ON THE STRONG CHROMATIC INDEX AND MAXIMUM INDUCED MATCHING OF TREE-COGRAPHS, PERMUTATION GRAPHS AND CHORDAL BIPARTITE GRAPHS

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Abstract. We show that there exist linear-time algorithms that compute the strong chromatic index and a maximum induced matching of tree-cographs when the decomposition tree is a part of the input. We also show that there exist efficient algorithms for the strong chromatic index of (bipartite) permutation graphs and of chordal bipartite graphs.

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1 Introduction

Definition 1 ([8]). An induced matching in a graph $G$ is a set of edges, no two of which meet a common vertex or are joined by an edge of $G$. The size of an induced matching is the number of edges in the induced matching. An induced matching is maximum if its size is largest among all possible induced matchings.

Definition 2 ([13]). Let $G = (V, E)$ be a graph. A strong edge coloring of $G$ is a proper edge coloring such that no edge is adjacent to two edges of the same color. A strong edge-coloring of a graph is a partition of its edges into induced matchings. The strong chromatic index of $G$ is the minimal integer $k$ such that $G$ has a strong edge coloring with $k$ colors. We denote the strong chromatic index of $G$ by $s\chi'(G)$.

Equivalently, a strong edge coloring of $G$ is a vertex coloring of $L(G)^2$, the square of the linegraph of $G$. The strong chromatic index problem can be solved in polynomial time for chordal graphs [8] and for partial $k$-trees [39], and can be solved in linear time for trees [16]. However, it is NP-complete to find the strong chromatic index for general graphs [8,36,41], or even for planar bipartite graphs [27]. In this paper, we show that there exist linear-time algorithms that compute the strong chromatic index and a maximum induced matching of tree-cographs when the decomposition tree is a part of the input. We also show that there exist efficient algorithms for the strong chromatic index of (bipartite) permutation graphs and of chordal bipartite graphs.

The class of tree-cographs was introduced by Tinhofer in [42].

Definition 3. Tree-cographs are defined recursively by the following rules.

1. Every tree is a tree-cograph.
2. If $G$ is a tree-cograph then also the complement $\overline{G}$ of $G$ is a tree-cograph.
3. For $k \geq 2$, if $G_1, \ldots, G_k$ are connected tree-cographs then also the disjoint union is a tree-cograph.

Let $G$ be a tree-cograph. A decomposition tree for $G$ consists of a rooted binary tree $T$ in which each internal node, including the root, is labeled as a join node $\otimes$ or a union node $\oplus$. The leaves of $T$ are labeled by trees or complements of trees. It is easy to see that a decomposition tree for a tree-cograph $G$ can be obtained in $O(n^3)$ time.

2 The strong chromatic index of tree-cographs

The linegraph $L(G)$ of a graph $G$ is the intersection graph of the edges of $G$ [5]. It is well-known that, when $G$ is a tree then the linegraph $L(G)$ of $G$ is a claw-free blockgraph [24]. A graph is chordal if it has no induced cycles of length more than three [14]. Notice that blockgraphs are chordal.

A vertex $x$ in a graph $G$ is simplicial if its neighborhood $N(x)$ induces a clique in $G$. Chordal graphs are characterized by the property of having a perfect elimination ordering, which is an ordering $[v_1, \ldots, v_n]$ of the vertices of $G$ such that
A perfect elimination ordering of a chordal graph can be computed in linear time \[38\]. This implies that chordal graphs have at most \( n \) maximal cliques, and the clique number can be computed in linear time, where the clique number of \( G \), denoted by \( \omega(G) \), is the number of vertices in a maximum clique of \( G \).

**Theorem 1** ([8]). If \( G \) is a chordal graph then \( L(G)^2 \) is also chordal.

**Theorem 2** ([10]). Let \( k \in \mathbb{N} \) and let \( k \geq 4 \). Let \( G \) be a graph and assume that \( G \) has no induced cycles of length at least \( k \). Then \( L(G)^2 \) has no induced cycles of length at least \( k \).

**Lemma 1.** Tree-cographs have no induced cycles of length more than four.

**Proof.** Let \( G \) be a tree-cograph. First observe that trees are bipartite. It follows that complements of trees have no induced cycles of length more than four.

We prove the claim by induction on the depth of a decomposition tree for \( G \). If \( G \) is the union of two tree-cographs \( G_1 \) and \( G_2 \) then the claim follows by induction since any induced cycle is contained in one of \( G_1 \) and \( G_2 \). Assume \( G \) is the join of two tree-cographs \( G_1 \) and \( G_2 \). Assume that \( G \) has an induced cycle \( C \) of length at least five. We may assume that \( C \) has at least one vertex in each of \( G_1 \) and \( G_2 \). As one of \( G_1 \) and \( G_2 \) has more than two vertices of \( C \), \( C \) has a vertex of degree at least three, which is a contradiction. \( \square \)

**Lemma 2.** Let \( T \) be a tree. Then \( L(\overline{T})^2 \) is a clique.

**Proof.** Consider two non-edges \( \{a, b\} \) and \( \{p, q\} \) of \( T \). If the non-edges share an endpoint then they are adjacent in \( L(\overline{T})^2 \) since they are already adjacent in \( L(\overline{T}) \). Otherwise, since \( T \) is a tree, at least one pair of \( \{a, p\}, \{a, q\}, \{b, p\} \) and \( \{b, q\} \) is a non-edge in \( T \), otherwise \( T \) has a 4-cycle. By definition, \( \{a, b\} \) and \( \{p, q\} \) are adjacent in \( L(\overline{T})^2 \). \( \square \)

If \( G \) is the union of two tree-cographs \( G_1 \) and \( G_2 \) then

\[
\omega(L(G)^2) = \max\{\omega(L(G_1)^2), \omega(L(G_2)^2)\}.
\]

The following lemma deals with the join of two tree-cographs.

**Lemma 3.** Let \( P \) and \( Q \) be tree-cographs and let \( G \) be the join of \( P \) and \( Q \). Let \( X \) be the set of edges that have one endpoint in \( P \) and one endpoint in \( Q \). Then

(a) \( X \) forms a clique in \( L(G)^2 \),
(b) every edge of \( X \) is adjacent in \( L(G)^2 \) to every edge of \( P \) and to every edge of \( Q \),
(c) every edge of \( P \) is adjacent in \( L(G)^2 \) to every edge of \( Q \).

**Proof.** This is an immediate consequence of the definitions. \( \square \)
For $k \geq 3$, a $k$-sun is a graph which consists of a clique with $k$ vertices and an independent set with $k$ vertices. There exist orderings $c_1, \ldots, c_k$ and $s_1, \ldots, s_k$ of the vertices in the clique and independent set such that each $s_i$ is adjacent to $c_i$ and to $c_{i+1}$ for $i = 1, \ldots, k-1$ and such that $s_k$ is adjacent to $c_k$ and $c_1$. A graph is strongly chordal if it is chordal and has no $k$-sun, for $k \geq 3$ [15].

**Lemma 4.** Let $T$ be a tree. Then $L(T)^2$ is strongly chordal.

**Proof.** When $T$ is a tree then $L(T)$ is a blockgraph. Obviously, blockgraphs are strongly chordal. Lubiw proves in [35] that all powers of strongly chordal graphs are strongly chordal. □

We strengthen the result of Lemma 4 as follows. Ptolemaic graphs are graphs that are both distance hereditary and chordal [28]. Ptolemaic graphs are gem-free chordal graphs. The following theorem characterizes ptolemaic graphs.

**Theorem 3 ([28]).** A connected graph is ptolemaic if and only if for all pairs of maximal cliques $C_1$ and $C_2$ with $C_1 \cap C_2 \neq \emptyset$, the intersection $C_1 \cap C_2$ separates $C_1 \setminus C_2$ from $C_2 \setminus C_1$.

**Lemma 5.** Let $T$ be a tree. Then $L(T)^2$ is ptolemaic.

**Proof.** Consider $L(T)$. Let $C$ be a block and let $P$ and $Q$ be two blocks that each intersects $C$ in one vertex. Since $L(T)$ is claw-free, the intersections of $P \cap C$ and $Q \cap C$ are distinct vertices. The intersection of the maximal cliques $P \cup C$ and $Q \cup C$, which is $C$, separates $P \setminus Q$ and $Q \setminus P$ in $L(T)^2$. Since all intersecting pairs of maximal cliques are of this form, this proves the lemma. □

**Corollary 1.** Let $G$ be a tree-cograph. Then $L(G)^2$ has a decomposition tree with internal nodes labeled as join nodes and union nodes and where the leaves are labeled as ptolemaic graphs.

From Corollary 1 it follows that $L(G)^2$ is perfect [12], that is, $L(G)^2$ has no odd holes or odd antiholes [34]. This implies that the chromatic number of $L(G)^2$ is equal to the clique number. Therefore, to compute the strong chromatic index of a tree-cograph $G$ it suffices to compute the clique number of $L(G)^2$.
Theorem 4. Let $G$ be a tree-cograph and let $T$ be a decomposition tree for $G$. There exists a linear-time algorithm that computes the strong chromatic index of $G$.

Proof. First assume that $G = (V, E)$ is a tree. Then the strong chromatic index of $G$ is

$$s\chi'(G) = \max \{ d(x) + d(y) - 1 \mid (x, y) \in E \}$$

(1)

where $d(x)$ is the degree of the vertex $x$. To see this notice that Formula (1) gives the clique number of $L(G)^2$.

Assume that $G$ is the complement of a tree. By Lemma 2 the strong chromatic index is the number of nonedges in $G$, which is

$$s\chi'(G) = \left(\frac{n}{2}\right) - (n - 1).$$

Assume that $G$ is the union of two tree-cographs $G_1$ and $G_2$. Then, obviously,

$$s\chi'(G) = \max \{ s\chi'(G_1), s\chi'(G_2) \}.$$

Finally, assume that $G$ is the join of two tree-cographs $G_1$ and $G_2$. Let $X$ be the set of edges of $G$ that have one endpoint in $G_1$ and the other in $G_2$. Then, by Lemma 3 we have

$$s\chi'(G) = |X| + s\chi'(G_1) + s\chi'(G_2).$$

The decomposition tree for $G$ has $O(n)$ nodes. For the trees the strong chromatic index can be computed in linear time. In all other cases, the evaluation of $s\chi'(G)$ takes constant time. It follows that this algorithm runs in $O(n)$ time, when a decomposition tree is a part of the input. \qed

3 Induced matching in tree-cographs

Consider a strong edge coloring of a tree-cograph $G$. Then each color class is an induced matching in $G$, which is an independent set in $L(G)^2$ \cite{8}. In this section we show that the maximal value of an induced matching in $G$ can be computed in linear time. Again, we assume that a decomposition tree is a part of the input.

Theorem 5. Let $G$ be a tree-cograph and let $T$ be a decomposition tree for $G$. Then the maximal number of edges in an induced matching in $G$ can be computed in linear time.

Proof. In this proof we denote the cardinality of a maximum induced matching in a graph $G$ by $i\nu(G)$.

First assume that $G$ is a tree. Since the maximum induced matching problem can be formulated in monadic second-order logic, there exists a linear-time algorithm to compute the cardinality of a maximal induced matching in $G$ \cite{7,19}.
Assume that $G$ is the complement of a tree. By Lemma 2 $L(G)^2$ is a clique. Thus the cardinality of a maximum induced matching in $G$ is one if $G$ has a nonedge and otherwise it is zero.

Assume that $G$ is the union of two tree-cographs $G_1$ and $G_2$. Then

$$i\nu(G) = i\nu(G_1) + i\nu(G_2).$$

Assume that $G$ is the join of two tree-cographs $G_1$ and $G_2$. Then

$$i\nu(G) = \max \{ i\nu(G_1), i\nu(G_2), 1 \}.$$

This proves the theorem. \qed

4 Permutation graphs

A permutation diagram on $n$ points is obtained as follows. Consider two horizontal lines $L_1$ and $L_2$ in the Euclidean plane. For each line $L_i$ consider a linear ordering $\prec_i$ of $\{1, \ldots, n\}$ and put points $1, \ldots, n$ on $L_i$ in this order. For $k = 1, \ldots, n$ connect the two points with the label $k$ by a straight line segment.

Definition 4 ([20]). A graph $G$ is a permutation graph if it is the intersection graph of the line segments in a permutation diagram.

![Permutation Graph and Diagram](image)

Consider two horizontal lines $L_1$ and $L_2$ and on each line $L_i$ choose $n$ intervals. Connect the left - and right endpoint of the $k^{th}$ interval on $L_1$ with the left - and right endpoint of the $k^{th}$ interval on $L_2$. Thus we obtain a collection of $n$ trapezoids. We call this a trapezoid diagram.

Definition 5. A graph is a trapezoid graph if it is the intersection graph of a collection of trapezoids in a trapezoid diagram.

Lemma 6. If $G$ is a permutation graph then $L(G)^2$ is a trapezoid graph.
Proof. Consider a permutation diagram for $G$. Each edge of $G$ corresponds to two intersecting line segments in the diagram. The four endpoints of a pair of intersecting line segments define a trapezoid. Two vertices in $L(G)^2$ are adjacent exactly when the corresponding trapezoids intersect (see Proposition 1 in [9]). □

**Theorem 6.** There exists an $O(n^4)$ algorithm that computes a strong edge coloring in permutation graphs.

Proof. Dagan, et al., [13] show that a trapezoid graph can be colored by a greedy coloring algorithm. It is easy to see that this algorithm can be adapted so that it finds a strong edge-coloring in permutation graphs. □

**Remark 1.** A somewhat faster coloring algorithm for trapezoid graphs appears in [17]. Their algorithm runs in $O(n \log n)$ time where $n$ is the number of vertices in the trapezoid graph. An adaption of their algorithm yields a strong edge coloring for permutation graphs that runs in $O(m \log n)$ time, where $n$ and $m$ are the number of vertices and edges in the permutation graph.

### 4.1 Bipartite permutation graphs

A graph is a *bipartite permutation graph* if it is not only a bipartite graph but also a permutation graph [40]. Let $G = (A, B, E)$ be a bipartite permutation graph with color classes $A$ and $B$.

**Lemma 7.** Let $G$ be a bipartite permutation graph. Then $L(G)^2$ is an interval graph.

Proof. We first show that $L(G)^2$ is chordal. We may assume that $L(G)^2$ is connected.

Let $x$ and $y$ be two non-adjacent vertices in a graph $H$. An $x$, $y$-separator is a set $S$ of vertices which separates $x$ and $y$ in distinct components. The separator is a minimal $x$, $y$-separator if no proper subset of $S$ separates $x$ and $y$. A set $S$ is a minimal separator if there exist non-adjacent vertices $x$ and $y$ such that $S$ is a minimal $x$, $y$-separator. Recall that Dirac characterizes chordal graphs by the property that every minimal separator is a clique [14].

Consider the trapezoid diagram. Let $S$ be a minimal separator in the trapezoid graph $L(G)^2$ and consider removing the trapezoids that are in $S$ from the diagram. Every component of $L(G)^2 - S$ is a connected part in the diagram. Consider the left-to-right ordering of the components in the diagram. Since $S$ is a minimal separator there must exist two consecutive components $C_1$ and $C_2$ such that every vertex of $S$ has a neighbor in both $C_1$ and $C_2$ [6].

Assume that $S$ has two non-adjacent trapezoids $t_1$ and $t_2$. Each of $t_i$ is characterized by two crossing line segments $\{a_i, b_i\}$ of the permutation diagram. Since $t_1$ and $t_2$ are not adjacent, any pair of line-segments with one element in $\{a_1, b_1\}$ and the other element in $\{a_2, b_2\}$ are parallel.

Each trapezoid $t_i$ intersects each component $C_1$ and $C_2$. Since pairs of line-segments are parallel, we have that, for some $i \in \{1, 2\}$
and the reverse inequalities hold for $C_2$. Each trapezoid $t_i$ has at least one line segment of $\{a_i, b_i\}$ intersecting with a line segment of $C_i$. By the neighborhood containments this implies that $G$ has a triangle, which contradicts that $G$ is bipartite. This proves that $S$ is a clique and by Dirac’s characterization $L(G)^2$ is chordal.

Lekkerkerker and Boland prove in [33] that a graph $H$ is an interval graph if and only if $H$ is chordal and $H$ has no asteroidal triple. It is easy to see that a permutation graph has no asteroidal triple [30]. Cameron proves in [9] (and independently Chang proves in [11]) that $L(H)^2$ is AT-free whenever a graph $H$ is AT-free. Thus, since $L(G)^2$ is chordal and AT-free, $L(G)^2$ is an interval graph.

This proves the lemma.

Chang proves in [11] that there exists a linear-time algorithm that computes a maximum induced matching in bipartite permutation graphs. We show that there is a simple linear-time algorithm that computes the strong chromatic index of bipartite permutation graphs.

**Theorem 7.** There exists a linear-time algorithm that computes the strong chromatic index of bipartite permutation graphs.

**Proof.** Let $G = (A, B, E)$ be a bipartite permutation graph and consider a permutation diagram for $G$. Let $[a_1, \ldots, a_s]$ and $[b_1, \ldots, b_t]$ be left-to-right orderings of the vertices of $A$ and $B$ on the topline of the diagram. Assume that $a_1$ is the left-most endpoint of a line segment on the topline. We may assume that the line segment of $a_1$ intersects the line segment of $b_1$.

Consider the set of edges in the maximal complete bipartite subgraph $M$ in $G$ that consists of the following vertices.

(a) $M$ contains $a_1$ and $b_1$,
(b) $M$ contains all the vertices of $A$ of which the endpoint on the topline is to the left of $b_1$,
(c) $M$ contains all the vertices of $B$ of which the endpoint on the bottom line is to the left of $a_1$.

Notice that $M$ is the set of edges in the complete bipartite subgraph in $G$ induced by

$$N[a_1] \cup N[b_1].$$

Extend the set of edges in $M$ with the edges in $G$ that have one endpoint in $M$. Call the set of edges in $M$ plus the edges with one endpoint in $N(a_1) \cup N(b_1)$ the extension $\bar{M}$ of $M$. That is,

$$\bar{M} = \{ \{p, q\} \in E \mid p \in M \text{ or } q \in M \}.$$
Notice that $\tilde{M}$ is the unique maximal clique that contains the simplicial edge $(a_1, b_1)$ in $L(G)^2$.

The second maximal clique in $L(G)^2$ is found by the process described above for the line segments induced by $A \setminus \{a_1\} \cup B$. Likewise, the third maximal clique in $L(G)^2$ is found by repeating the process for the line segments induced by $A \cup B \setminus \{b_1\}$.

Next, remove the vertices $a_1$ and $b_1$ and repeat the three steps described above. It is easy to see that the list obtained in this manner contains all the maximal cliques of $L(G)^2$. Notice also that this algorithm can be implemented to run in linear time. Since $L(G)^2$ is perfect the chromatic number is equal to the clique number, so it suffices to keep track of the cardinalities of the maximal cliques that are found in the process described above. □

5 Chordal bipartite graphs

Definition 6 ([21,29]). A bipartite graph is chordal bipartite if it has no induced cycles of length more than four.

In contrast to bipartite permutation graphs, $L(G)^2$ is not necessarily chordal when $G$ is chordal bipartite. An example to the contrary is shown in Figure 3.

Fig. 3. A chordal bipartite graph $G$ for which $L(G)^2$ is not chordal.

A graph is weakly chordal if it has no induced cycle of length more than four or the complement of such a cycle [25]. Weakly chordal graphs are perfect. Notice that chordal bipartite graphs are weakly chordal.

Cameron, Sritharan and Tang prove in [10] (and independently Chang proves in [11]) that $L(G)^2$ is weakly chordal whenever $G$ is weakly chordal. Thus, if $G$ is chordal bipartite then $L(G)^2$ is perfect and so, in order to compute the strong chromatic index of $G$ it is sufficient to compute the clique number in $L(G)^2$ (see also [11]).

It is well-known that the clique number of a perfect graph can be computed in polynomial time [22]. The algorithm presented in [26] to compute the clique

\^3 Actually, this paper shows that for any graph $G$ with $\omega(G) = \chi(G)$ the values of these parameters can be determined in polynomial time. The reason is that Lovász’ bound $\bar{e}(G)$ for the Shannon capacity of a graph can be computed in polynomial time for all graphs, via the ellipsoid method, and the parameter $\vartheta(G)$ is sandwiched between $\omega(G)$ and $\chi(G)$. 

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number of weakly chordal graphs runs in $O(n^3)$ time, where $n$ is the number of vertices in the graph.

A direct application of their algorithm to solve the strong chromatic index of chordal bipartite graphs $G$ involves computing the graph $L(G)^2$. This graph has $m$ vertices, where $m$ is the number of edges in $G$. This gives a timebound $O(n^6)$ for computing the strong chromatic index of a chordal bipartite graph (see also [1]).

In this section we show that there is a more efficient method.

**Definition 7 ([43]).** A bipartite graph $G = (A, B, E)$ is a chain graph if there exists an ordering $a_1, a_2, \ldots, a_{|A|}$ of the vertices in $A$ such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_{|A|})$.

Chain graphs are sometimes called *difference graphs* [23]. Equivalently, a graph $G = (V, E)$ is a chain graph if there exists a positive real number $T$ and a real number $w(x)$ for every vertex $x \in V$ such that $|w(x)| < T$ for every $x \in V$ and such that, for any pair of vertices $x$ and $y$, $(x, y) \in E$ if and only if $|w(x) - w(y)| \geq T$ (see Figure 4 for an example).

![Fig. 4. A difference graph with $T = 8$.](image)

Chain graphs can be characterized in many ways [23]. For example, a graph is a chain graph if and only if it has no induced $K_3$, $2K_2$, or $C_5$ [23, Proposition 2.6]. Thus a bipartite graph is a chain graph if it has no induced $2K_2$. Abueida, et al., prove that, if $G$ is a bipartite graph that does not contain an induced $C_6$, then a maximal clique in $L(G)^2$ is a maximal chain subgraph of $G$ [1]. Notice that $C_6$ has three consecutive edges that form a clique in $L(G)^2$, however, these edges do not form a chain subgraph.

Thus, computing the clique number of $L(G)^2$ for $C_6$-free bipartite graphs $G$ is equivalent to finding the maximal number of edges that form a chain graph in $G$. In [1] the authors prove that, if $G$ is a $C_6$-free bipartite graph, then

$$\chi(G^*) = ch(G),$$

where $G^*$ is the complement of $L(G)^2$ and $ch(G)$ is the minimum number of chain subgraphs of $G$ that cover the edges of $G$.

An *antimatching* in a graph $G$ is a collection of edges which forms a clique in $L(G)^2$ [36]. It is easy to see that finding a chain subgraph with the maximal
number of edges in general graphs is NP-complete. Mahdian mentions in his paper that the complexity of maximum antimatching in simple bipartite graphs is open \[36\].

**Lemma 8.** Let \( G \) be a bipartite graph. Finding a maximum set of edges that form a chain subgraph of \( G \) is NP-complete.

**Proof.** Let \( G = (A, B, E) \) be a bipartite graph. Let \( C(G) \) be the graph obtained from \( G \) by making cliques of \( A \) and \( B \). Notice that \( G \) is a chain graph if and only if \( C(G) \) is chordal. Yannakakis shows in \[43\] that adding a minimum set of edges to \( C(G) \) such that this graph becomes chordal is NP-complete.

Consider the bipartite complement \( G' \) of \( G \). Adding a minimum set of edges such that \( G \) becomes a chain graph is equivalent to removing a minimum set of edges from \( G' \) such that the remaining graph is a chain graph. This completes the proof. \( \Box \)

In the following theorem we present our result for chordal bipartite graphs.

**Theorem 8.** There exists an \( O(n^4) \) algorithm that computes the strong chromatic index of chordal bipartite graphs.

**Proof.** Let \( G \) be chordal bipartite with color classes \( C \) and \( D \). Consider the bipartite adjacency matrix \( A \) in which rows correspond with vertices of \( C \) and columns correspond with vertices of \( D \). An entry of this matrix is one if the corresponding vertices are adjacent and it is zero if they are not adjacent.

It is well-known that \( G \) is chordal bipartite if and only if \( A \) is totally balanced. Notice that a chain graph has a bipartite adjacency matrix that is triangular. So we look for a maximal submatrix of \( A \) which is triangular after permuting rows and columns.

Anstee and Farber and Lehel prove that a totally balanced matrix, which has no repeated columns, can be completed into a ‘maximal totally balanced matrix.’ \[4,32\]. If \( A \) has \( n \) rows then this completing has \( \binom{n+1}{2} \) + 1 columns. The rows and columns of a maximal totally balanced matrix can be permuted such that the adjacency matrix gets the following form.

One can easily deal with repeated columns in \( A \) by giving the vertices a weight. When the matrix has the desired form, one can easily find the maximal triangular submatrix in linear time. Anstee and Farber give a rough bound of \( O(n^3) \) to find the completion. But faster algorithms are given by Paige and Tarjan and by Lubiw \[37,35\]. \( \Box \)

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A = \begin{bmatrix}
A' & 0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 1 & \cdots & 1
\end{bmatrix}

Fig. 5. A totally balanced matrix.

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