The Color-Dipole Picture and $F_L$

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The prediction of $F_L(x, Q^2) = 0.27 F_2(x, Q^2)$ in the color-dipole picture, based on color-transparency and transverse-size reduction, is consistent with the experimental results from HERA.

We consider the photon-nucleon interaction at low $x_{bj} \gg Q^2 / W^2 < 0.1$, such that

$$\Delta E = \frac{1}{x_{bj} + \frac{M_p^2}{W^2}} \frac{1}{M_p} \gg \frac{1}{M_p}. \tag{1}$$

The covariant quantity in (1) is identical to the lifetime of a hadronic $q\bar{q}$ fluctuation of mass $M_{q\bar{q}}$ of the photon in the rest frame of the nucleon. The inequality (1) is the space-time condition $[1]$ for the validity of generalized vector dominance $[2]$.

The $\gamma^* p$ scattering process at low $x_{bj}$ proceeds via $q\bar{q}$ scattering. The $q\bar{q}$ state interacts via gluons coupled to both the quark and the antiquark, i.e. it interacts as a color-dipole state (color-dipole picture, CDP) $[3]$.

$\gamma^* p \to \gamma^* p \to (q\bar{q})_p \to (q\bar{q})_p$

The mass of a $q\bar{q}$ fluctuation, $M_{q\bar{q}}$, is restricted by

$$m_{\rho^0}^2 \leq M_{q\bar{q}}^2 \leq m_1^2(W^2), \tag{2}$$

where $m_1^2(W^2) \ll W^2$ approximately coincides with the upper end of the diffractive mass spectrum observed at HERA. The frequently adopted approximation of $m_1^2(W^2) \to \infty$ restricts the kinematic domain of validity of the CDP.

Consider a timelike photon of mass $M_{q\bar{q}}$. The structure of its $\gamma^*(q\bar{q})$ coupling implies an enhancement $[4]$ of the transverse size of the $(q\bar{q})_{J=1}^T$ state of mass $M_{q\bar{q}}$ and spin $J = 1$ originating from a transversely polarized photon, relative to the transverse size of the $(q\bar{q})_{L=1}^T$ state originating from a longitudinally polarized photon. The transverse-size enhancement implies an enhanced cross section,

$$\sigma_{(q\bar{q})_{J=1}^T}(M_{q\bar{q}}^2, W^2) = \rho \sigma_{(q\bar{q})_{L=1}^P}(M_{q\bar{q}}^2, W^2), \tag{3}$$

where $[4]$

$$\rho = \frac{4}{3}. \tag{4}$$

The factor $\rho$ is independent of the Lorentz boost from the $(q\bar{q})$ rest frame to the energy $W$ of the $(q\bar{q})p$ interaction; $\rho$ is independent of $W$.

The transition from the interaction of a timelike photon, $\gamma_{L,T}^*$, of mass $M_{q\bar{q}}$ to the interaction of a spacelike one of four momentum squared $q^2 = -Q^2 < 0$ in the imaginary

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part of the forward Compton amplitude requires integration over all masses of the incoming and outgoing $q\bar{q}$ fluctuations. Upon introducing the transverse size, $r_\perp$, of a $q\bar{q}$ fluctuation and upon introducing the $(q\bar{q})_{L,T}^{\perp=1} p$ scattering cross section for spin $J = 1$ quark-antiquark dipole states, the photoabsorption cross section in the CDP becomes \[ (5) \]

$$
\sigma_{\gamma_{L,T} p}(W^2, Q^2) = \frac{2\alpha R_{+,-}^z Q^2}{3\pi^2} \int d^2 r'_\perp K^2_{0,1}(r'_\perp Q) \sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(r'_\perp, W^2).
$$

Massless quarks are assumed in (5). Quark masses can be introduced via quark-hadron duality. The variable $r'_\perp$ is related to the transverse size of a $q\bar{q}$ state via $r'_\perp = r_\perp \sqrt{z(1-z)}$ where $0 \leq z \leq 1$, and $R_{+,-}^z$ is a sum of $Q_q^2$, where $Q_q$ denotes the quark charge, and $Q / \sqrt{Q^2}$. The representation (5) of the CDP factorizes the $\gamma_{L,T} p$ cross section into the $Q^2$-dependent (square of the) photon wave function, given by the modified Bessel function $K_{0,1}(r'_\perp Q)$, and the $W$-dependent dipole cross section $\sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(r'_\perp, W^2)$. Since $\gamma^* p$ interactions at low $x_{bj}$ proceed via (on-shell) $q\bar{q}$ scattering, the frequently employed factorization in $(Q^2, x_{bj})$ rather than in $(Q^2, W^2)$ can at most be of approximate validity \[ (6) \]. The transverse-size enhancement \[ (3) \] enters \[ (6) \] via

$$
\sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(r'_\perp, W^2) = \rho \sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(r'_\perp, W^2).
$$

The interaction of the $q\bar{q}$ state as a color-dipole state requires a representation of the dipole cross section in \[ (5) \] given by \[ (3) (5) \]

$$
\sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(r'_\perp, W^2) = \int d^2 l'_\perp \sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(l'_\perp, W^2) \left( 1 - e^{-d'_\perp \cdot l'_\perp} \right)
\quad \begin{cases} 
\int d^2 l'_\perp \sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(l'_\perp, W^2), & \text{for } r'_\perp \to \infty, \\

r'_\perp \frac{\alpha}{2} \int d^2 l'_\perp l'_\perp^2 \sigma_{(q\bar{q})_{L,T}^{\perp=1} p}(l'_\perp, W^2), & \text{for } r'_\perp \to 0,
\end{cases}
$$

where $l'_\perp = l'_\perp / \sqrt{z(1-z)}$, and $l'_\perp$ is the transverse momentum of the absorbed gluon.

The color-dipole cross section becomes $r'_\perp$-independent for $r'_\perp$ sufficiently large (“saturation”). It vanishes, as $r'_\perp^2$, for $r'_\perp$ sufficiently small (“color transparency”). Note that the scale for the $r'_\perp$ dependence is $W$-dependent. It is determined by the magnitude of the $l'_\perp^2$-moment of the dipole cross section in the third line of \[ (7) \].

An important conclusion on the ratio

$$
R(W^2, Q^2) = \frac{\sigma_{\gamma^* p}(W^2, Q^2)}{\sigma_{\gamma p}(W^2, Q^2)}
$$

follows immediately from \[ (5) \], \[ (6) \] and \[ (7) \]. Replacing the transverse dipole cross section in \[ (5) \] by \[ (6) \], and noting that for sufficiently large $Q^2$ and appropriate energy, $W$, the integral in \[ (5) \] is determined by the $r'_\perp^2 \to 0$ behavior of \[ (7) \], we obtain \[ (4) \]

$$
R(W^2, Q^2) = \frac{1}{\rho} \int d^2 r'_\perp r'_\perp^2 K^2_{0,1}(r'_\perp Q) = \frac{1}{2\rho},
$$

where the mathematical identity

$$
\int_0^\infty dy y^3 K^2_0(y) = \frac{1}{2} \int_0^\infty dy y^3 K^2_1(y)
$$

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was inserted.

We note that a suppression of the longitudinal relative to the transverse photoabsorption cross section by the factor 0.5 in (4) is due to the photon wave function, more precisely to the first moment of the photon wave function as a function of \( r' \) that enters as a consequence of color transparency. For \( \rho = 1 \) in (4), i.e. helicity independence of the interaction of the \((q\bar{q})^j=1\) fluctuation with the proton, \( R(W^2, Q^2) = 0.5 \). Any deviation from this value must be due to a helicity dependence of the \((q\bar{q})^j=1\) cross section, i.e. a dependence on whether the \((q\bar{q})^j=1\) fluctuation originates from a transversely or a longitudinally polarized photon. For the transverse-size enhancement (4) we find \( R(W^2, Q^2) = 0.375 \), i.e.

\[
R(W^2, Q^2) = \begin{cases} 
\frac{1}{3} = 0.5, & \text{for } \rho = 1, \text{ helicity independence,} \\
\frac{2}{3} = 0.375, & \text{for } \rho = \frac{2}{3}, \text{ transverse-size enhancement.}
\end{cases}
\]  

(11)

In terms of the structure functions \( F_L(x, Q^2) \) and \( F_2(x, Q^2) \), we have

\[
F_L(x, Q^2) = \frac{1}{1 + 2\rho} F_2(x, Q^2) = \begin{cases} 
0.33 F_2(x, Q^2), & (\rho = 1), \\
0.27 F_2(x, Q^2), & (\rho = \frac{2}{3}).
\end{cases}
\]  

(12)

We add the remark that the equalities (11) and (12) require sufficiently large \( Q^2 \). Quantitatively, in terms of the low-\( x_{bj} \) scaling variable \( \eta(W^2, Q^2) \) [7],

\[
\eta(W^2, Q^2) \equiv \frac{Q^2 + m_0^2}{\Lambda_{sat}(W^2)} > 1
\]  

(13)

is required, where \( m_0^2 \simeq 0.14 \text{ GeV}^2 \) and

\[
\Lambda_{sat}(W^2) \sim \int d\vec{r}_1^2 \vec{r}_2^2 \sigma_{(q\bar{q})}^{i=1}(\vec{r}_1^2, W^2) \sim (W^2)^{c_2}.
\]  

(14)

As seen in figs. 2 and 3, the experimental data are consistent with a transverse-size enhancement in (12).

The empirical validity of low-\( x_{bj} \) scaling, \( \sigma_{\gamma^*p}(W^2, Q^2) = \sigma_{\gamma^*p}(\eta(W^2, Q^2)) \), was established [7] in a model-independent analysis of the experimental data from HERA. Theoretically, low-\( x_{bj} \) scaling is a consequence of the general structure of the color-dipole interaction [7] combined with the (approximate) constancy of the \( r' \rightarrow \infty \) limit of the dipole-cross section in [7], and dimensional analysis.

For \( \eta(W^2, Q^2) > 1 \) or \( Q^2 > \Lambda_{sat}(W^2) \), where \( 2 \text{GeV}^2 \leq \Lambda_{sat}(W^2) \leq 7 \text{GeV}^2 \) at HERA energies, both \( F_2(x, Q^2) \) and the gluon distribution \( \alpha_s(Q^2)xg(x, Q^2) \), using \( x \equiv x_{bj} \), are proportional [8] to the saturation scale, \( \Lambda_{sat}(W^2) \),

\[
F_2(x, Q^2) \sim \alpha_s(Q^2)xg(x, Q^2) \sim \Lambda_{sat}(W^2) \sim (W^2)^{c_2}.
\]  

(15)

**Figure 2:** The prediction of \( F_L(x, Q^2) = 0.27 \) \( F_2(x, Q^2) \) compared with H1 experimental results (V. Chekelian, private communication).
Consistency with DGLAP evolution \[9\],

\[
\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{R_{\pi e^-}}{9\pi} \alpha_s(Q^2)xg(x, Q^2)
\]

requires \[8\]

\[
\frac{\partial}{\partial \ln W^2} \Lambda^2_{\text{sat}}(2W^2) = \frac{1}{2(2\rho + 1)} \Lambda^2_{\text{sat}}(W^2)
\]
or

\[(2\rho + 1)c_2^2c_2 = 1. \tag{18}\]

Relation \(17\) implies, respectively,

\[
\rho = \begin{cases} 1, & c_2^{\text{theor}} = 0.276, \\ \frac{4}{3}, & c_2^{\text{theor}} = 0.23. \end{cases} \tag{19}\]

The result \(19\) is consistent with the value from the model-independent analysis of the experimental data \[7\],

\[c_2^{\text{exp}}|_{\text{Model indep.}} = 0.28 \pm 0.06. \tag{20}\]

Supplementing the CDP by the evolution constraint \(18\) allows one to predict \(c_2\), i.e. the (strong) energy dependence, proportional to \((W^2)^{\gamma_2}\) of \(\sigma_{\gamma p}(W^2, Q^2)\) and \(F_2(x, Q^2)\) for \(Q^2 > \Lambda^2_{\text{sat}}(W^2)\) in agreement with the experimental data.

It is worth noting that the consistency of the evolution constraint \(18\) on \(c_2^{\text{theor}}\) with the experimental value of \(c_2^{\text{exp}}\) rules out values of \(\rho \gg 1\), as well as \(\rho \ll \frac{4}{3}\), compare Table 1. The experimental result for the longitudinal-to-transverse ratio \(R = 1/2\rho \simeq 0.375\) is indeed intimately related to the constant \(c_2\) that determines the energy dependence of \(\sigma_{\gamma p}(W^2, Q^2)\) and of \(F_2(x, Q^2)\).

Since \(c_2\) is correctly predicted by requiring \(16\) to be valid for the structure function \(F_2(x, Q^2) = (Q^2/4\pi^2\alpha)\sigma_{\gamma p}(\eta(W^2, Q^2))\) in the CDP, the experimentally observed low-\(x\) scaling does not require non-linear effects in the evolution equations, neither for \(\eta(W^2, Q^2) > 1\), nor for \(\eta(W^2, Q^2) < 1\). The saturation phenomenon for \(\eta(W^2, Q^2) < 1\), where \(\sigma_{\gamma p}(\eta(W^2, Q^2)) \sim \ln(1/\eta(W^2, Q^2))\), is a consequence of the dipole interaction \(7\). For sufficiently large energy, \(\Lambda^2_{\text{sat}}(W^2) \gg Q^2\), for any fixed \(Q^2\), the photoabsorption cross section is determined by the \(r'_\perp \to \infty\) limit of the dipole cross section.

\[\text{Figure 3: As fig. 2, but compared with the ZEUS experimental results (B. Reisert, private communication.) In case the originally yellow line } F_L = 0.27F_2 \text{ is not well reproduced, compare the slides of this presentation available under DIS2009.}\]
in (7). For $A^2_{\text{sat}}(W^2) \ll Q^2$, the color-dipole state interacts as a dipole of vanishing size, $r_\perp \rightarrow 0$, while for $A^2_{\text{sat}}(W^2) \gg Q^2$, it interacts as an ordinary hadron with the gluons in the nucleon.

Consistency of linear evolution and scaling at low $x$ has recently also been found [10] by examining the double-asymptotic scaling approximation of the DGLAP evolution equations.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\rho$ & $c^\text{theor.}_2$ & $\frac{\sigma_{\perp}}{\sigma_\parallel}$ & $F_2 \left( W^2 = \frac{Q^2}{x} \right)$ \\
\hline
$\rightarrow \infty$ & 0 & 0 & $\left( \frac{Q^2}{x} \right)^0 = \text{const}$ \\
0 & 0.65 & $\infty$ & $\left( \frac{Q^2}{x} \right)^{0.65}$ \\
\hline
\end{tabular}
\caption{The results for $c^\text{theor.}_2$ for the assumptions of a very large and a very small value of $\rho$.}
\end{table}

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