Construction of Lie Superalgebras from Triple Product Systems

Susumu Okubo

Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627

Abstract. Any simple Lie superalgebras over the complex field can be constructed from some triple systems. Examples of Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ are given by utilizing a general construction method based upon $(-1,-1)$ balanced Freudenthal-Kantor triple system.

1. LIE AND ANTI-LIE TRIPLE SYSTEMS

The triple products are perhaps a little unfamiliar in physics, although it has been utilized to find some solutions of Yang-Baxter equation [1] as well as of para-statistics [2]. Some other examples are also found in reference 3.

Before going into details, let us briefly sketch what a triple product is. Let $V$ be a vector space over a field $F$. Then, a bilinear product in $V$ is a linear map:

$V \otimes V \rightarrow V$

denoted as $xy = z$ for $x, y, z \in V$. If $e_1, e_2, \ldots, e_N$ is a basis of $V$, then its algebraic structure is completely determined by its multiplication table of

$$e_j e_k = \sum_{\ell=1}^{N} C_{jk}^{\ell} e_{\ell} \quad (j, k, \ell = 1, 2, \ldots, N) \quad (1.1)$$

for some structure constants $C_{jk}^{\ell} \in F$.

In contrast, a triple product defined in $V$ is a linear mapping

$$V \otimes V \otimes V \rightarrow V$$

and we write the triple product as $xyz$, $[x,y,z]$ or $x \cdot y \cdot z$ or any other symbol you would prefer. Then, the analogue of Eq. (1.1) is

$$[e_j, e_k, e_\ell] = \sum_{m=1}^{N} C_{jk\ell}^{m} e_{m} \quad (1.2)$$

for some structure constants $C_{jk\ell}^{m} \in F$, where we used the symbol of $[x,y,z]$ as the triple product here to be definite.

A simple example [4] of a triple system is obtained as follows. Let $<.>_{\ldots}$ be a bilinear form in $V$, satisfying a condition of
\[<y|x> = -\varepsilon <x|y>, \quad (x, y \in V) \quad (1.3)\]

\(\varepsilon = \pm 1\). We now introduce a triple product in \(V\) by

\[[x,y,z] := <x|z>y + \varepsilon <y|z>x.\quad (1.4)\]

It is easy to verify that it satisfies

(i) \([x,y,z] = \varepsilon [y,x,z]\) \quad (1.5a)

(ii) \([x,y,z] + [y,z,x] + [z,x,y] = 0\) \quad (1.5b)

(iii) \([u,v,[x,y,z]] = [[u,v,x],y,z] + [x,[u,v,y],z] + [x,y,[u,v,z]]\) \quad (1.5c)

for any \(u, v, x, y, z \in V\). We then say that any vector space \(V\) with a triple product \([x,y,z]\) satisfying Eqs. (1.5) is a Lie \([5]\) (for \(\varepsilon = -1\)) and an anti-Lie \([6]\) (for \(\varepsilon = +1\)) triple system, respectively.

As we will see shortly, Lie and anti-Lie triple systems are intimately related to Lie and Lie superalgebras, respectively.

### 2. CANONICAL CONSTRUCTION

It is well-known \([4]-[7]\) that we can construct Lie and Lie superalgebras, respectively from Lie and anti-Lie triple systems as follows.

We first introduce the Lie-multiplication operator \(L(.,.) : V \otimes V \rightarrow \text{End } V\) by

\[L(x,y)z := [x,y,z]. \quad (2.1)\]

We emphasize the fact that \(L(x,y) \in \text{End } V\) is a linear transformation operator in the vector space \(V\), so that they form an associative algebra in the ordinary sense. We note then that Eq. (1.5a) immediately gives

\[L(y,x) = \varepsilon L(x,y) \quad (2.2)\]

since for any \(z \in V\), we calculate

\[\{L(y,x) - \varepsilon L(x,y)\} z = [y,x,z] - \varepsilon [x,y,z] = 0\]

by Eq. (1.5a). Secondly, Eq. (1.5c) is rewritten as

\[[L(u,v),L(x,y)] = L([u,v,x],y) + L(x,[u,v,y])\quad (2.3)\]

where we have set

\[[L(u,v),L(x,y)] : = L(u,v)L(x,y) - L(x,y)L(u,v) \quad (2.4)\]

as the usual commutator. To see the validity of Eq. (2.3) we calculate...
\[ L(u,v)L(x,y)z = L(u,v)[x,y,z] = [u,v,[x,y,z]], \]
\[ L(x,y)L(u,v)z = L(x,y)[u,v,z] = [x,y,[u,v,z]], \]
\[ L([u,v,x],y)z = [[u,v,x],y], \]
\[ L(x,[u,v,y])z = [x,[u,v,y],z], \]

from the definition of \( L(x,y) \) acting on any \( z \in V \). Therefore, Eq. (1.5c) is rewritten as
\[
\{[L(u,v),L(x,y)] - L([u,v,x],y) - L(x,[u,v,y])\}z = 0 \tag{2.5}
\]
which leads to the validity of Eq. (2.4) since the linear transformation acting on \( z \in V \) in the left side of Eq. (2.5) is a null transformation. We then note that Eq. (2.4) gives a Lie algebra since \( L(u,v) \) and \( L(x,y) \) form an associative algebra.

So far, we did not utilize Eq. (1.5b). We consider a larger vector space
\[ W = L(V,V) \oplus V : = V_0 \oplus V_1 \tag{2.6} \]
where \( L(V,V) \) is a vector space consisting of all linear combination of \( L(x,y) \)'s \( x, y, \in V \). Note that the commutators such as \([x,y] \) and \([x,L(y,z)] \) are not defined in the theory. We can, nevertheless, introduce these commutators formally by relations,

(i) \([x,y] := L(x,y) = \varepsilon L(y,x) \tag{2.7a}\)
(ii) \([L(x,y),z] := -[z,L(x,y)] := [x,y,z]. \tag{2.7b}\)

We have now to consider two cases of \( \varepsilon = 1 \) and \( -1 \), separately. First, we discuss the Lie triple system with \( \varepsilon = -1 \). In that case, Eqs. (2.2) and (2.7a) give
\[ [x,y] = -[y,x]. \tag{2.8}\]
Then, \( W \) becomes a larger Lie algebra, i.e., we can prove to have
\[ [X,Y] = -[Y,X] \tag{2.9a}\]
\[ [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \tag{2.9b}\]
for any \( X, Y, Z \in W \), if we take Eq. (1.5b) into account.

On the contrary, the case of the anti-Lie triple system with \( \varepsilon = +1 \) leads to a Lie superalgebra as follows. First, this leads to
\[ [x,y] = [y,x] \tag{2.10}\]
instead of Eq. (2.8). Moreover, we note that
\[ V_0 = L(V,V), \quad L_1 = V \tag{2.11}\]
offers a $\mathbb{Z}_2$-graded space since

$$[V_0, V_0] \subseteq V_0, \quad [V_0, V_1] \subseteq V_0,$$

$$[V_1, V_1] \subseteq V_1 \tag{2.12}$$

by Eqs. (2.3) and (2.7). Then, we can introduce the grading function by

$$(−1)^X = \begin{cases} 
1, & \text{if } X = L(x, y) \in V_0 \\
-1, & \text{if } X = x \in V = V_1 \end{cases} \tag{2.13}$$

In this case, the resulting algebra is a Lie superalgebra satisfying \([8, 9]\)

(i) $[X, Y] = -(-1)^{XY}[Y, X]$ \hspace{1cm} \tag{2.14a}

(ii) $(-1)^{XZ}[[X, Y], Z] + (-1)^{YX}[[Y, Z], X] + (-1)^{ZY}[[Z, X], Y] = 0$ \hspace{1cm} \tag{2.14b}

instead of Eqs. (2.9).

It may be instructive to inspect the example of Eq. (1.4) for Lie and Lie supertriple systems. Consider first the case of $\varepsilon = -1$. Let $e_1, e_2, \ldots, e_N$ ($N = \dim V$) be a basis of $V$

$< e_j | e_k > = \delta_{jk}, \quad (j, k = 1, 2, \ldots, N). \tag{2.15}$

Then, setting

$$J_{jk} = -J_{kj} = L(e_j, e_k), \tag{2.16}$$

Eqs. (2.1) and (1.4) give

$$J_{jk}e_\ell = \delta_{j\ell}e_k - \delta_{k\ell}e_j \tag{2.17}$$

and Eq. (2.3) leads to the $so(N)$ Lie algebra of

$$[J_{jk}, J_{\ell m}] = \delta_{j\ell}J_{km} - \delta_{k\ell}J_{jm} + \delta_{jm}J_{k\ell} - \delta_{km}J_{j\ell} \tag{2.18}$$

since we calculate

$$[J_{jk}, J_{\ell m}] = [L(e_j, e_k), L(e_\ell, e_m)]$$

$$= L([e_j, e_k, e_\ell], e_m) + L(e_\ell, [e_j, e_k, e_m])$$

$$= L(\delta_{j\ell}e_k - \delta_{k\ell}e_j, e_m) + L(e_\ell, \delta_{jm}e_k - \delta_{km}e_j)$$

$$= \delta_{j\ell}L(e_k, e_m) - \delta_{k\ell}L(e_j, e_m) + \delta_{jm}L(e_\ell, e_k) - \delta_{km}L(e_\ell, e_j).$$

On the other side, Eqs. (2.7) give
\[ [e_j, e_k] = J_{jk}, \]  
\[ [J_{jk}, e_\ell] = -[e_\ell, J_{jk}] = \delta_{j\ell}e_k - \delta_{k\ell}e_j. \]  
\[ (2.19a) \]
\[ (2.19b) \]

Therefore introducing

\[ J_{0j} = -J_{j0} = e_j, \]
\[ J_{00} = 0, \]  
\[ (2.20) \]

for \( j = 1, 2, \ldots, N \), the relations of Eqs. (2.18) and (2.19) are combined into a single relation of

\[ [J_{AB}, J_{CD}] = \delta_{AC}J_{BD} - \delta_{BC}J_{AD} + \delta_{BD}J_{AC} - \delta_{AD}J_{AC}, \]  
\[ J_{AB} = -J_{BA} \]  
\[ (2.21a) \]
\[ (2.21b) \]

for \( A, B, C, D = 0, 1, 2, \ldots, N \). Thus, the larger Lie algebra \( W \) is now \( so(N + 1) \).

On the other case of \( \varepsilon = +1 \), the condition Eq. (1.3) must be modified as

\[ < e_j|e_k > = \varepsilon_{jk} = -\varepsilon_{k j} \]  
\[ (2.22) \]

for \( j, k = 1, 2, \ldots, N \), where \( \varepsilon_{jk} \) is the symplectic form \( (N = \text{even}) \). In that case, \( J_{jk} \) given by Eq. (2.16) now satisfies

\[ [J_{jk}, J_{\ell m}] = \varepsilon_{j\ell} J_{km} - \varepsilon_{k\ell} J_{jm} + \varepsilon_{jm} J_{\ell k} - \varepsilon_{km} J_{\ell j} \]  
\[ (2.23a) \]

with

\[ J_{jk} = J_{kj}, \]  
\[ (2.23b) \]

which is the symplectic Lie algebra \( sp(N) \). The larger vector space \( W \) now gives the Lie superalgebra \( \mathfrak{osp}(1,N) \), although we will not go into detail.

In this connection, we may note that if \( V \) is a super-space from the beginning satisfying

\[ < x|y > = (-1)^{xy} < y|x > \]  
\[ (2.24) \]

instead of Eq. (1.3), we could have obtained a more general Lie superalgebra \( \mathfrak{osp}(M,N) \). For details, see reference 2.

3. FREUDENTHAL-KANTOR TRIPLE SYSTEMS

It is known \[ [10, 11] \] that all simple Lie algebras over the complex field can be constructed from some suitable triple systems. Recently, we have shown \[ [12, 13] \] that all simple
Lie superalgebras over the complex field can also be constructed from triple systems. To this end, we must consider more general triple systems. As an example, we briefly sketch the notion of \((\varepsilon, \varepsilon)\) balanced Freudenthal-Kantor triple system (abbreviated as \((\varepsilon, \varepsilon)\) BFKTS). Let \(<x|y>\) to satisfy again Eq. (1.3), i.e.,

\[
<x|y> = -\varepsilon <y|x>
\]

for \(\varepsilon = \pm 1\). We write a triple product in \(V\) now as a juxtaposition \(xyz\). Suppose that it satisfies

\[
(i) \quad xyz - \varepsilon zyx = 2 <x|z>y
\]

\[
(ii) \quad xyz - \varepsilon yxz = 2 <x|y>z
\]

\[
(iii) \quad uv(xyz) = (uvx)yz + \varepsilon x(vuy)z + xy(uvz).
\]

Then, any vector space \(V\) with the triple product \(xyz\) satisfy Eqs. (3.2) is called a \((\varepsilon, \varepsilon)\) BFKTS.

A simple example \([4]\) is a triple product defined by

\[
xyz = <x|z>y - \varepsilon <x|y>z + \varepsilon <y|z>x.
\]

The reason why such a triple system is of interest is due to the fact that we can construct Lie and anti-Lie triple systems from them as follows. We consider a larger vector space \(W\) by

\[
W = V \oplus V.
\]

It is convenient to rewrite the generic element \(w = x \oplus y\) of \(W\) as

\[
w = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x, y \in V, \tag{3.5}
\]

and introduce a new triple product in \(W\) by

\[
\left[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] : = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.6}
\]

with

\[
w_1 : = x_1y_2z_1 - \varepsilon y_1x_2z_1 + 2\varepsilon <x_1|y_1>z_2,
\]

\[
w_2 : = \varepsilon y_2x_1z_2 - x_2y_1z_2 - 2\varepsilon <x_2|y_2>z_1. \tag{3.7}
\]

Then, as a special case of a theorem \([4]\) on a more general Freudenthal-Kantor triple system, \(W\) becomes a Lie triple system for \(\varepsilon = +1\), and an anti-Lie triple system for \(\varepsilon = -1\). Therefore, from any \((\varepsilon, \varepsilon)\) BFKTS, we can construct a Lie algebra for \(\varepsilon = +1\) and a Lie superalgebra for \(\varepsilon = -1\) by the canonical construction explained in section 2.
Note that we have to let $\varepsilon \to -\varepsilon$ in Eqs. (1.5) in order to now use the same symbols for Lie and anti-Lie triple system.

Since we are interested in Lie superalgebras, we will consider only the case of $(-1, -1)$ BFKTS hereafter. Setting $\varepsilon = -1$ in Eq. (3.3), the resulting $(-1, -1)$ BFKTS together with the canonical construction will give a Lie superalgebra $osp(N, 2)$ for $N = \dim V$. In order to obtain more interesting Lie superalgebras, we will consider the following examples in which the construction [12] of exceptional Lie superalgebras $D(2, 1; \alpha), G(3)$ and $F(4)$ are based.

**Example 1**

Let

$$V = \{e_1, e_2, e_3, e_4\}$$

with

$$\langle e_j | e_k \rangle = \delta_{jk}, \quad (j, k = 1, 2, 3, 4).$$

For an arbitrary constant $\sigma \in F$, a triple product defined by

$$e_j e_k e_\ell := \sigma \sum_{m=1}^{4} \epsilon_{jk\ell m} e_m - \delta_{k\ell} e_j + \delta_{j\ell} e_k + \delta_{jk} e_\ell$$

gives a $(-1, -1)$ BFKTS. The resulting Lie superalgebra is then $D(2, 1; \alpha)$ with

$$\alpha = \frac{1 - \sigma}{1 + \sigma}.$$  

**Example 2**

Let $x \cdot y$ be an octonionic product in the octonion algebra with

$$\bar{x} = 2 < e|x > - e - x.$$  

Then, a triple product given by

$$xyz := \frac{1}{3} (x \cdot \bar{y}) \cdot z - \frac{4}{3} < y | z > x + \frac{4}{3} < x | z > y - \frac{2}{3} < x | y > z$$

defines a $(-1, -1)$ BFKTS. The corresponding Lie superalgebra is the exceptional one $F(4)$ in the Kac’s notation.

We can also construct an equivalent $(-1, -1)$ BFKTS in terms of the 7-dimensional Dirac-Clifford algebra.

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \quad (\mu, \nu = 1, 2, \ldots, 7).$$

The multiplication table for the resulting $F(4)$ essentially gives the same one given by Frappat et al., [14] as has been explained in reference 12.
Example 3

Let \( x \cdot y \) again be the octonionic product but we restrict ourselves to a 7-dimensional sub-space

\[ V = \{ x \mid x = \text{octonion, with } < e|x> = 0 \}, \tag{3.15} \]

and set

\[ xyz := -\frac{1}{4} \{(x \cdot y) \cdot z - x \cdot (y \cdot z)\} - < y|z > x + < x|z > y + < x|y > z \]

which defines again a \((-1, -1)\) BFKTS. The resulting Lie superalgebra is \( G(3) \).

We can construct\(^{13}\) also the strange Lie superalgebras \( P(n) \), and \( Q(n) \) as well as the Cartan-type Lie superalgebras \( W(n) \) etc. from some other types of triple system. However we will not go into detail.

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