Some Properties of Alpha Interior and Alpha Closure in Double Bi-Topological Space

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Abstract. In this paper we will present a study of a new type of internal points and closing points based on $(\mathcal{D}\eta\eta\alpha$-open set) and $(\mathcal{D}\eta\eta\alpha$-open set) knowledge on a double bi-topological space, with an explanation of the most important characteristics of these concepts.

Keyword. $\mathcal{D}\eta\eta\alpha$-open set, $\mathcal{D}\eta\eta\alpha$-int$_{D}(\tilde{A})$, $\mathcal{D}\eta\eta$-Alpha-Closure, $\mathcal{D}\eta\eta$-Alpha-Interior.

1. Introduction

The first to present the subject of intuitionistic set or double set is the mathematician Coker [5]. In 1998, Coker contemplated the idea of topology [7] around a similar set. After that a great deal of learns about this set were referenced and we notice them. In 2000, a prologue to intuitionistic topological spaces [6] was considered. In 2014, summed up preregular shut sets were characterized by the double set [9]. In the very year, the double set was utilized to characterize the idea of semi open [12]. In 2016, notes were made about semi open double set [8], investigation of the idea of fixed point around double set [10], and meaning of intuitionistic $\beta$-open sets [1]. In 2017, a subject named basic fixed purposes of periodically pitifully viable in intuitionistic fuzzy measurement space was dispatched [11]. In 2019, the topic generalized closed set in intuitionistic fuzzy topology was considered [2]. In 2020, the subject of certain properties of double minimal space [3] was studied. In 2020, the Alpha Set concept is studied over double bi-topological space [4]. In this paper, we will study the concept of an inward and closed point on alpha open set and alpha closed set defined as double bi-topological space.

2. Preliminaries

In this section, we remember some basic concepts related to double sets.

2.1. Definition [5]

The Double set (DS for short) of $\chi \neq \emptyset$ is $\mathcal{A} = \langle A_1, A_2 \rangle$ where $A_1, A_2 \subseteq \chi$ and $A_1 \cap A_2 = \emptyset$.

2.2. Definition [2]

If $\mathcal{A} = \langle A_1, A_2 \rangle$ and $\mathcal{B} = \langle B_1, B_2 \rangle$ are double sets in $\chi$. Then

1. $\mathcal{A} \subseteq_{D} \mathcal{B}$ iff $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$.
2. $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq_{D} \mathcal{B}$ and $\mathcal{B} \subseteq_{D} \mathcal{A}$.
3. \( A \cup_D B = \langle A_1 \cup B_1, A_2 \cap B_2 \rangle \).
4. \( A \cap_D B = \langle A_1 \cap B_1, A_2 \cup B_2 \rangle \).
5. \( \text{cop.} A = \langle A_2, A_1 \rangle \).
6. \( \tilde{X}_D = \langle X, \emptyset \rangle \).
7. \( \tilde{D}_D = \langle \emptyset, X \rangle \).

2.3. Definition [5]
Double point of \( X \neq \emptyset \) is \( \tilde{p}_D = \langle \{p\}, \{p\}^c \rangle \), where \( p \in X \).

2.4. Definition [5]
A collection \( T_D \) of double sets of \( X \) called double topology (DT) if is satisfying the following axioms:
1. \( \tilde{D}_D, \tilde{X}_D \in T_D \).
2. \( A \cap_D B \in T_D \) for each \( \tilde{A}, \tilde{B} \in T_D \).
3. \( \cup_D A_i \in T_D \) for each \( \tilde{A}_i \in T_D \) and \( i \in I \).

the pair \( (X, T_D) \) is called a double topological space (DTS) and any double set belong to \( DT \) called \( D \)-open set. \( \text{cop.} A \) is \( D \)-closed if and only if \( \tilde{A} \) is \( D \)-open.

2.5. Definition [7]
Let \( (X, T_D) \) be an DTS and \( \tilde{A} \) is double set. Then
1. \( \text{int}_D(\tilde{A}) = \cup_D \{ \tilde{G} : \tilde{G} \in T_D \text{ and } \tilde{G} \subseteq_D \tilde{A} \} \).
2. \( \text{cl}_D(\tilde{A}) = \cap_D \{ \tilde{F} : \tilde{F} \text{ is } D \text{-closed set and } \tilde{A} \subseteq_D \tilde{F} \} \).

2.6. Definition [4]
Let \( (X, T_{D1}, T_{D2}) \) be a double bi-topological space. Then a double subset \( \tilde{A} \) of \( X \) is called \( D\eta\alpha \)-open set if \( \tilde{A} \subseteq_D \eta = \text{int}_D(\tilde{A}) \cap \text{cl}_D(\tilde{A}) \), where \( \eta \neq \eta \cdot \eta = 1,2 \). \( \tilde{A} \) is \( D\eta\alpha \)-closed set iff \( \text{cop.} \tilde{A} \) is \( D\eta\alpha \)-closed set.

\( D\eta\alpha \). O\( (X) = \{ \tilde{A} : \tilde{A} \text{ is } D\eta\alpha \text{-open set} \} \).
\( D\eta\alpha \). C\( (X) = \{ \tilde{A} : \tilde{A} \text{ is } D\eta\alpha \text{-closed set} \} \).

3. \( D\eta\alpha-\text{Alpha-Interior Via Double Bi-Topological Space} \)
In this section, we will define the concept of Interior based on \( D\eta\alpha \)-open set and clarify the most important characteristics of this concept.

3.1. Definition
Let \( (X, T_{D1}, T_{D2}) \) be a double bi-topological space, and \( \tilde{A} \) double set of \( X \). A double point \( \tilde{p}_D \) is said to be \( D\eta\alpha \)-interior point of \( A \) if and only if there exists an \( D\eta\alpha \)-open set \( \tilde{B} \) such that \( \tilde{p}_D \in \tilde{B} \subseteq \tilde{A} \), and the set of all \( D\eta\alpha \)-interior points of \( A \) denoted by \( D\eta\alpha-\text{int}_D(\tilde{A}) \).

3.2. Remark
\( D\eta\alpha-\text{int}_D(\tilde{A}) \) is not necessarily equal to \( D\eta-\text{int}_D(\tilde{A}) \).

3.3. Example
Let \( X = \{ x_1, x_2, x_3 \} \). \( T_{D1} = \{ \tilde{G}_D, \langle x_1 \rangle, \{ x_2, x_3 \} \} \) and \( T_{D2} = \{ \tilde{D}_D, \langle x_1 \rangle, \emptyset, \{ x_2, x_3 \} \} \). If we take \( \tilde{A} = \{ x_1, x_2 \} \), then
\( D12\alpha-\text{int}_D(\tilde{A}) = D21\alpha-\text{int}_D(\tilde{A}) = \tilde{A} \) and \( T_{D1}-\text{int}_D(\tilde{A}) = \{ x_1 \} \). Hence \( D12\alpha-\text{int}_D(\tilde{A}) \neq T_{D1}-\text{int}_D(\tilde{A}) \) and
\[ D21\alpha-int_D(\bar{A}) \neq T_{D2}-int_D(\bar{A}). \]

3.4. Remark
1. If \( P_D \in D\eta\alpha-int_D(\bar{A}) \), then \( \bar{A} \) is \( D\eta\alpha \)-nhd of \( P_D \).
2. Let \((\chi, T_{D1}, T_{D2})\) be a double bi-topological space, and \( \bar{A} \) double set of \( \chi \), then:
   \[ D\eta\alpha-int_D(\bar{A}) = \bigcup_D \{ B \in D\eta\alpha-O(\chi) : B \subseteq_D \}. \]

3.5. Proposition
Let \((\chi, T_{D3}, T_{D2})\) be a double bi-topological space, and \( \bar{A} \) double set of \( \chi \), then:
1. \( D\eta\alpha-int_D(\bar{A}) \) is an \( D\eta\alpha \)-open set.
2. \( \bar{A} \) is \( D\eta\alpha \)-open set iff \( D\eta\alpha-int_D(\bar{A}) = \bar{A} \).

Proof.
1. Is clear by (Remark 3.4) and proposition
2. Obvoise.

3.6. Theorem
Let \((\chi, T_{D1}, T_{D2})\) be a double bi-topological space, and let \( \bar{A}, \bar{B} \) be double sets of \( \chi \), then:
1. \( D\eta\alpha-int_D(\bar{X}_D) = \bar{X}_D \) and \( D\eta\alpha-int_D(\emptyset_D) \).
2. \( D\eta\alpha-int_D(\bar{A}) \subseteq_D \bar{A} \).
3. If \( \bar{A} \subseteq_D \bar{B} \), then \( D\eta\alpha-int_D(\bar{A}) \subseteq_D D\eta\alpha-int_D(\bar{B}) \).
4. \( D\eta\alpha-int_D(\bar{A} \cap_D \bar{B}) \subseteq_D D\eta\alpha-int_D(\bar{A}) \cap_D D\eta\alpha-int_D(\bar{B}) \).
5. \( D\eta\alpha-int_D(\bar{A}) \cap_D D\eta\alpha-int_D(\bar{B}) \subseteq_D D\eta\alpha-int_D(\bar{A} \cap_D \bar{B}) \).
6. \( D\eta\alpha-int_D(D\eta\alpha-int_D(\bar{A})) \subseteq_D D\eta\alpha-int_D(\bar{A}) \).

Proof.
1. Since \( \bar{X}_D \) and \( \emptyset_D \) are \( D\eta\alpha \)-open sets, so by (Proposition 3.5) part (2). \( D\eta\alpha-int_D(\bar{X}_D) = \bar{X}_D \) and \( D\eta\alpha-int_D(\emptyset_D) \).
2. If \( P_D \in D\eta\alpha-int_D(\bar{A}) \), then \( P_D \) is an \( D\eta\alpha \)-interior point of \( \bar{A} \), and \( \bar{A} \) is \( D\eta\alpha \)-nhd of \( P_D \).
3. Let \( P_D \in D\eta\alpha-int_D(\bar{A}) \), then \( P_D \) is an \( D\eta\alpha \)-interior point of \( \bar{A} \), and \( \bar{A} \) is \( D\eta\alpha \)-nhd of \( P_D \), since \( \bar{A} \subseteq_D \bar{B} \), so \( \bar{B} \) is also an \( D\eta\alpha \)-nhd of \( P_D \), this implies that \( P_D \in D\eta\alpha-int_D(\bar{B}) \). Hence \( D\eta\alpha-int_D(\bar{A}) \subseteq_D D\eta\alpha-int_D(\bar{B}) \).
4. Since \( \bar{A} \cap_D \bar{B} \subseteq_D \bar{A} \), and \( \bar{A} \cap_D \bar{B} \subseteq_D \bar{B} \), by part (3) above we have \( D\eta\alpha-int_D(\bar{A} \cap_D \bar{B}) \subseteq_D D\eta\alpha-int_D(\bar{A}) \) and \( D\eta\alpha-int_D(\bar{A} \cap_D \bar{B}) \subseteq_D \bar{B} \). Hence \( D\eta\alpha-int_D(\bar{A} \cap_D \bar{B}) \subseteq_D D\eta\alpha-int_D(\bar{A}) \cap_D D\eta\alpha-int_D(\bar{B}) \).
5. Similar part (4)
6. Direct from (Proposition 3.5).

3.7. Remark
The equality of part (2) and (3) of the previous theorem are not true as the following example.

3.8. Example
By (Example 3.3), if we take \( \bar{A} = \{X_1, X_2, \{X_2\}, \} \), then \( D12\alpha-int_D(\bar{A}) = D21\alpha-int_D(\bar{A}) = \emptyset_D \). Clearly \( \bar{A} \not\subseteq_D D12\alpha-int_D(\bar{A}) \), \( \bar{A} \not\subseteq_D D21\alpha-int_D(\bar{A}) \).
And if we take \( \bar{B} = \bar{X}_D \), then \( D12\alpha-int_D(\bar{B}) = D\eta\alpha-int_D(\bar{B}) = \bar{X}_D \). Clearly, \( \bar{A} \subseteq_D \bar{B} \) but \( D12\alpha-int_D(\bar{B}) \not\subseteq_D D12\alpha-int_D(\bar{A}) \) and \( D21\alpha-int_D(\bar{B}) \not\subseteq_D D21\alpha-int_D(\bar{A}) \).

3.9. Remark
The equality of part (5) of the previous theorem are not true as the following example.
4.3. Example
If we take \( A = (\{x_1\}, \{x_2, x_3\}) \) and \( B = (\{x_3\}, \{x_1, x_2\}) \), then \( A \cup B = (\{x_1, x_3\}, \{x_2\}) \). Direct from (Definition 4.1).

4.4. Example
By (Example 3.3), the family of all \( D12 \alpha \)-closed sets and \( D21 \alpha \)-closed sets are:
\( D12 \alpha - c(\chi) = D21 \alpha - c(\chi) = \{\emptyset, (x_2, x_3), \{x_3\}, \{x_1, x_2\}, \{x_2, x_3, x_1\}, \chi_d\} \)
If we take \( A = (\{x_2, x_3\}, \{x_1\}) \), then \( D12 \alpha - cl_d (A) = D21 \alpha - cl_d (A) = (\{x_2, x_3\}, \{x_1\}) \).

4.4. Theorem
Let \( (\chi, T_{D1}, T_{D2}) \) be a double bi-topological space, and \( \tilde{A} \) double set of \( \chi \). \( \tilde{A} \) is a \( D\eta\alpha - \)closed set iff \( D\eta\alpha - cl_d (\tilde{A}) = \tilde{A} \).

Proof.
1. Direct from (Theorem 4.3)
2. Direct from (Definition 4.1)
3. By part (2) above, \( \tilde{B} \subseteq D\eta\alpha - cl_d (\tilde{B}) \), since \( \tilde{A} \subseteq \tilde{B} \), then \( \tilde{A} \subseteq D\eta\alpha - cl_d (\tilde{B}) \) but \( D\eta\alpha - cl_d (\tilde{B}) \) is an \( D\eta\alpha - \)closed set, thus \( D\eta\alpha - cl_d (\tilde{B}) = D\eta\alpha - cl_d (\tilde{B}) \). Hence \( D\eta\alpha - cl_d (\tilde{A}) \subseteq D\eta\alpha - cl_d (\tilde{B}) \).
4. Since \( \tilde{A} \subseteq \tilde{D} \cup D \tilde{B} \) and \( \tilde{B} \subseteq \tilde{A} \cup D \tilde{B} \), by a part (3) above, we have \( D\eta\alpha - cl_d (\tilde{A}) \subseteq D\eta\alpha - cl_d (\tilde{D} \cup D \tilde{B}) \) and \( D\eta\alpha - cl_d (\tilde{B}) \subseteq D\eta\alpha - cl_d (\tilde{A} \cup D \tilde{B}) \). Hence \( D\eta\alpha - cl_d (\tilde{A}) \cup D\eta\alpha - cl_d (\tilde{B}) \).
5. Since \( \tilde{A} \cap D \tilde{B} \subseteq \tilde{A} \cap D \tilde{B} \) and \( \tilde{A} \cap D \tilde{B} \subseteq \tilde{A} \cap D \tilde{B} \), then \( D\eta\alpha - cl_d (\tilde{A} \cap D \tilde{B}) \subseteq D\eta\alpha - cl_d (\tilde{A}) \) and \( D\eta\alpha - cl_d (\tilde{A} \cap D \tilde{B}) \subseteq D\eta\alpha - cl_d (\tilde{B}) \). Hence \( D\eta\alpha - cl_d (\tilde{A} \cap D \tilde{B}) \).
6. \( D\eta\alpha - cl_d (\tilde{A}) \) is \( D\eta\alpha - \)closed set, we have by (Theorem 4.3) \( D\eta\alpha - cl_d (D\eta\alpha - cl_d (\tilde{A})) = D\eta\alpha - cl_d (\tilde{A}) \).
4.5. Remark
The equality of part (2), (3), (4) and (5) of above theorem are not true in general as the following example.

4.6. Example
(i) Let $(X, T_{D1}, T_{D2})$ be the same double bi-topological space of (Example 3.3).

1. If we take $\tilde{A} = \{(X_1), (X_2, X_3)\}$, then $D12\beta-cl_D(\tilde{A}) = D21\beta-cl_D(\tilde{A}) = \tilde{X}_D$. Hence it is clearly that $D12\beta-cl_D(\tilde{A}) \not\subseteq \tilde{A}$ and $D21\beta-cl_D(\tilde{A}) \not\subseteq \tilde{A}$.

2. If we take $\tilde{A} = \{(X_2), (X_1, X_3)\}$ and $\tilde{B} = \{(X_2, X_3), (X_1)\}$, then clearly $\tilde{A} \subseteq \tilde{D} \tilde{B}$, and $D12\beta-cl_D(\tilde{A}) = D21\beta-cl_D(\tilde{A}) = \{(X_2), (X_1, X_3)\}$. 

Hence $D12\beta-cl_D(\tilde{B}) \not\subseteq D12\beta-cl_D(\tilde{A})$ and $D21\beta-cl_D(\tilde{B}) \not\subseteq D21\beta-cl_D(\tilde{A})$.

3. If we take $\tilde{A} = \{(X_1), (X_2), (X_3)\}$ and $\tilde{B} = \{(X_2, X_3), (X_1)\}$, then $\tilde{A} \cap_D \tilde{B} = \{(X_2), (X_1, X_3)\}$.

Also $D12\beta-cl_D(\tilde{A} \cap_D \tilde{B}) = D21\beta-cl_D(\tilde{A} \cap_D \tilde{B}) = \{(X_2), (X_1, X_3)\}$.

Also $D12\beta-cl_D(\tilde{A}) = D21\beta-cl_D(\tilde{A}) = \tilde{X}_D$. 

Hence $D12\beta-cl_D(\tilde{B}) = \{(X_2, X_3), (X_1)\}$ and $D21\beta-cl_D(\tilde{A}) \cap_D D21\beta-cl_D(\tilde{B}) = \{(X_2, X_3), (X_1)\}$.

Therefore $D12\beta-cl_D(\tilde{A}) \cap_D D12\beta-cl_D(\tilde{B}) \not\subseteq D12\beta-cl_D(\tilde{A} \cap_D \tilde{B})$ and $D21\beta-cl_D(\tilde{A}) \cap_D D21\beta-cl_D(\tilde{B}) \not\subseteq D21\beta-cl_D(\tilde{A} \cap_D \tilde{B})$.

(ii) Let $X = \{X_1, X_2, X_3, X_4\}$,

$T_{D1} = \{\emptyset, (X_1), (X_2, X_3), (X_4), (X_1, X_2), (X_1, X_3), (X_1, X_4), (X_2, X_3, X_4), (X_1, X_2, X_3, X_4)\}$.

$T_{D2} = \{\emptyset, (X_3), (X_2, X_3), (X_3, X_4), (X_2, X_3, X_4), (X_2, X_4), (X_1, X_2, X_3, X_4)\}$, if we take $\tilde{A} = \{(X_1), (X_2, X_3, X_4)\}$. 

$\tilde{B} = \{(X_4), (X_1, X_2, X_3)\}$ are $D21\alpha$-closed sets but $\tilde{B} \cup_D \tilde{B} = \{(X_1, X_4), (X_2, X_3)\}$ is not $D21\alpha$-closed set. 

$D21\alpha-cl_D(\tilde{A}) = \tilde{A}$, $D21\alpha-cl_D(\tilde{B}) = \tilde{B}$ and $D21\alpha-cl_D(\tilde{A} \cup_D \tilde{B}) = \tilde{X}_D$.

Hence $D21\alpha-cl_D(\tilde{A}) \cup_D D21\alpha-cl_D(\tilde{B}) = \{(X_1, X_4), (X_2, X_3)\}$. Thus, $D21\alpha-cl_D(\tilde{A} \cup_D \tilde{B}) \not\subseteq D21\alpha-cl_D(\tilde{A}) \cup_D D21\alpha-cl_D(\tilde{B})$.

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