Modified Holographic Ricci Dark energy Cosmological Models in $f(R, T)$ Gravity

K. V. S. Sireesha$^1$ & V. U. M. Rao$^2$

$^1$Dept. of Engineering Mathematics, GITAM (deemed to be) University, Visakhapatnam, India
$^2$Department of Applied Mathematics, Andhra University, Visakhapatnam, India

sireeshakuppili@gmail.com$^1$ & umrao57@hotmail.com$^2$

Abstract: We have investigated spatially homogeneous anisotropic Bianchi type-II, VIII & IX cosmological models in the presence of pressure less matter and modified holographic Ricci dark energy in the frame work of $f(R, T)$ gravity in the context of late time accelerating expansion of the universe. The solutions of the field equations obtained using (i) a relation between metric potentials and (ii) the modified holographic Ricci dark energy. We have determined the cosmological parameters, namely, EoS parameter, matter energy density, anisotropic dark energy density, deceleration parameter. We discuss some physical and geometrical properties of the obtained models which are found to be consistent with recent observation.

Keywords: Bianchi type - II, VIII & IX metrics, Modified Holographic Ricci dark energy, $f(R,T)$ gravity.

1. Introduction:

The accelerated expansion of our universe is established by the recent cosmological observations (Riess et al. [1] and Perlmutter et al. [2]). It is said that an exotic type of unknown force with
positive energy density and huge negative pressure known as “dark energy” (DE) is responsible for this cosmic acceleration. Although around 70% of the mass and energy content of our universe is occupied with this type of DE, its nature, even today, is an open problem in cosmology. Usually the behavior of DE phenomena is characterized with the equation of state (EoS) parameter $w = \frac{p}{\rho}$, where $p$ is the pressure and $\rho$ is the energy density. The EoS parameter $w$ lying in the range (-1, -1/3) represents the quintessence DE model, $w = -1$ describes the vacuum DE, commonly known as cosmological constant or $\Lambda CDM$ model and $w < -1$ describes the DE model known as the phantom model. This Phantom DE model can lead to a future unavoidable singularity of the space time.

The cosmological constant cold dark matter $\Lambda CDM$ model is the simplest cosmological model of DE in which vacuum energy plays the role of DE but it has issues like the coincidence and fine tuning problems. This motivated various authors to find some alternatives to describe the nature of DE. In this scenario, the matter part of the Einstein-Hilbert action is modified and one proposed various dynamical models such as families of scalar fields (which include quintessence, phantom, k-essence etc.) {B.Ratra & J.Peebles [3], A.Jawad & S.Chattopadhyay [4], R.J.Scherrer [5] & T.Chiba, Phys.Rev. [6]}, the Chaplygin gas model {M.C. Bento et al. [7]} and holographic DE models {A.G.Cohen et al. [8]}. Padmanabhan [9],Copeland et al. [10] and Bamba et al. [11] have presented a comprehensive review of DE models and modified theories of gravity. Some of the modified theories of gravity are Brans-Dicke [12] scalar tensor theory, $f(R)$ and $f(R,T)$ theories {S.Nojiri & S.D.Odintsov [13], S.Capozziello & M.De Laurentis [14] & T. Harko et al. [15]}. (where R is the curvature scalar and T is the trace of the energy momentum tensor), etc. Even after all these attempts, the origin, evolution and true nature of DE have not been convincingly explained yet.

Among the above mentioned candidates for DE, a very interesting model which is attracting more and more attention is the so-called “holographic dark energy (HDE)”. This is based on the holographic principle which states that the number of degrees freedom of a physical system should scale with its bounding area $L^2$ rather than with its volume (G. Hoof’t, [16]) and it should be constrained by an infrared (IR) cut off. Cohen et al. [8], motivated by this principle,
suggested that the vacuum energy density is proportional to the Hubble scale \( L \approx H^{-1} \). Li [17] has defined the energy density of holographic dark energy as

\[
\rho_\Lambda = 3c^2 M_{pl}^2 L^{-2},
\]

where \( L \) is the IR cut-off radius, \( c \) is a constant and

\[
M_{pl}^2 = \frac{1}{8\pi G}
\]

is the Planck mass. The IR cut-off has been considered as the Hubble radius or future event horizon. Later Gao et al. [18] have assumed that the future event horizon is replaced by the inverse of the Ricci scalar curvature, i.e., \( L = R^{-\frac{1}{2}} \). In this case the model is called Ricci dark energy model. Also Granda and Oliveros [19, 20] proposed a new holographic Ricci dark energy model with energy density given by

\[
\rho_\Lambda = \frac{3}{8\pi G} \left( \xi H^2 + \eta \dot{H} \right),
\]

and subsequently, Chen and Jing [21] have modified this model and named it as modified holographic Ricci dark energy (MHRDE) with energy density given by

\[
\rho_\Lambda = \frac{3}{8\pi G} \left( \xi H^2 + \eta \dot{H} + \zeta \ddot{H} H^{-1} \right),
\]

where an overhead dot indicates differentiation with respect to \( t \) and \( G \) is the gravitational constant which is taken as a function of time. The time variation of the gravitational constant \( G \) is the natural consequence of Dirac’s “Large Number Hypothesis” (Dirac et al. [22]). Inspired by this hypothesis, many attempts have been made to obtain physical and cosmological results about the universe. Further, it was proven through various observations that \( G \) could be a function of time {Umezu et al. [23], S. Nesseris & L. Perivolaropoulos [24] & M. Biesiada, B. Malec [25]}. Setare [26, 27] and Jamil et al. [28] have investigated interacting HDE models with or without varying \( G \) in order to explain the current status of the universe. Sharif and Jawad [29] have explored interacting modified HDE in the Kaluza-Klein universe. Recently, many authors in the literature have investigated HDE and MHRDE cosmological models within the framework of Bianchi space-times using the constant deceleration parameter and the time varying
deceleration parameter \{Santhi et al. [30] \& Reddy et al. [31]\}. Jawad et al. [32] have discussed MHRDE in Chameleon BD cosmology with non-minimally matter coupling of the scalar field and its thermodynamic consequence. Rao and Prasanthi [33] have explored some Bianchi-type MHRDE models in Saez-Ballester scalar-tensor theory of gravity with a variable deceleration parameter. Reddy [34] has discussed the Bianchi-type-V MHRDE model of the universe in Saez-Ballester scalar-tensor theory of gravitation for different dynamical average scale factors. Naidu et al. [35] have obtained Dynamics of axially symmetric anisotropic modified holographic Ricci dark energy model in Brans-Dicke theory of gravitation. Aditya and Reddy [36] have investigated FRW type Kaluza-Klein MHRDE models in Brans-Dicke theory of gravitation. Singh and Ajay Kumar [37] have discussed Ricci dark energy model with bulk viscosity.

In \( f(R,T) \) gravity, the field equations are obtained from the Hilbert-Einstein type variation principle. The action principle for this modified theory of gravity is given by

\[
S = \frac{1}{16\pi G} \int f(R,T)\sqrt{-g} d^4x + \int L_m\sqrt{-g} d^4x
\]  
(1.5)

where \( L_m \) represent matter Lagrangian density. Here gravity Lagrangian \( f(R,T) \) admitting minimal coupling only with \( L_m \) (Harko et al. [14]). Here \( T \) is the trace of the energy-momentum tensor of matter and \( R \) is the Ricci scalar.

The stress energy momentum tensor of matter is

\[
T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\partial (\sqrt{-g})}{\partial g^{ij}} L_m, \Theta_{ij} = -2T_{ij} - pg_{ij},
\]  
(1.6)

Using gravitational units (by taking \( G \& c \) as unity) the corresponding field equations of \( f(R,T) \) gravity are obtained by varying the action principle (1.1) with respect to \( g_{ij} \) as

\[
f_R(R,T)R_{ij} - \frac{1}{2} f(R,T) g_{ij} + (g_{ij} \nabla^\kappa \nabla_\kappa - \nabla_i \nabla_j) f_R(R,T) =
8\pi T_{ij} - f_{,T}(R,T)T_{ij} - f_T(R,T)\Theta_{ij}
\]  
(1.7)
where \( f_r = \frac{\partial f(R,T)}{\partial R}, \ f_T = \frac{\partial f(R,T)}{\partial T} \) & \( \Theta_{ij} = g^{\alpha\beta} \frac{\partial T_{\alpha\beta}}{\partial g_{ij}} \)

Here \( \nabla_i \) is the covariant derivative and \( T_{ij} \) is usual matter energy-momentum tensor derived from the Lagrangian \( L_m \). It can be observed that when \( f(R,T) = f(R) \), then (1.7) reduce to field equations of \( f(R) \) gravity. Also he presented the field equations of several particular models, corresponding to some explicit forms of the function \( f(R,T) \). Sharif and Zubair [38, 39] have investigated thermodynamics and anisotropic universe models with perfect fluid and scalar field in \( f(R,T) \) gravity. Sharif and Zubair [40] have discussed energy conditions constraints and stability of power law solutions in \( f(R,T) \) gravity. Sharif and Zubair [41] have investigated cosmology of holographic and new age graphic \( f(R,T) \) models. Rao et al. [42] have obtained LRS Bianchi type-I with perfect fluid in a modified theory of gravity. Chakraborthy [43] has discussed \( f(R,T) \) gravity by considering three different cases. Rao et al. [44] have investigated Perfect fluid cosmological models in a modified theory of gravity. Rao et al. [45] have discussed Perfect Fluid Bianchi Universes in a Modified Theory of Gravity. Rao et al. [46] have obtained Interacting LRS Bianchi Type-I cosmological model in \( f(R,T) \) gravity with modified Chaplygin gas. Singh and Pankaj Kumar [47] have investigated State finder diagnosis for holographic dark energy models in modified \( f(R,T) \) gravity. Milan Srivastava and Singh [48] have obtained new holographic dark energy model with constant bulk viscosity in modified \( f(R,T) \) gravity theory. Recently, Reddy and Aditya [49] have discussed Dynamics of perfect fluid cosmological model in the presence of massive scalar field in \( f(R,T) \) gravity. The main aim of the present work is to study the spatially homogeneous Bianchi type-II, VIII & IX cosmological models in the presence of pressure less matter and modified holographic Ricci dark energy in the frame work of \( f(R,T) \) gravity proposed by Harko et al. [14]. Our work in this paper is organized as follows: In Sect.2, we derive the field equations of \( f(R,T) \) gravity in Bianchi type-II, VIII & IX cosmological models in the presence of pressure less matter and modified holographic Ricci dark energy. In Sect.3, we solve the field equations of the models.
Sect. 4 is concerned with physical and kinematical parameters of the models. In Sect. 5, the last section contains conclusions.

2. Metric and Field equations:

We consider a spatially homogeneous Bianchi type-II, VIII & IX metrics of the form

\[ ds^2 = dt^2 - R^2 [d\theta^2 + f^2(\theta)d\phi^2] - S^2 [d\psi + h(\theta)d\varphi]^2 \]  

(2.1)

where \((\theta, \phi, \psi)\) are the Eulerian angles, \(R\) and \(S\) are functions of \(t\) only.

It represents

- Bianchi type - II if \(f(\theta) = 1\) and \(h(\theta) = \theta\)
- Bianchi type - VIII if \(f(\theta) = Cosh \theta\) and \(h(\theta) = Sinh \theta\)
- Bianchi type - IX if \(f(\theta) = Sin \theta\) and \(h(\theta) = Cos \theta\)

Here we assumed the \(f(R,T)\) gravity field equations (Milan Srivastava and Singh [48]) with a special choice of function \(f(R,T) = R + 2f(T)\) are given by

\[ G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij} + 2f'(T)T_{ij} + f(T)g_{ij} \]  

(2.2)

where the prime indicates derivative with respect to the argument \(T\) and

\( T_{ij} = T_{ij}^m + T_{ij}^{de}, T_{ij}^m\) the energy-momentum tensor for matter and \( T_{ij}^{de}\) the energy momentum tensor for modified holographic Ricci dark energy.

They are defined as

\[ T_{ij}^m = \rho_m u_i u_j \]  

(2.3)

and \( T_{ij}^{de} = (\rho_\Lambda + p_\Lambda) u_i u_j - g_{ij}\rho_\Lambda \)  

(2.4)

where \( \rho_m \) is the energy density of matter, \( \rho_\Lambda \) is the energy density of modified holographic Ricci dark energy and \( p_\Lambda \) is the pressure of modified holographic Ricci dark energy.
In a co moving coordinate system, we get

\[ T_1 = T_2 = T_3 = -p_\Lambda \text{ and } T_4 = \rho_m + \rho_\Lambda \]  

(2.5)

where the quantities \( \rho_m, \rho_\Lambda \) and \( p_\Lambda \) are functions of ‘t’ only.

Now with the help of (2.3) to (2.5), the field equations (2.2) for the metric (2.1) can be written as

\[ \frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R} \dot{S}}{R S} + \frac{S^2}{4 R^4} = (8 \pi + 5 \lambda) p_\Lambda - \lambda (\rho_m + \rho_\Lambda) \]  

(2.6)

\[ 2 \frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R} \dot{S}}{R S} + \frac{3 S^2}{4 R^4} = (8 \pi + 5 \lambda) p_\Lambda - \lambda (\rho_m + \rho_\Lambda) \]  

(2.7)

\[ 2 \frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R} \dot{S}}{R S} + \frac{S^2}{4 R^4} = -(8 \pi + 3 \lambda) (\rho_m + \rho_\Lambda) + 3 \lambda p_\Lambda \]  

(2.8)

Here the overhead dot denotes differentiation with respect to ‘t’.

When \( \delta = 0, -1 \text{ and } +1 \) the field equations (2.6) to (2.8) correspond to the Bianchi types II, VIII & IX universes respectively.

3. Modified Holographic Ricci dark Energy cosmological models:

In this section, we present modified holographic Ricci dark energy cosmological models by solving the field equations using some physically valid conditions. The field equations (2.6) to (2.8) are only three independent equations with six unknowns \( R, S, \rho_m, \rho_\Lambda, p_\Lambda \text{ and } w_\Lambda \), which are functions of ‘t’. Since these equations are non-linear in nature, in order to get a deterministic solution we use the following physically viable conditions:

1. The shear scalar \( \sigma^2 \) is proportional to the scalar expansion \( \theta \), which gives us

\[ S = R^n \]  

(3.1)

2. In this case the energy density of modified holographic Ricci dark energy in the form
\[ \rho_\Lambda = \frac{3}{8\pi} \left( \xi H^2 + \eta \dot{H} + \zeta \ddot{H} H^{-1} \right) \]  \hspace{1cm} (3.2)

From equations (2.6), (2.7) and (3.1), we get

\[ \frac{\ddot{R}}{R} + (1 + n) \frac{\dot{R}^2}{R^2} + \frac{\delta}{(1-n)R^2} - \frac{R^{2n-4}}{(1-n)} = 0 \ , \ n \neq 1 \]  \hspace{1cm} (3.3)

### 3.1. Bianchi type-II (\( \delta = 0 \)) cosmological model:

If \( \delta = 0 \), equation (3.3) can be written as

\[ \frac{\ddot{R}}{R} + (1 + n) \frac{\dot{R}^2}{R^2} - \frac{R^{2n-4}}{(1-n)} = 0 \ , \ n \neq 1 \]  \hspace{1cm} (3.4)

From equation (3.4), we get

\[ R = (at + b)^{\frac{1}{2-n}} \ , \ n \neq 2 \]  \hspace{1cm} (3.5)

where \( a \) and \( n \) are related by \( 2a^2 n(1-n) - (2-n)^2 = 0 \)

and \( a \neq 0, b \) are arbitrary constants.

From equations (3.1) & (3.5), we get

\[ S = (at + b)^{\frac{n}{2-n}} \]  \hspace{1cm} (3.6)

The Hubble parameter \( H \) is

\[ H = \frac{1}{3} \left( \frac{2\ddot{R}}{R} + \frac{\dot{S}}{S} \right) = \frac{n+2}{3(2-n)} \left( \frac{a}{at+b} \right) \]  \hspace{1cm} (3.7)

Now the metric (2.1) can be written as
\[ ds^2 = dt^2 - [at + b]^{2/2-n} (d\theta^2 + d\phi^2) - [at + b]^{2/2-n} (d\varphi + \theta d\phi)^2 \]  

(3.8)

From eqs. (3.2) and (3.7), we get the energy density of the modified holographic Ricci dark energy in the universe is obtained as

\[ \rho_{\lambda} = \left( \frac{3a^2}{8\pi(at+b)^2} \right) \left( \frac{n+2}{9(2-n)^2} \left[ n(\xi + 3\eta) + 2(\xi - 3\eta) \right] + \zeta \right) \]  

(3.9)

From eqs. (2.6) – (2.8), (3.5) and (3.6), we get the pressure of the modified holographic Ricci dark energy in the universe is obtained as

\[ p_{\lambda} = \left( \frac{a^2}{8\pi(at+b)^2} \right) \left( \frac{3}{(2-n)^2} \left[ 3\lambda \left( \frac{2(n^2+n-1)}{(2-n)^2} - \frac{1}{2a^2} \right) - (16\pi + 10\lambda) \left( \frac{2n+1}{(2-n)^2} - \frac{1}{4a^2} \right) \right] \right) \]  

\[ \left( \frac{3a^2}{8\pi(at+b)^2} \right) \left( \frac{n+2}{9(2-n)^2} \left[ n(\xi + 3\eta) + 2(\xi - 3\eta) \right] + \zeta \right) \]  

(3.10)

From eqs. (3.1) – (3.3), (3.5) and (3.6), we get the energy density of the matter is

\[ \rho_m = \left( \frac{a^2}{8\pi(at+b)^2} \right) \left( \frac{3}{(2-n)^2} \left[ 3\lambda \left( \frac{2(n^2+n-1)}{(2-n)^2} - \frac{1}{2a^2} \right) - (16\pi + 10\lambda) \left( \frac{2n+1}{(2-n)^2} - \frac{1}{4a^2} \right) \right] \right) \]  

\[ \left( \frac{3a^2}{8\pi(at+b)^2} \right) \left( \frac{n+2}{9(2-n)^2} \left[ n(\xi + 3\eta) + 2(\xi - 3\eta) \right] + \zeta \right) \]  

(3.11)

From eqs. (3.9) and (3.10), we get EoS parameter in the model as

\[ w_{\lambda} = \frac{p_{\lambda}}{\rho_{\lambda}} = \left( \frac{3a^2}{8\pi(at+b)^2} \right) \left( \frac{n+2}{9(2-n)^2} \left[ n(\xi + 3\eta) + 2(\xi - 3\eta) \right] + \zeta \right) \]  

(3.12)

The coincident parameter is
3.2. Bianchi type-VIII ($\delta = -1$) cosmological model:

If $\delta = -1$, equation (3.3) can be written as

$$\frac{\dot{R}}{R} + (1 + n) \frac{\dot{R}^2}{R^2} - \frac{1}{(1 - n)R^2} - \frac{R^{2n-4}}{(1 - n)} = 0, n \neq 1$$

(3.14)

We can solve the above equation and get the deterministic solution only for $n = 2$. So, from equation (3.14) with a suitable substitution, we get

$$\dot{R}^2 = w^2 - \gamma^2 R^2$$

(3.15)

where $w^2 = \frac{1}{3}$ & $\gamma^2 = \frac{1}{4}$.

From equation (3.15), we get

$$R = \left(\frac{w}{\gamma}\right) \sin(\gamma t)$$

(3.16)

From equations (3.1) & (3.16), we get

$$S = \left[\left(\frac{w}{\gamma}\right) \sin(\gamma t)\right]^2$$

(3.17)

The Hubble parameter $H$ is
Now the metric (2.1) can be written as

\[
ds^2 = dt^2 - \left[\left(\frac{\pi}{\gamma}\right)\sin(\gamma t)\right]^2 \left(\frac{\pi^2}{\gamma^2} \cos^2 \theta d\phi^2 - \left[\left(\frac{\pi}{\gamma}\right)\sin(\gamma t)\right]^2 \left(d\varphi + \sin \theta d\phi\right)^2\right)
\]  

(3.19)

From eqs. (3.2) and (3.18), we get the energy density of the modified holographic Ricci dark energy in the universe is obtained as

\[
\rho_{\Lambda} = \left\{\frac{3}{32\pi}\right\}\left[\left[\frac{16}{9}\xi - \frac{4\eta}{3} + 2\zeta\right] \cot^2 \gamma t + \left(2\zeta - \frac{4\eta}{3}\right)\right]
\]  

(3.20)

From eqs. (2.6) – (2.8), (3.16) and (3.17), we get the pressure of the modified holographic Ricci dark energy in the universe is obtained as

\[
p_{\Lambda} = \frac{\cot^2 \gamma t (16\pi + 2\lambda) - (8\pi + 4\lambda)}{[16\pi + 10\lambda)(8\pi + 3\lambda) - 6\lambda^2]}
\]  

(3.21)

From eqs. (2.6) – (2.8), (3.16) and (3.17), we get the energy density of the matter is

\[
\rho_{m} = \frac{\cot^2 \gamma t (16\pi + 7\lambda) + 4(\pi + \lambda)}{3\lambda^2 - (8\pi + 5\lambda)(8\pi + 3\lambda)} - \left\{\frac{3}{32\pi}\right\}\left[\left[\frac{16}{9}\xi - \frac{4\eta}{3} + 2\zeta\right] \cot^2 \gamma t + \left(2\zeta - \frac{4\eta}{3}\right)\right]
\]  

(3.22)

From eqs. (3.20) and (3.21), we get EoS parameter in the model as

\[
w_{\Lambda} = \frac{p_{\Lambda}}{\rho_{\Lambda}} = \left\{\frac{3}{32\pi}\right\}\left[\left[\frac{16}{9}\xi - \frac{4\eta}{3} + 2\zeta\right] \cot^2 \gamma t + \left(2\zeta - \frac{4\eta}{3}\right)\right]
\]  

(3.23)

The coincident parameter is

\[
r = \frac{\rho_{\Lambda}}{\rho_{m}} = \left\{\frac{3}{32\pi}\right\}\left[\left[\frac{16}{9}\xi - \frac{4\eta}{3} + 2\zeta\right] \cot^2 \gamma t + \left(2\zeta - \frac{4\eta}{3}\right)\right]
\]  

(3.24)
3.3. Bianchi type-IX ($\delta = 1$) cosmological model:

If $\delta = 1$, equation (3.3) can be written as

$$\frac{\ddot{R}}{R} + (1 + n) \frac{\dot{R}^2}{R^2} + \frac{1}{(1 - n)R^2} - \frac{R^{2n-4}}{(1 - n)} = 0, \quad n \neq 1$$

(3.25)

We can solve the above equation and get the deterministic solution only for $n = 2$. So, from equation (3.25) with a suitable substitution, we get

$$\dot{R}^2 = w^2 - \gamma^2 R^2$$

(3.26)

where $w^2 = \frac{1}{2}$ & $\gamma^2 = \frac{1}{4}$.

From equation (3.26), we get

$$R = \left(\frac{w}{\gamma}\right) \sin(\gamma t)$$

(3.27)

From equations (3.1) & (3.27), we get

$$S = \left[\left(\frac{w}{\gamma}\right) \sin(\gamma t)\right]^2$$

(3.28)

The Hubble parameter H is

$$H = \frac{1}{3} \left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = \frac{4}{3} \gamma \cot \gamma t$$

(3.29)

Now the metric (2.1) can be written as

$$ds^2 = dt^2 - \left(\frac{w}{\gamma}\right) \sin(\gamma t) \left(\frac{d\theta^2}{\sin^2 \theta} + \sin^2 \theta d\phi^2\right) - \left(\frac{w}{\gamma}\right) \sin(\gamma t) \left(d\varphi + \cos \theta d\phi\right)^2$$

(3.30)

From eqs. (3.2) and (3.29), we get the energy density of the modified holographic Ricci dark energy in the universe is obtained as
\[ \rho_\Lambda = \left( \frac{3}{32\pi} \right) \left[ \left( \frac{16}{9} \xi - \frac{4\eta}{3} + 2\zeta \right) \text{Cot}^2 \gamma t + \left( 2\zeta - \frac{4\eta}{3} \right) \right] \]  

(3.31)

From eqs. (2.6) – (2.8), (3.27) and (3.28), we get the pressure of the modified holographic Ricci dark energy in the universe is obtained as

\[ p_\Lambda = \frac{\text{Cot}^2 \gamma t (16\pi + 2\lambda) - (8\pi + 4\lambda)}{(16\pi + 10\lambda)(8\pi + 3\lambda) - 6\lambda^2} \]  

(3.32)

From eqs. (2.6) – (2.8), (3.31) and (3.32), we get the energy density of the matter is

\[ \rho_m = \frac{\text{Cot}^2 \gamma t (16\pi + 7\lambda) + 4(\pi + \lambda)}{3\lambda^2 - (8\pi + 5\lambda)(8\pi + 3\lambda)} - \left( \frac{3}{32\pi} \right) \left[ \left( \frac{16}{9} \xi - \frac{4\eta}{3} + 2\zeta \right) \text{Cot}^2 \gamma t + \left( 2\zeta - \frac{4\eta}{3} \right) \right] \]  

(3.33)

From eqs. (3.31) and (3.32), we get EoS parameter in the model as

\[ w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = \frac{\text{Cot}^2 \gamma t (16\pi + 2\lambda) - (8\pi + 4\lambda)}{(16\pi + 10\lambda)(8\pi + 3\lambda) - 6\lambda^2} \left[ \left( \frac{16}{9} \xi - \frac{4\eta}{3} + 2\zeta \right) \text{Cot}^2 \gamma t + \left( 2\zeta - \frac{4\eta}{3} \right) \right] \]  

(3.34)

The coincident parameter is

\[ r = \frac{p_\Lambda}{p_m} = \frac{\text{Cot}^2 \gamma t (16\pi + 7\lambda) + 4(\pi + \lambda)}{3\lambda^2 - (8\pi + 5\lambda)(8\pi + 3\lambda)} - \left( \frac{3}{32\pi} \right) \left[ \left( \frac{16}{9} \xi - \frac{4\eta}{3} + 2\zeta \right) \text{Cot}^2 \gamma t + \left( 2\zeta - \frac{4\eta}{3} \right) \right] \]  

(3.35)

4. Some other important properties of the models:

4.1. Bianchi type-II cosmological model ( \( \delta = 0 \)):

The spatial volume for the model (3.8) is
\[ V = (at + b)^{\frac{n + 2}{2 - n}} \]  

(4.1)

The average scale factor for the model is

\[ a(t) = V^{\frac{1}{3}} = (at + b)^{\frac{n + 2}{3(2 - n)}} \]  

(4.2)

The expression for expansion scalar \( \theta \) calculated for the flow vector \( U^i \) is given by

\[ \theta = 3H = \left( \frac{n + 2}{2 - n} \right) \frac{a}{(at + b)} \]  

(4.3)

and the shear scalar \( \sigma \) is given by

\[ \sigma^2 = \frac{7}{18} \left( \frac{n + 2}{2 - n} \right)^2 \frac{a^2}{(at + b)^2} \]  

(4.4)

The deceleration parameter \( q \) is given by

\[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{4(1 - n)}{(n + 2)} , n \not= 1, 2 \text{ & } -2 . \]  

(4.5)

The recent observations of SN Ia reveal that the present universe is accelerating and the value of deceleration parameter lies somewhere in the range \(-1 < q < 0\). From equation (4.5) we can see that for \( 1 < (n \not= 2) < \infty \), \( q \) is negative which may be attributed to the current accelerated expansion of the universe.

The components of Hubble parameter \( H_x, H_y \text{ & } H_z \) are given by

\[ H_x = H_y = \frac{\dot{R}}{R} = \left( \frac{1}{2 - n} \right) \frac{a}{(at + b)} , H_z = \frac{\dot{S}}{S} = \left( \frac{n}{2 - n} \right) \frac{a}{(at + b)} \]
Therefore the generalized mean Hubble parameter (H) is

\[
H = \frac{1}{3} \left( H_x + H_y + H_z \right) = \frac{1}{3} \left( \frac{n+2}{2-n} \right) \frac{a}{(at + b)}. \tag{4.6}
\]

The average anisotropy parameter is defined by

\[
A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = \frac{2(n-1)^2}{(n+2)^2} \tag{4.7}
\]

where \( \Delta H_i = H_i - H \) \( (i = 1, 2, 3) \).

### 4.2. Bianchi type-VIII \((\delta = -1)\) & IX \((\delta = 1)\) cosmological models:

The spatial volume for both the models (3.19) & (3.30) is

\[
V = \left[ \left( \frac{W}{\gamma} \right) \sin(\gamma t) \right]^4 f(\theta) \tag{4.8}
\]

The average scale factor for the model is

\[
a(t) = V^{1/3} = V = \left[ \left( \frac{W}{\gamma} \right) \sin(\gamma t) \right]^4 f(\theta) \right]^{1/3} \tag{4.9}
\]

where \( f(\theta) = \cosh \theta \) & \( \sin \theta \) for Bianchi type-VIII & IX respectively.

The expression for expansion scalar \( \theta \) and the shear \( \sigma \) for the models (3.19) & (3.30) are given by

\[
\theta = 3H = 4\gamma \cot(\gamma t) \tag{4.10}
\]
\[ \sigma^2 = \frac{56}{9} \gamma^2 \cot^2(\gamma t) \] (4.11)

The deceleration parameter \( q \) for the models (3.19) & (3.30) is given by

\[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = \frac{3}{4} \sec^2(\gamma t) - 1 \] (4.12)

From equation (4.12), we can observe that the deceleration parameter \( q \) is always negative and hence they represent accelerating universe.

The components of the Hubble parameter \( H_x, H_y, \) & \( H_z \) for the models (3.19) & (3.30) are given by

\[ H_x = H_y = \frac{\dot{R}}{R} = \gamma \cot(\gamma t), H_z = \frac{\dot{S}}{S} = 2\gamma \cot(\gamma t) \]

Therefore the generalized mean Hubble parameter (H) is

\[ H = \frac{1}{3}(H_x + H_y + H_z) = \frac{4}{3} \gamma \cot(\gamma t) \] (4.13)

The average anisotropy parameter for the models (3.19) & (3.30) are defined by

\[ A_{\text{av}} = \frac{\frac{1}{3} \sum_i \left( \frac{\Delta H_i}{H} \right)}{3} = \frac{1}{8} \] (4.14)

where \( \Delta H_i = H_i - H \ (i = 1,2,3) \).

5. Discussion and Conclusions:

In this paper, we have investigated spatially homogeneous anisotropic Bianchi type-II, VIII & IX cosmological models in the presence of pressure less matter and modified holographic Ricci dark
energy in the frame work of \( f(R,T) \) gravity (Harko et al., Phys. Rev. D 84, 024020, 2011). For Bianchi type-II cosmological model, we observe that at \( t = \frac{-b}{a} \), the spatial volume vanishes and increases continuously with time while all other parameters diverge for \( 0 < (n \neq 1) < 2 \). This shows that at the initial epoch the universe starts with zero volume and expands continuously approaching to infinite volume. Also the model has initial singularity at \( t = \frac{-b}{a} \) when \( n < 0 \) and also for \( n > 2 \). From equation (4.5), we can observe that for \( 1 < (n \neq 2) < \infty \), \( q \) is negative which may be attributed to the current accelerated expansion of the universe. From (3.9) - (3.11), we can see that energy densities of matter & MHRDE and pressure of MHRDE will vanish with the increase of time. For Bianchi type-VIII & IX cosmological models, we can observe that the spatial volume increases with the increase of time and also the models have no initial singularity at \( t = 0 \). The expansion scalar \( \theta \), the shear scalar \( \sigma \) and Hubble parameter \( H \) decrease as time increases. From equation (4.12), we can observe that the deceleration parameter \( q \) is always negative and hence they represent accelerating universe. From (4.7)& (4.14), we can observe that \( A_m \neq 0 \), which indicates that these Bianchi type-II, VIII & IX models are always anisotropic.

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