M5 algebra and SO(5,5) duality

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Abstract

We present “M5 algebra” to derive Courant brackets of the generalized geometry of $T ⊕ Λ^2 T^* ⊕ Λ^5 T^*$: The Courant bracket generates the generalized diffeomorphism including gauge transformations of three and six form gauge fields. The Dirac bracket between selfdual gauge fields on a M5-brane gives a $C^{[3]}$-twisted contribution to the Courant brackets. For M-theory compactified on a five dimensional torus the U-duality symmetry is SO(5,5) and the M5 algebra basis is in the 16-dimensional spinor representation. The M5 worldvolume diffeomorphism constraints can be written as bilinear forms of the basis and transform as a SO(5,5) vector. We also present an extended space spanned by the 16-dimensional coordinates with section conditions determined from the M5 worldvolume diffeomorphism constraints.
1 Introduction

U-duality symmetry is a powerful guiding principle to define M-theory. Generalized manifold which extends diffeomorphism in T-duality covariant way is introduced in [1] and it is applied to generalized geometry for M-theory [2, 3]. A manifest T-duality is realized in [4] in a double field theory by duplicated coordinates. There are several approaches on the dualities; in string mechanics [4, 5], in Gaillard-Zumino approach [6], in string field theory [7] and in double field theory [8]. D-branes and RR fields are key ingredients to promote T-duality to U-duality, [9, 10, 11, 12, 13, 14, 15, 16, 17].

Recent progress on U-duality [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] has been exploring new description of the M-theory and its geometry. Berman, Perry and collaborators reformulate the low energy effective theory of M-theory using a coset construction with manifest U-duality symmetry [23, 24, 25, 26]. The U-duality symmetries of the M-theory with \( d \)-dimensional torus compactification depend on \( d \). It is SL(5) for \( d=4 \) and its coset construction is presented in [23]. The M2-brane origin is clarified in a worldvolume approach [27] and generalization of the Riemannian geometry is presented in [28]. In this paper we clarify M5-brane origin of the SO(5,5) duality for \( d=5 \) compactification in which the supergravity action is reformulated by using coset construction of SO(5,5)/[SO(5)×SO(5)] [24].

In generalization of Riemannian geometry the generalized diffeomorphism transformations are generated by Courant bracket in a duality covariant way. There are several approaches to construct such bracket; integrability conditions of the Dirac manifold structure [29], exceptional group extension [3, 22], generalized derivatives including gauge fields [31, 23, 25] and a brane algebra [4]. The authors have used the brane algebra approach for Dp-branes [15] and M2-brane [27]. In this paper we extend it to M5-brane. In the brane algebra approach an one parameter family of Courant brackets with parameter \( K \) are derived. Courant bracket \( (K = 0) \) is antisymmetric in two vectors, while Dorfman bracket \( (K = 1) \) is not antisymmetric but gives gauge transformation rules directly. Generators of a brane algebra are momentum and brane currents. For example the ones for the string algebra are \( Z_M = (p_m, \partial_\sigma x^m) \). The brane algebra basis \( Z_M \) is a representation of the duality symmetry. When the Hamiltonian constraint and worldvolume diffeomorphism constraints are written in terms of \( Z_M \), they should be also representation of the duality symmetry consistently. The generalized metric in the Hamiltonian is also a G/H coset representation in terms of supergravity fields in which G is duality symmetry and H is its subgroup.

To construct a theory with manifest duality symmetry a doubled space or an extended space is introduced, where its coordinates are in the representation of duality symmetry. Section conditions are required to obtain the physical space. It is natural to identify the section conditions with worldvolume diffeomorphism constraints of the probe brane, since the brane algebra basis becomes the conjugate momentum of the extended space coordinate preserving brane constraints. In the string case the \( \sigma \) diffeomorphism constraint is \( \mathcal{H}_\sigma = p_m \partial_\sigma x^m = \frac{1}{2} Z_M \eta^{MN} Z_N = 0 \) with O\((d,d)\) invariant metric \( \eta^{MN} \). The algebra basis \( Z_M \) is interpreted as \( \partial_M = \partial/(\partial X^M) \), conjugate to the doubled space coordinates \( X^M \). A physical space is obtained by the section condition \( \Delta = \partial_M \eta^{MN} \partial_N = 0 \).
Section conditions for M-theory in $d=4$ are obtained in [25] from the closure of the algebra of generalized diffeomorphism including gauge transformation [23]. The extended coordinates are $y_{mn} = -y_{nm}$ in addition to the usual coordinates $x^m$ with $m = 1, \cdots, 4$. They form SL(5) covariant coordinates $X^{\hat{m}\hat{n}} = \left( X^{m5} = x^m, X^{mn} = \frac{1}{2} \epsilon^{mnl_{1}l_{2}} y_{l_{1}l_{2}} \right)$ with $\hat{m} = (m, 5)$. The section conditions are

$$
\epsilon^{\hat{m}_1 \cdots \hat{m}_4} \frac{\partial}{\partial X^{\hat{m}_1 \hat{m}_2}} \frac{\partial}{\partial X^{\hat{m}_3 \hat{m}_4}} = 0 \iff \left\{ \begin{array}{l}
\epsilon^{m_1 \cdots m_4} \frac{\partial}{\partial y_{m_1 m_2}} \frac{\partial}{\partial y_{m_3 m_4}} = 0 \\
\frac{\partial}{\partial x^n} \frac{\partial}{\partial y_{mn}} = 0 \end{array} \right. \quad (1.1)
$$

This is nothing but the diffeomorphism constraints for a M2-brane [27]. For a M5-brane in $d=5$ there is a scalar coordinate $y$ in addition to $(x^m, y_{mn})$ with $m = 1, \cdots, 5$ to make 16-dimensional SO(5,5) spinor representation. The section condition will be modified from (1.1) to ones involving $y$. In this paper we derive the section conditions for M-theory in $d=5$ from the M5-brane constraints. The section conditions are a part of the BPS condition of eleven dimensional M-theory [30]. BPS D-branes and M-branes satisfy the BPS conditions of type II theories and M-theory respectively as well as constraints. There is a correspondence between the BPS projection and the $\kappa$-symmetry projection, which is roughly square root of bilinear constraints. This is a reflection of a correspondence between the global supersymmetry algebra and the local supersymmetry algebra for a supersymmetric brane system.

The organization of this paper is the following. In section 2 we reformulate M5-brane action given by Pasti, Sorokin and Tonin [32] in such a way that constraints of a M5-brane in the supergravity background are written in bilinear forms of some basis $Z_M$. The selfdual gauge field on a single M5-brane is treated by Dirac bracket preserving worldvolume covariance. The M5 algebra is calculated and a series of Courant brackets for $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*$ is obtained. It is shown that the obtained Courant bracket gives correct generalized diffeomorphism including gauge transformations for $C^{[3]}$ and $C^{[6]}$. In section 3 we focus on M-theory in $d=5$, where U-duality symmetry is SO(5,5). SO(5,5) spinor states are constructed and the SO(5,5) and SO(5)$\times$SO(5) transformation rules are presented. It is shown the worldvolume diffeomorphism constraints form a SO(5,5) vector, while the Hamiltonian constraint determines SO(5,5) transformation rules of the supergravity fields, $G_{mn}$ and $C^{[3]}_{mnl}$. In section 4 an extended space spanned by $X^M$ is presented with section conditions determined from the M5 diffeomorphism constraints. SO(5,5) covariant C-bracket is also presented.

## 2 M5 algebra and Courant bracket

A probe brane in supergravity background determines an algebra, whose representation basis $Z_M$ satisfies following conditions:

1. Transformation algebra generated by $Z_M$ is closed.

2. The Hamiltonian constraint for a probe brane is bilinear form in $Z_M$ as $H_\perp = \frac{1}{2} Z_M M^{MN} Z_N \approx 0$ where $M^{MN}$ is the generalized metric.
3. The set of worldvolume diffeomorphism constraints $H_i \approx 0$ can be written in bilinears in $Z_M$ as $Z_M \delta^{MN} Z_N \approx 0$, where $\delta^{MN}$ is a constant matrix.

4. Charges of $Z_M$, such as momentum charge and brane charges, are rotated covariantly under U-duality.

Once the brane algebra is found, Courant bracket is determined as an algebra between vectors in the space spanned by $Z_M$.

In this section local structure of the geometry generated by the M5 algebra is presented. We begin with the M5 action given by Pasti, Sorokin and Tonin [32] and we perform canonical analysis in the temporal gauge [33]. The selfdual condition on rank two gauge fields are mixture of first class constraints and second class constraints. We do not fix the first class constraint, Gauss law constraint, to preserve worldvolume five dimensional covariance necessary to compute M5 algebra and Courant bracket. The second class constraints are treated by using the Dirac bracket keeping the worldvolume five dimensional covariance. Using with the Dirac bracket the M5 algebra is obtained and the Courant brackets for M5 is obtained. It gives correct generalized diffeomorphism including gauge transformations of $C^{[3]}$ and its magnetic dual field $C^{[6]}$.

### 2.1 M5 constraints

We begin with an action for a M5 brane proposed by Pasti, Sorokin and Tonin [32]

$$I = \int d^6 \sigma \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{SD} + \mathcal{L}_{WZ}, \quad \text{(2.1)}$$

$$\mathcal{L}_{DBI} = -T \sqrt{-h} \mathcal{F} \mathcal{F}, \quad h = \det (h_{ij} + \mathcal{F}_{ij}), \quad h_{ij} = \partial_k x^m \partial_j x^n G_{mn}, \quad \mathcal{F}_{ij} = h_{ik} h_{jk} \mathcal{F}^{kk'},$$

$$\mathcal{L}_{SD} = \frac{T \sqrt{-h}}{2} \mathcal{F}^{ij} \mathcal{F}_{ijk} n^k, \quad h = \det h_{ij}, \quad i = 0, 1, \cdots, 5,$$

$$\mathcal{L}_{WZ} = T e^{i1 \cdots i6} \left( \frac{1}{6!} C_{i1 \cdots i6}^{[6]} + \frac{1}{2 \cdot 3!} \mathcal{F}_{i1 i2 i3} C_{i4 i5 i6}^{[3]} \right),$$

$$F_{ijk} = \partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}, \quad \mathcal{F}_{ijk} = F_{ijk} - \partial_i x^m \partial_j x^n \partial_k x^l C_{mnl}^{[3]}.$$
Using an identity which holds for a 5-dimensional symmetric matrix $S_{ij} = s_i \tilde{s}_j \tilde{a}$ and an antisymmetric matrix $T_{ij} = -T_{ji}$, with $i, \tilde{a} = 1, \ldots, 5$,}

\[
\det(S_{ij} + T_{ij}) = \det S_{ij} + \frac{1}{24} T_{\bar{a}bc} T_{\bar{a}bc} + \frac{1}{64} T_{\bar{a}} T_{\bar{a}},
\]

the determinant term in $\mathcal{H}_\perp$ is rewritten as

\[
\det(h_{ij} + \tilde{\mathcal{F}}_{ij}) = \frac{1}{5!} \left( \epsilon^{i_1 \cdots i_5} \partial_{i_1} x^{m_1} \cdots \partial_{i_5} x^{m_5} \right) G_{m_1 n_1} \cdots G_{m_5 n_5} \left( \epsilon^{j_1 \cdots j_5} \partial_{j_1} x^{m_1} \cdots \partial_{j_5} x^{m_5} \right) + \cdots
\]

where $h_{ij}$ is inverse of $h_{ij}$ and

\[
\begin{align*}
\tilde{p}_m &= \frac{\partial \mathcal{L}_{DBI}}{\partial (\partial_0 x^m)} \quad (2.2) \\
t^n &= \partial_i x^n h^{ij} \epsilon^{i_1 \cdots i_5} \quad (2.3)
\end{align*}
\]

Hamiltonian constraints are given as

\[
\begin{align*}
\mathcal{H}_\perp &= \frac{1}{2T} \left( \tilde{p}_m G^{mn} \tilde{p}_n + T^2 \det (h + \tilde{\mathcal{F}}) \right) = 0 \\
\mathcal{H}_i &= \tilde{p}_m \partial_i x^m = 0 \\
\Phi^{ij} &= \frac{1}{\sqrt{T}} E^{ij} - \frac{\sqrt{T}}{4} \epsilon^{i_1 i_2 i_3} \partial_{i_1} A_{i_2 i_3} = 0
\end{align*}
\]

The diffeomorphism constraint in (2.4) leads to

\[
\mathcal{H}_i h^{ij} \mathcal{F}_{j_1 j_2} \mathcal{F}_{j_3 j_4 j_5} \epsilon^{i_1 \cdots i_5} / 4! = \tilde{p}_m t^m + T t^m G_{mn} t^n = 0
\]

The diffeomorphism constraint in (2.4) leads to

\[
\mathcal{H}_i h^{ij} \mathcal{F}_{j_1 j_2} \mathcal{F}_{j_3 j_4 j_5} \epsilon^{i_1 \cdots i_5} / 4! = \tilde{p}_m t^m + T t^m G_{mn} t^n = 0
\]
where $\tilde{p}_m = \tilde{p}_m - TG_{mn}t^n$. The first term in the Hamiltonian in (2.4) is rewritten by (2.7) as

$$\tilde{p}_mG^{mn}\tilde{p}_n = \tilde{p}_mG^{mn}\tilde{p}_n + 2T\tilde{p}_m t^n + T^2 t^m G_{mn}t^n = \tilde{p}_mG^{mn}\tilde{p}_n - T^2 t^m G_{mn}t^n,$$

where the $t^m$ dependent term is cancelled out with the second term, the contribution in the 5-dimensional spatial determinant in (2.6). The basis for a M5 brane system is introduced as

$$Z_M = \left( \begin{array}{c} Z_m \\ Z^[2]_{m_1m_2} \\ Z^[5]_{m_1\ldots m_5} \end{array} \right) = \left( \begin{array}{c} p_m \\ 2E^{i_1i_2}\partial_{i_1}x^{m_1}\partial_{i_2}x^{m_2} \\ T\epsilon^{i_1\ldots i_5}\partial_{i_1}x^{m_1}\partial_{i_2}x^{m_2}\partial_{i_3}x^{m_3}\partial_{i_4}x^{m_4}\partial_{i_5}x^{m_5} \end{array} \right)$$

As a result Hamiltonian constraints become

$$\begin{cases} \mathcal{H}_\perp &= \frac{1}{2T} \left( \tilde{p}_mG^{mn}\tilde{p}_n + \frac{1}{2}Z^[2]_{m_1m_2}G_{[m_1n_31]G_{[m_2n_2]}Z^[2]_{n_1n_2} \right. \\
&+ \frac{1}{2}Z^[5]_{m_1\ldots m_5}C_{[m_1n_3]} \ldots C_{[m_5n_5]}Z^[5]_{n_1\ldots n_5} \right) = 0 \\
\mathcal{H}_i &= \tilde{p}_m\partial_i x^n + \frac{1}{2}E^{jk}F_{ij} = p_m\partial_i x^n + \frac{1}{2}E^{jk}F_{ij} \\
&= p_m\partial_i x^n + \frac{1}{2T}\epsilon^{i_1i_2i_3i_4i_5}E^{i_1i_2}E^{i_3i_4} = 0, \end{cases}$$

where

$$\tilde{p}_m = p_m + \frac{1}{2}C_{[m_1n_2]}^3Z^{[2]n_1n_2} - \frac{1}{5!}(C_{[m_1\ldots n_5]}^6 + 5C_{[m_1n_2}C_{n_3]}^3Z^[5]_{n_1\ldots n_5}$$

$$\tilde{Z}^{[2]n_1n_2} = 2\tilde{E}^{i_1i_2}\partial_{i_1}x^{n_1}\partial_{i_2}x^{n_2} = Z^{[2]n_1n_2} - \frac{1}{3!}C_{[m_1n_2m_3]}^3Z^[5]_{n_1n_2m_1m_2m_3}. \quad (2.10)$$

The constraint $\mathcal{H}_\perp$ can be written in terms of bilinear form of M5 brane basis, $Z_M$ in (2.9),

$$\mathcal{H}_\perp = \frac{1}{2T}Z_M\mathcal{M}^{MN}Z_N \quad (2.11)$$

$$\mathcal{M}^{MN} = (\mathcal{N}^T)LM_0^{LK}\mathcal{N}_K^N \quad (2.12)$$

$$\mathcal{M}_0^{ML} = \left( \begin{array}{ccc} G_{m_1l_1} & 0 & 0 \\ 0 & G_{[m_1n_3]G_{n_3]}l_2 & 0 \\ 0 & 0 & G_{[m_1n_3]} \ldots G_{[m_5n_5]} \end{array} \right)$$

$$\mathcal{N}_L^N = \left( \begin{array}{ccc} \delta_{l_1}^n & C_{[l_1n_2]}^3 & -C_{[l_1n_2]}^6 - \frac{1}{4!}C_{[l_1n_2]}^3C_{n_3n_4n_5]}^3 \\ 0 & \delta_{n_1}^l \delta_{n_2}^l & -\frac{1}{3!}C_{[n_1n_2n_3]}^3\delta_{n_4}^l \delta_{n_5}^l \\ 0 & 0 & \delta_{l_1}^n \ldots \delta_{l_5}^n \end{array} \right).$$

Indices are contracted as $U^M V_M = U^m V_m + \frac{1}{2} U^{m_1m_2} V_{m_1m_2} + \frac{1}{3!} U^{m_1\ldots m_5} V_{m_1\ldots m_5}.$
Worldvolume spatial diffeomorphism constraints $\mathcal{H}_i = 0$ are also written in bilinears of $Z_M$ by contracting with $\epsilon^{i_1\ldots i_4}\partial_{i_1}x^{m_1}\ldots\partial_{i_4}x^{m_4}$ and $E^{ij}\partial_j x^m$,

$$\mathcal{H}_i = 0 \Rightarrow Z_M\tilde{\rho}^{MN} Z_N = (Z_M\tilde{\rho}^{MN}[M_5]Z_N)^n a_n + (Z_M\tilde{\rho}^{MN}[M_5]Z_N)^{m_1\ldots m_4} b_{m_1\ldots m_4} = 0 \quad (2.13)$$

$$\tilde{\rho}^{MN} = \begin{pmatrix} 0 & a_{[n_1}\delta^{m}_{n_2]} & b_{[n_1\ldots n_4}\delta^{m}_{n_5]} \\ a_{[m_1}\delta^{m}_{m_2]} & b_{[m_1\ldots m_2 n_1 n_2]} & 0 \\ b_{[m_1\ldots m_4}\delta^{n}_{m_5]} & 0 & 0 \end{pmatrix},$$

where $a_m$ and $b_{m_1\ldots m_4}$ are arbitrary constants. Not only the M5-brane diffeomorphism constraints but also M2-brane diffeomorphism constraints obtained in [27] appear.

A single M5-brane system including the selfdual gauge field reduces into a single D4-brane system including a Dirac-Born-Infeld gauge field by the double dimensional reduction [34, 35]. The set of basis $Z_M$ for a M5 brane given in (2.9) corresponds to the one for a D4 brane in the IIA theory given in [15] as

| M5 mode | D4 mode |
|---------|---------|
| momentum | $Z_m$ | $p_m$ |
| M2 mode | $Z^{[2]mn}$ | $\frac{1}{2}E^i\partial_i x^m$ |
| momentum on M5 | $t^m$ | $\frac{1}{4!}\epsilon_{j_1j_2k_1k_2}F_{j_1j_2}F_{k_1k_2}$ |
| M2 mode on M5 | $Z^{[2]m_1m_2}$ | $\epsilon^{i_1i_2j_1j_2}F_{i_1i_2}\partial_j x^{m_1}\partial_{j_2} x^{m_2}$ |
| M5 mode | $Z^{[5]m_1\ldots m_5}$ | $\frac{1}{4!}\epsilon^{i_1\ldots i_4}\partial_{i_1} x^{m_1}\ldots\partial_{i_4} x^{m_4}$ |

The M2 mode $Z^{[2]}$ on M5 reduces into both string mode and D2 mode on D4 from selfdual property. The M5 algebra basis reduces into the one for D4 algebra except the uplifted D0 mode on D4, whose origin $t^m$ appears in the momentum $\tilde{p}_m$ only at the first stage.

### 2.2 Selfdual gauge field

A M5 brane includes M2 brane boundaries which are selfdual rank two antisymmetric gauge field. The gauge field $A_{ij}$ and its canonical conjugate $E^{ij}$ satisfy the Poisson bracket

$$\{E^{ij}(\sigma), A_{i'j'}(\sigma')\} = -i\delta^{ij}_{i'j'}\delta(\sigma - \sigma') \quad (2.15)$$

as well as the selfduality constraints in (2.4), $\Phi^{ij} = 0$. This conditions is a mixture of first class and second class constraints [36]. Its longitudinal modes are first class Gauss law constraints, $\partial_\nu \Phi^{ij} = \partial_i E^{ij} = 0$ and the transverse modes are second class

$$\Phi^{ij} = \mathcal{P}^{i}_{\perp j} \mathcal{P}^{j}_{\perp i} \Phi^{ij}_{\perp} = \delta^{ij} - \delta^{ij}_\Delta = 0, \quad \mathcal{P}^{j}_{\perp i} = \delta^j_i - \frac{\partial_i \partial^j}{\Delta} \quad (2.16)$$

satisfying

$$\{\Phi^{j_1j_2}_{\perp}(\sigma), \Phi^{j_3j_4}_{\perp}(\sigma')\} = -i\epsilon^{j_1j_2j_3j_4}\partial_\nu \delta(\sigma - \sigma') \equiv \Xi^{j_1j_2j_3j_4}(\sigma, \sigma') \quad (2.17)$$
The inverse of (2.17) exists only in the transverse directions

\[
(\Xi^{-1})_{\mu_1\mu_2\mu_3\mu_4}(\sigma,\sigma') = i\epsilon_{\mu_1\mu_2\mu_3\mu_4} \frac{1}{\Delta} \delta(\sigma - \sigma'),
\]

\[
\frac{1}{2} \int d\sigma' (\Xi^{-1})_{ij} \ i'j'(\sigma,\sigma')(\Xi)^{i'j'}_{kl}(\sigma',\sigma'') = \mathcal{P}_{\perp}^{[k} \mathcal{P}_{\perp]}^{j} \delta(\sigma - \sigma'').
\]

The Dirac bracket, verifying \(\{\Phi_{\perp}^{ij}(\sigma), \mathcal{O}(\sigma')\}_D = 0\) for any \(\mathcal{O}\), is defined as

\[
\{\mathcal{O}_1(\sigma), \mathcal{O}_2(\sigma')\}_D = \{\mathcal{O}_1(\sigma), \mathcal{O}_2(\sigma')\} - \int d\sigma'' \{\mathcal{O}_1(\sigma), \Phi_{\perp}^{ij}(\sigma'')\} \frac{i}{4} \epsilon_{ij}^{jk} \frac{1}{\Delta}\delta(\sigma'') \mathcal{O}_2'(\sigma').
\]

It leads to nontrivial bracket between two \(E^{ij}\)'s as

\[
\{E^{i_1i_2}(\sigma), E^{i_3i_4}(\sigma')\}_D = \frac{iT}{4} \epsilon_{i_1i_2i_3i_4} \partial_k \delta(\sigma - \sigma'). \quad (2.18)
\]

### 2.3 M5 algebra and Courant brackets

Now let us compute the M5 algebra. The Dirac bracket between \(Z_M\)'s in (2.9) is given by

\[
\{Z_M(\sigma), Z_{N'}(\sigma')\}_D = iT \rho^i_{MN} \partial_i \delta(\sigma - \sigma'),
\]

\[
\rho^i_{MN} = \begin{pmatrix}
0 & \frac{1}{T} E^{ij} \partial_j x^{[m_1} \delta_{m_2]} r_{i}{}^{m_1 \cdots m_5} \\
\frac{1}{T} E^{ij} \partial_j x^{[m_1} \delta_{m_2]} & r_{i}{}^{m_1 m_2 m_3} r_{i}{}^{m_4 m_5} \quad 0
\end{pmatrix},
\]

\[
r_{i}{}^{m_1 \cdots m_5} = \frac{1}{4!} \epsilon_{i_1 \cdots i_4} \partial_{i_1} x^{[m_1} \cdots \partial_{i_4} x^{m_4} \delta_{m_5]}, \quad (2.19)
\]

where (2.18) is used. The matrix \(\rho_{MN}\) is symmetric and satisfies the following relations

\[
\partial_i \rho^i_{MN} = 0, \quad \rho^i_{MN} \partial_i x^l \partial_l = f^i_{MN} Z_L \partial_M.
\]

by the Gauss law constraint. We introduce local vector \(\Lambda^M(\sigma)\) in the M5 basis (2.9) as

\[
\hat{\Lambda}(\sigma) = \Lambda^M Z_M = \lambda + \lambda^{[2]} + \lambda^{[5]} \in T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*
\]

\[
\begin{align*}
\lambda &= \Lambda^m Z_m \\
\lambda^{[2]} &= \frac{1}{2} \Lambda_{m_1 m_2} Z^{[2]}_{m_1 m_2} \\
\lambda^{[5]} &= \frac{1}{5!} \Lambda_{m_1 \cdots m_5} Z^{[5]}_{m_1 \cdots m_5}.
\end{align*}
\]

The M5 algebra is closed up to the brane-like anomalous terms as

\[
\{\hat{\Lambda}_1(\sigma), \hat{\Lambda}_2(\sigma')\}_D = -iT \hat{\Lambda}_{12} \delta(\sigma - \sigma') + iT \left(\frac{1 - K}{2} \Psi_{(12)}^i(\sigma) + \frac{1 + K}{2} \Psi_{(12)}^i(\sigma')\right) \partial_i \delta(\sigma - \sigma').
\]
For \( K \) gauge transformations of antisymmetric gauge fields automatically. Dorfman bracket. It is not antisymmetric in \( \Lambda_{1}^{a} \).

The general coordinate transformation compatible with the M5 brane background is given as

\[
\Psi_{(12)}^{i} = \Lambda_{1}^{a} \rho_{MN}^{i} \Lambda_{2}^{N} = \frac{1}{2} \Lambda_{1}^{a} \rho_{MN}^{i} \Lambda_{2}^{N}.
\]

\( K \) is an arbitrary constant reflecting an ambiguity of \( \partial_{i} \delta(\sigma - \sigma') \) term as shown in [15]. The vector \( \hat{\Lambda}_{12} \) in (2.22) is recognized as one parameter family of Courant bracket for \( T \oplus \Lambda^{2} T^{*} \oplus \Lambda^{5} T^{*} \) derived from the M5 algebra

\[
[\hat{\Lambda}_{1}, \hat{\Lambda}_{2}]_{M5,K} = \left( \Lambda_{1}^{a} \partial_{\mu} \Lambda_{2}^{a} - \frac{1}{2} f_{MN}^{L} \Lambda_{1}^{L} \partial_{\mu} \Lambda_{2}^{N} + \frac{K}{2} f_{LNP}^{M} \Lambda_{1}^{L} \partial_{\nu} \Lambda_{2}^{P} \right) Z_{M}.
\]

(2.23)

For \( K = 0 \) the bracket is the Courant bracket which is antisymmetric in \( \Lambda_{1} \leftrightarrow \Lambda_{2} \):

\[
[\hat{\Lambda}_{1}, \hat{\Lambda}_{2}]_{M5,K=0} = [\lambda_{1}, \lambda_{2}] + \mathcal{L}_{\lambda_{1}} \lambda_{2}^{[2]} + \mathcal{L}_{\lambda_{1}} \lambda_{2}^{[5]} - \frac{1}{2} d(\lambda_{1} \lambda_{2}^{[2]}) - \frac{1}{2} d(\lambda_{1} \lambda_{2}^{[5]})
\]

\[
- \frac{1}{2} \lambda_{1}^{[2]} \wedge \lambda_{2}^{[2]}
\]

(2.24)

This result is consistent with the one in [3]. A useful choice is \( K = 1 \) and the bracket is Dorfman bracket. It is not antisymmetric in \( \Lambda_{1} \leftrightarrow \Lambda_{2} \), but it gives totally antisymmetrized gauge transformations of antisymmetric gauge fields automatically.

\[
[\hat{\Lambda}_{1}, \hat{\Lambda}_{2}]_{M5,K=1} = [\lambda_{1}, \lambda_{2}] + \mathcal{L}_{\lambda_{1}} \lambda_{2}^{[2]} + \mathcal{L}_{\lambda_{1}} \lambda_{2}^{[5]} - \lambda_{1} \lambda_{2} d\lambda_{1}^{[2]} - \lambda_{2} d\lambda_{1}^{[5]}
\]

\[
+ \lambda_{2}^{[2]} \wedge \lambda_{1}^{[2]}
\]

(2.25)

with

\[
\lambda_{2} d\lambda_{1}^{[p]} = \frac{1}{p!} \Lambda_{2}^{a} \partial_{[p]} \Lambda_{1}^{a} \lambda_{2}^{[m_{1} \cdots m_{p}]} Z^{[p] m_{1} \cdots m_{p}}
\]

Using with the obtained Courant bracket for \( K = 1 \), Dorfman bracket in (2.25), the general coordinate transformation compatible with the M5 brane background is given as

\[
\delta_{\xi} \hat{E}^{M5}_{a} = [\hat{\xi}, \hat{E}^{M5}_{a}]_{M5,K=1}
\]

\[
(\hat{E}^{M5}_{a})^{M} = \begin{pmatrix}
\epsilon_{a}^{m} \\
\epsilon_{a}^{m} C^{[3]}_{m_{1}m_{2}} \\
\epsilon_{a}^{m} C^{[6]}_{m_{1} \cdots m_{5}}
\end{pmatrix}, \quad (\hat{\xi})^{m} = \begin{pmatrix}
\xi^{m} \\
\xi_{m_{1}m_{2}}^{[2]} \\
\xi_{m_{1} \cdots m_{5}}^{[5]}
\end{pmatrix}.
\]

(2.26)
By contracting the local Lorentz SO(5) index $a$, $e_m^a e_a^n = \delta_m^n$ and $e_m^a e_n^b \delta_{ab} = G_{mn}$, it gives the expected general coordinate transformations and gauge transformations as

$$
\begin{align*}
\delta \xi_{Gmn} &= \xi^l \partial G_{mn} + \partial_{[m} \xi^l G_{ln]} \\
\delta \xi_{C[3]}^{m_1 m_2 m_3} &= \xi^l \partial C^{[3]}_{m_1 m_2 m_3} + \frac{1}{2} \partial_{[m_1} \xi^l C^{[3]}_{l m_2 m_3]} - \frac{1}{2} \partial_{m_1} \xi^{[2]}_{m_2 m_3]} \\
\delta \xi_{C[6]}^{m_1 \ldots m_6} &= \xi^l \partial C^{[6]}_{m_1 \ldots m_6} + \frac{1}{2} \partial_{[m_1} \xi^l C^{[6]}_{l \ldots m_6]} - \frac{1}{2} \partial_{m_1} \xi^{[5]}_{\ldots m_6]} + \frac{1}{4} C^{[3]}_{m_1 m_2 m_3} \partial_{m_4} \xi^{[2]}_{m_5 m_6]}.
\end{align*}
$$

(2.27)

The last term in the gauge transformation of $C^{[6]}$ comes from the Dirac bracket between selfdual gauge fields.

3 SO(5,5) duality symmetry

We focus on a M-theory in $d=5$, where the duality symmetry group is SO(5,5). The SO(5,5;Z) duality mixes quantized momentum, M2 and M5-brane charges, while SO(5,5;R) duality symmetry rotates momentum and M2 and M5-brane currents in the low energy effective theory. In this section we consider SO(5,5;R) transformation of the M5 algebra basis (2.9), which is 16-dimensional spinor representation. The supergravity fields, metric and three form gauge field, are coset parameters of G/H with G=SO(5,5) and H=SO(5)×SO(5).

3.1 SO(5,5) spinor states

In the case of $d=5$ the five form $Z^{[5]}_{m_1 \ldots m_5}$ becomes a pseudo scalar $Z^{[5]}$, then the M5 basis becomes $5 + 10 + 1=16$-dimensional spinor representation of SO(5,5),

$$
Z_M = (Z_m, Z^{[2]}_{m_1 m_2}, Z^{[5]}_m) , \quad m = 1, \ldots, 5 .
$$

(3.1)

The 16-dimensional spinor representation of SO(5,5) duality group is constructed by fermionic oscillators $\psi^m$ and $\psi_m^\dagger$ [10]

$$
\{ \psi_m^\dagger, \psi^n \} = \delta_m^n \ , \ \{ \psi^m, \psi^n \} = \{ \psi_m^\dagger, \psi_n^\dagger \} = 0 .
$$

(3.2)

The SO(5,5) Clifford algebra is

$$
\Gamma_a = (\Gamma_a, \Gamma_a^\dagger) = \left\{ \begin{array}{ll}
\Gamma_a = \psi^m e_{ma} + e_a^m \psi_m^\dagger & a, \dot{a} = 1, \ldots, 5 \\
\Gamma_{\dot{a}} = \psi^m e_{m\dot{a}} - e_a^m \psi_m^\dagger & a, \dot{a} = 1, \ldots, 5
\end{array} \right.
$$

(3.3)

$$
\{ \Gamma_a, \Gamma_b \} = \delta_{ab} \ , \ \hat{\eta}_{ab} = \text{diag}(\delta_{ab}, -\delta_{ab})
$$

$$
\Gamma_{11} = \prod_a \Gamma_a = \prod_{m=1}^5 \begin{bmatrix} \psi_m & \psi_m^\dagger \end{bmatrix} .
$$

The SO(5,5) chiral spinor states are constructed by acting odd numbers of fermions on $|+\rangle$ defined by $\psi^{[m]} |+\rangle = 0$ for a choice $\Gamma_{11} = -1$. They are

$$
| M \rangle = (|m\rangle , |m_1 m_2\rangle , |\rangle)
$$
\[
\begin{bmatrix}
\delta v \\
|m_1 m_2\rangle = \psi_{m_1}^\dagger \psi_{m_2}^\dagger |+\rangle \\
|m_{1 m_2}\rangle = \frac{1}{3!} \epsilon_{m_1 m_2 m_3 m_4 m_5} \psi_{m_3}^\dagger \psi_{m_4}^\dagger \psi_{m_5}^\dagger |+\rangle \\
\langle - | = \frac{1}{3!} \epsilon_{m_1 m_2 m_3 m_4 m_5} \psi_{m_1}^\dagger \psi_{m_2}^\dagger \psi_{m_3}^\dagger \psi_{m_4}^\dagger \psi_{m_5}^\dagger |+\rangle
\end{bmatrix},
\]

\[
\langle N | M \rangle = \hat{\delta}_{NM} = \begin{pmatrix}
\delta_n^m & 0 & 0 \\
0 & \delta_{[m_1 m_2]}^{n_2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (3.4)

Infinitesimal SO(5,5) rotations are parameterized by \(\alpha_n^m\), \(\beta_{[nm]}\), \(\gamma^{[nm]}\) as

\[
\text{SO}(5,5) \ni g, \quad g^T \eta g = \eta, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\Rightarrow g \delta n = \begin{pmatrix}
(1 + \alpha)_n^m & \beta_{mn} \\
\gamma_{mn} & (1 + \alpha)_n^m
\end{pmatrix}, \quad \beta_{mn} = -\beta_{nm}, \quad \gamma_{mn} = -\gamma_{nm}. \quad (3.5)
\]

The chiral spinor states \(|M\rangle\) are transformed as

\[
\delta |M\rangle = \hat{S} |M\rangle = S_M^N |N\rangle, \quad \hat{S} = \frac{1}{2} \alpha_n^m [\psi_{m}^\dagger, \psi_n^\dagger] + \frac{1}{2} \beta_{mn} \psi_m \psi_n + \frac{1}{2} \gamma_{mn} \psi_m \psi_n^\dagger,
\]

\[
S_M^N = \begin{pmatrix}
-\frac{\hat{\alpha}}{2} \delta_{n}^{m} + \alpha_n^m & -\hat{\gamma}_{nm} & 0 \\
\beta_{m_1 n_2 m_2} & \frac{\hat{\alpha}}{2} \delta_{[m_1 m_2]}^{n_2} - \alpha_{[m_1 m_2]}^{n_2} & -\gamma_{n_1 n_2} \\
0 & \beta_{m_1 n_2 m_2} & \frac{\hat{\alpha}}{2}
\end{pmatrix}, \quad (3.6)
\]

where \(\hat{\alpha} = \alpha_{m n}, \quad \beta_{m_1 n_2 m_2} = \epsilon_{m_1 n_2 m k} k_2 \beta_{k_1 k_2} / 2\).

Under typical T-duality rotations a SO(5,5) vector \((v_m, \bar{v}^m)\) is transformed using \(g\) in (3.5) as \(\delta v_m = \beta_{mn} \bar{v}^n\) and \(\delta \bar{v}^m = \gamma^{mn} v_n\). The M5 basis, \(Z_M\), is transformed in the same way as the SO(5,5) spinor states \(|M\rangle\) as in (3.6),

\[
\delta Z_m = -\frac{1}{4} \epsilon_{m n_1 n_2 n_3} \gamma^{n_1 n_2} Z^{[2]}_{n_3 n_4}, \quad \delta Z^{[5]} = \frac{1}{2} \beta_{mn} Z^{[2]}_{mn}, \quad \delta Z^{[2]}_{mn} = \frac{1}{2} \epsilon_{m n_1 n_2 n_3} \beta_{l_1 l_2} p_l - \gamma^{mn} Z^{[5]}.
\] (3.7)

SO(5,5) contains a subgroup SO(5) \(\times\) SO(5) generated by

\[
\frac{1}{4} \omega_{-}^{ab} \Gamma_{[a} \Gamma_b] + \frac{1}{4} \omega_{+}^{ab} \Gamma_{[a} \Gamma_b] \quad \Rightarrow \quad \omega_{-}^{ab} = \omega_{ab} + \bar{\omega}_{ab}, \quad \omega_{+}^{ab} = \omega_{ab} - \bar{\omega}_{ab}
\] (3.8)

with infinitesimal parameters \(\omega_{ab}\) and \(\bar{\omega}_{ab}\),

\[
\text{SO}(5) \times \text{SO}(5) \ni h, \quad h^T \eta h = \eta, \quad h^T \delta h = \delta
\]

\[
\Rightarrow \quad h_{a b} = \begin{pmatrix} (1 + \omega)^a_b & \bar{\omega}_{ab} \\
\bar{\bar{\omega}}_{ab} & (1 + \omega)^a_b \end{pmatrix}, \quad \omega_{ab} = -\omega_{ba}, \quad \bar{\omega}_{ab} = -\omega_{ba}.
\]
Under the SO(5)\times SO(5) transformation spinor states are transformed as
\[
\delta_h |A\rangle = S_h A |B\rangle , \quad S_h A = \begin{pmatrix}
\omega^b_a & -\bar{\omega}_{ab_1b_2} & 0 \\
-\bar{\omega}_{a_1b_2} & -\omega_{b_1[a_1}\delta_{b_2]} & -\bar{\omega}^{a_1a_2} \\
0 & \bar{\omega}_{b_1b_2} & 0 \\
\end{pmatrix}
\] (3.9)
with $|A\rangle = \nu^M A |M\rangle$ for a coset element $\nu \in SO(5,5)/[SO(5)\times SO(5)]$. This is analogous to right/left separation of string modes. For the right mover SO(5) transformations are given with the parameter $\omega_+$ in (3.8) as
\[
\delta (V_a + \tilde{V}_a) = (\omega_+)_{ab}(V_b + \tilde{V}_b)
\]
\[
V_a = p_a Z^5 - \frac{1}{8} \epsilon_{aa_1...a_4}Z^{[2]a_1a_2}Z^{[2]a_3a_4}, \quad \tilde{V}^a = p_b Z^{[2]b\alpha},
\] (3.10)
where $Z_A = \nu^M A Z_M$. The left mover is transformed similarly by replacing the $\pm$ signs.

### 3.2 SO(5,5) transformation of M5 constraints

For a probe M5-brane system worldvolume diffeomorphism constraints must be covariant under the SO(5,5) transformations. Diffeomorphism constraints $\mathcal{H}_t = 0$ in (2.10) are recasted into the ones for M2-brane and M5-brane as in (2.13). In d=5 M5-brane diffeomorphism constraints $(Z_{\tilde{M}M} Z_{MN} Z_{m_1...m_4} \approx 0$ become a five dimensional vector. Together with the one for M2 it forms a SO(5,5) fundamental vector multiplet
\[
\begin{align*}
(Z_{\tilde{M}[M} Z_{N]} Z_{m} &= 2Z_m Z^5 - \frac{1}{4} \epsilon_{mm_1...m_4} Z^{[2]m_1m_2} Z^{[2]m_3m_4} \approx 0 \\
(Z_{\tilde{M}[M} Z_{N]} Z^{mn} &= 2Z_n Z^{[2]mn} \approx 0
\end{align*}
\] (3.11)
Under the SO(5,5) transformation given in (3.6)
\[
\begin{align*}
\delta Z_m &= \alpha_m n Z_n - \frac{\hat{\alpha}}{2} Z_m - \frac{1}{2} \gamma_{mm_1m_2} Z_{[2]m_1m_2} \\
\delta Z^{[2]m_1m_2} &= \tilde{\beta}^{m_1m_2 n} Z_n + \alpha_n [m_1 Z^{m_2 n} + \frac{\hat{\alpha}}{2} Z_{[2]m_1m_2} - \gamma_{m_1m_2} Z^5] \\
\delta Z^5 &= \frac{1}{2} \beta_{m_1m_2} Z_{[2]m_1m_2} + \frac{\hat{\alpha}}{2} Z^5
\end{align*}
\]
they make a covariant SO(5,5) vector, which are consistent with the constraints:
\[
\begin{align*}
\delta (Z_{\tilde{M}[M} Z_{N]} Z_m &= \alpha_m n (Z_{\tilde{M}[M} Z_{N]} Z_n + \beta_{mn} (Z_{\tilde{M}[M} Z_{N]} Z^n \\
\delta (Z_{\tilde{M}[M} Z_{N]} Z^{mn} &= - \alpha_n [m (Z_{\tilde{M}[M} Z_{N]} Z^n) - \gamma_{mn} (Z_{\tilde{M}[M} Z_{N]} Z)_n]
\end{align*}
\]
It is interesting to notice that the M5 diffeomorphism constraints $(Z_{\tilde{M}[M} Z_{N]} Z_m = 0$ is equal to $\tilde{p}_m = \tilde{p}_m + TG_{mtn} \approx 0$ in (2.2).

Now let us examine the SO(5,5) transformations of the supergravity background fields. The Hamiltonian constraint in (2.11) is written, for 5-dimensional part, as
\[
\mathcal{H}_\perp = \frac{e^{2/3}}{2T} Z_M M^{MN} Z_N
\]
where a normalization \( \det \nu = 1 \) is chosen. The generalized metric \( \mathcal{M} = \nu^T \nu \) is written in terms of the metric and the gauge field, \( (G_{mn}, C^{[3]}_{mnl}) \), which are 25 coset parameters of the coset \( \text{SO}(5,5)/\text{SO}(5) \times \text{SO}(5) \). Under the \( \text{SO}(5,5) \) transformations \( Z_M \) and \( \nu_A^M \) are transformed as

\[
Z_M \to (1 + S)_M^N Z_N, \quad \nu_A^M \to (1 + S_h)_A^B \nu_B^N (1 + S)^{-1}_N^M.
\] (3.13)

The transformation matrices \( (1 + S) \in \text{SO}(5,5) \) and \( (1 + S_h) \in \text{SO}(5) \times \text{SO}(5) \) are given in (3.6) and (3.9) where the pullback parameter is determined as \( \bar{\omega}_{ab} = e^{\phi/2} \bar{e}^m_a \bar{e}^n_b \beta_{mn} \). Although the \( \text{SO}(5,5) \) transformation of \( e_m^a \) depends on \( \text{SO}(5) \) parameter \( \omega \) as

\[
\delta e_m^a = \frac{3}{10} (-\dot{\alpha} + \beta_{nl} \bar{C}^{[3]nl}) e_m^a + \bar{C}^{[3]al} \beta_{lm} - \omega_m^a + \alpha_m^a,
\]

the transformations rules of \( \text{SO}(5) \) invariant gauge fields does not depend on it. The obtained \( \text{SO}(5,5) \) transformation rules of the metric and the gauge field are given as

\[
\begin{align*}
\delta G_{mn} &= \frac{3}{5} (-\dot{\alpha} + \beta_{1l} \bar{C}^{[3]1l}) G_{mn} + \beta_{1l} (m G_n)_{1l} \bar{C}^{[3]1l} + \alpha_{(m} G_{n)l}, \\
\delta C^{[3]}_{mnl} &= (-\dot{\alpha} + \frac{1}{2} \beta_{1l} \bar{C}^{[3]1l}) C^{[3]}_{mnl} + \frac{1}{2} \alpha_{(m} C^{[3]}_{n)l}, \\
\bar{\beta}_{mnl} &= \frac{1}{2} \epsilon_{mll1} C^{[3]}_{l1} G^{l2n2} \beta_{1n2}.
\end{align*}
\] (3.14)

The transformation rule of \( C^{[3]} \) is the fractional linear transformation as expected.

## 4 Extended space

An extended space with manifest \( \text{SO}(5,5) \) duality symmetry is proposed by introducing coordinates \( X^M = (x^m, y_{mn}, y) \). They are subject to subsidiary conditions on functions \( f(X^M) \)

\[
\partial_M \rho^{MN}_{[M_2]N} \partial_N = \partial_M \tilde{\rho}^{MN}_{[M_5]N} \partial_N = 0,
\]
where \( \partial_M = \frac{\partial}{\partial x^M} \) and \( \tilde{\rho}^{MN} \) and \( \tilde{\rho}^{MN}_{[M_2]} \) are given in (2.13). In components they are
\[
\frac{\partial}{\partial x^m} \frac{\partial}{\partial y_{mn}} = \frac{\partial}{\partial x^m} \frac{\partial}{\partial y} + \frac{1}{8} \delta_{mn12} \frac{\partial}{\partial y_{m1m2}} \frac{\partial}{\partial y_{m3m4}} = 0 \quad , \quad m = 1, \cdots, 5 .
\]
The extended space has manifest SO(5,5) symmetry, where SO(5,5) spinor coordinates are transformed as \( \delta X^M = -X^N S_N^M \) and \( \delta \partial_M = S_M^N \partial_N \) preserving the canonical bracket \( \{ \partial_M , X^N \} = \delta_M^N \). Extended coordinates are transformed as
\[
\begin{align*}
\delta x^m &= \frac{\hat{\alpha}}{2} x^m - x^\alpha n^m - \frac{1}{2} y_{n1n2} \tilde{n}_{1n2}^m \\
\delta y_{m1m2} &= x^n \tilde{n}_{m1m2} - \frac{\hat{\alpha}}{2} y_{m1m2} - y_{[m1} \alpha_{m2]} - y_{m1m2} \quad .
\end{align*}
\]
(4.1)

On the other hand under SO(5) \( \times \) SO(5) \( \ni h \) satisfies \( h^T \delta h = \delta \). Under the SO(5) \( \times \) SO(5) \( \ni h \) with “flat coordinates”, \( x^a = e^m_a x^m \), \( y_{ab} = e^m_a e^m_b y_{m1m2} \) and \( y \), are transformed as
\[
\begin{align*}
\delta x^a &= -\frac{1}{2} \tilde{r}^b \omega^a_{-b} + \frac{1}{4} y_{b1b2} \tilde{\omega}_+^{b1b2a} \\
\delta y_{a1a2} &= \frac{1}{2} \tilde{r}^b \tilde{\omega}_{+;b1a2} - \frac{1}{2} y_{b[a1} \tilde{\omega}^b_{+|a2]} + \frac{1}{2} y_{ba1a2} \\
\delta y &= \frac{1}{4} y_{b1b2} \tilde{\omega}_+^{b1b2} \\
\end{align*}
\]
(4.2)

A SO(5,5) covariant C-bracket in this extended space is proposed as
\[
\Lambda = \Lambda^M(X) Z_M = \Lambda^M(x^m, y_{m1m2}, y) Z_M \\
\left( [\Lambda_1, \Lambda_2]_C \right)^M = \Lambda_1^N \delta_N \Lambda_2^M + \Lambda_1^N f_{NLM}^{M;K} \partial_K \Lambda_2^L \quad .
\]
(4.3)

where \( f_{NLM}^{M;K} \) is SO(5,5) covariantized version of the symmetric structure constant \( f_{NLM}^{M;K} \) in (2.20). This bracket is reduced to the Courant bracket for the M5 brane obtained in (2.23) with the choice \( \partial/\partial y_{m1m2} = \partial/\partial y = 0 \) and \( K = -1 \).

5 Summary and discussion

We have presented a M5 algebra from canonical analysis of a M5-brane in the supergravity background. The M5 algebra is closed by the Gauss law constraint which is the first class part of the selfduality condition. The second class constraints of the selfduality condition are treated by the Dirac bracket. We have derived a series of Courant brackets.
on the generalized geometry $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*$ from the M5 algebra. The Dirac bracket between selfdual gauge fields gives a $C^{[3]}$-twisted term in the Courant bracket. By using it generalized diffeomorphism transformations including gauge transformations for $C^{[3]}$ and $C^{[6]}$ are derived.

The M-theory compactified on five dimensional torus has SO(5,5) duality symmetry, where the M5 algebra basis is in 16 dimensional SO(5,5) spinor representation. The worldvolume diffeomorphism constraints are written in bilinear forms in the M5 algebra basis, and they form a 10 dimensional SO(5,5) vector. The generalized metric of $\mathcal{H}_\perp$ contains the metric and three form gauge field, $G_{mn}$ and $C^{[3]}_{mnl}$, which are $15 + 10 = 25$ parameters of the coset SO(5,5)/[SO(5) × SO(5)].

We have also proposed an extended space with manifest SO(5,5) duality symmetry with section conditions determined from the M5 worldvolume diffeomorphism constraints. The SO(5,5) covariant C-bracket is also written down.

So far we have derived Courant brackets for M-theory in $d = 4, 5$ by M2 and M5 algebra, while exceptional generalized geometry in the brane algebra approach has not been discussed yet. We have analyzed only bosonic sector of supergravity of the M-theory. Inclusion of fermions is an interesting issue for which there are several prior researches e.g. [13]. U-duality is raised by supersymmetry while supersymmetric probe branes have $\kappa$-symmetry in addition to the worldvolume diffeomorphism constraints. It is interesting to clarify the role of these constraints in the generalized geometry. These issues are left for future investigations.

**Acknowledgements**

M.H. would like to thank Yutaka Matsuo, Warren Siegel and Maxim Zabzine for fruitful discussions especially on M5 and selfdual gauge field. She also thanks to Satoshi Iso and Takeshi Morita for valuable discussions. The work of M.H. is supported by Grant-in-Aid for Scientific Research (C) No. 24540284 from The Ministry of Education, Culture, Sports, Science and Technology of Japan.

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