LARGE GAPS BETWEEN THE ZEROS OF THE Riemann
Zeta Function

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Abstract. We show that the generalized Riemann hypothesis implies that there are infinitely many consecutive zeros of the zeta function whose spacing is 2.9125 times larger than the average spacing. This is deduced from the calculation of the second moment of the Riemann zeta function multiplied by a Dirichlet polynomial averaged over the zeros of the zeta function.

1. Introduction

If the Riemann hypothesis (RH) is true then the non-trivial zeros of the Riemann zeta function, \( \zeta(s) \), satisfy \( \frac{1}{2} + it_n \) with \( t_n \in \mathbb{R} \). Riemann noted that the argument principle implies that number of zeros of \( \zeta(s) \) in the box with vertices 0, 1, 1 + iT, and iT is \( N(T) \sim (T/2\pi) \log(T/2\pi) \). This implies that on average \( (\gamma_{n+1} - \gamma_n) \approx 2\pi/\log \gamma_n \) and hence the average spacing of the sequence \( \hat{\gamma}_n = \gamma_n \log \gamma_n / 2\pi \) is one. Montgomery [9] investigated the pair correlation of these numbers and he proposed the fundamental conjecture

\[
\frac{1}{N} \# \{1 \leq j \neq k \leq N \mid a \leq \hat{\gamma}_j - \hat{\gamma}_k \leq b \} \sim \int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx \tag{1}
\]

for \( 0 < a < b \) as \( N \to \infty \). Moreover, it is expected that the consecutive spacings, \( \hat{\gamma}_{n+1} - \hat{\gamma}_n \), have a limiting distribution function which agrees with the Gaussian Unitary Ensemble from random matrix theory. See Odlyzko [13] for extensive numerical evidence in favour of this conjecture and also see Rudnick-Sarnak [14] for a study of the n-level correlations of \( \hat{\gamma}_n \). In light of the expected distribution of the consecutive spacings of zeta Montgomery suggested in [9] that there exist arbitrarily large and small gaps between the zeros of the zeta function. That is to say

\[
\lambda = \limsup_{n \to \infty} (\hat{\gamma}_{n+1} - \hat{\gamma}_n) = \infty \quad \text{and} \quad \mu = \liminf_{n \to \infty} (\hat{\gamma}_{n+1} - \hat{\gamma}_n) = 0.
\]

In this article, we focus on the large gaps and we assume the generalized Riemann hypothesis (GRH) is true. This conjecture states that the non-trivial zeros of the Dirichlet L-functions are on the \( \text{Re}(s) = 1/2 \) line. We establish

Theorem 1. The generalized Riemann hypothesis implies \( \lambda > 2.9125 \).
Selberg was the first to establish that $\lambda > 1$ based on his work concerning moments of $S(t) = (1/\pi) \arg \zeta(1/2+it)$ in short intervals. Montgomery and Odlyzko [10] obtained $\lambda > 1.9799$ assuming the Riemann hypothesis. The current record due to Hall is $\lambda > 2.34$. Hall’s work makes use of Wirtinger’s inequality in conjunction with asymptotic formulae for continuous mixed moments of the zeta function and its derivatives. Moreover, Hall is currently attempting to show that the asymptotic evaluation of all mixed moments of zeta and its derivatives yields $\lambda = \infty$. It should be noted that the best published result [2] assuming the Riemann hypothesis is worse than Hall’s unconditional work. Theorem 1 extends earlier results of Conrey, Ghosh, and Gonek where they assume GRH to obtain $\lambda > 2.68$. In fact, their work is based on the following idea of J. Mueller [11]. Let $H : \mathbb{C} \to \mathbb{R}$ and consider the associated functions

\begin{align}
M_1(H, T) &= \int_1^T H(1/2 + it) \, dt, \\
m(H, T; \alpha) &= \sum_{T < \gamma < 2T} H(1/2 + i(\gamma + \alpha)), \\
M_2(H, T; c) &= \int_{-c/L}^{c/L} m(H, T; \alpha) \, d\alpha
\end{align}

where we put $L = \log(T/2\pi)$. This notation shall be used throughout the article. However, one notes that

\begin{equation}
\frac{M_2(H, 2T; c) - M_2(H, T; c)}{M_1(H, 2T) - M_1(H, T)} < 1
\end{equation}

implies $\lambda > \frac{T}{c}$. Mueller applied this idea with $H(s) = |\zeta(s)|^2$ and obtained $\lambda > 1.9$. We should note that the method of Montgomery and Odlyzko [10] is equivalent to the method of Mueller [11]. This was realized later by the authors of [2]. Now consider the Dirichlet polynomial

\begin{equation}
A(s) = \sum_{n \leq y} a(n)n^{-s}.
\end{equation}

Assuming the Riemann hypothesis, Conrey, Ghosh, and Gonek in [2] applied (5) to $H(s) = |A(s)|^2$ with $a(n) = d_{2\mu}(n)$, $y = T^{1-\epsilon}$ and obtained $\lambda > 2.337$ (and $\mu < 0.5172$). Here $d_{\mu}(n)$ is the coefficient of $n^{-s}$ in the Dirichlet series $\zeta(s)$. If $r$ is a natural number then $d_r(n)$ equals the number of representations of $n$ as a product of $r$ positive integers. In recent work [12], we have shown that the Riemann hypothesis implies $\lambda > 2.56$ (and $\mu < 0.5162$). In [3], Conrey, Ghosh, and Gonek applied (5) to $H(s) = |\zeta(s)A(s)|^2$ with $a(n) = 1$ and $y = (T/2\pi)^{2-\epsilon}$ and obtained $\lambda > 2.68$. However, in this situation it is necessary to assume GRH in order to evaluate the discrete mean value $m(H, T; \alpha)$. We continue this programme by considering a more general choice for the coefficient $a(n)$. Precisely, we choose as our function $H_r(s) = |\zeta(s)A(s)|^2$ where $A(s)$ has coefficients

\begin{equation}
a(n) = d_r(n) P \left( \frac{\log n}{\log y} \right)
\end{equation}

for $P$ a polynomial and for $r \in \mathbb{N}$. Furthermore, we put $y = (T/2\pi)^{2-\epsilon}$ and obtained $\lambda > 2.68$. However, we must take into account that $d_r$ is not a completely multiplicative function for $r \geq 2$. It should be noted that Chris
Hughes [8] has shown that if $H(s) = |\zeta(s)|^4$ is admissible then his random matrix theory conjectures yield $\lambda > 2.7$. In addition, he has shown (unpublished) that if $H(s) = |\zeta(s)|^k$ is admissible for arbitrarily large $k$ then the random matrix theory conjectures for $M_1(H, T)$ and $m(H, T; \alpha)$ yield $\lambda \geq f(k)$ where $f(k) \not\to \infty$ at a linear rate. By choosing $H_r(s) = |\zeta(s)A_r(s)|^2$ with coefficients $a(n) = d_r(n)$ we are hoping that $H_r(s)$ will mimic the larger moment $|\zeta(s)|^{2r+2}$. The work of [8] corresponds to the choice $r = 1$, $P(x) = 1$.

We now state the precise result. We define several functions that will appear in the course of the proof. Given a polynomial $P$ and $u \in \mathbb{Z}_{\geq 0}$ we define

$$Q_u(x) = \int_0^1 \theta^u P(x + \theta(1 - x)) \, d\theta .$$

(8)

Given $\vec{n} = (n_1, n_2, n_3, n_4, n_5) \in (\mathbb{Z}_{\geq 0})^5$ we define

$$i_P(\vec{n}) = \int_0^1 \int_0^{1-x} x^{r-1}(1-x)^{n_1}(1-y-x)^{n_2}y^{n_3}Q_{n_4}(x)Q_{n_5}(x+y)dydx .$$

(9)

For $\eta \in \mathbb{R}$ and $\vec{n} = (n_1, n_2, n_3) \in (\mathbb{Z}_{\geq 0})^3$ we define

$$k_P(\vec{n}) = k_P(\vec{n}; \eta) = \int_0^1 \int_0^{1-x} x^{r-1}(1-x)^{n_1}y^{n_2}Q_{n_3}(x+y)dydx .$$

(10)

Recall $\eta$ corresponds to the length of our Dirichlet polynomial. Given $r \geq 1$ we define the constants

$$a_r = \prod_p \left( 1 - p^{-1} \right)^2 \sum_{m=0}^\infty \left( \frac{\Gamma(r+m)}{\Gamma(r)m!} \right)^2 p^{-m} \quad \text{and} \quad C_r = \frac{a_{r+1}}{(r^2 - 1)!((r-1)!)^2} .$$

(11)

With all of these definitions in hand we present our result for $m(H_r, T; \alpha)$.

**Theorem 2.** Suppose $r \in \mathbb{N}$ and $\eta < 1/2$. GRH implies

$$m(H_r, T; \alpha) \sim \frac{C_rTL^{(r+1)^2+1}}{\pi} \Re \sum_{j=1}^\infty z^j \eta^{j+1+(r+1)^2+1} \left( \frac{\hat{i}(r, j, \eta)}{j!} + \hat{k}(r, j, \eta) \right)$$

(12)

where $z = i\alpha L$, $|z| \ll 1$,

$$\hat{i}(r, j, \eta) = -i_P(r, r, j, r-1, r-1, r-1)\eta^{-1} + i_P(r+1, r, j, r, r-1) + i_P(r, r+1, j, r-1, r) ,$$

(13)

$$\hat{k}(r, j, \eta) = -(r-1)! \sum_{n=-2}^{\min(j,r-2)} \frac{(-1)^n(j+r)}{(j-n)!(r+n+1)!}k_P(j-n, n, r+n+2, r+n+1) .$$

(14)

This result is valid up to an error term which is $O_{\epsilon, r}(T^{-1/2}(r+1)^2 + T^{1/2+\eta+r})$.

We note that it is probable that Theorem 2 can be proven only assuming the Generalized Lindelöf Hypothesis by following the work of Conrey, Ghosh, and Gonek [4] on simple zeros of $\zeta(s)$. Even this assumption may possibly be weakened further.
since the main theorem in [1] actually assumes an upper bound for the sixth integral moment of \( L(s, \chi) \) on average. Also we remark that the case \( r = 1, P(x) = 1 \) reduces, after some calculation, to

\[
m(H_1, T; \alpha) \sim \frac{6}{\pi^2} \frac{TL}{2\pi} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}(\alpha L)^{2j+2}}{(2j+5)!}.
\]

\[
\left( \frac{-3\eta^2 + (2j + 5)\eta^3}{3} \right) - \frac{2j + 5}{j + 3} \eta^{2j+6} + \eta^{2j+7} + \eta^2(1 - \eta)^{2j+5}
\]

which corresponds to Theorem 1 of [3].

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2. Theorem 2 implies Theorem 1

In this section, we deduce Theorem 1 from Theorem 2. The rest of the article will be devoted to establishing the discrete moment result of Theorem 1. Put \( \eta = 1/2 - \epsilon \) with \( \epsilon \) arbitrarily small. Since \( \text{Re}(z^j) = (-1)^k(\alpha L)^{2k} \) if \( j = 2k \) and zero otherwise, it follows from [12] that

\[
m(H_r, 2T; \alpha) - m(H_r, T; \alpha) = \phi(r, \eta, \alpha) \frac{C_rTL^{(r+1)^2+1}}{\pi} (1 + O(L^{-1}))
\]

where

\[
\phi(r, \eta, \alpha) = \eta^{(r+1)^2+1} \sum_{j=1}^{\infty} (-1)^j(\alpha L\eta)^{2j} \left( \frac{r^j(r, 2j, \eta)}{(2j+1)!} + \frac{\hat{k}(r, 2j, \eta)}{2j+1} \right).
\]

Integrating [16] with respect to \( \alpha \) over the interval \([-c/L, c/L]\) we have \( \mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c) \) equals

\[
\frac{2C_rTL^{(r+1)^2} \eta^{(r+1)^2+1}}{\pi} \sum_{j=1}^{\infty} (-1)^j e^{2j+1}\eta^{2j} \left( \frac{r^j(r, 2j, \eta)}{(2j+1)!} + \frac{\hat{k}(r, 2j, \eta)}{2j+1} \right)
\]

plus an error \( O(TL^{(r+1)^2}) \). In the above expression, we may replace \( \eta = 1/2 - \epsilon \) by \( 1/2 \) yielding

\[
\mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c) = \frac{2C_rTL^{(r+1)^2} \eta^{(r+1)^2+1}}{\pi}.
\]

\[
\sum_{j=1}^{\infty} \frac{(-1)^j e^{2j+1}}{2^{2j}} \left( \frac{r^j(r, 2j, \frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(r, 2j, \frac{1}{2})}{(2j+1)} \right) + O(\epsilon TL^{(r+1)^2})
\]

We now recall the following result of Conrey and Ghosh [1].

Lemma 1. If \( y = T^\eta \) with \( 0 < \eta < 1/2 \) then

\[
\mathcal{M}_1(H_r, T) \sim \frac{a_{r+1}}{(r - 1)!^2 (r^2 - 1)!} T(\log y)^{(r+1)^2}.
\]

\[
\int_0^1 \alpha^{r+1} (\eta^{-1}(1 - \alpha)^2 \eta^{-1} Q_{r-1}(\alpha)^2 - 2(1 - \alpha)^{2r+1} Q_r(\alpha) Q_{r-1}(\alpha)) d\alpha
\]

(17)
as $T \to \infty$. This is valid up to an error term which is $O(L^{-1})$ smaller than the main term.

Hence, we have

$$\mathcal{M}_1(H_r, 2T) - \mathcal{M}_1(H_r, T) = C_r T (L \eta)^{(r+1)^2}$$

$$\int_0^1 \alpha^{r^2-1} (\eta^{-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 - 2(1-\alpha)^{2r+1} Q_{r-1}(\alpha) Q_r(\alpha)) \, d\alpha + O(\epsilon T L^{(r+1)^2}) .$$

We deduce that

$$\frac{\mathcal{M}_2(H_r, 2T; c) - \mathcal{M}_2(H_r, T; c)}{\mathcal{M}_1(H_r, 2T) - \mathcal{M}_1(H_r, T)} = f_r(c) + O(\epsilon)$$

where

$$f_r(c) = \frac{1}{D} \sum_{j=1}^{\infty} \frac{(-1)^j e^{2j+1}}{2^{2j}} \left( \frac{r \hat{c}(2, 2j, \frac{1}{2})}{(2j + 1)!} + \hat{k}(r, 2j, \frac{1}{2}) \right)$$

and

$$D := \pi \int_0^1 \alpha^{r^2-1} (\eta^{-1}(1-\alpha)^{2r} Q_{r-1}(\alpha)^2 - 2(1-\alpha)^{2r+1} Q_{r-1}(\alpha) Q_r(\alpha)) \, d\alpha$$

We define $\lambda_r := \sup_{f_r(c) < 1} (c)$ and thus $\lambda \geq \frac{\lambda_r}{\pi}$. We may now compute for various choices of $r$ and $P(x)$. For example, we shall choose $c = 2.9125 \pi$, $r = 2$ and $P(x) = 1 - 0.1x + 100x^2 - 0.2x^3$. We compute the sum as follows: by a Maple calculation we have

$$D^{-1} \sum_{j=0}^{J} \frac{(-1)^j e^{2j+1}}{2^{2j}} \left( \frac{2 \hat{c}(2, 2j, \frac{1}{2})}{(2j + 1)!} + \hat{k}(2, 2j, \frac{1}{2}) \right) = 0.9999845837$$

for $J = 80$. On the other hand, we may bound the terms $j > J$. Since $|Q_u(x)| \leq ||P||_1$ we may establish the crude bound

$$|i P(\vec{n})| \leq \left| \frac{|P|^2}{(r^2 - 1)!} \right| \frac{(n_1 + n_3 + 1)!}{(n_1 + n_3 + r^2 + 1)!}$$

for $\vec{n} \in (\mathbb{Z}_{\geq 0})^5$. It thus follows that

$$|\hat{c}(r, 2j, 1/2)| \leq \frac{|P|^2}{2j + 1} \left( \frac{4(r + 2j + 1)!}{(r^2 + r + 2j + 1)!} \right)$$

and hence

$$\left| \frac{1}{D} \sum_{j>J}^{\infty} \frac{(-1)^j e^{2j+1}}{2^{2j}} \frac{2 \hat{c}(2, 2j, \frac{1}{2})}{(2j + 1)!} \right| \leq \frac{48c ||P||_1^2}{D(2J)} \sum_{j>J}^{(c/2)^2j(2j + 3)!} \frac{(c/2)^2j(2j + 7)!}{(2j + 1)!(2j + 7)!}$$

$$\leq \frac{48c ||P||_1^2}{\sqrt{2\pi D(2J)^{3/2}}} \sum_{j>J} e^{-2j(\log(2J) - (\log(c/2) + 1))}$$

$$< \frac{48c ||P||_1^2}{\sqrt{2\pi D(2J)^{3/2}}} e^{-2J(\log(2J) - \log(c/2) - 1)} < 10^{-184}$$
where we have applied \( n! > (n/e)^n \). A similar calculation establishes that

\[
\left| \frac{1}{D} \sum_{j > J} \frac{(-1)^j e^{2j+1}}{2^{2j}} \frac{k(2, 2j, \frac{1}{2})}{(2j+1)} \right| < 10^{-130}.
\]

We conclude that \( f_2(2.1925\pi) < 1 \) and hence establish Theorem 1. We made our choice of \( r \) and \( P(x) \) by a computer search. We note that there are many choices of \( r \) and \( P(x) \) that improve the work of [3]. For example, \( r = 3, P(x) = 1 \) yields \( \lambda > 2.78 \) whereas \( r = 2, P(x) = 1 \) yields \( \lambda > 2.86 \).

3. Some notation and definitions

Throughout this article we shall employ the notation

\[
[t]_y := \frac{\log t}{\log y}
\]

for \( t, y > 0 \). This will allow us to write several equations more compactly. In addition, we shall encounter a variety of arithmetic functions. We define \( j_\tau(n) \), \( \Lambda(n) \), and \( d_r(n) \) as follows:

\[
j_\tau(n) = \prod_{p \mid n} (1 + O(p^{-r}))
\]

for \( \tau > 0 \) and the constant in the \( O \) is fixed and independent of \( \tau \). Next \( \Lambda(n) \) and \( d_r(n) \) may be defined by their Dirichlet series generating functions:

\[
-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \text{ and } \zeta(s)^r = \sum_{n=1}^{\infty} \frac{d_r(n)}{n^s}.
\]

Since this article concerns the calculation of discrete mean values of \( m(H_r, T, \alpha) \) we need to invoke several properties of \( d_r \). Throughout this article we apply repeatedly the following facts concerning \( d_r \):

\[
\sum_{a=0}^{\infty} d_r(p^a)p^{-as} = (1 - p^{-s})^{-r},
\]

\[
\sum_{m \leq x} d_r(m) m^{-1} \ll \log^r x, \text{ and}
\]

\[
\sum_{m \leq x} d_r(m)^2 m^{-1} \ll \log^{r^2} x.
\]

In hindsight, we realize that there is nothing really special about the multiplicative function \( d_r \) and that the calculation of this article can be done for more general multiplicative functions \( f \) subject to certain simple assumptions.

4. Initial manipulations

In this section we set up the plan of attack for our evaluation of \( m(H_r, T; \alpha) \). Recall that \( T \) is large, \( L = \log(T/2\pi) \), and \( \epsilon \) can be made arbitrarily small. Let \( R \) denote the positively oriented contour with vertices \( a+i, a+i(T+\alpha), 1-a+i(T+\alpha), 1-a+i \), the top edge of which has a small semicircular indentation centred at
1/2 + i(T + α) opening downward and a = 1 + O(L^{-1}). By an application of Cauchy’s residue theorem, the reflection principle, and RH we have

\[ m(H_r, T; α) = \frac{1}{2\pi i} \int \zeta'(s - iα)ζ(s)ζ(1 - s)A(s)A(1 - s) ds. \]

For s in the interior or boundary of R we have \( A(s) \ll y^{1 - σ + ε} \) and \( ζ(s) \ll T^{1/2(1 - σ) + ε}. \) The first bound is elementary and the second is the convexity bound. These combine to give \( ζ(s)ζ(1 - s)A(s)A(1 - s) \ll yT^{1/2 + ε}. \) Now choose \( T' \) such that \( T - 2 < T' < T - 1 \) such that \( T' + α \) is not the ordinate of a zero of \( ζ(s) \) and \( (ζ'/ζ)(σ + iT') \ll L^2, \) uniformly for \(-1 ≤ σ ≤ 2.\) A simple argument using Cauchy’s residue theorem establishes that the top edge of the contour is \( yT^{1/2 + ε}.\) Similarly, the bottom edge of the contour is \( \ll yT^ε \) since \( |ζ(s)| \ll 1 \) for \(|s| ≤ 1\) and \(|s - 1| \gg 1.\) Differentiating the functional equation, \( ζ(1 - s) = χ(1 - s)ζ(s), \) we have

\[ \frac{ζ'}{ζ}(1 - s - iα) = \frac{ζ'}{χ}(1 - s - iα) - \frac{ζ'}{ζ}(s + iα). \] (22)

where \( χ(s) = 2π^{s-1} \sin(πs/2)Γ(1 - s). \) Now the right edge is

\[ I = \frac{1}{2\pi i} \int_{a + i}^{a + i(T + α)} \frac{ζ'}{ζ}(s - iα)ζ(s)ζ(1 - s)A(s)A(1 - s) ds \] (23)

and the left edge is by (22)

\[ \frac{1}{2\pi i} \int_{1 - a + iT + α}^{1 - a + i} \frac{ζ'}{ζ}(s - iα)ζ(s)ζ(1 - s)A(s)A(1 - s) ds = \frac{1}{2\pi i} \int_{a - i(T + α)}^{a - i} \left( \frac{ζ'}{ζ}(s + iα) - \frac{ζ'}{χ}(1 - s - iα) \right)ζ(s)ζ(1 - s)A(s)A(1 - s) ds = T - J \]

where \( J = \frac{1}{2\pi i} \int_{a + i}^{a + i(T + α)} \frac{ζ'}{χ}(1 - s + iα)ζ(s)(1 - s)A(s)A(1 - s) ds. \) (24)

Combining results we obtain

\[ m(H_r, T; α) = 2\text{Re}I - J + O_ε(yT^{1/2 + ε}). \] (25)

We begin with the evaluation of \( J \) since it is rather simple. By Stirling’s formula one has \( (χ'/χ)(1 - s + iα) = -\log(t/2π) + O(t^{-1}) \) for \( t ≥ 1, 1/2 ≤ σ ≤ 2, \) and \(|α| ≤ cL^{-1}.\) By moving the contour to the 1/2 line in (24) and then substituting the previous estimate we obtain

\[ J = -\frac{1}{2π} \int_{1}^{T} (\log t/(2π)) |ζA(1/2 + it)|^2 dt + O \left( \int_{1}^{T} |ζA(1/2 + it)|^2 \frac{dt}{t} + yT^{1/2 + ε} \right). \]

The last term comes from the horizontal integral. An integration by parts shows that the second integral is \( L^{(r + 1)^2 + 1} \) and therefore

\[ J = -\frac{L}{2π} \mathcal{M}_1(H_r, T) + \int_{1}^{T} \mathcal{M}_1(H_r, t) \frac{dt}{t} + O(L^{(r + 1)^2 + 1} + yT^{1/2 + ε}). \]
Initially simplifications. By the functional equation (23) becomes

\[ B(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s) ds \]

where \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \). However, Lemma 2 on pp. 504-506 of [3] deals with such integrals.

**Lemma 2.** Suppose \( B(s) = \sum_{j \geq 1} b(j)j^{-s} \) and \( A(s) = \sum_{k \leq y} \frac{a(k)}{k^s} \) where \( a(j) \ll d_r(j)(\log j)^{r_1} \) and \( b(j) \ll d_r(j)(\log j)^{r_2} \) for some non-negative integers \( r_1, r_2, l_1, l_2 \) and \( T^\varepsilon \ll y \ll T \) for some \( \varepsilon > 0 \). If

\[ I = \int_{c+i\infty}^{c+i\infty} \chi(1-s)B(s)A(1-s) ds \]

then

\[ I = \sum_{k \leq y} \frac{a(k)}{k} \sum_{j \leq \frac{y}{k}} b(j)e(-j/k) + O(yT^{\frac{1}{2} + \varepsilon}) \]

We deduce that

\[ I = \sum_{k \leq y} \frac{d_r(k)P([\log y])}{k} \sum_{j \leq \frac{y}{k}} b(j)e(-j/k) + O(yT^{\frac{1}{2} + \varepsilon}) \]

The goal of the rest of this paper is to evaluate the sum in (28). We now give a brief sketch how the proof shall proceed. We define the Dirichlet series

\[ Q^*(s, \alpha, k) = \sum_{j=1}^{\infty} b(j)e(-j/k)j^{-s} \]

Now the inner sum in (28) can be written by Perron’s formula as

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Q^*(s, \alpha, k) \left( \frac{kT}{2\pi} \right)^s ds = M(k) + E(k) \]

with \( c > 1 \). We shall move this contour left to \( \text{Re}(s) = 1/2 + L^{-1} \) and we will have a main term, \( M(k) \), arising from the residues of \( Q^*(s, \alpha, k) \) at \( s = 1, s = 1 + i\alpha \). Moreover, the contribution from the contour on the line \( \text{Re}(s) = 1/2 + L^{-1} \) will be
an error term denoted $E(k)$. Next $M(k)$ will be reinserted in (28) and this will give the main term in the evaluation of $I$. The rest of the proof concentrates on the calculation of

$$I = \sum_{k \leq y} \frac{d_r(k)P([k]y)}{k} M(k)$$

(30)

and this part of the calculation will be somewhat complicated. However, it should be noted that the evaluation of (30) will not require GRH as it is essentially an elementary arithmetic sum.

We now explain the connection to the Generalized Riemann Hypothesis and how it will be invoked in the argument. Note that the additive character $e(-j/k)$ may be written in terms of multiplicative characters. In particular, if $(j, k) = 1$ we have the nice formula

$$e(-j/k) = \frac{1}{\phi(k)} \sum_{\chi \text{ (mod } k)} \chi(-j)\tau(\chi).$$

(31)

By this identity we shall decompose $Q^*(s, \alpha, k)$ into combinations of $L(s, \chi)$ and its logarithmic derivative where $\chi$ is a character mod $l$ for $l \mid k$. Now by assuming GRH we guarantee that $Q^*(s, \alpha, k)$ has only the poles at $s = 1, 1 + i\alpha$. If GRH were false then there would be extra poles occurring at those zeros that violate GRH. This obviously would complicate the argument. Secondly, we require a Lindelöf type bound for $L(s, \chi)$ and $(L'/L)(s, \chi)$ in order to ensure that the error term $E(k)$ in (29) is small. Finally, we mention that many of the technicalities in evaluating (30) arise from the fact that (42) only holds for $(j, k) = 1$.

5. Lemmas

In this section we present the lemmas that will be required for the bounding the contribution coming from the error terms, $E(k)$, and for evaluating the main term (30). The next lemma is useful for analyzing Dirichlet series that are products of several other Dirichlet series.

Lemma 3. Suppose that $A_j(s) = \sum_{n=1}^{\infty} \alpha_j(n)n^{-s}$ is absolutely convergent for $\sigma > 1$, for $1 \leq j \leq J$, and that

$$A(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \prod_{j=1}^{J} A_j(s).$$

Then for any positive integer $d$,

$$\sum_{n=1}^{\infty} \frac{\alpha(dn)}{n^s} = \sum_{d_1 \cdots d_J = d} \prod_{j=1}^{J} \left( \sum_{n=1}^{\infty} \frac{\alpha_j(nd_j)}{n^s} \right)$$

where $P_j = \prod_{i<j} d_i$.

This is Lemma 3 of [4] pp.506.

In Lemmas 4 and 5 we consider two Dirichlet series, $D(s, h/k)$ and $Q(s, \alpha, h/k)$ which arise in the analysis of $Q^*(s, \alpha, k)$.
Lemma 4. For \((h, k) = 1 \text{ with } k > 0\) we define
\[
D(s, h/k) = \sum_{n=1}^{\infty} d(n)n^{-s}e(nh/k) \quad (\sigma > 1) .
\]
Then \(D(s, h/k)\) is regular in the entire complex plane except for a double pole at \(s = 1\). Moreover, it has the same meromorphic part as \(k^{1-2\sigma} \zeta^2(s)\).

This is proven in Estermann [5] pp.124-126.

Lemma 5. Let \((h, k) = 1\) and \(k = \prod p^\lambda > 0\). For \(\alpha \in \mathbb{R}\) and \(\sigma > 1\) define
\[
Q(s, \alpha, h/k) = -\sum_{m,n=1}^{\infty} d(m)\Lambda(n)e\left(-\frac{mnh}{k}\right) .
\]
Then \(Q(s, \alpha, h/k)\) has a meromorphic continuation to the entire complex plane. If \(\alpha \neq 0\), \(Q(s, \alpha, h/k)\) has

(i) at most a double pole at \(s = 1\) with same principal part as
\[
k^{1-2\sigma} \zeta^2(s) \left(\frac{\zeta'(s - i\alpha) - G(s, \alpha, k)}{\zeta(s - i\alpha)}\right) ,
\]

where
\[
G(s, \alpha, k) = \sum_{p|k} \log p \left(\sum_{a=1}^{\lambda-1} p^{\alpha(s-1+i\alpha)} + \frac{p^{\lambda(s-1+i\alpha)}}{1-p^{-s+i\alpha}} - \frac{1}{p^{s-i\alpha} - 1}\right) ;
\]

(ii) a simple pole at \(s = 1 + i\alpha\) with residue
\[
-\frac{1}{k^{1+\alpha} \phi(k)} \zeta^2(1 + i\alpha) R_k(1 + i\alpha) .
\]

where
\[
R_k(s) = \prod_{p^\lambda|k} (1 - p^{-1} + \lambda(1-p^{-s})(1-p^{s-1})) .
\]

Moreover, on GRH, \(Q(s, \alpha, h/k)\) is regular in \(\sigma > 1/2\) except for these two poles.

This is Lemma 5 of [3] pp.217-218.

In the proof of Lemma 5 of [3], the generating function \(Q(s, \alpha, h/k)\) is written as a linear combination of \((L'/L)(s, \chi)\) where \(L(s, \chi)\) is a Dirichlet L-function modulo \(k\). These L-functions contribute the pole at \(s = 1 + i\alpha\). Moreover, \(Q(s, \alpha, h/k)\) is regular for \(\sigma > 1/2\) since \((L'/L)(s, \chi)\) is regular in this region assuming GRH.

For an arbitrary variable \(x\) we define the following generating function for \(d_r\)
\[
T_r(x, \lambda) = \sum_{j \geq \lambda} d_r(p^j)x^j .
\]

Lemma 6. For \(r, \lambda \in \mathbb{N}\) and \(x\) an indeterminate we have
\[
(1 - x)^r T_r(x, \lambda) = \lambda d_r(p^\lambda) \int_0^x t^{\lambda-1}(1-t)^{r-1} dt .
\]
We define for \( \lambda, r \in \mathbb{N} \) the polynomial

\[
H_{\lambda,r}(x) := \lambda x^{-\lambda} \int_0^x t^{\lambda-1} (1-t)^{r-1} \, dt. \tag{39}
\]

Note that \( H_{\lambda,r}(x) \) is a degree \( r \) polynomial and \( H_{\lambda,r}(0) = 1 \). Consequently, the lemma may be rewritten as

\[
(1 - x)^r T_r(x, \lambda) = d_r(p^\lambda) x^\lambda H_{\lambda,r}(x).
\]

**Proof.** Define the generating functions

\[
A(x, y) := \sum_{\lambda=1}^{\infty} (1-x)^r T_r(x, \lambda) y^\lambda,
\]

\[
B(x, y) := \sum_{\lambda=1}^{\infty} \left( \lambda d_r(p^\lambda) \int_0^x t^{\lambda-1} (1-t)^{r-1} \, dt \right) y^\lambda.
\]

We will show that these generating functions are equal and hence we have (38).

Note that

\[
A(x, y) = (1-x)^r \sum_{j=1}^{\infty} d_r(p^j)x^j \sum_{\lambda=1}^{\infty} y^\lambda = \frac{y(1-x)^r}{y-1} \sum_{j=1}^{\infty} d_r(p^j)x^j(y^j - 1)
\]

and since \( \lambda d_r(p^\lambda) = rd_{r+1}(p^{\lambda-1}) \) for \( \lambda \geq 1 \)

\[
B(x, y) = r \int_0^x (1-t)^{r-1} \left( \sum_{\lambda=1}^{\infty} d_{r+1}(p^{\lambda-1}) t^{\lambda-1} y^\lambda \right) \, dt = ry \int_0^x \frac{(1-t)^{r-1}}{(1-ty)^{r+1}} \, dt.
\]

A calculation shows that \( A_x(x, y) = B_x(x, y) = \frac{ry(1-x)^{r-1}}{(1-xy)^{r+1}} \) and since \( A(0, y) = B(0, y) = 0 \) it follows that \( A(x, y) = B(x, y) \).

Our calculations require Perron’s formula.

**Lemma 7.** Let \( F(s) := \sum_{n \geq 1} a_n n^{-s} \) be a Dirichlet series with finite abscissa of absolute convergence \( \sigma_a \). Suppose there exists a real number \( \alpha \geq 0 \) such that

\[
\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha} (\sigma > \sigma_a)
\]

and that \( B \) is a non-decreasing function such that \( |a_n| \leq B(n) \) for \( n \geq 1 \). Then for \( x \geq 2, T \geq 2, \sigma \leq \sigma_a, \kappa := \sigma_a - \sigma + (\log x)^{-1} \), we have

\[
\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F(s+w) \frac{x^w}{w} \, dw + O \left( \frac{x^{\sigma_a - \sigma} (\log x)^{\alpha}}{T} + \frac{B(2x)}{x^\sigma} \left( 1 + x \frac{\log T}{T} \right) \right). \tag{40}
\]

This is Corollary 2.1 p.133 of [10].

The following Lemma is another place where GRH is invoked. This lemma gives bounds for \( Q^*(s, \alpha, k) \) in the critical strip. These bounds are required for estimating the left side of the contour in (29). In fact, GRH shall be invoked in the form of a Lindelöf type bound for Dirichlet \( L \)-functions.
Lemma 8. Assume GRH. Let \( y = (T/2\pi)^\eta \) where \( 0 < \eta < 1/2 \), \( k \in \mathbb{N} \) with \( k \leq y \), and \( \alpha \in \mathbb{R} \). Set

\[
Q^*(s, \alpha, k) = \sum_{j=1}^\infty b(j) j^{-s} e(-j/k) \quad (\sigma > 1) ,
\]

where

\[
b(j) = - \sum_{\substack{hmn=j \\ h \leq y}} d_r(h) P([h]_y) d(m) \Lambda(n)n^{\sigma} .
\]

Then \( Q^*(s, \alpha, k) \) has an analytic continuation to \( \sigma > 1/2 \) except possible poles at \( s = 1 \) and \( 1 + i\alpha \). Furthermore,

\[
Q^*(s, \alpha, k) = O_{\epsilon, P}(y^{\frac{1}{2}}\Gamma^\epsilon)
\]

where \( s = \sigma +it, \frac{1}{2}+L^{-1} \leq \sigma \leq 1+L^{-1}, |t| \leq T, |s-1| > 0.1, \) and \( |s-1-i\alpha| > 0.1 \). Note that the constant in the big-\( O \) depends on the polynomial \( P \).

Proof. If \( \chi \) is a character mod \( k \), its Gauss sum is \( \tau(\chi) = \sum_{h=1}^k \chi(h) e(h/k) \) from which it follows that

\[
e(-j/k) = \sum_{d|j} d^\sigma \sum_{\chi \mod \frac{\phi(k)}{d}} \tau(\chi) \chi(-j/d) .
\]

By inserting (12) in (11) we obtain

\[
Q^*(s, \alpha, k) = \sum_{d|k} \frac{1}{\phi(k/d)} d^s \sum_{\chi \mod \frac{\phi(k)}{d}} \tau(\chi) \chi(-d) B(s, d) \]

where for \( \sigma > 1 \), \( B(s, d) = \sum_{j=1}^\infty b(jd) \chi(jd) j^{-s} \). We now write \( P(x) = \sum_{i=0}^N c_i x^i \) and hence we obtain

\[
Q^*(s, \alpha, k) = \sum_{i=0}^N \frac{c_i}{(\log y)^i} Q_i^*(s, \alpha, k)
\]

where

\[
Q_i^*(s, \alpha, k) = \sum_{d|k} \frac{1}{\phi(k/d)} d^s \sum_{\chi \mod \frac{\phi(k)}{d}} \tau(\chi) \chi(-d) \frac{\partial^i}{\partial z^i} B(s, d; z) \bigg|_{z=0} ,
\]

\[
B(s, d; z) = \sum_{j=1}^\infty b_z(dj) \chi(dj) j^{-s} , \text{ and } b_z(j) = \sum_{\substack{hmn=j \\ h \leq y}} d_r(h) h^s d(m) \Lambda(n)n^{\sigma} .
\]

Since \( \chi \) is completely multiplicative we note that

\[
B(s, 1; z) = \left( \sum_{h \leq y} \frac{\chi(h)d_r(h)h^s}{h^s} \right) L(s, \chi) \left( \sum_{n \geq 1} \frac{\chi(n)\Lambda(n)}{n^{s-\sigma}} \right) .
\]

An application of Lemma 3 implies

\[
B(s, d; z) = \sum_{f_1f_2f_3} A_1(s, f_1; z) A_2(s, f_2, f_1) A_2(s, f_3, f_1f_2) A_3(s, f_4, f_1f_2f_3)
\]

(46)
where
\[ A_1(s, f; z) = \chi(f) \sum_{h \leq y/f} \frac{\chi(h)d_r(fh)(fh)^z}{h^s}, \]
\[ A_2(s, f, r) = \sum_{(n, r)=1} \frac{\chi(fn)}{n^s} = \chi(f)L(s, \chi) \prod_{p \mid r} (1 - \chi(p)p^{-s}), \tag{47} \]
\[ A_3(s, f, r) = -\sum_{(n, r)=1} \chi(fn)\Lambda(fn)(fn)^{i\alpha}n^{-s}. \]

We are aiming to show that uniformly for \(|z| \leq 0.1L^{-1}\)
\[ B(s, d; z) \ll_{\varepsilon} \begin{cases} y^{1/2} T^\varepsilon & \text{if } \chi \text{ is principal} \\ T^\varepsilon & \text{otherwise} \end{cases} \tag{48} \]
in the region \(\sigma \geq 1/2 + L^{-1}, |t| \leq T, \text{ and } |s-1|, |s-1-\alpha| > 0.1. \) If \(18\) holds then we have by applying the Cauchy integral formula with a circle of radius \(0.1L^{-1}\) that
\[ \frac{\partial^i}{\partial z^i} B(s, d; z)|_{z=0} \ll \begin{cases} y^{1/2} T^\varepsilon & \text{if } \chi \text{ is principal} \\ T^\varepsilon & \text{otherwise} \end{cases}. \tag{49} \]

By \(18\) and \(18\)
\[ Q^*_1(s, \alpha, k) \ll_{\varepsilon} T^\varepsilon \sum_{d \mid k} \frac{1}{\phi(k/d)d^{1/2}} \left( y^{1/2} |\tau(\chi_0)| + \sum_{\chi \neq \chi_0 (\text{mod } k/d)} |\tau(\chi)| \right). \]

Since
\[ \tau(\chi) \ll \begin{cases} (k/d)^{1/2} & \chi \neq \chi_0 (\text{mod } k/d) \\ 1 & \chi = \chi_0 (\text{mod } k/d) \end{cases} \]
it follows that
\[ Q^*_1(s, \alpha, k) \ll T^\varepsilon \left( \frac{y/k}{1/2} \sum_{d \mid k} d^{1/2} \phi(d)^{-1} + k^{1/2} \sum_{d \mid k} d^{-1} \right) \ll y^{1/2} T^\varepsilon \]
and hence by \(18\) the desired bound \(Q^*(s, \alpha, k) \ll_{\varepsilon, \psi} y^{1/2+\varepsilon} T^{\varepsilon} \) follows. It now suffices to establish \(18\). If \(\chi\) is principal (mod \(k/d\)) then
\[ A_1(s, f; z) \ll f^{1/2} \sum_{n \leq y/f} n^{-1/2} \ll y^{1/2}. \tag{50} \]

Now suppose \(\chi\) is non-principal. If \(y/f \ll y^\varepsilon\), we have trivially that \(|A_1(s, f)| \ll y^\varepsilon\). If we suppose \(y/f \gg y^\varepsilon\) then by Perron’s formula (Lemma 7)
\[ A_1(s, f; z) = \frac{\chi(f)f^z}{2\pi i} \int_{\kappa-2iT}^{\kappa+2iT} G(s + z + w) \frac{(y/f)^w}{w} dw + O(1) \tag{51} \]
for \(\sigma \geq 1/2 + L^{-1}, |t| \leq T, \kappa = 1 - \sigma + 2L^{-1}\) where \(G(w) = \sum_{n=1}^\infty d_r(fn) \chi(n)n^{-w}\). By multiplicativity we have
\[ G(w) = L(w, \chi)^r \prod_{p^r || f} \left( \frac{\sum_{a=0}^\infty \chi(p^a)d_r(p^{r+a})p^{-aw}}{\sum_{a=0}^\infty \chi(p^a)d_r(p^{r+a})p^{-aw}} \right). \tag{52} \]

By Lemma 6, it follows that
\[ G(w) = d_r(f)L(w, \chi)^r \prod_{p^r || f} H_{\lambda, r}(x_p). \tag{53} \]
with \( x_p = \chi(p)p^{-s} \). We have that \( |x_p| \leq p^{-\sigma} \) and since \( H_{\lambda,r}(0) = 1 \) it follows that
\[
\left| \prod_{p^r \mid |f|} H_{\lambda,r}(x_p) \right| \ll \prod_{p \mid f} (1 + O(p^{-1/2})) \ll f^\epsilon.
\]
In addition, GRH implies \( |L(w, \chi)| \ll (1 + |t|)^{\epsilon} (k/d)^{\epsilon} \) for \( \text{Re}(w) \geq 1/2 \) and any \( \epsilon > 0 \). We now move the contour in (51) to \( \text{Re}(w) = \kappa' \) line where \( \kappa' = 1/2 - \sigma + 2L^{-1} \) and have
\[
\mathcal{A}_1(s, f; z) = \frac{\chi(f) f^z}{2\pi i} \int_{\kappa' - 2iT}^{\kappa' + 2iT} G(s + z + w) \frac{(y/f)^w}{w} \, dw + O(T^\epsilon)
\]
Since \( 0.5 \leq \text{Re}(s + z + w) \) and \( \text{Re}(w) \leq L^{-1} \) it follows that
\[
\mathcal{A}_1(s, f; z) \ll f^\epsilon T^{\epsilon} (k/d)^{\epsilon} (y/f)^L T^{-1} \int_{\kappa' - 2iT}^{\kappa' + 2iT} \frac{dw}{|w|} \ll T^\epsilon.
\]
(54)
For \( f \) and \( r \) dividing \( d \), we have
\[
\mathcal{A}_j(s, f, r) \ll T^\epsilon
\]
(55)
for \( j = 2, 3 \). This is proven in [3] pp.219-220. By (56) in combination with the bounds (50), (54), and (55) we obtain (58) which finishes the lemma.

The purpose of the next five lemmas is to provide a variety of formulae for mean values of certain multiplicative functions which arise in our asymptotic evaluation of \( I \) [28]. Lemma 9 provides bounds for certain divisor sums. Lemmas 10, 11, and 13 give asymptotic formulae for divisor and other divisor-like sums. Lemma 12 provides a formula for simple prime number sums.

**Lemma 9.** For \( \alpha \in \mathbb{R} \) and \( j \in \mathbb{Z}_{\geq 0} \) we have
\[
\mathcal{G}(j)(1, \alpha, k) = \sum_{p \mid k} p^{\alpha} (\log p)^{j+1} + O(C_j(k))
\]
(56)
where \( \mathcal{G}(s, \alpha, k) \) is defined by (74) and
\[
C_j(k) = \sum_{p \mid k} \frac{\log^j p}{p} + \sum_{p^\alpha \mid k, \alpha \geq 2} a \log^j p.
\]
(57)
Moreover, we have
\[
\sum_{h, k \leq x} \frac{d_r(h)d_r(k)(h)\log k}{hk} C_j \left( \frac{k}{(h, k)} \right) \ll (\log x)^{r^2 + r}.
\]
(58)

**Proof.** We remark that (60) is proven in [3] pp.222-223. The sum in (68) is bounded by
\[
\sum_{h, k \leq x} \frac{d_r(h)d_r(k)}{hk}(C_j(k) + 1) \sum_{a \mid h, a \mid k} \phi(a)
\]
\[
\leq \sum_{a \leq x} \frac{d_r(a)^2\phi(a)}{a^2} \sum_{h, k \leq \frac{x}{a}} \frac{d_r(h)d_r(k)(C_j(ak) + 1)}{hk}
\]
\[
\leq (\log x)^{2r} \sum_{a \leq x} \frac{d_r(a)^2(C_j(a) + 1)}{a} + (\log x)^{r} \sum_{a \leq x} \frac{d_r(a)^2}{a} \sum_{k \leq \frac{x}{a}} \frac{d_r(k)C_j(k)}{k}
\]
Observe that
\[ \sum_{a \leq y} \frac{d_r(a)^2 C_j(a)}{a} = \sum_{p \leq y} \log^2 p \sum_{u \leq \frac{y}{p}} \frac{d_r(up)^2}{up} + \sum_{p^a \leq y, a \geq 2} a \log p \sum_{u \leq \frac{y}{p^a}} \frac{d_r(up^a)^2}{up^a} \]
\[ \ll (\log x)^2 \left( \sum_p \frac{(\log p)^2}{p^2} \right) \ll (\log x)^2 \]

where we have applied (21). A similar argument establishes that \( \sum_{k \leq x} d_r(k) C_j(k) k^{-1} \ll (\log x)^{r^2 + r} \). Putting together the results establishes the lemma.

We now introduce the arithmetic function \( \sigma_r(m, s) \) where \( r \in \mathbb{N} \) and \( s \in \mathbb{C} \). It is defined by
\[ \sigma_r(m, s) := \left( \sum_{n=1}^{\infty} \frac{d_r(mn)}{n^s} \right) \zeta(s)^{-r} = \prod_{p \Vert m} (1 - p^{-s})^r p^\lambda s \sum_{j \geq \lambda} d_r(p^j) p^{js}. \] (59)

The second equation is obtained by mutiplicativity. By Lemma 6, it follows that
\[ \sigma_r(m, s) = \prod_{p \Vert m} d_r(p^\lambda) H_{\lambda, r}(p^{-s}) . \]
The value \( s = 1 \) will have a special importance so we set \( \sigma_r(m) := \sigma_r(m, 1) \). In the following calculations we shall often employ the bound
\[ |\sigma_r(m, s)| \ll d_r(m) j_r(m) \text{ for } \mathrm{Re}(s) \geq \tau > 0 \] (60)
The function \( \sigma_r \) is a correction factor that arises due to the fact \( d_r \) is not completely mutiplicative. More precisely, we notice in all cases of the following lemma that
\[ \sum_{h \leq t} d_r(mh) f(h) \sim \sigma_r(m) \sum_{h \leq t} d_r(h) f(h) \]
where \( f \) is a smooth function.

**Lemma 10.** Suppose \( r, n \in \mathbb{N}, 1 \leq x, \frac{x}{r \tau} \), and \( F \in C^1([0, 1]) \). There exists an absolute constant \( \tau_0 = \tau_0(r) \) such that
\[ \sum_{h \leq x} \frac{d_r(nh)}{h} F((h)_r) = \sigma_r(n)(\log x)^r \frac{\theta^{-1} F(\theta) \, d\theta}{(r-1)!} + O(d_r(n) j_{\tau_0}(n)) \] (61)

where \( j_{\tau_0}(n) \) is defined by (24). In order to abbreviate notation we define
\[ \epsilon(n) = d_r(n) j_{\tau_0}(n) . \] (62)

Suppose \( m, u, v \in \mathbb{N}, 1 \leq y, m \leq \frac{x}{r \tau}, \) a prime with \( p \leq \frac{x}{r \tau} \), and \( P \in C^1([0, 1]) \). We now deduce the following formulae:
\[ (i) \]
\[ \sum_{h \leq \frac{y}{m}} \frac{d_r(mh)}{h} (\log h)^u P([mh]_y) \sim \frac{\sigma_r(m)}{(r-1)!} \log \left( \frac{y}{m} \right)^{r+u} \int_0^1 F_1(\theta, m) \, d\theta , \] (63)
\[ (ii) \]
\[ \sum_{h \leq \frac{y}{pm}} \frac{d_r(mph)P([mph]_y)(\log ph)^v}{h} \sim \frac{\sigma_r(pm)}{(r-1)!} \log \left( \frac{y}{pm} \right)^r \int_0^1 F_2(\theta, pm) \, d\theta , \] (64)
Proof. It was established in Lemmas 4 and 5 of \cite{ng2013} that
\[ \sum_{h \leq t \leq x} \frac{d_r(nh)}{h} = \frac{\sigma_r(n)(\log t)^{r}}{r!} + O(d_r(n)j_{\tau_0}(n)) \] (67)
for some \( \tau_0 = \tau_0(r) > 0 \). We abbreviate \( T(t) = M(t) + O(\epsilon(n)) \). If \( g \in C^1([0, 1]) \) we deduce
\[ \sum_{h \leq x} \frac{d_r(nh)}{h} g([t]_x) = \int_1^x M'(t)g([t]_x) \, dt 
+ O \left( \epsilon(n) \left( |g(0)| + |g(1)| + \frac{1}{\log x} \int_1^x |g'(t)| \frac{dt}{t} \right) \right). \]
The error term is \( \ll \epsilon(n) \) and the principal term is
\[ \frac{\sigma_r(n)}{(r-1)!} \int_1^x (\log t)^{r-1} g([t]_x) \, dt = \frac{\sigma_r(n)(\log x)^{r-1}}{(r-1)!} \int_0^1 \theta^{r-1} g(\theta) \, d\theta \]
by the variable change \( \theta = [t]_x \). Formulæ (i)-(iii) of this lemma correspond to the following choices of parameters \((n, g(\theta), x)\):
\[ \left( m, \theta^n P([m]_y + \theta), \frac{y}{m} \right), \left( pm, (p)_x + \theta)^n P([pm]_y + \theta), \frac{y}{pm} \right), \]
\[ \left( m, \frac{\log t}{\log x} - \theta \right)^n P([m]_y + \theta), \frac{y}{m} \).
Note that the error term in (ii) is \( \epsilon(pm) \ll \epsilon(m) \). Furthermore, part (iii) requires the variable change \( \theta \to [x]_y \theta \).

In the following lemma we consider averages of the expression \( \sigma_r(\cdot)^2 \). It is in this lemma that the constant \( a_{r+1} \) of Theorem 2 appears. It naturally arises upon considering the Dirichlet series \( \sum_{n \geq 1} \phi(n)\sigma_r(n)^2 n^{-s} \).

Lemma 11. Let \( r \in \mathbb{N} \) and \( g \in C^1([0, 1]) \).
(i) For \( p \leq y \) prime we have
\[ \sum_{m \leq \frac{y}{p}} \phi(m)\sigma_r(m)\sigma_r(pm) \frac{g([m]_y)}{m^2} = \frac{\sigma_r(p)a_{r+1}(\log y)^{r^2}}{(r^2 - 1)!} \int_0^{1-[p]_y} \delta^{r^2-1} g(\delta) \, d\delta 
+ O \left( (\log y)^{r^2} (p^{-1} + (\log y)^{-1}) \right) \] (68)
(ii) For $0 \leq \theta < 1$ we have
\[
\sum_{m \leq y^{1-\theta}} \frac{\phi(m)\sigma_r(m)^2}{m^2} g([m]_y) = \frac{a_{r+1}(\log y)^{r^2}}{(r^2 - 1)!} \int_0^{1-\theta} \delta^{r^2-1} g(\delta) d\delta (1 + O((\log y)^{-1})) .
\]
(69)

**Proof.** We only prove (i) since (ii) is similar. We begin by noting that
\[
\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} = \sigma_r(p) \sum_{m \leq t} \frac{\phi(m)\sigma_r(m)^2}{m^2} + \sum_{\substack{m \leq t \\text{p|m}}} \frac{\phi(m)\sigma_r(m)(\sigma_r(p)\sigma_r(m) - \sigma_r(pm))}{m^2} .
\]

Since, $\sigma_r(m) \ll d_r(m)j_1(m)$, $d_r(uv) \leq d_r(u)d_r(v)$, and $\phi(up) \leq \phi(u)p$, it follows that the second term is
\[
\ll \frac{d_r(p)}{p} \sum_{n \leq \frac{t}{p}} \frac{d_r(n)^2j_1(n)}{n} \ll p^{-1}(\log x)^{r^2} .
\]

By equations (36)-(38) of [1] in conjunction with Theorem 2 of [13] we deduce
\[
\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)^2}{m^2} = \frac{a_{r+1}(\log t)^{r^2}}{(r^2 - 1)!} (1 + O((\log t)^{-1}))
\]
and hence we arrive at
\[
\sum_{m \leq t} \frac{\phi(m)\sigma_r(m)\sigma_r(pm)}{m^2} = \frac{r^{a_{r+1}(\log t)^{r^2}}}{(r^2 - 1)!} + O\left((\log t)^{r^2} p^{-1} + \log t^{r^2} t^2\right) .
\]

We abbreviate this equation to $T(t) = M(t) + O(E(t))$. The sum in (i) may be expressed as the Stieltjes integral
\[
\int_{1-}^\infty g([t]_y) dT(t) = \int_{1-}^\infty g([t]_y) dM(t) + g([t]_y) E(t) |_{1-}^\infty - \int_{1-}^\infty g^\prime([t]_y) E(t) dt .
\]
(70)

The integral equals
\[
\frac{r^{a_{r+1}}}{(r^2 - 1)!} \int_1^\infty \delta^{r^2-1} g([t]_y) d\delta = \frac{r^{a_{r+1}}}{(r^2 - 1)!} \int_0^{1-[t]_y} \delta^{r^2-1} d\delta (1 + O((\log t)^{r^2} p^{-1} + (\log t)^{r^2-1})).
\]

Moreover, it is clear that the error term in (70) is $O\left((\log t)^{r^2} p^{-1} + (\log t)^{r^2-1}\right)$.

In the main calculation of this article we compute certain simple sums over primes. The following lemma provides the required result

**Lemma 12.** Suppose $w \geq 1$, $0 \leq \theta < 1$, and $g \in C^4([0,1])$ then
\[
\sum_{p \leq y^{1-\theta}} \frac{(\log p)^w}{p^{1+i\alpha}} g([p]_y) = \sum_{j=0}^{\infty} \frac{(i\alpha)^j}{j!} (\log y)^{j+w} \int_0^{1-\theta} \beta^{j+w-1} g(\beta) d\beta + O((\log y)^{w-1}) .
\]
(71)
where $\tau$ we must begin by analyzing the Dirichlet series

\[ \sum_{n \leq x} d_r(n) \tau(n) = 1 \]

Consequently, we could equivalently define $f$ by the rule

\[ f(p^a) := (1 + a k_p) p^{-a} \]

where

\[ k_p := k_p(\alpha) = (1 - p^{ia})(1 - p^{-1-ia})/(1 - p^{-1}) \]

which we obtain from (76). Moreover, note that $k_p(0) = 0$.

**Lemma 13.** Put $l = \log x$ and suppose $|\alpha| \ll (\log x)^{-1}$. For $1 \leq m \leq T$, $n$ squarefree and $n \mid m$ we have

\[ \sum_{k \leq x} d_r(mk) f(nk) = \frac{\sigma_r(m)}{n} \sum_{j=0}^{\tau} \binom{r}{j} \frac{1}{j!} (-i\alpha l)^j + O\left( \frac{d_r(m) \tau_0(m) l^{-1}}{n^{1-\epsilon}} \right) \]

where $\tau_0 = 1/3$ is valid and $\tau_0(m)$ is defined by (20).

**Proof.** This lemma will follow from an application of Perron’s formula. However, we must begin by analyzing the Dirichlet series $Z(s, \alpha)$. We put $m = \prod_p p^\lambda = uv$ with $u = \prod_{p < n} p^\lambda$ and hence by multiplicativity

\[ Z(s, \alpha) = \left( \prod_{p \mid u} \frac{\alpha_p(s, \alpha)}{h_p(s, \alpha)} \right) \left( \prod_{p \mid v} \frac{\beta_p(s, \alpha)}{h_p(s, \alpha)} \right) \left( \prod_p h_p(s, \alpha) \right) \]

where

\[ \alpha_p = \alpha_p(s, \alpha) = \sum_{a \geq 0} d_r(p^{a+\lambda}) f(p^{a+1}) p^{-as}, \]

\[ \beta_p = \beta_p(s, \alpha) = \sum_{a \geq 0} d_r(p^{a+\lambda}) f(p^a) p^{-as}, \]
\[ h_p = h_p(s, \alpha) = \sum_{a \geq 0} d_r(p^a) f(p^a) p^{-as}. \]  

(78)

In the above product we label

\[ Z_{11}(s, \alpha) = \prod_{p^k \mid u} \alpha_p(s, \alpha) \quad \text{and} \quad Z_{12}(s, \alpha) = \prod_{p^k \mid v} \beta_p(s, \alpha), \]  

(79)

and we set \( Z_1(s, \alpha) = Z_{11}(s, \alpha)Z_{12}(s, \alpha) \). Next we remark that the last product factors as

\[ \prod_p h_p(s, \alpha) = \frac{\zeta^{2r}(1+s)}{\zeta^r(1+s + i\alpha)} Z_3(s, \alpha) := Z_2(s, \alpha)Z_3(s, \alpha) \]  

(80)

with \( Z_3(s, \alpha) \) holomorphic in \( \Re(s) > -1/2 \). This shall follow from the expressions we derive for \( \alpha_p, \beta_p, \) and \( h_p \) in the next section. Thus we have the factorization

\[ Z(s, \alpha) = Z_1(s, \alpha)Z_2(s, \alpha)Z_3(s, \alpha). \]  

(81)

By Perron’s formula we have

\[ \sum_{k \leq x} d_r(mk)f(nk) = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} Z(s, \alpha) \frac{x^s}{s} ds + O \left( \frac{d_r(m)}{n^{1-\epsilon}} \left( \frac{\log x}{U} + 1 \right) \right). \]  

(82)

where \( c = (\log x)^{-1} \). Let \( \Gamma(U) \) denote the contour consisting of \( s \in \mathbb{C} \) such that

\[ \Re(s) = -\frac{\beta}{\log(|\Im(s)|+2)} \]  

where \( \beta \) is a sufficiently small fixed positive number and \( |\Im(s)| \leq U \). Our strategy will be to deform the contour in \( \mathcal{S}2 \) to \( \Gamma(U) \), thus picking up the pole at \( s = 0 \) which shall account for the main term in the lemma. However, we must also bound the contribution coming from \( \Gamma(U) \) and the horizontal parts of the contour. In the following section, we shall establish

\[ |Z_1(s, \alpha)| \ll \frac{d_r(m)j_{\alpha}(m)}{n^{1-\epsilon}} \]  

(83)

in the cases \( \Re(s) \geq -1/2, |\alpha| \leq cL^{-1} \) and also \( \Re(s) \geq -\epsilon, |\alpha| \leq \epsilon \). Moreover, we have \( |Z_2(s, \alpha)| \ll 1 \) in \( \Re(s) \geq -1/4 \) by the absolute convergence of its series. Furthermore, it is known that

\[ \zeta(1+s) - \frac{1}{s} = O(\log(|\Im(s)| + 2)) \]  

and

\[ \frac{1}{\zeta(1+s)} = O(\log(|\Im(s)| + 2)) \]

on \( \Gamma(U) \) and to the right of \( \Gamma(U) \). By \( \mathcal{S}1 \) and our previous estimates, we have on \( \Gamma(U) \) the bound

\[ |Z(s, \alpha)| \ll \log(|\Im(s)| + 2)^3 \frac{d_r(m)j_{\alpha}(m)}{n^{1-\epsilon}}. \]  

(84)
We now deform the above contour to $\Gamma(U)$ picking up the residue at $s = 0$. It follows that
\[
\frac{1}{2\pi i} \int_{\Gamma(U)} Z(s, \alpha) \frac{x^s}{s} ds \ll \frac{d_r(m)j_{\gamma_0}(m)}{n^{1-\epsilon}} \int_0^U x^{-\frac{\beta}{\log(U + 2)}} \frac{dt}{|t| + 1} \ll \frac{d_r(m)j_{\gamma_0}(m)}{n^{1-\epsilon}} (\log U)^{3r+1} \exp \left(-\frac{\beta \log x}{\log(U + 2)}\right) \exp(-\beta_1 \sqrt{\log x})
\]
by the choice $U = \exp(\beta_2 \sqrt{\log x})$ for a suitable $\beta_2$. Similarly, we can show that the horizontal edges connecting $\Gamma(U)$ to $[c - iU, c + iU]$ contribute an amount $d_r(m)j_{\gamma_0}(m)n^{-1}U^{-1}$. Collecting estimates we conclude
\[
\sum_{k \leq x} d_r(mk)f(nk) = \frac{r\beta_2}{\pi} \left( Z(s, \alpha)x^s s^{-1} \right) + O(d_r(m)j_{\gamma_0}(m)n^{-1}) \quad \text{.}
\]

In the next two subsections we establish the bound (83) and in the final subsection we will compute the residue in (86).

5.1. **Computing the local factors $h_p$, $\alpha_p$, and $\beta_p$.** We simplify notation by putting $u = p^{-s-1}$ and $s = \sigma + it$. By (73) and (77) we have
\[
h_p = \sum_{a=0}^{\infty} d_r(p^a\lambda^+u^a + k_p \sum_{a=0}^{\infty} ad_r(p^a\mu^a)u^a} = (1 - u)^{-r-1}(1 + (rk_p - 1)u) \quad \text{.} \quad (87)
\]
Note that we have used $ad_r(p^a\mu^a) = rd_{r+1}(p^{a-1})$ for $a \geq 1$. By (74), $k_p = 1 - p^\alpha + O(p^{-1+\epsilon})$ and it follows that
\[
h_p = (1 - p^{-s-1})^{-r-1} \left( 1 + \frac{r - 1}{p^{s+1}} - \frac{r}{p^{s+1-\epsilon}} + O(p^{-2-\sigma+\epsilon}) \right) \quad \text{.} \quad (88)
\]
Equation (80) now follows from (88). As before we have for $\lambda \geq 1$
\[
\beta_p = \sum_{a=0}^{\infty} d_r(p^{a+\lambda}\mu^a + k_p \sum_{a=0}^{\infty} ad_r(p^{a+\lambda})u^a} := \beta + k_p\beta' \quad \text{.}
\]
Note that by Lemma 6, $\beta = d_r(p^\lambda)(1 - u)^{-r}H_{\lambda,r}(u)$. and hence it follows that
\[
\beta = d_r(p^\lambda)(1 - u)^{-r-1}(1 + O_r(p^{-1-\sigma}) \quad \text{.}
\]
Similarly, we note that $\beta' = u\frac{d}{du}(\beta(u))$ from which it follows that
\[
\beta' = d_r(p^\lambda)(1 - u)^{-r-1}(1 - u)\frac{d}{du}H_{\lambda,r}(u) - rH_{\lambda,r}(u) \ll d_r(p^\lambda)(1 - u)^{-r-1}O(|u|) \quad \text{.}
\]
We conclude that
\[
\beta_p = d_r(p^\lambda)(1 - u)^{-r-1} \left( 1 + O \left( |k_p| p^{-1-\sigma} \right) \right) \quad \text{.} \quad (89)
\]
Likewise, we have
\[
\alpha_p = \frac{1}{p} \left( \sum_{a=0}^{\infty} d_r(p^{a+\lambda}\mu^a + k_p \sum_{a=0}^{\infty} ad_r(p^{a+\lambda})u^a} \right) = \frac{1}{p} \left( \beta(1 + k_p) + k_p\beta' \right) \quad \text{.}
\]
and it follows from our previous estimates that
\[
\alpha_p = d_r(p^\lambda)p^{-1}(1 - u)^{-r-1}O(|k_p| + 1) \quad \text{.} \quad (90)
\]
5.2. Establishing (83). With our estimates for \( \alpha_p, \beta_p, \) and \( h_p \) in hand, we are ready to estimate \( Z_{11}(s, \alpha) \). We have by (74), (77), and (79)

\[
|Z_{11}(s, \alpha)| \leq \prod_{p^\lambda \parallel u} \frac{\alpha_p(s, \alpha)}{|p^\lambda|} \leq \prod_{p^\lambda \parallel u} \frac{d_r(p^\lambda)(|k_p| + 1)}{p} \frac{1}{1 + (rk_p - 1)p^{s-1}|^{-1}}. \quad (91)
\]

In addition, by (74), (77), and (79) it follows that

\[
|Z_{12}(s, \alpha)| \leq \prod_{p^\lambda \parallel v} \frac{|\beta_p(s, \alpha)|}{|h_p(s, \alpha)|} \leq \prod_{p^\lambda \parallel v} d_r(p^\lambda)(1 + O(|k_p|^{p^{-1-\sigma}}))\frac{1}{1 + (rk_p - 1)p^{s-1}|^{-1}}. \quad (92)
\]

In order to finish bounding these terms, we require a bound for \( k_p \). We shall provide a bound for \( k_p \) and hence \( Z_{11}(s, \alpha) \) in each of the cases \( 0 < |\alpha| \leq cL^{-1} \) and \( 0 < |\alpha| \leq \epsilon \).

**Case 1:** \( 0 < |\alpha| \leq cL^{-1} \) and \( \Re(s) \geq -1/2 \).

By the definition (74) it follows that

\[
|k_p| \ll_c |1 - p^{i\alpha}| \ll_c \min \left(1, \frac{\log p}{L}\right) \quad (93)
\]

since we have the bounds \( |p^{i\alpha}| \leq \exp(|\alpha| \log p) \) and \( |1 - p^{i\alpha}| \leq (|\alpha| \log p)e^{|\alpha| \log p} \).

Let \( c_1, c_2, \ldots \) be effectively computable constants depending on \( c \) and \( r \). We have \( |(rk_p - 1)p^{-s-1}| \ll p^{-\frac{1}{2}} < 0.5 \) if \( p \geq c_1 \). If \( p \leq c_1 \) then we may choose \( T \) sufficiently large such that (83) yields \( |k_p| \leq 1/20r \). Thus \( |(rk_p - 1)p^{-s-1}| \leq 1.1p^{-\frac{1}{2}} < 0.8 \) for all primes \( p < c_1 \) as long as \( T \) is sufficiently large. By (74) and our aforementioned bounds we obtain,

\[
|Z_{11}(s, \alpha)| \leq \prod_{p^\lambda \parallel u} \frac{c_2d_r(p^\lambda)}{p} \leq \frac{d_r(u)c_2^{\nu(n)}}{n} \quad (94)
\]

where \( \nu(n) \) is the number of prime factors of \( n \) and

\[
Z_{12}(s, \alpha) = \prod_{p^\lambda \parallel v} d_r(p^\lambda) \left(1 + O(p^{s+\epsilon})\frac{1}{1 + O(p^{-1/2+\epsilon})}\right) = d_r(e)\prod_{p \parallel v}(1 + O(p^{-1/2+\epsilon})) \quad (95)
\]

Since \( c_2^{\nu(n)} \ll n^\epsilon \) and \( Z_1(s, \alpha) = Z_{11}(s, \alpha)Z_{12}(s, \alpha) \) we deduce that \( Z_1(s, \alpha) \ll d_r(m)\frac{1}{\log^2 v}n^{s-1} \) in the range \( \Re(s) \geq -1/2 \) and \( |\alpha| \leq cL^{-1} \).

**Case 2:** \( 0 < |\alpha| \leq \epsilon \) and \( \Re(s) \geq -\epsilon \).

In this case, it follows from (74) that

\[
|k_p| \leq 4|1 - p^{i\alpha}| \leq \min (8p^\epsilon, 4\epsilon L) \quad (96)
\]

by employing again the bounds \( |p^{i\alpha}| \leq \exp(|\alpha| \log p) \) and \( |1 - p^{i\alpha}| \leq (|\alpha| \log p)e^{|\alpha| \log p} \).

The first bound in (96) implies that \( |(rk_p - 1)p^{-s-1}| \leq (8r + 1)p^{-1+2\epsilon} < 0.5 \) if \( p \) is sufficiently large, say \( p > c_3 \). If \( p \leq c_3 \) then \( |(rk_p - 1)p^{-s-1}| \leq \frac{4\epsilon L \log p}{p^{1-\epsilon}} \frac{1}{1+\frac{1}{p^{1-\epsilon}}} \leq 0.51 \) for \( \epsilon \) sufficiently small. Thus

\[
Z_{11}(s, \alpha) = \prod_{p^\lambda \parallel u, p \leq c_3} \frac{c_4d_r(p^\lambda)}{p^{1-\epsilon}} \prod_{p^\lambda \parallel u, p > c_3} \frac{d_r(p^\lambda)}{p^{1-\epsilon}} \left(1 + O(p^{-1+\epsilon})\right) \ll \frac{d_r(u)}{n^{1-\epsilon}} J_{\gamma_0}(u).
\]
We conclude that if $\text{Re}(s) \geq -\epsilon$ and $|\alpha| \leq \epsilon$ then $|Z_1(s, \alpha)| \ll d_r(m) j_\tau(m) n^{r-1}$. This completes our calculation of (83). The lemma will be thus completed once the residue is computed.

5.3. The residue computation. We decompose

$$Z(s, \alpha)x^s s^{-1} = \zeta(1 + s - i\alpha)^{-r} Z_1(s, \alpha) Z_3(s, \alpha)x^s \zeta(1 + s)^{-2r} s^{-1}.$$  \hspace{1cm} (97)

We now compute the Laurent expansion of each factor. We have

$$
\zeta^{2r}(1 + s)s^{-1} = s^{-2r-1}(1 + a_1 s + a_2 s^2 + \cdots),
$$

$$
x^s = 1 + (\log x) s + (\log x^2) s^2 / 2! + \cdots,
$$

$$
\zeta(1 + s - i\alpha)^{-r} = f(-i\alpha) + f'(-i\alpha) s + f^{(2)}(-i\alpha) s^2 / 2! + \cdots
$$

where we put $f(z) = \zeta(1 + z)^{-r}$. Note that a simple calculation yields

$$f^{(j)}(-i\alpha) = \begin{cases} r(r-1) \cdots (r-(j-1))(-i\alpha)^{-r-j} + O(|\alpha|^{-j+1}) & 0 \leq j \leq r \\ c_j + O(|\alpha|) & j \geq r + 1 \end{cases} \hspace{1cm} (99)$$

and $c_j \in \mathbb{R}$. Next note that $Z_3(s, \alpha)$ has an absolutely convergent power series in $\text{Re}(s) > -1/2, |\alpha| \leq cL^{-1}$. It follows that $Z_3(0, \alpha) = Z_3(0, 0) + O(|\alpha|) = 1 + O(|\alpha|)$ and $Z_3^{(j)}(0, \alpha) \ll 1$ for $j \geq 0$. Combining these facts yields

$$Z_3(s, \alpha) = (1 + O(|\alpha|)) + O(1)s + O(1)s^2 + \cdots.$$ \hspace{1cm} (100)

We now compute the Taylor expansion of $Z_1(s, \alpha)$. Since $k_p(0) = 0$ it follows from (76), (77), (87), and (79) that

$$Z_1(s, 0) = \frac{\sigma_r(m, s + 1)}{n}. \hspace{1cm} (101)$$

By Cauchy’s integral formula with a circle of radius $\epsilon/2$, we establish a bound for $Z_1^{(j)}(0, \alpha)$:

$$Z_1^{(j)}(0, \alpha) = \frac{1}{2\pi i} \int_{|w-\alpha|=\epsilon/2} Z_1(0, w) dw (w-\alpha)^{-j+1} \ll \left(\frac{2}{\epsilon}\right)^{j+1} \frac{d_r(m) j_\tau(m) n^{j+1}}{n^{1-\epsilon}}$$ \hspace{1cm} (102)

by (83). By the Taylor series expansion and (102) it follows that

$$Z_1(0, \alpha) = \frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m) j_\tau(m) |\alpha|}{n^{1-\epsilon}}\right)$$ \hspace{1cm} (103)

since $Z_1(0, 0) = \sigma_r(m)/n$. Combining (102) and (103) we obtain

$$Z_1(s, \alpha) = \left(\frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m) j_\tau(m) |\alpha|}{n^{1-\epsilon}}\right)\right) + \sum_{j=0}^{\infty} O(d_r(m) j_\tau(m) n^{j-1}) s^j / j!.$$ \hspace{1cm} (104)
We are now in a position to compute the residue. It follows from (97), (98), (100), and (103) that the residue at \( s = 0 \) is
\[
res = \sum_{u_1+u_2+u_3+u_4+u_5=2r} \frac{l^{u_1} f^{(u_2)}(-i\alpha) Z_1^{(u_3)}(0,\alpha) Z_3^{(u_4)}(0,\alpha)\sigma}{u_1!u_2!u_3!u_4!}.
\] (105)

We first show that those terms with \( u_5 \geq 1 \) contribute a smaller amount. Since \( |f^{(u_2)}(-i\alpha)| \ll |\alpha|^{-u_2} \) for \( 0 \leq u_2 \leq r \) and \( |f^{(u_2)}(-i\alpha)| \ll 1 \) for \( r+1 \leq u_2 \leq 2r \) it follows that the terms with \( u_5 \geq 1 \) contribute
\[
\ll_r \frac{d_r(m) j_1(m)}{n^{1-\epsilon}} \sum_{u_1+u_2 \leq 2r-1} \frac{l^{u_1} f^{(u_2)}(-i\alpha)}{u_1!u_2!} \left( \sum_{u_1+u_2 \leq 2r-1} l^{u_1} |\alpha|^{r-u_2} + \sum_{0 \leq u_2 \leq r+1} l^{u_1} \right)
\ll \frac{d_r(m) j_1(m)}{n^{1-\epsilon}} (r^{-1} + r^{-2}).
\] (106)

We deduce that
\[
res = \sum_{u_1+u_2+u_3+u_4+u_5=2r} \frac{l^{u_1} f^{(u_2)}(-i\alpha) Z_1^{(u_3)}(0,\alpha) Z_3^{(u_4)}(0,\alpha)\sigma}{u_1!u_2!u_3!u_4!} + O\left(\frac{d_r(m) j_1(m)n^{r-1}}{n^{1-\epsilon}}\right).
\] (106)

The contribution from those terms in (106) satisfying \( u_1 + u_2 = 2r, u_2 \leq r \) is
\[
\left( \sum_{u_1+u_2=2r, u_2 \leq r} \frac{l^{u_1} f^{(u_2)}(-i\alpha)}{u_1!u_2!} \right) \left( \frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m) j_1(m) \sigma}{n^{1-\epsilon} |\alpha|}\right) \right) (1 + O(|\alpha|)).
\]
\[
= \sum_{u_2 \leq r} \frac{l^{2r-u_2}}{(2r-u_2)!} \left( \frac{r}{u_2} (-i\alpha)^{r-u_2} + O(|\alpha|^{r-u_2+1}) \right) \left( \frac{\sigma_r(m)}{n} + O\left(\frac{d_r(m) j_1(m) |\alpha|}{n^{1-\epsilon}}\right) \right)
= \frac{\sigma_r(m)}{n} \sum_{a=0}^r \frac{r}{(r+a)!} (\alpha)^{r} + O\left(\frac{d_r(m) j_1(m)}{n^{1-\epsilon}}\right).
\]

Those terms in (106) with \( u_1 \leq r-1 \) contribute
\[
\frac{d_r(m) j_1(m)}{n^{1-\epsilon}} \sum_{u_1+u_2+u_3+u_4=2r, u_1 \leq r-1} l^{u_1} |f^{(u_2)}(-i\alpha)| \ll \frac{d_r(m) j_1(m)}{n^{1-\epsilon}} r^{-1}
\]
since \( |\alpha| \leq cL^{-1} \ll 1 \) and the remaining terms in (106) are
\[
\ll \frac{d_r(m) j_1(m)}{n^{1-\epsilon}} \sum_{u_1+u_2+u_3+u_4=2r, u_1 \leq 2r-1, u_2 \leq 2r-1} l^{u_1} |\alpha|^{u_1+1-r} \ll \frac{d_r(m) j_1(m)}{n^{1-\epsilon}} r^{-1}.
\]

We thus conclude that
\[
res = \frac{\sigma_r(m)}{n} \sum_{a=0}^r \frac{r}{(r+a)!} (\alpha)^{r} + O\left(\frac{d_r(m) j_1(m) r^{-1}}{n^{1-\epsilon}}\right)
\] (107)
and the lemma follows from (99) and (104).
We have the Taylor series expansion
\[ R_k(1 + i\alpha) = R_k(1) + R_k'(1)(i\alpha) + R_k''(1)(i\alpha)^2/2 + \cdots. \]

We denote the truncated Taylor series expansion \( T_k_N(\alpha) = \sum_{j=0}^{N} R_k^{(j)}(1)(i\alpha)^j/j! \).

**Lemma 14.** We have for \( l = \log x, \ |\alpha| \ll (\log x)^{-1}, \) and \( \tau_0 = 1/3 \)
\[
\sum_{k \leq x} d(mk) T_{nk,r}(\alpha) \phi(nk) = \frac{\sigma_r(m)}{n} \sum_{j=0}^{r} \left( \frac{r}{j} \right) (-i\alpha)^j + O \left( \frac{d_r(m) j_{\tau_0}(v)^{l-1}}{n^{1-\epsilon}} \right). \tag{108}
\]

**Proof.** We begin by noting that it suffices to prove
\[
\sum_{k \leq x} d_r(mk) R_k^{(j)}(1) \phi(nk) = (-1)^j \frac{\sigma_r(m)}{n} \left( \frac{r}{j} \right) j! \frac{d_r(m) j_{\tau_0}(m)^{l+j-1}}{n}. \tag{109}
\]

This is since if we multiply \( (109) \) by \( (i\alpha)^j/j! \) and sum \( j = r \) we obtain the result. The Dirichlet series generating function for the sum in question is
\[
i^{-j} \sum_{k=1}^{\infty} d_r(mk) \frac{d^j}{d\alpha^j} R_{nk}(1 + i\alpha) \bigg|_{\alpha=0} = i^{-j} \frac{d^j}{d\alpha^j} Z(s, \alpha) \bigg|_{\alpha=0}. \tag{110}
\]

By Perron’s formula it follows that the sum in question is
\[
\frac{i^{-j}}{2\pi i} \int_{c-iU}^{c+iU} \frac{d^j}{d\alpha^j} Z(s, \alpha) \bigg|_{\alpha=0} \frac{x^s}{s} ds + O \left( \frac{d_r(m) j_{\tau_0}(v)^{l-1}}{n^{1-\epsilon}} \right). \tag{111}
\]

where \( c = (\log x)^{-1} \). As in Lemma 13 equations (82), (85), we want to deform the contour \([c - iU, c + iU]\) to \(\Gamma(U)\) and then pick up the residue at \(s = 0\). As this calculation is analogous to the preceding lemma we omit the details. This procedure yields
\[
\sum_{k \leq x} \frac{d_r(mk) R_k^{(j)}(1)}{\phi(nk)} = i^{-j} \frac{\sigma_r(m, s+1)}{n} \frac{x^s}{s} + O \left( \frac{d_r(m) j_{\tau_0}(v)^{l-1}}{n^{1-\epsilon}} \right). \tag{112}
\]

Recall that \( Z(s, \alpha) = Z_1(s, \alpha)Z_2(s, \alpha)Z_3(s, \alpha) \) where
\[
Z_1(s, \alpha) = \frac{\sigma_r(m, s+1)}{n}, \quad Z_2(s, \alpha) = \frac{\zeta^{2r}(1+s)}{\zeta'(1+s - i\alpha)}, \quad Z_3^{(j)}(0, 0) \ll 1 \tag{113}
\]
for all \( j \geq 0 \). By the product rule we have
\[
\frac{d^j}{d\alpha^j} Z(s, \alpha) \bigg|_{\alpha=0} = \sum_{u_1 + u_2 + u_3 = j} \binom{j}{u_1, u_2, u_3} Z_1^{(u_1)}(s, 0)Z_2^{(u_2)}(s, 0)Z_3^{(u_3)}(s, 0). \tag{114}
\]

Thus we need to compute
\[
\text{res}_{s=0} \left( Z_1^{(u_1)}(s, 0)Z_2^{(u_2)}(s, 0)Z_3^{(u_3)}(s, 0)x^{s-1} \right) \tag{115}
\]
for all \( u_1 + u_2 + u_3 = j \). In fact, it turns out that the main term arises from those triples \((u_1, u_2, u_3) = (0, j, 0)\). We now compute the residue arising from these
terms. We have the Laurent expansions,

\[ Z_1(s, 0) = \frac{\sigma_r(m)}{n} + \frac{\sigma_r^{(1)}(m, 1)}{n}s + \cdots, \]

\[ Z_2^{(j)}(s, 0) = \frac{r(r - 1) \cdots (r - (j - 1))(-i)^j}{s^{r+j}} + \frac{c_1}{s^{r+j-1}} + \cdots, \]

\[ Z_3(s, 0) = 1 + d_1 s + \cdots. \]

We further remark that by Cauchy’s integral formula we may establish \( \sigma_r^{(k)}(m, 1) \ll d_r(m)j_{\tau_0}(m) \) for some \( \tau_0 > 0 \). These terms contribute

\[ \text{res}_{s=0} Z_1(s, 0)Z_2^{(j)}(s, 0)Z_3(s, 0)x^s s^{-1} = \]

\[ \frac{\sigma_r(m)r(r - 1) \cdots (r - (j - 1))(-i)^j \log x^{r+j-1}}{n(r+j)!} \]

(116)

A similar calculation shows that for those triples \((u_1, u_2, u_3)\) such that \( u_2 \leq j - 1 \) then

\[ \text{res}_{s=0} Z_1^{(u_1)}(s, 0)Z_2^{(u_2)}(s, 0)Z_3^{(u_3)}(s, 0)x^s s^{-1} \ll \frac{d_r(m)j_{\tau_0}(m)\log x^{r+j-1}}{n}. \]

(117)

The lemma now follows by combining (112), (114), (115), (116), and (117).

We deduce the following corollary to Lemmas 13 and 14:

**Lemma 15.**

\[ \sum_{k \leq x} d_r(mk) \left( f(nk) - \frac{T_{nk,r}(\alpha)}{\phi(nk)} \right) \ll |\alpha|^{r+1}L^2 \frac{d_r(m)j_{\tau_0}(m)}{n^{1-\epsilon}}. \]

(118)

**Proof.** Note that

\[ f(nk) = \frac{T_{nk,r}(\alpha)}{\phi(nk)} + \alpha^{r+1}g(\alpha; nk) \]

where \( g \) is entire in \( \alpha \). Moreover, it follows that

\[ \sum_{k \leq x} d_r(mk) \left( f(nk) - \frac{T_{nk,r}(\alpha)}{\phi(nk)} \right) = \alpha^{r+1}g^*(\alpha; n, x) \]

(119)

where \( g^* \) entire in \( \alpha \). Combining Lemmas 13 and 14 we deduce that

\[ \max_{|\alpha| \leq cL^{-1}} |\alpha|^{r+1}g^*(\alpha; n, x) | \ll \frac{d_r(m)j_{\tau_0}(m)L^{r-1}}{n^{1-\epsilon}} \]

and hence by the maximum modulus principle

\[ \max_{|\alpha| \leq cL^{-1}} |g^*(\alpha; n, x)| \ll \frac{d_r(m)j_{\tau_0}(m)L^{2r}}{n^{1-\epsilon}}. \]

(120)

Hence, (119) and (120) imply the statement of the lemma.
6. Proof of Theorem 2

6.1. Initial manipulations. In this section we apply the lemmas to manipulate $I$ into a suitable form for evaluation. Recall that we had

$$I = \sum_{k \leq y} \frac{d_r(k)P([k]_y)}{k} \sum_{j \leq \frac{y}{k^2}} b(j)e(-j/k) + O(yT^{1/2+\varepsilon}).$$

By Perron’s formula with $c = 1 + L^{-1}$ the inner sum is

$$\sum_{j \leq \frac{y}{k^2}} b(j)e(-j/k) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} Q^*(s, \alpha, k) \left( \frac{kT}{2\pi} \right)^s \frac{ds}{s} + O(kT^{c-1})$$

where $Q^*(s, \alpha, k) = \sum_{j=1}^{\infty} b(j) j^{-s} e(-j/k)$. Pulling the contour left to $c_0 = 1/2 + L^{-1}$ we obtain

$$\sum_{j \leq \frac{y}{k^2}} b(j)e(-j/k) = R_1 + R_{1+i\alpha}$$

where $R_u$ is the residue at $s = u$. By Lemma 8 the left and horizontal edges contribute $yT^{1/2+\varepsilon}$. Moreover by (27) it follows that

$$Q^*(s, \alpha, k) = \sum_{h \leq y} \frac{d_r(h)P([h]_y)Q(s, \alpha, h/k)}{h^s}$$

where $Q(s, \alpha, h/k)$ is defined by (32). We will now invoke Lemma 5, however we require that $h, k$ be relatively prime. Therefore we set $\frac{h}{k} = \frac{H}{K}$ where $H = h/(h,k), K = k/(h,k)$, and $(H, K) = 1$. We deduce

$$R_1 = \sum_{h \leq y} d_r(h)P([h]_y) \sum_{s=1}^{r_x} \left( \zeta^2(s) \left( \frac{\zeta'(s-i\alpha)}{\zeta(s)} - G(s, \alpha, K) \right) \left( \frac{T}{2\pi HK} \right)^s s^{-1} \right).$$

By an application of Lemma 5(i) this is

$$R_1 = K \sum_{h \leq y} d_r(h)P([h]_y) \sum_{s=1}^{r_x} \left( \zeta^2(s) \left( \frac{\zeta'(s-i\alpha)}{\zeta(s)} - G(s, \alpha, K) \right) \left( \frac{T}{2\pi HK} \right)^s s^{-1} \right)$$

$$= \frac{T}{2\pi} \sum_{h \leq y} \frac{d_r(h)P([h]_y)}{H} \cdot \left( \left( \frac{\zeta'/\zeta(\tau)}{\zeta(1, \alpha, K)} \right) \log \left( \frac{T e^{2\gamma-1}}{2\pi HK} \right) + \left( \frac{\zeta'/\zeta(\tau)}{\zeta(1, \alpha, K)} \right) \phi(K) \right)$$

(124)

where we put $\tau = 1 + i\alpha$. Likewise Lemma 5(ii) implies

$$R_{1+i\alpha} = \sum_{h \leq y} d_r(h)P([h]_y) \sum_{s=1}^{r_x} \left( Q(s, \alpha, H/K) \left( \frac{T}{2\pi H} \right)^s s^{-1} \right)$$

$$= \frac{T}{2\pi} \sum_{h \leq y} \frac{d_r(h)P([h]_y)}{H} \left( \frac{T}{2\pi H} \right)^{i\alpha} \left( K^R_K \right) \phi(K).$$

(125)
Combining (121), (122), (124), and (125) we deduce
\[ I = \frac{T}{2\pi} \sum_{h, k \leq y} \frac{d_r(h) \cdot d_r(k) \cdot P([h]_y) \cdot P([k]_y)(h, k)}{hk} \left( \log \frac{T e^{2\gamma - 1}}{2\pi H K} \left( \frac{\zeta'}{\zeta}(\tau) - G(1, \alpha, K) \right) + \frac{\zeta'^2}{\zeta}(\tau) \frac{T}{2\pi H} \frac{K R_{K}(\tau)}{\phi(K)} + O(yT^{1/2+\epsilon}) \right) \]

where \( G(s, \alpha, K) \) is defined by (34). We may write for \( j = 0, 1 \) \( G^{(j)}(1, \alpha, K) = \sum_{p|K} p^{i\alpha} \log^{j+1} p + O(C_j(K)) \). By Lemma 9, the \( O(C_j(K)) \) terms contribute \( O(TL^{(r+1)^\epsilon}) \).

Whence
\[ I = \frac{T}{2\pi} \sum_{h, k \leq y} \frac{d_r(h) \cdot d_r(k) \cdot P([h]_y) \cdot P([k]_y)(h, k)}{hk} \left( \log \frac{T e^{2\gamma - 1}}{2\pi H K} \left( \frac{\zeta'}{\zeta}(\tau) - \sum_{p|K} p^{i\alpha} \log p \right) + \frac{\zeta'^2}{\zeta}(\tau) \frac{T}{2\pi H} \frac{K R_{K}(\tau)}{\phi(K)} + O(yT^{1/2+\epsilon}) \right) \]

where \( z = 1 + i\alpha \). Insertion of the identity
\[ f((h, k)) = \sum_{m|h} \sum_{n|m} \mu(n) f \left( \frac{n}{m} \right) \]
produces
\[ I = \frac{T}{2\pi} \sum_{h, k \leq y} \frac{d_r(h)P([h]_y) \cdot d_r(k)P([k]_y)}{hk} \sum_{m|h} \sum_{n|m} \frac{\mu(n)}{n} \]
\[ \cdot \left( \log \frac{T e^{2\gamma - 1} \cdot m^2}{2\pi hkn^2} \left( \frac{\zeta'}{\zeta}(\tau) - \sum_{p|\frac{n}{m}} p^{i\alpha} \log p \right) + \frac{\zeta'^2}{\zeta}(\tau) \frac{T m}{2\pi nh} \frac{R_{\frac{m}{n}}(\tau)}{\phi(\frac{n}{m})} \right) + O(yT^{1/2+\epsilon}) \]

Changing summation order and making the variable changes \( h \to hm \) and \( k \to km \) yields
\[ I = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h, k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y) \cdot d_r(mk)P([mk]_y)}{hk} \]
\[ \cdot \left( \log \frac{T e^{2\gamma - 1} \cdot m^2}{2\pi hkn^2} \left( \frac{\zeta'}{\zeta}(\tau) - \sum_{p|\frac{nk}{m}} p^{i\alpha} \log p \right) + \frac{\zeta'^2}{\zeta}(\tau) \frac{T m}{2\pi nh} \frac{R_{nk}(\tau)}{\phi(nk)} \right) + O(yT^{1/2+\epsilon}) \].

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Rearrange this as \( I = I_1 + I_2 + O(yT^{1/2+\epsilon}) \) where
\[
I_1 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left( -\log \frac{T e^{2\gamma-1}}{2\pi hkn_2} \sum_{p|nk} p^{\alpha} \log p - \sum_{p|nk} p^{\alpha} \log^2 p \right)
\]
and
\[
I_2 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left( \log \left( \frac{T e^{2\gamma-1}}{2\pi hkn_2} \right) \frac{\zeta'}{\zeta}(\tau) + \left( \frac{\zeta'}{\zeta}(\tau) \right)' - \frac{\zeta^2(\tau)}{\tau} \left( \frac{T}{2\pi nh}\right)^{\alpha} \frac{nk R_{nk}(\tau)}{\phi(nk)} \right).
\]
(126)
The first sum is
\[
I_1 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left( -\log \frac{T}{2\pi hk} \sum_{p|k} p^{\alpha} \log p - \sum_{p|k} p^{\alpha} \log^2 p + O(L \log n) \right).
\]
A calculation shows that the \( O(L \log n) \) contributes \( O(TL^{(r+1)^2}) \). Since \( \phi(m)m^{-1} = \sum_{n|m} \mu(n)n^{-1} \) we deduce that
\[
I_1 = \frac{T}{2\pi} \sum_{m \leq y} \frac{\phi(m)}{m^2} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left( -\log \frac{T}{2\pi hk} \sum_{p|k} p^{\alpha} \log p - \sum_{p|k} p^{\alpha} \log^2 p \right) + O(TL^{(r+1)^2}).
\]
(127)
This puts \( I_1 \) in a suitable form to be evaluated by the lemmas. We now simplify \( I_2 \) by substituting the Laurent expansions
\[
\left( \frac{\zeta'}{\zeta}(\tau) \right)' = (i\alpha)^{-1} + O(1),
\]
\[
\left( \frac{\zeta'}{\zeta}(\tau) \right)' = (i\alpha)^{-2} + O(1),
\]
\[
\zeta^2(\tau)^{-1} = (i\alpha)^{-2} + (2\gamma-1)(i\alpha)^{-1} + O(1)
\]
in (126). The \( O(1) \) terms of these Laurent expansions contribute
\[
TL \sum_{m \leq y} \frac{d_r(m)^2}{m} \sum_{n|m} \frac{1}{n} \sum_{h,k \leq \frac{y}{m}} \frac{d_r(h)d_r(k)}{hk} \ll TL^{(r+1)^2}
\]
by \((21)\) and Lemma 13. Thus we deduce

\[
T \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n \mid m} 1 \left( \sum_{h \leq \frac{n}{m}} \frac{d_r(h)}{h} \right) \sum_{k \leq \frac{n}{m}} d_r(mk) f(nk) \leq TL' \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n \mid m} \left( \frac{d_r(m) \sigma_r(m) L'}{n^{1-\epsilon}} \right) \leq TL(r+1)^2 - 1
\]

by \((21)\) and Lemma 13. Thus we deduce

\[
I_2 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{n}{m}} \frac{d_r(mh) P([mh]_y) d_r(mk) P([mk]_y)}{hk} \cdot \left( 1 + i\alpha \log \frac{T}{2\pi hkn} - \frac{\tau nk R_{\alpha k}(\tau)}{\phi(nk)} i\alpha \right)
\]

plus an error term \(O(TL(r+1)^2)\). In the above formula we replace \(\frac{R_{\alpha k}(\tau)}{\phi(nk)}\) by \(\frac{T_{nk,\tau}(\alpha)}{\phi(nk)}\)

and by \((118)\) this introduces an error of

\[
|\alpha|^{-2} \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n \mid m} \sum_{k \leq \frac{n}{m}} d_r(mk) \left( \frac{R_{nk}(\tau)}{\phi(nk)} - \frac{T_{nk,\tau}(\alpha)}{\phi(nk)} \right) \leq |\alpha|^{-2} TL' \sum_{m \leq y} \frac{d_r(m)}{m} \sum_{n \mid m} \left( |\alpha|^{-1} T L' d_r(m) j_{\text{opt}}(m) \right) \leq TL(r+1)^2.
\]

Therefore we have

\[
I_2 = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{n}{m}} \frac{d_r(mh) P([mh]_y) d_r(mk) P([mk]_y)}{hk} \cdot \left( 1 + i\alpha \log \frac{T}{2\pi hkn} - \frac{\tau nk R_{\alpha k}(\tau)}{\phi(nk)} i\alpha \right) + O(TL(r+1)^2).
\]

A calculation shows that \(R_k(1) = \phi(k)/k\), \(R_k'(1) = -\phi(k) \log k/k\) and thus it follows that

\[
\frac{T_{nk,\tau}(\alpha)}{\phi(nk)} = \frac{1}{nk} (1 - \log(nk)(i\alpha)) + \sum_{j=2}^{r} \frac{R_{nk}^{(j)}(1)(i\alpha)^j}{\phi(nk)j!}.
\]

We further decompose \(I_2 = I_{21} + I_{22} + O(TL(r+1)^2)\) where

\[
I_{21} = \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \sum_{h,k \leq \frac{n}{m}} \frac{d_r(mh) P([mh]_y) d_r(mk) P([mk]_y)}{hk} \cdot \left( 1 + i\alpha \log \frac{T}{2\pi hkn} - \frac{\tau nk R_{\alpha k}(\tau)}{\phi(nk)} i\alpha \right) (1 - (i\alpha) \log(nk))
\]

(128)
and \( I_{22} = \frac{T}{2\pi} \sum_{j=2}^{r} \frac{(i\alpha)^{j-2}}{j!} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \mu(n) \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{h} \left( \frac{T}{2\pi h n} \right)^{i\alpha} \frac{nkR^{(j)}_{nk}(1)}{\phi(nk)} \).  

6.2. Evaluation of \( I_1 \). By (127) it follows that

\[
I_1 = \frac{T}{2\pi} \left( -La_0,0,1 + a_1,0,1 + a_0,1,1 - a_{0,0,2} \right) + O(TL^{(r+1)^2})
\]

where for \( u, v, w \in \mathbb{Z}_{\geq 0} \) we define \( a_{u,v,w} \) to be the sum

\[
\sum_{mh,mk \leq y} \phi(m)d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)(\log h)^u(\log k)^v
\]

\[
\sum_{p|k} p^{i\alpha}(\log p)^w.
\]

By (129) it suffices to evaluate \( a_{u,v,w} \). Inverting summation we have

\[
a_{u,v,w} = \sum_{m \leq y} \phi(m)\frac{\sigma_r(m)^2 p^{i\alpha}(\log p)^w}{m^2 p} \left( \sum_{h \leq \frac{y}{m}} \frac{d_r(mh)P([mh]_y)(\log h)^u}{h} \right) \left( \sum_{k \leq \frac{y}{pk}} \frac{d_r(mpk)P([mpk]_y)(\log pk)^v}{k} \right)
\]

By Lemma 10, equations (128) and (129) we have

\[
a_{u,v,w} = \frac{1}{(r-1)!^2} \sum_{m \leq y} \phi(m)\sigma_r(m)^2 p^{i\alpha}(\log p)^w \log \left( \frac{y}{m} \right)^{r+u} \log \left( \frac{y}{pm} \right)^r
\]

\[
\cdot \int_0^1 \int_0^1 F_1(\theta_1, m)F_2(\theta_2, pm) d\theta_1 d\theta_2 + \epsilon_1 + \epsilon_2 + \epsilon_3
\]

where \( \epsilon_1 \ll \sum_{m \leq y} \sigma_r(m)\frac{L^{u+r}}{m} \sum_{p \leq y} (\log p)^w \epsilon(m) \),

\[
\epsilon_2 \ll \sum_{m \leq y} \frac{\epsilon(m)}{m} \sum_{p \leq y} (\log p)^w \frac{\sigma_r(pm)L^{u+r}}{p},
\]

\[
\epsilon_3 \ll \sum_{m \leq y} \frac{\epsilon(m)}{m} \sum_{p \leq y} (\log p)^w \frac{\epsilon(m)}{p}.
\]

By (125) it follows that

\[
\epsilon_1 \ll L^{u+w+r} \sum_{m \leq y} \frac{d_r(m)^2}{m} \ll L^{u+w+r} \sum_{m \leq y} \frac{d_r(m)^2}{m} \ll L^{u+w+r^2+r}.
\]

A similar calculation gives \( \epsilon_2 \ll L^{u+w+r^2+r} \) and \( \epsilon_3 \ll L^{u+r^2} \). Recalling (125) and rearranging a little, yields

\[
a_{u,v,w} = \left( \frac{(\log y)^2 r+u+v}{(r-1)!^2} \right) \int_0^1 \int_0^1 \frac{\theta_1^{u+1} \theta_2^{r-1}}{\theta_1 \theta_2} \sum_{p \leq y} p^{i\alpha}(\log p)^w
\]

\[
\cdot \sum_{m \leq y} \phi(m)\sigma_r(m)\frac{\sigma_r(pm)}{m^2} g_{u,v}([m]_y, [p]_y) d\theta_1 d\theta_2 + O(L^{\max(u,v)+r^2+r}).
\]
where \( g_{u,v}(\delta, \beta) = \)

\[
(1-\delta)^{r+u}(1-\beta-\delta)^r(\beta + \theta_2(1-\beta-\delta))^{u}P(\delta + \theta_1(1-\delta))P(\delta + \beta + \theta_2(1-\beta-\delta)).
\]

By Lemma 11(ii), (131) becomes

\[
a_{u,v,w} = rC_r((\log y))^{r^2+2r+u+v} \int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} \sum_{p \leq y} p^{\alpha} \frac{(\log p)^w}{p}.
\]

\[
\cdot \int_0^{1-\lfloor p \rfloor} \delta^{r-1} g_{u,v}(\delta, \lfloor p \rfloor) \, d\delta \, d\theta_1 + c_4 + O(L^{\max(u,v)+r^2+r})
\]

where \( C_r \) is defined by (11) and

\[
\epsilon_4 \ll L^{2r+u+v} \sum_{p \leq y} \frac{(\log p)^w}{p} (L^2(p-1+L^{-2}) \ll L^{2r+2r+u+v+w-1}
\]

since \( w \geq 1 \). Inverting summation

\[
a_{u,v,w} = rC_r((\log y))^{r^2+2r+u+v} \int_0^1 \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} \delta^{r-1} \sum_{p \leq y^{1-\delta}} p^{\alpha} \frac{(\log p)^w}{p} g_{u,v}(\delta, \beta) \, d\delta \, d\theta_1 \, d\theta_2 + O(L^{2r+2r+u+v+w-1}).
\]

An application of Lemma 12 yields

\[
a_{u,v,w} = rC_r((\log y))^{r^2+2r+u+v+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}.
\]

\[
\cdot \int_0^1 \theta_1^{r+u-1} \theta_2^{r-1} g_{u,v}(\delta, \beta) \, d\theta_1 \, d\theta_2 = (1-\delta)^{r+u}(1-\beta-\delta)^r Q_{r+u-1}(\delta) R_v(\delta, \beta)
\]

where

\[
Q_{r+u-1}(\delta) = \int_0^1 \theta_1^{r+u-1} P(\delta + \theta_1(1-\delta)) \, d\theta_1,
\]

\[
R_v(\delta, \beta) = \int_0^1 \theta_2^{r-1}(\beta + \theta_2(1-\delta-\beta))^r P(\delta + \beta + \theta_2(1-\delta-\beta)) \, d\theta_2,
\]

and hence

\[
a_{u,v,w} \sim rC_r((\log y))^{r^2+2r+u+v+w} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!}.
\]

\[
\cdot \int_0^1 \delta^{r-1}(1-\delta)^{r+u} Q_{u+r-1}(\delta) \int_0^{1-\delta} \beta^{j+w-1}(1-\beta-\delta)^r R_v(\delta, \beta) \, d\beta \, d\delta.
\]

Now note that

\[
R_0(\delta, \beta) = Q_{r-1}(\delta + \beta), R_1(\delta, \beta) = \beta Q_{r-1}(\delta + \beta) + (1-\delta-\beta)Q_r(\delta + \beta).
\]
We see that
\[ a_{u,0,w} \sim rC_r (\log y)^{r+2+u+w} \sum_{j=0}^{\infty} \frac{(\log y)^j}{j!} \]
\[ \cdot \int_0^1 \int_0^{1-\delta} \delta^{r-1}(1-\delta)^{r+u}(1-\beta-\delta)^{r+\beta+w-1} Q_{u+r-1}(\delta) Q_{r-1}(\delta+\beta) \, d\beta \, d\delta \]
and
\[ a_{u,1,w} \sim rC_r (\log y)^{r+2+1+u+w} \sum_{j=0}^{\infty} \frac{(\log y)^j}{j!} \]
\[ \cdot \left( \int_0^1 \int_0^{1-\delta} \delta^{r-1}(1-\delta)^{r+u}(1-\beta-\delta)^{r+\beta+w} Q_{u+r-1}(\delta) Q_{r-1}(\delta+\beta) \, d\beta \, d\delta \right) + \int_0^1 \int_0^{1-\delta} \delta^{r-1}(1-\delta)^{r+u}(1-\beta-\delta)^{r+1+\beta+w-1} Q_{u+r-1}(\delta) Q_{r}(\delta+\beta) \, d\beta \, d\delta \right) . \]

For \( \bar{n} = (n_1, n_2, n_3, n_4, n_5) \in (\mathbb{Z}_0)^5 \) we recall the definition (9)
\[ i_P(\bar{n}) = \int_0^1 \int_0^{1-x_1} x_1^{-1} (1-x_1)^{n_2} (1-x_1-x_2)^{n_4} x_2^{n_5} Q_{n_4}(x_1) Q_{n_5}(x_1 + x_2) \, dx_2 \, dx_1 \]
and hence
\[ a_{0,0,1} = rC_r L^{(r+1)} \sum_{j=0}^{\infty} \frac{z^j n^{j+(r+1)}_2}{j!} i_P(r, r, j, r-1, r-1) , \]
\[ a_{1,0,1} = rC_r L^{(r+1)}+1 \sum_{j=0}^{\infty} \frac{z^j n^{j+(r+1)}_1}{j!} i_P(r+1, r, j, r-1) , \]
\[ a_{0,0,2} = rC_r L^{(r+1)}+1 \sum_{j=0}^{\infty} \frac{z^j n^{j+(r+1)}_2}{j!} i_P(r, r, j+1, r-1, r-1) , \]
\[ a_{1,0,1} = rC_r L^{(r+1)}+1 \sum_{j=0}^{\infty} \frac{z^j n^{j+(r+1)}_1}{j!} i_P(r, r, j+1, r-1, r) + i_P(r, r+1, j, r-1, r) . \]

Combining these identities with (130) we arrive at
\[ I_1 \sim rC_r \frac{T}{2\pi} L^{(r+1)}+1 \sum_{j=0}^{\infty} \frac{z^j n^{j+(r+1)}_2}{j!} \]
\[ \cdot (-i_P(r, r, j, r-1, r-1) + \eta(i_P(r+1, r, j, r-1) + i_P(r, r+1, j, r-1, r))) \]
and this is valid up to an error which is smaller by a factor \( O(L^{-1}) \).

6.3. Evaluation of \( I_{21} \). We recall that
\[ I_{21} \sim \frac{T}{2\pi} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \leq \sqrt{y}} \frac{d_r(mh) P([mh]_y) d_r(mk) P([mk]_y)}{hk} \]
\[ \cdot \left( 1 + i\alpha \log \frac{T}{2\pi \sqrt{hkn}} - \frac{T}{2\pi hkn} \right) (1 - (i\alpha) \log n) \right) . \]
A little algebra shows that the expression within the brackets simplifies to
\[
\log \left( \frac{T}{2\pi h} \right) \log(nk) - (1 - (i\alpha) \log nk) \log \left( \frac{T}{2\pi h} \right) 2 \sum_{j=0}^{\infty} \frac{(i\alpha \log \left( \frac{T}{2\pi n} \right))^j}{(j + 2)!}.
\]

We may replace \( \log \frac{T}{2\pi n} \) by \( \log \frac{T}{2\pi} \) and \( \log(nk) \) by \( \log k \) up to an error of \( O(\log n) \). This error term contributes \( O(TL^{(r+1)^2}) \) as long as we use \( |\alpha| \leq cL^{-1} \). It thus follows that
\[
I_{21} \sim \frac{T}{2\pi} \sum_{m \leq y} \frac{\phi(m)}{m^2} \sum_{h, k \leq \frac{\pi}{T}} \frac{d_r(mh)P([mh]_y)d_r(mk)P([mk]_y)}{hk} \left( \log \left( \frac{T}{2\pi h} \right) \log(k - 1 - (i\alpha) \log k) \log \left( \frac{T}{2\pi h} \right) 2 \sum_{j=0}^{\infty} \frac{(i\alpha \log \left( \frac{T}{2\pi n} \right))^j}{(j + 2)!} \right) \cdot \left( \log \left( \frac{T}{2\pi} \right) \log k - (1 - (i\alpha) \log k) \log \left( \frac{T}{2\pi} \right) 2 \sum_{j=0}^{\infty} \frac{(i\alpha \log \left( \frac{T}{2\pi n} \right))^j}{(j + 2)!} \right)
\]
and hence
\[
I_2 \sim \frac{T}{2\pi} \left( b_{1,1} - \sum_{j=0}^{\infty} \frac{(i\alpha)^j}{(j + 2)!} b_{j+2,0} + \sum_{j=0}^{\infty} \frac{(i\alpha)^j+1}{(j + 2)!} b_{j+2,1} \right)
\]
where
\[
b_{u,v} = \sum_{m \leq y} \frac{\phi(m)}{m^2} \sum_{h, k \leq \frac{\pi}{T}} \frac{d_r(mh)P([mh]_y)\log \left( \frac{T}{2\pi n} \right)^u d_r(mk)P([mk]_y)(\log k)^v}{hk}
\]
for \( u, v > 0 \). It suffices to evaluate \( b_{u,v} \). Inserting equations (134) and (135) of Lemma 10 in (136) gives
\[
b_{u,v} \sim \frac{(\log y)^{u+r}}{(r-1)!^2} \sum_{m \leq y} \frac{\phi(m)\sigma_r(m)^2}{m^2} \log \left( \frac{y}{m} \right)^{r+v} \int_0^{1-|m|_y} F_3(\theta_1, m) d\theta_1 \int_0^1 F_1(\theta_2, m) d\theta_2
\]
where \( F_1, F_3 \) are given by (136). This is valid up to an error of \( O(L^{r^2+r+\max(u,v)}) \) and the calculation is analogous to the calculation we did in the last section for \( a_{u,v,w} \). Exchanging summation order and recalling (136) gives
\[
b_{u,v} = \frac{(\log y)^{2r+u+v}}{(r-1)!^2} \int_0^1 \int_0^1 \int_0^{1} \int_0^{1} \frac{\phi(m)\sigma_r(m)^2}{m^2} \frac{g([m]_y)}{d\theta_1 d\theta_2}
\]
where \( g(\delta) = (1 - \delta)^{r+u} P(\delta + \theta_1) P(\delta + (1 - \delta)\theta_2) \). By Lemma 11(ii) we have
\[
b_{u,v} = C_r (\log y)^{r^2+2r+u+v} \int_0^1 \int_0^1 \int_0^{1} \int_0^{1} \theta_1^{r-1}(\eta^{1-\theta_1})^u \theta_2^{r+1-v} \delta^{2-1} g(\delta) d\delta d\theta_1 d\theta_2
\]
plus an error \( O(L^{r^2+r+\max(u,v)}) \). Since \( Q_{r+v-1}(\delta) = \int_0^{1} \theta_2^{r+v-1} P(\delta + (1 - \delta)\theta_2) d\theta_2 \) it follows that
\[
b_{u,v} \sim C_r (\log y)^{r^2+2r+u+v} k_P(u, r, v, r + v - 1)
\]
where we recall \( 10 \)
\[
k_P(n_1, n_2, n_3) = \int_0^1 \int_0^{1} \int_0^{1} \theta_1^{r-1}(\eta^{1-\theta_1})^u \delta^{r^2-1}(1 - \delta)^{n_2} P(\theta_1 + \delta) Q_{n_3}(\delta) d\delta d\theta_1.
\]
We conclude

\[
I_{21} \sim C_r \frac{T}{2\pi} (\log y)^{r} 2^{r+2} \sum_{j=0}^{\infty} (z \eta)^j \left( \frac{k p(j+1,r+1,r)}{(j+1)!} - \frac{k p(j+2,r,r-1)}{(j+2)!} \right).
\]  

(136)

It can be checked that the error term \(O(L^{r+\max(u,v)})\) contributes an amount \(O(L^{-1})\) smaller than the main term.

6.4. Evaluation of \(I_{22}\). By [129]

\[
I_{22} = -\frac{T}{2\pi} \sum_{r=0}^{\infty} \frac{1}{j!} \sum_{u=0}^{\infty} \frac{(i \alpha)^u}{u!} c_{u,j}
\]

(137)

where

\[
c_{u,j} = \sum_{m \leq y} \frac{1}{m} \sum_{n \mid m} \frac{\mu(n)}{n} \left( \sum_{k \leq \frac{m}{n}} d_r(mh) P([mh]_y) \log \left( \frac{T}{2\pi h} \right)^u \right)
\]

\[
\times \left( \sum_{k \leq \frac{m}{n}} d_r(mk) P([mk]_y) \frac{\mathcal{R}^{(j)}_{nk}(1)}{\phi(nk)} \right),
\]

(138)

Applying partial summation to [109] yields

\[
\sum_{k \leq \frac{m}{n}} d_r(mk) P([mk]_y) \frac{\mathcal{R}^{(j)}_{nk}(1)}{\phi(nk)} = \frac{\sigma_r(m)(-1)^j j! (\frac{m}{n})^{r+j}}{n(r+j-1)!} \int_0^1 \theta_r^{r+j-1} P([m]_y + (1 - [m]_y) \theta)d\theta + O(E(y))
\]

(139)

(140)

where \(E(y)\) denotes the error term in [109]. We apply Lemma 10(iii) to the first factor in [138] and we apply [140] to the second factor of [138] to obtain

\[
c_{u,j} = \frac{(-1)^j j! (\frac{m}{n})^{u+r}}{(r-1)!(r+j-1)!} \sum_{m \leq y} \sigma_r(m)^2 \log \left( \frac{y}{m} \right)^{r+j} \left( \sum_{n \mid m} \frac{\mu(n)}{n^{1+i\alpha}} \right)
\]

\[
\times \int_0^{\frac{1}{\theta_1}} F_3(\theta_1, m)d\theta_1 \int_0^{\frac{1}{\theta_2}} \theta_1^{r+j-1} P([m]_y + (1 - [m]_y) \theta_2) d\theta_2
\]

where \(F_3(\theta_1, m) = \theta_1^{r-1}(\eta^{-1} - \theta_1)^u P([m]_y + \theta_1)\). Further simplification gives

\[
c_{u,j} = \frac{(-1)^j j! (\frac{m}{n})^{u+r+j}}{(r-1)!(r+j-1)!} \int_0^1 \int_0^1 \theta_1^{r-1} \theta_2^{r+j-1} (\eta^{-1} - \theta_1)^u \sum_{m \leq y} \frac{\sigma_r(m)^2}{m} \frac{\mu(n)}{n^{1+i\alpha}} (1 - [m]_y)^{r+j} P(\theta_1 + [m]_y) P([m]_y + (1 - [m]_y) \theta_2) d\theta_1 d\theta_2.
\]
Now note that $\sum_{n|m} n^{-1-i\alpha} = \frac{\phi(m)}{m} + O(\alpha|\sum_{n|m} n^{-1})$. Thus we have

$$c_{u,j} = \frac{(-1)^{j}j!(\log y)^{2r+u+j}}{(r-1)!(r+j-1)!} \int_{0}^{1} \int_{0}^{1} \theta_{1}^{-1}\theta_{2}^{r+j-1}(\eta^{-1} - \theta_{1})^{u} \, d\theta_{1}d\theta_{2}$$

(141)

plus an error term of the shape

$$\ll_{r,j,u} |\alpha|(|\log y|)^{2r+u+j} \sum_{m \leq y} \frac{\sigma_{r}(m)^{2}}{m^{2}} \sum_{n|m} \frac{1}{n} \ll_{r,j,u} \sigma_{r}(n)^{2} \sum_{k \leq y/m} \frac{\sigma_{r}(k)^{2}}{k} \ll_{r,j,u} \sigma_{r}(n)^{2} \sum_{k \leq y/m} \frac{\sigma_{r}(k)^{2}}{k} \ll_{r,j,u} \sigma_{r}(n)^{2} = \ll_{r,j,u} L^{2r+u+j-1}.\quad (142)$$

Note that we can write down the constant in the $O$ term explicitly in terms of $r, j,$ and $u$. Applying Lemma 11 to the inner sum we derive

$$c_{u,j} = \frac{a_{r+1}(-1)^{j}j!(\log y)^{2r+u+j}}{(r-1)!(r+j-1)!(r^2-1)!} \cdot \int_{0}^{1} \int_{0}^{1} \theta_{1}^{-1}\theta_{2}^{r+j-1}(\eta^{-1} - \theta_{1})^{u} \delta^{2-1}R(\delta)d\delta d\theta_{1}d\theta_{2}$$

where $R(\delta) = (1 - \delta)^{r+j}P(\theta_{1} + \delta)P(\delta + (1 - \delta)\theta_{2})$ and this is valid up to an error of $O_{r,j,u}(L^{2r+2r+u+j-1})$. If we recall the definition $Q_{u}(\delta) = \int_{0}^{1} \theta_{1}^{u}P(\delta + (1 - \delta)\theta_{2})d\theta_{2}$ and then execute the integration in the $\theta_{2}$-variable this becomes

$$c_{u,j} \sim \frac{a_{r+1}(-1)^{j}j!(\log y)^{2r+u+j}}{(r-1)!(r+j-1)!(r^2-1)!} \cdot \int_{0}^{1} \int_{0}^{1} \theta_{1}^{-1}\theta_{2}^{r+j-1}(\eta^{-1} - \theta_{1})^{u} \delta^{2-1}(1 - \delta)^{r+j}P(\theta_{1} + \delta)Q_{r+j-1}(\delta)d\delta d\theta_{1}.\quad (143)$$

Recalling definitions (10) and (11) we have

$$c_{u,j} = \frac{a_{r+1}(-1)^{j}j!(\log y)^{2r+u+j}}{(r-1)!(r+j-1)!} \cdot \int_{0}^{1} \int_{0}^{1} \theta_{1}^{-1}\theta_{2}^{r+j-1}(\eta^{-1} - \theta_{1})^{u} \delta^{2-1}(1 - \delta)^{r+j}P(\theta_{1} + \delta)Q_{r+j-1}(\delta)d\delta d\theta_{1}.\quad (143)$$

Combining (13) and (143) establishes that $I_{22}$ is $-(r - 1)!C_{r} \frac{|\log y|^{2r+2}}{2\pi}$ multiplied by the series

$$\sum_{j=2}^{r} \frac{(-1)^{j}j!}{(r+j-1)!} \sum_{u=0}^{\infty} \frac{(i\alpha \log y)^{u}}{u!} k_{p}(u, r + j, r + j - 1)$$

$$= \sum_{j=0}^{r-2} \frac{(-1)^{j}j!}{(r+j+1)!} \sum_{u=0}^{\infty} \frac{(i\alpha \log y)^{u}}{u!} k_{p}(u, r + j + 2, r + j + 1)$$

$$= \sum_{j=0}^{r-2} \frac{(-1)^{j}j!}{(r+j+2)!} \sum_{n=j}^{\infty} \frac{(\eta z)^{n}}{(n-j)!} k_{p}(n-j, r+j+2, r+j+1)$$
where we changed \( j - 2 \to j \) and then made the variable change \( n = u + j \) in the inner sum. Moreover, we can check that the error term \( O_{r,j,u}(L^{r^2 + 2r + u + j - 1}) \) when substituted in (137) is smaller than the main term by a factor of \( O(L^{-1}) \). We now write \( I_{22} = I'_{22} + I''_{22} \) where \( I'_{22} \) is the contribution from the \( j = 0 \) term and \( I''_{22} \) is the rest.

\[
I'_{22} = - \frac{(r-1)!C_r}{2(r+1)} \sum_{n=0}^{\infty} \frac{z^n \eta^{(r+1)^2+1} n!}{n} k_P(n, r+2, r+1) \quad (144)
\]

\[
I''_{22} = - (r-1)!C_r \frac{T}{2\pi} L^{(r+1)^2+1} \sum_{n=1}^{\infty} \frac{z^n \eta^{(r+1)^2+1}}{n!} \cdot \sum_{1 \leq j \leq \min(n, r-2)} \frac{(-1)^j (r+2)}{(n-j)! (r+j+1)!} k_P(n-j, r+j+2, r+j+1). \quad (145)
\]

6.5. **Evaluating \( I \).** We collect our estimates to conclude the evaluation of \( I \). Since \( I = I_1 + I_{21} + I'_{22} + I''_{22} \) plus error terms it follows from (133), (136), (144), and (145) that

\[
I \sim C_r \frac{T}{2\pi} L^{(r+1)^2+1} \left( \sum_{j=1}^{\infty} \frac{z^j \eta^{j+(r+1)^2+1}}{j!} \left( \frac{\hat{r}(r, \eta, j)}{j!} + \hat{k}_1(r, \eta, j) + \hat{k}_2(r, \eta, j) \right) \right) + \text{CT}(I) \quad (146)
\]

where \( \text{CT}(I) \) denotes the constant term in the above Taylor series,

\[
\hat{i}(r, \eta, j) = -i_P(r, r, j, r-1, r-1) \eta^{-1} + (i_P(r+1, r, j, r-1) + i_P(r, r+1, j, r-1)),
\]

\[
\hat{k}_1(r, \eta, j) = - \frac{k_P(j+2, r, r-1)}{(j+2)!} + \frac{k_P(j+1, r+1, r)}{(j+1)!} - \frac{(r-1) k_P(j, r+2, r+1)}{2(r+1) j!},
\]

\[
\hat{k}_2(r, \eta, j) = -(r-1)! \sum_{u=1}^{\min(j, r-2)} \frac{(-1)^u (r+2)}{(j-u)! (r+u+1)!} k_P(j-u, r+u+2, r+u+1).
\]

Next remark that we may conveniently combine \( \hat{k}(r, \eta, j) = \hat{k}_1(r, \eta, j) + \hat{k}_2(r, \eta, j) \) to obtain

\[
\hat{k}(r, \eta, j) = -(r-1)! \sum_{u=-2}^{\min(j, r-2)} \frac{(-1)^u (r+2)}{(j-u)! (r+u+1)!} k_P(j-u, r+u+2, r+u+1). \quad (147)
\]

This completes the evaluation of \( I \).

6.6. **The final details.** We now complete the proof of Theorem 2. In order to abbreviate the following equations we put

\[
\theta = C_r \frac{T}{2\pi} L^{(r+1)^2+1}, \ a = \eta^{(r+1)^2-1}, \ b = \eta^{(r+1)^2}, \text{ and } c = \eta^{(r+1)^2+2}. \quad (148)
\]

Recall that the discrete moment we are evaluating satisfies

\[
m(H_r, T; \alpha) = 2 \text{Re}(I) - J + O(yT^{1/2+\epsilon}) \quad (149)
\]
Moreover, we showed \(^{(148)}\) that \(J = \text{CT}(J)(1 + O(L^{-1}))\) where
\[
\text{CT}(J) = -\theta \left( a \int_0^1 \alpha^{r-1}(1 - \alpha)^{2r} Q_r(\alpha) \, d\alpha \right)
- \frac{2b}{\alpha} \int_0^1 \alpha^{r-1}(1 - \alpha)^{2r+1} Q_r(\alpha) \, d\alpha \ .
\]
(150)

We shall now combine \((146)\) and \((150)\) in \((149)\) to finish the proof. In particular we shall now prove that \(2\text{CT}(I) = \text{CT}(J)\) and hence \(\text{CT}(m(H_r, T, \alpha)) = 0\). This was expected since the constant term in the Taylor series of \(\zeta(\rho + \alpha)\) is zero for each \(\rho\). Moreover, the fact that the constant term must be zero provides a consistency check of our calculation. We now verify that \(2\text{CT}(I) = \text{CT}(J)\). Recall that \(\text{CT}(I) = \text{CT}(I_1) + \text{CT}(I_{21}) + \text{CT}(I_{22})\). From \((148)\) we have

\[
\text{CT}(I_1) = \sum_{j=0}^{n} \frac{2}{r+j} \left( \frac{1}{i} \right)^{r+j} \int_0^1 x^{r+j} (1 - x)^{2r+1} Q_r(\alpha) \, dx \ .
\]

Each of the above integrals has the form
\[
\int_0^1 \int_0^1 x^{r-1}(1 - x)^u(1 - y - x)^v Q_u-1(x)Q_v-1(x+y) \, dy \, dx
\]
(151)

for \((u, v) = (r, r), (r+1, r), (r, r+1)\). Note that we have the identity
\[
(1 - x)^{n+1}Q_n(x) = \int_0^1 \beta^n P(x + \beta) \, d\beta .
\]

(152)

One may deduce from \((152)\) that
\[
\frac{1}{v} (1 - x)^{u+1} Q_v(x) = \int_0^1 (1 - x - y)^v Q_{v-1}(x+y) \, dy .
\]

and hence
\[
\text{CT}(I_{21}) = \frac{1}{v} \int_0^1 x^{r-1}(1 - x)^u v^{+1} Q_u-1(x)Q_v(x) \, dx .
\]

(153)

It follows that

\[
\text{CT}(I_1) = \theta \cdot \left( a \int_0^1 x^{r-1}(1 - x)^{2r} Q_r(\alpha) \, dx \right)
+ b \left( \int_0^1 x^{r-1}(1 - x)^{2r+1} Q_r(\alpha) \, dx \right)
+ \frac{r}{r+1} \left( \int_0^1 x^{r-1}(1 - x)^{2r+2} Q_{r+1}(x) \, dx \right).
\]

(154)

(155)

By \((148)\) we have \(\text{CT}(I_{21}) = \theta \eta (r+1)^2 + \eta (1/2) \eta q(2, r+1, r) - (1/2) \eta q(2, r+1, r-1)\). Expanding out the factor \((\eta^{-1} - \eta_1)^2\) in the definition of \(\eta q\) we have

\[
\theta \eta (r+1)^2 + \eta q(2, r+1, r) - (1/2) \eta q(2, r+1, r-1) \sim \theta \eta (r+1)^2 .
\]

(156)
However, by (152) this simplifies to
\[ \theta_{\eta}^{(r+1)^2+1} k_p(2, r, r - 1) \sim \theta \left( a \int_0^1 \delta^{r-1}(1 - \delta)^{2r} Q_{r-1}(\delta)^2 \, d\delta - 2b \int_0^1 \delta^{r-1}(1 - \delta)^{2r+1} Q_{r-1}(\delta)Q_r(\delta) \, d\delta \right) = \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+2} Q_{r-1}(\delta)Q_{r+1}(\delta) \, d\delta. \] (156)

Moreover, a similar calculation establishes
\[ \theta_{\eta}^{(r+1)^2+1} k_p(1, r + 1, r) \sim \theta \left( b \int_0^1 \delta^{r-1}(1 - \delta)^{2r+1} Q_{r-1}(\delta)Q_r(\delta) \, d\delta - c \int_0^1 \delta^{r-1}(1 - \delta)^{2r+2} Q_r(\delta)^2 \, d\delta \right). \] (157)

Combining (156) and (157) establishes
\[ \text{CT}(I_2) = \theta \left( -\frac{a}{2} \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r} Q_{r-1}(\delta)^2 \, d\delta + 2b \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+1} Q_{r-1}(\delta)Q_r(\delta) \, d\delta - c \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+2} (Q_r(\delta)^2 + \frac{1}{2} Q_{r-1}(\delta)Q_{r+1}(\delta)) \, d\delta \right) \]

In a similar way, it follows from (144)
\[ \text{CT}(I_{22}) = -\theta \frac{(r - 1)}{2(r + 1)} c \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+2} Q_{r-1}(\delta)Q_{r+1}(\delta) \, d\delta. \]

Combining constant terms yields \( \text{CT}(I) = \theta(c_1a + c_2b + c_3c) \) where
\[ c_1 = -\frac{1}{2} \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r} Q_{r-1}(\delta)^2 \, d\delta, \]
\[ c_2 = \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+1} Q_{r-1}(\delta)Q_r(\delta) \, d\delta, \]
\[ c_3 = \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+2} (Q_r(\delta)^2(1 - 1) + Q_{r-1}(\delta)Q_{r+1}(\delta) \left( \frac{r}{r + 1} - \frac{1}{2(r + 1)} \right)) \, d\delta. \]

Observe that \( c_3 = 0 \) and hence we have shown that
\[ \text{CT}(J) = \theta \left( -\frac{a}{2} \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r} Q_{r-1}(\delta)^2 \, d\delta + b \int_0^1 \delta^{r-2-1}(1 - \delta)^{2r+1} Q_{r-1}(\delta)Q_r(\delta) \, d\delta \right). \]

However, glancing back at (150) we see that \( 2\text{CT}(I) = \text{CT}(J) \). By this fact, (146), (147), and (149) we finally deduce
\[ m(H_r, T; \alpha) \sim C \pi L^{(r+1)^2+1} \Re \left( \sum_{j=1}^{\infty} \left( i\alpha L \right)^j \eta^{j+(r+1)^2+1} \left( \frac{r \hat{g}(r, \eta, j)}{j!} + \hat{k}(r, \eta, j) \right) \right). \]
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