda)}$ in the second case, then we obtain other multivariate gamma distributions and other multivariate multi-factor gamma distributions in the sense that their marginal distributions are respectively $\gamma_{(p, \lambda)}$ and $\gamma_{(p, \lambda)}$. Their cdf and pdf are given by (1) and (2) respectively, and have a link with multivariate gamma distribution associated with $(P, \lambda)$ and multi-factor gamma distribution associated with $(P, \Lambda)$ respectively.

5 The bidimensional and tridimensional cases

For the bidimensional case, Theorem 15 and Theorem 16 give the following corollary.

Corollary 17 For the bivariate gamma distribution such that $L_{\gamma_{(P, \Lambda)}}(\theta) = (P(\theta))^{-\lambda}$, we have

$$C_{L_{\gamma_{(P, \Lambda)}}}(v_1, v_2) = v_1 v_2 [1 - r_{12} (1 - v_1^\Lambda + 1 - v_2^\Lambda)]^{-\lambda},$$

where, for $X = (X_1, X_2)$ with pd $\gamma_{(P, \lambda)}$, $r_{12} = -P(-p_1^{-1}, -p_2^{-1}) = 1 - p_{12}/(p_1 p_2)$ is the linear correlation coefficient of the random variables $X_1$, $X_2$; it checks $0 \leq r_{12} \leq 1$. It is the copula of the BB10 family p. 154 in the Joe’s book (Joe 1997). This result is due to the following equalities: $E(X_i) = \lambda p_i$, $\text{Var}(X_i) = \lambda p_i^2$, $\text{Cov}(X_1, X_2) = \lambda (-p_{12} + p_1 p_2)$ obtained by derivating $L_{\gamma_{(P, \Lambda)}}(\theta)$ at $0$.

For the bivariate multi-factor gamma distribution such that $L_{\gamma_{(P, \Lambda)}}(\theta) = (P(\theta))^{-\lambda} \prod_{i=1}^2 (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}$, we have a more general family

$$C_{L_{\gamma_{(P, \Lambda)}}}(v_1, v_2) = v_1 v_2 [1 - r_{12} (1 - v_1^\Lambda + 1 - v_2^\Lambda)]^{-\lambda}, \quad (22)$$

where, for $X = (X_1, X_2) = (Y_1, Y_2) + (Z_1, Z_2) = Y + Z$ as defined in Proposition 7 with pd $\gamma_{(P, \lambda)}$, and $0 \leq r_{12} = -P(-p_1^{-1}, -p_2^{-1}) = 1 - p_{12}/(p_1 p_2) \leq 1$, is the linear correlation coefficient of the random variables $Y_1, Y_2$ in the bivariate gamma distribution such that $L_{\gamma}(\theta) = [P(\theta)]^{-\lambda}$.

This bivariate family is not given in the Joe’s books (Joe, 1997, 2014). Bekrizadeh et al., 2012 give a similar formula for $\lambda$ being a negative integer.
We recall the formulas (Joe, 2014) for the computation of $\tau$, the Kendall’s tau, and $\rho_S$, the Spearman’s rho

\[
\tau = 1 - 4 \int_{[0,1]^2} \frac{\partial C}{\partial u} (u, v) \frac{\partial C}{\partial v} (u, v) \, du \, dv, \quad (23)
\]

\[
\rho_S = 12 \int_{[0,1]^2} C (u, v) \, du \, dv - 3. \quad (24)
\]

Then, we can give the following result

**Proposition 18** The Kendall’s tau and the Spearman’s rho of the Laplace copula of the bivariate multi-factor gamma distribution are given by the following formulas

\[
\tau = 1 - F_3^2 (2\lambda, 1, 1; 2\lambda_1 + 1, 2\lambda_2 + 1; r_{12}) + \frac{4\lambda}{(2\lambda_1 + 1)(2\lambda_2 + 1)} r_{12} F_3^3 (2\lambda + 1, 1, 2; 2\lambda_1 + 2, 2\lambda_2 + 2; r_{12}) - \frac{\lambda^2}{(2\lambda_1 + 1)(2\lambda_2 + 1)(\lambda_1 + 1)(\lambda_2 + 1)} F_3^4 (2\lambda + 2, 2; 2\lambda_1 + 3, 2\lambda_2 + 3; r_{12}), \quad (25)
\]

and

\[
\rho_S = 3 \left[ F_3^2 (1, 1, \lambda; 2\lambda_1 + 1, 2\lambda_2 + 1; r_{12}) - 1 \right] - \frac{3\lambda}{(2\lambda_1 + 1)(2\lambda_2 + 1)} r_{12} F_3^3 (1, 2, \lambda + 1; 2\lambda_1 + 2, 2\lambda_2 + 2; r_{12}). \quad (26)
\]

For the tridimensional case, Theorem 15 and Theorem 16 give the following corollary

**Corollary 19** For the trivariate gamma distribution $\gamma_{(p, \lambda)}$ such that $L_{\gamma_{(p, \lambda)}} (\theta) = (P (\theta))^{-\lambda}$, we have

\[
C_{L_{\gamma_{(p, \lambda)}}} (v_1, v_2, v_3) = v_1 v_2 v_3 [1 - r_{12} (1 - v_1^\frac{1}{\lambda}) (1 - v_2^\frac{1}{\lambda}) - r_{13} (1 - v_1^\frac{1}{\lambda}) (1 - v_3^\frac{1}{\lambda}) - r_{23} (1 - v_2^\frac{1}{\lambda}) (1 - v_3^\frac{1}{\lambda})] - \frac{3\lambda}{(2\lambda_1 + 1)(2\lambda_2 + 1)} r_{12} F_3^3 (1, 2, \lambda + 1; 2\lambda_1 + 2, 2\lambda_2 + 2; r_{12}), \quad (27)
\]

where, for $X = (X_1, X_2, X_3)$ with pd $\gamma_{(p, \lambda)}$, we denote by $r_{ij} = -P (\frac{\lambda}{p} \mathbf{1}_{(i,j)}) = 1 - p_{ij}/(p_ip_j)$, $1 \leq i \neq j \leq 3$, the linear correlation coefficient of the random variables $X_i, X_j$, they check $0 \leq r_{ij} \leq 1$, and we denote by $r_{123}$ the number defined by

\[
r_{123} = \frac{\mathbb{E} \left( \prod_{i=1}^{3} [X_i - \mathbb{E} (X_i)] \right) / \prod_{i=1}^{3} (\mathbb{E} ([X_i - \mathbb{E} (X_i)]^2))^{1/3}}{\prod_{i=1}^{3} (\mathbb{E} ([X_i - \mathbb{E} (X_i)]^2))^{1/2}} = \frac{-\frac{\lambda}{p} P (-p_1^{-1}, -p_2^{-1}, -p_3^{-1})}{\prod_{i=1}^{3} (\mathbb{E} ([X_i - \mathbb{E} (X_i)]^2))^{1/2}} \quad (28)
\]

(to compare with $r_{12} = \mathbb{E} \left( \prod_{i=1}^{2} [X_i - \mathbb{E} (X_i)] \right) / \prod_{i=1}^{2} (\mathbb{E} ([X_i - \mathbb{E} (X_i)]^2))^{1/2}$).

This last result is due to the following equalities: $\mathbb{E} (X_i) = \lambda p_i$, $\mathbb{E} ([X_i - \mathbb{E} (X_i)]^3) = 2\lambda p_i^3$, $\mathbb{E} \left( \prod_{i=1}^{3} [X_i - \mathbb{E} (X_i)] \right) = -\lambda p_1 p_2 p_3 P (-p_1^{-1}, -p_2^{-1}, -p_3^{-1})$ obtained by deriving $L_{\gamma_{(p, \lambda)}} (\theta)$ at $\theta = 0$.  

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For the trivariate multi-factor gamma distribution such that $L_{\gamma(P,\Lambda)}(\theta) = [P(\theta)]^{-\lambda} \Pi_{i=1}^{3} (1 + p_i \theta_i)^{-(\lambda_i - \lambda)}$ ($\lambda_i \geq \lambda$, $i = 1, 2, 3$), we have

$$C_{L_{\gamma(P,\Lambda)}}(v_1, v_2, v_3) = v_1 v_2 v_3 [1 - r_{12}(1 - v_1^\lambda)(1 - v_2^\lambda) - r_{13}(1 - v_1^\lambda)(1 - v_3^\lambda)]$$

$$- r_{23}(1 - v_2^\lambda)(1 - v_3^\lambda) + 2r_{123}(1 - v_1^\lambda)(1 - v_2^\lambda)(1 - v_3^\lambda)]^{-\lambda},$$

where, for $X = (X_1, X_2, X_3) = (Y_1, Y_2, Y_3) + (Z_1, Z_2, Z_3) = Y + Z$ as defined in Proposition 7 with pd $\gamma(P,\Lambda)$, we denote by $r_{ij} = -P(-\frac{1}{p_i} (1, 1, 1)) = 1 - \frac{p_i}{p_j}$, $1 \leq i \neq j \leq 3$, the linear correlation coefficient of the random variables $Y_i, Y_j$ in the trivariate gamma distribution such that $L_{\gamma}(\theta) = (P(\theta))^{-\lambda}$, they check $0 \leq r_{ij} \leq 1$, and $r_{123}$ is defined by equality (28).

6 Appendix

6.1 Proof of Theorem 3

Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be independent random variables. The pdf of $Z = (X_1 + Y_1, X_2 + Y_2)$ where $X_i$ has pd $\gamma(p_i, \lambda_i - \lambda)$, $i = 1, 2$ and $Y$ has pd $\gamma(P, \lambda)$, is obtained by convolution. By changing variables $1 - v_i/z_i = u_i$, $i = 1, 2$, and with the notation $c = b_{12}(\tilde{P}) = -p_{12} + p_1 p_2$, $d_i = c p_{12}^{-1}$, $i = 1, 2$ we obtain the pdf $f_Z$ of $Z$ by $f_Z(z_1, z_2) = 0$ if $(z_1, z_2) \notin (0, \infty)^2$ and for $(z_1, z_2) \in (0, \infty)^2$

$$f_Z(z_1, z_2) = \int_0^{z_1} \int_0^{z_2} \frac{1}{[\Gamma(\lambda)]^2 p_{12}^\lambda} e^{-\left(\frac{z_1 - v_1}{p_1} + \frac{z_2 - v_2}{p_2}\right)} (v_1 v_2)^{\lambda - 1} F_1(\lambda; c v_1 v_2)$$

$$\times e^{-\frac{z_1 - v_1}{p_1}} \Gamma(\lambda_1 - \lambda) \left(\frac{z_1 - v_1}{p_1}\right)^{\lambda_1 - \lambda} e^{-\frac{z_2 - v_2}{p_2}} \Gamma(\lambda_2 - \lambda) \left(\frac{z_2 - v_2}{p_2}\right)^{\lambda_2 - \lambda} \frac{dv_1}{z_1 - v_1} \frac{dv_2}{z_2 - v_2}$$

$$= \frac{(c z_1 z_2)^k}{\Gamma(\lambda) \Gamma(\lambda_1 - \lambda) \Gamma(\lambda_2 - \lambda) p_{12}^\lambda p_1^{\lambda_1 - \lambda} p_2^{\lambda_2 - \lambda}} \sum_{k \geq 0} (c z_1 z_2)^k$$

$$\times \int_0^1 e^{d_1 z_1 u_1} u_1^{\lambda_1 - \lambda - 1} (1 - u_1)^{\lambda + k - 1} du_1 \int_0^1 e^{d_2 z_2 u_2} u_2^{\lambda_2 - \lambda - 1} (1 - u_2)^{\lambda + k - 1} du_2. \quad (29)$$

As we have, with the notation $B(\alpha, \beta) = \int_0^1 u^{\alpha - 1} (1 - u)^{\beta - 1} du = \Gamma(\alpha) \Gamma(\beta) / \Gamma(\alpha + \beta)$, $\alpha, \beta > 0$ for the Euler’s Beta function,

$$\int_0^1 e^{su} u^{\alpha - 1} (1 - u)^{\beta - 1} du = B(\alpha, \beta) \sum_{n=0}^{\infty} \frac{\alpha^n}{(\alpha + \beta)^n} \frac{\delta^n}{n!},$$

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we obtain
\[
f_{\mathbf{z}}(z) = \frac{z^{\lambda_1-1}z_2^{\lambda_2-1}e^{-\left(\frac{z_1}{\lambda_1}z_1 + \frac{z_2}{\lambda_2}z_2\right)}}{\Gamma(\lambda_1) \Gamma(\lambda_1 - \lambda) \Gamma(\lambda_2 - \lambda)} \sum_{k \geq 0} \frac{(cz_2)^k}{\Gamma(\lambda_1 + k) \Gamma(\lambda_1 - k)}
\times \sum_{n=0}^{\infty} (\lambda_1 - \lambda)_n (d_1 z_1)^n \Gamma(\lambda_2 - \lambda) \sum_{m=0}^{\infty} (\lambda_2 - \lambda)_m (d_2 z_2)^m \frac{\Gamma(\lambda_1 + k + n) \Gamma(\lambda_1 - k + m) (cz_2)^k}{n! \cdot m!}.
\]

Hence, we have proved the formula (11).

6.2 Proof of Theorem [7]

We start from the Laplace transform of the trivariate gamma distribution \(\gamma(\lambda, P)\) associated to \((\lambda, P)\).

First, we write
\[
L_{\gamma(\lambda, P)}(\bm{\theta}) = (1 + p_1 \theta_1 + p_2 \theta_2 + p_3 \theta_3 + p_{12} \theta_1 \theta_2 + p_{13} \theta_1 \theta_3 + p_{23} \theta_2 \theta_3 + p_{123} \theta_1 \theta_2 \theta_3)^{-\lambda}
= (1 + p_3 \theta_3)^{-\lambda} \left(1 + \frac{p_1 + p_{13} \theta_3}{1 + p_3 \theta_3} \theta_1 + \frac{p_2 + p_{23} \theta_3}{1 + p_3 \theta_3} \theta_2 + \frac{p_{12} + p_{123} \theta_3}{1 + p_3 \theta_3} \theta_1 \theta_2\right)^{-\lambda}.
\]

Let \(Q(\theta_1, \theta_2) = 1 + q_1 \theta_1 + q_2 \theta_2 + q_{12} \theta_1 \theta_2\) where \(q_1 = (p_1 + p_{13} \theta_3) / (1 + p_3 \theta_3), q_2 = (p_2 + p_{23} \theta_3) / (1 + p_3 \theta_3), q_{12} = (p_{12} + p_{123} \theta_3) / (1 + p_3 \theta_3).\) With these notations, we have
\[
L_{\gamma(\lambda, P)}(\bm{\theta}) = (1 + p_3 \theta_3)^{-\lambda} (1 + q_1 \theta_1 + q_2 \theta_2 + q_{12} \theta_1 \theta_2)^{-\lambda}.
\]

Let \(\tilde{p}_T = -p_T \bar{p}_{123}, \tilde{b}_{ij} = \bar{p}_{ij} + \bar{p}_i \bar{p}_j,\) and \(\tilde{b}_{123} = \bar{p}_{123} + \bar{p}_1 \bar{p}_{23} + \bar{p}_2 \bar{p}_{13} + \bar{p}_3 \bar{p}_{12} + 2 \bar{p}_1 \bar{p}_2 \bar{p}_3,\) then we have
\[
b_{1,2}(\tilde{Q}) = \frac{\tilde{b}_{12} + \tilde{b}_{123}}{\tilde{Q}_{12}} \left(\tilde{Q}_{13} - \tilde{Q}_{12}\right)^{-\lambda}.
\]

Let \(\gamma(\lambda, Q)\) be the bivariate gamma distribution associated to \((\lambda, Q)\), its Laplace transform is
\[
(1 + q_1 \theta_1 + q_2 \theta_2 + q_{12} \theta_1 \theta_2)^{-\lambda}.
\]

We have
\[
\gamma(\lambda, Q)(dx) = \frac{\tilde{Q}_{12}^{\lambda}}{[\Gamma(\lambda)]^2} e^{-\frac{\tilde{Q}_{12} x_1}{\tilde{Q}_{12} x_2}} e^{-\frac{\tilde{Q}_{13} x_1}{\tilde{Q}_{12} x_2}} (x_1 x_2)^{\lambda-1} F_1[\lambda, b_{1,2}(\tilde{Q}) x_1 x_2] 1_{(0, \infty)^2}(x_1, x_2) \, dx_1 \, dx_2.
\]

Second, we are looking for the unidimensional distribution for which its Laplace transform with respect
to the variable $\theta_3$ is equal to

$$
(1 + p_3 \theta_3)^{-\lambda} \frac{q_{12}^{\lambda}}{(\Gamma (\lambda))^{\frac{k}{2}}} \exp \left(-\frac{1}{2} \theta_3 - \frac{1}{2} \frac{q_{12}^{\lambda}}{(\Gamma (\lambda))^{\frac{k}{2}}} (x_1, x_2)^{\lambda-1} F_1[\lambda, b_{12}(Qx_1, x_2)] \mathbb{I}_{(0, \infty)^2} (x_1, x_2) \, dx_1 dx_2
$$

$$
= \frac{p_{123}^{\lambda}}{[\Gamma (\lambda)]^2} \exp \left(\bar{p}_{12} (x_1 + \bar{p}_{23}) (x_1, x_2)^{\lambda-1} (\bar{b}_{12} \theta_3 - \bar{p}_{3}) \exp \left(\bar{b}_{13} x_1 + \bar{b}_{23} x_2 \right) \right)

\times F_1\{\lambda, \tilde{b}_{12} + \bar{b}_{13} x_1 + \bar{b}_{23} x_2 \} \mathbb{I}_{(0, \infty)^2} (x_1, x_2) \, dx_1 dx_2.
$$

(30)

From equality (30) and (Hladik, 1969), we obtain the unidimensional distribution for which its Laplace transform is

$$
\frac{p_{123}^{\lambda}}{[\Gamma (\lambda)]^2} \exp \left(\bar{p}_{12} (x_1 + \bar{p}_{23}) (x_1, x_2)^{\lambda-1} (\bar{b}_{12} \theta_3 - \bar{p}_{3}) \exp \left(\bar{b}_{13} x_1 + \bar{b}_{23} x_2 \right) \right)

\times F_1\{\lambda, \tilde{b}_{12} + \bar{b}_{13} x_1 + \bar{b}_{23} x_2 \} \mathbb{I}_{(0, \infty)^2} (x_1, x_2) \, dx_1 dx_2.
$$

(31)

Third, we are looking for the unidimensional distribution for which its Laplace transform with respect to the variable $\theta_3$ is equal to

$$
\theta_3^{-\lambda} \exp \left(\tilde{b}_{13} x_1 + \tilde{b}_{23} x_2 \right) F_1\{\lambda, \tilde{b}_{12} \theta_3 + \tilde{b}_{13} x_1 + \tilde{b}_{23} x_2 \}

= \sum_{k=0}^{\infty} \frac{1}{(\lambda)^k} (x_1, x_2)^k \sum_{\ell+m+n=k} \frac{1}{\ell!m!n!} \tilde{b}_{13}^{\ell} \tilde{b}_{23}^{m} \tilde{b}_{12}^{n} \theta_3^{(\lambda + 2 \ell + m)} \exp \left(\tilde{b}_{13} x_1 + \tilde{b}_{23} x_2 \right)
$$

(32)

From (Hadlik, 1986), we have the following equality

$$
L_{[\Gamma (\lambda)]^{-1} e^{\lambda-1} F_1(\lambda, at) \mathbb{I}_{(0, \infty)} (x_3) \, dx_3} = s^{-\lambda} \exp \left(\frac{a}{s} \right)
$$

(33)

We utilize the equality (33) in (32) and obtain the following unidimensional distribution

$$
\sum_{k=0}^{\infty} \frac{1}{(\lambda)^k} (x_1, x_2)^k \sum_{\ell+m+n=k} \frac{1}{\ell!m!n!} \tilde{b}_{13}^{\ell} \tilde{b}_{23}^{m} \tilde{b}_{12}^{n} \theta_3^{(\lambda + 2 \ell + m)} \mathbb{I}_{(0, \infty)^2} (x_3) \, dx_3
$$

$$
= \mathbb{I}_{(0, \infty)} (x_3)

\times F_1\left[\lambda + 2 \ell + m, (\tilde{b}_{13} x_1 + \tilde{b}_{23} x_2) x_3 \right] \mathbb{I}_{(0, \infty)} (x_3) \, dx_3
$$

$$
= \mathbb{I}_{(0, \infty)} (x_3)

\times \sum_{(\ell, m, n) \in \mathbb{N}^3} \frac{1}{(\lambda)^{\ell+m+n} (\lambda)^{2 \ell + m + n}} \left(\tilde{b}_{13} x_1 x_3 \tilde{b}_{23} x_2 x_3 \right)^{\ell} \left(\tilde{b}_{13} x_1 x_3 \tilde{b}_{23} x_2 x_3 \right)^{m} \left(\tilde{b}_{12} x_1 x_3 \right)^{n} \, dx_3
$$

(34)
We carry this last equality \((34)\) in \((31)\) and we obtain the pd \((15)\).

### 6.3 Proof of Theorem 15

We utilize the following notations, for \(i \in [n]\), \(L_{\gamma(P, \lambda)}(\theta_i) = (1 + p_i \theta_i)^{-\lambda} = v_i, 0 < \lambda_i < \lambda \) and \(1 + p_i \theta_i = u_i\). In particular \(\theta_i = (u_i - 1)/p_i = \left( v_i^{-1/\lambda} - 1 \right)/p_i \) and \(u_i = v_i^{-1/\lambda}\). We remark that \(u_i = 0 \iff \theta_i = -1/p_i\).

We start from the Laplace transform of the multivariate gamma distribution associated to \((P, \lambda)\). From the definition of \(\gamma(P, \lambda)\), we obtain

\[
L_{\gamma(P, \lambda)}(\theta) = \sum_{T \subset [n]} p_T \theta^T.
\]

Hence, we have

\[
[L_{\gamma(P, \lambda)}(\theta)]^{-\frac{1}{\lambda}} = \sum_{T \subset [n]} p_T \theta^T \left( \frac{1}{P} \right)^T = \sum_{T \subset [n]} q_T u^T = Q(u)
\]

where \(u = (u_1, \ldots, u_n)\), and

\[
Q(u) = \sum_{T \subset [n]} \alpha_T (u - 1_{[n]})^T u^T,
\]

indeed, the polynomials \((u - 1_{[n]})^T u^T, T \subset [n]\) form a basis of the vector space of affine polynomials of degree less or equal to \(n\). Because \(u_i = 0 \iff \theta_i = -1/p_i\), for \(i \in [n]\), we have

\[
Q(0) = P(- \frac{1}{P} 1_{[n]}) = (-1)^n \alpha_{[n]};
\]

consequently

\[
\alpha_{[n]} = (-1)^n P(- \frac{1}{P} 1_{[n]}).
\]

Since \(P\) is an affine polynomial, we have

\[
\alpha_T = (-1)^{|T|} P(- \frac{1}{P} 1_T),
\]

in particular \(\alpha_T = 0\), if \(|T| = 1\) and \(\alpha_\emptyset = 1\). Finally we obtain

\[
P(\theta) = \sum_{T \subset [n]} (-1)^{|T|} P(- \frac{1}{P} 1_T) (u - 1_{[n]})^T u^T
\]

\[(35)\]

\[
= u^{[n]} + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(- \frac{1}{P} 1_T) (u - 1_{[n]})^T u^T
\]
We start from the Laplace transform of the multivariate multi-factor gamma distribution associated to \((P, \Lambda)\), \(\Lambda = (\lambda, \lambda_1, \ldots, \lambda_n)\), where \(P\) is an affine polynomial with respect to \(\theta = (\theta_1, \ldots, \theta_n)\) and \(\lambda_i \geq \lambda\), for \(i \in [n]\). We use the following notations \(u_i = (1 + p_i \theta_i)^{-\lambda}\) and \(v_i = (1 + p_i \theta_i)^{-\lambda_i}\) so that \((1 + p_i \theta_i)^{-(\lambda_i - \lambda)} = u_i / u_i^\lambda\) and \((1 + p_i \theta_i)^{-1} = u_i^{1/\lambda} = v_i^{1/\lambda_i}\). According to equality \((\ref{eq:20})\) we have

\[
\begin{aligned}
L_{\gamma(P, \Lambda)}(\theta) &= [P(\theta)]^{-\lambda} \prod_{i \in [n]} (1 + p_i \theta_i)^{-(\lambda_i - \lambda)} \\
&= u^{[n]}[1 + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P} 1_T)(1 - u_i^\lambda)]^{-\lambda} \prod_{i \in [n]} \frac{v_i}{u_i} \\
&= v^{[n]}[1 + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P} 1_T)(1 - u_i^\lambda)]^{-\lambda} \\
&= C_{L_{\gamma(P, \Lambda)}}(v).
\end{aligned}
\]

### 6.4 Proof of Theorem 16

We start from the Laplace transform of the multivariate multi-factor gamma distribution associated to \((P, \Lambda)\), \(\Lambda = (\lambda, \lambda_1, \ldots, \lambda_n)\), where \(P\) is an affine polynomial with respect to \(\theta = (\theta_1, \ldots, \theta_n)\) and \(\lambda_i \geq \lambda\), for \(i \in [n]\). We use the following notations \(u_i = (1 + p_i \theta_i)^{-\lambda}\) and \(v_i = (1 + p_i \theta_i)^{-\lambda_i}\) so that \((1 + p_i \theta_i)^{-(\lambda_i - \lambda)} = u_i / u_i^\lambda\) and \((1 + p_i \theta_i)^{-1} = u_i^{1/\lambda} = v_i^{1/\lambda_i}\). According to equality \((\ref{eq:20})\) we have

\[
\begin{aligned}
L_{\gamma(P, \Lambda)}(\theta) &= [P(\theta)]^{-\lambda} \prod_{i \in [n]} (1 + p_i \theta_i)^{-(\lambda_i - \lambda)} \\
&= u^{[n]}[1 + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P} 1_T)(1 - u_i^\lambda)]^{-\lambda} \prod_{i \in [n]} \frac{v_i}{u_i} \\
&= v^{[n]}[1 + \sum_{T \subset [n], |T| > 1} (-1)^{|T|} P(-\frac{1}{P} 1_T)(1 - u_i^\lambda)]^{-\lambda} \\
&= C_{L_{\gamma(P, \Lambda)}}(v).
\end{aligned}
\]

### 6.5 Proof of Proposition 18

#### 6.5.1 Kendall’s tau

By injecting the given copula in formula \((\ref{eq:22})\) in formula \((\ref{eq:23})\), we obtain

\[
\frac{1 - \tau}{4} = \int_{[0,1]^2} u_1 u_2 [1 - r_{12}(1 - u_1^{1/\lambda})(1 - u_2^{1/\lambda})]^{-2\lambda - 2} \\
\{1 - r_{12}[1 - (1 - \frac{\lambda}{\lambda_1})u_1^{1/\lambda}](1 - u_2^{1/\lambda})\} \{1 - r_{12}[1 - (1 - \frac{\lambda}{\lambda_2})u_2^{1/\lambda}](1 - u_1^{1/\lambda})\} du_1 du_2.
\]
By changing the variables $t_i = 1 - u_i^{1/\lambda_i}$, $i = 1, 2$, we obtain

\[
\frac{1 - \tau}{4\lambda_1 \lambda_2} = \int_0^1 \int_0^1 (1 - r_{12} t_1 t_2)^{-2\lambda} (1 - t_1)^{2\lambda_1 - 1} (1 - t_2)^{2\lambda_2 - 1}
\]

\[
- \frac{\lambda}{\lambda_1} r_{12} \sum_{k=0}^\infty (2\lambda + 1)_k \frac{r_{12}^k}{k!} B (k + 1, 2\lambda_1) B (k + 1, 2\lambda_2)
\]

\[
- \frac{\lambda}{\lambda_2} r_{12} \sum_{k=0}^\infty (2\lambda + 1)_k \frac{r_{12}^k}{k!} B (k + 2, 2\lambda_1) B (k + 1, 2\lambda_2 + 1)
\]

\[
+ \frac{\lambda^2}{\lambda_1 \lambda_2} r_{12}^2 \sum_{k=0}^\infty (2\lambda + 2)_k \frac{r_{12}^k}{k!} B (k + 2, 2\lambda_1 + 1) B (k + 2, 2\lambda_2 + 1).
\]

By using equality $(1 - r_{12} t_1 t_2)^{-2\lambda} = \sum_{k=0}^\infty (2\lambda)_k \frac{r_{12}^k}{k!}$ in the last result, we obtain,

\[
1 - \tau = \sum_{k=0}^\infty \frac{(1)_k (1)_k (2\lambda)_k}{(2\lambda_1 + 1)_k (2\lambda_2 + 1)_k} \frac{r_{12}^k}{k!}
\]

\[
- \frac{4\lambda}{(2\lambda_1 + 1)(2\lambda_2 + 1)} r_{12} \sum_{k=0}^\infty \frac{(1)_k (2)_k (2\lambda + 1)_k}{(2\lambda_1 + 2)_k (2\lambda_2 + 2)_k} \frac{r_{12}^k}{k!}
\]

\[
+ \frac{\lambda^2}{(2\lambda_1 + 1)(2\lambda_2 + 1)(\lambda_1 + 1)(\lambda_2 + 1)} r_{12}^2 \sum_{k=0}^\infty \frac{(2)_k (2)_k (2\lambda + 2)_k}{(2\lambda_1 + 3)_k (2\lambda_2 + 3)_k} \frac{r_{12}^k}{k!}.
\]

With the equality $B (\alpha, \beta) = \Gamma (\alpha) \Gamma (\beta) / \Gamma (\alpha + \beta)$ for $\alpha, \beta > 0$, we obtain

\[
1 - \tau = \sum_{k=0}^\infty \frac{(1)_k (1)_k (2\lambda)_k}{(2\lambda_1 + 1)_k (2\lambda_2 + 1)_k} \frac{r_{12}^k}{k!}
\]

\[
- \frac{4\lambda}{(2\lambda_1 + 1)(2\lambda_2 + 1)} r_{12} \sum_{k=0}^\infty \frac{(1)_k (2)_k (2\lambda + 1)_k}{(2\lambda_1 + 2)_k (2\lambda_2 + 2)_k} \frac{r_{12}^k}{k!}
\]

\[
+ \frac{\lambda^2}{(2\lambda_1 + 1)(2\lambda_2 + 1)(\lambda_1 + 1)(\lambda_2 + 1)} r_{12}^2 \sum_{k=0}^\infty \frac{(2)_k (2)_k (2\lambda + 2)_k}{(2\lambda_1 + 3)_k (2\lambda_2 + 3)_k} \frac{r_{12}^k}{k!}.
\]

Finally, by using equality (36), we have proved equality (25).

### 6.5.2 Spearman’s rho

By injecting the given copula in formula (22) in formula (24), we obtain

\[
g_S = 12 \int_0^1 \int_0^1 u_1 u_2 [1 - r_{12} (1 - u_1^{1/\lambda}) (1 - u_2^{1/\lambda})]^{-\lambda} du_1 du_2 - 3. \tag{36}
\]
By changing the variables \( v_i = u_i^{-1}, i = 1, 2 \), we obtain

\[
\int_0^1 \int_0^1 u_1 u_2 [1 - r_{12} \left( (1 - u_1^{-1}) (1 - u_2^{-1}) \right)]^{-\lambda} \, du_1 \, du_2
\]

\[
= \int_0^1 \int_0^1 \lambda_1 \lambda_2 v_1^{2\lambda_1 - 1} v_2^{2\lambda_2 - 1} [1 - r_{12} (1 - v_1) (1 - v_2)]^{-\lambda} \, dv_1 \, dv_2
\]

\[
= \lambda_1 \lambda_2 \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \int_0^1 v_1^{2\lambda_1 - 1} (1 - v_1)^k \, dv_1 \int_0^1 v_2^{2\lambda_2 - 1} (1 - v_2)^k \, dv_2
\]

\[
= \lambda_1 \lambda_2 \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} r^2_{12} B(2\lambda_1, k + 1) B(2\lambda_2, k + 1)
\]

\[
= \frac{1}{4} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(2\lambda_1 + 1)_k (2\lambda_2 + 1)_k} \frac{r^2_{12}}{k!}
\]

\[
= \frac{1}{4} F_2^3 (1, 1, \lambda; 2\lambda_1 + 1, 2\lambda_2 + 1; r_{12})
\]

(37)

By injecting equality (37) in equality (36), we obtain equality (26). We remark that we have

\[
F_2^3 (\lambda, 1, 1; 2\lambda_1 + 1, 2\lambda_2 + 1; r_{12}) - 1 = \frac{\lambda}{(2\lambda_1 + 1) (2\lambda_2 + 1)} r_{12} \left[ \sum_{k=0}^{\infty} (k + 1) \frac{(\lambda + 1)_k}{(2\lambda_1 + 2)_k (2\lambda_2 + 2)_k} r_{12}^k \right].
\]

This gives equality (27).

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References

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