Can a circulating light beam produce a time machine?

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Abstract

In a recent paper, Mallett found a solution of the Einstein equations in which closed timelike curves (CTC’s) are present in the empty space outside an infinitely long cylinder of light moving in circular paths around an axis. Here we show that, for physically realistic energy densities, the CTC’s occur at distances from the axis greater than the radius of the visible universe by an immense factor. We then show that Mallett’s solution has a curvature singularity on the axis, even in the case where the intensity of the light vanishes. Thus it is not the solution one would get by starting with Minkowski space and establishing a cylinder of light.

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Mallett, in a recent article [1], wrote down a static, cylindrically symmetric solution to the Einstein field equations of General Relativity in the space surrounding an infinitely long axially symmetric circulating cylinder of light with radiation energy density $\epsilon$. We take the $z$-axis to be the symmetry axis of the cylinder. The solution is intended to approximate the solution in the physically meaningful case of a light beam carried by a wave guide of finite length spiraling around the $z$-axis. Obviously one could at best hope the approximation to be valid for $\rho < L$ where $L$ is the length of the spiral along the $z$-axis and $\rho$ is the radial distance from the axis of the cylinder, which we take to be equal to the radius of the cylindrical region containing the spiral. The solution obtained in [1] has the remarkable property that in the region exterior to the cylinder a plane of constant $z$ contains circular closed timelike curves (CTC’s) enclosing the cylinder of all radii $\rho > \rho_{\text{min}} > \rho_0$, where $\rho_0$ is the radius of the cylinder. The CTC’s occur even if $\epsilon$ becomes arbitrarily small, although in the limit $\epsilon \to 0$, $\rho_{\text{min}}$, the minimum radius of the CTC’s, $\to \infty$. This result seems unphysical, since it is difficult to believe that a gravitational source of arbitrarily small mass could distort space so drastically as to cause the formation of CTC’s.

We first summarize the argument in [1]. (Units are chosen such that $G = c = 1$, where $G$ is the gravitational constant.) Mallett begins with an axially symmetric metric of the canonical form

$$ds^2 = fdt^2 - 2wrd\phi - l\phi^2 - e^\nu(d\rho^2 + dz^2)$$ (I-5)

[Throughout, equation number I-n refers to Eq. (n) in [1].] As a source in the Einstein equation he inserts the energy momentum tensor of an axially circulating light beam

$$T_{\mu\nu} = \epsilon\eta_{\mu}\eta_{\nu}$$ (I-2)

where $\epsilon$ is the radiation energy density,

$$(x^0, x^1, x^2, x^3) = (t, \rho, z, \phi)$$ (1)

$$\eta_{\mu}\eta^{\mu} = 0$$ (I-3)

and

$$\eta_{\mu} = (\eta_0, 0, 0, \eta_3)$$ (I-4)

(Since $z$-independence is assumed, what one actually is talking about is an infinitely long cylindrical surface carrying a circulating beam described by its energy/unit length, $\mu$.)

Mallett then writes the Einstein equation corresponding to (I-2) and (I-3), obtaining three different component equations (I-12)–(I-14). From these he obtains the constraint equation

$$\partial^2 \Delta / \partial \rho^2 = 0$$ (I-17)

where

$$\Delta^2 = fl + w^2$$ (I-8)

Equation (I-17) has the solutions $\Delta = \rho$ and $\Delta = \text{constant}$. Mallett chooses the solution $\Delta = \rho$ which he says simplifies the field equations. He does not discuss this, but, from Eq. (I-8) in the limit $w = 0$, this is consistent with the usual Minkowski metric $f = 1$ and $l = \rho^2$, with $\phi$ the usual angular coordinate. The other choice would lead to $f = l = \text{constant}$ and

$$x_3 = \rho \phi = s$$ (2)
a linear coordinate equal to the arc length \( s \) along a circle of radius \( \rho \).

Next Mallett from (I-2), (I-3), and (I-4) obtains the result

\[ \eta^0 = \xi \eta^3 \]  

(I-19)

where

\[ \xi = (w + \rho)/f \]  

(I-20)

(The \( \rho \) in (I-20) is actually \( \Delta \) and would become a constant if one took \( \Delta = \) constant and \( x^3 = s \).) He then proceeds to solve the field equations using the ansatz \( \xi = \eta^0/\eta^3 = \) constant. This would describe the usual situation for an electromagnetic wave if \( \Delta = \) constant, so that \( \eta^3 \) represented the density of the linear momentum \( p^3 \) in the \( x^3 \)-direction. However, if \( \phi \) is an angular variable, the choice of \( \Delta = \rho \) means that, by analogy with Eq. (2), \( \eta^3 \) is the density of \( p^3/\rho \), and since the energy and momentum densities in an electromagnetic wave are equal, this would lead to \( \xi = \eta^0/\eta^3 = \rho \) for values of \( \rho \) for which the radiation density is nonzero. Hence the combined ansatz \( \Delta = \rho \) and \( \xi = \) constant implies that the coordinates do not have their conventional meaning. Thus, e. g., the metric obtained in [1] in the limit \( \epsilon = w = 0 \) has, from Eq. (I-20), the metric component \( f = g_{00} \) proportional to \( \rho \) rather than the expected Minkowski result \( f = -1 \). One’s first guess might be that the metric obtained in [1] from the ansatz \( \xi = \) constant differs from the metric of conventional cylindrical coordinates only by a coordinate transformation. We will, however, see presently that the situation is more complex than that.

Mallett proceeds by obtaining from the field equations in the region outside the light beam, where the energy-momentum tensor \( T_{\mu\nu} = 0 \), the differential equation

\[ \rho^2 \partial^2 w/\partial \rho^2 - \rho \partial w/\partial \rho + w = 0 \]  

(I-32)

This linear equation has the general solution \( w = c_1 \rho + c_2 \rho \ln \rho \). For the present case of interest this is rewritten in [1] as

\[ w = \lambda \rho \ln(\rho/\alpha) \]  

(I-33)

with \( \lambda \) and \( -\lambda \ln \alpha \) thus playing the role of the two arbitrary constants in the solution of (I-32). The parameter \( \lambda \) is an unspecified dimensionless constant proportional to \( \mu \), the radiation energy per unit length, and \( \alpha \) is an unspecified length. The order of magnitude of both of these constants can be obtained on dimensional grounds. The constant \( \lambda \) must depend on \( G \) as well as \( \epsilon \), and hence we expect \( \lambda \) to be of the order of the dimensionless quantity \( G \mu/c^4 \) (= \( \mu \) in our units); i. e., \( \lambda \approx \) the energy per unit length in the natural units of Planck energy per Planck length, where \( m_p c^2 = c^2 \sqrt{\hbar c/G} \approx 10^{19} \text{GeV} \approx 10^9 \text{J} \), and \( L_P = Gm_P/c^2 \approx 10^{-35} \text{m} \). As for \( \alpha \), in the case of an infinite cylinder, the only relevant geometrical quantity with the dimensions of length is \( \rho_0 \), the radius of the light cylinder, so we expect \( \alpha \approx \rho_0 \).

Taking \( w \) to be given by (I-33) outside of the circulating light beam, i. e., for \( \rho \geq \alpha \), Mallett obtains from the field equations together with the constraint \( \Delta^2 = \rho^2 \)

\[ l = \rho \alpha [1 - \lambda \ln(\rho/\alpha)] \]  

(I-36)

Eq. (I-36) implies that, for sufficiently large \( \rho \), \( l \) becomes negative and hence \( \phi \) becomes a timelike coordinate, so that traversing a circle at sufficiently large \( \rho \) allows one to traverse
a CTC and return to the same point in space at an earlier time. Thus one has a time
machine. Note that the Cauchy horizon enclosing the region containing CTC’s extends
to infinity and is thus not compactly generated. Thus the theorems of Tipler and
Hawking requiring violation of the weak energy condition can be evaded, and we have
CTC’s without the presence of regions of negative energy density. (These theorems would,
however, rule out the creation of CTC’s in any finite-size approximation to this spacetime.)

However, even accepting the results of [1], there is a serious practical problem. CTC’s
occur for regions where ln(ρ/α) > 1/λ, where α ∼ ρ₀, the radius of the circulating light
beam, and λ ∼ µ. Suppose the circulating light beam comes from the light from a laser of
average power P and beam radius r fed into a light pipe of the same diameter wound in a
tight spiral of radius ρ₀ around the z-axis. The energy density in the laser beam is thus

\[ \epsilon = \frac{P}{(\pi r^2 c)}, \]

Per unit length is given by

\[ \mu = \frac{\pi P \rho_0}{cr}. \]  \hspace{1cm} (3)

Let us take P = 1kW, ρ₀ = 0.5m and r = 1mm. (The values of P and ρ₀ appear relatively
realistic if perhaps a bit optimistic. But, as we will see, the specific numerical values are
essentially irrelevant, and could be changed by many orders of magnitude without altering
the conclusions.). Putting λ = Gµ/c^4, we obtain λ = πGPρ₀/(c^5r), which is of order 10^{-46}. Thus, from (I-36), CTC’s would only occur for ln(ρ/α) > 10^{46}, or

\[ \rho > 10^{(10^{46})}\rho_0. \] \hspace{1cm} (4)

So, because of the logarithmic dependence in (I-36), CTC’s are predicted to occur only out-
side a region whose radius is so fantastically large that it cannot even be sensibly compared
to the radius of the visible universe

To be more realistic, one should remember that, in the physically interesting case of a
cylinder of circulating light of finite length L the equations can at most be relevant only
for ρ < L. The prediction of CTC’s certainly cannot be regarded as reliable if they do not
occur in that range. For this to happen, the energy/unit length of the light beam in natural
units would be 1/N, where L/ρ₀ = N > 1, meaning the intensity of the light beam would
have to be of the order of that required to form a black hole unless the ratio of the length to
the width of the apparatus is huge. Note that the numerical value of µ/c^2, from Eq. (3) and
the numbers given above, is about 10^{-19} kg/m, emphasizing the extremely small amount of
mass in the rotating light cylinder in a realistic case, and how unexpected it would be for
such an object to give rise to any noticeable distortion, much less the production of CTC’s,
in the surrounding space.

From the above discussion there is no practical possibility of using an apparatus with a
circulating light beam to build a terrestrial time machine. However, the discussion in the
preceding paragraph might suggest that such an apparatus could, in principle, lead to the
production of CTC’s. If true, this would be of considerable significance, since the question
as to whether CTC’s can ever be produced, even in principle, is of fundamental importance.
Therefore we should examine further what light, if any, is shed on this question by [1].

Unfortunately, it appears that the metric of [1] is not the metric that one would get by
starting from Minkowski space and establishing a circulating cylinder of light. It is true that
it is almost everywhere a solution to Einstein’s equations with Eq. (12) as a source, but at
the origin ρ = 0 there is a line singularity. For example, the trtr component of the Riemann
tensor is
\[ R^t_{rr} = \frac{1}{8\rho^2} \]  
and so diverges at the origin. This is not a coordinate artifact, as we can see by taking the scalar
\[ R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{3}{4\alpha\rho^3} \]
which is also divergent. Worse yet, this divergence has no dependence on \( \lambda \), and so persists if one takes the limit in which the source intensity \( \epsilon \) goes to 0.

Thus it appears that the metric of \[1\] describes a cylinder of light circulating in a spacetime which is pathological even without the light. It is thus unlike the spacetime of van Stockum \[5\] studied by Tipler \[6\], which does go to Minkowski space as the source is removed.

Setting \( \epsilon = 0 \) and consequently \( \lambda = 0 \), we have the metric
\[ ds^2 = \left( \frac{\rho}{\alpha} \right) dt^2 - \rho d\phi^2 - \sqrt{\frac{\alpha}{\rho}} (dp^2 + dz^2) \]
It is straightforward to compute the connection and curvature from this metric and demonstrate the following properties.

1. Except for \( \rho = 0 \), it is a vacuum solution of Einstein’s equations. However, it is not flat anywhere, and it has a curvature singularity at \( \rho = 0 \).

2. The paths \( \rho = \text{constant}, \ z = \text{constant}, \ d\phi/dt = 1/\alpha \) are null geodesics, so the light does not require any external apparatus to keep it in circulation; the photonic crystals discussed in \[1\] would not be necessary. (In the van Stockum/Tipler spacetime \[5, 6\], the dust is kept in circulation by its gravitational attraction, but in the present case the light is circulating on the geodesics of the background metric, i. e., it is in orbit around the singularity.)

3. The length of the circle \( z = \text{constant}, \ \rho = \text{constant} \) is \( 2\pi \sqrt{\rho\alpha} \), but a photon can traverse this loop in either direction in time \( 2\alpha \) independent of \( \rho \). Thus at large distances the elapsed time is arbitrarily small as compared to the distance traveled. Therefore it is not surprising that a small modification of this metric yields CTC’s.

Thus it appears that the closed timelike curves appearing in \[1\] are the result of starting with a pathological spacetime instead of Minkowski space. There is no reason to believe on the basis of \[1\] that CTC’s could be produced in the laboratory, even if we had sufficient technology to control a density of electromagnetic radiation so large as to have measurable gravitational effects.

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