ON THE KLEIMAN-MORI CONE

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Abstract. The Kleiman-Mori cone plays important roles in the birational geometry. In this paper, we construct complete varieties whose Kleiman-Mori cones have interesting properties. First, we construct a simple and explicit example of complete non-projective singular varieties for which Kleiman’s ampleness criterion does not hold. More precisely, we construct a complete non-projective toric variety $X$ and a line bundle $L$ on $X$ such that $L$ is positive on $\mathcal{NE}(X) \setminus \{0\}$. Next, we construct complete singular varieties $X$ with $\mathcal{NE}(X) = N_1(X) \simeq \mathbb{R}^k$ for any $k$. These explicit examples seem to be missing in the literature.

1. Introduction

The Kleiman-Mori cone plays important roles in the birational geometry. In this paper, we construct complete varieties whose Kleiman-Mori cones have interesting properties. First, we construct a simple and explicit example of complete non-projective singular varieties for which Kleiman’s ampleness criterion does not hold. More precisely, we construct a complete non-projective toric variety $X$ and a line bundle $L$ on $X$ such that $L$ is positive on $\mathcal{NE}(X) \setminus \{0\}$.

Definition 1.1. Let $V$ be a complete algebraic scheme defined over an algebraically closed field $k$. We say that Kleiman’s ampleness criterion holds for $V$ if and only if the interior of the nef cone of $V$ coincides with the ample cone of $V$.

Note that Kleiman’s original statements are very sharp. We recommend the readers to see [Kl] Chapter IV §2 Theorems 1, 2]. Of course, our example is not “quasi-divisorial” in the sense of Kleiman (see [Kl] Chapter IV §2 Definition 4 and Theorem 2]). We do not repeat the definition of quasi-divisorial since we do not use it in this paper. Note that if $X$ is projective or $\mathbb{Q}$-factorial then $X$ is quasi-divisorial in the sense of Kleiman. Next, we construct complete singular varieties $X$ with $\mathcal{NE}(X) = N_1(X) \simeq \mathbb{R}^k$ for any $k$. We note that the condition

Date: 2005/1/5.

2000 Mathematics Subject Classification. Primary 14M25; Secondary 14E30.
NE(X) = N_1(X) is equivalent to the following one: a line bundle L is nef if and only if L is numerically trivial. These explicit examples seem to be missing in the literature. We adopt the toric geometry to construct examples.

**Notation.** We freely use the basic notation of the toric geometry throughout this paper. Let $v_i \in \mathbb{N}$ for $0 \leq i \leq k$. Then the symbol $\langle v_1, v_2, \ldots, v_k \rangle$ denotes the cone $\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \cdots + \mathbb{R}_{\geq 0}v_k$ in $\mathbb{N}\mathbb{R}$.

**Acknowledgments.** I would like to thank Professor Shigefumi Mori and Doctor Hiroshi Sato for fruitful discussions and useful comments.

## 2. On Kleiman’s ampleness criterion

In this section, we construct explicit examples for which Kleiman’s ampleness criterion does not hold. We think that the following example is the simplest one. It seems to be easy to construct a lot of singular toric varieties for which Kleiman’s ampleness criterion does not hold. The reader can find many examples of singular toric 3-folds in [FS2]. He can easily check that Kleiman’s ampleness criterion does not hold for $X_6$ in [FS2]. For the cone theorem of toric varieties, see [FS1, Theorem 4.1].

### 2.1 (Construction).

We fix $N = \mathbb{Z}^3$. We put

$$
\begin{align*}
v_1 &= (1, 0, 1), & v_2 &= (0, 1, 1), & v_3 &= (-1, -1, 1), \\
v_4 &= (1, 0, -1), & v_5 &= (0, 1, -1), & v_6 &= (-1, -1, -1).
\end{align*}
$$

We consider the following fans.

$$
\Delta_P = \left\{ \begin{array}{c}
\langle v_1, v_2, v_4 \rangle, \quad \langle v_2, v_4, v_5 \rangle, \quad \langle v_2, v_3, v_5, v_6 \rangle, \\
\langle v_1, v_3, v_4, v_6 \rangle, \quad \langle v_1, v_2, v_3 \rangle, \quad \langle v_4, v_5, v_6 \rangle,
\end{array} \right\}
$$

and their faces

$$
\Delta_Q = \left\{ \begin{array}{c}
\langle v_1, v_2, v_4, v_5 \rangle, \quad \langle v_2, v_3, v_5, v_6 \rangle, \quad \langle v_1, v_3, v_4, v_6 \rangle, \\
\langle v_1, v_2, v_3 \rangle, \quad \langle v_4, v_5, v_6 \rangle,
\end{array} \right\}.
$$

We recommend the reader to draw pictures of $\Delta_P$ and $\Delta_Q$ by himself.

**Lemma 2.2.** We have the following properties:

(i) $X_P := X(\Delta_P)$ is a non-projective complete toric variety with $\rho(X_P) = 1$,

(ii) $X_Q := X(\Delta_Q)$ is a projective toric variety with $\rho(X_Q) = 1$,

(iii) there exists a toric birational morphism $f_{PQ} : X_P \to X_Q$, which contracts a $\mathbb{P}^1$ on $X_P$. 

(iv) $X_P$ and $X_Q$ have only canonical Gorenstein singularities, and
(v)
\[
N_1(X_P) \cong N_1(X_Q) \\
\bigcup N_1(X_P) \cong N_1(X_Q).
\]

In particular, $NE(X_P) = NE(X_Q)$ is a half line.

**Proof.** It is easy to check that $X_Q$ is projective and $\rho(X_Q) = 1$ (cf. [E, Theorem 3.2, Example 3.5]). Assume that $X_P$ is projective. Then there exists a strict upper convex support function $h$. We note that
\[
v_1 + v_5 = v_2 + v_4, \\
v_2 + v_6 = v_3 + v_5, \\
v_3 + v_4 = v_1 + v_6.
\]
Thus, we obtain
\[
h(v_1) + h(v_5) < h(v_2) + h(v_4), \\
h(v_2) + h(v_6) = h(v_3) + h(v_5), \\
h(v_3) + h(v_4) = h(v_1) + h(v_6).
\]
This implies that
\[\sum_{i=1}^{6} h(v_i) < \sum_{i=1}^{6} h(v_i).\]
It is a contradiction. Therefore, $X_P$ is not projective. Thus $f_{PQ}$ is not a projective morphism. So, $L \cdot C = 0$ for every $L \in \text{Pic}(X_P)$, where $C \cong \mathbb{P}^1$ is the exceptional locus of $f_{PQ}$. We note that the condition (ii) (b) in [Kl, p.325 Theorem 1] does not hold. The other statements are easy to check. \qed

Let $H$ be an ample Cartier divisor on $X_Q$ and $D := (f_{PQ})^* H$. Then $D$ is positive on $NE(X_P) \setminus \{0\} = NE(X_P) \setminus \{0\}$. Thus, the interior of the nef cone of $X_P$ is non-empty (cf. [Kl] p.327 Proposition 2]). However, $D$ is not ample on $X_P$. Therefore, Kleiman’s ampleness criterion does not hold for $X_P$. Note that $X_P$ is not projective nor quasi-divisorial in the sense of Kleiman (see [Kl] p.326 Definition 4]).

**Corollary 2.3.** We put $X := X_P \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Then we obtain complete non-projective singular toric varieties with $\dim X \geq 4$ for which Kleiman’s ampleness criterion does not hold. Since every complete toric surface is $\mathbb{Q}$-factorial and projective, Kleiman’s ampleness criterion always holds for toric surfaces.
Remark 2.4. Let $X$ be a complete $\mathbb{Q}$-factorial algebraic variety. Then it is not difficult to see that $X$ is projective if $\rho(X) = 1$.

Remark 2.5. In [Ko, Chapter VI. Appendix 2.19.3 Exercise], Kollár pointed out that Kleiman’s ampleness criterion does not hold for smooth proper algebraic spaces.

Remark 2.6. In [FS1, Theorem 4.1], we claim that $\text{NE}(X/Y)$ is strongly convex if $f : X \to Y$ is projective. This is obvious. However, in the proof of Theorem 4.1 in [FS1], we say that it follows from Kleiman’s criterion. Sorry, it is misleading.

We note the following ampleness criterion, which works for complete toric varieties with arbitrary singularities.

**Proposition 2.7.** Let $X$ be a complete toric variety and $L$ a line bundle on $X$. Assume that $L \cdot C > 0$ for every torus invariant integral curve $C$ on $X$. Then $L$ is ample. In particular, $X$ is projective.

**Proof.** Since $\text{NE}(X)$ is spanned by the torus invariant curves on $X$, it is obvious that $L$ is nef. This implies that $L$ is generated by its global sections. Note that we can replace $X$ (resp. $L$) with its toric resolution $Y$ (resp. the pull-back of $L$ on $Y$) in order to check the freeness of $L$. Thus, the proof of the freeness is easy. We consider the equivariant morphism $f := \Phi_{|L|} : X \to Y$ associated to the linear system $|L|$. Then we obtain that $L = f^*H$ for a very ample line bundle $H$ on $Y$. Since $L \cdot C > 0$ for every torus invariant integral curve $C$ on $X$, we have that $f$ is finite. Thus, $L$ is ample. \hfill $\square$

3. **Singular varieties with $\text{NE}(X) = N_1(X)$**

In this section, we construct complete singular toric varieties with $\text{NE}(X) = N_1(X) \simeq \mathbb{R}^k$ for any $k \geq 0$.

**Remark 3.1.** The condition $\text{NE}(X) = N_1(X)$ is equivalent to the following one: a line bundle $L$ is nef if and only if $L$ is numerically equivalent to zero.

**3.2 (Construction).** We fix $N = \mathbb{Z}^3$ and $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^3$. We put

$\begin{align*}
v_1 &= (1, 0, 1), \\
v_2 &= (0, 1, 1), \\
v_3 &= (-1, -2, 1), \\
v_4 &= (1, 0, -1), \\
v_5 &= (0, 1, -1), \\
v_6 &= (-1, -1, -1), \\
v_7 &= (0, 0, -1).
\end{align*}$
First, we consider the following fan.

\[ \Delta_A = \left\{ \langle v_1, v_2, v_4, v_5 \rangle, \langle v_2, v_3, v_5, v_6 \rangle, \langle v_1, v_3, v_4, v_6 \rangle, \langle v_1, v_2, v_3 \rangle, \langle v_4, v_5, v_6 \rangle \right\}. \]

We recommend the reader to draw the picture of \( \Delta_A \) by himself. We put \( X_A := X(\Delta_A) \) and \( D_i := V(v_i) \) for every \( i \). This example \( X_A \) is essentially the same as [Example 3.5]. We consider the principal divisor

\[ D = D_1 + D_2 + D_3 - D_4 - D_5 - D_6 \]

that is associated to \( m = (0, 0, -1) \in M \). We put

\[ \sigma_1 = \langle v_1, v_2, v_4, v_5 \rangle, \quad \sigma_2 = \langle v_2, v_3, v_5, v_6 \rangle, \quad \sigma_3 = \langle v_1, v_3, v_4, v_6 \rangle. \]

Then, all points in \( \sum_i \text{AD}(\sigma_i) \) are linear combinations of the lines of the matrix

\[
\begin{pmatrix}
1 & -1 & 0 & 1 & -1 & 0 \\
0 & -1 & 2 & 0 & -3 & 2 \\
-2 & 0 & 1 & -1 & 0 & 2
\end{pmatrix}
\]

which has rank 3. Note that we use \( D \) to define \( \text{AD}(\sigma_i) \) and that \( \sigma_i \) is not simplicial for every \( i \) and all the other 3-dimensional cones in \( \Delta_A \) are simplicial. For the definition of \( \text{AD}(\sigma_i) \), see [Theorem 3.2]. Therefore, \( \text{Pic}X_A \simeq \mathbb{Z}^{6-3-3} = \{0\} \) by [Theorem 3.2].

Next, we consider the following fan.

\[ \Delta_B = \left\{ \langle v_1, v_2, v_4, v_5 \rangle, \langle v_2, v_3, v_5, v_6 \rangle, \langle v_1, v_3, v_4, v_6 \rangle, \langle v_1, v_2, v_3 \rangle, \langle v_4, v_5, v_6 \rangle, \langle v_1, v_3, v_4, v_6 \rangle, \langle v_4, v_6, v_7 \rangle, \langle v_5, v_6, v_7 \rangle \right\}. \]

We recommend the reader to draw the picture of \( \Delta_B \) by himself. We put \( X_B := X(\Delta_B) \). Then \( X_B \rightarrow X_A \) is the blow up along \( \langle v_7 \rangle \). We consider the principal divisor

\[ D' = D_1 + D_2 + D_3 - D_4 - D_5 - D_6 - D_7 \]

that is associated to \( m = (0, 0, -1) \in M \). Then, all points in \( \sum_{i=1}^{3} \text{AD}(\sigma_i) \) are linear combinations of the lines of the matrix

\[
\begin{pmatrix}
1 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 2 & 0 & -3 & 2 & 0 \\
-2 & 0 & 1 & -1 & 0 & 2 & 0
\end{pmatrix}
\]

which has rank 3. We note that we use \( D' \) to define \( \text{AD}(\sigma_i) \) and that \( \sigma_i \) is not simplicial for every \( i \) and all the other 3-dimensional cones in \( \Delta_B \) are simplicial. Thus, we have \( \text{Pic}X_B \simeq \mathbb{Z}^{7-3-3} = \mathbb{Z} \) by [Theorem 3.2].
Lemma 3.3. $D_7$ is a Cartier divisor.

Proof. It is because each 3-dimensional cone containing $v_7$ is non-singular. Thus, $D_7$ is Cartier. □

We put $C_1 := V(\langle v_4, v_5 \rangle) \simeq \mathbb{P}^1$ and $C_2 := V(\langle v_4, v_7 \rangle) \simeq \mathbb{P}^1$. The following lemma is a key property of this example.

Lemma 3.4. $C_1 \cdot D_7 > 0$ and $C_2 \cdot D_7 < 0$. More precisely, $C_1 \cdot D_7 = 1$ and $C_2 \cdot D_7 = -3$. Therefore, $D_7$ is a generator of $\text{Pic} X_B \simeq \mathbb{Z}$.

Proof. It is obvious that $C_1 \cdot D_7 = 1 > 0$. Since $v_4 + v_5 + v_6 - 3v_7 = 0$, we have $C_2 \cdot D_7 = -3C_2 \cdot D_5 = -3 < 0$. □

Therefore, $\text{NE}(X_B) = N_1(X_B) \simeq \mathbb{R}$. In particular, $X_B$ is not projective.

Corollary 3.5. Let $X := X_A$. Then $\text{NE}(X) = N_1(X) = \{0\}$. Let $X := X_B \times X_B \times \cdots \times X_B$ be the $k$ times product of $X_B$. Then $\text{NE}(X) = N_1(X) \simeq \mathbb{R}^k$.

Proof. The first statement is obvious by the above construction. It is not difficult to see that $\text{Pic} X \simeq \bigotimes_{i=1}^k p_i^* \text{Pic} X_B$, where $p_i : X \longrightarrow X_B$ is the $i$-th projection. Thus, we can check that any nef line bundle on $X$ is numerically trivial. Therefore, $\text{NE}(X) = N_1(X) \simeq \mathbb{R}^k$. □

In the above corollary, $\dim X = 3k$ if $\text{NE}(X) = N_1(X) \simeq \mathbb{R}^k$ for $k \geq 1$. We can construct 3-folds with $\text{NE}(X) = N_1(X) \simeq \mathbb{R}^k$ for every $k \geq 0$. We just note this fact in the next remark. The details are left to the reader.

Remark 3.6. We put $X_0 := X_A$, where $X_A$ is in 3.2. We define primitive vectors $\{v_k\}$ inductively for $k \geq 7$ as follows: $\langle v_k \rangle = \langle v_4 + v_5 + v_{k-1} \rangle$. Let $X_k \longrightarrow X_{k-1}$ be the blow up along $\langle v_{k+6} \rangle$ for $k \geq 1$. Note that $X_1 = X_B$, where $X_B$ is in 3.2. It is not difficult to see that $\text{NE}(X_k) = N_1(X_k) \simeq \mathbb{R}^k$ and $\dim X_k = 3$. We note that the numerical equivalence classes of $D_i = V(v_i)$’s for $7 \leq i \leq k + 6$ form a basis of $N^1(X_k)$.

4. Miscellaneous comments

4.1. Let $X$ be a complete normal variety. Assume that $X$ is $\mathbb{Q}$-factorial. We note that $\rho(X) \geq 1$ in this case. Our question is as follows.

Question 4.2. Are there any complete normal $\mathbb{Q}$-factorial varieties $X$ with $\text{NE}(X) = N_1(X)$?

The examples constructed in Section 3 are obviously non-$\mathbb{Q}$-factorial. We note that $\rho(X) \geq 2$ when $\text{NE}(X) = N_1(X)$ by Remark 2.4.
Remark 4.3. Let $X$ be a complete normal (not necessarily $\mathbb{Q}$-factorial) variety. If there exists a proper surjective morphism $f : X \to Y$ such that

(i) $f$ has connected fibers,
(ii) $Y$ is projective and $\dim Y \geq 1$, and
(iii) $f$ is not an isomorphism,
then it is obvious that $NE(X) \subset \overline{NE}(X) \subset N_1(X)$.

4.4 (Toric variety). We have the following conjecture on Question 4.2.

Conjecture 4.5. Let $X$ be a non-singular complete toric variety. Then $NE(X) = NE(X) \subset N_1(X)$.

Remark 4.6. If $X$ is a complete (not necessarily $\mathbb{Q}$-factorial) toric variety with $NE(X) \subset N_1(X)$, then there exists a non-trivial nef line bundle $L$ on $X$. Thus, we obtain the toric morphism $f : X \to Y := \text{Proj} \bigoplus_{k \geq 0} H^0(X, L^k)$ such that $L = f^*H$, where $H$ is an ample line bundle on $Y$. Note that $\dim Y \geq 1$.

4.7 (Algebraic space). I learned the following example from S. Mori, who call it Hironaka’s example. I have never seen this in the literature.

Example 4.8. Let $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be a non-singular quadric surface in $\mathbb{P}^3$. We take a non-singular $(3, d)$-curve $C$ in $Q$, where $d \in \mathbb{Z}_{>0}$. That is, $O_Q(C) \simeq p_1^*O_{\mathbb{P}^1}(3) \otimes p_2^*O_{\mathbb{P}^1}(d)$, where $p_1$ (resp. $p_2$) is the first (resp. second) projection from $Q$ to $\mathbb{P}^1$. Let $f_1$ (resp. $f_2$) be a fiber of $p_2 : Q \to \mathbb{P}^1$ (resp. $p_1$). We take the blow up $\pi : X \to \mathbb{P}^3$ along $C$. Let $Q'$ be the strict transform of $Q$ and $E$ the exceptional divisor of $\pi$. Then $Q' = f^*Q - E$. Let $f'_i$ be the strict transform of $f_i$ for $i = 1, 2$. We have

$$Q' \cdot f'_1 = f^*Q \cdot f'_1 - E \cdot f'_1 = Q \cdot f_1 - 3 = 2 - 3 = -1.$$ 

Thus we can blow down $X$ to $Y$ along the ruling $p_2 : Q' \simeq Q \to \mathbb{P}^1$ (cf. [N] Main Theorem], [FN]). Note that $Y$ is a compact Moishezon manifold. The Kleiman-Mori cone $\overline{NE}(X)$ is spanned by 2 rays $R$ and $Q$. We note that $X$ is non-singular projective and $\rho(X) = 2$. Let $l$ be a fiber of $\pi : X \to \mathbb{P}^3$. Then, one ray $R$ is spanned by the numerical equivalence class of $l$. We put $L := \pi^*O_{\mathbb{P}^3}(1)$. Then $L$ is non-negative on $\overline{NE}(X)$ and $R = (L = 0) \cap \overline{NE}(X)$. We have the following intersection numbers.

$$L \cdot l = 0, \quad L \cdot f'_1 = L \cdot f'_2 = 2,$$
$$E \cdot l = -1, \quad E \cdot f'_1 = 3, \quad E \cdot f'_2 = d.$$
From now on, we assume $d \geq 4$. We can write $f'_1 = af'_2 + bl$ in $N_1(X)$ for $a, b \in \mathbb{R}$. Thus we can easily check that $a = 1, b = d - 3 > 0$. Therefore, the numerical class of $f'_1$ is in the interior of the cone spanned by the numerical classes of $f'_2$ and $l$. Thus, we have $NE(Y) = \overline{NE}(Y) = N_1(Y) \simeq \mathbb{R}$. Therefore, $Y$ is a non-singular complete algebraic space with $\rho(X) = 1$. Note that $Y$ is not a scheme.

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