Nonlinear Spinor Field in Anisotropic Universes

Bijan Saha

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Moscow Reg., Russia.
e-mail: saha@thsun1.jinr.ru

Abstract

Evolution of an anisotropic universe described by a Bianchi type I (BI) model in presence of nonlinear spinor field has been studied by us recently in a series of papers. On offer the Bianchi models, those are both inhomogeneous and anisotropic. Within the scope of Bianchi type VI (BVI) model the self-consistent system of nonlinear spinor and gravitational fields are considered. The role of inhomogeneity in the evolution of spinor and gravitational field is studied.

Key words: Anisotropic universe, Nonlinear spinor field (NLSF)

PACS 04.20.Jb

1. INTRODUCTION

Several authors studied the nonlinear spinor fields (NLSF) since Ivanenko [1-3] showed that a relativistic theory imposes in some cases a fourth order self-coupling. Nonlinear spinor

*Talk, given at the International Conference “Scientific Reading devoted to 90 years anniversary of Professor Yakov Petrovich Terletskii”, July 1-3, 2002, Russian Peoples’ Friendship University, Moscow, Russia.
field in an anisotropic universe, namely in a Bianchi-type I universe (BI) is studied by us in a series of papers [4–11]. In these papers we considered the nonlinear spinor field, as well as a system of interacting spinor and scalar fields. Beside the spinor fields, we also study the role of a Λ term in the evolution of the universe. For the details of a NLSF in a BI universe one can consult [12]. In the light of results obtained in the previously mentioned papers the study of NLSF in other anisotropic universes presents great interest. In this report we consider the self-consistent system of nonlinear spinor and BVI gravitational fields. The results have been compared with those obtained for the BI universe.

2. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

We choose the Lagrangian for the self-consistent system of spinor and gravitational fields in the form

\[
L = \frac{R}{2\kappa} + \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - M \bar{\psi} \psi + L_N
\]  

(2.1)

with \( R \) being the scalar curvature and \( \kappa \) being the Einstein’s gravitational constant. The nonlinear term \( L_N \) describes the self-interaction of spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of spinor field having the form:

\[
S = \bar{\psi} \psi \quad \text{(scalar)},
\]

(2.2a)

\[
P = i\bar{\psi} \gamma^5 \psi \quad \text{(pseudoscalar)},
\]

(2.2b)

\[
u^\mu = (\bar{\psi} \gamma^\mu \psi) \quad \text{(vector)},
\]

(2.2c)

\[
A^\mu = (\bar{\psi} \gamma^5 \gamma^\mu \psi) \quad \text{(pseudovector)},
\]

(2.2d)

\[
Q^{\mu\nu} = (\bar{\psi} \sigma^{\mu\nu} \psi) \quad \text{(antisymmetric tensor)},
\]

(2.2e)

where \( \sigma^{\mu\nu} = (i/2)[\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu] \). Invariants, corresponding to the bilinear forms, are

\[
I = S^2,
\]

(2.3a)
\[ J = \rho^2, \quad (2.3b) \]
\[ I_v = v_\mu v^\mu = (\bar{\psi} \gamma^\mu \psi) g_{\mu\nu} (\bar{\psi} \gamma^\nu \psi), \quad (2.3c) \]
\[ I_A = A_\mu A^\mu = (\bar{\psi} \gamma^5 \gamma^\mu \psi) g_{\mu\nu} (\bar{\psi} \gamma^5 \gamma^\nu \psi), \quad (2.3d) \]
\[ I_Q = Q_{\mu\nu} Q^{\mu\nu} = (\bar{\psi} \sigma^{\mu\nu} \psi) g_{\mu\alpha} g_{\nu\beta} (\bar{\psi} \sigma^{\alpha\beta} \psi). \quad (2.3e) \]

According to the Pauli-Fierz theorem \[13\] among the five invariants only \( I \) and \( J \) are independent as all other can be expressed by them: \( I_V = -I_A = I + J \) and \( I_Q = I - J \). Therefore, we choose the nonlinear term \( L_N = F(I, J) \), thus claiming that it describes the nonlinearity in the most general of its form.

We choose the anisotropic inhomogeneous universe given by a Bianchi-type VI (BVI) metric:

\[ ds^2 = dt^2 - a^2 e^{-2mz} dx^2 - b^2 e^{2nz} dy^2 - c^2 dz^2, \quad (2.4) \]

with \( a, b, c \) being the functions of time only. Here \( m, n \) are some arbitrary constants and the velocity of light is taken to be unity. Note that the suitable choice of \( m, n \) as well as the metric functions \( a, b, c \) in the BVI given by (2.4) evokes the following Bianchi-type universes. Thus

- for \( m = n \) the BVI metric transforms to a Bianchi-type V (BV) one, i.e.,
  \[ m = n, \text{ BVI} \implies \text{BV} \in \text{open FRW}; \]

- for \( n = 0 \) the BVI metric transforms to a Bianchi-type III (BIII) one, i.e.,
  \[ n = 0, \text{ BVI} \implies \text{BIII}; \]

- for \( m = n = 0 \) the BVI metric transforms to a Bianchi-type I (BI) one, i.e.,
  \[ m = n = 0, \text{ BVI} \implies \text{BI}; \]

- for \( m = n = 0 \) and equal scale factor in all three direction the BVI metric transforms to a Friedmann-Robertson-Walker (FRW) universe, i.e.,
  \[ m = n = 0 \text{ and } a = b = c, \text{ BVI} \implies \text{FRW}. \]
Variation of the Lagrangian (2.1) with respect to field functions $\psi(\bar{\psi})$ gives the nonlinear spinor field equations:

$$i\gamma^\mu \nabla_\mu \psi - M\psi + D\psi + iG\gamma^5\psi = 0$$  \hspace{1cm} (2.5a)

$$i\nabla_\mu \bar{\psi}\gamma^\mu + M\bar{\psi} - D\bar{\psi} - iG\bar{\psi}\gamma^5 = 0$$  \hspace{1cm} (2.5b)

where $D = 2SF_I$ and $G = 2PF_J$.

Varying (2.1) with respect to metric function $(g_{\mu\nu})$ we find the Einstein equations for $a, b, c$ which for the metric (2.4) read

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} - \frac{n^2}{c^2} = -\kappa T^1_1,$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{c}}{ac} - \frac{m^2}{c^2} = -\kappa T^2_2,$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{mn}{c^2} = -\kappa T^3_3,$$

$$\frac{\dot{a}}{a} - \frac{\dot{b}}{b} - (m - n)\frac{\dot{c}}{c} = -\kappa T^0_0.$$

Here overdots denote differentiation with respect to time ($t$) and $T^\nu_\mu$ is the energy-momentum tensor of the material field and has the form

$$T^\rho_\mu = \frac{i}{4}g^{\rho\nu}(\bar{\psi}\gamma_\mu \nabla_\nu \psi + \bar{\psi}\gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma_\nu \psi - \nabla_\nu \bar{\psi}\gamma_\mu \psi) - \delta^\rho_\mu L_{sp}.$$  \hspace{1cm} (2.7)

Here $L_{sp}$ is the spinor field Lagrangian, which on account of spinor field equations (2.3) takes the form:

$$L_{sp} = -DS - G\mathcal{P} + F.$$  \hspace{1cm} (2.8)

In the expressions above $\nabla_\mu$ denotes the covariant derivative of spinor, having the form [14]:

$$\nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi$$  \hspace{1cm} (2.9)

where $\Gamma_\mu(x)$ are spinor affine connection matrices defined by the equality

$$\Gamma_\mu(x) = (1/4)g_{\rho\sigma}(x)(\partial_\mu e^\rho_b e^\sigma_b - \Gamma^\rho_{\mu b})\gamma^\sigma\gamma^\delta.$$. 
For the metric element (2.4) it gives

\[\Gamma_0 = 0,\]
\[\Gamma_1 = \frac{1}{2} [\dot{a} \bar{\gamma}^1 \gamma^0 - m a c \bar{\gamma}^1 \bar{\gamma}^3] e^{-mz},\]
\[\Gamma_2 = \frac{1}{2} [\dot{b} \bar{\gamma}^2 \gamma^0 + n b c \bar{\gamma}^2 \bar{\gamma}^3] e^{nz},\]
\[\Gamma_3 = \frac{1}{2} \dot{c} \bar{\gamma}^3 \gamma^0.\]

It is easy to show that

\[\gamma^\mu \Gamma_\mu = - \frac{1}{2} \dot{\tau} \bar{\gamma}^0 + \frac{m - n}{2c} \frac{\bar{\gamma}^3}{\gamma^3},\]

where we define

\[\tau = abc.\]  (2.10)

The Dirac matrices \(\gamma^\mu(x)\) of curved space-time are connected with those of Minkowski one as follows:

\[\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1 e^{mz}/a, \quad \gamma^2 = \bar{\gamma}^2 / b e^{nz}, \quad \gamma^3 = \bar{\gamma}^3 / c\]

with

\[\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \bar{\gamma}^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},\]

where \(\sigma_i\) are the Pauli matrices:

\[\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\]

Note that the \(\bar{\gamma}\) and the \(\sigma\) matrices obey the following properties:

\[\bar{\gamma}^i \bar{\gamma}^j + \bar{\gamma}^j \bar{\gamma}^i = 2 \eta^{ij}, \quad i, j = 0, 1, 2, 3\]
\[\bar{\gamma}^i \bar{\gamma}^5 + \bar{\gamma}^5 \bar{\gamma}^i = 0, \quad (\bar{\gamma}^5)^2 = I, \quad i = 0, 1, 2, 3\]
\[\sigma^i \sigma^k = \delta_{jk} + i \varepsilon_{jkl} \sigma^l, \quad j, k, l = 1, 2, 3\]

where \(\eta_{ij} = \{1, -1, -1, -1\}\) is the diagonal matrix, \(\delta_{jk}\) is the Kroneker symbol and \(\varepsilon_{jkl}\) is the totally antisymmetric matrix with \(\varepsilon_{123} = +1\). Let us consider the spinors to be functions of \(t\) and \(z\) only, such that
\[ \psi(t, z) = v(t) e^{ikz}, \quad \bar{\psi}(t, z) = \bar{v}(t) e^{-ikz} \quad (2.11) \]

Inserting (2.11) into (2.5) for the nonlinear spinor field we find

\[ \hat{s}^0 \left( \dot{v} + \frac{\dot{c}}{2 \tau} v \right) - \left( \frac{m-n}{2c} - i \frac{k}{c} \right) \hat{s}^3 v + i \Phi v + \mathcal{G} \bar{\sigma}^5 v = 0 \quad (2.12a) \]
\[ \left( \dot{\bar{v}} + \frac{\dot{c}}{2 \tau} \bar{v} \right) \hat{s}^0 - \left( \frac{m-n}{2c} + i \frac{k}{c} \right) \bar{v} \hat{s}^3 - i \Phi \bar{v} - \mathcal{G} \bar{\sigma}^5 = 0. \quad (2.12b) \]

Here we define \( \Phi = M - \mathcal{D} \). Introduce a new function \( u_j(t) = \sqrt{\tau} v_j(t) \), for the components of the NLSF from (2.12) one obtains

\[ \dot{u}_1 + i \Phi u_1 - \left[ \frac{m-n}{2c} - i \frac{k}{c} + \mathcal{G} \right] u_3 = 0, \quad (2.13a) \]
\[ \dot{u}_2 + i \Phi u_2 + \left[ \frac{m-n}{2c} - i \frac{k}{c} - \mathcal{G} \right] u_4 = 0, \quad (2.13b) \]
\[ \dot{u}_3 - i \Phi u_3 - \left[ \frac{m-n}{2c} - i \frac{k}{c} - \mathcal{G} \right] u_1 = 0, \quad (2.13c) \]
\[ \dot{u}_4 - i \Phi u_4 + \left[ \frac{m-n}{2c} - i \frac{k}{c} + \mathcal{G} \right] u_2 = 0. \quad (2.13d) \]

Using the spinor field equations (2.5) and (2.12) it can be shown that the bilinear spinor forms, defined by (2.2), i.e.,

\[ S = \bar{\psi} \psi = \bar{v} v, \quad P = i \bar{\psi} \bar{\sigma}^5 \psi = i \bar{v} \bar{\sigma}^5 v, \quad A^0 = \bar{\psi} \bar{\sigma}^5 \bar{\sigma}^0 \psi = \bar{v} \bar{\sigma}^5 \bar{\sigma}^0 v, \]
\[ A^3 = \bar{\psi} \bar{\sigma}^5 \bar{\sigma}^3 \psi = \bar{v} \bar{\sigma}^5 \bar{\sigma}^3 v, \quad V^0 = \bar{\psi} \bar{\sigma}^0 \psi = \bar{v} \bar{\sigma}^0 v, \quad V^3 = \bar{\psi} \bar{\sigma}^3 \psi = \bar{v} \bar{\sigma}^3 v, \]
\[ Q^{30} = i \bar{\psi} \bar{\sigma}^3 \bar{\sigma}^0 \psi = i \bar{v} \bar{\sigma}^3 \bar{\sigma}^0 v, \quad Q^{21} = \bar{\psi} \bar{\sigma}^2 \bar{\sigma}^5 \bar{\sigma}^1 \psi = i \bar{v} \bar{\sigma}^2 \bar{\sigma}^5 \bar{\sigma}^1 v, \]

obeying the following system of equations:

\[ \dot{S}_0 - \frac{k}{c} Q^{30}_0 - 2G A^0_0 = 0, \quad (2.14a) \]
\[ \dot{P}_0 - \frac{k}{c} Q^{21}_0 - 2G A^0_0 = 0, \quad (2.14b) \]
\[ \dot{A}^0_0 - \frac{m-n}{c} A^3_0 + 2 \Phi P_0 + 2 G S_0 = 0, \quad (2.14c) \]
\[ \dot{A}^3_0 - \frac{m-n}{c} A^0_0 = 0, \quad (2.14d) \]
\[ \dot{V}^0_0 - \frac{m-n}{c} V^3_0 = 0, \quad (2.14e) \]
\[ \dot{V}^3_0 - \frac{m-n}{c} V^0_0 + 2 \Phi Q^{30}_0 - 2G Q^{21}_0 = 0, \quad (2.14f) \]
\[ \dot{Q}^{30}_0 + 2 \frac{k}{c} S_0 - 2G V^3_0 = 0, \quad (2.14g) \]
\[ \dot{Q}^{21}_0 + 2 \frac{k}{c} P_0 + 2G V^3_0 = 0, \quad (2.14h) \]
where we denote $F_0 = \tau F$. Combining these equations together and taking the first integral one gets

$$(S_0)^2 + (P_0)^2 + (A_0^0)^2 - (A_0^3)^2 + (V_0^0)^2 + (V_0^3)^2 + (Q_0^{30})^2 + (Q_0^{21})^2 = C = \text{Const} \quad (2.15a)$$

Before dealing with the Einstein equations (2.6) let us go back to (2.13). From the first and the third equations of the system (2.13) one finds

$$\dot{u}_{13} = (\mathcal{G} - Q)u_{13}^2 - 2i\Phi u_{13} + (\mathcal{G} + Q), \quad (2.16)$$

where, we denote $u_{13} = u_1/u_3$ and $Q = [m - n - 2ik]/2c$. The equation (2.16) is a Riccati one [15] with variable coefficients. Further, setting $u_{13} = v_{13}\exp[-2i\int \Phi(t)dt]$, from (2.16) one obtains

$$\dot{v}_{13} = (\mathcal{G} - Q)v_{13}^2 e^{-2i\int \Phi(t)dt} + (\mathcal{G} + Q)e^{2i\int \Phi(t)dt}. \quad (2.17)$$

The general solution of (2.17) can be written as

$$v_{13} = -\left[\int (\mathcal{G} - Q)e^{-2i\int \Phi(t)dt}dt + C(t)\right]^{-1}, \quad (2.18)$$

with the integration constant be defined from

$$\int \left[\int (\mathcal{G} - Q)e^{-2i\int \Phi(t)dt}dt + C(t)\right]^{-2}dC = \int (\mathcal{G} + Q)e^{2i\int \Phi(t)dt}dt. \quad (2.19)$$

Thus given the nonlinear term in the Lagrangian and solution of the Einstein equations, one finds the relation between $u_1$ and $u_3$ ($u_2$ and $u_4$ as well), hence the components of the spinor field.

Now we study the Einstein equation (2.4). In doing so, we write the components of the energy-momentum tensor, which in our case read

$$T_0^0 = mS - F + \frac{k}{c}V^3, \quad (2.20a)$$

$$T_1^1 = T_2^2 = \mathcal{D}S + \mathcal{G}P - F, \quad (2.20b)$$

$$T_3^3 = \mathcal{D}S + \mathcal{G}P - F - \frac{k}{c}V^3, \quad (2.20c)$$

$$T_3^0 = -kV^0. \quad (2.20d)$$
Let us demand the energy-momentum tensor to be conserved, i.e.,

\[ T_{\nu\mu} = T_{\nu\mu} + \Gamma_{\beta\mu}^{\nu} T_{\nu}^{\beta} - \Gamma_{\nu\mu}^{\beta} T_{\beta}^{\mu} = 0 \]  

(2.21)

Taking into account that \( T^\nu_{\mu} \) is a function of \( t \) only, from (2.21) we find

\[ \Phi \dot{S}_0 - G \dot{P}_0 + \frac{k}{c} \dot{V}_0^3 - \frac{k}{c} \frac{m - n}{c} V_0^0 = 0, \]  

(2.22a)

\[ \dot{V}_0^0 - \frac{m - n}{c} V_0^3 = 0. \]  

(2.22b)

As one can easily verify, the equations (2.22) are consistent with those of (2.14).

Let us now deal with the Einstein equations (2.6). In view of (2.20), from (2.6e) one obtains

\[ a^m b^{-n} c^{m-n} = \mathcal{N} \exp[\kappa k \int V_0^0 dt], \quad \mathcal{N} = \text{const.}, \]  

(2.23)

whereas, subtraction of (2.6b) and (2.6a) leads to

\[ \frac{d}{dt} \left[ \tau \frac{d}{dt} \left\{ \ln \left( \frac{a}{b} \right) \right\} \right] = \frac{m^2 - n^2}{c^2} \tau. \]  

(2.24)

Note that, the two other equations, obtained by subtracting (2.6c) from (2.6a) and (2.6c)
from (2.6b), respectively, are identical to (2.24), that can be easily verified inserting (2.6e) and
(2.14) into the equations in question. Finally, summation of (2.6a), (2.6b), (2.6c) and
(2.6d), multiplied by 3, leads to the equation for \( \tau \), which in view of (2.20) takes the form

\[ \frac{\ddot{\tau}}{\tau} = 2 \frac{m^2 - mn - n^2}{c^2} - \frac{k}{2} \left[ 3(MS + DS + GP - 2F) + 2 \frac{k}{c} V_0^3 \right]. \]  

(2.25)

For the right-hand side of the equation (2.25) are some functions of \( \tau \) only, the solution to
this equation is well known [15]. It can be shown that the quantities, related to the spinor
field are indeed the functions of \( \tau \). Now, if we assume the metric function \( c \) also to be a
function of \( \tau \), both (2.24) and (2.25) can be solved explicitly. Thus both the nonlinear spinor
and the gravitational field equations are solved in general. In what follows, we consider some
special cases and compare the solutions obtained with those for a BI universe. In can be
emphasized that the introduction of inhomogeneity both in gravitational (through \( m \) and
\( n \)) and spinor (through \( k \)) significantly complicates the whole picture.
To begin with we consider the linear case setting $F(I,J) = 0$. As one sees, the structures of the spinor field equation (2.13), as well as the gravitational ones (2.24) and (2.25) in this case remain unaltered. Thus, the removal of the nonlinearity has little to offer in our cause. As it was mentioned earlier, the introduction of inhomogeneity is in the root of all these troubles. So, for some break-through we demand the spinor field completely space-independent setting $k = 0$. In this case for the components of the energy-momentum tensor immediately we find

\begin{align*}
T_0^0 &= mS - F, \quad (2.26a) \\
T_1^1 &= T_2^2 = T_3^3 = DS + GP - F, \quad (2.26b) \\
T_3^0 &= 0. \quad (2.26c)
\end{align*}

In view of (2.26) from (2.23) we obtain

\begin{equation}
\alpha^{mn}b^{-n}/c^{m-n} = \mathcal{N}, \quad (2.27)
\end{equation}

but the equations (2.13), (2.24) and (2.25) are still unchanged. Thus we see that even the elimination of spinor field inhomogeneity has little to offer.

Finally, we consider the case, when the spinor field is independent of space i.e., $k = 0$ and the gravitational field is given by a BV space-time with $m = n$ in (2.4). In this case for the spinor field we find

\begin{align*}
\dot{u}_1 + i\Phi u_1 - G u_3 &= 0, \quad (2.28a) \\
\dot{u}_2 + i\Phi u_2 - G u_4 &= 0, \quad (2.28b) \\
\dot{u}_3 - i\Phi u_3 + G u_1 &= 0, \quad (2.28c) \\
\dot{u}_4 - i\Phi u_4 + G u_2 &= 0. \quad (2.28d)
\end{align*}

The bilinear spinor forms in this case satisfy

\begin{align*}
\dot{S}_0 - 2GA_0^0 &= 0, \quad (2.29a) \\
\dot{P}_0 - 2\Phi A_0^0 &= 0. \quad (2.29b)
\end{align*}
\[ \dot{A}_0^0 + 2\Phi P_0 + 2\mathcal{G}S_0 = 0, \] (2.29c)
\[ \dot{A}_3^3 = 0, \] (2.29d)
\[ \dot{V}_0^0 = 0, \] (2.29e)
\[ \dot{V}_0^3 + 2\Phi Q_0^{30} - 2\mathcal{G}Q_0^{21} = 0, \] (2.29f)
\[ \dot{Q}_0^{30} - 2\Phi V_0^3 = 0, \] (2.29g)
\[ \dot{Q}_0^{21} + 2\mathcal{G}V_0^3 = 0, \] (2.29h)

with the relations

\begin{align*}
(S_0)^2 + (P_0)^2 + (A_0^0)^2 &= B_1, \quad (2.30a) \\
A_3^3 &= B_2, \quad (2.30b) \\
V_0^0 &= B_3, \quad (2.30c) \\
(V_0^3)^2 + (Q_0^{30})^2 + (Q_0^{21})^2 &= B_4, \quad (2.30d)
\end{align*}

with \( B_i \) being the constant of integration.

For the gravitational field we find,

\[ a = (\mathcal{N})^{(1/m)} b, \] (2.31)

with the equations

\[ \frac{d}{dt} \left[ \tau \frac{d}{dt} \{ \ln \left( \frac{b}{c} \right) \} \right] = -\frac{2m^2}{c^2} \tau, \] (2.32)

and

\[ \frac{\ddot{\tau}}{\tau} = -2\frac{m^2}{c^2} - \frac{k}{2}[3(MS + DS + GP - 2F)]. \] (2.33)

Note that the spinor field equation (2.28) completely coincides with those for a BI metric.

If the nonlinear term in the Lagrangian is given as \( F = F(I) \), then the components of the spinor field can be given as [12]

\[ \psi_1(t) = (C_1/\sqrt{\tau})\exp \left[ -i \int (M - D)dt \right], \] (2.34a)
$\psi_2(t) = (C_2/\sqrt{\tau})\exp[-i\int (M - D)dt], \quad (2.34b)$

$\psi_3(t) = (C_3/\sqrt{\tau})\exp[i\int (M - D)dt], \quad (2.34c)$

$\psi_4(t) = (C_4/\sqrt{\tau})\exp[i\int (M - D)dt], \quad (2.34d)$

with $C_1, C_2, C_3, C_4$ being the integration constants, such that

$$C_1^2 + C_2^2 - C_3^2 - C_4^2 = C_0,$$

with

$$S = C_0/\tau.$$ 

In case, the nonlinear term is given by $F = F(J)$, the components of the spinor field have the form

$$\psi_1 = \frac{1}{\sqrt{\tau}}(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad (2.35a)$$

$$\psi_2 = \frac{1}{\sqrt{\tau}}(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \quad (2.35b)$$

$$\psi_3 = \frac{1}{\sqrt{\tau}}(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad (2.35c)$$

$$\psi_4 = \frac{1}{\sqrt{\tau}}(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \quad (2.35d)$$

Here $\sigma = \int G dt$, and the integration constants $D_i$ obey

$$2 (D_1^2 + D_2^2 - D_3^2 - D_4^2) = D_0,$$

with $D_0$ to be determined from

$$P = D_0/\tau.$$ 

Contrary to the BI metric, where the metric functions $a, b, c$ and $\tau$ are easily determined given the concrete form of nonlinearity, in the case considered, the process is much complicated with the space inhomogeneity being an active player. Thus we see that the introduction of inhomogeneity in the space-time has a far reaching effect in the evolution of both spinor and gravitational field.
REFERENCES

[1] D. Ivanenko, Phys. Zs. Sowjetunion 13, 141 (1938).

[2] D. Ivanenko, Soviet Physics Uspekhi 32, 149 (1947).

[3] V. Rodichev, Soviet Physics JETP 13, 1029 (1961).

[4] Yu.P. Rybakov, B. Saha and G.N. Shikin, PFU Reports: Physics 2, (2), 61 (1994).

[5] Yu.P. Rybakov, B. Saha and G.N. Shikin, Communications in Theoretical Physics 3, 199 (1994).

[6] B. Saha and G.N. Shikin, Journal Mathematical Physics 38, 5305 (1997).

[7] R. Alvarado, Yu.P. Rybakov, B. Saha and G.N. Shikin, JINR Preprint E2-95-16, 11 p. (1995), Communications in Theoretical Physics 4, (2), 247 (1995), gr-qc/9603035.

[8] R. Alvarado, Yu.P. Rybakov, B. Saha and G.N. Shikin, Izvestia Vysshikh Uchebnikh Zavedenii: Fizika 38, (7) 53 (1995) [Russ. Phys. J. 38, 700 (1995)].

[9] B. Saha, and G.N. Shikin, General Relativity and Gravitation 29, 1099 (1997).

[10] Bijan Saha, and G.N. Shikin, PFU Reports: Physics 8, (1), 17 (2000); gr-qc/0102059.

[11] Bijan Saha, Modern Physics Letters A 16, 1287 (2001); gr-qc/0009002.

[12] Bijan Saha, Physical Review D 64, 123501 (2001); gr-qc/0107013.

[13] V.B. Berestetski, E.M. Lifshitz and L.P. Pitaevski, Quantum Electrodynamics (Nauka, Moscow, 1989).

[14] V.A. Zhelnorovich, Spinor theory and its application in physics and mechanics (Nauka, Moscow, 1982).

[15] E. Kamke, Differentialgleichungen losungmethoden und losungen (Akademische Verlagsgesellschaft, Leipzig, 1957).