Generalized gradient expansions in quantum transport equations

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Abstract

Gradient expansions in quantum transport equations of a Kadanoff–Baym form have been reexamined. We have realized that in a consistent approach the expansion should be performed also inside of the self-energy in the scattering integrals of these equations. In the first perturbation order this internal expansion gives new correction terms to the generalized Boltzmann equation. These correction terms are found here for several typical systems. Possible corrections to the theory of a linear response to weak electric fields are also discussed.
1 Introduction

Time dependent transport phenomena in quantum many–body systems can be described by the nonequilibrium Green’s function formalism (NGF) of Kadanoff and Baym [1] or Keldysh [2]. The Kadanoff–Baym transport equations for nonequilibrium correlation functions can be obtained by analytic continuation to real times of the Dyson equation for Matsubara Green’s functions in purely complex times [3, 4]. The differential form of these equations has been applied in many systems [3, 4, 8, 9]. Usually it is necessary to approximate the equations on several levels [1], although in some systems the equations can be directly solved by powerful numerics [10]. The integral form of the Kadanoff–Baym equations was less exploited, because approximations in a time domain are not so familiar here [11]. Both these approaches have been also used to develop a linear response theory for quantum systems in weak dc and ac–electric fields. Here the integral version [12] seems to be more direct than the older differential versions [13, 14].

All functions in the Kadanoff–Baym equations depend separately on two time and two space arguments \((r_1, r_2; t_1, t_2)\). This two–argument structure, which result in nonequilibrium many–body systems with time or space nonlocal scattering, is the main obstacle in solving the transport equations. An approximate one–argument form of the equations can be obtained in systems with not very strong interactions [15]. Most older transport methods have a one–argument structure, because they usually implicitly consider the presence of weak interactions.

A different simplification can be obtained in all types of systems if external excitation fields vary slowly in time and space. Then it is useful to subtract two equivalent sets of the Kadanoff–Baym equations, differentiated over the first \((r_1, t_1)\) and second variables \((r_2, t_2)\), and perform the so called gradient expansion [1] in the new equations transformed to the center–of–mass system (CMS) \(\xi = (x, X) = (r, t; R, T) = \left( r_1 - r_2, t_1 - t_2; \frac{r_1 + r_2}{2}, \frac{t_1 + t_2}{2} \right)\). In the transformed equations various terms have the second arguments with big variables \((R, T)\) shifted by different fractions of the small variables \((r', t')\). A Taylor expansion of these terms in the small variables arround the common values of the big variables is the gradient expansion. This expansion in powers of derivatives over \((R, T)\) multiplied by \((r', t')\) can be unambiguously and consistently stopped at chosen perturbation orders. In this way the two sets of CMS variables \((R, T)\) and \((r', t')\) can be step by step decoupled. By this decoupling memory effects are cuted, so that the nonequilibrium dynamics becomes quasi–local and the equations get a quasi–equilibrium form. Far from equilibrium this approach evidently fails, because the nonlocal scattering, leading (in equilibrium) to quasiparticles with a \((k, \omega)\)–dependent self–energy, should be reflected in the nonlocality of the nonequilibrium dynamics.
In the zeroth order of the gradient expansion the transport equations get a local dynamics. Stopping the gradient expansion in the first order gives corrections to the equations which partially restore memory effects peculiar to the nonlocal dynamics. These equations are called a generalized Boltzmann equation [1] (GBE). The GBE is not limited to weak interactions, but it can only describe slow dynamics close to equilibrium. The weaker is the scattering the better is the description far from equilibrium. When the scattering processes can be considered extremally weak, then the nonequilibrium correlation functions in the GBE can be substituted by a delta–like spectral function multiplied with a distribution function for momenta [1]. As a result the GBE reduces to the Boltzmann equation (BE), which differs from its classical counterpart only by the degeneracy of the described gases. It is interesting that the integral version of the quantum transport equations [16] approximated by low orders of the gradient expansion do not give the BE. It would be also good to mention that the gradient expansions in quantum transport equations are analogous to expansions in the classical Enskog’s equation or the more general BBGKY equations [17].

The gradient corrections in the GBE described in the past [1] do not fully reflect the character of scattering processes, because the self–energy in the scattering integrals of the transport equations is considered as a structure–less entity. We have realized [18] that consistently performed gradient expansion should include also expansion of the self–energy itself, as soon as the self–energy includes additional scattering events separated by internal vertices [4]. In the first order this internal expansion gives new correction terms to the GBE, which are determined by the character of many–body scattering processes.

In this work phenomenological rules are presented, which allow to perform these internal gradient expansions. The rules are applied to consistently derive the GBE with all correction terms. These correction terms are found in three examples of a self–energy: the averaged $T$–matrix approximation for a self–energy in electron scattering on local potentials, the local $T$–matrix approximation and the shielded potential approximation for a self–energy of interacting spinless Fermions. The importance of the internal gradient corrections for a linear response to weak electric fields is also briefly discussed.
2 Gradient expansions in quantum transport equations

In slowly changing fields the two differential forms of the Kadanoff–Baym equations in (3.1) can be subtracted and the gradient expansion can be performed in the resulting equations. The right sides of the subtracted equations include scattering terms of the form

\[ \Sigma^\alpha(x_1, \bar{x}_3) G^\beta(\bar{x}_3, x_2), \quad G^\alpha(x_1, \bar{x}_3) \Sigma^\beta(\bar{x}_3, x_2), \quad x_i = (r_i, t_i), \tag{1} \]

where the analytical structure of the self–energy \( \Sigma \) and the Green’s function \( G \) in (1) is determined by the index \( \alpha, \beta = r, a, <, > \) (see the Appendix A).

2.1 External expansions

Consider as an example the gradient expansion in the term \( \Sigma^\alpha(x_1, \bar{x}_3) G^\beta(\bar{x}_3, x_2) \) from (1), where the many–body structure of \( \Sigma^\alpha \) is neglected (1). The expansion results in the CMS coordinates \( \chi = (x, X) = (r, t; R, T) \) as follows

\[
\int dx_3 \Sigma^\alpha(x_1, x_3) G^\beta(x_3, x_2) = \int dx_3 \Sigma^\alpha \left( x_1 - x_3, \frac{x_1 + x_3}{2} \right) G^\beta \left( x_3 - x_2, \frac{x_3 + x_2}{2} \right) \\
= \int d\bar{x} \Sigma^\alpha \left( \bar{x}, X + \frac{x - \bar{x}}{2} \right) G^\beta \left( x - \bar{x}, X - \frac{\bar{x}}{2} \right) \\
= \int d\bar{x} \Sigma^\alpha(\bar{x}, X) G^\beta(x - \bar{x}, X) + \int d\bar{x} \frac{\partial \Sigma^\alpha(\bar{x}, X)}{\partial X} \left( \frac{x - \bar{x}}{2} \right) G^\beta(x - \bar{x}, X) \\
+ \int d\bar{x} \Sigma^\alpha(\bar{x}, X) \frac{\partial G^\beta(x - \bar{x}, X)}{\partial X} \left( \frac{x - \bar{x}}{2} \right) + \ldots, \tag{2} \]

The new coordinates are equal to \( x = x_1 - x_2, \bar{x} = x_1 - x_3, X = \frac{x_1 + x_2}{2} \). A Fourier transform of the last expression in (2) from the coordinates \( x = (r, t) \) to \( q = (k, \omega) \) gives the following expansion in a series of Poisson brackets \( \{ \} \)

\[
\exp \left( \frac{i}{2} D(\xi, \xi') \right) \Sigma^\alpha(\xi) G^\beta(\xi') \equiv \Sigma^\alpha(\xi) G^\beta(\xi) \\
+ \frac{i}{2} \left( \frac{\partial \Sigma^\alpha(\xi)}{\partial R} \frac{\partial G^\beta(\xi)}{\partial k} - \frac{\partial \Sigma^\alpha(\xi)}{\partial k} \frac{\partial G^\beta(\xi)}{\partial R} - \frac{\partial \Sigma^\alpha(\xi)}{\partial T} \frac{\partial G^\beta(\xi)}{\partial \omega} + \frac{\partial \Sigma^\alpha(\xi)}{\partial \omega} \frac{\partial G^\beta(\xi)}{\partial T} \right) + \ldots \\
= \Sigma^\alpha(\xi) G^\beta(\xi) + \frac{i}{2} \left[ \Sigma^\alpha(\xi), G^\beta(\xi) \right] + \ldots, \quad \xi = (q, X) = (k, \omega; R, T). \tag{3} \]

The gradient expansion of the scattering term \( \Sigma^\alpha(x_1, \bar{x}_3) G^\beta(\bar{x}_3, x_2) \) has been obtained in (3) without taking into account the many–body structure of the self–energy \( \Sigma^\alpha \). Therefore this expansion can be called external. The first order of this expansion includes only the first Poisson bracket in (3). Similarly can be performed the gradient expansion in driving terms of the quantum transport equations.
2.2 Internal expansions

In many systems the self–energy is approximated by Feynman diagrams formed by ladders or bubbles of Fermion and Boson Green’s functions [1, 8, 19]. This nontrivial functional of Green’s functions, connected by internal integrations, can substitute the self–energy in any step of calculations. The internal structure of the functional was not taken into account in the gradient expansion (3). Does it contribute by new terms in the gradient expansion? It is obvious that this question can be answered, if a systematic gradient expansion is performed in the expressions (1), where the self–energy is substituted by the functional of the Green’s functions.

Consider for simplicity that the self–energy describes electron scattering on the localized potentials \( V(r) = \sum_i V_0 \delta(r - r_i) \). Assume further that the value \( V_0 \) is of a moderate strength, but the number the random coordinates \( r_i \) is relatively small. Then the suitable self–energy for this problem results by the averaged T–matrix approximation [20] (ATA)

\[
\Sigma(t_1, t_2) = c \left\{ V_0 + V_0^2 \ G(t_1, t_2) + V_0^3 \ G(t_1, \tilde{t}_3) \ G(\tilde{t}_3, t_2) + \ldots \right\} = c \ \theta(t_1, t_2) .
\]  

Here \( c \) represents the weak concentration of the potentials \( V_0 \) and \( \theta \) is the local T–matrix. The Green’s functions in (4) depends only on the time variables, because the space variables have been integrated out due to the local scattering.

Analytical continuation of the self–energy (4) gives the propagator and correlation functions for the self–energy in the form (see the Appendix B)

\[
\Sigma^r(t_1, t_2) = c \left\{ V_0 + V_0^2 \ G^r(t_1, t_2) + V_0^3 \ G^r(t_1, \tilde{t}_3) \ G^r(\tilde{t}_3, t_2) + \ldots \right\} = c \ \theta^r(t_1, t_2) ,
\]

\[
\Sigma^{<}(t_1, t_2) = c \ \theta^r(t_1, \tilde{t}_3) \ G^<(\tilde{t}_3, \tilde{t}_4) \ \theta^a(\tilde{t}_4, t_2) = c \ \theta^{<}(t_1, t_2) .
\]

Each term from the expressions for \( \Sigma^{r<} \) in (3–4) is formed by several propagators or a correlation function with time arguments in a ‘series’ (see the Appendix B).

Let us study in details the gradient expansion of the scattering terms in (1) for the self–energy in (3–4). Assume for the beginning that the self–energy in the first expression from (1) is substituted by one of the terms from (3–4), which has two Green’s functions \( \Delta \Sigma(t_1, t_3) = c \ V_0^2 \ G^\alpha(t_1, \tilde{t}_2) \ G^\beta(\tilde{t}_2, t_3) \) (no index is used at the contribution to the self–energy \( \Delta \Sigma \) to show its analytical structure). Then the expression \( \Delta \Sigma(t_1, t_3) G^\gamma(t_2, t_3) \) can be transformed to the CMS coordinates as follows (we suppress the prefactor \( c V_0^3 \) and neglect the fact that the function \( G^\gamma \) depends also on space variables)

\[
G^\alpha(t_1, \tilde{t}_2) \ G^\beta(\tilde{t}_2, \tilde{t}_3) \ G^\gamma(\tilde{t}_3, t_4)
\]

\[= G^\alpha \left(t_1 - \tilde{t}_2, \frac{t_1 + \tilde{t}_2}{2} \right) G^\beta \left(\tilde{t}_2 - \tilde{t}_3, \frac{\tilde{t}_2 + \tilde{t}_3}{2} \right) G^\gamma \left(\tilde{t}_3 - t_4, \frac{\tilde{t}_3 + t_4}{2} \right) \]
\[ G^\alpha \left( \bar{\tau}_I, T + \frac{\tau - \bar{\tau}_I - \bar{\tau}_{II}}{2} + \frac{\bar{\tau}_{II}}{2} \right) G^\beta \left( \bar{\tau}_{II}, T + \frac{\tau - \bar{\tau}_I - \bar{\tau}_{II}}{2} - \frac{\bar{\tau}_I}{2} \right) G^\gamma \left( \tau - \bar{\tau}_I - \bar{\tau}_{II}, T - \frac{\bar{\tau}_I}{2} - \frac{\bar{\tau}_{II}}{2} + \bar{\tau}_I \right), \quad \alpha, \beta, \gamma = r, a, <, >, \tag{7} \]

where it holds \((\tau, T) \equiv \left( t_1 - t_4, \frac{t_1 + t_4}{2} \right)\) and \(\bar{\tau}_I = t_1 - \bar{t}_2, \bar{\tau}_{II} = \bar{t}_2 - \bar{t}_3\). In the expression (7) a Taylor expansion in the \(\tau\)-coordinates can be performed around the \(T\)-coordinates as in (2). In a first order each of the Green’s functions differentiated over \(T\) has multiplicative coefficients formed by the \(\tau\)-coordinates of the remaining two Green’s functions in a series. Therefore in this structure of arguments the Green’s functions are equivalent from the point of view of gradient expansions.

After a Fourier transform over the small variables a symmetrical expression in the three functions \(G^{\alpha,\beta,\gamma}\) with respect to the derivatives \(\frac{\partial}{\partial X} \frac{\partial}{\partial q}\) can be obtained. Therefore the complete gradient expansion up to the linear order can be shortly written as follows (the space variables have been included in \(G^\gamma\))

\[ c V_0^3 \left\{ G^\alpha(\omega, T) G^\beta(\omega, T) G^\gamma(\xi) + \frac{i}{2} \left[ G^\alpha(\omega, T) G^\beta(\omega, T) , G^\gamma(\xi) \right] + \frac{i}{2} \left[ G^\alpha(\omega, T), G^\beta(\omega, T) \right] G^\gamma(\xi) \right\}, \quad \xi = (k, \omega; R, T), \tag{8} \]

where the Poisson brackets have been used. Since the space coordinates \((k, R)\) are integrated out in the present self–energy, gradient expansions cannot be performed in these variables. But in a general case both pairs of coordinates contribute in the way shown in (8).

The second term in (8) resulted as in (3) by the (external) gradient expansion of the expression \(\Delta \Sigma(t_1, \bar{t}_2) G^\gamma(\bar{t}_2, t_3)\). The last term in (8) resulted by the gradient expansion of the internal structure of the self–energy contribution \(\Delta \Sigma\). Therefore this expansion and the resulting term can be called internal. In (8) only contributions to the self–energy diagrams with two Green’s functions were included. Analogously can be performed the gradient expansion in terms with any number of Green’s functions. This problem is solved in the next section in details.

The internal gradient corrections can be easily physically understood on the previous example, where scattering processes can be seen as many consequent events on the same center. If the system is excited by a time dependent field, then scattering conditions on the center can change between these consequent scattering events. Corrections to these changed scattering conditions are represented by the internal correction terms. For centers which are little smeared in space, internal gradient corrections result nonzero even in excitation by static fields.
3 Gradient expansions in general

A gradient expansion in quantum transport equations can be unambiguously represented by a series of Poisson brackets of an increasing order \( n \). Physically this is a consistent expansion in powers of space (time) inhomogeneities, since the \( n \)-th order Poisson brackets could be appreciated by terms of the form \((k\sigma)^n\), where \( k \) is the inverse mean free path and \( \sigma \) is the range of space inhomogeneity (analogous terms apply for the time inhomogeneity). If \( k\sigma \ll 1 \) then the expansion can be stopped in the first order \((n = 1)\), which is equivalent to inclusion of both external and internal corrections in the GBE (see also (8), (9) and (32)).

Gradient expansions can be performed also in classical transport equations describing dense systems. Such systems have been firstly approximately studied by the Enskog’s equation [17], which generalizes the Boltzman equation by taking finite volumes of scattering particles. Since only binary collisions are included here, like in the BE, gradient expansions in this equation are analogous to the external expansions in quantum transport equations, giving the uncomplete quantum GBE [1]. Later on classical dense systems have been described by the so called BBGKY hierarchy of kinetic equations [17], which can include multiple encounters of particles. Under some assumptions this set of equations can be reduced to a classical generalized Boltzman equation [17], which has scattering integrals with a structure analogous to that in the quantum transport equations. Therefore gradient expansions in this classical GBE are analogous to both the external and internal expansions leading to the complete quantum GBE (in the classical GBE gradient expansions can be performed, while the quantum GBE is the result of the gradient expansions).

3.1 General rules for gradient expansions

We formulate a set of phenomenological rules for performing external and internal gradient expansions in terms like in (1), appearing in quantum transport equations. From now the gradient expansions are stopped after the first perturbation order (higher order contributions can be found analogously). The following qualitatively different points specify and summarize the necessary steps for gradient expansions in the concret terms:

1. Analytic continuation to real times of the term is performed, to get an expression formed by propagators and correlation functions.

2. Each of the self–energy functions \( \Sigma^{r,a,<,>} \) is resolved into a functional of full Fermion and Boson Green’s functions \( G^{r,a,<,>} \) and undressed matrix elements.

3. The whole term is transformed into CMS coordinates, and the big coordinates \( X \) are linearized in the conjugated small coordinates \( x \). After a Fourier trans-
form over the coordinates $x$, the linearization prefactors $x$ become derivatives $\partial_q$, which produce a series of new terms. In each of these new terms just two derivatives appear (over $X$ and $q$). The expansion should be performed on a hierarchical structure of levels, going more and more inside to the structure of $\Sigma$ (the vertices are the landmarks). Corrections for higher levels are done on lower levels. All objects, which depend on some of the coordinates, must be differentiated. The lowest objects are the full Green’s functions and undressed coupling matrix elements.

4. ‘Parallel’ objects with the same coordinates $(x_i, x_j)$, but the order of $x_{i,j}$, can be considered in the derivatives as a single differentiated object. At the lower level some of the objects with the coordinates $(x_i, x_j)$ can have still other internal coordinates, which can give further gradient corrections.

5. ‘Serial’ objects with arguments like in (II) are differentiated in such a way that differentiation of one object over a big coordinate $X$ is accompanied by differentiation of the other objects in a series (one by one) over the conjugated small coordinates $q$. Sign prefactors depend on the sequence of objects. Realization of this rule for terms with many Green’s functions in a series can be done by a second functional derivative of the term over the objects in the series. This derivative is multiplied by a Poisson bracket of the two differentiating objects in the functional derivative.

In the previous section these rules have been already implicitly applied on a simple algebraic term from a self–energy (see (I)). Analogously can be dealt other such algebraic terms or complex recursive terms. Before we come to these expansions let us still present one theorem. Application of the above rules to scattering terms from transport equations, which include a self–energy as in (I), gives the following formal expansion (stopped in the first perturbation order)

$$G^\alpha(\xi) \Sigma(\xi) + \frac{i}{2} [G^\alpha(\xi), \Sigma(\xi)] + G^\alpha(\xi) F_i[\Sigma](\xi).$$

(9)

Here $F_i[\Sigma](\xi)$ represents the internal gradient expansion in the self–energy $\Sigma$. Since the form (I) is fully general, it can be taken as a theorem:

- **Internal** expansion of a term with a self–energy results by the substitution of $\Sigma(\xi)$ by $\Sigma(\xi) + F_i[\Sigma](\xi)$ in the zeroth order term.

From this theorem it follows that the *internal* gradient expansion in terms with a self–energy can be directly found from the function $F_i[\Sigma](\xi)$.

### 3.2 Expansions of a self–energy

We can concentrate on this function $F_i[\Sigma](\xi)$ and evaluate it for several typical examples of a self–energy.
3.2.1 Static averaged $T$–matrix approximation

The first example concerns the ATA self–energy \[ \Sigma(r, T) = c \frac{V_0}{1 - G^r(\omega, T) V_0} = c \theta^r(\omega, T) \, , \] (10)

\[ \Sigma^< (\omega, T) = c \theta^r(\omega, T) G^< (\omega, T) \theta^a (\omega, T) = c \theta^< (\omega, T) \, . \] (11)

Application of the rule 5. to the retarded part in (5) gives the first order term

\[ F_i[\Sigma^r](\xi) = \frac{\delta^2 \Sigma^r}{\delta G^r \delta G^a}(\xi) \frac{i}{2} [G^r(\xi), G^r(\xi)] = 0 \, , \] (12)

since only functional derivatives of $\Sigma^r$ over the propagator Green’s functions $G^r$ can be applied here. Because the Poisson bracket from equivalent objects is zero, the function $F_i[\Sigma^r](\xi)$ does not contribute to the internal expansion. Analogously can be found the internal term for the correlated part in (6)

\[ F_i[\Sigma^<](\xi) = \frac{\delta^2 \Sigma^<}{\delta G^r \delta G^a}(\xi) \frac{i}{2} [G^r(\xi), G^a(\xi)] + \frac{\delta^2 \Sigma^<}{\delta G^< \delta G^a}(\xi) \frac{i}{2} [G^< (\xi), G^a(\xi)] \]

\[ + \frac{\delta^2 \Sigma^<}{\delta G^r \delta G^a}(\xi) \frac{i}{2} [G^r(\xi), G^a(\xi)] = ... = \]

\[ = -c \text{ Im} \left( (\theta^r(\omega, T))^2 \theta^a (\omega, T) [G^r(\omega, T), G^< (\omega, T)] \right) - \frac{c}{2} |\theta^r(\omega, T)|^4 G^< (\omega, T) \]

\[ \times \int \frac{d\bar{\omega}}{2\pi} \frac{1}{\omega - \bar{\omega}} \left( \frac{\partial A(\bar{\omega}, T)}{\partial \bar{\omega}} \frac{\partial A(\omega, T)}{\partial T} - \frac{\partial A(\bar{\omega}, T)}{\partial T} \frac{\partial A(\omega, T)}{\partial \omega} \right) . \] (13)

The term $F_i[\Sigma^>](\omega, T)$ can be evaluated in the same way. In the first expression from (13) only the nonzero functional derivatives have been considered, which result by differentiation over Green’s functions of different analytical structures. The order of derivatives and terms in Poisson brackets is the same as the order of these functions in (13). The second expression results by per–partes integration in the Poisson bracket $[G^r(\xi), G^a(\xi)]$ and some simple algebra.

3.2.2 Dynamic $T$–matrix approximation

Another $T$–matrix approximation is used \[ 1, 19 \], if mutual interaction of electrons are studied. This approximation gives a self–energy of a very similar structure to the ATA self–energy in (13), but internal dynamics is more complicated here. For spinless Fermions the self–energy can be written as folows \[ 13 \]

\[ \Sigma(t_1, t_2) = -i \Theta(t_1, t_2) G(t_2, t_1) \, , \] (14)
where $\Theta$ is the dynamic $T$–matrix

$$
\Theta(t_1, t_2) = V \delta(t_1 - t_2) + V R_0(t_1, \bar{t}_3) \Theta(\bar{t}_3, t_2) ,
$$

$$
R_0(t_1, t_2) = i G(t_1, t_2) G(t_1, t_2) .
$$

(15)

In this self–energy the singular Hartree term is included ($t_2 \rightarrow t_1^-$), but the exchange terms have been neglected for simplicity.

Analytical continuation to real times of (15) gives the propagators and correlation functions (see the Appendix B)

$$
R^r_0(t_1, t_2) = G^r(t_1, t_2) G^< (t_1, t_2) - G^<(t_1, t_2) G^r(t_1, t_2) ,
$$

$$
R^<_0(t_1, t_2) = G^<(t_1, t_2) G^<(t_1, t_2)
$$

(16)

and

$$
\Theta^r(t_1, t_2) = V \delta(t_1 - t_2) + V R^r_0(t_1, \bar{t}_3) \Theta^r(\bar{t}_3, t_2) ,
$$

$$
\Theta^<(t_1, t_2) = V [\Theta^r(t_1, \bar{t}_3) R^<_0(\bar{t}_3, t_2) + \Theta^<(t_1, \bar{t}_3) R^a_0(\bar{t}_3, t_2) ] .
$$

(17)

After application of the rule 3. in (16–17) the zeroth order terms can be completed in a form similar to the $\theta$–functions in (10–11)

$$
\Theta^r(\omega, T) = \frac{V}{1 - R^r_0(\omega, T) V} , \quad \Theta^<(\omega, T) = \Theta^r(\omega, T) R^<_0(\omega, T) \Theta^a(\omega, T) .
$$

(18)

The zeroth order contributions to the propagator and correlated parts of the self–energy (14) results from these functions as follows

$$
\Sigma^r(\omega, T) = \Theta^r(\omega + \bar{\omega}, T) G^<(\bar{\omega}, T) - \Theta^<(\omega + \bar{\omega}, T) G^a(\bar{\omega}, T) ,
$$

$$
\Sigma^<(\omega, T) = \Theta^<(\omega + \bar{\omega}, T) G^>(\bar{\omega}, T) .
$$

(19)

We can continue with the internal gradient expansion in the self–energy (14). The separate Green’s function in (19) is on the highest level, so it is excluded from the internal expansion. The remaining functions $\Theta^r$ and $\Theta^<$ could be independently expanded after the rule 4. and give $F_i[\Theta^r](\xi)$ and $F_i[\Theta^<](\xi)$.

Time arguments of the objects $R_0(t_1, t_2)$ in the function $\Theta(t_1, t_2)$ from (15) are in a series, so that the rule 5. can be directly applied. The expansion of the propagator function in terms of $R^r_0$ results zero, similarly as in (12)

$$
F_i[\Theta^r](\xi) = \frac{\delta^2 \Theta^r}{\delta R_0^a \delta R_0^r} (\xi) \frac{i}{2} [ R^r_0(\xi), R^r_0(\xi) ] = 0 .
$$

(20)
The internal expansions of the correlation function $\Theta^<$ is formed by terms analogous to those in (13)

$$F_i[\Theta^<](\xi) = \frac{\delta^2 \Theta^<}{\delta R_0^r \delta R_0^a}(\xi) \frac{i}{2} [R_0^r(\xi), R_0^a(\xi)] + \frac{\delta^2 \Theta^<}{\delta R_0^< \delta R_0^a}(\xi) \frac{i}{2} [R_0^<(\xi), R_0^a(\xi)]$$

$$+ \frac{\delta^2 \Theta^<}{\delta R_0^r \delta R_0^a}(\xi) \frac{i}{2} [R_0^r(\xi), R_0^<(\xi)] + \cdots$$

$$= - \operatorname{Im} \left( (\Theta^r(\omega, T))^2 \Theta^a(\omega, T) [R_0^r(\omega, T), R_0^<(\omega, T)] \right) - 2 |\Theta^r(\omega, T)|^4 R_0^<(\omega, T)$$

$$\times \int \frac{d\omega}{2\pi} \frac{1}{\omega - \bar{\omega}} \left( \frac{\partial \operatorname{Im} R_0^r(\omega, T)}{\partial \omega} \frac{\partial \operatorname{Im} R_0^a(\omega, T)}{\partial T} - \frac{\partial \operatorname{Im} R_0^r(\bar{\omega}, T)}{\partial T} \frac{\partial \operatorname{Im} R_0^a(\omega, T)}{\partial \omega} \right). \quad (21)$$

The function $F_i[\Theta^>(\xi)]$ results by the change of the index $< \rightarrow >$ in all places of (21). Finally the internal expansions $F_i[\Sigma^r](\xi)$ and $F_i[\Sigma^<](\xi)$ can be obtained, if the functions $\Theta^r(\xi)$ and $\Theta^<(\xi)$ in the zeroth order self–energy (19) are substituted by $F_i[\Theta^r](\xi)$ and $F_i[\Theta^<](\xi)$ from (20–21)

$$F_i[\Sigma^r](\omega, T) = F_i[\Theta^<](\omega + \bar{\omega}, T) G^a(\bar{\omega}, T),$$

$$F_i[\Sigma^<](\omega, T) = F_i[\Theta^<](\omega + \bar{\omega}, T) G^>(\bar{\omega}, T). \quad (22)$$

Here the fact that $F_i[\Theta^r](\xi) = 0$ in (20) has been taken into account.

### 3.2.3 Shielded potential approximation

Particle interactions are often long range, like in the Coulomb potential. If free particles are available in the system, then screening can shorten the range of the interaction (give a much more localized potential). The shielded potential approximation of a self–energy [1] does not include ladders, like in the above studied examples, but rows of electron–hole bubbles screening the potential. Localization of the potential by screening has a dynamical character. Therefore it would be demanding to know time dependent behavior of screening in nonequilibrium processes. Recently dynamics of ultrafast screening processes have been studied by NGF [21, 22]. Close to equilibrium these systems can be described by the GBE, where the internal corrections to the screening dynamics can play an important role.

The shielded approximation for the self–energy can be written in the form [1]

$$\Sigma(1, 2) = i \, V_s(1, 2) \, G(1, 2), \quad (23)$$

where the singular Fock term is included ($t_2 \rightarrow t_1^+$). In the lowest order of a perturbation theory, the screened potential $V_s$ is related to the unscreened one $V$ by the polarization function $L_0$ as follows
\[ V_s(1, 2) = V(1, 2) - V(1, 3) \, L_0(3, 4) \, V_s(4, 2) \, , \quad V(1, 2) = V(r_1 - r_2) \delta(t_1 - t_2) \, , \]

\[ L_0(1, 2) = i \, G(1, 2) \, G(2, 1) \, . \] (24)

The propagators and correlation functions for the polarization function \( L_0 \) in (24) can be found by the rules in the Appendix B

\[ L_0^r(1, 2) = -(G^r(1, 2) \, G^<(2, 1) + G^<(2, 1) \, G^a(2, 1)) \, , \]

\[ L_0^<(1, 2) = -G^<(1, 2) \, G^>(2, 1) \, , \] (25)

and the functions \( V^r_s \) and \( V^<_s \) result analogously as \( \Theta^r \) and \( \Theta^< \) in (17).

Therefore after application of the rule 3. in (25), the zeroth order of the gradient expansion can be obtained as in (18) (the full CMS coordinates \( \xi = (k, \omega; R, T) \) are considered, since the interaction is nonlocal)

\[ V^r_s(\xi) = \frac{V(k)}{1 + L_0^r(\xi) \, V(k)} \, , \quad V^<_s(\xi) = V^r_s(\xi) \, L_0^<(\xi) \, V^a_s(\xi) \, . \] (26)

Similarly the zeroth order gradient contributions to the propagator and correlated parts of the self–energy (23) result

\[ \Sigma^r(k, \omega; R, T) = G^r(k - \bar{k}, \omega - \bar{\omega}; R, T) \, V^>_s(\bar{k}, \bar{\omega}; R, T) \]

\[ -G^<(k - \bar{k}, \omega - \bar{\omega}; R, T) \, V^a>_s(\bar{k}, \bar{\omega}; R, T) \, , \]

\[ \Sigma^<(k, \omega; R, T) = G^<(k - \bar{k}, \omega - \bar{\omega}; R, T) \, V^<_s(\bar{k}, \bar{\omega}; R, T) \, . \] (27)

The structure of the internal corrections to the self–energy (24) is also similar to the previous example. The main difference is in that here the potential \( V(k) \) depends on the wave vector \( k \), so that new corrections terms are included

\[ \frac{\delta^2 V^r_s}{\delta L^r_0 \delta L^r_0}(\xi) \frac{i}{2} [L^r_0(\xi), L^r_0(\xi)] \]

\[ + \frac{\delta^2 V^r_s}{\delta L^r_0 \delta V}(\xi) \frac{i}{2} [L^r_0(\xi), V(k)] + \frac{\delta^2 V^r_s}{\delta V^r \delta L^r_0}(\xi) \frac{i}{2} [V(k), L^r_0(\xi)] = 0 \, . \] (28)

The potential \( V \) in the last two terms is either more 'right' or more 'left' than \( L_0 \) in the series (24) (see also comment below (13)). Since the second functional derivative is symmetrical in the differentiating functions, these two terms diminish each other. In the Poisson brackets from (28) derivatives over space–momentum coordinates are also performed (see (3)), but the potential \( V(k) \) can be differentiated only over the momentum.
The correlation function $V^<_s$ gives more interesting internal contributions. We write only terms with nonzero Poisson brackets and also neglect terms analogous those in (28), where both functions originate from one of the propagators $V^r_{s,a}$. The fact that the differentiating functions $V, \ L^r_{0,a}$ originate from different full potential propagators $V^r_{s,a}$ is symbolized by a small index at the potentials $V^{(r)}, \ V^{(a)}$, which show the side they originate from. The nonzero terms are

$$F_i[V^<_s](\xi) = \frac{\delta^2 V^<_s}{\delta L^r_{0} \delta L^a_{0}}(\xi) \left( \frac{i}{2} \left[ L^r_{0}(\xi), L^a_{0}(\xi) \right] + \frac{\delta^2 V^<_s}{\delta L^r_{0} \delta L^a_{0}}(\xi) \frac{i}{2} \left[ L^r_{0}(\xi), L^a_{0}(\xi) \right] \right.$$ 

$$+ \frac{\delta^2 V^<_s}{\delta L^r_{0} \delta V^{(a)}}(\xi) \left( \frac{i}{2} \left[ L^r_{0}(\xi), V(k) \right] + \frac{\delta^2 V^<_s}{\delta V^{(r)} \delta L^a_{0}}(\xi) \frac{i}{2} \left[ V(k), L^a_{0}(\xi) \right] \right.$$ 

$$+ \frac{\delta^2 V^<_s}{\delta L^a_{0} \delta V^{(a)}}(\xi) \left( \frac{i}{2} \left[ L^r_{0}(\xi), V(k) \right] + \frac{\delta^2 V^<_s}{\delta V^{(r)} \delta L^a_{0}}(\xi) \frac{i}{2} \left[ V(k), L^a_{0}(\xi) \right] \right).$$

(29)

The first three terms in the right side of (29) are analogous those in (21), but the additional space–momentum derivatives present in the Poisson brackets here. The next two terms in (29) are evidently nonzero, because the derivatives and the Poisson brackets are different and nonzero. In the remaining two terms the Poisson brackets are opposite each other, but the functional derivatives are different, because the potentials are taken from different sides.

We can evaluate the above functional derivatives and collect all the terms

$$F_i[V^<_s](\xi) = \text{Im} \left\{ (V^r_{s}(\xi))^2 \ V^a_{s}(\xi) \left( \left[ L^r_{0}(\xi), L^a_{0}(\xi) \right] - \frac{1}{V(k)^2} \left[ V(k), L^a_{0}(\xi) \right] \right) \right\}$$

$$+ (V^r_{s}(\xi))^2 \ L^a_{0}(\xi) \ (V^a_{s}(\xi))^2 \frac{i}{2} \left( \left[ L^r_{0}(\xi), L^a_{0}(\xi) \right] + \frac{1}{V(k)^2} \left[ V(k), L^a_{0}(\xi) - L^a_{0}(\xi) \right] \right).$$

(30)

The function $F_i[V^>(\xi)]$ results by the change of the index $<$ by $>$ in all places of (30).

As before the internal expansions $F_i[\Sigma^r](\xi)$ and $F_i[\Sigma^<](\xi)$ can be obtained, if the functions $V^r_{s,a}(\xi)$ and $V^<_{s,a}(\xi)$ in the zeroth order self–energy (27) are substituted by $F_i[V^r_{s,a}](\xi)$ and $F_i[V^<_{s,a}](\xi)$ from (28 30)

$$F_i[\Sigma^r](k, \omega; R, T) = G^r(k - \bar{k}, \omega - \bar{\omega}; R, T) \ F_i[V^<_s](\bar{k}, \bar{\omega}; R, T),$$

$$F_i[\Sigma^<](k, \omega; R, T) = G^<(-k, \omega - \bar{\omega}; R, T) \ F_i[V^>_{s,a}](\bar{k}, \bar{\omega}; R, T).$$

(31)

The fact that $F_i[V^r_{s,a}](\xi) = 0$ in (28) has been again taken into account.
4 Transport equations with internal corrections

The internal corrections might appear in all kind of equations derived by the gradient expansions from the Kadanoff–Baym equations.

4.1 Generalized Boltzmann equation

The complete generalized Boltzmann equation with both the external and internal gradient corrections can be found directly, when the above theorem is implemented in its derivation from the Appendix B. The theorem says that the internal terms result by the substitution of $\Sigma^<(>)\!(\xi)$ by $\Sigma^<(>)\!(\xi) + F_i[\Sigma^<(>)\!(\xi)]$ in the zeroth order terms. Therefore after a Fourier transform the complete GBE gets the form

$$
\left( \frac{\partial}{\partial T} + \frac{p \cdot \nabla_R}{m} - \nabla_R U_{\text{eff}}(\xi) \cdot \nabla_p + \frac{\partial U_{\text{eff}}(\xi)}{\partial T} \frac{\partial}{\partial \omega} \right) G^<(\xi)
$$

$$
- [\text{Re}\Sigma^r(\xi), G^<(\xi)] + [\text{Re}G^r(\xi), \Sigma^<(\xi)]
$$

$$
= - (\Sigma^>(\xi) + F_i[\Sigma^>(\xi)]) G^<(\xi) + (\Sigma^<(\xi) + F_i[\Sigma^<(\xi)]) G^>(\xi) .
$$

(32)

The equation for the correlation function $G^>(\xi)$ can be found similarly. The singular (Hartree–Fock) contributions in (32) are included in the effective potential $U_{\text{eff}}(\xi)$. The propagator functions $G^{r(\alpha)}(\xi)$ and $\Sigma^{r(\alpha)}(\xi)$ in (32) are related to the correlated parts $G^<(\xi), G^>(\xi)$ and $\Sigma^<(\xi), \Sigma^>(\xi)$ by the Hilbert transform (A.7). Therefore it is not necessary to find separate transport equations for these propagator functions, which would eventually include the nonzero internal terms $F_i[\Sigma^{r(\alpha)}(\xi)]$. The internal terms in (32) contribute to the dynamics (not the renormalization) of the studied system, so that they cannot be neglected in the transport equations. Conservation laws for the GBE in (32) could be proven similarly as for the exact Kadanoff–Baym equations.

In the above example of the ATA self–energy, the expressions $F_i[\Sigma^<(>)\!(\xi)]$ in (32) should be substituted by (13). For the dynamic $T$–matrix approximation the terms (22) fulfill this role (the Hartree term should be taken once). In both these examples the CMS coordinates in the functions $F_i[\Sigma^<(>)\!(\omega, T)]$ are reduced with respect to the other terms in (32), where it is $\xi = (k, \omega; R, T)$. This reduction results from the local form of the $T$–matrix approximations studied here. In the shielded potential approximation for the self–energy, the expressions $F_i[\Sigma^<(>)\!(\xi)]$ from (31) with full CMS coordinates can be used in the GBE. In all these cases the complete GBE with internal correction terms results quite complicated. Nevertheless, we believe that it can be handled by some approximate numerical methods.
4.2 Linearized transport equations

The internal gradient corrections can appear also in linearized transport equations derived from the GBE in weak $dc$–electric fields [13]. It has been already mentioned [23] that these linearized equations should include further correction terms for complicated scattering.

The left side of the GBE in (32) looks in $dc$–electric fields as follows

\[
\left( \frac{\partial}{\partial T} + \frac{1}{m} (\mathbf{k} + eTE_0) \cdot \left( \nabla_R + eE_0 \frac{\partial}{\partial \omega} \right) \right) G^{<}(\xi) .
\]

To diminish the explicit time dependence in (33), not present in dissipative systems in weak $dc$–fields, the following transform should be performed [13]

\[
Q \rightarrow \mathbf{k} + eE_0 T , \quad \frac{\partial}{\partial T} \rightarrow \frac{\partial}{\partial T} + eE_0 \cdot \nabla Q ,
\]

where $E_0$ is the intensity of the $dc$–electric field. Application of this transform to the Poisson brackets in the second line of (32) gives mixed terms in the following way

\[
[A, B] \rightarrow [A, B] + eE_0 \cdot \left( \frac{\partial A}{\partial \omega} \nabla_Q B - \nabla_Q A \frac{\partial B}{\partial \omega} \right).
\]

If this transform is applied also to the Poisson brackets in the internal corrections $F_i[\Sigma^{<\langle\rangle}(\xi)]$ in (32), further new terms can result.

When the self–energy $\Sigma(\xi)$ depends only on $(\omega, T)$ variables, like in the space localized scattering (4) or (14), then only the external terms $\frac{\partial \text{Re} \Sigma'(\omega)}{\partial \omega} \frac{\partial G^{<\langle\rangle}(k, \omega)}{\partial k}$ and $\frac{\partial \Sigma^{<\langle\rangle}(\omega)}{\partial \omega} \frac{\partial \text{Re} G^{<\langle\rangle}(k, \omega)}{\partial k}$, result by the transform (33). Application of this transform in the terms $F_i[\Sigma^{<\langle\rangle}(\xi)]$ from (13), (22) give nothing, because only $k$–independent Green’s functions are present there. If a nonlocal ATA self–energy [24] is used, generalizing (for neutral smeared imperfections) the local form (4), then new terms would result from application of the transform (33) in the $k$–dependent internal corrections $F_i[\Sigma^{<\langle\rangle}(\xi)]$. The nonlocal scattering results for example also on charged impurities [25], where it is reasonable to screen the impurity potential [26], similarly as in the self–energy (23). In both these examples internal and external corrections in (32) are nonzero. Since evaluation of these terms is direct, we do not write them as well as the resulting complicated linearized equations.

The question is how the internal corrections can contribute in the case of the linear response to weak $ac$–electric fields, where we would expect that also the space–local interactions (4), (14) give new correction terms. The linearized transport equations in the $ac$–electric fields have been studied [14], but the transform to new coordinates [13] was not performed. Therefore no gradient corrections would seemingly contribute by new terms. In fact it is probably hard or not unique to find the above transform in the $ac$–case. Moreover in the $ac$–case it is not sufficient
to stop the gradient expansions in the lowest orders. Therefore a new gauge invariant approach has been developed \[27\] to study linear response to the weak ac and dc–electric fields, where no additional transforms are necessary. Unfortunately, the resulting equations are still quite complicated. Recently a relatively simple consistent approach has been devised \[12\], which starts from the integral version of the Kadanoff–Baym equations.

5 Conclusion

We have found new gradient corrections in the generalized Boltzmann equation \[1\]. These corrections result if the gradient expansion is performed also inside of the self–energy in scattering integrals of the quantum transport equations. We call these corrections internal, because they reflect the many–body character of scattering processes represented by the internal structure of the self–energy. Analogously the standard gradient corrections \[1\], which do not take into account the internal structure of the self–energy, are called here external.

The generalized Boltzmann equation with all correction terms has been derived. The internal corrections to the GBE have been calculated for electron scattering on localized static potentials, which is described by the ATA self–energy. More complex corrections have been obtained for interacting spinless Fermions, where the self–energy is described either by a local \(T\)–matrix approximation or by a nonlocal shielded potential approximation. We believe that the GBE with the new correction terms might be a proper tool for studies of relaxation to equilibrium in systems with nontrivial electron interactions. We are planning to investigate some of these systems in future.

We have also discussed the importance of the internal corrections in the linearized transport equations in weak electric fields \[13\], which can be derived from the GBE. The presence of new correction terms in the \(dc\)–version of these equations has been clarified in the above studied examples. The internal corrections might be important in many other physical problems, where the self–energy includes multiple scattering events.

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Appendix A

The causal Fermion ($O = \psi$) or Boson ($O = A$) Green’s functions in real times are defined by \[ [4, 5] \] (Matsubara Green’s functions in complex times look analogously)

$$G^t(1, 2) = -i \frac{\bar{\hbar}}{\hbar} < T[O(1) O^\dagger(2)] > , \quad j \equiv (r_j, t_j) , \quad (j = 1, 2) . \quad (A.1)$$

Correlation functions are related to the causal function as follows

$$i \bar{\hbar} G^t(1, 2) = G^>(1, 2) = <O(1) O^\dagger(2)> , \quad t_1 > t_2 ,$$

$$\mp i \bar{\hbar} G^t(1, 2) = G^<(1, 2) = <O^\dagger(2) O(1)>, \quad t_1 < t_2 , \quad (A.2)$$

where the upper (lower) sign applies to Fermions (Bosons).

The retarded and advanced Green’s functions are defined by

$$G^r(1, 2) = -i \frac{\bar{\hbar}}{\hbar} \theta(1 - 2) [G^>(1, 2) \pm G^<(1, 2)] ,$$

$$G^a(1, 2) = i \frac{\bar{\hbar}}{\hbar} \theta(2 - 1) [G^>(1, 2) \pm G^<(1, 2)] , \quad (A.3)$$

where the theta function is $\theta(t) = 0, \quad t < 0; \quad \theta(t) = 1, \quad t \geq 0$.

In equilibrium and space homogeneous systems the Green’s functions depend only on the difference of coordinates $(r, t) = (r_1 - r_2, t_1 - t_2)$, so that they can be easily Fourier transformed to the $(k, \omega)$–representation as follows

$$G(k, \omega) = \int d^n r \int dt \ \exp(i(\omega t - r \cdot k)) \ G(r_1 - r_2; t_1 - t_2) . \quad (A.4)$$

Then the Fermion and Boson correlation functions can be expressed as [1]

$$G^<(k, \omega) = n_{F,B}(\hbar \omega) A(k, \omega) , \quad G^>(k, \omega) = (1 \mp n_{F,B}(\hbar \omega)) A(k, \omega) , \quad (A.5)$$

where $n_F, n_B$ denote the Fermi–Dirac and Bose–Einstein distributions

$$n_{F,B}(\hbar \omega) = \frac{1}{e^{\frac{\bar{\hbar} \omega}{kT}} \pm 1}$$

and the spectral function is defined by

$$A(k, \omega) \equiv -2 \ \text{Im} \ G^r(k, \omega) = G^>(k, \omega) \pm G^<(k, \omega) . \quad (A.6)$$

The retarded Green’s function can be calculated from the spectral function (A.6) as follows ((A.6–A.7) hold also in full CMS coordinates $\xi = (k, \omega; R, T)$)

$$G^r(k, \omega) = \int_{-\infty}^{\infty} d\tilde{\omega} \frac{A(k, \tilde{\omega})}{2\pi} \frac{\tilde{\omega}}{\omega - \tilde{\omega} + i\delta} . \quad (A.7)$$

Similar formulas can be applied also for the self–energy.

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Appendix B

The nonequilibrium Green’s functions can be found by analytical continuation to real times of the Matsubara Green’s functions in complex times [1]. In NGF it is often necessary to find the propagator or correlation part of a combination of functions. An example is the product

\[ A(1,2) = B(1,2) \, C(1,2) \, , \quad (B.1) \]

where \( A, B, C \) are one-particle causal Green’s functions or self–energies. The required functions can be found by LW rules [3], where the signs and prefactors result from the definitions (A.2–A.3). We have found the expression for (B.1) in two cases, where the functions \( A, B, C \) correspond either to Fermions (F) or to Bosons (B) as follows: (1) \((A,B,C) = (F,F,B)\) (an example is an electron–phonon self–energy [5]) or (2) \((A,B,C) = (B,F,F)\) (an example is the function \( R_0 \) in (15)). In both these cases the expressions result in the form

\[
A^< (1, 2) = -i \, B^< (1, 2) \, C^< (1, 2) \, , \quad A^> (1, 2) = -i \, B^> (1, 2) \, C^> (1, 2) \, ,
\]

\[
A^\tau (1, 2) = -i \, (B^\tau (1, 2) \, C^> (1, 2) - B^< (1, 2) \, C^\tau (1, 2)) \, , \\
A^a (1, 2) = -i \, (B^a (1, 2) \, C^> (1, 2) - B^< (1, 2) \, C^a (1, 2)) \, . \quad (B.2)
\]

Similarly can be found the propagators and correlation functions for the expression

\[ A(1,2) = B(1,2) \, C(2,1) \, , \quad (B.3) \]

where the following possibilities have been chosen: (1) \((A,B,C) = (F,B,F)\) (an example is the self–energy (14)) or (2) \((A,B,C) = (B,F,F)\) (an example is the electron–hole bubble \( L_0 \) in (24)). These possibilities, which, but the order of \( B, F \) in (1), are the same as in (B.2), give the identities ((1) for (−), (2) for (+))

\[
A^< (1, 2) = i \, B^< (1, 2) \, C^> (2, 1) \, , \quad A^> (1, 2) = i \, B^> (1, 2) \, C^< (2, 1) \, , \\
A^\tau (1, 2) = i \, (B^\tau (1, 2) \, C^< (2, 1) = B^< (1, 2) \, C^\tau (2, 1)) \, , \\
A^a (1, 2) = i \, (B^a (1, 2) \, C^< (2, 1) = B^< (1, 2) \, C^a (2, 1)) \, . \quad (B.4)
\]

The following structure appears also in most formulas

\[ A(1,2) = B(1,\bar{3}) \, C(\bar{3},2) \, , \quad (B.5) \]

where the bars over the arguments mean integration over the whole real axis of these coordinates. The expressions result the same, irrespective of the types of involved functions

\[ A^< (1, 2) = B^\tau (1, \bar{3}) \, C^< (\bar{3}, 2) + B^< (1, \bar{3}) \, C^a (\bar{3}, 2) \, , \]
\[ A^>(1,2) = B^r(1,3) \ C^>(3,2) + B^>(1,3) \ C^a(3,2), \]
\[ A^r(1,2) = B^r(1,\bar{3}) \ C^r(\bar{3},2), \quad A^a(1,2) = B^a(1,\bar{3}) \ C^a(\bar{3},2). \]  

(B.6)

In the text we use the term 'parallel' for the structures of arguments in \( (B.1-4) \), while the structure \( (B.5-6) \) is termed 'serial'.

The Kadanoff–Baym equations can be found by application of the rules \( (B.6) \) to the differential Dyson equation, which can be written in two forms. If we take into account that \( (G^{-1}_0)^<(1,2) = 0 \) and \( \delta^<(1-2) = 0 \), then the two forms of the Kadanoff–Baym equations look as follows

\[
(G^{-1}_0)^<(1,3) \ G^<(3,2) = \Sigma^r(1,3) \ G^<(3,2) + \Sigma^<(1,3) \ G^a(3,2),
\]
\[
G^<(1,3) \ (G^a_0)^-(3,2) = G^r(1,3) \ \Sigma^<(3,2) + G^<(1,3) \ \Sigma^a(3,2). \]  

(B.7)

Analogous equations can be obtained for the correlation function \( G^> \).

The generalized Boltzmann equation \( \square \) can be derived by subtraction of the two sets of equations \( (B.7) \). In the resulting quantum transport equations for \( G^{<,>}_\square \) it is necessary to introduce the CMS coordinates and perform the gradient expansion up to the first order. Then a Fourier transform over the small coordinates is performed. It is helpful to resolve the propagators \( G^{r,a}, \Sigma^{r,a} \) from the right side of \( (B.7) \) into the real and imaginary parts. The imaginary parts of these propagators can be resolved with the help of the identity \( (A.8) \) (in the CMS coordinates). Then the terms with equal correlation signs \( <, < \) and \( >, > \) fall out from the scattering side of the new equations and the GBE easily results (see its completed form in \( (32) \)).

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