On chaos in mean field spin glasses

Silvio Franz(1), Muriel Ney-Nifle(2)

(1) Laboratoire de Physique Théorique,
Ecole Normale Supérieure
24, Rue Lhomond, 75231 Paris Cedex 05, France

(2) Laboratoire de Physique Théorique et Hautes Energies
Batiment 211, Université de Paris-Sud,
91405 Orsay, France

Abstract

We study the correlations between two equilibrium states of SK spin glasses at different temperatures or magnetic fields. The question, previously investigated by Kondor and Kondor and Végsö, is approached here constraining two copies of the same system at different external parameters to have a fixed overlap. We find that imposing an overlap different from the minimal one implies an extensive cost in free energy. This confirms by a different method the Kondor’s finding that equilibrium states corresponding to different values of the external parameters are completely uncorrelated. We also consider the Generalized Random Energy Model of Derrida as an example of system with strong correlations among states at different temperatures.

1Unité propre du CNRS, associée à l’Ecole Normale Supérieure et à l’Université de Paris Sud
2 present address: NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark; e-mail: franz@nordita.dk
3Laboratoire associé au CNRS
4present address : Laboratoire de Physique, Ecole Normale Supérieure, 46 allées d’Italie, 69364 Lyon cedex 07, France
1 Introduction

The structure of the equilibrium states of mean field spin glasses has been widely discussed in the literature [1]. At low temperature ergodicity is broken, and the contribution to the Boltzmann average comes from "valleys" separated by infinite barriers. The statistics of the correlations between the different valleys for equal values of the external parameters, is one of the most remarkable outcomes of the replica method. In the context of the Parisi ansatz, it is found that the function $q(x)$ is directly related to the statistics of couples of valleys, while the whole set of states is organized as an ultrametric tree. Much less information is available about the relations among the equilibrium states for different values of the external parameters. The object of this paper is an investigation of this relation.

The states of mean field spin glasses can be thought as low free energy solution of TAP equations. Changings in the external parameters can cause the appearance or the disappearance of solutions and modify the relative order of the free energy levels. There are models, for example the $p$-spin spherical spin glass, where the order of the levels is not affected by changes in temperature [2]. In this case the states at different temperatures are correlated. In a model where a change of an external parameter imply a reshuffling of the states of an extensive amount, we expect to find zero correlations between states at different values of the external parameters.

The problem of the correlations between low free energy states at different values of the external parameters has been addressed for the SK model by Kondor [3] and Kondor and Végső [4]. Within the frame of the replica method they considered the partition function of two realizations of the same system for different external parameters. In these papers there were assumed constant correlations; this constant was found to be zero in mean field and the Gaussian corrections to this situation were computed.

Here we reexamine the question without using this assumption. Instead we follow a method introduced in [5], and we take into account in the partition function $(Z)$ of the two systems only these couples of configurations having overlap equal to a fixed value $p_d$. It was shown in [5], in the case of two systems at equal external parameters, that if $p_d$ is in the support of the probability function $P(q)$, the logarithm of the constrained partition function is extensively equal to that of an unconstrained system. This result tells simply that the partition function is dominated by these couple of equilibrium states which satisfy the constraint. Conversely, if $p_d$ is out of the support of the $P(q)$ the system is forced out of equilibrium and this implies an extensive increase of the free energy $F = -\log Z$.

In the present case, an extensive increase of $F$ implies a reshuffling of the free energy level of an extensive amount, and zero correlation between states for different
external parameters. The method, relying on the saddle point approximation, is limited to the computation of the extensive part of the free energy. So, a zero cost in free energy density would not strictly imply no reshuffling, but just that the reshuffling is not extensive. We will refer to ”chaos” with respect to an external parameter to a situation in which the free energy increase is extensive.

The paper is organized as follows: In section 2 we state the basic definitions and our method. In section 3 we discuss the problem in the simple case of the Derrida Generalized Random Energy Model (GREM), where handwaving arguments show that there is no chaos with temperature in absence of a magnetic field, and along the lines of constant magnetization. We show how some modifications in those models produce chaos with temperature. In section 4 we study the SK model near the glassy transition. We show that no ultrametric solution exist for the problems with different magnetic fields or temperatures. We argue that this is the sign that chaos is present in both cases and give estimation for the free energy increase. Finally we draw our conclusions in section 5.

2 The model

Let us consider a system composed of two copies of a Sherrington Kirkpatrick (SK) model, having different temperatures and magnetic fields, \((T_1, h_1)\) and \((T_2, h_2)\) respectively. The partition function of such a system is

\[
Z = \sum_{\{S^1_i, S^2_i\}} \exp \left[ \beta_1 \sum_{i<j} J_{ij} S^1_i S^1_j + h_1 \sum_i S^1_i + \beta_2 \sum_{i<j} J_{ij} S^2_i S^2_j + h_2 \sum_i S^2_i \right] = Z_1 Z_2. 
\]

(1)

Where the couplings \(J_{ij} (i, j = 1, ..., N)\) are Gaussian independent variables of zero mean and variance \(1/N\), and the spins \(S^r_i\) are Ising variables. In the following we will improperly call free energy the quantity \(F = -\log Z\). To address the question of the correlations between the states dominating \(Z_1\), and \(Z_2\) respectively, let us consider \(Z(p_d)\) the sum \((\Pi)\) restricted to these configurations which verify

\[
p_d = \frac{1}{N} \sum_i S^1_i S^2_i. 
\]

(2)

It is clear from the definition that \(Z(p_d) \leq Z\). It is shown in \([\text{I}]\) in the case \(T_1 = T_2\) and \(h_1 = h_2\) that, in the low temperature phase, one has \((1/N) \log Z = (1/N) \log Z(p_d)\) as soon as \(p_d\) is in the support of the function \(P(q)\) of the free system. This is a consequence of the fact that the number of valleys dominating the
partition function grows less than exponentially with $N$, as it can be easily understood noting that $Z = \int dp dZ(p_d)$. An increase in free energy at an extensive level, implies the absence of low-lying states having overlap $p_d$.

In [5] it was shown how to deal with a problem of two coupled copies with the replica method. We shall not repeat here the derivation of formulae which is completely analogous to that ref.[5], but we shall just sketch the results. Instead of the usual order parameter matrix $Q_{ab}$ of the replica method, there are 3 matrices $Q_{ab}^{(1)}$, $Q_{ab}^{(2)}$ and $P_{ab}$ representing respectively

$$
Q_{ab}^{(r)} = \frac{1}{N} \sum_{i=1}^{N} S_i^{ra} S_i^{rb}; \quad r = 1, 2 \quad P_{ab} = \frac{1}{N} \sum_{i=1}^{N} S_i^a S_i^b. \quad (3)
$$

where $a, b = 1, \ldots, n$ and $n$ is ”the number of replicas”, which as usual has to be sent to zero. The constraint in the partition function implies that the elements $P_{aa}$ have to be set equal to $p_d$. Combining $Q_{ab}^{(1)}$, $Q_{ab}^{(2)}$, $P$ and its transposed $P^T$ into the matrix

$$
Q = \begin{pmatrix}
Q^{(1)} & P \\
P^T & Q^{(2)}
\end{pmatrix},
$$

and denoting its elements as $Q_{\alpha\beta}$, $\alpha = (r, a)$, $\beta = (s, b)$ it is possible to see that for both $T_1$ and $T_2$ near the critical temperature $T_c = 1$ and $h_1$ and $h_2$ small, $Q_{\alpha\beta} \sim 1 - T_s$ and the free energy admits the expansion up to the fourth order in $T_s - T_c$ ($s = 1, 2$):

$$
F(p_d) = \lim_{N \to \infty} - \frac{1}{N} \log Z(p_d) = \lim_{n \to 0} \frac{1}{2n} \left\{ \tau_1 \text{Tr} Q^{(1)}^2 + \tau_2 \text{Tr} Q^{(2)}^2 + 2\tau_{12} \text{Tr} P^2 + \frac{w}{3} \text{Tr} Q^3 \\
+ \frac{u}{6} \sum_{\alpha\beta} Q_{\alpha\beta}^4 + \frac{v}{4} \text{Tr} Q^4 - \frac{y}{2} \sum_{\alpha\beta\gamma} Q_{\alpha\beta}^2 Q_{\beta\gamma}^2
\right\}
+ h_1^2 \sum_{ab} Q_{ab}^{(1)} + h_2^2 \sum_{ab} Q_{ab}^{(2)} + 2h_1 h_2 \sum_{ab} P_{ab}
$$

where

$$
\tau_s = (1 - T_s^2)/2 \quad s = 1, 2 \quad \tau_{12} = (1 - T_1 T_2)/2. \quad (5)
$$

For the ”complete” SK model $y = u = v = w = 1$. It is customary in the study of the glassy transition to consider a ”truncated”, (or ”reduced”) model in which it is artificially posed $y = v = 0$, and retained only the term $\sum_{\alpha\beta} Q_{\alpha\beta}^4$ among all the quartic terms. Kondor and Végső have shown recently that this can give rise to instabilities in considering couples of systems with different temperatures when the magnetic field is zero. We anticipate that the argument showing the presence of chaos do not depend critically on which of the two models is used. So, we will
use the complete model to prove the presence of chaos and we will estimate the free energy increase within the truncated one. In any case (4) has to be maximized with respect to the values of the elements of the replica matrices.

The basic object of our investigation will be the free energy difference

\[ \Delta F = F(p_d) - F \]  

where \( F \) is the logarithm of the partition function of the two systems (at two different external parameters) without constraint, and is equal to the sum of the free energies of the two systems. In the following we will refer to \( \Delta F \) as ”free energy excess”, and to ”chaos” whenever this quantity is non zero.

Following [5] we will consider here an analytic continuation to \( n \rightarrow 0 \) for \( F(p_d) \) in which each of the matrices \( Q^{(r)} \) and \( P \) are parametrized according to the Parisi scheme, that is, specifying the value of the diagonal elements and a function of the interval \( [0,1] \).

\[ Q^{(r)} \rightarrow (0, q_r(x)); \quad P \rightarrow (p_d, p(x)) \quad 0 \leq x \leq 1. \]  

The usual restriction of the choice of the Parisi function to the space of non decreasing function, is here substituted by the requirement of semi-positive definiteness of the matrices

\[ \left( \begin{array}{cc} q_1'(x) & p'(x) \\ p'(x) & q_2'(x) \end{array} \right) \]  

for any \( x \), where the primes denote derivation with respect to \( x \) [6]. In particular, this implies that the functions \( q'_s(x) \) are both positive.

The saddle point equations of maximization of \( F_2(p_d) \) of the truncated model are written in terms of the \( q_s(x) \) and \( p(x) \) as

\[ \frac{\delta F}{\delta q_s(x)} = 2[\tau_s - \langle q_s \rangle]q_s(x) + 2[p_d - \langle p \rangle]p(x) - \int_0^x dy[q_s(x) - q_s(y)]^2 \\
- \int_0^x dy[p(x) - p(y)]^2 + \frac{2}{3}q_3^2(x) + h_x^2 = 0, \]  

\[ \frac{\delta F}{\delta p(x)} = [\tau_{12} - \langle q_1 + q_2 \rangle]p(x) + [p_d - \langle p \rangle][q_1(x) + q_2(x)] \\
- \int_0^x dy[q_1(x) - q_1(y) + q_2(x) - q_2(y)][p(x) - p(y)] + \frac{2}{3}p^3(x) + h_1h_2 = 0, \]  

We also write for further reference the expression for the derivative of \( F \) with respect to \( p_d \).

\[ \frac{\partial F}{\partial p_d} = -[\tau_{12}p_d - \langle p(q_1 + q_2) \rangle] + \frac{2}{3}p^3_d + h_1h_2 \]  

(11)
Our variational equations differ from the one considered in reference [3, 4] in the fact that there \( p_d \) was taken as a variational parameter and the function \( p(x) \) was constrained to a constant. We will see in the next section that models without chaos require non constant \( p(x) \).

Before starting the discussion of the maximization of (4) and the solutions of (9,10), we discuss the correlations between states at different temperature and magnetic field in the GREM.

3 The GREM: a model without chaos.

Let us briefly review the Derrida construction of the Generalized Random Energy Model. Without any pretension of being exhaustive on this point, we refer the reader to the original papers on the model [7, 8, 9].

In the GREM one considers \( 2^N \) configurations, associated to the "leaves" of an ultrametric tree. The tree, is composed of \( L \) levels of branching. At a level \( \alpha \), each branch generates \( N_\alpha = \exp(NS_\alpha) \) new branches, in such a way that \( 2^N = e^{N \sum_\alpha S_\alpha} \).

To each branch at a level \( \alpha \) is associated a random energy so that the total energy of a configuration is given by \( E = \sum_\alpha \epsilon_\alpha \). The \( \epsilon_\alpha \) are taken as independent Gaussian variables of zero mean and variance \( \epsilon_\alpha^2 = NJ_\alpha^2/2 \). Two configurations are conventionally said to have an overlap \( q_\alpha \), with \( 0 \leq q_\alpha \leq 1 \), if they coincide at a level \( \alpha \), and consequently have the same \( \epsilon_\beta \) for \( \beta \leq \alpha \). It can be eventually considered a "continuum limit" of infinite number of levels \( (L \to \infty) \) with infinitesimal spacings, where \( J_\alpha \to J(q) dq \quad S_\alpha \to S(q) dq \). This exhausts the construction in absence of a magnetic field. In presence of a magnetic field \( h \) one has to associate magnetization one to an arbitrarily selected state, in such a way that the configurations of magnetization \( m \) are those having an overlap equal to \( m \) with this state. For these states the energy gets an extra contribution equal to \( -Nh m \) where \( h \) is the magnetic field.

In the following we will limit our discussion to the case where \( J(q) \) and \( S(q) \) increase with \( q \), where the levels associated with small \( q \) freeze at higher temperature than the levels of high \( q \).

The absence of chaos with temperature in zero magnetic field is almost obvious; a change in temperature do not affect, by definition of the model, the order of the energy levels. So, two equilibrium states at different temperatures \( T \) and \( T' \) can be strongly correlated. In the same way it is easy to realize that there is chaos with the magnetic field. Two different magnetic fields, say \( h_1 \) and \( h_2 \), impose to the system to have magnetization respectively \( m_1 \) and \( m_2 \) with \( m_1 \neq m_2 \). By construction the overlap between two states with such different magnetizations is equal to \( q_{12} = \min\{m_1,m_2\} \). Imposing a different overlap would bring the magnetizations
out of their equilibrium values, implying an extensive cost in terms of free energy. Furthermore, in presence of a magnetic field, a change in temperature implies a change in the magnetization, and again we find chaos. Thus chaos is clearly absent along the lines \((T, h)\) of constant magnetization \([10]\).

Let us now show with the aid of the replica method that, for two different temperatures in zero magnetic field, it is cost-less to impose an overlap \(p_d\) in the support of the \(P(q)\) function. Note that the same results could be easily obtained with the Derrida probabilistic technique. The replicated partition function of the GREM in zero field, in the discrete formalism is

\[
Z^n = \sum_{\{\delta^\alpha_{a,s}\}} \exp \left( - \sum_{s,a,\alpha} \beta_s \epsilon_{j^\alpha_{a,s}} \right) \prod_{\alpha \leq \alpha_0} \delta_{j^\alpha_{a,1},j^\alpha_{a,2}}
\]

(12)

where the level \(\alpha_0\) corresponds to the overlap \(p_d\), \(\alpha = 1, \ldots, L\), \(s = 1, 2\) and \(a = 1, \ldots, n\) is the indice of replica. Upon performing the average over the values of the energy levels one gets

\[
\overline{Z^n} = \sum_{\{\delta^\alpha_{a,s}\}} \exp \left( \frac{N}{4} \sum_{a,r,s} J^2_{\alpha} \beta_r \beta_s \sum_{ab} \delta_{j^\alpha_{a,r},j^\alpha_{b,s}} \right) \prod_{\alpha \leq \alpha_0} \delta_{j^\alpha_{a,1},j^\alpha_{a,2}}.
\]

(13)

To evaluate the partition function we make the following ansatz on the arrangement of the replicas. We suppose that for the levels \(\alpha \leq \alpha_0\), where \(\delta^\alpha_{a,1} = \delta^\alpha_{a,2}\), the replicas are divided into \(n/x_\alpha\) groups of \(x_\alpha\) coinciding states (i.e. \(j^\alpha_{a,r} = j^\alpha_{b,s}\) for any \(r, s\) if \(a\) and \(b\) are in the same group). For the levels \(\alpha \geq \alpha_0\), one has \(\delta^\alpha_{a,1} \neq \delta^\alpha_{a,2}\), and one can divide, accordingly to the same scheme the replicas with \(s = 1\) and \(s = 2\) into \(n/x^1_\alpha\) and \(n/x^2_\alpha\) groups respectively. It is easy to see that the free energy is given by the expression

\[
\frac{1}{nN} \log \overline{Z^n} = \sum_{\alpha \leq \alpha_0} \left[ \frac{S_\alpha}{x_\alpha} + (\beta_1 + \beta_2)^2 J^2_{\alpha} x_\alpha \right]
\]

\[
\quad + \sum_{\alpha > \alpha_0} \left[ S_\alpha \left( \frac{1}{x^1_\alpha} + \frac{1}{x^2_\alpha} \right) + J^2_{\alpha} \left( \beta_1 x^1_\alpha + \beta_2 x^2_\alpha \right) \right]
\]

(14)

taken at the saddle point aver the various \(x\). Assuming that the level \(\alpha_0\) is frozen at the temperatures \(T_1\) and \(T_2\) one finds upon deriving with respect to \(x_\alpha, x^1_\alpha\) and \(x^2_\alpha\)

\[
\left\{ \begin{array}{ll}
(\beta_1 + \beta_2)^2 x^2_\alpha = \frac{S_\alpha}{J^2_{\alpha}} & \alpha \leq \alpha_0 \\
\beta_2^2 x^1_\alpha = \frac{S_\alpha}{J^2_{\alpha}} & \alpha > \alpha_0 \text{ and } S_\alpha/(J^2_{\alpha} \beta^2_{\alpha}) < 1 \\
x^1_\alpha = 1 & \alpha > \alpha_0 \text{ and } S_\alpha/(J^2_{\alpha} \beta^2_{\alpha}) > 1
\end{array} \right.
\]

(15)
Substituting in (14) one sees easily that \((1/N) \log |Z| = (1/N)[\log |Z_1| + \log |Z_2|]\), that is, we find no chaos with temperature.

It is interesting to notice that, in the continuum limit, inverting the functions \(x(q)\) and \(x_s(q)\), the functions \(q_s(x)\) and \(p(x)\) take the form:

\[
q_s(x) = \begin{cases} 
q_u((\beta_1 + \beta_2)x) & x \leq x_0 \\
p_d & x_0 \leq x \leq x_s \\
qu(\beta_s x) & x > x_s
\end{cases}
\]

\[
p(x) = \begin{cases} 
q_u((\beta_1 + \beta_2)x) & x \leq x_0 \\
p_d \\
qu(\beta_s x) & x > x_s
\end{cases}
\]

where \(q_u(\beta x)\) is the inverse function of \(\beta x(q) = S(q)/J^2(q)\), and the points \(x_0\) and \(x_s\) are defined by the relations

\[
qu((\beta_1 + \beta_2)x_0) = qu(\beta_s x_s) = p_d.
\]

The only solution with a constant \(p(x)\) is the one with \(p_d = p(x) = 0\).

So we have shown that, in a situation where the order of the levels do not depend on the temperature, imposing an overlap \(p_d\) in a suitable interval do not imply an extensive free energy cost i.e. there is not chaos with temperature. The situation is different if the ordering of the state depends on the temperature. In the context of the GREM model, such a dependence can be introduced upon considering one or two of the following modifications:

- choosing temperature dependent \(S_\alpha\) or \(J_\alpha\)
- not imposing the identity of the levels for different temperatures.

Here we choose to discuss the simplest possible case, namely the second point. We take a REM, (i.e. a GREM with only one level, \(L = 1\)) where the Gaussian energies depend on \(T\), and the correlations are specified by

\[
\epsilon_f(T_1)\epsilon_k(T_2) = \delta_{j;k}\frac{1}{2}C(T_1, T_2)N
\]

where \(C(T_1, T_2) \leq J^2 = C(T, T)\). Let us compute within the replica formalism, the partition function of 2 replicas, at temperatures \(T_1\) and \(T_2\) below the freezing transition, constrained to be in the same state

\[
\overline{Z}^n = \sum_{j^a} \exp\left[-\beta_1\epsilon^a_j(T_1) - \beta_2\epsilon^a_j(T_2)\right]
\]

\[
= \sum_{j^a} \exp\left\{N\left[(\beta_1^2 + \beta_2^2)J^2 + 2\beta_1\beta_2C(T_1, T_2)\right] \sum_{ab} \delta_{j^a,j^b}\right\}
\]
Proceeding as above for the GREM (dividing the replicas into groups) one finds that
\[
\frac{1}{nN} \log Z_n = \sqrt{\log 2((\beta_1^2 + \beta_2^2)J^2 + 2\beta_1\beta_2C(T_1, T_2))} \leq \frac{1}{nN} \log Z_1^n + \frac{1}{nN} \log Z_2^n.
\] (21)

The equality is recovered for \( C(T_1, T_2) = J^2 \) which corresponds to identical levels at the two temperatures. It would be interesting to understand if this mechanism which produce chaos with temperature is of any relevance in microscopic models.

4 Chaos in the SK model.

Let us now turn to the study of the SK model and investigate the possibility of absence of chaos. A possible scenario implying the absence of chaos has been proposed in [11]. The states at different temperatures are strongly correlated. Lowering the temperature the ultrametric tree of states undergoes multifurcations in such a way that the states at the new temperature are the descendent in the tree of the ones at the old temperature. This is what happens in the GREM, and it seems reasonable that whenever chaos is absent this must be the correct picture: the states at different temperatures must be part of the same ultrametric tree. In this case the total matrix \( Q_{\alpha\beta} \) should be ultrametric: for any given three distinct replicas \( \alpha, \beta, \gamma \) one should find \( Q_{\alpha\beta} \geq \min\{Q_{\alpha\gamma}, Q_{\beta\gamma}\} \). Specializing the relation to \( \alpha = (1, a), \beta = (2, a), \gamma = (1, c) \), that is, \( Q_{1a,2a} = P_{aa} = p_d \) it is easily found that
\[
Q_{1c,2a} = P_{ac} = \begin{cases} p_d & Q_{1a,1c} \geq p_d \\ Q_{1a,1c} & Q_{1a,1c} < p_d \end{cases}
\] (22)

If we suppose that, as in the case of coinciding external parameters, the functions \( q_s(x) \) and \( p(x) \) are continuous [5], we find that the condition (22) reflects on the functions \( q_s(x) \) and \( p(x) \) in the following way: \( q_1(x) \) and \( q_2(x) \) must be non decreasing in the whole interval \([0, 1]\), and it must exist a point \( \overline{x} \) in \([0, 1]\) such that
\[
q_s(x) = p(x) = q_<(x) \quad x \leq \overline{x}
\]
\[
p(x) = p_d \quad x > \overline{x}
\] (23)

By continuity one has \( q_<(\overline{x}) = p_d \).

The solution (16,17) for the GREM is obviously of the form proposed here. A solution of this form was found in [5] in the case with \( T_1 = T_2 = T \) and \( h_1 = h_2 = h \). It reads
\[
q_1(x) = q_2(x) = \begin{cases} q_F(2x) & 0 \leq x \leq x_0/2 \\ p_d & x_0/2 \leq x \leq x_0 \\ q_F(x) & x_0 \leq x \leq 1 \end{cases}
\] (24)
\[ p(x) = \begin{cases} 
q_F(2x) & 0 \leq x \leq x_0/2 \\
p_d & x_0/2 \leq x \leq 1. 
\end{cases} \]  

(25)

Where the function \( q_F(x) \) is the ”free” Parisi function

\[ q_F(x) = \begin{cases} 
q_{\text{min}} = \left( \frac{3h^2}{4} \right)^{\frac{1}{2}} & 0 \leq x \leq x_{\text{min}} \\
\frac{x}{2} & x_{\text{min}} \leq x \leq x_{\text{max}} \\
q_{\text{max}} = \frac{1-\sqrt{1-4\tau}}{2} & x_{\text{max}} \leq x \leq 1 
\end{cases} \]  

(26)

where \( x_0 \) is the point defined by \( q_F(x_0) = p_d \) and \( x_{\text{min}} \) and \( x_{\text{max}} \) are given by continuity. The interval of \( p_d \) for which this solution is well defined, and that will be considered in the following, is \( q_{\text{min}} \leq p_d \leq q_{\text{max}} \). The form (24,25) is not limited to this problem; just as a consequence of ultrametricity any model with replica symmetry breaking admits (24,25) as solution of the two replicas problem if the function \( q_F(x) \) solves the single replica one [12].

The main result of this paper is that as long as \( T_1 \neq T_2 \) or \( h_1 \neq h_2 \) it does not exist an ultrametric solution of the kind (23) both for the truncated and the complete models of section 2. A prove of this fact can be given assuming a form of the kind (23) and showing that it does not satisfy the saddle point equations. We postpone this prove to the appendix; despite its conceptual simplicity the prove, already rather technical for the reduced model, is complicated by the necessity of using the complete model if we want a full control of all the terms of order \( \tau_1^4 \) in the free energy.

We conclude that chaos must be present both in temperature and magnetic field.

We shall now give some estimate for the free energy excess to impose the constraint (2) in various situations. The solution of equations (9,10) for generic values of the temperatures, the magnetic fields and \( p_d \) is very difficult to find. We have seen that the situation simplifies for \( h_1 = h_2 \) and \( T_1 = T_2 \) where the solution for generic \( p_d \) is (24,25,26). Other simple cases, to be presented below, are found for \( T_1 \neq T_2 \) and \( h_1 \neq h_2 \) for special values of \( p_d \equiv p_0^d \) which allow for functions \( p(x) = \text{constant} \).

It is easy to find that in this last case the system verifies \( \partial F/\partial p_d = 0 \), that is, the free energy is an extremum with respect to \( p_d \). The only stable solution is the one which is a minimum with respect to \( p_d \) [3], and has a free energy excess equal to zero. It is easy to find that this solution must verify:

\[ q_d(x) = q_F(x) \quad \text{and} \quad p(x) = p_d. \]  

(27)

\footnote{The reader should not be confused at this point; we are by no means extremizing \( F \) with respect to \( p_d \), but we are claiming that it exists a special value \( p_0^d \) for which the free energy has a stable saddle point with \( p(x) = \text{constant} \).}
The values of $p_d$ for which this solution exists satisfy

$$\tau_{12} p_d - (q_1 + q_2) p_d + y p_d^3 + h_1 h_2 = 0$$

(28)

which coincides with $\partial F/\partial p_d = 0$ (see (11)). This solution was first found by Kondor in [3]. It is the only solution we found which has zero free energy excess and it implies minimal correlations among states corresponding to different parameters.

We shall use both the solutions (24,25) and (27,28) as starting points to compute the free energy excess perturbatively in some small parameter. We shall consider the three following limit situations:

- case (1) $T_1 = T_2$, $h_1 \neq h_2$, $p_d = p_0 + \delta p_d$ and we perturb for small $\delta p_d$ around the solution (23) with $p(x) = p^0_d$.

- case (2) $T_1 = T_2$, $h_2 = h_1 + \delta h$ fixed $p_d$ and we perturb for small $\delta h$ around the solution (24,25).

- case (3) $T_1 \neq T_2$, $h_1 = h_2$, $p_d = p_0 + \delta p_d$ and we solve perturbatively in $\delta p_d$.

In all cases instead of solving the equations (9,10) even in an approximated form, we will suitably parameterize the functions $q_d(x)$ and $p(x)$, and maximize the free energy functional with respect to these parameters. This variational procedure will enable us to obtain lower bounds for the free energy excess in the various situations. We expect however to obtain the correct order of magnitude of $\Delta F$ as a function of the various external parameters. The whole program is analogous to the one pursued in [4] to compute the free energy excess to have $p_d$ out of the support of the $P(q)$ for identical parameters or in [5] to study violations of ultrametricity. We refer the interested reader to these papers for a presentation more detailed than the present one.

Let us illustrate as an example the case (1). For $p_d = p_0^0 + \delta p_d$ ($\delta p_d << p_0^0$) we look for functions $q_d(x)$ and $p(x)$ equal to (27) plus some small variations. These variations, that we call $\delta q_d(x)$ and $\delta p(x)$, have to be of order of $\delta p_d$ in the saddle point solution. We choose to parameterize them as follows

$$\delta q_1(x) = \begin{cases} 
\delta q_1^1 & x < x_m/2 \\
\delta q_1^2 & x_m/2 < x < x_1 + \delta x_1 \\
0 & x > x_1 + \delta x_1 
\end{cases}$$

$$\delta q_2(x) = \begin{cases} 
\delta q_2^1 & x < x_m/2 \\
\delta q_2^2 & x_m/2 < x < x_2 + \delta x_2 \\
0 & x > x_2 + \delta x_2 
\end{cases}$$

(29)

$$\delta p(x) = \begin{cases} 
\delta p^1 & x < x_m/2 \\
\delta p^2 & x > x_m/2 
\end{cases}$$
Figure 1: The free energy excess $\Delta F/\delta p_d^2$ as a function of $q_{min}^2 = \left(\frac{3h_s^2}{4}\right)^{1/3}$ in two cases. Upper curve (a); $q_{min}^1 = 0.027, h_1 = 0.0038, \Delta F = 0$ for $q_{min}^2 = 0.027$. Lower curve (b): $q_{min}^1 = h_1 = 0 \Delta F = 0$ for $q_{min}^2 = 0$. The symmetry of $\Delta F$ as a function of $q_1$ and $q_2$ implies that two curves are equal in the opposite extremes.

where $x_m/2$ is arbitrarily choosen as the middle of the first plateau in $q_F$, i.e. $x_m = \min[x_1, x_2]$ with $x_s = 2q_{min}^s$ and $q_{min}^s = (3h_s^2/4)^{1/3}$, $(s = 1, 2)$. The various parameters appearing above are determined by maximization of the free energy functional supposing self-consistently that they are of order $\delta p_d$. It turns out that at the lowest order the free energy excess is of order $\delta p_d^2$. As we are only interested to the lowest order we can minimize the polynomial of order two obtained expanding up to second order the free energy functional in all the parameters of order $\delta p_d$. The resulting saddle point equations are linear equations in the (10) variational parameters, that we solved numerically for given values of $\tau$, $h_1$ and $h_2$. In figure 1 we present the result for the free energy excess at some values of the external parameters. We also solved analytically the equations in the two limit cases (1) $h_2 = 0$ and (2) $h_2 = h_1 + \delta h$ with $\delta h << h_1$. In this last case we just computed $\Delta F$ to the first order in $\delta h$. The results for these two cases are:

$$\Delta F = \begin{cases} \left(\frac{2187}{32}\right)^{1/3} \delta p_d^2 h_1^{8/3}/q_{max} & h_2 = 0 \\ \sqrt{2} \delta p_d^2 h_1 \delta h & \delta h << h_1 \end{cases}$$ (30)

In all cases the variational parameters turned out to be consistent with the hypothesis of being of order $\delta p_d$ and with the positivity condition [8], e.g. in the
case $h_2 = 0$, $h_1 \neq 0$, $x_m = 0$, we found $\delta q_1^2 = -\delta p_d q_{\min}/(4q_{\max} - 3q_{\min})$ and

$\delta p_1^1 = \delta p_2^2 = \delta p_d$ the other variables being zero.

The computation of case (2) for small field difference $\delta h = h_2 - h_1$ follows a very similar scheme. In this case we perturb around the solution of the problem with $\delta h = 0$ [24,25] with $q_{\min} \leq p_d \leq q_{\max}$. Without entering in the details of the solution, which is similar to the one of the previous case, we just give the result. Under the (self-consistent) hypothesis that all the variations are of order $\delta h$ one finds that the free energy excess is of order $\delta h^2$. We get

$$\Delta F = \frac{2}{3} \delta h^2 (p_d - q_{\min})(p_d + q_{\min}) \left( \frac{3p_d^2 + 3p_d q_{\min} + 2q_{\min}^2}{p_d^2 + p_d q_{\min} + 2q_{\min}^2} \right)$$

(31)

note that, as it should be, the free energy excess is zero for $p_d = q_{\min}$.

Very similar paths can be followed to study the case (3) of chaos with temperature. We found here for the free-energy excess

$$\Delta F = \delta p_d^2 (T_1 - T_2)^4/\tau_1.$$  

(32)

It is worth noticing that eq. (32) is derived from the truncated model in a magnetic field $h_1 = h_2 = h$ but does not depend on $h$. As it was noticed by Kondor and Végső [4] the truncated model presents a spurious instability in the fluctuation matrix. The result (32) shows that our calculation is insensitive to this instability.

Formulae (30,31,32) can be used to estimate the probability distribution for an overlap $p_d$ among states at different parameters in finite systems via the relation $P(p_d) \sim \exp(-N \Delta F)$. This relation allow for tests of (30,31,32) in numerical simulations.

**Finite dimensions**

Let us now briefly comment about the relevance of our results for finite dimensional spin glasses.

In addition to the spin-spin correlations an important quantity in finite dimension is the correlation overlap function $\langle S_i S_j \rangle_1 < \langle S_i S_j \rangle_2$. In a chaotic situation this decays exponentially with a characteristic length $\xi_{1,2}$ for large $|i - j|$. This quantity was studied in [3,4] where it was found that, when $d > 8$

$$\xi_{0,h} \sim h^{-2/3} \quad \text{and} \quad \xi_{T_1,T_2} \sim |T_1 - T_2|^{-1}.$$ 

(33)

This behaviour was confirmed in numerical simulations by Ritort [13]. The sensitivity to small variations of an external parameter $X$ is characterized by a ”chaos exponent” $\zeta$, in $\xi_{X_1,X_2} \sim |X_1 - X_2|^{-1/\zeta}$, first considered in the framework of the scaling theory of Bray and Moore [14] and the droplet theory of Fisher and Huse [15].
These results (33) may be compared with ours via the relation found in [13]

\[ N[\delta p_d^2]_{av} \sim \xi^4 \]  

(34)

where \([\cdots]_{av}\) denotes the average with respect to the distribution function of \(p_d\), \(P(p_d) \sim \exp(-N \Delta F)\). Upon substituting our results for the free energy excess \(\Delta F\) one recovers the results (33) for the dimension independent exponent \(\zeta\) in the corresponding cases, in the case of two non-zero magnetic fields, for small \(|h_1 - h_2| = \delta h\) our result is:

\[ \xi_{h_1,h_1+\delta h} \sim (h_1 \delta h)^{-1/4}. \]  

(35)

In lower dimensions it is possible to determine \(\xi_{T_1,T_2}\) [14, 15, 16] and \(\Delta F\) [17] within the scaling theories. The differences with mean-field are that the relation between the two is different from (34) that gives \(\Delta F \sim \delta p_d^2 |T_1 - T_2|^4/\zeta\), the exponent \(\zeta\) now depends on dimension and there are two regimes in temperature with two different behaviours of the two quantities mentioned [16] (one is the low-temperature phase, the other is the critical region). It could be interesting to see whether the latter happens in mean-field too.

## 5 Conclusions

We have studied in this paper the correlations among states at different magnetic fields and temperatures in some spin glass models. In the REM and in the GREM it is absent chaos with temperature, while there is chaos with magnetic field. This is understood in simple terms based on the ultrametric construction of temperature independent trees. As soon as a temperature dependence is assumed, considering correlated but not identical energy levels for different temperatures, chaos is present. This could provide a possible mechanism for chaos production in microscopic models.

In the SK model near \(T_c\) we find that a free energy excess has to be paid to constrain two systems to have an overlap greater than the one corresponding to zero correlations, if the magnetic fields or the temperatures in the two systems are different. This implies that all the possible couples of states with different external parameters and free energy density equal to the one of the states dominating the partition function, have minimal correlations. The scenario we find has implication on the physical picture of the low temperature phase of the model. The hypothesis of successive bifurcations of the ultrametric tree as the temperature is lowered [11] is incompatible with our results.

Let us conclude commenting on the fact that temperature cycling experiments in spin glass off-equilibrium relaxation [18] show strong correlations in the dynamics
at different temperatures on finite time scales. If the physics of experimental spin glasses were similar to that of the SK model in this respect, one could expect these correlations eventually to decay to zero for large times. It would be very interesting in this context to test the finite time behavior of the SK model in simulated temperature cycling experiments.

Acknowledgements

We thank H.J. Hilhorst, I. Kondor, J. Kurchan, M. Mézard, G. Parisi, F. Ritort, M.A. Virasoro for interesting discussions.

Appendix

In this appendix we show that no ultrametric solution of the kind discussed in section 2 exists for the SK model near \( T_c \) (4). We will show this by absurd, assuming an ultrametric solution with \( q_s(x) \) and \( p(x) \) continuous in \( x \). The discussion is done in the case of the complete model, the same argument could also be applied to the truncated model, with the same conclusions.

We discuss the case \( h_1 = h_2 = 0 \) and different temperatures, a similar (and simpler) prove leads to the conclusion that there is chaos with magnetic field. Let us write the variational equations for the complete model considering generic values of \( w, u, y, v \):

\[
2\tau_{rs}Q_{\alpha\beta} + w(Q^2)_{\alpha\beta} + \frac{2}{3}uQ_{\alpha\beta}^3 - y\sum_\gamma [Q^2_{\alpha\gamma} + Q^2_{\beta\gamma}]Q_{\alpha\beta} + v(Q^3)_{\alpha\beta} = 0.
\] (36)

Plugging in the Parisi form for the matrices \( Q_s \) and \( P \) one get a set of coupled integral equations for the functions \( q_s(x) \) and \( p(x) \) that can be solved by repeated differentiation with respect to \( x \).

For future reference we write the solution of the free case \[4\] at a temperature \( \tau = (1 - T^2)/2 \).

\[
q_F(x) = \left\{ \begin{array}{ll}
\frac{w}{2u} \frac{x}{\sqrt{1 + \frac{x^2}{2}}} & x < \pi \\
q(1) & x \geq \pi
\end{array} \right.
\] (37)

Where \( q(x) \) is continuous in \( \pi \) and \( q(1) \) is specified by the equation

\[
2\tau + 2y\langle q^2 \rangle - 2wq(1) + (3v + 2u)q(1)^2 = 0.
\] (38)

In order to solve the problem we have to compute the Parisi functions associated to the various terms of (36). In particular we need to compute the functions
associated to
\[ Q^2 = \begin{pmatrix} Q_1^2 + P^2 & P(Q_1 + Q_2) \\ P(Q_1 + Q_2) & Q_2^2 + P^2 \end{pmatrix} \] (39)

and
\[ Q^3 = \begin{pmatrix} Q_1^3 + P^2(2Q_1 + Q_2) & P^3 + P(Q_1^2 + Q_2 + Q_1Q_2) \\ P^3 + P(Q_1^2 + Q_2 + Q_1Q_2) & Q_2^3 + P^2(Q_1 + 2Q_2) \end{pmatrix}. \] (40)

To do that let us remind that the eigenvalues associated to a Parisi matrix \( A \rightarrow (a_d, a(x)) \) are:

\[ \lambda_0 = a_d - \langle a \rangle \quad \text{with multiplicity} \quad 1 \] (41)

\[ \lambda(x) = a_d - xa(x) - \int_x^1 dy \ a(y) \quad \text{with multiplicity} \quad -n \frac{dx}{x^2}. \] (42)

Observing that
\[ \lambda'(x) = -xa'(x) \] (43)

one can invert the relation (42) and get
\[ a(x) = a(1) + \int_x^1 dy \ \frac{\lambda'(y)}{y} \] (44)

Let us denote \( \lambda_1(x), \lambda_2(x), \lambda_p(x) \) the eigenvalues associated with \( Q_1, Q_2, P \) respectively. The eigenvalues associated to \( Q^2 \) will be:

\[ Q_s^2 + P^2 \rightarrow \lambda_s^2(x) + \lambda_p^2(x) = 2\Lambda_s(x) \] (45)

\[ P(Q_1 + Q_2) \rightarrow \lambda_p(x)[\lambda_1(x) + \lambda_2(x)] = 2\Lambda_p(x) \] (46)

\((s = 1, 2)\) the corresponding functions can be obtained from (44) noting that as the magnetic field is zero \( q_s(0) = p(0) = 0, \)

\[ 2A_s(x) = 2 \int_0^x dy \ [\lambda_s(y)q_s'(y) + \lambda_p(y)p'(y)] \] (47)

\[ 2A_p(x) = 2 \int_0^x dy \ \{p'(y)[\lambda_1(y) + \lambda_2(y)] + [q_1'(y) + q_2'(y)]\lambda_p(y)\} \] (48)

having made use of (43). The derivative with respect to \( x \) of these functions are:

\[ 2A_s'(x) = 2[\lambda_s(x)q_s'(x) + \lambda_p(x)p'(x)] \] (49)

\[ 2A_p'(x) = 2\{p'(x)[\lambda_1(x) + \lambda_2(x)] + [q_1'(x) + q_2'(x)]\lambda_p(x)\} \] (50)

In a completely analogous way one finds the functions \( 3A_s(x) \) and \( 3A_p(x) \) and their derivatives:

\[ 3A_s'(x) = 3q_s'(x)^2 + 2p'\lambda_p(2\lambda_1 + \lambda_2) + \lambda_p^2(2q_1' + q_2') \] (51)

\[ 3A_p'(x) = 3p'\lambda_p^2 + p'(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + \lambda_p(2\lambda_1q_1' + 2\lambda_2q_2' + \lambda_1q_2' + \lambda_2q_1') \] (52)
The formula for \(3A_2'(x)\) is obtained from (51) interchanging the indices ‘1’ and ‘2’.

Observe that

\[
\sum_\gamma [Q^2_{\alpha\gamma} + Q^2_{\beta\gamma}] = \begin{cases}
2[p_2^2 - < p^2 > - < q^2_s >] & \text{if } r = s \\
2[p_2^2 - < p^2 > - < q^2_1 >] & \text{if } r \neq s
\end{cases}
\]

and that the functions associated to \(Q^3_{\alpha\beta}\) are just \(q^3_s(x)\) and \(p^3(x)\) one finds that \(the derivatives with respect to x of the saddle point equations read\)

\[
[2\tau_1 - 2y(p_2^2 - < p^2 > - < q^2_1 >)]q_1' + 2w[p'(\lambda_1 + \lambda_2) + (q_1' + q_2')] + v(3q_1'(\lambda_1^2 + 2\lambda_1 + \lambda_2) + \lambda_p(2q_1' + q_2') + 2uq_1'q_1' = 0
\]

a similar equation with \(1 \leftrightarrow 2\), and

\[
[2\tau_1 - 2y(p_2^2 - < p^2 > - < q^2_2 >)]q_2' + w[p'(\lambda_1 + \lambda_2) + (q_1' + q_2')] + v(3q_2'(\lambda_1^2 + 2\lambda_1 + \lambda_2) + \lambda_p(2q_1' + q_2') + 2uq_1'q_1' = 0.
\]

Let us now study the possibility of an ultrametric solution. Consider the ‘small region’ defined in (23), where \(q_1(x) = q_2(x) = p(x) = f(x)\), and suppose \(f'(x) \neq 0\). One can the differentiate repeatedly (54) and find that

\[
f(x) = q_F(2x).
\]

In the ‘large \(x\) region’, where \(p(x) = p_d\) observing that \(\lambda_p(x) = 0\) there, one finds that the second equation is automatically satisfied, while the first two equations reduce to

\[
q_s'[2\tau - 2y(p_d^2 - < p^2 > - < q^2_s >)] + 2w\lambda_s + 3v\lambda^2_s + 2uq^2_s = 0.
\]

that is, we get two uncoupled equations for \(q_1\) and \(q_2\) in this region. Again by repeated differentiation, we find that if \(q'_s \neq 0\) then \(q_s(x) = q_F(x)\). Using the assumption of continuity we find

\[
q_s(x) = \begin{cases}
q_F(2x) & x \leq x_0/2 \\
p_d & x_0/2 < x \leq x_0 \\
q_F(x) & x_0 < x \leq \tau_s \\
q_s(1) & \tau_s < x \leq 1
\end{cases}
\]

\[
p(x) = \begin{cases}
q_F(2x) & x \leq x_0/2 \\
p_d & x_0/2 < x \leq 1
\end{cases}
\]

17
The only free parameter at this level are the values $q_1(1)$ and $q_2(1)$. These can be fixed e.g. considering eq. (57) in $x = x_s$ which gives

$$2\tau_s + 2y \left[ \int_0^{x_s} dx \, q_F(x)^2 + (1 - x_s)q_s(1)^2 \right] - 2wq_s(1) + (3v + 2u)q_s(1)^2 = 0. \quad (59)$$

showing that $q_s(1)$ is equal to the value $q_F(1)$ corresponding to $\tau = \tau_s$. If now one inserts the resulting functions in equations (54) one finds the absurd

$$(p_d - \langle p \rangle)(\langle q_1 \rangle - \langle q_2 \rangle) = 0 \quad (60)$$

showing the inconsistency of the hypothesis of an ultrametric solution except for the trivial one with $p_d = 0$.

References

[1] M. Mézard, G. Parisi, and M.A. Virasoro, ”Spin glass theory and beyond”, World Scientific (Singapore 1987).

[2] J.Kurchan, G.Parisi, M.A. Virasoro, J.Physique I 3 (1993) 1819

[3] I. Kondor, J.Phys. A 22 (1989) L163

[4] I.Kondor, A. Végsö, J.Phys. A 26 (1993) L641

[5] S.Franz, G.Parisi, M.A. Virasoro, J.Physique I 2 (1992) 1969

[6] S.Franz, G.Parisi, M.A. Virasoro, Europhys.Lett. 22 (1993) 405

[7] B.Derrida, J.Phys. (Paris) Lett. 46 (1985) 401

[8] B.Derrida, E. Gardner, J.Phys. C 19 (1986) 2253

[9] B.Derrida, E. Gardner, J.Phys. C 19 (1986) 5783

[10] We thank J. Kurchan for this observation.

[11] V.S.Dotsenko, J.Phys. C 20 (1987) 5473; J.Phys. Condens. Matter 2 (1990) 2721

[12] S.Franz, Ph.D. Thesis, unpublished.

[13] F.Ritort, Phys. Rev. B 50 (1994) 6844
[14] A.J. Bray and M.A. Moore, Phys. Rev. Lett. 58 (1987) 57
[15] D.S. Fisher and D.A. Huse, Phys. Rev. B 38 (1988) 386
[16] M. Nifle and H.J. Hilhorst, Phys. Rev. Lett. 68 (1992) 2992
[17] H.J. Hilhorst, private communication.
[18] F. Lefloch, J. Hamman, M. Ocio, E. Vincent, Europhys. Lett. 18 (1992) 647
[19] I. Kondor, C. Dedominicis, T. Temesvári, Physica A 185 (1992) 295