Abstract

In models of large extra dimensions, the fundamental Planck scale can be as low as TeV. Thus, in hadronic collisions interesting objects like black holes, string balls, or $p$-branes can be produced. In scenarios of fat brane or universal extra dimensions, the SM particles are allowed to propagate in the extra spatial dimensions, which leads to the enhancement of production cross sections of black holes and $p$-branes. Especially, the ratio of $p$-brane cross section to the black hole cross section increases substantially, in comparison with the original confined scenario. The ratio can be as large as 105 (for the case $n = 7, m = 5 = p = r = k$).

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I. INTRODUCTION

Trans-Planckian objects including black holes (BH) recently receive a lot of attentions in models of large extra dimensions. In an attractive model of large extra dimensions or TeV quantum gravity (ADD model) [1], the fundamental Planck scale can be as low as a few TeV. This is made possible by confining the SM particles on a 3-brane (using the idea of D-branes in Type I or II string theory), while gravity is free to propagate in all dimensions. The observed Planck scale (∼ 10^{19} \text{ GeV}) is then a derived quantity. Extensive phenomenological studies have been carried out in past few years. Signatures for the ADD model can be divided into two categories: sub-Planckian and trans-Planckian. The former is the one that was studied extensively, while the latter just recently receives more attentions, especially, black hole production in hadronic collisions.

In models of large extra dimensions, the properties of black holes are modified and interesting signatures emerge [2,3,4]. The fact that the fundamental Planck scale is as low as TeV also opens up an interesting possibility of producing a large number of black holes at collider experiments (e.g. LHC) [3,4]. Reference [4] showed that a BH localized on a brane will radiate mainly in the brane, instead of radiating into the Kaluza-Klein states of gravitons of the bulk. In this case, the BH so produced will decay mainly into the SM particles, which can then be detected in the detector. This opportunity has enabled investigation of the properties of BH at terrestrial collider experiments. There have been a number of such studies [5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21] at hadronic colliders. A typical signature of the BH decay is a high multiplicity, isothermal event, very much like a spherical “fireball”. On the other hand, BH production has also been studied in cosmic ray experiments or neutrino telescopes [22,23,24,25,26,27,28,29,30].

Another interesting trans-Planckian phenomenon is the $p$-brane ($p$-B) [31,32,33]. A BH can be considered as a 0-brane. In particle collisions, if one considers BH production, one should also consider $p$-brane production. In fact, the properties of $p$-branes reduce to those of BH in the limit $p \to 0$. In extra dimension models, in which there are large extra dimensions and small extra dimensions of the size of Planck length, let a $p$-brane wraps on $r$ small and $p - r$ large dimensions. It was found [31] that the production of $p$-branes is comparable to BH’s only when $r = p$, i.e., the $p$-brane wraps entirely on the small dimensions only. If $r < p$, the production of $p$-branes would be suppressed by powers of $(M_* / M_{Pl})$, where
\( M_* \) is the fundamental scale of the 4 + \( n \) dimensions and \( M_{\text{Pl}} \) is the 4-dimensional Planck scale. The decay of \( p \)-branes is not well understood. One interesting possibility is cascade into branes of lower dimensions until they reach the dimension of zero. Whether the zero brane is stable depends on the model. Another possibility is the decay into brane and bulk particles, thus experimentally the decay can be observed. Or it can be a combination of cascade into lower-dimensional branes and direct decays.

A naive picture of BH or \( p \)-brane production is as follows. Hadronic production of BH’s or \( p \)-branes at colliders is expected when the colliding partons with a center-of-mass energy \( \sqrt{s} \gtrsim M_{\text{BH}} \) and an impact parameter less than \( R_{\text{BH}} \) or \( R_{\text{pB}} \). Here \( R_{\text{BH}} \) is the Schwarzschild radius of a BH with mass \( M_{\text{BH}} \) and \( R_{\text{pB}} \) is the radius of the \( p \)-brane. This semi-classical argument calls for a geometric approximation for the cross section for producing a BH of mass \( M_{\text{BH}} \) as

\[
\sigma(M_{\text{BH}}) \approx \pi R_{\text{BH}}^2.
\]

This is valid if the colliding partons are confined to a 3-brane (the SM brane.)

There are other scenarios such as fat branes [34] or universal extra dimensions [35], in which the SM particles are allowed, to some extent, to propagate in the space dimensions other than the normal 3 + 1 dimensions. \(^1\) In the fat brane scenario, the original SM brane (with an infinitesimal thickness) is extended into a small but finite thickness (called a fat brane) in the extra dimension coordinates [34]. The SM fermions are localized in various locations on the fat brane in order that the Yukawa coupling can be interpreted as the overlap of the fermion wave-functions involved. Interesting phenomenology in flavor physics arises in this scenario. The scenario of universal extra dimensions [35] allows all SM particles to move in the entire extra dimensions, excluding those dimensions that are very large of mm size. The size of these extra dimensions that the SM can propagate can be as large as \((300 \text{ GeV})^{-1}\), depending on whether the lightest KK state is stable or not. Since the SM particles can move in the extra dimensions, the conservation of momentum in the extra dimensions turns into the conservation of the KK number. The phenomenology includes: (i) the KK states (\( \pm n \)) must be pair-produced, and (ii) the lightest KK state is stable (this

\(^1\) In these scenarios, there are \( \sim \text{TeV}^{-1} \) size bounds on the dimensions that the SM particles can propagate [35, 36]. There are at the same time some dimensions of very large size in order to bring the fundamental Planck scale down to TeV.
is rather similar in spirit to the $R_p$ parity conservation in the supersymmetry.)

We mentioned these two scenarios because we want to point out that if the SM particles also move in the extra dimensions, the cross section for producing black holes or $p$-branes will be modified, especially, the ratio of $p$-brane production to BH production will be enhanced much more significantly than the usual $3+1$ dimensions. This is the main result of the present work. We shall use “confined scenario” to stand for the scenario in which the SM particles are restricted to the SM brane, while “unconfined scenario” to stand for the scenario when the SM particles are allowed to propagate in the extra dimensions.

The purpose of this work is to investigate the enhancement in the ratio of $p$-brane cross section to BH cross section when the SM particles also propagate in some of the extra dimensions. We show that the ratio increases from $1 - 3.8$ to $1 - 105$ for $n = 7$ (the SM particles are allowed to propagate up to five dimensions of the $n = 7$ extra dimensions, i.e., $p = m \leq 5$.) Thus, $p$-brane production is more interesting in these fat brane and universal extra dimension scenarios.

II. SCATTERING IN $D > 3 + 1$ DIMENSIONS

In $D = 3 + 1$ dimensions, the production cross section of BH’s is given by

$$\sigma_{5+1}^{BH}(M_{BH}) = \pi R_{BH}^2,$$

where $R_{BH}$ is the Schwarzchild radius of the BH. This formula is based on a geometric argument. When two incoming partons with a center-of-mass energy $\sqrt{s} \geq M_{BH}$ collide each other at an impact parameter less than $R_{BH}$, a BH is formed. Thus, the production cross section is given by the above formula.

In the scenario that the SM particles also propagate in the extra dimensions, the incoming partons must be colliding at an impact parameter less than $R_{BH}$ in all dimensions, in order to produce a BH. This is understood as follows. In $D = 3 + 1$ dimensions, the distance between the incoming partons in the extra dimension coordinates is simply zero. In $D > 3+1$ dimensions, if the scattering distance between the incoming partons are larger than the $R_{BH}$, they cannot merge into a BH. Therefore, the cross section for producing a BH must scale as $R_{BH}^{k+2}$, where $k$ is the number of extra dimensions that the SM particles also propagate. We
therefore have
\[ \sigma_{3+1+k}^{\text{BH}}(M_{\text{BH}}) = A_{4+k} R_{\text{BH}}^{2+k}. \] (2)

Here \( k \) is not necessarily equal to \( n \), the total number of extra dimensions. \( A_{4+k} \) is a geometrical factor in \( 3 + 1 + k \) dimension. In fact, in the following we shall introduce some small extra dimensions of size \( M_D^{-1} \) and some large extra dimensions of mm size. It is the large extra dimensions that bring the fundamental Planck scale to TeV. The SM particles propagate in the small extra dimensions but not the large extra dimensions. Therefore, \( k \leq n - 2 \), where experimentally the number of large extra dimensions must be at least two.

**A. Space time configuration**

Let there be \( n \) total extra dimensions with \( m \) small extra dimensions and \( n - m \) large extra dimensions. When we say small extra dimensions, we mean the size is of order of \( 1/M_* \), the fundamental Planck scale. The observed 4D Planck scale \( M_{\text{Pl}} \) is then a derived quantity given by [1]
\[ M_{\text{Pl}}^2 = M_*^{2+n} V_m V_{n-m}, \] (3)
where \( V_m \) and \( V_{n-m} \) are the volumes of the extra \( m \) and \( n - m \) dimensions, respectively, given by
\[ V_m = L_m^m \equiv \left( \frac{l_m}{M_*} \right)^m; \quad V_{n-m} = L_{n-m}^{n-m} \equiv \left( \frac{l_{n-m}}{M_*} \right)^{n-m}, \] (4)
where we have expressed the lengths \( L_m, L_{n-m} \) in units of Planckian length \( 1/M_* \).

Suppose the small extra dimension has the size of \( L_m \sim 1/M_* \), i.e., \( l_m \sim 1 \) then
\[ M_{\text{Pl}}^2 = M_*^2 \left( l_{n-m} \right)^{n-m}. \] (5)

The fundamental Planck scale \( M_* \) is lowered to the TeV range if the size \( L_{n-m} \) is taken to be very large, of order \( O(\text{mm}) \).

In literature, another conventionally used definition of the fundamental Planck scale \( M_D \) is related to \( M_* \) by
\[ M_D^{n+2} = \left( \frac{2\pi}{8\pi} \right)^n G_{4+n} = \left( \frac{2\pi}{8\pi} \right)^n M_*^{n+2}, \] (6)
where \( G_{4+n} \) is the gravitational constant in \( D = 4 + n \) dimensions (used in the Einstein equation: \( \mathcal{R}_{AB} - \frac{1}{2} g_{AB} \mathcal{R} = -8\pi G_{4+n} T_{AB} \).)
B. Black hole production

A black hole is characterized by its mass, angular momentum, and charge. Here we consider only the uncharged and non-rotating case. The Schwarzschild radius $R_{BH}$ of a BH of mass $M_{BH}$ in $4 + n$ dimensions is given by

$$R_{BH}(M_{BH}) = \frac{1}{M_D} \left( \frac{M_{BH}}{M_D} \right)^{\frac{1}{n+1}} \left( \frac{2^n \pi^{\frac{n+3}{2}} \Gamma(\frac{n+3}{2})}{n+2} \right)^{\frac{1}{n+1}}, \quad (7)$$

which is much smaller than the size of the extra dimensions. Another important quantity that characterizes a BH is its entropy given by

$$S_{BH}(M_{BH}) = \frac{4\pi}{n+2} \left( \frac{M_{BH}}{M_D} \right)^{\frac{n+2}{n+1}} \left( \frac{2^n \pi^{\frac{n+3}{2}} \Gamma(\frac{n+3}{2})}{n+2} \right)^{\frac{1}{n+1}}. \quad (8)$$

To ensure the validity of the above classical description of BH, the entropy must be sufficiently large, of order 25 or so. In Ref. [9, 10, 11] it was shown that when $M_{BH}/M_D \gtrsim 5$, the entropy $S_{BH} \gtrsim 25$. Therefore, to avoid getting into the nonperturbative regime of the BH and to ensure the semi-classical validity, we restrict the mass of the BH to be $M_{BH} \geq 5M_D$. BH production is expected when the colliding partons with a center-of-mass energy $\sqrt{s} \gtrsim M_{BH}$ pass within a distance less than $R_{BH}$. A black hole of mass $M_{BH}$ is formed and the rest of energy, if there is, is radiated as ordinary SM particles. This semi-classical argument calls for a geometric approximation for the cross section for producing a BH of mass $M_{BH}$ as in Eq. (1). In the case that the SM particles also move in $k$ extra dimensions, the production cross section is modified to Eq. (2).

C. $p$-brane production

$p$-branes can also be formed in particle collisions; in particular, when there exist small extra dimensions of the size $\sim 1/M_*$ in addition to the large ones of the size $\gg 1/M_*$. It was pointed out by Ahn et al. [31] that the production cross section of a $p$-brane completely wrapped on the small extra dimensions is larger than that of a spherically symmetric black hole.

Consider an uncharged and static $p$-brane with a mass $M_{PB}$ in $(4 + n)$ dimensional space-time ($m$ small Planckian size and $n - m$ large size extra dimensions such that $n \geq p$).
Suppose the $p$-brane wraps on $r(\leq m)$ small extra dimensions and on $p-r(\leq n-m)$ large extra dimensions. Then the “radius” of the $p$-brane is

$$R_{pB}(M_{pB}) = \frac{1}{\sqrt{\pi M_*}} \gamma(n, p) V_{pB}^{1+n-p} \left( \frac{M_{pB}}{M_*} \right)^{1+n-p},$$  \hspace{1cm} (9)$$

where $V_{pB}$ is the volume wrapped by the $p$-brane in units of the Planckian length. Recall from Eq. (3), $M_{Pl}^2 = M_*^2 l_{n-m}^m l_m$, where $l_{n-m} \equiv L_{n-m} M_*$ and $l_m \equiv L_m M_*$ are the lengths of the size of the large and small extra dimensions in units of Planckian length ($\sim 1/M_*$). Then $V_{pB}$ is given by

$$V_{pB} = l_{n-m}^r l_m^r \approx \left( \frac{M_{Pl}}{M_*} \right)^{2(n-r) \over n-m},$$  \hspace{1cm} (10)$$

where we have taken $l_m \equiv L_m M_* \sim 1$. The function $\gamma(n, p)$ is given by

$$\gamma(n, p) = \left[ 8\Gamma \left( \frac{3+n-p}{2} \right) \sqrt{\frac{1+p}{(n+2)(2+n-p)}} \right]^{1 \over 1+n-p}.$$  \hspace{1cm} (11)$$

The $R_{pB}$ reduces to the $R_{BH}$ in the limit $p = 0$.

The production cross section of $p$-brane is similar to that of BH’s, based on a naive geometric argument [31]. When the partons collide with a center-of-mass energy $\sqrt{s}$ larger than the fundamental Planck scale and an impact parameter less than the size of the $p$-brane, a $p$-brane of mass $M_{pB} \leq \sqrt{s}$ can be formed. That is

$$\hat{\sigma}_{3+1}^{pB}(M_{pB}) = \pi R_{pB}^2,$$  \hspace{1cm} (12)$$

in the $D = 3+1$ scattering. In the scenario that the SM particles also propagate in $k$ extra dimensions, the production cross section for $p$-branes is modified to

$$\hat{\sigma}_{3+1+k}^{pB}(M_{pB}) = A_{4+k} F(s) R_{pB}^{2+k}.$$  \hspace{1cm} (13)$$

where $F(s)$ is a dimensionless form factor of order one. For simplicity we assume $F(s) = 1$. Therefore, the production cross section for $p$-brane is the same as BH’s in the limit $p = 0$ (i.e., a BH can be considered as a 0-brane.)

D. Ratio of $p$-brane to BH production

In Eq. (3) we can see that the radius of a $p$-brane is suppressed by some powers of the volume $V_{pB}$ wrapped by the $p$-brane. It is then obvious that the production cross section is
largest when \( V_{\rho B} \) is minimal, in other words, the \( p \)-brane wraps entirely on the small extra dimensions only, i.e, \( r = p \). When \( r = p \), \( V_{\rho B} = 1 \). We can also compare the production cross section of \( p \)-branes with BH’s. Assuming that their masses are the same and the production threshold \( M_{\text{min}} \) is the same, the ratio of cross sections in \( D = 3 + 1 + k \) scattering is given by

\[
R \equiv \frac{\sigma_{3+1+k}^{\rho B}(M_{\rho B} = M)}{\sigma_{3+1+k}^{\text{BH}}(M_{\text{BH}} = M)} = \left( \frac{M_*}{M_{\text{Pl}}} \right)^{\frac{2(2+k)(p-r)}{(n-m)(1+n-p)}} \left( \frac{M}{M_*} \right)^{\frac{(2+k)p}{(1+n)(1+n-p)}} \left( \frac{\gamma(n,p)}{\gamma(n,0)} \right)^{2+k},
\]  \( (14) \)

which reduces to the result in Ref. [31] in the limit \( k = 0 \). In the above equation, the most severe suppression factor is in the first parenthesis on the right hand side. Since we are considering a fundamental scale \( M_* \sim \text{TeV} \), the factor \( (M_*/M_{\text{Pl}}) \sim 10^{-16} - 10^{-15} \). Thus, the only meaningful production of \( p \)-brane occurs for \( r = p \), and then their production is comparable. In table I, we show this ratio for various values of \( n \) and \( p \). We choose \( k = m \), i.e., the SM particles propagate in all small extra dimensions. This is the maximal choice of \( k \). We note that \( p (= r) \leq m \). The maximal ratio occurs if \( p = m \). We also choose the mass of BH and \( p \)-brane to be \( 5M_D \). Under these choices, the only free parameters are \( n \) and \( m \) (with \( m \leq n - 2 \)).

III. DISCUSSIONS

Black hole and \( p \)-brane production only occurs at an energy scale substantially larger than the fundamental Planck scale. At this scale, the SM particles already feel the presence and move in the small TeV\(^{-1}\)-sized extra dimensions in the unconfined scenario. Dimensional analysis tells us that the cross section scales as \( R^{2+k} \). Therefore, the ratio of production of \( p \)-branes to BH is now enhanced by a much larger factor than in the confined scenario.

One typical question to ask is: will the BH or \( p \)-brane decay into SM particles such that it can be detected? In the scenario that the SM particles are confined to a 3-brane, it was shown [3] that the BH decays mostly into the SM particles. The naive argument is as follows. The BH decays via Hawking evaporation and the wavelength \( \lambda \) of the thermal spectrum corresponding to the Hawking temperature is much larger than the size of the BH. Therefore, the BH radiates like a point source in s-waves such that it decays equally into brane and bulk modes, and will not see the higher angular momentum states available in the extra dimensions. Since on the brane there are many more particles than in the bulk,
and therefore the BH decays dominantly into brane modes.

The scenario considered in this work is different as the SM particles are allowed to move off the SM brane when the energy scale is above a certain scale (\(\sim\) TeV). However, we argue that when a BH is formed by colliding two SM particles, the BH must be within a distance \(R_{\text{BH}}\) away from the SM brane. Suppose two SM particles are accelerated on the SM brane, which is at \(y = 0\) (here \(y\) denotes collectively the coordinates of the extra dimensions.) When the energy scale is above the compactification scale (\(\sim\) TeV), the SM particles begin to feel the extra dimensions and can move off the SM brane. However, for the two colliding particles to form a BH, the impact distance between them must be less than \(R_{\text{BH}}\). We argue that the chance that the two particles move far away from the SM brane to go to nearly the same spot in the extra dimensions (within \(R_{\text{BH}}\) in each dimension) is very tiny. Therefore, when a BH is formed it is most likely to be within \(y = 0 \pm R_{\text{BH}}\), i.e., the BH is at least touching or intersecting with the SM brane. Hence, when the BH decays into SM particles (each of only a few hundreds of GeV), they should be observable on the SM brane. A similar argument applies to the \(p\)-brane formation and decay.

In this paper, we have pointed out that the production of \(p\)-branes relative to black holes can be enhanced by a much larger factor in the unconfined scenario that the SM particles are allowed to move in the extra dimensions (e.g., fat-brane scenario and universal extra dimensions) than in the confined scenario. In the confined scenario, the ratio of \(p\)-brane production to BH production can at most be 3.8 (for the case \(n = 7, m = 5 = p\)), while in the unconfined case the ratio can be as large as 105 (for the case \(n = 7, m = 5 = p = k\)).

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TABLE I: The ratio $R \equiv \frac{\hat{\sigma}^{pB}_{3+1+k}(M_{pB} = M)}{\hat{\sigma}^{BH}_{3+1+k}(M_{BH} = M)}$ of Eq. (14) for various $n$ and $p$ with $m \leq n - 2$ in the unconfined scenario (here $k$ stands for the number of small extra dimensions that the SM particles can propagate.) We have used $M_{BH} = M_{pB} = 5M_D$. We have assumed that the $p$-brane wraps entirely on small extra dimensions, i.e., $r = p$. In order to obtain the largest ratio $R$ we have chosen $p = m$ and $k = m$. The numbers in the parenthesis are for the confined scenario, i.e. $k = 0$.  

|     | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|-----|---------|---------|---------|---------|---------|---------|
| $n = 2$ | 1       |         |         |         |         |         |
| $n = 3$ | 1       | 2.4(1.8)|         |         |         |         |
| $n = 4$ | 1       | 1.7(1.4)| 6.1(2.5)|         |         |         |
| $n = 5$ | 1       | 1.4(1.3)| 3.0(1.7)| 16 (3.0)|         |         |
| $n = 6$ | 1       | 1.3(1.2)| 2.0(1.4)| 5.3(1.9)| 41(3.5) |         |
| $n = 7$ | 1       | 1.2(1.1)| 1.6(1.3)| 2.9(1.5)| 9.3(2.1)| 105(3.8) |