HEAT FLOW ON 1-FORMS UNDER LOWER RICCI BOUNDS.
FUNCTIONAL INEQUALITIES, SPECTRAL THEORY, AND
HEAT KERNEL

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Abstract. We study the canonical heat flow \((H_t)_{t \geq 0}\) on the cotangent module \(L^2(T^*M)\) over an RCD\((K, \infty)\) space \((M, d, m)\), \(K \in \mathbb{R}\). We show Hess–Schrader–Uhlenbrock’s inequality and, if \((M, d, m)\) is also an RCD\(^*\)(\(K, N\)) space, \(N \in (1, \infty)\), Bakry–Ledoux’s inequality for \((H_t)_{t \geq 0}\) w.r.t. the heat flow \((P_t)_{t \geq 0}\) on \(L^2(M)\). Variable versions of these estimates are discussed as well.

In conjunction with a study of logarithmic Sobolev inequalities for 1-forms, the previous inequalities yield various \(L^p\)-properties of \((H_t)_{t \geq 0}\), \(p \in [1, \infty]\).

Then we establish explicit inclusions between the spectrum of its generator, the Hodge Laplacian \(\Delta\), of the negative functional Laplacian \(-\Delta\), and of the Schrödinger operator \(-\Delta + K\). In the RCD\(^*\)(\(K, N\)) case, we prove compactness of \(\Delta^{-1}\) if \(M\) is compact, and the independence of the \(L^p\)-spectrum of \(\Delta\) on \(p \in [1, \infty]\) under a volume growth condition.

We terminate by giving an appropriate interpretation of a heat kernel for \((H_t)_{t \geq 0}\). We show its existence in full generality without any local compactness or doubling, and derive fundamental estimates and properties of it.

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1. Introduction

Let \((M, d, m)\) be a metric measure space, i.e. a complete and separable metric space \((M, d)\) endowed with a locally finite, fully supported Borel measure \(m\). We always assume that \((M, d, m)\) is an RCD\((K, \infty)\) space for some \(K \in \mathbb{R}\). See Chapter 2 for a more detailed account on these.

In the seminal papers \([39, 40]\), based on the notion of \(L^\infty\)-modules over \((M, d, m)\), a first and, in particular, a second order differential structure on such, possibly quite singular, spaces has been built. In high analogy with the setting of a complete Riemannian manifold with Ricci curvature bounded from below by \(K\) endowed with its Riemannian volume measure, \([40]\) makes sense of differential geometric objects such as gradients, differentials, Hessians, vector fields, 1-forms, etc. — in fact, even in smooth situations, the axiomatization of \([40]\) covers certain Schrödinger-type operators on weighted Riemannian manifolds. In particular, building upon a notion of Hodge Laplacian \(\tilde{\Delta}\), a nonsmooth cohomology theory and the heat flow \((H_t)_{t \geq 0}\) with generator \(-\tilde{\Delta}\) acting on differential 1-forms in the cotangent module \(L^2(T^*M)\) have been introduced in \([40]\).

On Riemannian manifolds — weighted or not — \((H_t)_{t \geq 0}\) has been introduced long before and was studied extensively over the last decades. Let us mention e.g. \([18, 20, 22, 29, 61, 63]\) for geometric and analytic, and \([11, 33]\) for probabilistic studies on the heat flow on 1-forms, its integral kernel, etc. See also \([34]\) for the development of a Hodge theory on the Wiener space over a Riemannian manifold. On the other hand, apart from \([40]\), little is known about \((H_t)_{t \geq 0}\) in the more general RCD\((K, \infty)\) framework. This article aims in a thorough study of properties of \((H_t)_{t \geq 0}\) as well as its generator, which has not been the central objective of \([40]\). The final outcome of our discussion is an appropriate definition and the construction of a heat kernel for \((H_t)_{t \geq 0}\) in the nonsmooth setting.

Our motivation for deeply studying the heat flow on 1-forms over RCD spaces comes from different directions. First, we provide a further contribution to the large diversity of works generalizing important “classical smooth” statements to nonsmooth spaces. Second, we believe that RCD spaces (or more general spaces with “lower Ricci bounds” [12, 89, 36]) with the tensor language of \([39, 40]\) are the correct framework to develop nonsmooth notions of stochastic differential geometry, e.g. (damped) stochastic parallel transports, a project which we attack in the future. Therein, in establishing Bismut–Elworthy–Li-type derivative formulas for the functional heat flow \((P_t)_{t \geq 0}\) \([11, 33]\) — which in turn are expected to provide further regularity information about it — a good understanding of its 1-form counterpart \((H_t)_{t \geq 0}\) is essential \([13, 30]\). Lastly, in smooth contexts, heat kernel methods for 1-forms are useful in many important applications, all of which could lead to RCD analogues. Exemplary, let us quote

- a deeper understanding of the Hodge theorem (see \([40]\) for the RCD result) by the study of the heat kernel \([65]\),
- a proof variant of index theorems in Riemannian geometry \([11, 20, 55]\),
- the study of boundedness of the Riesz transform, see e.g. \([23, 24, 25, 29]\) and the references therein, and
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Theorem 3.17 opens the door for plenty of new insights into the behavior of \((\text{Theorem 4.1})\). Theorem 3.15

Hess–Schrader–Uhlenbrock’s inequality. Let us now start by outlining the content of the paper. If \(M\) is a Riemannian manifold with volume measure \(\text{vol}\), the heat flow \((H_t)_{t \geq 0}\) is well-known \([37, 50, 72, 82]\) to be tightly linked to the Ricci curvature \(\text{Ric}\) of \(M\) via \(\Delta\) through the vector Bochner formula

\[
\Delta \frac{|X|^2}{2} + \langle X, (\Delta X^\gamma) \rangle = |\nabla X|^2 + \text{Ric}(X, X),
\]

valid for every sufficiently regular vector field \(X\). Since this identity also involves the Laplace–Beltrami operator \(\Delta\) — which is the generator of the heat semigroup \((P_t)_{t \geq 0}\) acting on \(L^2(M)\) — it implies important comparison estimates between \((H_t)_{t \geq 0}\) and \((P_t)_{t \geq 0}\) as described now.

The main result in \([40]\) in the \(\text{RCD}(K, \infty)\) case tells us that the vector Bochner formula can be made sense of in a weak form involving the measure-valued Laplacian \(\Delta\) and Ricci tensor \(\text{Ric}\) — actually, the latter is defined via such an identity. A classical interpolation argument from Bakry–Emery’s theory then implies that for every \(\omega \in L^2(T^*M)\) and every \(t \geq 0\),

\[
|H_t \omega|^2 \leq e^{-2Kt} P_t (|\omega|^2) \quad \text{m-a.e.} \quad (1.1)
\]

This estimate, albeit being pointwise, is too weak to derive \(L^p\)-properties of \((H_t)_{t \geq 0}\) for the important range \(p \in [1, 2)\). On Riemannian manifolds with Ricci lower bound \(K\) and \(m := \text{vol}\), however, Bochner’s formula is known to entail the stronger Hess–Schrader–Uhlenbrock inequality

\[
|H_t \omega| \leq e^{-Kt} P_t |\omega| \quad \text{m-a.e.} \quad (1.2)
\]

for every \(\omega \in L^2(T^*M)\) and every \(t \geq 0\). In this context, \((1.2)\) is due to \([49, 50]\).

We prove the bound \((1.2)\) in full generality on \(\text{RCD}(K, \infty)\) spaces \((M, d, m)\).

Theorem A (see Theorem 3.15). For every \(\omega \in L^2(T^*M)\) and every \(t \geq 0\), \((1.2)\) is satisfied.

Theorem A opens the door for plenty of new insights into the behavior of \((H_t)_{t \geq 0}\) and its generator and is used at many places in this paper. For instance, \((H_t)_{t \geq 0}\) extends to a semigroup of bounded operators mapping \(L^p(T^*M)\) into \(L^p(T^*M)\) for every \(p \in [1, \infty]\), strongly continuous if \(p < \infty\), with operator norm of \(H_t\) no larger than \(e^{-Kt}\), see Theorem 4.1.

Bakry–Ledoux’s inequality. Before further commenting on the proof of Theorem A, we introduce improved versions of \((1.1)\) we will obtain if \((M, d, m)\) obeys the more restrictive \(\text{RCD}^*(K, N)\) condition, \(N \in (1, \infty)\). To the best of our knowledge and apart from exact 1-forms \([8, 35, 36]\), these seem to be new even in the smooth case. At this point, let us concentrate on the strongest — we say that a 1-form \(\omega\) obeys the strong Bakry–Ledoux inequality if for every \(t > 0\),

\[
|H_t \omega|^2 + \frac{2}{N} \int_0^t e^{-2Ks} P_s (|\text{div} H_{t-s} \omega|^2) \, ds \leq e^{-2Kt} P_t (|\omega|^2) \quad \text{m-a.e.}
\]

Theorem B (see Theorem 3.17). If \((M, d, m)\) is an \(\text{RCD}^*(K, N)\) space, the strong Bakry–Ledoux inequality holds for every \(\omega \in L^2(T^*M)\).

The link between the \(\text{RCD}^*(K, N)\) condition and the strong version of Bakry–Ledoux is created by the same interpolation technique as in the dimension-free case \([40]\) in combination with the dimensional vector 2-Bochner-inequality obtained by the study of \(\text{Ric}\) on \(\text{RCD}^*(K, N)\) spaces from \([45]\).
The Bakry–Ledoux inequality — already if valid for sufficiently many exact 1-forms — characterizes the RCD*(K, N) condition for (M, d, m) according to [35, 36] and the commutation relation between (Ht)t≥0 and (P⊥t)t≥0 from Lemma 3.3 below.

**Vector 1-Bochner inequality.** A crucial ingredient for Theorem A is that the vector 2-Bochner inequality — or, in clearer terms, vector Γ2-inequality — is self-improving. As proven in [40], this means that from the a priori weaker inequality

$$\Delta \frac{|X|^2}{2} + \langle X, (\tilde{A}X)^\flat \rangle m \geq K |X|^2 m$$

for a sufficiently large class of vector fields X, it already follows that

$$\Delta \frac{|X|^2}{2} + \langle X, (\tilde{A}X)^\flat \rangle m \geq [K |X|^2 + |\nabla X|^2] m,$$

This property has been crucial in defining Ric in [40].

Once having proven that |X| ∈ Dom(Δ) for sufficiently many X, the chain rule

$$\Delta |X|^2 = 2 |X| \Delta |X| + 2 |\nabla X|^2 m$$

then implies the *vector 1-Bochner inequality*

$$\Delta |X|^2 + |X|^{-1} \langle X, (\tilde{A}X)^\flat \rangle m \geq K |X| m,$$

as a byproduct, where the key feature canceling out the covariant term is

$$|\nabla X| \leq |\nabla X| \text{ m-a.e.}$$

The latter, known as Kato’s inequality, is well-known in the smooth framework, while its nonsmooth analogue, stated in Proposition 3.7, has recently appeared in [27]. On the other hand, the vector 1-Bochner inequality for vector fields not necessarily of gradient-type is new in the dimension-free setting.

In an integrated version, the link between this vector 1-Bochner inequality and L1-type bounds of the form (1.2) is known as *form domination*. See the classical references [10, 49, 80] for motivations. For Theorem A, we adopt a similar strategy.

It is worth noting — and shortly addressed in Remark 3.18, as the constant case will suffice for our purposes — that the above-mentioned key estimates also hold true in greater generality if (M, d, m) has a stronger variable lower Ricci bound $k \in L^1_{\log}(M)$ with $k \geq K$ on M in the sense of [12, 89]. For instance, the estimate (1.2) appearing in Theorem A can be phrased in terms of Brownian motion $(B_t)_{t \geq 0}$ on (M, d, m) through

$$|H_t \omega| \leq E \left[ e^{-\int_0^t k(B_r)/2 \, dr} |\omega|(B_{2t}) \right] \text{ m-a.e.}$$

More generally, we expect these results to hold in the metric measure space settings of [13, 63, 89] and even on tamed Dirichlet spaces [36] without any uniform lower Ricci bounds.

**Logarithmic Sobolev inequalities.** Logarithmic Sobolev inequalities for functions and their connections to the functional heat flow $(P_t)_{t \geq 0}$, initiated in [41], have been an active field of research in past decades. For an overview over the vast literature on this subject, see [9, 26]. In a similar manner, in this article we relate logarithmic Sobolev inequalities for 1-forms to certain further integral properties of $(H_t)_{t \geq 0}$ described below. There is some ambiguity in formulating the former depending on whether one regards 1-forms as vector fields or really as contravariant objects. For brevity, we only outline Definition 4.4, where we say that a sufficiently regular vector field X obeys the 2-logarithmic Sobolev inequality $\text{LSI}_2(\beta, \chi)$ with parameters $\beta > 0$ and $\chi \in \mathbb{R}$ if

$$\int_M |X|^2 \log |X| \, dm \leq \beta |\nabla X|^2 + \chi \|X\|_{L^2}^2 + \|X\|_{L^2} \log \|X\|_{L^2}.$$
The advantage of this form is that it follows from logarithmic Sobolev inequalities for functions, known to hold in various cases [16, 93], via (1.4), see Lemma 4.8. It also implies its contravariant pendant from Definition 4.5 for arbitrary exponents, see Proposition 4.10.

The integral properties of \((H_t)_{t \geq 0}\) to be derived are the following. We call \((H_t)_{t \geq 0}\)
- hypercontractive if there exist \(T \in (0, \infty]\) and a strictly increasing \(C^1\)-function \(p : [0, T) \rightarrow (1, \infty)\) such that \(H_t\) is bounded from \(L^{p(t)}(T^*M)\) to \(L^{p(t)}(T^*M)\) for every \(t \in (0, T)\), and
- ultracontractive if there exist \(p_0 \in (1, \infty)\) and \(T > 0\) such that \(H_T\) is bounded from \(L^{p_0}(T^*M)\) to \(L^\infty(T^*M)\).

In great generality, in Theorem 4.12 we study when certain logarithmic Sobolev inequalities imply hyper- or ultracontractivity of \((H_t)_{t \geq 0}\). We also treat a partial converse in Theorem 4.16.

Read in concrete applications, according to all these discussions and the known functional examples from [16, 93], we deduce the following hypercontractivity.

**Theorem C.** On any compact RCD\(^*\)\((K, N)\) space or, for \(K > 0\), any RCD\((K, \infty)\) space, there exists a constant \(\beta > 0\) such that for every \(p_0 \in (1, \infty)\), \(H_t\) is bounded from \(L^{p_0}(T^*M)\) to \(L^{p(t)}(T^*M)\) with operator norm no larger than \(e^{-Kt}\) for every \(t > 0\), where the function \(p : [0, \infty) \rightarrow (1, \infty)\) is given by \(p(t) := 1 + (p_0 - 1)e^{\alpha t/\beta}\).

Many of our arguments and results entailing Theorem C are inspired by the functional treatise [26]. In the case of non-weighted Riemannian manifolds, logarithmic Sobolev inequalities for 1-forms have been studied with similar results in [18].

**Spectral behavior of Hodge’s Laplacian.** Kato’s inequality (1.4) also explicitly connects the spectra of the Hodge and the functional Laplacian. The study of the former is our goal in Chapter 5.

The following is first shown in full generality.

**Theorem D** (see Theorem 5.3 and Corollary 5.4). If a positive real number belongs to the spectrum of \(-\Delta\), then it is also contained in the spectrum of \(\widetilde{\Delta}\). Similar inclusions hold between the respective point and essential spectra. In particular,

\[
\inf \sigma(-\Delta + K) \leq \inf \sigma(\widetilde{\Delta}) \leq \inf \sigma(\widetilde{\Delta}) \setminus \{0\} \leq \inf \sigma(-\Delta) \setminus \{0\}.
\]

The stated spectral inclusions are known in the non-weighted Riemannian setting by [19]. Our proof of the former adopts a similar strategy, relying on a suitable variant of Weyl’s criterion. The first stated spectral gap inequality follows by basic spectral theory and is well-known in the smooth setting. See e.g. [44] for a more general smooth treatise and further references.

On compact RCD\(^*\)\((K, N)\) spaces, similarly to the case of functions, the spectrum of \(\widetilde{\Delta}\) can be characterized much better. A key tool towards an explicit understanding of it in this case is the following Rellich-type compact embedding theorem.

**Theorem E** (see Theorem 5.8). If \((M, d, m)\) is a compact RCD\(^*\)\((K, N)\) space, the formal operator \(\widetilde{\Delta}^{-1}\) is compact.

For Ricci limit spaces, i.e. noncollapsed mGH-limits of sequences of non-weighted Riemannian manifolds with uniformly lower bounded Ricci curvatures, Theorem E is due to [52, 53]. In the very recent work [54], Theorem E has been proven independently in a different way using so-called \(\delta\)-splitting maps.

The proof of Theorem E uses several powerful properties of \((H_t)_{t \geq 0}\) on compact RCD\(^*\)\((K, N)\) spaces. Using that \((P_t)_{t \geq 0}\) admits a heat kernel which obeys Gaussian bounds [56, 92], together with Theorem A and Bishop–Gromov’s inequality, we see in Theorem 4.3 that the heat operator \(H_t\) maps \(L^p(T^*M)\) boundedly into \(L^\infty(T^*M)\).
for every $t > 0$ and every $p \in [1, \infty]$. In particular, $H_t$ is a Hilbert–Schmidt operator on $L^2(T^*M)$, and Theorem E as well as expected properties of the spectrum of $\Delta$ stated in Theorem 5.13 are then deduced by abstract functional analysis. We also establish $L^\infty$-estimates on eigenforms of $\Delta$, with an explicit growth rate for positive eigenvalues. See Corollary 5.14 and Theorem 5.15.

The last part of Chapter 5, especially Theorem 5.19, is devoted to the proof of the independence of the $L^p$-spectrum of $\Delta$ on $p \in [1, \infty]$, provided $(M, d, m)$ is an $\text{RCD}^+(K, N)$ space satisfying, for every $\varepsilon > 0$, the volume growth condition

$$\sup_{x \in M} \int_M e^{-\varepsilon d(x, y)} m[B_1(x)]^{-1/2} m[B_1(y)]^{-1/2} \, dm(y) < \infty.$$ 

On non-weighted Riemannian manifolds, this is shown in [17]. Our proof, based on a perturbation argument, Theorem A and functional heat kernel bounds, is inspired by similar results for the functional Laplacian [47, 48, 75, 84]. See also [21, 28, 59, 90, 91] for further works in this direction for Markov processes and Feynman–Kac semigroups.

**Heat kernel.** Up to now no general result ensuring the existence of a heat kernel for $(H_t)_{t \geq 0}$ was known in the setting of [40]. Outside the scope of noncompact, even weighted Riemannian manifolds [44, 68, 73], there are only few metric measure constructions under restrictive structural (existence of a continuous covector bundle with constant fiber dimensions) and volume doubling assumptions [23, 79]. Our axiomatization and existence proof of a heat kernel for $(H_t)_{t \geq 0}$ on $\text{RCD}(K, \infty)$ spaces is hoped to push forward research in the above areas on such spaces. Our general study applies to non-locally compact or non-doubling, possibly infinite-dimensional spaces.

Let us motivate our axiomatization via the heat kernel $p$ of $(P_t)_{t \geq 0}$ from [3]. Slightly abusing notation, it induces a map $p : (0, \infty) \times L^0(M)^2 \to L^0(M^2)$ sending $t > 0$ and $(g, f) \in L^0(M)^2$ to the $m^\otimes 2$-measurable function given by $p_t[g, f](x, y) := p_t(x, y) g(x) f(y)$ in such a way that for a sufficiently large class of functions $f$ and $g$, we have $p_t[g, f] \in L^1(M^2)$ as well as

$$g P_t f = \int_M p_t[g, f](\cdot, y) \, dm(y) \quad \text{m.a.e.}$$

Let us turn to 1-forms. Recall that a heat kernel for $(H_t)_{t \geq 0}$ in the smooth, possibly weighted setting is a jointly smooth map $h : (0, \infty) \times M^2 \to (T^*M)^* \otimes T^*M$ — i.e. for every $t > 0$ and every $(x, y) \in M^2$, $h_t(x, y)$ is a homomorphism mapping $T^*_x M$ to $T^*_y M$ — satisfying

$$H_t \omega = \int_M h_t(\cdot, y) \omega(y) \, dm(y) \quad \text{m.a.e.} \quad (1.5)$$

for every $\omega \in L^2(T^*M)$. The heat kernel for 1-forms has first been constructed on compact spaces by [68] using the so-called parametrix construction. See also [44, 73].

Since $\text{RCD}(K, \infty)$ spaces a priori do neither come with any covector bundle nor with a smooth structure, the fiberwise notion (1.5) is replaced by “testing (1.5) pointwise against sufficiently many 1-forms”. Motivated by our functional considerations, we understand a mapping $h : (0, \infty) \times L^0(T^*M)^2 \to L^0(M^2)$ to be a heat kernel for $(H_t)_{t \geq 0}$ if, for every $t > 0$, $h_t$ is $L^0$-bilinear, and for all sufficiently regular 1-forms $\omega$ and $\eta$, we have $h_t[\eta, \omega] \in L^1(M^2)$ and

$$\langle \eta, H_t \omega \rangle = \int_M h_t[\eta, \omega](\cdot, y) \, dm(y) \quad \text{m.a.e.}$$
Theorem F (see Theorem 6.5). The heat kernel for $(H_t)_{t \geq 0}$ in the indicated sense exists and is unique.

The proof strategy is the following. Motivated by similar functional results [85, 76], a crucial tool to obtain integral kernels for certain operators is a Dunford–Pettis-type theorem [31, 32], a very general $L^\infty$-module version of which we prove in Theorem 6.3. Boiled down to the 1-form setting, it states that any linear operator which is bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ in the Banach sense admits an integral kernel, the concept of which is similar to the axiomatization of the 1-form heat kernel. Now for $t > 0$, the heat operator $H_t$ is not bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ in this generality. But by [92], given any $\varepsilon > 0$ there exist constants $C_1, C_2 > 0$ with

$$p_t(x,y) \leq m[B_{\sqrt{t}}(x)]^{-1/2} m[B_{\sqrt{t}}(x)]^{-1/2} \exp\left(C_1(1 + C_2 t) - \frac{d^2(x,y)}{(4 + \varepsilon)t}\right)$$

for every $t > 0$ and $m^{\otimes}$-a.e. $(x,y) \in M^2$. By Theorem A, the perturbed operator $A_t := \phi_t H_t \phi_t$, where $\phi_t(x) := m[B_{\sqrt{t}}(x)]^{1/2}$, is thus bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ and thus admits an integral kernel — formally multiplying $A_t$ by $\phi_t^{-1}$ from both sides then yields the desired integral kernel $h_t$ for $H_t$. Note that for this argument, it is essential that $(P_t)_{t \geq 0}$ has a heat kernel. (This explains best our restriction to uniform lower Ricci bounds, a more general result is not available up to now.)

Having existence of $h$ at our disposal, further properties of $h$ such as symmetry, Hess–Schrader–Uhlenbrock’s inequality for the “pointwise operator norm” $|h_t|_\otimes$ of $h_t$, i.e. for every $t > 0$,

$$|h_t|_\otimes(x,y) \leq e^{-Kt} p_t(x,y)$$

for $m^{\otimes}$-a.e. $(x,y) \in M^2$, and Chapman–Kolmogorov’s formula are shown in Theorem 6.7.

Two further results are then finally given on the class of RCD$^*$($K, N$) spaces. In Theorem 6.10, for every $t > 0$ we first prove the trace inequality

$$\text{tr} H_t \leq (\dim_{d.m} M) e^{-Kt} \text{tr} P_t.$$ 

Here, $\dim_{d.m} M$, a positive integer not larger than $N$, is the essential dimension of $(M, d, m)$ in the sense of [15, 66]. This generalizes similar results on possibly weighted Riemannian manifolds [44, 50, 72]. Furthermore, our spectral analysis for $\tilde{\Delta}$ from Theorem F entails a spectral resolution identity for $h_t$ in Theorem 6.11 as soon as $M$ is also compact.

Organization. In Chapter 2, we collect basic preliminaries about RCD spaces and differential geometric notions on these.

In Chapter 3, we recall basic properties of the heat equation for 1-forms and its solution. We establish the vector 1-Bochner inequality, and prove the important pointwise properties of $A_t$ and Theorem B of $(H_t)_{t \geq 0}$.

Chapter 4 is devoted to integral consequences of the results from Chapter 3. We demonstrate that $(H_t)_{t \geq 0}$ extends to a semigroup acting on $L^p(T^*M)$ for every $p \in [1, \infty]$ and discuss the $L^p$-$L^\infty$-regularization property of $(H_t)_{t \geq 0}$. This chapter is then closed by a discussion on hyper- and ultraccontractivity properties of $(H_t)_{t \geq 0}$ and their relation to logarithmic Sobolev inequalities, for instance, the treatise of Theorem C.

Chapter 5 contains spectral properties of $\tilde{\Delta}$, including the proofs of Theorem D and Theorem E. Section 5.3 is devoted to the independence of the $L^p$-spectrum of $\tilde{\Delta}$ on $p \in [1, \infty]$. 
The most important part of this work, Chapter 6, consists of a careful axiomatization of the notion of integral kernels in the context of $L^\infty$-modules, the proof of Theorem F, and further basic properties of the heat kernel for $(H_t)_{t\geq 0}$, i.e. the previously described trace inequality and the spectral resolution identity for $h_t$ in Section 6.3.

2. Synthetic Ricci bounds and Ricci curvature

2.1. Basic preliminaries.

Notation. The triple $(M, d, m)$ consists of a complete and separable metric space $(M, d)$ and a Borel measure $m$ on $(M, d)$ with full support. We always assume that $(M, d, m)$ is an RCD($K, \infty$) space, $K \in \mathbb{R}$. In particular, we have the following volume growth condition according to [87]. For every $z \in M$, there exists $C < \infty$ such that for every $r > 0$,

$$m[B_r(z)] \leq C e^{Cr^2}. \tag{2.1}$$

By $L^0(M)$, we denote the space of (m-a.e. equivalence classes of) $m$-measurable functions $f : M \to \mathbb{R}$. Given any partition $(E_j)_{j \in \mathbb{N}}$ of $M$ into Borel sets of finite and positive $m$-measure, it is a complete and separable metric space w.r.t. the metric $d_{L^p}$ defined through

$$d_{L^p}(f, g) := \sum_{j=1}^{\infty} \frac{2^{-j}}{m[E_j]} \int_{E_j} \min\{|f - g|, 1\} \, dm.$$

The induced topology on $L^0(M)$ does not depend on the choice of $(E_j)_{j \in \mathbb{N}}$ — indeed, $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence w.r.t. $d_{L^p}$ if and only if it is a Cauchy sequence w.r.t. convergence in $m$-measure on any Borel set $B \subset M$ with $m[B] < \infty$. Similar facts hold — and definitions are used — for the space $L^p(M^2)$ of $m^\otimes 2$-measurable $f : M^2 \to \mathbb{R}$. We write $L^p(M)$ for the $p$-th order Lebesgue space w.r.t. $m$, $p \in [1, \infty]$, endowed with the norm $\| \cdot \|_{L^p}$. $C(M)$ and Lip($M$) denote the sets of $d$-continuous and $d$-Lipschitz continuous functions $f : M \to \mathbb{R}$ — their subspaces of bounded or boundedly supported functions are marked with the subscript $b$ or $bs$, respectively.

For $i \in \{1, 2\}$, the projection maps $p_i : M^2 \to M$ are $p_i(z, x_i) := x_i$. Similarly, we define $p_{i+1} : M^3 \to M$ for $i \in \{1, 2, 3\}$. The distance function from a given point $z \in M$ is denoted by $r_z : M \to \mathbb{R}$ and defined by $r_z(x) := d(z, x)$.

Let $\mathfrak{M}(M)$ be the space of signed Radon measures on $(M, d)$ with finite total variation.

For two Banach spaces $\mathcal{M}$ and $\mathcal{N}$, the operator norm of an operator $A : \mathcal{M} \to \mathcal{N}$ is $\|A\|_{\mathcal{M} \to \mathcal{N}}$.

Notions of RCD($K, \infty$) spaces. We assume the definition of RCD($K, \infty$) spaces, $K \in \mathbb{R}$, to be known to the reader. In this paragraph, we only collect basic properties of them — details can be found in [1, 2, 3, 40, 62, 77, 87] and the references therein.

Let $S^2(M)$ be the class of functions $f \in L^0(M)$ which have a (squared) minimal weak upper gradient $\Gamma(f) \in L^1(M)$, whose polarization is also denoted by $\Gamma$. By the RCD($K, \infty$) assumption, $(M, d, m)$ is infinitesimally Hilbertian — that is, $(W^{1,2}(M), \| \cdot \|_{W^{1,2}})$ is a Hilbert space, where

$$W^{1,2}(M) := S^2(M) \cap L^2(M),$$

$$\| \cdot \|_{W^{1,2}}^2 := \| \cdot \|_{L^2}^2 + \| \Gamma(\cdot) \|_{L^2}.$$

The infinitesimal generator of the Dirichlet form induced by $\Gamma$ (the so-called Cheeger energy) is the Laplacian $\Delta$. It is linear, nonpositive, self-adjoint and densely
defined on $L^2(M)$ with domain $\text{Dom}(\Delta)$. It also gives rise to the $m$-symmetric, mass-preserving (functional) heat flow $\{P_t\}_{t \geq 0}$ on $L^2(M)$ via $P_t := e^{\Delta t}$, which extends to a contractive semigroup of linear operators to $L^p(M)$ for every $p \in [1, \infty]$, strongly continuous if $p < \infty$. If $f \in W^{1,2}(M)$, the curve $t \mapsto P_t f$ is $W^{1,2}$-continuous on $[0, \infty)$. For every $t > 0$, $P_t \Delta = \Delta P_t$ on $\text{Dom}(\Delta)$, and for every $f \in L^2(M)$,

$$\|\Delta P_t f\|_{L^2} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^2}. \tag{2.2}$$

The RCD($K, \infty$) property entails further properties of $\{P_t\}_{t \geq 0}$ crucially exploited in this work. First, as a consequence of heat flow analysis, the Sobolev-to-Lipschitz property holds, i.e. every $f \in W^{1,2}(M)$ with $\Gamma(f) \in L^\infty(M)$ has an $m$-a.e. representative in $\text{Lip}(M)$ whose Lipschitz constant is no larger than $\|\Gamma(f)^{1/2}\|_{L^\infty}$. Second, $P_t$ has a version which maps $L^\infty(M)$ to $\text{Lip}(M)$ for every $t > 0$. Third, the 1-Bakry–Émery inequality holds, i.e. for every $f \in W^{1,2}(M)$ and every $t \geq 0$,

$$\Gamma(P_t f)^{1/2} \leq e^{-Kt} P_t(\Gamma(f)^{1/2}) \quad \text{m-a.e.} \tag{2.3}$$

Fourth, the functional heat flow admits a heat kernel, i.e. for every $t > 0$ there exists a symmetric $m^{\otimes 2}$-measurable map $p_t: M^2 \to (0, \infty)$ such that

$$\int_M p_t(\cdot, y) \, dm(y) = 1$$

and, for every $f \in L^2(M)$,

$$P_t f = \int_M p_t(\cdot, y) f(y) \, dm(y) \quad \text{m-a.e.}$$

The following result from [92] establishes a Gaussian upper bound for $p_t$, $t > 0$.

**Theorem 2.1.** For every $\varepsilon > 0$, there exist finite constants $C_1 > 0$, depending only on $\varepsilon$, and $C_2 \geq 0$, depending only on $K$, such that for every $t > 0$ and $m^{\otimes 2}$-a.e. $(x, y) \in M^2$,

$$p_t(x, y) \leq m[B_{\sqrt{t}}(x)]^{-1/2} m[B_{\sqrt{t}}(y)]^{-1/2} \exp\left(C_1(1 + C_2 t) \frac{d^2(x, y)}{(4 + \varepsilon)^2}\right).$$

If $K \geq 0$, the constant $C_2$ can be chosen equal to zero.

Closely related to $(P_t)_{t \geq 0}$ and its regularizing properties is the set of test functions

$$\text{Test}(M) := \{f \in \text{Dom}(\Delta) \cap L^\infty(M) : \Gamma(f) \in L^\infty(M), \ \Delta f \in W^{1,2}(M)\},$$

which is an algebra w.r.t. pointwise multiplication. For instance, the heat operator $P_t$ maps $L^2(M) \cap L^\infty(M)$ to $\text{Test}(M)$ for every $t > 0$, whence $\text{Test}(M)$ is dense in $W^{1,2}(M)$. Variants of this have widely been used in the literature, e.g. yielding the subsequent useful results [40, 77].

**Lemma 2.2.** For every $f \in W^{1,2}(M)$ with $a \leq f \leq b$ m-a.e., $a, b \in [-\infty, \infty]$, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $\text{Test}(M)$ converging to $f$ in $W^{1,2}(M)$ with $a \leq f_n \leq b$ m-a.e. and $\Delta f_n \in L^\infty(M)$ for all $n \in \mathbb{N}$. If $\Gamma(f) \in L^\infty(M)$, this sequence can be chosen such that $(\Gamma(f_n))_{n \in \mathbb{N}}$ is bounded in $L^\infty(M)$.

**Lemma 2.3.** Assume that $M$ is locally compact, and let $U, V \subset M$ be two open subsets with $U \subset V$. Suppose that $U$ and $V^c$ have positive $d$-distance to each other. Then there exists a continuous function $\psi_{U,V} \in \text{Test}(M)$ satisfying $\psi_{U,V}(M) \subset [0, 1]$, $\psi_{U,V} = 1$ on $U$ and $\psi_{U,V} = 0$ on $V^c$. 
Notions of $\text{RCD}^*(K,N)$ spaces. We recall useful properties of $\text{RCD}(K,\infty)$ spaces admitting a synthetic notion of “upper dimension bound” $N \in (1,\infty)$ in addition, the so-called $\text{RCD}^*(K,N)$ spaces. See [35, 62, 88] and the references therein for details.

The first is the following corollary of Bishop–Gromov’s inequality. For every $D > 0$, whenever $0 < r < R < D$, there exists a constant $C < \infty$ depending only on $K$, $N$ and $D$ such that
\[
\frac{m[B_R(x)]}{m[B_r(x)]} \leq C \left(\frac{R}{r}\right)^N
\] (2.4)
for every $x \in M$ — in particular, since $B_1(y) \subset B_{1+\delta(x,y)}(x)$, for every $y \in M$,
\[
m[B_1(y)] \leq C e^{N\delta(x,y)} m[B_1(x)].
\] (2.5)
A further consequence of (2.4) is that $m$ is locally doubling, that is, for every $x \in M$ and $r \in (0,D)$ we have
\[
m[B_{2r}(x)] \leq 2^N C m[B_r(x)].
\] (2.6)
In turn, this condition implies local compactness of $M$. In particular, every finite diameter $\text{RCD}^*(K,N)$ space is necessarily compact — this is in particular the case when $K > 0$.

Since $\text{RCD}^*(K,N)$ spaces also satisfy local (1,1)- and (2,2)-Poincaré inequalities [35, 69], the general study from [85, 86] yields the existence of a locally Hölder continuous representative of the heat kernel $p$ on $(0,\infty) \times M^2$. By [56], for every $\varepsilon > 0$, there exist constants $C_3, C_4 > 1$ depending only on $K$, $N$ and $\varepsilon$, such that for every $x, y \in M$ and every $t > 0$,
\[
p_t(x,y) \leq C_3 m[B_{\sqrt{t}}(x)]^{-1} \exp\left(C_4 \frac{d^2(x,y)}{(4 + \varepsilon)t}\right).
\] (2.7)

2.2. A glimpse on nonsmooth differential geometry.

$L^\infty$-modules. The next two paragraphs summarize [40, Ch. 1], creating the framework of spaces of higher order differential objects. This toolbox is particularly needed in Section 6.1.

Definition 2.4. Given any $p \in [1,\infty]$, we call a real Banach space $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ (or simply $\mathcal{M}$ if its norm is understood) an $L^p$-normed $L^\infty$-module over $(M,d,m)$ if it is endowed with

a. a bilinear multiplication mapping $\cdot : L^\infty(M) \times \mathcal{M} \to \mathcal{M}$ satisfying
\[
(fg) \cdot v = f \cdot (g \cdot v),
\]
\[
1_M \cdot v = v,
\]

b. a nonnegatively valued mapping $| \cdot |_{\mathcal{M}} : \mathcal{M} \to L^p(M)$, the pointwise norm, obeying
\[
\|v\|_{\mathcal{M}} = \|v\|_{\mathcal{M}L^p},
\]
\[
|f \cdot v|_{\mathcal{M}} = |f| \|v\|_{\mathcal{M}} \text{ m-a.e.}
\]
for every $v \in \mathcal{M}$ and every $f,g \in L^\infty(M)$.

We shall drop the $\cdot$ sign and leave out the subscript $\mathcal{M}$ from all pointwise norms — it is always clear from the context which one is considered. A simple example of an $L^p$-normed $L^\infty$-module is $L^p(M)$ itself. The pointwise norm $| \cdot |$ is local, i.e. for every $v \in \mathcal{M}$, $1_B v = 0$ if and only if $\|v\| = 0$ m-a.e. on $B$ for every Borel set $B \subset M$, and it satisfies the pointwise m-a.e. triangle inequality. The set of all $v \in \mathcal{M}$ such that $1_B v = 0$ for some bounded Borel set $B \subset M$ will be termed $\mathcal{M}_{0B}$. 

A Hilbert module is an $L^2$-normed $L^\infty$-module $\mathcal{M}$ which is a Hilbert space. In this case, $|\cdot|$ satisfies the pointwise m.a.e. parallelogram identity, hence induces a pointwise scalar product $\langle \cdot, \cdot \rangle: \mathcal{M}^2 \to L^1(M)$. The latter is $L^\infty$-bilinear, local in both components and obeys the pointwise m.a.e. Cauchy–Schwarz inequality.

The dual module $\mathcal{M}^*$ of $\mathcal{M}$ is the set of all linear maps $L: \mathcal{M} \to L^1(M)$ for which $\|L\|_{\mathcal{M}, \mathcal{M}^*} < \infty$ and $L(fv) = fLv$ for every $v \in \mathcal{M}$ and every $f \in L^\infty(M)$. We endow $\mathcal{M}^*$ with the usual operator norm. Whenever convenient, we denote the pairing $Lv$ of $L \in \mathcal{M}^*$ and $v \in \mathcal{M}$ by $\langle v \mid L \rangle$, $L(v)$ or $v(L)$. Then $\mathcal{M}^*$ is an $L^0$-normed $L^\infty$-module, where $\|q\| \in [1, \infty]$ satisfies $1/p + 1/q = 1$, its pointwise norm $|\cdot|: \mathcal{M}^* \to L^0(M)$ being given by

$$|L| := \text{esssup}\{\langle v \mid L \rangle : v \in \mathcal{M}, \|v\| \leq 1 \text{ m.a.e.}\}.$$ 

$L^0$-modules. Let $\mathcal{M}$ be an $L^0$-normed $L^\infty$-module, $p \in [1, \infty]$. By $\mathcal{M}^0$ we intend the $L^0$-module corresponding to $\mathcal{M}$ from [40, Sec. 1.3] — the completion of $\mathcal{M}$ w.r.t. the metric $d_{\mathcal{M}^0}$ given by

$$d_{\mathcal{M}^0}(v, w) := \sum_{j=1}^{\infty} 2^{-j} \frac{\min\{|v - w|, 1\}}{m(E_j)} \int_{E_j} \\, dm.$$ 

Here, $(E_j)_{j \in \mathbb{N}}$ is a partition of $M$ into Borel sets of finite and positive m-measure.

Roughly speaking, $\mathcal{M}^0$ is some larger space of "m-measurable elements $v$" for which $v \in \mathcal{M}$ if and only if $|v| \in L^p(M)$. Along with the construction of $\mathcal{M}^0$, the multiplication, the pointwise norm and the pointwise pairing operators on $\mathcal{M}$ uniquely extend to maps $\mathcal{M}^0 \times \mathcal{M}^0 \to \mathcal{M}^0$, $|\cdot|: \mathcal{M}^0 \to L^0(M)$ and $\langle \cdot \mid \cdot \rangle: \mathcal{M}^0 \times (\mathcal{M}^0)^* \to L^0(M)$ which satisfy similar properties as their former counterparts w.r.t. elements in $\mathcal{M}$ and $L^0(M)$ instead of $\mathcal{M}$ and $L^\infty(M)$, respectively.

Nonsmooth first order calculus. This paragraph surveys [40, Ch. 2], i.e. the introduction of the relevant spaces of $L^2$-1-forms and $L^2$-vector fields over $(M, d, \mathfrak{m})$.

We denote the cotangent module constructed in [40, Sec. 2.2] by $L^2(T^*M)$. Elements of it are called (differential) 1-forms. The tangent module, whose elements are called vector fields, is defined by $L^2(TM) := L^2(T^*M)^*$. The norms on both spaces are termed $\|\cdot\|_{L^2}$. $L^2(T^*M)$ and $L^2(TM)$ are separable Hilbert modules and are isometrically isomorphic. The isometry is even true pointwise m.a.e., via the musical isomorphisms $\iota: L^2(TM) \to L^2(T^*M)$ and $\sharp: L^2(T^*M) \to L^2(TM)$ with $X^\flat(Y) := \langle X, Y \rangle$ and $\langle \omega^\sharp, Y \rangle := \omega(Y)$.

The construction of $L^2(T^*M)$ comes with a linear and continuous differential map $d: S^2(M) \to L^2(T^*M)$ for which $|df| = |\nabla f|^{1/2}$ m.a.e. for every $f \in S^2(M)$. It is closed in the sense that if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $S^2(M)$ converging to $f \in L^0(M)$ pointwise m.a.e., and if $(df_n)_{n \in \mathbb{N}}$ converges to some $\omega \in L^2(T^0M)$ w.r.t. $\|\cdot\|_{L^2}$, then $f \in S^2(M)$ and $\omega = df$. The gradient $\nabla f$ of $f \in S^2(M)$ is then defined by $\nabla f := (df)^2 \in L^2(TM)$ — equivalently, $\nabla f$ is the unique element $Y \in L^2(TM)$ such that $df(Y) = |df|^2 = |Y|^2$ m.a.e. It obeys similar calculus rules as the following ones for $d$.

Lemma 2.5. (i) Locality. For every $L^1$-negligible Borel set $A \subset R$ and every $f \in S^2(M)$, we have $1_{f^{-1}(A)} df = 0$. In particular, $1_{[c]}(f) df = 0$ for every $c \in R$.

(ii) Chain rule. For every $f \in S^2(M)$ and every $\Phi \in \text{Lip}(R)$, define $\Phi'(f)$ arbitrarily on the preimage of all non-differentiability points of $\Phi$ under $f$. Then $\Phi(f) \in S^2(M)$ and $d\Phi(f) = \Phi'(f) \, df$. 
(iii) **Leibniz Rule.** For every \( f, g \in S^2(M) \cap L^\infty(M) \), \( f g \in S^2(M) \) with
\[
\int_M g f \, dm = - \int_M dg(X) \, dm.
\]

**Definition 2.6.** Let \( \text{Dom}(\text{div}) \) be the space of all \( X \in L^2(TM) \) for which there exists a function \( f \in L^2(M) \) such that for every \( g \in W^{1,2}(M) \),
\[
\int_M g f \, dm = - \int_M dg(X) \, dm.
\]
In case of existence, \( f \) is unique, denoted by \( \text{div} X \) and called the divergence of \( X \).

By the integration by parts formula for \( \Delta \), for every \( f \in \text{Dom}(\text{div}) \), we have \( \nabla f \in \text{Dom}(\text{div}) \) with the usual identity \( \text{div} \nabla f = \Delta f \). Moreover, given any \( X \in \text{Dom}(\text{div}) \) and any \( f \in S^2(M) \cap L^\infty(M) \) with \( |df| \in L^\infty(M) \), we have \( f X \in \text{Dom}(\text{div}) \) with
\[
\text{div}(f X) = df(X) + f \text{div} X \quad \text{m-a.e.} 
\]

**Further Lebesgue spaces and \( L^\infty \)-module tensor products.** We set
\[
L^0(T^* M) := L^2(T^* M)^0, \\
L^0(TM) := L^2(TM)^0.
\]

\( L^p(T^* M) \) is the class of all \( v \in L^0(T^* M) \) with \(|v| \in L^p(T^* M) \), where \( p \in [1, \infty] \). It is an \( L^p \)-normed \( L^\infty \)-module w.r.t. the canonical norm \( \| \cdot \|_{L^p} \), separable if \( p < \infty \). For every \( p, q \in [1, \infty] \) with \( 1/p + 1/q = 1 \), \( L^p(T^* M)^* \) and \( L^q(T^* M) \) are isometrically isomorphic as modules. Define \( L^0(TM) \) analogously for \( p \in [1, \infty] \).

Denote by \( L^2((T^*)^{\otimes 2} M) \) and \( L^2(T^{\otimes 2} M) \) the two-fold tensor products of \( L^2(T^* M) \) and \( L^2(TM) \) in the \( L^\infty \)-module sense of [40, Sec. 1.5]. These spaces are separable. Hilbert modules in which the linear spans of elements of the kind \( \omega_1 \otimes \omega_2 \) and \( X_1 \otimes X_2 \) with \( \omega_1, \omega_2 \in L^2(T^* M) \cap L^\infty(T^* M) \) and \( X_1, X_2 \in L^2(TM) \cap L^\infty(TM) \), are respectively dense. The corresponding pointwise norms : \( L^2((T^*)^{\otimes 2} M) \) and \( L^2(T^{\otimes 2} M) \) — which turn both spaces isometrically isomorphic — as well as \( (\cdot, \cdot) \) on \( L^2(\Lambda^2 T^* M) \) are initially defined — and then extended by approximation — by
\[
(\omega_1 \otimes \omega_2) : (X_1^1 \otimes X_2^2) := \omega_1(X_1) \omega_2(X_2), \\
(\omega_1 \wedge \omega_2, X_1^1 \wedge X_2^2) := \det \omega_1(X_2).
\]

**Hilbert–Schmidt operators and Hilbert space tensor products.** We call a linear operator \( S : L^2(T^* M) \to L^2(T^* M) \) a Hilbert–Schmidt operator if for some — or equivalently any — countable orthonormal bases \((\omega_i)_{i \in \mathbb{N}}\) and \((\eta_i')_{i' \in \mathbb{N}}\) of \( L^2(T^* M) \),
\[
\|S\|_{\text{HS}}^2 := \sum_{i, i' = 1}^{\infty} \left[ \int_M (S\omega_i, \eta_i') \, dm \right]^2 < \infty.
\]

The two-fold Hilbert space tensor product \( L^2(T^* M)^{\otimes 2} \) of \( L^2(T^* M) \) — see e.g. [40, 57] for the precise definition — which will be needed in Section 6.3, is isometrically isomorphic to the space of all Hilbert–Schmidt operators from \( L^2(T^* M)' \) to \( L^2(T^* M) \), endowed with the norm \( \| \cdot \|_{\text{HS}} \). Moreover, up to isomorphism, it is characterized by the following universal property [57, Thm. 2.6.4]. Given a real Hilbert space \( H \), a bilinear mapping \( G : L^2(T^* M)^2 \to H \) is termed weakly Hilbert–Schmidt if for some — or equivalently any — countable orthonormal bases \((\omega_i)_{i \in \mathbb{N}}\) and \((\eta_i')_{i' \in \mathbb{N}}\) as above,
\[
\|G\|_{\text{wHS}}^2 := \sup \left\{ \|h\|_H^2 \sum_{i, i' = 1}^{\infty} \|G(\omega_i, \eta_i') \|^2 : h \in H \setminus \{0\} \right\} < \infty.
\]
Theorem 2.7. The mapping $e: L^2(T^*M)^2 \to L^2(T^*M)^{\otimes 2}$ which is defined by $e(\eta, \omega) := \eta \otimes \omega$ is weakly Hilbert–Schmidt. Moreover, given any real Hilbert space $H$, for every weakly Hilbert–Schmidt mapping $G: L^2(T^*M)^2 \to H$, there exists a unique bounded operator $T: L^2(T^*M)^{\otimes 2} \to H$ such that

$$G = T(e) \quad \text{and} \quad \|T\|_{L^2,H} = \|G\|_{\text{HS}}.$$

From the $L^\infty$-module perspective, it is not difficult to prove the subsequent result which actually holds true for the Hilbert space tensor product of any two Hilbert modules over $(M, d, \mathfrak{m})$. Note that we use the $\otimes$ sign in both cases, although $L^2(T^*M)^{\otimes 2}$ differs from $L^2((T^*)^{\otimes 2}M)$. For instance, we always have $\omega_1 \otimes \omega_2 \in L^2(T^*M)^{\otimes 2}$ for every $\omega_1, \omega_2 \in L^2(T^*M)$, but in Gigli’s sense, $\omega_1 \otimes \omega_2$ does not necessarily belong to $L^2((T^*)^{\otimes 2}M)$ unless $\omega_i \in L^\infty(T^*M)$ for at least one $i \in \{1, 2\}$. (Indeed, one should rather formally think of $\omega_1 \otimes \omega_2 \in L^2(T^*M)^{\otimes 2}$ as section $(x, y) \mapsto \omega_1(x) \otimes \omega_2(y)$ and of $\omega_1 \otimes \omega_2 \in L^2((T^*)^{\otimes 2}M)$ as section $x \mapsto \omega_1(x) \otimes \omega_2(x)$.)

Proposition 2.8. The Hilbert space tensor product $L^2(T^*M)^{\otimes 2}$ has a natural structure of a Hilbert module over the product space $(M^2, d^2, \mathfrak{m}^{\otimes 2})$ such that the multiplication $:\cdot : L^\infty(M^2) \times L^2(T^*M)^{\otimes 2} \to L^2(T^*M)^{\otimes 2}$ and the pointwise norm $|\cdot|: L^2(T^*M)^{\otimes 2} \to L^2(M^2)$ satisfy

$$(f(pr_1)g(pr_2))(\omega_1 \otimes \omega_2) = (f \omega_1) \otimes (g \omega_2),$$

$$\omega_1 \otimes \omega_2 = |\omega_1||pr_1| \omega_2|pr_2|$$

for every $\omega_1, \omega_2 \in L^2(T^*M)$ and every $f, g \in L^\infty(M)$.

Nonsmooth second order calculus. The next two paragraphs summarize [40, Ch. 3]. In Gigli’s treatise, Hessian, covariant derivative and exterior differential are defined by integration by parts procedures. However, we do not need their precise defining formulas in these cases, which are thus omitted — we focus on their calculus rules.

We denote the space of test 1-forms and test vector fields, respectively, by

$$\text{Test}(T^*M) := \text{span}\{g df : f, g \in \text{Test}(M)\},$$

$$\text{Test}(TM) := \text{Test}(T^*M)^2.$$

Then $\text{Test}(T^*M)$ and $\text{Test}(TM)$ are dense in $L^2(T^*M)$ and $L^2(TM)$, respectively.

Define the linear space $W^{2,1}(M)$ as the space of all $f \in W^{1,2}(M)$ which admit a Hessian $\text{Hess} f \in L^2((T^*)^{\otimes 2}M)$. The Hessian of a fixed function is symmetric and $L^\infty$-bilinear. Its graph is a closed subset of $W^{1,2}(M) \times L^2((T^*)^{\otimes 2}M)$. As an important consequence of the RCD($K, \infty$) assumption, a class which is dense in $L^2(M)$ is contained in $W^{2,2}(M)$, and the integrated 2-Bochner inequality holds — more precisely, for every $f \in \text{Dom}(\Delta)$, we have $f \in W^{2,2}(M)$ with

$$\int_M |\text{Hess} f|^2 dm \leq \int_M |\Delta f|^2 - K |df|^2 \ dm.$$  \hspace{1cm} (2.9)

Next, let $W^{1,2}(TM)$ be the linear space of all $X \in L^2(TM)$ admitting a covariant derivative $\nabla X \in L^2(T^\otimes 2M)$. The graph of the operator $\nabla$ is a closed subset of $L^2(TM) \times L^2(T^\otimes 2M)$. We have $\text{Test}(TM) \subset W^{1,2}(TM)$ — more precisely, given any $f, g \in \text{Test}(M)$, we have $g \nabla f \in W^{1,2}(TM)$ with

$$\nabla(g \nabla f) = \nabla g \otimes \nabla f + g (\text{Hess} f)^\sharp.$$  \hspace{1cm} (2.10)

Moreover, $\nabla f \in W^{1,2}(TM)$ for every $f \in W^{2,2}(M)$ with $\nabla^2 f := \nabla \nabla f = (\text{Hess} f)^\sharp$. Lastly, $\nabla$ is compatible with $(\cdot, \cdot)$ on $L^2(TM)^2$ in the following way. Let

$$H^{1,2}(TM) = \|\cdot\|_{H^{1,2}} \text{Test}(TM),$$

$$\|\cdot\|_{H^{1,2}} := \|\cdot\|_{L^2}^2 + \|\nabla \cdot\|_{L^2}^2,$$
a separable Hilbert space, dense in \( L^2(TM) \). Given any \( X \in H^{1,2}(TM) \cap L^\infty(TM) \) and \( Z \in L^2(TM) \), let \( \nabla Z X \in L^0(TM) \) be the vector field uniquely defined by \( \left< \nabla Z X, V \right> := \nabla X : (Z \otimes V) \) for every \( V \in L^0(TM) \). Then for every \( Y \in H^{1,2}(TM) \cap L^\infty(TM) \), we have \( (X,Y) \in W^{1,2}(M) \) with
\[
d(X,Y)(Z) = \left< \nabla Z X, Y \right> + \left< \nabla Z Y, X \right> \quad \text{m-a.e.} \quad (2.10)
\]

We next turn to the contravariant picture. Let \( \text{Dom}(d) \) be the linear space of all \( \omega \in L^2(T^*M) \) which have an exterior differential \( d\omega \in L^2(\Lambda^2 T^*M) \). This gives rise to a closed operator \( d \) on \( L^2(T^*M) \). For every \( f, g \in \text{Test}(M) \), we have \( gdf \in \text{Dom}(d) \) with
\[
d(gdf) := dg \wedge df.
\]

Lastly, we have \( df \in \text{Dom}(d) \) for every \( f \in W^{1,2}(M) \) with \( d^2f := dd^2f = 0 \). The formal adjoint of the exterior differential is the codifferential \( \delta \). Its domain \( \text{Dom}(\delta) \) is the space of all differential 1-forms \( \omega \in L^2(T^*M) \) for which \( \omega^\delta \in \text{Dom}(\text{div}) \), in which case we define \( \delta \omega := -\text{div} \omega^\delta \). In particular, if \( f \in \text{Dom}(\Delta) \), then \( df \in \text{Dom}(\delta) \) with \( \delta df = -\Delta f \). The graph of \( \delta \) is a closed subset of \( L^2(T^*M) \times L^2(M) \).

Having now these two notions at our disposal, define the Hodge space
\[
H^{1,2}(T^*M) := \text{cl}_\perp \|\mu\|_{2,2} \text{Test}(T^*M),
\]
\[
\| \cdot \|_{2,2} := \| \cdot \|_{L^2}^2 + \| \cdot \|_{L^2}^2 + \| \delta \cdot \|_{L^2}^2,
\]
a separable Hilbert space which is dense in \( L^2(T^*M) \). The Hodge energy functional \( \mathcal{E} : L^2(T^*M) \to [0, \infty) \) is
\[
\mathcal{E}(\omega) := \begin{cases} 
\frac{1}{2} \int_M \left[ |d\omega|^2 + |\delta \omega|^2 \right] \, \text{dm} & \text{if } \omega \in H^{1,2}(T^*M), \\
\text{otherwise.} & 
\end{cases}
\]

We provide the subsequent two lemmata concerning the space \( H^{1,2}(T^*M) \). We give a proof for Lemma 2.9 for convenience. Lemma 2.10 follows by similar results for \( \text{Test}(T^*M) \) from [40, Thm. 3.5.2, Prop. 3.5.12] by approximation as in Lemma 2.2.

**Lemma 2.9.** For every \( f \in L^2(M) \) and every \( t > 0 \), we have \( dP_t f \in H^{1,2}(T^*M) \).

**Proof.** Given any \( f \in L^2(M) \) and \( n \in \mathbb{N} \), set \( f_n := \min\{n, \max\{f, -n\} \} \in L^2(M) \cap L^\infty(M) \). Then \( (dP_t f_n)_{n \in \mathbb{N}} \) is a sequence in \( \text{Test}(T^*M) \) which converges to \( dP_t f \) in \( L^2(T^*M) \) for every \( t > 0 \).

By the closedness of the exterior differential, it follows that \( dP_t f \in \text{Dom}(d) \) and \( d^2P_t f_n = d^2P_t f = 0 \) for every \( n \in \mathbb{N} \). Since \( \delta dP_t f_n = -\Delta P_t f_n \to -\Delta P_t f = \delta dP_t f \in L^2(M) \) as \( n \to \infty \) as a consequence of (2.2), we obtain the claim. \( \square \)

**Lemma 2.10.** Let \( f \in S^2(M) \cap L^\infty(M) \) and \( \omega \in H^{1,2}(T^*M) \). Assume moreover that \( df \in L^\infty(T^*M) \) or that \( \omega \in L^\infty(T^*M) \). Then \( f \omega \in H^{1,2}(T^*M) \) with
\[
\delta(f \omega) = f \delta \omega - \langle df, \omega \rangle.
\]

**Definition 2.11.** A differential 1-form \( \omega \in H^{1,2}(T^*M) \) belongs to \( \text{Dom}(\Delta) \) if there exists \( \alpha \in L^2(T^*M) \) such that for every \( \eta \in H^{1,2}(T^*M) \),
\[
\int_M \langle \eta, \alpha \rangle \, \text{dm} = \int_M \left[ \langle d\eta, d\omega \rangle + \delta \eta \delta \omega \right] \, \text{dm}.
\]

In case of existence, \( \alpha \) is unique, denoted by \( \Delta \omega \) and termed Hodge Laplacian of \( \omega \).
The induced operator $\tilde{\Delta}$ on $L^2(T^*M)$ is nonnegative, self-adjoint and closed. The space of harmonic 1-forms, i.e. those $\omega \in \text{Dom}(\Delta)$ with $\Delta \omega = 0$ — or equivalently, $\omega \in H^{1,2}(T^*M)$ with $d\omega = 0$ and $\delta \omega = 0$ — is called the space of harmonic 1-forms. Furthermore, for every $f,g \in \text{Test}(M)$, we have $gdf \in \text{Dom}(\Delta)$ — actually also with $\Delta(gdf) \in L^1(T^*M)$ — and

$$\Delta(gdf) = -g d\Delta f - \Delta g df - 2 \text{Hess} f(\nabla g, \cdot).$$

(2.12)

Measure-valued Ricci curvature. The main result of [40] is the appropriate definition of a measure-valued Ricci tensor $\text{Ric}$ on $\text{RCD}(K, \infty)$ spaces. Its final outcome is stated in Theorem 2.14. In the $\text{RCD}^*(K, N)$ framework, $N \in (1, \infty)$, a dimensional $N$-Ricci tensor has been introduced and studied in [45]. The main result therein is formulated in Proposition 2.15.

Definition 2.12. Let $\text{Dom}(\Delta)$ consist of all functions $f \in W^{1,2}(M)$ for which there exists a signed measure $\mu \in \mathcal{M}(M)$ such that for every $g \in \text{Lip}_{ba}(M)$,

$$\int_M g ~d\mu = -\int_M \langle \nabla g, \nabla f \rangle ~dm.$$

In case of existence, $\mu$ is unique, denoted by $\Delta f$ and called the measure-valued Laplacian of $f$.

This definition is compatible with the functional Laplacian in the sense that $f \in \text{Dom}(\Delta)$ if and only if $f \in \text{Dom}(\Delta)$ and $\Delta f$ has a density $h \in L^2(M)$ w.r.t. $m$, in which case $\Delta f = h$ $m$-a.e. The following is proven in [12, Lem. 3.1].

Lemma 2.13. For every $f \in \text{Dom}(\Delta) \cap L^\infty(M)$, every interval $I \subset \mathbb{R}$ with $f(M) \cup \{0\} \subset I$ and every $\Phi \in C^2(I)$ with $\Phi(0) = 0$, we have $\Phi(f) \in \text{Dom}(\Delta)$ with

$$\Delta \Phi(f) = \Phi'(f) \Delta f + \Phi''(f) \Gamma(f) m.$$

A crucial outcome of the $\text{RCD}(K, \infty)$ condition is that $|X|^2 \in \text{Dom}(\Delta)$ for every $X \in \text{Test}(TM)$.

Consider now the image $H^{1,2}(T^*M)^{\sharp}$ of $H^{1,2}(T^*M)$ under $\sharp$ and, abusing notation, equip it with the norm $\| \cdot \|_{H^{1,2}} := \| \cdot \|_{H^{1,2}}$. A key feature yielding Theorem 2.14 below is that the inclusion $H^{1,2}(T^*M)^{\sharp} \subset H^{1,2}(TM)$ is continuous, i.e. $X \in H^{1,2}(TM)$ for every $X \in H^{1,2}(T^*M)^{\sharp}$, and

$$\int_M |\nabla X|^2 ~dm \leq \int_M \left[ |dX|^2 + |\delta X|^2 - K |X|^2 \right] ~dm. \tag{2.13}$$

Theorem 2.14. There exists a unique continuous map $\text{Ric}: H^{1,2}(T^*M)^{\sharp} \rightarrow \mathcal{M}(M)$ which satisfies

$$\text{Ric}(X, Y) = \frac{1}{2} \Delta (X, Y) + \frac{1}{2} \langle X, (\tilde{\Delta} Y)^{\sharp} \rangle + \frac{1}{2} \langle Y, (\tilde{\Delta} X)^{\sharp} \rangle - \nabla X : \nabla Y \rangle$$

for every $X,Y \in \text{Test}(TM)$. It is symmetric and $\mathbb{R}$-bilinear. Furthermore, for every $X,Y \in H^{1,2}(T^*M)^{\sharp}$ it satisfies the following relations.

(i) Ricci bound. We have

$$\text{Ric}(X,X) \geq K |X|^2 m.$$

(ii) Integrated Bochner formula. We have

$$\text{Ric}(X,X)[M] = \int_M \left[ |dX|^2 + |\delta X|^2 \delta Y - \nabla X : \nabla Y \right] ~dm$$

Proposition 2.15. If $(M, d, m)$ satisfies the $\text{RCD}^*(K, N)$ condition, $N \in (1, \infty)$, then within the notation of Theorem 2.14, for every $X \in H^{1,2}(T^*M)^{\sharp}$ we have

$$\Gamma_2(X) \geq \left[ K |X|^2 + \frac{1}{N} |\text{div} X|^2 \right] m,$$
3. Pointwise estimates for the heat flow

3.1. Definition and basic properties. The negative Hodge Laplacian $-\Delta$ is nonpositive, self-adjoint and densely defined on $L^2(T^*M)$. By spectral calculus, it generates a family of bounded linear operators $(H_t)_{t \geq 0}$ on $L^2(T^*M)$ via $H_t := e^{-t\Delta}$ for which, given any $\omega \in L^2(T^*M)$, the map $t \mapsto H_t\omega$ defined on $[0, \infty)$ is uniquely characterized by the subsequent three properties.

a. **Initial value.** $H_0\omega = \omega$.

b. **Strong continuity.** The map $t \mapsto H_t\omega$ is strongly $L^2$-continuous on $[0, \infty)$.

c. **Kolmogorov forward equation.** For every $t > 0$, it holds that $H_t\omega \in \text{Dom}(\Delta)$. Moreover, the limit of $h^{-1} (H_t + h\omega - H_t\omega)$ as $h \to 0$ exists strongly in $L^2(T^*M)$ and satisfies

$$\frac{d}{dt} H_t\omega = -\Delta H_t\omega.$$

By uniqueness, it admits the semigroup property $H_{t+s}\omega = H_t H_s\omega$, $s, t \geq 0$.

**Definition 3.1.** We refer to $(H_t)_{t \geq 0}$ as the heat semigroup or heat flow on 1-forms.

**Remark 3.2.** One can equivalently define $(H_t\omega)_{t \geq 0}$ as the gradient flow of the functional $E$ from (2.11) starting in $\omega \in L^2(T^*M)$ via Brézis–Komura’s theory [14], as done in [40, Sec. 3.6].

Such flow satisfies the conditions a. and b. by construction. The link between c. and the usual differential inclusion appearing in the previous gradient flow approach is due to the fact that the subdifferential of $E$ at a given $\eta \in H^{1,2}(T^*M)$ is nonempty if and only if $\eta \in \text{Dom}(\Delta)$, in which case it only contains $\Delta \eta$. See [40, Subsec. 3.4.4] for a similar discussion.

Let us list some properties of $(H_t)_{t \geq 0}$. **Lemma 3.3** treats the link between $(H_t)_{t \geq 0}$ and $(P_t)_{t \geq 0}$ on exact forms, while Theorem 3.6 is due to standard semigroup theory [14, 71, 94] — item (vii) therein has been proven in [40, Prop. 3.6.10] using Theorem 2.14.

**Lemma 3.3.** For every $f \in W^{1,2}(M)$ and every $t > 0$, $dP_t f \in \text{Dom}(\Delta)$ and

$$H_t df = dP_t f.$$

**Proof.** Set $f_n := \min \{n, \max \{f, -n\}\} \in W^{1,2}(M) \cap L^\infty(M)$ for every $n \in \mathbb{N}$. Then the collection $(dP_t f_n)_{t \geq 0}$ in $L^2(T^*M)$ clearly satisfies a. above, and b. follows by continuity of $t \mapsto P_t f_n$ on $[0, \infty)$ w.r.t. strong $W^{1,2}$-convergence. To check c., given any $t > 0$, recall that $dP_t f_n \in \text{Test}(T^*M)$. The identity (2.12) and the closedness of $d$ [40, Thm. 2.2.9] together with $\Delta P_t f_n = P_{t/2} \Delta P_{t/2} f_n \in W^{1,2}(M)$ then imply $dP_t f_n \in \text{Dom}(\Delta)$ and

$$\frac{d}{dt} P_t f_n = \frac{d}{dt} P_t f_n = d \Delta P_t f_n = -\Delta dP_t f_n.$$

Therefore, $H_t df_n = dP_t f_n$ for every $n \in \mathbb{N}$ by uniqueness.

The claim follows by letting $n \to \infty$, using the boundedness of $H_t$ as well as the fact that $(P_t f_n)_{n \in \mathbb{N}}$ converges to $P_t f$ in $W^{1,2}(M)$. In particular, $dP_t f \in \text{Dom}(\Delta)$. □
Remark 3.4. Given any $k \in \mathbb{N}$ with $k \geq 2$, one can define a “canonical” heat flow $(H_t^k)_{t \geq 0}$ acting on $L^2(\Lambda^k T^* M)$ as the semigroup corresponding to the negative Hodge Laplacian on $k$-forms. However, we do not know if for every $\omega \in H^{1,2}(\Lambda^{k-1} T^* M)$ and every $t \geq 0$, we have

$$H_t^k \omega = dH_t^{k-1} \omega.$$

The subtle problem arising when mimicking the proof of Lemma 3.3 is that, to verify that $dH_t^{k-1} \omega \in \text{Dom}(\Delta^k)$, by definition of $\Delta^k$ one a priori has to know that $dH_t^{k-1} \omega \in H^{1,2}(\Lambda^{k-1} T^* M)$, i.e. to have an analogue of Lemma 2.9 for forms of higher degree, which is unclear to us.

Corollary 3.5. If $\omega \in \text{Dom}(\delta)$ and $t \geq 0$, then $H_t \omega \in \text{Dom}(\delta)$ with

$$\delta H_t \omega = \partial_t \delta \omega.$$

Theorem 3.6. The following properties of $(H_t)_{t \geq 0}$ hold for every $\omega \in L^2(T^* M)$ and every $t > 0$.

(i) Self-adjointness. The operator $H_t$ is self-adjoint in $L^2(T^* M)$.

(ii) Kolmogorov backward equation. If $\omega \in \text{Dom}(\Delta)$, we have

$$\frac{d}{dt} H_t \omega = -H_t \Delta \omega.$$

In particular, the identity $\Delta H_t = H_t \Delta$ holds on $\text{Dom}(\Delta)$.

(iii) $L^2$-contractivity. $\|H_t \omega\|_{L^2} \leq \|H_0 \omega\|_{L^2}$ for every $s \in [0,t]$.

(iv) Energy dissipation. The function $t \mapsto \mathcal{E}(H_t \omega)$ belongs to the class $C^1((0, \infty))$. Its derivative satisfies

$$\frac{d}{dt} \mathcal{E}(H_t \omega) = -\int_M |\Delta H_t \omega|^2 \, dm.$$

In particular, $\mathcal{E}(H_t \omega) \leq \mathcal{E}(H_0 \omega)$ for every $s \in [0,t]$.

(v) $H^{1,2}$-continuity. If $\omega \in H^{1,2}(T^* M)$, the map $t \mapsto H_t \omega$ is continuous on $[0, \infty)$ w.r.t. strong convergence in $H^{1,2}(T^* M)$.

(vi) A priori estimates. We have

$$\mathcal{E}(H_t \omega) \leq \frac{1}{4t} \|\omega\|_{L^2}^2,$$

$$\|\Delta H_t \omega\|_{L^2}^2 \leq \frac{1}{2t^2} \|\omega\|_{L^2}^2.$$

(vii) Contravariant Bakry–Émery inequality. We have

$$|H_t \omega|^2 \leq e^{-2Kt} P_t(|\omega|^2) \quad \text{m.a.e.}$$

3.2. Kato’s inequality. A further key to obtain vector 1-Bochner inequalities and to derive various functional inequalities for vector fields and 1-forms is Proposition 3.7 below. In the smooth framework, this estimate is known as Kato’s inequality. The elementary proof of its nonsmooth counterpart can be found in [27, Lem. 2.5]. It also easily provides us with the chain rule from Corollary 3.8.

Besides the rest of the current Chapter 3, further applications of Proposition 3.7 are treated in Lemma 4.8 and Corollary 5.4.

Proposition 3.7 (Kato inequality). For every $X \in H^{1,2}(TM)$, we have $|X| \in W^{1,2}(M)$ and

$$|\nabla |X|| \leq |\nabla X| \quad \text{m.a.e.}$$

Recalling that $H^{1,2}(T^* M)^2 \subset H^{1,2}(TM)$ by virtue of (2.13), Proposition 3.7 yields in particular that for every $\omega \in H^{1,2}(T^* M)$, we have $|\omega| \in W^{1,2}(M)$ and

$$|d|\omega|| \leq |\nabla |\omega|| \quad \text{m.a.e.}$$
Corollary 3.8. For every \( X \in H^{1,2}(TM) \cap L^\infty(TM) \), we have \(|X|^2 \in W^{1,2}(M)\) with
\[
\nabla |X|^2 = 2 |X| \nabla |X|.
\]

Remark 3.9. One can drop the assumption that \( X \in L^\infty(T^*M) \) in Corollary 3.8, still retaining the stated identity for \( \nabla |X|^2 \). In this case, one has to understand \( \nabla |X|^2 \) as \( L^1 \)-covariant derivative of the \( L^1 \)-function \(|X|^2\) belonging to the Sobolev space \( H^{1,1}(TM) \) [40, Subsec. 3.3.3, Prop. 3.4.6].

3.3. Vector 1-Bochner inequality. A first important consequence of Proposition 3.7 is a 1-Bochner inequality for vector fields, see Theorem 3.12, in the dimension-free case. The results proven on the go will also yield the very important \( L^1 \)-type estimate between \((H_t)_{t \geq 0}\) and \((P_t)_{t \geq 0}\) in Theorem 3.15. Similarly to the level of functions [77, Lem. 2.6, Cor. 4.3], the key point is to verify that \(|X| \in \text{Dom}(\Delta)\) for a sufficiently large class of vector fields \( X \). Theorem 3.12 is then essentially a consequence of the chain rule for the measure-valued Laplacian \( \Delta \) from Lemma 2.13.

In this section, we state our results commonly for the case when \((M,d,m)\) is an RCD\((K,\infty)\) space.

A crucial estimate is established in the subsequent lemma.

Lemma 3.10. Let \( X \in H^{1,2}(T^*M)^i \cap L^\infty(TM) \) satisfy \( X^\flat \in \text{Dom}(\hat{\Delta}) \), and let \( \psi \in W^{1,2}(M) \cap L^\infty(M) \) be nonnegative. Then
\[
K \int_M \psi |X|^2 \, dm \leq -\frac{1}{2} \int_M \langle \nabla \psi, \nabla |X|^2 \rangle \, dm + \int_M \psi \langle X, (\hat{\Delta}X)^2 \rangle \, dm - \int_M \psi |\nabla X|^2 \, dm.
\]

Proof. Let \((\psi_n)_{n \in \mathbb{N}}\) be a sequence of nonnegative functions in \( \text{Test}(M) \) converging to \( \psi \) in \( W^{1,2}(M) \) according to Lemma 2.2. Moreover, let \((X_i)_{i \in \mathbb{N}}\) be a sequence in \( \text{Test}(TM) \) converging to \( X \) in \( H^{1,2}(T^*M)^i \). For every \( n, i \in \mathbb{N} \), by the definition of \( \Delta \) as well as Theorem 2.14,
\[
K \int_M \psi_n |X_i|^2 \, dm \leq -\frac{1}{2} \int_M \langle \nabla \psi_n, \nabla |X_i|^2 \rangle \, dm + \int_M \psi_n \langle X_i, (\hat{\Delta}X_i)^2 \rangle \, dm - \int_M \psi_n |\nabla X_i|^2 \, dm.
\]

For every \( n \in \mathbb{N} \), the convergence of the first, second, and fourth term towards the desired quantities as \( i \to \infty \) is clear by boundedness of \( \psi_n \) and \( \nabla \psi_n \), Corollary 3.8 and (2.13). Moreover, integration by parts, Lemma 2.10 and the facts that \( \psi_n \in L^\infty(M) \) and \( d \psi_n \in L^\infty(T^*M) \) also entail the appropriate convergence of the third term as \( i \to \infty \).

The claim follows by letting \( n \to \infty \) in the resulting inequality. \( \square \)

Proposition 3.11. Let \( X \in H^{1,2}(T^*M)^i \cap L^\infty(TM) \) satisfy \( X^\flat \in \text{Dom}(\hat{\Delta}) \), and let \( \phi \in \text{Dom}(\Delta) \cap L^\infty(M) \) be nonnegative with \( \Delta \phi \in L^\infty(M) \). Then for every \( \varepsilon > 0 \), we have
\[
\int_M \Delta \phi \left[ (|X|^2 + \varepsilon)^{1/2} - \varepsilon^{1/2} \right] \, dm + \int_M \phi \langle |X|^2 + \varepsilon \rangle^{-1/2} \langle X, (\hat{\Delta}X)^2 \rangle \, dm
\]
\[
- \int_M \phi \langle |X|^2 + \varepsilon \rangle^{-1/2} |\nabla X|^2 \, dm
\]
\[
+ \int_M \phi \langle |X|^2 + \varepsilon \rangle^{-1/2} |\nabla X|^2 \, dm
\]
\[
\geq K \int_M \phi \langle |X|^2 + \varepsilon \rangle^{-1/2} |X|^2 \, dm.
\]
Proof. Given any $\varepsilon > 0$, note that the function $\Phi_\varepsilon \in C^\infty ([0, \infty))$ defined by

$$\Phi_\varepsilon (r) := 2(r + \varepsilon)^{1/2} - 2\varepsilon^{1/2}$$

obeys the inequality

$$-2 \Phi_\varepsilon'' (r) r \leq \Phi_\varepsilon' (r). \quad (3.1)$$

The stated inequality then follows by applying Lemma 3.10 to $\psi := \Phi_\varepsilon' (|X|^2)$, which belongs to $W^{1,2} (M) \cap L^\infty (M)$ by Corollary 3.8 and the Leibniz rule, noting that by (3.1),

$$-\frac{1}{2} \int_M \langle \nabla \psi, \nabla |X|^2 \rangle = -\frac{1}{2} \int_M \Phi_\varepsilon' (|X|^2) \langle \nabla \phi, \nabla |X|^2 \rangle \, dm - \frac{1}{2} \int_M \phi \langle \nabla \Phi_\varepsilon' (|X|^2), \nabla |X|^2 \rangle \, dm$$

$$= -\frac{1}{2} \int_M \nabla \phi, \nabla \Phi_\varepsilon (|X|^2) \rangle \, dm - 2 \int_M \phi \Phi_\varepsilon' (|X|^2) |X|^2 |\nabla |X|^2 \rangle \, dm$$

$$\leq \frac{1}{2} \int_M \Phi_\varepsilon (|X|^2) \Delta \phi \, dm + \int_M \phi \Phi_\varepsilon' (|X|^2) |\nabla |X|^2 \rangle \, dm. \quad \square$$

**Theorem 3.12** (Vector 1-Bochner inequality). Let $(M, d, m)$ be an RCD($K, \infty$) space for $K \in \mathbf{R}$. Let $X \in \text{Test}(TM)$. Then $|X| \in \text{Dom} (\Delta)$ and, after redefining $|X|^{-1} \langle X, (\Delta X)^f \rangle := 0$ on $|X|^{-1} \{0\}$,

$$\Delta |X| + |X|^{-1} \langle X, (\Delta X)^f \rangle \geq K |X| \, m.$$

**Proof.** We already know that $|X| \in W^{1,2} (M)$ under the given assumptions. Furthermore, given any nonnegative $\phi \in \text{Dom} (\Delta) \cap L^\infty (M)$ with $\Delta \phi \in L^\infty (M)$, via Proposition 3.11, letting $\varepsilon \to 0$ with Lebesgue’s theorem, we arrive at

$$\int_M \Delta \phi |X| \, dm \geq \int_M \phi [-|X|^{-1} \langle X, (\Delta X)^f \rangle] + K |X| \, dm.$$ 

Hence, it follows from [77, Lem. 2.6] that $|X| \in \text{Dom} (\Delta)$ with

$$\Delta |X| + |X|^{-1} \langle X, (\Delta X)^f \rangle \geq K |X| \, m.$$

**Remark 13.** Theorem 3.12 is a priori true for $X \in H^{1,2}(TM)^f \cap L^1 (TM) \cap L^\infty (TM)$ with $X^f \in \text{Dom} (\Delta)$ and $\Delta X^f \in L^1 (TM)$. We restricted ourselves to the assumption that $X \in \text{Test}(TM)$ to simplify the presentation.

It is worth to note that a posteriori, by Theorem 3.15 and Theorem 4.1 below, the assumption of Theorem 3.12 is satisfied for every vector field $X := (H_\omega)^f$ with $t > 0$ and $\omega \in L^1 (T^*M) \cap L^\infty (T^*M)$. See also Subsection 4.2.1.

### 3.4. Hess–Schrader–Uhlenbrock’s inequality.

The “almost” vector 1-Bochner inequality from Proposition 3.11 is also an indispensable tool for Theorem 3.15. It is an instance of form domination in smooth situations [49, 80], which also implicitly uses some sort of “integrated 1-Bochner inequality”.

A slight restatement of Proposition 3.11 and Theorem 3.12 is Lemma 3.14 below. Theorem 3.15 then directly follows from the very general form domination result [49, Thm. 2.15]. See also [80, 67] for comprehensive proofs of the previous implication.

**Lemma 3.14.** For every $\omega \in \text{Dom} (\Delta)$ and every nonnegative $\phi \in W^{1,2} (M)$,

$$\int_M \langle \nabla \phi, \nabla |\omega| \rangle \, dm + K \int_M \phi |\omega| \, dm \leq \int_M \phi |\omega|^{-1} \langle \omega, \Delta \omega \rangle \, dm. \quad (3.2)$$
Proposition 3.11. If $\omega$ also belongs to $L^\infty(T^*M)$, the claimed inequality originates from Proposition 3.11 by letting $\varepsilon \to 0$, dropping the contributions by $|\nabla \omega|^2$ and $|\nabla |\omega||^2$ via Proposition 3.7, and integrating by parts — the resulting inequality easily extends to the asserted class of $\phi$ by Lemma 2.2.

Observe that given any $\omega \in \text{Dom}(\Delta)$, there exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ in $\text{Dom}(\Delta) \cap L^\infty(T^*M)$ converging to $\omega$ in $H^{1,2}(T^*M)$ such that $\Delta \omega_n \to \Delta \omega$ in $L^2(T^*M)$ as $n \to \infty$ — this then already provides (3.2) by approximation, possibly after taking pointwise $m$-a.e. converging subsequences. Indeed, given any $z \in M$ and $R > 0$, set $\omega_R := 1_{B_R(z)} 1_{[0,R]}(|\omega|) \omega$. By Theorem 3.6, for every $t > 0$ we have $H_t \omega_R \to H_t \omega$ in $H^{1,2}(T^*M)$ and $\Delta H_t \omega_R \to \Delta H_t \omega$ in $L^2(T^*M)$ as $R \to \infty$. Since also $H_t \omega \to \omega$ in $H^{1,2}(T^*M)$ as well as $\Delta H_t \omega = H_t \Delta \omega \to \Delta \omega$ as $t \to 0$ thanks to Theorem 3.6 as well, the claimed existence of an approximation sequence follows by a diagonal argument.

**Theorem 3.15** (Hess–Schrader–Uhlenbrock inequality). Suppose that $(M,d,m)$ is an RCD$(K,\infty)$ space for some $K \in \mathbb{R}$. Then for every $\omega \in L^2(T^*M)$ and every $t \geq 0$, we have

$$|H_t \omega| \leq e^{-Kt} P_t |\omega| \quad m\text{-a.e.}$$

**Remark 3.16.** For Riemannian manifolds, Theorem 3.15 is due to [50, Ch. 3].

A similar approach to Theorem 3.15, by encoding uniform lower Ricci bounds via Bakry–Émery calculus, comes from the study of form domination for Hilbert space valued functions [78]. The analogy is created by the structural characterization of $L^2(T^*M)$ from [40, Thm. 1.4.11].

The setting of [78] is, however, more restrictive than ours. For instance, the assumptions $(\Gamma)$ and $(\Gamma_\lambda)$ made in [78, Ch. 3], see (3.1) and (3.7) therein, transferred to our notation require that $\langle \omega, \eta \rangle \in \text{Dom}(\Delta)$ for a sufficiently large class of $\omega, \eta \in \text{Dom}(\Delta)$, an assumption we can make sense of only in a weak form using the measure-valued Laplacian $\Delta$. (In [78], the previous regularity assumption has been used to define a contravariant $\Gamma$-operator.)

3.5. **Bakry–Ledoux’s inequality.** We turn to a version of (vii) in Theorem 3.6 under synthetic upper dimension bounds, namely the Bakry–Ledoux inequality. Our strategy for Theorem 3.17 closely follows its functional counterparts for [36, Thm. 3.6, Prop. 3.7] and in fact both uses and entails their equivalence with the RCD$(K,N)$ condition on RCD$(K,\infty)$ spaces (or, with the same arguments, on RCD$(K',\infty)$ spaces with $K' < K$).

**Theorem 3.17** (Bakry–Ledoux inequality). Let $(M,d,m)$ be an RCD$(K,\infty)$ space for some $K \in \mathbb{R}$, and let $N \in (1,\infty)$. Then $(M,d,m)$ is an RCD$(K,N)$ space if and only if any of the subsequent inequalities holds for every $\omega \in L^2(T^*M)$.

(i) **Strong Bakry–Ledoux inequality.** For every $t > 0$,

$$|H_t \omega|^2 + \frac{2}{N} \int_0^t e^{-2Ks} P_s (|\delta H_t - s \omega|^2) \, ds \leq e^{-2Kt} P_t (|\omega|^2) \quad m\text{-a.e.}$$

(ii) **Integral weak Bakry–Ledoux inequality.** For every $t > 0$,

$$|H_t \omega|^2 + \frac{2}{N} \int_0^t e^{-2Ks} |P_s \delta H_t - s \omega|^2 \, ds \leq e^{-2Kt} P_t (|\omega|^2) \quad m\text{-a.e.}$$

(iii) **Non-integral weak Bakry–Ledoux inequality.** For every $t > 0$, with the interpretation of the prefactor in the second term as $2t/N$ in the case $K = 0$,

$$|H_t \omega|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\delta H_t \omega|^2 \leq e^{-2Kt} P_t (|\omega|^2) \quad m\text{-a.e.}$$
Proof. It suffices to prove that the RCD\(^+(K, N)\) condition implies (i). Indeed, (i) easily implies (ii) by Jensen’s inequality. If (ii) holds, then also (iii) is satisfied, since then, by Jensen’s inequality, Corollary 3.5 and with the indicated interpretation in the case \(K = 0\),

\[
\frac{4Kt^2}{N(e^{2Kr} - 1)} |\delta H_t \omega|^2 = \frac{2t}{N} |\delta H_t \omega|^2 \left[ \frac{1}{t} \int_0^t e^{2Ks} \, ds \right]^{-1} \leq \frac{2}{N} \int_0^t e^{-2Ks} |\delta H_t \omega|^2 \, ds = \frac{2}{N} \int_0^t e^{-2Ks} |P_s \delta H_{t-s} \omega|^2 \, ds
\]

Moreover, under the given topological assumptions, it is known by [35, Thm. 4.19] that (iii), restricted to exact differential 1-forms, implies the RCD\(^+(K, N)\) condition on the metric measure space \((M, d, m)\).

For \(R > 0\), let \(\omega_R \in L^2(T^*M) \cap L^\infty(T^*M)\) and \(F: [0, t] \to \mathbb{R}\) be defined by

\[
\omega_R := 1_{[0, R]}(\omega) \omega,
\]

\[
F(s) := e^{-2Ks} \int_M P_s \phi |H_{t-s} \omega_R|^2 \, dm,
\]

where \(\phi \in \text{Test}(M)\) is nonnegative. By the respective continuity of \((P_t)_{t \geq 0}\) and \((H_t)_{t \geq 0}\) in \(L^2(M)\) and \(L^2(T^*M)\) and the boundedness of \(\phi\) and \(\omega_R\), \(F\) is continuous. Its restriction to \([0, t]\) has \(C^1\)-regularity, which is also inherited from the corresponding properties of \((P_t)_{t \geq 0}\) and \((H_t)_{t \geq 0}\) as well as item (vii) in Theorem 3.6.

Therefore, employing [45, Thm. 4.3], the claim follows by integration and the arbitrariness of \(\phi\) after observing that

\[
F'(s) = -2Ke^{-2Ks} \int_M P_s \phi |H_{t-s} \omega_R|^2 \, dm + e^{-2Ks} \int_M \Delta P_s \phi |H_{t-s} \omega_R|^2 \, dm
\]

\[
+ 2e^{-2Ks} \int_M P_s \phi (H_{t-s} \omega_R, \Delta H_{t-s} \omega_R) \, dm
\]

\[
\geq \frac{2}{N} e^{-2Ks} \int_M P_s \phi |\delta H_{t-s} \omega_R|^2 \, dm. \quad \square
\]

Remark 3.18 (Generalization to variable Ricci bounds). To simplify the presentation and since most later results do not require refinements of the previous facts, we reduced ourselves to the case of constant lower bounds \(K\) for the Ricci curvature in Chapter 3.

However, the arguments in Chapter 3 perfectly work in a similar manner if the RCD\((K, \infty)\) space \((M, d, m)\) obeys the stronger 2-Bakry–Émery inequalities BE\(_2(k, \infty)\) or BE\(_2(k, N)\) with variable curvature bound \(k\) in the sense of [12, Def. 1.4] or [89, Def. 3.3], respectively. Here, \(k: M \to \mathbb{R}\) is a locally \(m\)-integrable function with \(k \geq K\) on \(M\). (The assumption on lower semicontinuity on \(k\) in [12, 89] is not needed in the purely Eulerian perspective of [40].) In particular, Theorem 3.12 remains true by just replacing \(K\) by \(k\). In addition, Theorem 3.15 holds when replacing \(e^{-qKt} P_t, q \in \{1, 2\}\), by the operator \(P_t^{qk}\), where \((P_t^{qk})_{t \geq 0}\) is the \textit{Schrödinger semigroup} on \(L^2(M)\) with generator \(\Delta - qk\) [81]. One way to read Theorem 3.15 in terms of Brownian motion \(((P_t^\omega)_{\omega \in M}, (B_t)_{t \geq 0})\) on \((M, d, m)\) — defined w.r.t. the generator \(\Delta/2\) [3] — is thus

\[
|H_{t \omega}| \leq E \left[ e^{-\int_0^{B_t} k(B_s)/2 \, dr} |\omega| \left( |B_{2t}| \right) \right] \quad \text{m-a.e.}
\]

4. Integral estimates for the heat flow

4.1. Basic \(L^p\)-properties and \(L^p\)-\(L^\infty\)-regularization. From Theorem 3.15 and a standard procedure, the following is immediate by approximation. It is worth to
emphasize that the restriction of $H_t$ to $L^\infty(T^*M)$ is defined as the Banach space adjoint of the restriction of $H_t$ to $L^1(T^*M)$ for every $t \geq 0$. See also Section 5.3.

**Theorem 4.1.** Let $(M, d, m)$ be an RCD$(K, \infty)$ space for some $K \in \mathbb{R}$. For every $p \in [1, \infty]$, $(H_t)_{t \geq 0}$ then extends to a semigroup of bounded linear operators from $L^p(T^*M)$ into $L^p(T^*M)$, strongly continuous if $p < \infty$, which satisfies

$$
\|H_t \omega\|_{L^p} \leq e^{-Kt} \|\omega\|_{L^p}
$$

for every $\omega \in L^p(T^*M)$ and every $t \geq 0$.

**Remark 4.2.** This result implies the $L^p$-contractivity of $(H_t)_{t \geq 0}$ for every $p \in [1, \infty]$ under nonnegative lower Ricci bounds. However, we should not expect contractivity in larger generality, not even on Riemannian manifolds [82, 83].

On compact RCD$^*(K, N)$ spaces, $N \in (1, \infty)$, the heat operator $H_t$ is not only bounded from $L^p(T^*M)$ to $L^p(T^*M)$, but also from $L^p(T^*M)$ to $L^\infty(T^*M)$ for every $t > 0$ and every $p \in [1, \infty]$. This is the content of the following result which will be crucial in Section 5.2 and Section 6.3.

**Theorem 4.3** ($L^p$-$L^\infty$-regularization). Let $(M, d, m)$ be a compact RCD$^*(K, N)$ space, $K \in \mathbb{R}$ and $N \in (1, \infty)$. Furthermore, let $t > 0$ and $p \in [1, \infty]$. Then $H_t$ is bounded from $L^p(T^*M)$ to $L^\infty(T^*M)$.

**Proof.** By Hölder’s inequality, it suffices to prove boundedness of $H_t$ from $L^1(T^*M)$ to $L^\infty(T^*M)$.

Let $\omega \in L^1(T^*M) \cap L^2(T^*M)$ with $\|\omega\|_{L^1} \leq 1$ be arbitrary — the consideration of such 1-forms is enough by the density of $L^1(T^*M) \cap L^2(T^*M)$ in $L^1(T^*M)$. By (2.5), there exists a constant $C > 0$ such that $m[B, \tau(\cdot)]^{-1} \leq C m[B, \tau(z)]^{-1}$ on $M$. The conclusion follows by observing that by Theorem 3.15 and (2.7), there exist constants $C_3, C_4 > 1$ depending only on $K$ and $N$ such that

$$
|H_t \omega| \leq e^{-Kt} |\omega| = e^{-Kt} \int_M p_t(\cdot, y)|\omega|(y) \, dm(y)
$$

$$
\leq C_3 e^{-K-C_4 t} m[B, \tau(\cdot)]^{-1}
$$

$$
\leq C C_3 e^{-K-C_4 t} m[B, \tau(z)]^{-1}, \quad \text{m-a.e.} \quad \square
$$

4.2. Logarithmic Sobolev inequalities. We come to an important class of functional inequalities, namely logarithmic Sobolev inequalities for 1-forms and their relation to integral-type inequalities for $(H_t)_{t \geq 0}$. More precisely, following [18, 26] we show that the former imply, for certain $t > 0$ and every $p_0 \in (1, \infty)$, the boundedness of $H_t$ from $L^{p_0}(T^*M)$ into $L^p(0)(T^*M)$, where $p$ is a real-valued function with $p(0) = p_0$. This property is called hyperrcontractivity. Under more restrictive assumptions, for some finite $T > 0$ it is even possible to prove the boundedness of $H_T$ from $L^{p_0}(T^*M)$ to $L^\infty(T^*M)$, a property termed ultrahypercontractivity. See Theorem 4.12.

A certain reverse implication also holds, see Theorem 4.16.

**Definition 4.4.** Let $\beta > 0$ and $\chi \in \mathbb{R}$. We say that $X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM)$ satisfies the 2-logarithmic Sobolev inequality with constants $\beta$ and $\chi$, briefly $\text{LS}_2(\beta, \chi)$, if

$$
\int_M |X|^2 \log |X| \, dm \leq \beta \|\nabla X\|_{L^2}^2 + \chi \|X\|_{L^2}^2 + \|X\|_{L^2}^2 \log \|X\|_{L^2}.
$$
Theorem 3.15, where we identify Proposition 4.10 Theorem 3.6 18 Section 5.3 16, 

Remark 4.7. We do not discuss the case of 1-logarithmic Sobolev inequalities since it is not clear, even having an appropriate version of such inequality at our disposal, that for \( \omega \in L^1(T^*M) \cap L^\infty(T^*M) \), the function \( |\omega| \log |\omega| \) is integrable.

On the other hand, the integrability of \( |\omega|^p \log |\omega| \) for \( p \in (1, \infty) \) is clear by local boundedness of the function \( r \mapsto r^\beta \log r \) on \([0, \infty)\) for every \( \delta > 0 \).

Later, special interest will be devoted to the class

\[ V_{1,\infty} := \bigcup_{t>0} H_t\left( L^1(T^*M) \cap L^\infty(T^*M) \right) \]

By Theorem 3.6 and Theorem 3.15, \( V_{1,\infty} \) is contained in \( \text{Dom}(\Delta) \) as well as in \( L^p(T^*M) \) for every \( p \in [1, \infty) \), is invariant under the action of \( H_t \) for every \( t > 0 \), and it is strongly dense in the latter space if \( p < \infty \). Additionally, since the infinitesimal generator of the restriction of \( (H_t)_{t \geq 0} \) onto \( L^p(T^*M) \) applied to any \( \omega \in V_{1,\infty} \) coincides with \( \Delta \), we have \( \Delta \omega \in L^p(T^*M) \) for every \( p \in [1, \infty] \). (See Section 4.1 and Section 5.3 for the correct interpretation in the case \( p = \infty \).)

4.2.1. Relations between different logarithmic Sobolev inequalities. In view of Proposition 4.10, we first focus on the 2-logarithmic Sobolev inequality, in particular showing how to derive it from its functional counterpart in the next Lemma 4.8.

Given any \( \beta > 0 \), following (1.2) of [16] (replacing \( \alpha \) by \( 1/\beta \) therein), a nonnegative \( f \in \text{Lip}(M) \cap L^1(M) \) is said to obey the functional 2-logarithmic Sobolev inequality with constant \( \beta \) if, with the convention \( \nabla f^2/f := 0 \) on \( f^{-1}(\{0\}) \),

\[
2 \int_M f \log f \, dm - 2 \int_M f \, dm \log \int_M f \, dm \leq \beta \int_M \frac{|
abla f|^2}{f} \, dm. \tag{4.1}
\]

Lemma 4.8. Let \( \beta > 0 \) be given. Suppose that every nonnegative \( f \in \text{Lip}(M) \cap L^1(M) \) obeys the functional 2-logarithmic Sobolev inequality with constant \( \beta \). Then every \( X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM) \) satisfies \( \text{LSI}_2(\beta, 0) \).

Proof. Let \( R > 1 \) and \( z \in M \), and let \( \psi_R \in \text{Lip}_{ba}(M) \) be a cutoff function with \( \psi_R(M) = [0, 1] \), identically equal to 1 on \( B_R(z) \) and identically equal to 0 on \( M \setminus B_{R+1}(z) \).

Given \( X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM) \), observe that \( P_{1/n}[X] \in \text{Test}(M) \cap \text{Lip}(M) \cap L^1(M) \) for every \( n \in \mathbb{N} \) by Proposition 3.7, where we identify \( P_{1/n}[X] \) with its Lipschitz m.a.e. representative by the Sobolev-to-Lipschitz property of \( (M, d, m) \). We may and will assume that the sequence \( (g_n)_{n \in \mathbb{N}} \) in \( \text{Lip}_{ba}(M) \cap L^1(M) \), where \( g_n := \psi_R P_{1/n}[X] \), converges to \( \psi_R [X] \) pointwise m.a.e. and strongly in \( W^{1,2}(M) \).

Setting \( f_n := g_n^2 \) entails

\[
|\nabla f_n|^2 = 4 f_n |\nabla g_n|^2 \quad \text{m.a.e.}
\]
Lebesgue’s theorem as well as (4.1) applied to $f_n$ for every $n \in \mathbb{N}$ yield
\[
2 \int_M \psi_R^2 |X|^2 \log(\psi_R^2 |X|^2) \, dm - 2 \int_M \psi_R^2 |X|^2 \, dm \log \int_M \psi_R^2 |X|^2 \, dm \\
= \lim_{n \to \infty} \left[ \frac{1}{2} \int_M f_n \log f_n \, dm - 2 \int_M f_n \, dm \log \int_M f_n \, dm \right] \\
\leq \lim_{n \to \infty} \beta \int_M \frac{\nabla f_n^2}{f_n} \, dm \\
\leq \lim_{n \to \infty} 4\beta \int_M |\nabla g_n|^2 \, dm = 4\beta \int_M |\nabla (\psi_R |X|)|^2 \, dm.
\]

The claim follows by letting $R \to \infty$, using Lebesgue’s theorem and Proposition 3.7.

Example 4.9. By [93, Thm. 30.21] if $(M, d, m)$ is an RCD($K, \infty$) space with $K > 0$, or [16, Thm. 1.9] in the case when $(M, d, m)$ is a compact RCD$^*(K, N)$ space, $K \in \mathbb{R}$ and $N \in (1, \infty)$, the hypothesis of Lemma 4.8 is known to be satisfied for some finite $\beta > 0$.

The constant $\beta$ can explicitly be chosen to be $1/K$ and $(N - 1)/KN$ if $K > 0$, respectively.

Proposition 4.10. Let $\beta > 0$ and $\chi \in \mathbb{R}$. Define the functions $\varepsilon, \gamma \in C(1, \infty)$ by
\[
\varepsilon(p) := \frac{\beta p}{2(p - 1)}, \\
\gamma(p) := \frac{2\chi}{p} - \frac{K \beta p}{2(p - 1)}.
\]
Assume that every $X \in H^{1,2}(T^*M)^2 \cap L^1(TM) \cap L^{\infty}(TM)$ obeys $\text{LSI}_p(\beta, \chi)$ according to Definition 4.4. Then every $\omega \in V_{1,\infty}$ obeys $\text{fLSI}_p(\varepsilon(p), \gamma(p))$ for every $p \in (1, \infty)$.

Proof. The claim for $p = 2$ follows by (2.13). Thus we concentrate on the case $p \in (1, 2) \cup (2, \infty)$.

Given any $\tau > 0$, the function $\Phi_\tau \in C^\infty([0, \infty))$ given by
\[
\Phi_\tau(r) := (r + \tau)^{p/2 - 1}
\]
obey the inequalities
\[
0 \leq \frac{p^2}{p - 2} \Phi_\tau'(r) r \leq \Phi_\tau(r) + \Phi_\tau'(r) r. \tag{4.2}
\]

By Lemma 2.10 and Proposition 3.7, we have $\Phi_\tau(|\omega|) \in S^2(M) \cap L^\infty(M)$ as well as $\Phi_\tau(|\omega|) \omega^2 \in H^{1,2}(T^*M)^2 \cap L^1(TM) \cap L^{\infty}(TM)$ for every $\tau > 0$. By LSI$^2(\beta, \chi)$ applied to $\Phi_\tau(|\omega|) \omega^2$ and letting $\tau \downarrow 0$, employing Lebesgue’s theorem and (2.13), we infer that
\[
\int_M |\omega|^p \log |\omega| \, dm - \frac{2\chi}{p} \|\omega\|_{L^p}^p - \|\omega\|_{L^p}^p \log \|\omega\|_{L^p} \\
\leq \liminf_{\tau \downarrow 0} \left[ \frac{2}{p} \int_M |\Phi_\tau(|\omega|) \omega|^2 \log |\Phi_\tau(|\omega|) \omega| \, dm - \frac{2\chi}{p} \|\Phi_\tau(|\omega|) \omega\|^2_{L^2} \\
- \frac{2}{p} \|\Phi_\tau(|\omega|) \omega\|^2_{L^2} \log \|\Phi_\tau(|\omega|) \omega\|_{L^2} \right] \\
\leq \liminf_{\tau \downarrow 0} \frac{2\beta}{p} \int_M |\nabla (\Phi_\tau(|\omega|) \omega^2)|^2 \, dm. \tag{4.3}
\]
It remains to estimate the limit in (4.3). We start by recalling that, by definition,
\[ |d|ω| ∧ ω| = |d|ω|^2|ω|^2 − ⟨d|ω|, ω⟩. \]
Therefore, for every \( τ > 0 \), we observe by (2.13) and Lemma 2.10, taking into account that \( Φ^2(|ω|) \) does also belong to \( H^{1,2}(T^*M) \), and finally integration by parts that
\[
\int_M |\nabla (Φ_τ(|ω|) ω^2)|^2 \, dm + K ∥Φ_τ(|ω|) ω∥^2_{L^2} \tag{4.4}
\]
\[
\leq \int_M [|d(Φ_τ(|ω|) ω)|^2 + |δ(Φ_τ(|ω|) ω)|^2] \, dm
\]
\[
= \int_M [Φ^2_τ(|ω|) |dω|^2 + 2 Φ_τ(|ω|) Φ'_τ(|ω|) ⟨d|ω| ∧ ω, dω⟩] \, dm
\]
\[
+ \int_M [(Φ'_τ)^2(|ω|) |d|ω| ∧ ω|^2 + Φ^2_τ(|ω|) |δω|^2] \, dm
\]
\[
- \int_M [2 Φ_τ(|ω|) Φ'_τ(|ω|) δω ⟨d|ω|, ω⟩]
\]
\[
= \int_M [(d(Φ^2_τ(|ω|) ω), dω) + δ(Φ^2_τ(|ω|) ω) δω] \, dm
\]
\[
+ \int_M (Φ'_τ)^2(|ω|) |ω|^2 |d|ω|^2 \, dm
\]
\[
= \int_M Φ^2_τ(|ω|) ⟨ω, Δω⟩ \, dm + \int_M (Φ'_τ)^2(|ω|) |ω|^2 |d|ω|^2 \, dm. \tag{4.5}
\]
Thanks to (4.2), the Leibniz rule, the chain rule and Proposition 3.7, we have
\[
(Φ'_τ)^2(|ω|) |ω|^2 |d|ω|^2 \leq \frac{(p - 2)^2}{p^2} |\nabla (Φ_τ(|ω|) ω^2)|^2 \text{ m.a.e.}
\]
Rearranging the estimate resulting from this bound with the inequality between (4.4) and (4.5) and then sending \( τ \downarrow 0 \) yields
\[
\liminf_{τ \downarrow 0} \frac{2β}{p} \int_M |\nabla (Φ_τ(|ω|) ω^2)|^2 \, dm
\]
\[
\leq ε(p) \int_M |ω|^{p-2}(ω, Δω) \, dm + \frac{Kβp}{2(p - 1)} ∥ω∥^p_{L^p}.
\]
From (4.3), this readily provides the claim. \( □ \)

Remark 4.11. Let \( β > 0 \) and \( χ \in \mathbb{R} \), and define \( ε, γ \in C((1, ∞)) \) by
\[
ε(p) := \frac{βp}{2(p - 1)},
\]
\[
γ(p) := \frac{2χ}{p} - \frac{Kβ(p - 2)^2}{2p(p - 1)}.
\]

With a slight modification of the proof of Proposition 4.10, it is possible to show that if every element in \( \text{Dom}(Δ) \cap L^1(T^*M) \cap L^∞(T^*M) \) obeys fLSI\(_{(β, χ)}\), then every \( ω \in V_{1,∞} \) satisfies fLSI\(_p(ε(p), γ(p))\) for every \( p \in (2, ∞) \). Up to changing the involved constants, this assumption is weaker compared to the one of Proposition 4.10.
4.2.2. From logarithmic Sobolev inequalities to hyper- and ultracontractivity.

**Theorem 4.12.** Let $p_0 \in (1, \infty)$. Let $\varepsilon \in C([p_0, \infty))$ be a positive function, and $\gamma \in C([p_0, \infty))$. Suppose that the integrals

\[
T := \int_{p_0}^{\infty} \frac{\varepsilon(r)}{r} \, dr, \\
C := \int_{p_0}^{\infty} \frac{\gamma(r)}{r} \, dr
\]

exist with values in $(0, \infty]$ and $(-\infty, \infty]$, respectively. Define $p \in C^1([0, T])$ and $A \in C^1([0, \infty))$ through the relations

\[
\int_{p_0}^{p(t)} \frac{\varepsilon(r)}{r} \, dr := t, \\
A(t) := \int_{0}^{t} \frac{\gamma(p(r))}{\varepsilon(p(r))} \, dr.
\]

Assume $\text{fLSI}_{p}(\varepsilon(p), \gamma(p))$ for every $\omega \in V_{1, \infty}$ and every $p \in [p_0, \infty)$. Then the following hold.

(i) **Hypercontractivity.** For every $t \in [0, T)$, we have

\[
\|H_t\|_{L^{p_0}, L^p(t)} \leq e^{A(t)}.
\]

(ii) **Ultracontractivity.** If $T < \infty$ and $C < \infty$, we have

\[
\|H_t\|_{L^{p_0}, L^\infty} \leq e^C.
\]

**Proof.** First observe that $p(0) = p_0$, that $A(0) = 0$, and that $p$ is strictly increasing with $p(t) \to \infty$ as $t \to T$. Moreover, $A(t) \to C$ as $t \to T$ thanks to the relations

\[
p' = \frac{p}{\varepsilon(p)}, \\
A' = \frac{\gamma(p)}{\varepsilon(p)} = \frac{\gamma(p) p'}{p}.
\]

(4.6)

Owing to (i), given any $\omega \in V_{1, \infty} \setminus \{0\}$, we assume that $H_t \omega \neq 0$ for every $t \in [0, T)$, which is always true at least for small times. Otherwise, the following computations are performed until the heat flow dies out. We consider the positive function $F \in C^1([0, T))$ given by

\[
F(t) := e^{-A(t)} \|H_t \omega\|_{L^p(t)}.
\]

Note that the function $t \mapsto |H_t \omega|^2$ is continuously differentiable on $[0, \infty)$ in $L^2(T^*M)$ with derivative $-2 (H_t \omega, \Delta H_t \omega) \in L^2(T^*M)$ for every $t \geq 0$ thanks to Theorem 3.6. In particular,

\[
\frac{d}{dt} [H_t \omega]^{p(t)} = [H_t \omega]^{p(t)} [p'(t) \log |H_t \omega| - p(t)] [H_t \omega]^{-2} (H_t \omega, \Delta H_t \omega) \quad \text{m-a.e.},
\]

and the assertion on the regularity of $F$ indeed follows by the integrability assumptions on $\omega$, $C^1$-regularity of $p$, Theorem 3.15 and arguing as in Remark 4.7.
Moreover, for every $t \in [0, T)$, from \textit{fLSI}(p(t))(\varepsilon(p(t)), \gamma(p(t)))$ for $H_1 \omega \in V_{1, \infty}$ as well as (4.6),
\[
\frac{d}{dt} \log F(t) = -A'(t) + \|H_1 \omega\|_{L^p(t)}^{-1} \frac{d}{dt} \|H_1 \omega\|_{L^p(t)}
\]
\[
= -A'(t) - \frac{p'(t)}{p(t)} \log \|H_1 \omega\|_{L^p(t)} + \frac{1}{p(t)} \|H_1 \omega\|_{L^p(t)} \frac{d}{dt} \|H_1 \omega\|_{L^p(t)}
\]
\[
= -\gamma(p(t)) - \frac{1}{\varepsilon(p(t))} \log \|H_1 \omega\|_{L^p(t)} + \frac{1}{\varepsilon(p(t))} \|H_1 \omega\|_{L^p(t)} \int_M |H_1 \omega|^{p(t)} \log |H_1 \omega| \, dm
\]
\[
- \|H_1 \omega\|_{L^p(t)}^{-p(t)} \int_M |H_1 \omega|^{p(t)-2} (H_1 \omega, \tilde{\Delta} H_1 \omega) \, dm \leq 0.
\]
Since log is strictly increasing, $F$ is nonincreasing, yielding (i) by the density of $V_{1, \infty}$ in $L^{p_0}(T^* M)$.

Concerning (ii), invoking the strict increasingness of $p$ and Hölder’s inequality, for every $s, t \in [0, T)$ with $s < t$ and every bounded Borel set $B \subset M$ with positive $\mathfrak{m}$-measure we have
\[
\|1_B H_1 \omega\|_{L^p(t)} \leq m[B]^{-p(s)/p(t)} \|H_1 \omega\|_{L^p(t)}
\]
\[
\leq m[B]^{-p(s)/p(t)} e^{A(t) - A(s)} \|H_1 \omega\|_{L^p(t)}
\]
\[
\leq m[B]^{-p(s)/p(t)} e^{A(t)} \|\omega\|_{L^{p_0}}.
\]
The claim follows by letting $t \to T$ and $s \to T$ in such a way that $p(s)/p(t) \to 1$ and afterwards using the arbitrariness of $B$ as well as the density of $V_{1, \infty}$ in $L^{p_0}(T^* M)$.

\textbf{Example 4.13.} Given any $\beta > 0$, the functions $\varepsilon, \gamma \in C((1, \infty))$ with
\[
\varepsilon(p) := \frac{\beta p}{2(p-1)},
\]
\[
\gamma(p) := -\frac{K \beta p}{2(p-1)},
\]
are the coefficients in Proposition 4.10 arising from the setup of Lemma 4.8 and Example 4.9.

Retaining the notation from Theorem 4.12, subject to these coefficients and any $p_0 \in (1, \infty)$, the value $T$ is always infinite, while $C$ takes the values $-\infty$, 0 or $\infty$ depending on whether $K > 0$, $K = 0$ or $K < 0$. Moreover, the functions $p$ and $A$ from Theorem 4.12 read
\[
p(t) = 1 + (p_0 - 1) e^{2t/\beta},
\]
\[
A(t) = \int_{p_0}^{p(t)} \frac{\gamma(s)}{s} \, ds = -K t.
\]

\textbf{Corollary 4.14.} In the setting of Example 4.13, given any $p_0 \in (1, \infty)$, for every $t \geq 0$, we have
\[
\|H_1\|_{L^{p_0}, L^p(t)} \leq e^{-K t}.
\]

\textbf{Corollary 4.15.} In the setting of Example 4.13, let $\omega \in \text{Dom}(\tilde{\Delta})$ be an eigenform for $\tilde{\Delta}$ with eigenvalue $\lambda \geq 0$, i.e. $\tilde{\Delta} \omega = \lambda \omega$. Then $\omega \in L^q(T^* M)$ for every $q \in (2, \infty)$ with the inequality
\[
\|\omega\|_{L^q} \leq (q - 1)^{(\lambda - K)/2} \|\omega\|_{L^2}.
\]
Proof. Note that $H_t \omega = e^{-\lambda t} \omega$ for every $t \geq 0$. We apply Corollary 4.14 to $\rho_0 := 2$. Given any $q \in (1, \infty)$, since $p(t) = q$ if and only if $t = \log(q - 1)\beta/2$, for this value of $t$ we have
\[
\|\omega\|_{L^q} = e^{\lambda t} \|H_t \omega\|_{L^q(t)} \leq e^{(\lambda-K)t} \|\omega\|_{L^q(0)} = (q-1)^{(\lambda-K)/2} \|\omega\|_{L^2}. \qed
\]

4.2.3. From ultracontractivity to logarithmic Sobolev inequalities.

**Theorem 4.16.** Let $T \in (0, \infty)$ as well as $c \in C((0, T))$. Suppose that
\[
\|H_t\|_{L^2,L^\infty} \leq e^{c(t)}
\]
holds for every $t \in (0, T)$. Then every $\omega \in \text{Dom}(\tilde{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$ satisfies $\text{fLSI}_2(\varepsilon, c(\varepsilon))$ for every $\varepsilon \in (0, T)$.

**Proof.** Let $U := \{\xi \in C : \Re \xi \in [0, 1]\}$. Given any $\varepsilon \in (0, T)$ and $\xi \in U$, we consider the operator
\[
S_\xi := e^{-\varepsilon \xi \tilde{\Delta}}
\]
acting on $L^2(T^*M) + i L^2(T^*M)$. For every $\eta, \rho \in L^2(T^*M) + i L^2(T^*M)$, the canonical bilinear form in $L^2(T^*M) + i L^2(T^*M)$ induced by $S_\xi$ evaluated at $(\eta, \rho)$ is continuous in $U$, and its restriction to the interior of $U$ is holomorphic. For every $\eta \in L^2(T^*M) + i L^2(T^*M)$ and every $\varepsilon \in \mathbb{R}$, we have
\[
\|S_\varepsilon \eta\|_{L^2} \leq \|\eta\|_{L^2},
\]
\[
\|S_{\varepsilon+i\varepsilon} \eta\|_{L^2} \leq e^{-c(\varepsilon)} \|S_\varepsilon \eta\|_{L^2} \leq e^{-c(\varepsilon)} \|\eta\|_{L^2}.
\]

Given any $\omega \in \text{Dom}(\tilde{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$, for every $\tau \in (0, 1)$, via Stein’s interpolation theorem we infer
\[
\|H_{\varepsilon \tau} \omega\|_{L^2/(1-\tau)} = \|S_\tau \omega\|_{L^2/(1-\tau)} \leq e^{-c(\varepsilon)\tau} \|\omega\|_{L^2}. \quad (4.7)
\]

Now we define $p \in C^1([0, \varepsilon])$ by $p(t) := 2\varepsilon/(\varepsilon - t)$. Setting $\tau := t/\varepsilon$ in (4.7) translates into
\[
\|H_{\tau} \omega\|_{L^p(\tau)} \leq e^{-c(\varepsilon)\tau} \|\omega\|_{L^2},
\]
and the claim follows after differentiating both sides at 0 via
\[
\int_M \omega^2 \left[\frac{2}{\varepsilon} \log |\omega| - 2 |\omega|^{-2} \langle \omega, \tilde{\Delta} \omega \rangle \right] \, dm \leq \frac{2c(\varepsilon)}{\varepsilon} \|\omega\|_{L^2}. \quad \Box
\]

**Example 4.17.** If $P_t$ is bounded from $L^2(M)$ to $L^\infty(M)$, then so is $H_t$ by means of Theorem 3.15. Compare this with (the proof of) Theorem 4.3.

See [18, Ch. 4] for an application to certain Gaussian upper bounds for the heat kernel on 1-forms in the non-weighted smooth setting.

5. Spectral properties of the Hodge Laplacian

Next, we study properties of the spectrum $\sigma(\tilde{\Delta})$ of $\tilde{\Delta}$, i.e. the set of all $\lambda \in \mathbb{C}$ such that the operator $\tilde{\Delta} - \lambda$ fails to be bijective. We denote the *essential spectrum* of $\tilde{\Delta}$ by $\sigma_e(\tilde{\Delta})$. The *point spectrum* of $\tilde{\Delta}$ is denoted by $\sigma_p(\tilde{\Delta})$, and the *essential spectrum* of $\tilde{\Delta}$ will be termed $\sigma_e(\tilde{\Delta})$. The former is the set of all $\lambda \in \sigma(\tilde{\Delta})$ for which $\tilde{\Delta} - \lambda$ is not injective, and the second is the set of all $\lambda \in \sigma(\tilde{\Delta})$ which are not an eigenvalue of $\tilde{\Delta}$ with finite algebraic multiplicity. Similar notations are employed for the operators $-\Delta$ and $-\Delta + K$.

One immediately sees that since $\tilde{\Delta}$ is self-adjoint and nonnegative, we have
\[
\sigma(\tilde{\Delta}) \subset [0, \infty).
\]
In addition, eigenspaces w.r.t. different eigenvalues are mutually orthogonal in $L^2(T^*M)$. 

5.1. **Inclusion of spectra.** In this section, we show that, except the critical value 0, the spectrum of the negative functional Laplacian $-\Delta$ is contained in $\sigma(\Delta)$. Similar inclusions hold between the respective point and essential spectra. See Theorem 5.3. Our proof follows the smooth treatise for [19, Cor. 4.4, Cor. 4.5].

As an important application, in Corollary 5.4 we derive explicit relations between the spectral gaps of the Schrödinger operator $-\Delta + K$, $\Delta$ and $-\Delta$.

We shall need the subsequent characterization of points in the (essential) spectrum of $\Delta$. See [19, Prop. 2.5] and the references therein for a more general statement.

**Lemma 5.1.** For every $\lambda > 0$, we have $\lambda \in \sigma(\Delta)$ if and only if there exist $\alpha < 0$ and a sequence $(\omega_n)_{n \in \mathbb{N}}$ in $\text{Dom}(\Delta)$ such that

\[ \text{a. } ||\omega_n||_{L^2} = 1 \text{ for every } n \in \mathbb{N}, \text{ and} \]

\[ \text{b. for every } j \in \{1, 2\}, \text{ one has} \]

\[ \lim_{n \to \infty} \int_M \langle (\Delta - \alpha)^{-j} \omega_n, \Delta \omega_n - \lambda \omega_n \rangle \, dm = 0. \]

Moreover, a number $\lambda > 0$ belongs to the essential spectrum of $\Delta$ if and only if some sequence $(\omega_n)_{n \in \mathbb{N}}$ in $\text{Dom}(\Delta)$ satisfies the previous conditions a. and b. as well as

\[ \text{c. } \omega_n \to 0 \text{ in } L^2(T^*M) \text{ as } n \to \infty. \]

**Lemma 5.2.** Let $\alpha < 0$. Then for every $f \in W^{1,2}(M)$, we have

\[ (\Delta - \alpha)^{-1} df = d(-\Delta - \alpha)^{-1} f, \]

while for every $\omega \in \text{Dom}(\delta)$, we have

\[ (\Delta - \alpha)^{-1} \delta \omega = \delta(\Delta - \alpha)^{-1} \omega. \]

**Proof.** Any $\alpha < 0$ belongs to $\rho(\Delta)$ and $\rho(-\Delta)$, thus $\Delta - \alpha$ and $-\Delta - \alpha$ are invertible with bounded inverse. Furthermore, the second identity follows from the first by definition of $\delta$ and the self-adjointness of $(\Delta - \alpha)^{-1}$, since $\alpha$ is real — we thus concentrate on the proof of the first equality.

Given any $f \in W^{1,2}(M)$, let $u \in \text{Dom}(\Delta)$ be the unique solution to the equation $-\Delta u - \alpha u = f$ on $M$. By Lemma 3.3, for every $t > 0$ we have $dP_t u \in \text{Dom}(\Delta)$ and

\[ \tilde{\Delta} dP_t u - \alpha dP_t u = -d(\Delta P_t u + \alpha P_t u) = dP_t f. \]

Therefore $H_t du = (\Delta - \alpha)^{-1} H_t df$ again by Lemma 3.3, and the claim follows by letting $t \to 0$. \qed

**Theorem 5.3.** Let $(M, d, m)$ be an RCD($K, \infty$) space for some $K \in \mathbb{R}$. Then

\[ \sigma_p(-\Delta) \subset \sigma_p(\tilde{\Delta}), \]

\[ \sigma(-\Delta) \setminus \{0\} \subset \sigma(\Delta), \]

\[ \sigma_c(-\Delta) \setminus \{0\} \subset \sigma_c(\tilde{\Delta}). \]

**Proof.** The first inclusion is elementary, since for every $\lambda \in \sigma_p(-\Delta)$ and its corresponding eigenfunction $f \in \text{Dom}(\Delta)$, by Lemma 3.3, $dP_1 f \in \text{Dom}(\Delta)$ and

\[ \tilde{\Delta} dP_1 f = -d\Delta P_1 f = \lambda dP_1 f. \]

To prove the second inclusion, let $\lambda \in \sigma(-\Delta) \setminus \{0\}$. By Weyl’s criterion [51, Thm. 5.10] applied to $-\Delta$, for every $n \in \mathbb{N}$ there exists $g_n \in \text{Dom}(\Delta)$ with

\[ ||g_n||_{L^2} = 1, \]

\[ ||\Delta g_n + \lambda g_n||_{L^2} \leq 2^{-n}. \]
Moreover, for every $n \in \mathbb{N}$ there exists $t_n > 0$ such that $f_n := P_{t_n} g_n \in \text{Dom}(\Delta)$ satisfies the similar estimates

\[
\sqrt{2}^{-1} \leq \|f_n\|_{L^2} \leq 1,
\]

\[
\|\Delta f_n + \lambda f_n\|_{L^2} \leq 2^{-n}.
\]

Provided that $2^{-n} \leq \lambda/4$, from (5.1) we get

\[
\int_M |df_n|^2 \, dm = - \int_M f_n \Delta f_n \, dm
\]

\[
\geq -\|\Delta f_n + \lambda f_n\|_{L^2}^2 \geq \frac{\lambda}{4} > 0.
\]

Possibly relabeling $(f_n)_{n \in \mathbb{N}}$, we may and will assume that the sequence $(\|df_n\|^2_{L^2})_{n \in \mathbb{N}}$ is uniformly bounded from below by $\lambda/4$. In particular, for $j \in \{1, 2\}$ it follows from Lemma 3.3, Lemma 5.2, contractivity of $(-\Delta + 1)^{-j}$ in $L^2(M)$, (5.1) and (5.2) that

\[
\left| \int_M ((\tilde{\Delta} + 1)^{-j} d\tilde{f}_n, (\tilde{\Delta} - \lambda) d\tilde{f}_n) \, dm \right|
\]

\[
= \left| \int_M [\delta(\tilde{\Delta} + 1)^{-j} d\tilde{f}_n] (\Delta f_n + \lambda f_n) \, dm \right|
\]

\[
= \left| \int_M [(-\Delta + 1)^{-j} \Delta f_n] (\Delta f_n + \lambda f_n) \, dm \right|
\]

\[
\leq 2^{-n} \|\Delta f_n\|_{L^2} \leq 2^{-n} (\lambda + 2^{-n}) \leq \frac{4(\lambda + 1)}{\lambda} 2^{-n} \|df_n\|^2_{L^2}.
\]

In particular, the sequence $\omega_n := \|df_n\|_{L^2}^{-1} \, df_n$, \hfill (5.3)

which takes values in $\text{Dom}(\tilde{\Delta})$ by Lemma 3.3, obeys a. and b. from Lemma 5.1, whence $\lambda \in \sigma(\tilde{\Delta})$.

Turning to the last inclusion, if $\lambda \in \sigma_e(-\Delta)$, then the sequence $(f_n)_{n \in \mathbb{N}}$ from the previous step can be constructed to satisfy $f_n \to 0$ in $L^2(M)$ as $n \to \infty$ in addition to (5.1) above [51, Thm. 7.2]. Therefore, for every $\eta \in \text{Test}(T^*M)$ we obtain for the sequence $(\omega_n)_{n \in \mathbb{N}}$ defined in (5.3) that

\[
\lim_{n \to \infty} \left| \int_M (\omega_n, \eta) \, dm \right| \leq \lim_{n \to \infty} \frac{2}{\sqrt{\lambda}} \left| \int_M f_n \delta \eta \, dm \right| = 0.
\]

Since $\|\omega_n\|_{L^2} = 1$ for every $n \in \mathbb{N}$, this provides c. in Lemma 5.1. \hfill \Box

**Corollary 5.4.** Under the assumptions of Theorem 5.3, we have

\[
\inf \sigma(-\Delta + K) \leq \inf \sigma(\tilde{\Delta}) \leq \inf \sigma(\tilde{\Delta}) \setminus \{0\} \leq \inf \sigma(-\Delta) \setminus \{0\}.
\]

**Proof.** The last inequality follows from Theorem 5.3.

The proof of the first inequality basically reduces to an inequality between quadratic forms. Indeed, from Theorem 2.14 and Proposition 3.7 we obtain, for every $\omega \in H^{1,2}(T^*M)$ with $\|\omega\|_{L^2} = 1$,

\[
\int_M [d\omega]^2 + |\delta \omega|^2 \, dm \geq \int_M [\nabla \omega]^2 + K |\omega|^2 \, dm
\]

\[
\geq \int_M [\nabla |\omega|^2 + K |\omega|^2] \, dm \geq \inf \sigma(-\Delta + K),
\]

and we conclude by taking the infimum over $\omega$ as above. \hfill \Box
Remark 5.5. In the setting of Remark 3.18, without any change of the previous proof, under a variable, uniformly lower bounded lower Ricci bound \( k \in L^1_{\text{loc}}(M) \) for \((M, d, m)\), one verifies that

\[
\inf \sigma(-\Delta + k) \leq \inf(\Delta).
\]

5.2. The spectrum in the compact case. Much more about \( \sigma(\Delta) \) can be said if \((M, d, m)\) is a compact \( \text{RCD}^*(K, N) \) space. In this framework, adopted throughout this section, we prove that \( \sigma(\Delta) \) is discrete and only consists of eigenvalues, see Theorem 5.13. A closely related result is that the natural inclusion of \( H^{1,2}(T^*M) \) into \( L^2(T^*M) \) is compact, see Theorem 5.8. In turn, by abstract functional analysis, this follows if \( H_t \) is a Hilbert–Schmidt operator on \( L^2(T^*M) \) for every \( t > 0 \), which is the content of Corollary 5.7.

Afterwards, we establish the boundedness of eigenforms for \( \Delta \) with an explicit growth rate for their \( L^{\infty} \)-norms for positive eigenvalues, see Corollary 5.14 and Theorem 5.15. The entire discussion in this section heavily relies on the \( L^2-L^{\infty} \)-regularization property of \((H_t)_{t \geq 0}\) from Theorem 4.3.

The simple proof of the subsequent lemma is taken from [6, Subsec. 1.8.4].

**Lemma 5.6.** Suppose that \( S \) is a linear operator which maps \( L^2(T^*M) \) boundedly into \( L^{\infty}(T^*M) \). Then \( S \) is a Hilbert–Schmidt operator.

**Proof.** Let \((\omega_i)_{i \in \mathbb{N}}\) be any orthonormal basis of the separable Hilbert space \( L^2(T^*M) \). Given \( d \in \mathbb{N} \), we denote by \( B^d \) the closed unit ball of \( \mathbb{R}^d \). Let \( C^d \) be fixed countable and dense subset of \( B^d \).

It easily follows from the boundedness of \( S \) that, for every \( a \in C^d \),

\[
|S \left[ \sum_{i=1}^{d} a_i \omega_i \right]| \leq \|S\|_{L^2, L^\infty} \quad m\text{-a.e.} \quad (5.4)
\]

Hence, there exists an \( m \)-null set \( P \subset M \) such that for every \( x \in M \setminus P \), the inequality from (5.4) holds true for every \( a \in C^d \). It follows that

\[
\sum_{i=1}^{d} |S \omega_i|^2 = \sup \left\{ \left[ \sum_{i=1}^{d} a_i S \omega_i \right]^2 : a \in C^d \right\} \leq \|S\|_{L^2, L^\infty}^2 \quad \text{on} \quad M \setminus P.
\]

Integrating this inequality and using the arbitrariness of \( d \) concludes the proof. \( \square \)

**Corollary 5.7.** For every \( t > 0 \), \( H_t \) is a Hilbert–Schmidt operator on \( L^2(T^*M) \).

In particular, \( H_t \) is compact on \( L^2(T^*M) \) for every \( t > 0 \). Employing standard functional analytic results, see e.g. [60, Cor. 1.5], this entails the following crucial Rellich-type theorem. (The reader is also invited to consult [54, Thm. 6.1], where the same result has independently been proven by different, more geometric means via \( \delta \)-splitting maps.) It implies in particular that \( \text{Harm}(T^*M) \) is a closed subspace of \( L^2(T^*M) \), hence the \( L^2 \)-orthogonal projection \( T \) onto \( \text{Harm}(T^*M) \) is well-defined. Lemma 5.9 is then easily argued by contradiction.

**Theorem 5.8** (Compactness of \( \bar{\Delta}^{-1} \)). Let \((M, d, m)\) be a compact \( \text{RCD}^*(K, N) \) space with \( K \in \mathbb{R} \) and \( N \in (1, \infty) \). Then the natural inclusion of \( H^{1,2}(T^*M) \) into \( L^2(T^*M) \) is compact.

**Lemma 5.9.** There exists a constant \( C < \infty \) such that for every \( \omega \in L^2(T^*M) \),

\[
\|\omega - T \omega\|_{L^2}^2 \leq 2C \mathcal{E}(\omega). \quad (5.5)
\]
Remark 5.10. Lemma 5.9 can be seen as a qualitative global Poincaré inequality for \( \bar{\Delta} \). In contrast to logarithmic Sobolev inequalities, we did not derive local or global Poincaré inequalities for \( 1 \)-forms from the corresponding functional estimates. Combining Proposition 3.7 with \([69, \text{Thm. 1.1, Thm. 1.2}] \) or \([93, \text{Thm. 30.24}] \), this would be possible to some extent.

Paying the price of a less explicit constant, the point is however that the terms \( \int_{B_r(x)} f \, dm \) or \( \int_M f \, dm \) if \( m[M] < \infty \), respectively, appearing in the functional versions are the \( L^2 \)-orthogonal projections of \( f \) onto the space of harmonic functions on the respective \( L^2 \)-spaces, while their \( 1 \)-form counterparts \( \int_{B_r(x)} |\omega| \, dm \) or \( \int_M |\omega| \, dm \) appearing in the derived estimates arguing as for Lemma 4.8 would clearly lack this interpretation.

**Corollary 5.11.** Let \( C > 0 \) be any constant for which (5.5) holds, and let \( \lambda \in \sigma(\bar{\Delta}) \setminus \{0\} \). Then \( \lambda \) is an eigenvalue of \( \bar{\Delta} \) and satisfies the inequality

\[
\lambda \geq 1/C.
\]

**Proof.** Let \( \lambda \in \sigma(\bar{\Delta}) \setminus \{0\} \). As in the proof of Theorem 5.3, by Weyl’s criterion there exists a sequence \((\omega_n)_{n \in \mathbb{N}}\) in \( \text{Dom}(\bar{\Delta}) \) such that, for every \( n \in \mathbb{N} \),

\[
\|\omega_n\|_{L^2} = 1,
\]

\[
\|\bar{\Delta} \omega_n - \lambda \omega_n\|_{L^2} \leq 2^{-n}.
\]

(5.6)

To prove that \( \lambda \) is an eigenvalue, observe that \((\|\omega_n\|_{H^{1,2}_{\text{loc}}})_{n \in \mathbb{N}} \) is uniformly bounded by (5.6). According to Theorem 5.8, a non-relabeled subsequence of \((\omega_n)_{n \in \mathbb{N}} \) converges weakly in \( H^{1,2}(T^* M) \) and strongly in \( L^2(T^* M) \) to some \( \omega \in H^{1,2}(T^* M) \) with \( \|\omega\|_{L^2} = 1 \). Given any \( \rho \in \text{Test}(T^* M) \), since

\[
\lambda \int_M \langle \rho, \omega \rangle \, dm = \lim_{n \to \infty} \int_M \langle \rho, \bar{\Delta} \omega_n \rangle \, dm = \lim_{n \to \infty} \int_M [\langle d\rho, d\omega_n \rangle + \delta \rho \, \delta \omega_n] \, dm
\]

\[
= \int_M [\langle d\rho, d\omega \rangle + \delta \rho \, \delta \omega] \, dm,
\]

we also obtain that \( \omega \in \text{Dom}(\bar{\Delta}) \) with \( \bar{\Delta} \omega = \lambda \omega \), which is the claim.

The bound \( \lambda \geq 1/C \) then follows by inserting \( \omega \) into (5.5), recalling that eigenspaces w.r.t. different eigenvalues are orthogonal in \( L^2(T^* M) \).

**Remark 5.12.** The nonsmooth analogue of Hodge’s theorem \([40, \text{Thm. 3.5.15}] \) gives a one-to-one correspondence between the multiplicity of harmonic \( 1 \)-forms in terms of the dimension of the first de Rham cohomology group \( H^1_{\text{dR}}(M) \) as defined in \([40, \text{Def. 3.5.8}] \).

In view of Corollary 5.11, given \( \lambda \geq 0 \) we denote the eigenspace of \( \bar{\Delta} \) w.r.t. \( \lambda \) by

\[
E_\lambda(\bar{\Delta}) := \{ \omega \in \text{Dom}(\bar{\Delta}) : \bar{\Delta} \omega = \lambda \omega \}.
\]

The proofs of the following basic results are standard once having Theorem 5.8 and Corollary 5.11 at our disposal. We refer to \([52, \text{Thm. 4.3}] \) as well as \([74, \text{Lem. VI.3.6, Prop. VI.3.8}] \) for similar statements in the smooth setting and for comprehensive proofs.

**Theorem 5.13.** The spectrum \( \sigma(\bar{\Delta}) \) has the following properties.

(i) **Finite Dimensionality.** For every \( \lambda \geq 0 \), the vector space dimension of \( E_\lambda(\bar{\Delta}) \) is finite.

(ii) **Discreteness and Unboundedness.** The spectrum \( \sigma(\bar{\Delta}) \) is discrete (i.e. for every \( \lambda \in \sigma(\bar{\Delta}) \) there exists some \( r > 0 \) such that \( (\lambda - r, \lambda + r) \cap \sigma(\bar{\Delta}) = \{\lambda\} \) and unbounded. 

(iii) **Variational Principle.** Let \((λ_i)_{i∈N}\) be an increasing enumeration of the eigenvalues of \(\overline{Δ}\) counted with multiplicities. Let \(S\) denote the unit sphere in \(L^2(T^*M)\). Then for every \(i ∈ N\),
\[
λ_i = \inf \left\{ \sup_{ω ∈ E^i} 2\bar{E}(ω) : E ⊂ H^{1,2}(T^*M) \text{ subspace with dim } E = i \right\}.
\]

(iv) **Orthonormal Eigenbasis.** The direct sum
\[
E(\overline{Δ}) := \bigoplus_{λ ∈ σ(\overline{Δ})} E_λ(\overline{Δ})
\]
is dense both in \(H^{1,2}(T^*M)\) and \(L^2(T^*M)\), endowed with their respective norms. In particular, there exists a countable orthonormal basis \((ω_i)_{i∈N}\) of \(L^2(T^*M)\) such that, for every \(i ∈ N\), we have \(ω_i ∈ E_λ(\overline{Δ})\) for some \(λ_i ∈ σ(\overline{Δ})\).

Since \(ω = H_tω\) for every \(ω ∈ Harm(T^*M)\), Theorem 4.3 immediately provides Corollary 5.14 below. An argument as in the proof of Theorem 4.3 with a finer estimation then yields Theorem 5.15. For similar statements, see [5, Prop. 7.1] for functions — whose proof is adopted in our approach — and [52, Prop. 4.14] for arbitrary tensor fields in the Ricci limit framework.

**Corollary 5.14.** We have \(\text{Harm}(T^*M) ⊂ L^∞(T^*M)\).

**Theorem 5.15.** Let \((M, d, m)\) be a compact \(\text{RCD}^*(K, N)\) space for some \(K ∈ \mathbb{R}\) and \(N ∈ (1, ∞)\). Let \(D > 0\) obey \(\text{diam} \, M ≤ D\). Assume that \(ω ∈ \text{Dom}(\overline{Δ})\) is an eigenform with eigenvalue \(λ ∈ [D^{-2}, ∞)\) and \(||ω||_{L^2} = 1\). Then there exists a constant \(C < ∞\) depending only on \(K\), \(N\) and \(D\) such that
\[
||ω||_{L^∞} ≤ C \, λ^{N/4}.
\]

**Proof.** Since \(ω ∈ E_λ(\overline{Δ})\), it follows that \(H_tω = e^{-λt}ω\) for every \(t ≥ 0\). Thus, for \(t ∈ (0, D^2]\) to be determined later, Theorem 3.15 and then (2.7) for \(ε := 1\) yield the existence of constants \(C_1, C_2 < ∞\) depending only on \(K\) and \(N\) such that
\[
|ω| = e^{\lambda t} |H_tω| ≤ e^{(λ + K^-)t} \int_M p_2^2(\cdot, y) |ω|^2(y) \, dm(y)
\]
\[
≤ e^{(λ + K^-)t} \left[ \int_M p_2^2(\cdot, y) \, dm(y) \right]^{1/2}
\]
\[
≤ C_1 e^{(λ + K^- + C_2)t} \text{m}[B_2(\tau^{-1})]^{-1} \left[ \int_M e^{-2d^2(\cdot, y)/5t} \, dm(y) \right]^{1/2} \text{m.a.e. (5.7)}
\]

Arguing exactly as in the proof of [5, Prop. 7.1], using (2.4), under the given assumptions we find a constant \(C < ∞\) depending only on \(K\), \(N\) and \(D\) such that
\[
\text{m}[B_{2\tau}(\cdot)]^{-1} \left[ \int_M e^{-2d^2(\cdot, y)/5t} \, dm(y) \right]^{1/2} ≤ C \left( \frac{D}{\sqrt{t}} \right)^{N/2} \text{m.a.e.}
\]
The choice of \(t := 1/λ\) gives the desired estimate. □

5.3. **Independence of the \(L^p\)-spectrum on \(p\).** In this section, fix an \(\text{RCD}^*(K, N)\) space \((M, d, m)\), where \(K ∈ \mathbb{R}\) and \(N ∈ (1, ∞)\). Under a volume growth assumption stated in Definition 5.16, following [17, 47, 84] we show that the \(L^p\)-spectrum of \(\overline{Δ}\) is independent of \(p ∈ [1, ∞)\), see Theorem 5.19 below.

To keep the presentation clear, in this section we denote by \(\overline{Δ}_2 := \overline{Δ}\) the Hodge Laplacian acting on \(L^2(T^*M)\) and by \((H_{2,t})_{t ≥ 0}\) the associated semigroup \(H_{2,t} := H_t\). Recalling Theorem 4.1, by \((H_{p,t})_{t ≥ 0}\) we denote the strongly continuous extension of \((H_{2,t})_{t ≥ 0}\) to \(L^p(T^*M)\) for every \(p ∈ [1, ∞)\). Let \(\overline{Δ}_p\) be the infinitesimal generator of \((H_{p,t})_{t ≥ 0}\). We also define \(H_{∞,t}\) and \(\overline{Δ}_∞\) on \(L^∞(T^*M)\) as the adjoints of \(H_{1,t}\) and \(\overline{Δ}_1\), respectively.
Given any \( p \in [1, \infty] \) and \( n \in \mathbb{N} \), by [94, Thm. IX.4.1, Cor. IX.4.1] we know that, for every \( \xi \in \rho(\Delta_p) \) with \( \Re \xi < K^- \), we have
\[
(\Delta_p - \xi)^{-n} = \frac{1}{(n-1)!} \int_0^\infty e^{\xi t} t^{n-1} \mathcal{H}_{p,t} \, dt.
\] (5.8)

**Definition 5.16.** We say that the reference measure \( \mathfrak{m} \) on \( (M, d) \) is uniformly subexponentially integrable if for every \( \varepsilon > 0 \), we have
\[
\sup_{x \in M} \int_M e^{-\varepsilon d(x,y)} \mathfrak{m}[B_1(x)]^{-1/2} \mathfrak{m}[B_1(y)]^{-1/2} \, dm(y) < \infty.
\]

**Remark 5.17.** By the same argument as for [84, Prop. 1], \( \mathfrak{m} \) is uniformly subexponentially integrable if for every \( \varepsilon > 0 \), there exists \( C < \infty \) such that for every \( x \in M \) and every \( r > 0 \),
\[
\mathfrak{m}[B_r(x)] \leq C e^{\varepsilon r} \mathfrak{m}[B_1(x)].
\]

**Example 5.18.** If \((M, d, \mathfrak{m})\) is globally doubling (2.6), then \( \mathfrak{m} \) is uniformly subexponentially integrable. Indeed, by a well-known iteration argument starting from (2.6), the sufficient condition from the previous Remark 5.17 follows from the existence of finite constants \( \alpha, \beta > 0 \) such that
\[
\mathfrak{m}[B_r(x)] \leq \beta r^\alpha \mathfrak{m}[B_1(x)]
\]
holds for every \( x \in M \) and every \( r \geq 1 \).

Since \( \text{RCD}^*(K, N) \) spaces with a nonnegative lower Ricci bound are globally doubling [88, Cor. 2.4], uniform subexponential integrability of \( \mathfrak{m} \) is granted as soon as \( K \geq 0 \).

**Theorem 5.19.** Let \((M, d, \mathfrak{m})\) be an \( \text{RCD}^*(K, N) \) space for some \( K \in \mathbb{R} \) and \( N \in (1, \infty) \). Assume that \( \mathfrak{m} \) is uniformly subexponentially integrable. Then the spectrum \( \sigma(\Delta_p) \) of the operator \( \Delta_p \) acting on \( L^p(T^*M) \) is equal to \( \sigma(\hat{\Delta}_p) \) for every \( p \in [1, \infty] \). Furthermore, for every \( p, q \in [1, \infty] \), every isolated eigenvalue of \( \Delta_p \) with finite algebraic multiplicity is also an isolated eigenvalue of \( \hat{\Delta}_q \) with the same algebraic multiplicity.

The key point of the proof of Theorem 5.19 is a perturbation argument whose core we outsource into Lemma 5.21, Lemma 5.22 and Corollary 5.23 below. Before that, we quickly fix some notation.

We define the measurable function
\[
\phi_1 : M \rightarrow \mathbb{R}
\]
by
\[
\phi_1(x) := \mathfrak{m}[B_1(x)]^{1/2}.
\]

Given any \( \varepsilon > 0 \), we consider the class
\[
\Gamma_\varepsilon := \{ \psi \in W^{1,2}(M) \cap C_b(M) : |d\psi| \leq \varepsilon \text{ m-a.e.} \}
\]
and recall from [4, Thm. 4.17] that, for every \( x, y \in M \),
\[
\varepsilon \, d(x, y) = \sup\{ \psi(x) - \psi(y) : \psi \in \Gamma_\varepsilon \}.
\] (5.9)

Lastly, given \( \psi \in \Gamma_\varepsilon \), by \( e^{\psi} \hat{\Delta}_2 e^{-\psi} \) we intend the linear, densely defined operator on \( L^2(T^*M) \) given by setting, for arbitrary \( \omega, \eta \in \text{Test}(T^*M) \),
\[
\int_M (\eta, (e^{\psi} \hat{\Delta}_2 e^{-\psi}) \omega) \, dm
:= \int_M (\eta, \hat{\Delta}_2 \omega) \, dm - \int_M |\nabla \psi|^2 (\eta, \omega) \, dm + \int_M \left[ \nabla \omega^\sharp (\nabla \psi, \eta^\sharp) - \nabla \eta^\sharp (\nabla \psi, \omega^\sharp) \right] \, dm.
\] (5.10)
Remark 5.20. Observe that if \( \psi \) is sufficiently regular, say, \( \psi \in \Gamma_\varepsilon \cap \text{Test}(M) \) with \( \Delta \psi \in L^\infty(M) \), then \( e^\psi \Delta_2 e^{-\psi} \omega \) is pointwise well-defined on any \( \omega \in \text{Test}(T^*M) \) as composition of the multiplication operators \( e^\psi \) and \( e^{-\psi} \) as well as the Hodge Laplacian \( \Delta_2 \) in the indicated order by (2.12). In this case, (5.10) follows by a straightforward computation using Lemma 2.13 and (2.10).

The class of functions in \( \Gamma_\varepsilon \cap \text{Test}(M) \) with bounded Laplacian is dense in \( \Gamma_\varepsilon \) w.r.t. strong convergence in \( W^{1,2}(M) \), see Lemma 2.2.

Lemma 5.21. For every compact \( V \subset \rho(\Delta_2) \), there exist \( \varepsilon \in (0,1) \) and a constant \( C < \infty \) such that for every \( \xi \in V \) and every \( \psi \in \Gamma_\varepsilon , \xi \in \rho(e^\psi \Delta_2 e^{-\psi}) \) with
\[
\|(e^\psi \Delta_2 e^{-\psi} - \xi)\|_{L^2, L^2} \leq C. \tag{5.11}
\]

Proof. Given any \( \varepsilon \in (0,1) \) to be determined later and \( \psi \in \Gamma_\varepsilon \), the operator
\[
T_\psi := e^\psi \Delta_2 e^{-\psi} - \Delta_2
\]
is well-defined on \( \text{Test}(T^*M) + i \text{Test}(T^*M) \) and therefore a densely defined linear operator on the complexified vector space \( L^2(T^*M) + i L^2(T^*M) \). (5.10) yields
\[
\int_M \langle T_\psi \omega, \overline{\omega} \rangle \, dm = - \int_M |\nabla \psi|^2 |\omega|^2 \, dm + 2i \int_M \Im \langle \nabla \omega^\dagger (\nabla \psi, \overline{\omega}) \rangle \, dm
\]
for every \( \omega \in \text{Test}(T^*M) + i \text{Test}(T^*M) \). Young’s inequality and (2.13) thus give
\[
\left| \int_M \langle T_\psi \omega, \overline{\omega} \rangle \, dm \right| \leq \varepsilon^2 \|\omega\|_{L^2}^2 + 2\varepsilon \int_M |\nabla \omega|^2 \, dm \leq (\varepsilon + 1) \varepsilon \|\omega\|_{L^2}^2 + \varepsilon \int_M |\nabla \omega|^2 \, dm \leq (\varepsilon + 1 - K) \varepsilon \|\omega\|_{L^2}^2 + \varepsilon \int_M \langle \Delta_2 \omega, \omega \rangle \, dm.
\]

Given any compact \( V \subset \rho(\Delta_2) \), there exists \( \varepsilon \in (0,1) \) such that for every \( \xi \in V \),
\[
2 \|(\varepsilon + 1 - K) \varepsilon + \varepsilon \Delta_2 (\Delta_2 - \xi)^{-1}\|_{L^2, L^2} \leq 2\varepsilon (3 + |K| + |\xi|) \|(\Delta_2 - \xi)^{-1}\|_{L^2, L^2} + 2\varepsilon < 1.
\]
From the above form boundedness of \( T_\psi \) by \( \Delta_2 \), which is uniform in \( \psi \in \Gamma_\varepsilon \), and under the previous choice of \( \varepsilon \), [58, Thm. VI.3.9] both gives \( \xi \in \rho(e^\psi \Delta_2 e^{-\psi}) \) for every \( \psi \in \Gamma_\varepsilon \) and provides us with the existence of a finite constant \( C \) such that (5.11) holds uniformly in \( \xi \in V \) and \( \psi \in \Gamma_\varepsilon \). \( \square \)

Recall that every real \( \alpha < 0 \) belongs to \( \rho(\Delta_2) \). Therefore, using (2.5) in the first case, the operators \( (\Delta_2 - \alpha)^{-1/2} e^{-\psi} \phi_1 \) and \( (\Delta_2 - \alpha)^{-1/2} e^{-\psi} \) are well-defined on \( L^\infty(T^*M) \) (which is the space of all \( \omega \in L^\infty(T^*M) \) such that \( |\omega| \) has bounded support), hence densely defined on \( L^p(T^*M) \) for every \( p \in [1, \infty) \).

Lemma 5.22. There exists \( \alpha < 0 \) such that for every \( \varepsilon \in (0,1) \), there exist an even \( n \in \mathbb{N} \) and a constant \( C < \infty \) such that for every \( \psi \in \Gamma_\varepsilon \), we have
\[
\|e^\psi (\Delta_2 - \alpha)^{-n/2} e^{-\psi} \phi_1\|_{L^1, L^2} \leq C, \tag{5.12}
\]
\[
\|\phi_1 e^\psi (\Delta_2 - \alpha)^{-n/2} e^{-\psi}\|_{L^2, L^\infty} \leq C. \tag{5.13}
\]
Proof. Fix any real \(\alpha < 0\) to be determined later, an arbitrary \(\beta \in \mathbb{R}\) as well as an even \(n \in \mathbb{N}\) with \(n \geq [N+4]\). Employing the formula (5.8) and then Theorem 3.15, (2.7) as well as (2.4), there exist constants \(c, C_1, C_2, C_3 < \infty\) with

\[
|\tilde{\Delta}_2 - \alpha|^{-n/2} \eta| \leq c \int_0^\infty e^{\alpha t} t^{n/2-1} |H_2 t \eta| \, dt
\]

\[
\leq c \int_0^\infty e^{(\alpha-K)t} t^{n/2-1} \int_M p_1(\cdot, y) |\eta|(y) \, dm(y) \, dt
\]

\[
\leq c C_1 \int_0^\infty \left[ \int_M e^{(\alpha-K+C_2)t} t^{n/2-1} \left( m[B_1 t \eta](\cdot) \right)^{-1} \right] \times \int_M e^{-\beta t} |\eta|(y) \, dm(y) \, dt
\]

\[
\leq c C_1 C_3 \left[ \int_0^\infty e^{(\alpha-K+C_2+5\beta^2/4)t} t^{n/2-1} \max \{t^{-N/2}, 1\} \phi_1^{-2} \right] \times \int_M e^{-\beta t} |\eta|(y) \, dm(y) \, m.a.e.
\]

for every \(\eta \in L^2(T^*M)\). In the last inequality, we also used that

\[
\frac{5L}{4} \beta^2 - \beta d(x, y) + \frac{d^2(x, y)}{5L} \geq 0.
\]

Setting \(\alpha := \min \{-1 - K + C_2 + 5\beta^2/4, -1\}\) gives the existence of a constant \(C_4 < \infty\) together with

\[
|(\tilde{\Delta}_2 - \alpha)|^{-n/2} \eta| \leq C_4 \phi_1^{-2} \int_M e^{-\beta d(\cdot, y)} |\eta|(y) \, dm(y) \, m.a.e. \quad (5.12)
\]

The next step is to use (5.12) subject to a particular choice of \(\beta \in \mathbb{R}\) to be determined later. Let \(\varepsilon \in (0, 1)\) and \(\psi \in \Gamma_\varepsilon\) be arbitrary. Since \(e^\psi (\tilde{\Delta}_2 - \alpha)^{-n/2} e^{-\psi} \phi_1 \) is the formal adjoint of \(\phi_1 e^\psi (\tilde{\Delta}_2 - \alpha)^{-n/2} e^{-\psi}\), the first estimate will actually follow from the second inequality. To prove the latter, for every \(\omega \in L^\infty_{bs}(T^*M)\), inserting \(\eta := e^{-\psi} \omega\) into (5.12) for arbitrary \(\beta > \varepsilon\) and using (5.9) yields

\[
|\phi_1 e^\psi (\tilde{\Delta}_2 - \alpha)|^{-n/2} e^{-\psi} \omega| \leq C_4 \phi_1^{-1} e^\psi \int_M e^{-\beta d(\cdot, y)} e^{-\psi(y)} |\omega|(y) \, dm(y)
\]

\[
\leq C_4 \left[ \phi_1^{-2} \int_M e^{-2(\beta - \varepsilon) d(\cdot, y)} \, dm(y) \right]^{1/2} \|\omega\|_{L^2}
\]

\[
\leq C_4 \left[ \sum_{j=1}^\infty e^{-2(\beta - \varepsilon)(j-1)} m[B_j(\cdot)] m[B_1(\cdot)]^{-1} \right]^{1/2} \|\omega\|_{L^2} \, m.a.e. \quad (5.13)
\]

By (2.4), the last sum is uniformly bounded uniformly on \(M\) and in \(\varepsilon \in (0, 1)\) as soon as \(\beta > 0\) is chosen large enough.

The inequality (5.13) for arbitrary \(\omega \in L^2(T^*M)\) follows by density of \(L^\infty_{bs}(T^*M)\), after passing to pointwise m.a.e. convergent subsequences. \(\square\)

**Corollary 5.23.** For every compact \(V \subset \rho(\tilde{\Delta}_2)\), there exist \(\varepsilon \in (0, 1)\), an even \(n \in \mathbb{N}\) and a constant \(C < \infty\) such that for every \(\xi \in V\), one has

\[
\| (\tilde{\Delta}_2 - \xi)^{-n} \|_{L^\infty, L^\infty} \leq C \sup_{x \in M} \int_M e^{-d(x, y)} \phi_1^{-1} (x) \phi_1^{-1}(y) \, dm(y).
\]
Proof. Let $V \subset \rho(\Delta_2)$ be compact, and let $\varepsilon \in (0, 1)$ be as provided by Lemma 5.21 and $n \in \mathbb{N}$ be as in Lemma 5.22. For every $\xi \in V$ and every $\psi \in \Gamma_\varepsilon$, the first resolvent identity gives
\[
\phi_1 e^{\psi(\Delta_2 - \xi)^{-n}} e^{-\psi} \phi_1 \\
= \sum_{j=0}^{n-1} \left[ \left( \frac{n}{j} \right) (\xi - \alpha)^j \left( \phi_1 e^{\psi(\Delta_2 - \alpha)^{-n/2}} e^{-\psi} \right) \left( e^{\psi(\Delta_2 - \xi)^{-1}} e^{-\psi} \right)^j \right].
\]
By Lemma 5.21 and Lemma 5.22 we find a constant $C < \infty$ such that for every $\xi \in V$ and $\psi \in \Gamma_\varepsilon$,
\[
\|\phi_1 e^{\psi(\Delta_2 - \xi)^{-n}} e^{-\psi} \phi_1\|_{L^1, L^\infty} \leq C.
\]
Hence, $\phi_1 e^{\psi(\Delta_2 - \xi)^{-n}} e^{-\psi} \phi_1$ is representable as an integral operator in the sense of Theorem 6.3 — in particular, for every $\eta \in L^\infty_0(T^*M)$, we obtain
\[
|\phi_1 e^{\psi(\Delta_2 - \xi)^{-n}} e^{-\psi} \phi_1 \eta| \leq C \int_M |\eta|(y) \, d\mathcal{m}(y) \quad \text{m.a.e.}
\]
For clarity, we point out that here we used the mentioned Theorem 6.3 (which is proven only later), but no circular reasoning occurs since Corollary 5.23 will not be used in its proof. Setting $\eta := \phi_1^{-1} e^{\psi} \omega$, where $\omega \in L^\infty_0(T^*M)$ is arbitrary,
\[
|(\Delta_2 - \xi)^{-n} \omega| \leq C \int_M e^{-\psi} e^{\psi(y)} \phi_1^{-1} \phi_1^{-1}(y) |\omega|(y) \, d\mathcal{m}(y) \quad \text{m.a.e.}
\]
By the arbitrariness of $\psi \in \Gamma_\varepsilon$ and (5.9), we obtain
\[
|(\Delta_2 - \xi)^{-n} \omega| \leq C \int_M e^{-\varepsilon d(\cdot, y)} \phi_1^{-1} \phi_1^{-1}(y) |\omega|(y) \, d\mathcal{m}(y) \quad \text{m.a.e.}
\]
The latter estimate is indeed true for every $\omega \in L^\infty_0(T^*M)$ by an elementary cutoff argument, which establishes the desired assertion. 

Proof of Theorem 5.19. Fix an arbitrary $p \in [1, 2) \cup (2, \infty)$. We concentrate on the inclusion $\sigma(\Delta_p) \subset \sigma(\Delta_2)$. The inclusion $\sigma(\Delta_p) \supset \sigma(\Delta_2)$ follows as for [17, Prop. 9], and the argument for the isolated eigenvalues is the same as in (the references given in) the proof of [17, Prop. 2.2].

Let $V \subset \rho(\Delta_2)$ be compact with $V \cap (-\infty, 0) \neq \emptyset$. Let $n \in \mathbb{N}$ be as in Corollary 5.23. Since $\sigma$ is uniformly subexponentially integrable, by Corollary 5.23 and taking adjoints, we see that $(\Delta_2 - \xi)^{-n}$ is bounded from $L^p(T^*M)$ into $L^p(T^*M)$ for $p \in \{1, \infty\}$. By Riesz–Thorin’s interpolation theorem, $(\Delta_2 - \xi)^{-n}$ is actually bounded from $L^p(T^*M)$ into $L^p(T^*M)$ for every $p \in [1, \infty]$.

By (5.8) and since $H_2$ is $H_{p,t}$ on $L^2(T^*M) \cap L^p(T^*M)$ for every $t \geq 0$, we get
\[
(\Delta_2 - \xi)^{-n} = (\Delta_p - \xi)^{-n} \quad \text{on } L^2(T^*M) \cap L^p(T^*M) \quad (5.14)
\]
for every $\xi \in \rho(\Delta_2) \cap (-\infty, 0)$. Since $V \cap (-\infty, 0) \neq \emptyset$ and the map $\xi \mapsto (\Delta_2 - \xi)^{-n}$ is analytic on $\rho(\Delta_2)$, the identity (5.14) holds for every $\xi \in V$. In particular, $(\Delta_p - \xi)^{-n}$ extends to a bounded linear operator from $L^p(T^*M)$ into $L^p(T^*M)$ for every $\xi \in V$, with
\[
(\Delta_p - \xi)^{-n} = (\Delta_2 - \xi)^{-n} \quad \text{on } L^p(T^*M).
\]
It follows that $V \subset \rho(\Delta_p)$. Taking complements, we deduce the claimed inclusion.  \(\square\)
Example 5.24. Let $N \in \mathbb{N}$ and $K < 0$. The $N$-dimensional hyperbolic space $\mathbb{H}^N_K$ with constant sectional curvature $K$ is an RCD$^*(K(N-1), N)$ space when endowed with its Riemannian distance and Riemannian volume measure. In this situation, it is due to [17, Thm. 14] that the set $\sigma(\Delta_p)$ does depend on $p \in [1, \infty]$.

6. Heat kernel

6.1. Dunford–Pettis’ theorem. A crucial step in proving the existence of a heat kernel is a Dunford–Pettis-type theorem for co- or contravariant objects, see Theorem 6.3 below. See [17, Lem. 11] for a smooth analogue obtained via computations in local coordinates. Our Theorem 6.3 significantly enlarges the scope of the latter to the language of $L^1$-normed $L^\infty$-modules from Section 2.2.

The following definition provides the setting for our understanding of integral operators over an $L^p$-normed $L^\infty$-module $\mathcal{M}$, $p \in [1, \infty]$.

Definition 6.1. Given any $p, r \in [1, \infty]$, let $\mathcal{M}$ be a separable $L^p$-normed $L^\infty$-module, and $\mathcal{N}$ be a separable $L^r$-normed $L^\infty$-module. Let $\mathcal{M}^0$ and $\mathcal{N}^0$ be their corresponding $L^p$-modules as introduced in Section 2.2. We denote by $\mathcal{N}^0 \otimes \mathcal{M}^0$ the space of all $\mathcal{M}^0 \times \mathcal{N}^0 \to \mathcal{L}^0(M^2)$.

In the case $p = r$ and $\mathcal{M} = \mathcal{N}$, we briefly write $(\mathcal{M}^0)^{\otimes 2} := \mathcal{N}^0 \otimes \mathcal{M}^0$.

For $a \in \mathcal{N}^0 \otimes \mathcal{M}^0$, we define the $\mathcal{M}^{\otimes 2}$-measurable function $|a|_\otimes : M^2 \to [0, \infty]$ by

$$|a|_\otimes(x, y) := \operatorname{esssup}\{ |a(s, v)|(x, y) : s \in \mathcal{N}^0, v \in \mathcal{M}^0 \text{ with } |s|, |v| \leq 1 \text{ m-a.e.} \}.$$ 

A key ingredient for Theorem 6.3 is the subsequent result from [31, Thm. 2.2.5]. Its advantage compared to the more general result [32, Thm. VI.8.6] — providing a similar statement with the Banach dual of any separable Banach space as target domain — is described in Remark 6.4.

Proposition 6.2. Assume that $\mathcal{B} : L^1(M) \to L^\infty(M)$ is a linear and bounded map. Then there exists an $\mathcal{M}^{\otimes 2}$-measurable kernel $b : M^2 \to \mathbb{R}$ such that

$$\|b\|_{L^\infty} = \|\mathcal{B}\|_{L^1, L^\infty} < \infty$$

and, for every $g \in L^1(M)$,

$$\mathcal{B}g = \int_M b(\cdot, y) g(y) \, dm(y) \text{ m-a.e.}$$

Such a kernel is unique in the sense that if $\tilde{b} : M^2 \to \mathbb{R}$ is another $\mathcal{M}^{\otimes 2}$-measurable kernel fulfilling the foregoing obstructions, then $\tilde{b}$ does $\mathcal{M}^{\otimes 2}$-a.e. coincide with $b$.

Theorem 6.3 (Dunford–Pettis theorem for $L^\infty$-modules). Let $\mathcal{M}$ and $\mathcal{N}$ be separable $L^1$-normed $L^\infty$-modules defined over $(M, d, m)$. Suppose that $A : \mathcal{M} \to \mathcal{N}^*$ is a linear map with $\|A\|_{\mathcal{M}, \mathcal{N}^*} < \infty$. Then there exists $a \in \mathcal{N}^0 \otimes \mathcal{M}^0$ such that

$$\|a|_\otimes\|_{L^\infty} = \|A\|_{\mathcal{M}, \mathcal{N}^*}$$

and, for every $v \in \mathcal{M}$ and every $s \in \mathcal{N}$, we have $a(s, v) \in L^1(M^2)$ with

$$(s | Av) = \int_M a(s, v)(\cdot, y) \, dm(y) \text{ m-a.e.}$$

The element $a$ is unique in the sense that for any other $\tilde{a} \in \mathcal{N}^0 \otimes \mathcal{M}^0$ satisfying the foregoing obstructions, $a(s, v) = \tilde{a}(s, v)$ holds $\mathcal{M}^{\otimes 2}$-a.e. for every $v \in \mathcal{M}^0$ and every $s \in \mathcal{N}^0$.
Proposition 6.2.  We consider the distances \( |f - g| \) between \( f \) and \( g \), respectively, where \( f, g \in \mathcal{D} \), \( m \)-essentially bounded elements. We may and will assume that \( \mathcal{D} = \mathcal{D}_\infty \). Let \( \mathcal{D}_\infty \) be the smallest algebra of functions in \( L^0(M) \) w.r.t. pointwise multiplication, which contains \( \mathcal{D} \). Furthermore, let \( \mathcal{A}_\infty \) and \( \mathcal{N}_\infty \) be the sets of all finite linear combinations of elements of the form \( f \cdot v \) and \( f \cdot s \), respectively, where \( f \in \mathcal{D}_\infty \), \( v \in \mathcal{A}_\infty \) and \( s \in \mathcal{N}_\infty \). The classes \( \mathcal{D}_\infty \), \( \mathcal{A}_\infty \) and \( \mathcal{N}_\infty \) are all countable, consist of \( m \)-essentially bounded elements, and are dense in \( L^0(M) \), \( \mathcal{A}_\infty^0 \) and \( \mathcal{N}_\infty^0 \), respectively.

Given any \( v \in \mathcal{A}_\infty \) and any \( s \in \mathcal{N}_\infty \), define \( B(s,v) : L^1(M) \to L^\infty(M) \) by

\[
B(s,v)g := (s | A(gv)).
\]

The map \( B(s,v) \) is clearly linear, and it is well-defined and bounded since

\[
\|B(s,v)g\|_{L^\infty} \leq \|A(gv)\|_{\mathcal{A}_\infty}, \quad \|s\|_{L^\infty} \leq \|A\|_{\mathcal{A}_\infty, \mathcal{N}_\infty}, \quad \|g\|_{\mathcal{A}_\infty}, \quad \|s\|_{L^\infty}.
\]

By Proposition 6.2, there exists a kernel \( b(s,v) \in L^\infty(M^2), \) \( m_{\mathcal{O}_2} \)-a.e. uniquely determined in a proper way, such that for every \( g \in L^1(M) \),

\[
B(s,v)g = \int_M b(s,v)(\cdot, y) g(y) \, dm(y) \quad m\text{-a.e.} \tag{6.1}
\]

Step 2. Properties of the obtained integral kernel. An immediate property coming from the fact that \( B(s,c,v)g = B(s,v)[cg] \) and \( B(d,s,v)g = dB[s,v]g \) for every \( g \in L^1(M) \) and every \( c,d \in \mathcal{D}_\infty \), and the \( m_{\mathcal{O}_2} \)-a.e. uniqueness of the induced integral kernel is the following bilinearity. For every \( c,d \in \mathcal{D}_\infty \), every \( v,v' \in \mathcal{A}_\infty \) and every \( s,s' \in \mathcal{N}_\infty \), we have

\[
b(d(s+c)v+v', d(s)v + d(s')v) = c(pr_1) b(s,v) + c(pr_2) b(s',v) + d(pr_1) b(s,v) + d(pr_2) b(s',v). \tag{6.2}
\]

Moreover, for every \( v \in \mathcal{A}_\infty \) and every \( s \in \mathcal{N}_\infty \), we claim that

\[
|b(s,v)| \leq \|A\|_{\mathcal{A}_\infty, \mathcal{N}_\infty^*} |s|(pr_1) |v|(pr_2) \quad m_{\mathcal{O}_2} \text{-a.e.} \tag{6.3}
\]

Indeed, let \( g,h \in L^1(M) \) be nonnegative. Multiplying both sides of the identity (6.1) with \( h \) and integrating w.r.t. \( m \) yields

\[
\int_{M^2} b(s,v)(x,y) g(y) h(x) \, dm_{\mathcal{O}_2}(x,y) = \int_M B(s,v)g(x) h(x) \, dm(x) \tag{6.4}
\]

Changing the sign in both sides of (6.4), the claim follows by the arbitrariness of \( g \) and \( h \).

Step 3. Definition of \( a \). Since the topology of \( \mathcal{A}_\infty^0 \) and \( \mathcal{N}_\infty^0 \) is intrinsic, in the sense indicated in Section 2.2, we consider the distances \( d_{\mathcal{A}_\infty^0} \) and \( d_{\mathcal{N}_\infty^0} \) as defined w.r.t. a fixed partition \( (E_j)_{j \in \mathbb{N}} \) of \( M \) into Borel subsets of finite and positive \( m \)-measure. Then \( (E_j \times E_{j'})_{j,j' \in \mathbb{N}} \) is a partition of \( M^2 \) into Borel sets of finite and positive \( m_{\mathcal{O}_2} \)-measure. Let \( (F_k)_{k \in \mathbb{N}_{\geq 2}} \) be an enumeration of the latter sequence with the property that if \( F_k = E_j \times E_{j'} \), then \( j + j' \leq k \) (e.g. using Cantor’s diagonal procedure). We define the distance \( d_{\mathcal{O}_2} \) on \( L^0(M^2) \) w.r.t. this sequence \( (F_k)_{k \in \mathbb{N}_{\geq 2}} \).

If \( v \in \mathcal{A}_\infty \) and \( s \in \mathcal{N}_\infty \), we set

\[
a[s,v] := b[s,v].
\]
Next, let any \( v \in \mathcal{M}^0 \) and \( s \in \mathcal{N}^0 \) satisfy \(|v|, |s| \in L^\infty(M)\). By density of \( \mathcal{M}_o \) in \( \mathcal{M} \) and by the definition of \( \mathcal{M}_c \), there exists a sequence \((v_n)_{n \in \mathbb{N}} \) in \( \mathcal{M}_c \) converging to \( v \) w.r.t. \( d_{\mathcal{M}^0} \). Analogously, we extract a sequence \((s_i)_{i \in \mathbb{N}} \) in \( \mathcal{N}_c \) converging to \( s \) w.r.t. \( d_{\mathcal{N}^0} \). Define

\[
C := \max\{\|A\|_{\mathcal{M}_c, \mathcal{N}_c} + 1, ||v||_{L^\infty} + 1, ||s||_{L^\infty} + 1\}.
\]

Given any \( \varepsilon > 0 \), select \( L \in \mathbb{N} \) such that, for every \( n, n', i, i' \geq L \),

\[
\max\{d_{\mathcal{M}^0}(v_n, v_{n'}), d_{\mathcal{M}^0}(v, v_{n'}), d_{\mathcal{N}^0}(s_i, s_{i'}), d_{\mathcal{N}^0}(s_i, s)\} \leq \frac{\varepsilon}{6C^2}.
\]

Using the elementary fact that

\[
\min\{a + b, 1\} \leq \min\{a, 1\} + \min\{b, 1\},
\]

\[
\min\{ab, 1\} \leq \min\{a, 1\} + \min\{b, 1\}
\]

for every \( a, b \in [0, \infty) \) as well as (6.2) and (6.3) thus yields

\[
\sum_{k=2}^{\infty} \frac{2^{-k}}{m^{k/2}F_k} \int_{F_k} \min\{|a[s_i, v_n] - a[s_i', v_{n'}]|, 1\} \, dm \geq 2
\]

\[
\leq \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j'] \cdot m[E_j]} \int_{E_j \times E_j'} \min\{|a[s_i, v_n] - a[s_i', v_{n'}]|, 1\} \, dm \geq 2
\]

\[
\leq \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j'] \cdot m[E_j]} \int_{E_j \times E_j'} \min\{|a[s_i, v_n - v_{n'}]|, 1\} \, dm \geq 2
\]

\[
+ \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j'] \cdot m[E_j]} \int_{E_j \times E_j'} \min\{|a[s_i - s_i', v_{n'}]|, 1\} \, dm \geq 2
\]

\[
\leq C \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|s_i| |v_n - v_{n'}| (pr_1) \, |v_n - v_{n'}| (pr_2), 1\} \, dm \geq 2
\]

\[
+ C \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|s_i - s_i'| (pr_1) |v_n - v_{n'}| (pr_2), 1\} \, dm \geq 2
\]

\[
\leq C \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|s_i - s_i'|(pr_1) |v_n - v_{n'}| (pr_2), 1\} \, dm \geq 2
\]

\[
+ C^2 \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|v_n - v_{n'}|, 1\} \, dm \geq 2
\]

\[
+ C \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|s_i - s_i'| (pr_1) |v_n - v| (pr_2), 1\} \, dm \geq 2
\]

\[
+ C^2 \sum_{j,j'=1}^{\infty} \frac{2^{-j-j'}}{m[E_j] \cdot m[E_j']} \int_{E_j \times E_j'} \min\{|s_i - s_i'|, 1\} \, dm \leq \varepsilon.
\]

Thus \((b[s_i, v_n])_{n,i \in \mathbb{N}}\) is a Cauchy sequence in \( L^0(M^2) \) — we define \( a[s, v] \) as its \( m^{\otimes 2} \)-a.e. unique limit in \( L^0(M^2) \). A similar argument shows that this definition of \( a[s, v] \) is independent of the particularly chosen approximating sequences in \( \mathcal{M}_c \) and \( \mathcal{N}_c \), respectively. Moreover, the identities (6.2), for arbitrary \( c, d \in L^\infty(M) \), and (6.3) remain true for \( b \) replaced by \( a \).

Lastly, for arbitrary \( v \in \mathcal{M}^0 \) and \( s \in \mathcal{N}^0 \), the sequences \((v_n)_{n \in \mathbb{N}} \) and \((s_i)_{i \in \mathbb{N}} \) given by \( v_n := 1_{[a,n]}(|v|) \, v \) and \( s_i := 1_{[a_i]}(|s|) \, s \) converge to \( v \) and \( s \) in \( \mathcal{M}^0 \) and \( \mathcal{N}^0 \), respectively. Indeed, observe that \(|v - v_n| \to 0\) pointwise \( m \)-a.e. as \( n \to \infty \) and \(|s - s_i| \to 0\) pointwise \( m \)-a.e. as \( i \to \infty \), and the claim follows since pointwise
m-a.e. convergent sequences converge in measure on finite measure spaces. By (6.2) and (6.3), \((b[s, v_n])_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^0(M^2)\) — we again define \(a[s, v]\) as its \(m^\otimes\)-a.e. unique limit. Once again, it is easily seen that this definition does not depend on the chosen approximating sequences for \(v\) and \(s\), respectively, and that (6.2), for arbitrary \(c, d \in L^0(M)\), and (6.3) hold for \(a\) instead of \(b\).

**Step 4. Properties of \(a\).** From the previous step, we already know that \(a \in \mathcal{N}^0 \otimes \mathcal{M}^0\).

Moreover, from (6.3) for \(a\), it already follows that \(a[s, v] \in L^1(M^2)\) for every \(v \in \mathcal{M}\) and every \(s \in \mathcal{N}\), and that

\[
\|a\|_{L^\infty} \leq \|A\|_{\mathcal{M} \otimes \mathcal{N}^*}.
\]

To show the claimed integral identity, let \(z \in M\). For any sequences \((v_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}_0\) and \((s_i)_{i \in \mathbb{N}}\) in \(\mathcal{N}_0\) converging to \(v\) and \(s\) in \(\mathcal{M}\) and \(\mathcal{N}\), respectively, from (6.1) we get, for every \(n, i, k \in \mathbb{N}\),

\[
\langle s \mid A(1_{B_k(z)} v) \rangle = \int_{B_k(z)} a[s_i, v_n](z, y) \, dm(y) \quad \text{m-a.e.}
\]

Letting \(k \to \infty\) together with the continuity of \(A\) and then \(n \to \infty\) and \(i \to \infty\), employing that \(a[s_i, v_n] \to a[s, v]\) in \(L^1(M^2)\) by virtue of (6.3), the desired claim is deduced.

From this, the inequality

\[
\|a\|_{L^1} \geq \|A\|_{\mathcal{M} \otimes \mathcal{N}^*}
\]

simply follows by observing that, by definition of the pointwise norm in \(\mathcal{N}^*\),

\[
\|Ae\| \leq \int_M \text{esssup}\{a[s, v] : s \in \mathcal{N}^0, |s| \leq 1 \text{ m-a.e.}\} \, dm
\]

\[
\leq \|a\|_{L^\infty} \|v\|_{\mathcal{M}} \quad \text{m-a.e.}
\]

The uniqueness statement is clear by \(L^0\)-bilinearity of all considered mappings. \(\square\)

**Remark 6.4.** In the setting of Theorem 6.3, the general result [32, Thm. VI.8.6] would provide a map \(a\) on \(\mathcal{M}\), \(m\)-essentially uniquely determined in a proper way, such that \(a(y) : \mathcal{M} \to \mathcal{N}^*\) is linear for \(m\)-a.e. \(y \in M\) and, for every \(v \in \mathcal{M}\) and every \(s \in \mathcal{N}\), we have

\[
\int_M \langle s \mid A e \rangle \, dm = \int_M \int_M \langle s \mid a(y)v \rangle(x) \, dm(x) \, dm(y).
\]

However, it is not clear that the map \((x, y) \mapsto \langle s \mid a(y)v \rangle(x)\) is \(m^\otimes\)-measurable — a property which is implicitly used at many places in the proof of Theorem 6.3. Even in functional treatises, this is considered as a delicate detail [43, Ch. 3] and explains why we chose the formulation of Definition 6.1 with target space \(L^0(M^2)\).

**6.2. Explicit construction as integral kernel.** We are now in a position to introduce our main result. On weighted Riemannian manifolds with not necessarily uniform lower Ricci bounds, a version of it has been proven in [44, Thm. XI.1] using Lebesgue’s differentiation theorem and thus local compactness of the underlying space, an assumption we do not make. See also [42] for a thorough functional treatment.

**Theorem 6.5 (Heat kernel existence).** Let \((M, d, m)\) be an RCD\((K, \infty)\) space, \(K \in \mathbb{R}\). Then there exists a mapping \(h : (0, \infty) \to L^p(T^*M)^{\otimes 2}\) such that for all \(p, q \in\)
[1, \infty] \text{ with } 1/p + 1/q = 1, \text{ if } \omega \in L^p(T^*M) \text{ and } \eta \in L^q(T^*M), \text{ for every } t > 0 \text{ we have } h_t[\eta, \omega] \in L^1(M^2), \text{ and }

\langle \eta, H_t \omega \rangle = \int_M h_t[\eta, \omega](\cdot, y) \, dm(y) \text{ m-a.e.}

The previously mentioned mapping \( h \) is uniquely determined in the sense that for every mapping \( \tilde{h} \colon (0, \infty) \to L^0(T^*M)^{\mathbb{S}^2} \) satisfying the foregoing obstructions, for every \( \omega, \eta \in L^0(T^*M) \) and every \( t > 0 \), the identity \( h_t[\eta, \omega] = \tilde{h}_t[\eta, \omega] \) holds \( \mathfrak{m}^{\mathbb{S}^2} \)-a.e.

\textbf{Proof. Step 1. Kernel for a perturbation of } H_t. \text{ Let } t > 0. \text{ We define the weight } \phi_t \colon M \to \mathbb{R}, \text{ locally bounded by the volume growth property (2.1), and the operator } A_t \colon L^1_{bs}(T^*M) \to L^0(T^*M) \text{ as }

\phi_t(x) := m[B_{\sqrt{\nu}}(x)]^{1/2},

A_t := \phi_t H_t \phi_t.

By Theorem 3.15 and the functional heat kernel bound from Theorem 2.1, there exist constants \( C_1, C_2 < \infty \) such that for every \( \omega \in L^1_{bs}(T^*M) \),

\begin{align*}
|A_t \omega| & \leq e^{-Kt} \int_M \phi_t p_t(\cdot, y) \phi_t(y) |\omega|(y) \, dm(y) \\
& \leq e^{-Kt} e^{C_1(1+C_2t)} \|\omega\|_{L^1} \text{ m-a.e.}
\end{align*}

Therefore, \( A_t \) uniquely extends to a bounded and linear operator from \( L^1(T^*M) \) into \( L^\infty(T^*M) \), whose extension we still denote by \( A_t \).

Theorem 6.3 thus provides us with some element \( a_t \in L^0(T^*M)^{\mathbb{S}^2} \), uniquely determined in a proper way, such that for every \( \omega, \eta \in L^1(T^*M) \),

\begin{align*}
&\langle \eta, A_t \omega \rangle = \int_M a_t[\eta, \omega](\cdot, y) \, dm(y) \text{ m-a.e.}
\end{align*}

In fact, arguing as in the proof of Theorem 6.3, we obtain

\begin{align*}
|a_t|_\infty & \leq e^{-Kt} \phi_t(pr_1) \phi_t(pr_2) p_t \text{ m}\mathfrak{m}^{\mathbb{S}^2}\text{-a.e.} \quad (6.5)
\end{align*}

\textbf{Step 2. Removing the weights.} Given the element \( a_t \) extracted in the previous step and any \( \varepsilon, \iota > 0 \), we define \( h^{\varepsilon, \iota}_t \in L^0(T^*M)^{\mathbb{S}^2} \) through

\begin{equation}
\begin{aligned}
h^{\varepsilon, \iota}_t[\eta, \omega] & := a_t \left[ \frac{1}{\phi_t + \varepsilon} \eta, \frac{1}{\phi_t + \iota} \omega \right].
\end{aligned}
\end{equation}

It is clear from (6.5) and Theorem 2.1 that \( h^{\varepsilon, \iota}_t[\eta, \omega] \in L^1(M^2) \) for every \( \omega, \eta \in L^1(T^*M) \). Moreover, if in addition \( \omega \in L^1_{bs}(T^*M) \), then

\begin{align*}
\langle \eta, \frac{\phi_t}{\phi_t + \varepsilon} H_{t} \left[ \frac{\phi_t}{\phi_t + \iota} \omega \right] \rangle = \int_M h^{\varepsilon, \iota}_t[\eta, \omega](\cdot, y) \, dm(y) \text{ m-a.e.} \quad (6.6)
\end{align*}
Next, observe that for every $\omega, \eta \in L^0(T^*M)$ and every $\varepsilon', \iota' > 0$, by (6.5),

$$|h_t^{\varepsilon', \iota}[\eta, \omega] - h_t^{\varepsilon', \iota'}[\eta, \omega]|$$

$$\leq \left| a_t \left[ \frac{1}{\phi_t + \varepsilon} \eta, \frac{1}{\phi_t + \iota} \omega - \frac{1}{\phi_t + \iota'} \omega \right] \right|$$

$$+ \left| a_t \left[ \frac{1}{\phi_t + \varepsilon} \eta - \frac{1}{\phi_t + \iota'} \eta, \frac{1}{\phi_t + \iota} \omega \right] \right|$$

$$\leq e^{-Kt} \left| \eta \right| \left( \frac{1}{\phi_t} \right) \left| \omega \right| \left( \frac{1}{\phi_t} \right) \left( \frac{1}{\phi_t + \varepsilon} \right) \left( \frac{1}{\phi_t + \iota} \right)$$

$$\times \left[ \frac{\phi_t(\eta)}{\phi_t(\eta) + \varepsilon} - \frac{\phi_t(\eta)}{\phi_t(\eta) + \iota} \right] \frac{1}{\phi_t(\eta) + \iota}$$

Thus, independently of the choice of sequences $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(\iota_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ converging to 0 in place of $\varepsilon$ and $\iota$, the two-parameter family $(h_t^{\varepsilon', \iota}[\eta, \omega])_{\varepsilon', \iota > 0}$ has a unique limit in $L^0(M^2)$ — we define $h_t[\eta, \omega]$ to be this limit and denote by $h_t \in L^0(T^*M)^{\otimes 2}$ the induced element.

**Step 3. Properties of $h_t$.** Turning to the claimed integral representation of $H_t$, given any $\omega \in L^0(T^*M)$ and $\eta \in L^q(T^*M)$ where $p, q \in [1, \infty]$ are dual to each other, we integrate (6.6) and let $\varepsilon, \iota \to 0$. On the one hand, by Hölder’s inequality, **Theorem 4.1** and Lebesgue’s theorem,

$$\lim_{\varepsilon, \iota \to 0} \int_M \langle \eta, \frac{\phi_t(\eta)}{\phi_t + \varepsilon} H_t \left[ \frac{\phi_t(\eta)}{\phi_t + \iota} \omega \right] \rangle \, dm = \int_M \langle \eta, H_t \omega \rangle \, dm.$$

On the other hand, from the construction in the previous step,

$$|h_t|_{\infty} \leq e^{-Kt} \mathbb{P}_t \text{ m}^{\otimes 2} \text{-a.e.}$$

so that a further application of Lebesgue’s theorem to (6.6) entails

$$\int_M \langle \eta, H_t \omega \rangle \, dm = \int_M h_t[\eta, \omega] \, dm \otimes \mathbb{P}_t.$$

Replacing $\eta$ by $f \eta$ for arbitrary $f \in L^\infty(M)$ finally gives the claimed pointwise $\mathbb{P}_t$-a.e. equality.

The uniqueness statement is as clear as in the proof of **Theorem 6.3.**

**Definition 6.6.** We call the mapping $h$ from **Theorem 6.5** the 1-form heat kernel of $(M, d, m)$.

It is straightforward to check the following result using the symmetry and the semigroup property of $(H_t)_{t \geq 0}$, the Chapman–Kolmogorov formula for the functional heat kernel [3, Thm. 6.1] as well as **Theorem 3.15**, **Theorem 6.9** then follows from **Theorem 2.1** and (2.7).

**Theorem 6.7.** For every $\omega, \eta \in L^0(T^*M)$ and every $s, t > 0$, the 1-form heat kernel $h_s[\eta, \omega] \in L^0(T^*M)$ obeys the following relations at $\mathbb{P}_t$-a.e. $(x, y) \in M^2$.

1. **Symmetry.** We have

$$h_t[\eta, \omega](x, y) = h_t[\omega, \eta](y, x)$$

2. **Pointwise Hess–Schrader–Uhlenbrock Inequality.** We have

$$|h_t|_{\infty}(x, y) \leq e^{-Kt} \mathbb{P}_t(x, y)$$

3. **Chapman–Kolmogorov Equation.** We have

$$\int_M h_{t+s}[\eta, \omega](\cdot, y) \, dm(y) = \int_M h_s[\eta, H_t \omega](\cdot, y) \, dm(y) \text{ m-a.e.}$$
Remark 6.8. Since \((H_j)_{j \geq 0}\) does not localize, we cannot state Chapman–Kolmogorov’s formula from (iii) in Theorem 6.7 as a pointwise \(m^{\otimes 2}\)-a.e. equality in the previous sense.

However, if \((M,d,m)\) is an \(\text{RCD}^*(K,N)\) space, \(N \in (1, \infty)\), this can be circumvented by using the \textit{dimensional decomposition} of \(L^2(T^*M)\) [40, Prop. 1.4.5]. In this case (see Section 6.3 for details), there exists a unique \(n \in \mathbb{N}\) as well as a Borel set \(E_n \subseteq M\) with \(m[M \setminus E_n] = 0\) such that for every Borel set \(B \subseteq E_n\) with \(m[B] < \infty\), there exist local basis vectors \(\rho_1, \ldots, \rho_n \in L^2(T^*M)\) [40, Sec. 1.4] such that for every \(i,j \in \{1, \ldots, n\}\), we have

\[
1_B \cdot \rho_i = 0,
\]

\[
\langle \rho_i, \rho_j \rangle = \delta_{ij} \quad \text{m-a.e. on } B.
\]

In this framework, given any \(\omega, \eta \in L^0(T^*M)\), we claim that for every \(s, t > 0\),

\[
h_{t+s}[\eta, \omega] = \sum_{i=1}^n \int_B h_s[\eta, \rho_i](pr_1, z) h_t[\rho_i, \omega](z, pr_2) \, dm(z) \quad m^{\otimes 2}\text{-a.e. on } B^2.
\]

Indeed, let \(z \in B \) and \(R > 0\) be arbitrary, and set

\[
\omega_R := 1_B(z) \quad B|0, R| \langle \omega \rangle | \omega,
\]

\[
\eta_R := 1_B(z) \quad B|0, R| \langle \eta \rangle | \eta.
\]

By Theorem 6.5 and Theorem 6.7 applied to \(1_B \omega_R\) and \(1_B \eta_R\) in place of \(\omega\) and \(\eta\), we obtain

\[
\int_{B^2} h_{t+s}[\eta_R, \omega_R] \, dm^{\otimes 2} = \sum_{i=1}^n \int_B h_s[\eta_R, \rho_i](pr_1, z) h_t[\rho_i, \omega_R](z, pr_2) \, dm \tag{44}
\]

The integrands on both sides of this chain of equalities are local in their respective components, and the claim follows by letting \(R \to \infty\).

Theorem 6.9. Let \((M,d,m)\) be an \(\text{RCD}(K, \infty)\) space, and let \(\varepsilon > 0\). Then there exist finite constants \(C_1 > 0\), depending only on \(\varepsilon\), and \(C_2 \geq 0\), depending only on \(K\), with \(C_2 := 0\) if \(K \geq 0\), such that for \(m^{\otimes 2}\)-a.e. \((x, y) \in M^2\), we have

\[
|h_t|_\otimes(x, y) \leq m\left[B_{\sqrt{t}}(x)\right]^{-1/2} m\left[B_{\sqrt{t}}(y)\right]^{-1/2} \exp \left( C_1 (1 + (C_2 - K)t - \frac{d^2(x,y)}{(4 + \varepsilon)t} \right).
\]

In particular, if \((M,d,m)\) obeys the \(\text{RCD}^*(K, N)\) condition with \(N \in (1, \infty)\), there exist finite constants \(C_3, C_4 > 1\) depending only on \(\varepsilon\), \(K\) and \(N\) such that at \(m^{\otimes 2}\)-a.e. \((x, y) \in M^2\), we have

\[
|h_t|_\otimes(x, y) \leq C_3 m[B_{\sqrt{t}}(y)]^{-1} \exp \left( (C_4 - K)t - \frac{d^2(x,y)}{(4 + \varepsilon)t} \right).
\]
6.3. Trace inequality and spectral resolution. Let $(M, d, m)$ be an RCD$^*(K, N)$ space for some $N \in (1, \infty)$. First, relying on Theorem 6.7 and Remark 6.8, we prove a trace inequality between $H_t$ and $P_t$ in Theorem 6.10. In the smooth case, the corresponding bound is classical, see e.g. [44, Cor. XI.8], (1.1) in [50] or [72, Thm. 3.5]. A key feature of the RCD$^*(K, N)$ framework is that, by [38, 45], there exists precisely one $n \in N$ such that $m[E_n] > 0$ within the dimensional decomposition $(E_n)_{n \in N}$ of $L^2(T^*M)$ — actually, $n$ is equal to the essential dimension $\dim_{d, m} M \in \{1, \ldots, |N|\}$ of $(M, d, m)$. See [15, 38, 66] for comprehensive accounts on the latter from the structure theoretic point of view.

Via Theorem 5.13, if $M$ is additionally compact, we also prove a spectral resolution identity for $h_t$, $t > 0$, in Theorem 6.11. More precisely, we show that $h_t$ can be viewed as an element in the two-fold Hilbert space tensor product $L^2(T^*M)^\otimes 2$ of $L^2(T^*M)$ indicated in Section 2.2.

**Theorem 6.10.** Let $(M, d, m)$ be an RCD$^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Introducing the traces in the usual Hilbert space sense, for every $t > 0$,

$$\operatorname{tr} H_t \leq (\dim_{d, m} M) e^{-Kt} \operatorname{tr} P_t.$$

**Proof.** Abbreviate $d := \dim_{d, m} M$ and let $B \subset E_d$ be any bounded Borel set with $m[B] \in (0, \infty)$. Let $\rho_1, \ldots, \rho_d \in L^2(T^*M)$ be local basis vectors of $L^2(T^*M)$ on $B$ as in Remark 6.8. We set $\omega_k := m[B]^{-1/2} \rho_k$ with $k \in \{1, \ldots, n\}$ and complete this set of $1$-forms to a countable orthonormal basis $(\omega_k)_{k \in N}$ of $L^2(T^*M)$ — we may and will assume that $1_B (\omega_k, \rho_i) = 0$ for all $k > d$ and $i \in \{1, \ldots, d\}$. Given any $\alpha \in N$ with $\alpha > d$, by Theorem 6.7 and Cauchy–Schwarz’s inequality,

$$\sum_{k=1}^\alpha \int_M |H_t \omega_k, \omega_k| \, dm$$

$$= \sum_{k=1}^\alpha \sum_{i, i'=1}^d \int_{B^2} \langle \omega_k, \rho_i \rangle (pr_1) \langle \omega_k, \rho_{i'} \rangle (pr_2) h_t [\rho_i, \rho_{i'}] \, dm^\otimes 2$$

$$= m[B]^{-1} \sum_{i=1}^d \int_{B^2} h_t [\rho_i, \rho_i] \, dm^\otimes 2$$

$$= m[B]^{-1} \sum_{i=1}^d \int_{B^2} h_{t/2} [\rho_i, H_{t/2} \rho_i] \, dm^\otimes 2$$

$$\leq m[B]^{-1} \sum_{i=1}^d \int_{B^2} |h_{t/2} | \otimes |H_{t/2} \rho_i| (pr_2) \, dm^\otimes 2$$

$$\leq m[B]^{-1} d \int_{B^2} |h_{t/2} | \otimes (pr_3, pr_3) |h_{t/2} | \otimes (pr_3, pr_3) \, dm^\otimes 3 \quad (6.7)$$

$$\leq d \int_{B^2} |h_{t/2} |^2 \, dm^\otimes 2 \leq d e^{-Kt} \int_{M^2} p_{t/2}^2 \, dm^\otimes 2 = d e^{-Kt} \operatorname{tr} P_t.$$

In (6.7), we used Theorem 6.5 together with duality for the pointwise norm, see Section 2.2. The last identity follows from the self-adjointness of the functional heat flow in $L^2(M)$.

The asserted inequality then follows by letting $\alpha \to \infty$. \hfill \Box

Now, let $M$ be compact. Let $(\omega_i)_{i \in N}$ be an orthonormal basis of $L^2(T^*M)$ consisting of eigenforms for $\Delta$ provided by Theorem 5.13, i.e. $\omega_i \in E_{\lambda_i}(\Delta)$ for some $\lambda_i \in \sigma(\Delta)$ and every $i \in N$. 


Theorem 6.11. Let \((M, \mathfrak{d}, \mathfrak{m})\) be a compact \(\text{RCD}^*(K, N)\) space, with \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). For every \(t > 0\), there exists a unique element \(g_t \in L^2(T^* M) \otimes 2\) such that, for every \(\omega, \eta \in L^2(T^* M)\),

\[
(g_t, \eta \otimes \omega) = h_t[\eta, \omega] \quad \mathfrak{m}^{\otimes 2}\text{-a.e.}
\]

Moreover, w.r.t. strong convergence in \(L^2(T^* M) \otimes 2\), \(g_t\) admits the representation

\[
g_t = \sum_{i=1}^\infty e^{-\lambda_i t} \omega_i \otimes \omega_i.
\]

Proof. By virtue of (2.5) and Theorem 6.9, we have \(|h_t|_\otimes \in L^\infty(M^2)\). Hence, the bilinear map \(G_t \colon L^2(T^* M)^2 \to \mathbb{R}\) is well-defined, where

\[
G_t(\eta, \omega) := \int_{M^2} h_t[\eta, \omega] \, \mathfrak{m}^{\otimes 2}.
\]

Moreover, \(G_t\) is weakly Hilbert–Schmidt by Theorem 6.5 and Corollary 5.7. Therefore, by Theorem 2.7 there exists a unique bounded \(T_t \colon L^2(T^* M) \otimes 2 \to \mathbb{R}\) such that \(G_t(\eta, \omega) = T_t(\eta \otimes \omega)\) for every \(\omega, \eta \in L^2(T^* M)\). Recalling Proposition 2.8, by Riesz’ theorem for Hilbert modules [40, Thm. 1.2.24] there exists a unique \(g_t \in L^2(T^* M) \otimes 2\) such that for every \(\omega, \eta \in L^2(T^* M),\)

\[
\int_{M^2} (g_t, \eta \otimes \omega) \, \mathfrak{m}^{\otimes 2} = T_t(\eta \otimes \omega) = G_t(\eta, \omega) = \int_{M^2} h_t[\eta, \omega] \, \mathfrak{m}^{\otimes 2}.
\]

Replacing \(\omega\) and \(\eta\) by \(f \omega\) and \(g \eta\) for arbitrary \(f, g \in L^\infty(M)\), respectively, provides the claimed \(\mathfrak{m}^{\otimes 2}\text{-a.e. valid identity \((g_t, \eta \otimes \omega) = h_t[\eta, \omega]\).}

It remains to prove the series representation of \(g_t\). Since \((\omega_i \otimes \omega_j)_{i,j \in \mathbb{N}}\) is an orthonormal basis of \(L^2(T^* M) \otimes 2\), this simply follows by writing

\[
g_t = \sum_{i,j=1}^\infty c_{ij}(t) \omega_i \otimes \omega_j
\]

w.r.t. strong convergence in \(L^2(T^* M) \otimes 2\), where the coefficients are given by

\[
c_{ij}(t) = \int_{M^2} \langle g_t, \omega_i \otimes \omega_j \rangle \, \mathfrak{m}^{\otimes 2} = \int_M \langle \omega_i, H_t \omega_j \rangle \, \mathfrak{m} = e^{-\lambda_i t} \delta_{ij}. \quad \square
\]

Remark 6.12. By Theorem 6.11 and [73, Prop. 3.1], our notion of the 1-form heat kernel from Definition 6.6 is fully compatible with the so-called parametrix approach to it on compact, non-weighted Riemannian manifolds [68, Ch. 4]. More precisely, denoting the smooth heat kernel by \(h \colon (0, \infty) \times M^2 \to (T^* M)^* \otimes T^* M\) as in Chapter 1 by a slight abuse of notation, for every smooth 1-forms \(\omega\) and \(\eta\), we have \(g_t(\eta \otimes \omega)(x, y) = \langle \eta(x), h_t(x, y) \omega(y) \rangle\) for \(\mathfrak{m}^{\otimes 2}\text{-a.e. \((x, y) \in M^2\).}

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