Confinement and Effective Mass of Test Particles in Thick Branes Coupled Non Minimally With the Dilatonic Field

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Abstract: In this manuscript we study the confinement of test particles in smooth Randall-Sundrum models. It is a known fact that free test particles can not be localized neither in thin or thick branes. Recently a coupling of the particle to a scalar field has been used to localize a limited range of masses. It is found that the threshold mass depends on the specific model chosen. Up to now, no mechanism has been found that can trap test particles of any mass to the brane. In order to solve this problem, we first find a general expression for the effective mass observed on the brane. Next we show that by coupling the particle to the dilaton we obtain the trapping of particles of any mass. We further show the fact that this result is valid for any smooth version which recover the RS model asymptotically. For this, we just need of general aspects of the field equations of motion. As an example, we apply the method to the specific cases of deformed and undeformed topological defects.
1 Introduction

Until recently, the main theory of extra dimensions was due to Kaluza-Klein, [1, 2], in which is introduced an extra dimension in high energy Physics in order to unify electromagnetism with gravity. Whith the purpose of recovering the Physics of 4-dimensions, it is imposed that the extra dimension must be compactified with compactification radius of the order of the Planck length [2].

In the year of 1999, it was proposed by Randall and Sundrum the possibility of our 4-dimensional universe being a hypersurface, called brane, of a ADS 5-dimensional space-time with a non factorable metric [3]. In this model the Planck scale is not a fundamental scale, but a consequence of the large size of the new dimension. This idea was first considered by ADD model [4]. In this scenario, matter and non gravitational fields must be confined in a 4-dimensional space-time by some mechanism in a way such that the extra dimension can not be measured by direct observations in a energy level bellow the TeV scale, whereas the gravity permeates the whole space [4, 5]. Since these extra dimensions can not be detected, how can they help us to solve some of the problems in four dimensional physics? One of the
most famous work in this direction, was the one due to Lisa Randall and Ramman Sundrum, pursuing to solve the hierarchy problem \([5, 6]\). The Randall-Sundrum model is divided in two, RS-I \([5]\) and RS-II \([3]\). In the RS-I model, the extra dimension is compactified in a circle whose superior and inferior parts are identified. Formally, this means that we work in an orbifold \(S^1/\mathbb{Z}_2\), where \(S^1\) is an unidimensional sphere (e.t, a circle) and \(\mathbb{Z}_2\) is the discreet group \(\{-1, 1\}\). This construction implies in two fixed points, one in the origin \(\phi = 0\) and other in the opposite extremity of the circle \(\phi = \pi\) it is at these points that the 4-dimensional hypersurfaces, like the one live in, are located. By analogy with membranes that involve a volume, these \((3+1)\)-dimensional world are also called of 3-branes. The RS-I model explain how an exponential hierarchy between mass scales can be generated. While that in the RS-II it is assumed that the extra dimension is large (called \(y\)) and there is just one delta-like 3-brane located at \(y = 0\), where the gravity is confined. It also explain how the 4D gravity emerge in the Newtonian limit, becoming an alternative to compactification. In the RS-II model, the usual brane \([3]\) associated with such a metric has been replaced by a scalar field \(\phi\) residing in this space coupled through the standard action in which an scalar field is minimaly coupled with gravitation \([7–11]\).

Since the extra dimension is not compact in the RS-II model, all of the matter fields, not only the gravity, must be confined on the brane in order to reproduce a realistic model. However, the RS-II model does not show any arguments that guarantee the localization of others fields such as gauge and fermionic. Seeking the solution of this problem many works have been proposed in the literature \([11–19]\).

For the case of test particles there is a problem related to the confinement and stability on the brane, in both types of brane (delta \([20]\) and thick \([21]\)), it arise because the gravity by itself is not enough to confine the particle. Then, in the RS spacetime it is necessary some confinement mechanism for the two kinds of branes. The simplest one, for both cases, is to invert the signal of the brane tension, however it causes the non localizability of gravity \([12]\), making the mechanism not possible. For smooth brane spacetime it’s possible to create test particles confinement mechanisms as in \([21]\), the particle confinement is possible if it interact directly with a scalar field. This interaction is made by a modification in the free particle action through a redefinition of the canonic moment in 5-dimension \(P_A P^A = -(M_0^2 + h^2 \phi^2)\), based in a Yukawa interaction out of quantum level, allowing that test particles be confined on the brane obeying a condition in the mass.

Based in work \([22]\), we obtain a mechanism to obtain the effective mass of test particles subjected to effective potentials having maximum point such as cases of the free particle and particles coupled with scalar field \([21]\), where we obtain the maximum value of the mass of the particle that are observable.

Finally, we propose a mechanism for the confinement of the test particles based in the work \([23]\), where a conformal transformation of the Jordan to Einstein frame produces a new field, called dilaton. In this scenario all matter fields are coupled with this new field. In the simplest case, where matter is phenomenologically represented by a set of point particles of mass \(\tilde{m}\), the matter action becomes

\[
S_m = - \sum \int \tilde{m} e^{\lambda \pi} ds. \tag{1.1}
\]
This can yet be reformulated as a spacetime dependent mass in the conformal Einstein frame, \( m = \tilde{m} e^{\lambda \phi} \), where \( m \) is the constant mass in conformal Jordan frame [23, 24].

In section (2) we make a review of the RS-II type delta, thick branes, the model deformed by the dilaton and the model deformed by an control parameter. In section (3) we review and analyze the confinement of test particles in a RS II brane world with delta and thick branes, showing that the free test particle is not confined. That way it is necessary a mechanism that makes the confinement possible. In section (4) we obtain the effective mass of the test particles observed on the brane to cases where the effective potential have a maximum point. In section (5), we propose a mechanism where the test particles are coupled with the dilaton field in order to confine them.
2 Review of Randall-Sundrum Model

In this section we will make a brief description of the braneworld models such as: delta-like, thick, deformed by scalar field (dilaton) and the brane as topological defect. We shall use the Minkowsky metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ on the brane.

2.1 Delta-like branes

Delta type branes are defined by the action [20]

$$S = \int d^4x \int dy \sqrt{-g} (2M_P^{(3)} R - \Lambda) + \sigma \int_{y=0} d^4x \sqrt{-g_b},$$  \hspace{1cm} (2.1)

where $M_P^{(3)}$ is the Planck scale for theory, $R$ is the Ricci scalar, $\Lambda$ is the cosmological constant, $\sigma$ is the brane tension and $g_b$ is the induced metric on the brane, $g_b = g(x,y)|_{y=0}$.

The metric of theory is of the form:

$$g_{AB} = \text{diag}(e^{2a(y)}\eta_{\mu\nu},1).$$  \hspace{1cm} (2.2)

The obtained Einstein’s equations are

$$\Lambda = -24M_P^{(3)} a'^2,$$  \hspace{1cm} (2.3)

$$\sigma = 12M_P^{(3)} a'' + 24M_P^{(3)} a'^2 + \Lambda.$$  \hspace{1cm} (2.4)

Solving the equation (2.3) we obtain the solution

$$a(y) = -k|y|,$$  \hspace{1cm} (2.5)

this is a warp factor, where $k = \sqrt{-\Lambda/24M_P^{(3)}}$. The metric (2.2) then takes the form

$$g_{AB} = \text{diag}(e^{-2k|y|}\eta_{\mu\nu},1)$$  \hspace{1cm} (2.6)

2.2 Thick branes

In the presence of a gravitational field in a 5- dimensional spacetime, the action for a scalar field is [11]:

$$S = \int dx^5 \sqrt{-g} \left( 2M_P^{(3)} R - \frac{1}{2} \partial_A \phi \partial^A \phi - V(\phi) \right).$$  \hspace{1cm} (2.7)

Where $g$ is the metric determinant, $R$ is the curvature scalar relative $g_{AB}$, $V(\phi)$ corresponds to the potential of the scalar field and $M_P^{(3)}$ is the Planck constant for 5- dimensional spacetime. We obtain from the action (2.7), that the energy moment tensor of the model is

$$T_{AB} = \nabla_A \phi \nabla_B \phi - g_{AB} \left( \frac{1}{2} \nabla^A \phi \nabla_A \phi + V(\phi) \right),$$  \hspace{1cm} (2.8)
which allows us to obtain the Einstein equations
\[ G_{AB} = \frac{1}{4M_p^3} T_{AB}, \]
while the dynamics of the scalar field is governed by the equation
\[ \nabla^2 \phi - \frac{\partial V(\phi)}{\partial \phi} = 0. \] (2.9)

In the absence of gravity for a scalar potential of the double well type 
\[ V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 \]
the scalar field equation possesses bounce-like static solution depending only on the fifth, which the simplest one is \[ \phi_B(y) = v \tanh(cy), \] (2.10)
where \( a^2 \equiv \lambda v^2 / 2 \). When we introduce the metric (2.2) we obtain the Einstein equations
\[ \frac{1}{2}(\phi')^2 - V(\phi) = 24M_p^3 (a')^2, \] (2.11)
\[ \frac{1}{2}(\phi')^2 + V(\phi) = -12M_p^3 a'' - 24M_p^3 (a')^2. \] (2.12)

When we add (2.11) and (2.12), we obtain the second order differential equation
\[ (\phi')^2 = -12M_p^3 a''. \] (2.13)

Applying (2.10) in (2.13), we get the solutions
\[ a(y) = -\beta \ln \cosh^2(cy) - \frac{\beta}{2} \tanh^2(cy), \] (2.14)
where \( \beta \equiv \frac{v^2}{36M_p^3} \). Note that this represents a localized metric warp factor [11]:
\[ e^{-2a(y)} = \frac{e^{-\beta \tanh^2(cy)}}{(\cosh^2(cy))^{2\beta}} \propto e^{-4c\beta|y|} \quad y \to \infty \] (2.15)

### 2.3 Dilaton coupling

Let us consider the action [11]:
\[ S = \int dx^5 \sqrt{-g} \left\{ 2M_p^3 R - \frac{1}{2} (\partial^2 \phi)^2 - \frac{1}{2} (\partial \pi)^2 - V(\phi, \pi) \right\}, \] (2.16)
where \( \phi \) is the scalar field intended to obtain the bounce-like configuration and the other scalar field \( \pi \) is called dilaton, with the appropriated choice of the potential. In general, this potential is dependent of \( \phi \) and \( \pi \), so the resulting motion equations from (2.16) are
\[ R_{AB} - \frac{1}{2} g_{AB} = \frac{1}{4M_p^3} \left\{ \partial_A \phi \partial_B \phi + \partial_A \pi \partial_B \pi - g_{AB} \left( \frac{1}{2} (\partial \phi)^2 + (\partial \pi)^2 + V \right) \right\}, \] (2.17)
\[ \nabla^2 \phi - \frac{\partial V(\phi)}{\partial \phi} = 0, \] (2.18)
\[ \nabla^2 \pi - \frac{\partial V(\pi)}{\partial \pi} = 0. \] (2.19)
Introducing an ansatz for the metric in the form

\[ g_{AB} = \text{diag}(e^{2a(y)} \eta_{\mu\nu}, e^{2b(y)}) \]  

(2.20)

and whereas the solutions for the scalar fields depend only on the extra dimension \( y \), we obtain the set of Einstein equations

\[
\frac{1}{2} (\phi')^2 + \frac{1}{2} (\pi')^2 - e^{2b} V = 24M_P^{(3)} (a')^2, \\
\frac{1}{2} (\phi')^2 + \frac{1}{2} (\pi')^2 + e^{2b} V = -12M_P^{(3)} a'' - 24M_P^{(3)} (a')^2 + 12M_P^{(3)} ab', \\
\phi'' + (4a' - b')\phi' = \partial_\phi V, \\
\pi'' + (4a' - b')\pi' = \partial_\pi V.
\]

(2.21)

(2.22)

(2.23)

(2.24)

Choosing appropriately the dilaton and the metric deformation how

\[ \pi(y) = -\sqrt{3M_P^{(3)} a(y)}, \]

(2.25)

\[ b(y) = \frac{a(y)}{4}, \]

(2.26)

respectively, we reobtain the differential equations (2.11), (2.12) and (2.13), therefore, we obtain as solution the equation (2.14) and

\[ \pi(y) = \sqrt{3M_P^{(3)}} \beta \left( \ln \cosh^2 (cy) + \frac{1}{2} \tanh^2 (cy) \right) \]

(2.27)

for a dilaton field.

2.4 Deformed brane

The deformation method, approached in [25] is based in modifications of the potentials of models containing solitons in order to produce new and unexpected solutions [26], let’s use the metric ansatz

\[ g_{AB} = \text{diag}(e^{2a_s} \eta_{\mu\nu}, e^{2b_s}), \]

(2.28)

where \( s \) is a deformation parameter, that controls the kind of topology. The deformation parameter controls the kind of deformed topological defect we want in order to obtain different classes of membranes. To solve this problem, we use the so-called superpotential function \( W_s(\phi) \), defined by

\[ \phi' = \frac{\partial W_s}{\partial \phi}, \]

(2.29)

using, then, the same approach of Kehagias and Tamvakis [11]. The particular solution follows from choosing the potential

\[ V_s = e^{\sqrt{24M_P^{(3)}}} \left[ \frac{1}{2} \left( \frac{\partial W_s}{\partial \phi} \right)^2 - \frac{5}{32M_P^{(3)}} W_s(\phi)^2 \right], \]

(2.30)
and superpotential
\[ W_s(\phi) = c\phi^2 \left[ \frac{2}{2s-1} \left( \frac{v}{\phi} \right)^{\frac{1}{2}} - \frac{2}{2s-1} \left( \frac{\phi}{v} \right)^{\frac{1}{2}} \right], \] (2.31)
where \( c \) and \( v \) are parameters to adjust the dimensionality. As in [11], this potential gives us the desired soliton-like solution. In this way, it is easy to obtain first order differential equations whose solutions are the ones for the equation of motion above, namely
\[ a'_s = \frac{W_s}{12M_{P}^{(3)}}, \quad b_s = \frac{a_s}{4}, \quad \pi_s = -\sqrt{3M_{P}^{(3)}}a_s. \] (2.32)

The solution for these new set of equations are the following:
\[ \phi(y) = v \tanh(cy), \] (2.33)
\[ a_1(y) = -\beta_1 \ln \cosh^2(cy) - \frac{\beta_1}{2} \tanh^2(cy), \] (2.34)
for \( s = 1 \), and
\[ \phi(y) = v \tanh^s \left( \frac{cy}{s} \right) \] (2.35)
\[ a_s(y) = -\frac{\beta_s}{2} \tanh^{2s} \left( \frac{cy}{s} \right) - \frac{2s\beta_s}{2s-1} \left\{ \ln \cosh \left( \frac{cy}{s} \right) - \sum_{n=1}^{s-1} \frac{1}{2n} \tanh^{2n} \left( \frac{cy}{s} \right) \right\} \] (2.36)
for \( s > 1 \), where \( \beta_1 = \frac{v^2}{36M_{P}^{(3)}} \) and \( \beta_s = \frac{v^2}{12M_{P}^{(3)}} \frac{s}{2s+1} \) [25]. Then, we obtain a set of second order differential equations similar to equation (2.13),
\[ (\phi')^2 = -12M_{P}^{(3)} a_s''. \] (2.37)

And \( s \) assumes only odd integer values positives, once that the first derivative of scalar function must behave like an step function [7]. And note that this warp factor represents a localized metric warp factor
\[ e^{-2a_s(y)} \propto e^{-\frac{4c_s\beta_s}{2s-1}|y|}\equiv e^{-4c_s\beta_s|y|} \quad y \to \infty, \] (2.38)
with \( c_s = \frac{s}{2s-1} \).
3 Confinement of test particles in braneworld

In this section we will review what has been done in the literature about the confinement of test particles in the Randall-Sundrum scenario. The motion of test particle is described by a geodesic, and the most general action that includes massive and massless particles is achieved by introducing an auxiliary variable $E(\theta)$, which should, however, not introduce new dynamical degrees of freedom [27]. The action containing $x^A$ and $E$ is

$$S = \frac{1}{2} \int E \left( -E^{-2} g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + M^2(y) \right) d\theta$$

(3.1)

where $\theta$ is a parameter. The equations of motion in a 5-dimensional Randall-Sundrum spacetime are

$$\frac{\delta S}{\delta E} = 0 \rightarrow g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + E^2 M^2(y) = 0,$$

(3.2)

$$\frac{\delta S}{\delta x^\mu} = 0 \rightarrow \eta_{\mu\nu} \frac{d^2 x^\nu}{d\theta^2} + a' \eta_{\mu\nu} \frac{dy}{d\theta} \frac{dx^\nu}{d\theta} - E^{-1} \frac{dE}{d\theta} \eta_{\mu\nu} \frac{dx^\nu}{d\theta} = 0,$$

(3.3)

$$\frac{\delta S}{\delta y} = 0 \rightarrow g_{55} \frac{dy}{d\theta} + a' g_{\mu\nu} \frac{dx^\mu}{d\theta} \frac{dx^\nu}{d\theta} + \frac{1}{2} g_{55} \left( \frac{dy}{d\theta} \right)^2 - E^{-1} \frac{dE}{d\theta} g_{55} \frac{dy}{d\theta} + E^2 M(y) M'(y) = 0,$$

(3.4)

with $a' = da/dy$ and $M'(y) = dM(y)/dy$.

Since the equation of motion for $E$ is purely algebraic, $E$ does not represent a new dynamical degree of freedom. In general, for massive test particles we can fix the new variable as $E = \frac{1}{M_0(y)}$ and use the constraint $g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + 1 = 0$. For massless test particles we can fix the new variable as $E = 1$ and the constraint $g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} = 0$, for $y = 0$, $M^2(0) = M_0^2$, where $M_0^2$ is the bare mass of the test particle. So if the bare mass is null the action (3.1) is not well defined for the choice $E = \frac{1}{M_0(y)}$. Then to avoid this problem we can choose $E = 1$ and we use the constraint

$$g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + M^2(y) = 0,$$

(3.5)

where we define massive particles for non-null bare mass and massless particles for null bare mass.

We can write the equations (3.3) and (3.4) as total derivatives, that is, we can obtain conserved quantities in the directions $x^\mu$ and $y$. Initially we must multiply the equation (3.3) by $E^{-1} e^{2a}$ and write it as a total derivative, which provides us:

$$\frac{dx^\mu}{d\theta} = e^{-2a} \eta^{\mu\nu} E p_\nu.$$

(3.6)

So, when we replace (3.6) in (3.4) we obtain the conserved quantity

$$e^{2a} g_{55} E^{-2} \left( \frac{dy}{d\theta} \right)^2 + e^{-2a} p^2 + \int d\theta \left( \frac{dM^2}{d\theta} \right) e^{2a} = C,$$

(3.7)
where $p^2 = \eta^{\mu\nu}p_\mu p_\nu$.

Other conserved quantity is obtained from the equation (3.4) when we cast the parameter relationship (3.2) in the form $e^{2a} \eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = -\left[ M^2 E^2 + g_{55} \left( \frac{dy}{d\theta} \right)^2 \right]$, so the equation of motion (3.4) can be written only in terms of the extra dimension

$$g_{55} \frac{d^2 y}{d\theta^2} + \Theta \left[ E^2 M^2 + g_{55} \left( \frac{dy}{d\theta} \right)^2 \right] + \frac{1}{2} g_{55}' \left( \frac{dy}{d\theta} \right)^2 - E^{-1} \frac{dE}{d\theta} g_{55} \frac{dy}{d\theta} + MM' = 0, \quad (3.8)$$

and there its motion along the extra dimension is independent of the others. Multiplying (3.8) by a factor $E^{-2} e^{2a} \frac{dy}{d\theta}$, we obtain a conserved quantity

$$E^{-2} g_{55} e^{2a} \left( \frac{dy}{d\theta} \right)^2 + M^2 e^{2a} = \alpha, \quad (3.9)$$

that quantity can be interpreted as the total energy of the test particle. Therefore, based in the quantities (3.7) and (3.9) we can analyze the confinement and stability of the test particle analytically or by the effective potential method, respectively, as we will see below.

### 3.1 Analytical method

The confinement of free test particles in delta-like branes is described analytically in reference [20], based on a non-affine parameter transformation $t$ that allows us to obtain an analytical form to the equations of motion and to verify the behavior of the particle along the extra dimension. So, it is possible to evaluate the particle motion based on the non-affine parametrization

$$\frac{dt}{d\theta} = E e^{-2a}, \quad (3.10)$$

that only applies to $y \neq 0$, because in $y = 0$ the parametrization above is affine under certain conditions [22], hence, equations (3.6) can be written as first order EDOs. So, applying the chain rule in equation (3.6) and replacing (3.10) we obtain

$$\dot{x}^\mu \equiv \frac{dx^\mu}{dt} = p^\mu \to x^\mu = x^\mu_0 + p^\mu t, \quad (3.11)$$

that is, in the parameter $t$ the particle follows a flat geodesic. For free test particle the mass is constant and given by $M^2(y) = M^2_0$. Applying the chain rule to equation (3.7), using the non-affine parametrization (3.10) and choosing the parameter $E$ as unity, we find:

$$\left( \frac{dy}{dt} \right)^2 e^{-4a} + e^{-2a} p^2 = C. \quad (3.12)$$

Replacing (3.6) in (3.12) we obtain the parameter equation (3.5) for $M^2(y) = M^2_0$, therefore in the above equation, for parametrization in $\theta$ we have $C = -M^2_0$. Now, integrating the equation (3.6) we obtain

$$\int \frac{dy e^{-2a}}{(-M^2_0 - v^2 e^{-2a})^{\frac{1}{2}}} = \int dt, \quad (3.13)$$
which has analytical solution for delta-like branes, while for thick branes is less trivial. Taking (2.5) and with the new parameter, the equation (3.7), can be written as
\[ \dot{y}^2 e^{2k|y|} + e^{2k|y|} p^2 = -M_0^2, \] (3.14)
where \( \dot{y} \equiv \frac{dy}{dt} \) and we can define \( y_0 \) for a given initial time \( t_0 \), so that \(-M_0^2 = \dot{y}_0^2 e^{2k|y_0|} + e^{2k|y_0|} p^2\). After analogous mathematical manipulation we obtain an analytical form to equation
\[ e^{2k|y|} = e^{2k|y_0|} + 2k \dot{y}_0 e^{2k|y_0|} t - p^2 k^2 t^2. \] (3.15)
The above equation is valid when \( y \neq 0 \). So, for Randall-Sundrum model \((k > 0)\), one ordinary particle, \( p^2 < 0 \), is repelled from the brane even if we assume \(|y_0| \to 0\) at any time \( t \). The particle will not reach \( y = 0 \), in other words, the particle is repelled out of the brane, and in the limit that \(|y_0| \to 0\) and \(|\dot{y}_0| \to 0\) the equation (3.14) is valid if \( M_0^2 > 0 \), that is, the particle in the bulk also is ordinary. For tachionics particles, \( p^2 > 0 \), they are attracted for the brane and in the limit that \(|y_0| \to 0\) and \(|\dot{y}_0| \to 0\) the equation (3.14) is valid if \( M_0^2 < 0 \), that is, the particle in the bulk also tachionic. While massless particles, \( p^2 = 0 \), they will only be expelled from the brane if \( \dot{y}_0 \neq 0 \), since they are continuous, as we can infer from (3.4). One way to solve the problem of particle localization is to invert the brane tension \( k \to -k \) [20], but then the gravity is no more localized [12].

### 3.2 Effective potential method

In references [21, 22] the motion of free particle from the point of view of conserved quantities is discussed. Where in [22] the motion is evaluated from equations of motion in which an affine transformation which takes the particle is performed to a hypersurface \( y = Y \) and we are interested on the brane \((Y = 0)\), where the equation of motion can be written as Minkowski spacetime, leading thus to a conserved quantity. In [21], the particle motion along extra dimension is evaluated, from the standard action \( S = M_0 \int (-g_{AB} \frac{dx^A}{dt} \frac{dx^B}{dt})^{\frac{1}{2}} d\theta \) with the constraint \( g_{AB} \frac{dx^A}{dt} \frac{dx^B}{dt} = -\epsilon \), where \( \epsilon \) assume the values \(-1, 0, 1\) representing tachions, light and causal particles, respectively. But this action is not well defined for \( M_0 = 0 \). We fix this by using the action in the form (3.1) and the constraint (3.5).

So, evaluating the equation (3.9) in \( y = 0 \), we obtain the equation \( \alpha = \left( \frac{dy}{d\theta} \right)^2 |_{y=0} + M_0^2 \), where \( \left( \frac{dy}{d\theta} \right)^2 |_{y=0} \) can be interpreted as initial kinetic energy of the particle along the extra dimension, when the particle was initially on the brane. Then, replacing this result in (3.9) and using the metric (2.2), we obtain
\[ e^{2a} \left( \frac{dy}{d\theta} \right)^2 |_{y=0} - M_0^2(e^{2a} - 1), \] (3.16)
that is the equation of the total energy of the particle in the extra dimension where the last term acts as a effective potential. The particle can move in the region \( \left( \frac{dy}{d\theta} \right)^2 |_{y=0} - M_0^2(e^{2a} - 1) \geq 0 \), being \( \left( \frac{dy}{d\theta} \right)^2 |_{y=0} \) the maximum value of the effective potential
\[ V_{eff}(y_{max}) = \left( \frac{dy}{d\theta} \right)^2 |_{y=0}. \] (3.17)
The equation (3.16), therefore, represents the motion of free particle under influence of an effective potential

\[ V_{\text{eff}} = M_0^2(e^{2a} - 1), \]  

(3.18)

except for massless particles, since for \( M_0^2 = 0 \), those are not affected by the effective potential and stay on the brane if \( \dot{y}_0 = 0 \), otherwise the particle is expelled from the brane because the velocity of the massless test particle obey, by equation (3.16), the relation

\[ e^{2a} \left( \frac{dy}{d\theta} \right)^2 = \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0}, \]  

(3.19)

and in the limit \( y \to \infty \), \( \left( \frac{dy}{d\theta} \right)^2 \to \infty \), that is, the test particle does not return.

To investigate the motion of a test particle on the brane, in which we will see that the motion isn’t limited by a hypersurface \( \Sigma_0 \), we must analyze the first and second derivatives of the effective potential around \( y = 0 \) and \( M_0^2 \neq 0 \). The first derivative is

\[ V'_{\text{eff}} = 2M_0^2 a' e^{2a}. \]

One necessary condition for confinement is to have \( y = 0 \) as a critical point, that way it is required that \( a'(0) = 0 \). Thus, \( a(y) \) must satisfy an initial condition \( a(0) = a'(0) = 0 \). Once the two conditions are satisfied, the second derivative of the effective potential provides us information about stability of the critical point, being \( V''_{\text{eff}}(0) = 2M_0^2 a''(0) \). And according to equations (2.5) and (2.14), \( a''(y) \leq 0 \) for all \( y \). Therefore, we can evaluate the particle stability according to the value of \( M_0^2 \), being the point \( y = 0 \) an maximum point for causal particles \( (M_0^2 > 0) \) and minimum point for tachions \( (M_0^2 < 0) \).

For the thick brane warp factor defined in (2.14), we have

\[ V_{\text{eff}} = M_0^2 \left( \frac{\text{sech}^4 \beta (cy)}{e^{\beta \tanh^2 (cy)}} - 1 \right). \]

The tachions are stably confined, while causal particles are not, that is, the gravity alone is not sufficient to confine causal particles, as shown in the figure below.

Then, it is necessary to introduce another mechanism for test particles to be stably confined on the brane. In [21], the author proposed a mechanism where the particle is coupled non-minimally with the scalar field to solve the problem of the confinement.

### 3.3 Particle coupling with scalar field

The localization of general fields in the Randall-Sundrum model is only possible by implementation of specific mechanism which explain the field localization on the brane. As an example, in quantum regime, the Dirac spinor field can be trapped on the brane by a Yukawa-type interaction with the scalar field [28]. The action is expressed by

\[ S_\psi = \int dyd^4x (i \bar{\psi} \Gamma^4 \partial_A \psi - h \phi \bar{\psi} \psi). \]  

(3.20)

When the classical domain wall solution \( \phi_B \) is considered, the field equation for \( \psi \) is

\[ i\Gamma^4 \partial_A \psi - h \phi_B \psi = 0, \]  

(3.21)
by analogy to the Dirac equation, we note that the scalar field generates mass for the five-dimensional fermion. Admitting that the five-dimensional fermion possesses an rest mass $M_0$, due to the Yukawa interaction the following relation is obtained

$$P_A P^A = -(M_0^2 + h^2 \phi^2),$$

(3.22)

where $P_A$ represents the 5D-momentum of the fermion. So, assuming that this mechanism can emerge in a classical picture of test particles, the confinement is made possible by allowing an direct interaction between the test particle and scalar field [21]. The interaction is proposed by redefinition of the action (3.1) modifying the mass through the Yukawa interaction

$$M_0 \to (M_0^2 + h^2 \phi^2)^{\frac{1}{2}},$$

(3.23)

then the new action for the test particle assumes the form

$$S = \frac{1}{2} \int \left( -g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + M_0^2 + h^2 \phi^2 \right) d\theta,$$

(3.24)

where we choose $E = 1$. For the action (3.24), we obtain the equations of motion,

$$\frac{d^2 y}{d\theta^2} + \alpha \left( M_0^2 + h^2 \phi^2 + \left( \frac{dy}{d\theta} \right)^2 \right) + h^2 \phi \phi' = 0,$$

(3.25)

where the constraint $g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} = -(M_0^2 + h^2 \phi^2)$ was used, that is, for massive test particles leading to non-zero bare mass, but if $M_0 = 0$ in $y = 0$ we have a massless test particle, however this not a problem anymore.
Multiplying (3.25) by the factor $e^{2a \frac{dy}{d\theta}}$, the first integral can be obtained, then we obtain a conserved quantity
\[
\left[ \left( \frac{dy}{d\theta} \right)^2 + M^2_0 + h^2 \phi^2 \right] e^{2a} = \sigma. \tag{3.26}
\]
Evaluating in $y = 0$ we can rearrange it and obtain an energy equation in the direction of the extra dimension as below
\[
e^{2a} \left( \frac{dy}{d\theta} \right)^2 = \left( \frac{dy}{d\theta} \right)^2 \big|_{y=0} - \left[ M^2_0 (e^{2a} - 1) + e^{2a} h^2 \phi^2 \right], \tag{3.27}\]
the last term is the effective potential
\[
V_{\text{eff}}(y) = \left[ e^{2a} (M^2_0 + h^2 \phi^2) - M^2_0 \right], \tag{3.28}\]
which reduces to the free case (3.18), to $M^2_0 \neq 0$, when $h = 0$. We can obtain the extreme points of the effective potential by evaluating its derivatives, as in the previous section. Taking the first derivative of the effective potential we obtain
\[
V'_{\text{eff}}(y) = 2e^{2a} \left[ a' (M^2_0 + h^2 \phi^2) + h^2 \phi \phi' \right]. \tag{3.29}\]
The real values for $y$ that must satisfy $V'_{\text{eff}}(y) = 0$ are
\[
y = 0; \tag{3.30}\]
\[
|y_{\text{max}}| = \frac{1}{c} \text{arctanh} \left[ \left( \frac{3}{2} - \frac{M^2_0}{2h^2v^2} + \frac{1}{2\beta} \right. \right.
\nonumber\]
\[
+ \left. \left. \sqrt{-4h^2v^2 \beta (h^2v^2 - 3M^2_0 \beta) + (-h^2v^2 + M^2_0 \beta - 3h^2v^2 \beta)^2} \right) \right] \tag{3.31}\]

that represents the extreme points of the function. To evaluate the conditions for stable confinement of the test particle, we must evaluate the extreme points in $V''_{\text{eff}}(y)$,
\[
V''_{\text{eff}}(y) = 2e^{2a} \left\{ \phi'^2 \left[ h^2 - \frac{1}{12M^2_P} \left( M^2_0 + h^2 \phi^2 \right) \right] + 2h^2 \phi \phi' + h^2 \phi'' \right\} \nonumber\]
\[
+ 2a V'_{\text{eff}}(y), \tag{3.32}\]
where the relationship (2.13) has been used. We obtain one condition for the mass particle, so that $y = 0$ is a minimum point [21].
\[
M^2_0 < 12M^2_P h^2. \tag{3.34}\]
The point $y = 0$ will represent a stable equilibrium point when the mass in the bulk satisfies inequality (3.34). However, the author does not evaluate the mass of the test particle on the brane.

If the particle acquires velocity related to the extra dimension, the kinetic energy of the particle along the extra dimension should imply an increase in the effective mass, so the
motion of the particle in the extra dimension is characterized by the increasing of the mass. We can thus analyze the effective mass of the particle in RS-II models, and then evaluate the observable values. In the next section we will develop an mechanism showing we can determine either the particle can be confined just by knowing the function describing the particle mass in the bulk.

Figure 2. Confining behavior of the effective potential $V_{\text{eff}}$ when the interaction is taken into account (thin line) for a particular choice of the constants ($c = 1, v = 6, M_{P}^{(3)} = \sqrt{2}$ and $M_0^2 = 3$), compared to the case when there is no interaction (tick line), i.e. for $h = 0$. 
4 General conditions to confinement and effective mass of test particles in braneworld

As we said in the previous section, we need an expression that gives us the conditions to confine the test particle and the effective mass observed on the brane. In this direction the authors of [22] define the effective mass through the constraint \( g_{AB} \frac{dx^A}{dt} \frac{dx^B}{dt} + 1 = 0 \), but they do not find a general expression that provides it. In this section we will develop and extend the work [22] in order to obtain a general expression for the effective mass of test particles observed on the brane.

Different from what occur with fields, where the field localization is based in the Kaluza-Klein modes [3], with particles we obtain the effective mass of the particle by an affine parametrization in the motion parameter \( \theta \) to other parameter \( \lambda \). Such motion is observed on the hypersurface at \( y = 0 \), called brane. And this model represents our world [22]. Then, we need to find an affine transformation that allows us to write the equation of motion (3.3)

\[
\frac{d^2x^\mu}{d\lambda^2} = 0, \tag{4.1}
\]

Therefore, on the brane \( y = 0 \) we want the Minkowsky metric

\[
\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + m_{eff}^2 = 0, \tag{4.2}
\]

where \( m_{eff}^2 \) is the effective mass of the test particle on the brane. Then, by parametrization (3.5) we obtain the equation

\[
\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -e^{-2a} \left( \frac{d\theta}{d\lambda} \right)^2 \left[ M^2(y) + g_{55} \left( \frac{dy}{d\theta} \right)^2 \right] , \tag{4.3}
\]

and the action (3.1) is invariant under reparametrization \( E(\lambda) = \frac{d\theta}{d\lambda} E(\theta) \) for the new variable, and by constraints (3.5) and (4.2) we choose \( E(\theta) = E(\lambda) = 1 \), therefore the relation under the parameters is given by \( \left( \frac{d\theta}{d\lambda} \right)^2 = 1 \). Using the parametrization (4.2), the effective mass of the test particle on the brane is

\[
m_{eff}^2 = M^2(0) + \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} . \tag{4.4}
\]

So, by equation (4.4), the effective mass on the brane depends of the initial velocity of the test particle in the extra dimension direction. But the initial velocity of the test particle has a limit which should depends on the effective potential, as view in section 3. The effective potential for a free test particle (3.18) has an unstably equilibrium point on the brane \( y = 0 \) and therefore the test particle is not confined (see fig. 2, for \( h = 0 \)). Their effective mass can be cross the brane, but it is not confined.

For the case where the test particle is coupled with the scalar field, the effective potential (3.28) has a stable equilibrium point. Therefore is confined by an condition under the mass of the test particle in the bulk (3.34) and the initial velocity of the test particle has a limit not to exceed the maximum point of the effective potential (see fig. 2, for \( h = \sqrt{3} \)). We will see below that all this information can be obtained from the mass of the test particle in the bulk.
4.1 Confinement of the test particles in braneworld

In order to analyze the confinement of the most general mass of test particles, let’s assume in action (3.1) a test particle with mass \( M(y) \) where the bare mass of the test particle is coupled with any field dependent of the extra dimension. Thus, the conjugated momentum in 5-D dimensional spacetime is \( P_A P^A = g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} = -M(y)^2 \). In all cases studied here the equation for conservation of energy along the extra dimension can be written as

\[
e^{2a} \left( \frac{dy}{d\theta} \right)^2 = \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} - V_{eff}(y),
\]

(4.5)

which is different from the result obtained in the work [21], but both are equivalent, with the advantage of the potential obtained by the action (3.1) with \( E(\theta) = 1 \) does not have \( M_0^2 \) as singularity. Then, for a mass of the test particle in the bulk \( M(y) \), a conserved quantity as (4.5) is given by

\[
e^{2a} \left( \frac{dy}{d\theta} \right)^2 = \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} - \left[ e^{2a}M^2(y) - M^2(0) \right],
\]

(4.6)

where

\[
V_{eff}(y) = e^{2a}M^2(y) - M^2(0),
\]

(4.7)

is the effective potential. The confinement of the test particle is granted on the brane if this condition are satisfied: (i) \( V'_{eff}(y)|_{y=0} = 0 \) and (ii) \( V''_{eff}(y)|_{y=0} > 0 \). Then, the first derivative of effective potential

\[
V'_{eff}(y) = 2(a'M^2 + MM')e^{2a},
\]

(4.8)

when evaluated in \( y = 0 \), the condition (i) is satisfied if \( M'(0) = 0 \), once that \( a'(0) = 0 \). Taking the second derivative of the effective potential

\[
V''_{eff}(y) = 2(a''M^2 + 2a'M'M + M'^2 + MM'')e^{2a} + 2a''V'_{eff}(y),
\]

(4.9)

when evaluated in \( y = 0 \) the condition (ii) is satisfied if the following inequality is satisfied

\[
M(0)M''(0) + M^2(0)a''(0) > 0.
\]

(4.10)

The confinement of test particle on the brane depends only on its mass in the bulk, that is, the field which is coupled with the bare mass of the test particle. Therefore, we can obtain how the effective mass of the test particle is observed on the brane.

4.2 Effective mass

According to equation (4.5), the particle can move in the region where \( \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} - V_{eff}(y) \geq 0 \), therefore the maximum distance that the geodesic can reach \( (y = y_{max}) \) is given by the condition \( \left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} = V_{eff}(y_{max}) \) and replacing (4.7), the maximum initial velocity of the particle is

\[
\left( \frac{dy}{d\theta} \right)^2 \bigg|_{y=0} = e^{2a(y_{max})}M^2(y_{max}) - M^2(0).
\]

(4.11)
Then, assuming that the condition (4.10) is satisfied, we can replace (4.11) in the equation (4.4) and obtain the maximum effective mass of the test particle observed on the brane, which is given by

$$m_{\text{eff}}^2 = e^{2a(y_{\text{max}})} M^2(y_{\text{max}}). \quad (4.12)$$

This result agree with the result obtained in the work [5], where any mass parameter $m_0^2$ on the visible 3-brane in the fundamental higher-dimensional theory will correspond to a physical mass

$$M^2 = e^{2a} m_0^2. \quad (4.13)$$

Let’s now analyze the effective masses of test particles studied in the literature:

- Free test particle;

The case of the free particle with bare mass given by $M(y) = M_0$, we obtain from (4.10) that

$$M_0^2 a''(0) > 0, \quad (4.14)$$

but $a''(y) < 0$ for all $y$ (2.13). Then, the free massive particle is not confined, as view in 3.2 where the free test particle is subject to an effective potential $V_{\text{eff}} = M_0^2(e^{2a} - 1)$. So, $y = 0$ is an unstable equilibrium point (see fig 1). For a bare massless test particle the equation (4.10) is identically null implying $y = 0$ also, a unstable equilibrium point.

Therefore, free particles are not confined on the brane. Which agrees with the conclusion of [20, 21], where gravity alone is not able to confine test particles. Therefore, it’s required the coupling of the test particle with some field which depends of the extra dimension.

- Test particle coupled with scalar field:

For the test particle coupled with scalar field (3.22), its mass is given by $M^2(y) = M_0^2 + h^2 \phi^2$. Replacing it in the inequality (4.10) we obtain

$$h^2 - \frac{M_0^2}{12M_P^2} > 0, \quad (4.15)$$

and we obtain the same condition (3.34):

$$M_0^2 < 12M_P^2 h^2,$$

for non-null bare mass, $M_0^2 \neq 0$ and

$$h^2 > 0 \quad (4.16)$$

for null bare mass, $M_0^2 = 0$.

Then, a test particle in the bulk with non-null bare mass, $M_0 \neq 0$, is observed on the brane with effective mass (4.12) given by

$$m_{\text{eff}}^2 < [12M_P^2 + \phi^2(y_{\text{max}})] h^2 e^{2a(y_{\text{max}})}; \quad (4.17)$$
where $y_{\text{max}}$ is given by (3.32) and represents the maximum point to the effective potential (3.28) which is the point where the equation (3.17) is satisfied (see fig 2 for $h = \sqrt{3}$). Then if the particle has mass greater than that of the inequalities above, the particle can not be stably confined. And the greater the value of the maximum effective potential, the greater the range of the mass of the particle observed on the brane.

In the case of test particles with null bare mass, $M_0 = 0$, the effective mass on the brane is obtained by taking the limit when $M_0^2 \to 0$ in the equation (4.17) and condition (4.16), so we have

$$m_{\text{eff}}^2 < \hbar^2 \phi^2(y_{\text{max}});$$  

(4.18)

but $\phi(0) = 0$, therefore, on the brane, the test particle is observed as a massless test particle

$$m_{\text{eff}}^2 = 0.$$  

(4.19)

Besides that, the canonical momentum in the bulk (3.22), for $M_0^2 = 0$ is given by $P^A P_A = -\hbar^2 \phi^2$ is null on the brane and agrees with the results obtained in [21], and is non null out of the brane. In the work [21], the author does not obtain the value of the effective mass observed on the brane.
5 Confinement of test particles non minimally coupled with the dilaton

In the section (2.3), we have seen that this field acts in such a way that it modifies the space-time metric, allowing us to choose an appropriate field of the form \( \pi(y) = -\sqrt{3}M_p^3a(y) \) and \( b(y) = \frac{1}{4}a(y) \) such that the equation (2.13) remains unchanged in the new scenario. We emphasize that the free particle in this new scenario will also not be confined.

To solve this, we propose the test particle non-minimally coupled with the dilation field, \( M_0^2 \rightarrow e^{2\lambda \pi}M_0^2 \), (5.1)
whose conjugated momentum becomes \( P_A\dot{P}^A = -M_0^2e^{2\lambda \pi(y)} \), it makes possible to confine the particle on the brane.

5.1 Dilaton coupling in deformed RS spacetime

As shown in the introduction, the action for the test particle (3.1), for \( M^2(y) = e^{2\lambda \pi}M_0^2 \), can be write as:
\[
S = \frac{1}{2} \int \left( -g_{AB} \frac{dx^A}{d\theta} \frac{dx^B}{d\theta} + M_0^2 e^{2\lambda \pi} \right) d\theta, \tag{5.2}
\]
where we choose the new variable \( E(\theta) = 1 \), as saw previously. Using the condition for confinement of the test particle (4.10) and replacing the mass of this test particle given by (5.1), we obtain the inequality
\[
M_0^2(-\lambda \sqrt{3M_p^{(3)}} + 1)a''(0) \geq 0. \tag{5.3}
\]
But as \( a'' < 0 \) for all \( y \), the inequality above is satisfied if
\[
\lambda \geq \frac{1}{\sqrt{3M_p^{(3)}}}, \tag{5.4}
\]
for a non-null bare mass of the test particle in the bulk, \( M_0^2 \neq 0 \). For the case where the bare mass of the test particle is null, \( M_0^2 = 0 \), the inequality (5.3) is not satisfied, therefore the massless test particle is not confined.

Then, a test particle in the bulk with non-null bare mass, \( M_0 \neq 0 \), is observed on the brane with maximum effective mass (4.12) given by
\[
m_{eff}^2 = M_0^2 e^{2a(y_{\text{max}})(1-\lambda \sqrt{3M_p^{(3)}})}; \tag{5.5}
\]
where \( y_{\text{max}} \) can be obtained analyzing the effective potential equation (4.7), that is given by
\[
V_{eff}(y) = M_0^2 (e^{2a(1-\lambda \sqrt{3M_p^{(3)}})} - 1), \tag{5.6}
\]
where the case for a free test particle is obtained when we take \( \lambda = 0 \). However, if the relation (5.4) is satisfied. We analyze the second derivative of the effective potential (5.3)
\[
V_{eff}''(y) = 2(-\sqrt{3M_p^{(3)}} \lambda + 1)a'' e^{2(\lambda \pi + a)} + 4(\lambda \pi' + a')^2 e^{2(\lambda \pi + a)}.
\]
We note that the first term is positive, once that \((\phi')^2 = -12M_P^3a''\) for all values of \(y\) and the second term of the expression above is always positive, that is, the second derivative of the effective potential (5.3) is positive for all values of \(y\) implying in a maximum potential with \(y_{max} \to \infty\) and \(V_{eff}(y_{max}) \to \infty\). Therefore, the effective mass of the free particle coupled with the dilaton field on the brane can assume any value,

\[
m_{eff}^2 < \infty.
\]  

(5.7)

Besides there is no restrictions on the mass of the test particle in the bulk, and is valid to deformed brane for any value of \(s\). We are able to solve the problem of the limited mass in [21], but paying the price that light type particles are not stably confined on the brane.

**Figure 3.** Confining behavior of the deformed effective potential \(V_{eff}(y)\) when the interaction is take into account (thin line) for a particular choice of the constants \((M_0^2 = 3, b = \frac{1}{2}, c = 1, M_P^3 = 0.3, \lambda = 1.5)\), compared to the case when there is no interaction (tick line), i.e. for \(\lambda = 0\).
6 Conclusion

We revised the motion of test particles on the brane world RS-II type delta, thick and deformed models. It is well known that, in the models above, the gravity alone is not enough to trap a free particle if it is transversally disturbed [20]. Similarly, with some fields, where it is necessary the development of mechanisms that make it possible to trap the field on the brane.

For many cases, the most simple mechanism is to reverse the brane tension, including the case of test particle. The crucial problem in reversing the brane tension is the fact that gravity is not localized [12], which is not physically acceptable. Therefore, the confinement of test particles in models with type delta branes has not been solved. In the case of test particles in smooth models, a solution was proposed based in the Yukawa interaction, a mechanism that solves the localization problem of Dirac spinor field on the brane through an interaction between the spinor and the scalar field [21]. In this interaction the mass of the spinor becomes a function of the scalar field. Based on this idea and assuming this picture in classical level, a new Lagrangian of test particle is defined, assuming that mass depends of the scalar field in the form

\[ M_0 \rightarrow \left( M_0^2 + \hbar^2 \phi^2 \right)^{1/2}, \]

the mechanism allows the confinement of test particles with mass that obey the relation

\[ M_0^2 < 12M_P^{(3)} \hbar^2. \]

We further develop the result obtained in work [21] where we obtain the effective mass of the particle observed on the brane and we use a polynomial action for test particle (3.1), allowing to evaluate without problems as massless test particles are observed on the brane. By moving in the extra direction, part of the total kinetic energy that was initially all concentrated on the brane, is now converted into the energy of the extra dimension, being characterized by the gain of effective mass of the particle with maximum value given by

\[ m_{eff}^2 = M_0^2 e^{2a(y_{max})}. \]

The maximum values to the mass in the bulk \( M_0^2 < 12M_P^{(3)} \hbar^2 \) that obey the condition \( M(0)M''(0) + M(0)^2 a''(0) > 0 \) and guarantees the confinement of the test particle. Thus, the effective mass observed on the brane is

\[ m_{eff}^2 < [12M_P^{(3)} + \phi^2(y_{max})] \hbar^2 e^{2a(y_{max})} \] for a non-null bare mass and \( m_{eff}^2 < \hbar^2 \phi^2(y_{max}) \) for a null bare mass of the test particle.

Finally, we show how to solve all problems relative to the confinement of the massive test particle on the brane. We propose the coupling between the dilaton field and the mass of the test particle in the bulk as given by \( M_0 \rightarrow e^{\lambda \pi} M_0 \) making possible the confinement, because the new mass obeys the relationship \( M(0)M''(0) + M(0)^2 a''(0) > 0 \) if \( \lambda > \frac{1}{\sqrt{3M_0}}. \)

The condition to \( \lambda \) is also obtained evaluating the effective potential that acts on the test particle

\[ V_{eff} = e^{2(\lambda \pi + a)} - 1 \] such that its second derivative is positive at \( y = 0 \) ensuring a minimum point at this point. Taking the second derivative of the effective potential

\[ V_{eff}''(y) = 2(-\sqrt{3M_0^3} \lambda + 1)a'' e^{2(\lambda \pi + a)} + 4(\lambda \pi' + a')^2 e^{2(\lambda \pi + a)}, \]

we note that the second term is always positive and if the condition for \( \lambda \) is satisfied and knowing that, by equation \( (\phi')^2 = -12M_P^{(3)} a'', \) \( a'' < 0 \) for every \( y \) then the maximum point for the effective potential is given in \( y_{max} \rightarrow \infty \) (see fig 3). The effective mass of the test particle observed on the brane is given, as we have seen, by

\[ m_{eff}^2 = M(0)^2 [1 + \ldots]. \]
At the limit where \( y_{\text{max}} \to \infty \) we have that the effective potential diverges, \( V_{\text{eff}}(y_{\text{max}}) \to \infty \). Therefore the effective mass of the particle observed on the brane is

\[
m_{\text{eff}} < \infty.
\]

The problem is the confinement of massless test particles on the brane, whose confinement is not achieved. Without loss of generality, for the case of the topological defect we verified that the test particles coupled with dilaton are also located for any mass.

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