Discordlike correlation of bipartite coherence

Yu Guo\textsuperscript{1,2} and Sumit Goswami\textsuperscript{2}

\textsuperscript{1}Institute of Quantum Information Science, Shanxi Datong University, Datong, Shanxi 037009, China
\textsuperscript{2}Institute for Quantum Science and Technology, University of Calgary, Alberta T2N 1N4, Canada

Quantum discord has been studied extensively as a measure of non-classical correlations which includes entanglement as a subset. Although it is well known that non-zero discord can exist without entanglement, the origin of quantum discord is not well understood as compared to entanglement, which manifests itself more simply as inseparable higher dimensional quantum superposition. In this paper we establish the discordlike correlation of bipartite coherence and then compare it to quantum discord. Consequently, we show that the minimum of the discordlike correlation of coherence coincides with the original quantum discord. This demonstrates quantum discord as the irreducible correlated bipartite coherence. In addition, the discordlike correlated coherence is shown to admit the postulates of the quantum resource theory (QRT), although the original quantum discord is not a “good” candidate under the QRT. We also find that the relative entropy measure induced from the discordlike coherence is a well-defined coherence measure for bipartite states.

I. INTRODUCTION

Correlated information always lies at the heart of quantum information theory\textsuperscript{1,2}. Quantum discord was introduced in\textsuperscript{3} to quantify the total amount of quantum correlation present in a bipartite system. Entanglement, the most widely used quantum correlation, is included as a subset in quantum discord. It is shown that quantum discord is more robust against environment induced decoherence than entanglement\textsuperscript{4,5}. Moreover, it has been proven to be an important quantum resource in a plethora of quantum information processing tasks\textsuperscript{6,7,24}.

The relation between different quantum resources is of great importance\textsuperscript{24,26}. It is shown that entanglement is a minimal quantum discord over state extensions\textsuperscript{24}. Another fundamental concept in quantum physics which is closely connected to quantum superposition and quantum correlations is coherence. An algorithmic characterization of quantum coherence as a resource and a set of bona fide criteria for coherence monotones have been identified\textsuperscript{25,27,36}. Correlated coherence has been proposed to capture the mutual coherence between the two subsystems of a bipartite system\textsuperscript{26}. It is shown that coherence can be measured with entanglement\textsuperscript{28}, the correlated coherence is closely related to entanglement\textsuperscript{20}, and the basis-free relative entropy of coherence coincides with the relative entropy of the quantum discord\textsuperscript{37}. The main purpose of this paper is to investigate a discordlike correlation of bipartite coherence and then compare it to the original quantum discord. By replacing the von Neumann entropy with the relative entropy measure of coherence, replacing the mutual information by the correlated coherence and replacing the von Neumann measurements by the local rank-one projective physically incoherent operations we establish a discordlike correlation of coherence for bipartite states and then compare it to the original quantum discord: the original quantum discord turned out to be the minimal discordlike correlation of coherence over all possible reference bases.

Later we investigate the role of the discordlike correlation of coherence as quantum resources in context of the recently developed quantum resource theory (QRT)\textsuperscript{38}. QRT was developed to create a unifying theoretical framework for different quantum resources. Considerable work on formulating QRT has been done recently\textsuperscript{38–49}. A general structure of QRT has three ingredients: (1) the free states, (2) the resource states, and (3) the free operations. For example, in entanglement theory the resource states are the entangled states, the free states are the separable ones and the free operations are the local operations and classical communication (LOCC). In the theory of coherence, the resource states are the coherent states, the free states are the incoherent ones and the free operations are the incoherent operations. However, not all the free operations can be implemented physically. Hence, the physically consistent conditions for QRT was formulated very recently\textsuperscript{49}, particularly in the context of coherence. Physically incoherent operations turned out to be some special incoherent operations\textsuperscript{10}. Consequently, we show that although the original quantum discord does not obey the structure of QRT, the discordlike correlation of coherence demonstrated itself as a reasonable resource under QRT.

The rest of this paper is structured as follows. In Sec. II, we give a brief overview of the quantum discord and the coherence. The discordlike correlation of coherence is established in Sec. III and then we obtain a relative entropy of the discordlike correlation in Sec. IV. Section V discuss the discordlike correlation as a quantum resource under QRT. We conclude in Sec. VI.

II. PRELIMINARY NOTIONS

We recall the definitions of quantum discord and coherence at first. For a state $\rho_{ab}$ of a bipartite system, with finite dimensional subsystems $A$ and $B$, described by Hilbert space $H_A \otimes H_B$, the quantum discord of $\rho_{ab}$
(up to part A) is defined by

$$D^a(\rho_{ab}) := \min_{\Pi^a} \{I(\rho_{ab}) - I(\rho_{ab}|\Pi^a)\},$$

(1)

where the minimum is taken over all local von Neumann measurements $\Pi^a$ [i.e., $\Pi^a(\cdot) = \sum_{i} \Pi_i^a \otimes I_b(\Pi_i^a \otimes I_b)$ with $\Pi_i^a = |\psi_i\rangle \langle \psi_i|$ for some orthonormal basis $\{ |\psi_i\rangle \}$ of $H_a$]. $I(\rho_{ab}) := S(\rho_a) + S(\rho_b) - S(\rho_{ab})$ is interpreted as the quantum mutual information, $S(\cdot) := -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy, and $H$ is a state space. Henceforth, we call the fixed basis reference basis. If $\{|i\rangle\}$ and $\{|j\rangle\}$ are reference bases of $H_a$ and $H_b$, respectively, and $I_b$ denotes the identity map on part $B$. Coherence is defined and quantified along the approach in 27. Let $H$ be a finite-dimensional Hilbert space with $\text{dim} H = d$. Fixing a particular basis $\{|i\rangle\}^d_{i=0}$, we call all quantum states represented by density operators that are diagonal in this basis incoherent. This incoherent set of quantum states will be labeled by $I$, all density operators $\rho \in I$ are of the form

$$\rho = \sum_{i=0}^{d-1} \delta_i |i\rangle \langle i|.$$  

Henceforth, we call the fixed basis reference basis. If $\{|i\rangle\}$ and $\{|j\rangle\}$ are reference bases of $H_a$ and $H_b$, respectively, then $\{|i\rangle\} \otimes |j\rangle\}$ is the reference basis of $H_a \otimes H_b$. A quantum operation is incoherent if its Kraus operators fulfill $K_{ij} \rho K_{ij}^\dagger / \text{Tr}(K_{ij} \rho K_{ij}^\dagger) \in I$ for all $\rho \in I$ and for all $n$. For a state space $H$ we denote by $B(H)$ and $S(H)$ the space of all bounded linear operators on $H$ and the set of all quantum states on $H$, respectively.

III. DISCORDLIKE CORRELATION OF BIPARTITE COHERENCE

The relative entropy of coherence is defined by 27

$$C_r(\rho) := \min_{\sigma \in I} S(\rho||\sigma),$$

(4)

where $S(\rho||\sigma) = \text{Tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ is the relative entropy. $C_r(\rho)$ can be calculated to be 27

$$C_r(\rho) = S[\Delta(\rho)] - S(\rho),$$

(5)

where $\Delta(\rho)$ denotes the diagonal part of the $\rho$ in the reference basis, i.e.,

$$\Delta(\rho) = \sum_i |i\rangle \langle i| \rho |i\rangle \langle i|.$$  

(6)

$C_r$ admits the super-additive property 50

$$C_r(\rho_{ab}) \geq C_r(\rho_a) + C_r(\rho_b).$$

(7)

The equality holds whenever $\rho_{ab}$ is a product state 50. We remark here that $S(\rho_{ab}) \leq S(\rho_a) + S(\rho_b)$ and the equality holds if and only if $\rho_{ab}$ is a product state. However, $C_r(\rho_{ab}) = C_r(\rho_a) + C_r(\rho_b)$ whenever $\rho_{ab}$ is a product state but not vice versa. In fact, for any diagonal bipartite state $\rho_{ab}$, we have $C_r(\rho_{ab}) = C_r(\rho_a) + C_r(\rho_b)$. But $\rho_{ab}$ is not necessarily a product state. If $\rho_{ab}$ is a maximally coherent state, the equality in Eq. (7) holds if and only if it is a product state, provided that $\text{dim} H_a = \text{dim} H_b$. But it is not valid when $\text{dim} H_a \neq \text{dim} H_b$. 51.

Definition 1. For any $\rho_{ab} \in S(H_a \otimes H_b)$, we call the difference

$$I_{co}(\rho_{ab}) := C_r(\rho_{ab}) - C_r(\rho_a) - C_r(\rho_b)$$

(8)

the correlated coherence of $\rho_{ab}$ with respect to the relative entropy measure of coherence.

By definition, $I_{co}(\rho_{ab})$ is the amount of mutual coherence information contained in $\rho_{ab}$, which is similar to that of mutual information $I(\rho_{ab})$. We remark here that the correlated coherence $C_{co}(\rho_{ab}) := C(\rho_{ab}) - C(\rho_a) - C(\rho_b)$ was proposed first by the authors of 24, where $C$ is the $l_1$ norm measure of coherence, i.e., $C(\rho) := \sum_{i \neq j} |\langle i|\rho|j\rangle|$ with respect to the reference basis $\{|i\rangle\}$. We use $C_r$ instead of the $l_1$ norm here since, as will be shown, $I_{co}$ can induce the discordlike correlation (Definition 2) that connects with the original quantum discord closely (Theorem 2). However, one can easily check that the $l_1$ norm can not reveal such a relation.

In 19, the physically incoherent operation (PIO) is proposed according to the physically consistent QRT. A completely positive trace preserving (CPTP) map is proven to be a PIO if and only if it can be written as a convex combination of operations each with Kraus operators $\{K_j\}_{j=1}^m$ of the form

$$K_j = U_j P_j = \sum_y e^{i\theta_y} |\pi_j(y)\rangle \langle y| P_j,$$

(9)

where $\{|y\rangle\}$ is the reference basis of part $A$, $P_j$ forms an orthogonal and complete set of incoherent projectors on part $A$, $\sum_j K_j^\dagger K_j = I$ and $\pi_j$ are permutations. Namely, a quantum operation $\mathcal{E}$ is a PIO if and only if it can be written as $\mathcal{E}(\cdot) = \sum_{j=1}^m t_j \mathcal{E}_j(\cdot)$ with $\sum_j t_j = 1$, $t_j > 0$, $m \geq 1$. Here $\mathcal{E}_j(\cdot) = \sum_j K_j(i) \mathcal{E}_j(k) K_j^\dagger(k)$ with $K_j(i)$ as in Eq. (9). Particularly, we call a PIO as projective PIO (PPIO) if $m = 1$, i.e., $\mathcal{E}$ is a PPIO if and only if $\mathcal{E}(\cdot) = \sum_j K_j(i) K_j^\dagger(k)$ where $K_j$s admit the form in Eq. (9). In other words, any PIO is a convex combination of PPIOs. In a PPIO there is only one set of Kraus operators and the action of these Kraus operators $K_j$ is similar to that of the projective operators $P_j$ (as $K_j = U_j P_j$ is unitarily related to $P_j$). Hence, we call this special kind of PIO as “projective” PIO. Here we note that $U_j$ and $P_j$ are defined with respect to a fixed reference basis (that of coherence), as they are defined as incoherent unitary and incoherent projector, respectively. For the fixed reference basis $\{|y\rangle\}$, the form of $U_j$ is given in Eq. (9) and $\{P_j\}$ are any orthogonal and complete set of projectors in that basis.
Moreover, if all the projectors \( (P_j) \) of a PPIO are rank-one, we also have all Kraus operators (related by the incoherent unitary) of rank one. We define this special case of a PPIO as a rank-one PPIO. For example, in the case of a three-dimensional Hilbert space with reference basis \( \{ |1\rangle, |2\rangle, |3\rangle \} \) if we have all projectors \( (P_j) \) of rank one (i.e., \( P_1 = |1\rangle \langle 1|, P_2 = |2\rangle \langle 2|, P_3 = |3\rangle \langle 3| \)), then the corresponding PPIO would be a rank-one PPIO. However, if one of the projectors is not rank-one (e.g., \( \{ P_1 = |1\rangle \langle 1| + |2\rangle \langle 2|, P_2 = |3\rangle \langle 3| \} \)) then the corresponding PPIO would no more be a rank-one PPIO.

We now show that any local rank-one PPIO cannot increase correlated coherence, which is similar to the action of local von Neumann measurement on the mutual information (see Eq. (14) in [4]).

**Theorem 1.** Let \( \Pi_a \) be any given local rank-one PPIO (on the subsystem \( A \)), \( \Pi^a(\cdot) := \Pi_a \otimes I_b(\cdot) \), then

\[
I_{co}[\Pi^a(\rho_{ab})] \leq I_{co}(\rho_{ab})
\]

holds for any bipartite state \( \rho_{ab} \in S(H_a \otimes H_b) \).

**Proof.** Let \( \Pi_a \) be a PPIO on part \( A \). Then, for any \( \rho_{ab} \), we apply \( \Pi^a \) on \( \rho_{ab} \), the output state is \( \rho'_{ab} = (\Pi_a \otimes I_b)\rho_{ab} \). It follows that

\[
I_{co}(\rho_{ab}) - I_{co}(\rho'_{ab}) = C_r(\rho_{ab}) - C_r(\rho'_{ab})
\]

where \( C_r(\rho_{ab}) := \Gamma(\Pi_{ab}, \Pi_a) \). If \( \Pi_a \) is a rank-one PPIO with \( U_i = U_j \) for any \( i \) and \( j \), it is clear since \( I(\rho_{ab}) - I(\rho'_{ab}) \geq 0 \) and \( \Gamma(\Pi_{ab}, \Pi_a) = 0 \). We now turn to the case of \( U_i \neq U_j \) for some \( i \) and \( j \). We assume with no loss of generality that \( \dim H_a = 3 \) and \( \Pi_a(\cdot) = \sum_{i,j} U_i P_j(\cdot) U_j^\dagger \) with \( U_1 = |0\rangle \langle 1| + |1\rangle \langle 0| + |2\rangle \langle 2|, \quad U_2 = U_3 = I_3 \). It follows that, for any \( \rho_{ab} = \sum_{i,j} |i\rangle \langle j| \otimes B_{ij} \), where \( B_{ij} \in B(H_b) \),

\[
\Pi^a(\rho_{ab}) = |1\rangle \langle 1| \otimes (B_{11} + B_{22}) + |2\rangle \langle 2| \otimes B_{33}
\]

where \( p_1^1 \rho_{11} = B_{11} + B_{22}, \quad p_2^2 \rho_{22} = B_{33} \). Let \( B_{ii} = q_i \rho_{ii}, \quad i = 1, 2, 3 \), then

\[
I_{co}(\rho_{ab}) - I_{co}[\Pi^a(\rho_{ab})] \geq I(\rho_{ab}) - I[\Pi^a(\rho_{ab})] \geq 0
\]

and the equality holds whenever \( \Pi^a(\rho_{ab}) = \sum_i p_i |i\rangle \langle i| \otimes \rho_{ii} \) for some permutation \( \pi \) provided that

\[
I(\rho_{ab}) = I\left(\sum_i p_i |i\rangle \langle i| \otimes \rho_{ii}\right)
\]

We are now ready for giving our main concept, which is an analog of quantum discord: mutual information \( I \) is replaced by correlated coherence \( I_{co} \) and the von Neumann measurement is substituted by rank-one PPIO. We may readily verify that \( D_a^c(\rho_{ab}) = 0 \) if and only if \( \rho_{ab} = \sum_i p_i |i\rangle \langle i| \otimes \rho_{ii} \). By Theorem 3.1 in [22], \( D_a^c(\rho) = 0 \) for \( \rho = \sum_{i,j} A_{ij} \otimes |i\rangle \langle j| \) if and only if \( A_{ij} \) are mutually commuting normal operators which are diagonal under the reference basis. We denote the set of all states that with zero \( D_a^c \) by \( D_a^{c=0} \). Then \( D_a^{c=0} \) is a convex set. Let \( D_a^{c=0} \) be the set of all zero-discordant states (up to part \( A \)), then \( D_a^{c=0} \) is a proper subset of \( D_a^c \).

| Correlation | Free states | Physically free operation | Invariant operation | Measurement |
|-------------|-------------|---------------------------|--------------------|-------------|
| \( D^a \)  | \( D^a_c \)  | \( \mathcal{E}_a \otimes \mathcal{E}_{ab}^c \)  | Local UO | \( \Pi^a \)  |
| \( D_a^c \) | \( D_{a=0}^c \) | \( \mathcal{E}_{ia} \otimes \mathcal{E}_{ab}^b \) | Local IUO | \( \Pi^a \)  |

\( a \mathcal{E}_a \) denotes the unitary operation, \( \mathcal{E}_a \) is any given local operation on part \( B \).

\( b \mathcal{E}_{ia} \) denotes the incoherently unitary operation.
Let \{|y\}\rangle be the reference basis. An operation \(\mathcal{E}\) is called an incoherently unitary operation (IUO) if
\[
\mathcal{E}(\rho) = U \rho U^\dagger, \quad U = \sum_y e^{i\theta_y} |\pi(y)\rangle \langle y|.
\]
It is straightforward that \(D^a_c\) is invariant under local IUO. In addition, we can see that (i) \(D^a_c[\Pi^a(\rho_{ab})] = 0\) for any local rank-one PPIO \(\Pi^a\) and any \(\rho_{ab}\) and (ii) \(D^a_c\) can be generated under LOCC.

We now discuss the relation between \(D^a_c\) and \(D^c\). The following theorem is immediate from Eq. (11) and the proof of Theorem 1. Since \(D^a_c\) is defined to be the minimum over all \(\Pi^a\) it also contains the case \(\Pi^a(\cdot) = \sum_j P_j(\cdot) P_j^\dagger\) in which case [along with other cases discussed after Eq. (11)] the equality holds in Eq. (11).

**Theorem 2.** Let \(\Omega\) be the set of all orthonormal bases of \(H_a\) and \(H_b\). Then for any \(\rho_{ab} \in S(H_a \otimes H_b)\),
\[
D^a_c(\rho_{ab}) = \min_{\Omega} D^c(\rho_{ab}),
\tag{13}
\]
where the minimum is taken over all possible reference bases in \(\Omega\).

Equation (13) displays the relation between the quantum discord and the correlated coherence: quantum discord is the minimal correlated bipartite coherence. In addition, although the calculation of \(D^a_c\) is NP-complete [53], \(D^c\) can be obtained directly since it is not dependent on the choice of the rank-one PPIO. Furthermore, \(D^c(\rho_{ab})\) not only displays the quantum discord contained in \(\rho_{ab}\), but also reflects the correlated coherence of \(\rho_{ab}\). In other words, \(D^a_c\) reveals quantum correlation and coherence simultaneously.

Equation (13) is different from the Eq. (16) in [37]: The discord in [37] is not the original quantum discord, it is the relative entropy of the discord, and in addition, the basis-free measure of coherence \(C^{\text{free}}\) there is different from \(D^a_c\) (since \(C^{\text{free}}(\rho_{ab}) = 0\) iff \(\rho_{ab} = \sum_{i,j} \delta_{ij} |i\rangle \langle i'| |j\rangle \langle j'|\) where \(|i\rangle \langle i'\rangle\) is the reference basis). In addition, the correlated coherence was compared to the symmetric quantum discord as well in [24]. However, the result therein is far different from Eq. (13): it is based on \(C_{cc}\) while our relation is deduced from \(D^a_c\).

**IV. RELATIVE ENTROPY OF THE DISCORDLIKE COHERENCE**

We follow the axiomatic method for a reasonable measure of quantum coherence \(C(\rho)\) proposed in [24]. (C1) \(C(\delta) \geq 0\), \(\forall \delta \in \mathcal{I}\) [(C1') \(C(\delta) = 0\) iff \(\delta \in \mathcal{I}\)]. (C2a) \(C(\rho)\) is nonincreasing under incoherent operations, i.e., \(C(\mathcal{E}(\rho)) \leq C(\rho)\) for any IO \(\mathcal{E}\). (C2b) Monotonicity for average coherence under selective IO, i.e., \(C(\mathcal{E}(\rho)) \geq \sum_i p_i C(\rho_i)\), with probabilities \(p_i = Tr(K_i \rho K_i^\dagger)\), state \(\rho_i = K_i \rho K_i^\dagger / p_i\), and incoherent Kraus operators \(K_i\) obeying \(K_i K_i^\dagger \subseteq \mathcal{I}\). (C3) Convexity, i.e., \(\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)\) for any set \(\{p_i, \rho_i : \sum_i p_i = 1\}\).

The relative entropy of the resource is an important quantity in QRT [38]. We now discuss the relative entropy of discordlike coherence. We define
\[
C^r(\rho_{ab}) := \inf_{\sigma_{ab} \in D^c_{\rho_{ab}}} S(\rho_{ab}||\sigma_{ab}).
\tag{14}
\]
Let \(\sigma_{ab} = \sum_i p_i |i\rangle \langle i| \otimes \rho_i^b\) be any state in \(D^c_{\rho_{ab}}\). It follows that \(S(\rho_{ab}||\sigma_{ab}) = S(\rho_{ab}) - S(\rho_{ab} + S(\rho_{ab}) + S(\rho_{ab} + S(\rho_{ab}) + S(\rho_{ab}))\), and thus we get
\[
C^r(\rho_{ab}) = S(\rho_{ab}) - S(\rho_{ab}).
\tag{15}
\]
It is clear that \(C^r(\rho_{ab})\) fulfills (C1) (but it does not satisfy (C1’), that is, \(C^r(\rho_{ab})\) is not faithful). It satisfies (C2a) and (C3) since the relative entropy is contractive and jointly convex. It also fulfills (C2b) by Theorem 5 in [34] (also see the similar argument as that of (C2b) for the original relative entropy of coherence in [27]). That is, although \(D^c\) is not a well-defined coherence measure since they are not convex under the mixing of states, the relative entropy measure \(C^r\) is a measure of bipartite coherence. The defect of this coherence measure is that it is not faithful, which is similar to that of negativity [52] as an entanglement measure. So we can also consider the symmetric discordlike measure and the symmetric relative entropy of the discordlike quantity, denoted by \(\tilde{C}^r\) and \(\tilde{C}^r\), respectively. That is,
\[
\tilde{D}^c(\rho_{ab}) := \min_{\tilde{\Pi}_{ab} \otimes \tilde{\Pi}_b} \{I_{ca}(\rho_{ab}) - I_{ca}[(\tilde{\Pi}_a \otimes \tilde{\Pi}_b)\rho_{ab}]\},\tag{16}
\]
where the minimum is taken over all local PPIOs \(\tilde{\Pi}_a \otimes \tilde{\Pi}_b\). For this symmetric measure, \(\tilde{D}^c(\rho_{ab}) = 0\) if and only if \(\rho_{ab}\) is diagonal with respect to the reference basis \(|i\rangle \langle i|\).

Let \(D^c_{\rho_{ab}}\) be the set of all states with zero symmetric discordlike measure. The corresponding relative entropy measure is
\[
\tilde{C}^r(\rho_{ab}) := \inf_{\sigma_{ab} \in D^c_{\rho_{ab}}} S(\rho_{ab}||\sigma_{ab}).
\tag{17}
\]
That is, the symmetric measure of relative entropy coincides with the original relative entropy coherence measure.

**V. QRT OF THE DISCORDLIKE COHERENCE**

In what follows, we discuss whether \(D^a_c\) obeys the requirements of QRT. Let \(\mathcal{F}\) be the set of all free states (in all possible finite dimensions), and \(F_m = \mathcal{F} \cap S(H_m)\), where \(H_m = H_{m_1} \otimes H_{m_2} \otimes \cdots H_{m_s}\), \(\dim H_m = m_i\), \(i = 1, \ldots, s\). Brandão and Gour proposed the following postulates [38] for any QRT: (i) \(\mathcal{F}\) is closed under tensor products; (ii) \(\mathcal{F}\) is closed under the partial trace of spatially separated subsystems; (iii) \(\mathcal{F}\) is closed under permutations of spatially separated subsystems; (iv) each \(F_m\) is a closed set; (v) each \(F_m\) is a convex set; (vi) the set of free operations cannot generate a resource; they cannot convert free states into resource states.
When we process a quantum task, the interacting environment always consumes resources. To account for this very recently Chitambar and Gour investigated the physically consistent QRT \cite{Chitambar}. A QRT defined on some quantum system $S$ is physically consistent if any free operation $E$ on $S$ can be obtained by an auxiliary state $\rho_E$, a joint unitary $U_{S,E}$, and a projective measurement $\{P_i\}$ that are all free in an extended system $S + E$. We call the free operation in the physically consistent QRT physically free operation hereafter. The following is the main result of this section.

**Theorem 3.** A quantum operation $E : \mathcal{B}(H_a \otimes H_b) \to \mathcal{B}(H_a \otimes H_b)$ is a physically free operation of the discordlike correlation of coherence if and only if it can be expressed as a convex combination of maps each having Kraus operators $\{K_j\}_{j=1}^r$ of the form

$$K_j = U_a \otimes B_j,$$

where $U_a$ is an IUO on $H_a$ and $\{B_j\}_{j=1}^r$ is any Kraus operators that satisfy $\sum_j B_j^\dagger B_j = I_b$.

**Proof.** We assume that the environment is denoted by part $E$. Following the scenario in \cite{Chitambar}, any physically free operation of the discordlike correlation of coherence on this composite system can be decomposed into three steps: (i) a joint unitary $U_{ab,E}$ is applied on the input state $\rho_{ab}$ and some fixed state $\rho_E$, i.e., the state becomes $U_{ab,E}\rho_{ab}\otimes\rho_E U_{ab,E}^\dagger$, where $U_{ab,E} = U_a \otimes U_{B_E}$. $U_a$ is a IUO (note that the joint unitary operation is free, so it admits this form) and $U_{B_E}$ is an arbitrary given unitary operator on $H_b \otimes H_E$. (ii) a von Neumann measurement $\{P_j\}$ act on the environment encoding the measurement outcome as a classical index, i.e., the state after this process is

$$\sum_j (I_b \otimes P_j)(U_{ab,E}\rho_{ab}\otimes\rho_E U_{ab,E}^\dagger)(I_b \otimes P_j) = \sum_{j=1}^r \sum_{j'} \rho_{ab,j} \otimes |j\rangle E\langle j|,$$

where

$$\rho_{ab,j} = \text{Tr}_E[(I_b \otimes P_j)(U_{ab,E}\rho_{ab}\otimes\rho_E U_{ab,E}^\dagger)],$$

and (iii) a classical processing channel is applied to the measurement outcomes. That is, the final state is

$$\sum_{k=1}^t \rho_{ab,k} \otimes \ket{k}_E\bra{k},$$

where $\rho_{ab,k} = \sum_{j=1}^t p_{kj} \rho_{ab,j}$ for some classical channel $p_{kj}$.

From the discussion above, let $\rho_{ab} = \sum_{ij} p_{ij} |i\rangle \otimes B_{ij}$ with respect to the reference basis $\{|i\rangle\}$ of $H_a$, then we can conclude that

$$\rho_{ab,j} = \text{Tr}_E[(I_b \otimes P_j)(U_a \otimes U_{B_E}\rho_{ab}\otimes\rho_E U_{B_E}^\dagger U_{B_E}^\dagger)] = \text{Tr}_E \left[ \sum_{kl} U_a |k\rangle \langle l| U_a^\dagger \otimes I_b \otimes P_j U_{B_E}(B_{kl} \otimes \rho_E) U_{B_E}^\dagger \right]$$

Then the final state becomes

$$\rho_{ab} = \sum_{j} \rho_{ab,j} = \sum_{j} \sum_{kl} U_a |k\rangle \langle l| U_a^\dagger \otimes \text{Tr}_E[I_b \otimes P_j U_{B_E}(B_{kl} \otimes \rho_E) U_{B_E}^\dagger].$$

which completes the proof since any quantum operation on the system $B + E$ admits the form of $\text{Tr}_E[U_{B_E}(\cdot) U_{B_E}^\dagger]$. \hfill \Box

By Theorem 3, one can readily obtain that a quantum operation $E$ is a physically free operation for quantum discord if and only if it has the form as Eq. (18) with $U_a$ is any unitary operation on $H_a$ since the difference between $D_c^e$ and $D^a$ is in nature the fixed reference basis for $D_c^e$ comparing with the free bases for $D^a$.

When we regard $D^-_{c\rightarrow 0}$ as the free states, and consider the operations in Theorem 3 as physically free operations, we show below that it is a new physically consistent resource under QRT.

Since $D^a$ is a measure of bipartite systems, we assume that $H_{m_i} = H_a \otimes H_{b_i}, \ i = 1, \ldots, s$, and $\rho \in S(H_m)$ is free if any reduced state in $H_{m_i}$ is free. We check the postulates (i) through (v) for free states item by item. (i) With no loss of generality, we consider the case of $s = 1$. If $\rho_{ab} \in S(H_a \otimes H_b)$ with $D^a_c(\rho_{ab}) = 0$, then $\rho_{ab} = \sum_i p_i |i\rangle \langle i| \otimes \rho_{b_i}^k$ with respect to the reference basis $\{|i\rangle\}$ of $H_a$. It follows that $\rho_{ab} \otimes \rho_{a'b'} \in S(H_a \otimes H_b \otimes H_{a'} \otimes H_{b'})$ can be expressed as $\rho_{ab,a'b'} = (\sum_i p_i |i\rangle \langle i| \otimes \rho_{b_i}^k) \otimes (\sum_i p'_i |i'\rangle \langle i'| \otimes \rho_{b_i}^{k'})$ and thus it is a free state in $S(H_a \otimes H_{a'} \otimes H_b \otimes H_{b'})$. (ii) and (iii) are clear. It is easy to check (iv) according to Theorem 3.1 in \cite{Coffman}. Postulate (v) is clear and the properties for the free operations are guaranteed by Theorem 3. We thus conclude that the discordlike correlation of bipartite coherence can also be regarded as a quantum resource (the comparison between $D_c^e$ and $D^a$ is listed in Table 1).

It is worth mentioning that for both entanglement and coherence, the free states are closely related to the free operations. A state is entangled if and only if it can not be prepared by LOCC, and a state is incoherent if and only if it can not be created via incoherent operations \cite{Chang}. However, quantum discord is defined via local von Neumann measurements, which is only a proper set of its free operations (one can see that any quantum operation $E : \mathcal{B}(H_a \otimes H_b) \to \mathcal{B}(H_a \otimes H_b)$ that has the Kraus operators of the form $K_j = A_j \otimes B_j$ is a free operation of quantum discord, where $\sum_j A_j A_j^\dagger$ is a commutativity preserving operation \cite{Winter} and $\sum_j B_j B_j^\dagger$ is any quantum operation on part $B$). More remarkably, the free states of quantum discord is not convex. All these facts indicate that quantum discord as a quantum resource is not a “good” candidate.
Induced via measurements. We believe that the discordance of quantifying coherence in nature since the bipartite system. It is far different from all the previous measures. This measure can be presented as a coherence measure of QRT. Interestingly, we also prove that the relative entropy measure induced from discord of the bipartite state (thus it vastly improves both Theorem 2 in [37] and the results in [26]). This indicates the inherent nature of quantum discord as a bipartite coherence. Moreover, the discordlike correlation of coherence can be calculated in a straightforward manner and obeys all requirements of QRT. Interestingly, we also prove that the relative entropy measure induced from this measure can be presented as a coherence measure of the bipartite system. It is far different from all the previous ways of quantifying coherence in nature since $D^n$ is induced via measurements. We believe that the discordlike correlation of coherence would be more useful than the original quantum discord as a quantum resource in quantum information technology.

ACKNOWLEDGMENTS

This work was completed while Guo was visiting the Institute of Quantum Science and Technology of the University of Calgary under the support of the China Scholarship Council (Grant No. 201608140008). Guo thanks Professor C. Simon, Professor A. Lvovsky and Professor G. Gour for their hospitality. The authors thank Professor S. Wu for helpful discussions. We also would like to thank an anonymous referee for useful and important suggestions. Guo is supported by the National Natural Science Foundation of China under Grant No. 11301312 and the National Science Foundation of Shanxi under Grants No. 201701D121001.
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