A linearly implicit energy-preserving exponential integrator for the nonlinear Klein-Gordon equation

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Abstract

In this paper, we generalize the exponential energy-preserving integrator proposed in the recent paper [SIAM J. Sci. Comput. 38(2016) A1876-A1895] for conservative systems, which now becomes linearly implicit by further utilizing the idea of the scalar auxiliary variable approach. Comparing with the original exponential energy-preserving integrator which usually leads to a nonlinear algebraic system, our new method only involves a linear system with constant coefficient matrix. Taking the nonlinear Klein-Gordon equation as an example, we derive the concrete energy-preserving scheme and demonstrate its high efficiency through numerical experiments.

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1 Introduction

It is well-known that exponential integrators permit larger step sizes and achieve higher accuracy than nonexponential ones when the considered problem is a very stiff differential equation such as highly oscillatory ODEs or semidiscrete time-dependent PDEs. As to exponential integrators, the earlier attempts can date back to the original paper by Hersch [15], whereas the term “exponential integrators” was coined in the seminal paper by Hochbruck, Lubich, and Selhofer [21]. Readers are referred to Ref. [16] for details about exponential integrators. In recent years, there has been growing interest in structure-preserving exponential methods, which can preserve as much as possible the physical/geronomic properties of the dynamic system under consideration [14]. Due to the superior properties in the capability for the long-term computation, symplectic exponential methods have attracted much attention (e.g., see Refs. [24, 29, 33] and references therein). On the other hand, energy is the most important first integral of the dynamic system. Moreover, in many cases, the stability, convergence, existence and uniqueness of the solution of a numerical method can be established by directly using its discrete energy conservative property [19, 35]. Thus, devising energy-preserving schemes for conservative systems attracts a lot of interest (e.g., see Refs. [2, 4, 5, 7, 10, 11, 18, 13, 14, 23, 24] and references therein). Recently, combining the ideas of exponential integrators and discrete gradients [25, 23], Li and Wu proposed an
energy-preserving exponential scheme for conservative systems, which was revisited more recently by Shen and Leok [29]. However, such scheme is implicit. At each time step, one needs to solve a fully nonlinear system and thus it might be very time consuming. Compared with fully implicit schemes, linearly implicit schemes only require to solve a linear system, which leads to considerably lower costs than implicit ones [7]. However, as far as we know, there has been no reference considering linearly implicit exponential schemes for energy-conserving systems, which can inherit the energy.

In this paper, taking the nonlinear Klein-Gordon equation as an example, we propose a novel linearly implicit exponential scheme for conservative systems by combining the ideas of the exponential integrator and the scalar auxiliary variable (SAV) approach [27, 28]. The proposed scheme can inherit the energy and enjoy the same computational advantages as the one (see [3]) provided by the classical SAV approach. The SAV approach as well as the earlier invariant energy quadratization (IEQ) approach [34, 36] is developed based on the idea of the energy quadratization, which can result linearly implicit and energy stable schemes for gradient flows. To the best of our knowledge, there has been no reference considering the combination of the ideas of the SAV approach and the exponential integrator for developing linearly implicit energy-preserving schemes for energy-conserving systems. Taking the nonlinear Klein-Gordon equation for example, we first explore the feasibility.

The outline of this paper is organized as follows. In Section 2 based on the SAV approach, the NKGE (2.1) is reformulated into an equivalent form. In Section 3 we will concentrate on the construction for the linearly implicit energy-preserving exponential scheme. Several numerical experiments are reported in Section 5. We draw some conclusions in Section 6.

2 Reformation of the model equation through the SAV approach

The Klein-Gordon equation is frequently used in mathematical models for problems in many fields of science and engineering, particularly in quantum field theory and relativistic quantum mechanics. Here, we consider the following nonlinear Klein-Gordon equation (NKGE)

\[
\begin{align*}
\partial_{tt} u(x, t) - \omega^2 \Delta u(x, t) + G'(u(x, t)) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
u(x, 0) &= \phi_1(x), \quad \partial_t u(x, 0) = \phi_2(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

where \( t \) is time variable, \( x \in \mathbb{R}^d \) is the spatial variable, \( u := u(x, t) \) is a real-valued function, \( \omega \) is a real parameter, \( \Delta \) is the usual Laplace operator, \( G(u) \) is a smooth potential energy function with \( G(u) \geq 0 \), and \( \phi_1 := \phi_1(x) \) and \( \phi_2 := \phi_2(x) \) are two given real-value initial data. The NKGE (2.1) conserves the energy

\[
E(t) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\partial_t u|^2 + \frac{\omega^2}{2} |\nabla u|^2 + G(u) \right] dx = E(0), \quad t \geq 0.
\]

In the last few decades, various structure-preserving methods have been developed for solving the NKGE (2.1), including symplectic methods (e.g., see [9, 22, 24]), multisymplectic methods (e.g., see [17, 26, 50, 37]) and energy-preserving methods (e.g., see [11, 6, 12, 31]), local momentum-preserving methods (e.g., see [19, 32]), etc. However, there has been no reference considering a linearly implicit structure-preserving exponential scheme for the NKGE (2.1) to our knowledge.
Following the idea of the SAV approach, we introduce a scalar auxiliary variable, as follows:

\[ q := q(t) = \sqrt{(G(u), 1) + C_0}, \]

where \((f, g)\) is the inner product defined by \((f, g) = \int_{\mathbb{R}^d} f g dx\), and \(C_0\) is a constant large enough to make \(q \neq 0\). Here, we should note that, in our computation, when \(G(u) \equiv 0\), we choose \(C_0 = 1\), otherwise, we choose \(C_0 = 0\). The original equation \((2.2)\) can then be rewritten as

\[ E(t) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\partial_t u|^2 + \frac{\omega^2}{2} |\nabla u|^2 \right] dx + q^2 - C_0. \]

(2.3)

According to the energy variational formula, the NKGE \((2.1)\) can be reformulated into the following equivalent form

\[
\begin{align*}
\partial_t u &= v, \\
\partial_t v &= \omega^2 \Delta u - \frac{G'(u)}{\sqrt{(G(u), 1) + C_0}} q, \\
\partial_t q &= \frac{(G'(u), \partial_t u)}{2 \sqrt{(G(u), 1) + C_0}}, \\
u(x, 0) &= \phi_1(x), \quad \partial_t u(x, 0) = \phi_2(x), \quad q(0) = \sqrt{(G(u(x, 0)), 1) + C_0},
\end{align*}
\]

(2.4)

where \(x \in \mathbb{R}^d\) and \(t > 0\).

**Theorem 2.1.** The system \((2.4)\) possesses the following modified energy.

\[ E(t) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |v|^2 + \frac{\omega^2}{2} |\nabla u|^2 \right] dx + q^2 - C_0 = E(0), \quad t \geq 0. \]

(2.5)

**Proof.** Taking the inner products with \(v\) of the second equality of \((2.4)\), we have, together with the first equality of \((2.4)\)

\[ \frac{d}{dt} \int_{\mathbb{R}^d} \left[ \frac{1}{2} |v|^2 + \frac{\omega^2}{2} |\nabla u|^2 \right] dx + \int_{\mathbb{R}^d} \frac{G'(u) \partial_t u}{\sqrt{(G(u), 1) + C_0}} q dx = 0. \]

(2.6)

Multiplying the third equality of \((2.4)\) by \(q\) gives

\[ \frac{d}{dt} q^2 = \int_{\mathbb{R}^d} \frac{G'(u) \partial_t u}{\sqrt{(G(u), 1)}} q dx. \]

(2.7)

Combining \((2.6)\) and \((2.7)\), one obtains \((2.5)\) immediately. \(\square\)

### 3 Energy-preserving spatial semi-discretization

For simplicity of notation, we shall introduce our scheme in one space dimension, i.e. \(d = 1\) in \((2.4)\). Generalizations to \(d > 1\) are straightforward for tensor product grids and the results remain valid with modifications. For \(d = 1\), the NKGE \((2.4)\) is truncated on a bounded interval \((a, b)\) with the periodic boundary condition.

Choose the mesh size \(h = (b - a)/N\) with \(N\) an even positive integer, and denote the grid points by \(x_j = jh\) for \(j = 0, 1, 2, \cdots, N\); let \(u_j\) and \(v_j\) be the numerical approximations of \(u(x_j, t)\) and \(v(x_j, t)\) for \(j = 0, 1, \cdots, N\), respectively, and
$u := (u_0, u_1, \cdots, u_{N-1})^T$, $v := (v_0, v_1, \cdots, v_{N-1})^T$ be the solution vectors and define the following finite difference operators as
\[
\delta_x^+ u_j = \frac{u_{j+1} - u_j}{h}, \quad \delta_x^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \quad 0 \leq j \leq N - 1.
\]

In addition, for any $u$ and $v$, we define the discrete inner product and notions as follows
\[
\langle u, v \rangle_2 = h \sum_{j=0}^{N-1} u_j v_j, \quad ||v||_2^2 = h \sum_{j=0}^{N-1} |v_j|^2, \quad ||\delta_x^+ u||_2^2 = h \sum_{j=0}^{N-1} |\delta_x^+ u_j|^2.
\]

Then we apply the second-order centered difference scheme for spatial discretization
\[
\begin{cases}
\frac{d}{dt} u = v,
\frac{d}{dt} v = \omega^2 \delta_x^2 u - \frac{G'(u)}{\sqrt{\langle G(u), \mathbf{1} \rangle_2} + C_0} q,
\frac{d}{dt} q = \frac{\langle G'(u), \frac{d}{dt} u \rangle_2}{2\sqrt{\langle G(u), \mathbf{1} \rangle_2} + C_0},
\end{cases}
\tag{3.1}
\]
\[
u_j(0) = \phi_1(x_j), \quad v_j(0) = \phi_2(x_j), \quad q(0) = \sqrt{\langle G(u(0)), \mathbf{1} \rangle_2} + C_0,
\]
\[u_0 = u_N, \quad u_{-1} = u_{N-1},
\]
where $G(u) = (G(u_0, u_1, \cdots, u_{N-1}))^T$ and $0 \leq j \leq N$.

**Theorem 3.1.** The semi-discrete system \( (3.1) \) admits the semi-discrete modified energy
\[
E_h(t) = \frac{1}{2} ||v||_2^2 + \frac{\omega^2}{2} ||\delta_x^+ u||_2^2 + q^2 - C_0 = E_h(0), \quad t \geq 0.
\tag{3.2}
\]

**Proof.** Taking the discrete inner products with $v$ of the second equality of \( (3.1) \), we have, together with the first equality of \( (3.1) \),
\[
\frac{d}{dt} \left[ \frac{1}{2} ||v||_2^2 + \frac{\omega^2}{2} ||\delta_x^+ u||_2^2 \right] + \frac{\langle G'(u), \frac{d}{dt} u \rangle_2}{\sqrt{\langle G(u), \mathbf{1} \rangle_2} + C_0} q = 0.
\tag{3.3}
\]

Multiplying with the third equality of \( (3.1) \) by $q$ reads
\[
\frac{d}{dt} q^2 = \frac{\langle G'(u), \frac{d}{dt} u \rangle_2}{\sqrt{\langle G(u), \mathbf{1} \rangle_2} + C_0} q.
\tag{3.4}
\]

Combining \( (3.3) \) and \( (3.4) \), one obtains \( (3.2) \) immediately. \qed

### 4 Construction of the linearly implicit energy-preserving exponential scheme

Choose $\tau$ be the time step, and denote $t_n = n \tau$ for $n = 0, 1, 2, \cdots$; let $u^n_j$ be the numerical approximation of $u(x_j, t_n)$ for $j = 0, 1, \cdots, N$ and $n = 0, 1, 2, \cdots$; denote $u^n$ as the solution vector at $t = t_n$ and define
\[
\delta_t u_j^n = \frac{u_{j+1}^n - u_j^n}{\tau}, \quad u_{j+\frac{1}{2}}^n = \frac{u_{j+1}^n + u_j^n}{2}, \quad u_j^{n+\frac{1}{2}} = \frac{3 u_j^n - u_{j-1}^n}{2}, \quad 0 \leq j \leq N - 1.
\]
Then, we obtain the new scheme
\[
z(t) = \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right), \quad S = \left( \begin{array}{cc} O & I \\ -I & O \end{array} \right), \quad M = \left( \begin{array}{cc} -\omega^2 B_2 & O \\ O & I \end{array} \right),
\]
and
\[
f(u(t), q(t)) = \left( \frac{G'(u(t))}{\sqrt{\langle G(u(t)):1 \rangle_{L^2} + C_0}} q(t) \right).
\]

Here, matrix $B_2$ represents the operator $\partial_x^2$ under the periodic boundary condition.

Integrating the equation \(3.1\) from $t_n$ to $t_{n+1}$, we have
\[
z(t_n + \tau) = \exp(V)z(t_n) + \tau \int_0^1 \exp((1 - \xi)V)Sf(u(t_n + \xi\tau), q(t_n + \xi\tau))d\xi, \quad (4.1)
\]
and
\[
q(t_n + \tau) = q(t_n) + \tau \int_0^1 \frac{\langle G'(u(t_n + \xi\tau)), \frac{d}{dt}u \rangle_{L^2}}{2\sqrt{\langle G(u(t_n + \xi\tau)):1 \rangle_{L^2} + C_0}} d\xi, \quad (4.2)
\]
where $V = SM\tau$.

Replacing $f(u(t_n + \xi\tau), q(t_n + \xi\tau))$ and $\frac{\langle G'(u(t_n + \xi\tau)), \frac{d}{dt}u \rangle_{L^2}}{2\sqrt{\langle G(u(t_n + \xi\tau)):1 \rangle_{L^2} + C_0}}$ with the linearized Crank-Nicolson method $f(\hat{w}^{n+\frac{1}{2}}, w^{n+\frac{1}{2}})$ and $\frac{\langle G'(\hat{w}^{n+\frac{1}{2}}), \delta u \rangle_{L^2}}{2\sqrt{\langle G(\hat{w}^{n+\frac{1}{2}}):1 \rangle_{L^2} + C_0}}$, respectively, the integrals in (4.1) and (4.2) can be approximated by
\[
\int_0^1 \exp((1 - \xi)V)Sf(u(t_n + \xi\tau), q(t_n + \xi\tau))d\xi
\approx \int_0^1 \exp((1 - \xi)V)d\xi Sf(\hat{w}^{n+\frac{1}{2}}, q^{n+\frac{1}{2}}) = \phi(V)Sf(\hat{w}^{n+\frac{1}{2}}, q^{n+\frac{1}{2}}),
\]
and
\[
\int_0^1 \frac{\langle G'(\hat{w}^{n+\frac{1}{2}}), \delta u \rangle_{L^2}}{2\sqrt{\langle G(\hat{w}^{n+\frac{1}{2}}):1 \rangle_{L^2} + C_0}} d\xi \approx \frac{\langle G'(\hat{w}^{n+\frac{1}{2}}), \delta u \rangle_{L^2}}{2\sqrt{\langle G(\hat{w}^{n+\frac{1}{2}}):1 \rangle_{L^2} + C_0}}.
\]

where
\[
\phi(V) = \int_0^1 \exp((1 - \xi)V)d\xi.
\]

Then, we obtain the new scheme
\[
z^{n+1} = \exp(V)z^n + \tau \phi(V)Sf(\hat{w}^{n+\frac{1}{2}}, q^{n+\frac{1}{2}}), \quad (4.3)
\]
and
\[
q^{n+1} = q^n + \tau \frac{\langle G'(\hat{w}^{n+\frac{1}{2}}), \delta u \rangle_{L^2}}{2\sqrt{\langle G(\hat{w}^{n+\frac{1}{2}}):1 \rangle_{L^2} + C_0}}, \quad (4.4)
\]
for $n = 1, 2, \cdots$. The initial and boundary conditions in (3.1) are discretized as
\[
u_0^j = \phi_1(x_j), \quad v_0^j = \phi_2(x_j), \quad q_0^j = \sqrt{\langle G(u^0):1 \rangle_{L^2} + C_0}, \quad j = 0, 1, 2, \cdots, N, \quad (4.5)
\]
\[
u_0^n = u_n^N, \quad u_{n-1}^n = u_{N-1}^n, \quad n \geq 0. \quad (4.6)
\]

**Remark 4.1.** Note that the proposed scheme \(4.3, 4.4\) is a three level scheme and we obtain $z^1$ and $q^1$ by using $u^0$ instead of $\hat{w}^\frac{1}{2}$ for the first step.
Remark 4.2. With noting
\[ B_2 = F^H \Lambda F, \quad \Lambda = \text{diag} \left[ \lambda_0, \lambda_1, \cdots, \lambda_{N-1} \right], \quad \lambda_j = -\frac{4}{h^2} \sin^2 \frac{j\pi}{N}, \]
we have
\[ \exp(V) = \begin{pmatrix} F^H \Lambda_1 F & F^H \Lambda_2 F \\ F^H \Lambda_3 F & F^H \Lambda_4 F \end{pmatrix}, \]
where
\[
\begin{align*}
(A_1)_{ij} &= \begin{cases} 1, & j = 0, \\
\cosh(\tau \omega \lambda_j^{\frac{1}{2}}), & 1 \leq j \leq N - 1, 
\end{cases} \\
(A_2)_{ij} &= \begin{cases} \tau, & j = 0, \\
(\omega \lambda_j^{\frac{1}{2}})^{-1} \sinh(\tau \omega \lambda_j^{\frac{1}{2}}), & 1 \leq j \leq N - 1, 
\end{cases} \\
(A_3)_{ij} &= \begin{cases} 0, & j = 0, \\
(\omega \lambda_j^{\frac{1}{2}}) \sinh(\tau \omega \lambda_j^{\frac{1}{2}}), & 1 \leq j \leq N - 1. 
\end{cases}
\]

By the similar argument as above, we have
\[ \phi(V) = \begin{pmatrix} F^H \Sigma_1 F & F^H \Sigma_2 F \\ F^H \Sigma_3 F & F^H \Sigma_4 F \end{pmatrix}, \]
where
\[
\begin{align*}
(\Sigma_1)_{ij} &= \begin{cases} 1, & j = 0, \\
(\tau \omega \lambda_j^{\frac{1}{2}})^{-1} \sinh(\tau \omega \lambda_j^{\frac{1}{2}}), & 1 \leq j \leq N - 1, 
\end{cases} \\
(\Sigma_2)_{ij} &= \begin{cases} \frac{\tau}{2}, & j = 0, \\
(\tau \omega \lambda_j^{\frac{1}{2}})^{-1} (\cosh(\tau \omega \lambda_j^{\frac{1}{2}}) - 1), & 1 \leq j \leq N - 1, 
\end{cases} \\
(\Sigma_3)_{ij} &= \begin{cases} 0, & j = 0, \\
\tau^{-1} (\cosh(\tau \omega \lambda_j^{\frac{1}{2}}) - 1), & 1 \leq j \leq N - 1. 
\end{cases}
\]

Lemma 4.1. [20] For any symmetric matrix \( M \), and scalar \( \tau \geq 0 \), the matrix
\[ A = \exp(V)^T M \exp(V) - M \]
is a nilpotent matrix, when \( S \) is skew symmetric.

Theorem 4.1. The proposed scheme (4.3)-(4.4) can preserve the following discrete modified energy
\[ E_{n+1}^h = E_n^h, \quad E_n^h = \frac{1}{2} ||v^n||_2^2 + \frac{\omega^2}{2} ||\delta^+_x u^n||_2^2 + (q^n)^2 - C_0, \quad (4.7) \]
for \( n = 0, 1, 2, \cdots \).

Proof. We first note that the matrix \( M \) is singular, and assume that \( \{M_\epsilon\} \) is a series of symmetric and nonsingular matrices, which converge to \( M \) when \( \epsilon \to 0 \). Let \( z^n_\epsilon \) and \( q^n_\epsilon \) satisfy the perturbed scheme
\[ z^n_{\epsilon+1} = \exp(V_\epsilon) z^n_\epsilon + \tau \phi(V_\epsilon) S f(u^n_{\epsilon+\frac{1}{2}}, q^n_{\epsilon+\frac{1}{2}}), \quad (4.8) \]
\[ q^{n+1}_e = q^n_e + \tau \frac{\langle G'(u^{n+\frac{1}{2}}_e), \delta_t u^{n}_e \rangle}{2\sqrt{\langle G(u^{n+\frac{1}{2}}_e), 1 \rangle} + C_0}, \]  
(4.9)

where \( V_e = SM_e \) and \( n = 1, 2, \ldots \). Denote \( \bar{f}_e := M_e^{-1} f_e = M_e^{-1} f(\tilde{u}_e^{n+\frac{1}{2}}, q^{n}_e) \) and

\[ E^{n+1}_{\epsilon,h} = \frac{h}{2} (\epsilon^n)^T M_e z^n_e + (q^n_e)^2. \]  
(4.10)

Then we have

\[
\begin{align*}
\frac{1}{2} (z^{n+1}_e)^T M_e z^{n+1}_e &= \frac{1}{2} \left[ (z^n_e)^T \exp(V_e)^T + \tau f_e^T S^T \phi(V_e)^T \right] M_e \left[ \exp(V_e) z^n_e + \tau \phi(V_e) S f_e \right] \\
&= \frac{1}{2} (z^n_e)^T \exp(V_e)^T M_e \exp(V_e) z^n_e + (z^n_e)^T \exp(V_e)^T M_e \left[ \exp(V_e) - I \right] \bar{f}_e \\
&\quad + \frac{1}{2} f_e^T \left[ \exp(V_e)^T - I \right] M_e \left[ \exp(V_e) - I \right] \bar{f}_e \\
&= \frac{1}{2} (z^n_e)^T \exp(V_e)^T M_e \exp(V_e) z^n_e + (z^n_e)^T \left[ \exp(V_e)^T M_e \exp(V_e) - \exp(V_e)^T M_e \right] \bar{f}_e \\
&\quad + \frac{1}{2} f_e^T \left[ \exp(V_e)^T M_e \exp(V_e) - \exp(V_e)^T M_e \right] \bar{f}_e. \tag{4.11}
\end{align*}
\]

On the other hand, it follows from (4.9) that

\[
(q^{n+1}_e)^2 - (q^n_e)^2 = \frac{\langle G'(u^{n+\frac{1}{2}}_e), u^{n}_e - u^{n}_e \rangle}{2\sqrt{\langle G(u^{n+\frac{1}{2}}_e), 1 \rangle} + C_0} q^{n+\frac{1}{2}}_e \\
= h((z^{n+1}_e)^T - (z^n_e)^T) f_e \\
= h(z^n_e)^T \left[ \exp(V_e)^T - I \right] f_e + \tau h f_e^T S^T \phi(V_e)^T f_e \\
= h(z^n_e)^T \left[ \exp(V_e)^T M_e - M_e \right] \bar{f}_e + h f_e^T V_e^T \phi(V_e)^T M_e \bar{f}_e \\
= h(z^n_e)^T \left[ \exp(V_e)^T M_e - M_e \right] \bar{f}_e + h f_e^T \left[ \exp(V_e)^T M_e - M_e \right] \bar{f}_e. \tag{4.12}
\]

Then, we can deduce from (4.11) and (4.12) that

\[
\begin{align*}
E^{n+1}_{\epsilon,h} - E^{n}_{\epsilon,h} &= \frac{h}{2} (z^{n+1}_e)^T M_e z^{n+1}_e - \frac{h}{2} (z^n_e)^T M_e z^n_e + (q^{n+1}_e)^2 - (q^n_e)^2 \\
&= \frac{h}{2} (z^n_e)^T \left[ \exp(V_e)^T M_e \exp(V_e) - M_e \right] z^n_e + h(z^n_e)^T \left[ \exp(V_e)^T M_e \exp(V_e) - M_e \right] \bar{f}_e \\
&\quad + \frac{h}{2} f_e^T \left[ \exp(V_e)^T M_e \exp(V_e) - M_e \right] \bar{f}_e + \frac{h}{2} f_e^T \left[ \exp(V_e)^T M_e - M_e \exp(V_e) \right] \bar{f}_e \\
&= \frac{h}{2} (z^n_e + \bar{f})^T A_e (z^n_e + \bar{f}) + \frac{h}{2} f_e^T C_e \bar{f} = 0,
\end{align*}
\]

where \( A_e = \exp(V_e)^T M_e \exp(V_e) - M_e \) and \( C_e = \exp(V_e)^T M_e - M_e \exp(V_e) \). The last equality is from Lemma 1.1 and the skew symmetry of the matrix \( C_e \). Thus, when \( \epsilon \to 0 \), \( z^n_e \to z^n \), \( q^n_e \to q^n \) and (4.10) lead to

\[ E^{n+1}_h = E^n_h. \]

This completes the proof. \( \square \)
Corollary 4.1. Supposing $\phi_1 \in H^1(\mathbb{R})$ and $\phi_2 \in L^2(\mathbb{R})$, it then follows from (4.7) that

$$||v^n||_{l^2} \leq C, \ ||\delta^n_x u^n||_{l^2} \leq C, \ |q^n| \leq C, \ n = 0, 1, 2, \ldots,$$

which implies that the new scheme is unconditionally stable.

Besides its energy-preserving property, a most remarkable thing about the above scheme is that it can be solved efficiently. We rewrite (4.13) and (4.14) as

$$u^{n+1} = \exp_{11} u^n + \exp_{12} v^n - \tau \phi_{12} \frac{G'(\hat{u}^{n+\frac{1}{2}})}{\sqrt{\langle G(\hat{u}^{n+\frac{1}{2}}), 1 \rangle_{l^2} + C_0}} q^{n+\frac{1}{2}}, \quad (4.13)$$

$$v^{n+1} = \exp_{21} u^n + \exp_{22} v^n - \tau \phi_{22} \frac{G'(\hat{u}^{n+\frac{1}{2}})}{\sqrt{\langle G(\hat{u}^{n+\frac{1}{2}}), 1 \rangle_{l^2} + C_0}} q^{n+\frac{1}{2}}, \quad (4.14)$$

$$q^{n+1} = q^n + \tau \frac{\langle G'(\hat{u}^{n+\frac{1}{2}}), \delta u^n \rangle_{l^2}}{2\sqrt{\langle G(\hat{u}^{n+\frac{1}{2}}), 1 \rangle_{l^2} + C_0}}, \quad (4.15)$$

where $\exp(V)$ and $\phi(V)$ are partitioned into

$$\exp(V) = \left( \begin{array}{cc} \exp_{11} & \exp_{12} \\ \exp_{21} & \exp_{22} \end{array} \right), \quad \phi(V) = \left( \begin{array}{cc} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{array} \right).$$

Next, by eliminating $q^{n+\frac{1}{2}}$ in (4.13), we have

$$u^{n+1} + \gamma \langle G'(\hat{u}^{n+\frac{1}{2}}), u^{n+1} \rangle_{l^2} = g^n, \quad (4.16)$$

where

$$\gamma = \frac{\tau \phi_{12} G'(\hat{u}^{n+\frac{1}{2}})}{4\langle G(\hat{u}^{n+\frac{1}{2}}), 1 \rangle_{l^2} + 4C_0},$$

and

$$g^n = \exp_{11} u^n + \exp_{12} v^n - \tau \phi_{12} \frac{G'(\hat{u}^{n+\frac{1}{2}})}{\sqrt{\langle G(\hat{u}^{n+\frac{1}{2}}), 1 \rangle_{l^2} + C_0}} q^n + \gamma \langle G'(\hat{u}^{n+\frac{1}{2}}), u^n \rangle_{l^2}.$$ 

We take the discrete inner product of (4.16) with $G'(\hat{u}^{n+\frac{1}{2}})$ and have

$$\left(1 + \langle G'(\hat{u}^{n+\frac{1}{2}}), \gamma \rangle_{l^2}\right) \langle G'(\hat{u}^{n+\frac{1}{2}}), u^{n+1} \rangle_{l^2} = \langle G'(\hat{u}^{n+\frac{1}{2}}), g^n \rangle_{l^2}.$$ 

Notice $\langle G'(\hat{u}^{n+\frac{1}{2}}), \gamma \rangle_{l^2} \geq 0$, since $\phi_{12}$ is a symmetrical positive semidefinite matrix. We then obtain from the above that

$$\langle G'(\hat{u}^{n+\frac{1}{2}}), u^{n+1} \rangle_{l^2} = \frac{\langle G'(\hat{u}^{n+\frac{1}{2}}), g^n \rangle_{l^2}}{1 + \langle G'(\hat{u}^{n+\frac{1}{2}}), \gamma \rangle_{l^2}}. \quad (4.17)$$

After solving $\langle G'(\hat{u}^{n+\frac{1}{2}}), u^{n+1} \rangle_{l^2}$ from the linear system (4.17), $u^{n+1}$ is then updated from (4.16). Subsequently, $q^{n+1}$ is obtained from (4.15). Finally, we get $v^{n+1}$ from (4.14).
5 Numerical examples

In this section, we report the numerical performance in accuracy, CPU time and energy preservation of the proposed scheme (4.3)-(4.4) (denoted by ESAVS). The computation is carried out via Matlab 7.0 with AMD A8-7100 and RAM 4GB. Furthermore, we compare the ESAVS (4.3)-(4.4) with the exponential averaged vector filed scheme (denoted by EAVFS) which is given by

\[
\begin{align*}
  u^{n+1} &= \exp_{11} u^n + \exp_{12} v^n - \tau \phi_{12} \int_0^1 G' \left( (1 - \xi) u^n + \xi u^{n+1} \right) d\xi, \\
  v^{n+1} &= \exp_{21} u^n + \exp_{22} v^n - \tau \phi_{22} \int_0^1 G' \left( (1 - \xi) u^n + \xi v^{n+1} \right) d\xi.
\end{align*}
\]

Here, we should note that the above scheme is obtained by using the energy-preserving exponential integrator [20] and the finite difference method to solve the NKGE (2.1).

In our computation, the standard fixed-point iteration is used for the EAVFS and the iteration will terminate when the infinity norm of the error between two adjacent iterative steps is less than $10^{-14}$. In order to quantify the numerical solution, we use the $l^2$- and $l^\infty$-norms of the error between the numerical solution $u^n_j$ and the exact solution $u(x_j, t_n)$, respectively, as

\[
e_{h, 2}(t_n) = \sqrt{\frac{1}{h} \sum_{j=0}^{N-1} |u^n_j - u(x_j, t_n)|^2}, \quad e_{h, \infty}(t_n) = \max_{0 \leq j \leq N-1} |u^n_j - u(x_j, t_n)|, \quad n \geq 0.
\]

**Problem 1.** We first consider the nonlinear sine-Gordon equation in one dimension

\[
\partial_{tt} u - \partial_{xx} u + \sin(u) = 0, \quad -20 < x < 20, \quad t > 0,
\]

(5.1)

with initial conditions

\[
u(x, 0) = 0, \quad u_t(x, 0) = 4 \text{sech}(x), \quad -20 \leq x \leq 20,
\]

and the periodic boundary conditions. Equation (5.1) possesses the analytical solution

\[
u(x, t) = 4 \arctan(t \text{sech}(x)), \quad x \in \mathbb{R}, \quad t \geq 0.
\]

The error and convergence order of EAVFS and ESAVS at time $t = 1$ are given in Tab. 1 which can be observed that all schemes have second order accuracy in time and space and the error provided by ESAVS has the same order of magnitude as the one provided by EAVFS. Besides, we carry out comparison on the computational cost of the two schemes in Fig. 1 by refining the mesh size gradually, which shows that the cost of ESAVS is cheaper. Moreover, as the refinement of mesh sizes, the advantage of ESAVS emerges, which implies that our scheme shows the remarkable performance in the efficiency. The long-term energy deviations are plotted in Fig. 2. It is clear that ESAVS and EAVFS can exactly preserve the energy conservation law and our scheme admits much smaller energy error.
Table. 1: Numerical error and convergence rate for ESAVS and EAVFS under different grid steps at $t = 1$.

| Scheme | $(h, \tau)$ | $e_{h,2}$ | order | $e_{h,\infty}$ | order |
|--------|-------------|-----------|--------|----------------|--------|
| ESAVS  | $(\frac{1}{10}, \frac{1}{100})$ | 1.287e-03 | -      | 1.367e-03 | -      |
|        | $(\frac{1}{20}, \frac{1}{200})$ | 3.217e-04 | 2.00   | 3.413e-04 | 2.00   |
|        | $(\frac{1}{40}, \frac{1}{400})$ | 8.044e-05 | 2.00   | 8.531e-05 | 2.00   |
|        | $(\frac{1}{80}, \frac{1}{800})$ | 2.011e-05 | 2.00   | 2.133e-05 | 2.00   |
| EAVFS  | $(\frac{1}{10}, \frac{1}{100})$ | 1.104e-03 | -      | 1.050e-03 | -      |
|        | $(\frac{1}{20}, \frac{1}{200})$ | 2.761e-04 | 2.00   | 2.621e-04 | 2.00   |
|        | $(\frac{1}{40}, \frac{1}{400})$ | 6.902e-05 | 2.00   | 6.551e-05 | 2.00   |
|        | $(\frac{1}{80}, \frac{1}{800})$ | 1.725e-05 | 2.00   | 1.638e-05 | 2.00   |

Fig. 1: CPU time of the two schemes for the soliton with different mesh sizes till $t = 10$ under $\tau = 0.001$.

Fig. 2: The energy deviation with $h = 0.2$ and $\tau = 0.01$ over the time interval $t \in [0, 200]$.

Problem 2. We then consider the nonlinear sine-Gordon equation in two dimensions

$$\partial_t u - \partial_{xx} u - \partial_{yy} u + \sin(u) = 0, \quad -30 < x, y < 10, \quad t > 0,$$

with initial conditions [8]

$$u(x, y, 0) = 4 \tan^{-1} \left[ \exp \left( \frac{4 - \sqrt{(x + 3)^2 + (y + 7)^2}}{0.436} \right) \right],$$

$$u_t(x, y, 0) = \frac{4.13}{\cosh \left( \frac{4 - \sqrt{(x+3)^2+(y+7)^2}}{0.436} \right)}, \quad -30 \leq x, y \leq 10,$$

and the periodic boundary conditions.

In Fig. 3, we carry out comparison on the computational cost between two schemes by refining the mesh size gradually. It is clear to see that the cost of EAVFS is more expensive. Moreover, as the refinement of mesh sizes, the advantage of ESAVS emerges, which implies that our scheme is more preferable for large scale simulations than the exponential energy-preserving scheme EAVFS. Fig. 4 shows the collision precisely among four expanding circular ring solitons, which are in good agreement with those given in Ref. [8]. Here, the solution includes the extension across $x = -10$ and $y = -10$ by symmetry properties of the problem. Moreover, we also calculate the energy deviation for the two schemes over the time interval $t \in [0, 100]$ and plot it in Fig. 5. As is clear, our scheme is comparable with the energy-preserving scheme EAVFS.
Fig. 3: CPU time of the two schemes for the soliton with different mesh sizes till $t = 1$ under $\tau = 0.01$. 
Fig. 4: Collision of four ring solitons (mesh plot (left) and contour plot (right)) in terms of \( \sin(u/2) \) at times \( t = 0, 2.5, 5, 7.5, 10 \) with \( h = 0.2 \) and \( \tau = 0.1 \). (Continued).

Fig. 5: The energy deviation over the time interval \( t \in [0, 100] \) with \( h = 0.2 \) and \( \tau = 0.1 \).

**Problem 2.** We consider the nonlinear Klein-Gordon equation in two dimensions, as follows

\[
\partial_{tt}u - \partial_{xx}u - \partial_{yy}u + u^3 = 0, \quad -10 < x, y < 10, \quad t > 0,
\]

with initial conditions

\[
u(x, y, 0) = 2\text{sech}(\cosh(x^2 + y^2)), \quad u_t(x, y, 0) = 0, \quad -10 \leq x, y \leq 10,
\]

and the periodic boundary conditions.
Fig. 6: The Snapshots of numerical solution at times \( t = 0, 1, 3, 5, 7, t = 8 \) with \( h = 0.1 \) and \( \tau = 0.1 \).

Fig. 7: The energy deviation over the time interval \( t \in [0, 100] \) with \( h = 0.1 \) and \( \tau = 0.1 \).

Fig. 6 presents the initial condition as well as numerical solutions at different times, which shows the expansion and propagation of the initial soliton to the whole domain until getting the boundary at \( t = 8 \). The long time energy deviation of the two schemes is displayed in Fig. 7 which behaves similarly as that of Fig. 5. Here, we omit the comparison of the two schemes for the CPU time. This is due to the fact that the obtained results behave similarly as that of Fig. 3.

6 Concluding remarks

In this paper, we develop a novel linearly implicit energy-preserving exponential scheme for the nonlinear Klein-Gordon equation. The essential idea is first to reformulate the original equation (2.1) into an equivalent system, which inherits a quadratization energy, by utilizing the idea of the SAV approach. The resulting system is then discretized by the conventional finite difference method in space and a semi-discretized system is obtained, which can conserve the semi-discretized energy. Finally, the linearly implicit full discretized scheme is proposed by using the variation of constants formula and the linearized Crank-Nicolson method for the semi-discretized system. Several numerical examples are presented to illustrate the efficiency of our numerical scheme. Comparing with the exponential averaged vector filed scheme, our scheme shows remarkable...
efficiency. The strategy presented in this paper is rather general and useful so that it can be directly extended to develop linearly implicit energy-preserving exponential schemes for the other energy-conserving equation, such as the KdV equation, the non-linear Schrödinger equation, etc. In future work, we will focus on giving an a priori estimate for the proposed scheme.

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