BOUND OF AUTOMORPHISMS OF PROJECTIVE VARIETIES OF GENERAL TYPE

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Abstract

We prove that there exists a positive integer $C_n$ depending only on $n$ such that for every smooth projective $n$-fold of general type $X$ defined over $\mathbb{C}$, the automorphism group $\text{Aut}(X)$ of $X$ satisfies

$$\#\text{Aut}(X) \leq C_n \cdot \mu(X, K_X),$$

where $\mu(X, K_X)$ is the volume of $X$ with respect to $K_X$. MSC14E05,32J25.

1 Introduction

The automorphism group of a projective variety of general type is known to be finite. For every curve $C$ of genus $g \geq 2$, we have the estimate:

$$\#\text{Aut}(C) \leq 84(g - 1)$$

by well known Hurwitz’s theorem.

In the case of surfaces, G. Xiao proved that for every smooth minimal surface of general type

$$\#\text{Aut}(S) \leq 1764 \cdot K_S^2$$

holds \[17\]. The main purpose of this article is to prove the following theorem.
Theorem 1.1  There exists a positive number $C_n$ which depends only on $n$ such that for every smooth projective $n$-fold $X$ of general type defined over complex numbers, the automorphism group $\text{Aut}(X)$ of $X$ satisfies the estimate:

$$\sharp \text{Aut}(X) \leq C_n \cdot \mu(X, K_X),$$

where $\mu(X, K_X)$ is the volume of $X$ with respect to $K_X$ (cf. Definition 2.3).

The method of the proof of Theorem 1.1 is a combination of the ideas in \cite{17, 18} and \cite{15}. Let $X$ be a projective $n$-fold of general type and let $G$ denote the automorphism group of $X$. Since $G$ acts on the canonical ring $R(X, K_X)$ of $X$, by \cite{14} we may assume that $X$ is a canonical model, i.e. $X$ has only canonical singularity and $K_X$ is ample (our proofs of Theorem 1.1 and Theorem 1.2 below depend on the finite generation of canonical rings of varieties of general type in \cite{14} which has not yet published. For the safe side, one may restrict oneself to the case of $\dim X \leq 3$ (cf. \cite{8})) The quotient $X/G$ is a projective variety. Let $K_{X/G, orb}$ be the orbifold canonical divisor of $X/G$. Then we see that

$$| G | = \frac{K^n_X}{K^n_{X/G, orb}}$$

holds, where $| G |$ denotes the order of $G$. Since $\mu(X, K_X) = K^n_X$ holds in this case, we see that Theorem 1.1 follows from the following theorem.

Theorem 1.2  Let $X, G$ be as above. There exists a positive constant $c_n$ depending only on $n$ such that

$$K^n_{X/G, orb} \geq c_n$$

holds.

It is easy to see $c_1$ can be taken to be $1/42$. This leads to Hurwicz’s theorem. G. Xiao proved that $c_2$ can be taken as $1/1764$ (\cite{17, 18}).

The key ingredient of the proof of Theorem 1.2 is the subadjunction formula in \cite{6} which relates the canonical divisor of the minimal center of log-canonical singularities and the canonical divisor of the ambient space. Using this we see that $X/G$ with $\mu(X/G, K_{X/G, orb}) = K^n_{X/G, orb} \leq 1$ is birationally bounded by the inductive procedure in \cite{13}. Then Theorem 1.1 and Theorem 1.2 follows from a Diophantine consideration.

Theorem 1.1 and Theorem 1.2 are not effective in the sense that there exist no explicit estimates of $C_n$ and $c_n$. 

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2 Preliminaries

2.1 Orbifold canonical divisors

Let $X$ be a projective variety of general type with only canonical singularities. Let $G$ denote the automorphism group of $X$. It is well known that $G$ is a finite group. The quotient $X/G$ is a projective variety. Let $\tilde{X}$ be the equivalent resolution of $X$ with respect to $G$ such that $\tilde{X}/G$ is also smooth. We may take $\tilde{X}$ such that the ramification divisor $R$ of

$$\tilde{\pi} : \tilde{X} \longrightarrow \tilde{X}/G$$

and the branch locus $B = (\tilde{\pi}_*(R))_{\text{red}}$ is a divisor with normal crossings. Let $B = \sum_i B_i$ be the irreducible decomposition of $B$. Then there exists a set of positive integers $m_i$ such that

$$K_{\tilde{X}} = \tilde{\pi}^*(K_{\tilde{X}/G} + \sum_i \frac{m_i - 1}{m_i} B_i)$$

Let

$$\varpi : \tilde{X}/G \longrightarrow X/G$$

be the natural morphism. We set

$$K_{X/G,\text{orb}} := \varpi_*(K_{\tilde{X}/G} + \sum_i \frac{m_i - 1}{m_i} B_i)$$

and call it the orbifold canonical divisor of $X/G$. Let

$$\pi : X \longrightarrow X/G$$

be the natural morphism. Then

$$K_X = \pi^*K_{X/G,\text{orb}}$$

holds. The orbifold canonical ring is defined by

$$R(X/G, K_{X/G,\text{orb}}) := R(X, K_X)^G.$$ 

And the linear system $| mK_{X/G,\text{orb}} |$ is given by

$$| mK_{X/G,\text{orb}} | = | mK_X |^G.$$ 

Hence we have that

$$R(X/G, K_{X/G,\text{orb}}) = \oplus_{m \geq 0} \Gamma(X/G, \mathcal{O}_{X/G}([mK_{X/G,\text{orb}}]))$$

holds.
2.2 Multiplier ideal sheaves

In this section, we shall review the basic definitions and properties of multiplier ideal sheaves.

**Definition 2.1** Let $L$ be a line bundle on a complex manifold $M$. A singular hermitian metric $h$ is given by

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$-hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(M)$ is an arbitrary function on $M$.

The curvature current $\Theta_h$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\partial \bar{\partial}$ is taken in the sense of a current. The $L^2$-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\mathcal{L}^2(L, h) := \{ \sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{\text{loc}}(U) \},$$

where $U$ runs opens subsets of $M$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the multiplier ideal sheaf of $(L, h)$. If we write $h$ as

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. We have the following vanishing theorem.

**Theorem 2.1** (Nadel’s vanishing theorem [9, p.561]) Let $(L, h)$ be a singular hermitian line bundle on a compact Kähler manifold $M$ and let $\omega$ be a Kähler form on $M$. Suppose that $\Theta_h$ is strictly positive, i.e., there exists a positive constant $\varepsilon$ such that

$$\Theta_h \geq \varepsilon \omega$$

holds. Then $\mathcal{I}(h)$ is a coherent sheaf of $\mathcal{O}_M$-ideal and for every $q \geq 1$

$$H^q(M, \mathcal{O}_M(K_M + L) \otimes \mathcal{I}(h)) = 0$$

holds.
2.3 Analytic Zariski decomposition

To study a big line bundle we introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like a nef and big line bundles.

**Definition 2.2** Let $M$ be a compact complex manifold and let $L$ be a line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an analytic Zariski decomposition, if the followings hold.

1. $\Theta_h$ is a closed positive current,
2. for every $m \geq 0$, the natural inclusion

   $$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(M, \mathcal{O}_M(mL))$$

   is isomorphism.

**Remark 2.1** If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above.

**Theorem 2.2** ([11, 12]) Let $L$ be a big line bundle on a smooth projective variety $M$. Then $L$ has an AZD.

2.4 Volume of projective varieties

To measure the positivity of big line bundles on a projective variety we shall introduce a volume of a projective variety with respect to a line bundle.

**Definition 2.3** Let $L$ be a line bundle on a compact complex manifold $M$ of dimension $n$. We define the $L$-volume of $M$ by

$$\mu(M, L) := n! \cdot \limsup_{m \to \infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)).$$

**Definition 2.4** ([14]) Let $L$ be a big line bundle on a smooth projective variety $X$. Let $Y$ be a subvariety of $X$ of dimension $r$. We define the volume $\mu(Y, L)$ of $Y$ with respect to $L$ by

$$\mu(Y, L) := r! \cdot \limsup_{m \to \infty} m^{-r} \dim H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)/\text{tor}),$$

where $h$ is an AZD of $L$ and $\text{tor}$ denotes the torsion part of the sheaf $\mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)$. This definition can be easily generalized to the case that $L$ is a $\mathbb{Q}$-line bundle.
3 Stratification of varieties by multiplier ideal sheaves

Let $X$ be a smooth projective $n$-fold of general type. Then the canonical ring $R(X, K_X)$ is finitely generated by $[12]$. Let $X_{can}$ be the canonical model of $X$. $K_{X_{can}}$ is an ample $\mathbb{Q}$-Cartier divisor on $X_{can}$. We assume that the natural rational map

$$\varphi : X \dasharrow X_{can}$$

is a morphism. Let $h_{can}$ be a $C^\infty$-hermitian metric on $K_{X_{can}}$ induced from the Fubini-Study metric on the hyperplane bundle of a projective space by a projective embedding of $X_{can}$ associated with $| rK_{X_{can}} |$ where $r$ is a sufficiently large positive integer such that $rK_{X_{can}}$ is Cartier. Then $h_{can}$ has strictly positive curvature on $X_{can}$. $h_{can}$ induces a singular hermitian metric $h$ on $K_X$ in a natural manner. By the definition, $h$ is an AZD of $K_X$. To prove Theorem 1.1, we may replace $X$ by any birational model of $X$, we may assume that there exists an effective $\mathbb{Q}$-divisor $N$ such that $I(h^m) = \mathcal{O}_X([-mN])$ holds for every $m \geq 0$. In particular we may and do assume that $I(h^m)$ is locally free for every $m \geq 0$. Let us denote $\mu(X/G, K_{X/G, orb})$ by $\mu_0$. We set

$$X^\circ = \{ x \in X \mid \varphi \text{ is a local isomorphism around } x \}.$$

Let $G$ be the group of the birational automorphism of $X$. To prove Theorem 1.1, we may assume that $G$ acts $X$ regularly and $X/G$ is also smooth. Let

$$\pi : X \longrightarrow X/G$$

be the natural morphism. We set

$$(X/G)^\circ = \pi(X^\circ).$$

Lemma 3.1 Let $x, y$ be distinct points on $(X/G)^\circ$. We set

$$\mathcal{M}_{x,y} = \mathcal{M}_x \otimes \mathcal{M}_y$$

Let $\varepsilon$ be a sufficiently small positive number. Then

$$H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G, orb}) \otimes \mathcal{M}_{x,y}^{[\sqrt[\frac{1}{m}] (1-\varepsilon) \frac{m}{m^2}]}) \neq 0$$

for every sufficiently large $m$, where $\mathcal{M}_x, \mathcal{M}_y$ denote the maximal ideal sheaf of the points $x, y$ respectively.
Proof of Lemma 3.1. Let us consider the exact sequence:

\[ 0 \to H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G,orb}) \otimes \mathcal{M}_{x,y}^{\left[ \frac{\psi_{m_0}(1-\varepsilon)}{\sqrt{2}} \right]} ) \to H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G,orb})) \to H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G,orb})/\mathcal{M}_{x,y}^{\left[ \frac{\psi_{m_0}(1-\varepsilon)}{\sqrt{2}} \right]}). \]

Since \( n! \limsup_{m \to \infty} m^{-n} \dim H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G,orb})/\mathcal{M}_{x,y}^{\left[ \frac{\psi_{m_0}(1-\varepsilon)}{\sqrt{2}} \right]} ) = \mu_0(1-\varepsilon)^n < \mu_0 \)
hold, we see that Lemma 3.1 holds. Q.E.D.

Let us take a sufficiently large positive integer \( m_0 \) and let \( \sigma \) be a general (nonzero) element of \( H^0(X/G, \mathcal{O}_{X/G}(m_0K_{X/G,orb}) \otimes \mathcal{M}_{x,y}^{\left[ \frac{\psi_{m_0}(1-\varepsilon)}{\sqrt{2}} \right]} ) \). We define a singular hermitian metric \( h_0 \) on \( K_{X/G,orb} \) by

\[ h_0(\tau, \tau) := \frac{|\tau|^2}{|\sigma|^{2/m_0}}. \]

Then \( \Theta_{h_0} = \frac{2\pi}{m_0}(\sigma) \) holds, where \( (\sigma) \) denotes the closed positive current defined by the divisor \( (\sigma) \). Hence \( \Theta_{h_0} \) is a closed positive current. Let \( \alpha \) be a positive number and let \( \mathcal{I}(\alpha) \) denote the multiplier ideal sheaf of \( h_0^\alpha \), i.e.,

\[ \mathcal{I}(\alpha) = \mathcal{L}^2(\mathcal{O}_{X/G}, (\frac{h_0}{h_{X/G}})^\alpha), \]

where \( h_{X/G} \) is an arbitrary \( C^\infty \)-hermitian metric on \( K_{X/G,orb} \). Let us define a positive number \( \alpha_0(= \alpha_0(x, y)) \) by

\[ \alpha_0 := \inf\{ \alpha > 0 \mid (\mathcal{O}_{X/G}/\mathcal{I}(\alpha))_x \neq 0 \text{ and } (\mathcal{O}_{X/G}/\mathcal{I}(\alpha))_y \neq 0 \}. \]

Since \( (\sum_{i=1}^n |z_i|^2)^{-n} \) is not locally integrable around \( O \in \mathbb{C}^n \), by the construction of \( h_0 \), we see that

\[ \alpha_0 \leq \frac{n \sqrt{2}}{\sqrt{\mu_0(1-\varepsilon)}} \]

holds. Then one of the following two cases occurs.
Case 1.1: For every small positive number $\delta$, $\mathcal{O}_{X/G}/\mathcal{I}(\alpha_0 - \delta)$ has 0-stalk at both $x$ and $y$.

Case 1.2: For every small positive number $\delta$, $\mathcal{O}_{X/G}/\mathcal{I}(\alpha_0 - \delta)$ has nonzero-stalk at one of $x$ or $y$ say $y$.

First we consider Case 1.1. Let $\delta$ be a sufficiently small positive number and let $V_1$ be the germ of subscheme at $x$ defined by the ideal sheaf $\mathcal{I}(\alpha_0 + \delta)$. By the coherence of $\mathcal{I}(\alpha)(\alpha > 0)$, we see that if we take $\delta$ sufficiently small, then $V_1$ is independent of $\delta$. It is also easy to verify that $V_1$ is reduced if we take $\delta$ sufficiently small. In fact if we take a log resolution of $(X/G, \sigma)$, $V_1$ is the image of the divisor with discrepancy $-1$ (for example cf. [4, p.207]). Let $(X/G)_1$ be a subvariety of $X/G$ which defines a branch of $V_1$ at $x$. We consider the following two cases.

Case 2.1: $(X/G)_1$ passes through both $x$ and $y$,

Case 2.2: Otherwise

For the first we consider Case 2.1. Suppose that $(X/G)_1$ is not isolated at $x$. Let $n_1$ denote the dimension of $(X/G)_1$. Let us define the volume $\mu_1$ of $(X/G)_1$ with respect to $K_{X/G,\text{orb}}$ by

$$\mu_1 := \mu((X/G)_1, K_{X/G,\text{orb}}).$$

Since $x \in X/G^o$, we see that $\mu_1 > 0$ holds.

Lemma 3.2 Let $\varepsilon$ be a sufficiently small positive number and let $x_1, x_2$ be distinct regular points on $(X/G)_1 \cap X/G^o$. Then for a sufficiently large $m > 1$ divisible by $|G|$, $H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(mK_{X/G,\text{orb}}) \otimes \mathcal{I}(h^m) \otimes \mathcal{M}_{x_1,x_2}[^{n_1\mu_1/(1-\varepsilon)\cdot m}]) \neq 0$

holds.

The proof of Lemma 3.2 is identical as that of Lemma 3.1, since

$$\mathcal{I}(h^m)_{x_i} = \mathcal{O}_{X/G,x_i}(i = 1, 2)$$

hold for every $m$. 

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By Kodaira’s lemma there is an effective $\mathbb{Q}$-divisor $E$ such that $K_{X/G,\text{orb}} - E$ is ample. Let $\ell$ be a sufficiently large positive integer such that

$$L := \ell(K_{X/G,\text{orb}} - E)$$

is a line bundle and satisfies the property in Lemma 3.3.

**Lemma 3.3** If we take $\ell$ sufficiently large, then

$$\phi_m : H^0(X/G, \mathcal{O}_{X/G}(mK_{X/G,\text{orb}} + L) \otimes \mathcal{I}(h^m)) \to H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(mK_{X/G,\text{orb}} + L) \otimes \mathcal{I}(h^m))$$

is surjective for every $m \geq 0$ divisible by $|G|$.  

**Proof.** Let us take a locally free resolution of the ideal sheaf $\mathcal{I}_{(X/G)_1}$ of $(X/G)_1$:

$$0 \leftarrow \mathcal{I}_{(X/G)_1} \leftarrow \mathcal{E}_1 \leftarrow \mathcal{E}_2 \leftarrow \cdots \leftarrow \mathcal{E}_k \leftarrow 0.$$  

Then by the trivial extension of the case of vector bundles, if $r$ is sufficiently large, we see that

$$H^q(X/G, \mathcal{O}_{X/G}(mK_{X/G,\text{orb}} + L) \otimes \mathcal{I}(h^m) \otimes \mathcal{E}_j) = 0$$

holds for every $m \geq 1$, $q \geq 1$ and $1 \leq j \leq k$. In fact if we take $\ell$ sufficiently large, we see that for every $j$, $\mathcal{O}_{X/G}(L - K_{X/G}) \otimes \mathcal{E}_j$ admits a $C^\infty$-hermitian metric $g_j$ such that

$$\Theta_{g_j} \geq \text{Id}_{\mathcal{E}_j} \otimes \omega$$

holds, where $\omega$ is a Kähler form on $X/G$. By [2, Theorem 4.1.2 and Lemma 4.2.2] we have the desired vanishing.

Hence

$$H^1(X/G, \mathcal{O}_{X/G}(mK_{X/G,\text{orb}} + L) \otimes \mathcal{I}(h^m) \otimes \mathcal{I}_{(X/G)_1}) = 0$$

holds. This completes the proof of Lemma 3.3. Q.E.D.

Let $\tau$ be a general section in $H^0(X/G, \mathcal{O}_{X/G}(L))$.

Let $m_1$ be a sufficiently large positive integer divisible by $|G|$ and let $\sigma'_1$ be a general element of

$$H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(m_1K_{X/G,\text{orb}}) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}_{x_1, x_2}^{\left\lceil \frac{n_1 m_1 (1 - \varepsilon) m_1}{n_2} \right\rceil}),$$

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where \( x_1, x_2 \in (X/G)_1 \) are distinct nonsingular points on \((X/G)_1\).

By Lemma 3.2, we may assume that \( \sigma'_1 \) is nonzero. Then by Lemma 3.3 we see that
\[
\sigma'_1 \otimes \tau \in H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(m_1 K_{X/G,orb} + L) \otimes \mathcal{I}(h^{m_1}) \otimes M_{x_1, x_2}^{[\frac{n_1 \mu_1 (1 - \varepsilon \sqrt{2})}{\sqrt{2} \sqrt{2}}]})
\]
extends to a section
\[
\sigma_1 \in H^0(X/G, \mathcal{O}_{X/G}(m + \ell) K_{X/G,orb} \otimes \mathcal{I}(h^{m+\ell}))
\]
We may assume that there exists a neighbourhood \( U_{x,y} \) of \( \{x, y\} \) such that the divisor \((\sigma_1)\) is smooth on \( U_{x,y} - (X/G)_1 \) by Bertini’s theorem, if we take \( \ell \) sufficiently large, since by Theorem 2.1,
\[
H^0(X/G, \mathcal{O}_{X/G}(m K_{X/G,orb} + L) \otimes \mathcal{I}(h^m)) \to H^0(X/G, \mathcal{O}_{X/G}(m K_{X/G,orb} + L) \otimes \mathcal{I}(h^m))/\mathcal{O}_{X/G}(- (X/G)_1) \cdot \mathcal{M}_y)
\]
is surjective for every \( y \in X/G \) and \( m \geq 0 \) divisible by \( |G| \), where \( \mathcal{O}_{X/G}(- (X/G)_1) \) is the ideal sheaf of \((X/G)_1\). We define a singular hermitian metric \( h_1 \) on \( K_{X/G,orb} \) by
\[
h_1 = \frac{1}{|\sigma_1|^{\frac{1}{m_1 + \ell}}}.
\]
Let \( \varepsilon_0 \) be a sufficiently small positive number and let \( \mathcal{I}_1(\alpha) \) be the multiplier ideal sheaf of \( h_0^{\alpha_0 - \varepsilon_0} \cdot h_1^{\alpha} \), i.e.,
\[
\mathcal{I}_1(\alpha) = \mathcal{L}^2(\mathcal{O}_{X/G}, h_0^{\alpha_0 - \varepsilon_0} h_1^{\alpha} / h_1^{\alpha_0 + \varepsilon_0}).
\]
Suppose that \( x, y \) are nonsingular points on \((X/G)_1\). Then we set \( x_1 = x, x_2 = y \) and define \( \alpha_1 := \alpha_1(x, y) > 0 \) by
\[
\alpha_1 := \inf \{ \alpha \mid (\mathcal{O}_{X/G}/\mathcal{I}_1(\alpha))_x \neq 0 \text{ and } (\mathcal{O}_{X/G}/\mathcal{I}_1(\alpha))_y \neq 0 \}.
\]
By Lemma 3.3 we may assume that we have taken \( m_1 \) so that
\[
\frac{\ell}{m_1} \leq \varepsilon_0 \frac{n \sqrt{\mu_1}}{n_1 \sqrt{2}}
\]
holds.

Lemma 3.4
\[
\alpha_1 \leq n_1 \frac{n_1 \sqrt{2}}{\sqrt{\mu_1}} + O(\varepsilon_0)
\]
holds.
To prove Lemma 3.4, we need the following elementary lemma.

**Lemma 3.5** ([14], p.12, Lemma 6]) Let $a, b$ be positive numbers. Then

$$
\int_0^1 \frac{r_2^{2n_1-1}}{(r_1^2 + r_2^{2a})^b} dr_2 = r_1^{-2a} \int_0^1 \frac{r_3^{2n_1-1}}{(1 + r_3^{2a})^b} dr_3
$$

holds, where

$$
r_3 = r_2 / r_1^{1/a}.
$$

**Proof of Lemma 3.3.** Let $(z_1, \ldots, z_n)$ be a local coordinate on a neighbourhood $U$ of $x$ in $X/G$ such that

$$
U \cap (X/G)_1 = \{ q \in U \mid z_{n_1+1}(q) = \cdots = z_n(q) = 0 \}.
$$

We set $r_1 = (\sum_{i=n_1+1}^{n_1} |z_i|^2)^{1/2}$ and $r_2 = (\sum_{i=1}^{n_1} |z_i|^2)^{1/2}$. Then there exists a positive constant $C$ such that

$$
\| \sigma_1 \|^2 \leq C(r_1^2 + r_2^{2\left[n\sqrt{m(1-\varepsilon)} \cdot \frac{n_1}{\sqrt{2}}\right]})
$$

holds on a neighbourhood of $x$, where $\| \|$ denotes the norm with respect to $h_{X/G}^{m_1+\ell}$. We note that there exists a positive integer $M$ such that

$$
\| \sigma \|^2 = O(r_1^{-M})
$$

holds on a neighbourhood of the generic point of $U \cap (X/G)_1$, where $\| \|$ denotes the norm with respect to $h_{X/G}^{m_0}$. Then by Lemma 3.5, we have the inequality

$$
\alpha_1 \leq (m_1 + \ell) \cdot \frac{n_1}{m_1} \cdot \frac{n\sqrt{2}}{n\sqrt{\mu_1}} + O(\varepsilon_0)
$$

holds. By using the fact that

$$
\frac{\ell}{m_1} \leq \varepsilon_0 \cdot \frac{n\sqrt{\mu_1}}{n_1 \sqrt{2}}
$$

we obtain that

$$
\alpha_1 \leq \frac{n_1}{n\sqrt{\mu_1}} \cdot \frac{n\sqrt{2}}{n_1} + O(\varepsilon_0)
$$

holds. Q.E.D.

If $x$ or $y$ is a singular point on $(X/G)_1$, we need the following lemma.
Lemma 3.6 Let $\varphi$ be a plurisubharmonic function on $\Delta^n \times \Delta$. Let $\varphi_t(t \in \Delta)$ be the restriction of $\varphi$ on $\Delta^n \times \{t\}$. Assume that $e^{-\varphi_t}$ does not belong to $L^1_{\text{loc}}(\Delta^n, O)$ for every $t \in \Delta^*$.

Then $e^{-\varphi_0}$ is not locally integrable at $O \in \Delta^n$.

Lemma 3.6 is an immediate consequence of [10]. Using Lemma 3.6 and Lemma 3.5, we see that Lemma 3.4 holds by letting $x_1 \to x$ and $x_2 \to y$.

For the next we consider Case 1.2 and Case 2.2. We note that in Case 2.2 by modifying $\sigma$ a little bit, if necessary we may assume that $(\mathcal{O}_{X/G}/\mathcal{I}(\alpha_0 - \varepsilon))_y \neq 0$ and $(\mathcal{O}_{X/G}/\mathcal{I}(\alpha_0 - \varepsilon'))_x = 0$ hold for a sufficiently small positive number $\varepsilon'$. For example it is sufficient to replace $\sigma$ by the following $\sigma'$ constructed below.

Let $X/G'_1$ be a subvariety which defines a branch of

$$\text{Spec}(\mathcal{O}_{X/G}/\mathcal{I}(\alpha + \delta))$$

at $y$. By the assumption (changing $(X/G)_1$, if necessary) we may assume that $(X/G)_1'$ does not contain $x$. Let $m'$ be a sufficiently large positive integer divisible by $| G |$ such that $m'/m_0$ is sufficiently small (we can take $m_0$ arbitrary large).

Let $\tau_y$ be a general element of

$$H^0(X/G, \mathcal{O}_{X/G}(m'K_{X/G,\text{orb}}) \otimes \mathcal{I}_{(X/G)_1'})$$

where $\mathcal{I}_{(X/G)_1'}$ is the ideal sheaf of $(X/G)_1'$. If we take $m'$ sufficiently large, $\tau_y$ is not identically zero. We set

$$\sigma' = \sigma \cdot \tau_y.$$ 

Then we see that the new singular hermitian metric $h'_0$ defined by $\sigma'$ satisfies the desired property.

In these cases, instead of Lemma 3.2, we use the following simpler lemma.

Lemma 3.7 Let $\varepsilon$ be a sufficiently small positive number and let $x_1$ be a smooth point on $(X/G)_1$. Then for a sufficiently large $m > 1$ divisible by $| G |$, 

$$H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(mK_{X/G,\text{orb}}) \otimes \mathcal{I}(h^m) \otimes \mathcal{M}_{x_1}^{[n\sqrt{m(1-\varepsilon)m}])} \neq 0$$

holds.
Then taking a general $\sigma'_1$ in

$$H^0((X/G)_1, \mathcal{O}_{(X/G)_1}(m_1K_{X/G, orb}) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}^{[n_1/(1-\varepsilon)n_1]}_{x_1}),$$

for a sufficiently large $m_1$. As in Case 1.1 and Case 2.1 we obtain a proper subvariety $(X/G)_2$ in $(X/G)_1$ also in this case.

Inductively for distinct points $x, y \in X/G^o$, we construct a strictly decreasing sequence of subvarieties

$$X/G = (X/G)_0(x, y) \supset (X/G)_1(x, y) \supset \cdots$$

$$\supset (X/G)_r(x, y) \supset (X/G)_{r+1}(x, y) = \{x\} \text{ or } \{x, y\},$$

where $R_y$ (or $R_x$) is a subvariety such that $x$ does not belong to $R_y$ and $y$ belongs to $R_x$, and invariants:

$$\alpha_0(x, y), \alpha_1(x, y), \ldots, \alpha_r(x, y),$$

$$\mu_0, \mu_1(x, y), \ldots, \mu_r(x, y)$$

and

$$n > n_1 > \cdots > n_r.$$

By Nadel’s vanishing theorem (Theorem 2.1) we have the following lemma.

**Lemma 3.8** Let $x, y$ be two distinct points on $X/G^o$. Then for every $m \geq \lceil \sum_{i=0}^r \alpha_i(x, y) \rceil + 1$, $\Phi_{|mK_{X/G, orb}|}$ separates $x$ and $y$.

**Proof.** For simplicity let us denote $\alpha_i(x, y)$ by $\alpha_i$. Let us define the singular hermitian metric $h_{x,y}$ of the $\mathbb{Q}$-line bundle $(m - 1)K_{X/G, orb}$ defined by

$$h_{x,y} = \left( \prod_{i=0}^{r-1} h_{i}^{\alpha_i - \varepsilon_i} \right) \cdot h_{r}^{\alpha_r + \varepsilon_r} h^{(m - 1)\left(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i) - (\alpha_r + \varepsilon_r) - \delta_L\right)} \cdot h_{L}^{\delta_L},$$

where $h_{L}$ is a $C^\infty$-hermitian metric on $L$ with strictly positive curvature and $\delta_L$ be a sufficiently small positive number. Then we see that $\mathcal{I}(h_{x,y})$ defines a subscheme of $X/G$ with isolated support around $x$ or $y$ by the definition of the invariants $\{\alpha_i\}$’s. By the construction the curvature current $\Theta_{h_{x,y}}$ is strictly positive on $X/G$. Then by Nadel’s vanishing theorem (Theorem 2.1) we see that

$$H^1(X/G, \mathcal{O}_{X/G}(K_{X/G} + [(m - 1)K_{X/G, orb}]) \otimes \mathcal{I}(h_{x,y})) = 0.$$

Hence

$$H^0(X/G, \mathcal{O}_{X/G}(K_{X/G} + (m - 1)[K_{X/G, orb}])$$

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separates $x$ and $y$. We note that
\[ H^0(X/G, \mathcal{O}_{X/G}(K_{X/G} + (m - 1)[K_{X/G,orb}])) \]
is a subspace of
\[ H^0(X, \mathcal{O}_X(mK_X))^G \]
by the definition of $K_{X/G,orb}$. This implies that $\Phi_{[mK_{X/G,orb}]}$ separates $x$ and $y$. Q.E.D.

We note that for a fixed $x$, $\sum_{i=0}^r \alpha_i(x, y)$ depends on $y$. We set
\[ \alpha(x) = \sup_{y \in U_0} \sum_{i=0}^r \alpha_i \]
and let
\[
X/G = (X/G)_0 \supset (X/G)_1 \supset (X/G)_2 \supset \cdots \\
(X/G)_r \supset (X/G)_{r+1} = \{x\} \text{ or } \{x, y\}
\]
be the stratification which attains $\alpha(x)$. In this case we call it the maximal stratification at $x$. We see that there exists a nonempty open subset $U$ in countable Zariski topology of $X/G$ such that on $U$ the function $\alpha(x)$ is constant and there exists an irreducible family of stratification which attains $\alpha(x)$ for every $x \in U$.

In fact this can be verified as follows. We note that the cardinality of
\[
\{(X/G)_i(x, y) \mid x, y \in X/G, x \neq y (i = 0, 1, \ldots)\}
\]
is uncontably many, while the cardinality of the irreducible components of Hilbert scheme of $X/G$ is countably many. We see that for fixed $i$ and very general $x$, $\{(X/G)_i(x, y)\}$ should form a family on $X/G$. Similarly we see that for very general $x$, we may assume that the maximal stratification $\{(X/G)_i(x)\}$ forms a family. This implies the existence of $U$.

And we may also assume that the corresponding invariants $\{\alpha_0, \ldots, \alpha_r\}$, $\{\mu_0, \ldots, \mu_r\}$, $\{n = n_0 \ldots, n_r\}$ are constant on $U$. Hereafter we denote these invariants again by the same notations for simplicity. The proof of the following lemma is parallel to that of Lemma 3.4.

**Lemma 3.9**

\[ \alpha_i \leq \frac{n_i \sqrt{2}}{\sqrt[4]{\mu_i}} + O(\varepsilon_{i-1}) \]

hold for $1 \leq i \leq r$. 

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Proposition 3.1 For every 
\[ m > \left\lceil \sum_{i=0}^{r} \alpha_i \right\rceil + 1 \]
\[ | [mK_{X/G,orb}] | \] gives a birational rational map from \( X/G \) into a projective space.

Lemma 3.10 If \( \Phi_m |_{(X/G)_i} \) is birational rational map onto its image, then
\[ \text{deg} \Phi_m((X/G)_i) \leq m^{n_i} \mu_i \]
holds.

Proof. Let \( p : X/G \to X/G \) be the resolution of the base locus of \( | mK_{X/G,orb} | \) and let
\[ p^* | [mK_{X/G,orb}] | = | P_m | + F_m \]
be the decomposition into the free part \( | P_m | \) and the fixed component \( F_m \). Let \( p_i : X/G_i \to (X/G)_i \) be the resolution of the base locus of \( \Phi_{|mK_{X/G,orb}|} |_{(X/G)_i} \) obtained by the restriction of \( p \) on \( p^{-1}((X/G)_i) \). Let
\[ p_i^*(| mK_{X/G,orb} |_{(X/G)_i}) = | P_{m,i} | + F_{m,i} \]
be the decomposition into the free part \( | P_{m,i} | \) and the fixed part \( F_{m,i} \). We have
\[ \text{deg} \Phi_{|mK_{X/G,orb}|}((X/G)_i) = P_{m,i}^{n_i} \]
holds. Then by the ring structure of \( R(X/G, K_{X/G,orb}) \), we have that there exists a natural injection
\[ H^0(X/G, \mathcal{O}_{X/G}(\nu P_m)) \to H^0(X/G, \mathcal{O}_{X/G}([m\nu K_{X/G,orb}] \otimes \mathcal{I}(h^{\nu})) \]
for every \( \nu \geq 1 \). Hence there exists a natural morphism
\[ H^0((X/G)_i, \mathcal{O}_{(X/G)_i}(\nu P_{m,i})) \to H^0((X/G)_i, \mathcal{O}_{(X/G)_i}([m\nu K_{X/G,orb}] \otimes \mathcal{I}(h^{\nu})) \]
for every \( \nu \geq 1 \). This morphism is clearly injective. This implies that
\[ \mu_i \geq m^{-n_i} \mu((X/G)_i, P_{m,i}) \]
holds. Since $P_{m,i}$ is nef and big on $(X/G)_i$, we see that

$$\mu((X/G)_i, P_{m,i}) = P_{m,i}^{n_i}$$

holds. Hence

$$\mu_i \geq m^{-n_i} P_{m,i}^{n_i}$$

holds. This implies that

$$\deg \Phi_{|mK_{X/G, orb}|((X/G)_i)} \leq \mu_i m^{n_i}$$

holds. Q.E.D.

4 Proof of Theorem 1.1

To prove Theorem 1.1 we use the following subadjunction formula.

**Theorem 4.1** ([4]) Let $X/G$ be a normal projective variety. Let $D^o$ and $D$ be effective $\mathbb{Q}$-divisor on $X$ such that $D^o < D$, $(X, D^o)$ is logterminal and $(X, D)$ is logcanonical. Let $W$ be a minimal center of logcanonical singularities for $(X, D)$. Let $H$ be an ample Cartier divisor on $X$ and $\epsilon$ a positive rational number. Then there exists an effective $\mathbb{Q}$-divisor $D_W$ on $D$ such that

$$(K_X + D + \epsilon H) |_{W} \sim_{\mathbb{Q}} K_W + D_W$$

and $(W, D_W)$ is logterminal. In particular $W$ has only rational singularities.

Let us start the proof of Theorem 1.1. We prove Theorem 1.1 by induction on $n = \dim X$. Suppose that Theorem 1.1 holds for varieties of general type of dimension $< n$. Then there exists a positive constant $C(m)(m < n)$ depending only on $m$ such that for every smooth projective variety $Y$ of general type of dimension $m$

$$\mu(Y, K_Y)/\sharp \text{Aut}(Y) \geq C(m)$$

holds. Let $X$ be a smooth projective variety of general type as in Section 3. We use the same notations as in Section 3. Let $x, y$ be distinct points on $(X/G)^o$ and let

$$X/G = (X/G)_0 \supset (X/G)_1 \supset \cdots (X/G)_r \supset (X/G)_{r+1} = \{x\} \text{or} \{x, y\}$$

be the stratification constructed as in Section 3 and let

$$\mu_0, \ldots, \mu_r$$
be the invariants as in Section 3. Let
\[ X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} \]
be the corresponding stratification of \( X \). If we take \( x, y \) general, \( X_i (0 \leq i \leq r) \) are projective varieties of general type. Let
\[ X_{can} := \text{Proj} \ R(X, K_X) \]
be the canonical model of \( X \).

We have the corresponding stratification
\[ X_{can} = X_{0,can} \supset X_{1,can} \supset \cdots \supset X_{r,can} \supset X_{r+1,can} \]
on \( X_{can} \) (here we note that \( X_{i,can} \) does not denote the canonical model of \( X_i \) for \( i \geq 1 \)).

Then we see that
\[ \mu_i = \frac{1}{|G|} \mu(X_i, K_X) = \frac{1}{|G|} (K_{X_{can}})^{n_i} \cdot X_{i,can} \]
holds. Let \( H \) be an ample divisor on \( X \). By the subadjunction formula, we see that for every positive rational number \( \epsilon \)
\[ K_{X_{i,can}} < \mathbb{Q} (1 + \sum_{j=0}^{i-1} \alpha_j) K_{X_{can}} + \epsilon H \]
holds, where \( < \mathbb{Q} \) means that the righthandside minus the lefthandside is \( \mathbb{Q} \)-linear equivalent to an effective divisor and \( K_{X_{i,can}} \) denotes the pushforward of the canonical divisor of a nonsingular model of \( K_{X_{i,can}} \). This can be verified as follows. Let
\[ \pi : X \longrightarrow X/G \]
be the natural morphism. Let \( D_i \) be the divisor on \( X \) which corresponds to the singular hermitian metric
\[ \pi^*(h_0^{\alpha_0-\varepsilon_0} \cdots h_{i-1}^{\alpha_{i-1}-\varepsilon_{i-1}} \cdot h_i^{\alpha_i}). \]
\( D_i \) is a positive linear combinations of \( \{ \pi^*(\sigma_0), \ldots, (\sigma_j) \} \) by the constructions of \( h_0, \ldots, h_i \). Also we may assume that \( D_i \) is a \( \mathbb{Q} \)-divisor by perturbations of \( \varepsilon_0, \ldots, \varepsilon_{i-1} \). \( X_{i,can} \) may not be the minimal center of \( (X, D_i) \) and \( (X, D_i) \) may not be logcanonical. But if we take a suitable modification
\[ \pi_i : Y_i \longrightarrow X_{i,can}, \]
we may assume that there exists an effective \( \mathbb{Q} \)-divisor \( E_i \) such that
1. $\pi_i^* D_i - E_i$ is effective,

2. $(Y_i, \pi_i^* D_i - E_i)$ is logcanonical and the proper transform of $X_{i,\text{can}}$ is the minimal center of $(Y_i, \pi_i^* D_i - E_i)$.

Then by Theorem 4.1, we have that for every positive rational number $\epsilon$

$$K_{X_{i,\text{can}}} < Q (1 + \sum_{j=0}^{i-1} \alpha_j)K_{X_{\text{can}}} + \epsilon H$$

holds. By the inductive assumption this implies that

$$(1 + \sum_{j=0}^{i-1} \alpha_j)^{n_i} \cdot \mu_i \geq C(n_i)$$

holds. Since

$$\alpha_i \leq \frac{\sqrt{2n_i}}{\sqrt{\mu_i}} + O(\varepsilon_{i-1})$$

holds by Lemma 3.9, we see that

$$(*) \quad \frac{1}{\sqrt{\mu_i}} \leq (1 + \sum_{j=0}^{i-1} \frac{\sqrt{2n_j}}{\sqrt{\mu_j}}) \cdot C(n_i)^{-1}$$

holds for every $i \geq 1$. Inductively we see that if $\mu_0 \leq 1$ holds,

$$\frac{1}{\sqrt[\mu_i]} \leq \frac{1}{\sqrt{\mu_0}} C(C(1), \ldots, C(n-1))$$

holds where $C(C(1), \ldots, C(n-1))$ is a positive constant depending only on $C(1), \ldots, C(n-1)$. Hence if $\mu_0 < 1$ holds then we see that

$$\deg \Phi|_{(1+|\sum_{i=0}^{r} \alpha_i|)K_{X/G, orb}}(X) \leq C(C(1), \ldots, C(n-1))^n$$

holds. This implies that $X/G$ is birationally bounded, if

$$\mu_0 = \frac{1}{|G|} \mu(X, K_X) \leq 1$$

holds. We set

$$\alpha := \lceil \sum_{i=0}^{r} \alpha_i + 1 \rceil.$$ 

Then using Lemma 3.10, we have the following lemma.
Lemma 4.1 If $\mu_0 \leq 1$ holds, then there exists a positive constant $A(n)$ depending only on $n$ such that

$$1 \leq \alpha^n \mu_0 \leq A(n)$$

holds.

Let

$$|\alpha K_X|^G = |P| + F$$

be the decomposition of $|\alpha K_X|^G$ into the movable part $|P|$ and the fixed component $F$. Taking a suitable successive $G$-equivariant blowing ups, we may assume that $|P|$ is base point free. And also we may assume that the canonical birational map

$$f : X \longrightarrow X_{can}$$

is a morphism.

Lemma 4.2 There exists a positive constant $c_n$ depending only on $n$ such that

$$f^* K_{X_{can}} \cdot P^{n-1} \geq c_n |G|$$

holds. In particular

$$\alpha^{-1} K_{X_{can}/G, orb}^n \geq c_n$$

holds.

Proof. Let

$$f_G : X/G \longrightarrow X_{can}/G$$

be the natural morphism. Let us write

$$K_{X/G} = f_G^*(K_{X_{can}/G}) + \sum a_i E_i$$

where $\{E_i\}$ are irreducible exceptional divisor of $f_G$. We set

$$Y := \Phi_{[\alpha K_X]^G}(X).$$

and we set

$$\phi := \Phi_{[P]} : X \longrightarrow Y.$$ 

Let

$$\phi_G : X/G \longrightarrow Y$$

be the birational morphism induced by $\phi$. Then

$$f^* K_{X_{can}} \cdot P^{n-1} = \phi_* f^* K_{X_{can}} \cdot H^{n-1}$$
holds, where $H$ denotes the hyperplane section of $Y$. Also

$$
\phi_* f^* K_{\text{can}} \cdot H^{n-1} = |G| \cdot (\phi G)_* f_G^* K_{\text{can}/G,\text{orb}} \cdot H^{n-1}
$$

holds. On the other hand

$$(\phi G)_* f_G^* K_{\text{can}/G} \cdot H^{n-1} = (\phi G)_* (K_{X/G} - \sum a_i E_i) \cdot H^{n-1}$$

$$= K_Y \cdot H^{n-1} - \sum a_i (\phi G)_* E_i \cdot H^{n-1}$$

holds, where $K_Y$ denotes the pushforward of the canonical divisor of the normalization of $Y$ to $Y$. We note that $K_Y \cdot H^{n-1}(= K_{X/G} \cdot P^{n-1})$ is an integer. Since $E_i$’s appear as fixed components of $|\alpha K_{\text{can}/G,\text{orb}}|$, we see that

$$\sum_i (\phi G)_* E_i \cdot H^{n-1} \leq \alpha n \mu_0 = C(n)$$

hold. Hence $\sum_i (\phi G)_* E_i$ is bounded.

Since $\sum_i (\phi G)_* E_i$ is an exceptional divisor of the birational rational map

$$f_G \circ \phi_G^{-1} : Y \to X_{\text{can}}/G,$$

$\{a_i\}$ is of finitely many possibilities. Hence there exists a positive constant $K_n$ depending only on $n$ such that

$$(\sharp) \quad (\phi G)_* f_G^* (K_{\text{can}/G}) \cdot H^{n-1} \geq -K_n$$

holds. Let $\{D_j\}$ be the irreducible divisors such that

$$K_{\text{can}/G,\text{orb}} = K_{\text{can}/G} + \sum_j \frac{m_j - 1}{m_j} D_j$$

for some positive integers $\{m_j\}$. Then we see that

$$(\flat) \quad (f_G^* K_{\text{can}/G,\text{orb}}) \cdot \phi_G^* H^{n-1} = f_G^* K_{\text{can}/G} \cdot \phi_G^* H^{n-1} + \sum_j \frac{m_j - 1}{m_j} f_G^* D_j \cdot \phi_G^* H^{n-1}$$

$$\leq \alpha^n \mu_0$$

$$\leq A(n)$$

hold. By $(\sharp)$ this implies that $\sum_j (\phi G)_* f_G^* D_j$ is bounded and

$$\# \{j \mid (\phi G)_* f_G^* D_j \neq 0\}$$

is uniformly bounded by a positive integer, say $N$ depending only on $n$. 

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Lemma 4.3 Let $N$ and $B$ are fixed positive integers. Then

$$\left\{-\sum_{j=1}^{N} \frac{b_j}{a_j} \right\} \mid a_j, b_j \text{ are integers such that } b_j \leq B \} - \{0\}$$

is bounded below by a positive constant, where for a rational number $c$ \{c\} denotes the fractional part of $c$. i.e.

$$\{c\} := c - [c].$$

Proof. Suppose not. Then there exists a sequence of positive integers

$$\{a_{j,k}\}, \{b_{j,k}\} 1 \leq j \leq N, k = 1, 2, \ldots$$

such that

$$b_{j,k} \leq B,$$

$$\left\{-\sum_{j=1}^{N} \frac{b_{j,k}}{a_{j,k}} \right\} \neq 0,$$

$$\lim_{k \to \infty} \frac{b_{j,k}}{a_{j,k}}$$

exists for every $j$ and

$$\lim_{k \to \infty} \left\{-\sum_{j=1}^{N} \frac{b_{j,k}}{a_{j,k}} \right\} = 0$$

hold. We note that if

$$\lim_{k \to \infty} \frac{b_{j,k}}{a_{j,k}} \neq 0$$

then by the boundedness of $b_{j,k}$ the sequence is constant for every sufficiently large $k$ and if

$$\lim_{k \to \infty} \frac{b_{j,k}}{a_{j,k}} = 0$$

then $a_{j,k}$ tends to infinity $k$ goes to infinity. Since

$$\lim_{k \to \infty} \left\{-\sum_{j=1}^{N} \frac{b_{j,k}}{a_{j,k}} \right\} = 0$$

holds, there is no $j$ such that

$$\lim_{k \to \infty} \frac{b_{j,k}}{a_{j,k}} = 0$$
holds. Hence by the above observation we see that for every $j$ the sequence \( \{b_{j,k}/a_{j,k}\}_{k=1}^{\infty} \) is constant for every sufficiently large $k$ and $j$. This contradicts to the fact that 
\[
\{- \sum_{j=1}^{N} \frac{b_{j,k}}{a_{j,k}}\} \neq 0
\]
holds for every $k$. This completes the proof of Lemma 4.3. Q.E.D.

We note that by \((*)\), the finiteness properties of \(\{a_i\}\) and the boundedness of \(\sum_i(\phi_G)_*E_i\), we see that the rational number \(f_G^*K_{X_{can}/G} \cdot H^{n-1}\) is of finitely many possibilities. By (b), the boundedness of \(\sum_j(\phi_G)_*f_G^*D_j\) and Lemma 4.3, we see that there exists a positive constant $c_n$ depending only on $n$ such that
\[
f^*K_{X_{can}} \cdot P^{n-1} \geq c_n \left| G \right|
\]
holds. Since $R(X_{can}/G, K_{X_{can}/G,orb})$ is a ring,
\[
\alpha^{n-1}K^n_{X_{can}/G,orb} \geq c_n
\]
holds. This completes the proof of Lemma 4.1. Q.E.D.

By Lemma 4.1 and Lemma 4.2 we see that
\[
\alpha \leq \frac{A(n)}{c_n}
\]
holds. By Lemma 4.1, we see that
\[
\mu_0 \geq \frac{1}{\alpha^n}
\]
holds. Hence we have that
\[
\mu_0 \geq \left( \frac{c_n}{A(n)} \right)^n
\]
holds. This completes the proof of Theorem 1.2. Since
\[
\mu_0 = \frac{1}{\left| G \right|} \mu(X, K_X)
\]
holds, we have that
\[
\left| G \right| \leq \left( \frac{A(n)}{c_n} \right)^n \mu(X, K_X)
\]
holds. This completes the proof of Theorem 1.1.
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