Irreducible laminations for IWIP
Automorphisms of free products and
Centralisers

Dionysios Syrigos

October 31, 2014

Abstract

Let $G = G_1 \ast \ldots \ast G_q \ast F_r$ be a group which splits as free product, where $F_r$ is a finitely generated free group. For every such decomposition, we can associate some (relative) outer space $\mathcal{O}$. In this paper we develop the theory of irreducible laminations for free products of groups. In particular, we examine the action of $Out(G, \mathcal{O}) \leq Out(G)$ (of automorphisms which preserve the conjugacy classes of $G_i$’s) on the set of laminations. We generalise the theory of irreducible laminations corresponding to finitely generated free groups. The strategy is the same as in the classical case, but some statements are slightly different because of the non-trivial kernel of the action.

As a corollary, we prove that the centraliser of an IWIP modulo its intersection with the kernel of the action is virtually cyclic, which is a generalisation of a well-known theorem in the free case.
# Contents

1 Introduction

2 Preliminaries
   2.1 Outer space and $\mathcal{O}$-maps
   2.2 Train Track Maps
   2.3 Bounded Cancellation Lemma
   2.4 N-periodic paths
   2.5 Relative train-track maps
   2.6 Graph of Groups and Subgroups

3 Every two $\mathcal{O}$-maps coincide

4 Characterisation of the kernel of the action

5 Laminations
   5.1 Construction of the lamination and properties
   5.2 Lamination in different trees

6 Action

7 Subgroups carrying the lamination

8 $\lambda$-map

9 Kernel of the homomorphism
   9.1 Reducible case
   9.2 Irreducible case

10 Discreteness of the Image

11 Finite index subgroup of the Kernel

12 Main Results
1 Introduction

Let $G$ be a group with finite Kurosh rank and let’s fix a free product decomposition $G = G_1 \ast \ldots \ast G_q \ast F_r$. Guirardel and Levitt in [10] constructed an outer space relative to the Grushko decomposition (every $G_i$ is freely indecomposable and is not isomorphic to $\mathbb{Z}$) and later Francaviglia and Martino in [9] noticed that the outer space $\mathcal{O}$ can be constructed relative to any (finite) free product decomposition. Let $Out(G, \mathcal{O})$ be the subgroup of $Out(G)$, which is consisted of the automorphisms which preserve the conjugacy classes of $G_i$’s (note that in the case of the Grushko decomposition, $Out(G) = Out(G, \mathcal{O})$). We say that an element $\phi \in Out(G, \mathcal{O})$ is irreducible relative to $\mathcal{O}$, which corresponds to a free factor system $\mathcal{G}$, if $\mathcal{G}$ is a maximal proper, $\phi$-invariant free factor system. Therefore we can define the notion of irreducible with irreducible powers (or simply IWIP) automorphism relative to a fixed outer space, as in the special case where $G$ is a finitely generated free group.

In this paper, we study IWIP automorphisms and in particular we show that we can define the stable (and unstable) lamination associated to an IWIP, using exactly the same method as in the free case. In the classical case, it can be proved that the stabiliser of the lamination is virtually cyclic (see [1]). Our first main result is that the stabiliser modulo its intersection with the kernel of the action of $Out(G, \mathcal{O})$ on $\mathcal{O}$, which we denote by $KA$, is virtually cyclic. More precisely:

**Theorem 1.1.** Let $\phi \in Out(G, \mathcal{O})$ be an IWIP relative to $\mathcal{O}$ and let’s denote $\Lambda$ the associated stable lamination. Then $Stab(\Lambda)/(KA \cap Stab(\Lambda))$ is virtually cyclic.

This is a direct generalisation of the result in the free case, since the $Out(F_n)$ acts on $CV_n$ faithfully. Moreover, we will give a precise and simple characterisation of the kernel. In fact, we prove that the kernel of the action is the direct product of the automorphism groups of the factors. Note also that the theory is similar to that of the case of free groups, with the advantage that the relativity of the space covers at once both the general case of a free product as well as many relative cases of $CV_n$. However, if we restrict our attention to f.g. free groups then we actually study the relative...
free case (where all $G_i$ are free groups). E. Meucci studied this case in her thesis, see [19], but her approach is different since during her work uses the group of automorphisms that act to every $G_i$ by conjugation, while we use a bigger subgroup of outer automorphisms which preserve their conjugacy classes. Therefore even if our results look similar in the relative free case, they are not actually the same.

The laminations are very well studied in the free case, since we can find them in a lot of different forms in the literature and the study of them implies important results (for example see [1], [2], [3], [5], [6] and [11]), therefore it looks like interesting to generalise this notion in a more general context. In addition, further motivation is that we can find natural generalisations for a lot of facts for $CV_n$ in the general case, for example in [9], Francaviglia and Martino generalised a lot of tools like train track maps and the Lipschitz metric. But there are some very recent papers that show we can also use further methods of studying $Out(F_n)$ for $Out(G)$ (where $G$ is written as free product as above) such that the closure of outer space, the Tits alternatives for $Out(G)$ and the hyperbolic complex corresponding to $Out(G)$ (see [13], [14], [12] and [17]).

Given a group $G$ and an element $g \in G$, a natural question is study the centraliser $C(g)$ of $g$ in $G$. In several classes of groups, centralisers of elements are reasonably well-understood and sometimes they are useful to the study of the group. For example, Feighn and Handel in [8] classified abelian subgroups in $Out(F_n)$ by studying centralisers of elements. Moreover, a well known result for an IWIP automorphism of a free groups (there are several proofs, see [1], [18] or [16]) states that their centralisers are virtually cyclic. In the general case, we can obtain a similar result but again the kernel is appeared and more precisely:

**Theorem 1.2.** Let $\phi \in Out(G, O)$ be an IWIP relative to $O$ and $C(\phi)$ be the centraliser of $\phi$. Let’s denote $KA$ the kernel of the action of $Out(G, O)$ on $O$. Then $C(\phi)/(KA \cap C(\phi))$ is virtually cyclic.

Note that there are a lot of IWIP automorphisms that don’t commute with automorphisms of the $G_i$’s and in particular for them, the theorem above implies that their centralisers are virtually cyclic. On the other hand, there are examples of IWIP automorphisms which have
big centraliser (thus not virtually cyclic).

**Example 1.3.** Suppose that \( G = \langle a_1 \rangle \ast \langle a_2 \rangle \ast \langle b_1 \rangle \ast \langle b_2 \rangle = G_1 \ast \langle b_1 \rangle \ast \langle b_2 \rangle \) where \( a_i, b_i \) of infinite order, \( G_1 = \langle a_1 \rangle \ast \langle a_2 \rangle \) and \( F_2 = \langle b_1 \rangle \ast \langle b_2 \rangle \). Then in the corresponding outer space, which we denote by \( \mathcal{O} \), the quotient of every tree is a graph of groups with two loops and exactly one vertex with non-trivial vertex group. Then we define \( \phi(a) = a \) for every \( a \in G_1 \), \( \phi(b_1) = b_2, \phi(b_2) = b_1 b_2 \), which is an IWIP relative to \( \mathcal{O} \). But then every automorphism of \( G_1 \) commutes with \( \phi \). Therefore \( C(\phi) \) contains the subgroup \( \text{Aut}(G_1) \text{Inn}(G) \). So the centraliser is not virtually cyclic, but its quotient with the kernel of the action is (by the previous theorem).

**Strategy of the proof:** The paper is organized as follows:

In Section 2, we recall some preliminary definitions, facts and well known results about the outer space of free products. In Section 3, we prove a useful technical lemma for \( \mathcal{O} \)-maps, more specifically we prove that every two such maps are equal except possibly two bounded (depends only on the map, not the path) paths near the endpoints. In Section 4, we give an easy description of the kernel of the action using only the automorphism groups of free factors.

The next sections form the main part of this paper and we follow the same approach as in [1]. In section 5, we define the lamination using train track representatives, we extend the notion to any tree and then we list some useful properties. In Section 6, we explain the action on the stabiliser of the lamination. In Section 7 we define the notion of a subgroup which carries the lamination and then we prove that any such subgroup has finite index in the whole group. In Section 8, which is the most important, we construct a homomorphism from the stabiliser to the positive real numbers. Then in Section 9, we study the kernel of the homomorphism, we prove that any element of the kernel is non-exponentially growing and in the reducible case it has a very good form restricted to the lower strata. In Section 10, we prove the discreteness of the image which allows us to think the previous map, as a homomorphism from the stabiliser to the integers. Section 11 is devoted to prove that the intersection of the kernel of the action with the stabiliser of
2 Preliminaries

2.1 Outer space and $\mathcal{O}$-maps

In this subsection we recall the definitions of outer space and some basic properties. For example, the existence of $\mathcal{O}$-maps between every two elements of the space which is a very useful tool. Let $G$ be a group which splits as a finite free product of the following form $G = H_1 \ast \ldots \ast H_q \ast F_r$, where $q \geq 1, r \geq 0, r + q \geq 2$ (note that it is not necessary to assume that every $H_i$ is not isomorphic to $\mathbb{Z}$ or that every $H_i$ is freely indecomposable, but of course we assume that every $H_i$ is not trivial).

We define the (relative) outer space $\mathcal{O}$ which we can associate to the group $G$ relative to a fixed decomposition as above. These groups admit co-compact actions on $\mathbb{R}$-trees (and vice-versa). Note that we can apply the theory in the case that $G$ is free, and the $G_i$’s are certain free factors of $G$. This case is called the relative free case.

The elements of outer space can be thought as simplicial metric $G$-trees, up to $G$-equivariant isometry. Moreover, we require that these trees also satisfy the following:

- The action of $G$ on $T$ is minimal.
- The edge stabilisers are trivial.
- There are finitely many orbits of vertices with non-trivial stabiliser,
2.1 Outer space and $\mathcal{O}$-maps

more precisely for every $H_i$, $i = 1, \ldots, p$ (as above) there is exactly one vertex $v_i$ with stabiliser $H_i$ (all the vertices in the orbits of $v_i$’s are called non-free vertices).

- All other vertices have trivial stabiliser (and we call them free vertices)

These assumptions imply that the quotient $T/G$ is a finite graph of groups. We could also define the outer space as the space of ”marked metric graph of groups” as in the free case, but we won’t use this point of view since here it is easier to work in the space of trees. We would like to define a natural action of $Out(G)$ on $\mathcal{O}$, but this is not possible since it not always the case that the automorphisms preserve the structure of the trees. However, we can describe here the action of a specific subgroup of $Out(G)$ (it contains the automorphisms that preserve the decomposition or equivalently the structure of the trees) on $\mathcal{O}$. Let $Aut(G, \mathcal{O})$ be the subgroup of $Aut(G)$ preserve the set of conjugacy classes of the $G_i$ ’s. Equivalently, $\phi \in Aut(G)$ belongs to $Aut(G, \mathcal{O})$ if $\phi(G_i)$ is conjugate to one of the $G_i$ ’s. Note that in the case of a Grushko decomposition (which is a finite free decomposition as above, but we require every $H_i$ to be freely indecomposable and not isomorphic to $\mathbb{Z}$) $Aut(G) = Aut(G, \mathcal{O})$, since it is well known for the Grushko’s decomposition that every automorphism preserves the conjugacy classes of $H_i$’s. The group $Aut(G, \mathcal{O})$ admits a natural action on a simplicial tree (as above) by changing the action, that is, for $\phi \in Aut(G)$ and $T$ as above, then $\phi(T)$ has the same underlying tree with $T$ but the action is $g \cdot x = \phi(g)x$ (where the action in the right side is the action of the $G$-tree $T$). But since the set of inner automorphisms of $G$, $Inn(G)$ acts trivially on $\mathcal{O}$ we can define $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ which acts on $\mathcal{O}$ as above.

Note here that the above discussion can be replaced using the notion of free factor systems. We can define $Out(G, \mathcal{O})$ as the subgroup of $Out(G)$ which fixes the free factor system of the pre-chosen decomposition. Even if we won’t use the notion of free factor systems, we will give the precise definition latter.

We can now define some nice maps between the elements of the outer space. In fact, these maps, which are called $\mathcal{O}$-maps, play the role of the homotopy equivalences between every two marked metric graphs in the free case. We
say that a map between trees $A, B \in \mathcal{O}$, $f : A \to B$ is an $\mathcal{O}$-map, if it $G$-equivariant, Lipschitz continuous, surjective function. Note here that we denote by $\text{Lip}(f)$ the Lipschitz constant of $f$.

It is very useful to know that there are such maps between every two trees. This is true and, additionally, by their construction they coincide on the non-free vertices. More specifically by [9]:

**Lemma 2.1.** For every pair $A, B \in \mathcal{O}$; there exists a $\mathcal{O}$-map $f : A \to B$. Moreover, any two $\mathcal{O}$-maps from $A$ to $B$ coincide on the non-free vertices.

Let $f : A \to A$ be a simplicial $\mathcal{O}$-map, where $A \in \mathcal{O}$. Then $f$ induces a map (here we denote by $Df$ the map which sends every edge $e$ to the first edge of the edge path $f(e)$) on the set of turns, sending every turn $(e_1, e_2)$ to the turn $(Df(e_1), Df(e_2))$. Then we usually say that the turn $(e_1, e_2)$ is legal if for every $k$ the turn $(Df^k(e_1), Df^k(e_2))$ is non-degenerate. This induces a pre-train track structure on the set of edges at each vertex. There are also different pre-train track structures and one of which we will use later, therefore we need the general definition.

**Definition 2.2.** (i) A **pre-train track structure** on a $G$-tree $T$ is a $G$-invariant equivalence relation on the set of germs of edges at each vertex of $T$. Equivalence classes of germs are called **gates**.

(ii) A **train track structure** on a $G$-tree $T$ is a pre-train track structure with at least two gates at every vertex.

(iii) A **turn** is a pair of germs of edges emanating from the same vertex. A **legal turn** is called a turn for which the two germs belong to different equivalent classes. A **legal path**, is a path that contains only legal turns.

A pre-train track structure induced by some $\mathcal{O}$-map is not always a train track structure, but there are some $\mathcal{O}$-maps (we call them optimal maps) which induce train track structures. But firstly we need the notion of PL maps (which corresponds to piecewise linear homotopy equivalence in the free case). We call a map between two elements of the outer space $\mathcal{P}L$, if it is piecewise linear and $\mathcal{O}$-map. We denote by $A_{max}(f)$ the subgraph
of $A$ consisting on those edges $e$ of $A$ for which $S_{f,e} = \text{Lip}(f)$. In fact this is the set of edges which are maximally stretched by $f$. Note that $A_{\text{max}}$ is $G$-invariant and that in literature the set $A_{\text{max}}$ is often referred to as tension graph.

As we have seen in the discussion above, for every map there is an induced structure. More specifically, if $A, B \in \mathcal{O}$ and $f : A \to B$ is a PL-map, then the pre-train track structure induced by $f$ on $A$ is defined by declaring germs of edges to be equivalent if they have the same non-degenerate $f$-image.

We are now in position to define optimal maps:

**Definition 2.3.** Let $A, B \in \mathcal{O}$. A PL-map $f : A \to B$ is not optimal at $v$ if $A_{\text{max}}$ has only one gate at $v$ for the pre-train track structure induced by $f$. Otherwise, $f$ is **optimal at $v$**. The map $f$ is optimal, if it is optimal at all vertices.

**Remark.** A PL-map $f : A \to B$ is optimal if and only if the pre-train track structure induced by $f$ is a train track structure on $A_{\text{max}}$. In particular if $f : A \to B$ is an optimal map, then at every vertex $v$ of $A_{\text{max}}$ there is a legal turn in $A_{\text{max}}$.

Note also that by [9], every PL-map is optimal at non-free vertices and for every $A, B \in \mathcal{O}$ there exists an optimal map from $A$ to $B$. Therefore we can always choose our $\mathcal{O}$-maps to be optimal.

### 2.2 Train Track Maps

In this section we will define the notion of a good representative of an outer automorphism. As we have seen there are representatives of every outer automorphism, but sometimes we can find representatives with better properties. These maps, which are called train track maps, are very useful and every irreducible automorphism has such representative (we can choose it simplicial as well).

For $T \in \mathcal{O}$ we say that a Lipschitz surjective map $f : T \to T$ **represents** $\phi$ if for any $g \in G$ and $t \in T$ we have $f(gt) = \phi(g)(f(t))$. (In other words, if it is an $\mathcal{O}$-map from $T$ to $\phi(T)$.) We give below the definition of a train
2.2 Train Track Maps

A train track map representing some outer automorphism. We are interested in these maps because we can control the cancellation (it is not possible to avoid it).

**Definition 2.4.** If \( T \in \mathcal{O} \) then a PL-map \( f : T \to T \), which representing \( \phi \), is a train track map if there is a train track structure on \( T \) so that

1. \( f \) maps edges to legal paths (in particular, \( f \) does not collapse edges)
2. If \( f(v) \) is a vertex, then \( f \) maps inequivalent germs at \( v \) to inequivalent germs at \( f(v) \).

In the free case we have the notion of an irreducible automorphism \( \phi \), if there is no \( \phi \)-invariant free factor up to conjugation (or equivalently the topological representatives of \( \phi \) haven’t non-trivial proper invariant subgraphs). Here we define the irreducibility of some automorphism relative to the space \( \mathcal{O} \).

**Definition 2.5.** We say \( \Phi \in Aut(G; \mathcal{O}) \) is \( \mathcal{O} \)-irreducible (or simply irreducible) if for any \( T \in \mathcal{O} \) and for any \( f : T \to T \) representing \( \Phi \), if \( W \subseteq T \) is a proper \( f \)-invariant \( G \)-subgraph then \( W/G \) is a union of trees each of which contains at most one non-free vertex.

We can also give an alternative algebraic definition, but we need the notion of free factor system. Suppose that \( G \) can be written as a free product, \( G = G_1 \ast G_2 \ast ... G_k \ast G_\infty \), where we allow the possibility that \( G_\infty \) is trivial. Then we say that the set \( A = \{ [G_i] : 1 \leq i \leq k \} \) is a free factor system for \( G \), where \( [A] = \{ gA^{-1} : g \in G \} \) is the set of conjugates of \( A \). Now we define an order on the set of free factor systems for \( G \). More specifically, given two free factor systems \( \mathcal{G} = \{ [G_i] : 1 \leq i \leq k \} \) and \( \mathcal{H} = \{ [H_j] : 1 \leq j \leq m \} \), we write \( \mathcal{G} \sqsubseteq \mathcal{H} \) if for each \( i \) there exists a \( j \) such that \( G_i \leq gH_jg^{-1} \) for some \( g \in G \). The inclusion is strict, and we write \( \mathcal{G} \sqsubset \mathcal{H} \) if some \( G_i \) is contained strictly in some conjugate of \( H_j \). We can see \( \{ [G] \} \) as a free factor system and in fact, it is the maximal (under \( \sqsubseteq \) free factor system. Any free factor systems that is contained strictly to \( \mathcal{G} \) is called proper. Note also that the Grushko decomposition induces a free factor system, which is actually the minimal free factor system (relative to
We say that \( \mathcal{G} = \{ [G_i] : 1 \leq i \leq k \} \) is \( \phi \)-\text{invariant} for some \( \phi \in \text{Out}(G) \), if \( \phi \) preserves the conjugacy classes of \( G_i \)'s. We are only interested for free factor systems that \( G_\infty \) is a finitely generated free group. In particular, we suppose that \( G = G_1 * G_2 * ... G_k * G_\infty \), and \( G_\infty = F_k \) for some f.g. free group \( F_k \). Associated to such a free factor system \( \mathcal{G} = \{ [G_i] : 1 \leq i \leq k \} \) we have the outer space \( \mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^{p}, F_k) \) and any (outer) \( \phi \in \text{Out}(G) \) leaving \( \mathcal{G} \) invariant, will act on \( \mathcal{O} \) in the same way we have described earlier.

**Definition 2.6.** Let \( \mathcal{G} \) be a free factor system of \( G \) as above and suppose it is \( \Phi \)-invariant for some \( \Phi \in \text{Out}(G) \). Then \( \Phi \) is called \textit{irreducible relative to} \( \mathcal{G} \), if \( \mathcal{G} \) is a maximal (under \( \sqsubseteq \)) proper, \( \Phi \)-invariant free factor system.

The next lemma confirms that the two definitions are related.

**Lemma 2.7.** Suppose \( \mathcal{G} \) is a free factor system of \( G \) with associated space of trees \( \mathcal{O} \), and further suppose that \( \mathcal{G} \) is \( \phi \)-invariant. Then \( \phi \) is irreducible relative to \( \mathcal{G} \) if and only if \( \phi \) is \( \mathcal{O} \)-irreducible.

Note that for an irreducible automorphism we can give a characterization of train track maps using the axis of the hyperbolic elements. More specifically, if \( \phi \) is irreducible, then for a map \( f \) representing \( \phi \in \text{Out}(G, \mathcal{O}) \), to be a train track map is equivalent to the condition that there is \( g \in G \) (hyperbolic element) so that \( L = \text{axis}_T(g) \) (the axis of \( g \)) is legal and \( f^k(L) \) is legal \( k \in \mathbb{N} \).

**Definition 2.8.** An outer automorphism \( \phi \in \text{Out}(G, \mathcal{O}) \) is called \textit{IWIP} (irreducible with irreducible powers or fully irreducible), if every \( \phi^k \) is irreducible relative to \( \mathcal{O} \).

The next theorem is very important since we can always choose representatives of irreducible automorphisms with nice properties, as in the free case. In particular, we can apply it for every power of some IWIP.

**Theorem 2.9.** Let \( \phi \in \text{Out}(G, \mathcal{O}) \) be irreducible. Then there exists a (simplicial) train track map representing \( \phi \).
The discussion above implies that we can always find an optimal train track representative of an irreducible $\phi \in \text{Out}(G, \mathcal{O})$. This map has the property that the image of every legal path (in particular, of edges) is stretched by a constant number $\lambda$ which depends only on $\phi$.

We close this subsection with an interesting remark.

**Remark.** Every outer automorphism $\phi \in \text{Out}(G)$ is irreducible relative to some appropriate space (or relative to some free product decomposition).

Everything in the present and the previous subsection about the outer space, $\mathcal{O}$ - maps and the train tracks and more can be found in [9].

### 2.3 Bounded Cancellation Lemma

Let $T, T' \in \mathcal{O}$ and $f : T \to T'$ be a map. If we have a concatenation of legal paths $ab$ where the corresponding turn is illegal, then it is possible to have cancellation between $f(a)f(b)$. But if we choose the map $f$ to be $\mathcal{O}$-map then the cancellation is bounded, with some bound that depends only on $f$ and not on $a, b$. In particular, we can define the bounded cancellation constant of $f$ (denoted it by $\text{BCC}(f)$) to be the supremum of all real numbers $N$ with the property that there exist $A, B, C$ some points of $T$ with $B$ in the (unique) reduced path between $A$ and $C$ such that $d_T(f(B), [f(A), f(C)]) = N$ (the distance of $f(B)$ from the reduced path connecting $f(A)$ and $f(C)$), or equivalently is the lowest upper bound of the cancellation for a fixed $\mathcal{O}$-map.

The existence of such number is well known, for example a bound has given in [13]:

**Lemma 2.10.** Let $T \in \mathcal{O}$, let $T' \in \mathcal{O}$, and let $f : T \to T'$ be a Lipschitz map. Then $\text{BCC}(f) \leq \text{Lip}(f)\text{qvol}(T)$, where $\text{qvol}(T)$ the quotient volume of $T$, defined as the infimal volume of a finite subtree of $T$ whose $G$-translates cover $T$.

We can also, exactly as in the free case, define a critical constant, $C_{\text{crit}}$ corresponding to a train track map.

Now suppose that $f$ is train track map with expanding factor $\lambda$. If we
take $a, b, c$ legal paths and $abc$ is a path in the tree, and let’s denote $l = \text{length}(b)$ the length of the middle segment. If we suppose further that satisfies $\lambda l - 2BCC(f) > l$, then iteration and tightening of $abc$ will produce paths with the length of the legal leaf segment corresponding to $b$ to be arbitrarily long. This is equivalent to requiring $l > \frac{2BCC(f)}{\lambda - 1}$, and the number $C_{crit} = \frac{2BCC(f)}{\lambda - 1}$ is the critical constant for $f$. For every $C$ that exceeds the critical constant there is $m > 0$ such that $b$, as above, has length at least $C$ then the length of the legal leaf segment of $[f^k(abc)]$ corresponding to is $b$ is at least $m\lambda^k l e n g t h(b)$. Therefore we can see that any path which contains a legal segment of length at least $C_{crit}$, has the property that the lengths of reduced $f$-iterates of the path are going to infinity.

2.4 N-periodic paths

A difference between the free and the general case is that there are not finitely many orbits of paths of a specific length, but it is true that there are finitely many paths that have different projection. Therefore the role of Nielsen periodic paths play the N-periodic paths that we define below.

**Definition 2.11.** (i) Two paths $p, q$ in $S \in \mathcal{O}$ are equivalent if they project to the same path in the quotient $S/G$.

(ii) Let $h : S \rightarrow S$ be a representative of some outer automorphism $\psi$, then we say that a path $p$ (suppose that the endpoints of $p$ are $h^k$ periodic, which means that $h^k(x) = gx$ for some $g \in G$) in $S$ is N-periodic, if the paths $[h^k(p)], p$ are equivalent.

**Geometric and non-Geometric automorphisms:** We will define here some notions for automorphisms that have been motivated by the properties of geometric and non-geometric automorphisms, respectively. The terminology also comes from the free case. In that case, we say that $\phi$ is geometric as it can be represented as a (pseudo-Anosov) homeomorphism of a punctured surface. It is well known that for the non-geometric case there is an integer $m$ such that it is impossible to concatenate more than $m$ indivisible Nielsen paths for every map $f$ which represents $\phi$. We will generalise this property in order to give our definitions. In particular:
**Definition 2.12.** We say that some \( \phi \) has the NGC property, if it is impossible to concatenate more than \( m \) indivisible \( N \)-periodic paths for every \( \mathcal{O} \)-map \( f \) which represents \( \phi \). Otherwise, we say that \( \phi \) has the GC property.

### 2.5 Relative train-track maps

Having good representatives of outer automorphisms, is very useful. If our automorphism is irreducible, it is possible to find train track representatives, as we have seen. But even in the reducible case we can find relative train track representative. The existence of such maps it follows from [9] or [4].

That we have is that every automorphism can be represented as an \( \mathcal{O} \)-map \( f : T \to T \) such that \( T \) has a filtration \( T_0 \subseteq T_1 \subseteq ... \subseteq T_k = T \) by \( f \)-invariant \( G \)-subgraphs, where \( T_0 \) contains every non-free vertex, we denote by \( H_r = cl(T_r - T_{r-1}) \) and we suppose that the transition matrix (as in the free case but we count orbits of edges) of every \( H_r \) is irreducible (or zero matrix) so we can correspond in every \( H_r \) some PF eigenvalue (let’s denote it \( \lambda_r \)) . In addition, \( f \) has some train track properties (such as mixed turns are legal and the map is \( r \)-legal). There is a very interesting corollary that we will use: for every edge-path \( a \) in \( H_r \), the reduced image of \( a, [f(a)] \), can be written as a concatenation of non-degenerate edge-paths in \( T_i - 1 \) and \( H_i \) with the first and the last contained in \( H_i \).

For such \( a \), we can distinguish between two cases for the strata: if there exists some edge of \( e \) in \( H_r \) such that \( [f(e)] \) contains at least two copies (orbits) of \( e \), then we say that the stratum is exponentially growing and we can see the \( r \)-lengths of images of edges in \( H_r \) expands by \( \lambda_r > 1 \) and in particular the lengths of reduced \( f \)-iterates of edges in \( H_r \) are going to infinity (using the train track properties). Otherwise, the stratum called non-exponentially growing and the map \( f \) (if we ignore the lower strata) is just a permutation of edges of the same length. An automorphism is called *exponentially growing* if some representative has at least one exponentially growing stratum. In other case, it is called *non-exponentially growing* automorphism.
2.6 Graph of Groups and Subgroups

We will recall only some facts for the graph of groups. For more about graph of groups and their subgroups, see [20]. In the special case that we are interested, a graph of groups can be defined as a finite connected graph $X$ (let call $\Gamma$ the underlying graph) for which in every vertex $v$ we correspond some (vertex) group $G_v$. We call non-free the vertices for which the corresponding group is non-trivial. Then the fundamental group of $X$, $\pi_1(X)$ is the free product of $\pi_1(\Gamma)$ (which is a f.g. free group) and the vertex groups.

We will use a specific kind of subgroups of $\pi_1(X)$. Let $\gamma$ be a loop in $v_0 \in \Gamma$. Then starting from $v_o$ and following the path of $\gamma$ we meet some non-free vertices (we can return back also, but we have always follow $\gamma$). So we can read words of a fixed form, and this process produces words of the fundamental group (we can see it as the group which it is consisted of all the words constructed as above but without fixing some loop $\gamma$). In fact, the set of all such words is a subgroup, which corresponds to $\gamma$.

3 Every two $O$ - maps coincide

In [9] it has been proved the existence of $O$-maps. We will prove that even if in the construction of such maps there is a lot of freedom, the reduced images of all of them coincide, up to bounded error. As a consequence we obtain that their lengths are comparable.

Theorem 3.1. Let $f, h : A \to B$ be $O$ - maps. Then there exists $C$ (depends only on $f$, $h$ and $A$) so that for every path $L$ in $A$, if we delete an initial and a terminal subpath of $[f(L)]$ and $[h(L)]$ with length at most $C$, then the resulting paths are equal.

Proof. Firstly, we suppose that there is at least one orbit of non-free vertex. Let call it $v$. Then we have that $f(v) = h(v)$. Then if $L = [a, b]$ is an edge - path, then we can find in distance at most $C$, which is bounded by the $\text{vol}(A/G) = \text{vol}(\Gamma)$, some vertices $g_1v, g_2v$ near $a, b$ respectively such that $[a, b] \subseteq [g_1v, g_2v]$. Then $[f(L)]$ is contained in $[f(g_1v), f(g_2v)]$, except
possibly some segments near \(a, b\) of length at most \(C' = CLip(f)\). This constant is bounded above by \(vol(\Gamma)Lip(f)\). Similarly, for \([h(g_1 v), h(g_2 v)] = [f(g_1 v), f(g_2 v)]\) for a constant \(C'' = vol(\Gamma)Lip(h)\) and therefore \([f(L)] = [h(L)]\) except possibly some segments near \(a, b\) which are bounded by \(C_1 = max(C', C'')\) (by definition depends only on Lip(f), Lip(h), vol(\Gamma) = vol(A/G))

If there are no non-free vertices, we are in the free case and the result is well known.

Note also that it is easy to see that every \(O\)-map is a quasi-isometry.

\section{Characterisation of the kernel of the action}

If some automorphism of \(F_n\) acts on \(CV_n\) leaving every \(F_n\)-tree invariant then this implies that the automorphism must be the \(Id_{Out(F_n)}\), but this is not the true in the free product case. There are non-trivial automorphisms that act trivially on \(O\). We will give a characterisation of the kernel, proving that is generated by the automorphisms of the free factors.

Let’s denote \(KA\) the kernel of the action of \(Out(G;O)\) on \(O\).

\begin{proposition}
It holds that \(KA = \bigoplus_{i=1}^{m} (Aut(G_i))Inn(G)\).
Namely, \(\phi \in KA\) iff \(\phi\) is conjugate to an automorphism which is the identity restricted on \(F_r\) and it is an automorphism of \(G_i\) restricted to each \(G_i\).
\end{proposition}

\textbf{Proof.} Firstly let choose \(\phi \in ker(p)\), then for every \(T \in O\) it holds that \(T = \phi(T)\) (up to \(G\)-equivariant isometry).

We choose the tree \(T\) corresponding to the graph of groups with exactly one free vertex \(w\), \(m\) non-free vertices \(v_1, ..., v_m\) corresponding to \(G_i\), \(i = 1, ..., m\), each of them has valence one (the only edge that starting from \(v_i\) is the edge that connects \(v_i\) to \(w\)) and \(r\) loops at \(w\). So we have \(m + r\) edges, and we denote by \(e_i\) the edges from \(w\) to \(v_i\), for \(i = 1, ..., m\) and \(e_i\) are loops at \(w\), for \(i = m + 1, ..., m + r\). We put length \(i\) at each \(e_i\).

We will use the same notation for the lifts of vertices and edges in the tree \(T\). Then we have that if \(f\) is a \(G\)-equivariant isometry from \(T\) to \(\phi(T)\) in particular is a homeomorphism, and it must send \(w\) to some element in the same orbit as \(w\), and changing \(f\) with some appropriate representative in the same conjugacy class, we can suppose that \(f(w) = w\).
Now $f$ must fix every $v_i$, since must send $v_1$ to a non-free vertex with distance 1 from $w$ (in general, $i = d(w, v_i) = d(f(w), f(v_i)) = d(w, f(v_i))$) and therefore the only choice is $v_1$. Similarly, for the rest of $v_i$'s and so $f$ fix every $e_i, i = 1, \ldots, m$. Moreover, $w$ has at distance $m + i$ the vertices of the form $x_iw$, where $x_i$ are the elements of the basis of the free group, and using again that $f$ is isometry and homeomorphism (the neighborhood of a vertex is sent by $f$ to a neighborhood with the same number of edges emanating from there) we have that $f(x_iw) = x_iw$ and $f(e_i) = e_i$ is true for every $i$.

Now we note that $\phi(x_i)w = \phi(x_i)f(w) = f(x_iw) = x_iw$ and the fact that $w$ is free implies that $\phi(x_i) = x_i$ for every $i = 1, \ldots, r$, and so $\phi$ is identity restricted on $F_n$. For non-free vertices, we have that for $g \in G_i$ we look at $v_i$ and there since $f(v_i) = v_i$, we have that the path $f(ge_i)$ emanating from $f(ge_i) = f(v_i) = v_i$, and therefore is of the form $g'e_i$ for some $g' \in G_i$, so using the $G$-equivariance of $f$, $\phi(g)e_i = g'e_i$ so $\phi(g) = g' \in G_i$. This is true for every $g \in G_i$, and then if we restrict $\phi$ as an homomorphism from $G_i$ to $G_i$, but the fact that $\phi$ is automorphism implies that each restriction is surjective ($\phi$ is surjective), and so an automorphism of $G_i$. Therefore $\phi \in \bigoplus_{i=1}^{m} (Aut(G_i) \cdot Inn(G))$.

To prove the other direction, suppose that $\phi \in \bigoplus_{i=1}^{m} (Aut(G_i) \cdot Inn(G))$ and let $T$ be a tree of $O$. We can choose the representative of the same outer automorphism with the property that $\phi_{G_i} : G_i \rightarrow G_i$ for every $i$ and $\phi_{F_r} = id_{F_r}$. Let $T \in O$ and consider a map $f$ which is identity on the fundamental domain (one edge and vertex of each orbit), $f(v_i) = v_i, f(w_i) = w_i$ and $f(e_i) = e_i$, and we extend it $G$-equivariantly. This map is an $O$-map, since if $v$ is fixed by $g$ iff is fixed by $\phi(g)$, by choice of $\phi$ and by construction of $O$-maps in which it is enough to send the non-free vertices as above and free vertices anywhere we want. So $f$ is continuous and we will prove that it is an isometry.

Note that locally at non-free vertices of the fundamental domain if $f(v_i) = g_iv_i$ for $g_i \in G_i$, then $f(g_i e) = \phi(g_i) e$ for every edge $e$ emanating from $v_i$. In general every edge of $T$ is of the form $ge$, for $g \in G$ and $e$ edge of the fundamental domain, then $f(ge) = \phi(g) e$, so it sends edges to edges of the same orbit, in particular it is isometry on edges. Moreover,
5 Laminations

In this section we define the notion of the lamination associated to an IWIP. Firstly, we use the train track maps to define the lamination in a specific tree and the existence of $O$-maps between every two trees allows us to generalise it to every tree.

5.1 Construction of the lamination and properties

Let $\phi \in \text{Out}(G, \mathcal{O})$ be an irreducible automorphism, with irreducible powers and $f : A \to A$ for some $A \in \mathcal{O}$ be a train track map which represent $\phi (f(gx) = \phi(g)f(x))$ and we can suppose that $f$ expand the length of the edges by a uniform factor $\lambda > 1$ (This can be done if we choose an optimal train track that represents $f$, this is possible by [9]).

Let $x \in A$ be any periodic point ($f^k(x) = x$, for some $k$), in the interior of some edge (in general there exists $x$ s.t. $f^k(x) = gx$ since the quotient is finite, but we can change the space $A$ with changing isometrically the action with $\phi_g(A)$ and there the requested property holds). Now let $U$ some $\epsilon$-neighbourhood, for some small $\epsilon$ (we want the neighbourhood to be contained in the interior of the edge) and then there is some $N > 0$ s.t. $f^N(U) \supset U$.

We can choose an isometry $\ell : (-\epsilon, \epsilon) \to U$ and extend it to the unique isometry $\ell : \mathbb{R} \to A$ s.t. $\ell(\lambda^N t) = f^N(\ell(t))$ and then we say that $\ell$ is obtained by iterating a neighbourhood of $x$. 

$f$ doesn’t fold turns (this is the case by construction for edges of different orbits and for every turn of the form $e, ge$ for some $g \in G_{\phi(e)}$, is not possible to have $f(e) = \phi(g)f(e)$ since $\phi(g) \neq 1$ and the action on edges is free) and therefore we have that $f$ sends edge paths to edge paths with the same orbits of edges and in the same order ($e_1, ..., e_n$ to $g_1 e_1, ..., g_n e_n$ for some $g_i \in G_{\phi(e_i)}$), so in particular sends the unique reduced path connecting two points to the unique reduced path connecting their images and therefore $f$ is an isometry. 

\[ \square \]
Definition 5.1. We say that two isometric immersions $A : [a,b] \to A$ and $B : [c,d] \to A$ are equivalent, if there exists an isometry $q : [a,b] \to [c,d]$ s.t the triangle commutes ($Bq = A$). This relation is an equivalence relation on the set of isometric immersions from a finite interval to $A$.

- If $P$ is an equivalence class and we choose a representative of that class $\gamma : [a,b] \to A$, we can define $f(P)$ as the equivalence class of $f \gamma : [a,b] \to A$, pulled tight and scaled so it is an isometric immersion.

- A leaf segment of an isometric immersion $\mathbb{R} \to A$ is the equivalence class of the restriction to a finite interval.

Let $\ell$ be an isometric immersion, then we can correspond the $G$-set $I_\ell$ (of the leaf segments of $\ell$) to $\ell$. We can also define an equivalence relation on the set of isometric immersions from $\mathbb{R}$ to $A$. We say that two lines $\ell, \ell'$ are equivalent if $I_\ell = I_{\ell'}$.

Definition 5.2. Let $\ell, \ell'$ be two isometric immersions from $\mathbb{R}$ to $A$, then we say that they are equivalent if $I_\ell = GI_{\ell'}$. Namely, we say that are equivalent if for every leaf segment $P$ of $\ell$ there is an element $g \in G$ and $Q$ a leaf segment of $\ell'$ s.t. $P = gQ$ and vice versa (or equivalently every l.s. of $\ell$ is mapped by some $g$ to a l.s. of $\ell'$)

Remark. Here note that it is obvious that if $\ell(t) = g\ell'(t)$ ($\ell, \ell'$ are in the same orbit), then $\ell$ and $\ell'$ are equivalent.

We will prove that if we construct any other line obtained by iterating a neighbourhood of any other periodic point then it is equivalent with $\ell$.

Lemma 5.3. Let $y \in A$, be any other $f$-periodic point in the interior of some edge of $A$ and $\ell'$ is the obtained by iterating of some neighborhood of $y$. Then $\ell$ and $\ell'$ are equivalent.

Proof. We will show that any l.s. of $\ell$ is mapped by some element of $G$ to a l.s. of $\ell'$, then the converse follows by symmetry.

Since $f$ represents an irreducible automorphism (and the same holds for
every power of \( f \), \( \ell' \) contains some orbit of every edge, so in particular if \( x \) is contained in the interior of the edge \( e \) we have that there exists some \( g \in G \), s.t. \( gx \in ge \subseteq \ell' \). So there is an isometry \( \psi : (-\epsilon, \epsilon) \to (a - \epsilon, a + \epsilon) \) with the property \( \ell(t) = g\ell'(\psi(t)) \).

Let \( N' \) be a natural number s.t. \( \ell' (\lambda^N t) = f^{N'}(\ell(t)) \) and then for any \( t \in U \) (U as in the definition) we have that \( \ell(\lambda^k \lambda^N t) = f^{kN'}(\ell(t)) = f^{kN'}(g\ell'(\psi(t))) = \phi^{kN'}(g) f^{kN'}(\ell'(\psi(t))) = \phi^{kN'}(g) \ell' (\lambda^k \lambda^N \psi(t)) \).

But since every prechosen interval is contained in some interval of the form \( \lambda^k \lambda^N (-\epsilon, \epsilon) \) for large \( k \), we have that for every l.s. of \( \ell \) is mapped by some \( \phi^{kN'}(g) \in G \) to some l.s. of \( \ell' \).

We are now in position to define the stable lamination corresponding to \( A \).

**Definition 5.4.** The **stable lamination** in \( A \)-coordinates \( \Lambda = \Lambda^+_f(A) \) is the equivalence class of isometric immersions from \( \mathbb{R} \) to \( A \) containing some (and by previous lemma any) immersion obtained as above (by iterating a neighborhood of a periodic point). We call the immersions representing \( \Lambda \) **leaves** of \( \Lambda \) and the leaf segments (l.s.) of some leaf of \( \Lambda \) **leaf segments** of \( \Lambda \) (by definition of the equivalence relation, every leaf of \( \Lambda \) contains some orbit of every l.s. of \( \Lambda \)).

Note that the every leaf of the lamination project to the same bi-infinite path in the quotient.

We will list some useful properties of the stable lamination.

**Proposition 5.5.** (i) Any edge of \( A \) is a leaf segment of \( \Lambda \).

(ii) Any \( f \)-iterate of a leaf segment is a leaf segment.

(iii) Any subsegment of a leaf segment is a leaf segment.

(iv) Any leaf segment is a subsegment of a sufficiently high iterate of an edge.

(v) For any leaf segment \( P \) there is a leaf segment \( P' \) such that \( f(P') = P \).
(vi) Let $a$ be a periodic segment ($o(a) = gt(a)$ for some $g \in G$) which crosses $k$ edges (counted with multiplications). Then any $f$-iterate of $a$ (pulled tight) can be written as concatenation of less or equal $k$ leaf segments.

Proof.  
(i) This is clear by the proof of the previous lemma, since $f$ represents an irreducible automorphism and this implies that every $\ell$ contains orbits of every edge, so if $ge$ is contained in $\ell$ then $e$ is contained in $g^{-1}\ell$ which is equivalent to $\ell$ thus is a leaf of $\Lambda$, and as consequence $e$ is leaf segment of a leaf therefore it is l.s. of $\Lambda$.

(ii) Firstly, we note that if $x$ is $f$-periodic then $f(x)$ is $f$-periodic with the same period (in fact every $f^m(x)$ is periodic) and let’s denote $\ell'$ the isometric immersion constructed as above, so if $P$ is a l.s. of $\ell$, then $f(P)$ is a l.s. of $\ell'$ but since $\ell, \ell'$ are equivalent by lemma, we have that $\ell'$ is a leaf of $\Lambda$ and therefore $f(P)$ is a l.s. of $\Lambda$. So we can do it for every iterate of $f$.

(iii) This is obvious, since we restrict the isometric immersion to the subsegment and it is a l.s. of a leaf of $\Lambda$ and as a consequence a l.s. of $\Lambda$.

(iv) We have that $f$ expands the length of every edge by $\lambda$, but we can use for representative the isometric immersion constructed as above (by iterating a periodic neighborhood) and the edge in which the periodic point belongs, then by construction of $\ell$ every l.s. is contained in an high iterate of this edge. For any other representative $\ell'$ now we can translate $\ell$ as above (by some element $g \in G$) to have a common segment that contain the prechosen l.s. and the proof reduced to the first case.

(v) Let $P$ be a l.s. of $\Lambda$. By (iv) we have that there exists some iterate of an edge and so by $\ell$ an iterate of a l.s. $P''$ s.t. $P$ is contained in $f^m(P'')$ and since iterates of l.s. are l.s. and subsegments are l.s. as well, we have that there is $P'$ subsegment of $f^{m-1}(P'')$ with the property $P = f(P')$
(vi) This is obvious since edges are l.s. and $f$-iterates of l.s. are l.s.. □

We note that (ii) implies that $f^k(\ell)$ is a leaf of the lamination, for every $k$.

**Definition 5.6.** We say that a sequence $a_i$ of isometric immersions $[0, 1]_i \to A$ (where the metric on $[0, 1]_i$ is scalar multiple of the standard part which depends on $i$), (weakly) converges to $\Lambda$, if for every $L > 0$ the ratio,

$$\frac{m(\{x \in [0, 1]_i| \text{the } L-\text{nbhd of } x \text{ is a leaf segment}\})}{m([0, 1]_i)}$$

converges to 1.

**Proposition 5.7.** Suppose that $a$ is a periodic segment in $A$ (the period of the axis of some hyperbolic element) which is not $N$-periodic. Then the sequence (of tightenings of $f^i(a)$), $[f^i(a)]$ weakly converges to $\Lambda$.

Note that such hyperbolic elements always exist. For example the basis elements of the free group, are not $N$-periodic by definition of irreducibility.

**Proof.** Suppose that $a$ can be written as a concatenation of $k$ l.s. then we have $k-1$ illegal turns (we don’t count the endpoints) and since $f$ is train track we have the number of illegal turns in $[f^k(a)]$ is non-increasing so it contains less than or equal to $k-1$ l.s.. Therefore if the lengths of reduced iterates of $a$ is bounded, and since there are finitely many inequivalent paths with length less than or equal to a specific number, we have that $a$ is $N$-preperiodic and therefore periodic because $a$ corresponds to a group element, which leads to a contradiction to the hypothesis. Therefore some $[f^i(a)]$ contains arbitrarily long legal segments ($> C_{crit}$), and since the length of $[f^j(a)]$ expands for large $j$, we have that there are finitely many $L$-nbds contain points without the requested property (of the endpoints of the concatenation of l.s. so at most $k$) and the measure of these is at most $2Lk$, as a consequence the ratio converges to 1. □
Definition 5.8. An isometric immersion \( l : \mathbb{R} \rightarrow A \) is quasiperiodic (qp), if for every \( L > 0 \) there exists \( L' > 0 \) s.t. for every l.s. \( P \) of \( \ell \) of length \( L \) and for every l.s. \( Q \) of length \( L' \) there is \( g \in G \) s.t. \( gP \subseteq Q \) (\( P \) is mapped by \( g \) to a subsegment of \( Q \)).

Proposition 5.9. Every leaf of \( \Lambda \) is quasiperiodic.

Proof. We will first prove it for \( \ell \) that has constructed by iterating neighbourhood of a periodic point.

We first verify it for leaf segments \( \Pi \) that consists of only two edges.

If we choose \( L_0 > 2\max_e(len(e)) \), then if a l.s. \( P \) has length \( \geq L_0 \), then it contains a subleaf segment which is an edge. Then there is \( N \) (we can also choose it to be multiple of \( k \)) s.t. \( f^N \) restricted to any edge crosses some orbit of every turn that they crossed by leaves of \( \Lambda_f(A) \). So in particular for the chosen \( \Pi \) the iterate of \( f \) takes the orbit of that turn, so there exists \( g \in G \) such that \( \Pi \subseteq gf^N(P) \).

Now if \( P' \) is any l.s. of length \( \lambda^N L_0 \), then \( P' = f^N(P) \) for some \( P \) l.s. of length \( L_0 \) and therefore \( \Pi \subseteq gP' \).

For the general case, let \( L > 0 \) be given, then there is \( M > 0 \) (we choose it to have the property \( \lambda^{-M}L < 2\min(len(e)) \)) s.t. any l.s. of length \( \leq \lambda^{-M}L \) is a subsegment of a two-edge l.s. as above and let \( L' = \lambda^{M+N}L_0 \).

So let \( P \) be a l.s. of length \( L \) and \( P' \) be a l.s. of length \( L' \). Then by the properties we have that \( P = f^M(\Pi) \) where \( \Pi \) is contained to a l.s. as in the special case (by the choice of \( M \), since \( \Pi \) has length \( \lambda^{-M}L \)), and similarly \( P' = f^M(\Pi') \) for a l.s. \( \Pi' \) of length \( \lambda^N L_0 \). By the special case we have that \( \Pi \subseteq g\Pi' \) and this implies that \( P = f^M(\Pi) \subseteq \Phi^M(g)f^M(\Pi') = \Phi^M(g)P' \).

Since \( \ell \) is \( \Phi^M(g) \)-invariant, we have the requested property.

For any other equivalent isometric immersion \( \ell' \), if we have \( P \) l.s. of length \( L \) and \( Q \) l.s. of length \( L' \) then we can find an isometric immersion \( \ell \) like the first case with \( Q \) as common segment. Then by the equivalence there exists \( g_1 \in G \) s.t. \( g_1P \) is l.s. of \( \ell \), and by quasiperiodicity of \( \ell \), there is \( g_2 \) s.t. \( g_2P \subseteq Q \) and \( g_2P \) is a l.s. of \( \ell' \), so we have that \( \ell' \) is quasiperiodic. 

\( \square \)
5.2 Lamination in different trees

Suppose that \( f : A \to A \) and \( \Lambda^+_f(A) \) as above and \( B \in \mathcal{O} \). Then we know that there exists an optimal map (in particular \( \mathcal{O} \)-map) \( \tau : A \to B \). Then for any immersion \( \ell : \mathbb{R} \to A \) we denote by \( \tau(\ell) : \mathbb{R} \to B \) the unique (up to precomposition by an isometry of \( \mathbb{R} \)) pulled tight to be an isometric immersion corresponding to \( \tau \ell \).

Lemma 5.10. • If \( \ell, \ell' : \mathbb{R} \to A \) are equivalent leaves, then \( \tau(\ell), \tau(\ell') \) are equivalent.

• If \( \ell \) is quasiperiodic, then \( \tau(\ell) \) is quasiperiodic.

Proof. Every optimal map \( \tau \) by [9], can be factored as the composition of a homeomorphism and a finite sequence of folds. We have just to prove that the lemma is true for homeomorphism and folds.

Firstly, let suppose that \( \tau \) is homeomorphism. In particular \([\tau(\ell)] = \tau(\ell)\) and the same holds for \( \ell' \) as well.

Let \( P' \) is a l.s. of \( \tau(\ell) \), then there is some l.s. of \( \ell \) \( P \) s.t. \( P' = \tau(P) \), so there is a translation of \( P \) by some element of the group, \( gP \) which is contained in \( \ell' \), therefore \( \tau(gP) = gP' \) is contained in \( \tau(\ell') \). By symmetry, this argument holds and vice versa so \( \tau(\ell) \) and \( \tau(\ell') \) are equivalent.

Suppose now that \( \ell \) is quasiperiodic, fix a \( L > 0 \) let \( P' \) l.s. of \( \tau(\ell) \) of length \( L \). Then there is a l.s. \( P \) of length at most \( K \) (by Bounded Cancellation Lemma there exists such \( K \) which doesn’t depend on \( P \) but only on \( L \)) s.t. \( P' = \tau(P) \). Then we can define \( L'' = L'\text{Lip}(\tau) \), where \( L' \) is the constant corresponding by quasiperiodicity to \( K \) and we have that if we choose any \( Q' \) l.s. of \( \tau(\ell) \) of length \( L'' \) then there exists a l.s. \( Q \) of \( \ell \) of length at least \( L' \) s.t. \( \tau(Q) = Q' \). Then \( Q \) contains orbits of any l.s. of length at most \( K \), in particular it contains some translation of \( P \) for some \( g \in G \) and therefore as above \( Q' \) contains some translation of \( P' \). So \( \tau(\ell) \) is quasiperiodic.

We suppose that \( \tau \) is an equivariant isometric simple fold of some segments starting from the same point \( v \) and has the same \( \tau \)-image, let call them \( a, b \) and \( c \) be the corresponding segment in the quotient.

For the first statement, we note that is obvious for a l.s. of \([\tau(\ell)]\) which don’t contain some orbit of \( c \), since there \( \tau \) is the identity. On the other
hand, if $P'$ is l.s. of $\tau(\ell)$ which contains some orbits of $c$, then there exists $P$ which contain the same number of orbits as the folded turn and $[\tau(P)] = P'$ (it is concatenation of the segments before and after the folds). Since $\ell, \ell'$ are equivalent we have that we can find $g \in G$ s.t. $gP$ is contained in $\ell'$, then $[\tau(gP)]$ is a a l.s. of $[\tau(\ell')]$. But $[\tau(gP)]$ is just a translation (by $g$) of $\tau(P)$, and therefore as above we obtain that $[\tau(\ell)], [\tau(\ell')]$ are equivalent.

For the quasiperiodity of $[\tau(\ell)]$ we fix a number $L > 0$ and we say $M$ the maximum number of orbits of $v$ which there are in a segment of length $L$, and $L'$ is the number corresponds by quasiperiodicity for $L'' = L + 2M\text{len}(a)$. Now let $P'$ be a l.s. of length $L$, then there is $P$ which contains the same number of orbits of the the folded turn and $[\tau(P)] = P'$ as above. Then $P$ has length at most $L''$, some translation of it is contained in every l.s. of $\ell$ of length $L'$. Now let choose $Q$ any l.s. of $[\tau(\ell)]$ of length $L'$ then the preimage has length at least $L'$, and therefore the preimage have the requested property and so $Q$ contains a translation of $P'$ as above.

**Definition 5.11.** The stable lamination of $f : B \to B$ in the $B$-coordinates is the equivalence class $\Lambda_f^+(B)$ containing $\tau(\ell)$ for some (and by previous lemma any) leaf of $\Lambda_f^+(A)$.

Using again the property that $\tau$ is factored as the composition of a homeomorphism and a finite sequence of folds combined with the result for the $\Lambda_f^+(A)$, we have the following proposition.

**Proposition 5.12.** Let $a$ be a periodic segment in $A$, which is not $N$-periodic. Then the sequence $\{[\tau(f^i(a))]\}$ weakly converges to $\Lambda_f^+(B)$

**Lemma 5.13.** Suppose that $h : B \to B$ is any other train track map representing $\Phi$. Then $\Lambda_f^+(B) = \Lambda_h^+(B)$

**Proof.** Let $a$ be a periodic segment as in 5.7 and 5.12. Then the sequence $[\tau(f^i(a))], [h^i(\tau(a))]$ weakly converges by the propositions to $\Lambda_f^+(B)$ and to $\Lambda_f^+(B)$, respectively by the previous lemmas. But $\tau f^i, h^i \tau$ are $O$-maps from $A$ to $\phi(B)$, so their reduced images coincide in every path, after deleting some bounded segments near endpoints. Then there are leaves $\ell, \ell'$ of $\Lambda_h^+(B)$ and $\Lambda_f^+(B)$ respectively with arbitrarily long common leaf segments. Since they are both quasiperiodic, it follows that they are equivalent. Indeed, let
Let \( \phi \) be an IWIP and \( f : T \to T \) be an optimal train track representative of \( \phi \), in particular we suppose that \( f^k \) has irreducible transition matrix, for every \( k \).

We denote by \( \mathcal{IL} \) the set of stable laminations \( \Lambda_\phi^+ \), as \( \phi \) ranges over all IWIP automorphisms relative to \( \mathcal{O} \). The group \( \text{Out}(G, \mathcal{O}) \) acts on \( \mathcal{IL} \) via

\[
\psi \Lambda_\phi^+ = \Lambda_{\psi \phi \psi^{-1}}^+
\]  

(6.1)

More specifically, if \( \ell \) is a leaf of \( \Lambda_\phi^+ \) in the \( S \)-coordinates and \( h : S \to S \) an \( \mathcal{O} \) map representing \( \psi \), then \([h(\ell)]\) represents a leaf of \( \Lambda_{\psi \phi \psi^{-1}}^+ \).

We are interested to study the stabiliser of the action for a fixed laminate. Note that every element of the centraliser of \( \phi \) fixes \( \Lambda_\phi^+ \). Therefore, the centraliser \( C(\phi) \) of the IWIP \( \phi \) is a subgroup of \( \text{Stab}(\Lambda) \).

We will equip \( T \) with a specific train-track structure, the \textit{minimal train-track structure}; more specifically we declare a turn legal if it is crossed by some leaf of \( \Lambda_\phi^+ \). Equivalently, the properties of the laminate imply that a turn is legal iff there is a \( f \)-iterate of an edge of \( T \) that crosses the turn.

We fix the above notation for the rest of the sections.
7 Subgroups carrying the lamination

This section is devoted to prove that it is not possible for a proper subtree to contain every leaf of the lamination. Moreover, we will prove that every relative train track representative of some automorphism of the stabiliser, after passing to some power, induces the identity on the quotient restricted to any proper invariant subraph (which is union of strata).

Definition 7.1. Let $A$ be a subgroup of $G$ of finite Kurosh rank, and let’s denote $T \in \mathcal{O}$ and $T_A$ the minimal invariant $A$-subtree. We suppose that for every $v \in V(T_A)$, $\text{Stab}_A(v) = \text{Stab}_G(v)$. Then we say that $A$ carries the lamination $\Lambda$, if there exist some leaf $\ell$ of $\Lambda$ is contained in $T_A$.

Remark. (i) Every two leaves of the lamination project to the same bi-infinite path in $\Gamma$.

(ii) For every vertex $v$ of $T$ there exist $g \in G$ s.t. $gv \in T_A$.

Proposition 7.2. If $A$ is a subgroup of $G$, as in the previous definition, which carries $\Lambda^+_{\phi}$ then $A$ has finite index in $G$.

Proof. Let $f : T \rightarrow T$ be a train-track representative of $\phi$, and let $H \rightarrow \Gamma$ be an isometric immersion corresponding to $A \leq G$. Then by our assumptions $H$ is finite graph of groups and by the remarks contains every non-free vertex. Therefore (using also the assumption that the corresponding vertex groups are full), we can complete the immersion by adding vertices (with trivial vertex group) and edges to a connected finite-sheeted covering space $p : \Gamma' \rightarrow \Gamma$ and therefore we have that $T' = T$ (the universal trees are the same).

Now we know that if $A$ has infinite index, then we are really adding new edges in $\Gamma'$ or equivalently we add new orbits of edges in $T$. But then using irreducibility we can reach a contradiction.

More specifically, we choose $e$ (edge of $T$) such that $f(e)$ starts with $e$. Then for every $n$ the path $f^n(e)$ is a path of $T_A$. So if we choose any edge $e_1$ (lift of some edge in $\Gamma' - H$) there does not exist $n$ and $g \in G$ such that $f^n(ge)$ passes through $e_1$ (since $e_1$ is in different orbit of edges in $T_A$), but this contradicts the fact that the transition matrix corresponding to $f$, denote
it by $A(f)$, is irreducible. As a consequence, $A$ must have finite index in $G$. \hfill\qed

**Proposition 7.3.** Let $\psi \in \text{Stab}(\Lambda)$, and let $h : S \rightarrow S$ be a relative train-track representative of $\psi$. Then some iterate of $h$ induces the identity on the quotient on every proper $h$-invariant subgraph of $S$ (without free vertices of valence 1) that is a union of strata.

**Proof.** Let $\ell$ be a leaf in $S$-coordinates and let $S_0$ be a proper $h$-invariant subgraph. It is not possible for $\ell$ to contain arbitrarily long segments of a proper subgraph since then the quasiperiodicity implies that $\ell$ is contained in that subgraph which is a contradiction to the previous proposition. Therefore $\ell$ is a concatenation of non-degenerate segments in $S_0$ and in $S - S_0$ (otherwise would lift to a proper subgraph of $H$, which is impossible as we have noticed). Now we have that all $S_0$-segments are $N$-preperiodic or else $h$-iteration will produce arbitrarily long leaf segments contained in $S_0$ contradicting quasiperiodicity. The same argument implies that there is an upper bound to the length of both $S_0$ and $S - S_0$ segments, and hence only finitely many inequivalent segments occur (since there are finitely many lengths corresponding to edge-paths of bounded length in the quotient).

We can start with the disjoint union $X$ of copies of the segments and the natural immersion $X \rightarrow S$ and we identify two endpoints of $X$ if they are mapped to the same point of $S$. Then fold to convert the resulting map to an immersion $\pi : X' \rightarrow S$. But $\ell$ lifts to $X'$ (by construction) and so by previous proposition we have again that $X' = S$ (it corresponds to a finite covering space of graph of groups). In particular, any periodic segment (lift of some loop) in $S_0$ lifts to $X'$. Consequently, this segment is a concatenation of paths in $S_0$ each of which is $N$-preperiodic, and therefore the periodic segment is $N$-preperiodic, and so $N$-periodic (every $N$-preperiodic segment that corresponds to an element of the group is $N$-periodic, since every periodic segment corresponds to a group element and $h$ represents $\psi$). Thus every periodic segment $a$ in $S_0$ is equivalent to some power $h^k(a)$ (and there is some bound for the powers) and hence for some $k$, $h^k$ restricted to $S_0$ induces the identity on the quotient. \hfill\qed
8 \( \lambda \)-map

In this section we will see that we can define a homomorphism from the stabiliser of the lamination to \( \mathbb{R} \) and that the \( KA \) is contained in the kernel of this homomorphism.

**Lemma 8.1.** Suppose that \( h : S \to S \) an \( O\)-map represents \( \psi \in \text{Out}(G, O) \). Then there exists a positive number \( \lambda = \lambda(h, \Lambda) \) such that for every \( \epsilon > 0 \) there is \( N > 0 \) so that if \( L \) is a leaf segment of \( \Lambda \) of length \( > N \), then

\[
\left| \frac{\text{length}(\text{h}(L))}{\text{length}(L)} - \lambda \right| < \epsilon
\]

**Proof.** We note that since \( f \) is IWIP, we have that the transition matrix \( M = A(f) \) is irreducible (as it is every power of \( M \)) and therefore we can apply the Perron-Frobenius theorem to \( M \), as a consequence we have that long leaf segments of \( \Lambda \) cross orbits of edges of \( T \) with frequencies close to those determined by the components of the PF eigenvector.

Now fix large \( k \) and then large l.s. are concatenation of l.s. of the form \( f^k(e) \), for some edges of \( T \), each orbit of edges with definite frequency.(For \( k = 1 \) this is the statement above, for \( k > 1 \) apply P.F theorem for \( f^k \)).

If \( M \) is large enough, then for any l.s. \( L \) with \( \text{length}(L) > M \) we can think \( L \) as concatenation of l.s. of the form \( f^k(e) \) (there are possible some shorts segments contained in the first and the final segment, which are not of this form but we can ignore them since their contribution in lengths is negligible).

Now let \( C \) be the bounded cancellation constant for \( h : T \to T \), and let’s denote \( l_e = \text{len}(f^k(e)) \), \( l^h_e = \text{len}(h(f^k(e))) \), \( N_e \) be the number of occurrences of orbits of \( f^k(e) \) in \( L \) and \( N = \sum N_e \), then we have that \( \frac{N_e}{N} \to r_e \), as \( \text{len}(L) \to \infty \) (\( r_e \) is the PF component of the eigenvector that corresponds to \( e \)) by the PF theorem.

Note that the numbers \( N_e, l_e, l^h_e \) depends on \( k \), so we define \( a_k = \frac{\sum r_e l^h_e}{\sum r_e l_e} \). We have that \( \text{len}(L) = \sum N_e l_e \) and by bounded cancellation lemma:

\[
\frac{\sum N_e (l^h_e - 2C)}{\sum N_e l_e} \leq A_M = \frac{\text{len}(h(L))}{\text{len}(L)} \leq \frac{\sum N_e l^h_e}{\sum N_e l_e} \tag{8.1}
\]

and subdividing the sums by \( N \) we have that

\[
\frac{\sum \frac{N_e}{N} l^h_e - 2C \frac{N_e}{N}}{\sum \frac{N_e}{N} l_e} \leq A_M = \frac{\text{len}(h(L))}{\text{len}(L)} \leq \frac{\sum \frac{N_e}{N} l^h_e}{\sum \frac{N_e}{N} l_e} \tag{8.2}
\]
where the term $2C\sum_{N\in\mathcal{N}}$ converges to 0 as $k \to \infty$ and as we noted above $\frac{N}{N} \to r$, as $\text{len}(L) \to \infty$. As a consequence, for every $\epsilon$ for large $k = k(\epsilon)$ and for large $M = M(\epsilon, k)$, $a_k - \epsilon \leq A_M \leq a_k + \epsilon$.

Firstly, we send $M \to \infty$ and then for every $\epsilon > 0$ for large $k$,

$$a_k - \epsilon \leq \liminf A_M \leq \limsup A_M \leq a_k + \epsilon \quad (8.3)$$

Therefore sending $\epsilon$ to 0, $k$ to infinity, we have that, choosing a subsequence of $a_k$ that converges to $a$,

$$a \leq \liminf A_M \leq \limsup A_M \leq a \quad (8.4)$$

and therefore $\lim A_M = \liminf A_M = \limsup A_M = a$.

As consequence we have the requested property that there exists a positive number $\lambda$ s.t. $\frac{\text{len}([h(L)])}{\text{len}(L)} \to \lambda$, as $\text{len}(L)$ is going to infinity. □

**Lemma 8.2.** Using the notation as above and choosing any other representative $h'$ of $\psi$, then $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. In particular, the number doesn’t depend on the representative but only on $\psi$.

**Proof.** Let $h, h'$ $O$-maps which represent $\psi$ as in the previous lemma. Therefore by the proposition 3.1 for any $L$, $[h(L)] = [h'(L)]$, up to bounded error that doesn’t depend on $L$. Therefore for every $L$, $\text{len}([h(L)]) \leq \text{len}([h'(L)]) + C$, where $C$ is positive fixed and as a consequence

$$\left| \frac{\text{len}([h(L)]) - \text{len}([h'(L)])}{\text{len}(L)} \right| \leq \frac{C}{\text{len}(L)} \to 0$$

for large $\text{len}(L)$.

Therefore since $\frac{\text{len}([h(L)])}{\text{len}(L)} \to \lambda(h, \Lambda)$ and $\frac{\text{len}([h'(L)])}{\text{len}(L)} \to \lambda(h', \Lambda)$, we have as a consequence $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. □

**Lemma 8.3.** Using the notation above we have that $\sigma : \text{Stab}(\Lambda) \to \mathbb{R}^+$, where $\sigma(\psi) = \lambda(h, \Lambda)$, is a well defined homomorphism.

**Proof.** Since we have that $\psi \in \text{Stab}(\Lambda)$, this means that $[h(\ell)]$ is a leaf (for any leaf $\ell$) and as a consequence $\sigma$ is a well defined map.
We will prove that $\sigma$ is homomorphism.
So we have to prove that for any $\psi_1, \psi_2 \in Stab(\Lambda)$ it holds that $\sigma(\psi_1)\sigma(\psi_2) = \sigma(\psi_1\psi_2)$. We choose representatives $h_1, h_2$ of $\psi_1, \psi_2$ respectively, and by definitions $\frac{\text{len}(h_1(L))}{\text{len}(L)} \rightarrow \sigma(\psi_1)$ and $\frac{\text{len}(h_2(L))}{\text{len}(L)} \rightarrow \sigma(\psi_2)$. Moreover, $h_1h_2$ represents $\psi_1\psi_2$ (by previous lemma we can choose any representative).
Therefore since $\frac{\text{len}(h_1(h_2(L)))}{\text{len}(h_2(L))} \rightarrow \sigma(\psi_1)\psi_2)$, for $\text{len}(L) \rightarrow \infty$ and $\frac{\text{len}(h_1(h_2(L)))}{\text{len}(h_2(L))} = \frac{\text{len}(h_1(L))}{\text{len}(L)} \frac{\text{len}(h_2(L))}{\text{len}(h_2(L))}$ up to bounded error. But now sending $\text{len}(L)$ to infinity, it holds $\frac{\text{len}(h_1(h_2(L)))}{\text{len}(h_2(L))} \rightarrow \sigma(\psi_1)$ (as $\text{len}(h_2(L))$ converges to infinity when $\text{len}(L) \rightarrow \infty$ and the fact that $[h_1(h_2(L))]$ and $[h_1(h_2(L))]$ are in bounded distance and the bound doesn’t depend on $L$).
Therefore by uniqueness of the limit, we have that $\sigma(\psi_1\psi_2) = \sigma(\psi_1)\sigma(\psi_2)$.

9 Kernel of the homomorphism

Now we investigate the properties of the kernel, We would like to prove that $\ker(\sigma)$ contains as subgroup of finite index the intersection of the stabiliser with the kernel of the action. But firstly, we aim to prove that the subgroup $\ker(\sigma)$ contains only non-exponentially growing automorphisms. We will prove it separately for irreducible and reducible automorphisms.

9.1 Reducible case

In the reducible case we will see that the automorphisms of the $Stab(\Lambda)$, have representatives of a very specific form. More specifically, every stratum except the top, is non-exponentially growing and moreover the representative restricted to each stratum is just a permutation of edges. Therefore we can calculate the value of $\sigma$, using only the top stratum if it is exponentially growing.

**Proposition 9.1.** If $\psi \in Stab(\Lambda)$ is exponentially growing and there exists some $k$ s.t. $\psi^k$ reducible, then $\sigma \notin Ker(\sigma)$

**Proof.** Let $h : S \rightarrow S$ be a relative train track representative of $\psi$(we can change $h$ with some power if it is necessary).
Firstly, we note that every stratum, except possible the top stratum, is non-exponentially growing. This is true, since otherwise if some $H_r$ is exponentially growing and $e \in H_r$ we have that the lengths of tightenings of $h$-iterates of $e$ are arbitrarily long (by the train track property) and they are l.s. (by definition of the stabiliser of the lamination), but this means that we have arbitrarily long segments contained in some proper subgraph (since $h(G_r) \subseteq G_r$), which is impossible as we have seen in 7.2.

Therefore if $\psi$ is exponentially growing then we suppose, changing $h$ with some iterate if it is necessary, that there exists $H_0$ which is union of strata, all of them are non-exponentially growing, $h$ restricted to $H_0$ induces the identity on the quotient, and that the top stratum is exponentially growing, so if we have a leaf of the lamination and using the subgraph-overgraph decomposition of the leaf, it is implied that the lengths of long l.s. grow exponentially and in fact the actual value is the Perron-Frobenius eigenvalue that corresponds to the unique exponentially growing stratum.

9.2 Irreducible case

Now let’s suppose that $\psi$ is an IWIP. We have two cases and we will prove the theorem independently for automorphisms that have the NGC and the rest automorphisms that have the GC (the dichotomy is the same that as in the free case, but for the automorphisms with GC we need arguments of different nature). We will prove again that the value of $\sigma$ corresponds to the Perron-Frobenious eigenvalue of $\psi$ (or $\psi^{-1}$).

**Lemma 9.2.** Let $h : S \to S$ be a train track map representing some irreducible $\psi \in Out(G, \mathcal{O})$.

Then for every $C > 0$ there is a number $M > 0$ such that if $L$ is any path, then one of the following holds:

(i) $[h^M(L)]$ contains a legal segment of length $> C$

(ii) $[h^M(L)]$ has fewer illegal turns that $L$

(iii) $L$ is concatenation $x \cdot y \cdot z$, such that $y$ is N-preperiodic and $x, z$ have length $\leq 2C$ and at most one illegal turn.
Proof. Choose $M$ to be a natural number that exceeds the number of inequivalent legal edge paths of length $\leq 2C$.

Now assume that $L$ is a path such that the second statement fails, so $[h^M(L)]$ has the same number of illegal turns with $L$ (since $h$ is train track map, sends edges to legal paths and legal turns to legal turns so it is not possible the image of a path to have more illegal turns than the path). So each $h$-iteration of $L$ amounts to iterating maximal legal subsegments of $L$ and cancelling portions of adjacent ones.

If, in addition, the first fail as well, then each maximal legal segment (which has length $\leq C$) of $L$, except possibly the ones that contain the endpoints must have two iterates that after cancellation yield equivalent segments (otherwise we will have $M$ equivalent legal segments of length $\leq C$, but this contradicts to the choice of $M$).

Therefore, we have that each segment contains a preperiodic point so that these points subdivide $L$ as $x \cdot y_1 \cdot \ldots \cdot y_m \cdot z$, and we have that this path satisfies the third statement. \qed

Firstly we will prove a useful lemma for IWIP automorphisms which satisfy the property NGC and then we see that how we can use it for GC automorphisms.

**Lemma 9.3.** Let $\psi, \psi^{-1}$ irreducible automorphisms (IWIP’S), $h : S \to S$ train track map representing $\psi$, $h' : S' \to S'$ representing $\psi^{-1}$ and let’s suppose that there is an integer $m$ so that it is impossible to concatenate more than $m$ N- periodic in $S$ and in $S'$. Let $\tau : S \to S$, $\tau' : S' \to S'$, $O$-maps.

Then for any $C > 0$ there are constants $N_0 > 0$ and $L_0$ such that if $j$ is line or a path of length $\geq L_0$ and if $j'$ the isometric immersion obtained from $[\tau j]$, then one of the following holds:

(A) $[h^M(j)]$ contains a legal segment of length $> C$

(B) $[h'^M(j')]$ contains a legal segment of length $> C$

**Proof.** Without loss, we may assume that $C$ is larger than the critical constants for $h$ and for $h'$. Let $M$ be the larger of the two integers guaranteed by previous lemma applied to $h, C$ and $h', C$. We will fix a large integer
9.2 Irreducible case

Suppose that (A) does not hold with \( N_0 = sM \). We will apply the previous lemma only to \( hM \)-admissible segments (a segment \( L \subseteq j \) so that \( h^M(\partial L) \subseteq [h^M(j)] \)). By our assumption the first of the previous lemma doesn’t hold. If we further restrict to segments \( L \) with > \( m + 2 \) illegal turns, then we can’t have the third case either. So for such segments the second is always true. We can represent \( j \) as a concatenation of such segments of uniformly bounded length and the uniform bound does not depend on \( j \), but only on \( h, h', \tau, \tau', M \) (since we will apply the same argument using \( [\tau h(j)], h' \) instead of \( j, h \) respectively).

Say \( p \) is an upper bound to the number of illegal turns in each segment (there are finitely since they are of uniformly bounded length). Fix \( a \) with \( \frac{p-1}{p} < a < 1 \). For long enough segments \( L \) in \( j \) the ratio \( \frac{\text{number of illegal turns in } [h^M(L)]}{\text{number of illegal turns in } L} < a \) (since the number of illegal turns in \( L \) than \( p \) and number of illegal turns in \( [h^M(L)] \) is strictly less that the number of illegal turns in \( L \)).

By applying the same argument to \( h^M(j) \) and then to \( h^{2M}(j) \) etc, we see that for given \( s > 0 \) and long enough segments \( L \subseteq j \) (the length depends on \( s \) as well ), we have \( \frac{\text{number of illegal turns in } [h^sM(L)]}{\text{number of illegal turns in } L} < a^s \), or else (A) holds with \( N_0 = sM \). Since legal segments have length above by \( C \) and below by the length of the shortest edge(with the exception of the two containing the endpoints), the length can be compared with two inequalities to the number of illegal turns. Therefore if (A) fails, there exists a constant \( A = A(h, C) \) with the property \( \frac{\text{length}[h^sM(L)]}{\text{length}(L)} < Aa^s \). Similarly, we can use the same argument using \( [\tau h^{sM}j] \) in place of \( j \) and with \( h' \) in place of \( h \). If (B) fails as well,(with \( N_0 = sM \)) we reach a similar conclusion that \( \frac{\text{length}[h'^sM\tau h^{sM}(L)]}{\text{length}[\tau h^{sM}(L)]} < B a^s \) for some \( B \) depends only on \( h', C \).

Firstly, we note that \( h^sM \tau h^{sM} \), \( \tau \) are both \( \mathcal{O} \)-maps so they coincide to every path, except some bounded error near endpoints, in particular for long \( L \), we have that the ratio of their lengths is bounded above by 2 and below by \( 1/2 \). Therefore multiplying the above inequalities and changing \( h^sM \tau h^{sM} \) by \( \tau \) we have the inequality:

\[
\frac{\text{length}[\tau(L)]}{\text{length}[\tau h^{sM}(L)]} \frac{\text{length}[h^{sM}(L)]}{\text{length}(L)} < 2ABa^{2s}. \tag{9.1}
\]
9.2 Irreducible case

On the other hand, \( \frac{\text{length}[h^M(j)]}{\text{length}(j)} \frac{\text{length}[\tau(L)]}{\text{length}(L)} > \frac{1}{2\text{Lip}(\tau)Lip(\tau')} \) using again that \( \tau' \tau \) and the identity are both \( O \)-maps as above.

But sending \( s \) to infinity we have a contradiction, since \( a < 1 \).

**Geometric Case:** In the proof of the previous lemma we have used the property that there is an integer \( m \) so that it is impossible to concatenate more than \( m \) \( N \)-periodic paths in \( j \) (and the iterates \( [h^M(j)] \)) and the same is true for \( j' \) (and the iterates \( [h^M(j')] \)). The previous lemma is true for NGC automorphisms for every \( j \). But if we apply this when \( j \) is some leaf of the lamination and \( h \in \text{Stab}(\Lambda) \), we can prove that this always the case.

**Lemma 9.4.** If \( \ell \) is some leaf of the lamination, then there is an integer \( m \) so that it is not possible for \( \ell \) to contain a concatenation of \( m \) subpaths that each of them is \( N \)-periodic.

**Proof.** Choose \( f : T \to T \), stable train track representative (this is possible by [4], since \( N \)-periodic paths correspond to Nielsen periodic paths in the quotient), then there is exactly one path in \( \Gamma = T/G \) in which every (indivisible) \( N \)-periodic path projects. We suppose that there is no bound in the number of concatenation of INP in \( \ell \). So by quasiperiodicity we have that every leaf segment is contained in some concatenation of equivalent paths of the form \( P_1P_2...P_n \) (where every \( P_i \) is a path that projects to the loop \( P \)). But then the subgroup that is constructed by the graph of groups as concatenation of this loop, carries the lamination and therefore has finite index (by [7,2] in \( G \), which is possible.

Therefore the lemma [9.3] is true, in this case, if we restrict to \( h \in \text{Stab}(\Lambda) \) and \( \ell \) some leaf of the lamination.

**Definition 9.5.** We say that a sequence \( \{\Lambda_i\} \) of irreducible laminations in \( \mathcal{IL} \) if for some (any) tree \( H \) every leaf segment of \( \Lambda \) in \( S \)-coordinates is a leaf segment of \( \Lambda_i \) in \( S \)-coordinates for all but finitely many \( i \).

**Proposition 9.6.** Let \( \Lambda = \Lambda_\phi^+ \in \mathcal{IL} \) and let \( \psi \in \text{Aut}(G,\mathcal{O}) \) which is an IWIP. Suppose that \( \psi \in \text{Stab}(\Lambda) \), then \( \Lambda = \Lambda_\psi^+ \) or \( \Lambda = \Lambda_\psi^- \).

We note again that if a segment contains a legal segment with length larger than \( C_{\text{crit}} \) then the length of reduced iterates converge to infinity.
Proof. In the non-geometric case:
Using the notation of the previous lemmas. Let $\ell$ be a leaf of $\Lambda$ in the $S$-coordinates. We apply the lemma to $[h^K\ell]$ with $K > 0$ and $C$ larger the critical constants of $h$ and $h'$. If for some $K > 0$ (A) holds, then it follows from quasiperiodicity that the forward iterates weakly converges to $\Lambda_{\psi}^+$, since we have that the length of reduced images converges to infinity and so we have arbitrarily long legal segments and the quasiperiodicity implies that some translation of every leaf segment is finally contained in the reduced images.

The remaining possibility is that $[\tau h^K\ell]$ contains an $S'$ legal segment of length $> C$ for all $K > 0$. But this means that $[\tau \ell]$ which equals to $[h^K\tau h^K\ell]$ up to bounded error, contains an arbitrarily high $h'$-iterate of a legal segment and quasiperiodicity now implies that $\Lambda = \Lambda_h^-$. Now in the geometric case we use the same argument but only for $h \in \Stab(\Lambda)$ and we have the same result that $\Lambda = \Lambda_h^\pm$.

Note that we have proved that for automorphisms with the property NGC, it is true for every IWIP $\psi$ (relative to $\mathcal{O}$) either the forward $\psi$-iterates of $\Lambda$ weakly converges to $\Lambda_{\psi}^+$ or $\Lambda = \Lambda_{\psi}^-.

**Corollary 9.7.** If $\psi \in \Stab(\Lambda)$ is exponentially growing, then $\psi \notin \Ker(\sigma)$

Proof. For reducible automorphisms, we have already proved it in 9.1. For irreducible ones, we have by the previous proposition that $\Lambda = \Lambda_{\psi}^+$ (changing $\psi$ with $\psi^{-1}$, if it necessary) and so we can choose $f = h$, where $h$ is the train track representative of $\psi$, in the proof of 8.1 and then $\sigma(\psi)$ is obviously equal to the Perron-Frobenius eigenvalue which is greater than 1, since $\psi$ is exponentially growing (it is an IWIP).

10 Discreteness of the Image

We will prove that the image of the homomorphism $\sigma$ is discrete and therefore we can see $\sigma$ as a homomorphism $\sigma : \Stab(\Lambda) \to \mathbb{Z}$.
Lemma 10.1. $\sigma(Stab(\Lambda))$ is a discrete set.

Proof. This is true since by the proof of the propositions (9.1, 9.7), every $\sigma(\psi)$ other than 1, occurs as the Perron-Frobenius eigenvalue for an irreducible integer matrix of uniformly bounded size. It is well known then that the set of such numbers form a discrete set and as a consequence $\sigma(Stab(\Lambda))$ is an infinite discrete subset of $\mathbb{R}$ and is hence isomorphic to $\mathbb{Z}$. \qed

11 Finite index subgroup of the Kernel

Let $T \in \mathcal{O}$ and denote by $K_T$, the subgroup of $Stab(\Lambda)$ of automorphisms that have a representative $h : T \to T$ that preserves the orbit of every vertex of $T$ and the orbits of every germ, then $K_T$ is a finite index subgroup of $Stab(\Lambda)$ (since there are finitely many orbits of vertices and finitely many orbits of edges, therefore finitely many orbits of germs emanating from each vertex). Note that this property doesn't depend on $T$, but only on $\psi$.

Proposition 11.1. $KA \cap Stab(\Lambda)$ has finite index in $Ker(\sigma)$.

Proof. We prove that $KA \cap Stab(\Lambda) = K_T \cap Ker(\sigma)$. Note that it is obvious that $KA \cap Stab(\Lambda) \subseteq K_T \cap Ker(\sigma)$, since if some element of the kernel belong to the stabiliser then it has some representative which is isometry and preserves the orbits of vertices and edges.

It remains to be proved that any $\psi \in K_T \cap Ker(\sigma)$ belongs to $KA \cap Stab(\Lambda)$. Firstly, we will prove the theorem for irreducible automorphisms.

Let $\psi \in K_T \cap ker(\sigma)$ (and then $\psi \in Stab(\Lambda)$), by definitions and 9.7 there is a (train track) representative of $\psi$, $h : S \to S$ which preserves the orbits of vertices and the orbits of germs and (since it is non-exponentially growing) it is an isometry. This implies that preserves the orbits of edges (sends every edge $e$ to some edge of the form $ge$, where $g \in G_o(e)$). Therefore this implies that $h$ induces the identity in the quotient and since $\psi \in Ker(\sigma)$ it represents an element of $KA$. Therefore $\psi \in KA \cap Stab(\Lambda)$.

Now let prove the theorem for reducible automorphisms. By 9.7, we have that there is relative train track representative, $h : S \to S$ and a maximal proper $h$-invariant subgraph $S_0$ (and we denote by $H_0$ the
quotient $S_0/G$ s.t. the restriction of $h$ in $S_0$ induces a map that has finite order in the quotient. But since $h$ has no exponentially growing strata [9.1], and moreover $h^k$ induces just a permutation of edges of the same length (in particular it is isometry), and since it preserves the orbits of vertices and the orbits of germs, it is implied that it preserves the orbits of edges (sending an edge $e$ to some other $ge$) therefore $k = 1$ and the restriction of $h$ induces actually the identity in the quotient restricted on $H_0$.

Now for the top stratum we can suppose that it contains a single edge $e$ and that $h(e) = ea$, where $a$ is a segment of $S_0$. But then since $h$ is a relative train track and $h \in Stab(\Lambda)$, we have that $h$-iterates of $e$ produces arbitrarily long segments of the lamination that are contained in $S_0$ which contradicts quasiperiodicity, except if the leaf of the lamination is of the form (in the quotient, so every $a, e$ correspond to some orbits of them):

$$...ea^{b_i}e^{-1}e_{-1}ea^{b_0}ea^{b_i}e^{-1}x_0ea^{b_i}...$$

for some integers $b_i$ and $x_i$ are contained in $S_0$ (or $H_0$ in the quotient). In this case the lamination lifts to a graph of groups which is consisted by the disjoint union of a graph of groups corresponding to $H_0$ and a loop corresponding to $a$ joined by an edge corresponding to $e$. But this leads to contradiction since there is exactly one lift of $e$ and two lifts of $a$.

Therefore we have that $h(e) = e$ and then $h$ induces also the identity on $\Gamma - H_0$ ($\Gamma = T/G$).

Therefore $h$ induces the identity on the quotient and as above, we have that $\psi \in KA$.

12 Main Results

In this section, we will state and prove the main theorems. We use the same notation as in the sections above.

Firstly, we will see that $Stab(\Lambda)$ modulo its intersection with the direct product of $Aut(G_i)$ is virtually cyclic. More precisely :

**Theorem 12.1.** $Stab(\Lambda)/(KA \cap Stab(\Lambda))$ is virtually cyclic.
Proof. Let’s denote $B = KA \cap Stab(\Lambda)$ and then $Stab(\Lambda)/B$ contains the finite (by [11.1]) subgroup $Ker(\sigma)/B$ and

$\frac{Stab(\Lambda)/B}{(Ker(\sigma)/B)}$ is isomorphic to $Stab(\Lambda)/Ker(\sigma)$ which is isomorphic with a subgroup of $\mathbb{Z}$ and as a consequence it is cyclic. So $Stab(\Lambda)/KA$ is finite-by-cyclic and therefore virtually cyclic. □

Note that is possible for $Stab(\Lambda)$ to be virtually cyclic, for example in the case that it doesn’t contain any free factor automorphism or in the case that $KA$ is a finite group (equivalently every $Aut(G_i)$ is finite).

A direct corollary of the previous theorem is the following. Let’s denote $C(\phi)$ the centraliser of $\phi$.

**Theorem 12.2.** $C(\phi)/(KA \cap C(\phi))$ is virtually cyclic.

Note that in the case in which $\mathcal{O}$ corresponds to the Grushko’s decomposition of $G$, we have that the previous theorem generalises the theorem for the classical case that the centraliser of an IWIP (for f.g. free groups with the absolute notion of irreducibility) is virtually cyclic since $Out(F_n)$ acts on $CV_n$ faithfully (with trivial kernel). Additionally, we can take also relative results for the free and for the general case. This is possible since we can use the fact that every automorphism is irreducible relative to some appropriate space.

Moreover, note that if $\phi$ doesn’t commute with the automorphisms of the free factors then $C(\phi)$ is virtually cyclic. But as we have seen in the introduction, there are examples that this is not true. This example can be adjusted using any group in the place of the unique non-free vertex. In particular, we can find centralisers of IWIP automorphisms (relative to some space) which contain big subgroups and as a consequence they are not virtually cyclic.
References

[1] Mladen Bestvina, Mark Feighn, Michael Handel *Laminations, trees, and irreducible automorphisms of free groups*, Geometric and Functional Analysis GAFA, May 1997, Volume 7, Issue 2, pp 215-244

[2] Mladen Bestvina, Mark Feighn, and Michael Handel. *The Tits alternative for Out (Fn) I: Dynamics of exponentially-growing automorphisms.* Annals of Mathematics-Second Series 151.2 (2000): 517-624.

[3] Mladen Bestvina, Mark Feighn, and Michael Handel. "The Tits Alternative for II: A Kolchin Type Theorem." Annals of mathematics (2005): 1-59.

[4] D. J. Collins., E.C. Turner. *Efficient representatives for automorphisms of free products* The Michigan Mathematical Journal 41 (1994), no. 3, 443-464. doi:10.1307/mmj/1029005072. [http://projecteuclid.org/euclid.mmj/1029005072](http://projecteuclid.org/euclid.mmj/1029005072).

[5] Thierry Coulbois and Arnaud Hilion and Martin Lustig, *R-trees and laminations for free groups I: Algebraic laminations*, J. Lond. Math. Soc, (2008), No. 3, 723–736

[6] Thierry Coulbois and Arnaud Hilion and Martin Lustig, *R-trees and laminations for free groups II: The dual lamination of an R-tree* J. Lond. Math. Soc., 78 (2008), No. 3, 737754.

[7] Marc Culler and Karen Vogtmann, *Moduli of graphs and automorphisms of free groups* Invent. Math., 84(1):91119, 1986.

[8] Mark Feighn and Michael Handel, *Abelian subgroups of Out(F_n)*, Geometry and Topology 13.3 (2009): 1657-1727.

[9] Stefano Francaviglia, Armando Martino, *Stretching factors, metrics and train tracks for free products*, arXiv:1312.4172, 2013

[10] Vincent Guirardel and Gilbert Levitt, *The Outer space of a free product*, Proc. London Math. Soc. (3) 94 (2007) 695714
[11] Michael Handel, Lee Mosher, *Relative free splitting and free factor complexes I: Hyperbolicity*, 2014, [arXiv:1407.3508](https://arxiv.org/abs/1407.3508)

[12] Camille Horbez, *The Tits alternative for the automorphism group of a free product*, 2014, [arXiv:1408.0546](https://arxiv.org/abs/1408.0546)

[13] Camille Horbez, *Hyperbolic graphs for free products, and the Gromov boundary of the graph of cyclic splittings*, [arXiv:1408.0544v2](https://arxiv.org/abs/1408.0544v2) (2014).

[14] Camille Horbez, *The boundary of the outer space of a free product*, [arXiv:1408.0543v2](https://arxiv.org/abs/1408.0543v2) (2014), [arXiv:1408.0543](https://arxiv.org/abs/1408.0543)

[15] Ilya Kapovich, *Algorithmic detectability of iwip automorphisms*, Bull. London Math. Soc. (2014) 46 (2): 279-290 January 9, 2014

[16] Ilya Kapovich and Martin Lustig, *Stabilisers of \( \mathbb{R} \)-trees with isometric free actions* Journal of Group Theory. Volume 14, Issue 5, Pages 673-694, ISSN (Online) 1435-4446, ISSN (Print) 1433-5883, DOI: 10.1515/jgt.2010.070, August 2011

[17] Ilya Kapovich and Martin Lustig, *Invariant laminations for irreducible automorphisms of free groups* Q. J Math first published online January 30, 2014 doi:10.1093/qmath/hat056

[18] Martin Lustig, *Conjugacy and Centralizers for Iwip Automorphisms of Free Groups*, Geometric Group Theory, Trends in Mathematics, 2007, doi:10.1007/978-3-7643-8412-811, 197-224.

[19] Erika Meucci, *Relative Outer Automorphisms of Free Groups*, Ph.D. thesis, University of Utah (2011).

[20] Jean-Pierre Serre *Trees*. Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2003). APA