Intersection cohomology of the circle actions

Gabriel Padilla†
Universidad Central de Venezuela

Martintxo Saralegi-Aranguren‡
Université d’Artois

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Abstract

A classical result says that a free action of the circle $S^1$ on a topological space $X$ is geometrically classified by the orbit space $B$ and by a cohomological class $e \in H^2(B, \mathbb{Z})$, the Euler class. When the action is not free we have a difficult open question:

$$\Pi : \text{"Is the space } X \text{ determined by the orbit space } B \text{ and the Euler class?"}$$

The main result of this work is a step towards the understanding of the above question in the category of unfolded pseudomanifolds. We prove that the orbit space $B$ and the Euler class determine:

- the intersection cohomology of $X$,
- the real homotopy type of $X$.

In this work, we give an answer to the question $\Pi$ in the category of stratified pseudomanifolds (cf. 1.2). The object studied are the modelled actions $\Phi : S^1 \times X \to X$ (cf. 1.4). Here, the total space $X$ is a stratified pseudomanifold and the action $\Phi$ preserves this structure in such a way that the orbit space $B$ is still a stratified pseudomanifold.

A priori, the action $\Phi$ classifies the strata of $X$ in two types: the mobile strata (containing one-dimensional orbits), and the fixed strata (containing the fixed points). But we see in this work that we need a finer classification: a fixed stratum $S$ can be perverse or not perverse. The stratum $S$ is perverse when its link is cohomologically trivial (cf. 3.1).

On the other hand, notice that in our context the meaning of “Euler class” it is not clear: there are non trivial circle actions with $B = [0, 1]$! We show how to recover the Euler class by using the de Rham intersection cohomology $\mathbb{H}^2(\cdot)$ (cf. 2.3). We prove that, in fact, the Euler class $e$ lives in $\mathbb{H}^2_\ast(B)$ where the Euler perversity $\overline{e}$ takes the following values

$$\overline{e}(S) = \begin{cases} 
0 & \text{when } S \text{ mobile stratum}, \\
1 & \text{when } S \text{ not perverse fixed stratum} \\
2 & \text{when } S \text{ perverse stratum}
\end{cases}$$

(cf. 3.1). Notice that the Euler class contains the geometrical information about the nature of the strata.

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†Escuela de Matemática. Universidad Central Venezuela. Caracas 1010-Venezuela. gpadilla@euler.ciens.ucv.ve.

‡UPRES-EA 2462 Laboratoire de Mathématiques de Lens. Faculté Jean Perrin. Université d’Artois. Rue Jean Souvraz SP 18. 62 307 Lens Cedex - France. saralegi@euler.univ-artois.fr.
The intersection homology has been defined for any stratified pseudomanifold $X$, but the intersection cohomology with differential forms needs an extra datum on $X$. For this reason, we work with a particular type of stratified pseudomanifolds: the unfolded pseudomanifolds (cf. 1.4). The modelled actions preserve this structure.

The main result of this work is the following: the orbit space $B$ of a modelled action and the Euler class $e \in I H^2(B)$ determine the intersection cohomology of $X$ (cf. Corollary 4.3), the real homotopy type of $X$ (cf. Corollary 4.4) and the real homotopy type of $X$ (cf. Corollary 4.5). The main tool we use is the Gysin sequence constructed for $\Phi$ in [8].

In the first section we present the geometrical framework. We recall the intersection cohomology in the second section. The third section is devoted to treat the cohomology properties of the modelled actions. Finally, the main results of the work appear in section four.

In the sequel, any manifold will be considered connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$).

1 Geometrical preliminaries

We develop the geometrical tools of this work, that is, the stratified pseudomanifolds and the modelled actions.

1.1 Stratifications. A stratification of a paracompact space $X$ is a locally finite partition $\mathcal{S}_X$ of $X$ into disjoint connected smooth manifolds, called strata, such that

$$S \cap \overline{S'} \neq \emptyset \iff S \subset \overline{S'} \ (\text{and we write } S \preceq S').$$

We shall say that $X$ is a stratified space. Notice that $(\mathcal{S}_X, \preceq)$ is a partially ordered set (poset). The depth of $X$, written depth $(X)$, is defined to be the largest $i$ for which there exists a chain of strata $S_0 \prec S_1 \prec \cdots \prec S_i$. This number is finite since the family of strata is locally finite.

The minimal (resp. maximal) strata are the closed strata (resp. open). The open strata are the regular strata and the other ones are the singular strata. We shall write $\mathcal{S}_X^{\text{sing}}$ the family of singular strata. The union $\Sigma_X$ of singular strata is the singular part, which is a closed subset. The regular part $X \setminus \Sigma_X$ is an open dense subset. We ask the regular strata to have the same dimension, written dim $X$.

The 1-codimensional strata are allowed and the regular part is not necessarily connected. This is a difference with the original presentation of [6].

An open subset $U \subset X$ inherits the canonical stratification $\mathcal{S}_X$ whose elements are the connected components of each $S \cap U$, with $S \in \mathcal{S}_X$.

The main example of stratified space is given by the following conical construction. Consider $L$ a compact stratified space and write $cL$ the cone of $L$, that is, $cL = L \times [0,1]/L \times \{0\}$. This cone is naturally endowed with the following stratification:

$$\mathcal{S}_{cL} = \{\emptyset\} \cup \{S \times [0,1]/ S \in \mathcal{S}_L\},$$

where $\emptyset = [x,0]$ is the vertex of the cone. Here, the points of $cL$ are denoted by $[x,t]$. Notice that depth $(cL) = \text{depth } (L) + 1$.

A continuous map (resp. homeomorphism) $f : Y \to X$ between two stratified spaces is a stratified morphism (resp. stratified isomorphism) if it sends the strata of $Y$ to the strata of $X$. 


smoothly (resp. diffeomorphically). A stratified morphism $f: Y \to X$ induces a poset morphism $f_S: \mathcal{S}_Y \to \mathcal{S}_X$ by putting $f_S(S) \supset f(S)$.

1.2 Stratified pseudomanifolds. A stratified space $X$ is a stratified pseudomanifold when the strata of $\mathcal{S}_X$ verify a local conical equisingularity condition. Equidimensional Thom-Mather stratified spaces and, therefore, equidimensional complex or real analytic varieties endowed with suitable stratifications are examples of pseudomanifolds.

More explicitly, a couple $(X, \mathcal{S}_X)$ is a stratified pseudomanifold when for each point $x$ of a singular stratum $S \in \mathcal{S}_X$ there exists a stratified isomorphism

$$\varphi: U \longrightarrow \mathbb{R}^n \times cL_S,$$

where

(a) $U \subset X$ is an open neighborhood of $x$ endowed with the induced stratification,

(b) $L_S$ is a compact stratified pseudomanifold, called link of $S$,

(c) $\mathbb{R}^n \times cL_S$ is endowed with the stratification $\{\mathbb{R}^n\} \times \mathcal{S}_{cL_S}$, and

(d) $\varphi(x) = (0, \theta)$.

This definition makes sense because it is made by induction on depth $(X)$. The couple $(U, \varphi)$ is a chart of $x$. For a more complete study of these notions, we refer the reader to, for example, [7, 10].

The stratified pseudomanifold is said to be normal when the each link is connected. Notice that the regular part of a connected normal stratified pseudomanifold is connected.

1.3 Unfolded pseudomanifolds. The intersection homology has been defined for any stratified pseudomanifold (cf. [6]) but the intersection cohomology with differential forms needs an extra datum (see for example [4], [1], [2], [3], [5], [11], ...). In this work we use the unfolding of [11].

An unfolding of a stratified pseudomanifold $X$ is an onto, proper, continuous map $\mathcal{L}: \widetilde{X} \to X$, where $\widetilde{X}$ is a manifold, verifying:

1. The restriction $\mathcal{L}_X: \mathcal{L}_X^{-1}(X \setminus \Sigma_X) \longrightarrow X \setminus \Sigma_X$ is a smooth trivial finite covering.

2. There exist a family of unfoldings $\{\mathcal{L}_{L_S}: \widetilde{L}_S \to L_S\}_{S \in \mathcal{S}_X^{\text{sing}}}$ and an atlas $\mathcal{A}$ of $X$ such that for each chart $(U, \varphi) \in \mathcal{A}$ there exists a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}^n \times \tilde{L}_S \times \mathbb{R}^1 & \longrightarrow & \mathcal{L}_X^{-1}(U) \\
\varphi \downarrow & & \mathcal{L}_X \downarrow \\
\mathbb{R}^n \times cL_S & \longrightarrow & U
\end{array}$$

where

(a) $\tilde{\varphi}$ is a diffeomorphism onto $\mathcal{L}_X^{-1}(\text{Im } \varphi)$ and

(b) $Q(x, \tilde{\zeta}, t) = (x, [\mathcal{L}_{L_S}(\tilde{\zeta}), |t|]).$
When depth \((X) = 0\) then \(L_X\) is just a smooth trivial covering. We say that \(X\), endowed with the unfolding \(L: \tilde{X} \to X\), is an unfolded pseudomanifold. An unfolded morphism (resp. unfolded isomorphism) between two unfolded pseudomanifolds is a commutative diagram

\[
\begin{array}{c}
\tilde{X}_1 \xrightarrow{\tilde{f}} \tilde{X}_2 \\
\downarrow L_{\tilde{X}_1} \quad \downarrow L_{\tilde{X}_2} \\
X_1 \xrightarrow{f} X_2
\end{array}
\]

where \(f\) is a stratified morphism (resp. stratified isomorphism) and \(\tilde{f}\) is a smooth map. (resp. diffeomorphism).

### 1.4 Modelled actions

A reasonable action of the circle on a stratified pseudomanifold must produce a stratified pseudomanifold as orbit space. These are the \(S^1\)-pseudomanifolds of \([9]\).

In this work we shall use a variant of this concept, the modelled action of \([8]\), which has been introduced for the treatment of the intersection cohomology using the notion of unfolding.

We recall the properties we use in this work of a modelled action \(\Phi: S^1 \times X \to X\) of the circle on an unfolded pseudomanifold \(X\).

1. **(MA.i)** The isotropy subgroup \(S^1_x\) is the same for each \(x \in S\). It will be written \(S^1_S\).

2. **(MA.ii)** For each regular stratum \(R\) we have \(S^1_R = \{1\}\).

3. **(MA.iii)** For each fixed stratum \(S\) (i.e. \(S^1_S = S^1\)) there exists a modelled action \(\Phi_L_S: S^1 \times L_S \to L_S\) such that each point \(x \in S\) possesses a \(S^1\)-equivariant chart \(\varphi: U \to \mathbb{R}^n \times cL_S\), where the action on the right term is \(g \cdot (x, [y, t]) = (x, [g \cdot y, t])\).

4. **(MA.iv)** The orbit space \(B = X/S^1\) is an unfolded pseudomanifold relatively to the stratification \(\mathcal{S}_B = \{\pi(S) / S \in \mathcal{S}_X\}\).

5. **(MA.v)** The natural projection \(\pi: X \to B\) is a unfolded morphism and the induced map \(\pi_S: \mathcal{S}_X \to \mathcal{S}_B\) is a bijection.

6. **(MA.vi)** We have the commutative diagram

\[
\begin{array}{c}
\tilde{X} \xrightarrow{\tilde{\pi}} \tilde{B} \\
\downarrow L_X \quad \downarrow L_B \\
X \xrightarrow{\pi} B
\end{array}
\]

Given a modelled action \(\Phi: S^1 \times X \to X\) of the circle on a stratified pseudomanifold \(X\), a stratum \(S \in \mathcal{S}_X\) can be:

1. **mobile** when \(S^1_S\) is finite.
2. **fixed** when \(S^1_S = S^1\).

We also say that the stratum \(\pi(S) \in \mathcal{S}_B^{\text{sing}}\) is a fixed stratum (resp. mobile stratum) when \(S \in \mathcal{S}_X\) is a fixed stratum (resp. mobile stratum). Regular strata are mobile (cf. MA.ii).

## 2 Cohomological preliminaries

We present the intersection cohomology which has proven to be specially adapted for the study of stratified pseudomanifolds (with a extra datum!). We fix a unfolded pseudomanifold \(X\).
2.1 Perverse algebras. We introduce first the notion of perverse algebra. It resumes the formal properties of the intersection cohomology.

A perverse set is a triple \((\mathcal{P}, +, \leq)\) where \((\mathcal{P}, +)\) is an abelian semi-group with an unity element \(0\) and \((\mathcal{P}, \leq)\) is a poset verifying the compatibility condition:

\[
\overline{p} \leq \overline{q} \text{ and } \overline{p}' \leq \overline{q}' \implies \overline{p} + \overline{p}' \leq \overline{q} + \overline{q}', \quad \text{for } \overline{p}, \overline{q}, \overline{p}', \overline{q}' \in \mathcal{P}.
\]

In order to simplify the writing, we shall say that \(\mathcal{P}\) is a perverse set. The notion of isomorphism between two perverse sets is defined in a natural way.

A dgc perverse algebra (or simply a perverse algebra) is a quadruple \(E = (E, \iota, \wedge, d)\) where

- \(E = \bigoplus_{\overline{p} \in \mathcal{P}} E_{\overline{p}}\) where each \(E_{\overline{p}}\) is a graded (over \(\mathbb{Z}\)) vector space,
- \(\iota = \{ \iota_{\overline{p}, \overline{q}} : E_{\overline{p}} \to E_{\overline{q}} / \overline{p} \leq \overline{q} \}\) is a family of graded linear morphisms, and
- \((E, d, \wedge)\) is a dgc algebra,

verifying

\[
+ \iota_{\overline{p}, \overline{q}} = \text{Identity,}
+ \iota_{\overline{p}, \overline{q}} = \iota_{\overline{q}, \overline{p}},
+ \wedge (E_{\overline{p}} \times E_{\overline{q}}) \subset E_{\overline{p} + \overline{q}},
+ \iota_{\overline{p} \cdot \overline{p}, \overline{q}} (a \wedge a') = \iota_{\overline{p}, \overline{q}} (a) \wedge \iota_{\overline{p}, \overline{q}} (a'),
+ d (E_{\overline{p}}) \subset E_{\overline{p}}, \quad \text{and}
+ d (\iota_{\overline{p}, \overline{q}}) = \iota_{\overline{p}, \overline{q}} d.
\]

Here, \(\overline{p} \leq \overline{q}, \overline{p}' \leq \overline{q}', \overline{a} \in E_{\overline{p}}, \text{ and } \overline{a}' \in E_{\overline{p}}'.\)

Associated to a dgc perverse algebra \(E = (E, \iota, \wedge, d)\) we have another dgc perverse algebra, namely, its cohomology \(H(E) = \left( \bigoplus_{\overline{p} \in \mathcal{P}} H(E_{\overline{p}}, d), \iota, \wedge, 0 \right)\), where \(\iota\) and \(\wedge\) are induced by the previous \(\iota\) and \(\wedge\).

A dgc perverse morphism (or simply perverse morphism) \(f\) between two perverse algebras \(E = (E, \iota, \wedge, d)\) and \(E' = (E', \iota', \wedge', d')\) is given by a family \(f = \{ f_{\overline{p}} : E_{\overline{p}} \to E'_{\overline{p}} \}\) of differential graded morphisms verifying

(i) \(\iota'_{\overline{p}, \overline{q}} \circ f_{\overline{p}} = f_{\overline{q}} \circ \iota_{\overline{p}, \overline{q}}\), and

(ii) \(f_{\overline{p}, \overline{q}} (a \wedge b) = f_{\overline{p}} (a) \wedge f_{\overline{q}} (b)\).

Here, \(\overline{p} \leq \overline{q}, \overline{a} \in E_{\overline{p}}, \text{ and } \overline{b} \in E_{\overline{p}}'.\) We shall denote the perverse morphism by \(f : E \to E'\). It induces the perverse morphism \(f : H(E) \to H(E')\), defined by \(f_{\overline{p}} [a] = [f_{\overline{p}} (a)]\) for each \(\overline{p}\) and \([a] \in H(E_{\overline{p}}, d)\).

When each \(f_{\overline{p}}\) is an isomorphism, we shall say that \(f\) is a dgc perverse isomorphism (or simply perverse isomorphism). It induces the perverse isomorphism \(f : H(E) \to H(E')\).
2.2 Perverse forms. The intersection cohomology of a stratified pseudomanifold can be computed by using differential forms in different ways: \[4\], \[2\], \[3\], \[1\], \[5\], \[11]\ldots But, in any case we need an extra datum on the stratified pseudomanifold: a Thom-Mather system, a PL-structure, a riemannian metric, \ldots In this work we use the unfolding of \(X\) (see \[11\]). Three ingredients are needed to introduce these forms:

(i) The complex of allowable forms. It is the differential complex \(\Omega^*_\text{all}(X) \subset \Omega^*(X - \Sigma_X)\) made up with the differential forms possessing a lifting to \(\tilde{X}\).

(ii) The perverse degree (cf. op. cit.). It is in fact a family of maps

\[
||-||_S: \Omega^*_\text{all}(X) \to \mathbb{N} \cup \{-\infty\},
\]

with \(S\) running on the family of singular strata, verifying

- \(||\lambda||_S = 0\) if \(\lambda \in \mathbb{R}\setminus\{0\}\) and \(||0||_S = -\infty\),
- \(||\omega + \eta||_S \leq \max(||\omega||_S, ||\eta||_S)||,
- \(||\omega \wedge \eta||_S \leq ||\omega||_S + ||\eta||_S||,
- \(||\omega||_S \leq \deg \omega||,

(iii) The perversity. It is just a map \(\mathfrak{p}: S^\text{sing}_X \to \mathbb{Z}\). The constant perversities are \(\mathfrak{p}\) with \(a \in \mathbb{R}\). The top perversity \(\mathfrak{f}\) is defined by \(\mathfrak{f}(S) = \text{codim}_X(S) - 2\). Notice that the family of perversities \(\mathcal{P}_X\) has a partial order \(\leq\) and an abelian law \(+\) in such a way that \(\mathcal{P}_X\) is a perverse set. The perversity \(\mathfrak{f}\) is the unity element of this semi-group.

Finally, we define the complex of \(\mathfrak{p}\)-perverse forms by

\[
\Omega^*_\mathfrak{p}(X) = \left\{ \omega \in \Omega^*_\text{all}(X) / ||\omega||_S \leq \mathfrak{p}(S) \text{ and } ||d\omega||_S \leq \mathfrak{p}(S) \text{ for each } S \in S^\text{sing}_X \right\}.
\]

The order \(\mathfrak{p} \leq \mathfrak{q}\) induces the natural inclusion \(i_{\mathfrak{p},\mathfrak{q}}: \Omega^*_\mathfrak{p}(X) \hookrightarrow \Omega^*_\mathfrak{q}(X)\). In fact, the quadruple

\[
\Omega(X) = \left( \Omega(X) = \bigoplus_{\mathfrak{p} \in \mathcal{P}_X} \Omega^*_\mathfrak{p}(X), i, \wedge, d \right)
\]

is a dgc perverse algebra, the perverse de Rham algebra of \(X\).

2.3 Intersection cohomology. For each perversity \(\mathfrak{p}\) the \(\mathfrak{p}\)-intersection cohomology \(IH^*_\mathfrak{p}(X)\) is just the cohomology of the complex \(\left(\Omega^*_\mathfrak{p}(X), d\right)\). Notice that the dgc perverse algebra

\[
IH(X) = \left( IH(X) = \bigoplus_{\mathfrak{p} \in \mathcal{P}_X} IH^*_\mathfrak{p}(X), i, \wedge, 0 \right)
\]

is the cohomology of the perverse algebra \(\Omega(X)\), that is, \(IH(X) = H\left(\Omega(X)\right)\). It is the intersection cohomology algebra of \(X\).
3 Modelled actions

We consider a modelled action $\Phi : S^1 \times X \to X$. We know that the intersection cohomology of $X$ and that of the orbit space $B$ are related by a Gysin sequence (cf. [8]). We conclude that the intersection cohomology of $X$ can be computed in terms of basic data. In this section, we prove that the perverse de Rham algebra of $X$ can be also computed in terms of basic data.

Convention. The two posets $S^\text{sing}_B$ and $S^\text{sing}_X$ are isomorphic through the isomorphism $S \mapsto \pi(S)$. The perverse sets $P_B$ and $P_X$ are isomorphic through the map $p \mapsto p \circ \pi$ (see (MA.iv) and (MA.v)). In the sequel, we shall identify this two perverse sets.

3.1 Euler class. The Euler class plays a key rôle on the study of circle actions. It is defined from the characteristic form $\chi \in \Omega^1(X)$ of the action $\Phi$, which depends on the choice of a suitable metric $\mu$. For each singular stratum $S \in S^\text{sing}_X$, we have $||\chi||_S = \pi(S)$, where $\pi$ is the perversity defined by

$$\pi(S) = \begin{cases} 1 & \text{if } S \text{ is a fixed stratum} \\ 0 & \text{if } S \text{ is a mobile stratum.} \end{cases}$$

The differential $d\chi \in \Omega^2(X)$ is in fact a basic form relatively to $\pi$. So, there exists an unique differential form $\epsilon \in \Omega^2(B)$ with $d\chi = \pi^* \epsilon$. This form is a cycle.

The form $\epsilon$ is an Euler form, it depends on the choice of $\mu$. To describe its perverse degree we need to distinguish two different kinds of fixed strata: the perverse and the non perverse. A fixed stratum $S$ is perverse when one of the two following equivalent conditions is not fulfilled:

(a) The induced Euler form $\epsilon_S \in \Omega^1\left(L_S - \Sigma L_S/S^1\right)$ is a boundary.

(b) $H^\epsilon(L_S - \Sigma L_S) = H^\epsilon((L_S - \Sigma L_S)/S^1) \otimes H^\epsilon(S^1)$.

We also say that the stratum $\pi(S) \in S^\text{sing}_B$ is a perverse stratum when $S \in S^\text{sing}_X$ is a perverse stratum. There always exists a good metric $\mu$ such that $\epsilon \in \Omega^2(B)$, where the Euler perversity $\overline{\epsilon}$ is defined by

$$\overline{\epsilon}(\pi(S)) = \begin{cases} 2 & \text{if } S \text{ is a perverse stratum} \\ 1 & \text{if } S \text{ is a non-perverse fixed stratum} \\ 0 & \text{if } S \text{ is a mobile stratum.} \end{cases}$$

Since $\epsilon \in \Omega^2(B)$ is a cycle then it defines a class $e = [\epsilon] \in H^2_{\overline{\epsilon}}(B)$. This class does not depend on the choice of the good metric $\mu$. In fact, we have that the Euler class $e$ vanishes if and only if the Euler class of the (free) action $\Phi : S^1 \times (X - \Sigma X) \to (X - \Sigma X)$ vanishes. So, a fixed stratum $S^\text{sing}_X$ is non-perverse if and only if

(c) The Euler class $e_S = [\epsilon_S] \in H^2_{\overline{\epsilon}}(L_S/S^1)$ vanishes.

Consider two strata $S_1 \preceq S_2$ of $X$. If $S_2$ is a fixed stratum (resp. perverse stratum) then $S_1$ is a fixed stratum (resp. perverse stratum).

We fix a good metric $\mu$ and therefore a characteristic form $\chi$ and an Euler form $\epsilon$.\footnote{The results presented in this section are proved in [8].}
3.2 Examples. Consider $B = cS^2$. Essentially, there are three different modelled actions having $B$ as the orbit space.

- $\Phi_1 : S^1 \times cS^2 \rightarrow cS^3$ defined by $\Phi_1(z, [(u, v), t]) = [(z \cdot u, z \cdot v), t], \Phi_2(z, [(x, w), t]) = [(x, z \cdot w), t]$, and $\Phi_3(z, [(x, t), w]) = [(x, t), z \cdot w].$

A straightforward calculation gives $e_1 \neq 0$ but $e_2 = e_3 = 0.$ The difference between the second and third case lies on the geometrical nature of the singular stratum $\{\vartheta\}$ of $B.$ In fact, in the second case the stratum $\{\vartheta\}$ is a non-perverse fixed stratum $(\bar{e}_2(\{\vartheta\}) = 1)$ and in the third case the stratum $\{\vartheta\}$ is a mobile stratum $(\bar{e}_3(\{\vartheta\}) = 0).$

3.3 The invariant differential forms. Since the Lie group $S^1$ is connected and compact, the subcomplex of the invariant perverse forms computes the intersection cohomology. In fact, for any perversity $\overline{\pi}$, the inclusion $\left(\Omega^*_{\overline{\pi}}(X)\right)^{S^1} \hookrightarrow \Omega^*_X(X)$ induces an isomorphism in cohomology. This complex can described in terms of basic data as follows.

Consider the graded complex

$$I\Omega^*_X(X) = \left\{ (\alpha, \beta) \in \Omega^*_\text{alt}(B) \oplus \Omega^{*,-1}_{\overline{\pi}-\overline{\pi}}(B) \middle| |d\alpha + (-1)^{[\beta]}/(\beta \wedge \epsilon)|_{\pi(S)} \leq \overline{\pi}(S) \right\}$$

endowed with the differential $D(\alpha, \beta) = (d\alpha + (-1)^{[\beta]}/(\beta \wedge \epsilon, d\beta).$ Here $| \cdot |$ stands for the degree of the form. The assignment $(\alpha, \beta) \mapsto \pi^*\alpha + \pi^*\beta \wedge \chi$ establishes a differential graded isomorphism between $I\Omega^*_X(X)$ and $\left(\Omega^*_{\overline{\pi}}(X)\right)^{S^1}.$

The quadruple

$$I\Omega(X) = \left( I\Omega(X) = \bigoplus_{\overline{\pi} \in P_X} I\Omega^*_X(X), \iota, \wedge, D \right)$$

is a perverse algebra. Here, the wedge product is defined by $(\alpha, \beta) \wedge (\alpha', \beta') = (\alpha \wedge \alpha', (-1)^{|\alpha'|}(\beta \wedge \alpha' + \alpha \wedge \beta')).$ A straightforward calculation shows that the operator

$$\Delta = \{\Delta_{\overline{\pi}}\} : I\Omega(X) \hookrightarrow \Omega(X),$$

defined by $\Delta_{\overline{\pi}}(\alpha, \beta) = \pi^*\alpha + \pi^*\beta \wedge \chi,$ induces a perverse isomorphism in cohomology. We get that the perverse de Rham algebra of $X$ can be expressed in terms of basic data.

For each perversity $\overline{\pi}$ we have the linear morphism $\pi^*_\overline{\pi} : \Omega^*_X(B) \rightarrow I\Omega^*_\overline{\pi}(X)$ defined by $\pi^*_\overline{\pi}(\alpha) = (\alpha, 0).$ The operator $\pi = \{\pi_{\overline{\pi}}\} : \Omega^*_X(B) \rightarrow I\Omega^*_X(X)$ is a perverse morphism. It induces the perverse morphism $\pi : \mathcal{H}^*(B) \rightarrow \mathcal{H}^*(X).$

3.4 Gysin sequence. The relationship between the intersection cohomology of $X$ and $B$ is given the Gysin sequence of $[8].$ It can be constructed as follows.

Consider $\overline{\pi}$ a perversity. From (1) we have the short exact sequence

$$0 \rightarrow \Omega^*_{\overline{\pi}}(B) \xrightarrow{\pi^*_\overline{\pi}} I\Omega^*_\overline{\pi}(X) \xrightarrow{f^*_\overline{\pi}} G^*_\overline{\pi}(B) \rightarrow 0,$$

where
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- The Gysin term $G^*(B)$ is the differential complex

\[ \beta \in \Omega^*_{\text{sing}}(B) \Big/ \exists \alpha \in \Omega^*_{\text{sing}}(B) \text{ with } \begin{cases} ||\alpha||_{p} \leq \bar{p}(S) & \text{if } S \in S^\text{sing} \\ \|d\alpha + (-1)^{|\beta|}\beta \wedge e\|_{p} \leq \bar{p}(S) & \end{cases} \]

- $f_\pi(\alpha, \beta) = \beta$, and

- $\pi_\pi(\omega) = \pi^*(\omega)$.

The associated long exact sequence

\[ \cdots \longrightarrow H^i_{\pi}(X) \xrightarrow{f_\pi} H^i\left(G^*(B)\right) \xrightarrow{e_\pi} H^{i+2}(B) \xrightarrow{\pi_\pi} H^{i+2}(X) \longrightarrow \cdots, \]

where $e_\pi[\beta] = [d\alpha + (-1)^{|\beta|}\beta \wedge e]$, is the Gysin sequence. Notice that this Gysin sequence gives the intersection cohomology of $X$ in terms of basic data.

In the next Section, we shall use the following facts

\[ H^0\left(G^*_\pi(B)\right) \cong \mathbb{R} \quad \text{and} \quad e_\pi(1) = e, \]

when $p \geq \bar{p}$ and $X$ is connected and normal. Let us see that. The condition $p \geq \bar{p}$ implies $1 \in G^*_\pi(B)$. Since $X$ is connected and normal, then the regular part $B \setminus \Sigma_B$ is connected. This gives $H^0\left(G^*_\pi(B)\right) \cong \mathbb{R}$. Finally, the definition of $e$ provides $e_\pi(1) = [e] = e$.

4 Cohomological classification of modelled actions

We considered in this section a modelled action $\Phi: \mathbb{S}^1 \times X \to X$ whose orbit space is a fixed unfolded pseudomanifold $B$. We prove that the intersection cohomology algebra and the (perverse) real homotopy type of $X$ are determined by the Euler class.

4.1 Fixing the orbit space. We are going to deal with modelled actions having a common orbit space. We precise this notion. Consider $\Phi_1: \mathbb{S}^1 \times X_1 \to X_1$ and $\Phi_2: \mathbb{S}^1 \times X_2 \to X_2$ two modelled actions and write $B_1$ and $B_2$ the two orbit spaces.

Consider $f: B_1 \to B_2$ an unfolded isomorphism. The two posets $S^\text{sing}_{B_1}$ and $S_{B_2}^\text{sing}$ are isomorphic through the map $\pi_1(S) \mapsto f(\pi_1(S))$. The perversity sets $P_{B_1}$ and $P_{B_2}$ are isomorphic through the map $\pi \mapsto \pi f^{-1}$ (see (MA.v)). In the sequel, we shall identify this two perverse sets in order to compare the de Rham algebras of $X_1$ and $X_2$.

The induced map $f^*: \Omega^*_{\text{sing}}(B_2) \to \Omega^*_{\text{sing}}(B_1)$ is a well defined differential graded isomorphism. It preserves the perversity degree. For each perversity $\pi$ we write $f_\pi: \Omega^*_{\pi}(B_2) \to \Omega^*_{\pi}(B_1)$ the differential graded isomorphism defined by $f_\pi(\alpha) = f^*\alpha$. The operator $f = \{f_\pi\}: \Omega(B_2) \to \Omega(B_1)$, is a perverse isomorphism. It induces the perverse isomorphism $f: \mathcal{H}(B_2) \to \mathcal{H}(B_1)$.

The unfolded isomorphism $f$ is optimal when it preserves the nature of the strata, that is, when it sends the fixed (resp. perverse, resp. non-perverse) strata into fixed (resp. perverse, resp. non-perverse) strata. In this case, the two Euler perversities are equal: $\pi_1(\omega(S)) = \pi_2(f(\omega(S)))$ for each singular stratum $S \in S^\text{sing}_{X_1}$. We shall write $f$ for this Euler perversity.

Now we can compare the two Euler classes $e_1 \in \mathcal{H}^2(B_1)$ and $e_2 \in \mathcal{H}^2(B_1)$ We shall say that $e_1$ and $e_2$ are proportional if there exists a number $\lambda \in \mathbb{R}\setminus\{0\}$ such that $f_\pi(e_2) = \lambda \cdot e_1$. As we are going to see, this is the key test for the comparison between the de Rham algebras of $X_1$ and $X_2$. 
Finally, we say that the actions $\Phi_1$ and $\Phi_2$ have a common orbit space if there exists an optimal isomorphism between theirs orbit spaces.

The two main results of this work come from this Proposition.

**Proposition 4.2** Let $X_1$, $X_2$ be two connected normal unfolded pseudomanifolds. Consider two modelled actions $\Phi_1: S^1 \times X_1 \to X_1$ and $\Phi_2: S^1 \times X_2 \to X_2$. Let us suppose that there exists an unfolded isomorphism $f: B_1 \to B_2$ between the associated orbit spaces. Then, the two following statements are equivalent:

(a) The isomorphism $f$ is optimal and the Euler classes $e_1$ and $e_2$ are proportional.

(b) There exists a perverse isomorphism $F: \mathcal{I}H(X_2) \to \mathcal{I}H(X_1)$ verifying $F \circ \pi_2 = \pi_1 \circ f$.

**Proof.** We proceed in two steps.

\[(a) \Rightarrow (b)\] Since the isomorphism $f$ is optimal then $\overline{x}_1 = \overline{x}_2$ and we will denote by $\overline{\pi}$ this perversity. Since $f^* e_2 = f^* [e_2] = \lambda \cdot e_1 = \lambda \cdot [e_1], \text{ with } \lambda \in \mathbb{R}^*, \text{ then there exists } \gamma \in \Omega^*_\overline{\pi}(B_2) \text{ with } f^* e_2 = \lambda \cdot e_1 - d(f^* \gamma). \text{ For each perversity } \overline{\pi} \text{ we define } F_{\overline{\pi}}: I\Omega^*_{\overline{\pi}}(X_2) \longrightarrow I\Omega^*_{\overline{\pi}}(X_1) \text{ by} \]

$$F_{\overline{\pi}}(\alpha, \beta) = (f^* (\alpha - \beta \land \gamma), f^* (\lambda \cdot \beta)).$$

The map $F_{\overline{\pi}}$ is a well defined differential graded morphism. Let us see that. For each $(\alpha, \beta) \in I\Omega^*_{\overline{\pi}}(X_2)$ and for each $S \in S^2_{\overline{\pi}}$ we have

- $f^* (\alpha - \beta \land \gamma) \in \Omega^*_{\overline{\pi}}(B_1)$.
- $f^* (\lambda \cdot \beta) \in \Omega^*_{\overline{\pi}}(B_1)$.

- $\|f^* (\alpha - \beta \land \gamma)\|_{\pi(S)} = \|\alpha - \beta \land \gamma\|_{\pi(f(S))} \leq \max\{\|\alpha\|_{\pi(f(S))}, \|\beta\|_{\pi(f(S))} + \|\gamma\|_{\pi(f(S))}\} \leq \max\{\overline{\pi}(S), \overline{\pi}(S) - \pi(S) + \|\gamma\|_{\pi(f(S))}\} \leq \overline{\pi}(S)$ since $\|\gamma\|_{\pi(f(S))} \leq \overline{\pi}(S)$.

- $\|f^* d\alpha - f^* (d\beta \land \gamma) - (-1)^{|\beta|} f^* (\beta \land d\gamma) + (-1)^{|\beta|} f^* (\beta \land d\gamma)\|_{\pi(S)} = \|f^* (d\alpha + (-1)^{|\beta|} \beta \land \epsilon_2) - f^* (d\beta \land \gamma)\|_{\pi(S)} \leq \max\{|d\alpha + (-1)^{|\beta|} \beta \land \epsilon_2|_{\pi(f(S))}, |d\beta \land \gamma|_{\pi(f(S))}\}.\overline{\pi}(S).

- $D_1 F_{\overline{\pi}}(\alpha, \beta) = (f^* d\alpha - f^* (\beta \land \epsilon_2), f^* (\lambda \cdot d\beta)) = (f^* (d\alpha + (-1)^{|\beta|} \beta \land \epsilon_2) - f^*(d\beta \land \gamma), f^*(\lambda \cdot d\beta)) = F_{\overline{\pi}}(d\alpha + (-1)^{|\beta|} \beta \land \epsilon_2, d\beta) = F_{\overline{\pi}} D_2(\alpha, \beta).

The family $F = \{F_{\overline{\pi}}\}: I\Omega(X_2) \to I\Omega(X_1)$ is a perverse morphism since:

2.1(i) A straightforward calculation.

2.1(ii) Consider $(\alpha, \beta) \in I\Omega^*_{\overline{\pi}}(X_2)$ and $(\alpha', \beta') \in I\Omega^*_{\overline{\pi}}(X_2)$. Then

$$F_{\overline{\pi}}((\alpha, \beta) \land (\alpha', \beta')) = F_{\overline{\pi}}((\alpha \land \alpha', (-1)^{|\alpha'\beta|} \beta \land \alpha' + \alpha \land \beta')) = (f^* (\alpha \land \alpha' - (-1)^{|\alpha'\beta|} \beta \land \\
(\alpha' \land \gamma - \alpha \land \beta' \land \gamma), f^* (-(-1)^{|\alpha'| \lambda \cdot \beta \land \alpha' + \lambda \cdot \alpha \land \beta')) = (f^* (\alpha - \beta \land \gamma), f^* (\lambda \cdot \beta) \land (f^*(\alpha' - \beta' \land \gamma), f^* (\lambda \cdot \beta))).$$

In fact, the perverse morphism $F$ is a perverse isomorphism, the inverse is given by $F^{-1} = \{F_{\overline{\pi}}\}$, where $F^{-1}(\alpha, \beta) = (f^{-1} \alpha + \lambda^{-1} \cdot f^{-1} \beta \land \gamma, \lambda^{-1} \cdot f^{-1} \beta)$. We conclude that the induced operator $F: \mathcal{I}H(X_2) \to \mathcal{I}H(X_1)$ is a perverse isomorphism. Finally, the equality $F \circ \pi_2 = \pi_1 \circ f$ comes from

$$F_{\overline{\pi}}(\pi_2(\alpha)) = F_{\overline{\pi}}(\alpha, 0) = (f^* \alpha, 0) = (\pi_1)_{\overline{\pi}}(f^* \alpha) = (\pi_1)_{\overline{\pi}}(f_{\overline{\pi}}(\alpha)).$$
where $\mathfrak{p}$ is a perversity and $\alpha \in R^\bullet_{\mathfrak{p}}(B_2)$.

\((b) \Rightarrow (a)\) Write $f = \{ f_{\mathfrak{p}}: H^\bullet_{\mathfrak{p}}(B_2) \to H^\bullet_{\mathfrak{p}}(B_1) \}$ and $F = \{ F_{\mathfrak{p}}: H^\bullet_{\mathfrak{p}}(X_2) \to H^\bullet_{\mathfrak{p}}(X_1) \}$.

Consider now the Gysin sequences associated to the action $\Phi_1$ and $\Phi_2$. The two Gysin terms are written $\mathcal{G}$ and $\mathcal{G}$ respectively. Since $F_{\mathfrak{p}} \circ (\pi_2)_{\mathfrak{p}} = (\pi_1)_{\mathfrak{p}} \circ f_{\mathfrak{p}}$ we can construct a commutative diagram

\[
\begin{array}{cccc}
H^1_{\mathfrak{p}}(B_2) & \xrightarrow{\pi_2} & H^1_{\mathfrak{p}}(X_2) & \xrightarrow{f_2} & H^0(\mathcal{G}^*_{\mathfrak{p}}(B_2)) & \xrightarrow{(\pi_2)_{\mathfrak{p}}} & H^2_{\mathfrak{p}}(B_2) & \xrightarrow{\pi_2} & H^2_{\mathfrak{p}}(X_2) \\
\downarrow f_{\mathfrak{p}} & & \downarrow f_{\mathfrak{p}} & & \downarrow \ell & & \downarrow f_{\mathfrak{p}} & & \downarrow f_{\mathfrak{p}} \\
H^1_{\mathfrak{p}}(B_1) & \xrightarrow{\pi_1} & H^1_{\mathfrak{p}}(X_1) & \xrightarrow{f_1} & H^0(\mathcal{G}^*_{\mathfrak{p}}(B_1)) & \xrightarrow{(\pi_1)_{\mathfrak{p}}} & H^2_{\mathfrak{p}}(B_1) & \xrightarrow{\pi_2} & H^2_{\mathfrak{p}}(X_1),
\end{array}
\]

where $\ell: H^0(\mathcal{G}^*_{\mathfrak{p}}(B_2)) \to H^0(\mathcal{G}^*_{\mathfrak{p}}(B_1))$ is an isomorphism. From (3) we get that $H^0(\mathcal{G}^*_{\mathfrak{p}}(B_2))$ is (the constant functions) and therefore $\ell$ is the multiplication by a number $\lambda \in \mathbb{R}\setminus\{0\}$. We prove (a) in two steps.

If the isomorphism $f$ is optimal then the Euler classes $e_1$ and $e_2$ are proportional. We have $\mathfrak{e}_1 = \mathfrak{e}_2 = \mathfrak{e}$. The formula (3) and the diagram (4) give

\[\lambda \cdot e_1 = \lambda \cdot (e_1)_{\mathfrak{p}}(1) = f_{\mathfrak{p}}((e_2)_{\mathfrak{p}}(1)) = f_{\mathfrak{p}}(e_2).\]

The isomorphism $f$ is optimal. It suffices to prove that $\mathfrak{e}_1(\pi_1(S)) = \mathfrak{e}_2(f(\pi_1(S)))$ for each $S \in S_{X_1}^{\text{sing}}$. Since $H^0(\mathcal{G}^*_{\mathfrak{p}}(B_1)) = \mathbb{R}$ then $1 \in \mathcal{G}^*_{\mathfrak{p}}(B_1)$ and we get that $\mathfrak{e}_2 - \mathfrak{e}_1 \geq 0$. So, $\mathfrak{e}_1(\pi_1(S)) = 0$ if $\mathfrak{e}_2(f(\pi_1(S))) = 0$. By symmetry: $\mathfrak{e}_1(\pi_1(S)) = 0 \iff \mathfrak{e}_2(f(\pi_1(S))) = 0$.

The fixed strata are the same for both actions. If the perverse strata are different, then we can find a fixed stratum $S$ with $\mathfrak{e}_1(\pi_1(S)) \neq \mathfrak{e}_2(f(\pi_1(S)))$ and $\mathfrak{e}_1(\pi_1(S')) = \mathfrak{e}_2(f(\pi_1(S')))$. In particular, the fixed strata and the perverse strata are the same on $L_S$. We have proved that the Euler classes of the actions $\Phi_1, \Phi_2: S^i \times L_S \to L_S$ and $\Phi_2, L_S: S^i \times L_S \to L_S$ are proportional through a non-vanishing factor. So, they vanish or not simultaneously. This would give $\mathfrak{e}_1(\pi_1(S)) = \mathfrak{e}_2(f(\pi_1(S)))$ (cf. 3.1). Contradiction.

\subsection{Remark}

The connectedness and the formality of $X_1$ and $X_2$ have only been used in the proof of $(b) \Rightarrow (a)$.

The first result of this work shows how the Euler class of the action determines the intersection cohomology algebra of the unfolded pseudomanifold $X$.

\textbf{Corollary 4.3} Consider two modelled actions $\Phi_1: S^i \times X_1 \to X_1$ and $\Phi_2: S^i \times X_2 \to X_2$ having a common orbit space. If the Euler classes $e_1$ and $e_2$ are proportional then intersection cohomology algebra of $X_1$ and $X_2$ are isomorphic.

The second result of this work shows how the Euler class of the action determines the real homotopy type of the stratified unfolded $X$.

\textbf{Corollary 4.4} Consider two modelled actions $\Phi_1: S^i \times X_1 \to X_1$ and $\Phi_2: S^i \times X_2 \to X_2$ having a common orbit space. If the two Euler classes $e_1$ and $e_2$ are proportional than the real homotopy type of $X_1$ and $X_2$ are the same.
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Proof. The real homotopy type of $X_k$ is determined by the dgca $\Omega^*_\mathbb{R}(X_k)$ for $k = 1, 2$ (cf. [10]). The result comes from the following sequence of dgca quasi-isomorphisms:

$$\Omega^*_\mathbb{R}(X_2) \leftrightarrow I\Omega^*_\mathbb{R}(X_2) \xrightarrow{F_0} I\Omega^*_\mathbb{R}(X_1) \leftrightarrow \Omega^*_\mathbb{R}(X_1)$$

(cf. (2), Proposition 4.2).

Inspired by the notion of real homotopy type we can define the perverse real homotopy type of an unfolded pseudomanifold in the following way. Two unfolded pseudomanifolds $X_1$ and $X_2$ have the same perverse real homotopy type if there exists a finite family of perverse quasi-isomorphisms

$$X_1 \leftrightarrow \bullet \rightarrow \cdots \leftrightarrow \bullet \rightarrow X_2.$$  

Here, a perverse quasi-isomorphism is a perverse isomorphism inducing an isomorphism in cohomology. Notice that, in the Proposition 4.2, we have proved in fact the following result:

**Corollary 4.5** Consider two modelled actions $\Phi_1 : S^1 \times X_1 \to X_1$ and $\Phi_2 : S^1 \times X_2 \to X_2$ having a common orbit space. If the two Euler classes $e_1$ and $e_2$ are proportional then the perverse real homotopy type of $X_1$ and $X_2$ are the same.

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