Coherence evolution and transfer supplemented by the state-restoring

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Abstract

The evolution of quantum coherences comes with a set of conservation laws provided that the Hamiltonian governing this evolution conserves the spin-excitation number. At that, coherences do not intertwist during the evolution. Using the transmission line and the receiver in the initial ground state we can transfer the coherences to the receiver without interaction between them, although the matrix elements contributing to each particular coherence intertwist in the receiver’s state. Therefore we propose a tool based on the unitary transformation at the receiver side to untwist these elements and thus restore (at least partially) the structure of the sender’s initial density matrix. A communication line with two-qubit sender and receiver is considered as an example of implementation of this technique.

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I. INTRODUCTION

The multiple quantum (MQ) NMR dynamics is a basic tool of well developed MQ NMR spectroscopy studying the nuclear spin distribution in different systems [1, 2]. Working with spin polarization we essentially deal with the diagonal elements of the density matrix. However, the MQ NMR method allows us to split the whole density matrix into \( N + 1 \) parts, and each of these parts contributes into a specific observable quantity called coherence intensity. Thus studying the coherence intensities and the methods of manipulating them becomes an important direction in development of MQ NMR methods. For instance, the problem of relaxation of MQ coherences was studied in [3–7]. A similar problem in nonopore was considered in [8]).

In MQ NMR experiment, the special sequence of the magnetic pulses is used to generate the so-called two-spin/two-quantum Hamiltonian \( (H_{MQ}) \) which is the non-secular part of the dipole-dipole interaction Hamiltonian averaged over fast oscillations. It was shown in the approximation of nearest-neighbor interactions that the \( H_{MQ} \) Hamiltonian can be reduced to the flip-flop XX-Hamiltonian \( (H_{XX}) \) \([11]\) via the unitary transformation \([2]\). Notice, that \( H_{MQ} \) does not commute with the \( z \)-projection of the total spin momentum \( I_z \), while \( [H_{XX}, I_z] = 0 \).

In this paper we consider the evolution problem for the created MQ coherences. Therefore, after creating the coherences, we switch off the irradiation and allow the coherences to evolve independently under the Hamiltonian commuting with \( I_z \) (this can be, for instance, \( H_{dz} \) Hamiltonian \([9, 10]\) or \( H_{XX} \) flip-flop Hamiltonian). We show that the coherences do not interact during the evolution governed by the Hamiltonian conserving the \( z \)-projection of the total spin momentum. This fact gives rise to the set of conservation laws associated with such dynamics, namely, the coherence intensity of an arbitrary order conserves. But the density-matrix elements contributing into the same order coherence do intertwist.

In addition, the coherences, created in some subsystem (sender) can be transferred to another subsystem (receiver) through the transmission line without interaction between coherences if only the both receiver and transmission line are in the initial state having only the zero-order coherence. This process can be considered as a particular implementation of the remote state creation in spin systems \([12, 13]\). We show that the sender’s density-matrix elements in the receiver’s state can be untwisted using the method based on the unitary
transformation of the receiver or, more effectively, of the extended receiver. The theoretical arguments are supplemented with the particular model of communication line having two-node sender and receiver. Notice that the extended receiver was already used in the previous papers concerning the remote state creation [14] with the purpose of proper correcting the created state of the receiver and improving the characteristics of the remote state creation [12, 13].

The paper is organized as follows. In Sec. II we select the matrices \( \rho^{(n)} \) responsible for forming the \( n \)-order coherence intensity and study some extremal values of coherence intensities. The evolution of the coherence intensities is considered in Sec. III. The transfer of the coherences from the sender to the receiver is studied in Sec. IV. In Sec. V we apply the results of previous sections to a particular model of a chain with 2-qubit sender and receiver. The brief discussion of obtained results is given in Sec. VI.

II. DENSITY MATRIX AND COHERENCES

It was shown (for instance, see [15]) that the density matrix of a quantum state can be written as a sum

\[
\rho = \sum_{n=-N}^{N} \rho^{(n)},
\]

where each submatrix \( \rho^{(n)} \) consists of the elements of \( \rho \) responsible for the spin-state transitions changing the total \( z \)-projection of the spin momentum by \( n \). These elements contribute to the so-called \( n \)-order coherence intensity \( I_n \) which can be registered using the MQ NMR methods. To select the density matrix elements contributing to the \( n \)-order coherence we turn to the density-matrix representation in the multiplicative basis

\[
|i_1 \ldots i_N\rangle, \quad i_k = 0, 1, \quad k = 1, \ldots, N,
\]

where \( i_k \) denotes the state of the \( k \)th spin. Thus, the transformation from the computational basis to the multiplicative one reads

\[
\rho_{ij} = \rho_{ij_1 \ldots i_N j_1 \ldots j_N}, \quad i = \sum_{n=1}^{N} i_n 2^{n-1} + 1, \quad j = \sum_{n=1}^{N} j_n 2^{n-1} + 1.
\]

Then, according to the definition,

\[
I_n(\rho) = \text{Tr} \left( \rho^{(n)} \rho^{(-n)} \right) = \sum_{\sum_k (j_k - i_k) = n} |\rho_{i_1 \ldots i_N j_1 \ldots j_N}|^2, \quad |n| \leq N.
\]
A. Extremal values of coherence intensities

First of all we find the extremal values of the zero order coherence intensity of $\rho$ provided that all other coherences absent, so that $\rho = \rho_0$. By the definition (4),

$$I_0 = \text{Tr} \left( \rho_0 \rho_0 \right) = \text{Tr} \left( U_0 \Lambda_0 U_0^+ \right)^2 = \text{Tr} \Lambda_0^2 = \sum_{i=1}^{2^N} \lambda_{0i}^2,$$

where $N$ is the number of spins in the sender, $\Lambda_0 = \text{diag}(\lambda_{01}, \ldots, \lambda_{02^N})$ and $U_0$ are, respectively, the eigenvalue and eigenvector matrices of $\rho$. Therefore we have to find the extremum of $I_0$ with the normalization condition $\sum_{i=1}^{2^N} \lambda_{0i} = 1$. Introducing the Lagrange factor $\alpha$ we reduce the problem to constructing the extremum of the function

$$\tilde{I}_0 = \sum_{i=1}^{2^N} \lambda_{0i}^2 - \alpha \left( \sum_{i=1}^{2^N} \lambda_{0i} - 1 \right).$$

Differentiating with respect to $\lambda_{0i}$ and equating the result to zero we obtain the system of equations

$$2\lambda_{0i} = \alpha, \quad i = 1, \ldots, 2^N,$$

therefore, $\lambda_{0i} = \frac{\alpha}{2}$. Using the normalization we have $\alpha = \frac{1}{2^{2^N-1}}$, so that $\lambda_{0i} = \frac{1}{2^N}$. The second derivative of $\tilde{I}_0$ shows that this is a minimum. Thus, we have

$$I_0^{\text{min}} = \frac{1}{2N}, \quad \rho|_{I_0^{\text{min}}} = \frac{1}{2^N}E,$$

where $E$ is the $2^N \times 2^N$ identity matrix. To find the maximum value of $I_0$ we observe that

$$\sum_{i=1}^{2^N} \lambda_{0i}^2 = \left( \sum_{i=1}^{2^N} \lambda_{0i} \right)^2 - \sum_{i \neq j} \lambda_{0i} \lambda_{0j} = 1 - \sum_{i \neq j} \lambda_{0i} \lambda_{0j} \leq 1.$$ 

It is obvious that the unit can be achieved if there is only one nonzero eigenvalue $\lambda_{01} = 1$. Thus

$$I_0^{\text{max}} = 1, \quad \rho|_{I_0^{\text{max}}} = \text{diag}(1, 0, 0, \ldots, 0).$$

Now we proceed to the analysis of the $n$-order coherence intensity for the matrix having only three non-zero coherences of zero- and $\pm n$-order, assuming that the zero-order coherence intensity $I_0$ is minimal, i.e.,

$$\rho = \frac{1}{2^N}E + \bar{\rho}^{(n)} = U_n \left( \frac{1}{2^N}E + \Lambda_n \right) U_n^+; \quad \bar{\rho}^{(n)} = \rho^{(n)} + \rho^{(-n)}$$

(11)
where $\Lambda_n = \text{diag}(\lambda_{n1}, \ldots, \lambda_{n2N})$ and $U_n$ are the matrices of eigenvalues and eigenvectors of $\tilde{\rho}^{(n)}$. Of course, $U_n$ is also the eigenvector matrix for the whole $\rho$ in this case and

$$\sum_{i=1}^{2N} \lambda_{ni} = 0. \quad (12)$$

Now we proof one of the interesting property of the eigenvalues for the considered case.

**Proposition 1.** Eigenvalues $\lambda_{ni}$ appear in pairs:

$$\lambda_{n(2i-1)} = \eta_{ni}, \quad \lambda_{n(2i)} = -\eta_{ni}, \quad i = 1, \ldots, 2^{N-1}. \quad (13)$$

**Proof.** First we show that, along with $\tilde{\rho}^{(n)}$, the odd powers of this matrix are also traceless. For instance, let us show that

$$\text{Tr}(\tilde{\rho}^{(n)})^3 = \sum_{i,j,k} \tilde{\rho}^{(n)}_{ij} \tilde{\rho}^{(n)}_{jk} \tilde{\rho}^{(n)}_{ki} = 0. \quad (14)$$

Using the multiplicative basis for the density-matrix elements in the rhs of eq. (14), we remark that only such elements $\tilde{\rho}_{ij}$, $\tilde{\rho}_{jk}$ and $\tilde{\rho}_{ki}$ are nonzero that, respectively, $\sum_{m} i_m - \sum_{m} j_m = \pm n$, $\sum_{m} j_m - \sum_{m} k_m = \pm n$ and $\sum_{m} k_m - \sum_{m} i_m = \pm n$. However, summing all these equalities we obtain the identical zero in the lhs and either $\pm 3n$ or $\pm n$ in the RHS. This contradiction means that there must be zero matrix elements in each term of the sum (14), i.e., the trace is zero.

Similar consideration works for higher odd powers of $\tilde{\rho}^{(n)}$ (however, the sum $\tilde{\rho}^{(n)} + \tilde{\rho}^{(k)}$, $k \neq n$, doesn’t possesses this property, i.e., the trace of any its power is non-zero in general). Consequently, along with (12), the following equalities hold:

$$\sum_{i=1}^{2N} \lambda_{ni}^m = 0 \text{ for any odd } m. \quad (15)$$

Condition (15) holds for any odd $m$ if only the eigenvalues $\lambda_{ni}$ appear in pairs (13). To prove this statement, first we assume that all eigenvalues are non-degenerate and let the eigenvalue $\lambda_{n1}$ be maximal by absolute value. We divide sum (15) by $\lambda_{n1}^m$:

$$1 + \sum_{i=2}^{2N} \left(\frac{\lambda_{ni}}{\lambda_{n1}}\right)^m = 0, \text{ for odd } m. \quad (16)$$

Each term in the sum can not exceed one by absolute value. Now we take the limit $m \to \infty$ in eq. (16). It is clear that all the terms such that $\left|\frac{\lambda_{ni}}{\lambda_{n1}}\right| < 1$ vanish. Since this sum is zero,
there must be an eigenvalue \( \lambda_{n2} \) such that \( \lambda_{n2} = -\lambda_{n1} \). Then, the appropriate term in \((16)\) yields -1. So, two first terms in sum \((16)\) cancel each other which reduces \((16)\) to

\[
\sum_{i=3}^{2N} \lambda_{ni}^m = 0, \quad \text{for odd } m. \tag{17}
\]

Next, we select the maximal (by absolute value) of the remaining eigenvalues, repeat our arguments and conclude that there are two more eigenvalues equal by absolute value and having opposite signs. And so on. Finally, after \(2^{N-1}\), steps we result in conclusion that all eigenvalues appear in pairs \((13)\).

Let the \((2k+1)\)th eigenvalue on the \((2k+1)\)-step is \(s\)-multiple, i.e. \(\lambda_{n(2k+1)} = \cdots = \lambda_{n(2k+s)}\). Then the sum \((15)\) gets the form

\[
\sum_{i=2k+1}^{2N} \left( \frac{\lambda_{ni}}{\lambda_{n(2k+1)}} \right)^m + \sum_{i=2k+s+1}^{2N} \left( \frac{\lambda_{ni}}{\lambda_{n(2k+1)}} \right)^m, \quad s \in \mathbb{N}, \ s \leq N - 2k, \ odd \ m. \tag{18}
\]

Now, to compensate \(s\) we need an \(s\)-multiple eigenvalue, such that \(\lambda_{n(2k+s+1)} = \cdots = \lambda_{n(2k+2s)} = -\lambda_{n(2k+1)}\). Thus, if there is \(s\)-multiple positive eigenvalue, there must be an \(s\)-multiple negative eigenvalue. This ends the proof. \(\square\)

Next, since all the eigenvalues of \(\rho\) must be non-negative and the density matrix \(\rho\) has the structure \((11)\), the negative eigenvalues \(\eta_{ni}\) can not exceed \(\frac{1}{2^N}\) by absolute value. Therefore, the maximal \(n\)-order coherence intensity corresponds to the case

\[
\eta_{ni} = \frac{1}{2^N}. \tag{19}
\]

Consequently,

\[
I_{n}^{\max} + I_{-n}^{\max} = 2I_{n}^{\max} = \sum_{j=1}^{N_n} \lambda_{ni}^2 = \frac{N_n}{2^{2N}} \leq \frac{1}{2^N}, \tag{20}
\]

where \(I_{n}^{\max} = I_{-n}^{\max}\) and \(N_n\) is the number of nonzero eigenvalues of \(\tilde{\rho}^{(n)}\). This number equals to the rank of \(\tilde{\rho}^{(n)}\) which, in turn, can be found as follows.

**Proposition 2.** The rank of the matrix \(\tilde{\rho}^{(n)}\) can be calculated using the formula

\[
N_n = \text{ran} \ \tilde{\rho}^{(n)} = \sum_{k=0}^{N} \min \left( \binom{N}{k}, \binom{N}{k+n} + \binom{N}{k-n} \right), \tag{21}
\]

where the binomial coefficients \(\binom{N}{m} = 0\) for \(m < 0\).
Proof. For the $n$-order coherence, the number of states with $k$ excited spins equals $\binom{N}{k}$. The $\pm n$-order coherence collects the elements of $\rho$ responsible for transitions from the states with $k$ excited spins to the states with $k \pm n$ excited spins. All together, there are $\binom{N}{k+n} + \binom{N}{k-n}$ such transitions. These transitions can be collected into the matrix of $\binom{N}{k}$ columns and $\binom{N}{k+n} + \binom{N}{k-n}$ rows, whose maximal rank equals $\min\left(\binom{N}{k}, \binom{N}{k+n} + \binom{N}{k-n}\right)$. Obviously, the rank of $\tilde{\rho}^{(n)}$ equals the sum of calculated ranks for different $k = 0, \ldots, N$, i.e., we obtain formula (21). 

Consequence. For the coherence intensity of the first order ($n = 1$) eq. (21) yields:

$$N_1 = \sum_{k=0}^{N} \binom{N}{k} = 2^N. \tag{22}$$

Proof. We have to show that in this case

$$\binom{N}{k} \leq \binom{N}{k+1} + \binom{N}{k-1}, \quad 0 \leq k \leq N. \tag{23}$$

First we consider the case $k > 1$ and $k < N$. Then

$$\binom{N}{k+1} + \binom{N}{k-1} = \binom{N}{k} \left(\frac{N-k}{k+1} + \frac{k}{N-k+1}\right). \tag{24}$$

Let us show that the expression inside the parenthesis is $\geq 1$. After simple transformations, this condition takes the form

$$3k^2 - 3Nk + N^2 - 1 \geq 0, \tag{25}$$

where the lhs is a quadratic expression in $k$. The roots of the lhs read

$$k_{1,2} = \frac{3N \pm \sqrt{12 - 3N^2}}{6}, \tag{26}$$

which are imaginary for $N > 2$. Therefore the parabola $3k^2 - 3Nk + N^2$ lies in the upper half-plane $k$ for $N > 2$ and consequently condition (25) holds for $N \geq 2$. In our case, the minimal $N$ is 2, which corresponds to the 1-qubit sender and 1-qubit receiver without the transmission line between them.

If $k = 1$ then, instead of (24), we have

$$\binom{N}{2} + \binom{N}{0} = \binom{N}{2} + 1 = \binom{N}{1} \frac{N-1}{2} + 1 \geq \binom{N}{1}, \quad N \in \mathbb{N}. \tag{27}$$

Therefore condition (23) is also satisfied.
### TABLE I: The maximal coherence intensities $I_n^{\text{max}}$ of the $n$-order coherence and the rank $N_n$ of $\tilde{\rho}^{(n)}$ for the different numbers of nodes $N$ in a spin system.

| $N$ | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|
| $n$ | 1 2| 1 2| 1 2| 1 2|
| $N_n$ | 4 2| 8 4| 16| 12 4 2|
| $2I_n^{\text{max}}$ | 1 1| 1 1| 1 3| 1 1|

If $k = 0$, then $\binom{N}{1} = 1$ and

$$\binom{N}{1} + \binom{N}{-1} = \binom{N}{1} > \binom{N}{0},$$

therefore condition (23) is also satisfied.

The cases $k = N$ can be considered in a similar way. □

Thus, $N_1$ equals the maximal possible rank $N_1 = \text{ran} \tilde{\rho}^{(1)}$, so that $2I_1^{\text{max}} = \frac{1}{2N}$. Similarly, for the $N$-order coherence we have only two nonzero terms in (14) which give $N_N = 2$ and $2I_N^{\text{max}} = \frac{1}{2^N}$. For the intensities of the other-order coherences we do not give similar result for any $N$. The maximal coherence intensities of the $n$-order ($n > 0$) for $N = 2, \ldots, 5$ are given in Table I. This table shows the ordering of $I_n^{\text{max}}$:

$$I_0^{\text{max}} > I_1^{\text{max}} > \cdots > I_N^{\text{max}}.$$  

Regarding the minimum of any non-zero-order coherence intensity, its value is obvious:

$$I_n^{\text{min}} = 0.$$  

### III. EVOLUTION OF COHERENCES

#### A. Conservation laws

First of all we remind a famous conservation law which holds for any evolutionary quantum system.

**Proposition 3.** The sum of all coherence intensities conserves:

$$\frac{d}{dt} \sum_{n=-N}^{N} I_n = \frac{d}{dt} \text{Tr} (\rho^{(n)} \rho^{(-n)}) = 0.$$  

\[ \]
**Proof.** In fact, consider the Liouville equation

\[ i \frac{d\rho}{dt} = [\rho, H]. \]  

(32)

Using this equation we have

\[ i \text{Tr}\frac{d\rho^2}{dt} = \text{Tr}[\rho^2, H] = 0. \]  

(33)

Therefore

\[ \text{Tr}\rho^2 = \text{Tr}\left( \sum_{n=-N}^{N} \rho^{(n)}\rho^{(-n)} \right) = \sum_{n=-N}^{N} \text{Tr}(\rho^{(n)}\rho^{(-n)}) = \sum_{n=-N}^{N} I_n \equiv \text{const.} \]  

(34)

which is equivalent to eq. (31). \[\Box\]

In addition, if the system evolves under the Hamiltonian commuting with \( I_z \),

\[ [H, I_z] = 0, \]  

(35)

then there is a family of conservation laws specified as follows.

**Consequence.** If (35) holds then all coherences conserve, i.e.

\[ \frac{dI_n}{dt} = 0, \quad |n| \leq N. \]  

(36)

**Proof.**

From eq. (32) we have

\[ i\rho^{(n)}\frac{d\rho}{dt} + i\frac{d\rho}{dt}\rho^{(-n)} = \rho^{(n)}[H, \rho] + [H, \rho]\rho^{(-n)}. \]  

(37)

The trace of this equation reads

\[ \text{Tr}\left( i\rho^{(n)}\frac{d\rho}{dt} + i\frac{d\rho}{dt}\rho^{(-n)} \right) = i\frac{d}{dt}\text{Tr}\left( \rho^{(n)}\rho^{(-n)} \right) \equiv \]  

(38)

\[ i\frac{dI_n}{dt} = \text{Tr}\left( \rho^{(n)}H\rho - \rho H\rho^{(n)} \right) - \text{Tr}\left( \rho H\rho^{(-n)} - \rho^{(-n)}H\rho \right). \]

We can introduce factors \( e^{i\phi I_z} \) and \( e^{-i\phi I_z} \) under the trace, substitute expansion (1) for \( \rho \) and use commutation relation (35). Then we have

\[ \text{Tr}\left( e^{i\phi I_z}(\rho^{(n)}H\rho - \rho H\rho^{(n)})e^{-i\phi I_z} \right) - \text{Tr}\left( e^{i\phi I_z}(\rho H\rho^{(-n)} - \rho^{(-n)}H\rho)e^{-i\phi I_z} \right) = \]

(39)

\[ \sum_{k=-N}^{N} \left( \text{Tr}\left( e^{i\phi(n+k)}(\rho^{(n)}H\rho^{(k)} - \rho^{(k)}H\rho^{(n)}) \right) \right) - \text{Tr}\left( e^{i\phi(k-n)}(\rho^{(k)}H\rho^{(-n)} - \rho^{(-n)}H\rho^{(k)}) \right). \]
Since this trace must be independent on $\phi$ we have $k = -n$ and $k = n$ in the first and the second trace respectively. Therefore expression (39) is identical to zero and eq.(38) yields set of conservation lows (36). $\square$

Equalities (36) represent the set of conservation laws associated with the dynamics of a spin system under the Hamiltonian $H$ commuting with $I_z$.

B. On map $\rho^{(n)}(0) \rightarrow \rho^{(n)}(t)$

Here we derive an important consequence of conservation laws (36) describing the dependence of the elements of the evolutionary matrix $\rho^{(n)}(t)$ on the elements of the initial matrix $\rho^{(n)}(0)$. First of all we notice that the Hamiltonian commuting with $I_z$ has the following block structure:

$$H = \sum_{l=0}^{N} H^{(l)},$$

(40)

where the block $H_l$ governs the dynamics of states with $l$ excited spins ($l$-excitation block). Then any matrix $\rho^{(n)}$ can be also represented as

$$\rho^{(n)} = \sum_{l=0}^{N-n} \rho^{(l,l+n)}, \quad \rho^{(-n)} = \sum_{l=n}^{N} \rho^{(l,l-n)}, \quad n = 0, 1, \ldots, N. \quad (41)$$

Then, introducing the evolution operators

$$V(t) = e^{-iHt}, \quad V^{(l)}(t) = e^{-iH^{(l)}t}, \quad (42)$$

we can write the evolution of the density matrix as

$$\rho(t) = V(t)\rho(0)V^+(t) = \sum_{n=-N}^{N} V(t)\rho^{(n)}(0)V^+(t) = \sum_{n=0}^{N} \sum_{l=0}^{N-n} V^{(l)}(t)\rho^{(l,l+n)}(0)(V^{(l+n)}(t))^+ + \sum_{n=-N}^{N} \sum_{l=n}^{N} V^{(l)}(t)\rho^{(l,l-n)}(0)(V^{(l-n)}(t))^+. \quad (43)$$

Since the operators $V^{(l)}$ do not change the excitation number, we can write

$$\rho(t) = \sum_{n=-N}^{N} \rho^{(n)}(t), \quad (44)$$

$$\rho^{(n)}(t) = \sum_{l=0}^{N-n} V^{(l)}(t)\rho^{(l,l+n)}(0)(V^{(l+n)}(t))^+ \equiv P^{(n)} [t, \rho^{(n)}(0)] , \quad (45)$$

$$\rho^{(-n)} = (\rho^{(n)}(t))^+ = \sum_{l=n}^{N} V^{(l)}(t)\rho^{(l,l-n)}(0)(V^{(l-n)}(t))^+ \equiv P^{(-n)} [t, \rho^{(-n)}(0)] , \quad (46)$$
where we introduce the linear evolutionary operators $P^{(n)} (P^{(-n)})$ mapping the matrix $\rho^{(n)}(0)$ ($\rho^{(-n)}(0)$) into the evolutionary matrix $\rho^{(n)}(t)$ ($\rho^{(-n)}(t)$) responsible for the same $n$-order ($(-n)$-order) coherence, i.e., the operator $P^{(n)}$ applied to the matrix of the $n$-order coherence doesn’t generate coherences of different order. We notice that, in certain sense, formulas (45) are similar to the Liouville representation [16]. Hereafter we do not write $t$ in the arguments of $P^{(n)}$ for simplicity.

IV. COHERENCE TRANSFER FROM SENDER TO RECEIVER

A. Coherence transfer as map $\rho^{(S)}(0) \rightarrow \rho^{(R)}(t)$

Now we consider the process of the coherence transfer from the M-qubit sender $(S)$ to the M-qubit receiver $(R)$ connected by the transmission line $(TL)$. The receiver’s density matrix reads

$$\rho^{R}(t) = \text{Tr}_{/ R} \rho(t) = \sum_{n = -M}^{M} \rho^{(R;n)}(t),$$

where the trace is taken over all the nodes of the quantum system except the receiver, and $\rho^{(R;n)}$ means the submatrix of $\rho^{(R)}$ contributing into the $n$-order coherence.

To proceed further, we consider the tensor product initial state

$$\rho(0) = \rho^{(S)}(0) \otimes \rho^{(TL;R)}(0),$$

Obviously

$$\rho^{(n)}(0) = \sum_{n_1 + n_2 = n} \rho^{(S;n_1)}(0) \otimes \rho^{(TL;R;n_2)}(0),$$

where $\rho^{(S;n)}$ and $\rho^{(TL;R;n)}$ are matrices contributing to the $n$-order coherence of, respectively, $\rho^{(S)}$ and $\rho^{(TL)}$. Using expansion (44) and operators $P^{(n)}$ defined in (45) we can write

$$\rho^{(R)} = \text{Tr}_{/ R} \sum_{n = -N}^{N} P^{(n)} [\rho^{(n)}(0)] = \text{Tr}_{/ R} \sum_{n = -N}^{N} \sum_{n_1 + n_2 = n} P^{(n)} [\rho^{(S;n_1)}(0) \otimes \rho^{(TL;R;n_2)}(0)].$$

Next we need the following Proposition.

**Proposition 4.** The partial trace of matrix $\rho$ does not mix coherences of different order and, in addition,

$$\text{Tr}_{/ R} \rho^{(n)} = 0, \ |n| > M,$$
Proof. We split the whole multiplicative basis of quantum state into the $2^M$-dimensional sub-basis $B^{(R)}$ of the receiver’s states and the $2^N-M$-dimensional sub-basis of the subsystem consisting of the sender and the transmission line $B^{(S,TL)}$, i.e., $|i⟩ = |i^{S,TL}⟩ ⊗ |i^R⟩$. Then elements of the density matrix $ρ$ are enumerated by the double indexes $i = (i^{S,TL}, i^R)$ and $j = (j^{S,TL}, j^R)$, i.e.,

$$ρ_{ij} = ρ_{(i^{S,TL},i^R),(j^{S,TL},j^R)}.$$  (51)

Then eq.(46) written in components reads

$$\rho^{(R)}_{i_R j_R} = \text{Tr}_{/R}ρ = \sum_{i^{S,TL}}ρ_{(i^{S,TL},i^R),(j^{S,TL},j^R)}.$$  (52)

Therefore the coherences in the matrix $ρ^{(R)}$ are formed only by the transitions in the subspace spanned by $B^{(R)}$. Therefore, the matrix $ρ^{(R,n)}$ forming the $n$-order coherence of the receiver consists of the elements included into the $n$-order coherence of the whole quantum system. Consequently, trace does not mix coherences.

Since the receiver is an $M$-qubit subsystem, it can form only the coherences of order $n$ such that $|n| \leq M$, which agrees with justifies condition (50). □

This Proposition allows us to conclude that

$$ρ^{(R,n)} = \text{Tr}_{/R} \sum_{n_1+n_2=n} P^{(n)}[ρ^{(S,n_1)}(0) ⊗ ρ^{(TL,R,n_2)}(0)], \quad |n| \leq M.$$  (53)

Formula (53) shows that, in general, all the coherences of $ρ^{(S,n)}$ are mixed in any particular order coherence of the receiver’s density matrix $ρ^R$. However, this is not the case if the initial state $ρ^{TL,R}(0)$ consists of elements contributing only to the zero-order coherence. Then (53) gets the form

$$ρ^{(R,n)} = \text{Tr}_{/R} \left( P^{(n)}[ρ^{(S,n)}(0) ⊗ ρ^{(TL,R,0)}(0)] \right), \quad |n| \leq M.$$  (54)

In this case the elements contributing to the $n$-order coherence of $ρ^S(0)$ contribute only to the $n$-order coherence of $ρ^R(t)$.

B. Restoring of sender’s state at receiver’s side

In Sec.IVA we show that, although the coherences of the sender’s initial state are properly separated in the receiver’s state, the elements contributing to the particular $n$-order...
coherence of $\rho_0^S$ are mixed in $\rho_n^R$. But we would like to separate the elements of $\rho_0^S$ in $\rho^R(t)$, so that, in the ideal case:

$$\rho_{ij}^R(t) = f_{ij}(t)\rho_{ij}^S, \quad (i, j) \neq (2^M, 2^M), \quad (55)$$

$$\rho_{2^M2^M}^R(t) = 1 - \sum_{i=1}^{2^M-1} f_{ii}(t)\rho_{ii}^S.$$ 

We refer to the state with elements satisfying (55) as a completely restored state. Perhaps, relation (55) can not be realized for all elements of $\rho^R$, in other words, the complete sender’s state restoring is impossible, in general case. However, the simple case of a complete restoring is the transfer of the one-qubit sender state to the one-qubit receiver because in this case there is only one element $\rho_{12}^S$ in $\rho^S$ contributing to the first order coherence in $\rho^R$ and one independent element $\rho_{11}^S$ contributing to the zero-order coherence. In addition, we can notice that the highest order coherences have the form (55) in general case, because there is only one element of the density matrix contributing to the $\pm M$-order coherence. Regarding the other coherences, we can try to partially restore at least some of the elements using the local unitary transformation at the receiver side.

1. Unitary transformation of extended receiver as state-restoring tool

Thus we can use the unitary transformation at the receiver to (partially) restore the initial sender’s state $\rho^S(0)$ in the density matrix $\rho^R(t)$ at some time instant $t$ in the sense of definition (55). It is simple to estimate that the number of parameters in the unitary transformation $U^{(R)}$ of the receiver itself is not enough to restore all the elements of the density matrix $\rho^S(0)$. To make the complete restoring possible we must increase the number of parameters in the unitary transformation by extending the receiver to $M^{(ext)} > M$ nodes and use the transformation $U^{(ext)}$ of this extended receiver to restore the state $\rho^S(0)$.

Thus we consider the $M^{(ext)}$-dimensional extended receiver and require that the above mentioned unitary transformation does not mix different submatrices $\rho^{(n)}$. This is possible if $U$ commutes with the $z$-projection of the total extended receiver’s spin momentum. In this case the matrix $\rho^R$ can be obtained from $\rho$ in three steps: (i) reducing $\rho(t)$ to the density matrix of the extended receiver $\rho^{R_{ext}}(t)$, (ii) applying the restoring unitary transformation $U^{(ext)}$ and (iii) reducing the resulting density matrix $U^{(ext)}\rho^{R_{ext}}(t)(U^{(ext)})^+$ to $\rho^R$. To find
out the general form of the unitary transformation we consider this transformation in the basis constructed on the matrices $I^\pm_j$ and $I^z_j$. This basis reads:

for the one-qubit subsystem (ith qubit of the whole quantum system),

$$B^{(i)} : E, I_{zi}, I^+_i, I^-_i; \quad (56)$$

for the two-qubit subsystem (the ith and jth qubits),

$$B^{(ij)} = B^{(i)} \otimes B^{(j)}; \quad (57)$$

for the three-qubit subsystem (the ith, jth and kth qubits),

$$B^{(ijk)} = B^{(ij)} \otimes B^{(k)}; \quad (58)$$

for the four-qubit subsystem (the ith, jth, kth and mth qubits),

$$B^{(ijkm)} = B^{(ij)} \otimes B^{(km)}, \quad (59)$$

and so on. The elements of the basis commuting with $I_z$ are formed by the pairs $I^+_p I^-_q$ and by the diagonal matrices $I^z_k, E$. Thus, the one-qubit basis (56) involves two elements commuting with $I_z$:

$$B^{(C;i)} : E, I_{zi}. \quad (60)$$

The two-qubit basis (57) involves 6 such elements:

$$B^{(C;ij)} : E, I_{zi}, I_{zj}, I_{zi}I_{zj}, I^+_i I^-_j, I^+_j I^-_i. \quad (61)$$

The three-qubit basis (58) involves 20 such elements:

$$B^{(C;ijk)} : E, I_{zp}, I_{zp}I_{zs}, I_{zi}I_{zj}I_{zk}, I^+_p I^-_s, I^+_p I^-_s I_{zr}, p, s, r \in \{i, j, k\}, r \neq p \neq s. \quad (62)$$

The four-qubit basis (59) involves 70 such elements:

$$B^{(C;ijkm)} : E, I_{zp}, I_{zp}I_{zs}, I_{zp}I_{zs}I_{zr}, I_{zi}I_{zj}I_{zk}I_{zm}, I^+_p I^-_s, I^+_p I^-_s I_{zr}, I^+_p I^-_s I_{zr}I_{zq}, \quad (63)$$

and so on. However, there is a common phase which can not effect the elements of the density matrix. Therefore, the number of parameters in the above unitary transformations which can effect the density-matrix elements is less then the dimensionality of the bases (60-63) by one.
V. PARTICULAR MODEL

As a particular model, we consider the spin-1/2 chain with two-qubit sender and receiver and the tensor product initial state

$$\rho(0) = \rho^S(0) \otimes \rho^{TL,R}(0),$$

where $\rho^S(0)$ is an arbitrary initial state of the sender and $\rho^{TL,R}(0)$ is the initial thermal equilibrium state of the transmission line and receiver,

$$\rho^{TL,B} = e^{bI_z} Z, \quad Z = \left( 2 \cosh \frac{b}{2} \right)^{N-2},$$

where $b = \frac{1}{kT}$, $T$ is temperature and $k$ is the Boltzmann constant. Thus, both $\rho^S(0)$ and $\rho^R(0)$ are $4 \times 4$ matrices.

Let the evolution of the spin chain be governed by the nearest-neighbor $XX$-Hamiltonian [11]

$$H = \sum_{i=1}^{N-1} D(I_{ix}(i+1)x + I_{iy}(i+1)y),$$

where $D$ is a coupling constant. Obviously, $[H, I_z] = 0$. Using the Jordan-Wigner transformations [17, 18] we can derive the explicit formula for the density matrix of the two-qubit receiver (46) but we do not represent the details of this derivation for the sake of brevity.

To proceed further, let us write formulas (53) contributing into each particular coherence as follows. For the zero order coherence we have

$$\rho_{ij}^{(R \mid 0)} = \alpha_{ij;11} \rho_{11}^S + \alpha_{ij;22} \rho_{22}^S + \alpha_{ij;33} \rho_{33}^S + \alpha_{ij;44} \rho_{44}^S + \alpha_{ij;23} \rho_{23}^S + \alpha_{ij;32} \rho_{32}^S,$$

$$\rho_{44}^{(R \mid 0)} = 1 - \sum_{i=1}^{3} \rho_{ii}^{R}, \quad \alpha_{ii;32} = \alpha_{ii;23}^*,$$

there are 12 real parameters $\alpha_{ii;jj}$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and 9 complex parameters $\alpha_{ii;23}$, $i = 1, 2, 3$, $\alpha_{25;ii}$, $i = 1, 2, 3, 4$, $\alpha_{23;23}$ and $\alpha_{23;32}$, i.e., 30 real parameters. For the first order coherence:

$$\rho_{ij}^{(R \mid 1)} = \alpha_{ij;12} \rho_{12}^S + \alpha_{ij;13} \rho_{13}^S + \alpha_{ij;24} \rho_{24}^S + \alpha_{ij;34} \rho_{34}^S,$$

there are 12 real parameters $\alpha_{ij;jj}$, $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$, and 9 complex parameters $\alpha_{ij;23}$, $i = 1, 2, 3$, $\alpha_{ij;25}$, $i = 1, 2, 3, 4, 5$, $\alpha_{25;ij}$, $i = 1, 2, 3, 4, 5$, and $\alpha_{23;23}$ and $\alpha_{23;32}$, i.e., 39 real parameters.
there are 16 complex parameters, or 32 real ones. Finally, for the second order coherence we have

\[ \rho_{14}^R = \alpha_{14;12}\rho_{14}^S, \]

there is one complex parameter (two real ones). In all these formulas, \( \alpha_{ij;nm} \) are defined by the interaction Hamiltonian and they depend on the time \( t \).

A. Simple example of \( \rho^{(S;\pm1)} \)-restoring

We see that there are 64 real parameter we would like to adjust in eqs. (67–69). For the purpose of complete restoring of an arbitrary state we need the extended receiver of \( M = 4 \) nodes so that the number of the effective parameters in the unitary transformation described in Sec. IV B 1 would be 69. However, for the sake of simplicity, here we use the unitary transformation of the two-qubit receiver to perform a complete restoring of the ±1-order coherence matrices \( \rho^{(S;\pm1)}(0) \) of a special form, namely

\[
\rho^{(S;1)} + \rho^{(S;-1)} = \begin{pmatrix}
0 & a & a & 0 \\
0 & 0 & a & 0 \\
an & 0 & 0 & 0 \\
0 & a^* & 0 & 0
\end{pmatrix}.
\]

The unitary transformation constructed on the basis (61) reads:

\[
U = e^{i\phi_1(I_1^+I_2^+ + I_1^-I_2^-)}e^{i\phi_2(I_1^+I_2^- - I_1^-I_2^+)}e^{i\Phi},
\]

where \( \Phi = \text{diag}(\phi_3, \ldots, \phi_6) \) is a diagonal matrix and \( \phi_i, i = 1, \ldots, 6 \), are arbitrary real parameters. Eqs. (68) reduce to

\[
(\rho_1^R)_{ij} = \alpha_{ij}\alpha, \quad \alpha_{ij} = \alpha_{ij;12} + \alpha_{ij;13} + \alpha_{ij;24}, \quad (i, j) = (1, 2), (1, 3), (2, 4), (3, 4).
\]

We consider the chain of \( N = 20 \) nodes and set \( b = 10 \). The time instant for the state registration at the receiver is chosen by the requirement to maximize the maximal-order coherence intensity (the second order in this model) because this intensity has the least maximal possible value according to (29). This time instance was found numerically and it equals \( Dt = 24.407 \).
Next, using the parameters $\phi_i$ of the unitary transformation (71) we can put zero the coefficient $\alpha_{34}$ and thus obtain the completely restored matrices $\rho^{(R;\pm)}$ in the form

$$
\rho^{(R;1)} + \rho^{(R;-1)} = \begin{pmatrix}
0 & \alpha_{12}a & \alpha_{13}a & 0 \\
\alpha_{12}^*a^* & 0 & 0 & \alpha_{24}a \\
\alpha_{13}^*a^* & 0 & 0 & 0 \\
0 & \alpha_{24}^*a^* & 0 & 0
\end{pmatrix}.
$$

(73)

The appropriate values of the parameters $\phi_i$ are following:

$$\phi_1 = 2.41811, \quad \phi_2 = 1.57113, \quad \phi_k = 0, \quad k = 2, \ldots, 6.
$$

(74)

Thus, using the unitary transformation of the receiver we restore the sender’s initial matrices $\rho^{(S;\pm)}(0)$ in the sense of definition (55). This result holds for arbitrary admissible initial matrices $\rho^{(S;0)}(0)$ and $\rho^{(S;2)}(0)$.

VI. CONCLUSION

The MQ coherence intensities are the characteristics of a density matrix which can be measured in MQ NMR experiments. We show that the coherences evolve independently if only the Hamiltonian governing the spin dynamics conserves the total $z$-projection of the spin momentum. This is an important property of quantum coherences which allows us to store them in the sense that the family of the density-matrix elements contributing into a particular-order coherence do not intertwist with other elements during evolution. In addition, if we connect the spin system with formed coherences (called sender in this case) to the transmission line and receiver we can transfer these coherences without mixing them if only the initial state of $\rho^{(TL,R)}(0)$ has only the zero-order coherence.

We also describe the restoring method which could allow (at least partially) to reconstruct the sender’s initial state. This state-restoring is based on the unitary transformation at the receiver side involving, in general, the so-called extended receiver with the purpose to enlarge the number of parameters in the unitary transformation. The partial state-restoring of two-qubit receiver via the unitary transformation on it is performed as a simplest example.
Examples of more accurate restoring involving the extended receiver require large technical work and will be done in a specialized paper.

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