Abstract: We demonstrate that the celebrated Stückelberg formalism gets modified in the case of a massive four (3+1)-dimensional (4D) Abelian 2-form theory due to the presence of a self-duality discrete symmetry in the theory. The latter symmetry entails upon the modified 4D massive Abelian 2-form gauge theory to become a massive model of Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism where there is the existence of a set of (anti-)co-BRST transformations corresponding to the usual nilpotent (anti-)BRST transformations. The latter exist in any arbitrary dimension of spacetime for the Stückelberg-modified massive Abelian 2-form gauge theory. The modification in the usual Stückelberg technique is backed by the precise mathematical arguments from differential geometry where the (co-)exterior derivatives play decisive roles. The modified version of the Stückelberg technique remains invariant under the discrete duality symmetry transformations which also establish a precise and deep connection between the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations of our 4D theory.

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1 Introduction

The concepts of pure mathematics and their applications in the progress of theoretical physics have been intertwined together in a meaningful manner since the advent of physics in the realm of modern-day science. In particular, the recent developments in theoretical high energy physics owe a great deal to some of the key ideas and concepts behind pure mathematics. For instance, we know that the concepts of differential geometry have found decisive applications in the domain of theoretical research activities related to the gauge theories, gravitational theories, (super)string theories, topological field theories, etc. In this context, it is pertinent to point out that the celebrated Stückelberg-technique of compensating field(s) \([1]\), responsible for the massive field theories (e.g. Proca theory) to acquire the beautiful gauge symmetry invariance, is also based on the ideas of the differential geometry (see, e.g. \([2-5]\) for details). In particular, the exterior derivative \((d = dx^\mu \partial_\mu, d^2 = 0)\) plays a key role [see, e.g. Eq. (6) below] in the replacement/modification of the gauge field due to the presence of some compensating field(s) (e.g. a pure scalar field in the context of Proca theory) which converts the second-class constraints of the massive theory to the first-class constraints (see, e.g. \([6,7]\) for details). The latter appear in the expression of generator for the gauge symmetry transformations (of the Stückelberg-modified theory).

One of central purposes of our present endeavor is to demonstrate that the standard Stückelberg-technique gets modified in the context of massive Abelian \(p\)-form \((p = 1, 2, 3, ...\)) gauge theories in \(D = 2p\) dimensions of spacetime because such kinds of massive theories respect, in addition to the gauge symmetry transformations (that are generated by the first-class constraints in the terminology of Dirac’s prescription for the classification of the constraints \([6, 7]\)), the dual-gauge symmetry transformations which appear in the gauge-fixed Lagrangian densities of the above kinds of theories. In a very recent work \([8]\), we have been able to corroborate the above claim in the context of a 2D Proca (i.e. the massive Abelian 1-form) theory. In fact, we have been able to demonstrate that, due to the modified version of the Stückelberg formalism (SF), we obtain the nilpotent (anti-)BRST and (anti-)co-BRST symmetries for the gauge-fixed Lagrangian density of the Stückelberg-modified 2D Proca theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism (cf. Appendix A below). It is worthwhile to mention that the massless Abelian \(p\)-form \((p = 1, 2, 3, ...\)) gauge theories (in \(D = 2p\) dimensions of the spacetime) have been proven to be field-theoretic models of Hodge theory (see, e.g. \([9]\) and reference therein). Further, we have been able to show that the Stückelberg-modified massive 2D Abelian 1-form (i.e. Proca theory) and 4D Abelian 2-form theory are, once again, very interesting examples of Hodge theory within the framework of BRST formalism (see, e.g. \([10-12]\)).

The central theme of our present investigation is to show that the Stückenberg-modified Lagrangian density of the massive 4D Abelian 2-form theory respects the (anti-)BRST symmetry transformations in any arbitrary dimension of spacetime. However, in the physical four \((3+1)\)-dimensions of spacetime, it respects the (anti-)BRST as well as the (anti-)co-BRST symmetry transformations due to \((i)\) the modification in the standard Stückenberg technique [cf. Eq. (2)] where an axial-vector field \((\tilde{\phi}_\mu)\) also appears explicitly [cf. Eq. (7)] backed by the precise mathematical arguments, and \((ii)\) the existence of a set of discrete duality symmetry transformations under which the modified Stückenberg-technique [cf. Eq. (9)] as well as the 4D Lagrangian density \(L\) [cf. Eq. (29)] both remain invariant. The gen-
eralization of these discrete duality symmetry transformations, within the realm of BRST formalism [cf. Eqs. (48), (54)] also establish a precise connection between the (anti-)BRST and (anti-)co-BRST symmetry transformations [9]. We provide proper arguments, however, to demonstrate that the nilpotent (anti-)BRST and (anti-)co-BRST transformations have their own identities as they provide the physical realizations [12] of the (co-)exterior derivatives of the differential geometry [2-5] which are also independent of each-other.

In our present endeavor, for the sake of brevity, we consider only the (co-)BRST invariant Lagrangian density (cf. Sec. 6) that is the generalization of $L_{(b_1)}$ [cf. Eq. (43)] and establish a connection between the BRST and co-BRST symmetry transformations due to the existence of a couple of discrete duality symmetry transformations (48) plus (54). In exactly similar fashion, the generalization of the Lagrangian density $L_{(b_2)}$ [cf. Eq. (54)] can be obtained at the quantum level (within the framework of BRST formalism) as $L_{\overline{B}}$ which will be anti-BRST as well as anti-co-BRST invariant [12]. Once again, we shall be able to establish the interconnection between the anti-BRST and anti-co-BRST symmetry transformations by exploiting the theoretical potential and power of the discrete duality symmetry transformation (48) and (54) at the quantum level (see, e.g. [12, 9] for details). Besides this connection, there exists another relationship between the co-BRST and BRST symmetry transformations [cf. Eq. (60)] which provide the physical realization of the relationship between co-exterior and exterior derivative of differential geometry [2-5].

The following key factors have been at the heart of our present investigation. First and foremost, in a very recent work [8], we have discussed the modification of the standard St"uckelberg-formalism in the context of a massive Abelian 1-form (i.e. Proca) theory in two (1+1)-dimensions of spacetime. Hence, we have been curious to find out its analogue in the context of a massive Abelian 2-form theory in the physical four (3+1)-dimensions of spacetime. Second, we envisage to find out the existence of fields with negative kinetic terms on the basis of symmetry properties of our present theory because such kinds of exotic fields are the possible candidates for the dark matter and dark energy and they play an important role in the context of the cyclic, bouncing and self-accelerated cosmological models of Universe (see, e.g. [13-15] and references therein). Third, we desire to establish a connection between the nilpotent (anti-)BRST and (anti-)co-BRST transformations on the basis of discrete duality symmetry transformations [cf. Eqs. (48), (54)] in our theory. Fourth, the higher $p$-form ($p = 2, 3, ...$) gauge theories of massless and massive varieties are interesting from the point of view of (super)string theories as they appear in their excitations. Finally, we wish to find out the physical realizations of the Hodge duality operator of differential geometry [2-5] in terms of the discrete duality symmetry transformations.

The theoretical material of our present endeavor is organized as follows. In Sec. 2, we discuss the bare essentials of the gauge symmetry transformations for the standard St"uckelberg-modified Lagrangian density in any arbitrary $D$-dimension of spacetime. Our Sec. 3 deals with the modification of the St"uckelberg-formalism where the (co-)exterior derivatives of differential geometry play decisive roles. The subject matter of Sec. 4 concerns itself with the derivation of the 4D Lagrangian densities that respect the (dual-)gauge symmetry transformations together for the gauge-fixed Lagrangian densities provided exactly similar kinds of restrictions are imposed on the (dual-)gauge transformation parameters from outside. Our Sec. 5 contains the theoretical discussion on the linearized versions of the gauge-fixed Lagrangian densities and Curci-Ferrari (CF) type restrictions. In Sec.
6, we establish a relationship between the BRST and co-BRST symmetry transformations due to the discrete duality symmetry transformations [cf. Eqs. (48), (54)] in our BRST-invariant theory. Finally, in Sec. 7, we make some concluding remarks and point out a few future theoretical directions for further investigation(s).

In our Appendix A, we very briefly recapitulate the bare essentials of our earlier work [8] on the St"uckelberg-modified 2D Proca theory where modified-SF has been used. The theoretical contents of our Appendix B is devoted to the generalization of the classical symmetry transformations (37) and (35) to their quantum counterparts (co-)BRST symmetry transformations for the appropriate (co-)BRST invariant Lagrangian density.

Convention and Notations: We follow the convention of the left-derivative w.r.t. all the fermionic (i.e. $\bar{C}_\mu, C_\mu, \bar{C}, C, \rho, \lambda,$) fields of our theory in the context of the derivation of the equations of motions, definition of the conjugate momenta and deduction of the Noether conserved currents and charges. The 4D Levi-Civita tensor is denoted by $\varepsilon_{\mu\nu\lambda\xi}$ with conventions: $\varepsilon_{0123} = +1 = -\varepsilon_{0123}$ and $\varepsilon_{\mu\nu\lambda\xi} \varepsilon_{\mu\nu\lambda\rho} = -4! \delta^\rho_\xi, \varepsilon_{\mu\nu\lambda\xi} \varepsilon_{\mu\nu\rho\sigma} = -2! (\delta^\rho_\lambda \delta^\sigma_\xi - \delta^\rho_\xi \delta^\sigma_\lambda),$ etc., where the Greek indices $\mu, \nu, \lambda, ... = 0, 1, 2, 3$ stand for the time and space directions and the Latin indices $i, j, k,... = 1, 2, 3$ correspond to space directions only. Hence, the 3D Levi-Civita tensor is $\epsilon_{ijk} = \varepsilon_{0ijk}$. The background flat 4D Minkowskian spacetime manifold is endowed with a flat metric tensor $\eta_{\mu\nu} = \text{diag} (+1, -1, -1, -1)$ so that the dot product between two non-null 4-vectors $P_\mu$ and $Q_\mu$ is represented by: $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i$. We denote the (anti-)BRST transformations by $s(a)b$ and the notation $s(a)d$ stands for the (anti-)dual [i.e. (anti-)co]-BRST transformations. In the same way, the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges are denoted by $Q(a)b$ and $Q(a)d$, respectively (corresponding to the above nilpotent symmetries).

Standard Definition: On a compact manifold without a boundary, we have a set of three operators ($d, \delta, \Delta$) which are known as the de Rham cohomological operators of differential geometry. The operators ($\delta$) $d$ are called as the (co-)exterior derivatives that are connected with each-other by the relationship: $\delta = \pm \ast \ d$ where $\ast$ is the Hodge duality operator on the above manifold. These operators satisfy the algebra: $d^2 = \delta^2 = 0, \Delta = (d + \delta)^2 = \{d, \delta\}, \ [\Delta, d] = [\Delta, \delta] = 0$ where $\Delta$ is the Laplacian operator [2-5]. This algebra (which is not a Lie algebra) is popularly known as the Hodge algebra and $\Delta$ behaves like a Casimir operator for the whole algebra (but not in the Lie algebraic sense). We shall be frequently using the names of these cohomological operators in our present endeavor.

2 Preliminaries: St"uckelberg Formalism in Any Arbitrary Dimension of Spacetime

We begin with the D-dimensional Lagrangian density ($L_0$) for the massive Abelian 2-form ($B^{(2)} = [(d x^\mu \wedge d x^\nu)/2!] B_{\mu\nu}$) theory with the anti-symmetric tensor ($B_{\mu\nu} = -B_{\nu\mu}$) field (carrying the rest mass $m$) as follows (see, e.g. [16] and references therein)

$$L_0 = \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu}, \quad (1)$$
where $H^{(3)} = dB^{(2)} = [(dx^\mu \wedge dx^\nu \wedge dx^\lambda)/3!] H_{\mu\nu\lambda}$ defines the kinetic term (with $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$ for the anti-symmetric tensor field $B_{\mu\nu}$ where the Greek indices $\mu, \nu, \lambda, \ldots = 0, 1, \ldots, D - 1$). It is straightforward to check that the equation of motion (EoM): $\partial_\mu H^{\mu\nu\lambda} + m^2 B^{\nu\lambda} = 0$ implies the subsidiary conditions: $\partial_\nu B^{\nu\lambda} = \partial_\lambda B^{\nu\lambda} = 0$ which emerge out from it for $m^2 \neq 0$. As a consequence, we observe that $B_{\mu\nu}$ field obeys the Klein-Gordon equation $(\Box + m^2) B_{\mu\nu} = 0$. We note that the massive Lagrangian density (1) does not respect the gauge transformation due to the fact that it is endowed with the second-class constraints in the terminology of Dirac’s prescription for the classification of constraints (because the gauge symmetries are generated by the first-class constraints [6,7]).

The gauge symmetry transformations can be restored for the modified version of the standard Lagrangian density (1) if we exploit the basic theoretical methodology of the Stückelberg-formalism (SF) [17] related with compensating field(s). In other words, due to SF, we replace the basic antisymmetric Abelian 2-form field $B_{\mu\nu}$ as follows [16, 17]

$$B_{\mu\nu} \rightarrow B_{\mu\nu} \mp \frac{1}{m} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu),$$

(2)

where the Abelian 1-form $\Phi^{(1)}_\mu = dx^\mu \phi_\mu$ defines the vector field $\phi_\mu$. It is straightforward to check that $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$ remains invariant under (2). We note that the mass dimension of $B_{\mu\nu}$ and $\phi_\mu$ fields are same in the D-dimensional spacetime when we use the natural units: $\hbar = c = 1$. Hence, the rest mass $m$ should be present in the denominator of Eq. (2) to balance the mass dimension on the l.h.s and r.h.s. of Eq. (2) in the above natural units. The mass term in Eq. (1) changes as follows, due to (2), namely:

$$-\frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \rightarrow -\frac{m^2}{4} \left[ B_{\mu\nu} \mp \frac{1}{m} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \left[ B^{\mu\nu} \mp \frac{1}{m} (\partial_\mu \phi^\nu - \partial_\nu \phi^\mu) \right].$$

(3)

Let us define an Abelian 2-form $F^{(2)} = d\Phi^{(1)} = [(dx^\mu \wedge dx^\nu)/2!] \Phi_{\mu\nu}$ where $\Phi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$ is the antisymmetric field strength tensor for the vector field $\phi_\mu$. With all these inputs, we obtain the Stückelberg-modified Lagrangian density $\mathcal{L}_S$ from $\mathcal{L}_0$ as

$$\mathcal{L}_0 \rightarrow \mathcal{L}_S = \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \pm \frac{m}{2} B_{\mu\nu} \Phi^{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu},$$

(4)

which (i.e. the Lagrangian density $\mathcal{L}_S$) respects the following continuous and infinitesimal gauge symmetry transformations $\delta_g$, namely:

$$\delta_g H_{\mu\nu\lambda} = 0, \quad \delta_g \phi_\mu = \pm (\partial_\mu \Lambda - m \Lambda_\mu),$$

$$\delta_g B_{\mu\nu} = - (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu), \quad \delta_g \Phi_{\mu\nu} = \mp m (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu),$$

(5)

where the Lorentz-scalar $\Lambda(x)$ and Lorentz-vector $\Lambda_\mu(x)$ are the infinitesimal local gauge symmetry transformation parameters. It is important to point out that there is a stage-one reducibility in the theory because the transformation: $\phi_\mu \rightarrow \phi_\mu \pm \partial_\mu \Lambda$ can be accommodated in the standard Stückelberg-technique [considered in (2)] without changing it in any way. This is why, in the gauge transformation of $\phi_\mu$ field [cf Eq. (5)], we have the local Lorentz scalar transformation parameter $\Lambda(x)$. It is straightforward to check that: $\delta_g \mathcal{L}_S = 0$ implying that the Stückelberg-modified Lagrangian density $\mathcal{L}_S$ respects the infinitesimal local.
gauge symmetry transformations (5) in a perfect manner. In other words, the St"uckelberg modified Lagrangian density \( \mathcal{L}_S \) does not transform to a total spacetime derivative that will ensure the gauge symmetry invariance of the action integral. We mention, in passing, that the second-class constraints of \( \mathcal{L}_0 \) have been converted into the first-class constraints [due to the introduction of the St"uckelberg pure vector field \( \phi_\mu \) in (2)]. The ensuing first-class constraints are the generators for the infinitesimal gauge symmetry transformations \( \delta_g \) in (5). These statements are true in any arbitrary D-dimension of spacetime [17].

3 Massive 4D Abelian 2-Form Theory: Modified SF

In the differential-form terminology, the standard St"uckelberg-technique (2), defined for any arbitrary D-dimension of spacetime, can be re-expressed as [17]:

\[
B^{(2)} \rightarrow B^{(2)} + \frac{1}{m} F^{(2)} \equiv B^{(2)} + \frac{1}{m} d \Phi^{(1)}. \tag{6}
\]

This also establishes the invariance of \( H^{(3)} = d B^{(2)} \) under (2) because of the nilpotency \( (d^2 = 0) \) of the exterior derivative. In the physical four \((3+1)\)-dimensional \((4D)\) flat spacetime, the theoretical technique (6) of the standard St"uckelberg-formalism can be modified in the following manner (in the language of differential-forms), namely:

\[
B^{(2)} \rightarrow B^{(2)} + \frac{1}{m} d \Phi^{(1)} + \frac{1}{m} * \bar{\Phi}^{(1)}, \tag{7}
\]

where the first two terms of the r.h.s. have already been explained. In the third term on the r.h.s., we have taken the axial-vector 1-form \( \bar{\Phi}^{(1)} = d x^\mu \bar{\phi}_\mu \) with the axial-vector field \( \bar{\phi}_\mu \). A pseudo 2-form \( \bar{F}^{(2)} = d \bar{\Phi}^{(1)} = [(d x^\mu \wedge d x^\nu)/2!]\bar{\Phi}_{\mu\nu} \) has been constructed from \( \bar{\Phi}^{(1)} \) by applying an exterior derivative on it so that we obtain \( \bar{\Phi}_{\mu\nu} = \partial_\mu \bar{\phi}_\nu - \partial_\nu \bar{\phi}_\mu \). To bring the parity of \( B^{(2)} \), \( F^{(2)} = d \Phi^{(1)} \) and the pseudo 2-form \( \bar{F}^{(2)} \) on equal footing, it is essential to obtain an ordinary 2-form from the pseudo 2-form \( \bar{F}^{(2)} \) by operating a single Hodge duality operation * on it. This mathematical technique (on the 4D spacetime manifold) leads to

\[
* \bar{F}^{(2)} = * \left( \frac{d x^\mu \wedge d x^\nu}{2!} \right) \bar{\Phi}_{\mu\nu} = \frac{1}{2!} (d x^\mu \wedge d x^\nu) f_{\mu\nu}, \tag{8}
\]

where \( f_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \bar{\phi}^\lambda \bar{\phi}^\xi \). Thus, in the language of a set of antisymmetric tensors \( B_{\mu\nu}, \Phi_{\mu\nu}, f_{\mu\nu} \), we have obtained the following from the modified version of 4D St"uckelberg technique (7), namely;

\[
B_{\mu\nu} \rightarrow B_{\mu\nu} + \frac{1}{m} \left( \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \right) + \frac{1}{m} \varepsilon_{\mu\nu\lambda\xi} \partial^\lambda \bar{\phi}^\xi \\
\equiv B_{\mu\nu} + \frac{1}{m} \Phi_{\mu\nu} + \frac{1}{m} f_{\mu\nu} \equiv B_{\mu\nu} + \frac{1}{m} \left( \Phi_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \bar{\phi}^\lambda \bar{\phi}^\xi \right). \tag{9}
\]

*We assume that the parity symmetry is respected in our discussion on the massive 4D Abelian 2-form theory (unlike the parity violation in the realm of weak interactions).
It is very interesting to state that the above modified 4D St"uckelberg technique remains form-invariant under the following discrete duality symmetry transformations

\[ B_{\mu\nu} \to \mp i \tilde{B}_{\mu\nu} \equiv \mp \frac{i}{2!} \epsilon_{\mu\nu\lambda\xi} B^{\lambda\xi}, \quad \phi_\mu \to \pm i \tilde{\phi}_\mu, \quad \tilde{\phi}_\mu \to \mp i \phi_\mu, \quad (10) \]

where \( \tilde{B}_{\mu\nu} = \frac{1}{2!} \epsilon_{\mu\nu\lambda\xi} B^{\lambda\xi} \) emerges out from the self-duality condition: \( * B^{(2)} = \left[ (d x^\mu \wedge d x^\nu) / 2! \right] B_{\mu\nu} \) which leads to the derivation of dual Abelian 2-form (in 4D) as follows:

\[ \tilde{B}^{(2)} = \left( \frac{d x^\mu \wedge d x^\nu}{2!} \right) \left[ \frac{1}{2!} \epsilon_{\mu\nu\lambda\xi} B^{\lambda\xi} \right] \equiv \left( \frac{d x^\mu \wedge d x^\nu}{2!} \right) \tilde{B}_{\mu\nu}. \quad (11) \]

We shall see that the discrete duality symmetry transformations in (10) will play very important role, later on, as its generalization (within the framework of BRST formalism) will provide the analogue of Hodge duality * operation of differential geometry. We would like to lay emphasis on the fact that the root-cause behind the existence of the discrete duality symmetry transformations in (10) is the self-duality condition on the Abelian 2-form \( (B^{(2)} = \left[ (d x^\mu \wedge d x^\nu) / 2! \right] B_{\mu\nu} \) in the physical four \((3+1)\)-dimensions of spacetime.

It is an elementary exercise to note that the mass term of Eq. (1) transforms, under the modified St"uckelberg-technique [cf. Eq. (9)], as

\[ - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \to - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \pm \frac{m}{2} B_{\mu\nu} \left( \Phi^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \tilde{\Phi}_{\rho\sigma} \right) \]

\[ - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{4} \tilde{\Phi}_{\mu\nu} \tilde{\Phi}^{\mu\nu}, \quad (12) \]

modulo a total spacetime derivative \( \partial_\mu [- \epsilon^{\mu\nu\lambda\xi} \phi_\nu \partial_\lambda \tilde{\phi}_\xi] \) which emerges out from a term \( - \frac{1}{2} \tilde{f}_{\mu\nu} \Phi^{\mu\nu} \) that appears in equation (12) due to the substitution (9). We also point out that the kinetic term \( [(1/12) H_{\mu\nu\lambda} H^{\mu\nu\lambda}] \) also transforms under (9) because it is straightforward to note that we have the following

\[ H_{\mu\nu\lambda} \to H_{\mu\nu\lambda} \mp \frac{1}{m} (\partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu}) \mp \frac{1}{m} (\partial_\mu \tilde{f}_{\nu\lambda} + \partial_\nu \tilde{f}_{\lambda\mu} + \partial_\lambda \tilde{f}_{\mu\nu}), \quad (13) \]

where \( \Phi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \) and \( \tilde{f}_{\mu\nu} = \epsilon_{\mu\nu\lambda\xi} \partial_\lambda \tilde{\phi}_\xi \). We note that the second term on the r.h.s. of the above equation turns out to be zero. However, the third term exists as:

\[ \Sigma_{\mu\nu\lambda} = (\partial_\mu \tilde{f}_{\nu\lambda} + \partial_\nu \tilde{f}_{\lambda\mu} + \partial_\lambda \tilde{f}_{\mu\nu}) \equiv (\epsilon_{\mu\nu\rho\sigma} \partial_\lambda + \epsilon_{\nu\lambda\rho\sigma} \partial_\mu + \epsilon_{\lambda\mu\rho\sigma} \partial_\nu) (\partial^\rho \tilde{\phi}^\sigma). \quad (14) \]

Thus, we have to find out the explicit value of the following (for the changes in the kinetic term due to the modified version of 4D St"uckelberg technique [cf. Eq. (9)]), namely;

\[ \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \to \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \mp \frac{1}{6m} H_{\mu\nu\lambda} \Sigma^{\mu\nu\lambda} + \frac{1}{12m^2} \Sigma_{\mu\nu\lambda} \Sigma^{\mu\nu\lambda}; \quad (15) \]

where we have taken into account: \( H_{\mu\nu\lambda} \to H_{\mu\nu\lambda} \mp \frac{1}{m} \Sigma_{\mu\nu\lambda} \). We focus on the second term \( \mp (1/6m) H^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} \) which can be explicitly written as:

\[ \mp \frac{1}{6m} H^{\mu\nu\lambda} \left[ \epsilon_{\mu\nu\rho\sigma} \partial_\lambda + \epsilon_{\nu\lambda\rho\sigma} \partial_\mu + \epsilon_{\lambda\mu\rho\sigma} \partial_\nu \right] (\partial^\rho \tilde{\phi}^\sigma). \quad (16) \]
The first term on the r.h.s. of the above equation contributes the following (modulo a total spacetime derivative), namely;

$$\pm \frac{1}{6m} (\partial_\lambda H^{\lambda \mu \nu}) \varepsilon_{\mu \nu \rho \sigma} (\partial^\rho \tilde{\phi}^\sigma).$$

(17)

It is self-evident that there are three derivatives in the above expression because $H_{\mu \nu \lambda}$ contains one derivative. Thus, the expression in (17) belongs to a higher derivative term for our 4D theory. It is worthwhile to mention here that in our earlier work [12], the higher derivatives terms have been ignored. This is why, relevant terms in the Lagrangian density have been obtained by trial and error method. However, we note in our present endeavor that one can get rid of a single derivative by using the on-shell condition: $\partial_\lambda H^{\lambda \mu \nu} + m^2 B^{\mu \nu} = 0$. All the three terms, on the r.h.s. of (16), individually contribute to the same result which can be added together to yield the following

$$\mp \frac{m}{2} \varepsilon_{\mu \nu \lambda \xi} B^{\mu \nu} (\partial_\lambda \tilde{\phi}^\xi) \equiv \mp \frac{m}{4} \varepsilon_{\mu \nu \lambda \xi} B_{\mu \nu} \tilde{\Phi}_{\lambda \xi},$$

(18)

where $\tilde{\Phi}_{\lambda \xi} = \partial_\lambda \tilde{\phi}_\xi - \partial_\xi \tilde{\phi}_\lambda$. It is interesting to point out that the above term has been incorporated into the BRST invariant Lagrangian density of our earlier work [12] on the basis of the trial and error method. However, as it self-evident, we have derived this term correctly in our present endeavor which is motivated by our earlier work on 2D Proca theory [8] where we have exploited similar trick to get rid of the higher derivative terms in the 2D massive Abelian 1-form (i.e. 2D Proca) theory. The mass term in Eq. (18) is just like the topological mass term of the $B \wedge F$ theory. In the latter theory, the 4D Abelian 2-form gauge theory also incorporates the Maxwell Abelian 1-form ($A^{(1)} = d x^\mu A_\mu$) gauge field with curvature 2-form $F^{(2)} = d A^{(1)}$. There are many ways to derive (18) from (16). However, we have chosen one of the simplest methods to derive the equation (18) which is not a higher derivative culprit term for our 4D massive Abelian 2-form theory.

We now focus on the explicit computation of the third term on the r.h.s. of (15). It is evident that, for a 4D Abelian 2-form theory, this third term is a higher derivative term because it contains four derivatives in it. A close look at (14) shows that there will also be total nine terms when we take into account $[(1/12 m^2) \Sigma_{\mu \nu \lambda} \Sigma_{\mu \nu \lambda}^\ast]$ and write the expression for $\Sigma_{\mu \nu \lambda}$ from Eq. (16). However, it turns out that only three of them contribute to the Lagrangian density and the rest of six terms are found to be total spacetime derivatives. Let us focus on the first existing term that is equal to:

$$\frac{1}{12 m^2} \varepsilon^{\mu \nu \rho \sigma} \partial^\lambda (\partial_\rho \tilde{\phi}_\sigma) \varepsilon_{\mu \nu \alpha \beta} \partial_\lambda (\partial^\alpha \tilde{\phi}^\beta) = - \frac{1}{6 m^2} [\partial^\lambda (\partial_\alpha \tilde{\phi}_\beta - \partial_\beta \tilde{\phi}_\alpha)] [\partial_\lambda (\partial^\alpha \tilde{\phi}^\beta)]$$

$$= \frac{1}{6 m^2} \partial^\lambda (\partial^\alpha \tilde{\Phi}_{\alpha \beta}) (\partial_\lambda \tilde{\phi}^\beta),$$

(19)

where we have dropped a total spacetime derivative term as it will not affect the dynamics. At this stage, to get rid of derivatives, we use the EoM: $\partial_\alpha \tilde{\Phi}^{\alpha \beta} + m^2 \tilde{\phi}^\beta = 0$. Thus, the final expression on the r.h.s. of Eq. (19) is given by

$$- \frac{1}{6} (\partial^\lambda \tilde{\phi}^\beta) (\partial_\lambda \tilde{\phi}^\beta) \equiv \frac{1}{6} \tilde{\phi}^\beta \Box \tilde{\phi}^\beta,$$

(20)
where we have dropped a total spacetime derivative term. Using the Klein-Gordon equation\footnote{It can be seen that the equation of motion: $\partial_\alpha \tilde{\Phi}^{\alpha \beta} + m^2 \tilde{\Phi}^{\beta} = 0$ implies that $\partial \cdot \tilde{\Phi} = 0$ for $m^2 \neq 0$ due to the antisymmetric ($\tilde{\Phi}^{\alpha \beta} = - \tilde{\Phi}^{\beta \alpha}$) property of the $\tilde{\Phi}^{\alpha \beta}$. As a consequence, we obtain $(\Box + m^2) \tilde{\Phi}^{\beta} = 0$.} $(\Box + m^2) \tilde{\phi}_\mu = 0$, we obtain the following (from the first contribution), namely;

$$\frac{1}{12 m^2} \epsilon^{\mu \nu \rho \sigma} \partial^\lambda (\partial_\rho \tilde{\phi}_\sigma) \epsilon_{\mu \nu \alpha \beta} \partial_\lambda (\partial^\alpha \tilde{\phi}^\beta) \equiv - \frac{m^2}{6} \tilde{\phi}_\beta \tilde{\phi}^\beta. \quad (21)$$

It is obvious that there are three such contributions in the total evaluation of the third term $[(1/2 m^2) \Sigma_{\mu \nu \lambda} \Sigma^{\mu \nu \lambda}]$ on the r.h.s. of Eq. (15). Thus, ultimately, we obtain the following explicit expression from all three existing terms, namely;

$$\frac{1}{12 m^2} \Sigma_{\mu \nu \lambda} \Sigma^{\mu \nu \lambda} = - \frac{m^2}{2} \tilde{\phi}_\mu \tilde{\phi}^\mu. \quad (22)$$

It is clear that the above term is not a culprit term and it is useful for us for our further discussions. The total terms on the r.h.s. of Eq. (15) can be re-expressed as follows:

$$\frac{1}{12} H^{\mu \nu \lambda} H_{\mu \nu \lambda} + \frac{m}{2} \epsilon_{\mu \nu \lambda \xi} B^{\mu \nu} (\partial^\lambda \tilde{\phi}^\xi) - \frac{m^2}{2} \tilde{\phi}_\mu \tilde{\phi}^\mu$$

$$\equiv - \frac{1}{8} \epsilon^{\mu \nu \lambda \xi} (\partial_\nu B_{\lambda \xi}) \epsilon_{\mu \alpha \beta \gamma} (\partial^\alpha B^{\beta \gamma}) \pm \frac{m}{2} \epsilon_{\mu \nu \lambda \xi} (\partial_\mu B^{\nu \lambda}) \tilde{\phi}^\xi - \frac{m^2}{2} \tilde{\phi}_\mu \tilde{\phi}^\mu, \quad (23)$$

where we have dropped a total spacetime derivative term and used the following:

$$\frac{1}{12} H^{\mu \nu \lambda} H_{\mu \nu \lambda} = - \frac{1}{8} \epsilon^{\mu \nu \lambda \xi} (\partial_\nu B_{\lambda \xi}) \epsilon_{\mu \alpha \beta \gamma} (\partial^\alpha B^{\beta \gamma}). \quad (24)$$

The correctness of the above equality can be checked explicitly by using the well-known property of the 4D Levi-Civita tensor. It is straightforward to observe that the final expression for (23) can be written as

$$- \frac{1}{2} \left[ \frac{1}{4} \epsilon^{\mu \nu \lambda \xi} (\partial_\nu B_{\lambda \xi}) \epsilon_{\mu \alpha \beta \gamma} (\partial^\alpha B^{\beta \gamma}) \pm m \tilde{\phi}_\mu \epsilon^{\mu \nu \lambda \xi} (\partial_\nu B_{\lambda \xi}) + m^2 \tilde{\phi}_\mu \tilde{\phi}^\mu \right], \quad (25)$$

where we have used the following

$$\pm \frac{m}{2} \epsilon_{\mu \nu \lambda \xi} (\partial_\nu B^{\mu \lambda}) \tilde{\phi}^\xi = \mp \frac{m}{2} \epsilon_{\mu \nu \lambda \xi} (\partial_\nu B_{\lambda \xi}), \quad (26)$$

to express (15) [and/or (23) and/or (25)] as a squared-term, namely;

$$\frac{1}{12} H_{\mu \nu \lambda} H^{\mu \nu \lambda} + \frac{1}{6 m} H_{\mu \nu \lambda} \Sigma^{\mu \nu \lambda} + \frac{1}{12 m^2} \Sigma_{\mu \nu \lambda} \Sigma^{\mu \nu \lambda} = - \frac{1}{2} \left[ \frac{1}{2} \epsilon_{\mu \nu \lambda \xi} \partial^\nu B^{\lambda \xi} \pm m \tilde{\phi}_\mu \right]^2, \quad (27)$$

which is nothing but the explicit expression for Eq. (25). Thus, we note that the final version of the Lagrangian density (with the modified SF\footnote{It is worthwhile to mention here that the kinetic terms for the $\tilde{\phi}_\mu$ and $\tilde{\phi}^\mu$ fields have a relative sign difference. In other words, one of the above fields has a negative kinetic term which is interesting.}) is\footnote{It is worthwhile to mention here that the kinetic terms for the $\tilde{\phi}_\mu$ and $\tilde{\phi}^\mu$ fields have a relative sign difference. In other words, one of the above fields has a negative kinetic term which is interesting.}

$$\mathcal{L}_S^{(m)} = - \frac{1}{2} \left[ \frac{1}{2} \epsilon_{\mu \nu \lambda \xi} \partial^\nu B^{\lambda \xi} \pm m \tilde{\phi}_\mu \right]^2 - \frac{m^2}{4} B_{\mu \nu} B^{\mu \nu}$$

$$+ \frac{m}{2} \left[ \Phi^{\mu \nu} + \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \tilde{\Phi}_{\rho \sigma} \right] B_{\mu \nu} - \frac{1}{4} \Phi_{\mu \nu} \Phi^{\mu \nu} + \frac{1}{4} \tilde{\Phi}_{\mu \nu} \tilde{\Phi}^{\mu \nu}, \quad (28)$$
where we have taken the inputs from Eqs (12) and (27) and superscript \((m)\) on this Lagrangian density denotes that we have taken the help of modified SF [cf. Eq. (9)].

We end this section with the final remark that we can add a gauge-fixing term for the Abelian 2-form field \((B_{\mu\nu})\), the axial-vector field \((\tilde{\phi}_\mu)\) and the polar-vector field \((\phi_\mu)\) so that we can quantize the theory [described by the Lagrangian density (28)]. At this stage, the role of co-exterior derivative \((\delta = \pm \ast d \ast, \delta^2 = 0)\) becomes quite essential as we note that \(\delta B^{(2)} = (\partial^\nu B_{\nu\mu}) d\xi^\mu, \delta \Phi^{(1)} = (\partial \cdot \phi), \delta \tilde{\Phi}^{(1)} = (\partial \cdot \tilde{\phi})\) where \(\delta = - \ast d \ast\) is the co-exterior derivative defined on the 4D spacetime (which is an even dimensional spacetime manifold).

The full Lagrangian density, with the gauge-fixing terms, is

\[
\mathcal{L} = \mathcal{L}_S^{(m)} + \mathcal{L}_{gf} = \frac{1}{2} \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^{\lambda\xi} \pm m \tilde{\phi}_\mu \right]^2 - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \\
\pm \frac{m}{2} \left[ \Phi^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \tilde{\Phi}_{\rho\sigma} \right] B_{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{4} \tilde{\Phi}_{\mu\nu} \tilde{\Phi}^{\mu\nu} \\
+ \frac{1}{2} \left[ \partial^{\nu} B_{\nu\mu} \pm m \phi_\mu \right]^2 + \frac{1}{2} (\partial \cdot \phi)^2 - \frac{1}{2} (\partial \cdot \tilde{\phi})^2, \tag{29}
\]

where the gauge-fixing term \(\frac{1}{2} (\partial^\nu B_{\nu\mu} \pm m \phi_\mu)^2\) is like the t’Hooft gauge in the context of modified Proca theory [17, 8]. We point out that the above gauge-fixed Lagrangian density respects the duality symmetry transformations (10). The latter, it goes without saying, are also respected by the modified SF that has been defined in Eq. (9). The equations of motion, satisfied by the basic fields \((B_{\mu\nu}, \phi_\mu, \tilde{\phi}_\mu)\), are the Klein-Gordon equations: \((\Box + m^2) B_{\mu\nu} = 0, (\Box + m^2) \phi_\mu = 0, (\Box + m^2) \tilde{\phi}_\mu = 0\) which emerge out from the Lagrangian density (29) which shows that all the fields have the rest mass \(m\).

4 Final Forms of the Gauge-Fixed Lagrangian Densities: Massive 4D Abelian 2-Form Theory

A close look and careful observation of the modified SF in (7) demonstrates that we have the freedom to introduce a pure-scalar \((\phi)\) and pseudo-scalar \((\tilde{\phi})\) fields in our 4D massive theory. The nilpotency \((d^2 = 0)\) of the exterior derivative \((d = d x^\mu \partial_\mu)\) ensures that the following transformations are permitted in our theory, namely;

\[
\Phi^{(1)} \rightarrow \Phi^{(1)} \pm d \Phi^{(0)} \quad \Rightarrow \quad \phi_\mu \rightarrow \phi_\mu \pm \kappa \partial_\mu \phi, \\
\tilde{\Phi}^{(1)} \rightarrow \tilde{\Phi}^{(1)} \pm d \tilde{\Phi}^{(0)} \quad \Rightarrow \quad \tilde{\phi}_\mu \rightarrow \tilde{\phi}_\mu \pm \kappa \partial_\mu \tilde{\phi}, \tag{30}
\]

where \(\kappa\) is a numerical constant. The latter can be chosen to fit our understanding of the CF-type restrictions emerging out from the superfield approach to BRST formalism [18]. As a result of the existence of the above pure-scalar and pseudo-scalar fields, the kinetic and gauge-fixing terms of the gauge field \((B_{\mu\nu})\) and the gauge-fixing terms for \((\phi_\mu)\) and \((\tilde{\phi}_\mu)\) fields will be modified (where the mass dimension of these fields will be taken into consideration). It is pretty obvious that the mass dimension of \(\phi_\mu\) and \(\tilde{\phi}_\mu\) fields is \([M]\). Hence, the mass dimension of fields \(\phi\) and \(\tilde{\phi}\) is zero (i.e. \([M]^0\)). As a consequence, we have
the following modifications of the gauge-fixed Lagrangian density (29), namely;

\[ \mathcal{L} \rightarrow \mathcal{L}_{(1)} = -\frac{1}{2} \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \phi \right] + m \tilde{\phi} \mu \pm \frac{1}{2} \partial_\mu \tilde{\phi} \right]^2 - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \\
+ \frac{m}{2} \left[ \Phi_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \tilde{\Phi}_{\rho\sigma} \right] B_{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi_{\rho\sigma} + \frac{1}{4} \tilde{\Phi}_{\mu\nu} \tilde{\Phi}_{\rho\sigma} \\
+ \frac{1}{2} \left( \partial^\nu B_{\nu\mu} \pm m \phi \mu \pm \frac{1}{2} \partial_\mu \phi \right)^2 + \frac{1}{2} \left( \partial \cdot \phi + \frac{m}{2} \phi \right)^2 - \frac{1}{2} \left( \partial \cdot \phi - \frac{m}{2} \phi \right)^2, \]  

(31)

\[ \mathcal{L} \rightarrow \mathcal{L}_{(2)} = -\frac{1}{2} \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \phi \right] + m \tilde{\phi} \mu \pm \frac{1}{2} \partial_\mu \tilde{\phi} \right]^2 - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu} \\
+ \frac{m}{2} \left[ \Phi_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \tilde{\Phi}_{\rho\sigma} \right] B_{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi_{\rho\sigma} + \frac{1}{4} \tilde{\Phi}_{\mu\nu} \tilde{\Phi}_{\rho\sigma} \\
+ \frac{1}{2} \left( \partial^\nu B_{\nu\mu} \pm m \phi \mu \pm \frac{1}{2} \partial_\mu \phi \right)^2 + \frac{1}{2} \left( \partial \cdot \phi - \frac{m}{2} \phi \right)^2 - \frac{1}{2} \left( \partial \cdot \phi + \frac{m}{2} \phi \right)^2, \]  

(32)

where the mass dimensions of the fields have been taken into account and we have taken into consideration both the signs that are present in (30) and chosen the constant numerical factor to be 1/2. We shall corroborate the logic behind the choice of the terms containing \( \phi \) and \( \tilde{\phi} \) in the modified Lagrangian densities (31) and (32). We shall also dwell a bit on our choice of the factor (1/2) in the kinetic and gauge-fixing terms that contain fields \( \tilde{\phi} \) and \( \phi \), respectively. The latter have been incorporated into \( \mathcal{L}_{(1)} \) and \( \mathcal{L}_{(2)} \) at appropriate places (e.g. the kinetic and gauge-fixing terms) with proper mass dimensions. It is worthwhile to mention that the signs of the last two terms, corresponding to the gauge-fixing of the axial-vector and polar-vector fields \( \tilde{\phi} \mu \) and \( \phi \mu \), respectively, are fixed that lead to the EL-EoMs:\footnote{We lay emphases on the fact that the axial-vector field (\( \tilde{\phi} \mu \)) and pseudo-scalar field (\( \phi \)) possess negative kinetic terms. However, they satisfy proper Klein-Gordon equation. Hence, these fields correspond to the exotic relativistic particles with well-defined rest mass. As a consequence, they are the possible candidates of dark matter [19, 20] and they are like the “phantom” fields in the realm of cosmology [13-15].}

\( \Box + m^2 \phi \mu = 0, \ (\Box + m^2) \tilde{\phi} \mu = 0, \ (\Box + m^2) \phi = 0, \ (\Box + m^2) \tilde{\phi} = 0. \)

At this juncture, we would like to point out that the generalization of the discrete duality symmetry transformations (10), namely;

\[ B_{\mu\nu} \rightarrow \mp i \tilde{B}_{\mu\nu} \equiv \mp i \varepsilon_{\mu\nu\lambda\xi} B^\lambda, \quad \phi \mu \rightarrow \mp \tilde{\phi} \mu, \quad \tilde{\phi} \mu \rightarrow \mp \phi \mu, \]

(33)

is respected by the completely gauge-fixed Lagrangian densities \( \mathcal{L}_{(1)} \) and \( \mathcal{L}_{(2)} \) and all the fields (i.e. \( B_{\mu\nu}, \tilde{\phi} \mu, \phi \mu, \tilde{\phi} \tilde{\phi} \)) satisfy the following Klein-Gordon equation\footnote{We stress on the fact that the discrete duality symmetry transformations (33) and equations of motion (34) are true for both the Lagrangian densities \( \mathcal{L}_{(1)} \) and \( \mathcal{L}_{(2)} \) [cf. Eqs. (31),(32)].};

\[ \Box + m^2 B_{\mu\nu} = 0, \quad \Box + m^2 \phi \mu = 0, \quad \Box + m^2 \tilde{\phi} \mu = 0, \quad \Box + m^2 \phi = 0, \quad \Box + m^2 \tilde{\phi} = 0, \]

(34)
which is the signature of the completely and correctly gauge-fixed Lagrangian density. It should be noted that the mass term for the $B_{\mu\nu}$ field [i.e. $- (m^2/4) B_{\mu\nu} B^{\mu\nu}$] remains invariant under the transformation $[B_{\mu\nu} \rightarrow \mp (i/2) \varepsilon_{\mu\nu\lambda\xi} B^{\lambda\xi}]$. The latter has its origin in the self-duality condition [cf. Eq. (11)]. This observation is crucial because it forces the whole theory to have a single mass parameter $m$. We point out that both the signs, chosen in the kinetic and gauge-fixing terms as well as in the third term of (31) and (32), are allowed and they do not violate the Klein-Gordon equations in (34). It is very interesting to highlight the following infinitesimal and continuous gauge transformations ($\delta_g$) for the basic fields of the Lagrangian density $L$ (1), namely:

$$
\delta_g B_{\mu\nu} = - (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu), \quad \delta_g \phi = \pm (\partial_\mu \Lambda - m \Lambda_\mu),
$$

$$
\delta_g \Phi_{\mu\nu} = \mp m (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu), \quad \delta_g \phi = \pm 2 [(\partial \cdot \Lambda) + m \Lambda],
$$

$$
\delta_g [H_{\mu\nu\lambda}, \tilde{\phi}_\mu, \phi, \tilde{\Phi}_{\mu\nu}] = 0,
$$

(35)

which are nothing but the generalization of the gauge symmetry transformations (5). Under these transformations, we observe that the Lagrangian density $L$ (1) transforms as follows:

$$
\delta_g L_{(1)} = \partial_\mu \left[ \mp m \varepsilon^{\mu\nu\lambda\xi} \Lambda_\nu \partial_\lambda \tilde{\phi}_\xi \right] \mp \left( \partial \cdot \phi + \frac{m}{2} \phi \right) [\Box + m^2] \Lambda
$$

$$
- \left[ \partial_\nu B^{\nu\mu} \pm m \phi^{\mu} \mp \frac{1}{2} \partial^{\mu} \phi \right] [\Box + m^2] \Lambda_\mu.
$$

(36)

Thus, it is crystal clear that if we impose the restrictions on the gauge transformation parameters as: $(\Box + m^2) \Lambda = 0$, $(\Box + m^2) \Lambda_\mu = 0$ from outside, the transformations (35) will become the symmetry transformations for the Lagrangian density $L$ (1). We shall see that, under the BRST approach to this theory, there will be no imposition of any kind of restriction from outside on the theory. We christen the infinitesimal and continuous transformations (35) as the gauge transformations because we observe that the total kinetic terms (i.e. $\delta_g H_{\mu\nu\lambda} = 0$, $\delta_g \phi_\mu = 0$, $\delta_g \tilde{\phi} = 0$, $\delta_g \tilde{\Phi}_{\mu\nu} = 0$), owing their origin to the exterior derivative $d = d x^\mu \partial_\mu$ (with $d^2 = 0$) of differential geometry [2-5], remain invariant. In addition to the gauge transformations (35), we have another set of infinitesimal and continuous transformations ($\delta_{dg}$) in the theory, namely:

$$
\delta_{dg} B_{\mu\nu} = - \varepsilon_{\mu\nu\lambda\xi} \partial_\lambda \Sigma^\xi, \quad \delta_{dg} \tilde{\phi}_\mu = \pm (\partial_\mu \Omega - m \Sigma_\mu),
$$

$$
\delta_{dg} \phi = \pm \left[ \partial \cdot \Sigma + m \Omega \right], \quad \delta_{dg} \tilde{\phi}_{\mu\nu} = \mp m \left( \partial_\mu \Sigma_\nu - \partial_\nu \Sigma_\mu \right),
$$

$$
\delta_{dg} [\partial^{\mu} B_{\nu\mu}, \phi_\mu, \tilde{\phi}, \tilde{\Phi}_{\mu\nu}] = 0,
$$

(37)

which imply that the total gauge-fixing term [i.e. $\frac{1}{2} (\partial_\nu B^{\nu\mu} \pm m \phi^{\mu} \mp \frac{1}{2} \partial^{\mu} \phi)$], owing its origin to the co-exterior derivative: $\delta = \pm * d \ast$, remains invariant. Here the infinitesimal transformation parameters $\Sigma_\mu$ and $\Omega$ are the Lorentz axial-vector and pseudo-scalar, respectively. We observe that the Lagrangian density $L$ (1) transforms under the infinitesimal and continuous transformations (37) as follows

$$
\delta_{dg} L_{(1)} = \partial_\mu \left[ \mp m \varepsilon^{\mu\nu\lambda\xi} \Sigma_\nu \partial_\lambda \phi_\xi \right] \pm \left( \partial \cdot \tilde{\phi} + \frac{m}{2} \tilde{\phi} \right) [\Box + m^2] \Omega
$$

$$
+ \left[ \frac{1}{2} \varepsilon^{\mu\nu\lambda\xi} \partial_\nu B_{\lambda\xi} \pm m \tilde{\phi}_\mu \mp \frac{1}{2} \partial^{\mu} \tilde{\phi} \right] [\Box + m^2] \Sigma_\mu,
$$

(38)
which shows that, if we impose the conditions: \((\Box + m^2) \Sigma_\mu = 0\) and \((\Box + m^2) \Omega = 0\) from outside, the infinitesimal and continuous transformations (37) will become the symmetry transformations for the completely gauge-fixed Lagrangian density \(L_0\). We christen the transformations in (37) as the dual-gauge transformations (\(\delta_{dg}\)) because the gauge-fixing term for the \(B_{\mu\nu}\) (and associated fields \(\phi_\mu\) and \(\phi\)) remain invariant.

Before we end this section, we very concisely highlight a few key points connected with the continuous symmetries of the Lagrangian density \(L_0\) [cf. Eq. (32)]. In this context, it is very illuminating to point out that the following infinitesimal and continuous (dual-)gauge symmetry transformations \([\delta_{(dg)}]\), namely:

\[
\begin{align*}
\delta_{dg} B_{\mu\nu} &= -\varepsilon_{\mu\nu\lambda\xi} \partial^\lambda \Sigma^\xi, \quad \delta_{dg} \tilde{\phi}_\mu = \pm(\partial_\mu \Omega - m \Sigma_\mu), \\
\delta_{dg} \tilde{\phi} &= \mp 2 [\partial \cdot \Sigma + m \Omega], \quad \delta_{dg} \tilde{\Phi}_{\mu\nu} = \mp m (\partial_\mu \Sigma_\nu - \partial_\nu \Sigma_\mu), \\
\delta_{dg} [\partial^\rho B_{\rho\mu\nu}, \phi_\mu, \phi, \Phi_{\mu\nu\rho}] &= 0,
\end{align*}
\]

(39)

transform the Lagrangian density \(L_0\) as follows:

\[
\begin{align*}
\delta_{dg} L_0 &= \partial_\mu \left[ \mp m \varepsilon^{\mu\nu\lambda\xi} \Sigma_\nu \partial^\lambda \phi_\xi \right] \pm \left( \partial \cdot \tilde{\phi} - \frac{m}{2} \tilde{\phi} \right) [\Box + m^2] \Omega, \\
&\quad + \left[ \frac{1}{2} \varepsilon^{\mu\nu\lambda\xi} \partial_\nu B_{\lambda\xi} \pm m \partial_\mu \tilde{\phi} \mp \frac{1}{2} \partial_\mu \tilde{\phi} \right] [\Box + m^2] \Sigma_\mu \\
\delta_g L_0 &= \partial_\mu \left[ \mp m \varepsilon^{\mu\nu\lambda\xi} \Lambda_\nu \partial^\lambda \tilde{\phi}_\xi \right] \mp \left( \partial \cdot \phi - \frac{m}{2} \phi \right) [\Box + m^2] \Lambda, \\
&\quad - \left[ \partial_\nu B^{\nu\mu} \pm m \phi^\mu \mp \frac{1}{2} \partial^\nu \phi \right] [\Box + m^2] \Lambda_\mu.
\end{align*}
\]

(40)

It is evident that if we impose the restrictions

\[
\begin{align*}
(\Box + m^2) \Sigma_\mu &= 0, & (\Box + m^2) \Lambda_\mu &= 0, \\
(\Box + m^2) \Omega &= 0, & (\Box + m^2) \Lambda &= 0,
\end{align*}
\]

(42)

on the dual-gauge transformation parameters \((\Sigma_\mu, \Omega)\) and the gauge transformation parameters \((\Lambda_\mu, \Lambda)\) from outside, we obtain the (dual-)gauge symmetry transformations (39) and (40) for the Lagrangian density \(L_0\). We note that the outside restrictions (36), (38) and (42) are exactly same on the (dual-)gauge transformation parameters of our theory. Hence, when we elevate the Lagrangian densities \(L_1\) and \(L_2\) to their counterparts at the quantum level (within the framework of BRST formalism), we shall find out that the Faddeev-Popov ghost terms will be the same for the coupled (but equivalent) (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities. The (anti-)ghost fields will not be restricted from outside for the quantum version of our theory within the ambit of BRST formalism (as the EoMs for the (anti-)ghost fields will take care of them).
5 Linearized Versions of the Lagrangian Densities: Auxiliary Fields and CF-Type Restrictions

We linearize the kinetic term for the $B_{\mu\nu}$ (and associated fields) and all the gauge-fixing terms by invoking the Nakanishi-Lautrup type auxiliary fields. In this context, first of all, let us focus on the Lagrangian density $\mathcal{L}_{(1)}$ which can be written as

$$\mathcal{L}_{(1)} \rightarrow \mathcal{L}_{(b_1)} = \frac{1}{2} \mathcal{B}_\mu \mathcal{B} - B^\mu \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \varepsilon \pm m \phi_\mu \mp \frac{1}{2} \partial_\mu \phi \right] - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu}$$

which leads to the following equations of motion:

$$\mathcal{B}_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \varepsilon \pm m \phi_\mu \mp \frac{1}{2} \partial_\mu \phi, \quad \mathcal{B} = - (\partial \cdot \phi + \frac{m}{2} \tilde{\phi}).$$

$$B_\mu = \partial^\nu B_{\nu\mu} \pm m \phi_\mu \mp \frac{1}{2} \partial_\mu \phi, \quad B = - (\partial \cdot \phi + \frac{m}{2} \tilde{\phi}).$$

In the above, the auxiliary fields ($\mathcal{B}_\mu, B_\mu, \mathcal{B}, B$) are the Nakanishi-Lautrup auxiliary fields which have been invoked for the linearization purposes. For instance, the auxiliary field $\mathcal{B}_\mu$ has been invoked for the linearization of the kinetic term for the 2-form field $B_{\mu\nu}$ and associated fields. On the other hand, the auxiliary fields ($\mathcal{B}_\mu, B, \mathcal{B}$) have been introduced to linearize the gauge-fixing terms for the $B_{\mu\nu}, \phi_\mu$ and $\phi_\mu$ fields. In exactly similar fashion, we can linearize the Lagrangian density $\mathcal{L}_{(2)}$ by invoking a different set of Nakanishi-Lautrup type auxiliary fields ($\tilde{\mathcal{B}}_\mu, \tilde{B}_\mu, \tilde{\mathcal{B}}, \tilde{B}$) as follows:

$$\mathcal{L}_{(2)} \rightarrow \mathcal{L}_{(b_2)} = \frac{1}{2} \tilde{\mathcal{B}}_\mu \tilde{\mathcal{B}} + \tilde{B}^\mu \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \varepsilon \pm m \phi_\mu \mp \frac{1}{2} \partial_\mu \phi \right] - \frac{m^2}{4} B_{\mu\nu} B^{\mu\nu}$$

which leads to the following equations of motion w.r.t. the Nakanishi-Lautrup type auxiliary fields, namely:

$$\tilde{\mathcal{B}}_\mu = - \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda \varepsilon \pm m \phi_\mu \mp \frac{1}{2} \partial_\mu \phi, \quad \tilde{\mathcal{B}} = (\partial \cdot \phi - \frac{m}{2} \tilde{\phi}),$$

$$\tilde{B}_\mu = - \partial^\nu B_{\nu\mu} \pm m \phi_\mu \pm \frac{1}{2} \partial_\mu \phi, \quad \tilde{B} = (\partial \cdot \phi - \frac{m}{2} \tilde{\phi}).$$
It is crystal clear that we can derive the following very useful and interesting relationships amongst the Nakanishi-Lautrup type auxiliary fields and (pseudo-)scalar fields from the equations of motion (44) and (46), namely;

\[ B_\mu + \bar{B}_\mu \pm \partial_\mu \tilde{\phi} = 0, \quad B + \bar{B} + m \phi = 0, \]

\[ B_\mu + \bar{B}_\mu \pm \partial_\mu \phi = 0, \quad B + \bar{B} + m \tilde{\phi} = 0, \]

(47)

which are nothing but the CF-type restrictions on our theory (see, e.g. [12, 21] for details).

We end this section with the following remarks. First of all, the Lagrangian densities \( L(b_1) \) and \( L(b_2) \) have been derived in a completely different manner in our present endeavor if we compare our present derivation against the derivation in our earlier work [12] where we have exploited the method of trial and error. Second, the CF-type restrictions: \( B + \bar{B} + m \phi = 0 \) and \( B + \bar{B} + m \tilde{\phi} = 0 \) are same as in our earlier work [12, 21] but the other two restrictions in (47) are different. Third, if we stick with the CF-type restrictions that have been derived from the superfield approach to BRST formalism in context of 4D Abelian 2-form massless and massive gauge theories [18, 21], we find that the other two restrictions of (47) are:

\[ B_\mu + \bar{B}_\mu \pm \partial_\mu \tilde{\phi} = 0 \]

\[ B_\mu + \bar{B}_\mu \pm \partial_\mu \phi = 0. \]

(48)

Finally, in the next section, we shall take only the simplest choices of signs for the (pseudo-)scalar fields within the framework of BRST formalism where the Lagrangian density \( L(b_1) \) will be generalized to incorporate into them the Faddeev-Popov ghost terms.

6 Nilpotent (co-)BRST Invariant Lagrangian Density

We have generalized the Lagrangian densities \( L(b_1) \) and \( L(b_2) \) to their counterparts nilpotent (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities \( L_B \) and \( L_{\bar{B}} \) that incorporate the Faddeev-Popov ghost terms. Such a set of coupled (but equivalent) Lagrangian densities have been written in our earlier works [12, 21]. However, we shall focus on only one Lagrangian density and discuss the importance of discrete duality symmetry transformations (48) [and (54) (below)] which will connect the BRST transformations with the co-BRST transformations and vice-versa. This kind of connection exists for the anti-BRST and anti-co-BRST symmetries, too. However, we shall not dwell on them as it will be only an academic exercise. We would like to emphasize that, in our earlier works [12, 21], such kinds of relationships have not been performed.
Towards the above goal in mind, we begin with the following \((\text{co-})\text{BRST}\) invariant Lagrangian density,\(^\text{1}\) (where \(\mathcal{L}_{(b)} \rightarrow \mathcal{L}_{\mathcal{B}}\)) (see, e.g. [9, 12, 21] for details)

\[
\mathcal{L}_{\mathcal{B}} = \frac{1}{2} B_\mu B^\mu - B_\mu \left[ \frac{1}{2} \varepsilon_{\mu\nu\lambda\xi} \partial^\nu B^\lambda + m \bar{\phi}_\mu - \frac{1}{2} \partial_\mu \bar{\phi} \right] - \frac{m^2}{4} B_\mu B^\mu
\]

\[
- \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{m}{2} B_\mu \left[ \Phi^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \Phi_{\rho\sigma} \right]
\]

\[
+ B_\mu \left[ \partial^\nu B_{\nu\mu} + m \phi_\mu - \frac{1}{2} \partial_\mu \phi \right] - \frac{1}{2} B_\mu B_\mu + B \left( \partial \cdot \phi + \frac{m}{2} \phi \right)
\]

\[
+ \frac{1}{2} B^2 - B \left( \partial \cdot \bar{\phi} + \frac{m}{2} \bar{\phi} \right) - \frac{1}{2} \partial_\mu \bar{\beta} \partial^\mu \beta + \frac{m^2}{2} \beta \bar{\beta}
\]

\[
- \left( \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu \right) \left( \partial^\mu C^\nu \right) + \left( \partial_\mu \bar{C} - m \bar{C}_\mu \right) \left( \partial^\mu C - m C^\mu \right)
\]

\[
- \frac{1}{2} \left[ \partial \cdot C + m \bar{C} + \frac{\rho}{4} \lambda - \frac{1}{2} \left( \partial \cdot C + m C - \frac{\lambda}{4} \right) \rho \right],
\]

(49)

where \((\bar{\beta})\beta\) are the bosonic (anti-)ghost fields with ghost numbers \((-2) + 2\), respectively and \((\bar{C}_\mu)C_\mu\) are the fermionic \((\bar{C}_\mu C_\nu + C_\mu \bar{C}_\nu + C_\nu \bar{C}_\mu) = 0, \bar{C}_\mu \bar{C}_\nu + \bar{C}_\nu \bar{C}_\mu = 0, C_\mu^2 = \bar{C}_\mu^2 = 0, \) etc.) (anti-)ghost fields with ghost numbers \((-1) + 1\), respectively. In addition, we have Lorentz scalar fermionic \((C \bar{C} + \bar{C} C = 0, C^2 = \bar{C}^2 = 0, \) etc.) (anti-)ghost fields with ghost numbers \((-1) + 1\), respectively. Our theory also contains the auxiliary fermionic \((\rho^2 = \lambda^2 = 0, \rho \lambda + \lambda \rho = 0)\) fields \((\rho)\lambda\) that carry ghost numbers \((-1) + 1\), respectively.

The above Lagrangian density respects the following off-shell nilpotent \((s_b^2 = 0)\) \(\text{BRST}\) symmetry transformations \((s_b)\), namely:

\[
s_b B_{\mu\nu} = - \left( \partial_\mu C_\nu - \partial_\nu C_\mu \right), \quad s_b C_\mu = - \partial_\mu \beta, \quad s_b \bar{C}_\mu = B_\mu,
\]

\[
s_b \bar{\beta} = - \rho, \quad s_b \phi_\mu = + \left( \partial_\mu C - m C_\mu \right), \quad s_b \bar{C} = B, \quad s_b \phi = + \lambda,
\]

\[
s_b C = - m \beta, \quad s_b \left[ H_{\mu\nu\lambda}, B, \lambda, \rho, B_\mu, B_\mu, \beta, B, \bar{\phi}_\mu, \bar{\phi}_\mu, \bar{\phi} \right] = 0,
\]

(50)

because the Lagrangian density \(\mathcal{L}_{(\mathcal{B})}\) transforms as \([12]\)

\[
s_b \mathcal{L}_{\mathcal{B}} = \partial_\mu \left[ - m \varepsilon^{\mu\nu\lambda\xi} \partial_\nu \partial_\lambda \bar{C}_\xi - \left( \partial^\mu C^\nu - \partial^\nu C^\mu \right) B_\nu - \frac{1}{2} \lambda B_\mu \right.
\]

\[
+ B \left( \partial^\mu C - m C^\mu \right) + \frac{1}{2} \rho \left( \partial^\mu \beta \right),
\]

(51)

which implies that the action integral \(S = \int d^4 x \mathcal{L}_{(\mathcal{B})}\) remains invariant under the infinitesimal, continuous and nilpotent \(\text{BRST}\) symmetry transformations \((s_b)\). This happens because of Gauss’s divergence theorem due to which all the physical fields vanish-off as \(x \rightarrow \pm \infty\). In addition to \(s_b\), the Lagrangian density \(\mathcal{L}_{\mathcal{B}}\) also respects the infinitesimal, continuous and nilpotent \([s_d^2 = 0]\) co-BRST symmetry transformations \((s_d)\) \([12]\):

\[
s_d B_{\mu\nu} = - \varepsilon_{\mu\nu\lambda\xi} \partial^\lambda \bar{C}_\xi, \quad s_d \bar{C}_\mu = - \partial_\mu \bar{\beta}, \quad s_d C_\mu = B_\mu,
\]

\[
s_d \beta = - \lambda, \quad s_d \phi_\mu = + \left( \partial_\mu C - m C_\mu \right), \quad s_d \bar{C} = B, \quad s_d C = - m \bar{\beta},
\]

\[
s_d \bar{\phi} = - \rho, \quad s_d \left[ \partial^\mu B_{\mu\nu}, B_\mu, B_\mu, B, \bar{\phi}_\mu, \Phi_{\mu\nu}, \phi, \beta, \lambda, \rho \right] = 0.
\]

(52)

\(^1\)For the sake of brevity, we have taken only one specific sign in the kinetic energy and gauge-fixing terms for the \(B_{\mu\nu}\) and associated fields. This is true for the (anti-)ghost fields, too. In our Appendix B, we take the most general form of the \((\text{co-})\text{BRST}\) invariant Lagrangian density which respects the generalized forms of \((\text{co-})\text{BRST}\) symmetry transformations corresponding to classical transformations \((37)\) and \((35)\).
It is straightforward to check that $\mathcal{L}(\mathcal{B})$ transforms, under $(s_d)$, as the total spacetime derivative in the four $(3+1)$-dimensional spacetime, namely;

$$s_d \mathcal{L}_B = \partial_\mu \left[ -m \varepsilon^{\mu \lambda \xi} \phi_\nu \partial_\lambda \bar{C}_\xi + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - \frac{1}{2} \rho B^\mu 
- (\partial^\mu \bar{C} - m \bar{C}^\mu) B + \frac{1}{2} \lambda (\partial^\mu \bar{\beta}) \right]. \quad (53)$$

As a consequence of the above observation, we find that the action integral $S = \int d^4x \mathcal{L}(\mathcal{B})$ remains invariant under the co-BRST transformation $s_d$ for all the physical fields that vanish-off as $x \rightarrow \pm \infty$ due to Gauss’s divergence theorem.

In addition to discrete duality symmetry transformations (48) in the bosonic (i.e. non-ghost) sector of the Lagrangian density $\mathcal{L}(\mathcal{B})$, we have the following discrete symmetry transformations in the ghost-sector of our theory, namely;

$$C_\mu \rightarrow \pm i \bar{C}_\mu, \quad \bar{C}_\mu \rightarrow \pm i C_\mu, \quad C \rightarrow \pm i \bar{C}, \quad \bar{C} \rightarrow \pm i C, \quad \rho \rightarrow \mp i \lambda, \quad \lambda \rightarrow \mp i \rho, \quad \beta \rightarrow \pm i \bar{\beta}, \quad \bar{\beta} \rightarrow \mp i \beta. \quad (54)$$

Under the full discrete symmetry transformations (48) and (54), it can be checked that the (co-)BRST symmetry transformations (32) and (50) are interconnected. To corroborate this claim, let us begin with $s_b B_{\mu \nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu)$. If we apply the discrete symmetry transformations (48) and (54) on it and take the replacement $s_b \rightarrow s_d$, we obtain

$$s_b (* B_{\mu \nu}) = - * (\partial_\mu C_\nu - \partial_\nu C_\mu) \implies s_d B_{\mu \nu} = - \varepsilon_{\mu \nu \lambda \xi} \partial^\lambda \bar{C}^\xi, \quad (55)$$

where $*$ is nothing but the full discrete duality symmetry transformations (48) plus (54). In other words, we have obtained the co-BRST symmetry transformation $s_d$ operating on $B_{\mu \nu}$ field from the operation of $s_b$ on $B_{\mu \nu}$. In exactly similar fashion, we note the following (with replacement: $s_d \rightarrow s_b$), for the transformations $s_d B_{\mu \nu} = - \varepsilon_{\mu \nu \lambda \xi} \partial^\lambda \bar{C}^\xi$, namely:

$$s_d (* B_{\mu \nu}) = - * \varepsilon_{\mu \nu \lambda \xi} (\partial^\lambda \bar{C}^\xi) \implies s_b B_{\mu \nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad (56)$$

when, once again, the $*$ operation is nothing but the total discrete duality symmetry transformations (48) plus (54). This observation is not limited only to the bosonic antisymmetric tensor field. To corroborate this assertion, let us focus on the symmetry transformation: $s_b \phi_\mu = +(\partial_\mu C - m C_\mu)$ on a bosonic vector field ($\phi_\mu$). By exploiting the strength of the full discrete duality symmetry transformations (48) and (54), we observe the following transformations on the axial-vector field (with input: $s_b \rightarrow s_d$), namely;

$$s_b (* \phi_\mu) = + * (\partial_\mu C - m C_\mu) \implies s_d \bar{\phi}_\mu = +(\partial_\mu \bar{C} - m \bar{C}_\mu). \quad (57)$$

This happens because, under discrete duality symmetry transformations (48), we have: $\phi_\mu \rightarrow \pm i \bar{\phi}_\mu$ and $\bar{\phi}_\mu \rightarrow \mp i \phi_\mu$. In exactly, similar fashion, we obtain the reverse symmetry transformations as follows (with input: $s_d \rightarrow s_b$), namely;

$$s_d (* \bar{\phi}_\mu) = + * (\partial_\mu \bar{C} - m \bar{C}_\mu) \implies s_b \phi_\mu = +(\partial_\mu C - m C_\mu). \quad (58)$$

**It can be readily checked that the Faddeev-Popov ghost part of the Lagrangian density $\mathcal{L}_B$ remains invariant under a couple of discrete symmetry transformations (54).**
The above kind of exercise can be repeated with all the fields of our theory. We observe that the discrete duality symmetry transformations (48) and (54) are the generalization of our basic discrete duality symmetry transformations \((B_{\mu\nu} \rightarrow \pm (i/2) \varepsilon_{\mu\nu\kappa\xi} B^{\kappa\xi}, \phi_{\mu} \rightarrow \pm i \tilde{\phi}_{\mu}, \tilde{\phi}_{\mu} \rightarrow \mp i \phi_{\mu})\) of the modified Stückelberg formalism [cf. Eqs (9), (10)]. To complete our present discussion, let us focus on a transformation on a fermionic field \(s_d \bar{C} = -m \bar{\beta}\). Using the strength of the discrete duality symmetry transformations (54), we obtain the following (with the input: \(s_d \rightarrow s_b\)), namely:

\[
s_d [(\bar{C})] = -m * \bar{\beta} \quad \Rightarrow \quad s_b C = -m \beta. \tag{59}
\]

Thus, we are able to obtain the BRST symmetry transformation: \(s_b C = -m \beta\) from the co-BRST symmetry transformation \(s_d \bar{C} = -m \bar{\beta}\) by exploiting the strength of the discrete duality symmetry transformations (54). Hence, our observation is true for fermionic field, too. It goes without saying that, repeating the same procedure, we can obtain \(s_d \bar{C} = -m \bar{\beta}\) from the given BRST symmetry transformation: \(s_b C = -m \beta\). Thus, the discrete duality symmetry transformations (48) and (54) connect the BRST and co-BRST symmetry transformations for the bosonic as well as the fermionic fields.

We end this section with the following remarks. First, the discrete duality symmetry transformations (48) and (54) are able to provide a connection between the symmetry transformations \(s_b\) and \(s_d\). Second, it can been seen that the interplay of the discrete and continuous symmetry transformations provides the physical realization of \(\delta = \pm * d *\) that exist [2-5] between the (co-)exterior derivatives of differential geometry. This interesting and beautiful relationship between \(s_d\) and \(s_b\) is:

\[
s_d = \pm * s_b *, \tag{60}
\]

where * is nothing but the complete set of discrete duality symmetry transformations (48) and (54). Third, despite the above connections between the BRST and co-BRST symmetry transformations in the language of the symmetry properties of our theory, these symmetries are independent of each-other in the same manner as do the exterior \((d)\) and co-exterior \((\delta)\) derivatives of differential geometry [2-5] even though these derivatives are connected with each-other by the relationship: \(\delta = \pm * d *\). Finally, it can be seen that the exactly similar kind of relationships exist between the nilpotent anti-co-BRST symmetry and anti-BRST symmetry transformations that exist for the Lagrangian density \(L_\bar{B}\) which turns to be the generalization of \(L_{b_2}\) [cf. Eq. (45)] (see, e.g., [12, 9] for details).

7 Conclusions

The Stückelberg-modified massive 4D free Abelian 2-form theory has already been proven to be a massive model of Hodge theory [12] where its discrete and continuous symmetry transformations (and corresponding conserved charges) have been shown to provide the physical realizations of the de Rham cohomological operators [2-5] of the differential geometry at the algebraic level within the framework of BRST formalism [12]. However, the full

\footnote{The \((\pm)\) signs on the r.h.s. of (60) are dictated by the successive operations of the discrete duality symmetry transformations (48) and (54) on the generic field \(\Phi = B_{\mu\nu}, \phi_{\mu}, \tilde{\phi}_{\mu}, C_{\mu}, \bar{C}_{\mu}, \phi, \tilde{\phi}, \) etc. In other words, the signs on the r.h.s. of \(* (\pm \Phi) = \pm \Phi\) dictate the signs on (60) (see, e.g., [5, 12] for details).}
coupled (but equivalent) Lagrangian densities of this theory have been obtained by trial and error method. In our present investigation, we have derived the correct forms of the coupled (but equivalent) Lagrangian densities. To be precise, we have concentrated only on the (co-)BRST invariant Lagrangian density (cf. Sec. 6) for the sake of brevity but indicated the theoretical methodology for the derivation of the coupled (but equivalent) Lagrangian densities that respect six continuous and a couple of useful discrete duality symmetry transformations (see, e.g. [12]) within the framework of BRST formalism.

One of the key results of our present investigation is the modification [cf. Eqs. (7), (9)] of the Stückelberg-formalism (SF) in the 4D flat Minkowskian spacetime manifold where the ideas from the differential geometry have played very important roles. It has been demonstrated that the modified SF remains form-invariant under the discrete duality symmetry transformations [cf. Eq. (10)] whose generalizations [cf. Eqs. (48), (54)], within the realm of BRST formalism, provide the physical realizations of the Hodge duality * operation of the differential geometry. As the gauge-fixed Lagrangian density (29) remains invariant under the discrete duality symmetry transformations (10), in exactly similar fashion, the (co-)BRST invariant Lagrangian density (49) remains invariant under the generalization of the discrete duality symmetry transformations (10) (i) to the equation (48) in the non-ghost sector, and (ii) to the equation (54) in the ghost-sector of the Lagrangian density (49). In addition, we have been able to establish a connection between the BRST and co-BRST symmetry transformations (i.e. $s_b \leftrightarrow s_d$) due to the existence of the discrete duality symmetry transformations (48) and (54). The latter symmetry transformations also play an important role in providing the analogue of relationship: $\delta = \pm * d*$ in the terminology of symmetry transformations of our present massive 4D theory [cf. Eq. (60)].

It is worthwhile to mention that the modified SF [cf. Eq. (9)] is invariant under the discrete duality symmetry transformations (10) and they lead to the combination of the polar-vector and axial-vector fields ($\phi_\mu$ and $\tilde{\phi}_\mu$) in the form: $\partial_\mu \phi_\nu - \partial_\nu \phi_\mu + \epsilon_{\mu\nu\lambda\xi} \partial^\lambda \tilde{\phi}^\xi$. Exactly the same combination has been taken by Zwanziger [22] in the description of the electromagnetic global duality invariant 4D Maxwell theory of electrodynamics with double potentials with the field strength tensor as: $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + \epsilon_{\mu\nu\lambda\xi} \partial^\lambda A^\xi$ where $V_\mu$ and $A_\mu$ are the polar-vector and axial-vector potentials. We have discussed the local duality invariance [23] of the Maxwell’s theory with these potentials and shown the existence of an axial-photon which mediates the spin-spin universal long-range interaction (see, e.g. [24], [23]). However, we have not discussed the applications of the axial-vector potential $A_\mu$ in the context of dark energy/dark matter. On the contrary, a close and careful look at the Lagrangian densities (31) and (32) demonstrates that the fields $\tilde{\phi}_\mu$ and $\phi$ turn up with negative kinetic terms in our theory which are interesting in the sense that they belong to the class of exotic fields that are supposed to be one of possible candidates for the dark matter/dark energy [19, 20] and “phantom” and/or “ghost” fields in the context of cyclic, bouncing and self-accelerated cosmological models of Universe [13-15].

In a set of very nice works [25-27], the Stückelberg-modified (SUSY) quantum electrodynamics and other aspects of (non-)interacting Abelian gauge theories have been considered where an ultralight dark matter candidate has been proposed and Stückelberg-boson has been able to cure the infrared problem in QED. It will be an interesting idea to apply our BRST approach to the examples that have been considered in [25-27]. Furthermore, we have already established that the 6D Abelian 3-form gauge theory is a model of Hodge
theory within the ambit of BRST formalism [9]. It will be a nice future endeavor to extend our understandings of the 2D Stückelberg-modified Proca (i.e. massive Abelian 1-form) theory [8] as well as our present work on the Stückelberg-modified massive 4D Abelian 2-form theory to study the Stückelberg-modified massive 6D Abelian 3-form theory within the framework of BRST formalism. We shall report on our progress in the above mentioned theoretical directions in our future publication(s) [28].

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Appendix A: On Modified 2D Proca Theory

For our present paper to be self-contained, we dwell a bit on the massive 2D Abelian 1-form (i.e. Proca) theory which has been at the heart of our present investigation on the massive 4D Abelian 2-form theory. We start off with the Proca Lagrangian density \([L_{(P)}]\) for a vector boson \(A_\mu\) with rest mass \(m\) as follows (see, e.g. [17])

\[
L_{(P)} = - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu, \tag{A.1}
\]

where the 2-form \(F^{(2)} = d A^{(1)} = [(d x^\mu \wedge d x^\nu)/2!] F_{\mu\nu}\) defines the field strength tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) for the vector field \(A_\mu\) that is defined through an Abelian 1-form \((A^{(1)} = d x^\mu A_\mu)\). Here the symbol \(d = d x^\mu \partial_\mu\) (with \(d^2 = 0\)) stands for the exterior derivative of differential geometry [2-5]. The standard Stückelberg formalism (valid in any arbitrary D-dimensional spacetime) gets modified in 2D case as (see, e.g. [8] for details)

\[
A_\mu \rightarrow A_\mu \pm \frac{1}{m} (\partial_\mu \phi + \varepsilon_{\mu\nu} \partial_\nu \tilde{\phi}), \tag{A.2}
\]

where \(\phi\) is a pure-scalar field and \(\tilde{\phi}\) is a pseudo-scalar field in 2D spacetime which is endowed with the Levi-Civita tensor \(\varepsilon_{\mu\nu}\) (with \(\varepsilon_{01} = \varepsilon_{10} = +1\), \(\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2!, \varepsilon_{\mu\nu} \varepsilon^{\nu\rho} = -1! \delta^\rho_\nu, E = - \varepsilon^{\mu\nu} \partial_\mu A_\nu = F_{01}, \) etc.). It can be readily checked that the modified 2D Stückelberg formalism is invariant under the discrete symmetry transformations: \(A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu, \phi \rightarrow \mp i \tilde{\phi}, \tilde{\phi} \rightarrow \mp i \phi\) which play a very important role in establishing a relationship with the Hodge duality \(*\) operation of the differential geometry (see, e.g. [8]).
We observe that, under the modified Stückelberg formalism (A.2), the field-strength tensor transforms as (see, e.g. [8])

\[ F_{\mu \nu} \rightarrow F_{\mu \nu} \mp \frac{1}{m} (\varepsilon_{\nu \rho} \partial_{\mu} - \varepsilon_{\mu \rho} \partial_{\nu}) (\partial^\rho \tilde{\phi}). \]  

(A.3)

We can introduce a notation \( \Sigma_{\mu \nu} = (\varepsilon_{\mu \rho} \partial_{\nu} - \varepsilon_{\nu \rho} \partial_{\mu}) (\partial^\rho \tilde{\phi}) \) to re-express the above transformation for the field strength tensor as follows

\[ F_{\mu \nu} \rightarrow F_{\mu \nu} \pm \frac{1}{m} \Sigma_{\mu \nu}, \]  

(A.4)

which leads to the following transformations for the kinetic term \([- (1/4) F_{\mu \nu} F^{\mu \nu}] \) of the Proca (i.e. massive Abelian 1-form) theory, namely;

\[ - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \rightarrow - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \mp \frac{1}{2 m} F^{\mu \nu} \Sigma_{\mu \nu} - \frac{1}{4} \frac{\Sigma^{\mu \nu} \Sigma_{\mu \nu}}{m^2}. \]  

(A.5)

It is straightforward to note that the second and third terms, on the r.h.s. of (A.5), are higher order derivative terms for a 2D theory of a vector boson. This is due to the fact that there are three and four derivative terms in the second and third terms, respectively, of the transformation for the field strength tensor [c.f. Eq. (A.5)] under (A.2).

We can get rid of the higher derivative terms by exploiting the on-shell conditions:

\[ \partial_{\mu} F^{\mu \nu} + m^2 A^\nu = 0 \]  

and \((\Box + m^2) \tilde{\phi} = 0\). The latter implies that the on-shell condition \((\Box + m^2) \partial^\mu \tilde{\phi} = 0\) is also true. The second term, on the r.h.s. of (A.5), can be explicitly expressed as follows:

\[ \mp \frac{1}{2 m} F^{\mu \nu} (\varepsilon_{\mu \rho} \partial_{\nu} - \varepsilon_{\nu \rho} \partial_{\mu}) (\partial^\rho \tilde{\phi}). \]  

(A.6)

Dropping the total spacetime derivative terms, we note that we have the following explicit form of (A.6), namely;

\[ \pm \frac{1}{2 m} (\partial_{\nu} F^{\mu \nu}) \varepsilon_{\mu \rho} (\partial^\rho \tilde{\phi}) \mp \frac{1}{2 m} (\partial_{\mu} F^{\mu \nu}) \varepsilon_{\nu \rho} (\partial^\rho \tilde{\phi}), \]  

(A.7)

where both the terms are equal and they lead to the following (due to the use of \( E = - \varepsilon^{\mu \nu} \partial_{\mu} A_{\nu} \) and on-shell condition: \( \partial_{\mu} F^{\mu \nu} = - m^2 A^\nu \)), namely;

\[ \pm m A^\nu \varepsilon_{\nu \rho} \partial^\rho \tilde{\phi} \equiv \pm \varepsilon^{\mu \nu} (\partial_{\rho} A_{\nu}) \tilde{\phi} \equiv \mp m E \tilde{\phi}. \]  

(A.8)

In the above, we have dropped a total spacetime derivative term (as it is a part of the Lagrangian density and its presence does not change the dynamics).

We concentrate now on the third term (with four derivatives) on the r.h.s. of equation (A.5) which is explicitly expressed as:

\[ - \frac{1}{4} \frac{\Sigma^{\mu \nu} \Sigma_{\mu \nu}}{m^2} = - \frac{1}{4} \frac{m^2}{m^2} \left[ (\varepsilon^{\mu \rho} \partial^\nu - \varepsilon^{\nu \rho} \partial^\mu) (\partial_{\rho} \tilde{\phi}) \right] \left[ (\varepsilon_{\mu \sigma} \partial_{\nu} - \varepsilon_{\nu \sigma} \partial_{\mu}) (\partial^\sigma \tilde{\phi}) \right]. \]  

(A.9)

The above expression leads to the following (modulo total spacetime derivatives)

\[ - \frac{2}{4} \frac{m^2}{m^2} \left[ (\varepsilon^{\mu \rho} \partial^\nu - \varepsilon^{\nu \rho} \partial^\mu) (\partial_{\rho} \tilde{\phi}) \right] \equiv \frac{1}{2 m^2} \partial^\nu (\partial_{\sigma} \tilde{\phi}) \partial_{\nu} (\partial^\sigma \tilde{\phi}). \]  

(A.10)
where we have used $\varepsilon^{\mu\nu} \varepsilon_{\mu\sigma} = -\delta^\nu_\sigma$. Dropping, once again, the total spacetime derivative term, we obtain the following

$$
\frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \equiv +\frac{1}{2} m^2 \tilde{\phi}^2,
$$

(A.11)

where we have used the on-shell conditions: $(\Box + m^2) \partial^\sigma \tilde{\phi} = 0$, $(\Box + m^2) \tilde{\phi} = 0$. Since the field-strength tensor $F_{\mu\nu}$ has only one non-vanishing component in 2D (which is nothing but the pseudo-scalar electric field $E = F_{01} = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$), we note that the explicit form of (A.5), with the help of (A.8) and (A.11), is as follows

$$
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \longrightarrow \frac{1}{2} E^2 \mp m E \tilde{\phi} + \frac{1}{2} m^2 \tilde{\phi}^2 \equiv \frac{1}{2} (E \mp m \tilde{\phi})^2,
$$

(A.12)

which has been derived in a different manner in our earlier work [12]. It is straightforward to note that the mass term of (A.1) transforms under (A.2) as

$$
\frac{m^2}{2} A_\mu A^\mu \longrightarrow \frac{m^2}{2} A_\mu A^\mu \mp m A_\mu \partial^\mu \tilde{\phi} + \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \pm m E \tilde{\phi},
$$

(A.13)

modulo some total spacetime derivative terms. Here we have used $E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$. Thus, the total Lagrangian density for the modified version of 2D Proca theory is

$$
\mathcal{L}^{(2D)}_{(P)} = \frac{1}{2} (E \mp m \tilde{\phi})^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu \mp m A_\mu \partial^\mu \tilde{\phi} + \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi},
$$

(A.14)

which has been taken into account in our earlier works [8, 10]. It is important to point out that the kinetic terms of the pure-scalar and pseudo-scalar have positive and negative signs, respectively. The latter (i.e. the pseudo-scalar) field is interesting from the point of view of the fact that it provides a field for a possible candidate of dark matter/dark energy. Such exotic fields are also useful in the context of cyclic, bouncing and self-accelerated cosmological models of the Universe [13-15] where these (i.e. fields with negative kinetic terms) have been called as the “phantom” and “ghost” fields.

We end this Appendix with final comment that one can add the gauge-fixing term ($\mathcal{L}_{gf}$) to the above Lagrangian density (A.14) in the 't Hooft gauge as follows (see, e.g. [17])

$$
\mathcal{L}^{(2D)}_{(S)} + \mathcal{L}_{(gf)} = \frac{1}{2} (E \mp m \tilde{\phi})^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu

\mp A_\mu \partial^\mu \tilde{\phi} + \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{2} (\partial \cdot A \pm m \phi)^2,
$$

(A.15)

which respects the discrete duality symmetry transformations on the basic fields of the theory as: $A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu$, $\phi \rightarrow \mp i \tilde{\phi}$, $\tilde{\phi} \rightarrow \mp i \phi$. The Lagrangian density (A.15) has been taken into account in the BRST analysis in our earlier work [8, 10].

\footnote{In our earlier work [8], we have taken into account $F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$ and obtained $F_{01} = E \rightarrow (E \pm \frac{1}{m} \Box \tilde{\phi})$ which has been converted into: $E \rightarrow (E \mp m \tilde{\phi})$ because of the use of mass-shell condition: $\Box \tilde{\phi} = -m^2 \tilde{\phi}$. The latter emerges out as an EoM from the Lagrangian density (A.15).}
Appendix B: On the Generalized (co-)BRST Symmetries

The central purpose of our present Appendix is to generalize the classical (dual-)gauge symmetry transformations (37) and (35), respectively, to their counterparts quantum (co-)BRST symmetry transformations for the appropriate generalized form of the (co-)BRST invariant Lagrangian density [that is more general than the Lagrangian density (49)]. First of all, we generalize the classical gauge symmetry transformations (35) to the following off-shell nilpotent \((s_b^2 = 0)\) quantum BRST symmetry transformations, namely;

\[
s_b B_{\mu \nu} = - (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = - \partial_\mu \beta, \quad s_b \bar{C}_\mu = B_\mu, \]

\[
s_b \phi_\mu = \pm (\partial_\mu C - m C_\mu), \quad s_b C = - m \beta, \quad s_b \bar{C} = B, \]

\[
s_b \Phi_{\mu \nu} = \bar{\pm} m (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b \phi = \pm \lambda, \quad s_b \bar{\beta} = \mp \rho, \quad s_b [H_{\mu \nu \lambda}, \rho, \lambda, \beta, B_\mu, B_\nu, B, \bar{\phi}_\mu, \bar{\Phi}_{\mu \nu}] = 0, \tag{B.1}\]

which transform the following generalized (co-)BRST invariant Lagrangian density \(L^{(g)}_B\), with appropriate \((\pm)\) signs, namely;

\[
L^{(g)}_B = \frac{1}{2} B_\mu B^\mu - B^\mu \left[ \frac{1}{2} \xi_{\mu \lambda \xi} (\partial^\nu B^{\lambda \xi}) + m \tilde{\phi}_\mu - \frac{1}{2} \partial_\mu \tilde{\phi} \right] - \frac{m^2}{4} B_{\mu \nu} B^{\mu \nu} - \frac{1}{4} \Phi_{\mu \nu} \Phi^{\mu \nu} + \frac{1}{4} \bar{\Phi}_{\mu \nu} \bar{\Phi}^{\mu \nu} + \frac{m}{2} B_{\mu \nu} \left[ \Phi^{\mu \nu} + \frac{1}{2} \xi_{\mu \nu \lambda \xi} \right] \]

\[
+ B^\mu \left[ (\partial_\nu B_{\nu \mu}) + m \phi_\mu - \frac{1}{2} \partial_\mu \phi \right] - \frac{1}{2} B^\mu B_\mu + B \left( \partial \cdot \phi + \frac{m}{2} \phi \right) + \frac{1}{2} B^2 - B \left( \partial \cdot \tilde{\phi} + \frac{m}{2} \tilde{\phi} \right) - \frac{1}{2} B^2 - \frac{1}{2} \partial_\mu \bar{\beta} \partial^\mu \beta + \frac{m^2}{2} \bar{\beta} \beta \]

\[
- (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^\nu) \pm (\partial_\mu \bar{C} - m \bar{C}_\mu) (\partial^\mu C - m C^\mu) \]

\[
+ \frac{1}{2} \left( \partial \cdot \bar{C} + m \bar{C} + \frac{\rho}{4} \right) \lambda + \frac{1}{2} \left( \partial \cdot C + m C - \frac{\lambda}{4} \right) \rho, \tag{B.2}\]

to the total spacetime derivative as:

\[
s_b L^{(g)}_B = \partial_\mu \left[ \pm m \xi_{\mu \nu \lambda \xi} \phi_\nu \partial_\lambda C_\xi - (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu \pm \frac{1}{2} \lambda B^\mu \right. \]

\[
\pm B (\partial^\mu C - m C^\mu) \pm \frac{1}{2} \rho (\partial^\mu \beta), \tag{B.3}\]

As a consequence of the above observation, it is clear that the action integral \(S = \int d^4x L^{(g)}_B\) remains invariant \((s_b S = 0)\) under the infinitesimal, continuous and off-shell nilpotent \((s_b^2 = 0)\) BRST symmetry transformations (B.1). A noteworthy point, at this juncture, is the observation that \((\pm)\) signs, associated with \((\pm m \phi_\mu, \pm m \phi_\mu)\) in the kinetic term and gauge-fixing term, respectively, have been changed to \((\pm m \tilde{\phi}_\mu, \pm m \phi_\mu)\) because only this choice of sign is allowed by the nilpotent BRST symmetry transformations (B.1).
The generalized Lagrangian density \((\mathcal{L}^{(g)}_B)\) also respects a set of off-shell nilpotent \((s_d^2 = 0)\) dual-BRST (i.e. co-BRST) symmetry transformations \((s_d)\) as quoted below:

\[
\begin{align*}
    s_d B_{\mu\nu} &= -\varepsilon_{\mu\nu\lambda\xi} \partial^\lambda \bar{C}^\xi, \quad s_d \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_d C_\mu = \mathcal{B}_\mu, \\
    s_d \tilde{\phi} &= \mp \rho, \quad s_d \tilde{\phi}_\mu = \pm (\partial_\mu \bar{C} - m \bar{C}_\mu), \\
    s_d \beta &= \mp \lambda, \quad s_d C = \mathcal{B}, \quad s_d \bar{C} = -m \bar{\beta}, \\
    s_d [(\partial^\nu B_{\nu\mu}, B_\mu, B, \phi, \phi_\mu, \Phi_{\mu\nu}, \bar{\beta}, \lambda, \rho)] &= 0.
\end{align*}
\]

(B.4)

because \(\mathcal{L}^{(g)}_B\) transforms to a total spacetime derivative as follows:

\[
\begin{align*}
    s_d \mathcal{L}^{(g)}_B &= \partial_\mu \left[ \mp m \varepsilon^{\mu\nu\lambda\xi} \phi_\nu \partial_\lambda \bar{C}_\xi + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu \mp \frac{1}{2} \rho \mathcal{B}_\mu \right] \\
    &= \mp \mathcal{B} (\partial^\mu \bar{C} - m \bar{C}^\mu) \mp \frac{1}{2} \lambda (\partial^\mu \bar{\beta}).
\end{align*}
\]

(B.5)

As a consequence, it is crystal clear that the infinitesimal, continuous and off-shell nilpotent \((s_d^2 = 0)\) co-BRST transformations (B.4) are the symmetry transformation for the action integral \(S = \int d^4x \mathcal{L}^{(g)}_B\) due to the validity of the Gauss’s divergence theorem, because of which, all the physical fields vanish-off as \(x \rightarrow \pm \infty\).

We end this Appendix with the final remark that the modified SF [cf. Eq. (9)] and Lagrangian density (B.2) remain invariant under the discrete duality symmetry transformations [cf. Eqs. (48), (54)] at the quantum level. Furthermore, these latter discrete symmetry transformations provide a connection between the BRST symmetry transformations (B.1) and co-BRST symmetry transformations (B.4) in exactly the same manner as we have discussed such kind of relationship in the simpler case of Lagrangian density (49) in Sec. 6. To take a simple example, let us focus on \(s_b \phi = \pm \lambda\). If we take the input \(s_b \rightarrow s_d\) and the discrete symmetry transformations: \(\phi \rightarrow \pm i \tilde{\phi}\), \(\lambda \rightarrow \mp i \rho\) [cf. Eqs. (48), (54)], we obtain \(s_d \tilde{\phi} = \mp \rho\) from \(s_b \phi = \pm \lambda\). Reciprocal relationship, it can be readily checked, is also true where we obtain \(s_b \phi = \pm \lambda\) from \(s_d \tilde{\phi} = \mp \rho\) if we take into account: \(s_d \rightarrow s_b\) and the discrete symmetry transformations (48) and (54) together.

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