ON CONMUTATIVE LEFT-NILALGEBRAS OF INDEX 4*

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Abstract

We first present a solution to a conjecture of [2] in the positive. We prove that if \( A \) is a commutative nonassociative algebra over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( x(x(xx)) = 0 \), then \( L_{a^t_1}L_{a^t_2} \cdots L_{a^t_s} \equiv 0 \) if \( t_1 + t_2 + \cdots + t_s \geq 10 \), where \( a \in A \).

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1 Introduction

Throughout this paper the term algebra is understood to be a commutative not necessarily associative algebra. We will use the notations and terminology of [6]. Let \( \mathfrak{A} \) be an (commutative nonassociative) algebra over a field \( F \). We define inductively the following powers, \( \mathfrak{A}^1 = \mathfrak{A} \) and \( \mathfrak{A}^s = \sum_{i+j=s} \mathfrak{A}^i \mathfrak{A}^j \) for all positive integers \( s \geq 2 \). We shall say that \( \mathfrak{A} \) is nilpotent in case there is an integer \( s \) such that \( \mathfrak{A}^s = (0) \). The algebra \( \mathfrak{A} \) is called nilalgebra in case the subalgebra \( \text{alg}(a) \) of \( \mathfrak{A} \) generated by \( a \) is nilpotent, for all \( a \in \mathfrak{A} \). Therefore \( \mathfrak{A} \) is nilalgebra if and only if for every \( a \in \mathfrak{A} \) there exists an integer \( t = k(a) \) such that every product of at least \( t \) factors each of them equal to \( a \), in whatever association, vanishes. The (principal) powers of an element \( a \) in \( \mathfrak{A} \) are defined recursively by \( a^1 = a \) and \( a^{i+1} = a a^i \) for all integers \( i \geq 1 \). The algebra \( \mathfrak{A} \) is called right-nilalgebra if for every \( a \) in \( \mathfrak{A} \) there exists an integer \( k = k(a) \) such that \( a^k = 0 \). The smallest positive integer \( k \) which this property is the index. Obviously, every nilalgebra is right-nilalgebra. For any element \( a \) in \( \mathfrak{A} \), the linear mapping \( L_a \) of \( \mathfrak{A} \) defined by \( x \to ax \) is called multiplication operator of \( \mathfrak{A} \). An Engel algebra is an algebra in which every multiplication operator is nilpotent in the sense that for every \( a \in \mathfrak{A} \) there exists a positive integer \( j \) such that \( L_a^j = 0 \).

An important question is that of the existence of simple nilalgebras in the class of finite-dimensional algebras. In [6] we proved that every nilalgebra \( \mathfrak{A} \) of dimension \( \leq 6 \) over a field of characteristic \( \neq 2,3,5 \) is solvable and hence \( \mathfrak{A}^2 \not\subseteq \mathfrak{A} \). For power-associative nilalgebras of dimension \( \leq 8 \) over a field of characteristic \( \neq 2,3,5 \), we have shown in [7] that they are solvable, and hence there is no simple algebra in this subclass. See also [4] and [6] for power-associative nilalgebras of dimension \( \leq 7 \).

We show now the process of linearization of identities, which is an important tool in the theory of varieties of algebras. See [8], [9] and [10] for more information. Let \( P \) be the free commutative nonassociative polynomial ring in two generators \( x \) and \( y \) over a field \( F \). For every \( \alpha_1, \ldots, \alpha_r \in P \), the operator linearization \( \delta[\alpha_1, \ldots, \alpha_r] \) can be defined as follows: if \( p(x,y) \) is a monomial in \( P \), then \( \delta[\alpha_1, \ldots, \alpha_r] p(x,y) \) is obtained by making all the possible replacements of \( r \) of the \( k \) identical arguments \( x \) by \( \alpha_1, \ldots, \alpha_r \), and summing the resulting terms if \( x-\text{degree of } p(x,y) \) is \( \geq r \), and is equal to zero in other cases. Some examples of this operator are

\[
\begin{align*}
\delta[y](x^2(xy)) &= 2(xy)^2 + x^2 y^2 \\
\delta[x^2, y](x^2) &= 2x^2 y, \\
\delta[y, x^2y, x](x^2) &= 0.
\end{align*}
\]

For simplicity, \( \delta[\alpha : r] \) will denote \( \delta[\alpha_1, \ldots, \alpha_r] \), where \( \alpha_1 = \cdots = \alpha_r = \alpha \). We observe that
if \( p(x) \) is a polynomial in \( P \), then \( p(x + y) = p(x) + \sum_{j=1}^{\infty} \delta[y : j]p(x) \), where \( \delta[y : j]p(x) \) is the sum of all the terms of \( p(x + y) \) which have degree \( j \) with respect to \( y \).

**Lemma 1** ([10]). Let \( p(x, y) \) be a commutative nonassociative polynomial of \( x \)-degree \( \leq n \). If \( F \) is a field of characteristic either zero or \( \geq n \), and the \( F \)-algebra \( \mathfrak{A} \) satisfies the identity \( p(x, y) \), then \( \mathfrak{A} \) satisfies all linearizations of \( p(x, y) \).

## 2 Right-nilalgebras of index 4

Throughout this section \( F \) is a field of characteristic different from 2 or 3 and all the algebras are over \( F \). We will study right-nilalgebras of index \( \leq 4 \), that is the variety \( \mathcal{V} \) of algebras over the field \( F \) satisfying the identity

\[
x^4 = 0.
\]  

(1)

Let \( \mathfrak{A} \) be an algebra in \( \mathcal{V} \). For simplicity, we will denote by \( L \) and \( U \) the multiplication operators, \( L_x \) and \( L_{x^2} \) respectively, where \( x \) is an element in \( \mathfrak{A} \). The following known result is a basic tool in our investigation. See [2] and [3].

**Lemma 2.** Let \( \mathfrak{A} \) be a commutative right-nilalgebra of index 4. Then \( \mathfrak{A} \) satisfies the identities

\[
x^3 x^3 = -x(x^2 x^2), \quad x^3 x^3 = (x^2)^3 = x(x(x^2 x^2)),
\]  

(2)

and \( p(x) = 0 \), for every monomial \( p(x) \) with \( x \)-degree \( \geq 7 \). Furthermore, we have

\[
L_{x^3} = -LU - 2L^3,
\]  

(3)

\[
L_{x^2 x^2} = -U^2 - 2UL^2 - 2ULL + 4L^4,
\]  

(4)

\[
L_{x(x^2 x^2)} = -LU^2 - 2LU^2 L - 2UL^2 L - 4L^3 U - 12L^5,
\]  

(5)

\[
L_{2(x^2 x^2)} = 2L^2 U^2 + 4L^2 U L^2 + 4L^4 U + 8L^6,
\]  

(6)

and also

| Table i: Multiplication identities of degree 5 |
|---------------------------------------------|
| \( U L U \) \( U L^2 \) \( U L^3 \) \( L U L \) \( L^2 U L \) \( L^3 U \) \( L^5 \) |
| \( U^2 L \) \( 0 \) \( -1 \) \( 2 \) \( 0 \) \( 0 \) \( -2 \) \( -8 \) |

and two identities of \( x \)-degree 6 which may be written as

| Table ii: Multiplication identities of degree 6 |
|-----------------------------------------------|
| \( U L^2 U \) \((L U)^2\) \( L^2 U^2 \) \( U L^4 \) \( L U L^3 \) \( L^2 U L^2 \) \( L^3 U L \) \( L^4 U \) \( L^6 \) |
| \( U^3 \) \( -2 \) \(-2 \) \( 2 \) \(-8 \) \(-8 \) \( 0 \) \(-4 \) \( 8 \) \( 40 \) |
| \((UL)^2 \) \( -1 \) \(-1 \) \( 1 \) \(-4 \) \(-2 \) \( 2 \) \( 0 \) \( 4 \) \( 24 \) |
We note that, for example, Table i means that $U^2L = -LU^2 + 2UL^3 - 2L^3U - 8L^5$. From the identities (3-6) we get that for any $a \in \mathfrak{A}$ the associative algebra $\mathfrak{A}_a$ generated by all $L_c$ with $c \in \text{alg}(a)$ is in fact generated by $L_a$ and $L_a^2$. Furthermore, every algebra in $\mathcal{V}$ is a nilalgebra of index $\leq 7$.

We now pass to study homogeneous identities in $\mathfrak{A}$ with $x$-degree $\geq 7$ and $y$-degree 1. From the relation $0 = \delta[y, x, x^3, x^3](x^4) = 2y(x(x^3x^3)) + 4y(x^3(x^3x^3)) + 2y(x(x^3x^3)) + 4x^3(y(x^3x^3)) + 4x^3(x(y^3x^3)) + 4x^3(x(x^3x^3)) - 4x^3(x(x^3x^3)) + 4x^3(x(x^3x^3)) = 2x((x^3)^2y) + 4x(x^3(x^3x^3)) + 4x^3(x(x^3x^3)) + 4x^3(x(x^3x^3)) = 2[LL_{x^3x^3} + 2LL_{x^3Lx^3} + 2L_{x^3}L_{x^3}L](y)$ we have

$$L^3U^2 = -2L^3UL^2 - L^4UL - 5L^5U - 20L^7,$$

since we can use the reductions (3-6) and replace the occurrences of $(UL)^2$. Multiplying the identity of Table i by $U$ from the left, replacing first the occurrences of $U^3$ and next using reductions from Table i, Table ii and above identity we get a new identity as follows:

$$0 = U^3L + ULU^2 - 2U^2L^3 + 2UL^3U + 8UL^5 = [-2UL^2UL - 2(LU)^2L + 2L^2U^2L - 8UL^5
- 8UL^4 - 4L^3UL^2 + 8L^4UL + 40L^7] + ULU^2 - 2U^2L^3 + 2UL^3U + 8UL^5
= -2UL^2UL + [2UL^2UL + 2(LU)^2] + [4L^3UL^2 + 2L^4UL + 10L^5U + 40L^7]
+ 8UL^4 + 4L^2UL^3 - 4L^3UL^2 - 8L^5U + 48L^7] + 2L^2U^2L - 8UL^5 - 8UL^4
- 4L^3UL^2 + 8L^4UL + 40L^7 + ULU^2 + [[-2L^2UL + 4UL^4 - 4L^4UL - 16L^7]
- 4L^5 + 4L^3UL^2 + 16L^7] + 2UL^3U + 8UL^5,$$

that is,

$$ULU^2 = 2\left(U^2UL - UL^3U - UL^2U - L^2ULU +
2UL^5 - 2UL^4 - 2L^2UL^3 - 3L^4UL - L^5U - 16L^7\right).$$

Next, we can reduce the relation $0 = \delta[y, x^2, x^2x^2](x^4)$ using the above identities. This yields

$$UL^5 = -LUL^4 + \frac{1}{2}L^2UL^3 + \frac{3}{4}L^4UL + \frac{3}{4}L^5U + 8L^7.$$

Now combining (8) and (9) we obtain $ULU^2 = 2UL^2UL - 2UL^3U - 2LUL^2U - 2L^2ULU - 8UL^4 - 2L^2UL^3 - 3L^4UL + L^5U$. Thus, we have three identities of $x$-degree 7 and $y$-degree 1 which may be written as multiplication identities:

| Table iii: Multiplication identities of degree 7 |
|------------------------------------------------|
| $UL^2UL$ | $UL^3U$ | $UL^2U$ | $L(U)^2$ | $LUL^4$ | $L^2UL^3$ | $L^3UL^2$ | $L^4UL$ | $L^5U$ | $L^7$ |
| $L^3U^2$ | 0 | 0 | 0 | 0 | 0 | 0 | -2 | -1 | -5 | -20 |
| $UL^5$ | 0 | 0 | 0 | 0 | -1 | 1/2 | 0 | 3/4 | 3/4 | 8 |
| $ULU^2$ | 2 | -2 | -2 | -2 | -8 | -2 | 0 | -3 | 1 | 0 |
In an analogous way, using successively the identities

\[ 0 = \delta[y, x, x(x^2x^2)]x^4, \quad 0 = \delta[y, x^2, x^2, x^2x^2]x^4, \quad 0 = \delta[y, x, x^2, x^2x^2]x^4, \quad 0 = \delta[y, x, x^2, x^2x^2]x^4, \quad 0 = \delta[y, x^3, x^2x^2]x^4, \]

multiplying the second identity of Table ii with the operator \( U \) from the left and replacing the occurrences of \( UUL \), and finally using \( 0 = \delta[y, x, x^2, x^2]x^4, \) we obtain the following 5 multiplication identities:

| Table iv: Multiplication identities of degree 8 |
|-----------------------------------------------|
| \( L^3ULU \) | \( UL^2U \) | \( UL^2UL \) | \( LUL^2 \) | \( L^2UL^2 \) | \( L^3UL \) | \( L^4U \) | \( L^5U \) |
| 0 | 0 | 0 | 0 | 0 | \(-1/2\) | \(-2\) | \(-11/2\) | \(-20\) |
| \(-4\) | \(-2\) | \(-2\) | 0 | 2 | \(-5/2\) | 13 | 31/2 | 32 |
| 0 | 1 | 0 | \(-1\) | \(-12\) | \(-11/4\) | \(-7/2\) | 25/4 | 36 |
| \(-1\) | \(-1\) | \(-1\) | 0 | \(-4\) | \(-11/2\) | \(-3\) | 9/2 | 0 |
| 0 | 0 | 0 | 0 | 0 | \(-3/4\) | \(-3/2\) | \(-3/4\) | \(-8\) |

Now, relations \( 0 = \delta[y, x, x^3, x^2x^2]x^4, \quad 0 = \delta[y, x, x^2, x(x^2x^2)]x^4, \quad 0 = \delta[y, x^2, x^2, x^2x^2]x^4, \quad 0 = \delta[y, x^3, x^2x^2]x^4, \) and multiplying the relation determined by the last row of Table iii with the operator \( U \) from the left and first replacing the occurrences of \( UUL \), imply the following 6 multiplication identities:

| Table v: Multiplication identities of degree 9 |
|-----------------------------------------------|
| \( L^6UL \) | \( LUL^2 \) | \( (L^2U)^2L \) | \( L_2UL^3 \) | \( L^9 \) |
| 0 | 0 | 0 | 0 | 0 |
| \(-7\) | \(-7\) | \(-217\) | \(-4510/3\) |
| 1 | 0 | \(-587/2\) | \(-6155/3\) |
| 0 | 0 | \(29/3\) | 422/9 |
| 0 | 0 | \(1318/3\) | 27988/9 |
| 0 | 0 | \(-23\) | \(-496/3\) |

The author used a MAPLE language program to discover these identities. We now present a solution of a Conjecture of [2] in the positive. We see that for every \( a \in \mathfrak{A} \), the associative algebra \( \mathfrak{A}_a \), generated by the multiplication operators \( L_a \) and \( L_{a^2} \), is nilpotent of index \( \leq 10 \).

**Theorem 1.** Let \( \mathfrak{A} \) be an algebra over a field \( F \) of characteristic \( \neq 2, 3 \), satisfying \( x^4 = 0 \). Then every monomial in \( P \) of \( x \)-degree \( \geq 10 \) and \( y \)-degree 1 is an identity in \( \mathfrak{A} \). In particular, \( L_a^{10} = 0 \) for all \( a \in \mathfrak{A} \).

**Proof.** First we shall prove that every monomial of \( x \)-degree 10 and \( y \)-degree 1 is an identity in \( \mathfrak{A} \). Multiplying the operators in the first line of Table v with \( L \) from the left...
and from the right, and the operators in the first line of Table iv with \( U \) from the left
and from the right and next using reductions from Tables i-v we see that we only need
to prove that \( L^2UL^4U = 0, L^8U = 0 \) and \( L^{10} = 0 \) are multiplication identities in \( \mathfrak{A} \). Now,
for any \( x \) in \( \mathfrak{A} \) we have

\[
L^7UL = L(L^6UL) = -7L^5U - 48L^{10},
\]

\[
L^6UL^2 = (L^6UL)L = -7L^7UL - 48L^{10} = 49L^8U + 288L^{10},
\]

\[
L^6UL^2 = L(L^5UL^2) = -23L^8U - 496/3L^{10}.
\]

Therefore

\[
27L^8U + 170L^{10} = 0. \tag{10}
\]

Now,

\[
L^5UL^3 = (L^5UL^2)L = -23L^7UL - 496/3L^{10} = 161L^8U + 2816/3L^{10},
\]

\[
L^5UL^3 = L^2(3UL^3) = -3/4L^6UL^2 - 3/2L^7UL - 3/4L^8U - 8L^{10}
\]

\[
= -27L^8U - 152L^{10},
\]

and hence

\[
141L^8U + 818L^{10} = 0. \tag{11}
\]

Next

\[
L^3UL^3U = L(L^2UL^3U) = 29/3L^8U + 422/9L^{10},
\]

\[
L^3UL^3U = (L^3UL^3)U = -3/4L^4UL^2U - 3/2L^5ULU - 3/4L^6UU - 8L^{10}
\]

\[
= -3/4L(L^3UL^2U) - 3/2L^2(L^3ULU) - 3/4L^3(L^3U^2) - 8L^{10}
\]

\[
= 9/4L^6UL^2 + 15/4L^7UL + 667/4L^8U + 2345/2L^{10}
\]

\[
= 1003L^8U + 3281/2L^{10},
\]

so that

\[
17880L^8U + 28685L^{10} = 0. \tag{12}
\]

Combining (10-12) we obtain that \( L^8U = 0 \) and \( L^{10} = 0 \). Now, we have by Table v
that \( 0 = (L^2UL^3U)L = L^2(UL^3UL) = -L^2UL^4U - L^3UL^2UL - L^3UL^3U - 4L^4UL^4 - \\
11/2L^6UL^2 - 3L^7UL + 9/2L^8U = -L^2UL^4U - (L^3UL^2U)L - 4L(L^3UL^3)L = -L^2UL^4U. \)

Therefore, we have \( L^2UL^4U = 0. \)

In an analogous way, we can see that every monomial of \( x \)-degree 11 and \( y \)-degree 1
is an identity in \( \mathfrak{A} \). This proves the theorem.

Now we shall investigate two subvarieties of \( \mathcal{V} \). We start in Subsection 2.1 with
the class of all nilalgebras in \( \mathcal{V} \) of index \( \leq 5 \) and next in Subsection 2.2 we study the
multiplication identities of the variety of all the nilalgebras in \( \mathcal{V} \) of index \( \leq 6 \). 
\[\Box\]
2.1 The identity $x((xx)(xx))=0$

We will now consider the class of all algebras in $\mathcal{V}$ satisfying the identity $x(x^2x^2) = 0$. First, linearization $\delta [y] \{x(x^2)^2\}$ implies

$$L_{x^2x^2} = -4LUL,$$

and identity $\delta [y] \{x^2x^3\} = 0$ forces

$$UU = -2ULL + 2LUL + 4L^4. \quad (13)$$

Next, using above identity and $\delta [y, x^2] \{x^2\} = 0$ we get that $0 = 4UUL + 4LUU + 8LL_{x^3}L = 4(UUL + LUU - 2LLUL - 4L^5) = 8(-UL^3 + LULL + 2L^5 - LULL + LLUL + 2L^5 - LLUL - 2L^5) = 8(-UL^3 + 2L^5)$. Hence $UL^3 = 2L^5$. Now identity $L_{x^2x^2} = 0$ and relations (5) and (14) imply $L^2UL = -L^3U - 4L^5$. Thus, we have the following multiplication identities

| $UUL$ | $LU^2$ | $L^3U$ | $L^5$ |
|-------|--------|--------|--------|
| $UUL$ | 0      | 2      | 0      | 0      |
| $LUU$ | 0      | -2     | -2     | -4     |
| $L^2UL$ | 0    | 0    | -1    | -4     |
| $UL^3$ | 0    | 0    | 0     | 2      |

From Table ii, we can prove that

$$(UL)^2 = -UL^2U - (LU)^2 + 2L^3UL + 4L^4U + 16L^6,$$

and $\delta[x^2]\{x^2(x(x(x)))\} - 2x(x(x(x(x)))) = 0$ forces

$$(UL)^2 + UL^2U + 2L^3UL + 4L^6 = 0. \quad (16)$$

Combining (15) and (16), we have $(LU)^2 = 4L^6$ and $(UL)^2 = -UL^2U + 2L^4U + 4L^6$. Now, we can check easily the following multiplication identities

| $ULL$ | $L^4$ | $L^6$ |
|-------|-------|-------|
| $UUU$ | -2    | 4     | 8     |
| $ULL$ | 0     | 0     | 4     |
| $ULUL$ | -1   | 2     | 4     |
| $LUUL$ | 0    | 2     | 0     |
| $LULU$ | 0    | 0     | 4     |

| $ULLU$ | $L^4$ | $L^6$ |
|-------|-------|-------|
| $LLUU$ | 0     | -4   | -4   |
| $UL^4$ | 0     | 0    | 2    |
| $LUL^3$ | 0    | 0    | 2    |
| $L^2UL^2$ | 0   | 1    | 0    |
| $L^3UL$ | 0    | -1   | -4   |
Theorem 2. Let $\mathfrak{A}$ be an algebra over a field $F$ of characteristic $\neq 2$ or 3, satisfying the identities $x^4 = 0$ and $x(x^2x^2) = 0$. Then every monomial in $P$ of $x$-degree $\geq 7$ and $y$-degree 1 is an identity in $\mathfrak{A}$. In particular, $L_7^7 = 0$ for all $a \in \mathfrak{A}$. Furthermore, the algebra generated by $L_x$ and $L_{x^2}$ is spanned, as vector space, by $L, U, L^2, UL, LU, L^3, UL^2, LUL, L^2U, L^4, ULU, LUL^2, L^3U, L^5, ULI^2U, L^4U, L^6$.

Proof. We shall prove that every monomial of $x$-degree $\geq 7$ and $y$-degree 1 is an identity in $\mathfrak{A}$. Multiplying the operators in the first line of Table vii with $L$ and $U$ from the left and from the right, and next using reductions from Tables i-vii we see that we only need to prove that $LUL^2U = 0, L^5U = 0$ and $L^7 = 0$ are multiplication identities in $\mathfrak{A}$. Now, we have $0 = \delta[y, x^2x^2]\{x(x^2)^2\} = 4L_{x^2x^2}UL + 4LU_{x^2x^2} = -16LULUL - 16LULUL = -32LULUL = -32(LU)^2 L = -2^7L^7$, so that $L^7 = 0$. Also $0 = LULUL = L(UL)^2 = -LUL^2U + 2L^5U$. Therefore, $LUL^2U = 2L^5U$. Finally, from Table vi we have that $0 = (L^2UL + L^3U + 4L^5)L^2 = L^2UL^3 + L^3UL^2 = L^3UL^2 = L(L^2UL^2) = L^5U$. This proves the theorem. \[\square\]

2.2 The identity $x(x((xx)(xx)))) = 0$

In this subsection we consider the class of all algebras in $\mathcal{V}$ satisfying the identity $x(x(x^2x^2)) = 0$. Because we use linearization process of identities and $x(x(x^2x^2))$ has degree 6, we need consider the field $F$ of characteristic not 5 (2 or 3.)

From linearization $\delta[y]\{x(x^2x^2)\}$, we get the multiplication identity $L_{x(x^2x^2)} + LL_{x^2x^2} + 4L^2UL = 0$ and now Lemma 2 forces

$$LUU = -2LUL^2 - 2L^3U - 4L^5.$$  \hfill(17)

The relation $0 = \delta[y, x^2]\{x(x^2x^2)\} = ULL_{x^2x^2} + 4LL^2x^3L + 4ULUL + 4LUUL + 8L^2L_{x^2}L + 4L^2UU$ implies

$$LUL^3 = -\frac{1}{2}(L^2UL^2 + L^3UL),$$ \hfill(18)

since we can use identities from Tables i-v. Next, by $0 = \delta[y, x^2]\{x(x^2x^2)\}$ and $0 = \delta[y, x^2, x^2]\{x(x^2x^2)\}$ we get

$L^4UL = -3L^5U - 16L^7,$ \hfill(19)

$L^2ULU = -L^3UL^2 + 5L^5U + 28L^7,$ \hfill(20)
and identities $0 = \delta[y, x^2, x^2]\{x(x^2x^2)\}$ and $0 = \delta[y, x^2, x^3]\{x(x^2x^2)\}$ imply

\begin{align*}
UL^4U &= -\frac{1}{2}L^2UL^2U + 24L^6U + 62L^8, \quad (21) \\
L^2UL^2U &= 48L^6U + 156L^8. \quad (22)
\end{align*}

Now, identity $0 = \delta[y, x^2x^2]\{x^3x^3\}$ forces

$$L^6U = -2L^8. \quad (23)$$

**Theorem 3.** Let $\mathfrak{A}$ be a commutative algebra over a field $F$ of characteristic not $2, 3$ or $5$, satisfying the identities $x^4 = 0$ and $x(x(x^2x^2)) = 0$. Then every monomial in $P$ of $x$-degree $\geq 9$ and $y$-degree $1$ is an identity in $\mathfrak{A}$. In particular, $L^a = 0$ for all $a \in \mathfrak{A}$.

**Proof.** By Tables i-v, we only need to prove that $LUL^4U = 0$, $L^2UL^2UL = 0$, $L^7U = 0$ and $L^9 = 0$ are multiplication identities in $\mathfrak{A}$. From (19-23) may be deduced immediately $L^7U = -2L^9$ and $2L^9 = 2L^8L = -L^6UL = -L^2(L^4UL) = 3L^7U + 16L^9 = -6L^9 + 16L^9 = 10L^9$. Therefore $L^9 = 0$ and $L^7U = 0$ are identities in $\mathfrak{A}$. Now $L^2UL^2UL = (L^2UL^2U)L = 48L^6UL + 156L^9 = 0$ and $LUL^4U = L(UL^4U) = -(1/2)L^3UL^2U + 24L^7U + 62L^9 = -(1/2)L(L^2UL^2U) = -24L^7U - 78L^9 = 0$. This proves the theorem. \qed

**References**

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