Local Mixed Hodge Structure on Brill-Noether Stacks

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Abstract

On a smooth algebraic curve $X$ with genus greater than 1 we consider a flat principal bundle with a reductive structure group $S$ and a vector bundle associated with it. To this set of information we put in correspondence a pro-algebraic group on whose functional algebra we introduce a mixed Hodge structure. This construction, in fact, works for any smooth algebraic variety $X$ which, considered as an analytic space, has a nonabelian first homotopy group, and the rest are trivial. The Hodge structure defined in this way can be expressed in terms of iterated integrals.

Furthermore, considered in the context of previous work by C. Simpson, this MHS is the local mixed Hodge structure on a nonabelian cohomological space on $X$ with coefficients into a Brill-Noether stack, i.e., a stack with two non-trivial homotopy groups: a fundamental group isomorphic to the group $S$ and an $n$-th homotopy group represented by a vector space, the fiber of the vector bundle discussed above above.

My construction is compatible and generalizes the work of R. Hain on Hodge structure on relative Malcev completion of the fundamental group of $X$.

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1 Introduction

Let $X$ be a smooth complex algebraic curve. We study the geometry of $X$ through introducing a Hodge structure on the nonabelian cohomology space $\text{Hom}(X, T)$, where the coefficient space, $T$, is an algebraic stack. To simplify the problem, one can restrict $T$ to be a very presentable geometric $n$-stack, which, in the category of topological spaces, is analogous to considering a homomorphism with to an $n$-truncated CW-complex. The case of $X$ being a curve of positive genus is additionally simplified by the fact that $X$ is a topologically $K(\pi_1, 1)$-space.

This problem was studied before by Simpson, in e.g. [17, 18], where he defined a Hodge structure on the stack $\text{Hom}(X, \kappa(G, 1))$, which parametrizes $G$-local systems on $X$ up to homotopy equivalence. Simpson proves that each $\mathbb{C}$-valued point corresponds to a Higgs bundle with a holomorphic connection $(E, \theta)$ and, on the other hand, to a representation $\sigma: \pi_1(X, x) \to G$. The ring of functions $\mathcal{O}(E)$ reflects the local geometry of $\text{Hom}(X, k(G, 1))$ at the point $(E, \theta)$ and Simpson proves, that his Hodge structure restricts to a mixed Hodge structure on $\mathcal{O}(E)$, which, under some additional restrictions, coincides with the MHS defined by Hain [12].

In this text we suggest a MHS on $\text{Hom}(X, T)$, where $T$ is a Brill-Noether stack, i.e., a stack which has only 2 nontrivial homotopy groups: $\pi_1 = G$ and $\pi_n = V$ and $\pi_1$ acts on $\pi_n$ by a chosen representation $\rho: G \to GL(V)$. For that, to a $\mathbb{C}$-valued point of $\text{Hom}(X, T)$, which parametrizes a triple $(E, \theta, \eta)$ of a flat $G$-bundle, holomorphic connection on $E$ end a cohomology class $\eta \in H^n(X, E \times^G V)$, we define a MHS on a proalgebraic group $\mathcal{G}_{\rho}$ and we define an action of $G$ on $\mathcal{G}_{\rho}$, which reflects the Whitehead product on the homotopy type of $T$. The resultant MHS on $\text{Hom}(X, T)$ is comprise by 2 ingredients — Simpson’s nonabelian MHS on $\text{Hom}(X, \kappa(G, 1))$ an the usual abelian Hodge theory on the cohomology group $H^n(X, E \times^G V)$.

In section 2 we provide some background information about the theory of stacks.

Section 3 is dedicated to proving the following main theorems:

**Theorem 1.1** There is an isomorphism $\mathcal{O}(\mathcal{G}_{\rho}) \cong H^0(I^\bullet(X, V))$ between the ring of functions on $\mathcal{G}_{\rho}$ and the 0-th cohomology of a the d.g.a. of iterated integrals with coefficients in $V$. 
Theorem 1.2 Let the structure representation of $\kappa(S, \rho, n)$, $\rho : S \to GL(V)$ be of Hodge type. Denote by $\hat{\mathfrak{g}}_\rho$ the Lie algebra of the group $\mathcal{G}_\rho$. Then there is a MHS on the Lie algebra $\hat{\mathfrak{g}}_\rho$, such that the weight filtration is provided as a limit of the natural filtrations of each $\mathfrak{g}_V$ coming from the lower central series and Hodge filtration is induced from external differentiation.

Theorem 1.3 Let $\hat{\mathfrak{g}}$ be the Lie algebra of the group $\mathcal{G}$. There is a natural action of $\hat{\mathfrak{g}}$ on $\hat{\mathfrak{g}}_\rho$, which is given by a morphism of mixed Hodge complexes.

We provide a different approach to Hain’s mixed Hodge structure on $\hat{\mathfrak{g}}$ and, especially, we define in a more geometric way the weight filtration, which was originally introduced in [12] using an implementation of Saito’s work [16].

2 The Brill-Noether stack

In this paper we denote by $\mathbf{T}$ the category of topological spaces. For us a $n$-stacks on the site $\mathbf{T}$ we will mean $n$-truncated objects in the simplicially enriched category $St(\mathbf{T}) = LSPr(\mathbf{T})$, which is obtained in [21] by taking the category of simplicial presheaves on $\mathbf{T}$, applying simplicial localization with respect to the local equivalences and introducing simplicial structure on the arrows. In the language used by to [17], these are very presentable geometric $n$-stacks on $(\text{Aff})_{\text{et}}$, via a theorem of GAGA type. This means, that $\pi_1(T, t)$ is represented by an algebraic group-scheme of finite type, and $\pi_j(T, t)$ is represented by a vector space for $j > 1$.

If $G$ is a group-sheaf, represented by an algebraic scheme $G$, a simple example of a very presentable stack, one can define $\kappa(G, n)$, so that $\pi_j(\kappa(G, n))$ is trivial except $\pi_j = G$. ($G$ must be abelian if $n > 1$.)

The stacks $\kappa(G, n)$ are classifying for the group $G$ in the sense of the following proposition:

Proposition 2.1 (Simpson, [18]) For each $X \in \mathbf{T}$, $\text{Hom}(X, \kappa(G, n))$ is a stack, such that

$$\pi_j(\text{Hom}(X, \kappa(G, n))) = H^{n-j}(X, G).$$

Very presentable geometric stacks can be built out of $\kappa(G, n)$-s like Postnikov towers. The simplest example is the Brill-Noether stack, denoted $\kappa(G, \rho, n)$, whose $\pi_1 = G$, $\pi_n = V$, the representation $\rho : G \to GL(V)$ gives
the action of $\pi_1$ on $\pi_n$ and the rest of the homotopy groups being trivial. $\kappa(G, \rho, n)$ is a fibration over $\kappa(G, 1)$:

$$
\begin{array}{ccc}
\kappa(V, n) & \xrightarrow{i} & \kappa(G, \rho, n) \\
& \searrow & \downarrow \tau \\
& & \kappa(G, 1)
\end{array}
$$

Here, the map $\tau$ is truncation and $i$ is inclusion.

For each $X \in T$, $\text{Hom}(X, \kappa(G, \rho, n))$ parametrizes triples $(E, \theta, \eta)$ of a $G$-bundle $E$, flat connection $\theta$ on $E$ and a cohomology class $\eta \in H^n(X, E \times^G V)$ (cf. [17]). The diagram (1) induces a smooth fibration of geometric $n$-stacks

$$
\text{Hom}(X, \kappa(G, \rho, n)) \xrightarrow{\phi} \text{Hom}(X, \kappa(G, 1))
$$

Over each $\mathbb{C}$-point $(E, \theta)$ of $\text{Hom}(X, \kappa(G, 1))$, the fiber is $H^n(X, E \times^G V)$.

## 3 Local Mixed Hodge Structure

We assume, that $X$ is a smooth complex algebraic curve of positive genus. Whenever we talk about the homotopy type of $X$, we will mean the underlying analytic space. Most of these facts hold for any topological $K(\pi_1, 1)$ space. In the proof of our main results (section 3.2), we essentially use the fact that $\pi_1(X, x_0)$ is finitely presented and it has a trivial torsion free nilpotent residue.

Let $S$ be a reductive algebraic group defined over $\mathbb{C}$. We will denote by $S(\mathbb{R})$ the $\mathbb{R}$-valued points of $S$. We assume that $S$ is a group of Hodge type (cf. [17]) and all representations $S \rightarrow GL(V)$ which appear throughout this section are homomorphisms of groups of Hodge type.

We show, that to each $\mathbb{C}$-point of Hodge type in $\text{Hom}(X, \kappa(S, \rho, n))$ corresponds a Lie algebra $\mathfrak{g}_\rho$ with a natural mixed Hodge structure, which we call *local* (Section 3.4). We use Hain’s construction of MHS on the relative pro-unipotent completion of the fundamental group on $X$ (Hain [12]), which is the “local” MHS for the stack $\text{Hom}(X, \kappa(S, 1))$ (cf. Simpson, [19]). In particular, we show that Hain’s result extends to any complex algebraic group which is a central extension of a reductive group.
3.1 The MHS on a relative prounipotent completion of the fundamental group

We recall some results and techniques from [12] in order to apply them to our problem.

Choose a point $x_0 \in X$ and a group morphism $\sigma: \pi_1(X, x_0) \to S(\mathbb{R})$ with a Zariski dense image. Let $p: P \to X$ be the correspondent flat $S$-bundle. We assume that $S$ acts on $P$ on the left. Throughout this whole chapter $\tilde{x}_0$ will denote a chosen preimage of $x_0$ back to the fiber $P_{x_0}$ and we will assume that the coordinates on $P_{x_0} \cong \{x_0\} \times S$ are chose so that $\tilde{x}_0 = x_0 \times 1_s$.

3.1.1 The category of filtered vector bundles on $X$

Consider the category $G(X, S)$ whose objects are pairs $(G, \tau)$ of an algebraic group $G$ and a rational map $\tau: S \to G$ which defines a splitting of $G = U \rtimes S$ as a semidirect product of $S$ with the unipotent kernel $U$. We require additionally that each $(G, \tau)$ comes with a representation $\tilde{\sigma}: \pi_1(X, x_0) \to G$, and $\tilde{\sigma}$ is a preimage of $\sigma$:

\[
\begin{align*}
1 & \longrightarrow U & \xrightarrow{\phi} & G & \xrightarrow{\tau} & S & \longrightarrow 1.
\end{align*}
\]

The arrows in $G(X, S)$ are homomorphisms of groups $\phi: G_1 \to G_2$ which commute with the maps of (2).

**Definition 3.1** The relative prounipotent completion of $\pi_1(X, x_0)$ (relative to $\sigma$) is the projective limit $\mathcal{G} = \varprojlim_{G \in G(X, S)} G$.

There is an exact sequence:

$$1 \to U \to G \to S \to 1,$$

where $U$ is the unipotent radical of $\mathcal{G}$.

We will denote the Lie algebras of the groups $G, U,$ etc., by $\mathfrak{g}, \mathfrak{u},$ etc. and for the Lie algebras of the pro-groups $\mathcal{G}, U,$ etc., by $\hat{\mathfrak{g}}, \hat{\mathfrak{u}},$ etc.

Next, for each $(G, \tau) \in G(X, S)$ and any linear representation $V$ of $G$ we construct a flat vector bundle $V \to X$ (an important spacial case is
$V = u = \text{Lie } U$, considered as vector space). The splitting $\tau$ defines a representation $\bar{\tau} : S \to GL(V)$ and results in a representation

$$\bar{\tau} \circ \sigma : \pi_1(X, x_0) \to \text{End}(V),$$

which gives rise to a vector bundle $\mathbb{V} \to X$ and a flat connection on $\mathbb{V}$. More precisely, we have a commutative diagram, defined by the action of $S$:

$$
\begin{array}{ccc}
P \times V & \rightarrow & P \times_S V = \mathbb{V} \\
\downarrow & & \downarrow \\
P & \rightarrow & P/S = X
\end{array}
$$

From the exact sequence of the homotopy groups of the principal bundle $P \to X$, we have:

$$\cdots \to \pi_1(S, 1) \to \pi_1(P, \tilde{x}_0) \to \pi_1(X, x_0) \to \pi_0(S, 1) \to \cdots$$

The nilpotent representation $\bar{\tau} \circ \sigma$ gives then a monodromy map of the trivial bundle $P \times V \to P$ whose differential defines a connection on $P \times V$ with with a form $\omega \in \mathcal{A}^1(P; \mathcal{O}(P)) \otimes_S u$.

Here we consider $P$ as a left $S$-module (which turns $\mathcal{O}(P)$ into a right module) and take tensor product of $S$-modules with $u = \text{Lie } U$, on which $S$ acts via adjoint action. Because $\omega$ comes from a representation of $\pi_1(X, x_0)$ it satisfies the conditions:

$$d\omega + \omega \wedge \omega = 0 \quad (3)$$

$$s^*\omega = \text{Ad}(s)\omega, \quad \text{for all } s \in S,$$

i.e., it defines a flat connection on $\mathbb{V}$.

Let $V = V^0 \supset V^1 \supset \ldots V^n$ be a filtration of vector spaces which are invariant under the restriction of $\bar{\tau} \circ \sigma$. Like $V^0$, each $G$-module $V^i$ gives rise to a flat vector bundle $\mathbb{V}^i$ with connection $\omega_i = \omega|_{\mathbb{V}^i}$.

Because of the reductiveness of $S$, there is a splitting

$$V \cong W_1 \oplus \ldots W_n, \quad (4)$$

such that each $W_i$ is an irreducible $S$-module. Let $V^i$ be the subspace of $V$ obtained by pullback of $W_i \oplus \cdots \oplus W_n$. We have $\text{Gr}^i V^\bullet \cong W_i$. 

6
**Definition 3.2** A filtration $V^\bullet$ such that $Gr^i V^\bullet$ comes from irreducible $S$-module will be called a **full** filtration (invariant with respect to the monodromy) of $\omega$.

The splitting (4) is unique up to order and, therefore, there is only one full filtration of $V^\bullet$ with respect to $\omega$ is up to conjugation with elements of $U$.

**Lemma 3.3** Let $V$ be a vector bundle associated with $p : P \to X$ which comes from a rational representation of $G = U \times S$. Let $V^\bullet$ be a full filtration of $bV$ with respect to $\omega$. Then $p^* Gr^i V^\bullet$ is the trivial flat bundle and the pullback of $\omega$ with respect to the isomorphism $p^* V \cong \oplus p^* Gr^i V^\bullet$ is the trivial connection.

**Proof.** The representation $\pi_1(P, \tilde{x}_0) \to End W_i$ is both unipotent and semisimple. Therefore it is trivial. □

Next, notice, that the unipotent group $U$ has a natural filtration coming from its lower central series:

$$U_0 = U, \quad U_i = [U, U_{i-1}], \quad i \geq 1.$$ 

Since $S$ acts by conjugation on $U$, it leaves each $U_i$ invariant and, therefore, the corresponding Lie algebras form a filtration

$$u_0 \supset u_1 \supset \ldots u_n = (0)$$

of $G$-modules.

**Lemma 3.4** The filtration $V^\bullet$ of vector bundles on $X$ produced by taking $V^i = u_i$ is full with respect to $\omega$.

**Proof.** Comes obviously from the fact that the lower central series of a nilpotent group are maximal, i.e., have length equal to the degree of nilpotency of $U$. □

On the other hand, let $B(X, S)$ be the category whose objects are pairs $(V^\bullet, \omega)$ of a full filtration $V^\bullet$ of flat vector bundles, such that the quotients are associated with irreducible rational representations of $S$ and a connection $\omega$ on $V$ which satisfies. A morphism $f : (V^\bullet, \omega) \to (W^\bullet, \zeta)$ is a sequence of morphisms of vector bundles $f^i : V^i \to W^m$ with connections, which is compatible with the filtrations.
For each $V^\bullet$, the fibers over $x_0$, $V^i = \mathbb{V}^i_{x_0}$ form a filtration of vector spaces:

$$V = V^0 \supset V^1 \supset \ldots V^n.$$  

More precisely, we require, that the monodromy using $\omega \in \mathcal{A}^1(P; \mathcal{O}(P)) \otimes_S \mathfrak{u}$ induces an irreducible rational representation $\tau_i : S \to GL(V^i/V^{i-1})$.

Each $Gr^i \mathbb{V}$ is a vector bundle associated with $P$ and, thus, they all trivialize simultaneously if pulled back to $p^*Gr^i \mathbb{V} \to P$. Let $\omega_i$ be the canonical flat connection on $p^*Gr^i \mathbb{V} \to P$. We can consider $\omega$ as the connection on $p^*V$ coming from the pullback of $\omega_1 + \cdots + \omega_n$ via the isomorphism $\psi : V \cong \oplus Gr^i \mathbb{V}$.

$\omega \in \mathcal{A}^1(P; \mathcal{O}(P)) \otimes_S \mathfrak{u}$ be a connection on $\mathbb{V}$, such that the monodromy using $\omega$ induces a rational representation $\tau_i : S \to GL(V^i/V^{i-1})$. Because of the reductiveness of $S$, there as an isomorphism $\mathbb{V} \cong \oplus Gr^i \mathbb{V}$. Each $Gr^i \mathbb{V}$ is a vector bundle associated with $P$ and, thus, they all trivialize simultaneously if pulled back to $p^*Gr^i \mathbb{V} \to P$.

Furthermore, let $\tau : S \to \Pi GL(Gr^i V)$ be the product of all $\tau_i$. Each $\mathbb{V}^\bullet$ has a monodromy representation $\tilde{\sigma} : \pi_1(X, x_0) \to G$, where

$$G = \{ \phi \in GL(V) | \phi \text{ preserves } V^\bullet \text{ and } Gr^\bullet \phi \in \text{im } \tau \}. \quad (5)$$

Denote by

$$U = \{ \phi \in GL(V) | \phi \text{ preserves } V^\bullet \text{ and acts trivially on } Gr^\bullet V \}.$$  

$$1 \to U \to G \to \tau(S) \to 1,$$

If $V$ is not a faithful $S$ module, we still have an split exact sequence

$$1 \to U \to G \to S \to 1,$$

by taking $G = U \rtimes S$.

Observe that:

1. By (5) we constructed a functor $A : B(X, S) \to G(X, S)$;
2. Lemma 3.4 gives us a functor $B : G(X, S) \to B(X, S)$.
3. Both $A$ and $B$ are fully faithful embeddings.
Proposition 3.5 The functors $A$ and $B$ are adjoint to each other. In fact, there is a bijection:

$$\text{Hom}_{G(X,S)}((G,\tau),A(V^\bullet,\omega)) \cong \text{Hom}_{B(X,S)}(B(G,\tau),(V^\bullet,\omega)).$$

The following lemma will be useful for the proof of Proposition 3.5.

Lemma 3.6 Let $(G,\tau) \in G(X,S)$, and let $(u^\bullet,\tilde{\omega}) = B(G,\tau)$. Let $(W^\bullet,\omega) \in B(X,S)$ be a result of some monodromy representation of $(G,\tau)$, $G \to \text{End}(W)$. Then there exists an isomorphism $W \otimes_S u^\bullet \cong W^\bullet$ in $B(X,S)$, which is given by conjugation by an element from $U$. Moreover, there is a canonical map $h_W : u^\bullet \to W^\bullet$.

Proof. First, we construct a map $u \to W$. By construction, $W \cong W_1 \oplus \cdots \oplus W_r$, where $W_i$ are irreducible $S$-modules. Each the image of $U$ in $\text{End}W_i$ has an eigenvector $w_i$, which is unique up to scalar multiplication. Let $w = w_1 \oplus \cdots \oplus w_r \in W$. We construct the map $u \to W$ by taking the composition $u \xrightarrow{\exp} U \xrightarrow{\text{ev}} W$, where $\exp$ is the exponent map and $\text{ev}_w : f \mapsto f(w)$ is the evaluation map. Then, we extend the map $u \to W$ to $h_W : u^\bullet \to W^\bullet$, which is possible, because of the compatibility between the monodromy of $\omega$ and $\tilde{\omega}$.

Finally, by lemma 3.4, $u^\bullet$ is a full filtration and, therefore, $W \otimes_S u^\bullet$ is a full filtration as well. The full filtration of $W$ is unique up to a choice of the order of the irreducible factors, which can be changed using conjugation with a matrix from $U$. □

Proof (of 3.5). Let $h \in \text{Hom}_{G(X,S)}((G,\tau),A(V^\bullet,\omega))$. This means that there is a commutative diagram:

$$
\begin{array}{c}
1 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 1 \\
\downarrow l \hspace{1cm} \downarrow h \hspace{1cm} \downarrow \hspace{1cm} \\
1 \longrightarrow U_V \longrightarrow G_V \longrightarrow S \longrightarrow 1
\end{array}
$$

where $U_V$ and $G_V$ are the groups constructed in [5]. The fact, that the corresponding splittings $\tau$ and $\tau_V$ are compatible with the diagram implies, that the morphism $l$ maps the lower central series of $U$ to those of $U_V$. Therefore, $l$ gives rise to a tangent morphism $\lambda : u \to u_V$. By Lemma 3.6, there is a map $h_V(u_V,\tilde{\omega}) \to (V^\bullet,\omega) \in B(X,S)$. Then, $h_V \circ \lambda$ is a map from $\text{Hom}_{B(X,S)}(B(G,\tau),(V^\bullet,\omega))$.
The other way around, let $\lambda \in \text{Hom}_{\mathcal{B}(X,S)}(B(G, \tau), (\mathbb{V}^\bullet, \omega))$, and let $\lambda_0 : \mathfrak{u} \to V$ be the map on th fiber over $x_0$. Since $\lambda$ respects the filtrations, it follows that $U$ acts on the filtered vector space $V^\bullet$. This by definition implies a map $U \to U_V$ and therefore $G \to U_V \rtimes S = A(\mathbb{V}^\bullet, \omega)$. □

The following consequences of 3.5 will be used:

**Corollary 3.7 (A)** Let $G$ be the pronilpotent completion of $\pi_1(X, x)$ relative to $\sigma : \pi_1 : (X, x_0) \to S$. Then:

$$G = \lim_{\mathbb{V}^\bullet, \omega \in \mathcal{B}(X,S)} A(\mathbb{V}^\bullet, \omega).$$

**(B)** Let $(\mathbb{V}, \omega)$ be an object of $\mathcal{B}(X,S)$. Denote by $\mathcal{B}(X,S)/\mathbb{V}$ the full subcategory of $\mathcal{B}(X,S)$ with final object $(\mathbb{V}, \omega)$ and denote by $G(X,S)/\mathbb{V}$ the full subcategory of $G(X,S)$ with final object $A(\mathbb{V}, \omega)$. Denote

$$G_{\mathbb{V}} = \lim_{(G', \tau') \in G(X,S)/\mathbb{V}} G'.$$

Then

$$G_{\mathbb{V}} = \lim_{(\mathbb{W}^\bullet, \eta) \in \mathcal{B}(X,S)/\mathbb{V}} A(\mathbb{W}^\bullet, \eta).$$

**(C)**

$$G = \lim_{(\mathbb{V}^\bullet, \omega) \in \mathcal{B}(X,S)} G_{\mathbb{V}}.$$

**Remark.** In the case when the $(\mathbb{V}, \omega)$ is coming from a rational representation $\rho : S \to \text{GL}(V)$, $V = \mathbb{V}_{x_0}$, we often write $G_{\rho}$ instead of $G_{\mathbb{V}}$.

**3.1.2 Hain’s MHS on $G$.**

**Remark.** For any semisimple group $S$, one can think of $\mathcal{O}(S)$ as an infinite dimensional $(S,S)$-module. It is known (cf. e.g. [12]), that

$$\mathcal{O}(S) = \bigoplus_{V} V^* \otimes_S V,$$

(6)

where, in the sum on the right side $V$ runs through the full collection of non isomorphic finitely dimensional left $S$-modules. Furthermore, if $P \to X$ is an $S$-bundle, then we will denote by $\mathcal{O}(P) \to X$ the flat infinite dimensional
vector bundle associated with the linear representation of $S$ into $\mathcal{O}(S)$. Then (10) implies
\[ \mathcal{O}(P) = \bigoplus_V V^* \otimes_S V, \]
representing $\mathcal{O}(P)$ as a direct limit of its finitely dimensional $S$-sublocal systems. In particular, for any finitely dimensional $S$ module $V$, one can think of $\mathcal{A}^\bullet(X, \mathcal{O}(P)) \otimes_S V$ as a subsheaf of $\mathcal{A}^\bullet(X, \mathcal{O}(P))$.

The main point of Hain’s MHS is the following theorem:

**Theorem 3.8 (Hain, [12])** There is an isomorphism of Hopf algebras:

\[ \mathcal{O}(\mathcal{G}) \cong H^0(I(X, \mathcal{O}(S))) \]

between $\mathcal{O}(\mathcal{G})$, the algebra of functions on $\mathcal{G}$ and the algebra of locally constant iterated integrals on $X$, based at $x_0$ and with coefficients in $\mathcal{O}(S)$.

This isomorphism is used to define the structure of a real Mixed Hodge complex on $\mathcal{O}(\mathcal{G})$ and, correspondingly, a dual one on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$.

There are several equivalent definitions of mixed Hodge structure. Throughout this section we will use the following one (cf. [?]).

**Definition 3.9** A real mixed Hodge structure is a set of data

\[ ((A_\mathbb{R}, W_{\mathbb{R}}^\bullet), (A_\mathbb{C}, W_{\mathbb{C}}^\bullet, F^\bullet)) \]

consisting of:

- A $\mathbb{R}$-vector space $A_\mathbb{R}$ and an increasing (weight) filtration $W_\mathbb{R}^\bullet$ on $A_\mathbb{R}$;
- A $\mathbb{C}$ vector space $A_\mathbb{C}$ and an increasing (weight) filtration $W_\mathbb{C}^\bullet$ on $A_\mathbb{C}$, such that $(A_\mathbb{R}, W_\mathbb{R}^\bullet)$ is a realization of the pair $(A_\mathbb{C}, W_\mathbb{C}^\bullet)$.
- A decreasing (Hodge) filtration $F^\bullet$ on the vector space $A_\mathbb{C}$, such that:

\[ \text{Gr}^W_n A_\mathbb{C} = F^p \text{Gr}^W_n A_\mathbb{C} \oplus \overline{F^{n-p+1}} \text{Gr}^W_n A_\mathbb{C} \]

for all $n, p \in \mathbb{Z}$. Here $\overline{F^p}$ denotes complex conjugation with respect to the complex structure $A_\mathbb{R} \subset A_\mathbb{C}$.

**Definition 3.10** A mixed Hodge structure is called mixed Hodge complex if, additionally, $A_\mathbb{R}$ and $A_\mathbb{C}$ have the structures of differential graded algebras such that the products preserve all filtrations.
In the case of $\mathcal{O}(\mathcal{G})$, Theorem 3.8 gives us 
\[(\mathcal{O}(\mathcal{G})_R, W_*), (\mathcal{O}(\mathcal{G})_\mathbb{C}, W_*, F^*)\],
such that the Hodge filtration $F^*$ is inherited from $H^0(I(X, \mathcal{O}(S)))$, which has a natural Hodge filtration coming from external differentiation, and the weight filtration $W_*$ is defined through a construction by Saito [16].

Dually, the Lie algebra $\hat{\mathfrak{g}}$ of $\mathcal{G}$ has a mixed Hodge structure of nonpositive weights, as given in (3.27).

In light of 3.7, this MHS can be found as a limit:

**Proposition 3.11** In the notations of [3.7], let $\hat{\mathfrak{g}}$, and $\mathfrak{g}_V$ be the Lie algebras of the groups $\mathcal{G}$ and $A(V^\bullet, \omega)$, respectively. For each $(V^\bullet, \omega) \in B(X, S)$, which is of Hodge type $1$, there is a MHS on the Lie algebra $\hat{\mathfrak{g}}_V$, such that the weight filtration is provided as a limit of the natural filtrations of each $\mathfrak{g}_V$ coming from the lower central series.

**Proposition 3.12** In the category of MHS
\[\hat{\mathfrak{g}} = \lim_{\leftarrow} \hat{\mathfrak{g}}_V\]

We will derive the proofs of these statements as a consequence of the following theorem.

**Theorem 3.13** There is an isomorphism $\mathcal{O}(\mathcal{G}_V) \cong H^0(I^*(X, V))$ between the ring of functions on $\mathcal{G}_V$ and the 0-th cohomology of a the d.g.a. of iterated integrals with coefficients in $V$.

The precise definitions and the proofs will be provided in 3.2. We will show how Theorem 3.13 implies Hain’s MHS and, also we will derive as a corollary Theorem 3.29

We follow an alternative approach to the weight filtration as a limit of the lower central series of the nilpotent kernels (cf. [14]) this will be the topic of Section 3.4. A more general treatment of this will be given in Section ??.

**Corollary 3.14** In the case when $S = 1$ is the trivial group, this projective limit produces a mixed Hodge structure on the prounipotent completion of $\pi_1(X, x_0)$.

\[1\]This means that $V$ comes from an admissible complex variation of MHS. The precise definitions will be discussed in section ??.
3.2 The algebra of iterated integrals

In this section we consider the curve $X$ as a complex analytic space. In fact, the properties which we discuss are valid for a general differentiable space.

3.2.1 Iterated integrals on a differentiable space

Recall that the structure of a differentiable space is given by a an atlas of maps (plots) $\{\alpha_i : U_i \to X | i \in J\}$, where $U_i$-s are open convex regions in an Euclidean space, such that:

- if $\beta : V \to U$ is a smooth map and $\alpha : U \to X$ is a plot then $\alpha \circ \beta : V \to X$ is a plot;
- any map $\{0\} \to X$ is a plot.

A path on a differentiable space $X$, for us, will be piecewise smooth maps $I = [0,1] \to X$. By $P(X)$, $P(X,x)$ and $P(X,x,y)$ we denote respectively the differentiable spaces of paths on $X$, of loops based at a point $x \in X$ and paths joining $x$ and $y$. (Sometimes we will write $PX$ instead of $P(X)$.) The structure of differentiable space on $PX$ is given by all maps $\alpha : Y \times I \to X$ of $X$. One can think of $f_\alpha$ as a path on $X$ with parameters in $U$. We will denote by $u$ and $t$ the variables on $U$ and $I$ correspondingly.

Let $(V,\omega) \in B(X,S)$ and, in the notation of Section 3.1, let $\tilde{x}_0 \in P$ be a preimage from the fiber of $P \to X$ over the point $x_0$ and denote by $f_{\tilde{\alpha}} : U \times I \to P$ the unique lift of the the path $f_\alpha(u,\cdot) : I \to X$ using parallel transport with respect to $\omega$. The plot $\tilde{\alpha} : U \to PP$ of the path space of the bundle $P$ which corresponds to $f_{\tilde{\alpha}}$ is called the lift of $\alpha$ to $P$.

For each differentiable form $w \in A^\bullet(P) \otimes_S u$ on $P$ with coefficients in $V$ and each plot $\alpha : U \to PX$, we denote

$$w'_\alpha := i_\partial^* f\tilde{\alpha}_w.$$  

In particular, if $w \in A^\bullet(X,\mathcal{O}(P)) \otimes_S u \subset A^\bullet(P) \otimes_S u$ then $w'_\alpha$ defines a form on $X$ which coincides with:

$$w'_\alpha := i_\partial^* f\tilde{\alpha}_w.$$  

Finally, recall, that the differential forms on a differential space form a sheaf. A global section is defined by choosing compatible pullbacks $\alpha^*(w)$ for all possible plots $\alpha$. 

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Definition 3.15  1. Let \( w_i \in \mathbb{A}^*(P) \otimes_S u, \, i = 1, \ldots, r \) be differential forms on \( P \) with coefficients in \( \mathbb{V} \). The iterated integral \( \int w_1 \ldots w_r \) is the element of \( \mathbb{A}^*(P \times P) \otimes_S u \) which for any plot \( \alpha : U \to P \) is represented by the pullback:

\[
\alpha \ast \int w_1 \ldots w_r = \int_{0 < t_1 \leq \ldots \leq t_r < 1} (w'_{1\alpha}(u, t_1) \ldots w'_{r\alpha}(u, t_r)) \, dt_1 \ldots dt_r.
\]

2. For any \( f \in \mathcal{O}(S) \) and \( w_i \in \mathbb{A}^*(X, \mathcal{O}(P)) \otimes_S u \) we denote by \( \int (w_1, \ldots, w_r|f) \) the differential form on \( PX \) such that for any plot \( \alpha : U \to PX \) the pull back \( \alpha^* \int (w_1, \ldots, w_r|f) \) is given by

\[
\left[ (\tilde{\alpha})^* \int w_1 \ldots w_r \right] f(\sigma(\alpha(u, \cdot))).
\]

(Like before, \( \sigma : \pi_1(X, x_0) \to S \) is the representation corresponding to the bundle \( P \).)

3. An iterated integral with coefficients in \( \mathbb{V} \) is is a differential form \( \int (w_1, \ldots, w_r|f) \) which is a linear combination of forms of the type \( \int (w_1, \ldots, w_r|f) \).

We will often write \( \int_\alpha (w_1 \ldots w_r|f) \) instead of \( \alpha^* \int (w_1, \ldots, w_r|f) \).

For a plot \( \alpha : U \to X \) and an iterated integral \( \int (w_1, \ldots, w_r|f) \) we introduce:

\[
\left\langle \alpha, \int (w_1, \ldots, w_r|f) \right\rangle := \begin{cases} 
\int_U \int_\alpha (w_1 \ldots w_r|f), & \text{if } \dim U = \deg \int (w_1 \ldots w_r|f) \\
0 & \text{otherwise.}
\end{cases} 
\]

Proposition 3.16 gives a brief account of the properties of the iterated integrals which will be useful later (cf., e.g., [4], [12]).

Proposition 3.16  1. The degree of the differential form \( \int (w_1, \ldots, w_r|f) \) is \( \sum (\deg w_i - 1) \). Thus, iterated integrals of degree 0, are obtained only when \( \deg w_i = 1 \) for all \( i \).
2. All iterated integrals with coefficients in $O(S)$ form a sub-d. g. a. of $A^*(PX, O(S)) \otimes_S \mathfrak{u}$, which we denote by $I^*(X, \mathcal{V})$. Moreover,

$$d \int (w_1 \ldots w_r|f) = \sum_{1 \leq i \leq r} \int (Jw_1 \ldots Jw_{i-1}(dw_i)w_{i+1} \ldots w_r|f)$$

$$+ \sum_{1 \leq i < r} (-1)^{i+1} \int (Jw_1 \ldots Jw_{i-1}(Jw_i \wedge w_{i+1})w_{i+2} \ldots w_r|f)$$

$$+ \int (Jw_1 \ldots Jw_r df|1)$$

and

$$\int (w_1 \ldots w_r|f) \int (w_{r+1} \ldots w_{r+s}|g) = \sum_{\sigma \in Sh(r,s)} \int (w_{\sigma(1)} \ldots w_{\sigma(r+s)}|fg)$$

3. Any two plots $\gamma : U_1 \rightarrow P(X, x_0)$ and $\mu : U_2 \rightarrow P(X, x_0)$ can be multiplied (considered as paths dependent on parameters). Then

$$\int_{\gamma \mu} (w_1 \ldots w_r|f)$$

$$= \sum_{i=0}^{r} \sum_{j} \int_{\gamma} (w_1 \ldots w_i|f_i') \int_{\mu} (\rho(\gamma^{-1}) * w_{i+1} \ldots \rho(\gamma^{-1})^* w_r|f''_i),$$

assuming that $\Delta_S f = \sum_j f'_j \otimes f''_j$, where $\Delta_S : O(S) \rightarrow O \otimes O(S)$ is the coproduct in $O(S)$. Thus, $I^*(X, \mathcal{V})$ is a Hopf algebra.

Remark. As a side note, one can define formal power series with coefficients in $I^*(X, \mathcal{V})$ and formal power series connection, similarly to Chen [4] in terms of iterated integrals with coefficient in $\mathcal{V}$. The following proposition (cf. [12]) shows that the monodromy of $\omega$ can be expressed in terms of iterated integrals.

Proposition 3.17 The parallel transport map of the connection $\omega$ is given by the following iterated integral:

$$T = \left( 1 + \int \omega + \int \omega \omega + \ldots \right) \tau(\sigma).$$

Consequently, the map $\tilde{\sigma}$ in (2) is given by the monodromy of the bundle $\mathcal{V} \rightarrow X$. It takes $\gamma \in PX$ to

$$\tilde{\sigma}(\gamma) = \left( 1 + \int_{\tilde{\gamma}} \omega + \int_{\tilde{\gamma}} \omega \omega + \ldots \right) \tau(\sigma(\gamma)) \in G$$
3.2.2 The bar construction in $A^\bullet(X, \mathcal{O}(S)) \otimes_S u$

For any commutative d.g.a. $A^\bullet$ over a field $k$ and any two d.g.a.-s over $A^\bullet$, $M^\bullet$ and $N^\bullet$, there is classical reduced bar construction which produces a d.g.a. $B(M^\bullet, A^\bullet, N^\bullet)$ (cf., e.g., [1]). Notice, that one can obtain the (non-reduced) bar construction as $B(k, A^\bullet, k)$ by turning $k$ int an $A$-module using the standard augmentation $\epsilon : A^\bullet \to k$ (coming from degree).

We use bar construction twice to obtain two different d.g.a. as follows. Let $A^\bullet = A^\bullet(X, \mathcal{O}(P)) \otimes_S u$.

We consider $\mathbb{R}$ as $A^\bullet$ modules, using the standard augmentation:

$$\epsilon_{\bar{x}_0} : A^\bullet(X, \mathcal{O}(P)) \otimes_S u \to \mathbb{R},$$

which is the deg map and, also, can be induced by the projection:

$$P \to \{\bar{x}_0\}.$$ We denote $J^\bullet_V = B^\bullet(\mathbb{R}, A^\bullet, \mathbb{R})$.

At the same time, we can consider $\mathcal{O}(S)$ as a $A^\bullet$-module by constructing an augmentation $\delta_{x_0} : A^\bullet \to \mathcal{O}(S)$ as composition $\delta_{x_0} = j_u \circ \varphi_{x_0}$ is a composition such that $\varphi_{x_0} : A^\bullet \to u$ is induced by the map $P \to p^{-1}(x_0)$ and

$$j_u : u \exp \to U \to \text{End} V \to \mathcal{O}(S).$$

Denote by $I^\bullet_V = B^\bullet(k, A^\bullet, \mathcal{O}(S))$.

**Proposition 3.18** There is an isomorphism of commutative Hopf d.g. algebras:

$$I^\bullet_V \to I^\bullet(X; \mathcal{V}),$$

given by:

$$[w_1|\ldots|w_r]f \mapsto \int (w_1 \ldots w_r|f).$$

The proof is straightforward from the definitions and [8] and is similar to the analogous result in [12].

**Corollary 3.19** The bar-filtration on $H^0(I^\bullet(X, \mathcal{V}))$ is:

$$B_s = H^0(\text{iterated integrals of length } s).$$
3.3 The De Rham theorem for the vector bundle \( V \) on \( X \)

3.3.1 The relative Malcev completion \( G_V \) expressed through iterated integrals

We define
\[
\mathcal{U}_V^{DR} = \text{Spec } H^0(J_V^\bullet)
\]
and
\[
G_V^{DR} = \text{Spec } H^0(I_V^\bullet)
\]

Proposition 3.20 \( \mathcal{U}_V^{DR} \) is a pro-unipotent group and \( G_V^{DR} \) is a pro-algebraic group. Moreover, \( G_V^{DR} = \mathcal{U}_V^{DR} \rtimes S \) and:

\[
1 \rightarrow \mathcal{U}_V^{DR} \rightarrow G_V^{DR} \rightarrow q \rightarrow S \rightarrow 1
\]

In the proof we use the following technical lemma (cf. [20], [12]):

Lemma 3.21 As before, denote \( A^\bullet = A^\bullet(X, \mathcal{O}(P)) \otimes_S u \). There exists a sub-d.g.a \( C^\bullet \) of \( A^\bullet \), which is also a \( S \)-submodule, such that \( C^0 = \mathbb{R} \) and the inclusion map \( C^\bullet \rightarrow A^\bullet \) is a quasi-isomorphism.

Proof of (3.20). Let \( A^\bullet \) be the subalgebra from Lemma 3.21. It is a simple corollary of the definitions that the bar constructions of quasi-isomorphic algebras are quasi-isomorphic, therefore the natural maps:

\[
H^0(\mathcal{B}^\bullet(\mathbb{R}, C^\bullet, \mathcal{O}(S))) \rightarrow H^0(\mathcal{B}^\bullet(\mathbb{R}, A^\bullet, \mathcal{O}(S)))
\]

and

\[
H^0(\mathcal{B}^\bullet(\mathbb{R}, C^\bullet, \mathbb{R})) \rightarrow H^0(\mathcal{B}^\bullet(\mathbb{R}, A^\bullet, \mathbb{R}))
\]

are isomorphisms. Since \( C^0 = \mathbb{R} \), it is true that

\[
H^0(\mathcal{B}^\bullet(\mathbb{R}, C^\bullet, \mathcal{O}(S))) = H^0(\mathcal{B}^\bullet(\mathbb{R}, C^\bullet, \mathbb{R})) \otimes \mathcal{O}(S)
\]

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as d.g.a. However, the co-product in the Hopf algebra $H^0(B^*(\mathbb{R}, C^*, \mathcal{O}(S))))$, which is given by (10), is twisted by the action of $S$. Since $B^*(\mathbb{R}, C^*, \mathcal{O}(S))$ is in fact the usual (non-relative) bar construction of $C^*$, we can apply the result from [?] to see that $H^0(B^*(\mathbb{R}, C^*, \mathcal{O}(S))))$ is isomorphic to a (non-commutative) formal power series, that is, to a direct sum $\mathbb{R} \oplus S^1 \oplus S^2 \oplus \ldots$ of symmetric tensor algebras of its indecomposable elements. This implies that $H^0(B^*(\mathbb{R}, C^*, \mathcal{O}(S))))$ is direct limit of the coordinate rings of unipotent Lie groups.

Further, there is a map $\tilde{\sigma} : \pi_1(X, x_0) \to Hom(\mathcal{O}(\mathcal{G}^{DR}_V), \mathbb{R})$ defined by:

\[
\tilde{\sigma}(\gamma) = \{[w_1] \ldots [w_r] f \mapsto \int_\gamma (w_1 \ldots w_r | f)\}, \tag{12}
\]

such that, if $q : \mathcal{G}^{DR}_V \to S$ is the natural projection, then we have: $\rho = q \circ \tilde{\sigma}$. □

**Proposition 3.22** Any object $(\mathcal{W}, \omega) \in B(X, S)/\mathcal{V}$ determines a canonical homomorphism:

\[
\pi_1(X, x_0) \downarrow \downarrow \quad (\mathcal{G}^{DR}_V, \tilde{\tau}) \quad \longrightarrow \quad A(\mathcal{W}, \omega)
\]

**Proof.** Denote by $(G, \tau) = A(\mathcal{W}, \omega)$ and by $U$ the unipotent kernel of $G$. Then $G$ and $U$ satisfy (2) and $\omega \in A^*(X, \mathcal{O}(P)) \otimes_S u$ satisfies (3). We will construct a *polynomial* map $\mathcal{O}(G) \to H^*(I^*_V)$, which will induce the desired map of groups.

Since $U$ is unipotent, the exponential map $u \to U$ is a polynomial isomorphism, which extends to a natural isomorphism of Hopf algebras:

\[
\mathcal{O}(U) \xrightarrow{\cong} \mathbb{R}[u] \to \lim_{\longrightarrow} Hom(Uu/I^n, \mathbb{R}).
\]

Here $Uu$ is the universal enveloping algebra of $u$ and, by a version of the Poincaré-Birkhoff-Witt (cf. [?]) is isomorphic to $Su$ the symmetric Hopf algebra of $u$ and $I$ is the augmentation ideal. As before, denote by $T \in B^*(A^*(X, \mathcal{O}(P)) \otimes_S u)$ the parallel transport map of

\[
T = 1 + |\omega| + |\omega| |\omega| + \ldots
\]
Since $G = U \times S$ we can define a linear map $\Theta : \mathcal{O}(G) \to I^0_V$ so that for any $g \otimes f \in \mathcal{O}(U) \otimes \mathcal{O}(S)$

$$g \otimes f \mapsto <T, g > f.$$ 

This is well defined map of Hopf d.g.a., for $\omega$ satisfies the condition (3). The image of $\Theta$ consist only of closed forms, because of the flatness of $\omega$. Therefore, it induces a Hopf algebra morphism $\mathcal{O}(G) \to H^0(I^*_V)$. □

**Corollary 3.23** *(De Rham Theorem)* There are isomorphisms of pro-algebraic groups:

$$\mathcal{U}^\text{DR}_V \cong \mathcal{U}_V \text{ and } \mathcal{G}^\text{DR}_V \cong \mathcal{G}_V.$$

### 3.3.2 Pairing of iterated integrals and paths

In this section we give a geometric interpretation of the De Rham Theorem for $G_V$.

We consider the parallel transport map $T$ of the connection $\omega$ of $V$, as defined in 3.17. Using $T$ we define a map

$$\Psi : P(X, x_0) \to P(P, \tilde{x}_0, \ast),$$

which lifts any loop in $X$ based at $x_0$ to its unique parallel lift which starts at $\tilde{x}_0$ and ends in some point in the fiber $P_{x_0}$. Denote by

$$\psi : \pi_1(X, x_0) \to P(P, \tilde{x}_0, \ast)/\sim$$

the induced map on $\pi_1(X, x_0)$, where the factorization on the right hand side is modulo homotopy equivalence and, additionally, the action of $S$, which leaves the homotopy classes on the left hand side invariant. Observe that the (non-relative) Malcev completion of the image $\psi(\pi_1(X, x_0))$ and the pro-unipotent kernel of the relative Malcev completion $\mathcal{G}_V$, which we denoted before by $\mathcal{U}_V$, are both by definition the Malcev completion of the monodromy representation of $\pi_1(X, x_0)$, and, hence, they coincide.

Below, we give a second, geometric proof of Corollary 3.23. For that, we extend the evaluation of iterated integrals (7) to homology cycles of $P(P, \tilde{x}_0, \ast)$.

Denote by $C_\bullet(P(P, \tilde{x}_0, \ast); V)$ the chain complex spanned by all the equivalence (up to $C^\infty$-smooth homeomorphism) classes of smooth simplexes in the path space $P(P, \tilde{x}_0, \ast)$ with coefficients in the vector bundle $V$, together
with an element denoted by 1, which is set to be of degree 0. All the
elements of $C_\bullet(P(P, \tilde{x}_0, *); \mathbb{V})$ are linear combinations of plots, and, therefore,
they form a d.g.a. with 1. Particularly, every loop $\alpha \in \pi_1(X, x_0)$ gives rise
to a 0-dimensional simplex. In this text, we will be concerned only with the
0-th homology group $H_0(P(P, \tilde{x}_0, *); \mathbb{V})$. Considering the fact, that for any
smooth 1-simplex $u$ of $P(P, \tilde{x}_0, *)$, the boundary $\partial u$ is equal to $\tilde{\beta} - \tilde{\alpha}$ where $\beta$ and $\alpha$ are homotopic loops from $P(X, x_0)$, we have:

$$H_0(P(P, \tilde{x}_0, *); \mathbb{V}) = u \oplus \mathbb{R}\pi_1(X, x_0) \otimes_S u \xrightarrow{\cong} u \oplus U_{V, u}.$$  

The extra summand $u$ is due to the artificial element 1 in $C_0(P(P, \tilde{x}_0, *); \mathbb{V})$. Hence, there is a homomorphism

$$H^0(J^\bullet) = H^0(P(P, \tilde{x}_0, *); \mathbb{V}) = Hom_u(u \oplus U_{V, u}, u) \rightarrow Hom_u(U_{V, u}, u).$$

Further, we define an augmentation $\varepsilon : C_\bullet(P(P, \tilde{x}_0, *); \mathbb{V}) \rightarrow u$, by setting:

$$\varepsilon(1) = 1$$

$$\varepsilon(u) = \begin{cases} 1 & \text{if } \deg u > 0 \\ 0 & \text{if } \deg u = 0, \end{cases}$$

and let $J_\varepsilon = \ker \varepsilon$ be the augmentation ideal of $C_\bullet(P(P, \tilde{x}_0, *); \mathbb{V})$. Observe,
that, for $q > 0$, $J_q = C_q(P(P, \tilde{x}_0, *); \mathbb{V})$ and $J_0$ is spanned by

$$\{ \tilde{\alpha} - 1 | \alpha \in P(X, x_0) \}.$$  

It follows, then, that the powers of the augmentation ideal $J_0^s$ are spanned by

$$\{ (\tilde{\alpha}_1 - 1) \ldots (\tilde{\alpha}_s - 1) | \alpha_1 \ldots \alpha_s \in P(X, x_0) \}.$$  

We denote by $\tilde{x}_0$ the 0-simplex of the constant loop at the base point $x_0$ and notice, that for any $s \geq 1$, $(1 - \tilde{x}_0)^s = 1 - \tilde{x}_0$ is an element of $J_0^s$.

We define a filtration of subcomplexes

$$\cdots \subset u = B^\bullet(0) \subset B^\bullet(1) \cdots \subset B^\bullet(s) \subset \cdots \subset C^\bullet(P(P, \tilde{x}_0, *); \mathbb{V})$$

by setting for every $s \geq 0$

$$B^q(s) = \{ y \in C^q(P(P, \tilde{x}_0, *); \mathbb{V}) | \langle y, J^{s+1} \rangle = 0 \},$$

$$20$$
and letting for every \( s < 0 \), \( B^q(s) = 0 \) Let \( \bar{J} \) be the augmentation ideal of \( \mathcal{U}_Y = \mathbb{R} \psi(\pi_1(X, x_0)) \). Then \( \bar{J}^{s+1} \) has a basis consisting of all

\[
(\bar{\alpha}_1 - \bar{x}_0) \ldots (\bar{\alpha}_{s+1} - \bar{x}_0),
\]

where \( \alpha_1 \ldots \alpha_{s+1} \) are loops based at \( x_0 \) and \( [\bar{\alpha}_i] \) denotes the corresponding elements of \( \text{im} \psi \). We consider \( \text{Hom}(\mathcal{U}_Y/\bar{J}^{s+1}, u) \) as a subgroup of \( \text{Hom}(\mathcal{U}_Y, u) \).

The composite homomorphism

\[
H^0(B^{\bullet}(s)) \rightarrow H^0(P(P, \bar{x}_0, *); u) \rightarrow \text{Hom}(\mathcal{U}_Y, u)
\]

maps every \( f \in H^0(B^{\bullet}(s)) \) to an \( f \in \text{Hom}(\mathcal{U}_Y, u) \), such that \( f'([\bar{\alpha}]) = \langle f, \alpha \rangle \). This map is injective, for \( 1 - x_0 \in \bar{J}^{s+1} \) implies

\[
\langle f, 1 \rangle = \langle f, x_0 \rangle = f'([\bar{x}_0]) .
\]

**Proposition 3.24**

\[ \text{Hom}(\mathcal{U}_Y/\bar{J}^{s+1}, u) \cong H^0(B^{\bullet}(s)). \]

**Proof.** Notice that \( \bar{J}^{s+1} \) is spanned by

\[
\{(\alpha_1 - x_0) \ldots (\alpha_{s+1} - x_0), \alpha x_0 - \alpha, x_0 \alpha - \alpha, x_0 - 1|\alpha, \alpha_1, \ldots \alpha_{s+1} \in P(X, x_0)\}
\]

It follows that for any \( f \in H^0(B^{\bullet}(s)) \), it’s image \( f' \in \text{Hom}(\mathcal{U}_Y, u) \) vanishes on all the elements of the type

\[
(\bar{\alpha}_1 - \bar{x}_0) \ldots (\bar{\alpha}_{s+1} - \bar{x}_0)
\]

and consequently \( f' \in \text{Hom}(\mathcal{U}_Y/\bar{J}^{s+1}, u) \) \( \Box \)

We now extend \( (7) \) to obtain a pairing

\[ \left\langle , \right\rangle : H^0(J^{\bullet}_Y) \times \mathcal{U}_Y \rightarrow u. \] (13)

by setting

\[
\left\langle \int u \ldots w_n, u \right\rangle = \int \int u \ldots w_n
\]

\[
\left\langle \int u \ldots w_n, 1 \right\rangle = \begin{cases} 1 & \text{if } r = 0 \text{ or } r = 1 \\ 0 & \text{otherwise} \end{cases}
\]

for any element of \( J^{\bullet}_Y \) which can be represented with an iterated integral \( \int w_1 \ldots w_n \) and for any smooth cube \( u \) of \( P(P, x_0, *) \).
Proposition 3.25 The pairing (13) results in a canonical isomorphism:

\[ B_s(H^0(J^*_V)) \cong H^0(B^*(s)) \cong \text{Hom}(\mathcal{U}_V/\tilde{J}^{s+1},u) \]

\[ \square \]

We will illustrate illustrate (3.25) with the following example.

Recall that the natural decreasing filtration of the pro-unipotent group \( \mathcal{U}_V \) is defined by:

\[ \mathcal{U}_{1}^V = \mathcal{U}^V, \quad \mathcal{U}_{s}^V = [\mathcal{U}^V, \mathcal{U}_{s-1}^V] \quad s \geq 2. \quad (14) \]

By construction, \( \mathbb{R}\tilde{J}^s \cong \mathcal{U}_V^s \).

**Example.** Observe that:

1. \( \int_{[\alpha,\beta]} w_1 = 0 \), because for any loops \( \alpha \) and \( \beta \) their commutator is homologous to zero.

2. \( \int_{[\gamma[\alpha,\beta]]} w_1 w_2 = \int_{\gamma} w_1 \int_{[\alpha,\beta]} w_2 - \int_{[\alpha,\beta]} w_1 \int_{\gamma} w_2 = 0 \)

Finally, (14) allows us to define a natural filtration on \( \mathcal{G}_V \)

\[ \mathcal{G}^0 = \mathcal{G}, \quad \mathcal{G}^s = \mathcal{U}^s, \quad s \geq 1 \]

and (13) allows us to define a pairing

\[ \langle \cdot, \cdot \rangle : H^0(I^*_V) \times \mathcal{G}_V \to \mathbb{R}. \quad (15) \]

by setting

\[ \left\langle \int (w_1 \ldots w_n | f), g \right\rangle = \int_{\alpha} (w_1 \ldots w_n | f) (a) \]

for any \( g = (\tilde{\alpha}, a) \in \mathcal{G}_V = \mathcal{U}_V \times S \), such that \( \tilde{\alpha} = \psi(\alpha) \), for some \( \alpha \in P(X,x_0) \).

Similarly to the case of Chen [4], the structure of a mixed Hodge complex is given by the multiplication of the ring \( Gr^\bullet \mathcal{G}_V \) defined with:

\[ [f, g] = f^{-1} g f g^{-1} + \mathcal{G}^{s+s+1}_V \]

for any \( f \in \mathcal{G}_V^s \) and \( g \in \mathcal{G}_V^s \). The pairing (15) induces a pairing of d.g.a., which we, by abuse of the notation will denote in the same way:

\[ \langle \cdot, \cdot \rangle : H^0(I^*(X,V)) \times Gr^\bullet \mathcal{G}_V \to \mathbb{R}. \quad (17) \]

With this set, the De Rham Theorem for \( \mathcal{G}_V \) is equivalent to the following straightforward corollary of Proposition 3.25.
Proposition 3.26 The map (15) is exact pairing of d.g.a. Moreover,

\[ Gr^s G_V = \left\{ f \in G_V | \langle w, f \rangle = 0 \text{ for all } w \in B_s H^0(I^\bullet(X, V)) \right\}, \]

Proof of Theorem 3.11. Recall that the bar filtration \( B_s(H^0(I^\bullet(X, V))) \) is a pure Hodge structure of weight \( s \). It follows from Proposition 3.26 that \( Gr^s G_V \) is a pure Hodge structure of weight \(-s\), making the Lie ring \( Gr^\bullet G_V \) a Mixed Hodge complex with non-positive weights. The multiplication is given by (16).

Then the Lie algebra \( \hat{g}_V \), has a natural filtration

\[ \hat{g}_V^s = s, \quad \hat{g}_V^s = \hat{u}_V, \quad s \geq 1 \] (18)

of unipotent Lie algebras. The factors are of finite dimension and carry a pure Hodge structure of weight \(-s\). Moreover, (16) implies that there is a multiplicative structure given by the correspondent tangent map, i.e.,

\[ [X, Y] = X^{-1} Y X - Y + 
\]

for any \( X \in \hat{g}_V^r \) and \( Y \in \hat{g}_V^s \). Finally, notice, that the weight filtration (18) can be obtained as a projective limit of the full filtrations

\[ u_{x_0}^s \supset u_{x_0}^{s+1} \]

of fibers over \( x_0 \) in the vector bundles \( B(G, \tau) \), where the projective limit is taken over all \( G \in G(X, S)/\mathbb{V} \), using the notation of Section 3.1 □

As a limit (cf., Corollary 3.7) we obtain Hain’s MHS:

Corollary 3.27 There is a MHS on \( \hat{g} \), of non positive weights, such that the weight filtration is given by the natural filtration of the pro-unipoten kernel of \( U \) of \( G \)

\[ \hat{g}^s = s, \quad \hat{g}^s = \hat{u}, \quad s \geq 1 \]

and the multiplicative structure is given by:

\[ [X, Y] = X^{-1} Y X - Y + \hat{g}^{r+s+1}, \]

for any \( X \in \hat{g}^r \) and \( Y \in \hat{g}^s \).
3.4 Local MHS on a Brill-Noether stack

With the notation and assumptions of the previous section, choose a flat vector bundle \((V, \omega) \in B(X, S)\) of Hodge type, which corresponds to a rational representation \(\rho : S \to GL(V)\). Denote by \(B_\rho(X, S)\) the full subcategory of \(B(X, S)\) consisting of filtrations of \(V\) and by \(G_\rho(X, S)\) — the correspondent subcategory \(G(X, S)/\mathbb{V}\) of \(G(X, S)\). The projective limits \(G_\rho\) and \(\mathbb{G}\) on \(G_\rho(X, S)\) and \(G(X, S)\) have the structures of MH complexes.

**Definition 3.28** A morphism of MHC is a d.g.a homomorphism, which respects all the filtrations and the multiplications.

This means that a MHC morphism

\[ \Phi^\bullet : (A_1^\bullet, W_1^\bullet, F_1^\bullet) \to (A_2^\bullet, W_2^\bullet, F_2^\bullet) \]

is a map of d.g.a, such that

\[ \Phi^p : Gr^s_{W^1} A_1 \to Gr^{p+s}_{W^2} A_2, \]

such that the restrictions of \(\Phi^p\) is a morphism mapping a pure Hodge structure of weight \(s\) to one of weight \(p + s\).

Let \(H_\rho : \mathbb{G} \to \mathbb{G}_\rho\) be the canonical morphism which is produced by the universal property of inverse limit and let \(\eta_\rho : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}_\rho\). Denote by \(\Phi : \mathbb{G} \times \mathbb{G}_\rho \to \mathbb{G}_\rho\) the map \((g_1, g_2) \mapsto H_\rho(g_1^{-1}) g_2 H_\rho(g_1) g_2^{-1}\). where \(H_\rho : \mathbb{G} \to \mathbb{G}_\rho\) is the canonical morphism which is produced by the universal property of inverse limit. For the Lie algebras \(\hat{\mathfrak{g}}\) and \(\hat{\mathfrak{g}}_\rho\) there is an induced action \(\phi : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}}_\rho \to \hat{\mathfrak{g}}_\rho\) by \((X, Y) \mapsto -XY + YX - Y = [Y, X] - Y\). Finally, there is a dual map \(h_\rho : \mathcal{O}(\mathbb{G}_\rho) \to \mathcal{O}(\mathbb{G})\) (cf. the Remark in the beginning of Section 3.1.2) which, combined with the co-multiplication and the inverse in the Hopf algebra of the iterated integrals produces a map \(f : \mathcal{O}(\mathbb{G}_\rho) \to \mathcal{O}(\mathbb{G}) \otimes \mathcal{O}(\mathbb{G}_\rho)\). One can formulate and prove analogous results for \(\Phi\), \(\phi\) of \(f\). We will develop only the thread that follows from \(\Phi\), because it is simplest to write down. The following result is central for this work:

**Theorem 3.29** \(\Phi\) induces a map of MHS

\[ \Phi^\bullet : \mathbb{G} \otimes \mathbb{G}_\rho \to \mathbb{G}_\rho. \]

**Proof.** First, notice that \(H_\rho^p = 0\) unless \(p = 0\). It is easiest to see this in terms of the map \(h_\rho\) which is simply inclusion and it preserves the weight.
The same observation is correct for the identity map and, therefore $\Phi_\rho$ is expected to be a map of pure weight 0 as well.

Then, for any $g_1 \in G^p$ and $g_2 \in G^q_\rho$, the product $g_1 \otimes g_2$ is an element of weight $p + q$ in $G \otimes G_\rho$.

At the same time, it follows from Proposition 3.26 and the multiplicative Hodge structures on $G$ and $G_\rho$ that $\Phi(g_1 \otimes g_2) \in G^{p+q}_\rho$. The fact that the restriction of $\Phi$ on $\oplus_p G^p \otimes G^{n-p}$ is a morphism of pure weight structures of wight $n$ follows from the fact, that so are $H_\rho$, the identity map. □

Remarks.

1. We think of $G$ and $G_\rho$ as $\mathbb{R}$-valued points in the parametrizing stacks: $\text{Hom}(X, \kappa(S,1))$ and $\text{Hom}(X, \kappa(S,\rho,n))$. The action $\rho$ of $\pi_1(\kappa(S,\rho,n))$ on $\pi_n(\kappa(S,\rho,n))$ induces Whitehead’s product: $\pi_1 \times \pi_n \rightarrow \pi_n$, sending $(g,h) \mapsto (\rho(g)h) - h$. Translated to d.g.a. terms, this action induces the map $\Phi$.

2. Using proposition ??, we can interpret the MHS on $\hat{\mathfrak{g}}_\rho$ as a local MHS on $\text{Hom}(X, \kappa(G,1)) = \text{Hom}(X, \kappa(S,\rho,1))$, when $G = V \rtimes^\rho S$.

3.5 Example

Assume that $X$ is an elliptic curve, then $\pi_1(X,x_0) = \mathbb{Z} \times \mathbb{Z}$ is abelian. Take $S = \mathbb{C}^*$. The MHS defined above repeats the well known facts from the usual Hodge theory but with a slightly different twist. Denote by $\sigma: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}^*$ the representation: $(a,b) \mapsto e^{a+ib}$. The image of this map is Zariski dense and the completion of $\mathbb{Z} \times \mathbb{Z}$ relative $\sigma$ is $G = \mathbb{C}^*$ with the usual abelian Hodge structure on it.

Let $\rho: S \rightarrow GL(n,\mathbb{C})$ be the rational representation

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. $$

We choose a splitting of the representation $V = L_1 \oplus L_2$ and let $\nabla_{1x_0} = L_1$. The monodromy group is and the unipotent kernel are:

$$G = \{ \begin{pmatrix} t & \ast \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \}, \quad U = \{ \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \},$$

and we have:

$$1 \xrightarrow{U} G \xrightarrow{\rho} S \xrightarrow{q} 1.$$ 

There are only 2 non-isomorphic objects in the category $B_\rho(X,S)$:
\[ V'_\bullet = [V \supset V_1 \supset (0)]; \]
\[ V''_\bullet = [V \supset (0)]. \]

There is an inclusion map \( V'_\bullet \to V''_\bullet \), which implies that the \( V'_\bullet \) will determine the weight filtration in the MHS on the projective limit in \( G_\rho(X, S) \), which gives, \( G_\rho = G \) and \( U_\rho = U \).

The weight filtration on \( H^\bullet(X; V) \) is given by: \( W_0 = H^\bullet(X; V) \supset W_{-1} = H^\bullet(X; V_1) \supset (0) \) and the Hodge filtration is the usual one.

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