Synchronization of fractional order chaotic systems

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(Dated:)

The chaotic dynamics of fractional order systems begin to attract much attentions in recent years. In this brief report, we study the master-slave synchronization of fractional order chaotic systems. It is shown that fractional order chaotic systems can also be synchronized.

PACS numbers: 05.45.-a

Fractional calculus is a 300-year-old topic. Although it has a long mathematical history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1, 2]. Many systems are known to display fractional order dynamics, such as viscoelastic systems [3-5], dielectric polarization [6], electrode-electrolyte polarization [7] and electromagnetic waves [8]. More recently, many authors begin to investigate the chaotic dynamics of fractional order dynamical systems [9-17]. In [9], it has been shown that the fractional order Chua’s system of order as low as 2.7 can produce a chaotic attractor. In [10], it has been shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In [11], chaotic behaviors of the fractional order “jerk” model was studied, in which chaotic attractor was obtained with system orders as low as 2.1, and in [12] the chaos control of this fractional order chaotic system was reported. In [13], chaotic behavior of the fractional order Lorenz system was studied, but unfortunately, the results presented in this paper are not correct. In [14] and [15], bifurcation and chaotic dynamics of the fractional order cellular neural networks were studied. In [16], chaos and hyperchaos in the fractional order Rössler equations were studied, in which we showed that chaos can exists in the fractional order Rössler equation with order as low as 2.4, and hyperchaos exists in the fractional order Rössler hyperchaos equation with order as low as 3.8. In [17], we have studied the chaotic behavior and its control in the fractional order Chen system. In [18], the author present a broad review of existing models of fractional kinetics and their connection to dynamical models, phase space topology, and other characteristics of chaos.

On the other hand, synchronization of chaotic systems has attracted much attentions [19] since the seminal paper by Pecora and Carroll [20]. In this brief report, we study the synchronization of fractional order chaotic systems. The analysis of fractional order systems is by no means trivial. So, we will numerically investigate this topic here.

There are many definitions of fractional derivatives [1]. Perhaps the best known one is the Riemann-Liouville definition, which is given by

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau \tag{1}
\]

where \(\Gamma(\cdot)\) is the gamma function and \(n - 1 \leq \alpha < n\). The geometric and physical interpretation of the fractional derivatives was given in [21]. Upon considering the initial conditions to be zero, the Laplace transform of the Riemann-Liouville fractional derivative is \(L\left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^n L\{f(t)\}\). So, the fractional integral operator of order “\(\alpha\)” can be represented by the transfer function \(F(s) = \frac{1}{s^\alpha}\).

The standard definition of the fractional differintegral do not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate the fractional operators by using the standard integer order operators. In the following simulations, we will use the approximation method proposed in [22], which was also adopted in [9, 11, 14, 15, 16, 17]. In Table 1 of [9], the authors gave approximations for \(1/s^q\) with \(q = 0.1 - 0.9\) in steps 0.1 with errors of approximately 2dB. We will use these approximations in our following simulations.

Consider the master-slave synchronization scheme of two autonomous \(n\)-dimensional fractional order chaotic systems

\[
M: \frac{d^\alpha x}{dt^\alpha} = f(x) \tag{2}
\]

\[
S: \frac{d^\alpha y}{dt^\alpha} = f(y) + c\Gamma(x - y)
\]

with the master system \(M\) and the slave system \(S\). Where \(\alpha > 0\) is the fractional order, with which the individual dynamical systems are chaotic, \(c > 0\) is the coupling strength, and \(\Gamma \in \mathbb{R}^{n \times n}\) is a constant 0 - 1 matrix linking the coupling variables. For simplicity, we assume \(\Gamma = \text{diag}(r_1, r_2, \cdots, r_n)\) is a diagonal matrix. If there
is a coupling between the $i$th state variable of the two coupled chaotic systems, then $r_i = 1$; otherwise, $r_i = 0$. Define the error signal as $e = x - y$, the aim of the synchronization scheme is to design the coupling strength such that $\|e(t)\| \to 0$ as $t \to \infty$. This scheme is similar to the master-slave synchronization of classical integer-order chaotic systems.

Next, we numerically study the synchronization of fractional order chaotic systems via two examples. We first consider the fractional order Chua's system [9]

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= a \left[ y + \frac{x - y}{\alpha} \right] \\
\frac{d^\alpha y}{dt^\alpha} &= x - y + z \\
\frac{d^\alpha z}{dt^\alpha} &= -\frac{100}{\alpha} y
\end{align*}
\]

when $\alpha \geq 0.9$, this system can produce chaotic solutions [9]. Particularly, when $\alpha = 0.9$ and $a = 12.75$, the fractional order Chua's system is chaotic. The phase plot of $x$ and $z$ is shown in Fig.1.

![Fig.1: Phase plot of the fractional order Chua's system with $\alpha = 0.9$.](image1)

We let $\Gamma = \text{diag}(1, 0, 0)$, which implies that only the first variable $x$ is used to coupling the two fractional order chaotic systems. To obtain a critical value of $c$ to make the two systems synchronized, we continuously increase the coupling strength $c$, from $c = 0$, in step 0.5. When $c < 4$, no synchronous phenomenon is observed. When $c = 4$, the curve of the synchronization error $J(t) = \log(\|e(t)\|)$ is shown in Fig.2 (a), which indicate that the master-slave synchronization is achieved. In Fig.2 (b), we show the curve of the synchronization error when $c = 7$, in which the synchronization effect is better than that of $c = 4$.

For the purpose of comparison, we also plot the curves of synchronization error of the integer order Chua's systems ($a = 9.5$) in Fig.3. Comparing Fig. 2 with Fig.3, we can know that the synchronization rate of the fractional order Chua's systems is slower than its integer order counterpart.

We next consider the fractional order Rössler system [16]:

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= -(y + z) \\
\frac{d^\alpha y}{dt^\alpha} &= x + ay \\
\frac{d^\alpha z}{dt^\alpha} &= 0.2 + z(x - 10)
\end{align*}
\]

when $\alpha = 0.9$ and $a = 0.4$, the above system is chaotic. The phase diagram of the chaotic attractor is shown in Fig. 4.

![Fig.4: Phase plot of the fractional order Rössler system with $\alpha = 0.9$.](image4)

We also let $\Gamma = \text{diag}(1, 0, 0)$, and do the similar simulations as in the above example. When the coupling strength $c = 0.5$, the two fractional Rössler systems achieve synchronization. The curve of the synchronization error of the fractional order Rössler system is shown in Fig. 5 (a). In Fig. 5 (b), we also plot the curve of the synchronization error of the integer order Rössler systems ($a = 0.165$). From Fig. 5, we know that the synchronization rate of the fractional order Rössler systems is also slightly slower than its integer counterpart.
We have also tested the synchronization scheme (2) on several other fractional order chaotic systems [23]. Limited to the length of this brief report, we omit these results here.

In summary, in this brief report, we have studied the master-slave synchronization of fractional order chaotic systems. To our best knowledge, this is the first report on the synchronization of fractional order dynamical systems. We have shown that fractional order chaotic systems can be synchronized by utilizing the similar scheme as that of their integer order counterparts.

Future works regarding this topic include the investigation of some other types of synchronization of fractional order chaotic systems, such as the phase synchronization [24] and the projective synchronization [25], as well as the synchronization of fractional order hyperchaotic systems [16].

We acknowledge supports from the National Natural Science Foundation of China under Grant 60271019, and the Youth Science and Technology Foundation of UESTC under Grant YF020207.

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