The integrable $O(6)$ model and the correspondence: checks and predictions

Francesco Buccheri∗ and Davide Fioravanti†

Sezione INFN di Bologna, Dipartimento di Fisica, Università di Bologna,
Via Irnerio 46, Bologna, Italy

Abstract

We exactly compute the energy density of the integrable $O(n)$ non-linear sigma model as a convergent series. This series is specifically analysed for the very important $O(6)$ symmetry, since it was suggested to result as a peculiar limit of the AdS string theory by Alday and Maldacena [1]. In this respect, the $O(6)$ model gives also refined confirmations and predictions once compared with the SYM Bethe Ansatz [2, 3].

∗ Laurea student of the University of Bologna ”Alma Mater Studiorum”.
† E-mail: bucca.virus@tin.it and fioravanti@bo.infn.it.
1 The question

The planar $sl(2)$ sector of $\mathcal{N} = 4$ SYM contains local composite operators of the form

$$\text{Tr}(\mathcal{D}^s\mathcal{Z}^L) + \ldots, \quad (1.1)$$

where $\mathcal{D}$ is the (symmetrised, traceless) covariant derivative acting in all possible ways on the $L$ bosonic fields $\mathcal{Z}$. The spin of these operators is $s$ and $L$ is the so-called ‘twist’. Moreover, this sector would be described – via the AdS/CFT correspondence [4] – by rotating string states on the AdS$_5 \times $S$^5$ spacetime with AdS$_5$ and S$^5$ charges $s$ and $L$, respectively [5, 6]. Proper superpositions of the operators (1.1) have definite conformal dimension $\Delta$ depending on 't Hooft coupling $\lambda = 8\pi^2 g^2$

$$\Delta = L + s + \gamma(g, s, L), \quad (1.2)$$

with $\gamma(g, s, L)$ the anomalous part. In fact, the correspondence would assign to this dimension $\Delta$ the exact value of the energy density of a rotating string, provided the 't Hooft coupling is identified with the string tension: $\sqrt{\lambda} = \frac{R^2}{\alpha'}$. This is indeed a duality relation between the coupling constants involving the semi-classical expansion on the string side [5, 6].

A great boost in the evaluation of the anomalous $\gamma(g, s, L)$ has come from the discovery of integrability and thus of a bethe Ansatz, although in another sector, the purely bosonic $so(6)$, and at one loop of the gauge theory [7]. On the other hand, in the twist sector of one loop QCD the (integrable) Bethe Ansatz problem was at hand [8, 9] and later on it has been realised to be equivalent to its supersymmetric relative with the occurrence of the integrability extended to the whole theory and at all loops [10]. This is the scenario on the side of the SYM theory in the sense that, for instance, any operator of the form (1.1) is associated to one solution of some (asymptotic 1) Bethe Ansatz-like equations and then any anomalous dimension is expressed in terms of this solution.

As a confirmation of the correspondence, integrability has been also uncovered and studied in the superstring theory [12] and in this respect our interest will be limited to the semi-classical calculations. In this approach, the string tension diverges since it plays the rôle of the inverse of the quantum Planck constant. Therefore, the $\lambda \rightarrow +\infty$ limit yields a power expansion in $1/\sqrt{\lambda}$ [13]. Which, in particular means, that it needs to be implemented before any other limit and thus endowing the semiclassical calculations with a different limit order with respect to the gauge theory (cf. below for more details and [2, 3]).

On both the string and the gauge theory, an important double scaling may be considered:

$$s \rightarrow \infty, \quad L \rightarrow \infty, \quad j = \frac{L}{2\ln s} = \text{fixed}. \quad (1.3)$$

1This important limitation emerged because the Bethe Ansatz is realised by somehow using the on-shell S-matrix [11].
In fact, the relevance of this logarithmic scaling for the anomalous dimension has been pointed out and deeply studied in [13] and [1] within the semi-classical string theory (cf. also [14] within the one-loop SYM theory). Moreover, these long operators with large spin have been recently shown to satisfy the Sudakov scaling [1, 15]

\[ \gamma(g, s, L) = f(g, j) \ln s + O((\ln s)^{-\infty}), \]

which generalises the one loop result of [14]. Actually, in [15] this statement was argued by computing iteratively the solution of some integral equations and then, thereof, the generalised scaling function, \( f(g, j) \) at the first orders in \( j \) and \( g^2 \): more precisely the first orders in \( g^2 \) have been computed for the first generalised scaling functions \( f_n(g) \), forming

\[ f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n. \]

As a by-product, the reasonable conjecture has been put forward that the two-variable function \( f(g, j) \) should be analytic (in \( g \) for fixed \( j \) and in \( j \) for fixed \( g \)). In [18] similar results have been derived for what concerns the contribution beyond the leading scaling function \( f(g) = f_0(g) \), but with a modification which has allowed to neglect the non-linearity for finite \( L \) and to end-up with one linear integral equation (LIE). The latter does not differ from the BES one (which covers the case \( j = 0 \), cf. the last of [10]), but for the inhomogeneous term. Moreover, a suitable modification of this LIE has been applied in [2, 3] to derive still a LIE in the scaling (1.3) (for any \( g \) and \( j \)). This is indeed, a way to determine the generalised scaling function \( f(g, j) \) and also its constituents \( f_n(g) \) for all values of \( j \) and \( g \), thus interpolating from weak to strong coupling. Today, an interesting paper [16] appears which seems to have some equivalent equation, coming from [15], from which it apparently derives a map to the \( O(6) \) sigma model and the leading strong coupling behaviour of \( f_3 \) (as \( f_2 = 0 \) appeared already in [15] and \( f_1 \) in [2]).

In the following, we will constrain ourselves to the analysis of the \( O(N) \) energy density of the non linear sigma models, since one representative, the \( O(6) \) model, was suggested by Alday and Maldacena [1] to represent the limit theory with small \( SO(6) \) charge \( j \gg \sqrt{\lambda} \). For in this regime the masses of the fermions do not contribute to the energy density, but to \( f_0(g) \), where they give the natural UV cut-off. Therefore, we will show that the additional energy density, \( 2\Omega(g, j) \) (cf. (8.75)) contained in \( f(g, j) \) can be computed exactly at least at the leading mass gap (\( m \) of (2.7)) order by means of the \( O(6) \) computations. Hence, we will compute below the \( O(6) \) energy density as a convergent series in \( j/m \), checking a perfect agreement with the gauge theory computations of [3] up to the first interaction and model depending term \( f_4(g) \), i.e. \( \Omega_4 \). An all order explicit match would be desirable for the future and in this respect the following \( O(6) \) model results give a series of exact predictions.

\[^2\] \( O(\ln s^{-\infty}) \) means a remainder which goes faster that any power of \( \ln s \):

\[ \lim_{s \to \infty} (\ln s)^k O((\ln s)^{-\infty}) = 0, \forall k > 0. \]
2 The O(N) nonlinear sigma model

We want to show that the result in [19] for the energy density \( \Omega \) as a function of the particle density \( j \) in the strong coupling regime \( (g \gg 1) \) with \( j \ll m \), namely the “non-interacting fermion gas” approximation, keeps correct when all orders in \( B \) are retained in the calculation, i.e. that it is the correct result up to the third order \( j^3 \). In formulas, that

\[
\Omega = m^2 \left( \frac{j}{m} + \frac{\pi^2}{6} \left( \frac{j}{m} \right)^3 + O \left( \left( \frac{j}{m} \right)^4 \right) \right) \tag{2.6}
\]

where the mass gap is (see [21] and [1], with \( \sqrt{\lambda} = \frac{1}{t} \))

\[
m = \frac{2^{3/4} \pi^{1/4}}{\Gamma(5/4)} g^{1/4} e^{-\pi g} + \ldots \tag{2.7}
\]

We begin in full generality, by considering a O(N) nonlinear sigma model. A manipulation of BA equations [19] provides a "pseudoenergy" \( \varepsilon(\vartheta) \) as the solution of a linear integral equation of Fredholm type

\[
\varepsilon(\vartheta) - \int_{-B}^{B} K(\vartheta - \vartheta') \varepsilon(\vartheta') d\vartheta' = h - m \cosh \vartheta \tag{2.8}
\]

With the boundary condition

\[
\varepsilon(B) = 0 \tag{2.9}
\]

The condition (2.8) allows us, at least in principle, to determine the value of the parameter \( B \). The kernel \( K(\vartheta) \) comes from the [20] two-particle S-matrix through

\[
K(\vartheta) = \frac{1}{2\pi i} \frac{\partial}{\partial \vartheta} \log S(\vartheta) \tag{2.10}
\]

where \( \Delta = \frac{1}{N-2} \). What is more, the Bethe Ansatz procedure gives the Helmholtz free energy \( f \) as

\[
-\frac{2\pi}{m} f(h, m) = \int_{-B}^{B} \cosh \vartheta \varepsilon(\vartheta) d\vartheta \tag{2.11}
\]

while general Thermodynamics provides the density of particles

\[
j = -\frac{\partial}{\partial h} f(h, m) \tag{2.12}
\]

This matrix contains an ambiguity of the CDD kind [17], as underlined by Zamolodchikov and Zamolodchikov. What people generally do, is to use the most simple solution [20]
Gathering up all the elements, an optimistic procedure would be that of solving (2) for \( \epsilon \), put it into (2.11), then calculate \( j \) by (2.12) and, finally, \( \Omega \) by Legendre transform

\[
\Omega (j, m) = f(h, m) + jh \tag{2.13}
\]

The “non-interacting fermion gas” approximation corresponds to the limit \( B \to 0 \) in (2), i.e. \( \frac{h-m}{m} \to 0^+ \). Indeed, considering \( S(\vartheta) = -1 \) and \( K(\vartheta) = \frac{1}{2\pi} \frac{d}{d\vartheta} S(\vartheta) = 0 \), is equivalent to saying that \( B \to 0 \) allows one to retain only the forcing therm of (2), that is the zero-th order approximation of the Liouville-Neumann procedure.

3 The 1D non-interacting fermion gas

Despite its simplicity, this model is noteworthy because it represents a paradigmatic example of how Bethe Ansatz works and also because it produces results that can directly be compared with those coming from more advanced calculations from the SYM front. What we will call ”fermions” are introduced in a simple fashion by defining their two-particle S-matrix as

\[
S(\vartheta) = -1 \tag{3.14}
\]

now we can set in motion the calculation sketched above, with \( K(\vartheta) = 0 \) for (2.10). Then, by (2), (2.12), (2.11), (2.13) we have

\[
\epsilon(\vartheta) = h - m \cosh \vartheta \tag{3.15}
\]

\[
f(h) - f(0) = -\frac{m}{2\pi} \int_{-B}^{B} (h-m \cosh \vartheta ) \cosh \vartheta d\vartheta = \frac{m}{2\pi} [2h \sinh B - mB - m \cosh B \sinh B] \tag{3.16}
\]

\[
j = \frac{m}{2\pi} \int_{-B}^{B} \cosh \vartheta d\vartheta = \frac{m}{\pi} \sinh B \tag{3.17}
\]

\[
\Omega = f + jh = \frac{m^2}{2\pi} [B + \sinh B \cosh B] = \frac{m^2}{2\pi} \left[ \arcsin h \left( \frac{\pi}{m} j \right) + \frac{\pi}{m} j \left( 1 + \left( \frac{\pi}{m} j \right)^2 \right)^{1/2} \right] = \frac{m^2}{2\pi} \left[ \frac{\pi}{m} j - \frac{1}{6} \left( \frac{\pi}{m} j \right)^3 + \frac{3}{40} \left( \frac{\pi}{m} j \right)^5 + \frac{\pi}{m} j \left( 1 + \frac{1}{2} \left( \frac{\pi}{m} j \right)^2 - \frac{1}{8} \left( \frac{\pi}{m} j \right)^4 \right) \right] + O \left( \frac{j^7}{m^7} \right) \tag{3.18}
\]

The series contains only the odd powers of \( j \): we will see that turning on an interaction will turn on the coefficients of the even, powers from the fourth power on, together with affecting the odd terms.
4 The relativistic interacting gas: some preliminaries

The formal solution of (2) is easily seen to be
\[ \varepsilon(\vartheta) = h - m \cosh \vartheta + \int_{-B}^{B} K(\vartheta - \vartheta') \left( \varepsilon(\vartheta') - \varepsilon(\vartheta) \right) d\vartheta' = \]
\[ \ldots = h - m \cosh \vartheta + \]
\[ + \sum_{n=1}^{\infty} \int_{-B}^{B} \int_{-B}^{B} d\vartheta_{1} \ldots d\vartheta_{n} K(\vartheta - \vartheta_{1}) K(\vartheta_{1} - \vartheta_{2}) \ldots K(\vartheta_{n-1} - \vartheta_{n}) (h - m \cosh \vartheta_{n}) = (4.19) \]
\[ = 1 + \sum_{n=1}^{\infty} B^{n} \int_{-1}^{1} \ldots \int_{-1}^{1} dx_{1} \ldots dx_{n} K(Bx - Bx_{1}) K(Bx_{1} - Bx_{2}) \]
\[ \ldots K(Bx_{n-1} - Bx_{n}) (h - m \cosh(Bx_{n})) (4.20) \]

where
\[ K(\vartheta) = \frac{1}{4\pi^{2}} \left[ \psi \left(1 + \frac{i\vartheta}{2\pi}\right) - \psi \left(\frac{1}{2} + \frac{i\vartheta}{2\pi}\right) + \psi \left(\frac{1}{2} + \frac{i\vartheta}{2\pi}\right) - \psi \left(\Delta + \frac{i\vartheta}{2\pi}\right) + \psi \left(1 - \frac{i\vartheta}{2\pi}\right) - \psi \left(\frac{1}{2} - \frac{i\vartheta}{2\pi}\right) + \psi \left(\frac{1}{2} + \frac{i\vartheta}{2\pi}\right) - \psi \left(\Delta - \frac{i\vartheta}{2\pi}\right) \right] \quad (4.21) \]

with
\[ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx} \ln \Gamma(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+x} \right) \]

where \(\gamma\) is the Euler-Mascheroni constant. Its \(h\)-derivative, by the aid of (2) and (2.8) reads
\[ \frac{\partial \varepsilon_{h}(\vartheta)}{\partial h} = 1 + \left( K(\vartheta - B) \varepsilon(B) \right) \frac{\partial B}{\partial h} + \int_{-B}^{B} d\vartheta K(\vartheta - \vartheta') \frac{\partial \varepsilon_{h}(\vartheta')}{\partial h} \]
\[ = 1 + \int_{-B}^{B} d\vartheta K(\vartheta - \vartheta') \frac{\partial \varepsilon_{h}(\vartheta')}{\partial h} \quad (4.22) \]

that is, another Fredholm integral equation, with a constant forcing term. Its solution is
\[ \frac{\partial \varepsilon_{h}(\vartheta)}{\partial h} = 1 + \sum_{n=1}^{\infty} \int_{-B}^{B} \int_{-B}^{B} d\vartheta_{1} \ldots d\vartheta_{n} K(\vartheta - \vartheta_{1}) K(\vartheta_{1} - \vartheta_{2}) \ldots K(\vartheta_{n-1} - \vartheta_{n}) = \]
\[ = 1 + \sum_{n=1}^{\infty} B^{n} \int_{-1}^{1} \ldots \int_{-1}^{1} dx_{1} \ldots dx_{n} K(Bx - Bx_{1}) K(Bx_{1} - Bx_{2}) \]
\[ \ldots K(Bx_{n-1} - Bx_{n}) \quad (4.23) \]
By the use of the formula
\[
\psi \left( \frac{1}{2} + \frac{1}{2}x \right) - \psi \left( \frac{1}{2}x \right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k + x}
\]
we can rewrite the Kernel as
\[
K(x) = \frac{1}{2\pi} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi(k+1) + i(\vartheta - \vartheta')} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi(k+2\Delta) + i(\vartheta - \vartheta')} \right]
\]
\[
+ \sum_{k=0}^{\infty} \frac{(-1)^k}{\vartheta(k+1) - i(\vartheta - \vartheta')} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi(k+2\Delta) - i(\vartheta - \vartheta')} \right]
\]
\[
= \frac{1}{2\pi} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k 2\pi(k+1)}{\pi^2(k+1)^2 + (\vartheta - \vartheta')^2} + \sum_{k=0}^{\infty} \frac{(-1)^k 2\pi(k+1)}{\pi^2(k+2\Delta)^2 + (\vartheta - \vartheta')^2} \right]
\]
\[
= \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left[ \frac{1}{k+1} + \frac{(-1)^k}{(\vartheta - \vartheta')^2} + \frac{1}{k+2\Delta} + \frac{(-1)^k}{(\vartheta - \vartheta')^2} \right] \quad (4.24)
\]

Some comments are to be made. First of all, even powers of \( j \) do not appear in this expansion, because the energy density is written as the sum of two odd function. Of course, it follows that all even powers in the interacting case will vanish when we shut down the interaction. What is more, the first three term are somehow "stable" (see comments to (5.42) against the insertion of a less trivial S-matrix.

5 The very first orders in \( B \)

We recall that in the O(6) strong-coupling regime we have \( B << 1 \). This way written, the kernel allows an easier calculation of the first few orders in \( B \) of the quantity \( \varepsilon(\vartheta) \). The first, raw approximation consists in writing the solution \( \varepsilon(\vartheta) \) as composed of the forcing term alone, \( \varepsilon(\vartheta) = h - m \cosh \vartheta \). The boundary condition implies
\[
B = \arccosh \left( \frac{h}{m} \right) \simeq \sqrt{2 \frac{h - m}{m}} \quad (5.25)
\]

This bare result is already enough to obtain the fermion gas approximation, as already stated. We would like to know how much this result for the energy density is reliable, i.e. if calculations involving higher orders in \( B \) are likely to leave the first coefficient as they are, or else if they are destined to upset the solution. We will go on by brute-force Taylor expansion in \( B \). To begin, we illustrate the second order case, an then we will give the systematics of the method.
5.1 What we can do up to the order $B^2$

By defining the new variable $x = \frac{\vartheta}{B}$ and expanding the fractions that compose the kernel in $B$ we obtain, up to the order $B^2$

$$
\varepsilon(\vartheta) = h - m \cosh \vartheta + B \int_{-1}^{1} K(\vartheta - Bx)\varepsilon(By)dx = 
$$

$$
= h - m \cosh \vartheta + B \int_{-1}^{1} K(\vartheta - Bx) \left( h - m \cosh Bx + B \int_{-1}^{1} K(Bx - By)\varepsilon(By)dy \right) dx = 
$$

$$
= h - m \cosh \vartheta + \frac{B}{\pi^2} \int_{-1}^{1} dx \sum_{k=0}^{\infty} \left[ \frac{1}{k+1} + \frac{(-1)^k}{\pi(k+1)} \right] \varepsilon(By)dy 
$$

$$
= h - m \cosh \vartheta + \frac{B}{\pi^2} \int_{-1}^{1} dx \sum_{k=0}^{\infty} \left[ \frac{1}{k+1} + \frac{(-1)^k}{\pi(k+1)} \right] \varepsilon(By)dy 
$$

$$
= h - m \cosh \vartheta + \frac{B}{\pi^2} \sum_{k=0}^{\infty} \int_{-1}^{1} \left( \frac{(-1)^k}{k+1} + \frac{(-1)^k}{k+2\Delta} \right) \varepsilon(By)dy 
$$

$$
= h - m \cosh \vartheta + \frac{B}{\pi} \int_{-1}^{1} \left[ S_1 - S_3 (\vartheta - Bx)^2 \right] 
$$

where

$$
S_1 = \frac{1}{\pi} \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{k+1} + \frac{(-1)^k}{k+2\Delta} \right] \quad \text{and} \quad S_3 = \frac{1}{\pi^3} \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{(k+1)^3} + \frac{(-1)^k}{(k+2\Delta)^3} \right] 
$$

$$
\varepsilon(\vartheta) \approx h - m \cosh \vartheta + 
$$

$$
\frac{B}{\pi} \int_{-1}^{1} \left[ S_1 - S_3 (\vartheta - Bx)^2 \right] \left[ h - m \left( 1 + \frac{1}{2} B^2 x^2 \right) + \frac{B}{\pi} \int_{-1}^{1} S_1 (h - m) dy \right] dx = 
$$

$$
\approx h - m \cosh \vartheta + \frac{B}{\pi} \int_{-1}^{1} \left[ S_1 - S_3 (\vartheta - Bx)^2 \right] \left[ (h - m) - \frac{m}{2} B^2 x^2 + 2 S_1 (h - m) \right] dx = 
$$

$$
\approx h - m \cosh \vartheta + \frac{B}{\pi} \left( 2 S_1 (h - m) - \frac{m}{2} S_1 B^2 \int_{-1}^{1} x^2 dx + 4 \frac{B}{\pi} S_1^2 (h - m) + 
$$

$$
- S_3 (h - m) \int_{-1}^{1} (\vartheta - Bx)^2 dx + \frac{m}{2} S_3 B^2 \int_{-1}^{1} (\vartheta - Bx)^2 x^2 dx + 
$$

$$
- 2 \frac{B}{\pi} S_1 S_3 \int_{-B}^{B} (\vartheta - Bx)^2 \right) dx
$$

(5.27)
$$\varepsilon(B) = h - m \left(1 + \frac{1}{2}B^2\right) + \frac{2}{\pi} (h - m) S_1 B + \frac{4}{\pi^2} (h - m) S_1^2 B^2 + o(B^3)$$

$$= h - m + \frac{2}{\pi} (h - m) S_1 B + \left(-\frac{m}{2} + \frac{4}{\pi^2} (h - m) S_1^2\right) B^2 + o(B^3) \quad (5.28)$$

By solving the condition $\varepsilon(B) = 0$ we gain an expression for the extreme

$$B \simeq \frac{-(h - m) S_1}{\pi} + \sqrt{\frac{1}{\pi^2} S_1^2 (h - m)^2 + \frac{m}{2} (h - m) - \frac{4}{\pi^2} (h - m)^2 S_1^2}$$

$$= \left(-\frac{h - m}{\pi} S_1 - \sqrt{\frac{m}{2} (h - m) \left(1 - \frac{3}{2\pi^2} S_1^2 (h - m)^2\right)}\right) \frac{2}{-m} \left(1 + 8 \frac{S_1^2 h - m}{m}\right) + \ldots =$$

$$= \sqrt{\frac{2}{m} (h - m) + \frac{2}{\pi} S_1 \frac{h - m}{m} + O \left(\left(\frac{h - m}{m}\right)^{\frac{3}{2}}\right)} \quad (5.29)$$

where we have coherently omitted all contributions from $B^3$ on, that is $\left(\frac{h - m}{m}\right)^{\frac{3}{2}}$ and higher powers. For what the free energy is concerned, we can write the BA formula and approximate to the second order in $B$

$$-\frac{2\pi}{m} f(h, m) = \int_{-B}^{B} \cosh \varepsilon(\vartheta) \, d\vartheta = B \int_{-1}^{1} \cosh (B x_0) \varepsilon(B x_0) \, dx_0$$

$$\simeq B \int_{-1}^{1} \cosh (B x_0) \left(h - m \cosh (B x_0) + B \int_{1}^{1} S_1 (h - m) \, dx_1\right) \, dx_0$$

$$\simeq B \int_{-1}^{1} \left(1 + \frac{1}{2} B^2 x_0^2\right) \left(h - m - \frac{m}{2} B^2 x_0^2 + B \int_{1}^{1} S_1 (h - m) \, dx_1\right) \, dx_0$$

$$\simeq B \int_{-1}^{1} (h - m + 2BS_1 (h - m)) \, dx_0 = 2(h - m)B + 4(h - m)S_1 B^2 \quad (5.30)$$

the density of thermodynamics, up to the same order, is

$$j = -\frac{\partial}{\partial h} f(h, m) = \frac{m}{2\pi} \int_{-B}^{B} d\vartheta \varepsilon(\vartheta) \cosh \vartheta = \frac{m}{2\pi} B \int_{1}^{1} \left(1 + S_1 B \int_{1}^{1} dy\right) \, dx =$$

$$= \frac{m}{\pi} (B + 2S_1 B^2) \quad (5.31)$$

so that the energy density becomes

$$\Omega = \frac{m}{2\pi} B \int_{1}^{1} \left(1 + S_1 B \int_{1}^{1} dy\right) - \frac{m}{2\pi} (2(h - m)B + 4(h - m)S_1 B^2) + O(B^3) =$$

$$= \frac{m^2}{\pi} \left(B + 2S_1 B^2\right) + O(B^3) = mj + O(j^3) \quad (5.32)$$
Indeed, contributions of order $B^2$ are all included in the $mj$ term. Unfortunately, to catch the $j^3$ contribution we have to keep the $B^3$s, as well as to catch the $j^n$ contribution we have to keep the $B^n$s, and consider at least $(n-1)$ nested kernels in the pseudoenergy.

### 5.2 What we can do up to the order $B^3$

Once again, we briefly repeat the very same steps as before, reaching one higher power of $B$.

$$
\varepsilon(\vartheta) = h - m \cosh \vartheta + B \int_{-1}^{1} dK(\vartheta - Bx) \left[ h - m \cosh(Bx) + B \int_{-1}^{1} dyK(Bx - By)(h - m) \right] \tag{5.33}
$$

$$
f \simeq -\frac{m}{2\pi} B \int_{-1}^{1} dx \left( 1 - \frac{1}{2} B^2 x^2 \right) \left\{ h - m - \frac{m}{2} B^2 x^2 + B \int_{-1}^{1} dxK(Bx - By) \left[ h - m - \frac{m}{2} B^2 x^2 + 2B \frac{1}{\pi} S_1 (h - m) \right] \right\} \tag{5.34}
$$

$$
j = -\frac{\partial f}{\partial h} \simeq \frac{m}{2\pi} \int_{-B}^{B} d\vartheta \cosh \vartheta \left[ 1 + 2B \frac{1}{\pi} S_1 + 4B^2 \left( \frac{1}{\pi} S_1 \right)^2 \right] = \frac{m}{\pi} \left( B + \frac{1}{6} B^3 \right) \left[ 1 + 2B \frac{1}{\pi} S_1 + 4B^2 \left( \frac{1}{\pi} S_1 \right)^2 \right] = \frac{m}{\pi} \left( B + 2 \frac{1}{\pi} S_1 B^2 + 4 \frac{1}{\pi^2} S_1^2 B^3 + \frac{1}{6} B^3 \right) \tag{5.35}
$$

$$
\Omega = f + jh \simeq \frac{m^2}{2\pi} B \int_{-1}^{1} dx \left( 1 - \frac{1}{2} B^2 x^2 \right) \left[ 1 + \frac{1}{2} B^2 x^2 + 2B \frac{1}{\pi} S_1 + 4B^2 \left( \frac{1}{\pi} S_1 \right)^2 \right] = mj + \frac{m^2}{6\pi} B^3 \tag{5.36}
$$

now, the term with $B^3$ can come only from the third power of the density $j$, because we have already acknowledged the term linear in $j$, so we calculate the coefficient of the $j$-expansion of $\Omega$ as the ratio among $\frac{m^2}{6\pi}$ and the coefficient of the first power of $B$
in the expansion of \( j \):

\[
\Omega_3 = \frac{m^2}{(\pi)^2} = \frac{\pi^2}{6m}.
\]

So we have

\[
\Omega = mj + \frac{\pi^2}{6m}j^3 + O(j^4)
\]  

(5.37)

5.3 A small improvement

We saw in (5.29) that \( B \approx \sqrt{2 \frac{h-m}{m}} + O \left( \frac{h-m}{m} \right) \). We can use this to show the stability of the non-interacting fermion gas approximation and that the most crude approximation yields a correct coefficient of \( j^3 \).

\[
\varepsilon (\vartheta) = h - m \cosh \vartheta + \int_{-B}^{B} d\vartheta' K (\vartheta - \vartheta') \left[ h - m \cosh (\vartheta') + B \int_{-1}^{1} d\vartheta'' K (\vartheta' - \vartheta'') (h - m) \right] + O(B^3)
\]

\[
= h - m \cosh \vartheta + B \int_{-1}^{1} d\vartheta' \frac{1}{\pi} S_1 \left[ h - m \cosh (\vartheta') + 2B \frac{1}{\pi} S_1 (h - m) \right] + O(B^3) =
\]

\[
= h - m \cosh \vartheta + B \frac{1}{\pi} S_1 \int_{-1}^{1} dx \left[ h - m \cosh (Bx) + BS_1 \int_{-1}^{1} dy (h - m) \right] + O(B^3) =
\]

\[
= h - m \cosh \vartheta + R(B)
\]  

(5.38)

where \( R(B) = 2(hB - m \sinh B) + 4B^2(h - m) \simeq 2(h - m)(B + 2B^2) \) behaves as a modification of the chemical potential. Thus, we can expect that our result will not be different from the free one. Indeed

\[
f(h) = -\frac{m}{2\pi} \int_{-B}^{B} d\vartheta \left( h + R(B) - m \cosh \vartheta \right) \cosh \vartheta
\]

\[
= \frac{m}{2\pi} \int_{-B}^{B} d\vartheta \left( h + 2(h - m)(B + 2B^2) - m \left( 1 - \frac{\vartheta^2}{2}B^2 \right) \right) \left( 1 - \frac{\vartheta^2}{2}B^2 \right) + O(B^4) =
\]

\[
= -\frac{m}{\pi} \left( (h - m)B + 2(h - m)(B + 2B^2) \right) + O(B^4)
\]  

(5.39)

We easily calculate, from (4.22),

\[
\frac{\partial}{\partial h} \varepsilon (\vartheta) = 1 + 2S_1 B + 4S_1 B^2 + O(B^3)
\]  

(5.40)

and, from (2.12)

\[
j = -\frac{\partial}{\partial h} f = \frac{m}{2\pi} \int_{-B}^{B} \frac{\partial}{\partial h} \varepsilon (\vartheta) \cosh \vartheta d\vartheta = \frac{m}{\pi} B \left( 1 + 2S_1 B + 4S_1 B^2 \right) + O(B^4)
\]  

(5.41)
to conclude from (2.13), somehow more "efficiently", that
\[
\Omega = jm - \frac{m}{\pi} ((h - m)B + 2(h - m)(B + 2B^2) + (h - m) 2B (1 + 2S_1B + 4S_1B^2)) + O(B^4)
\]
(5.42)

now, by remembering that \( h - m \simeq mB^2 + O(B^3) \), it becomes evident how the structure of the forcing term ensures the stability of the free-fermi approximation: every correction affects the series from the \( B^4 \) (i.e. from the \( j^4 \)) term on. Moreover, due to the absence of the \( B_s \) and of the \( B^2_s \), it suffices the first-order approximation for \( B \) to fix the coefficient of \( j^3 \).

6 A method for calculating \( B \) to all orders

We have seen that we can express \( B \) in powers of \( x = \frac{h - m}{m} \) as
\[
B = \sum_{n=0}^{\infty} b_n x^n
\]
(6.43)
as well as we can write the pseudoenergy as a power series
\[
\varepsilon(B) = m \sum_{n=0}^{\infty} e_n B^n
\]
(6.44)

At the same time, the x-dependence of the coefficients \( e_n \) can be explicited by Taylor expansion. This means that
\[
\varepsilon(B(x)) = e^{(0)}_0 + e^{(1)}_0 x + e^{(2)}_0 x^2 + \ldots + \left( e^{(0)}_1 + e^{(1)}_1 x + e^{(2)}_1 x^2 + \ldots \right) \left( b_0 + b_1 x + b_2 x^2 + \ldots \right) + \left( e^{(0)}_2 + e^{(1)}_2 x + e^{(2)}_2 x^2 + \ldots \right) \left( b_0 + b_1 x + b_2 x^2 + \ldots \right)^2 + \ldots
\]
(6.45)

We saw that, up to the second order, the only nonzero coefficients were
\[
e^{(2)}_0 = 1 \quad e^{(2)}_1 = \frac{2}{\pi} S_1 \quad e^{(2)}_2 = -\frac{1}{2} \quad e^{(2)}_3 = \frac{4}{\pi^2} S_1^2
\]

so we can solve, order by order, the boundary condition (2.8)

\[
\begin{align*}
\text{zero} & \quad -\frac{1}{2} b_0 = 0 \quad \Rightarrow \quad b_0 = 0 \\
\text{one} & \quad 0 = 0 \quad \text{useless} \\
\text{two} & \quad e^{(2)}_0 + e^{(1)}_1 b_1 + e^{(0)}_2 b_2 + e^{(0)}_1 b_1^2 + b_0(\ldots) = 1 - \frac{1}{2} b_1^2 = 0 \quad \Rightarrow \quad b_1 = \sqrt{2} \\
\text{three} & \quad e^{(2)}_1 b_1 + 2 e^{(0)}_0 b_1 b_2 + e^{(0)}_1 b_1^2 + b_0(\ldots) \Rightarrow \quad b_2 = 2 \frac{S_1}{\pi} + 2 e^{(0)}_3 \\
\cdots & \quad \cdots
\end{align*}
\]

the zeroth order fixes \( b_0 \): this guarantees that it is possible to calculate \( b_k \) only with the first \( k \) orders in \( B \) of (2.8). In general
\[
b_{n-1} = -\frac{1}{2e^{(0)}_2 b_1} \sum_{m=1}^{n} \sum_{t=0,2} (1 - \delta_{m,2}\delta_{t,0}) e^{(t)}_m \sum_{j_1+\ldots+j_n=n-t} b_{j_1} \ldots b_{j_m}
\]
(6.46)
with

\[ e_n^{(0)} = \sum_{p=0}^{\infty} \sum_{n+2p+2k_1+\ldots+2k_n=N} \frac{(-1)^{k_1+\ldots+k_n}}{\pi^n(2p)!} \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_n (1 - x_1)^{2k_1} \ldots (x_{n-1} - x_n)^{2k_n} x_n^{2p} B^N \]

\[ e_n^{(2)} = \sum_{n+2k_1+\ldots+2k_n=N} \frac{(-1)^{k_1+\ldots+k_n}}{\pi^n} \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_n (1 - x_1)^{2k_1} \ldots (x_{n-1} - x_n)^{2k_n} B^N \]  

(6.47)

7 \( \Omega(j) \): a systematics to all orders

By substituting the explicit formula for the kernel in (4.19)

\[ K(\vartheta - \vartheta') = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \left[ \frac{(-1)^k}{1 + \left( \frac{\vartheta - \vartheta'}{\pi(k+1)} \right)^2} + \frac{1}{k+2\Delta} \frac{(-1)^k}{1 + \left( \frac{\vartheta - \vartheta'}{\pi(k+2\Delta)} \right)^2} \right] = \]

\[ = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{k+1} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\vartheta - \vartheta'}{\pi(k+1)} \right)^{2j} + \frac{(-1)^k}{k+2\Delta} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\vartheta - \vartheta'}{\pi(k+2\Delta)} \right)^{2j} \right] \]

we get

\[ \varepsilon(\vartheta) = h - m \cosh \vartheta + \]

\[ + \sum_{n=1}^{\infty} \frac{1}{\pi^{2n}} B^n \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_n \sum_{m=0}^{\infty} \left[ \frac{1}{k+1} \frac{(-1)^k}{1 + \left( \frac{\vartheta - Bx_1}{\pi(k+1)} \right)^2} + \frac{1}{k+2\Delta} \frac{(-1)^k}{1 + \left( \frac{\vartheta - Bx_1}{\pi(k+2\Delta)} \right)^2} \right] \]

\[ \ldots \]

\[ = h - m \cosh \vartheta + \sum_{n=1}^{\infty} \frac{1}{\pi^{2n}} B^n \int_{-1}^{1} dx_1 \ldots \int_{-1}^{1} dx_n \sum_{m=0}^{\infty} (-1)^m \sum_{j=0}^{\infty} (-1)^j \]

\[ \left[ \frac{1}{m_1+1} \left( \frac{\vartheta - \vartheta'}{\pi(m_1+1)} \right)^{2j_1} + \frac{1}{m_1+2\Delta} \left( \frac{\vartheta - \vartheta'}{\pi(m_1+2\Delta)} \right)^{2j_1} \right] \sum_{j_1=0}^{\infty} (-1)^j \sum_{j_1=0}^{\infty} (-1)^j \]

\[ \ldots \]

(7.48)

and by defining the series

\[ S_n = \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{\pi^n(k+1)^n} + \frac{(-1)^k}{\pi^n(k+2\Delta)^n} \right] \]
we can rewrite the previous result in a much more tangled way, but with a more transparent insight on the contributions of the different powers of $B$

$$\varepsilon(\vartheta) = h - m \cosh \vartheta +$$

$$\sum_{n=1}^{\infty} \frac{B^n}{\pi^n} \int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \sum_{j_1 + \cdots + j_n = 0}^{\infty} (-1)^{j_1 + \cdots + j_n} S_{j_1} B^{j_1} \left( \frac{\vartheta}{B} - Bx_1 \right)^{2j_1} S_{j_2} B^{j_2} (x_1 - x_2)^{2j_2} \cdots$$

$$\ldots S_{j_n} B^{j_n} (x_{n-1} - x_n)^{2j_n} \left( h - m \cosh (Bx_n) \right)$$

(7.50)

Please note that we are allowed to exchange the order integrations and summations over the $j_i$ indexes, because on the finite interval $[-1, 1]$ all powers of $x$ are integrable and the succession of reduced sum does converge to the actual kernel.

$$\varepsilon(\vartheta) = h - m \cosh \vartheta +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi^n} \sum_{j_1 + \cdots + j_n = 0}^{\infty} (-1)^{J_n} B^{n+2J_n} S_{2j_1+1} S_{2j_2+1} \cdots S_{2j_n+1} \int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \left( \frac{\vartheta}{B} - Bx_1 \right)^{2j_1} \cdots (x_{n-1} - x_n)^{2j_n} \left( h - m \sum_{p=0}^{\infty} \frac{B^n x_{2p}^n}{(2p)!} \right) =$$

$$= h - m \cosh \vartheta +$$

$$+ \sum_{n=1}^{\infty} \sum_{j_1 + \cdots + j_n = 0}^{\infty} \frac{(-1)^{J_n}}{\pi^n} S_{2j_1+1} S_{2j_2+1} \cdots S_{2j_n+1} \int_{-1}^{1} dx_1 \cdots \int_{-1}^{1} dx_n \left( \frac{\vartheta}{B} - Bx_1 \right)^{2j_1} (x_1 - x_2)^{2j_2} \cdots$$

$$\ldots (x_{n-1} - x_n)^{2j_n} \left( h - m \sum_{p=0}^{\infty} \frac{B^n x_{2p}^n}{(2p)!} \right) B^{n+2J_n}$$

(7.51)

where $J_n = j_1 + j_2 + \ldots + j_n$. In the following we will also define

$$S_n \{j\} = S_{2j_1+1} S_{2j_2+1} \cdots S_{2j_n+1}$$

(7.52)
following [19] we can compute the free energy as a function of the chemical potential $h$ and of the mass $m$ of the field

$$\frac{-2\pi}{m} f(h, m) = \int_{-B}^{B} \cosh \vartheta \varepsilon(\vartheta) d\vartheta = B \int_{-1}^{1} \cosh (B x_0) \varepsilon (B x_0) d x_0 =$$

$$= \left\{ \sum_{r=0}^{\infty} \frac{1}{(2r)!} \sum_{n=0}^{\infty} \frac{1}{\pi^n} \sum_{j_1j_2...j_n=0} (1)^n S_n (\{j\}) \int_{-1}^{1} d x_0 ... \int_{-1}^{1} d x_n \right\}$$

$$\ldots x_0^{2r} (x_0 - x_1)^{2j_1} \ldots (x_{n-1} - x_n)^{2j_1} x_n^{2p} B^{1+n+2J_n+2r+2p}$$

(7.53)

now it is to the density $j$, dual to the chemical potential $h$ through the Helmholtz free energy $f$.

$$j = -\frac{\partial}{\partial h} f(h, m) = \frac{m}{2\pi} \frac{(2p)!}{\partial h} \int_{-B}^{B} d\vartheta \varepsilon(\vartheta) \cosh \vartheta = \frac{m}{2\pi} \int_{-B}^{B} d\vartheta \left( 1 + \int_{-B}^{B} d\vartheta' K(\vartheta - \vartheta') \frac{\partial \varepsilon_h(\vartheta')}{\partial h} \right) \cosh \vartheta$$

$$= \frac{m}{2\pi} \sum_{r=0}^{\infty} \frac{1}{(2r)!} \sum_{n=0}^{\infty} \frac{1}{\pi^n} \sum_{j_1j_2...j_n=0} (1)^n S_n (\{j\}) \int_{-1}^{1} d x_0 ... \int_{-1}^{1} d x_n$$

$$\ldots x_0^{2r} (x_0 - x_1)^{2j_1} \ldots (x_{n-1} - x_n)^{2j_1} x_n^{2p} B^{1+n+2J_n+2r+2p}$$

(7.54)

We have used [4.23] and [2.8]. The Legendre transform follows straightforwardly

$$\Omega (j, m) = f(h, m) + jh =$$

$$= \frac{m^2}{2\pi} \sum_{r=0}^{\infty} \frac{1}{(2r)!} \sum_{n=0}^{\infty} \frac{1}{\pi^n} \sum_{j_1j_2...j_n=0} (1)^n S_n (\{j\}) \sum_{p=0}^{\infty} \frac{1}{(2p)!} \int_{-1}^{1} d x_0 ... \int_{-1}^{1} d x_n$$

$$x_0^{2r} (x_0 - x_1)^{2j_1} \ldots (x_{n-1} - x_n)^{2j_1} x_n^{2p} B^{1+n+2J_n+2r+2p} =$$

$$= mj + \frac{m^2}{2\pi} \sum_{r=0}^{\infty} \frac{1}{(2r)!} \sum_{n=0}^{\infty} \frac{1}{\pi^n} \sum_{j_1j_2...j_n=0} (1)^n S_n (\{j\}) \sum_{p=1}^{\infty} \frac{1}{(2p)!} \int_{-1}^{1} d x_0 ... \int_{-1}^{1} d x_n$$

$$x_0^{2r} (x_0 - x_1)^{2j_1} \ldots (x_{n-1} - x_n)^{2j_1} x_n^{2p} B^{1+n+2J_n+2r+2p}$$

(7.55)

Please note that the $p$ index in the last summation, having recognized the $p = 0$ term as the linear contribution in $j$, now runs from 1 to infinity. This, of course, allows us to exclude any term of order $j^2$, as the lowest power of $B$ in the summation is now three.
At this step, we have the two series

\[ \Omega(B) = \sum_{n=1}^{\infty} \omega_n B^n \]  
\[ j(B) = \sum_{n=1}^{\infty} j_n B^n \]  

(7.56)

(7.57)

In this situation, one would naturally appeal to Lagrange inversion formula and express \( \Omega \) in powers of \( j \) as

\[ \Omega(j) = \sum_{n=1}^{\infty} \Omega_n j^n = \sum_{n=1}^{\infty} \frac{1}{n!} \left[ \frac{d^{n-1}}{dB^{n-1}} (\phi^n(B)\Omega'(B)) \right]_{B=0} j^n \]  

(7.58)

where \( \phi(B) = \frac{B}{j(B)} \).

### 7.1 Some explicit coefficient

However, in order to perform some calculation, it may be simpler to extract the coefficient of the \( n \)-th power of \( j \), \( \Omega_n \), from the coefficient of the powers of \( B \) up to the \( n \)-th: \( \omega_1, \ldots, \omega_n \). In fact, having already isolated the term \( mj \), we easily calculate \( \Omega_3 \):

\[ \omega_1 = \frac{m}{2\pi} \int_{-1}^{1} dx_0 = \frac{m}{\pi} \]  

(7.59)

\[ \omega_3 = \frac{m^2}{2\pi^2} \int_{-1}^{1} x_0^2 dx_0 = \frac{m^2}{6\pi} \]  

(7.60)

\[ \Omega_3 = \frac{\omega_3}{j_1^3} = \frac{\pi^2}{6m} \]  

(7.61)

We can make another step without too much trouble

\[ \omega_4 = \frac{m^2}{2\pi^2} \frac{1}{\pi} \int_{-1}^{1} dx_0 \int_{-1}^{1} dx_1 x_1^2 = \frac{m^2}{6\pi^2} = \frac{m^2}{3\pi^2} S_1 \]  

(7.62)

\[ (j^3)_4 = 3 \frac{m^3}{\pi^2} \frac{m}{\pi} \frac{S_1}{\pi^2} \int_{-1}^{1} dx_0 \int_{-1}^{1} dx_1 = 3 \frac{2m^3}{\pi^4} S_1 \]  

(7.63)

we will now specialize the solution to the O(6) sigma model, i.e. we set \( N = 6 \), so that \( \Delta = \frac{1}{4} \)

\[ S_1 = \frac{1}{\pi} \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{k+1} + \frac{(-1)^k}{k+\frac{1}{2}} \right] = \frac{1}{\pi} \ln 2 + \frac{1}{2} \]  

(7.64)
\[
\Omega_4 = \frac{\omega_4 - \Omega_3 (j^3)_4}{j_1^4} = \frac{1}{m^4} \left[ \frac{m^2}{3\pi^2} S_1 - \frac{\pi^2}{6m} \frac{6m^3}{\pi^4} S_1 \right] = -\frac{2}{3} \frac{\pi^2}{m^2} \left( \frac{1}{\pi} \ln 2 + \frac{1}{2} \right) \quad (7.65)
\]

where we have written the coefficient of \( j^4 \) in the series in \( B \) of \( j^3 \) as \( (j^3)_4 \). Now we have warmed up, we can approach the next one:

\[
\begin{align*}
\omega_5 &= \frac{m^2}{\pi} \left( \frac{7}{120} + \frac{2}{3} \frac{1}{\pi^2} S_1^2 \right), \\
(j^4)_5 &= 4j_1^3 j_2 = 4 \left( \frac{m}{\pi} \right)^3 \frac{8}{2\pi^2} S_1, \\
(j^3)_5 &= 3j_1^2 j_3 + 3j_1 j_2^2 = \left( \frac{m}{\pi} \right)^3 \left( \frac{12}{\pi^2} S_1^2 + \frac{1}{2} + \frac{12}{\pi^2} S_1 \right) \quad (7.66)
\end{align*}
\]

\[
\Omega_5 = \frac{1}{j_1^5} (\omega_5 - \Omega_4 (j^4)_5 - \Omega_3 (j^3)_5) = \frac{1}{j_1^5} (\omega_5 - 3\Omega_4 j_1^2 j_3 - 3\Omega_3 j_1 j_2^2 - 4\Omega_4 j_1^3 j_2) = \\
= \frac{\pi^4}{m^3} \left( -\frac{1}{40} + \frac{2}{\pi^2} S_1^2 \right) \quad (7.67)
\]

On one hand, please note that the coefficient \( \Omega_5 \), as well as those of all odd powers of \( j \), does reduce to its free approximation if we switch off the interaction, i.e., if we send \( S_1 \to 0 \). On the other hand, since the oddness of the non-interacting gas series, the \( \Omega_4 \) correctly vanishes in this limit. Generally speaking, unfortunately, when considering the order \( B^n \), we need to subtract all the terms coming from lower powers of \( j \), which makes the calculation quite cumbersome.

### 7.2 Comparison with Lagrange formula

Of course, this results does match with those coming from Lagrange formula. The first coefficient is trivial, the second reads

\[
\Omega_2 = \frac{1}{2} \frac{d}{dB} \left[ \sum_{p=1}^{\infty} \frac{p \omega_p B^{p-1}}{(\sum_{k=1}^{\infty} j_k B^{k-1})^2} \right] = \frac{1}{2} \left[ \sum_{p=2}^{\infty} \frac{p(p-1) \omega_p B^{p-2}}{(\sum_{k=1}^{\infty} j_k B^{k-1})^2} - \sum_{p=1}^{\infty} \frac{p \omega_p B^{p-1}}{(\sum_{k=1}^{\infty} j_k B^{k-1})^3} \right]_{B=0} \\
= \frac{1}{2} \left( \frac{2\omega_2}{j_1} - \frac{2j_2}{j_1^2} \right) = 0 \quad (7.68)
\]

Checks. The third

\[
\Omega_3 = \frac{1}{6} \left[ \Omega'' \Phi^3 + 6 \Omega'' \Phi^2 \Phi' + 3 \Omega' \left( \Phi^2 \Phi'' + \Phi (\Phi')^2 \right) \right]_{B=0} \quad (7.69)
\]
\[
\begin{align*}
\Omega'(0) &= \omega_1 \\
\Omega''(0) &= 2\omega_2 \\
&\quad \ldots \\
\Omega^{(n)}(0) &= n!\omega_n 
\end{align*}
\]

\[
\phi'(0) = \frac{d}{dB} \frac{B}{J(B)}_{B=0} = -\left[ \sum_{p=2}^{\infty} (p-1) j_p B^{p-2} \right]_{B=0} = -\frac{j_2}{j_1^2}
\]

\[
\phi''(0) = \frac{2j_2^2 - j_1 j_3}{j_1^3}
\]

\[
\begin{align*}
\Omega_3 &= \frac{1}{6} \left[ \frac{\omega_3}{j_1^3} - 12 \frac{j_2^2}{j_1^4} + 3\omega_1 \left( \frac{2j_2^2}{j_1^3} + 2\frac{j_2^2 - j_1 j_3}{j_1^5} \right) \right]_{B=0} \\
&= \frac{1}{6} \left[ \frac{\omega_3 - j_3}{j_1^3} - 12 \frac{j_2^2}{j_1^4} + 12 \frac{\omega_1 j_2^2}{j_1^5} \right] = \\
&= \frac{1}{6} \frac{\pi^4}{m^4} \left( \frac{m}{2\pi} + \frac{8}{2\pi} S_1 - \frac{m}{2\pi} \frac{1}{S_1} \right) = \frac{\pi^2}{6m}
\end{align*}
\]

checks again. We will not go further.

### 7.3 Analitycity

The convergence of the two series in powers of \(B\) will be estimated by combinatorial considerations. We have \(k^N\) possibilities to arrange \(k\) nonnegative integers in such a way that their sum yields \(N\). Moreover, the number \(n\) of the factors appears in the exponent together with the factors \(j_n\) themselves and the two indices \(p, s\). So the general \(N\)-th coefficient grows less than

\[
\sum_{n=0}^{N} 2^{2n} (N-n)^{n+2}
\]

where we have used

\[
\int_{-1}^{1} dx_0 \ldots \int_{-1}^{1} dx_n \ldots (x_0 - x_1)^{2j_1} \ldots (x_{n-1} - x_n)^{2j_n} \ll 2^n \cdot 2^n
\]

and

\[
|S_n| = \left| \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{\pi^n (k+1)^n} + \frac{(-1)^k}{\pi^n (k+2\Delta)^n} \right] \right| \leq 1
\]
We find the maximum value of the addend
\[
\frac{d}{dx} (N-x)^{x+2} = e^{(x+2)\ln(N-x)} \left( \ln (N-x) - \frac{x+2}{N-x} \right) = 0 \quad \implies \quad (N-a) = e^{\frac{a+2}{N-a}}
\]
\[
\frac{d^2}{dx^2} (N-x)^{x+2} = \frac{d}{dx} e^{(x+2)\ln(N-x)} \left( \ln (N-x) - \frac{x+2}{N-x} \right) = e^{(x+2)\ln(N-x)} \left[ \left( \ln (N-x) - \frac{x+2}{N-x} \right)^2 - \left( \frac{2}{N-x} + \frac{x+2}{(N-x)^2} \right) \right] = -e^{(a+2)\ln(N-a)} \ln (N-a) \left( \frac{2}{N-a} + \frac{a+2}{(N-x)^2} \right) < 0 \quad \text{(in } x=a\text{)} \quad (7.73)
\]
and we substitute it in the sum
\[
\sum_{n=0}^{N} 2^{2n} (N-n)^{n+2} \leq 2^{2N} \sum_{n=0}^{N} e^{\frac{a+2}{N-a}}^{n+2} = 2^{2N} (N+1) e^{\frac{a+2}{N-a}}^{2} \leq 2^{2N} (N+1) e^{\frac{(N+2)^2}{N-a}} \sim N (4e)^N
\]
(7.74)
The general term grows with the \(N\)-th power, so that the two series of powers of \(B \ll 1\) easily converge. To conclude, Lagrange theorem expresses the analiticity of the series \[(7.58)\], when written in powers of \(B \ll 1\).

### 8 Checks and previsions for strong SYM\(_4\)

As noticed by \([1]\), the strong coupling limit \(j << g\) and the consequent reduction to the O(6) bosonic nonlinear sigma model in two dimensions allows to calculate exactly the anomalous dimension of high spin operators \(f(g,j)\) with \(j = \frac{J}{\log S}\): the SO(6) charge density modifies the scaling by a \(2\Omega\),

\[
f(g,j) = f_0(g) + 2\Omega(g,j)
\]
(8.75)
The possibility of calculating, at least in principle, many successive coefficient \(f_1, f_2, \ldots\) of the expansion (as announced in the footnote 10 of \([2]\)) allows a check of the first scaling functions from the SYM side, appearing in the Sudakov scaling \[(1.4)\]. At this purpose, we underline that, on the one hand, the density \(j\) that we have used is defined from the string side in (3.1) of \([1]\) as

\[
j = \frac{J}{2 \log S}
\]
(8.76)
while on the other hand, recently available calculations \([3]\) approach the problem with slightly different notations. From their point of view

\[
j_{\text{SYM}} = \frac{J}{\log S} = 2j
\]
(8.77)
It follows that, rescaling their results, we get an explicit link with the O(6) model, namely

\[ f_n = 2^{n-1} \Omega_n \]  

(8.78)

The first three coefficients from the sigma model point of view (\(\Omega_1, \Omega_2\) and \(\Omega_3\)), are exactly the same as for the free, non-relativistic theory. They were already given as an approximation in \([19]\), as we said above. Further, we have performed the calculation of \(\Omega_4\) and \(\Omega_5\), that contains trace of our interaction and the model-dependence, too, entering through \(S_1\) as in \((7.64)\). On the other front, the correspondence with \(f_1, f_2, f_3\) and \(f_4\) is a remarkable fact. In particular, this very last scaling function catches the nature of the system and selects the specific model.

**Acknowledgments** D.F. owe very much to Marco Rossi and Paolo Grinza for many suggestions, and thanks the INFN grant ”Iniziativa specifica PI14” and the international agreement INFN-MEC-2008 for travel financial support. The authors also thank D. Bombardelli, F. Ravanini for useful discussions.

**References**

[1] L.F. Alday, J.M. Maldacena, *Comments on operators with large spin*, JHEP**11** (2007) 019 and [arXiv:0708.0672 [hep-th]];

[2] D. Fioravanti, P. Grinza, M. Rossi, *Strong coupling for planar \(\mathcal{N} = 4\) SYM theory: an all-order result*, [arXiv:0804.2893 [hep-th]];

[3] D. Fioravanti, P. Grinza, M. Rossi, *The generalised scaling function: a note*, [arXiv:0805.4407 [hep-th]];

[4] J.M. Maldacena, *The large \(N\) limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 and [hep-th/9711200]; • S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys.Lett. **B428** (1998) 105 and [hep-th/9802109]; • E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 and [hep-th/9802150];

[5] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *A Semiclassical limit of the gauge / string correspondence*, Nucl. Phys. **B636** (2002) 99 and [hep-th/9802109];

[6] S. Frolov, A.A. Tseytlin, *Semiclassical quantization of rotating superstring in \(AdS_5 \times S^5\)*, JHEP**06** (2002) 007 and [hep-th/0204226];

[7] J.A. Minahan, K. Zarembo, *The Bethe Ansatz for \(\mathcal{N} = 4\) Super Yang-Mills*, JHEP**03** (2003) 013 and [hep-th/0212208];
L.N. Lipatov, *Evolution equations in QCD*, in “Perspectives in Hadron Physics”, Proceedings of the Conference, ICTP, Trieste, Italy, May 1997, World Scientific (Singapore, 1998);

V.M. Braun, S.E. Derkachov, A.N. Manashov, *Integrability of three particle evolution equations in QCD*, Phys. Rev. Lett. **81** (1998) 2020 and hep-ph/9805225

- V.M. Braun, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, *Baryon distribution amplitudes in QCD*, Nucl. Phys. **B553** (1999) 355 and hep-ph/9902375

- A.V. Belitsky, *Fine structure of spectrum of twist-three operators in QCD*, Phys. Lett. **B453** (1999) 59 and hep-ph/9902361

- A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, *Gauge / string duality for QCD conformal operators*, Nucl. Phys. **B667** (2003) 3 and hep-th/0304028

N. Beisert, C. Kristjansen, M. Staudacher, *The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory*, Nucl. Phys. **B664** (2003) 131 and hep-th/0303060

- N. Beisert, M. Staudacher, *The $\mathcal{N} = 4$ SYM integrable super spin chain*, Nucl. Phys. **B670** (2003) 439 and hep-th/0307042

- N. Beisert, M. Staudacher, *Long-range PSU(2,2$|$4) Bethe Ansatz for gauge theory and strings*, Nucl. Phys. **B727** (2005) 1 and hep-th/0504190

- B. Eden, M. Staudacher, *Integrability and transcendentality*, J.Stat.Mech. **11** (2006) P014 and hep-th/0603157

- N. Beisert, B. Eden, M. Staudacher, *Transcendentality and crossing*, J.Stat.Mech. **07** (2007) P01021 and hep-th/0610251

A. Rej, D. Serban, M. Staudacher, *Planar $\mathcal{N} = 4$ Gauge Theory and the Hubbard Model*, JHEP **03** (2006) 018 and hep-th/0512077

- G. Feverati, D. Fioravanti, P. Grinza, M. Rossi, *Hubbard’s Adventures in $\mathcal{N} = 4$ SYM-land? Some non-perturbative considerations on finite length operators*, J.Stat.Mech. **02** (2007) P001 and hep-th/0611186

I. Bena, J. Polchinski, R. Roiban, *Hidden symmetries of the AdS$_5 \times S^5$ superstring*, Phys. Rev. **D69** (2004) 046002 and hep-th/0305116

- V.A. Kazakov, A. Marshakov, J.A. Minahan, K. Zarembo, *Classical/quantum integrability in AdS/CFT*, JHEP **05** (2004) 024 and hep-th/0402207

- V.A. Kazakov, K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, JHEP **10** (2004) 060 and hep-th/0410105

- G. Arutyunov, S. Frolov, M. Staudacher, *Bethe Ansatz for quantum strings*, JHEP **10** (2004) 016 and hep-th/0406256

- R. Janik, *The AdS$_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, Phys. Rev. **D73** (2006) 086006 and hep-th/0603038

- N. Beisert, R. Hernandez, E. Lopez, *A crossing symmetric phase for AdS$_5 \times S^5$ strings*, JHEP **11** (2006) 070 and hep-th/0609044
[13] S. Frolov, A. Tirziu, A.A. Tseytlin, Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT, Nucl. Phys. B766 (2007) 232 and hep-th/0611269; R. Roiban, A.A. Tseytlin, Spinning superstrings at two loops: strong-coupling corrections to dimensions of large-twist SYM operators, Phys.Rev. D77 (2008) 066006 and arXiv:0712.2479 [hep-th];

[14] A.V. Belitsky, A.S. Gorsky, G.P. Korchemsky, Logarithmic scaling in gauge/string correspondence, Nucl. Phys. B748 (2006) 24 and hep-th/0601112;

[15] L. Freyhult, A. Rej, M. Staudacher, A Generalized Scaling Function for AdS/CFT, arXiv:0712.2743;

[16] B. Basso, G.P. Korchemsky, Embedding nonlinear O(6) sigma model into N=4 super-Yang-Mills theory, arXiv:0805.4194 [hep-th];

[17] L. Castillejo, R. H. Dalitz and F. Dyson, Phys. Rev. 101 (1956), 453

[18] D. Bombardelli, D. Fioravanti, M. Rossi, Large spin corrections in $\mathcal{N} = 4$ SYM sl(2): still a linear integral equation, arXiv:0802.0027;

[19] P. Hasenfratz, M. Maggiore and N. Niedermayer, The exact mass gap of the O(3) and O(4) nonlinear sigma models in $d=2$, Phys. Lett. B 245 (1990) 522; P. Hasenfratz, N. Niedermayer, The exact mass gap of the O(N) sigma model for arbitrary N;

[20] A. B. Zamolodchikov, Al. B. Zamolodchikov Relativistic Factorized S Matrix In Two Dimensions As The Exact Solutions of Certain Relativistic Quantum Field Theory Models Annals Phys. 120 (1979) 253 A. B. Zamolodchikov, Al. B. Zamolodchikov Relativistic Factorized S Matrix In Two Dimensions Having O(N) Isotropic Symmetry Nucl. Phys. B133 (1978) 525

[21] E. Brezin and J. Zinn-Justin, Spontaneous breakdown of continuous symmetries near two dimensions, Phys. Rev. B 14 (1976) 3110