GRAPHS WHOSE KRONECKER COVERS ARE BIPARTITE KNESER GRAPHS

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Abstract. We show that there are $k$ simple graphs whose Kronecker covers are isomorphic to the bipartite Kneser graph $H(n, k)$, and that their chromatic numbers coincide with $\chi(K(n, k)) = n - 2k + 2$. We also determine the automorphism groups of these graphs.

1. Introduction

A covering map of graphs is a surjective graph homomorphism $p: \tilde{G} \to G$ such that $p$ maps the neighborhood of each vertex $v$ in $\tilde{G}$ bijectively onto the neighborhood of $p(v)$. The Kronecker cover of $G$ is the categorical product $K_2 \times G$ by $K_2$ (see Section 2). When $G$ is connected and non-bipartite, then its Kronecker cover is the unique cover which is bipartite, and when $G$ is bipartite, its Kronecker cover is the disjoint union of two copies of $G$. Kronecker covers are fundamental objects in covering theory of graphs, and have appeared in different branches of combinatorics (see [1], [6], and [7]).

It is known that different graphs may have isomorphic Kronecker covers. Then it is natural to classify the all possible graphs whose Kronecker covers are isomorphic to a given bipartite graph. Such a problem was actually written in [3], and was settled in the cases of hypercubes [2] and generalized Petersen graphs [4].

The purpose in this paper is to classify the all graphs whose Kronecker covers are isomorphic to the bipartite Kneser graph $H(n, k)$. Moreover, we determine the automorphism groups of them and chromatic numbers. Here we recall the definition of $H(n, k)$. Let $n$ and $k$ be positive integers with $n > 2k$. The Kneser graph $K(n, k)$ is the graph consisting of $k$-subsets of $[n] = \{1, \ldots, n\}$, where two $k$-subsets are adjacent if and only if they have no intersection. The bipartite Kneser graph $H(n, k)$ is the Kronecker cover of $K(n, k)$.

Theorem 1. Suppose that $n > 2k$ and $k \geq 2$. Then there are $k$ simple graphs

$$G_0(n, k), G_1(n, k), \ldots, G_{k-1}(n, k),$$

where $G_0(n, k) = K(n, k)$, $K_2 \times G_i(n, k) \cong H(n, k)$, but any two of them are not isomorphic.
Here we recall some previous works concerning Theorem 1. Imrich and Pisanski constructed a graph $G$ such that $G \not\cong K(5,2)$ but $K_2 \times G \cong H(5,2)$. Moreover, the author constructed a graph $KG'_{n,k}$ such that $KG'_{n,k} \not\cong K(n,k)$ but $K_2 \times KG'_{n,k} = H(n,k)$, when $k \geq 2$. In fact, the graph $KG'_{n,k}$ is the graph $G_1(n,k)$ in our sense, and $G_1(n,k)$ is a generalization of it.

Next we study the automorphism groups of $G_1(n,k)$. Let $\mathbb{Z}_2$ denote the cyclic group of order 2, and $S_n$ the symmetric group of $[n] = \{1, \cdots, n\}$. The automorphism group of $G_1(n,k)$ is described as follows. Here we recall that the automorphism group of $K(n,k)$ is isomorphic to $S_n$.

**Theorem 2.** For $i = 0, 1, \cdots, k - 1$, there is a group isomorphism

$$\text{Aut}(G_1(n,k)) \cong (\mathbb{Z}_2^i \rtimes \varphi S_i) \times S_{n-2i}.$$

Here the action $\varphi : S_i \to \text{Aut}(\mathbb{Z}_2^i)$ is defined by $\varphi(x_1, \cdots, x_i) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(i)})$, and $\mathbb{Z}_2^i \rtimes \varphi S_i$ is the semi-direct product of groups with respect to $\varphi$.

Finally, we study the chromatic number of $G_1(n,k)$. Recall that $\chi(K(n,k)) = n - 2k + 2$ is called the Kneser conjecture and is proved by Lovász. In his outstanding proof, Lovász introduced the neighborhood complex $N(G)$ of a graph $G$, and the connectivity of $N(G)$ gives a lower bound for the chromatic number $\chi(G)$ of $G$. Since the Kronecker cover of $G$ determines the isomorphism type of $N(G)$ (see [7]) and Lovász determines the connectivity of $N(K(n,k))$, we have the same lower bound for $\chi(G_1(n,k))$. Using this, we can determine the chromatic number of $G_1(n,k)$.

**Theorem 3.** The chromatic number of $G_1(n,k)$ for $i = 0, 1, \cdots, k - 1$ coincides with $\chi(K(n,k)) = n - 2k + 2$.

The rest in this paper is organized as follows. In Section 2, we recall some terminology and facts concerning Kronecker coverings, and prove Theorem 1 and Theorem 2. In Section 3, we review some facts of neighborhood complexes and prove Theorem 3.

**Acknowledgement.** The author is supported by JSPS KAKENHI 19K14536.

2. **Proofs of Theorem 1 and Theorem 2**

We first fix our notation and terminology, and review some facts concerning Kronecker covers. A graph is a pair $G = (V(G), E(G))$ consisting of a set $V(G)$ together with a symmetric binary relation $E(G)$ of $V(G)$. We write $v \sim_G w$ or simply $v \sim w$ to mean that $v$ and $w$ are adjacent in $G$. A map $f : V(G) \to V(H)$ is a graph homomorphism if $v \sim_G w$ implies $f(v) \sim_H f(w)$. An $n$-coloring is a graph homomorphism from $G$ to $K_n$. An isomorphism is a graph homomorphism having an inverse which is a graph homomorphism. An automorphism of $G$ is an isomorphism from $G$ to $G$. Let $\text{Aut}(G)$ denote the automorphism group of $G$. An involution of $G$ is an automorphism $\alpha$ of $G$ such that $\alpha^2 = \text{id}_G$. 

A bigraph \cite{1} is a graph \( X \) equipped with a 2-coloring \( \varepsilon : G \to K_2 \). For a pair \( X \) and \( Y \) of bigraphs, a graph homomorphism \( f : V(X) \to V(Y) \) is even if \( \varepsilon f = \varepsilon \), and odd if \( \varepsilon f(x) \neq \varepsilon(x) \) for every \( x \in V(X) \).

For a pair \( G \) and \( H \) be graphs. The categorical product \( G \times H \) is the graph whose vertex set is \( V(G) \times V(H) \), where \((v, w) \sim_{G \times H} (v', w')\) if and only if \( v \sim_G v' \) and \( w \sim_H w' \). The Kronecker cover is the categorical product \( K_2 \times G \). Note that the Kronecker cover \( K_2 \times G \) has a 2-coloring \( K_2 \times G \to K_2, (i,v) \mapsto i \), and has an odd involution \((1,v) \leftrightarrow (2,v)\). In fact, every bigraph \( X \) equipped with an odd involution \( \alpha \) is isomorphic to the Kronecker cover over a certain graph \( X/\alpha \) defined as follows.

Let \( X \) be a bigraph with an odd involution \( \alpha \). Define the quotient graph \( X/\alpha \) by

\[
V(X/\alpha) = \{\{x, \alpha(x)\} \mid x \in V(X)\},
\]

\[
E(X/\alpha) = \{(\sigma, \tau) \in V(X/\alpha) \times V(X/\alpha) \mid (\sigma \times \tau) \cap E(X) \neq \emptyset\}.
\]

In other words, \( \sigma \sim_{X/\alpha} \tau \) if and only if there is \( x \in \sigma \) and \( y \in \tau \) such that \( x \sim_X y \). Note that \( X/\alpha \) is not simple in general. In fact, \( X/\alpha \) is simple if and only if there is no vertex \( x \) in \( X \) such that \( x \sim_X \alpha(x) \). The graph homomorphism

\[
(\varepsilon, \pi) : X \to K_2 \times (X/\alpha), x \mapsto (\varepsilon(x), \pi(x))
\]

is an even isomorphism (see \cite{7}). The following two lemmas are known (see \cite{3} and Theorem 3.1 of \cite{7}) and easily proved.

**Lemma 4.** Let \( X \) and \( Y \) be bigraphs, \( \alpha \) and \( \beta \) odd involutions of \( X \) and \( Y \) respectively, and \( f : X \to Y \) a (not necessarily even) graph homomorphism satisfying \( f\alpha = \beta f \). Then there is a unique graph homomorphism \( \overline{f} : X/\alpha \to Y/\beta \) satisfying \( \overline{f} \pi = \pi \overline{f} \). If \( f \) is an isomorphism, then \( \overline{f} \) is an isomorphism.

**Lemma 5.** Let \( X \) and \( Y \) be bigraphs, and \( \alpha \) and \( \beta \) odd involutions of \( X \) and \( Y \), respectively. For every graph homomorphism \( f : X/\alpha \to Y/\beta \), there is a unique even graph homomorphism \( \tilde{f} : X \to Y \) such that \( \pi \tilde{f} = f \pi \). If \( f \) is an isomorphism, then \( \tilde{f} \) is also an isomorphism.

Here we mention two important applications of these lemmas. Two odd involutions \( \alpha \) and \( \beta \) of \( X \) are **conjugate** if there is an automorphism \( f \) such that \( f\alpha = \beta f \). Similarly, \( \alpha \) and \( \beta \) are **evenly conjugate** if there is an automorphism \( f \) such that \( f\alpha = \beta f \). Using this terminology, we have the following classification result. Note that in the following corollary, the implication (3) \( \Rightarrow \) (1) is known (see Proposition 3 of \cite{3} for example).

**Proposition 6.** Let \( \alpha \) and \( \beta \) be odd involutions in a bigraph \( X \). Then the following are equivalent.

1. \( X/\alpha \) and \( X/\beta \) are isomorphic.
2. \( \alpha \) and \( \beta \) are evenly conjugate.
(3) \( \alpha \) and \( \beta \) are conjugate.

**Proof.** Suppose that there is an isomorphism \( f : X/\alpha \to X/\beta \). It follows from (2) of Theorem 4 that there is an even isomorphism \( \tilde{f} : X \to X \) satisfying \( f\alpha = \beta f \). It is clear that (2) implies (3). It follows from Lemma 4 that (3) implies (1). \( \square \)

**Proposition 7.** Let \( \alpha \) be an odd involution of a bigraph \( X \). Then \( \text{Aut}(X/\alpha) \) is isomorphic to the subgroup of \( \text{Aut}(X) \) consisting of even elements commuting with \( \alpha \).

**Proof.** Let \( \Gamma \) be the subgroup of \( \text{Aut}(X) \) consisting of even elements commuting with \( \alpha \). Define the group homomorphisms \( \Phi : \Gamma \to \text{Aut}(X/\alpha) \) and \( \Psi : \text{Aut}(X/\alpha) \to \Gamma \) as follows.

Let \( f \in \Gamma \). Since \( f\alpha = \alpha f \), Lemma 4 implies that \( f \) induces an isomorphism \( \Phi(f) = \tilde{f} : X/\alpha \to Y/\beta \). On the other hand, let \( g \in \text{Aut}(X/\alpha) \). It follows from Lemma 5 that there is a unique even automorphism \( \tilde{g} : X \to X \) satisfying \( \tilde{g}\alpha = \alpha \tilde{g} \), and put \( \Psi(g) = \tilde{g} \in \Gamma \). These correspondences are group homomorphisms and \( \Psi \) is the inverse of \( \Phi \). \( \square \)

Now we study the automorphism group of \( K_2 \times G \). For a pair of graphs \( G \) and \( H \), we have a monomorphism \( \text{Aut}(G) \times \text{Aut}(H) \to \text{Aut}(G \times H) \) which sends \( (f, g) \) to \( f \times g \). Here \( f \times g \) is the automorphism sending \( (v, w) \) to \( (f(v), f(w)) \). Since \( \text{Aut}(K_2) = \mathbb{Z}_2 \), there is a monomorphism

\[
\mathbb{Z}_2 \times \text{Aut}(G) \to \text{Aut}(K_2 \times G)
\]

In general, this monomorphism is not an isomorphism (see Remark 11 for example). However, when \( G = K(n, k) \), this monomorphism is an isomorphism:

**Theorem 8** (Mirafzal [8]). If \( n > 2k \), then, the group homomorphism

\[
\mathbb{Z}_2 \times \text{Aut}(K(n, k)) \to \text{Aut}(H(n, k))
\]

described in (*) is an isomorphism. In particular, \( \text{Aut}(H(n, k)) \cong \mathbb{Z}_2 \times S_n \).

When the monomorphism (*) is an isomorphism, then the classification of the graphs whose Kronecker covers are \( K_2 \times G \) is simpler. Here we write \( \tau \) to indicate the non-trivial involution of \( K_2 \).

**Proposition 9.** Let \( G \) be a graph and suppose that the monomorphism (*) is an isomorphism. Then the following hold:

1. For every odd involution \( \alpha \) of \( K_2 \times G \), there is an involution \( \alpha' \) of \( G \) with \( \alpha = \tau \times \alpha' \).
2. Let \( \alpha' \) and \( \beta' \) be involutions of \( G \). Then \( \tau \times \alpha' \) and \( \tau \times \beta' \) are evenly conjugate if and only if \( \alpha' \) and \( \beta' \) are conjugate, i.e. there is \( f \in \text{Aut}(G) \) with \( f\alpha' = \beta'f \).

**Proof.** Since the monomorphism (*) is an isomorphism, every involution \( \alpha \) of \( K_2 \times G \) is written by \( \text{id}_{K_2} \times \alpha' \) or \( \tau \times \alpha' \) for some \( \alpha' \in \text{Aut}(G) \). Since \( \alpha \) is an involution, we have that \( \alpha' \) is an involution. The involution \( \text{id}_{K_2} \times \alpha' \) is even and \( \tau \times \alpha' \) is odd. This follows (1). (2) follows from the fact that every even automorphism of \( K_2 \times G \) is written as \( \text{id}_{K_2} \times f \) for some \( f \in \text{Aut}(G) \) under our assumption. \( \square \)
We are now ready to prove Theorem 1

Proof of Theorem 1 For \( i = 0, 1, \ldots, [n/2] \), define \( \sigma_i \in S_n \) to be the composite of transpositions

\[
(1, 2)(3, 4) \cdots (2i - 1, 2i).
\]

By the classification of conjugacy classes of \( S_n \), every element in \( S_n \) of order 2 is conjugate to some \( \sigma_i \), and \( i \neq j \) implies that \( \sigma_i \) and \( \sigma_j \) are not conjugate. Define \( \alpha_i \) to be the odd involution \( \tau \times \sigma_i \) of \( K_2 \times G \), and put \( G_i = G_i(n, k) = H(n, k)/\alpha_i \). Then \( G_0 = K(n, k) \), \( i \neq j \) implies \( G_i \neq G_j \), and for every odd involution \( \alpha \) of \( H(n, k) \), there is \( i \) with \( G_i \cong H(n, k)/\alpha \).

To complete the proof, we prove that \( G_i \) is simple if and only if \( i < k \).

Suppose \( i \geq k \). Then put \( v = \{1, 3, \ldots, 2k - 1\} \subseteq K(n, k) \). Then \( v \sim \sigma_i(v) \) in \( K(n, k) \) implies that \( \alpha_i(1, v) \sim (2, \sigma_i(v)) \) in \( H(n, k) \). Thus \( G_i \) is not simple. On the other hand, suppose \( i < k \). Then for each \( v \in V(K(n, k)) \), we have that \( \sigma_i(v) \cap v \neq \emptyset \) and hence \( v \neq \sigma_i(v) \) for every \( v \in V(K(n, k)) \). This means that \( (i, v) \neq (1, \sigma_i(v)) \) for every \( (i, v) \in V(H(n, k)) \). This means that \( G_i \) is simple, and completes the proof.

Before giving the proof of Theorem 2 we introduce the following notation: Let \( G \) be a group and \( x \) an element in \( G \). We write \( Z_G(x) \) to indicate the subgroup of \( G \) consisting of the elements in \( G \), which commute with \( x \).

Proof of Theorem 2 Since \((*)\) is an isomorphism, for every automorphism \( f \) of \( H(n, k) \), there is a unique \( \tilde{f} \in \text{Aut}(K(n, k)) \) satisfying \( f = \text{id}_{K_2} \times \tilde{f} \). Thus Proposition 4 implies that the automorphism group of \( G_i(n, k) = X/\langle \tau \times \sigma_i \rangle \) is isomorphic to \( Z_{S_n}(\sigma_i) \).

Hence the following proposition completes the proof.

**Proposition 10.** For \( m = 0, 1, \ldots, [n/2] \), there is a following isomorphism:

\[
Z_{S_n}(\sigma_m) = (Z_2^m \rtimes \varphi S_m) \times S_{n-2m}
\]

Here the action \( \varphi : S_m \rightarrow \text{Aut}(Z_2^m) \) is defined by \( \varphi(\sigma)(\langle x_i \rangle_i) = \langle x_{\sigma^{-1}(i)} \rangle_i \).

**Proof.** Every element \( \sigma \) in \( Z_{S_n}(\sigma_m) \) does not send an element of \( \{1, \ldots, 2m\} \) to \( \{2m + 1, \ldots, n\} \), and hence we have \( Z_{S_n}(\sigma_i) \cong Z_{S_{2m}}(\sigma_m) \times S_{n-2m} \). In \( S_{2m} \), \( \sigma_m \) is conjugate with the element

\[
\tau = (1, m + 1) \cdots (m, 2m).
\]

Thus it suffices to show \( Z_{S_{2m}}(\tau) = Z_2^m \rtimes \varphi S_m \).

First we define the group homomorphism \( \Phi: Z_2^m \rtimes \varphi S_m \rightarrow Z_{S_{2m}}(\tau) \). For \( i = 1, \ldots, m \), set \( \varepsilon_i = (i, n + i) \in S_{2m} \). For \( \sigma \in S_m \), then define \( \tilde{\sigma} \in S_{2m} \) by

\[
\tilde{\sigma}(i) = \begin{cases} 
\sigma(i) & (i = 1, \ldots, m) \\
\sigma(i - m) + m & (i = m + 1, \ldots, 2m).
\end{cases}
\]

Let \( \Phi: Z_2^m \rtimes \varphi S_m \rightarrow Z_{S_{2m}}(\tau) \) be the map which sends \( \langle (x_1, \ldots, x_m), \sigma \rangle \) to \( \varepsilon_1^{x_1} \cdots \varepsilon_m^{x_m} \tilde{\sigma} \).

Using the relation \( \varepsilon_i \tilde{\sigma} = \tilde{\sigma} \varepsilon_{\sigma^{-1}(i)} \), we have that \( \Phi \) is a group homomorphism.
Since $\Phi$ is injective, it suffices to show that $\Phi$ is surjective. Let $\sigma \in Z_{2n}(\tau)$. We identify $\{1, \ldots, 2n\}$ with $\mathbb{Z}_{2n}$, and for $i = 1, \ldots, 2n$, define $k_i \in \mathbb{Z}_{2n}$ by $\sigma(i) = i + k_i$. Since $\sigma$ and $\tau$ commute, we have $k_i = k_{n+i}$. This means that $\sigma$ gives rise to a permutation the family of sets

$$s_1 = \{1, n + 1\}, s_2 = \{2, n + 2\}, \ldots, s_n = \{n, 2n\}.$$

Define $\sigma' \in S_{m}$ by $\sigma(s_i) = s_{\tilde{\sigma}(i)}$. For $i = 1, \ldots, m$, define $x_i \in \mathbb{Z}_2$ as follows:

- If $\sigma(i) = \sigma'(i)$, then $x_{\sigma(i)} = 0$.
- If $\sigma(i) = \sigma'(i) + n$, then $x_{\sigma(i)} = 1$.

Then we have $\Phi((x_1, \ldots, x_m), \sigma') = \sigma$. This completes the proof. \hfill \Box

Remark 11. If $i > 0$, the group homomorphism

$$\mathbb{Z}_2 \times (\mathbb{Z}_2 \times_\phi S_i) \times S_{n-2i} \cong \mathbb{Z}_2 \times \text{Aut}(G_i(n, k)) \rightarrow \text{Aut}(H(n, k)) \cong \mathbb{Z}_2 \times S_n$$

described by (*) is not an isomorphism in the case of $G = G_i(n, k)$.

3. Proof of Theorem 3

The purpose in this section is to prove Theorem 3. Namely, we want to show that $\chi(G_i(n, k)) = n - 2k + 2$ if $n > 2k$. We note that the proof given here is a straightforward generalization of the proof of $\chi(G_1(n, k)) = n - 2k + 2$ in [7].

Recall that Lovász [5] introduces neighborhood complexes of graphs to determine $\chi(K(n, k))$. We first review the definition and facts concerning neighborhood complexes. Let $G$ be a graph, and $v$ a vertex in $G$. Then the neighborhood complex $N(G)$ is the simplicial complex whose simplex is a subset of $V(G)$ having a common neighbor. Lovász showed the following two theorems in his proof of Kneser’s conjecture:

Theorem 12. If $N(G)$ is $m$-connected, then $\chi(G) \geq m + 3$.

Theorem 13. The neighborhood complex $N(K(n, k))$ of $K(n, k)$ is $(n - 2k - 1)$-connected.

On the other hand, the author noted in [7] that the Kronecker cover of a graph $G$ determines the neighborhood complex of $G$:

Lemma 14 (Theorem 1.2 of [7]. See also [1]). Let $G$ and $H$ be graphs. If $K_2 \times G \cong K_2 \times H$, then their neighborhood complexes $N(G)$ and $N(H)$ are isomorphic.

Combining the above results, we have the following corollary:

Corollary 15. The neighborhood complex $N(G_i(n, k))$ is $(n - 2k - 1)$-connected. In particular, the inequality $\chi(G_i(n, k)) \geq n - 2k + 2$ holds.

We now complete the proof of $\chi(G_i(n, k)) = n - 2k + 2$. This is proved by induction on $n$. First, note that $G_i(2k, k)$ is a disjoint union of copies of $K_2$, and hence it is clear that
\(\chi(G_i(2k, k)) = 2\). Suppose that \(n > 2k\) and \(\chi(G_i(n - 1, k)) = n - 2k + 1\). A vertex in \(G_i(n, k)\) which is not contained in \(G_i(n - 1, k)\) is written as 
\(\{ (1, s), (2, \sigma_1 s) \}\),
where \(s\) is a \(k\)-subset of \([n]\) containing \(n\). Note that \(\sigma_1 \in S_n\) fixes \(n\). Since \(G_i(n - 1, k)\) is an induced subgraph of \(G_i(n, k)\), we have that
\[n - 2k + 2 \leq \chi(G_i(n, k)) \leq \chi(G_i(n - 1, k)) + 1 = n - 2k + 2.\]
This completes the proof.

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