Derivation of the exact NSVZ $\beta$-function in $N = 1$ SQED regularized by higher derivatives by summation of Feynman diagrams

To cite this article: K V Stepanyantz 2012 J. Phys.: Conf. Ser. 343 012115

Related content
- Multiloop calculations in supersymmetric theories with the higher covariant derivative regularization
  K V Stepanyantz
- The beta function of $N = 1$ SYM in differential renormalization
  Javier Mas, Cesar Seijas and Manuel Pérez-Victoria
- Non-trivial Backgrounds in (non-perturbative) Yang-Mills Theory by the Slavnov-Taylor Identity
  A Quadri

Recent citations
- One-loop divergences in non-Abelian supersymmetric theories regularized by BRST-invariant version of the higher derivative regularization
  S. S. Aleshin et al
- The NSVZ $\beta$-function and the Schwinger-Dyson equations for $N = 1$ SQED with $N_f$ flavors, regularized by higher derivatives
  K. V. Stepanyantz
- The higher derivative regularization and quantum corrections in supersymmetric theories
  I.L. Buchbinder and K.V. Stepanyantz

View the article online for updates and enhancements.
Derivation of the exact NSVZ $\beta$-function in $N = 1$ SQED regularized by higher derivatives by summation of Feynman diagrams

K V Stepanyantz

1 Department of Theoretical Physics, Physical Faculty, Moscow State University, Moscow, Russia

Abstract. We describe some calculations, which demonstrate that the $\beta$-function in $N = 1$ supersymmetric theories, regularized by higher covariant derivatives, is given by integrals of double total derivatives. This allows to calculate one of the loop integrals analytically and obtain the NSVZ $\beta$-function in the considered approximation. For the $N = 1$ supersymmetric electrodynamics the factorization of integrals into integrals of double total derivatives can be proved in all orders using a special technique. As a result, the exact NSVZ $\beta$-function is obtained by explicit summation of Feynman diagrams without a redefinition of the coupling constant.

1. Introduction

Quantum corrections in supersymmetric theories were studied for a long time. For example, the $\beta$-function for the general renomalizable $N = 1$ supersymmetric Yang–Mills theory with matter superfields was calculated in the one- [1], two- [2], three- [3, 4], and four-loop [5] approximations. The results obtained in these papers agree with the exact $\beta$-function, proposed by Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) [6, 7, 8, 9] after a special redefinition of the coupling constant [4, 10, 11]. The exact NSVZ $\beta$-function relates the $\beta$-function in a certain loop with the anomalous dimensions of the matter superfields in the previous loops.

All calculations mentioned above were made with the dimensional reduction [12] in $\overline{MS}$-scheme [13]. (We also mention some other calculations made with the dimensional reduction [14, 15, 16].) It should be noted that the dimensional regularization [17] explicitly breaks the supersymmetry and is not convenient for the calculations of quantum corrections in supersymmetric theories. The dimensional reduction is a special modification of the dimensional regularization, which does not explicitly break the supersymmetry. However, it is well known that the dimensional reduction is inconsistent [18]. Attempts to remove the inconsistencies [19, 20] lead to the loss of the explicit supersymmetry [21]. As a consequence, the supersymmetry can be broken by loop corrections [22]. A three-loop calculation made in [22] (with a correction made in [23]) shows that the supersymmetry is really broken in the three-loop approximation for $N = 2$ supersymmetric Yang–Mills theory. (Although for the $N = 4$ SYM theory the dimensional reduction does not break the supersymmetry even in the four-loop approximation [24].) Therefore, the regularization of supersymmetric theories is a nontrivial problem [25].

In supersymmetric theories one can also use the higher covariant derivative regularization. It was proposed in [26, 27] and generalized to the supersymmetric case in Refs. [28, 29].
This regularization is explicitly supersymmetric and does not suffer from inconsistencies. However, due to the complicated structure of loop integrals obtained with the higher derivative regularization it was not often applied for explicit calculations of quantum corrections. In particular, the first calculation of the one-loop $\beta$-function for the (non-supersymmetric) Yang–Mills theory was made only in 1995 [30]. Taking into account the corrections made in subsequent papers [31, 32], the result coincides with the well-known one [33, 34]. (In the one-loop approximation the result obtained with the HD regularization always agrees with the one obtained with the dimensional reduction or regularization [35].)

Application of the higher covariant derivative regularization to calculations in supersymmetric theories in the lowest orders allows to reveal an interesting feature of quantum corrections: the $\beta$-function is given by integrals of total derivatives [36] and even integrals of double total derivatives [37]. As a consequence, it is possible to calculate one of the loop integrals analytically and to obtain in the lowest orders the NSVZ $\beta$-function, relating the $\beta$-function and the anomalous dimension of the matter superfield [37, 38, 39]. For $N = 1$ SQED this can be verified in all loops [40].

In this paper we present the results of some calculations made with the higher covariant derivative regularization and demonstrate the factorization of integrals defining the $\beta$-function into integrals of double total derivatives. Then we briefly describe the proof of this factorization in $N = 1$ SQED exactly to all orders.

The paper is organized as follows. In Sec. 2 we introduce the higher covariant derivative regularization for the supersymmetric theories. In Sec. 3 we describe the results of the three-loop calculation in the $N = 1$ SQED and the two-loop calculation in the general renormalizable SYM theory, which demonstrate that the $\beta$-function is given by integrals of double total derivatives and allow to obtain the exact NSVZ $\beta$-function. The proof of this fact for the $N = 1$ SQED is sketched in Sec. 4. The results are briefly summarized in the Conclusion.

2. Higher covariant derivative regularization in supersymmetric theories

We consider a massless $N = 1$ supersymmetric Yang–Mills theory, described by the action

$$S = \frac{1}{2e^2} \text{Re} \text{tr} \int d^4x \, d^2\theta \, W_a C^{ab} W_b + \frac{1}{4} \int d^4x \, d^4\theta \left( \phi^* i(\epsilon^2 V) i \phi_j + \left( \frac{1}{6} \int d^4x \, d^2\theta \lambda^{ijk} \phi_i \phi_j \phi_k + \text{h.c.} \right) \right), \quad (1)$$

where $\phi_i$ are chiral matter superfields in a representation $R$, $V$ is a real scalar gauge superfield, and

$$W_a = \frac{1}{8} \bar{D}^2 (e^{-2V} D_a e^{2V}). \quad (2)$$

(In our notation $D_a$ and $\bar{D}_a$ are the right and left supersymmetric covariant derivatives, respectively.) In order to obtain the gauge invariance, the condition

$$(T^A)_m i\lambda^{mjk} + (T^A)_m j\lambda^{imk} + (T^A)_m k\lambda^{jim} = 0 \quad (3)$$

should be imposed.

A special case of (1) corresponding to $G = U(1)$ is the $N = 1$ SQED

$$S = \frac{1}{4e^2} \text{Re} \int d^4x \, d^2\theta \, W_a C^{ab} W_b + \frac{1}{4} \int d^4x \, d^4\theta \left( \phi^* e^{2V} \phi + \bar{\phi} e^{-2V} \bar{\phi} \right). \quad (4)$$
For the calculation of quantum corrections it is convenient to use the background field method. In the supersymmetric case [41, 42] we make the substitution
\[ e^{2V} \rightarrow e^{2V'} \equiv e^{\Omega^+} e^{2V} e^{\Omega}, \tag{5} \]
in action (1), where \( \Omega \) is a background superfield, and introduce the background covariant derivatives \( D, \bar{D}, \) and \( D_\alpha \). Then we can fix the gauge by adding
\[ S_{gf} = -\frac{1}{32e^2} \text{tr} \int d^4x d^4\theta \left( V D^2 \bar{D}^2 V + V D^2 D^2 V \right) \tag{6} \]
to the action. The gauge fixing procedure also require introducing the Faddeev–Popov and Nielsen–Kallosh ghosts [41, 42]. For the regularization we add the terms with the higher covariant derivatives
\[ S_\Lambda = \frac{1}{2e^2} \text{tr} \text{Re} \int d^4x d^4\theta \frac{\left( D^2 \right)^{n+1}}{\Lambda^{2n}} - \frac{1}{4} \int d^4x d^4\theta (\phi^*)^i \left[ e^{\Omega^+} \frac{\left( D^2 \right)^m}{\Lambda^{2m}} e^{\Omega} \right] j^i \phi_j. \tag{7} \]

Other choices of the higher derivative terms are also possible. However, because the considered theory contains a nontrivial superpotential, it is also necessary to introduce the higher covariant derivative term for the matter superfields. In the Abelian case (for \( N = 1 \) SQED) it is possible to use a simpler regulator:
\[ S_\Lambda = \frac{1}{4e^2} \text{Re} \int d^4x d^2\theta W_a C^{ab} \left( R \left( \frac{\partial^2}{\Lambda^2} \right) - 1 \right) W_b, \tag{8} \]
where the function \( R \) satisfies the conditions \( R(0) = 1 \) and \( R(\infty) = \infty \). For example, it is possible to choose
\[ R \left( \frac{\partial^2}{\Lambda^2} \right) = 1 + \frac{\partial^{2n}}{\Lambda^{2n}}. \tag{9} \]

It is well-known [43] that the higher covariant derivative term does not remove divergences in the one-loop approximation. In order to cancel the remaining one-loop divergences, it is necessary to introduce into the generating functional the Pauli–Villars determinants
\[ \prod_I \left( \int D\phi_I D\bar{\phi}_I e^{iS_I} \right)^{-C_I}, \tag{10} \]
where \( S_I \) is the action for the Pauli–Villars fields:
\[ S_I = \frac{1}{4} \int d^4x d^4\theta (\phi_I^*)^i \left[ e^{\Omega^+} \frac{\left( D^2 \right)^m}{\Lambda^{2m}} e^{\Omega} \right] ^i_j \phi_I_j + \left( \frac{1}{4} \int d^4x d^2\theta M_I^{ij}(\phi_I)_i(\phi_I)_j + \text{h.c.} \right). \tag{11} \]

Here the masses of the Pauli–Villars fields are proportional to the parameter \( \Lambda \): \( M_I^{ij} = a_I^{ij} \Lambda \). This means that \( \Lambda \) is the only dimensionful parameter of the regularized theory. We also assume that the mass term does not break the gauge invariance and choose the masses so that
\[ M_I^{ij}(M_I^j)^k_i = M_I^2 \delta^{kj}. \tag{12} \]
The coefficients \( C_I \) in Eq. (10) satisfy the conditions
\[ \sum_I C_I = 1; \quad \sum_I C_I M_I^2 = 0. \tag{13} \]
Also it is necessary to introduce the Pauli–Villars determinants for all ghosts. Their masses are denoted by \( m_I \), and the corresponding degrees of the determinants are denoted by \( c_I \).

For \( N = 1 \) SQED the Pauli–Villars action can be chosen as

\[
S_I = \frac{1}{4} \int d^4x \, d^4\theta \left( \phi_I^* e^{2V} \phi_I + \tilde{\phi}_I^* e^{-2V} \tilde{\phi}_I \right) + \left( \frac{1}{2} \right) \int d^4x \, d^4\theta \, M_I \phi_I \tilde{\phi}_I + \ldots. \tag{14}
\]

In this paper we will use the following notation. Terms in the effective action, corresponding to the renormalized two-point Green function of the gauge superfield, are written as

\[
\Gamma^{(2)}_V = -\frac{1}{8\pi} \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{d^4\theta}{(2\pi)^4} \mathbf{V}(-p) \, \partial^2 \Pi_{1/2} \mathbf{V}(p) \, d^{-1}(\alpha, \lambda, \mu/p). \tag{15}
\]

where \( \alpha \) is a renormalized coupling constant, and \( \mathbf{V} \) is the background gauge superfield. (Using the background gauge invariance it is possible to choose \( \Omega = \Omega^+ = \mathbf{V} \).) We will calculate

\[
\frac{d}{d\ln \Lambda} \left( d^{-1}(\alpha_0, \lambda_0, \Lambda/p) - \alpha_0^{-1} \right) = -\frac{\alpha_0}{\ln \Lambda} \frac{\beta(\alpha)}{\alpha_0}. \tag{16}
\]

The anomalous dimension is defined similarly. First we consider the two-point Green function for the matter superfield in the massless limit:

\[
\Gamma^{(2)}_\phi = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} \frac{d^4\theta}{(2\pi)^4} \phi^*(-p, \theta) \, \phi_j(p, \theta) \, (\mathbf{ZG})^j(\alpha, \lambda, \mu/p), \tag{17}
\]

where \( Z \) denotes the renormalization constant for the matter superfield. Then the anomalous dimensions is defined by

\[
\gamma^j_\phi(\alpha_0, \lambda, \Lambda/\mu) = -\frac{\partial}{\partial \ln \Lambda} \left( \ln Z(\alpha, \lambda, \Lambda/\mu) \right)^j. \tag{18}
\]

3. \( \beta \)-function in the lowest orders with the higher covariant derivative regularization

3.1. \( N = 1 \) SQED, three loops

Explicit calculations made with the higher derivative regularization \([36, 37, 39, 40]\) show that the integrals for the \( \beta \)-function are integrals of (double) total derivatives. For the \( N = 1 \) SQED the three-loop \( \beta \)-function, calculated according to Eq. (16), can be presented as

\[
\frac{\beta(\alpha)}{\alpha_0} = 2\pi \frac{d}{d\ln \Lambda} \sum_I C_I \int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_{\mu}} \frac{\ln(q^2 + M_I^2)}{q^2} + 4\pi \frac{d}{d\ln \Lambda} \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{e^4}{k^2 R_k^2} \times \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_{\mu}} \left( \frac{1}{q^2(k + q)^2} - \sum_I C_I \left( \frac{1}{q^2 + M_I^2} \right) \right) \left[ R_k \left( 1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) \right] - 2e^2 \left( \int \frac{d^4t}{(2\pi)^4} \frac{1}{t^2(k + t)^2} \right)^2 \times \sum_J C_J \left( \frac{1}{(t^2 + M_J^2)} \right) \right] + 4\pi \frac{d}{d\ln \Lambda} \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{e^4}{k^2 R_k l^2 R_l} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_{\mu}} \left( \frac{1}{q^2(q + k)^2(q + l)^2(q + k + l)^2} \right). \]

\[ \frac{1}{2} \frac{2}{q^2 (q + k)^2 (q + l)^2} \left( - \sum_T C_l \left( \frac{2(k^2 + M_f^2)}{(q^2 + M_f^2)((q + k)^2 + M_f^2)((q + l)^2 + M_f^2)} \right. \right. \]
\[ \times \left. \left. \frac{1}{(q + k + l)^2 + M_f^2} + \frac{2}{(q^2 + M_f^2)((q + k)^2 + M_f^2)((q + l)^2 + M_f^2)} - \frac{1}{(q^2 + M_f^2)^2} \times \frac{4M_f^2}{((q + k)^2 + M_f^2)((q + l)^2 + M_f^2)} \right) \right\} , \]
\]
where \( R_k \equiv R(k^2 / \Lambda^2) \). We see that it is defined by the integrals of double total derivatives. In order to calculate these integrals it is possible to use the identity
\[ \int \frac{d^4 q}{(2\pi)^4} \frac{\partial}{\partial q^0} \frac{\partial}{\partial q_\mu} \left( \frac{f(q^2)}{q^2} \right) = \lim_{\varepsilon \to 0} \int \frac{dS_{\mu} (-2)q^\mu f(q^2)}{(2\pi)^4} q^4 = \frac{1}{4\pi^2} f(0), \]
where \( f \) is a nonsingular function, which rapidly decreases at the infinity. It is equivalent to the identity
\[ \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4} \frac{d}{dq^2} f(q^2) = \frac{1}{16\pi^2} \left( f(\infty) - f(0) \right) = - \frac{1}{16\pi^2} f(0). \]
(This is a total derivative in the four-dimensional spherical coordinates.) Calculating the integrals in Eq. (19) and comparing the result with the expression for the two-loop anomalous dimension of the matter superfield
\[ \gamma(a_0) = -2e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln k^2} \left[ R_k \left( 1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) - \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{t^2(k + l)^2} \right] \]
\[ + \sum_T C_l \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{(t^2 + M_f^2)((k + l)^2 + M_f^2)} - \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{4e^2 k_{\mu} l_{\mu}}{k^4 l^4 [k^2 + l^2]}; \]
in the considered approximation we obtain the exact NSVZ \( \beta \)-function:
\[ \beta(a_0) = \frac{a_0^2}{\pi} \left( 1 - \gamma(a_0) \right) + O(a_0^3). \]
This is an equality of some well-defined integrals (which were not calculated explicitly). It is obtained without any redefinitions of the coupling constant.

3.2. The general renormalizable \( N = 1 \) SYM theory, two loops

In the non-Abelian case the result for the \( \beta \)-function can be written in the following form:
\[ \beta(\alpha) = \alpha^2 C_2 (I_{FP} + I_{NK}) + \alpha^2 T(R) I_0 + \alpha^3 C_2^2 I_1 + \frac{\alpha^3}{r} C(R)^i j C(R)^j i_2 + \]
\[ + \alpha^3 T(R) C_2 I_3 + \alpha^2 C(R)^i j \frac{\lambda_{ijkl}^a \lambda_{ijkl}^b}{4\pi r} I_4 + \ldots , \]
where
\[ \text{tr} \left( T^A T^B \right) \equiv T(R) \delta^{AB}; \quad (T^A)^k (T^A)^j \equiv C(R)^i j; \]
\[ f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \quad r \equiv \delta_{AA}. \]
Taking into account contributions of the Pauli–Villars fields, the integrals defining the \( \beta \)-function are given by

\[
I_i = I_i(0) - \sum_I c_I I_i(M_I), \quad i = \text{FP, NK}; \quad I_i = I_i(0) - \sum_I C_I I_i(M_I), \quad i = 0, 2, 3,
\]

where

\[
I_{\text{NK}}(m) = \frac{1}{2} I_{\text{FP}}(m) = \pi \int \frac{d^4 q}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\mu} \left\{ \frac{1}{q^2} \ln \left( q^2 + m^2 \right) \right\};
\]

\[
I_0(M) = -\pi \int \frac{d^4 q}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\mu} \left\{ \frac{1}{q^2} \ln \left( q^2 (1 + q^{2n}/\Lambda^{2n})^2 + M^2 \right) \right\};
\]

\[
I_1 = -12\pi^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial k^\mu} \left\{ \frac{1}{k^2 (1 + k^{2n}/\Lambda^{2n})q^2 (1 + q^{2n}/\Lambda^{2n})(q + k)^2} \times \frac{1}{(1 + (q + k)^{2n}/\Lambda^{2n})} \right\};
\]

\[
I_2(M) = 8\pi^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\mu} \left\{ \frac{1}{k^2 (1 + k^{2n}/\Lambda^{2n})} \times \frac{(1 + q^{2n}/\Lambda^{2n})(1 + (q + k)^{2n}/\Lambda^{2n})}{(q^2 (1 + q^{2n}/\Lambda^{2n})^2 + M^2)((q + k)^2 (1 + (q + k)^{2n}/\Lambda^{2n})^2 + M^2)} \right\};
\]

\[
I_3(M) = 8\pi^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\mu} \left\{ \frac{1}{k^2 (1 + k^{2n}/\Lambda^{2n})^2 + M^2) \times \frac{(1 + k^{2n}/\Lambda^{2n})(1 + q^{2n}/\Lambda^{2n})}{(q^2 (1 + q^{2n}/\Lambda^{2n})^2 + M^2) \frac{(1 + (q + k)^{2n}/\Lambda^{2n})}{(q^2 (1 + q^{2n}/\Lambda^{2n})^2 + M^2)} \right\};
\]

\[
I_4 = -8\pi^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\mu} \left\{ \frac{1}{k^2 (1 + k^{2n}/\Lambda^{2n})q^2 (1 + q^{2n}/\Lambda^{2n})(q + k)^2} \times \frac{1}{(1 + (q + k)^{2n}/\Lambda^{2n})} \right\};
\]

All these integrals are again integrals of double total derivatives. Calculating them using Eq. (20) we obtain

\[
\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left( 3C_2 - T(R) \right) + \frac{\alpha^3}{(2\pi)^2} \left( -3C_2^2 + T(R)C_2 + \frac{2}{r} C(R)c^j C(R)j \right) - \frac{\alpha^2 C(R)c^j \lambda^{ijkl}_R \lambda^{jkl}_R}{8\pi^3 r} + \ldots
\]

This result should be compared with the one-loop anomalous dimension

\[
\gamma^j(\alpha) = -\frac{\alpha C(R)c^j}{\pi} + \frac{\lambda^{ijkl}_R \lambda^{jkl}_R}{4\pi^2} + \ldots
\]
Then we see that in considered approximation the \( \beta \)-function is given by the exact NSVZ \( \beta \)-function

\[
\beta(\alpha) = -\frac{\alpha^2 [3C_2 - T(R) + C(R)^2 \gamma_j \gamma^i(\alpha)]/r}{2\pi(1 - C_2 \alpha/2\pi)}.
\] (35)

4. The exact \( \beta \)-function in \( N = 1 \) SQED

The result of the three-loop calculation in the \( N = 1 \) SQED, presented above, show that
1. the integrals defining the \( \beta \)-function are integrals of double total derivatives;
2. after taking these integrals we obtain (up to an overall factor) integrals defining the anomalous dimension of the matter superfield and, as a consequence, the exact NSVZ \( \beta \)-function.

For the \( N = 1 \) SQED both these statements can be proved exactly in all loops. For this purpose, first, we make the substitution

\[
V \to \overline{\theta}_a \theta^a \theta_b \overline{\theta}_b \equiv \theta^4,
\] (36)

which allows to extract the transversal part of the two-point Green function of the gauge superfield:

\[
\int d^4 \theta \, V(x, \theta) \partial^2 \Pi_{1/2} \, V(x, \theta) \to -8.
\] (37)

(It is possible, because we make calculations in the limit of zero external momentum.) Then we will try to present the sum of Feynman diagrams as integrals of total derivatives, which in the coordinate representation are given by

\[
\text{Tr} \left( [x^\mu, \text{Something}] \right) = 0.
\] (38)

First, we perform the Gaussian integration over the matter superfields \( \phi \) and \( \tilde{\phi} \) in the generating functional. The result can be written as

\[
Z = \int DV \prod_I \left( \det PV(V, M_I) \right)^{C_I} \times \exp \left\{ i \int d^8 x \left( \frac{1}{4e^2} V \partial^2 R(\partial^2) V - j \frac{D^2}{4\partial^2} \ast \bar{D}^2 \frac{D^2}{4\partial^2} \ast j \ast - \bar{J} \frac{D^2}{4\partial^2} \ast \bar{D}^2 \frac{D^2}{4\partial^2} \ast \bar{J} \ast \right) \right\},
\] (39)

where

\[
\ast \equiv \frac{1}{1 - (e^{2V} - 1)D^2 D^2/16\partial^2}, \quad \tilde{\ast} = \frac{1}{1 - (e^{-2V} - 1)D^2 D^2/16\partial^2}
\] (40)

encode chains of propagators and vertexes. In terms of these operators it is possible to present the part of the effective action corresponding to the two-point Green function of the gauge superfield in the form

\[
\Delta \Gamma^{(2)}_V = \left\{ -2i \left( \text{Tr} (V J_0 \ast) \right)^2 - 2i \text{Tr} (V J_0 \ast V J_0 \ast) - 2i \text{Tr} (V^2 J_0 \ast) \right\} + \text{terms with } \tilde{\ast} + (PV),
\] (41)

where the angular brackets include the functional integration only over the superfield \( V \), and
Thus, the first term in Eq. (41) can be rewritten as [40]

\[
J_0 = e^{2\nu} \frac{D^2 D^2}{16\delta^2}
\]  

(42)

is the effective vertex. The first term in Eq. (41) is a sum of diagrams in which external lines are attached to different loops of the matter superfields. The second term is a sum of diagrams in which external lines are attached to a single line of the matter superfields. The last term is not transversal. All such terms are canceled due to the Ward identities.

After some algebraic transformations, which are made using the method proposed in [44], the first term in Eq. (41) can be rewritten as [40]

\[
-2i \frac{d}{d\ln \Lambda} \left< \left\{ \text{Tr} \left[ -2\theta^\nu \bar{\theta}_d \left[ \ln(*) - \ln(\bar{x}) \right] + i\bar{\theta}^\nu (\gamma^\nu \gamma^\beta) \bar{\theta}_d \left[ \ln(*) - \ln(\bar{x}) \right] \right] + (PV) \right\} \right> ^2. 
\]  

(43)

where \((PV)\) denotes a contribution of the Pauli–Villars fields. Similarly, the second term in Eq. (41) is given by

\[
i \frac{d}{d\ln \Lambda} \text{Tr} \left< \left\{ \left[ (y^\mu)^* + i \partial \right] \ln(*) + \ln(\bar{x}) \right\} \right> + (PV) - \text{terms with a } \delta\text{-function},
\]  

(44)

It is rather complicated task to derive this expression. In particular the derivation involves using the identity

\[
\text{Tr} \left\{ (\gamma_\mu)^{ab} [y^\mu, A] [\bar{\theta}_b, B] [\theta_d, C] + (\gamma_\mu)^{ab} (-1)^P \bar{\theta}_d [\theta_a, B] [\bar{\theta}_b, C] [y^\mu, A] \right. 
\]

\[
-4i \left[ (\gamma_\mu)^{ab} [\theta_a, B] [\bar{\theta}_b, C] \right] + \text{cyclic perm. of } A, B, C 
\]

(45)

where \(A, B, \) and \(C\) are operators constructed from the supersymmetric covariant derivatives and usual derivatives which do not explicitly depend on \(\theta\) and \(\bar{\theta}\). This identity was derived in [45].

Terms with the \(\delta\text{-function}\) in Eq. (44) appear due to the identity

\[
[x^\mu, \frac{\partial}{\partial p^\mu}] = \left[ -i \frac{\partial}{\partial p^\mu}, -\frac{i p^\mu}{p^4} \right] = -2\pi^2 \delta^4(p_E) = -2\pi^2 i \delta^4(p).
\]  

(46)

This \(\delta\text{-function}\) allows to calculate one of loop integrals analytically and, thus, to decrease a number of integrations. Therefore, the \(\beta\text{-function}\) in \(n\)-th loop is given an integral over \(n - 1\) momentums. Qualitatively, such \(\delta\text{-functions}\) correspond to cutting the matter loop [37]. It is possible to find their contributions exactly to all orders [40], carefully taking into account all possible \(\partial_p/\partial^4\) in the commutators. The result is the exact NSVZ \(\beta\text{-function}\) [9]

\[
\beta(\alpha) = \frac{\alpha^2}{\pi} \left( 1 - \gamma(\alpha) \right).
\]  

(47)

Thus, the \(\beta\text{-function}\) in \(n\)-th loop is related with the anomalous dimension of the matter superfield in \((n - 1)\)-th loop. It is important that in order to derive Eq. (47) it is not necessary to redefine the coupling constant, as in the case of the dimensional reduction [4, 10, 11]. Therefore, the NSVZ scheme for the \(N = 1\) SQED can be formulated using the higher derivative regularization.

The last term in Eq. (41) vanishes after the substitution (36), which extracts the transversal part of the two-point Green function of the gauge superfield. Expressions for the first term and
the second term (Eqs. (43) and (44), respectively) allow to conclude that the \(\beta\)-function of the \(N = 1\) SQED is given by integrals of double total derivatives in all orders. (Another proof of this fact is given in Ref. [37], using a method based on the covariant Feynman rules in the background field method [46, 47].)

5. Conclusion

In this paper we demonstrate that the \(\beta\)-function in the \(N = 1\) supersymmetric theories regularized by higher covariant derivatives is given by integrals of double total derivatives. Possibly, this is a general feature of supersymmetric theories, but so far this statement was proved only for the \(N = 1\) SQED. The factorization of integrands into (double) total derivatives allows to calculate one of the loop integrals analytically. For the \(N = 1\) SQED this relates the \(\beta\)-function in the \(n\)-th loop with the anomalous dimension of the matter superfield in the previous \((n - 1)\)-th loop. As a consequence, the exact NSVZ \(\beta\)-function is obtained without any redefinition of the coupling constant.

For the \(N = 1\) non-Abelian supersymmetric theories, regularized by higher covariant derivatives, the factorization of integrands into double total derivatives was verified in the two-loop approximation. This factorization again gives the exact NSVZ \(\beta\)-function. However, in the two-loop approximation the \(\beta\)-function is scheme independent, and so far it is impossible to say anything about the necessity of the coupling constant redefinition in this case.

Acknowledgments

This work was supported by Russian Foundation for Basic Research grants No 11-01-00296-a. I am very grateful to prof. A.L.Kataev for valuable discussions.

[1] Ferrara S and Zumino B 1974 Nucl.Phys. B 79 413.
[2] Jones D R T 1975 Nucl.Phys. B 87 127.
[3] Avdeev L V and Tarasov O V 1982 Phys.Lett. B 112 356.
[4] Jack I, Jones D R T and North C G 1996 Phys.Lett. B 386 138.
[5] Harlander R V, Jones D R T, Kant P, Mihaila L and Steinhauser M 2006 JHEP 0612 024.
[6] Novikov V A, Shifman M A, Vainshtein A I and Zakharov V I 1983 Nucl.Phys. B 229 381.
[7] Novikov V A, Shifman M A, Vainshtein A I and Zakharov V I 1985 Phys.Lett.B 166 329.
[8] Shifman M A and Vainshtein A I 1986 Nucl.Phys. B 277 456.
[9] Vainshtein A I, Zakharov V I and Shifman M A 1985 JETP Lett. 42 224.
[10] Jack I, Jones D R T and North C G 1997 Nucl.Phys. B 486 479.
[11] Jack I, Jones D R T and Pickering A 1998 Phys.Lett. B 435 61.
[12] Siegel W 1979 Phys.Lett. B 84 193.
[13] Bardeen W A, Buras A J, Duke D W and Muta T 1978, Phys.Rev. D18 3998.
[14] Abbott L F, Grisary M T and Zanon D 1984 Nucl.Phys. B 244 454.
[15] Parkes A and West P 1983 Phys.Lett. B 138 99.
[16] Jack I, Jones D R T and North C G 1996 Nucl.Phys. B 473 308.
[17] t'Hooft G and Veltman M 1972 Nucl.Phys. B 44 189.
[18] Siegel W 1980 Phys.Lett. B 94 37.
[19] Avdeev L V, Chochia G A and Vladimirov A A 1981 Phys.Lett. B 105 272.
[20] Stöckinger D 2005 JHEP 0503 076.
[21] Avdeev L V and Vladimirov A A 1983 Nucl.Phys. B 219 262.
[22] Avdeev L V 1982 Phys.Lett. B 117 317.
[23] Velizhanin V N 2009 Nucl.Phys. B 818 95.
[24] Velizhanin V N 2011 Phys.Lett. B 696 560.
[25] Jack I and Jones D R T 1997 Regularisation of supersymmetric theories Preprint hep-ph/9707278.
[26] Slavnov A A 1971 Nucl.Phys. B 31 301.
[27] Slavnov A A 1972 Theor.Math.Phys. 13 1064.
[28] Krivoshchekov V K 1978 Theor.Math.Phys. 36 745.
[29] West P 1986 Nucl.Phys. B 268 113.
[30] Martin C and Ruiz Ruiz F 1995 Nucl.Phys. B 436 645.
[31] Asorey M and Falceto F 1996 Phys.Rev. D 54 5290.
[32] Bakeyev T and Slavnov A A 1996 Mod.Phys.Lett. A 11 1539.
[33] Gross D J and Wilczek F 1973 Phys.Rev.Lett. 30 1343.
[34] Politzer H D 1973 Phys.Rev.Lett. 30 1346.
[35] Pronin P and Stepanyantz K 1997 Phys.Lett. B 414 117.
[36] Soloshenko A A and Stepanyantz K V 2004 Theor.Math.Phys. 140 1264.
[37] Smilga A and Vainshtein A 2005 Nucl.Phys. B 704 445.
[38] Stepanyantz K V 2005 Theor.Math.Phys. 142 29.
[39] Pimenov A B, Shevtsova E S and Stepanyantz K V 2010 Phys.Lett. B 686 293.
[40] Stepanyantz K V 2011 Nucl.Phys. B 852 71.
[41] West P 1986 Introduction to supersymmetry and supergravity (World Scientific).
[42] Buchbinder I L and Kuzenko S M 1998 Ideas and methods of supersymmetry and supergravity (Bristol and Philadelphia, Institute of Physics Publishing).
[43] Faddeev L D and Slavnov A A 1990 Gauge fields, introduction to quantum theory (Benjamin Reading).
[44] Stepanyantz K 2006 Theor.Math.Phys. 146 321.
[45] Stepanyantz K 2011 Factorization of integrals defining the $\beta$-function into integrals of total derivatives in $N = 1$ SQED, regularized by higher derivatives Preprint ArXiv:1101.2956 [hep-th].
[46] Grisaru M T and Zanon D 1985 Nucl.Phys. B 252 578.
[47] Grisaru M T, Milewski B and Zanon D 1986 Nucl.Phys. B 266 589.