Inverse spectral problems for Sturm–Liouville operators with singular potentials

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Abstract

The inverse spectral problem is solved for the class of Sturm–Liouville operators with singular real-valued potentials from the space \(W^{−1/2}_2(0, 1)\). The potential is recovered via the eigenvalues and the corresponding norming constants. The reconstruction algorithm is presented and its stability proved. Also, the set of all possible spectral data is explicitly described and the isospectral sets are characterized.

1. Introduction

The main aim of the present paper is to solve the inverse spectral problem for the class of Sturm–Liouville operators with singular real-valued potentials from the space \(W^{−1/2}_2(0, 1)\). Given a real-valued distribution \(q \in W^{−1}_2(0, 1)\), we define a Sturm–Liouville operator \(T\) acting in the Hilbert space \(\mathcal{H} := L^2_2(0, 1)\) and corresponding to the differential expression

\[
T = -\frac{d^2}{dx^2} + q
\]

and, say, the Dirichlet boundary conditions by means of the regularization method due to Savchuk and Shkalikov \([1]\). Namely, we take a real-valued \(\sigma \in \mathcal{H}\) such that \(\sigma' = q\) in the sense of distributions (thus \(\sigma\) is a distributional primitive of \(q\)) and put

\[
Tu = T_\sigma u = l_\sigma(u) := -(u' - \sigma u)' - \sigma u'
\]

on the domain

\[
\mathcal{D}(T_\sigma) = \{u \in W^1_2(0, 1)| u' - \sigma u \in W^1_2(0, 1), l_\sigma(u) \in \mathcal{H}, u(0) = u(1) = 0\}.
\]

Observe that, in the sense of distributions, \(l_\sigma(u) = -u'' + qu\) for all \(u \in \mathcal{D}(T_\sigma)\). In particular, the operator \(T_\sigma\) does not depend on the particular choice of the primitive \(\sigma\) and for regular (i.e., locally summable) potentials it coincides with the standard Dirichlet Sturm–Liouville operator corresponding to (1.1). Also \(T_\sigma\) depends continuously in the uniform resolvent sense
on the primitive $\sigma \in \mathcal{H}$ [1] and thus it is a natural Dirichlet Sturm–Liouville operator related to (1.1) for an arbitrary $q = \sigma' \in W_2^{-1}(0,1)$. Note that the class of potentials considered includes, e.g., the Dirac $\delta$-like and Coulomb $1/r$-like potentials that have been extensively used in quantum mechanics and mathematical physics [2, 3].

It is known [1] that for any real-valued $\sigma \in \mathcal{H}$ the operator $T_{\sigma}$ defined above is a selfadjoint operator with discrete simple spectrum $(\lambda_k^2)$, $k \in \mathbb{N}$, and that $\lambda_k$ have the asymptotics $\lambda_k = \pi k + \mu_k$ with an $\ell_2$-sequence $(\mu_k)$ [1, 4, 5]. We recall that for regular potentials $q$ the above asymptotics refines to having $\mu_k = O(1/k)$.

The present paper has arisen as an attempt to answer the following question: which sequences $(\lambda_k^2)$ consisting of real pairwise distinct numbers and obeying the above-mentioned asymptotics are indeed spectra of Sturm–Liouville operators with singular potentials from $W_2^{-1}(0,1)$? This has led us to the inverse spectral problem for the operators considered, i.e., to reconstruction of the potential $q$ based on the corresponding spectral data.

In the regular situation, knowing only the spectrum $(\lambda_k^2)$ is insufficient: there are many different potentials $q$ (called isospectral) producing Sturm–Liouville operators with the same Dirichlet spectrum. It was shown by Pöschel and Trubowitz [6] that the set of all potentials in $\mathcal{H}$ with a given admissible spectrum $(\lambda_k^2)$ (i.e., real, simple and obeying the asymptotics $\lambda_k = \pi k + O(1/k)$) is analytically diffeomorphic to the weighted space $\ell_2(w_n)$ with the weights $w_n = n$. (A rather exceptional situation, where the spectrum determines the potential uniquely, was pointed out by Ambartsumyan [7]; namely, he proved that for the Neumann Sturm–Liouville operator on $(0,1)$ the equalities $\lambda_k = \pi(k-1)$ for all $k \in \mathbb{N}$ imply $q \equiv 0$.)

Therefore, to recover the potential $q$ uniquely, some additional information besides the spectrum must be supplied. This can be, e.g., knowledge of the potential on half of the interval $(0,1)$ [8–11], or the spectrum of a Sturm–Liouville operator given by the same differential expression but different boundary conditions [12, 13], or three spectra—one for the whole interval and the others for its two halves [14, 15]. Another kind of information is the squared norms $a_k$ of properly normalized eigenfunctions (the so-called norming constants) [6, 12], and that is exactly the setting we shall treat in this paper. The aforementioned settings of the inverse spectral problem will be considered elsewhere.

A complete solution of the inverse spectral problem for a class $\mathcal{S}$ of Sturm–Liouville operators must consist of two parts: (1) an explicit description of the set $\mathcal{S}$ of spectral data for the operators in $\mathcal{S}$ and (2) development and justification of the method of recovering the operator in $\mathcal{S}$ that corresponds to arbitrary given spectral data in $\mathcal{S}$. The algorithm of recovering the potential $q$ from the spectral data of a regular Sturm–Liouville operator based on the transformation operators and the so-called Gelfand–Levitan–Marchenko (GLM) equation was developed by Marchenko [16] and Gelfand and Levitan [17] in the early 1950s. In particular, a reconstruction method via the spectrum $(\lambda_k^2)$ and the norming constants $(a_k)$ was suggested in [17] (see also [12]), though no precise description was given therein of the set of all spectral data generated by the considered class of regular Sturm–Liouville operators. An alternative method for reconstruction of the potential $q$ by two spectra was developed by Krein [18]. A different approach was suggested later by Pöschel and Trubowitz in [6] for the class of Sturm–Liouville operators with potentials from $\mathcal{H}$. The authors studied in detail the mapping between the potentials in $\mathcal{H}$ and the spectral data, proved solvability of the inverse spectral problem and, in particular, completely characterized the spectral data. We refer the reader to the review paper [19] for further historical comments and an extensive reference list on inverse spectral problems for regular Sturm–Liouville operators.

In 1967, Zhikov [20] considered the singular case where the potential $q$ is the derivative of a function of bounded variation (i.e., where $q$ is a signed measure). The corresponding Sturm–Liouville operator was defined through the equivalent integral equation, and complete
solution of the inverse spectral problem (in particular, necessary and sufficient conditions on spectral data) was given. In fact, in [20] the problem was reduced to recovering q by the corresponding spectral function.

Singlarities of a different kind were treated by Rundell and Sacks [21] and Coleman and McLaughlin [22]. These authors considered Sturm–Liouville operators $S$ in impedance form $Su = -\frac{1}{a}(au')'$ with a positive impedance $a \in W^1_2[0, 1]$. In [21] the operators $S$ were shown to possess transformation operators whose kernels satisfy a wave-type equation. The initial and boundary conditions for this equation are uniquely determined by the spectral data, and the resulting overdetermined problem can be proved to have a unique solution. In this way the authors solved the inverse spectral problem, suggested a numerical algorithm for solution and gave necessary and sufficient conditions on the spectral data.

In [22] the impedance eigenvalue equation $(av')' + \lambda^2 av = 0$ with $a \in W^1_2[0, 1]$ was recast in the form $u'' + bu' + \lambda^2 u = 0$ with $b := a'/a$ and then the mapping between $b \in L_2(0, 1)$ and the spectral data was studied in detail. The authors generalized the method of [6] and, in particular, described the set $SD$ of the spectral data for the considered class of Sturm–Liouville operators and proved unique solvability of the inverse spectral problem.

Observe that the impedance Sturm–Liouville operator $S$ is similar to $Tu = -u'' + qu$ with $q = (\sigma' + \sqrt{a'})/\sqrt{a}$, so that for $a \in W^1_2[0, 1]$ the potential $q$ is precisely of the form we shall consider in this paper—i.e., a real-valued distribution from $W^{-1}_2(0, 1)$. Conversely, given a real-valued potential $\sigma' \in W^{-1}_2(0, 1)$ generating a positive Neumann–Dirichlet Sturm–Liouville operator $T_{ND}$, one can recast (not uniquely) the equation $I_\sigma(u) = \lambda^2 u$ in the form $(av')' + \lambda^2 av = 0$ with a suitable positive $a \in W^1_2[0, 1]$. In fact, one can take $a = \sigma_0^2$, where $\sigma_0$ is any nowhere vanishing solution to the equation $I_\sigma(u) = 0$ (such a solution exists under the above positivity assumption). Since the operator $T_{ND}$ is bounded below for any $q \in W^{-1}_2(0, 1)$, it becomes positive after addition to $q$ of a suitable constant and thus the above positivity assumption is not very restrictive.

Therefore, the class of impedance Sturm–Liouville operators studied in [21, 22] is essentially the same as we treat here; however, our method is completely different. Namely, we generalize the classical approach due to Gelfand, Levitan and Marchenko and completely solve the inverse spectral problem for the class $SL$ of Sturm–Liouville operators with potentials from $W^{-1}_2(0, 1)$. Namely, we give an explicit description of the set $SD$ of spectral data, explain how to recover $q$ from an arbitrary element of $SD$ and study the dependence of $q$ on spectral data. As a by-product, we show that the set of potentials in $W^{-1}_2(0, 1)$ that are isospectral to a given one is analytically diffeomorphic to the space $L_2$. The main tool of the reconstruction procedure is the transformation operators for the class of singular Sturm–Liouville operators in $SL$ that were constructed in [5].

Our primary purpose in this paper is to treat the general singular case. We do not prove here that if the spectral data have better asymptotics, then the recovered potential is smoother—e.g., if the spectral data formally correspond to a regular potential from $H$, then the recovered potential indeed falls into $H$. In a subsequent work we shall justify a more general claim that the reconstruction algorithm suggested here solves the inverse spectral problem for the class of Sturm–Liouville operators with potentials from $W^{-1}_2(0, 1)$ for every fixed $a \in [0, 1]$.

Note also that other types of singularity (e.g. discontinuous $a$ for impedance Sturm–Liouville operators $S$, $1/\sqrt{x}$–like potentials etc) were treated by Hald [23], Andersson [24], Carlson [25], Hald and McLaughlin [26], Yurko [27], Freiling and Yurko [28] and others.

The organization of the paper is as follows. In the next section we exploit transformation operators to study the direct spectral problem, i.e., to give necessary conditions on the set $SD$ of spectral data for the class $SL$ of Sturm–Liouville operators considered. In section 3 we establish a connection between the spectral data and the transformation operators and, in
particular, derive the so-called GLM equation. In section 4 the GLM equation is proved to possess a unique solution for any element from \(SD\), and this solution is shown to be the transformation operator for some Sturm–Liouville operator from \(SE\) in section 5. Moreover, the element from \(SD\) that we have started with turns out to be the spectral data for the Sturm–Liouville operator found, so that the inverse spectral problem is completely solved. In section 6 we show the stability of the reconstruction algorithm and characterize isospectral sets, and in the last section we comment on changes to be made for the case of Dirichlet–Neumann or third type boundary conditions. Finally, two appendices contain necessary facts on Riesz bases of sines and cosines in functional Hilbert spaces and on Hilbert–Schmidt operators respectively.

Throughout the paper, \(\sigma \in \mathcal{H}\) will denote a real-valued distributional primitive of the potential \(q \in \mathcal{W}^{-1}_2(0, 1)\), \(u^{[1]}\) will stand for the quasi-derivative \(u' - \sigma u\) of a function \(u\) and \(\|u\|\) will denote the \(\mathcal{H}\)-norm of \(u\).

2. Transformation operators and direct spectral problem

In this section we shall solve the direct spectral problem for Dirichlet Sturm–Liouville operators \(T_\sigma\) with \(\sigma \in \mathcal{H}\), i.e., we shall describe the set \(SD\) of spectral data for \(T_\sigma\)—the sequences of eigenvalues \((\lambda_n^2)\) and norming constants \((\alpha_k)\) introduced below—when \(\sigma\) runs over \(\mathcal{H}\). The main tool will be the transformation operators.

Suppose that \(\sigma \in \mathcal{H}\) is real valued; we denote by \(\tilde{T}_\sigma\) a Sturm–Liouville operator \(-\frac{d^2}{dx^2} + \sigma^2\) with the Dirichlet boundary condition at the point \(x = 0\). More precisely, \(\tilde{T}_\sigma\) acts according to

\[
\tilde{T}_\sigma u = l_\sigma(u) = -(u^{[1]})' - \sigma u^{[1]} - \sigma^2 u
\]

on the domain

\[
\mathcal{D}(\tilde{T}_\sigma) = \{u \in W_1^2[0, 1]| u^{[1]} \in W_1^2[0, 1], l_\sigma(u) \in L_2(0, 1), u(0) = 0\}.
\]

(We recall that \(u^{[1]} := u' - \sigma u\) is the quasi-derivative of \(u\).) According to the definition of \(l_\sigma\), the equation \(l_\sigma(u) = v\) is understood in the sense that

\[
\frac{d}{dx}
\begin{pmatrix}
    u \\
    u^{[1]}
\end{pmatrix} =
\begin{pmatrix}
    \sigma & 1 \\
    -\sigma^2 & -\sigma
\end{pmatrix}
\begin{pmatrix}
    u \\
    u^{[1]}
\end{pmatrix} =
\begin{pmatrix}
    0 \\
    -v
\end{pmatrix},
\]

so that its solutions enjoy the standard uniqueness properties.

One of the main results established in [5] is that the operators \(\tilde{T}_\sigma\) and \(\tilde{T}_0\) possess the transformation operators, which perform their similarity.

**Theorem 2.1** ([5]). Suppose that \(\sigma \in \mathcal{H}\); then there exists an integral Hilbert–Schmidt operator \(K_\sigma\) of the form

\[
(K_\sigma u)(x) = \int_0^x k_\sigma(x, y)u(y) dy
\]

such that \(I + K_\sigma\) is the transformation operator for \(\tilde{T}_\sigma\) and \(\tilde{T}_0\), i.e.,

\[
\tilde{T}_\sigma(I + K_\sigma) = (I + K_\sigma)\tilde{T}_0.
\]

If \(\sigma\) is real valued, then the kernel \(k_\sigma\) of \(K_\sigma\) is real valued too. For every \(x \in [0, 1]\) and \(y \in [0, 1]\), the functions \(k_\sigma(x, \cdot)\) and \(k_\sigma(\cdot, y)\) belong to \(\mathcal{H}\) and the mappings \(x \mapsto k_\sigma(x, \cdot)\) and \(y \mapsto k_\sigma(\cdot, y)\) are continuous in the \(\mathcal{H}\)-norm. Moreover, the operator \(K_\sigma\) with properties (2.1)–(2.2) is unique.

**Remark 2.2.** Some other properties of the transformation operator \(K_\sigma\) established in [5] imply that \(I + K_\sigma\) preserves the initial conditions at \(x = 0\) in the sense that for any \(u \in W^2_2[0, 1]\) and \(v := (I + K_\sigma)u\) we have \(v(0) = u(0)\) and \(v^{[1]}(0) = u'(0)\).
Denote by $T_\sigma$ the restriction of $T_0$ by the Dirichlet boundary condition at $x = 1$, i.e., $T_\sigma := T_0|_{D_0}$, where $D_0 = \{u \in D(T_0) \mid u(1) = 0\}$. It is known [1] that $T_\sigma$ is a bounded below selfadjoint operator with simple discrete spectrum. Denote by $\lambda^2_k, k \in \mathbb{N}$, eigenvalues of $T_\sigma$ in increasing order. We may assume that all the eigenvalues are positive (and that such are $\lambda_k$), as otherwise this situation can be achieved by adding a suitable constant to $q$.

Let also $u_k, k \in \mathbb{N}$, be the eigenfunction of $T_\sigma$ corresponding to the eigenvalue $\lambda^2_k$ and normalized by the condition $u^{(1)}_k(0) = \sqrt{\lambda_k}$; then $\alpha_k := \|u_k\|^2$ is the corresponding norming constant. Notice that in view of remark 2.2 we have $u_k = (I + K_\sigma) v_k$, where $v_k(x) = \sqrt{\lambda_k} \sin \lambda_k x$. We also put $v_{k,0}(x) = \sqrt{\lambda_k} \sin \pi k x$.

The following statement was proved in [1, 4, 5], but we sketch its proof here for the sake of completeness.

**Lemma 2.3.** Suppose that $\sigma \in \mathcal{H}$ is real valued and that $\lambda^2_1 < \lambda^2_2 < \cdots$ are eigenvalues of $T_\sigma$ with $\lambda^2_1 > 0$. Then $\lambda_k = \pi k + \mu_k$, where the sequence $(\mu_k)_{k=1}^\infty$ belongs to $\ell_2$.

**Proof.** Since $u_k(x) = v_k(x) + \int_0^1 k_\sigma(x, y) v_k(y) \, dy$, the numbers $\lambda_k$ are zeros of the function
\[
\Phi(\lambda) := \sin \lambda + \int_0^1 k_\sigma(1, y) \sin \lambda y \, dy.
\]
Recall that $k_\sigma(1, \cdot) \in \mathcal{H}$; whence $\int_0^1 k_\sigma(1, y) \sin \lambda y \, dy \to 0$ as $\lambda \to \infty$ by the Riemann lemma. Rouche’s theorem then gives $\lambda_k = \pi k + \mu_k$ with $\mu_k \to 0$ as $k \to \infty$ (see details in [13, ch 1.3] and [5]). Therefore (see appendix A) $[\sin \lambda_k]_{k=1}^\infty$ is a Riesz basis of $\mathcal{H}$ and the numbers
\[
\sin \lambda_k = (-1)^k \sin \mu_k = -\int_0^1 k_\sigma(1, y) \sin \lambda_k y \, dy
\]
are the Fourier coefficients of $-k_\sigma(1, \cdot) \in \mathcal{H}$ in the biorthogonal basis. It follows that $(\sin \mu_k) \in \ell_2$, hence $(\mu_k) \in \ell_2$ and the proof is complete. $\square$

The next lemma gives the asymptotics of the norming constants $\alpha_k$.

**Lemma 2.4.** Suppose that $\sigma \in \mathcal{H}$; then the norming constants $\alpha_k$ are of the form $1 + \beta_k$, where the sequence $(\beta_k)$ belongs to $\ell_2$.

**Proof.** Observe that the vectors $u_k, k \in \mathbb{N}$, form a Riesz basis of $\mathcal{H}$ since so do $v_k$ and $u_k = (I + K_\sigma) v_k$ with bounded and boundedly invertible $I + K_\sigma$. Therefore, the $\mathcal{H}$-norms of $u_k$ are uniformly bounded, and in view of the relations
\[
|\beta_k| = \|u_k\|^2 - \|v_k,0\|^2 \leq (1 + \|u_k\|)\|u_k - v_k,0\|
\]
the lemma will be proved as soon as we show that
\[
\sum_{k=1}^\infty \|u_k - v_k,0\|^2 < \infty.
\]
This inequality follows from the representation
\[
u_k - v_{k,0} = (I + K_\sigma)(v_k - v_{k,0}) + K_\sigma v_{k,0}
\]
and the fact that both sequences $(\|v_k + K_\sigma v_{k,0}\|)$ and $(\|K_\sigma v_{k,0}\|)$ belong to $\ell_2$: the former due to the relation
\[
\nu_k(x) - v_{k,0}(x) = 2\sqrt{\lambda_k} \sin \frac{\mu_k}{2} \cos(\pi k + \mu_k/2) = O(\mu_k)
\]
and the inclusion $(\mu_k) \in \ell_2$ (see the previous lemma), and the latter because $K_\sigma$ is a Hilbert–Schmidt operator and $(v_{k,0})$ is an orthonormal basis of $\mathcal{H}$ (see appendix B). The proof is complete.
Denote by $SD$ the set of all pairs of sequences $\{(\lambda_k^2)_{k=1}^{\infty}, (\alpha_k)_{k=1}^{\infty}\}$ satisfying the following two conditions:

(A1) $\lambda_k$ are all positive, strictly increase with $k$ and obey the asymptotic relation $\lambda_k = \pi k + \mu_k$ with some $\ell_2$-sequence $(\mu_k)_{k=1}^{\infty}$;

(A2) $\alpha_k = 1 + \beta_k > 0$ with some $\ell_2$-sequence $(\beta_k)_{k=1}^{\infty}$.

Also $SL$ will stand for the set of all positive Dirichlet Sturm–Liouville operators $T_\sigma$ with $\sigma \in H$ real valued. Lemmata 2.3 and 2.4 demonstrate that the spectral data for the operators in $SL$ belong to $SD$. Our next task is to show that, conversely, any element of $SD$ is spectral data for some operator in $SL$.

3. Connection between spectral data and transformation operators

In this section we shall derive a relation between the spectral data for a Dirichlet Sturm–Liouville operator $T_\sigma$ with $\sigma \in H$ and the transformation operator $K_\sigma$. This relation will be used in the next section to find $K_\sigma$ given the spectral data and thus to solve the inverse spectral problem.

Suppose therefore that $\{(\lambda_k^2), (\alpha_k)\} \in SD$ is the spectral data for a Sturm–Liouville operator $T_\sigma$ with $\sigma \in H$. We recall that $\lambda_k = \pi k + \mu_k > 0$ and $\alpha_k = 1 + \beta_k > 0$ with some $\ell_2$-sequences $(\mu_k)$ and $(\beta_k)$. Put

$$U := s\text{-lim}_{N \to \infty} \sum_{k=1}^{N} \frac{1}{\alpha_k} (\cdot, v_k) v_k,$$

where $v_k(x) = \sqrt{2} \sin \lambda_k x$ and $s\text{-lim}$ stands for the limit in the strong operator topology of $H$. Since by proposition A.1 the system $(v_k)_{k=1}^{\infty}$ forms a Riesz basis of $H$ and $\inf_{k \in \mathbb{N}} 1/\alpha_k > 0$, the operator $U$ is bounded, selfadjoint and uniformly positive.

**Lemma 3.1.** For all $j, k \in \mathbb{N}$, the following relation holds: $(U^{-1}v_j, v_k) = \alpha_k \delta_{jk}$, where $\delta_{jk}$ is the Kronecker delta.

**Proof.** Note that

$$U^{-1} = s\text{-lim}_{N \to \infty} \sum_{k=1}^{N} \alpha_k (\cdot, w_k) w_k,$$

where $(w_k)$ is a basis biorthogonal to $(v_k)$ (see [29, ch VI] and also appendix A). Therefore

$$(U^{-1}v_j, v_k) = s\text{-lim}_{N \to \infty} \sum_{l=1}^{N} \alpha_l (v_j, w_l)(w_l, v_k) = \alpha_l \delta_{jl} \delta_{lk} = \alpha_k \delta_{jk},$$

and the proof is complete. \(\square\)

**Lemma 3.2.** $F := U - I$ is an integral operator of the Hilbert–Schmidt class with kernel

$$f(x, y) = \phi(x + y) - \phi(x - y),$$

where

$$\phi(s) = \sum_{k \in \mathbb{N}} \left( \cos \pi k s - \frac{1}{\alpha_k} \cos \lambda_k s \right)$$

is an $L_2(0, 2)$-function.
**Proof.** Recall that we have denoted by \( v_{k,0}, k \in \mathbb{N} \), the function \( \sqrt{2} \sin \pi k x \). Since \( I = \text{s-lim}_{N \to \infty} \sum_{k=1}^{N} (\frac{1}{\alpha_k}(\cdot, v_k) v_k - (\cdot, v_{k,0}) v_{k,0}) \), we have

\[
U - I = \text{s-lim}_{N \to \infty} \sum_{k=1}^{N} \left( \frac{1}{\alpha_k}(\cdot, v_k) v_k - (\cdot, v_{k,0}) v_{k,0} \right). 
\]

Observe that \( \frac{1}{\alpha_k}(\cdot, v_k) v_k - (\cdot, v_{k,0}) v_{k,0} \) is an integral operator with kernel

\[
f_k(x, y) = \frac{2}{\alpha_k} \sin \lambda_k x \sin \lambda_k y - 2 \sin \pi k x \sin \pi y 
= \cos \pi k(x + y) - \frac{1}{\alpha_k} \cos \lambda_k(x + y) - \cos \pi k(x - y) + \frac{1}{\alpha_k} \cos \lambda_k(x - y). 
\]

Letting

\[
\phi_N := \sum_{k=1}^{N} \left( \cos \pi k s - \frac{1}{\alpha_k} \cos \lambda_k s \right), 
\]

we see that

\[
F_N := \sum_{k=1}^{N} \left( \frac{1}{\alpha_k}(\cdot, v_k) v_k - (\cdot, v_{k,0}) v_{k,0} \right) 
\]

is an integral operator with kernel \( f_N(x, y) := \phi_N(x + y) - \phi_N(x - y) \).

Put \( \tilde{\beta}_k := \beta_k/\alpha_k \); then \( 1/\alpha_k = 1 - \tilde{\beta}_k \) with \( (\tilde{\beta}_k)_{k=1}^{\infty} \in \ell_2 \) and

\[
\cos \pi k s = \frac{1}{\alpha_k} \cos \lambda_k s = 2 \sin(\mu_k/2) \sin(\pi k + \mu_k/2)s) + \tilde{\beta}_k \cos \lambda_k s. 
\]

Recall (see corollary A.2) that both \( (\sin(\pi k + \mu_k/2)s)_{k=1}^{\infty} \) and \( (\cos \lambda_k s)_{k=1}^{\infty} \) constitute Riesz bases of their closed linear spans in \( L_2(0, 2) \). Since, moreover,

\[
2 \sin(\mu_k/2) = s \mu_k + O(|\mu_k|^3) 
\]

as \( k \to \infty \), the series for \( \phi \) is convergent in \( L_2(0, 2) \) and hence \( \phi_N \to \phi \in L_2(0, 2) \). Thus \( f_N \) converge in the \( L_2((0, 1) \times (0, 1)) \)-norm to \( f \) given by (3.1). This implies that \( F_N \) converge in the Hilbert–Schmidt norm to the integral operator \( F \) with kernel \( f \), and the lemma is proved. \( \square \)

**Lemma 3.3.** \((I + K_\sigma)(I + F)(I + K_\sigma^*) = I\).

**Proof.** Since \( (\mu_k/\sqrt{\alpha_k})_{k=1}^{\infty} \) is an orthonormal basis of \( \mathcal{H} \) and \( u_k = (I + K_\sigma)v_k \), we have

\[
I = \text{s-lim}_{N \to \infty} \sum_{k=1}^{N} \left( \frac{1}{\alpha_k}(\cdot, u_k) u_k \right) = \text{s-lim}_{N \to \infty} \sum_{k=1}^{N} \left( \frac{1}{\alpha_k}(\cdot, (I + K_\sigma)v_k) (I + K_\sigma)v_k \right) 
= (I + K_\sigma) \left( \text{s-lim}_{N \to \infty} \sum_{k=1}^{N} \left( \frac{1}{\alpha_k}(\cdot, v_k) v_k \right) \right) (I + K_\sigma^*) = (I + K_\sigma)(I + F)(I + K_\sigma^*). 
\]

The lemma is proved. \( \square \)

**Corollary 3.4.** The kernels \( k_\sigma(x, y) \) and \( f(x, y) \) of the operators \( K_\sigma \) and \( F \) satisfy the following GLM equation for a.e. \( x, y \) with \( x > y \):

\[
k_\sigma(x, y) + f(x, y) + \int_0^y k_\sigma(x, s) f(s, y) \, ds = 0. 
\]

(3.3)
In the previous section, we showed that the spectral data \( (\lambda_2^2, (\alpha_2)) \) for the Sturm–Liouville operator \( T_0 \) are connected with the transformation operator \( K_0 \) for the pair \( T_0 \) and \( T_0 \) through the GLM equation (3.3). In this section we shall prove that the GLM equation is uniquely soluble for \( K_0 \). Let \( \Omega \) and \( k \) be an orthoprojector in \( S_2 \), where \( P \) is the linear operator in \( X \) that \( P \) generates an operator \( T \) such that \( T \) is uniformly positive in \( S_2 \). Therefore, solvability of the GLM equation is closely connected with the properties of \( T \). Here, \( \mathcal{I} \) is the identity operator in \( S_2 \). Therefore, solvability of the GLM equation is closely connected with the properties of the operator \( \tilde{K} \).

**Lemma 4.1.** The operator \( \mathcal{P}_G \) depends continuously on \( X \in \mathcal{B}(\mathcal{H}) \).

**Proof.** This is a straightforward consequence of the fact that \( \mathcal{P}_G \) depends linearly on \( X \) and that

\[
\|\mathcal{P}_G Y\|_{\sigma_2} \leq \|Y X\|_{\sigma_2} \leq \|X\|\|Y\|_{\sigma_2},
\]

see [29, ch 3].

**Lemma 4.2.** Suppose that \( I + X \) is a uniformly positive operator in \( \mathcal{H} \). Then \( I + \mathcal{P}_G \) is uniformly positive in \( S_2 \).
Remark 4.4. K found a solution equivalent to uniform positivity of the operator $I$ is boundedly invertible in $Y$ to be soluble is that, for any $Y$, there is a unique solution of (4.1). The lemma is proved.

Proof. Given any $Y \in \mathcal{S}_2^+$, we observe that $$(I + P^+_X Y, Y)_2 = (Y, Y)_2 + (YX, Y)_2 = \text{tr}(Y(I + X)Y^*) - \text{tr}(YXY^*).$$ If $\varepsilon > 0$ is chosen so that $I + X \geq \varepsilon I$, then $Y(I + X)Y^* \geq \varepsilon YY^*$ and by monotonicity of the trace we get $$(I + P^+_X Y, Y)_2 \geq \varepsilon (Y, Y)_2.$$ The lemma is proved.

Lemma 4.3. Suppose that sequences $(\lambda^2_k)^{\infty}_{k=1}$ and $(\alpha_k)^{\infty}_{k=1}$ satisfy conditions (A1) and (A2) and that $F$ is an integral operator with kernel $f$ of (3.1). Then equation (4.1) has a unique solution $K$, which belongs to $\mathcal{S}_2^+$.

Proof. Under assumptions (A1) and (A2) the operator $F$ is of Hilbert–Schmidt class and
$$I + F = \lim_{N \to \infty} \sum_{k=1}^N \frac{1}{\alpha_k} f(\cdot, v_k) v_k$$
(cf the proof of lemma 3.2). Moreover, $U := I + F$ is uniformly positive as noted at the beginning of section 3. Therefore, the operator $I + P^+_F$ is uniformly positive in $\mathcal{S}_2^+$ by lemma 4.2 and
$$K := -(I + P^+_F)^{-1} P^+ F \in \mathcal{S}_2^+$$
is a unique solution of (4.1). The lemma is proved.

Remark 4.4. It is well known that a necessary and sufficient condition for the GLM equation to be soluble is that, for any $a \in [0, 1]$, the operator
$$u \mapsto u(x) + \int_0^x f(x, y) u(y) \, dy$$
is boundedly invertible in $\mathcal{H}$. For a symmetric kernel $f$ this condition is easily seen to be equivalent to uniform positivity of the operator $I + F$.

Also, the GLM equation is uniquely soluble if and only if $I + F$ can be factorized as $(I + X^*)(I + X^-)$ with some $X^* \in \mathcal{S}_2^+$ and $X^- \in \mathcal{S}_2^- := \mathcal{S}_2 \ominus \mathcal{S}_2^+$, see [30, ch IV].

5. The inverse spectral problem

In the previous two sections we showed that the operator $F$ constructed via the spectral data $\{(\lambda^2_k), (\alpha_k)\}$ of a Dirichlet Sturm–Liouville operator $T_\sigma$, $\sigma \in \mathcal{H}$, is connected with the transformation operator $I + K_\sigma$ for $\hat{T}_\sigma$ and $\hat{T}_0$ through the GLM equation (3.3) or (4.1) and then proved that the GLM equation is uniquely soluble for $K_\sigma$. Suppose that instead we have started with an arbitrary element $\{(\lambda^2_k), (\alpha_k)\} \in \mathcal{S}D$, constructed $F$ in the same manner and found a solution $K$ of the GLM equation; is then $I + K$ a transformation operator for some $\hat{T}_\sigma$, $\sigma \in \mathcal{H}$ and $\hat{T}_0$?

In this section we give an affirmative answer to this question and find an explicit formula for the corresponding $\sigma$. Moreover, we show that the element $\{(\lambda^2_0), (\alpha_0)\}$ we have started with is indeed the spectral data for the Sturm–Liouville operator $T_\sigma$ with the $\sigma$ found. This completes the solution of the inverse spectral problem for the considered class of Sturm–Liouville operators.

Theorem 5.1. Suppose that $\{(\lambda^2_k), (\alpha_k)\}$ is an arbitrary element of $\mathcal{S}D$ and that $F$ is an integral operator with kernel $f$ of (3.1). Let also $K$ be a (unique) solution of the GLM equation (4.1). Then there exists a (unique up to an additive constant) function $\sigma \in \mathcal{H}$ such that $I + K$ coincides with the transformation operator $I + K_\sigma$ for the pair of Sturm–Liouville operators $\hat{T}_\sigma$ and $\hat{T}_0$. 

**Proof.** We shall approximate $F$ in the $\mathcal{S}_2$-norm by a sequence $(F_n)_{n=1}^\infty$ of Hilbert–Schmidt operators with smooth (say, infinitely differentiable) kernels $f_n$ so that the following holds:

(a) the solutions $K_n$ of (4.1) for $F$ replaced with $F_n$ converge to $K$ in $\mathcal{S}_2$ as $n \to \infty$;
(b) for each $n \in \mathbb{N}$ there exists $\sigma_n \in \mathcal{H}$ such that $I + K_n$ is a transformation operator for the pair $\tilde{T}_n$ and $\tilde{T}_0$;
(c) $(\sigma_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$.

Put $\sigma := \lim \sigma_n$; then by [5] the operators $I + K_n$ converge in $\mathcal{H}$ to an operator $I + K_\sigma$, which is the transformation operator for the pair $\tilde{T}_\sigma$ and $\tilde{T}_0$. Thus $K = K_\sigma$ yielding the result. The uniqueness of $\sigma$ up to an additive constant is obvious.

The details are as follows. We put

$$f_n(x, y) := \phi_n(x + y) - \phi_n(x - y),$$

where

$$\phi_n(s) := \sum_{k=1}^n \left( \cos \pi k s - \frac{1}{\alpha_k} \cos \lambda_k s \right).$$

We recall (see the proof of lemma 3.2) that $\phi_n \to \phi$ in $\mathcal{H}$ as $n \to \infty$. This implies that $f_n$ converge to $f$ in $L_2((0, 1) \times (0, 1))$, i.e., that $F_n$ converge to $F$ in $\mathcal{S}_2$ as $n \to \infty$. The corresponding operators $I + F_n$ are uniformly positive so that the equations $K_n + P^+ F_n + P^+ (K_n F_n) = 0$ have unique solutions $K_n = - (I + P^+_n)^{-1} P^+ F_n \in \mathcal{S}_2^\mu$. Combining all these statements, we find that

$$\|(I + P^+_n)^{-1} - (I + P^+_F)^{-1}\|_{\mathcal{S}(\mathcal{H})} \to 0$$

and

$$\|K_n - K\|_{\mathcal{S}} = \|(I + P^+_F)^{-1} P^+ F_n - (I + P^+_F)^{-1} P^+ F\|_{\mathcal{S}}$$

$$\leq \|(I + P^+_F)^{-1} - (I + P^+_F)^{-1}\|_{\mathcal{S}(\mathcal{H})} \|P^+ F_n\|_{\mathcal{S}}$$

$$+ \|(I + P^+_F)^{-1}\|_{\mathcal{S}(\mathcal{H})} \|P^+ F - P^+ F\|_{\mathcal{S}} \to 0,$$

as $n \to \infty$ thus establishing (a).

Note that the kernel $k_n$ of $K_n$ solves the GLM equation

$$k_n(x, y) + f_n(x, y) + \int_0^x k_n(s, x) f_n(s, y) \, ds = 0, \quad x > y.$$

Since $f_n$ is smooth, the classical result [12, ch II], [13, ch 2.3] states that $k_n$ is at least once continuously differentiable and that $I + K_n$ is the transformation operator for the Sturm–Liouville operators $\tilde{T}_n$ and $\tilde{T}_0$, where $\sigma_n$ is a primitive of $q_n(x) := 2 \frac{d}{dx} k_n(x, x)$. The suitable choice of the primitive (suggested by the GLM equation above) is

$$\sigma_n(x) := 2k_n(x, x) + 2\phi_n(0) = -2\phi_n(2x) - 2 \int_0^x k_n(s, x) f_n(s, x) \, ds,$$

(5.1)

and (b) is fulfilled.

To prove (c), we observe that

$$\|\sigma_n - \sigma_m\|^2 \leq 12 \int_0^1 |\phi_n(2x) - \phi_m(2x)|^2 \, dx$$

$$+ 12 \int_0^1 \, dx \left| \int_0^x (k_n(s, x) - k_m(s, x)) f_n(y, x) \, dy \right|^2$$

$$+ 12 \int_0^1 \, dx \left| \int_0^x k_m(s, x)(f_n(y, x) - f_m(y, x)) \, dy \right|^2.$$
Since \( f_n(x, y) = \phi_n(x + y) - \phi_n(x - y) \) and \( \phi_n \) form a Cauchy sequence in \( L_2(0, 2) \), we find that
\[
\int_0^1 \left( \int_0^x (k_n(x, s) - k_m(x, s)) f_n(s, x) \, ds \right)^2 \, dx \leq C \int_0^1 \left( \int_0^x |k_n(x, s) - k_m(x, s)|^2 \, ds \int_0^x |f_n(s, x)|^2 \, ds \right) \, dx
\]
for some positive \( C \) independent of \( n \in \mathbb{N} \) and \( x \in [0, 1] \), and hence
\[
\int_0^1 \left( \int_0^x (k_n(x, s) - k_m(x, s)) f_n(s, x) \, ds \right)^2 \, dx \leq C \int_0^1 \left( \int_0^x |k_n(x, s) - k_m(x, s)|^2 \, ds \int_0^x |f_n(s, x)|^2 \, ds \right) \, dx
\]
as \( n, m \to \infty \). In the same manner it can be shown that
\[
\int_0^1 |f_n(s, x) - f_m(s, x)|^2 \, ds \leq 2 \int_0^2 |\phi_n(s) - \phi_m(s)|^2 \, ds = 2\|\phi_n - \phi_m\|_{L_2(0, 2)}^2
\]
and
\[
\int_0^1 \left( \int_0^x k_m(x, s) (f_n(s, x) - f_m(s, x)) \, ds \right)^2 \, dx \leq 2\|K_m\|^2_{\mathcal{H}} \|\phi_n - \phi_m\|_{L_2(0, 2)}^2 \to 0
\]
as \( n, m \to \infty \). Combining the above relations, we conclude that (\( \sigma_n \)) is a Cauchy sequence in \( \mathcal{H} \). Therefore, (c) is satisfied and the proof is complete.

**Remark 5.2.** The arguments used to prove (c) above also justify passage to the limit in the \( \mathcal{H} \)-sense in (5.1), and this results in the equality
\[
\sigma(x) = -2\phi(2x) - 2 \int_0^x k(s, x) f(s, x) \, ds,
\]
where \( k \) is the kernel of \( K \). If \( k \) and \( \phi \) are smooth, then the GLM equation implies that \( \sigma(x) = 2k(x, x) - 2\phi(0) \), thus yielding the classical relation
\[
q(x) = 2\frac{d}{dx}k(x, x)
\]
for the potential \( q \).

To complete the arguments, we have to show that the spectral data for \( T_\sigma \) with \( \sigma \in \mathcal{H} \) just found coincide with the data \( \{(\lambda_k^2), (\alpha_k)\} \in \mathcal{SD} \) that we have started with.

**Theorem 5.3.** \( \{(\lambda_k^2), (\alpha_k)\} \) are the spectral data for \( T_\sigma \).

**Proof.** Put \( w_k := (I + K)v_k \); then \( w_k(0) = 0 \), \( w_k^{[1]}(0) = \sqrt{2}\lambda_k \) and, due to the similarity of \( \tilde{T}_\sigma \) and \( T_0, T_\sigma w_k = \lambda_k^2 w_k \). The system \( (w_k)_{k=1}^\infty \) is complete in \( \mathcal{H} \) since such is \( (\alpha_k)_{k=1}^\infty \); and \( I + K \) is a homeomorphism. Also, lemmata 3.1 and 3.3 imply the orthogonality relation
\[
(w_j, w_k) = (I + K^*)(I + K)v_j, v_k) = (U^{-1}v_j, v_k) = \alpha_k \delta_{jk};
\]
in particular, \( \alpha_k = \|w_k\|^2 \).
It remains to show that \( w_{k} \) are eigenfunctions of \( T_{\sigma} \), i.e., that \( w_{k}(1) = 0 \). Observe that

\[
0 = (\tilde{T}_{\sigma}w_{j}, w_{k}) - (w_{j}, \tilde{T}_{\sigma}w_{k}) = -w_{j}^{(1)}(1)w_{k}(1) + w_{j}(1)w_{k}^{(1)}(1).
\]

(5.4)

If \( w_{j}(1) = 0 \) for some \( j \in \mathbb{N} \), then \( w_{j}^{(1)}(1) \neq 0 \) due to uniqueness of solutions of the equation \( l_{\sigma}(y) = \lambda_{j}^{2}y \), and the above relation shows that \( w_{k}(1) = 0 \) for all \( k \in \mathbb{N} \) as required.

Otherwise \( w_{j}(1) \neq 0 \) for all \( j \in \mathbb{N} \), and equation (5.4) implies that there exists a constant \( h \in \mathbb{R} \) such that \( w_{j}^{(1)}(1)/w_{j}(1) = h \) for all \( j \in \mathbb{N} \). Thus \( \lambda_{k}^{2} \) are eigenvalues of the Sturm–Liouville operator \( T_{\sigma,h} \), which is the restriction of \( \tilde{T}_{\sigma} \) by the boundary condition \( y^{(1)}(1) = hy(1) \). However, the eigenvalues \( v_{k}^{\sigma} \) of \( T_{\sigma,h} \) are known to obey the asymptotics \( v_{k} = \pi(k - 1/2) + o(1) \) as \( k \to \infty \) (see section 7), and this contradiction eliminates the possibility that \( w_{j}(1) \) do not vanish.

Thus we have proved that every function \( w_{j} \) satisfies the Dirichlet condition at \( x = 1 \) and so is an eigenfunction of the Sturm–Liouville operator \( T_{\sigma} \) corresponding to the eigenvalue \( \lambda_{j}^{2} \).

Moreover, \( T_{\sigma} \) has no other eigenvalues since the system \( (w_{k})_{k=1}^{\infty} \) is already complete in \( \mathcal{H} \). Henceforth the element \( \{\lambda_{k}^{2}, (\alpha_{k})\} \in SD \) is indeed the spectral data for the Sturm–Liouville operator \( T_{\sigma} \) with \( \sigma \) of (5.3), and the theorem is proved.

\( \square \)

To sum up, we have established the following.

**Corollary 5.4.** Two sequences \( \{\lambda_{k}^{2}\} \) and \( \{\alpha_{k}\} \) are the spectral data for a (positive) Sturm–Liouville operator \( T_{\sigma} \) with potential \( q = \sigma' \) from the space \( W_{2}^{-1}(0, 1) \) if and only if assumptions (A1) and (A2) are satisfied.

The corresponding operator \( T_{\sigma} \) is uniquely recovered from the spectral data through formula (5.3), in which the functions \( \phi \) and \( f \) are given by (3.2) and (3.1) respectively, and \( k \) is the kernel of the solution \( K \) of the GLM equation (4.1).

### 6. Stability and isospectral sets

In this section, we would like to study the correspondence between the Dirichlet Sturm–Liouville operators \( T_{\sigma} \) with real-valued \( \sigma \in \mathcal{H} \) and their spectral data in more detail. Namely, we shall show that the potential \( q = \sigma' \in W_{2}^{-1}(0, 1) \) (and thus the operator \( T_{\sigma} \in SC \)) depends continuously on the spectral data \( \{\lambda_{k}^{2}, (\alpha_{k})\} \in SD \) and that the isospectral sets are analytically diffeomorphic to the Hilbert space \( \ell_{2} \). Similar results were established in [6, 31] for the regular case \( q \in \mathcal{H} \), and in [22, 24] for impedance Sturm–Liouville operators.

First we have to introduce the topology on the set \( SD \). Recall that by definition any element \( \{\lambda_{k}^{2}, (\alpha_{k})\} \in SD \) is uniquely determined by two \( \ell_{2} \)-sequences \( (\mu_{k}) \) and \( (\beta_{k}) \) through the relations \( \lambda_{k}^{2} = \pi k + \mu_{k} \) and \( \alpha_{k} = 1 + \beta_{k} \). Therefore, we can identify \( SD \) with an open subset of the space \( \ell_{2} \times \ell_{2} \) in the standard coordinate system \( ((\mu_{k}), (\beta_{k})) \). In this way \( SD \) becomes a subset of a Hilbert space and inherits the topology of that space.

We shall study the correspondence between spectral data \( \{\lambda_{k}^{2}, (\alpha_{k})\} \in SD \) and operators \( T_{\sigma} \in SC \) through the chain

\[
\{(\lambda_{k}^{2}), (\alpha_{k})\} \leftrightarrow \phi \leftrightarrow K \leftrightarrow \sigma \leftrightarrow T_{\sigma},
\]

in which \( \phi \in L_{2}(0, 2) \) is the function of (3.2) and \( K \in \mathcal{S}_{2} \) is the operator of (4.1).

**Lemma 6.1.** The function

\[
\phi(s) = \sum_{k \in \mathbb{N}} \left( \cos \pi ks - \frac{1}{\alpha_{k}} \cos \lambda_{k}s \right)
\]

depends continuously in \( L_{2}(0, 2) \) on the spectral data \( \{(\lambda_{k}^{2}), (\alpha_{k})\} \in SD \).
**Proof.** We can rewrite the function $\phi$ in terms of the $\ell_2$-sequences $(\mu_k)$ and $(\hat{\mu}_k)$ with 

$$
\hat{\phi}_k := 1 - 1/\alpha_k = \hat{\mu}_k/\alpha_k
$$

as follows:

$$
\phi(s) = \sum_{k \in \mathbb{N}} \left( \cos \pi k s - \cos(\pi k s + \mu_k s) + \hat{\phi}_k \cos(\pi k s + \mu_k s) \right)
$$

$$
= \sum_{k \in \mathbb{N}} 2 \sin(\mu_k s/2) \sin[(\pi k + \mu_k)/2] s \sum_{k \in \mathbb{N}} \hat{\phi}_k \cos(\pi k s + \mu_k s).
$$

Therefore, it suffices to prove that the mappings from $\ell_2 \times \ell_2$ into $L_2(0, 2)$ given by

$$
\{(v_k), (\gamma_k) \} \mapsto \psi_1(s) := \sum_{k \in \mathbb{N}} \gamma_k \cos(\pi k s + \nu_k s)
$$

and

$$
\{(v_k), (\gamma_k) \} \mapsto \psi_2(s) := \sum_{k \in \mathbb{N}} \gamma_k \sin(\pi k s + \nu_k s)
$$

are continuous.

Suppose that $\{(\hat{v}_k), (\hat{\nu}_k)\}$ is another element of $\ell_2 \times \ell_2$ giving rise to a function $\hat{\psi}_1 := \sum_{k \in \mathbb{N}} \hat{v}_k \cos(\pi k s + \hat{\nu}_k s) \in L_2(0, 2)$; then

$$
\|\psi_1 - \hat{\psi}_1\|_{L_2(0, 2)} \leq \left\| \sum_{k \in \mathbb{N}} (\gamma_k - \hat{\gamma}_k) \cos(\pi k s + \nu_k s) \right\|_{L_2(0, 2)} + \left\| \sum_{k \in \mathbb{N}} \gamma_k [\cos(\pi k s + \nu_k s) - \cos(\pi k s + \hat{\nu}_k s)] \right\|_{L_2(0, 2)}.
$$

(6.1)

We may assume that the sequence $(\pi k + \nu_k)$ strictly increases; then $(\cos(\pi k s + \nu_k s))_{k \in \mathbb{N}}$ is a Riesz basic sequence in $L_2(0, 2)$, and the first summand above is bounded by $C \| (\gamma_k) - (\hat{\gamma}_k) \|_{\ell_2}$, with $C = C((\nu_k))$ being the corresponding Riesz constant (see appendix A). In view of the inequality

$$
|\cos(\pi k s + \nu_k s) - \cos(\pi k s + \hat{\nu}_k s)| \leq 2|\nu_k - \hat{\nu}_k|,
$$

the second summand in (6.1) is bounded by $2 \| (\hat{\gamma}_k) \|_{\ell_1} \| (\nu_k) - (\hat{\nu}_k) \|_{\ell_1}$, and the statement for $\psi_1$ is proved. Continuity of the second mapping is proved analogously. □

**Lemma 6.2.** The solution $K \in \mathfrak{S}_2$ of the GLM equation (4.1) depends locally analytically on $\phi \in L_2(0, 2)$.

**Proof.** The operator $F$ depends linearly on $\phi \in L_2(0, 2)$ and, in view of inequality (5.2),

$$
\|F\| \leq \|F\|_{\mathfrak{S}_2} \leq 2 \|\phi\|_{L_2(0, 2)},
$$

so that $F$ is an analytic function of $\phi$ (see [6, appendix A] or [32, ch 2] on analytic mappings of Banach spaces). Next, by lemma 4.1 the operator $P^*_F \in \mathfrak{S}(\mathfrak{S}_2)$ is continuous in $F \in \mathfrak{S}_2$ (and thus analytic in view of linearity), while the inverse function $(I - P^*_F)^{-1}$ is locally analytic in $P^*_F$. Combining these statements and recalling the formula

$$
K = -(I + P^*_F)^{-1}P^*F,
$$

we easily derive the claim. □

**Lemma 6.3.** The function $\sigma$ of (5.3) depends locally analytically in $\mathcal{H}$ on $\phi$ in $L_2(0, 2)$.

**Proof.** By definition $\sigma(x) = -2\phi(2x) - 2 \int_0^x k(x, s) f(s, x) \, ds$, where $k$ is the kernel of $K$ and $f(s, x) = \phi(s + x) - \phi(s - x)$. The second summand above is a continuous bilinear function of $K$ and $\phi$ (see the proof of theorem 5.1). Therefore, $\sigma$ is a jointly analytic function of $\phi$ and $K$ [32, ch 2], and in view of lemma 6.2 the result follows. □

We denote by $\Sigma^*$ the set of all real-valued $\sigma \in \mathcal{H}$, for which the operators $T_\sigma$ are positive. Observe that the operators $T_\sigma, K_\sigma$ and the spectral data do not change if $\sigma$ is replaced by $\sigma + c$ with any real $c$, so that these objects depend in fact on an equivalence class $\hat{\sigma}$ in $\Sigma^*/\mathbb{R}$ (i.e., on the common derivative $q = \sigma'$ for $\sigma \in \hat{\sigma}$) rather than on $\sigma \in \Sigma^*$. It is proved in [1, 5] that this dependence is continuous for $T_\sigma$ and $K_\sigma$; we shall show next that the spectral data also depend continuously on $\hat{\sigma} \in \Sigma^*/\mathbb{R}$.
Lemma 6.4 (cf [24, lemma 4.3]). The mapping $\Sigma^* / \mathbb{R} \ni \hat{\sigma} \mapsto (\lambda_n^2)$ is continuous.

**Proof.** We recall that $\lambda_k = \pi k + \mu_k$ for some $\ell_2$-sequence $(\mu_k)$ and that the continuity of $(\lambda_n^2)$ is understood as continuity of $(\mu_k)$ in the $\ell_2$-topology. Also, $\lambda_n$ are simple zeros of the function

$$\Phi(\lambda) = \Phi(\lambda, \sigma) := \sin \lambda + \int_0^1 k_\sigma (1, y) \sin \lambda y \, dy,$$

where $k_\sigma$ is the kernel of the corresponding transformation operator $K_\sigma$.

It can be shown (see also [13, ch 1.3]) that

$$|\Phi(\lambda) - \sin \lambda| = \left| \int_0^1 k_\sigma (1, y) \sin \lambda y \, dy \right| \rightarrow 0,$$

$$|\Phi'(\lambda) - \cos \lambda| = \left| \int_0^1 y k_\sigma (1, y) \cos \lambda y \, dy \right| \rightarrow 0$$

as $\lambda \to \infty$ inside the strip $[z \in \mathbb{C} | \text{Im } z | \leq 1]$. Therefore, there exist $\epsilon_0 > 0$ and a sequence $(C_n)$ of circles $C_n := [z \in \mathbb{C} | |z - \lambda_n| = r_n]$, $r_n \in (0, 1/2)$, such that the discs $D_n := [z \in \mathbb{C} | |z - \lambda_n| \leq r_n]$ are pairwise disjoint and for all $n \in \mathbb{N}$ we have

$$\min_{\lambda \in C_n} |\Phi(\lambda)| \geq \epsilon_0, \quad \min_{\lambda \in D_n} |\Phi'(\lambda)| \geq \epsilon_0. \quad (6.2)$$

It is proved in [5] that the mapping $\Sigma^* \ni \sigma \mapsto k_\sigma (1, \cdot) \in \mathcal{H}$ is continuous, whence for any $\epsilon \in (0, \epsilon_0)$ there exists $\delta > 0$ such that for every $\hat{\sigma} \in \Sigma^*$ from a $\delta$-neighbourhood of $\sigma$ we have $\|k_{\hat{\sigma}} (1, \cdot) - k_{\sigma} (1, \cdot)\| < \epsilon / 4$. We fix such $\epsilon$ and a function $\hat{\sigma}$, denote by $\lambda_n^2$ the corresponding eigenvalues and put $\hat{\Phi}(\lambda) := \Phi(\lambda, \hat{\sigma})$ and $g := k_{\hat{\sigma}} (1, \cdot) - k_\sigma (1, \cdot)$. Simple computation shows that

$$\max_{\lambda \in C_n} |\hat{\Phi}(\lambda) - \Phi(\lambda)| \leq 2 \int_0^1 |g(y)| \, dy \leq \epsilon_0/2, \quad (6.3)$$

$$\max_{\lambda \in D_n} |\hat{\Phi}'(\lambda) - \Phi'(\lambda)| \leq 2 \int_0^1 |yg(y)| \, dy \leq \epsilon_0/2. \quad (6.4)$$

Estimates (6.2) and (6.3) and Rouché’s theorem imply that $\lambda_n \in D_n$.

Since the functions $\hat{\Phi}$ and $\Phi$ assume real values for real $\lambda$, for every $n \in \mathbb{N}$ there is $\hat{\lambda}_n$ between $\lambda_n$ and $\hat{\lambda}_n$, for which

$$\hat{\Phi}(\lambda_n) = \hat{\Phi}(\lambda_n) - \hat{\Phi}(\hat{\lambda}_n) = (\lambda_n - \hat{\lambda}_n) \hat{\Phi}'(\hat{\lambda}_n).$$

We observe that $\hat{\lambda}_n \in D_n$ and that, in view of (6.2) and (6.4),

$$|\hat{\Phi}(\lambda_n)| \geq |\hat{\Phi}(\hat{\lambda}_n)| - \epsilon_0/2 \geq \epsilon_0/2.$$

On the other hand,

$$\hat{\Phi}(\lambda_n) = \hat{\Phi}(\lambda_n) - \Phi(\lambda_n) = s_n(g) := \int_0^1 g(y) \sin \lambda_n y \, dy,$$

so that we get

$$|\lambda_n - \hat{\lambda}_n| \leq \frac{2s_n(g)}{\epsilon_0}$$

for all $n \in \mathbb{N}$. Recall (see appendix A) that the system of functions $\{\sin \lambda_n x\}$ forms a Riesz basis of $\mathcal{H}$, so that

$$\sum_{n \in \mathbb{N}} |s_n(g)|^2 \leq C \|g\|^2,$$
C > 0 being the corresponding Riesz constant. Combining the above estimates, we conclude that
\[ \sum_{n \in \mathbb{N}} |\lambda_n - \tilde{\lambda}_n|^2 \leq \frac{4C}{\epsilon \delta_0^2} \| g \|^2 \leq \frac{C\epsilon^2}{4\delta_0^2}, \]
and the desired continuity follows.

**Lemma 6.5.** The sequence \((\alpha_k)\) of norming constants depends continuously on \(\tilde{\sigma} \in \Sigma^+ / \mathbb{R}\).

**Proof.** Suppose that \(\tilde{\sigma} \in \mathcal{H}\) is in the \(\epsilon\)-neighbourhood \(\mathcal{O}_\epsilon\) of \(\sigma\) and \(\tilde{\alpha}_k\) are the corresponding norming constants for the operator \(T_{\tilde{\sigma}}\). As in the proof of lemma 2.4 we see that
\[ \alpha_k - \tilde{\alpha}_k = \| u_k - \tilde{u}_k \| \| u_k + \tilde{u}_k \|, \]
where \(u_k = (1 + K_\sigma)v_k\) and \(\tilde{u}_k = (1 + K_{\tilde{\sigma}})\tilde{v}_k\). \(K_\sigma\) and \(K_{\tilde{\sigma}}\) are transformation operators and
\[ v_k = \sqrt{2} \sin \lambda_k x, \quad \tilde{v}_k = \sqrt{2} \sin \tilde{\lambda}_k x. \]
Since the norms \(\| u_k \|\) are bounded uniformly in \(\tilde{\sigma} \in \mathcal{O}_\epsilon\), it remains to show that the sequence \(\| u_k - \tilde{u}_k \|\) is in \(\ell_2\) and \(\sum_{k \in \mathbb{N}} \| u_k - \tilde{u}_k \|^2 \to 0\) as \(\epsilon \to 0\).

We have
\[ u_k - \tilde{u}_k = v_k - \tilde{v}_k + (K_\sigma - K_{\tilde{\sigma}})v_k + K_{\tilde{\sigma}}(v_k - \tilde{v}_k). \]
The system \((v_k)\) with \(\lambda_k = \pi k + \mu_k\) and \((\mu_k) \in \ell_2\) depends continuously on \((\mu_k)\) in the sense that
\[ \sum_{k \in \mathbb{N}} \| v_k - \tilde{v}_k \|^2 \leq 2 \sum_{k \in \mathbb{N}} \| \mu_k - \tilde{\mu}_k \|^2. \]
Since the norms \(\| K_{\tilde{\sigma}} \|\) are uniformly bounded in \(\tilde{\sigma} \in \mathcal{O}_\epsilon\), also
\[ \sum_{k \in \mathbb{N}} \| K_{\tilde{\sigma}}(v_k - \tilde{v}_k) \|^2 \leq C_1 \sum_{k \in \mathbb{N}} \| \mu_k - \tilde{\mu}_k \|^2. \]
Finally, we observe that the system \((v_k)\) is a Riesz basis of \(\mathcal{H}\), so that
\[ \sum_{k \in \mathbb{N}} \| (K_\sigma - K_{\tilde{\sigma}})v_k \|^2 \leq C_2 \| K_\sigma - K_{\tilde{\sigma}} \|^2_{\ell_2}. \]
see appendix B. Since \(K_\sigma\) depends continuously in \(\mathcal{O}_2\) on \(\sigma \in \mathcal{H}\) by the results of [5], the required continuity follows.

Combining the above lemmata, we arrive at the following result.

**Theorem 6.6.** The mapping
\[ \Sigma^+ / \mathbb{R} \ni \tilde{\sigma} \mapsto \{ (\lambda_k^2), (\alpha_k) \} \in \mathcal{D} \]
is a homeomorphism.

Fixing the spectrum \((\lambda_k^2)\), we can say even more about the corresponding isospectral set (cf [6, ch 4] for the regular case \(q \in \mathcal{H}\)).

**Theorem 6.7.** Suppose that the sequence \((\lambda_k^2)\) satisfies assumption (A1). Then the set of all isospectral potentials in \(W_{2}^{-}\) \((0, 1)\) with the Dirichlet spectrum \((\lambda_k^2)\) is analytically diffeomorphic to an open subset of the Hilbert space \(\ell_2\). The diffeomorphism is performed through the sequence \((\beta_k)\), where \(\beta_k = \alpha_k - 1\).

**Proof.** It suffices to note that the correspondence \((\tilde{\beta}_k) \mapsto \phi\),
\[ \phi(s) = \sum_{k \in \mathbb{N}} (\cos \pi ks - \cos \lambda_k s) - \sum_{k \in \mathbb{N}} \tilde{\beta}_k \cos \lambda_k s, \]
where \(\tilde{\beta}_k = \beta_k/(1 + \beta_k)\), is bounded and affine (thus analytic) in \((\tilde{\beta}_k) \in \ell_2\). Since \(\alpha_k = 1 + \beta_k\) are uniformly bounded away from zero, the mapping \((\beta_k) \mapsto \phi\) is analytic in \((\beta_k) \in \ell_2\), and the result follows from lemmata 6.2 and 6.3.

Analogous stability and isospectrality results hold true for other boundary conditions considered in the next section.
7. The case of other boundary conditions

The solution of the inverse spectral problem presented in sections 3–5 can easily be adapted to the case of other types of boundary condition, e.g., the boundary conditions of the third type at both endpoints, Dirichlet–Neumann or Neumann–Dirichlet ones. Below, we shall briefly discuss the modifications to be made for these cases.

For \( \sigma \in \mathcal{H} \) and \( H, h \in \mathbb{C} \), we consider a Sturm–Liouville operator \( T_{\sigma, H, h} \) given by the differential expression \( l_{\sigma} \) of (1.2) and the boundary conditions

\[
\begin{align*}
 u^{[1]}(0) - Hu(0) &= 0, \\
 u^{[1]}(1) + hu(1) &= 0.
\end{align*}
\]

(We recall that \( u^{[1]}(x) = u'(x) - \sigma(x) u(x) \) is the quasi-derivative of \( u \).) More precisely, \( T_{\sigma, H, h} \) is given by

\[
T_{\sigma, H, h} u = l_{\sigma}(u) := -(u^{[1]})' - \sigma u'
\]
on the domain

\[\mathcal{D}(T_{\sigma, H, h}) = \{ u \in W^1_1(0, 1) \mid (u^{[1]}(0), l_{\sigma}(u)) \in \mathcal{H}, \quad u^{[1]}(0) - Hu(0) = u^{[1]}(1) + hu(1) = 0 \} \times \mathcal{H}.\]

Observe that \( T_{\sigma, H, h} = T_{\sigma + H, 0, h} \), so that without loss of generality we may (and will) assume that \( H = 0 \).

It is known [1] that for all real-valued \( \sigma \in \mathcal{H} \) and \( h \in \mathbb{R} \) the operator \( T_{\sigma, 0, h} \) is selfadjoint, bounded below and has discrete simple spectrum \( \{ \lambda_k^2 \} \), \( k \in \mathbb{N} \). As earlier, upon adding a suitable constant to the potential \( q = \sigma' \), we can make all the eigenvalues positive. Denote by \( u_k \) the eigenfunction of \( T_{\sigma, 0, h} \) corresponding to the eigenvalue \( \lambda_k^2 \) and normalized in such a way that \( u_k(0) = \sqrt{2} \), and put \( \alpha_k := \|u_k\|^2 \).

Our aim is to solve the inverse spectral problem for \( T_{\sigma, 0, h} \), i.e., firstly, to describe the set \( \mathcal{SD} \) of all spectral data \( \{ (\lambda_k^2), (\alpha_k) \} \) that can be obtained by varying real-valued \( \sigma \in \mathcal{H} \) and \( h \in \mathbb{R} \) and, secondly, for given spectral data \( \{ (\lambda_k^2), (\alpha_k) \} \in \mathcal{SD} \), to find the corresponding operator \( T_{\sigma, 0, h} \) (i.e., to find the corresponding primitive \( \sigma \in \mathcal{H} \) of the potential \( q \) and the number \( h \in \mathbb{R} \)).

**Lemma 7.1.** Suppose that \( \sigma \in \mathcal{H} \) is real valued, \( h \in \mathbb{R} \), and that \( \lambda_1^2 < \lambda_2^2 < \cdots \) are eigenvalues of the operator \( T_{\sigma, 0, h} \) with \( \lambda_1^2 > 0 \). Then \( \lambda_k = \pi (k - 1) + \mu_k \), where \( (\mu_k) \in \ell_2 \).

**Proof.** Denote by \( \tilde{T}_{\sigma, 0} \) the extension of the operator \( T_{\sigma, 0, h} \) obtained by omitting the boundary condition at the point \( x = 1 \). By [5], the operators \( \tilde{T}_{\sigma, 0} \) and \( \tilde{T}_{0, 0} \) are similar, and the similarity is performed by the transformation operator \( I + K_{\sigma, 0} \); here \( K_{\sigma, 0} \) is an integral operator of Volterra type, \( (K_{\sigma, 0}u)(x) = \int_0^x k_{\sigma, 0}(x, y) u(y) dy \), and the kernel \( k_{\sigma, 0} \) has the property that \( k_{\sigma, 0}(x, \cdot) \) is an \( \mathcal{H} \)-function for every \( x \in [0, 1] \).

Put \( u(\cdot, \lambda) = (I + K_{\sigma, 0}) v(\cdot, \lambda) \), where \( v(x, \lambda) = \sqrt{2} \cos \lambda x \). Then \( u_k = u(\cdot, \lambda_k) \) and the numbers \( \pm \lambda_k \) are solutions of the equation \( u^{[1]}(1, \lambda) + Hu(1, \lambda) = 0 \). Using the properties of the transformation operator \( K_{\sigma, 0} \) (see [5, remark 4.3]), we find that

\[
u^{[1]}(1, \lambda) = -\sqrt{2} \lambda \sin \lambda - \sqrt{2} \int_0^1 g_1(x) \sin \lambda x \, dx + \int_0^1 g_2(x) \sqrt{2} \cos \lambda x \, dx + C
\]

for some functions \( g_1, g_2 \) from \( \mathcal{H} \) and a real constant \( C \). It follows that \( \lambda_k \) are zeros of the analytic function

\[
\Phi_1(\lambda) := -\lambda \sin \lambda + h \cos \lambda - \lambda \int_0^1 g_1(x) \sin \lambda x \, dx + \int_0^1 g_3(x) \cos \lambda x \, dx + C/\sqrt{2}.
\]
where $g_3 := g_2 + h\kappa_\sigma,0(1, \cdot) \in \mathcal{H}$. Now the standard analysis (cf the proof of lemma 2.3) yields the asymptotics required. □

We next establish the asymptotics of the norming constants $\alpha_k$.

**Lemma 7.2.** Suppose that $\sigma \in \mathcal{H}$ is real valued, $h \in \mathbb{R}$, and that $u_k$ are eigenfunctions of the operator $T_{\sigma,0,h}$ normalized as above. Then $\alpha_k = 1 + \beta_k$, where the sequence $\beta_k$ belongs to $\ell_2$.

Proof of this statement is completely analogous to the proof of lemma 2.4. The minor changes to be made concern notations, namely, we should put $v_k(s) = v(s, \lambda_k) = \sqrt{2} \cos \lambda_k s$ and $\nu_k(s) = \sqrt{2} \cos \pi (k-1)s$.

Suppose that the sequences $(\lambda_k^2)$ and $(\alpha_k)$ are the spectral data for an operator $T_{\sigma,0,h}$ with $\sigma \in \mathcal{H}$ and $h \in \mathbb{R}$ and put

$$F := \lim_{N \to \infty} \sum_{k=1}^{N} \left( \frac{1}{\alpha_k} \cos \lambda_k s - \cos \pi (k-1)s \right).$$

It is easily seen that $I + F$ is uniformly positive. As in lemma 3.2, it can be proved that $F$ has the following properties.

**Lemma 7.3.** The operator $F$ is a Hilbert–Schmidt integral operator with kernel

$$f(x, y) = \phi(x + y) + \phi(x - y)$$

where

$$\phi(s) = \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_k} \cos \lambda_k s - \cos \pi (k-1)s \right)$$

is an $L_2(0, 2)$-function.

Next we show that $F$ is related to the transformation operator $I + K_{\sigma,0}$ through the GLM equation (4.1), which can be used to determine $K_{\sigma,0}$ uniquely through the formula $K_{\sigma,0} = -(I + P^*)^{-1}P^*F \in \Theta_{\mathcal{H}}^+$, see section 4.

Suppose now that the sequences $(\lambda_k^2)$ and $(\alpha_k)$ consist of positive numbers and are as claimed in lemmata 7.1 and 7.2. We solve the GLM equation (4.1) for $K_{\sigma,0}$ and then in the same manner as in theorem 5.1 we show that there exists $\sigma \in \mathcal{H}$ such that $I + K$ is the transformation operator for the Sturm–Liouville operators $T_{\sigma,0}$ and $\tilde{T}_{\sigma,0}$. The only thing that remains to be proved is that our initial sequences $(\lambda_k^2)$ and $(\alpha_k)$ are the spectral data for the Sturm–Liouville operator $T_{\sigma,0,h}$ with some $h \in \mathbb{R}$.

We use identity (5.4) and replicate the arguments of the proof of theorem 5.3 to show that none of $u_j(1)$ vanish as otherwise all $u_j(1)$ would vanish and $\lambda_k$ would have a different asymptotics (that of the Neumann–Dirichlet boundary conditions case, see below). Therefore, $u_j(1) \neq 0$ for all $j \in \mathbb{N}$, and relation (5.4) implies that there exists a constant $h \in \mathbb{R}$ such that

$$u_j^{(1)}(1)/u_j(1) = h$$

for all $j \in \mathbb{N}$. Thus $(\lambda_k^2)$ are eigenvalues of the Sturm–Liouville operator $-\frac{d^2}{dx^2} + \sigma'$ subject to the boundary conditions $y^{(1)}(0) = 0$ and $y^{(1)}(1) = hy(1)$, and the proof is complete.

We summarize the above considerations in the following theorem.

**Theorem 7.4.** For two sequences $(\lambda_k^2)$ and $(\alpha_k)$ to be the spectral data of a (positive) Sturm–Liouville operator $T_{\sigma,0,h}$ with real-valued potential $q = \sigma'$ from $W^{-1}_2(0, 1)$ and $h \in \mathbb{R}$, it is necessary and sufficient that $\alpha_k$ are as in $(A2)$ and $\lambda_k$ satisfy the following assumption:

$(A1')$ $\lambda_k$ are all positive, strictly increase with $k$ and obey the asymptotic relation $\lambda_k = \pi (k-1) + \mu_k$ with some $\ell_2$-sequence $(\mu_k)_{k=1}^{\infty}$. 
The corresponding operator \( T_{\sigma,0,h} \) is uniquely recovered from the spectral data through formula (5.3), in which the functions \( \phi \) and \( f \) are given by (7.2) and (7.1) respectively, and \( k \) is the kernel of the solution \( K \) of the GLM equation (4.1). The number \( h \) in the boundary conditions is given by (7.3).

In a similar manner the case where one of the boundary conditions is a Dirichlet one and the other one is of the third type can be treated. It suffices to consider only the cases where \( H = \infty, h = 0 \), or \( H = 0, h = \infty \), as other situations reduce to one of these in view of the relation \( T_{\sigma,H,h} = T_{\sigma+\bar{\sigma},H-H',h+\bar{h}'} \).

The operators \( T_{\sigma,0,0} \) and \( T_{\sigma,0,\infty} \) can be uniquely recovered from the spectral data, the sequences of eigenvalues and norming constants. The eigenvalues \( \lambda_k^2 \) of the operators \( T_{\sigma,0,0} \) and \( T_{\sigma,0,\infty} \) obey the asymptotics \( \lambda_k = \pi(k - 1/2) + \mu_k \) for \( \ell_2 \)-sequences \( (\mu_k) \), while the norming constants \( C_k \) (defined as in section 2 or 7 according to \( H = \infty \) or 0) satisfy (A2). Thus the sets of spectral data for the families of Sturm–Liouville operators \( T_{\sigma,\infty,0} \) and \( T_{\sigma,0,\infty} \), where \( \sigma \) runs through \( \mathcal{H} \), admit explicit descriptions, while the reconstruction algorithm remains the same. The details can be recovered by analogy with the analysis of sections 2–7 (cf [12, section II.10] for the regular case).

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### Appendix A. Riesz bases

In this appendix we gather some well known facts about Riesz bases of sines and cosines (see, e.g., [29, 33] and references therein for a detailed exposition of this topic).

Recall that a sequence \((e_n)_{n=1}^{\infty}\) in a Hilbert space \( \mathcal{H} \) is a Riesz basis if and only if any element \( e \in \mathcal{H} \) has a unique expansion \( e = \sum_{n=1}^{\infty} c_n e_n \) with \((c_n) \in \ell_2 \). If \((e_n)\) is a Riesz basis, then in the above expansion the Fourier coefficients \( c_n \) are given by \( c_n = \langle e, e_n \rangle \), where \((e_n')_{n=1}^{\infty}\) is a system biorthogonal to \((e_n)\), i.e., which satisfies the equalities \( \langle e_k, e_n' \rangle = \delta_{kn} \) for all \( k, n \in \mathbb{N} \). Moreover, the biorthogonal system \((e_n')\) is a Riesz basis of \( \mathcal{H} \) as long as \((e_n)\) is, in which case for any \( e \in \mathcal{H} \) the expansion \( e = \sum (e, e_n) e_n' \) also takes place. In particular, if \((e_n)\) is a Riesz basis, then for any \( e \in \mathcal{H} \) the sequence \((e_n')\) with \( e_n' := (e, e_n) \) belongs to \( \ell_2 \).

Also, any Riesz basis \((e_n)\) is equivalent to an orthonormal one, i.e., there exists a homeomorphism \( U \) such that \((Ue_n)\) is an orthonormal basis of \( \mathcal{H} \). As a result, there exists a constant \( C > 0 \) (the Riesz constant) such that, for any sequence \((c_k) \in \ell_2 \),

\[
C^{-1} \sum |c_k|^2 \leq \left\| \sum c_k e_k \right\|^2 \leq C \sum |c_k|^2;
\]

in particular, the series \( \sum c_k e_k \) converges in \( \mathcal{H} \) for any \( \ell_2 \)-sequence \((c_k)\).

**Proposition A.1** ([33]). Suppose that real numbers \( \mu_k, k \in \mathbb{N} \), tend to zero and that the sequence \( (\pi k + \mu_k) \) strictly increases. Then each of the following systems forms a Riesz basis of \( L_2(0,1) \):

(a) \( \{\sin(\pi k x + \mu_k x)\}_{k=1}^{\infty} \);
(b) \( \{\sin(\pi (k - 1/2) x + \mu_k x)\}_{k=1}^{\infty} \);
(c) \( \{\cos(\pi k x + \mu_k x)\}_{k=0}^{\infty} \);
(d) \( \{\cos(\pi (k + 1/2) x + \mu_k x)\}_{k=0}^{\infty} \).
Moreover, holds. Also, ST is finite. Being endowed with the norm ∥·∥₂ bases of their closed linear spans in L₂(0, 2).

Appendix B. The ideal of Hilbert–Schmidt operators

In this appendix we recall some necessary facts about the Schatten–von Neumann 𝒮ₚ ideals (see details in [29]).

Suppose that T is a compact operator in a Hilbert space ℋ; then |T| = (T* T)₁/₂ is a non-negative selfadjoint compact operator. Denote by λ₁(|T|) ≥ λ₂(|T|) ≥ · · · the eigenvalues of |T| in non-decreasing order and repeated according to their multiplicity. Then λₖ(|T|) is called the k-th s-number of T and is denoted by sₖ(T).

The ideal 𝒮ₚ, p ∈ [1, ∞), consists of all compact operators for which the expression

\[ \|T\|_{𝒮_p} := \left( \sum_{k \in \mathbb{N}} s_k^p(T) \right)^{1/p} \]

is finite. Being endowed with the norm ∥·∥_{𝒮_p}, the ideal 𝒮ₚ becomes a Banach space.

In particular, 𝒮₁ is the ideal of trace class operators. For any T ∈ 𝒮₁ its matrix trace \( \text{tr } T := \sum_{k \in \mathbb{N}} (T e_k, e_k) \) with respect to any orthonormal basis \((e_k)\) is finite and coincides with the spectral trace, i.e., the sum of all eigenvalues of T repeated according to their algebraic multiplicity.

For p = 2 the ideal 𝒮₂ consists of all Hilbert–Schmidt operators. For T ∈ 𝒮₂ and any two orthonormal bases \((e_j)\) and \((e'_k)\) of ℋ, we have

\[ \|T\|_{𝒮₂}^2 = \sum_{j,k \in \mathbb{N}} |(T e_j, e'_k)|^2 = \sum_{j \in \mathbb{N}} \|T e_j\|^2 < \infty. \]

If \((e_j)\) is a Riesz basis of ℋ, then \((U e_j)\) is an orthonormal basis for some homeomorphism U, and for any T ∈ 𝒮₂ the estimate

\[ \sum_{j \in \mathbb{N}} \|T e_j\|^2 = \|T U^{-1}\|_{𝒮₂}^2 \leq \|T\|_{𝒮₂}^2 \|U^{-1}\|_2^2 \]

holds. Also, ST is a trace class operator whenever S and T are Hilbert–Schmidt ones. Moreover, 𝒮₂ is a Hilbert space under the scalar product given by

\[ \langle S, T \rangle_2 := \text{tr } S T^* = \text{tr } T^* S. \]

If ℋ = L₂(0, 1), any Hilbert–Schmidt operator T is an integral one with kernel t given by

\[ t(x, y) := \sum_{j,k \in \mathbb{N}} (T e_j, e_k) e_k(x) e_j(y); \]

here \((e_k)\) is any orthonormal basis of ℋ and the series converges in L₂((0, 1) × (0, 1)). Moreover,

\[ \int_0^1 \int_0^1 |t(x, y)|^2 \, dx \, dy = \|T\|_{𝒮₂}^2. \]

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