Jacobi-Maupertuis Randers-Finsler metric for curved spaces and the gravitational magnetoelectric effect

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Abstract

In this paper we return to the subject of Jacobi metrics for timelike and null geodesics in stationary spacetimes, correcting some previous misconceptions. We show that not only null geodesics, but also timelike geodesics are governed by a Jacobi-Maupertuis type variational principle and a Randers-Finsler metric for which we give explicit formulae. The cases of the Taub-NUT and Kerr spacetimes are discussed in detail. Finally we show how our Jacobi-Maupertuis Randers-Finsler metric may be expressed in terms of the effective medium describing the behaviour of Maxwell’s equations in the curved spacetime. In particular, we see in very concrete terms how the magnetolectric susceptibility enters the Jacobi-Maupertuis-Randers-Finsler function.

1 Introduction

In this paper we consider the motion of a neutral particle moving in a stationary spacetime. Because the metric is stationary there is a conserved energy and it is natural, following the time honoured procedure of Maupertuis and Jacobi, to convert the problem to a variational problem at fixed energy. If the metric is static this is straightforward and one finds that the motion may be obtained as the
the geodesics of a Riemannian metric which for massive particles depends on the conserved energy \cite{1}. For massless particles the metric is independent of the energy and is often referred to as the optical or sometimes the Fermat metric. For stationary metrics the cross term between the space and time components of the spacetime metric renders the situation more complicated.

For massless particles the motion may be obtained as the geodesic of a metric which is, however, not Riemannian but rather a Finsler metric of so-called Randers type \cite{2}. Randers-Finsler metrics arose originally in an attempt to provide a unified description of gravitation and electromagnetism. As a first step Randers started by considering a charged particle moving in curved spacetime metric and an electromagnetic field\cite{3}. Subsequently the particle trajectories were recognised as the geodesics of a particular example of a Finsler metric. In \cite{1} it was stated that “the motion of massive particles is governed by neither a Riemannian nor a Finslerian metric”. As pointed out in \cite{4} this statement is not correct. In fact, the motions of massive neutral particles are the geodesics of an energy dependent Randers-Finsler metric. In this paper we elaborate on the considerations of \cite{4} and the light they throw on the unsuccessful attempts of \cite{5}.

As pointed out above, the necessity of passing to a Randers-Finsler metric arises from the cross terms in the metric. This is perhaps not too surprising since the cross terms also give rise to the phenomenon sometimes referred to as the “rotation of inertial frames” or, more graphically, “gravito-magnetism”. In \cite{3} the Maxwell field was treated as a fixed background. However it is also of interest to ask how a general time dependent Maxwell field responds to a stationary background. One approach to this question is to write out Maxwell’s equations in a local coordinate system defining electric and magnetic fields and electric and magnetic inductions. The former are related to the latter by linear constitutive relations where the susceptibilities depend upon the components of the spacetime metric and are, in general, position dependent\cite{6,7}. In a stationary metric gravito-magnetism expresses itself as gravitational magnetoelectric effect\cite{8} in which the constitutive relations mix electric and magnetic fields. We are able to show how the susceptibilities enter the Jacobi-Randers-Finsler metric reflecting this phenomenon.

Furthermore, we know that the existence of a conserved quantity implies the reduction of a degree of freedom from the overall motion. Since the formulation of the Jacobi-Maupertuis metric is centred around the availability of a conserved quantity, it is essentially a reduction of the dimension of the moduli space of geodesics. When the conserved quantity is associated with a cyclical co-ordinate, we are reducing the dimensions of the configuration space of the geodesic by that cyclical co-ordinate, thus simplifying the mechanical problem.

The plan of the paper is as follows. Section 2 introduces constraints among the spatial momenta and the energy giving rise to the difficulties experienced in \cite{1,5}. In section 3 we treat the simpler electromagnetic case in preparation for the full stationary case in section 4. The results obtained in section 4 agree with a remark in \cite{9}, unknown to the authors at the time of writing \cite{1,5}, whose main purpose was to discuss relativistic brachistochrones. The analogy to the motion of a charged particle moving in electromagnetic field was noted in \cite{1} but not related to Randers-Finsler geometry. In section 5 we give some explicit examples including the physically important case of the Kerr metric for a rotating black hole. Finally, in section 6 we relate the previous sections to the gravitational magnetoelectric effect.
2 Constraint for momenta of a geodesic

If the relativistic Lagrangian for a free particle moving in a stationary spacetime is given by:

$$\mathcal{L} = -mc\sqrt{g_{\mu\nu}(\mathbf{x})\dot{x}^\mu\dot{x}^\nu} = -mc\sqrt{c^2g_{00}(\mathbf{x})\dot{t}^2 + 2cg_{0i}(\mathbf{x})\dot{t}\dot{x}^i + g_{ij}(\mathbf{x})\dot{x}^i\dot{x}^j},$$

(1)

where the metric signature is $+, -, -, -$; $\mathbf{x} = (x, y, z)$ specifies the spatial coordinates and the dot indicates differentiation with respect to an arbitrary parameter, then the canonical momenta are:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -mc\frac{g_{\mu\nu}(\mathbf{x})\dot{x}^\nu}{\sqrt{g_{\alpha\beta}(\mathbf{x})\dot{x}^\alpha\dot{x}^\beta}}$$

$$\begin{cases} p_0 = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -mc\frac{cg_{00}(\mathbf{x})\dot{t} + cg_{0i}(\mathbf{x})\dot{t}\dot{x}^i + cg_{ij}(\mathbf{x})\dot{x}^i\dot{x}^j}{\sqrt{g_{\alpha\beta}(\mathbf{x})\dot{x}^\alpha\dot{x}^\beta}} = -\frac{\mathcal{E}}{c}, \quad (2) \\
p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = -mc\frac{cg_{0i}(\mathbf{x})\dot{t} + cg_{ij}(\mathbf{x})\dot{x}^j}{\sqrt{g_{\alpha\beta}(\mathbf{x})\dot{x}^\alpha\dot{x}^\beta}} \end{cases}$$

where $\mathcal{E} = p_0\dot{t} - \mathcal{L}$ is the relativistic energy of the system. Thus, the canonical momenta of a geodesic satisfy the constraint:

$$g^{\mu\nu}(\mathbf{x})p_\mu p_\nu = (mc)^2 g^{\mu\nu}(\mathbf{x})\frac{(g_{\mu\alpha}(\mathbf{x})\dot{x}^\alpha)}{\sqrt{g_{\sigma\tau}(\mathbf{x})\dot{x}^\sigma\dot{x}^\tau}} \frac{(g_{\nu\beta}(\mathbf{x})\dot{x}^\beta)}{\sqrt{g_{\sigma\tau}(\mathbf{x})\dot{x}^\sigma\dot{x}^\tau}} = (mc)^2.$$  

(3)

The relativistic momenta are therefore constrained to the “mass shell” at all points of spacetime. On the other hand, for the Jacobi metric $ds_J = p_i dx^i$, one demands:

$$L_J \equiv \dot{s}_J = p_i\dot{x}^i = -mc\frac{g_{\mu\nu}(\mathbf{x})\dot{x}^\mu\dot{x}^\nu}{\sqrt{g_{\alpha\beta}(\mathbf{x})\dot{x}^\alpha\dot{x}^\beta}} = \sqrt{J_{ij}\dot{x}^i\dot{x}^j}.  \quad (4)$$

Thus, the momenta would be given by

$$p_i = \frac{J_{ij}\dot{x}^j}{\sqrt{J_{ab}\dot{x}^a\dot{x}^b}}\quad (5)$$

and the constraint equation for the momenta (5) for the Jacobi metric would be:

$$J^{ij}(\mathbf{x})p_i p_j = J^{ij}\frac{J_{ij}p_i\dot{x}^i}{\sqrt{J_{ab}\dot{x}^a\dot{x}^b}}\frac{J_{ij}p_j\dot{x}^j}{\sqrt{J_{ab}\dot{x}^a\dot{x}^b}} = \frac{(J^{ij}J_{ij})\dot{x}^i\dot{x}^j}{J_{ab}\dot{x}^a\dot{x}^b} = 1.  \quad (6)$$

This rule has been applied to static spacetimes in [11] (where $g_{0i}(\mathbf{x}) = 0$, $g^{00}(\mathbf{x}) = (g_{00}(\mathbf{x}))^{-1}$)

$$g^{00}(\mathbf{x})p_0^2 + g^{ij}(\mathbf{x})p_ip_j = (mc)^2 \Rightarrow g^{00}(\mathbf{x}) = \frac{g^{ij}(\mathbf{x})(mc)^2g_{00}(\mathbf{x}) - p_0^2 g^{ij}(\mathbf{x})}{g_{00}(\mathbf{x})} p_i p_j = 1$$

$$\Rightarrow J^{ij}(\mathbf{x}) = \frac{g_{00}(\mathbf{x})}{(mc)^2g_{00}(\mathbf{x}) - p_0^2 g^{ij}(\mathbf{x})} \Rightarrow J_{ij}(\mathbf{x}) = \frac{(mc)^2g_{00}(\mathbf{x}) - p_0^2}{g_{00}(\mathbf{x})} g_{ij}(\mathbf{x}),$$

therefore

$$ds_J = \sqrt{(mc)^2g_{00}(\mathbf{x}) - p_0^2 g_{ij}(\mathbf{x})}dx^i dx^j  \quad (7)$$

and attempted in [5]. The constraint for momentum of a stationary spacetime will be:

$$g^{00}(\mathbf{x})p_0^2 + 2g^{0i}(\mathbf{x})p_0p_i + g^{ij}(\mathbf{x})p_ip_j = (mc)^2,$$

from which it is evidently impossible to determine the constraint equation for the Jacobi metric [11], compelling us to return to first principles.
3 Jacobi metric for a Lagrangian with magnetic fields

In this section we shall proceed by considering a non-relativistic Lagrangian involving magnetic fields. Starting from the relativistic Lagrangian (1), defining \( g_{00}(x) \) and parametrizing with respect to time\
\[
g_{00}(x) = 1 + \frac{2U(x)}{mc^2}, \quad \dot{t} = 1,
\]
and taking non-relativistic approximation by expanding binomially up to first order as shown in \([10]\), we will have the non-relativistic Lagrangian involving magnetic fields:

\[
L = -mc \sqrt{g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu} = -mc^2 \sqrt{1 + \frac{ \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + mc g_{0i}(x) \dot{x}^i + U(x) }{mc^2}}
\]
\[
\approx -mc^2 \left[ 1 + \frac{1}{mc^2} \left( \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + mc g_{0i}(x) \dot{x}^i + U(x) \right) \right]
\]
\[
= \left( -\frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j - mc g_{0i}(x) \dot{x}^i - U(x) \right) - mc^2 = L - mc^2.
\]

Omitting the additive term \(-mc^2\), and writing \( G_{ij}(x) = -g_{ij}(x) \), \(-mc g_{0i}(x) = A_i(x)\), we get:

\[
L = \frac{m}{2} G_{ij}(x) \dot{x}^i \dot{x}^j + A_i(x) \dot{x}^i - U(x). \tag{8}
\]

Since in the presence of magnetic type interactions the optical geometry is no longer Riemannian \([2]\), we are led to replace (4) with the more general ansatz

\[
L_J \equiv \dot{s}_J = p_i \dot{x}^i = F(x, \dot{x}) \tag{9}
\]

with \( F \) an arbitrary homogeneous function of degree one in the second set of variables fulfilling appropriate regularity conditions.

From (8) the canonical momenta and energy are given by:

\[
p_i = \frac{\partial L}{\partial \dot{x}^i} = m G_{ij}(x) \dot{x}^j + A_i(x) \Rightarrow m G_{ij}(x) \dot{x}^j = p_i - A_i(x) = \pi_i
\]
\[
E = p_i \dot{x}^i - L = \frac{1}{2m} G^{ij}(x) \pi_i \pi_j + U(x) \tag{10}
\]

Thus, by direct formulation by applying (10) to (9), the Jacobi Lagrangian is given by:

\[
F(x, \dot{x}) = p_i \dot{x}^i = m G_{ij}(x) \dot{x}^i \dot{x}^j + A_i(x) \dot{x}^i = \sqrt{2m (E - U(x))} G_{ij}(x) \dot{x}^i \dot{x}^j + A_i(x) \dot{x}^i
\]

Thus, the Jacobi metric for a system with magnetic fields is a Finsler metric of Randers type given by:

\[
 ds_J = \sqrt{2m (E - U(x))} G_{ij}(x) dx^i dx^j + A_i(x) dx^i. \tag{11}
\]

This represents a generalization of the the original model of Randers \([3]\) which was used in \([2]\) to describe null geodesics in stationary spacetimes. We will compare this result to the non-relativistic limit of the relativistic Jacobi metric that we will deduce later.
4 Jacobi metric for a stationary spacetime

Upon applying (2), the Jacobi Lagrangian is given by:

\[ L_J = p_i \dot{x}^i = -mc \frac{c g_{0i}(x) \dot{x}^i + g_{ij}(x) \dot{x}^i \dot{x}^j}{\sqrt{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta}} = p_0 \frac{g_{0i}(x)}{g_{00}(x)} \dot{x}^i + mc \frac{\left( -g_{ij}(x) + \frac{g_{0i}(x) g_{0j}(x)}{g_{00}(x)} \right) \dot{x}^i \dot{x}^j}{\sqrt{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta}} \]

Here, let us define the spatial metric,

\[ \gamma_{ij}(x) := -g_{ij}(x) + \frac{g_{0i}(x) g_{0j}(x)}{g_{00}(x)}, \quad (12) \]

allowing us to write the Jacobi Lagrangian as:

\[ L_J = p_0 \frac{g_{0i}(x)}{g_{00}(x)} \dot{x}^i + mc \sqrt{\frac{\gamma_{ij}(x) \dot{x}^i \dot{x}^j}{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta}} \sqrt{\gamma_{ij}(x) \dot{x}^i \dot{x}^j}. \quad (13) \]

From the 1st equation of (2) for momentum \( p_0 \), we obtain the following relationship,

\[ p_0^2 = (mc)^2 g_{00}(x) \left( 1 + \frac{\gamma_{ij}(x) \dot{x}^i \dot{x}^j}{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta} \right) \Rightarrow \frac{\gamma_{ij}(x) \dot{x}^i \dot{x}^j}{g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta} = \frac{p_0^2 - (mc)^2 g_{00}(x)}{(mc)^2 g_{00}(x)}. \quad (14) \]

Now, applying (14) to the Jacobi Lagrangian (13), we have the form

\[ L_J = F(x, \dot{x}) = p_0 \frac{g_{0i}(x)}{g_{00}(x)} \dot{x}^i + \sqrt{\frac{p_0^2 - (mc)^2 g_{00}(x)}{g_{00}(x)}} \gamma_{ij}(x) \dot{x}^i \dot{x}^j, \quad (15) \]

so that we can write the Jacobi metric as:

\[ ds_J = \sqrt{\frac{p_0^2 - (mc)^2 g_{00}(x)}{g_{00}(x)}} \gamma_{ij}(x) dx^i dx^j + p_0 \frac{g_{0i}(x)}{g_{00}(x)} dx^i \equiv \sqrt{a_{ij}(x)} dx^i dx^j + b_i dx^i. \quad (16) \]

This result, which agrees with proposition 3.3 of [9], is a Finsler metric of Randers type characterized by a Riemannian metric \( a_{ij} \) and a one-form \( b_i \) subject to positivity and convexity of \( F \), which turn out to be satisfied provided that (e.g., [11], p. 283f)

\[ \sqrt{a^{ij} b_i b_j} < 1. \quad (17) \]

If we set \( g_{0i}(x) = 0 \), the result (16) will clearly match (7). Under the approximation of non-relativistic limit, with weak potentials, we will obtain the non-relativistic limit of the Jacobi metric (16) as shown in [5]:

\[ g_{00}(x) = 1 + \frac{2U(x)}{mc^2}, \quad -p_0 = \frac{E}{c} \approx mc + \frac{E}{c}, \quad E \ll mc^2, \quad 2U(x) \ll mc^2 \]

\[ \frac{p_0^2 - (mc)^2 g_{00}(x)}{g_{00}(x)} \approx 2m (E - U(x)) \quad , \quad \frac{p_0}{g_{00}(x)} \approx -\frac{mc}{2U(x)} \left( 1 + \frac{E - 2U(x)}{mc^2} \right) \approx -mc \]
Thus, the non-relativistic Jacobi metric will be
\[
  ds_J \approx \sqrt{-2m(E - U(x))g_{ij}dx^idx^j - mcg_{0i}(x)dx^i}
\]
which is comparable to the Jacobi metric derived for a classical Lagrangian system with magnetic fields involved if we write \( G_{ij}(x) = -g_{ij}(x) \) and \( A_i(x) = -mcg_{0i}(x) \).

5 Some examples of Jacobi metrics

Two of the examples to which the formulation was applied require correction: the Taub-NUT, and Kerr metrics. The correct Jacobi-Maupertuis metrics are as follows:

5.1 Taub-NUT metric

The Euclidean Taub-NUT metric is given by:
\[
  dl^2 = 4M^2 \left( \frac{r - M}{r + M} \right)^2 (d\psi + \cos \theta \ d\varphi)^2 + \frac{r + M}{r - M} dr^2 + (r^2 - M^2) \left( d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) .
\]

From which, we can deduce:
\[
  g_{00}(x) = 4M^2 \left( \frac{r - M}{r + M} \right)^2 , \quad g_{0\varphi}(x) = 4M^2 \left( \frac{r - M}{r + M} \right) \cos \theta , \quad \gamma_{ij}(x) dx^i dx^j = -\frac{r + M}{r - M} dr^2 - (r^2 - M^2) \left( d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) .
\]

Thus, according to (16), for \( p_\psi = -mc \frac{\partial}{\partial \psi} \left( \sqrt{\left( \frac{dl}{d\tau} \right)^2} \right) \) the Jacobi metric for Taub-NUT will be:
\[
  ds_J = \sqrt{ \left( \frac{(mc)^2 - \frac{p_\psi^2}{4M^2} \frac{r + M}{r - M}}{\left( \frac{r + M}{r - M} \right)^2 + (r^2 - M^2) \left( \frac{r + M}{r - M} \right) \left( d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) \right)} + p_\psi \cos \theta \ d\varphi .
\]

and in the low energy \((-p_\psi \approx 2M (mc - Q) , \ Q^2 \ll (mc)^2)\) and weak field limit \( M \ll r \), where
\[
  \frac{p_\psi^2}{4M^2} \approx (mc)^2 - 2mcQ,
\]
\[
  \frac{r + M}{r - M} \approx \left( 1 + \frac{M}{r} \right) \left( 1 - \frac{M}{r} \right) \approx 1 , \quad \frac{p_\psi^2}{4M^2} \left( \frac{r + M}{r - M} \right) - (mc)^2 \approx -2mcQ
\]
the Jacobi metric (21) for Taub-NUT will become:
\[
  ds_J = \sqrt{2mcQ \left[ dr^2 + (r^2 - M^2) \left( d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) \right] - 2M(mc - Q) \cos \theta \ d\varphi .}
\]
5.2 Kerr metric

The Kerr metric (for \( c = 1 \)) is given by:

\[
dl^2 = \left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{4GMar\sin^2\theta}{\rho^2}d\varphi dt
- \left[\frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left\{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right\}d\varphi^2\right],
\]

(22)

where \( \Delta(r) = r^2 - 2GMr + a^2 \), \( \rho^2(r,\theta) = r^2 + a^2\cos^2\theta \).

From which, we can deduce:

\[
g_{00}(x) = 1 - \frac{2GMr}{\rho^2}, \quad g_{0\varphi}(x) = \frac{2GMar\sin^2\theta}{\rho^2},
\]

\[
\gamma_{ij}(x) dx^i dx^j = \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\sin^2\theta}{\Delta - a^2\sin^2\theta}d\varphi^2\right)
\]

(23)

Thus, according to (16), for \( p_0 = -E \) the Jacobi metric for Kerr metric will be:

\[
d_{s_J} = \sqrt{\left(\frac{E^2\rho^2}{\Delta - a^2\sin^2\theta} - m^2\right)\rho^2\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\sin^2\theta}{\Delta - a^2\sin^2\theta}d\varphi^2\right) + \frac{2E GMar\sin^2\theta}{\Delta - a^2\sin^2\theta}d\varphi.}
\]

(24)

For \( m = 0 \) and in the ultra-relativistic limit \( E \gg m \), this metric reduces to the Kerr-Randers optical metric investigated in [12] up to a constant factor \( E \). For the non-relativistic low energy and weak potential limit (\( GMr \ll \rho^2 \))

\[
U(x) = -\frac{GMr}{\rho^2}, \quad |2U(x)| \ll m \Rightarrow \frac{2GMr}{\rho^2} \ll 1
\]

\[
\Rightarrow \quad (g_{0\varphi}(x))^2 = \left(\frac{GMr}{\rho^2}\right)^2 4a^2\sin^4\theta \ll 1
\]

\[
\Delta = \rho^2\left(1 - \frac{2GMr}{\rho^2}\right) + a^2\sin^2\theta \approx \rho^2 + a^2\sin^2\theta \Rightarrow \Delta - a^2\sin^2\theta \approx \rho^2
\]

\[
-g_{ij}(x) dx^i dx^j \approx \frac{\rho^2}{\rho^2 + a^2\sin^2\theta}dr^2 + \rho^2d\theta^2 + \left(\rho^2 + a^2\sin^2\theta\right)\sin^2\theta d\varphi^2
\]

we have according to (18):

\[
ds_J \approx \sqrt{2m\left(E + \frac{GMr}{\rho^2}\right)\left(\frac{\rho^2}{\rho^2 + a^2\sin^2\theta}dr^2 + \rho^2d\theta^2 + \left(\rho^2 + a^2\sin^2\theta\right)\sin^2\theta d\varphi^2\right)
- m\frac{2GMar\sin^2\theta}{\rho^2}d\varphi.}
\]
6 Relation to the gravitational magnetoelectric effect

The magnetoelectric effect refers to the property of certain materials (e.g., multiferroics) by which electric fields yield magnetization, and magnetic fields yields polarization. The linear magnetoelectric susceptibility \(\alpha^{ij} = -\alpha^{ji}\) can be defined by the constitutive relations\(^1\) between the electromagnetic fields,

\[
D^i = \varepsilon^{ij}E_j + \alpha^{ij}H_j, \quad (25)
\]
\[
B^i = \mu^{ij}H_j + \alpha^{ji}E_j, \quad (26)
\]

where \(\varepsilon^{ij}\) is the electric permittivity, and \(\mu^{ij}\) the magnetic permeability. Now it turns out that electromagnetism in spacetime with \(g_{0i} \neq 0\) is also subject to this effect, which is thus called the gravitational magnetoelectric effect. But since this involves the spatial electromagnetic fields, its precise form depends both on the chart used and on the definition of the electromagnetic fields, for which two conventions are in common use. These involve tensor densities and may be referred to as the zero weight formalism and the unit weight formalism (for details, see e.g. [8]).

We adopt the latter which has the feature that in any local coordinate system \((t, x^i)\) the source-free Maxwell’s equations take the familiar form\(^2\)

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \quad (27)
\]
\[
\nabla \times H = \frac{\partial D}{\partial t}, \quad \nabla \cdot D = 0. \quad (28)
\]

\[
\gamma_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}, \quad \gamma^{ij} = -g^{ij}, \quad (29)
\]

then the permittivity and permeability are found to be equal (a property known as impedance-matched)

\[
\varepsilon^{ij} = \mu^{ij} = \frac{\sqrt{-g} g^{ij}}{g_{00}}, \quad (30)
\]

and the gravitational magnetoelectric susceptibility becomes

\[
\alpha^{ij} = -\varepsilon^{ijk} \frac{g_{0k}}{g_{00}}, \quad (31)
\]

where \(\epsilon^{123} = \epsilon_{123} = 1\), whence

\[
g_{0k} = -\frac{1}{2} g_{00} \epsilon_{ijk} \alpha^{ij}. \quad (32)
\]

Moreover, from (30) we have

\[
\gamma_{ij} = \frac{g_{00}}{\sqrt{-g}} (\epsilon^{-1})_{ij} = \frac{g_{00}}{\sqrt{-g}} (\mu^{-1})_{ij}. \quad (33)
\]

---

\(^1\)In his section we use units in which the permeability and permittivity of the vacuum are set to unity and hence \(c = 1\).

\(^2\)The relation of the spatial electromagnetic fields \(E, B, H, D\) to the components of the Maxwell tensor \(F_{\mu\nu}\) is spelt out in detail in §II.C of [8], where quantities in the unit weight formalism are denoted by tildes, however.
As before, we can now consider the geodesic of a free particle of mass \( m \) in this spacetime. Then the Jacobi Lagrangian is the Finsler function of Randers form given by (16) and thus, using (30) and (31), we find,

\[
F(x, \dot{x}^i) = p_0 \frac{g_{0i}}{g_{00}} \dot{x}^i + \sqrt{\frac{p_0^2 - m^2 g_{00}}{g_{00}}} \gamma_{ij} \dot{x}^i \dot{x}^j = -\frac{1}{2} p_0 \epsilon_{ijk} \alpha^{ij} \dot{x}^k + \sqrt{\frac{p_0^2 - m^2 g_{00}}{-g}} (\epsilon^{-1})_{ij} \dot{x}^i \dot{x}^j. \tag{34}
\]

The inverse of the electric permittivity and the dual of the magnetoelectric susceptibility act therefore as the Riemannian metric and the one-form of the spatial Randers metric respectively. Of course by (30) we obtain an identical relation in terms of the magnetic permeability and the dual of the magnetoelectric susceptibility.

## 7 Conclusion

In this paper we have studied the motion of massive particles moving in a general stationary spacetime. We have obtained an explicit formula for the energy dependent Randers-Finsler metric from which the equations of motion may be obtained. This extends and unifies previous work on the static case when our result reduces to a Jacobi-Maupertuis style Riemannian metric and on the massless case in stationary spacetimes, in which case our expression reduces to the previously known energy independent Randers-Finsler metric. In the process we have corrected some misstatements in the literature. We have given explicit formulæ for our Jacobi-Maupertuis Randers-Finsler metric in the astrophysically important case of a Kerr black hole. Finally we have studied Maxwell’s equations in a general stationary metric and shown how the effective electric permittivity, magnetic permeability and magnetoelectric susceptibilities enter our Jacobi-Maupertuis Randers-Finsler metric. We believe that this should contribute to our understanding of polarization due to gravitational lensing.

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