Orbit Codes over $M_2(\mathbb{F}_q)$ and Their Homogeneous Distance

E C T Corro$^{1,*}$, J M T Lampos$^1$, H S Palines$^1$ and V P Sison$^1$

$^1$Institute of Mathematical Science and Physics, University of the Philippines Los Baños, College, Laguna 4031, Philippines

*etcorro@up.edu.ph

Abstract. Let $q$ be a power of a prime. The lattice of one-sided ideals of the finite unital non-commutative Frobenius ring $M_2(\mathbb{F}_q)$ of $2 \times 2$ matrices over the Galois field $\mathbb{F}_q$ is completely analyzed. It turns out that $M_2(\mathbb{F}_q)$ is a principal left semi-local ring in which each left ideal is generated by an idempotent element. The explicit forms of the non-trivial idempotents of $M_2(\mathbb{F}_q)$ are determined to give $q + 1$ proper non-trivial left maximal ideals each with $q^2$ elements. These are exactly the minimal left ideals as well. Using the structure of $M_2(\mathbb{F}_q)$ as a partial ordering of ideals, the generalized Möbius and Euler phi functions are applied to derive the explicit form of the homogeneous weight function on $M_2(\mathbb{F}_q)$. This weight depends on whether the element is the zero element, a zero divisor or a unit. A zero divisor gives the largest homogeneous weight. Moreover, orbit codes over $M_2(\mathbb{F}_q)$ are constructed via the action of the general linear group $GL(2,q)$ on $M_2(\mathbb{F}_q)$ by left translation. The orbit determined by a nonzero nonunit idempotent element of $M_2(\mathbb{F}_q)$ forms the nonzero elements of a minimal left ideal of $M_2(\mathbb{F}_q)$ which are all zero divisors. Consequently, it is shown that the minimum homogeneous distance of the orbit code generated by a nonzero nonunit idempotent element of $M_2(\mathbb{F}_q)$ approaches the Plotkin upper bound as the field size $q$ becomes larger. Analogous results are obtained when the lattice of right ideals is considered and the action of $GL(2,q)$ on $M_2(\mathbb{F}_q)$ by right translation is used instead.

1. Introduction

In this paper, we consider the finite noncommutative Frobenius ring $M_2(\mathbb{F}_q)$ which admits a homogeneous weight. Frobenius rings are appropriate choices as alphabet for codes since the MacWilliams extension theorem is valid in such rings.

Codes over commutative rings are widely studied, however, only a few codes, such as the ones presented in [1] and [2], use noncommutative rings as alphabet. The construction of orbit codes based from the action of $GL(2,q)$ on $M_2(\mathbb{F}_q)$ is presented in this study. The concept of an orbit of a group action as a code is discussed in [3]. The orbit codes constructed in this study, which are determined by a suitable choice of nonzero nonunit idempotent element, are observed to approach the Plotkin bound in terms of the homogeneous distance.

This paper is organized as follows. In Section 2, definitions and concepts relevant to the study are presented. The results are elaborated in Section 3 which consists of four subsections. First, the lattice of left ideals of $M_2(\mathbb{F}_q)$ is derived in Section 3.1. Then the results in Section 3.1 are used to come up with an explicit form of the homogeneous weight of $M_2(\mathbb{F}_q)$ as shown in Section 3.2. Section 3.3
discusses the properties of the action of $GL(2,q)$ on $M_2(\mathbb{F}_q)$ by left translation which are used for the construction of orbit codes in Section 3.4.

2. Preliminaries

The concept of homogeneous weight was first introduced by Constantinescu and Heise in [4]. Homogeneous weight can be seen as a natural generalization of the Hamming weight on finite fields and the Lee weight on $\mathbb{Z}_4$.

A homogeneous weight on an arbitrary finite ring $R$ with unity is defined as follows. Let $\mathcal{R}$ be the set of real numbers and $Rx$ be the principal (left) ideal generated by $x \in R$. A weight function $w : R \to \mathcal{R}$ is called (left) homogeneous if the following conditions are satisfied.

(i) If $Rx = Ry$, then $w(x) = w(y)$ for all $x, y \in R$.

(ii) There exists a real number $\Gamma \geq 0$ such that $\sum_{y \in Rx} w(y) = \Gamma \cdot |Rx|$, for all $x \in R \setminus \{0\}$.

The definition for the right homogeneous weight follows the same analogy. The weight $w$ is simply called a homogeneous weight if it satisfies both the left and right cases. The average value of $w$ is denoted by the constant $\Gamma$. If $\Gamma = 1$, then $w$ is a normalized homogeneous weight. The weight $w$ on $R^n$ is simply a natural extension from that on $R$, that is, $w(z) = \sum_{i=0}^{n-1} w(z_i)$ for $z = (z_0, z_1, ..., z_{n-1}) \in R^n$.

The homogeneous distance metric $\delta : R^n \times R^n \to \mathcal{R}$ is defined by $\delta(x,y) = w(x - y)$ for $x, y \in R^n$.

The following theorem which was proven in [5], shows the existence and uniqueness of the homogeneous weight on a finite ring.

**Theorem 2.1.** Let $w$ be a weight defined on the finite ring $R$. The weight $w$ is homogeneous if and only if there exists a positive real number $c$ such that $w(x) = c \left(1 - \frac{\mu(0,Rx)}{|R|x|}\right)$ for all $x \in R$.

The set $R^\times$ denotes the multiplicative group of units of $R$, and $\mu$ is the Möbius function on the poset of the principal left ideals of $R$ given by (i.) $\mu(Rx,Rx) = 1$ for all $x \in R$, (ii.) $\mu(Ry,Rx) = 0$ if $Ry \not\subseteq Rx$, and (iii.) $\sum_{Ry \subseteq Rx \subseteq Ry} \mu(Rz,Rx) = 0$ if $Ry \subseteq Rx$. Since the Möbius function, in this case, is defined on the poset of one-sided ideals of a finite ring, finding its values is not easy. Some related results involved deriving explicit formulas for the homogeneous weights of finite principal ideals in [6], and the noncommutative Frobenius ring $M_2(\mathbb{F}_p)$ of $2 \times 2$ matrices over the prime field $\mathbb{F}_p$ [2]. In this paper, we come up with an explicit formula of the homogeneous weight of the noncommutative Frobenius ring $M_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ is any finite field, and $q$ is a power of a prime.

Furthermore, this paper also includes a construction of matrix codes in the form of orbit codes based on the action of the general linear group $GL(2,q)$ on $M_2(\mathbb{F}_q)$. In [3], a code $C$ is an orbit code if it is an orbit of a group $G$ on a metric set $X$. The minimum homogeneous distance $\delta_{\text{min}}$ of $C$ is defined to be $\delta_{\text{min}} = \min \{\delta(x,y) | x, y \in C, x \neq y\}$. Applying the homogeneous distance on these orbit codes, it is observed that these codes are ‘near’ Plotkin optimal.

3. Results and Discussions

3.1. One-sided ideals of $M_2(\mathbb{F}_q)$

The ring $M_2(\mathbb{F}_q)$ is a left Artinian ring with unity whose (Jacobson) radical, $\text{rad} M_2(\mathbb{F}_q)$, the largest nilpotent ideal of $M_2(\mathbb{F}_q)$, is trivial, that is $\text{rad} M_2(\mathbb{F}_q) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$. It follows that each left ideal of $M_2(\mathbb{F}_q)$ is of the form $M_2(\mathbb{F}_q) e$, where $e$ is an idempotent element of $M_2(\mathbb{F}_q)$. Hence, it is important.
to know the idempotent elements of $M_2(\mathbb{F}_q)$ to have an idea about the properties and explicit forms of the left ideals of $M_2(\mathbb{F}_q)$.

In [7], it was proven that given a commutative local ring $R$, for each $A \in M_2(\mathbb{F}_q)$, $A^2 = A$ if and only if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix}$ with $bc = a - a^2$. This result applies to $M_2(\mathbb{F}_q)$ since $\mathbb{F}_q$ is a commutative local ring. Thus, we come up with the following corollary.

**Corollary 3.1.** The matrix ring $M_2(\mathbb{F}_q)$ has $q^2 + q + 2$ idempotent elements.

Particularly, each nonzero nonunit idempotent element of $M_2(\mathbb{F}_q)$ is of the form
\[
\begin{pmatrix} 1 & k_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k_3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ k_4 & 1 \end{pmatrix} \quad \text{where} \quad k_1, k_2, k_3, k_4 \in \mathbb{F}_q, \quad \text{and} \quad \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix} \quad \text{where} \quad a \in \mathbb{F}_q \setminus \{0, 1\} \quad \text{and} \quad b, c \in \mathbb{F}_q^\times, \quad \text{the group of units of} \quad \mathbb{F}_q.
\]

Using these idempotent elements and forming the left ideals $I_L$ for each idempotent element $\epsilon \in M_2(\mathbb{F}_q)$, we come up the following theorem.

**Theorem 3.2.** The matrix ring $M_2(\mathbb{F}_q)$ has $q + 1$ proper left ideals. Moreover, each proper left ideal is minimal with $q^2$ elements.

The lattice of left ideals of $M_2(\mathbb{F}_q)$ is given by Figure 1

\[
\begin{array}{cccc}
M_2(\mathbb{F}_q) \\
I_{L_1} & I_{L_2} & \cdots & I_{L_{q+1}} \\
(0) & & & \\
\end{array}
\]

**Figure. 1** Lattice of left ideals of $M_2(\mathbb{F}_q)$

where for each $i \in \{1, 2, \ldots, q + 1\}$, $I_{L_i}$ denotes a distinct minimal left ideal of $M_2(\mathbb{F}_q)$.

**Example 3.3.** The idempotent elements of $M_2(\mathbb{F}_2)$ are given by
\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then the minimal left ideals of $M_2(\mathbb{F}_2)$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively, are as follows:
\[
\begin{align*}
&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

Note that the treatment for right ideals gives analogous results.

3.2. Homogeneous weight on $M_2(\mathbb{F}_q)$

In the paper [5] by Grefeather and Schmidt, the homogeneous weight $w(x)$ of $x \in R$, where $R$ is a finite ring with identity, is given by $w(x) = \Gamma(1 - \mu(0Rx)/\varphi(x))$, where $\mu$ is the M"{o}bius function on the poset of the principal left ideals of $R$, and $\varphi(x)$ is the Euler phi function. Recall that for $x \in R$, $\varphi(x) = |\{y \in R : Rxy = Rx\}|$. Applying our results in Section 3.1, we come up with the following results on the homogeneous weight of $M_2(\mathbb{F}_q)$. In the succeeding discussions, we let $R = M_2(\mathbb{F}_q)$ and $0$ be the zero ideal of $R$. 


Lemma 3.4. If $Z$ is the zero matrix, then $\mu(0, RZ) = 1$ and $\varphi(Z) = 1$.

Proof:
$$\mu(0, RZ) = \mu(0, 0) = 1 \text{ and } \varphi(Z) = |\{ y \in R \mid Ry = 0 \}| = |\{0\}| = 1.$$ (1)

Lemma 3.5. If $I_L$ is a proper left ideal of $R$, then $\mu(I_L, R) = -1$.

Proof:
$$\text{Note that } \mu(I_L, R) + \mu(R, R) = 0. \text{ Thus, } \mu(I_L, R) = -1.$$ (2)

Lemma 3.6. If $U \in \text{GL}(2, q)$, then $\mu(0, RU) = q$ and $\varphi(U) = q(q - 1)(q^2 - 1)$.

Proof:
$$\text{Let } U \in \text{GL}(2, q). \text{ Then } \mu(0, RU) = \mu(0, R). \text{ By the definition of the Möbius function, } \mu(0, R) + \mu(I_1, R) + \cdots + \mu(I_{q+1}, R) + \mu(R, R) = 0, \text{ where } I_1, \cdots, I_{q+1} \text{ are minimal left ideals of } R. \text{ Applying Lemma 3.5, we have } \mu(0, R) + (-1)(q + 1) + \mu(R, R) = 0 \text{ which implies that } \mu(0, RU) = \mu(0, R) = q. \text{ On the other hand, } \varphi(U) = |\{ y \in R \mid Ry = R \}| = |\text{GL}(2, q)| = q(q - 1)(q^2 - 1).$$ (3)

Lemma 3.7. If $B$ is a zero divisor of $R$, then $\mu(0, RB) = -1$ and $\varphi(B) = q^2 - 1$.

Proof:
$$\text{Let } B \in R \text{ be a zero divisor. Thus, } RB \text{ is a minimal left ideal of } R. \text{ Hence, } \mu(0, RB) = -1. \text{ On the other hand, note that each minimal left ideal of } M_2(\mathbb{F}_q) \text{ has } q^2 - 1 \text{ nonzero elements. Thus, } \varphi(B) = |\{ y \in R \mid Ry = RB \}| = q^2 - 1.$$ (4)

With these results, we come up with an explicit formula of the homogeneous weight of $M_2(\mathbb{F}_q)$ as follows.

Theorem 3.8. The homogeneous weight of $M_2(\mathbb{F}_q)$ is given by
$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \Gamma \left( 1 - \frac{1}{(q - 1)(q^2 - 1)} \right) & \text{if } x \text{ is a unit} \\ \Gamma \left( \frac{q^2}{q^2 - 1} \right) & \text{if } x \text{ is a zero divisor} \end{cases}$$

Proof:
$$\text{The proof follows immediately from Lemma 3.4 to Lemma 3.7.}$$ (5)

Moreover, it is worth noting that $M_2(\mathbb{F}_q)$ is partitioned into the set containing the zero element, the multiplicative group of units, and sets of zero divisors.

3.3. Action of $\text{GL}(2, q)$ on $M_2(\mathbb{F}_q)$

Consider the action of the general linear group $\text{GL}(2, q)$ on $M_2(\mathbb{F}_q)$ by left translation. Note that this action is an equivalence relation. Given $A \in M_2(\mathbb{F}_q)$, we denote the $\overline{A}$ as the orbit containing $A$. Then the orbits under this action are given as follows.

Let $Z, U \in M_2(\mathbb{F}_q)$ be the zero matrix and a unit, respectively. Then $\overline{Z} = \{Z\}$ and $\overline{U} = \text{GL}(2, q)$. Now, consider the idempotent elements $B_k = \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}$ where $k \in \mathbb{F}_q$ and $B' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbb{F}_q)$. Then for each $k \in \mathbb{F}_q$,
$$\overline{B_k} = \left\{ \begin{pmatrix} u_{11} & u_{11}k \\ u_{21} & u_{21}k \end{pmatrix} \in M_2(\mathbb{F}_q) : u_{11}, u_{21} \in \mathbb{F}_q \text{ not both } 0 \right\}.$$ (6)
Here we have \( q \) distinct equivalence classes since we are considering the equivalence classes each containing one of \( B_k, k \in \mathbb{F}_q \). Moreover, notice that for each \( k \in \mathbb{F}_q, B_k \) has \( q^2 - 1 \) elements. On the other hand,
\[
\overline{B'} = \left\{ \begin{pmatrix} 0 & u_{12} \\ 0 & u_{22} \end{pmatrix} \in M_2(\mathbb{F}_q); u_{12}, u_{22} \in \mathbb{F}_q \text{ and } u_{12}, u_{22} \text{ not both } 0 \right\}.
\]
(7)

Similarly, \( \overline{B'} \) has \( q^2 - 1 \) elements.

**Example 3.9.** Consider the group action of \( GL(2,2) \) on the matrix ring \( M_2(\mathbb{F}_2) \) by left translation. Then, the following are the orbits of \( M_2(\mathbb{F}_2) \) under this action. Let \( Z \) be the zero matrix in \( M_2(\mathbb{F}_2) \). Then \( \overline{Z} = \{ Z \} \). Consider the identity element \( I \in M_2(\mathbb{F}_2) \) which is a unit in \( M_2(\mathbb{F}_2) \). So, \( \overline{I} = GL(2,2) \). Now, consider the idempotent elements \( B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Then \( \overline{B_0} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}, \overline{B_1} = \{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \} \) and \( \overline{B} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \).

### 3.4. Orbit codes over \( M_2(\mathbb{F}_q) \)

In this section, we construct nonlinear constant weight orbit codes over \( M_2(\mathbb{F}_q) \). A (nonlinear) code \( C \) is a constant weight code provided every nonzero codeword has the same weight. In the following theorem, we use the explicit forms of the orbits of the group action of \( GL(2,q) \) on \( M_2(\mathbb{F}_q) \) by left translation to construct nonlinear constant weight orbit codes.

**Theorem 3.10.** Let \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B_k = \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix} \) where \( k \in \mathbb{F}_q \). If \( C = \overline{B} = \left\{ \begin{pmatrix} a & b \\ 0 & b \end{pmatrix}; a, b \in \mathbb{F}_q \right\} \) or \( C = \overline{B_k} = \left\{ \begin{pmatrix} a & bk \\ b & bk \end{pmatrix}; a, b \in \mathbb{F}_q \right\} \), then \( C \) is a nonlinear constant weight orbit code over \( M_2(\mathbb{F}_q) \) with minimum homogeneous distance \( d_{\text{hom}} = \frac{q^2}{q^2 - 1} \Gamma \).

Proof:

Each of the idempotent elements \( B, B_k \in M_2(\mathbb{F}_q) \), where \( k \in \mathbb{F}_q \), determines an orbit under the action whose elements are the nonzero elements of a minimal left ideal of \( M_2(\mathbb{F}_q) \). Furthermore, note that each minimal left ideal of \( M_2(\mathbb{F}_q) \) is an additive group whose elements are zero divisors and the zero element since it is a principal left ideal generated by a zero divisor. Thus, for each \( k \in \mathbb{F}_q \), \( B \) and \( B_k \) are nonlinear constant weight orbit codes over \( M_2(\mathbb{F}_q) \).

Now, we investigate the optimality of the code constructed in Theorem 3.10. As shown in Theorem 3.11, this generalized Plotkin bound is discussed in [8]. Given a block code \( C \) over the finite Frobenius ring \( R \) which is not necessarily linear, and the homogeneous weight \( w \) defined on \( C \) with average value \( \Gamma \geq 0 \), we take \( C \) as an \( (n,M,d) \) code of length \( n \) with \( M \) codewords and minimum distance \( d \).

**Theorem 3.11.** If \( C \) is an \( (n,M,d) \) code, then
\[
M(M - 1)d \leq \sum_{x,y \in C} w(x,y) \leq \Gamma n M^2.
\]
(8)

It follows from this inequality that \( d \leq \frac{\Gamma n M}{M - 1} \). Hence, we come up with the following remark.

**Remark 3.12.** Let \( C \) be an orbit code over \( M_2(\mathbb{F}_q) \) as constructed in Theorem 3.10. Note that the code \( C \) is a \( \left( 1, q^2 - 1, \frac{q^2}{q^2 - 1} \right) \) code with respect to the normalized homogeneous weight. As the field size \( q \) becomes larger, the minimum normalized homogeneous distance of \( C \) approaches to the Plotkin bound. In this case, we say that \( C \) is ‘nearly’ Plotkin optimal.
Example 3.13. Let $C$ be the orbit of $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(F_2)$, that is, $C = \overline{A} = \{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \}$. Observe that each element of $C$ is a zero divisor of $M_2(F_2)$. Furthermore, the nonzero elements of the minimal left ideal generated by $A$ are the elements of $C$. Thus, $C$ is a (nonlinear) constant weight code with a common normalized homogeneous weight of $4/3$ and a minimum normalized homogeneous distance of $4/3$.

In Table 1, we illustrate how the minimum normalized homogeneous distance $d_{\text{min}}$ of the code $C$ over $M_2(F_q)$ as constructed in Theorem 3.10, approaches the Plotkin bound $d_P$ as we increase the field size $q$.

**Table 1.** Relative Values of the Homogeneous Distance and the Plotkin Bound.

| $q$ | $d_{\text{min}}$ | $d_P$ | Relative Difference (%) |
|-----|------------------|-------|------------------------|
| 2   | 4/3              | 3/2   | 11.1111                |
| 3   | 9/8              | 8/7   | 1.5625                 |
| 4   | 16/15            | 15/14 | 0.4444                 |
| 8   | 64/63            | 63/62 | 0.0252                 |
| 9   | 81/80            | 80/79 | 0.0156                 |
| 16  | 256/255          | 255/254 | 0.0015       |

The relative difference in Table 1 is the value $\frac{d_P - d_{\text{min}}}{d_P} \times 100$. It can be observed that as the field size $q$ becomes larger, the rate of convergence of $d_{\text{min}}$ to the Plotkin bound is reasonably fast. Practically, when $q \geq 16$, Plotkin optimality is safely achieved.

4. Summary and Conclusion

After determining the one-sided ideal structure of $M_2(F_q)$, an explicit form of the homogeneous weight of $M_2(F_q)$ was derived using the Möbius and Euler phi functions. Moreover, the action of the general linear group $GL(2,q)$ on $M_2(F_q)$ by left translation provides a novel construction of nonlinear constant weight orbit codes over $M_2(F_q)$ whose homogeneous distance approaches the Plotkin bound as the field size $q$ becomes larger.

Acknowledgement

The first author acknowledges the support from the DOST-ASTHRD Program of the Philippine government.

References

[1] Oggier F and Solé P 2012 *IEEE Trans. Inform. Theory* **58** 734
[2] Falcunit D and Sison V 2014 *Proc. of the 2014 Int. Zürich Seminar on Communications, Zürich, Switzerland* 91
[3] Trautmann A L, Manganiello F, Braun M and Rosenthal J 2013 *IEEE Trans. Inform. Theory* **59** 7386
[4] Constantinescu I and Heise W 1997 *Probl. Inform. Transm.* **33** 208
[5] Greferath M and Schmidt S E 2000 *J. Combin. Theory A* **92** 17
[6] Fan Y and Liu H 2010 *Math. Ann. (Chinese)* A **31** 355
[7] Chen H, Yang X and Zhou Y 2006 *J. Algebra* **301** 280
[8] Greferath M and O’Sullivan M E 2004 *Discrete Math.* **289** 11