DIMENSIONAL Crossover FOR THE 
BOSE – EINSTEIN CONDENSATION OF 
AN IDEAL BOSE GAS

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ABSTRACT

We study an ideal Bose gas of $N$ atoms contained in a box formed by two identical planar and parallel surfaces $S$, enclosed by a mantle of height $a$ perpendicular to them. Calling $r_0$ the mean atomic distance, we assume $S \gg r_0^2$, while $a$ may be comparable to $r_0$. In the bidimensional limit ($a/r_0 \equiv \gamma \ll 1$) we find a macroscopic number of atoms in the condensate at temperatures $T \sim \mathcal{O}(1/\log(N))$; therefore, condensation cannot be described in terms of intensive quantities; in addition, it occurs at temperatures not too low in comparison to the tridimensional case. When condensation is present we also find a macroscopic occupation of the low-lying excited states. In addition, the condensation phenomenon is sensitive to the shape of $S$. The former two effects are significant for a nanoscopic system. The tridimensional limit is slowly attained for increasing $\gamma$, roughly at $\gamma \sim 10^2 - 10^3$. 
1 INTRODUCTION

A system of $N$ non-interacting particles is in a condensed phase if a non-negligible fraction of them lie in the ground state; however, for small values of $N$, say $10^3$–$10^5$, the condensation boundary becomes blurred. As it is well-known, for an ideal gas obeying Boltzmann statistics, condensation occurs at vanishingly small temperatures, irrespective of the dimensionality $d$ of the system ($T \sim \mathcal{O}\{1/N^2\}$ for $d = 1$; $T \sim \mathcal{O}\{1/N\}$ for $d = 2$ and $T \sim \mathcal{O}\{1/N^{2/3}\}$ for $d = 3$). In contrast, for Bose statistics, condensation begins at a finite temperature in three-dimensional systems, while it is absent in one- and two-dimensional systems [1], at least in the mathematical limit $N \to \infty$ ($T \sim \mathcal{O}\{1/N\}$ for $d = 1$ and $T \sim \mathcal{O}\{1/\log(N)\}$ for $d = 2$, see Ref. [10]). The presence of an strong enough external potential can, however, bring about a Bose-Einstein condensation (BEC) in two and even one dimensions, at a finite temperature [2].

Since Bose and Einstein predicted the condensation phenomenon, there has been a great interest into achieving such a state in an actual system. The existence of superfluidity in $^4$He has been considered a sort of corroboration of BEC. However, superfluid Helium is a poor example of a non-interacting particle system, since the mean atomic separation is comparable with the range of the atomic potential. Therefore, a complete explanation of superfluidity must rely in Many-Body Theory.

The goal of obtaining BEC in a gas of nearly non-interacting atoms was recently achieved by Anderson et al. [3] who combined the techniques of laser and evaporative cooling of magnetically trapped atoms, implementing the latter one by means of an ingenious “time orbiting potential” magnetic trap. They obtained temperatures of $\sim 10^{-7}$ K for a sample of a few thousand atoms trapped in a region of linear dimensions $\sim 10^{-5}$ m and were thus able to achieve BEC. The nanoscopic character of the experiment motivates a more detailed study of BEC for such small systems.

The purpose of this work is to analyze the condensation process of an ideal Bose gas as a function of the dimensionality of the system and the shape of the container. Since a rigorous infinite two-dimensional gas is not a realistic system, we shall consider here a true tridimensional gas enclosed in a finite container; our aim is to understand how the properties of the gas change as the container is shrunken in one of its dimensions (say, $z$), expecting a continuous passage from tri to bidimensional behavior. In fact, at enough small temperatures, when the de Broglie wavelength $\lambda(T)$ becomes smaller than the container width, there is not enough thermal energy to excite the degree of
freedom associated to the \( z \) direction. Therefore, one dimension “freezes” and our system behaves like an “effective” two-dimensional gas, in spite of its actual tridimensional nature. If the container width is comparable to the interatomic mean distance, condensation must occur after the bidimensional regime has been established.

Interest in this problem dates back to London’s suggestion to use BEC as a qualitative model for understanding superfluidity in He\(^4\) \([4]\). Experiments showed that He\(^4\) remained a superfluid when forced to flow inside narrow capillaries. On the other hand, superconductivity was shown to persist in thin films. This prompted the study of “superproperties” in finite geometries. In one of the first studies\([5]\), an ideal bose gas was enclosed in a box of finite volume \((L \times D \times D)\) and the number of particles in the ground state was computed numerically as a function of the temperature. Large deviations from the bulk result were found in the one-dimensional limit \((L >> D)\). Attempts to ascertain the dependence of the superfluid transition temperature for He\(^4\) on the dimensions of narrow channels\([6]\), were followed by discussions on the ambiguity of the concept of a transition temperature in a finite geometry, where a rigorous phase transition is absent \([7, 8]\). Vanishing of quasiaverages in partially finite geometries (one or more dimensions finite while one or more dimensions extend to infinity) led to the suggestion\([9]\) that these geometries were not good approximations to thin films and pores found in laboratories. For a strictly finite geometry, BEC was shown to occur for an ideal gas\([9]\).

While in most of these earlier works, the goal was to achieve a qualitative understanding of superfluid properties, we are now genuinely concerned with describing the behavior of a finite number of nearly-independent bosons, inside a confining potential, such as in the recent experiments mentioned at the beginning\([3]\). The recent advances in nanotechnology and laser optics allow researchers now to manipulate molecules and their environment at the mesoscopic and nanoscopic levels, rendering the investigation of quantum effects in a system with a finite number of components (i.e., away from the thermodynamic limit), an interesting field of study.

In the present work we shall address the problem of crossover between two and tridimensional behaviour in BEC. We shall give both, exact numerical and asymptotic analytical results. For the sake of completeness we shall refresh some old results. However, our analytical expressions will give a compact and clear sight of the phenomenology under consideration; in particular, they show the non-intensive character of BEC near the two-dimensional limit, giving simple estimations for the temperature of condensation (we remember that the latter temperature is not sharply defined in this case). We shall also
clarify the effect of geometry in BEC, which can be summarized by some characteristic constants associated with the shape of the container, listing them for some particular cases; incidentally, the different boundary conditions will be considered like peculiarities of the topology of the container, affecting the values of former constants. The role of lower excited states and its macroscopic occupancy shall be also addressed. Finally, we shall also analyze the continuous transition between two and tridimensional geometries.

2 THE MODEL

We study an ideal Bose gas of \( N \) atoms contained in a box formed by two identical planar and parallel surfaces \( S \) (say, in the \( x, y \) plane), enclosed by a mantle of height \( a \) perpendicular to them. We call \( r_0 \) the mean atomic distance, and introduce the parameter \( \gamma = a/r_0 \). Since the container volume is \( Sa \), it holds \( r_0^3 = Sa/N \). We assume that \( S \gg a^2 \) (or equivalently \( N \gg \gamma^3 \)), while \( a \) may be comparable to \( r_0 \).

We introduce a characteristic temperature for the ideal gas, associated to the energy of a particle whose de Broglie wavelength \( \lambda(T) \) is comparable to the mean intermolecular distance \( r_0 \),

\[
T_0 \equiv \frac{\hbar^2}{2\pi m k_B r_0^2}
\]  

(1)

where \( m \) is the mass of a particle and \( h \), \( k_B \) are the Planck and Boltzmann constants, respectively. For \( T < T_0 \) the bidimensional behaviour can be associated with \( \gamma \sim O(1) \), while \( \gamma \gg O(1) \) corresponds to the tridimensional limit. We also define the “reduced” temperature \( \tau = T/T_0 \); now the (tridimensional) BEC critical temperature is \( \tau_c = 0.527201068760 \ldots \); see Ref. [1].

The energy dispersion relation for a particle enclosed there can be expressed like

\[
E(\nu, \ell) = k_B T \left\{ \eta \xi_\nu^2 + \sigma \ell^2 \right\}
\]  

(2)

here \( \sigma \equiv \pi/[(\pi \gamma^2(q + 1)^2) \right\}$, where \( q = 0 \) for cyclic boundary conditions (CBC) and \( q = 1 \) for impenetrable boundary conditions (IBC) in the \( z \) direction respectively; \( \ell = 0, \pm 1, \pm 2, \ldots \) for CBC, while \( \ell = 1, 2, 3 \ldots \) for IBC. \( \eta \equiv \pi/\tau N_s \), where \( N_s = N/\gamma = S/r_0^2 \) is the number of particles in a neighbourhood \( r_0 \) of container base \( S \). The numbers \( \{ \xi_\nu \} \) describe the spectrum of the bidimensional Helmholtz equation for the surface \( S \); they depend only
on the shape of $S$ and the boundary conditions used for the $x,y$ plane. We choose the ground state as energy zero; therefore we shift $\xi_\nu^2 \rightarrow \xi_\nu^2 - \xi_0^2$, while $\ell^2 \rightarrow \ell^2 - 1$ for IBC in the $z$ direction.

We shall use the asymptotic condition for the bidimensional energy

$$D(\xi) \, d\xi \equiv \{\text{Number of } \xi_\nu \in [\xi, \xi + d\xi]\} = (2\pi \xi - C_s) \, d\xi$$

valid for $\xi \, d\xi \gg 1$. Here $C_s$ is a constant that depends on the shape of $S$; therefore, the contribution $-C_s$ in Eq. (3) represents the finite size corrections to the density of states [11]; for example, $C_s = 0$ for a square with CBC, $C_s = 2$ for a square with IBC, and $C_s = \sqrt{\pi}$ for a circle with IBC.

The mean occupation of the state $E(\nu, \ell)$ is given by

$$N(\nu, \ell) \equiv N \cdot n(\nu, \ell) = \left\{ \zeta \exp \left[ \eta \xi_\nu^2 + \sigma (\ell^2 - q) \right] - 1 \right\}^{-1} .$$

(4)

Here $\zeta \equiv \exp(-\mu/k_BT)$, where $\mu$ is the chemical potential. According to the former relation, the ground state mean occupation is $N_0 \equiv N \cdot n_0 = 1/(\zeta - 1)$.

For IBC in the $z$ direction we shift the index $\ell \rightarrow \ell - 1$, in order to associate $\ell = 0$ to the ground state. Due to the fact that $N \gg \gamma^3$, it holds that $n(\nu, \ell \neq 0) \sim \mathcal{O}(1/N)$, while, for $\ell = 0$ the lower excitations in the $\{x,y\}$ plane may have a macroscopic population. In fact, when a macroscopic amount of particles lie in the ground state, relation (4) leads to

$$n(\nu, 0) \approx \left\{ \frac{1}{n_0} + \frac{\pi \gamma}{\tau} (\xi_\nu)^2 + \mathcal{O}(\frac{1}{N}) \right\}^{-1} .$$

(5)

The equation for $\zeta$ is $1 = \sum_{\nu,\ell} n(\nu, \ell)$. In order to study the condensation phenomenon, it is necessary to separate the former sum into contributions from states with macroscopic ($n(\nu, \ell = 0) \sim \mathcal{O}(1)$) and microscopic ($n(\nu, \ell \neq 0) \sim \mathcal{O}(1/N)$) occupancy. Only in the latter contributions we can safely approximate $\sum_{\nu} \rightarrow \int d\xi D(\xi)$ since then $n(\nu, \ell)$ is a smooth function of $\nu$. But the macroscopic contributions $n(\nu, 0)$ must be summed explicitly, at least for the lower energy states. Over a high enough value of $\nu$ it is safe to replace this sum by an integral; defining $\xi_L < R < \xi_{L+1}$, $R \gg 1$, the equation for $\zeta$ becomes

$$1 = \frac{1}{N(\zeta - 1)} + \sum_{\nu=1}^{L} n(\xi_\nu, 0)$$

$$+ \frac{2}{(q + 1)N} \int_{1}^{\infty} \int_{0}^{\infty} d\xi \, D(\xi) \left\{ \zeta \exp \left[ \eta \xi^2 + \sigma \ell (\ell + 2q) \right] - 1 \right\}^{-1} +$$
\[
\frac{1}{N} \int_{R}^{\infty} d\xi \, D(\xi) \left[ \zeta \exp \left( \eta \xi^2 \right) - 1 \right]^{-1}
\]

where \(1/N(\zeta - 1) \equiv n_0\) is the fractional ground state occupation. The bulk contribution to \(D(\xi)\) can be integrated in a closed form; the finite size correction to \(D(\xi)\) gives a negligible contribution, except for the case when condensation has begun and \(\ell = 0\); in the latter case we use Eq. (5). Thus, the equation for \(\zeta\) takes the form

\[
1 = \frac{1}{N(\zeta - 1)} + \frac{1}{N} \sum_{\nu=1}^{L} \left[ \zeta \exp \left( \eta \xi_{\nu}^2 \right) - 1 \right]^{-1} - \\
\frac{\tau}{\gamma} \log \left[ 1 - \frac{1}{\zeta} \exp \left( -\eta R^2 \right) \right] - \\
\frac{2\tau}{(q+1)\gamma} \sum_{\nu=1}^{\infty} \log \left\{ 1 - \frac{1}{\zeta} \exp \left[ -\sigma \ell (\ell + 2q) \right] \right\} - \\
C_s \sqrt{\frac{\tau n_0}{\pi \gamma}} \arctan \left[ \frac{1}{R} \sqrt{\frac{\tau}{\pi \gamma n_0}} \right]
\]

This expression is suitable for numerical computation. In practice, we have summed over the \(L = 9000\) lower states in the case of a rectangular base, and \(L = 1241\) states for the circle; for such values of \(R\) the error on approximating the sum by an integral is negligible. In addition, for a given \(L\), the specific choice of \(R\) in the range \(\xi_L < R < \xi_{L+1}\) is also irrelevant (the associated variation in \(n_0\) is lower than \(0.4 \times 10^{-4}\)).

**Some Approximate Results**

Before showing and discussing the numerical results, it is useful to obtain some asymptotic relations for the case \(n_0 \gg 1/N\) and a not too large value for \(\gamma\), say, \(\sigma = \pi/[\tau \gamma^2 (1 + q)^2] \gg 1/[N n_0]\). Then, it follows from Eq. (4), \(\zeta = 1 + 1/(N n_0) \approx 1\). In addition, replacing Eq. (4) into Eq. (7) and taking the limit \(R \to \infty\) we obtain

\[
1 = n_0 + \sum_{\nu=1}^{L} \frac{\tau n_0}{\tau + \pi \gamma n_0 \xi_{\nu}^2} - \frac{\tau}{\gamma} \left[ \log \left( \frac{\pi \gamma R^2}{\tau N} \right) - \Phi(\sigma) \right]
\]

where
\[
\Phi(\sigma) = \frac{-2}{(q+1)} \sum_{\ell=1}^{\infty} \log \left\{ 1 - \exp \left[ -\sigma \ell (\ell + 2q) \right] \right\}
\]
(i) **Small Temperature Case**— We shall assume that condensation is well established, and that the inequality \( n_0 \gg \frac{\tau}{\gamma} \) holds. We expand in a Taylor series the second term of Eq. (8), concluding the following relation for \( n_0 \):

\[
0 \approx n_0^2 - n_0 + \frac{n_0 \tau}{\gamma} \left[ \Phi(\sigma) + \log(N_s \tau \alpha) \right] - \left( \frac{\tau}{2\gamma} \right)^2 \kappa
\]

where
\[
\kappa \equiv \frac{4}{\pi^2} \left[ \sum_{0 < \xi_\nu < R} \left( \frac{1}{\xi_\nu} \right)^4 + \frac{\pi}{R^2} - \frac{C_s}{3R^2} \right]_{R \to \infty}
\]

and
\[
\alpha \equiv \frac{1}{\pi} \exp \left[ \frac{1}{\pi} \sum_{0 < \xi_\nu < R} \left( \frac{1}{\xi_\nu} \right)^2 - 2 \log(R) - \frac{C_s}{\pi R} \right]_{R \to \infty}
\]

(9)

Solving for \( n_0 \) we obtain

\[
n_0(\tau) \approx \frac{1}{2} \left\{ m + \left[ m^2 + \kappa \left( \frac{\tau}{\gamma} \right)^2 \right]^{1/2} \right\}
\]

where
\[
m(\tau) \equiv 1 - \frac{\tau}{\gamma} \left[ \log(\alpha \tau N_s) + \Phi(\sigma) \right]
\]

(10)

An estimation of the accuracy of this approximation is

\[
\Delta n_0 = n_0 - \{ n_0 \}_{\text{exact}} \sim C \tau^3 / (\gamma^3 n_0^2), \quad \text{where} \quad C \sim 0.2 , \quad \text{at least for square and circle geometries. Thus, the error strongly decreases as } n_0, \ N_s \ \text{or} \ \gamma \ \text{increases. However, the error is very small for } n_0 = 0.2, \ \text{where we have} \ \Delta n_0 \leq 0.01 \ \text{if} \ N \geq 10^4, \ \gamma \geq 1. \ \text{When} \ N > 10^8 \ \text{the error} \ \Delta n_0 \leq 0.003, \ \text{even for a very small amount of condensate}, \ n_0 \geq 0.08. \ \text{The accuracy of Eq. (10) decreases if the shape of } S \ \text{is very "oblong". It is important to remark that the constants } \alpha \ \text{and} \ \kappa \ \text{only depend on the shape of surface} \ S; \ \text{we include a table with the values of these constants for a circle and a rectangle} \ b_1 \times b_2; \ \text{in the latter case we use different boundary conditions and values for the ratio} \ \rho = b_1 / b_2 \ (\text{e.g.} \ \rho = 1 \ \text{corresponds to a square}).

As a consequence of these geometric constants, we conclude that condensation is inhibited when the shape of \( S \) is very oblong; the latter one is consistent with the absence of condensation in a nearly one–dimensional container [1], in accordance with our exact calculations (see Fig. 1).

(ii) **Intermediate Temperature Case**— Using Eq. (8) in the case \( 1/N \ll n_0 \ll 1 \), when condensation is scarcely incipient, we have concluded the asymptotic relation
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\[ n_0 \sim \mathcal{O}\left\{ \gamma \tau_I \left( \frac{1}{N} \right)^{(\tau - \tau_I)/\tau} \exp\left[ -2\gamma(1 + q)\sqrt{\frac{\tau}{\pi}} \right] \right\} \]  

where \( \tau_I = \gamma / \log(N) \) roughly corresponds to the temperature threshold for condensation appearance. In this way, the typical temperature for condensation in a quasi–bidimensional system roughly corresponds to \( \tau_I \sim \tau_c / \log(N) \), where \( \tau_c \) is the tridimensional critical temperature for BEC; the latter one implies that, typically, the quasi–bidimensional condensation occurs at temperatures one order of magnitude below the tridimensional case.

**Results and Discussion**

We have solved Eq. (7), determining \( n_0(\tau) \) for several values of \( N, \gamma \) and different shapes of \( S \): a circle, rectangles with different flatness \( \rho \). We have always used IBC in the normal direction with respect to \( S \). We display our numerical results in figs. 1–4. In what follows we summarize the principal consequences of these results.

(i) The shape of the base \( S \) influences the condensation phenomenon, specially in the nearly bidimensional limit, \( \gamma \sim \mathcal{O}\{1\} \). This effect is also enhanced for relatively small values of \( N \). In Fig. 1 we illustrate this effect for \( \gamma = 1 \) and \( N = 10000 \). We see that a IBC–square base yields the larger values of \( n_0(\tau) \) (at least for not too small values of \( n_0 \)); however, the circle gives only slightly lower values for \( n_0 \) in comparison to the IBC–square, while the CBC–square curve \( n_0(\tau) \) lies below the two former ones. Finally the lower values of \( n_0(\tau) \) in Fig. 1 correspond to IBC–rectangles; these curves descend on increasing the flatness \( \rho \), as the system approaches to a one–dimensional behaviour. The latter one is in accordance with the absence of BEC in that limit \([1]\).

This “shape effect” on \( n_0(\tau) \) is important in the 2–d limit, diminishing upon increasing \( \gamma \) or \( N \). Therefore, it can only be observed in truly nanoscopic systems. In the following item we shall explain this behaviour.

(ii) According to Eq. (5), the low–lying excited states also have a macroscopic occupancy once condensation begins. In Fig. 2 we display the occupancy of the lower seven excited levels in the case of circular base, \( N = 10000 \) and \( \gamma = 1 \). The associated quantum numbers are \( \nu = (m, j) = (0,1), (1,1), (2,1), (3,1), (1,2), (0,2) \) and \( (2,2) \). Here \( m \) is associated to the axial component of angular momentum \( L_z \), while \( j \) enumerates the eigenenergies for a given \( m \). Obviously, while \( m = 0 \) is not degenerate, \( m \neq 0 \) has
The effect of degeneracy becomes evident in the asymptotic behaviour \( \tau \to \infty \), where the occupancy of non-degenerate states is one-half of the degenerate ones. The maximal occupancy of an excited level corresponds to \((1,1)\) (say, the first excited state), giving \( n_{(1,1)}(\tau_{\text{Max}} = 0.13) = 0.093 \).

In the general case, the macroscopic occupancy of excited levels is enhanced as \( \gamma \) or \( N \) decreases. An approximate relation for the maximal occupation of these excited levels is given by

\[
n_{\nu}(\tau_{\text{Max}}) \approx \frac{D_\nu}{\theta \xi_\nu} \quad \text{with} \quad \theta = \pi \log \left[ \frac{N\alpha}{\log(N_s\alpha)} \right]
\]

and \( \tau_{\text{Max}} \approx \pi\gamma/\theta \); here \( D_\nu \) is the degeneracy. Thus, this maximal occupancy depends on \( N \) like \( 1/\log(N) \) (thus increasing as \( N \) decreases, being appreciable in truly nanoscopic systems), and it is inversely proportional to the bidimensional excitation energy \( E(\nu,0) - E(0,0) \) (see Eq. (12)). We remark that Eq. (12) is valid under the condition \( \gamma \sim 1 \); for larger values of \( \gamma \) the occupancy of the excited levels strongly decreases (for example, using the same \( N_s \) of Fig. 2, but \( \gamma = 10 \), we obtain a maximal occupancy \( n_{(1,1)}(\tau_{\text{Max}} = 0.47) = 0.037 \).)

According to relation (12), the macroscopic occupancy of the lower excited levels is strongly affected by the particular sequence \( \{\xi_\nu\} \), which, in turns, depends of the shape of \( S \). The latter one implies that the occupancy of the ground state is also affected by the base shape. In particular, the amount of condensate \( n_0 \) increases if the gap between the ground state and the first excited levels increases, i.e., when the level density decreases. For a fixed base area but different shapes, we can modify the former magnitudes and therefore modify \( n_0 \).

In order to obtain a further feeling for the effect of the shape in the amount of condensate \( n_0(\tau) \), it is very illustrative to note the following numerical result: The sum of occupancies of the ground state and the first excited level is roughly the same for the circle, the CBC–square and the IBC–square; the latter one, near the threshold temperature of condensation. Therefore, the shape effect on \( n_0 \) roughly corresponds to the difference in the populations of the first excited levels. Although this estimation is very crude, and certainly not applicable to arbitrary shapes (specially if one of them yield a quasi-degenerate first excited level), it gives a first estimate for the shape effect on the thermodynamics of the condensate.

The effect of shape in condensation is also evident from relations (9) and (10), since the geometrical constant \( \alpha \) increases as the density of levels in the neighbourhood of ground state decreases (see Eq. (10)). An increase
in \( \alpha \) implies a decrease of \( n_0(\tau) \), in accordance with Eq. (10).

The results of Fig. 1 is consistent with this explanation. In addition, since the occupancy of the low lying excited levels is only important for small values of \( \gamma \) or \( N \), the effect of shape in the ground state occupanncy \( n_0(\tau) \) is appreciable under the same conditions. Therefore, this “shape effect” can only be observed in truly nanoscopic systems.

The latter observation is consistent with the relation between the density of states \( D(E) \) an the system dimensionality. For example, in a fully tridimensional system, \( D(E) \sim \sqrt{E} \), vanishing as \( E \to 0 \); this low density is responsible of a genuine BEC at a relatively hight temperature. In the bidimensional case, \( D(E) \sim \{\text{Constant}\} \); while the onedimensional case corresponds to \( D(E) \sim 1/\sqrt{E} \), diverging as \( E \to 0 \). Accordingly, the intermediate density of states of the bidimensional case is consistent with condensation phenomenon at temperatures one order of magnitude below the tridimensional case. Finally, in the one–dimensional case, the high density of states near the ground state is in accordance with the absence of condensation in that case.

(iii) According to relation (10), the condensation phenomenon is highly non–intensive in the case of a small \( \gamma \), due to the dependence of \( n_0(\tau) \) on \( \log(N) \). Moreover, using that relation we can estimate an “effective critical temperature” \( \tau_{\text{eff}} \approx \gamma/\log[\alpha N/\log(N)] \sim \gamma/\log(N) \); this estimation is consistent with \( \tau_I \) of Eq. (11) and \( \tau_{\text{Max}} \) of Eq. (12).

In figs. 3 and 4 we plot \( n_0(\tau) \) for \( N = 10^4 \) and \( 10^8 \) for the case of a circle; in each figure we take different values of \( \gamma \). In general terms, these figures confirm that condensation is inhibited on decreasing \( \gamma \) or increasing \( N \). Inspection of these figures confirm the non–intensive character of \( n_0(\tau) \). For example, the curves \( \gamma = 1 \) of Fig. 3 and \( \gamma = 2 \) of Fig. 4 nearly coincide, in agreement with the fact that \( \tau_{\text{eff}} \sim \gamma/\log(N) \) has the same value in both cases.

The condensation phenomenon has a nearly 2–d character for small values of \( \gamma \), as long as the excitations in the (“thin”) \( z \) direction are very improbable at temperatures compatible with condensation of a macroscopic number of particles. An upper bound for this 2–d character can be roughly estimated as \( \gamma \leq 2 \) for a nanoscopic system \( (N_s \sim 10^4 \leftrightarrow 10^8) \). The tridimensional limit is slowly attained as \( \gamma \) increases; roughly, for \( \gamma \sim 10 \) if \( N = 10^4 \), and \( \gamma \sim 50 \) if \( N = 10^8 \). In this way, on increasing \( \gamma \), the tridimensional limit is attained faster for a small number of particles; this result seems natural, since for a fixed value of \( \gamma \) the base width \( (\sim \sqrt{S}) \) and container height \( (a) \) may be comparable if \( N \) is not too large, but \( (\sqrt{S} \gg a) \) for a greater value of \( N \).

Another important results illustrated in Fig. 3 are: (a) the finite size
effects, which yield a larger amount of condensate for $\gamma = 20$ in comparison to the tridimensional case. (b) The condensation phenomenon begins gradually for low values of $N$ and $\gamma$, in close accordance with the qualitative result (11); while larger values of $N$ and $\gamma$ yield a more sharp phase transition.

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Figure Captions

**Figure 1:** Number of particles in the ground state as a function of the reduced temperature for several different shapes of the container base. In all cases, we have IBC, except for the lower square, where CBC are used. ($\gamma = 1, N = 10^4$).

**Figure 2:** Occupancy of the lower seven excited states for the case of a circular base. Each state is labelled by $(m, j)$ where $m$ is associated to $L_z$ while $j$ enumerates the eigenenergies corresponding to a given $m$. ($\gamma = 1, N = 10^4$).

**Figure 3:** Circular base: Number of particles in the ground state as a function of the reduced temperature, for different values of $\gamma$. ($N = 10^4$).

**Figure 4:** Same as in fig.3, but for $N = 10^8$. 
| Surface | $\rho$ (Rectangle) | Boundary Conditions | $\alpha$      | $\kappa$      |
|---------|--------------------|---------------------|--------------|--------------|
| Circle  | –                  | I.B.C.              | 0.4019...    | 2.4967...    |
| Square  | 1                  | I.B.C.              | 0.35106...   | 2.280612...  |
| Rectangle | 1.5              | I.B.C.              | 0.391197...  | 2.854037...  |
| Rectangle | 2                | I.B.C.              | 0.489620...  | 4.159725...  |
| Rectangle | 3                | I.B.C.              | 0.888309...  | 8.319915...  |
| Rectangle | 5                | I.B.C.              | 3.76785...   | 22.1509...   |
| Rectangle | 10               | I.B.C.              | 232.3111...  | 87.65845...  |
| Square  | 1                  | C.B.C.              | 0.7247...    | 2.442575...  |
| Rectangle | 1.5              | C.B.C.              | 0.80982...   | 2.995687...  |
| Rectangle | 2                | C.B.C.              | 1.02498...   | 4.274506...  |
| Rectangle | 5                | C.B.C.              | 9.48733...   | 22.23856...  |
| Rectangle | 10               | C.B.C.              | 891.3993...  | 87.8829...   |
