The Onset of Deformation in Atomic Nuclei and Atomic Clusters

Mihai Horoi
Department of Physics, Central Michigan University, Mount Pleasant, MI 48859, USA
E-mail: horoi@phy.cmich.edu

Abstract. It is well known that atomic nuclei and atomic clusters exhibit properties that can be related to intrinsic deformation. The physical signals of deformation are different, but one can show that for these two mesoscopic systems one can identify the sizes for which they undergo shape transitions using a simple criteria. We give examples of shape transitions in these systems and we show how they can be identified experimentally looking to the binding energy per particle vs $n^{-1/3}$, where $n$ is the number of particles. It is also know that most mesoscopic systems exhibit prolate deformations. It is interesting to understand why the emergence of prolate deformation is dominant. We present some quantum mechanisms that could explain the dominance of the prolate shapes in nuclei.

1. Introduction
Mesoscopic systems occupy an intermediate size range between individual particles on the one hand and very large systems on the other. This intermediate range is interesting because here properties can change dramatically with the number of particles in the system. For example, in quantum systems like atomic nuclei or clusters of metal atoms, there are strong shell effects in the relative stability of mesoscopic systems, corresponding to the complete filling of energy shells. The shape of a mesoscopic system is also sensitive to the number of particles. In a simple picture, one expects a system of mutually attractive particles to form a compact, quasi-spherical shape to maximize the attractive potential energy. But in the mesoscopic size range, strong deviations from a spherical shape are known. Perhaps the most spectacular case is that of silicon clusters, $\text{Si}_n$. These clusters have been extensively studied both experimentally [1, 2, 3, 4, 5] and theoretically [6, 7, 8, 9, 10, 11, 12, 13] in recent years. Data from ion mobility experiments [14] indicate a roughly spherical shape at $n = 10$ that evolves into a highly elongated shape by $n = 24$. The ten-atom cluster is particularly stable and is a common by-product in the fragmentation of larger clusters. Over the range $n = 25$ to 28 prolate and compact structures are seen to coexist in the experiments and beyond $n = 28$ only compact structures are observed. The isotope chain of Samarium nuclei is another example of a mesoscopic shape transition. The isotope at $A = 144$ corresponds to a closed shell of neutrons and has a near spherical shape. As neutrons are added, the nucleus becomes deformed, slowly at first, and then abruptly over the range $A = 148 − 152$. At $A = 152$ there is experimental evidence for an excited nucleus with a compact shape coexisting with the elongated ground state [15]. In this paper we show that a horizontal plateau on a curve of binding energy per particle ($E_b/n$) vs. $n^{-1/3}$ is a signature of the shape transition in these type of mesoscopic systems.
The majority of deformed nuclei have axially symmetric prolate deformation in their ground state (g.s.). The presence of rotational bands with typical energy intervals is the first signature of stable deformation. The states in a band are connected by strong quadrupole transitions obeying simple rigid rotor intensity rules. The same rules [16] determine the expectation values of multipole operators. This is essentially a projection of nearly constant within the band intrinsic (body-fixed) quantities onto the laboratory (space-fixed) coordinate frame. The overwhelmingly positive sign of the intrinsic quadrupole moment manifests the dominance of prolate deformation. The Nilsson diagram shows a pattern of split spherical single-particle orbitals that seems to be more or less symmetric with respect to the sign of deformation; the difference in the liquid drop energy for the two signs of deformation is pretty small. While the results of the numerous mean field calculations for specific nuclei in general correctly predict the presence and the sign of deformation, the underlying physical reason for the predominance of prolate shapes is unclear.

In this contribution we present a novel approach to this problem that was previously attacked from the viewpoint of the deformed mean field and corresponding quasiparticle motion. As the mean field itself is generated by the nucleon-nucleon interactions, it makes sense to take a step back and consider the many-body problem rather than the resulting s.p. motion. This would allow us to avoid questions of self-consistency and take into account not only pairing and spin-orbit forces but the full inter-particle interaction. The natural framework is provided by the shell model (SM) with effective interactions; modern versions of the SM give good agreement with the data. The exact diagonalization of the Hamiltonian matrix, with all conservation laws strictly respected, provides the eigenstates in the space-fixed frame; the spectrum and observables can be analyzed similarly to the experimental data. The problem requires exploration of the parameter space of the SM Hamiltonians in a sufficiently large orbital space.

2. Signature of shape transition in atomic nuclei and atomic clusters

In the introduction we indicated that one could use the behavior of $E_b/n$ vs. $n^{-1/3}$ as a signal for shape transitions/shape coexistence in mesoscopic systems. It is easy to understand how a plot of $E_b/n$ vs. $n^{-1/3}$ could contain indications of shape change. For systems dominated by forces that saturate, the binding energy can be written generically in the form

$$E_b = -\gamma S + \epsilon V,$$

where $\gamma$ is the energy cost to create a unit area of surface, $S$ is the total surface area, $\epsilon$ is the energy per unit volume and $V$ is the total volume of the system. The number of particles in the system, $n$, is proportional to $V$, which in turn scales like $R^3$ in a quasi-spherical system. On the other hand, $S$ scales like $R^n$, or $n^{2/3}$. Therefore, the binding energy per particle is expected to behave as

$$\frac{E_b}{n} = \epsilon_b(\infty) - cn^{-1/3}$$

for a compact, quasi-spherical system. Here $\epsilon_b(\infty)$ is the binding energy per particle for an infinite system and $c$ is a constant, related to the surface tension of the system. For non-spherical particles, $S$ does not scale like $n^{2/3}$, provided that the shape is changing. Therefore, a deviation from the linear behavior with slope $c \neq 0$ predicted in Eq. (2) should be expected to accompany a departure from spherical shapes.

In Ref. [17] we analyzed three examples for which the plateau on $\frac{E_b}{n}$ vs. $n^{-1/3}$ represents a generic signature of shape change. There, we used a combination of theoretical calculations and data taken from relevant experiments to plot the binding energy per particle vs $n^{-1/3}$ for $^{20}_{\text{Si}}$, $^{4}_{\text{Sm}}$, and Lennard-Jones ($\text{LJ}_N$), demonstrating a common signature of shape evolution in each case. We then discussed these results, showing how the energetic signature can be understood in terms of a prolate growth pattern. We also contrasted the unusual behavior of $^{20}_{\text{Si}}$ clusters.
with the results for LJ$_N$. Here we examine two more examples, those of Morse clusters and Na clusters.

To further investigate the generality of the shape transition signature, we examine the case of heavy noble gas clusters whose binding energies can be described by a classical Morse potential,

$$V(R) = \epsilon \sum_{i<j} \exp \left( \rho \left( 1 - \frac{R_{ij}}{R_0} \right) \right) \left\{ \exp \left( \rho \left( 1 - \frac{R_{ij}}{R_0} \right) \right) - 2 \right\},$$

where $R$ represents a collective coordinate of all internuclear distances $R_{ij}$, and $R_0$ is the equilibrium distance for which the pair potential reaches the minimum value $\epsilon$ (i.e., the well depth); here, we apply the Morse potential in reduced coordinates by imposing $R_0 = 1$ and $\epsilon = 1$. The parameter $\rho$ describes the range of the potential, namely the potential well becomes wider as $\rho$ decreases. Here we use $\rho = 6$, which is in the middle of the range considered for this parameter.

As in the other cases, we do not attempt to describe the mechanism of formation of the two shape isomers. This process is complicated, and in general requires a more complex thermodynamical analysis [18]. Our point in treating Morse clusters is simply to show the generality of the shape transition signature described in the previous sections. The binding energies of the most stable Morse clusters are available for cluster sizes up to about a hundred [19]. The lowest two magic numbers for these clusters are 13 and 55, corresponding to completely filled geometrical shells of icosahedral symmetry. Plots of the binding energy per particle as a function of $n^{-1/3}$ for Morse clusters show clear inflections at $n$ around 13 and 55, also exhibiting a visible plateau (see e.g. Fig. 1). Examining the corresponding structures shows that over the plateau region, atoms are being added sequentially to one side of the icosahedral $n = 55$ structure. By $n = 60$, a full cap is formed, making the structure clearly prolate, as illustrated in Fig. 2.

Another example is that of Na clusters. Metallic clusters behave differently than the covalent bond clusters, such as Si$_n$ clusters. However, we were able to use our ”BigBang” algorithm [20]...
to calculate the lowest energy structure of Na\textsubscript{n} cluster with \( n \) up to 30. The same functional as for Si\textsubscript{n} clusters [21, 22] was used. The binding energies per atom vs. \( n^{-1/3} \) are presented in Fig. 3. The magic numbers 8 and 20 indicate the fermionic quantum behavior typical for an harmonic oscillator like mean field. The plot indicates once again that a plateau is developed after the magic number \( n = 20 \). Further indications that prolate structures appear at the end of the plateau, around \( n = 25 \), will be presented elsewhere, but the typical shape transition signature discussed above is also present in the case of Na\textsubscript{n} clusters.

The different mesoscopic systems described above exhibit similar growth features despite very different binding forces: (i) a prolate growth regime after a particularly stable structure is formed; (ii) a plateau on the curve of \( \epsilon_n(n) \) vs. \( n^{-1/3} \) accompanying the prolate growth; and (iii) a competition/coexistence of different shapes near the end of a long plateau. Certainly, one should not read too much in the analogy with the surface energy of macroscopic liquid drops. The shape of mesoscopic systems are also influenced by the barrier energies. However, in all cases studied we found this generic signature of phase transition to be present.

It is interesting to consider the origin of the common signature of shape transition described here. The simplest model of prolate growth is to consider a cylindrical shape with constant cross-section radius, \( R \). The volume of the system, \( \pi R^2 L \), is proportional to the number of particles in the system. The surface area \( S = (2\pi RL + 2\pi R^2) \) is also proportional to \( L \) (ignoring the endcaps), and thus to the number of particles. Therefore, according to Eq. (1), \( E_b/n \) will be independent of \( n \), for ideal prolate growth.

Among all examples of shape evolution discussed, Si\textsubscript{n} stands out as the most dramatic, as the prolate growth is far more pronounced [17]. Near \( n = 25 \), the aspect ratio of the clusters approaches 3:1. Furthermore, the end of the plateau region is marked by an abrupt transition back to compact shapes, whereas in \(^{\text{Am}}\text{Sm} \) and LJ\textsubscript{N} cases, in the linear region after the plateau, the clusters remain deformed, gradually decreasing in prolateness. It is instructive to contrast Si\textsubscript{n} and LJ\textsubscript{N} to understand the physical origin of this different behavior. In the case of LJ\textsubscript{N} atoms added to the magic LJ\textsubscript{147} core group together to form a cap, because of the attractive interaction between them. Once a cap is full, as in \( N = 160 \), the next atom can either go to a second layer on the cap, or to a cap edge to begin covering a new face of the core cluster. While

---

**Figure 2.** Shapes of the lowest energy Morse clusters (\( \rho = 6 \)) with \( n = 55 \) (compact) and \( n = 60 \) (deformed).

---
Figure 3. (Color online) Binding energies (B) per atom (in Hartree) vs $n^{-1/3}$ for the lowest and second lowest energy Na clusters.

the former would make the cluster still more prolate, the latter is energetically favored, as it maximizes the number of bonds to the ad-atom. As more atoms are added, more of the core is covered, and the cluster becomes less deformed. Finally, as the outer layer is completed, a new magic cluster is created with the previous magic cluster at its core.

In Si$_n$ the situation is different because the magic Si$_{10}$ structure that is the starting point for prolate growth does not form the core of larger compact structures. The ideal, four-fold covalent bonding of bulk Si cannot be realized in small clusters because of the number of dangling bonds that would be created on the cluster surface. Instead, the atoms rearrange to minimize the dangling bonds, leading to unusual bonding configurations in the small clusters. In larger compact structures, the interior atoms begin to adopt the tetrahedral arrangements very different from what is present in Si$_{10}$. As a result the growth in Si$_n$ is qualitatively different than in LJ$_N$ and Morse clusters. The first atoms beyond $n = 10$ form prolate caps, analogous to the situation in LJ$_N$, but as more atoms are added, their energy is lowered by creating stable subunits that stack up along a prolate axis, rather than by covering the Si$_{10}$ core. The stacking process continues out to $n = 25$, with very little change in the binding energy per particle, until compact clusters with new core structures become lower in energy. Further growth proceeds via compact structures following the behavior expected in Eq. 2.

3. The predominance of prolate deformation in nuclei

The majority of deformed nuclei have axially symmetric prolate deformation in their ground state. The presence of rotational bands with typical energy intervals is the first signature of stable deformation. The states in a band are connected by strong quadrupole transitions obeying simple rigid rotor intensity rules. The same rules [16] determine the expectation values of multipole operators. This is essentially a projection of nearly constant within the band intrinsic (body-fixed) quantities onto the laboratory (space-fixed) coordinate frame.
positive sign of the intrinsic quadrupole moment manifests the dominance of prolate deformation. The Nilsson diagram shows a pattern of split spherical single-particle orbitals that seems to be more or less symmetric with respect to the sign of deformation; the difference in the liquid drop energy for the two signs of deformation is pretty small. While the results of the numerous mean field calculations for specific nuclei in general correctly predict the presence and the sign of deformation, the underlying physical reason for the predominance of prolate shapes is unclear.

It was argued by Lemmer [23] that the kinetic energy should contain an additional contribution similar to the term $-D\ell^2$ introduced by Nilsson [24] for interpolation between the harmonic oscillator and potential box. Being split to time-conjugate pairs $(m, -m)$ by deformation, the large-$\ell$ spherical orbitals determine the single-particle (s.p.) occupancies in the way that makes the prolate case energetically favorable. Related arguments were given by Castel and Goeke [25], who showed that the collective inertial parameter is larger for the prolate deformation, and later by Castel, Rowe and Zamick [26] with the help of self-consistency conditions [16]. There are also ideas based on the semiclassical analysis of the s.p. level density [27], simplest periodic orbits [28] and their bifurcations in a deformed cavity [29]. The predominance of prolate deformation analogously emerges in s.p. motion in metallic clusters [30] and for many biological objects, from molecular level to pollen grains [31]. The latest analysis by Hamamoto and Mottelson [32] starting with the statement that “the nature of the parameters responsible for the prolate dominance has not yet been adequately understood” proceeds along similar lines. Detailed calculations and comparison between the harmonic oscillator potential and that of a spheroidal cavity show a different character of the mixing of spherical orbitals for the prolate and oblate cases. Being a surface effect, the difference should decrease in large systems. The authors of Ref. [32] mention additional factors which were not sufficiently accounted for including the role of the spin-orbit potential, pairing effects [33], and the presence of two kinds of particles.

Below we suggest a new approach to this problem that was previously attacked from the viewpoint of the deformed mean field and corresponding quasiparticle motion. As the mean field itself is generated by the nucleon-nucleon interactions, it makes sense to take a step back and consider the many-body problem rather than the resulting s.p. motion. This would allow us to avoid questions of self-consistency and take into account not only pairing and spin-orbit forces but the full inter-particle interaction. The natural framework is provided by the shell model (SM) with effective interactions; modern versions of the SM give good agreement with the data. The exact diagonalization of the Hamiltonian matrix, with all conservation laws strictly respected, provides the eigenstates in the space-fixed frame; the spectrum and observables can be analyzed similarly to the experimental data. The problem requires exploration of the parameter space of the SM Hamiltonians in a sufficiently large orbital space.

To explore the parameter space we will work with random interactions. All two-body matrix elements allowed by angular momentum, parity and isospin conservation are taken as random uncorrelated quantities. Among many realizations of such an ensemble we can identify those with definite rotational characteristics and study the abundance of such cases and their dynamic origin. Random interactions in many-body systems are usually analyzed for studies of quantum chaos [34, 35] in relation to the random matrix theory. In Ref. [36] the ensemble was constructed with rotationally invariant interactions, and only the magnitudes of allowed matrix elements were taken randomly. The unexpected output was the discovery of the predominance, for an even number of particles, of the zero ground state spin, in spite of relatively low multiplicity of $J = 0$ states in Hilbert space. This result, valid not only in a single-$j$ case but in the context of realistic SM space [37] as well, was fully understood in terms of geometry of the parameter space only for the exceptional case of the single $j = 7/2$ shell [38] and for the similar results in the interacting boson model [39]. In more realistic fermionic models the full theory is still absent; the most promising are the ideas of geometric chaoticity of random angular momentum coupling [40, 41].
The rotationally invariant random interaction models provide a wealth of information. Any given set of random interaction parameters gives rise to a specific version of the SM. It was noticed, for example, that some sequences of the lowest excited states, like $0^+ - 2^+ - 4^+ - ...$, appear with the enhanced probability. The energy ratios in such yrast sequences are spread around the values close to limiting cases of the collective models, such as vibrational and rotational bands [45]. In Ref. [42], the realizations with the lowest states $0^+ - 2^+$ were analyzed in terms of the Alaga ratio,

$$ A = \frac{Q(2^+)^2}{B(E2; 2^+ \rightarrow 0^+)} $$

of the squared expectation value of the quadrupole moment of the $2^+$ state to the reduced transition probability from this state to the g.s. The distribution of the Alaga ratio reveals two pronounced peaks, close to zero and at the value corresponding to the rigid rotor [16]. This means that the stable mean fields generated by random interactions with high probability correspond either to near-spherical or to well deformed shape. The situation is similar in the case of the interacting boson model [39].

We define the space-fixed quadrupole moment of the axially symmetric rotor with spin $J$ [16] as

$$ Q(J) = Q_0 \frac{3K^2 - J(J+1)}{(J+1)(2J+3)} \Rightarrow -Q_0 \frac{J}{2J+3}, $$

where $Q_0$ is the intrinsic (body-fixed) quadrupole moment, and we assume $K = 0$ for the yrast band of an even-even nucleus [46]. Prolate intrinsic shapes, $Q_0 > 0$, correspond to squeezed shapes around the axis of collective rotation, $Q < 0$. With the standard [16] definition of the $B(E2)$ probability for the yrast band of a well deformed rotor, the Alaga ratio (4) is equal to 4.10.

In the single-$j$ model [42], where the splitting of a $j$-level proceeds fan-like with approximate symmetry of oblate and prolate sides, the sign of $Q$ will be random. Therefore we expect the

|   | $\lambda$ | $N(0, 2)$ | $N(Q>0)/N(0, 2)$ | $N(E4/E2)$ | $N_{\text{rot}}$ | $N_{\text{rot}}^{\text{prolate}}$ |
|---|---|---|---|---|---|---|
| (a) | 0.05 | 1398 | 0.62 | 50 | 3 | 1.00 |
|   | 0.5  | 3320 | 0.54 | 322 | 100 | 0.74 |
|   | 1.0  | 3846 | 0.52 | 354 | 100 | 0.70 |
|   | 1.5  | 4056 | 0.52 | 378 | 119 | 0.72 |
|   | 2.0  | 4129 | 0.52 | 371 | 122 | 0.74 |
|   | 3.0  | 4196 | 0.52 | 367 | 125 | 0.71 |
|   | 4.0  | 4233 | 0.52 | 367 | 125 | 0.71 |
|   | 10.0 | 4295 | 0.53 | 368 | 112 | 0.74 |
| (b) | 1.0  | 3156 | 0.55 | 289 | 39 | 0.77 |
|   | 2.0  | 3153 | 0.53 | 264 | 34 | 0.79 |
|   | 3.0  | 3140 | 0.52 | 266 | 34 | 0.82 |
|   | 4.0  | 3156 | 0.53 | 240 | 35 | 0.89 |
| (c) | 1.0  | 4569 | 0.54 | 322 | 120 | 0.73 |
|   | 2.0  | 4530 | 0.52 | 339 | 116 | 0.75 |
|   | 3.0  | 4490 | 0.52 | 349 | 119 | 0.76 |
|   | 4.0  | 4461 | 0.52 | 371 | 124 | 0.81 |
| (d) | 1.0  | 2170 | 0.43 | 176 | 55 | 0.07 |
|   | 2.0  | 2185 | 0.37 | 193 | 70 | 0.06 |
|   | 3.0  | 2212 | 0.35 | 188 | 73 | 0.05 |

Table 1. The probabilities of occurrence of collective prolate configurations (see text for details).
crucial role of mixing of spherical orbitals. It was often suggested [47] that the spherical SM can realistically describe deformation if one uses at least two s.p. orbits with $\Delta j = 2$. This approach successfully produces collective bands, such as in $^{48}$Cr, and even in the double magic $^{56}$Ni [48]. The collectivity was tested not only by the $J(J+1)$ behavior, but also by large and consistent in band quadrupole moments and $B(E2)$ strengths. As a generic example, we consider a model with four neutrons and four protons in the space of two spherical orbitals, $f_{7/2}$ and $p_{3/2}$, of the same parity. This corresponds to the oversimplified SM description of $^{48}$Cr, the nucleus with well known collective properties. Without interactions, we would have half-filled $f_{7/2}$ orbitals with zero quadrupole moment. In this space the most general rotationally invariant two-body interaction is described by 30 matrix elements (see Table V of Ref. [49]). We chose the random ensemble defined by the uniform distribution of uncorrelated matrix elements between $-V$ and $+V$, and the only parameter is the ratio $\lambda = V/\epsilon$, where $\epsilon$ is the (non-random) spacing between the two orbitals. With the statistics of 10000 realizations for each value of $\lambda$, the results are presented in Table Ia.

The case of very small $\lambda = 0.05$ can be viewed as the single-$j$ result. The column $N(0,2)$ shows the number of realizations with the sequence $0^+, 2^+$ of the g.s. and the first excited state. The probability of such sequences is strongly enhanced. Before applying additional requirements we can find the sign of the expectation value (5) of the quadrupole moment $Q(2^+)$. As expected, the fraction $N(Q < 0)/N(0,2)$ of “prolate” cases, $Q < 0$, is stable near 50% (the third column). Among these states there are cases with rotational properties. The constraint that the ratio $E(4^+)/E(2^+)$ be between 3 and 3.6 (the rigid rotor would give 3.33) leaves the number of states indicated in the column $N(E4/E2)$. In the column $N_{rot}$ we count the states where, in addition to the energy criterion, the Alaga ratio (4) is between 3.90 and 4.30, an arbitrary but restrictive choice. The last column gives the fraction of states with negative sign of $Q$ among all “rotational” states. The predominance of prolate configurations is practically independent on the relative strength of mixing between the orbitals if this strength is sufficient for the onset of deformation. The full histogram for the distribution of the Alaga ratio is shown in Fig. 1 for two values of the interaction strength; this figure shows also the statistics without any rotational selection cuts. The prolate deformed peak is quite narrow. With the only energy criterion $(0^+, 2^+)$ applied, we see also a large peak for spherical configurations with $A$ close to zero, as well as intermediate background situations. In this large set, the prolate cases appear only in slightly more than 50% cases. We can also note that at very weak mixing, $\lambda = 0.05$, the numbers $N(0,2)$ and especially $N(E4/E2)$ fall sharply, and the number of rotational configurations satisfying our criteria is very small. This emphasizes the importance of orbital mixing.

Going from the isospin-symmetric case with $N = Z$ to asymmetric one, we consider next the system with six neutrons and four protons assuming the same set of two s.p. levels. The results are shown by Table Ib. The fraction of states with $Q < 0$ among all sequences $(0^+, 2^+)$ is still close to $1/2$. The total number of rotational cases is now significantly lower. At the same time, the fraction of prolate configurations is even higher than in the symmetric case. The results are essentially the same in the $N = Z$ case for the inverted level scheme when the spherical $p_{3/2}$ level is placed below the $f_{7/2}$ level, see Table Ic. Again for the very weak mixing, $\lambda = 0.01$, we do not find any rotational states; in this case all eight nucleons occupy the lowest $p_{3/2}$ level creating the closed shell. With strong level mixing, 45% of all states belong to the $(0^+, 2^+)$ sequence. Clearly, the occurrence of prolate deformation is even higher than for the normal $fp$ level sequence. This confirms the presence of the effect we mentioned in the introduction, namely the influence of the splitting of the level with higher $\ell$ when low-$|m|$ components steeply go down with the prolate deformation. The last case we consider here contains four protons and four neutrons in the space of two levels of opposite parity, $f_{5/2}$ and $g_{9/2}$, Table Id. Here the fraction of the sequences $0^+, 2^+$ is lower, and practically all rotational states have oblate deformation. It is clear that, due to parity conservation, direct mixing of split levels by the
The deformed mean field is impossible and only pairwise transfers of particles can play a role.

The main new dimension brought in by our approach is in the visualization of the entire parameter space. The mechanism of the prolate predominance is revealed as we identify the realizations of the ensemble responsible for the effect. Fig. 2. of Ref. [49] shows all 30 reduced matrix elements \( \langle j_1 j_2 | V | j_3 j_4 \rangle (J T) \) allowed by angular momentum and isospin for the strong coupling limit, \( \lambda = 4.0 \), in case (a) of Table I. The average value of each matrix element over the whole ensemble is zero, with small fluctuations (solid line). The dashed line shows the average over prolate rotational samples. Table V of Ref. [49] displays the ordering of the matrix elements and their average numerical values. Although all matrix elements are enhanced compared to the level of fluctuations, one can conclude that certain matrix elements are crucial for the transition from spherical shape to predominantly prolate axial deformation. We see the exceptional role of amplitudes 9-12 describing the transfer of a single nucleon \( f_{7/2} \leftrightarrow p_{3/2} \) in the interaction with another \( f_{7/2} \) nucleon, regardless of its isospin. Such a process is forbidden for the orbitals of opposite parity. This agrees with the idea discussed in Ref. [32]. The monopole pairing given by amplitudes 5,15 and 29 is not effective in this transition. Contrary to that, we see a large negative amplitude of quadrupole pair transfer 16. Large quadrupole-quadrupole forces in the particle-hole channel correspond to matrix elements 20-24 in the particle-particle channel. They induce the collectivization process and formation of the proper symmetry of the mean field after mixing \( p \)- and \( f \)-orbitals.

4. Conclusions
In this paper we analyzed three different mesoscopic systems, Silicon clusters, the Samarium isotope chain and Lennard-Jones clusters, that exhibit similar shape transition and coexistence phenomena. The forces involved in these three mesoscopic systems are very different: the Silicon clusters are bound by covalent bonds, Samarium nuclei by short range nuclear forces that create a quantum liquid drop, and noble gas clusters by classical interatomic potentials. Despite the diverse nature of the binding, each system exhibits the same signature of shape transition, namely a long plateau in a curve of \( e_b/n \) vs. \( n^{-1/3} \) following a size corresponding to a stable, compact structure. Further, we find that in each system, energetic competition between deformed and compact shapes occurs near the end of long plateaus, resulting in the observed coexistence of the two shape isomers in the case of \( \text{Si}_n \) and \( ^4\text{Sm} \). We conclude that these are generic features for mesoscopic systems.

We also investigate the predominance of prolate deformations in nuclei within the shell model approach. To achieve that goal we performed the exploration of the parameter space that serves as an arena for effective nucleon-nucleon interactions building the stable deformed mean field. We show the decisive role of the mixing between the valence spherical orbitals of the same parity split as a function of deformation. This mixing, different for the two sides of the axial deformation diagram, makes the prolate deformation energetically favorable with high probability. This picture is supported by the statistical analysis of the random interaction ensemble and by singling out the responsible components of the effective interaction. The process is amplified by the presence of two kinds of nucleons. Although we have shown in detail in this contribution only cases of simple configurations, the results are quite generic for a small system. The effect is driven by the surface orbitals but it still remains to see if it indeed disappears in the macroscopic limit as predicted in Ref. [32]. The random interaction ensemble provides a new powerful tool for understanding the many-body mechanisms of collective phenomena.

Acknowledgments
Support from the NSF grants No. PHY-0758099 and PHY-1068217 is acknowledged.
References

[1] R. R. Hudgins I. Motoharu and M. F. Jarrold 1999 *J. Chem. Phys.* **111** p 7865.
[2] M. F. Jarrold 1991 *Science* **252** p 1085.
[3] M. F. Jarrold and V. A. Constant 1991 *Phys. Rev. Lett.* **67** p 2994.
[4] E. C. Honee *et al.* 1933 *Nature* **366** p 42.
[5] Th. Bachels and R. Schäfer 2000 *Chem. Phys. Lett.* **324** p 365.
[6] K. M. Ho A. A. Shvartsburgh B. Pan, Z. Y. Lu C. Z. Wang J. G. Wacker J. L. Fye and M. F. Jarrold 1998 *Nature* **392** p 582.
[7] E. Kaxiras and K. A. Jackson 1993 *Phys. Rev. Lett.* **71** p 727.
[8] B. Liu Z. Y. Lu B. Pan C. Z. Wang K. M. Ho, A. A. Shvartsburg and M. F. Jarrold 1998 *J. Chem. Phys.* **109** p 9401.
[9] I. Rata A. A. Shvartsburgh M. Horoi Th. Frauenheim K. W. M. Liu and K. A. Jackson 2000 *Phys. Rev. Lett.* **85** p 546.
[10] L. R. Marin M. R. Lemes and A. Dal Pino Jr. 2003 *Phys. Rev. A* **67** p 033203.
[11] S. Yoo J. Zhao J. Wang and X. C. Zeng 2004 *J. Am. Chem. Soc.* **126** p 13845.
[12] S. Yoo and X. C. Zeng 2005 *Angew. Chem. Int. Ed.* **44** p 1491.
[13] A. Teken and B. Hartke 2005 *J. Theor. Comp. Chem.* **4** p 1119.
[14] The transition was observed for both cations and anions, although for anions it takes place for n=27-29 (see e.g. Fig. 2 of Ref. [1]).
[15] N. V. Zamfir *et al.* 1999 *Phys. Rev. C* **60** p 054312.
[16] A. Bohr and B.R. Mottelson 1989 *Nuclear Structure*, vol. 2 (Benjamin, New York).
[17] M. Horoi and K.A. Jackson 2006 *Chem. Phys. Lett.* **427** p 147.
[18] J. P. K. Doye M. A. Miller and D. J. Wales 1999 *J. Chem. Phys.* **110** p 6896.
[19] D. J. Wales and J. P. K. Doye 1997 *J. Phys. Chem. A* **101** p 5111.
[20] K.A. Jackson M. Horoi I. Chaudhuri Th. Frauenheim and A.A. Shvartsburg 2006 *Comp. Mat. Sci.* **35** p 232.
[21] K.A. Jackson M. Horoi I. Chaudhuri Th. Frauenheim and A. A. Shvartsburg 2004 *Phys. Rev. Lett.* **93** p 013401.
[22] J. P. Perdew K. Burke and M. Ernzerhof 1996 *Phys. Rev. Lett.* **77** p 3865.
[23] R.H. Lemmer 1960 *Phys. Rev.* **117** p 1551.
[24] S.G. Nilsson 1955 *Kgl. Dan. Vid. Selsk. Mat.-fys. Medd.* **29**, No. 16.
[25] B. Castel and K. Goeke 1976 *Phys. Rev. C* **13** p 1765.
[26] B. Castel D.J. Rowe and L. Zamick 1990 *Phys. Lett. B* **236** p 121.
[27] H. Frisk 1990 *Nucl. Phys.* **A511** p 309.
[28] M.A. Deleplanque *et al.* 2004 *Phys. Rev. C* **69** p 044309.
[29] K. Arita A. Sugita and K. Matsuyanagi 1998 *Czechoslovak J. Phys.* **48** p 821.
[30] I. Hamamoto B.R. Mottelson H. Xie and X.Z. Zhang 1991 *Zeitschr. Phys. D* **21** p 163.
[31] J. Martin M. Torrell A.A. Korobkov and J. Valles 2003 *Plant Biol. No. 5* p 85.
[32] I. Hamamoto and B.R. Mottelson 2009 *Phys. Rev. C* **79** p 034317.
[33] N. Tajima Y.R. Shimizu and N. Suzuki 2002 *Prog. Theor. Phys. Suppl.* **146** p 628.
[34] T.A. Brody *et al.* 1981 *Rev. Mod. Phys.* **53** p 385.
[35] V. Zelevinsky B.A. Brown N. Frazier and M. Horoi 1996 *Phys. Rep.* **276** p 85.
[36] C.W. Johnson G.F. Bertsch and D.J. Dean 1998 *Phys. Rev. Lett.* **80** p 2749.
[37] M. Horoi B.A. Brown and V. Zelevinsky 2001 *Phys. Rev. Lett.* **87** p 062501.
[38] P. Chau Hun-Tai A. Frank N.A. Smirnova and F. Van Isacker 2002 *Phys. Rev. C* **66** p 061302.
[39] R. Bijker and A. Frank 2002 *Phys. Rev. C* **65** p 044316.
[40] D. Mulhall A. Volya and V. Zelevinsky 2000 *Phys. Rev. Lett.* **85** p 4016.
[41] T. Papenbrock and H.A. Weidenmüller 2005 *Nucl. Phys.* **A757** p 422.
[42] V. Zelevinsky and A. Volya 2004 *Phys. Rep.* **391** p 311.
[43] Y.M. Zhao A. Arima and N. Yoshinaga 2004 *Phys. Rep.* **400** p 1.
[44] H.A. Weidenmüller and G.E. Mitchell 2008 *arXiv* 0807.1070.
[45] C.W. Johnson H. Nam 2007 *Phys. Rev. C* **75** p 047305.
[46] In Ref. [49] the similar Eq. (2) contains a missing 2 typo that is fixed here. However, the reported calculations were performed with the correct formula.
[47] E. Caurier *et al.* 2005 *Rev. Mod. Phys.* **77** p 427.
[48] M. Horoi *et al.* 2006 *Phys. Rev. C* **73** p 061305(R).
[49] M. Horoi and V. Zelevinsky 2010 *Phys. Rev. C* **81** p 034306.