STABILITY AND DICHOTOMY OF POSITIVE SEMIGROUPS ON $L_p$

Stephen Montgomery-Smith

Abstract. A new proof of a result of Lutz Weis is given, that states that the stability of a positive strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on $L_p$ may be determined by the quantity $s(A)$. We also give an example to show that the dichotomy of the semigroup may not always be determined by the spectrum $\sigma(A)$.

Consider a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ acting on a Banach space $X$ with unbounded generator $A$. It has long been known that the spectral mapping theorem $e^{t \sigma(A)} = \sigma(e^{tA}) \setminus \{0\}$ does not necessarily hold. (Here $\sigma(A)$ denotes the spectrum of an operator $A$.) Indeed, let $s(A) = \sup \Re(\sigma(A))$, and let $\omega(A) = \sup \Re(\log(\sigma(e^A))) = \inf \{\lambda : \|e^{\lambda A}\| \leq M e^{\lambda t}\}$. Then there are examples of semigroups for which $s(A) \neq \omega(A)$ (see [N]).

The purpose of this paper is to give one situation in which it is true that $s(A) = \omega(A)$. This next result has already been proved by Lutz Weis [We]. We will give a different, shorter proof. We refer the reader to [We] for a history of the problem.

Theorem 1. Let $e^{tA}$ be a strongly continuous positive semigroup on $L_p(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a sigma-finite measure space, and $1 \leq p < \infty$. Then $\omega(A) = s(A)$.

In order to show this result, we will make use of the following lemmas. The first result may be derived from [C], Theorem 7.4 (the reader may like to know that a proof of the ‘Pringsheim-Landau Theorem’ used in [C] may be found on page 59 of [Wi]).

Lemma 2. Let $e^{tA}$ be a strongly continuous positive semigroup on a Banach lattice $X$, and let $g \in X$. Then for any $\lambda > s(A)$ we have that

$$(\lambda - A)^{-1}g = \int_{0}^{\infty} e^{s(A - \lambda)}g \, ds.$$ 

Here the right hand side is taken in the sense of an improper integral.

The next result may be found in [LM1] and [LM2].

1991 Mathematics Subject Classification. Primary 47-02, 47D06, Secondary 35B40.

Research supported in part by N.S.F. Grant D.M.S. 9201357.
Lemma 3. Let $e^{tA}$ be a strongly continuous semigroup on a Banach space $X$, and let $1 \leq p < \infty$. Then $1 \notin \sigma(e^{2\pi A})$ if and only if $i\mathbb{Z} \cap \sigma(A) = \emptyset$ and there is a constant $c > 0$ such that for any $v_n$, $v_{n+1}, \ldots, v_n \in X$ we have

$$
\int_0^{2\pi} \left| \sum_{k=-n}^n (ik - A)^{-1} v_k e^{ikt} \right|^p dt \leq c \int_0^{2\pi} \left| \sum_{k=-n}^n v_k e^{ikt} \right|^p dt.
$$

For the next result, we specialize to a Banach lattice of functions on a sigma-finite measure space. In fact, this is really no loss of generality, and the interested reader should find no trouble making sense of this result for a general Banach lattice by applying the ideas in [LT] Chapter 1.4.

Lemma 4. Let $P$ be a positive operator on $X$, a Banach lattice of functions on a sigma-finite measure space, such that $|g| \leq f \in X$ implies that $g \in X$. Let $1 \leq p < \infty$. If $f : [0, 2\pi] \to X$ is a measurable, simple function, then

$$
\left( \int_0^{2\pi} |P(f(t))|^p dt \right)^{1/p} \leq P \left( \left( \int_0^{2\pi} |f(t)|^p dt \right)^{1/p} \right).
$$

Proof. Let us set $f = \sum_{k=1}^n v_k \chi_{A_k}$, where $v_k \in X$, and the sets $A_k \subseteq [0, 2\pi]$ are disjoint. Then, letting $f_k = v_k |A_k|^{1/p}$, the result reduces to showing that

$$
\left( \sum_{k=1}^n |P(f_k)|^p \right)^{1/p} \leq P \left( \left( \sum_{k=1}^n |f_k|^p \right)^{1/p} \right).
$$

However, we know that

$$
\left( \sum_{k=1}^n |f_k|^p \right)^{1/p} = \text{l.u.b.} \sum_{|a_k|^p \leq 1} \sum_{k=1}^n \text{Re}(a_k f_k).
$$

Here, l.u.b. denotes the least upper bound in the lattice. Now, since $P$ is positive, we have that

$$
P \left( \text{l.u.b.} \sum_{|a_k|^p \leq 1} \sum_{k=1}^n \text{Re}(a_k f_k) \right)
$$

is an upper bound for $\sum_{k=1}^n \text{Re}(a_k P(f_k))$ whenever $\sum |a_k|^q \leq 1$. Hence

$$
\left( \sum_{k=1}^n |P(f_k)|^p \right)^{1/p} = \text{l.u.b.} \sum_{|a_k|^p \leq 1} \sum_{k=1}^n \text{Re}(a_k P(f_k))
$$

$$
\leq P \left( \text{l.u.b.} \sum_{|a_k|^p \leq 1} \sum_{k=1}^n \text{Re}(a_k f_k) \right)
$$

$$
= P \left( \sum_{k=1}^n |f_k|^p \right)^{1/p}.
$$
Proof of Theorem 1. It is well known that \( s(A) \leq \omega(A) \) (see [N]). Thus by simple rescaling arguments, we see that it is sufficient to show that if \( s(A) < 0 \), then \( \mathbb{T} \cap \sigma(e^{2\pi A}) = \emptyset \).

We will show, under the assumption that \( s(A) < 0 \), that if \( f : \mathbb{R} \to L_p \) is a bounded, measurable function that is periodic with period \( 2\pi \), then for each \( N > 0 \) we have

\[
\left( \int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s) \, ds \right\|_{L_p}^p \, dt \right)^{1/p} \leq A^{-1} \left( \int_0^{2\pi} \| f(t) \|_{L_p}^p \, dt \right)^{1/p}.
\]

In order to show this, we may assume without loss of generality that \( f \) restricted to \([0, 2\pi]\) is a simple function. Fix \( N > 0 \). By the positivity of \( e^{sA} \), and Fubini’s Theorem, we have that

\[
\left( \int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s) \, ds \right\|_{L_p}^p \, dt \right)^{1/p} \leq \left( \int_0^{2\pi} \left\| \int_0^N e^{sA} |f(t-s)| \, ds \right\|_{L_p}^p \, dt \right)^{1/p} = \left\| \left( \int_0^{2\pi} \right) \left( \int_0^N e^{sA} |f(t-s)| \, ds \right)^p \, dt \right\|_{L_p}^{1/p}.
\]

By the integral version of Minkowski’s Theorem (see [HLP], Section 203), it follows that for each \( \omega \in \Omega \)

\[
\left( \int_0^{2\pi} \left( \int_0^N e^{sA} |f(t-s)(\omega)| \, ds \right)^p \, dt \right)^{1/p} \leq \int_0^N \left( \int_0^{2\pi} (e^{sA} |f(t-s)(\omega)|)^p \, dt \right)^{1/p} \, ds = \int_0^N \left( \int_0^{2\pi} (e^{sA} |f(t)(\omega)|)^p \, dt \right)^{1/p} \, ds.
\]

Finally, from Lemma 4, we see that

\[
\left( \int_0^{2\pi} (e^{sA} |f(t)|^p) \, dt \right)^{1/p} \leq e^{sA} \left( \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p}.
\]

Putting all of these together, and applying Lemma 2, we obtain

\[
\left( \int_0^{2\pi} \left\| \int_0^N e^{sA} f(t-s) \, ds \right\|_{L_p}^p \, dt \right)^{1/p} \leq \left\| \int_0^{2\pi} e^{sA} \left( \int_0^N |f(t)|^p \, dt \right)^{1/p} \, ds \right\|_{L_p} \\
\leq \left\| \int_0^\infty e^{sA} \left( \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p} \, ds \right\|_{L_p} \\
= \left\| A^{-1} \left( \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p} \right\|_{L_p} \\
\leq \left\| A^{-1} \left( \int_0^{2\pi} \| f(t) \|_{L_p}^p \, dt \right)^{1/p} \right\|_{L_p} \\
= \left\| A^{-1} \left( \int_0^{2\pi} \| f(t) \|_{L_p}^p \, dt \right)^{1/p} \right\|_{L_p},
\]
where the last equality uses Fubini’s theorem. Now, if \( f(t) = e^{i\beta t} \sum_{k=-n}^{n} v_k e^{ikt} \) for some \( \beta \in \mathbb{R} \), then by Lemma 2, we see that

\[
\int_0^N e^{sA} f(t - s) \, ds \to \sum_{k=-n}^{n} (ik + i\beta - A)^{-1} v_k e^{ikt}
\]

uniformly in \( t \) as \( N \to \infty \). Hence by Lemma 3 it follows that \( e^{i\beta} \notin \sigma(e^{2\pi A}) \).

One might conjecture that the spectrum of the generator of a positive semigroup \( e^{tD} \) on an \( L_p \) space might characterize the dichotomy of the semigroup, that is, if \( a \) is any real number, then \( (a + i\mathbb{R}) \cap \sigma(D) = \emptyset \) if and only if \( e^{iaT} \cap \sigma(e^{tD}) = \emptyset \). However, this is not the case, as the next result shows.

**Theorem 5.** There is a positive semigroup \( e^{tD} \) acting on an \( L_2 \) space such that \( (1 + i\mathbb{R}) \cap \sigma(D) = \emptyset \), but \( e^{2\pi} \in \sigma(e^{2\pi D}) \).

**Proof.** For each \( M \in \mathbb{N} \), let \( C_M \) be the contraction acting on \( \ell^2 \) by the matrix

\[
C_M = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Note that if \( \lambda \neq 0 \), then

\[
(\lambda - C_M)^{-1} = \sum_{j=0}^{M-1} \lambda^{-1-j} C^j_M = \\
\begin{bmatrix}
\lambda^{-1} & \lambda^{-2} & \lambda^{-3} & \cdots & \lambda^{-M} \\
0 & \lambda^{-1} & \lambda^{-2} & \cdots & \lambda^{-M+1} \\
0 & 0 & \lambda^{-1} & \cdots & \lambda^{-M+2} \\
0 & 0 & 0 & \lambda^{-1} & \cdots & \lambda^{-M+3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \lambda^{-1} \\
\end{bmatrix}
\]

Thus, if \( |\lambda| = 1 \), then \( \| (\lambda - C_M)^{-1} \| \geq \sqrt{M} \). Also, if \( |\lambda| > 1 \), then \( \| (\lambda - C_M)^{-1} \| \leq \sum_{j=0}^{M-1} |\lambda|^{-1-j} \leq 1/(|\lambda| - 1) \). In particular, if \( |\lambda| \geq 2 \), then \( \| (\lambda - C_M)^{-1} \| \leq 1 \).

Also note that

\[
e^{tC_M} = \begin{bmatrix}
1 & t & t^2/2 & t^3/6 & \cdots & t^{M-1}/(M-1)! \\
0 & 1 & t & t^2/2 & \cdots & t^{M-2}/(M-2)! \\
0 & 0 & 1 & t & \cdots & t^{M-3}/(M-3)! \\
0 & 0 & 0 & 1 & \cdots & t^{M-4}/(M-4)! \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]

Thus we see that \( e^{tC_M} \) is a positive operator. Clearly \( \| e^{tC_M} \| \leq e^t \| e^{tC_M} \| \leq e^t \).

Consider the positive semigroup acting on \( L_2([0, 2\pi]) \) by

\[
e^{tA_M} f(x) = (e^{4t} - 1) \int_0^{2\pi} f(x) \frac{dx}{2\pi} + f(x + Mt),
\]
so that its generator is the closure of
\[ A_M f(x) = 4 \int_0^{2\pi} f(x) \frac{dx}{2\pi} + M \frac{d}{dx} f(x). \]

Note that \( \|e^{tA_M}\| \leq e^{4t} \).

Now consider the positive semigroup \( e^{tB_M} = e^{tA_M} \otimes e^{tC_M} \) acting on
\[ X_M = L_2((0, 2\pi)) \otimes \ell^2_M = L_2((0, 2\pi) \times \{1, 2, \ldots, M\}). \]
We see that this semigroup is generated by \( B_M = A_M \otimes I + I \otimes C_M \). Also, \( \|e^{tB_M}\| \leq e^{5t} \).

Consider a typical element of \( X_M \) given by \( f(x) = \sum_{n=-\infty}^{\infty} v_n e^{inx} \in X_M \), where \( v_n \in \ell^2_M \), and \( \|f\|^2_{X_M} = 2\pi \sum_{n=-\infty}^{\infty} \|v_n\|^2_2 \). If \( \lambda \neq 4 \) and \( \lambda \notin MZ \setminus \{0\} \), then \( \lambda \notin \sigma(B_M) \), and
\[ (\lambda - B_M)^{-1} f(x) = (\lambda - 4 - C_M)^{-1} v_0 + \sum_{n \neq 0} (\lambda - inM - C_M)^{-1} v_n e^{inx}. \]
Thus
\[ \|(\lambda - B_M)^{-1}\| = \max \left\{ \|(\lambda - 4 - C_M)^{-1}\|, \sup_{n \neq 0} \|(\lambda - inM - C_M)^{-1}\| \right\}. \]
In particular, if \( \Re(\lambda) = 1 \) and \( |\lambda| \leq M - 2 \), then \( \|(\lambda - B_M)^{-1}\| \leq 1 \), whereas if \( \lambda = 1 + iM \), then \( \|(\lambda - B_M)^{-1}\| \geq \sqrt{M} \).

Now consider the semigroup \( e^{tD} = \bigoplus_{M=1}^{\infty} e^{tB_M} \) acting on
\[ \bigoplus_{M=1}^{\infty} X_M = L_2 \left( \bigvee_{M=1}^{\infty} ([0, 2\pi] \times \{1, 2, \ldots, M\}) \right). \]
Note that \( e^{tD} \) really is a strongly continuous semigroup, with \( \|e^{tD}\| \leq e^{5t} \). The generator \( D \) is the closure of \( \bigoplus_{M=1}^{\infty} B_M \), and hence its resolvent set consists of those \( \lambda \) such that
\[ \|(\lambda - D)^{-1}\| = \sup_{M \geq 1} \|(\lambda - B_M)^{-1}\| < \infty, \]
that is, \( \sigma(D) \subseteq \{z : |z - 4| \leq 1\} \cup iZ \setminus \{0\} \). In particular, if \( \Re(\lambda) = 1 \), then \( \lambda \notin \sigma(D) \). However, \( \sup_{\lambda \in 1+iZ} \|(\lambda - D)^{-1}\| = \infty \), and hence, by Gerhard’s Theorem (see [N], p. 95), \( e^{2\pi} \in \sigma(e^{2\pi D}) \).

REFERENCES

[C] Ph. Clement, H.J.A.M. Heijmans, et al., One Parameter Semigroups, North-Holland, 1987.
[HLP] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, 1952.
[LT] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces Volume II, Springer-Verlag, 1979.
[LM1] Y. Latushkin and S.J. Montgomery-Smith, Lyapunov theorems for Banach spaces, Bull. A.M.S. 31 (1994), 44–49.
[LM2] Y. Latushkin and S.J. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, J. Func. Anal. 127 (1995), 173–197.
[N] R. Nagel (ed.), One Parameter Semigroups of Positive Operators, Springer-Verlag, 1984.
[We] L. Weis, The stability of positive semigroups on \( L_p \) spaces, Proc. A.M.S. (to appear).
[Wi] D. Widder, The Laplace Transform, vol. 6, Princeton Math Series, 1946.

Math. Dept., University of Missouri, Columbia MO 65211, U.S.A.