ON EXTENSIONS OF MINKOWSKI’S THEOREM ON SUCCESSIVE MINIMA

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Abstract. Minkowski’s 2nd theorem in the Geometry of Numbers provides optimal upper and lower bounds for the volume of a \( o \)-symmetric convex body in terms of its successive minima. In this paper we study extensions of this theorem from two different points of view: either relaxing the symmetry condition, assuming that the centroid of the set lies at the origin, or replacing the volume functional by the surface area.

1. Introduction

Let \( \mathcal{K}^n \) be the set of all convex bodies, i.e., compact convex sets, in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with non-empty interior. Let \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) be the standard inner product and the Euclidean norm in \( \mathbb{R}^n \), respectively. We denote by \( \mathcal{K}^n_o \subset \mathcal{K}^n \) the set of all \( o \)-symmetric bodies, i.e., those \( K \in \mathcal{K}^n \) satisfying \( K = -K \), and let \( \mathcal{K}^n_c \subset \mathcal{K}^n \) be the set of all convex bodies with centroid at the origin, i.e.,

\[
\text{cen}(K) = \frac{1}{\text{vol}(K)} \int_{K} x \, d^n x = 0.
\]

Here, \( d^n x \) means integration with respect to the \( n \)-dimensional Lebesgue-measure and \( \text{vol}(K) = \int_{K} d^n x \) is the volume of \( K \). The surface area of \( K \in \mathcal{K}^n \) is denoted \( F(K) \), and for general information on the theory of convex bodies we refer to [12, 26].

We denote by \( \mathbb{Z}^n \) the integer lattice, i.e., the lattice of all points with integral coordinates in \( \mathbb{R}^n \). Then any lattice \( \Lambda \subset \mathbb{R}^n \) of rank \( n \) can be obtained as \( \Lambda = B \mathbb{Z}^n \) with \( B \in \text{GL}(n, \mathbb{R}) \), and the determinant of the lattice is defined as \( \det \Lambda = | \det B | \). As general references for lattices we refer to [12, 13].

For \( K \in \mathcal{K}^n_o \cup \mathcal{K}^n_c \) and a lattice \( \Lambda \) of rank \( n \), let

\[
\lambda_i(K, \Lambda) = \min \{ \lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i \}
\]

be the \( i \)-th successive minimum of \( K \) with respect to \( \Lambda \), \( 1 \leq i \leq n \). Minkowski’s 2nd theorem on successive minima [23] (cf. [12]) states that

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for $K \in K^n_o$, 

$$\frac{1}{n!} \prod_{i=1}^{n} \frac{2}{\lambda_i(K, \Lambda)} \leq \frac{\text{vol}(K)}{\det \Lambda} \leq \prod_{i=1}^{n} \frac{2}{\lambda_i(K, \Lambda)}.$$  

Both bounds are best possible; for instance, for $\Lambda = Z^n$, the upper bound is attained for the cube $C_n = [-1, 1]^n$ and the lower bound for its polar body, the cross-polytope $C^*_n = \text{conv}\{\pm e_i : 1 \leq i \leq n\}$, where $e_i$ denotes the $i$-th canonical unit vector, and $\text{conv} S$ is the convex hull of a set $S$.

Other special convex bodies that will appear throughout the paper are the standard simplex $S_n = \text{conv}\{0, e_1, \ldots, e_n\}$ and its homothetic copy $T_n = -1 + (n+1)S_n$, where $1 = (1, \ldots, 1)^T$ is the all-one-vector.

It is well known that via the difference body $DK = K - K \in K^n_o$, Minkowski’s results (1.1) can be generalized to arbitrary bodies (see, e.g., [13, p. 59]):

$$\frac{1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(DK, \Lambda)} \leq \frac{\text{vol}(K)}{\det \Lambda} \leq \prod_{i=1}^{n} \frac{1}{\lambda_i(DK, \Lambda)}.$$  

The upper bound is a combination of the upper bound in (1.1) and the Brunn-Minkowski inequality (see e.g. [12, Thm. 8.1]). The lower bound stems from the following well-known fact (see [4, Thm. 2] or [11]).

**Remark 1.1.** Let $v_1, w_1, \ldots, v_n, w_n \in K$. Then, the volume of $K$ is at least the volume of the $o$-symmetric cross-polytope $\text{conv}\{\pm (1/2)(v_i - w_i) : 1 \leq i \leq n\}$.

In particular, both bounds in (1.2) can only be realized for $K \in K^n_o$, and so they do not provide much more information than Minkowski’s original result for $o$-symmetric convex bodies. Therefore, we are interested in a variant of (1.1) that does not rely on the symmetrization $DK$. As a first result we obtain the following lower bound whose proof is given in Section 2.

**Theorem 1.2.** Let $K \in K^n_x$ and let $\Lambda$ be a lattice of rank $n$. Then

$$\frac{n+1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(K, \Lambda)} \leq \frac{\text{vol}(K)}{\det \Lambda}.$$  

Equality holds if and only if there are positive numbers $\mu_1, \ldots, \mu_n > 0$ such that $K = \text{conv}\{\mu_1z_1, \ldots, \mu_nz_n, -(\mu_1z_1 + \cdots + \mu_nz_n)\}$, where $\{z_1, \ldots, z_n\}$ is a basis of $\Lambda$.

A corresponding upper bound on the volume of $K \in K^n_x$ immediately relates to a longstanding conjecture of Ehrhart [9] on the maximal volume of a convex body $K \in K^n_c$ whose interior is free from non-zero lattice points. In this context, the best-known bound is based on a result of Milman and Pajor [22] showing that $\text{vol}(K) \leq 2^n \text{vol}(K \cap (-K))$ for $K \in K^n_o$. Hence with (1.1) applied to $K \cap (-K) \subseteq K$, we find

$$\frac{\text{vol}(K)}{\det \Lambda} \leq 4^n \prod_{i=1}^{n} \frac{1}{\lambda_i(K, \Lambda)},$$  

and in view of Ehrhart’s conjecture (see Conjecture 2.1) the optimal factor is conjectured to be $(n+1)^n/n!$ instead of $4^n$. In Propositions 2.3 and 2.4 we...
verify this conjecture for the special cases $n = 2$ and simplices of arbitrary dimension, respectively.

Another direction of extending Minkowski’s 2nd theorem is to replace the volume functional by other functionals, for instance, the lattice point enumerator (see, e.g., [5, 19, 20]) or the intrinsic volumes (see, e.g., [15, 28]). Here we are interested in inequalities analogous to (1.1) for the surface area. In [15] it was shown $F(K)/\operatorname{vol}(K) > \lambda_n(K, \mathbb{Z}^n)$ for $K \in \mathcal{K}_o^n$, and with the lower bound in (1.1) we get

$$F(K) > \frac{2^n}{n!} \frac{n-1}{\prod_{i=1}^{n} \lambda_i(K, \mathbb{Z}^n)}.$$

In order to present our improvement on this bound, we need the notation of the elementary symmetric functions

$$\sigma_k(\rho_1, \ldots, \rho_n) = \sum_{J \subseteq \{1, \ldots, n\}} \prod_{i \in J} \rho_i,$$

for $k \in \{1, \ldots, n\}$, and real numbers $\rho_1, \ldots, \rho_n$.

**Theorem 1.3.** Let $K \in \mathcal{K}_o^n$ and let $\lambda_i = \lambda_i(K, \mathbb{Z}^n)$, $1 \leq i \leq n$. Then

$$F(K) \geq \frac{2^n}{(n-1)!} \sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_n^{-2})},$$

and equality holds if and only if $K = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})C_n^*$, where $\text{diag}(\cdot)$ denotes the diagonal matrix.

The proof of this result is given in Section 3. Generalizations to arbitrary lattices are not so straightforward as those for the volume functional because the surface area is not $\text{SL}(n, \mathbb{R})$-invariant. Still we obtain meaningful results in the general situation that are presented in Theorem 3.5. We also note that the above inequality has the same structure as the one in [16, Thm. 1.3], where the surface area is related to the successive inner radii of a convex body.

In general, we cannot expect to find upper bounds on $F(K)$, or on the quotient $F(K)/\operatorname{vol}(K)$, in terms of $\lambda_i(K, \mathbb{Z}^n)^{-1}$ as Example 3.1 shows.

Hence, in order to obtain upper bounds, the convex bodies need to have more lattice structure, and this leads to the class of rational polytopes. Here, $P$ is called a rational polytope if all its vertices lie in $\mathbb{Q}^n$. For basic facts and notions about polytopes we refer to [29]. Given a rational polytope $P \in \mathcal{K}^n$ and a facet $F$, we denote by $\operatorname{aff}_o(F)$ the $(n-1)$-dimensional linear subspace parallel to the affine hull of $F$. We observe, by the rationality of $P$, that the intersection $\operatorname{aff}_o(F) \cap \mathbb{Z}^n$ is an $(n-1)$-dimensional lattice. With this notation, the lattice surface area can be described as

$$g_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\operatorname{vol}_{n-1}(F)}{\det(\operatorname{aff}_o(F) \cap \mathbb{Z}^n)},$$

where $\operatorname{vol}_{n-1}(\cdot)$ is the $(n-1)$-dimensional volume in $\mathbb{R}^{n-1}$. The notation $g_{n-1}(P)$ is taken from Ehrhart theory, where the lattice surface area of a
lattice polytope \( P \), i.e., all vertices lie in \( \mathbb{Z}^n \), appears as the coefficient of order \( n - 1 \) in its Ehrhart polynomial

\[
\#(kP \cap \mathbb{Z}^n) = \sum_{i=0}^{n} g_i(P)k^i, \quad k \in \mathbb{N}.
\]

For details and more information on Ehrhart theory, we refer to [2].

For \( o \)-symmetric lattice polytopes, and actually for \( o \)-symmetric rational polytopes, it was shown [18, Thm. 1.2] that

\[
g_{n-1}(P) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i(P, \mathbb{Z}^n).
\]

Equality holds, for example, for \( C_n \) and \( C_n^* \). Here we extend and complement this result by providing bounds for all rational polytopes as well as for rational polytopes with centroid at the origin.

**Theorem 1.4.**

i) Let \( P \in K^n \) be a rational polytope. Then

\[
\frac{g_{n-1}(P)}{\text{vol}(P)} \leq \frac{n+1}{2} \sum_{i=1}^{n} \lambda_i(DP, \mathbb{Z}^n),
\]

and the standard simplex \( S_n \) shows that the inequality is best possible.

ii) Let \( P \in K^n_c \) be a rational polytope and let \( n \geq 2 \). Then

\[
\frac{g_{n-1}(P)}{\text{vol}(P)} < \frac{n}{2} \sum_{i=1}^{n} \lambda_i(P, \mathbb{Z}^n).
\]

In Section 4, we discuss the proofs of these results, and moreover we show that the factor \( n/2 \) in the second inequality is almost tight. Now combining the above bounds on \( g_{n-1}(P)/\text{vol}(P) \) with the upper bounds in (1.1), (1.2), or (1.3), we immediately get

**Corollary 1.5.** Let \( P \in K^n \) be a rational polytope.

i) Then

\[
g_{n-1}(P) \leq \frac{n+1}{2} \sigma_{n-1} \left( \frac{1}{\lambda_1(DP, \mathbb{Z}^n)}, \ldots, \frac{1}{\lambda_n(DP, \mathbb{Z}^n)} \right).
\]

ii) If \( P \in K^n_c \), then

\[
g_{n-1}(P) < 4^n \frac{n}{2} \sigma_{n-1} \left( \frac{1}{\lambda_1(P, \mathbb{Z}^n)}, \ldots, \frac{1}{\lambda_n(P, \mathbb{Z}^n)} \right).
\]

iii) If \( P \in K^n_o \), then

\[
g_{n-1}(P) \leq 2^{n-1} \sigma_{n-1} \left( \frac{1}{\lambda_1(P, \mathbb{Z}^n)}, \ldots, \frac{1}{\lambda_n(P, \mathbb{Z}^n)} \right).
\]

However, only the last inequality is best possible, which has been pointed out before in [18]. Further immediate consequences of Theorem 1.4 are relations between the roots of the Ehrhart polynomial – when we regard the right hand side of (1.4) as a formal polynomial in a complex variable – and the successive minima (cf. Corollary 4.4). Those kind of relations were the main motivation for (1.5) in [18].
Finally, we remark that in contrast to the surface area, we now cannot expect lower bounds on $g_{n-1}(P)$ in terms of the successive minima as shown in Example 4.1.

2. Volume bounds for $K \in \mathcal{K}_c^n$

In this section, we discuss a variant of (1.1) for the class of convex bodies having their centroid at the origin, i.e., for $K \in \mathcal{K}_c^n$. A basic and beautiful result in this context is Grünbaum’s halfspace theorem. For a hyperplane $H$, we denote by $H^+$ and $H^-$ the two associated halfspaces.

**Theorem 2.1** (Grünbaum, [14]). Let $K \in \mathcal{K}_c^n$ and let $H^+$ be a halfspace containing the centroid of $K$. Then

$$\text{vol}(K \cap H^+) \geq \left(\frac{n}{n+1}\right)^n \text{vol}(K).$$

Our first aim is to prove Theorem 1.2, which is an immediate consequence of the following lemma.

**Lemma 2.2.** Let $K \in \mathcal{K}_c^n$ and let $u_1, \ldots, u_n \in K$ be linearly independent. Then

$$\text{vol}(K) \geq \frac{n+1}{n!}|\text{det}(u_1, \ldots, u_n)|.$$

Equality holds if and only if $K = \text{conv}\{u_1, \ldots, u_n, -(u_1 + \cdots + u_n)\}$.

**Proof.** Via a suitable linear transformation, we may assume that all the vectors $u_i$ have first coordinate equal to $-1$ and that $\text{det}(u_1, \ldots, u_n) = 1$. For $t \in \mathbb{R}$, let $H_t = \{x \in \mathbb{R}^n : \langle e_1, x \rangle = t\}$ be the family of hyperplanes orthogonal to the first unit vector $e_1$.

As in Grünbaum’s proof of Theorem 2.1, we first apply Schwarz-symmetrization to $K$ with respect to $\text{lin}\{e_1\}$, the linear hull of $e_1$ (see e.g. [12, Sect. 9.3]). Denoting by $B_n$ the $n$-dimensional unit ball, this means that for every $t \in \mathbb{R}$, we replace $K \cap H_t$ by the $(n-1)$-ball $t e_1 + r(t)(B_n \cap H_0)$ with center $t e_1$ and having the same volume as $K \cap H_t$, i.e.,

$$r(t) = \left(\frac{\text{vol}_{n-1}(K \cap H_t)}{\text{vol}_{n-1}(B_{n-1})}\right)^{1/(n-1)}.$$

The so created convex body $L$, say, is symmetric with respect to $\text{lin}\{e_1\}$, and we also have $\text{vol}(L) = \text{vol}(K)$ and $\text{cen}(L) = 0$. With $F = L \cap H_{-1}$, we get by the choice of the vectors $u_i$,

$$\text{vol}(\text{conv}(F, 0)) \geq \text{vol}(\text{conv}(0, u_1, \ldots, u_n)) = \frac{1}{n!}.\tag{2.1}$$

Now let $\hat{L} = K \cap \{x \in \mathbb{R}^n : \langle e_1, x \rangle \geq -1\}$ and let $\beta > 0$ be such that the pyramid $P = \text{conv}\{F, \beta e_1\}$ has the same volume as $\hat{L}$. Since $\hat{L}$ and $P$ are symmetric with respect to $e_1$ their centroids are on the line $\text{lin}\{e_1\}$ and so we may write $\text{cen}(P) = \gamma_P e_1$, $\text{cen}(L) = \gamma_{\hat{L}} e_1$ for suitable numbers $\gamma_P, \gamma_{\hat{L}}$ with $\gamma_{\hat{L}} \geq 0$. For pyramids we have $\text{vol}(P) = (n+1)\text{vol}(\text{conv}\{F, \gamma_P e_1\})$ (see e.g. [7, Sect. 34]) and so in view of (2.1)

$$\text{vol}(K) = \text{vol}(L) \geq \text{vol}(\hat{L}) = \text{vol}(P)$$

$$= (n+1)\text{vol}(\text{conv}\{F, \gamma_P e_1\}) = \frac{n+1}{n!}(1 + \gamma_P). \tag{2.2}$$
It remains to show \( \gamma_P \geq 0 \). Actually we will show \( \gamma_P \geq \gamma_{\hat{L}} \), which seems to be quite evident. But since we also want to discuss the equality case, we present a proof.

We may assume \( \hat{L} \neq P \). For \( t \in \mathbb{R} \), let \( l(t) \) be the radius of the \((n-1)\)-ball \( P \cap H_t \), i.e.,

\[
l(t) = \left( \frac{\text{vol}_{n-1}(P \cap H_t)}{\text{vol}_{n-1}(B_{n-1})} \right)^{1/(n-1)}.
\]

Then \( l(t) \neq 0 \) if and only if \( t \in [-1, \beta) \) and \( l(t) \) is an affine function.

Since \( r(-1) = l(-1) \) and \( r(t) \) is concave, \( \hat{L} \neq P \), and \( \text{vol}(P) = \text{vol}(\hat{L}) \), there exists a unique \( \alpha \in (-1, \beta) \) with

\[
r(t) > l(t) \quad \text{for} \quad t \in (-1, \alpha) \quad \text{and} \quad l(t) > r(t) \quad \text{for} \quad t \in (\alpha, \beta).
\]

Hence we know

\[
\langle e_1, x \rangle \leq \alpha \quad \text{for} \quad x \in \hat{L} \setminus P \quad \text{and} \quad \langle e_1, x \rangle \geq \alpha \quad \text{for} \quad x \in P \setminus \hat{L}.
\]

Finally, since \( \text{vol}(P) = \text{vol}(\hat{L}) \) it holds \( \text{vol}(\hat{L} \setminus P) = \text{vol}(P \setminus \hat{L}) \) and so we get

\[
\gamma_P = \int_P \langle e_1, x \rangle \, d^n x = \int_{P \setminus \hat{L}} \langle e_1, x \rangle \, d^n x + \int_{P \cap \hat{L}} \langle e_1, x \rangle \, d^n x
\]

\[
> \alpha \text{vol}(P \setminus \hat{L}) + \int_{P \cap \hat{L}} \langle e_1, x \rangle \, d^n x = \alpha \text{vol}(\hat{L} \setminus P) + \int_{P \cap \hat{L}} \langle e_1, x \rangle \, d^n x
\]

\[
> \int_{\hat{L} \setminus P} \langle e_1, x \rangle \, d^n x + \int_{P \cap \hat{L}} \langle e_1, x \rangle \, d^n x = \int_{\hat{L}} \langle e_1, x \rangle \, d^n x = \gamma_{\hat{L}}.
\]

Hence \( \gamma_P > \gamma_{\hat{L}} > 0 \) as desired, since we have assumed \( \hat{L} \neq P \).

If we have equality, then (2.2) gives \( L = \hat{L} \) and \( \gamma_P = 0 \), and in view of the above argumentation we must also have \( L = P \). Since we also must have equality in (2.1), we conclude \( K \cap H_{-1} = \text{conv}\{u_1, \ldots, u_n\} \). Let \( u \in K \) be the point whose image under the Schwarz-symmetrization is the apex \( \beta e_i \) of the pyramid. Since \( L = P \), we have \( K = \text{conv}\{u_1, \ldots, u_n, u\} \). Finally, since for a simplex the centroid coincides with the arithmetic mean of its vertices, we get \( u = -(u_1 + u_2 + \cdots + u_n) \).

The proof of Theorem (1.2) is now an immediate consequence of the Lemma above.

**Proof of Theorem (1.2)** We write \( \lambda_i = \lambda_i(K, \Lambda) \) and let \( z_1, \ldots, z_n \in \Lambda \) be linearly independent lattice points such that \( z_i/\lambda_i \in K \), \( 1 \leq i \leq n \). Lemma (2.2) applied to these vectors gives

\[
\text{vol}(K) \geq |\det(z_1, \ldots, z_n)| \frac{n+1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i} \geq \det \Lambda \frac{n+1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i},
\]

and equality holds if and only if \( |\det(z_1, \ldots, z_n)| = \det \Lambda \), i.e., \( \{z_1, \ldots, z_n\} \) is a basis of \( \Lambda \), and

\[
K = \text{conv} \left\{ \frac{1}{\lambda_1} z_1, \ldots, \frac{1}{\lambda_n} z_n, -\left( \frac{1}{\lambda_1} z_1 + \cdots + \frac{1}{\lambda_n} z_n \right) \right\}.
\]

In order to further discuss the equality case, it is no restriction to assume that \( \Lambda = \mathbb{Z}^n \) and \( z_i = e_i \), \( 1 \leq i \leq n \). We write \( \text{int} M \) to denote the interior of a set \( M \). Let \( K = \text{conv} \{\mu_1 e_1, \ldots, \mu_n e_n, -(\mu_1 e_1 + \cdots + \mu_n e_n)\} \) for real
numbers $\mu_1, \ldots, \mu_n > 0$. Assuming that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, we see that $\text{int}((1/\mu_i)K) \cap \mathbb{Z}^n \subset \text{lin}(e_1, \ldots, e_{i-1})$ and $e_i \in (1/\mu_i)K$, for $1 \leq i \leq n$. It means that $\lambda_i(K, \mathbb{Z}^n) = 1/\mu_i$ for $1 \leq i \leq n$, and thus $K$ attains equality. □

As mentioned in the introduction, the question about an upper bound as in (1.1) for $K \in \mathcal{K}_c^2$ is strongly related to Ehrhart’s conjecture (see [9], and also [3, 24]).

Conjecture 2.1 (Ehrhart, [9]). Let $K \in \mathcal{K}_c^2$ with $\text{int} K \cap \mathbb{Z}^n = \{0\}$. Then

$$\text{vol}(K) \leq \frac{(n + 1)^n}{n!},$$

and equality holds if and only if $K$ is – up to unimodular transformations – the simplex $T_n$.

Ehrhart [8, 10] proved his conjecture, among others, for two dimensional convex bodies. Here we follow his approaches in order to extend his results to successive minima inequalities.

Proposition 2.3. Let $K \in \mathcal{K}_c^2$ and let $\Lambda$ be a lattice of rank 2. Then

$$\frac{\text{vol}(K)}{\det \Lambda} \leq \frac{9}{2} \frac{1}{\lambda_1(K, \Lambda)} \frac{1}{\lambda_2(K, \Lambda)},$$

and for $\Lambda = \mathbb{Z}^2$, equality holds for the triangle $T_2$.

Proof. As always when dealing with the volume, we may assume $\Lambda = \mathbb{Z}^2$, and for short we write $\lambda_i = \lambda_i(K, \mathbb{Z}^2)$. We assume that $\lambda_1 \lambda_2 \text{vol}(K) > 9/2$, and let $H$ be a line passing through the centroid $0$ of $K$, such that $0$ is the midpoint of the corresponding chord $K \cap H$. Then, by a result of Ehrhart [8] we know that one of the sets $C^+ = (K \cap H^+) \cup (- (K \cap H^+))$ or $C^- = (K \cap H^-) \cup (- (K \cap H^-))$ is convex, and without loss of generality we assume that $C^+$ is convex. By the $o$-symmetry of $C^+$ we have, in particular, $\lambda_i \leq \lambda_i(C^+, \mathbb{Z}^2)$, $i = 1, 2$. Now, by Theorem 2.1 and our assumption we get

$$\text{vol}(K \cap H^+) \geq \frac{4}{9} \text{vol}(K) > \frac{4}{9} \frac{9}{2\lambda_1 \lambda_2} = \frac{2}{\lambda_1 \lambda_2},$$

and therefore,

$$\text{vol}(C^+) = 2 \text{vol}(K \cap H^+) > \frac{4}{\lambda_1 \lambda_2} \geq \frac{4}{\lambda_1(C^+, \mathbb{Z}^2) \lambda_2(C^+, \mathbb{Z}^2)},$$

contradicting Minkowski’s inequality (1.1). □

Proposition 2.4. Let $S \in \mathcal{K}_c^n$ be a simplex and let $\Lambda$ be a lattice of rank $n$. Then

$$\frac{\text{vol}(S)}{\det \Lambda} \leq \frac{(n + 1)^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(S, \Lambda)},$$

and for $\Lambda = \mathbb{Z}^n$, equality holds for the simplex $T_n$.

In [10], Ehrhart used a nice symmetrization that transforms the simplex into a parallelepiped. Let $S = \text{conv}\{v_0, v_1, \ldots, v_n\}$ be a simplex with
centroid at the origin, that is, \( v_0 = - \sum_{i=1}^{n} v_i \). We consider the \((n-1)\)-
dimensional subspace \( H \subset \mathbb{R}^n \) which is parallel to the facet \( \text{conv}\{v_1, \ldots, v_n\} \) of \( S \). Then \( S \cap H = \text{conv}\{w_1, \ldots, w_n\} \), where
\[
w_i = v_0 + \frac{n}{n+1}(v_i - v_0) = \frac{1}{n+1}v_0 + \frac{n}{n+1}v_i,
\]
\( 1 \leq i \leq n \). Now, we define the parallelepiped
\[
P_H(S) = \text{conv}\left\{v_0 + \sum_{i=1}^{n} \varepsilon_i(w_i - v_0) : (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n\right\}.
\]
The vertex of \( P_H(S) \) opposite to \( v_0 \) is \( v_0 + \sum_{i=1}^{n} (w_i - v_0) = -(n-1)v_0 \). Next, we translate \( P_H(S) \) by its center, \( \left( v_0 - (n-1)v_0 \right)/2 = -(n-2)/2v_0 \), and we define the “symmetral” of \( S \) by \( \Pi_H(S) = P_H(S) + (n-2)/2v_0 \).

**Lemma 2.5.** Let \( S = \text{conv}\{v_0, v_1, \ldots, v_n\} \) have its centroid at the origin and let \( H \) be the \((n-1)\)-
dimensional subspace parallel to \( \text{conv}\{v_1, \ldots, v_n\} \). Let \( H^- \) be the halfspace containing the vertex \( v_0 \). Then
\[
\Pi_H(S) \cap H^- \subseteq \frac{n}{2}S.
\]

**Proof.** Since the claim is invariant under affine transformations, we consider the simplex \( T_n \), which can be expressed as
\[
T_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq -1, 1 \leq i \leq n, \sum_{i=1}^{n} x_i \leq 1 \right\}.
\]
Here \( H = \{x \in \mathbb{R}^n : \langle 1, x \rangle = 0\} \) and \( T_n \cap H = -1 + \text{conv}\{ne_1, \ldots, ne_n\} \), and thus the symmetral is \( \Pi_H(T_n) = (n/2)[-1, 1]^n \). Since \( H^- = \{x \in \mathbb{R}^n : \langle 1, x \rangle \leq 0\} \), we get from the facet description of \( T_n \) that indeed it holds \( \Pi_H(T_n) \cap H^- \subseteq (n/2)T_n \). \( \Box \)

**Proof of Proposition 2.4**. Again, it suffices to consider the standard lattice \( \Lambda = \mathbb{Z}^n \). For \( i \in \{1, \ldots, n\} \), we write \( \lambda_i = \lambda_i(\Pi_H(S), \mathbb{Z}^n) \), and let \( z_i \in \mathbb{Z}^n \) be such that \( z_i \in \lambda_i \Pi_H(S) \). Since \( \Pi_H(S) \) is \( \varepsilon \)-symmetric, we can assume (after a suitable reflection of \( z_i \)) that \( z_i \in \lambda_i (\Pi_H(S) \cap H^-) \) – we follow the notation in Lemma 2.5. Then Lemma 2.5 implies that \( z_i \in \lambda_i (n/2)S \) and hence \( \lambda_i (S, \mathbb{Z}^n) \leq (n/2)\lambda_i, 1 \leq i \leq n \).

Now, we assume that \( \lambda_1(S, \mathbb{Z}^n) \cdot \ldots \cdot \lambda_n(S, \mathbb{Z}^n) \text{vol}(S) > (n+1)^n/n! \). By definition of \( \Pi_H(S) \), we have \( \text{vol}(\Pi_H(S)) = n! \text{vol}(S \cap H^-) \) and thus using Theorem 2.1 we get
\[
\text{vol}(\Pi_H(S)) = n! \text{vol}(S \cap H^-) \geq \frac{n! n^n}{(n+1)^n} \text{vol}(S)
\]
\[
> \frac{n^n}{\prod_{i=1}^{n} \lambda_i(S, \mathbb{Z}^n)} \geq \frac{2^n}{\prod_{i=1}^{n} \lambda_i}.
\]
It contradicts Minkowski’s 2nd theorem, (1.1), and proves the claim. \( \Box \)
3. Bounds for the surface area

In general, we cannot expect to find upper bounds on $F(K)$, or on the quotient $F(K)/\text{vol}(K)$, in terms of $\lambda_i(K, \mathbb{Z}^n)^{-1}$ as the following example shows.

**Example 3.1.** For $\ell \in \mathbb{N}$, we consider the cross-polytope

$$K_\ell = \text{conv}\{\pm e_1, \ldots, \pm e_{n-1}, \pm (\ell e_1 + e_n)\}.$$

Then $\lambda_i(K_\ell, \mathbb{Z}^n) = 1$, for $1 \leq i \leq n$ and all $\ell \in \mathbb{N}$, but both $F(K_\ell) \to \infty$ and $F(K_\ell)/\text{vol}(K_\ell) \to \infty$ as $\ell \to \infty$.

The proof of the lower bound, i.e., Theorem 1.3, is based on the following lemma which might be of independent interest.

**Lemma 3.2.** Let $Z \in \mathbb{Z}^{n \times n}$, $\det Z \neq 0$, and let $\alpha \in \mathbb{R}^n$ be with $\|\alpha\| = 1$. For $\varepsilon \in \mathbb{R}^n$, we write $\alpha_\varepsilon = (\varepsilon_1 a_1, \ldots, \varepsilon_n a_n)$. Then

$$\sum_{\varepsilon \in \{\pm 1\}^n} \|Z \alpha_\varepsilon\| \geq 2^n,$$

and, for $\alpha > 0$ equality holds if and only if up to column permutations and signs $-Z$ is the identity matrix.

**Proof.** After a suitable permutation of the columns $z_1, \ldots, z_n$ of $Z$ we may assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Let $g_1, \ldots, g_n$ be the Gram-Schmidt orthogonal basis associated to $z_1, \ldots, z_n$, i.e.,

$$g_i = z_i|\text{lin}\{z_1, \ldots, z_{i-1}\}^\perp, \quad 1 \leq i \leq n.$$

So $g_i$ is the orthogonal projection of $z_i$ onto the orthogonal complement of the $(i-1)$-dimensional space generated by $z_1, \ldots, z_{i-1}$. We observe that $g_1 = z_1$. First we claim that

$$(3.1) \sum_{\varepsilon \in \{\pm 1\}^n} \|Z \alpha_\varepsilon\| \geq 2^n \|\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n\|.$$  

Taking the symmetry of the vectors $\alpha_\varepsilon$ into account we have to show that

$$\sum_{\varepsilon \in \{\pm 1\}^n} \|Z \alpha_\varepsilon\| \geq 2^{n-1} \|\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_n g_n\|.$$  

Let $\tilde{g}_i = z_i|\text{lin}\{z_1\}^\perp$, for $2 \leq i \leq n$; in particular, $\tilde{g}_2 = g_2$. For each $\varepsilon \in \{\pm 1\}^n$, let

$$\varepsilon' = (1, -\varepsilon_2, \ldots, -\varepsilon_n)^T,$$

i.e., the last $n-1$ coordinates change their signs, and let

$$h = \varepsilon_2 \alpha_2 z_2 + \cdots + \varepsilon_n \alpha_n z_n.$$  

In view of the properties of Steiner-symmetrization (see e.g. [12, Prop. 9.1]), we see that the perimeter of the triangle $\text{conv}\{\pm \alpha_1 z_1, h|\text{lin}\{z_1\}^\perp\}$ is less
than or equal to the perimeter of conv\{±\alpha_1 z_1, h\}, i.e.,
\[ \| Z \alpha \| + \| Z \bar{\alpha} \| = \| \alpha_1 z_1 + h \| + \| \alpha_1 z_1 - h \| \]
\[ \geq \| \alpha_1 z_1 + h \| \text{lin} \{ z_1 \}^\perp + \| \alpha_1 z_1 - h \| \text{lin} \{ z_1 \}^\perp \]
\[ = 2 \| \alpha_1 z_1 + h \| \text{lin} \{ z_1 \}^\perp \]
\[ = 2 \| \alpha_1 g_1 + \varepsilon_2 \alpha_2 g_2 + \varepsilon_3 \alpha_3 g_3 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \|. \]

Hence we know that
\[ \sum_{\varepsilon \in \{1, \pm 1, \ldots, \pm 1\}^T} \| Z \alpha \| \]
\[ \geq 2 \sum_{\varepsilon \in \{1, \pm 1, \ldots, \pm 1\}^T} \| \alpha_1 g_1 + \alpha_2 g_2 + \varepsilon_3 \alpha_3 g_3 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \|. \tag{3.2} \]

Now we do the same with respect to \( g_2 = z_2 \| \text{lin} \{ z_1 \}^\perp \), i.e., we consider the points \( \bar{g}_i = g_i | \text{lin} \{ g_2 \}^\perp \), \( 1 \leq i \leq n, i \neq 2 \). By the orthogonality of \( g_1, g_2 \) we have \( \bar{g}_1 = g_1 = z_1 \), and by the definition of Gram-Schmidt orthogonal basis we also have \( \bar{g}_3 = g_3 \). Arguing as before but with respect to the triangle \( \text{conv} \{ \pm \alpha_2 g_2, \alpha_1 g_1 + h \} \), with \( h = \varepsilon_3 \alpha_3 \bar{g}_3 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \), we get
\[ \| \alpha_2 g_2 + \alpha_1 g_1 + h \| + \| \alpha_2 g_2 - (\alpha_1 g_1 + h) \| \]
\[ \geq 2 \| \alpha_2 g_2 + (\alpha_1 g_1 + h) | \text{lin} \{ g_2 \}^\perp \| \]
\[ = 2 \| \alpha_2 g_2 + \alpha_1 g_1 + \varepsilon_3 \alpha_3 \bar{g}_3 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \| \]
\[ = 2 \| \alpha_1 g_1 + \alpha_2 g_2 + \varepsilon_3 \alpha_3 \bar{g}_3 + \varepsilon_4 \alpha_4 \bar{g}_4 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \| , \]

and in view of the orthogonality of \( g_1 \) to \( g_2, h \) we conclude
\[ \| \alpha_1 g_1 + \alpha_2 g_2 + h \| + \| \alpha_1 g_1 + \alpha_2 g_2 - h \| \]
\[ = \| \alpha_2 g_2 + \alpha_1 g_1 + h \| + \| \alpha_2 g_2 - (\alpha_1 g_1 + h) \| \]
\[ \geq 2 \| \alpha_1 g_1 + \alpha_2 g_2 + \varepsilon_3 \alpha_3 \bar{g}_3 + \varepsilon_4 \alpha_4 \bar{g}_4 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \| . \]

Hence, together with (3.2), we get
\[ \sum_{\varepsilon \in \{1, \pm 1, \ldots, \pm 1\}^T} \| Z \alpha \| \]
\[ \geq 4 \sum_{\varepsilon \in \{1, \pm 1, \ldots, \pm 1\}^T} \| \alpha_1 g_1 + \alpha_2 g_2 + \varepsilon_3 \alpha_3 \bar{g}_3 + \varepsilon_4 \alpha_4 \bar{g}_4 + \cdots + \varepsilon_n \alpha_n \bar{g}_n \|. \]

Repeating this procedure we get (3.1) and so it suffices to show that
\[ \sum_{i=1}^{n} \alpha_i^2 \| g_i \|^2 \geq 1. \]

By the definition of Gram-Schmidt orthogonal basis we have
\[ Z = (g_1, \ldots, g_n) T, \]
where \( T \) is an upper triangular matrix whose diagonal elements are all equal to 1. Hence for \( 1 \leq i \leq n \) we have
\[ \det(Z_i^T Z_i) = \| g_i \|^2 \cdots \| g_i \|^2, \]
where $Z_i$ is the $(n \times i)$-submatrix of $Z$ consisting of the first $i$ columns; in particular, we have $Z_n = Z$. Let $m_i = \det(Z_i^\top Z_i)$, $1 \leq i \leq n$. Then $m_i \in \mathbb{N}$, $m_i \geq 1$, and since $\|\alpha\| = 1$, we may write, using the weighted arithmetic-geometric mean inequality, that

$$
\sum_{i=1}^{n} \alpha_i^2 \|g_i\|^2 = \alpha^2_1 m_1 + \alpha^2_2 \frac{m_2}{m_1} + \alpha^2_3 \frac{m_3}{m_2} + \cdots + \alpha^2_n \frac{m_n}{m_{n-1}} \\
\geq m_1 \left( \frac{m_2}{m_1} \right) \alpha^2_1 \cdots \left( \frac{m_n}{m_{n-1}} \right) \alpha^2_n \\
= m_1^{\alpha_1^2} m_2^{\alpha_2^2} \cdots m_{n-1}^{\alpha_{n-1}^2} m_n^{\alpha_n^2}.
$$

By assumption we have $\alpha_i \geq \alpha_{i+1}$ and since $m_i \geq 1$ we are done.

If equality holds then we have $m_i = 1$, $1 \leq i \leq n$, and so $\|g_i\| = 1$, $1 \leq i \leq n$. By the equality discussions of the Steiner-symmetrization we also know that the vectors $z_i$ have to be pairwise orthogonal and thus $g_i = z_i$, $1 \leq i \leq n$.

**Remark 3.3.** We observe that Lemma 3.2 does not restrict to integer matrices. It holds for any matrix $V$ with $\det(V_i^\top V_i) \geq 1$, $1 \leq i \leq n$, where $V_i$ is the $(n \times i)$-submatrix of $V$ consisting of the first $i$ columns. Equality is attained if only if $V$ is an orthogonal matrix.

The connection between the matrix problem and the surface area is based on the next calculation.

**Fact 3.4.** Let $P = \{ x \in \mathbb{R}^n : \langle a_j, x \rangle \leq b_j, 1 \leq j \leq m \}$ be a non-redundant representation of a polytope with $\|a_j\| = 1$, and let $\phi_j$ be the $(n-1)$-dimensional volume of the facet with normal vector $a_j$. Then, for $B \in \text{GL}(n, \mathbb{R})$, the polytope $BP$ has outer normal vectors $B^{-\top}a_j$, $1 \leq j \leq m$, and the $(n-1)$-dimensional volume of the facet with normal $B^{-\top}a_j$ is given by

$$
| \det B | \left\| B^{-\top}a_j \right\| \phi_j.
$$

Now we are ready to prove the main theorem of this section.

**Proof of Theorem 1.3.** Let $z_i \in \lambda_i K \cap \mathbb{Z}^n$, $1 \leq i \leq n$, be $n$ linearly independent lattice points. Let $Z$ be the matrix with columns $z_1, \ldots, z_n$. Then

$$
Z \text{ diag } (\lambda_1^{-1}, \ldots, \lambda_n^{-1}) C_n^* \subseteq K.
$$

The volume of each facet of the cross-polytope $\text{diag } (\lambda_1^{-1}, \ldots, \lambda_n^{-1}) C_n^*$ is

$$
\frac{1}{(n-1)!} \sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_n^{-2})},
$$

and writing

$$
\alpha = \frac{1}{\sqrt{\lambda_1^2 + \cdots + \lambda_n^2}} (\lambda_1, \ldots, \lambda_n)^\top,
$$
the $2^n$ outer unit normal vectors of the above cross-polytope are given by $\alpha_\epsilon$, for $\epsilon \in \{(+1, \ldots, +1)^t\}$. Hence, in view of Fact 3.4 we get

$$F(K) \geq F\left(Z \text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})C_n^*\right)$$

(3.4)  $$= \sum_{\epsilon \in \{(+1, \ldots, +1)^t\}} |\det Z||Z^{-T}\alpha_\epsilon| \frac{1}{(n-1)!}\sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_2^{-2})}.$$ 

Since $|\det Z|Z^{-T}$ is an integral matrix, the statement of the theorem follows from Lemma 3.2, as well as the characterization of the equality case. □

We notice that Theorem 1.3 together with the upper bound in (1.1) yields

$$\frac{F(K)}{\text{vol}(K)} \geq \frac{1}{(n-1)!}\sqrt{\lambda_1^2 + \cdots + \lambda_n^2}.$$ 

For $n = 2$ this bound improves the one obtained in [15], namely, that the ratio $F(K)/\text{vol}(K) > \lambda_n$; moreover, it is tight. But when $n \geq 3$ the above bound is worse. We conjecture the right bound of this type to be

$$\frac{F(K)}{\text{vol}(K)} \geq \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}.$$ 

In fact, it would be (asymptotically) sharp, as the example contained in [15] shows: For the parallelootope $P_\mu = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq \mu/2, 1 \leq i \leq n-1, |x_n| \leq 1/2\}$ it is easy to check that $\lambda_i(P_\mu, \mathbb{Z}^n) = 2/\mu, 1 \leq i \leq n-1$, $\lambda_n(P_\mu, \mathbb{Z}^n) = 2$, and $\text{vol}(P_\mu) = \mu^{n-1}$, $F(P_\mu) = 2\mu^{n-2}(n-1 + \mu)$. Hence,

$$\lim_{\mu \to \infty} \frac{F(P_\mu)}{\text{vol}(P_\mu)\sqrt{\sum_{i=1}^n \lambda_i^2}} = 1.$$ 

We conclude this section by discussing a generalization of Theorem 1.3 to arbitrary lattices. For it, we need the concept of minimal determinants of sublattices: For a lattice $\Lambda$ of rank $n$, and for $1 \leq i \leq n$, we define

$$D_i(\Lambda) = \min \{\det \Lambda_i : \Lambda_i \text{ an } i\text{-dimensional sublattice of } \Lambda\},$$

and we write $D(\Lambda) = \min \{D_i(\Lambda)^{1/i} : 1 \leq i \leq n\}$. Moreover,

$$\Lambda^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda\}$$

denotes the polar lattice of $\Lambda$. For more information on minimal determinants we refer to [27] and the references therein.

**Theorem 3.5.** Let $K \in \mathcal{K}_n^o$, $\Lambda$ be a lattice of rank $n$, and let $\lambda_i = \lambda_i(K, \Lambda)$, $1 \leq i \leq n$. Then

$$\frac{F(K)}{D(\Lambda^*) \det \Lambda} \geq \frac{2^n}{(n-1)!}\sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_n^{-2}).}$$

In particular, there exists an absolute constant $c > 0$ such that

$$\frac{F(K)}{D_{n-1}(\Lambda)} \geq \frac{c}{\sqrt{n}} \frac{2^n}{(n-1)!}\sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_n^{-2}).}$$
Proof. Let \( z_i \in \lambda_i K \cap \Lambda, 1 \leq i \leq n \), be \( n \) linearly independent lattice points.

Let \( Z \) be the matrix with columns \( z_1, \ldots, z_n \). Then, inequality (3.4) in the proof of Theorem 1.3 holds without modification.

In order to apply Lemma 3.2 in the general setting, we need to be a bit more careful. Let \( B \in \text{GL}(n, \mathbb{R}) \) be a basis of \( \Lambda \), i.e., \( \Lambda = BZ^n \). Then, there exists an integral matrix \( Y \in \mathbb{Z}^{n \times n} \) such that \( Z = BY \). Writing \( \bar{Z} = |\det Z|Z^{-\top} \), we get \( \bar{Z} = |\det B|B^{-\top}|\det Y|Y^{-\top} \), and \( \bar{Y} = |\det Y|Y^{-\top} \) is an integral matrix. With \( \bar{Y} = (\bar{y}_1, \ldots, \bar{y}_n) \) and using the notation of the proof of Lemma 3.2 we have

\[
\sqrt{m_i(\bar{Z})} = \sqrt{|\det(\bar{Z})^i_1Z_1|} \\
= \sqrt{|\det B|^{2i} \det((B^{-\top} \bar{y}_1, \ldots, B^{-\top} \bar{y}_i)^\top(B^{-\top} \bar{y}_1, \ldots, B^{-\top} \bar{y}_i))} \\
= (\det \Lambda)^i \det(\text{lattice spanned by } \{B^{-\top} \bar{y}_1, \ldots, B^{-\top} \bar{y}_i\}).
\]

Now, since \( B^{-\top} \) is a basis of the polar lattice \( \Lambda^* \), the lattice spanned by \( \{B^{-\top} \bar{y}_1, \ldots, B^{-\top} \bar{y}_i\} \) is an \( i \)-dimensional sublattice of \( \Lambda^* \), and so we get

\[
\sqrt{m_i(\bar{Z})} \geq (\det \Lambda)^i D_i(\Lambda^*).
\]

Since \( m_i(\bar{Z}) \) is homogeneous of degree \( 2i \), we need to multiply \( \bar{Z} \) by

\[
\left( \det \Lambda \min \left\{ D_i(\Lambda^*)^{1/i} : 1 \leq i \leq n \right\} \right)^{-1} = (\det(\Lambda^*) \det \Lambda)^{-1}
\]

in order to get a matrix with \( m_i \geq 1 \), for \( 1 \leq i \leq n \). Hence, by Remark 3.3 we get from (3.1) that

\[
F(K) \geq \frac{2^n D(\Lambda^*) \det \Lambda}{(n-1)!} \sqrt{\sigma_{n-1}(\lambda_1^{-2}, \ldots, \lambda_2^{-2})},
\]

which proves the first claimed inequality.

For the second estimate, let \( \Lambda_i \) be an \( i \)-dimensional sublattice of \( \Lambda^* \) with \( \det \Lambda_i = D_i(\Lambda^*) \). Applying (1.1) to the ball \( B_i \), we obtain

\[
\lambda_1(B_i, \Lambda^*)^i \leq \lambda_1(B_i, \Lambda_i)^i \leq \frac{2^i}{\text{vol}_i(B_i)} D_i(\Lambda^*).
\]

In view of \( \text{vol}_i(B_i) = \pi^{i/2}/\Gamma(i/2 + 1) \) and Stirling’s approximation of the \( \Gamma \)-function, we see that there exists an absolute constant \( c > 0 \) such that

\[
D_i(\Lambda^*)^{1/i} \geq \frac{c}{\sqrt{i}} \lambda_1(B_i, \Lambda^*) = \frac{c}{\sqrt{i}} \min_{\|z\| = \sqrt{i}} \lambda_1(z) = \frac{c}{\sqrt{i}} D_1(\Lambda^*).
\]

Therefore,

\[
D(\Lambda^*) \det \Lambda \geq \frac{c}{\sqrt{n}} D_1(\Lambda^*) \det \Lambda = \frac{c}{\sqrt{n}} D_{n-1}(\Lambda),
\]

where the last equality is a well-known relation between a lattice and its polar lattice (see e.g. [21, Cor. 1.3.5]). Using this inequality and (3.5), we obtain the desired estimate. \( \square \)

Remark 3.6. The investigation of lower bounds for the surface area of a convex body \( K \in \mathcal{K}^n_c \) or \( K \in \mathcal{K}^n \), in terms of the successive minima of \( D K \), leads to the question whether there is an analogous statement to Remark 1.1 for the surface area. We leave this as an open problem for subsequent studies.
4. Bounds for the lattice surface area

First of all, we argue that, in general, the lattice surface area cannot be bounded from below by the successive minima. For a different set of examples, that exclude lower bounds on the quotient $g_{n-1}(P)/\text{vol}(P)$ in terms of the sum of the $\lambda_i(P, \mathbb{Z}^n)$, see [18 Rem. 3.2].

**Example 4.1.** Let $\ell \in \mathbb{N}$ and consider $P_\ell^n = \text{diag}(\ell, 1, \ldots, 1)C^*_n$. Then, the volume of each facet $F$ of $P_\ell^n$ equals (see (3.3))

$$\frac{1}{(n-1)!}\sqrt{\sigma_{n-1}(P^2, 1, \ldots, 1)} = \frac{1}{(n-1)!}\sqrt{1 + (n-1)\ell^2}.$$

Moreover, we have

$$\det(\text{aff}_o(F) \cap \mathbb{Z}^n) = \|(1, \ell, \ldots, \ell)\| = \sqrt{1 + (n-1)\ell^2},$$

and hence $g_{n-1}(P_\ell^n) = 2^{n-1}/(n-1)!$. So, $g_{n-1}(P_\ell^n)$ does not depend on $\ell$, whereas $\lambda_1(P_\ell^n, \mathbb{Z}^n) = 1/\ell$ and $\lambda_2(P_\ell^n, \mathbb{Z}^n) = \cdots = \lambda_n(P_\ell^n, \mathbb{Z}^n) = 1$.

Next, we want to prove Theorem 1.4. It goes along the same lines as in [18, Thm. 1.2], and it is based on a generalized pyramid formula for the volume of a polytope which has been recently obtained in [17].

**Theorem 4.2** (Henk & Linke, [17]). Let $P \in \mathcal{K}_n^\R$ be a polytope with facets $F_j$ corresponding to normal vectors $a_j$, $1 \leq j \leq m$. Furthermore, let $L_k$ be a $k$-dimensional linear subspace. Then

$$\text{vol}(P) \geq \frac{n}{k} \sum_{a_j \in L_k} \text{vol}(\text{conv}\{0, F_j\}).$$

**Proof of Theorem 4.2 i):** First, we observe that the desired inequality is invariant under translations by rational vectors. Moreover, the centroid $\text{cen}(P)$ of the rational polytope $P$ has only rational entries. Indeed, for any triangulation of $P$ by rational simplices $S_1, \ldots, S_t$, we have

$$\text{cen}(P) = \frac{1}{t} \sum_{j=1}^t \frac{\text{cen}(S_j)}{\text{vol}(S_j)}$$

and

$$\text{cen}(S_j) = \frac{1}{n+1} \sum_{v \text{ vertex of } S_j} v \in \mathbb{Q}^n.$$

So, with $\text{vol}(S_j) \in \mathbb{Q}$, for all $j = 1, \ldots, t$, we get $\text{cen}(P) \in \mathbb{Q}^n$. Therefore, after a suitable translation of $P$ we can assume that $\text{cen}(P) = 0$.

Let $P = \{x \in \mathbb{R}^n : (a_j, x) \leq b_j, 1 \leq j \leq m\}$ with $b_j \in \mathbb{Q}_{>0}$ and primitive normal vectors $a_j \in \mathbb{Z}^n$, i.e., there is no lattice point on the interior of the line segment $[0, a_j]$. Writing $F_j$ for the facet of $P$ corresponding to the normal vector $a_j$, we have $\|a_j\| = \det(\text{aff}_o(F_j) \cap \mathbb{Z}^n)$, $1 \leq j \leq m$. Hence,

$$g_{n-1}(P) = \frac{1}{2} \sum_{j=1}^m \frac{\text{vol}_{n-1}(F_j)}{\det(\text{aff}_o(F_j) \cap \mathbb{Z}^n)} = \frac{1}{2} \sum_{j=1}^m \frac{\text{vol}_{n-1}(F_j)}{\|a_j\|}.$$

Writing $\lambda_i = \lambda_i(DP, \mathbb{Z}^n)$, $1 \leq i \leq n$, let $v_1, \ldots, v_n \in DP$ be linearly independent points such that $\lambda_i v_i = z_i \in \mathbb{Z}^n$, for every $i = 1, \ldots, n$, and let $L_k = \text{lin}\{v_1, \ldots, v_k\}$, $1 \leq k \leq n$, with $L_0 = \{0\}$.

Since the centroid of $P$ lies at the origin, it holds $DP \subseteq (n+1)P$ (cf. [7, Sect. 34]). Therefore

$$DP \subseteq \left\{x \in \mathbb{R}^n : \|a_j, x\| \leq (n+1)b_j, 1 \leq j \leq m\right\},$$
Theorem 1.4 ii) cannot be replaced by a constant smaller than \( n \).

For \( k = 0, \ldots, n \), we define \( V_k = \{ j : a_j \in L^+ \} \). Then, \( V_0 = \{ 1, \ldots, m \} \) and \( V_k \subseteq V_{k-1} \) for each \( k = 1, \ldots, n \). Furthermore, let \( q \) be the smallest index such that \( V_q = \emptyset \). Then, the integrality of the \( a_j \)'s and \( \mathbf{z}_i \)'s gives

\[
(n + 1) b_j \geq \frac{1}{\lambda_k} |\langle \mathbf{a}_j, \mathbf{z}_i \rangle| \quad \text{for all } j = 1, \ldots, m.
\]

So, writing \( F_j^q = \text{conv}\{0, F_j\} \) and get the estimate

\[
g_{n-1}(P) = \frac{1}{2} \sum_{k=1}^{q} \sum_{j \in V_{k-1} \setminus V_k} \frac{\text{vol}_n(F_j)}{\|a_j\|} \leq \frac{n(n+1)}{2} \sum_{k=1}^{q} \lambda_k \sum_{j \in V_{k-1} \setminus V_k} \frac{\text{vol}_{n-1}(F_j) b_j}{\|a_j\| n}
\]

\[
= \frac{n(n+1)}{2} \sum_{k=1}^{q} \lambda_k \left( \sum_{j \in V_{k-1}} \text{vol}(F_j^w) - \sum_{j \in V_k} \text{vol}(F_j^w) \right)
\]

\[
= \frac{n(n+1)}{2} \left( \lambda_1 \text{vol}(P) + \sum_{k=1}^{q-1} (\lambda_{k+1} - \lambda_k) \sum_{a_i \in L^+_k} \text{vol}(F_j^w) \right).
\]

In the last equality, we have used that \( \sum_{j \in V_q} \text{vol}(F_j^w) = \text{vol}(P) \) and \( V_q = \emptyset \).

Finally, by Theorem 4.2 and the monotonicity of the successive minima, we obtain

\[
g_{n-1}(P) / \text{vol}(P) \leq \frac{n+1}{2} \left( \lambda_1 + \sum_{k=1}^{q-1} \frac{n-k}{n} (\lambda_{k+1} - \lambda_k) \right)
\]

\[
= \frac{n+1}{2} \left( \sum_{k=1}^{q-1} \lambda_k + (n-q+1) \lambda_q \right) \leq \frac{n+1}{2} \sum_{k=1}^{n} \lambda_k.
\]

ii): We argue as above with only minor adjustments: Since for a polytope \( P \in \mathcal{K}_c^n \) it holds \( P \subseteq -nP \) (see [7] Sect. 34), we may replace (4.2) by \( nb_j \geq |\langle \mathbf{a}_j, \mathbf{z}_i \rangle| / \lambda_i \) and note that, for \( n \geq 2 \), this can never be an equality if \( \langle \mathbf{a}_j, \mathbf{z}_i \rangle \geq 0 \). Since there always exists a \( j \in \{1, \ldots, m\} \), with \( \langle \mathbf{a}_j, \mathbf{z}_i \rangle > 0 \), and hence \( j \in V_0 \setminus V_1 \), we see that there is at least one strict inequality in the argument. This implies that \( g_{n-1}(P) / \text{vol}(P) < (n/2) \sum_{i=1}^{n} \lambda_i(P, \mathbb{Z}^n) \), as desired. \( \square \)

It turns out that there are almost tight examples for Theorem 1.4 ii).

**Proposition 4.3.** There are lattice polytopes showing that the factor \( n/2 \) in Theorem 1.4 ii) cannot be replaced by a constant smaller than \( n^2 / (2(n+1)) \).

**Proof.** Let \( P \) be a reflexive \( (n−1) \)-polytope, i.e., \( P \in \mathcal{K}_c^{n−1} \) is a lattice polytope containing the origin in its interior such that its polar \( P^* = \{ x \in \mathbb{R}^{n−1} : \langle x, y \rangle \leq 1, y \in P \} \) is a lattice polytope as well. Reflexive polytopes were introduced in [1] (see [6] for more information and references). We
assume further that the centroid of \( P \) lies at the origin, and for \( \ell \in \mathbb{N} \), we consider the pyramid \( P_\ell = \text{conv}\{(n+1)\ell P \times \{-1\},ne_n\} \). By construction, the centroid of \( P_\ell \) is the origin as well, and the intersection of \( P_\ell \) with the hyperplane \( \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0 \} \) is the polytope \( n\ell P \). Since \( P \) has exactly one interior lattice point, we get

\[
\lambda_1(P_\ell, \mathbb{Z}^n) = \cdots = \lambda_{n-1}(P_\ell, \mathbb{Z}^n) = \frac{1}{n\ell} \quad \text{and} \quad \lambda_n(P_\ell, \mathbb{Z}^n) = \frac{1}{n},
\]

and therefore,

\[\sum_{i=1}^{n} \lambda_i(P_\ell, \mathbb{Z}^n) = \frac{\ell + n - 1}{n\ell}. \tag{4.3}\]

In order to compute the quotient \( g_{n-1}(P_\ell)/\text{vol}(P_\ell) \), we consider the pyramid \( Q' = \text{conv}\{Q \times \{0\}, e_n\} \) over an \((n-1)\)-dimensional lattice polytope \( Q \). As mentioned in the introduction, the lattice surface area of \( Q' \) is the coefficient of order \( n-1 \) of the Ehrhart polynomial \( #(kQ' \cap \mathbb{Z}^n) = \sum_{i=0}^{\infty} g_i(Q')k^i \), \( k \in \mathbb{N} \), of \( Q' \). Therein, the coefficient \( g_i(Q') \) is a homogeneous functional of degree \( i \), and \( g_n(Q') = \text{vol}(Q') \). Since \( Q' \) is a pyramid, we obtain (see e.g. [2, Sect. 2.4])

\[\#(kQ' \cap \mathbb{Z}^n) = \#(kQ \cap \mathbb{Z}^{n-1}) + \sum_{j=0}^{k-1} \#(jQ \cap \mathbb{Z}^{n-1}) = \sum_{j=0}^{n-1} g_j(Q)k^j + \sum_{j=1}^{n} \left( \sum_{i=j-1}^{n-1} g_i(Q) \binom{i+1}{j} \frac{b_{i-j+1}}{i+1} \right) k^j,\]

where \( b_j \) are the Bernoulli numbers. In particular, since \( b_0 = 1, b_1 = -1/2 \),

\[g_{n-1}(Q') = \frac{1}{n-1} g_n(Q) + \frac{1}{2} g_{n-1}(Q) = \frac{1}{n-1} g_{n-2}(Q) + \frac{1}{2} \text{vol}_{n-1}(Q).\]

Since the polytope \( P \) is reflexive, we have \( g_{n-2}(P) = ((n-1)/2) \text{vol}_{n-1}(P) \) (see [6, Lem. 3.1]) and thus using the homogeneity of the Ehrhart coefficients, we get for \( Q = \ell P \),

\[g_{n-1}(Q') = \frac{\ell^{n-2}}{n-1} g_{n-2}(P) + \frac{\ell^{n-1}}{2} \text{vol}_{n-1}(P) = \frac{\ell + 1}{2} \ell^{n-2} \text{vol}_{n-1}(P).\]

Moreover, by \( P_\ell = (n+1)Q' - e_n \), we obtain

\[\frac{g_{n-1}(P_\ell)}{\text{vol}(P_\ell)} = \frac{1}{n+1} \frac{\ell^{n-1} \text{vol}_{n-1}(P)}{\ell^{n-2} \text{vol}_{n-1}(P)} = \frac{n}{n+1} \frac{\ell + 1}{\ell}. \]

Hence, an inequality of the type \( g_{n-1}(P_\ell)/\text{vol}(P_\ell) \leq c_{n,\ell} \sum_{i=1}^{n} \lambda_i(P_\ell, \mathbb{Z}^n) \) implies, by (4.3), that

\[c_{n,\ell} \geq \frac{n^2}{2(n+1)} \frac{\ell + 1}{\ell + n - 1},\]

which goes to \( n^2/(2(n+1)) \) when \( \ell \to \infty \). \( \square \)

As mentioned in the introduction, Theorem 1.4 provides relations between the successive minima of a lattice polytope and the roots of its Ehrhart polynomial (for more information see [2, 18, 25] and the references inside).
To this end, for a lattice polytope $P \in \mathcal{K}^n$, we regard the right hand side of (1.4) as a polynomial in a complex variable $s \in \mathbb{C}$, and write
\[
\sum_{i=0}^{n} g_i(P)s^i = \prod_{i=1}^{n} \left( 1 + \frac{s}{\gamma_i(P)} \right).
\]
Hence, $-\gamma_1(P), \ldots, -\gamma_n(P)$ are the roots of the Ehrhart polynomial of $P$ and we have $g_i(P) = \sigma_i(1/\gamma_1(P), \ldots, 1/\gamma_n(P))$, $1 \leq i \leq n$. This implies that $g_{n-1}(P)/\text{vol}(P) = \sum_{i=1}^{n} \gamma_i(P)$, and therefore Theorem 1.4 can be reformulated as follows (for convenience we include the result for $\sigma$-symmetric lattice polytopes that already appeared in [18]).

**Corollary 4.4.** Let $P \in \mathcal{K}^n$ be a lattice polytope.

i) Then
\[
\sum_{i=1}^{n} \gamma_i(P) \leq \frac{n+1}{2} \sum_{i=1}^{n} \lambda_i(DP, Z^n),
\]
and the standard simplex $S_n$ shows that the inequality is best possible.

ii) If $P \in \mathcal{K}_c^n$ and $n \geq 2$, then
\[
\sum_{i=1}^{n} \gamma_i(P) < \frac{n}{2} \sum_{i=1}^{n} \lambda_i(P, Z^n),
\]
and Proposition 4.3 shows that the factor $n/2$ is of the right order.

iii) If $P \in \mathcal{K}_o^n$, then
\[
\sum_{i=1}^{n} \gamma_i(P) \leq \frac{1}{2} \sum_{i=1}^{n} \lambda_i(P, Z^n),
\]
and equality is attained, for example, by $C_n$ and $C_\ast^n$.

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