EXOTIC CLUSTER STRUCTURES ON $SL_n$ WITH
BELAVIN–DRINFELD DATA OF MINIMAL SIZE, I. THE
STRUCTURE

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Abstract. Using the notion of compatibility between Poisson brackets and cluster structures in the coordinate rings of simple Lie groups, Gekhtman Shapiro and Vainshtein conjectured a correspondence between the two. Poisson Lie groups are classified by the Belavin–Drinfeld classification of solutions to the classical Yang Baxter equation. For any non trivial Belavin–Drinfeld data of minimal size for $SL_n$, we give an algorithm for constructing an initial seed $\Sigma$ in $O(SL_n)$. The cluster structure $C = C(\Sigma)$ is then proved to be compatible with the Poisson bracket associated with that Belavin–Drinfeld data, and regular.

This is the first of two papers, and the second one proves the rest of the conjecture: the upper cluster algebra $A_C(C)$ is naturally isomorphic to $O(SL_n)$, and the correspondence of Belavin–Drinfeld classes and cluster structures is one to one.

1. Introduction

Since cluster algebras were introduced in [7], a natural question was the existence of cluster structures in the coordinate rings of a given algebraic variety $V$. Partial answers were given for Grassmannians $V = Gr_k(n)$ [11] and double Bruhat cells [2]. If $V = G$ is a simple Lie group, one can extend the cluster structure found in the double Bruhat cell to one in $O(G)$. The compatibility of cluster structures and Poisson brackets, as characterized in [9] suggested a connection between the two: given a Poisson bracket, does a compatible cluster structure exist? Is there a way to find it?

In the case that $V = G$ is a simple complex Lie group, R-matrix Poisson brackets on $G$ are classified by the Belavin–Drinfeld classification of solutions to the classical Yang Baxter equation [4]. Given a solution of that kind, a Poisson bracket can be defined on $G$, making it a Poisson–Lie group.

The Belavin–Drinfeld (BD) classification is based on pairs of isometric subsets of simple roots of the Lie algebra $\mathfrak{g}$ of $G$. The trivial case when the subsets are empty corresponds to the standard Poisson bracket on $G$. It has been shown in [11] that the extending the cluster structure introduced in [2] from the double Bruhat cell to the whole Lie group $V$ yields a cluster structure that is compatible with the standard Poisson bracket. This led to naming this cluster structure “standard”, and trying to find other cluster structures, compatible with brackets associated with non trivial BD subsets. The term “exotic” was suggested for these non standard structures [12].

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Gekhtman, Shapiro and Vainshtein conjectured a bijection between BD classes and cluster structures on simple Lie groups [11, 12]. According the conjecture, for a given BD class for \( G \), there exists a cluster structure on \( G \), with rank determined by the BD data. This cluster structure is compatible with the associated Poisson bracket. The conjecture also states that the structure is regular, and that the upper cluster algebra coincides with the ring of regular functions on \( G \). The conjecture was proved for the standard case and for \( G = SL_n \) with \( n < 5 \) in [11]. The Cremmer – Gervais case, which in some sense is the “furthest” from the standard one, was proved in [12]. It was also found to be true for all possible BD classes for \( SL_5 \) [5].

This paper proves parts of the conjecture for \( SL_n \) when the BD data is of minimal size, i.e., the two subsets contain only one simple root. Starting with two such subsets \( \{\alpha\} \) and \( \{\beta\} \), Section 3.1 describes an algorithm for construction of a set \( B_{\alpha\beta} \) of functions that will serve as the initial cluster. It is then proved that this set is log canonical with respect to the associated Poisson bracket \( \{\cdot,\cdot\}_{\alpha\beta} \). Adding a quiver \( Q_{\alpha\beta} \) (or an exchange matrix \( B_{\alpha\beta} \)) defines a cluster structure on \( SL_n \). It is shown in Section 4 that this structure is indeed compatible with the Poisson bracket. Then Section 5 proves that this cluster structure is regular.

This proves that for minimal size BD data for \( SL_n \) there exists a regular cluster structure, which is compatible with the associated Poisson bracket. The companion paper [6] will complete the proof of the conjecture: the bijection between cluster structures and BD classes of this type, the fact that the upper cluster algebra is naturally isomorphic to the ring of regular functions on \( SL_n \), and the description of a global toric action.

2. Background and main results

2.1. Cluster structures. Let \( \{z_1, \ldots, z_m\} \) be a set of independent variables, and let \( S \) denote the ring of Laurent polynomials generated by \( z_1, \ldots, z_m \):

\[
S = \mathbb{Z}\left[z_1^{\pm 1}, \ldots, z_m^{\pm 1}\right].
\]

(Here and in what follows \( z^{\pm 1} \) stands for \( z, z^{-1} \).) The ambient field \( \mathcal{F} \) is the field of rational functions in \( n \) independent variables (distinct from \( z_1, \ldots, z_m \)), with coefficients in the field of fractions of \( S \).

A seed (of geometric type) is a pair \((\mathbf{x}, B)\), where \( \mathbf{x} = (x_1, \ldots, x_n) \) is a transcendence basis of \( \mathcal{F} \) over the field of fractions of \( S \), and \( B \) is an \( n \times (n+m) \) integer matrix whose principal part \( B \) (that is, the \( n \times n \) matrix formed by columns \( 1 \ldots n \)) is skew-symmetric. The set \( \mathbf{x} \) is called a cluster, and its elements \((x_1, \ldots, x_n)\) are called cluster variables. Set \( x_{n+i} = z_i \) for \( i \in [1, m] \). The elements \( x_{n+1}, \ldots, x_{n+m} \) are called stable variables (or frozen variables). The set \( \tilde{\mathbf{x}} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \) is called an extended cluster. The square matrix \( B \) is called the exchange matrix, and \( \tilde{B} \) is called the extended exchange matrix. We sometimes denote the entries of \( \tilde{B} \) by \( b_{ij} \), or say that \( \tilde{B} \) is skew-symmetric when the matrix \( B \) has this property.

Let \( \Sigma = (\tilde{\mathbf{x}}, \tilde{B}) \) be a seed. The adjacent cluster in direction \( k \in [n] \) is \( \tilde{x}_k = (\tilde{\mathbf{x}} \setminus x_k) \cup \{x'_k\} \), where \( x'_k \) is defined by the exchange relation

\[
x_k \cdot x'_k = \prod_{b_{ij} > 0} x_j^{b_{ij}} + \prod_{b_{ij} < 0} x_j^{-b_{ij}}
\]  

(2.1)
A matrix mutation $\mu_k(\tilde{B})$ of $\tilde{B}$ in direction $k$ is defined by

$$b_{ij}' = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
\frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise.}
\end{cases}$$

Seed mutation in direction $k$ is then defined $\mu_k(\Sigma) = (\tilde{x}_k, \mu_k(\tilde{B}))$.

Two seeds are said to be mutation equivalent if they can be connected by a sequence of seed mutations.

Given a seed $\Sigma = (x, \tilde{B})$, the cluster structure $\mathcal{C}(\Sigma)$ (sometimes denoted $\mathcal{C}(\tilde{B})$, if $x$ is understood from the context) is the set of all seeds that are mutation equivalent to $\Sigma$. The number $n$ of rows in the matrix $\tilde{B}$ is called the rank of $\mathcal{C}$.

Let $\Sigma$ be a seed as above, and $\mathcal{A} = \mathbb{Z}[x_{n+1}, \ldots, x_{n+m}]$. The cluster algebra $A = A(\mathcal{C}) = A(\tilde{B})$ associated with the seed $\Sigma$ is the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds in $\mathcal{C}(\tilde{B})$. The upper cluster algebra $\mathfrak{A} = \mathfrak{A}(\mathcal{C}) = \mathfrak{A}(\tilde{B})$ is the intersection of the rings of Laurent polynomials over $\mathbb{A}$ in cluster variables taken over all seeds in $\mathcal{C}(\tilde{B})$. The famous Laurent phenomenon claims the inclusion $A(\mathcal{C}) \subseteq \mathfrak{A}(\mathcal{C})$.

It is sometimes convenient to describe a cluster structure $\mathcal{C}(\tilde{B})$ in terms of the exchange graph or exchange quiver $Q(\tilde{B})$; it is a directed graph with $n + m$ nodes labeled $x_1, \ldots, x_{n+m}$ (or just $1, \ldots, n + m$), and arrow pointing from $x_i$ to $x_j$ with weight $b_{ij}$ if $b_{ij} > 0$.

Let $V$ be a quasi-affine variety over $\mathbb{C}$, $\mathbb{C}(V)$ be the field of rational functions on $V$, and $\mathcal{O}(V)$ be the ring of regular functions on $V$. Let $C$ be a cluster structure in $\mathcal{F}$ as above, and assume that $\{f_1, \ldots, f_{n+m}\}$ is a transcendence basis of $\mathbb{C}(V)$. Then the map $\phi : x_i \rightarrow f_i$, $1 \leq i \leq n + m$, can be extended to a field isomorphism $\phi : \mathcal{F}_C \rightarrow \mathbb{C}(V)$ with $\mathcal{F}_C = \mathcal{F} \otimes \mathcal{O}$ obtained from $\mathcal{F}$ by extension of scalars. The pair $(\mathcal{C}, \phi)$ is then called a cluster structure in $\mathbb{C}(V)$ (or just a cluster structure on $V$), and the set $\{f_1, \ldots, f_{n+m}\}$ is called an extended cluster in $(\mathcal{C}, \phi)$. Sometimes we omit direct indication of $\phi$ and just say that $\mathcal{C}$ is a cluster structure on $V$.

A cluster structure $(\mathcal{C}, \phi)$ is called regular if $\phi(x)$ is a regular function for any cluster variable $x$. The two algebras defined above have their counterparts in $\mathcal{F}_C$ obtained by extension of scalars; they are denoted $A_C$ and $\mathfrak{A}_C$. If, moreover, the field isomorphism $\phi$ can be restricted to an isomorphism of $A_C$ (or $\mathfrak{A}_C$) and $\mathcal{O}(V)$, we say that $A_C$ (or $\mathfrak{A}_C$) is naturally isomorphic to $\mathcal{O}(V)$.

Let $\{., .\}$ be a Poisson bracket on the ambient field $\mathcal{F}$. Two elements $f_1, f_2 \in \mathcal{F}$ are log canonical if there exists a rational number $\omega_{f_1, f_2}$ such that $\{f_1, f_2\} = \omega_{f_1, f_2} f_1 f_2$. A set $F \subseteq \mathcal{F}$ is called a log canonical set if every pair $f_1, f_2 \in F$ is log canonical.

A cluster structure $\mathcal{C}$ in $\mathcal{F}$ is said to be compatible with the Poisson bracket $\{., .\}$ if every cluster is a log canonical set with respect to $\{., .\}$. In other words, for every cluster $x$ and every two cluster variables $x_i, x_j \in \tilde{x}$ there exists $\omega_{ij}$ s.t.

$$\{x_i, x_j\} = \omega_{ij} x_i x_j \quad (2.2)$$

The skew symmetric matrix $\Omega^x = (\omega_{ij})$ is called the coefficient matrix of $\{., .\}$ (in the basis $\tilde{x}$).
If $C(\tilde{B})$ is a cluster structure of maximal rank (i.e., $\text{rank} \tilde{B} = n$), one can give a complete characterization of all Poisson brackets compatible with $C(\tilde{B})$ (see [9], and also [10], Ch. 4). In particular, an immediate corollary of Theorem 1.4 in [9] is the following statement:

**Proposition 1.** If $\text{rank} \tilde{B} = n$ then a Poisson bracket is compatible with $C(\tilde{B})$ if and only if its coefficient matrix $\Omega^\mathbb{R}$ satisfies $\tilde{B}\Omega^\mathbb{R} |D0\rangle$, where $D$ is a diagonal matrix.

### 2.2. Poisson–Lie groups.

A Lie group $G$ with a Poisson bracket $\{\cdot, \cdot\}$ is called a **Poisson–Lie group** if the multiplication map $\mu : G \times G \to G$, $\mu : (x, y) \mapsto xy$ is Poisson. That is, $G$ with a Poisson bracket $\{\cdot, \cdot\}$ is a Poisson–Lie group if

$$\{f_1, f_2\}(xy) = \{\rho_y f_1, \rho_y f_2\}(x) + \{\lambda_x f_1, \lambda_x f_2\}(y),$$

where $\rho_y$ and $\lambda_x$ are, respectively, right and left translation operators on $G$.

Given a Lie group $G$ with a Lie algebra $\mathfrak{g}$, let $(\cdot, \cdot)$ be a nondegenerate bilinear form on $\mathfrak{g}$, and $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the corresponding Casimir element. For an element $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ denote

$$[r, r] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]$$

and $r^{21} = \sum_i b_i \otimes a_i$.

The **Classical Yang-Baxter equation (CYBE)** is

$$[r, r] = 0,$$

an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ that satisfies (2.3) together with the condition

$$r + r^{21} = \mathfrak{g}$$

is called a classical R-matrix.

A classical R-matrix $r$ induces a Poisson-Lie structure on $G$: choose a basis $\{I_\alpha\}$ in $\mathfrak{g}$, and denote by $\partial_\alpha$ (resp., $\partial'_\alpha$) the left (resp., right) invariant vector field whose value at the unit element is $I_\alpha$. Let

$$r = \sum_{\alpha, \beta} I_\alpha \otimes I_\beta,$$

then

$$\{f_1, f_2\}_r = \sum_{\alpha, \beta} r_{\alpha, \beta} (\partial_\alpha f_1 \partial_\beta f_2 - \partial'_\alpha f_1 \partial'_\beta f_2)$$

defines a Poisson bracket on $G$. This is called the **Sklyanin bracket** corresponding to $r$.

In [11] Belavin and Drinfeld give a classification of classical R-matrices for simple complex Lie groups: let $\mathfrak{g}$ be a simple complex Lie algebra with a fixed nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$. Fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Phi$ of $\mathfrak{g}$, and a set of positive roots $\Phi^+$. Let $\Delta \subseteq \Phi^+$ be a set of positive simple roots.

A Belavin–Drinfeld (BD) triple is two subsets $\Gamma_1, \Gamma_2 \subset \Delta$ and an isometry $\gamma : \Gamma_1 \to \Gamma_2$ with the following property: for every $\alpha \in \Gamma_1$ there exists $m \in \mathbb{N}$ such that $\gamma^j(\alpha) \in \Gamma_1$ for $j = 0, \ldots, m - 1$, but $\gamma^m(\alpha) \notin \Gamma_1$. The isometry $\gamma$ extends in a natural way to a map between root systems generated by $\Gamma_1, \Gamma_2$. This allows one to define a partial ordering on the root system: $\alpha \prec \beta$ if $\beta = \gamma^j(\alpha)$ for some $j \in \mathbb{N}$.

Select now root vectors $E_\alpha \in \mathfrak{g}$ that satisfy $(E_\alpha, E_{-\alpha}) = 1$. According to the Belavin–Drinfeld classification, the following is true (see, e.g., [11] Chap. 3).
Proposition 2. (i) Every classical \( R \)-matrix is equivalent (up to an action of \( \sigma \otimes \sigma \) where \( \sigma \) is an automorphism of \( g \)) to
\[ r = r_0 + \sum_{\alpha \in \Phi^+} E_{-\alpha} \otimes E_{\alpha} + \sum_{\alpha, \beta \in \Phi^+} E_{-\alpha} \wedge E_{\beta} \]  
(ii) \( r_0 \in h \otimes h \) in (2.6) satisfies
\[ (\gamma (\alpha) \otimes 1) r_0 + (1 \otimes \alpha) r_0 = 0 \]
for any \( \alpha \in \Gamma_1 \), and
\[ r_0 + r_0^{21} = t_0, \]
where \( t_0 \) is the \( h \otimes h \) component of \( t \).
(iii) Solutions \( r_0 \) to (2.7), (2.8) form a linear space of dimension \( k_T = |\Delta \setminus \Gamma_1| \).

Two classical \( R \)-matrices of the form (2.6) that are associated with the same BD triple are said to belong to the same Belavin–Drinfeld class. The corresponding bracket defined in (2.5) by an \( R \)-matrix \( r \) associated with a triple \( T \) will be denoted by \( \{ , \} _T \).

Given a BD triple \( T \) for \( G \), write
\[ h_T = \{ h \in h : \alpha(h) = \beta(h) \text{ if } \alpha \prec \beta \}, \]
and define the torus \( H_T = \exp h_T \subset G \).

2.3. Main results and outline. The following conjecture was given by Gekhtman, Shapiro and Vainshtein in [11]:

Conjecture 3. Let \( G \) be a simple complex Lie group. For any Belavin–Drinfeld triple \( T = (\Gamma_1, \Gamma_2, \gamma) \) there exists a cluster structure \( C_T \) on \( G \) such that

1. the number of stable variables is \( 2k_T \), and the corresponding extended exchange matrix has a full rank.
2. for any solution of CYBE that belongs to the Belavin–Drinfeld class specified by \( T \), the corresponding Sklyanin bracket is compatible with \( C_T \);
3. \( C_T \) is regular.
4. The corresponding upper cluster algebra \( \mathcal{A}_C(C_T) \) is naturally isomorphic to \( \mathcal{O}(G) \);
5. the global toric action of \( (\mathbb{C}^*)^{2k_T} \) on \( \mathcal{O}(G) \) is generated by the action of \( \mathcal{H}_T \otimes \mathcal{H}_T \) on \( G \) given by \( (H_1, H_2)(X) = H_1 X H_2 \);
6. a Poisson–Lie bracket on \( G \) is compatible with \( C_T \) only if it is a scalar multiple of the Sklyanin bracket associated with a solution of CYBE that belongs to the Belavin–Drinfeld class specified by \( T \).

The main result of this paper is the following theorem:

Theorem 4. For any Belavin–Drinfeld triple of the form \( T = (\{ \alpha \}, \{ \beta \}, \gamma : \alpha \rightarrow \beta) \), there exists a regular cluster structure on \( SL_n \) with \( 2k_T \) stable variables, that is compatible with the Sklyanin bracket associated with \( T \).

In other words, Theorem 4 states that parts [11] and [12] are true for \( SL_n \) for BD triple with \( |\Gamma_1| = 1 \).

For a given \( n \) and a BD triple \( T_{\alpha\beta} \), a set \( E_{\alpha\beta} \) of functions in \( \mathcal{O}(SL_n) \) is constructed in Section 3.1 The rest of Section [3] is dedicated to proving that this set is
log canonical with respect to the Sklyanin bracket \( \{ \cdot , \cdot \}_{\alpha \beta} \) associated with \( T_{\alpha \beta} \). After declaring some of these functions as frozen variables and introducing the quiver \( Q_{\alpha \beta} \) in Section 4.2 the initial seed \( (B_{\alpha \beta}, Q_{\alpha \beta}) \) determines a cluster structure \( C_{\alpha \beta} \). Theorem 10 states that \( C_{\alpha \beta} \) is compatible with the bracket \( \{ \cdot , \cdot \}_{\alpha \beta} \). Last, Section 5 proves that \( C_{\alpha \beta} \) is regular.

Parts 1, 2, and 4 of the conjecture will be proved in the companion paper [6].

3. A LOG CANONICAL BASIS

This section describes a log canonical set of function, that will serve as an initial cluster for the structure \( C_{\alpha \beta} \). After constructing this set in Section 3.1 the details of computing the bracket \( \{ \cdot , \cdot \}_{\alpha \beta} \) are given in Section 3.2. After some preparations in Section 3.3 it is then proved in Section 3.4 that this set is log canonical with respect to \( \{ \cdot , \cdot \}_{\alpha \beta} \).

Before moving on, note the following two isomorphisms of the BD data for \( SL_n \): the first reverses the direction of \( \gamma \) and transposes \( \Gamma_1 \) and \( \Gamma_2 \), while the second one takes each root \( \alpha_j \) to \( \alpha_{\omega_j(j)} \), where \( \omega_0 \) is the longest element in the Weyl group (which in \( SL_n \) is naturally identified with the symmetric group \( S_{n-1} \)). These two isomorphisms correspond to the automorphisms of \( SL_n \) given by \( X \mapsto -X' \) and \( X \mapsto \omega_0 X \omega_0 \), respectively. Since R-matrices are considered up to an action of \( \sigma \circ \sigma \), from here on we do not distinguish between BD triples obtained one from the other via these isomorphisms. We will also assume that \( \alpha < \beta \).

Slightly abusing the notation, we sometime refer to a root \( \alpha_i \in \Delta \) just as \( i \), and write \( \gamma : i \mapsto j \) instead of \( \gamma : \alpha_i \mapsto \alpha_j \).

3.1. Constructing a log canonical basis. For shorter notation, denote the BD triple \( \{ \alpha \}, \{ \beta \}, \gamma : \alpha \mapsto \beta \) by \( T_{\alpha \beta} \), and the corresponding Sklyanin bracket by \( \{ \cdot , \cdot \}_{\alpha \beta} \). For such a triple, we will construct a set of matrices \( M \) such that the set of all their trailing principal minors is log canonical with respect to \( \{ \cdot , \cdot \}_{\alpha \beta} \).

Following [13], recall the construction of the Drinfeld double of a Lie algebra \( g \) with the Killing form \( \langle \cdot , \cdot \rangle \) : define \( D(g) = g \oplus g \), with an invariant nondegenerate bilinear form
\[
\langle (\xi, \eta), (\xi', \eta') \rangle = \langle \xi, \xi' \rangle - \langle \eta, \eta' \rangle.
\]
Define subalgebras \( \mathfrak{d}_\pm \) of \( D(g) \) by
\[
\mathfrak{d}_+ = \{ (\xi, \xi) : \xi \in g \} , \quad \mathfrak{d}_- = \{ (R_+ (\xi), R_- (\xi)) : \xi \in g \} ,
\]
where \( R_\pm \in \text{End} \ g \) are defined for any R-matrix \( r \) by
\[
\langle R_+ (\eta), \zeta \rangle = - \langle R_- (\zeta), \eta \rangle = \langle r, \eta \otimes \zeta \rangle \otimes ,
\]
and \( \langle \cdot , \cdot \rangle \otimes \) is the corresponding Killing form on the tensor square of \( g \).

Start with an element \( (X,Y) \) in the double \( D(sl_n) \). Our building blocks will be submatrices of \( X \) and \( Y \).

Let \( X^C_R \) denote the determinant of submatrix of \( X \) with rows in the set \( R \) and columns in \( C \) (with \( R, C \subseteq \{ 1, \ldots , n \} \) and \( |R| = |C| \)).

Let \( \mu (i, j) = \min (n, n + 1 - i - j) \). So \( X^{[i \ldots n], [i \ldots n]} \) is the maximal submatrix of \( X \) with \( x_{ij} \) at the upper left corner. Take all such submatrices \( X^{[i \ldots n], [i \ldots n]} \) with \( i - 1 \neq \alpha \) and all \( Y^{[j \ldots n], [j \ldots n]} \) with \( j - 1 \neq \beta \). A submatrix of the first type has \( x_{\alpha \beta} \) at the lower right hand corner, while the second type has \( y_{\mu \eta} \). If this is a first type matrix with
\( \mu \neq \alpha \), put that matrix in the set \( \mathcal{M} \). The same goes for a second type matrix with \( \mu \neq \beta \). If not, there are two cases:

**case 1:** \( M \) is a first type matrix, with \( x_{n\alpha} \) at the lower right hand corner.

Then add the column \( \alpha + 1 \) to the right of \( M \). Add the submatrix \( M' = Y_{[\beta \ldots n]}^{[\beta \ldots \mu(\beta,1)]} \) under \( M \) so that \( y_{1\beta} \) is right under \( x_{n\alpha} \). The result is a matrix of the form

\[
\begin{bmatrix}
x_{i1} & \cdots & x_{i\alpha} & x_{i,\alpha+1} & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
x_{n1} & \cdots & x_{n\alpha} & x_{n,\alpha+1} & 0 & \cdots \\
0 & \cdots & y_{1\beta} & y_{1\beta+1} & \cdots & y_{1n} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & y_{\mu\beta} & \cdots & \cdots & y_{\mu n}
\end{bmatrix}
\]

This matrix is added to the set \( \mathcal{M} \).

**case 2:** Now \( M \) is a second type matrix, with \( y_{\beta n} \) at the lower right corner.

Add the row \( \beta + 1 \) under it, and then the submatrix \( M' = X_{[\alpha \ldots n]}^{[\alpha \ldots \mu(\alpha,1)]} \) to the right, such that \( x_{\alpha 1} \) is right next to \( y_{\beta n} \). The result is

\[
\begin{bmatrix}
y_{1,n+1-\beta} & \cdots & y_{1n} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
y_{\beta j} & \cdots & y_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\
y_{\beta+1,j} & \cdots & y_{\beta+1,n} & x_{\alpha+1,1} & \cdots & \vdots \\
0 & \cdots & 0 & \vdots & \ddots & \vdots \\
\vdots & \ddots & x_{n1} & \cdots & x_{n\mu}
\end{bmatrix}
\]

This matrix is also added to \( \mathcal{M} \).

When \( n \) is even there are two special cases - \( \alpha = \frac{n}{2} \) or \( \beta = \frac{n}{2} \). We discuss here the case \( \beta = \frac{n}{2} \), as the case of \( \alpha = \frac{n}{2} \) is symmetric (and isomorphic under \( \alpha \leftrightarrow \beta \)):

Instead of two matrices involving submatrices of \( X \) and of \( Y \), we now have only one such matrix, that can be viewed as a combination of two. It has the form

\[
M_1 = \begin{bmatrix}
x_{n+1-\alpha,1} & \cdots & x_{i,\alpha+1} & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
x_{n1} & \cdots & x_{n\alpha} & x_{n,\alpha+1} & 0 & \cdots & \vdots \\
0 & \cdots & y_{1\beta} & y_{1\beta+1} & \cdots & y_{1n} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & y_{\beta j} & \cdots & y_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\
y_{\beta+1} & y_{\beta+1,1} & \cdots & y_{\beta+1,n} & x_{\alpha+1,1} & \cdots & \vdots \\
0 & \cdots & 0 & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & x_{n1} & \cdots & x_{n\mu}
\end{bmatrix}
\]

Though it seems different, all of the following discussion is valid for this special case as well. Note that the function \( \det M_1 \) can breaks into pieces like in Section 3.4 and all the arguments based on analyzing these pieces still hold.
Remark 5. Further details about the construction of the set $\mathcal{M}$ can be found in [5]. The special case of $n = 5$ is addressed there, with any BD data, but it can be easily generalized to any $n$ with the restriction $|\Gamma_1| = |\Gamma_2| = 1$.

We now take the set of all trailing principal minors of every matrix $M \in \mathcal{M}$. These are all functions on the double $D(\mathfrak{sl}_n)$. The projections of these functions on the diagonal subgroup $(X, X) \subset D(\mathfrak{sl}_n)$ are functions on $\mathfrak{sl}_n$. Note that after this projection $X$ and $Y$, which are both elements in $\mathcal{M}$, coincide and give the same minors.

Define a map $\rho$ between these functions and $[n] \times [n]$ : it maps each minor to the pair of indices of the element in the upper left hand corner of the corresponding submatrix.

Lemma 6. The map $\rho$ is bijective.

Proof. Every pair $(i, j)$ occurs exactly once as a pair of indices on the main diagonal of a submatrix that is a “building block” of the set $\mathcal{M}$. If $i > j$ it is on the diagonal of a submatrix of $X$. If $i < j$ it is on the diagonal of a submatrix of $Y$, and when $i = j$ it is on both diagonals of $X$ and of $Y$, but it only contributes one minor after projection on $(X, X)$. Each building block is used exactly once: it could be used as a starting block, if it is $X_{1...\mu(1,i)}^{i...n}$ with $i - 1 \neq \alpha$ or $Y_{[1...\mu(j,1)]}^{[j...n]}$ with $j - 1 \neq \beta$. The other option is that it is added as a direct summand, if it is either $X_{[\alpha+1...n]}^{[1...\mu(1,\alpha+1)]}$ or $Y_{[\beta+1...n]}^{[1...\mu(\beta+1,1)]}$.

So the the notion of $F_{ij}(X, Y)$ for the function determined by the minor with either $x_{ij}$ or $y_{ij}$ at the upper left corner is well defined. The set of all functions $\varphi_{ij} = F_{ij}(X, X)$ will be denoted by $B_{\alpha\beta}$.

Theorem 7. The set $B_{\alpha\beta}$ is log canonical with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$.

The theorem will be proved in Section 3.4. The set $B_{\alpha\beta}$ will be the initial cluster for the cluster structure $C_{\alpha\beta}$.

3.2. The operator $R_+$. Following Lemma 4.1 in [12], we compute the Sklyanin bracket $\{f, g\}$ associated with an R-matrix $r$ through

\[ \{f, g\} (X) = \langle R_+ (Df(X)X), Dg(X)X \rangle - \langle R_+ (X[Df(X)]), X[Dg(X)] \rangle, \]

where $\langle X, Y \rangle = \text{Tr}(XY)$, $\nabla$ is the gradient with respect to the trace-form, and $R_+ \in \text{End}\mathfrak{gl}_n$ as defined in (3.2). For future computation, it will be easier to describe $R_+$ in a simpler way: for an element $\eta \in \mathfrak{gl}_n$, let $\eta_{>0}$ and $\eta_{\leq 0}$ be the projections of $\eta$ onto the subalgebra spanned by positive roots, and the Cartan subalgebra $\mathfrak{h}$, respectively. Let $h_i = e_i - e_{i+1}$ be a basis for $\mathfrak{h}$. The dual basis

(defined by $\langle \hat{h}_i, h_j \rangle = \delta_{ij}$ is then $\hat{h}_i = \frac{1}{n} \text{diag} \left( \frac{n-i}{i}, \ldots, \frac{n-i}{i} \right)$).

Defining

\[ s_h (j) = \begin{cases} n-j & j \geq k \\ -j & j < k \end{cases} \]

(3.4)
we can write \( \hat{h}_i \) \( s_k \). Next, define the operator \( R_{\text{diag}} \) by

\[
R_{\text{diag}}(e_{kk}) = \sum_{j=1}^{n-1} s_k(j) (\hat{h}_j - \hat{h}_{j-1}) + (s_k(\beta - 1) - s_k(\beta)) \hat{h}_\alpha.
\]

(3.5)

\( R_{\beta} \) in choosing the diagonal part \( R_{\beta} \).

The operator

\[
R(\alpha, \beta, \alpha) = E_{\alpha} \otimes \hat{h}_\alpha, \otimes \hat{h}_\alpha
\]

is determined by the coefficient matrix \( (a_{ij}) \), which is subject to the conditions

\[
a_{ij} + a_{ji} = (\alpha, \alpha) \quad \text{if} \quad \alpha, \alpha \in \Delta
\]

(3.7)

\[
a_{\gamma(i), j} + a_{ji} = 0 \quad \text{if} \quad \alpha_i \in \Gamma_1, \alpha_j \in \Gamma_2.
\]

Define two matrices,

\[
A_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j + 1 \end{cases}, \quad B_{ij} = \begin{cases} 1 & (\alpha, \alpha) \in \{(\alpha, \beta), (\beta - 1, \alpha), (\beta, \alpha + 1)\} \\ -1 & (\alpha, \alpha) \in \{(\beta, \alpha), (\alpha, \beta - 1), (\alpha + 1, \beta)\}. \end{cases}
\]

It is not hard to see that for the BD triple \( T_{\alpha \beta} = \{(\alpha), (\beta), \gamma : \alpha + \beta \}, \) the matrix \( A + B \) satisfies conditions \( 3.7 \) and \( 3.8 \). The case \( \beta = \alpha + 1 \) is different, and here \( A \) will be the coefficient matrix. We choose

\[
r_0^{\alpha \beta} = \sum_{i=1}^{n-1} \hat{h}_i \otimes \hat{h}_i - \sum_{i=1}^{n-2} \hat{h}_{i+1} \otimes \hat{h}_i + \hat{h}_\alpha \wedge \hat{h}_\beta + \hat{h}_{\beta - 1} \wedge \hat{h}_\alpha + \hat{h}_\beta \wedge \hat{h}_{\alpha + 1},
\]

(3.9)

or, if \( \beta = \alpha + 1 \)

\[
r_0^{\alpha \beta} = \sum_{i=1}^{n-1} \hat{h}_i \otimes \hat{h}_i - \sum_{i=1}^{n-2} \hat{h}_{i+1} \otimes \hat{h}_i.
\]

(3.10)

Note that in the standard case condition \( 3.8 \) is empty, so we can use \( r_0^{\alpha \beta} \) in the standard case as well.

To prove the Lemma, it is enough to show that \( 3.6 \) holds for all elements of the basis \( \{E_\delta\}_{\delta \in \Phi} \cup \{\hat{h}_k\}_{k=1}^{n-1} \). Recall that \( \langle E_i, E_j \rangle = \delta_{ij}; \)

1. \( \eta = E_\delta, \) with \( \delta \in \Phi \). so

\[
\langle r, \eta \otimes \zeta \rangle = \left\langle \left( E_\delta \otimes \sum_{\alpha < \beta} E_\alpha \wedge E_\beta \right), E_\delta \otimes \zeta \right\rangle
\]

\[
= \begin{cases} \langle E_\delta, \zeta \rangle & \delta \notin \Gamma_1 \\ \langle E_\delta, \zeta \rangle + \sum_{\delta \in \Gamma_1} \langle E_\beta, \zeta \rangle & \delta \in \Gamma_1. \end{cases}
\]

and \( R_+ (E_\delta) = \begin{cases} E_\delta & \delta \notin \Gamma_1 \\ \left( E_\delta - \sum_{\delta \in \Gamma_1} E_\beta \right) & \delta \in \Gamma_1, \end{cases} \) in accordance with \( 3.6 \).
2. \( \eta = E_{-\delta} \), with \( \delta \in \Phi^+ \).

\[
\langle r, \eta \otimes \zeta \rangle = \left\langle \left( \sum_{\alpha < \beta} E_{-\alpha} \wedge E_{\beta} \right), E_{-\delta} \otimes \zeta \right\rangle
\]

\[
= \begin{cases} 
0 & \delta \notin \Gamma_2 \\
-\sum_{\alpha < \delta} \langle E_{-\alpha}, \zeta \rangle & \delta \in \Gamma_2,
\end{cases}
\]
hence \( R_+ (E_{-\delta}) = \begin{cases} 
0 & \delta \notin \Gamma_2 \\
-\sum_{\alpha < \delta} \langle E_{-\alpha}, \zeta \rangle & \delta \in \Gamma_2,
\end{cases} 
\) which also fits (3.6).

3. \( \eta = h_k \).

\[
\langle r, \eta \otimes \zeta \rangle = \left\langle \sum_{i=1}^{n-2} \hat{h}_i \wedge \hat{h}_{i+1}, h_k \otimes \zeta \right\rangle
\]

\[
= \left\langle \hat{h}_{k+1}, \zeta \right\rangle - \left\langle \hat{h}_{k-1}, \zeta \right\rangle
\]

(with \( \hat{h}_0 = \hat{h}_n = 0 \)). Therefore \( R_+ (h_k) = \left( \hat{h}_{k+1} - \hat{h}_{k-1} \right) \). Expressing \( e_{kk} \) as a linear combination of \( \{ h_i \}_{i=1}^{n-1} \cup \{ 1 \} \) implies (3.6).

4. Last, look at \( \eta = 1 \). Here it is clear that

\[
\langle r, 1 \otimes \zeta \rangle = 0.
\]

This implies \( R_+ (1) = 0 \), and the proof is complete. \( \square \)

3.3. **Standard bracket computations.** We will need some preliminary results about special cases of \( \{ f, g \}_{\text{std}} \). We will use the properties of \( R_{\alpha \beta}^+ \) and (3.3). After that comparing the bracket \( \{ \cdot, \cdot \}_{\alpha \beta} \) with the standard one will help us computing \( \{ f, g \}_{\alpha \beta} \) for every pair \( f, g \in B_{\alpha \beta} \).

Since some of the proofs use the standard Poisson bracket and cluster algebra on \( SL_n \), we start with a reminder: there are multiple Poisson brackets on \( SL_n \) that correspond to the trivial BD data \( \Gamma_1 = \Gamma_2 = \emptyset \), since \( r_0 \) is not uniquely determined. For a pair \( \alpha, \beta \) we will use \( r_0 \) as defined in (3.9) and call the associated Poisson bracket the standard Poisson bracket on \( SL_n \). The corresponding cluster algebra on \( SL_n \) that will be called the standard one and denoted \( C_{\text{std}} \) is described in [2] and [11]. Note that this cluster structure is independent on the choice of \( r_0 \) and the Poisson bracket. The initial seed of this cluster algebra looks as follows: set \( \mu (i, j) = \min (n, n + i - j) \). Write

\[
(3.11) \quad f_{ij} = \det X_{i,j}^{[\mu(i,j)],[\mu(i,j)]},
\]

Note that for \( \varphi_{ij} \in B_{\alpha \beta} \cap B_{\text{std}} \) we have \( \varphi_{ij} = f_{ij} \).

The function \( f_{11} = \det X \) is constant on \( SL_n \). Take the set

\[
\{ f_{ij} \}_{i,j=1}^n \setminus \{ f_{11} \}
\]
as the set of cluster variables. Set the variables \( f_{i1} \) and \( f_{1j} \) to be frozen, so there are \( n^2 - 1 \) cluster variables with \( 2 (n - 1) \) of them frozen. Let \( Q_{\text{std}}^n \) be the quiver of \( C_{\text{std}} \). The vertices of \( Q_{\text{std}}^n \) are placed on an \( n \times n \) grid with \( f_{ij} \) corresponding to the vertex \( (i, j) \). There are arrows from each node \( (i, j) \) to \( (i, j + 1) \) (as long as \( j \neq n \)), from \( (i, j) \) to \( (i + 1, j) \) (when \( i \neq n \)) and from \( (i + 1, j + 1) \) to \( (i, j) \). Arrows connecting two frozen variables can be ignored. Figure [3.1] shows the initial quiver.
of the standard cluster structure on $SL_5$: square vertices represent frozen variables, while circles represent mutable ones.

Denote now the Sklyanin bracket associated with BD data ${\alpha} \to {\beta}$ by ${\cdot} \alpha\beta$, and the standard bracket by $\{\cdot,\cdot\}_{std}$. We will use the following notations:

\begin{align}
  f^{i \leftarrow j} &= (\nabla f : X)_{ij} = \sum_{k=1}^{n} \frac{\partial g}{\partial x_{kj}} x_{kj} \\
  f_{j \leftarrow i} &= (X \cdot \nabla f)_{ij} = \sum_{k=1}^{n} \frac{\partial g}{\partial x_{jk}} x_{ik}.
\end{align}

Note that if $f$ is a determinant of some $Y$ submatrix of $X$ then $f^{i \leftarrow j}$ (or $f_{j \leftarrow i}$) is the same determinant, with column (or row) $i$ replaced by column (row) $j$. If $Y$ does not have column (row) $i$ in it, then $f^{i \leftarrow j} = 0$ ($f_{j \leftarrow i} = 0$). If $f = f_{ij} = \det Y$, where $Y = X_{[i...k]}^{[j...l]}$ is a dense submatrix of $X$, we write

\begin{align*}
  f^{\uparrow} &= f_{i \leftarrow i+1} \\
  f^{\downarrow} &= f_{j \leftarrow j+1} \\
  f^{\uparrow \downarrow} &= f_{i \leftarrow j+1} \\
  f^{\downarrow \uparrow} &= f_{j \leftarrow i+1}.
\end{align*}

**Lemma 9.** For any two functions $f, g$ on $SL_n$,

\begin{align}
  \{f, g\}_{\alpha\beta} - \{f, g\}_{std} &= f^{\alpha \leftarrow \alpha+1} g^{\beta+1 \leftarrow \beta} - f^{\beta+1 \leftarrow \beta} g^{\alpha \leftarrow \alpha+1} \\
  &+ f^{\beta \leftarrow \beta+1} g^{\alpha+1 \leftarrow \alpha} - f^{\alpha+1 \leftarrow \alpha} g^{\beta \leftarrow \beta+1}.
\end{align}

**Proof.** Let $r_{\alpha\beta}$ and $r_{std}$ be the R-matrices associated with BD data ${\alpha} \to {\beta}$ and $\emptyset \to \emptyset$, respectively. Using (3.3), it is easy to see that the difference (3.14) comes from the difference $R_{+}^{\alpha\beta} - R_{+}^{std}$. According to Lemma (8), this is

\begin{align}
  R_{+}^{\alpha\beta}(\eta) - R_{+}^{std}(\eta) &= \eta_{\alpha,\alpha+1} e_{\beta,\beta+1} - \eta_{\beta+1,\beta} e_{\alpha+1,\alpha}.
\end{align}
Write $R_d = R_d^{\alpha\beta} - R_d^{std}$, so now
\[
\{f, g\}_{\alpha\beta} - \{f, g\}_{std}
\]
\[
= \langle R_d(\nabla f(X)X), \nabla g(X)X \rangle - \langle R_d(X\nabla f(X)X), \nabla g(X)X \rangle
\]
\[
= \langle \nabla f(X)X \rangle_{\alpha,\alpha+1}(\nabla g(X)X)_{\beta+1,\beta} - \langle \nabla f(X)X \rangle_{\beta+1,\beta}(\nabla g(X)X)_{\alpha,\alpha+1}
\]
\[
= \langle X\nabla f(X) \rangle_{\alpha,\alpha+1}(X\nabla g(X))_{\beta+1,\beta} - \langle X\nabla f(X) \rangle_{\beta+1,\beta}(X\nabla g(X))_{\alpha,\alpha+1}
\]
\[
= f^{\alpha\alpha+1} g^{\beta+1\beta} - f^{\beta+1\beta} g^{\alpha+1\alpha} - f^{\alpha+1\alpha} g^{\beta+1\beta} + f^{\beta+1\beta} g^{\alpha+1\alpha}.
\]
\[
= 0.
\]
\[
\triangleleft
\]

**Corollary 10.** If $f, g \in B_{\alpha\beta} \cap B_{std}$ then $\{f, g\}_{\alpha\beta} = \{f, g\}_{std}$.

**Proof.** All functions in $B_{std}$ are determinants of submatrices of $X$. Let $f_{ij}$ be such a function with $x_{ij}$ at the upper left hand corner of the corresponding submatrix $X_{[i, j]}^{[a, b]}$ (with $\mu(i, j)$ as defined in Section 3.1). For such a function $f_{ij}$, the term $f_{ik}^{pm}$ is the determinant of a similar submatrix, with column $m$ instead of column $k$ (i.e., every instance of $x_{pk}$ is replaced by $x_{pm}$). Therefore, for $f_{ij} \in B_{std}$, the function $f_{ij}^{\alpha+1\alpha+1}$ is non zero only if $f_{ij}$ is a determinant of a submatrix with column $\alpha$ but without column $\alpha + 1$. The only functions with this property in $B_{std}$ are determinants of submatrices of the form $X_{[i, j]}^{[a, b]}$, that is, the functions $f_{n+j-n, j}$ with $j \in \{1, \ldots, \alpha\}$. But these functions are not in $B_{\alpha\beta}$, because $\alpha \in \Gamma_1$ (see the construction in Section 3.1). Similarly, $f_{m-n, k}$ is the determinant of the matrix obtained by replacing the $m$-th of $X$ row by the $k$-th row. So the function $f_{ij}^{\beta+1\beta+1}$ is non zero only if $f_{ij}$ is the determinant of a submatrix with row $\beta$ and without row $\beta + 1$. The only functions in $B_{std}$ that satisfy this condition are $f_{i, n+i-\beta}$, and these functions are not in $B_{\alpha\beta}$ because $\beta \in \Gamma_2$ (see Section 3.1) again.

The next Lemma describes the “building blocks” of the functions in $B_{\alpha\beta} \cap \overline{B}_{std}$ and the Poisson coefficients of these functions with respect to the standard bracket. For a pair $(f, g)$ of log canonical functions, denote by $\omega_{f, g}$ the Poisson coefficient
\[
\omega_{f, g} = \frac{\{f, g\}_{std}}{fg}.
\]

**Lemma 11.** 1. The function $f_{n+k-n, k}^{\alpha \alpha}$ (with $k \in \{1, \ldots, \alpha\}$) is log canonical with all functions $g \in B_{\alpha\beta} \cap B_{std}$, provided that $g \neq f_{n+m-n, m}$ for some $m > k$, w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is
\[
\omega_{f_{n+k-n, k}^{\alpha \alpha}} = \omega_{f_{n+k-n, k}^{\alpha \alpha}} + \omega_{g^{\alpha \alpha}} - \omega_{g^{\alpha \alpha}}.
\]

2. The function $f_{1, \beta+1}^{\alpha}$ is log canonical with all functions $g \in B_{\alpha\beta} \cap B_{std}$, provided that $g \neq f_{m, \beta+1}$ for some $m \in \{2, \ldots, n\}$ w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is
\[
\omega_{f_{1, \beta+1}^{\alpha}} = \omega_{f_{1, \beta+1}^{\alpha}} + \omega_{g^{\beta+1}} - \omega_{g^{\beta+1}}.
\]

3. The function $f_{k, n+k-\beta}^{\alpha \beta}$ (with $k \in \{1, \ldots, \beta\}$) is log canonical with all functions $g \in B_{\alpha\beta} \cap B_{std}$, provided that $g \neq f_{m, n+m-\beta}$ for some $m > k$, w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is
\[
\omega_{f_{k, n+k-\beta}^{\alpha \beta}} = \omega_{f_{k, n+k-\beta}^{\alpha \beta}} + \omega_{g^{\beta+1}} - \omega_{g^{\beta+1}}.
\]
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4. The function $f^r_{\alpha+1,1}$ is log canonical with all functions $g \in B_{\alpha\beta} \cap B_{std}$, provided that $g \neq f_{\alpha+1,m}$ w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is

$$\omega^r_{f_{\alpha+1,1},g} = \omega_{f_{\alpha+1,1},g} + \omega_{x_{\alpha},g} - \omega_{x_{\alpha+1,1},g}.$$  

Proof. The proof will use the Desnanot–Jacobi identity (see [3]): for a square matrix $A$, denote by “hatted” subscripts and superscripts deleted rows and columns, respectively. Then

$$\det A \cdot \det A_{\hat{r}_1,\hat{r}_2} = \det A_{\hat{r}_1} \cdot \det A_{\hat{r}_2}^2 = \det A_{\hat{r}_2} \cdot \det A_{\hat{r}_1}^2,$$

By adding an appropriate row, we get a similar result for a non square matrix $B$ with number of rows greater by one than the number of columns:

$$\det B_{\hat{r}_1} \cdot \det B_{\hat{r}_2,\hat{r}_3} = \det B_{\hat{r}_2} \cdot \det B_{\hat{r}_1,\hat{r}_3}$$

and naturally, a similar identity can be obtained for a matrix with number of columns greater by one than the number of rows.

Start with statement 1. of the Lemma. Look at the initial quiver described in Section 3.3 and mutate in direction $f_{n\alpha}$. We can assume $\alpha > 1$ because if $\alpha = 1$ then $f_{n\alpha}$ is frozen. In this case statement 1 holds trivially, for $f_{n\alpha} - f_{n,\alpha+1} \in B_{std}$. For $\alpha > 1$ the exchange rule is

$$(3.22) f'_{n\alpha} = f_{n,\alpha} \cdot f_{n-1,\alpha} + f_{n-1,\alpha-1} f_{n,\alpha+1}$$

which implies $f'_{n\alpha} = \left| \begin{array}{cc} x_{n-1,\alpha} & x_{n-1,\alpha+1} \\ x_{n,\alpha-1} & x_{n,\alpha+1} \end{array} \right|$. The edges of the quiver take the following changes:

the edges $(f_{n,\alpha+1}, f_{n-1,\alpha})$, $(f_{n-1,\alpha-1}, f_{n,\alpha-1})$ and $(f_{n-1,\alpha-1}, f_{n-1,\alpha})$ are removed, and an edge $(f_{n,\alpha-1}, f_{n,\alpha+1})$ is added. All edges containing $f_{n\alpha}$ are inverted. Therefore the exchange rule at $f_{n-1,\alpha-1}$ is now

$$f_{n-1,\alpha-1} \cdot f'_{n-1,\alpha-1} = f_{n\alpha} f_{n-2,\alpha-2} + f_{n-2,\alpha} f_{n-1,\alpha-1}.$$ 

Proceed with the mutation sequence $(f_{n\alpha}, f_{n-1,\alpha-1}, \ldots, f_{n+k-1,\alpha-k-1})$. Assume that mutating at $f_{n+m-\alpha,m}$ the exchanged variable is

$$(3.24) f'_{n+m-\alpha,m} = f_{n+m-1,\alpha-\alpha} \cdot f'_{n+m-1,\alpha-\alpha}$$

and that the exchange rule at $f_{n+m-1,\alpha-\alpha}$ is now

$$f_{n+m-1,\alpha-\alpha} \cdot f'_{n+m-1,\alpha-\alpha} = f_{n+m-\alpha,m} f_{n+m-\alpha,m-2} + f_{n+m-\alpha,m-1} f_{n+m-\alpha,m-2}.$$ 

Write $A = X_{[n+m-2,\alpha-1]}$ with $p = n + m - 2 - \alpha$, so using (3.22) we get

$$f_{n+m-1,\alpha-\alpha} \cdot f'_{n+m-1,\alpha-\alpha} = \det A^m_{\alpha-1,\alpha} \cdot \det A^m_{\alpha-1,\alpha} + \det A^m_{\alpha-1,\alpha} \cdot \det A^m_{\alpha-1,\alpha}$$

$$= \det A^m_{\alpha-1,\alpha} \cdot \det A^m_{\alpha-1,\alpha}$$

$$= \det A^m_{\alpha-1,\alpha} \cdot \det A^m_{\alpha-1,\alpha}.$$
The quiver mutates as follows: edges \((f_{p,m-2}, f_{p,m-1})\) and \((f_{p,m-2}, f_{p+1,m-2})\) are removed, edges \((f_{p,m-1}, f_{p+2,m})\) and \((f_{p+1,m-2}, f_{p+2,m})\) are added, and all edges containing \(f_{n+m-1-n,m-1}\) are inverted. Therefore the mutation rule at the next cluster variable of the sequence will be now

\[
f_{n+m-2-n,m-2} \cdot f_{n+m-2-n,m-2} = f_{n+m-1-n,m-1} f_{n+m-3-n,m-3} + f_{n+m-3-n,m-2} f_{n+m-2-n,m-3}.
\]

That proves that after the mutation sequence \((f_{n\alpha}, f_{n-1,\alpha-1}, \ldots, f_{n+k-1,\alpha,k+1})\) we have \(f_n^{\alpha} = f_{n+k-1,\alpha,k+1} = f_{n+k-1,\alpha,k}\) and therefore it is log canonical with all cluster variables of the initial cluster, except for those that were mutated: \((f_{n\alpha}, f_{n-1,\alpha-1}, \ldots, f_{n+k-1,\alpha,k+1})\).

Now for \(g \neq f_{n+m-\alpha,m}\) with \(m > k\) the coefficient \(\omega_{f_{n+k-1,\alpha,k} g}\) can be computed: from the Leibniz rule for Poisson brackets, any triple of functions \(f_1, f_2, g\) such that \(\{f_1, g\} = \omega_1 f_1 g\) and \(\{f_2, g\} = \omega_2 f_2 g\), must satisfy

\[
\{f_1 f_2, g\} = (\omega_1 + \omega_2) f_1 f_2 g,
\]
or, in other words \(\omega_{f_1 f_2} = \omega_{f_1} + \omega_{f_2} g\). Applying this together with the linearity of the bracket to the exchange rule (3.23) we get

\[
\omega_{f_{n\alpha}} + \omega_{f_{n\alpha}'} = \omega_{f_{n-1,\alpha-1}} + \omega_{x_{n,\alpha+1}},
\]

which is

\[
(3.25) \quad \omega_{f_{n-1,\alpha-1}} = \omega_{f_{n-1,\alpha-1}} + \omega_{x_{n,\alpha+1}} - \omega_{f_{n\alpha}}.
\]

Again, we proceed inductively: assume that

\[
\omega_{f_{n+k-1,\alpha,k+1}} = \omega_{f_{n+k-1,\alpha,k}} + \omega_{x_{n,\alpha+1}} - \omega_{f_{n\alpha}}
\]

and the exchange rule at \(f_{n+k-1,\alpha,k}\) is

\[
f_{n+k-1,\alpha,k} \cdot f_{n+k-1,\alpha,k-1} = f_{n+k-1,\alpha,k+1} f_{n+k-1,\alpha,k-1} + f_{n+k-1,\alpha,k-1} f_{n+k-1,\alpha,k}.
\]

This means that

\[
\omega_{f_{n+k-1,\alpha,k+1}} = \omega_{f_{n+k-1,\alpha,k+1}} + \omega_{f_{n+k-1,\alpha,k-1}} - \omega_{f_{n+k-1,\alpha,k}},
\]

and recursively this leads to

\[
(3.26) \quad \omega_{f_{n+k-1,\alpha,k+1}} = \omega_{f_{n+k-1,\alpha,k-1}} + \omega_{x_{n,\alpha+1}} - \omega_{f_{n\alpha}}
\]

which complete the proof of statement 1.

Next, look at statement 2. Here also, we will show that \(f_{1,\beta+1}^{\alpha}\) is a cluster variable that can be obtained through a mutation sequence, which in this case is \((f_{n,\beta+1}, f_{n-1,\beta+1}, \ldots, f_{2,\beta+1})\). First, mutate at \(f_{n,\beta+1}\). It is easy to see, just like in (3.23) that

\[
f_{n,\beta+1}^{\alpha} = \begin{pmatrix} x_{n-1,\beta} & x_{n-1,\beta+2} \\ x_{n,\beta} & x_{n,\beta+2} \end{pmatrix} = f_{n-1,\beta+1}^{\alpha}.
\]

Just like we have already showed above, edges \((f_{n,\beta+2}, f_{n-1,\beta+1}), (f_{n-1,\beta}, f_{n,\beta})\) and \((f_{n-1,\beta}, f_{n+1,\beta+1})\) are removed from the quiver, and an edge \((f_{n,\beta}, f_{n,\beta+2})\) is added to it. In addition, all the edges adjacent to \(f_{n,\beta+1}\) are inverted. The exchange rule at \(f_{n-1,\beta+1}\) is now

\[
f_{n-1,\beta+1}^{\alpha} f_{n-1,\beta+1} = f_{n,\beta+1} f_{n-2,\beta+1} + f_{n-1,\beta+2} f_{n-2,\beta}.
\]
Apply now the mutation sequence \((f_{n,\beta+1}, f_{n-1,\beta+1}, \ldots)\). Assume that after mutating at \(f_{m+1,\beta+1}\) we get
\[
f'_{m+1,\beta+1} = f_{m,\beta+1}
\]
and the exchange rule at \(f_{m,\beta+1}\) is
\[
f_{m,\beta+1} \cdot f'_{m,\beta+1} = f_{m+1,\beta+1}f_{m-1,\beta+1} + f_{m,\beta+2}f_{m-1,\beta}.
\]
If \(m > \beta + 1\) then we can write \(\mu = \mu(\beta, m - 1)\) and \(B = X^{[\beta...\mu+1]}_{[m-1...n]}\). Then the exchange rule is
\[
f_{m,\beta+1} \cdot f'_{m,\beta+1} = \det B^\beta \det B^\beta_{m-1} + \det B^\mu_{m-1} \det B^\mu_{m-1} = \det B^\beta \det B^\beta_{m-1} = f_{m-1,\beta+1}f_{m,\beta+1}.
\]
If, on the other hand, \(m \leq \beta + 1\) we write \(\mu = \mu(\beta, m - 1)\) and \(A = X^{[\beta...n]}_{[m-1...\mu]}\) so that the exchange rule becomes
\[
f_{m,\beta+1} \cdot f'_{m,\beta+1} = \det A^\beta \det A^\beta_{m-1} + \det A^\mu \det A^\mu_{m-1} = \det A^\beta \det A^\beta_{m-1} = f_{m,\beta+1}f_{m-1,\beta+1}.
\]
hence
\[
f'_{m,\beta+1} = f_{m-1,\beta+1}.
\]
It is easy to see that the mutation of the quiver also agrees with the induction hypothesis, and we can conclude that after the mutation sequence
\[
f'_{2,\beta+1} = f_{1,\beta+1},
\]
and therefore \(f'_{m-1,\beta+1}\) is log canonical with all functions \(f_{ij} \in B_{std}\), excluding the functions \(f_{m,\beta+1}\) that were mutated on the way.

We can now compute the coefficients \(\omega f_{m-1,\beta+1}g\) recursively like we did in the first statement and get for every \(f_{m,\beta+1} \neq g \in B_{std}\),
\[
\omega f_{m-1,\beta+1}g = \omega f_{1,\beta+1}g + \omega x_{n,\beta}g - \omega x_{n,\beta+1}g.
\]
This completes the proof for statement 2. The proofs for statements 3. and 4. are similar, using symmetry.

We should add here a remark: when we use Lemma 11 to claim that \(\omega f_{m-1,\beta+1}g\) and \(\omega f'_{n+1,\beta+1}g\) are also log canonical with \(f_{k,n+k+1-\beta}\) it may seem that the proof does not hold, since the path \((f_{\beta,n}, f_{\beta-1,n-1}, \ldots)\) crosses the paths \((f_{\alpha+1,n}, f_{\alpha+1,n-1}, \ldots)\) and \((f_{n,\beta+1}, f_{n-1,\beta+1}, \ldots)\). However, this can be easily settled. First Apply the sequence \((f_{\beta,n}, f_{\beta-1,n-1}, \ldots)\). Now shift every vertex \(f'_{n+m-\beta}\) of the new quiver to the place \((m - 1, n + m - 1 - \beta)\) i.e., move it one row up and one column to the left. The quiver now looks locally just like the initial one, with two changes at \(f_{\beta,n}\) and at \(f_{1,n+1-\beta}\). Then, set \(k = 2\beta + 1 - n\). Note that if \(k \leq 1\) the paths do not cross each other, and there is no problem.

Now apply the sequence \((f_{n,\beta+1}, f_{n-1,\beta+1}, \ldots, f_{k+1,\beta+1})\). The quiver then reads the exact same exchange rules as the initial quiver. At \(f_{k+1,\beta+1}\) the exchange rule is then almost the same as it was in the proof above, with one change: the function \(f_{k,\beta+1}\) is now replaced by \(f'_{k,\beta+1}\). The exchange rule is
\[
f_{k+1,\beta+1} \cdot f'_{k+1,\beta+1} = f_{k,\beta}f'_{k+1,\beta+1} + f'_{k+2,\beta+1}f'_{k,\beta+1}.
\]
So write $A = X_{[\beta \ldots n]}^{\hat{\beta} \ldots \hat{n}]}$ and then

$$f_{k+1,\beta+1} \cdot f'_{k+1,\beta+1} = \det A \det A_{k\beta}^{\hat{\beta}+1} + \det A_k^{\hat{\beta}+1} \det A_{\beta}^\beta$$

$$= \det A_k^\beta \det A_{\beta}^\beta = f_{k+1,\beta+1} f'_{k,\beta+1} \cdot$$

The picture is slightly different in the special cases of $\beta + 1 \in \{n-1,n\}$, but it is not hard to see that even then the result is still

$$(3.30) f'_{k+1,\beta+1} = f_{k,\beta+1}^\dagger.$$  

Moving to the next step of the sequence, we mutate at $f_{k,\beta+1}$. The exchange rule here reads

$$f_{k,\beta+1} \cdot f'_{k,\beta+1} = f_{k-1,\beta} f'_{k,\beta+2} + f'_{k+1,\beta+1} f'_{k-1,\beta+1}$$

and (3.21) can be used again, with $A = X_{[k-1,\beta-1,\beta+1]}^{[\beta \ldots n]}$. The result is

$$f'_{k,\beta+1} = f_{k,\beta+1}^\dagger,$$

which is just what it was in the proof of Lemma (11). This means that by the end of the process we still get $f'_{2,\beta+1} = f_{1,\beta+1}^\dagger$, and so $f_{1,\beta+1}^\dagger$ is log canonical with all the functions of the form $f_{n+k-\beta,n}$.

This can be done symmetrically with the sequence $(f_{a+1,n}, \ldots, f_{a+1,2})$ to show that $f'_{a+1,1}$ is also log canonical with all $f_{n+k-\beta,n}$.

The following two Lemmas will be needed to compute the brackets of a function $f \in B_{\alpha,\beta} \cap B_{std}$ with certain families of functions in $B_{std}$. We will use the notation $\omega_{f,g} = \{f,g\}_{std}$ where $f$ and $g$ are two log canonical functions w.r.t. the standard bracket.

**Lemma 12.** 1. Let $g = f_{k,\beta+1}$ with $k > 1$. Then

$$(3.31) \{f_{1,\beta+1}^\dagger, g\}_{std} = (\omega_{f_1,\beta+1,g} + \omega_{x_1,\beta,g} - \omega_{x_1,\beta+1,g}) f_{1,\beta+1}^\dagger g + f_{1,\beta+1}^\dagger g^\dagger$$

2. Let $g = f_{n+m-\alpha,m}$ with $m > k$. Then

$$(3.32) \{g, f_{n+k-\alpha,k}\}_{std} = (\omega_{g,f_{n+k-\alpha,k}} - \omega_{g,x_n,\alpha} + \omega_{g,x_{n+1}}) g f_{n+k-\alpha,k} - f_{n+k-\alpha,k} g^\dagger$$

**Proof.** 1. Let $g = f_{k,\beta+1}$, we compute the bracket $\{f_{1,\beta+1}^\dagger, g\}_{std}$ directly using (3.3). Recall that

$$\langle \nabla f_{1,\beta+1}^\dagger \cdot X \rangle_{ij} = \sum_{m=1}^n \frac{\partial f_{1,\beta+1}^\dagger}{\partial x_{mi}} x_{mj} = (f_{1,\beta+1}^\dagger)^{i\beta}$$

and since $f_{1,\beta+1}^\dagger = \det X_{[1,\ldots,n-\beta]}^{[\beta,\beta+2,\ldots,n]}$ we have

$$f_{1,\beta+1}^\dagger = 0$$

for $i < \beta$ and $i = \beta + 1$. Similarly, the term

$$\langle \nabla g \cdot X \rangle_{ij} = \sum_{m=1}^n \frac{\partial g}{\partial x_{mi}} x_{mj} = g^{i\beta}$$

is not hard to see that even then the result is still
vanishes for \( i < \beta + 1 \). On the other hand, looking at the second trace form in (3.3),
\[
(X \cdot \nabla f_{1,\beta+1}^-)_{ij} = \sum_{m=1}^{n} \frac{\partial f_{1,\beta+1}^-}{\partial x_{jm}} x_{im} = (f_{1,\beta+1}^-)_{j-i},
\]
which vanishes for \( j > n - \beta \), and also
\[
(X \cdot \nabla g)_{ij} = \sum_{m=1}^{n} \frac{\partial g}{\partial x_{jm}} x_{im} = g_{j-i}
\]
is non zero only for \( k \leq j \leq n + k - \beta - 1 \). Applying \( R_+ \) to the matrices \( \nabla f_k \cdot X \)
and \( X \cdot \nabla f_k \) vanishes all entries below the main diagonal. On the main diagonal we
have only the original function with some coefficients \( \xi_i \). So we can write (3.3) as:
\[
\{ f_{1,\beta+1}^-, g \} = \langle R_+ (\nabla f_{1,\beta+1}^- \cdot X), \nabla g \cdot X \rangle - \langle R_+ (X \cdot \nabla f_{1,\beta+1}^-), X \cdot \nabla g \rangle = 
\sum_{i<j} (f_{1,\beta+1}^-)^{i+j} g^{i+j} + \sum_i \xi_i f_{1,\beta+1}^- g^{i+j} - 
\sum_{i<j} (f_{1,\beta+1}^-)_{j-i} g^{i+j} - \sum_i \xi_i f_{1,\beta+1}^- g^{i+j},
\]
Look at the term \( (f_{1,\beta+1}^-)^{i+j} \): whenever \( (i, j) \neq (\beta, \beta+1) \) it vanishes, because
\( f_{1,\beta+1}^- = \det X_{[\beta,\beta+2,\ldots,n]} \) and so \( (f_{1,\beta+1}^-)^{i+j} \) is the determinant of a submatrix with
two identical columns \( (j > i) \). The only non zero term here is then \( (f_{1,\beta+1}^-)^{\beta+\beta+1} = f_{1,\beta+1} \). Similarly, \( (f_{1,\beta+1}^-)^{j+i} \) must vanish when \( i < j \), because it is the determinant
of a submatrix with two identical rows. Therefore, the only non zero terms of the
trace form are \( f_{1,\beta+1}^- g^{\beta+\beta} \) and the diagonal ones. The latter are just the product
of the two functions multiplied by the coefficients \( \xi_i \) and \( \xi_i' \). Note that \( f_{1,\beta+1}^- \) vanishes
when \( i < \beta + 1 \), and \( f_{1,\beta+1}^- \) vanishes for \( i < \beta \) and for \( i = \beta + 1 \). Comparing these
coefficients with the coefficients of the bracket \( \{ f_{1,\beta+1}, g \} \), we see that the only
difference is the contribution of the elements in entries \((\beta, \beta)\) and \((\beta + 1, \beta + 1)\):
\[
\begin{align*}
(\nabla f_{1,\beta+1} \cdot X)_{\beta,\beta} &= 0 \\
(\nabla f_{1,\beta+1} \cdot X)_{\beta+1,\beta+1} &= f_{1,\beta+1} \\
(\nabla f_{1,\beta+1}^- \cdot X)_{\beta,\beta} &= f_{1,\beta+1}^- \\
(\nabla f_{1,\beta+1}^- \cdot X)_{\beta+1,\beta+1} &= 0
\end{align*}
\]
And this is just the same for \( x_{n,\beta} \) and \( x_{n,\beta+1} \):
\[
\begin{align*}
(\nabla x_{n,\beta} \cdot X)_{\beta,\beta} &= 0 \\
(\nabla x_{n,\beta+1} \cdot X)_{\beta+1,\beta+1} &= x_{n,\beta+1} \\
(\nabla x_{n,\beta} \cdot X)_{\beta,\beta} &= x_{1,\beta} \\
(\nabla x_{n,\beta} \cdot X)_{\beta+1,\beta+1} &= 0.
\end{align*}
\]
Hence, we can conclude
\[
(3.33) \quad \{ f_{1,\beta+1}^-, g \}_{\text{std}} = (\omega_{f_{1,\beta+1}, g} + \omega_{x_{n,\beta}, g} - \omega_{x_{n,\beta+1}, g}) f_{1,\beta+1}^- g + f_{1,\beta+1}^- g^-.
\]
2. The proof here follows a similar path: from (3.3) we have
\[ \{ g, f_{n+k-a,k} \}_{std} = \langle R_+ (\nabla g \cdot X), \nabla f_{n+k-a,k} \cdot X \rangle - \langle R_+ (X \cdot \nabla g), X \cdot \nabla f_{n+k-a,k} \rangle \]

and since \( R_+ \) annihilates all the entries below the main diagonal,

\[
\{ g, f_{n+k-a,k} \}_{std} = \sum_{i=m}^{\alpha} \sum_{j=i+1}^{\alpha-1} g^{i-j} (f_{n+k-a,k})^{j-i} \\
+ \sum_{i=m}^{\alpha} g^{i-a+1} (f_{n+k-a,k})^{a+1-i} + \sum_{j=1}^{n} \xi_j g f_{n+k-a,k} \\
+ \sum_{j=n+m-\alpha}^{\alpha} \sum_{i<j}^{n} g^{j-i} (f_{n+k-a,k})_{i-j} \\
+ \sum_{j=1}^{n} \xi_j' g f_{n+k-a,k} 
\]

where \( \xi_j \) and \( \xi_j' \) are some coefficients. But \( f_{n+k-a,k} = \det X_{[n+k-a,\ldots, n]}^{[\alpha-1,\ldots, 1]} \), and therefore, for every \( i \neq \alpha \) in \( \{ m, \ldots, \alpha \} \) and \( j \in \{ i+1, \ldots, \alpha-1 \} \) we get

\[
(f_{n+k-a,k})^{j-i} = 0
\]

because it is the determinant of a matrix with two identical columns. For the same reason, \( (f_{n+k-a,k})^{a+1-i} \) vanishes for every \( i \neq \alpha \). Similarly, the term \( (f_{n+k-a,k})_{i-j}^{\alpha} \) is zero for every \( j \in \{ n+m-\alpha, \ldots, n \} \) and \( i < j \), because this is also a determinant of a matrix with two identical columns.

So we are left with

\[
\{ g, f_{n+k-a,k} \}_{std} = \xi g f_{n+k-a,k} + \sum_{\alpha < j \leq \alpha + 1} g^{\alpha-j} (f_{n+k-a,k})^{j-\alpha} \\
= \xi g f_{n+k-a,k} + g^{\alpha-a+1} (f_{n+k-a,k})^{a+1-\alpha} \\
= \xi g f_{n+k-a,k} + g^a f_{n+k-a,k} 
\]

for some coefficient \( \xi \). Now, compare the coefficients \( \xi_j \) and \( \xi_j' \) in the bracket \( \{ g, f_{n+k-a,k} \} \) to those of \( \{ g, f_{n+k-a,k} \} \). The difference is equal to the difference between these coefficients in \( \{ g, x_{n,a+1} \} \) and \( \{ g, x_{na} \} \), to see that, note that these functions are determinants of submatrices of \( X \) that are distinguished only by the last column, which is \( \alpha + 1 \) in the first case and \( \alpha \) in the second. The result, like in (3.31) is

\[
(3.34) \quad \{ g, f_{n+k-a,k} \}_{std} = (\omega_1 - \omega_2 + \omega_3 - 1) f_{n+k-a,k} g + f_{n+k-a,k} g^a 
\]

with

\[
\omega_1 = \omega_{g,f_{n+k-a,k}} \\
\omega_2 = \omega_{g,x_{n,a}} \\
\omega_3 = \omega_{g,x_{n,a+1}}.
\]

\( \square \)
Lemma 13. 1. Let \( g \in B_{\text{std}} \) be a function of the initial standard cluster. Let
\[
(3.35) \quad s^\omega \alpha \beta (g) = \omega f_{n, \alpha} g - \omega f_{n+1, \alpha} g - \omega f_{n, \beta} g + \omega f_{n+1, \beta} g,
\]
then
\[
(3.36) \quad s^\omega \alpha \beta (g) = \begin{cases} 
1 & \text{if } g = f_{n+k-\alpha, k} \\
-1 & \text{if } g = f_{i, \beta+1} \\
0 & \text{otherwise}
\end{cases}
\]

2. Let \( g \in B_{\text{std}} \) be a function of the initial standard cluster. Write
\[
(3.37) \quad s'\omega \alpha \beta (g) = \begin{cases} 
1 & \text{if } g = f_{k,n+k-\beta} \\
-1 & \text{if } g = f_{\alpha+1, j} \\
0 & \text{otherwise}
\end{cases}
\]

Proof. 1. We will compute the coefficients through (3.3). Since \( \nabla x_{nk} = e_{kn} \) we have
\[
(\nabla x_{nk} X)_{ij} = \begin{cases} 
x_{nj} & i = k, \quad j > k \\
0 & i \neq k,
\end{cases}
\]
and \( (X \nabla x_{nk})_{ij} = \begin{cases} 
x_{ik} & j = n \\
0 & j \neq n.
\end{cases} \]

According to Lemma 8,
\[
R_+ (\nabla x_{nk} X)_{ij} = \begin{cases} 
x_{nj} & i = k, \quad j > k \\
x_{ik} x_{nk} & i = j
\end{cases}
\]
with some coefficients \( \xi_j \), and
\[
R_+ (X \nabla x_{nk})_{ij} = \begin{cases} 
x_{ik} & j = n \\
\xi'_j x_{nk} & i = j
\end{cases}
\]
with other coefficients \( \xi'_j \). Plugging this into (3.3) gives
\[
\{x_{nk}, g\}_{\text{std}} = \sum_{j=k+1}^{n} x_{nj} \sum_{i=1}^{n} \frac{\partial g}{\partial x_{ik}} x_{ik} + \sum_{j=1}^{n} \xi_j x_{nk} g
\]
\[
- \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^{n} x_{nj} \frac{\partial g}{\partial x_{ij}} - \sum_{j=1}^{n} \xi'_j x_{nk} g.
\]

Split the last term into the “diagonal” part
\[
D = \sum_{j=1}^{n} \xi_j x_{nk} g - \sum_{j=1}^{n} \xi'_j x_{nk} g
\]
and the “non diagonal” part
\[
N = \sum_{j=k+1}^{n} x_{nj} \sum_{i=1}^{n} \frac{\partial g}{\partial x_{ij}} x_{ik} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^{n} x_{nj} \frac{\partial g}{\partial x_{ij}}.
\]
Start with the diagonal part $D$. We need the coefficients $\xi_j$ and $\xi'_j$ for $k = \alpha, \beta, \alpha + 1, \beta + 1$. Recall (3.5):
\[
R_+ (e_{\alpha\alpha}) = \frac{1}{n} \sum_{j=1}^{n-1} s_\alpha (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) + \hat{h}_\alpha \\
+ \hat{h}_\beta - (n - \alpha) \hat{h}_{\beta - 1} + (n - \beta) \hat{h}_{\alpha + 1}
\]
\[
R_+ (e_{\beta\beta}) = \frac{1}{n} \sum_{j=1}^{n-1} s_\beta (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) \\
+ (1 - n) \hat{h}_\alpha + \alpha \hat{h}_{\beta - 1} + (n - \beta) \hat{h}_{\alpha + 1} \\
+ \hat{h}_\beta \begin{cases} 1 & \beta > \alpha + 1 \\
(n - n) & \beta = \alpha + 1 \end{cases}
\]
\[
R_+ (e_{\alpha+1,\alpha+1}) = \frac{1}{n} \sum_{j=1}^{n-1} s_{\alpha+1} (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) \\
+ (1 - n) \hat{h}_\beta + \alpha \hat{h}_{\beta - 1} + (n - \beta) \hat{h}_{\alpha + 1} \\
+ \hat{h}_\alpha \begin{cases} 1 & \beta > \alpha + 1 \\
(n - n) & \beta = \alpha + 1 \end{cases}
\]
\[
R_+ (e_{\beta+1,\beta+1}) = \frac{1}{n} \sum_{j=1}^{n-1} s_{\beta+1} (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) + \hat{h}_\alpha + \beta \hat{h}_1 \beta - 1 - \beta \hat{h}_{\alpha + 1}
\]
\[
R_+ (e_{nn}) = \frac{1}{n} \sum_{j=1}^{n-1} \left( \hat{h}_j - \hat{h}_{j-1} \right) + \hat{h}_\alpha + \hat{h}_\beta + \alpha \hat{h}_{\beta - 1} - \beta \hat{h}_{\alpha + 1}.
\]

Using $\left( \hat{h}_j - \hat{h}_{j-1} \right) = \frac{1}{n} \text{diag} (-1, \ldots, -1) + e_{jj}$ and the fact
\[
s_\alpha (j) - s_{\alpha+1} (j) = \begin{cases} n & j = \alpha \\
0 & j \neq \alpha \end{cases}
\]
we get
\[
\sum_{j=1}^{n-1} s_\alpha (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) - \sum_{j=1}^{n-1} s_{\alpha+1} (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) \\
= \text{diag} (-1, \ldots, -1) + ne_{\alpha\alpha}.
\]

and
\[
\sum_{j=1}^{n-1} s_{\beta+1} (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) - \sum_{j=1}^{n-1} s_\beta (j) \left( \hat{h}_j - \hat{h}_{j-1} \right) = \text{diag} (1, \ldots, 1) - ne_{\beta\beta}.
\]

Putting everything together gives
\[(3.38) \quad R_+ (e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1,\alpha+1} + e_{\beta+1,\beta+1})
\]
\[(3.39) \quad = n \left( \hat{h}_\alpha + \hat{h}_\beta - \hat{h}_{\alpha+1} - \hat{h}_{\beta-1} \right) + ne_{\alpha\alpha} - ne_{\beta\beta}.
\]

for $\beta > \alpha + 1$, or in the case $\beta = \alpha + 1$:
\[
R_+ (e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1,\alpha+1} + e_{\beta+1,\beta+1}) = e_{\alpha\alpha} - e_{\alpha+1,\alpha+1},
\]
and since $\hat{h}_\alpha - \hat{h}_{\alpha+1} = \frac{1}{n} \text{diag} (1, \ldots, 1) - e_{\alpha+1,\alpha+1}$, and

$\hat{h}_\beta - \hat{h}_{\beta-1} = \frac{1}{n} \text{diag} (-1, \ldots, -1) + e_{\beta\beta}$, (3.38) turns to

$$R_+ (e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1,\alpha+1} + e_{\beta+1,\beta+1}) = (e_{\alpha\alpha} - e_{\alpha+1,\alpha+1}).$$

Since $D$ is a trace of two matrices, we are only interested in products of the diagonal elements in $R_+ (\nabla x_{nk} \cdot X)$, $R_+ (X \cdot \nabla x_{nk})$ with the corresponding diagonal elements in $\nabla g \cdot X$ and $X \cdot \nabla g$. These products vanish for all $g \in B_{\text{std}}$ except $g = f_{i,\alpha+1}$ (which is a the determinant of a submatrix that has column $\alpha + 1$ but not col. $\alpha$) or $g = f_{n-\alpha+k,k}$ (a determinant of a submatrix that has column $\alpha$ but not column $\alpha + 1$). Write $\omega^D = \frac{D}{f_{nk}g}$. So the sum of coefficients of the diagonal part is

$$\omega^D_{f_{\alpha\alpha},g} - \omega^D_{f_{\beta\beta},g} - \omega^D_{f_{\alpha+1,\alpha+1},g} + \omega^D_{f_{\beta+1,\beta+1},g} = \begin{cases} 1 & g = f_{n-\alpha+k,k} \\ -1 & g = f_{i,\alpha+1} \\ 0 & \text{otherwise.} \end{cases}$$

We now turn to the non diagonal part $N$: recall

$$(\nabla x_{nk} X)_{ij} = (x_{nk})^{i\rightarrow j} = \begin{cases} x_{nj} & i = k \\ 0 & i \neq k, \end{cases}$$

$$(X \nabla x_{nk})_{ij} = (x_{nk})_{j\rightarrow i} = \begin{cases} x_{ik} & j = n \\ 0 & j \neq n, \end{cases}$$

and we have

$$R_+ (\nabla x_{nk} X) = \begin{bmatrix} 0 & \cdots & x_{n,k+1} & \cdots & x_{n,n} \\ & \ddots & \cdots & \cdots & \cdots \\ & & \ddots & \cdots & \cdots \\ & & & \ddots & \cdots \\ & & & & \ddots \end{bmatrix},$$

and

$$R_+ (X \nabla x_{nk}) = \begin{bmatrix} 0 & x_{1k} \\ \vdots & \vdots \\ \vdots & \vdots \\ & \ddots \\ & & \ddots \end{bmatrix}.$$
so when computing the bracket with (3.3),

\[
N = \sum_{j=k+1}^{n} x_{nj} \sum_{i=1}^{n} \frac{\partial g}{\partial x_{ij}} x_{ik} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^{n} x_{nj} \frac{\partial g}{\partial x_{ij}} 
\]

\[
= x_{nk} \sum_{j=k+1}^{n} x_{nj} \frac{\partial g}{\partial x_{nj}} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^{k} x_{nj} \frac{\partial g}{\partial x_{ij}} + x_{nk} \sum_{j=1}^{k} x_{nj} \frac{\partial g}{\partial x_{nj}} 
\]

\[
= x_{nk} g - \sum_{i=1}^{n} x_{ik} \sum_{j=1}^{k} x_{nj} \frac{\partial g}{\partial x_{ij}}. 
\]

Now, since \( g \) is a determinant of some submatrix \( A \) of \( X \), let \( g^{\max} \) and \( g^{\min} \) denote the maximal (right) and minimal (left) columns of \( A \). Similarly, let \( g_{\max} \) be the lower row of \( A \). Then

\[
\sum_{i=1}^{n} x_{ik} \sum_{j=1}^{k} \frac{\partial g}{\partial x_{ij}} = \begin{cases} 
0 & g^{\min} > k \\
x_{nk} g & g^{\min} \leq k \leq g^{\max} \\
x_{nk} g & g^{\max} < k \rightarrow g_{\max} = n 
\end{cases}
\]

so that

\[
N = \begin{cases} 
x_{nk} g & g_{\min} > k \\
0 & g_{\min} \leq k. 
\end{cases}
\]

Defining \( \omega_{f_{nk},g}^{N} = \frac{N}{f_{nk},g} \), summing over \( k = \alpha, \alpha + 1, \beta, \beta + 1 \) we get \( \sum \omega_{f_{nk},g}^{N} \neq 0 \) only when \( g = f_{i,\alpha + 1} \) or \( g = f_{i,\beta + 1} \), or in the “special” case \( \beta = \alpha + 1 \) : we can write the sum of these coefficients in a table:

| \( g^{\min} \) | \( \alpha \) | \( \alpha + 1 \) | \( \beta \) | \( \beta + 1 \) |
|----------------|----------|----------|----------|----------|
| \( \sum \omega_{f_{nk},g}^{N} \) | 0 | 1 | \( \beta = \alpha + 1 \) | \( \beta \) \( \geq \alpha + 1 \) | \( -1 \) |

We now add the \( D \)-coefficients, for \( s_{\omega_{\alpha\beta}}(g) = \sum \omega_{f_{nk},g}^{N} + \sum \omega_{f_{nk},g}^{D} \), so

1. If \( g = f_{i,\alpha + 1} \), then the \( N \)-coefficients sum is 1. We have seen that in this case the \( D \)-coefficients sum is \(-1\), and therefore \( s_{\omega_{\alpha\beta}}(g) = 0 \).
2. If \( g = f_{i,\beta} \) and \( \beta = \alpha + 1 \), just like in 1, it is \( s_{\omega_{\alpha\beta}}(g) = 0 \).
3. If \( g = f_{i,\beta + 1} \) then \( s_{\omega_{\alpha\beta}}(g) = -1 \).
4. If \( g = f_{n+k-\alpha,k} \) then \( s_{\omega_{\alpha\beta}}(g) = -1 \).
5. For any other \( g \in B_{std} \), \( s_{\omega_{\alpha\beta}}(g) = 0 \).

This completes the proof of part 1. of the Lemma. Part 2. is similar, using the symmetries \( x_{ij} \leftrightarrow x_{ji} \) (and therefore \( f_{ij} \leftrightarrow f_{ji} \)), and \( \alpha \leftrightarrow \beta \). \( \square \)

3.4. The set \( B_{\alpha\beta} \) is log canonical. Now we are ready to prove Theorem 7.

Proof: Compute the bracket \( \{ f, g \}_{\alpha\beta} \) for all \( f, g \in B_{\alpha\beta} \):

Let \( Z = B_{\alpha\beta} \cap B_{std} \). So first, by Corollary 10 if \( f, g \in Z \) then \( \{ f, g \}_{\alpha\beta} = \{ f, g \}_{std} \), and therefore \( f, g \) are log canonical with respect to \( \{ \cdot, \cdot \}_{\alpha\beta} \). We now turn to \( \{ f, g \}_{\alpha\beta} \).
where \( f \) or \( g \) are non standard basis functions. This means that \( f \) or \( g \) (or both) are of the form

\[
\begin{align*}
\theta_k &= f_{n+k-\alpha,k} \cdot f_{1,\beta+1} - f_{n+k-\alpha,k} \cdot f_{1,\beta+1}^- \\
\psi_j &= f_{m,n+m-\beta} \cdot f_{0,1+1} - f_{m,n+m-\beta} \cdot f_{1,1+1}^-
\end{align*}
\]

with \( k \in [\alpha] \) and \( m \in [\beta] \). Look at the bracket \( \{\theta_k, g\}_{\alpha\beta} \) with \( g \in S \). Assuming \( g \neq f_{m,\beta+1} \) for some \( m \in [2 \ldots n] \), and \( g \neq f_{n+m-\alpha,m} \) for some \( m > k \), we can write

\[
\{\theta_k, g\}_{\alpha\beta} = \{f_{n+k-\alpha,k} \cdot f_{1,\beta+1}, g\}_{\alpha\beta} - \{f_{n+k-\alpha,k} \cdot f_{1,\beta+1}^-, g\}_{\alpha\beta}
\]

(3.40)

According to Lemma 11 the functions \( f_{n+k-\alpha,k}^- \) and \( f_{1,\beta+1}^- \) are both log canonical with \( g \) (w.r.t. the standard bracket) with Poisson coefficients

\[
\begin{align*}
\omega_{f_{n+k-\alpha,k}^-} &= \omega f_{n+k-\alpha,k} + \omega x_{n,\alpha+1} - \omega x_{n,\alpha}, \\
\omega_{f_{1,\beta+1}^-} &= \omega f_{1,\beta+1} + \omega x_{\beta,n} - \omega x_{\beta,n+1}
\end{align*}
\]

so (3.40) turns to

\[
\begin{align*}
\{\theta_k, g\}_{\alpha\beta} &= \omega_1 f_{n+k-\alpha,k} \cdot f_{1,\beta+1} \cdot g \\
&\quad - (\omega_1 - s\omega_{\alpha\beta}(g)) (f_{n+k-\alpha,k}^- \cdot (f_{1,\beta+1}^-)^\top) \cdot g,
\end{align*}
\]

with

\[
\omega_1 = \omega f_{n+k-\alpha,k} + \omega f_{1,\beta+1}
\]

and

\[
s\omega_{\alpha\beta}(g) = \omega f_{n,\alpha} - \omega f_{n,\alpha+1} - \omega f_{n,\beta} + \omega f_{n,\beta+1},
\]

as defined in (3.35).

Now, using Lemma 13 we get

\[
\{\theta_k, g\}_{\alpha\beta} = (\omega f_{n+k-\alpha,k} + \omega f_{1,\beta+1}) \theta_k g.
\]

If \( g = f_{m,\beta+1} \) for some \( m \in [2 \ldots n] \), we write \( f_k = f_{n+k-\alpha,k} \), and then

\[
\begin{align*}
\{\theta_k, g\}_{\alpha\beta} &= \{f_k \cdot f_{1,\beta+1}, g\}_{\alpha\beta} - \{f_k^\top \cdot f_{1,\beta+1}^-, g\}_{\alpha\beta} \\
&= f_k \{f_{1,\beta+1}, g\}_{\alpha\beta} + f_{1,\beta+1} \{f_k, g\}_{\alpha\beta} \\
&\quad - f_k^\top \{f_{1,\beta+1}^-, g\}_{\alpha\beta} - f_{1,\beta+1}^\top \{f_k^\top, g\}_{\alpha\beta} \\
&= f_k \{f_{1,\beta+1}, g\}_{\text{std}} + f_{1,\beta+1} \{f_k, g\}_{\text{std}} \\
&\quad + f_{1,\beta+1} f_k^\top g^\top - f_k^\top \{f_{1,\beta+1}^-, g\}_{\text{std}} \\
&\quad - f_{1,\beta+1}^\top \{f_k^\top, g\}_{\text{std}}.
\end{align*}
\]
These brackets are of log canonical functions, except for \(\{f_{\alpha,\beta+1}^-; g\}_{\text{std}}\) which is given in [12] so it is

\[
\{\theta_k; g\}_{\alpha\beta} = (\omega_{f_{\alpha,\beta+1}^+} + \omega_{f_k} g) f_k \cdot f_{1,\beta+1} g + f_{1,\beta+1} f_k g \quad \leftarrow \quad \leftarrow \\
- (\omega_{f_{1,\beta+1}^+} + \omega_{x_{\alpha,\beta+1}} g - \omega_{x_{\alpha,\beta+1}} g) f_k f_{1,\beta+1} g - f_{1,\beta+1} f_k g \\
- (\omega_{f_k} g - \omega_{x_{\alpha,\beta+1}} g + \omega_{x_{\alpha,\beta+1}} g) f_{1,\beta+1} f_k g \\
= (\omega_{f_{1,\beta+1}^+} + \omega_{f_k} g) f_k \cdot f_{1,\beta+1} g - (\omega_{f_{1,\beta+1}^+} + \omega_{f_k} g - s\omega_{\alpha\beta} (g)) f_{1,\beta+1} f_k g,
\]

and with Lemma [13] this comes down to

\[
(3.41) \quad \{\theta_k; g\}_{\alpha\beta} = (\omega_{f_{1,\beta+1}^+} + \omega_{f_k} g) \theta_k g.
\]

Next, look at \(g = f_{n+m-\alpha,m}\) for some \(m > k\): with Lemma [12] we can compute

\[
\{\theta_k; g\}_{\alpha\beta} = \{f_k \cdot f_{1,\beta+1}^+; g\}_{\alpha\beta} - \{f_k \cdot f_{1,\beta+1}^-; g\}_{\alpha\beta} \\
= f_k \{f_{1,\beta+1}^+; g\}_{\alpha\beta} + f_{1,\beta+1} \{f_k; g\}_{\alpha\beta} \\
- f_{1,\beta+1} \{f_k; g\}_{\alpha\beta} \\
= f_k \{f_{1,\beta+1}^+; g\}_{\text{std}} - f_k f_{1,\beta+1}^+ g + f_{1,\beta+1} \{f_k; g\}_{\text{std}} \\
- f_{1,\beta+1} \{f_k; g\}_{\text{std}} \\
= (\omega_{f_{1,\beta+1}^+} + \omega_{f_k} g) f_k f_{1,\beta+1} g \\
- (\omega_{f_k} g + \omega_{f_{1,\beta+1}^+} - s\omega_{\alpha\beta} (g)) f_{1,\beta+1} f_k g,
\]

and with Lemma [13] this is

\[
(3.42) \quad \{\theta_k; g\}_{\alpha\beta} = (\omega_{f_{1,\beta+1}^+} + \omega_{f_k} g) \theta_k g.
\]

We now turn to look at \(\{\theta_k; \theta_m\}_{\alpha\beta}\): w.l.o.g. assume \(m > k\):

\[
\{\theta_k; \theta_m\}_{\alpha\beta} = \{f_k \cdot f_{1,\beta+1} - f_k \cdot f_{1,\beta+1}^+ + f_m \cdot f_{1,\beta+1}^+ - f_m \cdot f_{1,\beta+1}\}_{\alpha\beta} \\
= \{f_k \cdot f_{1,\beta+1}, f_m \cdot f_{1,\beta+1}^+\}_{\alpha\beta} - \{f_k \cdot f_{1,\beta+1}^+, f_m \cdot f_{1,\beta+1}\}_{\alpha\beta} \\
- \{f_k \cdot f_{1,\beta+1}, f_m \cdot f_{1,\beta+1}\}_{\alpha\beta} \\
+ \{f_k \cdot f_{1,\beta+1}^+, f_m \cdot f_{1,\beta+1}^+\}_{\alpha\beta}.
\]

The Poisson bracket satisfy the Leibniz rule:

\[
\{A \cdot B, C\} = A \cdot \{B, C\} + \{A, C\} \cdot B,
\]
so each of the four brackets above can break into four terms of the form $A \cdot B \cdot \{C, D\}$.

We have already seen that

$$
\begin{align*}
(3.44) \quad \{f_k, f_m^+\} &= (\omega_k f_m - \omega_k x_{n,\alpha} + \omega_m x_{n,\alpha+1}) f_k f_m^+ \\
(3.45) \quad \{f_k^-, f_m^-\} &= (\omega_k f_m - \omega_{x_{n,\alpha} f_m} + \omega_{x_{n,\alpha+1} f_m}) f_k^+ f_m^- - f_k f_m^- \\
(3.46) \quad \{f_k, f_{1,\beta+1}^-\} &= (\omega_k f_{1,\beta+1} + \omega_k x_{n,\beta} - \omega_k x_{n,\beta+1}) f_k f_{1,\beta+1}^- \\
(\text{and}) \quad \{f_k, f_{1,\beta+1}^-\} &= (\omega_k f_{1,\beta+1} - \omega_{x_{n,\alpha} x_{n,\beta}} + \omega_{x_{n,\alpha+1} x_{n,\beta+1}}) f_k f_{1,\beta+1}^- \\
\end{align*}
$$

and with (3.43) one at a time. The first one is

$$
\begin{align*}
\{f_k \cdot f_{1,\beta+1}, f_m \cdot f_{1,\beta+1}\}_{\alpha}\beta &= (f_{1,\beta+1}^+)^2 \{f_k, f_m\}_{\alpha}\beta + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\alpha}\beta \\
&\quad + f_m f_{1,\beta+1} \{f_k, f_{1,\beta+1}\}_{\alpha}\beta \\
&= (f_{1,\beta+1}^+)^2 \{f_k, f_m\}_{\text{std}} + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\text{std}} \\
&\quad - f_k f_{1,\beta+1} f_{1,\beta+1} f_m + f_m f_{1,\beta+1} \{f_k, f_{1,\beta+1}\}_{\text{std}} \\
&\quad + f_m f_{1,\beta+1} f_k f_{1,\beta+1} \\
&= \omega_{f_k f_m + f_k f_{1,\beta+1} + f_m f_{1,\beta+1}} (f_{1,\beta+1}^+)^2 f_k f_m \\
&\quad + f_k f_{1,\beta+1} f_m f_{1,\beta+1} - f_k f_{1,\beta+1} f_{1,\beta+1} f_m. \\
\end{align*}
$$

The second bracket:

$$
\begin{align*}
\{f_k^- \cdot f_{1,\beta+1}^-, f_m \cdot f_{1,\beta+1}\}_{\alpha}\beta &= (f_{1,\beta+1}^+) f_k f_{1,\beta+1} \{f_m, f_{1,\beta+1}\}_{\alpha}\beta \\
&\quad + f_m f_{1,\beta+1} \{f_k, f_{1,\beta+1}\}_{\alpha}\beta \\
&= f_{1,\beta+1}^- f_m \{f_k, f_{1,\beta+1}\}_{\alpha}\beta + f_{1,\beta+1}^- f_{1,\beta+1} \{f_k, f_m\}_{\alpha}\beta \\
&\quad + f_k f_m \{f_{1,\beta+1}, f_{1,\beta+1}\}_{\alpha}\beta + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\alpha}\beta \\
&= f_{1,\beta+1}^- f_m \{f_k, f_{1,\beta+1}\}_{\text{std}} + f_{1,\beta+1}^- f_{1,\beta+1} \{f_k, f_m\}_{\text{std}} \\
&\quad + f_k f_m \{f_{1,\beta+1}, f_{1,\beta+1}\}_{\text{std}} + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\text{std}}, \\
\end{align*}
$$

and with (3.44)–(3.48) and Lemma 13 this is

$$
\begin{align*}
(3.49) \quad \{f_k^- \cdot f_{1,\beta+1}^-, f_m \cdot f_{1,\beta+1}\}_{\alpha}\beta &= 2 f_k f_{1,\beta+1} f_m f_{1,\beta+1} - f_k f_{1,\beta+1} f_{1,\beta+1} f_m, \\
\end{align*}
$$

where

$$
\omega_2 = \omega_{f_k f_m + f_k f_{1,\beta+1} + f_m f_{1,\beta+1}} + 1.
$$

The third one is

$$
\begin{align*}
\{f_k \cdot f_{1,\beta+1}, f_m \cdot f_{1,\beta+1}\}_{\alpha}\beta &= f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_{1,\beta+1}\}_{\alpha}\beta + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\alpha}\beta \\
&\quad + f_{1,\beta+1} f_{1,\beta+1} \{f_k, f_{1,\beta+1}\}_{\alpha}\beta + f_{1,\beta+1} f_{1,\beta+1} \{f_k, f_m\}_{\alpha}\beta \\
&= f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_{1,\beta+1}\}_{\text{std}} + f_k f_{1,\beta+1} \{f_{1,\beta+1}, f_m\}_{\text{std}} \\
&\quad + f_{1,\beta+1} f_{1,\beta+1} \{f_k, f_{1,\beta+1}\}_{\text{std}} + f_{1,\beta+1} f_{1,\beta+1} \{f_k, f_m\}_{\text{std}}
\end{align*}
$$

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and with Lemma 13 it makes

\[ \{f_k \cdot f_{1,\beta+1}, f_m^- \cdot f_{1,\beta+1}^-\}_{\alpha\beta} = \omega_3 \ \ f_k f_{1,\beta+1} f_m^- f_{1,\beta+1}^- , \]

with

\[ \omega_3 = \omega_{f_k,f_m} + \omega_{f_k,f_{1,\beta+1}} + \omega_{f_{1,\beta+1},f_m} . \]

The last bracket is

\[
\begin{align*}
\{f_k^- \cdot f_{1,\beta+1}^- , f_j^- \cdot f_{1,\beta+1}^-\}_{\alpha\beta} &= f_k^- f_{1,\beta+1}^- \{f_{1,\beta+1}^- , f_j^-\}_{\alpha\beta} + f_{1,\beta+1}^- f_j^- \{f_k^- , f_{1,\beta+1}^-\}_{\alpha\beta} \\
&\quad + f_j^- f_{1,\beta+1}^- \{f_k^- , f_{1,\beta+1}^-\}_{\alpha\beta} + f_{1,\beta+1}^- f_j^- \{f_k^- , f_{1,\beta+1}^-\}_{\alpha\beta}
\end{align*}
\]

and again, Lemma 13 turns it to

\[ \{f_k^- \cdot f_{1,\beta+1}^- , f_j^- \cdot f_{1,\beta+1}^-\}_{\alpha\beta} = \omega_4 f_k^- \cdot f_{1,\beta+1}^- , f_j^- \cdot f_{1,\beta+1}^- , \]

with

\[ \omega_4 = \omega_{f_k,f_m} + \omega_{f_k,f_{1,\beta+1}} + \omega_{f_{1,\beta+1},f_m} . \]

Summing (3.49)–(3.52) proves that

\[ \{\theta_k,\theta_m\}_{\alpha\beta} = (\omega_{f_k,f_m} + \omega_{f_k,f_{1,\beta+1}} + \omega_{f_{1,\beta+1},f_m}) \theta_k \theta_m . \]

Last, we check that every pair \(\theta_k,\psi_m\) is log canonical w.r.t. \(\{\cdot,\cdot\}_{\alpha\beta}\). The process is pretty much like the one for \(\theta_k\) and \(\theta_m\): break the two functions into their components,

\[ \theta_k = f_{n+k-a,k} f_{1,\beta+1}^- - f_{n+k-a,k} f_{1,\beta+1}^- \]

and

\[ \psi_m = f_{m,n+m-\beta} f_{\alpha+1,1}^- - f_{m,n+m-\beta} f_{\alpha+1,1}^- . \]

Then compute all brackets of these components. Setting

\[
\begin{align*}
\omega_{\theta_k,\psi_m} &= \omega_{f_{n+k-a,k} f_{m,n+m-\beta} , f_{n+k-a,k} f_{\alpha+1,1}^-} + \omega_{f_{n+k-a,k} f_{m,n+m-\beta} , f_{\alpha+1,1}^-} \\
&\quad + \omega_{f_{1,\beta+1} , f_{m,n+m-\beta} f_{\alpha+1,1}^-} + \omega_{f_{1,\beta+1} , f_{n+k-a,k} f_{\alpha+1,1}^-} .
\end{align*}
\]

the result is

\[ \{\theta_k,\psi_m\}_{\alpha\beta} = \omega_{\theta_k,\psi_m} \theta_k \psi_m , \]

The other possible combinations are symmetric (e.g., \(\psi_k,\psi_m\) is symmetric to \(\{\theta_k,\theta_m\}\)).

4. The cluster structure \(C_{\alpha\beta}\)

4.1. Stable variables. Recall the set of functions \(\{F_{ij}\}\) defined in Section 3.1. Look at the set \(S = \{F_{i1}, F_{1j} | i \neq \alpha + 1, j \neq \beta + 1\}\). This is the set of determinants of full matrices \(M \in \mathcal{M}\). Let \(\bar{S}\) be the projections of these functions on the diagonal subgroup.

Though the following proposition is not required for the proof of the main theorem, it does give further information about the cluster structure: the set \(\bar{S}\) will be the set of stable variables. As indicated in [12], in all known cluster structures on Poisson varieties, the frozen variables have two important properties: they behave well under certain natural group actions, and they are log canonical with certain globally
defined coordinate functions. Proposition 14 states that these two properties hold in our case, and therefore supports the choice of \( \tilde{S} \) as the set of stable variables.

**Proposition 14.** 1. The elements of \( S \) are semi-invariants of the left and right action of \( D_- \) in \( D(\text{GL}_n) \).
2. The elements of \( \tilde{S} \) are log canonical with all matrix entries \( x_{ij} \).

**Proof.** 1. The subgroup \( D_- \) of \( D(\text{GL}_n) \) that corresponds to the subalgebra \( g_- \) of \( g \) is given by

\[
D_- = (U, L)
\]

with

\[
U = \begin{bmatrix}
a_1 & * & * & * & * & * \\
0 & \ddots & * & * & * & * \\
0 & 0 & a_{\alpha-1} & * & * & * \\
0 & 0 & 0 & A & * & * \\
0 & 0 & 0 & 0 & a_{\alpha+2} & * \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{bmatrix}
\]

and

\[
L = \begin{bmatrix}
da_{n+\beta+1} & 0 & 0 & 0 & 0 & 0 \\
* & \ddots & 0 & 0 & 0 & 0 \\
* & * & a_n & 0 & 0 & 0 \\
* & * & * & \ddots & 0 & 0 \\
* & * & * & * & a_{\alpha-1} & 0 \\
* & * & * & * & * & \ddots
\end{bmatrix}
\]

where \( A \in \text{GL}_2 \) and the indices of the diagonal entries \( a_i \) are taken modulo \( n \). The \(*\)'s will not play any role in further computations. The left and right action of \( D_- \) can be parametrized by

\[
(X, Y) \mapsto (A_1 X A_1', A_2 Y A_2')
\]

with matrices

\[
A_1 = \begin{bmatrix}
a_1 & * & * & * & * & * \\
0 & \ddots & * & * & * & * \\
0 & 0 & a_{\alpha-1} & * & * & * \\
0 & 0 & 0 & A & * & * \\
0 & 0 & 0 & 0 & a_{\alpha+2} & * \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{bmatrix}
\]

and

\[
A_1' = \begin{bmatrix}
a_1' & * & * & * & * & * \\
0 & \ddots & * & * & * & * \\
0 & 0 & a_{\alpha-1}' & * & * & * \\
0 & 0 & 0 & A' & * & * \\
0 & 0 & 0 & 0 & a_{\alpha+2}' & * \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{bmatrix}
\]
There are three kinds of functions in $S$: minors of $X$, minors of $Y$ and "mixed" functions. A function $f_X \in S$ that is a minor of $X$ is a semi-invariant of this action: $f_X$ is the determinant of a submatrix $X_{i...n}^{[1..\mu]}$ with $\mu = n - i + 1$. One has $i \notin \{\alpha + 1, n - \alpha + 1\}$ (see the construction in Section 3.1). The action of $D_-$ multiplies each row $k \in \{i, \ldots, n\}$ of $X$ by the corresponding entry $a_k$ and each column $\ell \in \{1, \ldots, \mu\}$ of $X$ by $a'_{\ell + n + \alpha - \beta - 1}$, with two exceptions: rows $\alpha$ and $\alpha + 1$ are multiplied together by $A$, and columns $\alpha$ and $\alpha + 1$ are multiplied by $A'$. So as long as one of these rows (or columns) does not occur in the submatrix $X_{i...n}^{[1..\mu]}$ without the other, $f_X$ is still a semi-invariant of the action. If $\alpha \in \{i, \ldots, n\}$ then clearly $\alpha + 1 \in \{i, \ldots, n\}$. On the other hand, if the only case with $\alpha + 1 \in \{i, \ldots, n\}$ and $\alpha \notin \{i, \ldots, n\}$ is when $i = \alpha + 1$. But this is not the case, since this minor is not in $S$. Looking at columns, it is easy to see that if $\alpha + 1 \in \{1, \ldots, \mu\}$ (that is, the column $\alpha + 1$ occurs in the submatrix $X_{i...n}^{[1..\mu]}$), then $\alpha \in \{1, \ldots, \mu\}$. The only way to have $\alpha \in \{1, \ldots, \mu\}$ and $\alpha + 1 \notin \{1, \ldots, \mu\}$ is $\mu = \alpha$. But this implies $i = n - \alpha + 1$, and this minor is also not in the set $S$.

The case of $f_Y \in S$ which is a minor of $Y$ is symmetric.

Look now at the function

$$
\theta = \det \begin{bmatrix}
x_{i1} & \cdots & x_{i\alpha} & x_{i,\alpha + 1} & 0 & \cdots \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\
x_{n1} & \cdots & x_{n\alpha} & x_{n,\alpha + 1} & 0 & \cdots \\
0 & \cdots & y_{1\beta} & y_{1,\beta + 1} & \cdots & y_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & y_{\mu\beta} & \cdots & \cdots & y_{\mu n}
\end{bmatrix},
$$

with $i = n + 1 - \alpha$. It is not hard to see that $\theta$ is a semi invariant of the action of $D_-$: the block of $x_{ij}$'s is subject to the same arguments as above, except for when $\alpha = \frac{n}{2}$, which will be treated later. The same holds for the block of $y_{ij}$'s, unless
\( \beta = \frac{n}{2} \). Therefore \( \theta \) is a semi-invariant of this action.

Symmetric arguments show that

\[
\psi = \det \begin{bmatrix}
y_{1,n+1-\beta} & \cdots & y_{1n} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
y_{\beta j} & \cdots & y_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\
y_{\beta+1,j} & \cdots & y_{\beta+1,n} & x_{\alpha+1,1} & \cdots & \vdots \\
0 & \cdots & 0 & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

is a semi-invariant.

Last, we look at the special case \( \alpha = \frac{n}{2} \): here there is only one matrix in \( \mathcal{M} \) with elements of both \( X \) and \( Y \). The “building blocks” of this matrix are submatrices of \( X \) and \( Y \) that satisfy the restrictions above, so the determinant of this matrix is also a semi-invariant of the action.

The case \( \beta = \frac{n}{2} \) is symmetric.

2. First, look at a function \( \varphi \in \tilde{S} \cap \mathcal{B}_{st} \). In this case it is not hard to see that \( \{ \varphi, x_{ij} \}_{\alpha \beta} = \{ \varphi, x_{ij} \}_{st} \) (according to Lemma \( \ref{lem9} \)) and therefore \( \varphi \) is log canonical with all \( x_{ij} \). There are only two other functions in \( \tilde{S} \): \( \theta = \varphi_{n-\alpha+1,1} \) and \( \psi = \varphi_{1,n-\beta+1} \). Start with \( \theta = f_{n-\alpha+1,1} f_{1,\beta+1} - f_{n-\alpha+1,1} f_{1,\beta+1} \). Following the line of the proof of Lemma \( \ref{lem13} \) it can be shown that \( f_{n-\alpha+1,1} \) is log canonical with every \( x_{ij} \) with \( j \neq \alpha \), and similarly \( f_{1,\beta+1} \) is log canonical with every \( x_{ij} \) with \( j \neq \beta + 1 \), with respect to the standard bracket. The cases \( j = \alpha \) and \( j = \beta + 1 \) are exactly the cases when the standard bracket and the \( \alpha \beta \) bracket do not coincide. Adding the difference that was described in Lemma \( \ref{lem9} \)

\[
\{ f_{n-\alpha+1,1}, x_{ij} \}_{\alpha \beta} = \begin{cases} 
\{ f_{n-\alpha+1,1}, x_{ij} \}_{st} & \text{if } j \neq \beta + 1 \\
\{ f_{n-\alpha+1,1}, x_{ij} \}_{st} + f_{n-\alpha+1,1} x_{\alpha j} & \text{if } j = \beta + 1,
\end{cases}
\]

and

\[
\{ f_{1,\beta+1}, x_{ij} \}_{\alpha \beta} = \begin{cases} 
\{ f_{1,\beta+1}, x_{ij} \}_{st} & \text{if } j \neq \alpha \\
\{ f_{1,\beta+1}, x_{ij} \}_{st} + f_{1,\beta+1} x_{i,\alpha + 1} & \text{if } j = \alpha,
\end{cases}
\]

shows that with respect to the bracket \( \{ \cdot, \cdot \}_{\alpha \beta} \), the pairs \( f_{n-\alpha+1,1} x_{\alpha i} \) and \( f_{1,\beta+1} x_{i,\beta+1} \) are log canonical. The coefficients \( \omega f_{n-\alpha+1,1}, x_{ij} = \frac{f_{n-\alpha+1,1}, x_{ij}}{f_{n-\alpha+1,1}} \) and \( \omega f_{1,\beta+1}, x_{ij} = \frac{f_{1,\beta+1}, x_{ij}}{f_{1,\beta+1}} \) can be computed like in the proof of Lemma \( \ref{lem13} \), showing that \( \theta \) is log canonical with \( x_{ij} \). Symmetric arguments hold for \( \psi \).

Note that a stable variable (an element of \( \tilde{S} \)) is a determinant of a matrix constructed from either \( X_{[1\ldots \mu (1,1)]} \) with \( i - 1 \neq \alpha \) or \( Y_{[j\ldots n] [1\ldots \mu (j,1)]} \) with \( j - 1 \neq \beta \). There are \( 2(n-1) - 2 \) such matrices, hence

\[
|\tilde{S}| = 2(n-1) - 2 = 2|\Delta \setminus \Gamma_1|,
\]

as in Statement \( \ref{stat1} \) of Conjecture \( \ref{conj3} \).
4.2. The quiver $Q_{\alpha\beta}$. To describe the quiver $Q^n_{\alpha\beta}$, start with the standard quiver $Q^n_{std}$ as given in Section 3.3. The vertex in the $i$-th row and $j$-th column corresponds to the cluster variable $f_{ij} = \det X_{\mu(i,j)}^{[\mu(i,j)]}$. The quiver of the exotic cluster structure with BD data $\alpha \rightarrow \beta$ is very close: a vertex $(i,j)$ now represents the cluster variable $\varphi_{ij}$, and the quiver takes these changes:

1. Vertices $(\alpha + 1, 1)$ and $(1, \beta + 1)$ are not frozen.
2. The arrows $(\alpha, 1) \rightarrow (\alpha + 1, 1)$ and $(1, \beta) \rightarrow (1, \beta + 1)$ are added.
3. The arrows $(n, \alpha + 1) \rightarrow (1, \beta + 1)$, $(1, \beta + 1) \rightarrow (n, \alpha)$ are added.
4. The arrows $(\beta + 1, n) \rightarrow (\alpha + 1, 1)$, $(\alpha + 1, 1) \rightarrow (\beta, n)$ are added.

The example of $1 \rightarrow 2$ on $SL_5$ is given in Figure 4.1.

Through this section we use $B, \omega, \Omega$ to denote the exchange matrix, Poisson coefficients and Poisson matrix in the standard case, and $\overline{B}, \overline{\omega}, \overline{\Omega}$ for their counterparts in the $\alpha \rightarrow \beta$ case. We now prove that the cluster structure described in the previous section is indeed compatible with the bracket $\{\cdot, \cdot\}_{\alpha\beta}$.

**Theorem 15.** The cluster structure $C(B_{\alpha\beta}, \overline{B})$ is compatible with the bracket $\{\cdot, \cdot\}_{\alpha\beta}$. That is,

$$\overline{B}\overline{\Omega} = [I \ 0].$$

**Proof.** Equation (4.1) can be rephrased as

$$\overline{(B\Omega)}_{ij} = \sum_{i \leftarrow k} \omega_{kj} - \sum_{i \rightarrow k} \omega_{kj} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

where the first sum is over vertices $k$ with an arrow pointing from $k$ to $i$, and the second sum is over vertices $k$ with an arrow pointing from $i$ to $k$. Recall that the standard case has

$$\sum_{i \leftarrow k} \omega_{kj} - \sum_{i \rightarrow k} \omega_{kj} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$
1. The $i$-th row of $\overline{B}$ is equal to the $i$-th row of $B$. This is true for almost all rows of $B$, or more precisely when

\begin{equation}
\phi_{1,\beta+1}, \phi_{\alpha\alpha}, \phi_{\alpha,\alpha+1}, \phi_{\alpha+1,1}, \phi_{\beta\beta}, \phi_{\beta+1,\alpha},
\end{equation}

and we have these possible situations:

(a) $i$ corresponds to a cluster variable in $B_S \cap B_{\alpha\beta}$.

i. Assume all cluster variables adjacent to $i$ are in $B_S \cap B_{\alpha\beta}$. If $j$ is also in $B_S \cap B_{\alpha\beta}$, this is just the same as the standard case, and $(B\Omega)_{ij} = (B\Omega)_{ij}$. If $j$ is not a standard basis function, then it is either of the form $j = \phi_{n+k-\alpha,k}$ and then

\[ \omega_{kj} = \omega_{kj} + \omega_{k,f_{n,a+1}} - \omega_{k,f_a} \]

or $j = \phi_{k,n+k-\beta}$, and

\[ \omega_{kj} = \omega_{kj} + \omega_{k,f_{b,n+1}} - \omega_{k,f_{\beta,n}}. \]

So in the first case,

\[ \sum_{i \rightarrow k} \omega_{kj} - \sum_{i \leftarrow k} \omega_{kj} = \sum_{i \rightarrow k} \omega_{kj} - \sum_{i \rightarrow k} \omega_{kj} + \sum_{i \rightarrow k} \omega_{k,f_{n,a+1}} - \sum_{i \rightarrow k} \omega_{k,f_{b,a}} + \sum_{i \rightarrow k} \omega_{k,f_{a}} \]

and since $i \neq \phi_{\alpha\alpha}, \phi_{\alpha,\alpha+1}$ (from (4.2)), the standard case tells us

\[ \sum_{i \rightarrow k} \omega_{k,f_{n,a+1}} - \sum_{i \rightarrow k} \omega_{k,f_{b,a+1}} = \sum_{i \rightarrow k} \omega_{k,f_{a}} - \sum_{i \rightarrow k} \omega_{k,f_{a}} = 0. \]

The case $j = \phi_{k,n+k-\beta}$ is symmetric.

ii. The cluster variable $i$ has at least one neighbor that is not in $B_S \cap B_{\alpha\beta}$.

A. $i$ has exactly one such neighbor. Looking closely at the quiver, there are only two such vertices: $i = (n, \alpha + 1)$ or $i = (\beta + 1, n)$. In both cases it means that the $i$-th row of $\overline{B}$ is different from that row of $B$, because the quiver $Q_{\alpha\beta}$ has arrows $(n, \alpha + 1) \rightarrow (1, \beta + 1)$ and $(\beta + 1, n) \rightarrow (\alpha + 1, 1)$, which $Q_{std}$ does not have. These cases will be handled later on.

B. $i$ has two such neighbors. In this case these two neighbors are connected to $i$ by arrows in opposite directions (i.e., one of them is pointing at $i$ and the other one from $i$). These two “non standard” neighbors must both belong to the same “family” of functions, either $\{\theta_k\}$ or $\{\psi_k\}$. We have seen that the Poisson coefficients of these function differ from their standard counterparts by a constant, e.g., for every function $g \in B_{std},$

\[ \omega_{\theta_k,g} = \omega_{\phi_{n+k-\alpha,k}g} = \omega_{f_{n+k-\alpha,k}g} + \omega_{f_{1,\beta+1}g}. \]

When summing over all neighbors of $i$, this constant is then added once, for the vertex with an arrow pointing at $i$, and subtracted once, for the vertex with an arrow pointing from $i$ to it. These cancel each other and the sum remains as it was in the standard case.

(b) $i$ is not in $B_S \cap B_{\alpha\beta}$, which means $i = \phi_{n+k-\alpha,k}$ or $i = \phi_{k,n+k-\beta}$. Assume $k < \alpha$ (for the first one) or $k < \beta$ (second), because $i = \phi_{\alpha\alpha}$ and $i = \phi_{\beta\beta}$ are in (4.2) and will be treated later. If $k = 1$ it is a frozen variable. Again, look at the first case (second is just the same): if $1 < k < \alpha$ then two neighbors of $i = \phi_{n+k-\alpha,k}$ are
“non standard”. These are \( \varphi_{n+k+1-\alpha,k+1} \) and \( \varphi_{n+k+1-\alpha,k-1} \) with edges pointing in opposite directions. Since we know that
\[
\mathbb{B} \varphi_{n+k-\alpha,k-1} = \omega f_{n+k-\alpha,k,j} + \omega f_{1,\beta+1,\beta+1},
\]
a constant is added to the sum for the vertex \( \varphi_{n+k+1-\alpha,k+1} \) and then subtracted for the vertex \( \varphi_{n+k-1-\alpha,k-1} \). In addition this constant is added to all \( \omega \)'s in the sum, and they cancel each other.

2. Here the \( i \)-th row of \( \overline{B} \) is different than the \( i \)-th row of \( B \).

(a) If \( i = \varphi_{1,\beta+1} \) then \( B \) does not have this row (it was a frozen variable). Its neighbors are now \( \varphi \): \( \varphi_{n,\alpha+1}, \varphi_{2,\beta+2}, \varphi_{1,\beta} \) with arrows pointing to \( i \) and \( \varphi_{2,\beta+1}, \varphi_{n \alpha} \) with arrows from \( i \) to them. So we have
\[
\sum_{i=1}^{k} \omega_{k_j} - \sum_{i=1}^{k} \omega_{k_j} = \omega \varphi_{1,\beta}, \varphi_j + \omega \varphi_{n,\alpha+1}, \varphi_j + \omega \varphi_{2,\beta+2}, \varphi_j - \omega \varphi_{2,\beta+1}, \varphi_j - \omega \varphi_{n,\alpha}, \varphi_j = \begin{cases} 
1 & j = (2, \beta+1) \\
0 & j \neq (2, \beta+1) 
\end{cases}
\]
and we continue, using standard exchange relation at \((i, \beta+1)\)
\[
\omega f_{i-1,\beta,j} + \omega f_{i,\beta+1,j} + \omega f_{i+1,\beta+1,j} \\
- \omega f_{i-1,\beta+1,j} - \omega f_{i,\beta,j} - \omega f_{i+1,\beta+2,j}
\]
and assuming \( j \neq (i, \beta+1) \)
\[
\omega f_{n,\beta,j} + \omega f_{n-1,\beta+1,j} - \omega f_{n-1,\beta,j} - \omega f_{n,\beta+2,j} = \begin{cases} 
1 & j = (n, \beta+1) \\
0 & j \neq (n, \beta+1) 
\end{cases}
\]
The standard exchange relation at \((n, \beta+1)\) implies
\[
\omega f_{n,\beta,j} + \omega f_{n-1,\beta+1,j} - \omega f_{n-1,\beta,j} - \omega f_{n,\beta+2,j} = 0
\]
or
\[
\omega f_{n,\beta,j} = \omega f_{n,\beta+2,j} - \omega f_{n-1,\beta+1,j} + \omega f_{n-1,\beta,j}.
\]
So
\[
\omega f_{n,\beta,j} - \omega f_{n,\beta+1,j} = \omega f_{n-2,\beta,j} + \omega f_{n-1,\beta+2,j} - \omega f_{n-1,\beta+1,j} - \omega f_{n-2,\beta+1,j},
\]
and using (4.3) recursively
\[
\omega f_{n,\beta,j} - \omega f_{n,\beta+1,j} = \omega f_{n-2,\beta,j} + \omega f_{n-1,\beta+2,j} - \omega f_{n-1,\beta+1,j} - \omega f_{n-2,\beta+1,j}.
\]
Now we only need \( \omega f_{n,\alpha+1,f_j} - \omega f_{n,\alpha,f_j} = \omega f_{n,\beta+1,f_j} - \omega f_{n,\beta,f_j} \). This is true from Lemma 13 and the assumption \( j \neq (i, \beta + 1) \), so \( \sum_{i \rightarrow k} \omega k_j - \sum_{i \leftarrow k} \omega k_j = 0, \forall j \neq (i, \beta + 1) \).

If, on the other hand, \( j = (1, \beta + 1) \), this still holds, but Lemma 13 now says 
\( \omega f_{n,\alpha+1,f_j} - \omega f_{n,\alpha,f_j} = \omega f_{n,\beta+1,f_j} - \omega f_{n,\beta,f_j} + 1 \), so that
\[
\sum_{i \rightarrow k} \omega k_j - \sum_{i \leftarrow k} \omega k_j = 1.
\]

Last, let \( j = (i, \beta + 1) \) with \( i > 1 \). So in \( \sum_{i \leftarrow k} \omega k_j \) we need to add 1 to the right hand side. This 1 is then added to the sum of coefficients over neighbors of \( (1, \beta + 1) \), but now according to Lemma 13 \( \omega f_{n,\alpha+1,f_j} - \omega f_{n,\alpha,f_j} = \omega f_{n,\beta+1,f_j} - \omega f_{n,\beta,f_j} + 1 \), so again
\[
\sum_{i \rightarrow k} \omega k_j - \sum_{i \leftarrow k} \omega k_j = 0.
\]

The special case \( \beta = n - 1 \) is different, because here vertices \( (i, \beta + 1) = (i, n) \) do not have neighbors on the right. However, the same arguments still hold, and since the exchange relations in the standard quiver are similar, the final conclusion is identical.

(b) Let \( i = (n, \alpha) \) then in the standard quiver its neighbors were \( (n, \alpha + 1) \) and \( (n-1, \alpha - 1) \) (with arrows from \( i \) to them), and \( (n, \alpha - 1) \) and \( (n-1, \alpha) \) (with arrows pointing to \( i \)). In \( Q_{\alpha \beta} \) an arrow is added from \( (1, \beta + 1) \) to \( i \). Let \( \theta_{\alpha - 1} = \varphi_{n-1,\alpha-1} \) be the function associated with the vertex \( (n-1, \alpha-1) \) in the quiver \( Q_{\alpha \beta} \), that is \( \theta_{\alpha - 1} = f_{n-1,\alpha-1} \cdot f_{1,\beta+1} - f_{n-1,\alpha-1} \cdot f_{1,\beta+1} \). So using \( \omega \theta_{n-1,g} = \omega f_{n-1,\alpha-1,g} + \omega f_{1,\beta+1,g} \) we write
\[
\sum_{i \leftarrow k} \omega k_j - \sum_{i \rightarrow k} \omega k_j = \omega \varphi_{n,\alpha-1,\beta+1} + \omega \varphi_{n-1,\alpha,\beta+1} + \omega \varphi_{n,\beta+1,\alpha} + \omega \varphi_{n-1,\alpha-1,\beta+1}
\]
and the last term is the one from the standard case, which equals \( \delta_{ij} \).

(c) \( i = \varphi_{n,\alpha+1} \). In the standard quiver there are arrows from \( i \) to \( (n, \alpha) \) and \( (n-1, \alpha + 1) \) and from \( (n, \alpha + 2) \) and \( (n-1, \alpha) \) to \( i \). In \( Q_{\alpha \beta} \) there is a new arrow \( (n, \alpha + 1) \rightarrow (1, \beta + 1) \). Again,
\[
\sum_{i \leftarrow k} \omega k_j - \sum_{i \rightarrow k} \omega k_j = \omega \varphi_{n,\alpha+2} + \omega \varphi_{n-1,\alpha} + \omega \varphi_{1,\beta+1}
\]
which is also equal to the standard.
Note that an immediate corollary from Theorem 15 is that the exchange matrix $B$ is of maximal rank, since $\text{rank}(B \Omega) \leq \min(\text{rank} B, \text{rank} \Omega)$, and (4.1) implies that $B \Omega$ has maximal rank.

5. Regularity

To prove that the cluster structure is regular we need the following Proposition, which is a weaker analogue of Proposition 3.37 in [10]:

**Proposition 16.** Let $V$ be a Zariski open subset in $\mathbb{C}^{n+m}$ and $(C = C \left( \hat{B} \right), \varphi)$ be a cluster structure in $C(V)$ with $n$ cluster and $m$ stable variables such that

1. $\text{rank} \hat{B} = n$;
2. there exists an extended cluster $\tilde{x} = (x_1, \ldots, x_{n+m})$ in $C$ such that $\varphi(x_i)$ is regular on $V$ for $i \in [n+m]$;
3. for any cluster variable $x'_k$, $k \in [n]$, obtained by applying the exchange relation (2.1) to $\tilde{x}$, $\varphi(x'_k)$ is regular on $V$;
   Then $C$ is a regular cluster structure. If additionally
4. for any stable variable $x_{n+i}$, $i \in [m]$, the function $\varphi(x_{n+i})$ vanishes at some point of $V$;
5. each regular function on $V$ belongs to $\varphi(\mathcal{A}_C(C))$.

Then $\mathcal{A}_C(C)$ is naturally isomorphic to $\mathcal{O}(V)$.

So proving that our cluster structure is regular reduces to proving the next theorem:

**Theorem 17.** For every exchangeable variable $f$ in the initial cluster, the exchanged variable $f'$ is a regular function.

**Proof.** We can use the similarity of the exchange quivers $Q_{\alpha \beta}$ and $Q_{\text{std}}$. The exchange relation (2.1) involves the cluster variable $f$ and its neighbors (i.e., cluster variables connected to $f$ by an arrow) in the exchange quiver.

Recall the following notation:

$$
\varphi_{ij} = \begin{cases} 
\theta_j & j \in \{1, \ldots, \alpha\} \text{ and } i = n + j - \alpha \\
\psi_i & i \in \{1, \ldots, \beta\} \text{ and } j = n + i - \beta \\
f_{ij} & \text{otherwise},
\end{cases}
$$

with

$$
f_{ij} = \det X_{[i, \ldots, \mu(i,j)]}^{[j, \ldots, \mu(i,j)]},
\theta_j = f_{n+j-\alpha,j} f_{1,\beta+1} - f_{n+j-\alpha,j} f_{1,\beta+1},
\psi_i = f_{i,n+i-g} f_{\alpha+1,1} - f_{i,n+i-g} f_{\alpha+1,1}.
$$

Consider the following cases:

1. $f$ is in $B_{\alpha \beta} \cap B_{\text{std}}$ and all its neighbors are also in $B_{\alpha \beta} \cap B_{\text{std}}$. This means the exchange rule is the same as in the standard case, and therefore the exchanged cluster variable is equal to the one in the standard case, which is regular.

2. $f$ is in $B_{\alpha \beta} \cap B_{\text{std}}$, but at least one of its neighbors is not in $B_{\alpha \beta} \cap B_{\text{std}}$.
   (a) Two neighbors of $f$ are not in $B_{\alpha \beta} \cap B_{\text{std}}$.
   Looking at the quiver as described in Section 4.2, it is clear that the two nonstandard neighbors of $f$ are $\varphi_{ij}$ and $\varphi_{i+1,j+1}$ (for some $i, j$) with arrows pointing...
at opposite directions (e.g., from $f$ to $\varphi_{ij}$ and from $\varphi_{i+1,j+1}$ to $f$). The exchange rule is now $f \cdot f' = \varphi_{ij} : p_1 + \varphi_{i+1,j+1} : p_2$ where $p_1, p_2$ are some monomials. Now recall that $\varphi_{ij} = f_{ij} h - \tilde{f}_{ij} \tilde{h}$ where $\tilde{f}_{ij}$ is either $(f_{ij})^{\alpha+1}$ or $(f_{ij})^{\beta+1}$, and $\tilde{h}$ is $h^{\beta+1+\alpha}$ or $h_{\alpha+1+\beta}$, respectively. The exchange rule is then

$$f \cdot f' = (f_{ij} h - \tilde{f}_{ij} \tilde{h}) p_1 + (f_{i+1,j+1} h - \tilde{f}_{i+1,j+1} \tilde{h}) p_2$$

the first part is just the standard exchange rule multiplied by $h$, so it is divisible by $f$. The term in the second parenthesis can be regarded as a Desnanot-Jacobi identity. It is equal to the standard one with just one change: the last column that was $\alpha$ in the standard case is now replaced by $\alpha+1$. Clearly, this also produces a product of $f$ and some other polynomial. Dividing by $f$ shows that $f'$ is a regular function.

(b) Only one neighbor of $f$ is not in $\mathcal{B}_{\alpha \beta} \cap \mathcal{B}_{\text{std}}$. There are only two such vertices: $\varphi_{n,\alpha+1}$ and $\varphi_{\beta+1,n}$. The vertex $\varphi_{n,\alpha+1} = x_{n,\alpha+1}$ has neighbors $\varphi_{n,\alpha}, \varphi_{n-1,\alpha}, \varphi_{n-1,\alpha+1}, \varphi_{n,\alpha+1}$ and $\varphi_{1,\beta+1}$. Figure 5.1 shows the relevant subquiver of $Q_{\alpha \beta}$. Since $\varphi_{n,\alpha} = x_{n,\alpha} f_{1,\beta+1} - x_{n,\alpha} f_{1,\beta+1}^{\beta+1-\alpha}$, we have

$$\varphi_{n,\alpha+1} \cdot \varphi'_{n,\alpha+1} = \varphi_{n,\alpha} \varphi_{n-1,\alpha+1} + \varphi_{n-1,\alpha} \varphi_{n,\alpha+2} \varphi_{1,\beta+1}$$

$$= f_{1,\beta+1} (x_{n,\alpha} f_{n-1,\alpha+1} + f_{n-1,\alpha} f_{n,\alpha+2}) - x_{n,\alpha+1} f_{1,\beta+1}^{\beta+1-\alpha} f_{n-1,\alpha+1}.$$

The term in parenthesis is the exchange rule in the standard case, so it is the product of $x_{n,\alpha+1}$ and some other regular function, and the second term is clearly divisible by $x_{n,\alpha+1}$. Therefore the exchanged variable is regular. The same arguments hold for the vertex $f_{\beta+1,n}$.
3. \( f \) is not in \( B_{\alpha\beta} \cap B_{\text{std}} \).

(a) \( f \) is either \( \varphi_{n,\alpha} \) or \( \varphi_{1,\beta} \).

Assume \( f = \varphi_{n,\alpha} = x_{n,\alpha} f_{1,\beta+1} - x_{n,\alpha+1} f_{1,\beta+1} \). Note that \( \alpha > 1 \), because for \( \alpha = 1 \) the variable \( \varphi_{n,1} \) must be frozen. The adjacent vertices are \( \varphi_{n+1,\alpha}, \varphi_{n-1,\alpha-1}, \varphi_{1,\beta+1} \) where \( \varphi_{n-1,\alpha-1} = f_{n-1,\alpha-1} f_{1,\beta+1} - f_{n-1,\alpha-1} f_{1,\beta+1} \), as shown in Figure 5.2. The exchange rule is

\[
\varphi_{n,\alpha} \cdot \varphi'_{n,\alpha} = x_{n,\alpha} f_{1,\beta+1} - x_{n,\alpha+1} f_{1,\beta+1} + x_{n,\alpha} f_{n-1,\alpha-1} f_{1,\beta+1} - x_{n,\alpha+1} f_{n-1,\alpha-1} f_{1,\beta+1} = f_{1,\beta+1} (x_{n,\alpha} f_{n-1,\alpha-1} + x_{n,\alpha+1} f_{n-1,\alpha-1}) - x_{n,\alpha} f_{n-1,\alpha} f_{1,\beta+1} + x_{n,\alpha+1} f_{n-1,\alpha-1} f_{1,\beta+1}
\]

and the term in parenthesis is just the standard exchange rule, which equals \( x_{n,\alpha} \cdot f_{n-1,\alpha-1} \). Therefore,

\[
\varphi_{n,\alpha} \cdot \varphi'_{n,\alpha} = (x_{n,\alpha} f_{1,\beta+1} - x_{n,\alpha+1} f_{1,\beta+1}) f_{n-1,\alpha-1} = \varphi_{n,\alpha} f_{n-1,\alpha-1},
\]

and \( \varphi'_{n,\alpha} = f_{n-1,\alpha-1} \) is regular.

Symmetric arguments show that \( \varphi'_{1,\beta} \) is also regular.

(b) \( f \) has two neighbors that are also not in \( B_{\alpha\beta} \cap B_{\text{std}} \).

This happens when \( f = \varphi_{ij} \), and the two non standard neighbors are

\[
\varphi_{i-1,j-1} = f_{i-1,j-1} f_{1,\beta+1} - f_{i-1,j-1} f_{1,\beta+1}, \\
\varphi_{i+1,j+1} = f_{i+1,j+1} f_{1,\beta+1} - f_{i+1,j+1} f_{1,\beta+1}
\]

The other neighbors are identical to those in the standard case. Denote the corresponding standard exchange rule at \( f_{ij} \) by \( e_{f_{ij}} \). This is a Desnanot–Jacobi identity 3.21 or the modified version of it 3.22. Let \( \tilde{e}_{f_{ij}} \) be the same identity with column \( \alpha \) (or row \( \beta \)) replaced by column \( \alpha + 1 \) (or row \( \beta + 1 \), respectively). In other words,
Assume $f = \varphi_{1, \beta+1}$ with neighbors $\varphi_{n, \alpha}, \varphi_{n, \alpha+1}, \varphi_{1, \beta}, \varphi_{2, \beta+1}, \varphi_{2, \beta+2}$ (Figure 5.3). The exchange rule is then

$\varphi_{1, \beta+1} \cdot \varphi'_{1, \beta+1} = \varphi_{n, \alpha} \varphi_{2, \beta+1} + \varphi_{n, \alpha+1} \varphi_{1, \beta} \varphi_{2, \beta+2}$.

If we put

$$A = \begin{bmatrix}
x_{n\alpha} & x_{n, \alpha+1} & 0 & \cdots & 0 \\
x_{1, \beta} & x_{1, \beta+1} & \cdots & \cdots & x_{1n} \\
\vdots & x_{2, \beta+1} & x_{2, \beta+2} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix},$$

then the exchange rule reads

$\varphi_{1, \beta+1} \cdot \varphi'_{1, \beta+1} = \det A \cdot \det A_{1, 2}^{1, n} + \det A_{2}^{1} \cdot \det A_{1}^{n}$.

Using 3.21 this reads

$\varphi_{1, \beta+1} \cdot \varphi'_{1, \beta+1} = \det A_{1}^{1} \cdot \det A_{2}^{n}$,

and since $\varphi_{1, \beta+1} = \det A_{1}^{1}$, we get $\varphi'_{1, \beta+1} = \det A_{2}^{n}$, which is regular. The case of $f = \varphi_{\alpha+1, 1}$ is symmetric.
This competes the proof, since in all cases $f'$ is a regular function. \qed

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