A generalization of Araki’s log-majorization

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Abstract

We generalize Araki’s log-majorization to the log-convexity theorem for the
eigenvalues of $\Phi(A^p)^{1/2}\Psi(B^p)\Phi(A^p)^{1/2}$ as a function of $p \geq 0$, where $A, B$
are positive semidefinite matrices and $\Phi, \Psi$ are positive linear maps between matrix
algebras. A similar generalization of the log-majorization of Ando-Hiai type is
given as well.

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1 Introduction

The Lieb-Thirring inequality \[16\] and its extension by Araki \[4\] are regarded as a
strengthening of the celebrated Golden-Thompson trace inequality, which can be writ-
ten, as explicitly stated in \[3\], in terms of log-majorization

$$\left(A^{1/2}BA^{1/2}\right)^r \prec_{(log)} A^{r/2}B^rA^{r/2}, \quad r \geq 1,$$

for matrices $A, B \geq 0$. Here, for $n \times n$ matrices $X, Y \geq 0$, the log-majorization
$X \prec_{(log)} Y$ means that

$$\prod_{i=1}^k \lambda_i(X) \leq \prod_{i=1}^k \lambda_i(Y), \quad k = 1, \ldots, n$$

with equality for $k = n$, where $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ are the eigenvalues of $X$ arranged
in decreasing order and counting multiplicities. The weak log-majorization $X \prec_{w(log)} Y$
is referred to when the last equality is not imposed. A concise survey of majorization
for matrices is found in, e.g., \[2\] (also \[12\] \[13\]).

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In the present paper we generalize the log-majorization in (1.1) to the log-convexity of the function

\[ p \in [0, \infty) \mapsto \lambda(\Phi(A^p)^{1/2}\Psi(B^p)^{1/2}) \]

in the sense of weak log-majorization order, involving positive linear maps \( \Phi, \Psi \) between matrix algebras. More precisely, in Theorem 3.1 of Section 3, we prove the weak log-majorization

\[ \lambda(\Phi(A^{p_\alpha})^{1/2}\Psi(B^{p_\alpha})^{1/2}) \prec_w \lambda(\Phi(A^{p_0})^{1/2}\Psi(B^{p_0})^{1/2})^{1-\alpha} \lambda(\Phi(A^{p_1})^{1/2}\Psi(B^{p_1})^{1/2})^{\alpha}, \]

(1.2)

where \( p_\alpha := (1-\alpha)p_0 + \alpha p_1 \) for \( 0 \leq \alpha \leq 1 \). In particular, when \( \Phi = \Psi = \text{id} \) and \( (p_0, p_1) = (0, 1) \), (1.2) reduces to

\[ \lambda(A^{\alpha/2}B^{\alpha/2}) \prec_w \lambda(A^{1/2}BA^{1/2}), \quad 0 \leq \alpha \leq 1, \]

(1.3)

which is equivalent to (1.1) by letting \( \alpha = 1/r \) and replacing \( A, B \) with \( A^r, B^r \). In Section 2 we show an operator norm inequality in a more general setting by a method using operator means. In Section 3 we extend this inequality to the weak log-majorization (1.2) by applying the well-known antisymmetric tensor power technique.

The recent paper of Bourin and Lee \([9]\) contains, as a consequence of their joint log-convexity theorem for a two-variable norm function, the weak log-majorization

\[ (A^{1/2}Z^*BZA^{1/2})^r \prec_w (A^{1/2}BZA^{1/2})^r, \quad r \geq 1, \]

which is closely related to ours, as explicitly mentioned in Remark 3.6 of Section 3.

The complementary Golden-Thompson inequality was first shown in \([14]\) and then it was extended in \([3]\) to the log-majorization

\[ A^r \#_\alpha B^r \prec (A \#_\alpha B)^r, \quad r \geq 1, \]

where \( \#_\alpha \) is the weighted geometric mean for \( 0 \leq \alpha \leq 1 \). In a more recent paper \([19]\) the class of operator means \( \sigma \) for which \( \lambda_1(A^r \sigma B^r) \leq \lambda_1(A \sigma B) \) holds for all \( r \geq 1 \) was characterized in terms of operator monotone functions representing \( \sigma \). In Section 4 of the paper, we show some generalizations of these results in \([3, 19]\) in a somewhat similar way to that of Araki’s log-majorization in Sections 2 and 3.

2 Operator norm inequalities

For \( n \in \mathbb{N} \) we write \( \mathbb{M}_n \) for the \( n \times n \) complex matrix algebra and \( \mathbb{M}_n^+ \) for the \( n \times n \) positive semidefinite matrices. For \( A \in \mathbb{M}_n \) we write \( A \geq 0 \) if \( A \in \mathbb{M}_n^+ \), and \( A > 0 \) if \( A \)
is positive definite, i.e., $A \geq 0$ and $A$ is invertible. The operator norm and the usual trace of $A \in \mathbb{M}_n$ is denoted by $\|A\|_\infty$ and $\mathrm{Tr} A$, respectively.

We denote by $\Omega_{\alpha,1}$ the set of non-negative operator monotone functions $f$ on $[0, \infty)$ such that $f(1) = 1$. In theory of operator means due to Kubo and Ando [15], a main result says that each operator mean $\sigma$ is associated with an $f \in \Omega_{\alpha,1}$ in such a way that

$$A \sigma B := A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

for $A, B \in \mathbb{M}_n^+$ with $A > 0$, which is further extended to general $A, B \in \mathbb{M}_n^+$ as

$$A \sigma B := \lim_{\varepsilon \downarrow 0}(A + \varepsilon I_n) \sigma (B + \varepsilon I_n).$$

We write $\sigma_f$ for the operator mean associated with $f \in \Omega_{\alpha,1}$. For $0 \leq \alpha \leq 1$, the operator mean corresponding to the function $x^\alpha$ in $\Omega_{\alpha,1}$ is called the weighted geometric mean denoted by $\#_\alpha$; more explicitly,

$$A \#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

for $A, B \in \mathbb{M}_n^+$ with $A > 0$. The case $\alpha = 1/2$ is the geometric mean $\#$, first introduced by Pusz and Woronowicz [18]. Let $\sigma^*_f$ be the adjoint of $\sigma_f$, i.e., the operator mean corresponding to $f^* \in \Omega_{\alpha,1}$ defined as $f^*(x) := f(x^{-1})^{-1}$, $x > 0$.

A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_l$ is said to be positive if $\Phi(A) \in \mathbb{M}_l^+$ for all $A \in \mathbb{M}_n^+$, which is furthermore said to be strictly positive if $\Phi(I_n) > 0$, that is, $\Phi(A) > 0$ for all $A \in \mathbb{M}_n$ with $A > 0$. In the rest of the paper, we throughout assume that $\Phi : \mathbb{M}_n \to \mathbb{M}_l$ and $\Psi : \mathbb{M}_m \to \mathbb{M}_l$ are positive linear maps. Recall the well-known fact, essentially due to Ando [1], that

$$\Phi(A \sigma B) \leq \Phi(A) \sigma \Phi(B)$$

for all $A, B \in \mathbb{M}_n^+$ and for any operator mean $\sigma$. This will be repeatedly used without reference in the sequel.

For non-negative functions $\varphi_0$ and $\varphi_1$ on $[0, \infty)$ a new non-negative function $\varphi := \varphi_0 \sigma f \varphi_1$ on $[0, \infty)$ is defined as

$$\varphi(x) = \varphi_0(x) \sigma_f \varphi_1(x) = \lim_{\varepsilon \downarrow 0}(\varphi_0(x) + \varepsilon)f\left(\frac{\varphi_1(x) + \varepsilon}{\varphi_0(x) + \varepsilon}\right), \quad x \in [0, \infty).$$

**Proposition 2.1.** Let $f \in \Omega_{\alpha,1}$. Let $\varphi_0$ and $\varphi_1$ be arbitrary non-negative functions on $[0, \infty)$ and define the functions $\varphi := \varphi_0 \sigma_f \varphi_1$ and $\tilde{\varphi} := \varphi_0 \sigma^*_f \varphi_1$ on $[0, \infty)$ as above. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$,

$$\||\Phi(\tilde{\varphi}(A))^{1/2}\Psi(\varphi(B))\Phi(\varphi(A))^{1/2}\|_\infty \leq \max\{\|\Phi(\varphi_0(A))^{1/2}\Psi(\varphi_0(B))\Phi(\varphi_0(A))^{1/2}\|_\infty, \|\Phi(\varphi_1(A))^{1/2}\Psi(\varphi_1(B))\Phi(\varphi_1(A))^{1/2}\|_\infty\}.\]$$
Proof. Letting
\[ \gamma_k := \| \Phi(\varphi_k(A))^{1/2} \Psi(\varphi_k(B)) \Phi(\varphi_k(A))^{1/2} \|_\infty, \quad k = 0, 1, \]
we may prove that
\[ \Phi(\bar{\varphi}(A))^{1/2} \Psi(\varphi(B)) \Phi(\bar{\varphi}(A))^{1/2} \leq \max\{\gamma_0, \gamma_1\}I_l. \tag{2.1} \]

First, assume that \( \Phi \) and \( \Psi \) are strictly positive and \( \varphi_0(x), \varphi_1(x) > 0 \) for any \( x \geq 0 \). Then \( \gamma_0, \gamma_1 > 0 \), and we have
\[ \Psi(\varphi_k(B)) \leq \gamma_k \Phi(\varphi_k(A))^{-1}, \quad k = 0, 1. \]

Since \( \varphi(B) = \varphi_0(B) \sigma_f \varphi_1(B) \) and \( \bar{\varphi}(A) = \varphi_0(A) \sigma_f^* \varphi_1(A) \), by the joint monotonicity of \( \sigma_f \) we have
\[ \Psi(\varphi(B)) \leq \Psi(\varphi_0(B)) \sigma_f \Psi(\varphi_1(B)) \leq (\gamma_0 \Phi(\varphi_0(A))^{-1}) \sigma_f (\gamma_1 \Phi(\varphi_1(A))^{-1}) \leq \max\{\gamma_0, \gamma_1\} \left\{ \Phi(\varphi_0(A)) \sigma_f^* \Phi(\varphi_1(A)) \right\}^{-1} \leq \max\{\gamma_0, \gamma_1\} \Phi(\varphi_0(A) \sigma_f^* \varphi_1(A))^{-1} = \max\{\gamma_0, \gamma_1\} \Phi(\bar{\varphi}(A))^{-1}, \tag{2.2} \]
which implies \( 2.1 \) under the assumptions given above.

For the general case, for every \( \varepsilon > 0 \) we define a strictly positive \( \Phi_\varepsilon : M_n \to M_l \) by
\[ \Phi_\varepsilon(X) := \Phi(X) + \varepsilon \text{Tr}(X)I_l. \]
and similarly \( \Psi_\varepsilon : M_m \to M_l \). Moreover let \( \varphi_{k,\varepsilon}(x) := \varphi_k(x) + \varepsilon, \quad k = 0, 1, \) for \( x \geq 0 \), and \( \varphi_\varepsilon := \varphi_{0,\varepsilon} \sigma_f \varphi_{1,\varepsilon}, \bar{\varphi}_\varepsilon := \varphi_{0,\varepsilon} \sigma_f^* \varphi_{1,\varepsilon} \). By the above case we then have
\[ \Phi_\varepsilon(\bar{\varphi}_\varepsilon(A))^{1/2} \Psi_\varepsilon(\varphi_\varepsilon(B)) \Phi_\varepsilon(\bar{\varphi}_\varepsilon(A))^{1/2} \leq \max\{\gamma_{0,\varepsilon}, \gamma_{1,\varepsilon}\}I_l, \tag{2.3} \]
where
\[ \gamma_{k,\varepsilon} := \| \Phi_\varepsilon(\varphi_{k,\varepsilon}(A))^{1/2} \Psi_\varepsilon(\varphi_{k,\varepsilon}(B)) \Phi_\varepsilon(\varphi_{k,\varepsilon}(A))^{1/2} \|_\infty, \quad k = 0, 1. \]
Since \( \bar{\varphi}_\varepsilon(A) \to \bar{\varphi}(A), \varphi_\varepsilon(B) \to \varphi(B) \) and \( \gamma_{k,\varepsilon} \to \gamma_k, \quad k = 0, 1, \) as \( \varepsilon \searrow 0 \), we have \( 2.1 \) in the general case by taking the limit of \( 2.3 \).

For non-negative functions \( \varphi_0, \varphi_1 \) the function \( \varphi_0^{1-\alpha} \varphi_1^\alpha \) with \( 0 \leq \alpha \leq 1 \) is often called the geometric bridge of \( \varphi_0, \varphi_1 \), for which we have
**Proposition 2.2.** Let $\varphi_0, \varphi_1$ be arbitrary non-negative functions on $[0, \infty)$ and $0 \leq \alpha \leq 1$. Define $\varphi(x) := \varphi_0(x)^{1-\alpha} \varphi_1(x)^{\alpha}$ on $[0, \infty)$ (with convention $0^0 := 1$). Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$,

$$
\| \Phi(\varphi_0(A))^{1/2} \Psi(\varphi_0(B)) \Phi(\varphi_1(A))^{1/2} \|_\infty \\
\leq \| \Phi(\varphi_0(A))^{1/2} \Psi(\varphi_0(B)) \Phi(\varphi_1(A))^{1/2} \|_\infty^{1-\alpha} \| \Phi(\varphi_1(A))^{1/2} \Psi(\varphi_1(B)) \Phi(\varphi_1(A))^{1/2} \|_\infty^\alpha.
$$

**Proof.** When $f(x) := x^\alpha = f^*(x)$ where $0 \leq \alpha \leq 1$, note that $\varphi_0 = \varphi_0 \sigma_f \varphi_1 = \varphi_0 \sigma_f^* \varphi_1$. With the same notation as in the proof of Proposition 2.1, inequality (2.2) is improved in the present case as

$$
\Psi(\varphi_0(B)) \leq \gamma_0^{1-\alpha} \gamma_1^\alpha \Phi(\varphi_0(A))^{-1}
$$

for every $\alpha \in [0, 1]$. Hence the asserted inequality follows as in the above proof. $\Box$

In particular, when $\varphi_0(x) = 1$ and $\varphi_1(x) = x$, since $\varphi_0 \sigma_f \varphi_1 = f$ and $\varphi_0 \sigma_f^* \varphi_1 = f^*$ in Proposition 2.1 we have

**Corollary 2.3.** Assume that $\Phi(I_n)^{1/2} \Psi(I_m) \Phi(I_n)^{1/2} \leq I_l$. If $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$ satisfy $\Phi(A)^{1/2} \Psi(B) \Phi(A)^{1/2} \leq I_l$, then

$$
\Phi(f^*(A))^{1/2} \Psi(f(B)) \Phi(f(A))^{1/2} \leq I_l
$$

(2.4)

for every $f \in \text{OM}_{+,1}$, and in particular,

$$
\Phi(A^\alpha)^{1/2} \Psi(B^\alpha) \Phi(A^\alpha)^{1/2} \leq I_l, \quad 0 \leq \alpha \leq 1.
$$

**Remark 2.4.** Assume that both $\Phi$ and $\Psi$ are sub-unital, i.e., $\Phi(I_n) \leq I_l$ and $\Psi(I_m) \leq I_l$. If $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$ satisfy $\Phi(A)^{1/2} \Psi(B) \Phi(A)^{1/2} \leq I_l$, then one can see (2.4) in a simpler way as follows: By continuity, one can assume that $\Phi$ is strictly positive and $A > 0$; then

$$
\Psi(f(B)) \leq f(\Psi(B)) \leq f(\Phi(A)^{-1}) = f^*(\Phi(A))^{-1} \leq \Phi(f^*(A))^{-1}.
$$

The above first and the last inequalities hold by the Jensen inequality due to [10] Theorem 2.1 and [11] Theorem 2.1. The merit of our method with use of the operator mean $\sigma_f$ is that it enables us to relax the sub-unitality assumption into $\Phi(I_n)^{1/2} \Psi(I_m) \Phi(I_n)^{1/2} \leq I_l$.

### 3 Log-majorization

When $\varphi_0$ and $\varphi_1$ are power functions, we can extend Proposition 2.2 to the log-majorization result in the next theorem. For $A \in \mathbb{M}_n^+$ we write $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$ for the eigenvalues of $A$ arranged in decreasing order with multiplicities. Also, for
Let $X \in \mathbb{M}_n$ let $s(X) = (s_1(X), \ldots, s_n(X))$ be the singular values of $X$ in decreasing order with multiplicities. For two non-negative vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ where $a_1 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq \cdots \geq b_n \geq 0$, the *weak log-majorization* (or the *log-submajorization*) $a \prec_w b$ means that

$$
\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i, \quad 1 \leq k \leq n,
$$

and the *log-majorization* $a \prec_{(\log)} b$ means that $a \prec_w b$ and equality hold for $k = n$ in (3.1). On the other hand, the *log-supremajorization* $a \prec_{w(\log)} b$ is defined as

$$
\prod_{i=n+1-k}^{n} a_i \geq \prod_{i=n-k+1}^{n} b_i, \quad 1 \leq k \leq n.
$$

**Theorem 3.1.** Let $p_0, p_1 \in [0, \infty)$ and $0 \leq \alpha \leq 1$, and let $p_\alpha := (1 - \alpha)p_0 + \alpha p_1$. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_n^+$,

$$
\lambda(\Phi(A^{p_\alpha})^{1/2}\Psi(B^{p_\alpha})\Phi(A^{p_\alpha})^{1/2})
\prec_{w(\log)} \lambda^{1-\alpha}(\Phi(A^{p_0})^{1/2}\Psi(B^{p_0})\Phi(A^{p_0})^{1/2})\lambda^\alpha(\Phi(A^{p_1})^{1/2}\Psi(B^{p_1})\Phi(A^{p_1})^{1/2}),
$$

or equivalently,

$$
s(\Phi(A^{p_\alpha})^{1/2}\Psi(B^{p_\alpha})^{1/2})
\prec_{w(\log)} s^{1-\alpha}(\Phi(A^{p_0})^{1/2}\Psi(B^{p_0})^{1/2})s^\alpha(\Phi(A^{p_1})^{1/2}\Psi(B^{p_1})^{1/2}).
$$

In particular, for every $A, B \in \mathbb{M}_n^+$,

$$
s(A^{p_\alpha}B^{p_\alpha}) \prec_{(\log)} s^{1-\alpha}(A^{p_0}B^{p_0})s^\alpha(A^{p_1}B^{p_1}).
$$

**Proof.** Let $C^*(I, A)$ be the commutative $C^*$-subalgebra of $\mathbb{M}_n$ generated by $I, A$. We may consider, instead of $\Phi$, the composition of the trace-preserving conditional expectation from $\mathbb{M}_n$ onto $C^*(I, A)$ and $\Phi|_{C^*(I, A)} : C^*(I, A) \to \mathbb{M}_d$, which is completely positive. Hence one can assume that $\Phi$ is completely positive and similarly for $\Psi$. The weak log-majorization means that

$$
\prod_{i=1}^{k} \lambda_i(\Phi(A^{p_\alpha})^{1/2}\Psi(B^{p_\alpha})\Phi(A^{p_\alpha})^{1/2})
\leq \prod_{i=1}^{k} \lambda_i^{1-\alpha}(\Phi(A^{p_0})^{1/2}\Psi(B^{p_0})\Phi(A^{p_0})^{1/2})\lambda_i^\alpha(\Phi(A^{p_1})^{1/2}\Psi(B^{p_1})\Phi(A^{p_1})^{1/2})
$$

for every $k = 1, \ldots, l$. The case $k = 1$ is Proposition 2.2 in the case where $\varphi_0(x) := x^{p_0}$ and $\varphi_1(x) := x^{p_1}$ so that $\varphi_\alpha(x) = x^{p_\alpha}$.
Next, for each $k$ with $2 \leq k \leq l$ we consider the $k$-fold tensor product
\[
\Phi^{\otimes k} : M_n^{\otimes k} = B((\mathbb{C}^n)^{\otimes k}) \to M_l^{\otimes k} = B((\mathbb{C}^l)^{\otimes k}),
\]
and similarly for $\Psi^{\otimes k}$. Let $P_\lambda$ be the orthogonal projection from $(\mathbb{C}^l)^{\otimes k}$ onto the $k$-fold antisymmetric tensor Hilbert space $(\mathbb{C}^l)^{\wedge k}$. Since $\Phi$ and $\Psi$ are assumed completely positive, one can define positive linear maps
\[
\Phi^{(k)} := P_\lambda \Phi^{\otimes k} (\cdot) P_\lambda : M_n^{\otimes k} \to B((\mathbb{C}^l)^{\wedge k}),
\]
\[
\Psi^{(k)} := P_\lambda \Psi^{\otimes k} (\cdot) P_\lambda : M_m^{\otimes k} \to B((\mathbb{C}^l)^{\wedge k}).
\]
For every $X \in M_n$ we note that $\Phi^{(k)}(X^{\otimes k}) = P_\lambda \Phi(X)^{\otimes k} P_\lambda$ is nothing but the $k$-fold antisymmetric tensor power $\Phi(X)^{\wedge k}$ of $\Phi(X)$. By applying the case $k = 1$ shown above to $A^{\otimes k}$ and $B^{\otimes k}$ we have
\[
\lambda_1(\Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha}))^{1/2} \psi^{(k)}((B^{\otimes k})^{\wedge p_\alpha}) \Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha})^{1/2} \leq \lambda_1^{1 - \alpha}(\Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha}))^{1/2} \psi^{(k)}((B^{\otimes k})^{\wedge p_\alpha}) \Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha})^{1/2} \lambda_1^{\alpha}(\Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha}))^{1/2} \psi^{(k)}((B^{\otimes k})^{\wedge p_\alpha}) \Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha})^{1/2}.
\]
Since $\Phi^{(k)}((A^{\otimes k})^{\wedge p_\alpha}) = \Phi((A^{p_\alpha})^{\wedge k})$ and $\Psi^{(k)}((B^{\otimes k})^{\wedge p_\alpha}) = \Psi((B^{p_\alpha})^{\wedge k}$, the above left-hand side is
\[
\lambda_1((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^1/2))^{\wedge k}) = \prod_{i=1}^{k} \lambda_i((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^{1/2}))^{\wedge k})
\]
and the right-hand side is
\[
\lambda_1^{1 - \alpha}((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^1/2))^{\wedge k})^{1/2} \lambda_1^{\alpha}((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^1/2))^{\wedge k})^{1/2} \prod_{i=1}^{k} \lambda_i^{1 - \alpha}((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^{1/2}))^{\wedge k})^{1/2} \lambda_i^{\alpha}((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^{1/2}))^{\wedge k})^{1/2}.
\]
Hence we have (3.5) for every $k = 1, \ldots, l$, so (3.2) follows.

Since $\lambda((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^{1/2}))^{\wedge k}) = s^2((\Phi((A^{p_\alpha})^{1/2} \psi((B^{p_\alpha})^{1/2})^{\wedge k})$, it is clear that (3.2) and (3.3) are equivalent. When $\Phi = \Psi = \text{id}$ and $A, B$ are replaced with $A^2, B^2$, (3.3) reduces to (3.4).

**Remark 3.2.** It is not known whether a modification of (3.3)
\[
s(\Phi((A^{p_\alpha}) \psi((B^{p_\alpha})^{ \wedge k})) \prec \log s(\Phi((A^{p_\alpha}) \psi((B^{p_\alpha})^{ \wedge k})) \quad \text{s}^{\alpha}(\Phi((A^{p_\alpha}) \psi((B^{p_\alpha})^{ \wedge k}))
\]
holds true or not.

By reducing (3.2) to the case $(p_0, p_1) = (0, 1)$ we have
Corollary 3.3. Let $0 \leq \alpha \leq 1$. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$,

$$
\lambda(\Phi(A^\alpha)^{1/2}\Psi(B^{\alpha})\Phi(A^\alpha)^{1/2})
\prec_{w(\log)} \lambda^{1-\alpha}(\Phi(I_n)^{1/2}\Psi(I_m)\Phi(I_n)^{1/2})\lambda^\alpha(\Phi(A)^{1/2}\Psi(B)\Phi(A)^{1/2}).
$$

(3.6)

Consequently, if $\Phi(I_n)^{1/2}\Psi(I_m)\Phi(I_n)^{1/2} \leq I_I$, then

$$
\lambda(\Phi(A^\alpha)^{1/2}\Psi(B^{\alpha})\Phi(A^\alpha)^{1/2}) \prec_{w(\log)} \lambda^\alpha(\Phi(A)^{1/2}\Psi(B)\Phi(A)^{1/2}).
$$

(3.7)

The last log-majorization with $\Phi = \Psi = id$ and also (3.4) with $(p_0, p_1) = (0, 1)$ give Araki’s log-majorization (1.3) or $s(A^{\alpha}B^{\alpha}) \prec_{(\log)} s^{\alpha}(AB)$ for $0 \leq \alpha \leq 1$. By letting $\alpha = 1/r$ with $r \geq 1$ and replacing $A, B$ with $A^r, B^r$ one can rephrase (3.6) as

$$
\lambda^r(\Phi(A)^{1/2}\Psi(B)\Phi(A)^{1/2}) 
\prec_{w(\log)} \lambda^{r-1}(\Phi(I_n)^{1/2}\Psi(I_m)\Phi(I_n)^{1/2})\lambda(\Phi(A^r)^{1/2}\Psi(B^r)\Phi(A^r)^{1/2})
$$

(3.8)

for all $r \geq 1$. Also, when $\Phi(I_n)^{1/2}\Psi(I_m)\Phi(I_n)^{1/2} \leq I_I$, (3.7) is rewritten as

$$
\lambda^r(\Phi(A)^{1/2}\Psi(B)\Phi(A)^{1/2}) \prec_{w(\log)} \lambda(\Phi(A^r)^{1/2}\Psi(B^r)\Phi(A^r)^{1/2}), \quad r \geq 1.
$$

(3.9)

A norm $\| \cdot \|$ on $\mathbb{M}_n$ is called a unitarily invariant norm (or a symmetric norm) if $\|UXV\| = \|X\|$ for all $X, U, V \in \mathbb{M}_n$ with $U, V$ unitaries.

Corollary 3.4. Let $p_0, p_1$ and $p_\alpha$ for $0 \leq \alpha \leq 1$ be as in Theorem 3.7. Let $\| \cdot \|$ be any unitarily invariant norm and $r > 0$. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$,

$$
\| \Phi(A^{p_0})^{1/2}\Psi(B^{p_0})^{1/2} \|^r
\leq \| \Phi(A^{p_0})^{1/2}\Psi(B^{p_0})^{1/2} \|^{1-\alpha} \| \Phi(A^{p_1})^{1/2}\Psi(B^{p_1})^{1/2} \|^\alpha.
$$

(3.10)

In particular, for every $A, B \in \mathbb{M}_n^+$,

$$
\| |A^{p_0}B^{p_0}| \| \| \leq \| |A^{p_0}B^{p_0}| \|^{1-\alpha} \| |A^{p_1}B^{p_1}| \|^\alpha.
$$

Proof. We may assume that $0 < \alpha < 1$. Let $\psi$ be the symmetric gauge function on $\mathbb{R}^l$ corresponding to the unitarily invariant norm $\| \cdot \|$, so $\|X\| = \psi(s(X))$ for $X \in \mathbb{M}_l$. Recall [5, IV.1.6] that $\psi$ satisfies the Hölder inequality

$$
\psi(a_1b_1, \ldots, a_ib_i) \leq \psi^{1-\alpha}(a_1^{1/\alpha}, \ldots, a_i^{1/\alpha}) \psi^\alpha(b_1^{1/\alpha}, \ldots, b_i^{1/\alpha})
$$

for every $a, b \in [0, \infty)^l$. Also, it is well-known (see, e.g., [13, Proposition 4.1.6 and Lemma 4.4.2]) that $a \prec_{w(\log)} b$ implies the weak majorization $a \prec_w b$ and so $\psi(a) \leq \psi(b)$. Hence it follows from the weak log-majorization in (3.3) that

$$
\| \Phi(A^{p_0})^{1/2}\Psi(B^{p_0})^{1/2} \|^r
$$

8
\[
\psi\left(s^r(\Phi(A^\alpha)1/2\Psi(B^{\alpha})1/2)\right)
\leq \psi\left(s^{(1-\alpha)r}(\Phi(A^\alpha)1/2\Psi(B^{\alpha})1/2)\right)\psi^{\alpha}\left(s^r(\Phi(A^\alpha)1/2\Psi(B^{\alpha})1/2)\right)
\leq \left\|\Phi(A^\alpha)1/2\Psi(B^{\alpha})1/2\right\|^{1-\alpha}\left\|\Phi(A^\alpha)1/2\Psi(B^{\alpha})1/2\right\|^{\alpha}.
\]

\[\square\]

The norm inequality in (3.10) is a kind of the Hölder type inequality, showing the log-convexity of the function

\[p \in [0, \infty) \mapsto \left\|\Phi(A^p)1/2\Psi(B^{p})1/2\right\|\]

**Corollary 3.5.** Let \(\| \cdot \|\) be a unitarily invariant norm. If \(\Phi(I_n)1/2\Psi(I_m)\Phi(I_n)1/2 \leq I_i\), then for every \(A \in M_n^+\) and \(B \in M_m^+\),

\[
\left\|\left\{\Phi(A^p)1/2\Psi(B^p)\Phi(A^p)1/2\right\}^{1/p}\right\| \leq \left\|\left\{\Phi(A^q)1/2\Psi(B^q)\Phi(A^q)1/2\right\}^{1/q}\right\| \quad \text{if } 0 < p \leq q.
\]

Furthermore, if \(\Phi\) and \(\Psi\) are unital and \(A, B > 0\), then

\[
\left\|\left\{\Phi(A^p)1/2\Psi(B^p)\Phi(A^p)1/2\right\}^{1/p}\right\|
\]

decreases to \(\|\exp\{\Phi(\log A) + \Psi(\log B)\}\|\) as \(p \searrow 0\).

**Proof.** Let \(0 < p \leq q\). By applying (3.7) to \(A^q\), \(B^q\) and \(\alpha = p/q\) we have

\[\lambda^{1/p}(\Phi(A^p)1/2\Psi(B^p)\Phi(A^p)1/2) \prec_{w(\log)} \lambda^{1/q}(\Phi(A^q)1/2\Psi(B^q)\Phi(A^q)1/2),\]

which implies the desired norm inequality. Under the additional assumptions on \(\Phi, \Psi\) and \(A, B\) as stated in the corollary, the proof of the limit formula is standard with use of

\[\Phi(A^p)^{1/2} = I_i + \frac{p}{2} \Phi(\log A) + o(p), \quad \Psi(B^p) = I_i + p\Psi(\log B) + o(p).\]

as \(p \to 0\). \[\square\]

**Remark 3.6.** When \(\Phi = \text{id}\) and \(\Psi = Z^r \cdot Z\) with a contraction \(Z \in M_n\), it follows from (3.9) that, for every \(A, B \in M_n^+\),

\[
\lambda^r(A^{1/2}Z^r BZ A^{1/2}) \preceq_{w(\log)} \lambda(A^{r/2}Z^r B^r Z A^{r/2}), \quad r \geq 1,
\]

which is [4 Corollary 2.3]. Although the form of (3.9) is seemingly more general than that of (3.11), it is in fact easy to see that (3.9) follows from (3.11) conversely. Indeed, we may assume as in the proof of Theorem 3.1 that \(\Phi\) and \(\Psi\) are completely positive. Then, via the Stinespring representation (see, e.g., [6 Theorem 3.1.2]), we may further
assume that \( \Phi = V^* \cdot V \) with an operator \( V : \mathbb{C}^l \to \mathbb{C}^n \) and \( \Psi = W^* \cdot W \) with an operator \( W : \mathbb{C}^l \to \mathbb{C}^m \). The assumption \( \Phi(I)^{1/2} \Psi(I) \Phi(I)^{1/2} \leq I \) is equivalent to \( \|WV^*\|_\infty \leq 1 \). One can see that

\[
\Phi(A^r)^{1/2} \Psi(B^r) \Phi(A^r)^{1/2} = (V^* A^r V)^{1/2} (W^* B^r W) (V^* A^r V)^{1/2}
\]

is unitarily equivalent to \( A^r W^* B^r W^* A^r \), and thus (3.11) implies (3.9). Here, it should be noted that the proof of (3.11) in [9] is valid even though \( Z = WV^* \) is an \( m \times n \) (not necessarily square) matrix. In this way, the log-majorization in (3.9) is equivalent to [9, Corollary 2.3]. Similarly, Corollary 3.5 is equivalent to [9, Corollary 2.2]. The author is indebted to J.-C. Bourin for the remark here.

4 More inequalities for operator means

The log-majorization obtained in [3] for the weighted geometric means says that, for every \( 0 \leq \alpha \leq 1 \) and every \( A, B \in \mathbb{M}_n^+ \),

\[
\lambda(A^r \ #_\alpha B^r) \prec_{(\log)} \lambda^r(A \ #_\alpha B), \quad r \geq 1,
\]

or equivalently,

\[
\lambda^q(A \ #_\alpha B) \prec_{(\log)} \lambda(A^q \ #_\alpha B^q), \quad 0 \leq q \leq 1.
\]

The essential first step to prove this is the operator norm inequality

\[
\|A^r \ #_\alpha B^r\|_\infty \leq \|A \ #_\alpha B\|_\infty, \quad r \geq 1,
\]

which is equivalent to that \( A \ #_\alpha B \leq I \Rightarrow A^r \ #_\alpha B^r \leq I \) for all \( r \geq 1 \). By taking the inverse when \( A, B > 0 \), this is also equivalent to that \( A \ #_\alpha B \geq I \Rightarrow A^r \ #_\alpha B^r \geq I \) for all \( r \geq 1 \). The last implication was recently extended in [19, Lemmas 2.1, 2.2] to the assertion stating the equivalence between the following two conditions for \( f \in \mathrm{OM}_{+;1} \):

(i) \( f(x)^r \leq f(x^r) \) for all \( x \geq 0 \) and \( r \geq 1 \);

(ii) for every \( A, B \in \mathbb{M}_n^+ \), \( A \sigma_f B \geq I \Rightarrow A^r \sigma_f B^r \geq I \) for all \( r \geq 1 \).

We note that the above conditions are also equivalent to

(iii) for every \( A, B \in \mathbb{M}_n^+ \),

\[
\lambda_n(A^r \sigma_f B^r) \geq \lambda_n^r(A \sigma_f B), \quad r \geq 1;
\]

or equivalently, for every \( A, B \in \mathbb{M}_n^+ \),

\[
\lambda_n(A^q \sigma_f B^q) \leq \lambda_n^q(A \sigma_f B), \quad 0 < q \leq 1.
\]
The next proposition extends the above result to the form involving positive linear maps. Below let \( \Phi \) and \( \Psi \) be positive linear maps as before.

**Proposition 4.1.** Assume that \( f \in \text{OM}_{+,1} \) satisfies the above condition (i). Then for every \( A \in \mathbb{M}_n^+ \) and \( B \in \mathbb{M}_n^+ \),

\[
\left( \max\{\|\Phi(I_n)\|_\infty, \|\Psi(I_m)\|_\infty \} \right)^{r-1} \lambda_l(\Phi(A^r) \sigma_f \Psi(B^r)) \geq \lambda_{l-1}(\Phi(A) \sigma_f \Psi(B)) \tag{4.2}
\]

for all \( r \geq 1 \).

**Proof.** By continuity we may assume that \( \Phi \) and \( \Psi \) are strictly positive. Let \( 0 < q \leq 1 \). Since \( \Phi(I_n)^{-1/2} \Phi(\cdot) \Phi(I_n)^{-1/2} \) is a unital positive linear map, it is well-known \([6, \text{Proposition 2.7.1}]\) that

\[
\Phi(I_n)^{-1/2} \Phi(A^q) \Phi(I_n)^{-1/2} \leq (\Phi(I_n)^{-1/2} \Phi(A) \Phi(I_n)^{-1/2})^q
\]

so that

\[
\Phi(A^q) \leq \Phi(I_n)^{1/2} (\Phi(I_n)^{-1/2} \Phi(A) \Phi(I_n)^{-1/2})^q \Phi(I_n)^{1/2} = \Phi(I_n) \#_q \Phi(A) \leq (\|\Phi(I_n)\|_\infty I_n) \#_q \Phi(A) = \|\Phi(I_n)\|^{1-q} \Phi(A)^q
\]

and similarly

\[
\Psi(B^q) \leq \|\Psi(I_m)\|^{1-q} \Psi(B)^q.
\]

By the joint monotonicity of \( \sigma_f \) we have

\[
\Phi(A^q) \sigma_f \Psi(B^q) \leq (\|\Phi(I_n)\|^{1-q} \Phi(A)^q) \sigma_f (\|\Psi(I_m)\|^{1-q} \Psi(B)^q) \leq \left( \max\{\|\Phi(I_n)\|_\infty, \|\Psi(I_m)\|_\infty \} \right)^{1-q} \Phi(A)^q \sigma_f \Psi(B)^q. \tag{4.3}
\]

Therefore,

\[
\lambda_l(\Phi(A^q) \sigma_f \Psi(B^q)) \leq \left( \max\{\|\Phi(I_n)\|_\infty, \|\Psi(I_m)\|_\infty \} \right)^{1-q} \lambda_l(\Phi(A)^q \sigma_f \Psi(B)^q) \leq \left( \max\{\|\Phi(I_n)\|_\infty, \|\Psi(I_m)\|_\infty \} \right)^{1-q} \lambda_{l-1}(\Phi(A) \sigma_f \Psi(B))
\]

by using the property (iii) above. Now, for \( 0 < r \leq 1 \) let \( q := 1/r \). By replacing \( A, B \) with \( A^r, B^r \), respectively, we obtain

\[
\lambda_l(\Phi(A) \sigma_f \Psi(B)) \leq \left( \max\{\|\Phi(I_n)\|_\infty, \|\Psi(I_m)\|_\infty \} \right)^{1-\frac{1}{r}} \lambda_{l-1}^{1/r}(\Phi(A^r) \sigma_f \Psi(B^r)),
\]

which yields \(4.2\). \(\Box\)

When \( \sigma_f \) is the weighted geometric mean \( \#_a \), one can improve Proposition 4.1 to the log-supermajorization result as follows:
Proposition 4.2. Let $0 \leq \alpha \leq 1$. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$,

$$(\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty)^{r-1} \lambda(\Phi(A^r) \#_\alpha \Psi(B^r)) \prec_{w(\log)} \lambda^r(\Phi(A) \#_\alpha \Psi(B)) \quad (4.4)$$

for all $r \geq 1$. Consequently, if $\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty \leq 1$, then

$$\lambda(\Phi(A^r) \#_\alpha \Psi(B^r)) \prec_{w(\log)} \lambda^r(\Phi(A) \#_\alpha \Psi(B)), \quad r \geq 1.$$ 

Proof. When $\sigma_f = \#_\alpha$, inequality (4.3) is improved as

$$\Phi(A^q) \#_\alpha \Psi(B^q) \leq (\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty)^{1-q}(\Phi(A)^q \#_\alpha \Psi(B)^q)$$

for $0 < q \leq 1$, and hence (4.2) is improved as

$$(\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty)^{r-1} \lambda_1(\Phi(A^r) \#_\alpha \Psi(B^r)) \geq \lambda_1^r(\Phi(A) \#_\alpha \Psi(B))$$

for all $r \geq 1$. One can then prove the asserted log-supermajorization result in the same way as in the proof of Theorem 3.1 with use of the antisymmetric tensor power technique, where the identity $\lambda_1(X^{\wedge k}) = \prod_{i=l-k+1}^n \lambda_i(X)$ for $X \in \mathbb{M}_n^+$ is used instead of $\lambda_1(X^{\wedge k}) = \prod_{i=1}^k \lambda_i(X)$ in the previous proof. The details may be omitted here. $\square$

In particular, when $\Phi = \Psi = \text{id}$, (4.4) reduces to (4.1) since, for $A, B > 0$, the log-supermajorization $\lambda(A^r \#_\alpha B^r) \prec_{w(\log)} \lambda^r(A \#_\alpha B)$ implies the log-majorization (4.1).

The notion of symmetric anti-norms was introduced in [7] with the notation $\| \cdot \|_1$. Recall that a non-negative continuous functional $\| \cdot \|_1$ on $\mathbb{M}_n^+$ is called a symmetric anti-norm if it is positively homogeneous, superadditive (instead of subadditive in case of usual norms) and unitarily invariant. Among others, a symmetric anti-norm is typically defined associated with a symmetric norm $\| \cdot \|$ on $\mathbb{M}_n$ and $p > 0$ in such a way that, for $A \in \mathbb{M}_n^+$,

$$\|A\|_1 := \begin{cases} \|A^{-p}\|^{-1/p} & \text{if } A \text{ is invertible,} \\ 0 & \text{otherwise.} \end{cases}$$

A symmetric anti-norm defined in this way is called a derived anti-norm, see [8] Proposition 4.6]. By Lemma [8, Lemma 4.10], similarly to Corollary 3.5, we have

Corollary 4.3. Let $0 \leq \alpha \leq 1$ and assume that $\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty \leq 1$. Then for every $A \in \mathbb{M}_n^+$ and $B \in \mathbb{M}_m^+$ and for any derived anti-norm $\| \cdot \|_1$ on $\mathbb{M}_n^+$,

$$\|\{\Phi(A^p) \#_\alpha \Psi(B^p)\}^{1/p}\|_1 \geq \|\{\Phi(A^q) \#_\alpha \Psi(B^q)\}^{1/q}\|_1, \quad \text{if } 0 < p \leq q.$$ 

Problem 4.4. It seems that our generalization of Ando-Hiai type log-majorization is not so much completed as that of Araki's log-majorization in Section 3. Although the form of (4.4) bears some resemblance to that of (3.8), they have also significant differences. For one thing, $\prec_{w(\log)}$ arises in (4.4) while $\prec_{w(\log)}$ in (3.8), which
should be reasonable since the directions of log-majorization are opposite between them. For another, the factor \(|\|\Phi(I_n)\|_\infty \#_\alpha \|\Psi(I_m)\|_\infty|^{-1}\) in (3.3) is apparently much worse than \(\lambda^{r-1}(\Phi(I_n)^{1/2} \Psi(I_m) \Phi(I_n)^{1/2})\) in (3.8). One might expect the better factor \(|\|\Phi(I_n)\#_\alpha \Psi(I_m)\|_\infty|^{-1}\) or even \(\lambda^{r-1}(\Phi(I_n) \#_\alpha \Psi(I_m))\). Indeed, a more general interesting problem is the \#_\alpha-version of (3.2), i.e., for \(p_0, p_1 \geq 0, 0 \leq \theta \leq 1\) and \(p_\theta := (1-\theta)p_0 + \theta p_1\),

\[
\lambda^{1-\theta}(\Phi(A^{p_0}) \#_\alpha \Psi(B^{p_0})) \lambda^{\theta}(\Phi(A^{p_1}) \#_\alpha \Psi(B^{p_1})) \prec w(\log) \lambda(A^{p_0} \#_\alpha B^{p_0})
\]

When \(\Phi = \Psi = \text{id}\), the problem becomes

\[
\lambda^{1-\theta}(A^{p_0} \#_\alpha B^{p_0}) \lambda^{\theta}(A^{p_1} \#_\alpha B^{p_1}) \prec (\log) \lambda(A^{p_0} \#_\alpha B^{p_0})? \tag{4.5}
\]

**Example 4.5.** Here is a sample computation of the last problem for \(A, B\) are \(2 \times 2\) and \(\alpha = 2\). Thanks to continuity and homogeneity, we may assume that \(A, B \in \mathbb{M}_2^+\) are invertible with determinant 1. So we write \(A = aI + x \cdot \sigma\) and \(B = bI + y \cdot \sigma\) with \(a, b > 0, x, y \in \mathbb{R}^3\), det \(A = a^2 - |x|^2 = 1\) and det \(B = b^2 - |y|^2 = 1\), where \(|x|^2 := x_1^2 + x_2^2 + x_3^2\) and \(x \cdot \sigma := x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3\) with Pauli matrices \(\sigma_i\), i.e., \(\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). For (4.3) in this situation, it suffices, thanks to [17] Proposition 3.11 (also [6] Proposition 4.1.12), to show that

\[
p \geq 0 \rightarrow \lambda_1\left(\frac{A^p + B^p}{\sqrt{\det(A^p + B^p)}}\right) = \left(\frac{\lambda_1(A^p + B^p)}{\lambda_2(A^p + B^p)}\right)^{1/2}\tag{4.6}
\]

is a log-concave function. Let \(e^\alpha = a + |x|\) and \(e^\beta = b + |y|\), so \(e^{-\alpha} = a - |x|, |x| = \sinh \alpha\), and similarly for \(|y|\). Then a direct computation yields

\[
A^p + B^p = (\cosh(\alpha p) + \cosh(\beta p))I + \begin{bmatrix} \sinh(\alpha p) \\ \sinh(\beta p) \end{bmatrix} \cdot \sigma,
\]

whose eigenvalues are

\[
\cosh(\alpha p) + \cosh(\beta p) \pm \left[\sinh^2(\alpha p) + \sinh^2(\beta p) + 2c\sinh(\alpha p)\sinh(\beta p)\right]^{1/2}
\]

with \(c := \frac{xy}{|x||y|} \in [-1, 1]\). Although numerical computations say that (4.6) is a log-concave function of \(p \geq 0\) for any \(\alpha, \beta \geq 0\) and \(c \in [-1, 1]\), it does not seem easy to give a rigorous proof.

In the rest of the paper we present one more log-majorization result. Let \(E \in \mathbb{M}_n\) be an orthogonal projection with \(\text{dim } E = l\). A particular case of (3.2) is

\[
\lambda(\Theta A^{(1-\theta)p_0 + \theta p_1} E) \prec w(\log) \lambda^{1-\theta}(\Theta A^{p_0} E) \lambda^{\theta}(\Theta A^{p_1} E), \quad 0 \leq \theta \leq 1\tag{4.7}
\]

for every \(A \in \mathbb{M}_n^+\). As a complementary version of this we show the following:
Proposition 4.6. Let $p_0, p_1 \geq 0$ and $0 \leq \theta \leq 1$. Then for every $\alpha \in (0, 1]$ and $A \in \mathbb{M}_n^+$,

\[
\left( \lambda_i(A^{(1-\theta)p_0 + \theta p_1} \#_\alpha E) \right)_{i=1}^l \prec_{(\log)} \left( \lambda_i^{1-\theta}(A^{p_0} \#_\alpha E) \lambda_i^\theta(A^{p_1} \#_\alpha E) \right)_{i=1}^l.
\]

(4.8)

The form of this log-majorization is similar to that of the problem (4.5). Although the directions of those are opposite, there is no contradiction between those two; indeed, the log-majorization of (4.5) is taken for matrices in $\mathbb{M}_n^+$ while that of (4.8) is for $l \times l$ matrices restricted to the range of $E$.

First, we give a lemma in a setting of more general operator means. Let $f$ be an operator monotone function on $[0, \infty)$ such that $f(0) = 0$, and let $\sigma_f$ be the operator mean corresponding to $f$ due to Kubo-Ando theory. An operator monotone function dual to $f$ is defined by $f^\perp(x) := x/f(x)$, $x > 0$, and $f^\perp(0) := \lim_{x \to 0} f^\perp(x)$.

Lemma 4.7. Let $f$ and $f^\perp$ be as stated above. Then for every $A \in \mathbb{M}_n^+$ with $A > 0$,

\[
A \sigma_f E = (Ef^\perp(EA^{-1}E)E)^{-1},
\]

where the inverse in the right-hand side is defined on the range of $E$ (i.e., in the sense of generalized inverse).

Proof. For $k = 0, 1, 2, \ldots$ we have

\[
A^{-1/2}E(EA^{-1}E)^kEA^{-1/2} = (A^{-1/2}EA^{-1/2})^{k+1}.
\]

Define a function $\hat{f}$ on $[0, \infty)$ by $\hat{f}(x) := f(x)/x$ for $x > 0$ and $\hat{f}(0) := 0$. Note that the eigenvalues of $EA^{-1}E$ and those of $A^{-1/2}EA^{-1/2}$ are the same including multiplicities. By approximating $\hat{f}$ by polynomials on the eigenvalues of $EA^{-1}E$, we have

\[
A^{-1/2}E\hat{f}(EA^{-1}E)EA^{-1/2} = A^{-1/2}EA^{-1/2}\hat{f}(A^{-1/2}EA^{-1/2}) = f(A^{-1/2}EA^{-1/2})
\]

since the assumption $f(0) = 0$ implies that $f(x) = x\hat{f}(x)$ for all $x \in [0, \infty)$. Therefore,

\[
E\hat{f}(EA^{-1}E)E = A^{1/2}f(A^{-1/2}EA^{-1/2})A^{1/2} = A \sigma_f E.
\]

Moreover, it is easy to verify that $(Ef^\perp(EA^{-1}E)E)^{-1} = E\hat{f}(EA^{-1}E)E$. \hfill \Box

Proof of Proposition 4.6. Since the result is trivial when $\alpha = 1$, we may assume that $0 < \alpha < 1$. Moreover, we may assume by continuity that $A$ is invertible. When $f(x) = x^\alpha$, note that $\sigma_f = \#_\alpha$ and $f^\perp(x) = x^{1-\alpha}$. Hence by Lemma 4.7 we have

\[
A^p \#_\alpha E = (EA^{-p}E)^{\alpha - 1}, \quad p \geq 0,
\]
where \((E \cdot E)^{-1}\) is defined on the range of \(E\). This implies that, for every \(k = 1, \ldots, l\),

\[
\prod_{i=l-k+1}^{l} \lambda_i(A^p \#_\alpha E) = \left( \prod_{i=1}^{k} \lambda_i(EA^{-p}E) \right)^{\alpha - 1}
\]

so that (4.8) immediately follows from (4.7) applied to \(A^{-1}\).

Similarly to Corollary 3.4 by Proposition 4.6 and [8, Lemma 4.10 and (4.4)] we see that if \(A \in \mathbb{M}_n^+\) and \(\| \cdot \|_1\) is a derived anti-norm on \(\mathbb{M}_n^+\), then \(\|A^p \#_\alpha E\|_1\) is a log-concave function of \(p \geq 0\), where \(A^p \#_\alpha E\) is considered as an \(l \times l\) matrix restricted to the range of \(E\).

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