The automorphisms of endomorphism semigroups of free Burnside groups

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Abstract
In this paper we describe the automorphism groups of the endomorphism semigroups of free Burnside groups $B(m,n)$ for odd exponents $n \geq 1003$. We prove, that the groups $\text{Aut}(\text{End}(B(m,n)))$ and $\text{Aut}(B(m,n))$ are canonically isomorphic. In particular, if the groups $\text{Aut}(\text{End}(B(m,n)))$ and $\text{Aut}(\text{End}(B(k,n)))$ are isomorphic, then $m = k$.

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The free Burnside group $B(m,n)$ is the free group of rank $m$ of the variety $B_n$ of all groups satisfying the identity $x^n = 1$. The group $B(m,n)$ is isomorphic to the quotient group of the absolutely free group $F_m$ of rank $m$ by normal subgroup $F^m_m$ generated by all $n$-th powers of its elements. It is well known (see [1, Theorem 2.15]) that for all odd $n \geq 665$ and rank $m > 1$ the group $B(m,n)$ is infinite (and even has exponential growth). According to one other theorem of S.I. Adyan (see [1, Theorem 3.21]) for $m > 1$ and odd periods $n \geq 665$ the center of $B(m,n)$ is trivial and hence, $B(m,n)$ is isomorphic to the inner automorphism subgroup $\text{Inn}(B(m,n))$ of the automorphism group $\text{Aut}(B(m,n))$. Other results on automorphisms and monomorphisms of the groups $B(m,n)$ appeared relatively recently in [2]-[10]. In this paper we describe the automorphism groups of the endomorphism semigroups of $B(m,n)$ for odd exponents $n \geq 1003$. In particular, we prove, that the groups $\text{Aut}(\text{End}(B(m,n)))$ and $\text{Aut}(\text{End}(B(k,n)))$ are isomorphic if and only if $m = k$. This is a particular problem about $\text{End}(A)$, for $A$ a free algebra in a certain variety, was raised by B.I. Plotkin in [14]. Analogous problems for $\text{End}(F)$ with $F$ a finitely generated free group or free monoid were solved by Formanek in [12] and Mashevitzky and Schein in [13] respectively.
For an arbitrary group $G$ consider a natural homomorphism

$$\tau_G : \text{Aut}(\text{End}(G)) \to \text{Aut}(\text{Aut}(G))$$

taking each automorphism from the endomorphism semigroup of the group $G$ to its restriction on the subgroup of all invertible elements $\text{Aut}(G)$ of this semigroup. Obviously, any inner automorphism of $\text{Aut}(G)$ extends to an automorphism of the semigroup $\text{End}(G)$ in the natural way. Therefore, if all automorphisms of a group $\text{Aut}(G)$ are inner, then $\tau_G$ is surjective homomorphism. In particular, this is true for complete groups $\text{Aut}(G)$. Recall, that a group is called complete, if its center is trivial and all its automorphisms are inner.

Moreover, if $\phi : \text{Inn}(G) \to \text{Inn}(G)$ is an automorphism of the inner automorphism group of $G$ and $\phi(i_g) = i_{\alpha(g)}$, then it is not hard to check that $\alpha : G \to G$ is an automorphism of $G$. By virtue of the relation $\alpha \circ i_g \circ \alpha^{-1} = i_{\alpha(g)}$ we get that the automorphism $\phi$ extends to the automorphism $i_\alpha$ of the semigroup $\text{End}(G)$. The subgroup $\text{Inn}(G)$ is characteristic in $\text{Aut}(G)$ for a complete groups $G$ by the criterion of Burnside. Hence, for any complete group $G$ the homomorphism

$$\iota_G : \text{Aut}(\text{End}(G)) \to \text{Aut}(\text{Inn}(G))$$

taking each automorphism from $\text{Aut}(\text{End}(G))$ to its restriction on the subgroup $\text{Inn}(G)$ also is surjective.

J. Dyer and E. Formanek in [11] proved that the automorphism group $\text{Aut}(F)$ is complete for any finitely generated non-abelian free group $F$. Therefore, we have an isomorphism $\text{Aut}(\text{Aut}(F)) \simeq \text{Aut}(F)$. Later on Formanek in [12] showed that the equality $\text{Ker}(\tau_F) = 1$ holds for the same free groups $F$. Thus, the groups $\text{Aut}(\text{End}(F))$ and $\text{Aut}(\text{Aut}(F))$ also are isomorphic and we have $\text{Aut}(\text{End}(F)) \simeq \text{Aut}(\text{Aut}(F)) \simeq \text{Aut}(F)$.

In [6] we have proved that for any $m > 1$ and odd $n \geq 1003$ the inner automorphism group $\text{Inn}(B(m, n))$ is the unique normal subgroup of $\text{Aut}(B(m, n))$ among all its subgroups, which is isomorphic to a free Burnside group $B(s, n)$ of some rank $s \geq 1$. From this it follows that $\text{Inn}(B(m, n))$ is a characteristic subgroup in $\text{Aut}(B(m, n))$ and hence, the group $\text{Aut}(B(m, n))$ is complete. As was noted above, the restriction of every automorphism of the endomorphism semigroup $\text{End}(B(m, n))$ on the subgroup $\text{Inn}(B(m, n))$ induces an automorphism, because $\text{Inn}(B(m, n))$ is a characteristic subgroup in $\text{Aut}(B(m, n))$.

The aim of this paper is to prove the following

**Theorem 1.** Let us $\Phi$ and $\Psi$ be arbitrary automorphisms of the endomorphism semigroup $\text{End}(B(m, n))$ of the free Burnside group $B(m, n)$ of odd period $n \geq 1003$ and rank $m > 1$. \[2\]
Then $\Phi = \Psi$ if and only if the restrictions of the automorphisms $\Phi$ and $\Psi$ on the subgroup $\text{Inn}(B(m,n))$ coincide, that is

$$\Phi \big|_{\text{Inn}(B(m,n))} = \Psi \big|_{\text{Inn}(B(m,n))}.$$ 

From Theorem \[\ref{thm:main} \] immediately follows

**Corollary 1.** The maps

$$\tau_{B(m,n)} : \text{Aut}(\text{End}(B(m,n))) \to \text{Aut}(\text{Aut}(B(m,n)))$$

and

$$\iota_{B(m,n)} : \text{Aut}(\text{End}(B(m,n))) \to \text{Aut}(\text{Inn}(B(m,n)))$$

are isomorphisms for any rank $m > 1$ and odd period $n \geq 1003$.

Taking into account that the groups $\text{Aut}(B(m,n))$ are complete for ranks $m > 1$ and odd periods $n \geq 1003$ we get

**Corollary 2.** For any automorphism $\Phi \in \text{Aut}(\text{End}(B(m,n)))$ there exists an automorphism $\alpha \in \text{Aut}(B(m,n))$ such that $\Phi(\varepsilon) = \alpha \circ \varepsilon \circ \alpha^{-1}$ for each endomorphism $\varepsilon \in \text{End}(B(m,n))$.

**Corollary 3.** For any odd $n \geq 1003$ the groups $\text{Aut}(\text{End}(B(m,n)))$ and $\text{Aut}(\text{End}(B(k,n)))$ are isomorphic if and only if $m = k$.

**Proof.** It follows from Corollary\[\ref{cor:tau} \] that $\text{Aut}(\text{End}(B(m,n)))$ is isomorphic to $\text{Aut}(B(m,n))$. By Theorem 1.3 from \[\cite{6} \] the automorphism groups $\text{Aut}(B(m,n))$ and $\text{Aut}(B(k,n))$ are isomorphic if and only if $m = k$. \qed

## 2 The proof of the main result

Obviously, to prove Theorem\[\ref{thm:main} \] it suffices to show that if the restriction of an automorphism $\Phi$ from $\text{End}(B(m,n))$ on the subgroup $\text{Inn}(B(m,n))$ is the identity automorphism, that is

$$\Phi \big|_{\text{Inn}(B(m,n))} = 1_{\text{Inn}(B(m,n))}, \quad (1)$$

then $\Phi$ is the identity automorphism of the semigroup $\text{End}(B(m,n))$.

Suppose that the equality (1) holds. We will prove that

$$\Phi = 1_{\text{End}(B(m,n))}$$
or equivalently we will prove that $\Phi(\varepsilon) = \varepsilon$ holds for each $\varepsilon \in \text{End}(B(m, n))$. More precisely we will show that the equality

$$\varepsilon(a)^{-1} \cdot \Phi(\varepsilon)(a) = 1$$

holds for any $a \in B(m, n)$ and $\varepsilon \in \text{End}(B(m, n))$.

The inner automorphism of $B(m, n)$ induced by element $a \in B(m, n)$ is denoted by $i_a$. Consider an arbitrary endomorphism $\varepsilon \in \text{End}(B(m, n))$ and apply the product of endomorphisms $\varepsilon \circ i_a$ to an element $x \in B(m, n)$. By definition we have

$$(\varepsilon \circ i_a)(x) = \varepsilon(i_a(x)) = \varepsilon(a)\varepsilon(x)\varepsilon(a)^{-1} = (i_{\varepsilon(a)} \circ \varepsilon)(x).$$

Hence, the equality

$$\varepsilon \circ i_a = i_{\varepsilon(a)} \circ \varepsilon$$

holds.

To both sides of the equality (3) applying the automorphism $\Phi$ and taking into account (1) we get the equality

$$\Phi(\varepsilon) \circ i_a = i_{\varepsilon(a)} \circ \Phi(\varepsilon).$$

Now the both sides of the equality (4) applying to an arbitrary element $x \in B(m, n)$ we obtain

$$\Phi(\varepsilon)(a) \cdot \Phi(\varepsilon)(x) \cdot \Phi(\varepsilon)(a)^{-1} = \varepsilon(a) \cdot \Phi(\varepsilon)(x) \cdot \varepsilon(a)^{-1}.$$
we have $\varepsilon = \varepsilon \circ \sigma$ and hence, $\Phi(\varepsilon) = \Phi(\varepsilon \circ \sigma) = \Phi(\varepsilon) \circ \sigma$, because $\Phi(\sigma) = \sigma$. On the other hand we have $(\Phi(\varepsilon) \circ \sigma)(b_j) = b^{-1} \neq b = \Phi(\varepsilon)(b_j)$, because $b^2 \neq 1$ provided $n$ is odd. This contradiction shows that the image of the trivial endomorphism is the trivial endomorphism.

Now suppose that $\varepsilon(b_i) = a^{k_i}$ for some element $a \in B(m, n)$ and for each $i \in I$, where $k_i \in \mathbb{Z}$. We assume also that $\varepsilon(b_j) \neq 1$ for some $j \in I$.

Let $d$ be a generator for the cyclic subgroup of additive group $\mathbb{Z}$ of integers generated by the set of the numbers $\{k_i\}_{i \in I}$. Obviously, applying some elementary transformations to the sequence of numbers $\{k_i\}_{i \in I}$ we can obtain a new sequence $\{s_i\}_{i \in I}$ such that $s_1 = d$ and $s_i = 0$ for $i \neq 1$. Note that to any Nielsen transformation of system of generators $\{b_i\}_{i \in I}$ corresponds a Nielsen transformation of the system $\{\varepsilon(b_i)\}_{i \in I}$. The Nielsen transformations of the system $\{\varepsilon(b_i)\}_{i \in I}$ lead to the corresponding elementary transformations of the exponents $\{k_i\}_{i \in I}$ and vise versa. Consequently, there exist such Nielsen transformations of the system of free generators $\{b_i\}_{i \in I}$ which lied to the system of new free generators $\{y_i\}_{i \in I}$ of $B(m, n)$ satisfying to the conditions $\varepsilon(y_1) = a^d$ and $\varepsilon(y_i) = 1$ for $i \neq 1$.

Case 1. Let the period $n$ of the group $B(m, n)$ is a prim number. Consider the endomorphism $\alpha$ given by the relations $\alpha(y_1) = y_1$ and $\alpha(y_i) = 1$ for $i \neq 1$. Then $\Phi(\alpha)$ is a non-trivial endomorphism, because $\alpha$ is non-trivial. According to (5) the element $\alpha(y_1)^{-1} \cdot \Phi(\alpha)(y_1)$ belongs to the centralizer of $\Phi(\alpha)(y_1)$ for all $i \in I$. The centralizer of $\Phi(\alpha)(y_1)$ is a cyclic group of order $n$. Therefor, the elements $\alpha(y_1)^{-1}$ and $\Phi(\alpha)(y_1)$ belongs to the centralizer of $\Phi(\alpha)(y_1)$ for each $i$. Consequently, for any $i \in I$ there is an integer $t_i$ such that the equality $\Phi(\alpha)(y_1) = y_1^{t_i}$ holds. Evidently, the equality $\alpha \circ \alpha = \alpha$ also holds. Hence, $\Phi(\alpha) \circ \Phi(\alpha) = \Phi(\alpha)$. So, we have $t_1 t_i \equiv t_i (mod n)$ for all $i \in I$. The integers $t_1$ and $t_1 - 1$ are relatively prime and $n$ is a prim number. Therefor, from $t_1^2 \equiv t_1 (mod n)$ it follows that $t_1 \equiv 0(mod n)$ or $t_1 \equiv 1(mod n)$. Since $\Phi(\alpha)$ is a non-trivial endomorphism, we obtain that the congruence $t_1 \equiv 1(mod n)$ holds. Now for any $j \neq 1$, $j \in I$ consider the Nielsen automorphism $\lambda_j$ given by equalities $\lambda_j(y_1) = y_1 y_j$ and $\lambda_j(y_i) = y_i$ for $i \neq 1$. It is easy to check that $\alpha \circ \lambda_j = \alpha$. Hence, we obtain $\Phi(\alpha \circ \lambda_j) = \Phi(\alpha) \circ \lambda_j = \Phi(\alpha)$ by virtu $\Phi(\lambda_j) = \lambda_j$. Therefor, $\Phi(\alpha)(\lambda_j(y_1)) = \Phi(\alpha)(y_1)$, that is $y_1^{t_1 + t_j} = y_1^{t_1}$ for $j \neq 1$. This means that $t_j \equiv 0(mod n)$ for $j \neq 1$. Thus, $\Phi(\alpha)(y_1) = y_1^{t_i}$, where $t_1 \equiv 1(mod n)$ and $t_i \equiv 0(mod n)$ for $i \neq 1$. So, we get $\Phi(\alpha) = \alpha$.

Now let $b$ be an arbitrary non-commuting with $a$ element of the group $B(m, n)$ and $\gamma$ be an endomorphism of $B(m, n)$ given on the free generators by the formulae $\gamma(y_1) = a^d$ and $\gamma(y_i) = b$ for $i \neq 1$. It is easy to verify that $\varepsilon = \gamma \circ \alpha$. Since $\text{Im}(\gamma)$ is a non-cyclic subgroup, then $\Phi(\gamma) = \gamma$. Therefor, $\Phi(\varepsilon) = \Phi(\gamma) \circ \Phi(\alpha) = \gamma \circ \alpha = \varepsilon$. In the Case 1 the
equality (2) proved.

**Case 2.** Let the period $n$ of $B(m, n)$ be a composite number and $n = n_1 n_2$, where $1 < n_1, n_2 < n$. Consider the endomorphism $\delta_1 : B(m, n) \to B(m, n)$ given on the generators by the equalities

$$\delta_1(y_1) = a^d \quad \text{and} \quad \delta_1(y_j) = y_k^{n_1} \quad \text{for} \quad j \neq 1,$$

where $y_k$ is a fixed and non-commuting with $a^d$ generator of $B(m, n)$. Consider also the endomorphism $\delta_2 : B(m, n) \to B(m, n)$ defined by the equalities

$$\delta_2(y_1) = y_1 \quad \text{and} \quad \delta_2(y_j) = y_j^{n_2} \quad \text{for} \quad j \neq 1.$$

Since the images of the endomorphisms $\delta_1, \delta_2$ are not cyclic, then $\Phi(\delta_i) = \delta_i, i = 1, 2$. From the definitions of endomorphisms $\delta_i$ immediately follows that $\delta_1 \circ \delta_2 = \varepsilon$. Therefore, we get $\Phi(\varepsilon) = \varepsilon$. Theorem 1 is proved.

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