Abstract—In this paper, we analyze the error probability of reconfigurable intelligent surfaces (RIS)-enabled communications with quantized channel phase compensation over Rayleigh fading channels. The probability density and characteristic functions of the received signal amplitude are derived and used to compute exact expressions for the bit error rate (BER). The resulting expressions are general, as they hold for an arbitrary number of reflecting elements \( N \), and quantization levels, \( L \). We introduce an exact asymptotic analysis in the high signal-to-noise ratio (SNR) regime, from which we demonstrate, in particular, that the diversity order is \( N/2 \) when \( L = 2 \) and \( N \) when \( L > 2 \). The theoretical frameworks and findings are validated with the aid of Monte Carlo simulations.

Index Terms—Reconfigurable intelligent surfaces, phase errors, BER analysis, Rayleigh fading, diversity order.

I. INTRODUCTION

Recently, a major design make-over of wireless communications enabled by reconfigurable intelligent surfaces (RISs), has become an extremely active and promising research topic [1], [2]. Thanks to their capability to easily manipulate the incident wave properties (frequency, amplitude, phase) to establish favourable channel responses [1]–[3] (and references therein), RIS offer undoubtedly an additional degree of freedom to boost capacity and enhance coverage with low cost and power-efficient infrastructure in 6G networks. Nevertheless, several hurdles must be overcome to enable RIS-assisted communications and make them work properly. One of the major challenges facing the implementation of RIS is their vulnerability to phase compensation errors at RIS elements [4]. In fact, while a suitable design of the phase shifts of the reflecting elements is necessary to reap the advantages of RIS-aided transmission [3]–[6], it is, however, difficult to implement these phases in practice due to hardware limitations. As a consequence, phase quantization errors inevitably arise.

Motivated by these considerations, recent attempts for studying RIS-assisted systems in the presence of phase errors include the use of approximate distributions and asymptotic analysis [7]–[11]. By using the central limit theorem (CLT), the authors of [7] and [8] obtained approximate BER expressions considering a large number of reconfigurable elements at the RIS. It was recently shown, however, that the approximation error attributed to the CLT can be significant for small number of elements with important discrepancies in the high signal-to-noise ratio (SNR) regime [7], [9]. In [10], the authors derived an approximate error performance of RIS in the presence of phase errors by leveraging the moment-based Gamma approximation. The Gamma-based framework appears, however, unsuitable for diversity analysis since it fails to extract the full diversity order even in the absence of phase errors. Recently, the authors of [9] and [11] identified sufficient conditions based on upper and lower bounds for ensuring that RIS-assisted systems achieve the full diversity order in the presence of phase noise. More specifically, it was shown in [9] that if the absolute difference between pairs of phase errors is less than \( \pi/2 \), RIS-assisted communications achieve full diversity.

Although the results from [7]–[11] are insightful, these works have been successfully tractable due to approximate SNR distributions and bounds yielding a diversity analysis in the presence of phase errors which is, so far, steadily inaccurate [9], [11]. Moreover, to the best of our knowledge, no exact error analysis for RIS with quantized phase shifts and arbitrary number of reconfigurable elements has been reported in the literature.

In this letter, we investigate the impact of quantized phase shifts on the performance of RIS-enabled communications. Analytical expressions for the bit error rate (BER) of binary phase shift keying (BPSK) modulated signals over Rayleigh fading channels, as a function of the number of phase quantization levels \( L \) (also known as phase resolution) and the number of reflecting elements \( N \), are derived based on exact expressions for the probability density function (PDF) and the characteristic function (CHF) of the combined signal amplitude. Approximate expressions for the BER are derived in the high-SNR regime yielding simple closed-form expressions which can be used to determine the coding gain, the diversity order, and the degradation in the system performance. The obtained results unveil, in particular, that for two level quantization, the diversity order is equal to half of the reconfigurable elements number. For higher quantization levels, RIS-assisted communications achieve full diversity order, despite the BER is not a linear function of the SNR (in dB scale).

II. SYSTEM MODEL

We consider an RIS with \( N \) reconfigurable elements which is deployed to assist data transmission to a single antenna receiver by reflecting an incident RF wave emitted by a single antenna transmitter. More specifically, we assume that
the direct transmission link between the transmitter and the receiver is blocked, and, thus, the RIS is deployed to relay the scattered signal and to leverage virtual line-of-sight (LOS) paths for enhancing the strength of the received signal. The received signal of the considered system is

\[ r = \sum_{i=1}^{N} h_i g_i e^{j(\theta_i - \theta_q^i) + \eta}, \]

where \( \alpha \) is the baseband transmitted symbol, \( \eta \) represents a zero mean complex additive white Gaussian noise (AWGN) with variance \( N_0 / 2 \), the channel amplitudes \( h_i \) and \( g_i \) are assumed to be Rayleigh distributed with \( \theta_i \) uniformly distributed between \([-\pi, \pi]\), and \( \theta_q^i \) is denoting the quantized phase shift induced by the \( i \)-th reflecting element at the RIS. For an \( L \)-level uniform quantizer the signal phase can be modeled as

\[ \theta_q^i = \theta_i - \theta_e^i, \]

where \( \theta_e^i \) represents the phase error which is uniformly distributed between \([-\pi / L, \pi / L]\).

In the rest of the work, we consider that \( \alpha \) is chosen from a BPSK signal constellation set \( \alpha \in \{-1, +1\} \). Hence, only the real part of the signal has an impact on the detection performance. This reduces (1) to

\[ r = \sum_{i=1}^{N} h_i g_i v_i + \text{Re}[\eta], \]

where \( v_i \triangleq \cos(\theta_e^i) \), and \( \text{Re}[\cdot] \) denotes the real part of a complex quantity. The instantaneous SNR is defined as

\[ \gamma = \rho \left( \sum_{i=1}^{N} h_i g_i v_i \right)^2, \]

with \( \rho \) denoting the transmit SNR.

### III. Average BER With Phase Noise

The evaluation of the error probability usually involves the computation of the PDF of \( x = \sqrt{\gamma} \) which is formulated in terms of a linear combination of the product of random variables. A common approach for analyzing the distribution of \( x \) is to leverage the CLT. However, this approach is accurate only for a large number of reconfigurable elements and the resulting analysis is usually not accurate in the high-SNR regime and, therefore, for analyzing the attainable diversity order. In general, the calculation of the exact error probability of RIS-assisted communications with phase noise is an open research issue, and is very intricate for arbitrary values of \( N \).

To tackle this issue, we proceed by writing the BER expression using the CHF-based approach as

\[ P_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t) \phi_x(t) dt, \]

where, under BPSK signalling, we have

\[ G(t) = \frac{1}{2t} \left( \frac{t}{\sqrt{\pi}} F_1\left(1, \frac{3}{2}; \frac{t^2}{4} \right) + j - je^{-\frac{t^2}{2}} \right), \]

where \( j = \sqrt{-1} \) and \( F_1(\cdot) \) represents the confluent hypergeometric function. Moreover, we denote \( \phi_x(t) \) the complex conjugate of the CHF of \( x \) defined as

\[ \phi_x(t) = \prod_{i=1}^{N} \phi_{z_i}(\sqrt{\rho t}), \]

where \( z_i = h_i g_i v_i, i = 1, \ldots, N \), and \( \phi_{z_i}(t) = \mathbb{E}_{z_i}[e^{jtzi}] \), with \( \mathbb{E}\{\cdot\} \) denoting the expectation operator. Thus, to evaluate the error probability, we need first to determine the CHF of \( z_i \), which is a product of three random variables.

**Proposition 1.** The PDF of \( z_i \) can be expressed as

\[ f_{z_i}(z) = L e^{-2z} - \frac{2L}{\pi} \int_{0}^{\pi/2} e^{-2z\left(\sqrt{1 + \frac{\tan(\pi/L)^2}{\sin(\psi)^2}}\right)} d\psi. \]

**Proof.** As noted earlier, \( \theta_e^i \) is uniformly distributed on the interval \([-\pi / L, \pi / L]\). As a result, the PDF of \( v_i \) is given by

\[ f_{v_i}(v) = \frac{L}{\pi \sqrt{1 - v^2}} \quad \text{for } \cos(\pi / L) \leq v \leq 1. \]

Recall that under independent and identically distributed (i.i.d.) Rayleigh fading \( f_{g_i}(x) = 2xe^{-x^2} \), for \( y \in \{h, g\} \), then we use (3) to calculate the PDF of \( z_i = h_i g_i v_i \) as

\[ f_{z_i}(z) = \int_{0}^{\infty} \frac{1}{x} f_{g_i}(x)f_{g_i}(v_i) \left( \frac{z}{x} \right) dx, \]

where

\[ f_{g_i}(x) = \frac{2L}{\pi} \int_{\cos(\pi/L)}^{1} e^{-2z^2} z^2 \sqrt{1 - z^2} dz. \]

By relabeling \( z = \sin(\theta) \) in (11), and applying [19] Eq. (2.33.2), we obtain

\[ f_{g_i}(x) = \frac{L}{\sqrt{\pi}} e^{-x^2} \text{erf} \left( x \tan \left( \frac{\pi}{L} \right) \right), \]

where \( \text{erf}(\cdot) \) is the error function. Substituting (12) into (10) while resorting to the alternative form

\[ \text{erf}(z) = 1 - \frac{2}{\pi} \int_{0}^{\pi/2} e^{-x^2} \sin(\psi) d\psi, \]

we obtain the PDF in [8] with the aid of [19]. This completes the proof. \( \square \)

**Remark 1.** Based on Proposition 1, we evince that for 1-bit quantization based RIS communications (i.e., when \( L = 2 \)), the PDF of \( z_i \) reduces to

\[ f_{z_i}(z) = 2e^{-2z}, \quad z \geq 0 \]

which is in agreement with [8] Eq. (17).

Using (8), the CHF of \( z_i \) follows from \( \phi_{z_i}(t) = \int_{0}^{\infty} f_{z_i}(z) e^{jtz} dz \) and is obtained as

\[ \phi_{z_i}(t) = \frac{L}{jt + 2} - \frac{2L}{\pi} \int_{0}^{\pi/2} \left( jt + 2 \left( \sqrt{1 + \frac{\tan(\pi)^2}{\sin(\psi)^2}} \right) \right)^{-1} d\psi. \]

[1] Although the analysis in this work is limited to BPSK modulation for tractability reasons, the proposed method can be used to evaluate the BER of other modulation schemes as well. For instance, for QPSK modulation, the term \( v_i \) in [3] can be replaced with \( v_i = \sqrt{2} \cos(\theta_e^i) \pm \frac{1}{2} \) [13]. The detailed analysis of higher-order QAM modulations is more involved, and is an important extension that we will address in future work.
Note that the CHF given in [15] can be efficiently estimated using the Gauss-Chebyshev quadrature (GCQ) [20 Eq.(25.4.38)] as
\[
\phi_{z_i}(t) \approx \frac{L}{jt + 2} \frac{L\pi}{2n} \sum_{k=0}^{n} \frac{\sqrt{1 - a_k^2}}{1 + \frac{\tan\left(\frac{\pi}{2}a_k\right)}{\sin\left(\frac{\pi}{2}a_k + 1\right)}},
\]  
(16)
where \(a_k = \cos\left(\frac{2\pi}{2n}(2k - 1)\right)\). It is important to note that the accuracy of the GCQ rule is extremely high; and a relative accuracy of \(10^{-15}\) is possible for all SNRs and RIS configurations (i.e., \(N, L\)). Thus using (16) and (7) together with (5) can be considered as a replacement for the closed-form solution to the BER. In particular, using GCQ leads to significant computational advantages as compared to the case when (5) is used with (15).

IV. HIGH-SNR BER-DIVERSITY ANALYSIS

In this section, we will analyze the communication robustness that can be achieved with \(L\)-level quantization-based RIS by focusing on the BER at high SNR. The objective is to quantify the diversity order and identify sufficient conditions for achieving it. This is a fundamental open issue for designing and optimizing RIS-aided systems. So far, by relying on the CLT, it was shown in [15], for example, that the diversity order is \(N\) in Rayleigh fading, which implies that the full diversity order cannot be obtained even in the absence of phase errors. By resorting to some bounds, however, the authors of [9] recently showed that the full diversity order is achievable in Rayleigh fading if the absolute difference between pairs of phase errors is less than \(\pi/2\).

In what follows, building upon the exact high-SNR analysis of BER for arbitrary \(N\), we compute the exact diversity order and coding gain of RIS-aided systems. In order to evaluate the BER at high SNR, according to [16], we are interested in analyzing the behavior of the PDF of \(x\) around the origin. Using (11) and resorting to the Mellin-Barnes integral representation of the exponential and error functions [17], it follows that
\[
f_{z_i}(z) = \frac{4L}{\pi(2\pi)^2} \int_{C_1} \int_{C_2} \Gamma(s_1) \Gamma(s_2 + \frac{1}{2}) \Gamma(-s_2) \Gamma(1 - s_2) \frac{z \tan\left(\frac{\pi}{2}z\right)}{x} \, dz \, ds_1 \, ds_2,
\]  
(17)
where \(C_1\) and \(C_2\) are two integral contours in the complex domain. Then, with the help of \(\int_{0}^{\infty} x^{2b} e^{-x^2} \, dx = \Gamma\left(\frac{b}{2} + \frac{1}{2}\right)/2\) and applying [17 Eq. (2.1)], we obtain
\[
f_{z_i}(z) \approx \frac{2L}{\pi} \int_{0}^{\infty} \int_{0,1,1.2}^{\infty,0} 2^z \tan\left(\frac{\pi}{2}z\right) \, dz \, ds_1 \, ds_2,
\]  
(18)
where \((a)\) follows from applying [17 Eq. (2.57)], and \(H[\cdot;\cdot]\) is the bivariate Fox’s H-function [17 Eq. (2.56)]. When \(z \to 0\), \((b)\) follows by computing residues at left poles of the two corresponding integrands [18 Theorem 1.3]. The Laplace Transform of the approximated \(f_{z_i}(z)\) (i.e., when \(z \to 0\)) is now given by
\[
L_{z_i}(s) = \frac{2L \tan\left(\frac{\pi}{2}z\right)}{\pi} \ln(s) - \frac{1}{s^2}.
\]  
(19)
Accordingly, the Laplace transform of the approximated PDF of \(x = \sqrt{\rho} \sum_{i=1}^{N} z_i\) can be formulated as
\[
L_x(s) = \left(\frac{2L \tan\left(\frac{\pi}{2}\right)}{\pi} \ln(\sqrt{\rho}s) \right)^N.
\]  
(20)
The PDF of \(x\) requires the computation of the inverse Laplace transform of (20) which is involved due to the term \(\ln(\sqrt{\rho}s)^N\). To proceed with the analysis, we use the following identity
\[
\ln(\sqrt{\rho}s)^N = \frac{1}{(2\pi)^N} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{N} \Gamma(1 + t_i) \Gamma(-t_i)^2 \Gamma(1 - t_i) \left(\sqrt{\rho}s - 1\right)^{-\sum_{i=1}^{N} t_i} \, dt_1 \cdots dt_N.
\]  
(21)
Then we plug the above equality into (20) and we compute the inverse Laplace transform with respect to \(s\) using [17 Eq. (2.21)] as
\[
f_x(x) = \left(\frac{2L \tan\left(\frac{\pi}{2}\right)}{\pi} \right)^N x^{2N-1} \Gamma(2N) \times \cdots \times \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{N} \Gamma(1 + t_i) \Gamma(-t_i)^2 \Gamma(1 - t_i) \times x^{-\sum_{i=1}^{N} t_i} \, dt_1 \cdots dt_N.
\]  
(23)
which can be expressed in terms of the multivariate Fox’s H-function [17, A.1] whose details are not provided due to space limitation. When \(x \to 0\), we apply the asymptotic expansion of the Mellin-Barnes integrals in (24) at the double poles \(t_i = 0\) using [18 Eqs. (1.8.14), (1.4.6)], thereby yielding
\[
f_x(x) \approx \frac{2L \tan\left(\frac{\pi}{2}\right)}{\rho \pi} x^{2N-1} \ln(\rho x)^N \Gamma(2N).
\]  
(24)
Using (24), we can now formulate the asymptotic BER as
\[
P_e = \frac{(2L \tan(\pi))}{2\sqrt{2\pi}} \int_{0}^{\infty} \text{erfc}(\sqrt{x}) x^{N-1} \ln(\rho x)^N \, dx.
\]  
(25)
where \(\text{erfc}(x)\) is the complementary error function. By rewriting the integrands in (25) as Fox’s H-functions using [17 Eq. (1.43)] and applying [17 Eq. (2.3)], we get (25) at the top of the page, where \(\Pi[\cdot]\) stands for the Fox’s H-function [17 Eq. (1.2)]. The asymptotic expression of the BER when \(\rho \to \infty\) can be obtained by representing the expression in (26) as a multiple Mellin-Barnes type integral and then computing the residues of the integrands using [18 Eq. (1.5.12)]. Hence from
we can finally compute the asymptotic BER as
\[ P_e = \left( \frac{2L \tan \left( \frac{\pi}{2} \right)}{\pi} \right)^N \frac{\ln(\rho)}{2\sqrt{\pi(N+1)\Gamma(2N)}} \rho^{-N}, \] (27)

from which the following important conclusions and performance trends are unveiled.

- Due to the presence of \( \tan \left( \frac{\pi}{2} \right) \), (27) holds for \( L > 2 \).
- (27) unveils a new scaling law of the BER which is \( \ln(\rho)/\rho \) as \( \rho \to \infty \). The BER is not a linear function with respect to \( \rho \) in dB, while the slope of BER changes very slowly at high SNR. This new scaling law generalizes the definitions of diversity order and coding gain typically used in wireless communications [16].
- (27) is, to the best of our knowledge, the first in the literature that yields the exact asymptotic BER for arbitrary \( N \). This is in contrast with the recently reported expressions in [1] Eqs. (4), (7), [7] Eq. (29) and [10] Eq. (39)], which are based on approximations (the CLT in [1], Eqs. (4), (7), [7, Eq. (29)] and [10, Eq. (39)], the moment-based Gamma approximation in [10]).

In order to better quantify the diversity order of RIS-assisted systems in the presence of quantized phase noise, it is important to study the case when \( L = 2 \). To this end, the asymptotic PDF of the normalized SNR \( \gamma/\rho \) follows by resorting to the fact that \( \mathcal{L}_{\tan}(s) = (2\mathcal{L}(e^{-2s}, s))^N = (\frac{s}{2} + 1)^{-N} \), and hence
\[ f_\gamma(\gamma) \approx \frac{\pi}{\Gamma(N)} e^{-\gamma/2}, \] (28)
which implies, according to [10], that the asymptotic error probability is
\[ P_e \approx \frac{2^{N-2}/\Gamma(N+1)}{\pi} e^{-2N/\rho}, \] (29)

Given (27) and (29), the diversity and coding gains of RIS-assisted communications with BPSK signalling and quantized phase noise are obtained, as stated in the following proposition.

**Proposition 2.** The diversity order and coding gains of RIS-assisted communications in the presence of phase noise are
\[ G_d = \begin{cases} N/2, & L = 2; \\ N, & L > 2. \end{cases} \] (30)

and
\[ G_c = \begin{cases} \frac{2^{N-1}N^{N+1/2}}{\sqrt{\pi \Gamma(N+1)}} e^{-2/N}, & L = 2; \\ \left( \frac{2L \tan \left( \frac{\pi}{2} \right)}{\pi} \right)^N \frac{\ln(\rho)N^{N+1/2}}{2\sqrt{\pi(N+1)\Gamma(2N)}} e^{-1/N}, & L > 2. \end{cases} \] (31)

The diversity analysis above helps to discover the effects of an \( L \)-level quantizer based RIS on the BER system performance. In particular, we conclude that

- For RIS-aided systems with BPSK signalling over Rayleigh fading, full diversity order can be ensured if at least three quantization levels (i.e. \( L > 2 \)) are used. This result is in agreement with [9], [11] where it was proved by using upper and lower bounds.
- For \( L = 2 \) the diversity order is only \( N/2 \). This is consistent with [11], where it is proved that the diversity order cannot exceed \( (N + 1)/2 \) for \( L = 2 \). To the best of the authors knowledge, this difference in the diversity gain between the scenarios \( L = 2 \) and \( L > 2 \) was never reported in the literature.
- The increase in the average BER as a quantization penalty is defined as
\[ \Psi(\rho, L) = 10 \log \left( \frac{P_e}{P_e^\infty} \right) = \begin{cases} 10 \log \left( \frac{2^{N-2}/\Gamma(N+1)}{\pi} e^{-2N/\rho} \right), & L = 2; \\ 10 \log \left( \frac{L \tan(\pi/L)}{\pi} \right), & L > 2, \end{cases} \] (32)

where \( P_e^\infty \) is the average SEP with infinite number of quantization bits obtained form (27) by using the small-angle approximation \( \tan(x) \approx x \) as \( x \to 0 \).

V. NUMERICAL RESULTS

In this section, we present analytical and simulated BER results for RIS-assisted communications using an \( L \)-level quantized phase shifts. The CGQ-based CHF in [16] is used to numerically evaluate the performance of BPSK signalling over Rayleigh fading channels.

Fig. 1 shows the average BER as a function of the average transmit SNR (i.e., \( \rho \)) with \( L = 2, 3, 4 \) and \( N = 5 \). We observe that the analytical expression of the BER using (16) and its corresponding high-SNR approximation in (27) and (29) are in close agreement with Monte Carlo simulations. In particular, with GCQ, a high level of accuracy is achieved with only \( n = 20 \). We observe a noteworthy improvement in the average BER when \( L \) changes from 2 to 3. This increase in the average BER performance is expected in the light of Proposition 2, which states that the full diversity order can be achieved as

2Considering the fact that energy consumption at RIS increases exponentially with the number of levels \( L \) (also known as quantizer resolution) [21], low-resolution quantizers, i.e., with small \( L \) values, could provide significant energy savings, notably, when the number of elements \( N \) at RIS is large.
long as $L \geq 3$. When $L = 2$, however, the quantization errors induce a diversity gain loss.

Fig. 2 shows a comparison between the exact and approximate BER expressions for several values of $N$ and $L$. We observe, in particular, that the BER scales with $\ln(\rho)/\rho$, as predicted by [26] and unveiled in [9] over Rayleigh fading channels and in the absence of phase noise. As expected, the BER decreases significantly as the number $N$ of reconfigurable elements of the RIS increases.

VI. CONCLUSION

This letter investigates the effect of low resolution quantization-based-RIS on the BER system performance. By conducting asymptotic analysis, this letter concisely unveils the diversity order, coding gain and the degradation in system performance with respect to quantization levels. It was shown, in particular, that for very low resolution quantizers (i.e. $L = 2$), RIS-assisted communications achieve a diversity order equal to half of the reconfigurable elements number. For higher values of $L$, however, a full diversity is extracted, despite the BER slowly changing slope in high SNRs.