NONDEGENERACY OF POSITIVE SOLUTIONS TO NONLINEAR HARDY-SOBOLEV EQUATIONS

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Abstract. In this note, we prove that the kernel of the linearized equation around a positive energy solution in $\mathbb{R}^n$, $n \geq 3$, to $-\Delta W - \gamma|x|^{-2}V = |x|^{-s}W^{2^*(s)-1}$ is one-dimensional when $s + \gamma > 0$. Here, $s \in [0,2)$, $0 \leq \gamma < (n-2)^2/4$ and $2^*(s) = 2(n-s)/(n-2)$.

We fix $n \geq 3$, $s \in [0,2)$ and $\gamma < (n-2)^2/4$. We define $2^*(s) = 2(n-s)/(n-2)$. We consider a nonnegative solution $W \in C^2(\mathbb{R}^n \setminus \{0\})$ to

\begin{equation}
-\Delta W - \frac{\gamma}{|x|^2} W = \frac{W^{2^*(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.
\end{equation}

Due to the abundance of solutions to (1), we require in addition that $W$ is an energy solution, that is $W \in D^1_2(\mathbb{R}^n)$, where $D^1_2(\mathbb{R}^n)$ is the completion of $C^\infty_c(\mathbb{R}^n)$ for the norm $u \mapsto \|\nabla u\|_2$. Linearizing (1) yields to consider

\begin{equation}
K := \left\{ \varphi \in D^1_2(\mathbb{R}^n)/ -\Delta \varphi - \frac{\gamma}{|x|^2} \varphi = (2^*(s) - 1) \frac{W^{2^*(s)-2}}{|x|^s} \varphi \text{ in } D^1_2(\mathbb{R}^n) \right\}
\end{equation}

Equation (1) is conformally invariant in the following sense: for any $r > 0$, define $W_r(x) := r^{n-2}W(rx)$ for all $x \in \mathbb{R}^n \setminus \{0\}$, then, as one checks, $W_r \in C^2(\mathbb{R}^n \setminus \{0\})$ is also a solution to (1), and, differentiating with respect to $r$ at $r = 1$, we get that

\[-\Delta Z - \frac{\gamma}{|x|^2} Z = (2^*(s) - 1) \frac{W^{2^*(s)-2}}{|x|^s} Z \text{ in } \mathbb{R}^n \setminus \{0\},
\]

where

\[Z := \frac{d}{dr}W_r|_{r=1} = \sum_i x_i \partial_i W + \frac{n-2}{2} W \in D^1_2(\mathbb{R}^n).
\]

Therefore, $Z \in K$. We prove that this is essentially the only element:

Theorem 0.1. We assume that $\gamma \geq 0$ and that $\gamma + s > 0$. Then $K = \mathbb{R}Z$. In other words, $K$ is one-dimensional.

Such a result is useful when performing Liapunov-Schmidt’s finite dimensional reduction. When $\gamma = s = 0$, the equation (1) is also invariant under the translations $x \mapsto W(x-x_0)$ for any $x_0 \in \mathbb{R}^n$, and the kernel $K$ is of dimension $n+1$ (see Rey [6] and also Bianchi-Egnell [1]). After this note was completed, we learnt that Dancer-Gladiali-Grossi [4] proved Theorem 0.1 in the case $s = 0$, and that their proof can be extended to our case, see also Gladiali-Grossi-Neves [5].

Date: December 29th 2016.

2010 Mathematics Subject Classification: 35J20, 35J60, 35J75.
This note is devoted to the proof of Theorem 0.1. Since \( \gamma + s > 0 \), it follows from Chou-Chu [3], that there exists \( r > 0 \) such that \( W = \lambda^{n-2} U_r \), where

\[
U(x) := \left( |x|^{-\frac{n-2}{2}} \alpha^{-\gamma} + |x|^{-\frac{n-2}{2}} \alpha^+ \right)^{-\frac{n-2}{2}}.
\]

with

\[
\epsilon := \sqrt{\frac{(n-2)^2}{4} - \gamma} \quad \text{and} \quad \alpha_\pm(\gamma) := \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}.
\]

As one checks, \( U \in D_1^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \) and

\[
- \Delta U - \frac{\gamma}{|x|^2} U = \lambda \frac{U^{2^*(s)-1}}{|x|^8} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad \text{with} \quad \lambda := \frac{n-2}{n-2-\epsilon^2}.
\]

Therefore, proving Theorem 0.1 reduces to prove that \( \bar{K} \) is one-dimensional, where

\[
\bar{K} := \left\{ \varphi \in D_1^2(\mathbb{R}^n) \mid - \Delta \varphi - \frac{\gamma}{|x|^2} \varphi = (2^*(s) - 1)\lambda \frac{U^{2^*(s)-2}}{|x|^8} \varphi \in D_1^2(\mathbb{R}^n) \right\}
\]

I. Conformal transformation.

We let \( S^{n-1} := \{ x \in \mathbb{R}^n \mid \sum x_i^2 = 1 \} \) be the standard \((n-1)\)-dimensional sphere of \( \mathbb{R}^n \). We endow it with its canonical metric can. We define

\[
\Gamma : \quad \mathbb{R} \times S^{n-1} \quad \mapsto \quad \mathbb{R}^n \setminus \{0\}
\]

The map \( \Phi \) is a smooth conformal diffeomorphism and \( \Phi^* \text{Eucl} = e^{-2t}(dt^2 + \text{can}) \). On any Riemannian manifold \((M, g)\), we define the conformal Laplacian as \( L_g := -\Delta_g + \frac{n-2}{4(n-1)} R_g \) where \( \Delta_g := \text{div}_g(\nabla) \) and \( R_g \) is the scalar curvature. The conformal invariance of the Laplacian reads as follows: for a metric \( g' = e^{2\omega} g \) conformal to \( g \) (\( \omega \in C^\infty(M) \)), we have that \( L_{g'} u = e^{-2\omega} L_g(e^{2\omega} u) \) for all \( u \in C^\infty(M) \).

It follows from this invariance that for any \( u \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \), we have that

\[
- (\Delta u) \circ \Phi(t, \sigma) = e^{-\frac{n+2t}{2}} \left( -\partial_t \ddot{u} - \Delta_{\text{can}} \ddot{u} + \frac{(n-2)^2}{4} \dddot{u} \right) (t, \sigma)
\]

for all \((t, \sigma) \in \mathbb{R} \times S^{n-1} \), where \( \ddot{u}(t, \sigma) := e^{-\frac{n+2t}{2}} u(e^{-t} \sigma) \) for all \((t, \sigma) \in \mathbb{R} \times S^{n-1} \).

In addition, as one checks, for any \( u, v \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \), we have that

\[
\int_{\mathbb{R}^n} (\nabla u, \nabla v) \, dx = \int_{\mathbb{R} \times S^{n-1}} \left( \partial_t \ddot{u} \partial_t \ddot{v} + (\nabla_\sigma \ddot{u}, \nabla_\sigma \ddot{v})_{\text{can}} \right) \frac{(n-2)^2}{4} \dddot{u} \ddot{v} \, dt \, d\sigma
\]

where we have denoted \( \nabla_\sigma \ddot{u} \) as the gradient on \( S^{n-1} \) with respect to the \( \sigma \) coordinate. We define the space \( H \) as the completion of \( C^\infty_c(\mathbb{R} \times S^{n-1}) \) for the norm \( \| \cdot \|_H := \sqrt{B(\cdot, \cdot)} \). As one checks, \( u \mapsto \dddot{u} \) extends to a bijective isometry \( D_1^2(\mathbb{R}^n) \rightarrow H \).

The Hardy-Sobolev inequality asserts the existence of \( K(n, s, \gamma) > 0 \) such that

\[
\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq K(n, s, \gamma) \int_{\mathbb{R}^n} \left( |\nabla u|^2 - \frac{s}{|x|^s} u^2 \right) \, dx \quad \text{for all} \quad u \in C^\infty_c(\mathbb{R}^n \setminus \{0\}).
\]

Via the isometry \( D_1^2(\mathbb{R}^n) \equiv H \), this inequality rewrites

\[
\left( \int_{\mathbb{R} \times S^{n-1}} |v|^{2^*(s)} \, dt \, d\sigma \right)^{\frac{2}{2^*(s)}} \leq K(n, s, \gamma) \int_{\mathbb{R} \times S^{n-1}} \left( (\partial_t v)^2 + |\nabla v|_{\text{can}}^2 + \epsilon^2 v^2 \right) \, dt \, d\sigma,
\]

for all \( v \in H \). In particular, \( v \in L^{2^*(s)}(\mathbb{R} \times S^{n-1}) \) for all \( v \in H \).
We define $H^2_\mathcal{R}(\mathbb{R})$ (resp. $H^2_\mathcal{S}(\mathbb{S}^{n-1})$) as the completion of $C_\infty^\infty(\mathbb{R})$ (resp. $C_\infty^\infty(\mathbb{S}^{n-1})$) for the norm

$$
\|u\|_{H^2_\mathcal{R}(\mathbb{R})} = \left( \int_\mathbb{R} (\hat{u}^2 + \hat{u}'^2) \, dx \right)^{1/2} \quad \text{resp.} \quad \|u\|_{H^2_\mathcal{S}(\mathbb{S}^{n-1})} = \left( \int_{\mathbb{S}^{n-1}} (|\nabla' u|^2 + u'^2) \, d\sigma \right)^{1/2}.
$$

Each norm arises from a Hilbert inner product. For any $(\varphi, Y) \in C_\infty^\infty(\mathbb{R}) \times C_\infty^\infty(\mathbb{S}^{n-1})$, define $\varphi \ast Y \in C_\infty^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ by $(\varphi \ast Y)(t, \sigma) := \varphi(t)Y(\sigma)$ for all $(t, \sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}$. As one checks, there exists $C > 0$ such that

$$
\|\varphi \ast Y\|_H \leq C \|\varphi\|_{H^2_\mathcal{R}(\mathbb{R})} \|Y\|_{H^2_\mathcal{S}(\mathbb{S}^{n-1})}
$$

for all $(\varphi, Y) \in C_\infty^\infty(\mathbb{R}) \times C_\infty^\infty(\mathbb{S}^{n-1})$. Therefore, the operator extends continuously from $H^2_\mathcal{R}(\mathbb{R}) \times H^2_\mathcal{S}(\mathbb{S}^{n-1})$ to $H$, such that (5) holds for all $(\varphi, Y) \in H^2_\mathcal{R}(\mathbb{R}) \times H^2_\mathcal{S}(\mathbb{S}^{n-1})$.

**Lemma 1.** We fix $u \in C_\infty^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ and $Y \in H^2_\mathcal{S}(\mathbb{S}^{n-1})$. We define

$$
u_Y(t) := \int_{\mathbb{S}^{n-1}} u(t, \sigma)Y(\sigma) \, d\sigma = \langle u(t, \cdot), Y \rangle_{L^2(\mathbb{S}^{n-1})} \quad \text{for all} \quad t \in \mathbb{R}.
$$

Then $\nu_Y \in H^2_\mathcal{R}(\mathbb{R})$. Moreover, this definition extends continuously to $u \in H$ and there exists $C > 0$ such that

$$
\|\nu_Y\|_{H^2_\mathcal{R}(\mathbb{R})} \leq C \|u\|_H \|Y\|_{H^2_\mathcal{S}(\mathbb{S}^{n-1})} \quad \text{for all} \quad (u, Y) \in H \times H^2_\mathcal{S}(\mathbb{S}^{n-1}).
$$

**Proof of Lemma 1.** We let $u \in C_\infty^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$, $Y \in H^2_\mathcal{S}(\mathbb{S}^{n-1})$ and $\varphi \in C_\infty^\infty(\mathbb{R})$. Fubini’s theorem yields:

$$
\int_{\mathbb{R} \times \mathbb{S}^{n-1}} (\partial_t u_Y \partial_t \varphi + u_Y \varphi') \, dt = \int_{\mathbb{R} \times \mathbb{S}^{n-1}} (\partial_t u \partial_t (\varphi \ast Y) + u \cdot (\varphi \ast Y')) \, dt d\sigma
$$

Taking $\varphi := u_Y$, the Cauchy-Schwartz inequality yields

$$
\|u_Y\|_{H^2_\mathcal{R}(\mathbb{R})}^2 \leq \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} ((\partial_t u)^2 + u^2) \, dt d\sigma \right) \times \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} ((\partial_t (u_Y \varphi))^2 + (u_Y \varphi')^2) \, dt d\sigma \right)
$$

$$
\leq C \|u\|_H \|u_Y\|_H \|Y\|_{H^2_\mathcal{S}(\mathbb{S}^{n-1})},
$$

and then $\|u_Y\|_{H^2_\mathcal{R}(\mathbb{R})} \leq C \|u\|_H \|Y\|_{H^2_\mathcal{S}(\mathbb{S}^{n-1})}$. The extension follows from density.

**II. Transformation of the problem.** We let $\varphi \in \hat{K}$, that is

$$
-\Delta \varphi - \frac{\gamma}{|x|^2} \varphi = (2^*(s) - 1)\lambda U^{2^*(s)-2} \varphi \quad \text{weakly in} \quad D^2_\mathcal{R}^\infty(\mathbb{R}^n).
$$

Since $U \in C_\infty^\infty(\mathbb{R}^n \setminus \{0\})$, elliptic regularity yields $\varphi \in C_\infty^\infty(\mathbb{R}^n \setminus \{0\})$. Moreover, the correspondence (6) yields

$$
- \partial_t \hat{\varphi} - \Delta \text{can} \hat{\varphi} + e^2 \hat{\varphi} = (2^*(s) - 1)\lambda U^{2^*(s)-2} \hat{\varphi}
$$

weakly in $H$. Note that since $\hat{\varphi}, \hat{U} \in H$ and $H$ is continuously embedded in $L^{2^*(s)}(\mathbb{R} \times \mathbb{S}^{n-1})$, this formulation makes sense. Since $\varphi \in C_\infty^\infty(\mathbb{R}^n \setminus \{0\})$, we get that $\hat{\varphi} \in C_\infty^\infty(\mathbb{R} \times \mathbb{S}^{n-1}) \cap H$ and equation (8) makes sense strongly in $\mathbb{R} \times \mathbb{S}^{n-1}$. As one checks, we have that

$$
\hat{U}(t, \sigma) = \left( e^{\frac{2^*(s)-2}{2}t} + e^{-\frac{2^*(s)-2}{2}t} \right)^{-\frac{n-2}{2}} \quad \text{for all} \quad (t, \sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}.
$$

In the sequel, we will write $\hat{U}(t)$ for $\hat{U}(t, \sigma)$ for $(t, \sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}$. 
integrating by parts and using Fubini's theorem yields
\[
\int_R (\partial_t \psi \partial_t \psi + (\mu + \epsilon^2) \psi \psi) \, dt = \int_R (2^*(s) - 1) \lambda \hat{U}^{2^*(s) - 2} \hat{\psi} \psi \, dt,
\]
where \( \hat{\psi} \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). Then
\[
A_\mu \hat{\psi} = 0 \quad \text{with} \quad A_\mu := -\partial_t + (\mu + \epsilon^2 - (2^*(s) - 1) \lambda \hat{U}^{2^*(s) - 2})
\]
where this identity holds both in the classical sense and in the weak \( H^1_\text{loc}(\mathbb{R}) \) sense. We claim that
\[
\hat{\psi} \equiv 0 \quad \text{for all eigenfunction} \quad Y \quad \text{of} \quad \mu \geq n - 1.
\]
We prove the claim by taking inspiration from Chang-Gustafson-Nakanishi (\cite{2}, Lemma 2.1). Differentiating with respect to \( i = 1, \ldots, n \), we get that
\[
-\Delta \hat{\psi} - \frac{\gamma}{|x|^2} \hat{\psi} + (2^*(s) - 1) \lambda \hat{U}^{2^*(s) - 2} \hat{\psi} = - \left( \frac{2\gamma}{|x|^4} U + \frac{s\lambda}{|x|^{s+2}} U^{2^*(s)-1} \right) x_i
\]
On \( \mathbb{R} \times S^{n-1} \), this equation reads
\[
-\partial_t \hat{\psi} - \Delta_{\text{can}} \hat{\psi} + (e^2 - (2^*(s) - 1) \lambda \hat{U}^{2^*(s) - 2}) \hat{\psi} = -\sigma_i e^t \left( 2\gamma \hat{U} + s\lambda \hat{U}^{2^*(s)-1} \right)
\]
Note that \( \partial_t \hat{\psi} = -V * \sigma_i \), where \( \sigma_i : S^{n-1} \to \mathbb{R} \) is the projection on the \( x_i \)'s and
\[
V(t) := -e^{-\frac{2\gamma}{|x|^4}} U'(e^{-t}) = e^{(1+\gamma)t} \left( \alpha_+(\gamma) + \alpha_-(\gamma) e^{2 \frac{|x|^4}{2\gamma} e^{-t}} \right) \left( 1 + e^{2 \frac{|x|^4}{2\gamma} e^{-t}} \right) > 0
\]
for all \( t \in \mathbb{R} \). Since \( -\Delta_{\text{can}} \sigma_i = (n-1) \sigma_i \) (the \( \sigma_i \)'s form a basis of the second eigenspace of \( -\Delta_{\text{can}} \)), we then get that
\[
A_\mu V \geq A_{n-1} V = e^t \left( 2\gamma \hat{U} + s\lambda \hat{U}^{2^*(s)-1} \right) > 0 \quad \text{for all} \quad \mu \geq n - 1 \quad \text{and} \quad V > 0.
\]
Note that for \( \gamma > 0 \), we have that \( \alpha_-(\gamma) > 0 \), and that for \( \gamma = 0 \), we have that \( \alpha_-(\gamma) = 0 \). As one checks, we have that
\[
(\gamma > 0 \quad \text{and} \quad \epsilon > 1) \quad \text{or} \quad (\gamma = 0 \quad \text{and} \quad s < \frac{n}{2}) \quad \Rightarrow \quad V \in H^1_\text{loc}(\mathbb{R})
\]
\[
(\gamma > 0 \quad \text{and} \quad \epsilon \leq 1) \quad \text{or} \quad (\gamma = 0 \quad \text{and} \quad s \geq \frac{n}{2}) \quad \Rightarrow \quad V \notin L^2((0, +\infty))
\]
Assume that case (i) holds: in this case, \( V \in H^1_\text{loc}(\mathbb{R}) \) is a distributional solution to \( A_\mu V > 0 \) in \( H^1_\text{loc}(\mathbb{R}) \). We define \( m := \inf \{ \int_R \varphi A_\mu \varphi \, dt \} \), where the infimum is taken on \( \varphi \in H^1_\text{loc}(\mathbb{R}) \) such that \( \| \varphi \|_2 = 1 \). We claim that \( m > 0 \). Otherwise, it follows from Lemma \( \text{[3]} \) below that the infimum is achieved, say by \( \varphi_0 \in H^1_\text{loc}(\mathbb{R}) \setminus \{ 0 \} \) that is a weak solution to \( A_\mu \varphi_0 = m \varphi_0 \) in \( \mathbb{R} \). Since \( \| \varphi_0 \| \) is also a minimizer, and due to the comparison principle, we can assume that \( \varphi_0 > 0 \). Using the self-adjointness of \( A_\mu \), we get that \( 0 \geq m \int_R \varphi_0 V \, dt = \int_R (A_\mu \varphi_0) V \, dt = \int_R (A_\mu V) \varphi_0 \, dt > 0 \), which is a
contradiction. Then $m > 0$. Since $A_\mu \varphi_Y = 0$, we then get that $\varphi_Y \equiv 0$ as soon as $\mu \geq n - 1$. This ends case (i).

Assume that case (ii) holds: we assume that $\varphi_Y \not\equiv 0$. It follows from Lemma 4 that $V(t) = o(e^{-\alpha t})$ as $t \to -\infty$ for all $0 < \alpha < \sqrt{c^2 + n - 1}$. As one checks with the explicit expression of $V$, this is a contradiction when $\epsilon < \frac{m - 2}{2}$. Since $\frac{\pi}{2} \leq s < 2$, we have that $n = 3$. As one checks, $(\mu + \epsilon^2 - (2^*(s) - 1)\lambda U^2(\lambda) - 2) > 0$ for $\mu \geq n - 1$ as soon as $n = 3$ and $s \geq 3/2$. Lemma 4 yields $\varphi_Y \equiv 0$, a contradiction. So $\varphi_Y \equiv 0$, this ends case (ii).

These steps above prove (11). Then, for all $t \in \mathbb{R}$, $\hat{\varphi}(t, \cdot)$ is orthogonal to the eigenspaces of $\mu_i$, $i \geq 1$, so it is in the eigenspace of $\mu_0 = 0$ spanned by 1, and therefore $\hat{\varphi} = \hat{\varphi}(t)$ is independent of $\sigma \in \mathbb{S}^{n-1}$. Then

$$-\hat{\varphi}'' + (\epsilon^2 - (2^*(s) - 1)\lambda U^2(\lambda)^{-2})\hat{\varphi} = 0 \text{ in } \mathbb{R} \text{ and } \hat{\varphi} \in H^2_1(\mathbb{R}).$$

It follows from Lemma 4 that the space of such functions is a most one-dimensional. Going back to $\varphi$, we get that $\hat{K}$ is of dimension at most one, and then so is $K$. Since $Z \in K$, then $K$ is one dimensional and $K = \mathbb{R}Z$. This proves Theorem 11.

III. Auxiliary lemmas.

Lemma 2. Let $q \in C^0(\mathbb{R})$. Then

$$\dim_{\mathbb{R}}\{\varphi \in C^2(\mathbb{R}) \cap H^2_1(\mathbb{R}) \text{ such that } -\hat{\varphi} + q \varphi = 0\} \leq 1.$$  

Proof of Lemma 2. Let $F$ be this space. Fix $\varphi, \psi \in F \setminus \{0\}$: we prove that they are linearly dependent. Define the Wronskian $W := \varphi \psi - \hat{\varphi} \psi$. As one checks, $W = 0$, so $W$ is constant. Since $\varphi, \hat{\varphi}, \psi \in L^2(\mathbb{R})$, then $W \in L^1(\mathbb{R})$ and then $W \equiv 0$. Therefore, there exists $\lambda \in \mathbb{R}$ such that $(\psi(0), \hat{\psi}(0)) = \lambda(\varphi(0), \hat{\varphi}(0))$, and then, classical ODE theory yields $\psi = \lambda \varphi$. Then $F$ is of dimension at most one. □

Lemma 3. Let $q \in C^0(\mathbb{R})$ be such that there exists $A > 0$ such that $\lim_{i \to \pm \infty} q(t) = A$, and define

$$m := \inf_{\varphi \in H^2_1(\mathbb{R}) \setminus \{0\}} \frac{\int_\mathbb{R} (\hat{\varphi}^2 + q \varphi^2) dt}{\int_\mathbb{R} \varphi^2 dt}.$$  

Then either $m > 0$, or the infimum is achieved.

Note that in the case $q(t) \equiv A$, $m = A$ and the infimum is not achieved.

Proof of Lemma 3. As one checks, $m \in \mathbb{R}$ is well-defined. We let $(\varphi_i)_i \in H^2_1(\mathbb{R})$ be a minimizing sequence such that $\int_\mathbb{R} \varphi_i^2 dt = 1$ for all $i$, that is $\int_\mathbb{R} (\hat{\varphi}_i^2 + q \varphi_i^2) dt = m + o(1)$ as $i \to +\infty$. Then $(\varphi_i)_i$ is bounded in $H^2_1(\mathbb{R})$, and, up to a subsequence, there exists $\varphi \in H^2_1(\mathbb{R})$ such that $\varphi_i \to \varphi$ weakly in $H^2_1(\mathbb{R})$ and $\varphi_i \to \varphi$ strongly in $L^2_{loc}(\mathbb{R})$ as $i \to +\infty$. We define $\theta_i := \varphi_i - \varphi$. Since $\lim_{i \to \pm \infty} (q(t) - A) = 0$ and $(\theta_i)_i$ goes to 0 strongly in $L^2_{loc}$, we get that $\lim_{i \to \pm \infty} \int_\mathbb{R} (q(t) - A) \theta_i^2 dt = 0$. Using the weak convergence to 0 and that $(\varphi_i)_i$ is minimizing, we get that

$$\int_\mathbb{R} (\hat{\varphi}_i^2 + q \varphi_i^2) dt + \int_\mathbb{R} (\hat{\theta}_i^2 + A \theta_i^2) dt = m + o(1)$$  

as $i \to +\infty$.

Since $1 - \|\varphi\|^2 = \|\theta_i\|^2 + o(1)$ as $i \to +\infty$ and $\int_\mathbb{R} (\hat{\varphi}_i^2 + q \varphi_i^2) dt \geq m \|\varphi\|^2$, we get

$$m \|\theta_i\|^2 \geq \int_\mathbb{R} (\hat{\theta}_i^2 + A \theta_i^2) dt + o(1)$$  

as $i \to +\infty$.

If $m \leq 0$, then $\theta_i \to 0$ strongly in $H^2_1(\mathbb{R})$, and then $(\varphi_i)_i$ goes strongly to $\varphi \not\equiv 0$ in $H^2_1$, and $\varphi$ is a minimizer for $m$. This proves the lemma. □
Lemma 4. Let $q \in C^0(\mathbb{R})$ be such that there exists $A > 0$ such that $\lim_{t \to \pm \infty} q(t) = A$ and $q$ is even. We let $\varphi \in C^2(\mathbb{R})$ be such that $-\ddot{\varphi} + q \varphi = 0$ in $\mathbb{R}$ and $\varphi \in H^1_0(\mathbb{R})$.

- If $q \geq 0$, then $\varphi \equiv 0$.
- We assume that there exists $V \in C^2(\mathbb{R})$ such that

$$-\dot{V} + qV > 0, \ V > 0 \text{ and } V \notin L^2((0, +\infty)).$$

Then either $\varphi \equiv 0$ or $V(t) = o(e^{-\alpha |t|})$ as $t \to -\infty$ for all $0 < \alpha < \sqrt{A}$.

Proof of Lemma 4 We assume that $\varphi \neq 0$. We first assume that $q \geq 0$. By studying the monotonicity of $\varphi$ between two consecutive zeros, we get that $\varphi$ has at most one zero, and then $\dot{\varphi}$ has constant sign around $\pm \infty$. Therefore, $\varphi$ is monotone around $\pm \infty$ and then has a limit, which is $0$ since $\varphi \in L^2(\mathbb{R})$. The contradiction follows from studying the sign of $\dot{\varphi}$, $\varphi$. Then $\varphi \equiv 0$ and the first part of Lemma 4 is proved.

We now deal with the second part and we let $V \in C^2(\mathbb{R})$ be as in the statement. We define $\psi := V^{-1} \varphi$. Then, $-\dot{\psi} + h\dot{\psi} + Q\psi = 0$ in $\mathbb{R}$ with $h, Q \in C^0(\mathbb{R})$ and $Q > 0$. Therefore, by studying the zeros, $\psi$ vanishes at most once, and then $\psi(t)$ has limits as $t \to \pm \infty$. Since $\varphi = V \psi$, $\dot{\varphi} \in L^2(\mathbb{R})$ and $V \notin L^2((0, +\infty))$, then $\lim_{t \to -\infty} \psi(t) = 0$. We claim that $\lim_{t \to +\infty} \psi(t) \neq 0$. Otherwise, the limit would be $0$. Then $\psi$ would be constant sign, say $\psi > 0$. At the maximum point $t_0$ of $\psi$, the equation would yield $\ddot{\psi}(t_0) > 0$, which contradicts the maximum. So the limit of $\psi$ at $-\infty$ is nonzero, and then $V(t) = O(\varphi(t))$ as $t \to -\infty$.

We claim that $\varphi$ is even or odd and $\varphi$ has constant sign around $+\infty$. Since $t \mapsto \varphi(-t)$ is also a solution to the ODE, it follows from Lemma 2 that it is a multiple of $\varphi$, and then $\varphi$ is even or odd. Since $\dot{\psi}$ changes sign at most once, then $\dot{\psi}$ changes sign at most twice. Therefore $\dot{\psi} = \psi V$ has constant sign around $+\infty$.

We fix $0 < A' < A$ and we let $R_0 > 0$ such that $q(t) > A'$ for all $t \geq R_0$. Without loss of generality, we also assume that $\varphi(t) > 0$ for $t \geq R_0$. We define $b(t) := C_0 e^{-\sqrt{A'} t} - \varphi(t)$ for all $t \in \mathbb{R}$ with $C_0 := 2\varphi(R_0) e^{\sqrt{A'} R_0}$. We claim that $b(t) \geq 0$ for all $t \geq R_0$. Otherwise $\inf_{t \geq R_0} b(t) < 0$, and since $\lim_{t \to +\infty} b(t) = 0$ and $b(R_0) > 0$, then there exists $t_1 > R_0$ such that $b(t_1) \geq 0$ and $b(t_1) < 0$. However, as one checks, the equation yields $\ddot{b}(t_1) < 0$, which is a contradiction. Therefore $b(t) \geq 0$ for all $t \geq R_0$, and then $0 < \varphi(t) \leq C_0 e^{-\sqrt{A'} t}$ for $t \to +\infty$. Lemma 2 follows from this inequality, $\varphi$ even or odd, and $V(t) = O(\varphi(t))$ as $t \to -\infty$.

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