Exponential Negation of a Probability Distribution

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Abstract—Negation operation is important in intelligent information processing. Different with existing arithmetic negation, an exponential negation is presented in this paper. The new negation can be seen as a kind of geometry negation. Some basic properties of the proposed negation are investigated, we find that the fix point is the uniform probability distribution. The negation is an entropy increase operation and all the probability distributions will converge to the uniform distribution after multiple negation iterations. The number of iterations of convergence is inversely proportional to the number of elements in the distribution. Some numerical examples are used to illustrate the efficiency of the proposed negation.

Index Terms—Negation; exponential negation; entropy; probability distribution; uniform distribution

I. INTRODUCTION

Knowledge representation and uncertainty measure are important issues in artificial intelligence. Probability distribution is an efficient way to describe uncertainty. As an abstract expression of the objective law of things, the idea of probability distribution has achieved remarkable results in artificial intelligence, quantum science and many other scientific fields. However, sometimes it is very difficult to express directly so that we need to do it from another perspective. Negative representation is an efficient way to present knowledge from opposite side. Negation means the opposite of the known knowledge, we can peek into some essences of things from it. For example, in the process of proving mathematical theorems, contradiction is usually an effective method. Through this example, we can find that negation can show us the other side of knowledge and help us indirectly acquire the knowledge we need. However, how to effectively represent the negation of probability distribution is still an open issue. Although there have been some studies in this area, the properties are still not perfect. Therefore, investigating a new effective method for negation of a probability distribution is of great significant meaning.

In previous researches, Zadeh first proposed the negation of probability distribution in his BISC blog. After that, Yager proposed an important method of negation which has the maximum entropy allocation [1]. The basic framework of Yager’s negation is subtracting a probability by 1. To some extent, Yager got the expression of negation through subtraction, can be seen as a kind of arithmetic inverse. Yager also indicate some basic properties of this negation which gets some superior properties that accord with our intuitive understanding of the laws of nature. Inspired by Yager’s negation, the negation of joint and marginal probability distributions in the binary case is proposed [2]. To study more about the nature of negative distribution, an uncertainty metrics is developed to measure the uncertainty associated with negative probability distributions [3]. A new method is also proposed based on Tsallis entropy [4] which will degenerate to Yager’s method in some special cases. In order to verify the new entropy with good performance in uncertainty measurement, a new Pythagoras fuzzy number inversion method is proposed and the new method can be applied in a service supplier selection system [5]. Combining the negative distribution of probability with evidence theory is also a popular application direction. According to this idea, a method for determining the weight of the sum of probability distribution based on MCDM is proposed recently [6]. Deng and Jiang proposed a negation transformation for belief structure which is based on maximum uncertainty allocation [7]. The negation of probability distribution is also applied to target recognition based on sensor fusion [8]. By introducing Dempster-Shafer theory, the negation of probability is extended to basically change the allocation. Some methods for the negation of BPA are proposed to be seen as a new perspective of uncertainty measure [9] [10]. Some matrix methods for the negation of BPA have also been proposed [11]. Based on the negation of BPA, new methods can be proposed for conflict management of evidence theory [12]. Go a step further, a new method of Z number negation combining probability and ambiguity is also presented based on the reliability of probability transmission [13].

All the previous negation transformations have the similar structure with Yager’s negation. Although all of them have many properties and applications, we attempt to provide a different perspective to represent the negation of the probability distribution. In this article, an exponential negation is proposed, which tries to divide 1 by probability to get a new inverse of the probability distribution. Considering zero probability, we solve this problem in the form of exponent. In short, Yager’s negation can be seen as arithmetic negation and the proposed method can be seen as geometry negation to some degree. Although the form of two transformations is different, the transformation we proposed have the same properties as the transformation of Yager in many fundamental ways. Specially, the two negations are completely different when we have a negation transformation on binary probability distribution. And the result of exponential inversion can better meet our intuitive understanding.

The paper is structured as follows: Section II introduces Yager’s negation and discusses it. The exponential negation is presented in Section III we also investigate some basic properties of the negation. In Section IV we give some numerical examples to support our point of view. The examples that the negative form conforms to the laws of nature is given. At last, we get the conclusion in Section V.
II. PRELIMINARIES

A. Yager’s Negation

Yager proposed a maximum entropy negation of a probability distribution [1]. Suppose the frame of reference is the set $X = \{x_1, x_2, \ldots, x_n\}$, $P = \{p_1, p_2, \ldots, p_n\}$ is a probability distribution on $X$. Yager’s negation can be expressed as follows [1]:

$$\overline{p_i} = \frac{1 - p_i}{n - 1} \quad (1)$$

Obviously this negation satisfies some basic properties of probability:

$$\sum_{i} \overline{p_i} = 1 \quad (2)$$
$$\overline{p_i} \in [0, 1] \quad (3)$$

For many other inversion operations like real number’s negation or the negation of a matrix, the final result which has been negative twice is equal to the initial one. However, Yager’s negation is not involutionary.

$$\overline{\overline{p_i}} \neq p_i \quad (4)$$

In the further research, Yager indicated the reason of this unusual property. Yager’s negation operation will increase the entropy of a distribution. The entropy of the two formulas:

$$H(P) = \sum_{i} (1 - p_i)(p_i) = 1 - \sum_{i} p_i^2 \quad (5)$$
$$H(\overline{P}) = \sum_{i} (1 - \overline{p_i})(\overline{p_i}) = 1 - \sum_{i} \overline{p_i}^2 \quad (6)$$

Then the entropy increasing in this negation process is shown as follows:

$$H(\overline{P}) - H(P) = \sum_{i} \overline{p_i}^2 - \sum_{i} p_i^2$$
$$= \sum_{i} \overline{p_i}^2 - \frac{1}{n(n-1)^2} \sum_{i} (1 - 2p_i + p_i^2) \quad (7)$$
$$= \frac{(n-2)}{(n-1)^2} (n \sum_{i} p_i^2 - 1)$$

We can easily find that Eq.(7) is positive and the negation is an entropy increasing operation.

III. EXPONENTIAL NEGATION

In this section, we propose a new method of negation of a probability distribution and discuss some necessary and efficacious properties of it.

Suppose there is a set of random variables are presented as $X = \{x_1, x_2, \ldots, x_n\}$. It is a mathematical abstract representation of event $A = \{A_1, A_2, \ldots, A_n\}$. And let $P = \{p_1, p_2, \ldots, p_n\}$ as a probability distribution on $X$. We can easily get some information about event set $A$ from the probability distribution $P$. $p_i$ represents the probability of occurrence of event $A_i$ and each $p_i$ corresponds to an event.

From the fundamental character of a probability distribution, it is easy to get the following two results: $1. \sum p_i = 1$ and $2. p_i \in [0, 1]$. Then, we use the distribution $\overline{P} = \{\overline{p_1}, \overline{p_2}, \ldots, \overline{p_n}\}$ to describe the negation of distribution $P$. $\overline{p_i}$ indicates the information or knowledge of ‘not $A_i$’. And each negation probability $\overline{p_i}$ has a corresponding ‘not $A_i$’.

**Definition III.1.** Given a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ on $X = \{x_1, x_2, \ldots, x_n\}$, the corresponding exponential negation of the probability distribution is defined as follows:

$$\overline{p_i} = e^{-p_i} \quad (8)$$

$$\sum_{i=1}^{n} e^{-p_i} = (\sum_{i=1}^{n} e^{-p_i})^{-1} e^{-p_i} = A e^{-p_i}$$

$A$ is a normal number, shown as follows:

$$A = (\sum_{i=1}^{n} e^{-p_i})^{-1} \quad (9)$$

$\overline{P}$ is still a probability distribution and satisfied as following conditions.

$$\sum_{i} \overline{p_i} = 1 \quad (10)$$

Here we have some examples to help understand.

**Example III.1.**

$P : p_1 = 1$

$$p_i = 0, (i \neq 1, i = 2, 3, \ldots, n)$$

$\overline{P} : \overline{p_i} = (e^{-1} + n - 1)^{-1} e^{-1}$

$$\overline{p_1} = (e^{-1} + n - 1)^{-1}, (i \neq 1, i = 2, 3, \ldots, n)$$

Considering $n = 2$, we can get the result as:

$$\overline{p_1} = (e^{-1} + 1)^{-1} e^{-1} = 0.2689$$
$$\overline{p_2} = (e^{-1} + 1)^{-1} = 0.7311, (i \neq 1, i = 2, 3, \ldots, n)$$

Specially, when $n = 2$, $p_1 = 1$, $p_2 = 0$, this distribution is a very special one. This binary distribution means a certain system which gets the minimum entropy and maximum information. Performing an iterative negation operation on this system will increase its entropy and converge to a uniform distribution. This is an excellent result following the laws of nature, we will explain it in detail later.

**Example III.2.**

$P : p_1 = p_2 = \frac{1}{2}$

$$p_i = 0, (i \neq 1, 2, i = 3, \ldots, n)$$

$\overline{P} : \overline{p_1} = p_2 = (2e^{-1} + n - 2)^{-1} e^{-\frac{3}{2}}$

$$\overline{p_i} = (2e^{-1} + n - 2)^{-1}, (i \neq 1, i = 2, 3, \ldots, n)$$
Considering \( n = 2 \), we can get the result as:

\[
P : p_1 = p_2 = \frac{1}{2}
\]

\[
\overline{P} : \overline{p_1} = p_2 = (2e^{-1})^{-1}e^{-1/2} = \frac{1}{2}
\]

The negation of \( P \) is also \( P \). From this result, we can guess that the uniform distribution is a fix point of the negation. We will use the entropy theory to explain this. The specific proof and discussion will also be carried out in detail.

Furthermore, the proposed exponential negation also has the property of order reversal. If \( p_i > p_j \), we can find that \( \overline{p}_i < \overline{p}_j \) from the definition. Intuitively, it is obvious to get such a result. If the probability of event \( A_1 \) is greater than event \( A_2 \), of course that the probability of ’not \( A_1 \)’ is smaller than ’not \( A_2 \)’ from experience.

Also, we note that this negation is not involutory, whereas Yager’s negation \([1]\) and Heyting intuitionistic logic \([14]\) is also not involutory.

\[
\overline{p}_i = A_1e^{P_i} = (\sum_i e^{-P_i})^{-1}e^{-p_i}
\]

\[
= (\sum_i e^{-p_i - P_i})^{-1}e^{-p_i}
\]

\[
\ne p_i
\]

This property can be understood by the previous examples, take the binary distribution we mentioned in Example 3.2 the distribution will converge to a uniform distribution. If the operation is involutory, we will never get the uniform distribution in this example.

Then, we figure out that uniform probability distribution is a fix point of the exponential negation, and we will use Shannon entropy \([15]\) to explain it. Also, we will give some numerical examples to show the details and help understand.

Considering a uniform probability distribution: \( P = \{p_i|p_i = \frac{1}{n}, (i = 1, 2, ..., n)\} \). By definition of the proposed exponential negation, we can calculate the negation distribution as:

\[
\overline{p}_i = e^{-p_i}/\sum_i e^{-p_i} = e^\frac{1}{n} = \frac{1}{n}
\]

So, we can get the negative distribution as: \( \overline{P} = \{\overline{p}_i|\overline{p}_i = \frac{1}{n}, (i = 1, 2, ..., n)\} \). From the first perspective, we can easily note that when \( p_1 = p_2 = ... = p_i = p_j = ... = p_n \), there is no doubt that \( \overline{P}_i = \overline{P}_2 = ... = \overline{P}_i = \overline{P}_j = ... = \overline{P}_n \) by the definition. And following the basic character, \( \sum_i \overline{P}_i = 1 \), so the negation of uniform distribution is also the same uniform distribution.

From the second perspective, we explain it by Shannon entropy \([15]\). Shannon’s entropy is defined as follows:

\[
H(P) = \sum_i p_i\ln(p_i)
\]

\( H(P) \) is a significant measurement for information of a probability distribution. For a probability distribution we obtained, the greater entropy means the less information. It is easy to prove that a uniform probability distribution has the maximum entropy in all distributions.

The proposed exponential negation is an entropy increase operation is proven mathematically as follows:

\[
H(P) = -\sum_i p_i\ln(p_i)
\]

\[
H(\overline{P}) = -\sum_i Ae^{-P_i}\ln(Ae^{-P_i})
\]

\[
H(P) - H(\overline{P}) = -\sum_i Ae^{-P_i}\ln(Ae^{-P_i}) + \sum_i p_i\ln(p_i)
\]

\[
= \sum_i [-Ae^{-P_i}\ln(Ae^{-P_i}) + p_i\ln(p_i)]
\]

\[
= \sum_i (-\sum_i e^{-P_i})^{-1}e^{-P_i}\ln((-\sum_i e^{-P_i})^{-1}e^{-P_i})
\]

\[
+ p_i\ln(p_i)
\]

\[
= \sum_i (\sum_i e^{-P_i})^{-1}e^{-P_i}(\ln(\sum_i e^{-P_i} + p_i) + p_i\ln p_i)
\]

\[
\geq 0
\]

Every added element in Eq. (14) is positive, the result which is always positive or equals to zero shows that every negative distribution will never have smaller entropy than the original one. When \( p_i = \frac{1}{n} \), \( H(\overline{P}) - H(P) = 0 \), the uniform distribution has the maximum entropy already, the entropy can’t increase anymore. So, after the negation operation, it will stay at the maximum and become the fix point of our negation.

This is also a strong evidence to show that our negation could reflect some essences of the real world. The second law of thermodynamics \([16]\) claims a theory that for an isolated system, it always changes in the direction of entropy increase. Our negation operation is consistent with this objective law.

IV. NUMERAL EXAMPLES

According to above reasoning, we can easily infer that after several negation iterations of entropy increase, the distribution will approach the fix point. In another perspective, all the probability distribution will converge to uniform distribution. In this section, we have three numeral examples to show this result, we give some figures and tables to show the details. Some same experiments for Yager’s negation is also shown to compare.

In all tables, \( P_i \) means the distribution, \( n \) means the number of negation iteration. Suppose three representative distribution for numerical calculation.

a. \( P_1 = \{p_{31}, p_{32}\} = \{0, 1\} \)

b. \( P_2 = \{p_{11}, p_{12}, p_{13}\} = \{0.1, 0.4, 0.5\} \)

c. \( P_3 = \{p_{21}, p_{22}, p_{23}, p_{24}, p_{25}\} = \{0.1, 0.13, 0.17, 0.3, 0.4\} \)

| TABLE I: \( P_i \) exponential negative iteration results | 1 |
|---|---|---|---|---|---|
| \( P_1 \) | 0 | 1 | 2 | 3 | 4 | 5 |
| \( p_{11} \) | 0.731 | 0.386 | 0.557 | 0.472 | 0.514 | |
| \( p_{12} \) | 1 | 0.269 | 0.614 | 0.443 | 0.528 | 0.486 |
TABLE II: $P_1$ exponential negative iteration results

| $P$ | $n$ | 6 | 7 | 8 | 9 | 10 |
|-----|-----|---|---|---|---|----|
| $p_{11}$ | 0.493 | 0.504 | 0.498 | 0.501 | 0.500 |
| $p_{12}$ | 0.507 | 0.496 | 0.502 | 0.499 | 0.500 |

Fig. 1: Change of probability distribution $P_1$ with the number of exponential negative iterations

TABLE III: $P_1$ Yager’s negative iteration results

| $P$ | $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|-----|---|---|---|---|---|----|
| $p_{11}$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $p_{12}$ | 1 | 0 | 1 | 0 | 1 | 0 |

Fig. 2: Change of probability distribution $P_1$ with the number of Yager’s negative iterations

For Yager’s negation, a special result appears in this special case because $\overline{p_{11}} = 1 - p_{21}$ and $\overline{p_{12}} = 1 - p_{22}$, no matter how many iterations, the distribution will always alternate between these two distributions which both have minimum entropy. In this distribution, Yager’s inverse operation will no longer cause entropy increase. However, Figure 2 shows that the distribution will finally converge to the uniform distribution. The exponential negation is an entropy increasing operation in this special case.

TABLE IV: $P_2$ exponential negative iterations results

| $P$ | $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|-----|---|---|---|---|---|---|----|
| $p_{21}$ | 0.100 | 0.414 | 0.306 | 0.342 | 0.330 | 0.334 | 0.333 |
| $p_{22}$ | 0.400 | 0.307 | 0.342 | 0.331 | 0.334 | 0.333 | 0.333 |
| $p_{23}$ | 0.500 | 0.278 | 0.352 | 0.327 | 0.335 | 0.332 | 0.333 |

Fig. 3: Change of probability distribution $P_2$ with the number of exponential negative iterations

TABLE V: $P_2$ Yager’s negative iterations results

| $P$ | $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|-----|---|---|---|---|---|---|----|
| $p_{21}$ | 0.100 | 0.450 | 0.275 | 0.363 | 0.319 | 0.341 | 0.330 |
| $p_{22}$ | 0.400 | 0.300 | 0.350 | 0.325 | 0.338 | 0.331 | 0.334 |
| $p_{23}$ | 0.500 | 0.250 | 0.375 | 0.313 | 0.344 | 0.328 | 0.336 |

Fig. 4: Change of probability distribution $P_2$ with the number of Yager’s negative iterations

In this example, both can converge to a uniform distribution. However, the convergence speed of Yager’s negation is obviously slower than the convergence speed of our proposed inverse operation. When the precision reaches three decimal places, the tenth iteration of Yager’s negation will converge to the uniform distribution, and our inverse operation will converge to the uniform distribution of this precision after six iterations.
TABLE VI: \( P_3 \) exponential negative iteration results

| \( P_3 \) | \( n \) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| \( p_{31} \) | 0.100 | 0.224 | 0.195 | 0.201 | 0.200 | 0.200 |
| \( p_{32} \) | 0.130 | 0.218 | 0.197 | 0.201 | 0.200 | 0.200 |
| \( p_{33} \) | 0.170 | 0.209 | 0.198 | 0.200 | 0.200 | 0.200 |
| \( p_{34} \) | 0.300 | 0.184 | 0.203 | 0.199 | 0.200 | 0.200 |
| \( p_{35} \) | 0.400 | 0.166 | 0.207 | 0.199 | 0.200 | 0.200 |

Fig. 5: Change of probability distribution \( P_3 \) with the number of exponential negative iterations

TABLE VII: \( P_3 \) Yager’s negative iteration results

| \( P_3 \) | \( n \) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| \( p_{31} \) | 0.100 | 0.225 | 0.194 | 0.202 | 0.200 | 0.200 |
| \( p_{32} \) | 0.130 | 0.218 | 0.196 | 0.201 | 0.200 | 0.200 |
| \( p_{33} \) | 0.170 | 0.208 | 0.198 | 0.200 | 0.200 | 0.200 |
| \( p_{34} \) | 0.300 | 0.175 | 0.206 | 0.198 | 0.200 | 0.200 |
| \( p_{35} \) | 0.400 | 0.150 | 0.213 | 0.197 | 0.201 | 0.200 |

Fig. 6: Change of probability distribution \( P_3 \) with the number of Yager’s negative iterations

It can be seen from the Figure 5 and 6 that the convergence images of the two negations are very similar which means the convergence process of the two is similar. It can be further found from the Table VII and Table VI that in this distribution, in the case of three bits, they all converge to a uniform distribution in the fourth iteration. But after comparing more accurate values, we found that Yager’s negation converged to a uniform distribution at the 10th time, and the exponential negation converged to a uniform distribution at the 8th iteration.

In general, the more elements in the distribution, the more similar the convergence process of the two negations, but for the special distribution \( P_1 \), the situation will become completely different. The entropy in the proposed exponential negation will converge to a uniform distribution in this special binary case which is common in the real world. This property means that our proposed mathematical model can better reflect the actual situation.

We can see that all the examples converge to the uniform distribution with our exponential negation. Combine our reasoning and numerical analysis, we can believe that all distributions will converge to uniform distribution. Also, we note the numbers of iteration are different, the more elements the distribution has, the smaller number is required to converge to uniform distribution. \( P_1 \) is required the biggest number in the three examples, we can infer that it also needs the biggest one in all distributions. It is facile to understand this result by entropy. As we mentioned above, the entropy of \( P_1 \) is \( H(P) = \sum_i p_{1i} \ln(p_{1i}) = 0 \), \( H(P) > 0 \), \( P_3 \) has the minimum entropy in all distribution. So it needs to do more negation calculations to reach the maximum entropy.

V. CONCLUSION

In this article, a new negation method, called as exponential negation, is presented. The proposed exponential negation has many desirable properties. For example, it is illustrated that all the probability distributions will converge to a uniform distribution after multiple negation iterations. In addition, it can still converge very well even in the special binary situations. Finally, it coincides with the second law of thermodynamics due to its entropy increase process.

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REFERENCES

[1] R. R. Yager, On the maximum entropy negation of a probability distribution, IEEE Transactions on Fuzzy Systems 23 (5) (2014) 1899–1902.
[2] A. Srivastava, S. Maheshwari, Some new properties of negation of a probability distribution, International Journal of Intelligent Systems 33 (6) (2018) 1133–1145.
[3] A. Srivastava, L. Kaur, Uncertainty and negation—information theoretic applications, International Journal of Intelligent Systems 34 (6) (2019) 1248–1260.
[4] J. Zhang, R. Liu, J. Zhang, B. Kang, Extension of yager’s negation of a probability distribution based on tsallis entropy, International Journal of Intelligent Systems 35 (1) (2020) 72–84.
[5] H. Mao, R. Cai, Negation of pythagorean fuzzy number based on a new uncertainty measure applied in a service supplier selection system, Entropy 22 (2) (2020) 195.
[6] C. Sun, S. Li, Y. Deng, Determining weights in multi-criteria decision making based on negation of probability distribution under uncertain environment, Mathematics 8 (2) (2020) 191.
[7] X. Deng, W. Jiang, On the negation of a dempster–shafer belief structure based on maximum uncertainty allocation, Information Sciences 516 (2020) 346–352.
[8] X. Gao, Y. Deng, The generalization negation of probability distribution and its application in target recognition based on sensor fusion, International Journal of Distributed Sensor Networks 15 (5) (2019) 1550147719849381.
[9] L. Yin, X. Deng, Y. Deng, The negation of a basic probability assignment, IEEE Transactions on Fuzzy Systems 27 (1) (2018) 135–143.
[10] K. Xie, F. Xiao, Negation of belief function based on the total uncertainty measure, Entropy 21 (1) (2019) 73.
[11] Z. Luo, Y. Deng, A matrix method of basic belief assignment’s negation in dempster-shafer theory, IEEE Transactions on Fuzzy Systems (2019).
[12] S. Li, F. Xiao, J. H. Abawajy, Conflict management of evidence theory based on belief entropy and negation, IEEE Access 8 (2020) 37766–37774.
[13] Q. Liu, H. Cui, Y. Tian, B. Kang, On the negation of discrete z-numbers, Information Sciences (2020).
[14] A. Heyting, Intuitionism: an introduction, Vol. 41, Elsevier, 1966.
[15] C. E. Shannon, A mathematical theory of communication, The Bell system technical journal 27 (3) (1948) 379–423.
[16] H. B. Callen, Thermodynamics and an introduction to thermostatics (1998).