FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS AND CONSUMER THEORY

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Abstract. In this paper, we show that the existence of a global solution of a standard first-order partial differential equation can be reduced to the extendability of the solution of the corresponding ordinary differential equation under the differentiable and locally Lipschitz environments. By using this result, we can produce many known existence theorems for partial differential equations. Moreover, we demonstrate that such a result can be applied to the integrability problem in consumer theory. This result holds even if the differentiability condition is dropped.

1. Introduction. Consider the following partial differential equation (PDE):

\[ Du(p) = f(p, u(p)). \]

In this paper, we show that the existence problem of a global solution of the above equation can be reduced to the corresponding extension problem for a solution of some parametrized ordinary differential equation (ODE) (Theorems 1, 2). Using this result, we can easily reproduce the results of [6], [1], and [4] (Corollaries 1, 2, 3).

Further, these results have an application in economics, and that is called the integrability problem. Consider the following problem:

\[
\begin{align*}
\max & \quad v(x) \\
\text{subject to.} & \quad x \geq 0, \\
& \quad p \cdot x \leq m,
\end{align*}
\]

and let \( f(p, m) \) be the unique solution of the above problem. The integrability problem considers a method for calculating the information of \( v \) from \( f \). In consumer theory, \( f \) is called the demand function, and represents the purchase behavior of consumer. On the other hand, \( v \) is called a utility function, and represents the preference of consumer. Thus, this problem concerns the problem of revealing the preference of consumer from their purchase behavior. We argue the relationship between the dual of the above problem and the standard partial differential equation (Theorem 3), and show that when there exists the global solution, the information of \( v \) can be recovered by \( f \) (Theorems 4, 5, Corollary 4).

Many results in this paper are derived from the differentiability and local Lipschitz assumption of \( f \). However, even if \( f \) is not differentiable, Theorems 1, 4, and 5 are applicable. We demonstrate such a case (Examples 1, 2).

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In [3], Theorem 2 is shown under the continuous differentiability of $f$. However, in this paper, the continuous differentiability is weakened to the differentiability and local Lipschitz condition. Therefore, this theorem and its corollaries are new.

In section 2, we recall some basic results on ODEs used in this paper. In section 3, we present our basic equivalence theorem, and show several applications. In section 4, we discuss the application of this theorem to economics. All results in section 2 are proved in the appendix.

2. Preliminaries: Notations, and results for ODEs. Throughout this paper, we use the following notations. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, and let $x \in \mathbb{R}^n$. Then, $x_i$ denotes $i$-th coordinate of $x$. Meanwhile, let $f : P \to \mathbb{R}^n$ be given. If $x = f(p)$, then $x_i$ is also denoted by $f_i(p)$. Next, for any $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$ for all $i$, and $x \gg y$ means $x_i > y_i$ for all $i$. $\mathbb{R}^n_{++}$ denotes the set of all $x \in \mathbb{R}^n$ such that $x \gg 0$, and is called the positive orthant. Meanwhile, $\mathbb{R}^n_{+}$ denotes the set of all $x \in \mathbb{R}^n$ such that $x \geq 0$, and is called the nonnegative orthant.

We now mention several basic results on solutions of ODEs. Our setting for the ODEs is different from that in standard textbooks, and thus we attach the proofs of all results in the appendix. Note that similar results as those in this subsection have been derived in many famous textbooks.\footnote{We mainly refer to section 0.4 of [5] and chapter 4 of [7]. The essence of the results in this section are also included in [2], [8], and many other textbooks.}

First, consider the following ODE:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

where $\dot{x}$ denotes the derivative of $x$ with respect to $t$. A solution of this equation is a $C^1$-class function $x$ defined on an interval $I$ containing $t_0$ such that $x(t_0) = x_0$ and $\dot{x}(t) = f(t, x(t))$ for any $t \in I$. We assume that $f : P \to \mathbb{R}^n$, where $P \subset \mathbb{R} \times \mathbb{R}^n$ is open and $(t_0, x_0) \in P$, and that $f$ is continuous in $(t, x)$ and locally Lipschitz in $x$: that is, for every compact set $C \subset P$, there exists $L > 0$ such that if $(t, x_1), (t, x_2) \in C$, then

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|.$$

Then, the following fact holds.

**FACT 1.** There is a solution of the above equation defined on some open interval including $t_0$. Moreover, for any two solutions $x, y$, $x(t) = y(t)$ for all $t$ included in the intersection of the domains of $x$ and $y$.\footnote{This fact is known as the Picard-Lindelöf’s theorem.}

A solution $y$ is called an extension of a solution $x$ if and only if the domain of $y$ includes the domain of $x$. A solution $x$ is nonextendable if and only if there is no extension of $x$ except for $x$ itself. Two facts on nonextendable solutions hold.

**FACT 2.** There uniquely exists a nonextendable solution defined on an open interval.

**FACT 3.** If $x$ is a nonextendable solution defined on $[a, b]$, then for any compact set $C \subset P$, there exist $\tilde{t}, \bar{t} \in [a, b]$ such that $(t, x(t)) \notin C$ if either $a < t \leq \tilde{t}$ or $\bar{t} \leq t < b$.\footnote{This fact is known as the Picard-Lindelöf’s theorem.}
Second, consider the following parametrized ODE:
\[ \dot{x} = f(t, x; y), \quad x(t_0; y) = x_0. \]
We assume that \( f : \tilde{P} \to \mathbb{R}^n \), where \( \tilde{P} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \) is open, and that \( f \) is continuous in \((t, x, y)\) and locally Lipschitz in \( x \). Then, the following fact holds.

**FACT 4.** There uniquely exists a function \( x(t; y) \) defined on some open set in \( \mathbb{R} \times \mathbb{R}^m \) such that if \( y \) is fixed, then \( x(t; y) \) is a nonextendable solution of the above problem. Moreover, \( x \) is continuous in \((t, y)\).

We also call this function \( x(t; y) \) a **nonextendable solution**. Then, the following fact holds.

**FACT 5.** If \( f \) is continuous in \((t, x, y)\) and continuously differentiable in \((x, y)\), then the function \( x(t; y) \) is also continuously differentiable in \( y \), and \( \frac{\partial^2 x}{\partial y^i}(t; y) \) can be defined and is equal to \( \frac{\partial^2 x}{\partial q^i}(t; y) \).

3. **Basic result.** Let \( n \geq 2 \), \( P \) be an open subset of \( \mathbb{R}^n \times \mathbb{R} \), \( \Omega \) be a subset of \( \mathbb{R}^n \), \((p, m) \in P \) and \( f : P \to \Omega \) be continuous. Moreover, suppose that \( f \) is locally Lipschitz in the last variable: that is, for any compact subset \( Q \subset P \), there exists a constant \( L > 0 \) such that for every \((q, w_1), (q, w_2) \in Q \),
\[ \|f(q, w_1) - f(q, w_2)\| \leq L|w_1 - w_2|. \]
If \( f \) is differentiable, define
\[ s_{ij}(q, w) = \frac{\partial f^i}{\partial q_j}(q, w) + \frac{\partial f^i}{\partial w_j}(q, w)f^j(q, w). \]
We say that \( f \) is **integrable** at \((q, w)\) if and only if \( s_{ij}(q, w) = s_{ji}(q, w) \). If \( f \) is integrable at every point in \( P \), then we say that \( f \) is integrable.

Then, the following theorem holds.

**Theorem 1.** Let \( U \) be a convex set in \( \mathbb{R}^n \) with nonempty interior, and \( p \in \text{int} \, U \), \((p, m) \in P \). Then, there exists a continuously differentiable solution \( u \) of the following PDE:
\[ Du(q) = f(q, u(q)), \quad u(p) = m, \] defined on \( U \) only if, for any \( q \in U \), there exists a solution \( c(t; q) \) of the following ODE:
\[ \dot{c}(t) = f\left((1-t)p + tq, c(t)\right) \cdot (q - p), \quad c(0) = m, \] defined on \([0, 1]\). If such a solution \( u : U \to \mathbb{R} \) exists, then it is unique, and \( u(q) = c(1; q) \). Moreover, if \( f \) is differentiable at \((p, m)\) and the solution of (1) exists, then \( f \) is integrable at \((p, m)\).

**Remark 1.** If \( q \) is on the boundary of \( U \), then \( u \) is differentiable at \( q \in U \) if and only if there exists a local differentiable extension \( \tilde{u} : V \to \mathbb{R} \) of \( u \): that is, there exists a differentiable function \( \tilde{u} \) defined on an open neighborhood \( V \) of \( q \), and if \( r \in V \cap U \), then \( u(r) = \tilde{u}(r) \). If \( u \) is differentiable at \( q \), then define \( Du(q) = D\tilde{u}(q) \).

**Proof.** If such a solution \( u : U \to \mathbb{R} \) of (1) exists, define
\[ c(t; q) = u((1-t)p + tq). \]
Then, $c(t;q)$ is a solution of (2) defined on $[0, 1]$. By FACT 2, $c(t;q)$ is the unique solution of (2) with parameter $q$, and thus the solution of (1) is also unique. Clearly $c(1;q) = u(q)$.

If $f$ is differentiable at $(p, m)$ and a continuously differentiable solution $u : U \to \mathbb{R}$ of (1) can be defined, then the right-hand side of (1) is differentiable at $p$. Hence, $u$ is twice differentiable. By a simple calculation, we have

$$s_{ij}(p, m) = \frac{\partial^2 u}{\partial p_i \partial p_j}(p),$$

and thus, the integrability of $f$ at $(p, m)$ is equivalent to

$$\frac{\partial^2 u}{\partial p_i \partial p_j}(p) = \frac{\partial^2 u}{\partial p_j \partial p_i}(p).$$

By Young’s theorem, if $V \subset \mathbb{R}^2$ is open, a function $g : V \to \mathbb{R}$ is differentiable at $(x_1, x_2)$, and both $\frac{\partial g}{\partial x_1}$ and $\frac{\partial g}{\partial x_2}$ are differentiable at $(x_1, x_2)$, then $\frac{\partial^2 g}{\partial x_1 \partial x_2} = \frac{\partial^2 g}{\partial x_2 \partial x_1}(x_1, x_2)$. Because $u$ is twice differentiable, we have (3) holds, and thus $f$ is integrable at $(p, m)$. This completes the proof.

Our aim is to prove the converse of the above theorem.³

**Theorem 2.** Let $p, m, U$ satisfy the conditions of Theorem 1. If $f$ is differentiable, locally Lipschitz, and integrable, and if the solution $c(t;q)$ of equation (2) can be extended to $[0, 1]$ for every $q \in U$, then $u(q) \equiv c(1;q)$ is a solution of equation (1).

**Proof.** By FACT 4, the domain of the function $c$ is open. Therefore, we can extend $u$ to some open set $\bar{U}$ including $U$. Thus, to prove this theorem, it suffices to show that $Du(q) = f(q, u(q))$ for any $q \in U$ such that $U$ includes some neighborhood of the segment $[p, q]$. Fix such a $q \in U$, and define $p(s) = (1 - s)p + sq$. First, suppose that for every $s \in [0, 1]$, there exists $v_s : V_s \to \mathbb{R}$ such that $V_s$ is an open ball centered at $p(s)$ and $v_s$ is a solution of the following differential equation:

$$Dv_s(r) = f(r, v_s(r)), \quad v_s(p(s)) = c(s; q).$$

If $s_1 < s_2$ and $V_{s_1} \cap V_{s_2} \neq \emptyset$, then there exists $s \in [s_1, s_2]$ such that $p(s) \in V_{s_1} \cap V_{s_2}$. Let $d_1(t) = c((1 - t)s_1 + ts_2; q)$ and $d_2(t) = v_s((1 - t)p(s_1) + tp(s_2))$. Then, both $d_i$ are solutions of the following ODE:

$$d_i(t) = f((1 - t)p(s_1) + tp(s_2), d_i(t)) \cdot (p(s_2) - p(s_1)), \quad d_i(0) = c(s_i; q).$$

By FACT 2, we have $d_1(t) = d_2(t)$ if both sides can be defined, and thus, if $s = (1 - t^*)s_1 + t^*s_2$, then $v_{s_1}(p(s)) = d_2(t^*) = d_1(t^*) = c(s; q) = v_s(p(s))$.

By the same reasoning, we have $v_{s_2}(p(s)) = v_s(p(s)) = v_{s_1}(p(s))$. Because $V_{s_1} \cap V_{s_2}$ is convex, Theorem 1 implies that $v_{s_1}(r) = v_{s_2}(r)$ for every $r \in V_{s_1} \cap V_{s_2}$. Therefore, if we define

$$v(r) = v_s(r)$$

for $r \in V \equiv \bigcup_{s \in [0, 1]} V_s$, then $v$ is well-defined and is a solution of (1) that satisfies $v(p(s)) = c(s; q)$.

³If $f$ is continuously differentiable, then the proof of this theorem is quite easy. We will demonstrate this fact in appendix.
Clearly, the domain $V$ of $v$ includes an open and convex neighborhood of the segment $[p, q]$. Therefore, there exists a convex neighborhood $W$ of $q$ such that if $r \in W$ and $t \in [0, 1]$, then $(1 - t)p + tr \in V$. We can assume that $V \subset U$ and $W \subset U$. If $r \in W$ and $t \in [0, 1]$, define $d_1(t) = c(t; r)$ and $d_2(t) = v((1 - t)p + tr)$. Then,

$$
\frac{d_i(t)}{dt} = f((1 - t)p + tr, d_i(t)) \cdot (r - p), \quad d_i(0) = m,
$$

and thus,

$$
u(r) = c(1; r) = d_1(1) = d_2(1) = v(r).
$$

This implies that

$$
Du(q) = Dv(q) = f(q, v(q)) = f(q, u(q)),
$$

and thus $u$ is a solution of (1), as desired.

Therefore, it suffices to show that for every $s \in [0, 1]$, there exists a local solution of (4). We will only argue the case $s = 0$, because the proof of the other cases is almost the same. Thus, it suffices to show that there exists a local solution of (1).

Choose $a > 0, b > 0$ such that

$$
\Pi_1 = \prod_{i=1}^n [p_i - a, p_i + a], \quad \Pi_2 = [m - b, m + b], \quad \Pi = \Pi_1 \times \Pi_2 \subset P,
$$

and let $L > 0$ be a constant such that if $(r_1, w_1), (r_2, w_2) \in \Pi$, then

$$
||f(r_1, w_1) - f(r_2, w_2)|| \leq L ||(r_1, w_1) - (r_2, w_2)||,
$$

and $M = \max_{i,(r,w) \in \Pi} |f_i(r, w)|$. Without loss of generality, we can assume that $a > 0$ is so small that $aM \leq b$.

Now, define $z^0(r) \equiv m$, and define recursively

$$
z^k(r) = m + \int_0^1 f((1 - t)p + tr, z^{k-1}((1 - t)p + tr)) \cdot (r - p) dt.
$$

Note that for every $r \in \Pi_1$, we have $(r, z^0(r)) \in \Pi$, and if $z^{k-1}(r)$ can be defined on $\Pi_1$ and $(r, z^{k-1}(r)) \in \Pi$ for every $r \in \Pi_1$, then $z^k(r)$ can also be defined on $\Pi_1$ and $(r, z^k(r)) \in \Pi$ for every $r \in \Pi_1$. Therefore it follows, by mathematical induction, that $z^k(r)$ is defined on $\Pi_1$, and $(r, z^k(r)) \in \Pi$ for every $r \in \Pi_1$. Now, let $|r|_1 = \sum_{i=1}^n |r_i|$. Then,

$$
|z^1(r) - z^0(r)| \leq \int_0^1 \max_i |f_i((1 - t)p + tr, m)||r - p||_1 dt 
\leq M ||r - p||_1,
$$

$$
|z^2(r) - z^1(r)| \leq \int_0^1 \max_i |f_i((1 - t)p + tr, z^1((1 - t)p + tr)) 
- f_i((1 - t)p + tr, z^0((1 - t)p + tr))||r - p||_1 dt 
\leq \int_0^1 L|z^1((1 - t)p + tr) - z^0((1 - t)p + tr)||r - p||_1 dt 
\leq \int_0^1 tLM||r - p||^2_1 dt = \frac{ML^2||r - p||^2_1}{2!},
$$

... 

$$
|z^k(r) - z^{k-1}(r)| \leq \frac{ML^k||r - p||^k_1}{k!}.
$$
Therefore, the sequence \((z^k(\cdot)) \in C(\Pi_1, \mathbb{R})\) is a Cauchy sequence with respect to the supremum norm, and thus it converges uniformly to a continuous function \(z : \Pi_1 \to \mathbb{R}\).

Next, suppose that the function \(z^{k-1}(\cdot)\) is differentiable and Lipschitz on \(\Pi_1\). Then, the function

\[ g^{k-1}(t, r) = f((1 - t)p + tr, z^{k-1}((1 - t)p + tr)) \cdot (r - p) \]

is differentiable and Lipschitz on \([0, 1] \times \Pi_1\). Thus, by the dominated convergence theorem,

\[
\lim_{r' \to 0} \left| \frac{z^k(r + r') - z^k(r) - r' \cdot \int_0^1 D_r g^{k-1}(t, r)dt}{\|r'\|} \right|
\]

\[
= \lim_{r' \to 0} \int_0^1 \left| g^{k-1}(t, r + r') - g^{k-1}(t, r) - D_r g^{k-1}(t, r)r' \right| dt
\]

\[
= \int_0^1 \lim_{r' \to 0} \left| g^{k-1}(t, r + r') - g^{k-1}(t, r) - D_r g^{k-1}(t, r)r' \right| dt
\]

\[
= 0,
\]

and thus, \(z^k(\cdot)\) is also differentiable and Lipschitz. Moreover,\(^4\)

\[
\frac{\partial z^k}{\partial r_j}(r) = \int_0^1 \left[ f_j + t \left( \frac{\partial f}{\partial q_j} + \frac{\partial f}{\partial w} \frac{\partial z^{k-1}}{\partial r_j} \right) \cdot (r - p) \right] dt.
\]

By mathematical induction, we have \(z^k(r)\) is differentiable and Lipschitz.

We will show that

\[
\|Dz^k(r) - f(r, z^k(r))\|_1 \leq nM \left( \frac{3Ln\|r - p\|_1}{k!} \right)^k.
\]

For \(k = 0\), this is obvious. Suppose that it is true for \(k\), and define\(^5\)

\[
h_j(t, r) = f_j + t \left( \frac{\partial f}{\partial q_j} + \frac{\partial f}{\partial w} \frac{\partial z^{k-1}}{\partial r_j} \right) \cdot (r - p)
\]

\[
= f_j + t \sum_{i=1}^n \left( \frac{\partial f_j}{\partial q_i} + \frac{\partial f_j}{\partial w} \frac{\partial z^{k-1}}{\partial r_i} \right) (r_i - p_i)
\]

\[
+ t \sum_{i=1}^n \frac{\partial f_j}{\partial w} \left( f_i - \frac{\partial z^{k}}{\partial r_i} \right) (r_i - p_i) + t \sum_{i=1}^n \frac{\partial f_i}{\partial w} \left( \frac{\partial z^{k}}{\partial r_j} - f_j \right) (r_i - p_i)
\]

\[
= \frac{d}{dt}[f_j] + t \sum_{i=1}^n \left[ \frac{\partial f_j}{\partial w} \left( f_i - \frac{\partial z^{k}}{\partial r_i} \right) + \frac{\partial f_i}{\partial w} \left( \frac{\partial z^{k}}{\partial r_j} - f_j \right) \right] (r_i - p_i),
\]

where the second equality follows from the integrability condition \(s_{ij} = s_{ji}\) of \(f\).

Thus,

\[
\|Dz^{k+1}(r) - f(r, z^{k+1}(r))\|_1
\]

\[
= \sum_{j=1}^n \left| \int_0^1 h_j(t, r)dt - f_j(r, z^{k+1}(r)) \right|
\]

\[
\leq \|f_j(r, z^k(r)) - f_j(r, z^{k+1}(r))\|_1
\]

\(^4\)We abbreviate the variables of functions.

\(^5\)We abbreviate \(f((1 - t)p + tr, z^k((1 - t)p + tr))\) as \(f\) and \(z^k((1 - t)p + tr)\) as \(z^k\).
\[ + \sum_{j=1}^{n} \left| \int_{0}^{1} \left( \sum_{i=1}^{n} \left[ \frac{\partial f_j}{\partial w} \left( f_i - \frac{\partial z^k}{\partial r_i} \right) + \frac{\partial f_i}{\partial w} \left( \frac{\partial z^k}{\partial r_j} - f_j \right) \right] (r_i - p_i) dt \right) \right| \equiv J_1 + J_2, \]

where

\[ J_1 \leq nL |z^{k+1}(r) - z^k(r)| \leq nM \frac{(L\|r - p\|_1)^{k+1}}{(k + 1)!}, \]
\[ J_2 \leq L\|r - p\|_1 \int_{0}^{1} \sum_{j=1}^{n} \left| f_j(t) - \frac{\partial z^k}{\partial r_j}(t) \right| dt \leq L\|r - p\|_1 \times 2(n - 1)nM \frac{(3L\|r - p\|_1)^k}{k!} \int_{0}^{1} t^k dt \]
\[ = 2 \times 3^k(n - 1)n^{k+1}M \frac{(L\|r - p\|_1)^{k+1}}{(k + 1)!}. \]

Hence,
\[ \|Dz^{k+1}(r) - f(r, z^{k+1}(r))\|_1 \leq nM \frac{(L\|r - p\|_1)^{k+1}}{(k + 1)!} [1 + (3^k \times 2)(n - 1)n^k] \]
\[ \leq nM \frac{(3L\|r - p\|_1)^{k+1}}{(k + 1)!}, \]

as desired.

Let \( r \) be in the interior of \( \Pi_1 \), and define
\[ \psi(s, k, r) = \begin{cases} \frac{z^k(r + se_j) - z^k(r)}{s}, & \text{if } s \neq 0, r + se_j \in \Pi_i, \\ \frac{\partial z^k}{\partial r_j}(r), & \text{if } s = 0. \end{cases} \]

If \( s \neq 0 \), by the mean value theorem,
\[ |\psi(s, k + \ell, r) - \psi(s, k, r)| = \left| \frac{z^{k+\ell}(r + se_j) - z^{k+\ell}(r)}{s} - \frac{z^{k}(r + se_j) - z^{k}(r)}{s} \right| \]
\[ = \left| \frac{(z^{k+\ell} - z^k)(r + se_j) - (z^{k+\ell} - z^k)(r)}{s} \right| \]
\[ = \left| \frac{\partial z^{k+\ell}}{\partial r_j}(r + s\theta e_j) - \frac{\partial z^{k}}{\partial r_j}(r + s\theta e_j) \right| \]
for some \( \theta \in [0, 1] \), where the right-hand side tends to 0 uniformly on \( s, \ell \) as \( k \to \infty \). Clearly this evaluation is valid even if \( s = 0 \). Therefore, the function \( s \to \psi(s, k, r) \) converges uniformly to some function \( \psi(s, r) \) as \( k \to \infty \), where \( \psi \) is continuous in \( s \) and, clearly,
\[ \psi(s, r) = \frac{z(r + se_j) - z(r)}{s}, \]

if \( s \neq 0 \). Thus,
\[ \lim_{s \to 0} \left| \frac{z(r + se_j) - z(r)}{s} - f_j(r, z(r)) \right| \]
\[ = \lim_{s \to 0} |\psi(s, r) - f_j(r, z(r))| \]
\[ \leq \lim_{s \to 0} [|\psi(s, r) - \psi(s, k, r)| + |\psi(s, k, r) - f_j(r, z^k(r))| + |f_j(r, z^k(r)) - f_j(r, z(r))|]] \]

6We abbreviate \( f((1 - t)p + tr, z^k((1 - t)p + tr)) \) as \( f(t) \), and \( z^k((1 - t)p + tr) \) as \( z^k(t) \).
Suppose that Corollary 1.

Further, because $z^k(p) \equiv m$ by definition, we have $z(p) = m$, and thus $z$ is a solution of (1) defined on some neighborhood of $p$. This completes the proof.

There are three corollaries to this theorem. The first is a revival of a classical result in [6].

Corollary 1. Suppose that $f : P \to \Omega$ is differentiable, locally Lipschitz, and integrable, and $P$ includes the cube $\Pi = (p, m) + \prod_{i=1}^{n}[-a, a] \times [-b, b]$. Let $Q = p + \prod_{i=1}^{n}[-c, c]$ and $c \leq \min\{a, b/nM\}$, where

$$M = \max_{i \in \{1, \ldots, n\}, (q, w) \in \Pi} |f^i(q, w)|.$$

Then, there exists a unique solution $u$ of (1) defined on $Q$.

Proof. By Theorem 2, it suffices to show that the solution $c(t; q)$ of (2) can be extended to $[0, 1]$ for all $q \in Q$. Suppose, on the contrary, that $c(t; q)$ can be defined only on $[0, t^*]$ for some $t^* \leq 1$. Define

$$\bar{t} = \sup\{t \geq 0 | \forall s \in [0, t], |c(s; q) - m| \leq b\}.$$

Then, $\bar{t} \in [0, t^*]$. Now, for any $t \in [0, \bar{t}]$,

$$|c(t; q) - m| = |c(t; q) - c(0; q)|$$

$$\leq \int_{0}^{t} \left| \sum_{i=1}^{n} f^i((1-s)p + sq, c(s; q))(q_i - p_i) \right| ds$$

$$\leq \int_{0}^{t} \sum_{i=1}^{n} M|q_i - p_i|ds$$

$$\leq bt.$$

If $\bar{t} < t^*$, then $\bar{t} < 1$, and thus $|c(\bar{t}; q) - m| < b$. However, in this case $|c(t; q) - m| < b$ for some $t$ such that $\bar{t} < t < t^*$, which contradicts the definition of $\bar{t}$. Therefore, $\bar{t} = t^*$. Thus, $(p(t), c(t; q)) \in \Pi$ for any $t \in [0, t^*]$, which contradicts FACT 3 and the definition of $t^*$. This completes the proof.

The second corollary is an extension of Theorem 10.9.4 of [1].

Corollary 2. Suppose that $f : P \to \Omega$ is differentiable, locally Lipschitz, and integrable. Then, for any $(p, m) \in P$, there exists an open and convex neighborhood $U$ of $p$ such that there exists a solution $u : U \to \mathbb{R}$ of (1).

Proof. Let $c(t; q)$ be the nonextendable solution of (2). Then, clearly

$$c(t; p) \equiv m.$$

Thus, the domain of the mapping $t \mapsto c(t; p)$ is $\mathbb{R}$ itself. By FACT 4, the domain $c : (t, q) \mapsto c(t; q)$ is open, and thus, we have that there exists an open and convex neighborhood $U$ of $p$ such that if $q \in U$, then the domain of $t \mapsto c(t; q)$ includes $[0, 1]$. Therefore, Theorem 2 can be applied and this completes the proof.

The third corollary is an extension of Theorem 2 of [4].
Corollary 3. Suppose that \( P = \mathbb{R}_+^n \times \mathbb{R}_+^m \) and \( \Omega = \mathbb{R}_+^n \). Let \( f: P \to \Omega \) be differentiable and locally Lipschitz, the matrix-valued function \( S_f(p, m) = (s_{ij}(p, m)) \) be negative semi-definite and symmetric,\(^7\) and Walras’ law \( p \cdot f(p, m) = m \) holds for every \((p, m) \in P\). Then, for every \((p, m) \in P\), there exists a global solution \( u: \mathbb{R}_+^n \to \mathbb{R}_+^n \) of (1).

Proof. By Theorem 2, this is equivalent to the extendability of the solution \( c(t; q) \) of (2) to \([0, 1]\). Suppose that \( c(t; q) \) is extendable only on \([0, t^*]\), where \( t^* \leq 1 \). By FACT 4, either \( \lim \inf_{t \to t^*} c(t; q) = 0 \) or \( \lim \sup_{t \to t^*} c(t; q) = +\infty \) holds.

Define \( p(t) = (1-t)p + tq \) and \( x = f(p(m)) \). We need the following lemma:

Lemma 1. Suppose that \( c(t; q) \) is a solution of (2) defined on \([0, \hat{t}]\). Let \( y = f(p(t), c(t; q)) \). Then, \( p \cdot y \geq m \) and \( p(t) \cdot x \geq c(t; q) \).

Proof of Lemma 1. Let \( d(t) = p \cdot f(p(t), c(t; q)) \). Then, by a simple calculation,
\[
\dot{d}(t) = p^T S_f(p(t), c(t; q))(q - p).
\]

Meanwhile, by differentiating the both side of Walras’ law
\[
q \cdot f(q, w) = w
\]
with respect to \( q \) and \( w \), we have
\[
\sum_{i=1}^{n} q_i \frac{\partial f_i}{\partial q}(q, w) + f_j(q, w) = 0,
\]
\[
\sum_{i=1}^{n} q_i \frac{\partial f_i}{\partial w}(q, w) = 1.
\]
Therefore, we can easily show that
\[
p(t)^T S_f(p(t), c(t; q))(q - p) = 0.
\]
Hence,
\[
\dot{d}(t) = -t(q - p)^T S_f(p(t), c(t; q))(q - p) \geq 0.
\]
Thus, we have \( p \cdot y = d(t) \geq d(0) = m \). The proof of the rest of the claim is symmetrical and is omitted here. \( \square \)

By Lemma 1, we have that \( p(t) \cdot x \geq c(t; q) \), and thus \( \lim \sup_{t \to t^*} c(t; q) < +\infty \). Therefore, \( \lim \inf_{t \to t^*} c(t; q) = 0 \), and hence there exists a sequence \((t_k)\) such that \( t_k \uparrow t^* \) and \( c(t_k; q) \to 0 \) as \( k \to \infty \). Let \( x^k = f(p(t_k), c(t_k; q)) \). By Lemma 1, we have \( p \cdot x^k \geq m \) and \( p(t_k) \cdot x \geq c(t_k; q) = p(t_k) \cdot x^k \). This implies that \( q \cdot x^k \leq q \cdot x \), and thus the sequence \((x^k)\) is bounded. Therefore, taking a subsequence, we can assume that \( x^k \to x^* \geq 0 \). Because \( p \cdot x^* \geq m \), we have \( x^* \neq 0 \). Then,
\[
0 < p(t^*) \cdot x^* = \lim_{k \to \infty} p(t_k) \cdot x^k = \lim_{k \to \infty} c(t_k; q) = 0,
\]
a contradiction. This completes the proof. \( \square \)

Is there a function \( f \) that admits the existence of the solution of (1) but is not covered by Theorem 2? The following simple example answers this question. Theorem 1 is essential for determining the solution in this example.

\(^7\) The symmetry of \( S_f(p, m) \) is exactly the same condition as the integrability of \( f \).
Example 1. Consider the following function
\[ f(p, m) = \begin{cases} \left( \frac{w}{q_1}, 0 \right), & \text{if } q_2^2 \geq 4q_1w, \\ \left( \frac{q_2^2 - 4q_1w - q_1^2}{4q_1}, \frac{4q_1w - q_2^2}{4q_1q_2} \right), & \text{otherwise.} \end{cases} \]

This function is not differentiable if \( p_2^2 = 4p_1m \), and thus Theorem 2 is not applicable. We will guess a solution of the equation (1) defined by the above \( f \), and verify that it is actually the (unique) solution. First, choose any \( q = (q_1, q_2) \). If \( u \) is a solution of (1), then \( c(t) = u((1-t)p + tq) \) satisfies the following ODE:
\[ c(t) = f((1-t)p + tq, c(t)) \cdot (q - p), \quad c(0) = m. \] (5)

Therefore, we can guess that if \( c(t) \) is a solution of (5) defined on \([0, 1]\), then \( c(1) \) coincides with \( u(q) \). Second, define
\[ f^1(q, w) = \left( \frac{w}{q_1}, 0 \right), \quad f^2(q, w) = \left( \frac{q_2^2 - 4q_1w - q_1^2}{4q_1}, \frac{4q_1w - q_2^2}{4q_1q_2} \right), \]
and consider
\[ \dot{c}_i(t) = f^i((1-t)p + tq, c_i(t)) \cdot (q - p). \] (6)

To solve (6), we have
\[ c_1(t) = c_1(s) \frac{p_1 + t(q_1 - p_1)}{p_1 + s(q_1 - p_1)}, \]
and if \( q_2 = p_2 \), then
\[ c_2(t) = c_2(s) - \frac{1}{4} \left[ \frac{(p_2 + t(q_2 - p_2))^2}{p_1 + t(q_1 - p_1)} - \frac{(p_2 + t(q_2 - p_2))^2}{p_1 + s(q_1 - p_1)} \right]. \]

Third, suppose that \( p_2 = q_2 \), \( 4p_1m \leq p_2^2 \), and \( 4q_1c_1(1) \leq q_2^2 \), where \( c_1(0) = m \). Note that \( 4(p_1 + t(q_1 - p_1))c_1(t) \) is monotone. Thus, in this case we have \( c(t) = c_1(t) \) on \([0, 1]\), and
\[ 4q_1c_1(1) = \frac{4q_2^2m}{p_1} \leq q_2^2. \]

Therefore, we obtain the following candidate for \( u(q) \):
\[ u(q) = \frac{q_1m}{p_1} \] (7)

if \( 2q_1 \sqrt{\frac{m}{p_1}} \leq q_2 \). Because \( f \) is homogeneous of degree zero, we can guess that \( u \) is homogeneous of degree one, and thus we can remove the assumption \( p_2 = q_2 \). By an easy calculation, we can confirm that \( u \) is actually a solution of equation (1) on the set \( \{ q | 2q_1 \sqrt{\frac{m}{p_1}} \leq q_2 \} \).

Fourth, suppose that \( p_2 = q_2 \), \( 4p_1m \leq p_2^2 \), and \( 4q_1c_1(1) > q_2^2 \), where \( c_1(0) = m \). Note that \( q_1 \neq p_1 \). If \( q_1 < p_1 \), then \( c_1(t) \) is decreasing and \( 4q_1c_1(1) < q_2^2 \), which is absurd. Thus, we have \( q_1 > p_1 \). We can guess that \( c(t) = c_1(t) \) on \([0, t^*]\), and \( c(t) = c_2(t) \) on \([t^*, 1]\), where \( c(t^*) = c_1(t^*) = c_2(t^*) \) and \( \dot{c}_2(t^*) = \dot{c}_1(t^*) \). Then,
\[ \frac{c_1(t^*)(q_1 - p_1)}{p_1 + t^*(q_1 - p_1)} = \dot{c}_1(t^*) = \dot{c}_2(t^*) = \frac{p_2^2(q_1 - p_1)}{4(p_1 + t^*(q_1 - p_1))^2}, \]
and thus
\[ c(t^*) = c_1(t^*) = c_2(t^*) = \frac{p_2^2}{4(p_1 + t^*(q_1 - p_1))}. \]
In particular, therefore, we obtain the following candidate for

\[ u = \frac{m(p_1 + t^*(q_1 - p_1))}{p_1} = \frac{p_2^2}{4(p_1 + t^*(q_1 - p_1))}, \]

and hence, we obtain

\[ t^* = \frac{p_1}{q_1 - p_1} \left[ \sqrt{\frac{p_2^2}{4p_1m}} - 1 \right]. \]

Check that \( t^* \in [0, 1] \) if and only if \( 2q_1 \sqrt{\frac{m}{p_1}} \geq q_2 \), which is equivalent to \( 4q_1c_1(1) \geq q_2^2 \). Because we have assumed \( 4q_1c_1(1) > q_2^2 \), this assumption holds. Then,

\[ c(t) = \begin{cases} m_1 + \frac{t(q_1 - p_1)}{p_1} & \text{if } t \leq t^*, \\ \left( p_2 + t(q_2 - p_2) \right) \sqrt{\frac{m}{p_1}} - \frac{(p_2 + t(q_2 - p_2))^2}{4(p_1 + t(q_1 - p_1))} & \text{if } t \geq t^*. \end{cases} \]

In particular,

\[ c(1) = q_2 \sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1}. \]

Therefore, we obtain the following candidate for \( u \):

\[ u(q) = q_2 \sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1}, \]

where this form is homogeneous of degree one. Thus, we can guess that

\[ u(q) = \begin{cases} \frac{2m}{p_1}, & \text{if } 2q_1 \sqrt{\frac{m}{p_1}} \leq q_2, \\ q_2 \sqrt{\frac{m}{p_1}} - \frac{q_2^2}{4q_1}, & \text{otherwise}. \end{cases} \]

We can check that this \( u \) is actually a solution of \( 1 \) with \( u(p) = m \).

By a similar argument, we obtain the following candidate for a solution \( u \) of \( 1 \) even if \( 4p_1m > p_2^2 \):

\[ u(q) = \begin{cases} q_1 \left( \frac{2p_1 + 4p_1m - p_2^2}{4p_1p_2} \right)^2, & \text{if } 2q_1 \left( \frac{2p_1 + 4p_1m - p_2^2}{4p_1p_2} \right) \leq q_2, \\ q_2 \left( \frac{2p_1 + 4p_1m - p_2^2}{4p_1p_2} \right) - \frac{q_2^2}{4q_1}, & \text{otherwise}. \end{cases} \]

It can easily be verified that this \( u \) is actually a solution of \( 1 \).

4. **Application in economics: The integrability problem.** Throughout this section, we assume that \( P = \mathbb{R}^n_{++} \times \mathbb{R}_{++} \) and \( \Omega = \mathbb{R}^n_+ \).

Consider the following (simple) optimization problem:

\[
\text{max} \quad v(x) \\
\text{subject to} \quad x \geq 0, \\
p \cdot x \leq m,
\]

where \( x \) is called the **consumption bundle**, \( p \) is called the **price vector**, and \( m \) is called the **income**. The value \( v(x) \) measures the goodness of the consumption bundle \( x \) for this consumer, and the function \( v \) is called the **utility function**. This problem represents the usual consumption problem. Under several conditions, for any \( p \gg 0 \) and \( m > 0 \) there uniquely exists a solution \( f(p, m) \) of the above problem, and the function \( f \) is called the **demand function** of \( v \). If \( v \) is increasing, then \( f \) clearly satisfies Walras' law \( p \cdot f(p, m) = m \).
Now, consider the following observation problem: we can observe consumer’s purchase behavior, and thus we can estimate the demand function $f$ by purchase data. In contrast, the utility function $v$ represents the consumer’s preference, which is hidden in the consumer’s mind. Therefore, to estimate $v$ is difficult. Our problem is the following: can we reversely calculate the information of $v$ hidden in the consumer’s mind. Therefore, to estimate $v$ is difficult. Our problem is the following: can we reversely calculate the information of $v$ by purchase data? If so, then we can estimate $v$ using the estimated value of $f$ and this reverse calculation method. In other words, we can reveal the consumer’s preference by purchase data. In economics, this problem is called the **integrability problem**.

Now, consider the following ‘dual’ problem:

$$\min \quad q \cdot y$$

subject to. \quad $y \geq 0$, 

$$v(y) \geq v(x).$$

Let $E^x(q)$ be the value of the above dual problem: that is,

$$E^x(q) = \inf\{q \cdot y | v(y) \geq v(x)\}.$$  

This function is called the **expenditure function** in consumer theory. We will show the following theorem.

**Theorem 3.** Suppose that the demand function $f : P \to \Omega$ of $v$ is differentiable and satisfies Walras’ law, and $x = f(p, m)$. Then, the function $E^x$ is concave and twice differentiable, and

$$DE^x(q) = f(q, E^x(q))$$

for every $q \in \mathbb{R}^n_+$.\(^8\) Moreover, $E^x(p) = m$, and if $v$ is continuous, then

$$v(y) \geq v(x) \Leftrightarrow E^y(q) \geq E^x(q)$$

for every $q \in \mathbb{R}^n_+$.

**Proof.** Choose any $q_1, q_2 \in \mathbb{R}^n_+, t \in [0, 1]$. Fix any $\varepsilon > 0$, and suppose $v(y) \geq v(x)$ and $q \cdot y \leq E^x(q) + \varepsilon$, where $q = (1 - t)q_1 + tq_2$. Then,

$$E^x(q) + \varepsilon \geq q \cdot y = (1 - t)q_1 \cdot y + tq_2 \cdot y \geq (1 - t)E^x(q_1) + tE^x(q_2).$$

Because $\varepsilon > 0$ is arbitrary, we have that $E^x$ is concave, and thus continuous.

Next, suppose that $v(y) \geq v(x)$ and $y \neq x$. Because $x = f(p, m)$, we have $p \cdot y > m$. Meanwhile, $v(x) \geq v(x)$ and $p \cdot x = m$. This implies that $E^x(p) = m$.

Define $x(q) = f(q, E^x(q))$. This function is continuous and $q \cdot x(q) = E^x(q)$. Fix any $\varepsilon > 0$ and define $x_\varepsilon(q) = f(q, E^x(q) + \varepsilon)$. By definition of $E^x(q)$, there exists $y \in \Omega$ such that $v(y) \geq v(x)$ and $q \cdot y < E^x(q) + \varepsilon$. This implies that $v(x_\varepsilon(q)) \geq v(y) \geq v(x)$. Hence, for any $q, r \in \mathbb{R}^n_+$, we have $q \cdot x(q) = E^x(q) \leq q \cdot x_\varepsilon(r)$. If $\varepsilon \downarrow 0$, then $x_\varepsilon(r) \to x(r)$ and thus $q \cdot x(q) \leq q \cdot x(r)$.

Now, let $e_i$ be the $i$-th unit vector and $q(t) = q + te_i$. Then,

$$E^x(q(t)) - E^x(q) = (q + te_i) \cdot x(q + te_i) - q \cdot x(q)$$

$$= q \cdot (x(q + te_i) - x(q)) + tx_i(q + te_i)$$

$$\geq tf_i(q + te_i, E^x(q + te_i)).$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} \frac{E^x(q(t)) - E^x(q)}{t} \geq f_i(q, E^x(q)) \geq \lim_{\varepsilon \downarrow 0} \frac{E^x(q(t)) - E^x(q)}{t},$$

\(^8\)In economics, this equation is called Shephard’s lemma.
where both limits exist and \( \lim_{t \to 0} \frac{E^x(q(t)) - E^x(q)}{t} \leq \lim_{t \to 0} \frac{E^y(q(t)) - E^y(q)}{t} \) because \( E^x \) is concave. This means that \( \frac{DE^x(q)}{\partial q_i}(q) = f_i(q, E^x(q)) \), and thus we have \( DE^x(q) = f(q, E^x(q)) \), as desired.

Therefore, \( DE^x(q) \) is continuous, and thus \( f(q, E^x(q)) \) is differentiable. Hence, \( DE^x(q) \) is differentiable, and thus \( E^x(q) \) is twice differentiable.

Next, let \( v \) be continuous, and define \( z = f(q, E^y(q)) \) and \( w = f(q, E^z(q)) \). By definition, for every \( \varepsilon > 0 \), there exists \( z' \) such that \( v(z') \geq v(y) \) and \( q \cdot z' < E^y(q) + \varepsilon \). Therefore,

\[
    v(f(q, E^y(q) + \varepsilon)) \geq v(z') \geq v(y),
\]

and by the continuity of \( v, f \), we have \( v(z) \geq v(y) \). Meanwhile,

\[
    v(f(q, E^y(q) - \varepsilon)) < v(y)
\]

by definition, and thus we have \( v(z) \leq v(y) \), which implies that \( v(z) = v(y) \). By the same arguments, we have \( v(x) = v(w) \). Then,

\[
    E^y(q) \geq E^x(q) \iff q \cdot z \geq q \cdot w \iff v(z) \geq v(w) \iff v(y) \geq v(x),
\]

as desired. This completes the proof. \( \square \)

This theorem includes a key idea for solving the integrability problem. We separate this idea into three steps.

- In the first step, we assume that \( f \) is a demand function of \( v \), and that we know \( v \). If \( v \) is continuous, then for any fixed \( \bar{p} \in \mathbb{R}^n_+ \), the mapping \( v_{f,\bar{p}} : x \mapsto E^x(\bar{p}) \) has the same information as \( v \).

- In the second step, we again assume that \( f \) is a demand function of \( v \), and that \( v \) is continuous. However, in this step, \( v \) is assumed to be unknown. In this case, the above \( v_{f,\bar{p}} \) still has the same information as \( v \). Because \( v \) is unknown, however, we cannot calculate \( v_{f,\bar{p}}(x) \) directly.

  In this case, Theorem 3 is important. Actually, we know that \( E^x(\bar{p}) = m \). Further, \( E^x \) is a solution of (1). Therefore, if \( f \) is locally Lipschitz in \( m \), we can apply the uniqueness result in Theorem 1, and thus \( E^x \) is the unique solution of (1). Hence, we can calculate \( v_{f,\bar{p}}(x) = E^x(\bar{p}) = u(\bar{p}) \) by solving the equation (1) or, alternatively, the equation (2) with \( q = \bar{p} \) and calculating \( c(1; \bar{p}) \).\(^9\)

- In the third step, only the function \( f \) is given. We do not know whether this function is a demand function of some utility function \( v \). In this case, there may be no solution of (1). If there is a unique solution \( E^x \) of (1) for every \( x = f(p, m) \), then \( v_{f,\bar{p}} : x \mapsto E^x(\bar{p}) \) can be calculated. However, it is not known whether \( f \) is a demand function of \( v_{f,\bar{p}} \). Therefore, we need a verification procedure for \( f \) to be a demand function of some utility function.

Actually, the following theorem holds.

**Theorem 4.** Suppose that \( f : P \to \Omega \) is continuous, locally Lipschitz in \( m \), and satisfies Walras’ law. Let the following two conditions hold:

(I) For every \((p, m)\), there exists a concave solution \( u : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) of (1).

(II) If \( x \neq y, x = f(p, m), y = f(q, w), \) and \( w \geq u(q) \), (where \( u : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) is a solution of (1),) then \( p \cdot y > m \).

Then, \( f \) is a demand function of some utility function.

---

\(^9\)We cannot calculate \( v_{f,\bar{p}}(x) \) for any \( x \) that is not included in the range of \( f \). This is a limit of this theory.
Corollary 4. Suppose that \( f \) is a demand function of some utility function \( v \), then (I) holds by Theorem 3. Moreover, if \( v \) is continuous and \( x = f(p, m), y = f(q, w), x \neq y \) and \( w \geq u(q) \), then \( E^v(q) = w \geq u(q) = E^v(q) \), and thus \( v(y) \geq v(x) \) by Theorem 3. If \( p \cdot y \leq m \), then \( v(y) \leq v(x) \), and thus \( v(y) = v(x) \). However, this contradicts that \( x = f(p, m) \) is the unique solution of the consumption problem with \((p, m)\). Therefore, we have \( p \cdot y > m \), and (II) holds. Thus, (I) and (II) are necessary conditions for the demand function of the continuous utility function. Theorem 4 says that, under several conditions, they are also sufficient conditions for the demand function.

Proof. Suppose that (I) and (II) hold. Let \( x \neq y, x = f(p, m), y = f(q, w), p \cdot y \leq m \), and \( u_1, u_2 : \mathbb{R}^n_+ \to \mathbb{R}_+^+ \) be solutions of

\[
Du(r) = f(r, u(r)) \tag{10}
\]

with \( u_1(p) = m \) and \( u_2(q) = w \), respectively. By contraposition of (II), we have that \( u_1(q) > w = u_2(q) \), and thus, we have \( u_1(r) > u_2(r) \) for every \( r \in \mathbb{R}^n_+ \). In particular, \( m = u_1(p) > u_2(p) \), and hence, we have \( q \cdot x > w \) by (II). To summarize this argument, if \( x \neq y, x = f(p, m), y = f(q, w) \) and \( p \cdot y \leq m \), then \( q \cdot x > w \). (In economics, this property is called the weak axiom of revealed preference.)

Suppose that \( x = f(p, m) = f(q, w) \). Let \( p(t) = (1 - t)p + tq \) and \( m(t) = (1 - t)m + tw \). If \( f(p(t), m(t)) = y \neq x \) for \( t \in [0, 1] \), then \( p(t) \cdot y = m(t) = p(t) \cdot x \), and thus by the weak axiom of revealed preference, we have \( p \cdot y > m \) and \( q \cdot x > w \), which implies that \( p(t) \cdot y > m(t) \), a contradiction. Therefore, \( f(p(t), m(t)) \equiv x \) on \([0, 1] \). Then,

\[
m(t) = \bar{p}(t) \cdot x = (q - p) \cdot f(p(t), m(t)), \quad m(0) = m.
\]

Meanwhile, if \( u \) is a solution of (10) with \( u(p) = m \), and \( c(t) = u((1 - t)p + tq) \), then

\[
c(t) = (q - p) \cdot f(p(t), c(t)), \quad c(0) = m.
\]

Therefore, both \( c(t) \) and \( m(t) \) are solutions of the same ODE. By the local Lipschitzian assumption in \( m \) and FACT 2, such a solution is unique, and thus \( c(t) \equiv m(t) \). In particular,

\[
u(q) = c(1) = m(1) = w.
\]

Fix any \( \bar{p} \in \mathbb{R}^n_+ \) and define \( v_{f, \bar{p}}(x) = 0 \) if \( x \) is not in the range of \( f \), and if \( x = f(p, m) \), then \( v_{f, \bar{p}}(x) = u(\bar{p}) \), where \( u : \mathbb{R}^n_+ \to \mathbb{R}_+^+ \) is a solution of (10) with \( u(p) = m \). Then, the above fact implies that the definition of \( v_{f, \bar{p}}(x) \) does not depend on the choice of \((p, m) \in f^{-1}(x) \).

Next, let \( x = f(p, m), x \neq y \) and \( p \cdot y \leq m \). If \( y \) is not in the range of \( f \), then \( v_{f, \bar{p}}(y) = 0 < v_{f, \bar{p}}(x) \). Otherwise, let \( u_1 \) (resp. \( u_2 \)) be the solution of (10) with \( u_1(p) = m \) (resp. \( u_2(q) = w \)). By contraposition of (II), we have \( u_1(q) > w = u_2(q) \). This implies that \( u_1(p) > u_2(\bar{p}) \), and thus \( v_{f, \bar{p}}(x) > v_{f, \bar{p}}(y) \). Therefore, we have that \( f \) is the demand function of the utility function \( v_{f, \bar{p}} \). This completes the proof. \( \square \)

By Corollary 3, we can obtain the following result.

Corollary 4. Suppose that \( f : P \to \Omega \) is differentiable, locally Lipschitz, and satisfies Walras’ law. Moreover, suppose that the matrix-valued function \( S_f(p, m) = (s_{ij}(p, m)) \) (hereafter, the Slutsky matrix) is negative semi-definite and symmetric. Then, \( f \) is a demand function of some utility function.

---

10If \( u_1(r) \leq u_2(r) \), then there exists \( r' \in [q, r] \) such that \( u_1(r') = u_2(r') \) by the intermediate value theorem, and Theorem 1 implies that \( u_1 \equiv u_2 \), a contradiction.
Remark 3. If \( f \) is a demand function of some utility function, then for every \((p,m)\) with \( x = f(p,m) \),
\[
DE^x(q) = f(q,E^x(q)), \quad E^x(p) = m,
\]
and thus,
\[
D^2E^x(p) = S_f(p,m).
\]
Because of Young’s theorem, \( S_f(p,m) \) is symmetric. Because \( E^x \) is concave, \( S_f(p,m) \) is negative semi-definite. Therefore, the above corollary states that the negative semi-definiteness and symmetry of the Slutsky matrix \( S_f(p,m) \) is the necessary and sufficient condition for \( f \) to be a demand function.

Proof. By Corollary 3, we have that for every \((p,m)\), there exists a solution \( u : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) of (10) with \( u(p) = m \). Then,
\[
D^2u(q) = S_f(q,u(q))
\]
is negative semi-definite, and thus \( u \) must be concave and statement (I) of Theorem 4 holds.

Let \( u_1 : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) be the solution of (10) with \( u_1(q) = w \), and define \( p(t) = (1-t)p + tq \) and \( d(t) = p \cdot f(p(t),u_1(p(t))) \). If \( u_1(q) = w > u(q) \), then \( u_1(p) > u(p) \).

In the proof of Lemma 1, we have shown that \( d \) is nondecreasing, and thus,
\[
p \cdot y = d(1) \geq d(0) = p \cdot f(p,u_1(p)) = u_1(p) > u(p) = m.
\]
Next, suppose that \( u_1(q) = w = u(q) \). Then, \( u_1(r) = u(r) \) for every \( r \in \mathbb{R}^n_+ \). Thus, \( p \cdot y = d(1) \) and \( m = p \cdot x = d(0) \). To show that statement (II) holds, it suffices to show that \( d(1) > d(0) \). Suppose not. Because \( d \) is nondecreasing, we have \( d(t) = d(0) = m \) for every \( t \in [0,1] \). We abbreviate \( S_f(p(t),u(p(t))) \) as \( S_t \). In the proof of Lemma 1, we showed that
\[
\dot{d}(t) = -t(q-p)^TS_t(q-p).
\]
Because \( S_t \) is symmetric and negative semi-definite, we have that there exists a symmetric, positive semi-definite matrix \( A_t \) such that \( S_t = -A_t^2 \).

Then,
\[
d(t) = t\|A_t(q-p)\|^2 = 0,
\]
for every \( t \in [0,1] \), and thus if \( t > 0 \), we have \( A_t(q-p) = 0 \). Define \( Y(t) = f(p(t),u(p(t))) \). Then, we have \( Y(0) = x, Y(1) = y \), and for every \( t \in [0,1] \),
\[
\dot{Y}(t) = S_t(q-p) = 0,
\]
which implies that \( x = y \), a contradiction. Therefore, we have \( d(1) > d(0) \), and (II) holds. This completes the proof. \( \square \)

\[11 \] If
\[
S_t = P^T \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix} P,
\]
where \( P \) is an orthogonal matrix and \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( S_t \), then
\[
A_t = P^T \begin{pmatrix}
\sqrt{-\lambda_1} & 0 & \ldots & 0 \\
0 & \sqrt{-\lambda_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sqrt{-\lambda_n}
\end{pmatrix} P.
\]
In general, statement (II) of Theorem 4 is rather hard to verify, and thus a more reasonable sufficient condition is needed. The following theorem gives such a condition.

**Theorem 5.** Suppose that there exists a finite family of $C^1$-class functions $f^1, ..., f^N : P \to \mathbb{R}^n$ that satisfy Walras' law and a partition $(A_1, ..., A_N)$ of $P$ such that $f(p, m) = f^i(p, m)$ for $(p, m) \in A_i$. Moreover, suppose that the Slutsky matrix $S_j(p, m)$ is symmetric and negative semi-definite for all $i$. Then, statement (I) of Theorem 4 implies statement (II) of Theorem 4.

**Proof.** It is clear that $f$ is locally Lipschitz. First, define 

$$df(p, m; q, w) = \left\{ \lim_{k \to \infty} \frac{f(p + t_kq, m + t_kw) - f(p, m)}{t_k} \right\}. $$

Note that because $f$ is locally Lipschitz, $df(p, m; q, w)$ is nonempty if $(p, m) \in P$. We will show that, for every $(p, m) \in P$ and $(q, w) \in \mathbb{R}^n \times \mathbb{R}$, there exists $v \in df(p, m; q, w)$ and a sequence $((p_k, m_k))$ such that $f$ is differentiable at $(p_k, m_k)$, $(p_k, m_k) \to (p, m)$, and 

$$v = \lim_{k \to \infty} Df(p_k, m_k)(q, w).$$

Fix $(q, w) \in \mathbb{R}^n \times \mathbb{R}$. Because $f$ is locally Lipschitz, we can use Rademacher's theorem, and thus, we have that $f$ is differentiable at almost every point. If $f$ is differentiable at $(p, m)$, there exists $i$ and a sequence $(t_k)$ of positive real numbers such that $t_k \downarrow 0$ and $(p + t_kq, m + t_kw) \in A_i$. By the continuity, we have $f(p, m) = f^i(p, m)$ and thus,

$$Df(p, m)(q, w) = \lim_{k \to \infty} \frac{f(p + t_kq, m + t_kw) - f(p, m)}{t_k} = \lim_{k \to \infty} \frac{f^i(p + t_kq, m + t_kw) - f^i(p, m)}{t_k} = Df^i(p, m)(q, w).$$

Therefore, if we define 

$$B_i = \{ (p, m) \in P | f(p, m) = f^i(p, m), Df(p, m)(q, w) = Df^i(p, m)(q, w) \},$$

then $\cup_i B_i$ is dense in $P$.

Choose any $(p, m) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ and $v \in df(p, m; q, w)$, and a sequence $(t_k)$ of positive real numbers such that $t_k \downarrow 0$ and 

$$v = \lim_{k \to \infty} \frac{f(p + t_kq, m + t_kw) - f(p, m)}{t_k}. $$

Taking a subsequence, we can assume that there exists $i$ such that for every $k$, $(p + t_kq, m + t_kw)$ is in the closure of $B_i$. Then, $(p, m)$ is also in the closure of $B_i$, and by the continuity of $f$, we have $f(p, m) = f^i(p, m)$ and $f(p + t_kq, m + t_kw) = f^i(p + t_kq, m + t_kw)$. Clearly, 

$$v = Df^i(p, m)(q, w),$$

and thus, if we choose any sequence $((p_k, m_k))$ in $B_i$ such that $(p_k, m_k) \to (p, m)$, then 

$$Df(p_k, m_k)(q, w) = Df^i(p_k, m_k)(q, w) \to Df^i(p, m)(q, w) = v,$$

as desired.
Moreover, in the proof of Lemma 1, we have already obtained
\( Df \)

On the other hand,
\( (1) \)

Choose any \( u \)

If \( p \), \( q \)

Suppose not. Then, there exists \( d \) and \( m \) such that \( \limsup_{t \downarrow t^*} \frac{c(t) - c(t^*)}{t - t^*} \geq 0. \)

This implies that
\[
\limsup_{t \downarrow t^*} \frac{d(t) - d(t^*)}{t - t^*} \leq \frac{d(t_2) - d(t_1)}{t_2 - t_1} < 0.
\]

Define \( w^* = \frac{d}{ds} u(p(s)) \). For \( t > t^* \),
\[
\frac{d(t) - d(t^*)}{t - t^*} - p \cdot f(p(t), u(p(t^*)) + (t - t^*)w^*) - f(p(t^*), u(p(t^*)))
\leq \|p\| \left| \frac{f(p(t), u(p(t))) - f(p(t), u(p(t^*)) + (t - t^*)w^*)}{t - t^*} \right|
\leq L \|p\| \left| \frac{u(p(t)) - u(p(t^*)) - (t - t^*)w^*}{t - t^*} \right|
\to 0 \text{ as } t \downarrow t^*,
\]

where \( L > 0 \) is some positive constant whose existence is ensured by the local Lipschitz property. By the above fact, there exists \( \ell \in df(p(t^*), u(p(t^*)); q - p, w^*) \) and a sequence \( (p_k, m_k) \) such that \( (p_k, m_k) \rightarrow (p(t^*), u(p(t^*))) \) and
\[
p \cdot Df(p_k, m_k)(q - p, w^*) \rightarrow p \cdot \ell \leq \limsup_{t \downarrow t^*} \frac{d(t) - d(t^*)}{t - t^*}. \tag{11}
\]

On the other hand,
\[
Df(p_k, m_k)(q - p, w^*) = [D_p f(p_k, m_k) + D_m f(p_k, m_k)f^T(p(t^*), u(p(t^*)))](q - p)
= S_f(p_k, m_k)(q - p)
+ D_m f(p_k, m_k)(f^T(p(t^*), u(p(t^*))) - f^T(p_k, m_k))(q - p).
\]

If \( u_k \) is a solution of \( (10) \) with the initial value condition \( u_k(p_k) = m_k \), then \( D^2 u_k(p_k) = S_f(p_k, m_k) \). Because \( u_k \) is concave, \( S_f(p_k, m_k) \) is negative semi-definite. Moreover, in the proof of Lemma 1, we have already obtained
\[
p_k^T S_f(p_k, m_k) = 0,
\]
by Walras’ law. Therefore,
\[ p^T D_f(p_k, m_k)(q - p, w^*) \]
\[ = -t^*(q - p)^T S_f(p_k, m_k)(q - p) + (p(t^*) - p_k)^T S_f(p_k, m_k)(q - p) \]
\[ + p^T D_m f(p_k, m_k)(f^T(p(t^*), u(p(t^*)))) - f^T(p_k, m_k)(q - p) \]
\[ \geq (p(t^*) - p_k)^T S_f(p_k, m_k)(q - p) \]
\[ + p^T D_m f(p_k, m_k)(f^T(p(t^*), u(p(t^*)))) - f^T(p_k, m_k)(q - p), \]
where the left-hand side tends to zero as \( k \to \infty \) because of the local Lipschitz condition of \( f \). This implies that
\[ \limsup_{t \downarrow t^*} \frac{d(t) - d(t^*)}{t - t^*} \geq 0, \]
a contradiction. Similarly, if \( c \) attains the maximum at \( t^* \), we can lead a contradiction. This completes the proof of Lemma 2. \( \square \)

Now, let \( x \neq y, x = f(p, m), y = f(q, w) \) and \( u_1 \) (resp. \( u_2 \)) be the solution of (10) with \( u_1(p) = m \) (resp. \( u_2(q) = w \)).

If \( u_2(q) = w > u_1(q) \), then \( u_2(p) > m \), and thus by Lemma 2,
\[ p \cdot y = p \cdot f(q, u_2(q)) = d(1) \geq d(0) = p \cdot f(p, u_2(p)) = u_2(p) > m, \]
where \( p(t) = (1 - t)p + tq \) and \( d(t) = p \cdot f(p(t), u_2(p(t))) \).

Next, suppose that \( u_2(q) = w = u_1(q) \). Then, we have \( u_1 \equiv u_2 \). Let \( p(t) = (1 - t)p + tq, d(t) = p \cdot f(p(t), u_1(p(t))) \). It suffices to show that \( m = d(0) < d(1) \).

Suppose not. By Lemma 2, we have \( d(t) \equiv d(0) \) on \([0, 1]\). Let \( X(r) = f(r, u_1(r)) \) and \( Y(t) = X(p(t)) \). Because \( Y(0) = x \neq y = Y(1) \) and \( Y \) is absolutely continuous, there exists \( t^* \in [0, 1] \) such that \( Y(t^*) \neq 0 \).

Let \( w^* = \frac{d}{ds} u_1(p(s)) \bigg|_{s=t^*}, v \in df(p(t^*), u_1(p(t^*)) \colon q - p, w^*), (p_k, m_k) \to (p(t^*), u_1(p(t^*))), \) and \( D_f(p_k, m_k)(q - p, w^*) \to v \) as \( k \to \infty \), and let \( S_k \) denote \( S_f(p_k, m_k) \).

Note that by Young’s theorem, we have that \( S_k \) is symmetric and negative semi-definite, and thus there exists a symmetric and positive semi-definite matrix \( A_k \) such that \( S_k = -A_k^2 \).

Then,
\[ -t^*(q - p) S_k(q - p) = t^*\|A_k(q - p)\|^2. \]

Because \( d(t^*) = 0 \), we have \( A_k(q - p) \to 0 \) as \( k \to \infty \) by (11) and (12).\( ^{12} \)

Meanwhile, by the same arguments as in Lemma 2, we have
\[ \frac{Y(t) - Y(t^*)}{t - t^*} - f(p(t), u_1(p(t^*))) + (t - t^*)w^* - f(p(t^*), u_1(p(t^*))) \]
\[ \leq L \left| \frac{u_1(p(t)) - u_1(p(t^*)) + (t - t^*) \frac{d}{ds} u_1(p(s)) \bigg|_{s=t^*}}{t - t^*} \right| \]
\[ \to 0 \text{ as } t \downarrow t^*, \]
where \( L > 0 \) is some constant. Therefore, we have
\[ \dot{Y}(t^*) = \lim_{k \to \infty} D_f(p_k, m_k)(q - p, w^*). \]

This implies that
\[ \dot{Y}(t^*) = v = \lim_{k \to \infty} S_k(q - p) = \lim_{k \to \infty} A_k(A_k(q - p)) = 0, \]

\( ^{12} \)Note that (11) holds with equality because \( d(t^*) \) exists.
which is absurd. This completes the proof.

Example 2. Consider the function

$$f(p, m) = \begin{cases} \frac{m}{p_1}, & \text{if } p_2^2 \geq 4p_1m, \\ \frac{p_2^2}{4p_1}, & \text{otherwise}, \end{cases}$$

in example 1. Then, the solution of (10) is generally solved by (8) and (9), and the solution \(u\) is concave in both cases. By Theorem 5, this function \(f\) satisfies both (I) and (II) of Theorem 4. Set \(\bar{p} = (1, 1)\). By an easy computation, we can obtain \(v_{f, \bar{p}}(x)\) in the proof of Theorem 4, and

$$v_{f, \bar{p}}(x) = \begin{cases} 0, & \text{if } x_1 = 0, \\ (v(x))^2, & \text{if } 2v(x) \leq 1, \\ v(x) - \frac{1}{4}, & \text{if } 2v(x) > 1, \end{cases}$$

where \(v(x) = \sqrt{x_1} + x_2\). Note that for \(x \geq 0\) with \(x_1 > 0\), this function \(v\) gives the same order as \(v_{f, \bar{p}}\). Readers can easily check that \(f(p, m)\) is the unique solution of

The maximum

subject to \(x \geq 0, p \cdot x \leq m.\)

Therefore, using Theorems 4 and 5, we have successfully recovered the information of \(v\) on \(\{x \in \mathbb{R}^2_+ | x_1 > 0\}\).

Appendix A. Proofs of FACTs.

Proof of FACT 1. Let \(a > 0, b > 0\) be sufficiently small and \(\Pi = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \subset P\), and let \(L > 0\) be a constant such that if \((t, x_1), (t, x_2) \in \Pi\), then

\[||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2||.\]

Define \(M = \max_{(t, x) \in \Pi} ||f(t, x)||\). Without loss of generality, we can assume that \(a \leq \frac{1}{L}\) and \(aL < 1\). Let \(\mathcal{X}^-\) be the space of all functions from \([t_0 - a, t_0 + a]\) into \(\prod_{i=1}^m [x_0 - b, x_0 + b]\), and define a function \(P : \mathcal{X}^- \to \mathcal{X}^-\) as

\[P(x(\cdot)) = x_0 + \int_{t_0}^t f(s, x(s))ds.\]

Note that the operator norm of \(A_k\) is less than the square root of the operator norm of \(S_k\), which is bounded by the local Lipschitz condition.

In general, the reverse-calculation procedure described in Theorem 4 can only restore the information of \(v\) on the range of \(f\).

Note that for any continuous function \(f\),

\[\left\| \int_{t_0}^t f(t)dt \right\| = \lim_{k \to \infty} \sum_{i=0}^{k-1} \left\| f(t_0 + \frac{t - t_0}{k}) \right\| \leq \lim_{k \to \infty} \sum_{i=0}^{k-1} \left\| f(t_0 + \frac{t - t_0}{k}) \right\| = \int_{t_0}^t \| f(t) \| dt.\]
Then, \( x : [t_0 - a, t_0 + a] \to \prod_{i=1}^{n} [x_{0i} - b, x_{0i} + b] \) is a solution of the ODE
\[
\dot{x}(t) = f(t, x(t)), \ x(t_0) = x_0,
\]
if and only if it is a fixed point of \( P \). Choose any \( x_1(\cdot), x_2(\cdot) \in \mathcal{X} \). Then,
\[
\|P(x_2(\cdot))(t) - P(x_1(\cdot))(t)\| \leq \int_{t_0}^{t} \|f(s, x_2(s)) - f(s, x_1(s))\|\,ds \\ \leq aL\|x_2(\cdot) - x_1(\cdot)\|_{\infty},
\]
where \( \| \cdot \|_{\infty} \) is the supremum norm. This means that \( P \) is a contraction with respect to the supremum norm, and by the contraction mapping theorem, there uniquely exists a fixed point \( x(\cdot) \in \mathcal{X} \). This function is a solution of the above ODE, and the unique solution in \( \mathcal{X} \) is connected.

Next, let \( x(\cdot), y(\cdot) \) be solutions, and \( I_x \) (resp. \( I_y \)) be the domain of \( x(\cdot) \) (resp. \( y(\cdot) \)). Let \( I = I_x \cap I_y \), and \( X = \{ t \in I \mid x(t) = y(t) \} \). Clearly \( t_0 \in X \), and thus \( X \) is nonempty. By the continuity of \( x(\cdot), y(\cdot) \), we have that \( X \) is closed in \( I \). Now, let \( t^* \in X \). Choose any \( a > 0, b > 0 \) such that there uniquely exists a solution \( z(\cdot) \in \mathcal{X} \) of the following ODE:
\[
\dot{z}(t) = f(t, z(t)), \ z(t^*) = x(t^*),
\]
where \( \mathcal{X} \) is the space of all functions from \([t^* - a, t^* + a]\) into \( \prod_{i=1}^{n} [x_{i}(t^*) - b, x_{i}(t^*) + b] \). Without loss of generality, we can assume that \( a > 0 \) is sufficiently small that \( x(\cdot), y(\cdot) \in \mathcal{X} \). Then, \( x(t) = z(t) = y(t) \) by the uniqueness, and thus \( X \) includes \([t^* - a, t^* + a]\). Therefore, \( X \) is open in \( I \), and thus we have that \( I = X \) by the connectedness of \( I \). This completes the proof.

**Proof of FACT 2.** Let \( \mathcal{X} \) be the set of all solutions of the ODE
\[
\dot{x}(t) = f(t, x(t)), \ x(t_0) = x_0.
\]
If \( x(\cdot) \in \mathcal{X} \), let \( I_x \) denote the domain of \( x(\cdot) \). Let \( I = \bigcup_x I_x \), and define \( x^*: I \to \mathbb{R}^n \) such that
\[
x^*(t) = x(t) \text{ if } t \in I_x.
\]
Then, \( x^*(\cdot) \) is a solution of the above ODE, and is clearly nonextendable. The uniqueness follows from FACT 1.

**Proof of FACT 3.** We will prove the existence of \( \bar{t} \) only. The rest can be treated symmetrically.

If \( b = +\infty \), we can choose \( \bar{t} = \max\{t \mid \exists x, (t, x) \in C\} \). Therefore, we can assume that \( b \in \mathbb{R} \). Let \( \rho = \inf_{(t_1, x_1) \in C, (t_2, x_2) \in P} \|f(t_1, x_1) - (t_2, x_2)\| \), and define \( C' = \{(t, x) \mid \exists (t', x'), (t' - t') < \frac{\varepsilon}{2}, \max |x_i - x_i'| < \frac{\varepsilon}{2}\} \). Because \( C \) is compact, \( \rho > 0 \), \( C' \) is open and its closure \( C'' \) is compact, and \( C \subset C' \subset C'' \subset P \). Let \( M = \max_{(t,x)\in C''} \|f(t, x)\| \) and \( L > 0 \) be a constant such that if \( (t_1, x_1), (t_2, x_2) \in C'' \), then
\[
\|f(t_1, x_1) - f(t_2, x_2)\| \leq L\|x_1 - x_2\|.
\]
Choose \( a' > 0 \) such that \( a'L < 1 \) and \( a' < \min \{ \frac{\varepsilon}{4M}, \frac{\varepsilon}{2L} \} \), and define \( \bar{t} = b - \frac{a'}{2} \). If there exists \( t^* \in [\bar{t}, b] \) such that \((t^*, x(t^*)) \in C\), then by repeating the arguments in the proof of FACT 1, we can show that there exists a solution of the ODE
\[
\dot{z}(t) = f(t, z(t)), \ z(t^*) = x(t^*),
\]
where \( z : [t^* - a', t^* + a'] \rightarrow \prod_{i=1}^{n}[x_i(t^*) - \frac{q_i}{2}, x_i(t^*) + \frac{q_i}{2}] \). Define

\[
x^*(t) = \begin{cases} x(t) & \text{if } t < b, \\ z(t) & \text{if } b \leq t + a', \end{cases}
\]

Then, \( x^*(\cdot) \) is an extension of \( x(\cdot) \), which contradicts the definition of \( x(t) \). This completes the proof. \( \square \)

**Proof of FACT 4.** The existence and uniqueness of \( x(t; y) \) follow immediately from FACT 2. Let \( U \subset \mathbb{R} \times \mathbb{R}^n \) be the domain of \( x(t; y) \).

Now, we introduce a lemma.

**Lemma 3.** Suppose that a continuous function \( u : [t_0, \bar{t}] \rightarrow \mathbb{R} \) satisfies

\[
u(t_0) = 0, \quad u(t) \leq \int_{t_0}^{t} [Au(s) + B]ds,
\]

for some \( A, B > 0 \). Then,

\[
u(t) \leq \frac{B}{A}(e^{A(t-t_0)} - 1).
\]

**Proof of Lemma 3.** Let \( u_0(t) = u(t) \), and when \( u_k(t) \) is already defined, define

\[
u_{k+1}(t) = \int_{t_0}^{t} [Au_k(s) + B]ds.
\]

Then, \( u_k(t) \) is increasing in \( k \), and

\[
u_2(t) - u_1(t) \leq A \int_{t_0}^{t} (u_1(s) - u_0(s))ds
\]

\[
\leq A(t - t_0)\|u_1 - u_0\|_{\infty},
\]

\[
u_3(t) - u_2(t) \leq A \int_{t_0}^{t} (u_2(s) - u_1(s))ds
\]

\[
\leq A^2(t - t_0)^2 \|u_1 - u_0\|_{\infty},
\]

\[
\cdots
\]

\[
u_{k+1}(t) - u_k(t) \leq \frac{A^k(t-t_0)^k}{k!} \|u_1 - u_0\|_{\infty}.
\]

Thus, the sequence \( (u_k) \) is a Cauchy sequence. Therefore, it converges uniformly to some function \( v \) and, clearly,

\[
v(t) = \int_{t_0}^{t} [Av(s) + B]ds.
\]

This integral equation has a unique solution \( v(t) = \frac{B}{A}(e^{A(t-t_0)} - 1) \), and obviously \( u(t) \leq v(t) \). This completes the proof. \( \square \)

Choose any \( y^* \) such that \( (t_0, x_0, y^*) \in \bar{P} \), and define \( x_y : t \mapsto x(t; y) \) and \( |a_y, b_y| \) as its domain. Let \( a_y < r_1 \leq t_0 \leq r_2 < b_y \). We will prove that there exists a neighborhood \( V \) of \( y^* \) such that if \( y \in V \), then \( a_y < r_1 \) and \( r_2 < b_y \), and for all \( t \in ]r_1, r_2[ \), \( x \) is continuous at \( (t, y^*) \).

Choose \( a > 0, b > 0 \) such that

\[
\bar{N} = \{(t, x, y) | r_1 \leq t \leq r_2, \|x - x(t; y^*)\| \leq a, \|y - y^*\| \leq b \} \subset \bar{P}.
\]
Because $\tilde{\Pi}$ is compact, there exists $L > 0$ such that if $(t, x_1, y), (t, x_2, y) \in \tilde{\Pi}$, then
\[
\|f(t, x_1, y) - f(t, x_2, y)\| \leq L\|x_1 - x_2\|.
\]
Moreover, because $f$ is continuous on $P$, it is uniformly continuous on $\tilde{\Pi}$, and thus there exists a nondecreasing nonnegative function $\beta(e)$ such that $\lim_{e \to 0} \beta(e) = 0$ and, if $(t, x_1, y), (t, x_2, y) \in \tilde{\Pi}$, then
\[
\|f(t, x_1) - f(t, x_2)\| \leq \beta(\|y_1 - y_2\|).
\]
Choose any $t \in [t_0, r_2]$. If $x(t; y)$ is defined and $(s, x(s, y), y) \in \tilde{\Pi}$ for all $s \in [t_0, t]$, then
\[
\|x(t; y) - x(t; y^*)\| \leq \int_{t_0}^{t} \|f(s, x(s; y), y) - f(s, x(s; y^*), y^*)\| ds
\]
\[
\leq \int_{t_0}^{t} \|f(s, x(s; y), y) - f(s, x(s; y^*), y)\|
\]
\[
+ \|f(s, x(s; y^*), y) - f(s, x(s; y^*), y^*)\| ds
\]
\[
\leq \int_{t_0}^{t} \left[L\|x(s; y) - x(s; y^*)\| + \beta(\|y - y^*\|)\right] ds.
\]
By Lemma 3,
\[
\|x(t; y) - x(t; y^*)\| \leq \frac{\beta(\|y - y^*\|)}{L}(e^{L(r_2 - t_0)} - 1) \equiv C_2 \beta(\|y - y^*\|),
\]
for some constant $C_2 > 0$. Choose any $\rho_2 > 0$ such that
\[
\rho_2 \leq b, \quad C_2 \beta(\rho_2) < a.
\]
Choose any $y$ with $\|y - y^*\| \leq \rho_2$ and define $\tilde{\ell} = \sup\{t \geq t_0 | (t; y) \in U, (s, x(s; y), y) \in \tilde{\Pi}\}$. If $\tilde{\ell} < r_2$, then for any $t \in [t_0, \tilde{\ell}]$,
\[
t \in [r_1, r_2], \quad \|x(t; y) - x(t; y^*)\| \leq C_2 \beta(\rho_2) < \|y - y^*\| \leq b,
\]
and thus, $(t, x(t; y), y) \in \tilde{\Pi}$. By FACT 3 and the continuity of $x_y(t)$, we have $x(\cdot; y)$ is defined at $\tilde{\ell}$, $(\tilde{\ell}, x(\tilde{\ell}; y), y) \in \tilde{\Pi}$, and $\|x(\tilde{\ell}; y) - x(\tilde{\ell}; y^*)\| \leq C_2 \beta(\rho_2) < a$. Hence, we have that for all $t > \tilde{\ell}$ such that $t - \tilde{\ell}$ is sufficiently small, $x(t; y)$ is defined and
\[
\|x(t; y) - x(t; y^*)\| < a,
\]
which contradicts the definition of $\tilde{\ell}$. Therefore, $\tilde{\ell} = r_2$ and $x_y(\cdot)$ is defined on $[t_0, r_2]$. Moreover,
\[
\|x(t; y) - x(t; y^*)\| \leq C_2 \beta(\|y - y^*\|).
\]
By symmetrical arguments, we can show that there exists $\rho_1 > 0$ such that if $\|y - y^*\| \leq \rho_1$, then $x_y(\cdot)$ is defined on $[r_1, t_0]$, and
\[
\|x(t; y) - x(t; y^*)\| \leq C_1 \beta(\|y - y^*\|).
\]
Define
\[
V = \{y | \|y - y^*\| < \min\{\rho_1, \rho_2\}\}.
\]
If $y \in V$, $(t, y) \in U$ for all $t \in [r_1, r_2]$. Moreover, if $r_1 \leq t \leq r_2$ and $(s, y) \in [r_1, r_2] \times \tilde{\Pi}$, then
\[
\|x(s; y) - x(t; y^*)\| \leq \|x(s; y) - x(s; y^*)\| + \|x(s; y^*) - x(t; y^*)\|
\]
\[
\leq \max\{C_1, C_2\} \beta(\|y - y^*\|) + (s - t)M,
\]
where $M = \max_{t \in [r_1, r_2]} \| f(t, x(t; y^*), y^*) \|$. Therefore, if $r_1 < t < r_2$, $x$ is continuous at $(t, y^*)$. Since $r_1, r_2$ are arbitrary, we have that $U$ is open and $x$ is continuous. This completes the proof. \hfill \Box

Proof of FACT 5. First, we will prove a lemma.

**Lemma 4** (Lemma 4 (Hadamard’s lemma)). Suppose that $Q \subset \mathbb{R}^k \times \mathbb{R}^n$ is open and if $0 \leq s \leq 1$ and $(t, y_1), (t, y_2) \in Q$, then $(t, sy_1 + (1-s)y_2) \in Q$. Moreover, suppose that a real-valued function $g(t, y)$ defined on $Q$ is continuous, and continuously differentiable in $y$. Then, there exists a $\mathbb{R}^n$-valued continuous function $h$ defined on the set of all $(t, y, y')$ with $(t, y') \in Q$ such that

$$g(t, y') - g(t, y^2) = h(t, y', y^2) \cdot (y^1 - y^2),$$

and $h_j(t, y, y') = \frac{\partial g}{\partial y_j}(t, y)$.

Proof of Lemma 4. Let $y(s) = sy^1 + (1-s)y^2$. Then,

$$g(t, y^1) - g(t, y^2) = g(t, y(1)) - g(t, y(0)) = \int_0^1 \sum_j \frac{\partial g}{\partial y_j}(t, y(s))(y^1_j - y^2_j)ds,$$

and thus, we can define

$$h_j(t, y^1, y^2) = \int_0^1 \frac{\partial g}{\partial y_j}(t, y(s))ds.$$

This completes the proof. \hfill \Box

Let $y^*$ be fixed, and the function $t \mapsto x(t; y^*)$ be defined on $|a, b|$. Choose any $r_1, r_2$ such that $a < r_1 \leq t_0 \leq r_2 < b$. By the proof of FACT 4, we have that the function $y \mapsto x(:, y)$ is continuous at $y^*$, where the metric of the range space $\mathcal{C} = \{x(:, x) : [r_1, r_2] \to \mathbb{R}^n, x \text{ is continuous}\}$ is the supremum norm. Choose $a > 0, b > 0$ such that, if $r_1 \leq t \leq r_2, \|x - x(t; y^*)\| < a$, and $\|y - y^*\| < b$, then $(t, x, y) \in P$. Then, there exists $\rho > 0$ such that $2\rho < b$ and if $r_1 \leq t \leq r_2$ and $\|y - y^*\| < 2\rho$, then $\|x(t; y) - x(t; y^*)\| < a$. Let

$$Q = \{(t, x, y) : r_1 < t < t_0, \|x - x(t; y^*)\| < a, \|y - y^*\| < 2\rho\}.$$ 

Then, $Q$ satisfies all requirements of Lemma 4. Now, choose any $\tau$ such that $\|y - y^*\| < \rho$ and $\|\tau\| < \rho$, and define $y^\tau = y + \tau e_j$, where $e_j$ is the $j$-th unit vector. Applying Lemma 4 to the function $f_j(t, x, y)$, we obtain

$$f_j(t, x(t; y^\tau), y^\tau) - f_i(t, x(t; y), y) = g_j(t, y, \tau) \cdot (x(t; y^\tau) - x(t; y)) + h_j(t, y, \tau),$$

where $g, h$ is continuous around $(t, y, 0)$, and

$$g_j(t, y, 0) = \frac{\partial f_i}{\partial x_j}(t, x(t; y), y), \quad h_j(t, y, 0) = \frac{\partial f_i}{\partial y_j}(t, x(t; y), y).$$

Now, define a function $\varphi$ for $\tau \neq 0$ as follows:

$$\varphi(t, y, \tau) = \frac{x(t; y^\tau) - x(t; y)}{\tau}.$$

Then,

$$\varphi_i(t, y, \tau) = g_i(t, y, \tau) \cdot \varphi(t, y, \tau) + h_i(t, y, \tau).$$

To summarize, the function $\varphi$ is the solution of the following linear differential equation:

$$\varphi(t, y, \tau) = g(t, y, \tau) \cdot \varphi(t, y, \tau) + h(t, y, \tau), \quad \varphi(t_0, y, \tau) = 0. \quad (13)$$
This equation can be solved even if \( \tau = 0 \), and by FACT 4, it is continuous on \( \tau \). Hence, we treat \( \varphi(t, y, \tau) \) as the solution of the above equation, and thus we consider that it is defined even if \( \tau = 0 \). In particular, 
\[
\frac{\partial x_i}{\partial y_j}(t; y) = \varphi_i(t, y, 0),
\]
where the right-hand side is continuous on \((t, y)\). Therefore, \( x(t; y) \) is continuously differentiable in \( y \) around \((t, y^*)\). Further,
\[
\frac{\partial^2 x_i}{\partial y_j \partial t}(t; y) = \varphi_i(t, y, 0) = g(t, y, 0) \cdot \varphi(t, y, 0) + h_j(t, y, 0),
\]
where the right-hand side is continuous. Therefore, both \( \frac{\partial^2 x_i}{\partial y_j \partial t}(t; y) \) and \( \frac{\partial^2 x_i}{\partial y_j \partial x} \) can be defined and are continuous, and thus they are the same. This completes the proof. \( \square \)

**Appendix B. The proof of Theorem 2 under the continuous differentiability.** We showed Theorem 2 under the differentiability and local Lipschitz condition of \( f \). If the continuous differentiability of \( f \) is assumed, the proof of Theorem 2 becomes quite easy. In this section, we demonstrate this fact.

As in the proof of Theorem 2, it suffices to show that \( Du(q) = f(q, u(q)) \) for any \( q \in U \) such that \( U \) includes some neighborhood of \([p, q]\). By FACT 5, \( c \) is differentiable in \( q \). Let
\[
h_i(t; q) = \frac{\partial c}{\partial q_i}(t; q) - tf_i((1 - t)p + tq, c(t; q)).
\]
Then, \( h_i(0; q) = 0 \) and,\(^{16}\)
\[
h_i(t; q) = \frac{\partial^2 c}{\partial t \partial q_i} - f_i - t \left[ \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} \times (q_j - p_j) + \frac{\partial f_i}{\partial w} \right]
\]
\[
= f_i + \sum_{j=1}^{n} \left( t \frac{\partial f_j}{\partial q_i} + \frac{\partial f_j}{\partial w} \frac{\partial c}{\partial w} \right) \times (q_j - p_j)
\]
\[
- f_i - t \left[ \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} \times (q_j - p_j) + \frac{\partial f_i}{\partial w} \sum_{j=1}^{n} f_j \times (q_j - p_j) \right]
\]
\[
= t \sum_{j=1}^{n} \left( \frac{\partial f_j}{\partial q_i} - s_{ij} \right) \times (q_j - p_j) + \sum_{j=1}^{n} \frac{\partial f_j}{\partial w} \frac{\partial c}{\partial w} \times (q_j - p_j)
\]
\[
= \left[ \frac{\partial c}{\partial q_i} - tf_i \right] \sum_{j=1}^{n} \frac{\partial f_j}{\partial w} (q_j - p_j)
\]
\[
= h_i(t; q) \sum_{j=1}^{n} \frac{\partial f_j}{\partial w} (q_j - p_j),
\]
\(^{16}\)We abbreviate \( f((1 - t)p + tq, c(t; q)) \) as \( f, c(t; q) \) as \( c \), and so on.
where the second equality comes from FACT 5. Thus, $h_i(t; q)$ is a solution of a linear differential equation $\dot{h}_i = a(t)h_i$ with $h_i(0; q) = 0$, where

$$a(t) = \frac{\partial f}{\partial w}((1 - t)p + tq, c(t; q)) \cdot (q - p).$$

By the uniqueness of the solution, we conclude that $h_i(t; q) \equiv 0$. Hence, $\frac{\partial c}{\partial q}(1; q) = f'(q, c(1; q))$, which implies that

$$Du(q) = f(q, u(q)).$$

This completes the proof.

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