Counting centralizers of a finite group with an application in constructing the commuting conjugacy class graph

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\section{Introduction}
Throughout this paper all groups are assumed to be finite and $C_G(x)$ denotes the centralizer of an element $x$ in $G$. If $G$ is a group containing two subgroups $H$ and $N$ such that $N \triangleleft G$, $H \cap N = 1$ and $G = HN$ then $G$ is said to be the semi-direct product of $H$ by $N$ and we write $G = N \ltimes H$. The group $G$ is called capable if there exists another group $H$ such that $G \cong H/Z(H)$. Our other notations are standard and taken mainly from [9] and our calculations are done with the aid of GAP [11].

The center of $G$ is denoted by $Z(G)$ and $\text{Cent}(G) = \{C_G(x) \mid x \in G\}$. The group $G$ is called $n$--centralizer if $n = |\text{Cent}(G)|$ [1]. The study of finite groups with respect to the number of distinct element centralizers was started by Belcastro and Sherman [3]. It is clear that $|\text{Cent}(G)| = 1$ if and only if $G$ is abelian and there is no finite group with exactly two or three element centralizers. They proved that if $\frac{G}{Z(G)} \cong Z_p \times Z_p$ then $|\text{Cent}(G)| = p + 2$ [3, Theorem 5].

Let $H$ be a graph with vertex set $\{1, 2, \ldots, k\}$ and $G_i$'s, $1 \leq i \leq k$, are disjoint graphs of order $n_i$. Following Sabidussi [10], the graph $H[G_1, G_2, \ldots, G_k]$ is formed by taking the graphs $G_1$, $G_2$, $\ldots$, $G_k$ and connect a vertex of $G_i$ to another vertex in $G_j$ whenever $i$ is adjacent to $j$ in $H$. The graph $H[G_1, G_2, \ldots, G_k]$ is called the $H$--join of the graphs $G_1$, $G_2$, $\ldots$, $G_k$.

The notion of the \textit{commuting graph} of a finite group was introduced by Brauer and Fowler [4] in an old paper in which the authors investigated groups of even orders. These authors did not use the graph theory language, but proved some elementary properties of this graph. Herzog, Longobardi and Maj [6] studied the simple graph $\Gamma(G)$ associated to a group $G$ with the set of conjugacy classes of nontrivial elements of $G$ as its vertex set. Two distinct vertices $C$ and $D$ of this graph are assumed to be adjacent if and only if $xy = yx$, for some $x \in C$ and $y \in D$. The notion of \textit{commuting conjugacy class graph} of a non-abelian group $G$, $\Gamma(G)$, was introduced by Mohammadian et al. [8]. This is a simple graph with...
non-central conjugacy classes of $G$ as its vertex set and two distinct vertices $A$ and $B$ are adjacent if and only if there are $x \in A$ and $y \in B$ such that $xy = yx$. In such a case, we also say that two conjugacy classes commute to each other. The authors of the mentioned paper obtained some interesting properties of this graph among them a classification of triangle-free commuting conjugacy class graph of a finite groups is given.

The following two theorems are the main results of this paper:

**Theorem 1.1.** Suppose $p$ is a prime number and $G$ is a group with center $Z$ such that $\frac{G}{Z} \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}$. Then $G$ has exactly $[(p+1)^2 + 1]$ element centralizers.

**Theorem 1.2.** Suppose $p$ is a prime number and $G$ is a group with center $Z$ such that $\frac{G}{Z} \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}$. The commuting conjugacy class graph of $G$ has one of the following types:

1. $\frac{G}{Z}$ is abelian. In this case, $\frac{G}{Z} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and the commuting conjugacy class graph of $G$ is isomorphic to $M_1[K_n(p-1), K_{n(p-1)}, K_{m(p^2-p)}]$, where there are $p+1$ copies of $K_n(p-1)$, $p^2 + p$ copies of $K_m(p^2-p)$ and $M_1$ is the graph depicted in Figure 1(a). Here, $m = \frac{|Z|}{p^2}$ and $n = \frac{|Z|}{p}$.

2. $\frac{G}{Z}$ is not abelian. In this case, the commuting conjugacy class graph of $G$ is isomorphic to $M_2[K_n(p-1), K_n(p-1), K_n(p^2-p)]$, where there are $p^2 + p + 1$ copies of $K_n(p-1)$, a copy of $K_m(p^2-p)$ and $M_2$ is depicted in Figure 1(b). Here, $p$ is an odd prime and $n = \frac{|Z|}{p}$.

![Image](a) The Graph $M_1$.  
(b) The Graph $M_2$.  

**2. Preliminary results**

The aim of this section is to prove some preliminary results which are crucial in next sections. The greatest common divisor of positive integers $r$ and $s$ is denoted by $(r, s)$. We start this section by the following simple lemma:

**Lemma 2.1.** Let $p$ be a prime number and $G$ be a group with center $Z$ such that $x^{p^2}, y^p \in Z$. Then, for each positive integer $m$ with this condition that $m \not\equiv 0 \pmod{p}$, we have $C_G(x^m) = C_G(x)$ and $C_G(y^m) = C_G(y)$. 


Lemma 2.2. Let $G$ be a finite group with center $Z$, $p \mid |G|$ and $a, b \in G \setminus Z$ such that $o(aZ) = o(bZ) = p^2$ and $G = \{a^ib^jz \mid z \in Z\}$. Then the following hold:

(a) Suppose $1 \leq i, j \leq p^2 - 1$ and at least one of $i, j$ is not divisible by $p$. Then $a^ib^j \neq b^ja^i$.
(b) $C_G(a^ib^j) = C_G(a^ib^j)$, $1 \leq s, t \leq p^2 - 1$, if and only if $s = t$.

The following result is a particular case of [7, Theorem 8.1(c)].

Theorem 2.3. The non-abelian split extension $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ is not capable.

Suppose $p$ is a prime number. Then it is easy to see that there are only two groups of order $p^4$ that can be written as a semidirect product of two cyclic groups of order $p^2$. One of these groups is the abelian group $L_1 = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and another one is a group presented as $L_2 = \langle x, y \mid x^{p^2} = y^{p^2} = 1, yx = x^{p+1}y \rangle$, see [5, pp. 145–146] for details. For simplicity of our argument, it is useful to write the presentations of $L_1$ and $L_2$ in the form $\langle x, y \mid x^{p^2} = y^{p^2} = 1, yx = x^{p+1}y \rangle$, where $r = 0, 1$.

Lemma 2.4. Suppose $G = L_1$ or $L_2$. Then for every $i, j, 0 \leq i, j \leq p^2 - 1$, which are not simultaneously zero, we have:

$$o(x^iy^j) = \begin{cases} p & (i,j) \\ p^2 & \text{Otherwise} \end{cases}.$$ 

Moreover, $y^ix^j = x^{ijp+i}y^j$ and $(x^iy^j)^k = x^{\frac{k(k-1)}{2}ijp+kiy^j}$, for each $k$.

Proof. The result is trivial for $L_1$, so assume $G = L_2$. Note that $G' = \langle x^p \rangle$ and $Z(G) = \langle x^p, y^p \rangle$ which shows that $G$ has nilpotency class two. Since $p$ is odd, we can see that the map $x \to x^p$ defines a homomorphism from $G$ onto $Z(G)$. Thus $Z(G)$ will be the kernel of this homomorphism and the elements of $G$ outside its center have order $p^2$.

3. Proof of Theorem 1.1

In this section, the proof of our first main result will be presented. To do this, we need some information about the group $G$ with this property that $|\frac{G}{Z(G)}| = p^4$, $p$ is prime, and $\frac{G}{Z(G)}$ can be generated by two elements of order $p^2$.

Lemma 3.1. Let $G$ be a finite group with center $Z$, $p$ be prime and $a, b \in G \setminus Z$. Moreover, we assume that $a^{p^2}, b^{p^2} \in Z$ and $(a^ib^j)^k = a^{k(k-1)p + ij + kj} b^k z$ where $z \in Z$, $r = 0, 1$ and $i, j, k$ are positive integers. The following statements hold:

i) If $1 \leq i, j \leq p - 1$, then there are $n_j$ and $s$ such that $a^ib^j = (a^ib^j)^{n_j}z$, where $n_j = 0$ (mod $p$), $1 \leq s \leq p - 1$ and $z \in Z$.
ii) If $1 \leq i \leq p - 1, 1 \leq j \leq p^2 - 1$ and $j \neq 0$ (mod $p$), then there are $n_i$ and $s$ such that $a^ib^j = (a^ib^j)^{n_i}z$, where $n_i = 0$ (mod $p$), $1 \leq s \leq p - 1$ and $z \in Z$.
iii) If $1 \leq j \leq p - 1, 1 \leq i \leq p^2 - 1$ and $i \neq 0$ (mod $p$), then there are $n_i$ and $s$ such that $a^ib^j = (a^ib^j)^{n_i}z$, where $n_i = 0$ (mod $p$), $1 \leq s \leq p - 1$ and $z \in Z$.
iv) If $1 \leq i, j \leq p^2 - 1, i \neq 0$ (mod $p$) and $j \neq 0$ (mod $p$), then there are $n_i$ and $s$ such that $a^ib^j = (a^ib^j)^{n_i}z$, where $n_i = 0$ (mod $p$), $s \neq 0$ (mod $p$), $1 \leq s \leq p^2 - 1$ and $z \in Z$.
v) If $1 \leq i, j \leq p^2 - 1, i \neq 0$ (mod $p$) and $j \neq 0$ (mod $p$), then there are $n_i$ and $s$ such that $a^ib^j = (a^ib^j)^{n_i}z$, where $n_i = 0$ (mod $p$), $s \neq 0$ (mod $p$), $1 \leq s \leq p^2 - 1$ and $z \in Z$.

Proof. The proof is straight forward, it is omitted.
Suppose $p$ is a prime number. By a result of Baer [2], the abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$ is capable. If $p$ is an odd prime, then the non-abelian group $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2$ is also capable [12]. This means that there exists a group $G$ such that $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \cong \mathbb{Z}_p^2 \times \mathbb{Z}_p^2$ or $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2$. In the next theorem, the number and also the structure of the centralizers of $G$ are obtained. By Theorem 2.3, the non-abelian group $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not capable. In the next theorem, if $p = 2$, then we will assume that $\mathbb{Z}_2 \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.

We are now ready to prove Theorem 1.1.

**Proof.** Observe that if $z \in Z = Z(G)$, then $C_G(g) = C_G(gz)$, for all $g \in G$. Hence, we only need be concerned about a representative for each coset $gZ$ of $\mathbb{Z}_2$. There are two different cases that need to be treated separately: $o(gZ) = p$ and $o(gZ) = p^2$.

1. $o(gZ) = p$. Let $Y = \langle a^p, b^p, Z \rangle$. From above, we see can find element $\hat{g} \in G$ so that $(\hat{g}Z)^p = gZ$ and in fact, if $x \in \hat{g}Y$, then $(xZ)^p = gZ$. Since $(xZ)$ is cyclic, we see that $(x, Z)$ is abelian. Hence, $x \in C_G(g)$. This implies that $C_G(g)$ contains the entire coset $\hat{g}Y$, and thus, $\langle \hat{g}, Y \rangle \leq C_G(g)$. Now, $\langle \hat{g}, Y \rangle$ has index $p$ in $G$ and $C_G(g) \leq G$, since $g$ is not central. Thus, $C_G(g) = \langle \hat{g}, Y \rangle$. It is not difficult to see that each subgroup of order $p$ in $\mathbb{Z}_2$ is the centralizer of an element of $G$ whose order modulo $Z$ is $p$.

2. $o(gZ) = p^2$. In this case, $\langle gZ \rangle$ is a maximal cyclic subgroup of $\mathbb{Z}_2$. Since $\langle gZ \rangle$ is cyclic, we see that $\langle g, Z \rangle$ is abelian. This implies that $\langle g, Z \rangle \leq C_G(g)$. We claim that $C_G(g) = \langle g, Z \rangle$. Notice by the previous paragraph that if $x \in Y$ and $x \not\in \langle g, Z \rangle$, then $g$ does not centralize $x$, and so $x$ does not centralize $g$. It follows that $C_G(g) = \langle g, Z \rangle$, and so, $C_G(g) = \langle g, Z \rangle$.

Finally, one observes that there are $p + 1$ subgroups of order $p$ in $\mathbb{Z}_2$ and then $\mathbb{Z}_2$ has $p^2 + p$ cyclic subgroups of order $p^2$ which give the count. The structure of non-trivial proper centralizers of $G$ is summarized in Table 1.

**Table 1.** The structure of non-trivial proper centralizers of $G$.

| $C_G(a^i)$ | $\{a^ib^p \mid 0 \leq i \leq p - 1, 0 \leq k \leq p^2 - 1, z \in Z\}$ |
| $C_G(a)$ | $\{a^i \mid 0 \leq i \leq p - 1, 0 \leq k \leq p^2 - 1, z \in Z\}$ |
| $C_G(b^i)$ | $\{a^ib^zk \mid 0 \leq i \leq p - 1, 0 \leq k \leq p^2 - 1, z \in Z\}$ |
| $C_G(b)$ | $\{b^i \mid 0 \leq i \leq p^2 - 1\}$ |

We end this section by the following conjecture:

**Conjecture 3.2.** Suppose $p$ is a prime number, $n$ is a positive integer and $G$ is a group with center $Z$ such that $\mathbb{Z}_2 \cong \mathbb{Z}_p^n \times \mathbb{Z}_p^n$. Then $|\text{Cent}(G)| = (p + 1)^n + 1$.

4. **Proof of Theorem 1.2**

In this section, we apply the results of Section 3 to obtain the structure of the commuting conjugacy class graph of $G$ when $\mathbb{Z}_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Section 4.1 is devoted to the proof of Theorem 1.2(1) and in Section 4.2, the proof of the second part will be presented.

4.1. **Proof of Theorem 1.2 (1)**

**Proof.** Since $\mathbb{Z}_2$ is abelian, $xgZ = xZgZ = gZxZ = gxZ$, where $x, g \in G \setminus Z$ are arbitrary. Hence there exists $z \in Z$ such that $xg = gxz$ and so for every $x \in G \setminus Z$,

$$x^G = \{g^{-1}xg \mid g \in G\} \subseteq \{g^{-1}gxz \mid z \in Z\} = \{xz \mid z \in Z\} = \{z \mid z \in Z\}.$$ (4.1)
Since \( \frac{G}{Z(G)} \cong Z_p^2 \times Z_p^2 \), there are \( a, b \in G \setminus Z \) such that
\[
G = \{a^ib^jz \mid 0 \leq i, j \leq p^2 - 1, \ ba = abz_i, \ z, z_i, a^p, b^p \in Z\}.
\]
Hence for each \( x \in G \setminus Z, x = a^ib^jz \), where \( 0 \leq i, j \leq p^2 - 1 \) and \( i, j \) are not simultaneously zero. To obtain the non-central conjugacy classes of \( G \), the following have to be investigated:

a) \( i \neq 0 \) and \( j = 0 \). We will consider two cases that \( p \mid i \) and \( p \nmid i \).

1. \( p \mid i \). In this case, \( i = sp \), where \( 1 \leq s \leq p - 1 \). By Table 1, \( |C_G(a^p)| = p^3 |Z| \) and so \( |(a^p)^G| = p \). By Equation 4.1, \( (a^p)^G = \{a^pz_1, a^pz_2, \ldots, a^pz_p\} \) is \( a^pH \), where \( H = \{z_1, z_2, \ldots, z_p\} \). Suppose \( |Z| \gtrsim p^2 \) and choose \( z_{r_1} \in Z \setminus H \). It is easy to see that \( a^pz_{r_1} \notin (a^p)^G \) and so \( (a^p)^G \neq (a^p)^G \). Since \( C_G(a^pz_{r_1}) = C_G(a^p), |(a^pz_{r_1})^G| = |(a^p)^G| = p \). Also, \( (a^p)^Gz_{r_1} = a^pz_{r_1} \) and \( (a^p)^G \cap (a^pz_{r_1})^G = \emptyset, H \cap Hz_{r_1} = \emptyset \) and \( H \cup Hz_{r_1} \subset Z \). We now choose an element \( z_{r_2} \in Z \setminus (H \cup Hz_{r_1}) \) and by above method we will see that \( p \mid |Z| \). Suppose \( n = \frac{|Z|}{p} \).

Thus, there are \( n \) distinct conjugacy classes of the form \( (a^p)^Gz_{r_1} \), \( (a^p)^Gz_{r_2} \), \ldots, \( (a^p)^Gz_m \). Since \( 1 \leq s \leq p - 1 \), there are \( n(p - 1) \) distinct conjugacy classes with \( m = \frac{|Z|}{p} \) and each of which has \( p \) elements.

b) \( i = 0 \) and \( j \neq 0 \).

3. For \( p \mid j \), then \( j = tp \), where \( 1 \leq t \leq p - 1 \). By Table 1, \( |C_G(a^t)| = p^2 |Z| \) and so \( |(a^t)^G| = p^2 \). By Equation 4.1 and a similar argument as (1), for each \( t \) there are \( n \) distinct conjugacy classes of the form \( (b^tp)^z_{r_1} \), \( (b^tp)^z_{r_2} \), \ldots, \( (b^tp)^z_m \), where \( n = \frac{|Z|}{p} \). Since \( 1 \leq t \leq p - 1 \), there are \( n(p - 1) \) distinct conjugacy classes with \( m = \frac{|Z|}{p} \) and each class has \( p \) elements.

4. If \( p \nmid j \), then by Table 1, \( |C_G(a^t)| = p^2 |Z| \) and so \( |(a^t)^G| = p^2 \). Apply again Equation 4.1 and a similar argument as (2) to result that for each constant \( j \), there are \( n \) distinct conjugacy classes of the form \( (b^t)^z_{r_1} \), \( (b^t)^z_{r_2} \), \ldots, \( (b^t)^z_m \), where \( m = \frac{|Z|}{p^2} \). Since \( 1 \leq j \leq p^2 - 1 \) and \( p \nmid j \), there are \( m(p^2 - p) \) distinct conjugacy classes with \( m = \frac{|Z|}{p} \) and each class has \( p^2 \) elements.

c) \( i \neq 0 \) and \( j \neq 0 \). This case can be separated into the following four cases:

5. \( p \mid i \) and \( p \mid j \). In this case, \( i = sp \) and \( j = tp \) in which \( 1 \leq s, t \leq p - 1 \). By Table 1, \( |C_G(a^s^bp^t)| = p^3 |Z| \) and so \( |(a^s^bp^t)^G| = p \). We now apply Equation 4.1 and a similar argument as (1) for constants \( s \) and \( t \) to deduce that there are \( n \) distinct conjugacy classes of the form \( (a^s^bp^t)^z_{r_1} \), \( (a^s^bp^t)^z_{r_2} \), \ldots, \( (a^s^bp^t)^z_m \), where \( n = \frac{|Z|}{p} \). Since \( 1 \leq s, t \leq p - 1 \), there are \( n(p - 1)^2 \) distinct conjugacy classes with \( n = \frac{|Z|}{p} \) and each class has \( p \) members.

6. \( p \nmid i \) and \( p \mid j \). Clearly \( j = tp, 1 \leq t \leq p - 1 \). By Table 1, \( |C_G(a^s^bp^t)| = p^2 |Z| \) and hence \( |(a^s^bp^t)^G| = p^2 \). On the other hand, Equation 4.1 and a similar argument as (2) for constants \( i \) and \( t \), show that there are \( m \) distinct conjugacy classes of the form \( (a^s^bp^t)^z_{r_1} \), \( (a^s^bp^t)^z_{r_2} \), \ldots, \( (a^s^bp^t)^z_m \), where
\[ m = \frac{|Z|}{p^t}. \] Since \( 1 \leq t \leq p - 1, 1 \leq i \leq p^2 - 1 \) and \( p \nmid i \), there are \( mp(p - 1)^2 \) distinct conjugacy classes with \( m = \frac{|Z|}{p^t} \) and each class has \( p^2 \) members.

7) \( p \mid i \) and \( p \nmid j \). In this case \( i = sp \), for some integer \( s \) such that \( 1 \leq s \leq p - 1 \). By Table 1, \( |C_G(a^p b^j)| = p^2|Z| \) and so \( |(a^p b^j)^G| = p^2 \). Then by Equation 4.1 and a similar argument as (2) for constants \( s \) and \( j \), one can prove that there are \( m \) distinct conjugacy classes of the form \( (a^p b^j z_{r_1})^G, (a^p b^j z_{r_2})^G, \ldots, (a^p b^j z_{r_m})^G \), where \( m = \frac{|Z|}{p^t} \). Since \( 1 \leq s \leq p - 1, 1 \leq j \leq p^2 - 1 \) and \( p \nmid j \), there are \( mp(p - 1)^2 \) distinct conjugacy classes with \( m = \frac{|Z|}{p^t} \) and each class has \( p^2 \) members.

8) \( p \nmid i \) and \( p \nmid j \). By Table 1, \( |C_G(a^i b^j)| = p^2|Z| \) and so \( |(a^i b^j)^G| = p^2 \). On the other hand, by Equation 4.1 and applying a similar argument as (2) for constants \( i \) and \( j \), one can see that there are \( m \) distinct conjugacy classes of the form \( (a^i b^j z_{r_1})^G, (a^i b^j z_{r_2})^G, \ldots, (a^i b^j z_{r_m})^G \), where \( m = \frac{|Z|}{p^t} \). Since \( 1 \leq i,j \leq p^2 - 1 \) and \( p \nmid i,j \), there are \( mp^2(p - 1)^2 \) distinct conjugacy classes with \( m = \frac{|Z|}{p^t} \) and each class has \( p^2 \) members.

The above discussion are summarized in Table 2.

| Type | The representatives of conjugacy classes | \(|G|^2| \) | # Conjugacy classes |
|------|----------------------------------------|----------------|------------------|
| 1    | \( a^{ip} \) \( 1 \leq s \leq p - 1 \) | \( p \)         | \( n(p - 1) \)   |
| 2    | \( a^i \) \( 1 \leq i \leq p^2 - 1, i \neq 0 \) | \( p^2 \)       | \( mp(p - 1) \)  |
| 3    | \( b^{ip} \) \( 1 \leq t \leq p - 1 \) | \( p \)         | \( n(p - 1) \)   |
| 4    | \( b^j \) \( 1 \leq j \leq p^2 - 1, j \neq 0 \) | \( p^2 \)       | \( mp(p - 1) \)  |
| 5    | \( a^{ip} b^{ip} \) \( 1 \leq s,t \leq p - 1 \) | \( p \)         | \( n(p - 1)^2 \) |
| 6    | \( b^{ip} b^j \) \( 1 \leq t \leq p - 1, 1 \leq i \leq p^2 - 1, i \neq 0 \) | \( p^2 \)       | \( mp(p - 1)^2 \) |
| 7    | \( a^{ip} b^j \) \( 1 \leq s \leq p - 1, 1 \leq j \leq p^2 - 1, j \neq 0 \) | \( p^2 \)       | \( mp(p - 1)^2 \) |
| 8    | \( a^i b^j \) \( 1 \leq i,j \leq p^2 - 1, i \neq 0, j \neq 0 \) | \( p^2 \)       | \( mp^2(p - 1)^2 \) |

We are now ready to obtain the commuting conjugacy class graph of \( G \). To do this, we consider the following cases:

(I) Suppose \( a^{ip} \) and \( a^{ip} \) are representatives of two conjugacy classes of Type 1 in Table 2. It is clear that \( a^{ip} a^{ip} = a^{ip} a^{ip} \) and so all such classes are commuting together. Hence, the commuting conjugacy class graph has a subgraph isomorphic to the complete graph \( K_{n(p - 1)} \).

We now assume that \( a^p \) is a representative of a conjugacy class of Type 1. By Table 1, \( C_G(a^p) \) contains representatives of the conjugacy classes of types 2, 3, 5, and 6. Therefore, the conjugacy classes of Type 1 are adjacent with the conjugacy classes of Types 2, 3, 5, and 6.

(II) It is clear that two conjugacy classes of Type 2 are commuted to each other and so the commuting conjugacy class graph has a subgraph isomorphic to \( K_{mp(p - 1)} \). We now determine the relationship between conjugacy classes of this and other types. To do this, we assume that \( a^i \) is a representatives of a conjugacy class of Type 2 and \( a^i b^j z \) is an arbitrary element of \( G \) such that \( (a^i)(a^i b^j z) = (a^i b^j z)(a^i) \). So \( a^i b^j = b^j a^i \) and since \( p \nmid i \), by Lemma 2.2(a) the latter is true if and only if \( v = 0 \). Therefore, \( a^i \) commutes with all elements in the form \( a^i z \). This shows that the conjugacy classes of Type 2 are commuted only with the conjugacy classes of Type 1.

(III) Suppose \( b^{ip} \) and \( b^{ip} \) are the representative of two conjugacy classes of Type 3. By a similar argument as (I), all conjugacy classes of Type 3 are commuting together. Hence, the commuting conjugacy class graph has a subgraph isomorphic to \( K_{n(p - 1)} \). Also, the conjugacy classes of Type 3 are adjacent with the conjugacy classes of Types 1, 4, 5, and 7.

(IV) Suppose \( b^i \) and \( b^j \) are the representatives of two conjugacy classes of Type 4. By a similar argument as (4.1), all conjugacy classes of Type 4 are commuted to each other. So, the commuting
conjugacy class graph has a subgraph isomorphic to $K_{mp(p-1)}$. Also, the conjugacy classes of Type 4 can be commuted only with the conjugacy classes of Type 3.

(V) Suppose $a^{i_1}b^{j_1}$ and $a^{i_2}b^{j_2}$ are the representatives of two conjugacy classes of Type 5. Since $a^p \in C_G(b^p)$, $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^{i_2}b^{j_2})(a^{i_1}b^{j_1})$ and so all conjugacy classes of Type 5 are commuted to each other. This gives us the complete graph $K_{n(p-1)^2}$ as a subgraph of the commuting conjugacy class graph. Hence, it is enough to determine the relationship between this and conjugacy classes of Types 6, 7, and 8. Suppose $a^p b^p$ and $a^u b^v$ are the representatives of the conjugacy classes of Types 5 and 6, respectively, such that they are commuting together. By Lemma 3.1, $a^p b^p = (a^p b)^{n_1} z_1$ and $a^u b^v = (a^u b)^{n_2} z_2$ and so by Lemma 2.1, $C_G(a^p b^p) = C_G(a^u b^v)$ and $C_G(a^u b^v) = C_G(ab^p)$. It is now easy to prove that $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $b^p a = ab^p$ which is a contradiction with Lemma 2.2(a). Therefore, a conjugacy class of Type 5 is not adjacent with a conjugacy class of Type 6. Similarly, a conjugacy class of Type 5 is not adjacent to another one of Type 7. We now assume that $a^p b^p$ and $a^u b^v$ are representatives of the conjugacy classes of Types 5 and 8, respectively which are commute to each other. By Lemma 3.1, $a^p b^p = (a^p b)^{n_1} z_1$ and $a^u b^v = (a^u b)^{n_2} z_2$. On the other hand, By Lemma 2.1, $C_G(a^p b^p) = C_G(a^u b^v)$ and $C_G(a^u b^v) = C_G(ab^p)$ and by Table 1, we can see that $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $j \equiv i (\text{mod} p)$. Since $1 \leq i \leq p - 1$, the conjugacy classes of Types 5 and 8 can be divided into $p - 1$ parts. Moreover, this process constructs a complete subgraph of size $n(p - 1)$ in the commuting conjugacy class graph.

(VI) Suppose $a^{i_1}b^{j_1}$ and $a^{i_2}b^{j_2}$ are the representative of two conjugacy classes of Types 6 such that they are commute to each other. By Lemma 3.1, $a^{i_1}b^{j_1} = (a^{i_1}b)^{n_1} z_1$ and $a^{i_2}b^{j_2} = (a^{i_2}b)^{n_2} z_2$. On the other hand, by Lemma 2.1, $C_G(a^{i_1}b^{j_1}) = C_G(a^{i_2}b^{j_2})$ and $C_G(a^{i_2}b^{j_2}) = C_G(ab^{i_1})$. Therefore, $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^{i_2}b^{j_2})(a^{i_1}b^{j_1})$ if and only if $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^{i_2}b^{j_2})(a^{i_1}b^{j_1})$ if and only if $ab^{i_2}s_1 = b^{i_2}s_1 a$. By Lemma 2.2(a), the last equality is satisfied if and only if $s_1 = s_2$. This proves that in the commuting conjugacy class graph, two conjugacy classes of Type 6 are adjacent, when the centralizers of their representatives is equal. By Table 1, the number of centralizer of Types 6 is $p - 1$ and so the conjugacy classes of this type can be divided into $p - 1$ parts. On the other hand, by Table 2, the number of conjugacy classes of Type 6 is $mp(p - 1)^2$. Hence, each part of Type 6 is a clique of size $mp(p - 1)$. Now it is enough to determine the relationship between conjugacy classes of Type 6 with other conjugacy classes of Types 7 and 8. To do this, we assume that $a^p b^p$ and $a^u b^v$ are the representatives of conjugacy classes of Types 6 and 7, respectively, such that they are commute to each other. Since $a^p \in C_G(b^p)$, $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $a^p b^p = a^u b^v$. Since $p \nmid i, j$, Lemma 2.2(a) implies that the last equality cannot be obtained. This means that a conjugacy class of Type 6 is not adjacent with another one of Type 7 in the commuting conjugacy class graph. Next, we assume that $a^p b^p$ and $a^u b^v$ are the representatives of two conjugacy classes of Types 6 and 8, respectively and they are commute to each other. By Lemma 3.1, $a^p b^p = (a^p b)^{n_1} z_1$ and $a^u b^v = (a^u b)^{n_2} z_2$ and by Lemma 2.1, $C_G(a^p b^p) = C_G(ab^p)$ and $C_G(ab^p) = C_G(a^u b^v)$. Therefore, $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $(a^p b^p)(a^u b^v) = (a^u b^v)(a^p b^p)$ if and only if $ab^{i_2}s_1 = b^{i_2}s_1 a$. We now apply Lemma 2.2(a) to prove that the last equality is satisfied if and only if $p^2 \nmid (sp - j)$ or $p \nmid j$ which is a contradiction. Since $p \nmid j$, the conjugacy classes of Type 6 are not adjacent with another one of Type 8 in the commuting conjugacy class graph.

(VII) Suppose that $a^{i_1}b^{j_1}$ and $a^{i_2}b^{j_2}$ are representatives of two conjugacy classes of Type 7. By a similar argument as (4.1), this type is divided into $p - 1$ parts and each part is a clique of size $mp(p - 1)$. Also, the conjugacy classes of Type 7 are not adjacent to any of Type 8 in the commuting conjugacy class graph.

(VIII) Suppose $a^{i_1}b^{j_1}$ and $a^{i_2}b^{j_2}$ are the representatives of two conjugacy classes of Type 8 such that they are commuting together. By Lemma 3.1, $a^{i_1}b^{j_1} = (a^i b)^{n_1} z_1$ and $a^{i_2}b^{j_2} = (a^i b)^{n_2} z_2$ and by Lemma 2.1, $C_G(a^{i_1}b^{j_1}) = C_G(a^i b)$ and $C_G(a^{i_2}b^{j_2}) = C_G(a^i b)$. Therefore, $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^i b)^{n_1} z_1 (a^i b)^{n_2} z_2 = (a^i b)^{n_1 + n_2} z_1 z_2 = (a^i b)^{n_1 + n_2} z_1 z_2$ and so by Lemma 2.1, $C_G(a^{i_1}b^{j_1}) = C_G(a^{i_2}b^{j_2}) = C_G(ab^{i_1})$. Therefore, $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^{i_2}b^{j_2})(a^{i_1}b^{j_1})$ if and only if $(a^{i_1}b^{j_1})(a^{i_2}b^{j_2}) = (a^{i_2}b^{j_2})(a^{i_1}b^{j_1})$ if and only if $ab^{i_2}s_1 = b^{i_2}s_1 a$. We now apply Lemma 2.2(a) to prove that the last equality is satisfied if and only if $p^2 \nmid (sp - j)$ or $p \nmid j$ which is a contradiction. Since $p \nmid j$, the conjugacy classes of Type 8 are not adjacent with another one of Type 8 in the commuting conjugacy class graph.
(a^1 b^1)(a^1 b^i) if and only if (a^s b)(a^t b) = (a^s b)(a^t b) if and only if a^{s-t} b = ba^{s-t}. By Lemma 2.2(a), the last equality is satisfied if and only if s = t. This means that two conjugacy classes of Type 8 are adjacent in the commuting conjugacy class graph when their centralizers is equal. Also, by Table 1, the number of centralizers of Type 8 is \( p(p-1) \) and so the conjugacy classes of this type can be divided into \( p(p-1) \) parts. On the other hand, by Table 2, the number of conjugacy classes of Type 8 is \( mp^2(p-1)^2 \) which proves that each part of Type 8 gives a clique of size \( mp(p-1) \) in the commuting conjugacy class graph. By our discussion in Case 5, all conjugacy classes of Type 8 are adjacent only with conjugacy classes of Type 5. Since the conjugacy classes of Types 5 and 8 are adjacent in the commuting conjugacy class graph when their centralizers are equal, also, by Table 1, the number of centralizers of Type 8 is \( s \). Thus, two conjugacy classes \( a_iZb_jZ \) and \( a_iZb_kZ \) with this condition that \( G \Gamma(G) \cong Z_{p^2} \times Z_{p^2} \). When the semidirect product is non-abelian. We recall that by Theorem 2.3, \( p \) is an odd prime number.

4.2. Proof of Theorem 1.2 (2)

**Proof.** Since \( G \cong Z_{p^2} \cong Z_{p^2} \times Z_{p^2} \), there are \( a, b \in G \setminus Z \) such that

\[ G = \{ a^ibz \mid 0 \leq i, j \leq p^2 - 1; ba = a^{p+1}bZ; z, z_1, a^2, b^2 \in Z \}. \]

Apply Lemma 2.4 to deduce that \( b^i a^j Z = a^i b^j + b^j Z \) which implies that \( b^j Z a^i = (a^p Z)^{i} a^i Z b^j Z \) and \( a^i Z b^j Z = (a^p Z)^{i} b^j Z a^i Z \). In last equality, we put \( i = p \) and \( j = 1 \). Since \( a^2 \in Z, bZa^p Z = a^p Z bZ \). Thus \( a^p Z \in Z(\Gamma(G)) \). Choose elements \( x \) and \( g \) in \( G \setminus Z \). Then \( x = a^i b^j z_1 \) and \( g = a^u b^v z_2 \), where \( z_1, z_2 \in Z, 0 \leq i, j, u, v \leq p^2 - 1 \) and \( i, j, u, v \), as well as \( u, v \), are not simultaneously zero. We can see that \( g^{-1} x g Z = (a^{i+u} z_1 b^j) Z \), and so, there exists \( z \in Z \) such that

\[ (a^i b^j)^{-1} (a^i b^j)^{a^u b^v} = a^{(i+u)(j+v)} b^j. \]
Suppose that $x = a^ib^jz$ is an arbitrary element of $G$ such that $0 \leq i, j \leq p^2 - 1$ and $i, j$ are not simultaneously zero. To compute the number of non-central conjugacy classes of $G$ and obtain their structure, we will consider three separate cases that all together can be divided into eight subcases:

(a) $i \neq 0$ and $j = 0$. This case is separated into two subcases that $p \mid i$ and $p \nmid i$ as follows:

1. $p \mid i$. Suppose $i = sp$, where $1 \leq s \leq p - 1$. By Equation 4.2 and the fact that $a^{p^2} \in Z$, $(a^{p^2})^{-1}(a^{sp}a^{sp}) = a^{-sp}a^{sp}z = a^pz$ and so $(a^{sp})^G = \{a^pz \mid z \in Z\}$. By a similar argument as the case of (1) in the proof of Theorem 1.2 (1), there are $n$ distinct conjugacy classes $(a^{sp}z_1)^G, (a^{sp}z_2)^G, \ldots, (a^{sp}z_n)^G$, where the number of distinct conjugacy classes is equal to $n(p - 1)$, $n = \frac{|Z|}{p}$ and each class has $p$ elements.

2. $p \nmid i$. By Equation 4.2, $(a^{sp})^{-1}(a^{sp}) = a^{-sp}a^{sp}z$. By Table 1, $|C_G(a^i)| = p^2 |Z|$ and so $(a^{sp})^G = \{a^pz \mid z \in Z\}$. Note that $0 \leq v \leq p^2 - 1$ and so there are $\nu$ and $k$ such that $-\nu = v' + k$ and $0 \leq k \leq p - 1$. Since $a^{k+1} \in Z$, $(a^{k')^{G} = \{a^{sp}z \mid 0 \leq k, l \leq p - 1, z(k,l) \in Z\}$. Suppose $A_i = \{a^{sp}z \mid 0 \leq k \leq p - 1, z \in Z\}$. Thus, we choose an element $z_r \in Z \setminus H$ and an element $a^i z_r \in A_i$. It can be easily seen that $a^i z_r \notin (a^{sp})^G$ and $(a^i z_r)^G \neq (a^{sp})^G$. Since $C(a^i z_r) = C(a^i)$, $(a^i z_r)^G = \{a^pz \mid z \in Z\}$. Therefore, there exists $z_r \in Z$ such that $(a^{sp})^G = (a^i z_r)^G$. This proves that the number of distinct conjugacy classes is equal to $n(p - 1)$ in which $n = \frac{|Z|}{p}$ and each class has $p^2$ elements.

(b) $i = 0$ and $j \neq 0$. Again there are two cases that $p \mid j$ and $p \nmid j$.

3. $p \mid j$. Suppose $j = tp$, where $1 \leq t \leq p - 1$. By Equation 4.2 and the fact that $a^{p^2} \in Z$, one can conclude that $(a^{ib^j})^{-1}(b^{tp}) = a^{ib^j}b^{tp}z = b^{tp}z$ and so $(b^{tp})^G = \{b^{tp}z \mid z \in Z\}$. By Table 1, $|C_G(b^t)| = p^2 |Z|$ and hence $(b^{tp})^G = p$. Now by a similar argument as in the Case 1, there are $n$ distinct conjugacy classes $(b^{tp}z_1)^G, \ldots, (b^{tp}z_n)^G$, where $n = \frac{|Z|}{p}$. So, the number of distinct conjugacy classes is equal to $n(p - 1)$ and each class has $p$ elements.

4. $p \nmid j$. By Equation 4.2, $(a^{ib^j})^{-1}(b^{tp}) = a^{ib^j}b^{tp}z$ and by Table 1, $|C_G(b^t)| = p^2 |Z|$. Hence $(b^{tp})^G = p^2$. A similar argument as in the Case 2 shows that $(b^{ip})^G = \{a^{sp}z \mid 0 \leq k, l \leq p - 1, z(k,l) \in Z\}$. Thus, there are $n$ distinct conjugacy classes $(b^i z_1)^G, \ldots, (b^i z_n)^G$, when $j$ is a fixed number and $n = \frac{|Z|}{p}$. Since $1 \leq j \leq p^2 - 1$ and $p \nmid j$, the number of distinct conjugacy classes is equal to $n(p^2 - p)$ and each class has $p^2$ elements.

(c) $i \neq 0$ and $j \neq 0$. We have four subcases as follows:

5. $p \mid i$ and $p \mid j$. Suppose $i = sp$ and $j = tp$ such that $1 \leq s, t \leq p - 1$. By Equation 4.2 and the fact that $a^{p^2} \in Z$, $(a^{ib^j})^{-1}(a^{ib^j}b^{tp}) = a^{ib^j}b^{tp}z = a^{ib^j}b^{tp}z$. Thus, $(a^{ib^j})^G = \{a^{ib^j}b^{tp}z \mid z \in Z\}$. By Table 1, $|C_G(a^{ib^j}b^{tp})| = p^3 |Z|$ which implies that $(a^{ib^j})^G = p$. Similar to the Case 1, for fixed non-negative integers $s$ and $t$, there are $n$ distinct conjugacy classes as $(a^{ib^j}b^{tp}z_1)^G, \ldots, (a^{ib^j}b^{tp}z_n)^G$ in which $n = \frac{|Z|}{p}$. By above discussion, we conclude that the number of the distinct conjugacy classes is equal to $n(p^2 - 1)^2$ and each class has $p$ elements.

6. $p \nmid i$ and $p \nmid j$. Suppose $j = tp$, where $1 \leq t \leq p - 1$. By Equation 4.2 and the fact that $a^{p^2} \in Z$, $(a^{ib^j})^{-1}(a^{ib^j}b^{tp}) = a^{ib^j}b^{tp}z = a^{ib^j}b^{tp}z$. By Table 1, $|C_G(a^{ib^j}b^{tp})| = p^2 |Z|$ and so
The relationship between conjugacy classes in Table 3. We consider the following cases:

(ii) Suppose \( G \) has \( p^2 \) distinct conjugacy classes and \( |G| = \frac{|Z|}{p} \). Hence, for fixed positive integers \( i \) and \( t \), there are \( n \) distinct conjugacy classes as \( (a^i b^j b^k z_{(k,l)} )^G \) in which \( n = \frac{|Z|}{p} \). Since \( 1 \leq i \leq p^2 - 1 \) and \( p \nmid i \), there are integers \( k' \) and \( i' \) such that \( i = k' p + i' \) and \( 1 \leq i' \leq p - 1 \). Therefore, \( (a^i b^j b^k z_{(k,l)} )^G \) for \( 1 \leq l \leq p - 1, z_{(k,l)} \in Z \). This shows that there exists \( z_{(i',t)} \in Z \) such that \( (a^i b^j b^k z_{(i',t)} )^G \). Since \( 1 \leq i' \leq p - 1 \), the number of the distinct conjugacy classes is equal to \( n(p - 1)^2 \) and each class has \( p^2 \) elements.

(7) \( p \mid i \) and \( p \nmid j \). Suppose \( i = sp \) such that \( 1 \leq s \leq p - 1 \). By Equation 4.2 and the fact that \( a^s \in Z \), we have \( (a^i b^j b^k)^G = (a^i b^k)^G = a^i (a^s b^j b^k ) = a^{i+j} b^k z_{(k,l)} \). By Table 1, \( |C_G(a^i b^j)| = p^2 \) and so \( |(a^i b^j)^G| = p^2 \) and the argument is similar to the Case 4.

(8) \( p \mid i \) and \( p \nmid j \). By Equation 4.2, \( (a^i b^j b^k)^G = (a^i b^k)^G = a^{i+j-k} b^k z_{(k,l)} \). By Table 1, \( |C_G(a^i b^j)| = p^2 \) and so \( |(a^i b^j)^G| = p^2 \) and the argument is similar to the Case 4.

We now investigate the commuting conjugacy class graph of \( G \). To do this, it is enough to determine the relationship between conjugacy classes in Table 3. We consider the following cases:

(i) Suppose \( a^{i_1 p} \) and \( a^{i_2 p} \) are the representatives of two classes of Type 1 in Table 3. By a similar argument as Case 1 in the proof of Theorem 1.2 (1), all conjugacy classes of Type 1 are commuting together. Hence, the commuting conjugacy class graph has a subgraph isomorphic to \( K_{n(p-1)} \).

(ii) Suppose \( a^{i_1} \) and \( a^{i_2} \) are the representatives of two conjugacy classes of Type 2. Obviously, \( a^{i_1} a^{i_2} = a^{i_2} a^{i_1} \) and so all conjugacy classes of Type 2 are commuted to each other. Hence, the commuting conjugacy class graph has a clique of size \( n(p - 1) \).

(iii) Suppose \( b^{i_1 p} \) and \( b^{i_2 p} \) are representatives of two conjugacy classes of Type 3. By a similar argument as Case 4.1 in the proof of Theorem 1.2 (1), all conjugacy classes of Type 3 are commuting together. Hence, the commuting conjugacy class graph has a subgraph isomorphic to \( K_{n(p-1)} \). Also, the conjugacy classes of Type 3 are adjacent with the conjugacy classes of Types 1, 4, 5 and 7.

| Type       | The representative of conjugacy classes | \( |G| \) | # Conjugacy classes |
|------------|----------------------------------------|--------|-------------------|
| 1          | \( a^i \)                                | \( 1 \leq i \leq p - 1 \) | \( p \) | \( n(p - 1) \)    |
| 2          | \( a^j \)                                | \( 1 \leq j \leq p - 1 \) | \( p^2 \) | \( n(p - 1) \)    |
| 3          | \( b^l \)                                | \( 1 \leq l \leq p - 1 \) | \( p \) | \( n(p - 1) \)    |
| 4          | \( a^i b^j \)                            | \( 1 \leq i \leq j \leq p - 1 \) | \( p^2 \) | \( n(p - 1) \)    |
| 5          | \( a^i b^j b^k \)                        | \( 1 \leq i \leq j \leq k \leq p - 1 \) | \( p^2 \) | \( n(p - 1)^2 \)  |
(iv) Suppose $b^i_1$ and $b^i_2$ are representatives of two conjugacy classes of Type 4. Since $b^i_1 b^i_2 = b^i_2 b^i_1$, all conjugacy classes of Type 4 are commuting together and hence the commuting conjugacy class graph has a clique of size $np(p - 1)$. By a similar argument as Case 4.1 in the proof of Theorem 1.2 (1), the conjugacy classes of Type 4 are commuting only with conjugacy classes of Type 3.

(v) Suppose $a^i_1 p b^i_1 p$ and $a^i_2 p b^i_2 p$ are representatives of two conjugacy classes of Type 5. The argument for this case is similar to Case 4.1 in the proof of Theorem 1.2 (1).

(vi) Suppose $a^i_1 b^i_1 P$ and $a^i_2 b^i_2 P$ are representatives of two commuting conjugacy classes of Type 6. The argument for this case is similar to Case 4.1 in the proof of Theorem 1.2 (1) and each part of Type 6 is a clique of size $n(p - 1)$.

(vii) Suppose $a^i_1 b^i_1$ and $a^i_2 b^i_2$ are representatives of two commuting conjugacy classes of Type 8. The argument for this case is similar to Case 4.1 in the proof of Theorem 1.2 (1) and each part of Type 8 is a clique of size $n(p - 1)$.

By above discussion, the commuting conjugacy class graph of $G$ is a connected graph with $n(p - 1)(p + 1)^2$ vertices. Suppose $M_2$ is the graph depicted in Figure 1(b). Then the commuting conjugacy class graph of $G$ can be written as a $M_2$-join, i.e.

$$\Gamma(G) = M_2[\{K_{np(p - 1)}\}] = M_2[\{K_{n(p - 1)}, K_{n(p - 1)}, K_{n(p - 1)}\}]$$

The graph $\Gamma(G)$ is depicted in Figure 3.

![Figure 3](image-url)  

Figure 3. The Graph $\Gamma(G)$, when $G_{\mathbb{Z}(G)}$ is non-abelian and $G_{\mathbb{Z}} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.

\[\square\]

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