ON MCQUILLAN’S “TAUTOLOGICAL INEQUALITY” AND
THE WEYL-AHLFORS THEORY OF ASSOCIATED CURVES

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Abstract. In 1941, L. Ahlfors gave another proof of a 1933 theorem of H. Cartan on
approximation to hyperplanes of holomorphic curves in \( \mathbb{P}^n \). Ahlfors’ proof built on earlier
work of H. and J. Weyl (1938), and proved Cartan’s theorem by studying the associated
curves of the holomorphic curve. This work has subsequently been reworked by H.-H.
Wu in 1970, using differential geometry, M. Cowen and P. A. Griffiths in 1976, further
emphasizing curvature, and by Y.-T. Siu in 1987 and 1990, emphasizing meromorphic
connections. This paper gives another variation of the proof, motivated by successive
minima as in the proof of Schmidt’s Subspace Theorem, and using McQuillan’s “tauto-
logical inequality.” In this proof, essentially all of the analysis is encapsulated within
a modified McQuillan-like inequality, so that most of the proof primarily uses methods
of algebraic geometry, in particular flag varieties. A diophantine conjecture based on
McQuillan’s inequality is also posed.

Cartan’s theorem on value distribution with respect to hyperplanes in \( \mathbb{P}^n \) in gen-
eral position [Ca] remains to this day as one of the pillars of value distribution of
holomorphic curves in higher-dimensional spaces.

**Theorem 0.1** (Cartan). Let \( n \in \mathbb{Z}_{>0} \), and let \( H_1, \ldots, H_q \) be hyperplanes in \( \mathbb{P}^n_{\mathbb{C}} \) in
general position (i.e., all collections of up to \( n + 1 \) of the corresponding linear
forms on \( \mathbb{C}^{n+1} \) are linearly independent). Let \( f: \mathbb{C} \to \mathbb{P}^n \) be a holomorphic curve
whose image is not contained in any hyperplane. Then

\[
\sum_{i=1}^{q} m_f(H_i, r) \leq_{\text{exc}} (n + 1)T_f(r) + O(\log^+ T_f(r)) + o(\log r)
\]

Here the notation \( \leq_{\text{exc}} \) means that the inequality holds for all \( r \) outside a union
of intervals of finite total length.

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Cartan’s proof is short and clever, but (so far) has not given much insight into the analogies with number theory [O], [V 1].

A few years after Cartan’s proof came out, H. and J. Weyl [W-W] and L. Ahlfors [A] developed a different proof of this theorem, based on a theory of associated curves modeled after the algebraic case. Briefly, if \( f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) is a holomorphic curve whose image is not contained in any hyperplane, and if \( x: \mathbb{C} \to \mathbb{C}^{n+1} \) is a lifting of \( f \), then for \( d = 1, \ldots, n \) the \( d \)th associated curve of \( f \) is the holomorphic curve from \( \mathbb{C} \) to the Grassmannian \( \text{Gr}_d(\mathbb{C}^{n+1}) \) given for general \( z \in \mathbb{C} \) by the linear subspace spanned by the vectors \( x(z), x'(z), \ldots, x^{(d-1)}(z) \). In contrast to Cartan’s proof, this proof is quite technical, and in fact Ahlfors notes in his Collected Works that a reviewer had described it as a “tour de force.”

This latter proof has been revisited over the years. In 1970 H.-H. Wu [Wu] revisited the theory, revising it with an emphasis on differential geometry. In 1976 M. Cowen and P. Griffiths [Co-G] reworked the proof with further emphasis on curvature (as part of a program of Griffiths to rework much of Nevanlinna theory using curvature). In the late 1980s, Y.-T. Siu reworked the proof again using meromorphic connections [S 1], [S 2].

Although it is longer and more technical, Ahlfors’ approach is better than Cartan’s from the point of view of comparison with the corresponding theorem in number theory, Schmidt’s Subspace Theorem. Schmidt’s theorem was proved using the theory of successive minima, and many of the constructions closely parallel those of associated curves. Because of this, it is my conviction that the derivative of a holomorphic function should translate into number theory as some sort of object involving successive minima. Most likely, the derivative of a holomorphic curve in a complex variety \( X \), modulo scalar multiplication (hence, a holomorphic curve in \( \mathbb{P}(\Omega^1_X) \), using Grothendieck’s convention on \( \mathbb{P}(\mathcal{F}) \) as described in Section 1) should correspond to the following object in Arakelov theory. Let \( Y = \text{Spec} \mathcal{O}_k \) for a number field \( k \) (or, let \( Y \) be a smooth projective curve over a field of characteristic zero, and let \( k \) be its function field), let \( X \) be an arithmetic variety over \( Y \), let \( P \in X_k(k) \) be a \( k \)-rational point, and let \( i: Y \to X \) be the corresponding arithmetic curve in \( X \). Then the derivative should correspond to a line subsheaf of maximal Arakelov degree in \( i^*\Omega^d_X/\mathcal{O}_k \). See Section 2.

This comparison (except for some additional details) was explored in Chapter 6 of [V 1]. That chapter presented Ahlfors’ and Schmidt’s proofs of their respective theorems from the point of view of describing their commonalities. This effort was hamstrung, however, by a key difference in their proofs. Let \( V \) be the vector space \( \mathbb{C}^{n+1} \) or \( k^{n+1} \) in the complex analytic and number theoretic contexts, respectively. When Schmidt worked in \( \bigwedge^d V \), he used hyperplanes corresponding to \( d \) of the original linear forms on \( V \), hence an element of \( \bigwedge^d V^\vee \). Ahlfors, on the other hand, always worked with a single hyperplane on \( V \), via a more general interior product, defined (in one special case) as follows. Let \( X \in \bigwedge^d V \) and let \( b \in V^\vee \). Then \( (X \cdot b) \) is the element of \( \bigwedge^{d-1} V \) characterized by the condition that

\[
(X \cdot b) \cdot Z = (X \cdot (b \wedge Z))
\]
for all $Z \in \bigwedge^{d-1} V^\vee$.

More recently, M. McQuillan formulated a tautological inequality (Theorem 1.2 below) which provides an elegant geometrical statement closely related to geometric generalizations of the lemma on the logarithmic derivative formulated by J. Noguchi [N 1] and [N 2], S. Lu [Lu], P.-M. Wong [Wo-S], K. Yamanoi [Y], and others. This significantly cleared up the picture for the situation of Cartan’s theorem (and Ahlfors’ proof thereof), and enabled me to translate Schmidt’s proof into the complex analytic case, obtaining a proof similar to Ahlfors’, but different in the sense that the associated curves in $\bigwedge^d V$ are now being compared with hyperplanes in $\bigwedge^d V^\vee$; i.e., the generalized interior product is no longer present. (Therefore, this proof could be regarded as being closer to Cartan’s than Ahlfors.’)

In addition, the proof in this paper is phrased almost entirely in terms of algebraic geometry: almost all of the analysis is encapsulated in McQuillan’s result.

Following [V 3], we actually prove a slightly stronger theorem than Theorem 0.1:

**Theorem 0.2.** Let $n \in \mathbb{Z}_{>0}$ and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ (not necessarily in general position). For $j = 1, \ldots, q$ let $\lambda_{H_j}$ be a corresponding Weil function; e.g.,

$$
\lambda_{H_j}(z) = -\frac{1}{2} \log \frac{|L_j(z_0, \ldots, z_n)|^2}{|z_0|^2 + \cdots + |z_n|^2},
$$

where $L_j$ is a linear form associated to $H_j$ and $[z_0 : \cdots : z_n]$ are homogeneous coordinates for $z \in \mathbb{P}^n(\mathbb{C}) \setminus H_j$. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve whose image is not contained in any hyperplane. Then

$$
\int_0^{2\pi} \max_J \sum_{i \in J} \lambda_{H_i}((fr e^{\sqrt{-1} \theta})) \frac{d\theta}{2\pi} \leq \text{exc}(n + 1)T_f(r) - N_W(r) + O(\log T_f(r)) + o(\log r),
$$

where the maximum is taken over all subsets $J$ of $\{1, \ldots, q\}$ for which the hyperplanes $H_i$, $i \in J$, lie in general position, and $N_W(r)$ is the counting function for the zeroes of the Wronskian of a choice of homogeneous coordinates of $f$ (with no common zeroes).

The paper is organized as follows. Section 1 gives McQuillan’s result. A little additional work is needed because a slightly more general theorem is needed here. Section 2 introduces a diophantine conjecture motivated by McQuillan’s result and the strengthening proved in Section 1. Section 3 contains a geometric discussion of Grassmannian and flag varieties. Section 4 discusses successive minima in the context of Schmidt’s theorem and proof, and uses this discussion to motivate the general outline of the proof that follows. Section 5 proves the main step of the proof in the case $d = 1$, and Section 6 gives the general case. The proof of Theorem 0.2 concludes in Section 7.

It would be more natural to phrase the proof in terms of jet spaces over $\mathbb{P}^n$ instead of spaces $\bigwedge^d V^\vee$, but attempts to do so did not succeed.
Most of the standard notations of Nevanlinna theory (proximity function $m_f(D, r)$, counting function $N_f(D, r)$, and height (characteristic) function $T_{f, \mathcal{L}}(r)$) are the standard ones, as in for example ([N 2], §1) (plus $T_{f, \mathcal{L}}(r) = T_{f, c_1(\mathcal{L})}(r)$).

§1. McQuillan’s “Tautological Inequality”

In this section let $X$ be a smooth compact complex variety and let $D$ be a normal crossings divisor (i.e., a divisor whose only singularities are locally of the form $z_1 \ldots z_r = 0$ for a suitable local holomorphic coordinate system $z_1, \ldots, z_n$ on $X$). Throughout this paper, normal crossings divisors are assumed to be effective and reduced.

If $E$ is a vector sheaf on $X$, then $P(E)$ is defined as $\text{Proj} \bigoplus_{d \geq 0} S^d E$, so that points on $P(E)$ lying over a point $x \in X$ correspond naturally to hyperplanes in the fiber $E_x/\mathfrak{m}_x E_x$ of $E$ at $x$.

Let $f: \mathbb{C} \to X$ be a non-constant holomorphic curve whose image is not contained in the support of $D$. Then $f$ lifts to a holomorphic map $f': \mathbb{C} \to \mathbb{P}(\Omega_X/\mathbb{C}(\log D))$.

Definition 1.1. The $D$-modified ramification counting function of $f$ is the counting function for vanishing of the pull-back $f^* \Omega_X/\mathbb{C}(\log D)$ (i.e., it counts the smallest order of vanishing of $f^*s$ at $z \in \mathbb{C}$ for local sections $s$ of $\Omega_X/\mathbb{C}(\log D)$ near $f(z)$).

McQuillan’s tautological inequality is then the following. It appeared in [McQ 1] with $D = 0$ and ([McQ 2], V.1.2) in general. See also ([V 4], Thm. A.6).

Theorem 1.2. Let $X, D, f$, and $f'$ be as above, and let $\mathcal{A}$ be a line sheaf on $X$ whose restriction to the Zariski closure of the image of $f$ is big. Then

$$T_{\mathcal{A}(1), f'}(r) \leq_{\text{exc}} N_f^{(1)}(D, r) - N_{\text{Ram}(D), f}(r) + O(\log^+ T_{\mathcal{A}, f}(r)) + o(\log r).$$

Let us compare Theorem 1.2 with the corresponding statement with $D = 0$. Of course $\mathcal{A}(1)$ is different, since $\mathbb{P}(\Omega_X/\mathbb{C})$ is not isomorphic to $\mathbb{P}(\Omega_X/\mathbb{C}(\log D))$. Also the term $N_f^{(1)}(D, r)$ goes away when $D = 0$. The difference in heights can be written in a canonical way as the height relative to a certain Cartier divisor on the closure of the graph of the canonical birational map $\mathbb{P}(\Omega_X/\mathbb{C}(\log D)) \dashrightarrow \mathbb{P}(\Omega_X/\mathbb{C})$, and the counting function associated to that divisor almost exactly cancels the term $N_f^{(1)}(D, r)$ (there are also differences in the ramification term). This makes sense, since McQuillan’s inequality is derived from a version of the lemma on the logarithmic derivative, which concerns only proximity functions.

This paper will need a variation of McQuillan’s inequality which refrains from moving things over into counting functions, and which considers a finite list of normal crossings divisors $D_1, \ldots, D_\ell$ in place of $D$. These divisors may be chosen independently, so that the support of their sum need not have normal crossings. In order to state this modified McQuillan inequality, we define a function on $\mathbb{P}(\Omega_X/\mathbb{C})$ as follows.
Lemma 1.3. Let \( X \) and \( D \) be as above. Let \( \Gamma \) be the closure of the graph of the canonical birational map \( \mathbb{P}(\Omega_X/\mathbb{C}(\log D)) \to \mathbb{P}(\Omega_X/\mathbb{C}) \), let \( p: \Gamma \to \mathbb{P}(\Omega_X/\mathbb{C}(\log D)) \) and \( q: \Gamma \to \mathbb{P}(\Omega_X/\mathbb{C}) \) be the projection morphisms, and let \( \partial_{\log}(1) \) and \( \partial(1) \) denote the tautological line sheaves on \( \mathbb{P}(\Omega_X/\mathbb{C}(\log D)) \) and \( \mathbb{P}(\Omega_X/\mathbb{C}) \), respectively. Then there is a Cartier divisor \( E \) on \( \Gamma \) and a canonical isomorphism \( p^*\partial_{\log}(1) \cong q^*\partial(1) \otimes \partial(E) \).

Moreover, let \( \lambda \) be a Weil function for \( E \) on \( \Gamma \), and let \( z_1, \ldots, z_n \) be local coordinates on \( X \) such that \( D \) is locally given by \( z_1 \cdots z_r = 0 \). Then \( \lambda \) corresponds to a function \( \mu \) on \( \mathbb{P}(\Omega_X/\mathbb{C}) \) satisfying

\[
(1.3.1) \quad \mu = \frac{1}{2} \log \frac{|dz_1|^2 + \cdots + |dz_n|^2}{|dz_1/z_1|^2 + \cdots + |dz_r/z_r|^2 + |dz_{r+1}|^2 + \cdots + |dz_n|^2} + O(1)
\]

on compact subsets of the coordinate patch, minus the support of \( D \).

Proof. By canonicity, we may work locally on \( X \), so let \( z_1, \ldots, z_n \) be local coordinates as above. If \( U = X \setminus \text{Supp} \, D \), then \( \Omega_X/\mathbb{C} \) and \( \Omega_X/\mathbb{C}(\log D) \) are canonically isomorphic over \( U \), so the above birational map gives an isomorphism \( \phi: \pi_{\log}^{-1}(U) \to \pi^{-1}(U) \), where \( \pi_{\log}: \mathbb{P}(\Omega_X/\mathbb{C}(\log D)) \to X \) and \( \pi: \mathbb{P}(\Omega_X/\mathbb{C}) \to X \) are the natural projections. This isomorphism naturally gives an isomorphism between \( \partial_{\log}(1)|_{\pi_{\log}^{-1}(U)} \) and \( \phi^*\partial(1)|_{\pi^{-1}(U)} \). This shows that \( E \) exists.

To construct the Weil function, let \( z_1, \ldots, z_n \) be a local coordinate system as above, and let \( s = a_1 dz_1 + \cdots + a_n dz_n \) be a local section of \( \Omega_X/\mathbb{C} \), where \( a_i \) are local functions on \( X \). Then, regarding \( s \) as a section of \( \partial(1) \) on \( \mathbb{P}(\Omega_X/\mathbb{C}) \), its divisor has a Weil function of the form

\[
-\frac{1}{2} \log \frac{|a_1 dz_1 + \cdots + a_n dz_n|^2}{|dz_1|^2 + \cdots + |dz_n|^2} + O(1) .
\]

Doing the same for \( s \) as a section of \( \partial_{\log}(1) \) gives a Weil function of the form

\[
-\frac{1}{2} \log \frac{|a_1 dz_1 + \cdots + a_n dz_n|^2}{|dz_1/z_1|^2 + \cdots + |dz_r/z_r|^2 + |dz_{r+1}|^2 + \cdots + |dz_n|^2} + O(1) .
\]

Pulling these back to \( \Gamma \) and subtracting then gives (1.3.1). \( \square \)

Remark 1.4. Since \( \mu \) is bounded from below, the pull-back of \( E \) to the normalization of \( \Gamma \) is effective.

Definition 1.5 ([V2], Def. 7.1). A generalized Weil function on a variety is a function on a dense open subset of the variety that pulls back to a Weil function on some blowing-up of the variety.

The function \( \mu \) in Lemma 1.3 is an example of a generalized Weil function that is not a Weil function. This function may be thought of as an approximate archimedean...
version of the truncated counting function at $D$. For example, near smooth points of $D$, where we may assume that $D$ is locally given by $z_1 = 0$, the graph $\Gamma$ in Lemma 1.3 is the blowing-up of $\mathbb{P}(\Omega_X/\mathbb{C})$ at $z_1 = d_2 z_1 = 0$, and $\mu$ is the proximity function of the strict transform of $D$ in that blowing-up. One may be tempted to denote it $m_f^{(1)}(D, r)$.

The version of McQuillan’s inequality to be used here is the following.

**Theorem 1.6.** Let $X$ be a smooth complete complex variety, and let $D_1, \ldots, D_\ell$ be normal crossings divisors on $X$ (whose sum need not have normal crossings support). Let $f : C \to X$ be a non-constant holomorphic curve whose image is not contained in the support of $\sum D_i$, let $f' : \mathbb{C} \to \mathbb{P}(\Omega_X/\mathbb{C})$ be the lifting of $f$, and let $\mathcal{A}$ be a line sheaf on $X$ whose restriction to the Zariski closure of the image of $f$ is big. Let $\mu_1, \ldots, \mu_\ell$ be generalized Weil functions on $\mathbb{P}(\Omega_X/\mathbb{C})$ obtained from $D_1, \ldots, D_\ell$, respectively, as in Lemma 1.3. Finally, let $N_{\text{Ram}, f}(r)$ be the counting function for the ramification of $f$. Then

$$T_{\theta(1), f'}(r) + \int_0^{2\pi} \max_{1 \leq i \leq \ell} \mu_i(f'(re^{\sqrt{-1}\theta})) \frac{d\theta}{2\pi} + N_{\text{Ram}, f}(r) \leq_{\text{exc}} O(\log^+ T_{\mathcal{A}, f}(r)) + o(\log r).$$

**Proof.** This proof is just a straightforward adaptation of the proof from ([V 4], Appendix), making obvious changes to accommodate the multiple divisors. We start with an enhanced version of the geometric lemma on the logarithmic derivative due to P.-M. Wong ([Wo-S], Thm A3):

**Lemma 1.6.2.** Let $X$, $D_1, \ldots, D_\ell$, $f$, and $\mathcal{A}$ be as in Theorem 1.6. Let $m \in \mathbb{Z}_{>0}$, and for each $i = 1, \ldots, \ell$ let $\| \cdot \|_i$ be a continuous pseudo jet metric on the jet space $J^m X(-\log D_i)$. Also let $j_D^m, f : C \to J^m X(-\log D_i)$ denote the $m^\text{th}$ jet lifting of $f$. Then

$$\int_0^{2\pi} \max_{1 \leq i \leq \ell} \| j_D^m, f(re^{\sqrt{-1}\theta}) \|_i \frac{d\theta}{2\pi} \leq_{\text{exc}} O(\log^+ T_{\mathcal{A}, f}(r)) + o(\log r).$$

**Proof.** This is proved by reducing to the special case $\ell = 1$, which is proved already ([Wo-S], Thm A3). Let $\pi : X' \to X$ be an embedded resolution of the set $\bigcup \text{Supp} D_i$, let $D'$ be the normal crossings divisor on $X'$ lying over this set, let $\| \cdot \|'$ be a continuous jet metric on $J^m X'(-\log D')$, and let $g : C \to X'$ be the holomorphic lifting of $f$. By compactness, we have

$$\| j_D^m, f(z) \|_i \leq \| j_D^{m'}, g(z) \|'$$

for all $z \in \mathbb{C}$ and all $i = 1, \ldots, \ell$, with a constant independent of $z$. Therefore

$$\int_0^{2\pi} \max_{1 \leq i \leq \ell} \| j_D^m, f(re^{\sqrt{-1}\theta}) \|_i \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \max_{1 \leq i \leq \ell} \| j_D^m, f(re^{\sqrt{-1}\theta}) \|' \frac{d\theta}{2\pi} + O(1) \leq_{\text{exc}} O(\log^+ T_{\mathcal{A}, f}(r)) + o(\log r) \leq O(\log^+ T_{\mathcal{A}, f}(r)) + o(\log r),$$
We now recall some notation from \([V 4]\). Let \(\Gamma\) be the closure of the rational map \(\mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X) \dashrightarrow \mathbb{P}(\Omega_{X/\mathbb{C}})\), and let \(p: \Gamma \rightarrow \mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\) and \(q: \Gamma \rightarrow \mathbb{P}(\Omega_{X/\mathbb{C}})\) be the canonical projections. The graph \(\Gamma\) is obtained from \(\mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\) by blowing up the zero section of \(\mathbb{V}(\Omega_{X/\mathbb{C}}) \subseteq \mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\); let \(\[0\]\) denote the exceptional divisor. The pull-backs of the tautological line sheaves on \(\mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\) and \(\mathbb{P}(\Omega_{X/\mathbb{C}})\) are related by

\[
q^* \mathcal{O}(1) \cong p^* \mathcal{O}(1) \otimes \mathcal{O}(-[0]) .
\]

Also let \([\infty]\) denote the divisor \(\mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X) \setminus \mathbb{V}(\Omega_{X/\mathbb{C}})\). Since \(\mathcal{O}([\infty]) \cong \mathcal{O}(1)\) on \(\mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\) and since the lifted curve \(\partial f: \mathbb{C} \rightarrow \mathbb{P}(\Omega_{X/\mathbb{C}} \oplus \mathcal{O}_X)\) never meets \([\infty]\), we have

\[
T_{\mathcal{O}(1), \partial f}(r) = m_{\partial f}([\infty], r) + O(1)
\]

for a continuous metric \(\| \cdot \|\) on the tangent bundle \(TX\), which we now fix. Let \(g: \mathbb{C} \rightarrow \Gamma\) be the lifting of \(f\) (so that \(p \circ g = \partial f\) and \(q \circ g = f'\)). By definition, \(N_{\text{Ram}, f}(r) = N_{g}([0], r)\), so (1.6.3) and (1.6.4) combine to give

\[
T_{\mathcal{O}(1), f'}(r) + N_{\text{Ram}, f}(r) = \int_0^{2\pi} \log^+ \| Tf(re^{\sqrt{-1} \theta})\| \frac{d\theta}{2\pi} - m_{g}([0], r) + O(1) .
\]

\textbf{Lemma 1.6.6.} Let \(D\) be a normal crossings divisor on \(X\); let \(\mu\) be the corresponding generalized Weil function, as in Lemma 1.3; fix continuous metrics \(\| \cdot \|\) and \(\| \cdot \|_D\) on \(TX\) and \(TX(−\log D)\), respectively; and let \(\lambda_{[0]}\) be a Weil function for the divisor \([0]\) on \(\Gamma\). Then

\[
\log^+ \| Tf(z)\| + \mu(f'(z)) - \lambda_{[0]}(g(z)) \leq \log^+ \| T_D f(z)\|_D + O(1)
\]

for all \(z \in \mathbb{C}\), where the constant in \(O(1)\) is independent of \(f\) and \(z\).

\textbf{Proof.} Choose local coordinates \(z_1, \ldots, z_n\) near \(f(z)\) such that \(D\) is given locally by \(z_1 \cdots z_r = 0\). Write \(f = (f_1, \ldots, f_n)\) in these coordinates. We then have

\[
\log^+ \| Tf(z)\| = \frac{1}{2} \log^+ (|f'_1(z)|^2 + \cdots + |f'_n(z)|^2) + O(1) ,
\]

\[
\log^+ \| T_D f(z)\|_D = \frac{1}{2} \log^+ \left( \frac{|f'_1(z)|^2}{f_1(z)} + \cdots + \frac{|f'_n(z)|^2}{f_r(z)} \right)
\]

\[
+ |f'_{r+1}(z)|^2 + \cdots + |f'_n(z)|^2 \right) + O(1) ,
\]

\[
\lambda_{[0]}(g(z)) = \frac{1}{2} \log^+ \frac{1}{|f'_1(z)|^2 + \cdots + |f'_n(z)|^2} + O(1) ,
\]
and, from (1.3.1),

\[
\mu(f'(z)) = -\frac{1}{2} \log(|f_1'(z)|^2 + \cdots + |f_n'(z)|^2) \\
+ \frac{1}{2} \log \left( \frac{|f_1'(z)|^2}{f_1(z)} + \cdots + \frac{|f_r'(z)|^2}{f_r(z)} \right) \\
+ |f_{r+1}'(z)|^2 + \cdots + |f_n'(z)|^2) + O(1).
\]

Therefore

\[
\log^+ \|T_D f(z)\|_D \geq \frac{1}{2} \log \left( \frac{|f_1'(z)|^2}{f_1(z)} + \cdots + \frac{|f_r'(z)|^2}{f_r(z)} \right) \\
+ |f_{r+1}'(z)|^2 + \cdots + |f_n'(z)|^2) + O(1) \\
= \mu(f'(z)) + \frac{1}{2} \log(|f_1'(z)|^2 + \cdots + |f_n'(z)|^2) + O(1) \\
= \mu(f'(z)) + \log^+ \|T f(z)\| - \lambda_{[0]}(g(z)) + O(1),
\]

as was to be shown. \(\square\)

Applying this lemma with \(D = D_i\) for \(i = 1, \ldots, \ell\), taking the max, and integrating gives

\[
\int_0^{2\pi} \log^+ \|T f(re^{\sqrt{-1}\theta})\|_D \frac{d\theta}{2\pi} + \int_0^{2\pi} \max_{1 \leq i \leq \ell} \mu_i(f'(re^{\sqrt{-1}\theta})) \frac{d\theta}{2\pi} - m_g([0], r)
\]

\[
\leq \int_0^{2\pi} \log^+ \max_{1 \leq i \leq \ell} \|T_{D_i} f(re^{\sqrt{-1}\theta})\|_i \frac{d\theta}{2\pi}
\]

and therefore

\[
T_{\theta(1), f'}(r) + N_{\text{Ram}, f}(r) + \int_0^{2\pi} \max_{1 \leq i \leq \ell} \mu_i(f'(re^{\sqrt{-1}\theta})) \frac{d\theta}{2\pi}
\]

\[
\leq \int_0^{2\pi} \log^+ \max_{1 \leq i \leq \ell} \|T_{D_i} f(re^{\sqrt{-1}\theta})\|_i \frac{d\theta}{2\pi}
\]

by (1.6.5). Combining this with Lemma 1.6.2 (with \(m = 1\)) then gives (1.6.1). \(\square\)

**Remark 1.7.** One can also formulate conjectural Second Main Theorems with multiple divisors \(D_1, \ldots, D_\ell\), and show that they would follow from the \(\ell = 1\) case by the same methods as were used in the proof of Lemma 1.6.2.

**Remark 1.8.** Theorem 1.6 can also be proved in the situation of a finite ramified covering \(p: Y \to \mathbb{C}\) and a holomorphic curve \(f: Y \to X\). In this case one would add a term \(N_{\text{Ram}, p}(r)\) on the right-hand side of (1.6.1). Details are left to the reader.
§2. A diophantine conjecture

As noted in the Introduction, in the number field (or function field) case, the derivative should be expressed in terms of successive minima, which in Arakelov theory corresponds to to successive maxima (of degrees of vector subsheaves). We describe this in more detail as follows, giving only the number field case since the translation to function fields is straightforward.

Let \( k \) be a number field, let \( Y = \text{Spec} \mathcal{O}_k \), and let \( \pi: X \to Y \) be a proper arithmetic variety. Points \( P \in X_k(\bar{k}) \) correspond bijectively to sections \( i: Y \to X \) of \( \pi \).

As noted in the Introduction, comparisons between Schmidt’s proof of his Subspace Theorem and Ahlfors’ proof of Cartan’s theorem suggest that derivatives of a holomorphic function should correspond somehow to a line subsheaf in \( i^*\Omega_{X/Y}^1 \) of maximal degree. Of course \( i^*\Omega_{X/Y} \) can only be assumed to be a vector sheaf if \( \pi \) is smooth, and this restriction is too strong for most applications. We can address this, however, as follows. A line subsheaf of \( i^*\Omega_{X/Y} \) of maximal degree corresponds to a quotient line sheaf of \( i^*\Omega_{X/Y} \) of minimal degree, and this corresponds to a lifting \( i': Y \to \mathbb{P}(\Omega_{X/Y}) \) of \( i: Y \to X \) for which \((i')^*\mathcal{O}(1)\) has minimal degree, where \( \mathcal{O}(1) \) is the tautological line sheaf. Of course this degree is none other than the height of the lifted rational point relative to \( \mathcal{O}(1) \). This leads to the following conjecture, which should give a number-theoretic analogue of Theorem 1.6.

**Conjecture 2.1.** Let \( k \) and \( Y \) be as above, let \( S \) be a finite set of places of \( k \) containing all of the archimedean places, let \( \pi: X \to Y \) be a proper arithmetic variety with smooth projective generic fiber \( X_k \), let \( D_1, \ldots, D_\ell \) be effective Cartier divisors on \( X \) whose restrictions to \( X_k \) are normal crossings divisors, and let \( \mu_1, \ldots, \mu_\ell \) be generalized Weil functions on \( X_k \) obtained from \( D_1, \ldots, D_\ell \), respectively, as in Lemma 1.3. Fix an ample line sheaf \( \mathcal{A} \) on \( X_k \), fix absolute heights \( h_{\mathcal{O}(1)} \) on \( \mathbb{P}(\Omega_{X/Y}) \) and \( h_{\mathcal{A}} \) on \( X \), and fix constants \( \epsilon > 0 \) and \( C \). For \( P \in X_k(k) \) let \( S_P \) be the set of places of \( k(P) \) lying over places in \( S \). Then, for all but finitely many \( P \in X_k(k) \) not lying in the support of any \( D_i \), there exists a point \( P' \in \mathbb{P}(\Omega_{X/Y})(k(P)) \) lying over \( P \) and satisfying the inequality

\[
(2.1.1) \quad h_{\mathcal{O}(1)}(P') + \frac{1}{[k(P):\mathbb{Q}]} \sum_{w \in S_P} \max_{1 \leq i \leq \ell} \mu_{i,w}(P') \leq d(P) + \epsilon h_{\mathcal{A}}(P) + C. 
\]

Here

\[
d(P) = \frac{1}{[k(P):\mathbb{Q}]} \log |D_{k(P)}|
\]

is as in ([V 1], pp. 57–58). Of course, if we restrict to rational points \( P \in X_k(k) \) then this term goes away. Also, if \( \ell = 0 \) then (2.1.1) becomes further shortened to

\[
(2.2) \quad h_{\mathcal{O}(1)}(P') \leq \epsilon h_{\mathcal{A}}(P) + O(1). 
\]

The restriction that \( X_k \) be projective is not essential, but eliminating it would require an adequate replacement for \( h_{\mathcal{A}} \), which would take some work.
The point $P'$ may not be uniquely determined by (2.1.1); however, Schmidt’s proof focuses on points where there is a gap in the successive minima, and such a gap would cause $P'$ to be uniquely defined. One could then drop the requirement that $P'$ be rational over $k(P)$.

This conjecture is somewhat reminiscent of Szpiro’s work on small points. Some differences include the (non-essential) fact that the above conjecture looks only at points rational over $k(P)$, the fact that this conjecture is phrased in a relative setting, and the fact that we are looking at the smallest height rather than a lim inf of heights.

This conjecture will be developed further in a subsequent paper.

§3. Some Geometry of Grassmann and Flag Varieties

This section provides a basic result about flag and Grassmann varieties (Proposition 3.7). Although we only need the result in the case $X = \text{Spec} \mathbb{C}$, we work over an arbitrary scheme $X$ since it is not any harder, and it may be useful in later work.

Throughout this section, $\mathcal{E}$ is a vector sheaf of rank $r$ over a scheme $X$.

The basic idea is as follows. Let $s \in \mathbb{Z}_{>0}$ and let $r \geq d_1 > \cdots > d_s \geq 0$ be integers. The flag variety is a scheme over $X$ whose fiber over $x \in X$ parametrizes flags of linear subspaces

$$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_s \subseteq \mathcal{E}(x),$$

where $\mathcal{E}(x)$ is the fiber of $\mathcal{E}$ over $x$, and where $W_i$ is a $k(x)$-vector subspace of the fiber $\mathcal{E}(x)$ of codimension $d_i$ for all $i$. However, it is useful to phrase this definition as a moduli problem.

**Definition 3.1.** Let $s \in \mathbb{Z}_{>0}$ and let $r \geq d_1 > \cdots > d_s \geq 0$ be integers. Then the flag bundle $\text{Fl}^{d_1, \ldots, d_s}(\mathcal{E})$ (if it exists) is the $X$-scheme $\pi: \text{Fl}^{d_1, \ldots, d_s}(\mathcal{E}) \to X$ representing the contravariant functor $F$ from $X$-schemes to sets, defined as follows. If $\phi: T \to X$ is an $X$-scheme, then $F(T)$ is the set of all flags

$$(3.1.1) \quad \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_s$$

of vector subsheaves of $\phi^*\mathcal{E}$ such that $\phi^*\mathcal{E}/\mathcal{F}_i$ is a vector sheaf of rank $d_i$ for all $i$. In the special case $s = 1$, the flag bundle $\text{Fl}^d(\mathcal{E})$ (for $0 \leq d \leq r$) is also called the Grassmann bundle, and is denoted $\text{Gr}^d(\mathcal{E})$. It represents the functor of vector subsheaves whose quotient is a vector sheaf of rank $d$.

**Remark 3.2.** Often it is more convenient to refer to chains of surjections

$$\phi^*\mathcal{E} \twoheadrightarrow \mathcal{G}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{G}_s,$$

where each $\mathcal{G}_i$ is a vector sheaf of rank $d_i$, in place of flags (3.1.1). The flag bundle also represents the functor of these objects, up to an obvious notion of isomorphism.
As is customary with Chow groups, superscripts indicate codimension and subscripts indicate dimension, so we also write
\[ \text{Fl}_{d_1, \ldots, d_s}(E) = \text{Fl}^{r-d_1, \ldots, r-d_s}(E) \quad \text{and} \quad \text{Gr}_d(E) = \text{Gr}^{r-d}(E), \]
where \( 0 \leq d_1 < \cdots < d_s \leq r \) and \( 0 \leq d \leq r \), respectively.

Finally, if \( X = \text{Spec } k \) for a field \( k \), and \( V \) is a finite-dimensional vector space over \( k \), then we write \( \text{Fl}^{d_1, \ldots, d_s}(V) = \text{Fl}^{d_1, \ldots, d_s}(\tilde{V}) \) and \( \text{Gr}^d(V) = \text{Gr}^d(\tilde{V}) \), etc. These are the \textbf{flag varieties} and \textbf{Grassmann varieties}, respectively.

**Proposition 3.3.** Let \( s \in \mathbb{Z}_{>0} \) and let \( r \geq d_1 > \cdots > d_s \geq 0 \) be integers. Then the flag bundle \( \pi: \text{Fl}^{d_1, \ldots, d_s}(E) \to X \) exists, and has universal sheaves
\[ \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{U}_s \subseteq \pi^* E \]
such that if \( f: T \to \text{Fl}^{d_1, \ldots, d_s}(E) \) corresponds to a flag \( (3.1.1) \), then \( \mathcal{F}_i = f^* \mathcal{U}_i \) for all \( i \).

**Proof.** Following ([F 2], Prop. 14.2.1), we use induction on \( s \). The case \( s = 1 \) (the Grassmannian) was already proved by Kleiman ([K], Prop. 1.2). If \( s > 1 \) then let \( \mathcal{U}_2' \subseteq \cdots \subseteq \mathcal{U}_s' \) be the universal sheaves on \( \text{Fl}^{d_2, \ldots, d_s}(E) \); then
\[ \text{Fl}^{d_1, \ldots, d_s}(E) = \text{Gr}^{d_2-d_1}(\mathcal{U}_2') \]
represents the functor of Definition 3.1, with universal sheaf \( \mathcal{U}_1 \) equal to the universal sheaf of the Grassmannian and \( \mathcal{U}_i \) equal to the pull-backs of \( \mathcal{U}_i' \) for \( i = 2, \ldots, s \).

We note that \( \text{Gr}^1(E) = \mathbb{P}(E) \), and that its universal sheaf is the kernel of the canonical map \( \pi^* E \to O(1) \) ([EGA], II 4.2.5).

If \( d_1, \ldots, d_s \) are as above, and if \( t \in \mathbb{Z}_{>0} \) and \( r \geq e_1 > \cdots > e_t \geq 0 \) are integers such that \( \{e_1, \ldots, e_t\} \) is a subset of \( \{d_1, \ldots, d_s\} \), then there is a \textbf{forgetful morphism}
\[ \text{Fl}^{d_1, \ldots, d_s}(E) \to \text{Fl}^{e_1, \ldots, e_t}(E) \]
over \( X \), and those universal bundles \( \mathcal{U}_i \) on \( \text{Fl}^{d_1, \ldots, d_s}(E) \) for which \( d_i = e_j \) for some \( j \), are pull-backs of the corresponding universal bundles \( \mathcal{U}_j' \) on \( \text{Fl}^{e_1, \ldots, e_t}(E) \). If \( 0 \leq i < j \leq s \), then the diagram
\[
\begin{array}{c}
\text{Fl}_{d_1, \ldots, d_s}(E) \longrightarrow \text{Fl}_{d_i, \ldots, d_s}(E) \\
\downarrow \quad \downarrow \\
\text{Fl}_{d_1, \ldots, d_j}(E) \longrightarrow \text{Fl}_{d_i, \ldots, d_j}(E)
\end{array}
\]
(3.4)
in which the arrows are forgetful morphisms, is Cartesian. Indeed, this is clear from the definition of flag varieties as objects representing functors, since if \( \mathcal{F}_{i+1}, \ldots, \mathcal{F}_j \) in (3.1.1) are fixed, then \( \mathcal{F}_1, \ldots, \mathcal{F}_i \) can be chosen independently of \( \mathcal{F}_{j+1}, \ldots, \mathcal{F}_s \).
We now look at line sheaves on flag bundles. For the case of flag varieties, a reference is ([F 1], Ch. 9).

For \(0 \leq d \leq r\) we define a morphism \(i: \text{Gr}^d(\mathcal{E}) \to \mathbb{P}(\wedge^d \mathcal{E})\), by specifying (via Remark 3.2) that a \(T\)-point corresponding to a surjection \(\phi^* \mathcal{E} \to \mathcal{G}\) is taken to the natural surjection \(\phi^* \wedge^d \mathcal{E} \to \wedge^d \mathcal{G}\); here \(\phi: T \to X\) is the structural morphism. By ([K], Prop. 1.5) this is a closed immersion. It is called the Plücker embedding. From this definition (in particular with \(T = \text{Gr}^d(\mathcal{E})\)) it follows that \(i^* \mathcal{O}(1) = \wedge^d (\pi^* \mathcal{E}/ \mathcal{U})\), where \(\mathcal{U}\) is the universal sheaf on \(\text{Gr}^d(\mathcal{E})\). This line sheaf \(i^* \mathcal{O}(1)\) is also denoted \(\mathcal{O}(1)\).

The Grassmann coordinates induce a cover of \(\text{Gr}^d(\mathcal{E}|_U)\) by open subsets isomorphic to \(\mathbb{A}^d(r-d)\), for all open \(U\) on \(X\) for which \(\mathcal{E}|_U\) is trivial ([K], Prop. 1.6). Therefore \(\text{Gr}^d(\mathcal{E})\) is smooth over \(X\) with connected fibers. By induction on \(s\), the same holds for flag bundles.

Returning to flag bundles, let \(r \geq d_1 > \cdots > d_s \geq 0\), let \(\text{pr}_{d_i}: \text{Fl}^{d_1, \ldots, d_s}(\mathcal{E}) \to \text{Gr}^{d_i}(\mathcal{E})\) denote the forgetful morphisms, and for all \(a_1, \ldots, a_s \in \mathbb{Z}\) define the line sheaf

\[
\mathcal{O}(a_1, \ldots, a_s) = \text{pr}_{d_1}^* \mathcal{O}(a_1) \otimes \cdots \otimes \text{pr}_{d_s}^* \mathcal{O}(a_s)
\]

\[
\cong \left(\wedge^{d_1} (\pi^* \mathcal{E}/ \mathcal{U}_1)\right)^{\otimes a_1} \otimes \cdots \otimes \left(\wedge^{d_s} (\pi^* \mathcal{E}/ \mathcal{U}_s)\right)^{\otimes a_s}
\]

on \(\text{Fl}^{d_1, \ldots, d_s}(\mathcal{E})\), where \(\mathcal{U}_i\) are the universal subsheaves.

**Lemma 3.6.** Let \(0 < d < r\), let \(G = \text{Gr}^d(\mathcal{E})\), let \(\pi: G \to X\) be its structural map, let \(\mathcal{F} \subseteq \pi^* \mathcal{E}\) be the universal subbundle, and let \(\mathcal{Q} = \pi^* \mathcal{E}/ \mathcal{F}\) be the quotient. Then

(a). There is a canonical isomorphism

\[
\alpha: \text{Fl}^{d+1, d}(\mathcal{E}) \sim \mathbb{P}(\mathcal{F})
\]

over \(G\), and a canonical isomorphism

\[
\alpha^* \mathcal{O}(1) \cong \mathcal{O}(1, -1) .
\]

(b). There is a canonical isomorphism

\[
\beta: \text{Fl}^{d, d-1}(\mathcal{E}) \sim \mathbb{P}(\mathcal{Q}^\vee)
\]

over \(G\), and a canonical isomorphism

\[
\beta^* \mathcal{O}(1) \cong \mathcal{O}(-1, 1) .
\]
Proof. The existence of $\alpha$ follows from the fact that
\[ \mathbb{P}(\mathcal{F}) = \text{Gr}^1(\mathcal{F}) = \text{Fl}^{d+1,d}(\mathcal{E}) \, . \]

Let $\pi': \text{Fl}^{d+1,d}(\mathcal{E}) \to X$ be the canonical map, and let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be the universal bundles on $\text{Fl}^{d+1,d}(\mathcal{E})$. Applying ([H], II Ex. 5.16d) to the short exact sequence
\[ (3.6.3) \quad 0 \to \mathcal{F}_2/\mathcal{F}_1 \to (\pi')^* \mathcal{E}/\mathcal{F}_1 \to (\pi')^* \mathcal{E}/\mathcal{F}_2 \to 0 \]
gives a canonical isomorphism
\[ \bigwedge^{d+1}((\pi')^* \mathcal{E}/\mathcal{F}_1) \cong (\mathcal{F}_2/\mathcal{F}_1) \otimes \bigwedge^d((\pi')^* \mathcal{E}/\mathcal{F}_2) \, . \]

Now let $\rho: \mathbb{P}(\mathcal{F}) \to G$ be the structural map; then $\alpha^* \rho^* \mathcal{F} = \mathcal{F}_2$ (as subbundles of $(\pi')^* \mathcal{E}$), and $\mathcal{F}_1 \cong \alpha^*(\ker(\rho^* \mathcal{F} \to \mathcal{O}(1)))$, so we have canonical isomorphisms
\[ \alpha^* \mathcal{O}(1) \cong \mathcal{F}_2/\mathcal{F}_1 \]
\[ \cong \bigwedge^{d+1}((\pi')^* \mathcal{E}/\mathcal{F}_1) \otimes \bigwedge^d((\pi')^* \mathcal{E}/\mathcal{F}_2)^\vee \]
\[ \cong \mathcal{O}(1,-1) \]
by (3.5). This gives (3.6.1).

Dually, the existence of $\beta$ follows from the fact that
\[ \mathbb{P}(\mathcal{Q}^\vee) = \text{Gr}^1(\mathcal{Q}) = \text{Fl}^{d,d-1}(\mathcal{E}) \, . \]
As before let $\pi': \text{Fl}^{d,d-1}(\mathcal{E}) \to X$ be the canonical map and $\mathcal{F}_1 \subseteq \mathcal{F}_2$ the universal subbundles. The short exact sequence (3.6.3) now gives a canonical isomorphism
\[ \bigwedge^d((\pi')^* \mathcal{E}/\mathcal{F}_1) \cong (\mathcal{F}_2/\mathcal{F}_1) \otimes \bigwedge^{d-1}((\pi')^* \mathcal{E}/\mathcal{F}_2) \, . \]
As in the first half of the proof, let $\rho: \mathbb{P}(\mathcal{Q}^\vee) \to X$ be the structural map; then we have a natural isomorphism $\beta^* \rho^* \mathcal{Q}^\vee \cong ((\pi')^* \mathcal{E}/\mathcal{F}_1)^\vee$, by which the subbundles $\beta^* \ker(\rho^* \mathcal{Q}^\vee \to \mathcal{O}(1))$ and $((\pi')^* \mathcal{E}/\mathcal{F}_2)^\vee$ correspond. Therefore,
\[ \beta^* \mathcal{O}(1) \cong ((\pi')^* \mathcal{E}/\mathcal{F}_1)^\vee/((\pi')^* \mathcal{E}/\mathcal{F}_2)^\vee \]
\[ \cong (\mathcal{F}_2/\mathcal{F}_1)^\vee \]
\[ \cong \bigwedge^{d-1}((\pi')^* \mathcal{E}/\mathcal{F}_2) \otimes \bigwedge^d((\pi')^* \mathcal{E}/\mathcal{F}_1)^\vee \]
\[ \cong \mathcal{O}(-1,1) \]
by (3.5) again, which gives (3.6.2). \qed
Proposition 3.7. Let $d$, $G$, $\pi$, $\mathcal{F}$, and $\mathcal{Q}$ be as in Lemma 3.6. Then there is a canonical map

$$\phi: \text{Fl}^{d+1,d,d-1}(E) \to \mathbb{P}(G_X)$$

and a canonical isomorphism

$$\phi^* \mathcal{O}(1) \cong \mathcal{O}(1,-2,1).$$

Proof. By ([H], II Prop. 7.12), it suffices to give a natural surjection

$$\text{pr}_d^* \Omega_{G/X} \to \mathcal{O}(1,-2,1),$$

where we recall that $\text{pr}_d: \text{Fl}^{d+1,d,d-1}(E) \to G$ is the forgetful map. By ([F 2], B.5.8), $\Omega_{G/X} \cong \mathcal{F} \otimes \mathcal{Q}^\vee$ (this was stated only for Grassmannians over fields, but the same proof is valid for general $X$). By (3.4) and Lemma 3.6,

$$\text{Fl}^{d+1,d,d-1}(E) \cong \text{Fl}^{d,d-1}(E) \times_G \text{Fl}^{d+1,d}(E) \cong \mathbb{P}(\mathcal{F}) \times_G \mathbb{P}(\mathcal{Q}^\vee).$$

Letting $p: \text{Fl}^{d+1,d,d-1}(E) \to \mathbb{P}(\mathcal{F})$ and $q: \text{Fl}^{d+1,d,d-1}(E) \to \mathbb{P}(\mathcal{Q}^\vee)$ be the projection morphisms via the above isomorphism, we have a canonical surjective map

$$\text{pr}_d^* \Omega_{G/X} \cong \text{pr}_d^* \mathcal{F} \otimes \text{pr}_d^* \mathcal{Q}^\vee$$

$$\to p^* \mathcal{O}(1) \otimes q^* \mathcal{O}(1)$$

$$\cong \mathcal{O}(1,-1,0) \otimes \mathcal{O}(0,-1,1)$$

$$\cong \mathcal{O}(1,-2,1)$$

by (3.6.1) and (3.6.2), as was to be shown. \qed

Corollary 3.8. Let $V$ be a complex vector space of dimension $r$, and let $P = \mathbb{P}(V^\vee) = \text{Gr}_1(V)$. Then there is a canonical isomorphism

$$\phi: \text{Fl}_{1,2}(V) \xrightarrow{\sim} \mathbb{P}(\Omega_{P/\mathbb{C}})$$

of schemes over $P$ and a canonical isomorphism

$$\phi^* \mathcal{O}(1) \cong \mathcal{O}(-2,1)$$

of line sheaves on $\text{Fl}_{1,2}(V)$.

Proof. Apply Proposition 3.7 with $X = \text{Spec} \mathbb{C}$, $\mathcal{E} = \bar{V}$, $d = r - 1$, and note that there is a canonical isomorphism

$$\text{Fl}^{r,r-1,r-2}(V) = \text{Fl}_{0,1,2}(V) \cong \text{Fl}_{1,2}(V).$$
taking $O(1,-2,1)$ to $O(-2,1)$ (note that $O(1,0,0)$ is trivial on $Fl_{0,1,2}(V)$). This gives the map $\phi$ and an isomorphism of line sheaves.

The fact that $\phi$ is an isomorphism is left to the reader, since it will not be used in the sequel. □

**Remark 3.9.** The intuition behind Proposition 3.7 is as follows. Let $X = \text{Spec} \mathbb{C}$ and $\mathcal{E} = \tilde{V}$ for a finite-dimensional complex vector space $V$. Let $G = \text{Gr}_d(V)$. Giving a closed point in $\mathcal{P}(\Omega G/\mathbb{C})$ corresponds to giving a line in the Plücker embedding of $G$ and a point on that line. One can show that this line is contained in the image of $G$ if and only if the deformation corresponding to the tangent direction in $G$ at the point corresponds to deforming the $d$-dimensional subspace in such a way that some $(d-1)$-dimensional subspace within the original subspace also lies within the deformations. If so, then all these deformations lie within some fixed $(d+1)$-dimensional subspace. This is exactly the situation of the associated holomorphic curves $X^d$ appearing in Section 6.

**Remark 3.10.** Proposition 3.7 can be used to prove the Plücker formulas ([G-H], p. 270). We also recall that [W-W] described its results as Plücker formulas for holomorphic curves.

§4. Motivation from Successive Minima

We start the main sequence of the proof of Theorem 0.2 by showing how the main step is motivated by the use of successive minima in Schmidt’s proof. First consider the simplest case, over $\mathbb{Q}$ and considering only the infinite place. Let $L_0, \ldots, L_n$ be linearly independent linear forms in $n+1$ variables $x_0, \ldots, x_n$ with algebraic coefficients. Let $x$ be a point close to the corresponding hyperplanes, and let $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $x$. We may assume that the $x_i$ are integers and that they are (collectively) relatively prime. Then the multiplicative and logarithmic heights of $x$ are $|x| = \max_{0 \leq i \leq n} |x_i|$ and $h(x) = \log |x|$, respectively, and the proximity functions relative to the hyperplanes $L_i = 0$ can be taken to be

$$m(L_i, x) = -\log \frac{|L_i(x)|}{\max_j |x_j|} = -\log |L_i(x)| + h(x).$$

Since there are $n + 1$ linear forms, Schmidt’s inequality

$$\sum m(L_i, x) \leq (n + 1 + \epsilon)h(x) + O(1)$$

reduces to $\sum -\log |L_i(x)| \leq \epsilon h(x) + O(1)$, or equivalently

$$\prod |L_i(x)| \gg |x|^{\epsilon}.$$  \hspace{1cm} (4.1)

Therefore it can be addressed by studying the successive minima of the parallelepiped

$$A_i |L_i(x)| \leq 1, \quad 0 \leq i \leq n$$
in $\mathbb{R}^{n+1}$ for suitable positive real constants $A_0, \ldots, A_n$. Recall that the successive minima of this parallelepiped are defined by the condition that the $d^{th}$ successive minimum is the smallest real number $\lambda_d$ such that the scaled parallelepiped $A_i|L_i(x)| \leq \lambda_d$ contains at least $d$ linearly independent lattice points in $\mathbb{Z}^{n+1}$.

The proof proceeds by contradiction, so assume that there are infinitely many counterexamples $x$ to (4.1). The constants $A_i$ are chosen, depending on $x$, such that $A_0 \cdots A_n = 1$, and so that the first successive minimum is small. Choosing the $A_i$ so that $x$ lands in the corner of the parallelepiped gives the best upper bound for $\lambda_1$, so we require $A_i|L_i(x)| = \lambda_1$ for all $i$, resulting in

$$A_i = \frac{n+1}{\prod_j |L_j(x)|} \quad \text{and} \quad \lambda_1 = \frac{n+1}{\prod_j |L_j(x)|}.$$  

Thus, assuming that (4.1) fails for $x$, this would imply $\lambda_1 \ll |x|^{-\varepsilon}$ (for a different $\varepsilon$). (There is a slight complication in case that some point other than $x$ gives rise to the first successive minimum, but that need not concern us here.)

The theory of successive minima implies that $\lambda_1 \cdots \lambda_{n+1}$ is bounded away from 0 by a constant depending only on $n$. Schmidt’s proof derives a contradiction by showing that

$$\frac{\lambda_{d+1}}{\lambda_d} \ll |x|^\varepsilon$$

for all $1 \leq d \leq n$ (and for a different $\varepsilon > 0$).

Translating the expression for $\lambda_1$ into Nevanlinna notation gives

$$- \log \lambda_1 = - \frac{\sum \log |L_i(x)|}{n+1} = \frac{\sum (m(L_i, x) - h(x))}{n+1} = \frac{\sum m(L_i, x)}{n+1} - h(x).$$

For higher successive minima, say for $\lambda_d$ with $1 \leq d \leq n+1$, we use the fact that the product $\lambda_1 \cdots \lambda_d$ arises (up to a bounded constant factor) as the first successive minimum in $\bigwedge^d \mathbb{R}^{n+1}$ for the parallelepiped $A_{i_1} \cdots A_{i_d} L_I(X) \leq 1$, where $I = \{i_1, \ldots, i_d\}$ varies over all $d$-element subsets of $\{0, \ldots, n\}$ and

$$L_I = L_{i_1} \wedge \cdots \wedge L_{i_d}.$$  

If $x, x', \ldots, x^{(d-1)}$ are lattice points in $\mathbb{Z}^{n+1}$ corresponding to the first $d$ successive minima, and if $X^d = x \wedge \cdots \wedge x^{(d-1)}$, then $X^d$ gives rise to the first successive minimum in $\bigwedge^d \mathbb{R}^{n+1}$ (or it comes within a constant factor of doing so), so for $1 \leq d \leq n+1$ (and also for $d = 0$) we have

$$\lambda_1 \cdots \lambda_d \gg \max_1 \frac{|L_I(X^d)|}{|L_{i_1}(x)| \cdots |L_{i_d}(x)|} \lambda_1^d.$$
Therefore the ratio of consecutive successive minima can be obtained as

\[
\frac{\lambda_{d+1}}{\lambda_d} = \frac{(\lambda_1 \cdots \lambda_{d-1})(\lambda_1 \cdots \lambda_{d+1})}{(\lambda_1 \cdots \lambda_d)^2} = \frac{\max_{\# I = d-1} |L_I(X^{d-1})| \cdot \max_{\# I = d+1} |L_I(X^{d+1})|}{\left(\max_{\# I = d} |L_I(X^d)|\right)^2}
\]

\[
\gg \ll \max_{\# I = d-1} |L_I(X^{d-1})| \cdot \max_{\# I = d+1} |L_I(X^{d+1})| \|
\]

for all \(1 \leq d \leq n\).

One would hope that the three maxima in the above expression occur at subsets that are nested in a reasonable way, but there is no obvious reason for why this should hold. Therefore, we will deviate a little from Schmidt’s proof by taking geometric means instead of maxima. So, in motivating the structure of the proof of Theorem 0.2, we assume that

\[
\lambda_1 \cdots \lambda_d = \left( \prod_{\# I = d} \left| \frac{L_I(X^d)}{|L_{i_1}(x)| \cdots |L_{i_d}(x)|} \right| \lambda_1^d \right)^{1/(n+1)} = \lambda_1^d \left( \prod_{\# I = d} |L_I(X^d)| \right)^{1/(n+1)} \frac{1}{\left( \prod_{i=0}^n |L_i(x)| \right)^{\frac{n+1}{n+1}}} = \left( \prod_{\# I = d} |L_I(X^d)| \right)^{1/(n+1)}.
\]

Then

\[
(4.2) \quad \frac{\lambda_{d+1}}{\lambda_d} = \left( \prod_{\# I = d-1} |L_I(X^{d-1})| \right)^{1/(n+1)} \left( \prod_{\# I = d+1} |L_I(X^{d+1})| \right)^{1/(n+1)} \left( \prod_{\# I = d} |L_I(X^d)| \right)^{2/(n+1)}.
\]

One would then bound this by \(H(x)^\epsilon\).

This derivation assumed that there was only one infinite place (and exactly \(n + 1\) hyperplanes—the fact that Schmidt’s theorem in this case is not a trivial consequence of the First Main Theorem stems from the fact that the coefficients are allowed to be algebraic). In the general case, we work with a collection of \(n + 1\) hyperplanes that is allowed to vary with the place (or, in the Nevanlinna case, to vary with \(z \in \mathbb{C}\) with finitely many possibilities).
§5. First Step of the Proof

The main step in the proof is Proposition 6.2. Here we prove a special case. We start with some notation.

Let \( V = \mathbb{C}^{n+1} \), with the usual norm \( |v|^2 = |v_0|^2 + \cdots + |v_n|^2 \) for \( v = (v_0, \ldots, v_n) \) in \( V \). Let \( y: \mathbb{C} \to V \) be a holomorphic map, not identically zero, with coordinate functions \( y_0, \ldots, y_n \). Define

\[
T_y(r) = \int_0^{2\pi} \log |y(re^{\sqrt{-1} \theta})| \frac{d\theta}{2\pi},
\]

and let \( N_y(r) \) be the counting function for simultaneous vanishing of the coordinates \( y_0, \ldots, y_n \) of \( y \).

Write \( P = \mathbb{P}(V^\vee) \), so that \( P \cong \mathbb{P}^n \) and \( y \) corresponds to a holomorphic curve \( f: \mathbb{C} \to \mathbb{P}^n \). Note that

\[
T_{\theta(1), f}(r) = T_y(r) - N_y(r).
\]

We also write \( T_f(r) = T_{\theta(1), f}(r) \).

Assume also that \( f \) is not constant. Then \( y \wedge y' \) is a holomorphic map \( \mathbb{C} \to \Lambda^2 V \), giving rise to a holomorphic curve \( f \wedge f': \mathbb{C} \to \mathbb{P}(\Lambda^2 V^\vee) \). We define \( T_{y \wedge y'}(r) \) and \( N_{y \wedge y'}(r) \) analogously to \( T_y(r) \) and \( N_y(r) \).

Finally, let \( L \) be a finite set, all of whose elements are \((n+1)\)-tuples \((L_0, \ldots, L_n)\) of linearly independent linear forms on \( V \). For each \( z \in \mathbb{C} \) pick an element \( (L_{z,0}, \ldots, L_{z,n}) \in L \).

We also write \( L_{r, \theta, i} = L_{r e^{\sqrt{-1} \theta}, i} \) for all \( i \), let

\[
\lambda_{r, \theta, i}(v) = -\log \frac{|L_{r, \theta, i}(v)|}{|v|}
\]

be a Weil function for \( L_{r, \theta, i} \), and let

\[
m_{1,Y}(L, r) = \frac{1}{n+1} \int_0^{2\pi} \sum_{i=0}^n \lambda_{r, \theta, i}(y(re^{\sqrt{-1} \theta})) \frac{d\theta}{2\pi}.
\]

Of course this depends not only on \( L \) but also on the chosen function \( \mathbb{C} \to L \). Note also that this is independent of the lifting \( y \) of \( f \), and that up to \( O(1) \) it is independent of multiplying the \( L_i \) (and therefore the \( L_{r, \theta, i} \)) by constants.

To help make sense of (5.2), we note that if \( L \) has only one element \((L_0, \ldots, L_n)\), then

\[
m_{1,Y}(L, r) = \frac{1}{n+1} \sum_{i=0}^n m_f(\{L_i = 0\}, r).
\]
By (5.1), we also have
\[ m_{1,y}(L, r) - \overline{T}_y(r) = \frac{1}{n+1} \int_0^{2\pi} \sum_{i=0}^n - \log |L_{r,\theta, i}(y(re^{-i\theta}))| \frac{d\theta}{2\pi}. \]

If \( I = \{ i, j \} \) is a two-element subset of \( \{ 0, \ldots, n \} \), then we let \( L_{z,I} \) denote the linear form \( L_{z,i} \wedge L_{z,j} \) on \( \bigwedge V \), let \( \lambda_{z,I} \) be the corresponding Weil function, and let \( \lambda_{r,\theta,I} = \lambda_{re^{-i\theta},I} \) as before.

**Definition 5.3.** We say that a collection \( \mathcal{I} \) of two-element subsets of \( \{ 0, \ldots, n \} \) is balanced if each \( i \in \{ 0, \ldots, n \} \) occurs in the same number of elements of \( \mathcal{I} \).

If \( \mathcal{I} \) is a nonempty collection of two-element subsets of \( \{ 0, \ldots, n \} \), then we define
\[
(5.4) \quad m_{\mathcal{I}, y \wedge y'}(L, r) = \frac{1}{\# \mathcal{I}} \int_0^{2\pi} \sum_{I \in \mathcal{I}} \lambda_{r,\theta,I}(y(re^{-i\theta}) \wedge y'(re^{-i\theta})) \frac{d\theta}{2\pi},
\]
which as before reduces to
\[
\frac{1}{\# \mathcal{I}} \sum_{I \in \mathcal{I}} m_{f \wedge f'}(\{ L_I = 0 \}, r)
\]
in the case \( L = \{ (L_0, \ldots, L_n) \} \).

**Lemma 5.5.** Let \( \mathcal{I} \) be a nonempty balanced collection of two-element subsets of \( \{ 0, \ldots, n \} \). Then
\[
(5.5.1) \quad 2m_{1,y}(L, r) - m_{\mathcal{I}, y \wedge y'}(L, r) \leq \text{exc} \ 2 \overline{T}_y(r) - \overline{T}_{y \wedge y'}(r) + O(\log^+ T_f(r)) + o(\log r).
\]

**Proof.** We may assume that the coordinate functions \( y_i \) of \( y \) never vanish simultaneously. Indeed, if all coordinates of \( y \) are divisible by an entire function \( g \), then all coordinates of \( y \wedge y' \) are divisible by \( g^2 \), so dividing \( y \) by \( g \) leaves both sides of (5.5.1) unchanged.

Since we now have \( \overline{T}_y(r) = T_f(r) \), (5.5.1) is equivalent to
\[
(5.5.2) \quad 2m_{1,y}(L, r) - m_{\mathcal{I}, y \wedge y'}(L, r) + N_{y \wedge y'}(r) \leq \text{exc} \ 2 T_f(r) - T_{f \wedge f'}(r) + O(\log^+ T_f(r)) + o(\log r).
\]

The strategy of the proof is to apply McQuillan’s Theorem 1.6 to \( P = \mathbb{P}(V^\vee) \).

First consider the first two height terms on the right-hand side. The holomorphic map
\[
\mathbb{C} \xrightarrow{(f, f \wedge f')} \text{Fl}_{1,2}(V) \xrightarrow{\phi} \mathbb{P}(\Omega_{P/\mathbb{C}})
\]
coincides with \( f' \), where \( \phi \) is the isomorphism of Corollary 3.8. By the second assertion of Corollary 3.8, we then have

\[
2T_f(r) - T_{f \wedge f'}(r) = -T_{\theta(1), f'}(r) + O(1) .
\]

We now consider the two proximity terms on the left-hand side of (5.5.2). Let \( D_1, \ldots, D_\ell \) be the divisors associated to the elements of \( L \), and let \( \mu_j \) be as in Theorem 1.6. Then we claim that

\[
\text{Fix} \ r (5.5.3), \text{it will suffice to show that} \ D
\]

\[
\text{then we have}
\]

\[
\text{where the constant in} \ O(1) \text{ depends only on} \ L . \text{From the definitions, we have}
\]

\[
2m_{1, y}(L, r) - m_{\not\exists, y \wedge y'}(L, r) \leq \int_0^{2\pi} \max_{1 \leq j \leq \ell} \mu_j (f' (re^{\sqrt{-1} \theta})) \frac{d\theta}{2\pi} + O(1) ,
\]

where the constant in \( O(1) \) depends only on \( L \). From the definitions, we have

\[
2m_{1, y}(L, r) - m_{\not\exists, y \wedge y'}(L, r) = \int_0^{2\pi} \left( \frac{2}{n + 1} \sum_{i=0}^{n} \lambda_{r, \theta, i}(y(re^{\sqrt{-1} \theta})) - \frac{1}{\# \mathcal{J}} \sum_{I \in \mathcal{J}} \lambda_{r, \theta, I}(y(re^{\sqrt{-1} \theta}) \wedge y'(re^{\sqrt{-1} \theta})) \right) d\theta .
\]

Fix \( r \) and \( \theta \), let \( L_i = L_{r e^{\sqrt{-1} \theta}, i} \) for all \( i \), let \( \lambda_i = \lambda_{r, \theta, i} \) for all \( i \), let \( \lambda_I = \lambda_{r, \theta, I} \) for all \( I \), and choose \( j \) so that \( D_j \) is the divisor associated to \( L_0, \ldots, L_n \). Then, to prove (5.5.4), it will suffice to show that

\[
\frac{2}{n + 1} \sum_{i=0}^{n} \lambda_{i}(y(re^{\sqrt{-1} \theta})) - \frac{1}{\# \mathcal{J}} \sum_{I \in \mathcal{J}} \lambda_{I}(y(re^{\sqrt{-1} \theta}) \wedge y'(re^{\sqrt{-1} \theta})) \leq \mu_j (f' (re^{\sqrt{-1} \theta})) + O(1) ,
\]

with a constant depending only on \( L_0, \ldots, L_n \). All of the terms in this expression are unchanged up to \( O(1) \) by a change of coordinates on \( V \), so we may assume that \( L_i(X_0, \ldots, X_n) = X_i \) for all \( i \). We also permute indices such that

\[
| y_0(re^{\sqrt{-1} \theta}) | \geq | y_i(re^{\sqrt{-1} \theta}) | \quad \text{for all} \ i .
\]

Then, letting \( z_i = y_i(re^{\sqrt{-1} \theta})/y_0(re^{\sqrt{-1} \theta}) \) and \( z'_i = (y_i/y_0)'(re^{\sqrt{-1} \theta}) \) for \( i = 0, \ldots, n \), we have \( |z_i| \leq 1 \) for all \( i \),

\[
\lambda_{i}(y(re^{\sqrt{-1} \theta})) = -\log |z_i| + O(1)
\]

for all \( i \), and

\[
\lambda_{I}(y(re^{\sqrt{-1} \theta}) \wedge y'(re^{\sqrt{-1} \theta})) = -\log \left| \frac{z_{i_1}^{z_{i_2}}}{z_{i_1}'} z_{i_2}' \right| + O(1)
\]

max\( 1 \leq k \leq n \ |z_{j_k}'| + O(1) \)
for all \( I = \{i_1, i_2\} \). Thus the left-hand side of (5.5.5) equals

\[
- \frac{2}{n+1} \sum_{i=0}^{n} \log |z_i| + \frac{1}{\# \mathcal{F}} \sum_{\{i_1, i_2\} \in \mathcal{F}} \log \max_{1 \leq k \leq n} \frac{|z_{i_1}|}{|z_{i_2}|} + O(1)
\]

\[
= \frac{1}{\# \mathcal{F}} \sum_{\{i_1, i_2\} \in \mathcal{F}} \log \frac{|z_{i_1}| |z_{i_2}|}{|z_{i_1}||z_{i_2}|} + O(1) .
\]

since \( \mathcal{F} \) is balanced. Substituting this and (1.3.1) into (5.5.5), it follows that to prove

(5.5.5) it will suffice to show that

\[
\log \frac{|z_{i_1}| |z_{i_2}|}{|z_{i_1}||z_{i_2}|} \leq - \frac{1}{2} \log \frac{|z_{i_1}^2 + \cdots + z_{i_2}^2|}{|z_{i_1}|^2 + \cdots + |z_{i_2}|^2} + O(1)
\]

for all \( 0 \leq i_1 < i_2 \leq n \), with a constant in \( O(1) \) depending only on \( n \). After exponentiating and noting that \( \max |z_k^\prime| \gg \max |z_k| \), this is equivalent to

\[
\frac{z_{i_2}^\prime}{z_{i_2}} - \frac{z_{i_1}^\prime}{z_{i_1}} \ll \sqrt{\frac{|z_{i_1}^2|}{z_{i_1}} + \cdots + \frac{|z_{i_2}^2|}{z_{i_2}}} ,
\]

which is easy to see. Thus (5.5.5) is proved, so (5.5.4) holds.

Finally, we note that

(5.5.6) \[ N_{Y \land Y'}(r) = N_{\text{Ram},f}(r) . \]

Indeed, we may suppose without loss of generality that \( y_0(z_0) \neq 0 \). We have

\[
\left( \frac{y_i}{y_0} \right)^\prime = \frac{y_i^\prime y_0 - y_0^\prime y_i}{y_0^2} = \frac{(y \land Y')_0}{y_0^2} .
\]

Pick \( i \in \{1, 2, \ldots, n\} \) such that \( \text{ord}_{z_0}(y_i/y_0)^\prime \) is minimal. This is the ramification order of \( f \) at \( z_0 \). It is also the order of vanishing of the \( 0i \) coordinate of \( y \land Y' \); therefore \( N_{Y \land Y'}(r) \leq N_{\text{Ram},f}(r) \). The opposite inequality is left to the reader (since it is not used here).

By (5.5.3), (5.5.4), (5.5.6), and Theorem 1.6, we then have

\[
- 2T_f(r) + T_{f \land f'}(r) + 2m_{1,Y}(L, r) - m_{\mathcal{F},Y \land Y'}(L, r) + N_{Y \land Y'}(r)
\]

\[
\leq T_{\theta(1), f}(r) + \int_0^{2\pi} \max_{1 \leq j \leq \ell} \mu_j(f'(r e^{-i\theta})) \frac{d\theta}{2\pi} + N_{\text{Ram},f}(r) + O(1)
\]

\[
\leq \text{exc } O(\log^{+} T_f(r)) + o(\log r) .
\]

This gives (5.5.2), as was to be shown. \( \square \)
§6. Main Step of the Proof

We begin with some notation. Let $V = \mathbb{C}^{n+1}$ as before, and let $x: \mathbb{C} \to V$ be a holomorphic map whose coordinate functions $x_0, \ldots, x_n$ are linearly independent over $\mathbb{C}$. This corresponds to a holomorphic curve $f: \mathbb{C} \to \mathbb{P}(V^\vee)$ whose image is not contained in any hyperplane.

As usual we let $x^{(j)} = (x_0^{(j)}, \ldots, x_n^{(j)})$ be the $j^{th}$ derivative of $x$ ($j \in \mathbb{N}$), and following Ahlfors we let

$$X^d = x \wedge x' \wedge \cdots \wedge x^{(d-1)} \in \wedge^d V,$$

$$T_{d,x}(r) = \int_0^{2\pi} \log |X^d(re^{-i\theta})| \frac{d\theta}{2\pi},$$

and let $N_{d,x}(r)$ be the counting function for the simultaneous vanishing of the coordinates of $X^d$, for $d = 0, \ldots, n+1$. (If $d = 0$ then $X^0: \mathbb{C} \to \wedge^0 V = \mathbb{C}$ is the constant map 1.) Let $F^d$ denote the corresponding map to $\mathbb{P}(\wedge^d V^\vee)$; we then have

$$T_{\Theta(1),F^d}(r) = T_{d,x}(r) - N_{d,x}(r).$$

We also write $T_{d,f}(r) = T_{\Theta(1),F^d}(r)$. Note that $T_{1,f}(r) = T_f(r)$ and $T_{0,f}(r) = 0$ for all $r$.

Let $L$, $L_{z,i}$, and $L_{r,\theta,i}$ be as before. If $I$ is a $d$-element subset of $\{0, \ldots, n\}$, then we let $L_{z,I} = L_{z,i_1} \wedge \cdots \wedge L_{z,i_d}$, where $I = \{i_1, \ldots, i_d\}$ with $i_1 < \cdots < i_d$. Also let $\lambda_{z,I}$ be the corresponding Weil function on $\mathbb{P}(\wedge^d V^\vee)$, and let $\lambda_{r,\theta,I} = \lambda_{re^{-i\theta},I}$ as before. Extending (5.2) in a manner similar to (5.4), we let

$$m_{d,f}(L,r) = \frac{(n+1)}{d} \int_0^{2\pi} \sum_{#I = d} \lambda_{r,\theta,I}(F^d(re^{-i\theta})) \frac{d\theta}{2\pi}. \tag{6.1}$$

The main step in the proof is then as follows.

**Proposition 6.2.** For all $d = 1, \ldots, n$, we have

$$m_{d-1,f}(L,r) + 2m_{d,f}(L,r) - m_{d+1,f}(L,r) \leq \text{exc} - T_{d-1,x}(r) + 2T_{d,x}(r) - T_{d+1,x}(r) + O(\log^+ T_{d,f}(r)) + o(\log r). \tag{6.2.1}$$

**Proof.** The proof works by applying Lemma 5.5 to $y := X^d$.

We start by noting that the derivative of $y = X^d = x \wedge x' \wedge \cdots \wedge x^{(d-1)}$ can be computed by a Leibniz-like relation. This gives $d$ terms, all but the last of which vanish, giving

$$y' = x \wedge x' \wedge \cdots \wedge x^{(d-2)} \wedge x^{(d)}.$$
When applying Lemma 5.5, we will use linear forms on $\bigwedge^d V$ obtained as wedge products of $d$ linear forms on $V$. When working with these forms, the classical formula

\begin{equation}
(L_1 \wedge \cdots \wedge L_d)(x_1 \wedge \cdots \wedge x_d) = \operatorname{det}(L_i(x_j))_{1 \leq i,j \leq n}
\end{equation}

is useful.

We will also use the following formula, also used by Ahlfors and Schmidt; see ([V 1], Lemma 6.3.14). Let $A$ be a $(d-1) \times (d-1)$ matrix, let $B$ and $B'$ be $(d-1) \times 1$ matrices, let $C$ and $C'$ be $1 \times (d-1)$ matrices, and let $d$, $e$, $f$, and $g$ be scalars. Then

\[
\begin{vmatrix}
A & B \\
C & d \\
\end{vmatrix}
\begin{vmatrix}
A & B' \\
C & e \\
\end{vmatrix}
= \begin{vmatrix}
A & B \\
C & d \\
\end{vmatrix}
\begin{vmatrix}
A & B' \\
C & f \\
\end{vmatrix}
\begin{vmatrix}
B & C \\
B' & C' \\
\end{vmatrix}.
\]

Combining this with (6.2.2) gives

\begin{equation}
(L_z,I \wedge L_z,J)(y \wedge y') = \begin{vmatrix}
L_z,I(y) & L_z,J(y) \\
L_z,I(y') & L_z,J(y') \\
\end{vmatrix} = \pm L_z,I \cap J(X^{d-1}) L_z,I \cup J(X^{d+1}),
\end{equation}

where $I$ and $J$ are $d$-element subsets of $\{0, \ldots, n\}$ that differ by exactly one element.

Therefore, for $I, J \subseteq \{0, \ldots, n\}$ with $\#I = \#J = d$, we let

$$\operatorname{dist}(I, J) = \#(I \setminus (I \cap J)) = \#(J \setminus (I \cap J))$$

be the number of elements in which they differ. Let $\mathcal{F}$ be the collection of sets $\{I, J\}$ with $I, J \subseteq \{0, \ldots, n\}$, $\#I = \#J = d$, and $\operatorname{dist}(I, J) = 1$. Then, with a suitable definition of $\mathcal{L}'$, we have

$$m_{\mathcal{F}, y, y'}(\mathcal{L}', r) = \frac{1}{\#\mathcal{F}} \int_0^{2\pi} \sum_{\substack{\#I = \#J = d \\
\operatorname{dist}(I, J) = 1}} \lambda_{r, \theta, \{I, J\}}(y(\sqrt{-1}r)^\theta \wedge y'(\sqrt{-1}r)^\theta) \frac{d\theta}{2\pi}.$$
and therefore, by \((5.1)\) and \((6.2.3)\),

\[
\mathcal{T}_{y,y'}(r) = m_{\mathcal{F},y,y'}(\mathbf{L}',r)
\]

\[
= \frac{1}{\# \mathcal{F}} \int_{0}^{2\pi} \sum_{\# I = \# J = d \atop \text{dist}(I,J) = 1} \left\lfloor \log((L_{r,\theta,I} \cap L_{r,\theta,J})(y(r e^{\sqrt{-1} \theta}) \wedge y'(r e^{\sqrt{-1} \theta}))) \right\rfloor \frac{d\theta}{2\pi}
\]

\[
= \frac{1}{\# \mathcal{F}} \int_{0}^{2\pi} \sum_{\# I = \# J = d \atop \text{dist}(I,J) = 1} \left( \log|L_{r,\theta,I \cap J}(X^{d-1}(r e^{\sqrt{-1} \theta}))| \right) \frac{d\theta}{2\pi}
\]

\[
+ \log|L_{r,\theta,I \cup J}(X^{d+1}(r e^{\sqrt{-1} \theta}))| \frac{d\theta}{2\pi}
\]

\[
= \left( \frac{n+1}{d-1} \right)^{-1} \int_{0}^{2\pi} \sum_{\# I = d-1} \log|L_{r,\theta,I}(X^{d-1}(r e^{\sqrt{-1} \theta}))| \frac{d\theta}{2\pi}
\]

\[
+ \left( \frac{n+1}{d+1} \right)^{-1} \int_{0}^{2\pi} \sum_{\# I = d+1} \log|L_{r,\theta,I}(X^{d+1}(r e^{\sqrt{-1} \theta}))| \frac{d\theta}{2\pi}
\]

\[
= T_{d-1,x}(r) - m_{d-1,f}(\mathbf{L},r) + T_{d+1,x}(r) - m_{d+1,f}(\mathbf{L},r).
\]

Here we also used the fact that, as \(\{I,J\}\) varies over \(\mathcal{F}\), the intersection \(I \cap J\) varies over all \((d-1)\)-element subsets of \(\{0,\ldots,n\}\) with equal frequency, and the same holds for the union \(I \cup J\).

Straight from the definitions we also have

\[
m_{1,y}(\mathbf{L}',r) = m_{d,f}(\mathbf{L},r), \quad T_{y}(r) = T_{d,x}(r), \quad \text{and} \quad T_{y}(r) = T_{d,f}(r),
\]

and therefore \((6.2.1)\) follows from \((5.5.1)\).

**Proposition 6.3.** In \((6.2.1)\), the error term \(O(\log^+ T_{d,f}(r))\) can be replaced by \(O(\log^+ T_f(r))\).

**Proof.** Let \(1 \leq d \leq n\) and let \(G = \text{Gr}_d(V)\). The composite map

\[
\mathbb{C} \xrightarrow{(X^{d-1},X^d,X^{d+1})} \mathbb{P}l_{d-1,d,d+1}(V) \xrightarrow{\phi} \mathbb{P}(\Omega_G/\mathbb{C}),
\]

where \(\phi\) is the map of Proposition 3.7, is just the map \((F^d)'.\) Applying McQuillan’s Theorem 1.2 with \(D = 0\) gives

\[
T_{\theta(1),(F^d)'}(r) \leq_{\text{exc}} O(\log^+ T_{d,f}(r)) + o(\log r).
\]

By Proposition 3.7, \(\phi^* \theta(1) \cong \theta(1, -2, 1)\), so by functoriality of the height (characteristic) function,

\[
T_{d-1,f}(r) - 2T_{d,f}(r) + T_{d+1,f}(r) \leq_{\text{exc}} O(\log^+ T_{d,f}(r)) + o(\log r).
\]

By induction on \(d\), we then have

\[
(6.3.1) \quad T_{d,f}(r) \leq_{\text{exc}} 2^{d-1}T_f(r) + O(\log^+ T_f(r)) + o(\log r).
\]

This implies the result. \(\square\)
§7. Conclusion of the Proof

Let \( f: \mathbb{C} \rightarrow \mathbb{P}^n \) be a holomorphic curve whose image is not contained in any hyperplane. Lift it to a holomorphic map \( x: \mathbb{C} \rightarrow \mathbb{C}^{n+1} = V \) such that the coordinate functions \( x_0, \ldots, x_n \) have no common zeroes. Then we have \( N_{1,f}(r) = 0 \) for all \( r \), so

\[
T_{1,x}(r) = T_f(r).
\]

Also, since \( \bigwedge^{n+1} V \cong \mathbb{C} \), we have

\[
T_{n+1,f}(r) = O(1).
\]

Now let \( H_1, \ldots, H_q \) be the hyperplanes in Theorem 0.2. Adding hyperplanes only strengthens the inequality, so we may assume that \( \bigcap H_i = \emptyset \). The Weil functions \( \lambda_{H_j} \) are bounded from below, so the left-hand side of (0.2.1) is changed by only \( O(1) \) if we assume that all subsets \( J \) have at least \( n+1 \) elements. Also, given a subset of \( H_1, \ldots, H_q \) in general position, no point in \( \mathbb{P}^n \) is close to more than \( n \) hyperplanes in the subset. Therefore, again up to \( O(1) \), we may assume that all subsets \( J \) have exactly \( n+1 \) elements.

Now choose linear forms \( L_1, \ldots, L_q \) defining the hyperplanes \( H_1, \ldots, H_q \), respectively, and let \( L \) be the collection of all tuples \( (L_{i_0}, \ldots, L_{i_n}) \) such that \( 1 \leq i_0 < i_1 < \cdots < i_n \leq q \) and \( H_{i_0}, \ldots, H_{i_n} \) are in general position. For each \( z \in \mathbb{C} \) pick \( (L_{z,0}, \ldots, L_{z,n}) \in L \) such that

\[
\max \sum_{j \in J} \lambda_{H_j}(f(z)) = \sum_{i=0}^{n} \lambda_{H_{z,i}}(f(z)),
\]

where \( H_{z,i} \) is the hyperplane determined by \( L_{z,i} \). We then have

\[
(7.1) \quad \int_0^{2\pi} \max \sum_{i \in J} \lambda_{H_i}(f(re^{i\theta})) \frac{d\theta}{2\pi} = (n + 1)m_{1,f}(L, r) + O(1),
\]

where the sets \( J \) are as in (0.2.1).

By Propositions 6.2 and 6.3, we have

\[
(7.2) \quad (n + 1)m_{1,f}(L, r) = -nm_{0,f}(L, r) + (n + 1)m_{1,f}(L, r) - m_{n+1,f}(L, r)
\]

\[
= \sum_{d=1}^{n} (n + 1 - d)(-m_{d-1,f}(L, r) + 2m_{d,f}(L, r) - m_{d+1,f}(L, r))
\]

\[
\leq \text{exc} \sum_{d=1}^{n} (n + 1 - d)(-T_{d-1,f}(r) + 2T_{d,f}(r) - T_{d+1,f}(r))
\]

\[
+ O(\log^+ T_f(r)) + o(\log r)
\]

\[
= -nT_{0,f}(r) + (n + 1)T_{1,f}(r) - T_{n+1,f}(r)
\]

\[
+ O(\log^+ T_f(r)) + o(\log r)
\]

\[
= (n + 1)T_f(r) - N_{n+1,f}(r) + O(\log^+ T_f(r)) + o(\log r).
\]
Combining (7.1) and (7.2) gives (0.2.1), upon noting that $X^{n+1} : \mathbb{C} \to \bigwedge^{n+1} V = \mathbb{C}$ is exactly the Wronskian used in (0.2.1).

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