On the emergence of the Lorentz signature in an expanding universe

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Abstract. A mechanism producing the transition from an Euclidean to a Minkowski manifold is described. A global Robertson-Walker symmetry is assumed from the large scale data of the visible universe. Allowing for the strain of the manifold as an additional field in the Lagrangian, we interpret the symmetry as a consequence of a global texture defect. The additional term gives rise to a boundary dividing the manifold into an Euclidean plus a Lorentzian region. It is also shown that the presence in the early epoch of homogeneous matter/energy fields preserves the horizon and the signature change across it. The horizon has properties much similar to the ones of the Big Bang of the Standard Model, including the need for a phase transition of the scalar field producing particles and fields as we know them now.

Keywords: Space-time signature, Robertson-Walker symmetry
1 Introduction

The commonly assumed background of special relativity is a Minkowski space-time, i.e. a flat four-dimensional manifold equipped with a Lorentzian signature. When generalizing the theory of relativity (GR) by the introduction of curvature in the actual space-time, the Minkowski manifold is typical of all tangent spaces to the curved manifold. Minkowski space-time as such does in a sense not exist, but is the asymptotic form of any real space-time when all kinds of matter/energy are taken out. This simple view, however, hides a puzzling feature. A Minkowski manifold is not the most general undifferentiated flat four-dimensional manifold, because of the light cones, i.e. of the Lorentzian signature. The structure of the light cones picks out a bunch of directions stemming from any given event in the manifold (the time-like worldlines) which cannot be confused with the rest. Where does this symmetry diminution come from? Indeed the most general four-dimensional manifold should be Euclidean: perfect isotropy and homogeneity. This is the question for which I shall try to find an answer in the present work.

The problem of the signature of space-time has already been considered in the past from different viewpoints and with different motivations. A role of Euclidean signature geometries in four dimensions has been introduced, for instance, in quantum cosmology where Euclidean path integrals over geometries have been generalized from flat quantum field theory to gravity and the universe \[1\] \[2\]. In a quantum approach the transition from Euclidean to Lorentzian signature is achieved by analytic continuation in the complex domain or formally just by means of a Wick’s rotation (introduction of imaginary time \(t \rightarrow it\)). The existence of a Euclidean patch with the geometry of a four-dimensional half sphere was intended, in Hawking’s approach, to remove the singularity in the remote past of the universe. How a real classical interpretation of the results emerges out of the quantum methods is however a non-trivial matter.

The approach I will adopt in this paper is entirely classical \textit{ab initio}. Even from a classical viewpoint the possibility of having different signatures in different regions of space-time has been discussed in the literature, especially in the 90’s of the past century \[3\] \[4\]. The discussion was soon concentrated on the general constraints posed by a possible signature flip somewhere in the manifold and on different viable approaches \[5\] \[6\]. The continuity conditions at the border between the Euclidean and the Lorentzian signature geometries were analyzed together with the conservation laws on the boundary \[7\] \[8\]. Though establishing the compatibility conditions for the co-existence of Euclidean and Lorentzian domains, the actual
presence of Euclidean regions is then left to further theories, without being considered as a logical implication of the symmetry breaking manifested by the presence of the light cones. The purpose of the present paper is precisely to provide a consistent classical framework wherein the matching between Euclidean and Lorentzian signature is obtained "naturally".

The question is: can we envisage a mechanism by which, pouring matter/energy (or something else) in a Euclidean manifold, the Lorentzian signature appears? In fact a way to induce a peculiar symmetry (to reduce the total symmetry) in a continuum is to introduce a defect in the sense used when describing material continua. This is what I meant by "something else" in the line above. However it is not clear how a defect can convert a $(++++)$ into a $(+-+-)$ signature. In order to find the way out I shall follow a path outlined in [11] and verified by various cosmological tests [12]; for it I shall use the name Strained State Cosmology (SSC) or Strained State Theory (SST). It consists in assuming that space-time behaves as a continuous deformable medium in four dimensions, extending the usual general relativistic approach by the introduction in the Lagrangian density of empty space-time of an "elastic potential" contribution built from the strain tensor; the final Lagrangian looks very similar to the ones used in the so called "massive gravity" but does indeed not coincide with them. The strain tensor of the manifold is intended as being half the difference between the actual metric tensor and the metric tensor of a reference undeformed manifold. According to the considerations I made above the reference manifold should be an Euclidean one. Mentioning two metric tensors gives the impression that I am presenting a bimetric theory. This however is not the case. Only one of the manifolds is actually existing: the one corresponding to our space-time (the natural manifold). The reference manifold is not present anywhere; it is not even a background. It simply is part of a logical description in which the universe is thought to behave as a deformed continuum. For this reason the mentioned metric tensor of the Euclidean manifold is no metric at all in the natural manifold.

The piece of evidence from which we start is that we know (or at least we think we know) that the universe, at scales of hundreds of Mpc or higher, is described by a Friedman-Lemaître-Robertson-Walker model, i.e. it has a Robertson-Walker (RW) symmetry (isotropic expansion of a homogeneous space). Since the presence of matter per se does not motivate the RW symmetry I shall attribute the symmetry fixing to the presence of a global defect, somewhere in the manifold.

In practice the initial assumptions will be:

1. an action integral of the empty space-time like the following:

$$\int \left[ R + \frac{1}{2} \lambda \varepsilon^2 + \mu \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} \right] \sqrt{-g} d^4x$$

(1.1)

where $\lambda$ and $\mu$ are the Lamé coefficients of space-time (fixed parameters) and $\varepsilon_{\mu\nu} = \frac{1}{2} (g_{\mu\nu} - E_{\mu\nu})$ is the strain tensor determined with respect to the symmetric tensor $E_{\mu\nu}$ that would correspond to the Euclidean metric on the reference manifold; it is also $\varepsilon = \varepsilon^{\alpha}_{\alpha}$ where all the raising and lowering of indices is performed by means of the unique metric tensor $g_{\mu\nu}$; no assumption is made about $\varepsilon_{\mu\nu}$ being small or not with respect to $E_{\mu\nu}$; $R$ plays the role of dynamical term for the strain tensor components;

2. a defect inducing a global RW symmetry.

Under these conditions we shall see that a Euclidean signature in a domain of the natural manifold matches a Lorentzian one in correspondence with an horizon in the manifold.
We shall also see that the presence of matter/energy does not spoil the effect I have just men-
tioned, provided the additional ingredient, under the horizon, is in the form of a completely
homogeneous field. Of course such a field should then undergo a phase transition giving rise
to the ingredients of the universe we observe now.

The SST whose essence has been outlined above is indeed a metric theory on a Rieman-
nian manifold which admits everywhere, excluding singularities, a flat tangent space-time.
In the Lorentzian domain the tangent space is Minkowskian; in practice this tells us that the
theory preserves the principle of equivalence and recovers locally the special relativity.

2 The expanding space-time

The assumptions presented in the introduction imply that the line element for the actual
space-time (natural line element) has the form:

\[ ds^2 = d\tau^2 - a^2(\tau) (dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (2.1)

The meaning of the symbols is the traditional one in GR and Cartesian coordinates
have been chosen for simplicity. The time dependence of the scale factor \( a \) implies the global
curvature of the manifold and indeed the RW global symmetry. The space has been assumed
to be flat, since this is what we conclude at the moment from the observation of the CMB.
In any case a space with positive or negative curvature would not modify the considerations
I am about to make and the final conclusions.

The line element on the reference Euclidean manifold is

\[ ds_r^2 = b^2(\tau) d\tau^2 + dx^2 + dy^2 + dz^2 \]  \hspace{1cm} (2.2)

where a gauge freedom has been allowed in the choice of the "time" coordinate identifying
the correspondence between points on the two manifolds; of course this coordinate is just
as space-like as the others in the flat Euclidean manifold. This gauge (in practice the lapse
function \( b \)) is motivated by the fact that the only variable entering the curvature of the RW
manifold is \( \tau \). In the discussion by other authors of the change of signature problem
the analog of the gauge function \( b \) with the corresponding degree of freedom is introduced in
the form of a lapse function in the natural line element (the only one they consider)[4].

From the definition of the natural and the reference line elements we immediately obtain
the strain tensor:

\[ \varepsilon_{00} = \frac{1 - b^2}{2} \]
\[ \varepsilon_{xx} = -\frac{a^2 + 1}{2} \]
\[ \varepsilon_{yy} = -\frac{a^2 + 1}{2} \]
\[ \varepsilon_{zz} = -\frac{a^2 + 1}{2} \]  \hspace{1cm} (2.3)

The second order scalars associated with the strain tensor are:

\[ \varepsilon^2 = \left( \frac{1 - b^2}{2} + 3\frac{a^2 + 1}{2a^2} \right)^2 \]  \hspace{1cm} (2.4)

and
\[ \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = \left( \frac{1 - b^2}{2} \right)^2 + 3 \left( \frac{a^2 + \frac{1}{2}a^2}{a^2} \right)^2 \] (2.5)

Introducing (2.4) and (2.5) into (1.1) we obtain the Lagrangian density

\[ \mathcal{L} = \left[ R + \frac{1}{2} \lambda e^2 + \mu \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} \right] \sqrt{-g} \]
\[ = a^3 \left( \frac{6}{a^2} \dot{a}^2 + \frac{1}{4} \left( \frac{\lambda}{2} + \mu \right) (1 - b^2)^2 + \frac{3}{4} \left( \frac{\lambda}{2} + \mu \right) \left( \frac{a^2 + \frac{1}{2}a^2}{a^2} \right)^2 + \frac{3}{4} \lambda (1 - b^2) \frac{a^2 + \frac{1}{2}a^2}{a^2} \right) \]

An integration by parts has been made in order to reduce the second order derivative present in the scalar curvature. The \( b \) function appears only in the terms containing the strain; the Euler-Lagrange equation for it reduces to \( \partial \mathcal{L}/\partial b = 0 \). Explicitly we find:

\[ \lambda \left( \frac{1 - b^2}{2} + \frac{3a^2 + \frac{1}{2}a^2}{a^2} \right) + 2\mu \left( \frac{1 - b^2}{2} \right) = 0 \] (2.6)
whose solution is:

\[ b^2 = 2 \frac{2\lambda + \mu}{\lambda + 2\mu} + 3 \frac{\lambda}{a^2 \lambda + 2\mu} \] (2.7)

For consistency \( (b^2 > 0 \text{ for any } a) \) this solution requires that it be:

\[ \mu > -\frac{\lambda}{2} \text{ if } \lambda > 0 \]
\[ \mu < -\frac{\lambda}{2} \text{ if } \lambda < 0 \]

The dynamics is fixed by the Euler-Lagrange equation for \( a \), after introducing (2.7) [12]:

\[ 2 \left( 2\ddot{a}a + \dot{a}^2 \right) - \frac{\mu}{2a^2} \frac{2\lambda + \mu}{\lambda + 2\mu} \left( 3a^4 + 2a^2 - 1 \right) = 0 \] (2.8)

A first integral of (2.8) is obtained by the energy condition:

\[ W = 6\dot{a}a^2 - \frac{3}{2a^2} \mu (a^2 + \frac{1}{2}a^2)^2 (2\lambda + \mu) \]
from which the Hubble parameter can be written

\[ H^2 = \frac{\dot{a}^2}{a^2} = \frac{1}{6a^3} \left( W + \frac{3}{2\mu} \frac{2\lambda + \mu}{\lambda + 2\mu} \frac{(a^2 + \frac{1}{2}a^2)^2}{a} \right) \]

In order to reproduce GR when \( \mu = 0 \), space-time is empty and space is flat it must be \( W = 0 \) so that

\[ \frac{\dot{a}}{a} = \pm \sqrt{\frac{\mu 2\lambda + \mu (a^2 + \frac{1}{2}a^2)}{4\lambda + 2\mu \frac{a^2}{a}}} \] (2.9)
In the absence of matter and excluding special values for the Lamé coefficients one has continued expansion or contraction only. Let us choose expansion, then let us solve (2.9). It comes:

\[ a = \sqrt{Ce^{\frac{\lambda+\mu}{2+\lambda+2\mu} \tau}} - 1 \]  

(2.10)

where only positive values of \( a \) have been considered and \( C \) is an integration constant. The variable \( \tau \) is the distance from the origin thought as being the seat of a ”defect” inducing the global RW symmetry; it is measured along a geodesic line everywhere perpendicular to the equal strain hypersurfaces; in the Lorentzian domain \( \tau \) corresponds to the cosmic time. \( C \) can be interpreted as being a feature, or ”strength”, of the defect; the situation for \( C = 0 \) corresponds to the flat reference Euclidean manifold.

For \( \tau = 0 \) it is:

\[ a(0) = \sqrt{C - 1} \]

A global Lorentzian signature in late times would imply \( C > 0 \), however if \( 0 < C < 1 \) we find an initial era during which the signature of the natural manifold is Euclidean. Then at

\[ \tau_h = \sqrt{\frac{\lambda + 2\mu}{\mu(2\lambda + \mu)}} \ln \frac{1}{C} \]

we find a horizon where \( a(\tau_h) = 0 \) and out of which (for \( \tau > \tau_h \)) a Lorentzian signature emerges.

\section{The effect of matter}

Let us now allow ”matter” to be present. In the Euclidean era a consistent presence could be in the form of a uniformly distributed field; for simplicity let us assume that its density scales as \( a^4 \), as afterwards would be the case for radiation. Of course if we want to have ordinary matter at late time we need some mechanism giving rise, on our side of the horizon, to matter terms scaling as \( a^3 \); let us say that some ”phase transition” is needed or that a matter component should be present already in the Euclidean era. In any case, as we shall see, the relevant domain is close to the horizon on both sides and there the dominant term is the \( \sim 1/a^4 \).

In order to study the behaviour before and in the vicinity of the horizon let us introduce in (2.9) an additional term that will lead to

\[ \frac{\dot{a}}{a} = \sqrt{\frac{B^2(a^2 + 1)^2}{a^4} + \frac{K^2}{a^4}} \]  

(3.1)

\[ B^2 = \frac{\mu 2\lambda + \mu}{4 \lambda + 2\mu} \]  

(3.2)

From (3.1) we have

\[ \int \frac{d(a^2)}{2\sqrt{B^2(a^2 + 1)^2 + K^2}} = \tau + T \]
i.e.

\[
\frac{1}{2B} \sinh^{-1} \left( \frac{B (a^2 + 1)}{K} \right) = \tau + T
\]

*T* is an integration constant that can be equalled to zero without loss of generality. Finally:

\[
a^2 = \frac{K}{B} \sinh 2B\tau - 1 \tag{3.3}
\]

We find a Euclidean era ending at

\[
\tau_h = \frac{1}{2B} \sinh^{-1} \left( \frac{B}{K} \right) \tag{3.4}
\]

The variable \(\tau_h\), under the boundary between the Euclidean and the Lorentzian domains, is not an observable. Our cosmic time is measured starting from the horizon, i.e. it is relative to \(\tau_h\). Afterwards the usual expansion in a Lorentzian space-time follows. If the content of the universe, after \(\tau = \tau_h\), would remain limited to the \(K/a^4\) component, the expansion would be continuous and accelerated. The situation changes if a component appears, proportional to \(1/a^3\). Should that component have been present right from the beginning in Eq. 3.1 I do not expect qualitative changes in the final result, but simply a shift in the position of the horizon.

If we introduce (2.10) into (3.1) we see that the latter is satisfied provided we let \(K \to 0\) thus recovering the solution for the empty space-time with a defect inducing the RW symmetry. Of course for \(K \to 0\) also a pure Euclidean manifold is a solution.

### 4 Junction conditions at the horizon

Let us have a closer look to what happens at the transition. Both in the case of an empty space-time and in presence of matter/energy the metric tensor is singular at the horizon since the scale factor \(a\) vanishes there. However \(a^2\) passes smoothly from negative to positive values while crossing the horizon; there are no jumps or discontinuities at any order. Of course the metric inverse is undefined on the horizon and this fact produces the divergence and a discontinuity in some of the Christoffels. The relevant quantity is the Einstein tensor \(G_{\mu\nu}\). Its elements diverge when \(a = 0\) but of course the four-divergence is identically zero: \(G_{\mu\nu} = 0\). The element \(G_{00}\), whose expression, in presence of a matter content, is

\[
G_{00} = 3B^2K^2 \frac{cosh^2 2B\tau}{(ksinh 2B\tau/B - 1)^2},
\]

has the same limit on the left and on the right of the junction. This is one of the conditions to be satisfied, according to the discussion that can be found in [7], in order not to violate the usual conservation of matter. By the way, this result is typical of Friedman-Lemaître-Robertson-Walker cosmologies with zero space curvature \((k = 0)\), as it is the case for the model I am discussing here.
5 Conclusion

As we have seen, it is possible to think of a four-dimensional manifold where both the Euclidean and the Lorentzian signature are present in different regions. The boundary between the two regions is smoothly crossed if the global manifold has a Robertson-Walker symmetry and the given symmetry can be there as a consequence of a texture defect. The simplest representation of the configuration of the manifold is obtained when the $\tau$ variable is chosen as originating from the defect and increasing along world-lines perpendicular to the equal strain hypersurfaces of the manifold. The shape of the latter hypersurfaces depends on the geometry of the defect; if the space in our universe is flat, as apparently it is, so has to be the singular submanifold corresponding to the defect. The boundary between the Euclidean and the Lorentzian areas is the hypersurface corresponding to a null scale factor, $a = 0$; this boundary is called a “horizon” from which the Lorentzian signature emerges, in analogy with the Schwarzschild horizon for a cylindrical stationary space-time (spherical in space).

Describing the situation I have now and then used terms as \textit{evolution}, \textit{transition}, even \textit{time}. Of course this way of depicting the manifold is, strictly speaking, inappropriate in the Euclidean domain. There, in fact, there is no time; all dimensions are space-like; nothing propagates. In the Euclidean domain the $\tau$ variable is an affine parameter along incomplete geodesics bounded by the cosmic defect on one side. The geodesics, besides being incomplete, are chosen so that they are everywhere perpendicular to the equal strain hypersurfaces of the manifold. It is only on the Lorentzian side of the boundary between the two domains that $\tau$ acquires the familiar role of \textit{cosmic time}. By the way in a consistent fully geometric view any description in terms of \textit{evolution} is somehow inappropriate: there is just one four-dimensional manifold described in terms of Gaussian coordinates, curved and locally warped according to a global symmetry and the local distribution of matter/energy. The manifold has two regions of different signature, matching each other in a reasonably smooth way on a three-dimensional boundary. The term \textit{evolution} is what we use for the way we label $3+1$ foliations of the manifold in the Lorentzian domain. Even though we would not use \textit{evolution} for it, a similar labeling for an analogous foliation is possible also on the Euclidean side: it would not be constrained by the light cones, but would be suggested by the symmetry. Of course this is so in an entirely classical approach, but so far the puzzle of the role of time and of some background in the attempts to quantize gravity remains unsolved.

It is important to remark that, even though the action integral (1.1) looks very much similar to the action for the so called ”massive gravity”, it is however different. In fact the original approach due to Fritz and Pauli [13] viewed (1.1) as a first order approximation of a perturbative treatment, whereas in our case (1.1) is ”exact”. Further developments of the massive gravity theory, introduced in order to cure various inconveniences present in the original version, do consider also non-linear approaches where the perturbative treatment is extended, in principle, to all orders, however in practice they are bimetric theories, where to the dynamical metric tensor a background non-dynamical metric is added, and the latter is used for building many scalars of the theory [14]. In the SSC instead, there exists just one metric and $E_{\mu \nu}$ is not a background and is not used as a metric tensor at all; raising and lowering of indices, then the construction of all scalars, are performed by means of the unique $g_{\mu \nu}$, just as in classical GR. Furthermore the problems whose presence is still a matter of debate in the massive gravity theories are absent from SSC, at least as far as the cosmic scale is concerned. An immediate example is the absence, in the cosmological solution, of the so called vDVZ (van Dam-Veltman-Zakharov [15][16]) discontinuity: when letting $\lambda$ and $\mu$ go
to zero ($B$ go to zero) in (2.9) and (3.1) the shear GR cosmological solutions are obtained.

The features described so far pertain to the only manifold, without calling in matter fields of any sort: in this case on the Lorentzian side we have an empty space-time whose space expands for ever according to formula (2.10). We have then considered the presence, in the Euclidean era, of matter/energy in the form of a fully homogeneous radiation-like field (in four dimensions). Provided the density of the field is small enough, the conclusion concerning the presence of a horizon does not substantially change, as we see in formula (3.4). However if we want to recover the present universe we must allow for a phase transition occurring in the primordial field in correspondence of the horizon, so that from it not only the Lorentzian signature appears but also the abundance of fields and particles we experience today. It is evident that a strong analogy exists between our horizon and the Big Bang of the Standard Model. In our case, under the horizon ("before" the Big Bang) we find an Euclidean era. Afterwards the present approach based on the physical role of the strain of the manifold has been successfully tested on a number of standard cosmological tests [12].

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