Measure and capacity of wandering domains in Gevrey near-integrable exact symplectic systems

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Abstract

A wandering domain for a diffeomorphism \(\Psi\) of \(\mathbb{A}^n = T^*\mathbb{T}^n\) is an open connected set \(W\) such that \(\Psi^k(W) \cap W = \emptyset\) for all \(k \in \mathbb{Z}^+\). We endow \(\mathbb{A}^n\) with its usual exact symplectic structure. An integrable diffeomorphism, i.e. the time-one map \(\Phi^h\) of a Hamiltonian \(h: \mathbb{A}^n \rightarrow \mathbb{R}\) which depends only on the action variables, has no nonempty wandering domains. The aim of this paper is to estimate the size (measure and Gromov capacity) of wandering domains in the case of an exact symplectic perturbation of \(\Phi^h\), in the analytic or Gevrey category. Upper estimates are related to Nekhoroshev theory, lower estimates are related to examples of Arnold diffusion. This is a contribution to the “quantitative Hamiltonian perturbation theory” initiated in previous works on the optimality of long term stability estimates and diffusion times; our emphasis here is on discrete systems because this is the natural setting to study wandering domains.

We first prove that the measure (or the capacity) of these wandering domains is exponentially small, with an upper bound of the form \(\exp(-c(\frac{1}{\epsilon})^\frac{1}{2n})\), where \(\epsilon\) is the size of the perturbation, \(\alpha \geq 1\) is the Gevrey exponent (\(\alpha = 1\) for analytic systems) and \(c\) is some positive constant depending mildly on \(h\). This is obtained as a consequence of an exponential stability theorem for near-integrable exact symplectic maps, in the analytic or Gevrey category, for which we give a complete proof based on the most recent improvements of Nekhoroshev theory for Hamiltonian flows, and which requires the development of specific Gevrey suspension techniques.

The second part of the paper is devoted to the construction of near-integrable Gevrey systems possessing wandering domains, for which the capacity (and thus the measure) can be estimated from below. We suppose \(n \geq 2\), essentially because KAM theory precludes Arnold diffusion in too low a dimension. For any \(\alpha > 1\), we produce examples with lower bounds of the form \(\exp(-c(\frac{1}{\epsilon})^{\frac{1}{2n-1}})\). This is done by means of a “coupling” technique, involving rescaled standard maps possessing wandering discs in \(\mathbb{A}\) and near-integrable systems possessing periodic domains of arbitrarily large periods in \(\mathbb{A}^{n-1}\). The most difficult part of the construction consists in obtaining a perturbed pendulum-like system on \(\mathbb{A}\) with periodic islands of arbitrarily large periods, whose areas are explicitly estimated from below. Our proof is based on a version due to Herman of the translated curve theorem.

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0 Introduction

0.1 Let $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ be the cotangent bundle of the torus $\mathbb{T}^n$, endowed with its usual angle-action coordinates $(\theta, r)$ and the usual exact symplectic form. What we call an integrable diffeomorphism is the time-one map of the flow generated by an integrable Hamiltonian, i.e. a Hamiltonian function which depends only on the action variables $r$; the phase space is then foliated into invariant tori $\mathbb{T}^n \times \{r^0\}$, $r^0 \in \mathbb{R}^n$, carrying quasiperiodic motions. We are interested in near-integrable systems, i.e. exact symplectic perturbations of an integrable diffeomorphism, and their wandering sets.

A wandering set for a diffeomorphism $\Psi$ of $\mathbb{A}^n$ is a subset $W \subset \mathbb{A}^n$ whose iterates $\Psi^k(W)$, $k \in \mathbb{Z}$, are pairwise disjoint. The Poincaré recurrence theorem shows that any wandering set of an integrable diffeomorphism has zero Lebesgue measure. It is more difficult to prove that wandering sets with positive measure may exist for near-integrable systems. In fact KAM theory shows that they cannot exist for $n = 1$ (at least when restricting to perturbations of non-degenerate integrable maps). In [MS04], examples of near-integrable systems possessing nonempty wandering domains, thus with positive measure, have been constructed for any $n \geq 2$.

The aim of the present work is to estimate, from above and from below, the possible “size” of the wandering sets of a near-integrable system as a function of the “size of the perturbation”. We define the size of the perturbation to be the distance between the near-integrable system and the integrable map of which it is a perturbation, assuming that all our functions belong to a Gevrey class and measuring this distance in the Gevrey sense. For Gevrey classes, we use the notation $G^\alpha$, where $\alpha \geq 1$ is a real parameter; recall that $G^1$ coincides with the analytic class, while the classes $G^\alpha$, $\alpha > 1$, are larger and more flexible, in particular they contain bump functions.

As for the size of wandering sets, we consider two natural candidates: the Lebesgue measure and the Gromov capacity. The former can be seen as the “maximal” possible one and is related to the theory of transport. The latter is the “minimal” possible one and is directly related to the symplectic character of our problem.

Our aim is to find explicit upper bounds for the measure of wandering Borel subsets and “test” their optimality by constructing examples of near-integrable systems possessing wandering sets whose capacity is estimated from below.

Our upper estimates are closely related to the long term stability estimates in perturbation theory, initiated by Nekhoroshev. Since usual estimates deal with continuous systems, we first have to transfer the whole theory to the discrete setting, which is done by adapted suspension techniques. The resulting estimates hold for near-integrable systems in all Gevrey classes $G^\alpha$, $\alpha \geq 1$: for such systems, the actions remain almost constant during exponentially long times. As a consequence, taking into account the measure-preserving character of symplectic diffeomorphisms, we prove that for any $G^\alpha$ near-integrable system the measure of a wandering Borel subset is exponentially small with respect to the size of the perturbation.

Our lower estimates deal with the capacity of the wandering sets. The interest of this is twofold: first, the measure of a set is always larger than a positive power of its capacity.

\[\text{We call domain an open connected subset of } \mathbb{A}^n.\]
(up to an explicit multiplicative factor), so lower bounds for the capacity entail lower bounds for the measure; second, capacity is a truly symplectic notion. The capacity of a set is in general extremely difficult to compute, however we will design our “unstable” examples so that they admit wandering polydiscs (i.e. products of discs in each factor of $\mathbb{A}^n$). In that case the capacity is just the minimum of the areas of the factors. Our constructions rely strongly on the existence of bump functions. So, as in [MS03] and [MS04], we produce our examples in the classes $G^\alpha$ for $\alpha > 1$ only. A striking fact is that the lower bounds on the capacity admit the same exponential form as the upper estimates deduced from Nekhoroshev theory. This is the main result we get here about instability in near-integrable systems. Related results on the existence of periodic domains with large periods will also be obtained in the course of the proof of the main instability result.

Before stating our results more precisely, let us now give a brief overview of the evolution of results in perturbation theory, in order to introduce the main tools we will use in the sequel.

0.2 We begin with stability results. The stability problem for perturbations of integrable Hamiltonian systems originated in the first investigations on the secular stability of the solar system. At this early stage, the main question was to understand the behaviour of the linearised equations along a particular solution. Then, under the influence of Poincaré, this purely local vision experienced a drastic metamorphosis towards a global qualitative understanding of the asymptotic behaviour of the orbits. He introduced a fundamental tool—amongst many others—for such a qualitative description: the method of normal forms, that is, the construction of simplified systems which nevertheless exhibit the pre-eminent features of the initial one. The theory of normal forms was further developed by Birkhoff, and thoroughly investigated since then by a number of authors.

It was also a fundamental contribution of Poincaré to distinguish between two different modes of “convergence” of series: convergence au sens des astronomes and convergence au sens des géomètres. The latter coincides with our usual notion of convergence, while the former is related to the notion of asymptotic expansion and does not exclude the possibility of performing “least term summation” (which is itself intimately related to the Gevrey nature of the series at hand).

This bunch of ideas was applied in particular to the fundamental problem of dynamics, that is, the study of the qualitative dynamical behaviour of analytic Hamiltonian systems on $\mathbb{A}^n$ which are perturbations of integrable Hamiltonians. The various problems of convergence of the series giving rise to the solutions of the perturbed problem were extensively examined by Poincaré, without however reaching a definitive conclusion.

The next major breakthrough was due to Kolmogorov, who understood in the 1950s how to take advantage of stability properties exhibited by the unperturbed quasiperiodic tori, provided that their fundamental frequencies are sufficiently nonresonant (i.e. satisfy Diophantine conditions). This approach was then generalised by Arnold and Moser and gave rise to the so-called KAM theory: in appropriate function spaces (analytic, $C^\omega$, finitely differentiable) the surviving tori form a subset whose relative measure tends to 1 when the size of the perturbation tends to 0. See [Du14] for a non-technical historical account and appropriate references on KAM theory.

Besides the KAM theorem, one major achievement occurred in the 1980s with the direct
proof by Eliasson of the convergence of the perturbative series under the usual assumptions of KAM theory. This yields directly the quasiperiodic solutions, and the KAM tori are nothing but their closure. So, after a somewhat surprising detour, the Poincaré convergence \textit{au sens des géomètres} indeed gives rise to invariant geometric objects.

In a different direction, the Poincaré convergence \textit{au sens des astronomes} can be considered as the mechanism at work behind “exponential stability”. Exponential stability means that, for a perturbation of an integrable Hamiltonian flow, the action variables of an arbitrary solution vary little over a time interval of the order of \(\exp\left(\frac{1}{\varepsilon^a}\right)\), where \(\varepsilon\) is the size of the perturbation and \(a\) is a positive exponent independent of the perturbation. This was established by Nekhoroshev in the 1970s for the analytic case. Again, the theory of normal forms revealed itself to be of crucial importance in this setting. The main idea there is to cover the whole phase space by a patchwork of domains with various resonant structures and perform in each of them a finite—but long—sequence of adapted normalising transformations. An additional geometric argument (steepness) then proves the confinement of the actions for all initial conditions over an exponentially long timescale.

In this text we will take advantage of the numerous improvements of the stability estimates after Nekhoroshev’s initial work, beginning with the work by Lochak [Lo92] where the question of the optimality of the stability exponent first appeared, with a first conjecture on its value. Then the “likely optimal” stability exponent was derived more precisely in the case of quasi-convex unperturbed systems by Lochak, Neishtadt and Niederman, and Pöschel (see [LNN92, Pö93] and references therein). Finally, based on Herman’s ideas, these works were later generalised to general Gevrey classes in [MS03]—which also clarified the connection with Gevrey asymptotics and least term summation—and the stability exponent was improved in [BMT11] to reach the probably optimal value. Our present study relies on these latter two works to produce upper estimates of the measure of wandering sets.

More surprisingly, the construction of our \textit{unstable} examples will also heavily rely on KAM techniques (in the form developed by Herman for invariant curves on the annulus). So stability results may also help produce unstable behaviour and transport in phase space. This has already been noticed in the context of Arnold diffusion (see below), but we will deal here with new phenomena.

0.3 Let us now pass to the description of some unstable systems, beginning with the seminal and highly inspiring Arnold example. In parallel with the evolution of stability theory, Arnold introduced in the 1960s a paradigm perturbed angle-action system exhibiting unstable behaviour [Ar64]. In his example (a non-autonomous nearly integrable Hamiltonian flow on \(\mathbb{A}^2\)), the action variables drift over intervals of fixed length \textit{whatever the size of the perturbation}. Arnold conjectured that this instability phenomenon (now called Arnold diffusion) should occur in the complement of the KAM tori for “typical” systems. Of course, due to Nekhoroshev theory, Arnold diffusion in analytic or Gevrey systems has to be exponentially slow with respect to the size of the perturbation.

The key idea in Arnold’s example is the possibility that a perturbation of an integrable Hamiltonian can create a continuous family of hyperbolic tori in a given energy level, whose invariant manifolds also vary continuously. An additional perturbation then makes the
stable and unstable manifolds of each torus intersect transversely in their energy level. It is therefore possible to exhibit ordered families \((T_m)_{1 \leq m \leq m_*}\) of hyperbolic tori, extracted from the continuous one, such that the unstable manifold \(W^u(T_m)\) intersects transversely the stable manifold \(W^s(T_{m+1})\), and such that the distance between the extremal tori \(T_1\) and \(T_{m_*}\) (in the action space) is independent of the size of the perturbation (the number \(m_*\) of tori in such a family tends to \(+\infty\) when the size of the perturbation tends to 0). Finally, one constructs orbits which shadow the consecutive heteroclinic orbits between the tori and pass close to both \(T_1\) and \(T_{m_*}\). The action variables of such orbits therefore experience a drift which is independent of the size of the perturbation.

Arnold’s example has been generalised in many ways, particularly in view of proving the “generic” occurrence of Arnold diffusion in nearly integrable systems on \(\mathbb{A}^3\). Notice that the existence of hyperbolic KAM tori (or more generally hyperbolic Mather sets) is an important tool to implement the previous scheme: this is a first example of how stability induces instability.

Another important development was the possibility of computing the drifting time of unstable orbits in examples of Arnold diffusion. This program was achieved in [MS03] for \(G^\alpha\) Gevrey systems with \(\alpha > 1\), and then in [LM05, Zh11] for analytic systems.

In this work, to produce examples of near-integrable systems possessing wandering polydiscs, we develop the method of [MS04], which itself builds on the techniques of [MS03]. The construction of unstable orbits in [MS03] is rather different from that of Arnold, even if, \textit{a posteriori}, one can see that these drifting orbits too shadow families of heteroclinically connected tori, which clearly shows the intrinsic complexity of their dynamics.

The key idea in [MS03] is to embed a well-controlled discrete dynamical system of \(\mathbb{A}\), namely a renormalised standard map, into a high iterate of a specific \(G^\alpha\) near-integrable system of \(\mathbb{A}^n\), assuming \(\alpha > 1\) (and then into a nearly integrable non-autonomous Hamiltonian flow). Taking advantage of the drifting points of the standard map, one can produce drifting orbits in this near-integrable system (and then in the corresponding Hamiltonian flow). In contrast with Arnold’s example, such a construction yields systems with orbits \textit{bi-asymptotic to infinity} in action. This proves to be a crucial feature of the construction when one builds on it to produce wandering sets with positive measure, since they cannot be confined inside compact subsets due to the preservation of volume.

In [MS04], the KAM theorem was used to produce wandering polydiscs surrounding drifting orbits in near-integrable systems of the same kind as those of [MS03], but without any quantitative estimate. It seems that such a coexistence of stable geometric objects (invariant tori) and highly unstable open sets had not been observed before, although it is reminiscent of the existence of the “periodic islands” in “chaotic seas” which are ubiquitous in the theory of two-dimensional symplectic maps. Our wandering domains coexist with (and are contained in the complement of) all the invariant compact subsets, including Lagrangian invariant tori, lower-dimensional invariant tori (like the hyperbolic ones used in Arnold’s mechanism) or Mather sets.

The novelty of the present paper is that, using more refined versions of the KAM theorem and a (much) better control of the normal forms, we are now able to estimate the capacity of the wandering polydiscs that we construct.
We can now informally describe the content of this paper. Our interest in wandering sets makes it essential that we deal with diffeomorphisms rather than flows, like in [MS04] and in contrast with most of the literature on Hamiltonian perturbation theory; in fact, it is the first time that wandering sets of near-integrable discrete systems are the object of such detailed investigation.

So we first have to transfer the known stability results for Hamiltonian flows to the setting of near-integrable discrete systems. The result is Theorem A whose simplified statement is the following. We fix a real $\alpha \geq 1$ and an integrable diffeomorphism of $\mathbb{A}^n$

$$\Phi^h : (\theta, r) \mapsto (\theta + \nabla h(r), r),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a $G^\alpha$ convex function. Then, given $\rho > 0$, for any $G^\alpha$ exact symplectic diffeomorphism $\Psi$ having $\varepsilon := \text{dist}_{G^\alpha}(\Psi, \Phi^h)$ small enough (see Section 1.1 for the precise definition of the distance in Gevrey classes), the iterates $\Psi^k(\theta^{[0]}, r^{[0]}) = (\theta^{[k]}, r^{[k]})$ of any initial condition satisfy

$$\|r^{[k]} - r^{[0]}\| \leq \rho \quad \text{for} \quad 0 \leq k \leq \exp \left( c \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2n\alpha}} \right),$$

where $c$ is a suitable positive constant depending mildly on $h$.

Other and more refined estimates are also available, for which the confinement radius of the action variables tends to 0 with $\varepsilon$.

Our proof makes use of a new suspension result, Theorem B which allows one to embed a $G^\alpha$ near-integrable system into a non-autonomous near-integrable Hamiltonian flow; the analytic case is essentially done in [KP94], while for the case $\alpha > 1$ we had to devise specific Gevrey techniques to adapt quantitatively Douady’s method [Dou82] based on generating functions.

Next, taking into account the preservation of the Lebesgue measure $\mu$ by symplectic diffeomorphisms, we prove in Theorem C that a wandering Borel set $W$ of $\Psi$, when it is contained in a bounded region of $\mathbb{A}^n$, must satisfy

$$\mu(W) \leq \exp \left( - c \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2n\alpha}} \right),$$

where $c > 0$ is a suitable constant depending mildly on $h$.

The rest of the paper is devoted to the construction of examples with wandering domains, with estimates of their Gromov capacity. This is the content of Theorem D there is a sequence $(\Psi_j)_{j \geq 1}$ of $G^\alpha$ diffeomorphisms of $\mathbb{A}^n$, with $\alpha > 1$, such that

$$\varepsilon_j := \text{dist}_{G^\alpha}(\Psi_j, \Phi^h) \left( r_j^2 + \cdots + r_n^2 \right)$$

tends to 0 when $j \to \infty$, each of which admits a wandering polydisc $\mathcal{W}_j$ whose Gromov capacity satisfies the inequality

$$C_G(\mathcal{W}_j) \geq \exp \left( - c' \left( \frac{1}{\varepsilon_j} \right)^{\frac{1}{2(n-1)(\alpha-1)}} \right),$$

where $c' > 0$ is a suitable constant.
The proof of Theorem C is based on a version of the “coupling lemma” introduced in [MS03] and [MS04], whose application requires the construction of several controlled dynamics on distinct subfactors of the annulus $\mathbb{A}^n$. Here is, in a few lines, the strategy:

– On the one hand, Theorem D provides us with a near-integrable system $G$ arbitrarily close to $\Phi^{\frac{1}{2}(r_2^2+\cdots+r_n^2)}$, possessing a $q$-periodic polydisc $\mathcal{V} \subset \mathbb{A}^{n-1}$ of arbitrarily large period $q$. Moreover the orbit of $\mathcal{V}$ under $G$ is controlled well enough and its capacity can be explicitly bounded from below.

– On the other hand, we choose a Gevrey function $U$ so that the “rescaled standard map”

$$\psi: (\theta_1, r_1) \mapsto (\theta_1 + qr_1, r_1 - \frac{1}{q}U'(\theta_1 + qr_1))$$

has a wandering domain $\mathcal{W} \subset \mathbb{A}$ whose area is of order $1/q$. We observe that this map can be written as the composition of the time-one map of the Hamiltonian $\frac{1}{q}U(\theta_1)$ with the $q$th iterate of the integrable map $\Phi^{\frac{1}{2}r_2^2}$.

– The aforementioned coupling lemma then produces an exact symplectic perturbation $\Psi$ of $\Phi^{\frac{1}{2}r_2^2} \times \Phi^{\frac{1}{2}(r_2^2+\cdots+r_n^2)} = \Phi^{\frac{1}{2}(r_1^2+r_2^2+\cdots+r_n^2)}$ whose $q$th iterate coincides with $\psi \times G^q$ when restricted to $\mathbb{A} \times \mathcal{V}$. In that situation, $\mathcal{W} \times \mathcal{V}$ is easily seen to be a wandering set of $\Psi$.

Choosing an appropriate function $U$ and estimating the area of the wandering domain $\mathcal{W}$ are rather easy.

The application of the coupling lemma requires a Gevrey function $g$ on $\mathbb{A}^{n-1}$ satisfying a “synchronization condition” with respect to the orbit of $\mathcal{V}$ under $G$. For this, we use a bump function, whose Gevrey norm is large, unavoidably, but somehow this can be compensated by choosing $q$ large enough, so as to ensure that $\Psi$ is indeed arbitrarily close to integrable.

Much more work will be needed to prove Theorem D. Another use of the coupling lemma will first allow us to reduce the problem to proving the two-dimensional version of the statement, which is essentially Theorem F. The proof of Theorem F is then the most technical part of the construction. It first necessitates the introduction of a “pseudo-pendulum” on $\mathbb{A}$, of the form

$$P(\theta, r) = \frac{1}{2}r^2 + \frac{1}{N^2}V(\theta),$$

where the potential $V$ is a flat-top bump function on $\mathbb{T}$ and $N$ is a large parameter. This pseudo-pendulum is then perturbed to produce elliptic periodic points of any period, surrounded by elliptic islands with controlled areas. The main task consists in estimating these areas, which requires the use of Herman’s quantitative version of the two-dimensional KAM theorem and necessitates the computation of high-order parametrised normal forms. Theorem F is interesting in itself; see for instance [Li04] for related questions on standard maps.

0.5 To conclude this introduction, let us mention that the present work can be seen as a contribution to the development of a “quantitative Hamiltonian perturbation theory”,

focused on the question of the size of wandering domains. Other studies should be devoted
to the numerous quantities one can associate with a nearly integrable exact symplectic
diffeomorphism or Hamiltonian flow: one may think of the separatrix splitting, the angles
of Green bundles, the topological entropy, the growth of isolated periodic orbits, and so
on; each of them should be estimated from above and below in an optimal way. While
the upper estimates may be based on normal form theory, the construction of “optimal”
examples may reveal itself to be extremely rich and difficult, as illustrated by the case
of wandering domains in this work. We see this problem as a challenging motivation to
pursue these quantitative studies and get a more developed vision of this domain—still in
its infancy.

0.6 The paper is organized as follows.

- Section 1 is dedicated to a precise formulation of our assumptions, notations and
results. Theorem A gives long time stability estimates for near-integrable systems
of $\mathbb{A}^n$. Theorem B and Theorem C respectively state our main results about the
upper and lower bounds for the measure and capacity of wandering sets of near-
integrable systems. Theorem D on the construction of near-integrable systems
of $\mathbb{A}^{n-1}$ possessing periodic domains with explicit lower bounds for their capac-
ity, is stated. It splits into two parts: Theorem D(i) deals with the two-dimensional
case, i.e. periodic domains in $\mathbb{A}$, while Theorem D(ii) is dedicated to systems on $\mathbb{A}^m$, $m \geq 2$.

- In Section 2 we state and prove Theorem E on the suspension of Gevrey near-
integrable systems. This enables us to deduce the stability theory for Gevrey diffeo-
morphisms from the stability theory for Gevrey Hamiltonian flows and thus prove
Theorem A and then Theorem B.

- Section 3 contains the most technical part of the paper, that is, the construction of
examples of near-integrable systems of $\mathbb{A}$ with periodic islands of arbitrarily large
period, whose area we are able to estimate from below. This is the content of Theo-
rem F which is a parametrised version of Theorem D(i). The proof of Theorem D(i)
is in Section 3.2, the rest of Section 3 is devoted to the proof of Theorem F.

- In Section 4 we explain the coupling lemma and its use to produce periodic or
wandering polydisks. The proof of Theorem D(ii) is thus obtained, by coupling
the periodic domains of Theorem F (suitably rescaled) with periodic domains of an
elementary perturbation of $\Phi^{\frac{1}{2}(r_1^2+\cdots+r_n^2)}$. Then, Theorem C is obtained by coupling
the wandering domain of a rescaled standard map and the periodic domains of
Theorem D.

- The paper ends with four appendices, dealing with some technicalities needed in the
course of the various proofs.
1  Presentation of the results

1.1  Perturbation theory for analytic or Gevrey near-integrable maps—Theorem A

1.1.1 Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $n \geq 1$ we denote by $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ the $2n$-dimensional annulus, viewed as the cotangent bundle of $\mathbb{T}^n$, with coordinates $\theta = (\theta_1, \ldots, \theta_n)$, $r = (r_1, \ldots, r_n)$, respectively called “angles” and “actions”, and Liouville exact symplectic form $\Omega = -d\lambda$, $\lambda := \sum_{j=1}^n r_j d\theta_j$. Recall that a map $\Psi$ is said to be exact symplectic (or globally canonical) if the differential 1-form $\Psi^* \Omega - \lambda$ is exact. Examples of exact symplectic maps are provided by the flows of Hamiltonian vector fields.

When the Hamiltonian vector field is generated by a function $H$ on $\mathbb{A}^n$
\[
\frac{\partial H}{\partial r_1} \frac{\partial}{\partial \theta_1} + \cdots + \frac{\partial H}{\partial r_n} \frac{\partial}{\partial \theta_n} - \frac{\partial H}{\partial \theta_1} \frac{\partial}{\partial r_1} - \cdots - \frac{\partial H}{\partial \theta_n} \frac{\partial}{\partial r_n}
\]
is complete, we denote by $\Phi^H_1: \mathbb{A}^n \to \mathbb{A}^n$ the time-one map of the Hamiltonian flow. We say that a diffeomorphism of $\mathbb{A}^n$ is integrable when it is of the form $\Phi^h_1$, where the function $h$ depends only on the action variable $r$, thus
\[
\Phi^h_1(\theta, r) = (\theta + \langle \nabla h(r) \rangle, r),
\]
where $\langle \cdot \rangle: \mathbb{R}^n \to \mathbb{R}^n$ is our notation for the canonical projection.

We are interested in near-integrable maps, i.e. exact symplectic maps close to an integrable map $\Phi^h_1$, closeness being intended in the analytic sense or Gevrey sense.

1.1.2 Let us introduce notations for the spaces of Gevrey functions. Given $n \geq 1$, we use the Euclidean norm in $\mathbb{R}^n$ and, for $R$ positive real or infinite, denote by $\mathcal{B}_R$ the closed ball of radius $R$ centred at 0 (so $\mathcal{B}_\infty = \mathbb{R}^n$). We set
\[
\mathbb{A}^n_R := \mathbb{T}^n \times \mathcal{B}_R,
\]
in particular $\mathbb{A}^n_{\mathbb{Z}} = \mathbb{A}^n$. Given $\alpha \geq 1$ and $L > 0$ real, and $0 < R \leq \infty$, we define the Banach spaces of real-valued functions
\[
G^\alpha,L(\mathbb{A}^n_R) := \left\{ h \in C^\infty(\mathbb{A}^n_R) \mid \|h\|_{\alpha,L,R} < \infty \right\}, \quad \|h\|_{\alpha,L,R} := \sum_{\ell \in \mathbb{N}^n} \frac{L^{\ell \alpha}}{\ell!^{\alpha}} \|\partial^\ell h\|_{C^0(\mathbb{A}^n_R)} \quad (1.3)
\]
\[
G^\alpha,L(\mathbb{A}^n_R) := \left\{ f \in C^\infty(\mathbb{A}^n_R) \mid \|f\|_{\alpha,L,R} < \infty \right\}, \quad \|f\|_{\alpha,L,R} := \sum_{\ell \in \mathbb{N}^{2n}} \frac{L^{\ell \alpha}}{\ell!^{\alpha}} \|\partial^\ell f\|_{C^0(\mathbb{A}^n_R)} \quad (1.4)
\]
We have used the standard notations $|\ell| = \ell_1 + \cdots + \ell_{2n}$, $\ell! = \ell_1! \cdots \ell_{2n}!$, $\partial^\ell = \partial_{x_1}^{\ell_1} \cdots \partial_{x_{2n}}^{\ell_{2n}}$, $\partial_k = \partial_{x_k}$ for $k = 1, \ldots, 2n$, where $(x_1, \ldots, x_{2n}) = (\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$.

\footnote{We also often call them “near-integrable systems” to emphasize that we are interested in the discrete dynamical systems consisting in iterating these maps. When we say “near-integrable”, the exact symplectic character is understood.}
We shall make use of the natural inclusion $G^{\alpha,L}(\mathbb{B}_R) \hookrightarrow G^{\alpha,L}(\mathbb{A}^n_R)$ without further notice, treating an $h \in G^{\alpha,L}(\mathbb{B}_R)$ indifferently as an element of any of the two spaces.

For $\alpha = 1$, one recovers real analytic functions of $\mathbb{A}^n_R$: any function $f \in G^{1,L}(\mathbb{A}^n_R)$ is real analytic in $\mathbb{A}^n_R$ and admits a holomorphic extension in $\mathcal{V} \mathbb{T}^n \times \mathcal{V} \mathbb{B}_R$, with complex neighbourhoods of $\mathbb{T}^n$ and $\mathbb{B}_R$ defined by

$$\mathcal{V} \mathbb{T}^n := \{ \theta \in (\mathbb{C}/\mathbb{Z})^n \mid (\text{Im} \theta_1, \ldots, \text{Im} \theta_n)_{|\infty} < L \},$$

$$\mathcal{V} \mathbb{B}_R := \bigcup_{r \in \mathbb{B}_R} \{ r \in \mathbb{C}^n \mid |r - r^*|_{|\infty} < L \}, \quad (1.5)$$

where $|\xi|_{|\infty} := \max\{||\xi||_{1}, \ldots, ||\xi||_{n}\}$ for $\xi \in \mathbb{R}^n$ or $\mathbb{C}^n$; conversely, for any function $f$ real analytic in $\mathbb{A}^n_R$, there exists $L > 0$ such that $f \in G^{1,L}(\mathbb{A}^n_R)$. For $\alpha > 1$, one gets non-quasianalytic spaces of Gevrey functions.

Recall that $\| \cdot \|_{\alpha,L}$ is an algebra norm for every $\alpha \geq 1$: $\|fg\|_{\alpha,L} \leq \|f\|_{\alpha,L} \|g\|_{\alpha,L}$; see [MS03]—some other useful properties of these norms are recalled in Appendix B.1.

1.1.3 Our maps will be analytic, i.e. Gevrey-1, or more generally Gevrey-$\alpha$ for some $\alpha \geq 1$, i.e. elements of one of the sets

$$G^{\alpha,L}(\mathbb{A}^n_R, \mathbb{A}^n) := \{ \Psi : \mathbb{A}^n_R \rightarrow \mathbb{T}^n \times \mathbb{R}^n \mid \exists \tilde{\Psi} : \mathbb{A}^n_R \rightarrow \mathbb{R}^n \times \mathbb{R}^n \text{ lifting } \Psi \}
\text{so that } \tilde{\Psi}_1, \ldots, \tilde{\Psi}_2n \in G^{\alpha,L}(\mathbb{A}^n_R) \},$$

with $L > 0$ and $0 < R \leq \infty$. We set, for any $\Delta \in G^{\alpha,L}(\mathbb{A}^n_R, \mathbb{A}^n)$,

$$|||\Delta|||_{\alpha,L,R} := \inf \left\{ ||\Delta_1||_{\alpha,L,R} + \cdots + ||\Delta_2n||_{\alpha,L,R} \mid \Delta : \mathbb{A}^n_R \rightarrow \mathbb{R}^n \times \mathbb{R}^n \text{ lift of } \Delta \right\} \quad (1.6)$$

(in fact the infimum is always attained); one can check that the formula

$$(\Psi_1, \Psi_2) \rightarrow |||\Psi_2 - \Psi_1|||_{\alpha,L,R}$$

defines a translation-invariant distance which makes $G^{\alpha,L}(\mathbb{A}^n_R, \mathbb{A}^n)$ a complete metric space.

1.1.4 Our first result is a version of the Nekhoroshev Theorem for Gevrey near-integrable exact symplectic maps in the convex case:

**Theorem A** (Exponential stability for maps). Let $n \geq 1$ be an integer. Let $\alpha \geq 1$ and $L, R, R_0 > 0$ be reals such that $R < R_0$. Let $h \in G^{\alpha,L}(\mathbb{B}_R)$ have positive definite Hessian matrix on $\mathbb{B}_R$. Then there exist positive reals $\varepsilon_*, c_\varepsilon$, and, for each positive $\rho < R_0 - R$, positive reals $\varepsilon'_\rho \leq \varepsilon_*$ and $c'_\rho \leq c_\varepsilon$, and, for each positive $\sigma < \frac{1}{n+1}$, positive reals $\varepsilon''_\sigma \leq \varepsilon_*$ and $c''_\sigma \leq c_\varepsilon$, satisfying the following:

For each exact symplectic map $\Psi \in G^{\alpha,L}(\mathbb{A}^n_R, \mathbb{A}^n)$ such that $\varepsilon := |||\Psi - \Phi h|||_{\alpha,L,R_0} \leq \varepsilon_*$, every point $(\theta^{[0]}, r^{[0]})$ of $\mathbb{A}^n_R$ has well-defined iterates $(\theta^{[k]}, r^{[k]})$ such that $\Psi^{[k]}(\theta^{[0]}, r^{[0]}) \in \mathbb{A}^n_R$ for all $k \in \mathbb{Z}$ such that $|k| \leq \exp \left( c_\varepsilon \left( \frac{1}{\varepsilon} \right)^{\frac{1}{1+\sigma}} \right)$, and

(i) $\varepsilon \leq \varepsilon'_\rho$ and $|k| \leq \exp \left( c'_\rho \left( \frac{1}{\varepsilon} \right)^{\frac{1}{1+\sigma}} \right)$ $\Rightarrow$ $|||r^{[k]} - r^{[0]}||| \leq \rho$, 

(ii) $\varepsilon''_\sigma$ and $|k| \leq \exp \left( c''_\sigma \left( \frac{1}{\varepsilon} \right)^{\frac{1}{1+\sigma}} \right)$ $\Rightarrow$ $|||r^{[k]} - r^{[0]}||| \leq \rho$, 

...
The case $\alpha = 1$ of (iii) is due to S. Kuksin and J. Pöschel [KP94]. The rest of the statement is, to the best of our knowledge, new. It relies on the most recent version of the Nekhoroshev Theorem for Gevrey near-integrable quasi-convex Hamiltonian vector fields due to A. Bounemoura and J.-P. Marco [BM11], which improves the possible exponents for the stability time (at the price of a less good control of the confinement of the orbits)——we reproduce Bounemoura-Marco’s statement in Section 2.3. To transfer it to the discrete dynamics induced by a near-integrable exact symplectic map $\Psi$, we will have to construct a non-autonomous time-periodic Gevrey Hamiltonian function, defined for $(\theta, r, t) \in \mathbb{A}^n_R \times \mathbb{T}$ with $R < R_1 < R_0$, whose flow interpolates the discrete dynamics—this is the content of Theorem 12 of Section 2.

The hypothesis that the Hessian matrix of the integrable part $h$ is positive definite is a strict convexity assumption: it amounts to the existence of a positive real $m$ such that $h$ is $m$-convex, in the sense that

$$\langle v, d\nabla h(r)v \rangle \geq m\|v\|^2 \quad \text{for all } r \in \mathbb{B}_{R_0} \text{ and } v \in \mathbb{R}^n,$$

where $d\nabla h(r)$ is the Hessian matrix of $h$ at the point $r$. In fact, the reals $\varepsilon_*, c_*, \varepsilon'_*, c'_*, \varepsilon_0, c_0$ depend on the integrable part $h$ only through $m$ and $\|h\|_{\alpha, L, R_0}$.

Remark 1.1 (On the time exponents). Beware that, as far as Nekhoroshev theory is concerned, exact symplectic maps in $\mathbb{A}^n$ behave like $N$-degree of freedom autonomous Hamiltonian systems with $N = n + 1$. So the “time exponent” and the “confinement exponent” in the case (iii) are simply $\frac{1}{2n\alpha}$ and $\frac{1}{2N\alpha}$, which have been familiar since the works by Lochak-Neishtadt and Pöschel in the analytic case, or Marco-Sauzin in the Gevrey case.

Bounemoura-Marco’s novel result of [BM11] was the obtention of better stability times at the price of releasing part of the control on the confinement property. The counterpart for discrete systems, as demonstrated by Theorem A, is that the time exponent can be taken as large as $a_\sigma = \frac{1}{2n\alpha}$ with arbitrary $\sigma$ such that $0 \leq \sigma \leq \frac{1}{n+1}$, and the corresponding confinement radius still tends to 0 as $\varepsilon \to 0$ if $\sigma > 0$, while we only get a fixed (but arbitrarily small) confinement radius $\rho$ if $\sigma = 0$; observe that $a_\sigma$ is a decreasing function of $\sigma$, so $\sigma$ close to $\frac{1}{n+1}$ yields worse stability exponents $a_\sigma$, close to the exponent $\frac{1}{2(n+1)\alpha}$ of case (iii) (but better confinement properties), while $\sigma = 0$ gives the best time exponent, namely $\frac{1}{2n\alpha}$, for general orbits. See Remark 2.8 for more comments.

Remark 1.2 (Stabilization by resonances). We leave it to the reader to devise a refined statement for orbits starting $O(\varepsilon^{1/2})$-close to a resonance of multiplicity $m \in \{1, \ldots, n\}$ by exploiting the well-known stabilizing effect of resonances available for Hamiltonian flows—see Remark 2.9. The time and confinement exponents then jump to $\frac{1}{2(n+1-m)\alpha}$ and $\frac{1}{2(n+1-m)\alpha}$. 

(i) $\varepsilon \leq \varepsilon_\sigma''$ and $|k| \leq \exp\left(c''_\sigma \left(\frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{2n\alpha}} \right)$ \Rightarrow \|r[k] - r[0]\| \leq \left\{ \begin{array}{ll} \frac{1}{c''_\sigma} \varepsilon^\frac{1}{\alpha} & \text{if } \alpha = 1, \\
 \frac{1}{c''_\sigma} \varepsilon^\frac{1}{n+\alpha} & \text{if } \alpha > 1, \end{array} \right.$
In particular, the time exponent $\frac{1}{2n\alpha}$ given in (i) for general orbits coincides with the time exponent available for the orbits starting close to a simple resonance, but the latter have a better confinement property (described by the positive exponent $\frac{1}{2n}$) than general orbits.

**Remark 1.3** (About the steep case). The original Nekhoroshev theorem was proved in the analytic case for a wider class of near-integrable Hamiltonian flows than just those with quasi-convex integrable part. Nekhoroshev only needed a non-degeneracy assumption called steepness, which turns out to be generic in quite a strong sense. This allowed S. Kuksin and J. Pöschel to give an exponential stability theorem for analytic near-integrable maps in the case where $h$ is supposed to be steep but not necessarily convex [KP94]. The same could be done for Gevrey near-integrable maps if the original Nekhorohev statement could be generalised to the Gevrey steep case.

**Remark 1.4** (KAM theorem for analytic or Gevrey near-integrable maps). The assumption that $h$ be non-degenerate in the sense of Kolmogorov (i.e. that $\nabla h$ be a local diffeomorphism, which is a weaker condition than strict convexity) is sufficient to apply the KAM theorem, in its analytic version if $\alpha = 1$, or in its $C^\infty$ version if $\alpha > 1$. For each $r_\ast \in B_{R_0}$ such that $\nabla h(r_\ast)$ is Diophantine, we obtain for the discrete dynamics $\Psi$ an invariant quasi-periodic torus $\cong \mathbb{T}^n$ located close to $\mathbb{T}^n \times \{r_\ast\}$ as soon as $|||\Psi - \Phi^h|||_{\alpha,L,R_0}$ is small enough. If $\alpha = 1$, then such a torus is known to be analytically embedded in $\mathbb{A}_R^n$. If $\alpha > 1$, then the embedding is known to be $C^\infty$ and one can prove that the embedding is in fact Gevrey-$\alpha$ by applying Popov’s KAM theorem for Gevrey near-integrable Hamiltonians [Po04] to the interpolating Hamiltonian flow constructed in Theorem E of Section 2.

### 1.2 Wandering sets of near-integrable systems—Theorems B and C

#### 1.2.1 The other results of this paper deal with wandering sets for near-integrable systems.

**Definition 1.5.** Given a diffeomorphism $\Psi$ of a manifold $M$, we say that $W \subset M$ is wandering if

$$\Psi^k(W) \cap W = \emptyset \quad \text{for all } k \in \mathbb{Z}\setminus\{0\}$$

or, equivalently, if $\Psi^k(W) \cap \Psi^\ell(W) = \emptyset$ for all $k, \ell \in \mathbb{Z}$ with $k \neq \ell$.

Notice that if $W' \subset W$ and $W$ is wandering, then $W'$ is wandering too. Beware that, when $W$ is reduced to a single point $x$, saying that the set $W = \{x\}$ is wandering is a less stringent condition than saying that the point $x$ is wandering in the usual sense (which amounts to the existence of a neighborhood $V$ of $x$ such that $\Psi^k(V) \cap V = \emptyset$ for $k \in \mathbb{Z}\setminus\{0\}$).

**Remark 1.6.** If $\Psi$ preserves a finite measure, then obviously any measurable wandering set must have zero measure (this is the key argument in the Poincaré recurrence theorem).

#### 1.2.2 We denote the canonical Lebesgue measure on $\mathbb{A}^n$ by $\mu$. Recall that a domain of $\mathbb{A}^n$ is a connected open subset of $\mathbb{A}^n$.

Before going further, we notice that, given $h$ of class $C^2$ on an open set $\Omega$ of $\mathbb{R}^n$, any measurable wandering set $W$ of the integrable diffeomorphism $\Phi^h : \mathbb{T}^n \times \Omega \ni$ has
zero Lebesgue measure. Indeed, formula (1.1) shows that each torus \( T_k \) is invariant, with the restriction of \( \Phi^h \) preserving the Haar measure \( \mu_r \) of \( T_k \), which is finite. Thus Remark 1.6 implies that the wandering set \( W \cap T_k \) has zero \( \mu_r \)-measure for each \( r \in \Omega \) and, by Fubini,

\[
\mu(W) = \int_{\Omega} \mu_r(W \cap T_k) \, dr = 0.
\]

In particular, the only wandering domain for \( \Phi^h \) is the empty set.

1.2.3 Another preliminary remark concerns the case \( n = 1 \): any measurable wandering set \( W \) of a near-integrable system of \( A \) has zero Lebesgue measure. More precisely, if \( 0 < R < R_0 < \infty \) and \( h \in G^{\alpha,L}([-R_0, R_0]) \) is Kolmogorov non-degenerate (i.e. its second derivative does not vanish), then for any exact symplectic diffeomorphism \( \Psi \) of \( A \) with a restriction to \( A_{R_0} = \mathbb{T} \times [-R_0, R_0] \) such that \( \|\Psi - \Phi^h\|_{\alpha,L,R_0} \) is small enough, any measurable wandering set contained in \( A_{R_0} \) has zero Lebesgue measure.

Indeed, the KAM theorem yields two invariant circles, one contained in \( A_{R_0} \) and the other one in \( \mathbb{T} \times [-R_0, -R_0] \), which bound a finite measure invariant region; any measurable wandering set contained in that region must have zero measure according to Remark 1.6. (This is the same argument which forbids Arnold diffusion in two degrees of freedom.)

1.2.4 We thus assume \( n \geq 2 \) from now on. The first examples of near-integrable systems possessing wandering sets of positive Lebesgue measure, namely wandering domains, were constructed in [MS04]. Although the construction was quite explicit, no estimate was given for the “size” of these wandering domains.

In this paper, we show that, for a perturbation \( \Psi \) of an integrable diffeomorphism \( \Phi^h \) with \( \varepsilon := \|\Psi - \Phi^h\|_{\alpha,L,R} \) small, the wandering sets have an exponentially small size. We shall provide general upper bounds and examples with explicit lower bounds.

1.2.5 We shall use two natural but essentially different notions of “size”: the Lebesgue measure and the Gromov capacity. Recall that the Gromov capacity (or width, or depth) \( C_G(W) \) of a subset \( W \) of a symplectic manifold is the supremum of the numbers \( \pi r^2 \), where \( r \geq 0 \) is such that the Euclidean ball \( B^{2n}(r) \) of radius \( r \) in \( \mathbb{R}^{2n} \) can be symplectically embedded in \( W \). As a consequence, for measurable subsets \( W \) of \( \mathbb{A}^n \),

\[
C_G(W) \leq \pi \left( \frac{\mu(W)}{\text{Vol}(B^{2n}(1))} \right)^{1/n}.
\]

The capacity of a domain in the 2-dimensional annulus \( \mathbb{A} \) equals its Lebesgue measure (i.e. its area in this case), but they are in general distinct for higher dimensional domains. As an extreme case, given a disc \( D \) in \( \mathbb{A} \), the capacity of \( W := D \times \mathbb{A}^{n-1} \subset \mathbb{A}^n \) is the area of \( D \), while the Lebesgue measure of \( W \) is infinite. We refer to [McDS95] for a more complete exposition of the notion of Gromov capacity.

\[\text{Notice that it is the exactness of near-integrable systems which makes the existence of such examples not obvious. If exactness is relaxed, then one trivially gets arbitrarily close to integrable symplectic maps with wandering domains by considering } \Psi_{\varepsilon} : (\theta, r) \in \mathbb{A} \mapsto (\theta + r, r + \varepsilon) \in \mathbb{A}, \text{ with small } \varepsilon > 0, \text{ and } W_{\varepsilon} := \mathbb{T} \times [0, \varepsilon[.\]
We are interested in estimates of the size of wandering subsets from above and from below. In view of inequality (1.8), we may content ourselves with using the Lebesgue measure for upper estimates and the Gromov capacity for the lower ones.

1.2.6 Our upper bound result consists in general exponentially small estimates, with explicit exponents stemming from Theorem A:

**Theorem B** (Upper bounds for wandering sets). Let \( n \geq 2 \) be integer. Let \( \alpha \geq 1 \) and \( L, R_0 > 0 \) be real. Let \( h \in G^{\alpha, L}(B_{R_0}) \) have positive definite Hessian matrix on \( B_{R_0} \). Then for \( 0 < R < R_0 \) there exist \( \varepsilon_*, c_* > 0 \) such that, for each exact symplectic diffeomorphism \( \Psi \) of \( \mathbb{A}^n \) whose restriction to \( \mathbb{A}^n_{\alpha, L, R_0} \) satisfies

\[
\Psi|_{\mathbb{A}^n_{\alpha, L, R_0}} \in G^{\alpha, L}(\mathbb{A}^n_{\alpha, L, R_0}), \quad \varepsilon := \|\Psi|_{\mathbb{A}^n_{\alpha, L, R_0}} - \Phi^h\|_{\alpha, L, R_0} < \varepsilon_*,
\]

any measurable wandering set \( W \) of \( \Psi \) contained in \( \mathbb{A}^n_{\alpha, L, R_0} \) has Lebesgue measure

\[
\mu(W) \leq \exp \left( - c_* \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2n\alpha}} \right). \tag{1.9}
\]

The proof is in Section 2.4. It is a pretty direct consequence of Theorem A and the preservation of the Lebesgue measure by symplectic maps. (It works for the case \( n = 1 \) as well but, as already mentioned, \( \mu(W) = 0 \) in that case.) Again, the reals \( \varepsilon_* \) and \( c_* \) depend on \( h \) only through \( \|h\|_{\alpha, L, R_0} \) and \( m \) such that \( h \) is \( m \)-convex in the sense of (1.7).

1.2.7 Our lower bound result consists in constructing examples which possess wandering domains whose Gromov capacity is estimated from below by an exponentially small quantity with explicit exponents:

**Theorem C** (Lower bounds in examples of wandering domains). Let \( n \geq 2 \) be integer. Let \( \alpha > 1 \) and \( L > 0 \) be real. Let \( h(r) := \frac{1}{2}(r_1^2 + \cdots + r_n^2) \). Then there exists a sequence \( (\Phi_j)_{j \geq 0} \) of exact symplectic diffeomorphisms of \( \mathbb{A}^n \) such that

- each \( \Phi_j \) has a wandering domain \( \mathcal{W}_j \) contained in \( \mathbb{A}^n_3 \),
- for \( 0 < R < \infty \) the maps \( \Phi_j \) belong to \( G^{\alpha, L}(\mathbb{A}^n_{R, R_0}, \mathbb{A}^n) \) and there exists \( c > 0 \) such that

\[
\varepsilon_j := \|\Phi_j - \Phi^h\|_{\alpha, L, R} \xrightarrow{j \to \infty} 0 \quad \text{and} \quad C_G(\mathcal{W}_j) \geq \exp \left( - c \left( \frac{1}{\varepsilon_j} \right)^{\frac{1}{2(n-1)(\alpha-1)}} \right) \tag{1.10}
\]

for all integers \( j \).

The proof is in Sections 3 and 4; see Section 1.3 for a description of the structure of the proof.

Observe that, putting together (1.8) and (1.9), we get

\[
C_G(\mathcal{W}_j) \leq K \mu(\mathcal{W}_j)^{1/n} \leq \exp \left( - c^* \left( \frac{1}{\varepsilon_j} \right)^{\frac{1}{2n\alpha}} \right),
\]

with appropriate \( K, c^* > 0 \), for \( j \) large enough, which is compatible with (1.10) because \( \frac{1}{2n\alpha} \leq \frac{1}{2(n-1)(\alpha-1)} \). Notice also that our examples are constructed only in the non-quasianalytic case \( \alpha > 1 \). See Section 1.4 for more comments on the previous inequalities and possible extensions to the analytic case.
Our method is related to the one developed in [MS03] for estimating the maximal speed of Arnold diffusion orbits and in [MS04] for constructing the first examples of near-integrable systems with wandering domains. A common feature of the examples in [MS04] and in Theorem C is that these wandering domains follow complicated paths in the phase space, located in the complement of the set of KAM tori.

1.3 Specific form of our examples and elliptic islands—Theorem D

We now indicate the structure of the proof of Theorem C to be found in Sections 3 and 4.

1.3.1 Given a function \( h \in C^\infty(\mathbb{R}^n) \) and real constants \( \alpha \geq 1 \) and \( L > 0 \), we set

\[
\mathcal{P}^0_m(L, \Phi^h) := \left\{ \Phi^{u_m} \circ \cdots \circ \Phi^{u_1} \circ \Phi^{h+u_0} \mid u_0, u_1, \ldots, u_m \in G^0(L^n) \right\}, \quad m \geq 1 \tag{1.11}
\]

\[
\mathcal{P}^0(L, \Phi^h) := \bigcup_{m \geq 1} \mathcal{P}^0_m(L, \Phi^h) \tag{1.12}
\]

(observe that the Hamiltonian functions \( h + u_0, u_1, \ldots, u_m \) generate complete vector fields because their partial derivatives with respect to the angles are bounded; the notation is well-defined, since giving the diffeomorphism \( \Phi^h \) allows one to compute the gradient \( \nabla h \) mod \( \mathbb{Z}^n \) and therefore the diffeomorphism \( \Phi^{h+u_0} \) for any smooth \( u_0 \)). Any \( \Psi \in \mathcal{P}^0(L, \Phi^h) \) is an exact symplectic map which can be viewed as a perturbation of \( \Phi^h \), with a “deviation” defined as

\[
\delta^0(L, \Phi^h) := \inf \left\{ \sum_{k=0}^{m} \| u_k \|_{\alpha,L,\infty} \mid m \geq 1, (u_0, u_1, \ldots, u_m) \in \mathcal{P}^0_m(L, \Phi^h) \right\}, \tag{1.13}
\]

where \( \mathcal{P}^0_m(L, \Phi^h) := \left\{ (u_0, u_1, \ldots, u_m) \in (G^0(L^n))^{m+1} \mid \Phi^{u_m} \circ \cdots \circ \Phi^{u_1} \circ \Phi^{h+u_0} = \Psi \right\} \).

One can check that the deviation vanishes if and only if \( \Psi = \Phi^h \).

1.3.2 If \( h \in G^0(L, B_{R_0}) \), then the elements of \( \mathcal{P}^0(L, \Phi^h) \) are Gevrey maps and the deviation can be compared to the distances \( \| \Psi - \Phi^h \|_{\alpha,L,R} \). More precisely,

**Proposition 1.7.** Let \( n \geq 1 \) be integer. Let \( \alpha \geq 1 \) and \( L, R_0 > 0 \) be real. Let \( h \in C^\infty(\mathbb{R}^n) \cap G^0(L, B_{R_0}) \). Then for \( 0 < R < R_0 \) there exist \( \varepsilon_s, L_s, C_s > 0 \), with \( L_s < L \), such that

\[
\Psi \in \mathcal{P}^0(L, \Phi^h) \text{ and } \delta^0(L, \Phi^h) < \varepsilon_s \Rightarrow \| \Psi - \Phi^h \|_{\alpha,L,R} \leq C_s \delta^0(L, \Phi^h). \tag{1.14}
\]

The proof of Proposition 1.7 is in Appendix B.3 (\( \varepsilon_s, L_s, C_s \) depend on \( h \) only through \( \| h \|_{\alpha,L,R_0} \)).

As a consequence, Theorems A and B apply to the maps of \( \mathcal{P}^0(L, \Phi^h) \) with \( \delta^0(L, \Phi^h) \) small enough, and the role of \( \varepsilon \) in the statements can be played by \( \delta^0(L, \Phi^h) \) instead of \( \| \Psi - \Phi^h \|_{\alpha,L,R} \).

1.3.3 Theorem C will follow from a more precise statement, Theorem C stated in Section 4.3.1. The unperturbed system \( h := \frac{1}{2}(r_1^2 + \cdots + r_n^2) \) and the constants \( \alpha > 1, L > 0 \) being fixed, Theorem C will yield very explicit maps \( \Phi_j \in \mathcal{P}_2^0(L, \Phi^h) \) when \( n = 2 \),
$\Phi_j \in \mathcal{P}^{n,L}_3(\Phi^h)$ when $n \geq 3$, with wandering domains $\mathcal{W}_j \subset \mathbb{A}^n_3$, and a real $c_\ast > 0$ such that

$$\varepsilon'_j := \delta^{n,L}(\Phi_j, \Phi^h) \xrightarrow{j \to \infty} 0, \quad C_G(\mathcal{W}_j) \geq \exp(-c_\ast \left(\frac{1}{\varepsilon'_j}\right)^{\frac{1}{3(n-1)(\alpha-1)}}). \quad (1.15)$$

By Proposition 1.7, (1.15) implies the property (1.10) for every finite $R$, hence Theorem C is an immediate consequence of Theorem C’.

The domains $\mathcal{W}_j$ will be polydiscs, i.e. product sets of the form $\mathcal{D}_j^{[1]} \times \cdots \times \mathcal{D}_j^{[n]}$ with discs $\mathcal{D}_j^{[1]}, \ldots, \mathcal{D}_j^{[n]} \subset \mathbb{A}$. This product structure is an essential feature in the use of the “coupling lemma” of Section 4.1, which is a basic ingredient of the proof of Theorem C’.

Note that the Gromov capacity of a polydisc is given by the formula

$$C_G(\mathcal{D}_1^{[1]} \times \cdots \times \mathcal{D}_1^{[n]}) = \min\{\text{area}(\mathcal{D}_1^{[1]}), \ldots, \text{area}(\mathcal{D}_1^{[n]})\}. \quad (1.16)$$

(One inequality follows from the fact that, in dimension 2, Gromov capacity and area coincide; the reverse inequality is a consequence of Gromov’s “non-squeezing theorem”—see [McDS95].)

1.3.4 As another ingredient of the proof of Theorem C, we shall have to devise an additional result on the construction of examples with periodic domains, which is interesting in itself and connected with other aspects of transport phenomena in near-integrable Hamiltonian systems.

To ease the comparison with Section 4, we present this result in $\mathbb{A}^{n-1}$ (still with $n \geq 2$), labelling the coordinates as $(\theta_2, r_2), \ldots, (\theta_n, r_n)$, and set

$$h(r) := \frac{1}{2}(r_2^2 + \cdots + r_n^2).$$

For an integer $q \geq 1$, we call $q$-periodic polydisc of a diffeomorphism $\phi$ of $\mathbb{A}^{n-1}$ a polydisc $\mathcal{D}$ of $\mathbb{A}^{n-1}$ such that $\phi^q(\mathcal{D}) = \mathcal{D}$. We introduce the notation

$$\mathbb{A}^+_d := \mathbb{T} \times [0, d] \subset \mathbb{A}, \quad \mathbb{B}_d := \{(\theta, r) \mid \theta \in [-d, d], \ r \in \mathbb{R}\} \subset \mathbb{A} \quad \text{for any real } d > 0. \quad (1.17)$$

**Theorem D** (Periodic domains in $\mathbb{A}^{n-1}$). Let $\alpha > 1$ and $L > 0$ be real, and let $n \geq 2$ be integer. Then there exist real numbers $c, C_1, C_2, C_3 > 0$, a non-negative integer $j_0$, and a sequence $(\Psi_{j,q})$ of exact symplectic diffeomorphisms of $\mathbb{A}^{n-1}$ belonging to $\mathcal{P}^{n,L}(\Phi^h)$ defined for

$$j, q \in \mathbb{N}, \quad j \geq j_0, \quad q \geq C_1 N_j, \quad (1.18)$$

with deviations

$$\delta^{n,L}(\Psi_{j,q}, \Phi^h) \leq \frac{C_2}{N_j^2}, \quad (1.19)$$

where

$$N_j := p_{j+2} \cdots p_{j+n}, \quad (1.20)$$

($p_j)_{j \geq 0}$ denoting the prime number sequence, so that:

\footnote{When we use the word “disc”, unless otherwise specified, we mean any bounded and simply connected domain in $\mathbb{A}$ or in $\mathbb{R}^2$.}
(i) If \( n = 2 \), each \( \Psi_{j,q} \) is in \( \mathcal{R}^{a,L}_1(\Phi^h) \) and has a \( q \)-periodic disc \( \mathcal{D} := \mathcal{D}_{j,q} \subset \mathbb{A}_3 \) with all its iterates also contained in \( \mathbb{A}_3 \), such that

\[
C_G(\mathcal{D}) \geq C_3 \min \left\{ \frac{N_j^2}{q^3}, \exp \left( - c N_j^{\frac{1}{n-1}} \right) \right\}
\]

and

\[
\mathcal{D} \subset \mathbb{A}_3^+ \cap \mathbb{B}_{1 \over \rho_{j+2}}, \quad \Psi_{j,q}^k(\mathcal{D}) \cap \mathbb{B}_{1 \over \rho_{j+2}} = \emptyset \quad \text{for } 1 \leq k \leq q - 1.
\]

(ii) If \( n \geq 3 \), each \( \Psi_{j,q} \) is in \( \mathcal{R}^{a,L}_2(\Phi^h) \) and, for \( q \) integer multiple of \( N_j \), \( \Psi_{j,q} \) has a \( q \)-periodic polydisc \( \mathcal{D} := \mathcal{D}_{j,q} \) whose iterates are polydiscs:

\[
\Psi_{j,q}^k(\mathcal{D}) = \mathcal{D}^{[2,k]} \times \cdots \times \mathcal{D}^{[n,k]}, \quad k \in \mathbb{Z},
\]

with \( \Psi_{j,q}^k(\mathcal{D}) \subset \mathbb{A}_3^{n-1} \) for all \( k \), such that

\[
C_G(\mathcal{D}) \geq C_3 \min \left\{ \frac{1}{q^5} N_j^{1-2 \over n-1}, \exp \left( - c N_j^{1 \over (n-1)(\alpha-1)} \right) \right\},
\]

the projections of the polydisc \( \mathcal{D} \) satisfy

\[
\mathcal{D}^{[2,0]} \subset \mathbb{A}_3^+ \cap \mathbb{B}_{1 \over \rho_{j+2}}, \quad \mathcal{D}^{[3,0]} \subset \mathbb{B}_{1 \over \rho_{j+3}}, \ldots, \quad \mathcal{D}^{[n,0]} \subset \mathbb{B}_{1 \over \rho_{j+n}},
\]

and, for \( 1 \leq k \leq q - 1 \), those of the polydisc \( \Psi_{j,q}^k(\mathcal{D}) \) satisfy

\[
\exists \ell \in \{2, \ldots, n\} \text{ such that } \mathcal{D}^{[\ell,k]} \cap \mathbb{B}_{1 \over \rho_{j+\ell}} = \emptyset.
\]

The proof of Theorem D is spread over Sections 3 and 4. More precisely, Case (i), i.e. the two-dimensional case, is proved in Section 3.2 based on an auxiliary result; this auxiliary result is also used in Section 4.2 together with the “coupling lemma” (Lemma 4.1), to prove Case (ii).

Theorem D is used in Section 4.3.3 (again with the help of the coupling lemma) to prove Theorem C, with an appropriate choice of \( q = q_j \) exponentially large with respect to \( N_j \).

**Remark 1.8.** Fix \( j \geq j_0 \) and \( q \) as in the statement of Theorem D. Because of condition (1.22) or conditions (1.24)–(1.25), the sets \( \Psi_{j,q}^k(\mathcal{D}_{j,q}) \), \( k = 0, 1, \ldots, q - 1 \), are pairwise disjoint. This implies that \( q \) is the minimal period of the periodic polydisc \( \mathcal{D}_{j,q} \). This also implies an upper bound for the Lebesgue measure of this polydisc:

\[
\mu(\mathcal{D}_{j,q}) \leq \frac{\mu(\mathbb{A}_3^{n-1})}{q}.
\]

Indeed, the \( q \) pairwise disjoint sets \( \Psi_{j,q}^k(\mathcal{D}_{j,q}) \) have the same Lebesgue measure and are all contained in \( \mathbb{A}_3^{n-1} \). It follows that the lower bound in (1.21) or (1.23) has to depend on \( q \), it cannot depend on \( j \) alone, because \( q \) is allowed to be arbitrarily large and (1.8) implies

\[
C_G(\mathcal{D}_{j,q}) \leq \frac{\pi}{q^{1 \over n-1}} \left( \frac{\mu(\mathbb{A}_3^{n-1})}{\text{Vol} (B^{2(n-1)}(1))} \right)^{1 \over n-1}.
\]
The aforementioned auxiliary result on which the proof of Theorem D(i) is based is Theorem F: this much more precise statement is the object of Section 3, it is the analytical core of our method.

The (quite lengthy) proof of Theorem F relies on the construction of a suitable perturbation of the time-one map of a “pseudo-pendulum” on $\mathbb{A}$, of the form

$$P(\theta, r) = \frac{1}{2} r^2 + \frac{1}{N^2} V(\theta),$$

(1.26)

where $V$ is a (specially designed) potential function on $\mathbb{T}$. Both $V$ and the perturbation can be made very explicit. The effect of the perturbation is to create elliptic islands around the periodic points located near the separatrix of the pseudo-pendulum. The main difficulty in estimating the size of these islands is that one has to use Herman’s quantitative version of the two dimensional KAM theorem ([He01]), whose implementation requires the computation of high order parametrized normal forms, the parameters being the size of the perturbation and the period of the island.

Another peculiarity of our systems is that the potential $V$ has degenerate maxima, which create degenerate stationary points for the Hamiltonian vector field generated by (1.26). This is crucial in order to find elliptic islands with “exponentially small” area: a nondegenerate situation would yield a double exponential in the estimates.

### 1.4 Further comments

#### 1.4.1

Observe that in Theorem B, we impose a priori that the wandering set $W$ be contained in a fixed compact $\mathbb{A}^n_R$.

Surprisingly enough, as soon as $n \geq 3$, this is necessary to ensure that the measure of $W$ is finite. Indeed, given $\alpha > 1$ and $L > 0$, for any $\varepsilon, R_0 > 0$ we can exhibit (by [MS04] or by Theorem C) a near-integrable system $\Psi$ on $\mathbb{A}^2$ with a non-empty wandering domain $W$, such that $|||\Psi - \Phi^{1/2}(r_1^2 + r_2^2)|||_{\alpha, L, R_0} < \varepsilon$. Therefore, when $n \geq 3$, the direct product $\hat{\Psi} = \Psi \times \Phi^{1/2}(r_3^2 + \cdots + r_n^2)$ on $\mathbb{A}^n$ admits the wandering domain $W \times \mathbb{A}^{n-2}$, which is of infinite measure, while $|||\Psi - \Phi^{1/2}(r_1^2 + \cdots + r_n^2)||| < \varepsilon$. As a consequence, by taking subsets of $W \times \mathbb{A}^{n-2}$, one may obtain for the near-integrable system $\hat{\Psi}$ wandering domains of arbitrary measure between 0 and $\infty$ inclusive.

#### 1.4.2

In any case, this leaves open the question of the existence of upper bounds for the Gromov capacity of an arbitrary wandering set $W$ (without the restriction $W \subset \mathbb{A}^n_R$): is it always finite? is it exponentially small?

Notice that a wandering set has empty intersection with the set of KAM tori, so a related question is the question of the finiteness of $C_G(\mathbb{T}^n \times (\mathbb{R}^n \setminus \mathcal{K}))$, where $\mathcal{K}$ is the set of all vectors satisfying a fixed Diophantine condition. Due to the intricate structure of this set, it could be worthwhile to produce a simpler model for this line of questions. For instance, what can be said on the finiteness of any symplectic capacity of the open subset

$$\mathbb{T}^n \times (\mathbb{R}^n \setminus \mathbb{Z}^n) \subset T^*\mathbb{T}^n?$$

---

5We insist on being able to take the period of the elliptic island arbitrarily large. If this requirement were dropped, a much simpler construction would be available—see the auxiliary Proposition 4.5.
This question seems to be completely open.

1.4.3 Another open question is that of the optimal exponents that one could obtain in inequalities such as (1.9) and (1.10): to sharpen Theorem [B] would mean to replace the exponent \( \frac{1}{2n\alpha} \) by a larger exponent \( a_{\text{up}} \) in (1.9), and to sharpen Theorem [C] would mean to replace the exponent \( \frac{1}{2(n-1)(\alpha-1)} \) by a smaller exponent \( a_{\text{low}} \) in (1.10); how large can one take the first exponent and how small can one take the second? Of course, one would still have \( a_{\text{up}} \leq a_{\text{low}} \); if the equality \( a_{\text{up}} = a_{\text{low}} \) could be realised, the resulting exponent should certainly be called “optimal”.

The problem is clearly related to the possibility of constructing examples in the analytic category \( \alpha = 1 \), since the factor \( \alpha - 1 \) (whose appearance is directly linked to our use of Gevrey bump functions) creates a major discrepancy between our lower and upper bounds when \( \alpha \to 1 \). We believe that such constructions are possible, at the cost of relaxing the constraint that our wandering subsets be domains.
2 Stability theory for Gevrey near-integrable maps

We develop in this section a perturbation theory for Gevrey discrete dynamical systems, based on the corresponding theory available for Gevrey Hamiltonian flows. To transfer the results from the latter to the former, we first prove a Gevrey suspension theorem (Theorem E), according to which any Gevrey near-integrable map can be viewed as the time-one map of a Gevrey near-integrable Hamiltonian vector field. This will allow us to prove in Section 2.3 the Nekhoroshev Theorem for Gevrey maps (Theorem A), from which we will derive upper bounds for the measure of their wandering sets (Theorem B) in Section 2.4.

2.1 Embedding in a Hamiltonian flow—Theorem E

Definition 2.1. Given an exact symplectic map \( \Psi: \mathbb{A}_R^n \to \mathbb{T}^n \times \mathbb{R}^n \), we call suspension of \( \Psi \) any 1-periodic time-dependent Hamiltonian function \( H: \Omega \times \mathbb{T} \to \mathbb{R} \), where \( \Omega \) is a neighbourhood of \( \mathbb{A}_R^n \), for which the flow map between the times \( t = 0 \) and \( t = 1 \) is well-defined on \( \mathbb{A}_R^n \) and coincides with \( \Psi \).

We adapt the definitions (1.3) and (1.4) to deal with \( C^\infty \) functions depending on an extra variable \( t \in \mathbb{T} \) or \( t \in [0,1] \):

\[
G^{\alpha,L}(\mathbb{T}) := \{ \eta \in C^\infty(\mathbb{T}) \mid \| \eta \|_{\alpha,L} < \infty \}, \quad \| \eta \|_{\alpha,L} := \sum_{\ell \in \mathbb{N}} \frac{L^{\ell\alpha}}{\ell!} \| \partial^\ell \eta \|_{C^0(\mathbb{T})}
\]

\[
G^{\alpha,L}(\mathbb{A}_R^n \times \mathbb{T}) := \{ f \in C^\infty(\mathbb{A}_R^n \times \mathbb{T}) \mid \| f \|_{\alpha,L,R} < \infty \}, \quad \| f \|_{\alpha,L,R} := \sum_{\ell \in \mathbb{N}^{2n+1}} \frac{L^{\ell\alpha}}{\ell!} \| \partial^\ell f \|_{C^0(\mathbb{A}_R^n)}
\]

and similarly for \( G^{\alpha,L}([0,1]) \) and \( G^{\alpha,L}(\mathbb{A}_R^n \times [0,1]) \).

Theorem E (Suspension theorem). Let \( n \) be a positive integer. Let \( \alpha \geq 1, L_0, R_0, E > 0 \) be reals such that \( R < R_0 \). Then there exist \( \varepsilon_\ast, L_\ast, C_\ast > 0 \) such that, for every \( h \in G^{\alpha,L_0}(\mathbb{B}_{R_0}) \) with \( \| h \|_{\alpha,L_0,R_0} \leq E \), the restriction to \( \mathbb{A}_R^n \) of any exact symplectic map \( \Psi \in G^{\alpha,L_0}(\mathbb{A}_R^n, \mathbb{A}_R^n) \) such that

\[
\varepsilon := \| \| \Psi - \Phi^h \| \|_{\alpha,L_0,R_0} \leq \varepsilon_\ast
\]

admits a suspension \( H = H(\theta, r, t) \in G^{\alpha,L_\ast}(\mathbb{A}_R^n \times \mathbb{T}) \) for which

\[
\| H - h \|_{\alpha,L_\ast,R} \leq C_\ast \varepsilon.
\]

Remark 2.2. In view of Proposition 1.7, Theorem E applies to the maps of \( \mathcal{P}^{\alpha,L}(\Phi^h) \) with \( \delta := \delta^{\alpha,L}(\Psi, \Phi^h) \) small enough, and the role of \( \varepsilon \) in the statement can be played by \( \delta \) instead of \( \| \| \Psi - \Phi^h \| \|_{\alpha,L,R_0} \).

In fact, the resulting statement can be proved directly if one restricts oneself to \( \Psi \in \mathcal{P}^{\alpha,L}_m(\Phi^h) \) with a fixed \( m \) (upon which the implied constants may depend) and \( \alpha > 1 \), by adapting the ideas of \( [MS03] \) §2.4.1 and \( [MS04] \) §5.2. Indeed, use the hypothesis \( \alpha > 1 \) to find non-negative functions \( \varphi_0, \varphi_1, \ldots, \varphi_m \in C^{\alpha,L}(\mathbb{T}) \) such that each \( \varphi_j \) has
total mass 1 and is supported on \( \left[ \frac{j}{m+1}, \frac{j+1}{m+1} \right] \mod \mathbb{Z} \) (use e.g. Lemma A.3 of \([MS03]\)), and set \( \tilde{\phi}_0(t) := \int_0^t (\phi_0(s) - 1) \, ds \). Then, for any \( u_0, u_1, \ldots, u_m \in G^{\alpha,L}(\mathbb{A}^n) \), the map \( \Psi = \Phi^{u_m} \circ \ldots \circ \Phi^{u_1} \circ \Phi^{h+u_0} \) admits an explicit suspension given by

\[
H(\theta, r, t) := h(r) + \phi_0(t)u_0(\theta) + \tilde{\phi}_0(t)\nabla h(r), r) + \sum_{j=1}^m \phi_j(t)u_j(\theta + (1 - t)\nabla h(r), r),
\]

and one can find \( \lambda \in (0, 1) \) and \( C > 0 \) independent of \( u_0, \ldots, u_m \) such that

\[
\|H - h\|_{\alpha, \lambda L, R} \leq C\left(\|u_0\|_{\alpha, L} + \cdots + \|u_m\|_{\alpha, L}\right).
\]

We now briefly indicate how to prove Theorem E in the analytic case, i.e. when \( \alpha = 1 \); the case \( \alpha > 1 \) is dealt with in Section 2.2.

**Proof of Theorem E in the case \( \alpha = 1 \).** This is due to Kuksin \([Ku93]\) and Kuksin-Pöschel \([KP94]\). There is only a slight difference in the way norms are measured, but this is immaterial: for a real analytic function \( \phi : \mathbb{A}^n \rightarrow \mathbb{R} \), \([KP94]\) defines \( |\phi|_\rho \) as the sup-norm of the holomorphic extension of \( \phi \) to a complex domain \( V_\rho \cap \mathbb{T}^n \times \mathbb{V}_\rho \mathbb{B}_R \) defined as in (1.5) but with \( \| \cdot \|_\infty \) replaced by \( \| \cdot \|_c \), with \( \| \xi \| := \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2} \) for \( \xi \in \mathbb{R}^n \) or \( \mathbb{C}^n \); this is related to our Gevrey-1 norms by

\[
c\|\phi\|_{1,L,R} \leq |\phi|_\rho \leq \|\phi\|_{1,\rho,R}
\]

for \( 0 < L < \rho/\sqrt{n} \), with \( c := (1 - L\rho^{-1} \sqrt{n})^{2n} \). With this in mind, when \( \alpha = 1 \), our Theorem E follows from Theorem 4 of \([KP94]\) by isoenergetic reduction, with the help of the Implicit Function Theorem (the same way their Theorem 1 follows from their Theorem 3).

\[ \square \]

### 2.2 Proof of Theorem E in the Gevrey non-analytic case

For the Gevrey non-analytic case, the proof will consist in a Gevrey quantitative adaptation of Douady’s method \([Dou82]\).

In all this section we fix a positive integer \( n \) and a real \( \alpha > 1 \). When dealing with a map \( \Psi \) taking its values in \( \mathbb{T}^n \times \mathbb{R}^n \) or \( \mathbb{R}^n \times \mathbb{R}^n \), we shall often denote its components by \( \Psi_1, \ldots, \Psi_{2n} \) and use the notation

\[
\Psi^{[1]} := (\Psi_1, \ldots, \Psi_n), \quad \Psi^{[2]} := (\Psi_{n+1}, \ldots, \Psi_{2n}).
\]

Similarly, we shall make use of the partial gradient operators \( \nabla^{[1]} \) and \( \nabla^{[2]} \) defined by (C.1).

#### 2.2.0 Overview

The construction is based on the classical formalism of generating functions for exact symplectic \( C^\infty \) maps, with mixed set of variables: we use the notation \( \mathcal{F}_A \) whenever we have a \( C^\infty \) function \( A \) defined on an open subset of \( \mathbb{A}^n \) such that the equation

\[
r = \theta + \nabla^{[1]} A(\theta, r')
\]
implicitly defines \( r' \in \mathbb{R}^n \) in terms of \( \theta \in \mathbb{T}^n \) and \( r \in \mathbb{R}^n \), so that we can set
\[
F_A(\theta, r) := (\theta', r'), \quad \theta' := \theta + \nabla^2 A(\theta, r').
\]
When it is defined, the map \( F_A \) is automatically an exact symplectic local diffeomorphism; moreover, all exact symplectic \( C^\infty \) maps close enough to identity are of this form. The reader is referred to Appendix C for more details.

Here is an overview of the construction of a suspension for a given exact symplectic Geyevry map \( \Psi \) close enough to \( \Phi^h \): following \cite{Dou82}, we write our map as
\[
\Psi = \Theta \circ F_A,
\]
while we pick \( \eta \in C^\infty([0,1]) \) such that \( \eta \equiv 0 \) on a neighbourhood of 0, \( \eta \equiv 1 \) on a neighbourhood of 1, and \( 0 \leq \eta \leq 1 \) on \([0,1]\); then the formula
\[
\Psi_t := \Theta \circ F_{\eta(t)} A
\]
defines an isotopy between the identity and \( \Psi \), which can be shown to be the flow map between time 0 and time \( t \) for a time-periodic Hamiltonian vector field \( H \) which is close to \( h \).

We will repeat the arguments in detail to check that one can find a small Geyevry function \( A \) such that (2.5) holds and that, provided we take a Geyevry function for \( \eta \) (which is possible because \( \alpha > 1 \)), we can find a suspension \( H \) Geyevry close to \( h \). The last point will follow from the very explicit formula that we shall obtain for \( H \): with the notation (2.4),
\[
H(\theta, r, t) = h(r) + \eta(t)A\left(\left(\frac{\partial}{\partial t}\right)_A(\nabla h(r), r)\right)
\]
(formula (2.24) below).

2.2.1 First step: finding a generating function

Proposition 2.3. Let \( L_0, R, R_0, E > 0 \) be reals such that \( R < R_0 \). Then there exist \( \varepsilon_*, L, C_* > 0 \) such that, for any \( h \in C^{\alpha,L_0}(\mathcal{F}_{R_0}) \) such that \( \|h\|_{\alpha,L_0,R_0} \leq E \) and any exact symplectic map \( \Psi \in C^{\alpha,L_0}(\mathbb{R}^n_{R_0},\mathbb{R}^n) \) such that
\[
\varepsilon := \||\Psi - \Theta h\|_{\alpha,L_0,R_0} \leq \varepsilon_*.
\]
there exist open subsets \( \Omega \) and \( \Omega' \) of \( \mathbb{R}^n_{R_0} \) which contain \( \mathbb{R}^n_R \) and a function \( A \in C^\infty(\Omega') \) such that
\begin{itemize}
  \item \( F_A : \Omega \to \mathbb{R}^n_{R_0} \) is a well-defined exact symplectic map,
  \item \( \Psi|_{\Omega} = \Theta \circ F_A \),
  \item \( A|_{\mathbb{R}^n_R} \in G^{\alpha,L}(\mathbb{R}^n_R) \) and \( \|A|_{\mathbb{R}^n_R}\|_{\alpha,L,R} \leq C_* \varepsilon \).
\end{itemize}

\footnote{Up to sign, the function \( A \) corresponds to what is called "generating function of type \( V \)" in \cite{McDS95} §9.2.}

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The proof of Proposition 2.3 relies on two auxiliary results. The first one is a straightforward Gevrey adaptation in $\mathbb{A}^n_R$ of the Poincaré lemma, the second one is a technical inversion result that will be needed in the second step too and whose proof is given in Appendix D.

**Lemma 2.4.** Let $R, L > 0$ and $\beta_1, \ldots, \beta_{2n} \in G^{\alpha,L}(\mathbb{A}^n_R)$. We denote the variables in $\mathbb{A}^n_R$ by $(\theta, r) = (x_1, \ldots, x_{2n})$ and assume that

- $\partial_x \beta_j = \partial_x \beta_i$ for $i, j = 1, \ldots, 2n$,
- for each $r \in \overline{B}_R$ and $i = 1, \ldots, n$, the function $\beta_i(\cdot, r)$ has mean value zero on $\mathbb{T}^n$.

Then there exists $A \in G^{\alpha,L}(\mathbb{A}^n_R)$ such that

$$\sum_{i=1}^{2n} \beta_i \, dx_i = dA \quad \text{and} \quad \|A\|_{\alpha,L,R} \leq C(\|\beta_1\|_{\alpha,L,R} + \cdots + \|\beta_{2n}\|_{\alpha,L,R}),$$

where $C := \max \{\frac{1}{2}, R, L^\alpha\}$.

**Lemma 2.5.** Let $R, R_0, L_0 > 0$ be reals such that $R < R_0$, and let $\eta \in G^{\alpha,L_0}([0,1])$ be a non-trivial function. Then there exist $\varepsilon_*, L > 0$ such that, for any $\psi = (\psi_1, \ldots, \psi_n) \in G^{\alpha,L_0}(\mathbb{A}^n_{R_0}, \mathbb{R}^n)$ satisfying

$$\varepsilon := \sum_{i=1}^{n} \|\psi_i\|_{\alpha,L_0,R_0} \leq \varepsilon_*$$

and for any $t \in [0,1]$, the map

$$(\theta, r) \in \mathbb{A}^n_{R_0} \mapsto (\theta, r') = (\theta, r + \eta(t)\psi(\theta, r)) \in \mathbb{A}^n$$

induces a $C^\infty$ diffeomorphism from $\mathbb{T}^n \times \overline{B}_{R_0}$ onto an open subset $\Omega_\varepsilon$ of $\mathbb{A}^n$ which contains $\mathbb{A}^n_{R_0}$ with an inverse map of the form

$$(\theta, r') \in \Omega_\varepsilon \mapsto (\theta, r) = (\theta, r' + \chi(\theta, r', t)) \in \mathbb{T}^n \times \overline{B}_{R_0},$$

where $\chi = (\chi_1, \ldots, \chi_n)$ is $C^\infty$ and restricts to $\mathbb{A}^n_{R_0} \times [0,1] \in G^{\alpha,L}(\mathbb{A}^n_{R_0} \times [0,1], \mathbb{R}^n)$ with

$$\sum_{i=1}^{n} \|\chi_i\|_{\alpha,L_0} \leq \varepsilon \|\eta\|_{\alpha,L_0}. \quad (2.9)$$

For $\varepsilon_*$ and $L$, one can take the values indicated in (D.1) and (D.2).

**Proof of Lemma 2.4.** The function

$$\tilde{A}(x) := \int_0^1 \sum_{i=1}^{2n} x_i \beta_i(tx) \, dt$$

is well defined on $\mathbb{R}^n \times \overline{B}_R$. An easy computation yields $\partial_x \tilde{A} = \beta_i$ for $i = 1, \ldots, 2n$. In particular, for each $r \in \overline{B}_R$, the functions $\partial_{\theta_1} \tilde{A}(\cdot, r), \ldots, \partial_{\theta_n} \tilde{A}(\cdot, r)$ are $\mathbb{Z}^n$-periodic and
have mean value zero, whence it follows that \( \hat{A}(\cdot, r) \) is itself \( \mathbb{Z}^n \)-periodic. Thus \( \hat{A} \) induces a function \( A \in C^\infty(A^n_R) \), and the differential of \( A \) is \( \beta_1 \, dx_1 + \cdots + \beta_n \, dx_n \).

Choosing \([-1, 1/2, \frac{1}{2}, 1]\) as a fundamental domain in \( \mathbb{R}^n \times B_R \), we get \( \|A\|_{C^0(A^n_R)} \leq \max \{ \frac{1}{2}, R \} \left( \|\beta_1\|_{C^0(A^n_R)} + \cdots + \|\beta_n\|_{C^0(A^n_R)} \right) \). Any \( \ell \in \mathbb{N}^n \) such that \( |\ell| \geq 1 \) can be written (usually in more than one way) as \( \ell = m + e_i \) with \( m \in \mathbb{N}^n \) and \( i \in \{1, \ldots, 2n\} \), moreover \( \partial^\ell A = \partial^m \beta_i \) and \( (m + e_i)! \geq m! \), hence

\[
\sum_{|\ell| \geq 1} \frac{L^{\ell}}{\ell!} \|\partial^\ell A\|_{C^0(A^n_R)} \leq \sum_{m \in \mathbb{N}^n} \sum_{i=1}^{2n} \frac{L^{1+|m|}}{(m + e_i)!} \|\partial^m \beta_i\|_{C^0(A^n_R)} \leq L \sum_{i=1}^{2n} \|\beta_i\|_{\alpha, L, R},
\]

which completes the proof.

Proof of Lemma 2.1.9. See Appendix D.

Proof of Proposition 2.1.10. Given \( L_0, L, R > 0 \) such that \( R < R_0 \), we set \( R' := \frac{R+R_0}{2} \) and

\[
\varepsilon_* := \min \left\{ \frac{R_0 - R'}{2}, \frac{L_0^\alpha}{(2\alpha+1)(2n+1)^{\alpha-1}} \right\}, \quad L := \frac{L_0}{(2\alpha+1)(2n+1)^{\alpha-1}}^{1/\alpha}.
\]

Let \( h \in C^{\alpha,L_0} (\partial B_{R_0}) \) and let \( \Psi \in C^{\alpha,L_0}(A^n_{R_0}, A^n) \) be exact symplectic and satisfy (2.6). Let us choose a lift \( \xi \in C^\infty(A^n_{R_0}, \mathbb{R}^n) \) of \( \Psi - \Phi^h \) so that \( \|\xi_1\|_{\alpha, L_R, R_0} + \cdots + \|\xi_n\|_{\alpha, L_R, R_0} = \varepsilon \).

Since \( \Phi^h(\theta, r) = (\theta + \langle \nabla h(r) \rangle, r) \), we have

\[
\Psi[1](\theta, r) = \theta + \langle \nabla h(r) \rangle, \quad \Psi[2](\theta, r) = r + \xi[2](\theta, r).
\]

We apply Lemma 2.1.9 with \( \eta = 1 \) and \( \psi = \xi[2] \): in view of (D.1) and (D.2), our choice (2.10) of \( \varepsilon_* \) and \( L \) implies the existence of an open subset \( \Omega_1 \) of \( A^n \) containing \( A^n_{R'} \) such that

\[
(\theta, r) \in \mathbb{T}^n \times B_{R_0} \mapsto (\theta, r') = (\theta, \Psi[2](\theta, r)) \in \Omega_1
\]

is a \( C^\infty \) diffeomorphism, the inverse of which can be written

\[
\Phi: (\theta, r') \in \Omega_1 \mapsto (\theta, r) = (\theta, r' + \chi(\theta, r')) \in \mathbb{T}^n \times B_{R_0},
\]

with \( \|\chi\|_{\alpha, L, R'} + \cdots + \|\chi\|_{\alpha, L, R'} \leq \varepsilon \). We set \( \Omega' := \Omega_1 \cap (\mathbb{T}^n \times B_{R_0}) \) and \( \Omega := \Phi(\Omega') \subset \mathbb{T}^n \times B_{R_0} \). Notice that \( A^n_{R'} \subset A^n_{R} \subset \Omega' \) and \( A^n_{R} \subset \Omega \) (because \( \|\xi[2](\theta, r)\| \leq R' - R \) for all \( (\theta, r) \in A^n_{R_0} \), thus \( \Phi^{-1}(A^n_{R'}) \subset A^n_{R} \)).

We now consider

\[
F(\theta, r) := \Phi^{-h} \circ \Psi(\theta, r) = (\Psi[1](\theta, r) - \langle \nabla h \circ \Psi[2](\theta, r) \rangle, \Psi[2](\theta, r))
\]

for \( (\theta, r) \in \Omega \) (which is possible since \( \Psi[2](\Omega) \subset B_{R_0} \)). This is an exact symplectic \( C^\infty \) local diffeomorphism, which can be written

\[
F(\theta, r) = (\theta + \langle f(\theta, r) \rangle, \Psi[2](\theta, r)), \quad f := \xi[1] + \nabla h - \nabla h \circ \Psi[2],
\]
and the map \((2.11)\) induces a \(C^\infty\) diffeomorphism from \(\Omega\) onto \(\Omega'\); therefore, following the recpee of Lemma [C.3] we know that the 1-form

\[
\beta := \sum_{i=1}^{n} \chi_i(\theta, r') \, d\theta_i + \sum_{i=1}^{n} f_i \circ \Phi(\theta, r') \, dr'_i
\]

is exact and \(F = \mathcal{F}_A\) on \(\Omega\), where \(A \in C^\infty(\Omega')\) is any primitive of \(\beta\).

We conclude by checking that we can apply Lemma 2.4 and get a primitive \(A \in G^{\alpha, L}(\mathbb{A}^n_R)\) whose norm we can bound. On the one hand, we have \(\chi_i \in G^{\alpha, L}(\mathbb{A}^n_R)\) for each \(i\) and \(\|\chi_1\|_{\alpha, L, R} + \cdots + \|\chi_n\|_{\alpha, L, R} \leq \varepsilon\). On the other hand, since \(\Psi[2] \circ \Phi(\theta, r') = r'\), we can write

\[
f_i \circ \Phi(\theta, r') = \xi_i \circ \Phi(\theta, r') + \eta_i(\theta, r'),
\]

\[
g_i(\theta, r') := \partial_i h \circ \Phi[2](\theta, r') - \partial_i h(r') = \sum_{j=1}^{n} \int_{\theta}^{r'} \partial_i \partial_j h(r' + s\chi(\theta, r')) \chi_j(\theta, r') \, ds.
\]

Let \(L_1 := L_0/2\). We can apply Proposition A.1 of [MS03] to the composition with \(\Phi(\theta, r') = (\theta, r' + \chi(\theta, r'))\) or, more generally, with \(U_s(\theta, r') := (\theta, r' + s\chi(\theta, r'))\) for \(s \in [0, 1]\), because

\[
\sum_{\ell \in \mathbb{N}, \ell \neq 0} \frac{L_{\|\alpha\|}}{L_{0}} \|\partial^\ell U_{s,i}\|_{C^0(\mathbb{A}^n_R)} \leq \frac{L_0^\alpha}{(2n)^{\alpha-1}}, \quad i = 1, \ldots, 2n
\]

(Indeed: this follows from \(L^\alpha + \|\chi_i\|_{\alpha, L, R} \leq \frac{L_0^\alpha}{2^\alpha(2n)^{\alpha-1}}\), and we get

\[
\|\xi_i \circ \Phi\|_{\alpha, L, R} \leq \|\xi_i\|_{\alpha, L_1, R_0}
\]

and \(\|\partial_i \partial_j h \circ U_s\|_{\alpha, L, R} \leq \|\partial_i \partial_j h\|_{\alpha, L_1, R_0}\), whence

\[
\|g_i\|_{\alpha, L, R} \leq \sum_j \|\partial_i \partial_j h\|_{\alpha, L_1, R_0} \|\chi_j\|_{\alpha, L, R}
\]

by the algebra norm property. Thus Lemma 2.4 gives us \(A \in G^{\alpha, L}(\mathbb{A}^n_R)\) with

\[
\|A\|_{\alpha, L, R} \leq C \left( \sum_i \|\xi_i\|_{\alpha, L_1, R_0} + \sum_{i,j} \|\partial_i \partial_j h\|_{\alpha, L_1, R_0} \|\chi_j\|_{\alpha, L, R} \right) \leq C \left( 1 + \sum_{i,j} \|\partial_i \partial_j h\|_{\alpha, L_1, R_0} \right) \varepsilon,
\]

and, using (B.1), we get the desired estimate with \(C_* := C(1 + \frac{2^{3n}}{L_0^\alpha})\).

\(\Box\)

### 2.2.2 Second step: constructing a Hamiltonian isotopy

**Proposition 2.6.** Let \(L_0, R, R_0 > 0\) be reals such that \(R < R_0\). Let \(\eta \in G^{\alpha, L_0}([0, 1])\). Then there exist \(\varepsilon_*, L, C > 0\) satisfying the following: for any \(A \in G^{\alpha, L_0}(\mathbb{A}^n_R)\) such that \(\|A\|_{\alpha, L_0, R_0} \leq \varepsilon_*\) and for any \(t \in [0, 1]\), there exists an open subset \(\Omega_t\) of \(\mathbb{A}^n\) containing \(\mathbb{A}^n_R\) such that

\[
\mathcal{F}_{\eta(t)} : \Omega_t \to \mathbb{T}^n \times B_{R_0}
\]
is a well-defined exact symplectic $C^\infty$ diffeomorphism, and for each $(\theta, r) \in A^n_R$,

$$\frac{d}{dt}\mathcal{F}_{\eta(t)}A(\theta, r) = X_f(\mathcal{F}_{\eta(t)}A(\theta, r), t), \quad t \in [0, 1],$$

(2.12)

where $X_f$ is the non-autonomous Hamiltonian vector field associated with

$$f(\theta, r, t) := \eta(t)A((\mathcal{F}_{\eta(t)}A)^{-1}[1](\theta, r), r), \quad (\theta, r, t) \in \mathbb{T}^n \times B_{R_0} \times [0, 1],$$

(2.13)

which is a $C^\infty$ Hamiltonian function whose restriction to $A^n_R \times [0, 1]$ is Gevrey-$(\alpha, L)$, with

$$\|f\|_{\alpha, L, R} \leq 2^n L_0^{-\alpha}\|\eta\|_{\alpha, L_0} \|A\|_{\alpha, L_0, R_0}.$$  

(2.14)

Proof. Let $L_1 := L_0/2$ and

$$\varepsilon_* := \frac{(L_0 - L_1)^\alpha}{\|\eta\|_{\alpha, L_0}} \min\left\{\sqrt{n}, \frac{R_0 - R}{2}, \frac{L_1^n}{2^{\alpha+1}(2n + 1)^{\alpha-1}}\right\}, \quad L := \frac{L_1}{(2^{\alpha+1}(2n + 1)^{\alpha-1})^{1/\alpha}}.$$

Let $A \in G^{\alpha, L_0}(A^n_R)$ such that $\varepsilon := \|A\|_{\alpha, L_0, R_0} \leq \varepsilon_*$. By Lemma C.1, we have $\sum_{i=1}^n \|\partial_i A\|_{\alpha, L_1, R_0} \leq \frac{1}{(L_0 - L_1)^\alpha} \varepsilon_*$, thus we can apply Lemma 2.5 and we get for each $t \in [0, 1]$ an open subset $\Omega_t$ of $A^n$ containing $A^n_R$ such that the map

$$(\theta, r') \in \mathbb{T}^n \times B_{R_0} \mapsto (\theta, r) = (\theta, r' + \eta(t)\nabla[1]A(\theta, r')) \in \Omega_t$$

is a diffeomorphism whose inverse is $C^\infty$ on $A^n_R \times [0, 1]$ in the variables $\theta$, $r$ and $t$. By Lemma C.1, $\eta(t)A$ is thus a generating function for $\Omega_t$, inducing an exact symplectic local diffeomorphism from $\Omega_t$ to $\mathbb{T}^n \times B_{R_0}$; given $(\theta, r) \in \Omega_t$ and $(\theta', r') \in \mathbb{T}^n \times B_{R_0},$

$$(\theta', r') = \mathcal{F}_{\eta(t)}A(\theta, r) \iff \begin{cases} r = r' + \eta(t)\nabla[1]A(\theta, r') \\ \theta' = \theta + \eta(t)\nabla[2]A(\theta, r'). \end{cases}$$

(2.15)

Moreover, $(\theta, r) \in A^n_R \times [0, 1] \mapsto \mathcal{F}_{\eta(t)}A(\theta, r)$ is $C^\infty$.

In order to check that $\mathcal{F}_{\eta(t)}A$ is in fact a diffeomorphism from $\Omega_t$ onto $\mathbb{T}^n \times B_{R_0}$, we consider the map

$$(\theta, r') \in B_{2\sqrt{n}} \times B_{R_0} \mapsto (\theta', r') = (\theta + \eta(t)\nabla[2]A(\theta, r'), r') \in \mathbb{R}^n \times \mathbb{R}^n.$$  

(2.16)

By Lemma C.1, we have $\sum_{i=1}^n \|\partial_{n+i} A\|_{\alpha, L_1, R_0} \leq \frac{1}{(L_0 - L_1)^\alpha} \varepsilon$, thus we can apply of Lemma 2.5 (or rather a variant of it in which the $\mathbb{Z}^n$-periodicity assumption is removed and the roles of $\theta$ and $r$ are exchanged): we get an open subset $\hat{\Omega}_t$ of $\mathbb{R}^n \times B_{R_0}$ containing $B_{\sqrt{n}} \times B_{R_0}$ such that the map (2.16) is a $C^\infty$ diffeomorphism from $B_{2\sqrt{n}} \times B_{R_0}$ to $\hat{\Omega}_t$, with an inverse of the form

$$(\theta', r') \in \hat{\Omega}_t \mapsto (\theta, r) = (\theta' + \hat{g}(\theta', r', t), r') \in B_{2\sqrt{n}} \times B_{R_0},$$

with Gevrey-$(\alpha, L)$ estimates on $B_{\sqrt{n}} \times B_{R_0} \times [0, 1]$ for the components of $\hat{g}$. Since $\nabla[2]A$ is $\mathbb{Z}^n$-periodic in $\theta$ and $B_{\sqrt{n}}$ contains $[-\frac{1}{2}, \frac{1}{2}]^n$, the vector-valued function $\hat{g}$ is $\mathbb{Z}^n$-periodic.
in $\theta'$ and extends by periodicity to the whole of $\mathbb{R}^n \times B_{R_0}$; we thus get a $C^\infty$ diffeomorphism $(\theta', r') \in \mathbb{T}^n \times B_{R_0} \mapsto (\theta, r') = (\theta' + g(\theta', r', t), r') \in \mathbb{T}^n \times B_{R_0}$ with

$$g_1, \ldots, g_n \in C^\alpha_L(\mathbb{A}_R^n \times [0, 1]), \quad \sum_{i=1}^n \|g_i\|_{\alpha, L, R} \leq \frac{\|\eta\|_{\alpha, L_0}}{(L_0 - L_1)^\alpha}. \varepsilon.$$  

In view of (2.15), we conclude that $\hat{\mathcal{F}}_{\eta(t), A}$ is invertible, with inverse

$$\hat{\mathcal{F}}_{\eta(t), A}^{-1}(\theta', r') = ((\theta' + g(\theta', r', t)), r' + \eta(t)\nabla^1 A(\theta' + g(\theta', r', t), r')), \quad (\theta', r') \in \mathbb{T}^n \times B_{R_0}.$$

Let us now consider the $C^\infty$ function

$$f(\theta, r, t) := \eta'(t) \hat{A}(\theta + g(\theta, r, t), r), \quad (\theta, r, t) \in \mathbb{T}^n \times B_{R_0} \times [0, 1].$$

By Proposition A.1 of [MS03] (cf. also Appendix B.1), it is Gevrey-$\alpha, L$ on $\mathbb{A}_R^n \times [0, 1]$ because

$$L^\alpha + \sum_{\ell \in \mathbb{N}^{2n+1}, \ell \neq 0} \frac{L[\ell]^\alpha}{\ell!^\alpha} \|\partial^\ell g_i\|_{C^\alpha(\mathbb{A}_R^n \times [0, 1])} \leq \frac{L_0^\alpha}{(2n)^{\alpha-1}}, \quad i = 1, \ldots, n,$$

and $\|f\|_{\alpha, L, R} \leq \|\eta\|_{\alpha, L, R} \|\hat{A}\|_{\alpha, L, R_0} \leq \frac{1}{(L_0 - L_1)^\alpha} \|\eta\|_{\alpha, L_0} \varepsilon$ (thanks to the algebra norm property and (B.1)), which yields (2.14). It only remains to be shown that, for each $(\theta, r) \in \mathbb{A}_R^n$, the $C^\infty$ curve $t \in [0, 1] \mapsto (\theta(t), r(t)) := \hat{\mathcal{F}}_{\eta(t), A}(\theta, r)$ satisfies the system of ordinary differential equations

$$\theta'(t) = \nabla^2 f(\theta(t), r(t), t), \quad r'(t) = -\nabla^1 f(\theta(t), r(t), t). \quad (2.17)$$

On the one hand, the relations

$$r = r(t) + \eta(t)\nabla^1 A(\theta, r(t)), \quad \theta(t) = \theta + \eta(t)\nabla^2 A(\theta, r(t))$$

entail

$$r'(t) = -\eta'(t) \left(1_n + \eta(t) \operatorname{d}^2(\nabla^1 A(\theta, r(t)))^{-1}\nabla^1 A(\theta, r(t)) \right)^{-1}\nabla^1 A(\theta, r(t)), \quad (2.18)$$

$$\theta'(t) = \eta'(t)\nabla^2 A(\theta, r(t)) + \eta(t)\operatorname{d}^2(\nabla^2 A(\theta, r(t))r'(t). \quad (2.19)$$

On the other hand, with the notation $\mathcal{G}_t := \hat{\mathcal{F}}_{\eta(t), A}^{-1}$, the formula (2.13) yields

$$\nabla^1 f(\theta, r, t) = \eta'(t) \left(\operatorname{d}[1] \mathcal{G}_t^1(\theta, r)\right) \nabla^1 A(\mathcal{G}_t^1(\theta, r), r)$$

$$\nabla^2 f(\theta, r, t) = \eta'(t)\nabla^2 A(\mathcal{G}_t^1(\theta, r), r) + \eta'(t) \left(\operatorname{d}[2] \mathcal{G}_t^1(\theta, r)\right) \nabla^1 A(\mathcal{G}_t^1(\theta, r), r)$$

for any $(\theta, r, t)$. We rewrite this at the point $(\theta(t), r(t), t)$, using the fact that the Jacobian matrix of $\mathcal{G}_t$ at $(\theta(t), r(t))$ is the inverse Jacobian matrix of $\mathcal{F}_{\eta(t), A}$ at $(\theta, r)$, whose first $n$ lines are given by (C.6), thus

$$\operatorname{d}[1] \mathcal{G}_t^1(\theta(t), r(t)) = \left(1_n + \eta(t)\operatorname{d}[1]\nabla^2 A(\theta, r(t)) \right)^{-1},$$

$$\operatorname{d}[2] \mathcal{G}_t^1(\theta(t), r(t)) = \operatorname{d}[1] \mathcal{G}_t^1(\theta(t), r(t))\operatorname{d}[2]\nabla^2 A(\theta, r(t)), \quad (2.16)$$

and

$$\nabla^1 f(\theta(t), r(t), t) = \eta'(t) \left(\operatorname{d}[1] \mathcal{G}_t^1(\theta(t), r(t))\right) \nabla^1 A(\mathcal{G}_t^1(\theta(t), r(t)), r(t)),$$

$$\nabla^2 f(\theta(t), r(t), t) = \eta'(t)\nabla^2 A(\mathcal{G}_t^1(\theta(t), r(t)), r(t)) + \eta'(t) \left(\operatorname{d}[2] \mathcal{G}_t^1(\theta(t), r(t))\right) \nabla^1 A(\mathcal{G}_t^1(\theta(t), r(t)), r(t)).$$

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and
\[ \nabla^{[1]} f(\theta(t), r(t), t) = \eta'(t) \left( 1_n + \eta(t) d^{[2]} \nabla^{[1]} A(\theta, r(t)) \right)^{-1} \nabla^{[1]} A(\theta, r(t)) = -r'(t) \]
by (2.18), and
\[ \nabla^{[2]} f(\theta(t), r(t), t) = \eta'(t) \nabla^{[2]} A(\theta, r(t)) + \eta(t) d^{[2]} \nabla^{[2]} A(\theta, r(t)) r'(t) = \theta'(t) \]
by (2.19), hence (2.17) is proved. \[ \square \]

**Remark 2.7.** The fact that \( t \mapsto \mathcal{F}_{\eta(t)} A \) is a Hamiltonian isotopy is standard result of basic symplectic topology, however the explicit formula (2.13) for the non-autonomous Hamiltonian function \( f \) is new. This explicit formula was needed to obtain the estimate (2.14).

### 2.2.3 Completion of the proof of Theorem [E]

We now prove Theorem [E]. We thus give ourselves reals \( L_0, R, R_0 > 0 \) such that \( R < R_0 \) and a function \( h \in C^{\alpha,L_0}(\overline{B_{R_0}}) \). We pick \( R_1 \in (R, R_0) \) and \( \eta \in C^{\alpha,L_0}([0,1]) \) such that \( \eta \equiv 0 \) on a neighbourhood of \( 0 \), \( \eta \equiv 1 \) on a neighbourhood of \( 1 \), and \( 0 \leq \eta \leq 1 \) on \([0,1]\) (e.g. \( \eta(t) = G(t)/G(1) \) with \( G(t) = \int_0^t F(s) \mathrm{d}s \), where \( F \in C^{\alpha,L_0}([0,1]) \) satisfies \( F \geq 0 \), \( F(\frac{1}{2}) = 1 \), \( F_{[0,\frac{1}{2}) \cup (\frac{1}{2},1]} = 0 \); such a function \( F \) is constructed in Lemma A.3 of [MS03].—see also Lemma 3.3 of [MS04] quoted in Appendix B.4).

Applying Proposition 2.3 with \( L_0, R_1, R_0 \) and \( h \), we get constants \( \varepsilon_1, L_1, C_1 \) such that, for any exact symplectic map \( \Psi \in C^{\alpha,L_0}(\mathbb{A}^{n}_{R_0}, \mathbb{A}^{n}) \) with
\[ \varepsilon := \left\| \Psi - \Phi^h \right\|_{\alpha,L_0,R_0} \leq \varepsilon_1, \tag{2.20} \]
there exists \( A \in C^{\alpha,L_1}(\mathbb{A}^{n}_{R_1}) \) such that \( \mathcal{F}_A : \mathbb{A}^{n}_{R_1} \to \mathbb{A}^{n}_{R_0} \) is a well-defined exact symplectic map,
\[ \Psi|_{\mathbb{A}^{n}_{R_1}} = \Phi^h \circ \mathcal{F}_A, \quad \| A \|_{\alpha,L_1,R_1} \leq C_1 \varepsilon. \tag{2.21} \]

Applying Proposition 2.6 with \( L_1, R, R_1 \) and \( \eta \): we get constants \( \varepsilon_2, L_2, C_2 \) such that, for any \( A \in C^{\alpha,L_1}(\mathbb{A}^{n}_{R_1}) \) with \( \| A \|_{\alpha,L_1,R_1} \leq \varepsilon_2 \) and for any \( t \in [0,1] \), there exists an open subset \( \Omega_t \) of \( \mathbb{A}^{n} \) containing \( \mathbb{A}^{n}_{R_1} \) such that \( \mathcal{F}_{\eta(t)} A : \Omega_t \to \mathbb{T}^{n} \times B_{R_1} \) is a well-defined exact symplectic \( C^{\infty} \) diffeomorphism, \( t \mapsto \mathcal{F}_{\eta(t)} A(\theta, r) \) satisfies the ordinary differential equation (2.12) for each \( (\theta, r) \in \mathbb{A}^{n}_{R_1}, \) with \( f \in C^{\infty}(\mathbb{T}^{n} \times B_{R_1} \times [0,1]) \) such that
\[ f(\theta, r, t) := \eta'(t) A \left( (\mathcal{F}_{\eta(t)} A)^{-1} \right)(\theta, r, t), \quad \| f \|_{\alpha,L_2,R} \leq 2^{\alpha} L_1^{-\alpha} \| \eta \|_{\alpha,L_1} \| A \|_{\alpha,L_1,R_1}. \tag{2.22} \]

Let us set
\[ \varepsilon_* := \min \left\{ \frac{1}{C_1}, \varepsilon_2 \right\}, \quad C_* := \frac{2^\alpha C_1}{L_1^\alpha} \| \eta \|_{\alpha,L_0} \]
and choose \( L_* > 0 \) small enough so that
\[ L_*^\alpha + \frac{L_*^\alpha (1 + L_2^2)}{(L_2 - L_*)^{\alpha}(L_0 - L_2)^2} \| h \|_{\alpha,L_0,R_0} \leq \frac{L_2^\alpha}{(2n + 1)^\alpha}. \tag{2.23} \]

Given an exact symplectic map \( \Psi \in C^{\alpha,L_0}(\mathbb{A}^{n}_{R_0}, \mathbb{A}^{n}) \) such that (2.20) holds, we get from Proposition 2.3 a function \( A \) satisfying (2.22). Since \( C_1 \varepsilon \leq \varepsilon_2 \), we can then apply
Proposition 2.3 to the generating function $A$ and get a non-autonomous Hamiltonian function $f \in C^\infty(\mathbb{T}^n \times B_{R_1} \times [0,1])$ as in (2.22), for which the flow between time 0 and time $t$ on $\mathbb{A}^n_R$ coincides with $\bar{\mathcal{F}}_{\eta(t)}A$ (because the differential equation (2.12) is satisfied and $\eta(0) = 0$, $\bar{\mathcal{F}}_0 = \text{Id}$). Notice that $\|f\|_{\alpha,L,2,R} \leq C_* \varepsilon$.

For $t \in [0,1]$, we define $\Psi_t := \Phi^{th} \circ \bar{\mathcal{F}}_{\eta(t)}A$ on $\mathbb{A}^n_R$: this is an isotopy from $\text{Id}$ to $\Psi$, and one checks easily that it gives the flow between time 0 and time $t$ on $\mathbb{A}^n_R$ for the Hamiltonian function

$$H(\theta, r, t) := h(r) + f \circ \Phi(\theta, r, t), \quad (\theta, r, t) \in \mathbb{T}^n \times B_{R_1} \times [0,1],$$

where $\Phi(\theta, r, t) := (\theta - t\nabla h(r), r, t)$ (because $\Phi^{th}$ is symplectic and $\Phi(x, t) = (\Phi^{-th}(x, t), t)$, hence $d\Phi^{th}(x)X_f(x, t) = X_{f \circ \Phi}(\Phi^{th}(x), t))$. Since $\eta'(t) \equiv 0$ in neighbourhoods of 0 and 1, the formula

$$H(\theta, r, t) = h(r) + \eta'(t) A((\bar{\mathcal{F}}^{-1}_{\eta(t)}A)[1](\theta - t\nabla h(r), r), r)$$

(2.24)

shows that $H$ can be extended by $\mathbb{Z}$-periodicity in $t$, so that we get $H \in C^\infty(\mathbb{T}^n \times B_{R_1} \times \mathbb{T})$, which can be viewed as a suspension of $\Psi|_{\mathbb{A}^n_R}$.

By Proposition A.1 of [MS03] (cf. also Appendix B.1), we have $H|_{\mathbb{A}^n_R \times \mathbb{T}} \in G^{\alpha,L_\ast}(\mathbb{A}^n_R \times \mathbb{T})$ and

$$\|H - h\|_{\alpha,L_\ast,R} = \|f \circ \Phi\|_{\alpha,L_\ast,R} \leq \|f\|_{\alpha,L,2,R} \leq C_* \varepsilon$$

because the components of $\Phi$ satisfy

$$\sum_{\ell \in \mathbb{N}^{2n+1}, \ell \neq 0} \frac{L^{|\ell|}}{\ell!} \|\partial^\ell \Phi_{|_{\mathcal{C}^0(\mathbb{A}^n_R \times [0,1])}} \| \leq \frac{L^{2n+1}}{(2n+1)^{\alpha-1}}, \quad i = 1, \ldots, 2n + 1$$

(2.25)

(indeed, the left-hand side is $\leq L^\alpha_\ast + \frac{L^a_\ast}{(L^2_\ast - L^\alpha_\ast)} \|t\partial_t h\|_{\alpha,L,2,R}$ by a $(2n+1)$-variable variant of Lemma B.2 which is $\leq L^\alpha_\ast + \frac{L^a_\ast(1+L^\alpha_\ast)^p}{(L^2_\ast - L^\alpha_\ast)^p} \|h\|_{\alpha,L_\ast,R_0}$ by (B.1), hence (2.25) follows from (2.23)). This ends the proof of Theorem E.

2.3 Proof of Theorem A (Nekhoroshev Theorem for maps)

We now prove Theorem A of Section 1.1. To this end, we first recall the exponential time theorem for near-integrable quasi-convex Hamiltonian flows in its most recent formulation. Theorem E will then allow us to transfer this result to near-integrable maps.

Theorem (Bounemoura-Marco [BM11]). Let $N \geq 2$ be an integer. Let $\alpha \geq 1$ and $L, R, R_0, m, E > 0$ be reals such that $R < R_0$. Then there exist positive reals $\varepsilon_\ast, c_\ast$, and, for each positive $R < R_0 - R$, positive reals $\varepsilon'_\rho \leq \varepsilon_\ast$ and $c'_\rho \leq c_\ast$, and, for each positive $\sigma < \frac{1}{N}$, positive reals $\varepsilon''_\sigma \leq \varepsilon_\ast$ and $c''_\sigma \leq c_\ast$, satisfying the following:

For each $h \in G^{\alpha,L}(\overline{B}_{R_0})$ such that $\|h\|_{\alpha,L,R_0} \leq E$, $\nabla h(r) \neq 0$ for every $r \in \overline{B}_{R_0}$ and

$$\int_0^1 d\nabla h(r) v \geq m \|v\| \quad \text{for all } v \in \mathbb{R}^N \text{ orthogonal to } \nabla h(r),$$

(2.26)

and for each $H \in G^{\alpha,L}(\mathbb{A}^n_{R_0})$ such that $\varepsilon := \|H - h\|_{\alpha,L,R_0} \leq \varepsilon_\ast$, every initial condition $(\theta[0], r[0])$ in $\mathbb{A}^n_{R_0}$ gives rise to a solution $t \mapsto (\theta(t), r(t))$ of $X_H$ which is defined at least for $|t| \leq \exp \left(c_\ast \left(\frac{1}{\varepsilon} \right)^{2(\alpha-1)N} \right)$, and
(i) $\varepsilon \leq \varepsilon'_{\rho}$ and $|t| \leq \exp \left( c'_\rho \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2(N-1)\rho}} \right)$ $\Rightarrow \|r(t) - r^{[0]}\| \leq \rho$

(ii) $\varepsilon \leq \varepsilon''_{\sigma}$ and $|t| \leq \exp \left( c''_{\sigma} \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2(N-1)\sigma}} \right)$ $\Rightarrow \|r(t) - r^{[0]}\| \leq \begin{cases} \frac{1}{c''_{\sigma}} \varepsilon^\frac{1}{2(N-1)\sigma} & \text{if } \alpha = 1, \\ \frac{1}{c''_{\sigma}} \varepsilon^\frac{1}{2(N-1)\sigma} \delta & \text{if } \alpha > 1, \end{cases}$

(iii) $\varepsilon \leq \varepsilon_\ast$ and $|t| \leq \exp \left( c_\ast \left( \frac{1}{\varepsilon} \right)^{\frac{1}{2N\alpha}} \right)$ $\Rightarrow \|r(t) - r^{[0]}\| \leq \frac{1}{c_\ast} \varepsilon^\frac{1}{2N\alpha}$.

**Remark 2.8.** This result is given in [BM11] in a slightly different presentation and we took the opportunity of correcting a slight mistake in the time exponent in the case $\alpha > 1$ of (ii): in [BM11], it should be $\frac{1}{2(N-1)\alpha} - \frac{\delta}{\alpha}$ with a parameter $\delta \in (0, \frac{1}{2N(N-1)})$ (and not $\frac{1}{2(N-1)\alpha} - \delta$ as is written there), and we introduced $\sigma = 2(N-1)\delta$.

It is a refined version of the Nekhoroshev theorem for analytic or Gevrey Hamiltonians in the case of an $m$-quasi-convex integrable part, i.e. in the case of a function $h(r)$ satisfying the condition (2.26). The article [BM11] is the last of a series of attempts to obtain the largest possible exponents $a$ in the stability time $\exp \left( \text{const} \left( \frac{1}{\varepsilon} \right)^{\alpha} \right)$ and $b$ in the corresponding confinement radius $\text{const} \varepsilon^b$, after the original work of Nekhoroshev in 1977 for analytic steep Hamiltonians, the refinement by Lochak-Neishtadt and Pöschel in 1992–94 for analytic quasi-convex Hamiltonians (which gave the exponents $a = b = \frac{1}{2N}$ as in (iii) in the case $\alpha = 1$), and the first Gevrey stability theorem by Marco-Sauzin in 2002 still in the quasi-convex case (which gave the exponents $a = \frac{1}{2N\alpha}$ and $b = \frac{1}{2N}$ as in (iii) in the case $\alpha \geq 1$).

Bounemoura-Marco’s article [BM11] focuses on the stability time (rather than the confinement radius, which is anyway a less important issue), for the Gevrey case ($\alpha \geq 1$); their discovery is that one can obtain a time exponent arbitrarily close to $\frac{1}{2(N-1)\alpha}$ at the price of a smaller confinement $b$, or even equal to that value at the price of accepting a weaker notion of confinement: there is an arbitrarily small confinement radius $\rho$ but it does not tend to 0 with $\varepsilon$. This weaker confinement property is all we need when studying wandering domains (see Section 2.4).

**Remark 2.9 (Stabilization by resonances).** The phenomenon of stabilization by resonances for quasi-convex Hamiltonians was first proved by Lochak-Neishtadt and Pöschel in the analytic case; Marco-Sauzin’s article [MS03] contains a generalization to the Gevrey case $\alpha \geq 1$ obtained by adapting Lochak’s periodic method. The result can be formulated as follows:

For any submodule $M$ of $\mathbb{Z}^n$ of rank $\text{mult}(M) \in \{1, \ldots, N-1\}$, consider the resonant surface

$$S_M := \{ r \in \overline{B}_{R_0} \mid \sum_{i=1}^n k_i \partial_i h(r) = 0 \text{ for all } k \in M \},$$

which is a $\text{mult}(M)$-codimensional submanifold of $\overline{B}_{R_0} \subset \mathbb{R}^N$. Then there is an improvement of the stability property whenever the initial condition lies at a distance $O(\varepsilon^{1/2})$ of $S_M$: for any real $\sigma > 0$, there exist $\varepsilon, \tilde{c} > 0$ (which depend on $\alpha, L, R_0, m, E, \sigma, M$) such that, for any $m$-quasi-convex $h \in G^{\alpha,L}(\overline{B}_{R_0})$ such that $\|h\|_{\alpha,L,R_0} \leq E$, for any
$H \in G^{\alpha,L}(A^n_R)$ such that $\varepsilon := \|H - h\|_{\alpha,L,R_0} \leq \tilde{\varepsilon}$, for any initial condition $(\theta^{[0]},r^{[0]})$ in $A^n_R$ such that

$$\text{dist}(r^{[0]},S_M) \leq \sigma \varepsilon^{1/2},$$

the solution $(\theta(t),r(t))$ of $X_H$ satisfies

$$|t| \leq \exp\left(\frac{\tilde{c}(\frac{1}{\varepsilon})}{a}\right) \Rightarrow \|r(t) - r^{[0]}\| \leq \frac{1}{\varepsilon} e^b$$

with $a := \frac{1}{2(N - \text{mult}(M))\alpha}$ and $b := \frac{1}{2(N - \text{mult}(M))}$.

**Proof of Theorem A** Let us give ourselves $n \geq 1$ integer and $\alpha \geq 1, L, R, R_0, m, E > 0$ reals such that $R < R_0$. Let $R_1 := \frac{R + R_0}{2}$.

Applying Theorem E with $n, \alpha, L, R_1, R_0, E$, we get positive reals $\varepsilon_1, L_1, C_1$ such that, for every $h \in G^{\alpha,L}(B_{R_0})$ such that $\|h\|_{\alpha,L,R_0} \leq E$ and every exact symplectic $\Psi \in G^{\alpha,L}(A^n_R, A^n)$ such that $\varepsilon := \|\Psi - \Phi^h\|_{\alpha,L,R_0} \leq \varepsilon_1$, there is a suspension $H \in G^{\alpha,L_1}(A^n_{R_1} \times \mathbb{T})$ such that $\|H - h\|_{\alpha,L_1,R_1} \leq C_1 \varepsilon$. Without loss of generality, we can assume $L_1 \leq L$ and $C_1 \geq 1$.

Let $N := n + 1$, $E_1 := R_1 + L_1 + E$, $m_1 := \frac{m}{1 - (L - R_1)}$. Applying Bounemoura-Marco’s theorem with $N, \alpha, L, R, R_1, m_1$, we get positive reals $\tilde{\varepsilon}_*, \tilde{c}_*$, and, for each positive $\rho < R_1 - R$, positive reals $\tilde{\varepsilon}_\rho \leq \tilde{\varepsilon}_*$ and $\tilde{c}_\rho \leq \tilde{c}_*$, and, for each positive $\sigma < \frac{1}{n + 1}$, positive reals $\tilde{\varepsilon}_\sigma \leq \tilde{\varepsilon}_*$ and $\tilde{c}_\sigma \leq \tilde{c}_*$, such that, for any $m_1$-quasi-convex $\tilde{h}$ and any $\tilde{H}$ in $G^{\alpha,L_1}(A^n_{R_1})$ such that $\|\tilde{h}\|_{\alpha,L_1,R_1} \leq E_1$ and $\tilde{\varepsilon} := \|\tilde{H} - \tilde{h}\|_{\alpha,L_1,R_1} \leq \tilde{\varepsilon}_*$, every initial condition $(\tilde{h}^{[0]}, \tilde{r}^{[0]})$ in $A_{R_1}^{n + 1}$ gives rise to a solution of $X_{\tilde{H}}$ defined at least for $|t| \leq \exp\left(\tilde{c}_*\left(\frac{t}{\varepsilon}\right)^{\frac{1}{\sigma}}\right)$, which satisfies the properties (i), (ii) and (iii) of Bounemoura-Marco’s theorem.

We now check the statement of Theorem A for an $m$-convex function $h \in G^{\alpha,L}(B_{R_0})$ such that $\|h\|_{\alpha,L,R_0} \leq E$ and an exact symplectic $\Psi \in G^{\alpha,L}(A^n_R, A^n)$ such that

$$\varepsilon := \|\Psi - \Phi^h\|_{\alpha,L,R_0} \leq \varepsilon_* := \min\{\varepsilon_1, \varepsilon_*/C_1\}.$$

Let $H \in G^{\alpha,L_1}(A^n_{R_1} \times \mathbb{T})$ denote the suspension of $\Psi$ obtained from Theorem E. We introduce the $(n + 1)$-degree of freedom autonomous Hamiltonian functions

$$\tilde{h}(r, r_{n+1}) := r_{n+1} + h(r), \quad \tilde{H}(\theta, r, \theta_{n+1}, r_{n+1}) := r_{n+1} + H(\theta, r, \theta_{n+1})$$

for $(\theta, r, \theta_{n+1}, r_{n+1}) \in A^n_{R_1} \times \mathbb{T} \times \mathbb{R} \simeq \mathbb{T}^{n+1} \times B_{R_1} \times \mathbb{R}$, which contains $A_{R_1}^{n + 1}$. One easily checks that $\|\tilde{h}\|_{\alpha,L_1,R_1} \leq E_1$ and $\tilde{h}$ is $m_1$-quasi-convex. Since $\|\tilde{H} - \tilde{h}\|_{\alpha,L_1,R_1} = \|H - h\|_{\alpha,L_1,R_1} \leq C_1 \varepsilon \leq \tilde{c}_*$, Bounemoura-Marco’s theorem ensures stability properties for all the solutions of $X_{\tilde{H}}$ starting in $A_{R_1}^{n + 1}$. The conclusion stems from the fact that the solutions of the autonomous vector field $X_{\bar{H}}$ are related to the solutions of the non-autonomous vector field $X_H$, which, in turn, interpolate the discrete dynamics induced by $\Psi$; in particular, if the initial condition is of the form $(\theta^{[0]}, r^{[0]}, \theta_{n+1}^{[0]}, r_{n+1}^{[0]}) = (\theta, r, 0, 0)$, then the value of the solution at any integer time $k$ such that $|k| \leq \exp\left(\tilde{c}_*\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\sigma}}\right)$ satisfies

$$\left(\theta(k), r(k)\right) = \Psi^k(\theta, r), \quad \theta_{n+1}(k) = k, \quad r_{n+1}(k) = H(\theta, r, 0) - H(\Psi^k(\theta, r), k),$$

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hence the properties (i), (ii) and (iii) in Bounemoura-Maro’s theorem imply the desired properties for the discrete orbits of $\Psi$ starting in $\mathbb{A}_R^n$, with $c_* := \tilde{c}_s\left(\frac{1}{C_1}\right)^{\frac{1}{2\alpha}}$, $\varepsilon'_* := \min\{\varepsilon_1, \varepsilon'/C_1\}$, $c'_* := \tilde{c}_s\left(\frac{1}{C_1}\right)^{\frac{1}{2\alpha}}$, $\varepsilon''_* := \min\{\varepsilon_1, \varepsilon''/C_1\}$, $c''_* := \tilde{c}_s\left(\frac{1}{C_1}\right)^{\frac{1}{2\alpha}}$.

2.4 Proof of Theorem B (upper bounds for wandering sets)

We now prove Theorem B of Section 1.2. Let us give ourselves $n \geq 1$ integer and $\alpha \geq 1$, $L, R, R_0, m, E > 0$ such that $R < R_0$. We take $\varepsilon_*$ and $c_*$ as in Theorem A.

Given an arbitrary $m$-convex function $h \in C^{\alpha,L}(\overline{B}_{R_0})$ such that $\|h\|_{\alpha,L,R_0} \leq E$, and a map $\Psi$ as in the statement of Theorem B with a measurable wandering set $W \subset \mathbb{A}_R^n$, we can apply Theorem A to $\Psi|_{\mathbb{A}_R^n}$. This shows that for a point $(\theta, r) \in \mathbb{A}_R^n$, all the iterates $\Psi^k(\theta, r)$ with $|k| \leq k_* := \exp (c_*\left(\frac{1}{C_1}\right)^{\frac{1}{2\alpha}})$ stay in $\mathbb{A}_R^n$. In particular, all the sets $\Psi^k(W)$ with $|k| \leq k_*$ are contained in $\mathbb{A}_R^n$. But these sets are pairwise disjoint and they all have the same Lebesgue measure, therefore $(2k_* + 1)\mu(W) \leq \mu(\mathbb{A}_R^n)$, which yields the desired estimate (diminishing the value of $\varepsilon_*$ and $c_*$ if necessary).
3 A quantitative KAM result – proof of Part (i) of Theorem D

As announced in Section 1.3, this section contains the proof of the two-dimensional case of Theorem D stated there. This proof is based on an auxiliary result, Theorem F, which will also be instrumental in the obtention of the full proof of Theorem D in Section 4.2.

3.1 Elliptic islands in $\mathbb{A}$ with a tuning parameter – Theorem F

We will take the liberty of identifying a 1-periodic function on $\mathbb{R}$ with a function on $\mathbb{T}$.

Here is the auxiliary result which has been alluded to:

**Theorem F.** Let $\alpha > 1$ and $L > 0$ be real numbers. Suppose, on the one hand, that $V \in C^\infty(\mathbb{R})$ is a 1-periodic function and that $L_0, \theta^*, \rho_0$ are positive reals such that $L_0 < \frac{1}{2} - \theta^*$ and

\[
\begin{align*}
(i) &\quad -L_0 \leq \theta \leq L_0 \quad \Rightarrow \quad V(\theta) = -\frac{1}{2} \rho_0^2 \\
(ii) &\quad \frac{1}{2} - \theta^* \leq \theta \leq \frac{1}{2} + \theta^* \quad \Rightarrow \quad V(\theta) = -\theta - 1/4 \\
(iii) &\quad \theta - \frac{1}{2} \notin \mathbb{Z} \quad \Rightarrow \quad V(\theta) < 0.
\end{align*}
\]

We use the notation

\[
P_{V/N^2}(\theta, r) := \frac{1}{2} r^2 + \frac{1}{N^2} V(\theta) \quad \text{for } N \in \mathbb{N}^*.
\] (3.1)

Suppose, on the other hand, that $\delta$ is a real number such that $0 < \delta < \rho_0/2$, and that $(W_N)_{N \in \mathbb{N}^*}$ is a sequence of 1-periodic functions of $C^\infty(\mathbb{R})$ such that

\[
\begin{align*}
(iv) &\quad -\frac{\delta}{2N} \leq \theta \leq \frac{\delta}{2N} \quad \Rightarrow \quad W_N(\theta) = \frac{1}{2} \theta^2 \\
v) &\quad \frac{\delta}{N} \leq \theta \leq 1 - \frac{\delta}{N} \quad \Rightarrow \quad W_N(\theta) = 0.
\end{align*}
\]

Then there exist positive reals $C_1, C_2, C_3, C_4$ such that, for any integers $q, N \in \mathbb{N}^*$ such that $q \geq C_1 N$ and for any real $\mu \in (0, C_2 N^4/q^5)$, the exact symplectic map of $\mathbb{A}$

\[
G_{N,\mu} := \Phi^{\mu W_N} \circ \Phi^{P_{V/N^2}}
\] (3.2)

admits a $q$-periodic disc $D_{q,N,\mu} \subset \mathbb{A}_3$, with all its iterates also contained in $\mathbb{A}_3$, such that

\[
C_3 \frac{\mu}{N^2} \leq \text{area}(D_{q,N,\mu}) \leq C_4 \frac{\mu}{N^2}
\] (3.3)

and

\[
D_{q,N,\mu} \subset \mathcal{B}_{\delta/(2N)} \cap \mathbb{A}_3^+, \quad G_{N,\mu}^k(D_{q,N,\mu}) \cap \mathcal{B}_{\delta/N} = \emptyset \quad \text{for } 1 \leq k \leq q - 1.
\] (3.4)
The proof of Theorem \( \text{F} \) is given in Sections 3.3–3.7. Recall that the notations \( \mathcal{A}_d^+ \) and \( \mathcal{B}_d \) were introduced in (1.17).

Observe that, when \( V, W_N \in G^{α, L}(\mathbb{R}) \), we have \( G_{N, μ} \in \mathcal{B}_{1, L}^{α, L}(Φ^{1/2^2}) \); this map can be viewed as a perturbation of the “pseudo-pendulum” \( Φ^{Fv/N^2} \). Here, we have two external parameters, \( N \) and \( μ \) (changing them amounts to changing the discrete dynamical system we are dealing with), and one internal parameter, \( q \) (we may vary it, e.g. taking it larger and larger, while keeping the same system \( G_{N, μ} \)). We call \( μ \) the “tuning parameter”; an appropriate choice of \( μ \) will yield Theorem \( \text{D}(i) \) in Section 3.2 and Theorem \( \text{D}(ii) \) in Section 4.2.

### 3.2 Theorem \( \text{F} \) implies Part (i) of Theorem \( \text{D} \)

Taking for granted Theorem \( \text{F} \) we now show how Theorem \( \text{D}(i) \) follows.

Let \( α > 1 \) and \( L > 0 \). With the help of “bump functions” as in Appendix B.3, we can easily choose \( V \in G^{α, L}(\mathbb{R}) \) satisfying conditions (i)–(iii) (for whatever choice of \( L_0, θ^*, p_0 \)). We choose \( p_0 > 2 \) and \( δ = 1 \). For the choice of the sequence \( (W_N)_{N \in \mathbb{N}^*} \) we apply Lemma B.5, which produces a real \( c(α, L) > 0 \) and a sequence of 1-periodic functions \( (η_N)_{N \in \mathbb{N}^*} \) in \( G^{α, L}(\mathbb{R}) \) such that

\[
-\frac{1}{2N} < θ < \frac{1}{2N} \Rightarrow η_N(θ) = 1, \quad \frac{1}{N} < θ < 1 - \frac{1}{N} \Rightarrow η_N(θ) = 0
\]

and

\[
\|η_N\|_{α, L} \leq \exp \left( c(α, L) \frac{N}{α - 1} \right) \quad \text{for all } N \in \mathbb{N}^*.
\] (3.5)

We then set \( W_N(θ) := \frac{1}{2} η_N(θ) \left( \text{dist}(θ, \mathbb{Z})^2 \right) \), so that

\[
\|W_N\|_{α, L} \leq C_0 \exp \left( c(α, L) \frac{N}{α - 1} \right) \quad \text{for all } N \in \mathbb{N}^*
\] (3.6)

with some constant \( C_0 > 0 \), and we apply Theorem \( \text{F} \).

We get \( C_1, C_2, C_3, C_4 > 0 \) fulfilling the conclusions of Theorem \( \text{F} \). Observe that formula (3.2) defines \( G_{N, μ} \in \mathcal{B}_{1, L}^{α, L}(Φ^{1/2^2}) \) with

\[
δ^{α, L} \left( G_{N, μ}, Φ^{1/2^2} \right) \leq \frac{1}{N^2} \|V\|_{α, L} + C_0 \mu \exp \left( c(α, L) \frac{N}{α - 1} \right)
\] (3.7)

for any integer \( N \geq 1 \) and real \( μ > 0 \). Recall that \( N_j := p_{j+2} \) is given by the prime number sequence. We set

\[
μ_{j, q} := \min \left\{ \frac{C_2 N_j^4}{2q^5}, \frac{1}{N_j^2} \exp \left( -c(α, L) N_j^{α - 1} \right) \right\}, \quad Ψ_{j, q} := G_{N_j, μ_{j, q}}, \quad \mathcal{D}_j := D_{q, N_j, μ_{j, q}}
\] (3.8)

for all \( j, q \in \mathbb{N}^* \) such that \( q \geq C_1 N_j \) (notice that \( D_{q, N_j, μ_{j, q}} \) is well defined because \( μ_{j, q} \in (0, C_2 N_j^4 / q^5) \) for such values of \( j \) and \( q \)).

Let us check that the conclusions of Theorem \( \text{D}(i) \) are fulfilled. Since (3.8) entails

\[
μ_{j, q} \exp \left( c(α, L) N_j^{α - 1} \right) \leq 1/N_j^2, \quad \text{we deduce from (3.7) that}
\]

\[
Ψ_{j, q} \in \mathcal{B}_{1, L}^{α, L}(Φ^{1/2^2}), \quad δ^{α, L}(Ψ_{j, q}, Φ^{1/2^2}) \leq \frac{\|V\|_{α, L} + C_0}{N_j^2}.
\]
According to Theorem F, $\mathcal{D}_{j,q}$ is a $q$-periodic disc for $\Psi_{j,q}$, whose orbit is localized precisely as desired, in particular (3.4) amounts exactly to (1.22). Now, by (3.3),

$$C_G(\mathcal{D}_{j,q}) = \text{area}(\mathcal{D}_{j,q}) \geq C_3 \frac{\mu_{j,q}}{N_j^2} = \frac{1}{2} C_2 C_3 \min \left\{ \frac{N_j^2}{q^2}, \frac{2}{C_2 N_j^4} \exp \left( - c(\alpha, L) \frac{1}{N_j^{1-\alpha}} \right) \right\}$$

$$\geq C'_3 \min \left\{ \frac{N_j^2}{q^2}, \exp \left( - c N_j^{\frac{1}{\alpha-\tau}} \right) \right\}$$

with $C'_3 := \frac{1}{2} C_2 C_3$ and $c := 2 c(\alpha, L)$ for $j$ large enough. This ends the proof of Theorem D(i).

3.3 Overview of the proof of Theorem F

The rest of Section 3 is devoted to the proof of Theorem F. We thus give ourselves once for all $\alpha, L, V$ and $(W_N)_{N \in \mathbb{N}^*}$ as in the statement. Let us begin with a brief overview of the method.

The pseudo-pendulum $\Phi_{V/N^2}$ has a degenerate equilibrium point at $(\frac{1}{2}, 0)$, with an “upper separatrix” $\{ (\theta, r) \in A \mid r > 0 \text{ and } P_{V/N^2}(\theta, r) = 0 \}$; see the figure on p. 37. It also has periodic points of arbitrarily high period located near this upper separatrix, and the effect of the perturbation $\Phi_{\mu W_N}$ in $G_{N,\mu}$ is to create elliptic islands around these periodic points.

We will estimate the size of these islands by means of Herman’s quantitative version of the two-dimensional KAM theorem [He01] recalled in Section 3.6. To do this, we have to compute parametrized normal forms of high order, the parameters being the size of the perturbation (measured by $1/N$ and $\mu$) and the period $q$ of the island.

More precisely, for $q$ large enough, we will study the $q$-th iterate of $G_{N,\mu}$ in a neighbourhood of a $q$-periodic point in Section 3.4 and, in Section 3.5, find normalizing coordinates in which Herman’s version of the invariant curve theorem can be applied to $G_{N,\mu}^q$.

3.4 Preliminary study near a $q$-periodic point $a_q$ of $G_{N,\mu}$

3.4.1 Localization

We begin with defining a suitable notion of adapted box, to be used in this section as well as in Section 4.2.

**Definition 3.1.** Let $q$ be an integer $\geq 2$ and fix $d \in (0, 1/2)$. Consider a Hamiltonian function $H : A \to \mathbb{R}$. A $q$-adapted box for $H$ and $\mathcal{B}_d$ is a rectangle $B = I \times [a, b] \subset \mathbb{T} \times \mathbb{R}$ contained in $\mathcal{B}_d$ such that

i) for $1 \leq t \leq q - 1$, $\Phi^t H(B) \cap \overline{\mathcal{B}_d} = \emptyset$;

ii) $\Phi^q H(B) \subset \overline{\mathcal{B}_{d/2}}$.

This section is devoted to the construction of an explicit $q$-adapted box centered at a $q$-periodic point $a_{q,N}$, as defined below, with respect to the system $P_{V/N^2}$ and $\mathcal{B}_{b/N}$. 

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In the same way as for the classical pendulum, the integral curve
\[ \mathcal{S}_e := \{(\theta, r) \in \mathbb{A} \mid r > 0, P_{V/N^2}(\theta, r) = e\} \]

is a closed curve if \( e > 0 \), so the flow \( t \mapsto \Phi^{tP_{V/N^2}}(\theta, r) \) is periodic on \( \mathcal{S}_e \), with a period \( T_{V/N^2}(e) \) given by
\[ T_{V/N^2}(e) = \int_0^1 \frac{du}{\sqrt{2(e - V(u)/N^2)}}. \tag{3.9} \]

**Definition 3.2.** Let \( q > 0 \) be a real number. Let \( a_{q,N} = (0, r_{q,N}) \) denote the unique point that satisfies \( r_{q,N} > 0 \) and \( T_{V/N^2}(a_{q,N}) = q \). We also set \( e_{q,N} = P_{V/N^2}(a_{q,N}) \) and \( \rho_N = \rho_0/N \).

![Diagram](image)

Note that \( \rho_N \) is the height of the separatrix \( \mathcal{S}_0 \) above \([-L_0, L_0] \times \{0\}\). We identify the time-1 flow \( \varphi = \Phi^{P_{V/N^2}} \) and its lift to \( \mathbb{R}^2 \) satisfying \( \varphi^q(0, r_{q,N}) = (1, r_{q,N}) \). We set
\[ B_q(\ell, \ell') := [-\ell, \ell] \times [r_{q,N} - \ell', r_{q,N} + \ell']. \]

If there is no source of confusion, we may identify \( B_q(\ell, \ell') \subset \mathbb{R}^2 \) with its image in \( \mathbb{A} \). We also denote \( \theta \) the projection to the first coordinate, from \( \mathbb{A} \) to \( T \), or from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

**Proposition 3.3.** Assume \( N \) is a positive integer and \( q \) is real number. If \( N \geq N_0 \) and \( q \geq q_0N \) then we have
\[ \rho_N \leq r_{q,N} \leq 2\rho_N, \quad C_1 \frac{N^2}{q^4} \leq e_{q,N} \leq C_2 \frac{N^2}{q^4} \quad \text{and} \quad C_3 \frac{N^3}{q^4} \leq r_{q,N} - \rho_N \leq C_4 \frac{N^3}{q^4}; \]
if \( q \geq 2 \) and \( 0 < \delta < \rho_0/2 \), if we set \( \ell = \delta/(4N) \) and \( \ell' = C_5 \frac{N^3}{q^4} \delta \) then \( B_q(\ell, \ell') \) is a \( q \)-adapted box with respect to \( P_{V/N^2} \) and \( \mathcal{B}_{\delta/N} \), where \( N_0, q_0, C_1, C_2, C_3, C_4 \) and \( C_5 \) are positive constants (which depend only on \( V \)).

Furthermore, if \( q \geq q_0', N \), with \((q_0' - q_0)N_0 \geq 1\), then we have
\[ \rho_N < r_{q+1,N} < r < r_{q-1,N} \leq 2\rho_N, \quad \text{for all} \quad (\theta, r) \in B_q(\ell, \ell'). \]

For the convenience of the reader, the proof of the proposition is split as follows. Lemma 3.4 provides a criterion for a rectangle to be \( q \)-adapted for the system \( P_{V/N^2} \) and Lemma 3.5 extracts from the system \( P_V \) all the properties we need. This eventually gives an explicite estimate of the size of a \( q \)-adapted box in Proposition 3.6 when \( N = 1 \).

In order to generalise these results to the system when \( N \geq N_0 \) is arbitrary, we note that the systems \( N^2P_{V/N^2} \) and \( P_V \) are equal up to a dilatation in the coordinate \( r \). From this, Lemma 3.7 gives explicit dependences between the two systems \( P_V \) and \( P_{V/N^2} \) for the main quantities we may compute.
Proof. Assuming Proposition 3.6, Lemma 3.5 and Lemma 3.7, we prove Proposition 3.3.

- Since \( e_{q,1} = (r_{q,1}^2 - \rho_0^2) / 2 \), we have \( \rho_0 < r_{q,1} \leq 2 \rho_0 \) if and only if \( 0 < e_{q,1} \leq \frac{1}{2} \rho_0^2 \). Let \( q_0 > 0 \) be the unique real number such that \( e_{q_0,1} = 3 \rho_0^2 / 2 \); Lemma 3.5 [iii] with \( k = 0 \) shows that

\[
c_0 / q \leq e_{q,1}^{1/4} \leq c_1 / q, \quad \text{for } q \geq q_0,
\]

where \( c_0 \) and \( c_1 \) are positive constants (which depend only on \( V \)). Therefore, for \( q \geq q_0 \), we have \( 0 < e_{q,1} \leq e_{q_0,1} \), so

\[
\rho_0 < r_{q,1} \leq 2 \rho_0 \quad ; \quad \frac{c_0^4}{q^4} \leq e_{q,1} \leq \frac{c_1^4}{q^4}.
\]

Furthermore, we have \( e_{q,1} = \frac{1}{2} (r_{q,1} + \rho_0) (r_{q,1} - \rho_0) \). Since \( \rho_0 < \frac{1}{2} (r_{q,1} + \rho_0) \leq 2 \rho_0 \), it follows that

\[
\frac{c_0^4}{2 \rho_0^4 q^4} \leq e_{q,1} \leq \frac{c_1^4}{\rho_0 q^4}.
\]

Since Lemma 3.7 shows that \( r_{q,N} = r_{q/N,1}/N \), we obtain that

\[
\rho_N < r_{q,N} \leq 2 \rho_N \quad \text{and} \quad C_3 \frac{N^3}{q^4} \leq r_{q,N} - \rho_N \leq C_4 \frac{N^3}{q^4}, \quad \text{provided } q/N \geq q_0,
\]

with \( C_3 := \frac{c_0^4}{(2 \rho_0)} \) and \( C_4 := \frac{c_1^4}{\rho_0} \).

In a similar way, since Lemma 3.7 shows that \( e_{q,N} = e_{q/N,1}/N^2 \), we obtain that

\[
C_1 \frac{N^2}{q^4} \leq e_{q,N} \leq C_2 \frac{N^2}{q^4}, \quad \text{provided } q/N \geq q_0,
\]

with \( C_1 := \frac{c_0^4}{q^4} \) and \( C_2 := \frac{c_1^4}{q^4} \).

- We now construct the \( q \)-adapted box. Proposition 3.6 with \( V/N^2 \) substituted to \( V \) shows that the rectangle \( B_q (\ell, \ell') \) is \( q \)-adapted with respect to \( P_{V/N^2} \) and \( \mathcal{B}_{\delta/N} \), provided we have \( 0 < \delta/N < \delta_{0,N}, \ell = (\delta/N)/4 \) and

\[
\ell' = \frac{\delta}{N} \cdot \left( \frac{r_{q,N} - \rho_N}{r_{0,N}} \right)^{5/4}.
\]

Here \( r_{0,N} \) and \( \delta_{0,N} \) are the corresponding constants for \( P_{V/N^2} \) given by (3.15), that is

\[
r_{0,N} := 2 \cdot (4 C_{0,N} \rho_N^2)^{4/5}; \quad (3.10a)
\]

\[
\delta_{0,N} = \min \left( L_0; \rho_N/2; \frac{r_{0,N}^{5/4}}{2 \rho_N^{1/4}} \right) = \min \left( L_0; \rho_N/2; 20 \cdot 2^{5/4} \rho_N^{7/4} \rho_0^{1/2} \right); \quad (3.10b)
\]

where \( C_{0,N} = C_0 N^{7/4} \) as it appears in (3.14). Since \( \rho_N = \rho_0/N \), we observe that \( C_{0,N} \rho_N^{7/4} = C_0 \rho_0^{7/4} \); so \( \delta_{0,N} = \rho_0/(2N) \) if \( N \geq N_0 \) with \( N_0 \) large enough (depending only on \( L_0, C_0 \) and \( \rho_0 \)), and we have

\[
r_{0,N} = \frac{C_6}{N^{1/5}}, \quad \text{with } C_6 = 2 \cdot (40 C_0 \rho_0^2)^{1/5}.
\]
This implies that
\[
\ell' \geq \frac{\delta}{N} \left( C_3 \frac{N^3 \sqrt{N^{1/5}}}{q^4} \right)^{5/4} \frac{1}{C_6} = \frac{C_5}{q^\delta}, \quad \text{with } C_5 = (C_3/C_6)^{5/4}.
\]
Since \( C_5 \) depends only on \( V \), this proves that \( B_q(\ell, \ell') \) is a suitable \( q \)-adapted box.

- Now we assume that \( q-1 \geq q_0N \) and we prove the estimates on \( r_{q-1,N} \) and \( r_{q+1,N} \). Notice that the assumption \( q \geq q_0N + (q_0' - q_0)N_0 \) implies that \( q-1 \geq q_0N \), hence \( \rho_N < r_{q+1} < r_{q-1,N} \leq 2\rho_N \).

Furthermore, since there exist \((q-1)\)-adapted and \((q+1)\)-adapted boxes respectively centered at \( a_{q-1,N} \) and \( a_{q+1,N} \), the image points \( \varphi^q(a_{q+1,N}) \) and \( \varphi^q(a_{q-1,N}) = \varphi(a_{q-1,N}) \) do not belong to \( \mathcal{B}_{\delta/N} \), so \( a_{q-1,N} \) and \( a_{q+1,N} \) are not in \( B_q(\ell, \ell') \). This implies that

\[
B_q(\ell, \ell') \subset \{(\theta, r) \mid r_{q+1} < r < r_{q-1,N}\}
\]
and the proof of the proposition is complete. \( \square \)

**Lemma 3.4.** Assume \( q \geq 2 \) and \( 0 < \delta < \min(\rho_0, L_0) \). We set \( m_q = a_{q,N} - (\ell, \ell') \) and \( M_q = a_{q,N} + (\ell, \ell') \). Then there exists \( \ell > 0 \) and \( \ell' > 0 \) satisfying the following properties.

(a) \( \ell < \delta/(2N) \) and \( \rho_N + \ell' < r_{q,N} \);

(b) \( \delta/N < \theta(\varphi(m_q)) \);

(c) \( \theta(\varphi^{q-1}(M_q)) < 1 - \delta/N \);

(d) \( |\theta(\varphi^q(m_q)) - 1| < \delta/(2N) \) and \( |\theta(\varphi^q(M_q)) - 1| < \delta/(2N) \).

Furthermore, if these conditions are fulfilled then \( B_q = B_q(\ell, \ell') \) is a \( q \)-adapted box.

**Proof.** We abbreviate \( a_q := a_{q,N} \) and \( r_q := r_{q,N} \). Note that \( r_q > \rho_N > \delta/N \) since we have chosen \( 0 < \delta < \rho_0/2 \). Assume first that \( \ell = \ell' = 0 \), so \( m_q = M_q = a_q \). Condition 2 in Lemma 3.3 implies that either \( r_q \leq L_0 \) and \( \theta(\varphi(a_q)) = r_q \), or \( r_q > L_0 \) and \( \theta(\varphi(a_q)) > L_0 \). In both cases, we have \( \theta(\varphi(a_q)) > \delta/N \). In the same way, either \( r_q \leq L_0 \) and \( \theta(\varphi^{q-1}(a_q)) = 1 - r_q \), or \( r_q > L_0 \) and \( \theta(\varphi^{q-1}(a_q)) < 1 - L_0 \). In both cases, we have \( \theta(\varphi^q(a_q)) < 1 - \delta/N \). At last, we have \( \theta(\varphi^q(a_q)) = 1 \). Thus Conditions (a), (b), (c) and (d) are fulfilled for \( \ell = \ell' = 0 \). Since these conditions are open, this implies that this already holds for \((\ell, \ell')\) close enough to zero.

Now we check that these conditions imply that \( B_q \) is \( q \)-adapted. Let assume that \((\ell, \ell')\) satisfies (a), (b), (c) and (d).

- We first observe that we have \( B_q \subset \mathcal{B}_{\delta/N} \) since \( \ell < \delta/(2N) \). Furthermore, we may observe that the condition \( \rho_0/N + \ell < r_{q,N} \) implies that \( B_q \) is completely above \( \mathcal{J}_0 \).

- We now prove (c) in Definition 3.1. Notice that \( \theta(\varphi^t(m)) \) is increasing with \( t > 0 \) if \( m \in B_q(\ell, \ell') \) since \( m \) is above the separatrix \( \mathcal{J}_0 \). Therefore (b) shows that \( \theta(\varphi^t(m_q)) > \delta/N \) for \( t \geq 1 \). From this, Lemma 3.3 [3] implies that \( \theta(\varphi^t(m)) > \delta/N \) for all \( m \in B_q(\ell, \ell') \) and \( t \geq 1 \). In the same way, (c) shows that \( \theta(\varphi^t(M_q)) < 1 - \delta/N \) for \( t \leq q - 1 \), so Lemma 3.3 [4] implies that \( \theta(\varphi^t(m)) < 1 - \delta/N \) for all \( m \in B_q(\ell, \ell') \) and \( t \leq q - 1 \). This proves that
\[ \delta/N < \theta(\varphi^t(m)) < 1 - \delta/N \], for \( 1 \leq t \leq q - 1 \) and \( m \in B_q(\ell, \ell') \), which implies (i).

At last, we prove (ii) in Definition 3.1. Notice that (d) shows that \( \varphi^t(m_q) \) and \( \varphi^t(M_q) \) belong to the band \((1, 0) + B_{5/(2N)}\). Therefore Condition (i) in Lemma 3.5 implies that
\[ 1 - \delta/(2N) < \theta(\varphi^t(m)) < 1 + \delta/(2N) \]
for all \( m \in B_q(\ell, \ell') \), which implies (i).

Thus Conditions (a), (b), (c) and (d) imply the required properties for \( B_q \) and the proof of the proposition is complete. \( \square \)

Lemma 3.5. The system defined by \( P_V \) satisfies the following properties.

i) Assume \( t > 0 \), \( \theta_0 \in \mathbb{R} \), \( r_0 \geq 0 \) and \( P_V(\theta_0, r_0) = 0 \). Then \( \theta \circ \varphi^{tP_V}(\theta_0, r) \) is an increasing function of \( r \in [r_0, +\infty) \).

ii) \( T_V \) is an increasing bijection from \((0, +\infty)\) onto itself;

iii) For each integer \( k \geq 0 \), we have
\[
T_V^{(k)}(e) \sim \frac{1}{e^{k + \frac{1}{2}}} \left( -\frac{1}{k} \right) \int_0^{+\infty} \frac{dx}{(1 + x^4)(k + 1/2)} \text{ as } e > 0 \text{ tends to zero.}
\]

iv) If we set \( \Delta T(r_2, r_1) = T_V(P_V(0, r_1)) - T_V(P_V(0, r_2)) \) then we have
\[
0 < \Delta T(r_2, r_1) \leq C_0 \frac{r_1(r_2 - r_1)}{(r_1 - \rho_0)^{5/4}}, \text{ provided } \rho_0 < r_1 < r_2 \leq 3\rho_0,
\]
where \( C_0 \) is a positive constant (which depends only on \( V \)).

Proof. (i). For every \( t > 0 \), we have
\[
t = \int_{\theta_0}^{\theta \circ \varphi^{tP_V}(\theta_0, r)} \frac{du}{\sqrt{2(e(r) - V(u))}}.
\]
Since the energy \( e(r) = P_V(\theta_0, r) = \frac{1}{2}r^2 + V(\theta_0) \) is increasing with \( r \geq 0 \), it follows that \( \theta \circ \varphi^{t}(\theta_0, r) \) is increasing with \( r \in [r_0, +\infty) \).

(ii). It follows from (3.9) that \( T_V \) is a continuous and decreasing function and converges to zero at infinity. We have \( T_V(0) = +\infty \) because \( V(\theta) = -(\theta - 1/2)^4 \) for \( |\theta - 1| \leq \theta^* \).

(iii). We compute
\[
T_V^{(k)}(e) = \left( -\frac{1}{k} \right) \int \frac{d\theta}{\sqrt{2\left(e - V(\theta)\right)^{1/2+k}}} = \frac{1}{\sqrt{2}} \left( -\frac{1}{k} \right) \int_{\frac{1}{2}+\theta^*}^{1/2} \frac{d\theta}{\left(e - V(\theta)^{1/2+k}\right)} + \left( -\frac{1}{k} \right) \sqrt{2} \int_0^{\theta^*} \frac{d\theta}{(e + \theta^4)^{k+1/2}}.
\]
Since
\[
\int_{\frac{1}{2}+\theta^*}^{1/2} \frac{d\theta}{(e - V(\theta))^{k+1/2}} \leq \int_{\frac{1}{2}+\theta^*}^{1/2} \frac{d\theta}{(e - V(\theta))^{k+1/2}} \text{ is bounded independently of } e, \text{ it follows that}
\]
\[
T_V^{(k)}(e) \sim \sqrt{2} \left( -\frac{1}{k} \right) \frac{d\theta}{(e + \theta^4)^{k+1/2}} = \sqrt{2} \left( -\frac{1}{k} \right) \frac{1}{e^{k+1/2}} \int_0^{\theta^*/e^{1/4}} \frac{d\theta}{(1 + \theta^4)^{k+1/2}}
\]
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as $e$ tends to zero, and this completes the proof of (ii).

Notice that $P_V(0, r) = (r^2 - \rho_0^2)/2 < 4\rho_0^2$ if $\rho_0 < r < 3\rho_0$. Furthermore, Condition (iii) shows that there exists a constant $B$ such that $0 < -T'_V(e) \leq Be^{-5/4}$ if $0 < e \leq 4\rho_0^2$; we set $\tau(r) = \Delta T(r, r_1)$, for $r_1 \leq r \leq r_2$, so we have:

$$0 \leq \tau'(r) = -rT'_V(H(0, r)) \leq Br((r^2 - \rho_0^2)/2)^{-5/4} \leq Br_1((r_1^2 - \rho_0^2)/2)^{-5/4},$$

hence the estimate as claimed, with $C_0 = B\rho_0^{-5/4}$ since $(r_1^2 - \rho_0^2)/2 \geq \rho_0(r_1 - \rho_0)$.

\[ \square \]

**Proposition 3.6.** There exist positive constants $\delta_0$ and $r_0$ (which depends only on $V$) such that for each integer $q \geq 2$ satisfying $\rho_0 < r_{q,1} \leq 2\rho_0$ and for $0 < \delta < \delta_0$, if we set

$$\ell = \delta/4 \quad \text{and} \quad \ell' = \delta \cdot \left(\frac{r_{q,1} - \rho_0}{r_0}\right)^{5/4}$$

then $r_{q,1} - \ell' > \rho_0$ and $B_q(\ell, \ell')$ is $q$-adapted with respect to $B_\delta$ and the system $P_V$.

\[ \text{Proof.} \]

Here we abbreviate $r_q := r_{q,1}$ and $a_q := a_{q,1}$, and we prove that $\ell, \ell'$, $m_q = a_q - (\ell, \ell')$ and $M_q = a_q + (\ell, \ell')$ satisfy Conditions (a),(b),(c) and (d) in Lemma 3.4 with $N = 1$, provided that $r_0$ and $\delta_0$ satisfy suitable conditions. We recall that we assume that $\delta < \rho_0/2$, so we may already set the constraint

$$\delta_0 \leq \rho_0/2. \tag{3.11}$$

We decompose the proof of the lemma in five steps: first we prove that $B_q(\ell, \ell')$ is above the separatrix $r = \rho_0$. Then we estimate the variation of the period inside $B_q(\ell, \ell')$ and, at last, we check (b), (c) and (d).

**Step 1:** we prove that $\ell' < (r_q - \rho_0)/2$. Notice that $0 < r_q - \rho_0 \leq \rho_0$, so we have

$$\frac{\ell'}{r_q - \rho_0} = \delta \frac{1}{\rho_0^{5/4}(r_q - \rho_0)^{1/4}} \leq \frac{1}{\rho_0^{5/4} \delta} < \frac{1}{2}, \quad \text{provided that} \quad \delta_0 \leq \frac{\rho_0^{5/4}}{2\rho_0^{1/4}}. \tag{3.12}$$

This implies that $r_q - \ell' > \frac{\ell}{2}(r_q + \rho_0) > \rho_0$. Since we have $\ell = \delta/4 < \delta/2$, we obtain Condition (a) of Lemma 3.4.

**Step 2:** we prove that $\ell + (r_q + \ell')\tau < \delta/2$, where $\ell = \delta/4$ and $\tau = \Delta T(r_q + \ell', r_q - \ell')$. Step 1 shows that $\ell' < \frac{1}{2}(r_q - \rho_0) \leq \frac{1}{2}\rho_0$. Set $r_q' = r_q - \ell'$; since $\rho_0 < r_q - \ell' < r_q + \ell' \leq 3\rho_0$, Lemma 3.5 shows that $\tau \leq 2C_0\ell' r_q'/(r_q' - \rho_0)^{-5/4}$. Since $r_q' - \rho_0 = r_q - \rho_0 - \ell' > \frac{r_q - \rho_0}{2}$ and $r_q + \ell' < \frac{3}{2}\rho_0$, since $r_q' \leq r_q \leq 2\rho_0$, it follows that

$$\tau(r_q + \ell') < 5\tau \rho_0/2 \leq \frac{10 C_0\ell' \rho_0^2}{((r_q - \rho_0)/2)^{5/4}} \leq \frac{10 C_0 \rho_0^2}{(r_0/2)^{5/4}} \delta \leq \frac{\delta}{4},$$

which proves the claimed estimate, provided

$$\frac{10C_0 \rho_0^2}{(r_0/2)^{5/4}} \leq \frac{1}{4}. \tag{3.13}$$
Step 3: we prove (d). Set $a_q^+ = (\ell, r_q)$, $a_q^- = (-\ell, r_q)$ and notice that $\theta(\varphi^q(a_q^+)) = 1 + \ell$, $\theta(\varphi^q(a_q^-)) = 1 - \ell$, provided that $\ell \leq L_0$; we assume that

$$\delta_0 \leq L_0.$$  \hfill (3.14)

Since $\theta(M_q) = \theta(a_q^+)$, $\theta(m_q) = \theta(a_q^-)$ and since Step 2 implies that $\ell + \tau(r_q + \ell') \leq \delta/2 \leq L_0$, Lemma 3.5 implies that

$$\begin{cases} 1 + \ell \leq \theta(\varphi^q(M_q)) \leq 1 + \ell + \tau(r_q + \ell'); \\ 1 - \ell - \tau(r_q - \ell') \leq \theta(\varphi^q(m_q)) \leq 1 - \ell. \end{cases}$$

This implies that $1 + \delta/4 \leq \theta(\varphi^q(M_q)) \leq 1 + \delta/2$ and $1 - \delta/2 \leq \theta(\varphi^q(m_q)) \leq 1 - \delta/4$, which proves (d).

Step 4: we prove (c). Recall that Step 3 implies that $1 \leq \theta(\varphi^q(M_q)) \leq 1 + \delta/2 \leq 1 + L_0$, so either $\delta - (r_q + \ell') < -L_0$ and $\theta(\varphi^{q-1}(M_q)) < 1 - L_0$, or $\theta(\varphi^{q-1}(M_q)) < 1 + \delta - (r_q + \ell')$. Since $L_0 > \delta$ and $r_q + \ell' > \rho_0 \geq 2\delta$, we obtain in both cases that $\theta(\varphi^{q-1}(M_q)) < 1 - \delta$, which completes the proof of (c).

Step 5: we prove (b). Either $-\ell + (r_q - \ell') > L_0$, so $\theta(\varphi(m_q)) > L_0 \geq \delta$, or $\theta(\varphi(m_q)) > -\ell + (r_q - \ell') > -\ell + \rho_0 > 2\delta - \ell = 7\delta/4 > \delta$. In both cases, this proves (b).

Thus we obtain Conditions (a), (b), (c) and (d) in Lemma 3.3 provided Conditions (3.11), (3.12), (3.13) and (3.14) are satisfied, hence the proposition holds true if we set

$$r_0 := 2 \cdot (40C_0\rho_0^2)^{4/5}, \quad \delta_0 = \min \left( L_0; \frac{\rho_0}{2}; \frac{r_0^{5/4}}{2\rho_0^{1/4}} \right) = \min \left( L_0; \frac{\rho_0}{2}; 20 \cdot 2^{5/4} \rho_0^{7/4} C_0 \right),$$

where $C_0$ is the constant in Lemma 3.5. Thus the constants $r_0$ and $\delta_0$ depend only on $V$ and the proof of the proposition is complete. \hfill \blacksquare

**Lemma 3.7.** Assume $N$ and $q$ are positive real numbers. Then the following holds true.

i) For all $e > 0$, we have $T_{V/N^2}(e) = NT_V(N^2e)$;

ii) for all $r \in \mathbb{R}$, we have $N^2P_{V/N^2}(0, r) = P_V(0, Nr)$;

iii) $e_{q,N} = \frac{1}{N^2} e_{q/N,1}$ and $r_{q,N} = \frac{1}{N} r_{q/N,1}$;

iv) for each integer $k \geq 0$, we have $T_{V/N^2}^{(k)}(e_{q,N}) = N^{2k+1} T_V^{(k)}(e_{q/N,1}) \sim (-1)^k \beta_k \frac{q^{2k+1}}{N^{2k}}$ uniformly as $q/N$ tends to infinity, where $\beta_k$ is a positive constant (which depends only on $k$);

v) with $\Delta_N T(r_2, r_1) = T_{V/N^2}(P_{V/N^2}(0, r_1)) - T_{V/N^2}(P_{V/N^2}(0, r_2))$, we have

$$0 < \Delta_N T(r_2, r_1) \leq C_0 \frac{N^{7/4} (r_2 - r_1)}{(r_1 - \rho_N)^{1/4}},$$

provided $\rho_N < r_1 < r_2 \leq 3\rho_N$,

where $C_0$ is a positive constant (which depends only on $V$).
Proposition 3.3, assuming $q$ (see Section 3.1) and $B$.

Here and in the following unless mentioned otherwise, the numbers $\ell, \ell', N_0$ and $q_1$ are as in Proposition 3.3, assuming $q \geq q_0'N$ and $N \geq N_0$. We abbreviate $G = G_{N,\mu} = \Phi_{\mu W_N} \circ \Phi_{P_V/N^2}$ (see Section 3.1) and $B_q = B_q(\ell, \ell')$.

Proof. The first two conditions follow directly from the formulas (3.1) and (3.9).

We use (i) to compute

$$T_V(e_{q/N,1}) = q/N = T_{V/N^2}(e_{q,N})/N = T_V(N^2e_{q,N}),$$

so $e_{q/N,1} = N^2e_{q,N}$ for $T_V$ is one to one. In a similar way, we use (ii) to compute

$$P_V(0, Nr_{q,N}) = N^2P_{V/N^2}(0, r_{q,N}) = N^2e_{q,N}e_{q/N,1} = P_V(0, r_{q/N,1}),$$

so $Nr_{q,N} = r_{q/N,1}$ since the function $P_V(0, \cdot)$ is one to one, and this proves (ii).

It follows from (i) that

$$T_{V/N^2}^{(k)}(e_{q,N}) = N \frac{d^k}{de^k} (T_V(N^2e)) = N^{2k+1}T_{V}^{(k)}(N^2e_{q,N}) = N^{2k+1}T_{V}^{(k)}(e_{q/N,1})$$

and this completes the proof of the equality in (ii). Furthermore, since $e_{q/N,1}$ tends to zero as $q/N$ tends to infinity, Statement (iii) implies that

$$N^{2k+1}T_{V}^{(k)}(e_{q/N,1}) \sim \frac{N^{2k+1}}{(e_{q/N,1})^{k+1/4}} \left( \int_0^{+\infty} \frac{dx}{(1 + x^4)^{k+1/2}} \right)^{4k+1}$$

$$\text{and } q/N = T_V(e_{q/N,1}) \sim \frac{1}{e_{q/N,1}^{1/4}} \int_0^{+\infty} \frac{dx}{(1 + x^4)^{1/2}}.$$

This implies (iii) with

$$\beta_k = \left( k - \frac{1}{2} \right) \int_0^{+\infty} \frac{dx}{(1 + x^4)^{k+1/2}} / \left( \int_0^{+\infty} \frac{dx}{(1 + x^4)^{1/2}} \right)^{4k+1}.$$

(IV) follows from Lemma 3.3 and (ii). This holds true because this assumptions on $r_1, r_2$ and (ii) imply that $\rho_0 < N \rho < 3 \rho_0, k = 1, 2,$ and

$$T_{V/N^2}(P_{V/N^2}(0, r_k)) = NT_V\left( N^2P_{V/N^2}(0, r_k) \right) = NT_V(P_V(0, N r_k)),$$

so $0 < \Delta_N(r_1, r_2) = N \Delta(Nr_1, Nr_2) \leq C_0 \frac{r_1(r_2 - r_1)N^{3-5/4}}{(r_1 - \rho_N)^{5/4}},$

which proves (iv) and completes the proof of the lemma.

3.4.2 Local form of $G^q$

Here and in the following unless mentioned otherwise, the numbers $\ell, \ell', N_0$ and $q_1$ are as in Proposition 3.3, assuming $q \geq q_0'N$ and $N \geq N_0$. We abbreviate $G = G_{N,\mu} = \Phi_{\mu W_N} \circ \Phi_{P_V/N^2}$ (see Section 3.1) and $B_q = B_q(\ell, \ell')$. 

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Proposition 3.8. In $B_q \subset \mathbb{R}$, the iterated map $G^3$ coincides with
$$F_{q,N,\mu} := \Phi^{\mu W_N} \circ \Phi^A_{q,N},$$
where $A_{q,N}(r) := \int_{eq,N}^{P_{V/N^2}(0,r)} (q - T_{V/N^2}(h)) \, dh$ on $B_q$.

Proof. We abbreviate $\varphi = \Phi^{P_{V/N^2}}$ and $V_N = V/N^2$. For $(\theta, r) \in \mathbb{R}^2$ satisfying $P_{V_N}(\theta, r) > 0$, let $\Psi(\theta, r) = (\tau, h)$ denote the time-energy coordinates with
$$\tau = \int_0^\theta \frac{du}{\sqrt{2(P_{V_N}(\theta, r) - V_N(u))}}; \quad h = P_{V_N}(\theta, r).$$

Note that $\Psi \circ \varphi \circ \Psi^{-1}(\tau, h) = (\tau + 1, h)$ and $\Psi(m + \theta, r) = \Psi(\theta, r) + (m T_{V_N}(h), 0)$, for all $m \in \mathbb{Z}$ and $h = P_{V_N}(\theta, r)$. Since $V_N$ is constant on $B_{L_0}$, we have
$$\tau = \int_0^\theta \frac{du}{\sqrt{2(P_{V_N}(\theta, r) - V_N(\theta))}} = \frac{\theta}{r}, \quad \text{for } |\theta| \leq L_0.$$ Since we have $\varphi^q(B_q) \subset B_{\delta/2} \subset B_{L_0}$ and $\varphi^q$ preserves $P_{V_N}$, which is an increasing function depending only on $r$ in $B_{L_0}$, there exist a continuous function $\theta_q : B_q \to (-L_0, L_0)$ and a constant integer $m_0 \in \mathbb{Z}$ such that $\varphi^q(\theta, r) = (m_0 + \theta_q(\theta, r), r)$ on $B_q \subset \mathbb{R}^2$. Furthermore, we have $m_0 = 1$ as it may be checked at $a_{q,N}$, since we have $\varphi^q(a_{q,N}) = (1, r_{q,N})$ and $\theta_q(a_{q,N}) = 0$. Therefore, we have proved that $\varphi^q$ coincides on $B_q$ with
$$\varphi^q(\theta, r) = (1 + \theta_q(\theta, r), r), \quad \text{so } \Psi \circ \varphi^q(\theta, r) = (\theta_q(\theta, r)/r + T_{V_N}(h), h), \quad \text{with } h = P_{V_N}(\theta, r).$$ Since we also have $\Psi \circ \varphi^q(\theta, r) = \Psi(\theta, r) + (q, 0) = (\theta/r + q, h)$, this implies that
$$\theta_q(\theta, r) = \theta + r(q - T_{V_N}(h)) = \theta - \partial_r A_{q,N}(r).$$ Thus, on the one hand, $\varphi^q$ coincides on $B_q$ with the time-1 flow $\Phi^A_q$ of the system
$$\begin{cases}
\theta' = -\partial_r A_{q,N}(r); \\
r' = 0 = \partial_\theta A_{q,N}(r).
\end{cases}$$

On the other hand, we have $\varphi^q(B_q) \cap B_{\delta/N} = \emptyset$, so $G^q = \varphi^k$ for $1 \leq k \leq q - 1$, and $G^q = \Phi^{\mu W_N} \circ \varphi^q$ on $B_q$.

From these two conclusions, the proposition follows. \hfill $\square$

3.4.3 The Taylor expansion of $G^q$ at $a_q$

This section carries out the first step in the proof of the existence of invariant curves in $B_q$ for the map $G^q = F_{q,N,\mu}$. The goal is to prove that we can find complex coordinates in which $a_{q,N}$ is mapped to zero and $F_{q,N,\mu}$ takes the form
$$F_{q,N,\mu}(z) = \lambda \left( z + \sum_{k=2}^n P_k(z) + \varepsilon(z) \right).$$
This is achieved in Proposition 3.13 and Corollary 3.14 below. Here \( n \) is an arbitrary large integer (but not depending on \( q, N \) and \( \mu \)), \( P_k \) is a homogeneous polynomial of degree \( k \) for \( 2 \leq k \leq n \) and the error term \( \varepsilon \) is small enough up to \( n \) derivatives. Note that the change of coordinates need not be symplectic in our setting.

**Asymptotic behaviours of \( A_{q,N} \) on \( B_q \)**— To achieve (3.16) in a quantitative way, we must control the derivatives of the map \( A_{q,N} \) near \( a_{q,N} \). For that purpose, it is convenient to introduce the following notation.

**Notation.** Here \( E \) denotes any set of parameters; for \( f_1 : E \to \mathbb{R}_+ \) and \( f_2 : E \to \mathbb{R}_+ \), we write \( f_1 = \Theta_E(f_2) \), or \( f_1(p) = \Theta_E(f_2(p)) \), or \( f_1(p) = \Theta(f_2(p)) \) uniformly for \( p \in E \), if there exists a constant \( C > 0 \) (which does not depend on \( p \)) such that

\[
\forall p \in E, \quad f_1(p) \leq C f_2(p).
\]

We write \( f_1 \asymp f_2 \), or \( f_1(p) \asymp f_2(p) \), or \( f_1(p) \asymp f_2(p) \) uniformly for \( p \in E \), if we have \( f_1 = \Theta_E(f_2) \) and \( f_2 = \Theta_E(f_1) \).

We recall that for \( q \geq q_0 N \), \( B_q \) is contained in the annulus \( \{ r_{q+1,N} \leq r \leq r_{q-1,N} \} \subset A \).

**Proposition 3.9.** We have

\[
\forall n \geq 0, \quad (-1)^n T^{(n)}_{V/N^2} \left( P_{V/N^2}(0,r) \right) \asymp q^{4n+1}/N^{2n}; \quad (3.17)
\]

\[
\forall n \geq 1, \quad \left| A^{(n)}_{q,N}(r) \right| = \Theta_E \left( q^{4n-3}/N^{3n-2} \right); \quad (3.18)
\]

\[
\forall n \geq 2, \quad (-1)^n A^{(n)}_{q,N}(r) \asymp q^{4n-3}/N^{3n-2}; \quad (3.19)
\]

where \( E_n = \{ (q,N,r) \mid q \geq q_n N, \ N \geq N_0, \ r_{q+1,N} \leq r \leq r_{q-1,N} \} \) and \( q_n \geq q_0 \), for \( n \geq 1 \), is a positive constant which depends only on \( V \). Furthermore, for \( n \geq 2 \), we have

\[
A^{(n)}_{q,N}(r_{q,N}) \sim (-1)^n \beta_{n-1} q^{4n-3}/N^{3n-2}, \quad \text{as } q/N \text{ tends to infinity},
\]

where \( \beta_{n-1} \) is a positive constant as defined in Lemma 3.7 (3).

**Proof.** We set \( q_1 \geq \max(2;q_0) \) and we prove (3.17).

First we assume that \( N = 1 \). A direct computation shows that \((-1)^n T^{(n)}_{V}(h) > 0 \) for \( h > 0 \). Furthermore, Lemma 3.7 (3) implies that

\[
T^{(n)}_{V}(e) \sim (-1)^n \beta_n T_{V}(e)^{4n+1}, \quad \text{as } e \text{ tends to zero}.
\]

This shows that there exist two positive constants \( c_n \) and \( d_n \), for each \( n \geq 1 \), such that

\[
\forall e \in (0;e_0), \quad c_n T_{V}(e)^{4n+1} \leq (-1)^n T^{(n)}_{V}(e) \leq d_n T_{V}(e)^{4n+1}, \quad \text{with } e_0 = P_{V}(a_{q_1-1,1}).
\]

Moreover, if \( r_{q-1,1} \leq r \leq r_{q+1,1} \) then we have \( e_{q+1,1} \leq P_{V}(0,r) \leq e_{q-1,1} \leq e_0 \). Since we have \( T_{V}(e_{q-1,1}) = q-1 \asymp q \asymp q+1 = T_{V}(e_{q+1,1}) \) uniformly for \( (q,1,r) \in E_1 \), it follows that

\[
\forall n \geq 0, \quad T^{(n)}_{V}(P_{V}(0,r)) \asymp q^{4n+1} \quad \text{uniformly for } (q,1,r) \in E_1.
\]
For \( N \geq N_0 \) and \( q \geq q_1 N \), we use Lemma 3.7, we observe that if we assume that
\[ r_{q-1,N} \leq r \leq r_{q+1,N} \]
then we have
\[ r_{q/N+1,1} \leq r_{(q+1)/N} = Nr_{q+1,N} \leq Nr \leq r_{q-1,N} = r_{(q-1)/N,1} \leq r_{q/N-1,1}. \]

Therefore, if \( e = P_V(0, r) = P_V(0, Nr)/N^2 \) then we have \( e_{q/N+1,1} \leq N^2e \leq e_{q/N-1,1} \), hence Lemma 3.7(3) and the discussion above when \( N = 1 \) show that
\[ T^{(n)}_{V/N^2}(e) = N^{2n+1}T^{(n)}_V(N^2e) \approx N^{2n+1}T^{(n)}_V(N^2e_{q/N,1}) \approx q^{n+1}/N^{2n} \]
uniformly for \( (q, N, r) \in E_1 \). This proves (3.17).

- We prove (3.18). First we assume that \( N = 1 \).
We set \( T_0(r) = q - T_V(P_V(0, r)) \) and \( T_k(t) = -T_V(P_V(0, r)) \), for \( k \geq 1 \); we observe that \( |T_0| \leq 1 \) and the point above shows that \( (-1)^{k+1}T_k(r) \approx q^{kn+1} \) uniformly for \( (q, 1, r) \in E_1 \).

An immediate induction over \( p \geq 1 \) shows that
\[
A_{q,1}^{(2p-1)}(r) = \sum_{k=0}^{p-1} C_{k,2p-1} r^{2k+1} T_{p+k-1}(r) \quad ; \quad A_{q,1}^{(2p)}(r) = \sum_{k=0}^{p} C_{k,2p} r^{2k} T_{p+k-1}(r),
\]
with
\[
C_{k,2p+1} = C_{k,2p} + (2k+2) C_{k+1,2p} \quad \text{if } p \geq 1 \text{ and } 0 \leq k \leq p-1, \\
C_{k,2p+2} = C_{k,2p+1} + (2k+1) C_{k,2p+1} \quad \text{if } p \geq 1 \text{ and } 1 \leq k \leq p, \\
C_{p-1,2p-1} = C_{p,2p} = 1 \quad \text{if } p \geq 1.
\]

Furthermore, we have
\[
r^{2k}(-1)^{p+k} T_{p+k-1}(r) \approx r^{2k+1}(-1)^{p+k} T_{p+k-1}(r) \approx q^{4p+4k-3}
\]
uniformly for \( (q, 1, r) \in E_1 \), since we have \( \rho_0 \leq r \leq 2\rho_0 \). Since we have \( q \geq q_1 \geq 2 \) on \( E_1 \), it follows that \( q^{4p+4k-3} = \mathcal{O}_E_1(q^{8p-3}) \) for \( 0 \leq k \leq k_0 \) and \( p \geq 1 \), hence
\[
\left| A_{q,1}^{(2p)}(r) \right| = \mathcal{O}_E_1(q^{8p-3}) \text{ and } \left| A_{q,1}^{(2p-1)}(r) \right| = \mathcal{O}_E_1(q^{8p-7}) \text{ for } p \geq 1,
\]
which proves (3.18) on \( E_1 \cap \{ N = 1 \} \). Since we have \( (q, N, r) \in E_1 \) if and only if \( (q/N, 1, r/N) \in E_1 \), this extends immediately to (3.18) for any \( N \geq N_0 \) according to Lemma 3.10 below.

- We prove (3.19). First we assume that \( N = 1 \).
It follows from (3.21) and (3.20) above that there exist positive constants \( c_{k,\ell} \) and \( d_{k,\ell} \) (depending only on \( V \)) such that
\[
\sum_{k=0}^{p-1} C_{k,2p-1} (-1)^{p+k} q^{4(p+k)-3} \leq A_{q,1}^{(2p-1)}(r) \leq \sum_{k=0}^{p-1} d_{k,2p-1} (-1)^{p+k} q^{4(p+k)-3}, \quad (3.22a)
\]
\[
\sum_{k=0}^{p} C_{k,2p} (-1)^{p+k} q^{4(p+k)-3} \leq A_{q,1}^{(2p)}(r) \leq \sum_{k=0}^{p} d_{k,2p} (-1)^{p+k} q^{4(p+k)-3}. \quad (3.22b)
\]
where $q_{2p} \geq q_{2p-1} \geq q_1$ are large enough (depending only on $V$). Therefore we have proved that $-A_{q,1}^{(2p-1)}(r) \approx q^{8p-7}$ and $A_{q,1}^{(2p)}(r) \approx q^{8p-3}$ uniformly for $(q,1,r)$ in $E_{2p-1}$ or $E_{2p}$ respectively, which is (3.19) on $\{ N = 1 \}$. This extends immediately to (3.18) for any $N \geq N_0$ according to Lemma 3.10 below.

• Since we have $C_{p-1,2p-1} = C_{p,2p} = 1$ in (3.21), we obtain with (3.20) that $A_{q,1}^{(2p)}(r_{q,1}) \sim r_{q,1}^{2p}T_{2p-1}(r_{q,1})$ and $A_{q,1}^{(2p-1)}(r_{q,1}) \sim r_{q,1}^{2p-1}T_{2p-2}(r_{q,1})$ as $q$ tends to infinity. But Lemma 3.7 shows that

$$T_{n-1}(r_{q,1}) = -T^{(n-1)}(P_V(0,r_{q,1})) \sim (-1)^n \beta_{n-1} q^{4n-3}.$$ 

Since Proposition 3.3 shows that $r_{q,1} \sim \rho_0$ as $q$ tends to infinity, we obtain that $A_{q,1}^{(n)}(r_{q,1}) \sim (-1)^n \rho_0^n \beta_{n-1} q^{4n-3}$ when $N = 1$. The announced equivalent for general $N \geq 1$ follows using Lemma 3.10 below and this completes the proof of the proposition.

**Lemma 3.10.** We have $A_{q,N}(r) = \frac{1}{N} A_{q/N,1}(Nr)$.

**Proof.** Using Lemma 3.7 we compute

$$A_{q,N}(r) = \int_{e_{q,N}}^{P_{V/N^2}(0,r)} (q - T_{V/N^2}(h)) \; dh = \int_{e_{q,N}}^{P_{V/N^2}(0,r)} (q - NT_V(N^2 h)) \; dh$$

$$= \frac{1}{N} \int_{N^2 e_{q,N}}^{N^2 P_{V/N^2}(0,r)} \left( \frac{q}{N} - T_V(h) \right) \; dh = \frac{1}{N} \int_{e_{q/N,1}}^{P_{V/(0,Nr)}} \left( \frac{q}{N} - T_V(h) \right) \; dh$$

$$= \frac{1}{N} A_{q/N,1}(Nr).$$

This proves the formula of the lemma. 

**Lineart part of $F_{q,N,\mu}$—** We recall that $B_q$ denotes a $q$-adapted box with respect to $P_{V/N^2}$ and $B_{q/N}$, as it appears in Proposition 3.3

**Proposition 3.11.** Set $\sigma_{q,N}(\theta_R) = (\theta,r_{q,N} + R)$. There exist a constant $\alpha_{q,N}$ and a function $S_{q,N}(R)$ satisfying for all $\mu > 0$ and $(\theta,R) \in \sigma_{q,N}^{-1}(B_q)$

$$\left| \sigma_{q,N}^{-1} \circ F_{q,\mu,N} \circ \sigma_{q,N}(\theta_R) \right| = \left( \frac{1}{-\mu} \right) \frac{\alpha_{q,N}}{1 - \mu \alpha_{q,N}} \left( \theta_R \right) + S_{q,N}(R) \left( -\frac{1}{\mu} \right),$$

and the following estimates hold true.

$$\alpha_{q,N} \approx \frac{q^5}{E_2 \; N^4}; \quad |S_{q,N}(R)| \approx \frac{R^2 \; q^2}{E_0 \; N^4}; \quad |S'_{q,N}(R)| \approx \frac{R^q \; q^4}{E_1 \; N^7}; \quad |R| \approx g_{E_1}(q^3 / N^5);$$

$$\forall n \geq 2, \quad ( -1 )^{n+1} S^{(n)}_{q,N}(R) \approx \frac{q^{4n+1} / N^{3n+1}}{E_n}. $$

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with \( E_n = \{(q,N,R) \mid q \geq q_1 N, N \geq N_0, (0,r,q,N + R) \in B_q\} \) and \( q_n \) a positive constants which depend only on \( V \), for \( n \geq 1 \).

Furthermore, we have
\[
\alpha_{q,N} \sim \beta_1 \rho_0^2 \frac{q^5}{N^4}, \quad S^{(2)}_{q,N}(0) \sim -\beta_2 \rho_0^3 \frac{q^6}{N^7} \quad \text{and} \quad S^{(3)}_{q,N}(0) \sim \beta_3 \rho_0^4 \frac{q^{13}}{N^{10}}
\]
as \( q/N \) tends to infinity.

**Proof.** \( \bullet \) Proposition 3.3 shows that \( \Phi^{A,N}(B_q) = \varphi^q(B_q) \subset B_{\delta/(2N)} \), so \( \mu W_N(\theta) = \frac{1}{2} \mu \theta^2 \) on \( \Phi^{A,N}(B_q) \). This implies that \( F_{q,N,\mu} \) coincides on \( B_q \) with
\[
F_{q,N,\mu}(\theta,r) = \left( \theta + A_{q,N}(r), r - \mu(\theta + A_{q,N}(r)) \right).
\]

Setting
\[
\alpha_{q,N} := A''_{q,N}(r_{q,N}) \quad \text{and} \quad S_{q,N}(R) := A'_{q,N}(r_{q,N} + R) - A'_{q,N}(r_{q,N}) - \alpha_{q,N} R, \quad (3.23)
\]
the annonced formula for \( F_{q,N,\mu} \) follows from a direct computation.

\( \bullet \) We prove the estimates of the proposition. Since we have \( \sigma_{q,N}(\theta,R) \in B_q \), it follows that \( r_{q+1,N} \leq r_{q,N} + R \leq r_{q-1,N} \). Now we apply Proposition 3.9

- with \( n = 2 \), Estimate (3.19) shows that \( \alpha_{q,N} = A''_{q,N}(r_{q,N}) = q^5/N^4 \).

- For \( \ell \geq 0 \), we have
\[
S_{q,N}(R) = \frac{R^2}{2} A''_{q,N}(r_{q,N} + \eta_0 R); \quad S'_{q,N}(R) = R A'_{q,N}(r_{q,N} + \eta_1 R);
\]
\[
S^{(\ell)}_{q,N}(R) = A^{(\ell+1)}_{q,N}(r_{q,N} + \eta_\ell R), \quad \text{for} \ \ell \geq 2;
\]
for some \( 0 < \eta_\ell < 1 \) (depending on \( R \)), hence Estimates (3.18) shows that
\[
S_{q,N}(R) = R^2 \frac{q^5}{N^4}; \quad |S'_{q,N}(R)| \asymp |R| \frac{q^9}{N^7}; \quad (-1)^{\ell+1} S^{(\ell)}_{q,N}(R) \asymp \frac{q^{4\ell+1}}{N^{3\ell+1}}, \quad \text{for} \ \ell \geq 2.
\]

Since we have \( |R| \leq \ell' \) and \( \ell' \asymp N^5/q^5 \) according to Proposition 3.3, the proof of the announced estimates is complete.

\( \bullet \) At last, we observe that
\[
\alpha_{q,N} = A''_{q,N}(r_{q,N}), \quad S''_{q,N}(0) = A''_{q,N}(r_{q}) \quad \text{and} \quad S^{(3)}_{q,N}(0) = A^{(4)}_{q,N}(r_{q,N}).
\]

Therefore these quantities as \( q/N \) tends to infinity may be estimated immediately from the last estimate of Proposition 3.9 which completes the proof of the proposition. \( \square \)

**Lemma 3.12.** We have \( \alpha_{q,N} = N \alpha_{q/N,1} \) and \( S_{q,N}(R) = NS_{q/N,1}(NR) \).
Proof. We have \( \alpha_{q,N} = A''_{q,N}(r_{q,N}) \). Therefore Lemma 3.10 and Lemma 3.7 imply that
\[
\alpha_{q,N} = NA''_{q,N}(Nr_{q,N}) = NA''_{q/N,1}(r_{q/N,1}) = N\alpha_{q/N,1}.
\]
In a similar way, Lemma 3.10 implies that
\[
S_{q,N}(R) = A'_q(r_{q,N} + R) - A'_q(r_{q,N}) - \alpha_q R
\]
\[
= NA''_{q,N}(Nr_{q,N} + NR) - NA''_{q/N,1}(Nr_{q,N}) - N\alpha_{q/N,1}R
\]
\[
= NA''_{q,N}(r_{q,N,1} + NR) - NA''_{q/N,1}(r_{q/N,1}) - N\alpha_{q/N,1}R
\]
\[
= NS_{q/N,1}(NR).
\]
This proves the second identity of the lemma and the proof is complete. \( \square \)

Diagonalization of the linear part and Taylor expansion

Notation. For \( n \in \mathbb{N}^* \) and \( \beta > 0 \), we set
\[
E_\beta := \{(q, N, \mu) \mid 0 < \mu \alpha_{q,N} < 1 \text{ and } q > \beta N, \ N \geq N_0\}, \quad (3.24a)
\]
\[
E_{\beta,n} := \{(q, N, \mu) \in E_\beta \mid \mu \alpha_{q,N} < \frac{1}{(n+1)^2}\} \quad (3.24b)
\]
(with the notation of Proposition 3.11 for \( \alpha_{q,N} \)). Unless mentioned otherwise, we shall abreviate \( f_1 \asymp f_2 \) if there exists a positive constant \( \beta \) (not depending on \( q, N, \mu \)) satisfying 
\[
f_1 \asymp E_{\beta,n} f_2.
\]

Notation. Let \( \lambda \in \mathbb{C} \) satisfy the following two conditions
\[
\lambda + \lambda^{-1} = 2 - \mu \alpha_{q,N}, \quad \lambda = \exp(i\gamma_0) \text{ with } -\frac{\pi}{3} < \gamma_0 < 0. \quad (3.25)
\]
so we have \(|\lambda^p - 1| \asymp |\lambda - 1|\) uniformly on \( E_{\beta,n} \) for \( 1 \leq p \leq 2n + 2 \) (see Lemma 3.15 below).

It follows immediately from (3.25) that
\[
1 - \lambda = 2\sin^2(\gamma_0/2) - i\sin \gamma_0 \text{ and } |\lambda - 1|^2 = \mu \alpha_{q,N} = (1 - \cos \gamma_0)^2 + \sin^2 \gamma_0 = 2(1 - \cos \gamma_0),
\]
hence \( \sin^2 \gamma_0 = \mu \alpha_{q,N}(1 - \mu \alpha_{q,N}/4) \) and (3.24a) implies that
\[
\frac{\sqrt{3}}{2} |\lambda - 1| \leq -\sin \gamma_0 \leq |\lambda - 1|, \quad \text{so } -\sin \gamma_0 \asymp |\lambda - 1| = \sqrt{\mu \alpha_{q,N}}.
\]

Notation. For all \( z \in \mathbb{C} \) we set \( \Psi(z) = \sigma_{q,N} \circ \psi(z) \) with \( \sigma_{q,N}(\theta, R) = (\theta, r_{q,N} + R) \) and
\[
\psi(z) = B \left( \frac{z}{\overline{z}} \right) \in \mathbb{R}^2, \quad B = \left( \frac{\alpha}{1} \frac{\alpha}{1} \right) \text{ and } \alpha = \alpha_{q,N}.
\]
Proposition 3.13. Assume $n \geq 2$ and set $\omega = q^{4}/N^{3}$. Then there exists a positive constant $\beta > 0$ such that for each $(q, N, \mu) \in E_{\beta}$ there exist $\rho > 0$ and $\kappa > 0$, $a_{\nu} \in \mathbb{R}$, for $2 \leq \nu \leq n$, a function $g : [-2\rho, 2\rho] \to \mathbb{R}$ satisfying the following properties.

i) $\mathbb{D}(0; \rho) \subset \Psi^{-1}(B_{q})$ and $\rho \approx \frac{|\lambda - 1|}{q \omega}$;

ii) $\Psi^{-1} \circ F_{q,N,\mu} \circ \Psi(z) = \lambda\left(z + i|\lambda - 1|g(z + \bar{z})\right)$ on $\mathbb{D}(0; \rho)$;

iii) $\Psi^{*}(dr \wedge d\theta) = \frac{\kappa}{2\iota} dz \wedge d\bar{z}$ and $\kappa \approx \frac{|\lambda - 1|}{\mu} \approx \omega q/N$;

iv) $g(x) = \sum_{\nu=2}^{n} a_{\nu} x^{\nu} + \varepsilon(x)$, where

\[
\begin{cases}
(1)^{\nu} a_{\nu} \approx \omega^{\nu-1}, & \text{for } 2 \leq \nu \leq n, \\
|\varepsilon^{(k)}(x)| = \mathcal{O}(\omega^{n}|x|^{n-k}) , & \text{for } 0 \leq k \leq n,
\end{cases}
\]

with $E = \{(x, q, N, \mu) \mid |x| \leq 2\rho, (q, N, \mu) \in E_{\beta}\}$ and $E_{\beta}$ as in (3.24a).

Proof. We abbreviate $F = F_{q,N,\mu}$, $S_{q,N} = S$ and $\alpha = \alpha_{q,N}$.

\[\blacklozenge\] We recall that $|\lambda - 1| = \sqrt{\alpha \mu}$, so $2Re\left(\frac{\alpha z}{\sqrt{\alpha \mu}}\right) \leq 2\sqrt{\alpha / \mu} |z|$. This shows that

$\forall (\ell, \ell') \in \mathbb{R}^{2}$, $|z| < \min\left(\ell, \frac{\ell}{2\sqrt{\alpha / \mu} \ell'}\right) \Rightarrow \psi(z) \in (-\ell, \ell) \times (-\ell', \ell')$.

Proposition 3.3 and Proposition 3.11 show that $\ell \approx \frac{1}{N}$, $\ell' \approx \frac{N^{3}}{q^{2}} = \frac{1}{q \omega}$ and $\alpha \approx \frac{q^{3}}{N^{3}}$, so

$\frac{\ell}{2\sqrt{\alpha / \mu} \ell'} \approx \sqrt{\alpha \mu} \frac{\ell}{\ell'} \approx |\lambda - 1| \frac{N^{3}}{q^{3}} = \frac{1}{q \omega}$.

Therefore, with $\rho \approx \min\left(\frac{|\lambda - 1|}{q \omega}; \frac{1}{q \omega}\right) = \frac{|\lambda - 1|}{q \omega}$, we obtain $\blacklozenge$.

\[\blacksquare\] Since $\left(\frac{\alpha i}{\sqrt{\alpha}}\right)$ and $\left(\frac{\alpha^{-1} i}{\sqrt{\alpha}}\right)$ are two eigenvectors of the operator $B$ corresponding to the eigenvalues $\lambda$ and $\bar{\lambda}$, we obtain by a direct computation that

$\Psi^{-1} \circ F \circ \Psi(z) = \lambda z + S(z + \bar{z}) \psi^{-1}\left(-\frac{1}{\mu}\right)$.

We notice that $\psi(i\lambda) = i(\lambda - \bar{\lambda}) \left(-\frac{1}{|\lambda - 1|^{2}}\right) = -2 \sin \gamma_{0} \left(-\frac{1}{\mu}\right) = \frac{2}{\mu} \sin \gamma_{0} \left(-\frac{1}{\mu}\right)$, hence

$\psi^{-1}\left(-\frac{1}{\mu}\right) = \frac{i\lambda \mu}{2 \sin \gamma_{0}}$.

This implies $\blacksquare$, with

$g(z) = \frac{\mu}{2|\lambda - 1| \sin \gamma_{0}} S(z)$. \hspace{1cm} (3.26)

\[\blacklozenge\] We have $\Psi^{*}(dr \wedge d\theta) = \frac{1}{2\iota} \det(\psi(1); \psi(i)) dz \wedge d\bar{z}$ and

\[\det(\psi(1); \psi(i)) = \begin{vmatrix} \Re\left(\frac{\alpha i}{\sqrt{\alpha}}\right) & \Re\left(\frac{i \alpha}{\sqrt{\alpha}}\right) \\ 1 & 0 \end{vmatrix} = -\frac{\alpha \sin \gamma_{0}}{|\lambda - 1|^{2}}.\]
Since \(- \sin \gamma_0 = |\lambda - 1| = \sqrt{\alpha \rho} \) and \(\alpha \approx q \omega / N\), this proves (ii).

Using (3.20) and the Taylor expansion \(S(x) = \sum_{\nu=2}^{n} \frac{1}{\nu!} S^{(\nu)}(0)x^\nu + R(x)\), we set

\[
a_\nu = \frac{\mu S^{(\nu)}(0)}{2\nu! |\lambda - 1| \sin \gamma_0} \quad \text{and} \quad \varepsilon(x) = \frac{\mu R(x)}{2 |\lambda - 1| \sin \gamma_0}, \quad \text{so} \quad g(x) = \sum_{\nu=2}^{n} a_\nu x^\nu + \varepsilon(x). \tag{3.27}
\]

Since \((-1)^{\nu-1} S^{(\nu)}(0) = q^{4\nu+1}/N^{3\nu+1} - \sin \gamma_0 \approx |\lambda - 1|, \alpha \approx q^5/N^4\), we have

\[
(-1)^{\nu}a_\nu \approx \frac{q^{4\nu+1}}{E_n N^{3\nu+1}} \frac{\mu}{|\lambda - 1|^2} = \frac{q^{4\nu+1}}{N^{3\nu+1}} \frac{1}{\alpha} \approx \frac{q^{4\nu+1}}{q^5} N^4 = \frac{q^{4\nu-4}}{N^{3\nu-3}} = \omega^{\nu-1}.
\]

Thus all that remains is to prove the estimates on \(\varepsilon(x)\). Notice that \(\psi(\rho) \in B(q, \ell')\), so \(2\rho \leq \ell'\). Furthermore, for \(0 \leq j \leq n\), the derivative \(R^{(j)}(x)\) is the remainder of the Taylor expansion at zero of \(S^{(j)}(x)\) up to order \(n - j\). Therefore the Taylor expansion theorem and Proposition 3.11 show that for \(x \in [-2\rho; 2\rho]\) we have

\[
\left| R^{(n+1-j)}(x) \right| \lesssim \frac{|x|^{n+1-j}}{(n+1-j)!} \max_{|x| \leq 2\rho} \left| S^{(n+1)}(y) \right| \lesssim |y|^{n+1-j} \frac{q^{4(n+1)+1}}{N^{3(n+1)+1}}.
\]

Moreover the estimates \(|\sin \gamma_0 | \approx |\lambda - 1| \) and \(\alpha \approx q^5/N^4\) imply that

\[
\frac{\mu}{|\lambda - 1| \sin \gamma_0} \approx \frac{\mu}{|\lambda - 1|^2} = \frac{1}{\alpha} \approx \frac{N^4}{q^5}, \quad \text{hence} \quad \left| \varepsilon^{(j)}(x) \right| = \mathcal{O}_E(q^{4n}/N^{3n} |x|^{n+1-j}),
\]

and the proof of (ii) is complete. \(\square\)

**Corollary 3.14** (Taylor expansion). Assume \(n \geq 0\); we set \(\omega = q^4/N^3\). Then for each \((q, N, \mu) \in E_{\beta, n}\), there exist \(\lambda \in \mathbb{C} \) (with \(\lambda = \exp(i\gamma_0)\) and \(-\pi/3 < \gamma_0 < 0\)), \(\rho > 0\), \(a_\nu \in \mathbb{R}\), for \(2 \leq 2\nu \leq 2n + 2\) and a function \(\varepsilon : [-2\rho; 2\rho] \to \mathbb{C}\) satisfying

\[
(\Psi^{-1} \circ F_{q, N, \mu} \circ \Psi)(z) = \lambda \left( z + i |\lambda - 1| \sum_{\ell=2}^{2n+2} a_\ell (z + \overline{z})^\ell \right) + \varepsilon(z + \overline{z}) \tag{3.28}
\]

with \(\left| \varepsilon^{(k)}(x) \right| = \mathcal{O}_E(\omega^{2n+2} |\lambda - 1| |x|^{2n+2-k})\), for \(0 \leq k \leq 2n + 2\),

and \(E = \{(x, q, N, \mu) \mid |x| \leq 2\rho, \ (q, N, \mu) \in E_{\beta}\}\).

Furthermore we have the following properties.

i) \(|\lambda^p - 1| \approx |\lambda - 1| \) for \(1 \leq p \leq 2n + 2\) and \(\rho \omega \approx |\lambda - 1|/q\);

ii) \((-1)^{\ell} a_\ell \approx \omega^{\ell - 1}\), for \(2 \leq \ell \leq 2n + 2\);

iii) \(2a_2^2 + 3a_3 |\lambda - 1| R(\lambda) \approx \omega^2\) uniformly on \(E_{\beta, n}\), with \(R(\lambda) = i \frac{1 + \lambda}{1 - \lambda} \frac{2 + \lambda + 2\lambda^2}{1 + \lambda + \lambda^2}\).
Proof. We prove \( \text{[3.13]} \), which is the only condition which does not follow directly from Proposition \( \text{[3.13]} \). Let \(( q, N, \mu ) \) be in \( E_{\beta, n} \), so \( 0 < \alpha \mu < 1/(n + 1)^2 \). Let \( 0 < \alpha_0 < \frac{\pi}{6} \) satisfy

\[
\sin(\alpha_0) = \frac{1}{2n + 2}.
\]

Since \( \alpha \mu = |\lambda - 1|^2 \) and \( -\sin(\gamma_0/2) = \frac{1}{2} |\lambda - 1| < \frac{1}{2n + 2} \), we obtain that \( -\gamma_0/2 < \alpha_0 \), so

\[
\cos(\alpha_0) |\lambda - 1| < \cos(\gamma_0/2) |\lambda - 1| = -\sin(\gamma_0) |\lambda - 1|
\]

As \( q/N \) tends to infinity, Proposition \( \text{[3.11]} \) and \( \text{[3.27]} \) show that

\[
a_2^2 = \frac{\mu^2 S''(0)^2}{16 \sin^2 \gamma_0 |\lambda - 1|^2} = \frac{\mu^2 S''(0)^2}{16 \cos^2(\frac{\gamma_0}{2}) |\lambda - 1|^2} = \frac{S''(0)^2}{16 \alpha^2 \cos^2(\frac{\gamma_0}{2})} \sim \frac{(\beta_2/\beta_1)^2}{16 \cos^2(\frac{\gamma_0}{2})} \rho_0^2 \omega^2,
\]

\[
a_3 = \frac{\mu S'''(0)}{-12 \sin \gamma_0 |\lambda - 1|} = \frac{\mu S'''(0)}{12 \cos(\frac{\gamma_0}{2}) |\lambda - 1|^2} = \frac{S'''(0)}{12 \alpha \cos(\frac{\gamma_0}{2})} \sim \frac{\beta_3/\beta_1}{12} \rho_0^2 \omega^2.
\]

Now we compute

\[
R(\lambda) = -\frac{\cos(\frac{\gamma_0}{2}) + 4 \cos \gamma_0}{\sin(\frac{\gamma_0}{2}) + 2 \cos \gamma_0}, \quad \text{hence} |\lambda - 1| R(\lambda) = 2 \cos(\frac{\gamma_0}{2}) \frac{8 \cos^2(\frac{\gamma_0}{2}) - 3}{4 \cos^2(\frac{\gamma_0}{2}) - 1}.
\]

Since we have \( \cos^2(\frac{\gamma_0}{2}) \geq 1 - \left( \frac{1}{2n+2} \right)^2 \geq \frac{15}{16} \) for \( n \geq 1 \), it follows that for \( q/N \) large enough we have

\[
\frac{72}{11} = 4 \cdot \frac{18}{11} \leq 2 \frac{|\lambda - 1| R(\lambda)}{\cos(\frac{\gamma_0}{2})} = 4 \frac{8 \cos^2(\frac{\gamma_0}{2}) - 3}{4 \cos^2(\frac{\gamma_0}{2}) - 1} \leq \frac{20}{3}, \quad \text{hence}
\]

\[
\left( \frac{18}{11} \beta_2^2 - 3 \beta_1 \beta_3 \right) \frac{\rho_0^2 \omega^2}{8 \beta_1^2} \leq \frac{1}{2} \left( \frac{72}{11} \beta_2^2 \right) - \frac{3 \beta_1 \beta_3}{12} \frac{\rho_0^2 \omega^2}{\beta_1^2 \cos(\frac{\gamma_0}{2})} \leq 2 a_2^2 |\lambda - 1| R(\lambda) + 3a_3 \leq \frac{5a_2^2}{12} \leq \omega^2.
\]

This holds true and implies the lemma because we can evaluate \( 18 \beta_2^2 - 11 \beta_1 \beta_3 > 0 \). \( \square \)

Lemma 3.15. If \( |\lambda - 1| < \frac{1}{n+1} \) then \( |\lambda - 1| \leq |\lambda^p - 1| \) for \( 1 \leq p \leq 2n + 2 \).

Proof. Notice that \( |\sin \gamma_0| \leq |\lambda - 1| < \frac{1}{n+1} \), so \( \gamma_0 > \frac{\pi}{2n+2} \). Therefore we have

\[
\frac{|\lambda^p - 1|}{|\lambda - 1|} \geq \Re \left( \sum_{j=0}^{p-1} \lambda^j \right) = \sum_{j=0}^{p-1} \cos(j \gamma_0) \geq \sum_{j=0}^{p-1} \cos \left( \frac{j \pi}{2n + 2} \right) \geq 1.
\]

\( \square \)

3.5 Normalisations

The goal of this section is to prove that we can find nearly symplectic coordinates in which \( F_{q, N, \mu} \) takes the form

\[
F_{q, N, \mu}(z) = \lambda z \exp(2 \pi i |z|^2 + \epsilon(z)),
\]

where the error term \( \epsilon \) is a real valued function and is small enough up to enough derivatives. For this purpose, our first step is to specify a suitable change of coordinates in which \( F_{q, N, \mu} \) appears as a Birkhoff’s normal form up to some order, namely

\[
F_{q, N, \mu}(z) = \lambda z \left( 1 + \sum_{p=1}^{n} b_p |z|^{2p} \right) + \tilde{\epsilon}(z).
\]

Note that the change of coordinates does not need to be symplectic in our setting.
3.5.1 Notations and statements

To achieve (3.30) and (3.29) in a quantitative way, we must deal with smooth functions on $D^*(0, \tau) = \{ z \in \mathbb{C} \mid 0 < |z| \leq \tau \}$ (but not necessarily smooth at zero) and control their behaviour near zero. To this end we introduce the following notations.

**Notation.** In the following, we use the operators $\tilde{\partial} = \frac{1}{2}(\partial_s + i\partial_t)$ and $\partial = \frac{1}{2}(\partial_s - i\partial_t)$, with $z = s + it$ and $(s, t) \in \mathbb{R}^2$. Assume $\tau > 0$ and $k \in \mathbb{N}$. A smooth function $f : D^*(0, \tau) \to \mathbb{C}$ is said to be controlled up to the $k$ derivatives, by $C \geq 0$ at order $\ell \in \mathbb{R}$, and we write $f \in \mathcal{O}_k(\ell; C, \tau)$ or $f(z) = \mathcal{O}_k(\ell; C, \rho)$ if

$$\forall z \in D^*(0, \tau), \quad \left| \partial_\alpha \tilde{\partial}_\beta f(z) \right| \leq C |z|^\ell - \alpha - \beta, \quad \text{for all } (\alpha, \beta) \in \mathbb{N}^2 \text{ such that } \alpha + \beta \leq k.$$  

**Notation.** For $(k, m) \in \mathbb{N}^2$, $\rho > 0$, two sets $E$ and $E'$ satisfying $E \subset E' \times \mathbb{C}$, two functions $f_1 : E \to \mathbb{C}$ and $f_2 : E' \to \mathbb{R}_+$, and a function $\rho : E' \to \mathbb{R}_+$, we write $f_1(\cdot, z) = \mathcal{O}_k(m; f_2, \rho)$ if there exists two constants $C \geq 0$ and $c > 0$ satisfying

$$\forall x \in E', \quad E' \times D(0; c\rho(x)) \subset E,$$

and $f_1(x, z) = \mathcal{O}_k(m; C f_2(x), \rho(x))$.

All the properties of the spaces $\mathcal{O}_k$ we need are listed in Appendix A. At last, we need to introduce analogous definitions in polar coordinates.

**Notation.** Assume $\rho > 0$, $\ell \in \mathbb{R}$ and $k \in \mathbb{Z}$. We recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

- A smooth function $f : (0; \rho] \times \mathbb{T} \to \mathbb{C}$ is said to be controlled up to $k$ derivatives, by $C \geq 0$ at order $\ell \in \mathbb{R}$, and we write $f \in \mathcal{O}_k^\mathbb{T}(\ell; C, \tau)$ or $f(r, \theta) = \mathcal{O}_k^\mathbb{T}(\ell; C, \rho)$ if

$$\left| \partial_\alpha \rho_\beta f(r, \theta) \right| \leq C r^\ell - \alpha, \quad \text{for } 0 < r \leq \rho, \theta \in \mathbb{T} \text{ and } \alpha + \beta \leq k.$$  

- For two sets $E$ and $E'$ satisfying $E \subset E' \times \mathbb{R}_+ \times \mathbb{T}$, two functions $f_1 : E \to \mathbb{C}$ and $f_2 : E' \to \mathbb{R}_+$, and a function $\rho : E' \to \mathbb{R}_+$, we write $f_1(\cdot, r, \theta) = \mathcal{O}_k^\mathbb{T}(\ell; f_2, \rho)$ if there exists two constants $C \geq 0$ and $c > 0$ satisfying

$$\forall x \in E', \quad E' \times (0; c\rho(x)) \times \mathbb{T} \subset E \quad \text{and} \quad f_1(x, r, \theta) = \mathcal{O}_k^\mathbb{T}(\ell; C f_2(x), \rho(x)).$$

Basically, we can rephrase Proposition 3.13 as follows

$$g(z + \overline{z}) = \sum_{\nu=2}^{n} a_\nu (z + \overline{z})^\nu + \mathcal{O}_{n, E_\beta, n}(n + 1; \omega^n, \rho), \quad (3.31)$$

where $E_{\beta, n}$ is defined by (3.24). The constants $n$ and $\beta$ do not depend on $(q, N, \mu)$. Here we introduce $E_{\beta, n}$ rather than $E_\beta$ (see (3.24)) for suitable estimates on the non resonant part of the conjugation of the transformation $F_{q, N, \mu}$ to its Birkhoff’s normal form (see Proposition 3.16 below). The constant $\beta > 0$ is chosen so $q/N$ is large enough for appropriate estimates of $a_2$ and $a_3$ (see Corollary 3.14 above).
Notation. From now on, unless mentioned otherwise, we shall abbreviate \( \Theta_{k,n} := \Theta_{k,E_{\beta,n}} \). In the following, \( h = h(.,q,N,\mu) \) denotes any family of symplectic maps from \( \mathbb{D}(0,\rho) \) into \( \mathbb{C} \). We assume that for \( 2n+2 \geq k \geq 1 \) we have

\[
h(z) = \lambda \left( z + i |\lambda - 1| \sum_{\ell=2}^{2n+2} a_{\ell}(z + \bar{z})^{\ell} \right) + \Theta_{k,n}(2n+3, |\lambda - 1| \omega^{2n+2}, \rho),
\]

where \( \omega = q^1 / N^3, \rho > 0, \lambda = \exp(i \gamma_0), a_\nu \in \mathbb{C} \) satisfy all the conditions in Corollary 3.14.

Birkhoff normal form

The next proposition is a the quantitative version of (3.30). It recalls a classical result of normal form theory. We construct polynomial coordinates in which the symplectic map \( h \) is put in its Birkhoff normal form up to a reminder of arbitrarily high order. The proof follows Moser’s strategy and is inductive in its nature: a sum of homogeneous polynomials is put in its Birkhoff normal form up to a reminder of arbitrarily high order. The proof is put in its Birkhoff normal form up to a reminder of arbitrarily high order. The proof follows Moser’s strategy and is inductive in its nature: a sum of homogeneous polynomials is used to normalize the Taylor expansion of \( h \) order by order. But for our purpose, we need to achieve this keeping a quantitative track of the operations involved. Therefore we provide below a complete proof of the statement.

Before proceeding to the precise statement, we need to introduce a few more notations. We shall consider a coordinates change of the form

\[
u = (\nu_1, \nu_2) \in \mathbb{N}^2 \text{ in the sum above runs over all the couples such that } 2 \leq |\nu| \leq 2n+2, \text{ with } |\nu| = \nu_1 + \nu_2.
\]

For such polynomial, we denote by \( [\Phi]_{\nu} \) the \( \nu \)-component \( \varphi_\nu z^\nu = \varphi_{\nu_1,\nu_2} z^{\nu_1} \bar{z}^{\nu_2} \). By extension, for any smooth function \( F \), we denote by \( [F]_{\nu} \) the \( \nu \)-component of its Taylor expansion at zero. For any integer \( p \), it is also convenient to denote by \( [F]_{\nu} \) its \( p \)-homogeneous part. Thus, for \( p \geq 2 \), we have

\[
[\Phi]_p(z) = \sum_{|\nu|=p} \varphi_\nu z^\nu \quad \text{and} \quad [f]_p(z) = i |\lambda - 1| a_p \left( z + \bar{z} \right)^p.
\]

Proposition 3.16 (Birkhoff normal form). Assume \( 2n+2 \geq k \geq 2 \). Then for each \( (q,N,\mu) \in E_{\beta,n} \) there exist \( \rho' > 0, b_j \in \mathbb{C} \) for \( 1 \leq j \leq n \), and \( \varphi_\nu \in \mathbb{C} \) for \( 2 \leq |\nu| \leq 2n+2 \) satisfying the following conditions.

i) The polynomial \( \Phi \) in (3.33) defines a diffeomorphism from a neighbourhood of zero onto a set that contains \( \mathbb{D}(0,\rho) \) and

\[
\Phi \circ h \circ \Phi^{-1}(u) = \lambda u \left( 1 + \sum_{p=1}^{n} b_p |u|^{2p} \right) + \Theta_{k,n}(2n+3; |\lambda - 1| \omega^{2n+2}, \rho);
\]

ii) \( J(\Phi)^2 - |\partial \Phi|^2 \) contains no \( (j,j) \)-component for \( 1 \leq j \leq n \);

iii) \( \text{Im} (\varphi_\nu) = 0 \) if \( \nu = (j+1,j) \) with \( 1 \leq j \leq n \);

iv) \( \rho' \ll \rho \);
\( v) \ b_1 \asymp |\lambda - 1| \omega^2 \) and \( |b_j| = O_{E_\beta,n}((\lambda - 1) \omega^{2j}) \) for \( 2 \leq j \leq n; \)

\( \text{vi}) \ |\Phi^{-1}(z)| \asymp |z| \text{ on } \mathbb{D}(0; r'); \)

\( \text{vii}) \ |\varphi| = O_{E_{n,\beta}}(|\nu|^{\gamma - 1}) \text{ for } 2 \leq |\nu| \leq 2n + 2; \)

**Herman normal form**

This is the quantitative version of (3.29). We first state this result in complex coordinates.

**Proposition 3.17.** Assume \( 2 \leq k \leq 2n \). Then there exist a diffeomorphism \( \psi \) from \( \mathbb{D}(0; \rho_1) \) into a set containing \( \mathbb{D}(0; \rho_2) \) and a function \( \varepsilon : \mathbb{D}(0; \rho_2) \to \mathbb{R} \) with the following properties.

- \( i) \ \psi \circ h \circ \psi^{-1}(z) = \lambda z \exp \left(2\pi i |z|^2 + \varepsilon(z)\right); \)
- \( ii) \ \varepsilon \) is a real valued function and \( \varepsilon(z) = O_{k,n}(2n; |\lambda - 1|^{-n}, \rho_2); \)
- \( iii) \ \rho_1 \asymp \rho, \rho_2 \asymp \rho \omega \sqrt{|\lambda - 1|} \) and \( |\psi(z)| \asymp |z| \omega \sqrt{|\lambda - 1|}. \)

We also give an equivalent result in polar coordinates, in order to apply the invariant curve theorem.

**Proposition 3.18** (Herman normal form). There exist \( \rho' > 0 \), a diffeomorphism \( \Psi \) from \( (0; \rho') \times \mathbb{T} \) into \( \mathbb{D}(0; \rho) \) and a function \( \varepsilon : (0; \rho') \times \mathbb{T} \to \mathbb{R} \) with the following properties.

- \( i) \ \Psi^{-1} \circ h \circ \Psi(r; \theta) = \left(r + \varepsilon(\theta, r); \frac{2n}{2\pi} + \theta + r \right); \)
- \( ii) \ \varepsilon \) is a real valued function and \( \varepsilon(r, \theta) = O_{k,n}^\theta(n + 1; |\lambda - 1|^{-n}, \rho'); \)
- \( iii) \ \rho' \asymp |\lambda - 1|^3/q^2, \ |\Psi(r, \theta)| \asymp \frac{1}{\omega} \sqrt{\frac{r}{|\lambda - 1|}} \text{ and area} \left(\Psi((0; r) \times \mathbb{T})\right) \asymp \frac{r}{\omega^2 |\lambda - 1|}. \)

3.5.2 **Proof of Proposition 3.16**

**Proof.** We construct a polynomial \( \Phi \) of degree \( 2n + 2 \) such that \( \Phi \circ h \) is of the form

\[
\Phi \circ h = \lambda \Phi \left(1 + i \sum_{\ell=1}^{n} b_{\ell} \left|\Phi\right|^{2\ell} \right) + O(\left|z\right|^{2n+3}).
\]  

(3.34)

Taking the \( \ell \)-homogeneous part of this, for \( 2 \leq \ell \leq 2n + 2 \), this is equivalent to

\[
\lambda [\Phi]_\ell(z) - [\Phi]_\ell(\lambda z) = [h]_\ell + \sum_{j=2}^{\ell-1} \left[[\Phi]_j \circ h\right]_\ell - i\lambda \sum_{j=1}^{\ell-1} b_j \left[\Phi \left|\Phi\right|^{2j}\right]_\ell.
\]  

(3.35)

**Computation of \( [\Phi]_2 \).** Since \( h(z) = \lambda z + i\lambda |\lambda - 1| a_2(z + \bar{z})^2 + O(|z|^3) \), we have

\[
[h]_2 = i\lambda a_2 |\lambda - 1|^2 |(z + \bar{z})|^2.
\]
so (3.35) implies that
\[
\lambda[\Phi]_2(z) - [\Phi]_2(\lambda z) = i\lambda a_2 |\lambda - 1| (z + \bar{z})^2.
\]

From this, we obtain that
\[
(\lambda - \lambda^2)\varphi_{2,0} = i\lambda |\lambda - 1| a_2 ; (\lambda - 1)\varphi_{1,1} = 2i\lambda |\lambda - 1| a_2 ; (\lambda - \lambda^2)\varphi_{0,2} = i\lambda |\lambda - 1| a_2,
\]

hence
\[
\begin{align*}
\varphi_{2,0} &= ia_2 |1 - \lambda|/(1 - \lambda), \\
\varphi_{1,1} &= 2ia_2 |1 - \lambda|/(\lambda - 1), \\
\varphi_{0,2} &= ia_2 |1 - \lambda|/(\lambda^2 - 1).
\end{align*}
\]

This shows in particular that if |\nu| = 2 then |\varphi_{\nu}| = |a_2| \geq \omega.

**Computation of b_1.** Equation (3.35) implies that
\[
\lambda[\Phi]_3(z) - [\Phi]_3(\lambda z) = [h]_3 + \left[ [\Phi]_2 \circ h \right]_3 - i\lambda b_1 \left[ [\Phi] |\Phi|^2 \right]_3.
\]

Therefore, we obtain that
\[
\lambda[\Phi]_3(z) - [\Phi]_3(\lambda z) = i\lambda |1 - \lambda| a_3(z + \bar{z})^3 + \left[ [\Phi]_2(\lambda z + i\lambda |1 - \lambda| a_2(z + \bar{z})^2) \right]_3 - i\lambda b_1 z |z|^2.
\]

Taking the (2, 1) part of this, we obtain that
\[
i\lambda b_1 = 3i\lambda |1 - \lambda| a_3 + \left[ [\Phi]_2(\lambda z + i\lambda |1 - \lambda| a_2(z + \bar{z})^2) \right]_{2,1} \quad \text{with } \left[ [\Phi]_2(\lambda z + i\lambda |1 - \lambda| a_2(z + \bar{z})^2) \right]_{2,1} = \varphi_{2,0} \lambda^2 |1 - \lambda| (2a_2 + \varphi_{0,2} |1 - \lambda| (-ia_2)) + \varphi_{0,2} \lambda^2 |1 - \lambda| (-2ia_2),
\]

\[
= -2a_2 \lambda |1 - \lambda|^2 \frac{1 + \lambda + 2 + \lambda + 2\lambda^2}{1 - \lambda^3}.
\]

This with Corollary 3.14 [22] implies that
\[
\frac{b_1}{|\lambda - 1|} = 3a_3 + 2a_2^2 |\lambda - 1| R(\lambda) \asymp \omega^2.
\]

**Computation of [\Phi]_3.** With \( (p, q) \in \mathbb{N}^2 \) satisfying \( p + q = 3 \) and \( \nu \neq (2, 1) \), Equation (3.35) implies that
\[
\lambda[\Phi]_\nu - \lambda^{p-q}[\Phi]_\nu = [h]_\nu + \left[ [\Phi]_2 \circ h \right]_\nu = i\lambda |\lambda - 1| a_3(\frac{3}{p}) z^{\nu'} + \left[ [\Phi]_2 \circ h \right]_\nu
\]

\[
= i\lambda |\lambda - 1| a_3(\frac{3}{p}) z^{\nu'} + \varphi_{2,0} \lambda^2 (2z i |\lambda - 1| a_2(z + \bar{z})^2)_{\nu'} + 2\varphi_{1,1} \text{ Re} (\bar{z} i |\lambda - 1| a_2(z + \bar{z})^2)_{\nu'} + \varphi_{0,2} \lambda^2 (-2z i |\lambda - 1| a_2(z + \bar{z})^2)_{\nu'}.
\]

Since \(|\lambda^{p-q} - \lambda| = |\lambda^{p-q-1} - 1| \geq |\lambda - 1| \) and \(|p - q - 1| \leq 4\), Lemma 3.15 implies that \(|\varphi_{\nu}| = \vartheta_{E_{3,n}}(a_2^2) = \vartheta_{E_{3,n}}(\omega^2)\).
We now compute \( \varphi_{2,1} \). We write
\[
\partial \Phi(z) = 1 + 2\varphi_{2,0}z + \varphi_{1,1} \bar{z} + 2\varphi_{2,1} |z|^2 + 3\varphi_{3,0}z^2 + \varphi_{1,2} \bar{z}^2 + O(|z|^3),
\]
\[
\partial \Phi(z) = 2\varphi_{0,2} \bar{z} + \varphi_{1,1} z + 2\varphi_{2,1} |z|^2 + 3\varphi_{0,3} \bar{z}^2 + \varphi_{2,1} z^2 + O(|z|^3).
\]
Therefore Constraint (3.35) of the proposition implies that
\[
0 = (J(\Phi))_{1,1}(z) = |z|^2 \left( 4|\varphi_{2,0}|^2 + |\varphi_{1,1}|^2 + 4\text{Re}(\varphi_{2,1}) - 4|\varphi_{0,2}|^2 - |\varphi_{1,1}|^2 \right).
\]
This with Constraint (3.35) shows that \( \varphi_{2,1} = |\varphi_{0,2}|^2 - |\varphi_{2,0}|^2 \). In particular, it implies that \( |\varphi_{2,1}| = \mathcal{O}_{E,\beta,n}(\omega^2) \).

**Estimates of \( \Phi \) and \( b_k \).** We compute \([\Phi]_p \) by induction over \( p \). Let assume that \( 4 \leq p \leq 2n+2 \) and that we have computed \([\Phi]_{p-1} \). We also assume that \( |\varphi_{\mu}| = \mathcal{O}_{E,\beta,n}(\omega^{|\mu|-1}) \) if \( |\mu| \leq p-1 \) and \( |b_k| = \mathcal{O}_{E,\beta,n}(|\lambda - 1| \omega^{2k}) \) if \( 2k \leq p - 2 \). Equation (3.35) implies that
\[
\lambda [\Phi]_p(z) - [\Phi]_p(\lambda z) = [h]_p + \sum_{\ell=2}^{p-1} \left[ [\Phi]_\ell \left( \lambda z + i\lambda |\lambda - 1| \sum_{m=2}^{p-\ell+1} a_m(z + \bar{z})^m \right) \right]_p
\]
\[
- i\lambda \sum_{\ell=1}^{(p-1)/2} b_\ell \left[ [\Phi]_\ell^{2\ell} \right]_p.
\]
(3.36)

We estimate each term of the right hand side of this. Let \( \nu \in \mathbb{N}^2 \) satisfy \( |\nu| = p \).
- We have \([h]_\nu = i\lambda |\lambda - 1| a_{|\nu|} |\nu| \) and \( |a_{|\nu|}| = \mathcal{O}_{E,\beta,n}(|\lambda - 1| \omega^{2|\nu|-1}) \).
- For \( 1 \leq \ell < (p-1)/2 \), we have
\[
[\Phi]_\ell = z^\nu \left( \sum_{\nu} \varphi_{\nu_0} \prod_{i=1}^{\ell} \varphi_{\nu_i} \bar{\varphi}_{\nu_{i+\ell}} \right),
\]
(3.37)
where the summation index \( \nu \) above runs over all the tuples \( \nu = (\nu_0, \nu_1, \ldots, \nu_\ell) \in (\mathbb{N}^2)^{2\ell+1} \) satisfying the condition
\[
\nu = \nu_0 + \sum_{j=1}^\ell (\nu_j + \nu_{j+\ell}), \quad \text{with } (\overline{r}, s) = (s, r) \text{ for all } (r, s) \in \mathbb{N}^2.
\]
Thus each term in the sum in (3.37) belongs to \( \mathcal{O}_{E,\beta,n}(\omega^{2\ell}) \), with
\[
n_N = |\nu_0| - 1 + \sum_{j=1}^\ell (2|\nu_j| - 2) = |\nu| - (1 + 2\ell).
\]
Since \( |b_\ell| = \mathcal{O}_{E,\beta,n}(|\lambda - 1| \omega^{2\ell}) \), it follows that \( b_\ell [\Phi]_\ell^{2\ell} \) is of the form \( c(\nu, \Phi, \ell) z^\nu \), with \( |c(\nu, \Phi, \ell)| = \mathcal{O}_{E,\beta,n}(|\lambda - 1| \omega^{2|\nu|-1}) \).
For $\mu \in \mathbb{N}^2$ satisfying $2 \leq |\mu| \leq p - 1$, we estimate the term
\[
\left[ (\Phi)_\mu (\lambda (z + i |\lambda| - 1) \sum_m a_m (z + \bar{z})^m) \right]_\nu = \varphi_\mu (\lambda (z + i |\lambda - 1| \sum_{\nu'} a_{|\nu'|} (|\nu'|) z^{\nu'}))^\mu,
\]
where $m$ in the sum in the left hand side runs over the intergers satisfying $2 \leq m \leq N_0 = p - |\mu| + 1$ and $\nu' \in \mathbb{N}^2$ in the sum of the right hand side runs over the couple satisfying $2 \leq |\nu'| \leq N_0$. A direct computation shows that the right hand side of this equality is of the form
\[
\varphi_\mu \lambda^{\mu} z^{\nu} \sum_{(\mu_0, \mu)} \prod_{\nu' \in \mathbb{N}} \left( i |\lambda - 1| a_{|\nu'|} (|\nu'|) \right)^{\mu_{\nu'}},
\]
where $\mathbb{N}$ denote the set of the couples $\nu' \in \mathbb{N}^2$ satisfying $|\nu'| \geq 2$ and the index in the sum above runs over all the tuples $(\mu_0, \mu)$, with $\mu_0 \in \mathbb{N}^2$ and $\mu = (\mu_\nu') \in (\mathbb{N}^2)^{\mathbb{N}}$ satisfying
\[
\mu_0 + \sum_{\nu' \in \mathbb{N}} \mu_{\nu'} \cdot \nu' = \nu \quad \text{and} \quad \mu_0 + \sum_{\nu' \in \mathbb{N}} \mu_{\nu'} = \mu,
\]
with $(r, s) \cdot (r', s') = (rr' + ss', rs' + r's)$ for all $(r, s, r', s') \in \mathbb{N}^4$. Note that we have $\mu_{\nu'} \neq 0$ for at least one index $\nu'$ since otherwise we should have $\nu = \mu_0 = \mu$, which is impossible since $|\mu| < p = |\nu|$. On the other hand, we have $\mu_{\nu'} \neq 0$ for at most $|\mu| \leq p - 1 \leq 2n + 1$ indices $\nu'$. This implies that
\[
\prod_{\nu' \in \mathbb{N}} |\lambda - 1|^{\mu_{\nu'}} = O_{E_{\beta, n}} (|\lambda - 1|).
\]
Furthermore, we have $|a_{|\nu'|}| = O_{E_{\beta, n}} (\omega^{|\nu'| - 1})$, so
\[
\prod_{\nu' \in \mathbb{N}} |a_{|\nu'|}|^{\mu_{\nu'}} = O_{E_{\beta, n}} \left( \prod_{\nu' \in \mathbb{N}} (\omega^{|\nu'| - 1})^{\mu_{\nu'}} \right) = O_{E_{\beta, n}} (\omega^{|\nu'| - |\mu|}).
\]
Since we have $|\varphi_\mu| = O_{E_{\beta, n}} (\omega^{|\nu'| - 1})$, it follows that
\[
|\varphi_\mu \lambda^{\mu} \sum_{(\mu_0, \mu)} \prod_{\nu' \in \mathbb{N}} \left( i |\lambda - 1| a_{|\nu'|} (|\nu'|) \right)^{\mu_{\nu'}}| = O_{E_{\beta, n}} (|\lambda - 1| \omega^{|\nu'| - 1}).
\]
Therefore, we obtain that $|\lambda - \lambda^{\nu'}| |\varphi_\nu| = O_{E_{\beta, n}} (|\lambda - 1| \omega^{|\nu'| - 1})$. With Lemma 3.13 this shows that $|\varphi_\nu| = O_{E_{\beta, n}} (\omega^{|\nu'| - 1})$ (if $\nu \neq (\ell_0 + 1, \ell_0)$ when $p = 2\ell_0 + 1$).

- If $p = 2\ell_0 + 1$ and $\nu_0 = (\ell_0 + 1, \ell_0)$ then \(3.36\) shows that
\[
\lambda b_{\ell_0} z |^2\ell_0 = [h]_{\nu_0} + \sum_{\ell=2}^{p-1} \left[ (\Phi)_{\ell} \circ h \right]_{\nu_0} - i\lambda \sum_{\ell=1}^{\ell_0 -1} b_{\ell} \left( \Phi \circ 2\ell \right)_{\nu_0},
\]
hence $|b_{\ell_0}| = O_{E_{\beta, n}} (|\lambda - 1| \omega^{|\nu_0| - 1}) = O_{E_{\beta, n}} (|\lambda - 1| \omega^{2\ell_0})$.
Furthermore, a direct computation shows that

\[ |z|^{-2\ell_0} (J(\Phi))_{(\ell_0,sl_0)} = 2(\ell_0 + 1) \text{Re} (\varphi_{\ell_0}) + \sum_{r+s=\ell_0+1 \atop (r,s) \neq (1,0), \ (r',s') \neq (1,0)} \frac{1}{rr'} (\varphi_{r,s} \bar{\varphi}_{r',s'}). \]

Since in the sum above we have \( |\varphi_{r,s}| = \Theta_{EB,n} (\omega^{r+s} + r' + s' - 2) = \Theta_{EB,n} (\omega^{2\ell_0}) \), Constraints (2) and (11) show that \( |\varphi_{\ell_0}| = \Theta_{EB,n} (\omega^{2\ell_0}) \).

Thus we have proved the announced estimates at the rank \( \rho \), and so at any order, and the proof of the points (13) and (22), (24) of the proposition is complete.

**Estimate of \( \Phi^{-1} \).** We shall apply the (inverse) axiom of Lemma A.1 (see Appendix A) to the polynomial \( \Phi \) with \( \varepsilon = |\lambda - 1| (\sqrt{3/2} - 1)/q \), \( C = 0 \) and \( \tau \leq \rho \) satisfying

\[ \| \tilde{\partial} \Phi - 1 \|_{\tau} + \| \tilde{\partial} \Phi \|_{\tau} \leq \varepsilon. \]  

Note that \( \varepsilon \leq \sqrt{3/2} - 1 \), so \( 2\varepsilon + \varepsilon^2 \leq 1/2 \). Furthermore, we may choose \( \tau \asymp \rho \). Indeed, we have

\[ \| \tilde{\partial} \Phi - 1 \|_{\tau} + \| \tilde{\partial} \Phi \|_{\tau} = \sum_{2 \leq |\nu| \leq 2n + 2} |\nu| |\varphi_{\nu}| \tau^{-1} = \sum_{\ell=1}^{2n+1} \Theta_{EB,n} (\omega^\ell) = \Theta_{EB,n} (\omega \tau) \text{ if } \omega \tau \leq 1. \]

We recall that \( \rho = \frac{|\lambda - 1|}{q} \), so \( \omega \rho = \frac{|\lambda - 1|}{q} \). This implies that there exists \( \tau > 0 \) verifying (3.38) with \( \tau \leq \rho \) and \( \tau \asymp \rho \). We set \( \rho' = (1 - \varepsilon) \tau \), so \( \rho' \asymp \rho \). The (inverse) axiom of Lemma A.1 shows that \( \Phi^{-1} \) exists from \( \mathbb{D}(0; \rho) \) into \( \mathbb{D}(0; \rho) \) and that there exists \( \Theta(q, N, \mu)(z) \), a polynomial in \( z \) of degree \( 2n + 2 \) and valuation \( 2 \), satisfying

\[ \begin{cases} \Phi^{-1}(z) = z + \Theta(z) + \Theta_{k,E_{\beta,n}} (2n + 3; \varepsilon/\rho^{2n+2}, \rho), \\ \| \tilde{\partial} \Theta \|_{\rho} + \| \tilde{\partial} \Theta \|_{\rho} = \Theta_{EB,n} (\varepsilon). \end{cases} \]  

At last, (3.38) implies that \( (1 - \varepsilon) |z| \leq |\Phi(z)| \leq (1 + \varepsilon) |z| \) on \( \mathbb{D}(0, \tau) \) and this completes the proof of (3.3) in the proposition.

**Estimate of the reminder.** We have shown that there exists a polynomial \( \Phi \) of degree \( 2n + 2 \) that verifies (3.34). We set

\[ P(z) = \lambda z \left( 1 + i \sum_{\ell=1}^{n} b_{\ell} |z|^{2\ell} \right) \quad \text{and} \quad V(z) = \lambda \left( z + i |\lambda - 1| \sum_{m=2}^{2n+2} a_{m} (z + \bar{z})^{m} \right), \]

so \( \Phi \circ h(z) = P \circ \Phi(z) + \Theta(|z|^{2n+3}) \) and \( h(z) = V(z) + \Theta_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho) \).

- We prove in two steps that \( \Phi \circ h - P \circ \Phi \) belongs to \( \Theta_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho) \).
- **Step 1.** We estimate \( \Phi \circ h \). We write \( h = V + \varepsilon_0 \) and

\[ \Phi \circ h = \Phi \circ V + \int_{0}^{1} \left( \tilde{\partial} (V + t \varepsilon_0) \cdot \varepsilon_0 + \tilde{\partial} (V + t \varepsilon_0) \cdot \bar{\varepsilon_0} \right) \, dt. \]  

The (Z-product) and the (restriction) axioms of Lemma A.1 imply that

\[ V(z) = \Theta_{k}(0; (k + 1)^{2n+1} \|V\|_{\rho}, \rho) \quad \text{and} \quad \varepsilon_0(z) = \Theta_{k}(0; |\lambda - 1| \omega^{2n+2} \rho^{2n+2}, \rho). \]
Since we have $|a_m| \rho^m = \Theta_{E_{\beta,n}}(\omega^{m-1} \rho^m)$ and $\omega \rho = \Theta_{E_{\beta,n}}(1)$, it follows that $V$ and $\varepsilon_0$ both lie in $\Theta_{k,n}(0, \rho, \rho)$. Therefore the (product) axiom shows that

$$\varphi_{\nu}(V + t\varepsilon_0)^{\nu'} = \Theta_{k,E_{\beta,n}}(0; |\varphi_{\nu}| \rho^{|\nu|-1}, \rho)$$

uniformly for $(q, N, \mu, t) \in E'_{\beta,n} = E_{\beta,n} \times [0; 1]$, with $\nu = (\nu_1, \nu_2)$ and $\nu' = (\nu_1 - 1, \nu_2)$ or $\nu' = (\nu_1, \nu_2 - 1)$.

This implies that $\partial \Phi \circ (V + t\varepsilon_0)$ and $\partial \Phi \circ (V + t\varepsilon_0)$ belong to $\Theta_{k,E_{\beta,n}}(0, 1, \rho)$, since we have $|\varphi_{\nu}| \rho^{|\nu|-1} = \Theta_{E_{\beta,n}}((\omega \rho)^{|\nu|-1})$ and $\Theta_{E_{\beta,n}}(1)$.

From this, since $\varepsilon_0(z) = \Theta_{k}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho)$, and the (product) axiom show that

$$\Phi \circ h = \Phi \circ V + \Theta_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho).$$

Furthermore, the ($Z$-product) axiom shows that

$$[\Phi \circ V]_{2n+3}(z) = \sum_{2 \leq |\nu| \leq 2n+2} \Theta_{k}(|\nu|; (k + 1)^{|\nu|-1} |\lambda - 1| \rho^{|\nu|-1} |a_m|, \rho),$$

where the indices in the sum runs over all the tuples $m = (m_\ell) \in (N^2)^{|\nu|}$ satisfying $m_\ell = (1, 0)$ if $|m_\ell| = 1$, $1 \leq |m_\ell| \leq 2n + 2$ for $1 \leq \ell \leq |\nu|$ and $|m| > 2n + 2$, with $|m| = \sum_{\ell \geq 0} |m_\ell|$, and where

$$p_m = \text{card}\{\ell \mid |m_\ell| > 1\} ; \ |a_m| = \prod_{\ell, |m_\ell| > 1} |a_{m_\ell}| \left(\frac{|m_\ell|}{m_\ell}\right).$$

We have $p_m \geq 1$ and $|m| \leq (2n + 2)^2$, so the (restriction) axiom shows that

$$\Theta_{k}(|m|; (k + 1)^{|m|-1} |\lambda - 1| p_m |a_m|, \rho) \subset \Theta_{k,E_{\beta,n}}(2n + 3; |\lambda - 1| |a_m| \rho^{|m|-2n+3}, \rho).$$

A direct computation shows that $|a_m| = \Theta_{E_{\beta,n}}(\omega^{|m|-|\nu|})$, so

$$|\varphi_{\nu}| |a_m| = \Theta_{E_{\beta,n}}(\omega^{|m|-1} \rho^{|m|-2n+3} \omega^{2n+2}) = \Theta_{E_{\beta,n}}(\omega^{2n+2}).$$

From this, it follows that $[\Phi \circ V]_{2n+3}$ lies in $\Theta_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho)$. Thus we have proved that

$$\Phi \circ h = [\Phi \circ V]_{2n+2} + \Theta_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho).$$

**Step 2.** We estimate $P \circ \Phi$. The ($Z$-product) axiom shows that

$$P \circ \Phi(z) = [P \circ \Phi]_{2n+2}(z) = \sum_{\ell = 1}^n |h_\ell| \sum_\nu \Theta_{k}(|\nu|; |\varphi_{\nu}| (k + 1)^{|\nu|-1}, \rho),$$

where the index in the sum runs over all the $\nu = (\nu_0, \ldots, \nu_{2\ell}) \in (N^2)^{2\ell+1}$ such that $|\nu| \geq 2n + 3$, with $|\nu| = \sum_{j = 0}^{2\ell} |\nu_j|$, and where

$$|\varphi_{\nu}| = \prod_{j = 0}^{2\ell} |\varphi_{\nu_j}| \subset \Theta_{E_{\beta,n}}(\prod_{j = 0}^{2\ell} \omega^{|\nu_j|-1}) = \Theta_{E_{\beta,n}}(\omega^{|\nu|-2\ell+1}).$$

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Since we have \(|b_ℓ| = \mathcal{O}_k(|λ - 1|\omega^{2ℓ})\), we obtain that
\[ |b_ℓ| \cdot |φ_ℓ| = O_k(|λ - 1|\omega^{1|φ|}). \]
This implies that
\[ |b_ℓ| \cdot \mathcal{O}_k\left( |ν|; |φ_ℓ| (k + 1)^{|ν| - 1}, p \right) < \mathcal{O}_k, n\left( |ν|; |λ - 1|\omega^{1|φ|}, p \right). \]

Thus we have proved that
\[ P \circ Φ(z) = [P \circ Φ]_{≤ 2n+2}(z) + \mathcal{O}_k, n(2n + 3; |λ - 1|\omega^{2n+2}, p). \]
Since \([P \circ Φ]_{≤ 2n+2} = [Φ \circ h]_{≤ 2n+2}\) by construction of \(Φ\), we obtain that
\[ Φ \circ h(z) = P \circ Φ(z) + \mathcal{O}_k, n(2n + 3; |λ - 1|\omega^{2n+2}, p) \]
and this completes the proof of the announced estimate of \(Φ \circ h - P \circ Φ\).

Now we may compute the estimate of the reminder \(Φ \circ h - Φ^{-1} - P\). Equation (3.39), the \((Z\text{-product})\) and \((\text{restriction})\) axioms imply that
\[ Φ^{-1}(z) = \mathcal{O}_k, n(1; 1, p). \]
Thus the (product) axiom shows that
\[ (Φ \circ h - P \circ Φ) \circ Φ^{-1}(z) = \mathcal{O}_k, n(2n + 3; |λ - 1|\omega^{2n+2}, p), \]
which proves \(3\) in Proposition 3.16 and this ends the proof of the proposition.

### 3.5.3 Proof of Proposition 3.17

Proposition 3.17 follows immediately from the Birkhoff normal form of \(h\) (Proposition 3.16) with Lemma 3.19 and Lemma 3.20.

Through the whole section, we assume that \(2n ≥ k ≥ 1\), \(|b_m| = \mathcal{O}_E, j, n\left( |λ - 1|\omega^{2m} \right)\) for \(1 ≤ m ≤ n\) and \(ωρ ≃ |λ - 1|/q\). We also recall that \(λ^p \neq 1\) for \(1 ≤ p ≤ 2n + 2\) and \(Φ\) is a diffeomorphism satisfying \(|Φ(z)| ≃ |z|\) and
\[ Φ \circ h \circ Φ^{-1}(z) = λzV(|z|^2) + \mathcal{O}_k, n(2n + 3; |λ - 1|\omega^{2n+2}, p), \text{ with } V(s) = 1 + i \sum_{m=1}^{n} b_m s^m. \]

**Lemma 3.19.** If \(h\) is symplectic on a neighbourhood of zero and the Taylor expansion of the Jacobian \(J(Φ) = |\partial Φ|^2 - |\partial Φ|^2\) does not contain any power of the form \(|z|^{2ℓ}\), for \(1 ≤ ℓ ≤ n\), then we have \(J(Φ)(z) = 1 + \mathcal{O}(|z|^{2n+2+2})\) and
\[ Φ \circ h \circ Φ^{-1}(z) = λz \exp\left( i \sum_{ℓ=1}^{n} γ_ℓ |z|^{2ℓ} + \mathcal{O}_k, n(2n + 2; |λ - 1|\omega^{2n+2}, p) \right), \]
where \(γ_ℓ ∈ \mathbb{R}\) for each \((q, N, μ) ∈ E, j, n\) and \(γ_ℓ = \mathcal{O}_E, j, n\left( |λ - 1|\omega^{2ℓ} \right)\) for \(1 ≤ ℓ ≤ n\). In particular, we have \(γ_1 = b_1\).
Proof. we split the proof of the lemma into 4 steps. Steps 1, 2 and 4 follow Moser’s arguments.

**Step 1:** We set \( g(z) = \lambda z V(|z|^2) \); we check that \( J(g)(z) \) is a polynomial in \(|z|^2\).
Indeed, we have \( \partial g(z) = \lambda (V(|z|^2) + |z|^2 V'(|z|^2)) \) and \( \partial g(z) = \lambda z^2 V'(|z|^2) \), so

\[
J(g)(z) = |\partial g(z)|^2 - |\partial g(z)|^2 = |V|^2 (|z|^2) + 2 |z|^2 \text{Re}(V V')(|z|^2) = \frac{d}{dx} (|V(x)|^2).
\]

**Step 2:** We prove that \( J(\Phi)(z) = 1 + \mathcal{O}(|z|^{2n+2}) \). Since \( \Phi(z) = \mathcal{O}_{k,n}(1; 1, \rho) \), we have

\[
(\Phi \circ h)(z) = g(\Phi(z)) + \mathcal{O}_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho).
\]

Since \( h \) is symplectic, we have \( J(\Phi \circ h) = J(\Phi) \circ h \) and from the form of \( \Phi \circ h \) above, it follows that

\[
J(\Phi) \circ f = (J(g) \circ \Phi) \cdot J(\Phi) + \mathcal{O}(|z|^{2n+2}),
\]

(3.41)

We write

\[
J(\Phi)(z) = 1 + J_p(z) + \mathcal{O}(|z|^{p+1}) \text{ and } \frac{d}{dx} (|V(x)|^2) = 1 + \alpha x^2 + \mathcal{O}(x^{q+1}),
\]

where \( J_p \) is an homogeneous polynomial of degree \( p \leq 2n + 1, \alpha \in \mathbb{R} \) and \( 0 < q < n + 1 \), so \( J(g)(z) = 1 + \alpha |z|^{2q} + \mathcal{O}(|z|^{2q+2}) = J(g)(\Phi(z)) + \mathcal{O}(|z|^{2q+1}). \)

Since \( h(z) = \lambda z + \mathcal{O}(|z|^2) \), (3.41) implies that

\[
1 + J_p(\lambda z) = (1 + \alpha |z|^{2q} + \mathcal{O}(|z|^{2q+1})) \cdot (1 + J_p(z)) + \mathcal{O}(|z|^{p+1}).
\]

(3.42)

Now we use the hypothesis that \( J_p \) does not contain any power of \(|z|^2\):

- If \( 2q > p \) then (3.42) implies that \( J_p(z) = J_p(\lambda z) \), so \( J_p = 0 \);
- If \( 2q = p \) then (3.42) implies that \( J_p(\lambda z) - J_p(z) = \alpha |z|^{2q} \), so \( J_p = \alpha = 0 \);
- If \( 2q < p \) then (3.42) implies that \( \alpha |z|^{2q} = \mathcal{O}(|z|^{2q+1}) \), so \( \alpha = 0 \).

In any case we obtain that \( J_p = 0 \) if \( p \leq 2n + 1 \), so \( J(\Phi)(z) = 1 + \mathcal{O}(|z|^{2n+2}) \). Furthermore, we should note that this also implies that \( J(g)(x) = 1 + \mathcal{O}(|x|^{n+1}) \).

**Step 3:** we prove the existence of the complex coefficients \( \gamma_{\ell} \), for \( 1 \leq \ell \leq n \). We have \( z^{-1} = \mathcal{O}_{k,(-1; k!), \rho} \). Therefore the (product) axiom shows that

\[
\Phi \circ h \circ \Phi^{-1}(z) = \lambda z(V(|z|^2) + \varepsilon_0(z)), \text{ with } \varepsilon_0(z) = \mathcal{O}_{k,n}(2n + 2; |\lambda - 1| \omega^{2n+2}, \rho).
\]

Set \( L(z) = \log (1 + z) \), where \log denotes the principal value of the logarithm; we estimate \( L(1 + V(|z|^2) + \varepsilon_0(z)) \). Since \( |b_m| = \mathcal{O}_{E_\beta,n}(|\lambda - 1| \omega^{2m}) \subset \mathcal{O}_{E_\beta,n}(1) \) and \( \omega \rho = \mathcal{O}_{E_\beta,n}(|\lambda - 1| / q) \), we have

\[
|1 + V(|z|^2) + \varepsilon_0(z)| = \mathcal{O}_{E_\beta,n}(|\lambda - 1| \omega \rho)^2) = \mathcal{O}_{E_\beta,n}(|\lambda - 1|^3 / q^2).
\]

Therefore, up to changing \( \rho \) to \( \rho' \), with \( \rho' \leq \rho \) and \( \rho' \approx \rho \) small enough, we may assume that \( |V(|z|^2)| + \varepsilon_0(z) - 1| \leq 1/2 \) on \( \mathbb{D}(0; \rho) \). Thus \( \log (V(|z|^2) + \varepsilon_0(z)) \) is well defined.
Moreover, we have \( |\mathcal{C}^{2n+2} L(z)| \leq (2n + 2)!2^{2n+2} \) on \( \mathbb{D}(0, 1/2) \). Therefore the (Taylor expansion) axiom shows that
\[
L(z) = \sum_{\ell=0}^{2n} \frac{(-1)^\ell}{\ell!} z^{\ell+1} + L_1(z), \quad \text{with } L_1(z) = \mathcal{O}_{k,n}(2n + 2; 1, 1/2).
\]
Since \( |b_m| = \mathcal{O}_{E_{\beta,n}}(|\lambda - 1| \omega^{2m}) \) and \( \omega \rho = \mathcal{O}_{E_{\beta,n}}(1) \), we have
\[
V(|z|^2) + \varepsilon_0(z) = 1 + \mathcal{O}_{k,n}(1; |\lambda - 1| \omega, \rho).
\]
Since \( 2n + 2 \geq k \), we may apply the (composition) axiom, which shows that
\[
L_1(V(|z|^2) + \varepsilon_0(z) - 1) = \mathcal{O}_{k,n}(2n + 2; |\lambda - 1|^{2n+2} \omega^{2n+2}, \rho).
\]
Now we write \( L_0 = L - L_1 \) and \( P(z) = V(|z|^2) - 1 \); we estimate \( L_0(P(z) + \varepsilon_0(z)) \). Since \( P(z) = \mathcal{O}_{k,n}(0; |\lambda - 1| \omega, \rho) \) and \( \varepsilon_0(z) = \mathcal{O}_{k,n}(2n + 2; |\lambda - 1| \omega^{2n+2}, \rho) \), the (product) axiom shows that
\[
(P(z) + \varepsilon_0(z))^\ell = P_{\ell}(z) + \mathcal{O}_{k,n}(2n + 2; |\lambda - 1|^\ell \omega^{2n+2}, \rho).
\]
Furthermore, we have
\[
(P(z))^\ell = (i)^\ell \sum_{\mathbf{m} = (m_1, \ldots, m_\ell) \in N_\ell \subset N^\ell} \prod_{j=\ell} b_m,
\]
where the summation index \( \mathbf{m} = (m_1, \ldots, m_\ell) \in N_\ell \subset N^\ell \) above runs over all the tuples satisfying \( \sum_{q=1}^\ell m_q = j \) and where \( b_m = \prod_{q=1}^\ell b_{m_q} \). Therefore we have \( |b_m| = \mathcal{O}_{E_{\beta,n}}(|\lambda - 1|^\ell \omega^{2j}) \) for \( m \in N_j \), so the (restriction) axiom shows that there exist \( \gamma_{j,\ell} \in \mathbb{C} \) for each \( (q, N, \mu) \in E_{\beta,n} \) satisfying \( |\gamma_{j,\ell}| = \mathcal{O}_{E_{\bar{E}_n}}(|\lambda - 1|^\ell \omega^{2j}) \) and
\[
(P(z))^\ell = i \sum_{j=\ell}^{n} \gamma_{j,\ell} |z|^{2j} + \mathcal{O}_{k,\bar{E}_n}(2n + 2; |\lambda - 1|^\ell \omega^{2n+2}, \rho).
\]
Thus, setting \( \gamma_j = \sum_{\ell=1}^j \frac{(-1)^{\ell-1}}{\ell} \gamma_{j,\ell} \), it follows from the estimates above that
\[
|\gamma_j| = \mathcal{O}_{E_{\bar{E}_n}}(|\lambda - 1| \omega^{2j});
\]
\[
L_0(P(z) + \varepsilon_0(z)) = i \sum_{j=1}^{n} \gamma_j |z|^{2j} + \mathcal{O}_{k,n}(2n + 2; |\lambda - 1| \omega^{2n+2}, \rho).
\]
In particular, we have \( \lambda z (1 + ib_1 |z|^2) + \mathcal{O}(|z|^5) = \lambda z \exp(i\gamma_1 |z|^2 + \mathcal{O}(|z|^4)) \), so \( ib_1 = i\gamma_1 \).

**Step 4:** all that remains is to check that \( \gamma_\ell \in \mathbb{R} \). We have \( J(g)(x) = \frac{d}{dx}(x |V(x)|^2) = 1 + \mathcal{O}(x^{n+1}) \), so \( |V(x)|^2 = 1 + \mathcal{O}(x^{n+1}) \) and
\[
\left| \exp \sum_{j=1}^{n} i\gamma_\ell |z|^{2j} \right| = V(|z|^2) + \mathcal{O}(|z|^{2n+2}) = 1 + \mathcal{O}(|z|^{2n+2}).
\]
This last estimate holds true if and only if \( \text{Re} (i\gamma_\ell) = 0 \) for \( 1 \leq \ell \leq n \), which means that \( \gamma_\ell \in \mathbb{R} \). This ends the proof of the existence of the coefficients \( \gamma_j \) with the announced properties and the proof of the lemma is complete. \( \square \)
Lemma 3.20. There exist $\rho' > 0$ and $\rho'' > 0$, with $\rho' \leq \rho$, a function $\varepsilon : \mathbb{D}(0; \rho') \to \mathbb{R}$ and a diffeomorphism $\varphi$ from $\mathbb{D}(0; \rho')$ into an open set that contains $\mathbb{D}(0; \rho'')$ satisfying

i) $\rho' \asymp \rho \asymp \rho''/\sqrt{b_1}$ and $|\varphi(z)| \asymp |z| \sqrt{b_1}$ uniformly on $E_{\beta,n} \times \mathbb{D}(0; \rho')$;

ii) $(\varphi \circ \Phi \circ h \circ \Phi^{-1})(z) = \lambda z \exp(-2\pi i |z|^2 + \varepsilon(z))$ on $\mathbb{D}(0; \rho'')$;

iii) $\varepsilon(z) = \mathcal{O}_{k,n}(2n, |\lambda - 1|^{-n}, \rho \sqrt{b_1})$.

Proof. Lemma [3.19] shows that

$$\Phi \circ h \circ \Phi^{-1}(z) = \lambda z \exp \left( i \sum_{\ell=1}^{n} \gamma_{\ell} |z|^{2\ell} + \varepsilon_0(z) \right),$$

with $\varepsilon_0 \in \mathcal{O}_{k,E_n}(2n + 2, |\lambda - 1| \omega^{2n+2}, \rho)$.

With $\varphi_0(z) = z \left( 1 + \sum_{\ell=2}^{n} \frac{\gamma_{\ell}}{\gamma_1} |z|^{2\ell-2} + \frac{1}{\gamma_1} \text{Im} \left( \frac{\varepsilon_0(z)}{|z|^2} \right) \right)^{1/2}$, this is equivalent to the equation

$$\Phi \circ h \circ \Phi^{-1}(z) = \lambda z \exp \left( i \gamma_1 |\varphi_0(z)|^2 + \text{Re} \left( \varepsilon_0(z) \right) \right). \tag{3.43}$$

We estimate $\Phi \circ h \circ \Phi^{-1} \varphi_0^{-1}(z)$.

We set $\varepsilon_1(z) = (1/\gamma_1) \text{Im} \left( |z|^{-2} \varepsilon_0(z) \right)$. Since $z^{-1} = \mathcal{O}_k(-1; k!, \rho)$, the (product) axiom shows that $\varepsilon_1(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho) \subset \mathcal{O}_k(0; (\rho \omega)^{2n}, \rho)$. Since $\rho \omega = \mathcal{O}_{E,n}(1)$ and $|\gamma_1/\gamma_1| = \mathcal{O}_{E_{\beta,n}}(\omega^{2\ell})$, it follows that

$$\sum_{\ell=2}^{n} \frac{\gamma_{\ell}}{|\gamma_1|} |z|^{2\ell-2} + |\varepsilon_1(z)| = \mathcal{O}_{E_{\beta,n}}((\rho \omega)^2).$$

This implies that there exists $\rho' \leq \rho$ such that $\rho' \asymp \rho$ and

$$\sum_{\ell=2}^{n} \frac{\gamma_{\ell}}{\gamma_1} |z|^{2\ell-2} + |\varepsilon_1(z)| \leq \frac{1}{2} \text{ on } \mathbb{D}(0; \rho').$$

In particular, $\varphi_0(z)$ is well defined for $z \in \mathbb{D}(0, \rho')$. Furthermore, the (Taylor) axiom shows that $\sqrt{1 + z} = \sum_{\ell=0}^{k} \left( 1/\ell ! \right) z^{\ell} + \mathcal{O}(k + 1; 1, 1/2)$, so, by composition, there exist coefficients $c_{\ell} \in \mathbb{R}$, $1 \leq \ell \leq n$ and a real values reminder $\varepsilon_2$ such that

$$\left( 1 + \sum_{\ell=2}^{n} \frac{\gamma_{\ell}}{\gamma_1} |z|^{2\ell-2} + \varepsilon_1(z) \right)^{1/2} = 1 + \sum_{\ell=1}^{n-1} c_{\ell} |z|^{2\ell} + \varepsilon_2(z), \quad \text{with} \quad \left\{ \begin{array}{l} |c_{\ell}| = \mathcal{O}_{E_{\beta,n}}(\omega^{2\ell}), \\ \varepsilon_2(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho') \end{array} \right.$$

Thus we have proved that

$$\varphi_0(z) = z + \sum_{\ell=1}^{n-1} c_{\ell} z \left| z \right|^{2\ell} + \mathcal{O}_{k,n}(2n + 1; \omega^{2n}, \rho'). \tag{3.44}$$

We write $\varphi_0(z) = P_0(z) + z \varepsilon_2(z)$; up to shrinking $\rho'$ (keeping $\rho' \asymp \rho$), we may assume that there exist $c > 0$ and $C > 0$ such that $c = 1$, $z \varepsilon_2(z) = \mathcal{O}_k(2n + 1; C \omega^{2n}, \rho')$ and

$$\|P_0 - 1\|_\rho + \|\varepsilon P_0\|_\rho + C(\rho')^{2n} \omega^{2n} \leq c \omega \rho^2 \leq \frac{1}{8}.$$
It follows with the (inverse) axiom applied with \( \varepsilon = c\omega^2\rho^2 \) that \( \varphi_0 \) is a diffeomorphism from \( \mathbb{D}(0, \rho') \) onto an open set that contains a disk of the form \( \mathbb{D}(0, \rho'') \), such that \( \rho'' \asymp \rho \) and there exists a polynomial \( Q \) of degree \( 2n \) and valuation \( 2 \) such that

\[
\varphi_0^{-1}(z) = z + Q(z) + \mathcal{O}_{k,n}(2n + 1; \varepsilon/(\rho'')^2n, \rho''), \quad \text{with } \|Q\|_{\rho''} = \mathcal{O}_{E_{\beta,n}}(\rho''\varepsilon).
\]

\( Q \) is a sum of monomials of the form \( d_\nu' z'\nu' \), with \( 2 \leq |\nu| \leq 2n \), and we have

\[
|d_\nu'| \rho'^{|\nu|} \leq \|Q\|_{\rho''} = \mathcal{O}_{E_{\beta,n}}(\rho'^{2n+2}),
\]

so \( |d_\nu'| = \mathcal{O}_{E_{\beta,n}}(\rho'^{2n-|\nu|} \omega^2) \) and \( d_\nu' z'\nu' = \mathcal{O}_{k,n}(2; \rho'' \omega^2, \rho'') \).

In a similar way, since we have \( \rho'' \asymp \rho \) and \( \varepsilon = \mathcal{O}_{E_{\beta,n}}(\rho^2 \omega^2) \), the (restriction) axiom shows that \( \mathcal{O}_{k,n}(2n + 1; \varepsilon/(\rho'')^{2n}, \rho'') \subset \mathcal{O}_{k,n}(2; \omega^2 \rho^2 \rho''^{2n-1}/\rho^{2n}, \rho'') = \mathcal{O}_{k,n}(2; \omega^2 \rho'', \rho'') \). Since \( \rho \omega = \mathcal{O}_{E_{\beta,n}}(1) \), we obtain that

\[
\varphi_0^{-1}(z) = z + \mathcal{O}_{k,n}(2; \omega^2 \rho'', \rho'') = z + \mathcal{O}_{k,n}(2; \omega, \rho'').
\] (3.46)

This with (3.43) and the (composition) axiom shows that

\[
\Phi \circ h \circ \Phi^{-1} \circ \varphi_0^{-1}(z) = \lambda \varphi_0^{-1}(z) \exp(i\gamma_1 |z|^2 + \varepsilon_3(z)),
\] (3.47)

with \( \varepsilon_3(z) = \text{Re}(\varepsilon_\theta \circ \varphi_0^{-1}(z)) = \mathcal{O}_{k,n}(2n + 2; |\lambda - 1| \omega^{2n+2}, \rho) \). (3.48)

We should also note that the (restriction) axiom shows that

\[
i\gamma_1 |z|^2 + \varepsilon_3(z) = \mathcal{O}_{k,n}(0; \omega^2 \rho^2, \rho) \subset \mathcal{O}_{k,n}(0, 1, \rho).
\]

Since \( \exp(z) = \sum_{\ell=0}^k z^\ell/\ell! + \mathcal{O}_{k,n}(k + 1, 1, 1) \), this implies that \( \exp(i\gamma_1 |z|^2 + \varepsilon_3(z)) = 1 + \mathcal{O}_{k,n}(0; \omega^2 \rho^2, \rho) \subset \mathcal{O}_{k,n}(0, 1, \rho) \). From this, (3.47) and (3.46), one can deduce that

\[
\Phi \circ h \circ \Phi^{-1} \circ \varphi_0^{-1}(z) = \mathcal{O}_{k,n}(1, 1, \rho).
\] (3.49)

We estimate \( \varphi_0 \circ \Phi \circ h \circ \Phi^{-1} \circ \varphi_0^{-1}(z) \).

We write \( f_0(z) = \Phi \circ h \circ \Phi^{-1} \circ \varphi_0^{-1}(z) \) and \( P_0(z) = z P_1(z) \), with \( P_1(z) = 1 + \sum_{\ell=1}^n c_\ell |z|^{2\ell} \) and \( \varphi_0(z) = z (P_1(z) + \varepsilon_2(z)) \). Since \( P_1 \) depends only on \( |z| \) and \( |f_0(z)| = |\varphi_0^{-1}(z)| \exp(\varepsilon_3(z)) \), we have

\[
P_1(f_0(z)) = P_1(\varphi_0^{-1}(z) e^{\varepsilon_3(z)}).
\]

Since \( |\lambda - 1| \omega^{2n+2} \rho^{2n+2} = \mathcal{O}_{E_{\beta,n}}(1) \), (3.48) shows with the (P-composition) and (restriction) axioms that \( \exp(\varepsilon_3(z)) = 1 + \mathcal{O}_{k,n}(2n + 2; |\lambda - 1| \omega^{2n+2}, \rho) \). Therefore the (product) axiom and (3.46) imply that

\[
\varphi_0^{-1}(z) e^{\varepsilon_3(z)} = \varphi_0^{-1}(z) + \mathcal{O}_{k,n}(2n + 3; |\lambda - 1| \omega^{2n+2}, \rho).
\]

From this and the estimates \( |c_\ell| = \mathcal{O}_{E_{\beta,n}}(\omega^{2\ell}) \), one may check that

\[
P_1(\varphi_0^{-1}(z) e^{\varepsilon_3(z)}) = P_1(\varphi_0^{-1}(z)) + \varepsilon_5(z),
\]

with \( \varepsilon_5(z) = \mathcal{O}_{k,n}(2n + 4; |\lambda - 1| \omega^{2n+4}, \rho) \). (3.50)
Since $P_1$ is real valued, so is $\varepsilon_5$. Moreover, (3.44), the (composition) axiom and (3.46) imply that

$$\varepsilon_2 \circ \varphi_0^{-1}(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho');$$

the estimate $|c_{\ell}| = \mathcal{O}_{E_{\beta,n}}(\omega^{2\ell})$ and (3.46) imply that

$$c_{\ell} |\varphi_0^{-1}(z)|^{2\ell} = \mathcal{O}_{k,n}(2\ell; \omega^{2\ell}, \rho') \subset \mathcal{O}_{k,n}(0; (\omega \rho)^{2\ell}, \rho).$$

Thus we obtain that $P_1(\varphi_0^{-1}(z)) + \varepsilon_2 \circ \varphi_0^{-1}(z) = 1 + \mathcal{O}_{k,n}(0; (\rho \omega)^{2}, \rho)$.

Since $\rho \omega = \mathcal{O}_{E_{\lambda,n}}(1)$, we may assume that $(P_1 + \varepsilon_2) \circ \varphi_0^{-1}(z)$ does not vanish, up to shrinking $\rho''$ (keeping $\rho'' = \rho$), and

$$((P_1 + \varepsilon_2) \circ \varphi_0^{-1}(z))^{-1} = 1 + \mathcal{O}_{k,n}(0; (\rho \omega)^{2}, \rho'') \subset \mathcal{O}_{k,n}(0,1, \rho).$$

From this discussion, it follows that

$$\varphi_0 \circ f_0(z) = f_0(z) \left( P_1(\varphi_0^{-1}(z)e^{3(z)}) + \varepsilon_4(z) \right)$$

$$= \lambda \varphi_0^{-1}(z) \exp \left( i \gamma_1 |z|^2 + \varepsilon_3(z) \right) \left( P_1(\varphi_0^{-1}(z)) + \varepsilon_5(z) + \varepsilon_4(z) \right)$$

$$= \lambda \varphi_0^{-1}(z) \exp \left( i \gamma_1 |z|^2 + \varepsilon_3(z) \right) \left( P_1(\varphi_0^{-1}(z)) + \varepsilon_2 \circ \varphi_0^{-1}(z) + \varepsilon_6(z) \right)$$

$$= \lambda \varphi_0^{-1}(z) \left( P_1 + \varepsilon_2 \right) (\varphi_0^{-1}(z)) \exp \left( i \gamma_1 |z|^2 + \varepsilon_3(z) \right) (1 + \varepsilon_7(z))$$

$$= \lambda z \exp \left( i \gamma_1 |z|^2 + \varepsilon_3(z) \right) (1 + \varepsilon_7(z))$$

with $\varepsilon_4 = \varepsilon_2 \circ f_0$, $\varepsilon_6 = \varepsilon_5 + \varepsilon_4 - \varepsilon_2 \circ \varphi_0^{-1}$ and $\varepsilon_7 = \varepsilon_6 \cdot \left( (P_1 + \varepsilon_2) \circ \varphi_0^{-1} \right)^{-1}$. Since $P_1$, $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon_5$ are real valued, so is $\varepsilon_7$. Furthermore, the (composition) axiom applied to (3.44) and (3.49) shows that $\varepsilon_4$ lies in $\mathcal{O}_{k,n}(2n; \omega^{2n}, \rho)$, and so $\varepsilon_6$ according to (3.50) and (3.51). With the (product) axiom, this estimate of $\varepsilon_6$ and (3.52) imply that $\varepsilon_7(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho)$.

Since $\log (1 + z) = \sum_{\ell = 0}^{k-1} (-1)^{\ell}/(\ell + 1) z^{\ell+1} + \mathcal{O}_{k,n}(k + 1; 1,1/2)$, the $(P \circ)$-composition shows that we may set $\varepsilon_8(z) = \log (1 + \varepsilon_7(z))$, up to shrinking $\rho''$ (keeping $\rho'' = \rho$), so $\varepsilon_8$ is real valued and $\varepsilon_8(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho)$ and

$$\varphi_0 \circ f_0(z) = \lambda z \exp \left( i \gamma_1 |z|^2 + \varepsilon_9(z) \right),$$

with $\varepsilon_9(z) = \varepsilon_3(z) + \varepsilon_8(z) = \mathcal{O}_{k,n}(2n; \omega^{2n}, \rho)$ according to (3.48) and the (restriction) axiom; we note that $\varepsilon_9$ is real valued.

Definition of $\varphi$. We set $\varphi_1(z) = z \sqrt{\gamma_1/2\pi}$ on a disk of the form $D(0; c\rho/\sqrt{\gamma_1})$ with $c > 0$ small enough (keeping $c \approx 1$) so $\varepsilon = \varepsilon_9 \circ \varphi_1^{-1}$ is well defined; we set $\varphi = \varphi_1 \circ \varphi_0$. Thus we have

$$\varphi \circ \Phi \circ h \circ \Phi^{-1} \circ \varphi^{-1}(z) = \varphi_1 \circ \varphi_0 \circ f_0 \circ \varphi_1^{-1}(z) = \lambda z \exp \left( -2\pi i |z|^2 + \varepsilon(z) \right).$$

Since we have $z \sqrt{2\pi/\gamma_1} = \mathcal{O}_{k,E_n}(1; \frac{1}{\sqrt{\gamma_1}}, \rho \sqrt{\gamma_1}) = \mathcal{O}_{k,E_n}(1; \frac{1}{\omega} \left| \lambda - 1 \right|^{-\frac{1}{2}}, \rho \sqrt{\gamma_1})$, the (composition) axiom shows that

$$\varepsilon(z) = \varepsilon_9(z \sqrt{2\pi/\gamma_1}) = \mathcal{O}_{k,E_n}(2n; 1/\left| \lambda - 1 \right|^n, \rho \sqrt{\gamma_1})$$

and this completes the proof of the lemma. \qed
3.5.4 Proof of Proposition 3.18

We now focus on the transformation of \( h \) in polar coordinates, as stated in Proposition 3.18.

Proof. We consider the diffeomorphism \( \psi \) of Proposition 3.17 and set \( \Psi = \psi^{-1} \circ p \), with \( p(r, \theta) = \sqrt{r} \exp(2\pi i \theta) \). Since \( \psi^{-1} \) is a diffeomorphism from \( \mathbb{D}(0; \rho_2) \) onto its image, so is \( \psi \) from \( (0; \rho_2^2] \times \mathbb{T} \). This proves the point 1 of the corollary, with \( \rho' = \rho_2^2 \) and \( \bar{\varepsilon}(r, \theta) = r(\exp(2\varepsilon \circ p(r, \theta)) - 1) \).

The function \( \bar{\varepsilon} \) is real valued by construction. We now estimate \( \bar{\varepsilon} \). Proposition 3.17 shows that \( \rho_2 = \rho \omega |\lambda - 1|^1/2 \), so \( \rho_2^{2n} |\lambda - 1|^{-n} \approx (\rho \omega)^{2n} = \mathcal{O}_{E_\beta,n}(1) \). Therefore, up to shrinking \( \rho_2 \) (keeping \( \rho_2 = \rho \omega |\lambda - 1|^1/2 \)), we may assume that \( 2\varepsilon(\mathbb{D}(0; \rho_2)) \subset \mathbb{D}(0; 1) \). The \((P\text{-composition)})\ axiom of Lemma A.1, applied to \( \exp(z) - 1 = P(z) + \mathcal{O}_{k,n}(2n; 1, 1) \) composed with

\[
2\varepsilon(z) = \mathcal{O}_{k,n}(2n; C_1/\rho_2^{2n}, \rho_2), \text{ where } C_1 = |\lambda - 1|^{-n} \rho_2^{2n} \text{ and } P(z) = \sum_{\ell=1}^{2n-1} z^\ell / \ell!,
\]

shows that \( \exp(2\varepsilon(z)) = 1 + \mathcal{O}_{k,n}(2n; C_{01}/\rho_2^{2n}, \rho_2) \), with

\[
C_{01} = C_1^{2n} + \|P\|_{C_1} = \mathcal{O}_{E_\beta,n} \left( \sum_{\ell=1}^{2n} C_1^\ell \right) = \mathcal{O}_{E_\beta,n}(C_1), \text{ since } C_1 = \mathcal{O}_{E_\beta,n}(1).
\]

Thus we have proved that \( \exp(2\varepsilon(z)) - 1 = \mathcal{O}_{k,n}(2n; |\lambda - 1|^{-n}, \rho_2) \), hence the \((Z\text{-product)})\ axiom of A.1 applied twice shows that

\[
|z|^2 \left( \exp(2\varepsilon(z)) - 1 \right) = \mathcal{O}_{k,n}(2n + 2; |\lambda - 1|^{-n}, \rho_2).
\]

Applying the \((T\text{-composition)})\ axiom of Lemma A.2 to this map composed with \( p \), the estimate of \( \bar{\varepsilon} \) of the lemma follows, with \( \rho' = \rho_2^2 \).

We already know that \( \rho' = \rho_2^2 \approx \rho^2 \omega^2 |\lambda - 1| \approx |\lambda - 1|^3 / q^2 \). Furthermore, we have \( |p(r, \theta)| = \sqrt{r} \) and Proposition 3.17 shows that

\[
|\psi^{-1}(z)| \approx \frac{|z|}{\omega \sqrt{|\lambda - 1|}}, \text{ hence } |\Psi(r, \theta)| \approx \frac{\sqrt{r}}{\omega \sqrt{|\lambda - 1|}}.
\]

At last, we observe that \( \psi(0) = 0 \). Therefore the diffeomorphism \( \psi^{-1} \) maps any Jordan curve with \( 0 \) in the interior onto a Jordan curve with \( 0 \) in the interior. But the estimate of \( |\psi^{-1}(z)| \) above implies that the Jordan curve \( \psi^{-1}(\mathbb{D}(0, \sqrt{r})) \) lies between two circles centered at zero with radii comparable to \( \frac{1}{\omega} \sqrt{r} / |\lambda - 1| \). Therefore we have

\[
\text{area}(\psi^{-1}(\mathbb{D}(0, \sqrt{r}))) \approx \left( \frac{\sqrt{r}}{\omega \sqrt{|\lambda - 1|}} \right)^2 = \frac{r}{\omega^2 |\lambda - 1|}.
\]

This implies the last estimate of the proposition since \( p((0; r] \times \mathbb{T}) = \mathbb{D}(0, \sqrt{r}) \setminus \{0\} \) and the proof of Proposition 3.18 is complete. \( \square \)
3.6 The invariant curve theorem

The following statement is taken from [He02], VII.11.3 and VII.11.11.A.1.

**Theorem (Herman [He02]).** Assume \( \delta > 0 \) and set \( \mathcal{A}_\delta = \mathbb{T} \times [-\delta, \delta] \). Let \( \gamma \in \mathbb{R} \) and \( \Gamma > 0 \) satisfy

\[
0 < \Gamma \leq \inf_{q \geq 1, \, p \in \mathbb{Z}} \{ q \, |q \gamma - p| \}. \tag{3.53}
\]

Then there exist two constants \( c_1 > 0 \) and \( C_2 \geq 0 \) such that for any embedding \( F : \mathcal{A}_\delta \rightarrow \mathcal{A} \) of the form

\[
F(\theta, r) = (\theta + \gamma + r, r + \varphi(\theta, r)), \quad \text{with } \varphi \in C^4(\mathcal{A}_\delta)
\]
satisfying the essential circle intersection property and

\[
\max_{1 \leq i + j \leq 4} \left| c_i^j \varphi \right|_{C^0(\mathcal{A}_\delta)} \leq c_1 \Gamma^2, \tag{3.54}
\]

there is an unique function \( \psi \in W^{3,2}(\mathbb{T}) \) and a diffeomorphism \( f \in C^1(\mathbb{T}) \) with rotation number \( \gamma \) such that \( F(\theta, \psi(\theta)) = (f(\theta), \psi(f(\theta))) \) and we have

\[
\| \psi \|_{W^{3,2}(\mathbb{T})} \leq C_2 \Gamma^{-1} \max_{1 \leq i + j \leq 4} \left| c_i^j \varphi \right|_{C^0(\mathcal{A}_\delta)}. \tag{3.55}
\]

If \( \delta / \Gamma \geq 0.123 \) then \( c_1 = 14 \) and \( C_2 = 0.097 \) are suitable for any \( \gamma \) satisfying (3.53).

This theorem requires a few comments.

1) We recall that \( F : \mathcal{A}_\delta \rightarrow \mathcal{A} \) satisfies the essential circle intersection property provided that each simple essential curve \( \mathcal{C} \subset \mathcal{A}_\delta \) (homotopy equivalent to the circle \( \{ r = 0 \} \)) satisfies \( F(\mathcal{C}) \cap \mathcal{C} = \emptyset \). In our case, since \( F_{q,N,\mu} \) is symplectic and fixes \( a_q \), this condition is automatically fulfilled.

2) Here \( W^{3,2}(\mathbb{T}) \) denotes the Sobolev space of all distributions \( \psi \in \mathscr{D}'(\mathbb{T}) \cap L^2(\mathbb{T}) \) with derivatives \( D^k \psi \) in \( L^2(\mathbb{T}) \) for \( 0 \leq k \leq 3 \), endowed with the norm

\[
\| \psi \|_{W^{3,2}(\mathbb{T})} = \left( \| D^3 \psi \|_{L^2(\mathbb{T})}^2 + \left\| \hat{\psi}(0) \right\|^2 \right)^{1/2}, \quad \text{with } \hat{\psi}(0) = \int_0^1 \psi(t) \, dt.
\]

In particular, it is standard to prove that \( W^{3,2}(\mathbb{T}) \) embeds in \( C^0(\mathbb{T}) \) (see Proposition IV.3.7 in [He02]) and

\[
\| \psi - \hat{\psi}(0) \|_{C^0(\mathbb{T})} \leq \frac{1}{12 \sqrt{210}} \| D^3 \psi \|_{L^2(\mathbb{T})}.
\]

Since here \( f(\theta) = \theta + \gamma + \psi(\theta) \) is a diffeomorphism of the circle with rotation number \( \gamma \), one can prove that \( f - \gamma \) has a fixed point (see [He01]). This implies that \( \psi \) should vanish at some point \( x_0 \in \mathbb{T} \). Therefore we have \( \left\| \hat{\psi}(0) \right\| = \| \psi(x_0) - \hat{\psi}(0) \| \leq \| \psi - \hat{\psi}(0) \|_{C^0(\mathbb{T})} \), hence

\[
\| \psi \|_{C^0(\mathbb{T})} \leq 2 \| \psi - \hat{\psi}(0) \|_{C^0(\mathbb{T})} \leq \frac{1}{6 \sqrt{210}} \| D^3 \psi \|_{L^2(\mathbb{T})} \leq \frac{1}{6 \sqrt{210}} \| \psi \|_{W^{3,2}(\mathbb{T})}. \tag{3.56}
\]

3) One says that the number \( \gamma \in \mathbb{R} \) is of constant type with Markoff constant at least \( \Gamma \) exactly when it satisfies (3.53). All we need to know about it is the following result (see IV.3.5 in [He02]).
Lemma 3.21 (Herman). There exists a constant \( c > 0 \) such that for all \( 0 < \eta < 1/2 \), if \( [a, b] \subset [0,1] \) satisfies \( |b - a| \geq \eta \) then \( [a, b] \) contains infinitely many numbers of constant type with Markoff constant at least \( c\eta \).

Notice that we may shrink \( c \) as we need; in the following, we shall take \( \frac{1}{c} \geq 0.123 \) in order to apply Herman’s theorem.

3.7 Conclusion of the proof of Theorem \( \mathcal{F} \)

Now we have all the ingredients to prove Theorem \( \mathcal{F} \) which follows immediately from the following proposition. We recall that \( B_{q,N} \) is a \( q \)-adapted box for \( G_{N,\mu} \) and that \( G_{N,\mu} = F_{q,N,\mu} \) on \( B_{q,N} \). The set \( E_{\beta,n} \) which appears in the statement is the one defined in (3.24b) on p. 49.

**Proposition 3.22.** There exists a real number \( \eta_0 > 0 \) and, for each \( (q, N, \mu) \in E_{\beta,n} \), a disc \( \Omega_{q,N,\mu} \subset B_{q,N} \) satisfying the following conditions.

i) If \( \mu q^5/N^4 < \eta_0 \) then \( F_{q,N,\mu}(\Omega_{q,N,\mu}) = \Omega_{q,N,\mu} \).

ii) \( \text{area}(\Omega_{q,N,\mu}) \approx \frac{\mu}{N^2} \).

**Proof.** We set \( n \geq 8 \), \( k = 4 \), \( \omega = q^4/N^3 \); we recall that the number \( \alpha_{q,N} \) introduced in Proposition 3.11 (on p. 17) satisfies \( \alpha_{q,N} \approx q^5/N^4 \), and \( \lambda = \exp(i\gamma_0) \) satisfies \( \lambda + \lambda^{-1} = 2 - \mu \alpha_{q,N} \) and \( |\lambda - 1| = \mu \alpha_{q,N} \).

- Since \( 2n + 2 \geq k \), Proposition 3.13 applied to \( F_{q,N,\mu} \) and \( 2n + 2 \) shows that for each \( (q, N, \mu) \in E_{\beta,n} \) there exist \( \rho > 0 \) and a map \( \Psi_0 \) from a disk \( \mathbb{D}(0; \rho) \) into \( B_{q,N} \) such that

\[
\Psi_0^{-1} \circ F_{q,N,\mu} \circ \Psi_0(z) = \lambda \left( \sum_{\nu = 2}^{2n+2} a_\nu (z + \bar{z})^{\nu} + O_{k,n} \left( 2n + 3, \omega^{2n+2}, \rho \right) \right),
\]

with \((-1)^{\nu-1} a_\nu \sim \omega^{\nu-1} \) and \( \rho \approx |\lambda - 1|^{-1} \). Since \( 2n \geq k = 4 \), we may apply Proposition 3.18 for each \( (q, N, \mu) \in E_{\beta,n} \), there exist \( \rho' > 0 \) and a map \( \Psi_1 \) from \( \mathbb{T} \times (0; \rho') \) into \( \mathbb{D}(0; \rho_{q,N,\mu}) \) satisfying

\[
\Psi_0^{-1} \circ F_{q,N,\mu} \circ \Psi_0 \circ \Psi_1(\theta, r) = \left( \frac{\gamma_0}{2\pi} + \theta + r, r + \bar{\epsilon}(r, \theta) \right),
\]

where \( \bar{\epsilon}(r, \theta) = O_{k,n} \left( n + 1; |\lambda - 1|^{-n}, \rho' \right) \) and \( \rho' \approx |\lambda - 1|^{-1} q^5/N^2 \).

- Since \( |\gamma_0| \approx |\lambda - 1| \approx \mu q^5/N^4 \), we may choose \( \eta_0 > 0 \) small enough so \( \rho' + \frac{|\gamma_0|}{2\pi} < 1 \). Lemma 3.21 shows that there exists \( r_0 \in [\rho'/3; 2\rho'/3] \) such that \( \gamma = \frac{\gamma_0}{2\pi} + r_0 \) is of constant type with Markoff constant \( \Gamma \geq \epsilon \rho'/3 \). We set \( \Psi_2(\theta, r') = (\theta, r_0 + r') \) and \( \delta = \rho'/3 \), so \( \Psi_2 \) defines an embedding from \( \mathbb{H}_\delta \) into \( \mathbb{T} \times (0; \rho') \) such that

\[
F(\theta, r') := \Psi_2^{-1} \circ \Psi_1^{-1} \circ \Psi_0^{-1} \circ F_{q,N,\mu} \circ \Psi_0 \circ \Psi_1 \circ \Psi_2(\theta, r') = \left( \gamma + \theta + r', r' + \bar{\epsilon}(\theta, r') \right),
\]

with \( \bar{\epsilon}(\theta, r') = \bar{\epsilon}(r_0 + r', \theta) \).
We apply Herman’s theorem of Section 3.6 to $F$ on $A_\delta$ with $c_1 = 14$, $C_2 = 0.097$ and $\Gamma = c \rho' / 3$. These constants $c_1$ and $C_2$ are suitable because $\delta / \Gamma = \frac{1}{c} \geq 0.123$, so we have

$$\max_{1 \leq i + j \leq 4} \left\| \frac{\partial^i \partial^j \psi}{\partial \theta^i \partial \theta^j} \right\|_{C^0(A_\delta)} \leq \max_{1 \leq i + j \leq 4} \left\| \frac{\partial^i \partial^j \psi}{\partial \theta^i \partial \theta^j} \right\|_{C^0(0, \rho') \times T} = O_{E, \beta, n}(|\lambda - 1|^{-n} \rho^{m-3}).$$

Since we have $|\lambda - 1| = \mu q^5 / N^4$, $|\lambda - 1|^{-n} \rho^{m-3} = |\lambda - 1|^{2n-9} / q^{2n-6}$ and $\Gamma^2 = \rho^2 = |\lambda - 1|^6 / q^4$ and since we assume $n \geq 8$, we may choose $\eta_0 > 0$ small enough so (3.54) is satisfied for all $(q, N, \mu) \in E_{\beta, n}$ verifying $\mu q^5 / N^4 \leq \eta_0$.

Herman’s theorem shows that there exists a map $\psi : \mathbb{T} \to \mathbb{R}$ such that $\mathcal{C} = \{ (\theta, \psi(\theta)) \}$ is globally invariant by $h(\theta, r) = (\gamma + \theta + r', r' + \varepsilon(\theta, r'))$ and (3.55) holds true. This with (3.56) implies that

$$\| \psi \|_{C^0(\mathbb{T})} = O_{E, \beta, n} \left( \max_{1 \leq i + j \leq 4} \left\| \frac{\partial^i \partial^j \psi}{\partial \theta^i \partial \theta^j} \right\|_{C^0(A_\delta)} \right) = O_{E, \beta, n}(|\lambda - 1|^{-n} \rho^{m-4}).$$

Since $|\lambda - 1|^{-n} \rho^{m-4} \asymp |\lambda - 1|^{2n-12} / q^{2n-4}$, $\delta \asymp |\lambda - 1|^3 / q^2$ and $n \geq 8$, we may choose $\eta_0 > 0$ small enough so $\| \psi \|_{C^0(A_\delta)} \leq \delta / 2$ for all $(q, N, \mu) \in E_{\beta, n}$ verifying $\mu q^5 / N^4 \leq \eta_0$.

Thus we have proved that the Jordan curve $\mathcal{C} = \{ \Psi_1 \circ \Psi_2(\theta, \psi(\theta)) \}$ is invariant by $F_0^{-1} \circ F_{\beta, n} \circ \Psi_0$. Since $\rho' / 3 \leq r_0 \leq 2 / 3 \rho'$ and $\| \psi \|_{C^0(\mathbb{T})} \leq \rho' / 6$, we have $\rho' / 6 \leq r_0 + \psi(\theta) \leq 5 \rho' / 6$ on $\mathbb{T}$, so the estimate of $\Psi_1$ in Proposition 3.13 shows that

$$\operatorname{area}(\operatorname{Int}(\mathcal{C})) \asymp \frac{\rho'}{\omega^2 |\lambda - 1|} \asymp \frac{|\lambda - 1|^2}{q^2 \omega^2}.$$ 

Therefore $\Omega_{q, N, \mu} = \Psi_0(\operatorname{Int}(\mathcal{C}))$ is invariant by $F_{q, N, \mu}$ and the point $[\mathbb{R}]$ in Proposition 3.13 indicates that

$$\operatorname{area}(\Omega_{q, N, \mu}) \asymp \frac{k |\lambda - 1|^2}{q^2 \omega^2} \asymp \frac{|\lambda - 1|}{q \omega N} \asymp \frac{\mu}{q \omega N} = \frac{\mu}{N^2};$$

this completes the proof of the proposition.
4 Coupling devices, multi-dimensional periodic domains, wandering domains

At this point of the paper, it only remains to be proven Theorem C stated in Section 1.2 and Part (ii) of Theorem D stated in Section 1.3. Both proofs will make use of a “coupling lemma” which is the object of Section 4.1.

4.1 Coupling devices

We quote here almost exactly Lemma 3.2 of [MS04], which was itself a simple adaptation of a result already present in [MS03]. Though very simple, this coupling lemma plays a crucial role in our constructions.

**Lemma 4.1.** Let \( m, m' \geq 1 \) be integers. Let \( F: \mathbb{A}^m \to \mathbb{R} \) and \( G: \mathbb{A}^{m'} \to \mathbb{R} \) be two diffeomorphisms, and let \( f: \mathbb{A}^m \to \mathbb{R} \) and \( g: \mathbb{A}^{m'} \to \mathbb{R} \) be two Hamiltonian functions which generate complete vector fields.

Suppose moreover that we are given \( q \geq 1 \) integer and \( \mathcal{V} \subset \mathbb{A}^{m'} \) such that \( \mathcal{V} \) is \( q \)-periodic for \( G \) (i.e. \( \mathcal{V} = G^q(\mathcal{V}) \)) and the “synchronization conditions”

\[
g(x') = 1, \quad dg(x') = 0, \quad g(G^s(x')) = 0, \quad dg(G^s(x')) = 0, \quad 1 \leq s \leq q - 1, \tag{4.1}
\]

hold for all \( x' \in \mathcal{V} \).

Then \( f \otimes g \) generates a complete Hamiltonian vector field and the diffeomorphism \( F := \Phi^{f \otimes g}(F \times G): \mathbb{A}^{m+m'} \to \mathbb{A}^{m+m'} \) satisfies

\[
F^{\ell q+s}(x, x') = \left( F^s \circ (\Phi^{f} \circ F^g)^{\ell}(x), \ G^{\ell q+s}(x') \right), \quad x \in \mathbb{A}^m, \ x' \in \mathcal{V}, \tag{4.2}
\]

for all integers \( \ell, s \in \mathbb{Z} \) such that \( 0 \leq s \leq q - 1 \).

We have denoted by \( f \otimes g \) the function \( (x, x') \mapsto f(x)g(x') \), and by \( F \times G \) the product diffeomorphism \( (x, x') \mapsto (F(x), G(x')) \).

**Proof.** See the proof of Lemma 3.2 in [MS04, p. 1631]. The point is that

\[
\Phi^{f \otimes g}(x, x') = \left( \Phi^g(x') f(x), \Phi^f(x) g(x') \right), \quad x \in \mathbb{A}^m, \ x' \in \mathbb{A}^{m'},
\]

(as proved in [MS03], using the invariance of both \( f \) and \( g \) by the Hamiltonian vector field generated by \( f \otimes g \)), so the synchronization conditions (4.1) easily imply (4.2). \( \Box \)

Notice that, under the assumptions of Lemma 4.1, the union

\[
\tilde{\mathcal{V}} := \mathcal{V} \cup G(\mathcal{V}) \cup \ldots \cup G^{q-1}(\mathcal{V}) \tag{4.3}
\]

is a disjoint union because, for any \( s \in \{1, \ldots, q - 1\} \), the synchronization conditions (4.1) say that \( \mathcal{V} \subset g^{-1}(1) \) and \( G^s(\mathcal{V}) \subset g^{-1}(0) \). Thus any \( x' \in \tilde{\mathcal{V}} \) can be written \( x' = G^s(x'_0) \) with uniquely determined \( s \in \{0, 1, \ldots, q - 1\} \) and \( x'_0 \in \mathcal{V} \); then (4.2) shows that 

\[
F^s(F^{k-s}(x), x'_0) = (x, x') \quad \text{and that}
\]

\[
F^k(x, x') = \left( F^{s_1} \circ (\Phi^f \circ F^g)^{\ell_1} \circ F^{k-s}(x), \ G^k(x') \right), \quad x \in \mathbb{A}^m, \ x' \in \tilde{\mathcal{V}},
\]

which completes the proof.
with $k + s = \ell_1 q + s_1$. In particular, the set $\mathbb{A}^m \times \hat{V}$ is invariant under $\mathcal{F}$ and the second projection makes $G|_{\hat{V}}$ a factor of $\mathcal{F}|_{\mathbb{A}^m \times \hat{V}}$. Note also that (4.2) yields

$$\mathcal{F}^q|_{\mathbb{A}^m \times \hat{V}} = (\Phi^q \circ F^q) \times (G|_{\hat{V}}).$$

(4.4)

Assuming furthermore that there is a subset $\mathcal{U} \subset \mathbb{A}^n$ which is periodic or wandering for $\Phi^q \circ F^q$, we easily obtain that $\mathcal{U} \times \hat{V} \subset \mathbb{A}^{m+m'}$ is periodic or wandering for $\mathcal{F}$. This is essentially the content of the following two corollaries.

**Corollary 4.2.** Let $\mathcal{F} = \Phi^q \circ (F \times G) : \mathbb{A}^{m+m'} \to \mathbb{A}^{m+m'}$ with $m, m', F, G, f, g, q$ and $\mathcal{V} \subset \mathbb{A}^{m'}$ as in Lemma 4.1 (in particular $\mathcal{V}$ is $q$-periodic for $G$ and the synchronization conditions (4.1) hold).

Assume now that the diffeomorphism $\Phi^q \circ F^q$ admits a $p$-periodic subset $\mathcal{U} \subset \mathbb{A}^n$, with a certain integer $p \geq 1$. Assume moreover that there exist sets $\mathcal{B} \subset \mathcal{B}_* \subset \mathbb{A}^m$ and $\mathcal{B}' \subset \mathcal{B}'_* \subset \mathbb{A}^{m'}$ such that

$$\mathcal{U} \subset \mathcal{B}, \quad (\Phi^q \circ F^q)^k(\mathcal{U}) \cap \mathcal{B}_* = \emptyset \quad \text{for } 1 \leq k \leq p - 1, \quad (4.5)$$

$$\mathcal{V} \subset \mathcal{B}', \quad G^k(\mathcal{V}) \cap \mathcal{B}'_* = \emptyset \quad \text{for } 1 \leq k \leq q - 1. \quad (4.6)$$

Then the product set $\mathcal{U} \times \mathcal{V} \subset \mathbb{A}^{m+m'}$ is $(pq)$-periodic for the diffeomorphism $\mathcal{F}$ and

$$\mathcal{U} \times \mathcal{V} \subset \mathcal{B} \times \mathcal{B}', \quad \mathcal{F}^k(\mathcal{U} \times \mathcal{V}) \cap (\mathcal{B}_* \times \mathcal{B}'_*) = \emptyset \quad \text{for } 1 \leq k \leq pq - 1. \quad (4.7)$$

*Proof.* Let $\psi := \Phi^q \circ F^q$. By (4.4), $\mathcal{F}^p(\mathcal{U} \times \mathcal{V}) = \left(\psi^p(\mathcal{U}), \; G^p(\mathcal{V})\right) = \mathcal{U} \times \mathcal{V}$ and this $(pq)$-periodic set is obviously contained in $\mathcal{B} \times \mathcal{B}'$.

Suppose that $k \in \mathbb{Z}$ and $\mathcal{F}^k(\mathcal{U} \times \mathcal{V}) \cap (\mathcal{B}_* \times \mathcal{B}'_*) \neq \emptyset$. We thus can find $(x, x') \in \mathcal{U} \times \mathcal{V}$ such that $z := \mathcal{F}^k(x, x') \in \mathcal{B}_* \times \mathcal{B}'_*$. By (4.2), the second projection of $z$ is $G^k(x')$, in view of (4.6) this implies that $k \in q\mathbb{Z}$, say $k = \ell q$. But, again by (4.2), the first projection of $z$ is thus $\psi^\ell(x)$, and (4.5) then implies $\ell \in p\mathbb{Z}$. Therefore $k \in pq\mathbb{Z}$ and (4.7) is proved. \hfill \square

**Corollary 4.3.** Let $\mathcal{F} = \Phi^q \circ (F \times G) : \mathbb{A}^{m+m'} \to \mathbb{A}^{m+m'}$ with $m, m', F, G, f, g, q$ and $\mathcal{V} \subset \mathbb{A}^{m'}$ as in Lemma 4.1 (in particular $\mathcal{V}$ is $q$-periodic for $G$ and the synchronization conditions (4.1) hold).

Assume now that the diffeomorphism $\Phi^q \circ F^q$ admits a wandering subset $\mathcal{U} \subset \mathbb{A}^n$. Then the product set $\mathcal{U} \times \mathcal{V} \subset \mathbb{A}^{m+m'}$ is wandering for the diffeomorphism $\mathcal{F}$.
see Figure 1.

Proof. Let \( \mathcal{W} := \mathcal{U} \times \mathcal{V} \). We show that \( \mathcal{F}^k(\mathcal{W}) \cap \mathcal{W} = \emptyset \) for arbitrary \( k \in \mathbb{Z}\setminus\{0\} \).

Suppose first that \( k \notin q\mathbb{Z} \). Then \( \mathcal{V} \cap G^k(\mathcal{V}) = \emptyset \), as already observed in (4.3). Thus \( x' \in \mathcal{V} \) implies \( G^k(x') \notin \mathcal{V} \), whence

\[
\mathcal{F}^k(x, x') \notin \mathcal{W} \quad \text{for all } x \in \mathbb{A}^m
\]

by (4.2), i.e., \( \mathcal{F}^k(\mathbb{A}^m \times \mathcal{V}) \cap (\mathbb{A}^m \times \mathcal{V}) = \emptyset \). In particular, \( \mathcal{F}^k(\mathcal{W}) \cap \mathcal{W} = \emptyset \) when \( k \notin q\mathbb{Z} \).

Suppose now that \( k = \ell q \) with \( \ell \in \mathbb{Z}\setminus\{0\} \). We have \( \mathcal{F}^k(\mathcal{W}) = ((\Phi^f \circ F^q)^\ell(\mathcal{W}), \mathcal{V}) \) by (4.4) and \( \mathcal{W} \) is wandering for \( \Phi^f \circ F^q \), hence \( (\Phi^f \circ F^q)^\ell(\mathcal{W}) \cap \mathcal{W} = \emptyset \), therefore \( \mathcal{F}^k(\mathcal{W}) \cap \mathcal{W} = \emptyset \) again. \( \square \)

4.2 Proof of Part (ii) of Theorem (periodic domains in \( \mathbb{A}^{n-1} \))

4.2.1 Overview of the method

For \( n \geq 3 \), we must construct an arbitrarily close to integrable system in \( \mathcal{B}^\alpha_L(\Phi_2^{1/2(r_2^2+\cdots+r_n^2)}) \) possessing a periodic polydisc of arbitrarily large period in \( \mathbb{A}^{n-1} \); the Gromov capacity of this polydisc must be bounded from below as below in (4.23) and “localization conditions” of the form (4.24)–(4.25) must hold for its orbit. The near-integrable system will be obtained by applying Corollary (4.2) with \( m = 1 \) and \( m' = n - 2 \). The period of the polydisc will be of the form \( Q = pq \), with

\[
p := \ell p_{j+2}, \quad \ell \in \mathbb{N} \text{ arbitrarily large, } \quad q := p_{j+3} \cdots p_{j+n}
\]

(recall that \( (p_j)_{j\geq1} \) is the prime number sequence), so that \( Q \) will be an integer multiple of \( N_j := p_{j+2}p_{j+3}\cdots p_{j+n} \), and the deviation of the system from \( \Phi_2^{1/2(r_2^2+\cdots+r_n^2)} \) will be \( O(1/N_j^2) \).

To apply Corollary (4.2), we must define a system \( F \), a function \( f \) and a \( p \)-periodic domain \( \mathcal{W} \) for \( \Phi^f \circ F \) in the first factor, \( \mathbb{A} \), and a system \( G \), a function \( g \) and a \( q \)-periodic domain \( \mathcal{V} \) for \( G \) in the second factor, \( \mathbb{A}^{n-2} \).

On the first factor, we will make use of Theorem (in a way very similar to the proof of Theorem (i)) in Section 3.2 to produce a system \( \Psi \in \mathcal{B}^\alpha_L(\Phi_2^{1/2r_2^2}) \) possessing a \( p \)-periodic disc \( \mathcal{W} \) in \( \mathbb{A} \), whose area admits a suitable bound from below and whose orbit is suitably localized. A simple rescaling of the action variable \( r_2 \) by the factor \( q \) will then yield a system of the form

\[
\psi = \Phi^f \circ F^q \quad \text{with } f \in G^\alpha_L(T) \text{ small, } \quad F \in \mathcal{B}_2^{\alpha,L}(\Phi_2^{1/r_2^2}),
\]

possessing a \( p \)-periodic disc \( \mathcal{W} \). The smallness of \( \|f\|_{\alpha,L} \) will be controlled by the choice of the “tuning parameter” \( \mu \) at the moment of using Theorem (i).

On the second factor, we will use a near-integrable system of the form

\[
G = G[3] \times \cdots \times G[n] \in \mathcal{B}_1^{\alpha,L}(\Phi_2^{1/2(r_2^2+\cdots+r_n^2)})
\]

(4.9)

where, for each \( \kappa, G[\kappa] \in \mathcal{B}_1^{\alpha,L}(\Phi_2^{1/2r_2^2}) \) has a \( p_{j+\kappa} \)-periodic disc \( \mathcal{V}[\kappa] \) with area suitably bounded from below and orbit suitably localized. Since \( p_{j+3}, \ldots, p_{j+n} \) are pairwise coprime.
and their product is \( q \), we shall have \( \mathcal{Y} = \mathcal{Y}^{[3]} \times \cdots \times \mathcal{Y}^{[n]} \) \( q \)-periodic for \( G \). Lemma [B.5] of the Appendix will then yield a “bump function” \( g \in G^{\alpha,L}(\mathbb{A}^{n-2}) \) satisfying the synchronization conditions relative to \( \mathcal{Y} \) and \( G \).

According to Corollary [4.2], the polydisc \( \mathcal{Y} \times \mathcal{Y} \) will thus be \( (pq) \)-periodic for \( \Phi^{p \otimes g} \circ (F \times G) \), which will be the desired near-integrable system. Notice that \( \| g \|_{\alpha,L} \) will be exponentially large, so we need to choose properly the tuning parameter \( \mu \) in the first step, so as to compensate the largeness of \( \| g \|_{\alpha,L} \) by the smallness of \( \| f \|_{\alpha,L} \) and ensure

\[
\delta^{\alpha,L}(\Phi^{p \otimes g} \circ (F \times G), \Phi^{\frac{1}{2}(r_1^2 + \cdots + r_n^2)}) = O(1/N_j^2).
\]

### 4.2.2 A \( p \)-periodic polydisc for a near-integrable system of the form \( \Phi^f \circ F^q \) in \( \mathbb{A} \)

Let \( \alpha > 1 \) and \( L > 0 \) be real.

We give ourselves reals \( \rho_0 > 2 \), \( L_0, \theta^* > 0 \) such that \( L_0 < \frac{1}{2} - \theta^* \), \( \delta := 1 \) and, as in Section 3.2, by means of Lemma [B.5] we pick \( 1 \)-periodic functions \( V \), \( (W_M)_{M \in \mathbb{N}^*} \) in \( G^{\alpha,L}(\mathbb{R}) \) which satisfy the assumptions (i)–(v) of Theorem [F]. In particular,

\[
W_M(\theta) := \frac{1}{2} \eta_M(\theta)(\text{dist}(\theta, \mathbb{Z}))^2, \quad \| W_M \|_{\alpha,L} \leq C_0 \exp\left(c(\alpha, L) M^{-\frac{1}{\alpha-1}}\right) \quad \text{for all } M \in \mathbb{N}^*,
\]

with some positive reals \( C_0 \) and \( c(\alpha, L) \).

We get \( C_1, C_2, C_3, C_4 > 0 \) fulfilling the conclusions of Theorem [F] setting

\[
P_{V/M^2}(\theta, r) := \frac{1}{2} r^2 + \frac{1}{M^2} V(\theta), \quad G_{M,\mu} := \Phi^{\mu W_M} \circ \Phi^{P_{V/M^2}}
\]

for every integer \( M \geq 1 \) and real \( \mu > 0 \) (as in (3.1)–(3.2)), Theorem [F] says that \( G_{M,\mu} \) has a \( p \)-periodic disc \( D_{p,M,\mu} \) for each integer \( p \geq C_1 M \) provided \( \mu < C_2 M^4/p^5 \), with area

\[
\text{area}(D_{p,M,\mu}) \geq C_3 \frac{\mu}{M^2}
\]

and orbit localized as in (3.4).

Let \( n \geq 3 \), \( j \geq 1 \) and \( \ell \geq C_1 \) be integers, and

\[
q := p_{j+3} \cdots p_{j+n}, \quad N_j := p_{j+2} q, \quad p := \ell p_{j+2},
\]

so that \( Q := pq = \ell N_j \) is an arbitrary multiple \( \geq C_1 N_j \) of \( N_j \). We define

\[
\mu_{j,\ell} := \min\left\{ \frac{C_2}{2\ell^3 p_{j+2}}, \frac{1}{(2p_{j+2})^n} \exp\left(-\frac{1}{2}(n-1)c(\alpha, L)(2p_{j+2})^{-\frac{1}{\alpha-1}}\right) \right\}.
\]

Notice that \( \mu_{j,\ell} < C_2 p_{j+2}^4/p^5 \). We may thus consider the map

\[
G_{p_{j+2},\mu_{j,\ell}} = \Phi^{\mu_{j,\ell} W_{p_{j+2}}} \circ \Phi^{\frac{1}{2}(r_1^2 + \cdots + r_n^2)}
\]

which has a well-defined \( p \)-periodic disc \( \mathcal{W}_{j,\ell} := D_{p_{j+2},\mu_{j,\ell}} \).
Lemma 4.4. Let
\[ \sigma: (\theta, r) \in \mathbb{A} \mapsto (\theta, qr) \in \mathbb{A}. \]
Then, for any Hamiltonian function of the form \( (\theta, r) \in \mathbb{A} \mapsto h(r) + v(\theta) \) with \( h(qr) = q^2 h(r) \), one has
\[ \sigma^{-1} \circ \Phi^{h+v} \circ \sigma = \Phi^{h(\theta + q^{-2}v)}. \]

Proof. This is a simple scaling property of the Hamiltonian flow already used in [MS03]. Since \( \sigma \) is not symplectic but conformal-symplectic, one needs to rescale the action variable \( r \) and the time: the identity \( \sigma^{-1} \circ \Phi^{h(\theta + v)} \circ \sigma = \Phi^{h(\theta + q^{-2}v)} \) is easily checked by differentiating both sides with respect to \( t \).

\[ \text{Applying Lemma 4.4 with } h(p) = \frac{1}{2} r^2, \text{ we get} \]
\[ \sigma^{-1} \circ \Phi^{\frac{1}{2} r^2 + p_j^2 V} \sigma = F^q_j, \quad F_j := \Phi^{\frac{1}{2} r^2 + \frac{1}{N^2} V} \] (4.16)

and, with \( h = 0 \),
\[ \sigma^{-1} \circ \Phi^{p_j V} \sigma = \Phi f_j, \quad f_j := q^{-1} \mu_j W_{p_j+2}. \] (4.17)

Therefore, the map
\[ \Phi f_j \circ F_j = \sigma^{-1} \circ G_{p_j+2} \mu_j \circ \sigma \] (4.18)
has a \( p \)-periodic disc \( \mathcal{U}_{j,l} := \sigma^{-1}(\mathcal{U}_{j,l}) \). Inequality (4.12) entails
\[ \text{area}(\mathcal{U}_{j,l}) \geq C_3 \frac{\mu_j}{q p_j^2} \] (4.19)
and, because of (3.14),
\[ \mathcal{U}_{j,l} \subset B_{1/(2p_j+2)} \cap \mathbb{A}_{\frac{q}{N}}^+, \quad (\Phi f_j \circ F_j)^k(\mathcal{U}_{j,l}) \cap B_{1/p_j+2} = \emptyset \text{ for } 1 \leq k \leq p - 1. \] (4.20)

4.2.3 A \( q \)-periodic polydisc for a near-integrable system \( G \) in \( \mathbb{A}^{n-2} \)
We now need a near-integrable system \( G = G_j \) of the form (4.9) possessing a \( q \)-periodic polydisc in \( \mathbb{A}^{n-2} \). We shall take each factor of the form described in

Proposition 4.5. For any integer \( p \geq 2 \) and positive real \( \nu < 1/p \), the exact-symplectic map of \( \mathbb{A} \)
\[ \Lambda_{p,\nu} = \Phi^{\nu W} \circ \Phi^{\frac{1}{2} r^2} \] (4.21)
(with the same sequence of functions \( W_M \) as in (4.11)) has a \( p \)-periodic disc \( E_{p,\nu} \)
such that
\[ \text{area}(E_{p,\nu}) = \frac{\pi \nu}{128 p} \] (4.22)
and
\[ E_{p,\nu} \subset B_{1/2p}, \quad \Lambda_{p,\nu}(E_{p,\nu}) \cap B_{1/p} = \emptyset \text{ for } 1 \leq k \leq p - 1. \] (4.23)
Indeed, we shall define

\[ G_j := \Lambda_{p_{j+3}^{-1}} \times \cdots \times \Lambda_{p_{j+n}^{-1}} \quad \text{with} \quad \nu_j^{[k]} := \frac{1}{N_j^{\kappa} \|W_j\|_{\alpha,L}} \quad \text{for} \quad \kappa = 3, \ldots, n \]

and, since \( p_{j+3}, \ldots, p_{j+n} \) are pairwise coprime and their product is \( q \),

\[ \mathcal{V}_j := E_{p_{j+3},\nu_j^{[n]}} \times \cdots \times E_{p_{j+n},\nu_j^{[n]}} \subset \mathcal{B}_{1/2p_{j+3}} \times \cdots \times \mathcal{B}_{1/2p_{j+n}} \]

will be a \( q \)-periodic polydisc for \( G_j \) whose iterates are polydiscs satisfying

\[ G_j^k(\mathcal{V}_j) \cap (\mathcal{B}_{1/p_{j+3}} \times \cdots \times \mathcal{B}_{1/p_{j+n}}) = \emptyset \quad \text{for} \quad 1 \leq k \leq q - 1. \]

**Proof of Proposition 4.3.** The disc \( E_{p,\nu} \) will be a \( p \)-periodic filled ellipse centred at \( O_p := (\langle 0 \rangle, 1/p) \in \Lambda \). Recall that

\[ \Phi^{pr^2}(\langle \theta \rangle, r) = (\langle \theta + r \rangle, r), \quad \Phi^{pr^2}(\langle \theta \rangle, r) = (\langle \theta \rangle, r - \nu W_p(\theta)). \]

Let us set

\[ B_p := \{ (\langle x \rangle, \frac{1}{p} + y) \mid |x| \leq \frac{1}{8p}, \ |y| \leq \frac{1}{8p^2} \} = O_p + \left[ -\frac{1}{8p}, \frac{1}{8p} \right] \times \left[ -\frac{1}{8p^2}, \frac{1}{8p^2} \right]. \]

We will sometimes omit the canonical projection \( \langle \cdot \rangle : \mathbb{R} \rightarrow \mathbb{T} \) in our notations and consider \((x, y)\) as local coordinates near \( O_p \).

We first note that \( B_p \) is a \( p \)-adapted box for \( h(r) = \frac{1}{2} r^2 \) and \( \mathcal{B}_{1/p} \) in the sense of Definition 3.11 of Section 3.4. Indeed, for \( O_p + (x, y) \in B_p \), we have \( \Phi^t(O_p + (x, y)) = (\langle x + t(\frac{1}{p} + y) \rangle, \frac{1}{p} + y) \) and a straightforward computation shows that

\[ 1 \leq t \leq p - 1 \quad \Rightarrow \quad \frac{1}{2p} < \frac{1}{8p} + \frac{1}{p} - \frac{1}{8p^2} \leq x + t(\frac{1}{p} + y) \leq \frac{1}{8p} + (p - 1)(\frac{1}{p} + \frac{1}{8p^2}) < 1 - \frac{1}{2p}, \]

hence

\[ 1 \leq t \leq p - 1 \quad \Rightarrow \quad \Phi^{tpr^2}(O_p + (x, y)) \notin \mathcal{B}_{1/p}, \]

while the first component of \( \Phi^{pr^2}(O_p + (x, y)) \) is \( \langle x + 1 + py \rangle = \langle x + py \rangle \) and \( |x + py| \leq \frac{1}{4p} \), hence

\[ \Phi^{tpr^2}(O_p + (x, y)) \notin \mathcal{B}_{1/p}. \]

We now observe that the restrictions to \( B_p \) of \( \Lambda_{p,\nu} \) and its iterates up to the \( p \)th

\[ A_{k,p,\nu} := \Lambda_{p,\nu} \big|_{B_p}, \quad k = 0, \ldots, p \]

are affine in the coordinates \((x, y)\), and even linear for the \( p \)th iterate. Indeed, one checks by induction on \( k \in \{0, \ldots, p - 1\} \) that \( A_{k,p,\nu} = \Phi^{\frac{1}{2}kr^2} \big|_{B_p} \) : this clearly holds for \( k = 0 \) and, assuming it for \( 0 \leq k \leq p - 2 \), we have

\[ A_{k+1,p,\nu} = \Phi^{pr^2} \circ \Phi^{\frac{1}{2}kr^2} \big|_{B_p} = \Phi^{\frac{1}{2}(k+1)r^2} \big|_{B_p} \]

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because, by (4.27), $\Phi^{(k+1)r^2}(B_p)$ lies away from the support of $W_p$. Now, (4.28) says that $\Phi^{\frac{1}{2}pr^2}(B_p)$ is contained in $\mathcal{F}$, and this is a part of $\mathcal{A}$ which we may identify with $[-\frac{1}{4p}, \frac{1}{4p}] \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$, in which $W_p = \frac{1}{2} \theta^2$ in the coordinates $(\theta, r)$, whence for $k = p$

$$A_{p,p}\nu = \Phi^{\nu W_p} \circ \Phi^{\frac{1}{2}r^2} \circ A_{p-1,p}\nu = \Phi^{\nu W_p} \circ \Phi^{\frac{1}{2}pr^2}|_{B_p} = \Phi^{\frac{1}{2}\nu \theta^2} \circ \Phi^{\frac{1}{2}pr^2}|_{B_p}.$$  

We thus end up with

$$A_{k,p}\nu(O_p + (x, y)) = O_p + \begin{cases} (\langle x + k(y + \frac{1}{p})\rangle, y) & \text{if } 0 \leq k \leq p - 1, \\ (\langle x + py\rangle, y - \nu(x + py)) & \text{if } k = p. \end{cases} \quad (4.29)$$

Let us consider the linear transformation $A: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$A \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x + py \\ y - \nu(x + py) \end{pmatrix} = \begin{pmatrix} 1 & p \\ -\nu & 1 - \nu p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

According to (4.29), if $\mathcal{E}$ is a filled ellipse centred at the origin and invariant by $A$ and $O_p + \mathcal{E} \subset B_p$, then $O_p + \mathcal{E}$ is a $p$-periodic disc for $A_{p}\nu$ which satisfies (4.23). Elementary linear algebra shows that

$$A = P \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} P^{-1},$$

where $0 < \gamma := \arccos \left(1 - \frac{\nu p}{2}\right) < \frac{\pi}{2}$ (recall that $0 < \nu p < 1$) and

$$P \begin{pmatrix} X \\ Y \end{pmatrix} := \frac{1}{p \sin \gamma} \begin{pmatrix} 1 & 0 \\ -1 + \cos \gamma & \sin \gamma \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$ 

Hence, for each $r > 0$, $\mathcal{E}(r) := P\left(\{ (\frac{X}{Y}) \in \mathbb{R}^2 \mid X^2 + Y^2 < r^2 \}\right)$ is a filled ellipse of area $\pi r^2$, centred at the origin and invariant by $A$. We choose

$$E_{p}\nu := O_p + \mathcal{E}(r_{p}\nu), \quad r_{p}\nu := \frac{1}{8}(\frac{\nu p}{2})^{1/2}.$$ 

Using $\sin \gamma > (\nu p/2)^{1/2}$ and $\sin \gamma > 1 - \cos \gamma$, the property $O_p + \mathcal{E}(r_{p}\nu) \subset B_p$ is easily checked and the desired conclusions are fulfilled, including (4.22). \hfill \Box

### 4.2.4 Applying Corollary 4.2

From now on, taking advantage of the Prime Number Theorem, we assume that the parameter $j$ is large enough so that

$$p_j + 2 < p_j + 3 < \cdots < p_j + n \leq 2p_j + 2 \quad (4.30)$$

(this is the interest of having taken successive prime numbers for our $n - 1$ pairwise coprime integers). Recall that the other parameter is $\ell \geq C_1$, so that $Q = pq$ is an arbitrary multiple $\geq C_1N_j$ of $N_j$.

On the one hand, in $\mathcal{A}^m = \mathcal{A}$, we have a $p$-periodic disc $\mathcal{M}_{j,\ell}$ for the map $\Phi_{j,\ell} \circ F_j$ defined by (4.16)–(4.17), satisfying the localization condition (4.20). On the other hand,
in \(\mathbb{A}^{m'} = \mathbb{A}^{n-2}\), we have a \(q\)-periodic polydisc \(\mathcal{V}_j\) for the map \(G_j\) defined by (4.24), with localization conditions (4.25)–(4.26). We can thus apply Corollary 4.2 with
\[
g_j := \eta_{p_{j+3}} \otimes \cdots \otimes \eta_{p_{j+n}}
\] (4.31)
and get a map
\[
\Psi := \Phi_{f_j, q(\mathbb{G}^j)} \circ (F_{j, \ell} \times G_j) = \Phi_{f_j, q(\mathbb{G}^j)} \circ \Phi_{\nu_j^{[1]}} W_{p_{j+3}} \cdots W_{p_{j+n}} \circ \Phi_{\nu_j^{[2]} (r_j^2 + \cdots + r_n^2) + \frac{1}{N_j} V}
\] (4.32)
possessing a \(Q\)-periodic polydisc
\[
\mathcal{D} := \mathcal{U}_{j, \ell} \times \mathcal{V}_j \subset \mathcal{B} \times \mathcal{B}',
\]
(recall that \(Q = pq\), with
\[
\Psi^k(\mathcal{D}) \cap (\mathcal{B}_* \times \mathcal{B}_*) = \emptyset \quad \text{for } 1 \leq k \leq Q - 1.
\]
The last two properties coincide with (1.24)–(1.25), since all iterates of \(\mathcal{D}\) are in fact polydiscs (in view of (4.2) and (4.24)).

To end the proof of Theorem (D)(ii), we just need to check that

(i) the map \(\Psi\), which clearly belongs to \(\mathcal{D}^{\alpha, L}(\Phi_{\nu_j^{[2]} (r_j^2 + \cdots + r_n^2) + \frac{1}{N_j} V})\), is indeed close to integrable, namely
\[
\delta^{\alpha, L}(\Psi, \Phi_{\nu_j^{[2]} (r_j^2 + \cdots + r_n^2) + \frac{1}{N_j} V}) \leq \frac{C_0 + n - 2 + \|V\|_{\alpha, L}}{N_j^2},
\] (4.33)
(ii) the Gromov capacity of \(\mathcal{D}\) is not too small, namely
\[
C_G(\mathcal{D}) \geq \tilde{C} \min \left\{ \frac{1}{Q^5 N_j^{4 - \frac{2}{n-1}}, \frac{1}{N_j^{2 - \frac{2}{n-1}}}} \exp \left(-\tilde{c} N_j^{\frac{1}{[m-1](n-1)}}\right) \right\},
\] (4.34)
with \(\tilde{C} := \min \left\{ \frac{C_0 C_3}{2}, 2^{-n-1} C_3 \right\}\) and \(\tilde{c} := 2^{\frac{1}{n-1}} (n-1) c(\alpha, L)\).

Indeed, this will yield (1.19) and (1.23), up to an obvious change of notation for the constants “\(C_2\)” and “\(C_3\)”, by taking for \(c\) a large enough function of \(\tilde{c}, \alpha, \) and \(n\).

**Proof of (i).** In view of (4.17) and (4.31), we can estimate the norm of \(f_{j, \ell} \otimes g_j\) thanks to (4.11) and (4.16):
\[
\|f_{j, \ell} \otimes g_j\|_{\alpha, L} \leq q^{-1} \mu_{j, \ell} C_0 \exp \left( p_j^{n-1} + \cdots + p_{j+n}^{n-1} \right)
\]
which, by (1.14) and (4.30), is \(\leq \frac{C_0}{(2p_{j+2})^q q} \leq \frac{C_0}{N_j^q}\). In view of (4.32) and the choice of \(\nu_j^{[3]}, \ldots, \nu_j^{[n]}\) in (4.24), this yields (4.33). \(\square\)
Proof of (ii). Since \( N_j = p_j + 2 q \) and \( q = p_j + 3 \cdot p_j + n \), (4.30) yields \( p_j^{n-1} < N_j < (2p_j+2)^{n-1} \), whence
\[
\frac{1}{2} N_j^{n-1} < p_j^{n-1} < N_j^{n-1}, \quad N_j^{n-1} < q < 2 N_j^{n-2}.
\]

Now, by (1.16), \( C_G(\mathcal{D}) = \min \{ \text{area}(\mathcal{U}_{j, \ell}), \text{area}(E_{p_{j+3}, p_j^{[\kappa]}}, \ldots, \text{area}(E_{p_{j+n}, p_j^{[\kappa]}}) \} \). In view of (4.22), for \( 3 \leq \kappa \leq n \),
\[
\text{area}(E_{p_{j+k}, p_j^{[\kappa]}}) = \frac{\pi p_j^{[\kappa]}}{128 p_j^{[\kappa]}} \geq \frac{\pi}{2^{1/3} C_0} \frac{1}{N_j^{1/2} p_j^{[\kappa]}} \exp \left( -c(\alpha, L) p_j^{n-1} \right)
\]
\[
\geq \frac{\pi}{2^{1/3} C_0} \frac{1}{N_j^{1/2} p_j^{[\kappa]}} \exp \left( -c(\alpha, L) N_j^{\frac{n-1}{(n-1)(\alpha-1)}} \right),
\]
while (4.19) yields \( \text{area}(\mathcal{U}_{j, \ell}) \geq \min\{A, B\} \) with
\[
A := \frac{C_2 C_3}{2 \ell^s p_j^{n+2} q} = \frac{C_2 C_3 N_j^2 q^2}{2 \ell^s} \geq \frac{C_2 C_3}{2} \frac{1}{Q^s} N_j^{4-\frac{2}{n-1}}
\]
and \( B := \frac{C_3}{2^{n+1} N_j^{2+\frac{2}{n-1}}} \exp \left( - (n-1)c(\alpha, L)(2p_j+2)^{n-1} \right) \) larger than
\[
\frac{C_3}{2^{n+1} N_j^{2+\frac{2}{n-1}}} \exp \left( - 2 \frac{1}{n-1} (n-1)c(\alpha, L) N_j^{\frac{1}{(n-1)(\alpha-1)}} \right),
\]
whence (4.31) follows. \( \square \)

The proof of Theorem D(ii) is now complete.

4.3 Proof of Theorem C (lower bounds for wandering domains in \( \mathbb{A}^n \))

4.3.1 Overview of the proof

Here is the more precise statement which, as explained in Section 1.3, implies Theorem C.

**Theorem C.** Let \( n \geq 2 \) be integer. Let \( \alpha > 1 \) and \( L > 0 \) be real, and let \( h(r) := \frac{1}{2}(r_1^2 + \cdots + r_n^2) \). Then there exist a positive real \( c_* \) and a sequence \( (\Phi_j)_{j \geq 0} \) of exact symplectic diffeomorphisms of \( \mathbb{A}^n \) which belong to \( \mathcal{P}_2^{\alpha, L}(h) \) if \( n = 2 \) and to \( \mathcal{P}_3^{\alpha, L}(h) \) if \( n \geq 3 \), such that each \( \Phi_j \) admits a wandering polydisc \( \mathcal{W}_j \subset \mathbb{A}^n \) and
\[
\varepsilon_j := \delta^{\alpha, L}(\Phi_j, \Phi^h) \to 0 \quad \text{and} \quad C_G(\mathcal{W}_j) \geq \exp \left( - c_* \left( \frac{1}{\varepsilon_j} \right)^{\frac{1}{(n-1)(\alpha-1)}} \right). \tag{4.35}
\]

The rest of Section 4.3 is devoted to the proof of Theorem C.
The idea is as follows. Each near-integrable system \( \Phi_j \) and wandering polydisc \( \mathcal{W}_j \) will be obtained by means of Corollary 4.3 in the form
\[
\Phi_j := \Phi^{j \otimes q} \circ (F \times G), \quad \mathcal{W}_j := \mathcal{U} \times \mathcal{V},
\]
where
\[
G := \Psi_{j,q_j} : \mathbb{A}^{n-1} \otimes, \quad \mathcal{V} := \mathfrak{D}_{j,q_j} \subset \mathbb{A}^{n-1}
\]
will be provided by Theorem with a suitably chosen large integer \( q_j \), and the function \( g \) will be chosen so as to satisfy the synchronization conditions (4.1) for the orbit of \( \mathfrak{D}_{j,q_j} \), while
\[
F := \Phi^{\frac{1}{2} r^2} : \mathbb{A} \otimes, \quad f := \frac{1}{q_j} U, \quad \mathcal{U} := W_{q_j} \subset \mathbb{A}
\]
stem from

**Proposition 4.6.** Let \( 0 < \rho < 1/2 \) and let \( U \in C^\infty(\mathbb{T}) \) be a function such that
\[
U'\langle x \rangle = -1 + x \quad \text{for } x \in [-\rho, \rho].
\]
Then there exist a real \( C_0 > 0 \) such that, for each integer \( q \geq 1 \), the diffeomorphism \( \Phi_{-U} \circ (\Phi^{\frac{1}{2} r^2})^q \) of \( \mathbb{A} \) admits a wandering disc \( W_q \subset \mathbb{A} \) such that
\[
\text{area}(W_q) = \frac{C_0}{q}.
\]

We give the proof of Proposition 4.6 in Section 4.3.2. Then, in Section 4.3.3 we indicate how to choose \( q_j \) and check that \( \Phi_j \) and \( \mathcal{W}_j \) have all the desired properties.

### 4.3.2 Standard maps with wandering discs in \( \mathbb{A} \)—Proof of Proposition 4.6

To prove Proposition 4.6, we first consider the so-called “standard map”
\[
\mathcal{S} := \Phi^U \circ \Phi^{\frac{1}{2} r^2} : \mathbb{A} \otimes
\]
\[i.e.
\mathcal{S}(\theta, r) = (\theta + r, r - U'(\theta + r)), \quad (\theta, r) \in \mathbb{T} \times \mathbb{R}.
\]
Since \( \mathcal{S}(\theta, r + 1) = \mathcal{S}(\theta, r) \), by passing to the quotient, \( \mathcal{S} \) induces a map \( \mathcal{S}^*: \mathbb{T} \times \mathbb{T} \otimes \). Our assumption on \( U \) entails that the origin \( (\langle 0 \rangle, \langle 0 \rangle) \) of \( \mathbb{T}^2 \) is a fixed point of \( \mathcal{S}^* \), in a neighbourhood of which \( \mathcal{S}^* \) is linear:
\[
\mathcal{S}^*(\langle x \rangle, \langle y \rangle) = (\langle x + y \rangle, \langle -x \rangle), \quad x \in [-\frac{\rho}{2}, \frac{\rho}{2}], \quad y \in [-\frac{\rho}{2}, \frac{\rho}{2}].
\]
The eigenvalues being \( e^{\pm \frac{i \pi}{4}} \), the origin is an elliptic fixed point surrounded by invariant ellipses. Let \( W^* \) denote any invariant filled ellipse contained in the projection onto \( \mathbb{T}^2 \) of \([-\frac{\rho}{2}, \frac{\rho}{2}] \times [-\frac{\rho}{2}, \frac{\rho}{2}] \), and let \( C_0 := \text{area}(W^*) \).

We define \( W \) to be the lift of \( W^* \) in \( \mathbb{A} \) which contains the point \( O := (\langle 0 \rangle, 0) \). Since \( U'(0) = -1 \), one sees that \( \mathcal{S}^*(\langle 0 \rangle, k) \) for all \( k \in \mathbb{Z} \), hence the orbit of \( W \) under \( \mathcal{S} \) consists of pairwise disjoint filled ellipses centred at the points \( \mathcal{S}^*(\langle 0 \rangle, k) \):
\[
\mathcal{S}^*(W) = (\langle 0 \rangle, k) + W, \quad k \in \mathbb{Z}.
\]
In particular, \( W \) is a wandering disc for \( \mathscr{S} \).

We now obtain a wandering disc for
\[
\mathcal{S}_q := \Phi^{\frac{1}{2}U} \circ (\Phi^{\frac{1}{2}r^2})^q,
\]
for any integer \( q \geq 1 \), by means of the scaling
\[
\sigma : (\theta, r) \in \mathbb{A} \mapsto (\theta, qr) \in \mathbb{A}.
\]
Indeed, Lemma [4.4] with \( h(r) = \frac{1}{2}r^2 \) and \( v = 0 \) yields \( \sigma^{-1} \circ \Phi^{\frac{1}{2}r^2} \circ \sigma = \Phi^{\frac{1}{2}qr^2} \) and, with \( h = 0 \) and \( v = U \), \( \sigma^{-1} \circ \Phi^U \circ \sigma = \Phi^{qr^{-2}U} \), whence
\[
\mathcal{S}_q = \sigma^{-1} \circ \mathcal{S} \circ \sigma
\]
and \( W_q := \sigma^{-1}(W) \) is a wandering disc for \( \mathcal{S}_q \). Clearly, \( \text{area}(W_q) = C_0/q \) and the proof of Proposition [4.6] is complete.

**Remark 4.7.** The diffeomorphism \( \mathcal{S} \) is “dynamically far” from the integrable map \( \Phi^{\frac{1}{2}r^2} \). Indeed, \( \mathcal{S} \) cannot possess any essential invariant curve \( \mathcal{C} \), otherwise the orbit of each point in the complement \( \mathbb{A} \setminus \mathcal{C} \) would be contained in a single connected components of \( \mathbb{A} \setminus \mathcal{C} \), and this is not the case for the orbit of \((0, 0)\). As a consequence, \( \mathcal{S}_q \) has no essential invariant curve. However, when \( q \to \infty \), \( \mathcal{S}_q \) is a small perturbation of the integrable map \( \Phi^{\frac{1}{2}qr^2} \). This is not in contradiction with the KAM theorem: the torsion of \( \Phi^{\frac{1}{2}qr^2} \) tends to infinity when \( q \to \infty \), which makes the KAM threshold tend to 0.

### 4.3.3 Proof of Theorem \( \mathbb{C} \)

Let \( n \geq 2 \) be integer. Let \( \alpha > 1 \) and \( L > 0 \) be real.

On the one hand, Theorem \( \mathbb{D} \) yields reals \( c, C_1, C_2, C_3 > 0 \) and a sequence \( (\Psi_{j,q}) \) in \( \mathcal{D}^{\alpha,L}((\Phi^{\frac{1}{2}(v_2^2 + \cdots + r_n^2)}) \) such that \( \delta^{\alpha,L}(\Psi_{j,q}, \Phi^{\frac{1}{2}(v_2^2 + \cdots + r_n^2)}) \leq \frac{C_2}{N_j^2} \), where \( N_j = p_{j+2} \cdots p_{j+n} \) is arbitrarily large, and each \( \Psi_{j,q} \) for \( q \) arbitrary integer multiple of \( N_j \) not smaller than \( C_1 N_j \) has a \( q \)-periodic polydisc \( \mathcal{D}_{j,q} \subset \mathbb{A}^{n-1} \) satisfying (1.22) or (1.24). In view of the localization conditions (1.22) or (1.24)–(1.25) satisfied by the orbit of \( \mathcal{D}_{j,q} \) independently of \( q \), we define
\[
g_j := \eta_{p_{j+2}} \otimes \cdots \otimes \eta_{p_{j+n}}
\]
(making use of the “bump functions” of Lemma [3.3]), so that \( g_j \) satisfies the synchronization conditions (1.1) for the orbit of \( \mathcal{D}_{j,q} \) under \( \Psi_{j,q} \).

On the other hand, choosing \( \rho := 1/6 \) and \( U \in C^{\alpha,L}(\mathbb{T}) \) satisfying (4.30) \( (e.g., U(\theta) := \eta_3(\theta)(-x + \frac{1}{2} x^2) \), where \( x \) is the lift of \( \theta \) in \((\frac{1}{2}, \frac{1}{2}]\)), we get from Proposition [4.6] a wandering disc \( W_{q} \subset \mathbb{A} \) for \( \Phi^{\frac{1}{2}U} \circ (\Phi^{\frac{1}{2}r^2})^q \) for each integer \( q \geq 1 \).

We take \( j \) large enough so that (4.30) holds (thanks to the Prime Number Theorem), and define
\[
q_j := M_j N_j, \quad M_j := \left[ N_j \|U\|_{\alpha,L} \|g_j\|_{\alpha,L} \right] + 1, \quad (4.41)
\]
where \([\ ]\) denotes the integer part. Applying Corollary [4.5] with the data (4.37)–(4.38), we obtain a wandering domain
\[
\mathcal{W}_j := W_{q_j} \times \mathcal{D}_{j,q_j}
\]
for the map
\[ \Phi_j := \Phi_{1/2}^\perp \circ (\Phi_{1/2} \times \Psi_{j,q_j}). \]

If \( n = 2 \), then \( \Psi_{j,q_j} \in \mathcal{P}_2^n(\Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \), hence \( \Phi_j \in \mathcal{P}_2^n(\Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \). If \( n = 3 \), then \( \Psi_{j,q_j} \in \mathcal{P}_3^n(\Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \), hence \( \Phi_j \in \mathcal{P}_3^n(\Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \). In all cases,
\[ \varepsilon_j := \delta^n(\Phi_j, \Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \leq \| \frac{1}{q_j} U \otimes g_j \|_{\alpha,L} + \delta^n(\Psi_{j,q_j}, \Phi_{1/2}^+(r_1^2 + \cdots + r_n^2)) \leq \frac{1 + C_2}{N_j^2}. \]

We conclude by bounding from below the Gromov capacity of \( W_j \) which, according to (1.16), is
\[ C_G(W_j) = \min \{ \text{area}(W_{q_j}), C_G(\mathcal{D}_{j,q_j}) \}. \]

We have \( q_j \leq 2 \| U \|_{\alpha,L} N_j^2 \| g_j \|_{\alpha,L} \) and, by (1.16) and (4.30),
\[ \| g_j \|_{\alpha,L} \leq \exp \left( c(\alpha,L)(\frac{1}{p_j} + \cdots + \frac{1}{p_j^q}) \right) \leq \exp \left( (n-1)c(\alpha,L)(2p_j+2)^{\frac{1}{n-1}} \right). \]

Since \( p_j^{n+1} < N_j \), we thus can find \( C, \tilde{c} > 0 \) independent of \( j \) such that
\[ q_j \leq C \exp \left( \tilde{c} N_j^{\frac{1}{n-1}(\alpha-1)} \right). \]

By (4.40), this yields
\[ \text{area}(W_{q_j}) \geq \frac{C_0}{C} \exp \left( - \tilde{c} N_j^{\frac{1}{n-1}(\alpha-1)} \right). \]

On the other hand,
\[ C_G(\mathcal{D}_{j,q_j}) \geq C_3 \min \left\{ \frac{1}{q_j} N_j^{4 - \frac{2}{n-1}}, \exp \left( - c N_j^{\frac{1}{n-1}(\alpha-1)} \right) \right\} \]

and, again by (4.42), one can find \( C', \tilde{c}' > 0 \) independent of \( j \) such that
\[ \frac{1}{q_j} N_j^{4 - \frac{2}{n-1}} \geq C' \exp \left( - c N_j^{\frac{1}{n-1}(\alpha-1)} \right). \]

We end up with
\[ C_G(W_j) \geq \min \left\{ \frac{C_0}{C}, C_3 C', C_3 \right\} \exp \left( - \max \{ \tilde{c}, \tilde{c}', c \} N_j^{\frac{1}{n-1}(\alpha-1)} \right) \]

and thus can find \( c_0 > 0 \) independent of \( j \) such that (4.35) holds.

This concludes the proof of Theorem C.'
Appendices

A Algebraic operations in $\mathcal{O}_k$

For $\ell \in \mathbb{R}$, we denote by $[\ell] \in \mathbb{Z}$ the integral part such that $[\ell] \leq \ell < [\ell] + 1$. For all $\nu = (\nu_1, \nu_2) \in \mathbb{N}^2$, we set $|\nu| = \nu_1 + \nu_2$, $\partial^\nu f = \partial^\nu_1 \partial^\nu_2 f$, $z^\nu = z_1^\nu_1 z_2^\nu_2$ and $\nu! = \nu_1! \nu_2!$.

If $P$ is a polynomial of the form $P(z) = \sum_{|\nu| \leq n-1} a_\nu z^\nu$ then we set $\|P\|_\tau = \sum_{|\nu| \leq n-1} |a_\nu| \tau^{|\nu|}$.

Lemma A.1. Assume $(n, m, \ell, \ell_1, \ell_2) \in \mathbb{N}^2 \times \mathbb{R}^3$. Then the spaces $\mathcal{O}_k$ of Section 3.5.1 satisfy the following axioms.

(Restriction) $\mathcal{O}_k(\ell; C, \tau) \subset \mathcal{O}_{k-1}(\ell; C, \tau) \cap \mathcal{O}_k(\ell - 1; C, \tau)$;

(Derivative) If $f \in \mathcal{O}_k(\ell; C, \tau)$ then $\partial^\alpha \partial^\beta f \in \mathcal{O}_{k-\alpha-\beta}(\ell - \alpha - \beta; C, \tau)$ for all $(\alpha, \beta) \in \mathbb{N}^2$ such that $\alpha + \beta \leq k$;

(Primitive) If $f \in \mathcal{O}_0(\ell + 1; C_0, \tau)$, $\partial f \in \mathcal{O}_0(\ell; C_1, \tau)$ and $\partial f \in \mathcal{O}_k(\ell; C_2, \tau)$ then $f \in \mathcal{O}_{k+1}(\ell + 1; C, \tau)$, with $C = \max(C_0, C_1, C_2)$;

(Product) If $f \in \mathcal{O}_k(\ell_1; C_1, \tau)$ and $g \in \mathcal{O}_k(\ell_2; C_2, \tau)$ then we have $fg \in \mathcal{O}_k(\ell_1 + \ell_2; 2^k C_1 C_2, \tau)$;

(Z-Product) If $f \in \mathcal{O}_k(\ell; C, \tau)$ then $zf(z) \in \mathcal{O}_k(\ell + 1; C(k + 1), \tau)$;

(Polynomial) If $P$ is a polynomial of degree $m$ and if $0 \leq n \leq m$ then we have $P = [P]_{n-1} + \mathcal{O}_k(n; (k + 1)^{m-n} \|P\|_\tau / \tau^n, \tau)$;

(P-product) If $f(z) = P(z) + \mathcal{O}_k(n+1, C/\tau^{n+1}, \tau)$ and $g = Q(z) + \mathcal{O}_k(n+1, C'/\tau^{n+1}, \tau)$, where $P$ et $Q$ are two polynomials of degree $n$, then we have

$$\begin{align*}
(fg)(z) = R(z) + \mathcal{O}_k(n + 1, C''/\tau^{n+1}, \tau),
\end{align*}$$

where $C'' \leq (k + 1)^n (\|P\|_\tau C' + \|Q\|_\tau C) + 2^k CC' + (k + 1)^{2n} \|P\|_\tau \|Q\|_\tau$ and $R$ is a polynomial of degree $n$ satisfying $\|R\|_\tau \leq \|P\|_\tau \|Q\|_\tau$;

(Lipschitz) If $\partial f \in \mathcal{O}_k(n; C_1, \tau)$ and $\partial f \in \mathcal{O}_k(n; C_2, \tau)$ then $f$ is Lipschitz continuous near zero. Furthermore, $f - f(0) \in \mathcal{O}_{k+1}(n + 1; C, \tau)$, with $C = C_0 + C_1$ if $n = 0$ and $C = \max(C_1, C_2)$ if $n \geq 1$;

(Composition) Assume $n \geq k$. If $h \in \mathcal{O}_k(n; C_0, \tau_0)$, $f \in \mathcal{O}_k(m; C_1, \tau_1)$ and $f(\mathbb{D}(0, \tau_1)) \subset \mathbb{D}(0, \tau_0)$ then $h \circ f \in \mathcal{O}_k(nm; \alpha_k C_0 C_1^{\alpha_k}, \tau_1)$, with $\alpha_k = 2^{-k(k+1)}$;

(P-Composition) Assume $n \geq k$. If $h(z) = P(z) + \mathcal{O}_k(n + 1; C_0, \tau)$ and $f(z) = Q(z) + \mathcal{O}_k(n + 1; C_1/\rho^{n+1}, \rho)$, where $P$ and $Q$ are polynomial of degree $n$ with $Q(0) = 0$ and $f(\mathbb{D}(0, \rho)) \subset \mathbb{D}(0, \tau)$ then there exist a polynomial $R$ of degree $n$ and a constant $C_{01}$ satisfying $\|R\|_\rho \leq \|P\|_\rho \|Q\|_\rho$ and

$$\begin{align*}
h \circ f(z) = R(z) + \mathcal{O}_k(n + 1; C_{01}/\rho^{n+1}, \rho),
\end{align*}$$

with $C_{01} \leq 2^{k(k+1)/2} C_0 (C_1 + (k + 1)^n \|Q\|_\rho)^{n+1} + \|P\|_\rho C_1 \|Q\|_\rho^{n+1}$;
(Taylor Expansion) Assume 0 < \eta < 1 and \ n \geq k. If \ f \ is holomorphic on \ \mathbb{D}(0, \tau) \ and \ \partial^n f \in \mathcal{O}_0(0; C, \tau) \ then \ f(z) = T^n_f(z) + \mathcal{O}_k(n; C/(n - k)!, \tau), \ with \ T^n_f(z) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} \partial^\ell f(0) z^\ell; \n \n (Inverse) Assume \ m > k \geq 1. Then there exists two constant \ \beta_m \geq 0 \ and \ B_m \geq 0 \ such \ that \ if \ \Phi(z) = z + P(z) + \mathcal{O}_k(m; C, \tau), \ where \ P \ is \ a \ polynomial \ of \ degree \ m - 1 \ and \ valuation \ 2 \ satisfying \ \|\partial P\|_\tau + \|\partial^2 P\|_\tau + C \tau^{m-1} \leq \epsilon, \ with \ 2 \epsilon + \epsilon^2 \leq 1/2, \ then \ \Phi \ is \ a \ diffeomorphism \ from \ \mathbb{D}(0, \tau) \ onto \ a \ set \ containing \ \mathbb{D}(0, \tau(1-\epsilon)) \ such \ that \ \Phi^{-1} \ is \ of \ the \ form \n \n \Phi^{-1}(z) = z + Q(z) + \mathcal{O}_k(m; B_m \epsilon / \tau^{m-1}, \tau(1-\epsilon)), \n \n where \ Q \ is \ a \ polynomial \ of \ degree \ m - 1 \ with \ valuation \ 2 \ and \ \|Q\|_\tau \leq \beta_m \epsilon \tau. \n
Proof. The proof of the axioms from (restriction) to (Z-product) follows directly from the definition or by easy inductions over \ k. \n
\bullet \ We \ prove \ the \ (Polynomial) \ axiom. \ Let’s \ write \n
\[ P(z) = [P]_{\leq n-1} + \sum_{n\leq|\nu| \leq m} a_\nu z^\nu. \n\]

Since \ z \ and \ \overline{z} \ belong to \ \mathcal{O}_k(1; 1, \tau), \ the \ (Z-product) \ axiom \ implies \ by \ an \ easy \ induction \ that \ z^\nu = \mathcal{O}_k(|\nu|; (k + 1)^{|\nu|-1}, \tau). \ Therefore, \ for \ \nu \in \mathbb{N}^2 \ satisfying \ m \geq |\nu| \geq n, \ the \ (restriction) \ axiom \ shows \ that \n
\[ z^\nu = \mathcal{O}_k(|\nu|; (k + 1)^{|\nu|-1}, \tau) \subset \mathcal{O}_k(n; (k + 1)^{m-1} \tau^{|\nu|-n}, \tau). \n\]

This implies that \ [P]_{\geq n} = \mathcal{O}_k(n; (k + 1)^{m-1} C / \tau^n, \tau), \ with \ C \leq \sum_{n\leq|\nu| \leq m} |a_\nu| \tau^{|\nu|}, \ so \ C \leq \|P\|_\tau, \ and \ the \ proof \ of \ the \ axiom \ is \ complete. \n
\bullet \ We \ prove \ the \ (Lipschitz) \ axiom. \ Assume \ that \ \partial f \in \mathcal{O}_k(n; C_1, \tau) \ and \ \partial^2 f \in \mathcal{O}_k(n; C_2, \tau). \ This \ implies \ that \n
\[ |df(z) \cdot Z| = |\partial f(z) Z + \partial^2 f(z) Z| \leq (C_1 + C_2) |z|^{n} |Z|. \n\]

Therefore the derivative of \ h \ is bounded by \ (C_1 + C_2) \tau^n \ on \ \mathbb{D}(0, \tau), \ so \ f \ extends to a \ Lipschitz-continuous \ function \ on \ this \ disc. \ Furthermore, \ we \ obtain \ that \n
\[ |f(z) - f(0)| \leq \frac{C_1 + C_2}{n + 1} |z|^{n+1} \leq C_0 |z|^{n+1}, \n\]

with \ C_0 = C_1 + C_2 \ if \ n = 0 \ and \ C_0 = \max(C_1, C_2) \ if \ n \geq 1. \ By \ the \ (primitive) \ axiom, \ this \ completes \ the \ proof \ of \ the \ (Lipschitz) \ axiom. \n
\bullet \ We \ prove \ the \ (composition) \ axiom \ by \ induction \ over \ k. \ The \ condition \ on \ f(\mathbb{D}(0; \tau_1)) \ shows \ that \ h \circ f \ is \ well \ defined \ on \ \mathbb{D}(0, \tau_1) \ and \ since \ n \geq 0, \ we \ have \ |h \circ f(z)| \leq C_0(C_1 |z|^m)^n = C_0 C_1^n |z|^{mn}. \ Thus \ we \ obtain \ that \ h \circ f \in \mathcal{O}_0(nm; \alpha_0 C_0 C_1^n, \tau_1), \ with \ \alpha_0 \leq 1, \ and \ the \ axiom \ is \ proved \ for \ k = 0.
Assume that $k \geq 1$, so $\hat{\partial} h(f)$ and $\hat{\partial} h(f)$ are in $\mathcal{O}_{k-1}((n-1)m; \alpha_{k-1} C_0 C_1^{n-1}, \tau_1)$, nI that case the (product) axiom shows that

$$
\begin{align*}
\hat{\partial}(h \circ f) &= \hat{\partial} h(f) \cdot \hat{\partial} f + \hat{\partial} h(f) \cdot \hat{\partial} f \in \mathcal{O}_{k-1}(nm - 1; 2^k \alpha_{k-1} C_0 C_1^n, \tau_1), \\
\hat{\partial}(g \circ f) &= \hat{\partial} g(f) \cdot \hat{\partial} f + \hat{\partial} h(f) \cdot \hat{\partial} g \in \mathcal{O}_{k-1}(nm - 1; 2^k \alpha_{k-1} C_0 C_1^n, \tau_1).
\end{align*}
$$

This implies that $h \circ f \in \mathcal{O}(nm; \alpha_k C_0 C_1^n, \tau_1)$, with $\alpha_k \leq \max(\alpha_0, 2^k \alpha_{k-1})$ and the (composition) axiom follows immediately.

• We prove the $(P\text{-composition})$ axiom. We write $h = P + \varepsilon_0$ and $f = Q + \varepsilon_1$, so $h \circ f = P \circ f + \varepsilon_0 \circ f$. Note that $1 \in \mathcal{O}_k(0; 1, \rho)$, so the $Z\text{-product}$ axiom shows that $z^\nu = \mathcal{O}(|\nu|; (k + 1)^{|\nu|}, \rho)$ for $\nu \in \mathbb{N}^2$. Now we write

$$
Q(z) = \sum_{1 \leq |\nu| \leq n} b_\nu z^\nu = \sum_{1 \leq |\nu| \leq n} \mathcal{O}_k(|\nu|; (k + 1)^{|\nu|} |b_\nu|, \rho)
$$

$$
= \sum_{1 \leq |\nu| \leq n} \mathcal{O}_k(1; (k + 1)^{|\nu|} |b_\nu| \rho^{|\nu| - 1}, \rho)
$$

$$
= \mathcal{O}_k(1; (k + 1)^n \frac{\|Q\|\rho}{\rho}, \rho).
$$

Thus we obtain that $f(z) = \mathcal{O}_k(1; \frac{(k+1)^n \|Q\|\rho + C_1}{\rho}, \rho)$ and the (composition) axiom implies that

$$
\varepsilon_0 \circ f(z) = \mathcal{O}_k \left( n + 1; 2^{\frac{2(k+1)^n \|Q\|\rho + C_1}{\rho}} \left( \frac{(k + 1)^n \|Q\|\rho + C_1}{\rho} \right)^{n+1}, \rho \right). \tag{A.1}
$$

Now we estimate $P \circ f$.

For $1 \leq j \leq \ell \leq n$ we have $Q^j = \mathcal{O}_k(0; (k + 1)^n \|Q\|\rho^j, \rho)$, so

$$
f^\ell = (Q + \varepsilon_0)^\ell = Q^\ell + \sum_{j=0}^{\ell-1} \binom{\ell}{j} Q^j \varepsilon_0^{\ell-j}
$$

$$
= Q^\ell + \sum_{j=0}^{\ell-1} \binom{\ell}{j} \mathcal{O}_k(\ell - j(n + 1); (k + 1)^n \|Q\|\rho) \left( \frac{2^k C_1}{\rho^{n+1}} \right)^{\ell-j}, \rho
$$

$$
= Q^\ell + \mathcal{O}_k \left( n + 1; \sum_{j=0}^{\ell-1} \binom{\ell}{j} (k + 1)^n \|Q\|\rho \left( \frac{2^k C_1}{\rho^{n+1}} \right)^{\ell-j} \rho^{n+1}, \rho \right).
$$

We can decompose $Q^\ell = [Q^\ell]_{\leq n} + [Q^\ell]_{> n}$ in its part of degree $n$ and its part of valuation $n + 1$. Since $[Q^\ell]_{> n} \in \mathcal{O}_k(n + 1; (k + 1)^n \|Q\|\rho / \rho^{n+1})$ and $\|Q^\ell\| \rho \leq \|Q\|\rho^n$, it follows that

$$
f^\ell = [Q^\ell]_{\leq n} + \mathcal{O}_k \left( n + 1; \left( (k + 1)^n \|Q\|\rho + 2^k C_1 \right)^{\ell} / \rho^{n+1}, \rho \right).
$$
Now we write \( P(z) = \sum_{|\nu| \leq n} a_{\nu} z^\nu \) and we note that \( \| Q^\nu \|_{\rho} \leq (\| Q \|_{\rho})^{\| \nu \|} \), so
\[
P \circ f = \sum_{|\nu| \leq n} a_{\nu} [Q^\nu]_{\leq n} + \mathcal{O}_k \left( n + 1; \sum_{|\nu| \leq n} |a_{\nu}| \left( \frac{(k + 1)^n \| Q \|_{\rho} + 2^k C_1}{\rho^{n+1}} \right), \rho \right) = R + \mathcal{O}_k \left( n + 1; \frac{1}{\rho^{n+1}} \| P \|_{(k+1)^n \| Q \|_{\rho} + 2^k C_1}, \rho \right),
\]
with \( R = \sum_{|\nu| \leq n} a_{\nu} [Q^\nu]_{\leq n} \) and \( \| R \|_{\rho} \leq \sum_{|\nu| \leq n} |a_{\nu}| (\| Q \|_{\rho})^{\| \nu \|} = \| P \|_{\rho} \).

This with (A.1) implies the \((P\text{-composition)}\) axiom.

- The \((\text{Taylor expansion})\) axiom directly follows from the Taylor expansion theorem, which shows that
\[
\left| \partial^j f(z) - \sum_{\ell=0}^{n-1-j} \frac{1}{\ell!} \partial^\ell f(0) z^\ell \right| \leq \frac{|z|^{n-j}}{(n-j)!} C.
\]

- The proof of the \((\text{inverse})\) axiom proceeds in several steps. We first prove the existence of the diffeomorphism, then we estimate its derivatives.

**Existence of \( \Phi^{-1} \).** Assume that \( |w| \leq \tau (1 - \varepsilon) \) and set \( \varphi_w(z) = w - \Phi(z) + z \). This implies that \( \| d \varphi_w(z) \| \leq \| \partial P \|_\tau + \| \partial^2 P \|_\tau + 2 C \tau^{m-1} \leq 2 \varepsilon \leq 1/2 \) and \( |\varphi_w(z)| \leq |w| + C \tau^m \leq |w| + \varepsilon \tau \leq \tau \) on \( \mathbb{D}(0, \tau) \). Therefore \( \varphi_w : \mathbb{D}(0, \tau) \to \mathbb{D}(0, \tau) \) is \((2\varepsilon)\)-Lipschitz. By Picard’s theorem, it follows that the equation \( \varphi_w(z) = z \), so \( \Phi(z) = w \), has an unique solution \( z \in \mathbb{D}(0, \tau) \) if \( w \in \mathbb{D}(0, \tau(1-\varepsilon)) \). Thus \( \mathbb{D}(0, \tau(1-\varepsilon)) \) lies in \( \Phi(\mathbb{D}(0, \tau)) \), so \( \Phi^{-1} \) is a diffeomorphism from \( \mathbb{D}(0, \tau(1-\varepsilon)) \) into \( \mathbb{D}(0, \tau) \).

We now prove by induction over \( k \geq 0 \) the existence of a constant \( \beta_{k,m} \) and of \( Q \) such that such that \( \Phi^{-1}(z) = z + Q(z) + \mathcal{O}_k(m; \beta_{k,m} \varepsilon \tau^{m-1}, \tau(1-\varepsilon)) \). We set \( \Phi_0(z) = \Phi(z) - z = w - z \), \( \Psi(w) = \Phi^{-1}(w) = z \) and \( \Psi_0(w) = \Psi(w) - w \).

**The boot strapping equation.** We write the derivatives of \( \Psi \circ \Phi(z) = z \) as
\[
\begin{align*}
1 &= (\partial \Psi) \circ \Phi \cdot \partial \Phi + (\partial \Psi) \circ \Phi \cdot \partial \Phi, \\
0 &= (\partial \Psi) \circ \Phi \cdot \partial \Phi + (\partial \Psi) \circ \Phi \cdot \partial \Phi.
\end{align*}
\]

Therefore we have
\[
\left( \frac{\partial \Psi}{\partial \Phi} \right) \circ \Phi = \left( 1 + \partial \Phi_0 1 + \partial \Phi_0 \right)^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \frac{1}{J(\Phi)} \left( \begin{array}{c} 1 + \partial \Phi_0 \\ -\partial \Phi_0 \end{array} \right), \tag{A.2}
\]

where \( J(\Phi) = |1 + \partial \Phi_0|^2 - |\partial \Phi_0|^2 = 1 + 2 \text{Re} (\partial \Phi_0) + |\partial \Phi_0|^2 - |\partial \Phi_0|^2 \).

We now estimate the right hand side of Equation (A.2). First we observe that \( \partial \Phi_0 \) and \( \partial \Phi_0 \) are bounded on \( \mathbb{D}(0; \tau) \) by \( \| \partial P \|_\tau + C \tau^{m-1} \) and \( \| \partial P \|_\tau + C \tau^{m-1} \) respectively, so they both lie in \( \mathcal{O}_0(0; \varepsilon, \tau) \). It follows that
\[
J(\Phi) = 1 + \mathcal{O}_0(0; 2 \varepsilon + \varepsilon^2, \tau) \subset 1 + \mathcal{O}_0(0; 1/2, \tau).
\]
We have $\partial \Phi_0 = \partial P + \partial \tilde{\Phi}_k (m-1; C, \tau)$ and $\partial \Phi_0 = \partial P + \partial \tilde{\Phi}_k (m-1; C, \tau)$, so the $(P$-product) shows that

$$|\partial \Phi_0|^2 = P_0 + \partial \tilde{\Phi}_k(m-1; C, \tau); \quad -|\partial \Phi_0|^2 = P_1 + \partial \tilde{\Phi}_k(m-1; C, \tau),$$

with $\|P_0\|_\tau \leq \|\partial P\|_\tau^2$, $\|P_1\|_\tau \leq \|\partial P\|_\tau^2$,

$$C_0 \leq 2k^{m-1}C_\tau C_\tau -2m^2 + k^{2m-2} + \|\partial P\|_\tau^2,$$

and $C_0' \leq 2k^{m-1}C_\tau C_\tau -2m^2 + k^{2m-2} + \|\partial P\|_\tau^2$.

Therefore the polynomial $J_0 = 2\text{Re}(\partial P) + P_0 - P_1$ has degree $m - 2$ and satisfies

$$J(\Phi) = 1 + J_0 + \partial \tilde{\Phi}_k(m-1; C'/\tau^{m-1}, \tau),$$

with $\|J_0\|_\tau \leq \|\partial P\|_\tau^2 + \|\partial P\|_\tau^2 \leq 2 \|\partial P\|_\tau + \varepsilon^2$

and $C' \leq 2C_\tau -2m^2 + C_0 + C_0' \leq 2C_\tau -2m^1 + C_0''$.

Here we have

$$C_0'' \leq C_0 + C_0'$$

$$\leq 2k^{m-1}C_\tau C_\tau -2m^2 + k^{2m-2} + \|\partial P\|_\tau^2 + \|\partial P\|_\tau^2$$

$$\leq \left( k^{m-1} + \|\partial P\|_\tau^2 \right)^2 \leq \max \left( 2k, k^{2m-2} \right) \varepsilon^2.$$

Now we estimate $1/J(\Phi)$. The (Taylor expansion) axiom shows that

$$\frac{1}{1+z} = \sum_{\ell=0}^{m-2} (-1)^\ell z^\ell + \partial \tilde{\Phi}_k(m-1; 2^m(m-1)!/(m-k)!/1/2).$$

Therefore the (P-composition) axiom applied to $(1 + z)^{-1} - 1$ and $J(\Phi) - 1$ shows that there exist a polynomial $J_1$ of degree $m - 2$ and a constant $C''$ such that

$$1/J(\Phi) = 1 + J_1 + \partial \tilde{\Phi}_k(m-1; C''/\tau^{m-1}, \tau),$$

with

$$\|J_1\|_\tau \leq \sum_{\ell=1}^{m-2} \|J_0\|_\tau^\ell \leq 2 \|J_0\|_\tau \leq 2(2\varepsilon + \varepsilon^2)$$

and

$$C'' \leq 2k^{k-1}2^{m-1}(m-k)! (C' + k^{m-2} \|J_0\|_\tau^m)^{m-1} + \sum_{\ell=1}^{m-2} (2k^{k-1} C' + k^{m-2} \|J_0\|_\tau^m)^{\ell}$$

$$\leq 2k^{k-1}2^{m-1}(m-k)! (C' + \|J_0\|_\tau^m)^m + \sum_{\ell=1}^{m-1} (2k^{k-1} C' + \|J_0\|_\tau^m)^{\ell}.$$
follows in both cases that
\[ C^n \leq 2^{\frac{k(k-1)}{2} - 2m + 1} (m - 1)! m^{(3m-4)(m-1)} (\varepsilon + \varepsilon^2) \]
\[ \leq 2^{\frac{m^2 + m + 2}{2}} (m - 1)! m^{(3m-4)(m-1)} (\varepsilon + \varepsilon^2) \]
\[ \leq D_m \varepsilon, \quad \text{with } D_m = 2^{\frac{(m+1)^2}{2}} (m - 1)! m^{(3m-4)(m-1)}. \]

If we set \( Q_0 = J_1 + \tilde{\partial}P + [J_1 \tilde{\partial}P]_{\leq m-2} \) and \( Q_1 = -\tilde{\partial}P - [J_1 \tilde{\partial}P]_{\leq m-2} \) then the estimates above, the \((P,\text{-product})\) axiom and Equation (A.2) imply that
\[
(\tilde{\partial} \Psi) \circ \Phi = 1 + Q_0(z) + \mathcal{O}_{k-1}(m-1; D_m \varepsilon / \tau^{m-1}, \tau), \quad (A.3)
\]
\[
(\tilde{\partial} \Psi) \circ \Phi = Q_1(z) + \mathcal{O}_{k-1}(m-1; D_m \varepsilon / \tau^{m-1}, \tau), \quad (A.4)
\]
where \( \|Q_0\|_\tau \leq \|J_1\|_\tau + \|\tilde{\partial}P\|_\tau + \|J_1\|_\tau \|\tilde{\partial}P\|_\tau \leq 2(2\varepsilon + \varepsilon^2) + \varepsilon + 2\varepsilon(2\varepsilon + \varepsilon^2) \leq \frac{13}{2} \varepsilon \) and
\[
\|Q_1\|_\tau \leq \|\tilde{\partial}P\|_\tau + \|\tilde{\partial}P\|_\tau \|J_1\|_\tau \leq 2\|\tilde{\partial}P\|_\tau \leq 2\varepsilon, \quad \text{as long as we have}
\]
\[
D'_m \varepsilon \geq C \tau^{m-1} + C'' + 2^{k-1} C'' \tau^{m-1} \]
\[ + k^{m-2} \left( \|\tilde{\partial}P\|_\tau \|J_1\|_\tau \right) + k^{2(m-2)} \|\tilde{\partial}P\|_\tau \|J_1\|_\tau \]
\[ \geq \varepsilon + D_m \varepsilon + 2^{k-1} D_m \varepsilon^2 + k^{m-2} D_m \varepsilon^2 + k^{2(m-2)} \varepsilon (2\varepsilon + \varepsilon^2) \]
\[ \geq \left( (1 + D_m + k^{2(m-2)}) + D_m (2^{k-1} + k^{m-2}) / 4 \right) \varepsilon, \]
\[
D''_m \varepsilon \geq C \tau^{m-1} + 2^{k-1} C \tau^{m-1} C'' \]
\[ + k^{m-2} \left( \|\tilde{\partial}P\|_\tau \|J_1\|_\tau \|C\tau^{m-1}\| \right) + k^{2(m-2)} \|J_1\|_\tau \|\tilde{\partial}P\|_\tau \]
\[ \geq \varepsilon + 2^{k-1} D_m \varepsilon^2 + k^{m-2} D_m \varepsilon^2 + k^{2(m-2)} \varepsilon (2\varepsilon + \varepsilon^2). \]

Therefore we may take
\[
D'_m = D''_m + D_m = 1 + m^{2(m-2)} + D_m \left( 1 + (2^{m-1} + m^{m-2}) / 4 \right). \]

**Computation of \( \beta_m \) and \( B_m \).** We have
\[
|z - w| = |\Phi_0(z)| \leq \left( \|\tilde{\partial}P\|_\tau + \|\tilde{\partial}P\|_\tau + 2C \tau^{m-1} \right) |z| \leq 2\varepsilon |z|, \quad \text{so } (1 - 2\varepsilon) |z| \leq |w|. \]

It follows that \( |\Psi(w) - w| = |z - w| \leq 2\varepsilon |z| \leq 2\varepsilon (1 - 2\varepsilon)^{-1} |w| \leq 4\varepsilon |w|. \) Therefore we have \( |\Psi(w)| \leq (1 + 4\varepsilon) |w| \leq 2 |w|. \) Furthermore, Equations (A.3) and (A.4) imply that
\[
\begin{aligned}
(\tilde{\partial} \Psi) \circ \Psi &= 1 + \mathcal{O}_0(1; \|Q_0\|_\tau + D_m \varepsilon)/\tau, \tau), \\
(\tilde{\partial} \Psi) \circ \Psi &= \mathcal{O}_0(1; \|Q_1\|_\tau + D_m \varepsilon)/\tau, \tau).
\end{aligned}
\]

Thus we obtain that
\[
\begin{aligned}
\tilde{\partial} \Phi &= 1 + \mathcal{O}_0(1; (1 + 4\varepsilon)(\|Q_0\|_\tau + D_m \varepsilon)/\tau, (1 - \varepsilon) \tau), \\
\tilde{\partial} \Phi &= \mathcal{O}_0(1; (1 + 4\varepsilon)(\|Q_1\|_\tau + D_m \varepsilon)/\tau, (1 - \varepsilon) \tau).
\end{aligned}
\]

This with the (Lipschitz) axiom implies that
\[
\Phi^{-1}(z) = z + \mathcal{O}_1(2; B_0^2 \varepsilon, (1 - \varepsilon)), \quad \text{with } B_0^2 = (1 + 4\varepsilon)(\|Q_0\|_\tau / \varepsilon + D_m) \leq 13 + 2D_m'.
\]

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Now we set $\beta_m^2 = 0$. Given $2 \leq \ell \leq m - 1$, we assume that there exist $B_m^\ell \geq 0$, $\beta_m^\ell \geq 0$ and a polynomial of degree $\ell - 1$ and valuation 2 satisfying $\|q_\ell\|_\tau \leq \beta_m^\ell \tau \varepsilon$ and $\Phi^{-1}(z) = z + q_\ell(z) + \sigma_{\min(k,\ell-1)}(\tau;B_m^\ell \varepsilon/\tau^{\ell-1};(1-\varepsilon)\tau)$. Note that Equations (A.3) and (A.4), the (polynomial) and (restriction) axioms imply that

$$\begin{align*}
(\partial^\ell \Psi) & \circ \Phi = 1 + \ell [Q_0]_{\leq \ell} + \sigma_{\min(k,\ell-1)}(\tau; (\sigma_{\min}(k,\ell) + 1) \beta_m^\ell \tau \varepsilon), \\
(\partial^\ell \Psi) & \circ \Phi = [Q_1]_{\leq \ell} + \sigma_{\min(k,\ell-1)}(\tau; (\sigma_{\min}(k,\ell) + 1) \beta_m^\ell \tau \varepsilon).
\end{align*}$$

The $(P$-composition) axiom and Equations (A.3) and (A.4) imply that there exist $C_{0\ell} \geq 0$ and $C_{1\ell} \geq 0$, two polynomials $R_{0\ell}$ and $R_{1\ell}$ of degree $\ell - 1$ satisfying

$$\begin{align*}
\|R_{0\ell}\|_\tau & \leq \|Q_0\|_{\tau + ||q_\ell||_\tau} \leq (1 + \beta_m^\ell \varepsilon)^{m-2} \|Q_0\|_\tau \leq \frac{13}{2} (1 + \beta_m^\ell /4)^{m-2} \varepsilon \quad \text{and} \quad \|R_{1\ell}\|_\tau \leq \|Q_1\|_{\tau + ||q_\ell||_\tau} \leq 2(1 + \beta_m^\ell /4)^{m-2} \varepsilon,
\end{align*}$$

and where

$$C_{0\ell} \leq 2 \frac{\ell (\ell-1)}{2} \min(k,\ell) \beta_m^\ell \tau \varepsilon \|Q_0\|_\tau + D_m^\ell \varepsilon \left(B_m^\ell \varepsilon \tau + \min(k,\ell) \ell - 1 (\tau + ||q_\ell||_\tau) \right) \ell + \|Q_0\|_{\ell} \|2^{\min(k,\ell-)1} B_m^\ell \varepsilon + \min(k,\ell) \ell - 1 (\tau + ||q_\ell||_\tau) \right) \ell \leq C_{0\ell} \varepsilon$$

with $C'_{0\ell} = \frac{\ell (\ell-1)}{2} \left( \frac{13}{2} \min(k,\ell) \ell - 1 + D_m^\ell \right) \left(B_m^\ell /4 + \min(k,\ell) \ell - 1 (1 + \beta_m^\ell /4) \right) \ell$

$$+ \frac{13}{2} \left( 2 \min(k,\ell) \ell - 1 \left( B_m^\ell /4 + \min(k,\ell) \ell - 1 (1 + \beta_m^\ell /4) \right) \ell.$$  

In a similar way, we obtain that $C_{1\ell} \leq C'_{1\ell} \varepsilon$, with

$$C'_{1\ell} = \frac{\ell (\ell-1)}{2} \left( 2 \min(k,\ell) \ell - 1 + D_m^\ell \right) \left(B_m^\ell /4 + \min(k,\ell) \ell - 1 (1 + \beta_m^\ell /4) \right) \ell$$

$$+ 2 \left( 2 \min(k,\ell) \ell - 1 B_m^\ell /4 + \min(k,\ell) \ell - 1 (1 + \beta_m^\ell /4) \right) \ell.$$  

Let apply the (Lipschitz) axiom to $\Psi(z) - q_{\ell+1}(z)$, where $q_{\ell+1}$ denotes the polynomial of degree $\ell$ and valuation 2 such that $\partial q_{\ell+1} = R_{0\ell}$ and $\partial q_{\ell+1} = R_{1\ell}$. We obtain that

$$\Psi(z) = z + q_{\ell+1}(z) + \sigma_{\min(\ell,k)}(\ell + 1; \max(C_{0\ell},C_{1\ell}) / \tau^\ell, (1-\varepsilon)\tau).$$

Since $||q_{\ell+1}\|_\tau \leq (\|R_{0\ell}\|_\tau + \|R_{1\ell}\|_\tau) \tau$, we may set $\beta_{m}^{\ell+1} = (2 + \frac{13}{2})(1 + \beta_m^\ell /4)^{m-2}$ and $B_m^{\ell+1} = \max(C'_{0\ell},C'_{1\ell}).$ This proves the (inverse) axiom by induction over $m$, with $\beta_m = \beta_m^\ell$ and $B_m = B_m^\ell.$

We have also used the following properties of the space $\sigma_k^T$. Since the proofs follow easily from the definitions and are very similar to those of the spaces $\sigma_k$, we omit them.
Lemma A.2. Assume \((k, m, n, \ell, \ell_1, \ell_2) \in \mathbb{N}^3 \times \mathbb{R}^3\). Then the spaces \(O_k^T\) satisfy the following axioms.

\((T\text{-Derivative})\) If \(p \in O_k^T(\ell; C, \tau)\) then \(\partial_\alpha \partial_\beta p \in O_{k-\alpha-\beta}(\ell - \alpha; C, \tau)\) for all \((\alpha, \beta) \in \mathbb{N}^2\) such that \(\alpha + \beta \leq k\);

\((T\text{-Primitive})\) If \(p \in O_0^T(\ell + 1; C_0, \tau)\), \(\partial p \in O_k^T(\ell; C_1, \tau)\) and \(\partial_\alpha p \in O_k^T(\ell; C_2, \tau)\) then we have \(p \in O_{k+1}^T(\ell + 1; C, \tau)\), with \(C \leq \max(C_0, C_1, C_2)\);

\((T\text{-Product})\) If \(p \in O_k^T(\ell_1; C_1, \tau)\) and \(g \in O_k^T(\ell_2; C_2, \tau)\) then we have \(pq \in O_k(\ell_1 + \ell_2; 2kC_1C_2, \tau)\);

\((T\text{-Composition})\) Assume \(n \geq k, \rho > 0\) and \(\tau > 0\). Let \(f : \mathbb{D}(0, \rho) \to \mathbb{C}\) and \(p : (0; \tau] \times T \to \mathbb{C}\) satisfy \(p((0; \tau] \times T) \subset \mathbb{D}(0, \rho), f(z) = O_k(n; C; \rho)\) and \(p = O_k^T(\ell; C_1, \tau)\). Then we have \(f \circ p(\nu, \theta) = O_k^T(n\ell; 2^{k(k+1)}CC_1^n, \tau)\).
B Estimates on Gevrey maps

We begin with preliminaries on the composition of Gevrey functions (Section B.1) and the flow of a Gevrey near-integrable Hamiltonian (Section B.2), then we prove Proposition B.7 in Section B.3. In all this part we omit the index \( \alpha \) in the Gevrey norms, writing for instance \( \| \cdot \|_{L,R} \) instead of \( \| \cdot \|_{\alpha,L,R} \).

We end Appendix B with a reminder of a result on Gevrey “bump” functions proved in [MS03] (Section B.4), used in Section 3.2 as well as in Sections 4.2 and 4.3.3.

B.1 Reminder on Gevrey maps and their composition

Let \( n \geq 1 \), \( L, R > 0 \), and \( \varphi \in G^{\alpha,L}(A^n_R) \). We first recall the analogue of the Cauchy inequalities for the Gevrey norms (1.4): if \( 0 < \Lambda < L \), then all the partial derivatives of \( \varphi \) belong to \( G^{\alpha,\Lambda}(A^n_R) \) and, for each \( k \in \mathbb{N} \),

\[
\sum_{m \in \mathbb{N}^{2n}; \ |m| = k} \| \partial^m \varphi \|_{\Lambda,R} \leq \frac{k!}{(L - \Lambda)^k} \| \varphi \|_{L,R} \tag{B.1}
\]

(Lemma A.2 from [MS03]).

To state the result on composition, we introduce a new notation:

\[
\mathcal{N}^*_L,R(\varphi) := \sum_{\ell \in \mathbb{N}^{2n}, \ \ell \neq 0} \frac{L^{(|\ell|_0)}}{\ell!} \| \partial^\ell \varphi \|_{G^0(A^n_R)},
\]

so that \( \| \varphi \|_{L,R} = \| \varphi \|_{G^0(A^n_R)} + \mathcal{N}^*_L,R(\varphi) \). Then, Proposition A.1 of [MS03] yields

**Proposition B.1.** Let \( n \geq 1 \), \( R, R_0 > 0 \), \( \Lambda, L > 0 \), and consider a map \( \phi: A^n_R \to A^n_{R_0} \), the 2n components of which belong\(^7\) to \( G^{\alpha,\Lambda}(A^n_R) \) and satisfy

\[
\mathcal{N}^*_L,R(\phi_1), \ldots, \mathcal{N}^*_L,R(\phi_{2n}) \leq \frac{L^\alpha}{(2n)^{\alpha - 1}}. \tag{B.3}
\]

Then, for any \( Y \in G^{\alpha,L}(A^n_{R_0}) \), we have \( Y \circ \phi \in G^{\alpha,\Lambda}(A^n_R) \) and \( \| Y \circ \phi \|_{\Lambda,R} \leq \| Y \|_{L,R_0} \).

When testing inequalities (B.3) to apply this result, the following may be useful:

**Lemma B.2.** Let \( n \geq 1 \) and \( R > 0 \). Suppose \( 0 < \Lambda < L \) and \( \varphi \in G^{\alpha,L}(A^n_R) \). Then

\[
\mathcal{N}^*_L,R(\varphi) \leq \frac{\Lambda^\alpha}{(L - \Lambda)^\alpha} \| \varphi \|_{L,R}. \tag{B.4}
\]

**Proof.** Bounding \( \mathcal{N}^*_L,R(\varphi) \) by the sum \( \sum_\mu \sum_m \Lambda^{(|\mu|_0)} \| \partial^{(\mu+m)} \varphi \|_{G^0(A^n_R)} \) over all multi-indices \( \mu, m \) with \( |\mu| = 1 \) and using \( (\mu+m)! \geq m! \), we get \( \mathcal{N}^*_L,R(\varphi) \leq \Lambda^\alpha \sum_\mu \| \partial^\mu \varphi \|_{\Lambda,R} \) and we conclude by (B.1). \( \square \)

\(^7\)In fact, the first \( n \) components are of the form \( \varphi: A^n_R \to \mathbb{T} \) and, for them, what we mean is that there is a lift \( \tilde{\varphi}: \mathbb{R}^n \times \mathbb{B}_R \to \mathbb{R} \) such that \( \tilde{\varphi}|_{\mathbb{F}_R} \in G^{\alpha,\Lambda}(\mathbb{F}_R) \), with \( \mathbb{F}_R := [-1,1]^n \times \mathbb{B}_R \); observe that \( \mathcal{N}^*_L,R(\varphi) \) stays well-defined.
### B.2 A lemma on the flow of a Gevrey near-integrable Hamiltonian

**Lemma B.3.** Let \( n \geq 1, \alpha \geq 1, L, R_0 > 0, \) and \( h \in C^{\alpha,L}(\overline{B}_{R_0}) \). Let \( R, \Lambda \) be such that

\[
0 < R < R_0, \quad 0 < \Lambda < \left(1 + 2^{2\alpha}L^{-2\alpha}\|h\|_{L,R_0}\right)^{-1/\alpha} \frac{L}{2 \cdot (2n)(n-1)/\alpha}.
\]  

(B.5)

Then there exist \( \varepsilon_0, C > 0 \) such that, for any \( u \in C^{\alpha,L}(\mathbb{A}^n_{R_0}) \) with \( \|u\|_{L,R_0} < \varepsilon_0 \) and any \( t \in [0,1] \), the time-\( t \) map \( \Phi^{(h+u)}: \mathbb{A}^n_R \to \mathbb{A}^n_{R_0} \) is well-defined and satisfies

\[
\|\Phi^{(h+u)} - \Phi^{th}\|_{\Lambda,R} \leq C\|u\|_{L,R_0}
\]  

(B.6)

(with the notation \((1.6)\)). The numbers \( \varepsilon_0 \) and \( C \) can be chosen as depending on \( h \) only through \( \|h\|_{L,R_0} \) and being respectively decreasing and increasing functions of this quantity.

**Remark B.4.** In fact \( \|\Phi^{(h+u)} - \Phi^{th}\|_{\Lambda,R} \leq Ct\|u\|_{L,R_0} \) for all \( t \in [0,1] \) (as can be seen by applying \((B.6)\) to \( \Phi+tu \) and \( \Phi+th \) themselves).

**Proof.** a) Let \( n, \alpha, L, R_0, R, \Lambda \) be as in the hypothesis, let \( h \in C^{\alpha,L}(\overline{B}_{R_0}) \). We set

\[
L' := \frac{\Lambda + L}{2}, \quad K := \max \left\{2 \cdot 2^{3\alpha}(L - \Lambda)^{-2\alpha}\|h\|_{L,R_0}, 1\right\}.
\]  

(B.7)

Let \( u \in C^{\alpha,L}(\mathbb{A}^n_{R_0}) \) and \( \varepsilon := \|u\|_{L,R_0} \). We shall work in the phase space \( \mathbb{R}^n \times \overline{B}_{R_0} \), denoting the variables by

\[
x = (\theta; r) = (\theta_1, \ldots, \theta_n; r_1, \ldots, r_n) = (x_1, \ldots, x_{2n}).
\]

We can consider that \( h \) and \( h+u \) generate Hamiltonian vector fields \( \tilde{X}_h \) and \( \tilde{X}_{h+u} \) which are defined on \( \mathbb{R}^n \times \overline{B}_{R_0} \) and 1-periodic in each of the first \( n \) variables. The flow of \( \tilde{X}_h \) is

\[
\tilde{\Phi}^{th}(\theta; r) = (\theta + t\nabla h(r); r).
\]  

(B.8)

It is \( \mathbb{Z}^n \)-equivariant, in the sense that \( \tilde{\Phi}^{th}(\theta + \ell;r) = \tilde{\Phi}^{th}(\theta;r) + (\ell;0) \) for any \( \ell \in \mathbb{Z}^n \). We shall study the flow over the time-interval \([0,1]\) of the vector field

\[
\tilde{X}_{h+u}(\theta; r) = \left(\nabla h(r) + \nabla[\theta]u(\theta; r); -\nabla[\theta]u(\theta; r)\right), \quad (\theta; r) \in \mathbb{R}^n \times \overline{B}_{R_0},
\]  

(B.9)

as the solution of a fixed-point equation in a complete metric space for which the contraction principle applies. We shall find a unique solution

\[
\tilde{\Phi}^{t(h+u)}: \mathbb{R}^n \times \overline{B}_{R} \to \mathbb{R}^n \times \overline{B}_{R_0}, \quad t \in [0,1],
\]

which is a \( \mathbb{Z}^n \)-equivariant lift to \( \mathbb{R}^n \times \overline{B}_{R_0} \) of the flow of \( X_{h+u} \) in \( \mathbb{A}^n_{R_0} \).

b) Let \( V := (C^{\alpha,L}(\mathbb{A}^n_{R_0}))^n \). For any \( \psi \in V \), we write

\[
\psi = (\psi_1, \ldots, \psi_n), \quad \|\psi\|_V := \|\psi_1\|_{\Lambda,R} + \cdots + \|\psi_n\|_{\Lambda,R}.
\]

Let \( W := V \times V \). For any \( \eta \in W \), we write

\[
\eta = (\eta[\theta]; \eta[\theta']) = (\eta_1, \ldots, \eta_n; \eta_{n+1}, \ldots, \eta_{2n}), \quad \|\eta\|_W := \frac{1}{K}\|\eta[\theta]\|_V + \|\eta[\theta']\|_V.
\]
Let \( E := C^{0}([0,1],W) \). For any \( \xi \in E \), we set
\[
\|\xi\|_{E} := \max_{t \in [0,1]} \|\xi(t)\|_{W}.
\]
We get a Banach space \((E, \|\cdot\|_{E})\).

Let us denote by \( \varphi \) the “unperturbed” flow over \([0,1]\), i.e.
\[
\varphi(t) := \Phi^{t}_{\text{th}}, \quad t \in [0,1]
\]
defined by (B.8). For every \( \xi \in E \) and \( t \in [0,1] \), \( \varphi(t) + \xi(t) \) can be considered as a \( \mathbb{Z}^{n} \)-equivariant map \( \mathbb{R}^{n} \times \overline{B}_{R} \rightarrow \mathbb{R}^{n} \times \mathbb{R}^{n} \) (identifying functions on \( A_{R}^{n} \) with functions on \( \mathbb{R}^{n} \times \overline{B}_{R} \) which are 1-periodic in the first \( n \) variables). We thus can view
\[
\varphi + E := \{ \varphi + \xi \mid \xi \in E \}
\]
as a complete metric space (with the distance \( \text{dist}(\varphi + \xi, \varphi + \xi^{*}) := \|\xi^{*} - \xi\|_{E} \) where the flow \( \Phi^{t(\text{h} + u)} \) is to be found. More specifically, we restrict ourselves to the closed ball
\[
\mathcal{B}_{\rho} := \{ \varphi + \xi \mid \|\xi\|_{E} \leq \rho \},
\]
with
\[
\rho := \frac{2^{\alpha+1} \varepsilon}{(L - \Lambda)^{\alpha}}, \quad \varepsilon < \varepsilon_{0} := \frac{(L - \Lambda)^{\alpha}}{2^{\alpha+1}} \min \left\{ R_{0} - R, \beta, \frac{(L - \Lambda)^{\alpha}}{2^{2\alpha} K} \right\} \tag{B.10}
\]
with \( \beta := \frac{L^{\alpha}}{(2n)^{\alpha-1}} - \left( 1 + 2^{2\alpha}(L - \Lambda)^{-2\alpha} \|h\|_{L,R} \right) \Lambda^{\alpha} \) (the positiveness of \( \beta \) is ensured by (B.5) because \( L - \Lambda > L/2 \)). Observe that
\[
\rho \leq \min \{ R_{0} - R, \beta \} \tag{B.11}
\]

The flow \( \Psi(t) = \Phi^{t(h + u)} \) that we are searching is characterised by \( \Psi(0) = \text{Id} \) and \( \frac{d\Psi}{dt}(t) = \tilde{X}_{h + u} \circ \Psi(t) \), or
\[
\Psi(t) = \text{Id} + \int_{0}^{t} \tilde{X}_{h + u} \circ \Psi(\tau) \, d\tau.
\]
Let us first check that, with our choice of \( \rho \), the formula
\[
\mathcal{F}(\Psi)(t) := \text{Id} + \int_{0}^{t} \tilde{X}_{h + u} \circ \Psi(\tau) \, d\tau, \quad t \in [0,1]
\]
defines a functional \( \mathcal{F} : \mathcal{B}_{\rho} \rightarrow \varphi + E \).

Assume \( \Psi = \varphi + \xi \in \mathcal{B}_{\rho} \). In view of (B.9), the components of \( \tilde{X}_{h + u} \) belong to \( G^{\alpha,L}(A_{R_{0}}^{n}) \). We thus only need to check that, for each \( \tau \in [0,1] \), \( \Psi(\tau) \) maps \( A_{R}^{n} \) in \( A_{R_{0}}^{n} \) and its components satisfy \( N_{\Lambda,R}^{*}(\Psi_{j}(\tau)) \leq L^{\alpha}/(2n)^{\alpha-1} \) so as to apply Proposition B.1.

The first condition is met because \( \Psi^{[r]}(\tau) = r + \xi^{[r]}(\tau) \) and the components of \( \xi^{[r]}(\tau) = (\xi_{n+1}(\tau), \ldots, \xi_{2n}(\tau)) \) satisfy
\[
\sum_{j=n+1}^{2n} \|\xi_{j}(\tau)\|_{C^{0}(A_{R})}^{2} \leq \|\xi(\tau)\|_{W}^{2} \leq \|\xi\|_{E}^{2} \leq \rho^{2} \leq (R_{0} - R)^{2}
\]
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by (B.11). The second condition is met because, for any \( 1 \leq j \leq 2n \), on the one hand 
\( \mathcal{N}^{*}_{\Lambda,R}(\xi_j(\tau)) \leq \| \xi_j(\tau) \|_{\Lambda,R} \leq \rho \), and on the other hand \( \mathcal{N}^{*}_{\Lambda,R}(\varphi_j(\tau)) = \Lambda^\alpha \) for \( j \geq n + 1 \),
while for \( j \leq n \), by (B.4) and (B.11),
\[
\mathcal{N}^{*}_{\Lambda,R}(\varphi_j(\tau)) \leq \Lambda^{\alpha} + \tau \mathcal{N}^{*}_{\Lambda,R}(\frac{\partial h}{\partial r_j}) \\
\leq \Lambda^{\alpha} + \frac{\Lambda^\alpha}{(L' - \Lambda)^{\alpha}} \left\| \frac{\partial h}{\partial r_j} \right\|_{L',R} \leq \Lambda^{\alpha} \left( 1 + 2^{2\alpha}(L - \Lambda)^{-2\alpha} \| h \|_{L,R} \right) = \frac{L^{\alpha}}{(2n)^{\alpha - 1}} - \beta,
\]
which is \( \leq \frac{L^{\alpha}}{(2n)^{\alpha - 1}} - \rho \) by (B.11).

**d)** Let us now check that \( \mathcal{F}(\mathcal{B}_\rho) \subset \mathcal{B}_\rho \). For \( \Psi = \varphi + \xi \in \mathcal{B}_\rho \), we write \( \mathcal{F}(\Psi) = \varphi + \eta \) and observe that, in view of (B.8) and (B.9), for each \( t \in [0, 1] \),
\[
\eta^{[\ell]}(t) = \int_0^t (g(\tau) + \nabla^{[r]} u \circ \Psi(\tau)) \, d\tau, \quad \eta^{[r]}(t) = -\int_0^t \nabla^{[\ell]} u \circ \Psi(\tau) \, d\tau
\]
with
\[
g(\tau)(\theta; r) := \nabla h(r + \xi^{[r]}(\tau)(\theta; r)) - \nabla h(r), \quad \tau \in [0, 1].
\]
We already checked that Proposition B.1 applies to \( \frac{\partial h}{\partial r_i} \circ \Psi(\tau) \) and \( \frac{\partial u}{\partial x_j} \circ \Psi(\tau) \). It yields
\[
\left\| \frac{\partial u}{\partial x_j} \circ \Psi(\tau) \right\|_{\Lambda,R} \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{L',R_0}, \quad \text{for } 1 \leq j \leq 2n, \tau \in [0, 1]
\]
whence
\[
\| \eta^{[\ell]}(t) - g(t) \|_V + \| \eta^{[r]}(t) \|_V \leq \sum_{j=1}^{2n} \left\| \frac{\partial u}{\partial x_j} \right\|_{L',R_0} \leq \frac{2^\alpha \varepsilon}{(L - \Lambda)^\alpha}
\]
(by B.1, recalling that \( \| u \|_{L,R_0} = \varepsilon \)). If \( 1 \leq i \leq n \), we can also apply Proposition B.1 to
\[
g_i(\tau) = \frac{\partial h}{\partial r_i}(r + \xi^{[r]}(\tau)) - \frac{\partial h}{\partial r_i}(r) = \left\langle \int_0^1 \nabla \frac{\partial h}{\partial r_i}(r + s\xi^{[r]}(\tau)) \, ds, \xi^{[r]}(\tau) \right\rangle,
\]
whence \( \| g_i(\tau) \|_{\Lambda,R} \leq \sum_{j=1}^n \left\| \frac{\partial^2 h}{\partial r_i \partial r_j} \right\|_{L',R_0} \| \xi_{n+j}(\tau) \|_{\Lambda,R} \) and
\[
\| g(\tau) \|_V \leq \sum_{j=1}^n \left( \sum_{i=1}^n \left\| \frac{\partial^2 h}{\partial r_i \partial r_j} \right\|_{L',R_0} \right) \| \xi_{n+j}(\tau) \|_{\Lambda,R} \leq \frac{2^{3\alpha} \| h \|_{L,R_0}^\rho}{(L - \Lambda)^{2\alpha}}
\]
by (B.1). Therefore, recalling that \( K \geq 1 \), we get
\[
\frac{1}{K} \| \eta^{[\ell]}(t) \|_{\Lambda,R} + \| \eta^{[r]}(t) \|_{\Lambda,R} \leq \frac{2^{3\alpha} \| h \|_{L,R_0}^\rho}{K(L - \Lambda)^{2\alpha}} + \frac{2^\alpha \varepsilon}{(L - \Lambda)^{\alpha}}.
\]
Both summands are \( \leq \rho/2 \), by our choices of \( K \) and \( \varepsilon_0 \), (B.7) and (B.10), whence \( \| \eta \|_E \leq \rho \) and \( \mathcal{F}(\Psi) \in \mathcal{B}_\rho \) as desired.
e) Similar computations show that $F$ induces a contraction on $B_\rho$: Let $\Psi = \varphi + \xi, \tilde{\Psi} = \varphi + \tilde{\xi} \in B_\rho$, and $F(\Psi) = \varphi + \eta, F(\tilde{\Psi}) = \varphi + \tilde{\eta}$. We get

$$\tilde{\eta}^{[\theta]}(t) - \eta^{[\theta]}(t) = \int_0^t (G(\tau) + U^{[\theta]}(\tau)) \partial \tau, \quad \tilde{\eta}^{[r]}(t) - \eta^{[r]}(t) = -\int_0^t U^{[r]}(\tau) \partial \tau,$$

with

$$G_i(\tau) = \frac{\partial h}{\partial r_i} (r + \tilde{\xi}^{[r]}(\tau)) - \frac{\partial h}{\partial r_i} (r + \xi^{[r]}(\tau)), \quad i = 1, \ldots, n$$

$$U_j(\tau) = \frac{\partial u}{\partial x_{j\pm n}} \circ \Psi(\tau) - \frac{\partial u}{\partial x_{j\pm n}} \circ \Psi(\tau), \quad j = 1, \ldots, 2n$$

(where $j \pm n$ stands for $j + n$ if $j \leq n$, for $j - n$ else). We obtain

$$\frac{1}{K} \|G(\tau)\|_V \leq \frac{1}{K} \sum_{j=1}^n \left( \sum_{i=1}^n \left\| \frac{\partial^2 h}{\partial r_i \partial r_j} \right\|_{L', R_0} \right) \|\tilde{\xi}_{n+j}(\tau) - \xi_{n+j}(\tau)\|_{\Lambda, R} \leq \lambda_0 \|\tilde{\xi} - \xi\|_E,$$

with $\lambda_0 := \frac{2^{2\alpha} \|h\|_{L, R_0}}{K(\Lambda - \alpha)^{2\alpha}} \leq \frac{1}{2}$, and

$$\|U(\tau)\|_W \leq \|U^{[\theta]}(\tau)\|_V + \|U^{[r]}(\tau)\|_V \leq \sum_{i=1}^2 \left( \sum_{j=1}^{2n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_{j\pm n}} \right\|_{L', R_0} \right) \|\tilde{\xi}_i(\tau) - \xi_i(\tau)\|_{\Lambda, R}$$

$$\leq \frac{2^{2\alpha} \varepsilon}{(\Lambda - \alpha)^{2\alpha}} \sum_{i=1}^{2n} \|\tilde{\xi}_i(\tau) - \xi_i(\tau)\|_{\Lambda, R} \leq \lambda_1 \|\tilde{\xi} - \xi\|_E,$$

with $\lambda_1 := \frac{2^{2\alpha} \varepsilon_0 K}{(\Lambda - \alpha)^{2\alpha}} < \frac{1}{2}$ by (3.10), whence the contraction property

$$\|\tilde{\eta} - \eta\|_E \leq (\lambda_0 + \lambda_1) \|\tilde{\xi} - \xi\|_E$$

with $\lambda_0 + \lambda_1 < 1$.

f) Finally, we get a unique fixed point $\Psi \in B_\rho$ for the functional $F$, which encodes the flow of $\tilde{X}_{h+u}$. The difference $\Delta(t) := \Psi(t) - \varphi(t)$, when viewed as a map $\Lambda_\rho \to \mathbb{R}^{2n}$, is a lift of the difference of flows $\Phi^{(h+u)} - \Phi^h$ and

$$\sum_{j=1}^{2n} \|\Delta_j(t)\|_{\Lambda, R} \leq K \rho = \frac{2^{\alpha+1} K}{(\Lambda - \alpha)^{\alpha} \varepsilon}.$$
Let \( n, \alpha, L, R, R_0 > 0 \) and \( h \) be as in the hypothesis of Proposition B.17. Let \( R' := \frac{R + R_0}{2} \). Lemma B.3 yields \( \varepsilon_0, \Lambda, C_* > 0 \) such that \( \Lambda \leq L/2 \) and, for any \( u \in G^{\alpha, L}(\mathbb{A}_n) \),

\[
\|u\|_{L,\infty} < \varepsilon_0 \Rightarrow |||\tilde{\Phi}^{h+u} - \tilde{\Phi}^h|||_{\Lambda, R'} + \|\tilde{\Phi}^h - \text{Id}\|_{\Lambda, R'} \leq C_*\|u\|_{L,\infty},
\]

where \( \tilde{\Phi}^{h+u}, \tilde{\Phi}^u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \) are the lifts of \( \Phi^{h+u}, \Phi^u \) obtained by flowing along the lifts to \( \mathbb{R}^n \times \mathbb{R}^n \) of the corresponding vector fields (which are complete in this case) and \( |||\phi|||_{\Lambda, R'} := \|\phi_1\|_{\Lambda, R'} + \cdots + \|\phi_{2\Lambda}\|_{\Lambda, R'} \) for a map \( \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \). We set

\[
\varepsilon_* := \min \left\{ \varepsilon_0, \frac{R_0 - R}{C_*}, 2 \left( 2 \cdot 2^{\alpha} \right)^{\frac{1}{\alpha}} \right\},
\]

\[
L_* := 2^{-\frac{1}{\alpha}} (2\pi)^{-\frac{1}{\alpha}} \left( 1 + 2^{4\alpha} L^{-2\alpha} \right)^{-\frac{1}{\alpha}} \Lambda.
\]

Let \( m \geq 1 \) and \( \Psi \in \mathcal{P}^{m, L}(\tilde{\Phi}^h) \) be such that \( \delta_m^{\alpha, L}(\Psi, \tilde{\Phi}^h) < \varepsilon_* \). Let \( \varepsilon \) be such that

\[
\delta_m^{\alpha, L}(\Psi, \tilde{\Phi}^h) < \varepsilon < \varepsilon_*.
\]

We shall prove that there is a lift \( \xi : \mathbb{A}_n \to \mathbb{R}^n \times \mathbb{R}^n \) of \( \Psi - \tilde{\Phi}^h \) such that \( |||\xi|||_{L_* R} \leq C_* \varepsilon \), which is sufficient to prove the proposition.

Let us choose \( u_0, u_1, \ldots, u_m \in G^{\alpha, L}(\mathbb{A}_n) \) such that \( \Psi = \Phi^{u_m} \circ \cdots \circ \Phi^{u_1} \circ \Phi^{h+u_0} \) and

\[
\|u_0\|_{L,\infty} + \|u_1\|_{L,\infty} + \cdots + \|u_m\|_{L,\infty} < \varepsilon.
\]

We observe that the formulae

\[
\xi^{[0]} := \tilde{\Phi}^{h+u_0} - \tilde{\Phi}^h, \quad \xi^{[j+1]} := \xi^{[j]} + (\tilde{\Phi}^{u_{j+1}} - \text{Id}) \circ (\Phi^h + \xi^{[j]})
\]

inductively define \( \xi^{[0]}, \xi^{[1]}, \ldots, \xi^{[m]} : \mathbb{A}_n \to \mathbb{R}^n \times \mathbb{R}^n \) so that \( \xi^{[m]} \) is a lift of \( \Psi - \tilde{\Phi}^h \). It is thus sufficient to check that

\[
|||\xi^{[j]}|||_{L_* R} \leq C_* \left( \|u_0\|_{L,\infty} + \|u_1\|_{L,\infty} + \cdots + \|u_j\|_{L,\infty} \right)
\]

for \( 0 \leq j \leq m \).

In view of (B.14), inequality (B.15) holds for \( j = 0 \) by (B.13), because \( \|u_0\|_{L,\infty} \leq \varepsilon < \varepsilon_0 \) and \( L_* < \Lambda, R < R' \).

Assume that (B.15) holds for a given \( j < m \). We observe that \( |||\tilde{\Phi}^{u_{j+1}} - \text{Id}|||_{\Lambda, R'} \leq C_* \|u_{j+1}\|_{L,\infty} \) by (B.13), because \( \|u_{j+1}\|_{L,\infty} \leq \varepsilon < \varepsilon_0 \). We can apply Proposition B.1 to check that the components of \( (\tilde{\Phi}^{u_{j+1}} - \text{Id}) \circ (\Phi^h + \xi^{[j]}) \) belong to \( C^{\alpha, L_*}(\mathbb{A}_n^\prime) \) and bound their norms, because the inequality

\[
\sum_{i=n+1}^{2N} \|\xi^{[i]}\|_{C^0(\mathbb{A}_n)} \leq |||\xi^{[j]}|||_{L_* R} \leq C_* \varepsilon < R_0 - R
\]

ensures that \( \Phi^h + \xi^{[j]} \) maps \( \mathbb{A}_n^\prime \) into \( \mathbb{A}_n^\prime \), and, for \( 1 \leq i \leq n \), both \( L_*^\alpha + M^{\alpha, L_* R}_{(2\Lambda)^{i-1}} + M^{\alpha, L_* R}_{(2\Lambda)^{i-1}} \) and \( L_*^\alpha + M^{\alpha, L_* R}_{(2\Lambda)^{i-1}} \) are \( \leq L_*^\alpha + 2^{2\alpha} L_*^\alpha \|h\|_{L, R} + \|\xi^{[j]}\|_{L_* R} \) (applying (B.4) between \( L_*^\alpha \) and \( L_*^\alpha + L_*^\alpha \), and (B.1) between \( \frac{L_* + L}{2} \) and \( L \)), which is \( \leq \frac{\Lambda^\alpha}{2} + C_* \varepsilon < \frac{\Lambda^\alpha}{2} \).

We thus get \( |||(\tilde{\Phi}^{u_{j+1}} - \text{Id}) \circ (\Phi^h + \xi^{[j]})|||_{L_* R} \leq |||\tilde{\Phi}^{u_{j+1}} - \text{Id}|||_{\Lambda, R'} \leq C_* \|u_{j+1}\|_{L,\infty} \), which implies (B.15) for the index \( j + 1 \) by virtue of (B.14).
B.4 Gevrey bump functions

We call “bump function” a function on $\mathbb{T}$ which vanishes identically outside a given interval $I$ and whose value is 1 at each point of a given subinterval of $I$ (so this is in fact a “flat-top bump function”). Of course, such a function can only exist in a non-quasianalytic functional space.

Dealing with Gevrey functions on $\mathbb{T}$, we use the notation (2.1) and quote without proof Lemma 3.3 of [MS04] on the existence of Gevrey bump functions on $\mathbb{T}$:

**Lemma B.5.** Let $\alpha > 1$ and $L > 0$. Then there exists a real $c(\alpha, L) > 0$ such that, for each real $p > 2$, the space $G^{\alpha, L}(\mathbb{T})$ contains a function $\eta_p$ which satisfies

$$\frac{-1}{2p} \leq \theta \leq \frac{1}{2p} \Rightarrow \eta_p(\langle \theta \rangle) = 1, \quad \frac{1}{p} \leq \theta \leq 1 - \frac{1}{p} \Rightarrow \eta_p(\langle \theta \rangle) = 0$$

and

$$\|\eta_p\|_{\alpha, L} \leq \exp \left( c(\alpha, L) p^{\frac{1}{\alpha - 1}} \right).$$

(B.16)

The proof can be found in [MS04, p. 1633].

Notice that, intuitively, higher values of $p$ must produce a larger norm, since the graph gets steeper between $\frac{1}{2p}$ and $\frac{1}{p}$ for instance, which makes the derivatives reach higher and higher values. In fact, one can prove that an exponential bound such as (B.16) is optimal.
C Generating functions for exact symplectic $C^\infty$ maps

In this appendix we fix $n \geq 1$ integer and review the classical formalism of generating functions of mixed sets of variables to define exact symplectic local diffeomorphisms of $\mathbb{R}^n$.

The coordinates in $T^n \times \mathbb{R}^n$ will be denoted indifferently $\{(\theta, r), (\theta_1, \ldots, \theta_n, r_1, \ldots, r_n)\}$, or simply $(x_1, \ldots, x_{2n})$. For instance, the Liouville 1-form which gives rise to the exact symplectic structure on $T^n \times \mathbb{R}^n$ can be written

$$\lambda = \sum_{i=1}^n r_i \, d\theta_i = \sum_{i=1}^n x_{n+i} \, dx_i.$$ 

We denote the partial gradient operators by

$$\nabla[1] := \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix}, \quad \nabla[2] := \begin{pmatrix} \hat{e}_{n+1} \\ \vdots \\ \hat{e}_{2n} \end{pmatrix},$$

and view $d[1] := (\hat{e}_1 \cdots \hat{e}_n)$ and $d[2] := (\hat{e}_{n+1} \cdots \hat{e}_{2n})$ as matrix-valued differential operators acting on vector-valued functions.

Recall that $\langle \cdot, \cdot \rangle : \mathbb{R}^n \to \mathbb{T}^n$ denotes the canonical projection.

Lemma C.1. Let $\Omega, \Omega' \subset T^n \times \mathbb{R}^n$ be open, and denote by $\overline{\Omega}$ and $\overline{\Omega}'$ their lifts in $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that $A \in C^\infty(\Omega')$ satisfies the property:

The map $(\theta, r') \mapsto (\theta, r + \nabla[1] A(\theta, r'))$ is a diffeomorphism from $\overline{\Omega}'$ onto $\overline{\Omega}$.

Denote by $F[2]$ the second group of components of the inverse diffeomorphism, so that, for each $(\theta, r) \in \overline{\Omega}$,

$$r' = F[2](\theta, r) \quad \iff \quad (\theta, r') \in \Omega' \text{ and } \nabla[1] S(\theta, r') = r,$$

where

$$S(\theta, r') := \sum_{i=1}^n \theta_i r'_i + A(\theta, r'), \quad (\theta, r') \in \Omega'.$$

Define $F[1](\theta, r) := \theta + \nabla[2] A(\theta, F[2](\theta, r)) \in \mathbb{R}^n$ for $(\theta, r) \in \Omega$, so that

$$\theta' = F[1](\theta, r) \quad \iff \quad \nabla[2] S(\theta, r') = \theta'.$$

Then $F = (F[1], F[2]) : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ is $C^\infty$ and induces an exact symplectic local diffeomorphism $F_A = (F[1], F[2]) : \Omega \to \mathbb{R}^n$. The inverse Jacobian matrix of $F_A$ at an arbitrary point $(\theta, r) \in \Omega$ is a block matrix

$$\begin{pmatrix} M^{1,1} & M^{1,2} \\ M^{2,1} & M^{2,2} \end{pmatrix}$$

with

$$M^{1,1} = (1_n + d[1] \nabla[2] A)^{-1}, \quad M^{1,2} = -(1_n + d[1] \nabla[2] A)^{-1} d[2] \nabla[2] A, \quad M^{2,1} = -d[2] \nabla[2] A,$$

where the partial derivatives of $A$ are evaluated at $(\theta, r') = (\theta, F[2](\theta, r))$. 

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We shall see in the course of the proof that
\[ \mathcal{F}_A^* \lambda - \lambda = d\Sigma, \]
where \( \Sigma \in C^\infty(\Omega) \) is defined by
\[ \Sigma(\theta, r) := \tilde{\Sigma}(\theta, F[2](\theta, r)), \quad \tilde{\Sigma}(\theta, r') := \sum_{i=1}^{n} r_i' \partial_{n+i} A(\theta, r') - A(\theta, r'). \] (C.7)
By abuse of language, we will call any function \( A \) satisfying (C.2) a generating function for \( \Omega' \) (although it is the function \( S(\theta, r') \) that is usually called generating function).

**Remark C.2.** It is easy to check that, if \( F = \mathcal{F}_A \), then the set of all possible generating functions of \( F \) coincide with the set of all functions
\[ (\theta, r') \mapsto A(\theta, r') + c + \sum_{i=1}^{n} \ell_i r_i', \quad c \in \mathbb{R}, \quad \ell \in \mathbb{Z}^n. \]

**Proof of Lemma C.1.** The Jacobian matrix of the diffeomorphism mentioned in (C.2) can be written as the block matrix
\[ \begin{pmatrix} 1_n & 0 \\ d[1] \nabla[1] S(\theta, r') & d[2] \nabla[1] S(\theta, r') \end{pmatrix}. \]
The hypothesis entails that the matrix \( d[2] \nabla[1] S(\theta, r') \) is invertible for each \( (\theta, r') \in \Omega' \); notice that its transpose is \( d[1] \nabla[2] S(\theta, r') \).

The definition (C.3) of the map \( F[2] \) shows that it is \( C^\infty \) on \( \Omega \), with
\[ d[1] \nabla[1] S + d[2] \nabla[1] S \cdot d[1] F[2] = 0, \quad d[2] \nabla[1] S \cdot d[2] F[2] = 1_n, \]
where it is understood that the partial derivatives of \( F[2] \) are evaluated on \( (\theta, r) \) and those of \( S \) on \( (\theta, r') = (\theta, F[2](\theta, r)) \). Moreover, \( F[2] \) can be equally viewed as a \( C^\infty \) function on \( \Omega \) (it is \( \mathbb{Z}^n \)-periodic in \( \theta \) because \( \nabla[1] S \) is). By (C.5), \( F[1] \) is \( C^\infty \) on \( \Omega \) and
\[ d[1] F[1] = d[1] \nabla[2] S + d[2] \nabla[2] S \cdot d[1] F[2], \quad d[2] F[1] = d[2] \nabla[2] S \cdot d[2] F[2]. \]
We have \( F[1](\theta + \ell, r) = F[1](\theta, r) + \ell \) for all \( \ell \in \mathbb{Z}^n \), thus \( F \) induces a \( C^\infty \) map \( \mathcal{F}_A : \Omega \to \mathbb{A}^n \).

A bit of calculus shows that the Jacobian matrix of \( F \), which is
\[ \begin{pmatrix} d[1] F[1] & d[2] F[1] \\ d[1] F[2] & d[2] F[2] \end{pmatrix}, \]
has the inverse
\[ \begin{pmatrix} 1_n & 0 \\ d[1] \nabla[1] S & d[2] \nabla[1] S \end{pmatrix} \begin{pmatrix} d[1] \nabla[2] S & 0 \\ 0 & d[1] \nabla[2] S \end{pmatrix}^{-1} \begin{pmatrix} 1_n & -d[2] \nabla[2] S \\ 0 & d[1] \nabla[2] S \end{pmatrix}, \]
so \( F \) and hence \( \mathcal{F}_A \) are local diffeomorphisms, and (C.6) is proved.

Let us denote by \( F_1, \ldots, F_{2n} \) the components of \( F \), so that (C.3) and (C.5) entail
\[ (\partial_i S)(\theta, F[2](\theta, r)) = r_i, \quad (\partial_{n+i} S)(\theta, F[2](\theta, r)) = F_i(\theta, r), \] (C.8)
for \((\theta, r) \in \Omega\) and \(i = 1, \ldots, n\), and \(E^* \lambda = \sum_{i=1}^{n} F_{n+i} \, dF_i\). Let us define \(\tilde{S} \in C^\infty(\Omega)\) by \(\tilde{S}(\theta, r) := \tilde{S}(\theta, F^{[2]}(\theta, r))\). Applying the chain rule and inserting (C.8), we get
\[
d\tilde{S} = \sum_{i=1}^{n} (\hat{e}_i \mathcal{S})(\theta, F^{[2]}(\theta, r)) \, d\theta_i + \sum_{i=1}^{n} (\hat{e}_{n+i} \mathcal{S})(\theta, F^{[2]}(\theta, r)) \, dF_{n+i}
\]
\[
= \sum_{i=1}^{n} r_i \, d\theta_i + \sum_{i=1}^{n} F_i(\theta, r) \, dF_{n+i} = \lambda - E^* \lambda + d\left(\sum_{i=1}^{n} F_{n+i} F_i\right) .
\]
Thus \(E^* \lambda - \lambda = d\chi\), with
\[
\chi(\theta, r) := \sum_{i=1}^{n} F_{n+i}(\theta, r) F_i(\theta, r) - \tilde{S}(\theta, r)
\]
\[
= \sum_{i=1}^{n} F_{n+i}(\theta, r) (\hat{e}_{n+i} \mathcal{S})(\theta, F^{[2]}(\theta, r)) - S(\theta, F^{[2]}(\theta, r))
\]
\[
= \tilde{\chi}(\theta, F^{[2]}(\theta, r)) ,
\]
where \(\tilde{\chi}(\theta, r') = \sum_{i=1}^{n} r_i^1 (\hat{e}_{n+i} \mathcal{S})(\theta, r') - S(\theta, r')\). Inserting (C.4), we see that \(\chi \in C^\infty(\Omega)\) is \(\mathbb{Z}^n\)-periodic in \(\theta\) and induces the function \(\Sigma \in C^\infty(\Omega)\) defined by (C.7).

Lemma C.3. Let \(\Omega \subset \mathbb{A}^n\) be open and connected. Let \(F: \Omega \to \mathbb{T}^n \times \mathbb{R}^n\) be an exact symplectic \(C^\infty\) local diffeomorphism of the form
\[
F(\theta, r) = (\theta + \langle f(\theta, r), F^{[2]}(\theta, r) \rangle), \quad (\theta, r) \in \Omega ,
\]
where \(f, F^{[2]} \in C^\infty(\Omega, \mathbb{R}^n)\). Assume that the map
\[
(\theta, r) \mapsto (\theta, r') = (\theta, F^{[2]}(\theta, r))
\]
is a \(C^\infty\)-diffeomorphism from \(\Omega\) onto an open set \(\Omega' \subset \mathbb{A}^n\).

Then there exists a generating function \(A \in C^\infty(\Omega')\) for \(\Omega\) such that \(F = \mathcal{F}_A\). It can be obtained as follows: let \(\Phi: \Omega' \to \Omega\) denote the inverse of the diffeomorphism (C.9) and set
\[
\beta := \sum_{i=1}^{n} (\Phi_{n+i}(\theta, r') - r_i^1) \, d\theta_i + \sum_{i=1}^{n} f_i \circ \Phi(\theta, r') \, dr_i
\]
then \(\beta\) is an exact \(C^\infty\) 1-form on \(\Omega'\) and any \(A \in C^\infty(\Omega')\) such that \(\beta = dA\) satisfies \(F = \mathcal{F}_A\).

Proof. The 1-form \(\beta\) can be written as
\[
\beta = \Phi^*(\lambda - F^* \lambda) + d\left(\sum_{i=1}^{n} r_i^1 (f_i \circ \Phi)\right)
\]
because \(\Phi^{[1]}(\theta, r') = \theta\) entails \(\Phi^* \lambda = \sum_{i=1}^{n} \Phi_{n+i} \, d\theta_i\) and \(F \circ \Phi(\theta, r') = (\langle \theta + f \circ \Phi(\theta, r'), r' \rangle, r')\) entails \(\Phi^*(F^* \lambda) = (F \circ \Phi)^* \lambda = \sum_{i=1}^{n} r_i^1 \, d(\theta_i + f_i \circ \Phi)\). Since \(F\) is exact symplectic, the 1-form \(F^* \lambda - \lambda\) is exact, and the formula (C.11) shows that \(\beta\) is thus exact too.
Pick any \( A \in C^\infty(\Omega) \) such that \( \beta = dA \). In view of (C.10), the map \( (\theta, r') \mapsto (\theta, r' + \nabla^1 A(\theta, r')) \) coincides with \( \Phi \) and is thus a \( C^\infty \) diffeomorphism \( \Omega' \to \Omega \), hence \( A \) is a generating function for \( \Omega \). We have \( \Phi^{-1}(\theta, r) = (\theta, F(\theta, r)) \) and, again thanks to (C.10), \( \theta + \nabla^2 A(\theta, F(\theta, r)) = \theta + f(\theta, r) \), therefore \( \mathcal{F}_A = F \).
D Proof of Lemma 2.5

D.1 Set-up

Let us give ourselves an integer \( n \geq 1 \), reals \( \alpha \geq 1 \), \( R, R_0, L_0 > 0 \) such that \( R < R_0 \), and a function \( \eta \in G^{\alpha,L_0}([0,1]) \). We assume that \( \eta \) is not identically zero (otherwise there is nothing to be proved). We set

\[
\varepsilon_* := \frac{1}{\|\eta\|_{\alpha,L_0}} \min \left\{ \frac{R_0 - R}{2}, \frac{L_0^{\alpha}}{2^{\alpha+1}(2n + 1)^{\alpha-1}} \right\}, \tag{D.1}
\]

\[
L := \frac{L_0}{(2^{\alpha+1}(2n + 1)^{\alpha-1})^{1/\alpha}}. \tag{D.2}
\]

Given \( \psi = (\psi_1, \ldots, \psi_n) \in G^{\alpha,L_0}(\mathbb{A}_R^n, \mathbb{R}^n) \) such that

\[
\varepsilon := \sum_{i=1}^n \|\psi_i\|_{\alpha,L_0,R_0} \leq \varepsilon_* , \tag{D.3}
\]

we define for each \( t \in [0,1] \) a \( C^\infty \) map

\[
\Psi_t: (\theta, r) \in \mathbb{A}_R^n \rightarrow (\theta, r') = (\theta, r + \eta(t)\psi(\theta, r)) \in \mathbb{A}_R^n. \tag{D.4}
\]

Our aim is to prove that \( \Psi_t \) induces a \( C^\infty \) diffeomorphism from \( \mathbb{T}^n \times B_{R_0} \) onto its image \( \Omega_t \), to check that \( \mathbb{A}_R^n \subset \Omega_t \) and to study the inverse map.

D.2 Diffeomorphism property

Let \( t \in [0,1] \). The Jacobian matrix of \( \Psi_t \) at an arbitrary \( (\theta, r) \in \mathbb{A}_R^n \) is the block matrix

\[
\begin{pmatrix}
1_{n} & 0 \\
M & 1_n + N
\end{pmatrix}
\]

where \( M := \eta(t)d[1] \psi(\theta, r) \) and \( N := \eta(t)d[2] \psi(\theta, r) \), with the notations of Appendix C.

The matrix norm of \( N \) subordinate to the Euclidean structure of \( \mathbb{R}^n \) is \( \leq \sum_{i,j=1}^n |N_{i,j}| \), and \( N_{i,j} = \eta(t)\tilde{c}_{a+j} \psi_i(\theta, r) \), thus this matrix norm is less than one by \( \text{D.1} \) and \( \text{D.3} \) (because \( \sum_{i,j} \|\eta\|_{C^0([0,1])}\|\tilde{c}_{a+j} \psi_i\|_{C^0(\mathbb{A}_R^n)} \leq \|\eta\|_{\alpha,L_0}\sum_{i} L_0^\alpha \|\psi_i\|_{\alpha,L_0,R_0} = L_0^{-\alpha} \varepsilon \|\eta\|_{\alpha,L_0} \) and \( 1_n + N \) is invertible. Therefore, by the Implicit Function Theorem, \( \Psi_t \) is a \( C^\infty \) local diffeomorphism on \( \mathbb{A}_R^n \).

Suppose that \( (\theta, r) \) and \( (\theta^*, r^*) \) have the same image by \( \Psi_t \). Then \( \theta^* = \theta \) and

\[
r^* - r = -\eta(t)(\psi(\theta, r^*) - \psi(\theta, r)) = -\eta(t) \int_0^1 d[2] \psi(\theta, (1-s)r + sr^*)(r^* - r) \, ds,
\]

whence \( \|r^* - r\| < \|r - r\| \) by the above remark on the matrix norm of \( \eta(t)d[2] \psi \), thus \( r^* = r \). Therefore, \( \Psi_t \) is injective on \( \mathbb{A}_R^n \) and induces a \( C^\infty \) diffeomorphism from \( \mathbb{T}^n \times B_{R_0} \) onto an open subset \( \Omega_t \) of \( \mathbb{A}_R^n \).
D.3 Study of the inverse map

We can write \( \Psi_t^{-1}(\theta, r') = (\theta, r' + \chi(\theta, r', t)) \), with \( \chi = (\chi_1, \ldots, \chi_n) \) and \( \chi_i(\cdot, \cdot, t) \in C^\infty(\Omega_t) \) for each \( i \). Given \( (\theta, r', t) \in \mathbb{R}^n \times [0, 1] \), the point \( (\theta, r') \) belongs to \( \Omega_t \) if and only if there exists \( u \in \mathbb{R}^n \) such that \( r' + u \in B_{R_0} \) and

\[
 u = -\eta(t)\psi(\theta, r' + u).
\]

This vector \( u \) is then unique and is \( \chi(\theta, r', t) \). We must prove that \( \mathbb{R}^n_R \subset \Omega_t \), that the restriction of the functions \( \chi_i \) to \( \mathbb{R}^n_R \) belong to \( G^{\alpha, L}(\mathbb{R}^n_R \times [0, 1]) \), and that their Gevrey norms satisfy (2.9). All this follows from Sub-Lemma. Consider the Banach space \( V := (G^{\alpha, L}(\mathbb{R}^n_R \times [0, 1]))^n \), with the norm

\[
 ||u||_V := ||u_1||_{\alpha, L, R} + \cdots + ||u_n||_{\alpha, L, R} \quad \text{for} \quad u = (u_1, \ldots, u_n) \in V.
\]

Let \( \mathcal{B} := \{ u \in V \mid ||u||_V \leq \varepsilon ||\eta||_{\alpha, L_0} \} \). Then, for any \( u \in \mathcal{B} \), the formula

\[
 v(\theta, r', t) := -\eta(t)\psi(\theta, r' + u(\theta, r', t)) \quad \text{(D.5)}
\]

makes sense for all \( (\theta, r', t) \in \mathbb{R}^n_R \times [0, 1] \) and defines a vector-valued function \( v = F(u) \), which belongs to \( \mathcal{B} \). Moreover, the functional \( F : \mathcal{B} \to \mathcal{B} \) satisfies

\[
 ||F(u^*) - F(u)||_V \leq \frac{1}{2}||u^* - u||_V, \quad u, u^* \in \mathcal{B}.
\]

Indeed, the contraction \( F \) has a unique fixed point, which is nothing but \( \chi|_{\mathbb{R}^n_R \times [0, 1]} \).

\textbf{Proof of Sub-Lemma.} Let \( u \in \mathcal{B} \). For each \( (\theta, r', t) \in \mathbb{R}^n_R \times [0, 1] \), we have \( ||u(\theta, r', t)|| \leq \sum ||u_i||_{C^0(\mathbb{R}^n_R \times [0, 1])} \leq ||u||_V \leq \varepsilon \eta ||\eta||_{\alpha, L_0} \leq R_0 - R \) by (D.1) and (D.3), thus

\[
 U(\theta, r', t) := (\theta, r' + u(\theta, r', t)) \in \mathbb{R}^n_R. \quad \text{(D.6)}
\]

Therefore, the function \( v : \mathbb{R}^n_R \times [0, 1] \to \mathbb{R}^n \) is well-defined as \( v = -\eta \cdot (\psi \circ U) \).

For each \( i = 1, \ldots, n \), we can apply Proposition A.1 of [MS03] to the composition \( \psi_i \circ U \): the function \( \psi_i \circ U \) belongs to \( G^{\alpha, L}(\mathbb{R}^n_R \times [0, 1]) \) and \( ||\psi_i \circ U||_{\alpha, L, R} \leq ||\psi_i||_{\alpha, L_0, R_0} \) because

\[
 \sum_{\ell \in \mathbb{N}^{2n+1}, \ell \neq 0} \frac{L^0_0^\ell ||\partial U_k||_{C^0(\mathbb{R}^n_R \times [0, 1])}}{(2n + 1)^{\alpha-1}} \leq \frac{L^0_0}{(2n + 1)^{\alpha-1}}, \quad k = 1, \ldots, 2n \quad \text{(D.7)}
\]

(indeed: for \( k \leq n \), the left-hand side of (D.7) is \( L^0 = \frac{L^0_0}{(2n + 1)^{\alpha-1}} \) by (D.2), and for \( k = n + j \) with \( 1 \leq j \leq n \), the left-hand side is \( \leq L^0 + ||u_j||_{\alpha, L, R} \leq L^0 + \varepsilon ||\eta||_{\alpha, L_0} \), and \( \varepsilon ||\eta||_{\alpha, L_0} \leq \frac{L^0_0}{(2n + 1)^{\alpha-1}} \)). Therefore, by the algebra norm property, \( v_i \in G^{\alpha, L}(\mathbb{R}^n_R \times [0, 1]) \) and \( ||v_i||_{\alpha, L, R} \leq ||\eta||_{\alpha, L_0}||\psi_i||_{\alpha, L_0, R_0} \), hence \( v \in \mathcal{B} \).

Let us now suppose that we are given \( u, u^* \in \mathcal{B} \) and consider the difference between \( v^* := F(u^*) \) and \( v := F(u) \). We have \( v_i^* - v_i = \sum_{j=1}^n M_{i,j} (u_j^* - u_j) \) for each \( i = 1, \ldots, n \), with

\[
 M_{i,j}(\theta, r', t) := -\eta(t) \int_0^1 \partial_n \psi_i \circ U_s(\theta, r', t) \, ds, \quad j = 1, \ldots, n.
\]
where, for each $s \in [0, 1]$, $U_s(\theta, r', t) := (\theta, r' + (1 - s)u(\theta, r', t) + su^*(\theta, r', t)) \in \mathbb{A}_{R_0}^n$ by (D.6).

On the one hand $\partial_{n+j}\psi_i \in G^{\alpha,L_0/2}(\mathbb{A}_{R_0}^n)$ and

$$
\sum_{j=1}^n \|\partial_{n+j}\psi_i\|_{\alpha, L_0, R_0} \leq \frac{2^\alpha}{L_0^\alpha} \|\psi_i\|_{\alpha, L_0, R_0} \tag{D.8}
$$

by (D.1). On the other hand

$$
\sum_{\ell \in \mathbb{N}^{2n+1}, \ell \neq 0} \frac{|\ell|!^\alpha}{\ell!^\alpha} \|\partial^\ell U_s, k\|_{C^0(\mathbb{A}_R^0 \times [0, 1])} \leq \frac{L_0^\alpha}{2^\alpha(2n + 1)^{\alpha-1}}, \quad k = 1, \ldots, 2n, \ s \in [0, 1]
$$

(same verification as for (D.7)). Thus, we can apply again Proposition A.1 of [MS03]: we get $\partial_{n+j}\psi_i \circ U_s \in G^{\alpha,L}(\mathbb{A}_R^0 \times [0, 1])$ and $\|\partial_{n+j}\psi_i \circ U_s\|_{\alpha, L_0, R} \leq \|\partial_{n+j}\psi_i\|_{\alpha, L_0, R_0}$.

Therefore, $M_{i,j} \in G^{\alpha,L}(\mathbb{A}_R^0 \times [0, 1])$ for each $(i,j)$ and, in view of the algebra norm property and (D.8),

$$
\sum_{i,j} \|M_{i,j}\|_{\alpha, L_0, R} \leq \|\eta\|_{\alpha, L_0} \sum_{i,j} \frac{2^\alpha}{L_0^\alpha} \|\psi_i\|_{\alpha, L_0, R_0} = \frac{2^\alpha}{L_0^\alpha} \|\eta\|_{\alpha, L_0} \leq \frac{1}{2}
$$

by (D.1) and (D.3), whence it follows that $\|v^* - v\|_V \leq \sum_{i,j} \|M_{i,j}\|_{\alpha, L_0, R} \|u_j^* - u_j\|_{\alpha, L_0, R} \leq \frac{1}{2} \|u^* - u\|_V$. \qed
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