Abstract

The purpose of this article is to prove a “Newton over Hodge” result for exponential sums on curves. Let $X$ be a smooth proper curve over a finite field $\mathbb{F}_q$ of characteristic $p \geq 5$ and let $V \subset X$ be an affine curve. For a regular function $f$ on $V$, we may form the $L$-function $L(f, V, s)$ associated to the exponential sums of $f$. In this article, we prove a lower estimate on the Newton polygon of $L(f, V, s)$. This estimate depends on the local monodromy of $f$ around each point $x \in X - V$. As a corollary, we obtain a lower estimate on the Newton polygon of a curve with an action of $\mathbb{Z}/p\mathbb{Z}$ in terms of local monodromy invariants.

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1
1 Introduction

1.1 Motivation

Let $p$ be a prime with $p \geq 5$ and let $q = p^a$. Let $V$ be an $d$-dimensional smooth affine variety over $\mathbb{F}_q$ and let $\overline{f}$ be a regular function on $V$. We define the exponential sum over the $\mathbb{F}_{q^k}$-points of $V$ to be

$$S_k(\overline{f}) = \sum_{x \in V(\mathbb{F}_{q^k})} \zeta_{p^{Tg_{q^k/p_q}(\overline{f}(x))}},$$

where $\zeta_p$ is a primitive $p$-th root of unity. A fundamental question in number theory is to understand the sequence of numbers $S_k(\overline{f})$. One approach is to study the generating $L$-function

$$L(\overline{f}, V, s) = \exp(\sum_{k=1}^{\infty} \frac{S_k(\overline{f})s^k}{k}).$$

By the work of Dwork and Grothendieck, we have

$$L(\overline{f}, V, s) = \prod_{\alpha_{i=1}}^{n_1}(1 - \alpha_is) \prod_{\beta_{i=1}}^{n_2}(1 - \beta_is) \in \mathbb{Z}(\zeta_p)(s),$$
which implies

\[ S_k(\mathcal{f}) = \sum_{i=1}^{n_1} \alpha_i^k - \sum_{i=1}^{n_2} \beta_i^k. \]

We are thus reduced to studying the algebraic integers \( \alpha_i \) and \( \beta_j \). What can be said about \( \alpha_i \) and \( \beta_j \) in general? Deligne’s work on the Weil conjectures provides us with two stringent conditions. First, for \( \ell \neq p \) these numbers are \( \ell \)-adic units:

\[ |\alpha_i|_\ell = |\beta_j|_\ell = 1. \]

Next, the \( \alpha_i \) and \( \beta_j \) are Weil numbers: there exists \( u_i, v_j \in \mathbb{Z} \cap [0, 2d] \), known as weights, such that for any Archimedean absolute value \( |\cdot|_\infty \) we have

\[ |\alpha_i|_\infty = q^{u_i/2} \quad \text{and} \quad |\beta_j|_\infty = q^{v_j/2}. \]

This leaves us with two natural questions. What are the \( p \)-adic valuations and what are the \( u_i, v_j \)? If we take \( V \) to be a multidimensional torus and make certain nonsingularity assumptions on \( \bar{f} \), there is a great deal known about both questions. Lower bounds on the \( p \)-adic valuations have been studied intensively by many (see, e.g., [2] and [24] for two monumental works). The weights have been computed by Adolphson-Sperber and Denef-Loeser (see [2] and [6]). When \( V \) is not a torus, it is possible to reduce to the toric case by an inclusion-exclusion argument. This allows us to write \( L(\mathcal{f}, V, s) \) as a ratio of \( L \)-functions of exponential sums on tori. However, it is difficult to deduce precise results about \( L(\mathcal{f}, V, s) \) from this ratio, as cancellation often occurs.

Now consider the case where \( V \) is a smooth curve. In this situation \( L(\mathcal{f}, V, s) \) is a polynomial and the weights are all one. Thus we are left with the question of the \( p \)-adic valuations. The purpose of this article is to prove lower bounds on the \( q \)-adic Newton polygon of \( L(\mathcal{f}, V, s) \).

1.2 Statement of main theorem

1.2.1 Exponential sums on curves

We now assume that \( V = \text{Spec}(B) \) is a smooth affine curve and we let \( X \) be its smooth compactification. Assume that \( \overline{f} \) is not of the form \( x^p - x \) with \( x \in B \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\text{alg}} \). We obtain a \( \mathbb{Z}/p\mathbb{Z} \)-cover of smooth proper curves \( r: C \to X \) from the equation

\[ Y^p - Y = \overline{f}, \tag{1} \]

and our condition on \( \overline{f} \) implies that \( C \) is geometrically connected. Let \( \{\tau_1, \ldots, \tau_m\} \in X \) be the points over which \( r \) ramifies and let \( Z = X - \{\tau_1, \ldots, \tau_m\} \). Note that \( V \) is contained in \( Z \). After increasing \( q \), we may assume that any point in \( X - V \) is defined over \( \mathbb{F}_q \). Consider a nontrivial character \( \rho: \text{Gal}(C/X) \to \mathbb{Z}_p[\zeta_p]^X \) and its Artin \( L \)-function

\[ L(\rho, s) = \prod_{x \in Z} \frac{1}{1 - \rho(Frob_x)s^{\deg(x)}}. \tag{2} \]

Let \( NP_q(L(\rho, s)) \) (resp. \( NP_q(L(\overline{f}, V, s)) \)) denote the \( q \)-adic Newton polygon of \( L(\rho, s) \) (resp. \( L(\overline{f}, V, s) \)). For some choice of \( \zeta_p \), we have

\[ S_k(\mathcal{f}) = \sum_{x \in V(\mathbb{F}_{q^k})} \zeta_p^{T_{x}} = \sum_{x \in V(\mathbb{F}_{q^k})} \rho(Frob(x)), \]

where \( T_{x} \) is the Newton polygon of \( \overline{f} \) at \( x \).
which gives the relation

$$L(f, V, s) = L(\rho, s) \prod_{x \in Z - V} (1 - \rho(Frob_x)s).$$

Therefore, we are reduced to studying $NP_q(L(\rho, s))$.

Our main result gives a lower bound on $NP_q(L(\rho, s))$ in terms of the Swan conductor of $\rho$ at each point. The Swan conductor $d_i$ at the point $\tau_i$ has a simple description. Let $T_i$ be a local parameter at $\tau_i$. Locally $r$ is given by an equation $Y^p - Y = g_i$, where $g_i \in F_q(T_i)$. We may assume that $g_i$ has a pole whose order is prime to $p$. The order of this pole is equal to the Swan conductor. That is, $g_i = \sum_{n \geq -d_i} a_n T_i^n$ and $a_{-d_i} \neq 0$.

**Theorem 1.1.** The polygon $NP_q(L(\rho, s))$ lies above the polygon whose slopes are

$$\{0, \ldots, 0, 1, \ldots, 1, \frac{1}{d_1}, \ldots, \frac{1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m}\}.$$ 

Let $c$ be the cardinality of $Z - V$. Then $NP_q(L(f, V, s))$ lies above the polygon whose slopes are

$$\{0, \ldots, 0, 1, \ldots, 1, \frac{1}{d_1}, \ldots, \frac{1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m}\}.$$ 

### 1.2.2 Zeta functions of Artin-Schreier covers

Theorem 1.1 also has interesting consequence about Newton polygons of $\mathbb{Z}/p\mathbb{Z}$-covers of curves. Let $r : C \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-cover ramified over $\tau_1, \ldots, \tau_m$ with Swan conductor $d_i$ at the point $\tau_i$. The zeta function of $C$ (resp. $X$) is a rational function of the form $Z(C, s) = \frac{P_C(s)}{(1-s)(1-qs)}$ (resp. $Z(X, s) = \frac{P_X(s)}{(1-s)(1-qs)}$). Let $NP_C$ (resp. $NP_X$) denote the $q$-adic Newton polygon of $P_C$ (resp. $P_X$). We are interested in the following question: to what extent can we determine $NP_C$ from $NP_X$ and the ramification invariants of $r$? One positive result is the Deuring-Shafarevich formula (see [9]), which allows us to completely determine the number of slope zero segments in $NP_C$. However, a precise formula for the higher slopes of $P_C$ is impossible. Instead, the best we may hope for are lower bounds. To connect this problem to Theorem 1.1 we recall the following decomposition:

$$Z(C, s) = Z(X, s) \prod_{\rho} L(\rho, s),$$

where $\rho$ varies over the nontrivial characters $Gal(C/X) \to \mathbb{Z}_{p}[\zeta_p]^\times$. This gives:

**Corollary 1.2.** The Newton polygon $NP_C$ lies above the polygon whose slopes are the multiset:

$$NP_X \sqcup \left( \bigsqcup_{i=1}^{p-1} \left\{0, \ldots, 0, 1, \ldots, 1, \frac{1}{d_1}, \ldots, \frac{1}{d_1}, \ldots, \frac{1}{d_m}, \ldots, \frac{d_m - 1}{d_m}\right\} \right),$$

where $\sqcup$ denotes a disjoint union.
1.3 Idea of proof and previous work

1.3.1 The Monsky trace formula

As our question is inherently $p$-adic in nature, it is necessary to utilize a $p$-adic Lefschetz trace formula. More specifically, we will utilize the Monsky trace formula (see [16]), which generalizes the trace formulas of Dwork and Reich. The Monsky trace formula works roughly as follows (see §7.2 for a more precise formulation): Let $X^{rig}$ be a rigid analytic lifting of $X$ defined over a finite extension $L$ of $\mathbb{Q}_p$ and let $V^{rig}$ be the tube of $V \subset X$. Let $B^\dagger$ denote the functions on $V^{rig}$ that overconverge in each tube $[\tau_i]$ and let $\sigma : B^\dagger \to B^\dagger$ be a ring homomorphism that lifts the $q$-th power Frobenius map of $V$. Using $\sigma$ we define an operator $U_q : B^\dagger \to B^\dagger$, which is the composition of a trace map $\text{Tr} : B^\dagger \to \sigma(B^\dagger)$ with $\frac{1}{q} \sigma^{-1}$. The Galois representation $\rho$ corresponds to an overconvergent $F$-isocrystal with rank one. This is a $B^\dagger$-module $M = B^\dagger e_0$ and a $B^\dagger$-linear isomorphism $\phi : M \otimes_B B^\dagger \to M$. Such an $F$-isocrystal is determined entirely some $\alpha \in B^\dagger$ such that $\phi(e_0 \otimes 1) = \alpha e_0$, which we refer to as the Frobenius structure of $M$. Let $L_\alpha : B^\dagger \to B^\dagger$ be the “multiplication by $\alpha$” map. In our specific setup (see §6), the Monsky trace formula can be written as follows:

$$L(\rho, s) = \frac{\det(1 - sU_q \circ L_\alpha|B^\dagger)}{\det(1 - sqU_q \circ L_\alpha|B^\dagger)}.$$  

(3)

To utilize (3), we need to understand $U_q \circ L_\alpha$. This breaks up into two questions:

**Question 1.3.** Is there a Frobenius $\sigma$ and a basis of $B^\dagger$, for which the operators $U_q$ are reasonable to understand?

**Question 1.4.** Can we understand the Frobenius structure of $M$? In particular, we need to understand the “growth” of the Frobenius structure in terms of a basis of $B^\dagger$.

1.3.2 Previous work

Let us now recount what was previously known and the previous approaches to these questions. When $X = \mathbb{P}_F^1$ and $m = 1, 2$, Theorem 1.1 is due to Robba (see [19]), building off of ideas of Dwork. In this case, it is trivial to address Question 1.3. Indeed, we have $B^\dagger = L(T, T^{-1})^\dagger$ and we may take $\sigma$ to be the map that sends $T \to T^q$. The Frobenius structure, which is known as a splitting function in this case, may be described explicitly using the Artin-Hasse exponential $E(T)$ (see §3.3). The Taylor expansion of $E(T)$ lies in $\mathbb{Z}_p[[T]]$, which makes it particularly easy to estimate the Frobenius structure. The work of Adolphson-Sperber and Wan for higher dimensional tori also utilizes the Artin-Hasse exponential. However, the cohomological calculations used to find $p$-adic estimates are significantly more nuanced.

The case where $X = \mathbb{P}_F^2$ and $m \geq 1$ has been studied by Zhu (see [26]). The key idea for addressing Question 1.3 is to decompose $B^\dagger$ using partial fractions. The operator $U_q$ can then be understood by modifying computations of Dwork (see [10], Chapter 5). To address Question 1.4, Zhu again utilizes the Artin-Hasse exponential, analogous to Robba’s splitting function. However, when $m > 2$ the Frobenius structure used in [26] does not correspond to the finite character $\rho$, and thus calculates a different $L$-function. In particular, Theorem 1.1 was unknown for $m > 2$. Nevertheless, Zhu’s idea of computing the $L$-function by looking locally around each pole has influenced this article.
1.3.3 The approach to Questions 1.3-1.4 in this article

We first discuss our approach to Question 1.3. We let \( T_i \) be a local parameter at \( \tau_i \) and let \( T_i \) be a local parameter that lifts \( T_i \). We need to find a Frobenius endomorphism \( \sigma: B^\dagger \to B^\dagger \), that behaves nicely with respect to \( T_i \). For a rational line with a global parameter \( T \), we may take \( \sigma \) to be the map that sends \( T \to T^q \). For a general \( C \), we bootstrap from the rational case by pulling back along a simply branched map \( C \to \mathbb{P}^1_{\overline{R}} \). This allows us to break up the action of \( \sigma \) on local parameters into three types of local Frobenius endomorphisms: \( T \to T^q \), \( T \to (T-b)^q + b \), and \( T \to \sqrt{(T-b)^q + b} \), where \( b \) is some constant. In §5 we estimate local versions of \( U_q \) for each type of local Frobenius endomorphism. This is one of the key technical obstacles for dealing with higher genus curves.

To utilize the local \( U_q \) operators, we need a systemic way to expand elements of \( B^\dagger \) in terms of the local parameters \( T_i \). The main idea is to consider “truncated expansions” at \( \tau_i \). That is, if we write \( f \in B^\dagger \) as a Laurent series \( \sum a_n T_i^n \), we consider its truncation in \( L(T_i^{-1})^\dagger \). We obtain an injection:

\[
\psi: B^\dagger \to R := \bigoplus_{i=1}^m L(T_i^{-1})^\dagger.
\]

We view \( B^\dagger \) as a subspace of \( R \) and we extend the operators \( U_q \circ L_\alpha \) found in §3 to all of \( R \). Using some delicate \( p \)-adic functional analysis (see Proposition 4.7), we can estimate the Frobenius determinant of \( U_q \circ L_\alpha \) on \( B^\dagger \) by estimating \( U_q \circ L_\alpha \) on all of \( R \).

Question 1.4 is also a major technical obstacle. We need to understand the Frobenius structure “around each \( \tau_i \)”. The Artin-Hasse exponential can only be used if the local Frobenius endomorphism is of the form \( T_i \to T_i^p \). Instead, we make use of the Dwork exponential function:

\[
\theta(x) = \exp(\mu(x^p - x)), \quad \mu^p - 1 = -p.
\]

In [7], Dwork proves that \( \theta(x) \) converges beyond the open unit disc. Furthermore, Dwork provides strong bounds for the first few terms in the series expansion of \( \theta(x) \). If \( z \) is a lift of a solution to the Artin-Schreier equation, then \( \alpha = \frac{\theta(z)}{\theta(z)^p} \) gives the Frobenius structure of the \( F \)-isocrystal corresponding to \( \rho \). This holds for any \( \alpha \). In §3.5 we establish precise bounds on the growth of \( z \) around each \( \tau_i \) depending on \( d_i \) (see Lemma 3.10). Proving these bounds is subtle difficulty not present in previous work. Since \( z \) lies in an extension of \( B^\dagger \), it does not have an expansion in \( T_i \). This makes it unclear how to measure \( z \) in a meaningful way. To overcome this obstacle, we embed our local rings into a much larger ring of Witt vectors (see §3.2). This large ring admits partial valuations, which allow us to compare \( z \) to elements of \( L(T_i^{-1}) \). This type of technique is present in Kedlaya’s proof of the \( p \)-adic local monodromy theorem (see [14]). Although the convergence of \( \theta(x) \) is not as good as the Artin-Hasse exponential function, we are able to use Dwork’s bounds on \( \theta(x) \) and our bounds on \( z \) to sufficiently estimate \( \alpha \).

1.4 Further work

There are many natural questions that arise from Theorem 1.1. The most pressing of which is to what extent Theorem 1.1 is optimal. Since \( NP_X \) has integer vertices we know that the vertices of \( NP_\rho(L(\rho, s)) \) have \( y \)-coordinates in \( \frac{1}{p-1} \mathbb{Z} \). This implies that if \( p \not\equiv 1 \mod d_i \) for some \( i \), Theorem 1.1 can be slightly improved. For \( V = \mathbb{A}^1 \), Blach and Fèrard show that after accounting for this natural restriction, the bound is optimal for a generic exponential sum (see
When $V$ is $G_m$, Robba proved (see [19]) that the bound in Theorem 1.1 is attained if and only if $p \equiv 1 \mod d_i$ for $i = 1, 2$. The obvious generalizations of both these results to higher genus curves using the bounds in Theorem 1.1 are false for general $X$. Indeed, if $X$ is not ordinary, then by the Deuring-Shafarevich formula we know that the bound in Theorem 1.1 has too many slope zero segments. If $X$ is ordinary, do these generalizations hold? When $X$ is not ordinary, can we replace some of the slope zero segments in Theorem 1.1 to obtain an optimal result?

Another question would be to generalize Theorem 1.1 to more general overconvergent $F$-isocrystals. Given an overconvergent $F$-isocrystal $M$ on $V$, can we bound $NP_q(L(\rho, s))$ in terms of the Swan conductor and perhaps the Frobenius slopes of $M$? In light of recent work on crystalline companions by Abe (see [1]), this would have profound consequences for the $L$-functions of $\ell$-adic sheaves and automorphic forms on function fields. The methods developed in this article give a very general solution to Question 1.3. Thus, to generalize Theorem 1.1 to general $\rho$, the main difficulty lies in bounding the Frobenius structure of $M$. If $M$ is unit-root, we suspect that Theorem 1.1 has a direct generalization using the Swan conductors of the corresponding $p$-adic representation. For $\mathbb{Z}/p^n\mathbb{Z}$-covers, there are generalizations of the Dwork exponential, which are known to be overconvergent by Matsuda (see [15]). In the case where $X = \mathbb{P}^1_{\mathbb{F}_q}$, $n \leq 2$, and $m = 1$, these generalized exponentials were used by Morofushi to prove bounds on $NP_q(L(\rho, s))$ (see [18]). Can the generalized Dwork exponentials be estimated well enough to prove optimal bounds on $NP_q(L(\rho, s))$ for any $X$?

Finally, we mention our requirement that $p \geq 5$. When $p = 3$ it is likely that the methods in this paper still work. The main difficulty is some of the estimates in §3.4 and §3.5 become more technical. When $p = 2$, a serious obstruction arises: we can no longer utilize a simply branched map $X \to \mathbb{P}^1_{\mathbb{F}_q}$. When $p = 2$, we can instead hope to find a map that is at most triply branched away from $\infty$. This appears to be possible, due to forthcoming work of Kiran Kedlaya, Daniel Litt, and Jakub Witaszek. With such a map, it seems likely that Theorem 1.1 may be extended to the case where $p = 2$.

1.5 Outline

In §3 we introduce local $F$-isocrystals and study the growth of local Dwork $F$-isocrystals. In §4 we give some preliminary results on Newton polygons of operators. We then apply these results to operators on certain spaces of power series. Next, in §5 we prove that $U_p$ satisfies certain growth conditions, depending on the Frobenius lift $\sigma$. We then carefully describe the global setup in §6. Finally, in §7 we complete the proof of Theorem 1.1.

1.6 Acknowledgments

Throughout the course of this work, we have benefited greatly from conversations with Daqing Wan, June Hui Zhu, James Upton, and Andrew Obus.

2 Conventions

The following conventions will be used throughout the article. Let $p \geq 5$ be prime and let $q = p^e$. Let $L_0$ be the unramified extension of $\mathbb{Q}_p$ whose residue field is $\mathbb{F}_q$. Let $E$ be a totally ramified finite extension of $\mathbb{Q}_p$ of degree $e$ with uniformizer $\pi$ and let $L = E \otimes_{\mathbb{Q}_p} L_0$. Let $\zeta_p$ be
a nontrivial $p$-th root of unity and let $\mu$ satisfy $\mu^{p-1} = -p$. We assume that $E$ contains both $\zeta_p$ and $\mu$. We define $\nu$ to be the endomorphism of $L$ defined by $\text{id} \otimes \text{Frob}$. For any $E$-algebra $R$ and $x \in R$, we let $L_x : R \to R$ denote the “multiplication by $x$” map. For any ring $R$ with valuation $v : R \to \mathbb{R}$ and any $x \in R$ with $v(x) > 0$, we let $v_x(\cdot)$ denote the normalization of $v$ satisfying $v_x(x) = 1$. Furthermore, for any $S \subset R$ we define $\mathcal{O}_S = \{ x \in S \mid v(x) \geq 0 \}$.

3 Local unit-root $F$-isocrystals of rank one

3.1 Basic definitions

We begin by definition some rings and modules, which will be used throughout this article. Let $F$ be $\mathbb{F}_q((T))$ and let $G_F$ denote its absolute Galois group. We define $L$-algebras:

$$
\mathcal{E}_L := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \mid \text{We have } a_n \in L, \lim_{n \to -\infty} v_p(a_n) = \infty, \text{ and } v_p(a_n) \text{ is bounded below.} \right\},
$$

$$
\mathcal{E}_L^\dagger := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{E}_L \mid \text{There exists } m > 0 \text{ such that } v_p(a_n) \geq -mn \text{ for } n \ll 0 \right\}.
$$

We define $\mathcal{E}_{L_0}$ and $\mathcal{E}_{L_0}^\dagger$ similarly. We refer to $\mathcal{E}_L$ (resp. $\mathcal{E}_{L_0}$) as the Amice ring over $L$ (resp. $L_0$) with parameter $T$. When there is no ambiguity, we will omit the $L$ or $L_0$. Note that $\mathcal{E}_L^\dagger$, $\mathcal{E}_L$, $\mathcal{E}_{L_0}^\dagger$ and $\mathcal{E}_{L_0}$ are local fields with residue field $F$. The valuation $v_p$ on $L$ extends to the Gauss valuation on each of these fields. We also define the $L$-vector spaces of truncated Laurent series:

$$
\mathcal{E}^\sim := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{E} \mid a_n = 0 \text{ for all } n > 0 \right\},
$$

$$
\mathcal{E}^{\leq k} := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{E} \mid a_n = 0 \text{ for all } n > k \right\}.
$$

The space $\mathcal{E}^\sim$ is a ring and $\mathcal{E}^{\leq k}$ is an $\mathcal{E}^\sim$-module. There is a natural projection $\mathcal{E} \to \mathcal{E}^{\leq k}$, given by truncating the Laurent series.

3.2 Extensions of $\mathcal{E}$

Let $\tilde{\mathcal{E}} = L \otimes_{\mathcal{O}_{L_0}} W(F^{\text{alg}})$, where $W(F^{\text{alg}})$ denotes the $p$-typical Witt vectors of the algebraic closure of $F$. There is an embedding

$$
\iota : \mathcal{E} \hookrightarrow \tilde{\mathcal{E}},
$$

which sends $T$ to $[T] \in \tilde{\mathcal{E}}$, where $[T]$ is the Teichmuller lift of $T \in F$. Now let $K$ be a finite separable extension of $F$. By [15] Theorem 2.2, there exists a unique unramified extension $\mathcal{E}^K$ of $\mathcal{E}$ contained in $\tilde{\mathcal{E}}$ whose residue field is $K$. Define

$$
\mathcal{E}^{\text{unr}} = \bigcup_{[K:F]<\infty} \mathcal{E}^K.
$$
and let \( \tilde{E}^{unr} \) be the \( p \)-adic completion of \( E^{unr} \). There is a continuous action of \( G_F \) on \( \tilde{E}^{unr} \) and \( \iota \) extends to an embedding \( \iota : \tilde{E}^{unr} \to \tilde{E} \). Let \( x \in \tilde{E} \) with Teichmuller expansion \( x = \sum_{i \gg \infty} [x_i] \pi^i \).

For any \( k \in \frac{1}{e} \mathbb{Z} \) we define the \( k \)-th partial valuation

\[
w_k(x) = \min_{v_p(\pi^n) \leq k} v_T(x_n).
\]

This definition does not depend on our choice of \( \pi \). If \( x \in \mathcal{O}_E \), then \( w_k(x) \) is the smallest power of \( T \) occurring in the reduction of \( x \) modulo \( \pi^{ke+1} \). These partial valuations satisfy:

\[
\begin{align*}
w_k(x + y) & \geq \min(w_k(x), w_k(y)), \\
w_k(xy) & \geq \min_{i+j \leq k} (w_i(x) + w_j(y)).
\end{align*}
\] (4)

Using these partial valuations we define

\[
\mathcal{O}^m_E(k) = \{ x \in \mathcal{O}_E \mid w_n(x) \geq -nm - k \text{ for all } n \in \mathbb{Q} \},
\]

\[
\mathcal{O}^m_E(k) = \mathcal{O}^m_E(k) \cap E.
\]

Note that

\[
x \in \mathcal{O}^m_E(k_1), y \in \mathcal{O}^m_E(k_2) \implies xy \in \mathcal{O}^m_E(k_1 + k_2).
\] (5)

An alternative definition of \( \mathcal{O}^m_E(k) \) is

\[
\mathcal{O}^m_E(k) := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{O}_E \mid \text{For all } n < 0 \text{ we have } v_p(a_{n-k}) \geq -\frac{1}{m} n \right\}.
\]

From these definitions we have:

**Lemma 3.1.** We have:

1. If \( x \in \mathcal{O}^m_E(k) \), then \( px \in \mathcal{O}^m_E(k - m) \).
2. If \( x \in \mathcal{O}^m_E(k) \) and \( p \mid x \) then \( \frac{x}{p} \in \mathcal{O}^m_E(k - m) \).

### 3.3 Rank one \( F \)-isocrystals and \( p \)-adic characters

We will now discuss the unit-root \( F \)-isocrystal associated to a \( p \)-adic character of \( G_F \).

**Definition 3.2.** Let \( R \) be \( E \) or \( E^1 \). A ring endomorphism \( \sigma \) of \( R \) is a Frobenius endomorphism if it reduces to the \( q \)-power map modulo \( \pi \):

\[
\sigma(x) \equiv x^q \mod \pi.
\]

A ring endomorphism \( \nu \) is a \( p \)-Frobenius endomorphism if it restricts to \( \nu \) on \( L \) and reduces to the \( p \)-power map modulo \( \pi \).

Let \( \nu \) be a \( p \)-Frobenius endomorphism. Then \( \sigma = \nu^f \) is a Frobenius endomorphism. Note that both \( \nu \) and \( \sigma \) extend to \( \tilde{E}^{unr} \) and both commute with the action of \( G_F \) (see e.g. [22 §2.6]).
Definition 3.3. A $\varphi$-module for $\sigma$ is a vector space $M$ over $R$ equipped with a $\sigma$-semilinear endomorphism $\varphi : M \to M$ whose linearization is an isomorphism. More precisely, we have $\varphi(cm) = \sigma(c)\varphi(m)$ for $c \in R$ and $\sigma^*\varphi : R \otimes_{\sigma} M \to M$ is an isomorphism.

Let $M = Re_0$ be a $\varphi$-module over $R$ whose underlying vector space has dimension one. The $\varphi$-module structure is then determined by $\varphi(e_0) = a_0e_0$, where $a_0 \in R^\times$. We refer to $a_0$ as a Frobenius structure of $M$. If $e_1 = ce_0$ and $\varphi(e_1) = \alpha_1 e_1$, then we have $\alpha_1 = \frac{c}{a}a_0$. In particular, the Frobenius structure of $M$ is well defined up to multiplication by an element of the form $\frac{c}{a}$. Thus, the $p$-adic valuation of the Frobenius structure does not depend on the choice of basis.

Definition 3.4. A unit-root $F$-isocrystal $M$ over $R$ of rank one is a $\varphi$-module over $R$ whose underlying vector space has dimension one and whose Frobenius structure is a $p$-adic unit.

The following theorem is a characteristic $p$ version of the Riemann-Hilbert correspondence.

Theorem 3.5 (Katz, see §4 in [13]). There is an equivalence of categories

$$\{\text{rank one unit-root $F$-isocrystals over } E\} \leftrightarrow \{\text{continuous characters } \rho : G_F \to L^\times\}.$$  

Katz’ equivalence can be made explicit. Let $V = Ve_0$ and consider a representation $\rho : G_F \to GL(V)$. The corresponding unit-root $F$-isocrystal $M$ is

$$\tilde{\mathcal{E}}^{unr} \otimes_L V)^{G_F},$$

where the Frobenius acts on $\tilde{\mathcal{E}}^{unr} \otimes_L V$ by $\sigma \otimes \text{id}$. Thus $M$ is in $\tilde{\mathcal{E}}^{unr} \otimes_L V$ and consists of elements $x_0 \otimes e_0$ satisfying $g(x_0) \otimes \rho(g)e_0 = x_0 \otimes e_0$. We refer to any such $x_0$ as a period of $\rho$. The periods of $\rho$ are well defined up to multiplication by $\mathcal{E}^\times$ and the Frobenius structure of $M$ is $\alpha = \frac{x_0}{a_0}$. We can recover the Galois representation by $\gamma \mapsto \frac{\gamma x}{x_0}$, for $\gamma \in G_F$.

Next, let $V_0 = Ee_0$ so that $V = V_0 \otimes_E L$. Assume that $\rho$ factors through a map $\rho_0 : G_F \to GL(V_0)$. The map $\tilde{\mathcal{E}}^{unr} \otimes_E V_0 \to \tilde{\mathcal{E}}^{unr} \otimes_L V$ is a $G_F$-equivariant isomorphism, so that we have an isomorphism

$$(\tilde{\mathcal{E}}^{unr} \otimes_E V_0)^{G_F} \to (\tilde{\mathcal{E}}^{unr} \otimes_L V)^{G_F}$$

sending $x_0 \otimes E e_0$ to $x_0 \otimes_L e_0$. Note that $\nu \otimes \text{id}$ acts on $(\tilde{\mathcal{E}}^{unr} \otimes_E V_0)^{G_F}$. Thus $\alpha_0 = \frac{x_0}{a_0} \in \mathcal{E}$ and we have the relation

$$\alpha = \prod_{i=0}^{a-1} \alpha_0^{n_i}.$$ 

3.4 Some auxiliary rings

We will now introduce some auxiliary rings and modules with precise growth conditions. Although these growth conditions appear arbitrary and complicated, they arise when studying local Dwork $F$-isocrystals and the $U_p$ (see §3.5 and §5). For $d \geq 2$, we define the following sequences of rational numbers:
For \( n \leq 0 \) we set \( t_d(n) = r_d(n) = 0 \). For \( d = 1 \) we define the sequences \( r_1(n) \) (resp. \( t_1(n) \)) to be the same as \( r_2(n) \) (resp. \( t_2(n) \)). We then define the following \( \mathcal{O}_L \)-modules:

\[
\mathcal{R}_d = \left\{ \sum_{i=1}^{\infty} a_i T^i \in \mathcal{O}_E \left| v_p(a_{-n}) \geq r_d(n) \right. \right\},
\]

\[
\mathcal{T}_d = \left\{ \sum_{i=1}^{\infty} a_i T^i \in \mathcal{O}_E \left| v_p(a_{-n}) \geq t_d(n) \right. \right\},
\]

\[
\mathcal{R}_d^{-} = \mathcal{R}_d \cap \mathcal{E}^{-}, \quad \mathcal{T}_d^{-} = \mathcal{T}_d \cap \mathcal{E}^{-}.
\]

Note that \( r_d(n_1) + r_d(n_2) \geq r_d(n_1 + n_2) \). In particular, \( \mathcal{R}_d \) is a ring. For \( k \in \mathbb{Z} \), we let \( \mathcal{R}_d(k) \) denote \( \mathcal{R}_d(k) \)-module \( T^{-k} \mathcal{R}_d \). We then define

\[
\mathcal{R}_d^-(k) = \mathcal{R}_d(k) \cap \mathcal{E}^{-}, \quad \mathcal{T}_d^-(k) = \mathcal{T}_d(k) \cap \mathcal{E}^{-}.
\]

The ring \( \mathcal{R}_d \) can also be defined using partial valuations:

\[
\mathcal{R}_d = \left\{ x \in \mathcal{O}_E \left| w_k(x) \geq -d(p-1) \frac{k}{p-1} \right. \right\}.
\]

This description will be useful in \( \S 3.5 \) The following two lemmas will be useful in \( \S 5.1 \)

**Lemma 3.6.** For \( n \geq 0 \) and \( k > 0 \) we have

\[
t_d(n + k) \leq t_d(n) + \frac{k}{2}.
\]

**Proof.** When \( d > 2 \), note that \( t_d(n) = r_d(p(n+1) - 1) \) for all \( n \geq 0 \). By (6), we see that \( r_d(n+p) \leq r_d(n) + \frac{p}{d(p-1)} \). This proves (9), since \( p \geq 5 \). When \( d = 2 \), we have \( t_2(n) = \frac{n+1}{2} \) if \( n = 0, 1 \). If \( n \geq 3 \), we have \( t_2(n) = r_d(p(n+1) - 1). \) We see that \( t_2(n+1) - t_2(n) \leq \frac{1}{d + 2} \), proving (9). \( \square \)

**Corollary 3.7.** Let \( b \in \mathcal{O}_L \) such that \( v_p(b) \geq t_d(n) \). Then

\[
b \mathcal{O}_E^2(n) \subset \mathcal{T}_d.
\]

**Lemma 3.8.** Let \( n \geq 0 \). We have

\[
t_d(n) - r_d(n) \geq \begin{cases} 
\frac{n+1}{d} & n < d-1 \\
\frac{1}{2} & n \geq d-1.
\end{cases}
\]

Furthermore \( t_d(n) - r_d(n) \) is unbounded as \( n \) goes to infinity.
Proof. It is immediate that \( t_d(n) - r_d(n) \) is unbounded. We will prove (10) for \( d \geq 2 \). The case where \( d = 1 \) will follow. Since \( r_d(n) \) is defined by three separate pieces, the proof of (10) will be broken up into different ranges of \( n \).

**Case 1.** \( n \leq d - 1 \): Here (10) is immediate.

**Case 2.** \( d \leq n \leq 3d - 1 \): In this case \( t_d(n) \geq \frac{p}{p-1} \) and \( r_d(n) \leq \frac{3}{p-1} \). The inequality follows, since \( p \geq 5 \) and \( d \geq 2 \).

**Case 3.** \( 3d \leq n \leq (p-1)(d+2) - 1 \): In this case \( t_d(n) \geq \frac{n}{d+2} \) and \( r_d(n) \leq \frac{n}{d(p-1)} \). Thus

\[
t_d(n) - r_d(n) \geq 3 \left( \frac{d}{d+2} - \frac{1}{p-1} \right),
\]

which is greater than \( \frac{1}{2} \) since \( p \geq 5 \) and \( d \geq 2 \).

**Case 4.** \( n \geq (p-1)(d+2) \): Here, \( t_d(n) \geq \frac{n(p-1)}{p(d+2)} \) and \( r_d(n) \leq \frac{n}{d(p-1)} \). The proof of (10) is then similar to the previous case.

\( \square \)

### 3.5 Local Dwork \( F \)-isocrystals

In this subsection we estimate the Frobenius structure of an \( F \)-isocrystal associated to a character of \( G_F \) factoring through an Artin-Schreier extension. Fix a \( p \)-Frobenius endomorphism \( \nu \) and set \( \sigma = \nu^f \). Recall the Dwork exponential

\[
\theta(x) = \exp(\mu(x^p - x)).
\]

Although the \( p \)-adic exponential function is only defined for \( v_p(x) > \frac{1}{p-1} \), Dwork miraculously proved that \( \theta(x) \) admits an analytic continuation (see [9, §9]). If we consider the expansion

\[
\theta(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{11}
\]

we have

\[
v_p(a_n) \geq n \frac{p-1}{p^2}. \tag{12}
\]

Thus \( \theta(x) \) is a function on the disc \( v_p(x) > \frac{1}{p-1} \), but we may only view it as the composition of \( \exp(x) \) and \( \mu(x^p - x) \) if \( v_p(x^p - x) > 0 \). Consider the Artin-Hasse exponential \( E(x) = \exp(\sum \frac{x^{p^i}}{p^i}) \), which is known to be a power series with coefficients in \( \mathbb{Z}_p \). Then

\[
\theta(x) \equiv E(\mu x) \mod x^{p^2} \mathcal{O}_L[[x]],
\]

which implies

\[
v_p(a_n) \geq \frac{n}{p-1} \text{ for } n < p^2. \tag{13}
\]
Let $K$ be a degree $p$ Galois extension of $F$ defined by the an Artin-Schreier equation $X^p - X = f$, with $f \in F$. Let $f \in \mathcal{O}_{E_{L_0}}$ be a lift of $f$ and let $z \in \tilde{E}^{unr}$ be a solution to the equation $X^p - X = f$. Note that we may regard $\theta(z)^p = \exp(p\mu(z^p - z))$ as the composition of $\exp(x)$ and $p\mu(z^p - z)$. Thus, for any $\gamma \in \text{Gal}(K/F)$, we have $(\theta(z)^p)^\gamma = \theta(z)^p$ and $\frac{\theta(z)^p}{\theta(z)}$ is a $p$-th root of unity. Following [9, §9], we obtain a nontrivial representation $\rho : \text{Gal}(K/F) \to \mathbb{Z}_p[\zeta_p]^\times$ defined by

$$\rho : \gamma \to \frac{\theta(z)^\gamma}{\theta(z)}.$$ 

From §3.3 we know that $\theta(z)$ is a period of $\rho$ and $\alpha_0 = \frac{\theta(z)^\nu}{\theta(z)}$ lies in $\mathcal{E}$. The Frobenius structure of the corresponding $F$-isocrystal is given by $\frac{\theta(z)^\nu}{\theta(z)}$. The main result of this subsection gives estimates for $\frac{\theta(z)^\nu}{\theta(z)}$, with some assumptions on $f(T)$ and the $p$-Frobenius endomorphism $\nu$.

**Proposition 3.9.** Let $d = -v_T(f)$ and assume that our lift $f$ has a pole of order $d$ at $T = 0$. We further assume $T^\nu \in \mathcal{O}_{\mathcal{E}}^\mathbb{Z}_p(-p)$. Then $\alpha_0 \in \mathcal{R}_d$.

To prove Proposition 3.9 we need to estimate $z$ and $z^\nu$. This is accomplished with:

**Lemma 3.10.** Let $h \in \mathcal{O}_{\mathcal{E}_{L_0}}$ and let $\lambda, m > 0$. Assume that $w_k(h) \geq -\lambda - km$ for all $k \geq 0$. If $y$ is a solution of $X^p - X = h$, then

$$w_k(y) \geq \begin{cases} \frac{-\lambda}{p} & 0 \leq k < 1 \\ -(\lambda + m)k & k \geq 1 \end{cases}.$$ 

**Proof.** Consider the Teichmüller expansion

$$y = \sum_{k=0}^{\infty} [y_k]p^k.$$ 

Note that we may use $p$ instead of $\pi$ in this expansion, because $h \in \mathcal{E}_{L_0}$. For $k = 0$, the lemma holds because $y_0$ is a solution to the Artin-Schreier equation $X^p - X = h$. Now let $k > 0$ and assume the result holds for all $n < k$. We have

$$-[y_k]p^k \equiv \left(\sum_{i=0}^{k-1} [y_i]p^i\right)^p - \sum_{i=0}^{k-1} [y_i]p^i + h \mod p^{k+1}.$$ 

By our assumption on $h$ and our inductive hypothesis we have $w_k(b_2) \geq -k(\lambda + m)$. Next, let $c_1 = [y_0]$ and $c_2 = \sum_{i=1}^{k-1} [y_i]p^i$. Note that $c_2 \in \mathcal{O}_E^{\lambda+m}$. It is clear that $w_k(c_1^p), w_k(c_2^p) \geq -k(\lambda + m)$.

By Lemma 3.1 we see that $(\begin{pmatrix} p \\ j \end{pmatrix})c_1^{p-j}c_2^j \in \mathcal{O}_E^{\lambda+m}$ for $0 < j < p$. Therefore $w_k(b_1) \geq -k(\lambda + m)$, from which we conclude $v_T(y_k) \geq -k(\lambda + m)$.

\qed
Corollary 3.11. For i > 0 we have
\[
\begin{align*}
w_k(z^i) & \geq \begin{cases} 
- \frac{id}{p} & 0 \leq k < \frac{p}{p-1} \\
- kd - \frac{(i-1)d}{p} & \frac{p}{p-1} \leq k
\end{cases} \\
w_k((z^\nu)^i) & \geq \begin{cases} 
- id & 0 \leq k < 1 \\
- kp(d+2) - (i-1)d & k \geq 1
\end{cases}
\end{align*}
\] (14) (15)

Proof. For i = 1, (14) follows from Lemma 3.10. For i > 1, we deduce (14) from Lemma 3.10 and (4). To prove (15), note that \( z^\nu \) is a solution to the equation \( X^p - X = f^\nu \). Our assumption that \( T^\nu \in \mathcal{O}_E^2(-p) \) implies that \( w_k(f^\nu) \geq -pd + 2pk \). Then (15) follows from Lemma 3.10.

Proof. (of Proposition 3.9) Using (11) and (8), it suffices to prove
\[
\begin{align*}
w_k(a_n(z^\nu)^n), w_k(a_nz^n) & \geq \begin{cases} 
-d(p-1)k & 0 \leq k < \frac{p}{p-1} \\
-(p-1)((d+2)) & \frac{p}{p-1} \leq k < p-1 \\
p^2(d+2) & p-1 \leq k
\end{cases} \\
\end{align*}
\] (16)

We will give the details for \( a_n(z^\nu)^n \). The same argument can be used for \( a_nz^n \). Let \( n < p^2 \). For \( k < \frac{p}{p-1} \) we know that (16) holds by Corollary 3.11 and (13). For \( k \geq \frac{p}{p-1} \), using Corollary 3.11 we have
\[
w_k(a_n(z^\nu)^n) \geq -(n-1)d - p(d+2)(k - \frac{n}{p-1}) \\
\geq -kp(d+2) \\
\geq -p^2d + 2p - 1,
\]
which implies (16). Next, consider \( n \geq p^2 \). By (12) and (13), we only need to consider \( k \geq p-1 \). We set \( c = \frac{n}{p^2} \). By (12) we know \( v(p(a_n)) \geq cn \). Write \( k = j + nc \). Again, by Corollary 3.11 we have
\[
w_k(a_n(z^\nu)^n) \geq - \max(jp(d+2) + (n-1)d, nd) \\
\geq - \frac{j(d+2)p^2}{p-1} - n(d+2) \\
= - \frac{p^2(d+2)}{p-1}k.
\]

\[ \square \]

4 Normed vector spaces and Newton polygons

4.1 Normed vector spaces and Banach spaces

We will introduce some basic facts about normed vector spaces over \( L \) and Banach spaces over \( L \). See [20] for a more detailed introduction. Let \( V \) be a vector space over \( L \) with a norm \( |.| \).
We say that $V$ is a Banach space if $V$ is complete with respect to its norm. An orthonormal basis of $V$ is a subset $B = \{e_i\}_{i \in I} \subset V$ such that every $x \in V$ can be written uniquely as

$$x = \sum_{i \in I} a_i e_i,$$

with $\lim_{i \in I} a_i = 0$ and $|x| = \sup_{i \in I} |a_i|^p$. All normed vector space will be assumed to have an orthonormal basis. Let $V_0$ denote the subset of $V$ satisfying $|x| \leq 1$ and let $\overline{V} = V_0/\pi V_0$. A subset $B \subset V_0$ forms an orthonormal basis of $V$ if and only if $B$ reduces to a basis of $\overline{V}$. When $V$ is a Banach space, this is [20, Lemme 1]. If $W$ is a subspace of $V$, we will automatically give $W$ the subspace norm and $V/W$ the quotient norm. Note that

$$(V/W)_0 \cong V_0/W_0.$$

The following lemma will be used in §7.3.

**Lemma 4.1.** Let $f : A \to V$ be a continuous map of Banach spaces such that $f(A_0) \subset V_0$.

1. If $\overline{f} : \overline{A} \to \overline{V}$ is surjective, then $f$ is surjective and $f(A_0) = V_0$. Furthermore,

$$\ker(f) = \ker(\overline{f}).$$

(17)

2. If $\overline{f} : \overline{A} \to \overline{V}$ is injective, then $f$ is injective.

**Proof.** Assume $\overline{f}$ is surjective. To prove $f(A_0) = V_0$, take $x \in V_0$, and successively approximate $x$ by elements of $f(A_0)$. This gives surjectivity as well. To prove (17), we write $\overline{A} = \ker(\overline{f}) \oplus \overline{M}$. The image of the map $\ker(f)_0 \to \overline{A}$ lies in $\ker(\overline{f})$ and has kernel $\pi \ker(f)_0$. This gives an injective map $\ker(f)_0 \to \ker(\overline{f})$. To show surjectivity, let $\overline{x} \in \ker(\overline{f})$ and let $x$ be a lift of $\overline{x}$ to $A_0$. Then $f(x) \in \pi V_0$. Write $f(x) = \pi y$, where $y \in V_0$. There exists $a \in A_0$ such that $f(a) = y$. We have $f(x - \pi a) = 0$. Thus, $x - \pi a$ is a lift of $x$ contained in $\ker(f)_0$. The second part of the lemma is similar.

We may regard $V$ as a vector space over $E$. Let $\zeta_1, \ldots, \zeta_a \in O_L$ be a elements that reduce to a basis of $F_q$ over $\mathbb{F}_p$ modulo $\pi$. Let $B' \subset B$ be a subset indexed by $I' \subset I$ and let $I'_E = I \times \{1, \ldots, a\}$. Then we define

$$B'_E = \{\zeta_j e_i\}_{(i,j) \in I'_E}.$$

In particular, $B_E$ forms an orthonormal basis of $V$ over $E$.

### 4.2 Generalities about Newton polygons

Let $\alpha \in \mathbb{Z}_{\geq 0} \cup \infty$. A Newton polygon $P$ of length $\alpha$ is the graph of points $(x, f(x))$ where $x \in [0, \alpha]$ and $f : [0, \alpha] \to \mathbb{R}$ satisfies the following properties

1. $f(0) = 0$

2. For any integer $i \in [0, \alpha]$, the function $f(x)$ is linear on the domain $x \in [i, i + 1]$ with slope $m_i \geq 0$. 

3. The \( m_i \) are nondecreasing, i.e., \( m_{i+1} \geq m_i \) for \( 0 \leq i < \alpha \).

4. If \( \alpha = \infty \), then the \( m_i \) are unbounded, i.e. \( \lim_{i \to \infty} m_i = \infty \).

We will refer to the multiset \( \{m_i\}_{i \in I} \) as the slope-set of \( P \) and its elements as the slopes of \( P \).

Note that the slope-set of \( P \) determine \( P \) entirely. In particular, let \( I \) be a countable set and let \( N = \{n_i\} \) be a multiset of nonnegative numbers. Assume that for any \( r > 0 \), the subset \( N_r = \{n \in N \mid n < r\} \)

is finite. Then there exists a unique Newton polygon \( P_N \) whose slope-set is \( N \).

We may compare two Newton polygons as follows. For \( i = 1, 2 \), let \( P_i \) be a Newton polygon of length \( \alpha_i \) determined by the function \( f_i : [0, \alpha_i] \to \mathbb{R} \). Then we have

\[ P_1 \succeq P_2 \]

if \( f_1(x) \geq f_2(x) \) for all \( 0 \leq x \leq \min(\alpha_1, \alpha_2) \). When \( P_1 \succeq P_2 \), we say that \( P_1 \) lies above \( P_2 \) (indeed, on the \( xy \)-plane \( P_1 \) does lie above \( P_2 \) where both are defined). If the slope set of \( P_2 \) is \( N \), we will occasionally write \( P_1 \succeq N \) instead of \( P_1 \succeq P_2 \).

Let us now describe some operations on the set of Newton polygons. If \( P \) and \( P' \) are Newton polygons whose slope-sets are \( N \) and \( N' \), we define the concatenation of \( P \) and \( P' \) to be the Newton polygon

\[ P \cup P' = P_{N \cup N'}. \]

That is, \( P \cup P' \) is the Newton polygon whose slope-set is the disjoint union of \( N \) and \( N' \). Next, let \( r > 0 \). We define the \( r \)-truncation \( P_r \) of \( P \) to be the Newton polygon whose slope-set is \( N_r \). Note that \( P_r \) necessarily has finite length. We may also scale Newton polygons. Let \( c = \frac{a}{b} \), where \( a \) and \( b \) are coprime. We define

\[ cP = \{(cx, cy) \in \mathbb{R}^2 \mid (x, y) \in P\}. \]

Note that \( cP \) is only a Newton polygon if the multiplicity of every slope of \( P \) is a multiple of \( b \).

Finally, we introduce Newton polygons associated to power series. Let \( Q(s) \in \mathcal{O}_L[[s]] \) and write \( Q(s) = \sum_{n=1}^{\infty} a_n s^n \). Let \( \alpha = \deg(Q) \). Then for \( * \) equal to \( p \) or \( q \), we define \( NP_*(Q(s^n)) \) to be the length \( \alpha \) Newton polygon that is the lower convex hull of the points \( (n, v_*(a_n)) \). We have the following relations:

\[ NP_*(Q(s^n)) = \{(ax, y) \mid (x, y) \in NP_*(Q(s))\} \]
\[ NP_p(Q(s)) = \{(x, ay) \mid (x, y) \in NP_q(Q(s))\}. \quad (18) \]

### 4.3 Newton polygons of operators

Let \( V \) be a vector space over \( L \) with norm \( |.| \) and let \( B = \{e_i\}_{i \in I} \) be an orthonormal basis of \( V \). We let \( \langle , \rangle_B \) (resp. \( \langle , \rangle_{BE} \)) be the inner product with respect to \( B \) (resp. \( BE \)). Let \( u : V \to V \) be an \( L \)-linear endomorphism and let \( v : V \to V \) be a \( E \)-linear endomorphism. We say that \( u \) (resp. \( v \)) is compact if it can be approximated arbitrarily well by \( L \)-linear (resp. \( E \)-linear)
operators whose image are finite dimensional. Let \((n_{i,j})\) be the matrix of \(u\) with respect to \(B\). The Fredholm determinant of \(u\) is defined to be

\[
\det(1 - su|V) = \sum_{n=0}^{\infty} c_n s^n
\]

where

\[
c_n = (-1)^n \sum_{|S|=n} \sum_{\sigma \in \text{Sym}(S)} \text{sgn}(\sigma) \prod_{i \in S} n_{i,\sigma(i)}.
\]

We define the Fredholm determinant \(\det(1 - sv|V)\) in an analogous manner using the matrix of \(v\) with respect to \(B_E\). In general the summation that defines \(c_n\) does not exist. However, if the operator is compact, then this summation converges and \(\lim_{n \to \infty} c_n = 0\).

**Definition 4.2.** Assume that the Fredholm determinant (19) exists and \(\lim_{n \to \infty} c_n = 0\). We define \(NP_*(u)\) (resp. \(NP_*(v)\)) to be \(NP_*(\det(1 - su|V))\) (resp. \(NP_*(\det(1 - sv|V))\)), where \(*\) is \(p\) or \(q\).

For the remainder of this subsection we restrict our attention to the \(E\)-linear map \(v\). We may find lower estimates of \(NP_p(v)\) as follows. For each \((i,j) \in I_E\), we define

\[
\text{slp}_{B}(v, e_i) = \inf_{x \in V_0} v_p(\langle v(x), e_i \rangle_B),
\]

and

\[
\text{slp}_{B_E}(v, \zeta_j e_i) = \inf_{x \in V_0} v_p(\langle v(x), \zeta_j e_i \rangle_{B_E}).
\]

**Definition 4.3.** Assume that \(v\) is compact. Let \(B' \subset B\) be a subset indexed by \(I' \subset I\). We define \(NP_{B'}(v)\) to be the Newton polygon whose slope-set is

\[
\left\{ \text{slp}_B(v, e_i) \right\}_{i \in I'}.
\]

We define \(NP_{B_E'}(v)\) to be the Newton polygon whose slope-set is

\[
\left\{ \text{slp}_{B_E}(v, \zeta_j e_i) \right\}_{(i,j) \in I'_E}.
\]

**Lemma 4.4.** Assume that \(v\) is a compact operator. Then \(NP(v) \succeq NP_{B_E}(v)\). Furthermore, for any \(B' \subset B\) indexed by \(I' \subset I\) we have \(\frac{1}{n} NP_{B_E}(v) \succeq NP_{B'}(v)\).

**Proof.** The first part of the lemma follows from (19) and (20). For the second part, we observe

\[
\text{slp}_{B_E}(v, \zeta_j e_i) \succeq \text{slp}_B(v, e_i).
\]

The remainder of this subsection is dedicated to proving Proposition 4.7. This proposition allows us to bound the Newton polygon of an operator by projecting onto a subspace.

**Lemma 4.5.** Adopt the notation from Definition 4.3 and assume that \(I \setminus I'\) is a finite set. Let \(V'\) be the subspace of \(V\) consisting of \(x \in V\) of the form

\[
x = \sum_{i \in I'} a_i e_i
\]
and let $pr : V \to V'$ be the projection map. Let $W \subset V$ be an $L$-linear subspace such that $v(W) \subset W$. Assume that $pr : W \to V'$ is a surjective map and let $d = \ker_L(pr|_W)$. We further assume that $pr(W_0) = V_0'$. Then

$$\frac{1}{a} NP_B(v|_W) \succeq \left\{ \begin{array}{c} 0, \ldots, 0 \\ d \end{array} \right\} \bigcup NP_{B'}(v).$$

Proof. For each $i \in I'$, let $f_i \in W_0$ satisfy $pr(f_i) = e_i$. Then

$$C = \{f_i\}_{i \in I'} \cup \{e_i\}_{i \in I-I'}$$

is an orthonormal basis of $V$ and for any $i \in I'$ we have

$$slp_B(v, e_i) = slp_C(v, f_i).$$

Extend $\{f_i\}_{i \in I'}$ to an orthonormal basis $D$ of $W$ over $L$. Then $D$ is of the form $\{f_i\}_{i \in I'} \cup \{r_1, \ldots, r_d\}$. Since $W_0 \subset V_0$, we know from (20) and (21) that $slp_D(v|_W, f_i) \geq slp_B(v, e_i)$ for all $i \in I'$. The lemma follows from Lemma 4.4.

We will be dealing with operators that are not compact. However, they become compact when restricted to a smaller subspace. These subspaces arise as follows. Let $b = \{b_i\}_{i \in I}$ be a multiset of elements in $O_L$ indexed by $I$ with $\lim_{i \in I} b_i = 0$. We let $V_b$ denote the subspace of $V$ consisting of elements $x$ that can be written

$$x = \sum_{i \in I} a_i b_i e_i, \quad \lim_{i \in I} a_i = 0.$$

Lemma 4.6. Assume that $v$ restricts to a compact endomorphism of $V_b$. Then the $\det(1 - sv|V)$ is well defined and

$$\det(1 - sv|V) = \det(1 - sv|V_b).$$

Proof. This is very similar to [16] Lemma 2.5. We know that the Fredholm determinant of $v|_V$ is well defined since it is compact. The result then follows by approximating $v|_V$ and $v$ with operators on finite dimensional vector spaces whose matrices are similar.

Proposition 4.7. Let $V, V', W, pr,$ and $d$ be as in Lemma 4.3 and let $b$ be as above. Let $v : V \to V$ be an $E$-linear operator that restricts to a compact operator on $V_b$ and on $W$. Let $b' = \{b_i\}_{i \in I'}$, so that $B'_0 = \{b_i e_i\}_{i \in I'}$ is an orthonormal basis of $V_0'$. We assume that $b_i \in O_L^\times$ for each $i \in I \setminus I'$. Then

$$\frac{1}{a} NP_B(v|_W) \succeq \left\{ \begin{array}{c} 0, \ldots, 0 \\ d \end{array} \right\} \bigcup NP_{B'_0}(v_b).$$

Proof. For $i \in I'$, let $f_i \in W_0$ with $pr(f_i) = e_i$ and let $g_1, \ldots, g_d$ be an orthonormal basis of $W \cap \ker(pr)$. Let $J$ be the disjoint union $I' \sqcup \{1, \ldots, d\}$. Then $\{f_i\}_{i \in I} \cup \{g_1, \ldots, g_d\}$ is an orthonormal basis of $W$ indexed by $J$. For each $j \in J$ we define

$$c_j = \begin{cases} b_j & j \in I' \\ 1 & j \in \{1, \ldots, d\} \end{cases}$$
and we let $c = \{c_j\}_{j \in J}$. By construction, we have

$$pr((W_c)_0) = (V'_0)_0, \text{ and } (W_c)_0 \subset (V'_0)_0.$$ 

Then by Lemma 4.5 we know that

$$\frac{1}{a} NP_p(v_c) \geq \left\{0, \ldots, 0\right\} \bigcup_{d} NP_B'(v_b).$$

The proposition follows from Lemma 4.6.

4.4 Application to the semi-local situation

In this subsection we give an application of Proposition 4.7 and Lemma 3.8. We point out that the $p$-adic estimates we prove in this subsection (see Proposition 4.8) closely resemble the $p$-adic estimates in Theorem 1.1. Let $S$ be a finite set. For each $i \in S$ let $\mathcal{E}_i$ be a copy of $\mathcal{E}$ with parameter $T_i$ and fix positive integers $d_i$ and $\kappa_i$. We will assume that $\mathcal{E}$ is large enough so that $v_p(E)$ contains $r_{d_i}(n)$ for all $i \in S$ and $n \in \mathbb{Z}$. Consider a continuous $\mathcal{E}$-linear operator

$$v : \bigoplus_{i \in S} \mathcal{E}_i^- \rightarrow \bigoplus_{i \in S} \mathcal{E}_i^-$$

that restricts to an $\mathcal{O}_E$-linear map

$$v : \bigoplus_{i \in S} \mathcal{R}_{d_i}^-(\kappa_i) \rightarrow \bigoplus_{i \in S} \mathcal{T}_{d_i}^-(\kappa_i).\quad (22)$$

**Proposition 4.8.** Let $A$ be an $L$-linear subspace of $\bigoplus_{i \in S} \mathcal{E}_i^-$ such that the projection $i : A \rightarrow \bigoplus_{i \in S} E_i^{<\kappa_i}$ is surjective. Let $d = \dim_L(v)$. If $v(A) \subset A$, then

$$\frac{1}{a} NP_p(v|_{A}) \geq \left\{0, \ldots, 0\right\} \bigcup_{d} \left(\bigcup_{i \in S} \left\{\frac{1}{d_i}, \ldots, \frac{{d_i-1}}{2d_i}\right\}\right).$$

**Proof.** Let $I = S \times \mathbb{Z}_{\geq 0}$. For each $i \in S$ and $n \in \mathbb{Z}_{\geq 0}$, select $b_{i,n} \in \mathcal{O}_L$ that satisfies $v_p(b_{i,n}) = r_{d_i}(n - \kappa_i)$ and define $e_{i,n} = b_{i,n} T_i^{-n}$. Let $V$ denote the Banach space over $L$ that has $B = \{e_{i,n}\}_{(i,n) \in I}$ as an orthonormal basis. In particular, $V \subset \bigoplus_{i \in S} \mathcal{E}_i^-$ and $V_0 \subset \bigoplus_{i \in S} \mathcal{R}_{d_i}^-(\kappa_i)$. Next, define

$$I' = \{(i, n) \in I \mid n \geq \kappa_i\},$$

and let $B' = \{e_{i,n}\}_{(i,n) \in I'}$. From (22) we know that

$$slp_B(v, e_{i,n}) \geq t_{d_i}(n - \kappa_i) - r_{d_i}(n - \kappa_i).$$

Therefore, from Lemma 3.8 we have

$$NP_{B'}(v) \geq \bigcup_{i \in S} \left\{\frac{1}{d_i}, \ldots, \frac{{d_i-1}}{2d_i}\right\}.$$ 

The proposition follows from Proposition 4.7.
5 Local $U_p$ operators

Let $\nu$ be a $p$-Frobenius of $E^\dagger$. We define $U_p$ to be the map:

$$\frac{1}{p} \nu^{-1} \circ \text{Tr}_{E^\dagger/\nu(E)^\dagger} : E^\dagger \to E^\dagger.$$

Note that $U_p$ is $\nu^{-1}$-semilinear (i.e. $U_p(\nu(y)x) = yx$ for all $y \in E^\dagger$). In this section, we will study the growth of $U_p$. We begin by introducing the notion of a $c$-moderate operator. Then we study properties of $U_p$ for arbitrary $\nu$. Finally, we consider $U_p$ for three specific $p$-Frobenius endomorphisms.

5.1 $c$-moderate operators

In this section we will introduce a class of operators on $E$, which we call $c$-moderate.

Definition 5.1. Let $D : E \to E$ be a $\nu^{-1}$-semilinear operator and let $c \in \mathbb{Z}$. We say that $D$ is $c$-moderate if for all $n \in \mathbb{Z}$ we have

$$D(T^n) \in O^2_E\left(c - \left\lfloor \frac{c + n}{p} \right\rfloor \right).$$

Definition 5.2. Let $D : E \to E$ be an operator. For $m \in \mathbb{Z}$ we define

$$D[k] = L_{T-k} \circ D \circ L_{Tk}.$$

Note that $D$ is $c$-moderate if and only if $D[k]$ is $(c+k)$-moderate.

Our main result about $c$-moderate operators is as follows:

Proposition 5.3. Let $D : E \to E$ be an $c$-moderate operator. Then

$$D(R_d(c+1)) \subset T_d(c+1).$$

Proof. By the remark at the end of Definition 5.2 it suffices to prove that if $D$ is $-1$-moderate, then $D(R_d) \subset T_d$. Therefore, it is enough to prove the following: if $b \in O_L$ with $v_p(b) \geq r_d(n)$, then $D(bT^{-n}) \in T_d$. To prove this, note that by (7) we have

$$t_d\left(\left\lfloor \frac{n+1}{p} \right\rfloor - 1\right) \leq r_d(n).$$

Using Corollary 3.7 and the fact that $[-x] = -[x]$, we see

$$D(bT^{-n}) \in b^{v_p^{-1}}O^2_E\left(\left\lfloor \frac{n+1}{p} \right\rfloor - 1\right) \subset T_d.$$  

$\blacksquare$
5.2 Local $U_p$ operators for general $p$-Frobenius endomorphisms

In this section we prove the following proposition:

**Proposition 5.4.** Let $m$ be a positive integer such that $T^\nu \in \mathcal{O}_E^m(-p)$. Then for all $n \in \mathbb{Z}$ we have

$$U_p(T^n) \in \mathcal{O}_E^m\left(m - \left\lceil \frac{1 + n}{p} \right\rceil \right).$$  \hspace{1cm} (23)

We will prove Proposition 5.4 for negative powers of $T$. The same proof will work for positive powers and (23) is obvious for $n = 0$. Let $\beta_1, \ldots, \beta_p$ be the Galois conjugates of $T^{-1}$ over $\nu(E)$. We will utilize the following symmetric functions in $\beta_1, \ldots, \beta_p$:

$$f_n = \beta_1^n + \cdots + \beta_p^n, \quad \text{for } n \geq 1,$$

$$e_n = \sum_{i_1 < \cdots < i_n} \beta_{i_1} \cdots \beta_{i_n} \quad \text{for } n = 1, \ldots, p.$$

Note that

$$U_p(T^{-n}) = \frac{f_n}{p}. \hspace{1cm} (24)$$

**Lemma 5.5.** (Newton identities) We have

$$f_n = (-1)^{n-1} ne_n + \sum_{i=1}^{n-1} (-1)^{n-1+i} e_{n-1} f_i \quad \text{for } n \leq p$$

$$f_n = \sum_{i=n-p}^{n-1} (-1)^{n-1+i} e_{n-i} f_i \quad \text{for } n > p.$$

**Proof.** For $n = 1$ the identity is immediate. Then proceed by induction. \hfill \Box

The Newton identities allows us to use the $e'_n$s to study the $f'_n$s. Thus, we must estimate $e_n$.

**Lemma 5.6.** Let $b \in \mathcal{O}_E^m(k)$. Then $b^\nu \in \mathcal{O}_E^{pm}(pk)$.

**Proof.** This lemma follows from (5), by considering the expansion of $b$. \hfill \Box

**Lemma 5.7.** We have $e_n^{p-1} \in \mathcal{O}_E^m$ for $n = 1, \ldots, p-1$ and $e_p^{p-1} \in \mathcal{O}_E^m(1)$.

**Proof.** Consider the minimal polynomial of $T^{-1}$ over $\nu(E)$:

$$Q(Y) = Y^p + a_1^\nu Y^{p-1} + \cdots + a_p^\nu,$$

where $a_i \in \mathcal{O}_E^1$ and $e_i = (-1)^i a_i^\nu$. We will prove inductively that there exists

$$Q_n(Y) = Y^p + a_{1,n}^\nu Y^{p-1} + \cdots + a_{p,n}^\nu,$$

such that the following hold: $Q_n(Y) \equiv Q(Y) \mod \pi^{n+1}$, $a_{i,n}$ is contained in $\mathcal{O}_E^m$ for $i = 1, \ldots, p-1$, and $a_{p,n} \in \mathcal{O}_E^m(1)$. 

For \( n = 0 \), we may take \( a_{i,1} = 0 \) for \( i = 1, \ldots, p - 1 \) and \( a_{p,1} = T^{-1} \). Next, let \( n \geq 0 \) and assume that \( Q_n(Y) \) exists. By our induction hypothesis and Lemma 5.6 we have

\[
a_{\nu, n}^\nu \in \mathcal{O}_{E}^{pm}(p) \\
a_{i, n}^\nu T^{-i} \in T^{-i} \mathcal{O}_{E}^{pm} \subset \mathcal{O}_{E}^{pm}(p).
\]

This implies \( Q_n(T^{-1}) \in \mathcal{O}_{E}^{pm}(p) \), which in turn means \( w_{n+1}(Q_n(T^{-1})) \geq -(n+1)p - p \). Let \( r = \frac{n+1}{e} \). Then

\[
Q_n(T^{-1}) \equiv \pi^{n+1} \sum_{k>rpm-p}^{\infty} c_k T^k \mod \pi^{n+2}
\]

\[
\equiv \pi^{n+1} \sum_{i=0}^{p-1} T^{-i} \sum_{k=d_i}^{\infty} c_{i,k}(T^k)^\nu \mod \pi^{n+2},
\]

where \( d_i \geq -rm \) for \( i = 1, \ldots, p - 1 \) and \( d_p \geq -rm - 1 \). We see that

\[
a_{i,n+1} = a_{i,n} - \pi^{n+1} \sum_{k=d_i}^{\infty} c_{i,k} T^k
\]

satisfies the desired properties. \( \square \)

**Proof.** (of Proposition 5.4) By apply Lemma 5.5 and Lemma 5.7 inductively, we see that \( f_{\nu}^{n-1} \in \mathcal{O}_{E}^{pm}(-\lceil \frac{1+n}{p} \rceil) \). The proposition follows from (24) and Lemma 3.1. \( \square \)

### 5.3 Local \( U_p \) operators for some specific \( p \)-Frobenius endomorphisms

Now we give estimates of \( U_p \) for three specific types of \( p \)-Frobenius endomorphisms. The first two occur when considering a global parameter of \( \mathbb{P}^1 \) localized at a point \( P \in \mathbb{P}^1 \). The third type will occur when we consider a simply branched cover of \( \mathbb{P}^1 \).

#### 5.3.1 Type 1: \( T \rightarrow T^p \)

The first type of \( p \)-Frobenius endomorphism is \( T^\nu = T^p \). In this case, the Galois conjugates of \( T \) over \( \nu(E^1) \) are \( T, \zeta_p T, \ldots, \zeta_p^{p-1} T \), where \( \zeta_p \) is a nontrivial \( p \)-th root of unity. Thus

\[
U_p(T^n) = \begin{cases} 
0, & \text{if } p \nmid n \\
T^n, & \text{if } p | n.
\end{cases}
\]

In particular, we see that \( U_p \) is a 0-moderate operator.

#### 5.3.2 Type 2: \( T \rightarrow (T + b)^p - b^p \)

The next type is of \( p \)-Frobenius endomorphism is of the from \( T^\nu = (T + b)^p - b^p \). This gives

\[
T^\nu = T^p + b^p - b^p + \sum_{i=1}^{p-1} \binom{p}{i} T^i b^{p-1}.
\]
Since \( p|b^p - b^r \) we know \( T^r \in \mathcal{O}_{\mathcal{E}}^p(-p) \). By Proposition 5.4 we see that \( U_p \) is 1-moderate. In fact, from (23) we may deduce a stronger property:

\[
U_p[-1]\left(\mathcal{O}_{\mathcal{E}}^p(n)\right) \subseteq \mathcal{O}_{\mathcal{E}}\left(-\left\lfloor \frac{n}{p} \right\rfloor\right). \tag{25}
\]

### 5.3.3 Type 3: \( T \rightarrow \sqrt{(T^2 + b)^p - b^r} \)

The last type of \( p \)-Frobenius endomorphism is of the form \( T^r = \sqrt{(T^2 + b)^p - b^r} \). We will prove \( U_p \) is 2-moderate. Note that \((T^2 + b)^p - b^r \in \mathcal{O}_{\mathcal{E}}^2(-2p) \). In particular, we have \( T^{-2r}((T^2 + b)^p - b^r) = 1 + py \), where \( py \in \mathcal{O}_{\mathcal{E}}^2 \). This gives

\[
T^{-p}T^r = \sqrt{1 + py}
= \sum_{n=0}^{\infty} \left(\frac{1}{n}\right)p^n y^n.
\]

This power series converges, since the binomials \( \left(\frac{1}{n}\right) \) have nonnegative \( p \)-adic valuation. Therefore \( T^{-p}T^r \in \mathcal{O}_{\mathcal{E}}^{2p} \) and \( T^r \in \mathcal{O}_{\mathcal{E}}^{2p}(-p) \).

Let \( S = T^2 \) and let \( \mathcal{E}_0 \) be the Amice over \( L \) with parameter \( S \). Then \( \mathcal{E} = \mathcal{E}_0 \oplus T^r \mathcal{E}_0 \). Note that \( T^{-p}T^r \in \mathcal{E}_0 \), since \( y \in \mathcal{E}_0 \) and therefore \( T^r \mathcal{E}_0 = T \mathcal{E}_0 \). We have

\[
\mathcal{O}_{\mathcal{E}}^{2m}(2n) \cap \mathcal{E}_0 = \mathcal{O}_{\mathcal{E}_0}^m(n). \tag{26}
\]

Let \( D = U_p[-2] \) and let \( D_0 = D|_{\mathcal{E}_0} = (U_p|_{\mathcal{E}_0})[-1] \). The restriction \( \nu|_{\mathcal{E}_0} \) is a \( p \)-Frobenius endomorphism of Type 2. Therefore, by (25) and (26) we know that for \( n \) even \( D(T^n) \in \mathcal{O}_{\mathcal{E}_0}^2(\left\lfloor \frac{n}{p} \right\rfloor) \). For \( n \) odd, we write \( T^n = T^r z \), with \( z \in \mathcal{E}_0 \). Let \( r \in \mathbb{Z} \) with \( 2r = n - p \). By (5) and (26) we have \( z \in \mathcal{O}_{\mathcal{E}_0}^p(-r) \). Then from (25) and (26) we have \( D(T^n) = TD(z) \in T\mathcal{O}_{\mathcal{E}_0}^1(\left\lfloor \frac{r}{p} \right\rfloor) \). Observing that \(-1 - 2\left\lfloor \frac{r}{p} \right\rfloor < -\left\lfloor \frac{n}{p} \right\rfloor \), we see that

\[
T\mathcal{O}_{\mathcal{E}_0}^2\left(-\left\lfloor \frac{n}{p} \right\rfloor\right) \subseteq \mathcal{O}_{\mathcal{E}}^2\left(-1 - 2\left\lfloor \frac{r}{p} \right\rfloor\right) \subseteq \mathcal{O}_{\mathcal{E}}^2\left(-\left\lfloor \frac{n}{p} \right\rfloor\right).
\]

Therefore \( D \) is 0-moderate.

### 5.3.4 Summary

We may summarize the content of §5.3.1, §5.3.2 and §5.3.3 with the following Proposition.

**Proposition 5.8.** Let \( \kappa = 1, 2, 3 \). Let \( \nu \) be a \( p \)-Frobenius endomorphism of type \( \kappa \). Then \( U_p \) is an \((\kappa - 1)\)-moderate operator.

### 6 Global setup

#### 6.1 Basic setup

Consider the projective line \( \mathbb{P}^1_{\mathbb{F}_q} = \text{Proj}(\mathbb{F}_q[x_1, x_2]) \) and let \( \mathbf{T} = \frac{x_1}{x_2} \) be a parameter at 0. Let \( P_1, \ldots, P_t \in \mathbb{P}^1_{\mathbb{F}_q} \) with \( P_1 = 0 \) and \( P_2 = \infty \). Let

\[
\eta : X \rightarrow \mathbb{P}^1_{\mathbb{F}_q}
\]
be a simple branched cover of degree $n$ ramified over the points $P_1, \ldots, P_w$, where $w \leq \ell$. Let $g$ be the genus of $X$. Then $w = 2g + 2n - 2$. For $1 \leq i \leq w$, let $\{Q_{i,1}, \ldots, Q_{i,n-1}\} = \eta^{-1}(P_i)$ where $Q_{i,1}$ has ramification index 2 over $P_i$ and $Q_{i,j}$ is unramified over $P_i$ for $j > 1$. For $w < i \leq \ell$, let $\{Q_{i,1}, \ldots, Q_{i,n}\} = \eta^{-1}(P_i)$. Set $W = \eta^{-1}\{\{P_1, \ldots, P_\ell\}\}$. We define $v = w(n-1) + (\ell - w)n$, which is the number points in $W$. Let $U = \mathbb{P}^1_{\mathbb{F}_q} - \{P_1, \ldots, P_\ell\}$ and $V = X - W$. Then $V \rightarrow U$ is a finite \'{e}tale map of degree $n$. Let $\overline{B}$ (resp. $\overline{A}$) be the $\mathbb{F}_q$-algebra such that $V = \text{Spec}(\overline{B})$ (resp. $U = \text{Spec}(\overline{A})$).

Let $\mathbb{P}^1_{\mathcal{O}_L}$ be the projective line over $\text{Spec}(\mathcal{O}_L)$ and let $\mathbb{P}^1_{\mathcal{O}_{L'}}$ be the formal projective line over $\text{Spf}(\mathcal{O}_L)$. Let $T$ be a global parameter of $\mathbb{P}^1_{\mathcal{O}_L}$ lifting $\overline{T}$. By the deformation theory of tame coverings (see \cite{12} Theorem 4.3.2]) there exists a tame cover $\eta : X \rightarrow \mathbb{P}^1_{\mathcal{O}_L}$ and by formal GAGA (see \cite{21} Tag 09ZT), there exists a morphism of smooth curves $X \rightarrow \mathbb{P}^1_{\mathcal{O}_L}$ whose formal completion is $\eta : X \rightarrow \mathbb{P}^1_{\mathcal{O}_{L'}}$. There exists local parameters $T_{P_i}$ and $S_{Q_{i,j}}$, which yield the diagram:

$$
\begin{array}{c}
\hat{\mathcal{O}}_{X,Q_{i,j}} \cong \mathcal{O}_L[[S_{Q_{i,j}}]] \\
\hat{\mathcal{O}}_{X,Q_{i,j}} \cong \mathbb{F}_q[[\overline{F}_{Q_{i,j}}]] \\
\end{array}
$$

$$
\begin{array}{c}
\hat{\mathcal{O}}_{\mathbb{P}^1_{\mathcal{O}_{L',P_i}}} \cong \mathcal{O}_L[[T_{P_i}]] \\
\hat{\mathcal{O}}_{\mathbb{P}^1_{\mathbb{F}_q,P_i}} \cong \mathbb{F}_q[[\overline{T}_{P_i}]]. \\
\end{array}
$$

Our assumptions on the branching of $\eta : X \rightarrow \mathbb{P}^1_{\mathbb{F}_q}$ allows us to choose $S_{Q_{i,j}}$ as follows:

1. If $1 \leq i \leq w$ and $j = 1$, then $Q_{i,j}$ has ramification index 2 and we take $S_{Q_{i,j}} = \sqrt{T_{P_i}}$.

2. Otherwise, $Q_{i,j}$ has ramification index 1, so we may take $S_{Q_{i,j}} = T_{P_i}$.

For each $P_i$ (resp. $Q_{i,j}$) we obtain an $\mathcal{O}_L$-point of $\mathbb{P}^1_{\mathcal{O}_L}$ (resp. $X$) by evaluating at $T_{P_i} = 0$ (resp. $S_{Q_{i,j}} = 0$):

$$
\begin{array}{c}
X \rightarrow \text{Spec}(\mathcal{O}_L[[S_{Q_{i,j}}]]) \\
\text{Spec}(\mathcal{O}_L) \\
\end{array}
$$

We denote the $\mathcal{O}_L$-point of $\mathbb{P}^1_{\mathcal{O}_L}$ (resp. $X$) by $[P_i]$ (resp. $[Q_{i,j}]$). After applying an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$, we may assume that $[P_i] = 0$ and $[P_2] = \infty$. Thus we may take $T_{P_1} = T$ and $T_{P_2} = \frac{1}{T}$.

For $i > 2$ we set $T_{P_i}$ to be $T - [P_i]$.

Let $U = \mathbb{P}^1_{\mathcal{O}_L} - \{[P_1], \ldots, [P_\ell]\}$ and $V = X - \{[R]\}_{R \in W}$, so that $\eta : V \rightarrow U$ is \'{e}tale. Similarly, we define $\overline{U} = \mathbb{P}^1_{\mathcal{O}_{L'}} - \{P_1, \ldots, P_\ell\}$ and $\overline{V} = X - \{R\}_{R \in W}$. Note that $U$ (resp. $V$) is the formal completion of $U$ (resp. $V$). Finally, we let $V^{\text{rig}}$ (resp. $U^{\text{rig}}$) be the rigid analytic fiber of $V$ (resp. $U$).

### 6.2 Local parameters and overconvergent rings

Let $A$ (resp. $\hat{A}$ and $\hat{\mathcal{E}}_{Q_{i,j}}$) be the ring of functions $\mathcal{O}_U(U)$ (resp. $\mathcal{O}_U(U)$ and $\mathcal{O}_{U^{\text{rig}}}(U^{\text{rig}})$) and let $B$ (resp. $\hat{B}$ and $\hat{\mathcal{E}}_{Q_{i,j}}$) be the ring of functions $\mathcal{O}_V(V)$ (resp. $\mathcal{O}_V(V)$ and $\mathcal{O}_{U^{\text{rig}}}(U^{\text{rig}})$). Let $S_{T_{P_i}}$ (resp. $S_{Q_{i,j}}$) be the Amice ring over $L$ with parameter $T_{P_i}$ (resp. $S_{Q_{i,j}}$). By expanding functions in terms of the $T_{P_i}$'s and $S_{Q_{i,j}}$'s, we obtain the following diagrams:
We let $A^\dagger$ (resp. $B^\dagger$) be the subring of $\hat{A}$ (resp. $\hat{B}$) consisting of functions that are overconvergent in the tube $|P_i|$ for each $i$ (resp. $|R|$ for all $R \in W$). In particular, $B^\dagger$ fits into the following Cartesian diagram:

\[
\begin{array}{ccc}
\hat{B} & \rightarrow & \bigoplus_{R \in W} \mathcal{O}_{\mathcal{E}_R} \\
\uparrow & & \uparrow \\
\hat{A} & \rightarrow & \bigoplus_{i=1}^\ell \mathcal{O}_{\mathcal{E}_{P_i}}
\end{array}
\]  

(27)

Note that $A^\dagger$ (resp. $B^\dagger$) is the weak completion of $A$ (resp. $B$) in the sense of \cite{17, §2}. In particular, we have $A^\dagger = \mathcal{O}_L \left( T, T^{-1}, \frac{1}{T-[P_1]}, \ldots, \frac{1}{T-[P_l]} \right)^\dagger$ and $B^\dagger$ is a étale finite $A^\dagger$-algebra.

Finally, we define $A^\dagger$ (resp. $B^\dagger$) to be $A^\dagger \otimes \mathbb{Q}_p$ (resp. $B^\dagger \otimes \mathbb{Q}_p$). Then $A^\dagger$ (resp. $B^\dagger$) is equal to the functions in $\hat{A}$ (resp. $\hat{B}$) that are overconvergent in the tube $|P_i|$ for each $i$ (resp. $|R|$ for all $R \in W$).

6.3 Global Frobenius and $U_p$ operators

Let $\nu : A^\dagger \rightarrow A^\dagger$ be the ring endomorphism that restricts to $\nu$ on $L$ and sends $T$ to $T^p$. Let $\sigma = \nu^f$. For each $i$, we may extend $\nu$ to a $p$-Frobenius endomorphism of $\mathcal{E}_{P_i}^\dagger$, which we refer to as $\nu_{P_i}$. In terms of the parameters $T_{P_i}$, these maps are given as follows:

1. For $i = 1, 2$, we have $T_{P_i} = T^{\pm 1}$. This means $T_{P_i}^{\nu_{P_i}} = T_{P_i}^p$ and thus $\nu_{P_i}$ is of type 1.

2. For $i > 2$, since $T_{P_i} = T - [P_i]$ we see that $\nu_{P_i}$ is of type 2.

Since the map $\hat{A} \rightarrow \hat{B}$ is étale and both rings are $p$-adically complete, we may extend $\sigma$ and $\nu$ to maps $\sigma, \nu : \hat{B} \rightarrow \hat{B}$. Note that $\nu$ lifts the absolute frobenius on $V$. This in turn gives $p$-Frobenius endomorphisms $\nu_{Q_{i,j}}$ of $\mathcal{E}_{Q_{i,j}}$, which make the diagrams in (27) $p$-Frobenius equivariant. Furthermore, since $\nu_{Q_{i,j}}$ extends $\nu_{P_i}$, we know that $\nu_{Q_{i,j}}$ induces a Frobenius endomorphism of $\mathcal{E}_{Q_{i,j}}^\dagger$. It follows from diagram (28) that $\sigma$ and $\nu$ restrict to an endomorphisms $\nu, \sigma : B^\dagger \rightarrow B^\dagger$. Following \cite{23, §3}, there is a trace map $Tr_0 : B^\dagger \rightarrow \nu(B^\dagger)$ (resp. $Tr : B^\dagger \rightarrow \sigma(B^\dagger)$). As in §3 we may define the $U_p$ operator on $B^\dagger$:

\[
U_p : B^\dagger \rightarrow B^\dagger \\
x \rightarrow \frac{1}{p} \nu^{-1}(Tr_0(x)).
\]
Similarly, we define \( U_q = \frac{1}{q} \sigma^{-1} \circ Tr \), so that \( U_p^f = U_q \). Note that \( U_p \) is \( E \)-linear and \( U_q \) is \( L \)-linear. The diagram (27) is equivariant for the \( U_p \) operator on \( B^f \) and the \( U_p \) operator defined on \( E_{Q,i,j}^1 \). In terms of the parameters \( S_{Q,i,j} \), the \( \nu_{Q,i,j} \) are of the following types:

1. Let \( i = 1, 2 \). Then for \( j = 1 \) we have \( S_{Q,i,j}^2 = T_{P_i} \) and for \( j > 1 \) we have \( S_{Q,i,j} = T_{P_i} \). Since \( \nu_{P_i} \) is of type 1, we see that \( \nu_{Q,i,j} \) is also of type 1.

2. Let \( 2 < i \leq w \) and \( j > 1 \). Then \( S_{Q,i,j} = T_{P_i} \), so that \( \nu_{Q,i,j} \) is also of type 2.

3. Let \( i > w \). Again, we find that \( \nu_{Q,i,j} \) is of type 2.

4. Finally, consider \( 2 < i \leq w \) and \( j = 1 \). We have \( S_{Q,i,j}^2 = T_{P_i} \). Since \( \nu_{P_i} \) is of type 2, we see that \( \nu_{Q,i,j} \) is of type 3.

We let \( \kappa_{Q,i,j} \) denote the type of \( \nu_{Q,i,j} \). Define \( k_n \) to be the number of \( Q_{i,j} \) such that \( \nu_{Q_{i,j}} \) is of type \( n \). Then we have

\[
\begin{align*}
  k_1 &= 2n - 2, \\
  k_2 &= (w - 2)(n - 2) + (\ell - w)n, \\
  k_3 &= w - 2.
\end{align*}
\]

Note that \( v = k_1 + k_2 + k_3 \). Using the formula \( w = 2g + 2n - 2 \) we obtain

\[
2(g + v - 1) = k_1 + 2k_2 + 3k_2 = \sum_{R \in W} \kappa_R.
\] (29)

7 Proof of main theorem

We will now finish the proof of Theorem 1.1. We begin with a description of the Dwork \( F \)-isocrystal corresponding to an Artin-Schreier cover. Next, we review the Monsky trace formula. Then using Proposition 4.8 and Proposition 5.8, we prove lower bounds for \( NP_q(L(\rho, s)) < 1 \). Finally, we explain how to deduce Theorem 1.1.

Define \( r : C \to X \), \( \tau_s \), \( d_i \) and \( f \) as in §1.2.1. By [11], there exists a simply branched cover \( X \to \mathbb{P}^1_{\mathbb{F}_q} \). We carry over all of the notation from §6 for this cover. Since we need to apply Proposition 4.8 we increase \( E \) so that \( \nu_{P_i}(E) \) contains \( r_{d_i}(n) \) for each \( i \) and \( n \). Furthermore, by increasing the number of \( P_1, \ldots, P_\ell \) in \( \mathbb{P}^1_{\mathbb{F}_q} \), we may assume \( \tau_1, \ldots, \tau_m \in W \).

7.1 The Dwork \( F \)-isocrystal associated to \( f \)

We let \( f \in \Gamma(U, O_U) \) be a lift of \( \overline{f} \) and let \( z \) be a solution to \( Y^p - Y = f \). As in §3.5 we obtain a nontrivial Galois representation \( \rho : \pi_{1}^f(V) \to \mathbb{Z}_p[\zeta_p]^\times \) given by

\[
\gamma \to \frac{\theta(z) \gamma}{\theta(z)},
\]

which factors through \( Gal(C/X) \). The corresponding unit-root \( F \)-isocrystal is called a global Dwork \( F \)-isocrystal. Following the discussion at the end of §3.3 we see that \( \alpha_0 = \frac{\theta(z) \gamma}{\theta(z)} \in \overline{B} \) and
that the Frobenius structure of the global Dwork $F$-isocrystal is
\[
\alpha = \alpha_0 \alpha_0^\nu \ldots \alpha_0^{\nu^{a-1}} = \frac{\theta(z)^\sigma}{\theta(z)}.
\]
By Proposition 3.9, we know that the image of $\alpha$ and $\alpha_0$ in $\mathcal{O}_{E_R}$ are contained in $\mathcal{O}_{E_R}^\dagger$ for each $R \in W$. Thus, by the Cartesian diagram (28) we know $\alpha, \alpha_0 \in B^\dagger$. We further note that $\alpha, \alpha_0 \equiv 1 \mod \pi B^\dagger$.

We now relate the global Dwork $F$-isocrystal to the local Dwork $F$-isocrystal introduced in §3.5. For $R \in W$, we define
\[
d_R = \begin{cases} 
d_k & \text{if } R = \tau_k \\
1 & \text{if } R \not\in \{\tau_1, \ldots, \tau_m\} \end{cases},
\]
In $\mathbb{F}_q((S_R))$, the global Artin-Schreier equation (1) can be transformed into an equation $Y^p - Y = g_R$, where $\text{ord}_R(g_R) \geq -d_R$. This inequality is an equality if and only if $R \in \{\tau_1, \ldots, \tau_m\}$. Let $g_R \in \mathcal{O}_{E_R}$ be a lift of $g_R$ such that $g_R \in S_R^{-d_R} \mathcal{O}_{La}[[S_R]]$. Let $z_R$ be a solution to the equation $Y^p - Y = g_R$. Then $\theta(z_R)$ is a period of the representation
\[
\rho_R : \text{Gal}(k((S_R)))^\text{sep}/k((S_R)) \to \pi_1^{\et}(V) \xrightarrow{\rho} \mathbb{Z}_p[\zeta_p]^\times;
\]
and by Proposition 3.9 we know
\[
c_R = \frac{\theta(z_R)^\nu}{\theta(z_R)} \in \mathcal{R}_{d_R}. \tag{30}
\]
Furthermore, the image of $\theta(z)$ in $\tilde{E}_R^\text{unr}$ is also a period of $\rho_R$. By the discussion after Definition 3.4 we know that there exists $b_R \in \mathcal{O}_{E_R}^\dagger$ such that $b_R^{-\nu}, b_R \alpha_0 = c_R$. Also, since $c_R \equiv 1 \mod \pi \mathcal{O}_{E_R}^\dagger$, we know that
\[
b_R \equiv 1 \mod \pi \mathcal{O}_{E_R}^\dagger. \tag{31}
\]
We let $\bar{c}$ (resp. $\bar{b}$ and $\alpha_0$) denote the element of $\bigoplus_{R \in W} \mathcal{O}_{E_R}^\dagger$ whose $R$-coordinate is $c_R$ (resp. $b_R$ and $\alpha_0$). Then we have
\[
\frac{\bar{b}^\nu}{\bar{b}} \bar{\alpha}_0 = \bar{c}, \tag{32}
\]
where each operation is done coordinate-wise.

### 7.2 The Monsky trace formula

We now give a brief overview of the Monsky trace formula for curves. For a complete treatment, see [16] or [25, §10]. Let $\Omega_{\mathcal{B}^\dagger}^i$ denote the space of $i$-forms of $\mathcal{B}^\dagger$ (see [16, §4]). The map $\sigma$ induces
a map $\sigma_i : \Omega_{B^\dagger}^i \to \Omega_{B^\dagger}^i$ that sends $xdy$ to $x^\sigma d(y^\sigma)$. Following \[23\] §3, there are trace maps $\text{Tr}_i : \Omega_{B^\dagger}^i \to \sigma(\Omega_{B^\dagger}^i)$. We then let $\Theta_i$ denote the map $\sigma_i^{-1} \circ \text{Tr}_i$. Note that for $\omega \in \Omega_{B^\dagger}$ we have

$$\Theta_1(x\omega^\sigma) = \Theta_0(x)\omega. \quad (33)$$

Now consider the $\mathcal{L}$-function

$$L(\rho, V, s) = \prod_{x \in V} \frac{1}{1 - \rho(\text{Frob}_x)s \deg(x)}, \quad (34)$$

which is a slight modification of (2). The Monsky trace formula states

$$L(\rho, V, s) = \det(1 - s\Theta_1 \circ L_\alpha|\Omega_{B^\dagger}) = \det(1 - s\Theta_0 \circ L_\alpha|\hat{B}^\dagger), \quad (35)$$

Thus we may estimate $L(\rho, V, s)$ by estimating operators on the space of 1-forms and 0-forms.

In our situation we may simplify (35). The ring homomorphism $\mathcal{A}^\dagger \to \hat{B}^\dagger$ is étale, which means $\Omega_{B^\dagger} = \pi^\ast \Omega_{\mathcal{A}^\dagger}$. Since $\Omega_{\mathcal{A}^\dagger}$ is equal to $\mathcal{A}^\dagger \frac{dT}{T}$, it follows that $\Omega_{B^\dagger} = \hat{B}^\dagger \frac{dT}{T}$. Furthermore, as $\frac{dT}{T} = \frac{1}{q} \frac{dT^\sigma}{T}$ we know by (33) that

$$\Theta_1(x \frac{dT}{T}) = \frac{1}{q} \Theta_0(x) \frac{dT}{T},$$

which means $\Theta_1 = U_q$ and $\Theta_0 = qU_q$. Therefore (35) becomes

$$L(\rho, V, s) = \frac{\det(1 - sU_q \circ L_\alpha|\hat{B})}{\det(1 - sqU_q \circ L_\alpha|\hat{B})}. \quad (36)$$

As $\det(1 - sU_q \circ L_\alpha|\hat{B}) \in \mathcal{O}_L[[s]]$, we obtain

$$NP_q(L(\rho, V, s))_{<1} = NP_q(U_q \circ L_\alpha)_{<1}. \quad (37)$$

**Lemma 7.1.** We have $NP_q(U_q \circ L_\alpha) = \frac{1}{a}NP_p(U_p \circ L_{\alpha_0})$. In particular, from (37) we obtain

$$NP_q(L(\rho, V, s))_{<1} = \frac{1}{a}NP_p(U_p \circ L_{\alpha_0})_{<1}. \quad (38)$$

**Proof.** This is a slight modification of an argument due to Dwork (see \[8\] §7 or \[4\] Lemma 2). For an $\mathcal{L}$-linear operator $u : V \to V$, we let $\det_F(1 - su)$ denote the Fredholm determinant when we view $u$ as an $E$-linear operator. Then we have

$$\det_F(1 - sU_q \circ L_\alpha) = N_{L/E} \det(1 - sU_q \circ L_\alpha),$$

where $N_{L/E}$ denotes the norm from $L$ to $E$. From (36) we know that $\det_F(1 - sU_q \circ L_\alpha) = \prod_{i=0}^{\infty} L(\rho, V, q^i s)$, and thus by (34) we know that $\det(1 - sU_q \circ L_\alpha)$ has coefficients in $E$. This implies

$$\det_F(1 - sU_q \circ L_\alpha|\hat{B}^\dagger) = \det_F(1 - sU_q \circ L_\alpha|\hat{B}^\dagger)^a. \quad (39)$$
Furthermore, we have \((U_p \circ \alpha_0)^a = U_q \circ \alpha\) as \(E\)-linear operators. Thus, (39) gives
\[
\det(1 - s^a U_q \circ L \alpha |B^\dagger)^a = \prod_{\zeta^a=1} \det(1 - \zeta s U_p \circ L \alpha_0 |B^\dagger).
\]
Each term in the product has the same Newton polygon, which means
\[
NP_q(\det(1 - s^a U_q \circ L \alpha |B^\dagger)) = NP_q(\det(1 - s U_p \circ L \alpha_0 |B^\dagger)).
\]
The lemma follows from (18).

7.3 Estimating \(NP_p(U_p \circ L \alpha_0) < \frac{1}{2}\).

Define \(\beta\) to be the composition of maps
\[
\beta : \wedge \longrightarrow \bigoplus_{R \in W} E_R \xrightarrow{L^\mathcal{E}} \bigoplus_{R \in W} E_R.
\]

We let \(\psi\) and \(\psi'\) to be the projection maps (see §3.1):
\[
\psi : \bigoplus_{R \in W} E_R \longrightarrow \bigoplus_{R \in W} E^-_R,
\]
\[
\psi' : \bigoplus_{R \in W} E_R \longrightarrow \bigoplus_{R \in W} E^\leq_{\kappa R}.
\]

We define \(\beta_0, \psi_0\) and \(\psi'_0\) to be integral versions of these maps, e.g.,
\[
\psi_0 : \bigoplus_{R \in W} \mathcal{O}_{E_R} \longrightarrow \bigoplus_{R \in W} \mathcal{O}_{E_R}.
\]

Similarly, we define \(\overline{\beta}, \overline{\psi}\) and \(\overline{\psi'}\) to be the reductions modulo \(\pi\).

**Lemma 7.2.** The map \(\beta \circ \psi\) is injective. The map \(\beta \circ \psi'\) is surjective. The kernel of \(\beta \circ \psi'\) has dimension \(v + g - 1\) as a vector space over \(L\).

**Proof.** By Lemma 4.1 we may prove the corresponding results for \(\overline{\beta} \circ \overline{\psi}\) and \(\overline{\beta} \circ \overline{\psi'}\). We know from (31) that the map \(L^\mathcal{E}\) is the identity map. Thus, \(\overline{\beta} \circ \overline{\psi}\) may be computed on each summand as follows. For \(g \in \wedge\), we first expand \(g\) in terms of the parameter \(S_R\). We then truncate the positive powers of \(S_R\):
\[
g = \sum_{n \gg -\infty} a_n S^n_R \rightarrow \sum_{n \gg -\infty} a_n S^n_R.
\]
This makes it clear that \(\overline{\beta} \circ \overline{\psi}\) is injective. Similarly, \(\overline{\beta} \circ \overline{\psi'}\) may be computed by truncating the powers of \(S_R\) after \(S^\leq_{\kappa R}\):
\[
g = \sum_{n \gg -\infty} a_n S^n_R \rightarrow \sum_{n \gg -\infty} a_n S^n_R.
\]

Let \(D\) be the effective divisor \(\sum_{R \in W} (\kappa R - 1)R\). The kernel of \(\overline{\beta} \circ \overline{\psi}\) is equal to \(H^0(X, \mathcal{O}_X(D))\). By (29), we know that \(D\) has degree \(2g + v - 2\), so by the Riemann-Roch theorem we have \(\ker(\overline{\beta} \circ \overline{\psi}) = v + g - 1\). Similarly, the Riemann-Roch theorem tells us that \(\overline{\beta} \circ \overline{\psi'}\) is surjective.
By Lemma 7.2, we see that there is a section $t$ of $\psi$ that makes the following diagram commute:

$\xymatrix{ \hat{B} \ar[r]^\beta \ar[d]_{\psi \circ \beta} & \bigoplus_{R \in W} \mathcal{E}_R \\
\bigoplus_{R \in W} \mathcal{E}_R^- \ar@{..>}[u]^t }$

Furthermore, we may take $t$ so that $t \left( \bigoplus_{R \in W} O_{E_R^-} \right) \subset \bigoplus_{R \in W} O_{E_R}$. Let $V = \psi \circ \beta(\hat{B})$. Since $L_{\tilde{b}} \circ U_p = U_p \circ L_{\tilde{b}}$, we know from (32) that

$$L_{\tilde{b}} \circ U_p \circ L_{\tilde{c}} = U_p \circ L_{\tilde{c}}.$$

We define $J = \psi \circ U_p \circ L_{\tilde{c}} \circ t$, to obtain

$$\det(1 - sU_p \circ L_{\alpha_0} | \hat{B}) = \det(1 - s\psi \circ L_{\tilde{b}} \circ U_p \circ L_{\alpha_0} \circ L_{\tilde{b}^{-1}} \circ t|V) = \det(1 - sJ|V). \quad (40)$$

**Proposition 7.3.** We have

$$\frac{1}{a} NP_p(U_p \circ L_{\alpha_0}) \geq \left\{ 0, \ldots, 0 \right\} \bigcup \left( \bigcup_{R \in W} \left\{ \frac{1}{d_R}, \ldots, \frac{\left\lfloor \frac{d_R - 1}{2} \right\rfloor}{d_R} \right\} \right)$$

**Proof.** By (40) it is enough to prove the bounds about $NP_p(J|V)$. First, we claim that

$$J \left( \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R) \right) \subset \bigoplus_{R \in W} \mathcal{T}_{d_R}^{-}(\kappa_R). \quad (41)$$

To see this, first note that

$$t \left( \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R) \right) \subset \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R),$$

$$L_{\tilde{c}} \left( \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R) \right) \subset \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R).$$

Then by Proposition 5.8 and Proposition 5.3 we know

$$U_p \left( \bigoplus_{R \in W} \mathcal{R}_{d_R}(\kappa_R) \right) \subset \bigoplus_{R \in W} \mathcal{T}_{d_R}(\kappa_R),$$

from which (41) follows. The proposition then follows from Proposition 4.8 and Lemma 7.2.
7.4 Finishing the proof

We now finish the proof of Theorem 1.1. From (38) and Proposition 7.3, we know that

\[ NP_q(L(\rho, V, s)) < \frac{1}{2} \geq \left\{ 0, \ldots, 0 \right\} \bigcup \left( \bigcup_{R \in W} \left\{ \frac{1}{d_R}, \ldots, \frac{d_R - 1}{2} \right\} \right) \]  \hspace{1cm} (42)

For \( R \notin \{\tau_1, \ldots, \tau_m\} \) we have \( d_R = 1 \), so the corresponding arithmetic sequence in (42) is empty. Thus, (42) becomes

\[ NP_q(L(\rho, V, s)) < \frac{1}{2} \geq \left\{ 0, \ldots, 0 \right\} \bigcup \left( \bigcup_{\tau_1 \leq i \leq \tau_m} \left\{ \frac{d_{\tau_i} - 1}{2} \right\} \right) \]  \hspace{1cm} (42)

From (2) and (34) we have

\[ L(\rho, V, s) = L(\rho, s) \cdot \prod_{R \notin \{\tau_1, \ldots, \tau_m\}} (1 - \rho(\text{Frob}_R)s). \]

There are \( v - m \) terms in this product, and each term accounts for a segment of slope zero. Thus,

\[ NP_q(L(\rho, s)) < \frac{1}{2} \geq \left\{ 0, \ldots, 0, \frac{1}{d_{\tau_1}}, \ldots, \frac{1}{d_{\tau_m}} \right\} \bigcup \left( \bigcup_{\tau_1 \leq i \leq \tau_m} \left\{ \frac{d_{\tau_i} - 1}{2} \right\} \right) \]

We know that the degree of \( L(\rho, s) \) is \( 2(m + g - 1) + \sum(d_{\tau_i} - 1) \). Then from the functional equation of \( L(\rho, s) \) we deduce:

\[ NP_q(L(\rho, s)) \geq \left\{ 0, \ldots, 0, 1, \ldots, 1, \frac{1}{d_1}, \ldots, \frac{1}{d_m} \right\} \bigcup \left( \bigcup_{\tau_1 \leq i \leq \tau_m} \left\{ \frac{d_{\tau_i} - 1}{2} \right\} \right) \]

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