Application Research on Reliability Modeling of Multi-failure Modes Correlation System Based on Time Varying Copula Model

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Abstract. The reliability modeling of dynamic correlation system with multi-failure modes is studied in this paper. Firstly, Patton model, one of the time-varying copula models, is introduced. Secondly, multi-dimensional time-varying copula function of multi-failure modes correlation system is transformed into two-dimensional time-varying copula functions based on vine Copula function, and then reliability modeling and solution procedure of multi-failure modes correlation system is presented based on time-varying copula model. Finally, the above reliability modeling and solution procedures are demonstrated through a vehicle suspension system engineering case, and their correctness are proved by the results obtained in engineering case.

1. Introduction
Now copula function has been more and more used in reliability modeling of correlation systems. In the initial research stage, the correlation parameters in copula function are constants, namely it is a static Copula function [1]. However, it is found that the correlation between failure modes in complex system changes with time in-depth study, i.e., correlation parameters in Copula function should not be constant, but change with time due to the changes of working environment, inherent properties and state of the system. Therefore, dynamic Copula function and model should be used for reliability modeling, analysis and evaluation of dynamic correlation systems. Zhihua Wang et al. propose a general framework and corresponding methods to deal with the time-dependent reliability analysis of a mechanism effectively [2]. Time-variant copula function is used to estimate the cumulative probabilities of failure by Wang Zhonglai et al. [3]. Zhao-Xia Xu et al. apply the copula-based sampling method firstly to study 3D reliability of seismic slopes as well as the cross correlation between the geotechnical parameters [4]. Árpád Rózsás et al. qualitatively and quantitatively analyze the impact of Gauss (normal or Gaussian) copula assumption on failure probability [5]. Y. T. Sun et al. present a new reliability analysis method in which the mixed Copula is constructed to describe the correlations among multi-failure modes [6]. Pan Yue et al. develops a hybrid Copula-Bayesian-based approach to model the structural health of an operational metro tunnel in a dependent system [7]. Dan Xu et al. propose a method of multivariate failure behavior modeling and reliability assessment based on vine-copula and accelerated degradation data [8]. Hu Qigu et al. propose the empirical distribution function-local maximum likelihood two-step method to estimate the time-varying parameters in the dynamic Copula function [9]. Yaping Wang et al. develop a dependent competing risk model for systems subject to multiple degradation processes and random shocks using time-varying copulas [10].
At present, the application research on dynamic copula function used in dynamic correlation system reliability modeling and analysis is rare, so it is urgent to be studied to meet the increasingly complex, huge and correlation system reliability modeling, prediction and analysis requirements.

2. Dynamic Copula Model and Time-varying parameter Estimation

Patton, who studied dynamic copula model earlier, proposed that the current correlation between variables can be explained by the previous correlation and the historical average of the cumulative probability of the two variables according to Hansen's autoregressive conditional density model. In Patton model, Copula’s correlation parameter is a function of lag data and a parameter transformed by autoregressive term. In Patton's article [11], based on Hansen autoregressive distribution model, dynamic correlation parameter in normal copula was presented as follows

\[
\rho_t = \Lambda \left( \omega_\rho + \beta_\rho \rho_{t-1} + \alpha_\rho \frac{1}{m} \sum_{i=1}^{m} \Phi^{-1}(U_{c^i}(t-i)) \Phi^{-1}(U_{c^i}(t-i)) \right)
\]

where \( \Lambda = (1 - e^{-\lambda}) \left(1 + e^{-\lambda} \right) \) is a transfer function, which is used to ensure correlation parameter \( \rho \) remains within the range of \([-1,1]\); \( \Phi^{-1}(\cdot) \) is inverse function of cumulative distribution function of the standard normal function; The parameters \( \omega_\rho, \beta_\rho \) and \( \alpha_\rho \) are constant parameters estimated by the time series; \( m \) is the window length value, which represents the number of lag items, and generally it is set as 10.

Time-varying correlation parameter estimation process of bivariate copula function based on SP method is as follows:

1) Through the empirical distribution formula \( u = \tilde{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x^{(i)} \leq x\} \), the original data series \( X, Y \) are transformed into \([0,1]\) interval sequence \( u, v \);
2) Time-varying correlation parameter is substituted into time-varying bivariate copula function;
3) The likelihood functions corresponding to time-varying bivariate copula function is obtained as follows

\[
l(\delta) = \sum_{k=1}^{n} \text{ln}(\tilde{F}(x^{(k)}) \tilde{F}(y^{(k)}) \delta)
\]

4) Time-varying correlation parameter is estimated by optimization method as follows

\[
\hat{\delta} = \arg \max_{\delta} l(\delta) = \min\{- l(\delta)\}
\]

3. D-Vine Copula and Reliability Formula of Serial System

Vine Copula function as a convenient mathematical tool can be used for reliability modeling problem of multi-failure mode correlation system [12]. Vine Copula function is used to decompose joint distribution functions of multiple variables into marginal distribution functions and two-dimensional Copula functions. C-Vine and D-Vine functions are two common types of Vine Copula functions.

In this paper, D-Vine function is used to decompose joint distribution functions of multiple variables. \( X = (X_1, X_2, \cdots, X_n) \) is an \( N \)-dimensional random vector. Based on D-Vine function, The decomposition form of probability density function is as follows

\[
f(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} f_k(x_k) \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} C_{i+j+i+1, \cdots, j+j-1} \left( F_{i+j+i+1, \cdots, j+j-1} (x_i | x_{i+1}, \cdots, x_{i+j-1}), F_{i+j+i+1, \cdots, j+j-1} (x_{i+j} | x_{i+1}, \cdots, x_{i+j-1}) \right)
\]
And Copula function of multi-failure mode correlation system is decomposed as follows

\[
c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) = \prod_{j=1}^{n-1} \prod_{i=1}^{j} c_{i,j} \left( F_{i+j+1,1,+,j-1}(x_i \mid x_{i+1}, 1, x_{i+j-1}) \right) F_{i+j+1,1,+,j-1}(x_i \mid x_{i+1}, 1, x_{i+j-1})
\]

where \( f_k(x_k) \) is marginal probability density function of each variable, \( k=1,2,\ldots, n \). The subscript \( j \) is the tree \( T_j \), and the subscript \( i \) is the edge in \( T_j \). \( c_{i,j} \) is a two-dimensional Copula function acting on \( F_{i+j+1,1,+,i+j-1}(\cdot) \) and \( F_{i+j+1,1,+,i+j-1}(\cdot) \).

In this paper, the reliability of series system will be calculated, and its expression is introduced firstly as follows

\[
R_S(t) = P(T > t) = P\left(\min(X_1, X_2, \ldots, X_n) > t\right)
\]

\[
= 1 - \sum_{i=1}^{n} (1 - R_i(t)) + \sum_{1 \leq i < j \leq n} C_{ij}(1 - R_i(t), 1 - R_j(t)) - \sum_{1 \leq i < j < k \leq n} C_{ijk}(1 - R_i(t), 1 - R_j(t), 1 - R_k(t)) + \ldots + (-1)^n C(1 - R_1(t), 1 - R_2(t), \ldots, 1 - R_n(t))
\]

where \( 2 \leq k \leq n \).

4. The Calculation Procedure of Multi-failure Mode Correlation System Reliability Based on Dynamic Copula

Based on Vine Copula function and bivariate time-varying Copula model, the reliability solution procedure of multi-failure mode correlation system is as follows:

1) According to the life data of each unit in the system, the marginal life distribution of each unit is fitted;
2) The reliability model of system is established according to the logical relationship between multiple failure modes, and then the calculation formula of system reliability is determined according to the reliability model of system. If it is a series system, the system reliability is expressed as formula (6);
3) The Vine Copula function is used to decompose the multiple Copula function into bivariate Copula function;
4) The correlation parameters in bivariate Copula functions are estimated by time-varying Copula model and static Copula model, and optimization Copula model is selected through the maximum values principle of logarithmic likelihood function, AIC principle and BIC principle.
5) The bivariate Copula model can be determined by substituting the time-varying correlation parameters;
6) The reliability degree of multi-failure mode correlation system can be obtained by substituting bivariate Copula model into the calculation formula of system reliability.

5. Application Case

The suspension system of a vehicle is composed of balance elbow, torque shaft and hydraulic shock absorber. Reference [1] provides 50 life data of each unit of the suspension system. The reliability calculation process of multi-failure mode correlation system based on dynamic Copula model and Vine Copula function is as follows:

1) Weibull distribution, which is commonly used in the life distribution of mechanical products, is used to describe the life distribution of balance elbow, torsion shaft and hydraulic shock absorber. According to the life data of reference [1], the Weibull distributions of balance elbow, torsion shaft and hydraulic shock absorber are fitted firstly;
2) According to the composition and working principle of the suspension system, the reliability model of suspension system is a series system. And reliability degree formula of the suspension system is as follows

\[
R_s(t) = P(T > t) = 1 - \sum_{i=1}^{3} F_i(t) + \sum_{1 \leq i < j \leq 3} C_2(F_i(t), F_j(t)) - \sum_{1 \leq i < j < k \leq 3} C_3(F_i(t), F_j(t), F_k(t))
\]  

(7)

where \( R_s(t) \) is system reliability degree, \( R_i(t) \) is reliability degree of each unit, \( F_i(t) \) is failure probability function of each unit, \( C_2 \) is bivariate Copula function, \( C_3 \) is triple Copula function.

3) Based on Vine Copula function decomposition technique, triple Copula functions in formula (7) are decomposed into bivariate Copula function as follows

\[
C_3(F_i(t), F_2(t), F_3(t)) = C_2(F_1(t), F_2(t)) C_2(F_2(t), F_3(t)) C_2(F_{12}(t), F_{32}(t))
\]  

(8)

where the formula of \( F_{12}(t) \) and \( F_{32}(t) \) are expressed as follows

\[
F_{12}(t) = \frac{\partial C(F_1(t), F_2(t))}{\partial F_2(t)} \quad F_{32}(t) = \frac{\partial C(F_3(t), F_2(t))}{\partial F_3(t)}
\]  

(9)

4) In view of the correlation between the life of mechanical parts are usually positive correlation, also it is shown that Gumbel copula function can accurately describe the correlation of mechanical parts in reference [1]. Thus, Gumbel copula function is selected to describe the correlations between balance elbow, torque shaft and hydraulic shock absorber. At the same time, normal copula function is adopt to compare with Gumbel copula function. Correlation parameters are estimated by time varying Gumbel copula model and static Gumbel copula model, as shown in Fig.1. The logarithmic likelihood function values, AIC values and BIC values of the static model and time-varying model of Gumbel Copula function and Normal Copula function are shown in Table 1. The results show that time-varying model of Gumbel Copula function is more appropriate to describe the correlation between units of suspension system.

5) When the optimal model of Copula function is determined to be time-varying model of Gumbel Copula function, the value of bivariate time-varying Gumbel Copula function can be calculated at a specific time point. For example, when \( t=6000 \text{km} \), bivariate time-varying Gumbel Copula function can be determined as follows

\[
C_2(F_1(t), F_2(t)) = \exp \left( \left( -\log(1 - F_1(t)) \right)^\alpha + \left( -\log(1 - F_2(t)) \right)^\alpha \right)^{1/\delta} = 9.7233 \times 10^{-4}
\]

\[
C_2(F_2(t), F_3(t)) = \exp \left( \left( -\log(1 - F_2(t)) \right)^\alpha + \left( -\log(1 - F_3(t)) \right)^\alpha \right)^{1/\delta} = 9.7403 \times 10^{-4}
\]

\[
C_2(F_1(t), F_3(t)) = \exp \left( \left( -\log(1 - F_1(t)) \right)^\alpha + \left( -\log(1 - F_3(t)) \right)^\alpha \right)^{1/\delta} = 0.1998
\]

\[
C_2(F_{12}(t), F_{32}(t)) = \exp \left( \left( -\log(1 - F_{12}(t)) \right)^\alpha + \left( -\log(1 - F_{32}(t)) \right)^\alpha \right)^{1/\delta} = 0.9973
\]  

(10)

6) Substituting above bivariate time-varying Gumbel Copula function into (8) and (7), and reliability degree of suspension system at \( t=6000 \text{km} \) can be obtained as follows

\[
R_s(t) = 1 - \sum_{i=1}^{3} F_i(t) + \sum_{1 \leq i < j \leq 3} C_2(F_i(t), F_j(t)) - \sum_{1 \leq i < j < k \leq 3} C_3(F_i(t), F_j(t), F_k(t))
\]

\[
= 1 - F_1(t) - F_2(t) - F_3(t) + C_2(F_1(t), F_2(t)) + C_2(F_2(t), F_3(t)) + C_2(F_1(t), F_3(t)) - C_3(F_1(t), F_2(t), F_3(t))
\]

\[
= 0.5208
\]  

(11)
(a) correlation parameter in $C_2(F_t(F_2(t)))$

(b) correlation parameter in $C_2(F_t(F_3(t)))$

(c) correlation parameter in $C_2(F_1(t),F_3(t))$

(d) correlation parameter in $C_2(F_2(t),F_3(t))$

Figure 1. Variation of correlation parameter in Gumbel Copula function.

Table 1. Logarithmic likelihood function values, AIC values and BIC values.

| Copula function | Copula model          | $l(\delta)$ | AIC        | BIC        |
|-----------------|-----------------------|-------------|------------|------------|
| $C_2(F_t,F_2(t))$ | Dynamic Gumbel copula | 5.6355      | 11.2411    | 11.1918    |
|                 | Static Gumbel copula  | -21.6195    | -43.2489   | -43.2653   |
|                 | Dynamic Normal Copula | -76.2430    | -152.5158  | -152.5651  |
|                 | Static Normal Copula  | -2.4840e+04 | -4.9681e+04| -4.9681e+04|
| $C_2(F_t,F_3(t))$ | Dynamic Gumbel copula | -0.0291     | -0.0881    | -0.1374    |
|                 | Static Gumbel copula  | -21.2691    | -42.5646   | -42.5646   |
|                 | Dynamic Normal Copula | -287.1480   | -574.3752  | -574.3752  |
|                 | Static Normal Copula  | -3.6206e+04 | -7.4212e+04| -7.4212e+04|
| $C_2(F_1(t),F_3(t))$ | Dynamic Gumbel copula | 164.2200    | 328.4102   | 328.3609   |
|                 | Static Gumbel copula  | 143.6518    | 287.2937   | 287.2772   |
|                 | Dynamic Normal Copula | -198.9406   | -397.9110  | -397.9603  |
|                 | Static Normal Copula  | -2.3250e+04 | -4.6500e+04| -4.6500e+04|

Reliability degree of suspension system is 0.5208 which is obtained by time-varying Gumbel Copula function, it is just in the interval between complete independence $R_s(t) = 0.425$ and perfect correlation $R_s(t) = 0.535$. It is conformed that time-varying Gumbel Copula function is more accurate to describe the correlation between each unit or each failure mode. Moreover, the actual engineering data...
provided in literature [1] show that failure probability of the suspension system at 6000km is 48%, which is equivalent to reliability degree of the suspension system at 6000km is 0.520, and it is slightly lower than the calculation value 0.5208. The results prove the correctness of time-varying Copula model used in this paper to describe the correlation between units or multi-failure modes of suspension system.

6. Conclusion
Based on time-varying copula model (Patton model) and vine Copula function, reliability modeling and solution procedure of multi-failure modes correlation system is presented in this paper. Reliability degree of suspension system is 0.5208 at 6000km which is obtained by the above reliability modeling and solution procedure. Moreover, the actual engineering data show that reliability degree of the suspension system is 0.520. Therefore, the results prove the correctness of the reliability modeling and solution procedure.

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