Contact transformations for difference schemes

Decio Levi\(^1\), Christian Scimiterna\(^2\), Zora Thomova\(^3\) and Pavel Winternitz\(^4\)

\(^1\) Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN Roma Tre, Via della Vasca Navale, 84, 00146 Roma, Italy
\(^2\) Dipartimento di Fisica and Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN Roma Tre, Via della Vasca Navale, 84, 00146 Roma, Italy
\(^3\) Department of Engineering, Sciences and Mathematics, SUNY Institute of Technology, 100 Seymour Road, Utica, NY 13502, USA
\(^4\) Centre de Recherches Mathématiques and Département de Mathématiques et de Statistique, Université de Montréal, Case postale 6128, succ. Centre-ville, Montréal, Québec H3C 3J7, Canada

E-mail: levi@roma3.infn.it, scimiterna@fis.uniroma3.it, thomovz@suniyit.edu and wintern@crm.umontreal.ca

Received 15 October 2011, in final form 21 November 2011
Published 13 December 2011
Online at stacks.iop.org/JPhysA/45/022001

Abstract
We define a class of transformations of the dependent and independent variables in an ordinary difference scheme. The transformations leave the solution set of the system invariant and reduces to a group of contact transformations in the continuous limit. We use a simple example to show that the class is not empty and that such ‘contact transformations for discrete systems’ genuinely exist.

PACS numbers: 02.20.−a, 02.30.Hq, 02.30.Ks, 03.65.Fd
Mathematics Subject Classification: 76M60, 58D19, 39-xx

1. Introduction
In a recent article [1], we proposed a definition of a contact transformation for an ordinary difference scheme (\(OΔS\)) of order \(K\):

\[
\begin{align*}
E_a(\{x_k\}, \{y_k\}, k = n + M, n + M + 1, \ldots, n + N) &= 0 \\
& a = 1, 2 \quad K = N - M + 1, \quad n, M, N \in \mathbb{Z}, \quad N > M.
\end{align*}
\]

The Lie algebra of the symmetry group of contact transformations in [1] was realized by vector fields of the form

\[
X = \xi_n \partial_{x_n} + \phi_n \partial_{y_n}.
\]

where the functions \(\xi_n\) and \(\phi_n\) satisfy the following conditions.

- They depend on \(J \in \mathbb{Z}^+\) points \((x_{n+k}, y_{n+k}), k = 0, 1, \ldots J - 1\) with \(J \geq 2\) for at least one infinitesimal coefficient of the vector field in the symmetry algebra.
The Lie algebra should be integrable to a Lie group. This implies that all coefficients in the $J$th prolongation of the vector fields $X$ should depend only on $(x_{n+k}, y_{n+k}), 0 \leq k \leq J,$ and not on any further points.

In the continuous limit, the algebra and the symmetry group of the $O\Delta S$ (1.1) should reduce to the Lie algebra and Lie group of the ordinary differential equation (ODE) that is the continuous limit of (1.1).

It was shown in [1] that contact transformations satisfying these conditions do not exist.

Theorems proven in [1] are difference analogs of Bäcklund’s famous theorem [7] from which it follows that contact transformations can only depend on the first derivatives. Indeed, those that depend on higher derivatives are prolongations of point transformations [7, 10].

The negative result presented in [1] leaves open the possibility that a more general class of transformations exists that takes solutions of the $O\Delta S$ (1.1) into solutions, and reduces to contact transformations in the continuous limit. The purpose of this article is to show that this is indeed the case.

In section 2, we consider a general $n$th order ODE and replace it by a system of two lower order equations. We show that some point symmetries of the system give rise to contact transformations for the original $n$th order equation. In particular, we show that the point symmetries of the system $u' = v, v'' = 0$ give rise to the ten-dimensional invariance group of contact transformations of the ODE $u''' = 0$.

In section 3, we apply the same approach to difference systems. We discretize the system $u' = v, v'' = 0$ in a manner that preserves a seven-dimensional subalgebra of its Lie point symmetry algebra. We then eliminate the variable $v$ from the system and obtain a third order $O\Delta S$ for the variable $u(x)$ allowing a group of transformations that in the continuous limit include contact transformations.

Section 4 is devoted to conclusions. We suggest there a less restrictive definition of contact transformations for $O\Delta S$ than the one given in [1].

2. Contact symmetries of an ODE as point symmetries of a system of ODEs

Let us consider an $n$th order ODE

$$u^{(m)} = F(x, u, u', \ldots, u^{(m-1)}), \quad m \geq 3.$$  \hspace{1cm} (2.1)

It can always be replaced by a system of lower order equations, for instance by putting

$$u' = v, \quad v^{(m-1)} = F(x, u, v, v', \ldots, v^{(m-2)}).$$  \hspace{1cm} (2.2)

With this choice, contact symmetries of (2.1) are reduced to point symmetries of the system (2.2). The Lie algebra of point symmetries of the system (2.2) is realized by vector fields of the form

$$X = \xi(x, u, v)\partial_v + \phi(x, u, v)\partial_v + \psi(x, u, v)\partial_v.$$  \hspace{1cm} (2.3)

such that the prolongation $\mu^{(m-1)}X$ of $X$ annihilates the system (2.2) on its solution set.

Since the solution sets of the single equation (2.1) and the system (2.2) coincide, the vector fields (2.3) will also generate symmetry transformations for the ODE (2.1). Returning to the original variables in (2.3), we have

$$X = \xi(x, u, u')\partial_v + \phi(x, u, u')\partial_v + \phi^{(1)}(x, u, u')\partial_v.$$ \hspace{1cm} (2.4)

Thus, if $\xi$ or $\phi$ in (2.3) depend on $v$, then (2.4) will correspond to the first prolongation of a contact transformation for the ODE (2.1) with

$$\psi(x, u, v) = \phi^{(1)}(x, u, u').$$ \hspace{1cm} (2.5)
Proceeding in this manner, we find all point symmetries of the system (2.2) and some if not all contact symmetries of the ODE (2.1).

Let us now consider the converse problem. Given a finite-dimensional Lie algebra of vector fields of the form (2.3), what is the most general system of two second order equations of the form (2.2), invariant under the corresponding group of point transformations? For simplicity, we concentrate on the case of a second order system, i.e. \( m = 3 \) in (2.1) and (2.2).

We choose a basis \( \{X_1,\ldots,X_s\} \) for the given Lie algebra and write the second order prolongations of the basis vectors as

\[
pr^{(2)} X_i = \xi_i \partial_x + \phi_i \partial_u + \psi_i \partial_v + \phi_i^1 \partial_u^2 + \psi_i^1 \partial_u v + \phi_i^2 \partial_v^2 + \psi_i^2 \partial_v u, \\
i = 1, \ldots, s.
\]  

To find the differential invariants, we must solve the system of quasilinear partial differential equations:

\[
pr^{(2)} X_i F(x, u, v, u', v', u'', v'') = 0, \quad i = 1, \ldots, s.
\]  

The system can be written as a system of equations in the matrix form as

\[
MU = 0, \quad U^r = (F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}),
\]  

where the subscripts denote partial derivatives. ‘Strong invariants’ are obtained if the matrix \( M \) has maximal rank \( r_M = \min(s, 7) \). The number of such invariants is \( N = 7 - r \) and they are invariant on the entire jet space.

The rank of \( M \) can be \( r < r_M \) on some submanifold and then further invariants, ‘weak invariants’ can exist. The number of all invariants is \( N = 7 - r \).

Let us now consider a specific example for which all contact symmetries are obtained from the point symmetries of a lower order system. Equation (2.1) is specified to

\[
u'' = 0
\]  

and the system (2.2) is

\[
u' = v, \quad v'' = 0.
\]  

Following a standard procedure [2], we find that the Lie point symmetry algebra of (2.10) is isomorphic to the de Sitter algebra \( \alpha(3, 2) \) realized by vector fields with the basis:

\[
\begin{align*}
X_1 &= \partial_u \\
X_2 &= x \partial_u + v \partial_x \\
X_3 &= \partial_x \\
X_4 &= x^2 \partial_u + 2x \partial_x \\
X_5 &= x \partial_x - v \partial_v \\
X_6 &= u \partial_u + v \partial_v \\
X_7 &= 2v \partial_a + v^2 \partial_u \\
X_8 &= x^2 \partial_x + 2uv \partial_u + 2u \partial_v \\
X_9 &= 2(xv - u) \partial_x + xv^2 \partial_u + v^2 \partial_v \\
X_{10} &= 2x(2u - xv) \partial_x + (4u^2 - x^2 v^2) \partial_u + 2v(2u - xv) \partial_v.
\end{align*}
\]  

We will denote the corresponding local Lie group \( G \).

By construction, (2.11) is the Lie algebra of point transformations of the system (2.10). If we eliminate \( v \) from (2.10), we return to the single third order ODE (2.9). Eliminating \( v \) in the same way from the vector field (2.11), we obtain the first prolongation of the Lie algebra of contact symmetries of (2.9) in the form (2.4).
Specifically for the vector fields (2.11), we see that $\xi$ and $\phi$ do not depend on $v$ for $X_1, \ldots, X_8$ so they will generate point symmetries for (2.1). On the other hand, $X_7, X_9$ and $X_{10}$ turn into contact symmetries of (2.1) after the substitution (2.5).

We mention that this is an alternative way of calculating the group of contact symmetries for (2.9) to the standard one, used for example in [3]. The result is of course the same.

Let us now consider the converse problem for the subalgebra $L_0 = \{X_1, \ldots, X_7\}$, where $X_7$ generates a contact transformation for the ODE (2.9).

The matrix $M$ of (2.8) has $r_M = 7$ so the group corresponding to $L_0$ has no strong invariants.

However, for $u' = v, \quad v'' = 0$ we have $r(M) = 5$ and on this manifold the system of ODEs (2.10) is invariant (or at least 'weakly invariant'). Moreover, applying the prolongations of $X_9, X_9$ and $X_{10}$ to the system (2.2), we verify that this system is invariant under the entire de Sitter group $G \sim O(3, 2)$. To sum up, the only invariants of $G \sim O(3, 2)$ and $G_0$ in this case are weak ones, namely

$$I_1 = u' - v = 0 \quad \text{and} \quad I_2 = v'' = 0.$$

(2.12)

All subgroups of $SO(3, 2)$ were classified into conjugacy classes in [4]. The subgroup $G_0 \subset G$ corresponding to the Lie algebra $L_0 = \{X_1, \ldots, X_7\}$ is isomorphic to one of the seven maximal subgroups of $SO(3, 2)$, called the 'optical group' $Opt(2, 1)$ of the (2+1)-dimensional Minkowski space. The algebra $L_0$ was already known to Lie [11, 12] as was its six-dimensional subalgebra $L_0 = \{X_1, X_2, X_3, X_4, X_5, X_7\}$. The algebras $L, L_0, L_0$ are the only finite-dimensional Lie algebras of contact transformations of a complex plane (other than point transformations) [11, 12].

3. Contact symmetries of an OΔS as point symmetries of a lower order system?

Let us now try to produce an analog of contact transformations for a difference system in the same manner as we did for the ODE (2.9) in section 2.

We again start from the subalgebra $L_0 = \{X_1, \ldots, X_7\}$ and look for a difference system that allows $L_0$ as a Lie point symmetry algebra. We use the formalism for symmetries of OΔS as outlined in [1, 5, 8]. The idea is to construct a 3-point difference system

$$E_{\alpha}(x_k, u_k, v_k)_{k=n+1,n+2} = 0, \quad \alpha = 1, 2, 3,$$

(3.1)

which has $L_0$ as its symmetry algebra and reduces to the system (2.10) in the continuous limit.

In order to facilitate the continuous limit, we use the variables

$$x_n, \quad h_{n+1} = x_{n+1} - x_n, \quad h_{n+2} = x_{n+2} - x_{n+1},$$

$$u_n, \quad p^{(1)}_{n+1} = \frac{u_{n+1} - u_n}{x_{n+1} - x_n}, \quad p^{(2)}_{n+2} = \frac{2P_{n+2} - P^{(1)}_{n+1}}{x_{n+2} - x_n},$$

$$v_n, \quad q^{(1)}_{n+1} = \frac{v_{n+1} - v_n}{x_{n+1} - x_n}, \quad q^{(2)}_{n+2} = \frac{2q_{n+2} - q^{(1)}_{n+1}}{x_{n+2} - x_n},$$

(3.2)

instead of $x_{n+k}, y_{n+k}, k = 0, 1, 2$.

The relevant prolongations of the vector fields in the new variables have the form

$$pX = \xi_0 \partial_{x_0} + \phi_0 \partial_{u_0} + \psi_0 \partial_{v_0} + \phi^{(1)}_{n+1} \partial_{p^{(1)}_{n+1}} + \psi^{(1)}_{n+1} \partial_{q^{(1)}_{n+1}} + \phi^{(2)}_{n+2} \partial_{p^{(2)}_{n+2}} + \psi^{(2)}_{n+2} \partial_{q^{(2)}_{n+2}} + \lambda^{(1)} \partial_{h_{n+1}} + \lambda^{(2)} \partial_{h_{n+2}},$$
Here \( \lambda^{(k)} \), \( \phi^{(k)} \) and \( \psi^{(k)} \) are calculated using the results given in [1], namely

\[
\lambda^{(k)} = \xi_{n+k} - \xi_{n+k-1}
\]

\[
\phi^{(k)} = \frac{k}{\sum_{j=1}^{k} h_{n+j}} h_{n+k} \Delta^T \phi^{(k-1)} = \frac{1}{h_{n+j}} \sum_{j=1}^{k} h_{n+j} \Delta^T \xi_{n+j-1}
\]

\[
\psi^{(k)} = \frac{k}{\sum_{j=1}^{k} h_{n+j}} h_{n+k} \Delta^T \psi^{(k-1)} = \frac{1}{h_{n+j}} \sum_{j=1}^{k} h_{n+j} \Delta^T \xi_{n+j-1}
\]

where \( \Delta^T \) is the total difference operator

\[
\Delta^T F(x_n, u_n, v_n, p^{(1)}_{n+1}, q^{(1)}_{n+1}, \ldots) = \frac{1}{h_{n+1}} \left\{ F(x_{n+1}, u_{n+1}, v_{n+1}, p^{(1)}_{n+2}, q^{(1)}_{n+2}, \ldots) - F(x_n, u_n, v_n, p^{(1)}_{n+1}, q^{(1)}_{n+1}, \ldots) \right\}
\]

(3.4)

More specifically we have

\[
prX_1 = \partial_{x_n}
\]

\[
prX_2 = \partial_{u_n} + \partial_{p_{n+1}}
\]

\[
prX_3 = \partial_{v_n}
\]

\[
prX_4 = \partial_{w_n} + 2 (2u_n + h_{n+1}) \partial_{p_{n+1}} + 2 \partial_{q_{n+1}} + 2 \partial_{q_{n+2}}
\]

\[
prX_5 = \partial_{x_n} + \partial_{v_n} + \partial_{w_n} + 2 (2u_n + h_{n+1}) \partial_{p_{n+1}} + 2 \partial_{q_{n+1}} + 2 \partial_{q_{n+2}}
\]

\[
prX_6 = \partial_{x_n} + \partial_{w_n} + 2 (2u_n + h_{n+1}) \partial_{p_{n+1}} + 2 \partial_{q_{n+1}} + 2 \partial_{q_{n+2}}
\]

\[
prX_7 = 2 v_n \partial_{x_n} + v_n^2 \partial_{u_n} + 2 h_{n+1} q_{n+1} \partial_{h_{n+1}}
\]

\[
+ 2 h_{n+2} \left( q_{n+2} + h_{n+1} + h_{n+2} q_{n+2} \right) \partial_{p_{n+2}}
\]

\[
+ 2 (2u_n q_{n+1} + 2 p^{(1)}_{n+1} + h_{n+1} (q^{(1)}_{n+1})^2) \partial_{p_{n+1}}
\]

\[
- 2 (2p^{(1)}_{n+1} q_{n+1}) \partial_{q_{n+1}^2}
\]

\[
- 2 \partial_{q_{n+2}^2}
\]

\[
\phi^{(2)} = \frac{2}{\sum_{j=1}^{k} h_{n+j}} h_{n+1} - h_{n+1} \partial_{x_n} + h_{n+2} \partial_{p_{n+2}} + \frac{1}{2} \left( q_{n+2}^2 \right) (h_{n+2})^2 + 2 \left( q^{(1)}_{n+1} \right)^2
\]

\[
+ 2 u_n q_{n+2} + 2 h_{n+1} q_{n+2} \partial_{h_{n+1}} - 2 h_{n+2} p^{(1)}_{n+1} q_{n+2}
\]

\[
+ \frac{1}{2} \left( q_{n+2}^2 \right)^2 \partial_{p_{n+1}^2}
\]

\[
- 4 p^{(1)}_{n+1} q_{n+2} \partial_{q_{n+2}}
\]

\[
\psi^{(2)} = - h_{n+1} \left( q_{n+2}^2 \right)^2 + 2 h_{n+2} \left( q_{n+2}^2 \right)^2 - 6 u_n q_{n+2}^2
\]

To calculate invariants of the subgroup corresponding to (3.5), we impose

\[
prX_a F(x_n, u_n, v_n, p^{(1)}_{n+1}, q^{(1)}_{n+1}, p^{(2)}_{n+2}, q^{(2)}_{n+2}, h_{n+1}, h_{n+2}) = 0,
\]

\( a = 1, \ldots, 7 \).

(3.6)

In the matrix form, (3.6) can be written as

\[
MU_d = 0, \quad U_d^T = \left( F_{x_n}, F_{u_n}, F_{v_n}, F_{p^{(1)}_{n+1}}, F_{q^{(1)}_{n+1}}, F_{p^{(2)}_{n+2}}, F_{q^{(2)}_{n+2}}, F_{h_{n+1}}, F_{h_{n+2}} \right),
\]

where the subscripts denote partial derivatives.
As in the continuous case, to get a sufficient number of invariants we must restrict to an invariant manifold on which the rank of \( M \) is \( r(M) = 5 \), rather than \( r(M) = 7 \) as in the generic case. The invariant manifold in this case is given by

\[
I_1 = p_{n+1}^{(1)} - v_n - \frac{1}{2} q_{n+1}^{(1)} h_{n+1} = 0, \quad I_2 = q_{n+2}^{(2)} = 0
\]

and on this manifold the invariants are

\[
I_1 = 0, \quad I_2 = 0
\]

\[
I_3 = \frac{h_{n+2}}{h_{n+1}}, \quad I_4 = (h_{n+1})^2 \left( p_{n+2}^{(2)} - q_{n+1}^{(1)} \right).
\]

An invariant difference scheme that reduces to the system (2.2) in the continuous limit \( h_{n+k} \to 0 \) is

\[
p_{n+1}^{(1)} - v_n - \frac{1}{2} h_{n+1} q_{n+1}^{(1)} = 0, \quad q_{n+2}^{(2)} = 0, \quad h_{n+1} = c h_{n+1},
\]

where \( c \) is an arbitrary real constant and in particular \( c = 1 \) corresponds to a uniform lattice.

The \( O(3, 2) \) is invariant under the group \( G_0 \). Let us now eliminate the variable \( v_n \) from the system (3.9). Taking the discrete derivative of the first equation in (3.9) and using the second equation \( q_{n+2}^{(2)} = 0 \), we obtain

\[
q_{n+1}^{(1)} = p_{n+1}^{(2)},
\]

thus, \( I_1 = I_2 = 0 \) implies \( I_3 = 0 \) in (3.8). Taking the discrete derivative of (3.10), we obtain

\[
q_{n+3}^{(2)} = \frac{2(h_{n+1} + h_{n+2} + h_{n+3})}{3(h_{n+1} + h_{n+2})} p_{n+3}^{(3)}
\]

and hence we have

\[
p_{n+3}^{(3)} = 0, \quad h_{n+2} = c h_{n+1}
\]

as a consequence of (3.9).

The difference system (3.9) together with its difference consequence (3.10) is thus invariant under the Lie group \( G_0 \). Moreover, the equations \( I_1 = I_2 = I_3 = 0 \) are invariant under the entire group \( G \sim O(3, 2) \); however, \( I_1 = c \) is invariant only under \( G_0 \sim O(2, 1) \). For the system (3.9), \( G \) and \( G_0 \) are groups of point transformations.

Now let us consider the third order difference scheme (3.12).

We can obtain its symmetry algebra from \( L_0 \) by eliminating \( v_n, q_{n+1}^{(1)} \) and \( q_{n+2}^{(2)} \) from all the expressions in (3.5). From (3.9) and (3.10), we have

\[
v_n = p_{n+1}^{(1)} - \frac{1}{2} h_{n+1} p_{n+2}^{(2)}.
\]

To get the actual vector fields of the symmetry algebra of the \( O(3, 2) \) (as opposed to their prolongations) we need to keep only the coefficients of \( \partial_{u_k} \) and \( \partial_{u_n} \). From (3.5), we see that \( \{X_1, \ldots, X_6\} \) remain as point transformations for the \( O(3, 2) \); however, \( X_7 \) corresponds to a contact transformation

\[
X_7 = \left( p_{n+1}^{(1)} - \frac{1}{2} h_{n+1} p_{n+2}^{(2)} \right) \partial_{u_k} + \left( p_{n+1}^{(1)} - \frac{1}{2} h_{n+1} p_{n+2}^{(2)} \right)^2 \partial_{u_n}.
\]

The third prolongation of \( X_7 \) will have the form

\[
pr X_7 = X_7 + \phi^{(1)} \partial_{v_{n+1}} + \phi^{(2)} \partial_{v_{n+2}} + \phi^{(3)} \partial_{v_{n+3}} + \lambda^{(1)} \partial_{u_{n+1}} + \lambda^{(2)} \partial_{u_{n+2}} + \lambda^{(3)} \partial_{u_{n+3}}.
\]

The coefficients \( \phi^{(k)} \) and \( \lambda^{(k)} \) in (3.15) were calculated using Maple and they are too long to be reproduced here. The important fact is that we have

\[
\phi^{(k)} = \phi^{(k)}(p_{n+j+2}^{(1)}, h_{n+j+2}), \quad 0 \leq j \leq k
\]
Thus, the algebra cannot be integrated into the Lie group as shown in general in our previous article [1].

However, on the solution set of the OΔS (3.12), we have \( p^{(3)}_{n+3} = 0 \) and \( pr^{(3)}X_\lambda \) (3.15) simplifies to

\[
prX_\lambda|_{p^{(3)}_{n+3}=0} = \left( p^{(1)}_{n+1} - \frac{1}{2} h_{n+1} p^{(2)}_{n+2} \right) \partial u_n + \left( p^{(1)}_{n+1} - \frac{1}{2} h_{n+1} p^{(2)}_{n+2} \right)^2 \partial u_n \\
+ \theta \partial_{u_n} = \left( p^{(2)}_{n+2} \right)^2 \partial_{u_{n+2}} + \theta \partial_{p^{(3)}_{n+3}} \\
+ p^{(2)}_{n+2} (h_{n+1} \partial u_{n+1} + h_{n+2} \partial u_{n+2} + h_{n+3} \partial u_{n+3}).
\]

This can be integrated to give a one-parameter group of contact transformations, namely

\[
\begin{align*}
\tilde{x}_n &= x_n + \lambda \left( p^{(1)}_{n+1} - \frac{1}{2} h_{n+1} p^{(2)}_{n+2} \right) \\
\tilde{u}_n &= u_n + \frac{1}{2} \lambda \left( p^{(1)}_{n+1} - \frac{1}{2} h_{n+1} p^{(2)}_{n+2} \right)^2 \\
\tilde{p}^{(1)}_{n+1} &= p^{(1)}_{n+1}, \\
\tilde{p}^{(2)}_{n+2} &= \frac{p^{(2)}_{n+2}}{1 + \lambda p^{(2)}_{n+2}}, \\
\tilde{p}^{(3)}_{n+3} &= p^{(3)}_{n+3} \\
\tilde{h}^{(k)}_{n+k} &= h_{n+k} (1 + \lambda p^{(2)}_{n+2}), \quad k = 1, 2, 3.
\end{align*}
\]

In the continuous case, we obtain

\[
\begin{align*}
\tilde{x} &= x + \lambda u_x, \\
\tilde{u} &= u + \frac{1}{2} \lambda (u_x)^2, \\
\tilde{u}_x &= u_x.
\end{align*}
\]

It is easy to see that (3.20) leaves the ODE (2.9) invariant and the OΔS (3.12) is invariant under the transformation (3.19).

4. Conclusions

The main conclusion is that the definition of contact symmetries for difference schemes given in [1] was too restrictive. It required that the Lie algebra of contact transformations be integrable to a Lie group on the entire jet space.

Let us propose a less restrictive definition.

**Definition 4.1.** The vector fields (1.2) where \( \xi_a \) and \( \phi_a \) are functions of

\[
\{x_{n+j}, y_{n+j}, j = 1, \ldots, J\}
\]

form a Lie algebra of contact symmetries of OΔS (1.1) if they satisfy the following conditions:

- \( pr^{(N)}X_x|_{E_i = E_j} = 0 \), \( a = 1, 2 \);
- at least one of the vector fields in the Lie algebra has \( J \geq 2 \);
- the Lie algebra should be integrable to a Lie group at least on the invariant surface defined by the OΔS (1.1);
- in the continuous limit, the symmetry algebra and Lie group reduce to the Lie algebra and group of contact symmetries of the corresponding ODE.

The algebra (3.5) constructed in section 3 is the algebra of contact symmetries of the OΔS (3.12). This provides an example of the fact that definition 4.1 is not empty: such contact symmetries of OΔS do exist!

Some more specific conclusions concerning the example of ODE \( y''' = 0 \) can be drawn.
The ODE (2.9) has a ten-dimensional symmetry group of contact transformations $G$. The OΔS (3.12) is only invariant under a seven-dimensional subgroup of contact transformations $G_0 \subset G$. The equation $p_{\eta+3}^{(1)} = 0$ is actually invariant under the larger group $G$, but the lattice condition $h_{\eta+2} = c_h_{\eta+1}$ is only invariant under $G_0$.

The situation is similar for the second order equation $y'' = 0$. The ODE is invariant under the group $\text{SL}(3, \mathbb{R})$ of point transformations. The corresponding OΔS is invariant under a six-dimensional subgroup of $\text{SL}(3, \mathbb{R})$ isomorphic to the group of general affine transformations of $\mathbb{R}^2$ [6].

The Lie algebras of point symmetries of the OΔS and the ODE are realized by identical vector fields (with the correspondence $(x, y) \leftrightarrow (x_n, y_n)$). The contact symmetry is however different: the derivative $p = u'$ is not replaced by $p_{\eta+3}^{(1)}$ but by $p_{\eta+1}^{(1)} - \frac{1}{2} h_{\eta} p_{\eta+2}^{(2)}$ (see (3.14) as opposed to $X_7$ in (2.11)).

The contact transformation $X_7$ for the OΔS (3.12) involves ‘second order contact’, i.e. $\xi_n$ and $\varphi_n$ in (3.14) depend on $p_{\eta+1}^{(1)}$ and $p_{\eta+2}^{(2)}$. In the continuous limit $h_{\eta+1} \to 0$, this reduces to first order contact ($p = u_\eta$ only). This is in agreement with Bäcklund’s theorem stating that contact transformations for an ODE are mostly of order 1 [7].

In [9] the authors introduced the concept of ‘internal’ and ‘external’ symmetries. External symmetries are defined on the entire jet space, internal ones only on the submanifold of the solutions of the equation. External symmetries refer only to strong invariants, internal symmetries also to weak ones. As stressed in [9], Bäcklund’s theorem [7] actually only applies to external symmetries.

In [1] we have shown that the only external symmetries of an OΔS are point ones. Here we have shown that the OΔS can allow a class of higher symmetries that reduces to contact ones in the continuous limit. In the terminology of [9], these are internal symmetries. They do not necessarily depend only on first order discrete derivatives.

It remains to determine whether this class of higher symmetries is actually useful, in particular whether it can be used to obtain solutions of ordinary difference schemes. To answer this question, we are planning to study symmetry-preserving discretizations of nontrivial ODEs that allow symmetry groups of genuine contact transformations [13].

Acknowledgments

We thank Vladimir Dorodnitsyn and Martin Thoma for helpful discussions. The research of PW was partly supported by a research grant from NSERC of Canada. LD and SC have been partly supported by the Italian Ministry of Education and Research, PRIN ‘Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps’ from 2010. ZT thanks the CRM, where parts of the research were carried out, for hospitality.

References

[1] Levi D, Thomova Z and Winternitz P 2011 Are there contact transformations for discrete equations? J. Phys. A: Math. Theor. 44 265201
[2] Olver P J 1993 Applications of Lie Groups to Differential Equations (Berlin: Springer)
[3] Hydon P E 1989 Differential Equations. Their Solution Using Symmetries (Cambridge: Cambridge University Press)
[4] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1977 Continuous subgroups of the fundamental groups of physics: III. The de Sitter groups J. Math. Phys. 18 2259–88
[5] Levi D and Winternitz P 2006 Continuous symmetries of difference equations J. Phys. A: Math. Gen. 39 R1–63
[6] Dorodnitsyn V, Kozlov R and Winternitz P 2000 Lie group classification of second-order ordinary difference equations J. Math. Phys. 41 480
[7] Bäcklund A V 1875 Über Flächentransformationen Math. Ann. 9 297–320
[8] Dorodnitsyn V A 2011 Applications of Lie Groups to Difference Equations (Boca Raton, FL: CRC Press)
[9] Anderson I M, Kamran N and Olver P J 1993 Internal, external and generalized symmetries Adv. Math. 100 53–100
[10] Ibragimov N H 1984 Transformation Groups Applied to Mathematical Physics (Dordrecht: Reidel)
[11] Lie S 1874 Begrundung einer Invariantentheorie der Berührungs transformationen Math. Ann. 8 215–88
[12] Lie S and Engel F 1890 *Theorie der Transformationsgruppen* vol 2 (Leipzig: Teubner)
[13] Wafo Soh C, Mahomed F M and Qu C 2002 Contact symmetry algebras of scalar ordinary differential equations Nonlinear Dyn. 28 215–30