Computation of the first Chow group of a Hilbert scheme of space curves

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Abstract

An earlier wrong formula for the dimension of $A_1(H_{d,g}) \otimes \mathbb{Q}$ is corrected.

Introduction

The results stated in ([T4], pp. 1) have to be corrected as follows: Let $H = H_{d,g} = \text{Hilb}^d(\mathbb{P}^3)$ be the Hilbert scheme, which parametrizes the curves in $\mathbb{P}_C^3$ of degree $d$ and genus $g$ (i.e., the closed subschemes of $\mathbb{P}_C^3$ with Hilbert polynomial $P(T) = dT - g + 1$). It is always assumed that $d \geq 3$ and $g$ is not maximal, i.e. that $g < (d - 1)(d - 2)/2$.

Theorem 0.1. Let $g(d) := (d - 2)^2/4$. Then $\dim \mathbb{Q} A_1(H_{d,g}) \otimes \mathbb{Q} = 3$ (resp. $= 4$), if $g \leq g(d)$ (resp. if $g > g(d)$).

Corollary 0.1. $NS(H) \simeq \mathbb{Z}^r$ and $\text{Pic}(H) \simeq \mathbb{Z}^r \oplus \mathbb{C}^s$, where $r := \dim_{\mathbb{C}} H^1(H, \mathcal{O}_H)$ and $\rho = 3$, if $g \leq g(d)$. If $g > g(d)$, then $\rho = 3$ or $\rho = 4$.

Theorem 0.2. Let $C \hookrightarrow H \times \mathbb{P}^3$ be the universal curve over $H$. Then $\dim \mathbb{Q} A_1(C) \otimes \mathbb{Z} = \dim \mathbb{Q} A_1(H) \otimes \mathbb{Z} + 1$.

Corollary 0.2. $NS(C) = \mathbb{Z}^{\rho+1}$ and $\text{Pic}(C) \simeq \mathbb{Z}^{\rho+1} \oplus \mathbb{C}^s$, where $s := \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$ and $\rho$ is defined as in Corollary 1.

That means, the formula $(d - 2)(d - 3)/2$ for the bound $g(d)$ in ([T4], p.1) is wrong and has to be replaced by the above formula.

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Summary of earlier results

1.1. Description of the starting situation

The notations are the same as in [T1]–[T4] and are summed up in Appendix A. The ground field is $\mathbb{C}$, and $H = H_{d,g}$ is the Hilbert scheme which parametrizes the curves $C \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $d$ and genus $g$ (i.e. the closed subschemes of $\mathbb{P}^3_{\mathbb{C}}$ with Hilbert polynomial $P(T) = dT - g + 1$). According to F.S. Macaulay, $H_{d,g}$ is not empty if and only if the “complementary” Hilbert polynomial $Q(T) = \left(\frac{T^4}{4} - P(T)\right)$ either has the form $Q(T) = \left(\frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{1}\right)$, or the form $Q(T) = \left(\frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{1}\right)$, where $a$ is an integer $\geq 1$, respectively $a$ and $b$ are integers (Macaulay coefficients), such that $2 \leq a \leq b$. Between the degree and genus on the one hand and the Macaulay coefficients on the other hand, one has the following relations $d = a, g = (d - 1)(d - 2)/2$, if $Q(T) = \left(\frac{T^3}{3} + \frac{T^2}{2}\right)$, and $d = a - 1, g = (a^2 - 3a + 4)/2 - b$, if $Q(T) = \left(\frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{1}\right)$, respectively. One sees that the first case occurs if and only if one is dealing with plane curves, in which case the groups $A_1(H)$ and $NS(H)$ both have the rank 2 (cf. [T1], Satz 2a, p. 91). Therefore in the following we always suppose that $d \geq 3$ and $g < (d - 1)(d - 2)/2$, that means, the complementary Hilbert polynomial has the form $Q(T) = \left(\frac{T^3}{3} + \frac{T^2}{2} + \frac{T}{1}\right)$, where $4 \leq a \leq b$.

We also write $H_Q$ instead of $H_{d,g}$ in order to express that this Hilbert scheme likewise parametrizes the ideals $I \subset \mathcal{O}_{\mathbb{P}^3}$ with Hilbert polynomial $Q(T)$, or equivalently, the saturated graded ideals in $\mathbb{C}[x, y, z, t]$ with Hilbert polynomial $Q(T)$.

In [T1]–[T4] it was tried to describe the first Chow group $A_1(H)$, where we always take rational coefficients, and we write $A_1(H)$ instead of $A_1(H) \otimes \mathbb{Q}$. The starting point is the following consideration: If the Borel group $B = B(4; k)$ operates on $H = H_Q$ in the obvious way, then one can deform each 1-cycle on $H$ in a 1-cycle, whose prime components are $B$-invariant, irreducible, reduced and closed curves on $H$. It follows that $A_1(H)$ is generated by such $B$-invariant 1-prime cycles on $H$. This is a partial statement of a theorem of Hirschowitz. (Later on we will have to use the general statement, whereas the partial statement can be proved in a simple way, see [T1], Lemma 1, p. 6.) Now such a $B$-invariant 1-prime cycle (i.e. closed, irreducible and reduced curve) $C$ on $H$ can be formally described as follows: Either each point of $C$ is invariant under $\Delta := U(4; k)$, or one has $C = \overline{G^i_a : \eta}$, where $\eta$ is a closed point of $H$, which is invariant under $T = T(4; k)$ and the group $G_i, i \in \{1, 2, 3\}$. Here $G^i_a$ is the group $G_a$, acting by

$$\begin{align*}
\psi^1_a : x &\mapsto x, y \mapsto y, z \mapsto z, t \mapsto \alpha z + t \\
\psi^2_a : x &\mapsto x, y \mapsto y, z \mapsto \alpha y + z, t \mapsto t \\
\psi^3_a : x &\mapsto x, y \mapsto \alpha x + y, z \mapsto z, t \mapsto t,
\end{align*}$$
respectively, on \( P = k[x, y, z, t] \), and \( G_i \) is the subgroup of \( \Delta \), which is complementary to \( G_{a'} \), that means, one defines

\[
G_1 := \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad G_2 := \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad G_3 := \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.
\]

If \( C \) has this form, then \( C \) is called a curve or a 1-cycle of type \( i \), where \( i \in \{1, 2, 3\} \). \( \mathcal{A}(H) := \text{Im}(A_1(H^\Delta) \to A_1(H)) \) is called the “algebraic part” and \( \overline{\mathcal{A}}_1(H) := A_1(H)/\mathcal{A}(H) \) is called the “combinatorial part” of the first Chow group of \( H \). Here \( H^\Delta \) denotes the fixed point scheme which, just as all other fixed point schemes that will occur later on, is supposed to have the induced reduced scheme structure. (This convention is valid also for the Hilbert scheme \( H^d := \text{Hilb}^d(\mathbb{P}^2) \), see below.)

In order to formulate the results obtained so far, one has to introduce the following "tautological" 1-cycles on \( H \):

\[
C_1 = \{ (x, y^a, y^{a-1}z^{b-a}(\alpha z + t)) \mid \alpha \in k \}^-
\]

\[
C_2 = \{ (x, y^{a-1}(\alpha y + z), y^{a-2}z^{b-a+1}(\alpha y + z)) \mid \alpha \in k \}^-
\]

\[
C_3 = \{ (x^a, \alpha x + y, x^{a-1}z^{b-a+1}) \mid \alpha \in k \}^-
\]

\[
D = \{ (x^2, xy, y^{a-1}, z^{b-2a+1}(y^{a-2} + \alpha xz^{a-3})) \mid \alpha \in k \}^-
\]

\[
E = \{ (x^2, xy, xz, y^a, y^{a-1}z^{b-a+1}, xt^{b-2} + \alpha y^{a-1}z^{b-a}) \mid \alpha \in k \}^-
\]

For the sake of simplicity, we now suppose \( d \geq 5 \) (i.e. \( a \geq 6 \)). (The cases \( d = 3 \) and \( d = 4 \) will be treated separately in Chapter 16.) Then one has the following results:

1. If \( b < 2a - 4 \), i.e. if \( g > \gamma(d) := (d - 2)(d - 3)/2 \), then \( \mathcal{A}(H) \) is generated by \( E \), and \( A_1(H) \) is generated by \( E, C_1, C_2, C_3 \).

2. If \( b \geq 2a - 4 \), i.e. if \( g \leq \gamma(d) \), then \( \mathcal{A}(H) \) is generated by \( E \) and \( D \) and \( A_1(H) \) is generated by \( E, D, C_2 \) and \( C_3 \) (see [T1], Satz 2, p. 91; [T3], Proposition 4, p. 22; [T4], Satz 1 and Proposition 2, p. 26).

From reasons of degree it follows that \( [C_2] \) can not lie in the vector space spanned by \( [E], [D], [C_3] \), so the problem is to decide, if \( [C_3] \in \mathcal{A}(H) \).

In ([T4], Proposition 3, p. 32) it was erroneously claimed that \( [C_3] \in \mathcal{A}(H) \), if \( b \geq 2a - 4 \). (The error is the wrong computation of the degree in ([T4], p. 28, line 21 to line 30.) Therefore the bound for the genus in ([T4], p. 1) is wrong.

Actually, in ([T2], 3.3.2) it had been proved, that \( [C_3] \in \mathcal{A}(H) \), if \( a \geq 6 \) is even and \( b \geq a^2/4 \), i.e. if \( d \geq 5 \) is odd and \( g \leq (d - 1)(d - 3)/4 \). In the case \( d \geq 6 \) even, in Conclusion 14.3 it will follow that \( [C_3] \in \mathcal{A}(H) \), if \( g \leq (d - 2)/4 \). (This means the bound of [T2], 3.3.3 is valid if \( d \geq 6 \), already . ). One sees that the condition for \( g \) in both cases can be summed up to \( g \leq (d - 2)^2/4 \).

The major part of the following text serves for the proof that this sufficient condition is a necessary condition, too (cf. Conclusion 14.1).
1.2. Technical tools

The formulas in ([T2], p. 134) and of ([T3], Anhang 2, p. 50) show that it is not possible to decide by means of the computation of degrees, whether \([C_3]\) lies in \(\mathcal{A}(\mathcal{H})\). Therefore we try to get a grasp of the relations among the \(B\)-invariant 1-cycles on \(\mathcal{H}\) with the help of the theorem of Hirschowitz ([Hi], Thm. 1, p. 87). We sketch the procedure.

1.2.1. The Theorem of Hirschowitz. There is a closed and reduced subscheme \(Z = Z(\mathcal{H})\) of \(\mathcal{H}\), such that \(Z(k) = \{x \in \mathcal{H}(k)|\dim \Delta \cdot x \leq 1\}\) (cf. [Ho], p. 412 and [T3], Lemma 1, p. 35). Then one can show, with the help of the theorem of Hirschowitz, that \(A_1(Z) \supseteq A_1(\mathcal{H})\) (cf. [T2], Lemma 24, p. 121). As was explained in (1.1), \(A_1(\mathcal{H})\) has a generating system consisting of \(B\)-invariant 1-cycles which lie in \(Z\), automatically. As \(\Delta\) is normalized by \(B\), \(B\) operates on \(Z\) and therefore one can form the so called equivariant Chow group \(A^B_1(Z)\), which is isomorphic to \(A_1(Z)\) ([Hi], loc. cit.). And the relations among \(B\)-invariant 1-cycles on \(Z\) are generated by relations among such cycles, which lie on \(B\)-invariant surfaces \(V \subset Z\) (see [Hi], Mode d’emploi, p. 89).

1.2.2. The Restriction morphism. Let \(U_t \subset \mathcal{H}\) be the open subset consisting of the ideals \(\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^3}\) with Hilbert polynomial \(Q\), such that \(t\) is not a zero divisor of \(\mathcal{O}_{\mathbb{P}^3}/\mathcal{I}\). Then there is a so called restriction-morphism \(h : U_t \rightarrow H^d := \text{Hilb}^d(\mathbb{P}^2_\mathbb{C})\), defined by \(\mathcal{I} \mapsto \mathcal{I}' := \mathcal{I} + t\mathcal{O}_{\mathbb{P}^3}(-1)/t\mathcal{O}_{\mathbb{P}^3}(-1)\). E.g., if \(G := \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array} \right\} \subset \mathbb{C}^3\) is contained in \(\Delta\), then the fixed point scheme \(F := \mathcal{H}^G\) is contained in \(U_t\), and the restriction of \(h\) to \(F\) is denoted by \(h\), again.

In ([G4], Abschnitt 6, p. 672f) the following description of \(\text{Im}(h)\) is given:

(i) There is a finite set \(\mathcal{F}\) of Hilbert functions of ideals of colength \(d\) on \(\mathbb{P}^2\) such that \(\text{Im}(h) = \bigcup \{ H_{\geq \varphi} | \varphi \in \mathcal{F} \}\).

(ii) If \(k = \mathbb{T}\) and if \(\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_k}\) is an ideal of colength \(d\) and Hilbert function \(\varphi\), then \(\mathcal{I} \in h(\text{Spec}(k)) \iff g^*(\varphi) := \sum_{d \geq 0} \varphi(n) - \binom{d+3}{3} + d^2 + 1 \geq g\).

(iii) If \(\varphi \in \mathcal{F}\) and if \(\psi\) is the Hilbert function of an ideal on \(\mathbb{P}^2\) of colength \(d\) such that \(\varphi(n) \leq \psi(n)\) for all \(n \in \mathbb{N}\), then \(\psi \in \mathcal{F}\).

(iv) Let be \(\varphi \in \mathcal{F}\) and \(\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_k}\) an ideal with Hilbert function \(\varphi\). Let \(\mathcal{I}^*\) be the ideal on \(\mathcal{O}_{\mathbb{P}^2_k}\) defined by \(H^0(\mathcal{I}^*(n)) = \bigoplus_{i=0}^n t^{n-i}H^0(\mathcal{I}(i))\), then \(V_+(\mathcal{I}^*) \subset \mathbb{P}^d_k\) is a curve of degree \(d\) and genus \(g^*(\varphi)\).

Here \(H_\varphi \subset H^d\) is the locally closed subscheme (with the reduced induced scheme structure), which parametrizes the ideals \(\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2}\) of colength \(d\) with Hilbert function \(\varphi\), and \(H_{\geq \varphi} := \bigcup \{ H_\psi | \psi \geq \varphi \}\) is a closed subscheme (with the induced reduced scheme structure). The image of \(C_3\) under \(h\) is the 1-cycle \(c_3 := \{(x^d, \alpha x + y)|\alpha \in k\}^\circ\). One has

**Theorem 1.1.** Let be \(d \geq 5\), \(\mathcal{H} := \bigcup \{ H_\varphi \subset H^d | g^*(\varphi) > g(d) \}\) and \(\mathcal{A}(\mathcal{H}) := \text{Im}(A_1(H^U(3k))) \rightarrow A_1(\mathcal{H}))\). Then \(c_3 \notin \mathcal{A}(\mathcal{H})\).
The proof extends over the chapters 2 to 10 and essentially rests on the apparently strong condition for an ideal $\mathcal{I}$ to have a Hilbert function $\varphi$ such that $g^*(\varphi) > g(d)$.

### 1.2.3. Standard cycles on $H^d$.

It has been shown, respectively it will be shown that $[C_3] \in \mathcal{A}(H_{d,g})$, if $g \leq g(d)$ (cf. 1.1). Therefore, we can suppose that $g > g(d)$. If $\mathcal{J} \in U_1$ and the restriction ideal $\mathcal{I} := \mathcal{J}'$ has the Hilbert function $\varphi$, then from (ii) in (1.2.2) it follows that $g^*(\varphi) > g(d)$. It will be shown in Chapter 2 that this implies there is a linear form $\ell \in S_1 - (0)$, an ideal $\mathcal{K} \subset O_{P^2}$ of colength $c$ and a form $f \in H^0(\mathcal{K}(m))$ such that $\mathcal{I} = \ell \mathcal{K}(-1) + f O_{P^2}(-m)$, $c + m = d$ and $m \geq c + 2$.

Let be $C = G_a \cdot \eta \subset H$ a 1-cycle of type 3 and let be $\mathcal{J} \leftrightarrow \eta$ the corresponding ideal in $O_{P^2}$ with Hilbert polynomial $Q$. Then the ideal $\mathcal{I} := \mathcal{J}' \leftrightarrow \eta' := h(\eta)$ is invariant under $T(3;k)$ and $\Gamma := \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} < U(3,k)$. It follows that either $\mathcal{I} = x \mathcal{K}(-1) + y^m O_{P^2}(-m)$ or $\mathcal{I} = y \mathcal{K}(-1) + x^m O_{P^2}(-m)$, where $\mathcal{K}$ is a monomial ideal. We say, $\mathcal{I}$ has $x$-standard form or $\mathcal{I}$ has $y$-standard form, respectively, and we call $C' := G_a \cdot \eta'$ a $x$-standard cycle or $y$-standard cycle on $H^d$, respectively. With the help of the theorem of Hirschowitz one can again try to describe the relations between $B(3;k)$-invariant $y$-standard cycles on $H$, and one obtains that such relations cannot make the $y$-standard cycle $c_3$ disappear modulo $A(\mathcal{H})$ (cf. Proposition 9.1) from which Theorem 0.1 will follow.

### 1.2.4. 1-cycles of proper type 3.

Let be $C = G_a \cdot \eta, \eta \leftrightarrow \mathcal{J}$, be a 1-cycle of type 3 on $H = H_{d,g}$, such that $d \geq 5$ and $g > g(d)$. $C$ is called a 1-cycle of proper type 3, if $C' := G_a \cdot \eta'$ is a $y$-standard cycle on $H$. Corresponding to Hirschowitz’s theorem one has to consider $B(4;k)$-invariant surfaces $V \subset Z(\mathcal{H})$, which contain a 1-cycle of proper type 3. It turns out that then $V$ is pointwise invariant under $G_3$ and therefore $V$ is contained in $U_1$. Then one can map relations between $B$-invariant 1-cycles on $V$ by $h_*$ into relations between $B(3;k)$-invariant 1-cycles on $h(V)$, and one obtains with the aid of Proposition 9.1 the main result of the second part of the paper (Theorem 14.1), which corresponds to Theorem 0.1. In Chapter 15 there is complete description of $A_1(H_{d,g})$ if $d \geq 5$, and in Chapter 16 this is done in the cases $d = 3$ and $d = 4$ (Theorem 15.1 and Theorem 16.1, respectively).
CHAPTER 2

Subschemes of points in $\mathbb{P}^2$ and their Hilbert functions

2.1. General properties

The ground field is $\mathbb{C}$ and $k$ denotes an extension field. A closed subscheme $Z \subset \mathbb{P}^2_k$ of length $d > 0$ is defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_k}$ with Hilbert polynomial $Q(n) = \binom{n+2}{2} - d$. If the Hilbert function $h^0(\mathcal{I}(n)) = \dim_k H^0(\mathbb{P}^2_k, \mathcal{I}(n)), n \in \mathbb{N}$, of $\mathcal{I}$ is denoted by $\varphi(n)$, then

$$\varphi'(n) := \varphi(n) - \varphi(n - 1), n \in \mathbb{N},$$

denotes the difference function. If $\varphi : \mathbb{N} \to \mathbb{N}$ is any function, such that $\varphi(n) = \binom{n+2}{2} - d$ for $n \gg 0$, then the ideals $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_k}$ with Hilbert function $\varphi$ form a locally closed subset $H_\varphi$ of the Hilbert scheme $H^d = \text{Hilb}^d(\mathbb{P}^2_\mathbb{C})$, and we take $H_\varphi$ as a subscheme of $H^d$ with the induced reduced scheme structure.

Iarrobino has shown ([I], Lemma 1.3, p. 8) that $H_\varphi \neq \emptyset$ if and only if the difference function fulfils the following two conditions:

(a) $\varphi'(n) \leq n + 1$, for all $n \in \mathbb{N}$ and
(b) $\varphi'(n) \leq \max(\varphi'(n + 1) - 1, 0)$, for all $n \in \mathbb{N}$.

If $\alpha = \alpha(\varphi) := \min\{n \in \mathbb{N} \mid \varphi(n) > 0\}$, then (b) is equivalent to:

(b') $\varphi'(n) + 1 \leq \varphi'(n + 1)$, for all $n \geq \alpha$.

The (Mumford-)regularity $e$ of an ideal $\mathcal{I}$ with Hilbert function $\varphi$ as before is characterized by $e = \text{reg}(\varphi) = \min\{n \in \mathbb{N} \mid \varphi'(n + 1) = n + 1\}$ (cf. Appendix B, Lemma 2). In principle, the graph of $\varphi '$ has the shape of Fig. 2.1. If $\emptyset \neq H_\varphi \subset H^d$, then $d$ is determined by the condition $\varphi(n) = \binom{n+2}{2} - d, n \gg 0$, and we call $d$ the colength of $\varphi$. It is known, that $\text{reg}(\varphi) \leq d$ ([G1], Lemma 2.9, p. 65), and $\text{reg}(\varphi) = d$ is equivalent with $\mathcal{I}$ being generated by a linear form $\ell \in S_1$ and a form $f \in S_d$, not divisible by $\ell$. Here $S = k[x, y, z]$ is the graded polynomial ring. Another characterization of $\text{reg}(\varphi) = d$ is that the graph of $\varphi'$ has the shape of Fig. 2.2. One notes that the colength of $\varphi$ is equal to the number of "monomials" between the graph of $\varphi'$ and the line $y = x + 1$. (For this and other properties, see [T1]-[T4].) In the following we write $\mathbb{P}^2$ instead of $\mathbb{P}^2_k$ and denote by $\mathcal{I}$ an ideal in $\mathcal{O}_{\mathbb{P}^2}$, whose finite colength (resp. whose Hilbert function) usually is denoted by $d$ (resp. by $\varphi$).

2.2. Numerical and algebraic properties

Lemma 2.1. Let be $k = \overline{k}, \mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2}$ an ideal with colength $d$, Hilbert function $\varphi$ and regularity $m$. We assume that there is a number $\varepsilon \in \mathbb{N}, 0 \leq \varepsilon < m - 2$, such that $\varphi'(n) = n$ for all $n \in \mathbb{N}$, such that $\varepsilon + 1 \leq n \leq m - 1$. Then there is a linear form $\ell = S_1$, an ideal $\mathcal{K} \subset \mathcal{O}_{\mathbb{P}^2}$ of colength $c$ and a form $f \in H^0(\mathcal{K}(m)), such that $\mathcal{I} = \ell \mathcal{K}(-1) + f \mathcal{O}_{\mathbb{P}^2}(-m)$. If
\(\ell_1, \ell_2\) are any linear forms in \(S_1\) such that \(\ell, \ell_1, \ell_2\) is a basis of the \(k\)-vector space \(S_1\) and if \(R := k[\ell_1, \ell_2]\) is the subring of \(S\), isomorphic to \(k[x, y]\), then \(d = c + m\) and 

\[
H^0(I(n)) = \begin{cases} 
\ell H^0(K(n-1)) & \text{if } n < m, \\
\ell H^0(K(n-1)) \oplus fR_{n-m} & \text{if } n \geq m. 
\end{cases}
\]

**Proof.** By assumption, the graph of \(\varphi'\) has the shape as in Fig.2.3. Then there is an \(\ell \in S_1 - (0)\) and an ideal \(K \subset O_{\mathbb{P}^2}\) of regularity \(\leq \varepsilon\) such that \(H^0(I(n)) = \ell H^0(K(n-1))\) for all \(n \leq m - 1\) (cf. [G4] and Appendix B, Lemma 1). If \(\psi\) is the Hilbert function of \(K\), then \(\varphi'(n) = \psi'(n-1)\) for \(1 \leq n \leq m - 1\), and because of the shape of the graphs of \(\varphi'\) and \(\psi'\) it follows that \(\varphi(n) = \psi(n - 1) + (n - m + 1)\) for all \(n \geq m\). Therefore \(H^0(I(m)) = \ell H^0(K(m-1)) \oplus f \cdot k\), where \(f \in H^0(I(m))\) is a suitable section. Because of the \(m\)-regularity of \(I\) it follows that \(H^0(I(n)) = \ell H^0(K(n-1)) + f S_{n-m}, n \geq m\). If \(n = m + 1\), then from \(\varphi(m + 1) = \psi(m) + 2\) it follows that \(S_1 f \cap \ell H^0(K(m))\) has the dimension 1. Thus there is an \(h \in S_1 - (0)\), such that \(hf \in H^0(K(m))\). If \(\ell\) would be a divisor of \(f\), then it would follow that \(I \subset \ell O_{\mathbb{P}^2}(-m)\) and thus \(I\) would not have a finite colength in \(O_{\mathbb{P}^2}\). Therefore we may suppose that \(h = \ell\), and it follows that \(f \in H^0(K(m))\).

We choose \(\ell_1, \ell_2 \in S_1\) such that \(\ell, \ell_1, \ell_2\) are linear independent and we put \(R := k[\ell_1, \ell_2]\). If there would be an \(\ell_1, \ell_2 \in S_1\) that \(\ell_1, \ell_2\) are any linear forms in \(S_1\) such that \(\ell_1, \ell_2\) is a basis of the \(k\)-vector space \(S_1\) and if \(R := k[\ell_1, \ell_2]\) is the subring of \(S\), isomorphic to \(k[x, y]\), then \(d = c + m\) and 

\[
H^0(I(n)) = \begin{cases} 
\ell H^0(K(n-1)) & \text{if } n < m, \\
\ell H^0(K(n-1)) \oplus fR_{n-m} & \text{if } n \geq m. 
\end{cases}
\]

**Corollary 2.1.** The assumptions and notations are as in Lemma 2.1. Then one has:

(i) \(\kappa := \text{reg}(K) \leq \varepsilon\), especially \(\kappa \leq m - 3\).
(ii) \(K\) (respectively the linear form \(\ell\)) is uniquely determined (respectively uniquely up to a factor out of \(k\) different from zero).
(iii) \(f\) is uniquely determined up to a factor out of \(k\) different from zero, modulo \(\ell H^0(K(m-1))\).
(iv) \(\kappa\) and \(\varepsilon\) are uniquely determined by \(\varphi\).

**Proof.** (i) \(\varepsilon + 1 = \varphi'(\varepsilon + 1) = \psi'(\varepsilon)\). From (Appendix B, Lemma 2) it follows that \(\kappa \leq \varepsilon\).

(ii) The regularity only depends on the Hilbert function, and therefore \(\kappa = \text{reg}(K_1) = \text{reg}(K_2) < m - 1\). Thus from \(\ell_1 H^0(K_1(m-2)) = \ell_2 H^0(K_2(m-2))\) it follows that \(\ell_1 K_1(-1) = \ell_2 K_2(-1)\).

If \(\ell_1\) would not be a divisor of \(\ell_2\) then one would have \(K_2 \subset \ell_1 O_{\mathbb{P}^2}(-1)\) contradiction. From this assertion (ii) does follow, and (iii) and (iv) are clear.

**Remark 2.1.** If \(\varphi\) and \(\psi\) are two Hilbert functions of colength \(d\), then from \(\varphi < \psi\) (that means \(\varphi(n) \leq \psi(n)\) for all \(n \in \mathbb{N}\) and \(\varphi(n) < \psi(n)\) for at least one \(n \in \mathbb{N}\)) it follows that \(g^*(\varphi) < g^*(\psi)\). This follows immediately from the definition of \(g^*(\varphi)\) in (1.2.2).

**Remark 2.2.** If \(e := \text{reg}(\varphi), d := \text{colength}(\varphi)\) and \(s(\varphi) := \sum_{i=0}^{e-2} \varphi(i)\), then \(g^*(\varphi) = s(\varphi) - \binom{e+1}{3} + d(e - 2) + 1\). Because of \(\varphi(n) = \binom{n+2}{2} - d\) for all \(n \geq e - 1\) this follows from a simple computation with binomial coefficients.
2.2.1. Hilbert functions of colength \( \leq 4 \). We use the formula of Remark 2.2 and orientate ourselves by the figures 2.4–2.7.

**d = 1** There is only one Hilbert function (cf. Fig. 2.4).
\[
e = 1, \ s(\varphi) = 0, \ g^*(\varphi) = 0 - \left(\frac{2}{3}\right) + 1 \cdot (1 - 2) + 1 = 0.
\]

**d = 2** There is again only one Hilbert function (cf. Fig. 2.5).
\[
e = 2, \ s(\varphi) = 0, \ g^*(\varphi) = 0 - \left(\frac{3}{3}\right) + 2 \cdot 0 + 1 = 0.
\]

**d = 3** There are two Hilbert functions (Fig. 2.6 a and Fig. 2.6 b).
\[
e_1 = 2, \ s(\varphi_1) = 0, \ g^*(\varphi_1) = 0 - \left(\frac{3}{3}\right) + 3 \cdot 0 + 1 = 0,
\]
\[
e_2 = 3, \ s(\varphi_2) = 1, \ g^*(\varphi_2) = 1 - \left(\frac{4}{3}\right) + 3 \cdot 1 + 1 = 1.
\]

**d = 4** There are two Hilbert functions (Fig. 2.7 a and Fig. 2.7 b).
\[
e_1 = 3, \ s(\varphi_1) = 0, \ g^*(\varphi_1) = 0 - \left(\frac{4}{3}\right) + 4 \cdot 1 + 1 = 1,
\]
\[
e_2 = 4, \ s(\varphi_2) = 4, \ g^*(\varphi_2) = 4 - \left(\frac{5}{3}\right) + 4 \cdot 2 + 1 = 3.
\]

2.2.2. Two special ideals. First case: If \( d \geq 6 \) is even, then let be \( e := d/2 + 1 \) and \( \mathcal{I} := (x^2, xy^{e-2}, y^e) \). The Hilbert function \( \chi \) can be read from Fig. 2.8. One notes that colength(\( \mathcal{I} \)) and reg(\( \mathcal{I} \)) really are equal to \( d \) and \( e \), respectively, and \( \chi(n) = \sum_{i=1}^{n-1} i = \left(\begin{array}{c} n \\ 2 \end{array}\right) \), if \( 1 \leq n \leq e - 2 \). Therefore \( s(\chi) = \sum_{i=1}^{e-2} \left(\begin{array}{c} i \\ 2 \end{array}\right) = \left(\begin{array}{c} e-1 \\ 3 \end{array}\right) \) and it follows that
\[
g^*(\chi) = \left(\begin{array}{c} e-1 \\ 3 \end{array}\right) - \left(\begin{array}{c} e+1 \\ 3 \end{array}\right) + 2(e - 1)(e - 2) + 1 = \left(\begin{array}{c} e-1 \\ 3 \end{array}\right) - \left(\begin{array}{c} e \\ 3 \end{array}\right) - \left(\begin{array}{c} e+1 \\ 3 \end{array}\right) + 2e^2 - 6e + 5
\]
\[
= -\frac{1}{2}(e - 1)(e - 2) - \frac{1}{2}e(e - 1) + 2e^2 - 6e + 5 = e^2 - 4e + 4
\]
\[
= (e - 2)^2 = \frac{1}{4}(d - 2)^2.
\]
Second case: If \( d \geq 5 \) is odd, then let be \( e := (d + 1)/2 \) and \( \mathcal{I} := (x^2, xy^{e-1}, y^e) \). The Hilbert function \( \chi \) can be read from Fig. 2.9. One notes that colength(\( \mathcal{I} \)) and reg(\( \mathcal{I} \)) are equal to \( d \) and \( e \), respectively, and \( \chi(n) = \left(\begin{array}{c} n \\ 2 \end{array}\right) \), if \( 1 \leq n < e \). Therefore \( s(\chi) = \sum_{i=2}^{e-2} \left(\begin{array}{c} i \\ 2 \end{array}\right) = \left(\begin{array}{c} e-1 \\ 3 \end{array}\right) \) and it follows that
\[
g^*(\chi) = \left(\begin{array}{c} e-1 \\ 3 \end{array}\right) - \left(\begin{array}{c} e+1 \\ 3 \end{array}\right) + 2(e - 1)(e - 2) + 1 = -\left(\begin{array}{c} e-1 \\ 2 \end{array}\right) - \left(\begin{array}{c} e \\ 2 \end{array}\right) + (2e - 1)(e - 2) + 1
\]
\[
= -(e - 1)^2 + 2e^2 - 5e + 3 = e^2 - 3e + 2 = \frac{1}{4}(d+1)^2 - \frac{3}{2}(d+1) + 2
\]
\[
= \frac{1}{4}(d^2 - 4d + 3).
\]
Definition 1. If \( d \geq 5 \), then we set
\[
g(d) := \begin{cases} 
\frac{1}{4}(d-2)^2 & \text{if } d \geq 6 \text{ is even,} \\
\frac{1}{4}(d-1)(d-3) & \text{if } d \geq 5 \text{ is odd.}
\end{cases}
\]

\( g(d) \) is called the deformation bound for ideals in \( O_{\varphi} \) of colength \( d \).

The rest of the article is to justify this notation.

2.2.3.

Lemma 2.2. Let be \( k = \overline{k}, \mathcal{I} \subset O_{\varphi} \) an ideal of colength \( d \geq 5 \) and regularity \( m \). Let be \( \varphi \) the Hilbert function of \( \mathcal{I} \). If \( \varphi' > g(d) \), then the assumptions of Lemma 2.1 are fulfilled by \( \mathcal{I} \).

Proof. Let be \( \chi \) the Hilbert function defined by Fig. 2.8 and Fig. 2.9, respectively. Let be \( m = \text{reg}(\varphi) \). If \( \varphi'(m) - \varphi'(m-1) > 2 \), then \( \varphi'(i) \leq \chi'(i) \) for all \( i \), and it would follow that \( g'(\varphi) \leq g(\chi) \) (Remark 1). If \( \varphi'(m) - \varphi'(m-1) = 1 \), then \( \varphi'(m-1) = \varphi'(m-1) = (m+1) - 1 = (m-1) + 1 \), therefore \( \text{reg}(\mathcal{I}) \leq m - 1 \) (cf. Appendix B, Lemma 2). It follows that \( \varphi'(m) - \varphi'(m-1) = 2 \), therefore \( \varphi'(m-1) = m - 1 \). If \( \varphi'(m-2) = m - 2 \), as well, then the assumptions of Lemma 2.1 are fulfilled with \( \varepsilon := m - 3 \), for instance. Thus without restriction of generality one can assume \( \varphi'(m-2) \leq m - 3 \).

Case 1: \( \varphi'(m-2) < m - 3 \). Figure 2.10 represents the Hilbert function \( \varphi \) as well as the \( B(3; k) \)-invariant ideal \( \mathcal{M} \) with Hilbert function \( \varphi \). Then one makes the deformation
\[
E(H^0(\mathcal{M}(m))) \mapsto E(H^0(\mathcal{M}(m))) - u \cup v =: E(H^0(\mathcal{N}(m))),
\]
where \( \mathcal{N} \) is a \( B(3; k) \)-invariant ideal with Hilbert function \( \psi > \varphi \). But then it follows \( g^*(\varphi) < g^*(\psi) \leq g^*(\chi) = g(d), \) contradiction.

Case 2: \( \varphi'(m-2) = m - 3 \). If the graph of \( \varphi' \) would have a shape different from that in Fig. 2.11 a, i.e., if the graph of \( \varphi' \) would have a “jumping place” \( n < m - 2 \) (cf. the terminology in [T1], p. 72), then as in the first case one could make a deformation
\[
E(H^0(\mathcal{M}(m))) \mapsto E(H^0(\mathcal{M}(m))) - u \cup v =: E(H^0(\mathcal{N}(m)))
\]
(cf. Fig. 2.11b) and would get a contradiction, again. It only remains the possibility represented in Fig. 2.11a. But then \( \varphi = \chi \), which contradicts the assumption \( g^*(\varphi) > g(d) \).

\( \square \)

2.3. Numerical conclusions from \( g^*(\varphi) > g(d) \)

2.3.1. At first we describe the starting situation: In this section we suppose that \( g(d) \) is defined, i.e. \( d \geq 5 \). Moreover let be \( \varphi \) a Hilbert function such that \( H_{\varphi} \neq \emptyset \), colength\( (\varphi) = d, \text{reg}(\varphi) = m \) and \( g^*(\varphi) > g(d) \). Then the assumptions of Lemma 2.2 are fulfilled for an ideal \( \mathcal{I} \), which can be supposed to be monomial. Therefore the assumption \( k = \overline{k} \) is superfluous. As \( m \) and \( d \) are uniquely determined by \( \varphi, c := d - m \) is uniquely determined, too.
The aim in this section is to prove the inequality \( m \geq c + 2 \). By Lemma 2.1 and Lemma 2.2, respectively, one can write \( \mathcal{I} = \ell(\mathcal{K}(-1) + f\mathcal{O}_{\mathbb{P}^2}(-m)) \), and \( c \) is equal to the colength of the Hilbert function \( \psi \) of \( \mathcal{K} \). (As \( \mathcal{I} \) is monomial, \( \mathcal{K} \) is monomial, too, and without restriction one has \( \ell \in \{x, y, z\} \).

**Lemma 2.3.** Let be \( \psi \) the Hilbert function of an ideal \( \mathcal{K} \) of colength \( c \geq 5 \), \( \kappa := \text{reg}(\psi) \), and \( m \geq \kappa + 2 \) an integer. If one defines \( \varphi \) by \( \varphi'(n) := \psi'(n-1), 0 \leq n \leq m-1, \varphi'(n) := n+1, n \geq m \), then \( H_\varphi \neq \emptyset \), colength(\( \varphi \)) = \( c + m \), reg(\( \varphi \)) = \( m \) and \( g^*(\varphi) = g^*(\psi) + \frac{1}{2}m(m-3) + c \).

**Proof.** We orientate ourselves by Figure 2.3, but the weaker assumption \( m \geq \kappa + 2 \) takes the place of the assumption \( m \geq \kappa + 3 \).

Without restriction one can assume that \( \mathcal{K} \) is \( B(3; k) \)-invariant. Then \( y^m \in H^0(\mathcal{K}(m)) \) and \( \mathcal{I} := x\mathcal{K}(-1) + y^m\mathcal{O}_{\mathbb{P}^2}(-m) \) has the Hilbert function \( \varphi \), the regularity \( m \) and the colength \( c + m \). This follows by considering the figure mentioned above (and has been shown in a more general situation in [G4], Lemma 4, p. 660). We compute \( g^*(\varphi) \) (cf. Remark 2.2):

\[
s(\varphi) = \sum_{i=0}^{m-2} \varphi(i) = \sum_{i=0}^{\kappa-1} \psi(i-1) + \sum_{i=\kappa}^{m-2} \psi(i-1) = \sum_{i=0}^{\kappa-2} \psi(i) + \sum_{i=\kappa}^{m-3} \psi(i) = s(\psi) + \sum_{i=\kappa-1}^{m-3} \left[ (\frac{i+2}{2}) - c \right]
\]

By Remark 2.2 it follows that:

\[
g^*(\varphi) = s(\psi) + \binom{m}{3} - \binom{\kappa+1}{3} - (m - \kappa - 1)c - \binom{m+1}{3} + (c + m)(m - 2) + 1
\]

\[
= s(\psi) - \binom{\kappa+1}{3} + c(\kappa - 2) + 1 + \binom{m}{3} - \binom{m+1}{3} - mc + c + (c + m)m - 2m
\]

\[
= g^*(\psi) - \binom{m}{2} + 5m^2 - 2m + c = g^*(\psi) + \frac{1}{2}m(m-3) + c.
\]

**2.3.2. The cases \( c \leq 4 \).** By Lemma 2.2 (resp. by Corollary 2.1 of Lemma 2.1) one has \( \varphi'(n) = \psi'(n-1), 0 \leq n \leq m-1, \varphi'(n) = n+1, n \geq m \), and \( \kappa = \text{reg}(\mathcal{K}) \leq m - 3 \). Then the assumptions of Lemma 2.3 are fulfilled.

We use the formula given there and orientate ourselves by the Figures 2.4 - 2.7. The regularity of the Hilbert function considered each time will now be denoted by \( \kappa \).

If \( c \in \{0, 1\} \), then because of \( d = m + c \) it follows that \( m \geq 4 \).

If \( c = 2 \), then \( \kappa = 2 \) and \( m \geq \kappa + 3 = 5 \).

If \( c = 3 \) and \( \kappa = 2 \) or \( \kappa = 3 \), then \( m \geq \kappa + 3 = 5 \).

If \( c = 4 \), then \( \kappa = 3 \) or \( \kappa = 4 \) and \( m \geq \kappa + 3 \geq 6 \).

Thus in the cases \( 0 \leq c \leq 4 \) one has \( m \geq c + 2 \).
2.3.3. The case $g^*(\psi) \leq g(c)$. This notation implies that $c \geq 5$. If $\kappa$ is the regularity and $c$ is the colength of any Hilbert function, then because of $1 + 2 + \cdots + \kappa \geq c$, one always has $\left(\frac{\kappa + 1}{2}\right) \geq c$, and therefore $\kappa \geq \sqrt{2c} - 1$. By Lemma 2.2 the assumptions of Lemma 2.1 are fulfilled, therefore by Corollary 1 it follows that $m \geq \kappa + 3 > 5.16$.

1st case: $c$ and $m$ are even. By the formulas for $g(d)$ and $g^*(\varphi)$ it follows that:

$$\frac{1}{4}(c^2 - 4c + 4) + \frac{1}{2}m(m - 3) + c > \frac{1}{4}[(c + m)^2 - 4(c + m) + 4]$$

$$\iff \quad \frac{1}{2}m(m - 3) + c > \frac{1}{4}[2cm + m^2 - 4m]$$

$$\iff \quad m^2 - 2(c + 1)m + 4c > 0.$$ 

The solutions of the corresponding quadratic equation are 0 and $2c$. Therefore $m \geq 2c + 1 > c + 2$.

2nd case: $c$ is even, $m$ is odd. One obtains the inequality:

$$\frac{1}{4}(c^2 - 4c + 4) + \frac{1}{2}m(m - 3) + c > \frac{1}{4}[(c + m)^2 - 4(c + m) + 3]$$

$$\iff \quad m^2 - 2(c + 1)m + 4c + 1 > 0.$$ 

The solutions of the corresponding quadratic equation are $m = c + 1 \pm \sqrt{c^2 - 2c} \geq 0$. Because of $c + 1 - \sqrt{c^2 - 2c} < 3$, if $c \geq 5$, it follows that $m \geq c + 1 + \sqrt{c^2 - 2c}$. Because of $c + 1 + \sqrt{c^2 - 2c} > 2c - 1$, if $c \geq 5$, it follows that $m \geq 2c$, therefore $m \geq 2c + 1 > c + 2$.

3rd case: $c$ is odd, $m$ is even. One obtains the inequality:

$$\frac{1}{4}(c^2 - 4c + 3) + \frac{1}{2}m(m - 3) + c > \frac{1}{4}[(c + m)^2 - 4(c + m) + 4]$$

$$\iff \quad m^2 - 2(c + 1)m + 4c > 0.$$ 

It follows that $m \geq 2c + 1 > c + 2$.

4th case: $c$ and $m$ are odd. One obtains the inequality:

$$\frac{1}{4}(c^2 - 4c + 3) + \frac{1}{2}m(m - 3) + c > \frac{1}{4}[(c + m)^2 - 4(c + m) + 4]$$

$$\iff \quad m^2 - 2(c + 1)m + 4c - 1 > 0.$$ 

The solutions of the corresponding quadratic equation are $m = c + 1 \pm \sqrt{(c - 1)^2 + 1}$. Because of $c + 1 - \sqrt{(c - 1)^2 + 1} < 2$ it follows that $m \geq c + 1 + \sqrt{(c - 1)^2 + 1} > 2c$, therefore $m \geq 2c + 1 \geq c + 2$.

2.3.4. The case $g^*(\psi) \geq g(c)$. As in the proof of Lemma 2.3 one can write $\mathcal{I} = xK(-1) + y^m \mathcal{O}_{\mathbb{P}^2}(-m)$, $K$ a $B(3; \kappa)$-invariant ideal of colength $c$, $\kappa = \text{reg}(K)$, $d = \text{colength}(\mathcal{I}) = c + m$, and again $m \geq \kappa + 3$ (Corollary 2.1). We represent the Hilbert function $\psi$ by the ideal $\mathcal{K} = x\mathcal{J}(-1) + y^\kappa \mathcal{O}_{\mathbb{P}^2}(-\kappa)$, $\mathcal{J}$ a $B(3; \kappa)$-invariant ideal of colength $b$ and of regularity $\varepsilon$, where $\kappa \geq \varepsilon + 3$ (cf. Corollary 2.1). If the Hilbert function of $\mathcal{J}$ is denoted by $\vartheta$, then in principle one has the situation represented by Fig. 2.12. If one assumes that $(m - 1) - \kappa \leq \text{colength}(\mathcal{J}) = b$, then one could bring the monomials denoted by $1, 2, 3, \ldots$
in the positions denoted by 1, 2, 3, ⋯ (cf. Fig. 2.12). In the course of this the Hilbert function increases and therefore \( g^*(\varphi) < g^*(\varphi_1) < \cdots < g(d) \), contradiction. Thus one has \((m-1)-\kappa > b\), i.e., \(m \geq \kappa + b + 2 = c + 2\).

### 2.3.5. Summary.

**Lemma 2.4.** *(Notations as in Lemma 2.1 and Lemma 2.2)* If \(g(d) < g^*(\varphi)\), then \(m \geq c + 2\). \(\square\)

From the proof of Lemma 2.4 we can conclude one more statement:

**Corollary 2.2.** Let be \(g^*(\psi) \leq g(c)\) *(which notation implies \(c \geq 5\)). Then \(m \geq 2c + 1\). \(\square\)

### 2.4. Additional group operations

#### 2.4.1. General auxiliary lemmas.

The group \(Gl(3; k)\) operates on \(S = k[x, y, z]\), and therefore on \(H^d = \text{Hilb}^d(P^2_k)\). If \(\rho = (\rho_0, \rho_1, \rho_2) \in \mathbb{Z}^3\) is a vector such that \(\rho_0 + \rho_1 + \rho_2 = 0\), then \(T(\rho) := \{(\lambda_0, \lambda_1, \lambda_2) \in (k^*)^3 \mid \lambda_0^\rho_0 \lambda_1^\rho_1 \lambda_2^\rho_2 = 1\}\) is a subgroup of \(T = T(3; k)\), and \(\Gamma := \left\{ \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\}\) is a subgroup of \(U(3; k)\). We let the additive group \(\mathbb{G}_a\) operate on \(S\) by

\[
\psi_\alpha : x \mapsto x, \quad y \mapsto \alpha x + y, \quad z \mapsto z, \quad \alpha \in k,
\]

and \(\sigma : \mathbb{G}_m \to T\) nearly always denotes the operation \(\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z, \lambda \in k^*\).

**Auxiliary Lemma 1.** If \(V \subset S_d\) is a vector space, invariant under \(G := \Gamma \cdot T(\rho)\) where \(\rho = (0, -1, 1)\) or \(\rho = (-1, 0, 1)\), then \(V\) is monomial, i.e. invariant under \(T\).

**Proof.** We first consider the case \(\rho = (0, -1, 1)\). We take a standard basis of \(V\) consisting of \(T(\rho)\)-semi-invariants (see Appendix E). Assuming that the assertion above is wrong, we conclude that there is a \(T(\rho)\)-semi-invariant \(f \in V\), such that the monomials occurring in \(f\) are not in \(V\). Then there is such a form with smallest \(z\)-degree. From the invariance of \(V\) under \(T(\rho)\) it follows that \(V\) is invariant under the \(G_m\)-action \(\tau(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z, \lambda \in k^*\). We write \(f = Mp,\) where \(M = x^\ell y^m, p = \sum_{i=0}^n a_i y^{n-i} z^i, \ell + m + n = d, n \geq 1\) and \(a_n \neq 0\). It follows that \(y \partial f / \partial z = yM \sum_{i=0}^n i a_i y^{n-i} z^{i-1} \in V\). Now \(y \partial f / \partial z\) is also a \(T(\rho)\)-semi-invariant with smaller \(z\)-degree than \(f\). According to the choice of \(f\) it follows that \(g := Myz^{n-1} \in V\), therefore \(y \partial g / \partial z = (n-1)Myz^{n-2} \in V\), etc. One gets \(My^i y^{n-i} \in V, 1 \leq i \leq n\), therefore \(My^n \in V\), contradiction.

In the case \(\rho = (-1, 0, 1)\) we write \(f = x^\ell y^m \sum_{i=0}^n a_i x^{n-i} z^i\). Because of \(x \partial f / \partial z = xM \sum_{i=1}^n i a_i x^{n-i} z^{i-1}\) we can argue as before. \(\square\)

**Auxiliary Lemma 2.** Let be \(\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2}\) an ideal of colength \(d\), which is invariant under \(G := \Gamma \cdot T(\rho)\). If \(\rho_0 + \rho_1 + \rho_2 = 0, \rho_0 < 0\) and \(\rho_1 < 0\), then \(\mathcal{I}\) is invariant under \(T\).

**Proof.** Let be \(n\) the smallest natural number, such that \(H^0(\mathcal{I}(n))\) is not \(T\)-invariant. Then we have without restriction that \(n \geq 1\). As \(H^0(\mathcal{I}(n))\) has a standard basis, there is a proper semi-invariant in \(H^0(\mathcal{I}(n))\), i.e. a form \(f \in H^0(\mathcal{I}(n))\) of the shape \(f = M(1 + a_1 X^p + a_2 X^{2p} + \cdots + a_r X^{rp})\), \(M\) a monomial, \(a_r \neq 0, r \geq 1\), and no monomial \(MX^{ip}\) is in \(H^0(\mathcal{I}(n))\), if \(a_i \neq 0\). If \(M\) would be divisible by \(z\), then \(g := z^{-1} f \in H^0(\mathcal{I}(n-1))\) would...
be a proper semi-invariant, too, because from \( z^{-1}MX^\rho \in H^0(\mathcal{I}(n-1)) \) it follows that \( MX^\rho \in H^0(\mathcal{I}(n)) \). Therefore, \( M \) is not divisible by \( z \). From the proper semi-invariants of \( H^0(\mathcal{I}(n)) \) we chose one, say \( f \), such that the \( z \)-degree is minimal. Now from \( f \in H^0(\mathcal{I}(n)) \), because of the \( \Gamma \)-invariance, it follows that \( x\partial f/\partial z = xM(a_1\rho_2z^{-1}X^\rho + \cdots + ra_r\rho_2z^{-1}X^\rho) \) and \( y\partial f/\partial z = yM(a_1\rho_2z^{-1}X^\rho + \cdots + ra_r\rho_2z^{-1}X^\rho) \) is in \( H^0(\mathcal{I}(n)) \), i.e., \( g := xMYz^{-1}p \) and \( h := yMX^\rho z^{-1}p \) are in \( H^0(\mathcal{I}(n)) \), where \( p(X^\rho) := a_1\rho_2 + 2a_2\rho_2X^\rho + \cdots + ra_r\rho_2X^{(r-1)\rho} \).

As the \( z \)-degree of \( g \) and of \( h \) is smaller than the \( z \)-degree of \( f, g \) and \( h \) are no longer proper semi-invariants, i.e., the monomials which occur in \( g \) or in \( h \), all are in \( H^0(\mathcal{I}(n)) \). It follows that \( u := z^{-1}xMX^\rho \) and \( v := z^{-1}yMX^\rho \) are in \( H^0(\mathcal{I}(n)) \). From the \( \Gamma \)-invariance it follows by applying the operators \( x\partial/\partial z \) and \( y\partial/\partial z \) repeatedly, that \( \frac{z^{\rho_1}}{z^{\rho_1}} X^\rho \in H^0(\mathcal{I}(n)) \).

Now \( X^\rho = x^{-|\rho_0|} y^{-|\rho_1|} z^{\rho_2} \) and \( \rho_0 + \rho_1 + \rho_2 = 0 \), therefore \( MX^{(r-1)\rho} \in H^0(\mathcal{I}(n)) \). Applying the operators mentioned before again gives \( MX^{(r-2)\rho} \in H^0(\mathcal{I}(n)) \), etc. It follows that \( MX^\rho \in H^0(\mathcal{I}(n)) \), \( 0 \leq i \leq r - 1 \), and therefore \( MX^\rho \in H^0(\mathcal{I}(n)) \), contradiction. \( \square \)

2.4.2. Successive construction of \( \Gamma \)-invariant ideals. At first we consider a general situation: Let be \( \mathcal{K} \subset \mathcal{O}_{\mathcal{P}_2} \) an ideal of colength \( c \) and of regularity \( e \); \( z \) is supposed to be a non-zero divisor of \( \mathcal{O}_{\mathcal{P}_2}/\mathcal{K} \); let be \( R := k[x, y] \) and \( \ell \in R_1 \) a linear form. Let be \( m > e \) an integer and \( f \in H^0(\mathcal{K}(m)) \) a section, whose leading term is not divisible by \( \ell \), i.e., if one writes \( f = f^0 + zf^1 + \cdots + zm^fm \), where \( f^i \in R_{m-i} \), then \( f^0 \) is not divisible by \( \ell \).

**Lemma 2.5.** The ideal \( \mathcal{I} := \ell \mathcal{K}(-1) + f \mathcal{O}_{\mathcal{P}_2}(-m) \) has the following properties:

(i) \( z \) is not a zero-divisor of \( \mathcal{O}_{\mathcal{P}_2}/\mathcal{I} \).

(ii) \( H^0(\mathcal{I}(n)) = \ell H^0(\mathcal{K}(n-1)) \), if \( n < m \), and

\[
H^0(\mathcal{I}(n)) = \ell H^0(\mathcal{K}(n-1)) \oplus k[x, z]_{n-m}, \text{ if } n \geq m \text{ and } \ell = \alpha x + y
\]

(respectively \( H^0(\mathcal{I}(n)) = \ell H^0(\mathcal{K}(n-1)) \oplus k[y, z]_{n-m} \), if \( n \geq m \) and \( \ell = x \)).

(iii) colength(\( \mathcal{I} \)) = \( c + m \), \( \text{reg}(\mathcal{I}) = m \).

**Proof.** If \( \ell = x \), these are the statements of ([G4], Lemma 4, p. 660). If \( \ell = \alpha x + y, \alpha \in k^* \), then let be \( u \) the automorphism \( x \mapsto \ell, y \mapsto y, z \mapsto z \) of \( S \). By applying (loc. cit) to

\[
\overline{\mathcal{K}} := u^{-1}(\mathcal{K}), \overline{\mathcal{I}} := u^{-1}(\mathcal{I}), \overline{f} := u^{-1}(f)
\]

one gets

\[
H^0(\overline{\mathcal{I}}(n)) = xH^0(\overline{\mathcal{K}}(n-1))) \oplus \overline{f}k[y, z]_{n-m}.
\]

Now applying \( u \) gives

\[
H^0(\mathcal{I}(n)) = \ell H^0(\mathcal{K}(n-1)) \oplus f[k[y, z]_{n-m}.
\]

As \( k[y, z] = k[\ell - \alpha x, z] \) and \( \ell f \in H^0(\mathcal{K}(m)) \), the statement (ii) follows, if \( \alpha \neq 0 \).

If \( \alpha = 0 \), we take the automorphism \( x \mapsto y, y \mapsto x, z \mapsto z \) and argue as before. \( \square \)

We would like to put the section \( f \) in a certain normal form. We first consider the case \( \ell = \alpha x + y \). Then we can write \( f^0 = x^m + \ell u, u \in R_{m-1} \), without restriction. As \( m-1 \geq e \), there is \( v = v^0 + zv^1 + z^2v^2 + \cdots \in H^0(\mathcal{K}(m-1)) \) such that \( v^0 = u \). As \( f \) is
determined only modulo $\ell H^0(\mathcal{K}(m-1))$, we can replace $f$ by $\tilde{f} := f - \ell v$, therefore we can assume without restriction, that $f = f^0 + z f^1 + \cdots + z^m f^m$, where $f^0 = x^m$.

We now suppose that $\mathcal{K}$ is invariant under $\Gamma$, and will formulate conditions that $\mathcal{I}$ is $\Gamma$-invariant, too. This is equivalent to the condition that $f$ is $\Gamma$-invariant modulo $\ell H^0(\mathcal{K}(m-1))$. By ([T2], Hilfsatz 1, p. 142) this is equivalent to the condition that $(x, y) \partial f/\partial z \subset \ell H^0(\mathcal{K}(m-1))$. It follows that $\ell$ is a divisor of $f^i, 1 \leq i \leq n$, i.e., one has $f = x^m + \ell z g, g \in S_{m-2}$.

Write $g = g^0 + zg^1 + z^2g^2 + \cdots$, where $g^i \in R_{m-2-i}$ and choose $u = u^0 + zu^1 + \cdots \in H^0(\mathcal{K}(m-2))$ such that $u^0 = g^0$. This is possible, if $m - 2 \geq e$. As $f$ is determined only modulo $\ell H^0(\mathcal{K}(m-1))$, one can replace $f$ by $\tilde{f} = f - \ell zu$. It follows that one can assume without restriction $f = x^m + \ell z^2 g, g \in S_{m-3}$.

Choose $u \in H^0(\mathcal{K}(m-3))$, where $u^0 = g^0$; this is possible, if $m - 3 \geq e$. If this is the case, replace $f$ by $\tilde{f} = f - \ell z^2 u$. It follows that one can assume without restriction $f = x^m + \ell z^3 g$, where $g \in S_{m-4}$, etc. Finally one obtains $f = x^m + z^{m-e} \ell g, \ell = ax + y, g \in S_{e-1}$, and the $\Gamma$-invariance of $f$ modulo $\ell H^0(\mathcal{K}(m-1))$ is equivalent to $(x, y)(m - e) z^{m-e-1} \ell g + z^{m-e} \ell \partial g/\partial z \subset \ell H^0(\mathcal{K}(m-1))$, i.e. equivalent to:

\[(1.1) \quad (x, y)((m - e) g + z \partial g/\partial z) \subset H^0(\mathcal{K}(e))\]

In the case $\ell = x$, because of $R_m = x R_{m-1} \oplus y^m \cdot k$, one can write $f^0 = y^m + xu$, and the same argumentation shows that one can write $f = y^m + z^{m-e} x g, g \in S_{e-1}$, and the $\Gamma$-invariance can again be expressed by the inclusion (1.1).

2.4.3. Standard forms. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2}$ have the colength $d$ and Hilbert function $\varphi$, and let $\mathcal{I}$ be invariant under $G := \Gamma \cdot T(\rho)$, where $\rho_2 > 0$. Moreover, we assume that $g^*(\varphi) > g(d)$. By Lemma 2.2 it follows that $\mathcal{I} = \ell \mathcal{K}(m-1) \oplus f \mathcal{O}_{\mathbb{P}^2}(-m)$, if $k = \mathcal{T}$ is supposed. As $H^0(\mathcal{I}(m-1)) = \ell H^0(\mathcal{K}(m-2))$ is then invariant under $G$ and $m - 1 > e = \text{reg}(\mathcal{K})$, it follows that $(\ell)$ and $\mathcal{K}$ are $G$-invariant. Assume that $(ax + by + z)$ is $\Gamma$-invariant. Then $(ax + by + z) = (a + \alpha) x + (b + \beta) y + z, \forall \alpha, \beta \in k$, which is not possible. Thus we have $\ell = ax + by$. From $(\lambda_0 ax + \lambda_1 by) = (ax + by), \forall (\lambda_0, \lambda_1, \lambda_2) \in T(\rho)$ it follows that $\lambda_0/\lambda_1 = 1 \forall (\lambda_0, \lambda_1, \lambda_2) \in T(\rho)$, if $a$ and $b$ both were different from 0. But then it would follow $T(\rho) \subset T(1, -1, 0)$, and therefore $\rho_2 = 0$, contradiction. Therefore we have $\ell = x$ or $\ell = y$, without restriction.

We consider the case $\ell = x$, for example. As it was shown in (2.4.2) we can write $f = y^m + z^{m-e} x g, e = \text{reg}(\mathcal{K}), g \in S_{e-1}$.

From Appendix E it follows that $x H^0(\mathcal{K}(m-1))$ has a standard basis of $T(\rho)$-semi-invariants $f_i = m_i p_i(X^\rho)$, i.e., $m_i$ is a monomial, $p_i$ is a polynomial in one variable with constant term 1, and such that $m_i$ does not occur in $f_j$ any longer, if $i \neq j$. Now each $f_i$ is divisible by $x$, therefore $y^m$ does not occur in $f_i$. If the initial monomial $m_i$ of $f_i$ appears in $f$, then $m_i$ has to appear in $z^{m-e} x g$. By choosing $\alpha \in k$ in a suitable way, one can achieve that $m_i$ does not occur in $\tilde{f} := f - \alpha f_i$. As $\rho_2 > 0$ and $f_i$ is divisible by $x$, $\tilde{f}$ still has the shape $y^m + z^{m-e} x \tilde{g}, \tilde{g} \in S_{e-1}$. By repeating this procedure one can achieve that none of the $m_i$ does occur in $f = y^m + z^{m-e} x g$ (and $f$ is still invariant under $\Gamma$ modulo...
The same argumentation as in the proof of the lemma in Appendix E then shows that $f$ is automatically a $T(\rho)$-semi-invariant with initial monomial $y^m$, and $f$ together with the $f_i$ forms a standard basis of $H^0(\mathcal{I}(m))$. We summarize:

**Lemma 2.6.** Let be $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2_k}$ an ideal of colength $d$, with Hilbert function $\varphi$, which is invariant under $G = \Gamma \cdot T(\rho)$, where $\rho_2 > 0$. Assume that $g(d) < g^*(\varphi)$. (It is not assumed that $k = \overline{k}$.) Then $\mathcal{I} = y \mathcal{K}(-1) + f \mathcal{O}_{\mathbb{P}^2}(-m)$ or $\mathcal{I} = y \mathcal{K}(-1) + f \mathcal{O}_{\mathbb{P}^2}(-m)$, where $\mathcal{K}$ is a $G$-invariant ideal with colength($\mathcal{K}$) = $c$, reg($\mathcal{K}$) = $e$, and $c+m = d$. Moreover $f \in H^0(\mathcal{K}(m))$ can be written in the form $f = y^m + z^{m-e}yg$ respectively in the form $f = x^m + z^{m-e}yg$, where $g \in S_{e-1}$. We have $(x, y)\partial f/\partial z \subset xH^0(\mathcal{K}(m-1))$ or $(x, y)\partial f/\partial z \subset yH^0(\mathcal{K}(m-1))$, respectively, and each of these inclusions are equivalent to the inclusion $(2.1)$ in section $(2.4.2)$. One has

$$H^0(\mathcal{I}(n)) = \begin{cases} xH^0(\mathcal{K}(n-1)) & \text{if } n < m, \\ xH^0(\mathcal{K}(n-1)) \oplus f k[y, z]_{n-m} & \text{if } n \geq m, \end{cases}$$

respectively

$$H^0(\mathcal{I}(n)) = \begin{cases} yH^0(\mathcal{K}(n-1)) & \text{if } n \leq m, \\ yH^0(\mathcal{K}(n-1)) \oplus f k[x, z]_{n-m} & \text{if } n \geq m. \end{cases}$$

If one chooses a basis $\{f_i\}$ of $xH^0(\mathcal{K}(m-1))$ or of $yH^0(\mathcal{K}(m-1))$, respectively, then one can choose $f$ in such a way that $f$ has the form and the properties mentioned above and together with the $f_i$'s forms a standard basis of $H^0(\mathcal{I}(m))$.

**Proof.** If $k = \overline{k}$ this follows from the foregoing discussion. One has, e.g. $\mathcal{I} \otimes \overline{k} = y \mathcal{K}(-1) + f \mathcal{O}_{\mathbb{P}^2}(-m)$, where $\mathcal{K} \subset \mathcal{O}_{\mathbb{P}^2_\overline{k}}$ and $f \in H^0(\mathcal{K}(m))$ have the properties mentioned above. One sees that $\overline{\mathcal{K}} = \mathcal{I} \otimes \overline{k} : y \mathcal{O}_{\mathbb{P}^2_\overline{k}}(-1)$ and therefore one has the exact sequence:

$$0 \rightarrow (\mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{K})(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} + y \mathcal{O}_{\mathbb{P}^2_\overline{k}}(-1) \rightarrow 0.$$ 

If $\mathcal{K} := \mathcal{I} = y \mathcal{O}_{\mathbb{P}^2}(-1)$, then the sequence

$$0 \rightarrow (\mathcal{O}_\mathbb{P^2}/\mathcal{K})(-1) \rightarrow \mathcal{O}_\mathbb{P^2}/\mathcal{I} \rightarrow \mathcal{O}_\mathbb{P^2}/\mathcal{I} + y \mathcal{O}_\mathbb{P^2}(-1) \rightarrow 0$$

is exact, too. Tensoring this sequence with $\overline{k}$ one obtains a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & (\mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{K} \otimes \overline{k})(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} + y \mathcal{O}_{\mathbb{P}^2_\overline{k}}(-1) \rightarrow 0 \\
0 & \rightarrow & (\mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{K})(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} \rightarrow \mathcal{O}_{\mathbb{P}^2_\overline{k}}/\mathcal{I} \otimes \overline{k} + y \mathcal{O}_{\mathbb{P}^2_\overline{k}}(-1) \rightarrow 0
\end{array}$$

with exact rows, where the first vertical arrow is obtained from the canonical injection $\mathcal{K} \otimes \overline{k} \hookrightarrow \mathcal{K}$ by tensoring with $\mathcal{O}_{\mathbb{P}^2_\overline{k}}(-1)$. It follows from this, that $\mathcal{K} \otimes \overline{k} \cong \mathcal{K}$. Because of $H^0(\mathcal{I}(m)) \otimes \overline{k} \cong H^0(\mathcal{I}(m) \otimes \overline{k})$ one obtains a standard basis of $T(\rho)$-semi-invariants of $H^0(\mathcal{I}(m) \otimes \overline{k})$ by tensoring a standard basis of $H^0(\mathcal{I}(m))$ with $\otimes \overline{1}_{\Gamma}$. As the elements of a standard basis are uniquely determined up to constant factors, it follows that $\overline{\mathcal{T}} = f \otimes_k 1_{\Gamma}$, where $f \in H^0(\mathcal{K}(m))$. Therefore $f$ has the form $x^m + z^{m-e}yg, g \in S_{e-1}$, if $\overline{\mathcal{T}}$ has the form $x^m + z^{m-e}y\overline{g}$, $\overline{g} \in S_{e-1} \otimes \overline{k}$. For reasons of dimension it follows
Lemma 2.6 follow by the same argumentation as in the case $G = H$.

Remark 2.3. If $I \neq J$ are two ideals of $\mathcal{O}_K$, then $\mathcal{O}_K$ is contained in $I$. Therefore $\mathcal{O}_K$ is contained in $I_n$. Hence $\mathcal{O}_K$ is contained in $I_n$.

Definition 2. The (uniquely determined) decomposition $I = x\mathcal{O}_K(I_{n-1}) + f\mathcal{O}_K(I_{n-1})$ or $I = y\mathcal{O}_K(I_{n-1}) + f\mathcal{O}_K(I_{n-1})$ of Lemma 2.6 is called $x$-standard form or $y$-standard form of $I$, respectively. (N.B. For the Hilbert function $\varphi$ of $I$ this definition implies that $g(d) < g^*(\varphi)$ is fulfilled.)

Corollary 2.3. Let be $R = k[x, y]$. If $I$ has $x$-standard form (resp. $y$-standard form), then $xR_{m-2}$ (resp. $yR_{m-2}$) is contained in $H^0(I_{m-1})$ and thus $xR_{m-1}$ (resp. $yR_{m-1}$) is contained in $H^0(I_{m})$.

Proof. One has $m - 2 \geq c$ by Lemma 2.4 and thus $xR_{m-2} \subset xH^0(K(m-2))$ (resp. $yR_{m-2} \subset yH^0(K(m-2))$) by Appendix C, Remark 2.

Remark 2.3. If $I$ has $x$-standard form or $y$-standard form, respectively, and if $G_m$ acts on $S$ by $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z, \lambda \in k^*$, then $I_0 := \lim_{\lambda \to 0} \sigma(\lambda)I$ again has $x$-standard form or $y$-standard form, respectively. This follows from $h^0(I_0(n)) = h^0(I(n))$, $n \in \mathbb{N}$ (cf. [G2], Lemma 4, p. 542), because then $H^0(I_0(n))$ is generated by the initial monomials of a standard basis of $H^0(I(n))$, $n \in \mathbb{N}$. The Figures 2.13a and 2.13b show the typical shape of the pyramid formed by the initial monomials.

Remark 2.4. An ideal cannot have $x$-standard form and $y$-standard form at the same time. For $m = \text{reg}(I)$ and $c = \text{colength}(I)$ are determined by the Hilbert function $\varphi$ of $I$. If $I$ would have $x$-standard form as well as $y$-standard form, then $E(H^0(I_0(d)))$ has the form shown in Figure 2.14a. As $I$ and $I_0$ have the same Hilbert function, $\varphi$ has the form shown in Figure 2.14b, and therefore $g^*(\varphi) \leq g^*(\chi) = g(d)$, where $\chi$ is the Hilbert function of (2.2.2).

Remark 2.5. (Notations and assumption as in Remark 2.3) $I_\infty := \lim_{\lambda \to \infty} \sigma(\lambda)I$ has again $x$-standard form or $y$-standard form, respectively. The reasoning is as follows: The Hilbert function $\vartheta$ of $I_\infty$ (respectively the regularity $\mu$ of $I_\infty$) is $\geq \varphi$ (respectively $\geq m$). This follows from the semicontinuity theorem. By Lemma 2.4 it follows that $m \geq c+2$, therefore $R_{m-2} \subset H^0(K(m-2))$ (cf. Appendix C, Remark 2). If $I$ has $x$-standard form (respectively $y$-standard form), then $xR_{m-2} \subset H^0(I_{m-1})$ (respectively $yR_{m-2} \subset H^0(I_{m-1})$) follows. Therefore $xR_{m-2} \subset H^0(I_{\infty}(m-1))$ (respectively $yR_{m-2} \subset H^0(I_{\infty}(m-1))$). It follows that $xR_{m-2} \subset H^0(I_{\infty}(\mu-1))$ (respectively $yR_{m-2} \subset H^0(I_{\infty}(\mu-1))$). Now $I_\infty$ has $x$-standard form or $y$-standard form, in any case, and the above inclusions show that $I$ and $I_\infty$ have the same kind of standard form.

2.5. The type of a $G$-invariant ideal

Let $I \subset \mathcal{O}_K$ be an ideal of colength $d$, with Hilbert function $\varphi$ and invariant under $G = \Gamma \cdot T(\rho)$, where $\rho_2 > 0$ and $k$ is an extension field of $\mathbb{C}$. 


2.5.1.

**Definition 3.** 1° \( \mathcal{I} \) has the type \((-1)\), if one of the following cases occurs:
1st case: \( g^*(\varphi) \leq g(d) \), where \( d \geq 5 \) by convention.
2nd case: \( 0 \leq d \leq 4 \)

2° We now assume \( g^*(\varphi) > g(d) \), which notation implies \( d \geq 5 \). Then one has \( \mathcal{I} = \ell_0 \mathcal{I}_1(-1) + f_0 \mathcal{O}_{P^2}(-m_0) \), where \((\ell_0, \mathcal{I}_1, f_0, m_0)\) is as in Lemma 2.6. If \( \mathcal{I}_1 \) has type \((-1)\), then we say \( \mathcal{I} \) has type 0. If \( \mathcal{I}_1 \) is not of type \((-1)\), then \( \mathcal{I}_1 = \ell_1 \mathcal{I}_2(-1) + f_1 \mathcal{O}_{P^2}(-m_1) \), where \((\ell_1, \mathcal{I}_2, f_1, m_1)\) is as in Lemma 2.6. If \( \mathcal{I}_2 \) has the type \((-1)\), then we say \( \mathcal{I} \) has the type 1, etc. As \( d = \text{collen}(\mathcal{I}) = \text{collen}(\mathcal{I}_1) + m_0, \) etc., the colengths of the ideals in question decrease and the procedure will terminate. We have

**Lemma 2.7.** If \( \mathcal{I} \) has not the type \((-1)\), then one has a sequence of decompositions

\[
\begin{align*}
\mathcal{I} &=: \mathcal{I}_0 = \ell_0 \mathcal{I}_1(-1) + f_0 \mathcal{O}_{P^2}(-m_0), \\
\mathcal{I}_1 &= \ell_1 \mathcal{I}_2(-1) + f_1 \mathcal{O}_{P^2}(-m_1), \\
\mathcal{I}_r &= \ell_r \mathcal{K}(-1) + f_r \mathcal{O}_{P^2}(-m_r).
\end{align*}
\]

(2.2)

For a given ideal, \( \mathcal{I}_i \) \((\ell_i, \mathcal{I}_{i+1}, f_i, m_i)\) is defined as in Lemma 2.6, where \( 0 \leq i \leq r \) and \( \mathcal{I}_{r+1} := \mathcal{K} \). If \( d_i \) and \( \varphi_i \) is the colength and the Hilbert function, respectively, of \( \mathcal{I}_i \), then the inequality \( g(d_i) < g^*(\varphi_i) \) is fulfilled. The ideal \( \mathcal{K} \) has the type \((-1)\). The colength (the Hilbert function, the regularity) of \( \mathcal{K} \) is denoted by \( c \) (by \( \psi \) and \( \kappa \), respectively).

3° We have already noted in Corollary 2.1 that the decompositions of \( \mathcal{I}_0, \mathcal{I}_1, \cdots \), are uniquely determined, in essence. Therefore the number \( r \) is determined uniquely. It is called the type of \( \mathcal{I} \). The types of the ideals occurring in (2.2), their Hilbert functions and the numbers \( m_0, \cdots, m_r \) are uniquely determined by the Hilbert function \( \varphi \). \( \square \)

2.5.2. **Properties of an ideal of type** \( r \geq 0 \). The assumptions and notations are as before, and we assume that \( \mathcal{I} \) has the type \( r \geq 0 \).

**Lemma 2.8.** In the decompositions (2.2) of \( \mathcal{I} \) one has:

(a) \( \text{collen}(\mathcal{I}_i) = \text{collen}(\mathcal{I}_{i+1}) + m_i \), i.e., \( \text{collen}(\mathcal{I}_i) = c + m_r + \cdots + m_i \), \( 0 \leq i \leq r \).
(b) If \( r = 0 \), then \( m_0 \geq c + 2 \), where \( c = \text{collen}(\mathcal{K}) \).
(c) If \( r \geq 1 \), then \( m_0 \geq c + 2 + m_r + \cdots + m_1 = \text{collen}(\mathcal{I}_1) + 2 \).
(d) If \( r \geq 0 \), then \( m_0 \geq 2^r(c + 2) \).

**Proof.** (a) follows from the decompositions (2.2) and from Lemma 2.6.
(b) follows from Lemma 2.4.
(c) As the statement only depends on the Hilbert functions of \( \mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{r+1} = \mathcal{K} \) one can assume without restriction \( \ell_i = x, 0 \leq i \leq r \), and \( \mathcal{I}_i \) is \( B(3;k) \)-invariant, \( 0 \leq i \leq r+1 \). Then one has

\[
\begin{align*}
\mathcal{I} &= x\mathcal{I}_1(-1) + y^{m_0} \mathcal{O}_{P^2}(-m_0) \quad \text{and} \quad \mathcal{I}_1 = x\mathcal{I}_2(-1) + y^{m_1} \mathcal{O}_{P^2}(-m_1),
\end{align*}
\]

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where $I_2 = \mathcal{K}$, if $r = 1$.

We argue as in (2.3.4) and we orientate ourselves by Figure 2.15. If one would have $m_0 - 1 - m_1 \leq \text{colength}(I_2)$, then one could make the deformations

\[ 1 \mapsto 1, \ldots, [m_0 - 1 - m_1] \mapsto m_0 - 1 - m_1. \]

Then we would get $g^*(\varphi) < \cdots \leq g(d)$, contradiction, because from type $r \geq 0$ it follows that $g^*(\varphi) > g(d)$. Therefore one has $m_0 - 1 - m_1 > \text{colength}(I_2)$. Now \text{colength}(I_2) = c, if $r = 1$ (respectively \text{colength}(I_2) = c + m_r + \cdots + m_2$, if $r \geq 2$) as was shown in (a).

(d) If $r = 0$, this is statement (b). If $r = 1$, then by (b) and (c) it follows that $m_0 \geq (c + 2) + m_1 \geq 2(c + 2)$. Now we assume $r \geq 2$. We argue by induction and assume that $m_r \geq c + 2, m_{r-1} \geq 2(c + 1), \ldots, m_1 \geq 2^{r-1}(c + 2)$. By (c) it follows that

\[ m_0 \geq (c + 2) + (c + 2) + 2(c + 2) + \cdots + 2^{r-1}(c + 2) = 2^r(c + 2). \]

In the case $r = 1$, the statement (c) is valid even if $\mathcal{K} = \mathcal{O}_{\mathbb{P}^2}$, i.e., if $c = 0$. Because then one can assume without restriction again

\[ I = xI_1(-1) + y^{m_0}\mathcal{O}_{\mathbb{P}^2}(-m_0) \quad \text{and} \quad I_1 = x\mathcal{O}_{\mathbb{P}^2}(-1) + y^{m_1}\mathcal{O}_{\mathbb{P}^2}(-m_1), \]

and then, because of \text{colength}(I_1) = m_1, by Lemma 2.4 it follows that $m_0 \geq m_1 + 2$, i.e., one gets the statement (c).

\[ \textbf{Corollary 2.4} \quad \text{(of the proof of Lemma 2.8). If} \ I \ \text{has type} \ r \geq 1, \ \text{then} \ m_j + j < m_i + i - 1 \ \text{for all} \ 0 \leq i < j \leq r. \]

\[ \textbf{Proof.} \ \text{We refer to Lemma 2.8 and use the same notations. If the sequence of decompositions (2.2) in (2.5.1) begins with} \ I_i \ \text{instead of} \ I_0, \ \text{then from Lemma 2.8c it follows that} \ m_{i-1} \geq c + 2 + m_r + \cdots + m_i. \ \text{It follows that} \ m_{i-1} - m_i \geq 2, \ \text{therefore} \ m_{i-1} + (i - 1) > m_i + i. \ \text{One gets} \]

\[ m_r + r < m_{r-1} + (r - 1) < \cdots < m_1 + 1 < m_0. \]

If $m_j + j = m_i + (i - 1)$, then it would follow $j = i + 1$ and therefore $m_i + 1 = m_i - 2$. We will show, that this is not possible.

The ideal $I_i$ has the type $r - i \geq 1$, the colength $c + m_r + \cdots + m_i = d_i$, and the Hilbert function $\varphi_i$ of $I_i$ fulfills the inequality $g(d_i) < g^*(\varphi_i)$ (cf. Lemma 2.7). As to the Hilbert function $\varphi_i$, one obtains the situation of Figure 2.16, which corresponds to the Hilbert function $\varphi$. But then it follows that $g^*(\varphi_i) \leq g^*(\chi') = g(d_i)$, where the graph of $\chi'$ is denoted by a dotted line (cf. the first case in (2.2.2), Figure 2.8, and the argumentation in the proof of Lemma 2.2). \qed
\#\{monomials between graph of $\varphi'_2(n-1)$ and line $y = x - 1\} = \text{colength}(I_2)$
A rough description of ideals invariant under $\Gamma \cdot T(\rho)$

It seems impossible to characterize ideals of colength $d$ in $\mathcal{O}_{p^2}$, which are invariant under $G := \Gamma \cdot T(\rho)$. If $\mathcal{I}$ is an ideal of type $r$ in the sense of the definition in (2.5.1), however, then one can give a rough description of the forms $f_0, \ldots, f_r$, which occur in the decomposition (2.2) in (2.5.1). This description will be used later on for estimating the so called $\alpha$-grade of $\mathcal{I}$ which will be defined in the next chapter.

3.1. Assumptions

Let $\mathcal{I} \subset \mathcal{O}_{p^2}$ be an ideal of colength $d$, invariant under $g \cdot T(\rho)$, where $\rho_2 > 0$, as usual. We assume that $\mathcal{I}$ has the type $r \geq 0$. Then according to Lemma 2.7 one has a decomposition:

$$
\mathcal{I} =: \mathcal{I}_0 = \ell_0 \mathcal{I}_1(-1) + f_0 \mathcal{O}_{p^2}(-m_0),
$$

$$
\mathcal{I}_1 = \ell_1 \mathcal{I}_2(-1) + f_1 \mathcal{O}_{p^2}(-m_1),
$$

\[\vdots\]

$$
\mathcal{I}_r = \ell_r \mathcal{I}_{r+1}(-1) + f_r \mathcal{O}_{p^2}(-m_r).
$$

Using the notations of (loc.cit.) the ideal $\mathcal{I}_{r+1} := \mathcal{K}$ has the colength $c$ and regularity $\kappa$. If $\ell_i = x$, then

$$
H^0(\mathcal{I}_i(n)) = x H^0(\mathcal{I}_{i+1}(n-1)) + f_i k[y, z]_{n-m_i}
$$

and if $\ell_i = y$, then

$$
H^0(\mathcal{I}_i(n)) = y H^0(\mathcal{I}_{i+1}(n-1)) + f_i k[x, z]_{n-m_i}
$$

(cf. Lemma 2.6). From (Z) follows

$$
H^0(\mathcal{I}_1(m_i + 1)) = \ell_1 H^0(\mathcal{I}_2(m_i + 1 - 2)),
$$

$$
H^0(\mathcal{I}_2(m_i + 1 - 2)) = \ell_2 H^0(\mathcal{I}_3(m_i + 1 - 3)),
$$

\[\vdots\]

$$
H^0(\mathcal{I}_{i-1}(m_i + 1)) = \ell_{i-1} H^0(\mathcal{I}_i(m_i)),
$$

$$
H^0(\mathcal{I}_i(m_i)) = \ell_i H^0(\mathcal{I}_{i+1}(m_i - 1)) + \langle f_i \rangle,
$$

for by Corollary 2.4 $m_i + 2 < m_{i-1}, i = 1, \ldots, r$. If one starts with $H^0(\mathcal{I}_1(m_i + 1 - 2))$, then one obtains a similar system of equations, whose last line is

$$
H^0(\mathcal{I}_i(m_i - 1)) = \ell_i H^0(\mathcal{I}_{i+1}(m_i - 2)).
$$
Conclusion 3.1. If \(2 \leq i \leq r\) (if \(1 \leq i \leq r\), respectively), then \(H^0(\mathcal{I}_1(m_i + i - 1)) = \ell_1 \cdots \ell_{i-1}H^0(\mathcal{I}_i(m_i)) (H^0(\mathcal{I}_1(m_i + i - 2)) = \ell_1 \cdots \ell_iH^0(\mathcal{I}_{i+1}(m_i - 2))\), respectively. \(\square\)

3.2. Notations

We orientate ourselves by Figure 3.1, which shows the initial monomials of \(H^0(\mathcal{I}(m_0))\). The set of all monomials in \(S_{m_0}\) with \(z\)-degree \(\geq m_0 - (c + r)\) (with \(z\)-degree \(\leq m_0 - (c + r + 1)\), respectively) is called the left domain and is denoted by \(\mathcal{LB}\) (is called right domain and is denoted by \(\mathcal{RB}\), respectively). The monomials in \(S_{c-1}\ in \(H^0(K(c - 1))\) form a basis of \(S_{c-1}/H^0(K(c - 1))\). If we put \(\ell = \ell_0 \cdots \ell_r\), then \(\ell[S_{c-1}/H^0(K(c - 1))]\) has a basis consisting of the \(c\) monomials in \(\ell S_{c-1} - \ell\ in \(H^0(K(c - 1))\)).

The initial monomial of \(\ell_0 \cdots \ell_{i-1} f_i \cdot z^{m_0 - m_i - i}\) is \(M_i^{\text{up}} := x^{i-\ell(i)} y^{m_i+\ell(i)} z^{m_0 - m_i - i}\) or \(M_i^{\text{down}} := x^{m_i+\ell(i)} y^{i} z^{m_0 - m_i - i}\), if \(\ell_i = x\) or if \(\ell_i = y\), respectively. Here we have put, if \(1 \leq i \leq r + 1, \ell (i) := \) number of indices \(0 \leq j < i\) such that \(\ell_j = y\). We also put \(\ell(0) = 0\), so the formulas give the right result for \(i = 0\), too.

For instance, in Figure 3.1 we have
\[r = 5, \ \ell_0 = x, \ \ell_1 = y, \ \ell_2 = \ell_3 = x, \ \ell_4 = \ell_5 = y,\]
and therefore
\[\ell(1) = 0, \ \ell(2) = \ell(3) = \ell(4) = 1, \ \ell(5) = 2, \ \ell(6) = 3.\]
(N.B. \(\ell_0 \cdots \ell_{i-1} = x^{i-\ell(i)} y^{i}\), if \(0 < i \leq r + 1\).)

As always we assume \(\rho_2 > 0\). If \(M_i^{\text{down}}\) occurs, then \(\mathcal{I}_i = y\mathcal{I}_{i+1}(-1) + f_i\mathcal{O}_{\mathbb{P}^2}(-m_i)\). If \(M_i^{\text{up}}\) occurs, then \(\mathcal{I}_i = x\mathcal{I}_{i+1}(-1) + f_i\mathcal{O}_{\mathbb{P}^2}(-m_i)\). By Lemma 2.8 \(m_i \geq (c + 2) + m_r + \cdots + m_{i+1}\), if \(r \geq 1 (m_0 \geq c + 2, \text{if } r = 0)\). The colength of \(\mathcal{I}_{i+1}\) equals \(c + m_r + \cdots + m_{i+1}\), and therefore \(R_n \subset H^0(\mathcal{I}_{i+1}(n))\) if \(n \geq m_i - 2 \geq c + m_r + \cdots + m_{i+1}\) in the case \(r \geq 1(R_n \subset H^0(K(n))\) if \(n \geq c\) in the case \(r = 0\). This follows from Remark 2 in Appendix C. The next generating element \(f_i\) has the initial monomial \(M_i^{\text{up}}\) or \(M_i^{\text{down}}\).

Conclusion 3.2. Suppose \(\rho_2 > 0\) and \(\rho_1\) is arbitrary. In order to somewhat simplify the notation, put \(F = x^{i-\ell(i)} y^{i} z^{m_0 - m_i - i}\). If \(M_i^{\text{up}}\) appears, then \(F \cdot x \cdot z \cdot R_{m_{i-2}}\) and \(F \cdot x \cdot R_{m_{i-1}}\) are contained in \(H^0(\mathcal{I}(m_0))\). If \(M_i^{\text{down}}\) appears, then \(F \cdot y \cdot z \cdot R_{m_{i-2}}\) and \(F \cdot y \cdot R_{m_{i-1}}\) are contained in \(H^0(\mathcal{I}(m_0))\). Thus \(xM_i^{\text{up}}\) or \(yM_i^{\text{down}}\), when it exists, is contained in \(\mathcal{I}\).

Proof. This follows from , Conclusion 3.1, the decompositions (Z) and the foregoing discussion. \(\square\)

As always we assume \(\rho_2 > 0\). We have to distinguish between two cases (cf. 2.4.1 Auxiliary Lemma 1 and Auxiliary Lemma 2):

Main Case I: \(\rho_0 > 0\) and \(\rho_1 < 0\)

Main Case II: \(\rho_0 < 0\) and \(\rho_1 > 0\)
3.3. Description of $f_0$ in Case I

If the initial monomial of $f_0$ equals $x^{m_0}$, then $f_0 = x^{m_0}$ and $\ell_0 = y$. Therefore we may assume that $f_0$ is a proper semi-invariant with initial monomial $y^{m_0}$. Then $I = I_0 = xI_1(1) + f_0O_{y^2}(-m_0)$, and we write $f_0 = y^{m_0} + z^\mu xG$, $\mu \in \mathbb{N}$ maximal, $G$ a $T(\rho)$-semi-invariant (cf. 2.4.3). If $G^0$ is the initial monomial of $G$, then $N := z^\mu \cdot x \cdot G^0$ is called the vice-monomial of $f_0$, which by assumption is not equal to zero.

The inclusion (2.1) of (2.4.2) reads
\[
\langle x, y \rangle [\mu G + z \partial G / \partial z] \subseteq H^0(I_1(m_0 - \mu))
\]

As for the initial monomial $G^0$ of $G$, one has
\[
\langle x, y \rangle G^0 \subseteq \text{in}(H^0(I_1(m_0 - \mu))) = H^0(\text{in}(I_1(m_0 - \mu)))
\]

where \text{in} denotes the subspace or the initial ideal, respectively, generated by the initial monomials. (Equality follows, for example, from $\text{in}(I) = \lim_{\lambda \rightarrow 0} \sigma(\lambda) I$ and [G2], Lemma 3 and 4.) Without restriction we may assume that $G^0 \notin \text{in}(H^0(I_1(m_0 - \mu - 1)))$, because we can reduce $f_0$ modulo $xH^0(I_1(m_0 - 1))$.

**Notabene:** The same statements are analogously valid in the Case II, that means, if $\rho_1 > 0$, $I = yI_1(-1) + f_0O_{y^2}(-m_0), f_0 = x^{m_0} + z^\mu yG$, etc.

It follows from this, that one of the following cases occurs, where $1 \leq i \leq r$:

1° $N = N_i^{\text{down}} := (z/x)M_i^{\text{down}} = x^{m_1 + i - i(i - 1)}y^{i(i)}z^{m_0 - m_1 - i + 1}$

2° $N = N_i^{\text{up}} := (z/y)M_i^{\text{up}} = x^{i - i(i)}y^{m_1 + i - i - 1}z^{m_0 - m_1 + i + 1}$

3° $\langle x, y \rangle N \subseteq \ell \text{in}(H^0(\mathcal{K}(m_0 - r)))$, where $\ell := \ell_0 \cdots \ell_r$ and $N$ is a monomial in $\mathcal{L} := \ell S_{m_0 - r + 1} - \ell \text{in}(H^0(\mathcal{K}(m_0 - r - 1)))$.

**Notabene:** One has $\mathcal{L} = [\ell S_{c - 1} - \ell \cdot \text{in}(H^0(\mathcal{K}(c - 1)))] \cdot z^{m_0 - r - c}$, because of
\[
\bigoplus_{\nu = c} z^{m_0 - r - 1 - \nu} R_\nu \subseteq H^0(\mathcal{K}(m_0 - r - 1)).
\]

As $\mu$ is maximal and $\rho_2 > 0$, $G^0$ is not divisible by $z$. In the cases 1° and 2° we therefore have $\mu = m_0 - m_1 - i + 1$. The case 3° will be treated in (3.3.6), and we assume the case 1° or 2°. Then (3.1) can be written as
\[
\langle x, y \rangle [\mu G + z \partial G / \partial z] \subseteq H^0(I_1(m_0 + i - 1)).
\]

If we put $h := \ell_1 \cdots \ell_{i-1}$, then $h = x^{i - i(i)}y^{i(i)}$, because of $\ell_0 = x$ one has $i(i) =$ number of indices $j$ such that $0 < j < i$ and $\ell_j = y$. If $i = 1$, then $h := 1$. By Conclusion 1 it follows from (3.1), that $G$ is divisible by $h$, that means, one has $G = hg, g \in S_{m_1-1}$. Therefore we can write
\[
\quad f_0 = y^{m_0} + z^\mu xhg
\]

and (3.1) is equivalent to:
\[
\langle x, y \rangle [\mu G + z \partial G / \partial z] \subseteq H^0(I_1(m_1)).
\]

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3.3.1. We assume the case 1°. As $N_i^{\downarrow}$ occurs, $I_i = yI_{i+1}(-1) + x^{m_i}O_{2z}(-m_i)$. If $g^0$ is the initial monomial of the form $g$ (cf. the inclusion (3)), then $z^\mu xG^0 = h_g g^0 xz^\mu = x^{\nu \cdot i}(i) y^i(i) z^\mu g^0 = N_i^{\downarrow}$. Because of $\mu = m_0 - m_i - i + 1$ it follows that $g^0 = x^{m_i-1}$.

Representing the initial monomials of $H^0(I_i(m_i))$ in the same way as the initial monomials of $H^0(I_0(m_0))$, one sees that southwest of $N_i^{\downarrow}$ there is no further monomial which occurs in $f_0$.

Conclusion 3.3. Let be $\rho_1 < 0$. If the vice-monomial of $f_0$ equals $N_i^{\downarrow}$, then $f_0 = y^{m_0} + \alpha N_i^{\downarrow}$, where $\alpha \in k$. As $f_i = M_i^{\downarrow}$ is a monomial, by Conclusion 3.2 it follows that all monomials which have the same $z$-degree as $M_i$ and are elements of $in(H^0(I_0(m_0)))$ also are elements of $H^0(I(m_0))$. Therefore we have $(x, y)g^{m_0} < I$.

3.3.2. We assume the case 2°. Then $y^{m_0}X^{\nu} = N_i^{\uparrow}$, where $\nu > 0$ and $1 \leq i \leq r$.

This statement is equivalent to the equations:

(3.4) \quad \nu \rho_0 = i - \nu(i), \quad \nu \rho_1 = m_i + \nu(i) - 1 - m_0, \quad \nu \rho_2 = m_0 - m_i - i + 1.

Lemma 3.1. If one has $f_0 = y^{m_0} + \alpha N_i^{\uparrow}$, where $1 \leq i \leq r$ and $\alpha \in k - (0)$, then $\rho_2 > m_{i+1} = \text{reg}(I_{i+1})$, where we put $I_{r+1} := K$ and $m_{r+1} := \kappa = \text{reg}(K)$. Especially $I_j$ is a monomial ideal for all $j > i$.

Proof. As we have assumed $\rho_1 < 0$, by (2.4.1 Auxiliary Lemma 2) one has $\rho_0 > 0$. Therefore $\nu \leq i$. On the other hand $\rho_2 = (m_0 - m_i - 1)/\nu \geq (m_0 - m_i - 1)/i$; thus it suffices to show $(m_0 - m_i - 1)/i > m_{i+1}$, i.e.

(3.5) \quad m_0 > m_i + i \cdot m_{i+1} + (i - 1).

We start with $i = r$, in which case one has to show $m_0 > m_r + r\kappa + (r - 1)$. This is valid if $r = 0$, so we may assume $r \geq 1$. By Lemma 2.8 one has $m_0 \geq c + 2 + m_r + \cdots + m_1$, thus it suffices to show $c + 2 + m_r + \cdots + m_1 > m_r + r\kappa + (r - 1)$. If $r = 1$, then this inequality is equivalent to $c + 2 > \kappa$, and this is true. Therefore one can assume without restriction that $r > 1$, and has to show

$$c + 2 + m_{r-1} + \cdots + m_1 > r\kappa + (r - 1).$$

Because of $\kappa \leq c$ it suffices to show $2 + m_{r-1} + \cdots + m_1 > (r - 1)(\kappa + 1)$ which is true because of $\kappa \leq c < m_{r-1} < \cdots < m_1$ (cf. Lemma 2.4 and Corollary 2 4).

Now we assume $1 \leq i < r$, in which case it suffices to show:

$$c + 2 + m_r + \cdots + m_1 > m_i + i \cdot m_{i+1} + (i - 1)$$

$$\iff (c + 2) + (m_r + \cdots + m_{i+2}) + (m_{i-1} + \cdots + m_1) > (i - 1)m_{i+1} + (i - 1).$$

On the left side of this inequality the second summand (the third summand, respectively) does not occur, if $i + 1 = r$ (if $i = 1$, respectively). If $i = 1$ and $r = 2$, the inequality reads $c + 2 > 0$. If $i = 1$ and $r \geq 3$, the inequality reduces to $(c + 2) + m_r + \cdots + m_3 > 0$. Thus the case $i \geq 2$ remains, and it suffices to show $m_{i-1} + \cdots + m_1 > (i - 1)(m_{i+1} + 1)$, which is true because of $m_{i+1} < m_i < \cdots < m_1$ (loc.cit.).
3.3.3. Description of the ideal $I_i$. The assumptions are the same as in Lemma 3.1, but we slightly change the notations and write $I = I_i$, $K = I_{i+1}$, $m = m_i$, $e = \text{reg}(K)$. (Thus $e = m_{i+1}$ or $e = \kappa$.) We have the following situation: $I = xK(-1) + fO_{Kz}(-m)$ is $\Gamma \cdot T(\rho)$-invariant, $K$ is monomial, $\rho_1 < 0$, $\rho_2 > e = \text{reg}(K)$. From the results of (2.4.2) and Lemma 2.6 it follows that $f = y^{m_i} + z^{-m_i+1}xu$, where $g \in S_{e-1}$ and

$$\langle x, y \rangle[(m - e)g + z\partial g/\partial z] \subset H^0(K(e))$$

where $f \in H^0(K(m))$ is determined only modulo $xH^0(K(m - 1))$ and we can assume that $f$ is a $T(\rho)$-semi-invariant. Then $g$ is a $T(\rho)$-semi-invariant, too. Thus we can write $g = N(1 + a_1X^e + \cdots)$, where $N \in S_{e-1}$ is a monomial. If we assume $a_1 \neq 0$ for example, then $NX^e \in S_{e-1}$ would have a $z$-degree $\geq \rho_2$. As $\rho_2 > e$ Lemma 3.1, this is impossible.

**Conclusion 3.4.** Under the assumptions mentioned above, the form $g$ in (3.6) equals a monomial $N \in S_{e-1} - H^0(K(e - 1))$ such that $\langle x, y \rangle N \subset H^0(K(e))$. It follows that $(x, y)y^m \subset I$. □

3.3.4. We go back to the notations of Lemma 3.1, i.e. one has $N = N_i^{up}$. As in the case $1^o$ one has $\mu = m_0 - m_i - i + 1$. As $y^{m_0}$ is the initial monomial of $f_0$, $\ell_0$ equals $x$ and $h := \ell_1 \cdots \ell_{i-1}$ equals $x^{i-1-\ell(i)}y^{\ell(i)}$ as before. If one puts $\overline{g} = \mu g + z\partial g/\partial z$, then (3.3) can be written as

$$\langle x, y \rangle \overline{g} \subset xH^0(I_{i+1}(m_i - 1)) + \langle f_i \rangle$$

where $f_i$ has the initial monomial $y^{m_i}$; thus

$$\overline{g} \in H^0(I_{i+1}(m_i - 1)).$$

The initial monomial of $\overline{g}$ equals the initial monomial of $g$; by construction this is equal to $y^{m_i-1}$. (Proof: The initial monomial of $xhg^\mu = xGz^\mu$ is equal to $N_i^{up}$ by assumption. Thus the initial monomial $g_0$ of $g$ fulfils the equation

$$xhg^0z^\mu = x^{i-\ell(i)}y^{m_i+\ell(i)-1}z^{m_0-m_i-i+1}.$$  

As $xh = x^{i-\ell(i)}y^{\ell(i)}$ and $\mu = m_0 - m_i - i + 1$, it follows that $g_0 = y^{m_i-1}$. From (3.7) it follows that

$$\overline{g} = xf + \alpha f_i, \; F \in H^0(I_{i+1}(m_i - 1)), \; \alpha \in k^*.$$  

Now we can write $f_i = y^{m_i} + z^{m_i-m_i-1}xu$ (cf. the last sentence in (2.4.2)), where $u$ is either a monomial in $S_{m_i+1} - H^0(I_{i+1}(m_{i+1} - 1))$ (cf. Conclusion 2.4), or $u = 0$.

We consider the first case. Then it follows that $z^{m_i-m_i-1}xu = \beta y^{m_i}X^\nu$, where $\beta \in k^*$ and $\nu \geq 1$. As $f_i$ can be reduced modulo $xH^0(I_{i+1}(m_i - 1))$, we have $z^{m_i-m_i-1}xu \notin xH^0(I_{i+1}(m_i - 1))$ without restriction. From (3.9) it follows that in $\overline{g}$, except for $y^{m_i-1}$, the monomial $\overline{u} := z^{m_i-m_i-1}xu/y$ occurs.

Suppose, there is another monomial $v$ in $\overline{g}$. Then one would have $yv \in xH^0(I_{i+1}(m_i - 1))$, and from (3.8) it would follow that $v$ is an element of the monomial subspace $H^0(I_{i+1}(m_i - 1))$. Figure 3.2 shows $H^0(I_{i+1}(m_i)) = xH^0(I_{i+1}(m_i - 1)) + \langle f_i \rangle$ marked with — and $H^0(I_{i+1}(m_{i+1}))$ marked with —. Suppose that $v \in H^0(I_{i+1}(m_i - 1)) = xH^0(I_{i+1}(m_i - 2))$. By construction $v$ occurs in $g$, therefore $xhv = \ell_0 \cdots \ell_i v$ occurs in $xG$.  

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and $z^uxhv$ occurs in $z^uxG$. On the other hand, $z^uxhv \in z^ux\ell_1 \cdots \ell_{i-1}xH^0(\mathcal{I}_{i+1}(m_i-2)) = z^uxH^0(\mathcal{I}_1(m_i + i - 2)) \subset xH^0(\mathcal{I}_1(m_0-1))$, because $\ell_i = x$ and so Conclusion 3.1 can be applied. As $f_0$ can be reduced modulo $xH^0(\mathcal{I}_1(m_0-1))$, one can assume without restriction, that $v$ does not occur in $g$ and therefore does not occur in the inclusions (3.7) or (3.8), which have to be fulfilled. Thus, we can assume without restriction, that $v \in H^0(\mathcal{I}_{i+1}(m_i-1)) - H^0(\mathcal{I}_1(m_i-1))$. From $yv \in xH^0(\mathcal{I}_{i+1}(m_i-1))$ it follows that $v$ is equal to one of the monomials in Figure 3.2, which are denoted by ?. Therefore the $z$-degree of $v$ is $\geq m_i - m_{i+1}$.

By construction, the $z$-degree of $\overline{v}$ is $\geq m_i - m_{i+1}$, too. As $\overline{v}$ and $v$ occur in the semi-invariant $\overline{g}$, both monomials differ by a factor of the form $X^\rho$. As $\rho > m_{i+1}$ (Lemma 3.1), it follows that $v = 0$, i.e., $\overline{v}$ and $v$ differ by a constant factor. Therefore one has

$$\overline{g} = \beta y^{m_i-1} + \gamma z^{m_i-m_{i+1}}xu/y ; \beta, \gamma \in k^*.$$

We have to describe the position of $xu$ more exactly. From $xu / \notin xH^0(\mathcal{I}_{i+1}(m_i+1-1))$ but $(x, y)xu \in xH^0(\mathcal{I}_{i+1}(m_i+1))$ and from the $\Gamma$-invariance of $\mathcal{I}_{i+1}$ it follows that $z^{m_0-m_i}xu$ equals $N_j^{\text{down}}$ or $N_j^{\text{up}}$, where $j$ is an index $r \geq j > i$, or equals a monomial $L \in \mathcal{L}$ (cf. 3.2).

Suppose $z^{m_0-m_i}xu = N_j^{\text{down}}$. Then $\overline{\mathcal{L}} = z^{m_0-m_i}xu/y$ is southwest of $N_j^{\text{down}}/z^{m_0-m_i}$ and does not occur in the monomial subspace $H^0(\mathcal{I}_{i+1}(m_i-1))$, which contradicts (8). Finally we note that the monomials of $\overline{g}$ agree with the monomials of $g$.

**Conclusion 3.5.** Assume that $f_0$ has the vice-monomial $N_j^{\text{up}}$ and that $f_i$ is not a monomial. Then (up to a constant factor) $f_i = M_i^{\text{up}} + \alpha N_j^{\text{up}}$, where $1 \leq i < j \leq r$ and $\alpha \in k^*$, or $f_i = M_i^{\text{up}} + \alpha L$, where $L \in \mathcal{L}$ is a monomial such $(x, y)L \subset \ell K(-r-1)$, and $\alpha \in k^*$.

Then it follows that $f_0 = y^{m_0} + \beta N_j^{\text{up}} + \gamma N_j^{\text{up}} \cdot (z/y)$ or $f_0 = y^{m_0} + \beta N_i^{\text{up}} + \gamma L \cdot (z/y)$, respectively. Here $\beta$ and $\gamma$ are elements of $k^*$. Finally, all monomials, which occur in $x f_0$ or in $y^2 f_0$ also occur in $\mathcal{I}$.

**Proof.** The shape of $f_0$ and of $f_i$ results from the forgoing argumentation. Conclusion 4 gives $(x, y)L \subset \ell K(-r-1)$. The statements concerning the monomials in $x f_0$ follow from Conclusion 3.2. By Lemma 3.1, $\mathcal{I}_j$ is monomial, thus $y N_j^{\text{up}} = z M_j^{\text{up}} \in \mathcal{I}$ and therefore $y M_i^{\text{up}} \in \mathcal{I}$. Furthermore $y^2 N_j^{\text{up}} \cdot (z/y) = y N_j^{\text{up}} z \in \mathcal{I}$ and as well $y^2 L \cdot (z/y) = y L z \in \mathcal{I}$. From this the assertion concerning the monomials of $y^2 f_0$ follows. \(\square\)

Now we come to the case that $f_i = y^{m_i}$. The above reasoning had shown that $y^{m_i-1}$ occurs in $\overline{g}$. Suppose that another monomial $v$ occurs in $\overline{g}$. The same argumentation as before shows that $v$ equals one of the monomials in Figure 3.2 denoted by ?. Thus $v$ is equal to $N_j^{\text{up}}$ or $N_j^{\text{down}}$ for an index $i < j \leq r$, or is equal to a monomial $L \in \mathcal{L}$, such that $(x, y)L \subset \ell K(-r-1)$. Furthermore, there can be only one such monomial $v$.

**Conclusion 3.6.** If $f_0$ has the vice-monomial $N_i^{\text{up}}$ and if $f_i = y^{m_i}$, then $f_0 = y^{m_0} + \alpha N_i^{\text{up}} + \beta N_j^{\text{up}}$ or $f_0 = y^{m_0} + \alpha N_i^{\text{up}} + \beta N_j^{\text{down}}$, where $1 \leq i < j \leq r$, or $f_0 = y^{m_0} + \alpha N_i^{\text{up}} + \beta L$, where $L \in \mathcal{L}$ is a monomial such that $(x, y)L \subset \ell K(-r-1)$. All monomials occurring in $x f_0$ or $y f_0$ also occur in $\mathcal{I}$.
**Proof.** The statements concerning the shape of \( f_0 \) follow from the foregoing discussion. As \( I_{i+1} \) is monomial by Lemma 3.1 and as \( f_i \) is a monomial, \( I_i \) is monomial and \((x,y)L, (x,y)N_{i}^{up} \) and \((x,y)N_{j}^{down} \) are contained in \( I \). 

**3.3.5.** Suppose in the forms \( f_0 \) and \( f_j \), where \( 1 \leq j \leq r \), as in Conclusion 3.5 or Conclusion 3.6 there are three monomials with coefficients different from zero. We call \( f_0 \) (and \( f_j \), respectively) a trinomial and we then have \( f_0 = y^{m_0} + \alpha N_{i}^{up} + \beta E_{0} \) and \( f_j = M_{j}^{up} + \gamma N_{k}^{up} + \delta E_{j} \), where the “final monomials” \( E_{0} \) and \( E_{j} \) have the shape described in Conclusion 3.5 and Conclusion 3.6, and where \( \alpha, \beta, \gamma, \delta \in k^{*} \). If \( i > k \), then \( m_i \leq k_{i+1} < \rho_2 \) (Lemma 3.1). As we are in the case \( \rho_0 > 0, \rho_2 > 0 \), it follows that \( |\rho_1| > m_i \), and as \( N_{i}^{up} \) and \( E_{0} \) both occur in the semi-invariant \( f_0 \), one has \( E_{0} = N_{1}^{up} \cdot X^{\nu} \), where \( \nu \geq 1 \). Looking at Figure 3.2, one sees that then \( E_{0} \) cannot be an element of \( S_{m_0} \). For \( 1 \leq j \leq r \), where 1 \( \leq i \leq j \) we conclude from this that 

\[
\nu \rho_0 = j - i(j), \quad \nu \rho_1 = m_j + i(j) - m_0, \quad \nu \rho_2 = m_0 - m_j - j,
\]

where \( \nu \leq 1 \). We want to show that this implies \( \rho_2 > m_{j+1} \). Similarly as in the proof of Lemma 3.1 it suffices to show \( m_0 > m_j + jm_{j+1} + j \). We start with the case \( j = r \). Then again \( m_{r+1} := \kappa \) and one has to show \( m_0 > m_r + r\kappa + r \). The case \( r = 0 \) cannot occur. If \( r = 1 \), the inequality reads \( m_0 > m_1 + \kappa + 1 \), and because of \( m_0 \geq c + 2 + m_1 \) (Lemma 2.8) and \( \kappa \leq c \) this is right. Therefore we can assume \( r > 2 \).

Because of (loc.cit.) it suffices to show \( c + 2 + m_r + \cdots + m_1 > m_r + r\kappa + r \). As \( \kappa \geq c \) it suffices to show \( 2 + m_{r-1} + \cdots + m_1 > (r-1)\kappa + r \iff 1 + m_{r-1} + \cdots + m_1 > (r-1)(\kappa+1) \). Because of \( \kappa \leq c < m_r < \cdots < m_1 \) this is true (cf. Lemma 2.4 and Corollary 2.4).

We now assume \( 1 \leq j < r \), in which case it suffices to show:

\[
c + 2 + m_r + \cdots + m_1 > m_j + jm_{j+1} + j
\]

If \( j = 1 \) this inequality reads \( c + 2 + m_r + \cdots + m_1 > m_1 + m_2 + 1 \), and this is true, because \( r \geq 2 \). Thus we can assume \( 2 \leq j < r \) and the inequality which has to be shown is equivalent to

\[
(c + 1) + (m_r + \cdots + m_{j+2}) + (m_{j-1} + \cdots + m_1) > (j - 1)(m_{j+1} + 1)
\]

Because of \( m_{j+1} < m_j < \cdots < m_1 \) (loc.cit.), this is true.

We thus have proved that from \( y^{m_0}X^{\nu} = M_{j}^{up} \)

\begin{equation}
(3.10) \quad \rho_2 > m_{j+1}
\end{equation}

follows. As \( k > j \) by definition, we have \( m_k \leq m_{j+1} < \rho_2 \) and the same argumentation as before carried out with \( N_{i}^{up} \) and \( E_{0} \) shows that \( N_{k}^{up} \) and \( E_{j} \) cannot simultaneously occur in \( f_j \).

**Conclusion 3.7.** Among the forms \( f_i \), \( 0 \leq i \leq r \), which have the shape described in Conclusion 3.5 and Conclusion 3.6, there can only occur at most one trinomial.
3.3.6. We now consider the case $3^\circ$. The following argumentation is first of all independent of the sign of $\rho_1$. We write $f_0 = f^0 + z^u g$, where $f^0 = y^m$ if $\ell_0 = x$ and $f^0 = x^m$, if $\ell_0 = y$. If we choose $\mu$ maximal then we have $\mu \geq m_0 - (\kappa + r)$. For as $R_\kappa \subset in(H^0(K(\kappa)))$, the initial monomial of $g$ has a $z$-degree $> m_0 - (\kappa + r + 1)$, i.e., the initial monomial occurs in a column of total degree in $x$ and $y$ smaller or equal $\kappa + r$.

From the $\Gamma$-invariance of $f_0$ modulo $\ell_0 H^0(\mathcal{I}_1(m_0 - 1))$ it follows that $\langle x, y \rangle \partial f/\partial z = \langle x, y \rangle [\mu z^{\mu - 1} + z^\mu \partial g/\partial z] \subset \ell_0 H^0(\mathcal{I}_1(m_0 - 1))$. From the decompositions in $(Z)$ (cf. 3.1) we conclude that $H^0(\mathcal{I}_i(n)) = \ell_i H^0(\mathcal{I}_{i+1}(n - 1))$ if $n < m_i$. Now

$$m_0 - \mu < \kappa + r + 1 < m_r + r < \cdots < m_1 + 1 < m_0$$

(cf. Corollary 2.4) and thus

$$H^0(\mathcal{I}_1(m_0 - \mu)) = \ell_1 H^0(\mathcal{I}_2(m_0 - \mu - 1)),$$

$$H^0(\mathcal{I}_2(m_0 - \mu - 1)) = \ell_2 H^0(\mathcal{I}_3(m_0 - \mu - 2)),$$

$$\ldots \ldots \ldots \ldots$$

$$H^0(\mathcal{I}_{r-1}(m_0 - \mu - r + 2)) = \ell_{r-1} H^0(\mathcal{I}_r(m_0 - \mu - r + 1)),$$

$$H^0(\mathcal{I}_r(m_0 - \mu - r + 1)) = \ell_r H^0(K(m_0 - \mu - r)).$$

It follows that $H^0(\mathcal{I}_1(m_0 - \mu)) = \ell_1 \cdots \ell_r H^0(K(m_0 - \mu - r))$ and therefore

$$\langle x, y \rangle [\mu g + z \partial g/\partial z] \subset H^0(K(m_0 - \mu - r)) \quad \text{where } \ell := \ell_0 \cdots \ell_r = x^a y^b$$

and $a$ (respectively $b$) is the number of $\ell_i = x$ (respectively the number of $\ell_i = y$). This implies that $g$ is divisible by $\ell$, and changing the notation we can write $f_0 = f^0 + \ell z^\mu g$, where $\ell = \ell_0 \cdots \ell_r, \mu \geq m_0 - (\kappa + r)$ is maximal, $g \in S_{m_0 - \mu - r - 1}$ and

$$\langle x, y \rangle [\mu g + z \partial g/\partial z] \subset H^0(K(m_0 - \mu - r)).$$

Now $f_0 \in H^0(K(m_0))$ or $f_0 \in H^0(\mathcal{I}_1(m_0))$ and $m_0 \geq \text{colength}(K) + 2$, if $r = 0$ or $m_0 \geq \text{colength}(\mathcal{I}_i) + 2$, if $r \geq 1$, respectively (cf. Lemma 2.6 and Lemma 2.8). Therefore $R_{m_0} \subset H^0(K(m_0))$ or $R_{m_0} \subset H^0(\mathcal{I}_1(m_0))$, respectively (cf. Appendix C, Remark 2). It follows $\ell z^\mu g \in H^0(K(m_0))$ (or $\ell z^\mu g \in H^0(\mathcal{I}_1(m_0))$) and thus $\ell g \in H^0(K(m_0 - \mu))$ (or $\ell g \in H^0(\mathcal{I}_1(m_0 - \mu)) = \ell_1 \cdots \ell_r H^0(K(m_0 - \mu - r))$, respectively).

From this we conclude that $\ell_0 g \in H^0(K(m_0 - \mu - r))$, in any case. We also note that $g \notin H^0(K(m_0 - \mu - r - 1))$ without restriction, because otherwise

$$\ell g \in \ell H^0(K(m_0 - \mu - r - 1)) = \ell_0 H^0(\mathcal{I}_1(m_0 - \mu - 1)),$$

and thus $z^\mu \ell g \in \ell_0 H^0(\mathcal{I}_1(m_0 - 1))$ would follow. But then $\ell g$ could be deleted.

**Conclusion 3.8.** If the vice-monomial of $f_0$ is different from all monomials $N^\uparrow_i$ and $N^\downarrow_j$, then we can write

$$f_0 = f^0 + z^\mu \ell g, \quad \text{where } \mu \geq m_0 - (\kappa + r)\text{is maximal, } \ell := \ell_0 \cdots \ell_r,$$

$$f^0 = y^m, \quad \text{if } \ell_0 = x \quad \text{(or } f = x^m, \text{if } \ell_0 = y, \text{respectively),}$$

$$g \in S_{m_0 - \mu - r - 1} - H^0(K(m_0 - \mu - r - 1)), \quad \ell_0 g \in H^0(K(m_0 - \mu - r))$$

and $\langle x, y \rangle [\mu g + z \partial g/\partial z] \subset H^0(K(m_0 - \mu - r))$. \(\square\)
3.4. Order of $f_0$ in the case $3^o$

We keep the notations of Conclusion 3.8, but now we write $h := \ell_0 \cdots \ell_r$. In order to simplify the notations, the cases $\rho_1 > 0$ and $\rho_1 < 0$ will be treated separately. As always we assume $\rho_2 > 0$.

3.4.1. Let be $\rho_1 > 0$ (and thus $\rho_0 < 0$). Conclusion 3.8 is still valid, if $x$ and $y$ are interchanged. Thus one can write $f_0 = x^{m_0} + z^\mu h g, \ell_0 = y$ and

$$H^0(I(d)) = y H^0(I_1(d-1)) \oplus \langle \{ f_0 x^n z^{d-m_0-n} | 0 \leq n \leq d - m_0 \} \rangle,$$

where $I_1 := K$ if $r = 0$ (cf. Lemma 2.6).

Suppose there is a $0 \leq \nu \leq d-m_0$ such that $x^{\nu+1} f_0 \equiv x^{m_0+\nu+1}$ modulo $H^0(M(\nu-r))$. Because of

$$h H^0(K(m_0 + \nu - r)) = y \ell_1 \cdots \ell_r H^0(K(m_0 + \nu - r)) \subset y H^0(I_1(m_0 + \nu))$$

then follows that

$$H^0(I(d)) = y H^0(I_1(d-1)) \oplus \langle \{ f_0 x^n z^{d-m_0-n} | 0 \leq n \leq \nu \} \rangle \oplus \langle \{ x^{m_0+n} z^{d-m_0-n} | \nu + 1 \leq n \leq d - m_0 \} \rangle.$$

If one wants to determine the so called $\alpha$-grade of

$$\hat{\lambda}_\alpha(H^0(I(d)))$$

(cf. Chapter 4), then it is easier to estimate the contribution coming from the monomials in $H^0(I(d))$. In the following definition we assume $\rho_2 > 0$ and the sign of $\rho_1$ is arbitrary.

Definition 4. The order of $f_0$ is the smallest natural number $\nu$ such that $x^{\nu+1} f_0 \equiv x^{m_0+\nu+1}$ modulo $h H^0(K(m_0 + \nu - r))$ (such that $y^{\nu+1} f_0 \equiv y^{m_0+\nu+1}$ modulo $h H^0(K(m_0 + \nu - r))$, respectively) if $\rho_1 > 0$ (if $\rho_1 < 0$, respectively).

Here we have put $h := \ell_0 \cdots \ell_r$ (cf. (Z) in 3.1).

Remark 3.1. If $\rho_1 > 0$, then $\rho_0 < 0$, and from Lemma 2.6 it follows that $f_0$ has the initial monomial $M_0^{\text{down}} = x^{m_0}$. Then Conclusion 3.2 gives $yx^{m_0} \in I$. On the other hand, if $\rho_1 < 0$ the same argumentation shows that $f_0$ has the initial monomial $y^{m_0}$ and $xy^{m_0} \in I$. □

We now take up again the assumption $\rho_1 > 0$ from the beginning of (3.4.1).

Subcase 1: $\ell_0 = x$. As $\rho_0 < 0$ one has $f_0 = y^{m_0}$.

Subcase 2: $\ell_0 = y$. Putting $h := x^a y^b$ one can write

$$f_0 = x^{m_0} + z^\mu h g = x^{m_0} (1 + x^{a-m_0} y^b z^\mu g).$$

As $f_0$ is a $T(\rho)$-semi-invariant, one has

$$x^{a-m_0} y^b z^\mu g = X^{\gamma p} p(X^p),$$

where $\gamma \in \mathbb{N} - (0)$, $p(X^p) = a_0 + a_1 X^p + \ldots + a_s X^{sp}$ and $a_i \in k$.

We now assume $f_0$ is not a monomial. If both $\mu$ and $\gamma$ are chosen maximal, then $a_0 \neq 0$, and we can write $g = Mp(X^p)$, where $M := x^{m_0-a+\gamma p_0} y^{\gamma p_1-b}$ and $\mu = \gamma p_2$. From the
\( \Gamma \)-invariance of \( f_0 \) modulo \( \ell_0 H^0(I_1(m_0) - 1) \) it follows that
\[
\langle x, y \rangle h z^{\mu-1} [\mu g + z \partial g / \partial z] \subset \ell_0 H^0(I_1(m_0 - 1)),
\]
thus
\[
\langle x, y \rangle \ell_1 \cdots \ell_r [\mu g + z \partial g / \partial z] \subset H^0(I_1(m_0 - \mu)).
\]

We had already obtained \( H^0(I_1(m_0 - \mu)) = \ell_1 \cdots \ell_r H^0(K(m_0 - \mu - r)) \) in (3.3.6).
We conclude that
\[
\langle x, y \rangle [\mu g + z \partial g / \partial z] \subset H^0(K(m_0 - \mu - r))
\]
and therefore:
\[
xg \equiv -\frac{1}{\mu} M \rho_2 a_1 X^\rho + 2 \rho_2 a_2 X^{2\rho} + \cdots + s \rho_2 a_s X^{s\rho} \bmod H^0(K(m_0 - \mu - r))
\]
Assume that \( s = \deg(p) > 0 \). Then \( a_s \neq 0 \) and \( j := \min \{ i > 0 | a_i \neq 0 \} \) is an integer between 1 and \( s \) inclusive. Then one can write:
\[
xg \equiv -\frac{1}{\mu} x M^i \rho_2 a_j + (j + 1) \rho_2 a_j X^\rho + \cdots + s \rho_2 a_s X^{(s-j)\rho} \bmod H^0(K(m_0 - \mu - r))
\]
From this it follows that
\[
x f_0 = x^{m_0 + 1} + h z^{\mu} xg \equiv \tilde{f} \bmod h H^0(K(m_0 - r)),
\]
where \( \tilde{f} \equiv x^{m_0 + 1} + h z^{\mu} M \tilde{p}(X^\rho) \) and \( \tilde{\mu} := \mu + j \rho_2 > \mu \), \( M := x^M x^{j_0} y^{j_1} \),
\[
\tilde{p}(X^\rho) := \tilde{a}_0 + \tilde{a}_1 X^\rho + \cdots + \tilde{a}_{s-j} X^{(s-j)\rho}
\]
and finally \( \tilde{a}_0 := -\frac{1}{\mu} j \rho_2 a_j \neq 0, \ldots, \tilde{a}_{s-j} := -\frac{1}{\mu} s \rho_2 a_s \neq 0 \).

**Remark 3.2.** \( \tilde{f} \) is again a \( T(\rho) \)-semi-invariant, because
\[
x^{-(m_0 + 1)} h z^{\mu} \tilde{M} = x^{-m_0 - 1} x^a y b z^{\mu + j \rho_2} \cdot x^{m_0 - a + \gamma_0} \cdot y^{\gamma_1 - b} \cdot x^{j_0} \cdot y^{j_1}
\]
\[
= x^{(\gamma + j) \rho_0} y^{(\gamma + j) \rho_1} z^{(\gamma + j) \rho_2}. \quad \Box
\]

**Remark 3.3.** \( \tilde{f} \) is only determined modulo \( \ell_0 H^0(I_1(m_0)) \), as \( h H^0(K(m_0 - r)) \subset \ell_0 H^0(I_1(m_0)) \).

We continue the above discussion and get at once \( x f_0 \equiv x^{m_0 + 1} \bmod h H^0(K(m_0 - r)) \), if \( s = 0 \). In any case we have degree \( (\tilde{p}) < \deg(p) \), and continuing in this way, we get finally \( \tilde{p}^{(s+1)} = 0 \). This means, there is an integer \( 0 \leq \nu \leq s \) such that
\[
x^{\nu+1} f_0 \equiv x^{m_0 + \nu + 1} \bmod h H^0(K(m_0 - r + \nu)). \tag{3.11}
\]

**3.4.2.** We now assume \( \rho_1 < 0 \). If \( f_0 \) is a monomial, then the order of \( f_0 \) is zero by definition. Thus we may assume without restriction that \( \rho_0 > 0 \) (c.f. 2.4.1 Auxiliary Lemma 2).

**Subcase 1:** \( \ell_0 = y \). Then \( f_0 = x^{m_0} \) (Lemma 2.6).

**Subcase 2:** \( \ell_0 = x \). With notations analogous to that of (3.4.1) one has: \( f_0 = y^{m_0} (1 + x a y b^{m_0} z^{\mu} g) \) is a semi-invariant, therefore \( x a y b^{m_0} z^{\mu} g = X^{\gamma \rho} p (X^\rho), \gamma \in \mathbb{N} - (0) \). If \( f_0 \) is not a monomial and \( \mu \) and \( \gamma \) are chosen maximal, then one can write \( g = M p (X^\rho), \mu = \gamma \rho_2, M = x^{\gamma \rho_0} a y^{m_0 - b + \gamma_0} \) and \( p (X^\rho) = a_0 + a_1 X^\rho + \cdots + a_s X^{s \rho} \). Furthermore one gets
\[
y f_0 \equiv \tilde{f} \bmod h H^0(K(m_0 - r)), \quad \tilde{f} := y^{m_0 + 1} + h z^{\mu} M \tilde{p}(X^\rho), \tilde{\mu} := \mu + j \rho_2, M := ...
\]

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yMx^{j_{\rho_0}}y^{j_{\rho_1}}$ and $\tilde{\rho}$ is defined as in (3.4.1). It is clear that one gets the corresponding statements as in (3.4.1), if $x$ and $y$ as well as $a$ and $b$ are interchanged. This means, there is an integer $0 \leq \nu \leq s$ such that

$$(3.12) \quad y^{\nu+1}f_0 \equiv y^{m_0+\nu+1} \text{ modulo } hH^0(\mathcal{K}(m_0-r+\nu)).$$

### 3.4.3. Estimate of the order of $f_0$. The order has been defined as the smallest natural number $\nu$ such that $x^{\nu+1}f_0 \equiv x^{m_0+\nu+1}$ (respectively $y^{\nu+1}f_0 \equiv y^{m_0+\nu+1}$) modulo $hH^0(\mathcal{K}(m_0-r+\nu))$. We keep the notations of (3.3.6) it follows that $\gamma \geq \lfloor (m_0-(\kappa+r))/\rho_2 \rfloor$. On the other hand,

$$x^{m_0-a+\gamma\rho_0} \cdot x^{s\rho_0} \text{ has to be a polynomial in case that } \rho_1 > 0 \text{ ( } y^{m_0-b+\gamma\rho_1} \cdot y^{s\rho_1} \text{ has to be a monomial in case that } \rho_1 < 0 \text{, respectively). That means } m_0-a+\gamma\rho_0 \geq 0 \text{ (respectively } m_0-b+\gamma\rho_1 \geq 0 \text{, respectively). Now one has } \rho_0 < 0 \text{, if } \rho_1 > 0 \text{, and } \rho_0 > 0 \text{, if } \rho_1 < 0 \text{ (cf. 2.4.1 Auxiliary Lemma 2). It follows that}

$$\gamma + s \leq (m_0-a)/|\rho_0| \text{ if } \rho_1 > 0,$$

$$\gamma + s \leq (m_0-b)/|\rho_1| \text{ if } \rho_1 < 0.$$

Here $a$ (respectively $b$) is the number of $\ell_i, 0 \leq i \leq r$, such that $\ell_i = x$ (respectively $\ell_i = y$).

In case that $\rho_1 > 0$ we obtain , because of $|\rho_0| = \rho_1 + \rho_2$:

$$s \leq \frac{m_0-a}{\rho_1 + \rho_2} - \frac{m_0-(\kappa+r)}{\rho_2},$$

And in case that $\rho_1 < 0$, because of $|\rho_1| = \rho_0 + \rho_2$, we obtain

$$s \leq \frac{m_0-b}{\rho_0 + \rho_2} - \frac{m_0-(\kappa+r)}{\rho_2}.$$

From the congruences (3.11) and (3.12) we get

**Conclusion 3.9.** The order $s$ of $f_0$ fulfils the inequality:

$$s \leq \begin{cases} \frac{m_0-a}{\rho_1 + \rho_2} - \frac{m_0-(\kappa+r)}{\rho_2} & \text{if } \rho_1 > 0, \\ \frac{m_0-b}{\rho_0 + \rho_2} - \frac{m_0-(\kappa+r)}{\rho_2} & \text{if } \rho_1 < 0. \end{cases}$$

### 3.5. Summary in the Case I

We recall that this means $\rho_0 > 0, \rho_2 > 0$ and $\rho_1 < 0$.

**Proposition 3.1.** Let be $\rho_2 > 0$ and $\rho_1 < 0$.

(a) The following cases can a priori occur:

1st case: One of the $f_i$, say $f_{i_0}$ has as vice-monomial an “upper empty corner” $N_{j_0}^{\text{up}}$. Then $\rho_2 > \kappa$, thus $\mathcal{K}$ is monomial, and the $f_i$ have one of the following forms:

1. $f_i = M_i^{\text{down}}$
2. $f_i = M_i^{\text{up}} + aN_j^{\text{down}}$
3. $f_i = M_i^{\text{up}} + aN_j^{\text{up}} + \beta N_k^{\text{up}}$
4. $f_i = M_i^{\text{up}} + aN_j^{\text{up}} + \beta N_k^{\text{down}}$
\( f_i = M^\text{up}_i + \alpha N^\text{up}_j + \beta L, L \in \mathcal{L} \) a monomial such that
\[(x, y) \cdot L \subset \ell K(-r - 1), \ell := \ell_0 \cdots \ell_r.\]

\( f_i = M^\text{up}_i + \alpha N^\text{up}_j + \beta N^\text{up}_k \cdot (z/y), \alpha, \beta \in k^* \)

\( f_i = M^\text{up}_i + \alpha N^\text{up}_j + \beta L \cdot (z/y), L \in \mathcal{L} \) a monomial such that
\[(x, y) \cdot L \subset \ell K(-r - 1), \alpha, \beta \in k^*.\]

Here \( \alpha \) and \( \beta \) are (a priori) arbitrary elements of \( k \) and \( 0 \leq i < j < k \leq r \).

2nd case: Not any of the \( f_i \) has as vice-monomial an “upper empty corner” \( N^\text{up}_j \). Then each of the \( f_i \) has one of the following forms:

(1) \( f_i = M^\text{up}_i + \alpha N^\text{down}_j \)

(2) \( f_i = M^\text{down}_i \)

(3) \( f_i = M^\text{up}_i + F, F = \sum \alpha_j L_j, L_j \in \mathcal{L} \) monomial, \( \alpha_j \in k^* \), such that \( xF \in \ell K(-r - 1) \), and for suitable \( \beta_j \in k^* \) and \( G := \sum \beta_j L_j \) one has \( (x, y) \cdot G \subset \ell K(-r - 1) \).

(b) In the 1st case there is only one trinomial at most, i.e. there is only one \( f_i \) at most, which has the shape as in one of the cases 1.3 - 1.7, with \( \alpha \) and \( \beta \) different from zero.

(c) Let be \( \mathcal{M} := \bigcup \{ \text{monomials in } H^0(\mathcal{I}(n)) \} \) and let be \( \langle \mathcal{M} \rangle \) the subspace of \( S \) generated by these monomials. In the 1st case one has:

- \( xf_i \in \langle \mathcal{M} \rangle \), for all \( 0 \leq i \leq r \);
- \( yf_i \in \langle \mathcal{M} \rangle \), if \( f_i \) has the shape as in one of the cases 1.1–1.5
- \( y^2f_i \in \langle \mathcal{M} \rangle \), if \( f_i \) has the shape as in one of the cases 1.6 and 1.7.

(d) In the 2nd case one has \( \langle x, y \rangle f_i \in \langle \mathcal{M} \rangle \), if \( f_i \) has the shape as in case 2.1.

If \( f_i \) has the shape as in case 2.3, then \( xM^\text{up}_i \in \mathcal{M} \). And the order \( s \) of \( f_i \) fulfills the inequality
\[ s \leq \frac{\kappa}{\rho_2} - m_i \left( \frac{1}{\rho_2} - \frac{1}{\rho_0 + \rho_2} \right) + \frac{r}{\rho_2}, \]
where \( 0 \leq i \leq r \).

**Proof.** (1.1) and (1.2) result from Conclusion 3.2 and Conclusion 3.3; (1.3)–(1.5) result from Conclusion 3.6; and (1.6), and (1.7) results from Conclusion 3.5. (2.1) results again from Conclusion 3.2 and Conclusion 3.3; (2.2) is clear, and (2.3) results from Conclusion 3.8, for if \( M^\text{up}_i \) occurs, then \( \ell_i = x \), and this corresponds to the linear form \( \ell_0 \) in Conclusion 3.8.

(b) results from Conclusion 3.7.

c results from the Conclusions 3.3, 3.5 and 3.6.

d results in the case (2.1) from the fact that \( f_j = M^\text{down}_j \) is a monomial and therefore all monomials with the same \( z \)-degree as \( M^\text{down}_j \), which occur in \( \text{in}(H^0(\mathcal{I}(m_0))) \), also occur in \( H^0(\mathcal{I}(m_0)) \) (Conclusion 3.2). In the case (2.3) the first part of the statement results from Conclusion 3.2, and the statement concerning the order results from Conclusion 3.9.
Remark 3.4. If \( r = 0 \), then only the cases (2.2) and (2.3) can occur, i.e., one has either \( f_0 = x^{m_0} \) or \( f_0 = y^{m_0} + F \), where \( F \) has the properties mentioned above.

3.6. Summary in the Case II

We recall that this means \( \rho_0 < 0, \rho_1 > 0, \rho_2 > 0 \).

We translate formally the results of (3.5): As always \( \rho_2 > 0 \) is assumed, one has \( \rho_0 < 0 \).

By definition \( \iota(i) = \#\{\ell_j = y|0 \leq j < i\} \) and therefore \( i - \iota(i) = \#\{\ell_j = x|0 \leq j < i\} \).

We conclude that if one interchanges \( x \) and \( y \) in the monomial \( M_i^{\text{up}} \) (respectively \( M_i^{\text{down}} \)) and if one simultaneously replaces \( \iota(i) \) by \( i - \iota(i) \) and vice versa \( i - \iota(i) \) by \( \iota(i) \), then one obtains \( M_i^{\text{down}} \) (respectively \( M_i^{\text{up}} \)). Therefore the statements of Proposition 3.1 can be translated to the Case II by interchanging \( x \) and \( y \) and “up” and “down”:

Proposition 3.2. Let be \( \rho_2 > 0 \) and \( \rho_1 > 0 \).

(a) The following cases can a priori occur:

1st case: One of the \( f_i \), say \( f_u \), has as vice-monomial a “lower empty corner” \( N_j^{\text{down}} \). Then \( \rho_2 > \kappa \), thus \( K \) is monomial and the \( f_i \) have one of the following forms:

1. \( f_i = M_i^{\text{up}} \)
2. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{up}} \)
3. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{down}} + \beta N_k^{\text{down}} \)
4. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{down}} + \beta N_k^{\text{up}} \)
5. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{down}} + \beta L, L \in L \) a monomial such that \( (x,y) \cdot L \subset K(-r - 1), \ell := \ell_0 \cdots \ell_r \)
6. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{down}} + \beta N_k^{\text{down}} \cdot (z/x), \alpha, \beta \in k^* \)
7. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{down}} + \beta L \cdot (z/x), L \in L \) a monomial such that \( (x,y) \cdot L \subset K(-r - 1), \alpha, \beta \in k^* \).

2nd case: Not any of the \( f_i \) has as vice-monomial a “lower empty corner” \( N_j^{\text{down}} \). Then each of the \( f_i \) has one of the following forms:

1. \( f_i = M_i^{\text{down}} + \alpha N_j^{\text{up}} \)
2. \( f_i = M_i^{\text{up}} \)
3. \( f_i = M_i^{\text{up}} + F, F = \sum \alpha_j L_j, L_j \in L \) monomial, \( \alpha_j \in k^* \), such that \( yF \in K(-r - 1) \), and for suitable \( \beta_j \in k^* \) and \( G := \sum \beta_j L_j \) one has \( (x,y) \cdot G \subset K(-r - 1) \).

(b) The same statement as in Proposition 3.1.
(c) The same statement as in Proposition 3.1, if \( x \) and \( y \) are interchanged.
(d) The same statement as in Proposition 3.1, if one replaces \( x M_i^{\text{up}} \) by \( y M_i^{\text{down}} \).

And the order \( s \) of \( f_i \) fulfils the inequality

\[
    s \leq \frac{\kappa}{\rho_2} - m_i \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right) + \frac{r}{\rho_2},
\]

where \( 0 \leq i \leq r \). \( \square \)

Remark 3.5. If \( r = 0 \), then only the cases (2.2) and (2.3) can occur.
Explanation: \( \text{reg}(\mathcal{K}) \leq c \Rightarrow R_n \subset H^0(\mathcal{K}(n)), n \geq c \)
\( S_{c-1}/H^0(\mathcal{K}(c-1)) \) has a basis of \( c \) monomials, namely the monomials
of \( S_{c-1} - inH^0(\mathcal{K}(c-1)) \Rightarrow \ell[S_{c-1}/H^0(\mathcal{K}(c-1))] \) has a basis
of \( c \) monomials, namely the monomials of \( \ell S_{c-1} - \ell \cdot inH^0(\mathcal{K}(c-1)) \)
They generate a subspace \( \mathcal{L} \). \( \ell := \ell_0 \cdots \ell_r = x^{r+1-(r+1)} y^{r+1} \)
\[ I_i = x I_{i+1}(-1) + f_i \mathcal{O}_{V^2}(-m_i) \]
CHAPTER 4

The \( \alpha \)-grade.

4.1. Notations.

We let \( \mathbb{G}_m \) (respectively \( \mathbb{G}_n \)) operate on \( S = k[x, y, z] \) by \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z \) (respectively by \( \psi_\alpha : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z \)).

Let be \( \mathbf{G} = \text{Grass}_m(S_n) \). If \( V \in \mathbf{G}(k) \), then \( \bigwedge V \) has the dimension 1 and \( V \mapsto \bigwedge V \) defines a closed immersion \( \mathbf{G} \overset{\rho}{\to} \mathbb{P}(\bigwedge S_n) = \mathbb{P}^N, N := \dim \bigwedge S_n - 1 \), the so called Plücker embedding.

If one numbers the monomials in \( S_n \), then one gets a basis \( \{e_1, e_2, \ldots \} \) of \( S_n \), and therefore \( e_{(i)} = e_{i_1} \wedge \cdots \wedge e_{i_m} \) is a basis of \( \bigwedge S_n \), where \( (i) = (i_1, \cdots, i_m) \) runs through all sequences of natural numbers, such that \( 1 \leq i_1 < \cdots < i_m \leq \binom{n+2}{2} \). If one puts

\[
\psi_\alpha(e_{(i)}) := \psi_\alpha(e_{i_1}) \wedge \cdots \wedge \psi_\alpha(e_{i_m}),
\]

then \( \mathbb{G}_n \) operates in an equivariant manner on \( \mathbf{G} \) and \( \mathbb{P}^N \), and the same is valid for the operation \( \sigma \) of \( \mathbb{G}_m \). If \( \xi \in \mathbb{G}(k) \) corresponds to the vector space \( V \subset S_n \), then \( C_\xi := \{ \bigwedge \psi_\alpha(V) \mid \alpha \in k \} \) is a point or a curve in \( \mathbb{P}^N \), and there are polynomials \( f_0, \ldots, f_N \) in one variable with coefficients in \( k \), such that \( C_\xi = \{(f_0(\alpha) : \cdots : f_N(\alpha)) \mid \alpha \in k \} \). At least one of the \( f_i \) is equal to 1, and if \( \xi \) is not invariant under \( \mathbb{G}_n \), then \( C_\xi \) is a \( \mathbb{G}_n \)-invariant closed curve in \( \mathbb{P}^N \) of degree equal to \( \max \{ \deg(f_i) \mid 0 \leq i \leq N \} \) (cf. [T1], Bemerkungen 2 und 3, p. 11). This number is denoted by \( \alpha \)-grade \( (V) \).

Let now be \( e_i, 1 \leq i \leq \ell := \binom{n+2}{2} \), the monomials in \( S_n \), ordered in the inverse lexicographic manner. If \( f_i = \sum_{j=1}^m a_{ij}e_j, 1 \leq i \leq m, \) is a basis of \( V \), then \( f_1 \wedge \cdots \wedge f_m = \sum_{(i)} P_{(i)}e_{(i)} \), where \( P_{(i)} = \det \begin{pmatrix} a_{i_11} \cdots a_{i_1m} \\ \cdots \\ a_{i_m1} \cdots a_{i_mm} \end{pmatrix} \) is the Plücker coordinate for the index \( (i) = (i_1, \cdots, i_m) \). It follows that \( \psi_\alpha(\bigwedge V) = \langle \sum_{(i)} P_{(i)}\psi_\alpha(e_{(i)}) \rangle \) and we conclude from this:

\[
\alpha \text{- grade } (V) \leq \max_{(i)} \{ \alpha \text{- grade } \psi_\alpha(e_{(i)}) \mid P_{(i)} \neq 0 \}
\]

where we define the \( \alpha \)-grade of \( \psi_\alpha(e_{(i)}) \) to be the \( \alpha \)-grade of the monomial subspace \( \langle e_{i_1}, \cdots, e_{i_m} \rangle \) of \( S_n \). This can be computed as follows: Write \( \psi_\alpha(e_{i_\nu}) = \sum_{j=1}^\ell p_{j\nu}(\alpha)e_j, 1 \leq \nu \leq m, \) where the \( p_{ij} \) are polynomials in one variable with coefficients in \( \mathbb{Z} \). Then \( \psi_\alpha(e_{(i)}) = \sum_{(j)} P_{(j)}(\alpha)e_{(j)} \), where the \( P_{(j)}(\alpha) \) are the Plücker coordinates of the vector space.
\begin{align}
\langle \psi \alpha(e_1), \cdots, \psi \alpha(e_m) \rangle. \text{ The } P_{(j)} \text{ are polynomials in one variable with coefficients in } \Z. \text{ As } P_{(j)}(\alpha) = 1, \text{ the } \alpha\text{-grade of } \langle e_{i_1}, \cdots, e_{i_m} \rangle \text{ is equal to max} \{\deg(P_{(j)})\}.
\end{align}

Whereas it seems practically impossible to find a formula for the \( \alpha \)-grade of an arbitrary vector space \( V \subset S_n \), the \( \alpha \)-grade of a monomial subspace \( V \subset S_n \) can be computed as follows: Write \( V = \bigoplus_{i=0}^{n} z^{n-i} V_i \) where \( V_i \subset R_i \) is a monomial subspace and \( R = k[x, y] \). If we put \( m(i) := \dim V_i \) then \( V_i \) has a basis of the form \( \{x^{i-a_{ij}} y^{a_{ij}} | 1 \leq j \leq m(i)\} \), where \( 0 \leq a_{i1} < \cdots < a_{im(i)} \leq i \) is a sequence of integers. As \( \alpha \)-grade \( (V) = \sum_{i=0}^{n} \alpha \)-grade \( (V_i) \), we can consider \( V \) as a graded vector space in \( R \), which has a basis of monomials of different degrees. In ([T1], 1.3, p. 12f.) it was shown:

If \( 0 \leq c_1 < \cdots < c_r \leq i \) are integers, and
\begin{align}
W := \langle x^{i-c_1} y^{c_1}, \cdots, x^{i-c_r} y^{c_r} \rangle \subset R_i,
\end{align}
then \( \alpha \)-grade \( (W) = (c_1 + \cdots + c_r) - (1 + \cdots + r - 1) \).

Later on we will need an estimate of the \( \alpha \)-grade of an ideal \( I \subset O_{p^2} \) of colength \( d \), which is invariant under \( G = \Gamma \cdot T(\rho) \). This will be done by estimating the \( \alpha \)-grade of the vector space \( V = H^0(I(n)) \), if \( n \) is sufficiently large. By means of (1) the estimate will be reduced to the computation of the \( \alpha \)-grade of monomial subspaces, and because of (2) this can be regarded as a combinatorial problem, the formulation of which needs some more notations.

### 4.2. The weight of a pyramid.

**Definition 5.** A pyramid with frame \( c \) and colength \( d \), \( 1 \leq d \leq c \), is a set \( P \) of monomials in \( R = k[x, y] \) with the following properties: The \( i \)-th “column” \( S_i \) of \( P \) consists of monomials \( x^{i-a_{ij}} y^{a_{ij}}, 0 \leq a_{i1} < \cdots < a_{im(i)} \leq i \), for all \( 0 \leq i \leq c - 1 \), such that the following conditions are fulfilled:

\begin{enumerate}
\item \( \bigcup_{i=0}^{c-1} S_i = P \)
\item \#(\{x^{i-j} y^j | 0 \leq j \leq i \leq c - 1\} \setminus P) = d
\end{enumerate}

Then \( w(S_i) := (a_{i1} + \cdots + a_{im(i)}) - (1 + \cdots + m(i) - 1) \) is called the weight of the \( i \)-th column \( S_i \), and \( w(P) := \sum_{i=0}^{c-1} w(S_i) \) is called the weight of the pyramid \( P \).

**Example.** Let be \( I \subset O_{p^2} \) an ideal of colength \( d \) which is invariant under \( T(3; k) \). Then \( \text{reg}(I) \leq d \), therefore \( h^0(I(d - 1)) = \binom{d+1}{2} - d \), and we can write \( H^0(I(d - 1)) = \bigoplus_{i=0}^{d-1} z^{d-1-i} V_i \), where \( V_i \subset R_i \) are monomial subspaces. If \( S_i \) is the set of monomials in \( V_i \), then \( P := \bigcup_{i=0}^{d-1} S_i \) is a pyramid with frame and colength equal to \( d \). From (2) one concludes that \( w(P) = \alpha \)-grade \( (H^0(I(d - 1))) \). (N.B. One has \( R_n \subset H^0(I(n)) \) if \( n \geq d \).)
Remark 4.1. Let $0 \leq c_1 < \cdots < c_r \leq i$ be a sequence of integers. $w(c_1, \cdots, c_r) := (c_1 + \cdots + c_r) - (1 + \cdots + r - 1)$ is maximal if and only if $c_\nu = i - r + \nu, 1 \leq \nu \leq r$, i.e. if $(c_1, \cdots, c_r) = (i - r + 1, \cdots, i)$. □

The aim is to determine those pyramid $P$ of type $(c, d)$, i.e., with frame $c$ and colength $d$, for which $w(P)$ is maximal. Because of Remark 4.1 we will consider without restriction only pyramids with $S_i = \{x^{i-a(i)}y^{a(i)}, \cdots, xy^{i-1}y\}$, where $a(i) := i + 1 - m(i)$ is a number between 0 and $i$ inclusive. We call $x^{i-a(i)}y^{a(i)}$ the initial monomial and $a(i)$ initial degree of $S_i$. For simplicity we write $S_i = (a(i), a(i) + 1, \cdots, i)$ and $P = (x^{i-a(i)}y^{a(i)}|0 \leq i \leq c - 1)$.

Remark 4.2. $w(S_j) = ia(i) + a(i) - a(i)\cdot i$.

Proof. $w(S_i) = (a(i) + \cdots + i) - (1 + \cdots + a(i)) = (1 + \cdots + i) - (1 + \cdots + a(i) - 1) - (1 + \cdots + i - a(i)) = (i+1) - \binom{a(i)}{2} - \binom{i-a(i)+1}{2}$, and a direct computation gives the assertion. □

Taking away $x^{i-a(i)}y^{a(i)}$ from $S_i$ and adding $x^{j-a(j)+1}y^{a(j)-1}$ to $S_j$, if $S_i \neq \emptyset, a(j) > 0$ and $j \neq i$, then gives a pyramid

$$P' = P - \{x^{i-a(i)}y^{a(i)}\} \cup \{x^{j-a(j)+1}y^{a(j)-1}\}.$$ 

We express this as $S_i(P) = (a(i), \cdots, i), S_j(P) = (a(j), \cdots, j), S_i(P') = (a(i) + 1, \cdots, i)$, $S_j(P') = (a(j) - 1, \cdots, j)$ and we get $w(S_i(P)) = ia(i) + a(i) - a(i)\cdot i; w(S_i(P')) = ia(i) + a(i) + (i+1) - a(i) + 1); w(S_j(P)) = ja(j) + a(j) - a(j)\cdot i; w(S_j(P')) = ja(j) - 1 + a(j) - 1 - (a(j) - 1)^2).$ It follows that $w(P') - w(P) = [w(S_i(P')) - w(S_i(P))] + [w(S_j(P')) - w(S_j(P))] = [i + 1 - 2a(i) - 1] + [-j - 1 + 2a(j) - 1].$ We get the following formula:

$$w(P') - w(P) = 2(a(j) - a(i)) - (j - i) - 2,$$

where we have made the assumption that $i \neq j, S_i(P) \neq \emptyset$, and $a(j) > 0$.

Now let $P$ be a pyramid of type $(c, d)$, such that $w(P)$ is maximal. Then for all deformations $P \mapsto P'$ as above, the right side of (3) is $\leq 0$, i.e. $a(j) - a(i) \leq \frac{1}{2}(j - i) + 1$. If $i < j$ (if $i > j$, respectively) this is equivalent to

$$\frac{a(j) - a(i)}{j - i} \leq \frac{1}{2} + \frac{1}{j - i}$$

and

$$\frac{a(j) - a(i)}{j - i} \geq \frac{1}{2} + \frac{1}{j - i},$$

respectively. Putting in $j = i + 1$ (respectively $j = i - 1$) gives $a(i + 1) - a(i) \leq 1.5$ (or $a(i) - a(i - 1) \geq -0.5$, respectively). As the left side of these inequalities are integers, it follows that $a(i + 1) - a(i) \leq 1$ (or $a(i) - a(i - 1) \geq 0$, respectively) for all $0 \leq i < c - 1$ (for all $0 < i \leq c - 1$, respectively). Note that we can apply (4.3) only under the assumption $a(i + 1) > 0$ (respectively $a(i - 1) > 0$). But if $a(i + 1) = 0$ (respectively $a(i - 1) = 0$), then the two last inequalities are true, too. So we get

Remark 4.3. If the pyramid $P$ of type $(c, d)$ has maximal weight, then one has $a(i) \leq a(i + 1) \leq a(i) + 1$, for all $0 \leq i \leq c - 2$. □

Remark 4.4. If $P = \{x^{i-a(i)}y^{a(i)}|i, j\}$ is a pyramid of type $(c, d)$, then $P' = yP := \{x^{i-a(i)}y^{a(i)+1}|i, j\}$ is called shifted pyramid, and $w(P') = w(P) + \#P$. 43
Remark 4.9. Each positive natural number \( P \) weight, too. But
\[
\{2(1 + m(i)) = w(S_i) + m(i) \Rightarrow w(P') = \sum_{i=0}^{c-1} (w(S_i) + m(i)) = w(P) + \#P.
\]

\[ \square \]

Remark 4.5. A pyramid with maximal weight does not contain a step of breadth \( \geq 4 \), i.e. it is not possible that \( a(j) = \cdots = a(i) > 0 \), if \( i - j \geq 3 \).

\[ \text{Proof.} \quad \begin{array}{l}
\text{If one would have the situation shown in Figure 4.1, then one could make the deformation } x^{i-a(i)}y^{a(i)} \Rightarrow x^{j-a(j)+y^{a(j)-1}} \text{ and then from (4.3) one would get: } w(P') - w(P) = 2 \cdot 0 - (j - i) - 2 = i - j - 2 \geq 1, \text{ contradiction.}
\end{array} \]

\[ \square \]

Remark 4.6. In any pyramid two consecutive “normal” steps can be replaced by a step of breadth 3, without a change of weight. In the course of this, the sum over the \( x \)-degrees of all monomials in the pyramid increases by 2, however.

\[ \text{Proof.} \quad \begin{array}{l}
\text{By assumption one has } a(i) + 1 = a(i + 1) \text{ and } a(i + 1) + 1 = a(i + 2). \text{ If one makes the deformation } x^{i-a(i)}y^{a(i)} \Rightarrow x^{j-a(j)+y^{a(j)-1}}, \text{ where } j = i + 2, \text{ then one gets } w(P') - w(P) = 2 \cdot 2 - 2 - 2 = 0.
\end{array} \]

\[ \text{N.B. The possibility } a(i) = 0 \text{ is not excluded.} \]

Remark 4.7. There is a pyramid of maximal weight, which does not contain two consecutive normal steps, that means, there is no index \( i \), such that \( a(i+1) = a(i) + 1, a(i+2) = a(i+1) + 1 \). This is called a “prepared” pyramid. (N.B. \( a(i) \) may equal zero.)

\[ \text{Proof.} \quad \text{Apply Remark 4.6 for several times.} \]

\[ \square \]

Remark 4.8. A prepared pyramid of maximal weight does not contain two steps of breadth 3.

\[ \text{Proof.} \quad \begin{array}{l}
\text{We consider two cases:}
\text{1st case: The steps of breadth 3 are situated side by side. Then one makes the deformation described in Fig. 4.3 and one gets: } w(P') - w(P) = 2[a(i - 5) - a(i)] + 5 - 2 = 1, \text{ contradiction.}
\end{array} \]

\[ \text{2nd case: Between the steps of breadth 3 there are } \nu \geq 1 \text{ double steps. One then makes the deformation described in Fig. 4.4. Putting } j = i - 2(\nu + 1) \text{ one gets: } w(P') - w(P) = 2(a(j) - a(i)) - (j - i) - 2 = 2(-\nu) + 2(\nu + 1) - 2 = 0. \text{ Then } P' \text{ would have maximal weight, too. But } P' \text{ contains a step of breadth 4, contradicting Remark 4.5.}
\]

\[ \square \]

Remark 4.9. Each positive natural number \( d \) can uniquely represented either in the form \( d = n(n+1) - r, 0 \leq r < n \) (1st case) or in the form \( d = n^2 - r, 0 \leq r < n \) (2nd case). Both cases exclude each other.

\[ \text{Proof.} \quad \text{Choosing } n \text{ sufficiently large, then the sequence } n(n+1), n(n+1)-1, \cdots, n(n+1) - (n-1), n(n+1) - n = n^2, n^2 - 1, \cdots, n^2 - (n-1), n^2 - n = (n-1)n, \cdots \text{ contains any given set } \{1, \cdots, m\} \subset \mathbb{N}. \]

\[ \square \]
We now assume that \( P \) is a prepared pyramid of type \((c,d)\), with \(d \geq 3\) and maximal weight. If \( P \) does not contain a step of breadth 3, then \( P \) has the form described either in Fig. 4.5a or in Fig. 4.5b, according to if either \( d = n(n+1) \) or \( d = n^2 \). (One has \( 2 \cdot \sum \nu = n(n+1) \) and \( \sum \nu(n-1) = n^2 \).) If there is 1 step of breadth 3, then from the Remarks 3,5,6,7 and 8 it follows that \( P \) has the form described either in Fig. 4.6a or in Fig. 4.6b. Here the step of breadth 3 may lie quite left or right in Fig. 4.6a or Fig. 4.6b, respectively. One sees that Fig. 4.6a and Fig. 4.6b result from Fig. 4.5a and Fig. 4.5b, respectively, by removing the monomials marked by \(-\).

We first compute the weights of the pyramids shown in Fig. 4.5a and Fig. 4.5b, and then the weights of the “reduced” pyramids \( \overline{P} \) in Fig. 4.6a and Fig. 4.6b.

1st case: (Fig. 4.5a)

| \( i \) | \( a(i) \) | \( w(S_i) = ia(i) - a(i)(a(i) - 1) \) |
|---|---|---|
| \( n-1 \) | \( n \) | \( (c-1)n - n(n-1) \) |
| \( n-2 \) | \( n \) | \( (c-2)n - n(n-1) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( n-2\nu \) | \( \nu \) | \( (c-2\nu)(n-\nu) - (n-\nu)(n-\nu-1) \) |
| \( n-2\nu-1 \) | \( \nu \) | \( (c-2\nu-1)(n-\nu) - (n-\nu)(n-\nu-1) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( n-2n+1 \) | \( 1 \) | \( (c-2n+1) \cdot 1 - 1 \cdot 0 \) |
| \( n-2n \) | \( 1 \) | \( (c-2n) \cdot 1 - 1 \cdot 0 \) |

2nd case: (Fig. 4.5b)

| \( i \) | \( a(i) \) | \( w(S_i) = ia(i) - a(i)(a(i) - 1) \) |
|---|---|---|
| \( n-1 \) | \( n \) | \( (c-1)n - n(n-1) \) |
| \( n-2 \) | \( n-1 \) | \( (c-2)(n-1) - (n-1)(n-2) \) |
| \( n-3 \) | \( n-1 \) | \( (c-3)(n-1) - (n-1)(n-2) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( n-2\nu \) | \( \nu \) | \( (c-2\nu)(n-\nu) - (n-\nu)(n-\nu-1) \) |
| \( n-2\nu-1 \) | \( \nu \) | \( (c-2\nu-1)(n-\nu) - (n-\nu)(n-\nu-1) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( n-2n+2 \) | \( 1 \) | \( (c-2n+2) \cdot 1 - 1 \cdot 0 \) |
| \( n-2n+1 \) | \( 1 \) | \( (c-2n+1) \cdot 1 - 1 \cdot 0 \) |

1st case: We sum up \( w(S_i) \) and \( w(S_{i-1}) \) if \( i = c-2\nu-1 \) and we get:

\[
\sum_{\nu=0}^{n-1} (n-\nu)(2c-2n-2\nu-1) = \sum_{\nu=1}^{n} \nu(2c-4n+2\nu-1) \\
= (2c-4n-1) \cdot \frac{1}{2}n(n+1) + \frac{1}{3}n(n+1)(2n+1) \\
= n(n+1)(c-2n-0.5 + \frac{2}{3}n + \frac{1}{3}) \\
= n(n+1)(c-\frac{4}{3}n - \frac{1}{6}).
\]

In the reduced pyramid the initial terms \( \overline{a}(i) \) of the column \( \overline{S} \), if \( i = c-2, c-4, \ldots, c-2r \), then are equal to \( n-1, n-2, \ldots, n-r \). This means, \( \overline{a}(i) = n-\nu \), if \( i = c-2\nu \), and \( w(\overline{S}_i) = (c-2\nu)(n-\nu) - (n-\nu)(n-\nu-1) \), \( 1 \leq \nu \leq r \). If \( i = c-2\nu \) we get:
Remark 4.12. The formulas (4.4) and (4.5) agree in the ends of the intervals. This is shown by putting in
\[ w(S_i) - w(S_i) = (c-2\nu)(n-\nu) - (n-\nu)(n-\nu-1) - [(c-2\nu)(n-\nu+1) - (n-\nu+1)(n-\nu)] = \]
\[ -(c-2\nu) + (n-\nu) \cdot 2 = 2n - c. \]

It follows that \( w(P) - w(P) = r(2n - c) \).

2nd case: We sum up \( w(S_i) \) and \( w(S_{i-1}) \) if \( i = c - 2\nu \) and we get:

\[
\sum_{\nu=1}^{n-1} (n-\nu)(2c-2n-2\nu+1) = \sum_{\nu=1}^{n-1} \nu(2c-4n+2\nu+1)
\]
\[ = (2c-4n+1) \cdot \frac{1}{2} (n-1)n + \frac{1}{3} (n-1)n(2n-1) \]
\[ = (n-1)n(c-2n+0.5+\frac{2}{3}n-\frac{1}{3}) \]
\[ = n(n-1)(c-\frac{4}{3}n+\frac{1}{6}). \]

Beside this, we have to add the weight \( (c-1)n - n(n-1) = n(c-n) \), if \( i = c-1 \), and we get \( w(P) = n[(c+0.5)n - \frac{4}{3}n^2 - \frac{1}{6}] \).

In the reduced pyramid the initial terms \( \overline{\pi}(i) \) of the column \( \overline{S}_i \), if \( i = c-1, c-3, \ldots, c-2r+1 \), then are equal to \( n-1, n-2, \ldots, n-r \). This means \( \overline{\pi}(i) = n-\nu \), if \( i = c-2\nu+1 \), and \( w(\overline{S}_i) = (c-2\nu+1)(n-\nu) - (n-\nu)(n-\nu-1), 1 \leq \nu \leq r \). If \( i = c-2\nu+1 \) we get:
\[ w(\overline{S}_i) - w(S_i) = (c-2\nu+1)(n-\nu) - (n-\nu)(n-\nu-1) - [(c-2\nu+1)(n-\nu+1) - (n-\nu+1)(n-\nu)] = 2n - c - 1, 1 \leq \nu \leq r. \]
We get \( w(\overline{P}) - w(P) = r(2n - c - 1) \).

Remark 4.10. The maximal weight of a pyramid \( P \) of type \( (c,d) \) is equal to

(4.4) \[ n[(c-1,5)n + (c+2r-1/6) - 4/3 \cdot n^2] - rc \]
if \( d = n(n+1) - r \) and \( 0 \leq r < n \) and it is equal to

(4.5) \[ n[(c+0.5)n + (2r-1/6) - 4/3 \cdot n^2] - r(c+1) \]
if \( d = n^2 - r \) and \( 0 \leq r < n \).

Proof. If \( d \geq 3 \), this follows from the foregoing computation. If \( d = 2 \) or if \( d = 1 \), then \( n = 1 \) and \( r = 0 \). The formula (4.4) and the formula (4.5) give the weights \( 2c - 3 \) and \( c - 1 \), which is confirmed by Fig. 4.7a and Fig. 4.7b, respectively.

Remark 4.11. The formulas (4.4) and (4.5) agree in the ends of the intervals. This is shown by putting in \( r = n \) in (4.4) and \( r = 0 \) in (4.5) and by putting in \( n - 1 \) instead of \( n \) and \( r = 0 \) in (4.4) and \( r = n \) in (4.5), respectively, and then checking equality.

We denote the maximal weight of a pyramid of type \( (c,d) \) by \( w(P_{c,d}) \).

Remark 4.12.

\[ w(P_{c,d}) = \begin{cases} 
-\frac{1}{3}n^3 - 1,5n^2 + (2r - \frac{1}{6})n + dc, & \text{if } d = n(n+1) - r, \\
-\frac{1}{3}n^3 + 0.5n^2 + (2r - \frac{1}{6})n - r + dc, & \text{if } d = n^2 - r
\end{cases} \]

Thus \( w(P_{c,d}) \) is a strictly increasing function of \( c \geq d \), if \( d \) is fixed.
Remark 4.13. Fixing the integer $c \geq 5$, then $w(P_{c,d})$ is a strictly increasing function of $1 \leq d \leq c$.

Proof. $w(P_{c,d}) = -\frac{4}{3}n^3 - 1, 5n^2 - \frac{1}{6}n + cn(n + 1) + r(2n - c)$, if $d = n(n + 1) - r, 0 \leq r \leq n$, and $w(P_{c,d}) = -\frac{4}{3}n^3 + 0, 5n^2 - \frac{1}{6}n + cn^2 + r(2n - c - 1)$, if $d = n^2 - r, 0 \leq r \leq n$. From $n(n + 1) - r = d \leq c$ and from $n^2 - r = d \leq c$ it follows that $n \leq \sqrt{c}$. From $c \geq 5$ we conclude $2n - c < 0$. From $d \geq 1$ it follows that $n \geq 1$. As a function of $r$ both terms for $w(P_{c,d})$ are strictly decreasing in the intervals $0 \leq r \leq n$, and the assertion follows from Remark 11 and Remark 12.

□

Remark 4.14. Fixing the integer $1 \leq c \leq 4$, $w(P_{c,d})$ is an increasing function of $1 \leq d \leq c$.

Proof. By drawing the possible patterns one finds:

| $d$ | $w(P_{2,d})$ | $w(P_{3,d})$ | $w(P_{4,d})$ |
|-----|--------------|--------------|--------------|
| 1   | 1            | 2            | 3            |
| 2   | 1            | 3            | 4            |
| 3   |              |              |              |

We define $P_c := P_{c,c}$ and $w(\emptyset) = 0$.

Proposition 4.1. $w(P_c) \leq (c - 1)^2$ for all $c \in \mathbb{N}$.

Proof. 1st case: $c = d = n(n + 1) - r, 0 \leq r < n$. Because of Remark 12 one has to show:

- $-\frac{4}{3}n^3 - 1, 5n^2 + (2r - \frac{1}{6})n + c^2 \leq (c - 1)^2$ ⇐
- $\frac{4}{3}n^3 + 1, 5n^2 - (2r - \frac{1}{6})n - 2[n(n + 1) - r] + 1 \geq 0$ ⇐
- $\frac{4}{3}n^3 - 0, 5n^2 - (2 + \frac{11}{6})n + 2r + 1 \geq 0$ ⇐
- $\frac{4}{3}n^3 - 0, 5n^2 - (2n + \frac{11}{6})n + 2n \geq 0$ ⇐
- $\frac{4}{3}n^3 - 2, 5n^2 + \frac{1}{6}n \geq 0$. This is true if $n \geq 2$. If $n = 1$, then $r = 0$, and by substituting one can convince oneself that the inequality is true in this case, too.

2nd case: $c = d = n^2 - r, 0 \leq r < n$. One has to show:

\[
-\frac{4}{3}n^3 + 0, 5n^2 + (2r - \frac{1}{6})n - r + c^2 \leq (c - 1)^2
\]
\[
\iff \frac{4}{3}n^3 - 0, 5n^2 - (2r - \frac{1}{6})n - r - 2[n^2 - r] + 1 \geq 0
\]
\[
\iff \frac{4}{3}n^3 - 2, 5n^2 - (2r - \frac{1}{6})n + 3r + 1 \geq 0.
\]

(4.6)

Assuming $n \geq 2$, this inequality follows from

\[
\frac{4}{3}n^3 - 2, 5n^2 - (2n - \frac{1}{6})n + 3n + 1 \geq 0
\]
\[
\iff \frac{4}{3}n^3 - 4, 5n^2 + 3\frac{1}{6}n \geq 0
\]
\[
\iff \frac{4}{3}n^2 - 4, 5n + 3\frac{1}{6} \geq 0.
\]

This is true if $n \geq 3$. Putting $n = 1$ and, $n = 2$ in (4.6) gives the inequalities $r \geq 0$ and $2 - r > 0$, respectively, which are true by assumption. As $P_{1,1} = \emptyset$, the assertion is true if $c = 0$ or $c = 1$, too.

□
Let be \( V \subset S_n \) a \( m \)-dimensional subspace and \( V \leftrightarrow \xi \in \mathbf{G}(k) \) the corresponding point in \( \mathbf{G} = \text{Grass}_m(S^n) \). We let \( \mathbb{G}_m \) and \( \mathbb{G}_a \) operate on \( S \) as described in (4.1). We assume \( V \) not to be invariant under \( \mathbb{G}_m \) or \( \mathbb{G}_a \). Then \( C := \{ \psi_\alpha(\xi) | \alpha \in k \} \) is a closed irreducible curve, which is to have the induced reduced scheme structure and which we imagine as a curve in \( \mathbb{P}^N \) by means of the Plücker embedding \( p \). Let be \( h \) the Hilbert polynomial of \( p(C) \subset \mathbb{P}^N \), \( \mathcal{X} = \text{Hilb}^h(\mathbf{G}) \leftrightarrow \text{Hilb}^h(\mathbb{P}^N) \) and \( \sigma : \mathbb{G}_m \to \mathcal{X} \) the morphism \( \lambda \mapsto \sigma(\lambda)C \). It has an extension \( \overline{\sigma} : \mathbb{P}^1 \longrightarrow \mathcal{X} \), which induces a family of curves

\[
C \hookrightarrow \mathbf{G} \times \mathbb{P}^1 \quad \overline{\sigma} \downarrow \quad p_2
\]

such that \( f \) is flat and \( C_\lambda := f^{-1}(\lambda) = \sigma(\lambda)C \) for all \( \lambda \in \mathbb{P}^1 - \{0, \infty\} \). As \( f \) is dominant, we get (cf. [Fu], p.15): If \( C_{0/\infty} := p_1(C_{0/\infty}) \), then \([C_0] = [C_{\infty}] \) in \( A_1(\mathbf{G}) \).

Let be \( \xi_{0/\infty} = \lim_{\lambda \to 0/\infty} \sigma(\lambda)\xi \) and \( C_{0/\infty} := \{ \psi_\alpha(\xi_{0/\infty}) | \alpha \in k \} \). The central theme of \([T1]-[T4]\) is the question, what is the connection of \([C_0] \) and \([C_{\infty}] \). The essential tool, which was already used in \([T1]\) is the \( \alpha \)-grade of \( V \), which is nothing else than the degree of \( C \), imbedded in \( \mathbb{P}^N \) by means of \( p \) (cf. \([T1]\), 1.3).

We paraphrase the recipe for estimating the \( \alpha \)-grade of \( V \): Let \( M_1 < \cdots < M_\ell, \ell = \binom{n+2}{2} \), be the inverse-lexicographically ordered monomials of \( S_n \), and let be \( f_i = \sum_{j=1}^{\ell} a_{ij} M_j \), \( 1 \leq i \leq m \), a basis of \( V \). If \( M_{j_1} < \cdots < M_{j_m} \) is a sequence of monomials in \( S_n \), then the Plücker coordinate \( P_V(M_{j_1}, \ldots, M_{j_m}) \) is defined to be the determinant of \( (a_{ij})_1 \leq i, j \leq m \).

In the following, \( V \) is a \( T(\rho) \)-invariant subspace of \( S_n \), and \( f_i = M_i(1 + a_i^1 X^\rho + a_i^2 X^{2\rho} + \cdots + a_i^{\rho}(\eta) X^{\nu(i)\rho}) \), \( 1 \leq i \leq m \), is a basis of \( T(\rho) \)-semi-invariants. From formula (1) in (4.1) it follows that

\[
\alpha \text{-grade}(V) \leq \max \{ \alpha \text{-grade} \langle M_1 X^{j(1)\rho}, \ldots, M_m X^{j(m)\rho} \rangle \}
\]

where \( (j) = (j(1), \ldots, j(m)) \in [1, \ell]^m \cap \mathbb{N}^m \) runs through all sequence such that \( P_V(M_1 X^{j(1)\rho}, \ldots, M_m X^{j(m)\rho}) \neq 0 \).

It is possible to choose the semi-invariants \( f_i \) so that the initial monomials \( M_i \) (the final monomials \( M_i X^{\nu(i)\rho} =: N_i \), respectively) are linearly independent, i.e. different from each other (cf. Appendix E or the proof of Hilfssatz 6 in \([T2]\), Anhang 1, p. 140). As the Plücker coordinates of a subvector space, up to a factor different from zero, do not depend on the basis which one has chosen, one has

\[
P_V(M_1, \ldots, M_m) \neq 0 \quad \text{and} \quad P_V(N_1, \ldots, N_M) \neq 0.
\]

Define \( V_0 = \langle M_1, \ldots, M_m \rangle \leftrightarrow \xi_0 \) and \( V_\infty := \langle N_1, \ldots, N_m \rangle \leftrightarrow \xi_\infty \). As the function \( \text{deg} \) is constant on flat families of curves, we get

\[
\alpha \text{-grade}(V) = \text{deg}(C) = \text{deg}(C_0) = \text{deg}(C_{\infty})
\]

As \( C_{0/\infty} \subset C_{0/\infty} \) we conclude that

\[
\alpha \text{-grade}(V) \geq \max(\alpha \text{-grade}(V_0), \alpha \text{-grade}(V_\infty)).
\]
Now it is scarcely possible to see if $P_{V}(M_{1}X_{j(1)}^{(1)}\rho, \ldots, M_{m}X_{j(m)}^{(m)}\rho)$ is different from zero. Therefore we introduce the number

$$\max -\alpha\text{-grade}(V) := \max_{(j)} \{\alpha - \text{grade} (a_{1}^{j(1)}M_{1}X_{j(1)}^{(1)}\rho, \ldots, a_{m}^{j(m)}M_{m}X_{j(m)}^{(m)}\rho)\}$$

where $(j)$ runs through all sequences $(j(1), \ldots, j(m)) \in \mathbb{N}^{m}$, such that $0 \leq j(i) \leq \nu(i)$ for all $1 \leq i \leq m$ and $a_{i}^{0} := 1$.

**Remark 4.15.** (a) Clearly $\alpha\text{-grade}(V) \leq \max -\alpha\text{-grade}(V)$.
(b) In the definition, the monomials need not be ordered.
(c) If one coefficient $a_{i}^{j(i)}$ is equal to zero or if two of the monomials $M_{i}X_{j(i)}^{(i)}\rho$ are equal for different indices $i$, then the $m$-times exterior product of the monomial space and its $\alpha$-grade are zero (cf. 4.1).

To say it differently, take from each semi-invariant $f_{i}$ a monomial $M_{i}X_{j(i)}^{(i)}\rho$, whose coefficient $a_{i}^{j(i)}$ is different from zero, form $\bigwedge_{i=1}^{m} \psi_{\alpha}(M_{i}X_{j(i)}^{(i)}\rho)$, and determine the highest power of $\alpha$ occurring in such an exterior product. Finally, define $\max -\alpha\text{-grade}(V)$ to be the maximum of these degrees, if $(j)$ runs through all such sequences.

Accordingly, one defines

$$\min -\alpha\text{-grade}(V) := \min_{(j)} \{\alpha\text{-grade}(a_{1}^{j(1)}M_{1}X_{j(1)}^{(1)}\rho, \ldots, a_{m}^{j(m)}M_{m}X_{j(m)}^{(m)}\rho)\}$$

where $(j)$ runs through all sequences of the kind described above and which give an $\alpha$-grade different from zero.

As $\alpha\text{-grade}(V_{0/\infty}) = \deg(C_{0/\infty})$, from (4.7) we conclude that

$$\min -\alpha\text{-grade}(V) \leq \min(\deg C_{0}, \deg C_{\infty}). \tag{4.9}$$

Later on, the vector space $V$ will always be equal to $H^{0}(\mathcal{I}(n)))$, where $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^{2}}$ is a $G = \Gamma \cdot T(\rho)$-invariant ideal of $y$-standard form (cf. 2.4.3 Definition 2). We will see that $\max -\alpha\text{-grade}(\mathcal{I}) := \max -\alpha\text{-grade}(H^{0}(\mathcal{I}(n)))$ and $\min -\alpha\text{-grade}(\mathcal{I}) := \min -\alpha\text{-grade}(H^{0}(\mathcal{I}(n)))$ not only are independent of $n \geq \text{colength} (\mathcal{I})$, but also can be computed with the help of smaller numbers $n$. The real aim of the following estimates is to prove the following inequality: If $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^{2}}$ is an ideal of $y$-standard form and if $\text{reg}(\mathcal{I}) = m$, then

$$Q(m - 1) + \min -\alpha\text{-grade}(\mathcal{I}) > \max -\alpha\text{-grade}(\mathcal{I}). \tag{!}$$

From this it will follow that $C_{0}$ and $C_{\infty}$ do not contain any $y$-standard cycle besides $C_{0}$ and $C_{\infty}$, respectively (cf. Lemma 9.2).
Estimates of the $\alpha$-grade in the case $\rho_1 < 0, \rho_2 > 0$.

5.1. Preliminary remarks.

We refer to Proposition 3.1 in section 3.5 and we treat case (I.1) at first. If in $f_i$ the vice-monomial $N^\text{up}_j$ occurs, then $I_k$ is monomial for all $k \geq j + 1$. Especially, $f_{j+1}, \ldots, f_r$ are monomials, which do not cause a deformation of the pyramid.

We show that the $z$-degree of such a monomial $N^\text{up}_j$ cannot be equal to the $z$-degree of an initial monomial $M_k$. For then it would follow $m_j + j - 1 = m_k + k$ for another index $k$, which is not possible by Corollary 2.4. For the same reason it is not possible that the $z$-degree of $N^\text{up}_j \cdot (z/y)$ is equal to the $z$-degree of $N^\text{down}_k$ or of $M_k$. The corresponding statements are true, if “up” and “down” are exchanged, as it follows from the corresponding definition in (3.2) and (3.3).

Finally, if there occurs a deformation of the form (1.6) of Proposition 3.1, then it can be that the final monomial $N^\text{up}_k \cdot (z/y)$ of $f_i$ has the same $z$-degree as the initial monomial $M_\ell$ of $f_\ell$. But then $\ell > k > j > i$, and therefore $I_\ell$ is a monomial ideal by Lemma 3.1. But then $f_\ell$ does not define a deformation, at all. It follows from this remarks that one can separately consider the deformations defined by the different $f_i$, if one wants to determine the changes of $\alpha$-grade caused by these deformations. (N.B. This statement is analogously valid in the situation described by Proposition 3.2, too).

At first we determine the change of the $\alpha$-grade, if in one $f_i$ the initial monomial $M^\text{up}_i$ is replaces by another monomial occurring in $f_i$:

1° $M^\text{up}_i \mapsto N^\text{down}_j$, if $0 \leq i < j \leq r$;
2° $M^\text{up}_i \mapsto N^\text{up}_j$, if $0 \leq i < j \leq r$;
3° $M^\text{up}_i \mapsto L, L \in L$ monomial such that $(x,y)L \subset \ell K(-r-1), 0 \leq i \leq r$;
4° $M^\text{up}_i \mapsto N^\text{up}_k \cdot (z/y) = M^\text{up}_k \cdot (z/y)^2, 0 \leq i < k \leq r$;
5° $M^\text{up}_i \mapsto L \cdot (z/y), L \in L$ monomial such that $(x,y)L \subset \ell K(-r-1), 0 \leq i \leq r$.

The deformation 4° (resp. 5°) comes from the case 1.6 (resp. 1.7) of Proposition 3.1, and therefore there is at most one such a deformation, whereas in the deformations 1° and 2° (resp. 3°) the index $i$ may a priori run through all integers $0 \leq i < r$ (resp. $0 \leq i \leq r$). Then for the index $j$ in the cases 1° and 2° (resp. for the monomial $L$ in the case 3°) there are several possibilities. But if one has chosen $i$, then one has to decide for an index $j$ (resp. for a monomial $L$), and we will give an uniform estimate of the corresponding changes of $\alpha$-grades.

We identify the set $L$, which was introduced in Section (3.2) with the vector space generated by the monomials in this set.
We denote by $LB$ (left domain) the vector space generated by all monomials in $S_{m_0}$ with $z$-degree $\geq m_0 - (c + r)$, i.e., generated by all monomials $x^ay^bz^{m_0-(a+b)}$, where $a + b \leq c + r$.

As to the deformation $4^\circ$ (resp. $5^\circ$), there is still the possibility $yM_i^{up} \mapsto yM_i^{up}(z/y)^2$ (resp. $yM_i^{up} \mapsto yL \cdot (z/y)$). This is because in the case 1.6 (resp. 1.7) of Proposition 3.1, $f_i$ has the order 1, whereas in the remaining cases $f_i$ has the order 0.

Remember that (cf. Figure 3.1 and 5.1):

1. $M_i^{up} = x^{i-i(i)}y^{m_i+i(i)}z^{m_0-m_i-i}$
2. $N_i^{up} = M_i^{up} \cdot (z/y) = x^{i-i(i)}y^{m_i+i(i)-1}z^{m_0-m_i-i+1}$
3. $M_i^{down} = x^{m_i+i(i)-i}y^{i(i)}z^{m_0-m_i-i}$
4. $N_i^{down} = M_i^{down} \cdot (z/x) = x^{m_i+i(i)-1}y^{i(i)}z^{m_0-m_i-i+1}$
5. $E_k^{up} := M_k^{up} \cdot (z/y)^2 = x^{k-i(k)}y^{m_k+i(k)-2}z^{m_0-m_k-k+2}$
6. $E_k^{down} := M_k^{down} \cdot (z/x)^2 = x^{m_k+i(k)-2}y^{i(k)}z^{m_0-m_k-k+2}$

5.2. Estimates in the case I.

We determine one after the other the changes of the $\alpha$-grade in the deformations:

1. $M_i^{up} \mapsto N_j^{down}$.
   - First we note $\varphi'(m_i + i) = m_i + 1$ and $\varphi'(m_j + j - 1) = m_j - 1$ (see Fig. 5.2). The $\alpha$-grade of the column, in which $M_i^{up}$ occurs changes by $-(m_i + i(i)) + \varphi'(m_i + i) - 1 = -i(i)$ (cf. the formula (4.2) in 4.1). The $\alpha$-grade of the column, to which $N_j^{down}$ is added, changes by $i(j) - \varphi'(m_j + j - 1) = i(j) - m_j + 1$ (loc. cit.). Therefore the $\alpha$-grade changes by $-m_i + \varphi'(m_i + i) - 1 + i(j) - \varphi'(m_j + j - 1) = i(j) - i(i) - m_j + 1$.
   - As $0 \leq i(i) \leq i(j) \leq j$ the absolute value of this difference is $\leq \max(j, m_j - 1)$, where $0 \leq i < j \leq r$.

2. $M_i^{up} \mapsto N_j^{up}$.
   - The $\alpha$-grade of the column, to which $M_i^{up}$ belongs changes by $-(m_i + i(i)) + \varphi'(m_i + i) - 1 = -i(i)$; the $\alpha$-grade of the column, to which $N_j^{up}$ is added, changes by $m_j + i(j) - 1 - \varphi'(m_j + j - 1) = i(j)$. Therefore the change of $\alpha$-grade is equal to $0 \leq i(j) - i(i) \leq j$, where $0 \leq i < j \leq r$.

3. $M_i^{up} \mapsto L \in \mathcal{L}$.
   - The $\alpha$-grade of the column, to which $M_i^{up}$ belongs, changes by $-i(i)$. From the domain $\mathcal{L}$ the monomial $L$ is removed, such that by Proposition 4.1 one gets the following estimate of the $\alpha$-grade: The $\alpha$-grade after the deformation $3^\circ$ of that part of the pyramid, which belongs to the left domain is $\leq (c-1)^2 + \iota(r+1)(\frac{c+1}{2}) - (c-1)$. For one has $\# \mathcal{L} = c$, and there are $(\frac{c+1}{2}) - c$ initial monomials of $\ell H^0(\mathcal{K}(c-1))$ in the left domain $LB$. Therefore the expression in the bracket gives the number of monomials in the corresponding part of the pyramid after the deformation. Besides this one has to take into account that the pyramid is pushed upwards by $\iota(r+1)$ units (cf. Remark 4.4). We recall that $LB$ (resp. $RB$) is the vector space generated by the monomials of total degree $\leq c + r$ (resp. of total degree $\geq c + r + 1$) in $x$ and $y$ (cf. Fig. 5.3). The change of $\alpha$-grade caused by the deformation $3^\circ$ can be expressed as follows: The change in the domain $RB$ is $-i(i)$;
the $\alpha$-grade of the left domain of the pyramid after the deformation is estimated as given above.

$4^\circ \ M_i^{up} \longmapsto E_k^{up}$.

At first we consider the case that $E_k^{up}$ belongs to the right domain, i.e. $m_k + k - 2 \geq c + r + 1$, and we orientate ourselves by Figure 5.1. The deformation $4^\circ$ changes the $\alpha$-grade of the column of $M_i^{up}$ by $-\iota(i)$ (cf. $3^\circ$), and the $\alpha$-grade of the column to which $E_k^{up}$ is added changes by $m_k + \iota(k) - 2 - \varphi'(m_k + k - 2)$. As we have remarked above, $\varphi'(m_k + k) = m_k + 1$ and therefore $\varphi'(m_k + k - 2) = m_k - 2$. Therefore the $\alpha$-grade of the column of $E_k^{up}$ changes by $\iota(k)$. Altogether the deformation $4^\circ$ causes a change of $\alpha$-grade by $0 \leq \iota(k) - \iota(i) \leq k$, where $0 \leq i < k \leq r$.

This deformation occurs only once, yet one has to take into account the deformation $4^\circ bis$ ($y/z \ M_i^{up} \longmapsto N_{k}^{up}$ (Proposition 3.1c). In the column of $y M_i^{up}$ this gives a change of the $\alpha$-grade by $-(m_i + \iota(i) + 1) + \varphi'(m_i + i + 1) - 1 = -m_i - \iota(i) - 1 + m_i + 2 - 1 = -\iota(i)$

In the column of $N_{k}^{up}$ the $\alpha$-grade changes by $m_k + \iota(k) - 1 - \varphi'(m_k + k - 1) = m_k + \iota(k) - 1 - (m_k - 1) = \iota(k)$. Altogether the deformation $4^\circ bis$ gives a change of $\alpha$-grade by $0 \leq \iota(k) - \iota(i) \leq k$.

Now to the case $m_k + k - 2 \leq c + r$. Due to the deformation $4^\circ$ (resp. $4^\circ bis$) the $\alpha$-grade in the right domain of the pyramid changes by $-\iota(i)$. In any case the deformation $4^\circ$ (resp. $4^\circ bis$) gives a change of $\alpha$-grade in the right domain of absolute value $\leq r$.

$5^\circ \ M_i^{up} \longmapsto L \cdot (z/y)$.

Removing $M_i^{up}$ (resp. $(y/z) M_i^{up}$) gives a change of $\alpha$-grade by $-\iota(i)$ in the corresponding column (cf. case $4^\circ$). The changes in the left domain will be estimated later on.

The deformations $1^\circ - 5^\circ$ exclude each other, i.e., there are at most $r + 1$ such deformations plus two deformations $4^\circ bis$ and $5^\circ bis$. The changes in the right domain can be estimated in the cases $1^\circ$ and $2^\circ$ by $\max(j, m_j - 1) \leq r + m_{i+1}$, where $i$ runs through the numbers $0, \ldots, r - 1$. The absolute value of the change in the case $3^\circ$ can be estimated by $r$, and the same is true for the deformations $4^\circ, 4^\circ bis, 5^\circ$ and $5^\circ bis$.

We now consider possible trinomials.

$6^\circ$ We assume there is a trinomial of the form $1.3$. We want to determine the change of $\alpha$-grade, if $N_{j}^{up}$ is replaced by $N_{k}^{up}$, where we start from a pyramid containing $N_{j}^{up}$ instead of $M_i^{up}$. The changes of $\alpha$-grade in the following diagram follow from the computation in $2^\circ$.

$$
\begin{array}{c}
\iota(j) - \iota(i) \quad \downarrow \\
N_{j}^{up} \quad \delta \\
N_{k}^{up}
\end{array}
\quad
\begin{array}{c}
\iota(k) - \iota(i) \\
M_i^{up}
\end{array}
$$

The change of $\alpha$-degree is therefore $0 \leq \delta := \iota(k) - \iota(i) \leq r$.

$7^\circ$ The trinomial has the form $1.4$.

$$
\begin{array}{c}
\iota(j) - \iota(i) \quad \downarrow \\
N_{j}^{up} \quad \delta \\
N_{k}^{down}
\end{array}
\quad
\begin{array}{c}
\iota(k) - \iota(i) - m_k + 1 \\
M_i^{up}
\end{array}
$$

(cf. $1^\circ$ and $2^\circ$) Therefore $\delta = \iota(k) - \iota(j) - m_k + 1$, and as in $1^\circ$ we obtain the estimate

$$
0 \leq |\delta| \leq \max(k, m_k - 1) \leq m_k + k < m_{i+1} + r.
$$
8° The trinomial has the form (1.5).

\[ \nu(j) - \nu(i) \xrightarrow{M_i^{up}} \begin{cases} M_i^{up} \\ N_j^{up} \rightarrow \delta \rightarrow L \end{cases} \]

(cf. 3°) Therefore \( \delta = -\nu(j) \) and \( 0 \leq |\delta| \leq r \).

9° The trinomial has the form (1.6).

\[ \nu(j) - \nu(i) \xrightarrow{M_i^{up}} \begin{cases} M_i^{up} \\ N_j^{up} \rightarrow \delta \rightarrow N_k^{up} \cdot (z/y) \end{cases} \]

(cf. 4°) It follows \( \delta = \nu(k) - \nu(j) \) (resp. \( \delta = -\nu(j) \)) and therefore \( |\delta| \leq r \).

10° The trinomial has the form (1.7).

\[ \nu(j) - \nu(i) \xrightarrow{M_i^{up}} \begin{cases} M_i^{up} \\ N_j^{up} \rightarrow \delta \rightarrow L \cdot (z/y) \end{cases} \]

(cf. 5°) It follows that \( \delta = -\nu(j) \) and \( |\delta| \leq r \).

**N.B.** Because of \( N_j^{up} \cdot (y/z) = M_j^{up} \) the cases 9° bis and 10° bis do not occur.

Summarizing the cases 1° – 10° one sees that the total change of \( \alpha \)-grade in the right domain has an absolute value \( \leq (r + 1)r + 2r + \sum m_i \). In order to formulate this result in a suitable manner, we have to introduce some notations.

We take up the decomposition (Z) of Section (3.1) and choose a standard basis of \( H^0(K(c)) \). Then we multiply the elements in this basis as well as the forms \( f_i \) by monomials in the variables \( x, y, z \) to obtain a basis of \( T(\rho) \)-semi-invariants of \( H^0(I(n)) \) with different initial monomials. By linearly combining one can achive that the initial monomial of each semi-invariant does not appear in any other of these semi-invariant, i.e., one gets a standard basis. (Fig. 3.1 is to show these initial monomials.) The set of all monomials which occur in this basis form a pyramid, which is denoted by \( P \). Here \( n \gg 0 \), e.g. \( n \geq d \).

From each element of the basis we take a monomial the coefficient of which is different from zero and such that the monomials are different from each other. Then we compute the \( \alpha \)-grade of the vector space generated by these monomials. The maximum and the minimum of the \( \alpha \)-grades which one obtains in this way had been denoted by \( \max -\alpha - \text{grade}(V) \) and by \( \min -\alpha - \text{grade}(V) \), respectively. One chooses from such a sequence of monomials, which gives the maximal \( \alpha \)-grade (which gives the minimal \( \alpha \)-grade , respectively) those monomials the total degree in \( x \) and \( y \) of which is \( \geq c + (r + 1) \). Then one forms the \( \alpha \)-grade of the subspaces generated by these monomials and denotes it by \( \max -\alpha - \text{grade}(P \cap RB) \) (by \( \min -\alpha - \text{grade}(P \cap RB) \), respectively ). If one chooses from such sequences of monomials those
monomials the total degree in $x$ and $y$ of which is $\leq c + r$, then $\max -\alpha$-grade ($\mathcal{P} \cap \mathcal{LB}$) and $\min -\alpha$-grade ($\mathcal{P} \cap \mathcal{LB}$) are defined analogously.

Of course this is valid in the case $\rho_1 > 0$, too, but the assumption $\rho_2 > 0$ is essential. We make the

**Definition 6.** $A := \max -\alpha$-grade ($\mathcal{P} \cap \mathcal{RB}$) $- \min -\alpha$-grade ($\mathcal{P} \cap \mathcal{RB}$).

Then we can formulate the result obtained above as

**Conclusion 5.1.** In the case I.1 one has

$$A \leq r(r + 3) + \sum_{i=1}^{r} m_i.$$  \[\square\]

**N.B.** If $r = 0$ one has actually $A = 0$.

Now to the case I.2 (cf. Proposition 3.1, 2nd case).

1° $M_i^{up} \mapsto N_j^{down}$ gives a change of $\alpha$-grade of absolute value $\leq \max(r, m_{i+1})$, where $0 \leq i \leq r - 1$ (cf. the case I.1).

2° $M_i^{up} \mapsto L \in \mathcal{L}$ gives a change of $\alpha$-grade in the right domain by $-\iota(i)$ (see above). Further possible deformations are $yM_i^{up} \mapsto \in \mathcal{L}, y^2 M_i^{up} \mapsto \in \mathcal{L}, \ldots, y^{\iota(i)} M_i^{up} \in \mathcal{L}$, so long as $m_i + i + \nu < m_{i-1} + (i - 1) - 1$ (cf. Conclusion 3.2). This gives in the column of $yM_i^{up}$ (of $y^2 M_i^{up}, \ldots, y^{\iota(i)} M_i^{up}$, respectively) a change of $\alpha$-grade by

$$-(m_i + \iota(i) + 1) + [\psi'(m_i + i + 1) - 1] = -(m_i + \iota(i) + 1) + [m_i + 1] = -\iota(i)$$

(by $-(m_i + \iota(i) + 2) + [\psi'(m_i + i + 2) - 1] = -(m_i + \iota(i) + 2) + [m_i + 2] = -\iota(i)$,

$\cdots$,

$-(m_i + \iota(i) + \nu) + [\psi'(m_i + i + \nu) - 1] = -(m_i + \iota(i) + \nu) - [m_i + \nu] = -\iota(i)$, respectively)

as long as $m_i + i + \nu(i) < m_{i-1} + (i - 2)$, see above.) This procedure can be repeated at most $c$ times, until $\mathcal{L}$ is full. As $\iota(r) \leq r$, the total change of $\alpha$-degree in the right domain caused by deformations 2° has an absolute value $\leq cr$. If $A$ is defined as before one gets

**Conclusion 5.2.** In the case I.2 one has

$$A \leq r(r + c) + \sum_{i=1}^{r} m_i.$$  \[\square\]

**N.B.** If $r = 0$, then one really has $A = 0$. For removing $M_0^{up} = y^{m_0}$ does not change the $\alpha$-grade of the column of $z$-degree 0 (this $\alpha$-grade is equal to 0), or one has $f_0 = x^{m_0}$.

5.3. The case $r \geq 1$.

We start with an ideal $\mathcal{I}$ of type $r \geq 0$ such that $\ell_0 = y$, and we refer to Proposition 3.1 again. The aim is to prove the inequality (!) in (4.3), where now $V = H^0(\mathcal{I}(n))$. In the course of the following computations it will turn out that $\alpha$-grade ($V$) is independent of $n$, if $n$ is sufficient large, e.g. if $n \geq d$.
If one simply writes \( \alpha \)-grade \((\mathcal{I})\) instead of \( \alpha \)-grade \((H^0(\mathcal{I}(n)))\), where \( n \) is sufficient large, then one has to show:

\[
Q(m_0 - 1) + \min -\alpha - \text{grade} (\mathcal{I}) > \max -\alpha - \text{grade} (\mathcal{I}) \quad (1)
\]

We orientate ourselves by Fig. 5.4. From the Remarks 4.4, 4.13, 4.14 and Proposition 4.1 it follows that

\[
\min -\alpha - \text{grade}(\mathcal{L}\mathcal{B} \cap \mathcal{P}) \geq \iota(r + 1)[\left(\frac{c + 1}{2}\right) - c]
\]

\[
\max -\alpha - \text{grade}(\mathcal{L}\mathcal{B} \cap \mathcal{P}) \leq (c - 1)^2 + \iota(r + 1)[\left(\frac{c + 1}{2}\right) - (c - s - 1)]
\]

where \( s + 1 \) = total number of all deformations \( M_i^{\text{up}} \rightarrow \mathcal{L}, \cdots, y' M_i^{\text{up}} \rightarrow \mathcal{L} \), even with different indices \( i \) (\( s = -1 \) means, there are no such deformations). For proving \((1)\) it is sufficient to show

\[
(\ast) \quad Q(m_0 - 1) > (c - 1)^2 + (s + 1) \cdot \iota(r + 1) + A
\]

where \( A = \max -\alpha - \text{grad} (\mathcal{P} \cap \mathcal{R}\mathcal{B}) - \min -\alpha - \text{grade} (\mathcal{P} \cap \mathcal{R}\mathcal{B}) \) by definition (cf. Section 5.2 for the notations).

**N.B.** If \( c = 0 \) there are no deformations into the left domain of the pyramid. Therefore the inequality \((\ast)\) reduces to

\[
(\ast \text{bis}) \quad Q(m_0 - 1) > A.
\]

These statements are independent of \( \rho_1 < 0 \) (Case I) or \( \rho_1 > 0 \) (Case II). Unfortunately one has to distinguish these two cases in the following estimates and we start with Case I.1 (cf. Proposition 3.1).

Because of \( 0 \leq s + 1 \leq c, 1 \leq \iota(r + 1) \leq r + 1 \) and the estimates of Conclusion 5.1 as well as Lemma 2.8 it is sufficient to show:

\[
\binom{m_0 + 1}{2} - (c + \sum_{i=0}^{r} m_i) > (c - 1)^2 + c(r + 1) + r(r + 3) + \sum_{i=1}^{r} m_i
\]

\[
(5.1) \quad \Longleftrightarrow \frac{1}{2}m_0(m_0 - 1) > c^2 + cr + r(r + 3) + 1
\]

The case \( r = 0 \) has to be treated separately (see below Section 5.4), such that we can assume \( r \geq 1 \). If \( c = 0 \), then \( m_0 \geq 2^{r+1} \) (Lemma 2.8), and \((5.1)\) follows from \( 2^r(2^{r+1} - 1) > r(r + 3) + 1 \), which inequality is true if \( r \geq 1 \). Therefore we can assume \( c > 0 \). Because of \( m_0 \geq 2^r(c+2) \) the inequality follows from \( 2^{r-1}(c+2) \cdot 2^r(c+1) > c^2 + cr + r(r+3) + 1 \Longleftrightarrow 2^{2r-1}(c^2 + 3c + 2) > c^2 + cr + r(r + 3) + 1 \), which is true if \( r \geq 1 \) and \( c > 0 \).

**Conclusion 5.3.** In the case I.1 the inequality \((\ast)\) is valid, if \( r \geq 1 \).

Now to the case I.2. Because of Conclusion 5.2 one has to replace in the inequality above \( r(r + 3) \) by \( r(r + c) \), i.e. one has to show that \( 2^{2r-1}(c^2 + 3c + 2) > c^2 + 2rc + r^2 + 1 \), if \( r \geq 1 \). This is true, if \( c \geq 0 \).

**Conclusion 5.4.** In the case I.2 the inequality \((\ast)\) is valid, if \( r \geq 1 \).

\[\square\]
5.4. The case $r = 0$.

As always we suppose $g^*(\phi) > g(d)$. As $\ell_0 = y$ and $\rho_1 < 0$ by assumption, one has $f_0 = x^{m_0}$ (cf. Fig. 5.5). Therefore there are no deformations into $\mathcal{L}$, i.e. $A = 0$ and (*) reads $Q(m_0 - 1) > (c - 1)^2$. If for simplicity one writes $m$ instead of $m_0$, then one has to show

$$m(m - 1) > 2c^2 - 2c + 2$$

Now $c = \text{colength } (K)$, $K$ an ideal of type $(-1)$ and the following cases can occur:

1. If $\psi$ is the Hilbert function of $K$, then $g^*(\psi) \leq g(c)$. Especially one has $c \geq 5$ and by Corollary 2.2 $m \geq 2c + 1$, from which (5.2) follows.
2. One has $c \leq 4$. If $c = 0$, then $\mathcal{I}$ is monomial and there are no deformations at all. It follows that ($\ast$ bis) is fulfilled.

A little consideration shows that because of $m \geq c + 2$ the inequality (5.2) is fulfilled in the cases $1 \leq c \leq 4$, too. Thus (*) and ($\ast$ bis) are proved in the case $r = 0$, also. Using Conclusion 5.4 gives

**Proposition 5.1.** Assume that $\mathcal{I}$ has the type $r \geq 0$ and has $y$-standard form; assume $\rho_1 < 0$ and $\rho_2 > 0$. Then (*) and ($\ast$ bis) are fulfilled, respectively. \hfill \Box

**N.B.** Hence the inequality (!) follows (see the corresponding argumentation in 5.3).
Fig. 5.1

\[
M_{k}^{up} \quad N_{k}^{up} \quad E_{k}^{up}
\]

\[m_{k+k-2} \quad m_{k+k}\]

Fig. 5.2

\[\varphi'(m_i + \bar{i}) = m_i + 1\]

\[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 8 \quad m_{2+2} \quad m_{1+1} \quad 10 \quad m_0\]
Fig. 5.3

#\{ monomials in the left domain \} = h^0(\mathcal{K}(c - 1)) = \binom{c+1}{2} - c
as \mathcal{K} has the Hilbert polynomial \( \binom{n+2}{2} - c \).

#\{ monomials in the right domain \} = Q(m_0 - 1) - #\{ monomials in the left domain \}
= \binom{m_0+1}{2} - d \left[ \binom{c+1}{2} - c \right]
Fig. 5.4a

\[ m_1 + 1 \cdots m_0 \]

Fig. 5.4b

\[ m_1 + 1 \cdots m_0 \]

Fig. 5.5 \((r = 0)\)

\[ \#\mathcal{L} = c \]

\(c\) \(c + 1\)

\(LB\) \(RB\)
CHAPTER 6

Estimates of the \( \alpha \)-grade in the case \( \rho_1 > 0, \rho_2 > 0 \) and \( r \geq 1 \).

We refer to Proposition 3.2 in (3.6) and take over the notations from there. As has been remarked in (5.1) one can compute the changes of \( \alpha \)-grade by the single deformations \( f_i \) separately.

6.1. Estimates in the case II

At first we compute the changes of the \( \alpha \)-grade, if we replace the initial monomial \( M_i \) of \( f_i \) by another monomial occurring in \( f_i \) (cf. 5.1):

1° \( M_i^{\text{down}} \rightarrow N_j^{\text{up}} \), if \( 0 \leq i < j \leq r \).

In the column of \( M_i^{\text{down}} \) there is a change of the \( \alpha \)-grade by \( -\iota(i)+\varphi'(m_i+i)-1 = m_i-\iota(i) \).

In the column of \( N_j^{\text{up}} \) there is a change of the \( \alpha \)-grade by \( m_j+\iota(j)-1-\varphi'(m_j+j-1) = m_j+\iota(j)-1-(m_j-1) = \iota(j) \).

Therefore the deformation 1° gives a change of the \( \alpha \)-grade by \( m_i+\iota(j)-\iota(i) \), \( 0 \leq i < j \leq r \).

2° \( M_i^{\text{down}} \rightarrow N_j^{\text{down}} \), \( 0 \leq i < j \leq r \).

In the column of \( M_i^{\text{down}} \) there is a change of the \( \alpha \)-grade by \( m_i-\iota(i) \); in the column of \( N_j^{\text{down}} \) is a change of \( \alpha \)-grade by \( \iota(j)-\varphi'(m_j+j-1) = \iota(j)-m_j+1 \). Therefore the deformation 2° gives a change of \( \alpha \)-grade, whose absolute value is \( |\iota(j)-\iota(i)+m_i-m_j+1| \leq \max(|m_i+\iota(j)|, |m_j+\iota(i)-1|) \leq m_i+j \), where \( 0 \leq i < j \leq r \).

3° \( M_i^{\text{down}} \rightarrow L \in \mathcal{L} \).

In the column of \( M_i^{\text{down}} \) is a change of \( \alpha \)-grade by \( 0 < m_i-\iota(i) \leq m_i+i \), \( 0 \leq i \leq r \). The change of \( \alpha \)-grade in the left domain can be estimated as in Case I.

4° \( M_i^{\text{down}} \rightarrow E_k^{\text{down}} = M_k^{\text{down}} \cdot (z/x)^2 \), \( 0 \leq i < k \leq r \).

At first we consider the case that \( E_k^{\text{down}} \) belongs to the right domain, i.e. \( m_k+k-2 \geq c+r+1 \). If the change of \( \alpha \)-grade in the column of \( M_i^{\text{down}} \) is \( m_i-\iota(i) \); the change of \( \alpha \)-grade in the column of \( E_k^{\text{down}} \) is \( \iota(k)-\varphi'(m_k+k-2) = \iota(k)-(m_k-2) \).

Therefore the absolute value of the change of \( \alpha \)-grade by the deformation 4° is \( |m_i-\iota(i)+\iota(k)-m_k+2| \leq \max(|m_i+\iota(k)|, |m_k+\iota(i)-2|) = m_i+\iota(k) \leq m_i+k \), if \( 0 \leq i < k \leq r \). This deformation can occur only once, yet one has to take into account the deformation

4° bis \( (x/z)M_i^{\text{down}} \rightarrow N_k^{\text{down}} \) (cf. Proposition 3.2c).

In the column of \( xM_i^{\text{down}} \) this gives a change of the \( \alpha \)-grade by \( -\iota(i)+\varphi'(m_i+i+1)-1 = m_i-\iota(i)+1 \). In the column of \( N_k^{\text{down}} \) the \( \alpha \)-grade changes by \( \iota(k)-\varphi'(m_k+k-1) = \iota(k)-m_k+1 \). Thus the absolute value of the change of \( \alpha \)-grade in the right domain due to 4° bis is \( |m_i-\iota(i)+1+\iota(k)-m_k+1| \leq \max(|m_i+\iota(k)|, |m_k+\iota(i)-2|) = m_i+\iota(k) \leq m_i+k \), where \( 0 \leq i < k < r \). This deformation occurs only once.

Now to the case that \( m_k+k-2 \leq c+r \). Removing \( M_i^{\text{down}} \) (resp. \( (x/z)M_i^{\text{down}} \)) gives a
change of $\alpha$-grade by $m_i - \iota(i)$ (by $m_i - \iota(i) + 1$, respectively), whose absolute value is bounded by $m_i + r$.

$5^\circ$ $M_i^{\text{down}} \mapsto L \cdot (z/x)$.

Removing $M_i^{\text{down}}$ (resp. $(x/z)M_i^{\text{down}}$) causes a change of $\alpha$-grade of the column of $M_i^{\text{down}}$ (resp. $(x/z)M_i^{\text{down}}$) by $m_i - \iota(i)$ (resp. by $m_i - \iota(i) + 1$), which are estimated by $m_i + i$ (resp. $m_i + i + 1$), where $0 \leq i \leq r$. The deformation $5^\circ$ (resp. $5^\circ \text{bis}$) can occur only once. The changes in the left domain will be estimated later on.

The deformation $1^\circ - 5^\circ$ exclude each other, i.e. there are at most $r + 1$ such deformation plus two deformations $4^\circ \text{bis}$ and $5^\circ \text{bis}$. The changes of $\alpha$-grade in the right domain in the cases $1^\circ - 3^\circ$ have an absolute value $\leq m_i + r, 0 \leq i \leq r$. The same estimate is valid for the deformations $4^\circ$ and $4^\circ \text{bis}$, even if $E_k$ belongs to the left domain, as we have assumed $r \geq 1$. As for the deformations $5^\circ$ (resp. $5^\circ \text{bis}$) we estimate the change of the $\alpha$-grade by $m_i + r$ (resp. $m_i + r + 1$).

We now consider possible trinomials. $6^\circ$ We assume there is a trinomial of the form $1.3$. Similarly as in the Case I in Chapter 5, we have a diagram

$$
\begin{array}{c}
M_i^{\text{down}} \\
\gamma(j) \bigleftarrow N_j^{\text{down}} \\
\delta \downarrow \\
\gamma(k) \bigleftarrow N_k^{\text{down}}
\end{array}
$$

where $\gamma(j) := m_i - m_j - \iota(i) + \iota(j) + 1$ and $\gamma(k) := m_i - m_k - \iota(i) + \iota(k) + 1$ (cf. $2^\circ$). Therefore $\delta = m_j - m_k - \iota(i)$. It follows that $|\delta| \leq \max(|m_j + \iota(i)|, |m_k + \iota(i)|) \leq m_i + r$.

$7^\circ$ The trinomial has the form $1.4$.

$$
\begin{array}{c}
M_i^{\text{down}} \\
\gamma(j) \bigleftarrow N_j^{\text{down}} \\
\delta \downarrow \\
\beta \bigleftarrow N_k^{\text{up}}
\end{array}
$$

where $\beta := m_i - \iota(i) + \iota(k)$ (cf. $1^\circ$ and $2^\circ$). It follows that $\delta = m_j - \iota(j) + \iota(k) - 1$ and $|\delta| \leq m_i + r$.

$8^\circ$ The trinomial has the form $1.5$.

$$
\begin{array}{c}
M_i^{\text{down}} \\
\gamma(j) \bigleftarrow N_j^{\text{down}} \\
\delta \downarrow \\
\beta \bigleftarrow L
\end{array}
$$

where $\beta := m_i - \iota(i)$ (cf. $3^\circ$). It follows that $\delta = m_j - \iota(j) - 1$ and $|\delta| \leq m_i + r$.

$9^\circ$ The trinomial has the form $1.6$.

$$
\begin{array}{c}
M_i^{\text{down}} \\
\gamma(j) \bigleftarrow N_j^{\text{down}} \\
\delta \downarrow \\
\beta \bigleftarrow N_k \cdot (z/x)
\end{array}
$$

where $\beta := m_i - m_k - \iota(i) + \iota(k) + 2$, respectively $\beta := m_i - \iota(i)$ (cf. $4^\circ$). It follows that $\delta = m_j - m_k - \iota(j) + \iota(k) + 1$ (resp. $\delta = m_j - \iota(j) - 1$) and $|\delta| \leq \max(|m_j - \iota(k)|, |m_k +
\( \iota(j) - 1 \) \( \leq m_j + r \).

10° The trinomial has the form 1.7.

\[
\begin{align*}
\gamma(j) & \sqrt{M_i^\text{down}} \\
N_j^\text{down} & \delta \rightarrow L \cdot (z/x)
\end{align*}
\]

(cf. 5°). It follows that \( \delta = m_j - \iota(j) - 1 \) and \( |\delta| < m_i + r \).

**Notabene.** Because of \( N_j^\text{down} \cdot (x/z) = M_j^\text{down} \) the case 9°bis or 10°bis does not occur.

Summarizing the cases 1° - 10° one sees that the total change of \( \alpha \)-grade in the right domain has an absolute value \( \leq (r + 1)r + 3m_0 + \sum_{i=1}^{r} m_i \). If one estimates the changes in the cases 4°bis and 5°bis by \( m_i + r \) and \( m_i + r + 1 \), respectively, one obtains \( A \leq r(r+3)+1+3m_0+\sum_{i=1}^{r} m_i \). As we have assumed that \( r \geq 1 \), we have \( m_0 \geq c+2+m_r+\cdots+m_1 \) (Lemma 2.8) and we obtain

**Conclusion 6.1.** If \( r \geq 1 \) is assumed, in the case II.1 one has \( A \leq 4m_0+r(r+3) - c-1 \). \( \square \)

Now we come to the case II.2 (cf. Proposition 3.2, 2nd case).

1° \( M_i^\text{down} \rightarrow N_j^\text{up} \) again gives a change of the \( \alpha \)-grade in the right domain, whose \( \alpha \)-grade value is \( \leq m_i + r, 0 \leq i \leq r-1 \) (see above).

2° \( M_i^\text{down} \rightarrow L \in \mathcal{L} \) gives a change of \( \alpha \)-grade in the right domain by \( m_i - \iota(i), 0 \leq i \leq r \) (see above). Further possible deformations are \( xM_i^\text{down} \rightarrow \mathcal{L}, x^2M_i^\text{down} \rightarrow \mathcal{L}, \ldots, x^\nu M_i^\text{down} \rightarrow \mathcal{L} \), as long as \( m_i + i + \nu < m_{i-1} + (i-2) \) (cf. Conclusion 3.2). This gives in the column of \( xM_i^\text{down} \) (of \( x^2M_i^\text{down}, \ldots, x^\nu M_i^\text{down} \), respectively) a change of \( \alpha \)-grade by \( -\iota(i) + \varphi'(m_i + i + 1) - 1 = -\iota(i) + (m_i + 2) - 1 = m_i - \iota(i) + 1 \) (by \( -\iota(i) + \varphi'(m_i + i + 2) - 1 = m_i - \iota(i) + 1, \ldots, -\iota(i) + \varphi'(m_i + i + \nu) - 1 = m_i - \iota(i) + \nu \), respectively), as long as \( m_i + i + \nu < m_{i-1} + (i-2) \) and \( \nu \leq c-1 \).

**Remark 6.1.** One has \( |m_i - \iota(i)| \leq m_0 \) for all \( 0 \leq i \leq r \).

**Proof.** The inequality \( -\iota(i) + m_i \leq m_i \leq m_0 \) is true, and \( \iota(i) - m_i \leq i - m_i \leq r - m_i \leq r \leq m_0 \) is true if \( r = 0 \). If \( r \geq 1 \) one has \( m_0 \geq c+2+m_r+\cdots+m_1 > r \) (Lemma 2.8). \( \square \)

From this we conclude: Replacing \( M_i^\text{down}, xM_i^\text{down}, \ldots, x^\nu(i)M_i^\text{down} \) by monomials in \( \mathcal{L} \), even with different indices \( i \) as long as \( \nu(i) \leq c-1 \), gives a change of \( \alpha \)-grade in the right domain whose absolute value is \( \leq \sum_{j=0}^{\nu(i)} (m_0 + j) \), because \( |m_i - \iota(i) + j| \leq m_0 + j \) by the remark above. One gets \( A \leq \sum_{i=0}^{r-1} (m_i+r) + \sum_{i=0}^{r} \nu(i) \sum_{j=0}^{m_0+j} \). As \( \nu(0)+1+\cdots+\nu(r)+1 \leq c \), it follows that \( A \leq \sum_{i=0}^{r-1} (m_i+r) + \sum_{j=0}^{c-1} (m_0+j) \). As \( m_0 \geq c+2+m_r+\cdots+m_1 \) and \( m_r \geq c+2 \),
we obtain \( \sum_{i=1}^{r-1} m_i \leq m_0 - 2(c+2) \). This estimate is valid if \( r \geq 1 \). In the case \( r = 0 \) one only has the deformations \( M_0^{\text{down}} \hookrightarrow \mathcal{L}, \cdots, x^s M_0^{\text{down}} \hookrightarrow \mathcal{L} \), and \( s \) can be estimated as in (Proposition 3.2).

If \( M_0^{\text{down}} \) occurs, the \( \alpha \)-grade in the column of \( M_0^{\text{down}}, \cdots, x^s M_0^{\text{down}} \) increases by \( m_0, \cdots, m_0 + s \), respectively.

**Conclusion 6.2.** In the case II.2 one has \( A \leq (c+2)m_0 + r^2 - 2(c+2) + \binom{c}{2} \), if \( r \geq 1 \). If \( r = 0 \), if \( \mathcal{I} \) has \( y \)-standard form and \( \kappa := \text{reg}(\mathcal{K}) \), then \( A \leq (s+1)m_0 + \binom{s+1}{2} \), where \( s \leq \kappa/\rho_2 - m_0(1/\rho_2 - 1/(\rho_1 + \rho_2)) \).

\( \Box \)

6.2. The case \( r \geq 2 \).

We recall that we started from an ideal \( \mathcal{I} \) of type \( r \geq 0 \) with \( y \)-standard form, and the aim was to show the inequalities

\[ Q(m_0 - 1) > (c - 1)^2 + (s + 1)\ell(r + 1) + A \quad (*) \]

and

\[ Q(m_0 - 1) > A \quad (*\text{bis}) \]

respectively (cf. Section 5.3).

At first we treat the case II.1, where \( a \leq 4m_0 + r(r + 3) - c - 1 \), if \( r \geq 1 \).

**Auxiliary Lemma 1.** If \( \mathcal{I} \) has the type \( r = 2 \) (the type \( r = 1 \), respectively), then \( m_0 \geq 14 \) (\( m_0 \geq 7 \), respectively).

**Proof.** We use the results of Lemma 2.8 and write in the case \( r = 2 \) : \( \mathcal{I} = \mathcal{I}_0 = y\mathcal{L}_1(-1) + f_0\mathcal{O}_{\mathbb{P}^2}(-m_0), \mathcal{I}_1 = \ell_1\mathcal{L}_2(-1) + f_1\mathcal{O}_{\mathbb{P}^2}(-m_1), \mathcal{I}_2 = \ell_2\mathcal{K}(-1) + f_2\mathcal{O}_{\mathbb{P}^2}(-m_2) \). \( \mathcal{I}_2 \) has the type 0, therefore colength \( (\mathcal{I}_2) = c + m_2 \geq 5 \) and it follows that \( m_2 \geq 5 - c \). As \( \mathcal{I}_1 \) has the type 1, we get \( m_1 \geq m_2 + c + 2 \geq 7 \). Because of \( m_0 \geq c + 2 + m_2 + m_1 \) it follows that \( m_0 \geq c + 2 + 5 - c + 7 \). □

To begin with, let \( c = 0 \). Then \( (*\text{bis}) \) reads \( \frac{1}{2}m_0(m_0 + 1) - (m_r + \cdots + m_0) > 4m_0 + r(r + 3) - 1 \). Because of \( m_0 - (c + 2) \geq \sum_{i=1}^{r} m_i \) it is sufficient to prove

\[ (6.1) \quad m_0(m_0 - 11) > 2r(r + 3) - 6 \]

If \( r = 2 \) (respectively \( r = 3 \)) the right side of (6.1) is equal to 14 (equal to 30, respectively). As \( m_0 \geq 14 \) by the Auxiliary Lemma 1 \( (m_0 \geq 2^{3}(c+2) = 16 \) by Lemma 2.8, respectively), (6.1) is true in these cases.

Let be \( r \geq 4 \). Then \( m_0 \geq 2^{r+1} \) and (1) follows from \( 2^{r}(2^{r+1} - 11) > r(r + 3) \). Now \( 2^{r} > r \) if \( r \geq 2 \), and \( 2^{r+1} - 11 > r + 3 \) is equivalent with \( 2^{r+1} > r + 14 \), which is true if \( r \geq 4 \).
Thus we can assume \( c \geq 1 \) in the following. Because of \( 1 \leq \ell(r+1) \leq r+1, 0 \leq s+1 \leq c, \sum_{i=1}^{r} m_i \leq m_0 - (c+2) \) the inequality (\( \ast \)) will follow from:

\[
\frac{1}{2} m_0(m_0 + 1) - (2m_0 - 2) > (c - 1)^2 + c(r + 1) + 4m_0 + r(r + 3) - c - 1
\]

(6.2) \( \iff m_0(m_0 - 11) > 2c^2 + 2(r - 2)c + 2r(r + 3) - 4 \)

Because of \( m_0 \geq 2^r(c+2) \) it suffices to show:

(6.3) \( 2^r(c + 2)(2c + 2r + 1 - 11) > 2c^2 + 2(r - 2)c + 2r(r + 3) - 4 \)

If \( r = 2 \) this inequality reads \( 4(c + 2)(4c - 3) > 2c^2 + 16 \iff 7c^2 + 10c - 20 > 0 \) and this is true if \( c \geq 2 \).

In the case \( r = 2, c = 1 \), the inequality (6.2) reads \( m_0(m_0 - 11) > 18 \), which is true because \( m_0 \geq 14 \) (cf. Auxiliary Lemma 1). Therefore we can now suppose without restriction \( r \geq 3, c \geq 1 \). But \( 2^{r+1} > 11 \) and thus (6.3) follows from \( 2^r(c + 2) \cdot 2^rc > 2c^2 + 2cr + 2r(r + 3) \) which is equivalent with:

(6.4) \( (2^{2r} - 2)c^2 + (2^{2r+1} - 2r)c > 2r(r + 3) \)

The left side of (6.4) is a monotone function of \( c \), and if \( c = 1 \), then (4) reads \( 2^{2r} - 2 + 2^{2r+1} - 2r > 2r(r + 3) \iff 2^{2r} + 2^{2r+1} > 2r^2 + 8r + 2 \iff 2^{2r-1} + 2^r > r^2 + 4r + 1 \). This is true if \( r \geq 3 \). Summarizing all subcases we obtain

**Conclusion 6.3.** In the case II.1 the inequality (\( \ast \)) is fulfilled for all \( r \geq 2 \).

We now consider the case II.2 and assume \( r \geq 2 \). With the help of Conclusion 6.2 and the estimates \( \sum_{i=1}^{r} m_i \leq m_0 - (c+2), s+1 \leq c, \ell(r+1) \leq r+1 \) one sees that (\( \ast \)) follows from

\[
\frac{1}{2} m_0(m_0 + 1) - (2m_0 - 2) \geq (c - 1)^2 + c(r + 1) + (c+2)m_0 + r^2 - 2(c+2) + \left(\frac{c}{2}\right).
\]

A simple computation shows that this is equivalent to

(6.5) \( m_0(m_0 - 2c - 7) > 3c^2 + 2cr - 7c - 10 + 2r^2 \).

Now we have \( m_0 \geq 2^r(c+2) \geq 4(c+2) \). If \( c = 0 \), then (6.5) follows from \( 2^{r+1} > -10 + r^2 \), which is true for all \( r \geq 2 \). Therefore we can assume \( c > 0 \). Then (6.5) follows from \( 2^r(c + 2)(2c + 1) > 3c^2 + 2cr + 2r^2 \iff 2^r(2c^2 + 5c + 2) > 3c^2 + 2cr + 2r^2 \) which is true for all \( c \geq 1 \) and \( r \geq 2 \). We get

**Conclusion 6.4.** In the case II.2 the inequality (\( \ast \)) is fulfilled for all \( r \geq 2 \).

### 6.3. The case \( r = 1 \).

Then \( I = yI_1(-1) + f_1O_{p^2}(-m_0), I_1 = \ell_1K(-1) + f_1O_{p^2}(-m_1) \), where \( I_1 \) has the type 0 and \( K \) has the type \(-1\).
6.3.1. We start with the case II.1 of Proposition 3.2.

Subcase 1: $\ell_1 = y$: Then one has the situation shown in Figure 6.1 and there are the following possibilities (case II 1.5 and II 1.7, respectively):
1° $f_0 = x^{m_0} + \alpha N_1^{\text{down}} + \beta L, L \in \mathcal{L}$ monomial such that $(x, y)L \subset \ell K(-2)$.
2° $f_0 = x^{m_0} + \alpha N_1^{\text{down}} + \beta L \cdot (z/x), L \in \mathcal{L}$ monomial such that $(x, y)L \subset \ell K(-2)$.

We treat the case 1°. At first, one has the possibility $x^{m_0} \mapsto N_1^{\text{down}}$. The $\alpha$-grade of the column of $x^{m_0}$ changes by $m_0$; the $\alpha$-grade of the column of $N_1^{\text{down}}$ changes by $\nu(1) - \varphi'(m_1) = 1 - (m_1 - 1) = 2 - m_1$. Therefore the change of $\alpha$-grade in the right domain is $m_0 - m_1 + 2$. The deformation $x^{m_0} \mapsto L$ gives a change of $\alpha$-grade by $m_0$ in the right domain. As the order of $f_0$ is equal to 0 in the case II 1.5 (cf. Proposition 3.2c), there are no other changes of $\alpha$-grade caused by $f_0$.

By Proposition 3.2 again, it follows that $f_1$ has the form of case II.1.5, where $\alpha = 0$, and the order of $f_1$ is equal to 0. The deformation $M_1^{\text{down}} \mapsto \in \mathcal{L}$ gives a change of $\alpha$-grade by $-\nu(1) + \varphi'(m_1 + 1) - 1 = -\nu(1) + m_1 - 1$. Thus in the case 1° one has $A \leq \max(m_0 - m_1 + 2, m_0) + m_1 - 1 = m_0 + m_1 - 1$, because $m_1 \geq c + 2$ (Lemma 2.4).

2° At first, $f_0$ defines a deformation as in the case 1° and gives a change of $\alpha$-grade $\leq \max(m_0, m_0 - m_1 + 2) = m_0$ in the right domain. But as $f_0$ has the order $\leq 1$ (Proposition 3.2c), there is still the possibility $x^{m_0+1} \mapsto \in \mathcal{L}$, which gives a change of $\alpha$-grade by $m_0 + 1$ in the right domain. As $f_1$ again has the same form as in the case 1°, it follows that $A \leq 2m_0 + m_1$.

Because of $s + 1 \leq c$ the inequality (*) follows from

$$\frac{1}{2}m_0(m_0 + 1) - (c + m_0 + m_1) > (c - 1)^2 + 2c + 2m_0 + m_1.$$ 

As $m_1 \leq m_0 - (c + 2)$, this inequality follows from $m_0(m_0 - 9) > 2c^2 - 2c - 6$. Because of $m_0 \geq 2c + 4$ it suffices to show $(2c + 4)(2c - 5) > 2c^2 - 2c - 6 \iff 2c^2 > 14$. This inequality is fulfilled, if $c \geq 3$. The cases $c = 0, 1, 2$ will be treated later on (see below).

Subcase 2: $\ell_1 = x$: Then from Figure 6.2 we conclude that only the second case of Proposition 3.2 can occur, and we have

**Conclusion 6.5.** In the case II.1 the inequality (*) is fulfilled except if $\ell_0 = y, \ell_1 = y$ and $c = \{0, 1, 2\}$.

6.3.2. We now treat the case II.2 of Proposition 3.2.

Subcase 1: $\ell_1 = y$: Figure 6.1 shows, that only the case II.2.3 is possible. Then there are $s + 1$ deformations $M_0 \mapsto \in \mathcal{L}, \ldots, x^sM_0 \mapsto \in \mathcal{L}$ (and $t + 1$ deformations $M_1 \mapsto \in \mathcal{L}, \ldots, x^tM_1 \mapsto \in \mathcal{L}$, respectively). The changes of $\alpha$-grade in the columns of $M_0, \ldots, x^sM_0$ (of $M_1, \ldots, x^tM_1$, respectively) is $m_0, \ldots, m_0 + s$ (and $m_1 - 1, m_1, \ldots, m_1 + t - 1$, respectively). Here $s$ and $t$ fulfil the inequalities of (Proposition 3.2d). Thus the total change of $\alpha$-grade in the right domain fulfils the inequality: $A \leq (s + 1)m_0 + \binom{s+1}{2} + \ldots$
\( (t + 1)m_1 + \binom{t}{2} - 1 \), where
\[
s \leq \frac{\kappa}{\rho_2} - m_0 \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right) + \frac{1}{\rho_2} \quad \text{and} \quad t \leq \frac{\kappa}{\rho_2} - m_1 \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right).
\]

**Estimate of \( s \):** Because of \( \kappa \leq c \) and \( m_0 \geq 2(c + 2) \) one obtains:
\[
s \leq \frac{c}{\rho_2} - 2(c + 2) \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right) - 1 = \left( c + 1 \right) \left( \frac{2}{\rho_1 + \rho_2} - \frac{1}{\rho_1 + \rho_2} \right) - 2 \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right).
\]

We first consider the possibility \( s \geq 0 \). This implies \( \frac{2}{\rho_1 + \rho_2} - \frac{1}{\rho_2} > 0 \), i.e. \( \rho_2 > \rho_1 \).

Let be \( f_a(x) = \frac{2}{x+a} - \frac{1}{x} \); i.e. \( x \) corresponds to \( \rho_2 \) and \( a \) corresponds to \( \rho_1 \), therefore \( 1 \leq a < x \). \( f_a'(x) = -2/(x+a)^2+1/x^2 < 0 \iff 2x^2 > (x+a)^2 \iff \sqrt{2}x > x+a \iff x > a(1+\sqrt{2}) \).

It follows that \( f_a(x) \) has the maximum for \( x = a(1+\sqrt{2}) \), and \( f_a(a(1+\sqrt{2})) = \frac{a}{2} \).

Therefore \( s \leq 0.172(c + 1) \).

**Estimate of \( t \):** Because of \( m_1 \geq c + 2 \) (cf. Lemma 2.4) one has
\[
t \leq \frac{c}{\rho_2} - (c + 2) \left( \frac{1}{\rho_1} - \frac{1}{\rho_1 + \rho_2} \right) = \frac{c}{\rho_2} - c \left( \frac{c}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right) - 2 \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right) = \frac{c - 2}{\rho_1 + \rho_2} - 2 \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right).
\]

Therefore \( t \leq c/2 \), if \( \rho_2 \leq \rho_1 \) and \( t \leq c/3 \), if \( \rho_2 > \rho_1 \).

**First possibility:** \( \rho_2 \leq \rho_1 \): Then \( s < 0 \), i.e. there are no deformations defined by \( f_0 \), and \( A \leq (c/2 + 1)m_1 + \binom{c}{2} - 1 \).

**Second possibility:** \( \rho_1 < \rho_2 \). Then \( s \leq 0.172(c + 1), t \leq c/3 \) and \( A \leq (0.172c + 1, 172)m_0 + (0.172c+1, 172) + (c/3 + 1)m_1 + \binom{c}{2} - 1 \).

As \( m_1 \leq m_0 - (c + 2) \) (Lemma 2.8), one obtains the following estimates:

**First possibility:** \( A \leq (c/2 + 1)[m_0 - (c + 2)] + c^2/8 - 1 \Rightarrow A \leq (0.5c + 1)m_0 - 3/8c^2 - 2c - 3 \)

**Second possibility:** \( A \leq (0.172c + 1, 172)m_0 + 0.5(0.172c + 1, 172)^2 + (c/3 + 1)[m_0 - (c + 2)] + c^2/18 - 1 \Rightarrow A \leq (0.51c + 2, 172)m_0 - 0.25c^2 - 1, 46c - 2, 3 \)

As we have assumed that \( \ell_0 = y, \ell_1 = y \) it follows that \( \nu(r + 1) = \nu(2) = 2 \). As mentioned above, only the case II.2.3 of Proposition 3.2 can occur. If \( c = 0 \), there are no deformations at all, so that one can assume \( c \geq 1 \). Replacing \( s + 1 \) by \( c \), one sees that it suffices to show:

\[(6.6) \quad Q(m_0 - 1) > c^2 + 1 + A \]
First possibility: $A \leq (0.5c + 1)m_1 + \left(\frac{0.5c}{2}\right) - 1$ (see above). One sees that it suffices to show:

$$\frac{1}{2}m_0(m_0 + 1) - (c + m_0 + m_1) > c^2 + 1 + (0.5c + 1)m_1 + \frac{1}{8}c^2 - 1 \iff \frac{1}{2}m_0(m_0 + 1) - m_0 - (0.5c + 2)m_1 > \frac{9}{8}c^2 + c.$$ 

Because of $m_1 \leq m_0 - (c+2)$ this follows from $\frac{1}{2}m_0(m_0+1)-(0.5c+3)m_0+(0.5c+2)(c+2) > \frac{9}{8}c^2 + c \iff m_0(m_0 - c - 5) > 1, 25c^2 - 4c - 8$. 

As $m_0 \geq 2c+4$ this follows from $(2c+4)(c-1) > 1, 25c^2 - 4c - 8 \iff 0.75c^2 + 6c + 4 > 0$. 

This inequality is fulfilled for all $c$.

Second possibility: $A \leq (0.51c+2,172)m_0 - 0.25c^2 - 1, 46c - 2, 3$. Putting this into (6), one has to show $m_0(m_0+1) - 2(c + m_0 + m_1) > 1, 5c^2 - 2, 92c - 2, 6 + (1.02c + 4, 344)m_0$. Because of $m_1 \leq m_0 - (c+2)$ this follows from $m_0(m_0+1)-2(2m_0-2)-(1,02c+4,344)m_0 > 1, 5c^2 - 2, 92c - 2, 6 \iff m_0(m_0 - 1, 02c - 7, 344) > 1, 5c^2 - 2, 92c - 6, 6$. 

As $m_0 \leq 2c + 4$ this follows from $(2c + 4)(0.98c - 3, 344) > 1, 5c^2 - 2, 92c - 6, 6 \iff 0.46c^2 + 0.152c - 6, 776 > 0$. 

This inequality is fulfilled if $c > 3, 676$ and the cases $c = 0, 1, 2, 3$ will be treated later on in (6.3.3).

Subcase 2: $\ell_1 = x$. Figure 6.2 shows that in Proposition 3.2 $f_1 = M_1^{\text{up}}$ has to be a monomial and that for $f_0$ one of the two cases 2.1 or 2.3 can occur. We first treat the case 2.1, i.e., one has the deformation $x^{m_0} \mapsto N_1^{\text{up}}$. The $\alpha$-grade of the column of $x^{m_0}$ changes by $m_0$ and the $\alpha$-grade of the column of $N_1^{\text{up}}$ changes by 1. Therefore $A \leq m_0 + 1$. There are no further deformations in the right domain. Now in the inequality (*) one has $s = 0$ and $\ell(2) = 1$ and therefore has to show:

$$\frac{1}{2}m_0(m_0 + 1) - (c + m_0 + m_1) > (c - 1)^2 + 1 + m_0 + 1.$$ 

Because of $m_1 \leq m_0 - (c + 2)$ this follows from $m_0(m_0 - 5) > 2(c - 1)^2$. As $m_0 \geq 2c + 4$ this follows from $(2c + 4)(2c - 1) > 2(c - 1)^2 \iff 2c^2 + 10c - 6 > 0$. This inequality is fulfilled, if $c \geq 1$. In the case $c = 0$ one has ($\ast$), i.e., one has to prove $Q(m_0 - 1) > A$, i.e. to prove $\frac{1}{2}m_0(m_0 + 1) - (c + m_1 + m_0) > m_0 + 1$. 

One sees that this follows from $m_0(m_0 - 5) > -2$. As one has $m_0 \geq 7$ by (6.2 Auxiliary Lemma 1), this inequality is fulfilled.

Now we treat the case 2.3, that means $f_0 = M_0 + F$ as in Proposition 3.2. The only possible deformations are $M_0 \mapsto L, xM_0 \mapsto L, \cdots, x^sM_0 \mapsto L$. As $c = 0$ implies that $L$ is monomial, we can assume $c > 0$.

We again distinguish two cases:

First possibility: $\rho_2 \leq \rho_1$. Then $s + 1 = 0$, i.e. there is no deformation at all, and therefore one has $A = 0$. Then (*) reads $\frac{1}{2}m_0(m_0 + 1) - (c + m_0 + m_1) > (c - 1)^2$. Because of $m_1 \leq m_0 - (c + 2)$ this follows from $m_0(m_0 - 3) > 2c^2 - 4c - 2$. Because of $m_0 \geq 2c + 4$ it suffices to show $(2c + 4)(2c + 1) > 2c^2 - 4c - 2 \iff 2c^2 + 14c + 6 > 0$ which is true for all $c$.

Second possibility: $\rho_1 < \rho_2$. Then $s \leq 0, 172(c + 1)$, as was shown above. We have already remarked at the beginning of (6.3.2) that the $s + 1$ deformations noted above give a change of $\alpha$-grade $A \leq (s + 1)m_0 + \left(\frac{s+1}{2}\right)$. It follows that $A \leq (0.172c + 1, 172)m_0 + \left(\frac{0.172c + 1, 172}{2}\right)$.
From Figure 6.3 it follows that

\[ (c - 1)^2 + (0.172c + 1, 172)[1 + m_0 + 0.5 \cdot (0.172c + 1, 172)]. \]

Because of \( m_1 \leq m_0 - (c+2) \) this follows from \( m_0(m_0 + 1) - 2(2m_1 - 2) > 2(c - 1)^2 + (0.344c + 2, 344)(m_0 + 0.086c + 1, 586) \iff m_0(m_0 - 0.344c - 5, 344) > 2, 029584c^2 - 3, 252832c + 1, 717584. \]

Now \( c \geq 1 \) and \( m_0 \geq 2c + 4 \) so that it suffices to show \( (2c + 4)(1656c - 1, 344) > 2, 029584c^2 - 3, 252832c + 1, 717584 \iff 1, 282416c^2 + 7, 188832c - 7, 093584 > 0. \]

This is true if \( c \geq 1 \). Therefore the inequality (*) is fulfilled in Subcase 2.

**Conclusion 6.6.** In the case II, if \( r = 1 \), then (*) is fulfilled except in the case II.1 if \( \ell_0 = y, \ell_1 = y \) and \( c \in \{0, 1, 2\} \) or in the case II.2 if \( \ell_0 = y, \ell_1 = y \) and \( c \in \{0, 1, 2, 3\} \).

**6.3.3. The cases \( 0 \leq c \leq 4 \).** From Figure 6.3 it follows that \( M_0 \leftrightarrow N_1 \) is the only possible deformation, which is the case II.1. The change of \( \alpha \)-grade is \( A = m_0 + m_1 - 2(m_1 - 1) = m_0 - m_1 + 2 \) and the inequality (*-bis) reads \( Q(m_0 - 1) > m_0 - m_1 + 2 \iff m_0(m_0 - 3) > 4 \), and this is true, as \( m_0 \geq 7 \) by (6.2 Auxiliary Lemma 1).

\( c = 0 \) At first, we note that \( \mathcal{K} \) is monomial. From Figure 6.4 it follows that there are three deformations, which can occur, with the “total” change of \( \alpha \)-grade \( B \):
1. \( M_0 \leftrightarrow L, B = m_0 + 2 \)
2. \( M_0 \leftrightarrow N_1 \) and \( M_1 \leftrightarrow L, B = (m_0 - m_1 + 2) + (2m_1 - m_1 - 1) + 2 = m_0 + 3 \)
3. \( M_1 \leftrightarrow L, B = m_1 + 1. \)

Then (*-bis) follows from \( Q(m_0 - 1) > m_0 + 3 \iff \frac{1}{2}m_0(m_0 + 1) - (1 + m_0 + m_1) > m_0 + 3. \) As \( m_1 \leq m_0 - 3 \) (Lemma 2.8), one sees that it suffices to show \( m_0(m_0 - 5) > 2 \). This is true, because of \( m_0 \geq 2(c + 2) = 6 \) (loc.cit.).

\( c = 2 \) Then \( \mathcal{K} \) is monomial, and the possible deformations are shown in Figure 6.5 a, b. If the case II.1 occurs, then \( f_0 = m_0 + \alpha N_1 + \beta F \) and \( f_1 = M_1 + \gamma L. \) One sees that the change of \( \alpha \)-grade in the right domain becomes maximal, if the deformations \( M_0 \leftrightarrow F \) and \( M_1 \leftrightarrow L \) occur. Therefore \( A \leq m_0 + 2m_1 - m_1 - 1 = m_0 + m_1 - 1 \). The inequality (*) follows from \( Q(m_0 - 1) > 1^2 + 2 \cdot 2 + (m_0 + m_1 - 1) \iff \frac{1}{2}m_0(m_0 + 1) - (2 + m_0 + m_1) > 4 + m_0 + m_1 \iff \frac{1}{2}m_0(m_0 + 1) - 2m_0 - 2m_1 > 6. \) Because of \( m_1 \leq m_0 - (c+2) = m_0 - 4 \) it suffices to show \( \frac{1}{2}m_0(m_0 + 1) - 4m_0 > -2 \iff m_0(m_0 - 7) > -4. \) This is true as \( m_0 \geq 2c + 4 = 8. \)

If the Figure 6.5a occurs, then only the case 1° of (6.3.1) is possible and the change of \( \alpha \)-grade in the right domain is \( A \leq m_0 + m_1 - 1 \). One gets the same estimate of \( A \) if Figure 6.5a occurs in the case II.2, because \( xF \) and \( xL \) are elements of \( \mathcal{K} \) and thus the order of \( f_0 \) and of \( f_1 \) is equal to 0. Then (*) reads \( Q(m_0 - 1) > 1^2 + 2 \cdot 2 + (m_0 + m_1 - 1), \) and this is true as was shown just before.

If Figure 6.5b occurs, then the order of \( f_0 \) can be equal to 1, but it is not possible that \( f_1 = M_1 + \alpha F \), where \( \alpha \neq 0 \), because \( \mathcal{K} \) is monomial and \( F \notin \mathcal{K} \), whereas \( f_1 \in \mathcal{K} \) by (Lemma 2.6).
We want to sharpen the estimate of (6.3). $M_0 \mapsto F$ and $xM_0 \mapsto L$ yield $A = 2m_0 + 1$; $M_0 \mapsto F, M_1 \mapsto L$ yield $A = m_0 + m_1 - 1$ and $M_0 \mapsto N_1, M_1 \mapsto L$ yield $A = (m_0 - m_1 + 2) + m_1 - 1$. Therefore $A \leq 2m_0 + 1$, and (*) reads $\frac{1}{2}m_0(m_0 + 1) - (2 + m_0 + m_1) > 1^2 + 2 \cdot 2 + 2m_0 + 1$. Because of $m_1 \leq m_0 - 4$ it suffices to show $m_0(m_0 - 7) > 8$. As $m_0 \geq 2 + 4 = 8$, the case $c = 2, m_1 = 4, m_0 = 8$ remains (cf. Fig. 6.5c). One sees that the deformations $M_0 \mapsto F, xM_0 \mapsto L$ give the maximal change of “total” $\alpha$-grade $B = (8 + 9) + 1 + 2 = 20$. As $Q(7) = \binom{9}{2} - 14 = 22$ the inequality (!) in (5.3) is fulfilled.

Because of Conclusion 6.6 one has only to consider deformations of the form $f_0 = M_0 + F, f_1 = M_1 + G$, where $F, G \in \mathcal{L}$ (cf. Proposition 3.2). From the computations in (6.3.2) it follows that the order of $f_0$ is $\leq 0.172(3 + 1) < 1$ and the order of $f_1$ is $\leq 1$. Therefore the change of $\alpha$-grade in the right domain is $A \leq m_0 + (m_1 - 1) + m_1$. If one replaces $m_1$ by $m_0 - 5 \geq m_1$ in the inequality (*) of (5.3), then one has to show: $\frac{1}{2}m_0(m_0 + 1) - (2m_0 - 2) > 2^2 + 3 \cdot 2 + m_0 + 2(m_0 - 5) - 1 \iff m_0(m_0 - 9) > -6$. This is true because of $m_0 \geq 2 \cdot (3 + 2) = 10$ (cf. Lemma 2.8).

**Conclusion 6.7.** Even in the cases $c \in \{0, 1, 2, 3\}$ the inequalities (*) and (!) respectively, are fulfilled.

Summarizing the Conclusions 6.1–6.7, we obtain:

**Proposition 6.1.** If $\rho_1 > 0, \rho_2 > 0, I$ has the type $r \geq 1$ and has $y$-standard form, then the inequality (!) of Section (5.3) is fulfilled.
CHAPTER 7

Estimates of the $\alpha$-grade in the case $\rho_1 > 0, \rho_2 > 0$ and $r = 0$.

Let $\mathcal{I}$ be an ideal of type $r = 0$ with $y$-standard form. Then one has by definition: $\mathcal{I} = y\mathcal{K}(-1) + f\mathcal{O}_{p_2}(-m)$, colength ($\mathcal{K}$) = $c$, reg($\mathcal{K}$) = $\kappa$, colength ($\mathcal{I}$) = $d = c + m$, $\mathcal{I}$ and $\mathcal{K}$ invariant under $G = \Gamma \cdot T(\rho)$, and if the Hilbert functions of $\mathcal{K}$ and $\mathcal{I}$ are $\psi$ and $\varphi$, respectively, then $g^*(\varphi) > g(d)$. By definition, $\mathcal{K}$ has the type $(-1)$, that means, the following cases can occur:

1st case: $g^*(\psi) \leq g(c)$, where $c \geq 5$ by convention.
2nd case: $0 \leq c \leq 4$.

As usual the aim is to prove (*) in (5.3), and we write $m$ and $f$ instead of $m_0$ and $f_0$.

7.1. The case $g^*(\psi) \leq g(c)$.

7.1.1. We first assume that there is no deformation into $\mathcal{L}$, at all. That means $f = x^m$ (cf. Fig. 7.1), $A = 0$ and one has to prove the inequality (*), i.e. $Q(m-1) > (c-1)^2$. Because of $Q(m-1) = \frac{1}{2}m(m+1)-(c+m)$, this is equivalent with $m(m-1) > 2c^2-2c+2$.

As $m \geq 2c+1$ by Corollary 2.2, it suffices to show $(2c+1) \cdot 2c > 2c^2-2c+2 \iff c^2+2c-1 > 0$, and this is true for $c \geq 5$.

7.1.2. We now assume that there are deformations $M_0 = x^m \mapsto \in \mathcal{L}, \cdots, x^sM_0 \mapsto \in \mathcal{L}$.

Auxiliary lemma 1. If $g^*(\psi) \leq g(c)$ and if there is the deformation $M_0 \mapsto \in \mathcal{L}$, then $\rho_1 < \rho_2$.

Proof. Because $M_0X^{\nu\nu} = L$ is a monomial in $\mathcal{L}$, the slope of the line connecting $M_0$ and $L$ is $\leq$ the slope of the line connecting $M_0$ and $L_0$ (see Figure 7.1, take into account the inclusion (1) in (3.3) and the following remark concerning the vice-monomial of $f$.) It follows that $\rho_1/\rho_2 \leq \kappa/(m-\kappa)$. Now $\kappa/(m-\kappa) < 1$ is equivalent with $2\kappa < m$, and because of $\kappa \leq c$ and Corollary 2.2 this is true. □

Auxiliary lemma 2. If $g^*(\psi) \leq g(c)$, then $\kappa = \text{reg}(\mathcal{K}) \leq c - 2$.

Proof. One has $\kappa \leq c$ and from $\kappa = c$ it follows that $g^*(\psi) = (c-1)(c-2)/2 > g(c)$ (cf. [T1], p. 92 and Anhang 2e, p. 96). By considering the figures 7.2a and 7.2b one can convince oneself that $\kappa = c-1$ implies $g^*(\psi) > g(c)$. □
It is clear that the change of $\alpha$-grade in the right domain caused by the deformations mentioned above is equal to

\[(7.1)\quad A = m + (m + 1) + \cdots + (m + s) = (s + 1)m + \binom{s + 1}{2} \].

Because of $\kappa \leq c - 2$ and $m \geq 2c + 1$ (Corollary 2.2) one has

\[ s \leq \frac{c - 2}{\rho_2} - (2c + 1) \left( \frac{1}{\rho_2} - \frac{1}{\rho_1 + \rho_2} \right). \]

It follows that

\[ s \leq \frac{c - 2}{\rho_2} - 2(c - 2) \left( \frac{1}{\rho_2} - \frac{1}{\rho_2 + \rho_2} \right) - 5 \left( \frac{1}{\rho_2} - \frac{1}{\rho_2 + \rho_2} \right) \]

and therefore

\[(7.2)\quad s < (c - 2) \left[ \frac{2}{\rho_1 + \rho_2} - \frac{1}{\rho_2} \right] \]

**Auxiliary lemma 3.** If $g^*(\psi) \leq g(c)$ and if there is a deformation $M_0 \mapsto \mathcal{L}$, then $s < \frac{1}{6}(c - 2)$.

**Proof.** As $1 \leq \rho_1 < \rho_2$ (cf. Auxiliary lemma 1), one has to find an upper bound for the function $f : [N - \{0\}] \times [N - \{0\}] \to \mathbb{R}$, $f(x, y) := \frac{2}{x+y} - \frac{1}{y}$, on the domain $1 \leq x < y$. One can convince oneself that $f(1, 2)$ is the maximal value. 

In the case $r = 0$ the inequality (*) reads $\frac{1}{7}m(m+1) - (c+m) > (c-1)^2 + (s+1) + A$. Putting in the expression (1), one gets $\frac{1}{7}m(m+1) - m > c^2 - c + s + 2 + (s+1)m + \left( \frac{s+1}{2} \right) \iff m(m+1) - 2m > 2c^2 - 2c + 2(s+2) + 2(s+1)m + s(s+1) \iff m(m-1) > 2c^2 - 2c + 2(s+2) + 2(s+1)m + s(s+1)$. Because of the Auxiliary lemma 3 it suffices to show: $m \left( m - \frac{1}{2} \right) \geq \frac{73}{36} c^2 - \frac{29}{18} c + \frac{28}{9}$. As $m \geq 2c + 1$ (Corollary 2.2) this follows from: $(2c + 1) \left( \frac{2}{3} c - \frac{1}{3} \right) > \frac{73}{36} c^2 - \frac{29}{18} c + \frac{28}{9} \iff \frac{73}{36} c^2 + \frac{14}{18} c - \frac{44}{9} > 0$. As this is true if $c \geq 2$, we have proven (*) in the 1st case.

**7.2. The cases $0 \leq c \leq 4$.**

If $c = 0$, then $\mathcal{I} = (y, x^m)$ is monomial and (*) is true, obviously. Unfortunately, one has to consider the cases $c = 1, \cdots, 4$ separately. The ideal $\mathcal{K}$ can have the Hilbert functions noted in (2.2.2).

**7.2.1. $[c = 1]$.** From Figure 7.3 we see that $\deg C_0 = 2 + \cdots + m - 1 = \binom{m}{2} - 1$ and $\deg C_\infty = 1 + \cdots + m = \binom{m+1}{2}$. The deformation of $C_0$ into $C_\infty$ is defined by $f_0 = x^m + y^m$, and this is a simple deformation in the sense of ([T1], 1.3).

One has $\alpha$-grade $(\mathcal{I}) = \deg(C) = \max(\deg C_0, \deg C_\infty)$ (loc.cit. Hilfssatz 2, p. 12). In order to prove (*) one has to show: $Q(m - 1) > \deg C_\infty - \deg C_0 = m + 1 \iff \frac{1}{2}m(m + 1) - (1 + m) > m + 1 \iff m^2 - 3m - 4 > 0$. This is true if $m \geq 5$.

**Conclusion 7.1.** If $c = 1$, then (*) is fulfilled except in the case $c = 1, m = 4$, which will be treated in (9.3).
7.2.2. \( c = 2 \). There are two subcases, which are shown in Fig. 7.4a and Fig. 7.4b. Here only simple deformations occur, again.

1st subcase (Fig. 7.4a). One gets \( \deg C_0 = 1 + 3 + 4 + \cdots + m - 1 = \binom{m}{2} - 2; \deg C_\infty = 2 + 3 + \cdots + m = \binom{m+1}{2} - 1 \). The same argumentation as in (7.2.1) shows that one has to show \( Q(m-1) > m + 1 \), i.e. \( m^2 - 3m - 6 > 0 \). This is fulfilled if \( m \geq 5 \).

In the case \( m = 4 \), by the formula of Remark 2.2, it follows that \( g^*(\varphi) = 4 \). As \( g(6) = 4 \) and \( g^*(\varphi) > g(d) \) by assumption, the case \( m = 4 \) cannot occur.

2nd subcase (Fig. 7.4b). \( \deg C_0 = 2 + \cdots + m - 1; \deg C_\infty = 2 + \cdots + m; Q(m-1) > \deg C_\infty - \deg C_0 = m \iff m^2 - 3m - 4 > 0 \). This is true if \( m \geq 5 \), and the case \( m = 4 \) cannot occur.

Conclusion 7.2. If \( c = 2 \), then (\( \ast \)) is fulfilled.

7.2.3. \( c = 3 \). Here 5 deformations are possible, which are all simple (Fig. 7.5a-7.5e).

1st subcase (Fig. 7.5a). \( \deg C_0 = 3 + \cdots + m - 1 = \binom{m}{2} - 3; \deg C_\infty = 2 + 4 + 4 + \cdots + m - 1 = \binom{m}{2} \). \( Q(m-1) > 3 \iff m^2 - m > 12 \). This is true because of \( m \geq c + 2 = 5 \).

2nd subcase (Fig. 7.5b). \( \deg C_0 = 3 + \cdots + m - 1; \deg C_\infty = 1 + 3 + 4 + \cdots + m \).

\( Q(m-1) > \deg C_\infty - \deg C_0 = m + 1 \iff m^2 - 3m - 8 > 0 \). This is true because of \( m \geq 5 \).

3rd subcase (Fig. 7.5c). \( \deg C_0 = 3 + \cdots + m - 1; \deg C_\infty = 2 + \cdots + m \).

\( Q(m-1) > \deg C_\infty - \deg C_0 = m + 2 \iff m^2 - 3m - 10 > 0 \). This is fulfilled if \( m \geq 6 \). As \( m \geq c + 2 = 5 \), the case \( m = 5 \) remains. But if \( m = 5 \), then from Remark 2.2 it follows that \( g^*(\varphi) = 8 < g(8) = 9 \), which contradicts the assumption.

4th subcase (Fig. 7.5d). \( \deg C_0 = 1 + 2 + 4 + \cdots + m - 1; \deg C_\infty = 1 + 3 + \cdots + m \).

\( Q(m-1) > \deg C_\infty - \deg C_0 = m + 1 \iff m^2 - 3m - 8 > 0 \). This is fulfilled if \( m \geq 5 \).

5th subcase (Fig. 7.5e). \( \deg C_0 = 2 + 4 + 4 + \cdots + m - 1; \deg C_\infty = 2 + \cdots + m; Q(m-1) > \deg C_\infty - \deg C_0 = m - 1 \iff m^2 - 3m > 4 \). This is fulfilled if \( m \geq 5 \).

Conclusion 7.3. If \( c = 3 \), then (\( \ast \)) is fulfilled.

7.2.4. \( c = 4 \). There are 8 subcases which are shown in Fig. 7.6a1-7.6e. At first we make the additional assumption that \( m \geq 7 \). The slope \( \rho_1/\rho_2 \) of the line connecting the monomials denoted by + and - is equal to 1/2 in Fig. 7.6b1, is \( \leq 1/4 \) in Chapter 6 Fig. 7.6b2 and is \( \leq 2/5 \) in Fig. 7.6b3. It follows that the deformations of Fig. 7.6b1 and of Fig. 7.6b2 (respectively the deformations of Fig. 7.6b1 and of Fig. 7.6b3 ) cannot occur simultaneously, that means, we have only simple deformations.

1st subcase (Fig. 7.6a1). \( \deg C_0 = 2 + 4 + \cdots + m - 1; \deg C_\infty = 1 + 2 + 4 + \cdots + m; Q(m-1) > \deg C_\infty - \deg C_0 = m + 1 \iff m^2 - 3m - 10 > 0 \). This is fulfilled if \( m \geq c + 2 = 6 \).
2nd subcase (Fig. 7.6a2). \( \deg C_0 = 2 + 4 + \cdots + m - 1; \deg C_\infty = 3 + 4 + \cdots + m; \deg C_\infty - \deg C_0 = m + 1 \), etc, as in the first subcase.

3rd subcase (Fig 7.6b1). \( \deg C_0 = 4 + 4 + 5 + \cdots + m - 1 = (m^2) - 2; \deg C_\infty = 2 + 4 + 6 + 5 + \cdots + m - 1; Q(m - 1) > \deg C_\infty - \deg C_0 = 4 \iff m(m - 1) > 16. \) This is fulfilled if \( m \geq 5 \).

4th subcase (Fig. 7.6b2). \( \deg C_0 = 4 + 4 + 5 + \cdots + (m - 1); \deg C_\infty = 2 + 4 + 4 + \cdots + m; Q(m - 1) > \deg C_\infty - \deg C_0 = m - 1 \iff m^2 - 3m - 6 > 0. \) This is fulfilled if \( m \geq 5 \).

5th subcase (Fig. 7.6b3). \( \deg C_0 = 4 + 4 + 5 + \cdots + m - 1; \deg C_\infty = 2 + 4 + 4 + \cdots + m; Q(m - 1) > m + 2 \iff m^2 - 3m - 12 > 0. \) This is fulfilled if \( m \geq 6 \).

6th subcase (Fig 7.6c). \( \deg C_0 = 3 + \cdots + m - 1; \deg C_\infty = 3 + \cdots + m; Q(m - 1) > m \iff m^2 - 3m - 8 > 0. \) This is fulfilled if \( m \geq 5 \).

7th subcase (Fig. 7.6d). \( \deg C_0 = 1 + 2 + 3 + 5 + \cdots + m - 1; \deg C_\infty = 1 + 2 + 4 + \cdots + m; Q(m - 1) > m + 1 \iff m^2 - 3m - 10 > 0. \) This is fulfilled if \( m \geq 6 \).

8th subcase (Fig. 7.6e). \( \deg C_0 = 2 + 4 + 6 + 5 + \cdots + m - 1 = (m^2) + 2; \deg C_\infty = 2 + 4 + 4 + \cdots + m = (m+1^2); Q(m - 1) > m - 2 \iff m^2 - 3m - 4 > 0. \) This is fulfilled, if \( m \geq 5 \).

As \( m \geq c + 2 = 6 \), the case \( m = 6 \) remains. All inequalities are fulfilled if \( m \geq 6 \); therefore the possibility remains that in Fig. 7.6b1 and Fig. 7.6b3, if \( m = 6 \) and \( \rho = (-3, 1, 2) \), the deformations \( f = (x^3y + ay^2z^2) \cdot z^2 \) and \( g = x^6 + by^2z^4 \) simultaneously occur. Now \( f \wedge g = x^3yz^2 \wedge x^6 + bx^3yz^2 \wedge y^2z^4 + ay^2z^4 \wedge x^6 \) and therefore 

\[
\max -\alpha - \text{grade} (I) \leq \max (\deg C_0 \text{ in Fig. 7.6b1, } \deg C_\infty \text{ in Fig. 7.6b3, } \deg C_\infty \text{ in Fig. 7.6b1}) = \max (\left(\frac{m}{2}\right) - 2, \left(\frac{m+1}{2}\right), \left(\frac{m}{2}\right) + 2) \text{ and } \min -\alpha - \text{grade} (I) = \min (\left(\frac{m}{2}\right) - 2, \left(\frac{m+1}{2}\right), \left(\frac{m}{2}\right) + 2).
\]

Now \( \left(\frac{m+1}{2}\right) > \left(\frac{m}{2}\right) + 2 \), if \( m \geq 3 \), and therefore \( \max -\alpha - \text{grade} (I) = \left(\frac{m+1}{2}\right) \), and \( \min -\alpha - \text{grade} (I) = \left(\frac{m}{2}\right) - 2 \). Thus (*) follows from \( Q(m - 1) > \left(\frac{m}{2}\right) - \left(\frac{m}{2}\right) + 2 \iff m^2 - 3m - 12 > 0 \), and this is true if \( m \geq 6 \).

Conclusion 7.4. If \( c = 4 \), then (*) is fulfilled.
first column filled with monomials of $H^0(\mathcal{K}(0))$

monomial domain

Fig. 7.2a

colength($\mathcal{K}$) = 10, reg($\mathcal{K}$) = 9

Fig. 7.2b

colength($\mathcal{K}$) = 11, reg($\mathcal{K}$) = 10
CHAPTER 8

Borel-invariant surfaces and standard cycles

8.1. Preliminary remarks.

The group $G_L(3; k)$ operates on $S$ by matrices, thus $G_L(3; k)$ operates on $H^d = \text{Hilb}^d(\mathbb{P}^2_k)$. We recall the subgroups $\Gamma = \left\{ \begin{pmatrix} 1 & 0 & \ast \\ 0 & 1 & \ast \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset U(3; k)$ and $T(\rho) < T := T(3; k)$, already introduced in (2.4.1). As each subspace $U \subset S_n$ is invariant under $D := \{ (\lambda, \lambda, \lambda) | \lambda \in k^* \}$, each ideal $I \subset \mathcal{O}_{\mathbb{P}^2}$ and each point of $H^d$ is invariant under $D$.

Instead of the operation of $T$ on $H^d$ one can consider the operation of the group $T/D$, which is isomorphic to $T(2; k)$.

According to the theory of Hirschowitz, the 1-cycles in $H^d$ which are invariant under $B = B(3; k)$, form a generating system of the first Chow group of $H^d$, and relations among them are realized in $B$-invariant surfaces $V \subset H^d$ ([Hi], Mode d’emploi, p. 89).

We distinguish between the cases, whether $V$ is pointwise invariant under the $G_m$-operation $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z$, or not. This we call the homogeneous case and the inhomogeneous case, respectively.

8.2. The inhomogeneous case.

We let $T = T(3; k)$ operate by diagonal matrices and let $G_a$ operate by $\psi_\alpha : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z$ on $S$. Then the group $G = G_a \cdot T$ operates on $H^d$. Let $V \subset H^d$ be a $G$-invariant, closed, two-dimensional variety, which is not pointwise invariant under $G_a$ and is not pointwise invariant under the $G_m$-operation $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z$. (For abbreviation we speak of an inhomogeneous surface.)

We suppose $k = \overline{k}$. If $\xi \in H^d(k)$ is a closed point, then $T_\xi$ and $G_\xi$ denote the inertia group of $\xi$ in $T$ and $G$, respectively.

8.2.1. Auxiliary lemma 1. There is a point $\xi \in V(k)$ such that $V = \overline{T \cdot \xi}$.

Proof. If $\dim T \cdot \xi < 2$ for all $\xi \in V(k)$, then $\xi$ is $T$-invariant or $\overline{T \cdot \xi}$ is a $T$-invariant, irreducible curve, $\forall \xi \in V(k)$. The first case can occur for only finitely many points; in the second case one has $T_\xi = T(\rho), \rho \in \mathbb{Z}^3 - (0)$, such that $\rho_0 + \rho_1 + \rho_2 = 0$ ([T1], Bemerkung 1, p.2).

There are only finitely many $\rho$, such that there exists an ideal $I \subset \mathcal{O}_{\mathbb{P}^2}$ of fixed colength $d$, which is invariant under $T(\rho)$, but not invariant under $T$ (see [T2], Hilfssatz 6, p. 140). In other words, there are only finitely many $\rho$, such that $(H^d)^T(\rho) \not\supset (H^d)^T$. 

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From the assumption follows $V = \bigcup \{ V^{\tau(\rho)} \mid i \in I \}$, where $I$ is a finite set of indices. As $V$ is irreducible, it follows that $V = V^{\tau(\rho)}$ for a certain $\rho$. Now one has
\[
G_{g(\xi)} = gG_\xi g^{-1} \forall \xi \in V(k), \forall g \in G(k),
\]
and therefore
\[
G_{g(\xi)} \supset T(\rho) \cup gT(\rho)g^{-1}, \forall \xi \in V(k), \forall g \in G(k).
\]
We show there are $\lambda_0 \neq \lambda_1$ in $k^*$ such that $\tau = (\lambda_0, \lambda_1, 1) \in T(\rho)$. We assume that $(\lambda_0, \lambda_1, \lambda_2) \in T(\rho)$ implies $\lambda_0/\lambda_2 = \lambda_1/\lambda_2$, i.e. $\lambda_0 = \lambda_1$. Then each element in $T(\rho)$ has the form $(\lambda, \lambda, \lambda)$, thus $T(\rho) \subset G \cdot D$, where $D = \{(\lambda, \lambda, \lambda) \mid \lambda \in k^*\}$. But then $\rho_2 = 0$ and $V$ is pointwise $G_m$-invariant, contradiction.

We thus take $\tau = (\lambda_0, \lambda_1, 1) \in T(\rho)$ such that $\lambda_0 \neq \lambda_1$ and take
\[
g = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G_a < G, \quad \text{where } \alpha \neq 0.
\]

Then
\[
g \tau g^{-1} = \begin{pmatrix} \lambda_0 & (\lambda_1 - \lambda_0)\alpha & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
is an element of $G_{g(\xi)}$ and thus
\[
\tau^{-1}g \tau g^{-1} = \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \beta := \lambda_0^{-1}(\lambda_1 - \lambda_0)\alpha \neq 0
\]
is an element of $G_{g(\xi)}$, too. It follows that $g(\xi)$ is invariant under $\psi_\beta$, thus invariant under $G_a$ and therefore $\xi$ is invariant under $G_a$. This is valid for all $\xi \in V(k)$, contradicting the assumption. \hfill \Box

8.2.2. If one lets $T(2; k)$ operate on $S$ by $x \mapsto \lambda_0x, y \mapsto \lambda_1y, z \mapsto z$ then one sees that $G = T \cdot G_a$ operates on $H^d$ in the same way as the group $T(2; k) \cdot G_a$ which for simplicity is again denoted by $G$ (and is equal to $B(2; k)$). If $\xi$ is as in the Auxiliary lemma 1, then from $V = G \cdot \xi$ it follows that $G_{\xi} \times G$ is 1-dimensional. There is the decomposition $G_{\xi} = \bigcup g, H, H := G_{\xi}^0$, in finitely many cosets. As $H$ is a 1-dimensional connected group, two cases can occur:

1st case: $H \simeq G_a$. Now the unipotent elements in $B(2; k)$ are just the matrices $\psi_\alpha = (\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix})$. It follows that there is $\alpha \neq 0$ such that $\psi_\alpha \in G_{\xi}$. But then $\xi$ is invariant under $G_a$. As $G_a$ is normalized by $T$, $T \cdot \xi$ is pointwise $G_a$-invariant. Because of $V = T \cdot \xi$; $V$ is pointwise $G_a$-invariant, contradiction.

2nd case: $H \simeq G_m$. As each element of $G_m$ is semi-simple, so is each element of the isomorphic image $H$. Thus the commutative group $H < GL(2; k)$ consists of semi-simple elements. Then there is an $g \in GL(2; k)$, such that $g^{-1}Hg$ consists of diagonal matrices ([K], Lemma 1, p. 150). Because of $G_m \simeq H \simeq g^{-1}Hg < T(2; k)$ one has a 1-psg $f : G_m \to T(2; k)$ Let $p : T(2; k) \to G_m$ be the projection onto the
first component. Then \( p \circ f : \mathbb{G}_m \to \mathbb{G}_m \) has the form \( \lambda \to \lambda^n \), \( n \) a suitable integer. Thus \( g^{-1}HG = \{ (\lambda^a 0 \ | \ \lambda \in k^* ) \} =: T(a,b) \), \( a \) and \( b \) suitable integers. It follows \( H = gT(a,b)g^{-1} \subset G = \mathbb{G}_a \cdot T(2;k) = B(2;k) \). Writing \( g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) gives 
\[
g^{-1} = D^{-1} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, D := \det(g). \]
We compute:
\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} = \begin{pmatrix} \lambda^a a_{11} & \lambda^b a_{12} \\ \lambda^a a_{21} & \lambda^b a_{22} \end{pmatrix}
\]
\[
g \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = D^{-1} \begin{pmatrix} \lambda^a a_{11} & \lambda^b a_{12} \\ \lambda^a a_{21} & \lambda^b a_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \]
\[
D^{-1} \begin{pmatrix} \lambda^a a_{11} a_{22} - \lambda^b a_{12} a_{21} \\ \lambda^a a_{21} a_{22} - \lambda^b a_{21} a_{22} \end{pmatrix} = -\lambda^a a_{11} a_{12} + \lambda^b a_{11} a_{12} = 0.
\]

This matrix is an upper triangular matrix if and only if \( (\lambda^a - \lambda^b) a_{21} a_{22} = 0 \), \( \forall \lambda \in k^* \).

Subcase 1. \( a = b \Rightarrow H = T(a,a) \).

Subcase 2. \( a \neq b \) and \( a_{21} = 0 \). Write
\[
g = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = u \tau, \quad \tau = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, u = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \] and \( c := a_{12}/a_{22} \).

Then \( H = gT(a,b)g^{-1} = uT(a,b)u^{-1} \).

Subcase 3. \( a \neq b \) and \( a_{21} \neq 0 \). Then \( a_{22} = 0 \) and
\[
g \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = D^{-1} \begin{pmatrix} -\lambda^b a_{12} a_{21} \\ 0 \end{pmatrix} \]
\[
(\begin{pmatrix} \lambda^b (\lambda^a - \lambda^b) c \\ \lambda^a \end{pmatrix}, \text{ where } c = a_{11} a_{12}/a_{21} a_{22} = a_{11}/a_{21}.
\]

Thus \( H = gT(a,b)g^{-1} = \{ (\lambda^b (\lambda^a - \lambda^b) c \ | \ \lambda \in k^* ) \}, \text{ where } c := a_{11}/a_{21}. \) If \( u := \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, \) then \( u \begin{pmatrix} \lambda^b (\lambda^a - \lambda^b) c \\ \lambda^a \end{pmatrix} u^{-1} = \begin{pmatrix} \lambda^b & 0 \\ 0 & \lambda^a \end{pmatrix} \) and thus \( H = u^{-1} T(b,a) u \).

We have proved

**Auxiliary lemma 2.** There is an element \( u \in \mathbb{G}_a \) such that \( H = uT(a,b)u^{-1} \), where \( a \) and \( b \) are suitable integers. \( \Box \)

**8.2.3.** Let \( \xi \) and \( u \) be as in Auxiliary lemma 1 and 2, respectively. Set \( \zeta := u^{-1}(\xi) \).

Then \( G_\xi = u^{-1} G_\xi u \supset u^{-1} H u = T(a,b) < T(2;k) \) and thus \( \dim T(2;k) \cdot \zeta \leq 1 \). If this dimension would be equal to 0, then \( G_\xi \) and \( G_\xi \) would have the dimension 2. Because of \( V = \mathbb{G} \cdot \xi \) the dimension of \( V \) would be 1, contradiction. Thus \( \dim T \cdot \zeta = 1 \) and (Appendix E, Corollary) gives

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Auxiliary lemma 3. The inertia group $T_\zeta$ of $\zeta$ in $T(3;k)$ has the form $T(\rho)$, where $\rho = (\rho_0, \rho_1, \rho_2)$ and $\rho_2 \neq 0$. □

8.2.4. Summary.

Proposition 8.1. Let $V \subset H^d$ be a closed 2-dimensional subvariety, invariant under $G = \mathbb{G}_a \cdot T(3;k)$, where $\mathbb{G}_a$ operates by $\psi_\alpha : x \mapsto x$, $y \mapsto \alpha x + y$, $z \mapsto z$ and $T(3;k)$ operates by diagonal matrices on $S$. We suppose that $V$ is not pointwise invariant under this $\mathbb{G}_a$-operation and is not pointwise invariant under the $G_m$-operation $\sigma(\lambda) : x \mapsto x$, $y \mapsto y$, $z \mapsto \lambda z$. Then there is a point $\xi \in V(k)$ and $u \in \mathbb{G}_a$ such that:

(i) $V = T(3;k) \cdot \xi$
(ii) The inertia group $T_\zeta$ of $\zeta := u(\xi)$ in $T(3;k)$ has the form $T(\rho)$, where $\rho = (\rho_0, \rho_1, \rho_2)$ and $\rho_2 \neq 0$. (iii) $V = \mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$.

Proof. The statements (i) and (ii) follow from (8.2.1) – (8.2.3). Put $G := \mathbb{G}_m \cdot \mathbb{G}_a = \mathbb{G}_m \times \mathbb{G}_a$. If the statement (iii) were wrong, then the inertia group $G_\zeta$ would have a dimension $\geq 1$ and thus would contain a subgroup $H$ isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$. If $H \simeq \mathbb{G}_m$, then $\zeta$ would be invariant under $p_1(H) = \{ (\lambda^n, 1) | \lambda \in k^* \}$, $n \in \mathbb{Z} - (0)$. Then $\zeta$ and $\xi$ would be invariant under $G_m$ and thus $V$ would be pointwise $G_m$-invariant, contradiction.

If $H \simeq \mathbb{G}_a$ then $\zeta$ and $\xi$ would be invariant under $G_a$. As $G_a$ is normalized by $T(3;k)$, $T(3;k) \cdot \xi$ would be pointwise $G_a$-invariant and the same would follow for $V$. □

8.3. The homogeneous case.

We now assume $V \subset H^d$ is a 2-dimensional subvariety, invariant under $G := \mathbb{G}_a \cdot T(3;k)$, not pointwise invariant under $\mathbb{G}_a$, but now pointwise invariant under the $G_m$-operation $\sigma$ as in (8.1). As there are only finitely many $T(3;k)$-fixed points in $H^d$, it follows that $V$ is not pointwise fixed by the $G_m$-operation $\sigma(\lambda) : x \mapsto \lambda x$, $y \mapsto y$, $z \mapsto z$.

Let $\xi \in V(k)$ be not $G_a$-invariant and not $G_m$-invariant. We define a morphism $f$ by $f : \mathbb{G}_a \times \mathbb{G}_m \to V$, $(\alpha, \lambda) \mapsto \psi_\alpha \sigma(\lambda) \xi$.

Assume that $f$ has an infinite fibre. Then there is an element $(\beta, \mu) \in \mathbb{G}_a \times \mathbb{G}_m$ such that $\psi_\alpha \sigma(\lambda) \xi = \psi_\beta \sigma(\mu) \xi$, i.e.

(8.1) $\psi_{(\alpha - \beta)\lambda^{-1}}(\xi) = \sigma(\lambda^{-1}\mu) \xi$

for infinitely many $(\alpha, \lambda)$ in $\mathbb{G}_m \times \mathbb{G}_a$.

By assumption, $C := \{ \psi_\alpha(\xi) | \alpha \in k \}$ and $D := \{ \sigma(\lambda)(\xi) | \lambda \in k^* \}$ are curves in $V$. If one assumes that only finitely many different $\lambda$ can occur in (1), then on the left side of (1) also only finitely many $\alpha$ can occur. For $\xi$ is not a fixed point of $\mathbb{G}_a$, so that from $\psi_\alpha(\xi) = \psi_\beta(\xi)$ it follows that $\alpha = \beta$. This contradicts the last assumption. Thus $C$ and $D$ have in common infinitely many points and hence are equal (as subschemes with the induced reduced scheme structure).

The fixed-point set of $C$ under $G_m$ consists of the two points $\xi_{0/\infty} := \lim_{\lambda \to 0/\infty} \sigma(\lambda) \xi$. As $C$ has an unique $G_a$-fixed-point, and as $G_a$ is normalized by $G_m$, one of the two points, say
\( \xi_\infty \), is fixed by \( \mathbb{G}_a \). Thus \( C = \{ \psi_\alpha(\xi_0) | \alpha \in k \} \) and \( \xi_0 \) corresponds to a monomial ideal. There are only finitely many \( T(3; k) \)-fixed points \( \xi_i, 1 \leq i \leq n, \) in \( H^d \). Set \( M := \bigcup_{i=1}^{n} \mathbb{G}_7 \cdot \xi_i \).

Then \( C \setminus M \) is open and non-empty, and choosing \( \xi \in C \setminus M \) it follows that \( f \) as defined above has no infinite fibre.

**Proposition 8.2.** Let \( V \subset H^d \) be a closed 2-dimensional subvariety, invariant under \( G = \mathbb{G}_a \cdot T(3; k) \), not pointwise \( \mathbb{G}_a \)-invariant, but pointwise invariant under the \( \mathbb{G}_m \)-operation \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z \). Then \( V \) is not pointwise invariant under the \( \mathbb{G}_m \)-operation \( \tau(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z \). And for all closed points \( \zeta \) in an open non-empty subset of \( V \) one has \( V = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta. \)

### 8.4. Standard cycles.

In the following we suppose \( d \geq 5 \) and we recall to memory the closed subscheme

\[
\mathcal{H} = \bigcup \{ H_\varphi \subset H^d | g^*(\varphi) > g(d) \}
\]

of \( H^d \) (cf. 1.2.2). As usual, we let \( \mathbb{G}_a \) operate by \( \psi_a : x \mapsto x, y \mapsto ax + y, z \mapsto z \) and we let \( \mathbb{G}_m \) operate by \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z \) (by \( \sigma(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z \)) on \( \mathcal{S} \) in the inhomogeneous case (in the homogeneous case, respectively).

#### 8.4.1.

**Definition 7.** Let \( C \subset \mathcal{H} \) be a \( B = B(3; k) \)-invariant curve, which is not pointwise \( \mathbb{G}_a \)-invariant. Then \( C = \{ \psi_\alpha(\xi) | \alpha \in k \} \), where \( \xi \leftrightarrow \mathcal{I} \) is an ideal, invariant under

\[
T = T(3; k) \text{ and } \Gamma = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} < \Delta := U(3; k), \text{ which is uniquely determined by } \mathcal{C} \].

(cf. [T1], Proposition 0, p. 3). We call \( C \) a \( x \)-standard cycle respectively a \( y \)-standard cycle, if \( \mathcal{I} \) has \( x \)-standard form respectively \( y \)-standard form (see 2.4.3 Definition 2).

#### 8.4.2. Let \( V \subset Z = Z(\mathcal{H}) \) be a 2-dimensional, \( B \)-invariant subvariety, where \( Z \) is defined as in (1.2.1). We suppose, that \( V \) contains a \( y \)-standard cycle. Then \( V \) is not pointwise \( \mathbb{G}_a \)-invariant, so that we can write \( V = \mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta \), where \( \zeta \in V(k) \) is as in Proposition 8.1 or Proposition 8.2, respectively. By the definition of \( Z \), the orbit \( \Delta \cdot \zeta \) has a dimension \( \leq 1 \). As \( \zeta \) is not \( \Delta \)-invariant, the inertia group \( \Delta \zeta \) of \( \zeta \) in \( \Delta \) has the form

\[
G(a : b) = \left\{ \begin{pmatrix} 1 & a & * \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in \Delta | aa + b\beta = 0 \right\}, \text{ where } a, b \in k \text{ and not both elements are zero (cf. Appendix D, Lemma 1). Let } \mathcal{I} \text{ be the ideal corresponding to } \zeta. \text{ If } \varphi \text{ is the Hilbert function of } \mathcal{I}, \text{ then } g^*(\varphi) > g(d) \text{ by the definition of } \mathcal{H} \text{ and thus } \mathcal{I} = \ell \mathcal{K}(-1) + f \mathcal{O}_\mathbb{P}^2(-m), \text{ where } \mathcal{K} \subset \mathcal{O}_\mathbb{P} \text{ has the colength } c, f \in S_m, c + m = d \text{ and } m = \text{reg}(\mathcal{I}) \geq c + 2 \text{ (see Lemma 2.1 - Lemma 2.4). If } \nu := \min \{ n | H^0(\mathcal{I}(n)) \neq (0) \}, \text{ then } \nu < m. \text{ This follows from Lemma 2.2 and Corollary 2.1. As } G(a : b) \text{ is unipotent, there is an eigenvector } f \in H^0(\mathcal{I}(\nu)). \text{ From } x \partial f / \partial z \in \langle f \rangle \text{ and } bx \partial f / \partial y - ay \partial f / \partial z \in \langle f \rangle \text{ we conclude that } f = x^\nu, \text{ if we assume } b \neq 0, \text{ which we do now. Let be } \eta \in V(k) \text{ any point. If } \mathcal{L} \text{ is the corresponding}
ideal, then $x^\nu \in H^0(\mathcal{L}(\nu))$. (Proof: This is first of all true for all $\eta \in W := \mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$. By means of the mapping $\mathcal{J} \mapsto H^0(\mathcal{J}(d))$ we can regard $H^d$ as a closed subscheme of $\mathbf{G} = \text{Grass}_{Q(d)}(S_d)$. If $\mathcal{J} \subset H^d$ is any ideal, the condition $x^\nu \in H^0(\mathcal{J}(\nu))$ is equivalent to the condition $x^\nu S_{d-\nu} \subset H^0(\mathcal{J}(d))$. An element of $\mathbf{G}(\text{Spec } A)$ is a subbundle $L \subset S_d \otimes A$ of rank $Q(d)$, and the condition $x^\nu S_{d-\nu} \subset L$ defines a closed subscheme of $\mathbf{G}$. Thus the condition $x^\nu \in H^0(\mathcal{L}(\nu))$ defines a closed subset of $V$. As $V = \overline{W}$, this condition is fulfilled for all points of $V$.) Assume that $\mathcal{L}$ has $y$-standard form. Then $\mathcal{L} = y \cdot \mathcal{M}(-1) + g \mathcal{O}_{P^2}(-n)$, where $e := \text{colength } (\mathcal{M}), n := \text{reg}(\mathcal{L}) \geq m$, and $e + n = d$. As $\nu < m \leq n$ we get $x^\nu \in H^0(\mathcal{L}(\nu)) = yH^0(\mathcal{M}(\nu - 1))$, contradiction.

**Lemma 8.1.** If $V \subset Z(\mathcal{H})$ is a $B$-invariant surface which contains a $y$-standard cycle, then $V$ is point-wise invariant under $\Gamma$.

**Proof.** From the foregoing reasoning it follows that $b = 0$, i.e., $\zeta$ is invariant under $\Gamma = G(1 : 0)$. As $\Gamma$ is normalized by $\mathbb{G}_a$ and $T$, it follows that $\mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$ is point-wise invariant under $\Gamma$, and the same is true for $V$. \hfill \Box

**8.4.3.** We suppose that $V \subset Z(\mathcal{H})$ is a $B$-invariant surface, which contains a $y$-standard cycle. We represent $V$ in the form $V = \mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$, according to Proposition 8.1 or Proposition 8.2, respectively.

**Lemma 8.2.** (a) One can assume without restriction that $\mathcal{J} \leftrightarrow \zeta$ has $y$-standard form.
(b) $\mathcal{I}_{0/\infty}$ are monomial ideals and have $y$-standard form.

**Proof.** Let be $\zeta \leftrightarrow \mathcal{I}$ as in Proposition 8.1 (resp. in Proposition 8.2).

(a) By Lemma 2.6, $\mathcal{I}$ has $x$- or $y$-standard form. First we consider the inhomogeneous case. From $\mathcal{I} = x\mathcal{K}(-1) + f\mathcal{O}_{P^2}(-m)$ and the $\Gamma$-invariance of $\mathcal{I}$ (Lemma 8.1), the $\Gamma$-invariance of $\mathcal{K}$ follows. By (Appendix C, Remark 2) we have $R_c \subset H^0(\mathcal{K}(c))$, and because of $m \geq c + 2$ we get $R_{m-2} \subset H^0(\mathcal{K}(m - 2))$, thus $xR_{m-2} \subset xH^0(\mathcal{K}(m - 2)) = H^0(\mathcal{I}(m - 1))$. We conclude that $xR_{m-2} \subset H^0(\mathcal{J}(m - 1))$, hence $xR_{m-2} \cdot S_{d-m+1} \subset H^0(\mathcal{J}(d))$, if $\mathcal{J} \leftrightarrow \eta$ is any point of $\mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$. The same reasoning as in (8.4.2) shows that $xR_{m-2} \cdot S_{d-m+1} \subset H^0(\mathcal{J}(d))$ and hence $xR_{m-2} \subset H^0(\mathcal{J}(m - 1))$ is true for all $\mathcal{J} \leftrightarrow \eta \in V$. As $\mathcal{J} = \mathcal{J}(1) + g\mathcal{O}_{P^2}(-n), e = \text{colength } (\mathcal{M}), n = \text{reg}(\mathcal{J}) \geq m, e + n = d$ and $e \in R_1 - (0)$, it follows that $H^0(\mathcal{J}(m - 1)) = \ell H^0(\mathcal{M}(m - 2)) \subset xR_{m-2}$, hence $\ell = x$. It follows, that no ideal in $V$ has $y$-standard form, contradiction. Thus the statement (a) is proved in the inhomogeneous case. If the homogeneous occurs and if we assume that $\zeta \leftrightarrow \mathcal{I}$ would have $x$-standard form, the same argumentation as in the inhomogeneous case gives a contradiction. By Lemma 2.2 we have a representation $\mathcal{I} = \mathcal{K}(-1) + f\mathcal{O}_{P^2}(-m)$, and if the homogeneous case occurs, because of the $\Gamma$-invariance of $\mathcal{I}$, we conclude that $\ell = \beta x + y$, where $\beta \in k$. Replacing $\mathcal{I}$ by $\psi_\beta(\mathcal{I}) = y\psi_\beta(\mathcal{K})(-1) + \psi_\beta(f)\mathcal{O}_{P^2}(-m)$, we can assume without restriction that $\mathcal{I}$ has $y$-standard form.

(b) If $\mathcal{I} = y\mathcal{K}(-1) + f\mathcal{O}_{P^2}(-m)$, colength $(\mathcal{K}) = c, \text{reg}(\mathcal{I}) = m, c + m = d, m \geq c + 2$, then $R_{m-2} \subset H^0(\mathcal{K}(m - 2))$ (Appendix C, Remark 2), thus $yR_{m-2} \subset H^0(\mathcal{I}(m - 1))$, hence $yR_{m-2} \subset H^0(\mathcal{J}(m - 1))$, $\lambda \in k^*$. As $yR_{m-2} \subset H^0(\mathcal{J}(m - 1))$ is equivalent to $yR_{m-2} \cdot S_{d-m+1} \subset H^0(\mathcal{J}(d))$, the condition $yR_{m-2} \subset H^0(\mathcal{J}(m - 1))$ defines a closed
subscheme of $V$, which is invariant under the $\mathbb{G}_m$-operation $\sigma$, as one sees by the same reasoning as in the proof of Lemma 8.1. It follows that $yR_{m-2} \subset H^0(I_{0/\infty}(m - 1))$. By Lemma 2.2 we can write $I_0 = \ell M(-1) + g O_{F_2}(-n)$, where $n = \text{reg}(I_0) \geq m$. But then $yR_{m-2} \subset H^0(I_0(m - 1)) = \ell H^0(M(m - 2))$, showing that $\ell = y$. The same argument shows that $I_{\infty}$ has $y$-standard form, too.

In the inhomogeneous case $\zeta$ is fixed by $T(\rho)$, where $\rho_2 \neq 0$ (Proposition 8.1), hence $\zeta_{0/\infty}$ are fixed by $T(\rho) \cdot \mathbb{G}_m = T(3; k)$.

In the homogeneous case $\zeta_{0/\infty}$ are invariant under the two $\mathbb{G}_m$-operations $\sigma$ and $\tau$, hence are invariant under $T(3; k)$.

\hfill $\Box$
CHAPTER 9

Relations between $B$-invariant 1-cycles

9.1. Generalities on relations between 1-cycles

We describe the method of Hirschowitz in our special situation. $B = B(3; k)$ operates on $H^d$ and we take a closed subscheme $X \subset H^d$, which is $B$-invariant. $Z^B_1(X)$ is the free group generated by $B$-invariant curves in $X$. It is easy to see that the canonical homomorphism $Z^B_1(X) \rightarrow A_1(X)$ is surjective (cf. [T1], Lemma 1, p. 6). By ([Hi], Theorem 1, p. 87) the kernel $\text{Rat}^B_1(X)$ can be described as follows. One has to consider all operations of $B$ on $P^1$ and all two dimensional subvarieties $\mathcal{B} \subset X \times_k P^1$ with the following properties:

(i) $p_2 : \mathcal{B} \rightarrow P^1$ is dominant, hence surjective and flat.
(ii) The operation of $B$ on $P^1$ fixes 0 and $\infty$.
(iii) $\mathcal{B}$ is invariant under the induced operation $(\xi, \mu) \mapsto (g\xi, g\mu), \xi \in X, \mu \in P^1, g \in B$.

N.B. According to a general theorem of Fogarty, the fixed-point scheme $(P^1)^U(3; k)$ is connected; hence from (iii) it follows that $U(3; k)$ operates trivially on $P^1$.

If $B_\mu := p_2^{-1}(\mu)$ is the fibre and $B_\mu := p_1(B_\mu)$ is the isomorphic image in $X$, then one has:

\[
g(B_\mu) = \{ (g\xi, g\mu) \mid \xi \in X, \text{ such that } (\xi, \mu) \in \mathcal{B} \} = \{ (\xi, g\mu) \mid \xi \in X, \text{ such that } (\xi, g\mu) \in \mathcal{B} \} = B_{g\mu}, \text{ for all } g \in B, \mu \in P^1
\]

We conclude that

(9.1) \[ gB_\mu = B_{g\mu}, \text{ for all } g \in B, \mu \in P^1. \]

From property (ii) it follows that $B_0$ and $B_\infty$ are $B$-invariant 1-cycles in $X$. Then $\text{Rat}^B_1(X)$ is defined to be the subgroup of $Z^B_1(X)$ generated by all elements of the form $B_0 - B_\infty$, and the theorem of Hirschowitz says that $A^B_1(X) := Z^B_1(X)/\text{Rat}^B_1(X)$ is canonically isomorphic to $A_1(X)$ (loc. cit.). We consider $V := p_1(B)$ as a closed subscheme of $X$ with the induced reduced scheme structure. As $\mathcal{B} \subset V \times_k P^1, \dim V \geq 1$. If $\dim V = 1$, then $\mathcal{B} = V \times P^1$ and $B_0 - B_\infty$ is equal to zero in $Z^B_1(X)$. Thus we can assume without restriction that $V$ is a $B$-invariant, 2-dimensional subvariety of $X$. 
9.2. The standard construction

Let $X \subset H^d$ be a $B(3; k)$-invariant subvariety. Start with a subvariety $B \subset X \times_k \mathbb{P}^1$ as in [9.1] such that $V := p_1(B) \subset H^d$ is a 2-dimensional subvariety, which is automatically $B(3; k)$-invariant. Assume $V$ is not pointwise invariant under the $G_m$-operation as in 8.2. Write

\begin{equation}
V = \overline{G_a \cdot G_m \cdot \xi}
\end{equation}

where $\xi \in V(k)$ is a suitable point and $G_m$ operates by $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z$ or by $\sigma(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z$, respectively (cf. Proposition 8.1 and Proposition 8.2). Then $G_m$ is a subgroup of $T$ and fixes $0 = (0 : 1)$ and $\infty = (1 : 0)$ by assumption. Assume that $\sigma$ operates trivially on $G_m$. From (9.1) it follows that $\sigma(\lambda)B_\mu = B_\mu, \forall \lambda \in k^*$. If $\xi$ is the point of (9.2), then one chooses $\mu$ in such a way that $(\xi, \mu) \in B_\mu$. Then $\psi_\alpha(\xi, \mu) = (\psi_\alpha(\xi), \mu) \in B_\mu, \forall \alpha \in k$, hence $\psi_\alpha(\xi) \in B_\mu, \forall \alpha \in k$. Because of (9.1) and (9.2) we would get $V \subset B_\mu$. Then the closed subscheme $B_\mu \subset B$ has the dimension two and it follows $B_\mu = B$, contradiction, as $p_2$ is dominant by assumption. This argumentation also shows $\xi \notin B_0 \cup B_\infty$, i.e. there is $\mu \in \mathbb{P}^1 - \{0, \infty\}$ such that $(\xi, \mu) \in B_\mu$. Thus one can find $\lambda \in k^*$ such that $\sigma(\lambda)\mu = (1 : 1)$. Replacing $\xi$ and $\mu$ by $\xi' = \sigma(\lambda)\xi$ and $\sigma(\lambda)\mu$, respectively, then (9.2) is fulfilled with $\xi'$ instead of $\xi$. Thus one can assume without restriction that $\mu = (1 : 1)$. As $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ is fixed by $\sigma, G_m$ operates by $\sigma$ on $\mathbb{A}^1$ and fixes $(0 : 1)$. Then there is $m \in \mathbb{Z}$ such that $\sigma(\lambda)(a : 1) = (\lambda^ma : 1)$ for all $a \in k, \lambda \in k^*$. As the action of $G_m$ on $\mathbb{P}^1$ is not trivial, $m \neq 0$.

$C := \{\psi_\alpha(\xi) | \alpha \in k\}^- \subset B_{(1:1)}$ is a curve in $V$; let $h$ be its Hilbert polynomial with respect to the usual embedding of $H^d$ in a projective space (cf. [T2], 4.1.1). The association $\lambda \mapsto \sigma(\lambda)C$ defines a morphism $G_m \rightarrow \mathcal{X} := \text{Hilb}^h(V)$. It has an extension to a morphism $\mathbb{P}^1 \rightarrow \mathcal{X}$, which defines a flat family of curves

\[
\begin{array}{ccc}
\mathcal{C} & \rightarrow & V \\
\downarrow p_2 & & \downarrow p_2 \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
\]

If $C_\lambda := p_2^{-1}(\lambda)$, then $C_\lambda := p_1(C_\lambda) = \sigma(\lambda)C, \forall \lambda \in k^*$ and

\begin{equation}
[p_1(C_0)] = [C] = [p_1(C_\infty)] \in A_1(V).
\end{equation}

The finite morphism $\mathbb{A}^1 - (0) \rightarrow \mathbb{A}^1 - (0)$ defined by $\lambda \mapsto \lambda^m$ has an extension $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $(\lambda : 1) \mapsto (\lambda^m : 1), \forall \lambda \in k$, and $(1 : 0) \mapsto (1 : 0)$. For simplicity, the morphism $1_V \times f : V \times \mathbb{P}^1 \rightarrow V \times \mathbb{P}^1$ is denoted by $f$, again. By construction $C \subset B_{(1:1)}$ and hence $\sigma(\lambda)C \subset \sigma(\lambda)B_{(1:1)} = B_{\sigma(\lambda)(1:1)}$. The fibre $C_\lambda = \sigma(\lambda)C \times \{(\lambda : 1)\}$ is mapped by $f$ into $\sigma(\lambda)C \times \{(\lambda^m : 1)\} = \sigma(\lambda)(C \times \{(1 : 1)\}) \subset \sigma(\lambda)(B_{(1:1)} \times \{(1 : 1)\}) = \sigma(\lambda)B_{(1:1)} = B_{\sigma(\lambda)(1:1)}, \forall \lambda \in k^*$.

This construction of the family $\mathcal{C}$ is called standard construction and the proof in ([T1], Lemma 1, p.6) shows that $\mathcal{C} \subset V \times_k \mathbb{P}^1$ is a subvariety. As $C_0$ and $C_\infty$ are closed in $\mathcal{C}$, $\mathcal{C} := \mathcal{C} - (C_0 \cup C_\infty)$ is open in $\mathcal{C}$. Because $\mathcal{C}$ and $\mathcal{B}$ are irreducible and $f$ is projective,
Proposition 8.1 and Proposition 8.2 that $W := C$. As $V \times \{0\}$ and $V \times \{\infty\}$ are mapped by $f$ into themselves, $C_0 \subset V \times \{0\}$ and $C_\infty \subset V \times \{\infty\}$ are mapped by $f$ into $B \cap V \times \{0\} = B_0$ and $B \cap V \times \{\infty\} = B_\infty$, respectively. As $f(C) = B$, we get $f(C_0) = B_0$ and $f(C_\infty) = B_\infty$, i.e. $C_0 = B_0$ and $C_\infty = B_\infty$. The curves $p_1(C_0)$ and $p_1(C_\infty)$ are called the limit curves (of the standard construction) and are denoted by $C_0 = \lim_{\lambda \rightarrow 0} \sigma(\lambda)C$ and $C_\infty = \lim_{\lambda \rightarrow \infty} \sigma(\lambda)C$, respectively, and one has

$$B_0 - B_\infty = C_0 - C_\infty \text{ in } Z_1^B(X).$$

We note that the relation (9.4) was derived under the assumption that $V$ is not pointwise invariant under $G_a$. If $V$ is pointwise invariant under $G_a$, the standard construction cannot be carried out. We get

Lemma 9.1. Let $X$ be as before. Elements of $\text{Rat}_1^B(X)$ are either of the form $\sum q_i C_i$, where $q_i \in Q$ and $C_i \subset X$ is a $B(3; k)$-invariant 1-prime cycle in $X^{G_a}$ or they occur in the following way: One considers all $B(3; k)$-invariant 2-dimensional subvarieties $V \subset X$, which are not pointwise invariant under $G_a$. Then there are points $\xi \in V(k)$ such that $V = \overline{G_m \cdot G_a \cdot \xi}$, where $G_m$ operates by $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z$ on $S$ (inhomogeneous case) or by $\sigma(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z$ (homogeneous case). $C := \{ \psi_a(\xi) | \alpha \in k^* \}$ is a curve in $V$ (with the reduced induced scheme structure). By means of the standard construction it defines a family of curves $C \subset V \times_k \mathbb{P}^1$, which is flat over $\mathbb{P}^1$ and with the fibres $C_\lambda = \sigma(\lambda)C$ for all $\lambda \in \mathbb{P}^1 - \{0, \infty\}$. $\text{Rat}_1^B(X)$ is generated by the relations in $X^{G_a}$ noted above as well as by the relations $C_0 - C_\infty$, where $C_0/\infty = \lim_{\lambda \rightarrow 0/\infty} \sigma(\lambda)C$ are the limit curves of $C$ in $V$ and these are $G_a$-invariant 1-cycles.

Proof. As $\sigma(\lambda) \psi_a = \psi_a \sigma(\lambda)$ in the homogeneous case, $\sigma(\lambda)C = \{ \psi_a \sigma(\lambda) \xi | \alpha \in k \}$ is $G_a$-invariant, $\forall \lambda \in k^*$. And the same is true in the inhomogeneous case .

9.3. Relations containing a $y$-standard cycle

Suppose $d \geq 5$. The closed subscheme $\mathcal{H} \subset H^d$ (cf. Section 8.4) is clearly $B$-invariant. As $U(3; k)$ is normalized by $T(3; k)$, the closed subscheme $X = Z(\mathcal{H})$ is $B$-invariant, too (cf. Section 1.2). From Lemma 8.1 it follows that relations in $Z_1^B(X)$ containing a $y$-standard cycle are defined by 2-dimensional $B$-invariant surfaces $V \subset X$, which are pointwise $\Gamma$-invariant but not pointwise $G_a$-invariant.

We can write $V = \overline{G_m \cdot G_a \cdot \zeta}$, where $\zeta \in V(k)$ is a point as in Proposition 8.1 or Proposition 8.2. By Lemma 8.2 we can assume that $\mathcal{I} \leftrightarrow \zeta$ has $y$-standard form. Let be $\zeta_{0/\infty} = \lim_{\lambda \rightarrow 0/\infty} \sigma(\lambda) \zeta$ and carry out the standard construction by using the curve $C := \{ \psi_a(\zeta) | \alpha \in k \}$. By Lemma 8.2 $C_{0/\infty}$ are $y$-standard cycles. We conclude from Proposition 8.1 and Proposition 8.2 that $W := G_m \cdot G_a \cdot \zeta$ is $B$-invariant and therefore $V - W$ is a union of $B$-invariant curves and points. As $C_{0/\infty}$ is invariant under $G_m$ and $G_a$, from $W \cap C_{0/\infty} \neq \emptyset$ it would follow that $V \subset C_{0/\infty}$. Hence $C_{0/\infty} \subset V - W$, and $C_{0/\infty}$ is a union of $B$-invariant curves, too. As $\zeta_{0/\infty} \in C_{0/\infty}$, one has $C_{0/\infty} \subset C_{0/\infty}$. Now from the Propositions 5.1, 6.1 and 7.1 it follows that the inequality (!) in (4.3) is fulfilled with one
Lemma 9.2. Let be \( m = \text{reg}(\mathcal{I}) \). Because of \( \max -\alpha\text{-grade}(\mathcal{I}) \leq \alpha\text{-grade}(\mathcal{I}) = \deg C = \deg C_0 = \deg C_\infty \) and \( \min(\deg C_0, \deg C_\infty) \geq \min -\alpha\text{-grade}(\mathcal{I}) \) (cf. the definitions and the inequality (4.9) in (4.3), from (!) it follows that

\[(**)
Q(m - 1) + \min(\deg C_0, \deg C_\infty) > \deg C_0 = \deg C_\infty.
\]

As \( V = \overline{W} \), each ideal \( \mathcal{J} \) in \( V \) has a regularity \( n \geq m \). If \( \mathcal{J} \) is monomial and has \( y \)-standard form then from Remark 4.4 it follows that the \( y \)-standard cycle generated by \( \mathcal{J} \) has a degree \( \geq Q(n - 1) \geq Q(m - 1) \).

**Lemma 9.2.** \( C_0 \) and \( C_\infty \) are the only \( y \)-standard cycles contained in the \( B \)-invariant cycles \( C_0 \) and \( C_\infty \), respectively, and they both occur with the multiplicity 1.

**Proof.** Except when \( c = 1, m = 4 \) the statements follow from (**) and the foregoing discussion. In the remaining exceptional case (cf. Proposition 7.1) we carry out the standard construction with \( \mathcal{I} = (y(x, y), x^4 + y^3) \leftrightarrow \zeta \). Then \( \zeta_0 \leftrightarrow (y(x, y), x^3), \zeta_\infty \leftrightarrow (y, x^5) \) and \( C_0 \) and \( C_\infty \) are \( y \)-standard cycles of degrees 5 and 10, respectively, as one can see from Figure 9.1.

If \( \eta := \lim_{\lambda \to \infty} \psi_\alpha(\zeta) \), the corresponding ideal \( \mathcal{L} \) is invariant under \( G_\alpha \), hence invariant under \( U(3; k) \). One sees that \( xz^2 \in \mathcal{L} \), and hence \( x \in \mathcal{L} \) follows. One concludes from this that \( \mathcal{L} = (x, y^5) \).

As \( \sigma \) and \( \psi_\alpha \) commute, it follows that \( \eta \) is the only point in \( \sigma(\lambda)C \) for all \( \lambda \), which is \( U(3; k) \)-invariant. From \( V = p_1(\mathcal{C}) = \bigcup\{p_1(C_\lambda) | \lambda \in \mathbb{P}^1\} = \bigcup\{\sigma(\lambda)C | \lambda \in k^*\} \cup C_0 \cup C_\infty \) it follows that

\[ V = W \cup \{\eta\} \cup C_0 \cup C_\infty \]

where \( W := G_m \cdot G_\alpha \cdot \zeta \). Now \( \eta_0 := \lim_{\lambda \to \infty} \psi_\alpha(\zeta_0) \leftrightarrow (x(x, y), y^4) \) and \( \eta_\infty := \lim_{\lambda \to \infty} \psi_\alpha(\zeta_\infty) \leftrightarrow (x, y^5) \) are the only \( B(3; k) \)-invariant points in \( C_0 \) and \( C_\infty \), respectively. By the theorem of Fogarty \( V^{U(3; k)} \) is connected, hence is a union of pointwise \( U(3; k) \)-invariant, closed and irreducible curves. If \( E \) is such a curve, then \( E \subset C_0 \cup C_\infty \) follows. Now \( \deg C_0 = \deg C_\infty = \deg C = 10 \). Hence \( C_\infty = C_\infty \) and \( E \subset C_0 \) follows. As each \( y \)-standard cycle in \( V \) has a degree \( \geq Q(3) = 5 \), \( C_0 \) is the only \( y \)-standard cycle contained in \( C_0 \), and \( C_0 \) occurs with multiplicity 1. \( \square \)

**Proposition 9.1.** Let \( X = Z(\mathcal{H}) \). Relations in \( \text{Rat}^B_1(X) \), which are defined by a \( B \)-invariant surface in \( X \) and which contain a \( y \)-standard cycle, are generated by relations of the form \( C_1 - C_2 + Z \) where \( C_1 \) and \( C_2 \) are \( y \)-standard cycles and \( Z \) is a 1-cycle in \( X \), whose prime components are \( B \)-invariant but are not \( y \)-standard cycles.

**Proof.** This follows from Lemma 9.1 and Lemma 9.2. \( \square \)
Proof of Theorem 1.1

This had been formulated in Chapter 1 and we repeat the notations and assumptions introduced there: \( d \geq 5, g(d) = (d - 2)^2/4, \mathcal{H} = \bigcup \{ H_{\geq \varphi} \subset H^d | g^*(\varphi) > g(d) \}, \mathcal{A}(\mathcal{H}) = \text{Im}(A_1(\mathcal{H}^{U(3,k)}) \to A_1(\mathcal{H})), c_3 = \{ (ax + y, x^d) | a \in k \}^- \).

We make the assumption \([c_3] \in \mathcal{A}(\mathcal{H})\). Then \([c_3] = \sum q_i U_i\) in \( A_1(\mathcal{H}) \), where \( q_i \in \mathbb{Q} \), and \( U_i \subset \mathcal{H} \) are \( T \)-invariant curves, which are pointwise \( U(3;k) \)-invariant. \((A_1(\mathcal{H}^{U(3,k)})\) is generated by such curves, as follows from the theorem of Hirschowitz or more directly from an application of Lemma 1 in [T1], p. 6.) If \( X := \mathcal{Z}(\mathcal{H}) \), then \( A_1(X) \to A_1(\mathcal{H})([T2], \text{Lemma 24}, \text{p.121}) \) and \( A_1^B(X) \to A_1(X) ([\text{Hi}], \text{Theorem 1}, \text{p.87}) \).

Hence we can regard \( \alpha := c_3 - \sum q_i U_i \) as a 1-cycle in \( Z_1^B(X) \), whose canonical image in \( A_1^B(X) := Z_1^B(X)/\text{Rat}_1^B(X) \) vanishes. (We recall that \( Z_1^B(X) \) is the free group generated by all closed, reduced and irreducible curves in \( X \).) Define \( \mathcal{B}(X) \) to be the subgroup of \( Z_1^B(X) \), which is generated by all \( B(3;k) \)-invariant curves in \( X \) (closed, reduced, irreducible), which are not \( y \)-standard cycles. Define:

\[
F_1(X) = Z_1^B(X)/\mathcal{B}(X), \quad R_1(X) = \text{Rat}_1^B(X) + \mathcal{B}(X)/\mathcal{B}(X)
\]

\( F_1(X) \) is the free group generated by the \( y \)-standard cycles, and by Proposition 9.1 \( R_1(X) \) is generated by elements of \( F_1(X) \) of the form \( C_1 - C_2 \), where \( C_1 \) and \( C_2 \) are (different) \( y \)-standard cycles. It follows that the canonical image of \( \alpha \) in \( F_1(X)/R_1(X) \) on the one hand vanishes and on the other hand is equal to the canonical image of \( c_3 \) in this \( \mathbb{Q} \)-vector space. Hence \( c_3 \in R_1(X) \). We will show that this is not possible. To each of the finitely many \( y \)-standard cycles we associate a canonical basis element \( e_i \in \mathbb{Q}^n, 1 \leq i \leq n \). Especially, we associate \( c_3 \) to the element \( e_n = (0, \ldots, 1) \). From \( c_3 \in R_1(X) \) it follows that \( e_n \in \{ e_i - e_j | 1 \leq i < j \leq n \} \), which is not possible. All in all we have

**Theorem 10.1.** \([c_3] \notin \mathcal{A}(\mathcal{H})\). \(\square\)

**Corollary 10.1.** \( \dim_{\mathbb{Q}} A_1(\mathcal{H}) \geq 3. \)

**Proof.** \( E := \{(x^2, xy, y^{d-1} + axz^{d-2}) | a \in k \}^- \) and \( F := \{(x, y^{d-1} (ay + z)) | a \in k \}^- \) are 1-cycles in \( \mathcal{H} \), which are pointwise invariant under \( U(3;k) \) and \( G(0,1) \), respectively (see the notation in Appendix D). If \([c_3] = q_1[E] + q_2[F] \), \( q_1, q_2 \in \mathbb{Q} \), we compute the intersection numbers with the tautological line bundles and get

\[
\binom{d}{2} = q_1 + q_2(n - d + 1)
\]

for all \( n \geq d - 1 \). Hence \( q_2 = 0 \). If \( q_1 \neq 0 \), then \([c_3] \in \mathcal{A}(\mathcal{H})\) would follow. \(\square\)
Surfaces in $H$ invariant under $\mathbb{G}_a \cdot T(4; k)$

We let $\mathbb{G}_a$ operate by $\psi_\alpha : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z, t \mapsto z$ and let $T(4; k)$ operate by diagonal matrices on $P = k[x, y, z, t]$, hence on $H = H_Q$. Let $V \subset H$ be a 2-dimensional subvariety, which is invariant under $\mathbb{G}_a \cdot T(4; k)$ but not pointwise invariant under $\mathbb{G}_a$.

### 11.1. The inhomogeneous case

We suppose that $V$ is not pointwise invariant under the $\mathbb{G}_m$-operation $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto \lambda t$.

#### 11.1.1. Auxiliary Lemma 1.

There is a $\xi \in V(k)$ such that $V = \overline{T(4; k) \cdot \xi}$.

**Proof.** If $\dim T(4; k) \cdot \xi \leq 1$, then the inertia group $T_\xi \subset T(4; k)$ has the dimension $\geq 3$. If the dimension is 4, then $\xi$ corresponds to a monomial ideal with Hilbert polynomial $Q$, and there are only finitely many such ideals. If $\dim T_\xi = 3$, then $T_\xi = T(\rho)$ ([T2], Hilfssatz 7, p.141). If $\xi$ corresponds to the ideal $I$ with Hilbert polynomial $Q$, then $H^0(I(b))$ has a basis of the form $f_i = m_i p_i(X^\rho)$, where $m$ is a monomial and $p_i$ is a polynomial in 1 variable. At least for one index $i$ the polynomial $p_i(X^\rho)$ contains a positive power of $X^\rho$. Then $m_i X^\rho \in P_\rho$. As $m_i \in P_\rho$ and $\rho = (\rho_0, \ldots, \rho_3) \in \mathbb{Z}^4 - (0)$ is a vector such that $\rho_0 + \cdots + \rho_3 = 0$, this implies $|\rho_i| \leq b$. Hence there are only finitely many such vectors $\rho$ such that the fixed-point scheme $H^{T(\rho)}$ does not consist of only finitely many $T(4; k)$-fixed points.

Suppose that $\dim T(4; k) \cdot \xi \leq 1$ for all $\xi \in V(k)$. Then $V = \bigcup_{i=1}^{n} T(\rho_i), \rho_i \in \mathbb{Z}^4 - (0)$, hence $V = V^{T(\rho)}, \rho \in \mathbb{Z}^4 - (0)$ suitable vector. We show there is $\tau = (\lambda_0, \ldots, \lambda_3) \in T(\rho)$ such that $\lambda_0 \neq \lambda_1$. If not, it follows that $T(\rho) \subset T(1, -1, 0, 0)$, hence $\nu \rho = (1, -1, 0, 0), \nu \in \mathbb{Z}$. But then $\rho_3 = 0$ and $V$ would be pointwise invariant under $\mathbb{G}_m$, contradiction.

Take this $\tau$ and an arbitrary $\alpha \neq 0$. Then $\tau^{-1} \psi_\alpha \tau \psi_\alpha^{-1} = \begin{pmatrix} 1 & \alpha(\lambda_0^{-1} \lambda_1 - 1) \\ 0 & 1 \end{pmatrix} = \psi_\beta$, where $\beta := \alpha(\lambda_0^{-1} \lambda_1 - 1) \neq 0$. The same argumentation as in the proof of (8.2.1 Auxiliary Lemma 1) gives a contradiction.

#### 11.1.1.1. If $0 \leq i < j \leq 3$ define $T_{ij} = \{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in T(4; k) | \lambda_\ell = 1 \text{ if } \ell \neq i \text{ and } \ell \neq j\}$. We let $G := \mathbb{G}_a \cdot T_{23}$ operate von $V$. If $\xi \in V(k)$ is as in Auxiliary Lemma 1, let $G_\xi$ be the inertia group of $\xi$ in $G$. From $1 \leq \dim G_\xi \leq 2$ it follows that there is a 1-dimensional connected subgroup $H < G_\xi^0$. Then $H \simeq \mathbb{G}_a$ or $H \simeq \mathbb{G}_m$. In the first case it follows that $H = \mathbb{G}_a$ and hence $T(4; k) \cdot \xi$ is pointwise $\mathbb{G}_a$-invariant, contradiction. Hence
$H \simeq \mathbb{G}_m$. In the diagram

$$\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{i} & G = \mathbb{G}_a \times T_{23} \\
p_1 \nearrow & & \searrow p_2 \\
\mathbb{G}_a & & T_{23}
\end{array}$$

$p_1 \circ i$ is the trivial map, hence $H = \{(1,1,\lambda',\lambda^s)|\lambda \in k^*\} =: T_{23}(r,s)$, where $r,s \in \mathbb{Z}$ are not both equal to zero.

11.1.1.2. Let be $G := \left\{ \left( \begin{array}{c} \lambda \\
\alpha \\
\mu \end{array} \right) \left| \lambda, \mu \in k^*, \alpha \in k \right. \right\}$ and let $\xi$ be as in Auxiliary Lemma 1. Then there is again a subgroup $H < G^0$ isomorphic to $\mathbb{G}_m$. It has the form $H = \left\{ \left( \begin{array}{c} \lambda^a \\
(\lambda^b - \lambda^a)c \\
\lambda^b \end{array} \right) \left| \lambda \in k^* \right. \right\}$, where $a,b \in \mathbb{Z}$ are not both equal to zero and $c \in k$ is a fixed element (cf. 8.2.2). Let be $u = \left( \begin{array}{ccc} 1 & -c & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{array} \right)$.

Then $uH_1^{-1} = \left\{ \left( \begin{array}{c} \lambda^a \\
\lambda^b \\
1 \end{array} \right) \left| \lambda \in k^* \right. \right\} =: T_{01}(a,b)$. If $\zeta := u(\xi)$ one gets $T_\zeta \subset T_{01}(a,b)$. From (11.1.2) it follows $T_\zeta \supset T_{23}(r,s)$, too. As $T_\zeta$ contains the diagonal group, $\dim T_\zeta \geq 3$ follows. If $\zeta$ were fixed by $T(4;k)$, then $\xi = u^{-1}(\zeta)$ would be fixed by $T_{23}$, contradiction. It follows that $T_\zeta = T(\rho)$. Here $\rho_3 \neq 0$, because $\zeta$ is not invariant under $\sigma$, for otherwise $V$ would be pointwise $\mathbb{G}_m$-invariant. If $c = 0$, then $T_\zeta = T(\rho)$, which contradicts the choice of $\xi$.

**Conclusion 11.1.** The element $u$ is different from 1, and putting $\zeta := u(\xi)$ one has $V = \overline{\mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta}$.

**Proof.** Put $G := \mathbb{G}_a \cdot \mathbb{G}_m = \mathbb{G}_a \times \mathbb{G}_m$. If $\dim G_\zeta \geq 1$, then $G^0_\zeta$ would contain a subgroup $H$ isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$. But then $\zeta$ would be invariant under $\mathbb{G}_m$ or $\mathbb{G}_a$, and then the same would be true for $\zeta$, which gives a contradiction as above.

**Conclusion 11.2.** $V \setminus \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$ is a union of points and curves which are invariant under $\mathbb{G}_a \cdot T(4;k)$.

**Proof.** If $\tau \in T(\rho)$, then $\tau \cdot \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \tau \cdot \zeta = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$, and if $\tau \in \mathbb{G}_m$, the same is true.

**11.1.2. The homogeneous case.**

11.1.2.1. We first suppose that $V$ is pointwise invariant under the $\mathbb{G}_m$-operation $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto \lambda t$, but not pointwise invariant under the $\mathbb{G}_m$-operation $\tau(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z, t \mapsto t$. Then $V$ is invariant under $T(3;k) \simeq \{(\lambda_0, \lambda_1, \lambda_2, 1)|\lambda_i \in k^*\}$ and we have the same situation as in Chapter 8. We use the same notations introduced there and get $V = \overline{T(2;k) \cdot \xi}$ (8.2.1 Auxiliary Lemma 1). We let $G := $
\( \mathbb{G}_a \cdot T(2; k) = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix} \bigg| \lambda, \mu \in k^*, \alpha \in k \right\} \) operate on \( V \). Then there is a subgroup \( H \) of \( G_\xi \), which is isomorphic to \( \mathbb{G}_a \) or \( \mathbb{G}_m \). In the first case \( \xi \) would be \( \mathbb{G}_a \)-invariant, hence \( V \) would be pointwise \( \mathbb{G}_a \)-invariant. In the second case \( H = \left\{ \begin{pmatrix} \lambda a & (\lambda^b - \lambda^a)c \\ 0 & \lambda^b \end{pmatrix} \bigg| \lambda \in k^* \right\} \).

The same argumentation as in (11.1.3) shows that the inertia group of \( \zeta = u(\xi) \) in \( T(3; k) \) has a dimension \( \geq 2 \). If \( T_\xi = T(3; k) \), then \( \zeta \) would be invariant under the \( \mathbb{G}_m \)-operation \( \tau \), hence \( \xi \) invariant under \( \tau \), too, and \( V \) would be pointwise invariant under \( \tau \). It follows that \( T_\xi = T(\rho) \), where \( \rho = (\rho_0, \rho_1, \rho_2, 0) \) and \( \rho_2 \neq 0 \). We note that \( u \neq 1 \), for otherwise the inertia group of \( \xi \) in \( T(4; k) \) would have a dimension \( \geq 3 \).

**Conclusion 11.3.** The element \( u \) is different from 1 and putting \( \zeta = u(\xi) \) one has \( V = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta \).

The same argumentation as in (11.1.3) gives

**Conclusion 11.4.** \( V \setminus \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta \) is a union of points and curves which are invariant under \( \mathbb{G}_a \cdot T(4; k) \).

The same argumentation as in (11.1.3) gives

**Conclusion 11.4.** \( V \setminus \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta \) is a union of points and curves which are invariant under \( \mathbb{G}_a \cdot T(4; k) \).

11.1.2.2. We now suppose \( V \) is pointwise invariant under \( T_{23} = \left\{ (1, 1, \lambda_2, \lambda_3) \big| \lambda_i \in k^* \right\} \).

Then \( V \) is not pointwise invariant under the \( \mathbb{G}_m \)-operation \( \sigma(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z, t \mapsto t \). Let \( \xi \in V \setminus (V^{\mathbb{G}_m} \cup V^{\mathbb{G}_a}) \) be a closed point, and put \( G := \mathbb{G}_a \ltimes \mathbb{G}_m = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & 1 \end{pmatrix} \bigg| \lambda \in k^*, \alpha \in k \right\} \).

Assume that \( \dim G_\xi \geq 1 \). Then \( H := G_\xi^0 \) is 1-dimensional and connected. As \( \xi \) is not \( \mathbb{G}_a \)-invariant, \( H \simeq \mathbb{G}_m \) hence \( H = \left\{ \begin{pmatrix} \lambda^a & (1 - \lambda^a)c \\ 0 & 1 \end{pmatrix} \bigg| \lambda \in k^* \right\}, a \in \mathbb{Z} - (0) \). Putting \( u = \begin{pmatrix} 1 & -c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) and \( \zeta = u(\xi) \) we get \( uHu^{-1} = \left\{ (\lambda^a, 1, 1, 1) \big| \lambda \in k^* \right\} \). It follows that \( \zeta \) is \( \mathbb{G}_m \)-invariant, hence \( T(4; k) \)-invariant. Thus \( \zeta \) corresponds to an ideal \( \mathcal{I} \) such that \( H^0(\mathcal{I}(b)) \) has a generating system of the form \( M_i(cx + y)^{n_i} \), where \( M_i \) is a monomial without \( y \) and \( n_i \in \mathbb{N} \). There are only finitely many points \( \zeta_i \in V(k) \) which are \( T(4; k) \)-invariant and it follows that \( \xi \in \bigcup \mathbb{G}_a \cdot \zeta_i \).

**Conclusion 11.5.** If \( \xi \in V \setminus \left[ \bigcup \mathbb{G}_a \cdot \zeta_i \cup V^{\mathbb{G}_a} \cup V^{\mathbb{G}_m} \right] \) is a closed point, then \( V = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \xi \).

**11.1.3. Summary.** Let \( V \subseteq H_Q \) be a 2-dimensional subvariety, invariant under \( G := \mathbb{G}_a \cdot T(4; k) \) but not pointwise invariant under \( \mathbb{G}_a \). We distinguish between three cases:

1. \( V \) is not pointwise invariant under the \( \mathbb{G}_m \)-operation \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto \lambda t \).
2. \( V \) is pointwise invariant under the \( \mathbb{G}_m \)-operation \( \tau(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z, t \mapsto t \).
3. $V$ is pointwise invariant under the $\mathbb{G}_m$-operations $\sigma$ and $\tau$ as in the 1th and 2nd case. Then $V$ is not pointwise invariant under the $\mathbb{G}_m$-operation $\omega(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z, t \mapsto t$.

Lemma 11.1. (a) In the 1st case (respectively in the 2nd case) there is $\zeta \in V(k)$ with the following properties:

(i) The inertia group $T_\zeta$ of $\zeta$ in $T(4; k)$ has the form $T(\rho)$ where $\rho_3 > 0$ (respectively $\rho_2 > 0$ and $\rho_3 = 0$).

(ii) $V = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$ and $V \setminus \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$ is a union of points and curves invariant under $\mathbb{G}_a \cdot T(4; k)$.

(iii) There is $u \in \mathbb{G}_a$, different from 1, such that $V = T(4; k) \cdot \xi$, where $\xi := u(\zeta)$.

(b) In the 3rd case, if one chooses $\zeta \in V(k)$ such that $\zeta$ is neither $\mathbb{G}_a$- nor $\mathbb{G}_m$-invariant and does not lie in the set \{ $\xi \in V(k) \mid \exists u \in \mathbb{G}_a \text{ and } \exists \mu \in V^{T(4; k)}(k)$ such that $\xi = u(\mu)$ \}, then $V = \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$. \qed
CHAPTER 12

Surfaces in \( H \) invariant under \( B(4; k) \)

12.1. The operation of the unipotent group on \( H \)

12.1.1. Let be \( p = (a : b : c) \in \mathbb{P}^2(k) \) and

\[
G(p) := \left\{ \begin{pmatrix} 1 & \alpha & * \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \bigg| a\alpha + b\beta + c\gamma = 0 \right\}.
\]

This is a 5-dimensional subgroup of \( \Delta := U(4 : k) \) and each 5-dimensional subgroup of \( \Delta \) has this form, where \( p \in \mathbb{P}^2(k) \) is uniquely determined (Appendix D, Lemma 1).

Especially one has the groups \( G_i = G(p_i) \), where \( p_1 = (0 : 0 : 1), p_2 = (0 : 1 : 0), p_3 = (1 : 0 : 0) \) and \( G_1 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, G_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, G_3 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \).

Remark 12.1. If \( \psi_\alpha = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), then \( \psi_\alpha G(p) \psi_\alpha^{-1} = G(p) \).

Remark 12.2. If \( \tau = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in T(4; k) \), then \( \tau G(p) \tau^{-1} = G(\tau p) \), where \( \tau p := (a\lambda_0^{-1}\lambda_1 : b\lambda_1^{-1}\lambda_2 : c\lambda_2^{-1}\lambda_3) \).

12.1.2. In ([T2], 3.2.1) and ([T3], 10.2) we had introduced a closed, reduced sub-scheme \( Z = Z(H_Q) \) of \( H_Q \) such that \( Z(k) = \{ x \in H_Q(k) \mid \dim \Delta \cdot x \leq 1 \} \). From the theorem of Hirschowitz it follows that \( A_1(Z) \xrightarrow{\sim} A_1(H_Q) \) ([T2], Lemma 24, p. 121). In the following we consider a surface \( V \subset Z \) (i.e. a closed 2-dimensional subvariety) which is \( B(4; k) \)-invariant, but not pointwise invariant under \( \Delta = U(4; k) \).

First case: \( G_2 \) operates by \( x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto \alpha z + t \).

Second case: \( G_3 \) operates by \( x \mapsto x, y \mapsto \alpha x + y, z \mapsto z, t \mapsto t \).

12.1.3. Auxiliary Lemma 1. Let \( V \subset Z \) be a \( B(4; k) \)-invariant surface. In the first case we assume that \( V \) is not pointwise invariant under \( G_2 \) and not pointwise invariant
under $T(0,0,−1,1)$. In the second case we assume that $V$ is not pointwise $\mathbb{G}_a$-invariant and not pointwise invariant under $T(1,−1,0,0)$. Then there is a $\xi \in V(k)$ such that $V = \overline{T(4;k) \cdot \xi}$.

**Proof.** Assume $\dim T_\xi \geq 3$ for all $\xi \in V(k)$. Then it follows as in the proof of (11.1.1 Auxiliary lemma 1) that $V = V^{T(\rho)}$. We consider the first case. Because of $T(\rho) \neq T(0,0,−1,1)$ there is $\tau = (\lambda_0,\lambda_1,\lambda_2,\lambda_3) \in T(\rho)$ such that $\lambda_2 \neq \lambda_3$. The proof of (8.2.1 Auxiliary lemma 1) shows that this gives a contradiction. The same argumentation also shows in the second case that the above assumption is wrong. □

12.1.4. Auxiliary Lemma 2. Let $V \subset Z$ be a $B(4;k)$-invariant surface which is not pointwise invariant under any of the groups $G_1, G_2, G_3$. Then there is a closed point $\xi \in W := V \setminus (V^{G_1} \cup V^{G_2} \cup V^{G_3})$ such that $V = \overline{T(4;k) \cdot \xi}$.

**Proof.** If $\xi \in W(k)$, the inertia group $\Delta_\xi$ has the form $G(p)$, where $p \in \mathbb{P}^2(k) − \{p_1,p_2,p_3\}$. If $\tau \in T_\xi$, then $\tau \rho(p) \tau^{-1} \xi = \tau \xi$ and thus $G(\tau \rho) \cdot \xi = \xi$. If $\tau \rho \neq p$, then $G(\tau \rho) \neq G(p)$ and $\xi$ would be fixed by the closed subgroup of $\Delta$, which is generated by $G(p)$ and $G(\tau \rho)$, i.e., $\xi$ would be fixed by $\Delta$, contradiction. Thus one has $\tau \rho = p$ for all $\tau \in T_\xi$.

1st case: $a,b,c \neq 0$. Then $\lambda_0^{-1}\lambda_1 = \lambda_1^{-1}\lambda_2 = \lambda_2^{-1}\lambda_3$ for all $\tau = (\lambda_0,\lambda_1,\lambda_2,\lambda_3) \in T_\xi$, hence $\dim T_\xi \leq 2$ and $V = \overline{T \cdot \xi}$.

2nd case: $p = (0 : b : c)$, where $b$ and $c \neq 0$ (respectively $p = (a : 0 : c)$, where $a$ and $c \neq 0$). It follows that $\xi$ is not invariant under the first $\mathbb{G}_a$-operation (respectively the second $\mathbb{G}_a$-operation) as in the first (respectively in the second) case of Auxiliary lemma 1. If $\xi$ would be invariant under $T(0,0,−1,1)$ in the case $p = (0 : b : c)$, then $T(0,0,−1,1) \subset T(0,1,−2,1)$ would follow, what is impossible. If $\xi$ would be invariant under $T(−1,1,0,0)$ in the case $p = (a : 0 : c)$, then $T(−1,1,0,0) \subset T(1,−1,−1,1)$ would follow, which is impossible. Thus the assumptions of Auxiliary lemma 1 are fulfilled.

3rd case: $p = (a : b : 0)$, where $a$ and $b \neq 0$. Then $\xi$ is not invariant under the $\mathbb{G}_a$-operation as in the second case of Auxiliary lemma 1. If $\xi$ would be invariant under $T(−1,1,0,0)$, then $T(−1,1,0,0) \subset T(1,−2,1,0)$ follows, which is impossible. Thus the assumptions of the second case of Auxiliary Lemma 1 are fulfilled. □

12.2. The case $p = (a : b : c)$ where $a,b,c \neq 0$.

Let be $V \subset Z$ a $B$-invariant surface and $\xi \in V(k)$ a point such that $\Delta_\xi = G(a : b : c)$, where $a,b,c \neq 0$. Then $T_\xi \subset \Lambda := \{(\lambda_0,\lambda_1,\lambda_0^{-1}\lambda_1^2,\lambda_0^{-2}\lambda_1^3)|\lambda_0,\lambda_1 \in k^*\}$. As $\dim T(4;k) \cdot \xi \leq 2$, one has $\dim T_\xi \geq 2$ and thus $T_\xi = \Lambda$.

We let $G := \mathbb{G}_a \cdot T_{01} = \left\{ \begin{pmatrix} \lambda_0 & \alpha & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} | \alpha \in k, \lambda_0, \lambda_1 \in k^*$ operate on $V$.

**Remark 12.3.** $T(4;k)_\xi \cap G = (1)$.
Remark 12.4. $V = \overline{T_{01} \cdot \xi}$.

**Proof.** Because of Remark (12.3) dim $T_{01} \cdot \xi < 2$ is not possible. \(\square\)

From Remark (12.4) it follows that $G_{\xi} < G$ is 1-dimensional. If $H := G_{\xi}^0$ would be isomorphic to $G_a$, then $\xi$ would be invariant under $G_a$, hence invariant under $\Delta$, contradiction. Thus $H = \left\{ \left( \begin{array}{c} \lambda^m \\
 \end{array} \right) \right\}$, $m, n \in \mathbb{Z}$. From (8.2.2 Auxiliary Lemma 2) if $u := \left( \begin{array}{c} 1 \\
 -c \\
 1 \end{array} \right)$, then $uH^{-1} = \left( \begin{array}{c} \lambda^m \\
 0 \\
 \lambda^n \end{array} \right) \lambda \in k^{*} =: T_{01}(m, n)$. Putting $\zeta := u(\xi)$ one gets $T_{\zeta} \supset T_{01}(m, n)$. Because of Remark (12.1) from $\Delta_{\zeta} = G(p)$ it follows that $\Delta_{\zeta} = G(p)$, too. Hence $T_{\zeta} = \Lambda \supset T_{01}(m, n)$, which is not possible. We have proved

**Lemma 12.1.** If $V \subset Z$ is a $B(4; k)$-invariant surface, which is not pointwise $\Delta$-invariant, then there is no point $\xi \in V(k)$, whose inertia group $\Delta_{\xi}$ in $\Delta$ has the form $G(a : b : c)$, where $a, b, c \neq 0$. \(\square\)

### 12.3. 1-cycles of proper type 3.

#### 12.3.1. Recalling the restriction morphism $h$. The ideals $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^3}$ with Hilbert polynomial $Q$ such that $t$ is not a zero divisor of $\mathcal{O}_{\mathbb{P}^3}/\mathcal{I}$ form an open non empty subset $U_t \subset H_Q$, and $\mathcal{I} \mapsto \mathcal{I}' := \mathcal{I} + t\mathcal{O}_{\mathbb{P}^3}(-1)/t\mathcal{O}_{\mathbb{P}^3}(-1)$ defines the so called restriction morphism $h : U_t \rightarrow H^4 = \text{Hilb}^d(\mathbb{P}^2)$. If $\Gamma := \left\{ \left( \begin{array}{ccc} 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \end{array} \right) \right\} < \Delta$, then $H_Q^\Gamma$ is contained in $U_t$.

#### 12.3.2. 1-cycles of proper type 3. We recall, respectively introduce, the notations: An ideal $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^3}$ with Hilbert polynomial $Q$ corresponds to a point $\xi \in H_Q(k)$. $\mathcal{J}$ has the type 3, if $\mathcal{J}$ is invariant under $T(4; k)$ and $G_3 = G(1 : 0 : 0)$, but not invariant under $\Delta$. The curve $C = G_{\xi} \cdot \xi = \{ \psi_{\alpha}(\xi) | \alpha \in k \}^{-1}$ in $H_Q$ is called the 1-cycle of type 3 defined by $\xi$. We say $C$ is a 1-cycle of proper type 3, if $\mathcal{I} := \mathcal{J}' = h(\mathcal{J})$ has $y$-standard form (cf. 2.4.3 Definition 2). If $\varphi$ is the Hilbert function of $\mathcal{I} \leftrightarrow \mathcal{I}' \in H^d(k)$, then $g'(\varphi) > g(d)$ by definition, and one has $\mathcal{I} = yK(-1) + x^m\mathcal{O}_{\mathbb{P}^2}(-m)$, where $K \subset \mathcal{O}_{\mathbb{P}^2}$ has the colength $c$ and is invariant under $T(3; k)$ and $G_3' := \left\{ \left( \begin{array}{c} 1 \\
 0 \\
 0 \end{array} \right) \right\} < U(3; k)$. Moreover one has $d = m + c$ and $m \geq c + 2$.

If $Q(T) = (T^{-1+3}) + (T^{-a+2}) + (T^{-b+1})$ is the Hilbert polynomial of $\mathcal{J}$, then the closed subscheme $V_+(\mathcal{J}) \subset \mathbb{P}^3$ defined by $\mathcal{J}$ has the “complementary” Hilbert polynomial $P(n) = dn - g + 1$, where $d = a - 1, g = g(\mathcal{J}) = (a^2 - 3a + 4)/2 - b$ (cf. [T1], p. 92).
Lemma 12.2. Let $\mathcal{J}$ be of proper type 3 and put $\nu := \min \{n | H^0(\mathcal{J}(n)) \neq 0\}$. If $x^\nu \in H^0(\mathcal{J}(\nu))$, then $g(\mathcal{J}) < 0$.

Proof. We start with an ideal $\mathcal{J}$ fulfilling these conditions. There are subspaces $U_i \subset S_i$, invariant under $T(3; k) \cdot G_2$ such that $S_i U_i \subset U_{i+1}$, $i = 0, 1, 2, \ldots$ and $H^0(\mathcal{J}(n)) = \bigoplus_{i=0}^{n} t^{n-i} U_i$, for all $n \in \mathbb{N}$. Besides this, $U_n = H^0(\mathcal{I}(n))$, at least if $n \geq b$, where $\mathcal{I} = \mathcal{J}'$ is the restriction ideal of $\mathcal{J}$ (cf. [G78], Lemma 2.9, p. 65). We replace $U_n$ by $H^0(\mathcal{I}(n))$ for all $\nu \leq n \leq b-1$ and we get an ideal $\mathcal{J}^* \supset \mathcal{J}$ such that $H^0(\mathcal{J}^*(n)) = (0)$, if $n < \nu$, and $H^0(\mathcal{J}^*(n)) = \bigoplus_{i=\nu}^{n} t^{n-i} H^0(\mathcal{I}(i))$, if $n \geq \nu$. (N.B.: $x^\nu \in H^0(\mathcal{J}(\nu)) \Rightarrow x^\nu \in \mathrm{Im}(H^0(\mathcal{J}(\nu)) \rightarrow H^0(\mathcal{I}(\nu)))$). Then $(\mathcal{J}^*)'$ is equal to $\mathcal{I}$, $\mathcal{J}^*$ is of proper type 3, the Hilbert polynomial of $\mathcal{J}^*$ is equal to $Q^*(n) = \binom{n+1+3}{3} + \binom{n-a+2}{2} + \binom{n-b+1}{1}$, the length of $\mathcal{J}^*/\mathcal{J}$ is equal to $Q^*(n) - Q(n) = b - b^* \geq 0$, and because of $g(\mathcal{J}^*) = (a^2 - 3a + 4)/2 - b^*$ one has $g(\mathcal{J}^*) \geq g(\mathcal{J})$. Thus it suffices to show $g(\mathcal{J}^*) < 0$. We thus can assume without restriction $\mathcal{J} = \mathcal{J}^*$, i.e. $H^0(\mathcal{J}(n)) = \bigoplus_{i=\nu}^{n} t^{n-i} H^0(\mathcal{I}(i))$ for all $n \geq \nu$, and $x^\nu \in H^0(\mathcal{I}(\nu))$.

Using the terminology of [T1]–[T4], one can say the pyramid $E(\mathcal{J})$ is complete up to the level $\nu$ over the platform $H^0(\mathcal{I}(n))$, where $n \geq b$, for instance (cf. Fig. 12.1). Further note that

\begin{equation}
H^0(\mathcal{I}(n)) = yH^0(\mathcal{K}(n-1)) \oplus x^m k[x, z]_{n-m}
\end{equation}

for all $n \in \mathbb{N}$ (cf. Lemma 2.6). We associate to $\mathcal{I}$ the ideal $\tilde{\mathcal{I}}$ represented in Figure 12.2; that means $\tilde{\mathcal{I}}$ arises from $\mathcal{I}$ by shifting $yH^0(\mathcal{K}(n-1))$ into $yH^0(\mathcal{K}(n-1))$, where the Hilbert functions of $\mathcal{K}$ and $\mathcal{K}$ agree and hence $\mathcal{I}$ and $\tilde{\mathcal{I}}$ have the same Hilbert function $\varphi$. Then $\tilde{\mathcal{J}}$ is defined by $H^0(\tilde{\mathcal{J}}(n)) = \bigoplus_{i=\nu}^{n} t^{n-i} H^0(\mathcal{I}(i))$ and $H^0(\tilde{\mathcal{J}}(n)) = (0)$, if $n < \nu$. Then $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{J}}$ fulfils the same assumptions as $\mathcal{I}$ and $\mathcal{J}$ do, and $g(\mathcal{I}) = g(\tilde{\mathcal{I}})$. Thus we can assume without restriction that $\mathcal{I}$ has the shape as represented by Fig. 12.2. Then one makes the graded deformations $\Box \ast \rightarrow \bigoplus (\Box \ast \rightarrow \bigoplus)$, etc. in $E(\mathcal{J})$. Each of the orbits which are to be exchanged, have the same length. One gets an ideal $\tilde{\mathcal{J}}$ with the same Hilbert polynomial, which is again of proper type 3. If $\tilde{\varphi}$ is the Hilbert function of $\tilde{\mathcal{I}} := (\tilde{\mathcal{J}})'$, then $\tilde{\varphi} > \varphi$, hence $g^\ast(\tilde{\varphi}) > g^\ast(\varphi) > g(\varphi)$ (cf. Remark 2.1). As $\mathcal{J}$ and $\tilde{\mathcal{J}}$ have the same Hilbert polynomial, one has $g(\tilde{\mathcal{J}}) = g(\mathcal{J})$. The colength of $\tilde{\mathcal{I}}$ in $\mathcal{O}_{\mathcal{P}^2}$ is the same as the colength $d$ of $\mathcal{I}$ in $\mathcal{O}_{\mathcal{P}^2}$, hence the coefficient $a$, remains unchanged. Now one can again pass to $(\tilde{\mathcal{J}})^*$ and has again $x^\nu \in H^0((\tilde{\mathcal{J}})^*(\nu))$, $\nu = \min \{n | H^0((\tilde{\mathcal{J}})^*(n)) \neq (0)\}$, $(\tilde{\mathcal{J}})^*$ of proper type 3. Thus it suffices to show $g((\tilde{\mathcal{J}})^*) < 0$. Continuing in this way one sees that one can assume without restriction: $\mathcal{J}$ is of proper type 3, $\mathcal{I} = \mathcal{J}'$ has the shape of Figure 12.4, $H^0(\mathcal{J}(n)) = \bigoplus_{i=\nu}^{n} t^{n-i} H^0(\mathcal{I}(i))$, for all $n \geq \nu$, $H^0(\mathcal{J}(n)) = (0)$, if $n < \nu$ and $x^\nu \in H^0(\mathcal{J}(\nu))$, i.e., $x^\nu \in H^0(\mathcal{I}(\nu))$. From (12.1) it follows that $\nu \geq m$. As $m \geq c + 2$, we have $h^0(\mathcal{K}(n)) = \binom{n+2}{2} - c, n \geq m - 1$, hence $h^0(\mathcal{I}(n)) = h^0(\mathcal{K}(n-1)) + (n-m+1) = \binom{n+1}{2} - c, n \geq m - 1$.}

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$$(n-1+2) - c + (n-m+1) = (\binom{n-1+2}{2}) + (\binom{n-1+1}{1}) + 1 - (c+m) = (\binom{n+2}{2}) - d, n \geq m.$$ But then

$$h^0(J(n)) = \sum_{i=0}^{n} h^0(I(i)) = \sum_{i=0}^{n} \binom{i+2}{2} - d$$

$$= \sum_{i=0}^{n} \binom{i+2}{2} - \sum_{i=0}^{\nu-1} \binom{i+2}{2} - (n-\nu+1)d$$

$$= \binom{n+3}{3} - \binom{\nu+2}{3} - (n-\nu+1)d.$$

From this we get $P(n) = (n-\nu+1)d + (\binom{\nu+2}{3})$, thus:

(12.2) $$g(J) = (\nu-1)d - (\binom{\nu+2}{3}) + 1.$$ We regard $g(J)$ as a function of $\nu \geq m$, and we have to determine the maximum of

$$g(x) := (x-1)d - \frac{1}{6}x^3 - \frac{1}{2}x^2 - \frac{1}{3}x + 1, \quad x \geq m.$$ We have $g'(x) = -\frac{1}{2}x^2 - x + (d - \frac{1}{3}) = 0 \Leftrightarrow x = -1 \pm \sqrt{2d+\frac{1}{3}}$, and we show $m \geq -1 + \sqrt{2d+\frac{1}{3}}$. This last inequality is equivalent to $(m+1)^2 > 2d + \frac{1}{3} \Leftrightarrow m^2 + \frac{2}{3} \geq 2c$ (because of $d = c + m$). As $m \geq c + 2$, this is true.

It follows that $g'(x) \leq 0$, if $x \geq m$, hence $g(x)$ is monotone decreasing if $x \geq m$. Now

$$g(m) = (m-1)d - \frac{1}{6}m^3 - \frac{1}{2}m^2 - \frac{1}{3}m + 1$$

$$= (m-1)(c+m) - \frac{1}{6}m^3 - \frac{1}{2}m^2 - \frac{1}{3}m + 1$$

$$\leq (m-1)(2m-2) - \frac{1}{6}m^3 - \frac{1}{2}m^2 - \frac{1}{3}m + 1.$$ The right side of this inequality is smaller than 0 if and only if $18 < m^3 - 9m^2 + 26m$, which is true as $m \geq c + 2 \geq 2$. 

12.4. $B(4;k)$-invariant surfaces containing a 1-cycle of proper type 3.

Let be $V \subset Z = Z(H_Q)$ a $B(4;k)$-invariant surface containing a 1-cycle $D$ of proper type 3. Then $V$ is not pointwise invariant under the $G_a$-operation $\psi_a : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z, t \mapsto t$. According to Lemma 11.1 one can write $V = G_a \cdot G_m \cdot \zeta$. The inertia group $\Delta_\zeta$ has the form $G(p)$ and by Lemma 12.1 it follows that $p = (a : b : c), a, b, c \neq 0$, is not possible.

12.4.1. The case $p = (a : 0 : c), a, c \neq 0$. Then $V$ is not pointwise invariant under the $G_m$-operation $\sigma$ (notations as in Lemma 11.1), as the following argumentation will show: Let $J \leftrightarrow \zeta$ be the corresponding ideal, let $P$ be an associated prime ideal, which is $G(p)$-invariant. If $t \in P$, then from (Appendix D, Lemma 2) it follows that $P = (x, y, z, t)$, contradiction. Thus $t$ is a non zero divisor of $O_{x^1}/\mathcal{J}$. If $\mathcal{J}$ would be invariant under $\sigma$, then $\mathcal{J}$ would be generated by elements of the form $f \cdot t^n$, where $f \in S_m, m, n \in \mathbb{N}$. It
follows that $f \in \mathcal{J}$ and thus $\mathcal{J}$ is invariant under $\Gamma$ (cf. 12.3.1). But by assumption
$\Delta \psi = G(a : 0 : c) \not\supset \Gamma$.

The inertia group $T_\lambda \subset T(4; k)$ has the form $T(\rho)$, where $\rho_3 > 0$ (Lemma 11.1 a). By
Appendix E $H^0(\mathcal{J}(b))$ has a standard basis $f_i = M_i p_i (X^\nu)$.

Let be $W := \mathbb{C}_a \cdot \mathbb{C}_m \cdot \zeta$. The morphism $h$ is defined on $V \cap U_t \supset W$. As $\overline{V} = V$, it
follows that $h(W) = \{ \psi_{\alpha}(z) | \alpha \in k \}$ is dense in $h(V \cap U_t)$, where $\zeta' = h(\zeta) \leftrightarrow \mathcal{J}'$. As
$\zeta \in U_t$, it follows that $\zeta_0 = \lim_{\lambda \to 0} \sigma(\lambda) \zeta \in U_t$ too (see [G87], Lemma 4, p. 542, and [G89],
1.6, p.14). As $\rho_3 > 0$ it follows that $\zeta'_0 = \zeta'$. Now write $D = \{ \psi_{\alpha}(\eta) | \alpha \in k \}$, where
$\eta \in V(k)$ is invariant under $T(4; k)$ and $G_3$, hence $C \subset V \cap U_t$ and $h(\eta) \in \{ \psi_{\alpha}(z') | \alpha \in k \}$.
As $\eta$ is not $\mathbb{G}_a$-invariant, $h(\eta)$ is not $\mathbb{G}_a$-invariant, hence $h(\eta) \in \{ \psi_{\alpha}(z') | \alpha \in k \}$. Then

(Appendix D, Lemma 3) shows $h(\eta) = \zeta' = \zeta'_0$. $H := \begin{pmatrix} 1 & 0 & \ast & \ast \\ 0 & 1 & \ast & \ast \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} < G(p)$ is
normalized by $\sigma$, hence $\zeta_0$ is $H$-invariant. As $\zeta_0 \in U_t$ is $\mathbb{G}_m$-invariant, it follows that $\zeta_0$
is $\Gamma$-invariant. But then $\zeta_0$ is invariant under $G_3$, and as $\zeta'_0 = \eta'$ corresponds to an ideal in
$y$-standard form, $\mathcal{J}_0 \leftrightarrow \zeta_0$ is of proper type 3.

As $G(p)$ is unipotent there is an eigenvector $f \neq 0$ in $H^0(\mathcal{J}(\nu)), \nu := \min \{ n | H^0(\mathcal{J}(\nu)) \neq (0) \}$. From $x \partial f / \partial t \in \langle f \rangle$ it follows that $\partial f / \partial t = 0$. From $y \partial f / \partial z \in \langle f \rangle$ it follows
$\partial f / \partial z = 0$ and from $c x \partial f / \partial \eta - a z \partial f / \partial t \in \langle f \rangle$ we deduce $f = x^\nu$ (cf. Appendix D,
Lemma 2). But then $x^\nu \in \mathcal{J}_0$, too. Now $h^0(\mathcal{J}_0(n)) = h^0(\mathcal{J}(n)), n \in \mathbb{Z}$ (cf. [G87], Lemma 4, p.542 and [G89], 1.6, p.14). But then from Lemma 12.2 it follows that $g < 0$.

**Conclusion 12.1.** In the case $p = (a : 0 : c), a, c \neq 0$, $V$ does not contain a 1-cycle of
proper type 3, if $g > g(d)$ is supposed.

12.4.2. The case $p = (a : b : 0), a, b \neq 0$. As $T_\lambda \subset T(1,-2,1,0)$ (cf. 12.1.4, 3rd
case) it follows from Lemma 11.1 that $V$ is pointwise invariant under $\sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto \lambda t$. As $V$ is pointwise invariant under $\Gamma$, one has $V \subset G_3$, where $\Phi$ is the
Hilbert function of $\mathcal{J} \leftrightarrow \zeta$ and $G_3$ is the corresponding “graded Hilbert scheme”. This
had been defined in ([G4], Abschnitt 2) as follows: $\mathbb{G}_m$ operates on $H^0 Q$ by $\sigma$. Then it is
shown in (loc. cit.) that $G := (H_Q)^{\mathbb{G}_m} \cap U_t$ is a closed subscheme of $H_Q$, and $G$ is a
disjunct union of closed subschemes $G_\Phi$, where $G_\Phi$ parametrizes the corresponding ideals
with Hilbert function $\Phi$.

Now suppose $D = \{ \psi_{\alpha}(\eta) | \alpha \in k \} \subset V$ is a 1-cycle of proper type 3, where $\eta$
corresponds to an ideal $\mathcal{L}$ of proper type 3. Then $\mathcal{L}$ and $\mathcal{J} \leftrightarrow \zeta$ have the same Hilbert
function.

Let $\nu$ and $f \in H^0(\mathcal{J}(\nu))$ be defined as in (12.4.1). From $x \partial f / \partial z \in \langle f \rangle$ and $y \partial f / \partial t \in \langle f \rangle$ it follows that $\partial f / \partial z = \partial f / \partial t = 0$. But then from $b x \partial f / \partial \eta - a y \partial f / \partial z \in \langle f \rangle$ it
follows that $\partial f / \partial y = 0$, hence $f = x^\nu$. (cf. Appendix D, Lemma 2). If $\mathcal{L}$ corresponds to
any point $\xi \in \mathbb{G}_a \cdot \mathbb{G}_m \cdot \zeta$, then $x^\nu \in H^0(\mathcal{I}(\nu))$, and the same argumentation as in (8.4.2)
shows this is true for all points in $V$. But from $x^\nu \in H^0(\mathcal{L}(\nu))$ it follows that $g < 0$.  

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Conclusion 12.2. In the case $p = (a : b : 0), a, b \neq 0$, $V$ does not contain a 1-cycle of proper type 3, if $g > g(d)$ is supposed.

12.4.3. If $V$ contains a 1-cycle of proper type 3, then $V$ cannot be pointwise $G_\alpha$-invariant, so the case $p = (0 : b : c)$ cannot occur, at all.

Lemma 12.3. Suppose $g > g(d)$. If $V \subset Z(H_Q)$ is a $B(4; k)$-invariant surface containing a 1-cycle of proper type 3, then $V$ is pointwise invariant under $G_3 = G(1 : 0 : 0)$.

Fig.: 12.1

![Diagram 12.1](image1.png)

$c$ monomials are missing

$x^m z^{n-m}$

Fig.: 12.2

![Diagram 12.2](image2.png)

A monomial diagram with a red line indicating a missing set of monomials.
Fig.: 12.3

Fig.: 12.4

$c$ monomials are missing
CHAPTER 13

Relations in $B(4; k)$-invariant surfaces

We suppose $g > g(d)$ and let $V \subset X = Z(H_Q)$ be a $B(4; k)$-invariant surface, which contains a $1$-cycle of proper type $3$. From Lemma 12.3 it follows that $V$ is pointwise $G_3$-invariant. Then from ([T1], Proposition 0, p.3) we conclude that any $B$-invariant $1$-prime cycle $D \subset V$ is either pointwise $\Delta$-invariant or a $1$-cycle of type $3$. The aim is to describe the relations in $Z_1^B(X)$ defined by the standard construction of Section 9.2 carried out with regard to $V$.

13.1. First case: $V$ is not pointwise invariant under the $G_m$-operation $\sigma$

We use the notation of (11.1). According to Lemma 11.1 $V = \overline{G_m \cdot G_x \cdot \zeta}$, where $T_\zeta = T(\rho)$ and $\rho_3 > 0$, for otherwise $\zeta$ would be $G_m$-invariant and hence $V$ would be pointwise $G_m$-invariant. Let $J \leftrightarrow \zeta$ and $C := \{\psi_\alpha(\zeta)|\alpha \in k\}^-$. If one chooses a standard basis of $H^0(J(b))$ consisting of $T(\rho)$-semi-invariants, then one sees that $h(V) = C' := \{\psi_\alpha(\zeta')|\alpha \in k\}^-$, where $\zeta' = h(\zeta)$. As $V$ contains a $1$-cycle of proper type $3$, $C'$ is a $y$-standard cycle, generated by $\zeta' \leftrightarrow J'$.

Now if $D \subset V$ is a $1$-cycle of proper type $3$, then $h(D) = C'$. If $D$ is of type $3$, but not of proper type $3$, then $h(D)$ is not a $y$-standard cycle. Write $D = \overline{G_x \cdot \eta, \eta \in V(k)}$ invariant under $T(4; k)$. It follows that $\eta' = h(\eta)$ is one of the two $T(3; k)$-fixed points of $C'$ (cf. Appendix D, Lemma 3). If $\eta' = \zeta'$, then $D$ would be of proper type $3$. Hence $\eta'$ is the unique point of $C'$, invariant under $B(3; k)$.

Let $C_\infty = \lim_{\lambda \to \infty} \sigma(\lambda)C$ be the $B$-invariant limit curves of $C$ coming from the standard construction. We write in $Z_1^B(V)$:

$$C_0 = \sum m_i A_i + Z_0 \quad \text{and} \quad C_\infty = \sum n_j B_j + Z_\infty$$

where $A_i$ and $B_j$ are the $1$-cycles of proper type $3$, occurring in $C_0$ and $C_\infty$, respectively, and $m_i, n_j \in \mathbb{N} - (0)$. All the other prime cycles occurring in $C_0$ and $C_\infty$ are summed up in $Z_0$ and $Z_\infty$, respectively.

Let $L_n$ be a tautological line bundle on $H_Q$. Then $(L_n \cdot A_i) = \delta_n + a_i$, $(L_n \cdot B_j) = \delta_n + b_j$. Here $\delta \in \mathbb{N} - (0)$ is independent of $i$ and $j$, as $h(A_i) = h(B_j) = C'$ for all $i, j$, whereas the constants $a_i$ and $b_j$ still depend on $A_i$ and $B_j$, respectively. From $[C_0] = [C_\infty]$ it follows that $\sum m_i (\delta_n + a_i) = \sum n_j (\delta_n + b_j)$ for all $n \gg 0$, hence $\sum m_i = \sum n_j$. Therefore we can write

$$C_0 - C_\infty = \sum_k (E_k - F_k) + \sum_k G_k$$

(13.1)
where \((E_1, E_2, \ldots) := (A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots)\) and \((F_2, F_2, \ldots) := (B_1, \ldots, B_1, B_2, \ldots, B_2, \ldots)\). Here \(A_1\) (respectively \(B_1\)) are to occur \(m_1\)-times (respectively \(n_1\)-times) etc. By the way, the arbitrary association \(E_k \mapsto F_k\) is possible because of \(\sum m_i = \sum n_j\). If \(E_k = F_k\), the summand \(E_k - F_k\) is left out. \(G_k\) is composed of either pointwise \(U(4; k)\)-invariant curves or 1-cycles of type 3, which are not proper.

13.2. Second case: \(V\) is pointwise invariant under the \(\mathbb{G}_m\)-operation \(\sigma\), but not pointwise invariant under the \(\mathbb{G}_m\)-operation \(\tau\)

We use the same notations as in (11.3).

1st subcase: \(h(V)\) is not 2-dimensional. By assumption \(V\) contains a 1-cycle \(D\) of proper type 3, hence \(h(V) = h(D) := D'\) is a \(y\)-standard cycle and all the other 1-cycles in \(V\), which are of proper type 3, are mapped by \(h\) onto \(D'\). Then one carries out the standard construction by means of the operation \(\tau\) and one gets formally the same relations as (13.1).

2nd subcase: \(h(V) = V' \subset H^d\) is a \((3; k)\)-invariant surface, which is pointwise invariant under \(G'_3 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset U(3; k)\). As \(V'\) contains a \(y\)-standard cycle, \(V'\) is not pointwise \(\mathbb{G}_a\)-invariant.

At first, the standard construction is carried out in \(V = \overline{\mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta}\) (cf. Lemma 11.1): \(C := \{\psi_\alpha(\zeta) | \alpha \in k\}^-\) is a closed curve in \(V\) with Hilbert polynomial \(p\) and \(\lambda \mapsto \tau(\lambda)C\) defines a morphism \(\mathbb{G}_m \rightarrow \text{Hilb}^p(V)^{\mathbb{G}_a}\), whose extension to \(\mathbb{P}^1\) defines a flat family \(C \hookrightarrow V \times \mathbb{P}^1\) over \(\mathbb{P}^1\) such that \(C_\lambda := p^{-1}(\lambda) = \tau(\lambda)C \times \{\lambda\}\), if \(\lambda \in k^*\). \(C_{0/\infty} := p_1(C_{0/\infty})\) are called the limit curves of \(C\) and \([C_0] - [C_\infty]\) is the "standard relation" defined by \(V\).

Put \(U := \{\tau(\lambda)\psi_\alpha(\zeta) | \alpha \in k, \lambda \in k^*\}\).

Put \(U' := \{\tau(\lambda)\psi_\alpha(\zeta') | \alpha \in k, \lambda \in k^*\}\).

Then \(\overline{U} = V\) and \(\overline{U'} = V'\). Carrying out the standard construction by means of \(C' := \{\psi_\alpha(\zeta') | \alpha \in k\}^-\) one gets a flat family \(C' \hookrightarrow V' \times \mathbb{P}^1 \xrightarrow{p_2} \mathbb{P}^1\). One has a morphism \(C \hookrightarrow V \times \mathbb{P}^1 \xrightarrow{h \times \text{id}} V' \times \mathbb{P}^1\), which is denoted by \(f\).

Put \(U := \{\tau(\lambda)\psi_\alpha(\zeta), \lambda) | \alpha \in k, \lambda \in k^*\} \subset C\).

Put \(U' := \{\tau(\lambda)\psi_\alpha(\zeta'), \lambda) | \alpha \in k, \lambda \in k^*\} \subset C'\).

\(C\) and \(C'\) are reduced and irreducible (see [T1], proof of Lemma 1, p.6). Hence \(\overline{U} = C\) and \(\overline{U'} = C'\). As \(f(U) = U'\) and \(f\) is projective, \(f(C) = C'\) follows. As the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
p_2 \searrow & & \swarrow p_2 \\
& \mathbb{P}^1 & \\
\end{array}
\]
is commutative, \( f(C_0) = C'_0 \) and \( f(C_\infty) = C'_\infty \) follow. As the diagram

\[ V \times \mathbb{P}^1 \xrightarrow{h \times id} V' \times \mathbb{P}^1 \]

\[ p_1 \downarrow \quad \downarrow p_1 \]

\[ V \xrightarrow{h} V' \]

is commutative, it follows that \( C'_{0/\infty} := p_1(C'_{0/\infty}) = p_1f(C_{0/\infty}) = h(C_{0/\infty}) \). Let be \( \zeta_{0/\infty} := \lim_{\lambda \to 0/\infty} \tau(\lambda)\zeta, \zeta'_{0/\infty} := \lim_{\lambda \to 0/\infty} \tau(\lambda)\zeta' \).

Now from Lemma 9.2, it follows that \( C'_{0/\infty} := \{\psi_\alpha(\zeta'_{0/\infty})|\alpha \in k\}^- \) are the only \( y \)-standard cycles in \( C'_{0/\infty} \) and they both occur with multiplicity 1. We want to show, that \( C_{0/\infty} := \{\psi_\alpha(\zeta_{0/\infty})|\alpha \in k\}^- \) are the only 1-cycles of proper type 3 in \( C_{0/\infty} \), and they both occur with multiplicity 1. In order to show this, we consider the extension of \( \tau : \lambda \mapsto \tau(\lambda)\zeta \) to a morphism \( \mathcal{T} : \mathbb{A}^1 \to V \). Then \( \zeta_0 = \mathcal{T}(0) \). As there is a commutative diagram

\[ \mathbb{G}_m \xrightarrow{\tau} V \]

\[ \tau' \downarrow \quad \downarrow h \]

\[ V' \]

where \( \tau'(\lambda) := \tau(\lambda)\zeta' \) and as the extensions are determined uniquely, it follows that \( \mathcal{T}' = h \circ \mathcal{T} \), hence \( h(\zeta_0) = \zeta'_0 \). By constructing the corresponding extensions to \( \mathbb{P}^1 \), it follows in the same way that \( h(\zeta_\infty) = \zeta'_{\infty} \). Hence we get \( h(C_{0/\infty}) = C'_{0/\infty} \). Therefore \( C_0 \) and \( C_\infty \) are 1-cycles of proper type 3, and from \( \zeta_{0/\infty} \in C_{0/\infty} \) we conclude \( C_{0/\infty} \subseteq C_{0/\infty} \).

Assume there is another 1-cycle \( D \) of proper type 3 contained in \( C_0 \), or assume that \( C_0 \) occurs with multiplicity \( \geq 2 \) in \( C_0 \). Then there is an equation \( C_0 = [C_0] + [D] + \cdots \) in \( Z_0^1(V) \), where \( D = C_0 \), if \( C_0 \) occurs with multiplicity \( \geq 2 \). From Lemma 9.2 it follows that \( h(D) = C'_0 \). It follows that

\[
\begin{align*}
  h_*(\cdot C_0) & = h_*(\cdot C_0) + h_*(\cdot D) + \cdots \\
  & = \deg(h|C_0)[h(C_0)] + \deg(h|D)[h(D)] + \cdots \\
  & = h_*(\cdot C') = \deg(h|C)[C'].
\end{align*}
\]

As \( C = \{\psi_\alpha(\zeta)|\alpha \in k\}^- \) and \( \alpha \mapsto \psi_\alpha(\zeta) \) is injective and the same is true for \( C' \) and \( \alpha \mapsto \psi_\alpha(\zeta') \), \( h|C \) is an isomorphism. The same argumentation shows that \( h|C_0 \) and \( h|C_\infty \) are isomorphisms. Hence we get \( [C'_0] + [C'_0] + \cdots = [C'] = [C'_0] \), which means that \( C'_0 \) occurs with multiplicity \( \geq 2 \) in \( C'_0 \), contradiction.

The same argumentation shows that \( C_\infty \) is the only 1-cycle of proper type 3 in \( C_\infty \), and it occurs with multiplicity 1. This gives a relation of the form

\[
(13.2) \quad C_0 - C_\infty = C_0 - C_\infty - Z
\]

where \( C_0 \) and \( C_\infty \) are 1-cycles of proper type 3 and \( Z \) is a 1-cycle whose components are either pointwise \( U(4;k) \)-invariant curves or 1-cycles of type 3, which are not proper.
13.3. Third case: $V$ is pointwise invariant under $\sigma$ and $\tau$

Then we write $V = \mathbb{G}_m \cdot \mathbb{G}_a \cdot \zeta$ as in the 3rd case of Lemma 11.1.

1st subcase: $h(V)$ is not 2-dimensional. The same argumentation as in the first subcase of (13.2), using the $\mathbb{G}_m$-operation $\omega$ as in the third case of Lemma 11.1 instead of the $\mathbb{G}_m$-operation $\tau$, gives relations of the form (13.1).

2nd subcase: If $h(V)$ is 2-dimensional, the same argumentation as in the second subcase of (13.2), with $\omega$ instead of $\tau$, gives relations of the form (13.2).

**Proposition 13.1.** Assume $g > g(d)$ and let $V \subset X := Z(\mathbb{H}_Q)$ be a $B(4; k)$-invariant surface containing a 1-cycle of proper type 3. Then each relation in $Z_1^B(X)$ defined by $V$, which contains a 1-cycle of proper type 3 is a sum of relations of the form $C_1 - C_2 + Z$. Here $C_1$ and $C_2$ are 1-cycles of proper type 3, and $Z$ is a 1-cycle whose prime components either are pointwise $\Delta$-invariant or of proper type 3.

**Proof.** This follows from the foregoing discussion. \hfill \Box
CHAPTER 14

Necessary and sufficient conditions

We take up the notations introduced in Chapter 1. Let be \(d \geq 5\), \(g(d) = \frac{(d-2)^2}{4}\), \(H = H_{d,g}\), \(A(H) = \text{Im}(A_1(H^{\nu(4k)}) \rightarrow A_1(H))\). Obviously, the cycle \(C_3 = \{(a^2, \alpha x + y, x^{a-1}z^{b-a+1}) | \alpha \in k\}\), where \(d = a-1\), \(g = \frac{(a^2 - 3a + 4)}{2} - b\), is a 1-cycle of proper type 3.

14.1. The necessary condition.

In the proof of Theorem 1.1 in Chapter 10 we replace \(H, U(3; k), B(3; k)\) and “y-standard cycle” by \(H, U(4; k), B(4; k)\) and “1-cycle of proper type 3”, respectively. Then using Proposition 13.1 instead of Proposition 9.1, the same reasoning as in the proof of Theorem 1.1 gives the

Conclusion 14.1. If \(d \geq 5\) and \(g > g(d)\), then \([C_3] \notin A(H)\). \(\square\)

14.2. The sufficient condition.

14.2.1. The case \(d \geq 5\) and \(d\) odd. In ([T2], 3.3.2, Folgerung, p.129) it had been shown \([C_3] \in A(H)\), if \(a \geq 6\) and \(b \geq a^2/4\). We express this condition by means of the formulas in (1.1):

\[
b \geq a^2/4 \iff g \leq \frac{1}{2}[(d+1)^2 - 3(d+1) + 4] - \frac{1}{4}(d+1)^2 = \frac{1}{4}(d^2 - 4d + 3).
\]

As \(d\) is odd, this is equivalent to \(g \leq g(d) = \frac{1}{4}(d-2)^2\).

Conclusion 14.2. If \(d \geq 5\), \(d\) is odd and \(g \leq g(d)\), then \([C_3] \in A(H_{d,g})\). \(\square\)

14.2.2. The case \(d \geq 6\) and \(d\) even. We will show that the bound given in ([T2], 3.3.3 Folgerung, p.129) is already valid, if \(a \geq 7\).

1° We consider the Hilbert function \(\varphi\) of the monomial ideal \((x^2, xy^{e-2}, y^e), e := d/2 + 1 \geq 4\), which is represented in Fig. 14.1. In Section (2.2.3) this Hilbert function had been denoted by \(\chi\) and it had been shown that \(g'(\varphi) = \frac{1}{4}(d-2)^2 =: g(d)\). The figures 14.1 - 14.6 show all possible monomial ideals with Hilbert function \(\varphi\). Besides this, we consider the ideal \(\mathcal{I} := (xy, x^e, y^e, x^{e-1} + y^{e-1})\) which corresponds to a closed point \(\xi\) in the Iarrobino variety \(I_{\varphi}\). This variety parametrizes all sequences \((U_0, U_1, \ldots)\) of subspaces \(U_i \subset R_i\) with dimension \(\varphi'(i) = \varphi(i) - \varphi(i-1)\), such that \(R_1 U_i \subset U_{i+1}, i \in \mathbb{N}\). Here \(R\) is the polynomial ring in two variables.

2° We show that \(V = \mathbb{G}_a \times \mathbb{G}_m \cdot \xi\) is 2-dimensional, where \(\mathbb{G}_m\) operates by \(\tau(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto z\). If this would not be the case, then \(\mathbb{G}_a \times \mathbb{G}_m \rightarrow I_{\varphi}\) defined by
$(\alpha, \lambda) \mapsto \psi_\alpha \tau(\lambda) \xi$ would have a fibre with infinitely many points. The argumentation in (8.3) then showed that one of the points $\xi_{0/\infty} = \lim_{\lambda \to 0/\infty} \tau(\lambda) \xi$ is invariant under $\mathbb{G}_a$. As $\xi_0 \mapsto I_0 = (xy, x^e, y^{e-1})$ and $\xi_\infty \mapsto I_\infty = (xy, x^{e-1}, y^e)$, this is wrong as $e \geq 4$. Thus we can carry out the standard construction with $C = \{ \psi_\alpha(\xi)|\alpha \in k \}^-$, if $e \geq 4$. As we have already noted in the proof of Lemma 9.1, the curves $\tau(\lambda)C$ are $\mathbb{G}_a$-invariant, hence the construction of the family $C$ takes place in $Hilb^p(I_{\varphi}^{G_a})$. Therefore the limit curves $C_{0/\infty}$ are invariant under $B(3;k)$. 

As usual, we put $C_{0/\infty} = \{ \psi_\alpha(\xi_{0/\infty})|\alpha \in k \}^-$. 

3° Let $I \mapsto (U_0, U_1, \cdots)$, where $U_i = xyR_{i-2}$, if $0 \leq i \leq e-2, U_{e-1} = xyR_{e-3} + (x^{e-1} + y^{e-1})$, $U_i = R_i, i \geq e$. We get

$$
\psi_\alpha(U_{e-1}) = x(\alpha x + y)R_{e-3} + (x^{e-1} + (\alpha x + y)(\alpha x + y)^{e-2}) = x(\alpha x + y)R_{e-3} + (x^{e-1} + (\alpha x + y)y^{e-2})$$

$$\therefore \hat{\psi}_\alpha(U_{e-1}) = \alpha x^{e-1} \land \alpha x^{e-2} y \land \cdots \land \alpha x^2 y^{e-3} \land \alpha xy^{e-2} + \text{terms with smaller powers of } \alpha.
$$

Hence $\alpha$-grade $(U_{e-1}) = e - 1$. On the other hand, by formula (2) in (4.1) we get $\alpha$-grade $(x^{e-2}y, \ldots, y^{e-1})) = e - 1$, too. From this it follows that $C_0 = C_0$. 

4° Besides $C_\infty$, the limit cycle $C_\infty$ contains other prime components $D_1, D_2, \ldots$ which are $B(3;k)$-invariant curves in $I_\varphi$.

As $\text{char}(k) = 0$ is supposed, if $m \leq n, G := \text{Grass}_m(R_n)$ has only one $\mathbb{G}_a$-fixed point, namely the subspace $\langle x^n, \cdots, x^{n-m+1}y^{m-1} \rangle$. Then ([T1], Proposition 0, p.3) shows that each $B(2;k)$-invariant curve in $G$ has the form $G_a \cdot \eta$, where $\eta \in G(k)$ corresponds to a monomial subspace of $R_n$. It follows that each $B(3;k)$-invariant curve in $I_\varphi$ also has the form $G_a \cdot \eta$, where $\eta \in I_\varphi(k)$ corresponds to a monomial ideal.

5° Figure 14.1 - Figure 14.6 show all possible monomial ideals with Hilbert function $\varphi$, and using formula (4.2) we compute the degrees of the corresponding 1-cycles:

$$\deg c_2 = (\binom{e-2}{2}), \deg c_3 = (\binom{e-2}{2}) + (e - 1), \deg c_4 = 2(\binom{e-2}{2}) + (e - 1), \deg c_5 = 2(\binom{e-2}{2}) + (e - 2), \deg c_6 = 1. \n$$

We see that $c_2$ (respectively $c_3$) is equal to $C_\infty$ (respectively $C_0$). If $C_0$ (respectively $D_i$) occurs in $C_\infty$ with the multiplicity $n$ (respectively $n_i$), then from $\deg C_\infty = \deg C_0 = \deg C_0$ it follows that $(\binom{e-2}{2}) + (e - 1) = n(\binom{e-2}{2}) + n_i \deg D_1 + \cdots$ where $n \geq 0, n_i \geq 0$.

1st case: $e \geq 6$. Then $(\binom{e-2}{2}) > e - 1$, hence $n = 1$. From $D_1 \in \{c_4, c_5, c_6\}$ and $e - 1 \geq n_1 \deg D_1$ we conclude that $D_1 = c_6$ and

$$[C_0] = [C_\infty] + (e - 1)[c_6].$$

2nd case: $e = 5$. Then $\deg C_0 = 7, \deg C_\infty = 3, \deg c_4 = 10, \deg c_5 = 9$, and we get $[C_0] = 2[C_\infty] + [c_6]$ or $[C_0] = [C_\infty] + 4[c_6]$.

3rd case: $e = 4$. Then $\deg C_0 = 4, \deg C_\infty = 1, \deg c_4 = 5, \deg c_5 = 4$. Therefore one gets

$$[C_0] = n[C_\infty] + m[c_6]$$

where $n, m \in \mathbb{N}, n \geq 1, n + m = 4$.

6° One has a closed immersion $I_\varphi \to H_{d,g(d)}$ defined by $I \mapsto I^*$, where $I^*$ is the ideal.
generated by $\mathcal{I}$ in $\mathcal{O}_{\mathbb{P}^3}$. That means $H^0(\mathbb{P}^3; \mathcal{I}^*(n)) = \bigoplus_{i=0}^{n} t^{n-i} H^0(\mathbb{P}^2; \mathcal{I}(i)), n \in \mathbb{N}$. In any case, one gets an equation

$$[C_0^*] = n[C_\infty^*] + m[c_6^*], \quad m, n \in \mathbb{N}, n \geq 1.$$

Now from ([T1], Lemma 5, p.45) it follows that one can deform $C_0^*$ (respectively $C_\infty^*$) by a sequence of graded deformations of type 3 (cf. [T1], p.44) modulo $\mathcal{A}(\mathcal{H})$ in the cycle $C_3$ (respectively in the zero cycle). The cycle $c_6^*$ is equal to the cycle $D_\alpha, \alpha = (d + 2)/2 = e$, which had been introduced in ([T4], p.20f). By ([T4], Lemma 5, p.25) one gets $D_e \equiv D_2$ modulo $\mathcal{A}(\mathcal{H})$, and in ([T4], Abschnitt 3.3, p.25) it had been noted that $[D_2] = [D]$, where $D$ is the cycle introduced in (1.1). From ([T4], Satz 1, p.26) it follows $[C_3] \in \mathcal{A}(\mathcal{H}_{d,g})$.

Applying the shifting morphism (cf. [T3], Folgerung, p.55 and Proposition, p.56) we get:

**Conclusion 14.3.** If $d \geq 6$ is even and $g \leq g(d)$, then $[C_3] \in \mathcal{A}(\mathcal{H}_{d,g})$.

**14.3. Summary.**

**Theorem 14.1.** Let be $d \geq 5$. Then $[C_3] \in \mathcal{A}(\mathcal{H}_{d,g})$ if and only if $g \leq g(d)$.

**Proof.** This follows from Conclusion 14.1- 14.3.
We suppose \( d \geq 5 \), i.e. \( a \geq 6 \), and \( g < \binom{d-1}{2} \), i.e. \( g \) not maximal.

From ([T1], Satz 2, p.91) and ([T4], Proposition 2, p.26) it follows that \( A_1(H_{d,g})/\mathcal{A}(H_{d,g}) \) is generated by the so-called combinatorial cycles \( C_1, C_2, C_3 \). Using the formulas in (1.1), one shows that \( g \leq \gamma(d) := \binom{d-2}{2} \) is equivalent to \( b \geq 2a - 4 \).

1st case: \( g > \gamma(d) \). This is equivalent to \( b < 2a - 4 \). By ([T4], Satz 1, p.26) one has \( \mathcal{A}(H_{d,g}) = \langle E \rangle \). Assume there is a relation \( q_0[E] + q_1[C_1] + q_2[C_2] + q_3[C_3] = 0, q_i \in \mathbb{Q} \).

Computing the intersection numbers with the tautological line bundles \( L_n \) by means of the formulas (1)-(5) in ([T2], p.134f) and the formula in ([T3], Hilfssatz 1, p.50), one sees that \( q_2 = 0 \). Put \( r := 2a - 4 - b \) and denote the \( r \)-times applied shifting morphism by \( f \) (see [T3], p.52 ). The images of \( E, C_1, C_3 \) under \( f \) in \( H_{d,\gamma(d)} \) are denoted by \( e, c_1, c_3 \). By ([T2], 3.2.2 Folgerung, p.124) and ([T3], Anhang 2, p.54f) it follows that \( \langle f(E) \rangle = \langle e \rangle \), \( \langle f(C_1) \rangle = \langle c_1 \rangle \) and \( \langle f(C_3) \rangle = \langle c_3 \rangle \) modulo \( \mathcal{A}(H_{d,\gamma(d)}) \). By ([T3], Proposition 4, p. 22) \( c_1 \notin \mathcal{A}(H_{d,\gamma(d)}) \). As \( \gamma(d) > g(d) \) if \( d \geq 5 \), from Theorem 14.1 it follows that \( C_3 \notin \mathcal{A}(H_{d,\gamma(d)}) \). Hence \( f(C_3) \) is not a single point, hence \( \text{deg}(f)C_3 \neq 0 \). Applying \( f \) to the relation \( q_0[E] + q_1[C_1] + q_3[C_3] = 0 \) then gives \( q_3 \text{deg}(f)C_3 \cdot C_3 \in \mathcal{A}(H_{d,\gamma(d)}) \). If \( q_3 \neq 0 \) it would follow that \( [c_3] \in \mathcal{A}(H_{d,\gamma(d)}) \), contradiction. But from \( q_0[E] + q_1[C_1] = 0 \) it follows \( q_0 = q_1 = 0 \) (cf. [T2], 4.1.3).

2nd case: \( g < g \leq \gamma(d) \). Then \( b \geq 2a - 4 \) and in (1.1) it was explained that in this case \( A_1(H_{d,g}) \) is generated by \( E, D, C_2 \) and \( C_3 \). If \( q_0[E] + q_1[D] + q_2[C_2] + q_3[C_3] = 0 \), then \( q_2 = 0 \) by ([T2], loc.cit.). If one assumes that \( q_3 \neq 0 \), it follows that \( [C_3] \in \langle E, D \rangle \subset \mathcal{A}(H_{d,g}) \), contradicting Theorem 14.1. As in the first case we get \( q_0 = q_1 = 0 \).

3rd case: \( g \leq g(d) \). Then \( \mathcal{A}(H_{d,g}) = \langle E, D, C_1 \rangle \subset \mathcal{A}(H_{d,g}) \) and \( [C_3] \in \mathcal{A}(H_{d,g}) \) (see [T3], Proposition 4, p.22 ; [T4], Satz 1, p.26 ; and finally Theorem 14.1 ). Thus \( A_1(H_{d,g}) = \langle E, D, C_2 \rangle \).

All in all we get: Suppose that \( d \geq 5 \) and \( g < \binom{d-1}{2} \), i.e. \( g \) is not maximal. Put \( g(d) := (d - 2)^2/4 \) and \( \gamma(d) := \binom{d-2}{2} \).

**Theorem 15.1.**

(i) If \( g > \gamma(d) \), then \( A_1(H_{d,g}) \) is freely generated by \( E, C_1, C_2, C_3 \).

(ii) If \( g(d) < g \leq \gamma(d) \), then \( A_1(H_{d,g}) \) is freely generated by \( E, D, C_2, C_3 \).

(iii) If \( g \leq g(d) \), then \( A_1(H_{d,g}) \) is freely generated by \( E, D, C_2 \). \( \square \)

N.B.: If \( g = \binom{d-1}{2} \), then \( A_1(H_{d,g}) \) is freely generated by \( C_1 := \{(x, y^d(\alpha y + z))|\alpha \in k\}^- \) and \( C_2 := \{(\alpha x + y, x^{d+1})|\alpha \in k\}^- \) ([T1], Satz 2, p.91).
CHAPTER 16

The cases \( d = 3 \) and \( d = 4 \)

16.1. The case \( d = 3 \).

If \( g \) is not maximal, i.e., if \( g \leq 0 \), then \( Q(T) = (T^{-1+3}) + (T^{-4+2}) + (T^{-b+1}) \), where \( b \geq 4 \). If \( b = 4 \), then in ([T2], p.137) it had been shown that \( [C_3] \in \mathcal{A}(H) \). Applying the shifting morphism, (see [T3], p.54) then shows that this statement is true for all \( g < 0 \) (cf. [T3], Anhang 2, Folgerung, p.55 and Proposition, p.56). By ([T1], Satz, p.91, [T3], Prop. 4, p.22 and Satz 1, p.26) it follows that \( A_1(H,g) \) is generated by \([E],[D],[C_2]\), if \( g \leq 0 \). The three cycles are linear independent, as already was noted in Chapter 15.

16.2. The case \( d = 4 \)

If \( g \) is not maximal, i.e., if \( g < 3 \), then \( Q(T) = (T^{-1+3}) + (T^{-5+2}) + (T^{-b+1}) \) and \( b \geq 5 \). If \( b = 5 \), then \( g = 2 \), and then \( A_1(H) = Q^2 \), as had been shown in ([T3], Anhang 1).

We now treat the case \( b = 6 \), i.e., \( g = 1 \). As the condition \( b \geq 2a - 4 \) is fulfilled, \( \mathcal{A}(H) = \langle E,D \rangle \) by ([T4], Satz 1, p.26). The quotient \( A_1(H)/\mathcal{A}(H) \) is generated by \( C_1,C_2 \) and further cycles \( C_3,C_4,C_5 \) of type 3 (cf. [T1], Satz 2c, p.92). By ([T3], Proposition 4, p.22) \([C_1]\in\mathcal{A}(H)\), and we are going to simplify the reductions of \( C_3,C_4,C_5 \) described in ([T3], Abschnitt 8).

16.2.1. The cycle \( C_3 \). By definition a cycle of type 3 has the form \( C = \overline{G_a} \cdot \xi \), where \( \xi \) corresponds to a monomial ideal \( J \) with Hilbert polynomial \( Q \), which is invariant under the subgroup \( G < U(4;k) \) (cf. Chapter 1). Here \( G_a \) operates as usual by \( \psi_a : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z, t \mapsto t \). Let \( \mathcal{I} := \mathcal{I}' = H^4(k) \) be the image of \( \mathcal{J} \) under the restriction morphism. If \( \mathcal{I} \) is \( G_a \)-invariant, then \( \mathcal{I} \) is \( B(3;k) \)-invariant, hence \( \deg(L_a|C) \) is constant and \([C]\in\mathcal{A}(H)\) (cf. [T3], Anhang 2, Hilfssatz 2, p.50). Therefore we can assume without restriction that \( \mathcal{I} \) is not \( G_x \)-invariant. As the colength of \( \mathcal{I} \) in \( \mathcal{O}_{p^2} \) is equal to 4, the Hilbert function of \( \mathcal{I} \) is equal to one of the functions represented in Figure 2.7a and Figure 2.7b. In (2.2.1) we had obtained \( g^*(\varphi_1) = 1 \) and \( g^*(\varphi_2) = 3 \). The Hilbert function \( \varphi_1 \) leads to two possible 1-cycles of proper type 3, namely to

\[
F_1 := \{\psi_\alpha(\xi_1) | \alpha \in k\}^-, \quad \xi_1 \leftrightarrow (xy, y^2, x^3), \quad \text{and}
F_2 := \{\psi_\alpha(\xi_2) | \alpha \in k\}^-, \quad \xi_2 \leftrightarrow (x^2, y^2).
\]

The Hilbert function \( \varphi_2 \) leads to different 1-cycles of proper type 3, which however by means of admissible \( G_3 \)-invariant deformations all can be deformed modulo \( \mathcal{A}(H) \) in \( C_3 = \{\psi_\alpha(\xi_3) | \alpha \in k\}^-, \xi_3 \leftrightarrow (x^5, y, x^4z^2) \).
With such admissible $G_3$-invariant deformations we can deform $C_3$ in the cycle generated by $(x^4, xy, y^2, z^2)$, and afterwards this cycle can be deformed by the graded deformation $(yz^2, yz^2, \ldots) \mapsto (x^3, x^3z, \ldots)$ in the cycle $F_1$.

We deform $F_1$ into $F_2$ in the following way: Put $K := (x^3, xy, y^2, y^3)$. If $L \subset R_2 = k[x, y]$ is a 2-dimensional subspace, then $(x, y)L \subset H^0(K(3)) = R_3$ and $z^nL \cap H^0(K(n + 2)) = (0)$ for all $n \geq 0$. It follows that the ideal $(L, K) \subset \mathcal{O}_{\mathbb{P}^3}$ has the Hilbert polynomial $Q$, and $L \mapsto (L, K)$ defines a closed immersion $\mathbb{P}^2 \cong \text{Grass}_2(R_2) \to H_q$. Let $\langle x, y^3 \rangle \leftrightarrow \eta_1 \in \mathbb{P}^1, \langle x^2, y^3 \rangle \leftrightarrow \eta_2 \in \mathbb{P}^2, \ell_i := \{\psi_\alpha(\eta_i) | \alpha \in k\}^\sim, i = 1, 2$. Because of reasons of degree one has $\{\ell_1\} = 2[\ell_2]$ in $A_1(\mathbb{P}^2)$, hence $[F_1] = 2[F_2]$ in $A_1(H_q)$. Now $F_2 = \{(x^2, xy, y^3, (\alpha x + y)^2 | \alpha \in k\}^\sim = \{(x^2, xy^2, y^3, \lambda xy + \mu y^2) | (\lambda : \mu) \in \mathbb{P}^1\}$ is equal to the cycle $D_3$ which had been introduced in ([T4], Abschnitt 3.2, p.20). By Lemma 5 in (loc.cit.) $D_3 \equiv D_2$ modulo $\mathcal{A}(H)$, where $D_2 := \{(x^2, xy, y^4, \lambda xz^2 + \mu y^3) | (\lambda : \mu) \in \mathbb{P}^1\}$ is the cycle, which had been denoted by $D$ in Chapter 1. It follows that $[C_3] \in \mathcal{A}(H)$, and applying the shifting morphism gives

**Conclusion 16.1.** $[C_3] \in \mathcal{A}(H_{4g})$ for all $g \leq 1$. \hfill \Box

**Remark 16.1.** If $d = 3$ and $g > g(3) = 1/4, C_3$ does not occur at all. If $d = 4$ and $g > g(4) = 1$, then $\mathcal{A}(H_{4,2}) = \langle E \rangle$ ([T4], Satz 1, p.26) and hence $[C_3] \notin \mathcal{A}(H_{4.2})$.

16.2.2. $[C_4] \in \mathcal{A}(H)$ is true for arbitrary $d$ and $g$ ([T4], Proposition 2, p.26).

16.2.3. The cycle $C_5$. In (loc.cit.) $[C_5] \in \mathcal{A}(H)$ was shown, too, but the proof required tedious computations, which can be avoided by the following argumentation. One has only to treat the cases $g = 0$ and $g = -2$, that means, the cases $b = 7$ and $b = 9$ ([T1], Satz 2c(iii), p.92).

We start with $b = 7$. Then $C_5 = \overline{G_m} \cdot \xi_0$, where $\xi_0 \leftrightarrow J_0 := (y^2, K), K := (x^3, x^2y, xy^2, y^3, x^2z, y^2z)$. Put $J := (x^2 + y^2, K) \leftrightarrow \xi$. As $J_0$ and $J$ have the same Hilbert function, $\xi_0$ and $\xi$ both lie in the same graded Hilbert scheme $H_\varphi$ (cf. Appendix A). If $\mathbb{C}_m$ operates by $\tau(\lambda) : x \mapsto \lambda x, y \mapsto y, z \mapsto t, t \mapsto t$, then one sees that indeed $\xi_0 = \lim_{\lambda \to 0} \tau(\lambda)\xi$ and $\xi_{\infty} := \lim_{\lambda \to \infty} \tau(\lambda)\xi \leftrightarrow J_{\infty} := (x^2, K)$. Put $C := \{\psi_\alpha(\xi) | \alpha \in k\}^\sim, C_{0/\infty} := \{\psi_\alpha(\xi_0/\infty) | \alpha \in k\}^\sim$. One sees that $\alpha$-grade $H^0(J(n)) = \alpha$-grade $H^0(J_0(n)) = \alpha$-grade $H^0(J_{\infty}(n)) + 2$, for all $n \geq 2$. Hence $\deg C = \deg C_0 = \deg C_{\infty} + 2$, relative to an appropriate imbedding of $H_\varphi$ into $\mathbb{P}^N$.

Put $V := \overline{G_m} \cdot \xi_0$. As $\xi$ is invariant under $G := G_3 \cdot T_{23}$ and as $G_3$ and $T_{23} = \{(1, 1, 2, -3) | \lambda_2, \lambda_3 \in k^*\}$ are normalized by $\mathbb{C}_m \cdot \xi$, $V$ is pointwise $G$-invariant. Let $p$ be the Hilbert polynomial of $C \subset \mathbb{P}^N$. Then the standard construction by means of $V$ takes place in $\mathcal{X} := \text{Hilb}^p(V)^{\mathbb{C}_m}$, as $C$ is $\mathbb{C}_m$-invariant. Thus the limit curves $C_{0/\infty}$ are pointwise $G$-invariant and invariant under $\mathbb{C}_m$, hence they are $B(4, k)$-invariant. Now $C_{0/\infty} \subset C_{0/\infty}$ and the above computation of the degrees shows that $C_0 = C_{\infty}$, whereas $C_{\infty}$ except from $C_{\infty}$ has to contain a further 1-cycle $Z$ of degree 2, i.e., $C_{\infty}$ has to contain an irreducible curve $F$ of degree $\geq 1$, which is $B(4; k)$-invariant. As $V$ is pointwise $G$-invariant, $F$ is either pointwise $U(4; k)$-invariant or an 1-cycle of type 3. As
deg(Ł_n|F) is constant it follows that [F] ∈ 𝒜(𝐻) (see [T3], Anhang 2, Hilfssatz 2, p.50).
It follows that Z ∈ 𝒜(𝐻).

From \([C_5] = \[C_0] = \[C_∞] + Z\) and \(C_∞ = C_4\) (cf. [T1], Satz 2, p. 91), because of (16.2.2) it follows that \([C_5] ∈ 𝒜(𝐻)\).

It follows that \(Z ∈ 𝒜(𝐻)\).

From \([C_5] = \[C_0] = \[C_∞] + Z\) and \(C_∞ = C_4\) (cf. [T1], Satz 2, p. 91), because of (16.2.2) it follows that \([C_5] ∈ 𝒜(𝐻)\).

We now treat the case \(b = 9\). Here \(C_5 = G_3 \cdot η\), where \(η ↔ (x^3, x^2y, y^3, x^2z^3)\). By the \(G_3\)-admissible deformation \(y^2 \mapsto x^2z^2\) \(C_5\) is deformed in the cycle \(C'_5 := G_3 \cdot ξ_0\), where \(ξ_0 ↔ J_0 := (x^3, x^2y, xy^2, y^3, y^2z, x^2z^2)\). By ([T1], 1.4.4) \([C_5] = q[C'_5]\) modulo \(𝒜(𝐻)\). Hence we can argue with \(C'_5\) and \(J_0\) in the same way as in the case \(b = 7\): Let \(K := (x^3, x^2y, xy^2, y^3, x^2z^2, y^2z^2)\) and \(J := (x^2z + y^2z, K) ↔ ξ\). If \(ξ_0/∞ := \lim_{λ→0/∞} τ(λ)ξ\), then \(ξ_0\) is as above and \(ξ_∞ ↔ J_∞ := (x^3, x^2y, xy^2, y^3, x^2z, y^2z^2)\).

Obviously, one has the same computation of degree as in the case \(b = 7\) and the same argumentation shows that \([C'_5] = [C_∞]\) modulo \(𝒜(𝐻)\). \(C_∞\) is deformed by the \(G_3\)-admissible deformation \(y^2z^2 \mapsto x^2\) in the cycle \(G_3 \cdot ζ\), where \(ζ ↔ (x^2, xy^2, y^3, y^2z^3)\). As this is the cycle \(C_4\), by 16.2.2 we get \([C_5] ∈ 𝒜(𝐻)\).

**Conclusion 16.2.** \([C_5] ∈ 𝒜(𝐻_{4g})\) if \(g = 0\) or \(g = -2\). □

### 16.2.4. Summary
The results of 16.1 and 16.2 give

**Theorem 16.1.** (i) If \(g \leq 0\), then \(A_1(𝐻_{3g})\) is freely generated by \([E], [D], [C_2]\).

(ii) \(A_1(𝐻_{4,2}) \simeq \mathbb{Q}^4\) and if \(g \leq 1\), then \(A_1(𝐻_{4,g})\) is freely generated by \([E], [D], [C_2]\). □
Correction of the results of [T4] and summary

In [T4] the computation of the degree on page 28 is wrong, and hence Conclusions 1-3 and Proposition 3 on the pages 29-32 are wrong. As well ([T4], Lemma 7, p.33) is wrong. The statement of ([T4], Proposition 4, p.33) is right, but with regard to Theorem 15.1(i) it is irrelevant. The statement in ([T4], Satz 2, p.35) is wrong and has to be replaced by the statements of Theorem 15.1 and Theorem 16.1, respectively. The results of the sections 8 and 9 in [T4] remain valid, if one replaces the old bound by the bound \( g(d) = \frac{(d - 2)^2}{4} \).

Furthermore, ([T4], Satz 3, p.35) has to be replaced by:

**Theorem 17.1.** Let be \( d \geq 3 \) and \( g \) not maximal, let \( C \) be the universal curve with degree \( d \) and genus \( g \) over \( H_{d,g} \).

(i) If \( g > \gamma(d) := \left(\frac{d-2}{2}\right) \), then \( A_1(C) \) is freely generated by \( E^*, C_1^*, C_2^*, C_3^* \) and \( L^* \).

(ii) If \( g(d) < g \leq \gamma(d) \), then \( A_1(C) \) is freely generated by \( E^*, D^*, C_2^*, C_3^* \) and \( L^* \).

(iii) If \( g \leq g(d) \), then \( A_1(C) \) is freely generated by \( E^*, D^*, C_2^* \) and \( L^* \).

The statements of ([T4], Satz 4 and Satz 5, p. 36) are correct, if the bound mention there is replaced by \( g(d) = \frac{(d - 2)^2}{4} \). The reason is that the arguments used in (loc.cit.) formally do not depend on the bound.

All in all, one gets the results which had been stated in the introduction.

Concluding Remark: Having arrived at this point, it is not so difficult any more to explicitly determine the cone of curves and the ample cone of \( H_{d,g} \) (and of \( C \)).
APPENDIX A

Notations

The ground field is \( \mathbb{C} \); all schemes are of finite type over \( \mathbb{C} \); \( k \) denotes an extension field of \( \mathbb{C} \). \( P = k[x, y, z, t], S = k[x, y, z], R = k[x, y] \) are the graded polynomial rings.

\( T = T(4; k) \) group of diagonal matrices
\( \Delta = U(4; k) \) unitriangular group
\( B = B(4; k) \) Borel group

\( T(\rho) \) subgroup of \( T(3; k) \) or of \( T(4; k) \) (cf. 2.4.1 and [T1], p.2).

\[ \Gamma = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} < U(4; k) \]

\( G_1, G_2, G_3 \) subgroups of \( U(4; k) \) (cf. 1.1)

\( H = H_{d,g} \) Hilbert scheme of curves in \( \mathbb{P}^3 \) with degree \( d \geq 1 \) and genus \( g \), i.e. \( H = \text{Hilb}^P(\mathbb{P}^3_k) \), where \( P(T) = dT - g + 1 \).

\( Q(T) = (T^4+3) - P(T) \) complementary Hilbert polynomial

\( H_Q = \text{Hilb}^Q \) Hilbert scheme of ideals \( I \subset O_{\mathbb{P}^3} \) with Hilbert polynomial \( Q(T) \), i.e. \( H = H_{d,g} = H_Q \).

\( H_Q \neq \emptyset \) if and only if \( Q(T) = (T^4+3) + (T-2)^2 \) or \( Q(T) = (T^4+3) + (T-a+2) + (T-b+1) \), where \( a \) and \( b \) are natural numbers \( 1 \leq a \leq b \). The first case is equivalent with \( d = a \) and \( g = (d-1)(d-2)/2 \), i.e., equivalent with the case of plane curves. We consider only the case \( g < (d-1)(d-2)/2 \). In this case we have the relations \( d = a - 1 \) and \( g = (a^2 - 3a + 4)/2 - b \).

It was not possible to reserve the letter \( d \) for denoting the degree of a curve. If necessary \( d \) denotes a number large enough, e.g. \( d \geq b = \text{bound of regularity of all ideals in } O_{\mathbb{P}^3} \) with Hilbert polynomial \( Q \) (cf. [G1], Lemma 2.9, p.65).

\( G = \text{Grass}_m(P_d) \) Grassmann scheme of \( m \)-dimensional subspaces of \( P_d \).

Let \( \varphi : \mathbb{N} \to \mathbb{N} \) be a function with the following properties: There is an ideal \( I \subset O_{\mathbb{P}^2} \) of finite colength with Hilbert function \( h(n) = h^0(I(n)) \), such that \( 0 \leq \varphi(n) \leq h(n) \) for all \( n \in \mathbb{N} \) and \( \varphi(n) = h(n) \) for \( n \geq d \), where \( n \) is large enough, e.g. \( n \geq d := \text{colength}(I) \).

On the category of \( k \)-schemes a functor is defined by

\[ H_\varphi(\text{Spec } A) = \{(U_0, \cdots, U_d)|U_n \subset S_n \otimes A \text{ subbundle of rank } \varphi(n) \text{ such that } S_1 U_{n-1} \subset U_n, 1 \leq n \leq d \} \]

\( H_\varphi \) is a closed subscheme of a suitable product of Grassmann schemes; it is called graded Hilbert scheme.
To each ideal $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^3}$ with Hilbert polynomial $Q$ corresponds a point $\xi \in \mathbb{H}(k)$, which we denote by $\xi \leftrightarrow \mathcal{J}$.

$h(\mathcal{J})$ denotes the Hilbert function of $\mathcal{J}$, that means $h(\mathcal{J})(n) = \dim_k H^0(\mathcal{J}(n)), n \in \mathbb{N}$.

If $\varphi$ is the Hilbert function of an ideal in $\mathcal{O}_{\mathbb{P}^2}$ of colength $d$, then

$$H_{\varphi} := \{ \mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2} | h^0(\mathcal{I}(n)) = \varphi(n), n \in \mathbb{N} \}$$

is a locally closed subset of $\text{Hilb}^d(\mathbb{P}^2)$, which we regard to have the induced reduced scheme structure.

If $G$ is a subgroup of $GL(4; k)$, then $\mathbb{H}^G$ denotes the fixed-point scheme, which is to have the induced reduced scheme structure. The same convention is to be valid for all fixed-point subschemes of $H^d = \text{Hilb}^d(\mathbb{P}^2)$.

If $C \hookrightarrow \mathbb{H}$ is a curve, then by means of the Grothendieck-Plücker embedding $\mathbb{H} \to \mathbb{P}^N$ we can regard $C$ as a curve in a projective space, whose Hilbert polynomial has the form $\deg(C) \cdot T + c$. Here $\deg(C)$ is defined as follows: If $\mathcal{I}$ is the universal sheaf of ideals on $X = \mathbb{H} \times \mathbb{P}^3$, then $\mathfrak{I} := \mathcal{O}_X/\mathcal{I}$ is the structure sheaf of the universal curve $\mathcal{C}$ over $\mathbb{H}$, and the direct image $\pi_*(\mathfrak{I}(n))$ is locally free on $\mathbb{H}$ of rank $P(n)$ for all $n \geq b$. The line bundles $\mathcal{M}_n := \wedge^d \pi_*(\mathfrak{I}(n))$ are called the tautological line bundles on $\mathbb{H}$, which are very ample and thus define the Grothendieck-Plücker embeddings in suitable projective spaces. Here $\wedge$ is to denote the highest exterior power. Then $\deg(C)$ is the intersection number $\deg(\mathcal{M}_n|C) := (\mathcal{M}_n \cdot C)$. (If $C$ is a so called tautological or basis cycle one can compute this intersection number directly, see [T2], Section 4.1.)

After these more or less conventional notations we introduce some notations concerning monomial ideals. If $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^3}$ is $T$-invariant, then $H^0(\mathcal{O}_{\mathbb{P}^3}, \mathcal{J}(d)) \subset \mathcal{O}_{\mathbb{P}^3}$ is generated by monomials. To each monomial $x^{a+b+c}y^at^b$ in $H^0(\mathcal{J}(d))$ we associate the cube $[a,a+1] \times [b,b+1] \times [c,c+1]$ in a $y - z - t$-coordinate system, and the union of these cubes gives a so called pyramid, which is denoted by $E(\mathcal{J}(d))$. Usually we assume that $\mathcal{J}$ is invariant under $\Delta$ or $\Gamma$. Then we can write $H^0(\mathcal{J}(d)) = \bigoplus_{n=0}^{d} t^{d-n} U_n$, where $U_n \subset S_n$ are subspaces such that $S_1 \cdot U_n \subset U_{n+1}, 0 \leq n \leq d - 1$, which we call the layers of the pyramid. (In [T1]–[T4] we made extensive use of this concept, but here it occurs only once in 12.3.3.)

$A_1(\cdot)$ denotes the group of rational equivalence classes with coefficients in $\mathbb{Q}$. 
Hilbert functions without Uniform Position Property

Lemma B.1. Let be \( k \) be an algebraically closed field, \( \mathcal{I} \subset \mathcal{O}_{\mathbb{P}_k^2} \) an ideal of finite colength with Hilbert function \( \varphi \) and difference function \( \varphi'(n) = \varphi(n) - \varphi(n-1) \). Let \( m \) be a natural number such that \( \varphi'(m+1) = \varphi'(m) + 1 \). The ideal \( \mathcal{J} \subset \mathcal{O}_{\mathbb{P}_k^2} \) generated by \( H^0(\mathcal{I}(m)) \) has the following properties:

(i) \( \mathcal{J} \) is \( m \)-regular;

(ii) \( H^0(\mathcal{J}(n)) = H^0(\mathcal{I}(n)) \) for all \( n \leq m + 1 \);

(iii) If \( \delta := m + 1 - \varphi'(m) > 0 \), then there is a form \( f \in S_\delta \) and an ideal \( \mathcal{L} \subset \mathcal{O}_{\mathbb{P}_k^2} \) of finite colength such that \( \mathcal{J} = f \cdot \mathcal{L}(-\delta) \).

Proof. Let be \( I_n := H^0(\mathcal{I}(n)) \), \( I := \bigoplus_{n=0}^{\infty} I_n \). The ideal \( I \) is called Borel-normed, if \( \text{in}(I) \) is invariant under \( B(3; k) \), where \( \text{in}(I) \) is the ideal generated by the initial monomials of all forms in \( I \). According to a theorem of Galligo, there is a \( g \in GL(3; k) \) such that \( g(I) \) is Borel-normed. (In [G4], Anhang IV, in the special case of three variables, there is an ad-hoc-proof.) Therefore we can assume without restriction that \( I \) is Borel-normed. Then Fig.A.1 shows not only the graph of \( \varphi' \), but also the monomials in \( H^0(\mathcal{I}_0(n)) \), where \( \mathcal{I}_0 := [\text{in}(I)]^\sim \) (cf. [G4], Anhang V, Hilfssatz 1, p.116). One sees that \( S_1 \text{in}(I_m) = \text{in}(I_{m+1}) \) and this implies the statements (i) and (ii) (cf. loc.cit., Lemma, p.116 or [Gre], Proposition 2.28, p.41).

Let \( \psi \) be the Hilbert function of \( \mathcal{J} \). Then \( \psi \) is also the Hilbert function of \( \mathcal{J}_0 = \text{in}(\mathcal{J}) \) (cf. [G4], Hilfssatz 1, p.114), and the further development of the graph of \( \psi' \) is marked by \( \cdots \) in Fig. A.1. The line \( l : y = x - \delta + 1 \) is marked by \( - - - - \). If \( c \) is the number of monomials between the graphs of \( \psi' \) and \( l \), then \( \psi(n) = \binom{n-\delta+2}{2} - c, n \geq m \). Then the Hilbert polynomial of \( \mathcal{O}_{\mathbb{P}_k^2}/\mathcal{J} \) is equal to \( p(n) = \binom{n+2}{2} - \psi(n) = \delta n + 1.5\delta - 0.5\delta^2 + c \). Hence \( V_+(\mathcal{J}) \subset \mathbb{P}_k^2 \) is 1- codimensional and there is an irreducible component which is equal to a hypersurface \( V \), \( f \in S_\delta \) an irreducible form (Hauptidealsatz of Krull). From \( \mathcal{J} \subset \text{Rad}(\mathcal{J}) \subset (f) \) it follows \( \mathcal{J} = f \cdot \mathcal{K}(-\nu) \), where \( \mathcal{K} \subset \mathcal{O}_{\mathbb{P}_k^2} \) is an ideal with Hilbert function \( \chi(n) := \psi(n+\nu) = \binom{n-\delta-\nu+2}{2} - c \). If \( \delta - \nu > 0 \), one argues with \( \mathcal{K} \) in the same way as with \( \mathcal{J} \), and one finally gets the statement (iii).

Lemma B.2. Let the assumptions and the notations be as before. Then \( \text{reg}(\mathcal{I}) = \min \{ n \in \mathbb{Z} | \varphi'(n) = n + 1 \} \).

Proof. As in the proof of Lemma 1, we can assume that \( \mathcal{I} \) is Borel-normed. We let \( G_m \) operate on \( S \) by \( \sigma(\lambda) : x \mapsto x, y \mapsto \lambda^q y, z \mapsto \lambda^{q^2} z \), where \( q \) is a high enough natural number. Then \( \lim_{\lambda^{q^2}} \sigma(\lambda) \mathcal{I} \) is equal to the ideal \( \mathcal{I}_0 \) as in the proof of Lemma 1, \( \mathcal{I} \) and \( \mathcal{I}_0 \) have the same Hilbert function, and \( \text{reg}(\mathcal{I}) = \text{reg}(\mathcal{I}_0) \) (cf. [Gre], Theorem 2.27). Hence
one can assume without restriction that $\mathcal{I}$ is monomial. But then the statement follows from ([T1], Anhang 2), for instance.
Ideals with many monomials

If \( k \) is a field, let be \( S = k[x_1, \ldots, x_r, t] \) and \( R = k[x_1, \ldots, x_r] \). \( \mathbb{G}_m \) operates on \( S \) by \( \sigma(\lambda) : x_i \mapsto x_i, 1 \leq i \leq r, \) and \( t \mapsto \lambda t, \lambda \in k^* \). Let \( H \) be the Hilbert scheme of ideals \( \mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r} \) with Hilbert polynomial \( Q \), i.e., \( H = \text{Hilb}^P(\mathbb{P}^r_\mathbb{k}) \), where \( P(n) = \binom{n+r}{r} - Q(n) \) is the complementary Hilbert polynomial of the subscheme \( V_+(\mathcal{I}) \subset \mathbb{P}^r \). We suppose that \( H \) is not empty. Then the ideals \( \mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r} \) with Hilbert polynomial \( Q \), for which \( t \) is a non-zero divisor of \( \mathcal{O}_{\mathbb{P}^r}/\mathcal{I} \), form an open, non-empty subset \( U_t \subset H \).

If \( K/k \) is an extension field and if \( \mathcal{I} \in H(K) \), then the limit ideals \( \mathcal{I}_{0/\infty} := \lim_{\lambda \to 0/\infty} \sigma(\lambda) \mathcal{I} \) are in \( H(K) \) again, and if \( \mathcal{I} \in U_t \), then \( \mathcal{I}_0 \in U_t \), too (cf. [G2], Lemma 4). We say that \( \mathcal{I} \) fulfils the limit condition, if \( \mathcal{I}_\infty \in U_t \).

Remark C.1. If \( \mathcal{I} \) is fixed by the subgroup \( \Gamma : x_i \mapsto x_i; t \mapsto \alpha_1 x_1 + \cdots + \alpha_r x_r + t \) of \( U(r+1;k) \), then \( \mathcal{I} \) does fulfil the limit condition (cf. [G2], proof of Lemma 3, p. 541).

If \( \mathcal{I} \in U_t \), then \( \mathcal{I}' := \mathcal{I} + t\mathcal{O}_{\mathbb{P}^r}(-1)/t\mathcal{O}_{\mathbb{P}^r}(-1) \) can be regarded as an ideal in \( \mathcal{O}_{\mathbb{P}^r-1} \) with Hilbert polynomial \( Q'(T) = Q(T) - Q(T-1) \).

C.0.5. Lemma. Let \( \mathcal{I} \in H(k) \cap U_t \) be an ideal which fulfils the limit condition.

(i) If \( d \geq \max(\text{reg}(\mathcal{I}_0), \text{reg}(\mathcal{I}_\infty)) \), then \( H^0(\mathbb{P}^r_k, \mathcal{I}(d)) \cap R_d \) has the dimension \( Q'(d) \).

(ii) If \( d \geq \text{reg}(\mathcal{I}') \) and \( H^0(\mathcal{I}(d)) \cap R_d \) has a dimension \( \geq Q'(d) \), then \( d \geq \max(\text{reg}(\mathcal{I}_0), \text{reg}(\mathcal{I}_\infty)) \).

Proof. (i) There is a basis of \( M := H^0(\mathcal{I}(d)) \) of the form \( g_i = t^{e_i} g_i^0 + t^{e_i-1} g_i^1 + \cdots \), such that \( 0 \leq e_1 \leq e_2 \leq \cdots \leq e_m \), \( m := Q(d), g_i^0 \in R \) and \( g_i^0 \in R_{d-e_i}, 1 \leq i \leq m \), linear independent. Then \( M_\infty := \lim_{\lambda \to 0} \sigma(\lambda) M = \{t^{e_i} g_i^0 | 1 \leq i \leq m \} \) (\( \text{limit in Grass}_m(S_d) \)) has the dimension \( m \). As \( d \geq \text{reg}(\mathcal{I}_\infty) \) by assumption, it follows that \( Q(d) = h^0(\mathcal{I}_\infty(d)) \), and hence \( M_\infty = H^0(\mathcal{I}_\infty(d)) \). Now \( t \) is a non-zero divisor of \( S/ \bigoplus_{n \geq 0} H^0(\mathcal{I}_\infty(n)) \) by assumption, thus it follows that \( H^0(\mathcal{I}_\infty(n)) = \{t^{e_i-(d-n)} g_i^0 | e_i \geq d-n \} \) for all \( 0 \leq n \leq d \). If \( n = d-1 \) one gets \( H^0(\mathcal{I}_\infty(d-1)) = \{t^{e_i-1} g_i^0 | e_i \geq 1 \} \), hence \( Q(d-1) = |\{i | e_i \geq 1 \}|. \) It follows that \( Q'(d) = |\{i | e_i = 0 \}|. \) Thus \( M \cap R_d \supset \{g_i^0 | e_i = 0 \} \) has a dimension \( \geq Q'(d) \). Because of \( \text{reg}(\mathcal{I}') \leq \text{reg}(\mathcal{I}) \) one has \( h^0(\mathcal{I}(d)) = Q'(d) \) and the canonical restriction mapping \( \rho_d : M = H^0(\mathcal{I}(d)) \to H^0(\mathcal{I}'(d)) \) is injective on \( M \cap R_d \). It follows that the dimension of \( M \cap R_d \) can not be greater than \( Q'(d) \).

(ii) From the exact sequence

\[
0 \to \mathcal{I}(-1) \xrightarrow{t} \mathcal{I} \to \mathcal{I}' \to 0
\]
it follows that \( H^i(I(n-i)) = (0) \) if \( i \geq 2 \) and \( n \geq e := \text{reg}(I') \) (see [M], p.102). The sequence
\[
0 \longrightarrow H^0(I(d-1)) \longrightarrow H^0(I(d)) \xrightarrow{\rho_d} H^0(I'(d)) \longrightarrow H^1(I(d-1)) \longrightarrow H^1(I(d)) \longrightarrow 0
\]
is exact as \( d \geq e \), where \( \rho \) is induced by the canonical restriction mapping \( S \longrightarrow S/tS(-1) = R \). As \( \rho_d \) is injective on \( H^0(I(d)) \cap R_d \) and \( h^0(I'(d)) = Q'(d) \), it follows from the assumption that \( \rho_d \) is surjective. From the \( e \)-regularity of \( I' \) it follows that \( R_1H^0(I'(n)) = H^0(I'(n+1)) \), for all \( n \geq e \). Hence \( \rho_n \) is surjective for all \( n \geq d \). Hence \( 0 \longrightarrow H^1(I(n-1)) \longrightarrow H^1(I(n)) \longrightarrow 0 \) is exact for all \( n \geq d \), thus \( H^1(I(n-1)) = (0) \) for all \( n \geq d \). It follows that \( \text{reg}(I) \leq d \). One again has the exact sequences:
\[
(2) \quad 0 \longrightarrow I_{0/\infty}(-1) \xrightarrow{-\cdot} I_{0/\infty} \longrightarrow I'_{0/\infty} \longrightarrow 0
\]
As \( (I')_{0/\infty} = (I_{0/\infty})' \supseteq I' \) and all these ideals have the Hilbert polynomial \( Q' \), it follows that \( (I')_{0/\infty} = I' \). As \( H^0(I(d)) \cap R_d \) is fixed by \( \sigma(\lambda) \), it follows that \( H^0(I(d)) \cap R_d \subset H^0(I'_{0/\infty}(d)) \). Then one argues as before, using (2) instead of (1).

**Remark C.2.** Let \( I \subset O_{\mathbb{P}^2} \) be an ideal of colength \( d \), let be \( S = k[x,y,z], R = k[x,y] \), and let \( G_m \) operate by \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z \). We assume \( I \) to be invariant under \( \Gamma \) (see above). As \( d \geq \text{reg}(I) \) for all ideals \( I \subset O_{\mathbb{P}^2} \) of colength \( d \), the assumption of part (i) of the lemma is fulfilled, hence \( H^0(I(n)) \cap R_n \) has the dimension \( Q'(n) = \binom{n+1}{1} \) for all \( n \geq d \) and therefore:
\[
(3) \quad H^0(I(n)) \supseteq R_n \quad \text{for all} \quad n \geq d.
\]
This inclusion has been used in the text for several times, e.g. in Section (2.2).

**Remark C.3.** Let \( I \subset O_{\mathbb{P}^2} \) be an ideal of finite colength, with Hilbert function \( \varphi \), which is invariant under \( \Gamma \cdot T(\rho) \). Let \( G_m \) operate on \( S \) by \( \sigma(\lambda) : x \mapsto x, y \mapsto y, z \mapsto \lambda z \). If \( \text{in}(I) \) is the initial ideal with regard to the inverse lexicographical order, then \( \text{in}(I) \) is equal to the limit ideal \( I_0 = \lim_{\lambda \to 0} \sigma(\lambda)I \). As \( h^0(I_0(n)) = \varphi(n) \), it follows that \( h^0(I_0(n)) = H^0(I_0(n)) \), for all \( n \in \mathbb{N} \) (cf. [G2], Lemma 3 and Lemma 4, pp. 541). Thus the number of the initial monomials and of the monomials in \( H^0(I_0(n)) \), which are represented in our figures, can be determined by means of the Hilbert function, alone.
APPENDIX D

Unipotent groups acting on polynomial rings

Lemma D.1. The 5-dimensional subgroups of $\Delta = U(4; k)$ have the form

$$G(p) := \left\{ \begin{pmatrix} 1 & \alpha & * & * \\ 0 & 1 & \beta & * \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg| a\alpha + b\beta + c\gamma = 0 \right\}$$

where $p = (a : b : c) \in \mathbb{P}^2(k)$ is uniquely determined.

Proof. $N := \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset \Delta$ is a normal subgroup. Let $G$ be a 5-dimensional subgroup of $\Delta$. Then $G/G \cap N \to \Delta/N$ is an injective homomorphism and $\Delta/N \cong \mathbb{G}_a^3$.

First case: $\dim G \cap N = 2$. Then $G \cap N = \left\{ \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in k, ax + by + cz = 0 \right\}$

where $(a : b : c) \in \mathbb{P}^2(k)$ is suitable point. It follows that $G = \left\{ \begin{pmatrix} 1 & \alpha & x & y \\ 0 & 1 & \beta & z \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$, where $\alpha, \beta, \gamma$ are any element of $k$, and $x, y, z \in k$ have to fulfil the conditions noted above. If

$$\begin{pmatrix} 1 & \alpha' & x' & y' \\ 0 & 1 & \beta' & z' \\ 0 & 0 & 1 & \gamma' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is any other element of $G$, then

$$a(x' + \alpha\beta' + x) + b(y' + \alpha z' + x\gamma' + y) + c(z' + \beta\gamma' + z) = 0.$$ 

As $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are any elements of $k$, we conclude that $a = b = c = 0$, contradiction.

Second case: $N \subset G$. Then $G/N \cong \mathbb{G}_a^3$ is 2-dimensional, and one concludes from this that $G$ has the form noted above and that $p \in \mathbb{P}^2(k)$ is uniquely determined. Furthermore it is easy to see that $G(p)$ is a subgroup. \hfill $\Box$

Lemma D.2. Let be $P = k[x, y, z, t]$, let $V \subset P$ be a subspace which is invariant under $G(p)$. If $f \in P$ is invariant under $G(p)$ modulo $V$, then the polynomials

$$x\partial f/\partial z, y\partial f/\partial t, x\partial f/\partial t, by\partial f/\partial y - ay\partial f/\partial z, cx\partial f/\partial y - az\partial f/\partial t, cy\partial f/\partial z - bz\partial f/\partial t$$

all lie in $V$. 137
This gives $y \partial f / \partial z$. 

If $g(f) - f \in V$, then it follows that $\alpha x \partial f / \partial y + \beta y \partial f / \partial z + \gamma \partial f / \partial t +$ terms containing $\alpha^i, \beta^i, \gamma^i$, where $i \geq 2$, lies in $V$.

First case: $c \neq 0$. Then $\gamma = -(aa/c + b\beta/c)$, hence $\alpha x \partial f / \partial y + \beta y \partial f / \partial z - (aa/c + b\beta/c)z \partial f / \partial t +$ terms containing $\alpha^2, \alpha \beta, \beta^2$, etc. in $V$. It follows that $\alpha(cx \partial f / \partial y - ax \partial f / \partial t) + \beta(cy \partial f / \partial z - bz \partial f / \partial t) +$ terms containing $\alpha^2, \alpha \beta, \beta^2$, etc. in $V$. Put $\alpha = 0$ and $\beta = 0$, respectively. It follows that $cy \partial f / \partial z - bz \partial f / \partial t \in V$ and $cx \partial f / \partial y - az \partial f / \partial t \in V$. 

Second case: $c = 0, b \neq 0$. Then we can choose $\gamma \in k$ arbitrarily and $-aa/b = \beta$. It follows that $\alpha x \partial f / \partial y - (aa/b)y \partial f / \partial z + \gamma z \partial f / \partial t +$ terms containing $\alpha^2, \gamma^2$, etc. in $V$. Putting $\alpha = 0$ gives $z \partial f / \partial t \in V$. Putting $\gamma = 0$ gives $bx \partial f / \partial y - ay \partial f / \partial z \in V$.

Third case: $b = 0, c = 0$. Then $a = 1$ and $\beta$ and $\gamma$ are any elements of $k$, whereas $\alpha = 0$. This gives $y \partial f / \partial z$ and $z \partial f / \partial t \in V$. As $N = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\subset G(p)$, the same reasoning as in the proof of ([T2], Hilfssatz 1, p. 142) shows that $x \partial f / \partial z, x \partial f / \partial t, y \partial f / \partial t \in V$. 

D.0.6. Lemma. Let $I \subset \mathcal{O}_k$ be a monomial ideal of colength $d > 0$ and let $\xi$ be the corresponding point in $H^d(k)$. If $\xi$ is not invariant under the $G_a$-operation $\psi_\alpha : x \mapsto x, y \mapsto \alpha x + y, z \mapsto z$, then $C := \overline{\xi}$ contains exactly two fixed points under $T(3; k)$, namely the point $\xi$ and the $G_a$ - fixed point $\psi_\infty(\xi) := \lim_\alpha \psi_\alpha(\xi)$.

Proof. Embedding $H^d$ in Grass$^q(S_d)$, where $q := \binom{d+2}{2} - d$, one sees that it is sufficient to prove the corresponding statement for a $T(3; k)$ - invariant $q$ - dimensional subspace $U \subset S_d$ and the corresponding point in Grass$^q(S_d)$. As one can write $U = \bigoplus_{i=0}^d z^{d-i}U_i$, where $U_i \subset R_i$, is a subspace, it suffices to prove the corresponding statement for a $r$ - dimensional subspace $U \subset R_n$, which is invariant under $T(2; k)$, but not invariant under $G_a$. As $\lim_\alpha \psi_\alpha(U)$ is a $G_a$ - invariant subspace and as $\text{char}(k) = 0$, it follows that this subspace is equal to $\langle x^n, x^{n-1}y, \ldots, x^{n-r+1}yr^{-1} \rangle$. It follows that $C$ has two fixed-points under $T(2; k)$. In order to prove there are no more fixed-points, it suffices to show the following: If there is an element $\alpha \neq 0$ in $k$ such that $\psi_\alpha(U)$ is $T(2; k)$ - invariant, then $\psi_\alpha(U)$ is $T(2; k)$ - invariant for all $\alpha \neq 0$. If one takes a monomial $M = x^{n-r}y^r \in U$, then $\psi_\alpha(M) = x^{n-r}(\alpha x + y)^r \in \psi_\alpha(U)$. As this is a monomial subspace by assumption,
it follows that $x^{n-r}x^iy^{r-i} \in \psi_\alpha(U)$, $0 \leq i \leq r$. Thus one has $M \in \psi_\alpha(U)$ and it follows that $U = \psi_\alpha(U)$. But this implies $\psi_{\alpha n}(U) = U$ for all $n \in \mathbb{N}$ and thus $\psi_\alpha(U) = U$ for all $\alpha \in k$. \qed
APPENDIX E

Standard bases

Let $k$ be an extension field of $\mathbb{C}$. If $\rho = (\rho_0, \ldots, \rho_r) \in \mathbb{Z}^{r+1} - (0)$, then $T(\rho) := \{ \lambda = (\lambda_0, \ldots, \lambda_r) | \lambda_i \in k^* \text{ and } \lambda^\rho := \lambda_0^{\rho_0} \cdots \lambda_r^{\rho_r} = 1 \}$ is a subgroup of dimension $r$ of $G_m^{r+1}$.

Auxiliary lemma: If $\sigma, \tau \in \mathbb{Z}^{r+1} - (0)$ such that $T(\sigma) \subset T(\tau)$, then there is an integer $n$ such that $\tau = n \cdot \sigma$.

Proof. Write $\sigma = (a_0, \ldots, a_r), \tau = (b_0, \ldots, b_r)$. As the dimension of $T(\sigma)$ is equal to $r$, there is an index $i$ such that $a_i \neq 0$ and $b_i \neq 0$. Choose $p, q \in \mathbb{Z}$ such that $pa_0 = qb_0$ and $(p, q) = 1$. Then $\lambda^{pa_1 - qb_1} \cdots \lambda^{pa_r - qb_r} = 1$ for all $\lambda \in T(\sigma)$ follows. Because of $\dim T(\sigma) = r$ one gets $pa_i - qb_i = 0$, $0 \leq i \leq r$, and thus $\sigma = \rho \tau$, $\tau = p \rho$, where $\rho \in \mathbb{Z}^{r+1} - (0)$ is a suitable vector. If $\epsilon$ is any $q$-th root of unity in $\mathbb{C}$, one can choose $\lambda \in (\mathbb{C}^*)^{r+1}$ such that $\lambda^\rho = \epsilon$. From $\lambda^\rho = \lambda^\sigma = 1$ it follows that $\epsilon^p = \lambda^\rho = \lambda^\tau = 1$, too, and $q = 1$ follows.

We let $G_m^{r+1}$ operate on $S = k[X_0, \ldots, X_r]$ by $X_i \mapsto \lambda_i X_i$. If $\rho = (\rho_0, \ldots, \rho_r)$, then $X^\rho := X_0^{\rho_0} \cdots X_r^{\rho_r}$.

Lemma E.1. Let $V \subset S_d$ be a $T(\rho)$ - invariant subspace. Then $V$ has a basis of the form $f_i = m_i \cdot p_i(X^\rho)$, where the $m_i$ are different monomials, $p_i$ is a polynomial in one variable with constant term 1 and $m_i$ does not appear in $m_j \cdot p_j(X^\rho)$ if $i \neq j$.

Proof. By linearly combining any basis of $V$ one obtains a basis $f_i = m_i + g_i$, where each $m_i$ is a sum of monomials, each of which is greater than $m_i$ in the inverse lexicographic order, and $m_i$ does not appear in $g_j$. If $g \in T(\rho)$, then $g(f_i)$ contains the same monomials as $f_i$ and from $g(f_i) \in \langle \{f_i\} \rangle$ we conclude that each $f_i$ is a semi-invariant, i.e., $\langle g(f_i) \rangle = \langle f_i \rangle$ for all $g \in T(\rho)$.

Now let be $f = \sum a_i X^{\alpha_i}$ any $T(\rho)$-semi-invariant. Let be $\lambda \in T(\rho)$. Then $\sum a_i \lambda^{\alpha_i} X^{\alpha_i} = c(\lambda) \cdot \sum a_i X^{\alpha_i}$, where $\lambda^{\alpha_i} := \lambda_0^{\alpha_0(0)} \cdots \lambda_r^{\alpha_r(0)}$. It follows that $\lambda^{\alpha_i} = c(\lambda)$, if $a_i \neq 0$, and therefore $\lambda^{\alpha_i - \alpha_j} = 1$ for all $i, j$, such that $a_i \neq 0$ and $a_j \neq 0$. Thus $T(\rho) \subset T(\alpha_i - \alpha_j)$, and the Auxiliary lemma gives $\alpha_i - \alpha_j = n_{ij} \rho$, $n_{ij} \in \mathbb{Z}$, if $a_i \neq 0$ and $a_j \neq 0$. One sees that there is an exponent $\alpha_0 \in \{\alpha_i\}$ and natural numbers $n_i$, such that $f = X^{\alpha_0} \cdot \sum a_i X^{n_i \rho}$. □

Corollary E.1. Let $V \subset S_d$ be a $m$- dimensional subspace, let $x \in \text{Grass}_m(S_d)$ be the closed point of Grass$_m(S_d)$ defined by $V$. If the orbit $T \cdot x$ has the dimension 1, then the inertia group $T_x$ of $x$ has the form $T(\rho)$, where $\rho \in \mathbb{Z}^{r+1} - (0)$.
Proof. This follows by similar argumentations as before (see [T2], Hilfssatz 7, p.141). □
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