We show that a system of particles on the lowest Landau level can be coupled to a probe U(1) gauge field $A_\mu$ in such a way that the theory is invariant under a noncommutative U(1) gauge symmetry. While the temporal component $A_0$ of the probe field is coupled to the projected density operator, the spatial components $A_i$ are best interpreted as quantum displacements, which distort the interaction potential between the particles. We develop a Seiberg-Witten-type map from the noncommutative U(1) gauge symmetry to a simpler version, which we call “baby noncommutative” gauge symmetry, where the Moyal brackets are replaced by the Poisson brackets. The latter symmetry group is isomorphic to the group of volume preserving diffeomorphisms. By using this map, we resolve the apparent contradiction between the noncommutative gauge symmetry, on the one hand, and the particle-hole symmetry of the half-filled Landau level and the presence of the mixed Chern-Simons terms in the effective Lagrangian of the fractional quantum Hall states, on the other hand. We outline the general procedure which can be used to write down effective field theories which respect the noncommutative U(1) symmetry.

I. INTRODUCTION

The problem of the fractional quantum Hall effect is often formulated in the limit of a single Landau level (e.g., the lowest Landau level (LLL)), in which the nontriviality of the problem becomes most stark. In this limit, the drift motion of a single electron in an external potential can be captured by a Hamiltonian formalism in which the two Cartesian coordinates $x$ and $y$ of the electrons do not commute with each other. It has been suggested that the quantum field theory describing the fractional quantum Hall fluids has to be a noncommutative (NC) field theory [1–5]. However, noncommutativity of space in the first-quantized description does not translate directly to a noncommutativity of space in the second-quantized formalism. Some progress in deriving noncommutative field theories for the quantum Hall effect has recently been made for bosonic quantum Hall states near filling factor $\nu = 1$ [6], based on previous works by Pasquier and Haldane [7] and Read [8].

In this paper we try to address the question of whether the quantum Hall states are governed by a noncommutative field theory. Instead of following a constructive route, we will base our approach on symmetry constraints. We show that the LLL electrons can be coupled to an external probe which can be identified with a NC U(1) gauge potential $A_\mu$ in a
way that partition function of the theory is invariant under NC U(1) gauge transformation. However, $A_\mu$ is not the usual electromagnetic probe. While the scalar potential $A_0$ is coupled simply to the projected density operator, the vector components $A_i$ do not couple to the charge current. In fact, $A_i$ is best interpreted as a displacement, disturbing the shape of the potential between two electrons. So in addition to the NC gauge symmetry, there is an additional symmetry where $A_i$ is shifted by a functions of time which are independent of space [Eq. (34) below].

Armed with the symmetry information, one can then proceed to a construction of effective field theories describing various fractional quantum Hall states. The simplest way to guarantee the invariance of the theory with respect to the NC U(1) symmetry is to try to promote field theories of the fractional quantum Hall effect to noncommutative field theories. However, one immediately encounters serious problems along the way. First, one sees that the naive noncommutative version of the Dirac composite fermion theory now breaks particle-hole symmetry. Furthermore, the mixed Chern-Simons term, crucial for the construction of the effective field theory for many quantum Hall states (the “hierarchy states”) [9], does not have an obvious noncommutative extension.

We solve this problem by developing a map, inspired by the Seiberg-Witten (SW) map [10], which we term the “SW’ map,” which maps the noncommutative probe field $A_\mu$ into what we call “baby noncommutative” (bNC) gauge field $A_\mu^b$. In the transformation laws of the bNC gauge field, the Moyal brackets are replaced by the Poisson brackets. The bNC U(1) gauge group is isomorphic to the group of volume-preserving diffeomorphisms (VPDs). Thus the task of writing down an action invariant under the gauge symmetry reduces to ensuring volume-preserving diffeomorphism invariance, which can be accomplished by using an appropriate geometric formalism, for example, the Newton-Cartan formalism [11, 12]. In this way one can write down a particle-hole-symmetric effective field theory for the half-filled Landau level as well as a Chern-Simons theory with a general $K$-matrix.

The paper is organized as follows. In Sec. II we review the basic formulas of noncommutative space and noncommutative gauge theory. In Sec. III we couple the electrons on a single Landau level to a NC U(1) gauge field. In Sec. IV we outline the ways one can construct effective field theories which respect the NC U(1) gauge symmetry. Section V contains final remarks.

II. NONCOMMUTATIVE SPACE AND NONCOMMUTATIVE GAUGE SYMMETRY

For convenience and further reference, in this Section we collect some formulas related to noncommutative space and noncommutative gauge symmetry.

Consider particles in a magnetic field $B = \partial_1 A_2 - \partial_2 A_1$. The gauge-invariant momenta are

$$\hat{p}_i = -i(\partial_i - iA_i) = \hat{k}_i - A_i(\hat{x}).$$

(1)
They satisfy the commutation relation

\[ [\hat{p}_i, \hat{p}_j] = i\varepsilon_{ij}B. \]  

(2)

Projecting to the LLL effectively sets \( p_i \approx 0 \) (the kinetic energy is \( \hat{p}^2/2m \), LLL projection corresponds to taking \( m \to 0 \)). This introduces the Dirac brackets between \( x_i \):

\[ [\hat{x}^i, \hat{x}^j]_D = -[\hat{x}^i, \hat{p}_k](\hat{p}, \hat{p})^{-1}_{kl}[\hat{p}_l, \hat{x}^j] = -i\ell^2 \varepsilon^{ij}. \]  

(3)

So effectively the particle lives in a noncommutative space, with the noncommutative parameter \( \theta^{ij} = \theta \varepsilon^{ij} \) with

\[ \theta = -\ell^2. \]  

(4)

Any operator in the Heisenberg algebra (i.e., can be expanded in Taylor series over \( \hat{x}_i \)) can be put into correspondence it Weyl symbol by the following rule

\[ e^{iq_i\hat{x}^i} \to e^{iq_i x^i}. \]  

(5)

Taking all possible linear combinations of the exponential one can put any function \( f(x) \) into correspondence with an operator \( \hat{f}(\hat{x}) \) in a unique way. (We will use the same letter for the operator and its Weyl symbol, putting a hat on top of the the symbol for the operator.)

The Moyal product between two functions \( f(x) \) and \( g(x) \), \( f \ast g \), is defined so that the Weyl symbol of \( \hat{f} \hat{g} \) is \( f \ast g \).

\[ f \ast g(x) = \exp \left( \frac{i}{2} \varepsilon^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x)g(y) \bigg|_{y \to x} = f(x) \exp \left( \frac{i}{2} \varepsilon^{ij} \partial_i \partial_j \right) g(x). \]  

(6)

The Moyal bracket is defined as

\[ \{f, g\} = \frac{1}{i}(f \ast g - g \ast f) = 2f \sin \left( \frac{1}{2} \varepsilon^{ij} \partial_i \partial_j \right) g, \]  

(7)

and is (up to the factor \(-i\)) the Weyl symbol of the commutator \( [\hat{f}, \hat{g}] \). The factor of \(-i\) makes sure that if \( f \) and \( g \) are real functions then \( \{f, g\} \) is also real. To leading order in \( \theta \) the Moyal bracket becomes the Poisson bracket

\[ \{f, g\} = \theta \{f, g\} + O(\theta^2), \]  

(8)

where

\[ \{f, g\} = \varepsilon^{ij} \partial_i f \partial_j g. \]  

(9)

We now consider noncommutative gauge symmetry. For the purposes of this paper, we can limit ourselves to the U(1) case and only to adjoint fields. An adjoint field \( \psi \) transforms under a noncommutative U(1) gauge transformation as

\[ \delta_x \psi = \{\psi, \lambda\}, \]  

(10)
with \( \lambda \) being the parameter of the infinitesimal gauge transform. The covariant derivative is defined as

\[
D_\mu \psi = \partial_\mu \psi + \{ A_\mu, \psi \},
\]

and the gauge potential transforms as

\[
\delta_\lambda A_\mu = \partial_\mu \lambda + \{ A_\mu, \lambda \}.
\]

The gauge invariant gauge field is

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{ A_\mu, A_\nu \}.
\]

\section{III. SYMMETRY OF THE LANDAU LEVEL PROBLEM}

\subsection{A. Introduction of external probe fields}

The starting point of our discussion is the Hamiltonian describing a system of \( N \) particles on a single Landau level, interacting with each other through a two-body potential \( V(x - y) \),

\[
H = \sum_{\langle ab \rangle} \hat{V}(\hat{x}_a - \hat{x}_b) = \sum_{\langle ab \rangle} \int \frac{dq}{(2\pi)^2} V(q) e^{iq(\hat{x}_a - \hat{x}_b)},
\]

where \( V(q) \) is the projected version of the inter-particle potential \( V(q) \)

\[
V(q) = V(q) e^{-q^2\ell^2/2} L_n \left( \frac{q^2\ell^2}{4} \right),
\]

with \( L_n \) being the \( n \)th Laguerre polynomial and \( n \) is the quantum number of the Landau level on which the electrons live (for a quick derivation of Eq. (14) see, e.g., Ref. [13]).

We will couple the theory to a set of probe fields. These probes are functions in spacetime and act on all particles in the same ways. The first probe is the scalar potential, or the temporal component of the electromagnetic field \( A_0(x) \). The Hamiltonian now contains a one-body term

\[
H = \sum_{\langle ab \rangle} \hat{V}(\hat{x}_a - \hat{x}_b) - \sum_a \hat{A}_0(\hat{x}_a),
\]

where, on the lowest Landau level, \( A_0(q) = e^{-q^2\ell^2/4} A_0(q) \).

Now we introduce two more probe fields which we call \( u^x(x) \) and \( u^y(x) \), forming a vector \( u(x) \). These probes replace the coordinates of the particles in the interaction potential \( V(x_a, x_b) \) by new coordinates

\[
x \rightarrow X = x + u(x).
\]

The perturbed Hamiltonian is now

\[
\tilde{H} = \sum_{\langle ab \rangle} \hat{V}(\hat{X}_a - \hat{X}_b) - \sum_a \hat{A}_0(\hat{x}_a) = \sum_{\langle ab \rangle} \hat{V}(\hat{x}_a + \hat{u}(\hat{x}_a) - \hat{x}_b - \hat{u}(\hat{x}_b)) - \sum_a \hat{A}_0(\hat{x}_a).
\]

Note that we do not shift the coordinates in the one-body term \( \hat{A}_0 \). The probes can also be made time-dependent: \( A_0 = A_0(t, x) \), \( u = u(t, x) \).
B. Detour: time dependent unitary transformation

Consider a system described by a Hamiltonian $\hat{H}$ (which, in general can be time-dependent, $\hat{H} = \hat{H}(t)$). Suppose we know how to find all solutions $|\psi(t)\rangle$ to the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}(t)|\psi\rangle.$$  \hfill (19) 

If now $\hat{U}$ is a (time-independent) unitary operator, and $\hat{H}_U = \hat{U} \hat{H} \hat{U}^{-1}$, then $|\psi\rangle_U = \hat{U} |\psi\rangle$ is the solution to the Schrödinger equation with $\hat{H}$ replaced by $\hat{H}_U$.

But one can also perform a time-dependent unitary transformation with time-dependent $\hat{U}$. It is easy to see that if one sets

$$\hat{H}_U(t) = \hat{U} \hat{H} \hat{U}^{-1} + i \partial_t \hat{U} \hat{U}^{-1},$$ \hfill (20) 

then the state $|\psi\rangle_U = \hat{U}(t) |\psi\rangle$ still solves the time-dependent Schrödinger equation. Therefore the new Hamiltonian is completely equivalent to the old Hamiltonian. In particular, if $\hat{U}(t)$ is not equal to 1 only in a finite time interval $t_i < t < t_f$, then the evolution operator $T \exp \left[ -i \int_{t_i}^{t_f} dt \hat{H}(t) \right]$ does not change when $\hat{H}(t)$ is replaced by $H_U(t)$.

We now consider an infinitesimal unitary transformation with $\hat{U} = e^{i\hat{\lambda}}$ where $\hat{\lambda} \ll 1$. The transformed Hamiltonian is then

$$\hat{H}_\lambda = \hat{H} + i[\hat{\lambda}, \hat{H}] - \partial_t \hat{\lambda}.$$ \hfill (21) 

C. Unitary transformation in the Landau-level problem

We now act on our Hamiltonian describing $N$ particles noncommutative plane with the following infinitesimal unitary transformation

$$\hat{U} = \exp \left[ i \sum_a \hat{\lambda}(t, \hat{x}_a) \right],$$ \hfill (22) 

where $\hat{\lambda}(t, \hat{x})$ is an infinitesimal function of $t$ and $\hat{x}$. Note that all particles are subjected to the same unitary transformation.

The transformed Hamiltonian is

$$\hat{H}_\lambda = \hat{U} \hat{H} \hat{U}^{-1} + i \partial_t \hat{U} \hat{U}^{-1} = \sum_{(ab)} \mathcal{V}(\hat{X}_a^\lambda - \hat{X}_b^\lambda) - \sum_a \hat{A}_0^\lambda(t, \hat{x}_a).$$ \hfill (23) 

where

$$\hat{X}_a^\lambda = \hat{X} + i[\hat{\lambda}, \hat{X}],$$ \hfill (24) 

$$\hat{A}_0^\lambda = \hat{A}_0 + i[\hat{\lambda}, \hat{A}_0] + \partial_t \hat{\lambda}.$$ \hfill (25)
So the new Hamiltonian has the same form as the old Hamiltonian but with the new values of the probe fields. The transformation laws of the corresponding Weyl symbols are

\[ X^\lambda = X + \{X, \lambda\}, \]
\[ A^\lambda_0 = A_0 + \{A_0, \lambda\} + \partial \lambda. \]  (26, 27)

Knowing how \( X^i \) transform we can find how \( u^i \) transforms. We have \( X = x + u \), so Eq. (26) reads

\[ x^i + (u^i)^\lambda = x^i + u^i + \{x^i + u^i, \lambda\}. \]  (28)

But

\[ \{x^i, \lambda\} = -\ell^2 \varepsilon^{ij} \partial_j \lambda, \]  (29)

which means

\[ (u^i)^\lambda = u^i + \{u^i, \lambda\} - \ell^2 \varepsilon^{ij} \partial_j \lambda. \]  (30)

Now if we introduce \( A_i \) so that

\[ u^i = -\ell^2 \varepsilon^{ij} A_j, \]  (31)

then \( A_i \) transforms as

\[ A^\lambda_i = A_i + \{A_i, \lambda\} + \partial_i \lambda. \]  (32)

That means we can combine \( A_0 \) with \( A_i \) into \( A_\mu \), all components of which transform in the same way

\[ \delta_\lambda A_\mu = \{A_\mu, \lambda\} + \partial_\mu \lambda. \]  (33)

This is exactly the gauge transformation of the noncommutative \( U(1) \) gauge symmetry.

So the system of particles on a Landau level can be coupled to a set of gauge potentials \( A_\mu \) so that the physics is invariant under the noncommutative \( U(1) \) gauge transformation. It follows that any low-energy effective theory should also couple to the \( A_\mu \) probe in a way that respects this invariance. Note that while \( A_0 \) is related to the temporal component of the electromagnetic field, \( A_i \) is not at all the usual vector potential of electromagnetism.

Beside the noncommutative \( U(1) \) gauge symmetry, translational invariance of the two-body potential term leads to a symmetry

\[ A_i(t, x) \rightarrow A_i(t, x) + \alpha_i(t), \]  (34)

where \( \alpha_i(t) \) are functions of time only. This additional symmetry should also be respected by any low-energy effective field theory.

IV. NONCOMMUTATIVE SYMMETRY IN EFFECTIVE FIELD THEORY

A. Problem with particle-hole symmetry

The effective field theory describing the half-filled Landau level [14, 15] is a \( U(1) \) gauge theory which involves, as dynamical degree of freedom, a composite fermion \( \psi \) and an
emergent gauge field $a_\mu$. The effective action has a $U(1)_a$ emergent gauge symmetry, and also the conventional $U(1)_{em}$ gauge symmetry, encoded in the coupling of the theory to external background gauge field $A_\mu$.

One now asks how one can couple the theory to the noncommutative probe field $A_\mu$. One may expect that the field theory should be promoted to a noncommutative field theory. In this scenario the $U(1)_a$ gauge symmetry would become a $U(1)_a$ noncommutative gauge symmetry. In order to ensure the time-dependent shift symmetry (34), one may require all fields to transform in the adjoint representation of the $U(1)$ NC gauge symmetry.

Such noncommutative field theory of the half-filled Landau level can be constructed (see, e.g., Ref. [16]) but one sees an immediate problem. Namely, the NC $U(1)_a$ gauge symmetry is in conflict with the particle-hole (PH) symmetry (or $CT$ symmetry) of the physics on a single Landau level. To show that, let us write down the transformation laws for the gauge fields under gauge transformations [15],

$$U(1)_a^{NC}: \quad \delta_\alpha a_\mu = \{a_\mu, \alpha\} + \partial_\mu \alpha,$$

$$U(1)^{NC}_{em}: \quad \delta_\lambda A_\mu = \{A_\mu, \lambda\} + \partial_\mu \lambda.$$  

Under particle-hole symmetry the fields transform as follows:

$$A_0(t, x) \rightarrow -A_0(-t, x), \quad A_i(t, x) \rightarrow A_i(-t, x),$$

$$a_0(t, x) \rightarrow a_0(-t, x), \quad a_i(t, x) \rightarrow -a_i(-t, x).$$

For this symmetry to be consistent with the noncommutative gauge symmetry, we need to be able to assign transformation laws to the gauge transformation parameters $\lambda$ and $\alpha$ under PH conjugation. While we can do this for $\lambda$ by postulating that $\lambda(t, x) \rightarrow \lambda(-t, x)$, we cannot do the same for $\alpha$: the two terms in the transformation law for $a_0$ in Eq. (35) always transform differently under PH, no matter which PH conjugation rule for $\alpha$. Thus one concludes that emergent noncommutative $U(1)$ gauge symmetry is fundamentally incompatible with the particle-hole symmetry of the half-filled Landau level.

There is one more obstacle in writing down a fully noncommutative description of quantum Hall states. The general Chern-Simons description of an abelian quantum Hall state involves, in general, multiple internal gauge fields with mutual Chern-Simons terms [9]

$$L \sim \sum_{I,J} K_{IJ} e^{\mu \nu \lambda} a_\mu^I \partial_\nu a_\lambda^J.$$  

However, since the NC $U(1)$ symmetry is nonabelian, it is not possible to write a mutual CS term involving two different NC $U(1)$ gauge fields. This problem has been noted recently revisited by Goldman and Senthil [17], where a partial solution, valid to next order in the expansion over the noncommutativity parameter $\theta$ is presented. Here we present a different approach, capable of solving the problem of the mutual Chern-Simons terms and the problem of particle-hole symmetry in one scoop, to all orders in the noncommutativity parameter.
B. Volume preserving diffeomorphism as a “baby” version of noncommutative gauge symmetry

To solve the problems identified above, we first note a close analogy between the noncommutative gauge symmetry $U(1)_A$ and volume preserving diffeomorphism $[12]^1$. We recall (see Ref. [12] for details) that the fractional quantum Hall problem has invariance under time-dependent diffeomorphisms that preserve the spatial volume:

$$x^i \rightarrow x^i + \xi^i(t, x), \quad \xi^i = \ell^2 \epsilon^{ij} \partial_j \lambda,$$

(40)

where $\ell^2 = 1/B$ is the magnetic length, if it is coupled to a scalar potential $A_0$ and a metric $g_{ij}$ that transform as follows:

$$\delta \lambda A_0 = \partial_0 \lambda + \theta \{A_0, \lambda\}, \quad \{A_0, \lambda\} = \epsilon^{ij} \partial_i A_0 \partial_j \lambda,$$

(41)

$$\delta \lambda g_{ij} = -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k, \quad \xi^k = \ell^2 \epsilon^{kl} \partial_l \lambda.$$

(42)

$A_0$ and $g_{ij}$ are the background fields for volume preserving diffeomorphisms. Note that the transformation law for $A_0$ is the long-wavelength limit of that for the noncommutative potential $A_0$, Eq. (12) Furthermore, one can limit oneself to flat metrics (which obviously remain flat under diffeomorphisms), which can be parametrized in terms of the coordinates $X^a(x)$, defined as the coordinates in which the metric tensor is $\delta_{ab}$:

$$g_{ij} = \delta_{ab} \partial_i X^a \partial_j X^b.$$

(43)

The transformation law for $X^a$ is

$$\delta X^a = -\xi^i \partial_i X^a = \theta \{X^a, \lambda\}.$$

(44)

If we now write the covariant coordinates as $O(\theta)$ perturbations around the background coordinates

$$X^i = x^i + \theta \epsilon^{ij} A_j,$$

(45)

then the transformation law for the metric induces a transformation law for the background fields $A^b_j$ which look like

$$\delta \lambda A_i = \partial_i \lambda + \theta \{A_i, \lambda\}.$$

(46)

Once again, this is the long-wavelength limit of the transformation law for the spatial components $A_i$ of the noncommutative gauge field. Combining $A_0$ and $A_i$ into a spacetime vector $A_\mu$ (with $A_0 = A_0$), the transformation law of the latter,

$$\delta \lambda A_\mu = \partial_\mu \lambda + \theta \{A_\mu, \lambda\},$$

(47)

$^1$ A nontrivial connection between area-preserving diffeomorphism and the $W_\infty$ algebra is also found by Dung X. Nguyen [18].
is the long wavelength limit of the noncommutative gauge transformation law where the Moyal brackets are replaced by the Poisson brackets. We will call this type of gauge symmetry the “baby noncommutative” (bNC) gauge symmetry.

There exists a well-defined procedure to write down field theories that are VPD-invariant. In these theories, each field transforms in a certain representation (scalar, vector, tensor, etc.) of the VPD group. For example, in the Dirac composite fermion theory, the composite fermion field transforms as

\[ \delta \psi = \theta \{ \psi, \lambda \} \] (48)

(here we ignore the possible coupling of \( \psi \) to the spin connection) and the emergent gauge field transforms as a one-form under VPD,

\[ \delta_\lambda a_\mu = -\xi^k \partial_k a_\mu - a_k \partial_\mu \xi^k = \theta \{ a_\mu, \lambda \} + \theta \epsilon^{kl} a_k \partial_\mu \partial_l \lambda. \] (49)

We note that this is \textit{not} what one would expect for the transformation law of an adjoint field. Under the internal \( U(1)_a \) gauge symmetry, the transformation law is simply

\[ \delta_\alpha a_\mu = \partial_\mu \alpha. \] (50)

The absence of the term \( \theta \{ a_\mu, \alpha \} \) allows the transformation law to be consistent with PH symmetry.

For completeness, we write down the simplest Lagrangian of the effective field theory of the Dirac composite fermion,

\[ \mathcal{L} = \frac{i}{2} \bar{\psi} D_\mu \psi - D_\mu \bar{\psi}^\dagger \psi + \frac{i}{2} \bar{\psi} e^i_a (D_i \psi - D_i \bar{\psi}^\dagger) - \frac{a_0}{4 \pi \ell^2} + A_0 \left( \frac{1}{4 \pi \ell^2} - \frac{b}{2 \pi} \right), \] (51)

where (neglecting the coupling of \( \psi \) to the spin connection) \( D_\mu \psi = (\partial_\mu - i a_\mu) \psi \), \( v^\mu = (1, \ell^2 \epsilon^{ij} \partial_j A_0) \), and \( e^i_a \) is the vielbein, which can be defined by \( e^i_a \partial_i X^b = \delta^a_b \). One can check that this theory is invariant under VPDs.

### C. SW’ map

We now would like to use the insights obtained from the last Section to find a way to write down a composite fermion theory that is both PH-symmetric and can be coupled to an external NC U(1) gauge field \( A_\mu \).

Our method is inspired by the Seiberg-Witten map between the NC and commutative gauge symmetries. We want to derive an analogous map that maps the NC gauge field \( A_\mu \) to a new gauge field \( A^b_\mu \)

\[ A^b_\mu = A^b_{\mu[A_\mu]}, \] (52)

such that the field \( A^b_\mu \) transforms under gauge transformations as a bNC gauge field, similarly to the field \( A_\mu \) discussed in the previous Section:

\[ \delta A^b_\mu = \partial_\mu \lambda^b + \theta \{ A^b_\mu, \lambda^b \}. \] (53)
The gauge parameter \( \lambda^b \) can depend on the gauge field

\[
\lambda^b = \lambda^b[\lambda, A_\mu]. 
\]  

(54)

If such a map can be found, then the rules for writing down the coupling of a composite fermion theory to the NC probe \( A_\mu \) is as follows. We first couple the theory to the probe \( A_\mu^b \) in a way that respect the bNC gauge symmetry. Then we restrict ourselves to flat metrics and parametrize the metric through the coordinates \( X^a \), or equivalently \( A_\mu^b \). Now we have an action coupled to the baby-NC gauge fields \( A_\mu^b \) and is invariant under baby-NC U(1) gauge transformations. We now use the SW' map (52) to replace the probe field \( A_\mu^b \) by the NC gauge field \( A_\mu \). The result is an effective theory coupled to the \( A_\mu \) field with NC U(1) gauge symmetry.

One may wonder if one can use the original Seiberg-Witten map to map the NC field \( A_\mu \) to a commutative gauge field \( A_\mu^c \), which can be coupled easily to any EFT of the FQHE. The reason we cannot do so is the time-dependent shift symmetry (34). The map from \( A_\mu \) to \( A_\mu^c \) involves \( A_i \) without spatial derivatives, and a generic coupling of the effective theory to the commutative field \( A_\mu^c \), when rewritten in terms of \( A_\mu \), does not generally have the shift symmetry (34).

To find the map from the NC theory to the bNC theory, we first develop a map from the bNC theory to the commutative theory, then use the known Seiberg-Witten map to map the commutative theory to the NC theory. It is possible that there is a more efficient algorithm to find the map between bNC and NC theories, but we leave finding it to future work.

D. Map between baby-NC theory and commutative theory

Suppose we have a commutative theory \( A_\mu^c \) which gauge transforms as

\[
\delta A_\mu^c = \partial_\mu \lambda^c. 
\]  

(55)

We wish to construct a field and a gauge parameter:

\[
A_\mu^b = A_\mu^b[A_\mu^c], \quad \lambda^b = \lambda^b[\lambda^c, A_\mu^c],
\]  

(56)

so that \( A_\mu^b \) transforms as a baby-NC gauge field with parameter \( \lambda^b \),

\[
\delta A_\mu^b = \partial_\mu \lambda^b + \theta\{A_\mu^b, \lambda^b\},
\]  

(57)

To find these, we will promote \( \theta \) to a parameter which we will call \( \tau \) and try to find a family of gauge field \( A_\mu(\tau) (= A_\mu(\tau)[A_\mu^c]) \) and \( \lambda(\tau) (= \lambda(\tau)[\lambda^c, A_\mu^c]) \) so that for each value of \( \tau \)

\[
\delta A_\mu(\tau) = \partial_\mu \lambda(\tau) + \tau\{A_\mu(\tau), \lambda(\tau)\}. 
\]  

(58)

Then the commutative and the bNC fields are just \( A_\mu(\tau) \) at two special values of \( \tau \): \( A_\mu^c = A_\mu(0) \) and \( A_\mu^b = A_\mu(\theta) \). We differentiate Eq. (58) over \( \tau \) to find that \( A_\mu(\tau) \) and \( \lambda(\tau) \) must depend on \( \tau \) in such away that the following condition is satisfied:

\[
\delta \dot{A}_\mu^b(\tau) = \partial_\mu \dot{\lambda}^b(\tau) + \{A_\mu(\tau), \lambda(\tau)\} + \tau\{\dot{A}_\mu(\tau), \lambda(\tau)\} + \tau\{A_\mu(\tau), \dot{\lambda}(\tau)\}, 
\]  

(59)
where dot denotes derivative with respect to $\tau$. One can check that this is satisfied if one requires

$$
\dot{A}_\mu(\tau) = \frac{1}{2} \epsilon^{ij} \left[ \partial_i A_\mu(\tau) + F_{i\mu}(\tau) \right] A_j(\tau), \quad (60a)
$$

$$
\dot{\lambda}(\tau) = \frac{1}{2} \epsilon^{ij} \partial_i \lambda(\tau) A_j(\tau), \quad (60b)
$$

where $F_{i\mu}(\tau) = \partial_i A_\mu(\tau) - \partial_\mu A_i(\tau) + \tau \{ A_i(\tau), A_\mu(\tau) \}$. To verify that, one just needs to replace, in Eq. (59), $A_\mu(\tau)$ and $\dot{\lambda}(\tau)$ by the expression given in Eqs. (60), then use Eq. (58) to convince oneself that two sides of the equation coincide.

To express the baby-NC field and gauge parameter $A^b_\mu$ and $\lambda^b$ in terms of the commutative field $A_\mu$ and gauge parameter $\lambda$, one solves Eq. (60a) with the initial condition $A_\mu(0) = A^c_\mu$, $\lambda(0) = \lambda^c$. Then at $\tau = \theta$ we find the bNC field and gauge parameter: $A_\mu(\theta) = A^b_\mu$, $\lambda(\theta) = \lambda^b$. Inversely, to express the commutative field $A^c_\mu$ in terms of the bNC field $A^b_\mu$, one solves this equation with the initial condition at $\tau = \theta$ and extract the solution at $\tau = 0$.

E. Seiberg-Witten map between NC theory and commutative theory

For convenience we review here the original Seiberg-Witten map between the NC and commutative field and gauge parameter:

$$
A_\mu = A_\mu^c[A_\mu^b], \quad \lambda = \lambda^c[A_\mu^b]. \quad (61)
$$

We now construct a one-parameter family of gauge field and gauge parameter $A_\mu(\tau)$ and $\lambda(\tau)$ so that the gauge transformation law is that of a NC theory with the noncommutativity parameter $\tau$:

$$
\delta A_\mu(\tau) = \partial_\mu \lambda(\tau) + \{ A_\mu(\tau), \lambda(\tau) \}_\tau, \quad (62)
$$

where we introduce the notations

$$
(f \star g)_\tau = f \exp \left( i(\tau) \epsilon^{ij} \partial_i \partial_j \right) g, \quad (63)
$$

$$
\{ f, g \}_\tau = \frac{1}{i} [(f \star g)_\tau - (g \star f)_\tau], \quad \{ f, g \}^+_\tau = \frac{1}{2} [(f \star g)_\tau + (g \star f)_\tau]. \quad (64)
$$

Differentiating Eq. (62) over $\tau$, we find

$$
\delta \dot{A}_\mu(\tau) = \partial_\mu \dot{\lambda}(\tau) + \epsilon^{ij} \{ \partial_i A_\mu(\tau), \partial_j \lambda(\tau) \}^+_\tau + \{ \dot{A}_\mu(\tau), \lambda(\tau) \}_\tau + \{ \dot{A}_\mu(\tau), \dot{\lambda}(\tau) \}^+_\tau. \quad (65)
$$

This condition is satisfied if one requires that $A_\mu(\tau)$ and $\lambda(\tau)$ satisfy the evolution equations

$$
\dot{A}_\mu(\tau) = \frac{1}{2} \epsilon^{ij} \{ \partial_i A_\mu(\tau) + F_{i\mu}(\tau), A_j(\tau) \}^+_\tau, \quad (66a)
$$

$$
\dot{\lambda}(\tau) = \frac{1}{2} \epsilon^{ij} \{ \partial_i \lambda(\tau), A_j(\tau) \}^+_\tau, \quad (66b)
$$

where $F_{i\mu}(\tau) = \partial_i A_\mu(\tau) - \partial_\mu A_i(\tau) + \{ A_i(\tau), A_\mu(\tau) \}_\tau$. Solving this system of equations between $\tau = 0$ and $\tau = \theta$ one can then establish a mapping between the NC gauge field and a commutative gauge field.
F. From NC to bNC

To express the bNC field $A^b_{\mu}$ and gauge parameter $\lambda^b$ in terms of the NC field $A_{\mu}$ and gauge parameter $\lambda$, one can do a two-step process: first one solves Eqs. (60) for $\tau$ running from $\theta$ to 0 to $A^c_{\mu}$ and $\lambda^c$, and then solves Eqs. (66) with $\tau$ running back from 0 to $\theta$. The result is an expression relating $A^b_{\mu}$ and $\lambda^b$ with $A_{\mu}$ and $\lambda$. Since Eqs. (60) and (66) differ from each other only by terms of order $\theta^2$ and higher on the right-hand sides, the difference between $A^b_{\mu}$ and $A_{\mu}$ starts at order $\theta^3$

$$A^b_{\mu} = A_{\mu} + \frac{\theta^3}{48} \varepsilon^{ij} \varepsilon^{i1j1} \varepsilon^{i2j2} \partial_{i1} \partial_{i2} (2 \partial_i A_\mu - \partial_\mu A_i) \partial_{j1} \partial_{j2} A_j + O(\theta^4),$$  \hspace{1cm} (67)

$$\lambda^b = \lambda + \frac{\theta^3}{48} \varepsilon^{ij} \varepsilon^{i1j1} \varepsilon^{i2j2} \partial_{i1} \partial_{i2} \lambda \partial_{j1} \partial_{j2} A_j + O(\theta^4).$$  \hspace{1cm} (68)

Since the difference between $A^b_{\mu}$ and $A_{\mu}$ involves spatial derivatives of $A_i$, time-dependent shifts on $A_i$ [Eq. (34)] simply become time-dependent shifts of $A^b_i$.

One can now require that fields in the effective field theories of the FQHE transform as tensors (or spinor) under VPD with parameter $\lambda^b$. Written in terms of $\lambda$ and $A$ these transformation laws are rather complicated and unnatural from the point of view of noncommutative field theory. But as the SW’ map demonstrates, there is no requirement that the correct theory of the FQHE should be a simple noncommutative field theory.

V. CONCLUSION

In this paper we show that the fact that the quantum Hall state lives on a single Landau level implies that such a system can be coupled to an external noncommutative U(1) gauge field in a way that preserves a NC U(1) gauge symmetry. We show that the task of writing down such a theory can be simplified by transforming the NC U(1) gauge symmetry into a “baby-NC” U(1) gauge symmetry, which is isomorphic to volume-preserving diffeomorphism. The task of enforcing VPD invariance in an effective field theory can be accomplished quite easily with existing tools, for example using the Newton-Cartan formalism [11, 12].

It appears from the discussion above that the NC U(1) gauge symmetry does not place any additional constraint on the effective field theory of the FQHE besides those which can be seen from the VPD.

It would be nice to find a direct map between the NC and bNC theories, bypassing the need to go through the intermediate commutative theory. Another remaining open question is on the precise relationship between the U(1) noncommutative gauge symmetry of the effective field theory and the GMP algebra [19]. We leave these questions to future work.
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