Pulsar Timing Response to Gravitational Waves from a Massive Compact Source

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ABSTRACT

Pulsar timing arrays (PTAs) are searching for nanohertz-frequency gravitational waves (GWs) through cross-correlation of pulse arrival times from a set of radio pulsars. PTAs have relied upon a frequency-shift formula of the pulse, where planar GWs are usually assumed. Phase corrections due to the wavefront curvature have been recently discussed. In this paper, we derive a frequency-shift formula for GWs from a compact source such as a binary of supermassive black holes, where the differences in the GW amplitude and direction between the Earth and the pulsar are examined in the quadrupole approximation. By using the new formula, effects beyond the plane-wave approximation are discussed and nearby relevant GW source candidates are also mentioned.

Keywords: Gravitational waves (678); Pulsar timing method (1305); Gravitational wave astronomy (675); Millisecond pulsars (1062); Supermassive black hole (1663)

1. INTRODUCTION

The method of using radio pulse timing to search for gravitational waves (GWs) can be dated back to (Estabrook & Wahlquist 1975; Sazhin 1978; Detweiler 1979; Hellings & Downs 1983). A possible deviation from the expected noise has been reported by the NANOGrav team (Arzoumanian et al. 2020; Antoniadis 2022), and has been argued by several teams (Pol et al. 2021; Alam et al. 2021a,b; Arzoumanian et al. 2021a,b; Kaiser 2022; Goncharov et al. 2022). It is expected that the first detection by the International Pulsar Timing Array consortium may come soon (Castelvecchi 2022).

PTA studies have relied upon a frequency-shift formula of a radio pulse, where planar GWs are usually assumed (Estabrook & Wahlquist 1975; Detweiler 1979; Hellings & Downs 1983). The wavefront curvature for a distant GW source has been discussed as a correction; the Fresnel approximation is discussed (Deng & Finn 2011; McGrath & Creighton 2021). Toward PTA cosmology, the importance of distinguishing the comoving distance from the luminosity distance has been examined (D’Orazio & Loeb 2021; McGrath et al. 2022).

One may ask how a compact GW source affects PTA observational signatures. The main purpose of this paper is to discuss a PTA detector response to GWs from a compact source such as binary supermassive black holes (SMBHs), which are thought to harbor in galactic centers. In section 2, we examine the PTA response to GWs from a compact source. In section 3, possible effects beyond the plane-wave approximation are discussed. Section 4 summarizes this paper. Throughout this paper, c = 1 and the Latin indices i, j run from 1 to 3.

2. PTA RESPONSE: FROM A PLANAR WAVE TO A SPHERICAL WAVE

2.1. PTA response to GWs

We begin with a derivation of the PTA response function (Creighton & Anderson 2013; Maggiore 2018). In particular, we do not assume planar GWs such that our result can be applied also to a spherical GW as shown in next subsection.

We suppose that a radio pulse is emitted by a pulsar (P) at time \( t_P \) and arrives at the Earth (E) at \( t_E \). The radio signal obeys the null condition as

\[
0 = -dt^2 + (\delta_{ij} + h_{ij}^{TT})dx^idx^j,
\]

where the transverse and traceless (TT) gauge is used and \( h_{ij}^{TT} \) is GW perturbations. The unit vector along the pulse is \( dx^i/d\ell = -n_P^i \), where \( \ell \) denotes the spatial length and \( n_P^i \) denotes the unit vector from E to P.

Eq. (1) is rearranged as

\[
d\ell = \left( 1 - \frac{1}{2} n_P^i n_P^j h_{ij}^{TT}(t, x(t)) \right) dt + O(h^2),
\]

2.2. PTA response to GWs from a compact source

In this section, we examine the PTA response to GWs from a compact source for a binary of supermassive black holes. The wavefront curvature for a distant GW source has been discussed as a correction; the Fresnel approximation is discussed (Deng & Finn 2011; McGrath & Creighton 2021). Toward PTA cosmology, the importance of distinguishing the comoving distance from the luminosity distance has been examined (D’Orazio & Loeb 2021; McGrath et al. 2022).

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2.2.1. PTA response to GWs from a compact source

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where \( O(h^2) \) denotes the second order terms in \( h_{ij}^{TT} \).

The distance \( L \) between E and P is

\[
L = \int_P^E \, dl \\
= t_E - t_P - \frac{1}{2} n^i_p n^j_p \int_{t_P}^{t_E} \, dt' h_{ij}^{TT}(t', \mathbf{x}(t')) \\
+ O(h^2).
\]

(3)

It follows that \( t_E = t_P + L + O(h) \).

The spatial position of the radio signal can be written at the lowest order as

\[
\mathbf{x}(t) = \mathbf{x}_E + (t_P + L - t) \mathbf{n}_P + O(h).
\]

(4)

Substituting Eq. (4) into Eq. (3) leads to

\[
L = t_E - t_P - \frac{1}{2} n^i_p n^j_p \int_{t_P}^{t_E} \, dt' h_{ij}^{TT}(t', \mathbf{x}_E + (t_P - t' + L) \mathbf{n}_P) \\
+ O(h^2).
\]

(5)

This agrees with e.g. Eq. (23.5) in Maggiore (2018).

For a radio pulse emitted at \( t_{em} \) and observed at \( t_{obs} \), we obtain

\[
L = t_{obs} - t_{em} - \frac{1}{2} n^i_p n^j_p \int_{t_{em}}^{t_{obs} + L} \, dt' h_{ij}^{TT}(t', \mathbf{x}_E + (t_{em} - t' + L) \mathbf{n}_P) \\
+ O(h^2).
\]

(6)

For the next pulse emitted at \( t'_{em} \) and observed at \( t'_{obs} \),

\[
L = t'_{obs} - t'_{em} - \frac{1}{2} n^i_p n^j_p \int_{t'_{em}}^{t'_{obs} + L} \, dt' h_{ij}^{TT}(t' + T_P, \mathbf{x}_E + (t_{em} - t' + L) \mathbf{n}_P) \\
+ O(h^2).
\]

(7)

The observed period and intrinsic one of the radio pulse are \( T_E = t_{obs} - t_{em} \) and \( T_P = t'_{em} - t_{em} \), respectively. In the TT gauge, Eqs. (6) and (7) have the same separation \( L \). Thereby, the deviation of the observed period from the intrinsic one is obtained as

\[
\Delta T = T_E - T_P \\
= \frac{1}{2} T_P n^i_p n^j_p \\
\times \int_{t_{em}}^{t_{obs} + L} \, dt' \frac{\partial}{\partial t'} h_{ij}^{TT}(t' + T_P, \mathbf{x}_E + (t_{em} - t' + L) \mathbf{n}_P),
\]

(8)

where the GW period \( T_{GW} \gg T_P \sim 1 \text{ msec.} \) is used.

**Figure 1.** Configuration of the Earth (E), a pulsar (P) and a GW source (S). The black solid arrows denote GW paths to E or P, where red (in color) dashed arrows indicate the unit vectors along the GW propagation, \( N_E \) and \( N_P \). The blue (in color) dotted arrow means the radio signal, where the unit vector from E to P is a red (in color) arrow \( n_P \). The distances between E and S, between E and P, and between P and S are \( D, L \) and \( D_P \), respectively.

The frequency-shift formula is thus obtained as

\[
z = \frac{\Delta T}{T_P} \\
= \frac{1}{2} n^i_p n^j_p \int_{t_{em}}^{t_{obs} + L} \, dt' \frac{\partial}{\partial t'} h_{ij}^{TT}(t' + T_P, \mathbf{x}_E + (t_{em} - t' + L) \mathbf{n}_P).
\]

(9)

The partial differentiation \( \partial / \partial t' \) acts only on the time argument in \( h_{ij}^{TT} \) but not on the spatial argument. Therefore, the integrand in Eq. (9) cannot be recast into a total differentiation, except for planar GWs. In a general situation without any approximation, therefore, we need perform the integral along the radio path.

2.2. PTA response to spherical GWs

Figure 1 shows a configuration of the Earth, a pulsar and a GW source (S). We suppose \( \lambda_P \ll \lambda_{GW} < D \sim D_P \), where \( \lambda_P \) and \( \lambda_{GW} \) are wavelengths of the radio pulse and the GW, respectively.

In the quadrupole approximation, the GW at a radio pulse position \( x_R(t) \) is expressed as

\[
h_{ij}^{TT}(t, x_R(t)) = \frac{q_{ij}^{TT}(U, N_R(t))}{|x_R(t) - x_S|},
\]

(10)

where \( q_{ij}^{TT} \) denotes the radiative part (the TT part of the second time derivative of the mass quadrupole moment multiplied by \( 2G \)) of the GW source at \( x_S \) and \( U \equiv t - |x_R(t) - x_S| \) is the retarded time.

The GW propagation direction at \( R \) is

\[
N_R(t) = \frac{x_R(t) - x_S}{|x_R(t) - x_S|}
\]

(11)
Note that \( q_{ij}^{TT} \) depends on \( N_R(t) \) via the TT projection operator, where \( N_R(t) \) as a function of time causes a deviation from a plane-wave case.

From Eq. (10), we obtain

\[
\frac{\partial}{\partial t} h_{ij}^{TT}(t, x_R(t)) = \frac{1}{|x_R(t) - x_S|} \frac{\partial q_{ij}^{TT}(U, N_R(t))}{\partial U}.
\]

(12)

By direct calculations, one can see

\[
\frac{d}{dt} q_{ij}^{TT}(U, N_R(t)) = (1 + n_P \cdot N_R(t)) \frac{\partial q_{ij}^{TT}(U, N_R(t))}{\partial U} \left[ 1 + O\left(\frac{\lambda_{GW}}{D}\right) \right],
\]

(13)

\[
\frac{d}{dt} \frac{1}{|x_R(t) - x_S|} = O\left(\frac{1}{D|x_R(t) - x_S|}\right).
\]

(14)

By using Eqs. (10), (13) and (14), we find

\[
\frac{d}{dt} h_{ij}^{TT}(t, x_R(t)) = \frac{1 + n_P \cdot N_R(t)}{|x_R(t) - x_S|} \frac{\partial h_{ij}^{TT}(U, N_R(t))}{\partial U} \left[ 1 + O\left(\frac{\lambda_{GW}}{D}\right) \right],
\]

(15)

By combining Eqs. (12) and (15), we obtain

\[
\frac{\partial}{\partial t} h_{ij}^{TT}(t, x_R(t)) = \frac{d}{dt} \left( \frac{1}{1 + n_P \cdot N_R(t)} h_{ij}^{TT}(t, x_R(t)) \right) \times \left[ 1 + O\left(\frac{\lambda_{GW}}{D}\right) \right],
\]

(16)

where we use

\[
\frac{d}{dt} \frac{1 + n_P \cdot N_R(t)}{|x_R(t) - x_S|} = O\left(\frac{1}{D}\right).
\]

(17)

Substituting Eq. (16) into Eq. (9) leads to

\[
z = \frac{1}{2} h_{ij}^{TT}(t_E, x_E) \int_{t_m}^{t_m+L} dt' \frac{d}{dt'} \left( \frac{1}{1 + n_P \cdot N_R(t')} h_{ij}^{TT}(t', x_R(t')) \right) \times \left[ 1 + O\left(\frac{\lambda_{GW}}{D}\right) \right],
\]

\[
= \frac{1}{2} h_{ij}^{TT}(t_E, x_E) \left[ \frac{1}{1 + n_P \cdot N_E} h_{ij}^{TT}(t_E, x_E) - \frac{1}{1 + n_P \cdot N_P} h_{ij}^{TT}(t_P, x_P) \right] + O\left(\frac{\lambda_{GW}}{D}\right),
\]

(18)

where the remainder term is \( \sim 10^{-7}(\lambda_{GW}/10\text{pc})(100\text{Mpc}/D) \times \) roughly \( 10^{-3} \) for \( L \sim 10 \text{ kpc} \) and \( D > 10 \text{ Mpc} \), for which the distance correction is less important. They can be important only for nearby GW sources.

The existence of a binary of SMBHs in M31 is suggested (Lauer et al. 1993; Bender et al. 2005). For such

Figure 2. Frequency shift in PTA due to GWs from an edge-on circular binary. The vertical axis denotes the frequency shift \( z = \Delta T/T_P \). The colored dotted curves include only the phase correction by the Fresnel model (McGrath & Creighton 2021). The colored solid curves take account of also the GW amplitude and direction corrections via Eq. (18). For the curves to be recognized by eye, \( L/D = \lambda_{GW}/L = 0.13 \) is chosen.
a nearby case, $L/D$ is $\sim 10^{-2}$, for which the corrections can be at the several percent level.

There could exist a hidden companion to Sagittarius A* (Naoz et al. 2020). If the hidden companion is orbiting around Sagittarius A*, one may use Eq. (18) for PTAs to GWs from this system ($L \sim D$) as it is.

Before closing this section, we mention a relation of the amplitude and direction corrections at $O(L/D)$ to the Fresnel phase correction at $O(L^2/(\lambda_{GW}D))$ (Deng & Finn 2011; McGrath & Creighton 2021; D'Orazio & Loeb 2021). For $\lambda_{GW} \ll L$, the former must be smaller than the latter. The ratio between them is $\sim 0.03(\lambda_{GW}/30\text{pc})(1\text{kpc}/L)$ for a millisecond pulsar at $L \sim 1\text{ kpc}$. For most of known millisecond pulsars, the $L/D$ correction is thus smaller by two or more digits than the Fresnel correction. However, the nearest millisecond pulsar J0437-4517 is located at $L = 156$ pc (Deller 2008), for which the ratio is $\sim 0.2(\lambda_{GW}/30\text{pc})(156\text{pc}/L)$ and hence the amplitude and direction corrections are comparable to the Fresnel correction. In Figure 2, in principle, the Fresnel approximation is not rigorously valid because $L^2/(\lambda_{GW}D) = O(1)$.

4. SUMMARY

The frequency-shift formula in PTA was derived for spherical GWs from a compact source. We confirmed that the Fresnel correction is a leading one under assumptions. As a next-leading correction, both the GW amplitude and direction corrections are at $O(L/D)$. A possible relation of nearby GW source candidates to the new formula was also mentioned. It is left for future to investigate the present formula for a larger parameter space in a more general situation.

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