Finite–size scaling properties and Casimir forces in an exactly solvable quantum statistical–mechanical model

H. Chamati\textsuperscript{a}, D.M. Danchev\textsuperscript{b} and N.S. Tonchev\textsuperscript{a}

\textsuperscript{a}Georgy Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
\textsuperscript{b}Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev St. bl. 4, 1113 Sofia, Bulgaria

Abstract

A $d$–dimensional finite quantum model system confined to a general hypercubical geometry with linear spatial size $L$ and “temporal size” $1/T$ ( $T$ - temperature of the system) is considered in the spherical approximation under periodic boundary conditions. Because of its close relation with the system of quantum rotors it represents an effective model for studying the low–temperature behaviour of quantum Heisenberg antiferromagnets. Close to the zero–temperature quantum critical point the ideas of finite–size scaling are used for studying the critical behaviour of the model. For a film geometry in different space dimensions $\frac{1}{2} \sigma < d < \frac{3}{2} \sigma$, where $0 < \sigma \leq 2$ controls the long–ranginess of the interactions, an analysis of the free energy and the Casimir forces is given.

1 Introduction and brief overview of the problem

According to the present understanding the Casimir effect is a phenomenon common to all systems characterized by fluctuating quantities on which some external boundary conditions are imposed [1,2]. Casimir forces arise from the influence of one portion of the system on the fluctuations that occur in a portion some distance away.

The Casimir force in statistical–mechanical systems is usually characterized by the excess free energy due to the finite–size contributions to the free energy

\textsuperscript{1} Published in: Journal of Theoretical and Applied Mechanics, Sofia (1998) 78–87
of the system. The simplest and the most often case of interest is that one of a film geometry.

A simple $O(n)$ symmetric $n \geq 1$ system with a geometry $L \times \infty^2$ (and under given boundary conditions $\tau$ imposed across the direction $L$) is a standard statistical mechanical model for describing a fluid, or a magnet, confined between two parallel plates of infinite area. One important quantity which arises naturally in the thermodynamics of these confined systems is

$$F_{\tau}(T, L) = -\frac{\partial f_{\tau}^e(T, L)}{\partial L},$$

where $f_{\tau}^e(T, L)$ is the excess free energy

$$f_{\tau}^e(T, L) = f_{\tau}(T, L) - Lf_{\text{bulk}}(T)$$

Here $f_{\tau}(T, L)$ is the full free energy per unit area (and per $k_B T$) of such a system and $f_{\text{bulk}}(T)$ is the bulk free energy density.

In the case of a fluid (then one actually has to consider the excess grand potential per unit area and the derivative is performed at constant chemical potential $\mu$ and temperature $T$) $F_{\tau}(T, L)$ is termed the solvation force \[3,4\] $F_{\text{solvation}}$, (sometimes called also the disjoining pressure) whereas in the case of a magnet one speaks instead about the Casimir force \[1,5\] $F_{\text{Casimir}}$ (where the derivative is performed at constant temperature and magnetic field $H$). In the remainder we will use only the term Casimir force.

According to the definition given by Eq. (1) the Casimir force is a generalized force conjugated to the distance between the surfaces bounding the system with the general property $F_{\text{Casimir}}(T, L) \rightarrow 0$, when $L \rightarrow \infty$. We will be interested in the behaviour of $F_{\text{Casimir}}$ when $L \gg 1$, which is a condition for the applicability of the finite–size scaling theory. In general the sign of the Casimir force is of particular interest. It is believed that if the boundary conditions $\tau$ are the same at the both surfaces, $F_{\text{Casimir}}$ will be negative (see, e.g., \[4,6\]; strictly speaking, for an Ising–like system this should happen above the wetting transition temperature $T_W$ \[4,6,7\]). For boundary conditions that are not identical at both confining the system surface planes the Casimir force is expected to be positive \[4,6\]. For a fluid confined between walls this implies that the net force between the plates for large separations will be attractive in the former and repulsive in the last case.

In general, the full free energy of a $d$–dimensional critical system in the form of a film with thickness $L$, area $A$, and boundary conditions $a$ and $b$ on the two surfaces, at the bulk critical point $T_c$, has the asymptotic form

$$f_{a,b}(T_c, L) \cong Lf_{\text{bulk}}(T_c) + f_a^{\text{surface}}(T_c) + f_b^{\text{surface}}(T_c) + L^{-(d-1)}\Delta_{a,b} + \cdots$$
in the limits $A \to \infty$, $L \gg 1$. Here $f^{(s)}_{\text{surface}}$ is the surface free energy contribution and $\Delta_{a,b}$ is the amplitude of the Casimir interaction. The $L$ dependence of the Casimir term (the last one in Eq. (3)) follows from the scale invariance of the free energy and has been derived by Fisher and de Gennes [8]. The amplitude $\Delta_{a,b}$ is universal, depending on the bulk universality class and the universality classes of the boundary conditions [1,5].

Equation (3) is valid for both fluid and magnetic systems at criticality. Prominent examples are, e.g., one–component fluid at the liquid–vapour critical point, the binary fluid at the consolute point, and liquid $^4$He at the $\lambda$ transition point [1]. The boundaries influence the system to a depth given by the bulk correlation length $\xi_\infty(T,\cdots)$ (here $\cdots$ stays for the dependence on other essential parameters, e.g. an external magnetic field $H$). In the vicinity of the bulk critical point $\xi_\infty(T) \sim |T - T_c|^{-\nu}$, where $\nu$ is its critical exponent. When $\xi_\infty(T) \ll L$ the Casimir force, as a fluctuation induced force between the plates, is negligible. The force becomes long–ranged when $\xi_\infty(T)$ diverges which takes place near and below the bulk critical point $T_c$ in an $O(n)$, $n \geq 2$ model system in the absence of an external magnetic field [9]. Therefore in the statistical–mechanical systems one can turn on and off the Casimir effect merely by changing, e.g., the temperature of the system.

The effect is of particular experimental interest in studies of wetting of a wall by binary liquid mixtures close to their critical end point (see, e.g., [10]). Whereas the free energy of magnetic films cannot be measured directly, the free energy of liquid films is of relevance for surface tension measurements and wetting phenomena. In wetting phenomena thin films are formed such that at least one confining boundary is not rigid but determined by thermal equilibrium [7,10]. The Casimir force enters into the force balance and thus forms the equilibrium thickness of the wetting films, which is accurately measurable.

The temperature dependence of the Casimir force for two–dimensional systems is investigated exactly only on the example of Ising strips [4]. The upper critical temperature dependence of the force in $O(n)$ models has been considered in [5]. The only example where the force is investigated exactly as a function of both the temperature and the magnetic field scaling variables is that of the three–dimensional spherical model under periodic boundary conditions [9]. There exact results for the Casimir force between two walls with a finite separation in an $L \times \infty^2$ mean–spherical model has been derived. The force is consistent with an attraction of the plates confining the system. The most results available at the moment are for the Casimir amplitudes. For $d = 2$ by using conformal–invariance methods they are exactly known for a large class of models [1]. In addition to the flat geometries recently some results about the Casimir amplitudes between spherical particles in a critical fluid have been derived too [11]. For $d \neq 2$ results are available via field-theoretical renormalization group theory in $4 - \varepsilon$ dimensions [1,5,11], Migdal–Kadanoff real–space
renormalization group methods [12], and, relatively recently, by Monte Carlo methods [13].

In addition to the statistical mechanics the Casimir forces are object of investigations also in the quantum electrodynamics. Nowadays the effect is also presented in studies of topics like nonplanar geometries of conducting or dielectric macroscopic bodies immersed in fluids, the bag model for the description of quark confinement in hadrons according to the quantum chromodynamics, Casimir effect in general curved space-time in cosmology, models of the early Universe, etc. For a review on the Casimir effect in the aforementioned fields one can consult, e.g., [2]. A relatively recent review on the Casimir effect in statistical–mechanical systems can be found in [1].

In recent years there has been a renewed interest [14,15] in the theory of zero-temperature quantum phase transitions. Distinctively from temperature driven critical phenomena, these phase transitions occur at zero temperature as a function of some non–thermal control parameter (or a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature.

It is well known from the theory of critical phenomena that for the temperature driven phase transitions quantum effects are unimportant near critical points with \( T_c > 0 \). It could be expected, however, that at rather low (as compared to characteristic excitations in the system) temperatures, the leading \( T \) dependence of all observables is specified by the properties of the zero–temperature critical points, which take place in quantum systems. The dimensional crossover rule asserts that the critical singularities of such a quantum system at \( T = 0 \) with dimensionality \( d \) are formally equivalent to those of a classical system with dimensionality \( d+z \) (\( z \) is the dynamical critical exponent) and critical temperature \( T_c > 0 \). This makes it possible to investigate low–temperature effects (considering an effective system with \( d \) infinite space and \( z \) finite time dimensions) in the framework of the theory of finite–size scaling (FSS). The idea of this theory has been applied to explore the low–temperature regime in quantum systems [14–16], when the properties of the thermodynamic observables in the finite–temperature quantum critical region have been the main focus of interest. The most famous model for discussing these properties is the quantum nonlinear \( O(n) \) sigma model (QNL\( \sigma \)M) [14,16].

Let us note that an increasing interest related with the spherical approximation (or large \( n \)–limit) generating tractable models in quantum critical phenomena has been observed in the last few years [17–22]. There are different possible ways of quantization of the spherical constraint. In general they lead to different universality classes at the quantum critical point [17–20].

In this paper a theory of the scaling properties of the free energy and the
Casimir forces of a quantum spherical model [17] with nearest–neighbour and some special cases of long–range interactions (decreasing at long distances \( r \) as \( 1/r^{d+\sigma} \)) is presented. Only the film geometry \( L \times \infty \times L \times \infty \) (where \( L \sim \hbar/(k_BT) \) is the finite–size in the imaginary time direction) will be considered. The plan of the paper is as follows: we start with a brief review of the model and the basic equations for the free energy and the quantum spherical field in the case of periodic boundary conditions (Section 2). Since we would like to exploit the ideas of the FSS theory, the bulk system in the low–temperature region is considered like an effective \((d+z)\) dimensional classical system with \( z \) finite (temporal) dimensions. This is done to make possible a comparison with other results based on the spherical type approximation, e.g., in the framework of the spherical model and the QNL\(\sigma\)M in the limit \( n \rightarrow \infty \). The scaling forms for the excess free energy, the spherical field equation and the Casimir force are derived for a \( \frac{1}{2} \sigma < d < \frac{3}{2} \sigma \) dimensional system with a film geometry in Section 3. In Section 4 we present some results for the Casimir amplitudes in the case of short–range interactions and in some special cases of long–range interactions. The paper closes with concluding remarks given in Section 5.

2 The model

The model we will consider here describes a magnetic ordering due to the interaction of quantum spins. It is characterized by the Hamiltonian [17]

\[
\mathcal{H} = \frac{1}{2} g \sum_{\ell} P_\ell^2 - \frac{1}{2} \sum_{\ell \ell'} J_{\ell \ell'} S_\ell S_{\ell'} + \frac{1}{2} \mu \sum_{\ell} S_\ell^2 - H \sum_{\ell} S_\ell,
\]

where \( S_\ell \) are spin operators at site \( \ell \). The operators \( P_\ell \) play the role of “conjugated” momenta (i.e., \([S_\ell, S_{\ell'}] = 0\), \([P_\ell, P_{\ell'}] = 0\), and \([P_\ell, S_{\ell'}] = i\delta_{\ell \ell'}\), with \( \hbar = 1 \)). The coupling constant \( g \) measures the strength of the quantum fluctuations (below it will be called quantum parameter), \( H \) is an ordering magnetic field, and the spherical field \( \mu \) is introduced so as to ensure the constraint

\[
\sum_{\ell} \langle S_\ell^2 \rangle = N.
\]

Here \( N \) is the total number of the quantum spins located at sites “\( \ell \)” of a finite hypercubical lattice \( \Lambda \) of size \( L_1 \times L_2 \times \cdots \times L_d = N \) and \( \langle \cdots \rangle \) denotes the standard thermodynamic average taken with the Hamiltonian \( \mathcal{H} \). In (4) the coupling constants \( J_{\ell \ell'} \) are decreasing at large distances \(|\ell - \ell'|\) as \( 1/|\ell - \ell'|^{d+\sigma} \), where \( \sigma \) determines the range of the interaction: i) \( 0 < \sigma < 2 \) for long–range interaction and ii) \( \sigma \geq 2 \) for short–range interaction.

The free energy of the model in a finite region \( \Lambda \) under periodic boundary
conditions applied across the finite dimensions has the form [22]

\[ \beta f_A (\beta, g, H) = \sup_\mu \left\{ \frac{1}{N} \sum_q \ln \left[ 2 \sinh \left( \frac{1}{2} \beta \omega (q; \mu) \right) \right] - \frac{\mu}{2} \beta - \frac{\beta g J}{2 \omega^2 (0; \mu)} H^2 \right\}. \]  

Here the vector \( q \) is a collective symbol, which for \( L_j \) odd integers has the components \( \left\{ \frac{2\pi n_1}{L_1}, \ldots, \frac{2\pi n_d}{L_d} \right\} \), \( n_j \in \left\{ \frac{-L_j-1}{2}, \ldots, \frac{L_j-1}{2} \right\} \), and \( \beta \) is the inverse temperature with the Boltzman constant \( k_B = 1 \). In (6) the spectrum is \( \omega^2 (q; \mu) = g (\mu + U(q)) \) with \( U(q) = 2J \sum_{i=1}^d (1 - \cos q_i) \) for nearest neighbour interactions and will be taken of the form \( U(q) \approx J \rho \sigma |q|^\sigma \), \( 0 < \sigma < 2 \), for long range interactions (\( \rho \sigma > 0 \) is a parameter to be taken equal to one in the remainder). In the above expressions \( U(q) \) is the Fourier transform of the interaction matrix where the energy scale has been fixed so that \( U(0) = 0 \).

The supremum in Eq. (6) is attained at the solutions of the mean-spherical constraint, Eq. (5), that reads

\[ 1 = \frac{t}{N} \sum_{m=-\infty}^{\infty} \sum_{q} \frac{1}{\phi + U(q) / J + b^2 m^2 + \frac{h^2}{\phi^2}}, \]  

where we have introduced the notations: \( b = (2\pi t) / \lambda \), \( \lambda = \sqrt{g / J} \) is the normalized quantum parameter, \( t = T / J \) - the normalized temperature, \( h = H / \sqrt{J} \) - the normalized magnetic field, and \( \phi = \mu / J \) is the scaled spherical field. Eqs. (6) and (7) provide the basis of the study of the critical behaviour of the model under consideration.

In the thermodynamic limit it has been shown [17] that for \( d > \sigma \) the long-range order exists at finite temperatures up to a given critical temperature \( t_c (\lambda) \). Here we shall consider the low-temperature region for \( \frac{1}{2} \sigma < d < \frac{3}{2} \sigma. \) We remind that \( \frac{1}{2} \sigma \) and \( \frac{3}{2} \sigma \) are the lower and the upper critical dimensions, respectively, for the quantum critical point of the considered system.

3 Scaling form of the excess free energy and the Casimir force at low temperatures

For a system with a film geometry \( L \times \infty^{d-1} \times L_\tau \) (where \( \frac{1}{2} \sigma < d < \frac{3}{2} \sigma \)), after taking the limits \( L_2 \to \infty, \ldots, L_d \to \infty \) in Eq. (6) and by using the Poisson summation formula for the only remaining finite space dimensionality \( L_1 = L \), we receive the following expression for the full free energy density
\begin{equation}
(8) \ f(t, \lambda, h; L) = -\frac{\lambda}{4\sigma/\sqrt{\pi}} \Gamma \left( \frac{d}{\sigma} \right) \Gamma \left( -\frac{d}{\sigma} - \frac{1}{2} \right) \phi^{\frac{d}{2} + \frac{1}{2}} + \left( \frac{\lambda}{\lambda_c} - 1 \right) \phi - \frac{h^2}{2\phi}
\end{equation}

\begin{equation}
-\frac{\lambda}{\sigma/\sqrt{\pi}} \Gamma \left( \frac{d}{\sigma} \right) \phi^{\frac{d}{2} + \frac{1}{2}} \sum_{m=1}^{\infty} K_{\frac{d}{2} + \frac{1}{2}} \left( m \frac{\lambda}{2t} \phi^{\frac{1}{2}} \right) \left( m \frac{\lambda}{2t} \phi^{\frac{1}{2}} \right)^{-(\frac{d}{2} + \frac{1}{2})}
\end{equation}

\begin{equation}
+2 \frac{tL^{-\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}} \sum_{m=1}^{\infty} \int_{x_D}^{\infty} dx \, x^{\frac{d}{2}} J_{\frac{d}{2}-1}(mLx) \ln \left[ 2 \sinh \left( \frac{\lambda}{2t} \sqrt{\phi + x^\sigma} \right) \right],
\end{equation}

where \( k_d^{-1} = \frac{1}{2}(4\pi)^{\frac{d}{2}} \Gamma(d/2) \), \( x_D \) is the radius of the sphericalized Brillouin zone, \( K_{\nu}(x) \) and \( J_{\nu}(x) \) are the MacDonald and Bessel functions, respectively, and the critical value of \( \lambda = \lambda_c \) is

\begin{equation}
(9) \ \lambda_c^{-1} = \frac{1}{2}(2\pi)^{-d} \int d^d q (U(q)/J)^{-\frac{1}{2}}.
\end{equation}

In Eq. (8) \( \phi \) is the solution of the corresponding spherical field equation that follows by requiring the partial derivative of the r.h.s. of Eq. (8) with respect to \( \phi \) to be zero. The bulk free energy \( f_{\text{bulk}}(t, \lambda, h) \) results from \( f(t, \lambda, h; L) \) by merely taking the limit \( L \to \infty \) in it. Let us denote the solution of the corresponding bulk spherical field equation by \( \phi_{\infty} \). Then for the excess free energy it is possible to obtain the finite size scaling form

\begin{equation}
(10) \ f^{ex}(t, \lambda, h; L)/L = \lambda L^{-(d+z)} X(x_1, x_2, a),
\end{equation}

with scaling variables \( x_1 = L^{-1/\nu} \left( 1/\lambda - 1/\lambda_c \right) \), \( x_2 = hL^{\Delta/\nu} \) and \( a = tL^{z/\lambda} \). Here \( \nu^{-1} = d - \frac{1}{2}\sigma \), \( \Delta/\nu = \frac{1}{2} \left( d + \frac{3}{2}\sigma \right) \) and \( z = \frac{1}{2}\sigma \) are the critical exponents of the model [17]. In Eq. (10) the universal scaling function of the excess free energy may obtained in an explicit form (this will be presented in a subsequent article).

For the Casimir forces in the considered system

\begin{equation}
(11) \ F_{\text{Casimir}}(T, \lambda, h; L) = \lambda L^{-(d+z)} X_{\text{Casimir}}(x_1, x_2, a),
\end{equation}

where the universal scaling functions of the Casimir force \( X_{\text{Casimir}}(x_1, x_2, a) \) is related to that one of the excess free energy \( X \equiv X(x_1, x_2, a) \) by

\begin{equation}
(12) \ X_{\text{Casimir}}(x_1, x_2, a) = -(d+z)X - \frac{1}{\nu} x_1 \frac{\partial X}{\partial x_1} + \frac{\Delta}{\nu} x_2 \frac{\partial X}{\partial x_2} + za \frac{\partial X}{\partial a}.
\end{equation}

4 Casimir amplitudes

In this section we determine the Casimir amplitudes of the model for the case of short range interactions at \( d = 2 \) and for some special cases of long range interactions.
4.1 short range interactions \((d = 2)\)

It can be shown that the solution \(y_0\) of the spherical field equation for the finite system with a film geometry \(L \times \infty \times L_\tau\) at zero temperature (i.e. \(1 \ll L \ll \infty, L_\tau = \infty\)) is \(y_0 = \ln(\sqrt{5}/2 + 1/2)\) at the quantum critical point \(\lambda = \lambda_c, h = 0\) (at this point \(y_\infty = 0\)) [21,22]. Setting this value of \(y_0\) in the scaling function (10) of the excess free energy, taking into account that 
\[
K_2 = \sqrt{\pi/(2x)} \exp(-x)(1 + 1/x),
\]
the identity 
\[
\ln \left[2\sinh \left(\sqrt{y}/2\right)\right] = \sqrt{y}/2 - \text{Li}_1 \left[\exp(-\sqrt{y})\right]
\]
and the properties of the polylogarithm functions \(\text{Li}_p(x)\) [23], we obtain from Eqs. (3) and (10) that the Casimir amplitude is
\[
\Delta_{\text{periodic b.c.}} = -\frac{2\zeta(3)}{5\pi} \approx -0.153051.
\]
Here \(\zeta(3)\) is the Riemann zeta function.

4.2 long range interactions \((d = \sigma)\)

Let us consider the “temporal Casimir amplitude” in a system with a geometry \(\infty^d \times L_\tau\) with \(d/\sigma = 1\) at the quantum critical point \(\lambda = \lambda_c, h = 0\). In a way, similar to that one explained for the case considered previously one obtains \((0 < \sigma \leq 2)\)
\[
f(t, \lambda_c, 0; \infty) - f(0, \lambda_c, 0; \infty) = -\lambda_c \frac{16}{5\sigma} \frac{\zeta(3)}{(4\pi)^{\sigma/2}} \frac{t^3}{\Gamma(\sigma/2)}.
\]
From here one can identify the “temporal Casimir amplitude” to be
\[
\Delta_t(\sigma) = -\frac{16}{5\sigma} \frac{\zeta(3)}{(4\pi)^{\sigma/2}} \frac{1}{\Gamma(\sigma/2)}.
\]

5 Concluding remarks

In the present article the free energy of a system with a geometry \(L \times \infty^{d-1} \times L_\tau\) (where \(\frac{1}{2} \sigma < d < \frac{3}{2} \sigma\)), is derived (see Eq. (8)). For \(\sigma = 2\) this new result reduces to the one reported in [22] where only the case of short–range interactions has been considered. A general expression for the Casimir force in the quantum spherical model is obtained (see Eqs. (11,12). In the classical limit \((\lambda = 0)\) for a system with short–range interaction it coincides with the corresponding one derived in [9] for the classical spherical model. Except for some two–dimensional systems the only model within which the Casimir effect is
investigated in an exact manner for \( d \neq 2 \) is that one of the spherical model. The investigations presented here complement the aforementioned ones for the case when quantum fluctuations are present in the system. In order to derive in a simple closed form the Casimir amplitudes some particular cases have been considered \((d = \sigma)\). For \( d = 2 \) this amplitude is given in Eq. (13). This amplitude is equal to the "temporal Casimir amplitude" (i.e. the corresponding temperature corrections in an \( \infty^2 \times L_\tau \) system to the ground state of the bulk system) for the \( \mathcal{O}(n) \) sigma model in the limit \( n \to \infty \) [23]. We have demonstrated here explicitly that the two models, due to the fact that they belong to the same universality class, indeed possess equal Casimir amplitudes as it is to be expected on the basis of the hypothesis of the universality. We note that instead of considering a finite system with a film geometry at the zero temperature one can consider a bulk system at low temperatures. As it is already clear from above this leads to the same result for the Casimir amplitudes. Note, that in accordance with the general expectations these amplitudes are negative. The correction to the ground state energy of the bulk system due to the nonzero temperature is determined for the general case \( d = \sigma \) by Eq. (15). Note, that the defined there "temporal Casimir amplitude" \( \Delta_t(\sigma) \) reduces for \( \sigma = 2 \) to the "normal" Casimir amplitude, given by Eq. (13). This reflects the existence of a special symmetry for that case between the "temporal" and the space dimensionalities of the system.

Acknowledgements

This work is supported by The Bulgarian Science Foundation (Projects F608/96 and MM603/96).

References

[1] Krech M. *The Casimir Effect in Critical Systems*, World Scientific, Singapore, 1994.

[2] Mostepanenko, V. M., N. N. Trunov. *The Casimir effect and its applications*, Moscow, Energoatomizdat, 1990, in russian; English version: New York, Clarendon Press, 1997.

[3] Evans, R. In: *Liquids at interfaces*, Les Houches Session XLVIII (Eds. J. Charvolin, J. Joanny and J. Zinn-Justin), Amsterdam, Elsevier, 1990.

[4] Evans R., J. Stecki. *Solvation force in two-dimensional Ising strips*. Phys. Rev. B, 49 (1994), 8842.
[5] Krech M., S. Dietrich. *Free energy and specific heat of critical films and surfaces.* Phys. Rev. A, 46 (1992), 1886.

[6] Parry, A. O., R. Evans. *Novel phase behaviour of a confined fluid or Ising magnet.* Physica A, 181 (1992), 250.

[7] Dietrich, S. In: *Phase Transitions and Critical Phenomena*, Vol. 12 (Eds. Domb C., J. L. Lebowitz) London, Academic Press, 1988.

[8] Fisher, M. E., P. G. de Gennes. *Phénomènes aux parois dans un mélange binaire critique.* C. R. Acad. Sci. Paris B 287 (1978) 207.

[9] Danchev, D. M. *Finite–size scaling Casimir force function: Exact spherical model results.* Phys. Rev. E, 53 (1996), 2104.

[10] Krech, M., S. Dietrich. *The specific heat of critical films, the Casimir force and wetting films near critical end points.* Phys. Rev. A 46 (1992), 1922.

[11] Eisenriegler, E., U. Ritschel. *Casimir forces between spherical particles in a critical fluid and conformal invariance.* Phys. Rev. B 51 (1995), 13 717.

[12] Indekeu, J. O., M. P. Nightingale, W. V. Wang. *Finite–size interaction amplitudes and their universality: Exact, mean–field and renormalization–group results.* Phys. Rev. B 34 (1986), 330.

[13] Krech, M., D. P. Landau. *Casimir effect in critical systems: A Monte Carlo simulation.* Phys. Rev. E 53 (1996), 4414.

[14] Sachdev, S., In: *Proceedings of the 19th IUPAP Int. Conf. Of Stat. Phys.* (Ed. B. L. Hao) Singapore, World Scientific, 1996.

[15] Sondhi, S. L., S. M. Girvin, J. P. Carini, D. Shahar, *Continuous quantum phase transitions.* Rev. Mod. Phys. 69 (1997) 315.

[16] Chakravarty S., B. I. Halperin, D. R. Nelson. *Two–dimensional quantum Heisenberg antiferromagnet at low temperatures.* Phys. Rev. B 39 (1989), 2344.

[17] Vojta T. *Quantum version of a spherical model: Crossover from quantum to classical critical behaviour.* Phys. Rev. B 53 (1996), 710.

[18] Tu Y., P. B. Weichman, *Quantum spherical models for dirty phase transitions.* Phys. Rev. Lett. 73 (1994), 6.

[19] Nieuwenhuizen, T. W. *Quantum description of spherical spins.* Phys. Rev. Lett. 74 (1995), 4289.

[20] Nieuwenhuizen, T. W., F. Ritort. *Quantum phase transition in spin glasses with multi–spin interactions.* [cond-mat/9706244](https://arxiv.org/abs/cond-mat/9706244).

[21] Chamati H., E. S. Pisanova, N. S. Tonchev. *Theory of a spherical quantum rotors model: low–temperature regime and finite–size scaling.* [cond-mat/9701159](https://arxiv.org/abs/cond-mat/9701159).

[22] Chamati H., D. M. Danchev, E. S. Pisanova, N. S. Tonchev. *Low–temperature regimes and finite–size scaling in a quantum spherical model.* Preprint No. IC/97/82, Trieste, Italy, July 1997. [cond-mat/9707280](https://arxiv.org/abs/cond-mat/9707280).
[23] Sachdev S. *Polylogarithm identities in a conformal field theory in three dimensions.* Phys. Lett. B 309 (1993), 285.