ON THE EVENTUAL STABILITY OF ASYMPTOTICALLY AUTONOMOUS SYSTEMS WITH CONSTRAINTS

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Dedicated to Professor Peter E. Kloeden on the occasion of his 70th birthday.

Abstract. In this paper we first give a criterion on stability of equilibrium solutions for autonomous systems with constraints. Then we discuss the relationship between asymptotic behaviors of an asymptotically autonomous system with constraint and its limit system. Finally as an example, we revisit an extreme ideology model proposed in the literature and give a more detailed description on the dynamics of the system.

1. Introduction. In [1] Aldila et al. proposed a mathematical model on extreme ideology in a closed population which reads as

\[
\begin{align*}
\dot{x}_1 &= E - \frac{\beta_1 x_1 x_3}{N} - \frac{\beta_2 x_1 x_5}{N} - \delta_1 x_1 - \mu x_1, \\
\dot{x}_2 &= \frac{\beta_1 x_1 x_3}{N} - \delta_1 x_2 - \gamma x_2 - \mu x_2, \\
\dot{x}_3 &= \gamma x_2 - \delta_2 x_3 - \mu x_3, \\
\dot{x}_4 &= \delta_1 (x_1 + x_2) - \mu x_4, \\
\dot{x}_5 &= \frac{\beta_2 x_1 x_5}{N} + \delta_2 x_3 - \mu x_5.
\end{align*}
\]

(1)

The total population \( N = N(t) \) is divided into five sub-populations:

\[ N(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t), \]

(2)

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where \( x_1, x_2 \) and \( x_3 \) denote the sub-populations of the virgin, the semi-fanatic and the fanatic respectively, \( x_4 \) denotes the population of the people aware to the ideology, and \( x_5 \) the recovered sub-population. They also assumed that the total population \( N \) satisfies an independent differential equation:

\[
\dot{N}(t) = E - \mu N(t), \tag{3}
\]

where \( E \) and \( \mu \) are recruitment rate from new born and death rate, respectively.

It is trivial to verify that all solutions of (3) approach to the equilibrium solution \( N^* = E/\mu \) as \( t \to +\infty \). Therefore (1) has a limit system

\[
\begin{align*}
\dot{x}_1 &= E - \frac{\beta_1 x_1 x_3}{N^*} - \frac{\beta_2 x_1 x_5}{N^*} - \delta_1 x_1 - \mu x_1, \\
\dot{x}_2 &= \frac{\beta_1 x_1 x_3}{N^*} - \delta_1 x_2 - \gamma x_2 - \mu x_2, \\
\dot{x}_3 &= \gamma x_2 - \delta_2 x_3 - \mu x_3, \\
\dot{x}_4 &= \delta_1 (x_1 + x_2) - \mu x_4, \\
\dot{x}_5 &= \frac{\beta_2 x_1 x_5}{N^*} + \delta_2 x_3 - \mu x_5.
\end{align*}
\]

with

\[
x_1 + x_2 + x_3 + x_4 + x_5 = N^*. \tag{5}
\]

In [1], the authors calculated the equilibria of (4) and discussed their stability properties. However, they neglected the effect of the constraint (5) while performing the stability analysis. Specifically, the authors only considered the stability and instability of the equilibria with respect to the free system (4). On the other hand, the stability property of equilibrium points of a system with constraint can differ significantly from that of the same system without constraint. For instance, the zero solution \( \theta = (0,0) \) of the planar system

\[
\begin{align*}
\dot{x} &= x + xy, \\
\dot{y} &= x - y, \tag{6}
\end{align*}
\]

is unstable. However, if we add in (6) a constraint, \( x = 0 \), then \( \theta \) becomes an asymptotically stable equilibrium. Besides finite dimensional cases mentioned above, the system [6] can be treated as a infinite dimensional one with constraint to some degree, since the total population size is constant in time.

There is also another natural question concerning the above mentioned system: Suppose the dynamics of the limit system (4)-(5) are clear. What can we say about the asymptotic behavior of the original system (1)-(3)?

The usual way to study system (1)-(3) is as follows. First, since (3) is a linear equation, we can solve \( N = N(t) \) explicitly. Then using (2) one may eliminate, say, \( x_5 \) from (1). This allows us to transform (1)-(3) into a (nonautonomous) system without constraint and apply the classical theory on dynamical systems to obtain information on the dynamics of system; see e.g. Castillo-Chavez and Thieme [4] for details. However, sometimes it may not be easy (or even impossible) to solve \( N \) explicitly from the corresponding equation as in the case where, say for instance, the population \( N \) is supposed to satisfy the nonlinear equation \( \dot{N} = rN(1-N) \).

In this paper we take a slightly different point of view and treat such problems in an abstract framework of the following system

\[
\begin{align*}
\dot{x} &= f(t, x), \quad x = x(t) \in \mathbb{R}^n; \\
g(t, x) &= 0. \tag{7}
\end{align*}
\]
The investigation of asymptotic behavior of (7) in the general case seems to be a difficult problem. Here we focus on a particular but important case where $f$ and $g$ satisfy the *asymptotically autonomous* condition

$$
\lim_{t \to +\infty} f(t, x) = f(x), \quad \lim_{t \to +\infty} g(t, x) = g(x).
$$

Accordingly (7) has an autonomous limit system

$$
\begin{cases}
\dot{x} = f(x), & x = x(t) \in \mathbb{R}^n; \\
g(x) = 0.
\end{cases}
$$

Note that (7) and (9) (and their infinite dimensional versions) cover a large number of important mathematical models not only from biology (see e.g. [4, 17]), but also from many other areas. (A system $\dot{x} = f(x)$ on a cylindrical surface in $\mathbb{R}^n$ can be naturally viewed as a system with constraint $x_1^2 + \cdots + x_{n-1}^2 = R^2$.) The main purpose of this work is as follows.

First, in Section 2 we give a criterion on stability of equilibrium solutions for the limit system (9), which allows us to study stability properties of equilibria of such systems without solving the constraint equations. Then we discuss in Sections 3 and 4 the relationship between asymptotic behaviors of (7) and (9) near an equilibrium solution $e$ of (9). Specifically, we show that if $e$ is asymptotically stable, then it is eventually uniformly asymptotically stable w.r.t (7). On the other hand, if $e$ is hyperbolic and unstable, we prove that system (7) has at least a solution $\gamma(t)$ on some interval $[\tau_0, \infty)$ forward converging to $e$. However, $\gamma$ is eventually unstable. (Note that $e$ may fail to be a solution of (7).) Finally in Section 5, as an example we revisit the extreme ideology model mentioned above and give a more detailed description on its dynamics.

The relationship between the dynamics of asymptotically autonomous systems without constraints and their limit systems has already been addressed by many authors in the past decades; see [2, 4, 7, 9, 10, 13, 14, 15, 16, 12] to name a few. To the authors’ knowledge, some of our results given here (particularly the ones on instability) seem to be new even if we come back to the situation of non-constrained systems.

The most general form of a constrained (finite dimensional) system may be as follows:

$$
\begin{cases}
\dot{x} = f(t, x), & x = x(t) \in \mathbb{R}^n; \\
x \in C(t),
\end{cases}
$$

where $C(t)$ is a subset of the phase space for each $t \in \mathbb{R}$. For instance, for many mathematical models from applications, only positive solutions are of practical sense. Therefore any such a model can be viewed as a constrained system as above with $C(t) \equiv P$. Here $P$ denotes the positive cone in $\mathbb{R}^n$. It seems to be interesting to establish a general theory to discuss the dynamical behavior of (10), particularly near some equilibrium-like points at the boundary of $C(t)$.

2. Preliminaries. Denote $|\cdot|$ the usual norm of the space $\mathbb{R}^n$. Given a linear operator $A \in L(\mathbb{R}^n)$, the norm of $A$ is denoted by $||A||$. For computational convenience, we also make a convention that a vector $x \in \mathbb{R}^n$ will be written as a column one, $x = (x_1, x_2, \cdots, x_n)^T$. Denote $D_x h(x)$ or $Db(x)$ the differential (Jacobian matrix) of a map $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. In the case that $h$ also depends on another variable $y$, i.e., $h = h(x, y)$, we will rewrite $D_x h(x, y)$ as $\partial_x h(x, y)$ to emphasize that to which variable the differential is taken.
Let \( f(t,x), g(t,x), f(x) \) and \( g(x) \) be the maps in (7) and (9), where
\[
f(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n), \quad f(x) \in C^1(\mathbb{R}^n; \mathbb{R}^n),
\]
and
\[
g(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m), \quad g(x) \in C^1(\mathbb{R}^n; \mathbb{R}^m)
\]
for some \( m < n \). We will always assume that
\[
\partial_t g(t,x) + \partial_x g(t,x) f(t,x) = 0, \quad Dg(x) f(x) = 0 \tag{11}
\]
for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). Hence the initial value problems of (7) and (9) are well-posed. Indeed, let \( x(t) \) be a solution of the equation \( \dot{x} = f(t,x) \) on an interval \((a, b)\) satisfying that \( g(t_0, x(t_0)) = 0 \). Then by (11) one always has
\[
g(t, x(t)) = \int_0^t \partial_s g(s, x(s)) + \partial_x g(s, x(s)) f(s, x(s)) \, ds = 0 \tag{12}
\]
for \( t \in (a, b) \). Similarly if \( x(t) \) is a solution of \( \dot{x} = f(x) \) with \( g(x(0)) = 0 \) then
\[
g(x(t)) = \int_0^t Dg(x(s)) f(x(s)) \, ds = 0, \quad t \geq 0. \tag{13}
\]
Instead of (8), in this work we will impose on the aforementioned maps a stronger asymptotically autonomous condition (C1): For any \( R > 0 \),
\[
\lim_{t \to +\infty} ||f(t, \cdot) - f(\cdot)||_{C^1(\mathbb{R})} = 0, \tag{14}
\]
and
\[
\lim_{t \to +\infty} ||g(t, \cdot) - g(\cdot)||_{C^1(\mathbb{R})} = 0. \tag{15}
\]
Since we are interested in the asymptotical behavior of (7) near an equilibrium of (9), without loss of generality we assume
\[
f(0) = 0, \quad g(0) = 0,
\]
thereby the origin \( \theta = 0 \) is an equilibrium solution of (9). We also impose on system (9) condition (C2):
\[
\text{rank}(Dg(0)) = m.
\]

2.1. A fundamental lemma. Set
\[
U = \{ u : \ Dg(0)u = 0 \}.
\]
By (C2) we know that \( \mathbb{R}^n \) has a direct sum decomposition \( \mathbb{R}^n = U \oplus V \) with \( \dim(V) = m \). For any \( x \in \mathbb{R}^n \), it can be represented as \( x = u + v, \ (u \in U, v \in V) \). Then \( g(x) = g(u + v) \), and
\[
g_v(0) : V \to \mathbb{R}^m
\]
has a bounded inverse, where \( g_v = \partial_v g(u + v) \), and \( g_u = \partial_u g(u + v) \).

For \( M, N \subset \mathbb{R}^n \), the Hausdorff semi-distance and Hausdorff distance of \( M \) and \( N \) are defined, respectively, as
\[
d_H(M, N) = \sup_{x \in M} d(x, N),
\]
\[
\delta_H(M, N) = \max \{ d_H(M, N), d_H(N, M) \}.
\]
Denote
\[
\mathcal{M}(t) = \{ x : \ g(t,x) = 0 \}, \quad \mathcal{M} = \{ x : \ g(x) = 0 \},
\]
\[
\mathcal{M}_r(t) = \mathcal{M}(t) \cap B(r), \quad \mathcal{M}_r = \mathcal{M} \cap B(r),
\]
where \( B(r) \) is the ball in \( \mathbb{R}^n \) centered at 0 with radius \( r \). As \( \mathcal{M}(t) \) and \( \mathcal{M} \) are closed, we clearly have

\[
\mathcal{M}_r(t) = \mathcal{M}(t) \cap \overline{B}(r), \quad \mathcal{M}_r = \mathcal{M} \cap \overline{B}(r). \tag{16}
\]

**Lemma 2.1.** Assume (15) and (C2). Then there exists \( r_0 > 0 \) such that

\[
\lim_{t \to +\infty} \delta_H(\mathcal{M}_r(t), \mathcal{M}_r) = 0.
\]

**Proof.** At first, we define a map \( h : R^+ \times U \times V \to \mathbb{R}^m \) as

\[
h(s, u, v) = \begin{cases} 
g(1/s, u + v), & s > 0, 
g(u + v), & s = 0. \end{cases}
\]

From the hypotheses, \( h \) and \( h_v \) are continuous on \( R^+ \times U \times V \). Since \( h_v(0, 0, 0) = g_v(0) \) has a bounded inverse, we set \( T = [g_v(0)]^{-1} \) and

\[
H(s, u, v) = v - Th(s, u, v).
\]

Then the fixed points of \( H \) are the solutions of \( h = 0 \). Because \( H_v(0, 0, 0) = 0 \), by the continuity of \( H_v \), there is a neighborhood \( [0, s_0] \times \Omega \times V \) of \( (0, 0, 0) \) in \( R^+ \times U \times V \) satisfying

\[
||H_v(s, u, v)|| \leq \alpha < 1
\]

for all \( (s, u, v) \in [0, s_0] \times \Omega \times V \). As \( H(0, 0, 0) = 0 \), this neighborhood can be picked so that

\[
H : [0, s_0] \times \overline{\Omega} \times V \to \overline{V}_1.
\]

It follows from the uniform contraction principle (see, e.g., [5, Theorem 2.2]) that there is a continuous map \( \varphi : [0, s_0] \times \overline{\Omega} \to \overline{V}_1 \) such that

\[
H(s, u, \varphi(s, u)) = \varphi(s, u);
\]

that is, \( h(s, u, \varphi(s, u)) = 0 \). By virtue of the uniform contraction principle, we can also prove that \( \varphi_u \) is continuous on \([0, s_0] \times \overline{\Omega}\).

In the following, we denote \( t_0 = 1/s_0 \). For \( t \geq t_0 \) and \( u \in \overline{\Omega} \), let \( \xi(t, u) = \varphi(1/t, u) \). It is obvious that

\[
g(t, u + \xi(t, u)) = 0. \tag{17}
\]

Also, for \( u \in \overline{\Omega} \), we set \( \xi(u) = \varphi(0, u) \) and have

\[
g(u + \xi(u)) = 0. \tag{18}
\]

Furthermore, it follows from the uniform continuity of \( \varphi \) and \( \varphi_u \) on \([0, s_0] \times \overline{\Omega}\) that

\[
\lim_{t \to +\infty} ||\xi(t, \cdot) - \xi(\cdot)||_{C_1(\overline{\Omega})} = 0. \tag{19}
\]

By (17) and (18), we deduce that there exist \( r_0 > 0 \) and \( t_1 \geq t_0 \) such that

\[
\{u + v : u \in \overline{\Omega}, \ v = \xi(t, u)\} \supset \mathcal{M}(t) \cap \overline{B}(r_0) = \mathcal{M}_{r_0}(t)
\]

for all \( t \geq t_1 \), and

\[
\{u + v : u \in \overline{\Omega}, \ v = \xi(u)\} \supset \mathcal{M} \cap \overline{B}(r_0) = \mathcal{M}_{r_0}.
\]

Therefore we see for \( t \geq t_1 \),

\[
\mathcal{M}_{r_0}(t) = \{u + v : u \in \overline{\Omega}, \ v = \xi(t, u), \ |u + v| \leq r_0\},
\]

and that

\[
\mathcal{M}_{r_0} = \{u + v : u \in \overline{\Omega}, \ v = \xi(u), \ |u + v| \leq r_0\}.
\]
Now let $\varepsilon > 0$. Then there exists $\tau > t_1$ such that

$$|\xi(t,u) - \xi(u)| < \varepsilon/2,$$

for all $t > \tau$ and $u \in \overline{\Omega}$. Thus if $x = u + v = u + \xi(t,u) \in \mathcal{M}_{r_0}(t)$, then $y = u + \xi(u) \in \mathcal{M}$ satisfies

$$d(x,y) = |\xi(t,u) - \xi(u)| < \varepsilon/2.$$  \hspace{1cm} (20)

This implies $y \in \mathcal{M}_{r_0 + \varepsilon}$ and that $d(x,\mathcal{M}_{r_0}) \leq \varepsilon$. Since $x$ is arbitrary, we conclude that

$$d_H(\mathcal{M}_{r_0}(t), \mathcal{M}_{r_0}) \leq \varepsilon, \quad t \geq \tau.$$  \hspace{1cm} (21)

Similarly it can be shown that $d_H(\mathcal{M}_{r_0}, \mathcal{M}_{r_0}(t)) \leq \varepsilon$ for $t \geq \tau$. Thus

$$\delta_H(\mathcal{M}_{r_0}(t), \mathcal{M}_{r_0}) \leq \varepsilon, \quad t \geq \tau,$$

which completes the proof of what we desired. \hfill \Box

### 2.2. Stability and instability of autonomous system

Now we give a criterion on the stability and instability of equilibria for autonomous system (9). This result seems to be of independent interest from the point of view of application.

Assume $g$ satisfies (C2). Let $U \oplus V$ be the direct sum decomposition of $\mathbb{R}^n$ given in Subsection 2.1, and let $P : \mathbb{R}^n \rightarrow U$ be the projection. Write $x = u + v$, and set

$$p(u,v) = Pf(x), \quad A = (p_u - p_v g^{-1} g_u)|_{x=0}. \hspace{1cm} (22)$$

Then $A$ is a linear operator on $U$.

**Theorem 2.2.** Denote $\sigma(A)$ the spectrum of $A$. Then

1. the origin $\theta = 0$ is asymptotically stable w.r.t system (9) if $\sigma(A) \subset \{ z \in \mathbb{C} : \text{Re} z < 0 \}$; and
2. $\theta$ is unstable w.r.t (9) if $\sigma(A) \cap \{ z \in \mathbb{C} : \text{Re} z > 0 \} \neq \emptyset$.

**Proof.** By definition $\mathcal{M}$ is an invariant manifold of system

$$\dot{x} = f(x), \quad x = x(t) \in \mathbb{R}^n \hspace{1cm} (23)$$

and (9) is equivalent to the reduction of (23) on $\mathcal{M}$ near $\theta$ (note that $\theta \in \mathcal{M}$).

The coordinate form of the reduction of (23) on $\mathcal{M}$ in $\Omega$ reads as

$$\dot{u} = p(u,v), \quad u \in \Omega \hspace{1cm} (24)$$

with $v = \xi(u)$, where $\Omega$ is the neighborhood of $u = 0$ in $U$ given in the proof of Lemma 2.1. Simple calculations show that

$$D_u p(u, \xi(u)) = p_u + p_v \xi_u. \hspace{1cm} (25)$$

Differentiating (18) it yields

$$g_u + g_v \xi_u = 0.$$  \hspace{1cm} (26)

Therefore for $x = u + v$ near $\theta$ one finds that $\xi_u = -g_v^{-1} g_u$. Thus we have

$$D_u p(u, \xi(u))|_{u=0} = (p_u - p_v g^{-1}_v g_u)|_{u=0}$$

$$= (p_u - p_v g^{-1}_v g_u)|_{x=0} = A.$$  \hspace{1cm} (27)

Now the conclusions follow immediately from classical results on stability and instability of equilibrium solutions (see, e.g., [8, Chap. 5]). \hfill \Box
3. **Eventual asymptotic stability of (7).** In this section we consider the eventual stability of the origin \( \theta \) with respect to system (7). For convenience, we rewrite

\[
f(t, x) = f(x) + q(t, x).
\]  

(26)

Then for any \( R > 0 \),

\[
\lim_{t \to +\infty} \|q(t, \cdot)\|_{C^1(B(R))} = 0.
\]  

(27)

Denote \( \psi(t, \tau) \) the (local) process generated by the system

\[
\begin{cases}
\dot{x} = f(t, x), & t \geq \tau, \ \tau \in \mathbb{R}, \\
x(\tau) = x_0,
\end{cases}
\]  

(28)

and \( \phi(t) \) the (local) semiflow associated with

\[
\begin{cases}
\dot{y} = f(y), & t \geq 0, \\
y(0) = y_0.
\end{cases}
\]  

(29)

Namely, \( x(t) = \psi(t, \tau)x_0 \ (t \geq \tau) \) and \( y(t) = \phi(t)y_0 \ (t \geq 0) \) are, respectively, solutions of (28) and (29) for all \( x_0, y_0 \in \mathbb{R}^n \).

**Remark 1.** Note that (12) and (13) imply, respectively, that \( \psi(t, \tau)M_m(\tau) \subset M_m(t) \ (t \geq \tau) \) and \( \phi(t)M \subset M \ (t \geq 0) \).

As in Section 2, denote \( M_r(t) = M(t) \cap B(r) \). The following notions are adapted from [15].

**Definition 3.1.** The origin \( \theta = 0 \) is called eventually uniformly stable w.r.t system (7), if for every \( \varepsilon > 0 \) there exist \( r > 0 \) and \( t_0 > 0 \) such that \( \psi(t, \tau)x_0 \) is defined for all \( t \geq \tau \geq t_0 \) and \( x_0 \in M_r(\tau) \); furthermore,

\[
|\psi(t; \tau)x_0| < \varepsilon, \quad \forall \ t \geq \tau \geq t_0.
\]

**Definition 3.2.** \( \theta \) is called eventually uniformly attracting w.r.t (7), if there exist \( r > 0 \) and \( t_0 > 0 \) such that \( \psi(t, \tau)x_0 \) is defined for all \( t \geq \tau \geq t_0 \) and \( x_0 \in M_r(\tau) \); furthermore, for any \( \varepsilon > 0 \) there is a \( T > 0 \) such that

\[
|\psi(t, \tau)x_0| < \varepsilon, \quad \forall \ t > \tau + T, \ \tau \geq t_0.
\]

**Definition 3.3.** \( \theta \) is called eventually uniformly asymptotically stable w.r.t (7) if it is both eventually uniformly stable and attracting.

The main result in this section is summarized in the following theorem.

**Theorem 3.4.** Assume that (C1) and (C2) hold. If the origin \( \theta \) is asymptotically stable w.r.t (9), then it is eventually uniformly asymptotically stable with respect to (7).

**Proof.** (1) We first show that \( \theta \) is eventually uniformly stable w.r.t (7).

Let \( N \) be a neighborhood of \( \theta \) in \( \mathbb{R}^n \). For \( \delta > 0 \), denote

\[
N_\delta = \{ x \in N : d(x, \partial N) > \delta \}.
\]

Fix a \( \delta > 0 \) sufficiently small so that \( N_\delta \) is a neighborhood of \( \theta \). Then since \( \theta \) is asymptotically stable under the system (9), there exists \( r > 0 \) such that

\[
\phi(t)M_r \subset N_\delta, \quad t \geq 0.
\]  

(31)
Furthermore, by very standard argument (see, e.g., Li [11]) it can be shown that the origin $\theta$ attracts $\mathcal{M}_r$ under the system $\phi(t)$, that is, for any $\varepsilon > 0$, there is a positive number $T = T(\varepsilon)$ such that

$$\phi(t)|\mathcal{M}_r \subset B(\varepsilon), \quad t > T.$$ 

Since $\mathcal{M}_r \subset \mathcal{M}$, it follows from the second relation in (30) that $\phi(t)|\mathcal{M}_r \subset \mathcal{M}$ for all $t \geq 0$. Hence one can find a $T > 0$ such that

$$\phi(T)|\mathcal{M}_r \subset (\mathcal{M} \cap B(r/2)) = \mathcal{M}_{r/2}. \quad (32)$$

In what follows we show that there is a $t_0 > 0$ such that

$$\psi(t,\tau)\mathcal{M}_r(\tau) \subset N, \quad t \geq \tau \geq t_0, \quad (33)$$

thus proving what we desired. For this purpose, we first pick an $\varepsilon > 0$ such that

$$\sqrt{2}e^{(L+1/2)T} < \min(\delta, r/2),$$

where $L$ is the Lipschitz constant of $f(x)$ in $N$. By (27) and Lemma 2.1 there exists $t_0 > 0$ such that for $t \geq t_0$, we have

(i) $|q(t, x)| < \varepsilon$ for all $x \in N$, where $q(t, x) = f(t, x) - f(x)$; and

(ii) for any $x \in \mathcal{M}_r(t)$, there is an $y \in \mathcal{M}_r$ such that $|x - y| < \varepsilon$.

Now we check that for any $\tau \geq t_0$ and $x_0 \in \mathcal{M}_r(\tau)$, it holds that

$$\psi(t, \tau)x_0 \in N, \quad t \in [\tau, \tau + T], \quad (34)$$

$$\psi(T + \tau, \tau)x_0 \in \mathcal{M}_r(T + \tau), \quad (35)$$

from which (33) immediately follows.

Take an $y_0 \in \mathcal{M}_r$ with $|x_0 - y_0| < \varepsilon$ and set

$$x(t) = \psi(t + \tau, \tau)x_0, \quad y(t) = \phi(t)y_0, \quad t \geq 0.$$ 

Then $x(t)$, $y(t)$ and $w(t) := x(t) - y(t)$ solve, respectively, the equations

$$\dot{x} = f(x) + q(\tau + t, x), \quad \dot{y} = f(y)$$

and

$$\dot{w} = f(x) - f(y) + q(\tau + t, x). \quad (36)$$

To prove (34) we argue by contradiction and suppose the contrary. Then there would exist $0 < s \leq T$ such that

$$x(t) \in N \quad (t \in [0, s]), \quad \text{and} \quad x(s) \in \partial N.$$ 

By (31) we deduce that $|w(s)| = |x(s) - y(s)| \geq \delta$.

On the other hand, taking the inner product of (37) with $w$ it yields

$$\frac{1}{2} \frac{d}{dt}|w|^2 \leq |f(y) - f(x)||w| + \varepsilon|w| \leq L|w|^2 + \frac{1}{2}|w|^2 + \frac{1}{2}e^2, \quad t \in [0, s].$$

The classical Gronwall Lemma then implies that

$$|w(t)|^2 \leq |w(0)|^2e^{(2L+1)t} + \frac{e^2}{2L + 1}e^{(2L+1)t} < 2\varepsilon^2e^{(2L+1)t}, \quad t \in [0, s].$$

In particular,

$$|w(s)| < \sqrt{2}\varepsilon e^{(L+1/2)T} < \min(\delta, r/2) \leq \delta,$$

which leads to a contradiction and verifies the validity of (34).

Having (34) in hand, one can repeat the above argument and conclude

$$|w(t)|^2 < 2\varepsilon^2e^{(2L+1)T}, \quad t \in [0, T].$$
Thus
\[ |x(T) - y(T)| = |w(T)| < \sqrt{2} e^{(L+1/2)T} < \min(\delta, r/2) \leq r/2. \]
Therefore by (32) we see that
\[ |\psi(T + \tau, \tau)x_0| = |x(T)| \leq |x(T) - y(T)| + |y(T)| \leq r/2 + r/2 = r. \]
Recalling that \( x_0 \in \overline{M}_r(\tau) \subset M(\tau) \) (by (16)), we also have by Remark 1 that
\[ \psi(T + \tau, \tau)x_0 \in \mathcal{M}(T + \tau). \]
Hence
\[ \psi(T + \tau, \tau)x_0 \in B(r) \cap M(T + \tau) = M_r(T + \tau), \]
which completes the proof of (35).

(2) Now we examine the attraction property of \( \theta \). It can be assumed that the neighborhood \( N \) of \( \theta \) in the first step is chosen sufficiently small so that \( x(t) \equiv \theta \) is the unique bounded complete solution of \( \phi \) (or (9)) in \( N \cap M \). We need to check that
\[ \lim_{t \to +\infty} \psi(t + \tau, \tau)x = \theta \]
uniformly with respect to \( x \in \overline{M}_r(\tau) \) and \( \tau \geq t_0 \).
Suppose the contrary. There would exist \( \eta > 0 \) and sequences \( \tau_n \geq t_0, x_n \in \overline{M}_r(\tau_n) \) and \( t_n \to +\infty \) (\( t_n \geq 0 \)) such that for all \( n \)
\[ |\psi(t_n + \tau_n, \tau_n)x_n| \geq \eta. \]
For \( t \in J_n := (-t_n/2, \infty) \), set
\[ w_n(t) = \psi(t + t_n + \tau_n, \tau_n)x_n. \]
Clearly \( |w_n(0)| = |\psi(t_n + \tau_n, \tau_n)x_n| \geq \eta \) for all \( n \), and
\[ w_n(t) \in N, \quad t \in J_n. \]
Thus \( w_n = w_n(t) \) solves the equation
\[ \dot{x} = f(x) + q(\tau_n + t_n + t, x), \quad t \in J_n. \]
As \( w_n(t) \in N \) for \( t \in J_n \), by very standard argument we know that, up to a subsequence, \( w_n \) converges uniformly on any compact interval to a function \( w \) defined on \( \mathbb{R} \). Clearly \( w \) is contained in \( N \).
Since \( \tau_n + t_n + t > t_n/2 \) for all \( t \in J_n \) and \( t_n \to +\infty \), we see that
\[ \sup_{x \in N, t \in J_n} |q(\tau_n + t_n + t, x)| \to 0 \]
as \( n \to \infty \). Passing to the limit in (40) one finds that \( w \) is a full solution of the system (9) in \( N \cap M \). On the other hand, since \( |w_n(0)| \geq \eta > 0 \) for all \( n \), we deduce that
\[ w(0) = \lim_{n \to \infty} w_n(0) \neq \theta. \]
This leads to a contradiction because the zero solution is the unique full solution of (9) in \( N \cap M \).
4. Instability of system (7). In this section we make some discussions on instability properties of (7) near equilibrium points of the limit system (9). Therefore as above, we assume the origin $θ = 0$ is an equilibrium of (9). To the authors’ knowledge, part of the results given here are new even if we come back to the situation of an asymptotically autonomous system without constraint.

Note that $θ$ may fail to be an equilibrium of (7). For such a non-equilibrium point, it seems to be of little sense to define its instability. (For instance, for the scalar system $\dot{x} = x^2$, it makes no sense to say that the point $x = 1$ is unstable w.r.t. the system.) To overcome this difficulty, we introduce the following more general notion.

**Definition 4.1.** A solution $γ = γ(t)$ on $[τ_0, +∞)$ of (7) is called eventually unstable, if there exist $δ > 0$ and sequences $τ_n → +∞$, $x_n ∈ M(τ_n)$ with $|x_n - γ(τ_n)| → 0$ and $t_n ≥ τ_n$ such that
\[
|ψ(t_n, τ_n)x_n - γ(t_n)| ≥ δ, \quad ∀ n ≥ 1.
\]

**Remark 2.** Note that $γ(t_n) = ψ(t_n, τ_n)γ(τ_n)$.

Our main results in this part are summarized in the theorem below.

**Theorem 4.2.** Assume that (C1) and (C2) hold. Suppose $θ = 0$ is an unstable hyperbolic equilibrium of (9). Then

1. (7) has a solution $γ(t)$ with $\lim_{t → +∞} γ(t) = θ$; and
2. $γ$ is eventually unstable.

**Proof.** (1) We first verify the existence of a solution $γ$ forward approaching $θ$. For this purpose, consider the truncated system:
\[
\begin{aligned}
&\dot{x} = f^∗(t, x), \\
g^∗(t, x) = 0,
\end{aligned}
\]  

(41)

where $f^*$ and $g^*$ are defined as
\[
f^∗(t, x) = \begin{cases} f(t, x), & t > τ, \\
f(τ, x), & t ≤ τ,
\end{cases}
\]

and
\[
g^∗(t, x) = \begin{cases} g(t, x), & t > τ, \\
g(τ, x), & t ≤ τ,
\end{cases}
\]

respectively.

Let $\mathbb{R}^n = U ⊕ V$ be the direct sum decomposition given in Section 2.1, and let $Ω$ be the neighborhood of $u = 0$ in $U$ given in the proof of Lemma 2.1. If $x(t)$ is a solution of system (41) in a sufficiently small neighborhood of the origin, then clearly
\[
x(t) ∈ M(t) \quad (t > τ), \quad \text{and} \quad x(t) ∈ M(τ) \quad (t ≤ τ).
\]

We infer from the proof of Lemma 2.1 that
\[
x(t) = u(t) + ξ^∗(t, u(t)),
\]

where
\[
ξ^∗(t, u) = \begin{cases} ξ(t, u), & t > τ, \\
ξ(τ, u), & t ≤ τ,
\end{cases}
\]
and \( \xi \) is the map appearing in (17). Hence (41) is equivalent to the following reduced system:

\[
\dot{u} = P \dot{f}(t, u + \xi(t, u))
\]

near the origin \( \theta = 0 \), where \( P : \mathbb{R}^n \to U \) is the projection. Let us rewrite (42) as

\[
\dot{u} = Au + h^\tau(t, u),
\]

where

\[
h^\tau(t, u) = Pf^\tau(t, u + \xi(t, u)) - Au = [Pf^\tau(t, u + \xi(t, u)) - Pf(u + \xi(t, u))] + [p(u, \xi(t, u)) - p(u, \xi(u))] + [p(u, \xi(u)) - Au] =: w_1(t, u) + w_2(t, u) + z(u),
\]

\( p(u, v) \) and \( A \) are defined by (22).

Note that

\[
\sup_{t \in \mathbb{R}} \|w_1(t, \cdot)\|_{C^1(\overline{\Omega})} = \sup_{t \geq \tau} \|w_1(t, \cdot)\|_{C^1(\overline{\Omega})} =: k_1(\tau).
\]

By (C1) and (19) it is easy to deduce that \( k_1(\tau) \to 0 \) as \( \tau \to +\infty \). Hence

\[
|w_1(t, u_1) - w_1(t, u_2)| \leq k_1(\tau)|u_1 - u_2|, \quad u_1, u_2 \in \overline{\Omega}, \quad t \in \mathbb{R},
\]

and

\[
|w_1(t, u)| \leq k_1(\tau)|u| + |w_1(t, 0)| \leq k_1(\tau)|u| + \eta_1(\tau)
\]

for all \( u \in \overline{\Omega} \) and \( t \in \mathbb{R} \), where

\[
\eta_1(\tau) := \sup_{t \geq \tau} |f(t, \xi(t, 0)) - f(\xi(t, 0))| \\
\geq \sup_{t \geq \tau} |Pf(t, \xi(t, 0)) - Pf(\xi(t, 0))| \\
= \sup_{t \in \mathbb{R}} |w_1(t, 0)|,
\]

\[
\eta_2(\tau) := \sup_{t \geq \tau} |f(\xi(t, 0))| \\
= \sup_{t \geq \tau} |f(\xi(t, 0)) - f(\xi(0))| \\
\geq \sup_{t \geq \tau} |Pf(\xi(t, 0)) - Pf(\xi(0))| \\
= \sup_{t \in \mathbb{R}} |w_2(t, 0)|.
\]

Here we have used the fact that \( \xi(0) = 0 \) (recall \( g(0) = 0 \) and that \( \xi(u) \) solves (18)). By (C1) it is obvious that \( \eta_1(\tau) \to 0 \) as \( \tau \to +\infty \). Since \( \xi(t, 0) \to \xi(0) = 0 \) and \( f(0) = 0 \), we also have \( \eta_2(\tau) \to 0 \) as \( \tau \to +\infty \).

By (25) we see that \( D_uz(0) = 0 \). Let \( \sigma(\rho) = \max_{u \in B_U(\rho)} |D_u z(u)| \), where \( B_U(\rho) \) denotes the ball in \( U \) centered at 0 with radius \( \rho \). Clearly \( \sigma(\rho) \to 0 \) as \( \rho \to 0 \). As \( z(0) = 0 \), we have \( |z(u)| \leq \sigma(\rho)|u| \) for \( u \in B_U(\rho) \). Combining this with (44) and (45) one concludes that for \( u, u_1, u_2 \in B_U(\rho), t \in \mathbb{R} \)

\[
|h^\tau(t, u_1) - h^\tau(t, u_2)| \leq (k(\tau) + \sigma(\rho))|u_1 - u_2|,
\]

\[
|h^\tau(t, u)| \leq (k(\tau) + \sigma(\rho))|u| + \eta(\tau).
\]
Here
\[ k(\tau) = k_1(\tau) + k_2(\tau), \quad \eta(\tau) = \eta_1(\tau) + \eta_2(\tau). \]

In what follows, we will look for a complete bounded solution in a small neighborhood of the origin for (43). The following procedure mainly uses the idea from [3, Theorem 2.1].

As \( \theta \) is a hyperbolic equilibrium of (9), the spectrum of \( A \) has a decomposition \( \sigma(A) = \sigma_1 \cup \sigma_2 \) with
\[ \sigma_1 \subset \{ \text{Re} \lambda > 0 \}, \quad \sigma_2 \subset \{ \text{Re} \lambda < 0 \}. \]

Accordingly, the space \( U \) from (43) has a unique fixed point provided
\[ ||e^{At}|| \leq Me^{\beta t} \quad (t \leq 0), \quad ||e^{At}|| \leq Me^{-\beta t} \quad (t \geq 0). \]

Let \( u \) be a bounded complete solution (a solution on \( \mathbb{R} \)) of (43). Then
\[ u(t) = e^{A(t-t_0)}u(t_0) + \int_{t_0}^{t} e^{A(t-s)}h^\tau(s, u(s))ds \]
for any \( t_0 \in \mathbb{R} \). Projecting (49) to \( U_1 \) (resp. \( U_2 \)) and setting \( t_0 \to +\infty \) (resp. \( t_0 \to -\infty \)) we obtain that
\[ P_1 u(t) = -\int_{t}^{+\infty} e^{A_1(t-s)}P_1 h^\tau(s, u(s))ds, \]
\[ P_2 u(t) = \int_{-\infty}^{t} e^{A_2(t-s)}P_2 h^\tau(s, u(s))ds. \]

Hence
\[ u(t) = -\int_{t}^{+\infty} e^{A_1(t-s)}P_1 h^\tau(s, u(s))ds + \int_{-\infty}^{t} e^{A_2(t-s)}P_2 h^\tau(s, u(s))ds. \]

Conversely if a function \( u : \mathbb{R} \to U \) satisfies (50), one can directly verify that it is a complete solution of (43).

Let
\[ X = \{ u \in C(\mathbb{R}, U) : \text{ } u \text{ is bounded on } \mathbb{R} \}. \]
\( X \) is equipped with the norm \( || \cdot ||_X \) defined by \( ||u||_X = \sup_{t \in \mathbb{R}} |u(t)| \). Define a map \( T : X \to X \) as
\[ T(u)(t) = -\int_{t}^{+\infty} e^{A_1(t-s)}P_1 h^\tau(s, u(s))ds + \int_{-\infty}^{t} e^{A_2(t-s)}P_2 h^\tau(s, u(s))ds. \]

We show that \( T \) has a unique fixed point in \( X \) near \( u = 0 \) provided \( \tau \) is sufficiently large, which is a bounded complete solution of (43).

The procedure is as follows. First, since \( \sigma(A) = 0 \) as \( \rho \to 0 \) and \( k(\tau), \eta(\tau) \to 0 \) as \( \tau \to +\infty \), using (47) one can find positive numbers \( \rho_0, \tau_0 > 0 \) such that \( T \) maps
\[ B_X(\rho) = \{ u \in X : ||u||_X \leq \rho \} \]
into itself provided \( \rho < \rho_0 \) and \( \tau > \tau_0 \). Further by (46) we can pick a positive number \( \rho < \rho_0 \) sufficiently small and \( \tau > \tau_0 \) sufficiently large so that \( T \) is contracting. Now applying the classical Banach fixed-point theorem one immediately concludes that \( T \) has a unique fixed point \( u^* \) in \( \overline{B}_X(\rho) \). As the argument involved here is quite standard, we omit the details.
Let \( u^* \) be the unique fixed point of \( T \) in \( \overline{B}_\rho(\rho) \). Then \( u^* \) is a bounded complete solution of (43). Therefore \( \gamma(t) = u^*(t) + \xi^*(t, u^*(t)) \) is a solution of (41). Since (41) and (7) coincide on \( [\tau, \infty) \), \( \gamma(t) \) is a solution of (7) on \( [\tau, \infty) \).

Because \( \theta_u = 0 \) is a hyperbolic equilibrium of (43), it can be assumed that \( \rho \) is chosen sufficiently small so that the singleton \( S_0 = \{\theta_u\} \) is the unique maximal compact invariant set of (43) in \( \overline{B}_\delta(\rho) \). By [13, Theorem 1.8] we then deduce that
\[
\lim_{t \to +\infty} u^*(t) = \theta_u. \quad \text{Consequently} \quad \gamma(t) \to \theta \quad \text{as} \quad t \to +\infty.
\]

(2) To prove the second conclusion, it suffices to check that there exist \( \delta > 0 \) and sequences \( \tau_n \to +\infty, t_n \geq 0 \) and \( y_n \in \mathcal{M}(\tau_n) \) with \( |y_n| \to 0 \) such that
\[
|\psi(t_n + \tau_n, \tau_n)y_n| \geq \delta, \quad \forall n \geq 1.
\]

By instability of the origin \( \theta \) w.r.t system (9), one easily deduces that there is an \( \eta > 0 \) such that for some sequences \( x_n \in \mathcal{M} \) with \( |x_n| \to 0 \) and \( t_n > 0 \), it holds that
\[
|\phi(t)x_n| < \eta \quad (t \in [0, t_n]), \quad \text{and} \quad |\phi(t_n)x_n| = \eta.
\]

Choose a sequence \( \varepsilon_n \downarrow 0 \) with
\[
\sqrt{2} \varepsilon_n e^{(L+1/2)t_n} < \eta/4 \tag{51}
\]
for each \( n \), where \( L \) is the Lipschitz constant of \( f(x) \) in \( \overline{B}(\eta) \).

We may assume that \( |x_n| \leq r_0 \) for all \( n \), where \( r_0 \) is the same number as given in Lemma 2.1. Then for each \( n \), by (27) and Lemma 2.1 we deduce that there exists \( \tau_n \to +\infty \) such that
\[
\begin{align*}
(\text{i}) & \quad |q(t, x)| < \varepsilon_n \quad \text{for all} \quad x \in \overline{B}(\eta) \quad \text{and} \quad t \geq \tau_n; \quad \text{and} \\
(\text{ii}) & \quad \text{there is a} \quad y_n \in \mathcal{M}(\tau_n) \quad \text{such that} \quad |x_n - y_n| < \varepsilon_n.
\end{align*}
\]

Note that \( y_n \to \theta \).

Denote
\[
x_n(t) = \phi(t)x_n, \quad y_n(t) = \psi(t + \tau_n, \tau_n)y_n, \quad t \geq 0.
\]

Then \( x_n(t) \) and \( y_n(t) \) solve the equations
\[
\dot{x} = f(x), \quad \dot{y} = f(y) + q(t_n + t, y),
\]
respectively.

We show that for each \( n \), \( |y_n(t_n)| > \eta/2 \). For this purpose, we may assume that \( |y_n(t)| < \eta \) for all \( t \in [0, t_n] \) (otherwise we are done). Set \( w_n(t) = x_n(t) - y_n(t) \). Then
\[
\dot{w}_n(t) = f(x_n(t)) - f(y_n(t)) - q(t_n + t, y_n(t)).
\]

Taking the inner product of the equation with \( w_n = w_n(t) \), it yields
\[
\frac{1}{2} \frac{d}{dt} |w_n|^2 \leq |f(x_n(t)) - f(y_n(t))||w_n| + \varepsilon_n |w_n|
\]
\[
\leq L |w_n|^2 + \frac{1}{2} |w_n|^2 + \frac{1}{2} \varepsilon_n^2.
\]

By virtue of the classical Gronwall Lemma we get
\[
|w_n(t)|^2 \leq \frac{|w_n(0)|^2 e^{(2L+1)t}}{2L + 1} + \frac{\varepsilon_n^2}{2L + 1} e^{(2L+1)t} < 2\varepsilon_n^2 e^{(2L+1)t_n},
\]
for all \( t \in [0, t_n] \). Therefore by (51) we find that
\[
|y_n(t_n)| \geq |x_n(t_n)| - |w_n(t_n)| \geq \eta - \eta/4 > \eta/2.
\]
Let $\delta = \eta/2$ and thus

$$|\psi(t_n + \tau_n, \tau_n)y_n| = |y_n(t_n)| > \delta,$$

which completes the proof of the theorem.

5. Application to the extreme ideology model. As an example, in this section we revisit the extreme ideology model

$$\dot{x}_1 = E - \frac{\beta_1 x_1 x_3}{N} - \frac{\beta_2 x_1 x_5}{N} - \delta_1 x_1 - \mu x_1,$$
$$\dot{x}_2 = \frac{\beta_1 x_1 x_3}{N} - \delta_1 x_2 - \gamma x_2 - \mu x_2,$$
$$\dot{x}_3 = \gamma x_2 - \delta_2 x_3 - \mu x_3,$$
$$\dot{x}_4 = \delta_1 (x_1 + x_2) - \mu x_4,$$
$$\dot{x}_5 = \frac{\beta_2 x_1 x_5}{N} + \delta_2 x_3 - \mu x_5,$$

proposed in [1], where all the parameters are assumed to be positive, and

$$x_1 + x_2 + \cdots + x_5 = N.$$

Denote $N_0$ the total population at the initial time, say, $t = 0$. As $N(t) = N(t, N_0)$ is supposed to satisfy equation $\dot{N} = E - \mu N$, it is trivial to see that

$$\lim_{t \to +\infty} N(t) = E/\mu =: N^*.$$

This allows us to view the model as an asymptotically autonomous system with constraint:

$$\begin{cases}
\dot{x} = f(t, x), \\
g(t, x) = 0,
\end{cases}$$

where $g(t, x) = x_1 + \cdots + x_5 - N(t)$.

The limit system of (54) reads

$$\begin{cases}
\dot{x} = f(x), \\
g(x) = 0.
\end{cases}$$

Here $g(x) = x_1 + \cdots + x_5 - N^*$. (55) has exactly two equilibrium points:

$$e_1 = \left( \frac{E}{\delta_1 + \mu}, 0, 0, 0, 0 \right)^T,$$
$$e_2 = \left( \frac{E\delta_1}{\beta_2 \mu}, 0, \frac{E\beta_2 (\delta_2 - \delta_1 - \mu)}{\beta_2 \mu} \right)^T.$$

Since $\frac{\partial g}{\partial x} = (1, 1, 1, 1, 1)$, as in Section 2 we can decompose the phase space $\mathbb{R}^5$ as $\mathbb{R}^5 = U \oplus V$ with $U = \text{span}\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ and $V = \text{span}\{\zeta_5\}$, where

$$\zeta_1 = (1, -1, 0, 0, 0)^T, \quad \zeta_2 = (1, 0, -1, 0, 0)^T,$$
$$\zeta_3 = (1, 0, 0, -1, 0)^T, \quad \zeta_4 = (1, 0, 0, 0, -1)^T,$$

and $\zeta_5 = (1, 1, 1, 1, 1)^T$. Consequently a point $x \in \mathbb{R}^5$ can be represented as

$$x = (y_1 \zeta_1 + \cdots + y_4 \zeta_4) + y_5 \zeta_5 := u + v,$$
where $y = (y_1, y_2, \cdots, y_5)^T$ satisfies $x = My$ with

$$M = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}.$$ 

Simple computations show that

$$g_u(x) = (0, 0, 0, 0), \quad \text{and} \quad g_u(x) = 5$$

for all $x \in \mathbb{R}^5$.

Let $P : \mathbb{R}^5 \to U$ be the projection, and let $p(u, v) = Pf(x) \ (x = u + v)$. Denote $f(x) = (f_1, \cdots, f_5)^T$. Noticing that $f_1 + \cdots + f_5 = 0$, we find that

$$p(u, v) = -(f_2 \zeta_1 + f_3 \zeta_2 + f_4 \zeta_3 + f_5 \zeta_4).$$

Therefore, we obtain $p_u$ as

$$p_u - p_v g_v^{-1} g_u = p_u.$$

Set $\Gamma_1 = \frac{\beta_2}{\delta_1 + \mu}$, $\Gamma_2 = \frac{\beta_1 \mu \gamma}{\delta_1 + \mu}(\delta_1 + \mu + \gamma)$, and $\Gamma_3 = \frac{\beta_1 \mu \gamma}{\delta_1 + \mu}(\delta_1 + \mu + \gamma)$.

**Theorem 5.1.** Assume $\Gamma_1, \Gamma_2 < 1$. Then $e_1$ is asymptotically stable w.r.t system (55), and hence is eventually uniformly asymptotically stable w.r.t (54).

**Proof.** By (56) we have

$$A = (p_u - p_v g_v^{-1} g_u)|_{x = e_1} = p_u|_{y = M^{-1} e_1}.$$

Hence

$$A = \begin{pmatrix}
-\delta_1 - \gamma - \mu & \frac{\beta_1 \mu}{\delta_1 + \mu} & 0 & 0 \\
\gamma & -\delta_2 - \mu & 0 & 0 \\
0 & -\delta_1 & -\delta_1 - \mu & -\delta_1 \\
0 & \delta_2 & 0 & \frac{\beta_2 \mu}{\delta_1 + \mu} - \mu
\end{pmatrix}.$$ 

The characteristic polynomial of $A$ is given by

$$\det(\lambda I - A) = \lambda^2 + (\delta_1 + \delta_2 + \gamma + 2\mu)\lambda + (\delta_1 + \gamma + \mu)(\delta_2 + \mu) - \frac{\beta_1 \mu \gamma}{\delta_1 + \mu},$$

$$(\lambda + \delta_1 + \mu)(\lambda - \frac{\beta_2 \mu}{\delta_1 + \mu} + \mu).$$
If \( \Gamma_1, \Gamma_2 < 1 \) then the real parts of all eigenvalues are negative. The conclusion of the theorem immediately follows from Theorems 2.2 and 3.4.

**Theorem 5.2.** Assume either \( \Gamma_1 > 1 \) and \( \Gamma_2 \neq 1 \), or \( \Gamma_2 > 1 \) and \( \Gamma_1 \neq 1 \). Then system (54) has a solution \( \sigma(t) \) with \( \lim_{t \to +\infty} \sigma(t) = e_1 \); furthermore, \( \sigma \) is eventually unstable.

*Proof.* In any case \( A \) is hyperbolic and has at least one eigenvalue whose real part is positive, and the conclusion of the theorem immediately follows from Theorem 4.2.

**Remark 3.** Note that \( e_1 \) is not an equilibrium solution of (54) unless the initial population \( N_0 = N^* \).

A fully analogous argument as above applies to \( e_2 \) to obtain the following results. We omit the details.

**Theorem 5.3.** Assume that \( \Gamma_1 > 1 \) and \( \Gamma_3 < 1 \). Then \( e_2 \) is asymptotically stable w.r.t system (55), and hence is eventually uniformly asymptotically stable w.r.t (54).

**Theorem 5.4.** Assume either \( \Gamma_1 < 1 \) and \( \Gamma_3 \neq 1 \), or \( \Gamma_3 > 1 \) and \( \Gamma_1 \neq 1 \). Then system (54) has a solution \( \sigma(t) \) with \( \lim_{t \to +\infty} \sigma(t) = e_2 \); furthermore, \( \sigma \) is eventually unstable.

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