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Semi-Analytical and Finite Element Investigations of the Vibration of a Stepped Beam on an Elastic Foundation

Hakan ERDOĞAN*, Safa Bozkurt COŞKUN

Abstract

In this study free vibration behavior of a stepped beam on an elastic foundation is considered. The vibration of uniform beams on an elastic foundation has been previously studied extensively and various solutions are available in the literature. However, the problem considered in current study appears not to have been widely covered in the literature and analytical solutions are strictly limited. To this aim, semi-analytical solutions are obtained first by using Adomian decomposition method, then finite element solutions are computed via structural finite element analysis software (SAP 2000). The free vibration analysis of stepped beam considering the combinations of different support conditions at each end are performed employing semi-analytical and finite element methods. The findings of the analysis are compared and discussed in detail.

Keywords: segmented beam, elastic foundation, vibration, Adomian decomposition method

1. INTRODUCTION

Beam on elastic foundation problems are of great interest for researchers in the fields of civil, mechanical and aeronautical engineering related to the design of structural members of buildings, aircrafts, pipes, railroads, etc.

In the literature, numerous studies were conducted on the vibration analysis of beams [1-3]. Researchers also focused on the special cases such as stepped beams [4] and beams on elastic foundations [5]. Wang [6], Kukla [7] and Belles et al. [8] are three interesting contributions to the technical studies about the subject. Thambiratnam and Zhuge [9] developed a simple finite element method and applied to treat the free vibration of
In order to find the eigenfrequencies $\omega$ of the beam one may assume

$$y(x, t) = w(x) \ e^{i\omega t}$$  \hspace{1cm} (2)$$

Substituting Eq.(2) in Eq.(1) yields

$$EI \frac{d^4w(x)}{dx^4} - \rho A \omega^2 w(x) + kw(x) = 0$$  \hspace{1cm} (3)$$

Eq.(3) can be rearranged as

$$\frac{d^4w(x)}{dx^4} - \left( \lambda^4 - \frac{k}{EI} \right) w(x) = 0$$  \hspace{1cm} (4)$$

where $\lambda^4 = \omega^2 \rho A/EI$. Introducing the parameter $\beta$, such that, $\beta^4 = \lambda^4 - \frac{k}{EI}$ solution to Eq.(4) is

$$w(x) = C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$  \hspace{1cm} (5)$$

when $\beta^4 \geq 0$ which describes that foundation stiffness has no effect in the solution. However, if the vibration frequency of the beam is relatively low, i.e., $\beta^4 < 0$ the solution of Eq.(4) becomes

$$w(x) = C_1 \sin \left( \frac{\beta x}{\sqrt{2}} \right) \sin \left( \frac{\beta x}{\sqrt{2}} \right) + C_2 \sinh \left( \frac{\beta x}{\sqrt{2}} \right) \cos \left( \frac{\beta x}{\sqrt{2}} \right) + C_3 \cos \left( \frac{\beta x}{\sqrt{2}} \right) \sinh \left( \frac{\beta x}{\sqrt{2}} \right) + C_4 \cos \left( \frac{\beta x}{\sqrt{2}} \right) \cosh \left( \frac{\beta x}{\sqrt{2}} \right)$$  \hspace{1cm} (6)$$

which is a different solution when compared to Eq.(5). The coefficients $C_1$, $C_2$, $C_3$, and $C_4$ in both solutions can be evaluated according to boundary conditions at the supports. These conditions are given as follows:

- For free end $w''(x) = 0$ and $w'''(x) = 0$
- For hinged end $w(x) = 0$ and $w''(x) = 0$
- For clamped end $w(x) = 0$ and $w'(x) = 0$

There are also four boundary conditions due to continuity at the junction of two segments of the stepped beam. These conditions impose the equality of the displacement, the slope, the moment and the shear force at the junction and given as follows:

- $w_1(x) = w_2(x)$
\begin{itemize}
  \item \(w_1'(x) = w_2'(x)\)
  \item \(w_2''(x) = \alpha w_2'(x)\)
  \item \(w_1'''(x) = \alpha w_2''(x)\)
\end{itemize}

where \(\alpha = I_2/I_1\).

3. ADOMIAN DECOMPOSITION METHOD

In Adomian decomposition method (ADM) a general form of the following differential equation is assumed.

\[ Lu + Nu + Ru = g(x) \quad (7) \]

where \(u(x)\) is the unknown solution, \(g(x)\) is the source term, \(L\) is the linear operator, \(N\) is the nonlinear operator and \(R\) is the operator for remainder terms. The solution to Eq.(7) is

\[ u(x) = f(x) - L^{-1}(Nu) - L^{-1}(Ru) \quad (8) \]

where \(L^{-1}\) is the inverse linear operator and \(f(x) = L^{-1}(g(x))\). The solution is constructed with an infinite series in the following form

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (9) \]

The nonlinear term \(Nu\) is represented by so-called Adomian polynomials given below.

\[ Nu = \sum_{n=0}^{\infty} A_n(s_0, s_1, \cdots, s_n) \quad (10) \]

where \(A_n\) is the \(n^{th}\) Adomian polynomial defined as the following term.

\[ A_n = \frac{1}{n!} \frac{d^n}{d \lambda^n} N(\sum_{k=0}^{\infty} \lambda^k u_k) \quad (11) \]

The method leads to successive approximations as follows:

\[ u_0(x) = f(x) \quad (12) \]

\[ u_n(x) = -L^{-1}(Ru_{n-1} - A_{n-1}) \quad (13) \]

Finally, the solution is calculated by adding the successive approximations given in Eqs.(12-13). An \(N^{th}\) order analytical approximation includes the terms up to \(u_N\) as given below

\[ u(x) = \sum_{n=0}^{N} u_n(x) \quad (14) \]

For further details of the method, the reader may refer to [11].

4. ADM SOLUTION OF THE PROBLEM

An initial approximation based on Eq.(12) may be obtained as

\[ w_0(x) = Ax^3 + Bx^2 + Cx + D \quad (15) \]

where \(A = y''(0)/6, B = y''(0)/2, C = y'(0)\) and \(D = y(0)\). Successive approximations for an \(N^{th}\) order solution may be computed according to Eq.(13) as

\[ w_n(x) = L^{-1}(\beta^4 y_{n-1}) \quad , \quad n > 0 \quad (16) \]

Since there are two segments in the stepped beam, an initial approximation of the form given in Eq.(15) is assumed for both segments of the beam.

\[ w_0^{(1)}(x) = A_1 x^3 + B_1 x^2 + C_1 x + D_1 \quad (17) \]

\[ w_0^{(2)}(x) = A_2 x^3 + B_2 x^2 + C_2 x + D_2 \quad (18) \]

Eight boundary conditions are required to determine eight unknowns introduced in Eqs.(17) and (18). These conditions are four boundary conditions at the supports and four continuity conditions. Hence eight equations in eight unknowns are produced can be represented in the following matrix form.

\[ [K]_{8x8} \{\Lambda\}_{8x1} = \{0\}_{8x1} \quad (19) \]

where \([K]\) includes the term \(\beta\) which is the function of vibration frequency \(\omega\) and the unknown vector \(\{\Lambda\}\) includes unknown coefficients in the initial approximations in Eqs.(17) and (18). The trivial solution of Eq.(19) corresponds the undeformed beam. Hence, a nontrivial solution to the problem can be obtained by equating the determinant of coefficient matrix to zero that lead to free vibration frequencies of the stepped beam on elastic foundation considered.

5. NUMERICAL APPLICATION

Wang [6] calculated analytical solutions for natural frequencies of two stepped beams on...
elastische fundamente, eine einfach befestigte und eine mit freien Enden. Es gibt keine analytischen Lösungen für den Balken mit beiden Enden befestigt, den Balken mit einem Ende befestigt und eine Enden befestigt und ein Ende freies. Für die Einfachheit folgende Abkürzungen werden für verschiedene Kombinationen von Randbedingungen.

- FF Frei – Frei
- SS Beide Enden einfach befestigten
- CC Befestigte – Befestigte
- CS Befestigte – einfach befestigen

Wang [6] durchgeführt die Analyse für die folgende Daten: \( E = 6.50 \times 10^{11} \text{ Pa}, \rho = 213.60 \text{ kg/m}^2, H_1 = 0.10 \text{ m}, H_2 = 0.15 \text{ m}, B = 0.08 \text{ m}, L = 5.00 \text{ m}. \) Zwei verschiedene Fundamentmodul wurden verwendet in den Berechnungen, \( \nu = 1/100 \) und \( \nu = 1/200 \) wo \( \nu \) definiert als \( k/EI. \) \( \mu \) ist die Ratio of the length of thinner segment having depth of \( H_1 \) to total length \( L \).

SAP2000 [12] wurde verwendet für die Ermittlung von endlichen Element (FE) Lösungen. Line Springs, die in einer beliebigen lokalen Richtung eines Rahmenobjekts in vertikale Richtung definiert wird, simulieren das elastische Fundament. SAP2000 [12] verteilt die Springs an allen Knoten (Abbildung 2).

Abbildung 2 FE Modell für einfach befestigte Stepped Balken auf elastischem Fundament (50 Balkenelementen)

Tabelle 1 vergleicht die vorherigen Ergebnisse (analytische [6], HPM [10]) mit den Ergebnissen der aktuellen Studie (ADM, FE Lösung) für natürliche Frequenzen des Stepped FF Balken auf elastischem Fundament. ADM Lösungen werden konzipiert für 12. Ordnung und Computations sind identisch zu den analytischen [6] und HPM [10] Lösungen. FE Lösungen sind auch in ausgezeichneten Übereinstimmung für die ersten zwei Frequenzwerte.

In Tabelle 2 nur die ersten zwei natürliche Frequenzen sind verfügbar für SS Balken [6, 10] und verwendet für die Vergleich. Jedoch, die ersten drei Frequenzen werden für Stepped SS Balken auf elastischem Fundament mit ADM und FEM. ADM Ergebnisse sind in ausgezeichneten Übereinstimmung mit den vorherigen Ergebnissen, während FE Lösungen in sehr gute Übereinstimmung mit den gleichen Ergebnissen für welche erste zwei Frequenzen haben die gleiche Genauigkeit von analytischer Lösung.

Analytische Lösungen für die ersten drei Frequenzen für Stepped CC und CS Balken auf elastischem Fundament sind nicht verfügbar in der Literatur. Nur HPM Lösungen [10] existieren und die Ergebnisse dieses Studien für diese Fälle sind mit den nur HPM Ergebnissen verglichen.

In Tabelle 3 und Tabelle 4 scheint, dass ADM Lösungen in sehr gute Übereinstimmung mit HPM Ergebnissen [10]. FE Lösungen sind in sehr gute Übereinstimmung mit beiden Ergebnissen in vorherigen Fällen.

| Table 1 |
|-------------------------------|
| Natural frequencies of FF stepped beam on elastic foundation |
| \( \nu = 1/100 \) |
| \( H_1 \) & \( H_2 \) & \( F_1 \) & \( F_2 \) & \( F_3 \) & \( F_1 \) & \( F_2 \) & \( F_3 \) & \( F_1 \) & \( F_2 \) & \( F_3 \) |
| 0  & 5.8531 & 5.8531 & 96.4081 & 5.8531 & 5.8531 & 96.4081 & 5.8531 & 5.8531 & 96.2888 & 5.8531 & 5.8531 & 96.3783 |
| 0.1 & 5.8534 & 6.2201 & 101.0194 & 5.8534 & 6.2201 & 101.0194 & 5.8534 & 6.2201 & 100.8885 & 5.8534 & 6.2201 & 100.9867 |
| 0.2 & 5.8557 & 6.5291 & 100.7912 & 5.8557 & 6.5291 & 100.7912 & 5.8557 & 6.5291 & 100.6555 & 5.8557 & 6.5290 & 100.7573 |
| 0.3 & 5.8635 & 6.7692 & 93.4247 & 5.8635 & 6.7692 & 93.4247 & 5.8635 & 6.7692 & 93.2962 & 5.8635 & 6.7691 & 93.3926 |
| 0.4 & 5.8821 & 6.9403 & 82.1686 & 5.8821 & 6.9403 & 82.1686 & 5.8821 & 6.9403 & 82.1676 & 5.8821 & 6.9402 & 82.1895 |
| 0.5 & 5.9194 & 7.0513 & 72.8477 & 5.9194 & 7.0513 & 72.8477 & 5.9194 & 7.0513 & 72.7549 & 5.9194 & 7.0512 & 72.8245 |
| 0.6 & 5.9876 & 7.1161 & 66.7402 & 5.9876 & 7.1161 & 66.7402 & 5.9876 & 7.1161 & 66.6565 & 5.9876 & 7.1160 & 66.7193 |
| 0.7 & 6.1046 & 7.1494 & 63.2525 & 6.1046 & 7.1494 & 63.2525 & 6.1046 & 7.1494 & 63.1733 & 6.1047 & 7.1494 & 63.2327 |
| 0.8 & 6.2993 & 7.1637 & 61.5574 & 6.2993 & 7.1637 & 61.5574 & 6.2993 & 7.1637 & 61.4814 & 6.2994 & 7.1637 & 61.5384 |
| 0.9 & 6.6214 & 7.1800 & 61.4321 & 6.6214 & 7.1800 & 61.4321 & 6.6214 & 7.1800 & 61.3585 & 6.6214 & 7.1800 & 61.4137 |
| 1.0 & 7.1685 & 7.1685 & 64.5528 & 7.1685 & 7.1685 & 64.5528 & 7.1685 & 7.1685 & 64.4735 & 7.1685 & 7.1685 & 64.5330 |
Table 2
Natural frequencies of SS stepped beam on elastic foundation

| µ   | f_1 (Hz) | f_2 (Hz) | f_3 (Hz) | f_4 (Hz) | f_5 (Hz) | f_6 (Hz) | f_7 (Hz) |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0   | 38.6517  | 169.8519 | 42.6517  | 169.8519 | 382.0578 | 42.6515  | 169.8511 |
| 0.1 | 42.3838  | 166.0034 | 42.3838  | 166.0034 | 366.1278 | 42.3825  | 166.0136 |
| 0.2 | 40.8661  | 152.2257 | 40.8663  | 152.2257 | 341.8615 | 40.8074  | 152.2212 |
| 0.3 | 37.9155  | 143.6983 | 37.9155  | 143.6983 | 324.0765 | 37.9161  | 143.6809 |
| 0.4 | 34.7788  | 134.3927 | 34.7788  | 134.3927 | 323.7259 | 34.7785  | 134.3800 |
| 0.5 | 32.1732  | 124.3627 | 32.1732  | 124.3629 | 300.8481 | 32.1611  | 124.5659 |
| 0.6 | 30.4553  | 114.6785 | 30.4553  | 114.6785 | 299.3743 | 30.4537  | 114.6389 |
| 0.7 | 29.1741  | 104.1574 | 29.1741  | 104.1574 | 291.5020 | 29.4157  | 104.1546 |
| 0.8 | 28.5141  | 94.1947  | 28.5141  | 94.1947  | 281.3500 | 28.5147  | 94.1947  |
| 0.9 | 28.8769  | 84.8769  | 28.8769  | 84.8769  | 265.8024 | 28.8765  | 84.8765  |
| 1.0 | 28.7056  | 74.3144  | 28.7056  | 74.3144  | 254.7527 | 28.7052  | 74.3143  |

| µ   | f_1 (Hz) | f_2 (Hz) | f_3 (Hz) | f_4 (Hz) | f_5 (Hz) | f_6 (Hz) | f_7 (Hz) |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0   | 38.6517  | 169.8519 | 42.6517  | 169.8519 | 382.0578 | 42.6515  | 169.8511 |
| 0.1 | 42.3838  | 166.0034 | 42.3838  | 166.0034 | 366.1278 | 42.3825  | 166.0136 |
| 0.2 | 40.8661  | 152.2257 | 40.8663  | 152.2257 | 341.8615 | 40.8074  | 152.2212 |
| 0.3 | 37.9155  | 143.6983 | 37.9155  | 143.6983 | 324.0765 | 37.9161  | 143.6809 |
| 0.4 | 34.7788  | 134.3927 | 34.7788  | 134.3927 | 323.7259 | 34.7785  | 134.3800 |
| 0.5 | 32.1732  | 124.3627 | 32.1732  | 124.3629 | 300.8481 | 32.1611  | 124.5659 |
| 0.6 | 30.4553  | 114.6785 | 30.4553  | 114.6785 | 299.3743 | 30.4537  | 114.6389 |
| 0.7 | 29.1741  | 104.1574 | 29.1741  | 104.1574 | 291.5020 | 29.4157  | 104.1546 |
| 0.8 | 28.5141  | 94.1947  | 28.5141  | 94.1947  | 281.3500 | 28.5147  | 94.1947  |
| 0.9 | 28.8769  | 84.8769  | 28.8769  | 84.8769  | 265.8024 | 28.8765  | 84.8765  |
| 1.0 | 28.7056  | 74.3144  | 28.7056  | 74.3144  | 254.7527 | 28.7052  | 74.3143  |

Table 3
Natural frequencies of CC stepped beam on elastic foundation

| µ   | f_1 (Hz) | f_2 (Hz) | f_3 (Hz) | f_4 (Hz) | f_5 (Hz) | f_6 (Hz) | f_7 (Hz) |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0   | 96.4081  | 265.3269 | 502.0532 | 96.4081  | 265.3269 | 502.0532 | 96.4080  |
| 0.1 | 83.6134  | 244.1429 | 489.1152 | 83.6134  | 244.1429 | 489.1152 | 83.6146  |
| 0.2 | 83.6454  | 237.3657 | 463.9575 | 83.6454  | 237.3657 | 463.9575 | 83.6402  |
| 0.3 | 83.6227  | 224.3318 | 461.3383 | 83.6227  | 224.3318 | 461.3383 | 83.6227  |
| 0.4 | 78.6661  | 222.4427 | 442.1322 | 78.6661  | 222.4427 | 442.1322 | 78.6661  |
| 0.5 | 76.2090  | 221.3049 | 410.8388 | 76.2090  | 221.3049 | 410.8388 | 76.2090  |
| 0.6 | 76.4158  | 208.0189 | 409.6846 | 76.4158  | 208.0189 | 409.6846 | 76.4158  |
| 0.7 | 78.2861  | 196.9283 | 391.5603 | 78.2861  | 196.9283 | 391.5603 | 78.2861  |
| 0.8 | 77.9781  | 198.5024 | 373.5053 | 77.9781  | 198.5024 | 373.5053 | 77.9781  |
| 0.9 | 78.2944  | 196.4500 | 377.8517 | 78.2944  | 196.4500 | 377.8517 | 78.2944  |
| 1.0 | 64.5528  | 176.9686 | 346.7543 | 64.5528  | 176.9686 | 346.7543 | 64.5527  |
Table 4  
Natural frequencies of CS stepped beam on elastic foundation

\[
\begin{array}{cccccccccccc}
\mu & f_1 (Hz) & f_2 (Hz) & f_3 (Hz) & f_1 (Hz) & f_2 (Hz) & f_3 (Hz) & f_1 (Hz) & f_2 (Hz) & f_3 (Hz) \\
0 & 66.5734 & 244.1077 & 489.0976 & 38.5108 & 244.1077 & 489.0976 & 38.5123 & 244.1219 & 489.1513 \\
0.1 & 56.5407 & 195.8263 & 420.3049 & 56.5407 & 195.8263 & 420.3049 & 56.5407 & 195.8263 & 420.3049 \\
0.2 & 56.2305 & 192.4146 & 399.6111 & 56.2305 & 192.4146 & 399.6111 & 56.2305 & 192.4146 & 399.6111 \\
0.3 & 55.8497 & 181.3330 & 393.1700 & 55.8497 & 181.3330 & 393.1700 & 55.8497 & 181.3330 & 393.1700 \\
0.4 & 53.7662 & 176.7855 & 346.3749 & 53.7662 & 176.7855 & 346.3749 & 53.7662 & 176.7855 & 346.3749 \\
0.5 & 50.7342 & 172.4997 & 319.2751 & 50.7342 & 172.4997 & 319.2751 & 50.7342 & 172.4997 & 319.2751 \\
0.6 & 47.9739 & 159.6724 & 319.2751 & 47.9739 & 159.6724 & 319.2751 & 47.9739 & 159.6724 & 319.2751 \\
0.7 & 45.1264 & 148.8735 & 319.2751 & 45.1264 & 148.8735 & 319.2751 & 45.1264 & 148.8735 & 319.2751 \\
0.8 & 42.8322 & 144.0672 & 319.2751 & 42.8322 & 144.0672 & 319.2751 & 42.8322 & 144.0672 & 319.2751 \\
0.9 & 40.7878 & 143.4492 & 319.2751 & 40.7878 & 143.4492 & 319.2751 & 40.7878 & 143.4492 & 319.2751 \\
1.0 & 40.7878 & 143.4492 & 319.2751 & 40.7878 & 143.4492 & 319.2751 & 40.7878 & 143.4492 & 319.2751 \\
\end{array}
\]

Figure 3  Effect of foundation stiffness on natural frequencies of stepped FF beam

![Figure 3](image-url)
Effect of foundation stiffness on first two frequencies is clearly illustrated in Fig.3. However, foundation effect for the third natural frequency is indistinguishable.

For stepped SS, CC and CS beams there are no significant difference in the natural frequencies for $\nu = 1/100$ and $\nu = 1/200$. Hence, between Figs. 4 and 6 only the variation of first three frequencies is plotted for $\nu = 1/100$.

Figure 4 Variation of natural frequencies of stepped SS beam

Figure 5 Variation of natural frequencies of stepped CC beam

It can be mentioned that natural frequencies of the stepped SS, CC and SS beams decreases with increasing $\mu$ values considering the variations shown in figures 4-6.

6. CONCLUSIONS

In this study, natural frequencies of stepped beams on elastic foundations are investigated via ADM and FEM. Analytical solutions for this problem are available for the beam with free ends and for simply supported beam. There are no other available analytical solutions for the beam with both ends clamped and for the beam with one end clamped and one end simply supported. All four cases previously were solved using HPM. ADM solutions of this study are in perfect agreement with analytical and HPM solutions. FE solutions are computed employing SAP 2000 software by using 50-element and 100-element models. Both FE models produced reliable results when compared ADM and previously available solutions.

Effect of foundation modulus is found to be distinguishable only for stepped FF beam considering the two different foundation modulus investigated in the scope of study. Variation of frequencies with the position of the intersection point of the two segments is also depicted graphically for stepped SS, CC and CS beams for which it is observed that natural frequencies decreases while $\mu$ increases. This result makes sense; As $\mu$ value increases, the frequency is...
mostly dominated with segment one that is having relatively low moment of inertia.

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**The Declaration of Conflict of Interest/Common Interest**

No conflict of interest or common interest has been declared by the authors.

**Authors' Contribution**

In this study, the contributions of the authors during the research, analysis, submission, review and editing stages are equal.

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The authors declare that this document does not require an ethics committee approval or any special permission.

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