Connecting T-duality invariant theories

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Abstract. I will show that the vanishing of the one-loop beta-functional of the doubled formalism (which describes string theory on a torus fibration in which the fibres are doubled) is the same as the equation of motion of the recently proposed generalised metric formulation of double field theory restricted to this background: both are the vanishing of a generalised Ricci tensor. That this tensor arises in both backgrounds indicates the importance of a new doubled differential geometry for understanding both constructions.

1. Introduction
When string theory is compactified on a circle a special duality symmetry appears which is known as T-duality[1]. This tells you that if the circle has radius $R$, the physics is exactly the same as if the circle has radius $1/R$ (or rather $\alpha'/R$ if we do not set $\alpha' = 1$). When the radius is inverted one also has to exchange the momentum in the compact direction (which is of course quantised) with the string winding number around the compact direction, intimating that T-duality is special to string theory and can not be present in a theory of point particles. When there are $D$ toroidal directions the T-duality group is enlarged to $O(D, D)$. Just as momentum is dual to the ordinary string targetspace co-ordinate $X$ we can introduce a co-ordinate $\tilde{X}$ whose dual is the winding number.

Many attempts have been made to promote this symmetry to a manifest symmetry of the string action, involving doubling the toroidal directions by introduceing the additional co-ordinates dual to the winding numbers. However, there is usually some form of constraint which has to be imposed on the fields and a price must be paid, such as the loss of manifest worldsheet Lorentz invariance. We examine the connection between two different T-duality invariant formalisms, one of them a sigma model and the other a field theory. We present the result of [2] that they are equivalent on background where they can both be formulated. The key object connecting them is a doubled analogue of the Ricci tensor, indicating that both pictures should be better understood in terms of some new double differential geometry. We describe both models individually before proceeding to show the connection.

2. The doubled formalism
String theory on a background with $D$ toroidal directions has $O(D, D)$ T-duality symmetry and the doubled formalism[3, 4, 5, 6] makes that manifest (earlier work in this area includes

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Specifically, on a target space which is locally a $T^D$ bundle the fibre co-ordinates are doubled to $T^{2D}$ and non-geometric backgrounds such as T-folds (where transition functions are allowed to include T-dualities) become geometric. We have base co-ordinates $Y^a$ and the fibre co-ordinates $X^i$ are doubled to $X^A = (X^i, \tilde{X}^i)$. The essential parts of the Lagrangian (as used in [12]) are

$$L = \frac{1}{4} \mathcal{H}_{AB}(Y) dX^A \wedge *dX^B + \mathcal{L}(Y),$$

where $\mathcal{L}(Y)$ is the standard string sigma model Lagrangian on the base directions. The object $\mathcal{H}(Y)$ plays the role of metric on the fibre but contains both $h$ and $b$, the ordinary metric and anti-symmetric tensor on the fibre, in a combination which has simple $O(D,D)$ transformation properties:

$$\mathcal{H}_{AB}(Y) = \begin{pmatrix} h^{-1} & -h^{-1}b \\ bh^{-1} & h - bh^{-1}b \end{pmatrix}.$$  

The special form of $\mathcal{H}$ means that if we raise the indices with the $O(D,D)$ invariant metric

$$L_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we get its inverse, that is $\mathcal{H}^{-1} = L^{-1}H L^{-1}$, for general symmetric $h$ and antisymmetric $b$ it is the most general metric with this property.

The apparent doubling of the degrees of freedom is compensated by the imposition of a constraint, which tells us the 2D doubled co-ordinates can be thought of as $D$ bosons and $D$ anti-chiral bosons. The constraint is

$$dX^A = L^{AB} \mathcal{H}_{BC} *dX^C.$$  

It is also helpful to introduce a vielbein $V^A_A$ which facilitates shifting to the chiral frame where both metrics are diagonal

$$\mathcal{H}_{AB}(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

One can see the simple form the constraint takes in this frame. It is just a chirality constraint on the co-ordinates. To facilitate performing the background field expansion we would like to integrate the constraint into the action. This was achieved in [12] where the PST method[13] was used to impose the constraint in this chiral frame leading to Floreanini-Jackiv actions for the scalars. On returning to the original frame the fibre part of the action takes the new form without manifest Lorentz invariance:

$$S = \frac{1}{2} \int d^2\sigma \left[ -\mathcal{H}_{AB} \partial_1 X^A \partial_1 X^B + L_{AB} \partial_1 X^A \partial_0 X^B \right].$$

The equation of motion for the fibre co-ordinates is

$$\partial_1 (\mathcal{H} \partial_1 X) = L \partial_1 \partial_0 X,$$

which integrates to give the constraint (4) (we use the gauge invariance of the action under $X^A \rightarrow X^A + f(\sigma_0)$ to remove an integration function of $\sigma_0$). The fibre part of the action remains the same but is not decoupled, its equation of motion involves the metric $\mathcal{H}$ and the doubled co-ordinates since $\mathcal{H}$ depends on $Y^a$. 

2
3. Quantum consistency of the doubled formalism

The doubled formalism can classically be shown equivalent to the string sigma model, and a number of methods were used to show quantum consistency and equivalence (that the quantising and the correct doubling/undoubling of the fibre commute)[14, 15, 16]. We will be interested in the calculation performed in [12] where the one-loop background field equations were calculated. Recall the result of the equivalent string sigma model calculation (for metric only) is that the Ricci tensor should vanish. The equivalent doubled calculation expresses the background field equations for metric and anti-symmetric tensor as the vanishing of a tensor constructed from the generalised metric $H$ and the base metric $g$. This is not quite the Ricci tensor of the total space, though the similar structures raised the question of whether it could be thought of as some differential geometric object on the doubled space. We will see that this is the case.

Performing the background field expansion is a two-fold check on the consistency of the model. If we show that the we can set the beta-functionals to zero we show there are no UV divergences, but also that the classical Weyl invariance of the worldsheet extends to one-loop. To perform the expansion[17, 18, 19] one writes the target space fields as quantum fluctuations around a classical background, with a careful choice of expansion parameter we can write the expansion in a covariant way. A handy algorithmic method of performing the expansion was devised in [20] so that to perform the expansion to $n$th order involves simply acting $n$ times on the Lagrangian with a differential operator. Terms at first order in the expansion are proportional to the equation of motion so vanish (as is always the case in the background field expansion) so it is the terms second order in the expansion that interest us.

The new effective action contains kinetic terms for the fluctuations and we would like to find their propagators so that we can Wick contract the remaining fluctuations out of the action. In order to find the propagators we introduce a vielbein and move to the chiral frame as in (5). This turns the kinetic terms on the fibre into those of chiral and anti-chiral bosons in flat space, for which the propagator is known, at the expense of modifying the connection in the derivative acting on fluctuations to a spin connection including a vielbein part. For the ordinary string there is a general argument that this new connection cannot contribute to the Weyl divergence and it can be dropped, but it was explicitly shown in [2] that it must be kept on the doubled fibre in our case. Shifted back to the original frame the fibre fluctuation propagator for fluctuations $\xi^A$ is

$$
\langle \xi^A(z)\xi^B(z)\rangle = \Delta_0 H^{AB} + \theta L^{AB}, \tag{8}
$$

where $\Delta_0$ contains the same divergence as an ordinary boson propagator and $\theta$ is a parameter which keeps track of a possible violation of worldsheet Lorentz invariance.

A further complication is that terms with two fluctuations, one of which is acted upon by a derivative, contribute to the one-loop divergence at second-order in the expansion of the exponential of the effective action so we must also Wick contract these terms (which have 4 fluctuations with 2 derivatives acting on them). These contractions were also found in [12] and with those and a fair amount of calculating we can reduce the terms which contribute at one-loop to the form

$$
L_{eff} = \frac{1}{2} W_{\alpha\beta} \partial_\mu X^\alpha \partial^\mu X^\beta \Delta_0 , \tag{9}
$$

where

$$
W_{ab} = -\hat{R}_{ab} - \frac{1}{8} \partial_a \mathcal{H}^{-1} \partial_b \mathcal{H} = -(R_{ab} - \frac{1}{8} \partial_a \mathcal{H}^{-1} \partial_b \mathcal{H}) , \tag{10}
$$

and $W_{AB} = -R_{AB}$ ($R_{AB}$ being the fibre components of the whole space Ricci tensor constructed using $\mathcal{H}$ as the metric on the fibre, whereas $\hat{R}_{ab}$ is the Ricci tensor of the base metric $g$). Terms proportional to $\theta$ drop out confirming that worldsheet Lorentz invariance is maintained. The remaining term is the same form as those in the original action and we can proceed to regularise
the divergence in $\Delta_0$; the equation for the vanishing of the beta functional is simply the vanishing of $W_{\alpha\beta}$. In [12] it was shown that this background field equation reproduces the background field equations of the standard string in this fibred set-up. Although this calculation includes only a metric like object it describes the background field equations with anti-symmetric Kalb-Ramond field $b$ included as well. The dilaton can also be incorporated into the calculation[12, 21]. In the ordinary string theory case the background field equations from the sigma model could also be obtained as the equations of motions of a field theory of $g, h$ and $\phi$. This action for this field theory is thus the string effective action, and performing the background field expansion to higher order would give higher order corrections to this action. The question becomes what is the effective doubled action for the doubled metric $H$ (though we are restricted to the fibred case where $H$ only depends on the base co-ordinates)?

4. Double field theory

Double field theory was proposed by Hull and Zwiebach in [22] (some earlier work on doubled field theories can be found in [8, 23, 24]). Their starting point was closed string field theory, where in the presence of toroidal directions the vertex operators must also depend on the coordinate dual to the winding as well as the original coordinate dual to the momentum. They were able to write down a field theory for the massless modes of the closed string to third order in perturbations around a background complete with a doubled gauge symmetry. The fields are allowed to depend on both the original and dual co-ordinates up to a constraint that follows from level matching for the string. The theory was further developed in [25, 26, 27]. For real progress a strong form of the level matching constraint had to be implemented. If $X^M = (\tilde{x}_i, x^i)$ are the doubled co-ordinates\(^2\) then the constraint is that the operator

$$\Delta = \partial_M \partial^M = 2\partial_i \partial_{\tilde{i}}$$

should annihilate not just the fields and gauge parameters (which would be the weak version of the constraint), but also products of the fields and gauge parameters. With the strong constraint in place a background independent action was found in terms of $E = h + b$ the sum of the metric and antisymmetric 2-form. The formulation that will interest us however is that of [26] in terms of the generalised metric we have already encountered in (2),

$$H_{MN} = \begin{pmatrix} h^{-1} & -h^{-1}b \\ bh^{-1} & h - bh^{-1}b \end{pmatrix}.$$ (12)

This action can further be written as an Einstein-Hilbert term for a generalised Ricci-like scalar $R$ which is a function of the generalised metric $H$ and the doubled dilaton, $d$, which is defined by the relation

$$e^{-2d} = \sqrt{R} e^{-2\phi}.$$ (13)

In detail,

$$S = \int dx \, d\tilde{x} \, e^{-2d} \, R.$$ (14)

\(^2\) Although only originally posited to describe toroidal directions, the doubling can be extended at least formally to the non-compact dimensions as well. However there only the original duality frame has a physical undoubled interpretation. Transformations in the $d$ compact directions are actually restricted to $O(d, d : \mathbb{Z})$ to preserve the periodic boundary if they are to be T-dualities which lead to an equivalent string background (undoubling can then be performed in the new frame).
for the generalised Ricci scalar
\[ \mathcal{R} = 4 \mathcal{H}^{MN} \partial_M \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} \]
\[ - 4 \mathcal{H}^{MN} \partial_M d \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d , \]
\[ + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} \].
\[ (15) \]

Since the \( O(D, D) \) indices \( M, N, \ldots \) are contracted properly \( O(D, D) \) invariance is manifest. The dilaton equation of motion is the vanishing of \( \mathcal{R} \), and the variation of the action with respect to \( \mathcal{H}^{AB} \) is proportional to \( \mathcal{K}_{AB} \delta \mathcal{H}^{AB} \), where

\[ \mathcal{K}_{MN} = \frac{1}{8} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{4} (\partial_L - 2(\partial_L d))(\mathcal{H}^{L,K} \partial_K \mathcal{H}_{MN}) + 2 \partial_M \partial_N d \]
\[ - \frac{1}{2} \delta (\partial_M \mathcal{H}^{KL} \partial_L \mathcal{H}_{NK}) + \frac{1}{2} (\partial_L - 2(\partial_L d))(\mathcal{H}^{KL} \partial_K \mathcal{H}_{MN}) + \mathcal{H}^{K} (\partial_K \mathcal{H}^{L,N}) . \]
\[ (16) \]

However, the field equation is not simply the vanishing of \( \mathcal{K}_{AB} \); \( \mathcal{H} \) is the most general matrix satisfying \( \mathcal{H}^{AB} \mathcal{H}^{CD} = L^{AD} \), and the field after variation \( \mathcal{H}' = \mathcal{H} + \delta \mathcal{H} \) must satisfy the same relation. This constrains the form of the variation and thus the field equation is the vanishing of

\[ \mathcal{R}_{MN} = \frac{1}{4} (\delta_M^P - \mathcal{H}_M^P) \mathcal{K}_{PQ} (\mathcal{H}_Q^N + \mathcal{H}_Q^N) + \frac{1}{4} (\delta_M^P + \mathcal{H}_M^P) \mathcal{K}_{PQ} (\mathcal{H}_Q^N - \mathcal{H}_Q^N) \]
\[ + \frac{1}{2} (\mathcal{K}_{MN} - \mathcal{H}_M^P \mathcal{K}_{PQ} \mathcal{H}_Q^N) . \]
\[ (17) \]

This gives the ‘generalised Ricci tensor’ that we can compare with the result of the background field expansion of the sigma model. The action (14) is invariant under doubled gauge transformations which includes diffeomorphisms of \( h \) and ordinary gauge transformations of \( b \) (in fact \( R \) is a double gauge scalar). The gauge transformations act linearly on \( \mathcal{H} \) (unlike their action on \( \mathcal{E} \)) which leads to the notion of a generalised Lie derivative whose commutator also introduces a modified Courant bracket for the doubled fields.

5. Connecting the doubled formalism and double field theory

So far we have learned that that the equation of motion of doubled field theory can be expressed as the vanishing of a ‘generalised Ricci tensor’, \( \mathcal{R}_{MN} \), while in the doubled formalism the background field equations can also be expressed in terms of the vanishing of some tensor \( W_{\alpha \beta} \), whose rôle is similar to that of the Ricci tensor in the undoubled string theory. We cannot directly compare these objects as the doubled formalism is only defined in a reduced class of backgrounds compared to the double field theory, in particular the doubled metric can only depend on the base co-ordinates, which remain undoubled. We now proceed to reduce the doubled field theory generalised Ricci tensor on the fibred background of the doubled formalism. Explicitly:

- We split into a base and fibre parts using the notation \( \mathcal{X}^\alpha = (\mathcal{X}^A, Y^\alpha) \) where the original (undoubled) co-ordinates \( x^A \) of the \( \mathcal{X}^A \) are compact and the original co-ordinates \( y^a \) of the \( Y^\alpha \) are in general non-compact. This requires the reordering the indices so that the base co-ordinates and their duals sit beside each other in \( Y^\alpha \). We take the metric to have block-diagonal form

\[ \mathcal{H}_{\alpha \beta} = \begin{pmatrix} \mathcal{H}_{AB} & 0 \\ 0 & \mathcal{H}_{ab} \end{pmatrix} . \]
\[ (19) \]
• We allow none of the fields to depend on the fibre co-ordinates, i.e.  \( \partial_{A} \mathcal{H}_{AB} = \partial_{A} \mathcal{H}_{ab} = 0 \).

• We undouble the base co-ordinates, similar to procedure for showing the equivalence of the double field theory to the ordinary string. Splitting \( Y^2 = (\bar{y}^a, y^a) \) this amounts to setting \( \partial^{a} = 0 \) on all fields. Once the base has been undoubled we will normally have the same index conventions as for the doubled formalism, with the splitting \( X^{a} = (X^{A}, Y^{a}) \).

• We take the anti-symmetric field \( b \) to vanish on the base so that

\[
H_{ab} = \begin{pmatrix}
g^{ab} & 0 \\
0 & g_{ab}
\end{pmatrix}.
\]  

(20)

On the base \( L_{\alpha \beta} \) takes the form

\[
L_{ab} = \begin{pmatrix}
0 & \delta^{a}_{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}.
\]  

(21)

• The doubled dilaton is defined through

\[
e^{-2d} = \sqrt{h} e^{-2\phi},
\]  

(22)

where \( h \) is the determinant of the undoubled metric on the whole space. From the block-diagonal form of \( h' \) we deduce

\[
\partial_{A} d = -\frac{1}{4} g^{ab} \partial_{A} g_{ab} + \partial_{A} \Phi,
\]  

(23)

where \( \Phi \) is the doubled dilaton for the doubled theory on the fibre only, defined through

\[
e^{-2\Phi} = \sqrt{h'} e^{-2\phi},
\]  

(24)

where this time \( h' \) is the determinant of the undoubled metric on the fibre only. A dilaton Fradkin-Tseytlin term in the sigma model is a counterterm which shifts the divergence maintaining the Weyl invariance \([28]\) and the dilaton terms in \( \mathcal{R}_{\alpha \beta} \) follow from this or introducing vanishing combinations of terms proportional to the equation of motion.

The result is as we expect\([2]\), that the vanishing of \( \mathcal{R}_{MN} \) and \( W_{\alpha \beta} \) are equivalent in this case, with

\[
\mathcal{R}_{\alpha \beta} = -\frac{1}{2} W_{\alpha \beta}.
\]  

(25)

6. Discussion

While the equivalence is not surprising in itself (both tensors are know to be equivalent to there ordinary string background field equations for \( g \) and \( b \) in their regime of validity) what is important is the central rôle of the generalised Ricci tensor in proceedings and how we can reformulate the ordinary string calculation in a manifestly T-duality invariant manner with no reference to the fields \( g \) and \( b \). The Ricci tensor is a central object in the ordinary differential geometry on the space with metric \( g \). The generalised Ricci tensor is a central object on the \( O(D, D) \) doubled space with co-ordinates in terms of the metric \( \mathcal{H} \) and the doubled dilaton \( d \), and thus of the massless closed string fields \( g, b \) and \( \phi \). Recent work has focussed on trying to decode this differential geometry\([29, 30, 31, 32]\).

What this also shows, is that in the fibred set-up in which we can work, doubled field theory is the effective field theory for the doubled formalism. One can then ask if it is possible to extend the doubled formalism to a more general background, namely any that can be described by doubled field theory, and whether this relation extends in that case. We hope to report in
future on this question. If such a sigma model were in place one could then perform a two-loop background field expansion to obtain higher order corrections to doubled field theory. The efficiency of such a calculation could perhaps be improved by performing the background field expansion in a different derivative more suited to the double differential geometry, perhaps that of [31]. If this proved efficient enough one could use this as a route to calculating higher order corrections in the undoubled theory. Formulation in terms of \( \mathcal{H} \) means that the \( b \) contributions are automatically included as well and in some remnant of the \( O(D, D) \) symmetry relating the equations for \( b \) and \( g \) may be seen (we know they can be combined into the \( O(D, D) \) invariant equation for \( \mathcal{H} \)).

Although T-duality is only a symmetry if we have toroidal directions, and it is only in this case that we have a clear interpretation of the \( \tilde{X} \) co-ordinates as a dual to winding. The doubled formalism can be extended, at least formally to non-compact directions. It would be nice to understand better what is going on in these cases and if there is some physical interpretation to the \( O(D, D) \) double geometry after \( O(D, D) \) rotation in the non-toroidal directions, which must include dependence on the \( \tilde{X} \) co-ordinates.

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