A CHARACTERIZATION OF QUINTIC HELICES

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Abstract. A polynomial curve of degree 5, α, is a helix if and only if both \(||\alpha'|||\) and \(||\alpha' \wedge \alpha''|||\) are polynomial functions.

1. INTRODUCTION

In [4] the authors study the notion of polynomial curves which made a constant angle with a fixed direction. Curves with constant slope are called there helices and we will use here the same terminology in spite of the fact that in other contexts they are also named generalized helices. The term “helices” is reserved to curves in a cylinder with the same property.

We refer to the introduction of the cited paper and some other papers like [2] or [3] for the relationship between such curves and some problems in the realm of computer-aided design of curves and surfaces. The only fact we want to recall here is that, for real applications, it seems clear that the suitable curves are the quintic helices.

As it is said in [4], any polynomial helix, α, must be a Pythagorean hodograph (PH) curve, i.e., \(||\alpha'|||^2\) is a perfect square of a polynomial. Moreover, this condition is sufficient in the cubical case: all PH cubics are helices.

As it is also said in the cited paper, (along the lines after formula (11)) another necessary condition is the fact that \(||\alpha' \wedge \alpha''|||^2\) (denoted there by \(p^2\)) must also be a perfect square of a polynomial. The easiest way to see this is the following: The argument that shows that a polynomial helix must be PH, there applied to the tangent vector, \(\vec{t} = \frac{\alpha'}{||\alpha'||}\), can also be applied to the binormal vector, \(\vec{b} = \frac{\alpha' \wedge \alpha''}{||\alpha' \wedge \alpha''||}\). (See the first part of the proof of Th. 1 for details.) The consequence now is that \(||\alpha' \wedge \alpha''|||\) is polynomial.

In this short paper we will go a little bit forward and show that both conditions are sufficient in the quintic case. Moreover, we will show an example of polynomial curve of degree 7 verifying both conditions but being not a helix.

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2. Spatial Pythagorean hodograph curves

We will use the quaternion representation of spatial PH curves. Given a quaternion polynomial
\[ A(t) = u(t) + iv(t) +jp(t) + kq(t), \]
the product
\[ \alpha'(t) = A(t)iA^*(t) \]
defines a spatial Pythagorean hodograph, \( \alpha' \), whose components are
\[
\begin{align*}
x' &= u^2 + v^2 - p^2 - q^2, \\
y' &= 2(uq + vp), \\
z' &= 2(vp - up),
\end{align*}
\]  
(2.1)
and such that \( ||\alpha'||^2 = (x')^2 + (y')^2 + (z')^2 = (u^2 + v^2 + p^2 + q^2)^2 \).

In terms of the Hopf map (see [2], theorem 4.2)
\[ H : \mathbb{C}^2 \to \mathbb{R}^3 \]
defined by \( H(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 \bar{z}_2) \), and taking
(2.2)
\[ z_1(t) = u(t) + iv(t), \quad z_2(t) = q(t) + ip(t), \]
the derivative of the curve can be written as
\[ \alpha'(t) = H(z_1(t), z_2(t)). \]

**Definition 1.** A curve \( \alpha \) is said a Pythagorean hodograph curve of second class (2-PH curve) if both \( ||\alpha'|| \) and \( ||\alpha' \wedge \alpha''|| \) are polynomial functions.

It is easy to check that 2-PH curves are examples of curves with a rational Frenet-Serret frame (see [7]).

**Lemma 1.** The Frenet-Serret frame of any 2-PH curve is made of rational vectorial functions.

**Proof.** Simply recall that
\[
\begin{align*}
\vec{t} &= \frac{\alpha'}{||\alpha'||}, \\
\vec{b} &= \frac{\alpha' \wedge \alpha''}{||\alpha' \wedge \alpha''||}, \quad \text{and} \\
\vec{n} &= \vec{b} \wedge \vec{t}.
\end{align*}
\]
\( \square \)

3. Characterization of 2-PH curves

Let \( \alpha \) be a spatial PH curve whose tangent vector is defined by the functions \( u, v, p, q \) as in (2.1) and let \( z_1, z_2 \) be the associated complex functions as in (2.2).

**Proposition 1.** \( ||\alpha' \wedge \alpha''|| \) is a polynomial function if and only if there is a complex polynomial function \( z(t) \) and a real polynomial function \( \omega(t) \) such that
\[
(3.1) \quad z_2^2 \left( \frac{z_1}{z_2} \right)' = \omega z^2.
\]
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Proof. A straightforward computation shows that
\[ ||\omega \alpha' \wedge \alpha''||^2 = 4|||\alpha'||^2((u'q - u'p - v'p + vp')^2 + (u'p - up' + v'q - vq')^2).\]

Therefore, \( ||\omega \alpha' \wedge \alpha''|| \) is a polynomial function if and only if \((u'q - u'p - v'p + vp')^2 + (u'p - up' + v'q - vq')^2\) is a perfect square of a polynomial. Since both terms, \(u'q - u'p - v'p + vp'\) and \(u'p - up' + v'q - vq'\), are polynomial, we can apply the well known result about Pythagorean curves, see [3] section 17.2: there is a polynomial function, \(\omega(t)\), and a complex polynomial function, \(z(t)\), such that
\[
(u'q - u'p - v'p + vp') + i(u'p - up' + v'q - vq') = \omega z^2.
\]

An algebraic manipulation using the functions \(z_1\) and \(z_2\) defined in (2.2) allows to write the left hand member as
\[
(u' + iv')(q + ip) - (u + iv)(q' + ip') = z'_1 z_2 - z_1 z'_2 = z^2 \left(\frac{z_1}{z_2}\right)',
\]
and the statement follows. \(\Box\)

Example 1. Let us check this result in the two examples shown in [4].

The first example is defined by the four quadratic polynomials
\(u(t) = t^2 - 3t, \quad v(t) = t^2 - 5t + 10, \quad p(t) = -2t^2 + 3t + 5, \quad q(t) = t^2 - 9t + 10.\)

Therefore, the complex functions \(z_1(t) = (t^2 - 3t) + i(t^2 - 5t + 10)\) and \(z_2(t) = (t^2 - 9t + 10) + i(-2t^2 + 3t + 5)\), verify expression (3.1) for \(\omega(t) = 1, \quad z(t) = \sqrt{1 - i}(t - (1 + 2i)).\)

The second example is defined by
\(u(t) = -19t^2 + 12t + 5, \quad v(t) = -22t^2 + 18t + 1, \quad p(t) = 15t^2 - 12t - 1, \quad q(t) = -31t^2 + 24t + 3.\)

Now, expression (3.1) holds for
\(\omega(t) = 26(3 - 7t + 3t^2), \quad z(t) = \sqrt{-1 + i}.\)

4. 2-PH CURVES OF DEGREE 5.

In this case \(z_1(t)\) and \(z_2(t)\) are quadratic polynomials and the term on the left of the expression
\[ z'_1 z_2 - z_1 z'_2 = \omega z^2 \]
is a polynomial of degree 2. Therefore, there are two possibilities for the pair \(\omega(t), z(t)\). The first is \(\omega(t)\) be a quadratic function, and \(z(t)\) a constant function. The second, \(\omega(t)\) be a constant and \(z(t)\) a linear polynomial. As we will see, each possibility correspond to one of the two classes of quintic helices studied in [4]: the general helices and the monotone helices.

We study first the case when \(\omega(t)\) is constant, and without loss of generality we can suppose that \(\omega(t) = 1.\)
Lemma 2. Monotone helices are characterized by a constant $\omega(t)$.

Proof. The complex polynomials $z_1(t)$ and $z_2(t)$ are of degree less or equal than two. The first possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$ z_1(t) = a(t - r_1)(t - r_2), $$

$$ z_2(t) = b(t - r_3)(t - r_4), $$

where $a, b, r_i \in \mathbb{C}$. An easy computation gives us that

$$ z'_1 z_2 - z'_1 z'_2 = ab((r_1 + r_2 - r_3 - r_4)t^2 + 2(r_3 r_4 - r_1 r_2)t + (r_3 + r_4)r_1 r_2 - (r_1 + r_2)r_3 r_4). $$

If $\omega(t) = 1$ this expression is the square of a complex polynomial function of degree 1, $z(t) = mt + n$, if and only if

$$ ab(r_1 + r_2 - r_3 - r_4) = m^2, $$

$$ ab(r_3 r_4 - r_1 r_2) = mn, $$

$$ ab((r_3 + r_4)r_1 r_2 - (r_1 + r_2)r_3 r_4) = n^2. $$

From the first two equations we get

$$ m = \pm \sqrt{ab\sqrt{r_1 + r_2 - r_3 - r_4}}, \quad n = \frac{ab(r_3 r_4 - r_1 r_2)}{m}. $$

Substituting in the last equation

$$ r_1 r_2 r_3 + r_1 r_2 r_4 - r_1 r_3 r_4 - r_2 r_3 r_4 = \frac{r_1^2 r_2^2 - 2r_1 r_2 r_3 r_4 + r_3^2 r_4^2}{r_1 + r_2 - r_3 - r_4}. $$

After some algebraic manipulation we can rewrite this equation as

$$ (r_1 - r_3)(r_2 - r_3)(r_1 - r_4)(r_2 - r_4) = 0. $$

Therefore, $z'_1 z_2 - z'_1 z'_2 = z^2$ if and only if $z_1(t)$ and $z_2(t)$ share a linear factor. In this case $\gcd(z_1(t), z_2(t)) \neq \text{constant}$ which is the characterization of monotone helices (see [4], section 3.1). Indeed,

$$ \gcd(x', y', z') = |\gcd(u + iv, p - iq)|^2 = |\gcd(z_1, -iz_2)|^2 = |\gcd(z_1, z_2)|^2. $$

The second possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$ z_1(t) = a(t - r_1)(t - r_2), $$

$$ z_2(t) = b(t - r_3), $$

where $a, b, r_i \in \mathbb{C}$. A similar analysis shows that $r_3 = r_1$ or $r_3 = r_2$, and the same conclusion holds.

The last possibility is that polynomials $z_1(t)$ and $z_2(t)$ are given by

$$ z_1(t) = a(t - r_1)(t - r_2), $$

$$ z_2(t) = b, $$

where $a, b, r_i \in \mathbb{C}$. It is easy to check that this case deals to a contradiction. \qed
5. Characterization of quintic helices

Let us recall first a description of PH quintics based on quaternions, see [4]. A spatial quintic helix is defined by a quadratic polynomial

\[ \mathcal{A}(t) = A_0 + A_1 t + A_2 t^2, \]

with quaternion coefficients

\[ \begin{align*}
A_0 &= a + a_x i + a_y j + a_z k, \\
A_1 &= b + b_x i + b_y j + b_z k, \\
A_2 &= c + c_x i + c_y j + c_z k.
\end{align*} \]

In terms of the functions \( u, v, p, q \): \( u(t) = a + bt + ct^2 \), \( v(t) = a_x + b_x t + c_x t^2 \), \( p(t) = a_y + b_y t + c_y t^2 \), and \( q(t) = a_z + b_z t + c_z t^2 \).

**Lemma 3.** Let \( z_1(t) = u(t) + iv(t) \) and \( z_2(t) = q(t) + ip(t) \) be the quadratic polynomials of a quintic PH curve defined by three quaternions \( \{A_i\}, i = 0, 1, 2 \). If \( z(t) \) in (3.1) is constant then \( A_1 = c_0 A_0 + c_2 A_2 \), for suitable real scalars \( c_0, c_2 \).

**Remark 1.** In [4] the authors use a Bézier quadratic polynomial

\[ \mathcal{A}(t) = A_0 (1 - t)^2 + 2 A_1 t (1 - t) + A_2 t^2. \]

We have used here the usual basis of polynomials instead of the Bernstein basis because computations are easier. The statement of the previous Lemma remains true for Bézier quaternion coefficients due to just a change of basis.

**Proof.** Let us suppose that

\[ \omega(t) = m_0 + m_1 t + m_2 t^2, \quad z(t) = e^{i\theta}, \]

where, \( m_0, m_1, m_2, \theta \in \mathbb{R} \) and without loss of generality, we assume that \( |z| = 1 \).

We use now Proposition [4] and compute the expression

\[ z'_1 z_2 - z_1 z'_2 = \omega z^2. \]

The real part of the left hand term can be written as

\[ u'q - uq' - v'p + vp' = (a_x b_y - a_y b_x + a_z b - a_y b_x) + 2(a_x c_y - a_c z - a_y c_x + a_z c) t + (b_x c - b_y c_x + b_y c_y - b_z c_z) t^2, \]

and the imaginary part as

\[ u'p - up' + v'q - vq' = (a_y b + a_z b_x - ab_y - a_x b_z) + 2(a_y c + a_z c_x - ac_y - a_x c_z) t + (b_y c + b_z c_x - bc_y - b_x c_z) t^2. \]

Analogously, the real part of the right hand term can be written as

\[ (m_0 + m_1 t + m_2 t^2) \cos(2\theta), \]

and the imaginary part as

\[ (m_0 + m_1 t + m_2 t^2) \sin(2\theta). \]
Therefore, the condition \( z_1'z_2 - z_1z_2' = \omega z^2 \) can be translated into a set of six equations. By equating the coefficients of \( t \) we can deduce that
\[
\theta = \frac{1}{2} \arctan \left( \frac{a_y c + a_z c_x - a c_y - a x c_z}{a_z c - a_y c_x + a_x c_y - a c_z} \right)
\]
and
\[
m_1 = 2 \sqrt{(a_y c + a_z c_x - a c_y - a x c_z)^2 + (a_z c - a_y c_x + a_x c_y - a c_z)^2}.
\]

Substituting these values into the other four equations and solving the resulting linear system we obtain
\[
\mathcal{A}_1 = \frac{2m_0}{m_1} A_0 + \frac{2m_2}{m_1} A_2.
\]

\[\square\]

**Theorem 1.** A quintic polynomial curve is a helix if and only if it is a 2-PH curve.

**Proof.** (Necessary conditions)

Let us recall that if a curve is a helix, then, not only the tangent vector makes a constant angle with the axis, but also the binormal vector, see the classical references [1],[6]. In fact, if \( \overrightarrow{u} \) is a unitary vector that determines the axis of the helix then
\[
< \overrightarrow{u}, \overrightarrow{t} > = c, \quad < \overrightarrow{u}, \overrightarrow{b} > = \sqrt{1 - c^2},
\]
where \( c \in \mathbb{R} \) is a constant, \( \overrightarrow{t} = \frac{\alpha'}{||\alpha'||} \) is the tangent vector and \( \overrightarrow{b} = \frac{\alpha' \wedge \alpha''}{||\alpha' \wedge \alpha''||} \) is the binormal vector of the curve.

The previous expressions are equivalent to
\[
< \overrightarrow{u}, \alpha' > = c ||\alpha'||, \quad < \overrightarrow{u}, \alpha' \wedge \alpha'' > = \sqrt{1 - c^2} ||\alpha' \wedge \alpha''||.
\]

If our curve \( \alpha \) is polynomial then it is a Pythagorean hodograph and also the norm of \( ||\alpha' \wedge \alpha''|| \) is polynomial, indeed
\[
||\alpha'|| = \frac{1}{c} < \overrightarrow{u}, \alpha' >, \quad ||\alpha' \wedge \alpha''|| = \frac{1}{\sqrt{1 - c^2}} < \overrightarrow{u}, \alpha' \wedge \alpha'' > .
\]

(Sufficient conditions) If \( ||\alpha' \wedge \alpha''|| \) is polynomial then by Proposition [1] we know that \( z_2^2 \left( \frac{\alpha}{z_2} \right)' = \omega z^2 \). In the quintic case, the only possibilities are \( \omega \) constant or \( z \) constant. If \( \omega \) is constant then by Lemma [2] the curve is a monotone helix.

If \( z \) is constant, then by Lemma [3] the quaternions defining the curve are linear dependents and by Proposition 1 in [1] we know that the curve is a helix. \[\square\]

**Remark 2.** In higher dimensions it is possible to find 2-PH curves being not helices. For example:
\[
\alpha(t) = (-3t + t^3 + \frac{t^5}{5} + \frac{t^7}{21}, 3t^2 - \frac{t^4}{2}, -2t^3)
\]
is a polynomial curve of degree 7 verifying

\[ ||\alpha'|| = \frac{1}{3}(9 + 9t^2 + 3t^4 + t^6), \quad ||\alpha' \wedge \alpha''|| = 2(1 + t^2)(9 + 9t^2 + 3t^4 + t^6) \]

but

\[ \frac{\tau}{\kappa} = \frac{-9 + 9t^4 + 2t^6}{9(1 + t^2)^2}, \]

so, it does not satisfy the Lancret’s theorem (see for example [6]) and the curve is not a generalized helix.

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