HIERARCHY OF GRAPH MATCHBOX MANIFOLDS

OLGA LUKINA

Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

Abstract. We study a class of graph foliated spaces, or graph matchbox manifolds, initially constructed by Kenyon and Ghys. For graph foliated spaces we introduce a quantifier of dynamical complexity which we call a level. We develop the fusion construction, which allows us to associate to every two graph foliated spaces a third one which contains the former two in its closure. Although the underlying idea of the fusion is very simple, it gives us a powerful tool to study graph foliated spaces. Using fusion, we prove that there is a hierarchy of graph foliated spaces at infinite levels. We also construct examples of graph foliated spaces with various dynamical and geometric properties.

1. Introduction

A matchbox manifold is a compact connected metrizable space $M$ such that each point $x \in M$ has a neighborhood homeomorphic to a product space $U_x \times N_x$, where $U_x \subset \mathbb{R}^n$ is open and $N_x$ is a compact totally disconnected space. The term originates from the works of Aarts and Martens [2], Aarts and Oversteegen [1] for the case when $n = 1$, when local charts can be thought of as 'boxes of matches'. The most well-studied classes of examples of matchbox manifolds are weak solenoids [29, 18], generalized solenoids [39], and tiling spaces with finite local complexity (see, for instance, [33], or [5] for a more general type of tilings). In this paper we consider a third class of examples, which we call graph matchbox manifolds. This construction was introduced by R. Kenyon and É. Ghys [19], and later generalized by E. Blanc [6], A. Lozano Rojo [26], F. Alcalde Cuesta, A. Lozano Rojo and M. Macho Stadler [3].

Let $G$ be a finitely generated group with a non-symmetric set of generators $G_0$, that is, if $h \in G_0$ then $h^{-1} \notin G_0$. Let $\mathcal{G}$ be the Cayley graph of $G$, and $X$ be the set of all infinite connected subtrees of $\mathcal{G}$ containing the identity $e$. Each subtree $T$ is equipped with a standard complete length metric $d$, and the pair $(T, e)$ is a pointed metric space. The set $X$, endowed with the Gromov-Hausdorff metric $d_{GH}$ [7], is a compact totally disconnected space [19, 6, 26]. One can define a partial action of the free group $F_n$ on $X$, where $n$ is the cardinality of the set of generators $G_0$. This action gives rise to a pseudogroup $\mathcal{G}$ on $X$, and an important feature of the construction is that the pseudogroup dynamical system $(X, \mathcal{G})$ can be realised as the holonomy pseudogroup of a smooth foliated space $\mathcal{M}_G$ with 2-dimensional leaves [19, 6, 26]. By this construction, for $(T, e) \in X$ the corresponding leaf $L_T \subset \mathcal{M}_G$ can be thought of as the two-dimensional boundary of the thickening of a quotient graph of $T$, where the quotient map is determined by the geometry of $T$.

DEFINITION 1.1. A graph matchbox manifold is the closure $\mathcal{M} = \overline{L}$ of a leaf $L$ in $\mathcal{M}_G$, that is, $\mathcal{M}$ is a connected closed saturated transitive subset of $\mathcal{M}_G$. 

E-mail address: ollukina940@gmail.com.

Key words and phrases. matchbox manifold, laminations, ends of leaves, growth of leaves, theory of levels, Gromov-Hausdorff metric.

2000 Mathematics Subject Classification. Primary 57R30. Secondary 37B05, 54H20.

Version date: December 22, 2011.
Dynamical properties of graph matchbox manifolds are determined by the space of pointed trees \((X,d_{GH})\) with the action of the pseudogroup \(\mathcal{G}\). The goal of this work is to understand the class of matchbox manifolds which can be obtained by the generalized construction of Kenyon and Ghys, and investigate their dynamical and geometrical properties.

In previous works the construction of Kenyon and Ghys was mostly used to produce examples of matchbox manifolds with specific geometric and ergodic properties. Ghys [19], see also [3], showed that if \(G = \mathbb{Z}^2\) then \(\mathcal{M}_{\mathbb{Z}^2}\) contains a leaf \(L\) such that the matchbox manifold \(\mathcal{M} = L\) is minimal and has leaves with different conformal structures. Lozano Rojo [27] studied minimal examples in the case \(G = \mathbb{Z}^2\) from the point of view of ergodic theory. In the case where \(G = F_3\), a free group on three generators, Blanc [6] found an example of a graph matchbox manifold containing leaves with any possible number of ends.

In this paper we study the hierarchical structure on the set of graph matchbox manifolds given by inclusions. A basic observation is as follows.

**Proposition 1.3.** The partially ordered set \((S_n, \preceq)\) of graph matchbox manifolds in the foliated space \(\mathcal{M}_n\), \(n > 1\), has the following properties.

1. **Statement of main results.** Let \(\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{M}_n\), then the rule
   \[
   \mathcal{M}_1 \preceq \mathcal{M}_2 \quad \text{if and only if} \quad \mathcal{M}_1 \subseteq \mathcal{M}_2
   \]
   defines a partial order on the set \(S_n\) of graph matchbox manifolds in \(\mathcal{M}_n\). Compact leaves and minimal subsets of \(\mathcal{M}_n\) are minimal elements in \(S_n\) with respect to this order. The following proposition describes the hierarchy of graph matchbox manifolds in \(\mathcal{M}_n\).

Recall [6] that a leaf \(L \subseteq \mathcal{M}\) is recurrent if and only if \(L\) is transitive and accumulates on itself. A leaf \(L\) is proper if it does not accumulate on itself.

**Proposition 1.3.** The partially ordered set \((S_n, \preceq)\) of graph matchbox manifolds in the foliated space \(\mathcal{M}_n\), \(n > 1\), has the following properties.

1. The set \(C = \{L \subseteq \mathcal{M}_n \mid L \text{ is compact}\}\) is a dense meager subset of \(\mathcal{M}_n\). Moreover, \(C \cap X\) is countable, where \(X\) is a canonical embedding of \(X\) into \(\mathcal{M}_n\).
2. \((S_n, \preceq)\) is a directed partially ordered set, i.e. given \(\mathcal{M}_1, \mathcal{M}_2 \in S_n\) there exists \(\mathcal{M}_3 \in S_n\) such that \(\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{M}_3\).
3. \((S_n, \preceq)\) contains a unique maximal element \(\mathcal{M}_{\max} = \mathcal{M}_n\) which has a recurrent leaf. Therefore, \(\mathcal{M}_n\) contains a residual subset of recurrent leaves.

In order to prove Proposition 1.3(2), we introduce the ‘fusion’ construction which associates to any two transitive subsets \(\mathcal{M}_1\) and \(\mathcal{M}_2\) of \(\mathcal{M}_n\) a transitive subset \(\mathcal{M}_3\) such that \(\mathcal{M}_3 \supseteq \mathcal{M}_1 \cup \mathcal{M}_2\). More precisely, given pointed graphs \((T_1, e)\) and \((T_2, e)\) such that \(\mathcal{M}_1 = \mathcal{T}_{T_1}\) and \(\mathcal{M}_2 = \mathcal{T}_{T_2}\) we give a recipe to construct a graph \((T_3, e)\) such that \(\mathcal{M}_3 = \mathcal{T}_{T_3}\) satisfies the required property. The underlying idea of the construction is very simple, but it gives us a powerful tool which allows us to obtain a lot of information about hierarchy and properties of graph matchbox manifolds. Propositions 1.3(2) and 1.3(3) are the first applications of the fusion.

Proposition 1.3(2) is a direct consequence of the fusion. The next important observation is that in the space \(\mathcal{M}_n\) fusion enables us to construct infinite increasing chains of graph matchbox manifolds. Using [6] Theorem 3.5 with slightly eased assumptions, we conclude that the closure of such a chain contains a dense leaf, and if every element in the chain is distinct, this dense leaf accumulates on itself. The existence of a maximal closed recurrent subset in the space of graph matchbox manifolds
follows by standard topological arguments, and, using fusion again, one argues that if a maximal recurrent subset exists, then it is unique, which is the statement of Proposition 1.3.(3). Then the following questions arise naturally.

**QUESTION 1.4.**

1. Is it possible to construct an infinite increasing chain of distinct graph matchbox manifolds such that its closure is a proper subset of the space $\mathcal{M}_n$?
2. Is there a hierarchy of recurrent subsets containing an infinite increasing chain of graph matchbox manifolds?

In order to answer these questions and rigorously describe the hierarchy of graph matchbox manifolds, we develop the theory of levels for graph matchbox manifolds, inspired by the ideas of Cantwell and Conlon [11, 12], Nishimori [31, 32], Tsuchiya [37] and Hector [21, 22]. Our main tool is the fusion technique of Proposition 1.3.(2). Thus, besides contributing to a direct proof of Proposition 1.3.(3), fusion provides the means to study the hierarchy of graph matchbox manifolds in $\mathcal{M}_n$ given by inclusions.

The study of the partial ordering given by inclusions of transitive subsets is a natural problem in topological dynamics. The consideration of the extension of the Poincaré Recurrence Theorem for flows, to the closures of leaves of foliations began in the 1950’s. In foliation theory, the strongest results have been obtained for codimension 1 transversally $C^2$-differentiable foliations by Cantwell and Conlon [11, 12], Nishimori [31, 32], Tsuchiya [37] and Hector [21, 22]. Later, the extent to which these ideas carry on to the codimension 1 $C^0$ case was studied by Salhi [34, 35, 36], and, for foliations of higher codimensions satisfying certain additional conditions, by Marzougui and Salhi [28]. Cantwell and Conlon [11] introduced the notion of a level of a leaf or a transitive subset, which can be seen as a quantifier of dynamical complexity. A similar, but not the same notion of ‘depth’ of a leaf was considered by Nishimori [31, 32]. The theory of levels for $C^2$-foliations relies heavily on the Kopell lemma [11, 8], which does not apply in our setting. However, similar notions can be pushed through to some extent by topological methods. We introduce the level of a graph matchbox manifold, which is similar to the notion of a level in [11].

**DEFINITION 1.5.** Let $\mathcal{M} \subset \mathcal{M}_n$ be a graph matchbox manifold.

1. $\mathcal{M}$ is said to be at level 0 if either $\mathcal{M}$ is a compact leaf, or $\mathcal{M}$ is a minimal foliated space. In that case all leaves of $\mathcal{M}$ are also at level 0.
2. $\mathcal{M}$ is at level $k$ if the closure of the union of leaves which are not dense in $\mathcal{M}$, is a proper closed subset of $\mathcal{M}$, every such leaf is at level at most $k-1$, and there is at least one leaf at level $k-1$. A leaf $L$ is at level $k$ if it is dense in $\mathcal{M}$.
3. A leaf $L$ is at infinite level if it is not at finite level.

By Proposition 1.3.(3) the space $\mathcal{M}_n$ contains a dense leaf $L$ and so $\mathcal{M}_n$ is a graph matchbox manifold. Since by Proposition 1.3.(1) the set of compact leaves is dense in $\mathcal{M}_n$, $\mathcal{M}_n$ cannot be at finite level, and so is at infinite level, and every dense leaf in $\mathcal{M}_n$ is at infinite level. By Proposition 1.3.(3) $\mathcal{M}_n$ contains a residual subset of dense leaves, therefore, we have the following corollary.

**COROLLARY 1.6.** In a foliated space $\mathcal{M}_n$, $n > 1$, leaves at infinite level form a residual subset.

Leaves at infinite level in Corollary 1.6 are leaves which are dense in $\mathcal{M}_n$, $n > 1$, and so contain any other leaf in their closure. It turns out that leaves at infinite level are much more diverse than that, namely, there exists a whole hierarchy of leaves at infinite levels, as the following theorem shows.

**THEOREM 1.7.** For a space of graph matchbox manifolds $\mathcal{M}_n$, $n > 1$, the following holds.

1. There exists an infinite increasing chain
   
   $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$

   of graph matchbox manifolds such that $\mathcal{M} = \bigcup_i \mathcal{M}_i$ is at infinite level and $\mathcal{M}$ is a proper subset of $\mathcal{M}_n$. Such a $\mathcal{M}$ is at infinite level of Type 1.
(2) Let $\mathcal{M}$ be a graph matchbox manifold at infinite level and suppose $\mathcal{M}$ is a proper subset of $\mathcal{M}_n$. Then there exists a graph matchbox manifold $\tilde{\mathcal{M}}$ such that
\[ \mathcal{M} \subset \tilde{\mathcal{M}} \subset \mathcal{M}_n \]
are proper inclusions. The space $\tilde{\mathcal{M}}$ is at infinite level of Type 2.

The fusion technique described earlier lies at the heart of the proof of Theorem 1.7. It is interesting to compare the result of Theorem 1.7 with the hierarchy of leaves at infinite levels for codimension 1 foliations. Namely, [11, 21] state that if the transverse differentiability of a codimension 1 foliation is $C^0$, then there exists a hierarchy of infinite levels, as in Theorem 1.7, but if the transverse differentiability is $C^2$, there is only one infinite level. In the space of graph matchbox manifolds, transversals are totally disconnected and one cannot increase the degree of differentiability. However, the following question is of interest.

**QUESTION 1.8.** Let $\mathcal{M} \subset \mathcal{M}_n$ be a graph matchbox manifold which contains leaves at more than one infinite levels. Does there exist a foliated embedding $\mathcal{M} \to \hat{\mathcal{M}}$ into a smooth foliated manifold $\hat{\mathcal{M}}$ with a foliation of codimension $q \geq 2$?

Indeed, comparing Theorem 1.7 with the results of [11] we conclude that graph matchbox manifolds containing leaves at infinite level of Type 2 cannot embed as foliated subsets of codimension 1 smooth foliations. It was shown in [27] that a minimal graph matchbox manifold can always be embedded as a codimension 2 subset of a manifold topologically. An embedding of $\mathcal{M}$ in a smooth way and, moreover, as a subset of a smooth foliation of $\mathcal{M}$, is an infinitely more subtle and technically demanding procedure. In general, not much is known about embeddings of foliated spaces transversely modeled on Cantor sets as saturated subsets of smooth foliations and, to the best of our knowledge, there are only a few works devoted to that question. Namely, Williams [39] showed that the expansive dynamical system associated to a transverse section of a generalized solenoid is conjugate to the dynamics of an expanding attractor, but it is an open question whether such a solenoid embeds as a subsets of the corresponding foliation of a manifold. Clark and Hurder [13] give sufficient conditions under which certain homogeneous solenoids embed as subsets of smooth foliations. Question 1.8 asks a similar question for graph matchbox manifolds. It is not answered in the present paper, but it indicates a direction of future work.

Following [11] we denote by $S(L)$ the union of non-dense leaves in a graph matchbox manifold $\mathcal{M} = \mathcal{T}$, and call $S(L)$ the substructure of a leaf $L$. In codimension 1 $C^2$ foliations leaves at infinite levels have a substructure of leaves at finite levels, and each of the finite levels is represented in $S(L)$ by at least one leaf [11]. Theorem 1.7 states that in the space of graph matchbox manifolds leaves at infinite level of Type 2 necessarily contain a leaf at infinite level in their substructure, and there exist leaves at infinite levels of Type 1 with the same property. Therefore, leaves of Type 1 in Theorem 1.7 can be subdivided into subtypes depending on what kind of non-dense leaves (at finite or infinite levels) they contain in their closure. Therefore, we ask the following questions.

**QUESTION 1.9.**
(1) In the space $\mathcal{M}_n$, does there exist a graph matchbox manifold $\mathcal{M} = \mathcal{T}$ at infinite level such that $S(L)$ has a substructure of leaves at finite levels?
(2) Given $k > 0$, give an example of a graph matchbox manifold $\mathcal{M}_k$ at level precisely $k$.

As it has already been mentioned, using fusion one can construct increasing chains of graph matchbox manifolds of arbitrary finite or infinite length, and Theorem 1.7 permits to conjecture that Questions 1.9 have positive answers. However, given $k > 2$, and an increasing chain $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots \subseteq \mathcal{M}_k$, the conclusion that $\mathcal{M}_k$ is at level precisely $k$ is a complicated problem. Orbits of graphs in $X_n$ can accumulate on other orbits in many different ways. For example, even a totally proper leaf (i.e. such that $\mathcal{L}$ contains only proper leaves) at level 1 can contain an infinite number of distinct leaves in its closure.

**THEOREM 1.10.** In a foliated space $\mathcal{M}_n$, $n > 1$, there is a totally proper graph matchbox manifold $\mathcal{M}$ at level 2 which contains a countably infinite number of compact leaves.
Every distinct leaf in a graph matchbox manifold \( \mathcal{M} \) adds a new pattern in the set of graphs describing the transverse dynamics, and adds to the complexity of the problem. The computation of a precise level of a graph matchbox manifold \( \mathcal{M} \) is a problem with strong combinatorial flavor, and we will not give a general algorithm here.

The remaining part of the article is devoted to dynamical properties of graph matchbox manifolds at finite levels. We first notice that, in terms of admissible types of transverse dynamics, graph matchbox manifolds do not exhibit much variety, as the following proposition shows.

Let \( \mathcal{O} \) be a clopen subset of a transversal space \( \mathcal{X} \) where the Gromov-Hausdorff topology on \( \mathcal{X} \) is realised by the ball metric \( d_\mathcal{X} \) defined in Section 2. Recall [24] that the pseudogroup \( \Theta|_\mathcal{O} \) is \( \epsilon \)-expansive if there exists \( \epsilon > 0 \) so that for all \( w \neq w' \in \mathcal{O} \) with \( d_\mathcal{X}(w, w') < \epsilon \) there exists a holonomy homeomorphism \( h \in \Theta|_\mathcal{O} \) with \( w, w' \in \text{dom}(h) \) such that \( d_\mathcal{X}(h(w), h(w')) \geq \epsilon \). A pseudogroup \( \Theta|_\mathcal{O} \) is equicontinuous if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( h \in \Theta|_\mathcal{O} \) and all \( w, w' \in \text{dom}(h) \) such that \( d_\mathcal{X}(w, w') < \delta \) we have \( d_\mathcal{X}(h(w), h(w')) < \epsilon \).

**PROPOSITION 1.11.** Let \( \mathcal{M} \) be a graph matchbox manifold in a foliated space \( \mathcal{M}_n \), and \((\mathcal{X}, d_\mathcal{X})\) be a complete transversal for \( \mathcal{M} \). Then the following holds.

1. \( \mathcal{M} \) contains a non-compact leaf if and only if for all \( \epsilon < \epsilon^{-2} \) the restriction of the pseudogroup \( \Theta \) to \( \mathcal{X} \) is \( \epsilon \)-expansive.
2. If there exists a clopen \( \mathcal{O} \subset \mathcal{X} \) such that \( \Theta|_\mathcal{O} \) is equicontinuous, then \( \mathcal{M} \) contains a recurrent leaf. The converse is false, since the restriction of the pseudogroup of any minimal matchbox manifold to any clopen subset of the transversal is expansive.

The result of Proposition 1.11 is not surprising, since the ball metric on \( X_n \) is similar to the metric on tiling spaces, which is well-studied and, in particular, it is known that for a non-periodic tiling, their transverse dynamics is expansive (see, for example, Benedetti and Gambaudo [3]). Leaves in tiling spaces are usually copies of the Euclidean space, and it is customary to consider tiling spaces where each leaf is dense. To the best of our knowledge, the most general type of tiling spaces which has been described in the literature is that in Benedetti and Gambaudo [5], where leaves are homogeneous spaces and are homeomorphic. Leaves in graph matchbox manifolds exhibit a much wider variety of topological properties, in particular, one can realise a surface of any genus as a leaf in \( \mathcal{M}_n \). Proposition 1.11(1) uses similar arguments to those in tilings to investigate transverse dynamics for graph matchbox manifolds.

Proposition 1.11(1) allows us to restrict the range of spaces which can be modeled by the method of Kenyon and Ghys, as the following corollary shows.

**COROLLARY 1.12.** The foliated space \( \mathcal{M}_n \) does not contain a weak solenoid.

Indeed, a weak solenoid is a minimal space with equicontinuous transverse dynamics. In particular, Corollary 1.12 shows that the range of foliated spaces treated in Clark and Hurder [13] and the range of foliated spaces in Question 1.8 are disjoint, and [13] does not provide even a partial answer to Question 1.8.

If a graph matchbox manifold \( \mathcal{M} \) is at level \( k = 0 \), then it is either minimal and has expansive dynamics, or it is a compact leaf and has trivially equicontinuous dynamics. If \( \mathcal{M} \) is at level \( k > 0 \), various types of dynamics can mix. In particular, the set of graph matchbox manifolds satisfying Proposition 1.11(2), is non-empty, as the following theorem shows.

**THEOREM 1.13.** In a foliated space \( \mathcal{M}_n \), there exists a graph matchbox manifold \( \mathcal{M} \) which contains an uncountable number of leaves, all but a finite number of which are at level 2. The complete transversal \( \mathcal{X} \) of \( \mathcal{M} \) has a clopen neighborhood \( \mathcal{O} \) such that the restriction of the pseudogroup \( \Theta \) to \( \mathcal{O} \) is equicontinuous. Every leaf in \( \mathcal{M} \) which is at level 2 is recurrent.

By Proposition 1.11(1), the clopen set \( \mathcal{O} \) in Theorem 1.13 is necessarily not a complete transversal. Recall [11] that a closed saturated subset of a foliated space is totally proper if it consists only of proper leaves. For codimension 1 \( C^2 \) foliations the growth function of a totally proper leaf at level \( k \)
is a polynomial of degree \( k \) and, if the growth function of a leaf is dominated by a polynomial, then the leaf is totally proper. There is no such relation between the level and the growth function of a leaf in the case of graph matchbox manifolds, as the following proposition shows.

**Proposition 1.14.** In a foliated space \( \mathcal{M}_n \), there exists a totally proper graph matchbox manifold \( \mathcal{M}_1 \) at level 1 with a transitive leaf \( L \) with linear growth, and there is also a totally proper graph matchbox manifold \( \mathcal{M}_2 \) at level 1 with a transitive leaf \( L' \) with exponential growth. Besides, \( \mathcal{M}_n \) contains a recurrent graph matchbox manifold \( \mathcal{M}_3 \) with leaves of polynomial growth.

Theorems 1.13 and 1.14 provide examples of graph matchbox manifolds at levels 1 and 2.

The rest of the paper is organized as follows. In Section 2 we give details of the generalized construction of Ghys and Kenyon, describe some relations between topology of trees in \( G \) and the graphs of their orbits under the pseudogroup \( G \), and give first examples of graph matchbox manifolds. We prove Proposition 1.3 in Section 3 and Theorems 1.7, 1.10 and 1.13 in Section 4. Proposition 1.14 is proved in Section 5 and Proposition 1.11 is proved in Section 6.

**Acknowledgements:** The author thanks Alex Clark and Steve Hurder for their encouragements to pursue this research topic and for offering many useful comments.

## 2. Construction of a foliated space after Kenyon and Ghys

In this section we give an outline of the construction of Ghys and [19, 26, 3], and define graph matchbox manifolds.

We also obtain Proposition 1.2, that is, that the foliated space \( \mathcal{M}_n \) obtained by the construction of Kenyon and Ghys in the case \( G = F_n \) has a universal property.

### 2.1. Preliminaries and notation.

Given a graph \( T \), denote by \( V(T) \) the set of vertices of \( T \), and by \( E(T) \) the set of edges of \( T \). An edge \( w \in E(T) \) is given an orientation by specifying its starting and its ending vertex, denoted by \( s(w) \) and \( t(w) \) respectively. A subgraph of \( T \) is a graph \( T' \) with the set of vertices \( V(T') \subset V(T) \) and the set of edges \( E(T') \subset E(T) \), where an edge \( w \in E(T') \) is and only if \( s(w) \in V(T') \) and \( t(w) \in V(T') \). A labeling of a graph \( T \), or of the set \( E(T) \) of edges of \( T \), by a set \( A \), is given by a function \( a : E(T) \to A \). If it is necessary to keep track of labeling of edges in \( E(T) \), we use the notation \( w_{a(w)} \) for an edge \( w \in E(T) \).

Let \( G \) be a finitely generated group acting on itself on the right, and let \( G_0 \) denote a set of generators of \( G \). We assume that \( G_0 \) is not a symmetric set, that is, if \( h \in G_0 \) then \( h^{-1} \notin G_0 \). Let \( \mathcal{G} \) be the Cayley graph of \( G \) with the set of generators \( G_0 \); more precisely, set \( V(\mathcal{G}) = G \), and to each pair \( g_1, g_2 \in V(\mathcal{G}) \) such that \( g_1 h = g_2 \) for some \( h \in G_0 \), associate an edge \( w_h \in E(\mathcal{G}) \) with \( s(w_h) = g_1 \) and \( t(w_h) = g_2 \). Thus \( \mathcal{G} \) is an oriented graph labeled by the set \( G_0 \).

We define a length structure \( \ell : P(\mathcal{G}) \to \mathbb{R} : \delta \mapsto \ell(\delta) \), where \( P(\mathcal{G}) \) the set of all paths in \( \mathcal{G} \), in the standard manner [7], so that edges in \( \mathcal{G} \) is parametrized in such a way that each of them has length 1. Associated to \( \ell \), there is a complete length metric \( d \) on \( \mathcal{G} \) defined by [7]

\[
D(x, y) = \inf_{\delta} \{ \ell(\delta) \mid \delta : [0, 1] \to \mathcal{G}, \delta(0) = x, \delta(1) = y \}.
\]

Thus \((\mathcal{G}, D)\) becomes a length metric space.

### 2.2. Space of pointed trees with Gromov-Hausdorff metric.

We call a subgraph \( T \subset \mathcal{G} \) an infinite tree if it is non-compact, connected and simply connected. The last condition implies that any loop in \( T \) is homotopic with fixed end-points to a trivial loop. Let \( X \) be the set of all infinite trees in \( \mathcal{G} \) containing the identity \( e \in G \). As a subset of \( \mathcal{G} \), a tree \( T \subset X \) has an induced length structure \( \ell \) and an induced metric \( D \). The induced metric \( D \) need not be geodesic, since \( T \) need not contain shortest paths between its points. However, \( D \) induces a length structure \( \ell' \) on \( T \) which coincides with the restriction of the length structure \( \ell \) from \( \mathcal{G} \), and they define a length metric \( d \) on \( T \). The pair \((T, e) \in X \) with metric \( d \) is a pointed metric space, or a pointed tree.
Let $X$ be the set of all infinite pointed trees in a locally compact Cayley graph $G$. Let $T,T' \in X$, and define the distance between $T$ and $T'$ by

$$(1) \quad d_X(T,T') = e^{-r(T,T')}, \quad r(T,T') = \max\{r \in \mathbb{N} \cup \{0\} \mid \exists \text{ isomorphism } B_T(e,r) \to B_{T'}(e,r)\}.$$ 

The metric $d_X$ is called the ball metric.

**DEFINITION 2.2.** Let $T = \{(T_n,e)\}_{n=1}^\infty$ be a sequence in $(X,d_X)$. Then $T$ converges to $(S,e) \in X$ if and only if for every $r > 0$ there exists $n_r > 0$ such that for any $n > n_r$ there is an isomorphism $B_{T_n}(e,r) \to B_S(e,r)$.

Using the ball metric $d_X$ on $X$ and the fact that $G$ is locally compact or, alternatively, the group $G$ is finitely generated, one can prove the following proposition.

**PROPOSITION 2.3.** The metric space $(X,d_X)$ is compact and totally disconnected. The space $(X,d_X)$ is perfect if and only if for every $r > 0$ and every $A \subset B_G(e,r)$ such that for some $(T,e) \in X$ there exist an isomorphism $A \to B_T(e,r)$, there is a least one $T' \neq T$ such that there is an isomorphism $A \to B_{T'}(e,r)$.

### 2.3. Pseudogroup action on the space of pointed trees.

We define a partial action of $F_n$ on the space of pointed trees $X$, which gives rise to a pseudogroup $\mathcal{F}$ on $X$.

Let $P_e(T)$ be the set of paths $\delta : [0,1] \to T$ such that $\delta(0) = e$, $\delta(1) = g \in V(T)$ and $\delta$ is the shortest path between $e$ and $g$ in $T$. The image of $\delta$ in $T$ is the union of edges

$$w_{h_{i_1}} \cup w_{h_{i_2}} \cup \cdots \cup w_{h_{i_m}}$$

where $h_{i_k} \in G_0$ for $1 \leq k \leq m$.

Thus $\delta$ defines a word $\tilde{h}_{i_1} \tilde{h}_{i_2} \cdots \tilde{h}_{i_m} \in F_n$, where

$$\begin{cases} 
\tilde{h}_{i_k} = h_{i_k}, & \text{if } \delta^{-1}(s(w_{h_{i_k}})) < \delta^{-1}(t(w_{h_{i_k}})), \\\n\tilde{h}_{i_k} = h_{i_k}^{-1}, & \text{if } \delta^{-1}(s(w_{h_{i_k}})) \geq \delta^{-1}(t(w_{h_{i_k}})).
\end{cases}$$

We note that $g = \tilde{h}_{i_1} \tilde{h}_{i_2} \cdots \tilde{h}_{i_m}$ is a representation in the set $G_0$ of generators of $G$. The procedure we have just described defines an injective map

$$p : P_e(T) \to F_n : \delta \mapsto \tilde{h}_{i_1} \tilde{h}_{i_2} \cdots \tilde{h}_{i_m},$$

and the action of $F_n$ on $X$ is defined as follows.

---

**Figure 1.** The action of $F_2$ on subtrees of the Cayley graph of $\mathbb{Z}^2$ with a set of generators $G_0 = \{a,b\}$, where $a$-edges are directed to the right, $b$-edges are directed upwards: a) $(T,e) \cdot aba^{-1}$ is defined, $(T,e) \cdot b$ is not defined, b) $(T,e) \cdot b$ is defined, $(T,e) \cdot aba^{-1}$ is not defined.
DEFINITION 2.4. Let \( n < \infty \) be the cardinality of a set \( G_{0} \) of generators of \( G \), and \((X, d_{X})\) be the corresponding set of pointed trees. An action of \( g \in F_{n} \) on \( X \) is defined as follows.

1. \((T, e) \cdot g \) is defined if and only if there exists a path \( \delta \in \mathcal{P}_{c}(T) \) such that \( p(\delta) = g \).
2. \((T', e) = (T, e) \cdot g \) if and only if there is an isomorphism of pointed spaces \( \alpha : (T, g) \rightarrow (T', e) \).

To a partial action of \( F_{n} \) on \( X \) we can associate a pseudogroup of local homeomorphisms \( \mathcal{G} \) as follows. For \( r > 0 \) denote by \( D_{X}(T, r) \) a clopen subset of diameter \( e^{-r} \) about \((T, e)\), that is,
\[
D_{X}(T, r) = \{(T', e) \mid d_{X}(T, T') \leq e^{-r}\}.
\]
For each \( g \in F_{n} \) let \( \ell_{g} = d_{F_{n}}(e, g) \). The action of \( g \) is defined on the union of clopen subsets
\[
D = \bigcup \{ D_{X}(T, \ell_{g}) \mid T \in X, \delta \in \mathcal{P}_{c}(T) \text{ such that } p(\delta) = g \},
\]
which is clopen since \( G_{0} \) is a finite set and so \( D \) is a finite union. The mapping
\[
\gamma_{g} : \text{dom}(\gamma_{g}) = D \rightarrow X_{n} : (T, e) \mapsto (T, e) \cdot g
\]
is a homeomorphism onto its image, and a pseudogroup \( \mathcal{G}_{n} \) is defined to be a collection
\[
\mathcal{G}_{n} = \bigsqcup \{ \gamma_{g} \mid g \in F_{n} \}
\]
of local homeomorphisms. A subset \( \mathcal{G}^{0}_{n} \) is defined by \( \mathcal{G}^{0}_{n} = \{ \gamma_{g} \in \mathcal{G} \mid g \in G_{0} \} \) is a generating set of \( \mathcal{G} \).

DEFINITION 2.5. \([6] \) Let \((X, d_{X})\) be a space of pointed trees, and \( \mathcal{G} \) be the pseudogroup on \( X \). Then \((T, e)\) and \((T', e)\) are \( \mathcal{R} \)-equivalent
\[
(T, e) \sim_{\mathcal{R}} (T', e)
\]
if and only if there exists \( \gamma_{g} \in \mathcal{G} \) such that \( \gamma_{g}(T, e) = (T', e) \). An equivalence class of \((T, e) \) in \( X \) with respect to \( \mathcal{R} \) is denoted by \( \mathcal{R}(T) \) and is called the orbit of \((T, e)\) under the action of \( \mathcal{G} \).

2.4. Foliated space \( \mathcal{M}_{G} \) with foliation by Riemann surfaces. We realize the pseudogroup dynamical system \((X, \mathcal{G})\) as the holonomy system of a smooth foliated space \( \mathcal{M}_{G} \).

THEOREM 2.6. \([19, 26] \) Let \( G \) be a finitely generated group, and \((X, d_{X})\) be the corresponding set of pointed trees with the action of a pseudogroup \( \mathcal{G} \). Then there exists a compact metric space \( \mathcal{M}_{G} \), and a finite smooth foliated atlas \( \mathcal{V} = \{ \phi_{i} : V_{i} \rightarrow X_{i} \} \), where \( U_{i} \subset \mathbb{R}^{2} \) is open, with associated holonomy pseudogroup \( \mathcal{P} \), such that the following holds.

1. The leaves of \( \mathcal{M}_{G} \) are Riemann surfaces.
2. There is a homeomorphism onto its image
\[
t : X \rightarrow \bigcup_{1 \leq i \leq \nu} X_{i},
\]
such that \( t(X) \) is a complete transversal for the foliation \( \mathcal{F} \), and \( \mathcal{P}|_{t(X)} = \mathcal{t}, \mathcal{G} \) is the pseudogroup induced on \( t(X) \) by \( \mathcal{G} \).

For completeness we give a sketch of the proof \([19, 26] \).

Sketch of proof. Let \( A \) be the finite set of connected subtrees of an open ball \( B_{G}(e, 1) \), excluding the subgraph consisting of a single point. Denote by \( n_{a} \) the number of edges of \( a \in A \). Let \( \Sigma_{a} \) be a compact surface with boundary homeomorphic to a 2-sphere with \( n_{a} \) disks taken out, and label the connected components of the boundary as follows: each boundary component corresponds to a labeled edge \( w_{h} \) of \( a \), a boundary component is labeled by \( h \) if \( s(w_{h}) = e \), and it is labeled by \( h^{-1} \) if \( t(w_{h}) = e \). Choose a Riemannian metric on \( \Sigma_{a} \) in such a way that each connected component of the boundary has a closed neighborhood \( Y_{a}^{n} \) isometric to \( \mathbb{R}/\mathbb{Z} \times [0, 1/2] \), and fix these metrics. For each \( a \in A \) choose a base point \( p_{a} \) in the interior of \( \Sigma_{a} \).

Form a disjoint union \( \bigsqcup_{a \in A} D_{X}(a, 1) \times \Sigma_{a} \), and identify neighborhoods of boundary components as follows. Suppose \((T, e) \) is \( D_{X}(a, 1) \) and \( \Sigma_{a} \) contains a boundary component marked by \( h \in G_{0} \cup G_{0}^{-1} \), where \( G_{0}^{-1} = \{ h^{-1} \mid h \in G_{0} \} \). Then there is a pointed tree
\[
(T', e) = (T, e) \cdot h \in D_{X}(a', 1),
\]
and $\Sigma_a$ has a boundary component marked by $h^{-1}$. We identify $\{(T, e)\} \times Y^a_h \subset \{(T, e)\} \times \Sigma_a$ and $\{(T', e)\} \times Y^{a'}_{h'} \subset \{(T', e)\} \times \Sigma_{a'}$ by setting

$$((T, e), (\theta, s)) \sim ((T', e), (-\theta, 1/2 - s)), \quad \theta \in [0, 1], \ s \in [0, 1/2].$$

Taking the quotient by this equivalence relation one obtains a space $\mathcal{M}_G := \bigsqcup_{a \in \mathbb{A}} D_X(a, 1) \times \Sigma_a / \sim.$

The obtained space $\mathcal{M}_G$ is compact foliated space [19, 26], with leaves Riemann surfaces. Given a clopen cover $Z_a$ of $D_X(a, 1)$, and a geodesically convex open cover $U_a$ of a surface $\Sigma_a$, by taking products of charts one constructs the required foliation cover $\mathcal{V}$, with tranverse space given by

$$\mathcal{X} = \bigsqcup \{Z_j^a \mid j \in J_a, \ a \in \mathbb{A}\}.$$

To obtain an embedding of $X$ notice that there is an embedding

$$i : X \to \mathcal{M}_G : (T, e) \mapsto i_a ((T, e), p_a),$$

and define $t : X \to \mathcal{X}$ as the obvious composition of (4) with the chart maps of $\mathcal{V}$. Define a metric $d_X$ on $\mathcal{X}$ as follows: since each $Z_j^a$ is a subset of $X$ we can define the metric $d_X$ on $Z_j^a$ by restricting the metric $d_X$. If $w \in Z_j^a$, $w' \in Z_j^a$, and $w, w' \in t(X)$, then set

$$d_X(w, w') = d_X(t^{-1}(w), t^{-1}(w')) ,$$

otherwise set $d_X(w, w') = 1$. The properties of $t$ stated in the formulation of the theorem follow straightforwardly from the construction.

2.5. Universal property. Proposition 1.2 is a consequence of Theorem 2.6. Given a pointed tree $(T, e) \in X$, we denote by $L_T$ a leaf such that $i(T, e) \in L_T$.

Proof. (of Proposition 1.2). Denote by $n$ the cardinality of the generating set $G_0$ of $G$. Identify $G_0$ with a generating set of a free group on $n$ generators $F_n$. Let $\langle X, d_X \rangle$ be a space of infinite pointed trees contained in the Cayley graph $\mathcal{G}$ of $G$, and $(X_n, d_X)$ be a space of infinite pointed trees contained in the Cayley graph $\mathcal{F}_n$ of $F_n$. Then every tree in $\mathcal{G}$ can be identified with a unique tree in $\mathcal{G}_n$, and so $X$ is identified with a subset of $X_n$. Thus there is an embedding $\Phi : X \to X_n$ such that $\Phi(X)$ is closed in $X_n$. It is then clear that the restricted pseudogroup $\Phi_* \mathcal{G} = \mathcal{G}_n|_{\Phi(X)}$, and the result follows as a consequence of Theorem 2.6.

2.6. Graph matchbox manifolds. The definition of a graph matchbox manifolds was previously given in Section 1, and we recall it here.

DEFINITION 2.7. Let $G$ be a finitely generated group, and $\mathcal{M}_G$ be a smooth foliated space obtained by the construction of Kenyon and Ghys. Then a graph matchbox manifold is the closure $\mathcal{M} = \overline{L}$ of a leaf $L$ in $\mathcal{M}_G$.

Given a pointed tree $(T, e) \in X$, we denote by $L_T$ a leaf such that $i(T, e) \in L_T$. It follows from the proof of Theorem 2.6 that $L_T \cap i(X) = i(\overline{\mathcal{R}(T)})$, where $\mathcal{R}(T)$ denotes the orbit of $(T, e) \in X$ under the action of the pseudogroup $\mathcal{G}$. As usual in foliation theory, one aims to relate asymptotic properties of orbits in $X$ with asymptotic properties of leaves in $\mathcal{M}_G$, and thus study the dynamics of graph matchbox manifolds via orbits of points in $X$. Using the construction in [14, section 2.2] of a cover well-adapted to the metrics $d_{\mathcal{M}}$ and $d_X$, and uniform metric estimates which measure distortions between $d_{\mathcal{M}}$ and $d_X$ [14, section 2.2], one obtains the following lemma.

LEMMA 2.8. $\mathcal{R}(T') \subset \overline{\mathcal{R}(T)}$ if and only if $L_{T'} \subset \overline{L_T}$.

In particular, if $(T, e) \subset X$ is a pointed tree in $X$, and $\mathcal{M}_T = \overline{L_T}$ is the corresponding graph matchbox manifold, then

$$\mathcal{M}_T = \left\{ L \subset \mathcal{M}_G \mid i\left(\overline{\mathcal{R}(T)}\right) \cap L \neq \emptyset \right\}.$$
As a consequence of Lemma 2.8, one obtains the relation between asymptotic properties of leaves in $\mathcal{M}_T$ and the topology of the set $\mathcal{R}(T)$. We first recall some definitions.

**Definition 2.9.** [8] Let $\{K_\alpha\}$ be the set of all compact subsets of $L_T$ and $W_\alpha = L_T - K_\alpha$ be the complement of $K_\alpha$ in $L_T$. Then the limit set, or the asymptote, of $L_T$ is the set

$$\lim L_T = \bigcap_\alpha W_\alpha.$$ 

A leaf $L_T$ is recurrent if $L_T \subset \lim L_T$. A graph matchbox manifold $\overline{L_T}$ is recurrent if $L_T$ is recurrent.

In a compact foliated space $\mathfrak{M}$, the limit set $\lim L$ is compact, non-empty and saturated [8]. The following corollary follows straightforwardly from Lemma 2.8 and from definitions.

**Corollary 2.10.** A graph matchbox manifold $\mathcal{M}_T = \overline{L_T}$ is recurrent if and only if $\mathcal{R}(T)$ is a Cantor set. If a matchbox manifold $\mathcal{M}$ has a recurrent leaf, then it contains an uncountable number of leaves. If a transitive leaf $L \subset \mathfrak{M}$ is not recurrent, then every point of $\mathcal{R}(T)$ is isolated.

Finally, it is convenient to formulate the following criterion of recurrence in terms of balls in graphs.

**Lemma 2.11.** A graph matchbox manifold $\mathcal{M}_T$ is recurrent if and only if there exists a sequence $\{g_i\} \in V(T)$, $g_i \neq e$, such that for every $r > 0$ there exists $l_r > 0$ such that for all $l \geq l_r$ there is an isomorphism

$$\alpha_r^T : D_T(e, r) \to D_T(g_i, r).$$

We also cite the minimality criterion for graph matchbox manifolds from Blanc [6].

**Lemma 2.12.** [6] Lemme 2.28] Given a graph $(T, e) \in X$ and any $(T', e) \in R(T)$, the equivalence class $R(T')$ is dense in $R(T)$ if and only if for every integer $r > 0$ there exists an integer $R > r$ such that for every vertex $g \in (T, e)$ there is a vertex $g' \in D_T(g, R - r)$ and an isomorphism

$$\alpha_r : D_T(e, r) \to D_T(g', r).$$

**2.7. Ends of leaves.** In this technical section, we make precise the relationship between the ends of a graph $T$ and the ends of the leaf $L_T$. A tool for that is the graph of the $\mathfrak{G}$-orbit of $(T, e) \in X$ which was considered in [20], and which we define now.

**Definition 2.13.** Let $(T, e) \in X$. A graph $\Gamma_T$ of the $\mathfrak{G}$-orbit $\mathcal{R}(T)$ is defined as follows: we set $V(\Gamma_T) = \mathcal{R}(T)$, and we join $(T', e)$ and $(T'', e) \in \mathcal{R}(T)$ by an oriented edge $w$ with $s(w) = (T', e)$ and $t(w) = (T'', e)$ if and only if there exists $\gamma_h \in \mathfrak{G}^0$ such that $\gamma_h(T', e) = (T'', e)$.

The graph $\Gamma_T$ is given a length structure in the usual manner so that the length of each edge is 1. We denote by $D$ the associated complete length metric.

Ends are the means to study asymptotic properties of a topological space $S$, and are a form of compactification of $S$. The following definition of ends also gives a way to compute the number of ends for $S$.

**Definition 2.14.** [8] Let $S$ be a Hausdorff locally compact locally connected separable topological space. Let $K_1 \subset K_2 \subset \ldots$ be an increasing sequence of compact subsets $K_i$ such that $\bigcup_i K_i = S$, and

$$\{U_{\alpha_i} : U_{\alpha_1} \supset U_{\alpha_2} \supset \ldots, \}$$

be a decreasing sequence, where $U_{\alpha_i}$ is an unbounded connected component of $S - K_i$. We say that $\{U_{\alpha_i}\}$ and $\{V_{\beta_i}\}$ are equivalent, $\{U_{\alpha_i}\} \sim \{V_{\beta_i}\}$, if for every $i > 0$ there exists $j > i$ such that

$$(U_{\alpha_i} \cap V_{\beta_i}) \supset (U_{\alpha_j} \cup V_{\beta_j}).$$

An end $e$ of $S$ is an equivalence class of $\{U_{\alpha_i}\}$ with respect to the equivalence relation $\sim$. 

We denote by $\mathcal{E}(S)$ the set of ends of $S$ and topologize $S^* = S \cup \mathcal{E}(S)$ as follows [8]. For every $e$ and $U_\alpha, e \in \{U_\alpha\}$, an open set $U_\alpha$ is called the fundamental neighborhood of $e$. The topology on $S^*$ is given by all open sets of $S$, plus all fundamental neighborhoods of ends together with ends contained in each neighborhood. We say that a sequence $\{x_n\} \in S^*$ converges to an end $e \in \mathcal{E}(S)$ if every fundamental neighborhood of $e$ contains all but a finite number of elements of $\{x_n\}$.

It was proved in \cite{9} Section 3.2], that the number of ends of a leaf in a foliated space is the same as the number of ends of a corresponding path-connected component of the holonomy graph of the foliated space. A very similar argument to that in \cite{9} Section 3.2], relying on the existence of embeddings

$$i_a : D_X(a, 1) \times \Sigma_a \to \M_G, \ a \in A,$$

which provide a cover of $\M_G$, and which allow us to associate to each orbit of $\mathcal{G}$ in $X$ a ‘plaque chain’ in $L_T$ made up of compact subsets isometric to $\Sigma_a, a \in A$, one obtains the following conclusion.

**Lemma 2.15.** Let $L_T \in \M_G$ be a leaf, and $\Gamma_T$ be the graph of $R(T)$. Then there is a homeomorphism $\mathcal{E}(L_T) \to \mathcal{E}(\Gamma_T)$.

Define a map $v_T : T \to \Gamma_T$ by $v_T(g) = (T, e) \cdot g$ on the sets of vertices $V(T)$ and $V(\Gamma_T)$, and map an edge $m \in E(T)$ isometrically onto the oriented edge starting at $v_T(s(m))$ and ending at $v_T(t(m))$.

**Lemma 2.16.** The map $v_T : T \to \Gamma_T$ is a covering projection. If $v_T$ is a homeomorphism, there is an induced homeomorphism $v_T^* : \mathcal{E}(T) \to \mathcal{E}(\Gamma_T)$ of end spaces.

2.8. Examples. Let $G = F_2$ and $G_0 = \{a, b\}$. As before, the Cayley graph of $F_2$ is denoted by $\mathcal{F}_2$.

**Genus two surface and a torus.** The graph $(\mathcal{F}_2, e)$ is invariant under the action of the pseudogroup $\mathcal{G}$, that is, for any $g \in G$ we have $\gamma_g(F_2, e) = (F_2, e)$. It follows that $L_{\mathcal{F}_2}$ is homeomorphic to a genus two surface.

Let $A \subset \mathcal{F}_2$ be a subgraph with the set of vertices $V(A) = \bigcup_{n \geq 0} \{a^n, a^{-n}\}$. Then $(A, e) \in \text{dom}(\gamma_{a^{\pm 1}})$, and $\gamma_{a^{\pm 1}}(A, e) = (A, e)$, and $L_A$ is homeomorphic to a genus two surface.

**Example of Kenyon and Ghys.** This example is described in detail in \cite{19} \cite{3}. Let $C_0$ be a subgraph of $\mathcal{F}_2$ with the set of vertices $V(C_0) = \{e, a, a^{-1}, b, b^{-1}\}$ (see fig. 2). Given a graph $(C_i, e)$, let $A_i^\delta$ and $B_i^\delta$, $\delta = \{-1, 1\}$ be subgraphs of $\mathcal{F}_2$ containing vertices $a^{\delta 2}$ and $b^{\delta 2}$ respectively and such that there are isometries

$$\alpha_i^\delta : A_i \to C_i \text{ with } \alpha_i^\delta(a^{\delta 2}) = e, \quad \beta_i^\delta : B_i \to C_i \text{ with } \beta_i^\delta(b^{\delta 2}) = e.$$

Then set $C_{i+1} = C_i \cup A_i^{-1} \cup A_i^1 \cup B_i^{-1} \cup B_i^1$, and let $K = \bigcup_{i \in \mathbb{N}} C_i$. The foliated subspace $\mathcal{M}_K$ is minimal \cite{3} \cite{19}. The graph $(K, e)$ is not invariant under the action of any $g \in \mathcal{F}_2$, and the corresponding leaf $L_K$ is homeomorphic to a sphere with four holes and has the hyperbolic plane as the universal cover. Every other leaf in $K$ has either 2 or 1 ends \cite{3} \cite{19}. Two-ended leaves are homeomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ and one-ended leaves are homeomorphic to $\mathbb{C}$.

3. Hierarchy of graph matchbox manifolds

Let $\M_G$ be a foliated space obtained by the construction of Kenyon and Ghys, and $\mathcal{S}$ be the set of graph matchbox manifolds, that is, the set of transitive saturated closed subsets of $\M_G$. In this section we study a partial order $\preceq$ on $\mathcal{S}$ given by inclusions, and prove Proposition 1.3.

**Definition 3.1.** Let $\M_G$ be a foliated space and $\mathcal{S}$ be the set of transitive saturated closed subsets in $\M_G$. A partial order $\preceq$ on $\mathcal{S}$ is given by the following rule:

Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{S}$, then $\mathcal{M}_1 \preceq \mathcal{M}_2$ if and only if $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

By Proposition 1.2 a partially ordered set $(\mathcal{S}, \preceq)$ is a subset of $(\mathcal{S}_n, \preceq)$, where $\mathcal{S}_n$ is the partially ordered set of graph matchbox manifolds in $\M_n$, $n = \text{card}(G_0)$. We can then restrict our study to $\M_n$. 
3.1. Compact leaves. We prove Proposition 1.3.1, that is, that the set

\[ C = \{ L \subset M_n | L \text{ is compact} \} \]

is a dense meager subset and \( C \cap X \) is countable, where \( X = i(X) \) is an embedding of \( X \) into \( M_n \) given by \( i \). The proof is contained in Lemmas 3.2 and 3.3.

We denote by \( \partial U \) the boundary of a subset \( U \subset F_n \). We notice that if \( r > 0 \) is an integer then the boundary \( \partial D_{F_n}(e,r) \) contains only vertices.

**Lemma 3.2.** The subset \( C = \{ L \subset M_n | L \text{ is compact} \} \), \( n > 1 \), is dense in \( M_n \).

**Proof.** By Lemma 2.8 it is enough to prove that the set \( C = \{ (S,e) \in X | L_S \in C \} \) is dense in \( X \), that is, for every \( (T,e) \in X \) and every \( \delta > 0 \) there is an \( (S,e) \in C \) such that \( d_X(T,S) < \delta \).

Given \( \delta > 0 \), choose \( 0 < r < \log \delta^{-1} \). The procedure of constructing \( S \) consists of attaching copies of a ball \( D_T(e,r) \) to a two-ended subgraph of \( F_n \) at regular intervals.

More precisely, since \( T \) is infinite, it has at least 1 end. Then there exists an edge \( w_h, h \in G_0 \), such that \( w_h \subset D_T(e,r) \) and \( w_h \cap \partial D_T(e,r) \neq \emptyset \). Set

\[ \beta = \begin{cases} 
1, & t(w_h) \in \partial D_T(e,r), \\
-1, & s(w_h) \in \partial D_T(e,r).
\end{cases} \]

Let \( v_2 = \partial D_T(e,r) \cap w_h \), and let \( v_1 \) be the other vertex of \( w_h \).

Choose \( h \in G_0 \) such that \( h \neq h \). Let \( c(t) \) be the shortest path joining \( e \in T \) with \( v_2 \). Then \( w_h \subset c(t) \). We associate to \( c(t) \) a finite word \( h_1 \cdots h_{k-1} h^\beta \in F_n \), where \( k = d_T(e,v_2) \), and \( h_s \in G_0 \) is such that...
for every \( 1 \leq s < k \) \( g = h_1 \cdots h_s \) is a vertex in \( c(t) \cap T \). Now let \( f = h_1 \cdots h_k h^{2\beta} \) and

\[
g^n = (fh^{-3\beta}(f)^{-1})^n, \quad g^{-n} = (fh^{-3\beta}(f)^{-1})^{-n}.
\]

Denote by \( \delta^+ \) and \( \delta^- \) respectively the subgraphs of \( F_n \) with the vertex sets

\[
V(\delta^+) = \bigcup_{i=1}^{3r} fh^i, \quad V(\delta^-) = \bigcup_{i=1}^{3r} fh^{-i}
\]

that is, \( \delta^\pm \) are paths comprised of \( 3r \) edges marked by \( \tilde{h} \) and starting at \( f \). The path \( \delta^+ \) traverses oriented \( \tilde{h} \)-edges in the positive direction, and \( \delta^- \) in the negative direction. Denote by \( V_0^+ = D_r(e, r) \cup \{ \tilde{w}_h \} \cup \delta_\pm \), where \( \tilde{w}_h \) is the edge with vertices \( g \) and \( f \). Let \( V^{\pm n} \) be a subgraph of \( F_n \) such that there is an isomorphism

\[
\alpha_{\pm n} : (V^{\pm n}, g^{\pm n}) \to (V_0^\pm, e).
\]

Then define

\[
S = (V_0^+ \cup V_0^-) \bigcup \left( \bigcup_{n \in \mathbb{Z} - \{0\}} V^n \right).
\]

The resulting infinite graph is connected and has two ends. By construction \( (S, e) \) is invariant under the action of \( g^{\pm n}, n \in \mathbb{N} \), and \( \Gamma_S \) is a finite graph. As desired, \( d_X ((T, e), (S, e)) < \delta. \)

Recall [25] that a set \( A \subset M \) is nowhre dense if \( M - \overline{A} \) is dense in \( M \). A subset \( A \subset M \) is meager (or of the first category) if it is the union of a countable sequence of nowhere dense sets.

Let \( (T, e) \in X \) be a graph and let \( \Gamma_T \) be the graph of \( R(T) \). We construct a section of the covering projection \( \pi_T : T \to \Gamma_T \) (see Section 2.7) on the set of vertices \( V(T) \) as follows. For every \( (T', e) \in V(\Gamma_T) \) let \( g_{T'} \in V(T) \) be a vertex such that

\[
d(e, g_{T'}) = \min \{ d(e, g) \mid g \in v_{T'}^{-1}(T', e) \},
\]

and let \( F(T) \) be a subgraph of \( T \) with the set of vertices \( V(F) = \bigcup_{(T', e) \in V(\Gamma_T)} g_{T'} \), that is, \( F(T) \) is a maximal subgraph of a fundamental domain of \( v_T \).

**PROPOSITION 3.3.** The set \( C \) of compact leaves is a meager subset of \( \mathfrak{M}_n \). Moreover, \( C \cap \iota(X) \) is countable.

**Proof.** Notice that \( C = \iota^{-1}(C \cap \iota(X)) \), where \( C \) is defined as in the proof of Lemma 3.2. By a similar argument as in [10], and with the help of Lemma 2.8 one can show that \( C \) is meager if and only if \( C \) is meager.

A leaf \( L_T \) is compact if and only if \( R(T) \) is a finite set. In this case \( F(T) \) is compact. For each \( r \in \mathbb{N} \) define

\[
A_r = \{(T, e) \in X \mid \text{card}(R(T)) \leq r \text{ and } F(T) \subset D_r(e, r) \}.
\]

Since \( D_{\mathfrak{F}_n}(e, r) \) contains at most a finite number of distinct subgraphs, there is a finite number of distinct \( F(T) \subset D_r(e, r) \). Thus the set \( A_r \) is finite and, therefore, nowhere dense in \( X \). Since \( C = \bigcup_{r \geq 0} A_r \), the set \( C \) is meager and countable.

**REMARK 3.4.** Alternatively, one can argue that given a compact leaf, one can easily construct a one-ended leaf accumulating on the former, which shows that the set \( C \) contains only leaves with non-trivial holonomy. Since the set of leaves without holonomy in every foliated space is residual by the result of Epstein, Millet, and Tischler [17] (and also observed independently by Hector [22]) the set \( C \) must be meager.
3.2. Directed partially ordered set and fusion of graph matchbox manifolds. In this section we prove Proposition 1.3.(2), that is, that $(S, \leq)$ is a partially ordered set, by means of the fusion technique, which associates to any given graph matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ a graph matchbox manifold $\mathcal{M}_3$ such that $\mathcal{M}_3 \supseteq \mathcal{M}_1 \cup \mathcal{M}_2$.

Proof. (of Proposition 1.3.(2)). By Lemma 2.7 it is enough to show that if $(T, e), (T', e) \in X$, then there exists $(S, e)$ such that $(T, e), (T', e) \in \mathcal{R}(S)$. The procedure of constructing $S$ consists of taking a 4-ended graph $C_1$ and attaching to it copies of $D_T(e, r)$ and $D_{T'}(e, r)$, $r > 0$, in a certain order, so that the graph $Γ_S$ of the orbit $\mathcal{R}(S)$ has 4 ends, there is at least one end accumulating on $(T, e)$, and at least one end accumulating on $(T', e)$. Recall from Definition 2.13 that the set of vertices $V(Γ_S) = \mathcal{R}(S)$.

Let $(S, e) \in X$. The leaf topology on $\mathcal{R}(S)$ is the topology induced from $Γ_S$. A point $(T, e) \in \mathcal{R}(S)$ is in the limit set $\lim_s Γ_S$ of an end $e \in E(Γ_S)$ if and only if there exists a sequence $\{(S_k, e)\} \in Γ_S$ converging to $e \in E(Γ_S)$ in the leaf topology, and to $(T, e)$ in the ambient topology. Since $\mathcal{R}(S)$ is compact, $\lim_s Γ_T$ is a non-empty compact saturated subset. In the rest of the proof we use Lemma 2.16, which allows us to identify the spaces $E(S)$ and $E(Γ_S)$ in the case when $v_T : S → Γ_S$ is homeomorphism.

Choose $h, \tilde{h} \in F^0_n$, and let $C_1 \subset \mathcal{F}_n$ be a subgraph with the set of vertices $V(C_1) = \bigcup_{n \in \mathbb{Z}} \{h^n, \tilde{h}^n\}$. Then $C_1$ has 4 ends. For $v \in \{h, \tilde{h}\}$ denote

$e^+_v = \lim_{n \to \infty} v^n,$

$e^-_v = \lim_{n \to -\infty} v^{-n},$

that is, $\{h^n\}_{n \in \mathbb{N}}$ represents an end $e^+_h$ and $\{\tilde{h}^{-n}\}_{n \in \mathbb{N}}$ represents an end $e^-_h$. Let $c^+_v$ and $c^-_v$ respectively be infinite edge paths starting at $e$ and comprised from the shortest paths between the elements of the sets $\{v^n\}$, and $\{v^{-n}\}$ so that $C_1 = c^+_{e_h} \cup c^-_{e_h} \cup c^+_{e_{\tilde{h}}} \cup c^-_{e_{\tilde{h}}}$. By Lemma 2.16 the graph $Γ_C_1$ has four ends. We will obtain the graph $S$ by attaching finite decorations to $c^+_{e_h}$ and $c^+_{e_{\tilde{h}}}$ so that

$Γ_T \subset \lim_s Γ_S \cup \lim_s Γ_S,$

and to $c^-_{e_h}$ and $c^-_{e_{\tilde{h}}}$ so that

$Γ_{T'} \subset \lim_s Γ_S \cup \lim_s Γ_S.$

If $T \neq T'$ there exists $r_0 > 0$ such that $D_T(e, r_0) \neq D_{T'}(e, r_0)$. Otherwise set $r_0 = 2$. Since $T$ and $T'$ are infinite, each of them has at least one end. Choose infinite edge paths $δ$ in $T$, and $δ'$ in $T'$ without intersections, such that the vertices of edges in the paths are sequences of points in $T$ and $T'$ converging to ends.

For $i > 0$ let

$r_i = 2^{r_0+i} - 1$ and $R_i = \sum_{k=1}^{i-1} r_k.$

Let $w_i$ be an edge of $δ$ such that $w_i \subset D_T(e, r_i)$ and $w_i \cap ∂D_T(e, r_i) ≠ \emptyset$. Set

$β = \begin{cases} 1, & \text{if } t(w_i) \in ∂D_T(e, r_i), \\ -1, & \text{if } s(w_i) \in ∂D_T(e, r_i). \end{cases}$

Let $v_i \in G_0$ be a letter labeling $w_i$. The intersection $δ_i = D_T(e, r_i) ∩ δ$ is a finite path in $T$ of length $k_i$, and there is a finite word $E_i = h_{1} \cdots h_{k_i-1} v^β_i$ such that $g ∈ δ_i$ is a vertex in $δ_i$ if and only if $g = h_1 \cdots h_s$ for $s < k_i$. Set

$E_i = \begin{cases} ̂h R_i^{+r}, E_i^{-1}, & \text{if } v_i ≠ ̂h, \\ h R_i^{+r}, E_i^{-1}, & \text{if } v_i = ̂h. \end{cases}$
Let $C_i$ be a subgraph of $F_n$ containing $\bar{E}_i$ and such that there is an isomorphism $\alpha_i : C_i \to D_T(e, r_i)$ with $\alpha_i(\bar{E}_i) = e$. Then the union

$$S' = \bigcup_{i > 0} C_i$$

is a connected subgraph of $F_n$ such that the $\mathfrak{G}_n$-action has trivial isotropy group at $S'$. By construction

$$(T, e) \in \lim_{\leftarrow} \Gamma_{S'} \cup \lim_{\to} \Gamma_{S'}.$$ 

Implementing a similar algorithm for $(T', e)$, that is, attaching balls isomorphic to $D_T(e, r_i)$, $r_i > r_0$, to the ends $e_h$ and $e_h'$ of $S'$, one obtains a graph $S$ satisfying conditions [5] and [6].

3.3. Maximal transitive components. We prove Proposition 1.3.(3), which says that $\mathfrak{M}_n$ contains recurrent leaves and so is a maximal element of $(\mathcal{S}_n, \preceq)$. We need [6] Theorem 3.5, whose proof we give here for convenience of the reader.

**Lemma 3.5.** Let $S' = \{M_i\}_{i \in \mathbb{N}}$ be a totally ordered infinite subset of $(\mathcal{S}_n, \preceq)$. Then

$$\mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i$$

contains a recurrent leaf. Therefore, $\mathcal{S}_n$ contains a maximal element.

**Proof.** [6] Theorem 3.5] Without loss of generality we can assume that every $M_{i-1}$ is a proper subset of $M_i$, and so no leaf $L \subset M_i$ is dense in $\mathcal{M}$. Let $\{x_i\}$ be a sequence of points such that $x_i \in M_i - M_{i-1}$. Since $\mathcal{M}$ is closed it is compact and $\{x_i\}$ has a limit point $x$. By construction $x \notin \bigcup_{i \in \mathbb{N}} M_i$, so $\mathcal{M} - \bigcup_{i \in \mathbb{N}} M_i$ is non-empty.

Let $\{U_i\} \subset \mathcal{M}$ be a system of open neighborhoods of $x$, such that $\bigcap_{i} U_i = x$. For every $U_i$ denote by $\hat{U}_i$ its saturation. The set $\hat{U}_i$ is open, and we claim that $\hat{U}_i$ is dense in $\mathcal{M}$.

Indeed, let $V \subset \mathcal{M}$ be open and let $\hat{V}$ be its saturation. Then there exists $M_{k_i}, M_{k_i'} \in S'$ such that $\hat{U}_i \cap M_{k_i} \neq \emptyset$, and $\hat{V} \cap M_{k_i'} \neq \emptyset$. Since $S'$ is totally ordered, we have either $M_{k_i} \preceq M_{k_i'}$ or $M_{k_i'} \preceq M_{k_i}$. For definitiveness assume the former. Let $L \subset M_{k_i'}$ be a transitive leaf, then $L \subset \hat{U}_i \cap V$, which implies that $\hat{U}_i$ is dense in $\mathcal{M}$.

The intersection $\bigcap_{i} \hat{U}_i$ of a countable family of open dense subsets is exactly the leaf $L_x \ni x$. Since $\mathcal{M}$ is compact and Hausdorff, it is a Baire space [35], and $L_x$ is dense in $\mathcal{M}$. Since $L_x$ is in the boundary of $\bigcup_{i \in \mathbb{N}} M_i$, $L_x$ is recurrent and $\mathcal{M} \in \mathcal{S}_n$.

It follows that every totally ordered subset of $(\mathcal{S}_n, \preceq)$ has an upper bound which is a recurrent graph matchbox manifold if $S'$ is an infinite chain, and a transitive graph matchbox manifold in the case when $S'$ is finite. Applying Zorn’s lemma we conclude that $\mathcal{S}_n$ contains a maximal element. □

Recall [25] that a set $A \subset \mathcal{M}$ is residual if its complement $\mathcal{M} - A$ is meager.

**Proof.** (of Proposition 1.3.(3)) Let $\mathcal{M}_{\text{max}}$ be a maximal element in $(\mathcal{S}_n, \preceq)$, and $\mathcal{M}$ be any graph matchbox manifold. By Proposition 1.3.(2) there exists a matchbox manifold $\mathcal{M}'$ such that $\mathcal{M}_{\text{max}} \cup \mathcal{M}$ is a subset of $\mathcal{M}'$. Since $\mathcal{M}_{\text{max}}$ is maximal, $\mathcal{M}_{\text{max}} = \mathcal{M}'$ and, therefore, $\mathcal{M} \subset \mathcal{M}_{\text{max}}$. It follows that there is a unique maximal element which contains every graph matchbox manifold $\mathcal{M}$. Then necessarily $\mathcal{M}_{\text{max}} = \mathfrak{M}_n \in (\mathcal{S}_n, \preceq)$. Since $X_n$ is a Cantor set, $\mathfrak{M}_n$ is recurrent. □
4. Theory of levels for graph matchbox manifolds

In this section, inspired by ideas of Cantwell and Conlon [11] [12], and also of Nishimori [31] [32], Tsuchiya [37] and Hector [21] for codimension 1 transversally $C^2$-differentiable foliations, we introduce a quantifier of dynamical complexity of a set in $(S_n, \preceq)$ which, following [11], we call level. We repeat now Definition 1.5.

**DEFINITION 4.1.** Let $\mathcal{M} \subset \mathfrak{M}_n$ be a graph matchbox manifold.

1. $\mathcal{M}$ is said to be at level 0 if either $\mathcal{M}$ is a compact leaf, or $\mathcal{M}$ is a minimal foliated space. In that case all leaves of $\mathcal{M}$ are also at level 0.

2. $\mathcal{M}$ is at level $k$ if the closure of the union of leaves which are not dense in $\mathcal{M}$, is a proper closed subset of $\mathcal{M}$, every such leaf is at level at most $k-1$, and there is at least one leaf at level $k-1$. If $\mathcal{M}$ is at level $k$, a leaf $L \subset \mathcal{M}$ is at level $k$ if and only if $L$ is dense in $\mathcal{M}$.

3. A leaf $L$ is at infinite level if it is not at finite level.

The goal of this section is to prove Theorems 1.7, 1.10 and 1.13.

4.1. Leaves at infinite levels. We investigate leaves at infinite levels and prove Theorem 1.7. We first notice that Definition 4.1 allows for two types of leaves at infinite levels.

Let $\mathcal{M} = \overline{T}$. Then $\overline{T} = \mathfrak{M}_n$ is not at a finite level if one of the following holds:

1. **Type 1.** The union of leaves which are not dense in $\mathcal{M}$ is dense in $\mathcal{M}$, and $\mathcal{M}$ is a proper subset of $\mathfrak{M}_n$.

2. **Type 2.** The union of leaves which are not dense in $\mathcal{M}$ is a proper subset of $\mathcal{M}$ and contains a leaf at infinite level.

Theorem 1.7 states that $\mathfrak{M}_n$ contains leaves of both types. A proof of that is given in the following two propositions.

**PROPOSITION 4.2.** In the space of graph matchbox manifolds $\mathfrak{M}_2$ there exists a leaf $L$ and a graph matchbox manifold $\mathcal{M} = \overline{T}$ at infinite level of Type 1.

**Proof.** We have to prove that there exists an infinite increasing chain

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$$

of distinct graph matchbox manifolds such that $\mathcal{M} = \bigcup_i \mathcal{M}_i$ is a proper subset of $\mathfrak{M}_n$. Then $\mathcal{M}$ is a graph matchbox manifold at infinite level of Type 1.

Let $F_2^2 = \{a, b\}$, and let $T_0$ be a subgraph of $F_2$ with the vertex set $V(T_0) = \cup_{n \geq 0} \{a^n, a^{-n}\}$. then $L_0 = L_{T_0}$ is a compact leaf. Let $T_1$ be a subgraph of $F_2$ with the vertex set $V(T_1) = \cup_{n \geq 0} a^n$. Then $L_1 = L_{T_1}$ is a proper leaf, and $\mathcal{M}_1 = \overline{L_1}$ contains $L_1$ and $L_0$. For $n > 2$ construct a graph $T_n$ by applying the fusion construction of Section 3.2 with $r_0 = 2$. Define the depth $k(T_n)$ to be the maximal number of changes from an edge labeled by $a$ to an edge labeled by $b$ or vice versa in a finite path in $T_n$, starting at the origin. It is easy to see that $k(T_n) \geq k(T_{n-1}) + 1$. This implies that for $R > 0$ large enough to contain a path of depth $k(T_n)$, there is no isomorphism from a closed ball in $T_n$, $i < n$, onto a closed ball of radius $R$ around the origin in $e$, and so $T_n \notin \bigcup_{0 \leq i < n} B(T_i)$. Therefore, all $T_n$ and $\mathcal{M}_n = \overline{L_{T_n}}$ are distinct and the complement of $\cup_{n \geq 0} \mathcal{M}_n$ in the closure

$$\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n$$

is non-empty. Then by Lemma 3.3 $\mathcal{M}$ is a recurrent graph matchbox manifold at infinite level of Type 1. We notice that $\mathcal{M}$ is a proper subset of $\mathfrak{M}_2$ since it is easy to find a finite subgraph $S \subset B_{F_2} (e, 3)$ which is isomorphic to no subgraph of $T_n$ for every $n \geq 0$. □
PROPOSITION 4.3. Let $\mathcal{M}$ be a graph matchbox manifold at infinite level and suppose $\mathcal{M}$ is a proper subset of $\mathcal{M}_n$. Then there exists a graph matchbox manifold $\tilde{\mathcal{M}}$ such that

$$\mathcal{M} \subset \tilde{\mathcal{M}} \subset \mathcal{M}_n$$

are proper inclusions. The space $\tilde{\mathcal{M}}$ is at infinite level and of Type 2.

Proof. Let $L \subset \mathcal{M}$ be a dense leaf, and $(T_L, e) \in i^{-1}(L \cap i(X))$. By assumption the complement $\mathcal{M}_n - \mathcal{M}$ is open, then we can find a compact leaf $C$ such that $C \cap \mathcal{M} = \emptyset$. Let $T_C \in i^{-1}(C \cap i(X))$, and let $T_R$ be a graph obtained by fusion on $T_L$ and $T_C$ (see Section 3.2), and $\Phi$ be the corresponding leaf. Then $\Phi \supset L \cup C$. The complement of $\Phi$ in $\mathcal{M}_n$ is open, and to prove the proposition we have to show that $\Phi$ can be chosen in such a way that the complement of $\Phi$ is non-empty.

The leaf $\Phi$ can either be recurrent or proper. If $\Phi$ is proper, then every point of $i^{-1}(\Phi \cap i(X_n))$ is isolated in $X_n$, and since $X_n$ is a Cantor set, this implies that the complement $\mathcal{M}_n - \Phi$ is non-empty. We show that $\Phi$ is proper.

Since $C \cap \mathcal{M} = \emptyset$, and $\Phi \supset C$, $\tilde{T}$ is a proper subset of $\Phi$. Suppose $\Phi$ accumulates on itself, then by Lemma 2.11 there exists a sequence $\{g_r\} \in V(T_R)$, $g_r \neq e$, such that for every $r > 0$ there exists $\ell_r > 0$, and for all $\ell \geq \ell_r$ there is an isomorphism

$$\alpha_r^r : DT_R(e, r) \rightarrow DT_R(g_r, r).$$

In the fusion construction of Section 3.2 let $h, \tilde{h} \in F_n^0$ and let $A$ and $B$ be subgraphs of $\mathcal{F}_n$ with $V(A) = \cup_{n \in \mathbb{Z}} h^n$ and $V(B) = \cup_{n \in \mathbb{Z}} \tilde{h}^n$. Recall that $T_R$ was obtained by attaching copies of closed balls $DT_L(e, n)$ and $DT_C(e, n)$ to the union of $A$ and $B$ in $\mathcal{F}_n$. Suppose there exists a subsequence $\{g_{k_r}\}$ such that for $r > r_k$ there is an isomorphism onto its image

$$DT_R(g_{k_r}, r) \subset DT_L(e, n_k),$$

$n_k \geq r$. Then $\Phi \subset \tilde{T}$ which contradicts the construction of $\Phi$. Therefore, without loss of generality we can assume that $DT_R(g_r, r)$ is not isomorphic to a subset of $DT_L(e, n)$ for any $\ell, r, n > 0$. By a similar argument we can assume that $DT_R(g_r, r)$ is not isomorphic to a subset of $DT_C(e, n)$ for any $\ell, r, n > 0$.

Suppose $g_r$ is a vertex in the copy of $DT_L(e, n_k)$ attached to $A \cup B$, and let $a_k$ be the length of the shortest path between $g_r$ and the point of the attachment. Then either $\{a_k\}$ contains a monotonically increasing subsequence, or $\{a_k\}$ converges to an integer $a$. In the first situation occurs, then it is possible to choose a subsequence of $\{g_r\}$ satisfying the property [7], which contradicts the choice of $\Phi$. The second situation cannot occur either by the following argument. Without loss of generality we can assume that $g_r$ is at distance $a$ from $A$, and $v_t \in A$ is such that $d_{T_R}(g_r, v_t) = a$. Then for $r > a$ the isomorphism between $DT_R(g_r, r)$ and $DT_R(e, r)$ implies that $v_t$ has 2 adjacent $b$-edges, which is impossible by the construction. By a similar argument $\{g_r\}$ cannot contain a subsequence of vertices lying in the copies of $DT_C(e, n)$.

Therefore, $\{g_r\} \in V(A \cup B)$. Passing to a subsequence without loss of generality we can assume that $g_r \in V(A)$. But such a sequence cannot exist since every point of $A$ except the origin is a vertex of at most three edges, while the origin is a vertex of four edges. It follows that $\Phi$ is a proper leaf, and the complement $\mathcal{M}_n - \Phi$ is non-empty. Then $\tilde{\mathcal{M}} = \Phi$ is at infinite level of Type 2. \qed

4.2. Proof of Theorem 1.10. Recall [8] that a graph matchbox manifold $\mathcal{M}$ is totally proper if it contains only proper leaves [11]. Theorem 1.10 states that there exists a totally proper graph matchbox manifold $\mathcal{M}$ at level 2 which contains a countably infinite number of distinct compact leaves. First we need a few auxiliary lemmas. Let $F_n^0 = \{a, b\}$.

LEMMA 4.4. Given $m \geq 0$, there exists $(B_m, e) \in X$ such that the leaf $L_m = L_{B_m} \subset \mathcal{M}_n$ is compact, and all leaves $L_m$ are distinct.
Figure 3. Graphs $B_m$, $m = 0, 1, 2$.

Figure 4. Graph $C$.

**Proof.** Let $B_0$ be a subgraph of $F_2$ with the set of vertices $V(B_0) = \bigcup_{k \geq 0}\{b^k, b^{-k}\}$ (see Fig. 3). Given $m \in \mathbb{N}$ let $B_m$ be a subgraph of $F_2$ with the set of vertices $V(B_m) = V(B_0) \cup \bigcup_{k \in \mathbb{Z}} \{b^m_k, b^{-m}_{k-1}\}$. Then $B_m$ is invariant under the action of $b^{m_k}$, and the corresponding leaf $L_m$ is compact and homeomorphic but not isometric to a standard 2-torus. □

**Lemma 4.5.** There exists a graph $(C, e) \in X$ such that the leaf $L_C \subset M_n$ contains a single leaf $L_0$ in its limit set. It follows that $M_C$ is at level 1.

**Proof.** Let $A_0$ be a subgraph of $F_2$ with the set of vertices $V(A_0) = \bigcup_{k \geq 0}\{a^k, a^{-k}\}$. For every $k > 0$ let $\tilde{B}_{\delta k}$, $\delta = \pm 1$, be a subgraph of $F_2$ containing $a^k$ such that there is an isometry $\alpha_{\delta k} : B_0 \to \tilde{B}_{\delta k}$ with $\alpha_{\delta k}(e) = a^k$. Set (see Fig. 4)

$$C = A_0 \cup B_0 \cup \left(\bigcup_{k > 0}\{\tilde{B}_k, \tilde{B}_{-k}\}\right).$$

The pseudogroup $\mathfrak{G}_C$ of the corresponding matchbox manifold $M_C$ is generated by $\gamma_a$ and $\gamma_b$, and the graph $(C, e)$ is invariant under $\gamma_a$. We have

$$\text{dom}(\gamma_a) \cap \mathcal{R}(C) = (C, e), \quad \mathcal{R}(C) \subset \text{dom}(\gamma_b).$$

Clearly $(B_0, e) \subset \overline{\mathcal{R}(C)}$, so $L_0 \subset \lim L_C$. We now have to show that there are no other leaves in $L_C$.

Suppose $(T, e) \in \overline{\mathcal{R}(C)}$. Then there exists a sequence $(C, g_s)$, $s \in \mathbb{N}$, such that for every $r > 0$ there exists $s_r$ such that for all $s > s_r$ there is an isometry

$$\alpha_s^r : D_C(g_s, r) \to D_T(e, r).$$

Since $\gamma_a(C, e) = (C, e)$ we can assume that $g_s = b^{k_s}$, $k_s \in \mathbb{Z}$. Suppose there exists $m \in \mathbb{N}$ such that for $s > m$ and all $r > 0$ we have $D_C(b^{k_s}, r - 1) \cap A_0 = \emptyset$. Then $(T, e) = (B_0, e)$. If not, let $m > 0$ be such that $D_C(b^{k_m}, r - 1) \cap A_0 = \emptyset$. Then for all $s > m$ we have $d_C(b^{k_s}, e) = d_C(b^{k_m}, e)$ and $b^{k_s} = b^{k_m}$. Then $(T, e) = (C, b^{k_m})$. Thus $M_C$ contains only 2 leaves, one of which is at level 0, and it follows that $M_C$ is at level 1. □
REMARK 4.6. Of course, there are many examples which satisfy requirements of Lemma 4.5. Let $B'_0$ be a subgraph of $\mathcal{F}_2$ with the set of vertices $V(B'_0) = V(B_0) \cup \{a, a^{-1}\}$. Then $L_0 \subset \mathcal{M}_{B'_0}$, and an argument similar to the one in Lemma 4.5 shows that $\mathcal{M}_{B'_0}$ is at level 1.

The following proposition completes the proof of Theorem 1.10.

PROPOSITION 4.7. There exists $(T, e) \in X$ such that the graph matchbox manifold $\mathcal{M}_T$ has the following properties.

1. $\bigcup_m L_m \subset \mathcal{M}_T$, where $L_m$ are as in Lemma 4.4.
2. For every $m \in \mathbb{N}$ there is a distinct end $e_m$ such that $L_m \subset \lim_{e_m} L_T$.
3. $\mathcal{M}_T$ is at level 2.

Proof. Let $A_0$ be a subgraph of $\mathcal{F}_2$ with the set of vertices $V(A_0) = \bigcup_{k \geq 0} \{a^k, a^{-k}\}$. For every $k > 0$ let $\tilde{B}_{ik}, \delta = \pm 1$, be a subgraph of $\mathcal{F}_2$ containing $a^{\delta k}$ such that there is an isometry $\alpha_{\delta k} : B_k \rightarrow \tilde{B}_{ik}$ with $\alpha_{\delta k}(e) = a^{\delta k}$, where $B_k$ was constructed in Lemma 4.4. Set

$$T = A_0 \cup B_0 \cup \left( \bigcup_{k \geq 0} \{\tilde{B}_k, \tilde{B}_{-k}\} \right).$$

Then (1) and (2) are clearly satisfied. We have to show that $\mathcal{M}_T$ is at level 2.

We first notice that $\mathcal{M}_T$ is at level at least 2, since $\mathcal{M}_T \supset \mathcal{M}_C$ where $\mathcal{M}_C$ was obtained in the proof of Lemma 4.5. To see that notice that, given $r > 0$, for every $s \geq s_r = r$ there is an isometry $\alpha_s^T : D_T(a^{2s}, r) \rightarrow D_C(e, r)$, and so the sequence $\{T, e \cdot a^{2r}\}$ converges to $(C, e)$.

We now show that non-dense leaves in $\mathcal{M}_T$ are at level at most 1, and so $\mathcal{M}_T$ is at level 2. Suppose $(T', e) \in \mathcal{K}(T)$. Then there is a sequence $(T, g_s)$ converging to $(T', e)$, i.e. for every $r \geq 2$ there exists $s_r > 0$ such that for every $s \geq s_r$ there is an isometry $\alpha_s^T : D_T(g_s, r) \rightarrow D_T(e, r)$. The following argument is just a more complicated version of the one in Lemma 4.5. Let

$$\ell = \min_s \{ \text{dist}(g_s, A_0) \mid s \in \mathbb{N} \},$$

and let $s_1$ be such that $\text{dist}(g_{s_1}, A_0) = \ell$. Consider the following cases.

Case 1. Suppose $\ell = 0$. Then $g_{s_1} = a^{m_1}, m_1 \in \mathbb{Z}$. Since $r \geq 2$, $D_T(g_{s_1}, r)$ contains an edge path of $a$-edges of length at least 4, and it follows that for all $r \geq r_1$ and all $s \geq s_1$ we have

$$g_s = a^{m_s}, m_s \in \mathbb{Z}.$$

Then two situations are possible: either there exists $r_2 \geq r_1$ and $s_2 \geq s_1$ such that $D_T(g_{s_2}, r_2)$ contains another path of $a$-edges which necessarily has length 2, or such an $r_2$ does not exist. In the first case it follows that for all $r \geq r_2$ and all $s \geq s_2$ we have $g_s = a^{m_{s_2}}, m_2 \in \mathbb{Z}$. Since $\{T, e \cdot g_s\}$ converges, there is an $s_3 \geq s_2$ such that for all $s \geq s_3$ either $g_s = g_{s_3} = a^{m_2}$ or $g_s = g_{s_3} = a^{-m_2}$. If the former is true, then $(T', e) = (T, e) \cdot a^{m_2}$, otherwise $(T', e) = (T, e) \cdot a^{-m_2}$. In the second case for every $r \geq r_1$ and $s \geq s_1$ one constructs an isometry $D_T(g_s, r) \rightarrow D_C(e, r)$, and it follows that $(T', e) = (C, e)$.

Case 2. Suppose $\ell > 0$, and set $\ell_s = \text{dist}(g_s, A_0), s \in \mathbb{N}$. Let $m_s$ be such that $\ell_s = d_T(a^{m_s}, g_s).

Case 2.1. Suppose there exists $r_2 \geq 2$ and $s_1 > 0$ such that for all $r \geq r_1$ and all $s \geq s_1$ we have $\ell_s \geq r$. Then either

$$g_s = a^{m_s}b^k, |k| > 1,$$

or

$$g_s = a^{m_s}b^ka^\beta, |k| > 1, \beta = \pm 1.$$
where \( d_H \) is the distance between these sets. Then for every \( r > r_2 \) the ball \( D_T(g_s, r) \) must contain two distinct paths of \( a \)-edges such that the distance between them is exactly \( \ell_s \). Then either \( g_{s_2} \in V(B_{\ell_s}) \) or \( g_s \in V(B_{-\ell_s}) \), and the same is true for \( g_s \) for \( s \geq s_2 \). It follows that \( (T', e) \in \mathcal{R}(B_{\ell_s}) \), as we show now.

Indeed, if \( g_s = a^{m_s}b^k \), let \( \tilde{g} \in V(T) \) be such that

\[
d_T(g_s, g_s \tilde{g}) = \text{dist}(g_s, c_1 \cup c_2),
\]

and if \( g_s = a^{m_s}b^k a^{\beta} \), let \( \tilde{g} = a^{-\beta} \). Then for every \( r \geq r_2 \) and every \( s \geq s_2 \) there is an isometry

\[
\alpha^r_s : D_T(g_s, r) \to D_{B_{\ell_s}}(\tilde{g}^{-1}, r),
\]

which implies \( (T', e) = (B_{\ell_s}, e) \cdot \tilde{g}^{-1} \).

**Case 2.1.2.** Second, there may exist \( r_2 \geq r_1 \) and \( s_2 \geq s_1 \) such that for all \( r \geq r_2 \) and all \( s \geq s_2 \) the ball \( D_T(g_s, r) \) contains exactly one path of \( a \)-edges \( c_1 \), which is necessarily of length 2. Then \( r' > r \) implies \( s_{r'} > s_r \), that is, there is a subsequence \( \{g_{s_i}\} \subset \{g_s\} \) such that distinct \( g_{s_i} \) lie in distinct \( B_{\ell_i} \). We claim that in this case \( (T', e) \in \mathcal{R}(B'_{\ell_0}) \), where \( B'_{\ell_0} \) is described in Remark 4.6.

Indeed, if \( g_s = a^{m_s}b^k \), let \( \tilde{g} \in V(T) \) be such that

\[
d_T(g_s, g_s \tilde{g}) = \text{dist}(g_s, c_1),
\]

and if \( g_s = a^{m_s}b^k a^{\beta} \), let \( \tilde{g} = a^{-\beta} \). Then for every \( r \geq r_2 \) and every \( s \geq s_2 \) we can construct an isometry

\[
\alpha^r_s : D_T(g_s, r) \to D_{B_{\ell_0}}(\tilde{g}^{-1}, r),
\]

which implies \( (T', e) = (B_{\ell_0}, e) \cdot \tilde{g}^{-1} \).

**Case 2.1.3.** Third, if for all \( r \geq r_1 \) and all \( s \geq s_1 \) the closed ball \( D_T(g_s, r) \) does not contain any \( a \)-edges, then there is an isometry

\[
D_T(g_s, r) \to D_{B_0}(e, r),
\]

and it follows that \( (T', e) = (B_0, e) \).

**Case 2.2.** Suppose for all \( r \geq 2 \) and all \( s > 0 \) we have \( \ell_s < r \). Then \( D_T(g_s, r) \cap A_0 \neq 0 \) and two situations are possible.

**Case 2.2.1.** There exists \( r_1 \geq 2 \) and \( s_1 > 0 \) such that for all \( s \geq s_1 \) and all \( s \geq s_1 \) we have \( \ell_s < s - 1 \). Then \( D_T(g_s, r) \) contains a path of \( a \)-edges of length at least 4, and for \( s > s_1 \) we have \( \ell_s = s_1 \). If there exists \( r_2 \geq r_1 \) and \( s_2 \geq s_1 \) such that \( D_T(g_{s_2}, r) \) contains another path of \( a \)-edges, which is of length necessarily 2, then for all \( s \geq s_2 \) either \( g_s = g_{s_2} \) or \( g_s = g_{s_2}^{-1} \). By increasing \( r \) and possibly \( s_2 \) we exclude one of these options, and it follows that either \( (T', e) = (T, e) \cdot g_{s_2} \) or \( (T', e) = (T, e) \cdot g_{s_2}^{-1} \). If \( r_2 \) with the property specified above does not exist, then \( g_s = a^{m_s}b^k \) for some \( k \in \mathbb{Z} \), and \( (T', e) = (C, e) \cdot b^k \).

**Case 2.2.2.** If an \( r_1 \) as specified in Case 2.2.1 does not exist, then for all \( r \geq 2 \) and all \( s > 0 \) we have \( \ell_s = r - 1 \), and \( D_T(g_s, r) \) contains at least one path of \( a \)-edges of length 2. If there exists \( r_2 \geq 2 \) and \( s_2 > 0 \) such that \( D_T(g_{s_2}, r_2) \) contains at least 2 distinct \( a \)-edge paths, then we are in the situation of the Case 2.1.1, and \( (T', e) \in \mathcal{R}(B_{\ell_2}) \) for some \( \ell_2 \geq 1 \). Otherwise \( D_T(g_s, r) \) contains exactly one \( a \)-edge path of length 2, and we are in the situation of the Case 2.1.2. Then \( (T', e) \in \mathcal{R}(B'_{\ell_0}) \).

This exhausts the distinct possibilities for choosing a sequence \( \{g_s\} \). We have shown that \( \lim L_T \) contains leaves at level at most 1. It follows that \( \mathcal{M} \) is at level 2.

### 4.3. Proof of Theorem \[\ref{1.13}\]

We prove Theorem \[\ref{1.13}\] which states that there exists a graph matchbox manifold at level 2 exhibiting interesting dynamics, namely, one can find a clopen subset of a transversal such that the restricted pseudogroup is equicontinuous.

We construct such a matchbox manifold for \( n = 2 \), i.e. in \( \mathcal{M}_2 \). Let \( F_n = \{a, b\} \) be a set of generators. We first give a construction of a tree \( (K, e) \in X \), and then prove that \( \mathcal{M}_K \) satisfies the requirements of the theorem in a series of propositions and lemmas.
Let $C_0 = L_0$ be a subgraph of $F_2$ with the set of vertices $V(C_0) = V(L_0) = \{e, a, a^{-1}, b, b^{-1}\}$. Given $C_i$ and $L_i$ define $A_i^\delta$, $\delta = \pm 1$, to be a subgraph of $F_2$ containing the vertex $a^{\delta 2^i}$ such that there is an isomorphism

$$\alpha_i^\delta : A_i^\delta \to C_i$$

with $\alpha_i^\delta(a^{\delta 2^i}) = e$,

and $B_i^\delta$ be a subgraph of $F_2$ containing the vertex $b^{\delta 2^i}$ such that there is an isomorphism

$$\beta_i^\delta : B_i^\delta \to L_i$$

with $\beta_i^\delta(b^{\delta 2^i}) = e$.

Then set $C_{i+1} = C_i \cup A_i^1 \cup A_i^{-1} \cup B_i^1 \cup B_i^{-1}$, and $L_{i+1} = L_i \cup B_i^1 \cup B_i^{-1}$. Define $K = \bigcup_{i \in \mathbb{N}} C_i$ (see Figure 5). We show that $\mathcal{M}_K$ contains an uncountable number of leaves, and each leaf is at finite level.

**Lemma 4.8.** The graph matchbox manifold $\mathcal{M}_K$ is recurrent, and, therefore, it has an uncountable number of leaves.

**Proof.** Let $g_i = a^{2^i}$, $i \in \mathbb{N}$, so all $g_i$ are distinct. Then for $k \geq i$ there is an isometry

$$\alpha_k^i : D_K(g_k, 2^i - 1) \to D_K(e, 2^i - 1),$$

and by Lemma 2.11, $\mathcal{M}_K$ is recurrent. Therefore, $\mathcal{M}_K$ contains an uncountable number of leaves. \qed

Let $A \subset K$ be a subgraph with the set of vertices $V(A) = \bigcup_{i \in \mathbb{Z}} \{a^i\}$. We call a connected subgraph $F \subset K$ a *vertical decoration* of $K$, if the set of vertices $V(F)$ satisfies the following two conditions.

1. There exists a vertex $a^n$, $n \in \mathbb{Z}$, called the *point of attachment* of $F$ to $A$, such that $a^n \in V(F)$ and for every $m \neq n$ we have $a^m \notin V(F)$.
2. If $a^m b^k \in V(F)$ then $a^n b^{-k} \notin V(F)$. 
A vertical decoration $F$ is a one-ended connected subtree of $K$ with exactly one vertex $a^n$ lying in $A$. We say that $F$ is finite of length $\ell$ if $\ell$ is the length of a longest path without self-intersections contained in $F$. If $F$ is not finite, we say that $F$ is infinite. We notice that $K$ has 2 distinct infinite decorations attached to $A$ at the vertex $e$, and all other vertical decorations are finite.

To prove Theorem 1.13 we need the following two lemmas.

**LEMMA 4.9.** If $(T,e) \in \overline{R}(K)$ has more than 2 ends, then $(T,e) \in R(K)$.

Lemma 4.9 states that $L_K$ is the only 4-ended leaf in $M_K$, and every other leaf is either 2-ended, 1-ended or compact.

**Proof.** Let $(T,e) \in \overline{R}(K)$, and notice that $(T,e)$ cannot have more than two infinite vertical decorations. Indeed, suppose there are more than two such decorations, then there are at least two distinct vertices $v_1$ and $v_2$ where these decorations are attached to the subgraph $A$. Since $(T,e) \in \overline{R}(K)$, there exists a sequence $\{g_k\} \subseteq V(K)$ such that for every $r > \max\{d_T(e,v_1), d_T(e,v_2)\}$ there is $s_r > 0$ such that for all $s > s_r$ there is an isometry $B_K(g_s, r) \to D_T(e,r)$, which implies that for any $m > 0$ we should be able to find a pair $F_1^{(m)}, F_2^{(m)}$ of vertical decorations in $K$ of length at least $m$ such that the distance between their points of attachment to $A$ is precisely $d_T(v_1, v_2)$. This is not possible, since if $F_1, F_2$ and $F_3$ are vertical decorations in $K$ of distinct lengths, then the set of distances between their points of attachment to $A \subseteq K$ contains at least two elements. Therefore, $(T,e)$ has at most two infinite vertical decorations. Now suppose $(T,e)$ has two infinite vertical decorations, and let $v_1 \in T$ be the vertex at which the decorations are attached. Then for any $r > 0$ one can construct an isometry $D_K(e,r) \to D_T(v_1,r)$, which implies $(T,e) = (K,e) \cdot v_1^{-1}$. 

To keep track of leaves in $M_K$, we construct a section of $\overline{R}(K) - R(K)$ using the coding procedure, employed in [3 Section 3.2] to study the example of Kenyon and Ghys.

**LEMMA 4.10.** Let $Q_4 = \{a,a^{-1}, b,b^{-1}\}^N$ be the set of one-sided infinite sequences. Then there exists a subset $Q \subseteq Q_4$ and a map $P : Q \to X$ such that for every $(T,e) \in \overline{R}(K) - R(K)$ we have $P(Q) \cap R(T) \neq \emptyset$.

**Proof.** Consider a subset

$$Q = \bigcup_{n \in \mathbb{N}} \{(a_0 \ldots a_n(aa^{-1})) | \ a_i = b^\delta, \ \delta \in \{-1,1\}, \ 0 \leq i \leq n\} \cup \{(bb^{-1})\} \cup \{(b)\} \cup \{(b^{-1})\} \subseteq Q_4,$$

and obtain the map $P : Q \to X$ as follows. Let $x_0 = e$, and $x_1 = x_0a_0 \in V(F_2)$. Set $E_0 = L_0$ to be the subgraph of $F_2$ with the set of vertices $V(E_0) = V(L_0) = \{x_1, x_1a, x_1a^{-1}, x_1b, x_1b^{-1}\}$. For $i \geq 2$ obtain the graphs $(E_i, x_i)$ are by the following inductive procedure. Let $x_{i-1}, L_{i-1}$ and $E_{i-1}$ be given. Let $x_i = x_{i-1}a_{i-1}^{-1}$, then there is an edge $w \subseteq F_2$ such that $E_{i-1} \cap w$ is a vertex, and such that $x_i$ is another vertex of $w$. Set $L'$ and $E'$ to denote the subgraphs of $F_2$ containing the vertex $x_i$ such that there are isometries

$$\alpha_i : L' \to L_{i-1}, \ \text{with} \ \alpha_i(x_i) = x_{i-1},$$

and

$$\beta_i : E' \to E_{i-1}, \ \text{with} \ \beta_i(x_i) = x_{i-1}. $$

Set $L'_\delta$ and $E'_\delta$ to denote the subgraphs of $F_2$ containing the vertices $x_i b^{2\delta i-1}$ and $x_i a^{2\delta i-1}$ respectively, such that there are isometries

$$\alpha_i^\delta : L'_\delta \to L', \ \text{with} \ \alpha_i^\delta(x_i b^{2\delta i-1}) = x_i, \ \delta \in \{-1,1\},$$

and

$$\beta_i^\delta : E'_\delta \to E', \ \text{with} \ \beta_i^\delta(x_i a^{2\delta i-1}) = x_i, \ \delta \in \{-1,1\}.$$
Let $E_i = E_i' \cup E_i^1 \cup E_i'_{-1} \cup L_i' \cup L_i'_{-1}$, and

$$P(\alpha) = \bigcup_{i \in \mathbb{N}} E_i.$$ 

We always have $x_0 \in E_i$. If the sequence $\alpha \in \mathcal{Q}$ is eventually periodic with period 2, then for large $i$ the complement of $E_i$ in $P(\alpha)$ has 2 unbounded connected components, and $P(\alpha)$ has 2 ends. There are also two sequences which are periodic with period 1, that is, $\alpha = (b)$ and $\alpha = (b^{-1})$. In this case the complement of $E_i$ has one unbounded connected component, and $P(\alpha)$ has 1 end.

We show that $P(\mathcal{Q})$ is a section of $\overline{R(K)} - \mathcal{R}(K)$. Let $V_N = D_X(K, 2^{N-1}) \cap X$, that is,

$$(T, e) \in V_N \text{ if and only if there is an isomorphism } D_K(e, 2^{N-1}) \rightarrow D_T(e, 2^{N-1}).$$

Then $V_1$ hits every orbit in $\overline{R(K)}$. Consider $V_2 \subset V_1$. We show that if $(T, e) \in V_2 - \mathcal{R}(K)$, there exists $\alpha = (\alpha_0, \ldots, \alpha_n(aa^{-1}))$ such that $P(\alpha) \in \mathcal{R}(T)$.

Indeed, if $(T, e) \in V_2 - \mathcal{R}(K)$ there is necessarily an isometry $D_K(a^m, 2) \rightarrow D_T(e, 2), m \in \mathbb{Z}$. Let $F$ be a vertical decoration attached to $e$. Since $(T, e) \notin \mathcal{R}(K)$, by Lemma 4.9 $F$ is of finite length $\ell = 2^i - 1, i \in \mathbb{N}$. Let $x_0$ be the vertex in $V(F)$ such that $d_T(x_0, e) = \ell$, and let $d(x_0, x_1) = (w_k)$, where $w_k$ is an edge adjacent to $x_0$. For definiteness assume that $t(w_k) = x_0$. Then for $0 < k \leq i$ set

$$x_k = x_{k-1} \cdot b^{-2^{k-1}}, \alpha_{k-1} = b^{-1}.$$ 

Then $x_0 = e$. For $k > i$, implement the following inductive procedure, which uses the fact that $(T, e) \in \overline{R(K)}$, and so every pattern in $T$ must be replicated in $K$. The boundary of the ball $B_T(x_k, 2^{k-1})$ has two adjacent $a$-edges $v_1$ and $v_2$, such that $t(v_1), s(v_2) \in \partial B_T(x_k, 2^{k-1})$ and $s(v_1), t(v_2) \notin B_T(x_k, 2^{k-1})$. Let $F_1$ and $F_2$ be vertical decorations attached to $p_1 = s(v_1)$ and $p_2 = t(v_2)$ respectively. Then one of them must have length $2^{k+1} - 1$, and another one must have length $2^{k+2} - 1$. If $p_1$ is the vertex with decorations of length $2^{k+1} - 1$, set $x_k = p_1$ and $\alpha_{k-1} = a^{-1}$. Otherwise set $x_k = p_2$ and $\alpha_{k-1} = a$. In this way we obtain $\alpha \in \mathcal{Q}$ such that $P(\mathcal{Q}) = (T, e) \cdot b^{2^i-1}$.

Now suppose $(T, e) \in V_1$ and $\mathcal{R}(T) \cap V_2 = \emptyset$. Let $\{g_s\} \in V(K)$ be a sequence of verticals such that for every $r > 0$ there is $s_r > 0$ such that for every $s > s_r$ there is an isometry

$$a_r^s : D_K(g_s, 2^r - 1) \rightarrow D_T(e, 2^r - 1).$$

By choosing a bigger $s_r$, if necessary, we can assume that $D_K(g_s, 2^r + 1) \cap A = \emptyset$, which means that $D_K(g_s, 2^r + 1)$ is contained in a vertical decoration $F_s$ of $K$. Then there are two situations: either there exists $r_1 > 0$ and $s_1 > 0$ such that there is an edge $w_0 \subset F_s$, such that

$$w_0 \cap D_K(g_{s_1}, 2^{r_1} - 1) \neq \emptyset, \text{ and } w_0 \cap (F_{s_1} - D_K(g_{s_1}, 2^{r_1} - 1)) \neq \emptyset,$$

or such an $r_1$ does not exist. In the first case the same condition is true for every $r > r_1$ and $s > s_1$, and it follows that either $(T, e) = P((b^{-1})) \cdot b^{-m}$ or to $P((b)) \cdot b^m$ for some $m \in \mathbb{Z}$. In the second case $(T, e) = P((bb^{-1}))$.

The following proposition completes the proof of Theorem 1.13. Recall that a pseudogroup $\mathcal{G}$ of local homeomorphisms of $X$ is equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $\gamma \in \mathcal{G}$ and every $x, y \in \text{dom}(\gamma)$ with $d_X(x, y) < \delta$ we have $d_X(\gamma(x), \gamma(y)) < \epsilon$.

**Proposition 4.11.** The restriction $\mathcal{G}|_{V_2}$ is equicontinuous. Every leaf in a graph matchbox manifold $\mathcal{M}_K$ is at level at most 2. A leaf $L_T$ is at level 0 or 1 if and only if $R(T) \cap V_2 = \emptyset$, and there is only a finite number of such leaves.

**Proof.** Let $L_T$ be a leaf such that $R(T) \cap V_2 = \emptyset$. Then by Lemma 4.10 either $R(T) \supset P((bb^{-1}))$, or $R(T) \supset P((b))$, or $R(T) \supset P((b^{-1}))$. In the first case $L_T$ is a compact leaf and so is at level 0. In the second and the third case an argument similar to those in Section 4.2 shows that $L_T$ is a one-ended leaf with a single compact leaf in its closure. Such a leaf is at level 1.

We now show that if $L_T$ is such that $(T, e) \in V_2$, then its orbit under the restricted pseudogroup $\mathcal{G}_2 = \mathcal{G}|_{V_2}$ is dense in $V_2$. If that is true, then $L_T$ is dense in $\mathcal{M}_K$ and so is at level 2.
To prove this statement we show that $\mathcal{G}_2$ is equicontinuous. Then by [14 Theorem 4.12] the $\mathcal{G}_2$-orbits of points in $V_2$ are dense in $V_2$.

First we notice that $V_2$ is invariant under the action of the sub-pseudogroup $\mathcal{A}_4 = \langle \gamma_a \rangle$, and

\begin{equation}
V_2 \subset \text{dom}(\gamma_g), \quad \gamma_g \in \mathcal{A}_4.
\end{equation}

Indeed, let $g = (a^4)^n$ and $(T,e) \in V_2$. Then $(T,e) = R(P(\alpha_0 \ldots \alpha_n(aa^{-1})))$ and there exists an isomorphism

$$\alpha : D_K(e,2) \to D_T(e,2).$$

It follows that $e$ lies on the bi-infinite line of $a$-edges and the action of $g$ on $(T,e)$ is defined. Since $(T,e) \in R(K)$ there exists $(K,g_n)$ such that there is an isomorphism

$$\alpha : D_T(e,4(n+1)) \to D_K(g_n,4(n+1)) \in V_2,$$

and which implies that $g_n = a^{4m}$. It follows that $(T,e) \cdot g \in V_2$. Next, we claim that $\mathcal{G}_2 = \mathcal{A}_4|_{V_2}$.

Let $(T,e) \in V_2$ and $g \in F_2$ such that $(T,e) \cdot g \in V_2$. Since $(T,e) \in R(K)$, then there exists $a^{4m}$, $m \in \mathbb{Z}$, such that there is an isomorphism $B_K(a^{4m},\ell(g)+3) \to B_T(e,\ell(g)+3)$, and so $(K,e) \cdot a^{4m} g$ is defined. Moreover, necessarily

$$(K,e) \cdot a^{4m} g = (K,e) \cdot a^{4n}.$$ 

Since $K$ is simply connected and translations by $b$ take $(K,e) \cdot a^{4m}$ out of $V_2$, then $g = a^{4(n-m)}$. We now show that $\mathcal{G}_2$ is equicontinuous. Let $(T,e) \in V_2$, let $N > 0$ and suppose there is an isomorphism $D_K(a^{4m},N) \to D_T(e,N)$, $m \in \mathbb{Z}$. We claim that there is an isomorphism

$$D_K(a^{4(m+n)},N) \to D_T(a^{4n},N)$$

for any $n \in \mathbb{N}$. Indeed, to see that it is enough to notice that by Lemma 4.10 that any 2-ended graph $(T,e) \in V_2$ has decorations of different lengths attached in the same order as in $(K,a^{4n})$. This completes the proof of the proposition. \hfill $\square$

5. Growth of leaves at finite levels

In this section we prove Proposition 1.14 which states that $\mathcal{M}_n$ contains a leaf of a linear growth, and a leaf of exponential growth. It is enough to construct such an example in $\mathcal{M}_2$. Let $G_0 = \{a,b\}$.

We first recall some definitions [4]. For $(T,e),(T',e) \in X$ define the ‘plaque distance’ by

$$PD(T',T) = \min \{ \ell_w(g) \mid \gamma_g(T,e) = (T',e) \},$$

where $\ell_w(g)$ is the length of $g$ in the word metric on $F_2$. Then $PD(T,T') = d_{\gamma_g}(T,T')$. For $k \geq 0$ let

$$\Gamma_k(R(T)) = \{(T',e) \in R(T) \mid PD(T',T) \leq k\},$$

and define the growth function of $R(T)$ by

$$H_{R(T)} : \mathbb{Z}^+ \to \mathbb{R}^+ : k \mapsto \text{card}(\Gamma_k(R(T))).$$

Recall from [8 Proposition 12.2.35] that the growth function $H_{R(T)}$ can be considered to be the growth function of the leaf $L_T$.

**Proof. (of Proposition 1.14).** Let $F_1$ be a subgraph of $F_2$ with the set of vertices

$$V(F_1) = \{a^n \mid n \in \mathbb{N} \cup \{0\}\}.$$

Then $\mathcal{M}_{F_1}$ is at level 1, and $L_{F_1}$ has linear growth, that is,

$$H_{R(F_1)}(k) = k + 1.$$

Let $F_2$ be a subgraph of $F_2$ with the set of vertices

$$V(F_2) = \{e\} \cup \{mg \mid m = a, \ g \in F_2\}.$$
By an argument similar to that in Lemma 4.5 the matchbox manifold $M_{F_2}$ consists of two leaves, namely $L_{F_2}$ and the genus two surface $L_{F_2}$. Thus $M_{F_2}$ is at level 1. We also have

$$H_{R(F)}(k) = 1 + 3^{k-1},$$

so $L_{F_2}$ has exponential growth. An example of a leaf with polynomial growth which is not totally proper is given in Theorem 1.13.

6. Pseudogroup dynamics of matchbox manifolds

In this section we study pseudogroup dynamics of graph matchbox manifolds in $M_n$ and prove Proposition 1.11. We first recall a definition of a foliation with expansive dynamics.

**Definition 6.1.** [§3] Let $(M, F)$ be a foliated space with a foliated atlas

$$\mathcal{U} = \{ \varphi_i : U_i \to [-1, 1]^n \times X_i \},$$

and let $\mathfrak{G}$ be the holonomy pseudogroup associated to $\mathcal{U}$. The dynamics of $F$ is $\epsilon$-expansive, or $\mathfrak{G}$ is $\epsilon$-expansive, if there exists $\epsilon > 0$ so that for all $\gamma \neq \gamma'$ in $\mathfrak{G}$ there exists a holonomy homeomorphism $h \in \mathfrak{G}$ with $w, w' \in \text{dom}(h)$ such that $d_X(h(w), h(w')) \geq \epsilon$.

Proof of statement (1) of Proposition 1.11 is contained in the following lemma.

**Lemma 6.2.** Let $L \subset M_n$ be a non-compact leaf and $M_T = L_T$ be a graph matchbox manifold. Then for every $0 < \epsilon < \epsilon^{-2}$ the foliation of $M_T$ has $\epsilon$-expansive dynamics. If $L_T$ is compact, then $M_T$ is equicontinuous.

**Proof.** Fix $\epsilon < \epsilon^{-2}$ and recall that the pseudogroup $\mathfrak{G}$ restricted to $X$ is finitely generated with the set of generators $\mathfrak{G}^0$. Suppose there exists $(T, e) \neq (T', e) \in X$ such that for all $\gamma \in \mathfrak{G}$ with $(T, e), (T', e) \in \text{dom}(\gamma)$ and $d_X((T, e), (T', e)) < \epsilon$ we have $d_X(\gamma(T, e), \gamma(T', e)) < \epsilon$. Notice that since $\epsilon < \epsilon^{-2}$ and $d_X((T, e), (T', e)) < \epsilon$, then for all $\gamma_i \in \mathfrak{G}^0(T, e) \in \text{dom}(\gamma_i)$ if and only if $(T', e) \in \text{dom}(\gamma_i)$. We show that in fact this implies that $(T, e) \in \text{dom}(\gamma)$ and only if $(T', e) \in \text{dom}(\gamma)$ for all $\gamma \in \mathfrak{G}$.

We have $\gamma = \gamma_{i_1} \circ \cdots \circ \gamma_{i_k}$. The proof is by induction on $\ell$, $1 \leq \ell \leq k$. Suppose $(T, e), (T', e) \in \text{dom}(\gamma_{i_1} \circ \cdots \circ \gamma_{i_{\ell}})$. Set

$$(T_{\ell}, e) = \gamma_{i_1} \circ \cdots \circ \gamma_{i_{\ell}}(T, e), \text{ and } (T'_{\ell}, e) = \gamma_{i_1} \circ \cdots \circ \gamma_{i_{\ell}}(T', e).$$

An assumption that $d_X((T_{\ell}, e), (T'_{\ell}, e)) < \epsilon$ means that there is an isometry

$$\alpha_0 : D_{T_{\ell}}(e, 2) \to D_{T'_{\ell}}(e, 2),$$

and $(T_{\ell}, e) \in \text{dom}(\gamma_{i_{\ell+1}})$ if and only if $(T'_{\ell}, e) \in \text{dom}(\gamma_{i_{\ell+1}})$. Therefore, $(T, e) \in \text{dom}(\gamma_{i_1} \circ \cdots \circ \gamma_{i_{\ell+1}})$ if and only if $(T', e) \in \text{dom}(\gamma_{i_1} \circ \cdots \circ \gamma_{i_{\ell+1}})$.

We now show that this implies $(T, e) = (T', e)$. Let $R \in \mathbb{N}$. We want to show that there is an isomorphism $\alpha_R : D_T(e, R) \to D_{T'}(e, R)$. The proof is by induction on $r$, $1 \leq r \leq R$. Suppose there is an isomorphism

$$\alpha_r : D_T(e, r) \to D_{T'}(e, r)$$

and let $\gamma_{g_1}, \ldots, \gamma_{g_n}$ be a collection of homeomorphisms such that $\ell(g_k) = r$ and $(T, e), (T', e) \in \text{dom}(\gamma_{g_k})$. Notice that

$$D_T(e, r + 1) \subset D_T(e, r) \cup \left( \bigcup_{1 \leq k \leq n_r} D_T(g_k, 2) \right)$$

and similarly for $T'$, so if for every $1 \leq k \leq n_r$ we have $D_T(g_k, 2) = D_{T'}(g_k, 2)$, then we are done. But the latter condition follows from the fact that

$$d_X(\gamma_{g_k}(T, e), \gamma_{g_k}(T', e)) < \epsilon < \epsilon^{-2}.$$
which contradicts the assumption. If \( L_T \) is compact, then \( \mathcal{R}(T) \) is a finite set, and so there exists \( \epsilon > 0 \) such that for any \( (T', e) \in \mathcal{R}(T) \) the \( \epsilon \)-neighborhood of \( (T', e) \) in \( \overline{\mathcal{R}(T)} = \mathcal{R}(T) \) contains exactly one point. Then \( \mathcal{M}_T \) is trivially equicontinuous.

We now restate and proof corollary \[1.12\]

**COROLLARY 6.3.** The foliated space \( \mathcal{M}_n \) does not contain a weak solenoid.

\[\text{Proof.} \ A \text{ weak solenoid must contain a non-compact leaf and have an equicontinuous pseudogroup, which is not possible by Lemma } 6.2.\]

The proof of statement (2) of Theorem \[1.11\] is given by the following lemma.

We say that the pseudogroup \( \mathcal{G} \) restricted to a transversal \( X \) has a non-trivial equicontinuous factor if and only if there exists a clopen subset \( C \subset X \) such that \( \mathcal{G}_C = \mathcal{G}|_C \) is equicontinuous.

**LEMMA 6.4.** Let \( \mathcal{G}_T \) be the holonomy pseudogroup associated to the transversal \( \overline{\mathcal{R}(T)} \) of a graph matchbox manifold \( \mathcal{M}_T \). If \( \mathcal{G}_T \) has a non-trivial equicontinuous factor, then \( L_T \) is a recurrent leaf. The converse is false, i.e. there exists a matchbox manifold \( \mathcal{M}_{T'} \) where \( L_{T'} \) is recurrent and \( \mathcal{G}_{T'} \) has no nontrivial equicontinuous factor.

\[\text{Proof.} \ Let \( C \subset \overline{\mathcal{R}(T)} \) be a neighborhood with equicontinuous restricted pseudogroup \( \mathcal{G}_C \). Since \( \mathcal{R}(T) \) is dense in \( \overline{\mathcal{R}(T)} \) there exists \( (T', e) \in \mathcal{R}(T) \cap C \), and by \[14\] Theorem 4.12 the \( \mathcal{G}_C \)-orbit of \( (T', e) \) is dense in \( C \). Therefore, \( (T', e) \in \overline{\mathcal{R}(T)} \) which yields \( L \subset \lim \mathcal{L} \).

Conversely, let \( \mathcal{M}_{T'} \) be minimal, and suppose there is a clopen neighborhood \( C \subset \overline{\mathcal{R}(T')} \) with equicontinuous restricted pseudogroup \( \mathcal{G}_C \). It is shown in \[15\] that in this case \( \mathcal{G} \) must be equicontinuous which contradicts Lemma \[6.2\]. \]

**References**

[1] J. Aarts and L. Oversteegen, *Matchbox manifolds*, In Continua (Cincinnati, OH, 1994), Lecture Notes in Pure and Appl. Math., Vol. 170, Dekker, New York, 1995, pages 3–14.

[2] J.M. Aarts and M. Martens, *Flows on one-dimensional spaces*, Fund. Math., 131:39–58, 1988.

[3] F. Alcalde Cuesta, A. Lozano Rojo and M. Macho Stadler, *Dynamique transverse de la lamination de Ghys-Keen-Kenyon*, Astérisque, 323:1–16, 2009.

[4] O. Attie and S. Hurder, *Manifolds which cannot be leaves of foliations*, Topology, 35(2):335–353, 1996.

[5] R. Benedetti and J.-M. Gambaudo, *On the dynamics of \( \mathcal{G} \)-solenoids. Applications to Delone sets*, Ergodic Theory Dyn. Syst., 23:673–691, 2003.

[6] E. Blanc, *Propriétés génériques des laminations*, PhD Thesis, Université de Claude Bernard-Lyon 1, Lyon, 2001.

[7] D. Burago, Yu. Burago and S.Ivanov, *A Course in Metric Geometry*, GSM 33, American Mathematical Society, Providence, Rhode Island 2001.

[8] A. Candel and L. Conlon, *Foliations I*, Amer. Math. Soc., Providence, RI, 2000.

[9] A. Candel and L. Conlon, *Foliations II*, Amer. Math. Soc., Providence, RI, 2003.

[10] J. Cantwell and L. Conlon, *Generic leaves*, Comment. Math. Helv., 73(2):306–336, 1998.

[11] J. Cantwell and L. Conlon, *Poincaré-Bendixson theory for leaves of codimension one*, Trans. Amer. Math. Soc., 265(1):181–209, 1981.

[12] J. Cantwell and L. Conlon, *Growth of leaves*, Comment. Math. Helv., 53:93–111, 1978.

[13] A. Clark and S. Hurder, *Embedding solenoids in foliations*, Topology Appl., 158:1249–1270, 2011.

[14] A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, to appear in Transactions AMS, version 2011.

[15] A. Clark, S. Hurder and O. Lukina, *Shape of matchbox manifolds*, preprint, 2011.

[16] P.R. Dippolito, *Codimension one foliations of closed manifolds*, Ann. of Math., 107: 403-453, 1978.

[17] D.B.A. Epstein, K.C. Millet, and D. Tischler, *Leaves without holonomy*, Jour. London Math. Soc., 16:548–552, 1977.

[18] R. Fokkink and L. Oversteegen, *Homogeneous weak solenoids*, Trans. Amer. Math. Soc., 354:3743–3755, 2002.

[19] É Ghys, *Laminations par surfaces de Riemann*, In Dynamique et Géométrie Complexes, Panoramas & Synthèses, 8:49–95, 1999.

[20] C. Godbillon, *Éléments de Topologie Algébrique*, Hermann, Paris 1971.

[21] G. Hector, *Architecture des feuilletages de classe \( C^2 \)*, Asterisque, 107-108, Société Mathématique de France, 1983: 243-258.

[22] G. Hector, *Feuilletages en cylindres*, Lecture Notes in Math., 597, Springer-Verlag, 1977, pp. 252-270.
[23] G. Hector, *Quelques exemples des feuilletages - Espèces rares*, Ann. Inst. Fourier, 26(1), 1976, pp. 239-264.
[24] S. Hurder, *Lectures on foliation dynamics: Barcelona 2010*, preprint 2011.
[25] K. Kuratowski, *Topology I*, Academic Press Inc., 1966, 547pp.
[26] A. Lozano Rojo, *Foliated spaces defined by graphs*, Rev. Semin. Iberoam. Mat., 3(4):21–38, 2007.
[27] A. Lozano Rojo, *An example of a non-uniquely ergodic lamination*, preprint 2009.
[28] H. Marzougui and E. Salhi, *Structure of foliations of codimension greater than one*, Comment. Math. Helv., 78(4):722–730, 2003.
[29] C. McCord, *Inverse limit sequences with covering maps*, Trans. Amer. Math. Soc., 114:197–209, 1965.
[30] C.C. Moore and C. Schochet, *Analysis on Foliated Spaces*, Math. Sci. Res. Inst. Publ. vol. 9, Second Edition, Cambridge University Press, New York, 2006.
[31] T. Nishimori, *Behaviour of leaves of codimension-one foliations*, Tôhoku Math. J., 29(2):255–273, 2977.
[32] T. Nishimori, *Ends of leaves of codimension-one foliations*, Tôhoku Math. J. (2), 1979(1):1–22, 1979.
[33] N. Priebe Frank and L. Sadun *Topology of some tiling spaces without finite local complexity*, Discrete Contin. Dyn. Syst., 23(3):847–865, 2009.
[34] E. Salhi, *Sur les ensembles minimaux locaux*, C. R. Acad. Sci. Paris Sér. I Math., 295(12):691–694, 1982.
[35] E. Salhi, *Niveau des feuilles*, C. R. Acad. Sci. Paris Sér. I Math., 301(5):219–222, 1985.
[36] E. Salhi, *Sur un théorème de structure des feuilletages de codimension 1*, C. R. Acad. Sci. Paris Sér. I Math., 300(18):635–638, 1985.
[37] N. Tsuchiya *Leaves of finite depth*, Japan. J. Math., 6(2):343–364, 1980.
[38] S. Willard *General Topology*, Dover Publications, Inc., Mineola, New York, 2004.
[39] R.F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math., 43:169–203, 1974.