Construction of a non ultra-static Ellis wormhole in general relativity

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Several traversable wormholes (T-WHs) of the Morris-Thorne type have been presented as exact solutions of Einstein-nonlinear electrodynamics gravity (GR-NLED), e.g. [19, 20, 23, 25–27]. However, none of these solutions is support by a nonlinear electrodynamics model satisfying plausible conditions. In this work, the first traversable wormhole solution of Einstein-nonlinear electrodynamics gravity coupled to a self-interacting scalar field (GR-NLED-SF) and with nonlinear electrodynamics model such that in the limit of weak field becomes the Maxwell electrodynamics, is presented.

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I. INTRODUCTION

The wormholes are fascinating predictions arising from the geometrical description of gravity. They involve a topological spacetime configuration as a shortcut between distant points or regions in spacetime. The first wormhole interpretation originally came from the work of Einstein and Rosen in 1935, with their solution known as the Einstein-Rosen bridge \cite{1}, which is, in essence, the maximally extended Schwarzschild black hole solution \cite{2}. However, the “throat” of this wormhole is dynamic and hence non-traversable, meaning that its radius expands to a maximum and quickly contracts to zero so fast that even a photon cannot pass through \cite{2}. Further, in 1988, a solution to the wormhole traversability problem was established by Morris and Thorne \cite{3}. They obtained one type of wormhole metric and the necessary conditions (absence of horizons and the flare-out condition) that can guarantee the traversability of a wormhole spacetime. Moreover, they showed that in the context of general relativity (GR), the throat of these types of wormholes only could be kept open by some form of “exotic” matter \cite{4} having negative energy density and whose energy-momentum tensor violates the null-energy condition (NEC). An example of this exotic type of

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matter is a phantom scalar field having a negative kinetic term in the gravitational action. Although an explanation about the fundamental origin of phantom fields is still in discussion, they are frequently used in Cosmology \cite{5–7}, and Astrophysics \cite{8}. In particular, a phantom scalar field could be used to generate the negative kinetic energy density that allows traversable wormholes. For instance, one of the first and simplest examples of traversable wormholes is the static, spherically symmetric and asymptotically flat (SSS-AF) Ellis wormhole \cite{9} whose energy-momentum tensor can be represented by a massless phantom scalar field. This solution has been extensively studied, and its properties like gravitational lensing \cite{10}, quasi-normal modes \cite{11}, shadows \cite{12} and stability \cite{13} have been thoroughly investigated. Recently, \cite{14} shows that the source of the Ellis wormhole as a perfect fluid with negative energy density and a source-free radial electric or magnetic field is also possible.

The construction of traversable wormholes has been studied using non-linear electrodynamics (NLED) as source. NLED theories are derived from Lagrangians $L = L(F, G)$ that depend arbitrarily on the two electromagnetic invariants, $F = 2(E^2 - B^2)$ and $G = E \cdot B$, where $E$ and $B$ are the electric and magnetic fields, respectively. Albeit this form is arbitrary, there exist two Lagrangians that are outstanding: the Born-Infeld theory (BI)

$$L_{\text{BI}}(F, G) = 4b^2 \left( -1 + \sqrt{1 + \frac{F^2}{2b^2} + \frac{G^2}{16b^4}} \right)$$ (1)

where $b$ is a constant which has the physical interpretation of a critical field strength \cite{15}; and the Euler-Heisenberg theory (EH),

$$L_{\text{EH}}(F, G) = -\frac{1}{2} F + \frac{\mu}{2} F^2 + \frac{7\mu}{8} G^2$$ (2)

which corresponds to the weak field approximation of \cite{16, 17}, and the coupling constant $\mu$ is written as $\mu = 2\alpha^2/(45m_e^2)$, where $m_e$ is the mass of the electron and $\alpha$ is the fine structure constant. Considering NLED Lagrangian of the form $L(F)$ coupled to gravity, interesting solutions arise, like regular black holes or traversable wormholes, among others, for instance, \cite{18–22}.

By using NLED as a source in the Einstein field equations, two kind of T-WH have been investigated: dynamic \cite{19, 20} and static \cite{23}. While, dynamic T-WHs are possible in the GR context by using NLED as the only source, the SSS T-WHs static wormholes are not possible in the NLED context \cite{24}. To date, the SSS T-WHs studied requires an additional scalar field \cite{23}. However, the common NLED models used for the construction of dynamical and static traversable wormholes do not satisfy plausible physical conditions, like how to reduce them to Maxwell electrodynamics in the weak field limit \cite{20, 23, 25–27}.

In this paper we will construct a SSS-AF T-WH, to do this we consider a Euler-Heisenberg-like electrodynamics model as the NLED source, with the Lagrangian density defined by:

$$L(F) = -\frac{1}{2} F + \mu_0 F^2 + \mu_1 |F|^\frac{3}{2},$$ (3)

where $\mu_0$ and $\mu_1$ are real parameters of the model. Moreover, we use a self-interacting scalar field, described by the
potential,
\[
\mathcal{V}(\phi) = \mathcal{V}_0 + \beta_0 \left( \frac{\phi}{2} \right)^4 + \frac{2\beta_1}{3} \left( \frac{\phi}{2} \right)^6 + \frac{\beta_2}{4} \left( \frac{\phi}{2} \right)^8,
\]
(4)
being \( \mathcal{V}_0 = \mathcal{V}(0) \), whereas \( \beta_0, \beta_1 \), and \( \beta_2 \) are real parameters. This power-law potential has interesting applications in cosmology [28–31].

This paper is structured as follows: In Section II we derive the field equations for the GR-NLED-SF, in Section III we present the canonical metric of the traversable wormhole spacetime and discuss the Ellis wormhole solution, in Section IV we construct our novel traversable wormhole solution, and study some of its properties. In the last section final conclusions are presented. Through this paper we will use the system of units where \( G = k_B = c = \hbar = 1 \), and the metric signature \( (-+++). \)

II. EINSTEIN-NONLINEAR ELECTRODYNAMICS GRAVITY COUPLED TO A SELF-INTERACTING SCALAR FIELD

The GR-NLED-SF theory is defined by the following action,
\[
S[g_{ab}, \phi, A_c] = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - 2 \mathcal{V}(\phi) \right) + \frac{1}{4\pi} \mathcal{L}(F) \right\},
\]
(5)
where \( R \) is the scalar curvature, \( \phi \) is a scalar field coupled to gravity, and \( \mathcal{V} = \mathcal{V}(\phi) \) is the scalar potential; whereas \( \mathcal{L} = \mathcal{L}(F) \) is a function of the electromagnetic invariants \( F \equiv F_{\alpha\beta}F^{\alpha\beta} \), being \( F_{ab} = 2\partial_{[a}A_{b]} \) the components of the electromagnetic field tensor \( F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta \) and \( A_a \) are the components of the electromagnetic potential.

Using the notation \( \mathcal{L}_F \equiv \frac{d\mathcal{L}}{dF} \) and \( \dot{\mathcal{V}} = \frac{d\mathcal{V}}{d\phi} \), the GR-NLED-SF field equations arising from (5) takes the form,
\[
G_{\alpha\beta} = 8\pi(E_{\alpha\beta})_{SF} + 8\pi(E_{\alpha\beta})_{NLED}, \quad \nabla_\mu (\mathcal{L}_F F^{\mu\nu}) = 0 = dF, \quad \nabla^2 \phi = 2 \dot{\mathcal{V}},
\]
(6)
where, \( G_{\alpha\beta} = R_{\alpha\beta} - \frac{4}{3} \delta_{\alpha\beta} \) are the components of the Einstein tensor, \( (E_{\alpha\beta})_{SF} \) are the components of the energy-momentum tensor of self-interacting scalar field,
\[
8\pi(E_{\alpha\beta})_{SF} = -\frac{1}{4}(\partial_\mu \phi \partial^\mu \phi)\delta_{\alpha\beta} + \frac{1}{2}\partial_\alpha \phi \partial^\beta \phi - \mathcal{V}\delta_{\alpha\beta},
\]
(7)
and \( (E_{\alpha\beta})_{NLED} \) are the components of the NLED energy-momentum tensor,
\[
8\pi(E_{\alpha\beta})_{NLED} = 2\mathcal{L}_F F_{\alpha\mu}F^{\beta\mu} - 2\mathcal{L}\delta_{\alpha\beta}.
\]
(8)

Our aim is to find a static, spherically symmetric, and asymptotically flat, charged wormhole solution for the set Eqs. (9) with a non-trivial scalar field. To do this, we assume the scalar field is static and spherically symmetric, \( \phi = \phi(r) \), and also the metric has the static and spherically symmetric form,
\[
ds^2 = -e^{A(r)} dt^2 + e^{B(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]
(9)
with \( A = A(r) \) and \( B = B(r) \) unknown functions to be determined.

In terms of \( A \) and \( B \), the non-vanishing components of the Einstein tensor are,
\[
G^t_t = \frac{e^{-B}}{r^2} (-rB' - e^B + 1), \quad G^r_r = \frac{e^{-B}}{r^2} (rA' - e^B + 1), \quad G^\theta_\theta = G^\varphi_\varphi = \frac{e^{-B}}{4r^2} (rA'^2 - rA'B' + 2rA'' + 2A' - 2B'),
\]
where \( ' \) denotes the derivative respect to the radial coordinate \( r \), i.e. \( A' = \frac{dA}{dr} \). Whereas, for the non-trivial components of the energy-momentum tensor of self-interacting scalar field we have,
\[
8(E^t_t)_{SS} = 8\pi (E^\theta_\theta)_{SS} = 8\pi (E^\varphi_\varphi)_{SS} = -\frac{1}{4} e^{-B} \phi'^2 - \mathcal{W}, \quad 8\pi (E^r_r)_{SS} = -\frac{1}{4} e^{-B} \phi'^2 - \mathcal{W}.
\]

Regarding the electromagnetic field tensor, since the spacetime is SSS, we can restrict ourselves to purely magnetic field, i.e., \( \mathcal{E} = 0 \) and \( \mathcal{B} \neq 0 \).

With this restriction, the electromagnetic field tensor has the form, \( F_{\alpha\beta} = \mathcal{B} \left( \delta^\alpha_\theta \delta^\beta_\varphi - \delta^\alpha_\varphi \delta^\beta_\theta \right) \). In this way, for a static and spherically symmetric spacetime with line element [1], the general solution of the equations \( \nabla_\mu (\mathcal{L}_F F^{\mu\nu}) = 0 \) is,
\[
F_{\theta\varphi} = r^4 Q(r) \sin \theta.
\]

This means \( F = r^4 Q(r) \sin \theta \, d\theta \wedge d\varphi \), and therefore \( dF = 0 = (r^4 Q(r))' \sin \theta \, dr \wedge d\theta \wedge d\varphi \). This implies \( Q(r) = \sqrt{2} q/r^4 \), where \( \sqrt{2} q \) is an integration constant, which plays the role of the magnetic charge. Hence, the components of the electromagnetic field tensor \( F_{\alpha\beta} \) and the invariant \( \mathcal{F} \) are given by;
\[
F_{\alpha\beta} = \sqrt{2} q \sin \theta \left( \delta^\alpha_\theta \delta^\beta_\varphi - \delta^\alpha_\varphi \delta^\beta_\theta \right), \quad \mathcal{F} = \frac{q^2}{r^4}.
\]

Finally, the energy-momentum tensor components for NLED, assuming the SSS spacetime with metric [1], the purely magnetic field [13], and a generic Lagrangian density \( \mathcal{L}(F) \), are written as
\[
8\pi (E^t_t)_{NLED} = 8\pi (E^r_r)_{NLED} = -2\mathcal{L}, \quad 8\pi (E^\theta_\theta)_{NLED} = 8\pi (E^\varphi_\varphi)_{NLED} = 2(2\mathcal{F} \mathcal{L}_F - \mathcal{L}).
\]

Inserting the above components in the field equations \( C^\alpha_\beta = G^\alpha_\beta - 8\pi \left[ (\alpha^\alpha)^{SS} + (\alpha^\alpha)^{NLED} \right] \), we obtain the GR-NLED-SF field equations for the metric ansatz [10] and the magnetic field [13]:
\[
C^t_t = 0 \implies \frac{e^{-B}}{r^2} (-rB' - e^B + 1) + \frac{1}{4} e^{-B} \phi'^2 + \mathcal{W} + 2\mathcal{L} = 0,
\]
\[
C^r_r = 0 \implies \frac{e^{-B}}{r^2} (rA' - e^B + 1) - \frac{1}{4} e^{-B} \phi'^2 + \mathcal{W} + 2\mathcal{L} = 0,
\]
\[
C^\theta_\theta = C^\varphi_\varphi = 0 \implies \frac{e^{-B}}{4r^2} (rA'^2 - rA'B' + 2rA'' + 2A' - 2B') + \frac{1}{4} e^{-B} \phi'^2 + \mathcal{W} - 2(2\mathcal{F} \mathcal{L}_F - \mathcal{L}) = 0.
\]

whereas the scalar field must satisfies,
\[
2r \phi'' + (4 + rA' - rB') \phi' - 4re^B \phi = 0.
\]

This ends the general treatment of the static, spherically symmetric and pure magnetic solutions. In what follows we will discuss the general properties of the wormhole spacetime solution we have obtained.
III. THE CANONICAL METRIC OF A WORMHOLE SPACETIME AND TRAVERSABILITY

The canonical metric of a (3+1)-dimensional SSS-WH solution \[3, 4\] is given by,

\[\text{ds}^2_{\text{WH}} = -e^{2\Phi(r)} \text{d}t^2 + \frac{\text{d}r^2}{1 - \frac{b(r)}{r}} + r^2 \text{d}\Omega^2,\]

(19)

where \(\Phi(r)\) and \(b(r)\) are smooth functions, known as redshift and shape functions respectively. The domain for radial coordinate \(r\) has a minimum at \(r = r_0\), where the WH throat is defined by \(b(r_0) = r_0\), and is unbounded for \(r > r_0\).

This coordinate has a special geometric interpretation, as \(4\pi r^2\) is the area of a sphere centered on the WH throat.

On the other hand, for the WH to be traversable, one must demand:

\textbf{Wormhole domain:} \[1 - \frac{b(r)}{r} > 0 \quad \forall r > r_0\] (20)

\textbf{Absence of horizons:} \[e^{2\Phi(r)} \in \mathbb{R}^+ - \{0\} \quad \forall r \geq r_0, \quad \text{and} \quad \Phi(r \to \infty) = 0\] (21)

\textbf{Flare-out condition:} \[b'(r)\big|_{r = r_0} < 1\] (22)

with \('\) denoting derivative with respect to \(r\), are satisfied (see \[3, 4\] for details).

\textbf{Traversability and violation of Null Energy Condition.}

Let us consider the null vector \(n = (e^{-\Phi(r)}, \sqrt{1 - b(r)/r}, 0, 0)\), identify \(e^A(r) = e^{2\Phi(r)}\) and \(e^B(r) = \frac{1}{1 - \frac{b(r)}{r}}\) in the spacetime \[19\]. Using \[10\] and assuming the flaring out condition is fulfilled, after contracting the Einstein tensor with \(n\) and evaluating at \(r = r_0\), yields:

\[G_{\alpha\beta}n^\alpha n^\beta\big|_{r = r_0} = (G_r r - G_t t)\big|_{r = r_0} = \frac{1}{r_0^2} [b'(r_0) - 1] < 0.\] (23)

Thus, in GR, \(G_{\alpha\beta} = \kappa T_{\alpha\beta}\), from the balance between the matter and the curvature quantities, the fulfillment of the flaring out condition implies that the NEC (which states that \(T_{\alpha\beta}n^\alpha n^\beta \geq 0\) for any null vector \(n^\alpha\)) is violated, therefore, the presence of exotic matter is unavoidable for having a T-WH in GR.

A. Ultra-static spherically symmetric and asymptotically flat solution in GR-NLED-massless scalar field theory: The Ellis wormhole

A spacetime is called ultra-static if it admits an atlas of charts in which the metric tensor takes the form,

\[\text{ds}^2 = -\text{d}t^2 + g_{ab}\text{d}x^a\text{d}x^b,\]

(24)

where the metric coefficients \(g_{ab}\) are independent of the time coordinate \(t\), and in where the Latin indices running over the spatial coordinates only. These spacetimes have interesting properties \[32–34\]. Setting \(A(r) = 0\) in \[11\] one arrive to the canonical metric for the ultra-static spherical symmetric spacetime.

Now, if we consider GR-NLED-SF with \(\mathcal{V}(\phi) = 0\) and with \(\mathcal{E} \neq 0\) and \(\mathcal{B} \neq 0\) see Appendix \[C\] the equation of motion for the scalar field \[13\] yields,

\[2r\phi'' + (4 - rB')\phi' = 0 \quad \Rightarrow \quad \phi'^2 = \frac{\gamma e^B}{r^4} \quad \text{being} \quad \gamma = \text{constant} \in \mathbb{R}.\] (25)
Subtracting (C5) from (C4), yields,

$$\phi'^2 = \frac{2B'}{r}. \tag{26}$$

Now, equating (25) with (26), gives,

$$\frac{\gamma e^B}{2r^3B'} = 1. \tag{27}$$

Identifying (19) with (19), yields

$$A = 2\Phi(r) = 0, \quad B = -\ln \left( 1 - \frac{b(r)}{r} \right)$$

and substituting them in (27) yields,

$$\frac{\gamma}{2r^2 [b' - \frac{B}{r}]} = 1. \tag{28}$$

Evaluating in the WH throat yields,

$$\frac{\gamma}{2r_0^2 [b'(r_0) - 1]} = 1 \quad \text{and} \quad b'(r_0) < 1 \quad \text{(flare-out condition)} \quad \Rightarrow \quad \gamma = -4q^2 < 0. \tag{29}$$

Replacing $\gamma$ in (28) and solving for $b$, gives

$$b = \tilde{q} r + \frac{q^2}{r} \quad \text{being} \quad \tilde{q} = \text{constant} \in \mathbb{R}. \tag{30}$$

Hence, in this case the metric (19) has the form,

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \tilde{q} - \frac{q^2}{r^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{31}$$

In order to this metric be asymptotically flat is necessary $\tilde{q} = 0$, and then we arrive to the ultra-static SS-AF metric given by,

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{q^2}{r^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{32}$$

Then, for this spacetime metric the solution of (18) is

$$\phi(r) = \phi_* + 2i \tan^{-1} \left( \sqrt{\frac{r^2 - q^2}{\tilde{q}^2}} \right), \tag{33}$$

where $\phi_*$ is an integration constant which can be fixed to zero without loss generality. By substituting (31) and (33) in (C4), (C5) and (C6) with $\mathcal{W}(\phi) = 0$, yields $F_{tr}^r F_{tr}^r \mathcal{L} = -\mathcal{L} = 0$ and $F_{\theta\phi} F^{\theta\phi} \mathcal{L} = -\mathcal{L} = 0$, these equations imply.

$$F_{tr}^r F_{tr}^r = F_{\theta\phi} F^{\theta\phi}, \tag{34}$$

On the other hand we can calculate the quantities $F_{tr}^r F_{tr}^r$ and $F_{\theta\phi} F^{\theta\phi}$, obtaining in the spacetime region $r > |q|,$

$$F_{tr}^r F_{tr}^r = g^{tt} g^{rr} (F_{tr})^2 = - \left( 1 - \frac{q^2}{r^2} \right) (F_{tr})^2 < 0 \quad \text{and} \quad F_{\theta\phi} F^{\theta\phi} = g^{\theta\theta} g^{\phi\phi} (F_{\theta\phi})^2 = \frac{(F_{\theta\phi})^2}{r^4 \sin^2 \theta} > 0. \tag{35}$$

Then, the equation (34) is valid only if $F_{tr} = F_{\theta\phi} = 0$, this means the electromagnetic field and the associated Lagrangian energy density $\mathcal{L}$ must be both zero.
The metric (32), originally introduced in [9], admits a T-WH interpretation since satisfies the properties (20)-(22), and is known as the Ellis wormhole metric.

Indeed, defining a new scalar field by
$$\psi = i\phi \quad \text{(phantom field)},$$
and using
$$U(\phi) = \mathcal{L}(\mathcal{F}) = 0 \quad \text{in the action (5),}$$
the wormhole metric (32) becomes a solution to the theory with gravitational action,
$$S[g_{ab}, \psi] = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi}\left(R + \frac{1}{2}\partial_\mu\psi\partial^\mu\psi\right)\right\}, \quad (36)$$
with $\psi$ given by,
$$\psi = 2\tan^{-1}\left(\sqrt{\frac{r^2 - q^2}{q^2}}\right) \in \mathbb{R}. \quad (37)$$
This is the action that was used by Ellis in Ref. [9] to get the ultra-static wormhole solution (32).

### IV. NEW T-WH SOLUTION: A NON-ULTRA-STATIC ELLIS WORMHOLE

In this section we will study the NLED-SF theory defined by NLED model and a scalar potential, given respectively by,
$$\mathcal{L}(\mathcal{F}) = -\frac{1}{2}\mathcal{F} + \mu_0\mathcal{F}^2 + \mu_1|\mathcal{F}|^\frac{3}{2}, \quad (38)$$
$$\mathcal{U}(\phi) = U_0 + \beta_0\left(\frac{\phi^4}{2}\right) + \frac{2\beta_1}{3}\left(\frac{\phi^6}{2}\right) + \frac{\beta_2}{4}\left(\frac{\phi^8}{2}\right), \quad (39)$$

admits the following metric,
$$ds^2 = -e^{-\frac{q^2}{r^2}}dt^2 + \frac{dr^2}{1 - \frac{q^2}{r^2}} + r^2d\Omega^2 \quad (40)$$
for $\mu_0 = -q^2/8$, $\mu_1 = 2|q|/3$, $U_0 = -1/(12q^2)$, $\beta_0 = \beta_1 = \beta_2 = 1/q^2$, together with the scalar field,
$$\phi(r) = 2i\sqrt{1 - \frac{q^2}{r^2}} \quad (41)$$
as a pure magnetic exact solution of the GR-NLED-SF field equations (14)-(18). The metric (40) admits a T-WH interpretation since satisfies the properties (20)-(22), because this metric has the form (32) but with non vanishing redshift function given by $\Phi(r) = -\frac{q^2}{r^2}$, can be interpreted as a non-trivial redshift modification of the Ellis WH metric, with WH throat at $r = r_0 = |q|$. In other words, the line element (40) is not of type (24), therefore it describes a non-ultra-static spherically symmetric asymptotically flat traversable wormhole.

The absence of curvature singularities in the WH domain, $r \geq |q|$, can be deduced from the analytical expressions of its curvature invariants,
$$R = \frac{2q^4}{r^8},$$
$$R_{\alpha\beta}R^{\alpha\beta} = \frac{2q^4}{r^{16}}\left(q^8 - 8q^6r^2 + 20q^4r^4 - 10q^2r^6 + 2r^8\right), \quad (42)$$
$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{4q^4}{r^{16}}\left[q^8 - 10q^6r^2 + 33q^4r^4 - 34q^2r^6 + 14r^8\right], \quad (43)$$
which are all regular in the whole WH domain \( r \geq |q| \).

Here, it is important to emphasise that the Lagrangian density reduces to Maxwell theory in the limit of weak field, i.e. \( \mathcal{L} \to \kappa \mathcal{F} \) and \( \mathcal{L}_\phi \to \kappa \) (being \( \kappa \) a constant) as \( \mathcal{F} \to 0 \).

Besides the Maxwell limit condition, another important physical requirement is desired for the electromagnetic Lagrangian to fulfill, it is the WEC. We can guarantee the validity of WEC in a limited region of the spacetime determined by,

\[
\frac{92 - 32\sqrt{7}}{9} \leq q^2 \mathcal{F} \leq 1 \quad \Rightarrow \quad \frac{r}{|q|} \in \left[ 1, \left( \frac{9}{92 - 32\sqrt{7}} \right)^{\frac{1}{4}} \right] \tag{44}
\]

which includes the wormhole throat \( r = r_0 = |q| \). See Appendix B for details.

### A. Behavior of null geodesics and capture cross-section for light

Now, we study the behaviour of the null geodesics in the T-WH geometry, using as a guide the equivalent problem of a particle in a potential well. We work with the metric written in terms of a new radial coordinate defined by \( \rho = \pm \sqrt{r^2 - q^2} \), where the plus (minus) sign is related to the upper (lower) part of the wormhole. According to [35], the geodesic motion of a test particle in this geometry is described by the following Lagrangian density,

\[
2\mathcal{L} = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -e^{-\frac{q^2}{\rho^2 + q^2}} \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{d\rho}{d\lambda} \right)^2 + (\rho^2 + q^2) \left[ \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\lambda} \right)^2 \right], \tag{45}
\]

where \( \lambda \) represents an affine parameter of the geodesic. The equations of motion for the test particle can be derived from the Euler-Lagrange equations, \( \frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial (dx^\alpha/d\lambda)} = 0 \) where \( p^\alpha = \frac{dx^\alpha}{d\lambda} \). Additionally, for geodesic motion of photons, the Lagrangian has to fulfill the condition \( \mathcal{L}(x^\alpha, p^\alpha) = 0 \). The Lagrangian density (45) does not depend explicitly on the variables \( t \) and \( \varphi \), then, there exists two conserved quantities associated to them: \( E = \frac{\partial \mathcal{L}}{\partial (d\lambda/dt)} = -e^{-\frac{q^2}{\rho^2 + q^2}} \frac{dt}{d\lambda} \), and \( \ell = \frac{\partial \mathcal{L}}{\partial (d\lambda/d\varphi)} = (\rho^2 + q^2) \sin^2 \theta \frac{d\varphi}{d\lambda} \), we can call them the energy and the angular momentum, respectively. To study the motion of test particles in the spacetime geometry, it is convenient to use the fact that the geodesic motion is always confined to a single plane, because the spherical symmetry. Without loss of generality we will restrict ourselves to the study of equatorial trajectories in the \( \theta = \pi/2 \) plane.

With this choice, the equation of motion for photons reduces to,

\[
\left( \frac{d\rho}{d\lambda} \right)^2 + \frac{\ell^2}{\rho^2 + q^2} - \left[ e^{-\frac{q^2}{\rho^2 + q^2}} - 1 \right] E^2 = E^2, \tag{46}
\]

which can be written as \( \left( \frac{d\rho}{d\lambda} \right)^2 + \mathcal{V}_{\text{eff}}(\rho) = E^2 \), with the effective potential \( \mathcal{V}_{\text{eff}}(\rho) \) given by,

\[
\mathcal{V}_{\text{eff}}(\rho) = \frac{\ell^2}{\rho^2 + q^2} + \left( 1 - e^{-\frac{q^2}{\rho^2 + q^2}} \right) E^2. \tag{47}
\]
The last potential goes to zero as \( \rho \to \pm \infty \), and we can verify it has three extreme points \( \rho_e = \{0, \rho_\pm\} \), such that

\[
\frac{dV_{\text{eff}}}{d\rho} \bigg|_{\rho = \rho_e} = 0,
\]

with \( \rho_\pm \) given by,

\[
\rho_\pm = \pm \sqrt{\ln \left( \frac{e^2 E^2}{\ell^2} \right)} |q| \quad \text{being} \quad \ln(c) = 1.
\]  

(48)

However, \( \rho_\pm \) become real only if,

\[
E^2 = \frac{\ell^2}{nq^2} \quad \text{where} \quad n \in (1, e).
\]  

(49)

On the other hand, the images of the extreme points of the effective potential are,

\[
V_{\text{eff}}(0) = \frac{\ell^2}{q^2} + (1 - e) E^2, \quad V_{\text{eff}}(\rho_\pm) = \frac{\ell^2}{q^2} \left( \frac{q^2 E^2}{\ell^2} - \ln \left( \frac{e^2 q^2 E^2}{\ell^2} \right) \right).
\]  

(50)

In order to determine if \( \rho_e = \{0, \rho_\pm\} \) are minimum or maximum of the effective potential, we must study the behavior of the signs of \( \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = \rho_e} \), finding,

\[
\frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = 0} = \frac{2\ell^2}{q^4} \left( \frac{e^2 q^2 E^2}{\ell^2} - 1 \right)
\]  

(51)

\[
\frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = \rho_\pm} = -4\ell^2 \frac{q^2}{q^4} \ln \left( \frac{e^2 q^2 E^2}{\ell^2} \right) \ln \left( \frac{\ell^2}{q^2 E^2} \right).
\]  

(52)

Depending on the energy of the photon, we have three cases for the values of \( E^2 \):

i) If \( E^2 = \frac{\ell^2}{nq^2} \) where \( n \in (0, 1] \), according to (49) the only real critical point is \( \rho_e = 0 \), which from (51) yields \( \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = 0} > 0 \), implying \( V_{\text{eff}}(0) \) is a minimum value of the potential. This, together with the fact \( V_{\text{eff}}(\rho) \to 0 \) as \( \rho \to \pm \infty \) imply the potential is negative everywhere. For the photon energy we are dealing with the relation \( E^2 > V_{\text{eff}}(\rho) \) holds, this means all of them can pass above this effective potential. See Fig. 1.

ii) If \( E^2 = \frac{\ell^2}{nq^2} \) being \( n \in (1, e) \), according to (49) we have three real critical points in this case, because \( \rho_\pm \in \mathbb{R} \). From (52) we have \( \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = \rho_\pm} < 0 \) then \( V_{\text{eff}}(\rho_\pm) \) are local maximum values of the potential, i.e. \( V_{\text{eff}}(\rho_\pm) = V_{\text{eff}}^{\text{Max}} \).

On the other hand from (51) we have \( \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho = 0} > 0 \), implying \( V_{\text{eff}}(0) \) is a local minimum value of the potential.

By looking at the asymptotic behavior \( V_{\text{eff}}(\rho) \to 0 \) as \( \rho \to \pm \infty \), we can conclude \( V_{\text{eff}}^{\text{Max}} = V_{\text{eff}}(\rho_\pm) \) is the global maximum value of the potential, whereas \( V_{\text{eff}}(0) \) is the global minimum value. Now, using (50), for \( \rho_\pm \) with \( E^2 = \frac{\ell^2}{nq^2} \) and \( n \in (1, e) \), we have,

\[
V_{\text{eff}}^{\text{Max}} = V_{\text{eff}}(\rho_\pm) = \frac{\ell^2}{nq^2} - \frac{\ell^2}{q^2} \ln \left( \frac{e}{n} \right) \quad \Rightarrow \quad E^2 = V_{\text{eff}}^{\text{Max}} + \frac{\ell^2}{q^2} \ln \left( \frac{e}{n} \right).
\]  

(53)

From the last relation we conclude that for this photon \( E^2 > V_{\text{eff}}^{\text{Max}} \) because \( \frac{\ell^2}{q^2} \ln \left( \frac{e}{n} \right) \in \mathbb{R}^+ \), and it always can pass above this effective potential. See Fig. 2.
FIG. 1: Effective potential for a massless test particle with $E^2 \in (0, 1]$, in the spacetime geometry (40) with
$$\rho = \pm \sqrt{r^2 - q^2}.$$ The ordinate is $V_{\text{eff}} = \frac{\ell^2 q^2}{E^2} \left( \frac{\rho^2}{q^2} + 1 \right)^{-1} + 1 - e^{\left(\frac{\rho^2}{q^2}+1\right)^{-1}}$; the abscissa is $\rho/|q|$.

FIG. 2: Effective potential for a massless test particle with $E^2 \in (1, e)$, in the spacetime geometry (40) with
$$\rho = \pm \sqrt{r^2 - q^2}.$$ The ordinate is $V_{\text{eff}} = \frac{\ell^2 q^2}{E^2} \left( \frac{\rho^2}{q^2} + 1 \right)^{-1} + 1 - e^{\left(\frac{\rho^2}{q^2}+1\right)^{-1}}$; the abscissa is $\rho/|q|$.
iii) Finally, if \( E^2 = \frac{\ell^2}{n q^2} \) where \( n \in [e, \infty) \), according to (51) we have 
\[
\left| \frac{d^2 V_{\text{eff}}}{d \rho^2} \right|_{\rho=0} = \frac{2e^2}{\ell^2} \left( \frac{e}{n} - 1 \right) \leq 0 
\]
this implies the potential has a local maximum located at \( \rho = 0 \). Moreover, since \( V_{\text{eff}}(\rho) \to 0 \) as \( \rho \to \pm \infty \) we conclude that \( V_{\text{eff}}(0) = V_{\text{eff}}^{\text{Max}} \) is the maximum value of the effective potential. Now, using (50), for \( \rho = 0 \), with \( E^2 = \frac{\ell^2}{n q^2} \), we obtain,
\[
V_{\text{eff}}^{\text{Max}} = V_{\text{eff}}(0) = \frac{\ell^2}{q^2} + (1 - e)E^2 = (1 + n - e)E^2 \quad \Rightarrow \quad E^2 = \frac{V_{\text{eff}}^{\text{Max}}}{1 + n - e} \leq V_{\text{eff}}^{\text{Max}}. \tag{54}
\]

The equation \( E^2 = V_{\text{eff}}^{\text{Max}} \) can only be satisfied for photons with \( E^2 = \frac{\ell^2}{cq^2} \), this corresponds to an unstable circular orbit. Whereas, photons with \( \frac{e^2}{q^2 E^2} \in (e, \infty) \) cannot pass through this effective potential because for them \( E^2 < V_{\text{eff}}^{\text{Max}} \). See Fig. 3.

**Photon sphere:** We have shown that in the spacetime geometry \( \text{(10)} \) is possible that some photons to follow circular orbits. Specifically, according to (54), a photon with \( E^2 = \frac{\ell^2}{cq^2} \) feel a potential so that \( E^2 = V_{\text{eff}}(0) = V_{\text{eff}}^{\text{Max}} = \frac{\ell^2}{cq^2} \), which implies that this could follow a circular geodesic. This orbit is called the photon circle (for details see [37]). Due to the spherical symmetry, the condition \( E^2 = V_{\text{eff}}^{\text{Max}} \) defines a collection of infinitely many such orbits, therefore the last photon orbit is also called photon sphere [37]. According to [38], the impact parameter \( b \) will be \( b = \ell / \sqrt{V_{\text{eff}}^{\text{Max}}} = \ldots \).
**Fig. 4:** Effective potential curves for several values of $n = \frac{\ell^2}{q^2}$, horizontal lines represent important values of the parameter $n$.

$\ell/|E| = e^{1/2}|q|$, and hence the capture cross-section for a light beam is $\sigma = \pi b^2 = e \pi q^2$, or in terms of the WH throat, $r = r_0 = |q|$, this becomes $\sigma = e \pi r_0^2$. The photon sphere can cast a wormhole shadow for an observer at infinity. This shadow is a disk specified by its radius $r_{sh}$ and it gives an apparent size and shape of WH throat. For a static spherically symmetric asymptotically flat wormhole $r_{sh}$ is just the impact parameter $b$. So for example, for the Ellis WH $r_{sh} = |q|$ which in terms of the Ellis WH throat $r_0 = |q|$, becomes $r_{sh} = r_0$, see [12] for details. In our case for the WH geometry (40) we get $r_{sh} = e^{1/2}|q| = e^{1/2}r_0$ which is bigger that of the Ellis WH.

**Asymptotically behavior of the metric (40) at infinity:** Expanding the metric in powers of $q/r$ around $r \to \infty$ give us

$$- g_{tt} = e^{-\frac{q^2}{r^2}} = 1 - \frac{q^2}{r^2} + \mathcal{O}\left(\frac{q^4}{r^4}\right). \quad (55)$$

This allow us to write the line element as,

$$ds^2 = - \left(1 - \frac{q^2}{r^2} + \mathcal{O}\left(\frac{q^4}{r^4}\right)\right) dt^2 + \frac{dr^2}{1 - \frac{q^2}{r^2}} + r^2 d\Omega^2, \quad (56)$$

which behaves asymptotically as the exterior region of the Reissner-Nordström black hole: $-g_{tt} = 1/g_{rr} = 1 - 2M_{\text{ADM}}/r + Q^2/r^2$, without Arnowitt-Deser-Misner (ADM) mass ($M_{\text{ADM}} = 0$) and with imaginary charge ($Q = iq$). The latter is a immediate consequence of the violation of the null and weak energy conditions in the region of weak field $\mathcal{F} \to 0 \equiv r/|q| \to \infty$ by the NLED Lagrangian density [38].

**Phantom scalar field:** Defining a new scalar field by $\psi = i\phi$ (phantom field), the theory for which the metric (40) is a pure magnetic exact solution, arises from the action,

$$S[g_{ab}, \phi, \partial_c] = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left(R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - 2\mathcal{L}(\psi)\right) + \frac{1}{4\pi} \mathcal{L}(\mathcal{F}) \right\}, \quad (57)$$
with nonlinear electromagnetic field described by (38) and scalar field potential,
\[ \mathcal{U}(\psi) = -\frac{\beta_0}{48} (3\psi^2 + 4) \left( 1 - \frac{\psi^2}{4} \right)^3, \] (58)
with \( \psi \) given by,
\[ \psi(r) = 2\sqrt{1 - \frac{q^2}{r^2}} \in \mathbb{R} \] (59)

V. CONCLUSIONS

In this work, in the Einstein’s gravity context, we have constructed a new SSS-AFT-WH solution which can be interpreted as a non-trivial redshift modification of the Ellis WH. The sources are; a self-interacting phantom scalar field, which was introduced to satisfy the flare-out condition; and a nonlinear electrodynamics field which becomes the Maxwell theory in the limit of weak field. Moreover, this nonlinear electrodynamics satisfying the WEC in a limited region of the spacetime which contains the WH throat. Our T-WH metric is determined by only one parameter \( q \), which can be associated to the magnetic charged, and defines the WH throat as \( r_\text{w} = |q| \). Thus, in the limit of zero magnetic charge, the Minkowski metric is recovered. Moreover, we found that this solution has a WH shadow of radius \( r_{\text{sh}} = e^{1/2} r_\text{w} \) which is bigger than the shadow radius of the Ellis WH.

To a better characterization of the wormhole we have just presented is necessary the study of their quasinormal modes, its corresponding Penrose diagram and its stability. This last behavior is essential in order to guarantee the traversability of this wormhole. We hope to return to these issues in a future work.

Appendix A: Null and weak energy conditions in GR

For an energy-momentum tensor \( T_{\mu\nu} \), the null energy condition (NEC), stipulates that for every null vector, \( n^\alpha \), yields \( T_{\mu\nu} n^\mu n^\nu \geq 0 \). Following [36], for a diagonal energy-momentum tensor \( (T_{\alpha\beta}) = \text{diag} (T_{tt}, T_{rr}, T_{\theta\theta}, T_{\phi\phi}) \), which can be conveniently written as,
\[ T_{\alpha\beta} = -\rho_t \delta_\alpha^t \delta_\beta^t + P_r \delta_\alpha^r \delta_\beta^r + P_\theta \delta_\alpha^\theta \delta_\beta^\theta + P_\varphi \delta_\alpha^\varphi \delta_\beta^\varphi, \] (A1)
where \( \rho_t \) may be interpreted as the rest energy density of the matter, whereas \( P_r, P_\theta \) and \( P_\varphi \) are respectively the pressures along the \( r, \theta \) and \( \varphi \) directions. In terms of (A1) the NEC implies:
\[ \rho_t + P_a \geq 0 \quad \text{with} \quad a = \{r, \theta, \varphi\}. \] (A2)

The weak energy condition (WEC) states that for any timelike vector \( k = k^\mu \partial_\mu \), (i.e., \( k_\mu k^\mu < 0 \)), the energy-momentum tensor obeys the inequality \( T_{\mu\nu} k^\mu k^\nu \geq 0 \), which means that the local energy density \( \rho_{\text{loc}} = T_{\mu\nu} k^\mu k^\nu \) as measured by any observer with timelike vector \( k \) is a non-negative quantity. For an energy-momentum tensor of the form (A1), the WEC will be satisfied if and only if,
\[ \rho_t = -T_t^t \geq 0, \quad \rho_t + P_a \geq 0 \quad \text{with} \quad a = \{r, \theta, \varphi\}. \] (A3)
• NEC and WEC for a self-interacting scalar field \((E^\alpha_\beta)_{SF}\)

Identifying (7) with (A1), and using (11), yields,

\[
8\pi (\rho_t)_{SF} = -8\pi (P_\theta)_{SF} = -8\pi (P_\varphi)_{SF} = \frac{1}{4} e^{-B \phi'^2} + \mathcal{U}, \quad (A4)
\]

\[
8\pi (P_r)_{SF} = \frac{1}{4} e^{-2 B \phi'^2} - \mathcal{U} \quad (A5)
\]

since \((\rho_t)_{SF} + (P_a)_{SF} = 0\) for all \(a = \theta, \varphi\), the tensor \((E^\alpha_\beta)_{SF}\) satisfies the NEC if

\[
8\pi (\rho_t)_{SF} + 8\pi (P_r)_{SF} = \frac{1}{2} e^{-B \phi'^2} \geq 0. \quad (A6)
\]

In addition to (A6) if,

\[
8\pi (\rho_t)_{SF} = \frac{1}{2} e^{-B \phi'^2} + \mathcal{U} \geq 0, \quad (A7)
\]

holds, the WEC is satisfied. We can see the WEC is more restrictive than the NEC, this is the reason we only use WEC in our work.

• NEC and WEC for the nonlinear electrodynamics field \((E^\alpha_\beta)_{NLED}\); pure-magnetic case

By using (8) and (14),

\[
8\pi (\rho_t)_{NLED} = -8\pi (P_\theta)_{NLED} = 2\mathcal{L}, \quad 8\pi (P_\theta)_{NLED} = 8\pi (P_\varphi)_{NLED} = 2(2\mathcal{F}\mathcal{L}_F - \mathcal{L}). \quad (A8)
\]

since \(\rho_t + P_\tau = 0\), the tensor \((E^\alpha_\beta)_{NLED}\) satisfies the NEC if

\[
8\pi (\rho_t)_{NLED} + 8\pi (P_\theta)_{NLED} = 8\pi (\rho_t)_{NLED} + 8\pi (\rho_\varphi)_{NLED} = 4\mathcal{F}\mathcal{L}_F \geq 0. \quad (A9)
\]

In addition to (A9) if,

\[
8\pi (\rho_t)_{NLED} = 2\mathcal{L} \geq 0, \quad (A10)
\]

the WEC is satisfied.

• NEC and WEC for the effective energy-momentum tensor \((E^\alpha_\beta)_{eff} = (E^\alpha_\beta)_{SF} + (E^\alpha_\beta)_{NLED}\)

\[
8\pi (\rho_t)_{eff} = \frac{1}{4} e^{-B \phi'^2} + \mathcal{U} + 2\mathcal{L} \quad (A11)
\]

\[
8\pi (P_r)_{eff} = \frac{1}{4} e^{-2 B \phi'^2} - \mathcal{U} - 2\mathcal{L} \quad (A12)
\]

\[
8\pi (P_\theta)_{eff} = 8\pi (P_\varphi)_{eff} = -\frac{1}{4} e^{-B \phi'^2} - \mathcal{U} + 2(2\mathcal{F}\mathcal{L}_F - \mathcal{L}) \quad (A13)
\]

So, the tensor \((E^\alpha_\beta)_{eff}\) satisfies the NEC if,

\[
8\pi (\rho_t)_{eff} + 8\pi (P_r)_{eff} = \frac{1}{2} e^{-B \phi'^2} \geq 0, \quad (A14)
\]

\[
8\pi (\rho_t)_{eff} + 8\pi (P_\theta)_{eff} = 8\pi (\rho_t)_{eff} + 8\pi (P_\varphi)_{eff} = 4\mathcal{F}\mathcal{L}_F \geq 0, \quad (A15)
\]
whereas, in addition to (A14) and (A15), if,

\[ 8\pi(\rho_t)_{\text{eff}} = \frac{1}{4} e^{-B} \phi^4 + \mathcal{V} + 2\mathcal{L} \geq 0 \]  

the WEC is satisfied.

**Appendix B: Domain of validity of the WEC and NEC for the GR-NLED-SF model**

To find the region where the WEC and the NEC are valid, let’s notice the following:

\[ q^2 F = \frac{q^4}{r^4} \in (0, 1], \quad q^2 \mathcal{L} = \left( -\frac{1}{2} + \frac{2}{3}|q^2 F|^\frac{1}{2} - \frac{1}{8}(q^2 F) \right) q^2 F, \]  

then,

\[ q^2 \mathcal{L} \geq 0 \quad \text{only if} \quad \frac{92 - 32\sqrt{7}}{9} \leq q^2 F \leq \frac{92 + 32\sqrt{7}}{9}. \]  

Whereas,

\[ \mathcal{L}_F = -\frac{1}{2} + |q^2 F|^\frac{1}{2} - \frac{1}{4}(q^2 F). \]

hence,

\[ \mathcal{L}_F \geq 0 \quad \text{only if} \quad 6 - 3\sqrt{2} \leq q^2 F \leq 6 + \sqrt{32} \]

Thus, according to (A9), (A10) and given that for the pure-magnetic \( F \) is positive defined (B3), the NLED model holds the WEC (i.e. \( \mathcal{L} \geq 0 \) and \( \mathcal{L}_F \geq 0 \)) only if,

\[ \frac{92 - 32\sqrt{7}}{9} \leq q^2 F \leq 6 + \sqrt{32} \]  

However, since in the WH domain \( r \in [|q|, \infty) \), or in terms of the electromagnetic invariant \( q^2 F = \frac{q^4}{r^4} \in (0, 1] \), then in the wormhole spacetime (B4) the NLED holds the WEC only if,

\[ \frac{92 - 32\sqrt{7}}{9} \leq q^2 F \leq 1. \]

**Appendix C: Field equations of GR-NLED-SF theory**

In this appendix we include the more general form of the equations of motion for GR-NLED-SF theory that are satisfied by a SSS metric.

Since the spacetime is static and spherically symmetric, the more general form of the electromagnetic field tensor is given by,

\[ F_{\alpha\beta} = (\delta^i_{\alpha} \delta^j_{\beta} - \delta^i_{\alpha} \delta^j_{\beta}) F_{i\tau} + \left( \delta^i_{\alpha} \delta^j_{\beta} - \delta^i_{\alpha} \delta^j_{\beta} \right) F_{\theta \phi}. \]  

(C1)
Hence, for an arbitrary NLED Lagrangian density $\mathcal{L}(F)$, the non-vanishing components of the NLED energy-momentum tensor, assuming the SSS metric (9) and the more general SSS electromagnetic field tensor (C1), are given by,

$$8\pi (E^t_t)_{\text{NLED}} = 8\pi (E^r_r)_{\text{NLED}} = 2(F_{tr} F^{tr} \mathcal{L}_F - \mathcal{L}),$$  \hspace{1cm} (C2)

$$8\pi (E^\theta_\theta)_{\text{NLED}} = 8\pi (E^\phi_\phi)_{\text{NLED}} = 2(F_{\theta\phi} F^{\theta\phi} \mathcal{L}_F - \mathcal{L}).$$  \hspace{1cm} (C3)

Inserting the above components in the Einstein field equations $C^{\alpha\beta} = G^{\alpha\beta} - 8\pi [(E^{\alpha\beta})_{SF} + (E^{\alpha\beta})_{\text{NLED}}] = 0$, for the metric ansatz (9) and scalar field energy-momentum tensor (11), yield,

$$C^t_t = 0 \implies \frac{e^{-B}}{r^2} (-r B' - e^B + 1) + \frac{1}{4} e^{-B} \phi'^2 + \mathcal{U} - 2(F_{tr} F^{tr} \mathcal{L}_F - \mathcal{L}) = 0,$$  \hspace{1cm} (C4)

$$C^r_r = 0 \implies \frac{e^{-B}}{r^2} (r A' - e^B + 1) - \frac{1}{4} e^{-B} \phi'^2 + \mathcal{U} - 2(F_{tr} F^{tr} \mathcal{L}_F - \mathcal{L}) = 0,$$  \hspace{1cm} (C5)

$$C^\theta_\theta = C^\phi_\phi = 0 \implies \frac{e^{-B}}{4r} (r A'^2 - r A'B' + 2r A'' + 2A' - 2B') + \frac{1}{4} e^{-B} \phi'^2 + \mathcal{U} - 2(F_{\theta\phi} F^{\theta\phi} \mathcal{L}_F - \mathcal{L}) = 0.$$  \hspace{1cm} (C6)

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