Short-distance charmonium correlator on the lattice with Möbius domain-wall fermion and a determination of charm quark mass

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Abstract

We calculate charmonium correlators on the lattice with 2+1-flavors of sea quarks and charm valence quark both described by the Möbius domain-wall fermion. Temporal moments of the correlators are calculated and matched to perturbative QCD formulae to extract the charm quark mass $m_c(\mu)$ and strong coupling constant $\alpha_s(\mu)$. Lattice data at three lattice spacings, 0.044, 0.055, and 0.080 fm, are extrapolated to the continuum limit. The correlators in the vector channel are confirmed to be consistent with the experimental data for $e^+e^- \rightarrow c\bar{c}$, while the pseudo-scalar channel is used to extract $m_c(\mu)$ and $\alpha_s(\mu)$. We obtain $m_c(3 \text{ GeV}) = 1.003(10) \text{ GeV}$ and $\alpha_s^{\overline{MS}}(3 \text{ GeV}) = 0.253(13)$. Dominant source of the error is the truncation of perturbative expansion at $\alpha_s^3$.

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I. INTRODUCTION

Numerical simulation of lattice QCD offers non-perturbative calculation of correlation functions on the Euclidean lattice. While one usually uses the long-distance correlators to extract the mass and matrix elements of hadrons, the same correlators at short distances also provide a rich source of information. The vector current correlator, for instance, may be used to test QCD by comparing the lattice calculation with the experimental data available for the $R$ ratio $\sigma_{e^+e^-\rightarrow q\bar{q}}/\sigma_{e^+e^-\rightarrow \mu^+\mu^-}$. The correlator becomes mostly perturbative at high energy scales, but the non-perturbative effect is still important. Another important use of the short-distance regime is the application of perturbation theory, from which one can extract the fundamental parameters such as the strong coupling constant $\alpha_s$ and charm quark mass $m_c$.

The HPQCD and Karlsruhe collaboration used the pseudo-scalar charmonium correlator to achieve a precise determination of $m_c$ and $\alpha_s$ [1], which has further been improved and extended to include the determination of the bottom quark mass [2, 3]. The basic idea is to use a perturbative QCD calculation performed at the order of $\alpha_s^3$ to express temporal moments of the charmonium correlator calculated non-perturbatively on the lattice. Since the perturbative expansion is given as a function of $\alpha_s$ and $m_c$, one can solve the equations to determine these parameters. The precision achieved is among the best for these important fundamental parameters of QCD.

In this work we utilize the same method to extract $m_c$ and $\alpha_s$. Our lattice data are independent from those used by the HPQCD collaboration. We use the lattice ensembles generated with 2+1 flavors of light sea quarks described by the Möbius domain-wall fermion formulation [4]. The valence charm quark is also treated by the same fermion formulation. Discretization effects expected for relatively large charm quark mass compared to the lattice spacing are largely removed by extrapolating to the continuum limit using the data at three lattice spacings, $a \simeq 0.080$, 0.055, and 0.044 fm. The light quark masses in the simulations are in the range corresponding to the pion mass of 230–500 MeV, which do not cover the physical value but their effect on the charmonium correlator is minor.

On the perturbative side, we use the same perturbative coefficients as those in the previous works [1–3]. We estimate the truncation error by examining the dependence on the renormalization scale $\mu_\alpha$ to define the coupling constant $\alpha_s(\mu_\alpha)$ as well as that on $\mu_m$ that
defines the running charm quark mass $m_c(\mu_m)$ appearing in the perturbative expansion.

Our results are in reasonable agreement with those of [1–3]. The estimated error is slightly larger, because of different systematic effect as well as different error estimates. We also try to validate the lattice calculation by providing a comparison to the experimental data available for the vector channel through the $R$-ratio. It mainly serves as a test of the discretization effects, which is an important source of the systematic error for heavy quarks. We find that the continuum extrapolation is nearly flat, confirming that the discretization error for charm quark is well under control in our setup.

This paper is organized as follows. In Section II we review the method of [1–3] as well as the formulae to compare the temporal moments with the experimental data. Some details of our lattice calculation are given in Section III. Lattice results for the vector current correlator and the comparison with the experimental data are given in Section IV which is followed by corresponding results for pseudo-scalar correlator in Section VI. The issues in the matching to perturbative results and its possible uncertainties are discussed in Section VII and results for charm quark mass and strong coupling constant are finally given in Section VIII. Our conclusions are in Section VIII.

II. CHARMONIUM CORRELATORS AND THEIR TEMPORAL MOMENTS

A. Charmonium correlators

We calculate the pseudo-scalar and vector charmonium correlators with vanishing spatial momentum

$$G^{PS}(t) = a^6 \sum_x (am_c)^2 \langle 0 | j_5(x,t)j_5(0,0)|0 \rangle,$$

$$G^V(t) = \frac{a^6}{3} \sum_{k=1}^{3} \sum_x Z^2_V \langle 0 | j_k(x,t)j_k(0,0)|0 \rangle,$$

on the lattice. The currents are defined as $j_5 = i\bar{\psi}_c\gamma_5\psi_c$ and $j_k = \bar{\psi}_c\gamma_k\psi_c$ with charm quark field $\psi_c$ on the lattice. Given the factor $a^6$, both $G^{PS}(t)$ are $G^V(t)$ are dimensionless. The pseudo-scalar density operator $j_5$ is multiplied by a (bare) charm quark mass $m_c$ such that the correlator becomes renormalization scale invariant, while a possible renormalization factor $Z_V$ for the vector current $j_k$ defined on the lattice is explicitly multiplied in (II.2).
FIG. 1. \((t/a)^n G(t)\) on the lattice of size 64\(^3\) × 128 at \(a = 0.044\) fm. The function is normalized by its peak. The data for \(n = 4\) (filled circle), 8 (open circle), and 12 (square) are shown. The long-dashed line around \(t/a \sim 48\) represents the point of \(\pi/(300\) MeV\), which is the distance that non-perturbative effect dominates. Three vertical dashed lines show the position of peak \(n/M\) (\(n = 4, 8\) and 12) for the single exponential function \(e^{-Mt}\). Here, \(Ma = 0.6656\).

We then construct the temporal moments as

\begin{align}
G_n^{PS} &= \sum_t \left(\frac{t}{a}\right)^n G^{PS}(t), \quad (II.3) \\
G_n^{V} &= \sum_t \left(\frac{t}{a}\right)^n G^{V}(t), \quad (II.4)
\end{align}

with \(n\) an even integer equal to or larger than four. (The correlator \(\langle 0 | j(x) j(0) |0 \rangle\) diverges as \(1/|x|^6\) in the small-separation limit, and the lower moments contain ultraviolet divergences.) On the lattice, the time coordinate \(t/a\) runs between \(-T/2a + 1\) and \(T/2a\) with \(T\) the temporal extent of the lattice.

Since the charmonium correlators \(G^{PS}(t)\) and \(G^{V}(t)\) decay exponentially at large \(t\) by the mass of the corresponding lowest energy states \(\eta_c\) and \(J/\psi\), respectively, the temporal moments \((II.3)\) and \((II.4)\) are sensitive only to the relatively short-range correlations. For an exponential function \(e^{-Mt}\), where \(M\) represents the mass of \(\eta_c\) or \(J/\psi\), the largest
contribution to the \( n \)-th moment comes from the region of \( t \sim n/M \). In the presence of excited state contributions, the dominant region is slightly shifted to smaller \( t \)'s. Figure I illustrates typical examples of the integrand \((t/a)^n G^{PS}(t)\) to construct the \( n \)-th moments. Lattice data at \( a = 0.044 \) fm are taken and data for \( n = 4, 8, \) and \( 12 \) are shown. The lowest moment, \( n = 4 \), receives a significant contribution from small \( t \) range, \((t/a) \sim 1–2\), where the discretization effect could be substantial. For higher moments \( n = 8 \) and \( 12 \), the sum is not affected much by the small \( t \) range.

The vector correlator and its moments may be related to those in the continuum theory and to the experimental data. The vacuum polarization function \( \Pi^V(q^2) \) is defined through

\[
(q^n q^n - q^2 \eta^\mu\nu) \Pi^V(q^2) = i \int d^4x e^{iqx} \langle 0|j^\mu(x)j^\nu(0)|0\rangle. \tag{II.5}
\]

Derivatives of \( \Pi^V(q^2) \) with respect to \( q^2 \),

\[
g_{2k+2}^V = (2m(\mu))^2k \frac{12\pi^2 Q_f^2}{k!} \left( \frac{\partial}{\partial q^2} \right)^k \Pi^V(q^2) \big|_{q^2=0}, \tag{II.6}
\]

may be related to the experimental data for the \( e^+e^- \to c\bar{c} \) process, \( i.e. \) the \( R \)-ratio \( R(s) \equiv \sigma_{e^+e^- \to c\bar{c}}(s)/\sigma_{e^+e^- \to \mu^+\mu^-}(s) \), as

\[
\frac{12\pi^2 Q_f^2}{k!} \left( \frac{\partial}{\partial q^2} \right)^k \Pi(q^2) \big|_{q^2=Q_0^2} = M_k \equiv \int_{s_0}^{\infty} ds \frac{1}{(s - Q_0^2)^{k+1}} R(s). \tag{II.7}
\]

Here \( Q_f \) stands for the electromagnetic charge of charm quark. The lower end of the integral \( s_0 \) should be set below the \( J/\psi \) mass. The reference scale \( Q_0^2 \) is arbitrary but is often taken at \( Q_0^2 = 0 \). Using this notation we may write the relation between the temporal moments on the lattice and the observable as

\[
G_n^V = \frac{g_n^V}{(am(\mu))^{n-2}}. \tag{II.8}
\]

A direct comparison of the lattice results with the experimental values for \( M_n \) (or their phenomenological estimates) is given in Section IV. The phenomenological estimates of \( M_n \) can be found in \[5–8\].

For the pseudo-scalar density correlator

\[
q^2 \Pi^{PS}(q^2) = i \int d^4x e^{iqx} \langle 0|j_5(x)j_5(0)|0\rangle, \tag{II.9}
\]

there is no such experimental information available, while the relation between the temporal moments and the derivatives of the vacuum polarization function may be written as

\[
G_n^{PS} = \frac{g_n^{PS}}{(am(\mu))^{n-4}} \tag{II.10}
\]
with $g_n^{PS}$ analogously defined as in (II.6).

The continuum vacuum polarization functions can be parametrized as

$$\Pi^{PS}(q^2) = \frac{3}{16\pi^2} \sum_{k=-1}^{\infty} C_k^{PS} z^k, \quad (II.11)$$

$$\Pi^V(q^2) = \frac{3}{16\pi^2} \sum_{k=-1}^{\infty} C_k^{V} z^k, \quad (II.12)$$

with $z = q^2/(2m_c(\mu))^2$. In perturbation theory, the coefficients $C_k^{PS}$ and $C_k^{V}$ are expanded in terms of $\alpha_s(\mu)/\pi$:

$$C_k = C_k^{(0)} + \frac{\alpha_s(\mu)}{\pi} \left( C_k^{(10)} + C_k^{(11)} l_m \right)$$

$$+ \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left( C_k^{(20)} + C_k^{(21)} l_m + C_k^{(22)} l_m^2 \right)$$

$$+ \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \left( C_k^{(30)} + C_k^{(31)} l_m + C_k^{(32)} l_m^2 + C_k^{(33)} l_m^3 \right) + \ldots \quad (II.13)$$

with $l_m = \log (m_c^2(\mu)/\mu^2)$. (Here, $C_k$ and its expansion coefficients are those of either $C_k^{PS}$ or $C_k^{V}$.) The perturbative calculation has been performed up to $O(\alpha_s^3)$. The calculation is conventionally performed in the $\overline{\text{MS}}$ renormalization scheme, and the coupling constant $\alpha_s(\mu)$ and running quark mass $m_c(\mu)$ are given in that scheme at a renormalization scale $\mu$. The relevant coefficients for $n_f = 4$ are summarized in Table I for convenience.

| $n$ | $k$ | $C_k^{(0)}$ | $C_k^{(10)}$ | $C_k^{(11)}$ | $C_k^{(20)}$ | $C_k^{(21)}$ | $C_k^{(22)}$ | $C_k^{(30)}$ | $C_k^{(31)}$ | $C_k^{(32)}$ | $C_k^{(33)}$ |
|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 4   | 1   | 1.33333    | 3.11111    | 0.00000     | 0.115353    | -6.48148    | 0.00000     | -1.22241    | 2.50084     | 13.5031     | 0.00000     |
| 6   | 2   | 0.533333   | 2.06420    | 1.06667     | 7.23618     | 1.590947    | -0.0444444  | 7.06380     | 14.5789     | 14.7645     | 11.0345     |
| 8   | 3   | 0.304762   | 1.21171    | 1.21905     | 5.99920     | 4.33726     | 1.16825     | 14.5789     | 7.36258     | 4.25232     | -0.0649030  |
| 10  | 4   | 0.203275   | 0.712756   | 1.21905     | 4.26701     | 4.80644     | 2.38730     | 13.3285     | 14.7645     | 11.0345     | 1.45891     |
| 12  | 5   | 0.1478     | 0.4013     | 1.1821      | 2.9149      | 4.34728     | 3.49922     | 14.1098     | 19.9049     | 8.7485      |
| 14  | 6   | 0.1137     | 0.1944     | 1.1366      | 1.9656      | 3.4173      | 4.49922     | 14.1098     | 19.9049     | 8.7485      |
| 16  | 7   | 0.0909     | 0.0500     | 1.0912      | 1.3353      | 2.2995      | 5.4104      | 10.7755     | 20.3500     | 14.1272     |
| 18  | 8   | 0.0749     | -0.0545    | 1.0484      | 0.9453      | 1.0837      | 6.2466      | 7.2863      | 17.9597     | 20.4750     |
| $n$ | $k$ | $C_k^{(0)}$ | $C_k^{(10)}$ | $C_k^{(11)}$ | $C_k^{(20)}$ | $C_k^{(21)}$ | $C_k^{(22)}$ | $C_k^{(30)}$ | $C_k^{(31)}$ | $C_k^{(32)}$ | $C_k^{(33)}$ |
|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 4   | 1   | 1.06667     | 2.55473     | 2.13333     | 2.49671     | 3.31303     | -0.0888889  | -5.640343   | 4.06686     | 0.959031    | 0.0641975   |
| 6   | 2   | 0.457142    | 1.10956     | 1.82857     | 2.77702     | 5.14888     | 1.75238     | -3.49351    | 6.72161     | 6.49161     | -0.0973544  |
| 8   | 3   | 0.270899    | 0.519396    | 1.62540     | 1.63882     | 4.72072     | 3.18307     | -2.8395     | 13.1654     | 1.94521     |
| 10  | 4   | 0.1847      | 0.2031      | 1.4776      | 0.7956      | 3.6440      | 4.3713      | -3.349      | 17.4612     | 5.5856      |
| 12  | 5   | 0.1364      | 0.0106      | 1.3640      | 0.2781      | 2.3385      | 5.3990      | 0.9026      | 18.7458     | 10.4981     |
| 14  | 6   | 0.1061      | -0.1158     | 1.2730      | 0.0070      | 0.9553      | 6.3121      | -3.1990     | 16.9759     | 16.4817     |
| 16  | 7   | 0.0856      | -0.2033     | 1.1982      | -0.0860     | -0.4423     | 7.1390      | -6.5399     | 12.2613     | 23.4000     |
| 18  | 8   | 0.0709      | -0.2660     | 1.1351      | -0.0496     | -1.8261     | 7.8984      | -8.6310     | 4.7480      | 31.1546     |

TABLE II. Perturbative coefficients for the vector correlator. The results for $n_f = 4$ are summarized from [8, 9, 11].

B. Formulae for the extraction of $m_c$ and $\alpha_s$

For the extraction of charm quark mass and strong coupling constant, we impose the equality between the lattice and perturbative moments, following the method introduced in [1, 2]. In the following we consider the pseudo-scalar channel unless otherwise stated and suppress the superscript $PS$.

In order to reduce the discretization effects, we define the reduced moment $R_n$ using the moment $G_n^{(0)}$ evaluated at tree level using the same lattice formulation. Namely,

$$R_n = \begin{cases} 
\frac{G_n}{G_n^{(0)}} & \text{for } n = 4, \\
\frac{am_{\eta_c}}{2\tilde{m}_c} \left( \frac{G_n}{G_n^{(0)}} \right)^{1/(n-4)} & \text{for } n \geq 6.
\end{cases} \quad (II.14)$$

$$R_n^V = \frac{am_{J/\psi}}{2\tilde{m}_c} \left( \frac{G_n^V}{G_n^{V(0)}} \right)^{1/(n-2)} \quad (II.15)$$

Here $m_{\eta_c}$ ($m_{J/\psi}$) represents the mass of the $\eta_c$ ($J/\psi$) meson calculated on the lattice, and $\tilde{m}_c$ is the charm quark pole mass at the tree-level on the same lattice ensemble. For domain-wall fermions, the pole mass at tree-level is given by

$$\tilde{m}_c = am_c \left[ 1 - \frac{1}{6}(am_c)^2 - \frac{7}{40}(am_c)^4 - \frac{5}{112}(am_c)^6 + \frac{53}{1152}(am_c)^8 + ... \right] \quad (II.16)$$

as a function of the input quark mass $am_c$ on the lattice. Details are in Appendix A. The correction term starts at $(am_c)^2$, and its size is 3.9% at $am_c = 0.4404$, which corresponds to
the input charm quark mass on our coarsest lattice. This correction is expected to partly cancel the discretization effect in the calculation of $am_{\eta_c}$. Overall, in the ratios of (II.14), the discretization effects cancel between numerator and denominator at the leading order, i.e. $O(\alpha_s^0)$, and the remaining error starts at $O(\alpha_s a^2)$ for $O(a)$-improved lattice actions.

Another definition of the reduced moment $\tilde{R}_n$ is used in [3]:

$$\tilde{R}_n = \frac{a}{am_c} \left( \frac{G_n}{G_n^{(0)}} \right)^{1/(n-4)}$$  \quad \text{for } n \geq 6. \tag{II.17}$$

It does not involve the meson mass $am_{\eta_c}$, and thus is free from the fitting error of the correlator using the exponential function $\exp(-am_{\eta_c}(t/a))$. On the other hand, it contains an explicit factor of the lattice spacing $a$, and the error of the input for the lattice scale directly reflects in the result of $m_c$. The advantage of having the factor $m_{\eta_c}/\bar{m}_c$ (or $m_{J/\psi}/\bar{m}_c$) in (II.14) (or in (II.15)) is that the meson mass $m_{\eta_c}$ (or $m_{J/\psi}$) effectively plays the role of the input scale to determine $m_c$. With $\tilde{R}_n$, the error in setting the lattice spacing, which is about 1.7% in our case, directly appears in the final result for $m_c$. We analyzed the data for both $R_n$ and $\tilde{R}_n$, and it turned out that $R_n$ gives more precise determination. Only the results with $R_n$ are presented in this paper.

On the continuum side, one defines the reduced moment $r_n$ from the derivatives of $q^2 \Pi(q^2)$ with respect to $q^2$

$$g_{2k} \equiv \frac{12\pi^2 Q_f^2}{k!} \left( \frac{\partial}{\partial z} \right)^k (z\Pi(q^2))\big|_{q^2=0} = \frac{12\pi^2 Q_f^2}{(k-1)!} \left( \frac{\partial}{\partial z} \right)^{k-1} (\Pi(q^2))\big|_{q^2=0}, \tag{II.18}$$

as

$$r_n = \begin{cases} 
\frac{g_4}{g_4^{(0)}} = C_1/C_1^{(0)} & \text{for } n = 4, \\
\left( \frac{g_n}{g_n^{(0)}} \right)^{1/(n-4)} = \left( \frac{C_{n/2}}{C_{n/2}^{(0)}} \right)^{1/(n-4)} & \text{for } n \geq 6.
\end{cases} \tag{II.19}$$

$$r_n^V = \left( \frac{g_n^V}{g_n^V^{(0)}} \right)^{1/(n-2)} = \left( \frac{C_{n/2}}{C_{n/2}^{(0)}} \right)^{1/(n-2)} \quad \text{for } n \geq 4. \tag{II.20}$$

The tree-level moment $g_n^{(0)}$ can be explicitly written as [14]

$$g_{2n+2}^{(0)} = 12\pi^2 Q_f^2 \frac{3}{8\pi^2} \frac{2^n(n-1)!}{(2n+1)!!}. \tag{II.21}$$

and $g_{2k+2}^{V(0)}$ as

$$g_{2n+2}^{V(0)} = 12\pi^2 Q_f^2 \frac{1}{4\pi^2} \frac{2^n(n+1)(n-1)!}{(2n+3)!!}. \tag{II.22}$$
Then, the equality (II.10) may be rewritten as

\[ R_n = \frac{m_n^{\exp}}{2m_c(\mu)} r_n(\alpha_s(\mu), m_c(\mu)). \]  

(II.23)

Here, \( r_n \) is a function of \( \alpha_s(\mu) \) and \( m_c(\mu) \), and the equation is understood as a condition to be satisfied by the parameters \( \alpha_s(\mu) \) and \( m_c(\mu) \) when a numerical value for \( R_n \) is non-perturbatively calculated on the lattice. We can also use a ratio of the reduced moments,

\[ \frac{R_n}{R_{n+2}} = \frac{r_n(\alpha_s(\mu), m_c(\mu))}{r_{n+2}(\alpha_s(\mu), m_c(\mu))}, \]  

(II.24)

which may play a complementary role to (II.23), since the truncation error of its perturbative expansion is different from that of individual \( r_n \).

In QCD, the perturbation theory is reliable only in the relatively short-distance regime compared to the hadronic scale \( 1/\Lambda_{\text{QCD}} \). In order to avoid the non-perturbative regime, \( n \) has to be small to satisfy a condition \( n/M \ll \pi/\Lambda_{\text{QCD}} \), which implies an upper limit for \( n \), i.e. \( n \ll \pi M/\Lambda_{\text{QCD}} \). For the charmonium of \( m \simeq 3 \text{ GeV} \), this means that \( n \) has to be of order of 10 or smaller. As shown in Section VI.B, the leading non-perturbative effect in the Operator Product Expansion (OPE) appears as a contribution of the gluon condensate. Its coefficient \( a_{n/2} \) in (VI.4) rapidly grows for larger \( n \).

Combined with the lower limit for \( n \) to avoid the large discretization effect, as discussed earlier in this section, there is a limited window of \( n \) for this method to be useful. In our analyses, we chose \( n = 6, 8, \) and 10. There is a practical limitation for \( n \geq 12 \), i.e. the \( O(\alpha_s^3) \) coefficients in the perturbative expansion of \( r_n \) are not available.

III. LATTICE DETAILS

We have performed a set of lattice QCD simulations with 2+1 flavors of dynamical quarks. The gauge action is that of tree-level Symanzik improved, and the fermion formulation is the Möbius domain-wall fermions [4]. The gauge links are smeared by applying the stout smearing [15] three times. With this choice, the residual mass, which quantifies the violation of the Ginsparg-Wilson relation, is under good control, i.e. the residual mass is of \( O(1 \text{ MeV}) \) on our coarsest lattice and much smaller on finer lattices. The effect of such a small violation can be neglected for the charmonium correlators. Light sea quark masses are extrapolated to the physical value such that the physical pion and kaon masses are reproduced. Since the sea quark mass dependence of \( R_n \) is minor, this is not a major source of uncertainty.
There are 15 ensembles of different lattice spacings and quark masses as listed in Table III. Lattice spacings are $a = 0.080, 0.055,$ and $0.044$ fm. The spatial size of these lattices is $L/a = 32, 48,$ and $64,$ respectively, to keep the physical lattice size $L$ approximately constant, $\sim 2.6-2.8$ fm. The temporal size $T/a$ is always twice longer than $L/a$. Each ensemble consists of 10,000 molecular dynamics trajectories, out of which we chose 50-100 gauge configurations equally separated and calculated the charmonium correlators eight or twelve times starting from different time slices on each configuration with a $Z_2$ noise. The number of measurement “#meas” is thus 400-800 depending on the ensemble as listed in the table.

The $Z_2$ noise is introduced to improve the statistical signal. Namely, the $Z_2 \ (\pm 1)$ noise is scattered over a time-slice as a source to calculate the charm quark propagator; only the local $Z_2$ invariant contribution survives after averaging over the noise, so that the desired contraction of charm and anti-charm propagators survive and other gauge non-invariant

| $\beta$ | $a^{-1}$ | $L^3 \times T(\times L_5)$ | $N_{\text{src}}$ | #meas | $am_{ud}$ | $am_s$ | $m_\pi$ | $m_\pi L$ | id |
|---------|---------|-------------------------|-----------------|------|---------|--------|---------|----------|----|
|         |         | [GeV]                   |                 |      |         |        |         |          |    |
| 4.17    | 2.453(4)| $32^3 \times 64(\times 12)$ | 8               | 800  | 0.0035  | 0.040  | 230(1)  | 3.0      | C-ud2-sa |
|         |         |                         |                 |      | 0.007   | 0.030  | 310(1)  | 4.0      | C-ud3-sb |
|         |         |                         |                 |      | 0.007   | 0.040  | 309(1)  | 4.0      | C-ud3-sa |
|         |         |                         |                 |      | 0.012   | 0.030  | 397(1)  | 5.2      | C-ud4-sb |
|         |         |                         |                 |      | 0.012   | 0.040  | 399(1)  | 5.2      | C-ud4-sa |
|         |         |                         |                 |      | 0.019   | 0.030  | 498(1)  | 6.5      | C-ud5-sb |
|         |         |                         |                 |      | 0.019   | 0.040  | 499(1)  | 6.5      | C-ud5-sa |
|         |         | $48^3 \times 96(\times 12)$ | 8               | 800  | 0.0035  | 0.040  | 226(1)  | 4.4      | C-ud2-sa-L |
|         |         |                         |                 |      | 0.0042  | 0.0180 | 296(1)  | 3.9      | M-ud3-sb |
|         |         |                         |                 |      | 0.0042  | 0.0250 | 300(1)  | 3.9      | M-ud3-sa |
|         |         |                         |                 |      | 0.0080  | 0.0180 | 407(1)  | 5.4      | M-ud4-sb |
|         |         |                         |                 |      | 0.0080  | 0.0250 | 408(1)  | 5.4      | M-ud4-sa |
|         |         |                         |                 |      | 0.0120  | 0.0180 | 499(1)  | 6.6      | M-ud5-sb |
|         |         |                         |                 |      | 0.0120  | 0.0250 | 501(1)  | 6.6      | M-ud5-sa |
|         |         | $4.47 \ 4.496(9) \ 64^3 \times 128(\times 8)$ | 8               | 400  | 0.0030  | 0.015  | 284(1)  | 4.0      | F-ud3-sa |

TABLE III. Lattice ensembles used in this study.
contributions vanish. In spite of the noise introduced, the signal is improved by averaging over the source points.

Each ensemble has an “id” name, which distinguishes coarse (C), medium (M), and fine (F) lattices, as well as the mass of \( ud \) and \( s \) quark masses. In the main ensembles (C and M), two values of strange quark mass are chosen to sandwich the physical value from above (a) or from below (b). On the coarse lattice at the lightest \( ud \) quark mass, there is an ensemble of larger volume of size \( 48^3 \times 96 \), which is indicated by “-L”. The difference between C-\( ud2\)-sa and C-\( ud2\)-sa-L is used to estimate the possible finite volume effect, as they have the smallest \( ud \) quark mass and the effect of finite spatial volume is expected to be most significant in our ensembles.

The lattice spacing is set through the Wilson-flow scale \( t_0 \) [16]. For its physical value, we input \( t_0^{1/2} = 0.1465(21)(13) \) fm [17]. The resulting values of \( a^{-1} \) are listed in Table III. The table lists the central values and the statistical error in our measurement of \( t_0 \). The error in this input value is to be added for each value of \( a^{-1} \).

Some details of the ensemble generation are available in [18, 19]. The same gauge ensembles have so far been used for a calculation of the \( \eta' \) meson mass [20], an analysis of short-distance current correlator [21], and a calculation of heavy-light meson decay constants [22]. The numerical calculations are performed using the IroIro++ code set for lattice QCD [23].

For the vector current, we multiply the renormalization constant \( Z_V \) obtained from the analysis of short-distance current correlator of light quarks [24]. The numerical values are 0.9553(92) at \( \beta = 4.17 \), 0.9636(58) at \( \beta = 4.35 \), and 0.9699(47) at \( \beta = 4.47 \), where errors include statistical and systematic ones added in quadrature.

On each ensemble, we calculate the charmonium correlator at a bare charm quark mass 0.4404, 0.2723 or 0.2105 at \( \beta = 4.17 \), 4.35 and 4.47, respectively. They are slightly mistuned to the physical charm quark mass, which we set by the spin-averaged mass of the 1S charmonium states \((m_{\eta_c} + 3m_{J/\psi})/4\). We correct this minor shift by using supplemental data set taken at three values of bare charm quark mass sandwiching the physical value. The supplemental data are obtained with a local source and therefore less precise, but only used for a small interpolation of the main data to the physical charm quark mass.

In the calculation of the charmonium correlator, we do not take account of the contribution of disconnected quark-loop diagrams, which may exist in the nature for the flavor-singlet operators like \( j_5 = \bar{\psi}_c \gamma_5 \psi_c \). For the correspondence between the lattice and perturbative
calculations, this does not cause any problem because one can omit the corresponding diagrams also in perturbation theory. For the input to tune the charm quark mass on the lattice, this could lead to some bias, as the physical input parameter, a mass of $\eta_c$ or $J/\psi$, includes such effect. Furthermore, the electromagnetic correction which is neglected in our lattice calculation could also be a source of systematic error. These sources of uncertainties are discussed in some detail in Section VII.

IV. TEMPORAL MOMENTS OF VECTOR CURRENT CORRELATOR

As described in Section II, the temporal moments of the charmonium vector-current correlator can be compared with the experimental value.

Analogous to the reduced moments defined for the pseudo-scalar channel (II.14), we define the reduced moments $R_{V_n}^V$ for the vector moments (II.4). We can then write the correspondence between the lattice and continuum as

$$R_{V_n}^V = m_{J/\psi} \left( \frac{M_k}{g_{2k+2}} \right)^\frac{1}{2}, \hspace{1cm} (IV.1)$$

which is obtained from (II.8).

Numerical results for $Z_{V}^{-\frac{\pi^2}{2}} R_{V_n}^V$ are summarized in Table IV for $n = 6, 8, 10$ and 12. For each ensemble, the results are interpolated to the physical charm quark mass; the statistical error is propagated by the bootstrap method.

The results are linearly extrapolated to the physical light quark mass and plotted as a function of $a^2$ in Figure 2. The lattice results are nearly constant in $a^2$, and the continuum extrapolation as discussed below is also shown.

We extrapolate $R_{V_n}^V$ assuming the form

$$R_{V_n}^V = R_{V_n}^V(0) \left( 1 + c_1 (am_c)^2 \right) \times \left( 1 + f_1 \frac{m_u + m_d + m_s}{m_c} \right), \hspace{1cm} (IV.2)$$

with free parameters $R_{V_n}^V(0)$, $c_1$, and $f_1$. The error of $O(a^2)$ is eliminated by an extrapolation with this form, while the effect of $O(a^4)$ still needs to be estimated. We attempt two continuum extrapolations assuming a linear dependence on $a^2$ with and without the point of the coarsest lattice. The three-point fit yields a $\chi^2$/dof = 0.17 (0.40) for $n = 6$ (8). The value of $\chi^2$/dof is slightly underestimated since the correlated systematic error for $Z_V$ among different $\beta$ values is not taken into account. We take the mean value of these two
extrapolations as a central value, and estimate the remaining discretization error using the deviation from the mean value.

The quark mass dependence of non-perturbative origin, which is assumed to be linear in \( m_u + m_d + m_s \), turned out to be tiny \( (f_1 \sim 0) \), and we do not consider its higher order effects.

Since the lattice calculation is performed with three light flavors \( (n_f = 3) \), we estimate the effect of charm quark loop by perturbative theory. Namely, we correct the lattice result of \( n_f = 3 \) to that of \( n_f = 4 \), by multiplying \( r_n^V(n_f = 4)/r_n^V(n_f = 3) \). The perturbative coefficients are calculated to \( O(\alpha_s^2) \) \cite{9} and partly to \( O(\alpha_s^3) \) \cite{8}. We set the number of heavy
flavors $n_h = 1$ (or 0) for $n_f = 4$ (or 3) to calculate the ratio $r^V_{n_f=4}(n_f = 4)/r^V_{n_f=3}(n_f = 3)$. We also take account of the small difference of $\alpha^{n_f=4}(\mu)$ and $\alpha^{n_f=3}(\mu)$ as well as that of $m_c(\mu)$. The correction is numerically small, i.e. the factor is 0.9992(26), 1.0026(68), 1.0156(342) for $n = 6, 8, \text{and} 10$, respectively.

Table IV summarizes the results for $R^V_n(0)$. The perturbative error is estimated by taking a range of the scale $\mu = 2–4 \text{ GeV}$. The large error for $n = 10$ is due to the lack of $O(\alpha_s^3)$ formula. The results in the continuum limit are compared with the phenomenological estimates [7, 25–28]. The agreement of the lattice data and the phenomenological estimates is remarkable. In particular, our data are consistent with the updated estimates with reduced error of [7, 27], and the size of total error is comparable.

|                | $Z_V^{-\frac{n^2}{2}} R^V_6$ | $Z_V^{-\frac{n^2}{2}} R^V_8$ | $Z_V^{-\frac{n^2}{2}} R^V_{10}$ | $Z_V^{-\frac{n^2}{2}} R^V_{12}$ |
|----------------|-------------------------------|-------------------------------|----------------------------------|----------------------------------|
| C-ud2-sa       | 1.3563(5)                     | 1.3101(5)                     | 1.2722(5)                        | 1.2429(5)                        |
| C-ud3-sb       | 1.3562(5)                     | 1.3101(5)                     | 1.2721(5)                        | 1.2428(5)                        |
| C-ud3-sa       | 1.3563(5)                     | 1.3102(5)                     | 1.2722(5)                        | 1.2430(5)                        |
| C-ud4-sb       | 1.3564(5)                     | 1.3103(5)                     | 1.2723(5)                        | 1.2430(5)                        |
| C-ud4-sa       | 1.3576(5)                     | 1.3112(5)                     | 1.2731(5)                        | 1.2437(5)                        |
| C-ud5-sb       | 1.3589(5)                     | 1.3125(5)                     | 1.2742(5)                        | 1.2448(5)                        |
| C-ud5-sa       | 1.3594(5)                     | 1.3130(5)                     | 1.2747(5)                        | 1.2452(5)                        |
| C-ud2-sa-L     | 1.3559(5)                     | 1.3099(4)                     | 1.2721(4)                        | 1.2432(4)                        |
| M-ud3-sb       | 1.3461(7)                     | 1.2919(6)                     | 1.2553(6)                        | 1.2285(6)                        |
| M-ud3-sa       | 1.3475(6)                     | 1.2932(6)                     | 1.2564(6)                        | 1.2296(5)                        |
| M-ud4-sb       | 1.3483(7)                     | 1.2939(6)                     | 1.2571(6)                        | 1.2302(6)                        |
| M-ud4-sa       | 1.3489(6)                     | 1.2944(6)                     | 1.2575(6)                        | 1.2306(6)                        |
| M-ud5-sb       | 1.3499(7)                     | 1.2953(6)                     | 1.2583(6)                        | 1.2312(6)                        |
| M-ud5-sa       | 1.3511(6)                     | 1.2964(6)                     | 1.2594(6)                        | 1.2323(6)                        |
| F-ud3-sa       | 1.3435(7)                     | 1.2892(6)                     | 1.2536(6)                        | 1.2275(6)                        |

TABLE IV. Reduced moment $Z_V^{-\frac{n^2}{2}} R^V_n$ for each ensemble.
This work Phenomenological estimates

| $n_f = 3$ | $n_f = 4$ | [27] | [7] | [25] | [26] |
|-----------|-----------|------|-----|------|------|
| $r_0^V(0)$ | 1.3191(33)(12)(4)(34) | 1.3181(33)(13)(4)(33)(34) | 1.3143(61) | 1.3185(59) | 1.2994(184) | 1.2978(176) |
| $r_8^V(0)$ | 1.2680(22)(7)(2)(28) | 1.2714(22)(8)(2)(28)(86) | 1.2732(44) | 1.2749(44) | 1.2620(135) | 1.2596(120) |
| $r_{10}^V(0)$ | 1.2365(16)(13)(0)(22) | 1.2558(16)(13)(0)(22)(423) | 1.2439(35) | 1.2447(34) | 1.2352(104) | 1.2330(91) |

TABLE V. Reduced moments $R_n^V(0)$ extrapolated to the continuum limit at physical light quark masses. The errors in “this work” are from statistical, discretization, finite volume, and the input value of $t_0^{1/2}$, respectively. The numbers for $n_f = 3$ are the lattice data with 2+1 flavors of dynamical quarks, while those for $n_f = 4$ are after the correction by $r_n(n_f = 4)/r_n(n_f = 3)$. The last error for “$n_f = 4$” is from this perturbative correction factor. Phenomenological estimates from [7, 25–27] are shown with the estimated error in these references.

V. TEMPORAL MOMENTS OF PSEUDOSCALAR CURRENT CORRELATOR

The reduced moments $R_n$ ($n = 6, 8, 10$ and 12) for the pseudo-scalar channel obtained at each ensemble are listed in Table VI and their ratios $R_n/R_{n+2}$ are in Table VII.

By comparing the data at two different volumes, which are available for the coarse lattice with the lightest sea quarks ($\beta = 4.17, \text{am}_{ud} = 0.0035$), we observe that the results on the larger volume $48^3 \times 96$ (C-ud2-sa-L) are lower than those on $32^3 \times 64$ (C-ud2-sa) by about two standard deviations for $R_4$ and $R_6$. For $R_8$, $R_{10}$ and $R_{12}$, on the other hand, the data at different volumes coincide within the statistical error. We estimate the systematic error due to finite volume effect by taking these differences and applying them for all the other ensembles assuming similar values for each. This should give a conservative estimate because the finite volume effect is expected to be significantly less for heavier sea quarks. We note that the value of $m_\pi L$ is small ($\sim 3.0$) only for this ensemble (C-ud2-sa); others satisfy $m_\pi L > 3.9$. As listed in the table of systematic errors in final results (Table IX), the estimated error from this source is an order of magnitude smaller than other sources, and any combined error of the finite volume effect with other sources is negligible.

We interpolate $R_n$ in $m_c$ to the physical point by tuning until the spin-averaged mass $(m_{\eta_c} + 3m_{J/\psi})/4$ reproduces the experimental value, 3.0687 GeV. Figure 3 shows an extrapolation of below. the spin-averaged mass to the physical pion mass. A fit is done assuming
that the slope in $m_c^2$ is independent on $\beta$, which seems reasonable as the plot shows. The $\chi^2$/dof of this fit is 1.9.

Our lattice results extrapolated to the physical pion mass are slightly lower than the experimental data by about 0.1–0.3% depending on $\beta$ because of slight mistuning of the input $m_c$. We correct for them by using the supplemental data taken at three different $m_c$'s for each $\beta$ as discussed.

Figure 4 is an example of the $m_c$ dependence of $R_6$ obtained at $\beta = 4.47$. Our main data point (filled square) is slightly off the target physical value of the physical $(m_{\eta_c} + 3m_{J/\psi})/4$ shown by a dashed line. We correct the data using a slope obtained from the supplemental data at three values of $m_c$ shown in Figure 4. The supplemental data have significantly larger statistical error, but are sufficiently precise to determine the slope needed for the correction. (The fit to obtain the slope is uncorrelated. The effect of ignoring the correlation among three data points should have little impact on the final result, since the correction itself is

|     | $R_6$   | $R_8$   | $R_{10}$ | $R_{12}$ |
|-----|---------|---------|----------|----------|
| C-ud2-sa | 1.4689(6) | 1.3681(5) | 1.3087(4) | 1.2679(4) |
| C-ud3-sb | 1.4696(5) | 1.3686(5) | 1.3090(4) | 1.2682(4) |
| C-ud3-sa | 1.4692(5) | 1.3683(5) | 1.3089(4) | 1.2681(4) |
| C-ud4-sb | 1.4696(6) | 1.3687(5) | 1.3091(4) | 1.2683(4) |
| C-ud4-sa | 1.4706(5) | 1.3693(5) | 1.3097(4) | 1.2687(4) |
| C-ud5-sb | 1.4720(5) | 1.3705(5) | 1.3107(4) | 1.2696(4) |
| C-ud5-sa | 1.4722(6) | 1.3708(5) | 1.3109(4) | 1.2699(4) |
| C-ud2-sa-L | 1.4693(5) | 1.3684(4) | 1.3091(4) | 1.2685(4) |
| M-ud3-sb | 1.4869(6) | 1.3598(5) | 1.2977(5) | 1.2582(4) |
| M-ud3-sa | 1.4882(6) | 1.3609(5) | 1.2986(5) | 1.2590(4) |
| M-ud4-sb | 1.4888(7) | 1.3611(6) | 1.2987(5) | 1.2590(5) |
| M-ud4-sa | 1.4896(6) | 1.3618(5) | 1.2994(5) | 1.2596(4) |
| M-ud5-sb | 1.4899(7) | 1.3621(5) | 1.2996(5) | 1.2598(5) |
| M-ud5-sa | 1.4912(6) | 1.3631(5) | 1.3005(4) | 1.2605(4) |
| F-ud3-sa | 1.4961(6) | 1.3616(5) | 1.2987(5) | 1.2590(4) |

TABLE VI. Reduced moment $R_n$ in each ensemble. The errors shown are statistical.
FIG. 3. Spin-averaged mass $(m_{\eta_c} + 3m_{J/\psi})/4$ as a function of $m^2_{\pi}$. The experimental value, 3.072 GeV, is shown by a filled circle. Data at $\beta = 4.17$ (square), $\beta = 4.35$ (circle), and $\beta = 4.47$ (triangle) are plotted. At each $\beta$, the extrapolation to the physical pion mass slightly misses the experimental value since the input $m_c$ is not exactly tuned. This tiny difference is corrected when we analyze the temporal moments.

FIG. 4. $R_6$ as a function of the spin-averaged mass $a(m_{\eta_c} + 3m_{J/\psi})/4$ from different $m_c$. Data at $\beta = 4.47$ are shown. The dashed line represents the physical spin-averaged mass. Three data points shown by open square are from the supplemental data set obtained without using the $Z_2$ noise source. The filled square is our main data point calculated with $Z_2$ noise source.
TABLE VII. Ratios of the reduced moment $R_n/R_{n+2}$ for each ensemble. The errors represent that of statistical.

|                | $R_6/R_8$          | $R_8/R_{10}$     | $R_{10}/R_{12}$ |
|----------------|--------------------|------------------|-----------------|
| C-ud2-sa       | 1.07365(8)         | 1.04540(3)       | 1.03216(2)      |
| C-ud3-sb       | 1.07381(7)         | 1.04548(3)       | 1.03220(2)      |
| C-ud3-sa       | 1.07368(7)         | 1.04542(3)       | 1.03217(2)      |
| C-ud4-sb       | 1.07377(7)         | 1.04547(3)       | 1.03221(2)      |
| C-ud4-sa       | 1.07396(7)         | 1.04555(3)       | 1.03226(2)      |
| C-ud5-sb       | 1.07406(7)         | 1.04563(3)       | 1.03234(2)      |
| C-ud5-sa       | 1.07400(8)         | 1.04563(3)       | 1.03235(2)      |
| C-ud2-sa-L     | 1.07369(5)         | 1.04529(2)       | 1.03204(1)      |
| M-ud3-sb       | 1.09346(11)        | 1.04783(5)       | 1.03144(3)      |
| M-ud3-sa       | 1.09360(9)         | 1.04792(4)       | 1.03151(2)      |
| M-ud4-sb       | 1.09380(12)        | 1.04801(5)       | 1.03157(3)      |
| M-ud4-sa       | 1.09384(12)        | 1.04802(5)       | 1.03158(3)      |
| M-ud5-sb       | 1.09388(12)        | 1.04808(5)       | 1.03162(3)      |
| M-ud5-sa       | 1.09401(9)         | 1.04814(4)       | 1.03167(2)      |
| F-ud3-sa       | 1.09882(10)        | 1.04839(4)       | 1.03150(3)      |

very small.) The correction factor on this ensemble is tiny, i.e. $\sim 0.03\%$.

When we interpolate to the physical point of $m_c$, we need to incorporate the uncertainty of the lattice spacing originating from the input value of $t_0^{1/2}$. This error is propagated to the following analysis by repeating the same analysis with the lattice spacing $a$ set to the upper and lower limits of its uncertainty.

We extrapolate $R_n$ to the continuum limit assuming the form similar to (IV.2):

$$ R_n = R_n(0) \left(1 + c_1(a m_c)^2\right) \times \left(1 + f_1 m_u + m_d + m_s \right). \quad (V.1) $$

This continuum extrapolation is shown in Figure 5.

Remaining discretization error is estimated as in the vector channel by taking the difference between the extrapolations with two and three data points. The lattice data at different values of $a$ and sea quark masses are statistically independent. We use the stan-
Continuum extrapolation of $R_n(a)$. Data points correspond to $R_6$, $R_8$, $R_{10}$, $R_{12}$, and $R_{14}$ from top to bottom. The continuum extrapolation assuming the form (V.1) is shown by lines. The points at $a = 0$ represent our estimate obtained from a mean of the extrapolated values with and without the coarsest lattice data.

Table VIII summarizes the results for $R_n(0)$. Systematic error due to finite volume is estimated as described above.

![FIG. 5. Continuum extrapolation of $R_n(a)$. Data points correspond to $R_6$, $R_8$, $R_{10}$, $R_{12}$, and $R_{14}$ from top to bottom. The continuum extrapolation assuming the form (V.1) is shown by lines. The points at $a = 0$ represent our estimate obtained from a mean of the extrapolated values with and without the coarsest lattice data.](image)

|            | $n_f = 3$                  | $n_f = 4$                  |
|------------|---------------------------|---------------------------|
| $R_6(0)$   | $1.5048(5)(5)(4)(66)$     | $1.5094(5)(5)(4)(66)$     |
| $R_8(0)$   | $1.3570(4)(22)(3)(39)$    | $1.3589(4)(22)(3)(39)$    |
| $R_{10}(0)$| $1.2931(4)(27)(5)(27)$   | $1.2965(4)(27)(5)(27)$   |
| $R_6(0)/R_8(0)$ | $1.1089(1)(13)(0)(17)$ | $1.1108(1)(13)(0)(17)$ |
| $R_8(0)/R_{10}(0)$ | $1.0494(0)(5)(1)(8)$   | $1.0481(0)(5)(1)(8)$   |

The errors represent statistical, discretization effect, finite volume effect, and the input value of $t_0^{1/2}$ in the order given.

Standard $\chi^2$ fitting; the value of $\chi^2$/dof is 2.1, 4.1, 5.1, 4.6, and 3.9 for $R_6$, $R_8$, $R_{10}$, $R_{12}$, and $R_{14}$, respectively.
Again we correct the lattice result of $n_f = 3$ to that of $n_f = 4$, by multiplying by $r_n(n_f = 4)/r_n(n_f = 3)$. This numerical factor is 1.0031, 1.0014, and 1.0026 for $n = 6, 8,$ and 10, respectively. Table VII lists the data before and after this correction.

VI. SYSTEMATIC ERRORS ON THE CONTINUUM SIDE

As summarized in Section II, one may use (II.23) and (II.24) to extract $\alpha_s(\mu)$ and $m_c(\mu)$ with the lattice inputs for $R_n$ obtained in the previous section. Several sources of systematic errors mainly on the perturbative side, are discussed in this section.

A. Truncation of perturbative series

Perturbative coefficients for $r_n$ are available up to $O(\alpha_s^3)$ as listed in Table I and the remaining error is $O(\alpha_s^4)$. Since the left-hand side of (II.23) is independent of the renormalization scale $\mu$, we estimate the truncation error from the residual $\mu$ dependence of the combination $r_n(\alpha_s(\mu), m_c(\mu))/m_c(\mu)$ on the right-hand side. We take $\mu = 3$ GeV for a central value and consider the variation in the range of $\pm 1$ GeV for the estimate of the truncation error. Figure 6 shows an example for $n = 8$. The $\mu$ dependence of $r_n(\alpha_s(\mu), m_c(\mu))$ is almost canceled by the dependence of $m_c(\mu)$, and the remnant $\mu$ dependence is tiny but non-zero which we take as the truncation error.

We generalize this argument by taking the scale to define $\alpha_s(\mu)$ and $m_c(\mu)$ differently. Namely, we reorganize the perturbative series in terms of $\alpha_s(\mu_\alpha)$ and $m_c(\mu_m)$ with $\mu_\alpha \neq \mu_m$. This can be done by inserting an expansion of $\alpha_s(\mu = \mu_m)$ in terms of $\alpha_s(\mu_\alpha)$ into the formula of $r_n(\alpha_s(\mu), m_c(\mu))$ and rearranging the perturbative series. The terms of $O(\alpha_s^4(\mu_\alpha))$ are then truncated.

After this extension, we estimate the truncation error by taking a variation in the range $\mu_\alpha = \mu_m \pm 1$ GeV with $2$ GeV $\leq \min\{\mu_\alpha, \mu_m\}$ and $\max\{\mu_\alpha, \mu_m\} \leq 4$ GeV. This provides more conservative estimate of the truncation error than simply taking $2$ GeV $\leq \mu_\alpha = \mu_m \leq 4$ GeV. Because of this choice, our estimate for the truncation error is larger than those in the previous works.
FIG. 6. Residual scale dependence of the ratio $r_n(\mu)/m_c(\mu)$. The case for $n = 8$ is plotted as a typical example.

**B. Non-perturbative corrections**

The perturbative expansion is supplemented by non-perturbative power corrections in the operator product expansion. Such power corrections should be carefully examined before applying the perturbative expansion for the current correlators.

At the lowest non-trivial order, which is of the order of $1/m_c^4$, the gluon condensate $\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle$ appears [29]. At the two-loop order, that is written as

$$\frac{\partial}{\partial q^2} (z\Pi(q^2)_{GG}) = \frac{\partial}{\partial q^2} \left( \frac{\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle}{(2m_{OS})^4} \sum_\ell (a_\ell + \frac{\alpha_s}{\pi} c_\ell) z^\ell \right),$$

where $m_{OS}$ is an on-shell heavy quark mass, and $a_\ell$ and $c_\ell$ are numerical coefficients. The lowest order coefficients $a_\ell$ for the pseudoscalar (PS) and vector (V) correlators are

$$a_\ell^{PS} = -\frac{\ell - 4}{12} \left( \frac{2}{3} \right)^\ell, \quad a_\ell^{V} = -\frac{2\ell - 2}{15} \left( \frac{7}{2} \right)^\ell,$$

with $(p)_\ell = \Gamma(p + \ell)/\Gamma(\ell)$. The higher order coefficients $c_\ell$ may be found in [29]. The on-shell mass $m_{OS}$ appearing in (VI.1) is related to $m_c(\mu)$ as $m_{OS} = m_c(\mu)[1 + \alpha_s/\pi(4/3 - \log m_c(\mu)/\mu)]$ up to $O(\alpha_s^2)$ corrections.

The contribution of this term to the moment $g_{2\ell}^{GG}$ is simply written as

$$g_{2\ell}^{GG} = \frac{\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle}{(2m_{OS})^4} \left( a_\ell + \frac{\alpha_s}{\pi} c_\ell \right),$$

(6.3)
and the reduced moment $r_n$ is modified as

$$r_n^{n-4} = \frac{1}{C_{n/2-1}^{(0)}} \left( C_{n/2-1} + \frac{16\pi^2 \langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle}{3(2m_{OS})^4} \left( a_{n/2} + \frac{\alpha_s}{\pi} c_{n/2} \right) \right).$$

The numerical coefficients $a_{n/2}$ are 0.179, 0.0, −0.208, −0.449 for $n = 6, 8, 10, 12$, respectively.

The uncertainty for the condensate $\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle$ is large, i.e. $\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle = 0.006 \pm 0.012 \mathrm{GeV}^4$ based on $\tau$ decay analysis [30], or from charmonium moments $\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle = 0.005 \pm 0.004 \mathrm{GeV}^4$ [31, 32], 0.022 $\pm 0.004 \mathrm{GeV}^4$ [33]. In our analysis, we treat $\langle (\alpha_s/\pi)G_{\mu\nu}^2 \rangle$ as a free parameter and determine from the charmonium temporal moments together with $m_c(\mu)$ and $\alpha_s(\mu)$. Thus we avoid further uncertainty from this source.

C. Effect of charm sea quark

Our lattice simulations do not contain dynamical charm quark, which is expected to be small since the leading contribution from this effect is $O(\alpha_s^2)$ and further suppressed by a factor of $1/m_c^2$. As already discussed, we estimate this contribution from perturbative calculation of $r_n(n_f = 4)/r_n(n_f = 3)$. We correct our lattice calculation $R_n(n_f = 3)$ by multiplying this correction evaluated perturbatively at $O(\alpha_s^3)$ with $m_c(\mu = 3 \mathrm{GeV}) = 0.9791 \mathrm{GeV}$, and $\alpha_s(\mu = 3 \mathrm{GeV}) = 0.2567$, which are taken from PDG. The numerical factor is 1.0031, 1.0014, and 1.0026 for $n = 6, 8, 10$, respectively for pseudo-scalar.

VII. DETERMINATION OF $m_c(\mu)$ AND $\alpha_s(\mu)$

We combine the non-perturbative calculation of $R_n$ with the perturbative expansion discussed in the previous sections.

An important issue in the precise determination is that the lattice calculation does not exactly correspond to the experimentally observable $\eta_c$ and $J/\psi$ mesons. This is because the electromagnetic interaction and the disconnected diagram contributions are missing. Their masses are used to tune the charm quark mass in the lattice calculation, and the mismatch is a potential source of systematic error.

Instead of including the effects of disconnect diagrams and electromagnetic force in the lattice calculation, we correct the meson masses for these effects. Namely, for the value of
Δ\(J/\psi - \eta_c\) calculated on the lattice and its continuum extrapolation. The error of lattice scale \(a\) from \(t^{1/2}\) is added for each data point.

\[m_{\eta_c}^{\text{exp}} \text{ in (II.23)} \]

we input the experimental value 2,983.6(7) MeV after subtracting the corrections due to disconnected and electromagnetic effects. The effect of disconnected diagrams reduces the \(\eta_c\) mass by 2.4(8) MeV according to a lattice study \cite{34}. The electromagnetic force is also expected to reduce the \(\eta_c\) mass by 2.6(1.3) MeV \cite{35}.

Including these potential systematic effects for the \(\eta_c\) meson mass, we use an input value

\[m_{\eta_c}^{\text{exp}} = 2983.6(0.7) + 2.4(0.8)_{\text{Disc.}} + 2.6(1.3)_{\text{EM}} \text{ MeV}.\]

Discretization effect may also affect the charmonium mass spectrum calculated on the lattice. The hyperfine splitting \(\Delta J/\psi - \eta_c = m_{J/\psi} - m_{\eta_c}\) is known to be sensitive to this source of error. Figure 7 shows \(\Delta J/\psi - \eta_c\) as a function of \(a^2\). A significant \(a^2\) dependence is visible on the lattice data especially for the coarsest lattice, for which the value of \(\Delta J/\psi - \eta_c\) is about 12\% lower than those at two finer lattices. We attempt a continuum extrapolation assuming a linear dependence on \(a^2\). The extrapolation yields 111.4(1.8) MeV, which is consistent with the experimental value, 110.9(2.1) MeV. It provides another evidence that the discretization effect for the charmonium correlator is under good control after the extrapolation by a linear extrapolation in \(a^2\).

Finally, we extract the charm quark mass \(m_c(\mu)\), strong coupling constant \(\alpha_s(\mu)\), as well as the gluon condensate \(\langle(\alpha/\pi)G^2\rangle/m_{\text{OS}}^4\), using (II.23) with three temporal moments \(R_6\), \(R_8\) and \(R_{10}\) as inputs. We also use the ratio of the moments \(R_6/R_8\) as in (II.24), which
FIG. 8. Constraints on $m_c(\mu)$ and $\alpha_s(\mu)$ from the moments $R_6$ (dotted curve), $R_8$ (dashed curve), $R_{10}$ (long dashed curve), and $R_6/R_8$ (solid curve). For each curve, the band represents the error due to the truncation of perturbative expansion.

is not independent from the individual moments but provides a consistency check as the truncation of perturbative expansion is different.

Figure 8 shows the constraints on $m_c(\mu)$ and $\alpha_s(\mu)$ at $\mu = 3$ GeV given by $R_6$, $R_8$, $R_{10}$, and $R_6/R_8$. The value of the gluon condensate is tuned such that the combination $R_6/R_8$, $R_8$, and $R_{10}$ give a simultaneous solution. The plot demonstrates that each moment $R_n$ has a sensitivity to a certain combination of $m_c(\mu)$ and $\alpha_s(\mu)$. The ratio $R_6/R_8$, on the other hand, is sensitive only to $\alpha_s(\mu)$, because by definition (II.24) the ratio depends on $m_c(\mu)$ only logarithmically.

Table IX lists the numerical results for the three parameters including the breakdown of estimated errors. They include those from the truncation of perturbative expansion, statistical, discretization error of $O(a^4)$ (or $O(\alpha_s a^2)$), finite volume, experimental data for $m_{\eta_c}^{\text{exp}}$, disconnected contribution, electromagnetic effect. The estimation of these individual errors is already described in previous sections.

Clearly, the truncation of the perturbative expansion is the dominant source of error for
TABLE IX. Numerical results for $m_c(\mu)$ (top panel), $\alpha_s(\mu)$ (mid panel) and $\langle (\alpha_s/\pi)^2 \rangle/\langle m^4 \rangle$ (bottom panel). The scale dependent quantities, $m_c(\mu)$ and $\alpha_s(\mu)$, are renormalized at $\mu = 3$ GeV. The results are listed for different choices of three input quantities out of $R_6$, $R_8$, $R_{10}$ and $R_6/R_8$. In addition to the central values with combined errors, the breakdown of the error is presented. They are the estimated errors from the truncation of perturbative expansion, the input value of $t_0^{1/2}$, statistical, discretization effect of $O(a^4)$ (or $O(\alpha_s a^2)$), finite volume, experimental data for $m_{\eta_c}^{\text{exp}}$, disconnected contribution, electromagnetic effect, in the order given. The total error is estimated by adding the individual errors in quadrature.

The gluon condensate cannot be determined precisely. In fact, our results are consistent with zero within estimated errors. This is not surprising because this quantity is obtained as a small difference between the perturbative and non-perturbative calculations. It would
FIG. 9. Constraint from the vector-current moments $R_V^6$ on the $(m_c(\mu), \alpha_s(\mu))$ plane. Dotted, dashed, long-dashed, and solid curves correspond to that of $R_6$, $R_8$, $R_{10}$, and $R_6/R_8$, respectively.

strongly depend on the order of perturbative expansion. Still, it shows a reasonable agreement with previous phenomenological estimates [30–33].

One may also use the vector channel to extract $m_c(\mu)$ and $\alpha_s(\mu)$ by performing the same analysis. Unfortunately, it was not very successful in our case. As one can see in Figure 9, the constraints on the $\{m_c(\mu), \alpha_s(\mu)\}$ plane given by different moments $R_6$, $R_8$, $R_{10}$ are similar to each other and we are not able to disentangle $m_c(\mu)$ and $\alpha_s(\mu)$. (The situation may be different if one can include $R_4$, but it contains too large discretization effect as we discussed.) Therefore, unless we use an input for $\alpha_s(\mu)$ for instance, we are not able to use it to determine $m_c(\mu)$. The statistical error is also 3–4 times larger for the vector channel.

Therefore, instead of using the vector channel to extract $m_c(\mu)$ and $\alpha_s(\mu)$, we attempt to determine $Z_V$ in (II.2) with inputs for $m_c(\mu)$ and $\alpha_s(\mu)$ obtained from the pseudo-scalar channel. We obtain $0.925(19)$, $0.937(22)$ and $0.942(31)$ for $\beta = 4.17$, 4.35 and 4.47, respectively. These results are to be compared with the determination using the light quark hadron correlators: $0.955(9)$, $0.964(6)$, $0.970(5)$ [24]. The determination with the charm correlator is slightly lower and has larger errors. The ratio between the two determinations
In this work, we extract the charm quark mass $m_c(\mu)$ and the strong coupling constant $\alpha_s(\mu)$ through the temporal moments of charmonium correlator calculated on lattice ensembles with 2+1 flavors of sea light quarks described by the Möbius domain-wall fermion. The method was originally introduced and developed by the HPQCD-Karlsruhe collaboration [1–3], and we apply it for the lattice data obtained with a different lattice formulation.

The temporal moments in the vector channel can be related to the experimentally available moments of the spectral function, and provide the means to validate or to calibrate the lattice calculations. For the determination of $m_c(\mu)$ and $\alpha_s(\mu)$, we use the pseudo-scalar channel, since the vector channel does not show enough sensitivity to determine $m_c(\mu)$ and $\alpha_s(\mu)$ separately.

For charm quark, the discretization effect could be sizable. Our lattice simulations are carried out at sufficiently small lattice spacings in the range 0.044–0.080 fm, and the continuum extrapolation of the temporal moments is under good control.

Our final results are compared with the PDG numbers [36] in Table X and a comparison with other collaborations are shown in Figure 10. For our results, we take the values of the smallest total uncertainties from Table IX. The charm quark mass $m_c(\mu = 3 \text{ GeV})$ is converted to $m_c(\mu = m_c)$, and the strong coupling constant $\alpha_s(3 \text{ GeV})$ is converted to the value at the $Z$ boson mass using four-loop running formulae. The threshold effect at the

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**TABLE X.** Comparison of our results with the values in the Review of Particle Properties (2014) [36]. All the quantities are understood to be given in the $\overline{\text{MS}}$ scheme.

|                      | this work | PDG (2014) |
|----------------------|-----------|------------|
| $m_c(\mu = 3 \text{ GeV})$ | 1.0033(96) GeV |             |
| $m_c(\mu = m_c)$     | 1.2871(123) GeV | 1.275(25) GeV |
| $\alpha_s(\mu = 3 \text{ GeV})$ | 0.2528(127) | 0.2567(34)  |
| $\alpha_s(\mu = M_Z)$ | 0.1177(26)  | 0.1185(6)   |
| $\Lambda_{n_f=4}^{\text{MS}}$ | 286(37) MeV  | 297(8) MeV  |
| $\Lambda_{n_f=5}^{\text{MS}}$ | 205(32) MeV  | 214(7) MeV  |

is consistent with 1, after taking the continuum limit.

**VIII. CONCLUSION**

In this work, we extract the charm quark mass $m_c(\mu)$ and the strong coupling constant $\alpha_s(\mu)$ through the temporal moments of charmonium correlator calculated on lattice ensembles with 2+1 flavors of sea light quarks described by the Möbius domain-wall fermion. The method was originally introduced and developed by the HPQCD-Karlsruhe collaboration [1–3], and we apply it for the lattice data obtained with a different lattice formulation.

The temporal moments in the vector channel can be related to the experimentally available moments of the spectral function, and provide the means to validate or to calibrate the lattice calculations. For the determination of $m_c(\mu)$ and $\alpha_s(\mu)$, we use the pseudo-scalar channel, since the vector channel does not show enough sensitivity to determine $m_c(\mu)$ and $\alpha_s(\mu)$ separately.

For charm quark, the discretization effect could be sizable. Our lattice simulations are carried out at sufficiently small lattice spacings in the range 0.044–0.080 fm, and the continuum extrapolation of the temporal moments is under good control.

Our final results are compared with the PDG numbers [36] in Table X and a comparison with other collaborations are shown in Figure 10. For our results, we take the values of the smallest total uncertainties from Table IX. The charm quark mass $m_c(\mu = 3 \text{ GeV})$ is converted to $m_c(\mu = m_c)$, and the strong coupling constant $\alpha_s(3 \text{ GeV})$ is converted to the value at the $Z$ boson mass using four-loop running formulae. The threshold effect at the
FIG. 10. Charm quark mass obtained in this work is compared with previous lattice determination. The previous results are separately shown for different number of sea quarks. HPQCD 14 \cite{3}, ETM a 14 \cite{38} and ETM b 14 \cite{39} for $n_f = 2 + 1 + 1$, Maezawa et al. 16 \cite{40}, HPQCD 10 \cite{2}, HPQCD 08 \cite{1}, and $\chi$QCD 14 \cite{41} for $n_f = 2+1$, and ALPHA 13 \cite{42}, ETM a 11 \cite{43}, and ETM b 11 \cite{44} for $n_f = 2$.

The previous results are separately shown for different number of sea quarks. HPQCD 14 \cite{3}, ETM a 14 \cite{38} and ETM b 14 \cite{39} for $n_f = 2 + 1 + 1$, Maezawa et al. 16 \cite{40}, HPQCD 10 \cite{2}, HPQCD 08 \cite{1}, and $\chi$QCD 14 \cite{41} for $n_f = 2+1$, and ALPHA 13 \cite{42}, ETM a 11 \cite{43}, and ETM b 11 \cite{44} for $n_f = 2$.

bottom quark mass is incorporated at one-loop. The resulting value of $\alpha_s(M_Z)$ is consistent with the PDG.

The result of the HPQCD collaboration \cite{3} for the charm quark mass is $m_c(\text{3 GeV}) = 0.9851(63) \text{ GeV}$. Our result is $1.8\pm1.2\%$ higher. Since the perturbative part of the method is common, a part of the error may be correlated among us.

Among various sources of the systematic error, the dominant one is the truncation of perturbative expansion, which is currently known up to $O(\alpha^3_s)$. In order to improve the precision on $m_c(\mu)$, therefore, higher order perturbative calculation has a primary importance, as well as the reduction of the scale uncertainty, which is common for all dimensionful parameters.
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Appendix A: Tree-level pole mass of domain-wall fermion

At the tree-level, the propagator of domain-wall fermion formulation on the lattice is written as

\[
\langle q(-p)q(p) \rangle = \frac{-i\gamma_\mu \sin p_\mu + m(1 - W e^{-\alpha})}{-(1 - We^\alpha) + m^2(1 - We^{-\alpha})},
\]

(A.1)

where the Wilson term \( W(p) \) is

\[
W(p) = 1 - M - r \sum_\mu (1 - \cos p_\mu).
\]

(A.2)

We take the parameters \( M = 1 \) and \( r = -1 \), according to the choice adopted in our simulations.

We obtain the pole mass at the tree-level \( \tilde{m}_1 \) by finding a pole of \( \langle q(-p)q(p) \rangle \). For zero spatial momentum, we solve the equation to define the pole with \( p_0 = i\tilde{m}_1 \). The result is

\[
\tilde{m}_1 = \cosh^{-1}\left( \frac{1 - Q + \sqrt{3Q + Q^2}}{2} \right)
\]

(A.3)

with \( Q = ((1 + m^2)/(1 - m^2))^2 \).

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