Almost-vector spaces in the intersection of translates of the frame space in finite dimension

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Abstract

We proved in a previous article a path-connectedness property of the intersection of translates of the space of finite-dimensional Hilbert space frames, which we formulated in the language of Stiefel manifolds \( St(n,H) \). Yet, the notion of a Hilbert space frame has already been successfully extended to \( C^* \)-algebras and Hilbert modules. In this article, we prove a new property of the intersection of translates of the space of frames both in the finite-dimensional Hilbert space and finite-dimensional \( C^* \)-algebraic cases. This property expresses that there are many almost-vector spaces in these intersections, and it relies crucially on the hypothesis of finite dimension. When the translating \( l \)-tuple is made of frames, we see that the intersection almost contains the vector space spanned by these frames. We finally conjecture that the property and its consequence are true in a finite-dimensional left-Hilbert module because of the simple and related structure of the latter.

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1 Introduction

Duffin and Shaeffer introduced in 1952 [4] the notion of a Hilbert space frame to study some interesting problems in nonharmonic Fourier series. The general idea of signal decomposition in terms of elementary signals was however known to Gabor [7] in 1946. The landmark paper of Daubechies, Grossmann, and Meyer [3] gave a new boost to the theory of frames in 1986 which then became more widely known to the community. Nowadays, frames have a wide range of applications in both engineering science and mathematics. Frames have found applications in signal processing, image processing, data compression, fault-tolerant data transmission, and sampling theory. They are also used in Banach space theory. There are many generalizations of frames in the literature, for instance frames in $C^*$-algebras and Hilbert $C^*$-modules [5]. A general introduction to frame theory can be found in ([1], [2]).

We proved in a previous article [6] a path-connectedness property of the intersection of translates of the space of finite-dimensional Hilbert space frames, which we formulated in the language of Stiefel manifolds $St(n,H)$. Yet, the notion of a Hilbert space frame has already been successfully extended to $C^*$-algebras and Hilbert modules. In this article, we prove a new property of the intersection of translates of the space of frames both in the finite-dimensional Hilbert space and finite-dimensional $C^*$-algebraic cases. This property expresses that there are many almost-vector spaces in these intersections, and it relies crucially on the hypothesis of finite dimension (see propositions 2.2 and 2.5). When the translating $l$-tuple is made of frames, we see that the intersection almost contains the vector space spanned by these frames. We finally conjecture that this property is true in a finite-dimensional left-Hilbert module since it can be shown that such an object is isomorphic to direct sums of $\mathbb{F}^m \otimes \mathbb{F}^n$’s and that the underlying $C^*$-algebra in which the scalar product is valued is finite-dimensional, i.e. isomorphic to a direct sum of $M_m(\mathbb{F})$’s.

Plan of the article. This article is organized as follows. In section 2, we set some notations, and recall some definitions and facts in frame theory in the setting of finite-dimensional Hilbert spaces, $C^*$-algebras and Hilbert modules. Section 3 contains the main results and corollaries, precisely formulated and proved in the the context of finite-dimensional Hilbert spaces and $C^*$-algebras, and left as conjectures in the context of finite-dimensional left-Hilbert modules.
2 Preliminaries

2.1 Notation

The following notations are used throughout this article. The notions of continuous Bessel and frame families mentioned here are introduced in the next subsection 2.2. We denote by \( \mathbb{N} \) the set of natural numbers including 0. \( \mathbb{N}^* \) denotes the set of natural numbers excluding 0. We denote by \( n \) an element of \( \mathbb{N}^* \) and by \( F \) one of the fields \( \mathbb{R} \) or \( \mathbb{C} \).

If \( K \) is a Hilbert space and \( U = (u_x)_{x \in X} \) with \( u_x \in K \) for all \( x \in K \) is a continuous Bessel family in \( K \), we define its analysis operator \( T_U : K \to L^2(X, \mu; \mathbb{F}) \) by

\[
\forall v \in K : T_U(v) := (\langle v, u_x \rangle)_{x \in X}.
\]

The adjoint of \( T_U \) is an operator \( T_U^* : L^2(X, \mu; \mathbb{F}) \to K \) given by

\[
\forall c \in L^2(X, \mu; \mathbb{F}) : T^*_U(c) = \int_X c(x)u_x d\mu(x).
\]

The composition \( S_U = T_U^* T_U : K \to K \) is given by

\[
\forall v \in K : S_U(v) = \int_X \langle v, u_x \rangle u_x d\mu(x)
\]

and called the frame operator of \( U \). Since \( U \) is Bessel, \( T_U, T^*_U, \) and \( S_U \) are all well defined and continuous. If \( U \) is a frame in \( K \), then \( S_U \) is a positive self-adjoint operator satisfying \( 0 < A \leq S_U \leq B \) and thus, it is invertible. If \( \mathcal{H} \) is a Hilbert module, the analysis, synthesis and frame operator of a Bessel family are defined similarly, where the old scalar product with values in \( \mathbb{F} \) is replaced by the scalar product of \( \mathcal{H} \) with values in the underlying \( C^* \)-algebra.

If \( K \) is a Hilbert space, we denote by \( L(K) \) and \( B(K) \) respectively the set of linear and bounded operators in \( K \). \( \text{Id}_K \) is the identity operator of \( K \).

If \( K \) is a Hilbert space, \( m \in \mathbb{N}^* \), and \( \theta_1, \ldots, \theta_m \in H \), the Gram matrix of \( (\theta_1, \ldots, \theta_m) \) is the matrix \( \text{Gram}(\theta_1, \ldots, \theta_m) \) whose \( k,l \)-coefficient is \( \langle \theta_k, \theta_l \rangle \).

If \( \sigma, \tau \in \mathbb{N}^* \), we denote by \( M_{\sigma,\tau}(\mathbb{F}) \) the algebra of matrices of size \( \sigma \times \tau \) over the field \( \mathbb{F} \). When \( \sigma = \tau \), we denote this algebra \( M_{\sigma}(\mathbb{F}) \).

An element \( x \in \mathbb{F}^n \) is a \( n \)-tuple \((x^1, \ldots, x^n)\) with \( x^k \in \mathbb{F} \) for all \( k \in [1, n] \).

If \( S \in L(\mathbb{F}^n) \), we denote by \( [S] \in M_n(\mathbb{F}) \) the matrix of \( S \) in the standard basis of \( \mathbb{F}^n \), and we write \( \text{Id}_n \) as a shorthand for \( [\text{Id}_{\mathbb{F}^n}] \).

If \( U = (u_x)_{x \in X} \) is a family in \( \mathbb{F}^n \) indexed by \( X \), then for each \( k \in [1, n] \), we denote by \( U^k \) the family \((u^k_x)_{x \in X}\).

2.2 Continuous frames with values in finite dimensional spaces

2.2.1 Finite-dimensional Hilbert spaces \( K = \mathbb{F}^n \)

Let \( K \) be a Hilbert space and \((X, \Sigma, \mu)\) a measure space.
Definition 2.1. [2] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in K$ for all $x \in X$ is a continuous frame in $K$ if

$$\exists 0 < A \leq B : \forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2$$

A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if $\Sigma$ is the discrete $\sigma$-algebra and $\mu$ is the counting measure. We denote by $\mathcal{F}_{(X,\mu),K}$ and $\mathcal{F}_{(X,\mu),\mathbb{F}}^n$ respectively the set of continuous frames with values in $K$ and the set of continuous frames with values in $\mathbb{F}^n$.

Now, we recall some elementary facts on Bessel families and frames in the context of $\mathbb{F}^n$.

Proposition 2.1. [6] A family $U = (u_x)_{x \in X}$ with $u_x \in \mathbb{F}^n$ for all $x \in X$ is a continuous Bessel family if and only if it belongs to $L^2(X,\mu,\mathbb{F}^n)$.

Lemma 2.1. [6] If $U \in L^2(X,\mu;\mathbb{F}^n)$, then $[S_U] = \text{Gram}(U^1, \cdots, U^n)$.

Proposition 2.2. [2] Suppose $\Phi = (\varphi_x)_{x \in X}$ is a family in $\mathbb{F}^n$. Then

$\Phi$ is a continuous frame $\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $S_\Phi$ is invertible

$\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $\det(\text{Gram}(\Phi^1, \cdots, \Phi^n)) > 0$

$\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $\{\Phi^1, \cdots, \Phi^n\}$ is free.

Proposition 2.3. [2] Suppose $\Phi = (\varphi_x)_{x \in X}$ is a family in $\mathbb{F}^n$ and let $a > 0$. Then

$\Phi$ is a measurable $a$-tight frame $\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $S_\Phi = aI_n$

$\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $\text{Gram}(\Phi^1, \cdots, \Phi^n) = aI_n$

$\iff \Phi \in L^2(X,\mu;\mathbb{F}^n)$ and $(\Phi^1, \cdots, \Phi^n)$ is an orthogonal family of $L^2(X,\mu;\mathbb{F}^n)$ and $(\forall i \in [1,n]) : \|\Phi^i\| = \sqrt{a}$.

2.2.2 Finite-dimensional $C^*$-algebras $\mathcal{A} = \bigoplus_{j=1}^r M_{n_j}(\mathbb{F})$

Let $\mathcal{A}$ be a $C^*$-algebra and $(X,\Sigma,\mu)$ a measure space. We denote by $I$ the identity of $\mathcal{A}$.

Definition 2.2. [2] [5] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in \mathcal{A}$ for all $x \in X$ is a continuous frame in $\mathcal{A}$ if

$$\exists 0 < A \leq B : \forall v \in \mathcal{A} : A.vv^* \leq \int_X v\varphi_x^*v\varphi_x d\mu(x) \leq B.vv^*$$

A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if $\Sigma$ is the discrete $\sigma$-algebra and $\mu$ is the counting measure. We denote by $\mathcal{F}_{(X,\mu),\mathcal{A}}$ the set of continuous frames with values in $\mathcal{A}$.
The definition of a continuous frame $\Phi$ in a unitary $C^*$-algebra is of course equivalent to:

$$
\exists 0 < A \leq B : \forall v \in A : A.I \leq \int_X \varphi_x^* \varphi_x d\mu(x) \leq B.I
$$

From now on, we consider $A_0 = \bigoplus_{j=1}^{r} M_{n_j}(\mathbb{F})$. The identity of $A_0$ is $I = \bigoplus_{j=1}^{r} I_{n_j}$ where $I_{n_j}$ is the identity matrix of size $n_j$. If $m = \{m_j\}_{j=1}^{r} \in A_0$, we denote $\det(m) = (\det(m_j))_{j=1}^{r}$. In what follows, we will recall some important facts about continuous Bessel families and frames in this setting.

**Proposition 2.4.** If we see $A_0$ as a left $A_0$ Hilbert module, then according to the definition of the frame operator of a continuous Bessel family $U = (u_x)_{x \in X}$ in this setting (see subsection 2.1), the frame operator of $U$ should be $S_U : \left\{ \begin{array}{l}
A_0 \to A_0 \\
v \mapsto \int_X vv^*_x u_x d\mu(x)
\end{array} \right.$.

However, this operator is invertible if and only if $\int_X u_x^* u_x d\mu(x) \in A_0$ is invertible and we shall denote $S_U = \int_X u_x^* u_x d\mu(x) \in A_0$.

**Proof.** This follows from the classical result that for every $C \in M_n(\mathbb{F})$, $T : \left\{ \begin{array}{l}
M_n(\mathbb{F}) \to M_n(\mathbb{F}) \\
M \mapsto MC
\end{array} \right.$ is invertible if and only if $C$ is invertible. \hfill $\square$

**Proposition 2.5.** Suppose $\Phi = (\varphi_x)_{x \in X}$ is a family in $A_0$. Then

$\Phi$ is a continuous frame $\iff \int_X \varphi_x^* \varphi_x d\mu(x)$ converges and $S_\Phi$ is invertible

$\iff \int_X \varphi_x^* \varphi_x d\mu(x)$ converges and $\det(S_\Phi) > 0$.

**Proposition 2.6.** Suppose $\Phi = \{\varphi_x\}_{x \in X}$ is a family in $A_0$ and let $a > 0$. Then

$\Phi$ is a measurable $a$-tight frame $\iff \int_X \varphi_x^* \varphi_x d\mu(x)$ converges and $S_\Phi = aI$.

### 2.2.3 Finite-dimensional Hilbert modules $\mathcal{H}$

If $\mathcal{H}$ is a Hilbert module and is finite-dimensional as a vector space, then it can be shown that it is isomorphic to direct sums of $\mathbb{F}^m \otimes \mathbb{F}^n$'s, and that the underlying $C^*$-algebra is isomorphic to a direct sum of $M_m(\mathbb{F})$'s, each matrix of size $m$ acting on the $(m, n)$-component as $M.(x \otimes y) = Mx \otimes y$. The scalar product has also a convenient form; most importantly, it takes its values in a direct sum of matrix algebras. We refer to the mathoverflow question [8]. This relatively simple and controllable structure will lead us to conjecture that the results of the next section, true in $\mathbb{F}^n$ and $\bigoplus_{j=1}^{r} M_{n_j}(\mathbb{F})$, generalize to the case of a finite-dimensional left-Hilbert module.

**Definition 2.3.** [2] [5] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in \mathcal{H}$ for all $x \in X$ is a continuous frame in $\mathcal{H}$ if

$$
\exists 0 < A \leq B : \forall v \in \mathcal{H} : A.\langle v, v \rangle_A \leq \int_X \langle \varphi_x, \varphi_x \rangle_A d\mu(x) \leq B.\langle v, v \rangle_A.
$$
A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if $\Sigma$ is the discrete $\sigma$-algebra and $\mu$ is the counting measure. We denote by $\mathcal{F}(X,\mu,H)$ the set of continuous frames with values in $H$.

3 Almost-vector spaces in the intersection of translates of the frame space in finite dimension

We will start with the case of the finite-dimensional Hilbert space $\mathbb{F}^n$ and prove the main proposition and corollary relative to this case. Similar results will be obtained for the second case $\bigoplus_{j=1}^{r} M_{n_j}(\mathbb{F})$, and we will finally conjecture that they stay true for the general case of a finite-dimensional Hilbert module.

3.1 Case 1: $\mathbb{F}^n$

To start this section, let $(X, \Sigma, \mu)$ be a measure space.

**Proposition 3.1.** Let $l \in \mathbb{N}^*$. Let $U(i) \in L^2(X, \mu; \mathbb{F}^n)$ for all $i \in [1, l]$. Let $A(i)$ such that $A(i) \in \mathcal{F}(X,\mu,n)$ for all $i \in [1, l]$. Then there exists $l$ hypersurfaces $(D_e)_i^l$ (which depend on the $U(i)$’s and $A(i)$’s) in $\mathbb{F}^l$ such that

$$\left( \sum_{i=1}^{l} c_i (A(i) - U(i)) \right) \in \bigcap_{i=1}^{l} (\mathcal{F}(X,\mu,n) - U(i)) \iff (c_i)_i^l \in \bigcap_{i=1}^{l} D_e^c.$$

**Proof.** Consider for all $e \in [1, l]$ the polynomial:

$$P_e(c_1, \cdots, c_l) = \det(\text{Gram}(\{U(e)^j + \sum_{i=1}^{l} c_i (A(i)^j - U(i)^j)\}_{j=1}^{n}))$$

We have for all $e \in [1, l] : P_e(0, \cdots, 0, 1, 0, \cdots, 0) > 0$ where the 1 is in the $e$-th position, because $A(e)$ is a frame (see proposition 2.2). Therefore, for all $e \in [1, l]$, $P_e \neq 0$ and so there exists a hypersurface $D_e = \{(c_1, \cdots, c_l) \in \mathbb{F}^l : P_e(c_1, \cdots, c_l) = 0\}$ in $\mathbb{F}^l$ such that

$$\left( \sum_{i=1}^{l} c_i (A(i) - U(i)) \right) \in \mathcal{F}(X,\mu,n) - U(e) \iff (c_i)_i^l \in D_e^c,$$

from which the result follows.

**Remark 3.1.** By Baire’s theorem and the facts that the translation maps are homeomorphisms and $\mathcal{F}(X,\mu,n)$ is open and dense in $L^2(X, \mu; \mathbb{F}^n)$ (see [6]), the set $\bigcap_{i=1}^{l} (\mathcal{F}(X,\mu,n) - U(i))$ is open and dense in $L^2(X, \mu; \mathbb{F}^n)$. By the result that we have just proved, for each choice of $A(i)$’s such that $A(i) \in \mathcal{F}(X,\mu,n)$ for all $i \in [1, l]$, we almost
have that \( \text{span}( (A(i) - U(i))_{i=1}^l \subseteq \bigcap_{i=1}^l \left( \mathcal{F}_{(X,\mu),n} - U(i) \right) \) except for a measure zero set of \( l \)-tuple of coefficients \((c_i)_{i=1}^l\), since \( \bigcap_{i=1}^l D_i^l = \bigcup_{i=1}^l D_i \subseteq \mathbb{F} \) has Lebesgue measure zero. This explains the title of this section: \( \bigcap_{i=1}^l \left( \mathcal{F}_{(X,\mu),n} - U(i) \right) \) contains infinitely many almost-vector spaces of finite dimension, one for each selection of \( A(i) \in \mathcal{F}_{(X,\mu),n} \) for all \( i \in [1,l] \). Since moreover \( \mathcal{F}_{(X,\mu),n} \) is open and dense, this gives us a stronger result than the fact that \( \bigcap_{i=1}^l \left( \mathcal{F}_{(X,\mu),n} - U(i) \right) \) is dense.

**Remark 3.2.** The proposition can be extended to the case \( l = +\infty \) but we need to make sure that there exists a cofinite set \( I \subseteq \mathbb{N} \) such that \( U(i) \) is a frame and \( A(i) = U(i) \) for all \( i \in I \), in order for the sums to be finite.

The following corollary can be derived from proposition 3.1.

**Corollary 3.1.** Let \( l \in \mathbb{N}^* \). Let \( U(i) \in \mathcal{F}_{(X,\mu),n} \) for all \( i \in [1,l] \). Then there exists \( l \) hypersurfaces \( (D_i)_{i=1}^l \) (which depend on the \( U(i) \)'s) in \( \mathbb{F}^l \) such that

\[
\left( \sum_{i=1}^l c_i U(i) \in \bigcap_{i=1}^l \left( \mathcal{F}_{(X,\mu),n} - U(i) \right) \right) \Leftrightarrow (c_i)_{i=1}^l \in \bigcap_{i=1}^l D_i^l.
\]

**Proof.** Let \( A(i) = 2U(i) \) for all \( i \in [1,l] \) and apply proposition 3.1. \( \square \)

**Remark 3.3.** Reasoning as in remark 3.1, we can see from this corollary that when the translating \( l \)-tuple is made of frames, the intersection almost contains the vector space spanned by these frames.

### 3.2 Case 2: \( \oplus_{j=1}^r M_{n_j}(\mathbb{F}) \)

Let \( \mathcal{A} = \oplus_{j=1}^r M_{n_j}(\mathbb{F}) \) be a finite-dimensional \( C^* \)-algebra and \((X,\Sigma,\mu)\) a measure space.

**Proposition 3.2.** Let \( l \in \mathbb{N}^* \). Let \( U(i) = (U(i)_x)_{x \in X} \) such that \( U(i)_x \in \mathcal{A} \) for all \( x \in X \) and \( \int_X U(i)_x^* U(i)_x d\mu(x) \) converges in \( \mathcal{A} \) for all \( i \in [1,l] \). Let \( A(i) \) such that \( A(i) \in \mathcal{F}_{(X,\mu),\mathcal{A}} \) for all \( i \in [1,l] \). Then there exists \( l \) manifolds \( (D_i)_{i=1}^l \) of codimension \( r \) (which equations can be written down explicitly in terms of the \( U(i) \)'s and \( A(i) \)'s) in \( \mathbb{F}^l \) such that

\[
\left( \sum_{i=1}^l c_i(A(i) - U(i)) \in \bigcap_{i=1}^l \left( \mathcal{F}_{(X,\mu),\mathcal{A}} - U(i) \right) \right) \Leftrightarrow (c_i)_{i=1}^l \in \bigcap_{i=1}^l D_i^l.
\]

We refer to remark 3.1 for the intended meaning of this proposition. Here, the \( D_i \)'s are strict manifolds rather than hypersurfaces, but the Lebesgue measure of \( \bigcup_{i=1}^l D_i \) is still 0. We also refer to remark 3.2.
Proof. Consider for all $e \in [1, l]$ the polynomials:

$$(P_e(c_1, \cdots, c_l))^r_{j=1} = \det(S_{U(e) + \sum_{i=1}^l c_i(A(i) - U(i))}).$$

We have for all $e \in [1, l]$ and $j \in [1, r]$, $P_e(0, \cdots, 0, 1, 0, \cdots, 0)_j > 0$ where the 1 is in the $e$-th position, because $A(e)$ is a frame (see proposition 2.5). Therefore, for all $e \in [1, l]$ and $j \in [1, r]$, $(P_e)_j \neq 0$ and so there exists a manifold $D_e = \{(c_1, \cdots, c_l) \in F^l : (P_e(c_1, \cdots, c_l))^r_{j=1} = (0, \cdots, 0)\}$ in $F^l$ of codimension $\in [1, r]$ such that

$$\left(\sum_{i=1}^l c_i(A(i) - U(i)) \in \mathcal{F}(X, \mu, A) - U(e)\right) \iff (c_i)_{i=1}^l \in D_e^c,$$

from which the result follows.

\[\square\]

**Corollary 3.2.** Let $l \in \mathbb{N}^*$. Let $U(i) \in \mathcal{F}(X, \mu, A)$ for all $i \in [1, l]$. Then there exist $l$ manifolds $(D_i)_{i=1}^l$ (which equations can be written down explicitly in terms of the $U(i)$’s) in $F^l$ of codimension $\in [1, r]$ such that

$$\left(\sum_{i=1}^l c_iU(i) \in \bigcap_{i=1}^l \left(\mathcal{F}(X, \mu, A) - U(i)\right)\right) \iff (c_i)_{i=1}^l \in \bigcap_{i=1}^l D_i^c.$$

**Proof.** Let $A(i) = 2U(i)$ for all $i \in [1, l]$ and apply proposition 3.2. \[\square\]

We refer to remark 3.3 for the intended meaning of this corollary. We also refer to remark 3.2.

### 3.3 Case 3: Finite-dimensional Hilbert modules $\mathcal{H}$

Let $\mathcal{H}$ be a left-Hilbert module which is a finite-dimensional vector space. We conjecture that the same results are valid in $\mathcal{H}$ because of two things: $\mathcal{H}$ is isomorphic to direct sums of $\mathbb{F}^m \otimes \mathbb{F}^n$’s, and the underlying C*-algebra (that in which the scalar product is valued) is isomorphic to a direct sum of matrix algebras $M_m(\mathbb{F})$’s. We refer to subsection 2.2.3 for a more detailed comment.

**Conjecture 3.1.** Let $l \in \mathbb{N}^*$. Let $U(i) = (U(i)_x)_{x \in X}$ such that $U(i)$ is a continuous Bessel family in $\mathcal{H}$ for all $i \in [1, l]$. Let $A(i)$ such that $A(i) \in \mathcal{F}(X, \mu, \mathcal{H})$ for all $i \in [1, l]$. Then there exists $l$ manifolds $(D_i)_{i=1}^l$ (which equations can be written down explicitly in terms of the $U(i)$’s and $A(i)$’s) in $F^l$ of codimension at least 1 such that

$$\left(\sum_{i=1}^l c_i(A(i) - U(i)) \in \bigcap_{i=1}^l \left(\mathcal{F}(X, \mu, \mathcal{H}) - U(i)\right)\right) \iff (c_i)_{i=1}^l \in \bigcap_{i=1}^l D_i^c.$$

We refer to remark 3.1 for the intended meaning of this conjecture. Here, the $D_i$’s are strict manifolds rather than hypersurfaces, but the Lebesgue measure of $\bigcup_{i=1}^l D_i$ is still 0. We also refer to remark 3.2.
Conjecture 3.2. Let $l \in \mathbb{N}^*$. Let $U(i) \in \mathcal{F}(X,\mu,\mathcal{H})$ for all $i \in [1, l]$. Then there exists $l$ manifolds $(D_i)_{i=1}^l$ (which equations can be written down explicitly in terms of the $U(i)$’s) in $\mathcal{F}$ of codimension at least 1 such that

$$ \left( \sum_{i=1}^l c_i U(i) \in \bigcap_{i=1}^l \left( \mathcal{F}(X,\mu,\mathcal{H}) - U(i) \right) \right) \iff (c_i)_{i=1}^l \in \bigcap_{i=1}^l D_i^c.$$  

We refer to remark 3.3 for the intended meaning of this conjecture. We also refer to remark 3.2.

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