Monadic Second-Order Logic with Path-Measure Quantifier is Undecidable

Raphaël Berthon  
Université libre de Bruxelles, Belgium  
Emmanuel Filiot  
Université libre de Bruxelles, Belgium  
Shibashis Guha  
Université libre de Bruxelles, Belgium  
Bastien Maubert  
Università degli Studi di Napoli Federico II, Italy  
Aniello Murano  
Università degli Studi di Napoli Federico II, Italy  
Laureline Pinault  
ENS Lyon, France  
Jean-François Raskin  
Université libre de Bruxelles, Belgium  
Sasha Rubin  
Università degli Studi di Napoli Federico II, Italy

Abstract

We consider an extension of monadic second-order logic, interpreted over the infinite binary tree, by the qualitative path-measure quantifier. This quantifier says that the set of infinite paths in the tree satisfying a formula has Lebesgue-measure one. We prove that this logic is undecidable. To do this we prove that the emptiness problem of qualitative universal parity tree automata is undecidable. Qualitative means that a run of a tree automaton is accepting if the set of paths in the run that satisfy the acceptance condition has Lebesgue-measure one.

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1 Introduction

Monadic Second-Order logic (MSO) is an extension of first-order logic with quantification on sets. The fundamental result about MSO is that its theory on the infinite binary tree is decidable [19]. There are a number of ways to extend this result: to structures generated by certain operations (see the survey [20]), by certain additional unary predicates [11], and by certain generalised quantifiers [2, 15]. In this note we consider MSO+∀=1 path, the extension of MSO by the measure-theoretic quantifier ∀=1 path X.ϕ, introduced in [14]. Here, ∀=1 path X.ϕ states that the set of paths of the tree that satisfy ϕ has Lebesgue-measure equal to 1. This means, intuitively, that a random path almost surely satisfies ϕ, where a random path is generated by repeatedly flipping a fair coin to decide to go to left or right. A weak fragment of this

* We became aware that this result was already proved by Mikołaj Bojańczyk, Edon Kelmendi and Michał Skrzypczak through a work that was conducted independently of ours [6]. Our proof technique relies on probabilistic automata, which differs from the approach of Bojańczyk et al.
logic is known to be decidable \cite{15}, and a more general one (in which the measure quantifier ranges over all sets instead of just paths) was proved undecidable in \cite{15}.

We prove that MSO+\textsubscript{path} is undecidable by encoding the emptiness problem of qualitative universal parity tree automata (the encoding is direct). Such an automaton accepts a tree \( t \) if every run \( \rho \) on \( t \) has the property that the Lebesgue-measure of the set of branches of \( \rho \) satisfying the parity condition is equal to 1. Thus, the main technical contribution of this note is that this emptiness problem is undecidable (Theorem \ref{t:undec}).

\section{Preliminaries}

Given a finite non-empty set \( \Sigma \), called an alphabet, we write \( \Sigma^* \) and \( \Sigma^\omega \) for the set of finite and infinite words over \( \Sigma \), respectively. For a finite word \( w = a_0 \ldots a_{n-1} \) we write \( |w| = n \) for its \textit{length}, and if \( w \) is an infinite word we let \( |w| = \omega \). For a word \( w \) and index \( i < |w| \), we let \( w_i \) be the letter at position \( i \) in \( w \). For a finite word \( w \in \Sigma^* \), the set \( \text{Cone}(w) = w \cdot \Sigma^\omega \) of infinite words is called the \textit{cone} of \( w \). For a function \( f : A \to B \), we write its \textit{codomain} \( \text{codom}(f) = \{ f(a) \mid a \in A \} \).

The set \( \{0, 1\}^* \) is called the \textit{infinite binary tree}. A \( \Sigma \)-\textit{tree} (or simply \textit{tree}) is a mapping \( t : \{0, 1\}^* \to \Sigma \). We write \( T_\Sigma \) for the set of \( \Sigma \)-trees. The elements of \( \{0, 1\}^* \) are called \textit{nodes}. We call \( \epsilon \) the \textit{root} node, and for every node \( u \in \{0, 1\}^* \), \( u \cdot 0 \) and \( u \cdot 1 \) are called the \textit{children} of \( u \). A \textit{branch} in a tree is an infinite sequence of nodes \( u_0 u_1 u_2 \ldots \) such that \( u_0 = \epsilon \) and for all \( i \geq 0 \), \( u_{i+1} \) is a child of \( u_i \). Alternatively, a branch \( u_0 u_1 u_2 \ldots \) can be seen as an infinite word \( \tau = \lim_{i \to \infty} u_i \in \{0, 1\}^\omega \). This way, each node \( u \) induces a cone \( \text{Cone}(u) = u \cdot \{0, 1\}^\omega \) of branches. Finally, given a branch \( \tau \in \{0, 1\}^\omega \), we let \( t(\tau) = t(\epsilon)t(\tau_0)t(\tau_0\tau_1)\ldots \) be the sequence of labels along this branch, i.e., we lift \( t \) to be a function \( t : \{0, 1\}^\omega \to \Sigma^\omega \).

Next, we recall various kinds of automata on words and trees that involve probabilistic aspects: in their transitions and/or their acceptance conditions.

\subsection{Probabilistic word automata}

A \textit{probabilistic word automaton} \( \mathcal{B} \) is a tuple \( (Q, \Sigma, \delta, q_i, \text{Acc}) \) where

\begin{itemize}
  \item \( Q \) is a finite set of states,
  \item \( \Sigma \) is an alphabet,
  \item \( \delta : Q \times \Sigma \times Q \to [0, 1] \) is a probabilistic transition function, i.e. for all \( q \in Q \) and \( \sigma \in \Sigma \), we have \( \sum_{p \in Q} \delta(q, \sigma, p) = 1 \),
  \item \( q_i \) is the initial state,
  \item \( \text{Acc} \subseteq Q^\omega \) is an acceptance condition.
\end{itemize}

A \textit{run} \( r \) of \( \mathcal{B} \) on \( w \in \Sigma^\omega \) is an infinite word over \( Q \) such that \( r_0 = q_i \) and for all \( i \geq 0 \), \( \delta(r_i, w_i, r_{i+1}) > 0 \). A run \( r \) is \textit{accepting} if \( r \in \text{Acc} \). We write \( \text{Runs}_\mathcal{B}^w \) and \( \text{AccRuns}_\mathcal{B}^w \) for the sets of runs and accepting runs, respectively, of \( \mathcal{B} \) on \( w \). We recall here certain \( \omega \)-regular acceptance conditions \cite{17}, i.e., Büchi, co-Büchi, Rabin and parity. We denote by \( \text{inf}(r) \) the set of states that are visited infinitely often along the run \( r \). The Büchi and co-Büchi acceptance conditions are given in terms of a set of \textit{accepting states} \( \alpha \subseteq Q \). A run \( r \) is accepting for the Büchi acceptance condition if \( \text{inf}(r) \cap \alpha \neq \emptyset \) and a run \( r \) is accepting for the co-Büchi acceptance condition if \( \text{inf}(r) \cap \alpha = \emptyset \). The Rabin acceptance condition is given in terms of Rabin pairs \( \{ (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k) \} \) for some \( k \in \mathbb{N} \) with \( \alpha_i, \beta_i \subseteq Q \), and a run \( r \) is accepting if for some \( 1 \leq i \leq k \), we have that \( \text{inf}(r) \cap \alpha_i \neq \emptyset \) and \( \text{inf}(r) \cap \beta_i = \emptyset \). The parity acceptance condition is defined by a parity function \( \alpha : Q \to \{0, 1, \ldots, k\} \) for some \( k \in \mathbb{N} \).
A run is accepting for the parity condition iff \( \min_{q \in \text{inf}(r)} \{\alpha(q)\} \) is even, that is, the minimum value seen infinitely often is even.

For every word \( w \in \Sigma^\omega \), the automaton induces a probability distribution \( \mu_w \) on \( \text{Runs}_w^A \), via cones, \( \sigma \)-algebras and Carathéodory’s unique extension theorem (see e.g. [3, 7] for more details). For convenience we will write \( B(w) = \mu_w(\text{AccRuns}_w^B) \), and we call \( B(w) \) the value of \( B \) on \( w \). While nondeterministic (resp. universal) automata accept a word if some (resp. every) run is accepting, probabilistic automata allow for more involved semantics that depend on the value of the automaton on its input.

**Probable semantics.**

A word \( w \) is **probably accepted** by \( B \) if it is accepted with non-zero probability, i.e. \( B(w) > 0 \). The language \( L_{>0}^B \) is the set of words \( w \in \Sigma^\omega \) that are probably accepted by \( B \). The emptiness problem for \( B \) with probable semantics asks whether \( L_{>0}^B = \emptyset \).

**Almost-sure semantics.**

A word \( w \) is **almost-surely accepted** by \( B \) if the set of accepting runs has measure 1, that is, \( B(w) = 1 \). The language \( L_{=1}^B \) is the set of words \( w \in \Sigma^\omega \) that are almost-surely accepted by \( B \). The emptiness problem for \( B \) with almost-sure semantics asks whether \( L_{=1}^B = \emptyset \).

### 2.2 Tree automata

We first recall non-probabilistic tree automata together with their classical semantics and the recent **qualitative semantics** of [7].

A tree automaton is a tuple \( A = (Q, \Sigma, \Delta, q_\iota, \text{Acc}) \) where:

- \( Q \) is a finite set of states,
- \( \Sigma \) is a finite alphabet,
- \( \Delta \subseteq Q \times \Sigma \times Q \times Q \) is a transition relation,
- \( q_\iota \in Q \) is an initial state,
- and \( \text{Acc} \subseteq Q^\omega \) is an acceptance condition.

A run of \( A \) on a \( \Sigma \)-tree \( t \) is a \( Q \)-tree \( r \) such that:

- \( r(\varepsilon) = q_\iota \),
- \( \forall u \in \{0, 1\}^* \), we have \( (r(u), t(u), r(u \cdot 0), r(u \cdot 1)) \in \Delta \).

A branch \( \tau \in \{0, 1\}^\omega \) of a run \( r \) is **accepting** if \( r(\tau) \in \text{Acc} \), and a run is accepting if all its branches are accepting. A run \( r \) is **qualitatively accepting** if

\[
\mu(\{\tau \in \{0, 1\}^\omega \mid r(\tau) \in \text{Acc}\}) = 1,
\]

where \( \mu \) is the coin-flipping probability measure defined on cones as follows: for \( u \in \{0, 1\}^* \),

\[
\mu(\text{Cone}(u)) = \frac{1}{2^{|u|}} \quad \text{(see [3, 7, 4] for more details)}.
\]

**Tree languages**

We define the **qualitative nondeterministic** language of a tree automaton \( A \) as follows:

\[
L_{\exists}^A(\text{Qual}) = \{ t \mid \exists r \text{ s.t. } r \text{ is a run of } A \text{ on } t \text{ and } r \text{ is qualitatively accepting} \}.
\]

Similarly, we define the **qualitative universal** language of \( A \) as follows:

\[
L_{\forall}^A(\text{Qual}) = \{ t \mid \forall r \text{ s.t. } r \text{ is a run of } A \text{ on } t, r \text{ is qualitatively accepting} \}.
\]
2.3 Probabilistic tree automata

We now recall the probabilistic tree automata introduced in [7].

A probabilistic tree automaton is a tuple $A = (Q, \Sigma, \delta, q_0, \text{Acc})$ where:

- $Q$ is a finite set of states,
- $\Sigma$ is a finite alphabet,
- $\delta : Q \times \Sigma \times Q \rightarrow [0, 1]$ is a probabilistic transition function, i.e. for all $q \in Q$ and $\sigma \in \Sigma$, we have $\sum_{q_0, q_1 \in Q} \delta(q, \sigma, q_0, q_1) = 1$,
- $q_0 \in Q$ is an initial state,
- and $\text{Acc} \subseteq Q^\omega$ is an acceptance condition.

A run of a probabilistic tree automaton $A$ on a $\Sigma$-tree $t$ is a $Q$-tree $r$ such that the root is labelled with $q_0$ and for every $u \in \{0, 1\}^*$, it holds that $\delta(r(u), t(u), r(u \cdot 0), r(u \cdot 1)) > 0$. Accepting and qualitatively accepting runs are defined as before, and the set of runs of $A$ (resp. accepting runs and qualitatively accepting runs) on input tree $t$ is written $\text{Runs}^A_{\text{Acc}}$ (resp. $\text{AccRuns}^A$ and $\text{QualAccRuns}^A$). Given a tree $t$, one can define a probability measure $\mu_t$ on the space of runs (see [7]).

> **Remark 1.** The definition of runs for probabilistic tree automata in [7] allows for transitions with probability zero, while we disallow them. But the set $R_0$ of all runs that contain at least one such transition is a countable union of cones of partial runs of measure zero (this follows directly from the definitions of partial runs and cones of runs and their measures, see [7] Section 4.1.1 for details). Therefore $R_0$ has measure zero, and the restriction of the probability measure on $\text{Runs}^A_{\text{Acc}} \cup R_0$ to $\text{Runs}^A_{\text{Acc}}$ is a probability measure on $\text{Runs}^A_{\text{Acc}}$.

We define the almost-sure and qualitative almost-sure languages of $A$ as follows:

$$L_{\text{ass}}^1(A) = \{ t \mid \mu_t(\text{AccRuns}^A) = 1 \}.$$  
$$L_{\text{ass}}^1_{\text{Qual}}(A) = \{ t \mid \mu_t(\text{QualAccRuns}^A) = 1 \}.$$

As shown in [7], acceptance of trees for the qualitative almost-sure semantics can be characterised via Markov chains, which will be useful later on.

> **Definition 2.** A Markov chain is a tuple $M = (S, s_i, \delta, \text{Acc})$ where

- $S$ is a countable set of states,
- $s_i$ is an initial state,
- $\delta : S \times S \rightarrow [0, 1]$ is a probabilistic transition function such that for all $s \in S$, we have $\sum_{s' \in S} \delta(s, s') = 1$, and
- $\text{Acc} \subseteq S^\omega$ is an objective.

A run is an infinite sequence of states, and $M$ induces a probability measure on runs. A Markov chain $M$ almost-surely fulfils its objective if the set of runs in $\text{Acc}$ has measure one.

> **Definition 3.** Given a probabilistic tree automaton $A = (Q, \Sigma, \delta, q_0, \text{Acc})$ and a $\Sigma$-tree $t$, we define the (infinite) Markov chain $M^A_t = (S, s_i, \delta', \text{Acc}')$ where:

- $S = Q \times \{0, 1\}^* \cup Q \times Q \times \{0, 1\}^*$,
- $s_i = (q_i, \epsilon)$,
- for all $q, u, q_0, q_1$,
  - $\delta'((q, u), (q_0, q_1, u)) = \delta(q, t(u), q_0, q_1)$,
  - $\delta'((q, q_0, q_1, u), (q_0, u \cdot 0)) = \delta'((q, q_0, q_1, u), (q_1, u \cdot 1)) = 1/2$, and
  - $\delta'(s, s') = 0$ in all other cases;
\textbf{Acc}' is inherited from Acc: a run is in Acc if, after removing states of the form 
\((q, q_0, q_1, x)\) and projecting states of the form \((q, x)\) on \(Q\), we obtain a run in Acc.

The following result is established in \cite{7} Proposition 45).

\textbf{Proposition 4.} Let \(A\) be a probabilistic tree automaton with \(\omega\)-regular acceptance condition, and let \(t\) be a tree. It holds that \(t \in L_{\text{Qual}}(A)\) iff \(M_A^t\) almost-surely fulfils its objective.

3 \(L_{\text{Qual}}\)-emptiness is undecidable for parity tree automata

In this section we prove our main undecidability result on tree automata, from which we will derive the undecidability on MSO in Section 4. The undecidability result on tree automata comes from some undecidability result on word automata. In a few words, undecidability of the almost-sure emptiness problem was known to be undecidable for Rabin word automata \cite{1}. We strengthen this result to parity word automata with binary branching (for every input letter, each state has exactly two outgoing transitions with \(\frac{1}{2}\) probability). Then, we exploit this result to show undecidability of the emptiness problem for qualitative parity tree automata. The main lines of this proof were sketched in an internship report \cite{18}. The main addition we bring is to prove that we can indeed restrict attention to automata with binary branching, a central assumption in the proof of the main result that was not justified in \cite{18}. Further, we do not know whether this assumption can be made in the case of co-Büchi as claimed in \cite{18}, but we prove that for Rabin and parity automata one can indeed assume binary branching while retaining undecidability of the emptiness problem.

3.1 Restricting to binary branching

We recall the notion of simple automata considered in \cite{13}, and introduce its restriction to binary branching, and a more general class of semi-simple automata, whose emptiness problem we prove to be reducible to the emptiness problem for binary-branching automata.

\textbf{Definition 5.} A probabilistic word automaton \(B = (Q, \Sigma, \delta, q_i, Acc)\) is:

- binary branching if \(\text{codom}(\delta) = \{0, \frac{1}{2}\}\);
- simple if \(\text{codom}(\delta) = \{0, \frac{1}{2}, 1\}\);
- semi-simple if \(\text{codom}(\delta) \subseteq \{\frac{p}{q} \mid p, q \in \mathbb{N}\}\).

In this section we strengthen the following known theorem to binary-branching parity word automata. It will be used in Section 3.2 to establish an undecidability result for parity tree automata.

\textbf{Proposition 6 (\cite{11}).} The problem whether \(L_{\text{Qual}}^{=1}(B) = \emptyset\) is undecidable for Rabin word automata.

To strengthen this result to binary-branching parity automata, we need a series of lemmas. The following result strengthens a result known from \cite{11} to simple automata.

\textbf{Lemma 7.} The problem whether \(L_{\text{Qual}}^{>0}(B) = \emptyset\) is undecidable for simple Büchi word automata.

\textbf{Proof.} It is proved in \cite{13} \cite{12} \cite{16} that the emptiness problem for simple probabilistic automata on finite words is undecidable \cite{16} Theorem 6.12. This result is used to prove that the value 1 problem for probabilistic automata on finite words is undecidable \cite{16} Theorem 6.23.
Lemma 9. In [1, Theorem 5.3], it has been proved that for every probabilistic Büchi word automaton on finite words to the emptiness problem for Büchi automata with probable semantics. Once again, this reduction only introduces transitions with probability one, hence the result.

Let $A$ be a word automaton with a set of accepting states $a$, and we note $A_{\text{Bü}}$ and $A_{\text{coBü}}$ the Büchi and coBüchi interpretations of $A$, respectively. Then clearly $\mathcal{L}^{-1}(A_{\text{Bü}}) = \mathcal{L}^{-1}(A_{\text{coBü}})$. It is known that probabilistic Büchi word automata with probable semantics are closed under complement [1], therefore there exists a Büchi automaton $A'_{\text{Bü}}$, such that $\mathcal{L}^{>0}(A'_{\text{Bü}}) = \mathcal{L}^{-1}(A_{\text{coBü}})$. While this implies the undecidability of the emptiness problem for coBüchi word automata with the almost-sure semantics, the automaton $A'_{\text{Bü}}$ obtained by the complementation procedure of [1] is neither simple nor semi-simple in general. To the best of our knowledge it is open whether the almost-sure emptiness problem for simple Rabin and parity word automata is indeed undecidable.

Lemma 8. For every simple Büchi word automaton $B$ one can construct a semi-simple Rabin word automaton $B'$ such that $\mathcal{L}^{>0}(B) = \mathcal{L}^{>0}(B')$ and for every $w \in \Sigma^\omega$, $B'(w) \in \{0, 1\}$.

Proof. In [1, Theorem 5.3], it has been proved that for every probabilistic Büchi word automaton $B$, there exists a probabilistic Rabin word automaton $B'$ for which for every $w \in \Sigma^\omega$, we have $B'(w) \in \{0, 1\}$ and also $\mathcal{L}^{>0}(B) = \mathcal{L}^{>0}(B')$ holds. In the proof of the above theorem, the probabilities of the transitions in the Rabin word automaton that is constructed are finite sums of finite products of transition probabilities in the original Büchi automaton, hence the result.

Note that in Lemma 8, $B'$ satisfies that $\mathcal{L}^{-1}(B') = \mathcal{L}^{>0}(B')$. Hence from Lemma 7 and Lemma 8, we get:

Lemma 9. The problems whether $\mathcal{L}^{>0}(B) = \emptyset$ and whether $\mathcal{L}^{-1}(B) = \emptyset$ are undecidable for semi-simple Rabin word automata.

Now, we show how to obtain a simple automaton from a semi-simple automaton while preserving language emptiness.

Lemma 10. For every semi-simple Rabin word automaton $B$ one can construct a simple Rabin word automaton $B'$ such that $\mathcal{L}^{-1}(B) = \emptyset$ iff $\mathcal{L}^{-1}(B') = \emptyset$, and $\mathcal{L}^{>0}(B) = \emptyset$ iff $\mathcal{L}^{>0}(B') = \emptyset$.

Proof. Let $B = (Q, \Sigma, \delta, q_0, \text{Acc})$ be a semi-simple word automaton, i.e. for all $q, q' \in Q$ and $a \in \Sigma$, $\delta(q, a, q') = c/2^d$ for some $c, d \in \mathbb{N}$. Since there are finitely many states we can assume that $d$ is the same for all $q, a, q'$ by taking $d$ as the largest of all $d'$ occurring on the transitions and multiplying the constants $c$ accordingly. For every $q \in Q$ and $a \in \Sigma$, we simulate the possible transitions from $q$ when reading $a$ with a full binary tree of transitions of depth $d$, where the root is $q$ and the leaves are the destination states (see Figure 1, here $d = 3$). To do so we introduce a set of $2^d - 2$ fresh states $(q, a)_b$ for the internal nodes of the binary tree of transitions. They are indexed by all finite words $b \in \{0, 1\}^*$ of length at most $d - 1$, and the transitions are as follows: first, they all have probability one half, except for the last level. Second, in state $q$ when reading $a$, the two possible transitions
Consider a simple word automaton

Lemma 11. For every simple Rabin word automaton $A$, one can construct a binary-branching Rabin word automaton $B$ such that $L_1(A) = L_1(B)$ and $L^{>0}(A) = L^{>0}(B)$.

Proof. Consider a simple word automaton $A = (Q, \Sigma, \delta, q_0, \text{Acc})$ with Rabin (resp. parity) acceptance condition. We construct a binary-branching automaton with Rabin (resp. parity) acceptance condition from $A$. First we define $\delta_1 \subseteq \delta$, the set of transitions that have probability 1: $\delta_1 = \{(p,a,q) \in Q \times \Sigma \times Q \mid \delta(p,a,q) = 1\}$. We define similarly $\delta$, to be
the set of transitions with probability $\frac{1}{2}$. Note that since $\mathcal{A}$ is simple, for all $(p, a, q)$ that is not in $\delta_1 \cup \delta_2$, we have that $\delta(p, a, q) = 0$. We also let $Q_1$ be the set of destination states of some transition in $\delta_1$, that is, $Q_1 = \{ q \mid \exists \beta \in Q_1, \exists a \in \Sigma, (p, a, q) \in \delta_1 \}$. For each state $q \in Q_1$, in the binary-branching automaton, we create a fresh state $q'$ (the primed version of $q$) and every transition $(p, a, q) \in \delta_1$ is split into two transitions $(p, a, q)$ and $(p, a, q')$, each with probability $\frac{1}{2}$.

Formally, let $Q'_1 = \{ q' \mid q \in Q_1 \}$ be a set of fresh states. We construct the binary-branching Rabin (resp. parity) word automaton $B = (Q', \Sigma, \delta', q_0, \text{Acc}')$, where $Q' = Q \cup Q'_1$, and $\delta'$ is defined as follows:

- For every $(p, a, q) \in \delta_1$ such that $p \notin Q_1$,
  \[ \delta'(p, a, q) = \delta'(p, a, q') = \frac{1}{2} \]

- For every $(p, a, q) \in \delta_1$ such that $p \in Q_1$,
  \[ \delta'(p, a, q) = \delta'(p, a, q') = \delta'(p', a, q) = \delta'(p', a, q') = \frac{1}{2} \]

- For every $(p, a, q) \in \delta_2$ such that $p \notin Q_1$,
  \[ \delta'(p, a, q) = \frac{1}{2} \]

- For every $(p, a, q) \in \delta_2$ such that $p \in Q_1$,
  \[ \delta'(p, a, q) = \delta'(p', a, q) = \frac{1}{2} \]

and all other transitions are assigned probability 0 by $\delta'$.

Now we define Acc' for each of $\mathcal{A}$ being a simple Rabin automaton or $\mathcal{A}$ being a simple parity automaton. First, let $\mathcal{A}$ be a Rabin automaton. Let Acc be defined in terms of $\{ \langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_k, \beta_k \rangle \}$. We define Acc' in terms of the pairs $\{ \langle \alpha'_1, \beta'_1 \rangle, \ldots, \langle \alpha'_k, \beta'_k \rangle \}$, where $\alpha'_i = \alpha_i \cup \{ q' \mid q \in \alpha_i$ and $q' \in Q' \setminus Q \}$ and $\beta'_i = \beta_i \cup \{ q' \mid q \in \beta_i$ and $q' \in Q' \setminus Q \}$ for all $1 \leq i \leq k$.

From the construction of $B$, we see that for every word $w \in \Sigma^\omega$, the measure of the set of accepting runs on input $w$ is the same in both $\mathcal{A}$ and $B$, hence the result. \hfill $\blacktriangle$

Now from Lemma 9, Lemma 10 and Lemma 11 we obtain the following.

**Corollary 12.** The problems whether $L^{=1}(\mathcal{B}) = \emptyset$ and whether $L^{>0}(\mathcal{B}) = \emptyset$ are undecidable for binary-branching Rabin word automata.

Finally, in the classical (non-probabilistic) setting, Rabin and parity word automata are known to have the same expressive power. We show that it also holds under the probabilistic almost-sure and positive semantics, while preserving binary branching, and therefore we get the following result:

**Theorem 13.** The problems whether $L^{=1}(\mathcal{B}) = \emptyset$ and whether $L^{>0}(\mathcal{B}) = \emptyset$ are undecidable for binary-branching parity word automata.

**Proof.** We show that any binary-branching Rabin word automaton $\mathcal{B}$ can be converted into an equivalent binary-branching parity word automaton $\mathcal{B}'$. Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, \text{Acc})$ where Acc $\subseteq Q^\omega$ is a Rabin condition (explicitly given as a set of Rabin pairs). We know that any (non-probabilistic) Rabin word automaton is
effectively equivalent to some deterministic parity automaton. Therefore, there exists a
deterministic parity automaton $P$ over the alphabet $Q$ such that its language $L(P) = \text{Acc}$.
Let $P = (Q, \Sigma, \delta, i_p, \alpha)$ where $\alpha$ is a parity function. We construct the probabilistic
parity word automaton $B' = (Q \times Q_P, \Sigma, \delta', (q, p), \alpha')$ where

$\delta'(w, q, a) = \delta(q, a, q')$ if $p' = \delta(p, q')$, and $0$ otherwise.

Note that this construction preserves binary branching, and in particular we have $\delta'(w, p, a) = \{(q_1, \delta\tau(p, q_1)), (q_2, \delta\nu(p, q_2))\}$ if $\delta(q, a) = \{q_1, q_2\}$.

To show that $L^{{\geq}1}(B) = L^{{\geq}1}(B')$ holds, consider a word $w \in \Sigma^*$ and an arbitrary linear
order $<$ on $Q$. Consider the tree $t_w : \{0,1\}^* \to Q$ defined by $t_w(\epsilon) = q_0$ and for $u \in \{0,1\}^*$,
if $\delta(t_w(u), w[u]) = \{q_0, q_1\}$ with $q_0 < q_1$, then let $t_w(u \cdot i) = q_i$ for $i = 0,1$. We call $t_w$
the tree of runs on $w$, and let $\text{Acc}_{w} = \{\tau \in \{0,1\}^\omega \mid t_w(\tau) \in \text{Acc}\}$. The tree $t_w$, with
probability $\frac{1}{2}$ on all edges, equipped with the acceptance condition $\text{Acc}_{w}$, can be seen as an
infinite Markov chain which almost-surely fulfills its objective iff $w \in L^{{\geq}1}(B)$, and fulfills it
with positive probability iff $w \in L^{>{\geq}1}(B)$.

Similarly, we can define the infinite tree $t'_w : \{0,1\}^* \to Q \times Q_P$ as the tree of runs of $B'$
on $w$, using any partial order such that $(q_1, p_1) < (q_2, p_2)$ implies $q_1 < q_2$. Let also define the
acceptance condition $\text{Acc}'_{w} = \{\tau \in \{0,1\}^\omega \mid t'_w(\tau) \models \alpha'\}$, which by definition of $\alpha'$
is equal to $\{\tau \in \{0,1\}^\omega \mid \text{proj}_Q(t'_w(\tau)) \in \text{Acc}\}$, where $\text{proj}_Q(t'_w(\tau))$ is the
letter-by-letter projection of $t'_w(\tau)$ on the $Q$-component. Equipped with $\frac{1}{2}$ probabilities on edges
and this acceptance condition, $t'_w$ can be seen as an infinite Markov chain which almost-surely fulfills its
objective iff $w \in L^{{\geq}1}(B')$.

Finally, note that $t_w$ and $t'_w$ are isomorphic, and the projection $\text{proj}_Q : Q \times Q_P \to Q$
allows to get $t_w$ from $t'_w$ (by projecting its labels). Moreover, by definition of $t'_w$, we also have
that $\text{Acc}'_{w} = \text{Acc}_{w}$. Hence, seen as infinite Markov chains, $t_w$ and $t'_w$ are the same (up
to isomorphism). As a consequence, $w \in L^{{\geq}1}(B)$ iff $w \in L^{{\geq}1}(B')$, and for the same reason
also $w \in L^{>{\geq}1}(B)$ iff $w \in L^{>{\geq}1}(B')$.

### 3.2 From words to trees

In this section we use Theorem [13] to establish an undecidability result for tree automata,
but before we recall the following result which we will use in the proof.

For every probabilistic parity word automaton (PPW) $B = (Q, \Sigma, \delta, q_0, \text{Acc})$, we define the
probabilistic parity tree automaton (PPT) $A_B = (Q, \Sigma, \delta', q_0, \text{Acc})$ such that for all
$p, q \in Q$ and $a \in \Sigma$,

$\delta'(p, a, q, q') = \delta(p, a, q')$, and

$\delta'(p, a, q, q') = 0$ for $q \neq q'$.

\begin{proposition}
Let $L^{{\geq}1}(B) = \emptyset$ iff $L^{{\geq}1}_{\text{Qual}}(A_B) = \emptyset$.
\end{proposition}

\begin{proof}
In [7], Proposition 43], it is shown that $L^{{\geq}1}_{\text{Qual}}(A_B)$ is equal to the set of $\Sigma$-trees $t$
such that the measure of the branches $\tau$ of $t$ such that $t(\tau) \in L^{{\geq}1}(B)$ is 1. This immediately
yields the result. Indeed, if $L^{{\geq}1}(B) = \emptyset$, then no such tree $t$ exists.

Conversely, if $L^{{\geq}1}(B)$ contains one word $w$, it suffices to construct the $\Sigma$-tree $t$ such that for all $\tau \in \{0,1\}^\omega$, we
have $t(\tau) = w$. Clearly, the measure of the branches $\tau$ of $t$ such that $t(\tau) \in L^{{\geq}1}(B)$ is 1, and
therefore $t \in L^{{\geq}1}(A_B)$.
\end{proof}

We now describe a different construction that translates a binary-branching PPW $B = (Q, \Sigma, \delta, q_0, \text{Acc})$
to a PPT $A_B$, which we then show to be equivalent to $A_B$. The PPT $A_B$ is
defined as the tuple $(Q, \Sigma, \delta', q_0, \text{Acc})$ where for all states $q, q_1, q_2 \in Q$ and $a \in \Sigma,$
Lemma 15. Let $B$ be a binary-branching probabilistic parity word automaton. Then

\[ L_{\text{Qual}}(A_B) = L_{\text{Qual}}(A'_B). \]

Proof. The only difference between $A_B$ and $A'_B$ is that transitions in $A_B$ of the form

\[(q, a, q_1, q_2)\text{ and } (q, a, q_2, q_1), \text{ each with probability } \frac{1}{2}.\]

become in $A'_B$ transitions of the form

\[(q, a, q_1, q_2)\text{ and } (q, a, q_2, q_1), \text{ each with probability } \frac{1}{2}.\]

We show that for every tree $t$, the acceptance Markov chains $M_t$ and $M'_t$ are essentially the same. To do so, we construct a Markov chain $M$ that almost-surely fulfills its objective iff $M_t$ almost-surely does, and similarly $M_t$ almost-surely fulfils its objective iff $M'_t$ does. As a consequence, $M_t$ almost-surely fulfils its objective iff $M'_t$ does. Hence, by Proposition 11 we get that $t \in L_{\text{Qual}}(A'_B)$ iff $t \in L_{\text{Qual}}(A_B)$.

Let us now show how to construct $M$. We let

\[ M = (Q \times \{0, 1\}, \{q, q, 0, 0, 1\}, \delta, \text{Acc}_M) \]

where $\delta_M((q, u), s) = \frac{1}{2}$ for $s \in \{(q_1, u \cdot 0), (q_1, u \cdot 1), (q_2, u \cdot 0), (q_2, u \cdot 1)\}$, with $\delta(q, t(u)) = \{q_1, q_2\}$, and $\text{Acc}_M = \{\rho \in (Q \times \{0, 1\})^\ast \mid \text{proj}_Q(\rho) \in \text{Acc}\}.$

Observe that $M_t$ can be obtained either from $M_t$ or $M'_t$ by removing states of type $Q^3 \times \{0, 1\}$ and, for each such state, attaching its children to its parent, as illustrated in Figure 12. Indeed, since we have binary branching, and by construction of $A'_B$, in $M'_t$ each state of the form $(q, u)$ has exactly two successors with probability $\frac{1}{2}$, of the form $(q, q_1, q_2, u)$ and $(q, q_2, q_1, u).$ From $(q, q_1, q_2, u)$, we have two $\frac{1}{2}$ probability transitions, one to $(q_1, u \cdot 0)$ and one to $(q_1, u \cdot 1)$. Thus from state $(q, u)$ we have probability $\frac{1}{4}$ to reach each of the states in $\{(q_1, u \cdot 0), (q_2, u \cdot 1), (q_2, u \cdot 0), (q_1, u \cdot 1)\}$, and it is also the case in $M_t$. Finally, the acceptance condition of $M_{t,3}$ and $M_{t,4}$ are the same as in $M_t$, modulo projecting paths on states of type $Q \times \{0, 1\}$.

Therefore, one gets that $M_{t,3}$ almost-surely fulfills its objective iff $M_t$ almost-surely fulfills its objective iff $M_{t,4}$ almost-surely fulfills its objective.

We now establish the main result of this section.

Theorem 16. The problem whether $L_{\text{Qual}}(A) = \emptyset$ is undecidable for parity tree automata.

Proof. We reduce the almost-sure emptiness problem of probabilistic parity word automata with binary branching, which is undecidable by Theorem 13. Let $B = (Q, \Sigma, \delta, q_0, \text{Acc})$ be a probabilistic parity word automaton with binary branching. Construct a (non-probabilistic) parity tree automaton $A = (Q, \Sigma, \Delta, q_0, \text{Acc})$ where

\[ \Delta = \{(q, a, q_1, q_2), (q, a, q_2, q_1) \mid \delta(q, a) = \{q_1, q_2\}\}. \]

We claim that $L_{\text{Qual}}(B) = \emptyset$ iff $L_{\text{Qual}}(A) = \emptyset.$
1. \( \exists w \in L^{=1}(B) \implies \exists t \in L^\forall_{\text{Qual}}(A) \): Assume that \( w \in L^{=1}(B) \). Construct the tree \( t \) such that for all branches \( \tau \), we have \( t(\tau) = w \). Take any run \( r \) of \( A \) on \( t \), and define the set \( Y = \{ r \in \{0,1\}^* \mid r(\tau) \in \text{Acc} \} \) of accepting branches in \( r \). By definition of \( A \), the run \( r \) (lifted to infinite sequences) is a bijection between \( \{0,1\}^* \) and \( \text{Runs}^B_w \) that preserves acceptance (i.e., \( r(Y) = \text{AccRuns}^B_w \)), and it also induces a bijection \( f : u \mapsto r(1)r(u_0) \ldots r(u_0 \ldots u_{|u|-1}) \) between \( \{0,1\}^* \) and finite prefixes of runs in \( \text{Runs}^B_w \). We show that \( r \) is measurable, and that \( \mu_w \) is the image measure of \( \mu \) under \( r \), i.e. \( \mu \circ r^{-1} = \mu_w \).

We then conclude that \( \mu(Y) = \mu \circ r^{-1}(\text{AccRuns}^B_w) = \mu_w(\text{AccRuns}^B_w) = 1 \), as required.

To see that \( r \) is measurable, it is enough to see that for every cone \( \text{Cone}(\rho) \subseteq \text{Runs}^B_w \), where \( \rho \) is a finite prefix of a run in \( \text{Runs}^B_w \), we have \( r^{-1}(\text{Cone}(\rho)) = \text{Cone}(r^{-1}(\rho)) \).

We now show that \( \mu \circ r^{-1} \) and \( \mu_w \) coincide on cones. Then, by Carathéodory’s unique extension theorem, we get that they coincide on all measurable sets. Let \( u \in \{0,1\}^* \), and recall that \( f \) is a bijection between \( \{0,1\}^* \) and finite prefixes of runs in \( \text{Runs}^B_w \). On the one hand, because all (non-zero) transitions in \( B \) have probability \( \frac{1}{2} \) and by definition of \( f \), we have \( \mu_w(\text{Cone}(f(u))) = \frac{1}{2^{|u|}} \). On the other hand, by definition of \( r \) and \( f \), we have \( \mu \circ r^{-1}(\text{Cone}(f(u))) = \mu(\text{Cone}(u)) = \frac{1}{2^{|u|}} \), which concludes the proof.

2. \( \exists t \in L^\forall_{\text{Qual}}(A) \implies \exists w \in L^{=1}(B) \): Assume that \( t \in L^\forall_{\text{Qual}}(A) \). We show that also \( t \in L^{=1}(A_B) \), where \( A_B \) is defined from \( B \) as in Proposition 14, from which we get the existence of some \( w \in L^{=1}(B) \).

Consider automaton \( A_B^t \), defined from \( B \) as in Lemma 15 and observe that it is a probabilistic version of \( A \) with binary branching. In particular they have same states, transitions (except for probabilities), runs, and acceptance conditions. Since \( t \in L^\forall_{\text{Qual}}(A) \), we also have \( t \in L^{=1}(A_B^t) \). Indeed, the set of qualitatively accepting runs of \( A_B^t \) over \( t \) is equal to the set of qualitatively accepting runs of \( A \) over \( t \). Since \( A \) accepts with a universal condition, all runs of \( A \) over \( t \) are qualitatively accepting, hence the set of qualitatively accepting runs has measure 1. Finally, by Lemma 15 we know that \( L^{=1}(A_B^t) = L^{=1}(A_B) \), hence we get that \( t \in L^{=1}(A_B) \), concluding the proof.

Note that a close result was established in [9]. The difference with ours is that it considers alternating co-Büchi automata, while we consider universal parity ones.

## 4 MSO+\( \forall=1 \) Path on Trees

The logic MSO+\( \forall=1 \), introduced and studied in [14] [15], extends MSO with a probabilistic operator \( \forall=1 X.\varphi \) which states that the set of sets satisfying \( \varphi \) has Lebesgue-measure 1. It is proved in these papers that the MSO+\( \forall=1 \)-theory of the infinite binary tree is undecidable. They also considered a variant of this logic, denoted by MSO+\( \forall=1 \) path, in which the quantification in the probabilistic operator is restricted to sets of nodes that form a path. They proved that, in terms of expressivity, this logic is between MSO and MSO+\( \forall=1 \), with a strict gain in expressivity compared to MSO. However, they left open the question of the decidability of its theory [15] Problem 4]. In this section we establish that it is in fact undecidable, as a direct consequence of Theorem 16.

We recall, from [14], the syntax and semantics of MSO+\( \forall=1 \) path on the infinite binary tree. The syntax of MSO+\( \forall=1 \) path is given by the following grammar:

\[
\varphi ::= \text{succ}_0(x,y) \mid \text{succ}_1(x,y) \mid x \in X \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \forall x.\varphi \mid \forall X.\varphi \mid \forall=1 \text{path} X.\varphi
\]

where \( x \) ranges over a countable set of \emph{first order variables}, and \( X \) ranges over a countable
Theorem 17. The qualitative universal language of a parity tree automaton automaton $\mathcal{A}$ (over alphabet $\Sigma \subseteq \{0, 1\}^n$ for a suitably large $n$) can be expressed in $\text{MSO}+\forall_{\text{path}}$ over the infinite binary tree, i.e., we can construct an $\text{MSO}+\forall_{\text{path}}$ formula $\varphi_{\mathcal{A}}(\vec{X})$ such that the set of $\{0, 1\}^n$-trees $\vec{X} = (X_1, \cdots, X_n)$ satisfying $\varphi_{\mathcal{A}}$ is equal to $\mathcal{L}_{\text{Qual}}^p(\mathcal{A})$. The formula $\varphi_{\mathcal{A}}(\vec{X})$ is of the

![Figure 2](image-url)
form
\[ \forall \vec{X}. (\vec{Y} \text{ is a run of } \mathcal{A} \text{ on } \vec{X} \rightarrow \forall_{\text{path}} Z. (\vec{Z} \text{ is an accepting path of } \vec{Y})), \]

where “\( \vec{Y} \) is a run of \( \mathcal{A} \) on \( \vec{X} \)” and “\( \vec{Z} \) is an accepting path of \( \vec{Y} \)” can be expressed in MSO for parity acceptance conditions (a similar encoding appears in [15] for qualitative nondeterministic languages, and in [19] for nondeterministic Muller tree automata). Now, note that \( \forall \vec{X}. \neg \varphi_{\mathcal{A}}(\vec{X}) \) holds in the infinite binary tree if and only if \( \mathcal{L}_{\mathcal{A}}(\vec{X}) = \emptyset \). Thus, we have reduced the problem of whether the qualitative universal language of a given parity tree automaton \( \mathcal{A} \) is empty, which is undecidable by Theorem [16] to deciding if the MSO+\( \forall_{\text{path}} \) sentence \( \forall \vec{X}. \neg \varphi_{\mathcal{A}}(\vec{X}) \) holds in the infinite binary tree. Thus, the MSO+\( \forall_{\text{path}} \)-theory of the infinite binary tree is undecidable.

\[ \blacksquare \]

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