Anick resolution and Koszul algebras of finite global dimension

Vladimir Dotsenko and Soutrik Roy Chowdhury

School of Mathematics, Trinity College, Dublin, Ireland; Departamento de Matemáticas, CINVESTAV-IPN, México D.F., Mexico

ABSTRACT
We show how to study a certain associative algebra recently discovered by Iyudu and Shkarin using the Anick resolution. This algebra is a counterexample to the conjecture of Positselski on Koszul algebras of finite global dimension.

ARTICLE HISTORY
Received 9 August 2016
Communicated by E. Kirkman

KEYWORDS
Anick resolution; global dimension; Koszul duality; Koszulness

2010 MATHEMATICS SUBJECT CLASSIFICATION
Primary: 16S37; Secondary: 13P10, 16E05, 16Z05, 18G10

1. Introduction

In the definitive treatment of quadratic algebras, the monograph [8], the following conjecture of Positselski is recorded:

Conjecture (Conjecture 2 in [8, Chapter 7]). Any Koszul algebra $A$ of finite global homological dimension $d$ has $\dim A_1 \geq d$. Dually, for a Koszul algebra $B$ with $B_{d+1} = 0$ and $B_d \neq 0$, one has $\dim B_1 \geq d$.

In a recent preprint of Iyudu and Shkarin [7] where a classification result for Hilbert series of Koszul algebras with three generators and three relations is proposed, one of the algebras from the classification theorem provides a counterexample to this conjecture. Namely, the following result holds.

Theorem. Consider the following algebra with three generators and three quadratic relations:

$$A = k\langle x, y, z \mid x^2 + yx, xz, zy \rangle.$$  

The algebra $A$ is Koszul. Its global dimension is equal to 4. Hence, the conjecture above is false.

The proof of Koszulness of the algebra $A$ in [7, Proposition 4.2] contains a typo of a sort: what is claimed to be the reduced Gröbner basis for $A$ is not really a Gröbner basis. The strategy of [7] appears to be valid if one uses the correct reduced Gröbner basis; however, we would like to present a completely different proof of this result which illustrates the power of the Anick resolution for associative algebras [1].

Proof. Throughout this proof, we use the degree-lexicographic ordering of variables with $x > y > z$. It turns out that the reduced Gröbner basis of $A$ for this ordering, though infinite, has a very simple and pleasant description. Namely, it consists of the following polynomials:

$$xy^k x + y^{k+1} x (k \geq 0), xz, zy.$$
To establish that, we use the Diamond Lemma [3]. First of all, we note that the ideal generated by the relation $x^2 + xy$ alone already contains all the elements $xy^k x + y^{k+1} x$ above, and that the overlaps of their leading terms give S-polynomials that can be reduced to zero modulo these elements; it is a computation essentially identical to the one of [4, Example 3.6.2]. Finally, we note that the S-polynomial of $xy^k x + y^{k+1} x$ and $xz$ is

$$(xy^k x + y^{k+1} x)z - xy^k(xz) = y^{k+1} x z,$$

which is divisible by $xz$ and hence can be reduced to zero, and the S-polynomial of $xz$ and $zy$ is actually equal to zero, as those relations are monomial.

Furthermore, it turns out that the Anick resolution [1] for the Gröbner basis we computed is surprisingly manageable. We shall use the description of that resolution due to Ufnarovski [10, Section 3.6], which we recall here for the sake of completeness. That definition recursively defines Anick chains and their tails for any algebra $A = \mathbb{k}(X \mid R)$, where the set of defining relations is a Gröbner basis:

- elements of $X$ are the only 0-chains, each of them coincides with its tail;
- each $q$-chain is word $c$ in the alphabet $X$ which is the concatenation $c't$, where $t$ is the tail of $c$, and $c'$ is a $(q - 1)$-chain;
- if we denote by $t'$ the tail of $c'$ in the above decomposition, there exists a “factorization” $t' = m_1 m_2$ with $m_2 t$ is the leading term of an element of $R$, and there are no other divisors of $t't$ that are leading terms of $R$.

In our case, a direct inspection of the leading terms $xy^k x, xz, zy$ of the reduced Gröbner basis we found instantly produces the list of all Anick chains as follows.

- the set of 0-chains $C_0$ consists of $x, y, z$;
- the set of 1-chains $C_1$ consists of $xy^k(x(k \geq 0), xz, zy$;
- for $n \geq 2$, the set of $n$-chains $C_n$ consists of $xy^{k_1} x \cdots xy^{k_n} x(k_1, \ldots, k_n \geq 0), xy^{k_1} x \cdots xy^{k_n-1} xz(k_1, \ldots, k_{n-1} \geq 0), xy^{k_1} x \cdots xy^{k_{n-2}} zy(k_1, \ldots, k_{n-2} \geq 0)$.

It is known [1, 10] that there exists a resolution of the (right) augmentation $A$-module $\mathbb{k}$ by free right $A$-modules of the form

$$\cdots \to \mathbb{k} C_n \otimes A \to \mathbb{k} C_{n-1} \otimes A \to \cdots \to \mathbb{k} C_0 \otimes A \to A \to 0.$$

The general recipe for the computation of the differential of that resolution in our case gives the following results. First, the formulas for $d_0$ and $d_1$ are standard:

$$d_0(x \otimes 1) = x, \quad d_0(y \otimes 1) = y, \quad d_0(z \otimes 1) = z,$$

$$d_1(xy^k x \otimes 1) = x \otimes y^k x + y \otimes y^k x, \quad d_1(xz \otimes 1) = x \otimes z,$$

$$d_1(zy \otimes 1) = z \otimes y.$$

Computing $d_n$ with $n \geq 2$ is very similar to the computation of [4, Proposition 4.3.1]. Namely, the following formulas hold:

$$d_n(xy^{k_1} x \cdots xy^{k_n} x \otimes 1)$$

$$= xy^{k_1} x \cdots xy^{k_{n-1}} x \otimes y^{k_n} x + \sum_{i=1}^{n-1} (-1)^{n-1-i} xy^{k_1} x \cdots xy^{k_{i-1}} x y^{k_i+k_{i+1}+1} x y^{k_{i+2}} x \cdots xy^{k_n} x \otimes 1,$$

$$d_n(xy^{k_1} x \cdots xy^{k_{n-1}} xz \otimes 1)$$

$$= xy^{k_1} x \cdots xy^{k_{n-1}} x \otimes z + \sum_{i=1}^{n-2} (-1)^{n-2-i} xy^{k_1} x \cdots xy^{k_{i-1}} x y^{k_i+k_{i+1}+1} x y^{k_{i+2}} x \cdots xy^{k_{n-1}} xz \otimes 1,$$

$$d_n(xy^{k_1} x \cdots xy^{k_{n-2}} zy \otimes 1)$$

$$= xy^{k_1} x \cdots xy^{k_{n-2}} zy \otimes y + \sum_{i=1}^{n-3} (-1)^{n-3-i} xy^{k_1} x \cdots xy^{k_{i-1}} x y^{k_i+k_{i+1}+1} x y^{k_{i+2}} x \cdots xy^{k_{n-2}} zy \otimes 1.$$
To compute the bar homology of $A$, or equivalently $\text{Tor}^A_*(k, k)$, we should compute the homology of the complex

$$\left\{ \left( kC_n \otimes A \right) \otimes_A k, \bar{d}_n \right\},$$

where the induced differential $\bar{d}_n = d_n \otimes 1$ only retains the terms that are not annihilated by the augmentation of $A$. In other words, under the identification $(kC_n \otimes A) \otimes_A k \cong kC_n$, we have

$$\bar{d}_0(x) = 0, \quad \bar{d}_0(y) = 0, \quad \bar{d}_0(z) = 0,$$

$$\bar{d}_1(xy^k x) = 0, \quad \bar{d}_1(xz) = 0, \quad \bar{d}_1(zy) = 0,$$

$$\bar{d}_n(xy^k x \cdots y^k x) = \sum_{i=1}^{n-1} (-1)^{n-1-i} xy^{k_1} x \cdots xy^{k_{i-1}} xy^{k_i+k_{i+1}+1} xy^{k_{i+2} x} \cdots xy^{k_n x},$$

$$\bar{d}_n(xy^k x \cdots y^{k_{n-1}} xz) = \sum_{i=1}^{n-2} (-1)^{n-2-i} xy^{k_1} x \cdots xy^{k_{i-1}} xy^{k_i+k_{i+1}+1} xy^{k_{i+2} x} \cdots xy^{k_{n-1} xz},$$

$$\bar{d}_n(xy^k x \cdots y^{k_{n-2}} xzy) = \sum_{i=1}^{n-3} (-1)^{n-3-i} xy^{k_1} x \cdots xy^{k_{i-1}} xy^{k_i+k_{i+1}+1} xy^{k_{i+2} x} \cdots xy^{k_{n-2} xzy}.$$

These formulas show that the chain complex we are dealing with decomposes as a direct sum of the subcomplex

$$0 \to k[x^2 y] \to k[x^2 z, xzy] \to k[x^2, xz, zy] \to k[x, y, z] \to k \to 0$$

which has zero differential, the subcomplexes $U^p_\bullet$, $p \geq 1$, spanned by the elements $xy^{k_1} x \cdots y^{k_n} x$, $n \geq 1$, the subcomplexes $V^p_\bullet$, $p \geq 1$, spanned by the elements $xy^{k_1} x \cdots y^{k_{n-1}} xz$, $n \geq 2$, and the subcomplexes $W^p_\bullet$, $p \geq 1$, spanned by the elements $xy^{k_1} x \cdots y^{k_{n-2}} xzy$, $n \geq 3$; in each of these cases, the superscript $(p)$ imposes the constraint $\sum (k_i + 1) = p + 1$. We note that $V^p_\bullet \cong U^p_{p+1}$ and $W^p_\bullet \cong U^p_{p+2}$, so if we prove that each subcomplex $U^p_\bullet$, $p \geq 1$, is acyclic, this will prove that all other complexes we consider are acyclic as well.

Note that the subcomplex $U^p_\bullet$ appears when computing the bar homology of the algebra $A' = k(x, y \mid x^2 = yx)$ via the Anick resolution. Indeed, after tensoring the Anick resolution of the augmentation module for $A'$ with the augmentation module, the resulting complex decomposes as a direct sum of the complex

$$0 \to k[x^2] \to k[x, y] \to k \to 0$$

with zero differential and complexes $U^p_\bullet$ with $p \geq 1$. At the same time, one can note that for the algebra $A'$, the degree-lexicographic order with $x < y$ gives a quadratic Gröbner basis, yielding the complex

$$0 \to k[yx] \to k[x, y] \to k \to 0$$

with zero differential which represents the bar homology of $A'$. This implies that $A'$ is Koszul, and that its bar homology for the ordering we are interested in must be represented by the classes of the Anick chains $1, x, y, x^2$. Therefore, all complexes $U^p_\bullet$ with $p \geq 1$ must be acyclic.

It follows that the subcomplex

$$0 \to k[x^2 y] \to k[x^2 z, xzy] \to k[x^2, xz, zy] \to k[x, y, z] \to k \to 0$$

we noted above represents the bar homology of $A$. Thus, the bar homology is concentrated on the diagonal, so $A$ is Koszul. Moreover, its Koszul dual coalgebra vanishes in degrees higher than 4 and
is nonzero in degree 4, which by the classical results on the Koszul duality [2] proves the claim on the global dimension, therefore completing the proof.

Remark 1. The fact that the complexes $U_{(p)}^k$ are acyclic can be also established by identifying them with graded pieces of the Shafarevich complex [5, 6] for the subset $\{t\}$ of $k(t)$, but we chose the argument above to emphasize how one can use the Anick resolutions for two different orderings simultaneously.

Acknowledgments

We are grateful to Leonid Positselski for drawing our attention to the fact that the list of algebras in [7] contains a counterexample to his conjecture. Ivan Yudin informed us that it is also possible to establish the claim on Koszulness by a careful examination of the two Anick resolutions for the orderings $x > y > z$ and $y > x > z$ for the original algebra; his argument utilizes Ufnarovski’s graph for generating Anick chains [9, 10]; we are very thankful for that remark.

References

[1] Anick, D. J. (1986). On the homology of associative algebras. Trans. Am. Math. Soc. 296(2):641–659.
[2] Beilinson, A., Ginsburg, V., Schechtman, V. (1988). Koszul duality. J. Geom. Phys. 5(3):317–350.
[3] Bergman, G. (1978). The diamond lemma for ring theory. Adv. Math. 29(2):178–218.
[4] Chowdhury, S. R. (2015). Gröbner bases: Connecting linear algebra with homological and homotopical algebra. M.Sc. Thesis. Trinity College Dublin, Preprint arXiv:1510.01542.
[5] Golod, E. S. (1988). Standard Bases and Homology. Lecture Notes in Mathematics, Vol. 1352, pp. 88–95.
[6] Golod, E. S., Shafarevich, I. R. (1964). On the class field tower. Izv. Akad. Nauk SSSR Ser. Mat. 28(2):261–272.
[7] Iyudu, N., Shkarin, S. (2016). One question from the Polishchuk and Positselski book on Quadratic algebras. Preprint IHES/M/16/16, \url{http://preprints.ihes.fr/2016/M/M-16-16.pdf} (Last accessed on May 18, 2016).
[8] Polishchuk, A., Positselski, L. (2005). Quadratic Algebras. University Lecture Series, Vol. 37. Providence, RI: AMS.
[9] Ufnarovski, V. A. (1991). On the use of graphs for computing a basis, growth and Hilbert series of associative algebras. Math. USSR – Sbornik 68(2):417–428.
[10] Ufnarovski, V. A. (1995). Combinatorial and asymptotic methods in algebra. In: Algebra, VI. Encyclopaedia of Mathematical Science, Vol. 57. Berlin: Springer, pp. 1–196.