A SUPERALGEBRA MORPHISM OF $U_q[osp(1/2N)]$ ONTO THE DEFORMED OSCILLATOR SUPERALGEBRA $W_q(N)$

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Abstract. We prove that the deformed oscillator superalgebra $W_q(n)$ (which in the Fock representation is generated essentially by $n$ pairs of $q$-bosons) is a factor algebra of the quantized universal enveloping algebra $U_q[osp(1/2n)]$. We write down a $q$-analog of the Cartan-Weyl basis for the deformed $osp(1/2n)$ and give also an oscillator realization of all Cartan-Weyl generators.

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I. Introduction

In Refs. 1, 2 a realization of the quantized universal enveloping algebra $U_q \equiv U_q[osp(1/2n)]$ of the orthosymplectic Lie superalgebra $osp(1/2n)$ has been constructed in terms of $n$ pairs of independent (mutually commuting) $q$ oscillators $b^\pm_i, i = 1, \ldots, n$, i.e., in terms of $q$ deformed Bose creation and annihilation operators as introduced in [3-5]. In other words it has been shown that there exists an algebra morphism $\pi$ of $U_q[osp(1/2n)]$ into the deformed Weyl (or oscillator) algebra $W_q(n)$. The main purpose of the present note is to show that $\pi$ is in fact a morphism of $U_q[osp(1/2n)]$ onto $W_q(n)$ in the category of associative superalgebras. The latter implies that $W_q(n)$ is a factor algebra of $U_q[osp(1/2n)]$. In this way we solve the conjecture, formulated in [6], and give a partial answer to the more general hypothesis [7], namely that there exists a morphism of $U_q[osp(1/2n)]$ into the deformed Weyl (or oscillator) algebra $W_q(n)$. We proceed to recall the definitions of $W_q(n)$ and of $U_q[osp(1/2n)]$ in the notation we are going to use. Following [12], we consider $W_q(n)$ as an associative algebra with unity 1, free generators $b^\pm_i, k_i, k_i^{-1}$ and relations $(i, j = 1, \ldots, n)$

\[ k_i^{-1}k_i = k_i k_i^{-1} = 1 \]

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\[ k_i b_i^\pm = q^{\pm 1} b_i^\pm k_i \]
\[ b_i^- b_i^+ - q^2 b_i^+ b_i^- = k_i^{-2} \]  
\[ b_i^- b_i^+ - q^{-2} b_i^+ b_i^- = k_i^2 \]
\[ a_i a_j = a_j a_i, \quad i \neq j. \]

where \( a_i = b_i^\pm, k_i^{\pm 1} \). To turn \( W_q(n) \) into a superalgebra (\( \mathbb{Z}_2 \)-graded algebra) we set

\[ \text{deg}(b_i^\pm) = 1 \in \mathbb{Z}_2, \quad \text{deg}(k_i^{\pm 1}) = 0 \in \mathbb{Z}_2, \quad i = 1, \ldots, n. \]  

In the Fock representation of \( W_q(n) \), namely when \( k_i = q^{N_i}, b_i^\pm \) are the deformed \( q \)-bosons and \( N_i \) is the \( i \)th boson number operator [3-5]. With respect to the grading induced from (2) \( W_q(n) \) is an infinite-dimensional associative superalgebra.

The superalgebra \( U_q[osp(1/2n)] \) can be defined in different equivalent ways [1,2,13]. We choose the Cartan matrix \( (\alpha_{ij}) \) as in [13], i.e., this is a \( n \)-by-\( n \) symmetric matrix with

\[ \alpha_{nn} = 1, \quad \alpha_{ii} = 2, \quad i = 1, \ldots, n - 1, \quad \alpha_{j,j+1} = \alpha_{j+1,j} = -1, \quad j = 1, \ldots, n - 1, \]  

and all other \( \alpha_{ij} = 0 \). Then \( U_q[osp(1/2n)] \) is the free associative superalgebra with Chevalley generators \( E_i, F_i, K_i, \) \( i = 1, \ldots, n \), graded as

\[ \text{deg}(E_n) = \text{deg}(F_n) = 1 \in \mathbb{Z}_2, \quad \text{deg}(E_i) = \text{deg}(F_i) = \text{deg}(K_j) = 0 \in \mathbb{Z}_2, \quad i = 1, \ldots, n - 1, \quad j = 1, \ldots, n, \]  

which satisfy the Cartan relations

\[ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad i, j = 1, \ldots, n, \]  

\[ K_i E_j = q^{\alpha_{ij}} E_j K_i, \quad K_i F_j = q^{-\alpha_{ij}} F_j K_i, \quad i, j = 1, \ldots, n, \]  

\[ [E_n, F_n] = \frac{K_n^2 - K_n^{-2}}{q - q^{-1}}, \quad [E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}} \quad \forall \ i, j = 1, \ldots, n \text{ except } i = j = n, \]  

the Serre relations for the simple positive root vectors

\[ [E_i, E_j] = 0, \quad \text{if} \quad i, j = 1, \ldots, n \text{ and } |i - j| > 1, \]  

\[ E_i^2 E_{i+1} - (q^2 + q^{-2}) E_i E_{i+1} E_i + E_{i+1} E_i^2 = 0, \quad i = 1, \ldots, n - 1, \]  

\[ E_i^2 E_{i-1} - (q^2 + q^{-2}) E_i E_{i-1} E_i + E_{i-1} E_i^2 = 0, \quad i = 2, \ldots, n - 1, \]  

\[ E_n^3 E_{n-1} + (1 - q^2 - q^{-2})(E_n^2 E_{n-1} E_n + E_n E_{n-1} E_n^2) + E_{n-1} E_n^3 = 0. \]
and the Serre relations obtained from (8)-(11) by replacing everywhere $E_i$ by $F_i$.

Here and throughout the paper

\[ [a, b] = ab - (-1)^{\text{deg}(a)\text{deg}(b)} ba, \quad (12) \]

\[ [a, b]_{q^n} = ab - (-1)^{\text{deg}(a)\text{deg}(b)} q^n ba, \quad (13) \]

for any two homogeneous elements $a, b \in U_q$ and it is assumed that the deformation parameter $q$ is any complex number except $q = 0, q^2 = 1$ and $q^4 = 1$.

The action of the coproduct $\Delta : U_q \to U_q \otimes U_q$, antipode $S : U_q \to U_q$ and counit $\varepsilon : U_q \to \mathbb{C}$ can be given, for instance, as:

\[ \Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad (14) \]

\[ S(E_i) = -q^{\alpha_i} E_i, \quad S(F_i) = -q^{-\alpha_i} F_i, \quad S(K_i) = K_i^{-1}, \quad (15) \]

\[ \varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(K_i) = 0, \quad \varepsilon(1) = 1. \quad (16) \]

**PROPOSITION 1.** The linear map $\pi : U_q[osp(1/2n)] \to W_q(n)$, given as

\[ \pi(E_i) = -b_i^+ b_{i+1}^+, \quad \pi(F_i) = -b_{i+1}^- b_i^+, \quad \pi(K_i) = k_i^{-1} k_{i+1}, \quad (17) \]

and extended by associativity defines a morphism (in the category of the associative superalgebras) of $U_q[osp(1/2n)]$ into the oscillator superalgebra $W_q(n)$.

The proof is an immediate consequence of the results of [2] and the observation that the map $\rho : W_q(n) \to W_q(n)$

\[ b_i^+ \to \xi b_i^+, \quad b_i^- \to -\xi b_i^+, \quad k_i \to q^{-1} k_i^{-1}, \quad \xi = 1 or -1, \quad i = 1, \ldots, n \quad (18) \]

defines an automorphism in $W_q(n)$.

We wish to show that $\pi$ is a map of $U_q[osp(1/2n)]$ onto $W_q(n)$. To this end consider the following $2n$ elements from $U_q[osp(1/2n)] (i = 1, \ldots, n - 1)$:

\[ B_i^- = -\sqrt{\frac{2q}{(q^q - 1)}} E_i K_i K_{i+1} \ldots K_n, \quad (19) \]

\[ B_n^- = -\sqrt{\frac{2q}{(q^q - 1)}} E_n K_n \]

and

\[ B_i^+ = \sqrt{\frac{2q}{(q^q - 1)}} \ldots [F_n, F_{n-1}, q^2, F_{n-2}, q^2, \ldots, q^2, F_{i+1} q^2, F_{i+2} q^2, F_{i+3} q^2, \ldots, q^2, F_i q^2, K_{i+1}^{-1} K_{i+2}^{-1} \ldots K_n^{-1}, \quad (20) \]

\[ B_n^+ = \sqrt{\frac{2q}{(q^q - 1)}} F_n K_n^{-1}. \]
From (17), (19) and (20) one derives expressions for the $q$-oscillator realization of $B_{1}^{\pm}, \ldots, B_{n}^{\pm}$,

$$
\pi(B_{i}^{\pm}) = q^{-2(n-i)} \sqrt{\frac{2}{(q + q^{-1})^{2}b_{i}^{-1}k_{i+1}^{-2}k_{i+2}^{-2} \ldots k_{n}^{-2}}}, \quad i = 1, \ldots, n,
$$

(21)

$$
\pi(B_{i}^{\pm}) = q^{2(n-i)+1} \sqrt{\frac{2}{(q + q^{-1})^{2}b_{i}^{+}k_{i+1}^{2}k_{i+2}^{2} \ldots k_{n}^{2}}}, \quad i = 1, \ldots, n.
$$

(22)

At $q = 1$ the images $\pi(B_{i}^{\pm})$ of $B_{i}^{\pm}$ reduce to the usual nondeformed Bose creation and annihilation operators, whereas the nondeformed $B_{1}^{\pm}, \ldots, B_{n}^{\pm}$ satisfy the trilinear relations ($\{x, y\} \equiv xy + yx$)

$$
\{B_{i}^{\xi}, B_{j}^{\eta}\} = (\epsilon - \xi)\delta_{ik}B_{j}^{\eta} + (\epsilon - \eta)\delta_{jk}B_{i}^{\xi}, \quad i, j, k = 1, \ldots, n, \quad \xi, \eta, \epsilon = \pm or \pm 1.
$$

(23)

In quantum field theory the operators (23) are known as para-Bose operators. They were introduced by Green [14] as a possible generalization of the Bose statistics. The para-Bose operators are a good substitute for the Chevalley generators in the sense that they also define completely the Lie superalgebra $osp(1/2n)$ [15]. More precisely, consider the operators

$$
B_{i}^{\xi}, \quad i = 1, \ldots, n,
$$

(24a)

$$
H_{i} = -\frac{1}{2}\{B_{i}^{+}, B_{i}^{-}\}, \quad i = 1, \ldots, n,
$$

(24b)

$$
\frac{1}{2}\{B_{j}^{+}, B_{k}^{-}\}, \quad j \neq k = 1, \ldots, n,
$$

(24c)

$$
\frac{1}{2}\{B_{p}^{\xi}, B_{q}^{\xi}\}, \quad p \leq q = 1, \ldots, n, \quad \xi = \pm or \pm 1.
$$

(24d)

Then the operators $H_{i}$ constitute a basis in a Cartan subalgebra $H$ of $osp(1/2n)$, the para-Bose operators (24a) — a basis in the odd subspace of $osp(1/2n)$, all anticommutators (24b — d) — a basis in the even subalgebra $sp(2n)$ and (24b, c) — a basis in the subalgebra $gl(n)$. The operators (24a, c, d) are root vectors (of $H$) and all operators (24) give (one possible) Cartan-Weyl basis in $osp(1/2n)$. Replacing in (24) the para-Bose operators with Bose operators, one obtains the usual ladder realizations of $osp(1/2n), sp(2n)$ and $gl(n)$.

The deformed operators $B_{1}^{\pm}, \ldots, B_{n}^{\pm}$, which are in fact deformed para-Bose operators, possess similar properties. Elsewhere we shall show that these operators together with the ”Cartan” elements $K_{1}^{\pm1}, \ldots, K_{n}^{\pm1}$ define uniquely $U_{q}[osp(1/2n)]$ and hence give an alternative definition of this quantum algebra in terms of generators, which have more immediate physical significance (see for more discussions along this line Ref.6). Also without a proof we formulate a proposition, which solves in a simple way the problem for constructing a $q$—analogue of the Cartan-Weyl basis in the deformed $osp(1/2n)$ superalgebra.

**PROPOSITION 2.** The $q$-analogue of the Cartan-Weyl basis of $osp(1/2n)$ is given with the set of all operators as follows:
\[ K_i^\pm, \quad i = 1, \ldots, n, \quad (25a) \]
\[ B_i^\pm, \quad i = 1, \ldots, n, \quad (25b) \]
\[ \frac{1}{2} \{ B_j^+, B_k^- \}, \quad j \neq k = 1, \ldots, n, \quad (25c) \]
\[ \frac{1}{2} \{ B_p^\xi, B_q^\xi \}, \quad p < q = 1, \ldots, n, \quad \xi = \pm \quad (25d) \]
\[ (B_i^\pm)^2, \quad r = 1, \ldots, n. \quad (25e) \]

The set of all ordered monomials (with respect to a certain normal order [13]) of the operators \((25a - d)\) constitute a basis in \(U_q[osp(1/2n)]\).

Remark 1. The operators \((25a, c)\) appear as a \(q\)-analog of the Cartan-Weyl basis of the deformed \(gl(n)\). These operators generate a Hopf subalgebra of \(U_q[osp(1/2n)]\), which is isomorphic to \(U_q[gl(n)]\). The latter follows from the observation that the generators and the relations of \(U_q[gl(n)]\) are among the generators and the relations of \(U_q[osp(1/2n)]\).

Remark 2. The operators \((25a, c - d)\) generate a subalgebra of \(U_q[osp(1/2n)]\), which is a deformation of the universal enveloping algebra of \(sp(2n)\). This subalgebra, however, is not a Hopf subalgebra of \(U_q[osp(1/2n)]\) even in the simplest case of \(n = 1\) [16].

The \(q\)-oscillator realization of \((25a, b)\) was already written (see eqs.\((17), (21)\) and \((22)\)). The realization of the rest of the Cartan-Weyl generators reads:

\[
\pi(\{ B_i^-, B_j^+ \}) = 2q^{2(i-j)}b_i^-b_j^+k_i^1k_j^2\ldots k_{j+1}^2k_{j-1}^2k_j^-k_{j+1}^-k_{j-1}^1k_j^1, \quad i < j = 1, \ldots, n, \quad (26) \\
\pi(\{ B_i^-, B_j^+ \}) = 2q^{2(i-j)}b_i^-b_j^+k_j^2\ldots k_{i-2}^2k_{i-1}^1k_i, \quad i > j = 1, \ldots, n, \quad (27) \\
\pi(\{ B_i^-, B_j^- \}) = 2q^{2(i+j-2n)+1}h_i^-h_j^-k_i^1k_j^1k_{i+1}^2k_{j+1}^2k_{i-1}^2k_{j-1}^1k_i^1k_j^1k_{i+1}^2k_{j+1}^2k_{i-1}^2k_{j-1}^1, \quad i \neq j = 1, \ldots, n, \quad (28) \\
\pi(\{ B_i^+, B_j^+ \}) = 2q^{2(2n-i-j)+3}b_i^+b_j^+k_i^1k_j^1k_{i+1}^2k_{j+1}^2\ldots k_n^2k_{i+1}^2k_{j+1}^2k_n^2, \quad i \neq j = 1, \ldots, n. \quad (29) \\
\]

**PROPOSITION 3.** The oscillator algebra \(W_q(n)\) is a factor algebra of \(U_q \equiv U_q[osp(1/2n)]\) (in the sense of associative superalgebras).

Proof. For a proof it suffices to show that the generators of \(W_q(n)\), namely \(b_i^\pm, k_i^\pm, \ i = 1, \ldots, n\) are among the images of \(U_q\) under the map \(\pi\) (see proposition 1).

Keeping in mind that \(\pi\) is a linear map, which preserves the multiplication, i.e., \(\pi(ab) = \pi(a)\pi(b)\) and as a consequence also \(\pi(c^{-1}) = (\pi(c))^{-1}\), one derives from \(\pi(K_i) = k_i^{-1}k_{i+1}, \ i = 1, \ldots, n - 1\) and \(\pi(K_n) = q^{-\frac{1}{2}}k_n\) [see \((17)\)] that

\[
k_i = \pi(q^{-\frac{1}{2}}k_i^{-1}K_{i+1}^{-1}\ldots K_n^{-1}), \quad k_i^{-1} = \pi(q^{\frac{1}{2}}K_iK_{i+1}\ldots K_n), \quad i = 1, \ldots, n \quad (30)
\]
and therefore

\[ k_i, k_i^{-1} \in \pi(U_q), \quad i = 1, \ldots, n. \tag{31} \]

From (21) we have that for each \( i = 1, \ldots, n \)

\[ b_i^- = q^{2(n-i)} \sqrt{q + q^{-1}} \pi(B_i^-)k_i k_{i+1}^2 t_{i+2} \ldots t_n^2. \]

Inserting here the expressions for \( \pi(B_i^-) \) and \( k_i \) from (19) and (30) we obtain after some rearrangement of the order of the generators

\[ b_i^- = \pi(-q^n - i[E_i, E_{i+1}, [E_{i+2}, \ldots [E_{n-2}, [E_{n-1}, E_n]_q]_q \ldots]_q]_q^{-2} \prod_{k=1}^{n-i} K_{i+k}^{-2k}), \quad i = 1, \ldots, n - 1. \tag{32} \]

In a similar way one has

\[ b_i^+ = \pi(q^{i-n}[F_n, F_{n-1}]_q^2, F_{n-2}]_q^2, \ldots]_q^2, F_{i+2}]_q^2, F_{i+1}]_q^2, F_i]_q^2 \prod_{k=1}^{n-i} K_{i+k}^{2k}), \quad i = 1, \ldots, n - 1. \tag{33} \]

From (17), (31), (32) and (33) we conclude that

\[ k_i, k_i^{-1}, b_i^\pm \in \pi(U_q), \quad i = 1, \ldots, n, \tag{34} \]

which completes the proof.

In a forthcoming paper we will discuss in more details the description of \( U_q \equiv U_q[osp(1/2n)] \) in terms of \( B_i^\pm, K_i^{\pm 1}, i = 1, \ldots, n \). Here we first fix the terminology.

**DEFINITION.** The generators \( B_i^\pm, K_i^{\pm 1}, i = 1, \ldots, n \) of \( U_q \) will be called pre-oscillator generators.

The description of \( U_q[osp(1/2n)] \) (or of any subalgebra \( A \subset U_q \)) in terms of the pre-oscillator generators will be referred to as pre-oscillator or pre-ladder form (realization, description) of \( U_q \) (resp. of \( A \)).

The pre-oscillator form of \( U_q \) gives an alternative description to the usual Chevalley realization, where all elements of \( U_q \) are functions of \( E_i, F_i, K_i^{\pm 1}, i = 1, \ldots, n \). The name pre-oscillator comes to remind that the image of any subalgebra \( A \subset U_q \) under the map \( \pi \) [see (17)],

\[ \pi : A \rightarrow W_q(n), \tag{35} \]

gives a ladder (or oscillator) realization of \( A \).

The pre-oscillator form of the subalgebra \( A \), which need not necessarily be a Hopf subalgebra of \( U_q \), can be used for construction of representations of \( A \) in any tensorial power of Fock spaces \( F_q(n) \) of the oscillator algebra \( W_q(n) \). This stems from the observation that the action of the comultiplication \( \Delta \) is well defined on the pre-oscillator generators, whereas it is not defined on their \( \pi \)-images, i.e., on the deformed \( q \)-bosons. In order to be more concrete, define by induction a morphism

\[ \Delta^{(k)} = [(\otimes_{i=1}^{k-2} id) \otimes \Delta] \circ \Delta^{(k-1)}, \quad \Delta^{(2)} = \Delta, \quad \Delta^{(1)} = id. \tag{36} \]
of $U_q$ into $\otimes_{i=1}^k U_q$. The map

$$\pi^{(k)} = (\otimes_{i=1}^k \pi) \circ \Delta^{(k)} : U_q[osp(1/2n)] \to \otimes_{i=1}^k W_q(n)$$

(37)

is a morphism of $U_q[osp(1/2n)]$ into $\otimes_{i=1}^k W_q(n)$. Consider $W_q(n)$ in its Fock space $F_q(n)$ representation, $W_q(n) \subset End[F_q(n)]$. Then $\pi^{(k)}(A)$ gives a representation of $A$ in $\otimes_{i=1}^k F_q(n)$. In particular if $A$ is given in a pre-oscillator form, then for any $a = a(B_1^\pm, \ldots, B_n^\pm, K_1^\pm, \ldots, K_n^\pm) \in A$ the map $\pi^{(k)}$, defined on each element $a \in A$ as

$$\pi^{(k)}(a) = a(\pi^{(k)} B_1^\pm, \ldots, \pi^{(k)} B_n^\pm, \pi^{(k)} K_1^\pm, \ldots, \pi^{(k)} K_n^\pm),$$

(38)

is a representation of $A$ in $\otimes_{i=1}^k F_q(n)$. Certainly the same statement holds if $A$ were given in a Chevalley form. The point is however that in the practical applications $A$ is usually given directly in terms of $q$-deformed bosons. From this realization it is fairly evident how to reconstruct the pre-oscillator form of $A$, to which one can apply the above technique in order to construct tensor products of Fock representations. The main difficulty will be then to decompose the tensor product $\otimes_{i=1}^k F_q(n)$, considered as an $A$-module, into a direct sum of irreducible (and, may be, also indecomposable) $A$-modules.

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