DIOPHANTINE EQUATIONS IN PRIMES: DENSITY OF PRIME POINTS ON AFFINE HYPERSURFACES II

SHUNTARO YAMAGISHI

Abstract. Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous form of degree $d \geq 2$, and let $V^*_F$ denote the singular locus of the affine variety $V(F) = \{z \in \mathbb{A}^n_\mathbb{C} : F(z) = 0\}$. In this paper, we prove the existence of integer solutions with prime coordinates to the equation

$$F(x_1, \ldots, x_n) = 0$$

provided $F$ satisfies suitable local conditions and

$$n - \dim V^*_F \geq 7d(2d - 1)4^d + 4(d-1)(12d-1)2^d + 12d.$$

The result is obtained by using the identity $\Lambda = \mu^* \log$ for the von Mangoldt function and optimizing various parts of the argument in the author’s previous work, which made use of the Vaughan identity and required $n - \dim V^*_F \geq 283^4 5^2 2^d 3 (2d - 1)^2 4^d$.

1. Introduction

Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous form of degree $d \geq 2$. In this paper, we are interested in the problem of finding prime solutions, which are integer solutions with prime coordinates, to the equation

$$F(x_1, \ldots, x_n) = 0.$$  

First we introduce some notation in order to state our result. Let $\mathcal{P}$ denote the set of prime numbers. Let $\mathbb{Z}_p^\times$ be the units of $p$-adic integers. We consider the following condition.

Local conditions ($\star$): The equation (1.1) has a non-singular real solution in $(0, 1)^n$, and also has a non-singular solution in $(\mathbb{Z}_p^\times)^n$ for every $p \in \mathcal{P}$.

Let $V^*_F$ denote the singular locus of $V(F) = \{z \in \mathbb{A}^n_\mathbb{C} : F(z) = 0\}$, i.e. it is the affine variety defined by

$$V^*_F = \{z \in \mathbb{A}^n_\mathbb{C} : \nabla F(z) = 0\},$$

where $\nabla F = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)$. We also define the following class of smooth weights.

Definition 1.1. Let $\delta, c > 0$ and $M_0 \in \mathbb{Z}_{\geq 0}$. We define $S^+(\delta; M_0; c)$ to be the set of smooth functions $\omega : \mathbb{R} \to [0, \infty)$ satisfying

(i) $\operatorname{supp} \omega = [-\delta, \delta]$,
(ii) for any $k \in \{0, \ldots, M_0\}$ we have $\|\partial^k \omega / \partial x^k\|_{L^\infty(\mathbb{R})} \leq c$.

Let $X > 1$. Let $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in (0, 1)^n$ and $\omega \in S^+(\delta; n; c)$, and we define

$$\psi_i(x_i) = \omega \left(\frac{x_i}{X} - x_{0,i}\right) \quad (1 \leq i \leq n).$$

Date: Revised on November 12, 2021.

2010 Mathematics Subject Classification. 11D45, 11D72, 11P32, 11P55.

Key words and phrases. Hardy-Littlewood circle method, Diophantine equations, primes.
Let $\Lambda$ denote the von Mangoldt function, where $\Lambda(x) = \log p$ if $x$ is a power of $p \in \wp$ and 0 otherwise. Given $\mathcal{X} \subseteq \mathbb{C}^n$, we let $1_{\mathcal{X}}$ be the characteristic function of $\mathcal{X}$, i.e. $1_{\mathcal{X}}(x) = 1$ if $x \in \mathcal{X}$ and 0 otherwise.

In this paper, we prove the following result by the Hardy-Littlewood circle method.

**Theorem 1.2.** Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous form of degree $d \geq 2$ satisfying
\begin{equation}
 n - \dim V_F^* \geq 7d(2d-1)4^d + 4(d-1)(2d-1)^2 + 12d, \tag{1.4}
\end{equation}
and the local conditions $(\star)$. Let $x_0 \in (0,1)^n$ be a non-singular real solution to the equation (1.1). Let $\delta, c > 0$ and $\omega \in S^+(\delta; n; c)$, where $\delta$ is sufficiently small with respect to $F$ and $x_0$, and let $\psi_1, \ldots, \psi_n$ be as in (1.3). Then for any $A > 0$ we have
\begin{equation}
\sum_{x \in [0,X]^n} \prod_{1 \leq i \leq n} \psi_i(x_i) \Lambda(x_i) \cdot 1_{V(F)}(x) = c(F; \omega, x_0) X^{n-d} + O\left( \frac{X^{n-d}}{(\log X)^{A}} \right), \tag{1.5}
\end{equation}
where $c(F; \omega, x_0) > 0$ is a constant depending only on $F, \omega$ and $x_0$.

The result improves on the author’s previous work [25] in which the above asymptotic formula (1.5) was obtained with
\begin{equation}
 n - \dim V_F^* \geq 2^8 3^4 5^2 6^3 (2d-1)^2 4^d
\end{equation}
instead of (1.4). The first result in this direction for a general degree $d$ was obtained by Cook and Magyar [5], where they required $n - \dim V_F^*$ to be an exponential tower in $d$. For results when $d = 2$, we refer the reader to see [4], [9], [16] and [26]. One day after the author’s work [25] appeared on the arXiv in May 2021 (we note that [25] was submitted in January 2019 and accepted in March 2021), a preprint by Liu and Zhao [17] appeared on the arXiv, where they developed an intricate, and different, method establishing the result with
\begin{equation}
 n \geq 16d^2 4^d
\end{equation}
when $\dim V_F^* = 0$. We remark that the results in [5] and [17] are obtained for systems of forms of equal degrees and without smooth weights. As mentioned in [25], we expect that these differences can be overcome with additional technical effort. Also we may deduce the existence of prime solutions from Theorem 1.2 we refer the reader to see [25] for the details. It may be possible to further improve on Theorem 1.2 by combining the work of this paper with the method developed by Liu and Zhao. Another potential approach to further improve on Theorem 1.2 is described in the final paragraph of this paper.

The main ideas of this paper go back to the author’s previous work [25]. The result is proved by incorporating several technical simplifications and improvements to optimize various parts of the argument in [25], one of the changes being the use of the identity $\Lambda = \mu * \log$, i.e.
\begin{equation}
\Lambda(x) = \sum_{m \mid x} \mu(m) \log(x/m),
\end{equation}
where $\mu$ denotes the Möbius function (defined in (4.9)), instead of the Vaughan identity. In Sections 2 and 4 we prepare for the major and minor arcs analyses respectively, and treat them in the subsequent sections. We complete the proof of Theorem 1.2 in Section 6. The following exponential sum
\begin{equation}
S(\alpha) = \sum_{x \in [0,X]^n} \prod_{1 \leq i \leq n} \psi_i(x_i) \Lambda(x_i) \cdot e(\alpha F(x)), \tag{1.6}
\end{equation}
where \( e(z) = e^{2\pi iz} \) for \( z \in \mathbb{R} \), is central to our study. In particular, it will be relevant in Sections 3-6.

**Notation.** Throughout the paper, we use \( \ll \) and \( \gg \) to denote Vinogradov’s well-known notation, i.e. the statement \( f \ll g \) means there exists a positive constant \( C \) (it may depend on parameters which are regarded as fixed) such that \( |f| \leq Cg \) for all values under consideration, and the statement \( g \gg f \) is equivalent to \( f \ll g \). We also make use of the \( O \)-notation; the statement \( f = O(g) \) is equivalent to \( f \ll g \). All the implicit constants in \( \ll, \gg \) and the \( O \)-notation are independent of \( X \). Given a homogeneous form \( G \in \mathbb{C}[x_1, \ldots, x_m] \), we denote

\[
\text{codim} V_G^* = m - \dim V_G^*.
\]

Also for \( f \in \mathbb{C}[x_1, \ldots, x_m] \), we let \( f^{[j]} \) denote the degree \( j \) homogeneous portion of \( f \). By \( A_m^\infty \) we mean the affine \( m \)-space over \( \mathbb{C} \), i.e. the set \( \mathbb{C}^m \) with the Zariski topology, and for any \( f \in \mathbb{C}[x_1, \ldots, x_m] \) we denote

\[
V(f) = \{ x \in A_m^\infty : f(x) = 0 \}.
\]

Given a function \( \varrho : \mathbb{R} \to \mathbb{R} \), we let \( \text{supp} \varrho = \overline{\{ z \in \mathbb{R} : \varrho(z) \neq 0 \}} \), i.e. the closure of the set \( \{ z \in \mathbb{R} : \varrho(z) \neq 0 \} \). We shall refer to \( \mathcal{B} \subseteq \mathbb{R}^m \) as a box if \( \mathcal{B} \) is of the form \( \mathcal{B} = I_1 \times \cdots \times I_m \), where each \( I_j \) is a closed, open or half open/closed interval. Finally, \( \varepsilon \) will always be a sufficiently small (with respect to relevant parameters) positive number, even when it is not explicitly stated so, and also we do not assume it to be fixed line by line. In other words, we allow it to vary from one expression/inequality to the next. For example, we may write \( X^{2^\varepsilon} \ll X^\varepsilon \ll X^{1-\varepsilon} \) and the three \( \varepsilon \) that appear should not necessarily be interpreted as the same real number.

**Acknowledgements.** The author was supported by the NWO Veni Grant 016.Veni.192.047 while working on this paper.

## 2. Preliminaries for the major arcs analysis

Let \( q \in \mathbb{N} \) and \( \chi^0 \) be the principal character modulo \( q \). We consider \( \chi^0 \) as a primitive character modulo \( q \) only when \( q = 1 \), and not otherwise. We will use the following zero-free region estimate of the Dirichlet \( L \)-functions. The proof is given in [25, Theorem 7.1] with the main input being [13].

**Theorem 2.1.** Let \( s = \sigma + it, \; M \geq 3, \; T \geq 0 \) and \( \mathcal{L} = \log M(T + 3) \). Then there exists an absolute constant \( c_1 > 0 \) such that \( L(s, \chi) \neq 0 \) whenever

\[
\sigma \geq 1 - \frac{c_1}{\log M + (\mathcal{L}\log 2\mathcal{L})^{3/4}} \quad \text{and} \quad |t| \leq T
\]

for all primitive characters \( \chi \) of modulus \( q \leq M \), with the possible exception of at most one primitive character \( \widetilde{\chi} \) modulo \( \widetilde{r} \). If such \( \widetilde{\chi} \) exists, then \( L(s, \widetilde{\chi}) \) has at most one zero in \( \{2, 3\} \) and the exceptional zero \( \widetilde{\beta} \) is real and simple, and \( \widetilde{r} \) satisfies \( M \geq \widetilde{r} \gg_A (\log M)^A \) for any \( A > 0 \).

**Remark 2.2.** Let \( M = X^{\psi_0} \) for a fixed \( \psi_0 > 0 \) and \( X \) sufficiently large. Then we have \( \widetilde{r} \gg_A (\log X)^A \) for any \( A > 0 \) (the subscript in \( \gg_A \) is to indicate that the implicit constant depends on \( A \)). In particular, the exceptional zero will not occur for the Dirichlet \( L \)-functions associated to primitive characters of modulus \( 1 \leq q \leq (\log X)^D \) with \( D > 0 \). It also follows
from Siegel’s theorem (see [25, Remark 7.2]) that
\begin{equation}
0 < 1 - \tilde{\beta} < \frac{1}{2}. \tag{2.2}
\end{equation}

Let
\begin{equation}
B_T = \{ s = \sigma + it : 0 \leq \sigma \leq 1, |t| \leq T \}.
\end{equation}

Let \( N_\chi(\alpha, T) \) denote the number of zeros, with multiplicity, of \( L(s, \chi) \) in the rectangle \( \alpha \leq \sigma \leq 1 \) and \( |t| \leq T \). For the remainder of this paper, we let \( \sum'_\rho \) denote the sum over the non-exceptional zeros (with respect to Theorem 2.1), with multiplicity, of \( L(s, \chi) \) in \( B_T \) and let \( \Re \rho = \beta \). Let \( \sum^*_{\chi(\mod r)} \) denote the sum over the primitive characters \( \chi \) modulo \( r \). The proof of the following lemma is given in [25, Lemma 7.3] with the main inputs being [12] and [15].

**Lemma 2.3.** Let \( \xi, D, A > 0 \). Let \( M = X^{\vartheta_0} \) and \( T = X^{\gamma} \), where \( \vartheta_0, \gamma > 0 \) satisfy \( 2\vartheta_0 + \gamma < \frac{5}{12} \). Then we have
\begin{align*}
i) & \sum_{1 \leq r \leq M} \sum^*_{\chi(\mod r)} \sum'_{\rho}(\xi X)^{\beta - 1} \ll 1, \\
j) & \sum_{1 \leq r \leq \left(\log X\right)^D} \sum^*_{\chi(\mod r)} \sum_{\rho \in B_T}_{L(\rho, \chi) = 0}(\xi X)^{\beta - 1} \ll \left(\log X\right)^{-A},
\end{align*}
where the sum \( \sum_{\rho \in B_T}_{L(\rho, \chi) = 0} \) is over all the zeros, with multiplicity, of \( L(s, \chi) \) in \( B_T \). Here the implicit constants may depend on \( \xi, D \) and \( A \).

Let \( \varphi \) be Euler’s totient function. Let \( U_q = (\mathbb{Z}/q\mathbb{Z})^* \), the group of units in \( \mathbb{Z}/q\mathbb{Z} \). The following lemma is proved in [25, Lemma 7.4].

**Lemma 2.4.** Let \( F \in \mathbb{Z}[x_1, \ldots, x_n] \) be a homogeneous form of degree \( d \geq 2 \). Let \( q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) be such that \( \gcd(a, q) = 1 \). Let \( \chi_1, \ldots, \chi_n \) be any Dirichlet characters modulo \( q \), and let
\begin{equation}
S = \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^n} \chi_1(h_1) \cdots \chi_n(h_n) e \left( \frac{aq}{q} F(h_1, \ldots, h_n) \right). \tag{2.3}
\end{equation}

Then for any \( \varepsilon > 0 \) we have
\begin{equation}
|S| \ll q^{-\frac{1}{2(2d-1)d^2 \text{codim} V^*_F + \varepsilon}},
\end{equation}
where the implicit constant is independent of the choice of \( \chi_1, \ldots, \chi_n \).

The following proposition on oscillatory integrals is proved as the main result in [24].

**Proposition 2.5.** Let \( F \in \mathbb{Z}[x_1, \ldots, x_n] \) be a homogeneous form of degree \( d \geq 2 \) satisfying \( n - \dim V^*_F > 4 \). Let \( r_j \in [-1, 0] \) and \( t_j \in \mathbb{R} \) \( (1 \leq j \leq n) \). Suppose \( x_0 = (x_{0, 1}, \ldots, x_{0, n}) \in (0, 1)^n \) is a non-singular real solution to the equation \( F(x) = 0 \). Let \( \omega \in S^+(\delta; n; c) \). Then provided \( \delta \) is sufficiently small, we have
\begin{equation}
\left| \int_0^\infty \cdots \int_0^\infty \prod_{1 \leq j \leq n} \omega(x_j - x_{0,j}) \cdot x_1^{r_1 + it_1} \cdots x_n^{r_n + it_n} e(\tau F(x)) \, dx \right| \ll \min \{ 1, |\tau|^{-1} \},
\end{equation}
where the implicit constant is independent of \( r_1, \ldots, r_n, t_1, \ldots, t_n \) and \( \tau \).
The key feature of the result is that the bound is uniform in \( t \); the estimate can be deduced easily for a fixed \( t \in \mathbb{R}^n \).

3. Major arcs

Let \( F \in \mathbb{Z}[x_1, \ldots, x_n] \) be a homogeneous form of degree \( d \geq 2 \) satisfying (1.4) and the local conditions (*). Let \( x_0, \delta, \omega \) and \( \chi_1, \ldots, \chi_n \) be as in the statement of Theorem 1.2. Let

\[
\Lambda^*(x) = \begin{cases} \log x & \text{if } x \in \wp, \\ 0 & \text{otherwise}. \end{cases}
\]

We define

\[
S^*(\alpha) = \sum_{x \in \mathbb{N}^n} \varpi(x) \prod_{1 \leq i \leq n} \Lambda^*(x_i) \cdot e(\alpha F(x)),
\]

where

\[
\varpi(x) = \prod_{1 \leq i \leq n} \psi_i(x_i) = \prod_{1 \leq i \leq n} \omega_i \left( \frac{x_i}{X} - x_{0,i} \right).
\]

Clearly we have

\[
S(\alpha) = S^*(\alpha) + O(X^{n-\frac{5}{12}}).
\]

We define

\[
\kappa = \frac{\text{codim } V_F^*}{2(2d - 1)4^d} - 1 - \kappa_0,
\]

where \( \kappa_0 > 0 \) is sufficiently small. In particular, we have \( \kappa > 2 \). Let \( \vartheta_0, \gamma, \lambda > 0 \) be such that

\[
2\vartheta_0 + \gamma < \frac{5}{12} \quad \text{and} \quad \gamma > 2\vartheta_0 + 2\lambda.
\]

We set

\[
M = X^{\vartheta_0} \quad \text{and} \quad T = X^{\gamma},
\]

and let \( \tilde{\chi}, \tilde{r} \) and \( \tilde{\beta} \) be as in Theorem 2.1. We define the following extended major arcs

\[
\mathcal{M}^+(\vartheta_0) = \bigcup_{1 \leq q \leq X^{\vartheta_0}} \bigcup_{0 \leq a \leq q} \mathcal{M}^+_{q,a}(\vartheta_0),
\]

where

\[
\mathcal{M}^+_{q,a}(\vartheta_0) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| < X^{\vartheta_0 + \lambda - d} \right\}.
\]

It can be verified that the arcs \( \mathcal{M}^+_{q,a}(\vartheta_0) \) are disjoint for \( X \) sufficiently large (see for example [1, Lemma 4.1] for the argument). In this section, we prove the following.

**Proposition 3.1.** Suppose \( F \) is as in the statement of Theorem 1.2. Let \( \vartheta_0, \gamma, \lambda > 0 \) be such that (3.6) is satisfied. Then for any \( A > 0 \) we have

\[
\int_{\mathcal{M}^+(\vartheta_0)} S(\alpha) \, d\alpha = c(F; \omega, x_0) X^{n-d} + O \left( \frac{X^{n-d}}{(\log X)^A} \right),
\]

where \( c(F; \omega, x_0) > 0 \) is a constant depending only on \( F, \omega \) and \( x_0 \).
Let \( m, \ell \in \mathbb{Z}_{\geq 0} \) be such that \( m + \ell \leq n \). We denote \( j = (j_1, \ldots, j_m), k = (k_1, \ldots, k_\ell) \) and \( i = (i_1, \ldots, i_{n-m-\ell}) \) satisfying

(3.9) \[ \{1, \ldots, n\} = \{i_1, \ldots, i_{n-m-\ell}\} \cup \{j_1, \ldots, j_m\} \cup \{k_1, \ldots, k_\ell\}. \]

For each such triple \((i, j, k)\) and Dirichlet characters \( \chi_1, \ldots, \chi_m \) modulo \( q \), we define

(3.10) \[ A(q, a; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) = \sum_{\chi \in \mathbb{U}_q} \chi_1(h_{j_1}) \cdots \chi_m(h_{j_m}) \chi_0(h_{k_1}) \cdots \chi_0(h_{k_\ell}) e\left(\frac{aq}{q}\right) \]

if \( \overline{r} \nmid q \), then we only consider \((i, j, k)\) with \( k = \emptyset \), i.e. \( \ell = 0 \). We also define

(3.11) \[ W(\tau; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \sum_{x_{j_1}, \ldots, x_{j_m} \in \mathbb{N}} \overline{\varpi}(x) \prod_{1 \leq v \leq m} \chi_v(x_{j_v}) A^*(x_{j_v}) \cdot x_{k_1}^{\overline{\tau}-1} \cdots x_{k_\ell}^{\overline{\tau}-1} e(\tau F(x)) dx_{j_1} \cdots dx_{j_m} dx_{k_1} \cdots dx_{k_\ell}, \]

where for each \( 1 \leq v \leq m \) we replace \( \chi_v(x_{j_v}) A^*(x_{j_v}) \) with \( (\chi^0(x_{j_v}) A^*(x_{j_v}) - 1) \) when \( \chi_v = \chi^0 \), and with \( (\overline{\chi} \chi^0(x_{j_v}) A^*(x_{j_v}) + x_{j_v}^{\overline{\tau}-1}) \) when \( \chi_v = \overline{\chi} \). We extend this definition to allow each \( \chi_v \) to be a primitive character modulo \( r_v \) as follows

(3.12) \[ W(\tau; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \sum_{x_{j_1}, \ldots, x_{j_m} \in \mathbb{N}} \overline{\varpi}(x) \prod_{1 \leq v \leq m} \chi_v(x_{j_v}) A^*(x_{j_v}) \cdot x_{k_1}^{\overline{\tau}-1} \cdots x_{k_\ell}^{\overline{\tau}-1} e(\tau F(x)) dx_{j_1} \cdots dx_{j_m} dx_{k_1} \cdots dx_{k_\ell}, \]

where for each \( 1 \leq v \leq m \) we replace \( \chi_v(x_{j_v}) A^*(x_{j_v}) \) with \( (A^*(x_{j_v}) - 1) \) when \( r_v = 1 \), and with \( (\overline{\chi}(x_{j_v}) A^*(x_{j_v}) + x_{j_v}^{\overline{\tau}-1}) \) when \( \chi_v = \overline{\chi} \). With these notation we have the following technical estimate; the result is essentially [25, Lemma 8.1] except that the range of \( \tau \) is \(|\tau| < X^{\vartheta_0+\lambda-d}\) instead of \(|\tau| < X^{\vartheta_0-d}\).

**Lemma 3.2.** Let \( \alpha \in [0, 1) \), where \( \alpha = \frac{a}{q} + \tau \), \( 0 \leq a \leq q \leq X^{\vartheta_0} \), \( \gcd(a, q) = 1 \) and \(|\tau| < X^{\vartheta_0+\lambda-d}\). Then we have

(3.13) \[ S^*(\alpha) = \frac{1}{\varphi(q)^n} A(q, a; (1, \ldots, n); \emptyset; \emptyset) W(\tau; (1, \ldots, n); \emptyset; \emptyset) + \sum_{(j, k)} \frac{(-1)^\ell}{\varphi(q)^n} S_{j, k}(\alpha) + O(X^{n+\vartheta_0+\lambda-1}), \]

where

\[ S_{j, k}(\alpha) = \sum_{\chi_1, \ldots, \chi_m \pmod{q}} A(q, a; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) W(\tau; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k), \]

and the sum \( \sum_{(j, k)} \) in (3.13) is over all \((j, k)\) satisfying \((m, \ell) \neq (0, 0)\) and (3.9) with an additional condition \( \ell = 0 \) if \( \overline{r} \nmid q \) or the exceptional zero does not exist.
Proof. Here we only consider the case where the exceptional zero does exist and \( r | q \); the proof for the cases \( r \nmid q \) or the exceptional zero does not exist are identical to this case with only slight modifications. First note if \( x_u \in \varphi \cap \{x_{0,u} - \delta \}X, (x_{0,u} + \delta)X \), then \( \gcd(x_u, q) = 1 \) because \( q \leq X^{0 \circ} \). Therefore, we obtain the following via the orthogonality relation of the Dirichlet characters

\[
S^*(\alpha) = \frac{1}{\phi(q)^n} \sum_{\chi_1, \ldots, \chi_n \pmod{q}} \sum_{h \in \mathbb{Z}_n} \overline{\chi_1(h_1)} \cdots \overline{\chi_n(h_n)} e\left( \frac{q}{h} F(h) \right) \cdot \sum_{\chi \in \mathbb{Z}_n} \varpi(\chi) \prod_{1 \leq u \leq n} \chi_u(x_u) \Lambda^*(x_u) \cdot e(\tau F(x)) .
\]

Let \( i' = (i'_1, \ldots, i'_{s'}) \), \( j' = (j'_1, \ldots, j'_{m'}) \), \( k' = (k'_1, \ldots, k'_{\ell'}) \), \( k = (k_1, \ldots, k_\ell) \) and \( i = (i_1, \ldots, i_{n-s'-m'-\ell'-\ell}) \), where

\[
\{1, \ldots, n\} = \{i_1, \ldots, i_{n-s'-m'-\ell'-\ell}\} \cup \{i'_1, \ldots, i'_{s'}\} \cup \{j'_1, \ldots, j'_{m'}\} \cup \{k'_1, \ldots, k'_{\ell'}\} \cup \{k_1, \ldots, k_\ell\} .
\]

We remark that \( s', m', \ell', \ell \) and \( n - s' - m' - \ell' - \ell \) are allowed to be 0. Now we break up the summands of the inner sum \( \sum_{\chi \in \mathbb{Z}_n} \) using the identities

\[
\chi^0(x_u) \Lambda^*(x_u) = (\chi^0(x_u) \Lambda^*(x_u) - 1) + 1
\]

when \( \chi_u = \chi^0 \), and

\[
\widetilde{\chi} \chi^0(x_u) \Lambda^*(x_u) = \left( \widetilde{\chi} \chi^0(x_u) \Lambda^*(x_u) + x_u^{\beta-1} \right) - x_u^{\beta-1}
\]

when \( \chi_u = \widetilde{\chi} \chi^0 \) (If \( r \nmid q \) or the exceptional zero does not exist, then we simply ignore this second identity); we do this for each \( 1 \leq u \leq n \) and obtain

\[
\sum_{x \in \mathbb{Z}_n} \varpi(x) \prod_{1 \leq u \leq n} \chi_u(x_u) \Lambda^*(x_u) \cdot e(\tau F(x))
\]

\[
= \sum_{(i,i',j',k',k)} \sum_{x \in \mathbb{Z}_n} \varpi(x) \prod_{w \in \{i'_1, \ldots, i'_{s'}\}} \chi_w(x_v) \Lambda^*(x_v) \cdot \prod_{v \in \{j'_1, \ldots, j'_{m'}\}} \widetilde{\chi} \chi^0(x_u) \Lambda^*(x_u) \cdot \prod_{u \in \{k'_1, \ldots, k'_{\ell'}\}} \chi_u(x_u) \Lambda^*(x_u) \cdot (-1)^{\ell} x_{k_1}^{\beta-1} \cdots x_{k_\ell}^{\beta-1} e(\tau F(x)) ,
\]

where the sum \( \sum_{(i,i',j',k',k)} \) is over all \((i_i, i', j', k', k)\) satisfying

\[
\{w : \chi_w = \chi^0\} = \{i_1, \ldots, i_{n-s'-m'-\ell'-\ell}\} \cup \{i'_1, \ldots, i'_{s'}\},
\]

\[
\{v : \chi_v \neq \chi^0, \widetilde{\chi} \chi^0\} = \{j'_1, \ldots, j'_{m'}\}
\]

and

\[
\{u : \chi_u = \widetilde{\chi} \chi^0\} = \{k'_1, \ldots, k'_{\ell'}\} \cup \{k_1, \ldots, k_\ell\},
\]

and the convention regarding \( \chi^0(x_w) \Lambda^*(x_w) \) and \( \widetilde{\chi} \chi^0(x_u) \Lambda^*(x_u) \) described in the sentence following (3.11) is being used. By substituting (3.16) into (3.14) and changing the order of
where the sum \( \sum_{i,i',j',k',k} \) is over all \((i, i', j', k', k)\) satisfying (3.15). Let us set \( j = (i', j', k') \). Then for fixed \( h \in \mathbb{U}_q^n, x \in \mathbb{N}^n \) and \((i, k)\), it follows that

\[
\sum_{(i,j',k',k)} \sum_{\chi_{j_1, \ldots, j_{m'}} \neq \chi_{\overline{0}}, \overline{\chi}_{\overline{0}}} \sum_{h \in \mathbb{U}_q^m} \prod_{u \in \{k_1' \ldots k_{r'}\}} \overline{\chi}_0^0(h_u) \cdot \prod_{v \in \{j_1' \ldots j_{r'}\}} \chi_v(x_v) \Lambda^*(x_v) \cdot \prod_{u \in \{k_1' \ldots k_{r'}\}} \overline{\chi}_0^0(h_u) \overline{\chi}_0^0(x_u) \Lambda^*(x_u) = \sum_{(i,j',k',k)} \prod_{v \in \{j_1' \ldots j_{r'}\}} \chi_v(x_v) \Lambda^*(x_v) \cdot \prod_{u \in \{k_1' \ldots k_{r'}\}} \overline{\chi}_0^0(h_u) \chi_v(x_v) \Lambda^*(x_v) \cdot \prod_{u \in \{k_1' \ldots k_{r'}\}} \overline{\chi}_0^0(h_u) \overline{\chi}_0^0(x_u) \Lambda^*(x_u) = \sum_j \sum_{\chi_{j_1, \ldots, j_{m'}}(\text{mod } q) = j_{j_1, \ldots, j_{m'}}} \prod_{v \in \{j_1' \ldots j_{r'}\}} \overline{\chi}(h_v) \chi_v(x_v) \Lambda^*(x_v),
\]
where the sum \( \sum_j \) is over all \( j \) satisfying (3.9). Therefore, we obtain

\[
S^*(\alpha) = \sum_{(i,j,k)} \sum_{\chi_{j_1, \ldots, j_{m'}}(\text{mod } q)} \sum_{h \in \mathbb{U}_q^m} \prod_{u \in \{k_1, \ldots, k_l\}} \overline{\chi}_0^0(h_u) \cdot \prod_{v \in \{j_1, \ldots, j_{m'}\}} \chi_v(x_v) \Lambda^*(x_v) \cdot (-1)^{\ell} \prod_{v \in \{j_1, \ldots, j_{m'}\}} x_{k_1}^{\overline{3}-1} \cdots x_{k_l}^{\overline{3}-1} e(\tau F(x)) = \sum_{x \in \mathbb{N}^n} \overline{\omega}(x) \prod_{v \in \{j_1, \ldots, j_{m'}\}} \chi_v(x_v) \Lambda^*(x_v) \cdot (-1)^{\ell} \prod_{v \in \{j_1, \ldots, j_{m'}\}} x_{k_1}^{\overline{3}-1} \cdots x_{k_l}^{\overline{3}-1} e(\tau F(x))
\]
where the sum \( \sum_{(j,k)} \) is over all \((j, k)\) satisfying \((m, \ell) \neq (0, 0)\) and (3.9). We also switched the notation from \( \chi_{j_{m'}^m} \) to \( \chi_m \) to get the second equality. Each summand of the sum \( \sum_{(j,k)} \)
in (3.18) can also be expressed as
\[
\frac{(-1)^\ell}{\phi(q)^m} \sum_{h \in \mathbb{U}_q^n} \overline{\chi^0}(h_{k_1}) \cdots \overline{\chi^0}(h_{k_t}) e\left(\frac{a}{q} F(h)\right) \\
\sum_{x_j \in \mathbb{N}} \prod_{1 \leq v \leq m} \Lambda^*(x_{j_v}) \sum_{\chi_v \bmod q} \overline{\chi}(h_{j_v}) \chi_v(x_{j_v}) \sum_{x_u \in \mathbb{N}} \varpi(x) x_{k_1}^{\beta_1-1} \cdots x_{k_t}^{\beta_t-1} e(\tau F(x)).
\]
where we used the orthogonality relation of the Dirichlet characters to obtain the latter expression. Next we apply the following estimate which can be deduced from the mean value theorem along with (2.2).

**Lemma 3.3.** For each \(x_1, \ldots, x_m \in \mathbb{N}\), we have
\[
\sum_{x_u \in \mathbb{N}} \varpi(x) x_{k_1}^{\beta_1-1} \cdots x_{k_t}^{\beta_t-1} e(\tau F(x)) \quad \leq \quad \int_0^\infty \cdots \int_0^\infty \varpi(x) x_{k_1}^{\beta_1-1} \cdots x_{k_t}^{\beta_t-1} e(\tau F(x)) dx_{i_1} \cdots dx_{i_m} + O(X^{n-m+\theta_0+\lambda-1}),
\]
where the implicit constant is independent of \(x_1, \ldots, x_m\).

**Proof.** We deal with one variable at a time. By the mean value theorem, it follows that
\[
\sum_{x_{k_1} \in \mathbb{N}} \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x))
\]
\[
= \sum_{t \in \mathbb{N}} \int_t^{t+1} \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x)) dx_{k_1} + O\left(\sup_{x_{k_1} \in [t, t+1]} \left| \frac{\partial}{\partial x_{k_1}} \left( \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x)) \right) \right| \right)
\]
\[
= \int_0^\infty \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x)) dx_{k_1} + O\left( X \sup_{x_{k_1} \in [1, X]} \left| \frac{\partial}{\partial x_{k_1}} \left( \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x)) \right) \right| \right).
\]
By (2.2) and the assumption \(|\tau| < X^{\theta_0+\lambda-d}\), we have
\[
\sup_{x_{k_1} \in [1, X]} \left| \frac{\partial}{\partial x_{k_1}} \left( \varpi(x) x_{k_1}^{\beta_1-1} e(\tau F(x)) \right) \right| \ll |\tau| X^{d-1} \ll |X^{\theta_0+\lambda-1}|.
\]
Therefore, we obtain
\[
\sum_{x_u \in \mathbb{N}} \varpi(x) x_{k_1}^{\beta_1-1} \cdots x_{k_t}^{\beta_t-1} e(\tau F(x))
\]
\[
= \int_0^\infty \sum_{x_u \in \mathbb{N}} \varpi(x) x_{k_1}^{\beta_1-1} \cdots x_{k_t}^{\beta_t-1} e(\tau F(x)) dx_{k_1} + O(X^{n-m+\theta_0+\lambda-1}).
\]
The result follows by repeating this procedure for each of the remaining variables. □

We now apply Lemma 3.3 to both terms on the right hand side of (3.18); the first term becomes

\[
\frac{1}{\phi(q)^n} A(q, a; (1, \ldots, n); \varnothing; \varnothing) \sum_{x \in \mathbb{N}^n} \omega(x) e(\tau F(x))
\]

\[
= \frac{1}{\phi(q)^n} A(q, a; (1, \ldots, n); \varnothing; \varnothing) W(\tau; (1, \ldots, n); \varnothing; \varnothing) + O(X^{n+\vartheta_0+\lambda-1}),
\]

and each summand of the sum \(\sum_{(j,k)}\) (see the expression above Lemma 3.3) becomes

\[
\frac{(-1)^\ell}{\phi(q)^n} \sum_{x_1, \ldots, x_m \equiv (j_1, \chi_1), \ldots, (j_m, \chi_m) \pmod{q}} A(q, a; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) \cdot \\
\sum_{x \in \mathbb{N}^n} \omega(x) \chi_1(x_{j_1}) \Lambda^*(x_{j_1}) \cdots \chi_m(x_{j_m}) \Lambda^*(x_{j_m}) x_{k_1} \cdots x_{k_\ell} e(\tau F(x))
\]

\[
= \frac{(-1)^\ell}{\phi(q)^n} \sum_{x_1, \ldots, x_m \equiv (j_1, \chi_1), \ldots, (j_m, \chi_m) \pmod{q}} A(q, a; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k) \cdot W(\tau; i; (j_1, \chi_1), \ldots, (j_m, \chi_m); k)
\]

\[
+ O \left( \frac{X^{n-m+\vartheta_0+\lambda-1}}{\phi(q)^{n-m}} \sum_{h \in U_q} \prod_{1 \leq v \leq m} \sum_{1 \leq x_{j_v} \leq X \atop x_{j_v} \equiv h_{j_v} \pmod{q}} \Lambda^*(x_{j_v}) \right).
\]

Therefore, we see that we have obtained the result apart form the error term. For each \((j, k)\), we have by the prime number theorem that

\[
\frac{X^{n-m+\vartheta_0+\lambda-1}}{\phi(q)^{n-m}} \sum_{h \in U_q} \prod_{1 \leq v \leq m} \sum_{1 \leq x_{j_v} \leq X \atop x_{j_v} \equiv h_{j_v} \pmod{q}} \Lambda^*(x_{j_v})
\]

\[
= X^{n-m+\vartheta_0+\lambda-1} \prod_{1 \leq v \leq m} \sum_{h_{j_v} \in U_q} \sum_{1 \leq x_{j_v} \leq X \atop x_{j_v} \equiv h_{j_v} \pmod{q}} \Lambda^*(x_{j_v})
\]

\[
\leq X^{n-m+\vartheta_0+\lambda-1} \left( \sum_{1 \leq x \leq X} \Lambda(x) \right)
\]

\[
\ll X^{n+\vartheta_0+\lambda-1}.
\]

From this bound it follows that the error term is as in the statement of the lemma. □
It follows from Lemma 3.2 that

\begin{equation}
\int_{\mathbb{R}^+} S^*(\alpha) \, d\alpha
= \sum_{1 \leq q \leq X^{0_0}} \sum_{1 \leq a \leq q \atop \gcd(a,q)=1} \frac{1}{\varphi(q)^n} A(q,a; (1, \ldots, n); \varnothing; \varnothing) \cdot \int_{|\tau| < X^{0_0+\lambda-d}} W(\tau; (1, \ldots, n); \varnothing; \varnothing) \, d\tau
\end{equation}

\begin{align*}
&+ \sum_{(j, \varnothing)} \sum_{1 \leq q \leq X^{0_0}} \sum_{1 \leq a \leq q \atop \gcd(a,q)=1} \frac{1}{\varphi(q)^n} \int_{|\tau| < X^{0_0+\lambda-d}} S_{j,\varnothing} \left( \frac{a}{q} + \tau \right) \, d\tau \\
&+ \sum_{(j,k) \neq (\varnothing)} \sum_{1 \leq q \leq X^{0_0}} \sum_{1 \leq a \leq q \atop \gcd(a,q)=1} \frac{(-1)^{\ell}}{\varphi(q)^n} \int_{|\tau| < X^{0_0+\lambda-d}} S_{j,k} \left( \frac{a}{q} + \tau \right) \, d\tau \\
&+ O(X^{n-d+4\theta_0+2\lambda-1}),
\end{align*}

where the sum \( \sum_{(j,k)} \) is as in the statement of Lemma 3.2 (If the exceptional zero does not exist, then the sum \( \sum_{k \neq \varnothing} \) is an empty sum.). We prove that the first term on the right hand side of (3.19) contributes the main term, while the remaining terms are error terms. We consider the contribution from each \((j,k)\) separately, where we deal with the case \( k = \varnothing \) in Section 3.1 and the case \( k \neq \varnothing \) in Section 3.2. Finally, Proposition 3.1 is established in Section 3.3.

3.1. Case \( k = \varnothing \). Since \((m, \ell) \neq (0,0)\), we necessarily have \( m > 0 \). Without loss of generality let \( j = (1, \ldots, m) \). In this case, we have

\begin{equation}
\left| \sum_{1 \leq q \leq X^{0_0}} \sum_{1 \leq a \leq q \atop \gcd(a,q)=1} \frac{1}{\varphi(q)^n} \int_{|\tau| < X^{0_0+\lambda-d}} S_{j,\varnothing} \left( \frac{a}{q} + \tau \right) \, d\tau \right|
= \left| \sum_{1 \leq q \leq X^{0_0}} \sum_{1 \leq a \leq q \atop (a \equiv x_0 \pmod{q}, \gcd(a,q)=1)} \frac{1}{\varphi(q)^n} A(q,a; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing) \cdot \int_{|\tau| < X^{0_0+\lambda-d}} W(\tau; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing) \, d\tau \right|
\end{equation}

Let us denote \( \chi^0_v = \chi^0 + \delta \), where \( \chi^0 \) is the primitive character modulo \( r_v \) which induces \( \chi^0_v \), and \( \chi^0 \) is the principal character modulo \( q \). We also denote

\begin{equation}
R = \text{lcm}(r_1, \ldots, r_m).
\end{equation}

We let

\( \delta_{\text{min}} = \min_{1 \leq u \leq n} (x_{0,u} - \delta) \).

Let \( p \in [(x_{0,u} - \delta)X, (x_{0,u} + \delta)X] \) be a prime. Since \( q \leq X^{0_0} < \delta_{\text{min}}X \), we have \( \gcd(p,q) = 1 \) and it follows that \( \chi^0_v(p)\chi^0(p) = \chi^0_v(p) \). Consequently, we obtain

\( W(\tau; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing) = W(\tau; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing) \)

\begin{align*}
&= W(\tau; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing) \\
&= W(\tau; (1, \chi^0_1, \ldots, \chi^0_m); \varnothing).
\end{align*}
Recall the definition of $\kappa$ given in \cite{25}. Since $q/\phi(q) \ll \varepsilon q^\varepsilon$ for any $\varepsilon > 0$, it follows from Lemma 2.4 that

\begin{equation}
\sum_{1 \leq q \leq X_0 \atop R \mid q} \sum_{1 \leq a \leq q \atop \gcd(a, q) = 1} \frac{1}{\phi(q)^n} A(q, a; i; (1, \chi_1 \chi^0), \ldots, (m, \chi_m \chi^0); \emptyset) \ll \sum_{1 \leq q \leq X_0 \atop R \mid q} \sum_{1 \leq a \leq q \atop \gcd(a, q) = 1} \frac{q^{-\operatorname{codim} V^*_{q, \emptyset}}}{} \frac{1}{\phi(q)^n} A(q, a; i; (1, \chi_1 \chi^0), \ldots, (m, \chi_m \chi^0); \emptyset) \ll \sum_{1 \leq q \leq X_0 \atop R \mid q} q^{-\operatorname{codim} V^*_{q, \emptyset} + 1 + \kappa_0} \ll R^{-\kappa}.
\end{equation}

Therefore, the term on the right hand side of \eqref{3.20} can be rewritten as

\begin{equation}
\frac{1}{\phi(q)^n} A(q, a; i; (1, \chi_1 \chi^0), \ldots, (m, \chi_m \chi^0); \emptyset) \cdot \int_{|\tau| < X_0^{\chi_0 + \lambda - d}} |\mathcal{W}(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); \emptyset)| d\tau \ll \sum_{1 \leq r_1, \ldots, r_m \leq X_0^{\chi_0 \atop (1 \leq v \leq m)}} R^{-\kappa} \sum_{\chi_v \left(\text{mod } r_v\right) \atop \leq a \leq q \atop \gcd(a, q) = 1} \frac{1}{\phi(q)^n} A(q, a; i; (1, \chi_1 \chi^0), \ldots, (m, \chi_m \chi^0); \emptyset) \ll R^{-\kappa} \sum_{1 \leq r_1, \ldots, r_m \leq X_0^{\chi_0 \atop (1 \leq v \leq m)}} R^{-\kappa} \sum_{\chi_v \left(\text{mod } r_v\right) \atop \leq a \leq q \atop \gcd(a, q) = 1} |\mathcal{W}(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); \emptyset)| d\tau.
\end{equation}

We will make use of the following explicit formula (for example, it can be deduced from \cite{27} §17 and §19): Let $1 \leq q \leq Y$ and $2 \leq T' \leq Y^{1/2}$. For any primitive character $\chi$ modulo $q$, we have

\begin{equation}
\sum_{y \leq Y} \Lambda^*(y) \chi(y) = \delta_{\chi = \chi^0} Y - \sum_{\rho \in B_{T'}} \frac{Y^\rho}{\rho} + \mathfrak{R}(Y),
\end{equation}

where $\delta_{\chi = \chi^0} = 1$ if $\chi = \chi^0$ and 0 otherwise, and

\begin{equation}
|\mathfrak{R}(Y)| \ll E(Y) = \frac{Y (\log Y)^2}{T'}.
\end{equation}

Here the sum $\sum_{\rho \in B_{T'}}$ is over all the zeros, with multiplicity, of $L(s, \chi)$ in $B_{T'}$. We will be using \eqref{3.24} with $T' = T$ and $X \ll Y \ll X$.

With these notation (also recall the notation introduced in the paragraph preceding Lemma 2.3) we obtain the following lemma, which is essentially \cite{25} Lemma 8.2; the result is obtained by replacing $\vartheta_0$ in the proof of \cite{25} Lemma 8.2 with $\vartheta_0 + \lambda$. Note we do not assume that $k = \emptyset$ for this lemma, and recall the notation introduced in the paragraph following \eqref{2.2}.

**Lemma 3.4.** Without loss of generality let $j = (1, \ldots, m)$ and $k = (m + 1, \ldots, m + \ell)$, where $m > 0$ and $\ell \geq 0$. Suppose $\chi_v$ is a primitive character modulo $r_v$ ($1 \leq v \leq m$). Let
where

\[ \tilde{E} = \sum_{\epsilon \in \{0,1\}^{m}\setminus\{0\}} X^n X^{(\theta_0 + \lambda - 1)\epsilon_1 + \cdots + \epsilon_m} E(X)^{\epsilon_1 + \cdots + \epsilon_m} \prod_{\epsilon_j = 0}^{} \sum_{\rho_j}^{'} \left( \delta_{\min} X \right)^{\beta_j - 1} \]

and Re \( \rho_j = \beta_j \).

Recall the definition of \( \varpi \) given in (3.3). We substitute (3.26) into (3.23), and apply Proposition 2.5 to the first of the resulting terms in the following manner.

\[
\begin{align*}
(3.27) \int_{|\tau| < X^{\theta_0 + \lambda - d}} \left| \int_{\tau_0}^{\infty} \cdots \int_{\tau_0}^{\infty} \prod_{v=1}^{m} \sum_{\rho_v}^{'} x_{\tau_v - 1} \cdot \varpi(x) e(\tau F(x)) \, dx \right| \, d\tau \\
= \left| X^{n-d} \int_{|\tau| < X^{\theta_0 + \lambda}} \left| \int_{\tau_0}^{\infty} \cdots \int_{\tau_0}^{\infty} \prod_{v=1}^{m} \sum_{\rho_v}^{'} (X x_{\tau_v})^{\rho_v - 1} \cdot \prod_{1 \leq u \leq n} \omega(x - x_{\tau_0}) \cdot e(\tau F(x)) \, dx \right| \, d\tau \\
\ll X^{n-d} \sum_{\rho_1, \ldots, \rho_m}^{'} \left| X^{\rho_1 - 1} \cdots X^{\rho_m - 1} \right| \cdot \\
\int_{|\tau| < X^{\theta_0 + \lambda}} \left| \int_{\tau_0}^{\infty} \cdots \int_{\tau_0}^{\infty} x_{\tau_1}^{\rho_1 - 1} \cdots x_{\tau_m}^{\rho_m - 1} \prod_{1 \leq u \leq n} \omega(x - x_{\tau_0}) \cdot e(\tau F(x)) \, dx \right| \, d\tau \\
\ll X^{n-d} (\log X) \sum_{\rho_1, \ldots, \rho_m}^{'} X^{\beta_1 - 1} \cdots X^{\beta_m - 1}.
\end{align*}
\]

Therefore, by combining (3.20), (3.23), (3.26) and (3.27), we obtain

\[
(3.28) \left| \sum_{1 \leq q \leq X^{\theta_0}} \sum_{1 \leq a \leq q}^{\gcd(a,q) = 1} \frac{1}{\phi(q)^n} \int_{|\tau| < X^{\theta_0 + \lambda - d}} S_{1, \omega} \left( \frac{a + \tau}{q} \right) \, d\tau \right| \\
\ll \sum_{1 \leq x_v (\mod \, \tau_v) \leq X^{\theta_0}} \sum_{(1 \leq v \leq m)}^{*} X^{n-d} (\log X) \sum_{\rho_1, \ldots, \rho_m}^{'} X^{\beta_1 - 1} \cdots X^{\beta_m - 1} \\
+ \sum_{1 \leq x_v (\mod \, \tau_v) \leq X^{\theta_0}}^{*} \sum_{(1 \leq v \leq m)}^{*} \int_{|\tau| < X^{\theta_0 + \lambda - d}} \tilde{E} \, d\tau.
\]

We begin by bounding the first term on the right hand side of (3.28). By the definition of \( R \) given in (3.21), we clearly have \( R^{-\kappa} \leq r_1^{-\kappa} \). Let \( D > 1 \) and \( A > 0 \). Then by Lemma 2.3
and Remark 2.2 we obtain

\begin{equation}
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} R^{-\kappa} \sum_{\chi \pmod{r_i}} X^{n-d} \sum_{\rho_1, \ldots, \rho_m} X^{\beta_1-1} \ldots X^{\beta_m-1}
\end{equation}

\[
\ll X^{n-d} (\log X) \left( \sum_{1 \leq r_1 \leq X^{\alpha_0}} r_1^{-\kappa} \sum_{\chi \pmod{r_1}} \sum_{\rho_1} X^{\beta_1-1} \right) \prod_{2 \leq v \leq m} \sum_{1 \leq r_v \leq X^{\alpha_0}} \sum_{\chi \pmod{r_v}} \sum_{\rho_v} X^{\beta_v-1}
\]

\[
\ll X^{n-d} (\log X) \sum_{1 \leq r_1 \leq X^{\alpha_0}} r_1^{-\kappa} \sum_{\chi \pmod{r_1}} \sum_{\rho_1} X^{\beta_1-1}
\]

\[
\ll X^{n-d} (\log X) \sum_{1 \leq r_1 \leq (\log X)^\rho} \sum_{\chi \pmod{r_1}} \sum_{\rho_1 \in B_{\gamma}} X^{\beta_1-1}
\]

\[
+ X^{n-d} (\log X)^{-D_{\kappa+1}} \sum_{1 \leq r_1 \leq X^{\alpha_0}} \sum_{\chi \pmod{r_1}} \sum_{\rho_1} X^{\beta_1-1}
\]

\[
\ll X^{n-d} (\log X)^{-A} + X^{n-d} (\log X)^{-D_{\kappa+1}}.
\]

Next we bound the second term on the right hand side of (3.28). We define

\[
R_{\epsilon} = \text{lcm}(r_{\epsilon_1}, \ldots, r_{\epsilon_{\gamma}+\ldots+\epsilon_m}),
\]

where

\[
\{\epsilon_1, \ldots, \epsilon_{\gamma}+\ldots+\epsilon_m\} = \{1 \leq i \leq m : \epsilon_i = 1\}.
\]

We have from (3.7) and (3.25) that

\[
E(X) \ll (\log X)^2 X^{1-\gamma}.
\]

Then by Lemma 2.3 it follows that

\begin{equation}
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} R^{-\kappa} \sum_{\chi \pmod{r_i}} X^{n-d+\varrho_0+\lambda} \sum_{\rho_1, \ldots, \rho_m} X^{\beta_1-1} \sum_{j=1}^{\delta_{\text{min}} X} \beta_j-1
\end{equation}

\[
\ll \sum_{\epsilon \in \{0,1\}^m \setminus \{0\}} X^{n-d+\varrho_0+\lambda} X^{(\varrho_0+\lambda-1)(\epsilon_1+\ldots+\epsilon_m)} E(X)^{\epsilon_1+\ldots+\epsilon_m}.
\]

\[
\ll R_{\epsilon}^{-\kappa} \prod_{\epsilon_i=1} X \sum_{\epsilon_j=0} \sum_{1 \leq r_j \leq X^{\alpha_0}} \sum_{\rho_j} \sum_{\delta_{\text{min}} X} \beta_j-1
\]

\[
\ll \sum_{\epsilon \in \{0,1\}^m \setminus \{0\}} X^{n-d+\varrho_0+\lambda} X^{(\varrho_0+\lambda-1)(\epsilon_1+\ldots+\epsilon_m)} E(X)^{\epsilon_1+\ldots+\epsilon_m} \sum_{1 \leq r \leq X^{\alpha_0}} R_{\epsilon}^{-\kappa} \prod_{\epsilon_i=1} \phi(r_i)
\]

\[
\ll (\log X)^2 \sum_{\epsilon \in \{0,1\}^m \setminus \{0\}} X^{n-d+\varrho_0+\lambda} X^{(\varrho_0+\lambda-\gamma)(\epsilon_1+\ldots+\epsilon_m)} \sum_{1 \leq r \leq X^{\alpha_0}} R_{\epsilon}^{-\kappa} \prod_{\epsilon_i=1} \phi(r_i).
\]

In order to estimate the final expression, we use the following.

**Lemma 3.5.** Let \( t \in \mathbb{R}_{>1} \) and \( h \in \mathbb{N} \). Then

\[
\sum_{1 \leq x_1 \leq B} \ldots \sum_{1 \leq x_h \leq B} \frac{\phi(x_1) \cdot \phi(x_h)}{\text{lcm}(x_1, \ldots, x_h)^t} \ll \begin{cases} B^\varepsilon & \text{if } t \geq h+1, \\ B^{h-\varepsilon+t+1} & \text{if } t < h+1. \end{cases}
\]
Proof. First we recall the following well-known estimate for the divisor function

$$\sum_{m \mid x} 1 \ll x^\varepsilon.$$ 

Suppose $t \geq h + 1$. Then since

$$\phi(x_1) \cdots \phi(x_h) \leq \frac{1}{\operatorname{lcm}(x_1, \ldots, x_h)^t},$$

it follows that

$$\sum_{1 \leq x_1 \leq B} \cdots \sum_{1 \leq x_h \leq B} \frac{\phi(x_1) \cdots \phi(x_h)}{\operatorname{lcm}(x_1, \ldots, x_h)^t} \leq \frac{\sum_{m \mid x} 1}{x} \ll \frac{B^h}{x^\varepsilon}.$$ 

On the other hand, suppose $t < h + 1$. Then since

$$\phi(x_1) \cdots \phi(x_h) \leq \frac{\phi(x_1) \cdots \phi(x_{h-[t-1]})}{\operatorname{lcm}(x_1, \ldots, x_h)^{1+(t-1)-[t-1]}} \leq \frac{x_1 \cdots x_{h-[t-1]}}{\operatorname{lcm}(x_1, \ldots, x_h)x_1^{1-[t-1]}},$$

it follows that

$$\sum_{1 \leq x_1 \leq B} \cdots \sum_{1 \leq x_h \leq B} \frac{\phi(x_1) \cdots \phi(x_h)}{\operatorname{lcm}(x_1, \ldots, x_h)^t} \leq \frac{B^{h-[t-1]-1}}{x^{[t-1]-1}} \sum_{1 \leq x_1 \leq B} \cdots \sum_{1 \leq x_h \leq B} \frac{x_1}{\operatorname{lcm}(x_1, \ldots, x_h)x_1^{[t-1]-1}} \leq B^{h-[t-1]-[t-1]} \frac{1}{\operatorname{lcm}(x_1, \ldots, x_h)} \sum_{1 \leq x_1 \leq B} \cdots \sum_{1 \leq x_h \leq B} \frac{1}{\operatorname{lcm}(x_1, \ldots, x_h)} \leq B^{h-[t-1]} \sum_{1 \leq x \leq B^h} \frac{\# \{(x_1, \ldots, x_h) \in \mathbb{N}^h : \operatorname{lcm}(x_1, \ldots, x_h) = x\}}{x} \leq B^{h-t+1} \frac{\left(\sum_{m \mid x} 1\right)^h}{x} \ll B^{h-t+1+\varepsilon}.$$
We now estimate the summands in (3.30). Suppose $\kappa \geq 1 + \epsilon_1 + \cdots + \epsilon_m$. In this case, by Lemma 3.5 we have

$$
X^{n-d + \vartheta_0 + \lambda} X^{(\vartheta_0 + \lambda - \gamma)(\epsilon_1 + \cdots + \epsilon_m)} \sum_{1 \leq r_i \leq X^{\theta_0}} R_{\epsilon_i}^{-\kappa} \prod_{\epsilon_i = 1} \phi(r_i)
$$

(3.31)

$$
\ll X^{n-d + \vartheta_0 + \lambda + (\vartheta_0 + \lambda - \gamma)(\epsilon_1 + \cdots + \epsilon_m) + \epsilon}
$$

$$
\ll X^{n-d - \epsilon},
$$

because $\gamma > 2\vartheta_0 + 2\lambda$. On the other hand, suppose $\kappa < 1 + \epsilon_1 + \cdots + \epsilon_m$. In this case, by Lemma 3.5 we have

$$
X^{n-d + \vartheta_0 + \lambda} X^{(\vartheta_0 + \lambda - \gamma)(\epsilon_1 + \cdots + \epsilon_m)} \sum_{1 \leq r_i \leq X^{\theta_0}} R_{\epsilon_i}^{-\kappa} \prod_{\epsilon_i = 1} \phi(r_i)
$$

(3.32)

$$
\ll X^{n-d + \vartheta_0 + \lambda + (2\vartheta_0 + \lambda - \gamma)(\epsilon_1 + \cdots + \epsilon_m) - (\kappa - 1)\vartheta_0 + \epsilon}
$$

$$
\ll X^{n-d + \vartheta_0 + \lambda + (2\vartheta_0 + \lambda - \gamma)(\kappa - 1) - (\kappa - 1)\vartheta_0 + \epsilon}
$$

$$
\ll X^{n-d + \vartheta_0 + \lambda + (\vartheta_0 + \lambda - \gamma)(\kappa - 1) + \epsilon}
$$

$$
\ll X^{n-d - \epsilon},
$$

because $\gamma > 2\vartheta_0 + 2\lambda$ and $\kappa > 2$. Therefore, by combining (3.28), (3.29), (3.30), (3.31) and (3.32), we obtain

$$
\left| \sum_{1 \leq q \leq X^{\vartheta_0}} \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} \frac{1}{\phi(q)^n} \int_{|\tau| < X^{\theta_0 + \lambda - d}} S_{j,\varnothing} \left( \frac{a}{q} + \tau \right) d\tau \right| \ll X^{n-d} (\log X)^{-C}
$$

(3.33)

for any $C > 0$.

3.2. Case $k \neq \emptyset$. In this case, we only need to consider $q$ divisible by $\bar{r}$. Without loss of generality let $j = (1, \ldots, m)$ and $k = (m + 1, \ldots, m + \ell)$. First we suppose $m = 0$. By the same calculations as in (3.22), we have

$$
\sum_{1 \leq q \leq X^{\vartheta_0} \atop \bar{r}|q} \sum_{1 \leq a \leq q \atop \gcd(a,q) = 1} \frac{1}{\phi(q)^n} |A(q, a; \emptyset; (1, \ldots, \ell))| \ll \bar{r}^{-\kappa}.
$$
Therefore, by Proposition 2.3, we obtain

\begin{equation}
\left| \sum_{1 \leq q < X^{\delta_0}} \sum_{\substack{1 \leq a \leq q \atop \gcd(a, q) = 1}} \frac{(-1)^\ell}{\phi(q)^n} \int_{|\tau| < X^{\delta_0 + \lambda - d}} S_{\varphi, k} \left( \frac{a}{q} + \tau \right) d\tau \right|
\end{equation}

\begin{align*}
&\leq \sum_{1 \leq q < X^{\delta_0}} \sum_{\substack{1 \leq a \leq q \atop \gcd(a, q) = 1}} \frac{(-1)^\ell}{\phi(q)^n} |A(q, a; \varnothing; (1, \ldots, \ell))| \\
&\times \int_{|\tau| < X^{\delta_0 + \lambda - d}} \left| \int_0^\infty \cdots \int_0^\infty x_1^\gamma \cdots x_\ell^\gamma \omega(x) e(\tau F(x)) \right| d\tau \\
&\ll \tilde{r}^{-\kappa} X^{n-d+\ell(\tilde{\beta}-1)} \int_{|\tau| < X^{\delta_0 + \lambda}} \left| \int_0^\infty \cdots \int_0^\infty x_1^\gamma \cdots x_\ell^\gamma \right| d\tau \\
&\ll \tilde{r}^{-\kappa} X^{n-d+\ell(\tilde{\beta}-1)} (\log X) \\
&\ll X^{n-d}(\log X)^{-A}
\end{align*}

for any $A > 0$, where the final inequality follows from Remark 2.2.

3.2.1. Case $\ell > 0$ and $m > 0$. In this case, we have

\begin{equation}
\left| \sum_{1 \leq q < X^{\delta_0}} \sum_{\substack{1 \leq a \leq q \atop \gcd(a, q) = 1}} \frac{(-1)^\ell}{\phi(q)^n} \int_{|\tau| < X^{\delta_0 + \lambda - d}} S_{j, k} \left( \frac{a}{q} + \tau \right) d\tau \right|
\end{equation}

\begin{align*}
&= \sum_{1 \leq q < X^{\delta_0}} \sum_{\substack{1 \leq a \leq q \atop (\mod q) \chi \equiv \chi_1, \ldots, \chi_m \mod q \atop \gcd(a, q) = 1}} \sum_{\substack{1 \leq a \leq q \atop \gcd(a, q) = 1}} \frac{(-1)^\ell}{\phi(q)^n} A(q, a; i; (1, \chi_1), \ldots, (m, \chi_m); k) \\
&\times \int_{|\tau| < X^{\delta_0 + \lambda - d}} W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k) d\tau.
\end{align*}

As in Section 3.1, let us denote $\chi'_v = \chi_v \chi^0$, where $\chi_v$ is the primitive character modulo $r_v$ which induces $\chi'_v$, and $\chi^0$ is the principal character modulo $q$. We also denote

$$R = \text{lcm}(r_1, \ldots, r_m).$$

We let

$$\delta_{\text{min}} = \min_{1 \leq u \leq n} (x_{0,u} - \delta).$$

Let $p \in [(x_{0,u} - \delta)X, (x_{0,u} + \delta)X]$ be a prime. Since $q \leq X^{\delta_0} < \delta_{\text{min}} X$, we have $\gcd(p, q) = 1$ and it follows that $\chi_v(p) \chi^0(p) = \chi_v(p)$. Consequently, we obtain

$$W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k) = W(\tau; i; (1, 1), \ldots, (m, \chi_m); k).$$

By the same calculations as in (3.22), we have

$$\sum_{1 \leq q < X^{\delta_0}} \sum_{\substack{1 \leq a \leq q \atop \gcd(a, q) = 1}} \frac{1}{\phi(q)^n} |A(q, a; i; (1, \chi_1 \chi^0), \ldots, (m, \chi_m \chi^0); k)| \ll R^{-\kappa}.$$
Therefore, the term on the right hand side of (3.35) can be rewritten as

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\leq \sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} R^{-\kappa} \sum_{\chi_v \pmod{r_v}} \int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau.
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]

We substitute (3.26) into (3.36), and apply Proposition 2.5 to the first of the resulting terms in the following manner

\[
\sum_{1 \leq r_1, \ldots, r_m \leq X^{\alpha_0}} \sum_{\chi_v \pmod{r_v}} \sum_{1 \leq a \leq q_0} \sum_{\text{gcd}(a, q) = 1} \frac{(-1)^d}{\phi(q)^n} A(q, a; (1, \chi_1), \ldots, (m, \chi_m); k) \cdot \\
\int_{|\tau| < X^{\alpha_0 + \lambda - d}} |W(\tau; i; (1, \chi_1), \ldots, (m, \chi_m); k)\rangle d\tau
\]
for any $C > 0$.

3.3. **Proof of Proposition 3.1** We combine \((3.1)\) and \((3.19)\) with the estimates from Sections \(3.1\) and \(3.2\), namely \((3.33)\), \((3.34)\) and \((3.39)\). As a result, we obtain

\[(3.40)\]

\[
\int_{\mathbb{R}^+} S(\alpha) \, d\alpha = \int_{\mathbb{R}^+} S^*(\alpha) \, d\alpha + O\left(X^{n-d+3\theta_0+\lambda-\frac{1}{2}}\right)
\]

\[
= \sum_{1 \leq q \leq X^{\theta_0}} \sum_{1 \leq a \leq q} \frac{1}{\phi(q)^n} A(q, a; (1, \ldots, n); \emptyset; \emptyset) \cdot \int_{|\tau| < X^{\theta_0+\lambda-d}} \mathcal{W}(\tau; (1, \ldots, n); \emptyset; \emptyset) \, d\tau
\]

\[
+ O\left(X^{n-d+4\theta_0+2\lambda-1} + X^{n-d+3\theta_0+\lambda-\frac{1}{2}} + \frac{X^{n-d}}{(\log X)^A}\right)
\]

for any $A > 0$. We now estimate the singular series and the singular integral by standard arguments; we keep the details to a minimum. We begin with the singular series (see for example [5, Section 7] for more details). Since $\text{codim} \, V_F^* > 4(2d-1)4^d$, by invoking\(^1\) Lemma

---

\(^1\)We may invoke [5, Lemma 10] instead, but since the proof contains a minor oversight and Lemma 2.4 suffices for our purpose, we take this approach. In the final paragraph in the proof of [5, Lemma 10], they make the following statement under the assumptions that $t \in \mathbb{Z}_{>d}$ and $a_1, \ldots, a_r \in U_q$: Set $P = p^{t-1}$ and $q_1 = p^{t-d}$. Then, for each $i = 1, \ldots, r$,

\[2|q' a_i - a'_i q_1| \leq P^{-(d-1)+(d-1)r\theta}\]

and

\[1 \leq q' \leq P^{(d-1)r\theta}\]

cannot be satisfied if $\theta < 1/(d-1)r$.

However, this statement can be seen to be not true by the following counterexample. Let $d \geq 2$, $r = 1$ and $t = d-1$. Then we can choose

\[q' = q_1 = p^{t-d} = p^{(t-1)(d-1)} \leq P^{(d-1)\theta} \quad \text{and} \quad a'_1 = a_1,
\]

which yields $q' a_1 - a'_i q_1 = 0$. 

---
it follows that there exists $\xi_1 > 0$ such that
\[
\sum_{1 \leq q \leq X^{\theta_0}} \sum_{1 \leq a \leq q \atop \gcd(a, q) = 1} \frac{1}{\phi(q)^n} A(q, a; (1, \ldots, n); \emptyset; \emptyset)
= \sum_{1 \leq q \leq X^{\theta_0}} \sum_{1 \leq a \leq q \atop \gcd(a, q) = 1} \frac{1}{\phi(q)^n} \sum_{h \in U_q^n} e\left(\frac{a}{q} F(h)\right)
= c_1(F) + O \left( \sum_{q > X^{\theta_0}} \sum_{1 \leq a \leq q \atop \gcd(a, q) = 1} \frac{1}{\phi(q)^n} q^{-\frac{\operatorname{codim} V_F^*}{2(2d-1)d^\alpha} + \varepsilon} \right)
= c_1(F) + O \left( \sum_{q > X^{\theta_0}} q^{-\frac{\operatorname{codim} V_F^*}{2(2d-1)d^\alpha} + \varepsilon} \right)
= c_1(F) + O(X^{-\xi_1}),
\]
where $c_1(F) \geq 0$ is a constant depending only on $F$.

Next we deal with the singular integral (see for example [3] Section 5 or [21, §2] for more details). The same argument to establish [3, (5.5)] yields
\[
\int_0^\infty \cdots \int_0^\infty \prod_{1 \leq u \leq n} \omega(x_u - x_{0,u}) \cdot e(\tau F(x)) \, dx \ll \min \left(1, |\tau|^{-\frac{\operatorname{codim} V_F^*}{(d-1)2d^\alpha} + \varepsilon} \right).
\]
Therefore, since $\operatorname{codim} V_F^* > (d-1)2^{d-1} + 1$, it follows that there exists $\xi_2 > 0$ such that
\[
\int_{|\tau| < X^{\theta_0 + \lambda - \varepsilon}} \mathcal{W}(\tau; (1, \ldots, n); \emptyset; \emptyset) \, d\tau
= X^{n-d} \int_{|\tau| < X^{\theta_0 + \lambda}} \int_0^\infty \cdots \int_0^\infty \prod_{1 \leq u \leq n} \omega(x_u - x_{0,u}) \cdot e(\tau F(x)) \, dx \, d\tau
= c_2(F; \omega, x_0) X^{n-d} + O \left( X^{n-d} \int_{|\tau| \geq X^{\theta_0 + \lambda}} \min \left(1, |\tau|^{-\frac{\operatorname{codim} V_F^*}{(d-1)2d^\alpha} + \varepsilon} \right) \, d\tau \right)
= c_2(F; \omega, x_0) X^{n-d} + O(X^{n-d-\xi_2}),
\]
where $c_2(F; \omega, x_0) \geq 0$ is a constant depending only on $F$, $\omega$ and $x_0$. The constant $c_1(F)c_2(F; \omega, x_0)$ is a product of local densities; in fact, $c_1(F)c_2(F; \omega, x_0) > 0$ provided $F$ satisfies the local conditions (\#) and $\varpi$ is as in this section. Therefore, by combining these estimates with [3,40], we obtain Proposition 3.1 with $c(F; \omega, x_0) = c_1(F)c_2(F; \omega, x_0)$.

4. Preliminaries for the minor arcs analysis

We collect some definitions and results from [23], which we will need in the next section. Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous form. Given partitions of variables $x = (z, w)$ and $z = (s, t)$, let us denote
\[
F_z(z) = F(z, 0) = F(x)|_{w=0}
\]
and
\[
\mathcal{G}(s, t) = F_z(s, t) - F_z(s, 0) - F_z(0, t).
\]
It is clear that \( G(s, 0) \) and \( G(0, t) \) are the zero polynomials. With these notation we consider two cases based on the structure of \( F \).

**Definition 4.1.** We define a structural dichotomy of \( F \) with respect to \( C_0 > 0 \) as follows:

(I) There exist partitions of variables \( x = (z, w) \) and \( z = (s, t) \) such that \( \text{codim } V^*_G > C_0 \).

(II) Given any partitions \( x = (z, w) \) and \( z = (s, t) \), we have \( \text{codim } V^*_G \leq C_0 \).

Let us set
\[
C_0 = 4(d - 1)2^d + 1.
\]

The situation when \( F \) satisfies (I) is easier to handle, and this is dealt with in Section 6. We deal with the more challenging case when \( F \) satisfies (II) in Section 5. The following result \([25, Lemma 4.3]\) played a crucial role in the author’s previous work \([25]\).

**Lemma 4.2.** Let \( F \in \mathbb{Z}[x_1, \ldots, x_n] \) be a homogeneous form of degree \( d \geq 2 \) satisfying (II) of Definition 4.1. Suppose we have a partition of the \( x \) variables into \( H \) sets \( x = (y_0, \ldots, y_{H-1}) \). For each \( 0 \leq i \leq H - 1 \), let us denote
\[
F_i(y_i) = F(x)|_{y_i=0 (i \neq i)}.
\]

Then there exists \( j_0 \in \{0, 1, \ldots, H - 1\} \) satisfying
\[
\text{codim } V^*_{F_{j_0}} \geq \frac{\text{codim } V^*_{F} - (H - 1)C_0}{H}.
\]

Let \( G(u; v) \) be a polynomial in variables \( u = (u_1, \ldots, u_h) \) and \( v = (v_1, \ldots, v_h) \), and say it is bihomogeneous of bidegree \((d_1, d_2)\) if
\[
G(au; bv) = a^{d_1}b^{d_2}G(u; v)
\]
for all \( a, b \in \mathbb{C} \). We define the following affine variety
\[
V^*_{G, 2} = \left\{(u, v) \in \mathbb{A}^{2h}_\mathbb{C} : \frac{\partial G}{\partial v_i}(u; v) = 0 \ (1 \leq i \leq h)\right\}.
\]

We take the partial derivatives with respect to the second set of variables (in the notation of \( G \)) for \( V^*_{G, 2} \). The following was the key estimate in establishing the main result in \([23]\); we present a different proof which improves \([23, Theorem 5.1]\) by almost a factor of 2 under the additional assumption that \( F \) satisfies (II) of Definition 4.1.

**Proposition 4.3.** Let \( F \in \mathbb{Z}[x_1, \ldots, x_n] \) be a homogeneous form of degree \( d \geq 2 \) satisfying (II) of Definition 4.1. Let \( z = (z_1, \ldots, z_h) \) with \( h \geq 1 \). We let \( x = (z, w) \) be a partition of variables and denote
\[
\mathcal{F}(z) = F_z(z) = F(z, 0) = F(x)|_{w=0}.
\]

Let us define a bihomogeneous form
\[
G(u; v) = \mathcal{F}(u_1v_1, \ldots, u_hv_h).
\]

Then
\[
\text{codim } V^*_{G, 2} \geq \text{codim } V^*_{\mathcal{F}} - C_0.
\]

**Proof.** We begin by proving the following result.
Lemma 4.4. Let $G_j \in \mathbb{C}[v_1, \ldots, v_k]$ be a homogeneous form for each $1 \leq j \leq s$. We define

\[ L = \{(v_1, \ldots, v_k) \in \mathbb{A}^k_C : G_j(v_1, \ldots, v_k) = 0 \quad (1 \leq j \leq s)\} \]

and

\[ \tilde{L} = \{(u_1, \ldots, u_k, v_1, \ldots, v_k) \in \mathbb{A}^{2k}_C : G_j(u_1v_1, \ldots, u_kv_k) = 0 \quad (1 \leq j \leq s)\}. \]

Then

\[ \dim \tilde{L} \leq \dim L + k. \]

Proof. Let $W$ be an irreducible component of $\tilde{L}$. Suppose there exists $1 \leq i \leq k$ such that

\[ (4.5) \quad W \subseteq V(u_i) \cap V(v_i). \]

Without loss of generality let us suppose $i = 1$. Then $W$ is of the form

\[ W = \{(0, u', 0, v') \in \mathbb{A}^{2k}_C : (u', v') \in W'\}, \]

where $u' = (u_2, \ldots, u_k)$, $v' = (v_2, \ldots, v_k)$ and $W' \subseteq \mathbb{A}^{2k-2}_C$ is an irreducible affine variety. It is easy to see that

\[ W \subseteq \{(0, u', v_1, v') \in \mathbb{A}^{2k}_C : v_1 \in \mathbb{A}_C, (u', v') \in W'\} \subseteq \tilde{L}, \]

and this is a contradiction because

\[ \{(0, u', v_1, v') \in \mathbb{A}^{2k}_C : v_1 \in \mathbb{A}_C, (u', v') \in W'\} \cong \mathbb{A}_C \times W' \]

is irreducible; here $\cong$ denotes an isomorphism of affine varieties. Therefore, we obtain that $(4.5)$ does not hold for any $1 \leq i \leq k$. In particular, this means that there exists $(\eta_1, \ldots, \eta_k, \zeta_1, \ldots, \zeta_k) \in W$ such that either $\eta_i \neq 0$ or $\zeta_i \neq 0$ for each $1 \leq i \leq k$ by the following reason. Let us suppose otherwise, in which case it follows that

\[ W = \bigcup_{1 \leq i \leq k} W \cap (V(u_i) \cap V(v_i)). \]

Then

\[ W = W \cap (V(u_{i_0}) \cap V(v_{i_0})) \]

for some $1 \leq i_0 \leq k$, because $W$ is irreducible; however, this contradicts the fact that $(4.5)$ does not hold for any $1 \leq i \leq k$. Without loss of generality let us suppose $\eta_i \neq 0$ for all $1 \leq i \leq k$. Then we obtain

\[ W \cap \bigcap_{1 \leq i \leq k} V(u_i - \eta_i) \]

\[ \subseteq \{(\eta_1, \ldots, \eta_k) \times (v_1, \ldots, v_k) \in \mathbb{A}^k_C : G_j(\eta_1v_1, \ldots, \eta_kv_k) = 0 \quad (1 \leq j \leq s)\} \]

\[ \cong L, \]

and by [10] Proposition I.7.1 that

\[ \dim W - k \leq \dim \left(W \cap \bigcap_{1 \leq i \leq k} V(u_i - \eta_i)\right). \]

Therefore, we have

\[ \dim W - k \leq \dim L. \]

Since this holds for any irreducible component $W$ of $\tilde{L}$, the result follows. \qed
By the definitions of $G$ and $V_{G,2}^*$, we have

$$V_{G,2}^* = \left\{ (u, v) \in \mathbb{A}^{2h}_{\mathbb{C}} : u_i \frac{\partial \mathcal{F}}{\partial v_i}(u_1v_1, \ldots, u_kv_h) = 0 \quad (1 \leq i \leq h) \right\} = \bigcup_{I \subseteq \{1, \ldots, h\}} Z_I,$$

where

$$Z_I = \left\{ (u, v) \in \mathbb{A}^{2h}_{\mathbb{C}} : \frac{\partial \mathcal{F}}{\partial v_i}(u_1v_1, \ldots, u_kv_h) = 0 \quad (i \in I) \right\} \bigcap \bigcap_{i \notin I} V(u_i).$$

Let us fix a choice of $I$ and without loss of generality we assume $I = \{1, \ldots, k\}$, where $0 \leq k \leq h$ (we let $I = \emptyset$ if $k = 0$). Let us suppose $k \geq 1$. Then

$$Z_I \cong \left\{ (u_1, \ldots, u_k, v_1, \ldots, v_k) \in \mathbb{A}^{2k}_{\mathbb{C}} : \frac{\partial \mathcal{F}}{\partial v_i}(u_1v_1, \ldots, u_kv_h, 0) = 0 \quad (1 \leq i \leq k) \right\} \times \mathbb{A}^{h-k}_{\mathbb{C}}.$$

Let us define

$$T_k = \left\{ (v_1, \ldots, v_k) \in \mathbb{A}^k_{\mathbb{C}} : \frac{\partial \mathcal{F}}{\partial v_i}(v_1, \ldots, v_k, 0) = 0 \quad (1 \leq i \leq k) \right\}$$

and

$$T'_k = \left\{ (v_{k+1}, \ldots, v_h) \in \mathbb{A}^{h-k}_{\mathbb{C}} : \frac{\partial \mathcal{F}}{\partial v_i}(0, v_{k+1}, \ldots, v_h) = 0 \quad (k+1 \leq i \leq h) \right\}.$$ 

Then by Lemma 4.4 it follows that

$$\dim Z_I \leq \dim T_k + h.$$

The result follows on proving

$$(4.6) \quad \dim T_k \leq \dim V_{\mathcal{F}}^* + C_0.$$ 

With this estimate it follows that

$$\dim Z_I \leq \dim V_{\mathcal{F}}^* + h + C_0;$$

note the inequality is trivial when $k = 0$, because $Z_I \cong \mathbb{A}^h_{\mathbb{C}}$ in this case. Since this holds for all $I \subseteq \{1, \ldots, h\}$, we then obtain

$$\dim V_{G,2}^* \leq \dim V_{\mathcal{F}}^* + h + C_0,$$

and equivalently

$$2h - \dim V_{G,2}^* \geq h - \dim V_{\mathcal{F}}^* - C_0.$$

We now prove (4.6) and complete the proof. We assume $1 \leq k < h$ as the inequality is trivial when $k = h$. Let us consider the following partition of variables $v = (s, t)$, where $s = (v_1, \ldots, v_k)$ and $t = (v_{k+1}, \ldots, v_h)$. Let

$$\mathcal{F}_1(v_1, \ldots, v_k) = \mathcal{F}(v_1, \ldots, v_k, 0), \quad \mathcal{F}_2(v_{k+1}, \ldots, v_h) = \mathcal{F}(0, v_{k+1}, \ldots, v_h)$$

and

$$\mathcal{G}(v) = \mathcal{F}(v) - \mathcal{F}_1(v_1, \ldots, v_k) - \mathcal{F}_2(v_{k+1}, \ldots, v_h).$$

Then on recalling the definition of $\mathcal{F}$, we see that

$$\mathcal{G}(v) = \mathcal{G}(s, t) = \mathcal{F}(s, t) - \mathcal{F}(s, 0) - \mathcal{F}(0, t)$$

as defined in (4.1). Therefore, since $F$ satisfies (II) of Definition 4.1 it follows that

$$(4.7) \quad h - \dim V_{\mathcal{G}}^* = \text{codim } V_{\mathcal{G}}^* \leq C_0.$$
We define the following homogeneous forms:
\[ \lambda_i(v) = \frac{\partial \mathcal{F}}{\partial v_i}(v) \quad (1 \leq i \leq h), \quad \tau_i(v) = \frac{\partial \mathcal{G}}{\partial v_i}(v) \quad (1 \leq i \leq h), \]
\[ \varphi_i(v) = \frac{\partial \mathcal{H}}{\partial v_i}(v_1, \ldots, v_k) \quad (1 \leq i \leq k), \quad \varphi_i(v) = \frac{\partial \mathcal{H}}{\partial v_i}(v_{k+1}, \ldots, v_h) \quad (k + 1 \leq i \leq h), \]
Clearly \( \lambda_i = \varphi_i + \tau_i \) (\( 1 \leq i \leq h \)). Let us define the following ideals:
\[ I_1 = (\varphi_1, \ldots, \varphi_k), \quad I_2 = (\varphi_{k+1}, \ldots, \varphi_h) \quad \text{and} \quad J = (\tau_1, \ldots, \tau_h). \]
Then we have
\[ V(I_1) \cong T_k \times \mathbb{A}_{\mathbb{C}}^{h-k}, \quad V(I_2) \cong T'_k \times \mathbb{A}_{\mathbb{C}}^{k}, \quad V(J) = V_{\Phi}^* \]
and
\[ V(I_1) \cap V(I_2) \cap V(J) = V(I_1 + I_2 + J) \subseteq V(\lambda_1, \ldots, \lambda_h) = V_{\Phi}^* \subseteq \mathbb{A}_{\mathbb{C}}^h. \]
Therefore, by [10, Proposition I.7.1] we obtain
\[
\dim V_{\Phi}^* \geq \dim (V(I_1) \cap V(I_2) \cap V(J)) \\
\geq \dim V(I_1) + \dim V(I_2) + \dim V(J) - 2h \\
= \dim T_k + (h - k) + \dim T'_k + k - h - (h - \dim V_{\Phi}^*) \\
\geq \dim T_k + \dim T'_k - (h - \dim V_{\Phi}^*) \\
\geq \dim T_k - C_0,
\]
where the final inequality follows from (4.7).

Next we recall the following identity for the von Mangoldt function
\[
(4.8) \quad \Lambda(x) = \sum_{m \ell = x} \mu(m) \log \ell,
\]
where \( \mu \) denotes the Möbius function, i.e.
\[
(4.9) \quad \mu(m) = \begin{cases} 0 & \text{if } p^2 \mid m \text{ for some } p \in \varphi, \\ 1 & \text{if } m = 1, \\ (-1)^s & \text{if } m \text{ is a product of } s \text{ distinct primes.} \end{cases}
\]
In order to apply this identity, we mainly follow the exposition in [19, Section 3] for the remainder of this section. We fix a constant \( \Theta > 1 \). Let \( \Psi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( \supp \Psi = [-\Theta, \Theta] \),
\[ \Psi(y) = 1 \quad (y \in [-1, 1]) \]
and obeys the derivative estimates
\[ |\Psi^{(s)}(y)| \ll 1 \quad (y \in \mathbb{R}) \]
for any fixed \( s \geq 0 \), where the implicit constant depends only on \( s \). We then have a smooth partition of unity
\[ 1 = \sum_{T \in \mathcal{D}} \Psi_T(x) \quad (x \in \mathbb{N}) \]
indexed by the multiplicative semigroup
\[ \mathcal{D} = \{ \Theta^t : t \in \mathbb{Z}_{\geq 0} \}, \]
where
\[ \Psi_T(x) = \Psi\left(\frac{x}{T}\right) - \Psi\left(\frac{\Theta x}{T}\right). \]
Let \( x \) and \( \psi_1, \ldots, \psi_n \) be as in the statement of Theorem 1.2 and let
\[
\text{supp } \psi_i = [a_i X, b_i X] \quad (1 \leq i \leq n).
\]
Let \( 1 \leq i \leq n \). Then
\[
(4.11) \quad \Lambda(x) \psi_i(x_i) = \sum_{m_i n_i = x_i} \mu(m_i)(\log n_i) \psi_i(x_i)
\]
\[
= \sum_{(M_i, N_i) \in \mathcal{D}^2} \sum_{m_i n_i = x_i} \mu(m_i) \Psi_M(m_i)(\log n_i) \Psi_N(n_i) \psi_i(m_i n_i).
\]
For each \( (M_i, N_i) \in \mathcal{D}^2 \), the summand vanishes unless
\[
(4.12) \quad a_i X \Theta^{-2} \leq M_i N_i \leq b_i X \Theta^2.
\]
We define \( \Xi(a_i X, b_i X) \) to be the set of all \( (M_i, N_i) \in \mathcal{D}^2 \) satisfying (4.12). We can easily deduce that
\[
(4.13) \quad \# \Xi(a_i X, b_i X) \ll (\log X)^2,
\]
where the implicit constant depends only on \( \Theta \). We also have by the well-known estimate for the divisor function that
\[
(4.14) \quad \sum_{\Theta^{-1} M_i \leq m_i \leq \Theta M_i} \sum_{\Theta^{-1} N_i \leq n_i \leq \Theta N_i} 1_{[a_i X, b_i X]}(m_i n_i) \ll X^{1+\epsilon}
\]
for \( (M_i, N_i) \in \Xi(a_i X, b_i X) \).

For each \( 1 \leq i \leq n \), let us write (4.11) as
\[
(4.15) \quad \Lambda(x) \psi_i(x_i) = \sum_{(M_i, N_i) \in \Xi(a_i X, b_i X)} \sum_{\Theta^{-1} U_i \leq u_i \leq \Theta U_i} K_i(u_i) L_i(v_i) \psi_i(u_i v_i),
\]
where \( U_i = \min(M_i, N_i) \) and \( V_i = \max(M_i, N_i) \),
\[
u_i = n_i, \quad L_i(v_i) = (\log n_i) \Psi_N(n_i)
\]
if \( M_i \leq N_i \), and
\[
u_i = m_i, \quad L_i(v_i) = (\log n_i) \Psi_M(m_i)
\]
if \( M_i > N_i \). In particular, it follows from (4.12) that
\[
(4.16) \quad 1 \leq U_i \ll X^{\frac{1}{2}}, \quad U_i \leq V_i, \quad \frac{X}{U_i} \ll V_i \ll \frac{X}{U_i},
\]
where the implicit constants depend only on \( a_i, b_i \) and \( \Theta \). We also have
\[
K_i(u_i), L_i(v_i) \ll \log X
\]
for all \( \Theta^{-1} U_i \leq u_i \leq \Theta U_i \) and \( \Theta^{-1} V_i \leq v_i \leq \Theta V_i \). For clarity we note that the definitions of \( K_i \) and \( L_i \) depend on \( U_i \) and \( V_i \) respectively, though we do not make this explicit in the notation.
In this section, we prove the following result.

**Proposition 5.1.** Suppose $F$ is as in the statement of Theorem 4.12. Furthermore, suppose $F$ satisfies (II) of Definition 4.1. Then there exist $\vartheta_0, \gamma, \lambda > 0$ satisfying (5.6) such that

$$
\int_{[0,1]^{2d}(\vartheta_0)} |S(\alpha)| \, da \ll X^{n-d-\varepsilon}.
$$

We begin by substituting (4.15) into the definition of $S$ given in (1.6), and obtain

$$
S(\alpha) = \sum_{(M,N) \in \Xi(aX,bX)} S(M,N; \alpha),
$$

where

$$
S(M,N; \alpha) = \sum_{\Theta^{-1} \leq U_i \leq \Theta U_i, 1 \leq i \leq n} \prod_{\Theta^{-1} \leq V_i \leq \Theta V_i, 1 \leq i \leq n} K_i(u_i)L_i(v_i)\psi_i(u_i)v_i) \cdot e(\alpha F(u_1v_1, \ldots, u_nv_n)),
$$

and

$$
\Xi(aX,bX) = \Xi(a_1X,b_1X) \times \cdots \times \Xi(a_nX,b_nX)
$$

We prove that given any $(M,N) \in \Xi(aX,bX)$, the sum $S(M,N; \alpha)$ satisfies the statement of Proposition 3.1 in place of $S(\alpha)$. Then Proposition 5.1 follows by noting (see (4.13)) that

$$
#\Xi(aX,bX) \ll (\log X)^{2n}.
$$

Let $H \in \mathbb{N}$ to be chosen later and set

$$
\sigma = \frac{1}{2H}.
$$

We consider the partition of $x = (x_1, \ldots, x_n)$ into $H$ sets $y = (y_0, \ldots, y_{H-1})$, where $y_j$ is the collection of $x_i$ such that $i$ satisfies

$$
X^\sigma \ll U_i \ll X^{(j+1)\sigma},
$$

where the implicit constants depend only on $a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\Theta$. Then by Lemma 4.2 there exists $j_0 \in \{0, 1, \ldots, H-1\}$ satisfying

$$
\text{codim} V_{F_{j_0}}^* \geq \frac{\text{codim} V_F^* - (H-1)C_0}{H},
$$

where

$$
F_{j_0}(y_{j_0}) = F(x)|_{y_i = a (i \neq j_0)}.
$$

Without loss of generality let $y_{j_0} = (x_1, \ldots, x_h)$. In particular, it then follows from (4.16) and (5.5) that

$$
X^{j_0\sigma} \ll U_i \ll X^{(j_0+1)\sigma} \quad \text{and} \quad X^{1-(j_0+1)\sigma} \ll V_i \ll X^{1-j_0\sigma} \quad (1 \leq i \leq h).
$$

By estimating every other variable trivially, we obtain

$$
S(M,N; \alpha) \ll X^{n-h+\varepsilon} \max_{\Theta^{-1} \leq U_i \leq \Theta U_i} \sum_{\Theta^{-1} \leq V_i \leq \Theta V_i, 1 \leq i \leq h} \prod_{\Theta^{-1} \leq U_i \leq \Theta U_i, (h+1 \leq i \leq n)} K_i(u_i)L_i(v_i)\psi_i(u_i)v_i) \cdot e(\alpha G(u,v)),
$$

\[26\] SHUNTARO YAMAGISHI

5. Minor Arcs
where \( \mathbf{u} = (u_1, \ldots, u_h), \, \mathbf{v} = (v_1, \ldots, v_h) \),

\[
G(\mathbf{u}, \mathbf{v}) = F(u_1v_1, \ldots, u_hv_h, \overline{x})
\]

and

\[
(5.9) \quad \overline{x} = (u_{h+1}v_{h+1}, \ldots, u_nv_n).
\]

In particular, the coefficients of the lower degree terms of \( G \) may depend on \( \overline{x} \). The degree 2\( d \) homogeneous portion of \( G(\mathbf{u}, \mathbf{v}) \) is \( G^{[2d]}(\mathbf{u}, \mathbf{v}) = F_{j_0}(u_1v_1, \ldots, u_hv_h) \). With this set-up it follows from Proposition 4.3 and (5.6) that

\[
(5.10) \quad \text{codim } V^{*}_{G^{[2d]},2} \geq \text{codim } V^{*}_F - \mathcal{C}_0 \geq \frac{\text{codim } V^{*}_F - (2H - 1)\mathcal{C}_0}{H}.
\]

Next step is a technical improvement that simplifies the argument by the author in [25]. Let

\[
U_{\min} = \min_{1 \leq j \leq h} U_j, \quad W_{i,\ell_i} = \Theta^{-1} U_i + \ell_i \Theta U_{\min},
\]

\[
V_{\min} = \min_{1 \leq j \leq h} V_j, \quad W'_{i,\ell'_i} = \Theta^{-1} V_i + \ell'_i \Theta V_{\min}
\]

for each \( 0 \leq \ell_i \leq \frac{U_i}{U_{\min}}, \quad 0 \leq \ell'_i \leq \frac{V_i}{V_{\min}} \) and \( 1 \leq i \leq h \). In particular, we have

\[
U_i V_{\min} \leq U_i V_i \ll X \quad (1 \leq i \leq h).
\]

We partition the range of summation in (5.8) for each \( 1 \leq i \leq h \) as follows

\[
\sum_{\Theta^{-1} U_i \leq u_i \leq \Theta U_i} \sum_{\Theta^{-1} V_i \leq v_i \leq \Theta V_i} K_i(u_i)L_i(v_i)\psi_i(u_i,v_i)
\]

\[
= \sum_{0 \leq \ell_i \leq \frac{U_i}{U_{\min}}} \sum_{W_{i,\ell_i} \leq u_i < W_{i,\ell_i} + \Theta U_{\min}} \sum_{0 \leq \ell'_i \leq \frac{V_i}{V_{\min}}} \sum_{W'_{i,\ell'_i} \leq v_i < W'_{i,\ell'_i} + \Theta V_{\min}} K_i(u_i)L_i(v_i)\mathbb{1}_{[\Theta^{-1} U_i, \Theta U_i]}(u_i)\mathbb{1}_{[\Theta^{-1} V_i, \Theta V_i]}(v_i)\psi_i(u_i,v_i).
\]

Let us denote \( \ell = (\ell_1, \ldots, \ell_h), \quad \ell' = (\ell'_1, \ldots, \ell'_h), \quad U = \Theta U_{\min}, \quad V = \Theta V_{\min} \) and

\[
\mathcal{E}_{\overline{x},\ell,\ell'}(\alpha)
\]

\[
= \prod_{W_{i,\ell_i} \leq u_i < W_{i,\ell_i} + U \quad 1 \leq i \leq h} K_i(u_i)L_i(v_i)\mathbb{1}_{[\Theta^{-1} U_i, \Theta U_i]}(u_i)\mathbb{1}_{[\Theta^{-1} V_i, \Theta V_i]}(v_i)\psi_i(u_i,v_i) \cdot c(\alpha G(\mathbf{u}, \mathbf{v})).
\]
Then (5.8) becomes

\[
(5.13) \quad |S(M, N; \alpha)| \ll X^{n-h+\varepsilon} \max_{\theta^{-1}U_j \leq u_j \leq \theta U_j} \sum_{0 \leq h \leq \frac{V_{h min}}{\varepsilon}} |\xi_{\alpha, h}^{\rho}(\alpha)|
\]

\[
\ll X^{n-h+\varepsilon} \frac{U_1 \cdots U_h}{V_{h min}^{\rho}} \max_{\theta^{-1}V_j \leq v_j \leq \theta V_j} \sum_{0 \leq h \leq \frac{V_{h min}}{\varepsilon}} |\xi_{\alpha, h}^{\rho}(\alpha)|
\]

\[
\ll \frac{X^{n+\varepsilon}}{U V} \max_{\theta^{-1}U_i \leq u_i \leq \theta U_i} |\xi_{\alpha, h}^{\rho}(\alpha)|
\]

Let us fix a choice of $\tilde{\alpha}$ (defined in (5.9)), $\ell$ and $\ell'$ as in the maximum of (5.13). We define $w, w' \in \mathbb{Z}^h$ by setting

\[
w_i = [W_{i\ell_i}] \quad \text{and} \quad w'_i = [W'_{i\ell'_i}]
\]

for each $1 \leq i \leq h$. In particular, it follows easily from the definition that

\[
w_i \ll W_{i\ell_i} \ll U_i \quad (1 \leq i \leq h).
\]

Then we have

\[
E_{\alpha}(\alpha) = \xi_{\alpha}^{\rho}(\alpha) = \sum_{u \in [0, U]^h \atop v \in [0, V]^h} K(u) L(v) \psi(u; v) \varepsilon(\alpha g(u, v)),
\]

where

\[
g(u, v) = G(u + w, v + w'),
\]

\[
K(u) = \prod_{1 \leq i \leq h} K_i(u_i + w_i) 1_{[\theta^{-1}U_i - u_i, \theta U_i - u_i]}(u_i) 1_{[0, W_{i\ell_i} + U - w_i]}(u_i),
\]

\[
L(v) = \prod_{1 \leq i \leq h} L_i(v_i + w'_i) 1_{[\theta^{-1}V_i - v_i, \theta V_i - v_i]}(v_i) 1_{[0, W'_{i\ell'_i} + V - w'_i]}(v_i)
\]

and

\[
\psi(u; v) = \prod_{1 \leq i \leq h} \psi_i(u_i + w_i)(v_i + w'_i).
\]

It is clear that $g(u, v)$ is a degree $2d$ polynomial in $u$ and $v$, and its degree $2d$ homogeneous portion is

\[
g^{[2d]}(u; v) = G^{[2d]}(u; v) = F_{j_0}(u_1 v_1, \ldots, u_h v_h) = F(u_1 v_1, \ldots, u_h v_h, 0, \ldots, 0).
\]

Therefore, by (5.10) we have

\[
codim V_{g^{[2d]}, 2}^* = \frac{\varepsilon V_{F^*} - (2H - 1)C_0}{H}.
\]
We now estimate $\mathcal{E}(\alpha)$ by the standard argument in [1]; we use a slight variant of the bihomogeneous version of the argument developed by Schindler in [20].

5.1. Weyl Differencing and Geometry of Numbers. We let $j = (j_1, \ldots, j_d)$ and $k = (k_1, \ldots, k_d)$, and denote

$$g^{[2d]}(u; v) = \sum_{j_1=1}^{h} \cdots \sum_{j_d=1}^{h} \sum_{k_1=1}^{h} \cdots \sum_{k_d=1}^{h} G_{j,k} \ u_{j_1} \cdots u_{j_d} \ v_{k_1} \cdots v_{k_d}$$

$$= \sum_{1 \leq j \leq h} \sum_{1 \leq k \leq h} G_{j,k} \ u_{j_1} \cdots u_{j_d} \ v_{k_1} \cdots v_{k_d}$$

with each $G_{j,k} \in \mathbb{Q}$ symmetric in $(j_1, \ldots, j_d)$ and also in $(k_1, \ldots, k_d)$. Note we have $(d!)^2 G_{j,k} \in \mathbb{Z}$. Also $G_{j,k} = 0$ unless $(j_1, \ldots, j_d)$ is a permutation of $(k_1, \ldots, k_d)$; this is because $g^{[2d]}(u; v) = F(u_1 v_1, \ldots, u_k v_k, 0, \ldots, 0)$.

Let us define

$$U = [0, U]^h \quad \text{and} \quad V = [0, V]^h.$$

By Hölder’s inequality we obtain

$$|\mathcal{E}(\alpha)|^{2d-1} \ll X^\varepsilon V^{h(2d-1)} \sum_{v \in V} |T_v(\alpha)|^{2d-1},$$

where

$$T_v(\alpha) = \sum_{u \in U} K(u) \psi(u; v) e(\alpha g(u, v)).$$

Next we use a form of Weyl’s inequality as in [22, Lemma 11.1] to bound $|T_v(\alpha)|^{2d-1}$. Given a subset $\mathcal{X} \subseteq \mathbb{R}^h$, we denote $\mathcal{X}^D = \mathcal{X} - \mathcal{X} = \{z - z' : z, z' \in \mathcal{X}\}$. Also for any $z_1, \ldots, z_t \in \mathbb{R}^h$, we let

$$\mathcal{X}(z_1, \ldots, z_t) = \cap_{\varepsilon_1 \in \{0, 1\}} \cdots \cap_{\varepsilon_t \in \{0, 1\}} (\mathcal{X} - \epsilon_1 z_1 - \cdots - \epsilon_t z_t).$$

In particular, it follows that

$$\mathcal{X}(z_1, \ldots, z_t) = \mathcal{X}(z_1, \ldots, z_{t-1}) \cap (\mathcal{X}(z_1, \ldots, z_{t-1}) - z_t).$$

Let

$$\tilde{F}(u) = \alpha g(u, v)$$

and

$$F(u) = \alpha g^{[2d]}(u; v) = \alpha \sum_{1 \leq j \leq h} \left( \sum_{1 \leq k \leq h} G_{j,k} \ v_{k_1} \cdots v_{k_d} \right) u_{j_1} \cdots u_{j_d}.$$

Then, viewing these as polynomials in $u$, we have

$$\tilde{F}^{[d]}(u) = F(u) + G_v(u),$$

where $G_v(u)$ is a polynomial of degree at most $d - 1$ in $v$ for a fixed $u$. For each $t \in \mathbb{N}$ we denote

$$\tilde{F}_t(u_1, \ldots, u_t) = \sum_{\varepsilon_1 \in \{0, 1\}} \cdots \sum_{\varepsilon_t \in \{0, 1\}} (-1)^{\varepsilon_1 + \cdots + \varepsilon_t} \tilde{F}(\varepsilon_1 u_1 + \cdots + \varepsilon_t u_t),$$
and let \( \tilde{F}_0 \) be the zero polynomial. In particular, it follows that

\[
(5.23) \quad \tilde{F}_t(u_1, \ldots, u_t) = \tilde{F}_{t-1}(u_1, \ldots, u_{t-1}) - \sum_{\epsilon_1 \in \{0,1\}} \cdots \sum_{\epsilon_{t-1} \in \{0,1\}} (-1)^{\epsilon_1 + \cdots + \epsilon_{t-1}} \tilde{F}(\epsilon_1 u_1 + \cdots + \epsilon_{t-1} u_{t-1} + u_t)
\]

\[
= \tilde{F}_{t-1}(u_1, \ldots, u_{t-1}) - \tilde{F}_{t-1}(u_1, \ldots, u_{t-2}, u_{t-1} + u_t) + \tilde{F}_{t-1}(u_1, \ldots, u_{t-2}, u_t).
\]

We let

\[
\psi_{u_1}(z; v) = \psi(z; v) \psi(u_1 + z; v)
\]

and recursively define

\[
(5.24) \quad \psi_{u_1, \ldots, u_t}(z; v) = \psi_{u_1, \ldots, u_{t-1}}(z; v) \psi_{u_1, \ldots, u_{t-1}}(u_t + z; v) = \prod_{\epsilon \in \{0,1\}} \psi(\epsilon_1 u_1 + \cdots + \epsilon_t u_t + z; v)
\]

for \( t \geq 2 \). We also define \( K_{u_1, \ldots, u_t} \) in a similar manner. By following the proof of [22, Lemma 11.1], while taking into account the weights, we obtain

\[
(5.25) \quad |T_v(\alpha)|^{d-1} \leq |U|^d |U|^d \sum_{u_1 \in U} \cdots \sum_{u_{d-1} \in U} \sum_{u_t \in U} K_{u_1, \ldots, u_{d-1}}(u_t) \psi_{u_1, \ldots, u_{d-1}}(u_t; v) e \left( \tilde{F}_t(u_1, \ldots, u_t) \right)
\]

for each \( t \geq 1 \). Then it follows by taking the square of the inequality (with \( t = d - 1 \)) and applying the Cauchy-Schwarz inequality that

\[
(5.26) \quad |T_v(\alpha)|^{d-1} \leq |U|^d |U|^d \sum_{u_1 \in U} \cdots \sum_{u_{d-1} \in U} \sum_{u_{d-2} \in U} \left| \sum_{z \in U(u_1, \ldots, u_{d-2})} K_{u_1, \ldots, u_{d-2}}(z) \psi_{u_1, \ldots, u_{d-2}}(z; v) e \left( \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, z) \right) \right|^2
\]

\[
= |U|^d |U|^d \sum_{u_1 \in U} \cdots \sum_{u_{d-1} \in U} \sum_{u_{d-2} \in U} \sum_{z, z' \in U(u_1, \ldots, u_{d-2})} K_{u_1, \ldots, u_{d-2}}(z) K_{u_1, \ldots, u_{d-2}}(z') \psi_{u_1, \ldots, u_{d-2}}(z; v) \psi_{u_1, \ldots, u_{d-2}}(z'; v) \cdot e \left( \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, z) - \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, z') \right).
\]

Given \( z, z' \in U(u_1, \ldots, u_{d-2}) \), let us set \( u_{d-1} = (z' - z) \in U(u_1, \ldots, u_{d-2}) \) and

\[
u_d = z \in U(u_1, \ldots, u_{d-2}) \cap (U(u_1, \ldots, u_{d-2}) - u_{d-1}) = U(u_1, \ldots, u_{d-1}).
\]

It also follows from (5.23) that

\[
\tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, z) - \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, z')
\]

\[
= \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, u_d) - \tilde{F}_{d-1}(u_1, \ldots, u_{d-2}, u_{d-1} + u_d)
\]

\[
= \tilde{F}_{d}(u_1, \ldots, u_{d-1}, u_d) - \tilde{F}_{d-1}(u_1, \ldots, u_{d-1}).
\]
Therefore, we obtain

\[
\left| \sum_{z \in U(u_1, \ldots, u_{d-2})} K_{u_1, \ldots, u_{d-2}}(z) \psi_{u_1, \ldots, u_{d-2}}(z, \nu) e\left( \xi_{d-1}(u_1, \ldots, u_{d-2}, z) \right) \right|^2
\]

\[
= \sum_{u_{d-1} \in U(u_1, \ldots, u_{d-2})^D} \sum_{u_d \in U(u_1, \ldots, u_{d-1})} K_{u_1, \ldots, u_{d-1}}(u_d) \psi_{u_1, \ldots, u_{d-1}}(u_d; \nu) \cdot e\left( \xi_{d-1}(u_1, \ldots, u_{d-1}) - \xi_{d-1}(u_1, \ldots, u_{d-1}) \right),
\]

and (5.26) becomes

\[
|T(\alpha)|^{2^{d-1}} \leq |U^D|^{2^{d-1}-d} \sum_{u_1 \in U^D} \cdots \sum_{u_{d-2} \in U^D} \sum_{u_{d-1} \in U(u_1, \ldots, u_{d-2})^D} \sum_{u_d \in U(u_1, \ldots, u_{d-1})} K_{u_1, \ldots, u_{d-1}}(u_d) \psi_{u_1, \ldots, u_{d-1}}(u_d; \nu) \cdot e\left( \xi_{d-1}(u_1, \ldots, u_{d-1}) - \xi_{d-1}(u_1, \ldots, u_{d-1}) \right).
\]

We note that \( U^D, U(u_1, \ldots, u_{d-2})^D \) are boxes contained in \([-U, U]^h \), and \( U(u_1, \ldots, u_{d-1}) \) is a box contained in \([0, U]^h \).

Let

\[
\xi_d(u_1, \ldots, u_d) = \sum_{(j_1, \ldots, j_d) \in \mathbb{Z}^h} (-1)^{\epsilon_1 + \cdots + \epsilon_d} \xi \left(\epsilon_1 u_1 + \cdots + \epsilon_d u_d\right)
\]

and

\[
(\xi_d)(u_1, \ldots, u_d) = \sum_{(j_1, \ldots, j_d) \in \mathbb{Z}^h} (-1)^{\epsilon_1 + \cdots + \epsilon_d} (\xi)(\epsilon_1 u_1 + \cdots + \epsilon_d u_d).
\]

By [22] Lemma 11.4, the polynomial \( \xi_d \) is the unique symmetric multilinear form associated to \( \xi_d \), i.e. \( \xi_d \) satisfies

\[
(5.28) \quad \xi_d(u_1, \ldots, u_d) = (-1)^d d! \xi_{d-1}(u_1, \ldots, u_d).
\]

Recalling (5.21) and making use of [22] Lemma 11.2 and [22] Lemma 11.4, it follows that

\[
\tilde{\xi}_d(u_1, \ldots, u_d) = \xi_d(u_1, \ldots, u_d) + (\xi_d)(u_1, \ldots, u_d)
\]

\[
= (-1)^d d! \alpha \sum_{1 \leq j \leq h} \left( \sum_{1 \leq k \leq h} G_{j,k} v_k \cdots v_k \right) u_{1,j} \cdots u_{d,j} + (\xi_d)(u_1, \ldots, u_d),
\]

where (\( G_d \))(u_1, \ldots, u_d) is a polynomial of degree at most \( d - 1 \) in \( v \) for fixed \( u_1, \ldots, u_d \). Let us denote

\[
-\tilde{\xi}_{d-1}(u_1, \ldots, u_{d-1}) = \alpha \sum_{1 \leq k \leq h} v_k \cdots v_k \tilde{H}_k(u_1, \ldots, u_{d-1}) + (\xi_d)(u_1, \ldots, u_{d-1}),
\]

where (\( \xi_d \))(u_1, \ldots, u_{d-1}) is a polynomial of degree at most \( d - 1 \) in \( v \) for fixed \( u_1, \ldots, u_{d-1} \), and each \( \tilde{H}_k(u_1, \ldots, u_{d-1}) \) is symmetric in \((k_1, \ldots, k_d)\). Clearly \( \tilde{H}_k \) and \( \xi_d \) are polynomials of degrees at most \( d \) in \((u_1, \ldots, u_{d-1})\).

Let us write \( u_j = (u_1, \ldots, u_d) \). We substitute the inequality (5.21) into (5.18), and we interchange the order of summation moving the sum over \( v \) inside the sums over \( u_j \). Then
we apply Hölder’s inequality to obtain
\[(5.29) \quad |E(\alpha)|^{2d-2} \ll X^\varepsilon U^h(2^{d-2} - d) V^h(2^{d-2} - d) \sum_{u_1 \in U} \cdots \sum_{u_d \in U} \sum_{u_{d-1} \in U} |F_u(\alpha)|^{2d-1},\]
where
\[F_u(\alpha) = \sum_{v \in V} \psi_{u_1, \ldots, u_{d-1}}(u_d; v) e \left( \alpha \sum_{1 \leq k \leq h} v_{k_1} \cdots v_{k_d} H_k(\mathbf{u}) + (G_v)_d(\mathbf{u}) + \mathfrak{F}_v(u_1, \ldots, u_{d-1}) \right)\]
and
\[(5.30) \quad H_k(\mathbf{u}) = \sum_{1 \leq j \leq h} G_{j,k} (-1)^d d! u_{1,j_1} \cdots u_{d,j_d} + \tilde{H}_k(u_1, \ldots, u_{d-1}).\]
Let us set \(\tilde{u} = (u_1, \ldots, u_{d-1})\). Let \(\tilde{\psi}_{\tilde{u}}(u_d; v) = \psi_{u_1, \ldots, u_{d-1}}(u_d; v)\). Similarly as before we let
\[\psi_{\tilde{u}, v_1}(u_d; z) = \psi_{\tilde{u}}(u_d; z) \psi_{\tilde{u}}(u_d; v_1 + z)\]
and recursively define
\[(5.31) \quad \psi_{\tilde{u}, v_1, \ldots, v_t}(u_d; z) = \psi_{\tilde{u}, v_1, \ldots, v_{t-1}}(u_d; z) \psi_{\tilde{u}, v_1, \ldots, v_{t-1}}(u_d; v_t + z) = \prod_{e \in \{0,1\}^t} \psi_{\tilde{u}}(u_d; e_1 v_1 + \cdots + e_t v_t + z)\]
for \(t \geq 2\). We also set \(\tilde{v} = (v_1, \ldots, v_{d-1})\) and \(v = (v_1, \ldots, v_d)\). Let \(\tilde{\psi}_{\tilde{u}, v}(u_d; z) = \psi_{\tilde{u}, v_1, \ldots, v_{d-1}}(u_d; z)\). We now apply the same differencing process as before to \(T_u(\alpha)\). This time, instead of \(5.20\), we use
\[(5.32) \quad \tilde{F}(v) = \alpha \sum_{1 \leq k \leq h} H_k(\mathbf{u}) v_{k_1} \cdots v_{k_d} + (G_v)_d(\mathbf{u}) + \mathfrak{F}_v(u_1, \ldots, u_{d-1}).\]
Then
\[(5.33) \quad \tilde{F}^{[d]}(v) = \alpha \sum_{1 \leq k \leq h} H_k(\mathbf{u}) v_{k_1} \cdots v_{k_d}\]
and we let
\[(5.34) \quad L(\mathbf{u}; \mathbf{v}) = \tilde{F}(v_1, \ldots, v_d) = (-1)^d d! \alpha \sum_{1 \leq k \leq h} H_k(\mathbf{u}) v_{1,k_1} \cdots v_{d,k_d},\]
where the second equality follows by \cite{22} Lemma 11.4]. Therefore, by the same argument as in obtaining \(5.25\) and substituting \(t = d\) yield
\[|T_u(\alpha)|^{2d-1} \leq |V^D|^{2d-2} \sum_{v_1 \in V^D} \cdots \sum_{v_{d-1} \in V^D} \left| \sum_{v_d \in V(v_1, \ldots, v_{d-1})} \psi_{\tilde{u}, v}(u_d; v_d) e(L(\mathbf{u}; \mathbf{v})) \right|,\]
and \(5.29\) becomes
\[(5.35) \quad |E(\alpha)|^{2d-2} \ll X^\varepsilon U^h(2^{d-2} - d) V^h(2^{d-2} - d) \sum_{u_1 \in U} \cdots \sum_{u_d \in U} \sum_{u_{d-1} \in U} \sum_{v_1 \in V^D} \cdots \sum_{v_{d-1} \in V^D} \left| \sum_{v_d \in V(v_1, \ldots, v_{d-1})} \psi_{\tilde{u}, v}(u_d; v_d) e(L(\mathbf{u}; \mathbf{v})) \right|\].
We note that $\mathcal{V}^D$ is a box contained in $[-V, V]^h$, and $\mathcal{V}(v_1, \ldots, v_{d-1})$ is a box contained in $[0, V]^h$. We now change the order of summation in (5.35), and bound the exponential sum
\begin{equation}
(5.36) \quad \sum_{u_d \in \mathcal{U}(u_1, \ldots, u_{d-1})} \left| \sum_{v_d \in \mathcal{V}(v_1, \ldots, v_{d-1})} \psi_{\mathcal{U}, \mathcal{V}}(u_d; v_d) e(\mathcal{L}(u; v)) \right|.
\end{equation}
Recall (1.3), (5.15), (5.24) and (5.31). First we have
\begin{equation}
\psi_{\mathcal{U}, \mathcal{V}}(u_d; v_d) = \prod_{\epsilon \in \{0,1\}^{d-1}} \prod_{\epsilon' \in \{0,1\}^{d-1}} \psi(\epsilon_1 u_1 + \cdots + \epsilon_{d-1} u_{d-1} + u_d; \epsilon'_1 v_1 + \cdots + \epsilon'_{d-1} v_{d-1} + v_d)
\end{equation}
where each $\psi$ is the $v$-th unit vector in $\mathbb{R}^h$. Given $z \in \mathbb{R}$, let
\begin{equation}
\|z\| = \min_{y \in \mathbb{Z}} |z - y|.
\end{equation}
It is clear from (5.34) that
\begin{equation}
\mathcal{L}(u; \mathbf{v}, v_d, e_i) = \mathcal{L}(u; \mathbf{v}, e_i)v_d.
\end{equation}
Therefore, by partial summation we obtain
\begin{equation}
(5.38) \quad \left| \sum_{v_d \in \mathcal{V}(v_1, \ldots, v_{d-1})} \Omega_i(v_d) e(\mathcal{L}(u; \mathbf{v}, v_d, e_i)) \right| \ll \left( \sup_{v_d \in [0, V]} \Omega_i(v_d) + \int_0^V \left| \frac{d\Omega_i}{dz}(v_d) \right| dv_d \right) \cdot \min \left( V, \|\mathcal{L}(u; \mathbf{v}, e_i)\|^{-1} \right).
\end{equation}
Recall (5.12) and (5.13). Then since
\begin{equation}
\left( \sum_{1 \leq s \leq d-1} \epsilon_s u_{s,i} + u_{d,i} + w_i \right) \frac{V}{X} \ll \frac{(U + U_i)V}{X} = \frac{(\Theta U_{\min} + U_i)V_{\min}}{X} \ll \frac{U_i V_{\min}}{X} \ll 1
\end{equation}
for any $\epsilon$ and $u$ under consideration, we have
\begin{equation}
\int_0^V \left| \frac{d\Omega_i}{dz}(v_d) \right| dv_d \ll \frac{U + U_i}{X} \int_0^V 1 dv_d \ll 1.
\end{equation}
Let
\[
\psi_{\tilde{u}_{\tilde{v}}}(u_d; v_d) \in (L(u; \tilde{v}))
\]
Then we obtain that the right hand side of (5.39) is bounded by
\[
\sum_{u_d \in U(u_1, \ldots, u_{d-1})} \prod_{1 \leq i \leq h} \min \left( V, \| L(u_i; \tilde{v}, e_i) \|^{-1} \right),
\]
where the implicit constant is independent of \(u\) and \(\tilde{v}\).

For \(z \in \mathbb{R}\) we define its fractional part to be
\[
\{z\} = z - \max_{y \leq z} y.
\]

Given \(c = (c_1, \ldots, c_h) \in \mathbb{Z}^h\) with \(0 \leq c_i < V\) \((1 \leq i \leq h)\), we let \(\mathcal{R}(\tilde{u}; \tilde{v}; c)\) be the set of \(u_d \in U(u_1, \ldots, u_{d-1})\) satisfying
\[
\frac{c_i}{V} \leq \{L(u_i; \tilde{v}, e_i)\} < \frac{c_i + 1}{V} \quad (1 \leq i \leq h).
\]

Then we obtain that the right hand side of (5.39) is bounded by
\[
\# \mathcal{R}(\tilde{u}; \tilde{v}; c) \cdot \prod_{1 \leq i \leq h} \min \left( V, \max \left( \frac{V}{c_i}, \frac{V}{V - c_i - 1} \right) \right).
\]

Next we obtain a bound for \#\(\mathcal{R}(\tilde{u}; \tilde{v}; c)\). We define the multilinear form
\[
\Gamma(u; \tilde{v}) = (d!)^2 \sum_{1 \leq i \leq h} \sum_{1 \leq k \leq h} G_{j,k} u_{1,j_1} \cdots u_{d,j_d} v_{1,k_1} \cdots v_{d,k_d}.
\]

When \(u = u_1 = \cdots = u_d\) and \(v = v_1 = \cdots = v_{d-1}\), we have
\[
\Gamma((u_1, \ldots, u_1); (v_1, \ldots, v_1)) = (d!)^2 \frac{\partial |d|}{\partial u_i} (u; v) \quad (1 \leq i \leq h).
\]

If \#\(\mathcal{R}(\tilde{u}; \tilde{v}; c)\) = 0, then there is nothing to prove. Thus we suppose \#\(\mathcal{R}(\tilde{u}; \tilde{v}; c) > 0\) and let \(z \in \mathcal{R}(\tilde{u}; \tilde{v}; c)\). Then for any \(z' \in \mathcal{R}(\tilde{u}; \tilde{v}; c)\), we have
\[
z - z' \in U(u_1, \ldots, u_{d-1})^D \subseteq [-U, U]^h
\]
and
\[
\| L(\tilde{u}; z; \tilde{v}, e_i) - L(\tilde{u}; z'; \tilde{v}, e_i) \| < V^{-1} \quad (1 \leq i \leq h).
\]

Let \(M(\tilde{u}; \tilde{v})\) be the number of integral vectors \(u \in [-U, U]^h\) such that
\[
\| \alpha \Gamma(\tilde{u}; u; \tilde{v}, e_i) \| < V^{-1} \quad (1 \leq i \leq h).
\]
Given any $v_d \in \mathbb{R}^h$, we have
\[
\mathcal{L}(\hat{u}, z; \hat{v}, v_d) - \mathcal{L}(\hat{u}, z'; \hat{v}, v_d) = (-1)^d \alpha \sum_{1 \leq k \leq h} (H_k(\hat{u}, z) - H_k(\hat{u}, z')) v_{1,k} \cdots v_{d,k} \\
= (d!)^2 \alpha \sum_{1 \leq k \leq h} \sum_{1 \leq j \leq h} G_{j,k} (u_{1,j_1} \cdots u_{d-1,j_{d-1}} z_{j_d} - u_{1,j_1} \cdots u_{d-1,j_{d-1}} z'_{j_d}) v_{1,k} \cdots v_{d,k} \\
= \alpha \Gamma(\hat{u}, z - z'; \hat{v}, v_d),
\]
where we recall (5.30) for the second equality. Thus (5.42) becomes
\[
\|\alpha \Gamma(\hat{u}, z - z'; \hat{v}, e_i)\| < V^{-1} \quad (1 \leq i \leq h),
\]
and it follows that the vector $z - z'$ is counted by $M(\hat{u}; \hat{v})$ for all $z' \in \mathcal{R}(\hat{u}; \hat{v}; c)$; therefore, we have
\[
\# \mathcal{R}(\hat{u}; \hat{v}; c) \leq M(\hat{u}; \hat{v})
\]
for any $c$ under consideration. Therefore, by combining (5.39) and (5.40), we obtain
\[
\sum_{u_d \in U(u_1, \ldots, u_{d-1})} \left| \sum_{v_d \in V(v_1, \ldots, v_{d-1})} \psi_{\hat{u}, \hat{v}}(u_d; v_d) e(\mathcal{L}(u; v)) \right| \ll \sum_{0 \leq c_1, \ldots, c_h < V} M(\hat{u}; \hat{v}) \prod_{1 \leq i \leq h} \min \left( V, \max \left( \frac{V}{c_i}, \frac{V}{V - c_i - 1} \right) \right) \ll M(\hat{u}; \hat{v}) \prod_{1 \leq i \leq h} V \log V.
\]
Let us define $M(\alpha; U'; V'; P)$ to be the number of integral vectors
\[
\mathbf{u} \in [-U', U']^{hd} \quad \text{and} \quad \hat{v} \in [-V', V']^{h(d-1)}
\]
satisfying
\[
\|\alpha \Gamma(\mathbf{u}, \hat{v}, e_i)\| < P \quad (1 \leq i \leq h).
\]
By substituting (5.43) into (5.35), it then follows that
\[
\mathcal{E}(\alpha)^{2d-2} \ll X^h U^{h(2d-2-d)} V^{h(2d-2-d+1)} M(\alpha; U; V; V^{-1}).
\]

The following lemma on geometry of numbers was obtained in [21]; this is a generalization of [6] Lemma 12.6.2.

Lemma 5.2. [21] Lemma 2.4] Let $\mathcal{L}_1, \ldots, \mathcal{L}_h$ be symmetric linear forms given by $\mathcal{L}_i = c_{i,1} y_1 + \cdots + c_{i,h} y_h$ ($1 \leq i \leq h$), i.e. $c_{i,j} = c_{j,i}$ ($1 \leq i, j \leq h$). Let $\gamma_1, \ldots, \gamma_h \in \mathbb{R}_{>1}$. We denote by $\mathcal{Y}(Z)$ the number of integer solutions $y_1, \ldots, y_{2h}$ to the system of inequalities
\[
|y_i| < \gamma_i Z \quad (1 \leq i \leq h) \quad \text{and} \quad |\mathcal{L}_i - y_{h+i}| < \gamma_i^{-1} Z \quad (1 \leq i \leq h).
\]
Then for $0 < Z_1 \leq Z_2 \leq 1$ we have
\[
\frac{\mathcal{Y}(Z_2)}{\mathcal{Y}(Z_1)} \ll \left( \frac{Z_2}{Z_1} \right)^h,
\]
where the implicit constant depends only on $h$.2.
Let $0 < Q_1, Q_2 \leq 1$ to be set in due course. By applying Lemma 5.2 (with $\gamma_1 = \cdots = \gamma_h$) $(d - 1)$-times with $Q_1$ and then $d$-times with $Q_2$, we obtain

\begin{equation}
\mathcal{M}(\alpha; U; V; V^{-1}) \ll Q_1^{h(d-1)} Q_2^{-h} \mathcal{M}(\alpha; Q_2 U; Q_1 V; Q_1^{-1} Q_2^d V^{-1}).
\end{equation}

With this estimate we obtain the following lemma. For simplicity let us denote

\begin{equation}
V_2^* = V_{g[2d]; 2}.
\end{equation}

**Lemma 5.3.** Let $0 < \vartheta < 1$. Let $\varepsilon > 0$ be sufficiently small. Then for $X$ sufficiently large, at least one of the following alternatives holds:

i) One has the upper bound

$$|\mathcal{E}(\alpha)| \ll X^{\varepsilon} U^h V^{h} V^{- \vartheta \text{codim} V_2^* / 2^{d-2}}.$$ 

ii) There exist $1 \leq q \leq V^{\vartheta(2d-1)}$ and $a \in \mathbb{Z}$ with gcd$(a, q) = 1$ such that

$$|qa - a| \leq X^{-d+\vartheta d} V^{\vartheta(2d-1)}.$$ 

**Proof.** If

$$X^{\frac{\vartheta}{4}} \geq V^{\vartheta \text{codim} V_2^* / 2^{d-2}},$$

then the result is trivial. Therefore, let us suppose otherwise and set

$$\vartheta' = \vartheta - \frac{\varepsilon}{2} \cdot \frac{\log X}{\log V} \cdot \frac{2^{d-2}}{\text{codim} V_2^*}.$$ 

Note from (5.7) and (5.11) we have

\begin{equation}
X^{\frac{\vartheta}{4}} \ll V \ll X.
\end{equation}

Consider the affine variety $Z$ defined by

$$Z = \left\{ (u, \hat{v}) \in \mathbb{A}_C^{h(2d-1)} : \Gamma(u, \hat{v}, e_i) = 0 \quad (1 \leq i \leq h) \right\}.$$ 

Let us define

$$\mathcal{N}(Z) = \left\{ (u, \hat{v}) \in \mathbb{Z}^{h(2d-1)} \cap Z : u_1, \ldots, u_d \in [-Q_2 U, Q_2 U]^h, v_1, \ldots, v_{d-1} \in [-Q_1 V, Q_1 V]^h \right\}.$$ 

In this proof, we set $Q_1 = V^{\vartheta'}$ and $Q_2 = V^{\vartheta} U^{-1}$.

Suppose we have that every point counted by $\mathcal{M}(\alpha; Q_2 U; Q_1 V; Q_1^{-1} Q_2^d V^{-1})$ is contained in $\mathcal{N}(Z)$. Then we apply [2, (3.1)]; this bound is independent of the coefficients of the polynomials defining the affine variety (depending only on the dimension and the degree). As a result, we obtain

\begin{equation}
\mathcal{M}(\alpha; Q_2 U; Q_1 V; Q_1^{-1} Q_2^d V^{-1}) \leq \# \mathcal{N}(Z) \ll V^{\vartheta' \text{dim} Z}.
\end{equation}

Therefore, it follows from (5.44), (5.45) and (5.48) that

\begin{equation}
|\mathcal{E}(\alpha)|^{2^{d-2}} \ll X^{\varepsilon} U^{h(2d-2)} V^{h(2d-2) - d} V^{h(d-1) - h(d-1) \vartheta'} U^{hd} V^{h(2d-2) - d} V^{\vartheta' \text{dim} Z} = X^{\varepsilon} U^{h2^{d-2}} V^{h2d-2} V^{\vartheta' (-h(d-1) - hd + \text{dim} Z)}.
\end{equation}

Let

\begin{equation}
\mathcal{D} = \left\{ (u, \hat{v}) \in \mathbb{A}_C^{h(2d-1)} : u_1 = \cdots = u_d, v_1 = \cdots = v_{d-1} \right\}.
\end{equation}
Then from (5.41) and (4.4) we have

$$\dim V_2 = \dim(\mathbb{Z} \cap D) \geq \dim \mathbb{Z} - h(d-1) - h(d-2) = -h(d-1) - hd + \dim \mathbb{Z} + 2h.$$  

With this inequality, (5.49) becomes

$$|E(\alpha)|^{2d-2} \ll X^{\varepsilon} U^{h2d-2} V^{h2d-2} V^{-\vartheta' \cdot \text{codim} V_2^*},$$

and the estimate in i) follows immediately from the definition of $\vartheta'$.

On the other hand, suppose there exists $(u, \hat{v})$ counted by $M(\alpha; Q_2 U; Q_1 V; Q_1^{d-1}Q_2^{d-1}V^{-1})$ which is not contained in $\mathcal{N}(\mathbb{Z})$, i.e. there exists $1 \leq i_0 \leq h$ such that

$$\Gamma(u, \hat{v}, e_{i_0}) \neq 0.$$  

Let us write

$$\alpha \Gamma(u, \hat{v}, e_{i_0}) = a + \xi,$$

where $a \in \mathbb{Z}$ and $|\xi| < Q_1^{d-1}Q_2^{d-1}V^{-1}$. Let $q$ be the absolute value of $\Gamma(u, \hat{v}, e_{i_0})$. Then

$$1 \leq q \ll Q_1^{d-1}Q_2^{d-1}U^d V^{d-1} = V^{\vartheta'(2d-1)}.$$  

It follows from (5.7) and (5.11) that $X^{1-\vartheta} \ll UV$. Therefore, we obtain

$$|\xi| < Q_1^{d-1}Q_2^{d-1}V^{-1} = \frac{V^{\vartheta'(2d-1)}}{U^d V^{d-1}} \ll X^{-d+\sigma} V^{\vartheta'(2d-1)}.$$  

Finally, since $\vartheta' < \vartheta$, it follows that $1 \leq q \leq V^{\vartheta'(2d-1)}$ and $|\xi| \leq X^{-d+\sigma} V^{\vartheta'(2d-1)}$ for $X$ sufficiently large; we have obtained the statement in ii).

By combining (5.13) and Lemma 5.3 we obtain the following.

**Proposition 5.4.** Let $0 < \vartheta < 1$. Let $\varepsilon > 0$ be sufficiently small. Then for $X$ sufficiently large, at least one of the following alternatives holds:

i) One has the upper bound

$$|S(M, N; \alpha)| \ll X^{n+\varepsilon} V^{-\vartheta' \cdot \text{codim} V_2^*}.$$  

ii) There exist $1 \leq q \leq V^{\vartheta'(2d-1)}$ and $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$ such that

$$|q\alpha - a| \leq X^{-d+\sigma} V^{\vartheta'(2d-1)}.$$  

We let $\nu = \frac{\log V}{\log X}$. Then from (5.47) it follows that

$$\frac{1}{2 + \varepsilon} < \nu < 1 + \varepsilon.$$  

We define $\mathfrak{M}(\vartheta)$ to be the set of $\alpha \in [0, 1)$ satisfying ii) in Proposition 5.4 i.e. there exist $1 \leq q \leq X^{(2d-1)\vartheta}$ and $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$ such that

$$|q\alpha - a| \leq X^{(2d-1)\vartheta+\sigma-d}.$$
5.2. Sliding scale argument. We begin by setting \( K = \frac{\text{codim} \, V^{*}_2}{2^{2d-2}} \).

For any \( 0 < \vartheta < 1 \), the Lebesgue measure of \( M^{(\vartheta)} \) is bounded by the following quantity

\[
\text{meas}(M^{(\vartheta)}) \ll \sum_{1 \leq q \leq X^{(2d-1)\vartheta}} \sum_{0 \leq a \leq q \atop \gcd(a, q) = 1} q^{-1} X^{-d+ds+(2d-1)\vartheta}.
\]

We then define a sequence \( 0 < \vartheta < \vartheta_1 < \cdots < \vartheta_J < 1 \) satisfying

\[
\frac{\tau}{2} > 2(2d-1)\vartheta_1 - \vartheta, \quad \frac{\tau}{2} > 2(2d-1)\vartheta_{j+1} - \vartheta_j \quad (1 \leq j \leq J - 1)
\]

and

\[
1 - \tau < \vartheta_J < 1,
\]

where \( \tau > 0 \) is sufficiently small. In particular, \( J \ll 1 \). Then, under the assumption that \( K > 2(2d-1) \) holds, by Proposition 5.4 and (5.52) we have

\[
\int_{[0,1) \backslash M^{(\vartheta)}} |S(M, N; \alpha)| \, d\alpha \ll X^{n-K\vartheta + \varepsilon} + X^{n-d+ds-(K-2(2d-1))\vartheta + \varepsilon}
\]

\[
\ll X^{n-d-\varepsilon} + X^{n-d+ds-(K-2(2d-1))\vartheta + \varepsilon}.
\]

Now we define \( 0 < \vartheta'_0 < 1 \) by

\[
\vartheta'_0 = \frac{(2 + \varepsilon)\sigma}{K - 2(2d-1)}.
\]

On recalling (5.51) we see that

\[
\frac{d\sigma}{K - 2(2d-1)} < \vartheta'_0
\]

is satisfied. We further need \( \vartheta'_0 \) to satisfy

\[
2(2d-1)\vartheta'_0 + d\sigma < \frac{5}{24},
\]

or equivalently

\[
K > \frac{4d\sigma(2d-1)}{5} + 2(2d-1) > 2(2d-1).
\]
Recall the definition of $C_0$ given in (4.2). Let us set $H = td$ with $t \in \mathbb{N}$ so that $\sigma = 1/(2td)$ (recall (5.4)). Then from (5.16), which states
\[
\text{codim} V_2^* \geq \frac{\text{codim} V_F^* - (2H - 1)C_0}{H},
\]
we see that (5.56) is satisfied provided
\[
\text{codim} V_F^* > H 4^{d-1} \left( \frac{4d\sigma(2d - 1)}{2t} - d\sigma + 2(2d - 1) \right) + (2H - 1)C_0
\]
\[
= d(2d - 1)4^d \left( \frac{H\sigma}{2t} - \frac{H}{2d} \right) + (2td - 1)C_0
\]
\[
= d(2d - 1)4^d \left( \frac{\frac{t}{2}}{2t - 1} + \frac{t}{2} \right) + (2td - 1)C_0
\]
\[
= d(2d - 1)4^d \frac{t(5t + 12)}{2(5t - 12)} + (2td - 1)C_0.
\]

Let us choose $t = 6$, in which case (5.57) becomes
\[
\text{codim} V_F^* > 7d(2d - 1)4^d + (12d - 1)C_0,
\]
which is precisely (1.4); therefore, it follows that (5.55) and (5.56) hold.

Let us set $\lambda = d\sigma = 1/12$ in the definition of $M^+(\vartheta_0)$ given in (3.8). Let $\vartheta_0 = (2d - 1)\vartheta'_0(1 + \varepsilon)$. Then from (5.55), which we now know to hold assuming (1.4), it follows that
\[
2\vartheta_0 + (2\vartheta_0 + 2\lambda + \varepsilon) < \frac{5}{12},
\]
i.e. (3.6) is satisfied with $\vartheta_0$ and $\lambda$ as in this section and $\gamma = 2\vartheta_0 + 2\lambda + \varepsilon$. It is easy to see from the definition of $M^{(II)}(\vartheta)$ and (5.51) that
\[
M^{(II)}(\vartheta'_0) \subseteq M^+(\vartheta_0),
\]
hence
\[
m^+(\vartheta_0) = [0, 1) \setminus M^+(\vartheta_0) \subseteq [0, 1) \setminus M^{(II)}(\vartheta'_0).
\]

Therefore, by (5.53), (5.54) and (5.56) we obtain
\[
\int_{m^+(\vartheta_0)} |S(M, N; \alpha)| \, d\alpha \leq \int_{[0, 1) \setminus M^{(II)}(\vartheta'_0)} |S(M, N; \alpha)| \, d\alpha \ll X^{n-d-\varepsilon}.
\]

As mentioned in the first paragraph of this section, it then follows from (5.11), (5.3) and (5.58) that
\[
\int_{m^+(\vartheta_0)} |S(\alpha)| \, d\alpha \leq \sum_{(M, N) \in \Xi(aX, bX)} \int_{m^+(\vartheta_0)} |S(M, N; \alpha)| \, d\alpha \ll X^{n-d-\varepsilon},
\]
this completes the proof of Proposition 5.1.
6. Conclusion

By combining Propositions 3.1 and 5.1 we obtain Theorem 1.2 when $F$ satisfies (II) of Definition 4.1. We now deal with the remaining case. Suppose $F$ satisfies (I) of Definition 4.1. Let us denote $w = (w_1, \ldots, w_t)$, $s = (s_1, \ldots, s_m)$ and $t = (t_1, \ldots, t_{n-m})$. We also let $s' = (s'_1, \ldots, s'_m)$ and $t' = (t'_1, \ldots, t'_{n-m})$. Let 

$$g_w(s, s', t, t') = F(s, t, w) - F(s, t', w) - F(s', t, w) + F(s', t', w).$$

Then

$$g_w^d(s, s', t, t') = \mathcal{G}(s, t) - \mathcal{G}(s, t') - \mathcal{G}(s', t) + \mathcal{G}(s', t')$$

for each fixed $w$, where $\mathcal{G}$ is defined in (4.1). By applying the Cauchy-Schwarz inequality twice, we obtain

$$|S(\alpha)|^4 \ll X^{2n+\ell+\varepsilon} \sum_{w \in [0,X]^{\ell}} \sum_{s, s' \in [0,X]^m} \sum_{t, t' \in [0,X]^{n-m}} e(\alpha g_w(s, s', t, t'))$$

$$\ll X^{2n+2\ell+\varepsilon} \max_{w \in [0,X]^{\ell}} \sum_{s, s' \in [0,X]^m} \sum_{t, t' \in [0,X]^{n-m}} e(\alpha g_w(s, s', t, t')).$$

Since $g_w^d(s, 0, t, 0) = \mathcal{G}(s, t)$, it follows from [25] Lemma 3.1 that

$$\text{codim } V_{\mathcal{G}}^* \geq \text{codim } V_{\mathcal{G}}^* > C_0$$

for each fixed $w$. Then we obtain the following as an immediate consequence of [1] Lemma 4.3 (additional explanation can be found in the proof of [25] Lemma 4.2).

Lemma 6.1. Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous form of degree $d \geq 2$ satisfying (I) of Definition 4.1. Let $0 < \zeta < 1$. Let $\varepsilon > 0$ be sufficiently small. Then for $X$ sufficiently large, at least one of the following alternatives holds:

i) One has the upper bound

$$|S(\alpha)| \ll X^{n-\varepsilon -2 - \zeta C_0 + \varepsilon}.$$

ii) There exist $1 \leq q \leq X^{(d-1)\zeta}$ and $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$ such that

$$|qa - a| \leq X^{-d+(d-1)\zeta}.$$

We define $\mathcal{M}^{(I)}(\zeta)$ to be the set of $\alpha \in [0, 1)$ satisfying ii) in Lemma 6.1 i.e. there exist $1 \leq q \leq X^{(d-1)\zeta}$ and $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$ such that

$$|qa - a| \leq X^{-d+(d-1)\zeta}.$$

6.1. Sliding scale argument. For any $0 < \zeta < 1$, the Lebesgue measure of $\mathcal{M}^{(I)}(\zeta)$ is bounded by the following quantity

$$\text{meas}(\mathcal{M}^{(I)}(\zeta)) \ll \sum_{1 \leq q \leq X^{(d-1)\zeta}} \sum_{0 \leq a < q, \gcd(a, q) = 1} q^{-1} X^{-d+(d-1)\zeta} \ll X^{-d+(d-1)\zeta}.$$
where \( \tau > 0 \) is sufficiently small. In particular, \( J \ll 1 \). From the definition of \( C_0 \) given in (4.2), it is clear that

\[
2^{-1-d}C_0 > d \quad \text{and} \quad 2^{-1-d}C_0 - 2(d-1) > 0.
\]

Then by Lemma 6.1 and (6.3) we have

\[
\int_{[0,1) \setminus \mathcal{M}^0(\varsigma)} |S(\alpha)| \, d\alpha = \int_{[0,1) \setminus \mathcal{M}^0(\varsigma)} |S(\alpha)| \, d\alpha + \sum_{j=0}^{J-1} \int_{\mathcal{M}(\varsigma_{j+1}) \setminus \mathcal{M}(\varsigma_j)} |S(\alpha)| \, d\alpha \ll X^{n-2^{-1-d}C_0+\varepsilon} + X^{n-d-(2^{-1-d}C_0-2(d-1))\varsigma+\varepsilon} \ll X^{n-d-\varepsilon}.
\]

Let \( \vartheta_0, \gamma, \lambda > 0 \) be such that (3.6) is satisfied. Let us set \( 0 < \varsigma < \vartheta_0/(d-1) \) so that

\[
\mathcal{M}(\varsigma) \subseteq \mathcal{M}(\vartheta_0),
\]

hence

\[
m^+(\vartheta_0) = [0, 1) \setminus \mathcal{M}(\vartheta_0) \subseteq [0, 1) \setminus \mathcal{M}(\varsigma).
\]

Therefore, we obtain from (6.4) that

\[
\int_{m^+(\vartheta_0)} |S(\alpha)| \, d\alpha \leq \int_{[0,1) \setminus \mathcal{M}(\varsigma)} |S(\alpha)| \, d\alpha \ll X^{n-d-\varepsilon}.
\]

By combining this estimate with Proposition 3.1 we obtain Theorem 1.2 when \( F \) satisfies (I) of Definition 4.1 as well; this completes the proof of Theorem 1.2.

Finally, we remark that it may be possible to further improve on Theorem 1.2 by using the Heath-Brown identity. The strategy would be to combine it with a combinatorial result such as [19, Lemma 3.1] or its extensions (The range of \( \sigma \) in [19, Lemma 3.1] can be extended at the cost of additional cases as explained in [19, Remark 3.2]), and deal with the resulting exponential sums. The method of this paper is suitable when the weights correspond to the Type I/II alternative in [19, Lemma 3.1]. However, in order to make further progress with this approach, one would need to be able to deal with the exponential sums of the form \( S(\alpha) \) with (variants of) divisor functions as weights without resorting to the methods of this paper, that is to make use of the bihomogeneous structure to remove all the weights. If this can be done, then it may be possible to establish Theorem 1.2 with \( \text{codim} V_F = O(d^2-\Delta 4^d) \) for some \( \Delta > 0 \).

References

[1] B. J. Birch, *Forms in many variables*. Proc. Roy. Soc. Ser. A 265 1961/1962, 245–263.
[2] T. D. Browning, *Quantitative arithmetic of projective varieties*, Progress in Mathematics, vol. 277, Birkhäuser Verlag, Basel, 2009.
[3] T. D. Browning, and S. M. Prendiville, *Improvements in Birch’s theorem on forms in many variables*. J. reine angew. Math. 731 (2017), 203–234.
[4] B. Cook, *Equations in the primes*. Ph.D. thesis, University of British Columbia, Vancouver 2012.
[5] B. Cook and Á. Magyar, *Diophantine equations in the primes*. Invent. Math. 198 (2014), 701–737.
[6] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*. Second edition. Cambridge University Press, Cambridge, 2005.
[7] H. Davenport, *Multiplicative Number Theory*. Third edition. Springer-Verlag, New York, 2000.
[8] P. X. Gallagher, *A large sieve density estimate near \( \sigma = 1 \)*. Invent. Math. 11 (1970), 329–339.
[9] B. Green, *Quadratic forms in 8 prime variables*. arXiv:2108.10401.
[10] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.
[11] D. R. Heath-Brown, *Prime numbers in short intervals and a generalized Vaughan identity*. Canad. J. Math. 34 (1982), no. 6, 1365–1377.

[12] M. N. Huxley, *Large values of Dirichlet polynomials. III*. Acta Arith. 26 (1974/75), no. 4, 435–444.

[13] H. Iwaniec, *On Zeros of Dirichlet’s L Series*. Invent. Math. 23 (1974), 97–104.

[14] H. Iwaniec and E. Kowalski, *Analytic Number Theory*. American Mathematical Society Colloquium Publications Vol. 53, 2004.

[15] M. Jutila, *On Linnik’s constant*. Math. Scand. 41 (1977), no. 1, 45–62.

[16] J. Liu, *Integral points on quadrics with prime coordinates*. Monatsh. Math. 164 (2011), no. 4, 439–465.

[17] J. Liu and L. Zhao, *On forms in prime variables*. arXiv:2105.12956.

[18] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach’s problem*. Acta Arith. 27 (1975), 353–370.

[19] D. H. J. Polymath, W. Castryck, É. Fouvry, G. Harcos, E. Kowalski, P. Michel, P. Nelson, E. Paldi, J. Pintz, A. V. Sutherland, T. Tao and X.-F. Xie, *New equidistribution estimates of Zhang type*. Algebra Number Theory 8 (2014) 2067–2199.

[20] D. Schindler, *Bihomogeneous forms in many variables*. J. Théorie Nombres Bordeaux 26 (2014), 483–506.

[21] D. Schindler and E. Sofos, *Sarnak’s saturation problem for complete intersections*. Mathematika 65 (2019), no. 1, 1–56.

[22] W. M. Schmidt, *The density of integer points on homogeneous varieties*. Acta Math. 154 (1985), no. 3-4, 243–296.

[23] S. Yamagishi, *Diophantine equations in semiprimes*. Discrete Analysis 2019:17, 21 pp.

[24] S. Yamagishi, *On an oscillatory integral involving a homogeneous form*. Funct. Approx. Comment. Math. 62 (2020), 21–58.

[25] S. Yamagishi, *Diophantine equations in primes: density of prime points on affine hypersurfaces*. Duke Math Journal, to appear.

[26] L. Zhao, *The quadratic form in nine prime variables*. Nagoya Math. J., 223 (1) (2016), 21–65.

Mathematisch Instituut, Universiteit Utrecht, Budapestlaan 6, NL-3584 CD Utrecht, The Netherlands

Email address: s.yamagishi@uu.nl