New bounds and efficient algorithm for sparse difference resultant

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Abstract: Let \( \mathbb{P} = \{ \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_n \} \) be a generic Laurent transformally essential system and \( \mathbb{P}_T = \{ \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_m \} (m \leq n) \) be its super essential system. We show that the sparse difference resultant of a simplified system of \( \mathbb{P}_T \) by setting the selected \( n - m \) variables to one is the same to the one of \( \mathbb{P} \). Moreover, new order bounds of sparse difference resultant are obtained. Then we propose an efficient algorithm to compute sparse difference resultant which is the quotient of two determinants whose elements are the coefficients of the polynomials in the strong essential system. We analyze complexity of the algorithm. Experimental results show the efficiency of the algorithm.

Keywords: Sparse difference resultant, Laurent transformally essential system, Sparse algebraic resultant

1 Introduction

It is well-known that the resultant, as a basic concept in algebraic geometry and a powerful tool in elimination theory, gives conditions for an over-determined system of polynomial equations to have common solutions \(^1\). Since most polynomials are sparse in that they only contain certain fixed monomials, Gelfand, Kapranov, Sturmfels, and Zelevinsky introduced the concept of sparse resultant \(^5\). Later various effective algorithms are proposed to compute sparse resultant. In particular, Canny and Emiris showed that the sparse resultant is a factor of the determinant of a Macaulay style matrix and gave an efficient algorithm to compute the sparse resultant based on this matrix representation \(^7\). Andrea further showed the sparse resultant is the quotient of two determinants where the denominator is a minor of the numerator \(^2\).

With the resultant and sparse resultant theories becoming more mature, extending the algebraic results to differential and difference cases is a natural way. However, such results in differential and difference cases are not complete parallel with algebraic case, even not hold. Since our paper concentrates on the difference case, though differential resultant and sparse differential resultant are studied successively by many researchers, we will not state them in detail and refer to \(^11\)\(^13\) and references therein.

For the difference case, Li, Yuan and Gao introduced the concept of sparse difference resultant for a Laurent transformally essential system consisting of \( n + 1 \) Laurent difference polynomials in \( n \) difference variables and its basic properties are proved \(^4\). Based on the degree and order bound, they proposed a single exponential algorithm in terms of the number of variables, the Jacobi number, and the size of the Laurent transformally essential system, which is essentially to search for sparse difference resultant with order and degree bound.

In this paper, we further explore efficient algorithms to find the sparse difference resultant of the given difference polynomial system. We show that the sparse difference resultant of a Laurent transformally essential system consisting of \( n + 1 \) Laurent difference polynomials in \( n \) difference variables is the same to the one of a simple system consisting of \( m + 1 \) polynomials.

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in \( m \) difference variables, where \( m \) is the rank of the symbolic support matrix of the super essential system. Moreover, new order bound of sparse difference resultant is given. Then we propose an efficient algorithm to compute sparse difference resultant, which is based on the results that sparse difference resultant is shown to be the algebraic sparse resultant for a certain generic polynomial system. It starts with the given sparse difference polynomial system and directly obtain a strong essential polynomial system of the original system, then one can regard it as sparse algebraic polynomial system and use the algorithm in \([2]\) to construct the matrix representation whose determinant is the required sparse difference resultant. Furthermore, the whole computations of the algorithm are compiled to the function \textbf{SDResultant} implemented with Mathematica.

The rest of the paper is arranged as follows. In Section 2, we review some preliminary results which contains definitions and theorems of sparse resultant and sparse difference resultant. Section 3 concentrates on the main results of the paper involving the theoretical preparation of the algorithm, algorithm implementation and an illustrated example. The last section concludes the results.

## 2 Preliminaries

### 2.1 Sparse resultant

We first introduce several basic notions and properties on sparse resultant which are needed in the algorithm. We refer to \([4,5]\) for more details.

Let \( \mathcal{B}_1, \ldots, \mathcal{B}_n \) be finite subsets of \( \mathbb{Z}^n \). Assume \( \mathbf{0} \in \mathcal{B}_i \) and \( |\mathcal{B}_i| \geq 2 \) for each \( i \). For algebraic indeterminates \( \mathbb{X} = \{x_1, \ldots, x_n\} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), denote \( \mathbb{X}^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \). Let

\[
F_i(x_1, \ldots, x_n) = c_{i0} + \sum_{\alpha \in \mathcal{B}_i \setminus \{ \mathbf{0} \}} c_{i\alpha} \mathbb{X}^\alpha \quad (i = 0, \ldots, n) \tag{1}
\]

be generic sparse Laurent polynomials. We call \( \mathcal{B}_i \) the support of \( F_i \) and \( \omega_i = \sum_{\alpha \in \mathcal{B}_i} c_{i\alpha} \alpha \) is called the\textit{ symbolic support vector} of \( F_i \). The smallest convex subset of \( \mathbb{R}^n \) containing \( \mathcal{B}_i \) is called the \textit{Newton polytope} of \( F_i \). For any subset \( I \subseteq \{0, \ldots, n\} \), the matrix \( M_I \) whose row vectors are \( \omega_i \) (\( i \in I \)) is called the \textit{symbolic support matrix} of \( \{F_i : i \in I\} \). Denote \( c_i = (c_{i\alpha})_{\alpha \in \mathcal{B}_i} \), \( c_I = \bigcup_{i \in I}(c_i) \) and by \( \text{rk}(M_I) \) the rank of matrix \( M_I \).

**Definition 2.1.** Follow the notations introduced above.

- A collection of \( \{F_i\}_{i \in I} \) is said to be weak essential if \( \text{rk}(M_I) = |I| - 1 \).
- A collection of \( \{F_i\}_{i \in I} \) is said to be essential if \( \text{rk}(M_I) = |I| - 1 \) and for each proper subset \( J \) of \( I \), \( \text{rk}(M_J) = |J| \).

A polynomial system \( \{F_i\}_{i \in I} \) is weak essential if and only if \( (F_i : i \in I) \cap \mathbb{Q}[c_I] \) is of codimension one. In this case, there exists an irreducible polynomial \( \mathbf{R} \in \mathbb{Q}[c_I] \) such that \( (F_i : i \in I) \cap \mathbb{Q}[c_I] = (\mathbf{R}) \) and \( \mathbf{R} \) is called the \textit{sparse resultant} of \( \{F_i : i \in I\} \). Furthermore, the system \( \{F_i\}_{i \in I} \) is essential if and only if \( (F_i : i \in I) \cap \mathbb{Q}[c_I] = (\mathbf{R}) \) and \( \mathbf{R} \) appears effectively in \( \mathbf{R} \) for each \( i \in I \).

Suppose an arbitrary total ordering of \( \{F_0, \ldots, F_n\} \) is given, say \( F_0 < F_1 < \cdots < F_n \). Now we define a total ordering among subsets of \( \{F_0, \ldots, F_n\} \). For any two subsets \( \mathcal{D} = \{D_0, \ldots, D_s\} \) and \( \mathcal{C} = \{C_0, \ldots, C_t\} \) where \( D_0 > \cdots > D_s \) and \( C_0 > \cdots > C_t \), \( \mathcal{D} \) is said to be of higher ranking than \( \mathcal{C} \), denoted by \( \mathcal{D} > \mathcal{C} \), if 1) there exists an \( i \leq \min(s, t) \) such that \( D_0 = C_0, \ldots, D_{i-1} = C_{i-1}, D_i > C_i \) or 2) \( s > t \) and \( D_i = C_i (i = 0, \ldots, t) \). Note that if \( \mathcal{D} \) is a proper subset of \( \mathcal{C} \), then \( \mathcal{C} > \mathcal{D} \). Thus for the system \( \mathcal{F} = \{F_i : i = 0, \ldots, n\} \) given in \( \{I\} \), if \( \text{rk}(M_\mathcal{F}) \leq n \), then \( \mathcal{F} \) has an essential subset with minimal ranking.
Lemma 2.2. (4) Suppose $\mathbb{F}_I = \{ \mathbb{F}_i : i \in I \}$ is an essential system. Then there exists an $I' \subset \{1, \ldots, n\}$ with $|I'| = n - |I| + 1$, such that by setting $x_i, i \in I'$ to 1, the specialized system $\tilde{\mathbb{F}}_I = \{ \mathbb{F}_i : i \in I \}$ satisfies

(1) $\mathbb{F}_I$ is still essential.

(2) $\text{rk}(M_{\mathbb{F}_I}) = |I| - 1$ is the number of variables in $\tilde{\mathbb{F}}_I$.

(3) $(\mathbb{F}_I) \cap \mathbb{Q}[c] = (\mathbb{F}_I) \cap \mathbb{Q}[c_I]$, where $\mathbb{F}_I = F_i | x_i = 1, i \in I'$.

An essential system $\{\mathbb{F}_i\}_{i \in I}$ is said to be variable-essential if there are only $|I| - 1$ variables appearing effectively in $\mathbb{F}_i$. Clearly, if $\{\mathbb{F}_i : i = 0, \ldots, n\}$ is essential, then it is variable-essential.

Lemma 2.3. (4) Let $\mathbb{F} = \{ \mathbb{F}_i : i = 0, \ldots, n\}$ be an essential system of the form (4). Then we can find an invertible variable transformation $x_1 = \prod_{j=1}^{m_1} z_j^{m_1}, \ldots, x_n = \prod_{j=1}^{m_n} z_j^{m_n}$ such that the image $\mathbb{G}$ of $\mathbb{F}$ under the above transformation is a generic sparse system satisfying

(1) $\mathbb{G}$ is essential.

(2) $\text{Span}_{\mathbb{Z}}(\mathbb{B}) = \mathbb{Z}^n$, where $\mathbb{B}$ is the set of all supports of $\mathbb{G}$.

(3) $(\mathbb{F}) \cap \mathbb{Q}[c] = (\mathbb{G}) \cap \mathbb{Q}[c]$.

We call a variable-essential system $\mathbb{F} = \{ \mathbb{F}_i : i = 0, \ldots, n\}$ strong essential if $\mathbb{F}$ also satisfies condition (2) in Lemma 2.3.

2.2 Sparse difference resultant

This section will review the results associated with sparse difference resultant, for details please refer to reference [4].

An ordinary difference field $\mathcal{F}$ is a field with a third unitary operation $\sigma$ satisfying that for any $a, b \in \mathcal{F}$, $\sigma(a + b) = \sigma(a) + \sigma(b)$, $\sigma(ab) = \sigma(a)\sigma(b)$, and $\sigma(a) = 0$ if and only if $a = 0$. We call $\sigma$ the transforming operator of $\mathcal{F}$. If $a \in \mathcal{F}$, $\sigma(a)$ is called the transform of $a$ and is denoted by $a^{(1)}$. And for $n \in \mathbb{Z}^+$, $\sigma^n(a) = \sigma^{n-1}(\sigma(a))$ is called the $n$-th transform of $a$ and denoted by $a^{(n)}$, with the usual assumption $a^{(0)} = a$. By $a^{[n]}$ we mean the set $\{a, a^{(1)}, \ldots, a^{(n)}\}$. If $\sigma^{-1}(a)$ is defined for each $a \in \mathcal{F}$, $\mathcal{F}$ is reflexive. Every difference field has an inversive closure [3]. In this paper, all difference fields are assumed to be inversive with characteristic zero. A typical example of reflexive difference field is $\mathbb{Q}(x)$ with $\sigma(f(x)) = f(x + 1)$.

A subset $\mathcal{S}$ of a difference extension field $\mathcal{G}$ of $\mathcal{F}$ is said to be transformally dependent over $\mathcal{F}$ if the set $\{\sigma^k a | a \in \mathcal{S}, k \geq 0\}$ is algebraically dependent over $\mathcal{F}$, otherwise, is called transformally independent over $\mathcal{F}$, or a family of difference indeterminates over $\mathcal{F}$. In the case $\mathcal{S}$ consists of one element $\alpha$, we say that $\alpha$ is transformally algebraic or transformally transcendental over $\mathcal{F}$, respectively. The maximal subset $\Omega$ of $\mathcal{G}$ which are transformally independent over $\mathcal{F}$ is said to be a transformal transcendence basis of $\mathcal{G}$ over $\mathcal{F}$. We use $\text{tr.deg.}\mathcal{G}/\mathcal{F}$ to denote the difference transcendence degree of $\mathcal{G}$ over $\mathcal{F}$, which is the cardinal number of $\Omega$.

For every Laurent difference polynomial $F \in \mathcal{F}\{Y, Y^{-1}\}$, there exists a unique Laurent difference monomial $M$ such that $M \cdot F$ is the norm form of $F$, denoted by $N(F)$, which satisfies 1) $M \cdot F \in \mathcal{F}\{Y\}$ and 2) for any Laurent difference monomial $T$ with $T \cdot F \in \mathcal{F}\{Y\}$, $T \cdot F$ is divisible by $M \cdot F$ as polynomials. The order and degree of $N(F)$ is defined to be the order and degree of $F$, denoted by $\text{ord}(F)$ and $\text{deg}(F)$.

Let $\mathcal{F}$ be an ordinary difference field with a transforming operator $\sigma$ and $\mathcal{F}\{Y\}$ the ring of difference polynomials in the difference indeterminates $\mathcal{Y} = \{y_1, \ldots, y_n\}$. Let $f$ be a difference polynomial in $\mathcal{F}\{Y\}$. The order of $f$ w.r.t. $y_i$ is defined to be the greatest number $k$ such that $y_i^{(k)}$ appears effectively in $f$, denoted by $\text{ord}(f, y_i)$. If $y_i$ does not appear in $f$, then we set $\text{ord}(f, y_i) = -\infty$. The order of $f$ is defined to be $\max_i \text{ord}(f, y_i)$, that is, $\text{ord}(f) = \max_i \text{ord}(f, y_i)$.

A Laurent difference monomial of order $s$ is in the form $\prod_{i=1}^{n} \prod_{k=0}^{s} (y_i^{(k)})^{d_{ik}}$ where $d_{ik}$ are integers which
can be negative. A **Laurent difference polynomial** over \( \mathcal{F} \) is a finite linear combination of Laurent difference monomials with coefficients in \( \mathcal{F} \).

Suppose \( \mathcal{A}_i = \{M_{10}, M_{11}, \ldots, M_{il_i}\} \) \( (i = 0, 1, \ldots, n) \) are finite sets of Laurent difference monomials in \( \mathcal{Y} \). Consider \( n + 1 \) generic Laurent difference polynomials defined over \( \mathcal{A}_0, \ldots, \mathcal{A}_n \):

\[
P_i = \sum_{k=0}^{l_i} u_{ik} M_{ik} \quad (i = 0, \ldots, n),
\]

where all the \( u_{ik} \) are transformally independent over \( \mathcal{F} \). Denote

\[
u_i = (u_{i0}, u_{i1}, \ldots, u_{il_i}) \quad (i = 0, \ldots, n)
\]

and

\[
u = \bigcup_{i=0}^{n} \nu_i \setminus \{u_{i0}\}.
\]

The number \( l_i + 1 \) is called the size of \( P_i \) and \( \mathcal{A}_i \) is called the support of \( P_i \). To avoid the triviality, \( l_i \geq 1 \) \( (i = 0, \ldots, n) \) are always assumed in this paper.

**Definition 2.4.** A set of Laurent difference polynomials of the form (2) is called **Laurent transformally essential** if there exist \( k_i \) \( (i = 0, \ldots, n) \) with \( 1 \leq k_i \leq l_i \) such that \( \sigma.\text{tr.deg} \mathbb{Q}\langle \frac{M_{ik_i}}{M_{i0}}, \ldots, \frac{M_{ik_n}}{M_{i0}} \rangle / \mathbb{Q} = n \). In this case, we also say that \( \mathcal{A}_0, \ldots, \mathcal{A}_n \) form a Laurent transformally essential system.

Let \( \mathfrak{m} \) be the set of all difference monomials in \( \mathcal{Y} \) and \([N(\mathcal{P}_0), \ldots, N(\mathcal{P}_n)]\) the difference ideal generated by \( N(\mathcal{P}_i) \) in \( \mathbb{Q}\{\mathcal{Y}, \nu_0, \ldots, \nu_n\} \). Let

\[
\mathcal{I}_{\mathcal{Y}, \nu} = ([N(\mathcal{P}_0), \ldots, N(\mathcal{P}_n)] : \mathfrak{m}),
\]

\[
\mathcal{I}_\nu = \mathcal{I}_{\mathcal{Y}, \nu} \cap \mathbb{Q}\{\nu_0, \ldots, \nu_n\}.
\]

Now suppose \( \mathcal{P} = \{\mathcal{P}_0, \ldots, \mathcal{P}_n\} \) is a Laurent transformally essential system. Since \( \mathcal{I}_\nu \) defined in (3) is a reflexive prime difference ideal of codimension one, there exists a unique irreducible difference polynomial \( \mathbf{R}(\nu; u_{00}, \ldots, u_{0n}) = \mathbf{R}(\nu_0, \ldots, \nu_n) \in \mathbb{Q}\{\nu_0, \ldots, \nu_n\} \) such that \( \mathbf{R} \) can serve as the first polynomial in each characteristic set of \( \mathcal{I}_\nu \) w.r.t. any ranking endowed on \( \nu_0, \ldots, \nu_n \). Thus the definition of sparse difference resultant is given as follows:

**Definition 2.5.** The above \( \mathbf{R}(\nu_0, \ldots, \nu_n) \in \mathbb{Q}\{\nu_0, \ldots, \nu_n\} \) is defined to be the sparse difference resultant of the Laurent transformally essential system \( \mathcal{P}_0, \ldots, \mathcal{P}_n \), denoted by \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \) or \( \text{Res}_{\mathcal{P}_0, \ldots, \mathcal{P}_n} \), where \( \mathcal{A}_i \) is the support of \( \mathcal{P}_i \) for \( i = 0, 1, \ldots, n \). When all the \( \mathcal{A}_i \) are equal to the same \( \mathcal{A} \), we simply denote it by \( \text{Res}_\mathcal{A} \).

The first step to find sparse difference resultant of a Laurent difference polynomial system \( \mathcal{P}_0, \ldots, \mathcal{P}_n \) is to determine whether the system is Laurent transformally essential or not. In \( \mathcal{A} \), the authors build a one-to-one correspondence between a difference polynomial system and so-called symbolic support matrix and then use the matrix to check whether a Laurent difference system is transformally essential.

Specifically, let \( B_i = \prod_{j=1}^{n} \prod_{k=0}^{s} (y_j^{(k)})^{d_{ijk}} \quad (i = 1, \ldots, m) \) be \( m \) Laurent difference monomials. Introduce a new algebraic indeterminate \( x \) and let

\[
d_{ij} = \sum_{k=0}^{s} d_{ijk} x^k \quad (i = 1, \ldots, m, j = 1, \ldots, n)
\]

be univariate polynomials in \( \mathbb{Z}[x] \). If \( \text{ord}(B_i, y_j) = -\infty \), then set \( d_{ij} = 0 \). The vector \( (d_{i1}, d_{i2}, \ldots, d_{im}) \) is called the **symbolic support vector** of \( B_i \). The matrix \( M = (d_{ij})_{m \times n} \) is called the **symbolic support matrix** of \( B_1, \ldots, B_m \).
Consider the set of generic Laurent difference polynomials defined in \( [2] \):

\[
P_i = u_{i0}M_{i0} + \sum_{k=1}^{l_i} u_{ik}M_{ik} \quad (i = 0, \ldots, n).
\]

Let \( \beta_{ik} \) be the symbolic support vector of \( M_{ik}/M_{i0}, k = 1, \ldots, l_i \). Then the vector \( w_i = \sum_{k=1}^{l_i} u_{ik}\beta_{ik} \) is called the symbolic support vector of \( P_i \) and the matrix \( M_P \) whose rows are \( w_0, \ldots, w_n \) is called the symbolic support matrix of \( P_0, \ldots, P_n \). Therefore, we have

**Theorem 2.6.** \( [4] \) A sufficient and necessary condition for \( P_0, \ldots, P_n \) form a Laurent transformally essential system is the rank of \( M_P \) is equal to \( n \).

Furthermore, we can use the symbolic support matrix to discriminate certain \( P_i \) such that their coefficients will not occur in the sparse difference resultant, which leads to the following definition:

**Definition 2.7.** Let \( T \subset \{0, 1, \ldots, n\} \). Then we call \( T \) or \( P_T \) super-essential if the following conditions hold: (1) \( \text{card}(T) - \text{rk}(M_{P_T}) = 1 \) and (2) \( \text{card}(\mathcal{J}) = \text{rk}(M_{P_\mathcal{J}}) \) for each proper subset \( \mathcal{J} \) of \( T \).

The existence of super-essential system of a difference polynomial system is given by the following theorem \( [4] \).

**Theorem 2.8.** If \( \{P_0, \ldots, P_n\} \) is a Laurent transformally essential system, then for any \( T \subset \{0, 1, \ldots, n\} \), \( \text{card}(T) - \text{rk}(M_{P_T}) \leq 1 \) and there exists a unique \( T \) which is super-essential.

Therefore, let \( R \) be the sparse difference resultant of a Laurent transformally essential system \( [2] \). Then a strong essential system \( S \) whose sparse resultant is equal to \( R \) can be obtained from \( [2] \).

Now, we introduce some notations which is needed to bound the order of \( R \). Let \( A = (a_{ij}) \) be an \( n \times n \) matrix where \( a_{ij} \) is an integer or \( -\infty \). A diagonal sum of \( A \) is any sum \( a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)} \) with \( \sigma \) a permutation of \( 1, \ldots, n \). If \( A \) is an \( m \times n \) matrix with \( k = \min\{m, n\} \), then a diagonal sum of \( A \) is a diagonal sum of any \( k \times k \) submatrix of \( A \). The Jacobi number of a matrix \( A \) is the maximal diagonal sum of \( A \), denoted by \( \text{Jac}(A) \).

Let \( s_{ij} = \text{ord}(\text{N}(P_i), y_j)(i = 0, \ldots, n; j = 1, \ldots, n) \) and \( s_i = \text{ord}(\text{N}(P_i)) \). We call the \( (n+1) \times n \) matrix \( A = (s_{ij}) \) the order matrix of \( P_0, \ldots, P_n \). By \( A_i \), we mean the submatrix of \( A \) obtained by deleting the \( (i+1) \)-th row from \( A \). We use \( P \) to denote the set \( \{\text{N}(P_0), \ldots, \text{N}(P_n)\} \) and by \( P_i \), we mean the set \( P \setminus \{\text{N}(P_i)\} \). We call \( J_i = \text{Jac}(A_i) \) the Jacobi number of the system \( P_i \), also denoted by \( \text{Jac}(P_i) \).

**Theorem 2.9.** Let \( P \) be a Laurent transformally essential system and \( R \) the sparse difference resultant of \( P \). Then

\[
\text{ord}(R, u_i) = \begin{cases} -\infty & \text{if } J_i = -\infty, \\ h_i & \text{if } J_i \geq 0. \end{cases}
\]

### 3 Main results

In this section, we first present some theoretical results, and then give an efficient algorithm to compute sparse difference resultant. We analyze the algorithm complexity and present an example to illustrate the algorithm.
Lemma 3.2. If the symbolic support matrix of \( P_T \) is an algebraically essential system, we have that the co-row rank of its symbolic support matrix \( M_{P_T} \) is \( 1 \). Since \( \tilde{P} \) is algebraically essential, we have \( \sum_{\delta^{i0} P_0} \tilde{f}_{ij} \delta^{i0} \tilde{P}_0 = 0 \) where \( 0 \neq \tilde{f}_{ij} \in Q \{ u_0, \ldots, u_n \} [x] \).
Assume that \( \mathcal{P} \) does not form an algebraically essential system, then its symbolic support matrix \( M_\mathcal{P} \) has full rank, thus we have that \( \sum_{\delta f_j \in \mathcal{P}} f_{ij} \omega_{ij} \neq 0 \) where \( \omega_{ij} \) is the symbolic support vector of \( \delta f_j \), or equivalently, \( \sum_{\delta f_j \in \mathcal{P}} f_{ij} x^i \tilde{\omega}_{ij} = 0 \) where \( \tilde{\omega}_{ij} \) is the symbolic support vector of \( \mathcal{P}_j \).

By the definition of \( \mathcal{P} \), we have that \( \mathcal{R} \) is the sparse resultant of \( \mathcal{P} \). By Lemma 3.2, \( \mathcal{P} \) is an algebraically essential system. Let \( \mathcal{R}' \) be the sparse resultant of \( \mathcal{P} \), then \( \mathcal{R}' \in (\mathcal{P}) \) since \( \mathcal{P} \) is obtained by setting \( y_j \) to 1 in \( \mathcal{P} \) for \( j = m + 1, \ldots, n \). Hence \( \mathcal{R} \in (\mathcal{P}) \) and \( \mathcal{R}|\mathcal{R}' \). Then, \( \mathcal{R} = \mathcal{R}' \) up to a sign since they are irreducible.

Now, we show that \( \mathcal{R} = \mathcal{R}' \) up to a sign. By the definition of \( \mathcal{P} \), \( \mathcal{R}' \) has the lowest rank in \( [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_n\} \) for each \( u_i \). We claim that \( \mathcal{R}' \) has the lowest rank in \( [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_n\} \) for each \( u_i \). If it is not the case, then \( \mathcal{R} \) has lower rank than \( \mathcal{R}' \) in \( u_i \) for some \( i \). Since \( [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_n\} \subset [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_i\} \), then \( \mathcal{R} \in [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_n\} \). It contradicts to the fact that \( \mathcal{R}' \) has the lowest rank in \( [\mathcal{P}_T] \cap \mathbb{Q}\{u_0, \ldots, u_n\} \) for each \( u_i \). Then by the uniqueness of sparse difference resultant, \( \mathcal{R} = \mathcal{R}' = \mathcal{R} \) up to a sign.

Now we give some new order bounds for the sparse difference resultant which is obviously true according to the above results. Let \( A_{\mathcal{P}_T} \) be the order matrix of the system \( \mathcal{P}_T \).

**Proposition 3.4.** The order bound of \( \mathcal{R} \) for each set \( u_i \) can be bounded by \( \text{Jac}(A_{\mathcal{P}_T}) \), where \( A_{\mathcal{P}_T} \) is any \((m + 1) \times m\) full rank sub-matrix of \( A_{\mathcal{P}_T} \).

By above proposition, one may take the order bound by the minimal Jacobi number of the corresponding \( m \times m \) full rank sub-matrices of \( A_{\mathcal{P}_T} \). Furthermore, we have

**Proposition 3.5.** Let \( M_{\mathcal{P}_T} \) be an \((m + 1) \times m\) full rank sub-matrix of \( M_{\mathcal{P}_T} \). Let \( f_i \) be the greatest common divisor of the \( i \)-th column of \( M_{\mathcal{P}_T} \) for \( i = 1, \ldots, m \). Then the order bound of \( \mathcal{R} \) for each set \( u_i \) can be bounded by \( J_i = \text{Jac}(A_{\mathcal{P}_T}), -\sum_{i=1}^{m} \deg(f_i) \).

**Proof:** Let \( \mathcal{P}_T \) be a super essential system and \( M_{\mathcal{P}_T} \) its symbolic support matrix. We denote by \( c_i \) the \( i \)-th column of \( M_{\mathcal{P}_T} \) and \( c'_i = c_i / f_i \). Then \( c_i \) and \( c'_i \) are linearly dependent. Let \( M_{\mathcal{P}_T} \) be the \((m + 1) \times (m + 1)\) matrix by adding a last column \( c'_i \) to \( M_{\mathcal{P}_T} \) and \( \mathcal{P}_T \) its corresponding difference system. Then by Theorem 3.3 \( \mathcal{P}_T \) and \( \mathcal{P}_T \) have the same sparse difference resultant. Let \( M_{\mathcal{P}_T} \) be the sub-matrix of \( M_{\mathcal{P}_T} \) by deleting the first column and \( \mathcal{P}_T \) its corresponding difference system.

By Theorem 3.3 again, \( \mathcal{P}_T \) and \( \mathcal{P}_T \) have the same sparse difference resultant. Inductively, one may take an \((m + 1) \times m\) matrix \( M_{\mathcal{P}_T} \) with \( c'_i \) as its \( i \)-th column and its corresponding difference system \( \mathcal{P}_T \), such that \( \mathcal{P}_T \) and \( \mathcal{P}_T \) have the same sparse difference resultant. Thus the order bound of \( \mathcal{R} \) for each set \( u_i \) equals the \( i \)-th Jacobi number of the order matrix \( A_{\mathcal{P}_T} \), which is \( J_i - \sum_{i=1}^{m} \deg(f_i) \) where \( J_i = \text{Jac}(A_{\mathcal{P}_T}), i \).

In what follows, we state a proposition which will accelerate the algorithm to search for a simple algebraic polynomial system induced by \( \mathcal{P}_T \).
Proposition 3.6. Let \( \hat{P} = \{P_0^{[J_0]}, P_1^{[J_1]}, \ldots, P_m^{[J_m]}\} \) be a polynomial system obtained from a transformally essential difference system, \( D_\hat{P} \) the algebraic symbolic support matrix of \( \hat{P} \), Jacobi numbers \( J_i \) are given as above, then its algebraic essential system is contained in \( \{P_0^{[J_0-p]}, P_1^{[J_1-p]}, \ldots, P_m^{[J_m-p]}\} \), where \( p = |\hat{P}| - \text{rank}(D_\hat{P}) - 1 \).

Proof: Let \( \hat{P} = \{P_0^{[J_0-p]}, P_1^{[J_1-p]}, \ldots, P_m^{[J_m-p]}\} \) and \( D_P \) the algebraic symbolic support matrix of \( P \). We only need to show that \( D_P \) is not of row full rank. If it is not the case, let \( R \) be the sparse difference resultant of the original system, we have that there exists an \( i \), such that \( \text{ord}(R, u_i) > J_i - p \). Then, the algebraic symbolic support matrix of \( P' = \{P_0^{[J_0-p]}, P_1^{[J_1]}, \ldots, P_i^{[J_i-1]}, P_i^{[J_i-p]}, P_i^{[J_{i+1}]} , \ldots, P_m^{[J_m-p]}\} \) is of full rank. Compare \( P' \) with \( \hat{P} \), we find that the co-rank of \( D_\hat{P} \) is no more than \( p \) which contradicts to the definition of \( p \). \( \square \)

3.2 A new algorithm for sparse difference resultant

Based on the new bounds and propositions for the sparse difference resultant, we propose an improved algorithm to compute sparse difference resultant for any given Laurent transformally essential difference polynomial system. The algorithm is motivated by the fact that sparse difference resultant is equal to the algebraic sparse resultant of a strong essential polynomial system which is derived from the original difference polynomial system. Thus one can transform the computation of sparse difference resultant to the computation of algebraic sparse resultant which has mature algorithms such as subdivision method initiated by Canny and Emiris [7].

Therefore, the algorithm is divided into two parts. The first part is to find the strong essential polynomial system. The main strategy for this part is to use the symbolic support matrix of the given difference polynomial system to determine the existence of sparse difference resultant and if yes, to obtain the unique super-essential system, and then simplify the super-essential system based on Theorem 3.3 and use algebraic tools to find the strong essential system. The second one is to use the mixed subdivision method to construct matrix representation of sparse resultant of the strong essential system whose determinant is the required sparse difference resultant up to a sign.

In order to present a better understanding of the whole procedure, we give the flow chart of the algorithm in Figure 1.
Algorithm 1 — SDResultant(\(\mathcal{P}\))

**Input:** A generic difference system \(\mathcal{P} = \{\mathcal{P}_0, \ldots, \mathcal{P}_n\}\).

**Output:** The sparse difference resultant \(R(u_0, \ldots, u_n)\) of \(\mathcal{P}\).

1. Construct the symbolic support matrix \(D_\mathcal{P}\) of \(\mathcal{P}\). If rank\((D_\mathcal{P}) = n\), then proceed to compute SDResultant; else, return “No SDResultant for \(\mathcal{P}\”).

2. Set \(\mathcal{T} = \{0, 1, \ldots, n\}\), \(S = \emptyset\).

3. Let \(S = \emptyset\), choose an element \(i \in \mathcal{T}\).

   3.1. Let \(S = S \cup \{i\}\), \(\mathcal{T}' = \mathcal{T} \setminus S\).

   3.2 If rank\((D_{\mathcal{P}_e}) = |\mathcal{T}'| - 1\), set \(\mathcal{T} = \mathcal{T}'\), go back to step 3.

   3.2.1 Else if \(\mathcal{T} = S\), go to the next step.

   3.2.2 else choose an \(i \in \mathcal{T} \setminus S\) go back to step 3.1.

   Note \(\mathcal{P}_\mathcal{T}\) is a super-essential system.

4. Assume \(\mathcal{T} = \{0, 1, \ldots, m\}\). Compute the rank of the symbolic support matrix \(D_{\mathcal{P}_e}\) of \(\mathcal{P}_\mathcal{T}\) by Gauss elimination. We obtain \((i_1, \ldots, i_m)\) such that the \((m + 1 \times m)\) matrix which corresponds to the \(i_1\)-th, \(\ldots\), \(i_m\)-th columns of \(D_{\mathcal{P}_e}\) has rank \(m\). We assume these columns are the first \(m\) columns. Set the variables \(\{y_{m+1}, \ldots, y_n\}\) in \(\mathcal{P}_\mathcal{T}\) to 1, we denoted by \(\mathcal{P}_\mathcal{T}'\) the new system under this substitution.

   Compute the order matrix \(A_{\mathcal{P}_\mathcal{T}}\) of \(\mathcal{P}_\mathcal{T}'\) and \(f_i\) the common factor of the \(i\)-th column of \(M_{\mathcal{P}_\mathcal{T}}\), compute the Jacobi number \(J_i = \text{Jac}(A_{\mathcal{P}_\mathcal{T}}(i)) - \sum_{i=1}^{m} \deg(f_i)\).

   Construct a new algebraic system \(\mathcal{P}_e = \{P_0^{[J_0]}, P_1^{[J_1]}, \ldots, P_m^{[J_m]}\}\).

5. Compute the algebraic symbolic support matrix \(D_{\mathcal{P}_e}\) of \(\mathcal{P}_e\), let \(p = |\mathcal{P}_e| - \text{rank}(D_{\mathcal{P}_e}) - 1\).

   Find the algebraic essential system with minimal ranking \(\mathcal{P}\) from \(\mathcal{P}_e = \{P_0^{[J_0-p]}, P_1^{[J_1-p]}, \ldots, P_m^{[J_m-p]}\}\).

6. Select \(n - \text{rank}(D_{\mathcal{P}_e})\) variables in \(\mathcal{P}\) to 1, denoted by \(\mathcal{P}_e\).

   Take a variable transformation for \(\mathcal{P}_e\) to make it be a strong essential system.

7. Use mixed subdivision algorithm to obtain algebraic sparse resultant of \(\mathcal{P}_e\).

//* in step 4, by Gauss elimination, we may obtain the row echelon form. The indexes are corresponding to the columns of the pivots in the row echelon form.

//* Steps 5 ~ 7 are performed in algebraic circumstance.

**Theorem 3.7.** The algorithm is correct.

**Proof:** The termination of the algorithm is obvious. The correctness of the algorithm is guaranteed by Lemma 2.2, Theorem 2.8, Lemma 3.1, Lemma 3.2, Theorem 3.3 and Proposition 3.6. □

### 3.3 Complexity of the algorithm

We divide the complexity analysis of Algorithm 1 into two parts: the first six steps and the mixed subdivision algorithm in step 7, and then combine these two parts to estimate the overall complexity. In complexity bounds, we sometimes ignore polylogarithmic factors in the parameters appearing in polynomial factors; this is denoted by \(O^*(z)\).

We first recall the complexity analysis of the mixed subdivision algorithm. The complexity of Newton matrix construction is analyzed in [14] and recalled as follows.

**Theorem 3.8.** Given polynomials \(g_1, \ldots, g_n\), the algorithm computes an implicit representation
of Newton matrix $M$ with worst-case bit complexity

$$O^* \left( |\mathcal{E}|n^{9.5} \mu^{6.5} \log^2 d \log \frac{1}{\epsilon \epsilon \delta} \right)$$

where $|\mathcal{E}|$ is the cardinality of the set that indexes the rows and columns of Newton matrix $M$, $\mu$ is the maximum point cardinality of the $n + 1$ supports, $d$ is the maximum degree of any polynomial in any variable, and $\epsilon, \epsilon \delta \in (0, 1)$ are the error probabilities for the lifting scheme and the perturbation, respectively.

Before we estimate the complexity of Algorithm 1, the following lemma is needed for the complexity analysis.

**Lemma 3.9.** For a symbolic matrix $M = (m_{i,j})$ with $p$ rows and $q$ columns, where $m_{i,j} \in \mathbb{C}[y_1, \ldots, y_k]$. The time complexity to compute the rank of $M$ with probability 1 is bounded by $\max(p, q)^3$.

**Proof:** Assume the rank of $M$ is $r$. Then, there exists an $r \times r$ sub-matrix $M_r$ of $M$, such that $\det(M_r) = g(y_1, \ldots, y_k) \neq 0$. Hence, we set $y_1, \ldots, y_k$ to concrete values $a_1, \ldots, a_k$ in $\mathbb{Z}$, the probability of $g(a_1, \ldots, a_k) \neq 0$ is 1. Now, let $\hat{M}$ be the matrix obtained by substituting $y_i$ by $a_i$ in $M$. Then, $\text{rk}(\hat{M}) \leq \text{rk}(M)$. If $g(a_1, \ldots, a_k) \neq 0$, the rank of $\hat{M}$ is $r$ and the time complexity to compute the rank of $\hat{M}$ is bounded by $O(\max(p, q)^3)$.

Next we analyze the complexity of Algorithm 1 for each step.

**Step 1.** The first step is to compute the rank of symbolic support matrix $D_\mathbb{P}$ whose size is $(n + 1) \times n$. By Lemma 3.9, the complexity is bounded by $O(n^3)$.

**Step 2.** The complexity of this step can be ignored.

**Step 3.** This step is to find the super essential system of $\mathbb{P}$. Consider the worst case which means the super essential system only contains two polynomials, thus we need $(n - 1)$ loops from $(n + 1)$ to 2. In the $k$-th loop, one need compute rank of the matrix with size $(n + 1 - k) \times n$ one time, then, by Lemma 3.9, the total computational cost of this step is $\sum_{k=0}^{n-1} [(n + 1 - k)^2 n] = O(n^3)$.

**Step 4.** By [5], it needs at most $O(d(\log d)^2)$ steps to compute the gcd of two polynomials in $\mathbb{Z}[x]$, so the complexity of computing the gcd of $F \in \mathbb{Z}[x]$ is bounded by $(m - 1)d(\log d)^2$. By [6], using Jacobi’s algorithm, the computational complexity of Jacobi number is bounded by $O(n^3)$.

**Step 5.** The fifth step is to look for the algebraic essential system with minimal ranking $\hat{\mathbb{P}}$. Consider the worst case. Denote by $s = \max_i(\text{ord}(\mathbb{P}_i))$ the maximal order of $\mathbb{P}_i$. The worst case is the Jacobi number $J_i = ms$ and $p = 0$, then the number of polynomials in the set $\hat{\mathbb{P}}$ is bounded by $(ms + 1)(m + 1)$. Then the symbolic support matrix of $\hat{\mathbb{P}}$ has the size bounded by $(ms + 1)(m + 1) \times ((ms + 1)(m + 1) - 1)$, and thus the complexity of computing the rank is bounded by $O(m^6 s^3)$ for each loop by Lemma 3.9. Similar as Step 3, the total computational time in this step is bounded by $O(m^8 s^4)$. Hence, the complexity in this step is bounded by $O(n^8 s^4)$ since $m \leq n$.

**Step 6.** The complexity of this step can be ignored.

**Step 7.** Now, consider the degree and size of $\hat{\mathbb{P}}$. The number of polynomials of $\hat{\mathbb{P}}$ is bounded by $(ns + 1)(n + 1)$. Compare to the original system $\mathbb{P}$, the degree and the maximum point cardinality of the $n + 1$ supports keep unchanged. Hence, by Theorem 3.3 the time complexity to compute the sparse resultant of $\hat{\mathbb{P}}$ is bounded by $O^* \left( |\mathcal{E}|n^{19} s^{9.5} \mu^{6.5} \log^2 d \log \frac{1}{\epsilon \epsilon \delta} \right)$ where $|\mathcal{E}|$ is the cardinality of the set that indexes the rows and columns of Newton matrix $M$ corresponds to $\hat{\mathbb{P}}$, $\mu$ is the maximum point cardinality of the $n + 1$ supports w.r.t. $\mathbb{P}$, $d$ is the maximum
degree of any polynomial \( P_i \) in any variable, and \( \epsilon_i, \epsilon_\delta \in (0, 1) \) are the error probabilities for the lifting scheme and the perturbation, respectively.

Summarizing the above complexity analysis yields the following results.

**Theorem 3.10.** The total complexity of Algorithm 1 is bounded by

\[
O^* \left( |E| n^{10} s^{9.5} \mu^{6.5} (\log^2 d) \epsilon_r \log \frac{1}{\epsilon_\delta} \right),
\]

where \( |E| \) is the cardinality of the set that indexes the rows and columns of Newton matrix \( M \) corresponds to \( \hat{P} \), \( \mu \) is the maximum point cardinality of the \( n + 1 \) supports w.r.t. \( P \), \( d \) is the maximum degree of any polynomial \( P_i \) in any variable, and \( \epsilon_i, \epsilon_\delta, \epsilon_r \in (0, 1) \) are the error probabilities for the lifting scheme, the perturbation and the rank computations for the symbolic support matrices, respectively.

### 3.4 Implementation in Mathematica

The procedure for compute sparse difference resultant described above has been implemented in the computer algebraic system Mathematica, named by SDResultant. The user interface only need two arguments: the difference polynomial system and the difference indeterminates. The compiled function SDResultant will automatically check whether the input difference polynomial system exists sparse difference resultant or not. If yes, the function SDResultant returns the required sparse difference resultant which corresponds to the algebraic sparse resultant of the algebraic essential system with minimal ranking where the number of polynomials is more one than the number of variables after some necessary simplifications.

One merit of the package SDresultant is that, by transforming the difference polynomials into the corresponding symbolic support matrix, it only requires the technique of linear algebra, such as the rank of matrix, row reduction and so on, to discriminate the related conditions and finally to compute sparse difference resultant. Another one is that the algorithm finally gives the matrix representation of sparse difference resultant which may facilitate to show the properties and construct fast algorithms for sparse difference resultant.

### 3.5 An example

In this section, we illustrate the Algorithm 1 by a difference polynomial system \( \mathbb{P} = \{ P_0, P_1, P_2, P_3, P_4 \} \), where \( y_{ij} = y_i^{(j)} \) and

\[
\begin{align*}
P_0 &= u_{00} + u_{01} y_{11}^2 y_{21}^2 y_{31} + u_{02} y_{11}^2 y_{21} y_{31} y_{41}, \\
P_1 &= u_{10} + u_{11} y_{11}^2 y_{21}^2 y_{31} + u_{12} y_{11}^2 y_{21} y_{31} y_{41} y_{42}, \\
P_2 &= u_{20} + u_{21} y_{11}^2 y_{21}^2 y_{32} + u_{22} y_{11}^2 y_{21}^2 y_{31} + u_{23} y_{11}^2 y_{21} y_{31} y_{41}, \\
P_3 &= u_{30} + u_{31} y_{11} y_{21} + u_{32} y_{11}^2 y_{21} y_{31} y_{42}, \\
P_4 &= u_{40} + u_{41} y_{11} y_{32} y_{41} + u_{42} y_{11}^2 y_{21} y_{42}.
\end{align*}
\]

#### 3.5.1 Concrete computations

The first step is to check whether or not the difference system \( \mathbb{P} \) is transformally essential. The symbolic support matrix of \( \mathbb{P} \) is

\[
D_{\mathbb{P}} = \begin{pmatrix}
2xu_{01} + 2u_{02} & 2xu_{01} + u_{02} & xu_{01} + u_{02} & (x + 1)u_{02} \\
2xu_{11} + 2xu_{12} & 2xu_{11} + xu_{12} & xu_{11} + xu_{12} & (x^2 + x)u_{12} \\
2u_{21} x^2 + 2u_{22} x + 2u_{23} & 2u_{21} x^2 + 2u_{22} x + u_{23} & v_{21} x^2 + u_{22} x + u_{23} & (x + 1)u_{23} \\
xu_{31} + 2xu_{32} & xu_{31} + xu_{32} & xu_{32} & x^2 u_{32} \\
xu_{41} + 2xu_{42} & x^2 u_{42} & x^2 u_{41} & xu_{41} + u_{42}
\end{pmatrix}.
\]
rk($D_P$) = 4, thus $P$ is transformally essential.

By the third step of the Algorithm 1, the super essential system of $P$ is $P_T = \{P_0, P_1, P_2\}$ with $T = \{0, 1, 2\}$, which is independent of $P_3$ and $P_4$. The symbolic support matrix of $P_T$ is

$$D_{P_T} = \begin{pmatrix}
2xu_{01} + 2u_{02} & 2xu_{01} + u_{02} & xu_{01} + u_{02} & (x + 1)u_{02} \\
2xu_{11} + xu_{12} & 2xu_{11} + xu_{12} & xu_{11} + xu_{12} & (x^2 + x)u_{12} \\
2u_{21}x^2 + 2u_{22}x + 2u_{23} & 2u_{21}x^2 + 2u_{22}x + u_{23} & u_{21}x^2 + u_{22}x + u_{23} & (x + 1)u_{23}
\end{pmatrix}.$$ 

Since the submatrix $M$ of $D_{P_T}$ by deleting the middle two columns is

$$A_{14} = \begin{pmatrix}
2xu_{01} + 2u_{02} & (x + 1)u_{02} \\
2xu_{11} + xu_{12} & (x^2 + x)u_{12} \\
2u_{21}x^2 + 2u_{22}x + 2u_{23} & (x + 1)u_{23}
\end{pmatrix},$$

whose rank is 2. Then by Theorem 3.3, we set $y_2$ and $y_3$ and their differences to 1, then $P_T = \{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$, where

- $\tilde{P}_0 = u_{00} + u_{01} y_1^2 + u_{02} y_2^2 y_4 y_{41}$,
- $\tilde{P}_1 = u_{10} + u_{11} y_1^2 + u_{12} y_2^2 y_4 y_{41}$,
- $\tilde{P}_2 = u_{20} + u_{21} y_1^2 + u_{22} y_2^2 y_4 y_{41}$.

Note that by Theorem 3.3, one can delete any two columns to find the submatrix with rank 2. For example, by deleting the last two columns of $D_{P_T}$, one also obtain the submatrix $A_{12}$. The rank of $A_{12}$ is 2 and one can set $y_3$ and $y_4$ and their differences to 1 to get the new simplified super essential difference system.

The order matrix of $P_T$ is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$, then the Jacobi numbers are $J_0 = 4, J_1 = J_2 = 3$. Since the last column of $A_{14}$ has a common factor $(x + 1)$, thus by Proposition 3.3, the modified Jacobi numbers are $\tilde{J}_0 = J_0 - 1 = 3, \tilde{J}_1 = J_1 - 1 = 2, \tilde{J}_2 = J_2 - 1 = 2$. Then we use the modified Jacobi numbers $\tilde{J}_i (i = 0, 1, 2)$ to construct new difference system $\hat{P} = \{\hat{P}_0, \hat{P}_1, \hat{P}_2\}.$

The following steps are performed in algebraic circumstance. For the system $\hat{P}$, we first search for an essential system with minimal ranking, and then perform a variable transformation for the essential system, i.e.,

$$\hat{P} = \{P_0, \delta P_0, \delta^2 P_0, P_1, \delta P_1, P_2, \delta P_2\}$$

$$= \{u_{00} + z_1 u_{01} + z_1 u_{02}, \delta u_{00} + z_1 \delta u_{01} + z_2 \delta u_{02}, \delta^2 u_{00} + z_1 \delta^2 u_{01} + z_3 \delta^2 u_{02}, u_{10} + z_1 u_{11} + z_2 u_{12}, \delta u_{10} + z_1 \delta u_{11} + z_3 \delta u_{12}, u_{20} + z_1 u_{21} + z_1 u_{22}, \delta u_{20} + z_1 \delta u_{21} + z_3 \delta u_{22} + z_2 \delta u_{23}\},$$

where $z_1 = y_1^2 y_{41}, z_2 = y_1^2 y_{42} y_{41}, z_3 = y_1^2 y_{42}, z_4 = y_1^2, z_5 = y_2^2, z_6 = y_2^2$.

Finally, regard $\hat{P}$ as the algebraic polynomial system of $z_1, \ldots, z_6$ and use the mixed subdivision algorithm in [2] to obtain the algebraic sparse resultant of $\hat{P}$ which is the required sparse difference resultant $R$. This step cannot be done by hands and perform on a computer in next subsection.

**Remark 3.11.** The modified Jacobi numbers reduce the dimensions of symbolic support matrices from $13 \times 11$ to $11 \times 8$ in searching the algebraic essential system with minimal ranking. Moreover, the modified Jacobi numbers will drop the complexity of the searching algorithm for sparse difference resultant in [2].
3.5.2 Implemented by SDResultant

We use the compiled package \textbf{SDResultant} to automatically compute the sparse difference resultant of $\mathbb{P}$.

Firstly, we transform the target difference polynomial system $\mathbb{P}$ to the given form. Input the difference polynomial system $\mathbb{P} = \{P_0, P_1, P_2, P_3, P_4\}$ in the form

\[
P_0 = u_{00}[i] + u_{01}[i] y[1, i + 1]^2 y[2, i + 1]^2 y[3, i + 1] + u_{02}[i] y[1, i]^2 y[2, i] y[3, i] y[4, i] y[4, i + 1],
\]
\[
P_1 = u_{10}[i] + u_{11}[i] y[1, i + 1]^2 y[2, i + 1]^2 y[3, i + 1] + u_{12}[i] y[1, i + 1]^2 y[2, i + 1] y[3, i + 1] y[4, i + 1] y[4, i + 2],
\]
\[
P_2 = u_{20}[i] + u_{21}[i] y[1, i + 2]^2 y[2, i + 2]^2 y[3, i + 2] + u_{22}[i] y[1, i + 1]^2 y[2, i + 1] y[3, i + 1] y[4, i] y[4, i + 1],
\]
\[
P_3 = u_{30}[i] + u_{31}[i] y[1, i + 1] y[2, i + 1] + u_{32}[i] y[1, i + 1]^2 y[2, i + 1] y[3, i + 1] y[4, i + 1] y[4, i + 2],
\]
\[
P_4 = u_{40}[i] + u_{41}[i] y[1, i + 1] y[3, i + 2] y[4, i + 1] + u_{42}[i] y[1, i + 1]^2 y[2, i + 2] y[4, i] y[4, i].
\]

The \textbf{SDResultant} gives the sparse difference resultant which is the quotient of determinants of two square matrices $M_1$ and $M_2$, where $\delta u_{jk} = u_{jk}[i + 1]$, $\delta^2 u_{jk} = u_{jk}[i + 2]$,

\[
M_1 = \begin{pmatrix}
\delta u_{10} & 0 & \delta u_{11} & 0 & 0 & 0 & \delta u_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & u_{10} & 0 & 0 & 0 & u_{11} & 0 & 0 & 0 & u_{12} & 0 & 0 & 0 \\
\delta u_{20} & \delta u_{21} & \delta u_{22} & 0 & 0 & 0 & 0 & 0 & \delta u_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & \delta^2 u_{00} & \delta^2 u_{01} & 0 & 0 & 0 & \delta^2 u_{02} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta u_{20} & \delta u_{21} & \delta u_{22} & 0 & 0 & 0 & 0 & \delta u_{23} & 0 & 0 & 0 \\
0 & 0 & \delta u_{20} & \delta u_{21} & \delta u_{22} & 0 & 0 & 0 & 0 & \delta u_{23} & 0 & 0 & 0 \\
0 & 0 & \delta u_{10} & 0 & \delta u_{11} & 0 & 0 & \delta u_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta u_{10} & 0 & \delta u_{11} & 0 & 0 & \delta u_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta u_{10} & 0 & \delta u_{11} & 0 & 0 & \delta u_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta u_{00} & \delta u_{01} & 0 & 0 & 0 & \delta u_{02} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and

\[
M_2 = \begin{pmatrix}
\delta u_{10} & \delta u_{11} & \delta u_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta u_{20} & \delta u_{22} & 0 & \delta u_{23} & 0 & 0 & 0 & 0 & 0 \\
\delta^2 u_{00} & \delta^2 u_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta u_{10} & \delta u_{11} & \delta u_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta u_{20} & \delta u_{22} & 0 & \delta u_{23} & 0 & 0 \\
0 & 0 & 0 & \delta^2 u_{00} & \delta^2 u_{02} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta u_{00} & \delta u_{01} & 0 & \delta u_{02} & 0 & 0 \\
\end{pmatrix}
\]
Then the require sparse difference resultant $R = |M_1|/|M_2|$, i.e.,

\[
R = -\delta u_00 \delta^2 u_{01} u_{02} u_{11} \delta u_{12} u_{21} \delta u_{23} + \delta u_00 u_{01} \delta^2 u_{02} \delta u_{11} u_{12} \delta u_{21} u_{23} + u_{00} \delta^2 u_{01} \delta u_{02} u_{11} \delta u_{12} \delta u_{22} u_{23} - \delta^2 u_00 u_{01} \delta u_{02} u_{12} \delta u_{12} \delta u_{21} u_{22} - \delta u_00 \delta^2 u_{01} u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - u_{00} \delta u_{01} \delta^2 u_{02} u_{11} \delta u_{12} \delta u_{21} u_{23} + \delta^2 u_00 u_{01} \delta u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - \delta^2 u_00 u_{02} \delta u_{01} \delta u_{12} \delta u_{12} \delta u_{21} u_{23} - u_{00} \delta^2 u_{01} \delta u_{02} u_{11} \delta u_{12} \delta u_{22} u_{23} + \delta^2 u_00 u_{01} \delta u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - \delta u_00 \delta^2 u_{01} u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - u_{00} \delta^2 u_{01} \delta u_{02} u_{11} \delta u_{12} \delta u_{22} u_{23} + \delta^2 u_00 u_{01} \delta u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - \delta u_00 \delta^2 u_{01} u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - u_{00} \delta^2 u_{01} \delta u_{02} u_{11} \delta u_{12} \delta u_{22} u_{23} + \delta^2 u_00 u_{01} \delta u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23} - \delta u_00 \delta^2 u_{01} u_{02} u_{12} \delta u_{12} \delta u_{22} u_{23}. 
\]

4 Conclusion

Sparse difference resultant for a Laurent transformally essential system is further studied and new order bounds are obtained. We use the difference specialization technique to simplify the computation of the sparse difference resultant. Based on these results, we give an efficient algorithm to compute sparse difference resultant. We also analyze the complexity of the algorithm and implement it on a computer with Mathematica. The algorithm gives a matrix representation of sparse difference resultant which partially solves one of open problems in [4].

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