Transforming Particular Stabilizer Codes into Hybrid Codes

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Abstract—In this paper, we prove how to extend a subset of quantum stabilizer codes into a qudit hybrid code storing \( \log_2 p \) classical bits over a qudit space with dimension \( p \), with \( p \) prime. Our proof also gives an explicit procedure for finding the entire collection of stabilizer algebras for all of the subcodes of the hybrid code. This allows extra classical information to be transmitted without having to arduously search for additional codes and their associated codewords, and also provides a first lower bound to the amount of classical information able to be transmitted in a qudit hybrid code, but unfortunately only allows for \( \log_2 p \) classical bits to be decoded by a receiver.

Coding of quantum information has made a great deal of progress since its early inception. Still, there is a need for a scheme that can efficiently encode classical information with the quantum information without incurring additional costs \([1]\). Codes which can accomplish this are called hybrid codes. In this piece, we propose (and prove) a scheme for encoding classical information using a specific family of hybrid codes as direct extensions of qubit stabilizer codes. In this paper we begin by defining the relevant background of progress since its early inception. Still, there is a need for a scheme that can efficiently encode classical information with the quantum information without incurring additional costs \([1]\). Codes which can accomplish this are called hybrid codes. In this piece, we propose (and prove) a scheme for encoding classical information using a specific family of hybrid codes as direct extensions of qubit stabilizer codes. In this piece, we propose (and prove) a scheme for encoding classical information using a specific family of hybrid codes as direct extensions of qubit stabilizer codes.

Definitions

Before addressing the solution to the problem, we begin by defining our tools. First we must define our basic operations:

**Definition 1.** Generalized Paulis for a space over \( p \) orthogonal levels, where we assume throughout \( p \) is prime, is given by:

\[
\omega = 1^{1/p} \quad X[j] = |(j + 1) \mod p \rangle \\
Z[j] = \omega^j |j\rangle
\]

where \( j \in \mathbb{Z}_p \). These Paulis, quotiented by the leading coefficient, form a group \([2]\).

Clearly, taking the tensor of compositions of these operators will also form a group, where each element will have order \( p \), with a leading coefficient.

**Definition 2.** A \( n \)-qudit stabilizer is an \( n \)-length string of tensored generalized Pauli operators, such that there exists at least one state, \( |\psi\rangle \) such that:

\[
s|\psi\rangle = |\psi\rangle
\]

where \( |\psi\rangle \in \mathbb{C}^n \). We call the set of states \( |\psi\rangle \) which satisfy this condition the codewords of the stabilizer \( s \).

A collection of these \( s \) that commutes with each other, and still leave at least a single state, form a subgroup of all the generalized Pauli over \( n \) qudits, where each element will have order \( p \). This forms what we call the stabilizer algebra \( S \) for these stabilizers. This algebra will have \( k \) elements, where \( k \) is the number of compositionally independent stabilizers in the collection, also known as the generators for the stabilizer algebra. We recall for the reader, the well-known result:

**Theorem 3.** For any stabilizer code with \( k \) qudit stabilizers and \( n \) physical qudits, there will be \( p^{n-k} \) stabilizer states.

**Definition 4.** A stabilizer code, specified by its stabilizers and stabilizer states, is characterized by a set of values:

- \( n \): the number of qudits that the states are over
- \( n - k \): the number of encoded (logical) qudits, where \( k \) is the number of stabilizers
- \( d \): the distance of the code, given by the lowest weight of an undetectable generalized Pauli error

These values are specified for a particular code as: \([n, n - k, d]_p\), where \( p \) is the dimension of the qudit space.

**Definition 5** (\( \phi \) representation of a qudit operator). We define the bijective map

\[
\tilde{\phi} : P^p_n \mapsto \mathbb{Z}_p^{2n+1}
\]

which carries a qudit Pauli \( P^p_n \) over \( n \) qudits to a \( 2n + 1 \) vector mod \( p \), where we define the action as:

\[
\tilde{\phi}(\omega^a \otimes_{i=1}^n I \otimes X_i \otimes Z_i^b \otimes_{i=1}^n I) = \langle \alpha | \otimes_{i=1}^n i \otimes a | \otimes_{i=1}^n i \rangle b \rangle \otimes_{i=1}^n (n-i) \rangle
\]

This mapping is also an isomorphism if we define: \( \hat{\phi}(s_1 \otimes s_2) = \hat{\phi}(s_1) \oplus \hat{\phi}(s_2) \), where \( \oplus \) is addition mod \( p \). At this point, we remove the bijective nature of this mapping to produce the more familiar mapping where we delete the coefficient entry and call this new injective homomorphism \( \hat{\phi} \).

**Definition 6.** A hybrid code is a collection of quantum stabilizer codes \( C_1, C_2, ..., C_m \) such that each code corresponds to a different set of codewords, and thus we can associate each word space with a different classical result. These codes are denoted: \([n, n - k, d]_p\), where \( d \) is the quantum code’s distance and we have assumed all of these codes to have the same number of stabilizers and distances \([3]\).

In this piece we consider a particular kind of qudit hybrid code utilizing the eigenvalues of the stabilizers. Let \( S \) be a stabilizer code with generators \( s_i \) and parameters \([n, n - k, d]_p\).
Let the corresponding codewords be given by $|c_w⟩_0$, where the 0 subscript is just foreshadowing and the $w$ value specifies the codeword and has $p^n - k$ values. By definition, we have:

$$s_1|c_w⟩_0 = |c_w⟩_0$$  \hfill (5)

We now investigate codewords $|c_w⟩_j$ such that:

$$s_i|c_w⟩_j = ω^{-j}|c_w⟩_j$$  \hfill (6)

where $ω = 11/p$. These $|c_w⟩_j$ codewords are equivalently the states stabilized by the stabilizer code where $s_i ↦ ω^j s_i$. However, a priori, we have no reason to believe that these codewords ought to exist nor have any method to convert from one to the next. Our first goal in this piece is to prove these facts.

**CLASSICAL PROMOTION OPERATORS**

In this section we define our new tool classical promotion operators, show its operation, and provide a constructive proof of useful existence for a certain subset of all stabilizer codes.

**Definition 7 (Classical Promotion Operator).** Let $f$ be a qudit Pauli operator such that:

$$f|c_w⟩_j = |c_w⟩_{j+1}$$  \hfill (7)

we call this $f$ the Classical Promotion Operator.

**Theorem 8.** These classical promotion operators preserve the space of correctable errors.

**Proof.** We must consider the action of an error on our system near the time of applying a classical promotion operator. We may assume that the state $|ψ⟩$ is perfect up until nearly this time (or has already been corrupted). Then there are cases to consider:

1) $S′fES|ψ⟩$: in this case the error $E$ will be corrected only if it has weight less than $d$.

2) $S′ES|ψ⟩$: in this case the error $E$ will be corrected so long as it’s also in the correctable space for $S′$, which, since $S′ = ω S$, will be the same space with a different phase for the syndromes.

3) $S′fES|ψ⟩$: this is the somewhat tricky case. We begin by noting that we know $f$ and are selecting it, however, we do not know $E$, thus commuting it past $f$ we pick up a "random" phase, say $ω^k$. This $E$ can then be resolved by $S′$ only if it’s also in the correctable space and we know which error it is. This random phase relabels our table, however, is a constant shift for all syndrome values. If we perform purposeful errors (and undo them), knowing what syndromes we expect, we can determine this "random" phase and undo it, thus allowing us to also correct $E$ if it was in the correctable space at first.

From this we see that the space of correctable errors is left unchanged not only in distance, but also in the exact errors that can be corrected.

**Lemma 9.** We only need a single $f$ in order to produce all code word sets.

**Proof.** Since this operator promotes by only a single word space at a time, and $f$ is a qudit operator, we will scan through the entire space of codewords using this single operator by simply taking higher powers of $f$, up to $f^p = I$.

**Lemma 10.** Let $f$ be defined as above. Then the following must be true: $sf = ωfs$.

**Proof.** WLOG we assume we’re doing the 0 to 1 promotion on some codeword $c$. Then:

$$f|c⟩_0 = |c⟩_1 ⇔ fs|c⟩_0 = |c⟩_1 ⇔ fs|c⟩_0 = ω^{-1}s|c⟩_1$$  \hfill (8)

and the result follows from: $f = ωs^{-1}fs$.

**Lemma 11.** Let $s_i$ be a set of independent stabilizers which satisfy the following constraints for $f$:

$$φ(s_i) ⊙ φ(f) = 1$$  \hfill (9)

where $⊙$ is the symplectic product, defined by:

$$φ(s_i) ⊙ φ(f) = φ(f)zφ(s_i)x - φ(f)xφ(s_i)x$$  \hfill (10)

Then these are the only conditions that $f$ needs to satisfy, and these thus fully specify $f$.

**Proof.** Consider the case where $s = s_is_j$, where $s_i$ and $s_j$ satisfy the given condition. Since $φ$ is a homomorphic map, this gives:

$$φ(s) ⊙ φ(f) = (φ(s_i) ⊙ φ(s_j)) ⊙ φ(f)$$

$$= φ(s_i) ⊙ φ(f) ⊙ φ(s_j) ⊙ φ(f)$$  \hfill (11)

which can easily be extended to larger compositions via induction. This proves that all higher order multiplications by stabilizers are simply addition mod $p$, as needed, thus all those constraints are satisfied if the first order ones are. We note that this proof also works for the isomorphic $φ$, thus meaning this has a bijective correspondence to the generalized Paulis.

**Theorem 12.** For any stabilizer code with a non-single codeword space, there exists such an $f$ operator for a special set of generators $S^*$ chosen from the total stabilizer algebra.

**Proof.** We begin by specifying $S^*$. We write the entire stabilizer algebra in the $φ$ representation. We call the matrix generated so $A$. This matrix $A$ has rank $k$, where $k$ is the number of independent stabilizers in the space, also known as the generators of the algebra. We may put this matrix into reduced row echelon form (RREF) (which is valid due to the isomorphism of $φ$), where exactly $k$ rows start with a 1 (where this can be accomplished since for any nonzero number in mod $p$, there exists a multiplicative inverse) and those places columns with a single 1. We define $S^* = RREF(A)$. This procedure works since $φ$ is bijective, so we know that we have a collection of stabilizers still to refer to.

Now we show that we may construct an $f$ that satisfies our conditions for the generators in $S^*$. We call these generators $s_i$ as before. Then we need $f$ to satisfy the system of equations:

$$[φ(s_i)z φ(s_i)x] \begin{bmatrix} -φ(f)x \\ φ(f)x \end{bmatrix} = [1]$$

Since we know that there are $k$ columns with a single 1, we may simply select those entries as being 1 in the $f$ column vector. This gives us the values of $(-f_x|f_z)$, which is easily
changed to \( f = (f_z | f_z) \). This completes the proof, since we may invert the \( \phi \) map to generate the qudit Pauli operator \( f \).

We note that this is not the only way to construct the classical promotion operator \( f \), but is a way to construct one such operator. We address this point later in this paper.

**Corollary 13.** The minimal weight of the classical promotion operator \( f \) is equal to the number of stabilizers \( k \) for the original code.

**Remark 14.** Since we would like these \( f \) to not be confused for errors, thus we need these to have higher weight than the distance of the code and to use a hard distance \( d \) for the code (whereby any errors outside this radius, even if they’re correctable by the original code, cause the code to annihilate the word), then we simply need \( k > d \). The Quantum Singleton bound provides \( k \geq 2(d - 1) \), which means this condition is often satisfied.

**Corollary 15.** Let \( S \) be a stabilizer algebra for a stabilizer code with \( [[n, n - k, d]]_p \), with \( p \) prime, then using the scheme described above, we may create a \( [[n, n - k : \log_p 2, d]]_p \) hybrid code.

**Proof.** This follows since we may encode any of \( 1 \rightarrow p \), which takes on \( \log_p 2 \) bits to represent, and our scheme preserves the code, and thus preserves the number of logical qudits and the distance of the code.

The implication of this is that in a \( p \) level qudit system, we can algorithmically turn any stabilizer code into \( p \) codes and thus encode any number from \( 1 \rightarrow p \), or equivalently \( \log_p 2 \) bits. This procedure also produces all the other space’s codewords, which means knowing a single space’s codewords generates all the rest. Therefore, knowing a single stabilizer code and its codewords is sufficient for this procedure, which removes the sometimes onerous task of discovering these for each code in the hybrid code.

**CLASSICAL PROMOTION OPERATOR DEGENERACY**

As we mentioned earlier, the classical promotion operator \( f \) was not unique. We in fact have a collection of possible operators. These form their own subalgebra of the generalized Pauli group over \( n \) qudits. Thus, for two classical promotion operators \( f_1 \) and \( f_2 \) we need them to commute and to be independent. Equivalently, we require:

\[
\phi(f_1) \odot \phi(f_2) = 0 \tag{12}
\]

and \( \phi(f_1) \) and \( \phi(f_2) \) linearly independent over \( \mathbb{Z}_p^{2n} \). This gives us:

**Lemma 16.** Suppose we have a collection of \( f \) of cardinality \( |f| \), such that for any pair \( f_i, f_j \) in the collection the following are both true:

- \( \phi(f_i) \odot \phi(f_j) = 0 \)
- \( \phi(f_i) \) and \( \phi(f_j) \) are linearly independent

then the code admitting these \( f \) supports \( |f| \) classical \( p \)-ary numbers.

**Proof.** The validity of this statement is proven above and via a trivial induction argument. \( \square \)

**Definition 17 (Nice Stabilizer Code).** A stabilizer code is nice if it corrects \( X \) and \( Z \) type errors on all qudits.

Given this definition, we know that we have \( 2n \) variables in \( f \). These variables are constrained by \( k|f| \) constraints for the commutation relations with the \( k \) stabilizers, as well as:

\[
\sum_{i=2}^{l} |f_i| - |f| - 1 \tag{13}
\]

for the requirement of all levels of commutators being 0, and lastly by one final constraint from needing these \( f \) to be linearly independent. Then this gives us the following condition:

\[
(k - 1)|f| + 2|f| \leq 2n \tag{14}
\]

thus the number of classical bits permitted in a particular stabilizer code is upper bounded by the largest \( f \) satisfying this.

We may also lower bound this value. Assuming we have a nice code, the rows are \( k \)-wise full rank (meaning that taking \( k \) columns at a time in the \( n \) columns representing the same Pauli operation, has these \( k \) columns still having full rank), and thus each of these \( k \)-wise matrices have eigenvectors with integer entries and thus have a solution in \( f \) space that only involves those \( k \) entries for the \( Z \) component. Upon finding the \( f \) that only use the \( Z \) operators, we can use these eigenvectors to produce the same number of \( f \) operators that using \( X \) operators composed with \( Z \) operators on only the same \( k \) indices in \( f \) space. Thus we have:

\[
2 \left\lfloor \frac{n}{k} \right\rfloor \leq f \tag{15}
\]

This is clearly a bijective encoding up to errors that the are beyond the distance of the code, however, decoding these states is not unique since determining the exact state held is generally not possible. For this reason, we must consider a more subtle and complex decoding scheme, thus reducing the set of stabilizer codes that we can use.

**DECODING PROCEDURE**

We begin by restating the Quantum Hamming bound:

\[
p^{n-k} \leq \sum_{j=0}^{t} \binom{n}{j} (p^2 - 1)^j \iff \sum_{j=0}^{t} \left( \binom{n}{j} (p^2 - 1)^j \leq p^k \right) \tag{16}
\]

where \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \), which can be seen from a simple counting argument and using our definitions. Now, if not all the syndromes are being used, we aim to use some of that spare space to transmit classical information. Suppose that we satisfy the particular condition:

\[
\sum_{j=0}^{t} \binom{n}{j} (p^2 - 1)^j \leq p^{|\frac{k}{2}|} \tag{17}
\]

which is the case if less than half the stabilizers are used for the syndrome of each correctable error. In this case, causing a
total shift to all syndrome values will still leave a correctable error and allow for the receiver to know which classical bit was being conveyed by simply taking a majority vote in the syndromes.

This condition is possible to satisfy! We use as an explicit example, the family of $[[2^j, 2^j - 2j - 2, 4]]$ extended quantum Hamming code[2]. If we embed this code in a prime dimensional space, we have:

$$1 + 2^j (p^2 - 1) \leq p^j$$

Which is satisfied for sufficiently large $j$ for a given value of $p$ (see appendix for proof). Therefore, there exists at least one family where our procedure works.

**Scheme**

In this section we describe our coding scheme. There are two players/phases to our coding scheme: the encoder (Alice) and the decoder (Bob). They are merely protecting themselves against noise and Alice is trying to transmit a quantum code to Bob as well as at least a single classical bit. This is accomplished in two phases as described below:

- Prior to the procedure, Alice and Bob both know which stabilizer code is being used and which members of the algebra have been selected as the generators. They also both know all possible classical promotion operators, and Bob knows which classical promotion operator Alice could apply to transmit classical information.

- Alice:
  1) Alice inputs some stabilizer state of her choosing.
  2) Alice applies a classical promotion operator.
  3) Alice reads the stabilizer syndrome to correct any incurred errors.

- The state is then transported to Bob who has no idea what operations Alice has performed.

- Bob:
  1) Bob immediately performs his own syndrome measurements with the generators and corrects any errors deviating from some classical promotion operator syndrome result.
  2) Bob then determines what classical promotion operators were applied on the state (which will manifest as a table relabel shift), then undoes such operations to yield the quantum state.

- Bob now has the classical promotion operations that were applied (and thus the classical information) as well as the initial state.

Since at each step the space of correctable errors is preserved, this code is still impervious to the same distance of errors but now carries classical information along with it.

**Remark 18.** Since there are multiple classical promotion operators, this can even be used as a way for Alice to protect her classical information from an eavesdropper Eve, since, if Alice tells Bob which classical promotion operator she will apply, he’ll know which to look for, while Eve will have to guess, thus somewhat protecting the information from snoopers.

**Remark 19.** Given the way that this code is constructed, we can easily concatenate stabilizer codes to encode multiple classical $p$-ary numbers, but the classical rate is still fixed at one $p$-ary number per block.

**Conclusion and Discussion**

In this piece, we have presented and proved a procedure to transmit classical information using a subset of stabilizer codes embedded in a higher dimensional space. Using this procedure, we can transmit $\log_2 p$ classical bits per qudit stabilizer code of dimension $p$, with $p$ prime. This is a simple first procedure and can quite easily be extended, but immediately provides a way to extend many quantum stabilizer codes into a qudit hybrid quantum stabilizer code. This takes out part of the craft of determining a collection of codes, and instead can automatically generate them using a programmable algorithm.

We remark here that all of the proofs in this paper also carry over to the traditional qubit case of $p = 2$, we just chose to deal in higher spaces since our explicit example required that and we do not know of any family of codes satisfying our conditions while remaining in $p = 2$ space. This procedure opens the door for even further extensions and allows additional classical information to be simply included along with quantum information.

Future work would include trying to find a better decoding procedure in order to transmit more than a single $p$-ary number for each block of stabilizer codes, as well as further development of these operators. These tools may be able to provide a lower bound on the upper bound of classical information able to be transmitted in a hybrid code. At the moment though, this procedure allows at least one family of stabilizer codes to be extended into hybrid codes.

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APPENDIX

In the main text, we make reference to embedding stabilizer codes from qubit space into qudit space. In this appendix, we provide a brief proof of this claim as well as the stronger generalization of this claim. This result both supports the primary text and also provides a tool for creating qudit stabilizer codes.

Notation

Throughout, we will use the Pauli group mod phase (which is physically irrelevant). We recall our notation from the main text before proceeding.

The action of these operators in the Pauli group can be expressed as:

\[
X|j\rangle = (|j + 1\rangle \mod 2) \quad Z|j\rangle = (-1)^{|j|}|j\rangle
\]  

(19)

Let \( p \) be a prime greater than 2, then the extension of these operators to qudits over a space with \( p \) orthonormal basis states is given by operators \( X \) and \( Z \) such that:

\[
\omega = e^{2\pi i / p} \quad X|j\rangle = (|j + 1\rangle \mod p) \quad Z|j\rangle = \omega^{|j|}|j\rangle
\]  

(20)

We call these operators generalized one-qudit Pauli operators, denoted \( P^n_P \), where \( p \) is a prime greater than 2. The group \( P^n_P \) is formed from all the possible compositions of the \( X \) and \( Z \) (both of which have order \( p \)), while the global phase is ignored. From here on out, we will only work with the generalized Pauli group, and often drop the superscript since it’s implicit.

We can extend this group over tensor products to create \( P^n \) where: \( P_n = \otimes^n_{i=1} P_1 \). \( P_1 \) forms an algebra, and thus so does \( P_n \). Thus, with some independent generating set \( S \) of commuting \( P_n \), we form a finite closed algebra since each \( P_n \) has order \( p \), we will form an algebra \( S \) of size \( p^{|S|} \). When this algebra has an associated state or set of states who are all +1 eigenvectors of all \( S \), we call this \( S \) the stabilizer generators and \( S \) the stabilizer algebra. We define the set of all stabilizers codes over qubits as \( S[C^2] \).

Embedding Stabilizer Codes in Higher Dimensional Spaces

From our \( \phi \) homomorphism the operators in \( S[C^2] \) are composed of \( X, Z \) with an associated \( \{0, 1\}^{2n} \) string representing the powers of each of these operators. Likewise operators in \( S[C^n] \) are composed of \( X, Z \) with an associated string in \( \mathbb{Z}_2^n \). Let \( C \) be a codeword. Each codeword is composed as a superposition of codeletters \( c \) which take the form of a \( n \) digit \( q \)-ary number. The action of a stabilizer on each codeletter is to either map it to itself again or to map it to another codeletter in the same codeword with perhaps a different coefficient.

Theorem 20 (Embedding Theorem). Let \( S[C^n] \) be the collection of stabilizer codes over the field \( C^n \) for some integer \( n \geq 2 \). Then:

\[
S[C^2] \subset S[C^n], \quad q \geq 3, \quad n > 2
\]  

(21)

And so any stabilizer code over qubits may be transferred over to the qudit case and the stabilizer code space is not a strict subset unless \( n \geq 3 \).

Proof. Let the stabilizer be specified by \( k \) generators. Using our prior remark regarding the code letters \( l \) in a codeword \( C \), we partition the codeword \( C \) into its \( q \)-length cyclical letter subgroups \( l_i \) for a particular stabilizer \( s \) (assumed to be in \( XZ \) form) in the stabilizer algebra. We define \( \xi = |S_Z \cdot S_X| \) as the number of simultaneous \( XZ \) terms. There will be \( q^{k-1} \) such cycles. The action of this stabilizer on this letter cycle is given by:

\[
s \sum_i \omega^{a_i}|l_i\rangle = s \sum_i \omega^{a_i}|l_0 + iS_x\rangle
\]

\[
= \sum_i \omega^{a_i + \sum s_x \wt(l_{0+i})}|l_0 + (i + 1)S_x\rangle
\]

(22)

which means that in order for this cycle to be stabilized, we must have:

\[
a_1 = a_0 + \sum wt(0),
\]

\[
a_2 = a_1 + \xi = a_0 + \sum wt(0) + (2 - 1)\xi,
\]

\[
a_t = a_0 + \sum wt(0) + (i - 1)\xi
\]

(23)

This gives us an equation set that each cycle must satisfy for this stabilizer. Now, consider that each codeword will have \( q^k \) codeletters. Each cycle has length \( q \) and for each stabilizer the cycles must be composed from purely unique codeletters as otherwise having a repeated element would imply the same cycle (same \( X \) part and could be cosetted out) and thus the same stabilizer. This means that for each stabilizer we form \( q^{k-1} \) disjoint, unique cycles.

We wish for there to be coefficients to satisfy our condition above for each of the \( k \) stabilizers. Before proceeding, we note a reduction:

\[
a_q = a_0 + \sum wt(0) + (q - 1)\xi
\]

\[
\Rightarrow \sum wt(0) + (q - 1)\xi = 0
\]

(24)

\[
\Rightarrow \sum wt(0) = \xi
\]

with this conclusion, generally: \( a_t = a_0 + i|S_Z| \). This generally means that we only have 1 degree of freedom per cycle unless we consider that we also have an additional freedom from taking \( s^b \) as our stabilizer instead of \( s \). This is still an operator with order \( q \) and is equivalent to \( s \) being a stabilizer. Using this, we have:

\[
a_t^b = a_0^b + bi\xi
\]

(25)

meaning that we have 2 parameters, or in total \( 2k \) variables which need to satisfy \( k \) equations and since each cycle may overlap with another cycle only once. Since \( \xi \) is constant for each stabilizer across all cycles, we will always have a solution so long as the original \( s \) had one for qubits.

Lastly, since there are qudit codes which use non-uniform powers of \( X, Z \), we can say that \( S[C^2] \subset S[C^n], \quad q \geq 3 \). □

This proof is easily converted into the general case:

Corollary 21. Using the same notation as above, we have:

\[
S[C^n] \subset S[C^r], \quad q > r, \quad n > r
\]

(26)

otherwise the inclusion is not per se strict.
Proof. The argument above carries over using \( \xi = |S_Z \cdot S_X| \), where:

\[
|S_Z \cdot S_X| = \sum_{X^*, Z^2 = S_i} a_i b_i
\]  

(27)

is the new weight function definition. We also note that, so long as \( n > r \) we can trivially form a stabilizer which uses \( r + 1 \) orders of \( Z \) and thus is in \( S[C^q] \), but not in \( S[C^r] \). \( \square \)

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