LOOKING AT THE QCD CORRECTIONS
FOR LARGE $M_t$:
AN EFFECTIVE LAGRANGIAN POINT OF VIEW*

S. Peris†
TH Division, CERN, 1211 Geneva 23, Switzerland

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$\Delta r$ and to the process $Z \rightarrow b\bar{b}$ as two of the most representative examples. This
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the issue of what $\mu$ scale is the appropriate one at every stage and argue that,
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†On leave from Grup de Física Teorica and IFAE, Universitat Autonoma de Barcelona, Barcelona,
Spain. peris@surya11.cern.ch
Looking at the QCD Corrections for Large $m_t$:
An Effective Field Theory Point of View

S. Peris†
Theory Division, CERN
CH-1211 Geneva 23, Switzerland
E-mail: peris@surya11.cern.ch

Abstract

We discuss the QCD corrections to the large-$m_t$ electroweak contributions to $\Delta r$ and to the process $Z \rightarrow b \bar{b}$ as two of the most representative examples. This needs the construction of an effective field theory below the top quark. We discuss the issue of what $\mu$ scale is the appropriate one at every stage and argue that, while matching corrections do verify the simple prescription of taking $\mu \simeq m_t$ in $\alpha_s(\mu)$, logarithmic (i.e. $\sim \log m_t$) corrections do not, and require the use of the running $\alpha_s(\mu)$ in the corresponding renormalization group equations. In particular we obtain the $\alpha_s$ correction to the non-universal log $m_t$ contribution to the $Zb\bar{b}$ vertex.

1. Introduction

Electroweak (EW) radiative corrections are presently achieving an extremely high degree of sophistication and complexity. After the high-precision experiments recently performed at LEP and the SLC † there is a clear need for increasingly higher-order calculations, even if only for assessing the size of the theoretical error when comparing to the experiment. Currently two-loop EW corrections (pure or mixed with QCD) are being analyzed rather systematically ‡ and, sometimes, even up to three loops are being accomplished §. Needless to say these calculations are extremely complicated and usually heavily rely on the use of the computer. In this paper we would like to point out that in some situations thinking in terms of effective field theories (EFTs) ¶ can help in this development.

Built as a systematic approximation scheme for problems with widely separated scales †, EFTs organize the calculation in a transparent way dealing with one scale at a time and clearly separating the physics of the ultraviolet from the physics of the infrared. They are based on the observation that, instead of obtaining the full answer and then taking the appropriate interesting limits, a more efficient strategy consists in taking the limit first, whereby considerably reducing the amount of complexity one has to deal with, right from the start. For this kind of problems EFTs are never more complicated than the actual loop-wise perturbative calculation and in some specific cases they may even be more advantageous, being even able to render an extremely

†On leave from Grup de Fisica Teorica and IFAE, Universitat Autonoma de Barcelona, Barcelona, Spain.
complicated calculation something very simple.

By EFT we specifically mean the systematic construction of the effective Lagrangian that results when a heavy particle is integrated out. The procedure goes as follows\footnote{One could also match S-matrix elements.}. Let us imagine we are interested in studying the physics at an energy scale $E_0$. Starting at a scale $\mu >> E_0$ one uses the powerful machinery of the renormalization group equations (RGEs) to scale the initial Lagrangian from the scale $\mu$ down to the energy $E_0$ one is interested in. If in doing so one encounters a certain particle with mass $m$, one must integrate this particle out and find the corresponding matching conditions so that the physics below and above the scale $\mu = m$ (that is to say the physics described by the Lagrangian with and without the heavy particle in question) is the same. This is technically achieved by equating the one-particle irreducible Green functions (with respect to the other light fields) in both theories to a certain order in inverse powers of the heavy mass $m$.\footnote{There is a typographical error in the log $M_W^2/m_t^2$ term of $\Delta r(top)$ in ref. \cite{11}, which appears with an overall minus sign with respect to our expression. We thank F. Jegerlehner for confirming this.}

This usually requires the introduction of local counterterms in the effective Lagrangian for $\mu < m$. Once this is done, one keeps using the RGEs until the energy $E_0$ is reached. If another particle’s threshold is crossed, the above matching has to be performed again. All this procedure is most efficiently carried out by using the MS renormalization scheme, where the RGEs are mass-independent and can be gotten directly from the $1/\epsilon$ poles of dimensional regularization. Schematically, the standard strategy in the case of the top quark is the following:

1. Matching the effective theory to the full theory at $m_t$.
2. Running the effective Lagrangian from $m_t$ down to $M_Z$.
3. Calculating matrix elements with the effective Lagrangian at the scale $M_Z$.

2. $\Delta r$

According to refs. \cite{8,9,10,11}, the top contribution to $\Delta r$ can be expressed as

$$\Delta r(top) \approx -\frac{c^2}{s^2} \frac{3m_t^2G_F\sqrt{2}}{(4\pi)^2} \left(1 - \frac{\alpha_s(\mu)}{\pi} \frac{6 + 2\pi^2}{9}\right) +$$

$$+ \frac{g^2}{(4\pi)^2} \frac{1}{2} \left(\frac{c^2}{s^2} - \frac{1}{3}\right) \log \left(\frac{M_W^2/m_t^2}{m_t^2}\right) \left(1 + \frac{\alpha_s(\mu)}{\pi}\right),$$

(2.1)

where we have only kept the leading and next-to-leading $m_t$ dependence.
Along with these results, there has appeared the discussion of the \( \mu \) scale at which one is supposed to evaluate \( \alpha_s(\mu) \) in these expressions, and the parameters in terms of which one ought to express the result, i.e., whether \( \overline{\text{MS}} \), or on-shell, etc. For this, a prescription has been designed that says that corrections coming from the \((t, b)\) doublet should be computed with \( \alpha_s(m_t) \). This prescription would then say that in all of the above expressions \( \alpha_s(\mu) \) should be taken as \( \alpha_s(m_t) \). We would like to explain what an effective field theory (EFT) point of view shows about this issue. We shall see that while this prescription works for the power-like terms (those that go like \( m_t^2 \)), the renormalization group (RG) supplies us with a different result for the logarithmic terms (those that go like \( \log m_t \)).

Here we shall be concerned with the QCD corrections to the large-\( m_t \) one-loop electroweak corrections. Therefore, for all practical purposes, one may think as if the top quark were the heaviest particle in the SM, much heavier than the Higgs boson, which is taken to be nearly degenerate with the W and Z. This automatically kills the \( \log M_H/M_W \) contributions and leaves the \( m_t^2 \) and the \( \log m_t/M_W \) ones, which are those we are interested in.

The general philosophy will parallel that so successfully used in the context of grand unified theories. There is of course a very important difference, namely that, upon integration of the top quark, the resulting effective theory will no longer exhibit an explicit linear \( \text{SU}_2 \times \text{U}_1 \) invariance. This would make a full account of the corresponding RGEs very cumbersome. Luckily we may keep only those contributions that are strictly relevant.

\[ \begin{align*}
W^+ & \quad W^+ \\
 b & \quad W^+ \\
 t & \quad W^+ \\
 b & \quad W^+ \\
 t & \quad W^+ \\
 B & \quad B \\
 t & \quad B \\
 t & \quad B \\
 t & \quad B \\
 t & \quad B \\
 t & \quad B \\
 t & \quad B \\
 t & \quad B
\end{align*} \]

Figure 1: Diagrams contributing to the matching conditions, eqs. (2.4-2.5).

Let us start with the full SM at \( \mu > m_t \). At \( \mu = m_t \), one integrates the top out,

\(^c\)In principle three scales appear in these loops: \( m_t, M_Z, m_b \).
\(^d\)From an effective field theory point of view these two types of contributions are totally different. While the former (i.e. power-like) comes from “matching”, the latter (i.e. logarithmic) comes from “running”. See below.
obtaining

\[ \mathcal{L} = W_\mu^+ \partial^2 W^- \mu + \frac{g_2^2(\mu)}{4} \left( v^2 + \delta v_+^2(\mu) \right) W_\mu^+ W^- \mu + \frac{1}{2} W_\mu^3 \partial^2 W_3^\mu + \frac{1}{2} B_\mu \partial^2 B^\mu + \]

\[ + \frac{1}{2} (g_3(\mu) W_3^\mu - g'(\mu) B^\mu) \left[ \frac{1}{4} \left( v^2 + \delta v_3^2(\mu) \right) - \delta Z_3 Y\cdot(\mu) \right] \left( g_3(\mu) W_3^\mu - g'(\mu) B_\mu \right) \]

\[ + \bar{\psi} i D( g_+^\dagger W^+, \ g_3 W_3, \ g' B) \psi , \]

(2.2)

from the diagrams of fig. 1 after a trivial field redefinition. Here \( \psi \) stands for all the fermions but the top. Notice that we have dealt with \( W_3 - B \) mixing by including a \( \partial^2 \) operator in the form of a “mass term” in eq. (2.2). This will make the subsequent diagonalization very simple since the neutral mass eigenstate is still of the form \( g W_3 - g' B \), like at tree level. Certainly, there will also be a tower of higher dimensional operators suppressed by the corresponding inverse powers of the top quark mass, but we shall neglect them. Possible four-fermion operators are irrelevant to the discussion that follows and are also disregarded. We also postpone the study of the \( Zb\bar{b} \) vertex to the next sections.

To the order we are working, i.e. one loop:

\[ g_+^2(\mu) \approx g^2 \left( 1 - g^2 \delta Z_+(\mu) \right) \]

\[ g_3^2(\mu) \approx g^2 \left( 1 - g^2 \delta Z_3(\mu) - g^2 \delta Z_3 Y\cdot(\mu) \right) \]

\[ g^2(\mu) \approx g^2 \left( 1 - g^2 \delta Z Y\cdot(\mu) - g^2 \delta Z_3 Y\cdot(\mu) \right) \].

(2.3)

Notice that below the top quark mass the initially unique coupling constant \( g \) has split into \( g_+ \) and \( g_3 \). Similarly \( v_+^2(\mu) \equiv v^2 + \delta v_+^2(\mu) \) and \( v_3^2(\mu) \equiv v^2 + \delta v_3^2(\mu) \) are also different. The matching conditions are very easily obtained since they are nothing else than the diagrams of fig. 1 evaluated at \( \mu = m_t \). This means that

\[ \delta v_+^2(m_t) = \frac{N_c}{(4\pi)^2} m_t^2 , \quad \delta v_3^2(m_t) = 0 . \]

(2.4)

Analogously,

\[ \delta Z_3 Y(m_t) = 0 , \quad g_+(m_t) = g_3(m_t) = g \quad \text{and} \quad g'(m_t) = g' . \]

(2.5)

Because of lack of space the reader interested in the details regarding this section is referred to ref.
Equation (2.5) says that the coupling constants are continuous across the threshold. This is true as long as one keeps only the leading logarithms. In general there are non-logarithmic pieces that modify (2.5) such as, for instance, the non-log term in the first of eqs. (2.4). The point is that this term in (2.4) is multiplied by \( m_t^2 \) (i.e. a large non-decoupling effect) and therefore contributes (in fact dominates) for large \( m_t \), whereas the same does not happen in (2.5). Therefore, non-log corrections to (2.5) do not affect the large-\( m_t \) discussion that follows.

In order to obtain the effective Lagrangian at the relevant lower scales \( \mu \simeq M \equiv M_W, M_Z \) one has to scale this Lagrangian down using the RGE for each ‘coupling’ \( g_+(\mu), g_3(\mu), g'(\mu), \delta v^2_+(\mu), \delta v^2_3(\mu) \) and \( \delta Z_{3Y}(\mu) \). The running of \( \delta v^2_{+,3}(\mu) \) is zero since it must be proportional to a light fermion mass, which we neglect. \(^7\) Therefore,

\[
\delta v^2_{+,3}(m_t) = \delta v^2_{+,3}(M) \quad .
\]

After including \( \alpha_s \) corrections, one immediately obtains (\( t \equiv \log \mu^2 \))

\[
\frac{dg^2_+}{dt} = \frac{g^4_+}{(4\pi)^2} \left[ 2 \left( 1 + \frac{\alpha_s(t)}{\pi} \right) + 1 \right] + ... \nonumber \\
\frac{dg^2_3}{dt} = \frac{g^4_3}{(4\pi)^2} \left[ \frac{7}{3} \left( 1 + \frac{\alpha_s(t)}{\pi} \right) + 1 \right] + ... \nonumber \\
\frac{dg^2}{dt} = \frac{g'^4}{(4\pi)^2} \left[ \frac{23}{9} \left( 1 + \frac{\alpha_s(t)}{\pi} \right) + 3 \right] + ... \nonumber \\
\frac{d}{dt}\delta Z_{3Y} = \frac{1}{6(4\pi)^2} \left( 1 + \frac{\alpha_s(t)}{\pi} \right) + ... \quad (2.7)
\]

from the diagrams of fig. 1, but with gluon corrections. Ellipses in eq. (2.7) stand for the contribution of the gauge bosons and the Higgs.\(^b\)\(\alpha_s(\mu)\) corrections do not affect eqs. (2.5), (2.6). Equations (2.7) are to be supplemented with the running of \( \alpha_s(t) \),

\[
\frac{d\alpha_s}{dt} = - \frac{\beta_0}{(4\pi)} \alpha_s^2 \quad , \quad \beta_0 = 11 - \frac{2}{3} n_f \quad , \quad n_f = 5 \text{ flavors} \quad . \quad (2.8)
\]

The boundary conditions (2.4) read\(^f\)

\[
\frac{v^2_+(M) - v^2_3(M)}{v^2_+(M)} = \frac{v^2_+(m_t) - v^2_3(m_t)}{v^2_+(m_t)} = \\
= \frac{3}{4\pi^2} \frac{m_t^2}{v^2_+(m_t)} \left[ 1 - \frac{2}{9} \frac{\alpha_s(m_t)}{\pi} (\pi^2 - 9) + \mathcal{O}(\alpha_s^2) \right] \quad . \quad (2.9)
\]

\(^f\)Hence we neglect possible terms \( \sim \log M_W/M_Z \).

\(^g\)We also neglect the contribution of the gauge bosons and the Higgs since they do not have QCD corrections. This simplifies the analysis enormously.

\(^h\)We note again that this contribution will not have QCD corrections to the order we are working.
where, as nicely explained in refs. 14, 13, the scale in $\alpha_s(\mu)$ and $m_t(\mu)$ clearly has to be $\sim m_t$ (and not $M_Z$ or $m_b$) because it is nothing but a matching condition at $\mu = m_t$. This is the $\rho$ parameter. We shall see below that $\nu_\tau^2(m_t) = (\sqrt{2}G_F)^{-1}$, where $G_F$ is the $\mu$-decay constant. Recently Sirlin 15 has noted certains virtues in an expression like eq. (2.9). Within the EFT approach it comes out very naturally.

Given that $g_+, g_3^2$ and $g^2$ are all rather smaller than $g_s^2 \equiv 4\pi\alpha_s$, a reasonable approximation is to take into account the running of $\alpha_s$ in eqs. (2.7) while keeping the $g_+, g_3$ and $g'$ frozen at a given value. This is tantamount to resumming the leading log’s accompanying powers of $\alpha_s$ but not those accompanied by powers of $g_+, g_3$ and $g'$.

With all this, one can now go about computing a typical physical quantity like for instance $\Delta r_W$, which is the same as the more familiar parameter $\Delta r$ defined by Marciano and Sirlin 17 but without the running of $e(\mu)$. In the EFT language this is obtained in the following way. According to the Lagrangian (2.2) the physical W and Z masses are given by the equations

$$M^2_W = \frac{g_+^2(M)}{4} \nu_+^2(M)$$
$$M^2_Z = \frac{g^2_3(M)}{4\nu^2(M)} \left(\nu_3^2(M) + 4M^2_Z\delta_{3\gamma}(M)\right), \quad (2.10)$$

where $\nu^2(M) \equiv \cos^2 \theta_W(M)$ and $\tan \theta_W(M) \equiv g'(M)/g_3(M)$.

Following the EFT technique, at the scale of the W mass one should integrate out the W boson. This gives rise to the appearance of 4-fermion operators that mediate $\mu$ decay, with strength $G_F(M)/\sqrt{2}$. The matching condition therefore becomes

$$\frac{G_F(M)}{\sqrt{2}} = \frac{g_+^2(M)}{8M^2_W} = \frac{1}{2\nu_+^2(M)}, \quad (2.11)$$

but since $G_F(\mu)$ does not run 19 one can see that actually $\nu_+^2(M) = \sqrt{2}G_F$, where $G_F$ is the Fermi constant as measured in $\mu$ decay. Therefore

$$\frac{G_F}{\sqrt{2}} = \frac{g_+^2(M)}{8M^2_W} = \frac{e^2(M)}{8M^2_W} \left[\frac{g_3^2(M)}{g_3^2(M)} \frac{1}{s^2(M)}\right], \quad (2.12)$$

where $e^2(\mu)$ is the running electromagnetic coupling constant. The quantity $\Delta r_W$ is defined as

$$\frac{G_F}{\sqrt{2}} = \frac{e^2(M)}{8M^2_W s^2(1 + \Delta r_W)}. \quad (2.13)$$

Consequently,

$$1 + \Delta r_W = \frac{s^2}{s^2(M)} \frac{g_+^2(M)}{g_3^2(M)}, \quad (2.14)$$
where \( s^2 \equiv 1 - M_Z^2/M_t^2 \) is Sirlin’s combination\(^3\).

Since we are only interested in resumming \( \alpha_s \) corrections we can approximate \( \Delta r_W \) in eq. (2.14) by

\[
\Delta r_W \approx \frac{c^2 - s^2}{s^2} \frac{g_3^2(M) - g_3^2(M_t)}{g^2} - \frac{c^2 v^2(m_t) - v^2(m_t)}{v^2 s^2} + \frac{4M_Z^2 c^2}{s^2 v^2} \delta Z_{3Y}(M). \tag{2.15}
\]

Integration of eqs. (2.7) and (2.8), with the boundary conditions (2.5), yields

\[
g_3^2(M) \approx 1 + \frac{g^2}{(4\pi)^2} \left[ \frac{-1}{3} \log \left( \frac{M^2}{m_t^2} \right) + \log \left( \frac{\alpha_s(M)}{\alpha_s(m_t)} \right) - \frac{\beta_0}{\alpha_s} \right]
\]

\[
\delta Z_{3Y}(M) \approx \frac{1}{6(4\pi)^2} \left[ \log \left( \frac{M^2}{m_t^2} \right) + \log \left( \frac{\alpha_s(M)}{\alpha_s(m_t)} \right) - \frac{\beta_0}{\alpha_s} \right] \tag{2.16}
\]

so that \( \Delta r_W \) is, finally,

\[
\Delta r_W \approx -\frac{c^2}{s^2} \frac{3}{(4\pi)^2} m_t^2(m_t) G_F \sqrt{2} \left[ 1 - \frac{2}{3} \frac{\alpha_s(m_t)}{\alpha_s(m_t)} \left( \frac{\pi^2 - 9}{9} \right) \right] + \frac{g^2}{(4\pi)^2} \frac{1}{2} \left( \frac{2}{s^2} - \frac{1}{3} \right) \log \left( \frac{M^2}{m_t^2} \right) + \log \left( \frac{\alpha_s(M)}{\alpha_s(m_t)} \right) - \frac{\beta_0}{\alpha_s}. \tag{2.17}
\]

with \( \beta_0 = 23/3 \). In the second term of eq. (2.17) one has actually resummed all orders in \( \alpha_s^a \log^a \). It is here that the powerfulness of the RG and EFT has proved to be very useful. Therefore we learn that while the term proportional to \( m_t^2(m_t) \) comes from matching, and has therefore a well-defined scale \( \mu \approx m_t \); the term proportional to \( g^2 \) comes from running, which in turn means that it has to depend on the two scales between which it is running, \( \mu \approx M \) and \( \mu \approx m_t \). From the point of view of an EFT aficionado, eq. (2.17) is somewhat unconventional in that it considers matching conditions (the \( \alpha_s(m_t) \) term) together with running (the \( \alpha_s(M)/\alpha_s(m_t) \) term) both at one loop. From the QCD point of view the former is a next-to-leading-log term whereas the latter is a leading-log one. The reason for taking both into account is of course that the \( \alpha_s(m_t) \) term is multiplied by the \( m_t^2 G_F \) combination, which is large.

If one takes the \( \alpha_s(M)/\alpha_s(m_t) \) logarithmic term, expands it in powers of \( \alpha_s \) and uses

\[
m_t \approx m_t(m_t) \left( 1 + \frac{4\alpha_s(m_t)}{3\pi} \right) \tag{2.18}
\]

to rewrite eq. (2.17) in terms of the pole mass, one of course reobtains eq. (2.1) to the given order.

3. \( Z \to b\bar{b} \)

The decay width \( Z \to b\bar{b} \) can be written as \(\underline{E}\underline{E}\).
\[ \Gamma(Z \to \bar{b}b) = N_c \frac{M_Z^3 \sqrt{2} G_F}{48\pi} \rho \ R_{QCD} \ R_{QED} \ [A^2 + V^2] \ , \]  
with
\[ A = 1 + \frac{1}{2} \Delta \rho_{\text{vertex}} \ ; \quad V = 1 + \frac{1}{2} \Delta \rho_{\text{vertex}} - \frac{4}{3} \kappa s_0^2 \ , \]
\[ \kappa \approx 1 - \frac{c^2}{c^2 - s^2} \Delta \rho + \frac{g^2}{(4\pi)^2} \frac{1}{6 \left(c^2 - s^2\right)} \log \frac{M_W^2}{m_t^2} \left(1 + \frac{\alpha_s(\mu)}{\pi}\right) \ , \]
\[ \rho = 1 + \Delta \rho \ , \quad \Delta \rho \approx \frac{3}{(4\pi)^2} m_t^2(m_t) G_F \sqrt{2} \left(1 - \frac{\alpha_s(\mu)}{\pi} \frac{2}{9} (\pi^2 - 9)\right) \ , \]
and
\[ s_0^2 = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4\pi \alpha(M_Z)}{\sqrt{2} G_F M_Z^2}}\right) \ , \]
where
\[ R_{QCD} \approx 1 + \frac{\alpha_s(\mu)}{\pi} \ , \quad R_{QED} \approx 1 + \frac{\alpha(\mu)}{12\pi} \ , \]
\[ \Delta \rho_{\text{vertex}} \approx -\frac{4m_t^2(m_t) G_F \sqrt{2}}{(4\pi)^2} \left(1 - \frac{\alpha_s(\mu)}{\pi} \frac{\pi^2 - 8}{3}\right) + \]
\[ + \frac{g^2}{(4\pi)^2} \log \left(\frac{M_W^2}{m_t^2}\right) \left(\frac{8}{3} + \frac{1}{6c^2}\right) \left(1 + \frac{C \alpha_s(\mu)}{\pi}\right) \ . \]

We employed the running \(\overline{\text{MS}}\) \(m_t(\mu = m_t)\).

Here we shall describe an effective field theory calculation of the physical process \(Z \to \bar{b}b\). As a result we shall obtain the value of the coefficient \(C\) in eq. (3.7). This coefficient has also been recently obtained in ref. \[24\] and our result agrees with theirs. Moreover, our construction of the EFT will also yield the value for the natural scale \(\mu\) that appears in the different terms of eqs. (3.3),(3.7). Again because of limitations of space we refer the reader to ref. \[23\] for any detail regarding this section.

Integrating the top quark out affects the coupling to the \(W\) and \(Z\) gauge bosons of every lighter fermion through vacuum polarization as we saw in the previous section. Moreover it also affects specifically the coupling of the bottom quark to the \(Z\) boson. The integration of the top quark is done in several steps. Firstly, at tree level, there is the contribution given by the diagram of fig. \[2\]. This contribution gives rise to an effective operator that is suppressed by two inverse powers of the top mass. Since ultimately this fact is due to dimensional analysis, it cannot change once QCD is switched on and one-loop \(\alpha_s\) corrections to the diagram of fig. \[2\] are also considered in the matching conditions. We shall consistently neglect this type of contributions since they can never give rise to the terms we are interested in, i.e. eq. (3.7). This is the only contribution in the unitary gauge, which is the one we shall employ.\(^1\) In any

\(^1\)In the previous section we did not need to fix the gauge since the fermion vacuum polarization is gauge invariant.
other gauge other effective operators arise because the would-be Nambu–Goldstone bosons couple proportionally to the top mass and may compensate the $m_t^2$ factor in the denominator.

The effective Lagrangian below the top quark mass reads:

$$\mathcal{L} = \mathcal{L}_4 + \mathcal{L}_6,$$

$$\mathcal{L}_4 = \bar{b} i \not{D} b - \frac{1}{2} c_L(\mu) \bar{b} Z P_L b + \frac{1}{3} c_V(\mu) \bar{b} Z b +$$

$$+ \bar{e} i \not{D} e - \frac{1}{2} c_L(\mu) \bar{e} \not{Z} P_L e + c_V(\mu) \bar{e} \not{Z} e + \frac{c_+(\mu)}{\sqrt{2}} (\bar{e} W^+ P_L \nu + \text{h.c.}) ;$$

$$\mathcal{L}_6 = \frac{1}{\Lambda_F^2} \sum_i c_i(\mu) \mathcal{O}_i , \quad (3.8)$$

where $P_L$ is the lefthanded projector and $\not{D}$ stands for the QED and QCD covariant derivatives. The $c(\mu)$'s of the electron are actually common to all the fermions but the bottom quark. For instance, the $Z\nu\bar{\nu}$ would be $+ c_L(\mu)/2$ since the neutrino has no vector coupling $c_V(\mu)$. Notice that we have decomposed the $Z f \bar{f}$ vertex in terms of a lefthanded and vector couplings instead of the more conventional left and righthanded, or vector and axial counterparts. In eq. (3.8) $\Lambda_F = 4\pi v$, $v = (\sqrt{2}G_F)^{-1/2} = 246$ GeV and the $\mathcal{O}_i$'s are a set of dimension-six operators involving the (lefthanded) bottom quark and three (covariant) derivatives; or the bottom quark, the $Z$ and two derivatives. They arise from the longitudinal part of the $W$ propagators. This is why the scale $\Lambda_F$ appears: it is the combination of the ordinary $1/m_t^2$ suppression of any six-dimensional operator in an effective field theory and the fact that the would-be Nambu–Goldstone bosons couple proportionally to the top mass.

For convenience we have changed here the notation for the effective couplings with respect to the previous section. The connection is given by

$$c_L(\mu) = \frac{g_3(\mu)}{c(\mu)} ; \quad c_V(\mu) = \frac{g_3(\mu)s^2(\mu)}{c(\mu)} ; \quad c_+(\mu) = g_+ (\mu) \quad , \quad (3.9)$$
where \( s^2(\mu) = \sin^2 \theta_W(\mu) \) and \( \tan \theta_W(\mu) = g'/(g_3(\mu)) \).

\[
W^b t \bar{b} \Rightarrow g b_L b_L
\]

(a)

\[
W^g t \bar{b} Z \downarrow Z^g b_L b_L
\]

(b)

**Figure 3**: One-loop matching due to QCD.

We can select the non-universal part of the \( Zb\bar{b} \) vertex by comparing the \( c_L^b(\mu) \) coupling on shell with the analogous coupling for the electron \( c_L(\mu) \) at the scale \( \mu \sim M_Z \sim M_W \equiv M \). One defines\(^j\)

\[
1 + \frac{1}{2} \Delta \rho_{vertex} = \frac{c_L^b(M)}{c_L(M)}.
\]  

(3.10)

In order to make contact with the physics at the scale \( \mu = M \), one has to scale the Lagrangian (3.8) down to this particular \( \mu \). In this process of scaling, \( c_L^b(\mu) \) and

\(^j\)This ratio is called \( 1 + \epsilon_b \) in ref. 24.
The calculation can be done by setting the external particles on shell.

Clearly the effect of integrating the top quark out affects only the lefthanded projection of the bottom-quark field, i.e. \( c^b_L(\mu) \), but leaves untouched the coefficient \( c^b_V(\mu) \). As a matter of fact \( c^b_V(\mu) = c_V(\mu) \).

The diagrams of fig. 3 give rise to the dimension-six operators that appear in eq. (3.8). In principle one should now calculate how all these operators mix back into the \( Zb\bar{b} \) operators of eq. (3.8) and make the coefficients \( c^b_L, V(\mu) \) evolve with \( \mu \) as one runs from \( m_t \) down to \( M_Z \). However, a clever use of the equations of motion helps us get rid of almost all the operator structures that are generated in the matching and leaves us with only one operator that is interesting. This one is

\[
O_1 = \bar{b}_L \gamma^\nu \frac{\lambda^A}{2} b_L \ g_s \ D^\mu G^A_{\mu\nu} .
\] (3.11)

Of course this operator only affects the running of \( c^b_L(\mu) \) and not of \( c^b_V(\mu) \).

An explicit straightforward evaluation of the diagrams of fig. 3b yields for the coefficient \( c_1(\mu) \) accompanying the operator \( O_1 \) the value

\[
c_1(m_t) = -\frac{7}{18} .
\] (3.12)

In order to make contact with the physics at the scale \( \mu \simeq M \) one has to find how \( c^b_L(\mu) \) scales with \( \mu \). We use the Feynman gauge propagator for the gluon. One obtains

\[
\frac{dc^b_L(t)}{dt} = (\text{something}) + \left( \frac{g}{c} \frac{g^2}{(4\pi)^2} \gamma_1 \ c_1(t) \frac{\alpha_s(t)}{\pi} \right) ,
\] (3.13)

where the second term comes from fig. 4b and “something” stands for a certain contribution common to the running of \( c^b_L(\mu) \) that will cancel in the final ratio (3.16) (see below). We obtain the following value for the coefficient \( \gamma_1 \):

\[
\gamma_1 = -\frac{1}{9c^2} \left( 1 - \frac{2}{3}s^2 \right) .
\] (3.14)

Since \( O_1 \) only involves the lefthanded bottom quark it is clear why the coefficient \( \gamma_1 \) turns out to be proportional to the lefthanded bottom coupling to the Z, i.e. the combination \( 1 - \frac{2}{3}s^2 \).

Now we would like to integrate eq. (3.13). In first approximation, one may take \( \alpha_s(t) \) and \( c_1(t) \) as constants independent of \( t \), i.e. \( \alpha_s(\mu) \simeq \alpha_s(m_t) \simeq \alpha_s(M) \equiv \alpha_s \) and \( c_1(\mu) \simeq c_1(m_t) \simeq c_1(M) \equiv c_1 = -7/18 \). The integration over \( t \) between \( \log m_t^2 \) and \( \log M^2 \) gives

\[k\] One could still use the equations of motion for the gluon field but we found more convenient not to do so.
It is in principle possible to improve on this approximation by considering the \( \mu \)-dependence of \( \alpha_s(\mu) \) and \( c_1(\mu) \) in eq. (3.13). The \( \mu \)-dependence of \( \alpha_s(\mu) \) is given by the usual one-loop \( \beta \) function. However the \( \mu \)-dependence of \( c_1(\mu) \) is more complicated to obtain because it requires performing a complete operator mixing analysis of the penguin operator along the lines of, for instance, the work carried out in the studies of \( b \to s\gamma \) or \( K^0 - \bar{K}^0 \) mixing \(^{27}\) whence most of the results could be taken over to our case. However, the fact that \( \gamma_1 c_1(m_t) = \frac{7}{16\pi^2} (1 - \frac{2}{3} s^2) \approx 0.05 \) turns out to be so small renders this improvement moot and we shall content ourselves with eq. (3.15) as it is. As we shall see later on, there are other sources of QCD corrections that are numerically more important.

One obtains (see ref. \(^{24}\) for details):

\[
\frac{c_L^b(M)}{c_L(M)} \approx \frac{c_L^b(m_t)}{c_L(m_t)} \left[ 1 + \frac{g^2}{4\pi^2} \left( \frac{4}{3} + \frac{1}{12\pi^2} \right) \log \frac{M^2}{m_t^2} + \frac{g^2}{(4\pi)^2} \gamma_1 c_1 \frac{\alpha_s}{\pi} \log \frac{M^2}{m_t^2} \right].
\] (3.16)

This fixes the coefficient \( \mathcal{C} \) in eq. (3.10) to be (remember eq. (3.10))
\[ C = 2 \gamma_1 c_1 \left( \frac{8}{3} + \frac{1}{6c^2} \right)^{-1} \approx 0.03 \] (3.17)

The boundary condition at \( m_t \) can be borrowed from the literature. Translated into our context it amounts to

\[ \frac{c^b_L(m_t)}{c_L(m_t)} = 1 - 2 \frac{m_t^2(m_t)}{(4\pi v)^2} \left[ 1 - \frac{\alpha_s(m_t)}{\pi} \left( \frac{\pi^2 - 8}{3} \right) \right] . \] (3.18)

Again, what the EFT tells us is that the \( \mu \) scale of \( \alpha_s(\mu) \) in this equation has to be \( m_t \) since it originates at the matching condition when the top is integrated out. Therefore we get to eq. (3.7) with \( \alpha_s(\mu = m_t) \) in the \( m_t^2 \)-dependent term.

However this is not yet all. Up to now all the physics has been described with RGEs (i.e. running) and their initial conditions (i.e. matching) which is only ultraviolet physics, and no reference to infrared physics has been made. For instance, where are the infrared divergences that appear when a gluon is radiated off a bottom-quark leg? As we shall now see, this physics is in the matrix element for \( Z \to b\bar{b} \). After all, we have only obtained the effective Lagrangian (3.8) at the scale \( \mu = M; \) we still have to compute the physical matrix element with it, and here is where all the infrared physics takes place.

Indeed, when computing the matrix element for \( Z \to b\bar{b} \) with the effective Lagrangian (3.8) expressed in terms of \( c^b_L,V(\mu) \) at \( \mu = M \), one has the contribution of the diagrams of figs. 5a, 5b, where the \( \otimes \) stands for the effective vertices proportional to \( c^b_L,V(M) \). These diagrams give rise to infrared divergences. These divergences disappear in the standard way once bremsstrahlung diagrams like those of fig. 5c are (incoherently) added.

As is well known, the net result of all this (a similar calculation can be performed for the QED corrections) is the appearance of the factors \( R_{QCD} \) and \( R_{QED} \) of eqs. (3.1) and (3.6), where \( b \)-quark mass effects can also be included if needed.

The EFT technology adds to this the choice of scale for \( \mu \), namely \( \mu = M \), in these factors:

\[ R_{QCD} \simeq 1 + \frac{\alpha_s(M)}{\pi} , \quad R_{QED} \simeq 1 + \frac{\alpha(M)}{12\pi} , \] (3.19)

and naturally leads to the factorized expression (3.1)–(3.7) with the value of \( C \) given by eq. (3.17). As previously stated, our result agrees with that of ref. 1. Since the “intrinsic” \( \alpha_s \) contribution of \( \Delta \rho^{vertex} \) is, due to the smallness of the coefficient \( C \), much less important than that of \( R_{QCD} \) one sees that the QCD corrections to the

1 Another advantage is that matching conditions are free from infrared divergences, which is a nice simplification. For some more discussion on infrared divergences, see below.

This has been previously suggested by D. Bardin (private communication).
Figure 5: Diagrams contributing to the matrix element of $Z \rightarrow b\bar{b}$ in the effective theory.
non-universal log $m_t$ piece of the $Zb\bar{b}$ vertex are, to a very good approximation, of the form one-loop QCD ($m_t << M_Z$) times one-loop electroweak ($m_t >> M_Z$).

4. Conclusions

We hope we have been able to show that the effective field theory construction can be very useful for multiloop calculations in the Standard Model when only a few terms in a large mass expansion are needed. Because the powerfulness of the RGEs is naturally implemented in the EFT, we achieve full “logarithmic” control over the relevant scales $\mu$ of the problem at hand. For instance the “large” logarithms are obtained through the beta functions of certain effective couplings. This only requires the calculation of simple poles in $1/\varepsilon$ which is a major simplification. We have also shown that the EFT framework answers quite naturally the question of the renormalization points to be used for the coupling constants in the different terms.

In addition it is important to remark that in the EFT language all the physics above $M$ is absorbed (in particular, all $m_t$ effects) in the coefficients of the effective operators so that infrared physics is relegated to the calculation of the physical process one is interested in. With our effective Lagrangian one could in principle compute any physical quantity, and not only the $Z$ width, like for example jet production (i.e. where cuts are needed), forward–backward asymmetries, etc. This is to be compared with more standard methods in which, in order to avoid problems with infrared divergences, one computes the imaginary part of the $Z$ self-energy to obtain the $Z$ width. In this case it is not at all clear how one can tailor to one’s needs the entire phase space.

The EFT calculation clearly separates ultraviolet from infrared physics and as a consequence it is more flexible. And it is also simpler since, after all, we never had to compute anything more complicated than a one-loop diagram.

Of course, our results become more accurate as the top mass becomes larger. In practice it is unlikely that the top quark be much heavier than, say, 200 GeV so due caution is recommended in the phenomenological use of eq. (3.1), for instance. In the lack of a (very hard!) full $\mathcal{O}(g^2\alpha_s)$ calculation, this is the best one can offer. Furthermore, we think it is interesting that at least there exists a limit (i.e. $m_t >> M_Z$) where the various contributions are under full theoretical control.

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