I. INTRODUCTION

Intermetallic compounds containing rare-earth or actinide elements are known to exhibit a variety of low-temperature phases, including antiferromagnetism, superconductivity, and Kondo-insulator phases. The Kondo lattice model is one of the standard models describing such physics of heavy electron systems. The low-temperature properties of the Kondo lattice model at half-filling, describing a group of compounds called Kondo-insulators, are governed by competition between the Kondo- and the Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions. While the Kondo interaction favors the formation of intra-atomic quantum disordered (Kondo singlet) phase, the RKKY interaction favors an antiferromagnetic long-range order. In one-dimension, the Kondo singlet phase is stabilized in the presence of Kondo interaction of any strength, causing an energy gap in the charge and spin excitations, the spin gap being always smaller than the charge gap.\[2\]

The Kondo lattice model is a simplified version of the Kondo necklace model in which the charge degree of freedom is frozen out in order to describe the above-mentioned competition.\[3\] In fact, the Kondo singlet gap $\Delta_K$ always opens in the Kondo necklace model as long as any finite Kondo interaction is present.\[4\] Thus the magnetization of this model starts to grow at a finite critical magnetic field $h = \Delta_K$, and continuously increases with increase in the external field. Note that these results for the Kondo necklace model are consistent with the magnetization curve of the Kondo lattice model.\[5\]

However, the above situation may be drastically changed when the next-nearest-neighboring conduction electron spins coupled antiferromagnetically. In our previous paper,\[6\] we studied a magnetization process of the Kondo necklace model with next-nearest-neighbor interaction $J_2$ which partially frustrates the nearest-neighbor interaction $J_1$, making use of a finite size scaling analysis of the numerical exact-diagonalization data.

In Ref. 10 we showed that for $\lambda = J_2/J_1 = 1/2$, the magnetization plateaus appear at zero ($m = 0$) and a half saturation magnetization ($m = 1/2$), satisfying the necessary condition for the appearance of plateaus.\[7\] The origin of these plateaus for $\lambda = 1/2$ can be understood as follows. The spin configuration in the plateau phase at $m = 0$ is interpreted as the aggregate of intra-atomic Kondo singlets. On the other hand, the spin configuration in the plateau phase at $m = 1/2$ represents the aggregate of intrachain dimers with broken translational symmetry, resulting from the frustrations due to competition between $J_1$ and $J_2$. However, the $m = 1/2$ plateau may be completely broken by quantum fluctuations with decreasing $J_2$ in fact, the $m = 1/2$ plateau does not appear at a magnetization curve of the Kondo necklace model without next-nearest-neighbor interaction. This suggests that a quantum phase transition between the plateau and nonplateau phases may occur in $0 < \lambda < 1/2$.

In this paper, we are going to investigate critical properties of frustrated Kondo necklace model

\[
\mathcal{H}_{\text{FKN}} = \sum_{j=1}^{L} \left\{ (\sigma_j \cdot \sigma_{j+1})_{\Delta} + \lambda (\sigma_j \cdot \sigma_{j+2})_{\Delta} \right\} + J_K \sum_{j=1}^{L} \sigma_j \cdot S_j, \tag{1}
\]

removing the restriction $\lambda = 1/2$ imposed in our previous work.\[8\] Here $(\sigma_j \cdot \sigma_{j+\delta})_{\Delta}$ with $\delta = 1, 2$ denotes the anisotropic spin interaction of the XXZ type,

\[
(\sigma_j \cdot \sigma_{j+\delta})_{\Delta} = \sigma_j^x \sigma_{j+\delta}^x + \sigma_j^y \sigma_{j+\delta}^y + \Delta \sigma_j^z \sigma_{j+\delta}^z \tag{2}
\]
We are also interested in whether or not the above-mentioned non-plateau phase belongs to the universality class of Tomonaga-Luttinger (TL) liquid. This problem is nontrivial in the case of the Kondo necklace model, although it is well known that many one-dimensional quantum spin systems exhibit the TL liquid phase with central charge $c = 1$ in the conformal field theory.

The paper is organized as follows. In §II, we map model (4) onto the quantum sine-Gordon model by means of bosonization technique, limiting ourselves to the Hilbert space with the half-magnetized states. In §III A, the phase diagram in the plane of Ising anisotropy $\Delta$ and the frustration parameter $\lambda$ will be constructed, making use of the method of level spectroscopy, on the basis of the assumption that our model (1) is described as the quantum sine-Gordon model. In §III B, the validity of this assumption will be checked by calculating scaling dimensions and the central charge. It will be shown that the central charge $c$ is very close to 1. The final section (§IV) is devoted to a summary and discussion.

II. BOSONIZATION AND EFFECTIVE HAMILTONIAN

Let us consider the effective Hamiltonian that describes our model in the $m = 1/2$ plateau phase. We showed in our previous paper [4] that inner core spins of the Kondo necklace model with $\lambda = 1/2$ in the $m = 1/2$ plateau phase are almost ferromagnetically aligned along the external field ($z$-axis). This enables us to replace the inner core spin operators $S_j$ with an average $\langle S_j^z \rangle = 1/2$, leading to the effective Hamiltonian of the following form:

$$\mathcal{H}_{\text{eff}} = \sum_{j=1}^{L} \left\{ (\sigma_j \cdot \sigma_{j+1})_\Delta + \lambda (\sigma_j \cdot \sigma_{j+2})_\Delta \right\} - h_{\text{eff}} \sum_{j=1}^{L} \sigma_j^z. \quad (3)$$

Here $h_{\text{eff}} = (h - J_K/2)$ is the effective magnetic field including the effect of Kondo coupling. Since the total magnetization of the $\sigma$ spins is vanishing ($J_K/2 (h_{\text{eff}} \approx 0)$, the system can be effectively regarded as a $\sigma = 1/2$ XXZ chain with next-nearest-neighbor interaction in the zero field. The ground state properties of the model have been extensively studied by many authors by means of various methods [13, 14, 15, 16, 17, 18]. It is known that the model is transformed into the quantum sine-Gordon model

$$\mathcal{H}_{SG} = \int dx \left\{ \frac{\pi v_F K}{2} \Pi^2(x) + \frac{v_s}{2\pi K} (\partial_x \phi(x))^2 - \frac{2g}{(2\pi \alpha)^2} \cos \left( \sqrt{2} \phi(x) \right) \right\}, \quad (4)$$

in terms of the Jordan-Wigner transformation and bosonization technique. Here, $\Pi(x)$ is the momentum density conjugate to the bosonic field $\phi(x)$, satisfying the commutation relation $[\phi(x), \Pi(x')] = i\delta(x-x')$, and $\alpha$ is the lattice constant. The spin wave velocity $v_s$, the Luttinger liquid parameter $K$, and the umklapp scattering amplitude $g$ are respectively given as

$$v_F = 2\sqrt{AC}, \quad K = \frac{1}{2\pi} \sqrt{\frac{C}{A}}, \quad g = 2\pi^2 \lambda^2 D, \quad (5)$$

where the coefficients $A$, $C$, and $D$ are given by

$$A = \frac{\alpha}{8\pi} \left( 1 + \frac{3\Delta}{\pi} + \frac{(6 + \Delta)\lambda}{\pi} \right),$$

$$C = 2\pi \alpha \left( 1 - \frac{\Delta}{\pi} - \frac{(2 - \Delta)\lambda}{\pi} \right),$$

$$D = \frac{1}{2\alpha} (\Delta - (2 + \Delta)\lambda). \quad (6)$$

Thus the critical properties of our system with $m = 1/2$ are well described in terms of the quantum sine-Gordon model, although the above expressions for $A$, $C$, and $D$ are valid only when $\Delta$, $\lambda$ and $h_{\text{eff}}$ are small. It is well known that the quantum sine-Gordon model exhibits two gapful (corresponding to dimer plateau and Néel plateau) phases and a gapless (TL liquid) phases, depending on the value of $K$. These results are derived from the flow diagram of renormalization group equations derived from the scaling of the cut-off $\alpha \rightarrow e^{\Omega} \alpha$ where $\Omega = \ln L$.

Thus the frustrated Kondo necklace model belongs to a universality class of TL liquid if the nonlinear term is renormalized to zero as $L \rightarrow \infty$. In the TL phase the $\sigma$-spin excitation is gapless, and the $\sigma$-spin correlation functions algebraically decay as

$$(-1)^r \langle \sigma_0^z \sigma_r^z \rangle - \langle \sigma^z \rangle^2 \sim r^{-K},$$

$$(-1)^r \langle \sigma_0^+ \sigma_r^- \rangle \sim r^{-1/K}. \quad (7, 8)$$

On the other hand, the dimer plateau phase is characterized by the $\sigma$-spin excitation gap, the exponential decay of the $\sigma$-spin correlation, and dimer long-rang order. The Néel plateau phase is characterized by the Ising gap for $\sigma$-spin and antiferromagnetic long-rang order (Néel order). Also, it is known that the dimer-TL and the Néel-TL transitions are of the Berezinskii-Kosterlitz-Thouless (BKT) type, and the dimer-Néel transition is of the Gaussian type.

III. NUMERICAL RESULTS

A. Phase diagram

In this section, we are going to determine the boundaries between TL, dimer, and Néel phases on the plane of Ising anisotropy $\Delta$ and the frustration parameter $\lambda$, making use of the method of level spectroscopy. This method is known to be powerful in the determination of the BKT transition points, which is difficult by the standard finite size scaling analysis due to logarithmic corrections [10, 17, 18].
TABLE I: Symmetry classification of the ground state and three (spin-wave type, dimer and Néel excitations) excitations in the Kondo necklace model with \( m = 1/2 \) and \( L = 4 \times \) (integer). The numbers in round brackets are quantum numbers in case of \( L = 4 \times \) integer + 2.

| Quantum Numbers                  | \( M \) | \( k \) | \( \mathcal{P} \) |
|----------------------------------|--------|--------|-------------------|
| ground state                    | \( L/2 \) | 0 (\( \pi \)) | +1 (−1) |
| spin-wave excited state         | \( L/2 \pm 1 \) | \( \pi \) (0) | −1 (+1) |
| dimer excited state             | \( L/2 \) | \( \pi \) (0) | +1 (−1) |
| Néel excited state              | \( L/2 \) | \( \pi \) (0) | −1 (+1) |

Since the Hamiltonian \([\hat{H}]\) is invariant under spin rotation around the \( z \)-axis, translation \( j \rightarrow j + 1 \), and space inversion \( j \rightarrow L - j + 1 \), the eigenvalues of total magnetization \( M \), wave number \( k = 2\pi n/L \), and parity \( \mathcal{P} = \pm 1 \) are good quantum numbers to label the eigenvalues and eigenvectors of \( \hat{H}_{\text{KFN}} \).

As expected from bosonization, the ground state with \( m = M/L = 1/2 \) in all three phases (i.e., TL, dimer and Néel phases) is shown to be labeled by \((M, k, \mathcal{P}) = (L/2, 0, 1)\) for the lattice with \( L = 4 \times \) (integer) and \((M, k, \mathcal{P}) = (L/2, \pi, -1)\) for the lattice with \( L = 4 \times \) (integer) + 2. On the other hand, the lowest excited state in TL, dimer, and Néel phases are labeled by \((M, k, \mathcal{P}) = (L/2 \pm 1, \pi, -1)\), \((L/2, \pi, 1)\), and \((L/2, \pi, -1)\) for \( L = 4 \times \) (integer) and \((M, k, \mathcal{P}) = (L/2 \pm 1, 0, 1), (L/2, 0, -1), \) and \((L/2, 0, 1)\) for the lattice with \( L = 4 \times \) (integer) + 2, respectively. Hereafter, we call each of the excitations in TL, dimer, and Néel phases spin-wave type, dimer, and Néel excitations. Table I summarizes the eigenvalues of these symmetry operators in the ground state and three (spin-wave type, dimer and Néel excitations) excitations in the frustrated Kondo necklace model with \( m = 1/2 \).

In the case of \( L = 4 \times \) (integer), the (finite size) critical points are determined from the crossing of the minimum excitation energies of the above three types of excitations

\[
\Delta E_{\text{sw}} = \frac{1}{2} \left\{ E(M = L/2 + 1, k = \pi, P = -1) + E(M = L/2 - 1, k = \pi, P = -1) - 2E(M = L/2, k = 0, P = 1) \right\}, \quad (9)
\]

\[
\Delta E_{\text{dimer}} = E(M = L/2, k = \pi, P = 1) - E(M = L/2, k = 0, P = 1), \quad (10)
\]

\[
\Delta E_{\text{Néel}} = E(M = L/2, k = \pi, P = -1) - E(M = L/2, k = 0, P = 1). \quad (11)
\]

Note that these minimum excitation energies correspond to the scaling dimensions \( x_{\text{sw}}, x_{\text{dimer}}, \) and \( x_{\text{Néel}} \) by

\[
x_i = \frac{L}{2\pi v_s} \Delta E_i \quad (i = \text{sw, dimer and Néel}), \quad (12)
\]

where \( v_s \) is the spin wave velocity defined as

\[
v_s = \lim_{L \to \infty} \frac{L}{2\pi} \left\{ E(M = L/2, k = 2\pi/L) - E(M = L/2, k = 0) \right\}. \quad (13)
\]

We defined the gap in the spin-wave type excitation \( \Delta E_{\text{sw}} \) as in eq. \( (9) \) so that the Zeeman energy due to the external magnetic field

\[
\mathcal{H}_{\text{Zeeman}} = -\hbar \sum_{j=1}^{L} (\sigma_j^x + S_j^z) \quad (14)
\]

is canceled out.

Let us consider the TL-dimer critical point for \( \Delta = J_K = 1.0 \) in order to illustrate how our method of level-crossing works. In Fig. 2 spin-wave, dimer and Néel excitation gaps for \( L = 8 \) are shown as functions of the parameter \( \lambda \). We see from Fig. 2 that the spin-wave type excitation is the lowest for \( \lambda < 0.35105 \) and so is the dimer excitation for \( \lambda > 0.35105 \). On the other hand, the Néel excitation is always higher than spin-wave type and/or dimer excitations.

We have estimated the values of \( \lambda \) at the crossing point of spin-wave type and dimer excitations for various sizes \( L = 6, 8, 10, 12 \) and 14. These are extrapolated to the thermodynamic limit \( L \to \infty \), assuming a formula \( \lambda_c(L) = \lambda_c + A/L^2 + B/L^4 \). Thus the TL-dimer critical point \( \lambda_c = 0.3419 \pm 0.0001 \) (See Fig. 3) is obtained in the limit \( L \to \infty \). We have performed similar procedures for various \( \Delta \)'s to construct the phase diagram on \( \Delta - \lambda \) plane, as shown in Fig. 4. We assumed that \( m = 1/2 \) and \( J_K = J_1 \) in these calculations.

![Fig. 1: The dimer, Néel, and spin-wave type excitation energies as functions of the frustration parameter \( \lambda \). The crossing point of dimer and spin-wave type excitation energies represents the BKT transition point for \( L = 8 \) and \( \Delta = 1.0 \).](image-url)
0.00 0.01
0.02
0.03
0.34
0.345
0.35
1/
L
2
(L = 6, 8, 10, 12 and 14) and
Δ = 1.0. In the thermodynamic limit, the BKT transition point is obtained as \( \lambda_c = 0.3419 \pm 0.0001 \), assuming an extrapolation formula \( \lambda_c(L) = \lambda_c + A/L^2 + B/L^4 \), assuming an extrapolation formula \( \lambda_c(L) = \lambda_c + A/L^2 + B/L^4 \).

![Graph showing crossing points of dimer and spin-wave type excitation energies for various sizes](image)

**FIG. 2:** The crossing points of dimer and spin-wave type excitation energies for various sizes (\( L = 6, 8, 10, 12 \) and 14) and \( \Delta = 1.0 \). In the thermodynamic limit, the BKT transition point is obtained as \( \lambda_c = 0.3419 \pm 0.0001 \), assuming an extrapolation formula \( \lambda_c(L) = \lambda_c + A/L^2 + B/L^4 \), assuming an extrapolation formula \( \lambda_c(L) = \lambda_c + A/L^2 + B/L^4 \).

![Graph showing phase diagram on the \( \Delta - \lambda \) plane](image)

**FIG. 3:** Phase diagram on the \( \Delta - \lambda \) plane of the frustrated Kondo necklace model with \( m = 1/2 \) for \( J_K = 1.0 \). The open circle denotes a multicritical point between the dimer, Néel and TL phases.

**B. Consistency Check**

In §II, we assumed that the effective Hamiltonian of our model at \( m = 1/2 \) is described by the quantum sine-Gordon model. Our analysis, making use of the method of level spectroscopy, is based on this assumption, according to which the scaling dimension without logarithmic corrections,

\[
x = \frac{x_{\text{dimer}} + x_{\text{Néel}} + 2x_{\text{sw}}}{4}
\]

should tend to 1/2 on the BKT critical line in the thermodynamic limit \( L \to \infty \). We confirmed this relation \( x \approx 1/2 \) on the TL-dimer and dimer-Néel transition lines, as clearly shown in Fig.4. Furthermore, Table II indicates that the relations between the scaling dimensions

\[
x_{\text{Néel}} = x_{\text{dimer}},
\]

\[
x_{\text{sw}}x_{\text{dimer}} = 1/4
\]

are satisfied on the dimer-Néel (Gaussian) transition points.

We have also estimated the central charge \( c \) along the TL-dimer and dimer-Néel critical lines from the finite size correction to the ground-state energy

\[
E_0 \approx \epsilon_0 L - \frac{\pi v_s}{6L} c,
\]

where \( \epsilon_0 \) is the ground-state energy per site (including two spins) in the thermodynamic limit. As shown in Fig.5, we obtained the value \( c \approx 1 \) with small errors. These results provide sufficient evidence that our model can be described by the quantum sine-Gordon model.

**IV. CONCLUSION**

In this paper, we have studied critical properties of the frustrated Kondo necklace model with a half-saturation...
FIG. 5: The central charge $c$ on the TL-dimer and Néel-dimer critical lines, yielding the result $c = 1$ within errors of 0.03%.

magnetization ($m = 1/2$), making use of the method of level spectroscopy. The phase diagram on the plane of the Ising anisotropy $\Delta$ and the frustration parameter $\lambda$ is constructed exhibiting two plateau (dimer and Néel) and a TL phases. We also confirmed that the values of scaling dimension without logarithmic corrections [Eq. (15)] are close to 1/2 at the TL-dimer (BKT) and dimer-Néel (Gaussian) transition points. The relations between the scaling dimensions [Eqs. (16) and (17)] are satisfied as the Gaussian transition points. Furthermore, the central charge $c \simeq 1$ is obtained along the BKT and Gaussian transition lines, by estimating the finite-size correction for the ground state energy [Eq. (18)]. Thus we can conclude that the present system belongs to the same universality class as the quantum sine-Gordon model.

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