Research Article

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A new analytical method to simulate the mutual impact of space-time memory indices embedded in (1 + 2)-physical models

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Abstract: In the present article, we geometrically and analytically examine the mutual impact of space-time Caputo derivatives embedded in (1 + 2)-physical models. This has been accomplished by integrating the residual power series method (RPSM) with a new trivariate fractional power series representation that encompasses spatial and temporal Caputo derivative parameters. Theoretically, some results regarding the convergence and the error for the proposed adaptation have been established by virtue of the Riemann–Liouville fractional integral. Practically, the embedding of Schrödinger, telegraph, and Burgers’ equations into higher fractional space has been considered, and their solutions furnished by means of a rapidly convergent series that has ultimately a closed-form fractional function. The graphical analysis of the obtained solutions has shown that the solutions possess a homotopy mapping characteristic, in a topological sense, to reach the integer case solution where the Caputo derivative parameters behave similarly to the homotopy parameters. Altogether, the proposed technique exhibits a high accuracy and high rate of convergence.

Keywords: Caputo derivative, time-space partial differential equations, fractional RPSM

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1 Introduction

Fractional derivatives have proven their capability to describe several phenomena associated with memory or aftereffects due to their nonlocality property [1,2]. Such phenomena are commonplace in physical processes, biological structures, and cosmological phenomena. For instance, the fractional electrodiffusion equations have been successfully used to describe the transport processes of charge carriers in systems with a hierarchical structure [3], the fractional Cattaneo equations have been used to study the transport process of electrolytes in media where subdiffusion occurs [4], the fractional Kelvin–Voigt rheological models have been employed to examine the hydropolymer dynamics at low applied force frequencies [5], the fractional rheological model of the cell has been developed to study the relationship between the dynamic viscoelastic behavior of the cytoskeleton and the static contractile stress that it bears [6], the fractional rumor spreading dynamical model in a social network has been studied and analyzed in ref. [7], and several other fractional complex models have been utilized in turbulent [8], viscoelastic [9], kinetic and reaction–diffusion processes [10], and quantum mechanics [11].

For this reason, it became necessary to illuminate and find the solutions to the models that describe these phenomena. In this context, several numerical and analytical methods have been presented for solving hybrid models with fractional derivatives. Most of these approaches were accommodations for the existing methods of the integer case, which is considered a natural approach since the fractional derivative generalizes the classical derivative to an arbitrary order. Some of the most popular methods have been driven by Taylor’s power series method (TPSM) [12–16], the Adomian decomposition method [17,18], the homotopy perturbation method [19–21], the q-homotopy analysis with Elzaki transform method [22,23], the reduced differential transform method (RDTM) [24–26], the spectral-
collocation with quadratic and cubic B-splines [27–30], the Laplace and Sumudu transform methods [31,32], and the variational uniqueness method [33]. Further, the existence and uniqueness analysis of the solution of some time-fractional models have been examined. See, for example, refs [34,35].

The functionality of the aforementioned methods is mainly to examine influences of either space- or the time-fractional derivatives. In contrast, several notable studies have shown that the power-law memory can be ingrained in both the spatial and temporal coordinates [36]. Motivated by these facts, several techniques related to the celebrated Taylor’s series, namely, TPSM, RDTM, and residual power series method (RPSM) have been adapted to furnish the solutions of models endowed with spatial and temporal fractional derivatives [37–48].

By proceeding in this direction, our motivation in this work is to present a new semi-analytical technique to simulate the mutual impact of space-time Caputo derivatives embedded in (1 + 2)-physical models. For this purpose, we will consider and adapt the RPSM by combining it with a new trivariate power series that comprised spatial and temporal Caputo fractional derivatives and provide the necessary convergence and error analysis related to this adaptation. The proposed method will be called by $(\alpha, \beta, \gamma)$-fractional residual power series method (FRPSM). Further, we will also provide a geometric interpretation for the role of the Caputo fractional derivative parameters. It should be noted here that the method’s applicability and efficiency require a high Caputo differentiability for the solution. In other words, the solution needs to be analytic in the sense of the Caputo fractional derivatives. We should mention here that all derivatives are defined in the Caputo sense due to their role in modeling phenomena with nonlocal properties and problems that possess interactions with the past [49].

It is worth mentioning here that the RPSM was first developed by a Jordanian researcher in ref. [50] to provide a series solution for the fuzzy differential equations under strongly generalized differentiability. In fact, the mechanism of the RPSM is a reformulation of the celebrated TPSM where the series coefficients are obtained by minimizing the residual error for the truncated series solution. This, in turn, implies that the series coefficients can be obtained by a successive differentiation of the truncated series solution. Recently, the RPSM has been successfully utilized to acquire approximate solutions for various problems in many areas [51–55].

The remainder of this article is presented as follows. An adaptation of the RPSM for handling fractional embedding of (1 + 2)-physical models is presented in Section 2 along with some convergence and error results. In Section 3, the solution for the embedding of Schrödinger, telegraph, and Burgers’ equations has furnished by means of the proposed method. Finally, concluding remarks are presented in Section 4.

2 The methodology of $(\alpha, \beta, \gamma)$-FRPSM

As mentioned earlier, our main goal is to combine the RPSM with a new trivariate power series expansion that is endowed with three Caputo derivative parameters $\alpha, \beta, \gamma \in (0, 1)$ to study their mutual impact. We start this section by recalling the notion of $(\alpha, \beta, \gamma)$-FPS and some of its relevant properties and convergence results.

Definition 2.1. [43]. An $(\alpha, \beta, \gamma)$-fractional power series centered at the origin (simply, $(\alpha, \beta, \gamma)$-fractional power series [FPS]) with constant coefficients $[\lambda_{i,j,k}]$ is a power series endowed with three fractional derivative parameters $\alpha, \beta, \gamma \in (0, 1)$ in the following Cauchy form:

$$
\sum_{i+j+k=0}^{\infty} \lambda_{i,j,k} t^{|i\alpha|} x^{|j\beta|} y^{|k\gamma|} = \lambda_{0,0,0} + \lambda_{0,0,1} t^{|\alpha|} + \lambda_{0,1,0} y^{|\beta|} + \lambda_{0,0,2} y^{|\gamma|} + \sum_{i+j+k=1}^{\infty} \sum_{r+s=0}^{n} \lambda_{r,s,t} t^{|\alpha|^r} x^{|\beta|^s} y^{|\gamma|^t} + \cdots,
$$

where $i, j, k \in \mathbb{N}^*$ and $t, x, \text{ and } y$ are nonnegative variables.

Proposition 2.2. [43]. If there exists $t_0, x_0, y_0 \in \mathbb{R}_{\geq 0}$ such that $\{\lambda_{i,j,k} t^{|i\alpha|} x^{|j\beta|} y^{|k\gamma|} : i, j, k \in \mathbb{N}^*\}$ is a bounded set, then the $(\alpha, \beta, \gamma)$-FPS is absolutely convergent on $[0, t_0) \times [0, x_0) \times [0, y_0)$.

Theorem 2.3. [43]. The $(\alpha, \beta, \gamma)$-FPS is either absolutely convergent for all $(t, x, y) \in \mathbb{R}_{\geq 0}^3$ or there exists $R_t, R_x, R_y \in \mathbb{R}_{\geq 0}$ such that it is absolutely convergent on $[0, R_t) \times [0, R_x) \times [0, R_y)$. Moreover, the set $\{\lambda_{i,j,k} t^{|i\alpha|} x^{|j\beta|} y^{|k\gamma|} : (i, j, k) \in \mathbb{N}^3\}$ is unbounded otherwise.

Definition 2.4. The triple $(R_t, R_x, R_y) \in \mathbb{R}_{\geq 0}^3$ in Theorem 2.3 is called the radius of convergence for $(\alpha, \beta, \gamma)$-FPS. Otherwise, the radius of convergence is said to be infinite.

Remark 2.5. It is worth mentioning here that the $(\alpha, \beta, \gamma)$-FPS can be rewritten as the following Cauchy product form:

...
Theorem 2.6. [43]. Let \( \sum_{i=0}^{\infty} b_i t^i \), \( \sum_{i=0}^{\infty} c_i x^i \), and \( \sum_{k=0}^{\infty} d_k y^k \) be three absolutely convergent series at \( t_0, x_0 \), and \( y_0 > 0 \), respectively. Then their Cauchy product (2.2) is absolutely convergent on \( D \), where, in this case, \( \lambda_{i,j,k} = a_i b_j c_k \). Moreover,

\[
\sum_{i+j+k=0}^{\infty} \lambda_{i,j,k} t^i x^j y^k = \left( \sum_{i=0}^{\infty} a_i t^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right) \left( \sum_{k=0}^{\infty} c_k y^k \right). \tag{2.3}
\]

Next, we recall some basic knowledge regarding the Caputo fractional derivative and the Riemann–Liouville fractional integral operators that will be employed in this work. The Caputo time-fractional derivative of order \( a \in (n - 1, n] \), \( n \in \mathbb{N} \) is defined for an appropriate function \( u(t, x, y) \) by ref. [56]

\[
D_a^\alpha [u(t, x, y)] = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n - a)} \int_0^t (t - \tau)^{n-a-1} u(t, x, y) d\tau, & a \in (n - 1, n), \ t > 0 \\
\frac{\partial^n u(t, x, y)}{\partial t^n}, & a = n, \ t > 0 \\
\frac{1}{\Gamma(a + 1)} t^{a-1}, & a \geq a, \ a \in \mathbb{R}, \ t > 0 \\
0, & a < 0,
\end{array} \right. \tag{2.4}
\]

With a direct implementation of (2.4) and using the integration by parts, we particularly obtain for \( a \in (0, 1) \)

\[
D_a^\alpha [t^a] = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(a + 1)} t^{a-1}, & a \geq a, \ a \in \mathbb{R}, \ t > 0 \\
0, & a = 0,
\end{array} \right. \tag{2.5}
\]

which will be intensively used in this work to derive our main results.

Remark 2.7. We can enforce the Caputo-fractional derivative order \( a \) to be in \( (0, 1) \) since \( D_a^{a-1} [D_a^{a-1} [u(t, x, y)]] = D_a^a [u(t, x, y)] \) for any order \( a \in (n - 1, n) \), \( n \in \mathbb{N} \).

The Riemann–Liouville time-fractional integral operator of order \( a \in (n - 1, n) \), \( n \in \mathbb{N} \) is defined for an appropriate function \( u(t, x, y) \) by

\[
\mathcal{J}_a^m [u(t, x, y)] = \frac{1}{\Gamma(m + 1)} \int_0^t (t - \tau)^{m+a-1} u(\tau, x, y) d\tau. \tag{2.6}
\]

It should be noted here that the Riemann–Liouville time-fractional integral operator is a right inverse for the Caputo time-fractional derivative operator but not a left inverse. More precisely, for \( a \in (n - 1, n) \), \( n \in \mathbb{N} \), we have

\[
D_a^m \mathcal{J}_a^m [u(t, x, y)] = u(t, x, y), \tag{2.7}
\]

and

\[
\mathcal{J}_a^m D_a^m [u(t, x, y)] = u(t, x, y) - \sum_{m=0}^{n-1} \partial_i^m u(0^+, x, y) \frac{t^m}{m!}. \tag{2.8}
\]

Notation 2.8. For the sake of shortening the mathematical equations, we will denote \( \Gamma(ia + 1) \) by \( \Gamma_d(i) \).

Now, assume that \( u(t, x, y) \) has an \((a, \beta, \gamma)\)-FPS representation with radius of convergence \((R_t, R_x, R_y)\). Then the mixed Caputo-fractional derivatives of \( u(t, x, y) \) is given in ref. [43] by

\[
D_a^m D_x^\beta D_y^\gamma [u(t, x, y)] = \sum_{r+s+m=0}^{\infty} \lambda_{r+s+m} t^r \frac{\Gamma(\beta)(s + \beta)\Gamma(m + \gamma)\Gamma_a(\beta\gamma m + \gamma)}{\Gamma_d(m)} t^s \frac{\Gamma(\gamma)(\gamma + \beta)\Gamma_a(\beta\gamma m + \gamma)}{\Gamma_d(\beta)} \tag{2.9}
\]

Consequently, by plugging \((t, x, y) = (0, 0, 0)\) into (2.9), we have the following form for the series coefficients in terms of the mixed Caputo-fractional derivatives:

\[
\lambda_{r+s+m} = \frac{D_a^m D_x^\beta D_y^\gamma [u(0, 0, 0)]}{\Gamma_d(m)} t^s, \tag{2.10}
\]

and, therefore,

\[
u(t, x, y) = \sum_{i+j+k=0}^{\infty} \lambda_{i,j,k} t^i x^j y^k. \tag{2.11}
\]

The last representation of \( u(t, x, y) \) will be recognized as the \((a, \beta, \gamma)\)-Maclaurin series due to its similarity with the celebrated classical Maclaurin series.

Next, to achieve our goal, we extend the mechanism of the RPSM into \((a, \beta, \gamma)\)-fractional space. Consider the following embedding of differential equations in \((a, \beta, \gamma)\)-fractional space

\[
\Psi(u(t, x, y), D_t^a u(t, x, y), D_x^\beta u(t, x, y), D_y^\gamma u(t, x, y), \ldots) = 0. \tag{2.12}
\]

Assume the existence of the solution in the form of \((a, \beta, \gamma)\)-FPS with radius of convergence \((R_t, R_x, R_y)\). Thus, the \(n\)-th truncated series of \( u \) is

\[
u_n(t, x, y) = \sum_{i+j+k=0}^{n} \lambda_{i,j,k} t^i x^j y^k. \tag{2.13}
\]
and an approximate solution of (2.12) will be obtained when the coefficients \( \lambda_{ij,k} \) are determined for all permutations \( i + j + k = 0, 1, 2, \ldots, n \).

We define the residual function for the solution of Eq. (2.12) by

\[
\text{Res}_u(t, x, y) = \Psi(u(t, x, y), D_t^\alpha u(t, x, y), D_x^\beta u(t, x, y), D_y^\gamma u(t, 2.14), (t, x, y), \ldots),
\]

and the residual function for the \( n \)-th truncated series solution of \( u \) by

\[
\text{Res}^n_u(t, x, y) = \Psi(u_n(t, x, y), D_t^\alpha u_n(t, x, y), D_x^\beta u_n(t, x, y), D_y^\gamma u_n(t, x, y), \ldots),
\]

Since \( u \) is a solution of (2.12) and the Caputo-fractional derivative of a constant function is zero, then we have the following apparent properties:

\[
(a) \quad \text{Res}_u(t, x, y) = 0,
\]

\[
(b) \quad \lim_{n \to \infty} \text{Res}^n_u(t, x, y) = \text{Res}_u(t, x, y),
\]

\[
(t, x, y) \in (0, R_t) \times (0, R_x) \times (0, R_y),
\]

\[
(c) \quad D_t^\alpha D_x^\beta D_y^\gamma \text{Res}^n_u(t, x, y) = 0, \quad \text{for each } i + j + k = 0, 1, 2, \ldots, n.
\]

Therefore, by inserting (2.13) into (2.15) and solving the resultant system of the following algebraic equations:

\[
D_t^\alpha D_x^\beta D_y^\gamma \text{Res}^n_u(0, 0, 0) = 0,
\]

throughout all the permutations of \( i + j + k = n, n \in \mathbb{N} \), and we obtain the wanted coefficients \( \lambda_{ij,k} \).

Next, we provide a formula for the remainder (or the error term) of the \( (\alpha, \beta, \gamma) \)-Maclaurin series solution in terms of the Riemann–Liouville fractional integral:

**Theorem 2.9.** Let \( D_t^\alpha D_x^\beta D_y^\gamma [u(t, x, y)] \in C((0, R_t) \times (0, R_x) \times (0, R_y)) \) for each \( i + j + k = 0, 1, 2, \ldots, n + 1 \). Then \( u(t, x, y) \) can be expressed as follows

\[
u(t, x, y) = \sum_{i+j+k=0}^{n} \frac{D_t^\alpha D_x^\beta D_y^\gamma [u(0, 0, 0)]}{\Gamma(i)\Gamma(j)\Gamma(k)} \times t^i x^j y^k + \text{Res}_u(t, x, y),
\]

where \( \text{Res}_u(t, x, y) \) is the remainder given in terms of the Riemann–Liouville fractional integral as follows

\[
R_n(t, x, y) = \sum_{j+k=0}^{n} J^{(n+1)-(j+k)\alpha}_{i} D_t^\alpha D_x^\beta D_y^\gamma [u(t, 0, 0)] + \sum_{k=0}^{n} J^{(n+1)\beta}_{i} D_x^\beta D_y^\gamma [u(t, 0, 0)]
\]

\[
\times [u(t, x, 0)] + J^{(n+1)\beta}_{i} D_x^\beta D_y^\gamma [u(t, x, 0)].
\]

**Proof.** First, by the help of Eq. (2.8), one can show by the mathematical induction, as in [16, Lemma 3.1], that

\[
J^{(n+1)\beta}_{i} D_x^\beta D_y^\gamma [u(t, x, 0)] = u(t, x, y) - \sum_{k=0}^{n} \frac{D_y^\gamma [u(t, x, 0)]}{\Gamma(k)} y^k.
\]

Using the fact that for a constant function \( c \),

\[
J^{\beta}_{i} c = \frac{c}{\Gamma(k)} y^k,
\]

and the sense of Eq. (2.22) with respect to \( t \), we have

\[
J^{(n+1)\beta}_{i} D_x^\beta D_y^\gamma [u(t, x, 0)] = u(t, x, y) - \sum_{k=0}^{n} \frac{D_y^\gamma [u(t, x, 0)]}{\Gamma(k)} y^k.
\]
\[
\sum_{j_k=0}^{n} f_i^{(n+1-(j_k+1)+a)}d_x^{(n+1-(j_k+1)+a)} f_i^{\beta y}d_y^{\alpha y}[u(t, 0, 0)]
\]
= \[
\sum_{j_k=0}^{n} f_i^{(n+1-(j_k+1)+a)}d_x^{(n+1-(j_k+1)+a)} f_i^{\beta y}d_y^{\alpha y}[u(t, 0, 0)] \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right]
\]
= \[
\sum_{j_k=0}^{n} f_i^{(n+1-(j_k+1)+a)}d_x^{(n+1-(j_k+1)+a)} f_i^{\beta y}d_y^{\alpha y}[u(t, 0, 0)] \times \left[ d_y^{\beta y}[u(t, 0, 0)] \times \Gamma_y(k) \right]
\]
= \[
\sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right) \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right] - \sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right)
\]
= \[
\sum_{j_k=0}^{n} d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right] - \sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right)
\]
= \[
\sum_{j_k=0}^{n} d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right] - \sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right)
\]
= \[
\sum_{j_k=0}^{n} d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right] - \sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right)
\]
= \[
\sum_{j_k=0}^{n} d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \times \left[ d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right] - \sum_{j_k=0}^{n} \left( d_y^{\beta y}[u(t, 0, 0)] / \Gamma_y(k) \right)
\]
By adding Eqs. (2.22), (2.24), and (2.25) we obtain the required formula of \( R_n(t, x, y) \).

**Theorem 2.10.** Suppose that \( D_\alpha^\beta D_\gamma^\gamma D^\beta y D^\beta y[u(t, x, y)] \in C([0, R_x] \times [0, R_y]) \) and there exists \( M > 0 \) such that \( |D_\alpha^\beta D_\gamma^\gamma D^\beta y D^\beta y[u(t, x, y)]| \leq M \) for all permutations \( i+j+k = 0, 1, 2, \ldots, n+1 \) in \( \tilde{D} = [0, R_x] \times [0, R_y] \times [0, R_y] \). Then
(a) the solution
\[
u(t, x, y) = \sum_{i+j+k=0}^{\infty} d_i^\alpha d_j^\beta d_k^\gamma u(0, 0, 0) / \Gamma_y(k) t^i x^j y^k,
\]
is absolutely convergent on \( \tilde{D} \).
(b) for all \( (t, x, y) \in \tilde{D} \), the remainder \( R_n(t, x, y) \) satisfies the bound
\[
|R_n(t, x, y)| \leq M \sum_{i+j+k=n+1} \frac{t^i x^j y^k}{\Gamma_y(k)}.
\]

**Proof.** The proof of (a) follows directly from 2.2 and 2.3. For (b), from the remainder definition, we have \( D_i^\alpha D_j^\beta D_k^\gamma R_n(0, 0, 0) = 0 \) for each \( i+j+k = 0, 1, 2, \ldots, n \) and \( D_i^\alpha D_j^\beta D_k^\gamma R_n(t, x, y) = D_i^\alpha D_j^\beta D_k^\gamma [u(t, x, y)] \) for all \( i+j+k \geq n+1 \) and \( (t, x, y) \in \tilde{D} \). By assumption, for each \( i+j+k = n+1 \) and \( (t, x, y) \in \tilde{D} \), we have
\[-M \leq \mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)] \leq M.\]

Therefore, for all \((t, x, y) \in \hat{D}\),

\[
\sum_{i+j+k=n+1} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [-M] \leq \sum_{i+j+k=n+1} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)]] \\
\leq \sum_{i+j+k=n+1} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [M].
\]

(2.28)

Since \(\mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [M] = \frac{M^{i+j+k} \mathcal{J}_{i}^{m} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha}}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(k)}\) for all \(i + j + k = n + 1\), then

\[
\sum_{i+j+k=n+1} \frac{-M^{i+j+k} \mathcal{J}_{i}^{m} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha}}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(k)} \leq \sum_{i+j+k=n+1} \frac{M^{i+j+k} \mathcal{J}_{i}^{m} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha}}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(k)}. \tag{2.29}
\]

Thus,

\[
\left| \sum_{i+j+k=n+1} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)]] \right| \leq M \sum_{i+j+k=n+1} \frac{t^{i+j+k} \mathcal{J}_{i}^{m} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha}}{\Gamma(i) \Gamma(j) \Gamma(l) \Gamma(k)}. \tag{2.30}
\]

Now, from Definition 2.1, the term \(i + j + k = n + 1\) in the aforementioned left sum can be rewritten for all \((t, x, y) \in \hat{D}\) as follows:

\[
\sum_{i+j+k=n+1} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)]] \\
= \sum_{j+k=0}^{n} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{i}^{(n+1-j-k)} \mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)]] \\
= \sum_{j+k=0}^{n} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{i}^{(n+1-j-k)} \mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)] + \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)]]] \\
= \sum_{j+k=0}^{n} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{i}^{(n+1-j-k)} \mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)] \\
+ \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)] + \mathcal{J}_{y}^{(n+1)} \mathcal{D}_{y}^{(n+1)} [u(t, x, y)]]. \tag{2.31}
\]

Therefore, the inequality (2.30) becomes

\[
\left| \sum_{j+k=0}^{n} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha} \mathcal{J}_{i}^{m} [\mathcal{D}_{i}^{(n+1-j-k)} \mathcal{D}_{x}^{\alpha} \mathcal{D}_{x}^{\beta} \mathcal{D}_{y}^{\gamma} [u(t, x, y)] + \mathcal{J}_{y}^{(n+1)} \mathcal{D}_{y}^{(n+1)} [u(t, x, y)]] \right| \leq M \sum_{j+k=0}^{n} \frac{t^{j+k} \mathcal{J}_{j+k} \mathcal{J}_{y}^{\beta} \mathcal{J}_{x}^{\alpha}}{\Gamma(j) \Gamma(k)}. \tag{2.32}
\]

Since (2.32) is valid for all \((t, x, y) \in \hat{D}\), in particular, we have the desired bound (2.27) of the remainder.

### 3 Application models

In this section, the declared \((\alpha, \beta, \gamma)\)-FRPSM will be employed to offer the solutions of the \((\alpha, \beta, \gamma)\)-embedding of Schrödinger and telegraph equations. These solutions will be presented in terms of a rapidly convergent series of the form \((\alpha, \beta, \gamma)\)-Maclaurin series, which eventually will have closed-form fractional functions. In addition, a graphical analysis has been provided to study the behavior of the solutions when the fractional derivative parameters vary in the interval \((0, 1)\). In all our applications, we assume that the fractional derivative parameters \(\alpha, \beta, \gamma \in (0, 1)\).

**Application 1.** Consider the following \((\alpha, \beta, \gamma)\)-Schrödinger problem:
\[ \mathbf{i D}^{\alpha}_{\mathbf{t}}[u(t, x, y)] = D^{\beta}_{\mathbf{x}}[u(t, x, y)] + D^{\gamma}_{\mathbf{y}}[u(t, x, y)], \]

with initial condition

\[ u(0, x, y) = \sin(\theta x) + \sin(\phi y), \]

where \( \sin(x) = \sum_{n=0}^{\infty} (-1)^{n}(2^{2n+1}x^{2n+1})/(2n+1)! \). We assume the existence of a solution for (3.1)–(3.2) in the form \( (\alpha, \beta, \gamma) \)-FPS. To find the coefficients \( \{\alpha_{i,j,k}\}_{i+j+k=0} \) by our proposed method, we first construct the residual function for the \( \text{nth} \)-truncated series solution of Eq. (3.1):

\[ \text{Res}_{n}(t, x, y) = \sum_{i+j+k=0}^{n} \alpha_{i,j,k} t^{i} x^{j} y^{k} - \sum_{i+j+k=0}^{n} \frac{\Gamma(j+2)}{\Gamma(j)} t^{i} x^{j} y^{k}. \]

From the fractional initial condition (3.2), we have the initial coefficients for \( j, k \geq 0 \)

\[ \lambda_{0,j+1,0} = \frac{(-1)^{j}}{\Gamma(j+1)}, \lambda_{0,k,0} = 0, \quad \text{otherwise}. \]

Next, we solve the system of algebraic equations \( \{\mathbf{D}^{\alpha}_{\mathbf{t}}\mathbf{D}^{\beta}_{\mathbf{x}}\mathbf{D}^{\gamma}_{\mathbf{y}}[\text{Res}_{n}(0, 0, 0) = 0] \} \) for each \( i+j+k = n \in \mathbb{N}^{*} \) with taking into consideration the initial coefficients (3.4), that is, by solving the following system:

When \( n = 0 \), the system \( \{\mathbf{D}^{\alpha}_{\mathbf{t}}\mathbf{D}^{\beta}_{\mathbf{x}}\mathbf{D}^{\gamma}_{\mathbf{y}}[\text{Res}_{n}(0, 0, 0) = 0] \}_{i+j+k=0} \) yields:

\[ i\mathbf{I}^{\alpha}_{n}(1)\lambda_{i,j},0 - \frac{\Gamma(j)}{\Gamma(j+1)}\lambda_{i,j,0} = 0. \]

When \( n = 1 \), the system \( \{\mathbf{D}^{\alpha}_{\mathbf{t}}\mathbf{D}^{\beta}_{\mathbf{x}}\mathbf{D}^{\gamma}_{\mathbf{y}}[\text{Res}_{n}(0, 0, 0) = 0] \}_{i+j+k=1} \) yields:

\[ i\mathbf{I}^{\alpha}_{n}(1)\lambda_{i,j+1,0} - \frac{\Gamma(j+1)}{\Gamma(j+2)}\lambda_{i,j+1,0} = 0. \]

When \( n = 2 \), the system \( \{\mathbf{D}^{\alpha}_{\mathbf{t}}\mathbf{D}^{\beta}_{\mathbf{x}}\mathbf{D}^{\gamma}_{\mathbf{y}}[\text{Res}_{n}(0, 0, 0) = 0] \}_{i+j+k=2} \) yields:

\[ i\mathbf{I}^{\alpha}_{n}(1)\lambda_{i,j+2,0} - \frac{\Gamma(j+2)}{\Gamma(j+3)}\lambda_{i,j+2,0} = 0. \]

When \( n = 3 \), the system \( \{\mathbf{D}^{\alpha}_{\mathbf{t}}\mathbf{D}^{\beta}_{\mathbf{x}}\mathbf{D}^{\gamma}_{\mathbf{y}}[\text{Res}_{n}(0, 0, 0) = 0] \}_{i+j+k=3} \) yields:

\[ i\mathbf{I}^{\alpha}_{n}(1)\lambda_{i,j+3,0} - \frac{\Gamma(j+3)}{\Gamma(j+4)}\lambda_{i,j+3,0} = 0. \]

and so forth. Solving the aforementioned sets of linear equations recursively leads to:
\[
\begin{align*}
\lambda_{1,0,0} &= 0, & \lambda_{2,0,0} &= 0, & \lambda_{3,0,0} &= 0, \\
\lambda_{1,1,0} &= \frac{i}{\Gamma_d(1)\Gamma(1)}, & \lambda_{2,1,0} &= \frac{i^2}{\Gamma_d(2)\Gamma(1)}, & \lambda_{3,1,0} &= \frac{i^3}{\Gamma_d(3)\Gamma(1)}, \\
\lambda_{1,0,1} &= \frac{i}{\Gamma_d(1)\Gamma(1)}, & \lambda_{2,0,1} &= \frac{i^2}{\Gamma_d(2)\Gamma(1)}, & \lambda_{3,0,1} &= \frac{i^3}{\Gamma_d(3)\Gamma(1)}, \\
\lambda_{1,2,0} &= 0, & \lambda_{2,2,0} &= 0, & \lambda_{3,2,0} &= 0, \\
\lambda_{1,0,2} &= 0, & \lambda_{2,0,2} &= 0, & \lambda_{3,0,2} &= 0, \\
\lambda_{1,1,1} &= 0, & \lambda_{2,1,1} &= 0, & \lambda_{3,1,1} &= 0, \\
\lambda_{1,3,0} &= -\frac{i}{\Gamma_d(1)\Gamma(1)}, & \lambda_{2,3,0} &= -\frac{i^2}{\Gamma_d(2)\Gamma(1)}, & \lambda_{1,0,3} &= -\frac{i}{\Gamma_d(1)\Gamma(1)}, & \lambda_{2,0,3} &= -\frac{i^2}{\Gamma_d(2)\Gamma(1)}. \\
\end{align*}
\] (3.9)

We continue in this fashion until we obtain the rest of all coefficients. In general,

\[
\lambda_{1,2j+1,0} = \frac{(-1)^j(i)^j}{\Gamma_d(2j+1)}, \lambda_{1,2j+1,2k+1} = \frac{(-1)^k(i)^k}{\Gamma_d(2k+1)}, \lambda_{0,0,k} = 0, \quad \text{otherwise}. 
\] (3.10)

Compensating (3.10) in (2.1) and using Theorem 2.6, to obtain the following closed-form solution to the \((\alpha, \beta, \gamma)\)-Schrödinger problem:

\[
\begin{align*}
&u(t, x, y) = \sum_{i+j+k=0}^{\infty} \lambda_{i,j,k} t^i x^j y^k + \sum_{i+j+k=0}^{\infty} \lambda_{i,0,2k+1} t^i x^j y^{2k+1} \\
&= \sum_{i+j+k=0}^{\infty} \frac{(-1)^j(i)^j}{\Gamma_d(2j+1)} t^i x^j y^k + \sum_{i+j+k=0}^{\infty} \frac{(-1)^k(i)^k}{\Gamma_d(2k+1)} t^i x^j y^{2k+1} \\
&= \sum_{i=0}^{\infty} \frac{(it)^i}{\Gamma_d(i)} \left( \sum_{j=0}^{\infty} \frac{(-1)^j x^j y^k}{\Gamma_d(2j+1)} + \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{\Gamma_d(2k+1)} \right) \\
&= E_d(it^\alpha)(\sinh(\beta x^\delta) + \sin(\gamma y^\delta)).
\end{align*}
\] (3.11)

It is worth mentioning here that when \(\alpha, \beta, \gamma \to 1\), we obtain the closed-form solution \(u(t, x, y) = e^{it} (\sin(x) + \sin(y))\) for the classical Schrödinger model.

Figure 1 exhibits the behavior of some cross-sections of the 10th-approximate series solution \(u_{10}(t, x, y)\) of the Eq. (3.11) at various values of \(\alpha, \beta, \gamma \in (0, 1)\). Because of the solution symmetry, the cross-sections, when \(\gamma\) varies, are similar to the case of \(\beta\). In all cases, it is evident that the cross-sections form a continuous sequence, as long as the fractional derivative parameter approaching 1, to reach the cross-section for the integer case solution. In other words, the solution (3.11) behaves like the homotopic mapping in the topological sense.

**Application 2.** Consider the following hyperbolic \((\alpha, \beta, \gamma)\)-telegraph problem:

\[
D_t^{2\alpha}[u(t, x, y)] + 2D_t^{\beta}[u(t, x, y)] + u(t, x, y) = \frac{1}{2}(D_x^{2\beta}[u(t, x, y)] + D_y^{2\gamma}[u(t, x, y)]),
\] (3.12)

with initial conditions

\[
u(0, x, y) = \sinh(\beta x^\delta) \sinh(\gamma y^\delta), \quad D_t^\alpha[u(0, x, y)] = -2 \sinh(\beta x^\delta) \sinh(\gamma y^\delta).
\] (3.13)

Again, we assume the existence of a solution for (3.12) and (3.13) in the form \((\alpha, \beta, \gamma)\)-FPS. To find the coefficients \(\{\lambda_{i,j,k}\}_{i+j+k=0}^{\infty}\) by our proposed method, we first construct the residual function for the \(n\)-th-truncated series solution of Eq. (3.12) as follows:
\[
\text{Res}_n^\alpha(t, x, y) = \sum_{i+j+k=0}^n \lambda_{i,j,k} \frac{\Gamma(i+2)}{i!} t^{i+2} x^i y^j + 2 \sum_{i+j+k=0}^n \lambda_{i+1,j,k} \frac{\Gamma(i+1)}{i!} t^{i+1} x^i y^j + \sum_{i+j+k=0}^n \lambda_{i,j+2,k} \frac{\Gamma(j+2)}{j!} t^{j+2} x^i y^j,
\]

From the fractional initial condition (3.13), we have the initial coefficients for \( j, k \geq 0 \)

\[
\lambda_{0,j+1,2k+1} = \frac{1}{\Gamma_B(2j)\Gamma_A(2k)} - \lambda_{i,j,k} = \lambda_{1,i,j,k} = 0, \quad \text{otherwise}.
\]

Next, we solve the system of algebraic equations \( \{D_i D_j D_k \} [\text{Res}_n^\alpha(0, 0, 0)] = 0 \) for each \( i + j + k = n \in \mathbb{N}^* \) with taking into consideration the initial coefficients (3.15). That is, solving the following system:

When \( n = 0 \), the system \( \{D_i D_j D_k \} [\text{Res}_n^\alpha(0, 0, 0)] = 0 \) for \( j, k = 0 \) yields:

![Figure 1](image-url)

**Figure 1**: The cross-section behavior of \( u_{\alpha \beta}(t, x, y) \) for (3.11) at various values of \( \alpha, \beta, \gamma \in (0, 1) \). (a) \( \beta = \gamma = 1 \). (b) \( \alpha = \gamma = 1 \).
When \( n = 1 \), the system \( \{ D_{ij}^k D_{ij}^k D_{ij}^k [\text{Res}_1^k (0, 0, 0)] = 0 \}_{i+j+k=1} \) yields:

\[
\begin{align*}
\Gamma_\alpha(3)\lambda_{2,0,0} + 2\Gamma_\alpha(2)\lambda_{1,0,0} + \lambda_{0,0,0} - \frac{1}{2} \Gamma_\beta(2)\lambda_{0,2,0} - \frac{1}{2} \Gamma_\gamma(2)\lambda_{0,0,2} &= 0, \\
\end{align*}
\]

When \( n = 2 \), the system \( \{ D_{ij}^k D_{ij}^k D_{ij}^k [\text{Res}_2^k (0, 0, 0)] = 0 \}_{i+j+k=2} \) yields:

\[
\begin{align*}
\Gamma_\alpha(4)\lambda_{4,0,0} + 2\Gamma_\alpha(3)\lambda_{3,0,0} + \Gamma_\alpha(2)\lambda_{2,0,0} - \frac{1}{2} \Gamma_\beta(2)\Gamma_\beta(2)\lambda_{2,2,0} - \frac{1}{2} \Gamma_\beta(2)\Gamma_\beta(2)\lambda_{0,2,0} &= 0, \\
\end{align*}
\]

When \( n = 3 \), the system \( \{ D_{ij}^k D_{ij}^k D_{ij}^k [\text{Res}_3^k (0, 0, 0)] = 0 \}_{i+j+k=3} \) yields:

\[
\begin{align*}
\Gamma_\alpha(5)\lambda_{5,0,0} + 2\Gamma_\alpha(4)\lambda_{4,0,0} + \Gamma_\alpha(3)\lambda_{3,0,0} - \frac{1}{2} \Gamma_\beta(3)\Gamma_\beta(2)\lambda_{3,2,0} - \frac{1}{2} \Gamma_\beta(3)\Gamma_\beta(2)\lambda_{3,0,2} &= 0, \\
\end{align*}
\]

and more of the same. Solving the aforementioned sets of linear equations recursively leads to:

Analytical Simulation of \((1+2)\)-physical models [531]
\[ \lambda_{2,0,0} = 0, \quad \lambda_{3,0,0} = 0, \quad \lambda_{4,0,0} = 0, \quad \lambda_{5,0,0} = 0. \]
\[ \lambda_{2,1,0} = 0, \quad \lambda_{3,1,0} = 0, \quad \lambda_{4,1,0} = 0, \quad \lambda_{5,1,0} = 0. \]
\[ \lambda_{2,0,1} = 0, \quad \lambda_{3,0,1} = 0, \quad \lambda_{4,0,1} = 0, \quad \lambda_{5,0,1} = 0. \]
\[ \lambda_{2,1,1} = \frac{4}{\Gamma_6(2)\Gamma_1(1)\Gamma_1(1)}, \quad \lambda_{3,1,1} = \frac{-8}{\Gamma_6(3)\Gamma_1(1)\Gamma_1(1)}, \]
\[ \lambda_{2,2,0} = 0, \quad \lambda_{3,2,0} = 0, \quad \lambda_{4,2,0} = 0, \quad \lambda_{5,2,0} = 0, \]
\[ \lambda_{2,3,0} = 0, \quad \lambda_{3,3,0} = 0, \quad \lambda_{4,3,0} = 0, \quad \lambda_{5,3,0} = 0. \]

We continue in this fashion until we obtain the rest of all coefficients. In general, we obtain

\[ \lambda_{i,j+k+1} = \frac{(-2)^j}{\Gamma_6(i)\Gamma_1(2j)\Gamma_1(2k)}, \quad \lambda_{i,j,k} = 0, \quad \text{otherwise}. \] (3.21)

Figure 2: The cross-section behavior of \( u_b(t, x, y) \) for (3.22) at various values of \( \alpha, \beta, \gamma \in (0,1) \). (a) \( \beta = \gamma = 1 \). (b) \( \alpha = \gamma = 1 \).
Compensating (3.21) in (2.1) and using Theorem 2.6, to obtain the following closed-form solution to the \((\alpha, \beta, \gamma)\)-telegraph problem:

\[
\begin{align*}
    u(t, x, y) &= \sum_{i+j+k=0}^{\infty} \frac{(-2)^j}{\Gamma_i(2i)\Gamma_j(2j)\Gamma_k(2k)} t^{i+1}x^{(2i+1)\beta}y^{(2k+1)\gamma} \\
    &= \left( \sum_{i=0}^{\infty} \frac{(-2)^i}{\Gamma_i(2i)} \right) \left( \sum_{j=0}^{\infty} \frac{x^{(2j+1)\beta}}{\Gamma_j(2j+1)} \right) \times \left( \sum_{k=0}^{\infty} \frac{y^{(2k+1)\gamma}}{\Gamma_k(2k+1)} \right) \\
    &= E\left(-2^\alpha t\right) \sin h\beta x \sin h\gamma y. 
\end{align*}
\]  

(3.22)

If \(\alpha, \beta, \gamma \to 1\), we obtain the closed-form solution \(u(t, x, y) = e^{-2t} \sin h(x) \sin h(y)\) for the classical integer telegraph problem.

Figure 2 shows the cross-section behavior for the 8th-approximate series solution \(u_8(t, x, y)\) of the Eq. (3.22) at various values of \(\alpha, \beta, \gamma \in (0, 1)\). Because of the solution symmetry, the cross-sections, when \(y\) varies, are similar to the case of \(\beta\). Again, it is clear that the cross-section solutions continuously deformed into the cross-section for the integer case solution. In other words, the solution (3.22) acts like the homotopic mapping in the topological sense.

**Figure 3:** The cross-section behavior of \(u_8(t, x, y)\) for (3.11) at various values of \(\alpha, \beta, \gamma \in (0,1)\). (a) \(\beta = \gamma = 1\). (b) \(\alpha = \gamma = 1\).
Application 3. Consider the following \((\alpha, \beta, \gamma)\)-Burgers' problem:

\[
D^\gamma_{t}[u(t, x, y)] = D^\beta_{x}[u(t, x, y)] + D^\gamma_{y}[u(t, x, y)] + u(t, x, y)D^\alpha_{x}[u(t, x, y)],
\]  

(3.23)

with initial condition

\[
u(0, x, y) = x^\beta + y^\alpha.
\]  

(3.24)

We assume the existence of a solution for (3.23)–(3.24) in the form \((\alpha, \beta, \gamma)\)-FPS. To find the coefficients \([A_{i,j,k,n}]_{t+j,k=0}^{n}\) by our proposed method, we first construct the residual function for the \(n\)th-truncated series solution of Eq. (3.23):

\[
\text{Res}(t, x, y) = \sum_{i+j,k=0}^{n} A_{i,j,k} \frac{\Gamma(i + 1)}{\Gamma(i)} \tau^{\alpha} \lambda^{\beta} \gamma^{ky} - \sum_{i+j,k=0}^{n} A_{i,j,k} \frac{\Gamma(j + 2)}{\Gamma(j)} \tau^{\alpha} \lambda^{\beta} \gamma^{ky} - \sum_{i+j,k=0}^{n} A_{i,j,k} \frac{\Gamma(k + 2)}{\Gamma(k)} \tau^{\alpha} \lambda^{\beta} \gamma^{ky}
\]

\[
- \left( \sum_{i+j,k=0}^{n} A_{i,j,k} \tau^{\alpha} \lambda^{\beta} \gamma^{ky} \right) \times \frac{\Gamma(j + 1)}{\Gamma(j)} \tau^{\alpha} \lambda^{\beta} \gamma^{ky} \right.
\]

(3.25)

From the fractional initial condition (3.24), we have the initial coefficients for \(j, k \geq 0\)

\[
A_{0,0,0} = \begin{cases} 
1: & j = 1, k = 0, \\
1: & j = 0, k = 1, \\
0: & \text{otherwise}.
\end{cases}
\]  

(3.26)

Next, we solve the system of algebraic equations \([D^\alpha_{x}D^\beta_{x}D^\gamma_{y}][\text{Res}(0, 0, 0)] = 0\) for each \(i + j + k = n \in \mathbb{N}^*\) with taking into consideration the initial coefficients (3.26). That is, by solving the following system:

When \(n = 0\), the system \([D^\alpha_{x}D^\beta_{x}D^\gamma_{y}][\text{Res}(0, 0, 0)] = 0\) yields:

\[
\Gamma(1)A_{1,0,0} - \Gamma(2)A_{2,0,0} - \Gamma(2)A_{0,0,0} = 0.
\]  

(3.27)

When \(n = 1\), the system \([D^\alpha_{x}D^\beta_{x}D^\gamma_{y}][\text{Res}(0, 0, 0)] = 0\) yields:

\[
\Gamma(2)A_{2,0,0} - \Gamma(2)A_{2,0,0} - \Gamma(2)A_{0,0,0} = 0,
\]  

(3.28)

When \(n = 2\), the system \([D^\alpha_{x}D^\beta_{x}D^\gamma_{y}][\text{Res}(0, 0, 0)] = 0\) yields:

\[
\Gamma(3)A_{3,0,0} - \Gamma(2)A_{3,0,0} - \Gamma(2)A_{0,0,0} = 0,
\]  

(3.29)

When \(n = 3\), the system \([D^\alpha_{x}D^\beta_{x}D^\gamma_{y}][\text{Res}(0, 0, 0)] = 0\) yields:
\( \Gamma_2(4) \alpha_{4,0,0} - \Gamma_2(3) \Gamma_2(2) \alpha_{2,2,0} - \Gamma_2(3) \Gamma_2(2) \alpha_{2,0,2} - \Gamma_2(3) \Gamma_2(1) (\alpha_{1,1,0} \alpha_{2,0,0} + \lambda_{1,0,0} \alpha_{2,1,0} + \lambda_{0,0,0} \alpha_{3,1,0}) = 0, \)
\( \Gamma_2(3) \Gamma_2(1) \alpha_{3,1,0} - \Gamma_2(2) \Gamma_2(3) \alpha_{3,3,0} - \Gamma_2(2) \Gamma_2(1) \Gamma_2(2) \alpha_{2,2,2} \)
\( \Gamma_2(2) \Gamma_2(1) \Gamma_2(1) \Gamma_2(2) \Gamma_2(2) \alpha_{2,2,2} - \Gamma_2(2) \Gamma_2(1) \Gamma_2(3) \Gamma_2(2) \alpha_{1,1,1} - \Gamma_2(2) \Gamma_2(1) \Gamma_2(3) \Gamma_2(2) (\alpha_{1,0,1} \alpha_{1,0,1} + \alpha_{0,0,0} \alpha_{1,0,1} + \alpha_{0,0,0} \alpha_{1,0,1}) = 0, \)
\( \Gamma_2(2) \Gamma_2(1) \Gamma_2(1) \Gamma_2(1) \Gamma_2(3) \alpha_{1,1,1} - \Gamma_2(1) \Gamma_2(3) \Gamma_2(2) \alpha_{0,0,0} = 0. \)

and so forth. Solving the above sets of linear equations recursively leads to:

\( \lambda_{i,0,0} = 0, \quad \lambda_{i,0,0} = 0, \quad \lambda_{i,0,0} = 0, \)
\( \lambda_{i,1,0} = \Gamma_2(1) \Gamma_2(1) \Gamma_2(4) \alpha_{2,2,0} + \lambda_{i,0,0} \alpha_{2,1,0} + \lambda_{0,0,0} \alpha_{i,1,0}, \)
\( \lambda_{i,0,1} = \Gamma_2(1) \Gamma_2(1) \Gamma_2(4) \alpha_{2,2,0} + \lambda_{0,0,0} \alpha_{2,1,0} + \lambda_{0,0,0} \alpha_{0,1,0}, \)
\( \lambda_{0,0,0} = 0, \quad \lambda_{0,0,0} = 0, \quad \lambda_{0,0,0} = 0, \quad \lambda_{0,0,0} = 0, \quad \lambda_{0,0,0} = 0. \)

We continue in this fashion until we obtain the rest of all coefficients. In general, we find out that the coefficients are recursively given by

\[ \lambda_{i,0,1} = \lambda_{i,1,0}, \quad i \geq 0, \quad \lambda_{i,i,k} = 0, \quad \text{otherwise}, \]
\[ \lambda_{i,1,0} = \begin{cases} 1, & i = 0 \\ \frac{\Gamma(i-1) - \Gamma(i)}{\Gamma(d-1)} \sum_{k=1}^{n} 2k^{\alpha-1,0,1,0^*} \lambda_{k,1,0} & i = 1 \\ \frac{\Gamma(i) - \Gamma(i-1)}{\Gamma(d)} \sum_{k=1}^{n} 2k^{\alpha-1,1,0,0^*} \lambda_{k,1,0} & i = 2n \\ \frac{\Gamma(i) - \Gamma(i-1)}{\Gamma(d)} \left( \lambda_{i-1,1,0} + \sum_{k=1}^{n} 2k^{i-1,1,0,0^*} \lambda_{k,1,0} \right) & i = 2n + 1. \end{cases} \] (3.33)

Compensating (3.33) in (2.1) to obtain the following series solution to the \((\alpha, \beta, \gamma)\)-Burgers’ problem:

\[ u(t, x, y) = \sum_{i+j+k=0}^{\infty} \lambda_{i,1,0} t^{i} x^{j} y^{k} \]

\[ = \sum_{i=0}^{\infty} \lambda_{i,1,0} t^{i} x^{\alpha} y^{\beta} + \sum_{i=0}^{\infty} \lambda_{i,0,1} t^{i} y^{\gamma} \] (3.34)

\[ = (x^{\alpha} + y^{\beta}) \sum_{i=0}^{\infty} \lambda_{i,1,0} t^{i}. \] (3.35)

It should be pointed out here that \(\lambda_{1,0,1} = 1\) when \(\alpha, \beta, \gamma \to 1\). Thus, the solution for the integer Burgers’ problem is given for \(0 \leq t < 1\) by

\[ u(t, x, y) = (x + y) \sum_{i=0}^{\infty} t^{i} = \frac{x + y}{1 - t}. \] (3.35)

Finally, as future work, this idea of research can be tested on various embeddings of more physical models, expanded to approximate solutions in a bounded space, and adapted with different fractional derivative operators.

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