ON CERTAIN PROJECTIONS OF C*-MATRIX ALGEBRAS

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Communicated by T. Loring

Abstract. In 1955, H. Dye defined certain projections of a C*-matrix algebra by

\[ P_{i,j}(a) = (1 + aa^*)^{-1} \otimes E_{i,i} + (1 + aa^*)^{-1}a \otimes E_{i,j} + a^*(1 + aa^*)^{-1}a \otimes E_{j,j}, \]

which was used to show that in the case of factors not of type I_{2n}, the unitary group determines the algebraic type of that factor. We study these projections and we show that in \( M_2(\mathbb{C}) \), the set of such projections includes all the projections. For infinite C*-algebra \( A \), having a system of matrix units, we have \( A \cong M_n(A) \). M. Leen proved that in a simple, purely infinite C*-algebra \( A \), the *-symmetries generate \( U_0(A) \). Assuming \( K_1(A) \) is trivial, we revise Leen’s proof and we use the same construction to show that any unitary close to the unity can be written as a product of eleven *-symmetries, eight of such are of the form \( 1 - 2P_{i,j}(\omega) \), \( \omega \in U(A) \). In simple, unital purely infinite C*-algebras having trivial \( K_1 \)-group, we prove that all \( P_{i,j}(\omega) \) have trivial \( K_0 \)-class. Consequently, we prove that every unitary of \( O_n \) can be written as a finite product of *-symmetries, of which a multiple of eight are conjugate as group elements.

1. Introduction and preliminaries

Let \( A \) be a unital C*-algebra. The set of projections and the group of unitaries of \( A \) are denoted by \( \mathcal{P}(A) \) and \( \mathcal{U}(A) \), respectively. Recall that the C*-matrix algebra over \( A \) which is denoted by \( \mathbb{M}_n(A) \) is the algebra of all \( n \times n \) matrices \( (a_{i,j}) \) over \( A \), with the usual addition, scalar multiplication, and multiplication of matrices and the involution (adjoint) is \( (a_{i,j})^* = (a_{j,i}^*) \). As in Dye’s viewpoint of \( \mathbb{M}_n(A) \), let \( S_n(A) \) denote the direct sum of \( n \) copies of \( A \), considered as a left
A-module. Addition of n-tuples $\bar{x} = (x_1, x_2, \ldots, x_n)$ in $S_n(A)$ is componentwise and $a \in A$ acts on $\bar{x}$ by $a(\bar{x}) = (ax_1, ax_2, \ldots, ax_n)$. Then $S_n(A)$ is a Hilbert $C^*$-algebra module, with the inner product defined by

$$<\bar{x}, \bar{y}> = \sum_{i=1}^{n} x_i y_i^*.$$  

By an $A$-endomorphism $T$ of $S_n(A)$, we mean an additive mapping on $S_n(A)$ which commutes with left multiplication: $a(\bar{x}T) = (a\bar{x})T$. In a familiar way, assign to any $T$ a uniquely determined matrix $(t_{ij})$ over $A$ $(1 \leq i, j \leq n)$ so that $\bar{x}T = (\sum_i x_i t_{i1}, \ldots, \sum_i x_i t_{in})$.

If $p$ is a projection in $M_n(A)$, then $p$ is a mapping on $S_n(A)$ having its range as a sub-module of $S_n(A)$. Then two projections are orthogonal means their sub-module ranges are so. The $C^*$-algebra $M_n(A)$ contains numerous projections. For each $a \in A$ and each pair of indices $i, j(i \neq j), 1 \leq i, j \leq n)$, H. Dye in [7] defined the projection $P_{i,j}(a)$ in $M_n(A)$, whose range consists of all left multiples of the vector with 1 in the $i^{th}$-place, $a$ in the $j^{th}$-place and zeros elsewhere. As a matrix, it has the form

$$P_{i,j}(a) = \begin{pmatrix} 0 & \cdots & 0 & (1 + aa^*)^{-1} & \cdots & (1 + aa^*)^{-1}a & \cdots & 0 \\
0 & \cdots & 0 & a^*(1 + aa^*)^{-1} & \cdots & a^*(1 + aa^*)^{-1}a & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}$$

Recall that (see [7], p.74) a system of matrix units of a unital $C^*$-algebra $A$ is a subset $\{e_{i,j}^r\}, 1 \leq i, j \leq n$ and $1 \leq r \leq m$ of $A$, such that

$$e_{i,j}^r e_{j,k}^r = e_{i,k}^r, \ e_{i,j}^r e_{k,l}^s = 0 \text{ if } r \neq s \text{ or } j \neq k, \ (e_{i,j}^r)^* = e_{j,i}^r, \ \sum_{i,r} e_{i,i}^r = 1$$

and for every $i$, $e_{i,i}^r \in \mathcal{P}(A)$. For the $C^*$-complex matrix algebra $M_n(\mathbb{C})$, let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard system of matrix units of the algebra, that is $E_{i,j}$ is the $n \times n$ matrix over $\mathbb{C}$ with 1 at the place $i \times j$ and zeros elsewhere. It is also known that $M_n(A)$ is $*$-isomorphic to $A \otimes M_n(\mathbb{C})$ (see [11]). We will see that having a system of matrix units is a necessary condition in order that a $C^*$-algebra $A$ is $*$-isomorphic to a $C^*$-matrix algebra $M_n(B)$. Using the notion of a system of matrix units, we write

$$P_{i,j}(a) = (1 + aa^*)^{-1} \otimes E_{i,i} + (1 + aa^*)^{-1}a \otimes E_{i,j} + a^*(1 + aa^*)^{-1} \otimes E_{j,i} + a^*(1 + aa^*)^{-1}a \otimes E_{j,j} \in \mathcal{P}(M_n(A)).$$

If $a = 0$, then $P_{i,j}(a)$ is the $i^{th}$ diagonal matrix unit of $M_n(A)$, which is $1 \otimes E_{i,i}$, or simply $E_i$.

Also in [10], M. Stone called the projection $P_{i,j}(a)$ by the characteristics matrix of $a$. 
H. Dye used these projections as a main tool to prove that an isomorphism between the discrete unitary groups of von Neumann factors not of type $I_n$, is implemented by a $*$-isomorphism between the factors themselves [[7], Theorem 2]. Indeed, let us recall main parts of his proof. Let $A$ and $B$ be two unital $C^*$-algebras and let $\varphi : \mathcal{U}(A) \to \mathcal{U}(B)$ be an isomorphism. As $\varphi$ preserves self-adjoint unitaries, it induces a natural bijection $\theta_\varphi : \mathcal{P}(A) \to \mathcal{P}(B)$ between the sets of projections of $A$ and $B$ given by

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p), \quad p \in \mathcal{P}(A).$$

This mapping is called a projection orthoisomorphism, if it preserves orthogonality, i.e. $pq = 0$ iff $\theta(p)\theta(q) = 0$.

Now, let $\theta$ be an orthoisomorphism from $\mathcal{P}(\mathbb{M}_n(A))$ onto $\mathcal{P}(\mathbb{M}_n(B))$. In [[7], Lemma 8] when $A$ and $B$ are von Neumann algebras, Dye proved that for any unitary $u \in \mathcal{U}(A)$, $\theta(P_{i,j}(u)) = P_{i,j}(v)$, for some unitary $v \in \mathcal{U}(B)$. A similar result is proved in the case of simple, unital $C^*$-algebras by the author in [1]. Afterwards, Dye in [[7], Lemma 6], proved that there exists a $*$-isomorphism (or $*$-antiisomorphism) from $\mathbb{M}_n(A)$ onto $\mathbb{M}_n(B)$ which coincides with $\theta$ on the projections $P_{i,j}(a)$. In fact, he proved that $\theta$ induces the $*$-isomorphism $\phi$ from $A$ onto $B$ defined by the relation $\theta(P_{i,j}(a)) = P_{i,j}(\phi(a))$.

In this paper, we study the projections $P_{i,j}(a)$ of a $C^*$-matrix algebra $\mathbb{M}_n(A)$, for some $C^*$-algebra $A$, and we deduce main results concerning such projections. The paper is organized as follows: In Section 2, we show that every projection in $\mathbb{M}_2(\mathbb{C})$ is of the form $P_{1,2}(a)$, for $a \in \mathbb{C}$. In Section 3, we show that some infinite $C^*$-algebra $A$ is isomorphic to its matrix algebra $\mathbb{M}_n(A)$, such as the Cuntz algebra $\mathcal{O}_n$, so the projections $P_{i,j}(a)$ can be considered as projections of $A$.

In a simple, unital purely infinite $C^*$-algebra $A$, M. Leen proved that self-adjoint unitaries (also called $*$-symmetries, or involutions) generate the connected component $\mathcal{U}_0(A)$ of the unitary group $\mathcal{U}(A)$. In Section 4, assuming in addition that $K_1(A)$ is trivial, we revise Leen’s proof, we fix certain projections and then following the same construction, we show that every unitary which is close to the unity, can be written as a product of eleven $*$-symmetries, eight of which are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$.

Consequently, since every unitary in the connected component of the unity can be written as a finite product of unitaries that are close to the unity (see [11], § 4.2), we have the following result:

**Theorem 1.1.** Let $A$ be a simple, unital purely infinite $C^*$-algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of $A$, with $e_{1,1} \sim 1$. Then every unitary of $A$ can be written as a finite product of $*$-symmetries, of which a multiple of eight have the form $1 - 2P_{i,j}(\omega)$, for some $\omega \in \mathcal{U}(A)$.

Finally in Section 5, we compute the $K_0$-class of such certain projections, and we prove that in simple, unital purely infinite $C^*$-algebras (assuming $K_1 = 0$), all projections of the form $P_{i,j}(u)$, $u \in \mathcal{U}(A)$ have trivial $K_0$-class. As a good
application for $O_n$, we have that every unitary can be written as a finite product of *-symmetries, of which a multiple of eight have the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(O_n)$. Hence using [2] (Lemma 2.1), all such involutions of the form $1 - 2P_{i,j}(\omega)$ are in fact conjugate, as group elements of $\mathcal{U}(O_n)$.

2. The $2 \times 2$-Complex Algebra Case

Let $A$ be a unital $C^*$-algebra, and let $P^n_{i,j}(A)$ denote the family of all projections in $M_n(A)$ of the form $P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Also, let $U^n_{i,j}(A)$ denote the set of all self-adjoint unitaries in $M_n(A)$ of the form $1 - 2P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Notice that $P^n_{i,j}(A)$ contains non-trivial projections. In this small section, we show that in the case of $M_2(\mathbb{C})$, the set $P^2_{i,j}(\mathbb{C})$ includes all the non-trivial projections $P(M_2(\mathbb{C}))$, i.e. every non-trivial projection is of the form $P_{i,j}(a)$, for some complex number $a$.

**Proposition 2.1.** If $p \in P(M_2(\mathbb{C})) \setminus \{0, 1\}$, then $p \in P^2_{i,j}(\mathbb{C})$.

**Proof.** Let $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-trivial projection in $P(M_2(\mathbb{C}))$. Then $a$ and $d$ are real numbers. If $b = 0$, then $p$ is either the diagonal matrix unit $E_{1,1}$ or $E_{2,2}$. Otherwise, we have $a + b = 1, a = a^2 + |b|^2$ and $d = d^2 + |b|^2$, therefore $|b|^2 \leq \frac{1}{4}$. By straightforward computations, one can deduce that $p$ is of the form

$$P_{1,2} \left( \frac{2b}{1 + \sqrt{1 - 4|b|^2}} \right), \text{ or } P_{1,2} \left( \frac{2b}{1 - \sqrt{1 - 4|b|^2}} \right).$$

\[ \square \]

**Remark 2.2.** The projections in $P^2_{i,j}(\mathbb{C})$ are all of rank one by definition, this implies that in the case of $M_3(\mathbb{C})$, the set $P^3_{i,j}(\mathbb{C})$ does not cover all the non-trivial projections. Indeed, there are projections in $P(M_3(\mathbb{C}))$ of rank one which do not belong to $P^3_{i,j}(\mathbb{C})$, since every projection in this latest family projects into a subspace of $\mathbb{C}^3$ which lies entirely in one coordinate plan.

3. Some Results for infinite $C^*$-algebras

Let $A$ be a unital $C^*$-algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$, for some $n \geq 3$. Recall that $e_{1,1}Ae_{1,1}$ is a $C^*$-algebra (corner algebra) which has $e_{1,1}$ as a unit. This system of matrix units implements a *-isomorphism between $A$ and $M_n(e_{1,1}Ae_{1,1})$. Indeed, let us define the mapping

$$\eta_1 : M_n(e_{1,1}Ae_{1,1}) \to A$$

by

$$\eta_1((a_{i,j})^n) = \sum_{i,j=1}^n e_{i,1}a_{i,j}e_{1,1}.$$ 

Moreover if $e_{1,1}$ is equivalent to 1 (i.e. $A$ is assumed to be an infinite $C^*$-algebra), then there exists a partial isometry $v$ of $A$ such that $v^*v = e_{1,1}$ and $vv^* = 1$, and this defines the *-isomorphism $\Delta_v : A \to e_{1,1}Ae_{1,1}$ by $\Delta_v(x) = v^*xv$. The
isomorphism $\Delta_v$ can be used to decompose a projection as a sum of orthogonal equivalent subprojections.

**Proposition 3.1.** Let $A$ be a unital $C^*$-algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. If $p$ is equivalent to the unity, then $p$ can be written as a sum of orthogonal equivalent subprojections.

**Proof.** As $p$ equivalent to 1, we consider the isomorphism $\Delta_v$, then apply it to the equality $1 = \sum_{i=1}^n e_{i,i}$, to get $p = \sum_{i=1}^n v^* e_{i,i} v$. Then $p_i = v^* e_{i,i} v$, for all $1 \leq i \leq n$, are equivalent subprojections of $p$. \hfill $\square$

Recall that, for two unital $C^*$-algebras $A$ and $B$, if $\alpha : A \to B$ is a $*$-isomorphism, then $\alpha$ induces the $*$-isomorphism $\hat{\alpha} : M_n(A) \to M_n(B)$, which is defined by $(a_{i,j}) \mapsto (\alpha(a_{i,j}))$. Then we have the following result.

**Proposition 3.2.** Let $A$ be an infinite unital $C^*$-algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. If $e_{1,1}$ is equivalent to 1, then $M_n(A)$ is $*$-isomorphic to $A$.

**Proof.** Let $\Delta_v : A \to e_{1,1} A e_{1,1}$ and $\eta_1 : M_n(e_{1,1} A e_{1,1}) \to A$ be defined as above. Then the mapping $\eta = \eta_1 \circ \Delta_v$ is a $*$-isomorphism from $M_n(A)$ onto $A$. Moreover,

$$\eta(a_{i,j})^n = \sum_{i,j} e_{i,1} v^* a_{i,j} v e_{1,j}, \text{ and }$$

$$\eta^{-1}(x) = (ve_{1,i} x e_{j,1} v^*)_{i,j}^n.$$ \hfill $\square$

As a main example of purely infinite $C^*$-algebras, let us recall the Cuntz algebra $O_n$; $n \geq 2$, is the universal $C^*$-algebra which is generated by isometries $s_1, s_2, \ldots, s_n$, such that $\sum_{i=1}^n s_i s_i^* = 1$ with $s_i^* s_j = 0$, when $i \neq j$ and $s_i^* s_i = 1$ (for more details, see [5], [[6], p.149]). Let

$$e_{i,j} = s_i s_j^*, \quad 1 \leq i, j \leq n.$$ Then $\{e_{i,j}\}_{i,j=1}^n$ forms a system of matrix units for $O_n$. As $s_i^*$ partial isometry between $e_{1,1}$ and the unity, then Proposition 3.2 shows that the mapping

$$\eta : M_n(O_n) \to O_n, \quad (a_{i,j})_{i,j} \mapsto \sum_{i,j=1}^n s_i a_{i,j} s_j^*$$

is a $*$-isomorphism. Moreover, for $x \in O_n$, $\eta^{-1}(x) = (s_i^* x s_j)_{i,j} \in M_n(O_n)$.

Therefore, we have proved the following result, which is in fact known, but for sake of completeness:

**Proposition 3.3.** The Cuntz algebra $O_n$ is isomorphic to the $C^*$-algebra $M_n(O_n)$. Then for $a \in O_n$, $P_{i,j}(a)$ are considered as projections of $O_n$ by applying the mapping $\eta$. Therefore,

$$P_{i,j}(a) = s_i (1 + aa^*)^{-1} s_i^* + s_i (1 + aa^*)^{-1} as_j^* + s_j a^* (1 + aa^*)^{-1} s_i^* + s_j a^* (1 + aa^*)^{-1} as_j^*.$$
4. Unitary Factors in Purely Infinite $C^*$-Algebras

Recall that in a unital $C^*$-algebra $A$, every self-adjoint unitary $u$ can be written as $u = 1 - 2p$, for some projection $p \in \mathcal{P}(A)$, let us say" the self-adjoint unitary $u$ is associated to the projection $p"$. In this section, we assume that $A$ is purely infinite simple $C^*$-algebra, and we study the factorizations of unitaries of $A$. In order to prove our main theorem (Theorem 4.2), let us first recall the following result of M. Leen.

**Theorem 4.1** ([9], Theorem 3.8). Let $A$ be a simple, unital purely infinite $C^*$-algebra. Then the $*$-symmetries (self-adjoint unitaries) generate the connected component of the unity $\mathcal{U}_0(A)$.

Now, consider a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$ of $A$, with $e_{1,1} \sim 1$. Let us recall the $*$-isomorphisms $\eta_i : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \to A$, and $\eta = \eta_i \circ \tilde{\Delta}_v$ from $\mathbb{M}_n(A)$ onto $A$. In this section we revise Leens’ proof of Theorem 3.5 in [9] and we fix some projections, then by following the same construction, we prove the following main theorem, which shows that every unitary of $A$ which lies within a neighborhood of the unity can be factorized as a product of eleven self-adjoint unitaries moreover, eight of such factors are associated to the projections $P_{1,i}(\mu)$, for some $\mu \in \mathcal{U}(A)$.

**Theorem 4.2.** Let $A$ be a simple, unital purely infinite $C^*$-algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of $A$, with $e_{1,1} \sim 1$. Then there exists $\epsilon > 0$ such that every unitary $a$ of $A$ with $\|a - 1\| < \epsilon$ can be written as a product of eleven self-adjoint unitaries, of which eight have the form:

$$
1 - 2\eta(P_{1,2}(-\alpha)), \quad 1 - 2\eta(P_{1,2}(-1))
$$

$$
1 - 2\eta(P_{1,3}(-\alpha)), \quad 1 - 2\eta(P_{1,3}(-1))
$$

$$
1 - 2\eta(P_{1,2}(-\gamma)), \quad 1 - 2\eta(P_{1,2}(-1))
$$

$$
1 - 2\eta(P_{1,3}(-\gamma)), \quad 1 - 2\eta(P_{1,3}(-1))
$$

for some $\alpha, \gamma \in \mathcal{U}(A)$.

Consequently, as the Cuntz algebra is simple, unital purely infinite $C^*$-algebra with $K_1(\mathcal{O}_n) = 0$ (see [4]) and using Proposition 3.3, we have the following result.

**Corollary 4.3.** Let $n$ be given, there is a positive number $\epsilon$ such that if $u \in \mathcal{U}(\mathcal{O}_n)$ with $\|u - 1\| < \epsilon$, then

$$
\begin{align*}
\hat{u} &\equiv z_1(1 - 2P_{1,2}(-\alpha))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\alpha))(1 - 2P_{1,3}(-1)) \\
&\quad (1 - 2P_{1,2}(-\gamma))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\gamma))(1 - 2P_{1,3}(-1))z_2z_3,
\end{align*}
$$

for some self-adjoint unitaries $z_1, z_2, z_3$ and $\alpha, \gamma \in \mathcal{U}(\mathcal{O}_n)$.

Let us introduce the following lemma which is used by M. Leen in his proof, and we shall use it as well.
Lemma 4.4. Let $A$ be a simple, unital purely infinite $C^*$-algebra, and let $\rho$ be a non-trivial projections of $A$. There is a positive number $\epsilon$ such that if $a \in \mathcal{U}_0(A)$ with $\|a - 1\| < \epsilon$, then there exist self-adjoint unitaries $z_1, z_2, z_3$ of $A$ and $x \in \mathcal{U}_0(\rho Ap)$ such that

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1 - \rho \end{pmatrix}.$$

Proof. Mimic the first part of the proof of Theorem 3.5 in [9], with replacing symmetries by $*$-symmetries and invertible by unitaries.

Proof of Theorem 4.2:

Proof. Since $A$ is a simple, unital purely infinite $C^*$-algebra, using [4], we have $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$. As $K_1(A)$ is assumed to be trivial, we have $\mathcal{U}(A) = \mathcal{U}_0(A)$.

Let $p = e_{1,1}$, as $p \sim 1$, use Proposition 3.1 and the isomorphism $\Delta_u (u^* u = e_{1,1}, uu^* = 1)$ to find a projection $p_1 < p$ (precisely, $p_1 = u^* e_{1,1} u$) which is equivalent to $p$ moreover, set the partial isometry $v = u e_{1,1}$, and put $p = p - p_1$, so $p$ is a non-trivial projection. Therefore applying Lemma 4.4, there is a positive number $\epsilon$ such that if $a \in \mathcal{U}(A)$ with $\|a - 1\| < \epsilon$, then there exist self-adjoint unitaries $z_1, z_2$ and $z_3$ such that

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1 - \rho \end{pmatrix},$$

where $x \in \mathcal{U}(\rho Ap)$.

Now, we shall use Leen’s approach to exhibit the desired factorization of $a$. Choose $q = e_{2,2}, r = e_{3,3}$ and put $v_1 = p + q + r$, then we have $q \sim r < 1 - p - q$. Following Leen’s notations, we choose $v_1 = e_{2,1}, v_2 = e_{3,2}$ and $v_3 = e_{1,3}$, so $v_1, v_2$ and $v_3$ are partial isometries such that

$$v_1^* v_1 = p, \quad v_1 v_1^* = q, \quad v_2^* v_2 = q, \quad v_2 v_2^* = r, \quad v_3^* v_3 = r, \quad \text{and } v_3 v_3^* = p.$$

Let $w = v_1 + v_2 + v_3$. Then following the construction in Leen’s proof, we get

$$z_1 a z_2 z_3 = \left(1 - 2\eta_1 (P_{1,2}(-\alpha_p))\right) \left(1 - 2\eta_1 (P_{1,2}(-p))\right)$$

$$\left(1 - 2\eta_1 (P_{1,3}(-\alpha_p))\right) \left(1 - 2\eta_1 (P_{1,3}(-p))\right)$$

$$\left(1 - 2\eta_1 (P_{1,2}(-\gamma_p))\right) \left(1 - 2\eta_1 (P_{1,2}(-p))\right)$$

$$\left(1 - 2\eta_1 (P_{1,3}(-\gamma_p))\right) \left(1 - 2\eta_1 (P_{1,3}(-p))\right)$$

where $\alpha_p$ and $\gamma_p$ are in $\mathcal{U}(\rho Ap)$. Notice that the factors in the right hand side are self-adjoint unitaries in $A$. Hence using the mapping $\eta$, we then get

$$a = z_1 \left(1 - 2\eta (P_{1,2}(-\alpha))\right) \left(1 - 2\eta (P_{1,2}(-1))\right)$$

$$\left(1 - 2\eta (P_{1,3}(-\alpha))\right) \left(1 - 2\eta (P_{1,3}(-1))\right)$$

$$\left(1 - 2\eta (P_{1,2}(-\gamma))\right) \left(1 - 2\eta (P_{1,2}(-1))\right)$$

$$\left(1 - 2\eta (P_{1,3}(-\gamma))\right) \left(1 - 2\eta (P_{1,3}(-1))\right) z_3 z_2$$

where $\alpha$ and $\gamma$ are unitaries in $A$, and this ends the proof.

Finally, let us finish this section by presenting the following open question:

Q. In the Cuntz algebra $\mathcal{O}_n$, do self-adjoint unitaries of the form $\{1 - 2P_{ij}(a)\}$ generate the unitary group $\mathcal{U}(\mathcal{O}_n)$?
5. K-Theory of Certain Projections

In this section, we study the $K_0$-class of the projections $P_{i,j}(u)$, where $u$ is a unitary of some unital $C^*$-algebra $A$. In particular, if $A$ is a simple purely infinite $C^*$-algebra, with $K_1(A) = 0$, or $A$ is a von Neumann factor of type $II_1$, or $III$, then for any unitary $u$ of $A$, $P_{i,j}(u)$ has trivial $K_0$-class. Afterwards, we present an application of Theorem 4.2, to the case of Cuntz algebras.

**Proposition 5.1.** Let $A$ be a unital $C^*$-algebra. If $v$ is a unitary in $A$ of finite order, then $[P_{i,j}(v)] = [1]$ in $K_0(A)$.

*Proof.* Consider a unitary $v$ in $A$, such that $v^m = 1$, for some positive integer $m$. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(v \otimes E_{i,i} + v \otimes E_{j,j} + E_{j,i} + E_{i,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}),$$

then $W^* = \frac{1}{\sqrt{2}}(v^{m-1} \otimes E_{i,i} + v \otimes E_{j,j} + v^{m-1} \otimes E_{j,i} + E_{i,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k})$, therefore $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$W^*P_{i,j}(v)W = \frac{1}{4}(2v^{m-1} \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W)$$

$$= \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\quad \text{(1 at the i-th place)}$$

$$= E_{i,i}.$$  

This implies that the projection $P_{i,j}(v)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(v)] = [1]$ in $K_0(A)$, hence the proposition has been checked. \hfill $\square$

**Proposition 5.2.** Let $A$ be a unital $C^*$-algebra. If $w_1$, $w_2$ and $v$ are unitaries of $A$ such that $v$ has order $m$, then $[P_{i,j}(w_1vw_2)] = [1]$ in $K_0(A)$.

*Proof.* As $w_1$ and $w_2$ are unitaries in $A$, then for all $i \neq j$, $W = w_1 \otimes E_{i,i} + w_2^* \otimes E_{j,j} + \sum_{k \notin \{i,j\}} E_{k,k} \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover, $WP_{i,j}(v)W^* = P_{i,j}(w_1vw_2)$, therefore by Proposition 5.1 we have $[P_{i,j}(w_1vw_2)] = [P_{i,j}(v)] = [1]$. \hfill $\square$

**Proposition 5.3.** Let $A$ be a unital $C^*$-algebra. If $u$ and $v$ are self-adjoint unitaries in $A$, then $[P_{i,j}(uv)] = [1]$ in $K_0(A)$.

*Proof.* For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(uv \otimes E_{i,i} + uv \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}),$$

then $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,
\[ W^* P_{i,j} (uv) W = \frac{1}{4} (2uv \otimes E_{i,i} + 2 \otimes E_{i,j}) (\sqrt{2} W) \]

and this implies that the projection \( P_{i,j} (uv) \) is unitarily equivalent to \( E_{i,i} \) in \( \mathbb{M}_n(A) \), therefore we have that \([P_{i,j}(uv)] = [1]\) in \( K_0(A)\), hence the proposition has been checked. \( \square \)

Combining the previous results, we have the following theorem concerning the \( K_0 \)-class of those projections \( P_{i,j}(u) \) in \( \mathcal{P}(\mathbb{M}_n(A)) \), evaluated at any unitary \( u \) of \( A \).

**Theorem 5.4.** Let \( A \) be a simple, unital purely infinite \( C^* \)-algebra, such that \( K_1(A) \) is the trivial group. If \( u \in \mathcal{U}(A) \), then \([P_{i,j}(u)] = [1]\) in \( K_0(A)\).

**Proof.** Consider a unitary \( u \) of \( A \). As \( K_1(A) = 0 \), and we know by \([4], \text{p.188}\) that \( K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A) \) then using M. Leen’s result (Theorem 4.1), we have that \( u = \prod_{k=1}^n v_k \), where \( v_k \) is a self-adjoint unitary (\( * \)-symmetry) of \( A \). If \( n = 1 \), then the result holds by using Proposition 5.1. Proposition 5.3 proves the case \( n = 2 \). If \( n \geq 3 \), then the result is done by Proposition 5.2, hence the proof is completed. \( \square \)

Moreover, as M. Broise in \([3], \text{Theorem 1}\) proved that in the case of von Neumann factors of either type \( II_1 \) or \( III \), the unitaries are generated by the self-adjoint unitaries, then a similar result in the case of von Neumann factors can be deduced as follows:

**Theorem 5.5.** Let \( A \) be a von Neumann factor of type \( II_1 \) or \( III \). If \( u \in \mathcal{U}(A) \), then \([P_{i,j}(u)] = [1]\) in \( K_0(A)\).

**Proof.** Let \( u \) be a unitary of \( A \). By \([3], \text{Theorem 1}\), \( u \) can be written as a finite product of self-adjoint unitaries of \( A \), then mimic the proof of Theorem 5.4. \( \square \)

Consequently, we have the following results concerning the \( K_0 \)-class of some certain projections.

**Corollary 5.6.** Let \( A \) be a unital \( C^* \)-algebra which is either:

(1) simple, purely infinite, with \( K_1(A) = 0 \), or

(2) von Neumann factor of type \( II_1 \), or \( III \).

If \( v \) is a unitary of \( A \), and \( p \) is the projection of \( \mathbb{M}_n(A) \) defined by

\[ p = \frac{1}{2} \otimes E_{1,1} + \frac{v}{2} \otimes E_{1,2} + \frac{v^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} + E_{3,3} + E_{4,4} \cdots + E_{m,m} \]

for some positive integer \( m \leq n - 2 \), then \([p] = (m - 1)[1]\) in \( K_0(A)\).

**Proof.** As the projection \( p \) is the orthogonal sums of \( P_{1,2}(v) + E_{3,3} + E_{4,4} \cdots + E_{m,m} \), then by either Theorem 5.4 or 5.5,

\[ [p] = [1] + ([1] + \cdots + [1]) = (m - 1)[1]. \]

\( \square \)
Corollary 5.7. Let $A$ be a unital $C^*$-algebra which is either:
(1) simple, purely infinite, with $K_1(A) = 0$, or
(2) von Neumann factor of type $II_1$, or $III$.
If $v_1, v_2 \cdots v_n$ are unitaries of $A$, and $p$ is the projection of $\mathbb{M}_{2n}(A)$ defined by
\[
[1]
\]
then $[p] = n[1]$, in $K_0(A)$.

Proof. Using Theorem 5.4 (or Theorem 5.5), we have
\[
[p] = [P_{1,2}(v_1)] + [P_{3,4}(v_2)] + \cdots + [P_{2n-1,2n}(v_n)] = n[1].
\]

Now let us prove the following lemma, which will be used in order to prove our main result in this section (Theorem 5.9), which is in fact a consequence application of Theorem 4.2, to the case of Cuntz algebras $O_n$.

Lemma 5.8. Let $A$ be a unital, simple purely infinite $C^*$-algebra, with $K_1(A) = 0$, and let \{e_{i,j}\}$_n$, with $e_{1,1} \sim 1$ be a system of matrix units of $A$. Then for any unitary $u \in U(A)$ we have $[\eta(P_{i,j}(u))] = [1]$ in $K_0(A)$.

Proof. As we have seen in the proof of Propositions 5.1, 5.2, 5.3 and Theorem 5.4, there exists a unitary $W \in U(\mathbb{M}_n(A))$, such that $W^*P_{i,j}(u)W = E_{i,i}$. Therefore,
\[
\eta(W)^*\eta(P_{i,j}(u))\eta(W) = \eta(E_{i,i}) = \eta_1\hat{\Delta}_e(E_{i,i}) = \eta_1(e_{1,1} \otimes E_{i,i}) = e_{i,i}.
\]
Then
\[
\eta(P_{i,j}(u)) \sim_u e_{i,i} \sim e_{1,1} \sim 1
\]
hence $\eta(P_{i,j}(u))$ and 1 have the same class in $K_0(A)$.

Finally, let us consider the case of the Cuntz algebra $O_n$. Let $u$ be a self-adjoint unitary (involution), so $u = 1 - 2p$, for some $p \in \mathcal{P}(O_n)$. We recall the concept type of involution which is introduced by the author in [2], as follows: Since $K_0(O_n) \cong \mathbb{Z}_{n-1}$ (see [4]), then the type of $u$ is defined to be the element $[p]$ in $K_0(O_n)$. By ([2], Lemma 2.1), two involutions are conjugate as group elements in $U(O_n)$ if and only if they have the same type.

As a consequence of Theorem 4.2, and the results concerning the $K_0$-group of the projections $P_{i,j}(u)$, which are deduced in this section, we have the following result.

Theorem 5.9. Let $n$ be given. There is a positive number $\epsilon$ such that every unitary of $O_n$ that lies within $\epsilon$-neighborhood of 1 can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in U(O_n)$ and consequently, all such eight involutions are conjugate group elements of $U(O_n)$. 


Proof. Using [4] and [5], the Cuntz algebra $O_n$ is a simple, unital purely infinite $C^*$-algebra with trivial $K_1$-group. Then by Theorem 4.2, there exists $\epsilon > 0$ such that for every $u \in U(O_n)$ with $\|u - 1\| < \epsilon$, then $u$ can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in U(O_n)$. The type of the involution $(1 - 2\eta P_{i,j}(\omega))$ is $[\eta P_{i,j}(\omega)]$ and by Lemma 5.8 equals 1 in $K_0(O_n)$. Hence, by [[2], Lemma 2.1], all these involutions are conjugate indeed, to the trivial involution $-1$. \hfill \Box

Consequently, and as every unitary (precisely in the connected component of unity) can be written as a finite product of unitaries that are close to the unity (see for example [11], § 4.2), we have the following:

**Corollary 5.10.** Every unitary of $O_n$ can be written as a finite product of involutions, of which a multiple of eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in U(O_n)$ and consequently, all such multiple of eight involutions are conjugate group elements of $U(O_n)$.

**Acknowledgement.** The author would like to thank the referee and the editor for their valuable comments and suggestions.

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