Theoretical results and modeling under the discrete Birnbaum-Saunders distribution

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ABSTRACT
In this paper, we discuss some theoretical results and properties of a discrete version of the Birnbaum-Saunders distribution. We present a proof of the unimodality of this model. Moreover, results on moments, quantile function, reliability and order statistics are also presented. In addition, we propose a regression model based on the discrete Birnbaum-Saunders distribution. The model parameters are estimated by the maximum likelihood method and a Monte Carlo study is performed to evaluate the performance of the estimators. Finally, we illustrate the proposed methodology with the use of real data sets.

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1. Introduction

Despite the increasing number of works on discrete distributions in reliability, one can note that in many practical cases there is the need of more flexible distributions to model lifetime data. One way to develop new discrete distributions is by generating the discrete analogous of usual distributions for continuous lifetimes; see Alzaatreh, Lee, and Famoye (2012). Some interesting discrete distributions are for example the discrete gamma distribution (Abouammoh and Alhazzani 2015) and the discrete Weibull distribution (Vila, Nakano, and Saulo 2019). It is well known that in reliability, the continuous Birnbaum-Saunders (BS) distribution, proposed by Birnbaum and Saunders (1969), takes advantage than most continuous probability distributions, including the continuous gamma and Weibull distributions; see Leiva (2016). The continuous BS distribution is a positively skewed model that is closely related to the normal distribution. Despite its origin in material fatigue, it has been considered in business, industry, insurance, inventory, finance, quality control, among others; see, for example, Lio and Park (2008), Balakrishnan et al. (2009), Ahmed et al. (2010), Vilca et al. (2010), Paula et al. (2012), Marchant et al. (2013), Rojas et al. (2015), Wanke and Leiva (2015), Leiva et al. (2011, 2014a, 2014b, 2017), Saulo et al. (2019), Desousa et al. (2018), Leão et al. (2018), Ventura et al. (2019), and Saulo, Balakrishnan, and Vila (2023). Good recent references
on this distribution are Leiva (2016) and Balakrishnan and Kundu (2019). In particular, a positive random variable $T$ is said to follow a continuous BS distribution if its cumulative distribution function (CDF) is given by

$$F_T(t; \theta) = \Phi[a(t; \theta)], \ t > 0,$$

where $\theta = (\alpha, \beta)^T$, with $\alpha > 0$ and $\beta > 0$ denoting the shape and scale parameters, respectively, $a(t; \theta) = \left(\sqrt{t/\beta} - \sqrt{\beta/t}\right)/\alpha$, and $\Phi[.]$ is the standard normal CDF. This distribution is usually denoted by $T \sim \text{BS}(\theta)$. Even though the number of applications of the usual continuous BS distribution has been growing, there is a big number of applications where a discrete version of this distribution could be more appropriate. For example, to model the number of cycles or runs that a material or equipment supports before failing or breaking, the number of sessions of a treatment until the cure of a patient, or even the shelf life (in days) of a food product; see Vila, Nakano, and Saulo (2019).

In this paper, we study in more depth a discrete version of the continuous BS distribution, which was initially introduced by Sen, Maiti, and Dey (2010). The primary objectives of this paper are: (i) to discuss novel theoretical results and properties of this discrete BS (BSd) distribution, such as $p$-th quantile, shape properties, order statistics, mean and variance residual life function, moments among others; and (ii) to introduce the corresponding regression model. The secondary objectives are: (i) to obtain the maximum likelihood estimates of the model parameters; (ii) to carry out Monte Carlo simulations to evaluate the performance of the maximum likelihood estimators; and (iv) to discuss real data applications of the proposed methodology.

The rest of the paper proceeds as follows. In Section 2, we present the BSd model and discuss some of its mathematical properties. In Section 3, a BSd regression model is proposed, and the model parameter estimation is approached by using the maximum likelihood method. In Section 4, we carry out Monte Carlo simulation studies to evaluate the performance of the estimators and we illustrate the proposed methodology with two real data sets. Finally, in Section 5, we make some concluding remarks.

### 2. Discrete Birnbaum-Saunders distribution

Before defining the proposed discrete distribution, we present the probability density function (PDF) and quantile function of the continuous BS distribution. If $T \sim \text{BS}(\theta)$, then its PDF is given by

$$f_T(t; \theta) = \phi[a(t; \theta)]a'(t; \theta), \ t > 0,$$

where $\phi[.]$ is the PDF of the standard normal distribution, $a(t; \theta)$ is as in (1) and $a'(t; \theta) = (t + \beta)/(2t^{3/2} \beta^{1/2})$ is the derivative of $a(t; \theta)$ with respect to $t$. Moreover, the $p$-th quantile of $T \sim \text{BS}(\theta)$ is given by

$$Q_p = \frac{\beta}{4} \left\{\alpha \Phi^{-1}(p) + \sqrt{\left[\alpha \Phi^{-1}(p)\right]^2 + 4}\right\}^2,$$

where $p \in (0, 1)$. 


Now, we are ready to present a discrete random variable associated to the positive \( T \) as follows \( S = \lfloor T \rfloor \), where \( \lfloor t \rfloor \) denotes the largest integer contained in \( t \). As the set of all possible values of \( T \) is the set \((0, \infty)\), then \( S = s \) if \( s < T \leq s + 1 \), \( s = 0, 1, \ldots \). Consequently, the probability mass function (PMF) of \( S \) can be expressed by

\[
\mathbb{P}(S = s) = \mathbb{P}(s < T \leq s + 1) = \begin{cases} \Phi[a(1; \theta)], & \text{if } s = 0, \\ \Phi[a(s + 1; \theta)] - \Phi[a(s; \theta)], & \text{if } s = 1, 2, \ldots, \end{cases}
\]  

(3)

where \( a(\cdot; \theta) \) is as in (1). We can show that \( \sum_{s=0}^{\infty} \mathbb{P}(S = s) = \mathbb{P}(T > 0) = 1 \), so \( \mathbb{P}(S = s) \) is a PMF. On the other hand, the CDF of \( S \) is given by

\[
F(s; \theta) = \mathbb{P}(S \leq s) = \Phi[a(\lfloor s \rfloor + 1; \theta)]_{\{s \geq 0\}}.
\]  

(4)

The distribution of the discrete random variable \( S \) will be denoted by \( S \sim \text{BS}_d(\theta) \) and will be called \( \text{BS}_d \) distribution.

The reliability function (RF) and hazard rate function (HR) of \( S \sim \text{BS}_d(\theta) \) are, respectively, given by

\[
R(s; \theta) = 1 - F(s; \theta) = 1 - \Phi[a(\lfloor s \rfloor + 1; \theta)]_{\{s \geq 0\}},
\]  

(5)

\[
H(s; \theta) = \frac{\mathbb{P}(S = s)}{\mathbb{P}(S = s) + R(s; \theta)} = \begin{cases} \Phi[a(1; \theta)], & \text{if } s = 0, \\ \frac{\Phi[a(s + 1; \theta)] - \Phi[a(s; \theta)]}{1 - \Phi[a(s; \theta)]}, & \text{if } s = 1, 2, \ldots, \\ 1 - R(0; \theta), & \text{if } s = 0, \\ \frac{1 - R(s; \theta)}{R(s - 1; \theta)}, & \text{if } s = 1, 2, \ldots. \end{cases}
\]

From the above identity, we have

\[
R(s; \theta) = \prod_{y=0}^{s-1} \frac{R(y; \theta)}{R(y - 1; \theta)} = \prod_{y=0}^{s-1} \left[1 - H(y; \theta)\right], \quad s = 0, 1, 2, \ldots,
\]

with the convention that \( \prod_{y=0}^{s-1} b_y = 1 \) and that \( R(-1; \theta) = 1 \).

Figures 1 and 2 display different shapes of the \( \text{BS}_d \) PMF and HR for different choices of parameters. From these figures, we observe that the \( \text{BS}_d \) distribution possesses unimodal shapes for the PMF and HR.

### 2.1. Properties

We present some properties of the \( \text{BS}_d \) distribution, many of the results can be easily derived from the definition of the \( \text{BS}_d \) distribution.

**Proposition 1.** If \( S \sim \text{BS}_d(\theta) \) and \( t > 0 \), the following holds:

(a) \( \sum_{s=0}^{\infty} \Phi[a(s + 1; \theta)] = 1 + \sum_{s=0}^{\infty} \Phi[a(s; \theta)] \), where we assume that \( \Phi[a(0; \theta)] = 0; \)

(b) \( \mathbb{P}(S \leq t) = \mathbb{P}(S \leq \lfloor t \rfloor) = \mathbb{P}(T \leq \lfloor t \rfloor + 1); \)

(c) \( \mathbb{P}(S < t) = \mathbb{P}(T \leq \lfloor t \rfloor); \)
\(P(S/C_{21}) = P(T/C_{21b}t_{c} + 1)\).

**Proof.** By (3),
\[
1 = \sum_{s=0}^{\infty} P(S = s) = \sum_{s=0}^{\infty} \{\Phi[a(s + 1; \theta)] - \Phi[a(s; \theta)]\},
\]
where we used the convention \(\Phi[a(0; \theta)] = 0\). Since both series \(\sum_{s=0}^{\infty} \Phi[a(s + 1; \theta)]\) and \(\sum_{s=0}^{\infty} \Phi[a(s; \theta)]\) converge, the proof of Item (a) follows.

Since \(S\) is discrete, it is clear that \(P(S \geq t) = P(T \geq \lfloor t \rfloor + 1)\); \(P(S \geq \lfloor t \rfloor) = P(T > \lfloor t \rfloor)\).

**Proof.** By (3), \(1 = \sum_{s=0}^{\infty} P(S = s) = \sum_{s=0}^{\infty} \{\Phi[a(s + 1; \theta)] - \Phi[a(s; \theta)]\}\), where we used the convention \(\Phi[a(0; \theta)] = 0\). Since both series \(\sum_{s=0}^{\infty} \Phi[a(s + 1; \theta)]\) and \(\sum_{s=0}^{\infty} \Phi[a(s; \theta)]\) converge, the proof of Item (a) follows.

Since \(S\) is discrete, it is clear that \(P(S \leq t) = P(S \leq \lfloor t \rfloor)\), where \(P(S \leq \lfloor t \rfloor) = \Phi[a(\lfloor t \rfloor + 1; \theta)]\). But \(P(T \leq \lfloor t \rfloor + 1) = \Phi[a(\lfloor t \rfloor + 1; \theta)]\), with \(T \sim BS(\theta)\), then the proof of Item (b) follows.

Analogously to the reasoning of the previous item, since \(P(S < t) = P(S \leq t - 1) = \Phi[a(\lfloor t \rfloor + 1; \theta)] = P(T \leq \lfloor t \rfloor)\), the statement of Item (c) follows. The proof of Item (d) follows by applying in Item (c) the complementary property of a probability measure. The statement of Item (e) follows by using Item (d).
2.1.1. p-Th quantile

Proposition 2. Let $S \sim \text{BS}_d(\theta)$ and $Q_p$ the quantile function in (2), $p \in (0, 1)$. Then,

(a) If $Q_p > 0$ is a natural number, then $Q_p - 1$ is the $p$-th quantile of the distribution of $S$;

(b) If $Q_p > 0$ is not a natural number, then $p$-th quantile of the distribution of $S$ can be represented by any value in the interval $[[Q_p], [Q_p] + 1)$.

Proof. Since $Q_p$ is the $p$-th quantile for the continuous random variable $T \sim \text{BS}(\theta)$, from Proposition 1, Item (e), we have

$$P(S < Q_p - 1) \leq P(S \leq Q_p - 1) \leq P(T \leq Q_p) = p, \quad S \sim \text{BS}_d(\theta),$$

whenever $Q_p > 0$ is a natural number. So, we have that $Q_p - 1$ is the $p$-th quantile of the distribution of $S$. This proves the first item.

Now, let $t = Q_p > 0$ be not a natural number. From Items (d) and (c) of Proposition 1 and from inequalities $[Q_p] \leq Q_p \leq [Q_p] + 1$, we have the following

$$P(S < Q_p) \leq P(T \leq [Q_p]) \leq P(T \leq Q_p) = p,$$

and consequently $P(S < Q_p) \leq p \leq P(S \leq Q_p)$. This will be true for any $[Q_p] \leq t < [Q_p] + 1$. So, we have that, at the percentage point $p$, the quantile for $S$ can be represented by any value in $[[Q_p], [Q_p] + 1)$. Thus we complete the proof. \hfill \Box

Remark 1. If $p = 0.5$, then $Q_p = \beta$. If $\beta$ is a natural number, by Proposition 2(a), $m = \beta - 1$ is the median of the distribution of $S$. Already, if $\beta$ is not a natural number, by Proposition 2(b), each $y \in [[\beta], [\beta] + 1)$ represents a median for $S$.

2.1.2. Shape properties

The next two results are related to the unimodality of the $\text{BS}_d$ distribution.

Proposition 3. The $\text{BS}_d$ distribution is unimodal.

Proof. Let $T \sim \text{BS}(\theta)$ be a random variable with continuous BS distribution. Let $f_T(t; \theta)$, $t > 0$ be their respective PDF. It is well-known that this distribution is unimodal (see Proposition 7 in Vila et al. (2020)), then there exists a unique point $t_0 > 0$ such that its PDF satisfies the following inequalities:

$$f_T(t; \theta) \geq f_T(t - 1; \theta), \quad \text{for all } t \leq t_0,$$

and

$$f_T(t; \theta) \geq f_T(t + 1; \theta), \quad \text{for all } t \geq t_0.$$

If $s$ is a natural number such that $s \leq [t_0] - 1$, then

$$P(S = s) = \int_s^{s+1} f_T(t; \theta) \, dt \geq \int_s^{s+1} f_T(t - 1; \theta) \, dt = P(S = s - 1),$$
or equivalently,
\[ P(S = s) - P(S = s - 1) \geq 0 \quad \text{for all } s \leq [t_0] - 1. \]

Similarly, for \( s \geq [t_0] + 1 \), we obtain
\[ P(S = s + 1) - P(S = s) \leq 0. \]

It follows that \( \{ P(S = s) : s = 0, 1, 2, \ldots \} \) is unimodal, whatever sign \( P(S = [t_0]) - P(S = [t_0] - 1) \) may have.

**Remark 2.** As a sub-product of the proof of Proposition 3, the mode of the BS\(_d\) distribution is \([t_0]\), where \( t_0 \) is the mode of the corresponding continuous BS distribution.

**Proposition 4.** The BS\(_d\) distribution has a unique mode in the set \( \{ s = 0, 1, \ldots : 0 \leq s \leq [\beta] \} \).

**Proof.** Proposition 3 guarantees the uniqueness of mode. It remains to prove that the BS\(_d\) distribution \( P(S = s) \) is decreasing for all \( s \geq [\beta] + 1 \). We prove this by comparing the continuous BS distribution with the corresponding BS\(_d\) distribution. Indeed, in Lemma 2.1. of Vila et al. (2020) is proved that the PDF \( f_T(t; \theta) \) of the continuous BS distribution is a decreasing function when \( t > \beta \). However this extends to every \( t \geq \beta \) because \( \{ \frac{d}{dt} \log[f_T(t; \theta)] \}_{t=\beta} = \frac{\alpha''(\beta; \theta)}{\alpha'(\beta; \theta)} \leq 0 \). Hence, as a sub-product of the proof of Proposition 3, it follows that the BS\(_d\) PMF (3) is decreasing for all \( s \geq [\beta] + 1 \). Thus, we have completed the proof.

2.1.3. Order statistics

**Proposition 5.** If \( S_1, \ldots, S_n \) is a sequence of independent and identically distributed random variables such that \( S_1 \sim \text{BS}_d(\theta) \), then, the \( i \)-th; \( i = 1, 2, \ldots, n \); order statistic of the BS\(_d\) distribution, denoted \( S_{(i)} \), can be written as
\[ P(S_{(i)} \leq s) = \sum_{k=i}^{n} \frac{n!}{k!(n-k)!} (-1)^{k} F[a([s]; \theta); k+j], \quad s \geq 1, \]

where \( F[:,:,k+j] \) denotes the CDF of the power normal distribution (PND). Different properties of the PND have been discussed by Gupta and Gupta (2008).

**Proof.** It is well-known that \( P(S_{(i)} \leq s) = \sum_{k=i}^{n} \binom{n}{k} [R(s; \theta)]^{n-k}, s \geq 1 \) (see Item (2.7) of Shahbaz et al. (2016)). Using the Newton binomial formula and the definition of a PND, the proof follows.

**Remark 3.** By using the identity \( \sum_{k=i}^{n} \binom{n}{k} p^k(1-p)^{n-k} = i \binom{n}{i} \int_{t=0}^{p} t^{i-1} (1-t)^{n-i} \, dt \), the distribution function of \( S_{(i)} \) can also be written as
\[ P(S_{(i)} \leq s) = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} \int_{t=0}^{\Phi[a([s]; \theta)]} t^{i-1} (1-t)^{n-i} \, dt, \quad s \geq 1, \]

where \( \Gamma(\cdot) \) is the Gamma function.
2.1.4. Mean residual life function and variance residual life function

Let $S \sim \text{BS}_d(\theta)$, the mean residual life function (MRLF) and variance residual life function (VRLF) are defined by

$$\mu_S(k) = \mathbb{E}(S - k | S \geq k) = \frac{\sum_{s=k}^{\infty} R(s; \theta)}{R(k-1; \theta)}$$

and

$$\sigma^2_S(k) = \text{Var}(S - k | S \geq k) = 2 \sum_{s=k}^{\infty} \frac{s R(s; \theta)}{R(k-1; \theta)} - (2k - 1) \mu_S(k) - \mu^2_S(k),$$

respectively, where $R(s; \theta)$ is given in (5) and $k = 0, 1, ..., .$

**Proposition 6.** Let $S \sim \text{BS}_d(\theta)$ with $\theta$ belongs to the set $\Theta = \{ \theta_s \in (0, \infty)^2 : C(t; \theta_s) = \phi[a(t; \theta)]|a'(t; \theta)]^2/(1 - \Phi[a(t; \theta)]) + a''(t; \theta_s) - a(t; \theta_s)[a'(t; \theta_s)]^2 > 0, \forall t > 0 \}$.

(a) $S$ has decreasing MRLF;
(b) $S$ has increasing HR;
(c) $S$ has decreasing VRLF.

**Proof.** Let $R_T(t; \theta)$ and $R(s; \theta)$ be the RFs of $T \sim \text{BS}(\theta)$ and $S \sim \text{BS}_d(\theta)$, respectively. For $\theta \in \Theta$, we have

$$\frac{d^2}{dt^2} \log[R_T(t; \theta)] = -\frac{\phi[a(t; \theta)]}{1 - \Phi[a(t; \theta)]} C(t; \theta) < 0 \quad \text{for all } t > 0.$$

In other words, the function $\log[R_T(t; \theta)]$ is concave. This condition implies that

$$\log[R_T(t_1; \theta)] \geq \frac{1}{2} \log[R_T(t_1; \theta)] + \frac{1}{2} \log[R_T(t_2; \theta)]$$

or equivalently that $[R_T(t_1; \theta)]^2 \geq R_T(t_1; \theta)R_T(t_2; \theta)$ for all $t_1, t_2 > 0$. Hence, taking $t_1 = [s] + 2$ and $t_2 = [s], s > 0$, the last inequality yields $[R_T([s] + 1; \theta)]^2 \geq R_T([s] + 2; \theta)R_T([s]; \theta)$, So, applying Proposition 1, Item (b), we get (for $s > 0$)

$$[R(s; \theta)]^2 \geq R(s + 1; \theta)R(s - 1; \theta) \iff H(s; \theta) \leq H(s + 1; \theta).$$

That is, $S$ has increasing hazard rate $H(\cdot; \theta)$. Then, by Theorem 2.1 of Gupta (2015), it follows that $S$ has decreasing mean residual life function. This proves the statement in Items (a) and (b). Finally, the proof of Item (c) follows directly by combining Item (a) with Theorem 2.2 in Gupta (2015). 

2.1.5. Moments properties

**Proposition 7.** The distribution of a random variable $S$ with BS$_d$ distribution has all moments.

**Proof.** The proof is straightforward since $S^p = |T|^p \leq T^p$, for $T \sim \text{BS}(\theta)$ and $p > 0$, and since $\mathbb{E}(T^p)$ always exists.

**Proposition 8.** If $S \sim \text{BS}_d(\theta)$ is a random variable, for each natural number $r$, we have
(a) \( \mathbb{E}(S') = \sum_{s=0}^{\infty}[(s + 1)' - s']\{1 - \Phi[a(s + 1; \theta)]\}; \)
(b) \( \mathbb{E}(S') = \sum_{s=0}^{\infty} \sum_{k=0}^{s'} \sum_{i=0}^{r-k-1} (r-k)_{i}^{k+i} \{1 - \Phi[a(s + 1; \theta)]\}; \)
(c) \( \text{Var}(S) = 2 \sum_{s=0}^{\infty} s\{1 - \Phi[a(s + 1; \theta)]\} + \sum_{s=0}^{\infty} \{1 - \Phi[a(s + 1; \theta)]\} [1 - \sum_{s=0}^{\infty} \{1 - \Phi[a(s + 1; \theta)]\}]. \)

Proof. The whole proof follows closely Proposition 2 of Saulo et al. (2021) and we present it for the sake of completeness. We emphasize that the statements of Items (a), (b), and (c) are valid for any discrete random variable \( S \) with support \( \{0, 1, \ldots \} \).

By using the telescopic series \( \sum_{s=0}^{l-1}[(x + 1)' - x'] = \tilde{r}' \), we have
\[
\mathbb{E}(S') = \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} 1_{\{s<\tilde{i}\}} [(s + 1)' - s'] \mathbb{P}(S = i) = \sum_{s=0}^{\infty} [(s + 1)' - s'] \sum_{i=0}^{\infty} 1_{\{s<\tilde{i}\}} \mathbb{P}(S = i) = \tilde{r}' \mathbb{P}(S = i),
\]
where in the second equality we exchange the orders of the summations because
\[
\sum_{s=0}^{\infty} 1_{\{s<\tilde{i}\}} [(s + 1)' - s'] \mathbb{P}(S = i) = \sum_{s=0}^{\infty} \mathbb{P}(S = i) = \tilde{r}' \mathbb{P}(S = i),
\]
is finite for each \( i = 0, 1, \ldots \); and because \( \sum_{i=0}^{\infty} \tilde{r}' \mathbb{P}(S = i) = \mathbb{E}(S') \) always exists (see Proposition 7). This proves Item (a). The second item follows by combining Item (a) with the polynomial identity \( a^n - b^n = (a - b) \sum_{k=0}^{n} a^{r-k} b^k \) and the binomial expansion. Already, the proof of Item (c) is obtained by using Item (a) and simple algebraic manipulations. \( \square \)

3. Discrete Birnbaum-Saunders regression model

In the context of count data, the BS\(_d\) distribution may be an interesting alternative distribution to usual discrete distributions or to those discrete distributions have been derived from continuous distributions. Then, for the BS\(_d\) distribution we are also going to consider its associated regression model, which will be the goal of this part of the study. The associated BS\(_d\) regression model that we are going to introduce is inspired by continuous BS regression model developed by Balakrishnan and Zhu (2015), where they considered the scale parameter depending on covariates.

Suppose that we observe independent failure times \( S_1, \ldots, S_n \), such as
\[
S_i \sim \text{BS}_d(\theta_i), \quad (6)
\]
where \( \theta_i = (\alpha, \beta_i) \), \( i = 1, \ldots, n \). The distribution depends on covariates \( x_i = (x_{i1}, \ldots, x_{ip})^T \) associated with \( \beta_i \) thought \( \beta_i = \exp(x_i^T \eta) \), with \( \eta = (\eta_0, \eta_1, \ldots, \eta_p)^T \) being a vector of unknown parameters. The corresponding PMF associated with (6) is
\[
\mathbb{P}(S_i = s_i) = \Phi[a(s_i; \theta_i)]1_{\{s_i = 0\}} + \{\Phi(a(s_i + 1; \theta_i)) - \Phi(a(s_i; \theta_i))\}1_{\{s_i \geq 1\}},
\]
\( i = 1, \ldots, n. \)
3.1. Maximum likelihood estimation

The log-likelihood function for \( \theta = (x, \eta^\top)\top \) is given by

\[
l(\theta) = \sum_{i=1}^{n} \log \left\{ \Phi(a(s_i; \theta_1)) \mathbb{I}_{\{s_i=0\}} + \{ \Phi(a(s_i + 1; \theta_1)) - \Phi(a(s_i; \theta_1)) \} \right\}, \tag{7}
\]

Then, the first derivatives of the log-likelihood function (7), with \( \theta = (x, \eta^\top)\top \), can be written as

\[
\hat{l}_u(\theta) = \sum_{i=1}^{n} \sum_{j=0}^{1} (-1)^{j+1} \frac{\partial a(s_i + j; \theta)}{\partial z_i} \phi_j(s_i, \theta), \quad u \in \{x, \eta\}, z_i \in \{x, \beta_i\},
\]

where \( \hat{l}_u(\theta) = \partial l(\theta) / \partial u \),

\[
\frac{\partial a(s_i; \theta)}{\partial x} = -\frac{1}{x} a(s_i; \theta), \quad \frac{\partial a(s_i; \theta)}{\partial \beta_i} = -\frac{1}{2x} \frac{1}{\beta_i^{3/2}} (s + \beta_i) \quad \text{and} \quad \phi_j(s_i, \theta) = \frac{\phi[a(s_i + j; \theta)]}{P(S = s_i)}.
\]

Specifically,

\[
\hat{l}_x(\theta) = \sum_{i=1}^{n} \sum_{j=0}^{1} (-1)^{j+1} \frac{\partial a(s_i + j; \theta)}{\partial x} \phi_j(s_i, \theta),
\]

\[
\hat{l}_\eta(\theta) = \sum_{i=1}^{n} \sum_{j=0}^{1} (-1)^{j+1} \frac{\partial a(s_i + j; \theta)}{\partial \beta_i} \phi_j(s_i, \theta) \frac{\partial \beta_i}{\partial \eta},
\]

where \( \partial \beta_i / \partial \eta = \beta_i x_i; i = 1, \ldots, n \). From the likelihood equations \( \hat{l}_x(\theta) = 0 \) and \( \hat{l}_\eta(\theta) = 0 \), we can see that there is no closed-form solution to the maximization problem, so we implement two algorithms in software R to find the maximum likelihood estimates of \( x, \beta \) and \( \eta_i, i = 0, \ldots, p \), by using the function \texttt{optim()}; see R Core Team (2020). These procedures are evaluated and used in the next section.

Furthermore, the Hessian matrix of \( l(\theta) \) is given by

\[
\left[ \hat{l}_{vu}(\theta) \right]_{p \times p} = \left[ \begin{array}{cc} \frac{\partial^2 l(\theta)}{\partial x^2} & \frac{\partial^2 l(\theta)}{\partial x \partial \eta} \\ \frac{\partial^2 l(\theta)}{\partial \eta \partial x} & \frac{\partial^2 l(\theta)}{\partial \eta^2} \end{array} \right],
\]

where for each \( v, u \in \{x, \eta\} \) and \( w_i, z_i \in \{x, \beta_i\} \), the elements of the Hessian matrix are given by

\[
\hat{l}_{vu}(\theta) = \sum_{i=1}^{n} \sum_{j=0}^{1} (-1)^{j+1} \left[ \frac{\partial^2 a(s_i + j; \theta)}{\partial w_i \partial z_i} \phi_j(s_i, \theta) \right. + \left. \frac{\partial a(s_i + j; \theta)}{\partial z_i} \frac{\partial w_i}{\partial v} \phi_j(s_i, \theta) \right]
\]

\[
- \sum_{i=1}^{n} \sum_{j=0}^{1} a(s_i + j; \theta) \frac{\partial a(s_i + j; \theta)}{\partial w_i} \frac{\partial z_i}{\partial u} \frac{\partial w_i}{\partial v} \phi_j(s_i, \theta)
\]

\[
- \sum_{i=1}^{n} \sum_{j=0}^{1} (-1)^{j+1} \frac{\partial a(s_i + j; \theta)}{\partial z_i} \phi_j(s_i, \theta) \sum_{k=0}^{1} (-1)^{k+1} \frac{\partial a(s_i + k; \theta)}{\partial w_i} \frac{\partial z_i}{\partial u} \frac{\partial w_i}{\partial v} \phi_k(s_i, \theta),
\]
where
\[
\frac{\partial^2 a(s_i; \theta)}{\partial x^2} = \frac{2}{x^2} a(s_i), \quad \frac{\partial^2 a(s_i; \theta)}{\partial \beta_i \partial x} = -\frac{1}{x} \frac{\partial a(s_i; \theta)}{\partial \beta_i}, \quad \frac{\partial^2 a(s_i; \theta)}{\partial \beta_i \partial \beta_i} = \frac{1}{4x s_i^{1/2}} \frac{\partial}{\partial \beta_i} \left(3s_i + \beta_i \right),
\]
\[
\frac{\partial \beta_i}{\partial \eta} = \beta_i x_i \quad \text{and} \quad \frac{\partial^2 \beta_i}{\partial \eta \partial \eta} = \beta_i x_i x_i^\top.
\]

Note that the equation \( \hat{l}_u(\theta) = 0 \) does not provide analytic solutions for \( \hat{\alpha} \) and \( \hat{\eta}_j \), \( j = 0, \ldots, p \). Therefore, we have implemented two algorithms in software \( \mathbb{R} \) to find the maximum likelihood estimates of \( \alpha \) and \( \eta_j \), \( j = 0, \ldots, p \), by using the function \( \text{optim()} \); see R Core Team (2020). Note that when \( \beta_i = \exp(\eta_0), \ i = 1, \ldots, n \), we have the BSd(\( \theta \)) distribution in (3).

4. Numerical evaluation

In this section, we carry out a simulation study to evaluate the performance of both the maximum likelihood estimators and residuals. Moreover, we analyze two real data sets. All numerical evaluations were done in the \( \mathbb{R} \) software; see R Core Team (2019). The \( \mathbb{R} \) codes are available upon request from the authors.

4.1. Simulation

We first evaluate the performance of the maximum likelihood estimators for the \( S \sim \text{BS}_d \) model. Then, we consider a BSd regression model where the parameter \( \beta \) is associated with a covariate, that is,

\[
\beta_i = \exp(\eta_0 + \eta_i x_i) \quad i = 1, \ldots, n.
\]  

In (8), the covariate values were randomly generated from the uniform distribution in the interval \( (0,1) \). The simulation scenario considers: sample size \( n \in \{10, 50, 150, 400\} \) and the values of the shape parameter as \( \alpha \in \{0.50, 1.50, 1.50, 3.00\} \), with 1,000 Monte Carlo replications for each sample size. The values of \( \alpha \) have been chosen to cover the performance under low, moderate and high skewness. The BSd samples were generated using the Proposition 2.

The maximum likelihood estimation results for the BSd model are presented in Table 1. We report the following sample statistics for the maximum likelihood estimates: empirical bias and mean squared error (MSE). Note that the results in Table 1 allow us to conclude that, as the sample size increases, the bias and MSE of the estimators \( \hat{\alpha} \) and \( \hat{\beta} \) decrease, indicating that they are asymptotically unbiased, as expected.

Table 2 reports the simulation results for the BSd regression model. A look at the results in Table 2 allows us to conclude that, as the sample size increases, the empirical bias and MSE decrease, as expected. Moreover, we note that, as the value of the parameter \( \alpha \) increases, the performances of the estimators of \( \beta_0, \beta_1 \) and \( \alpha \), deteriorate.
4.2. Examples

The BS_d distribution and its regression model proposed in Section 3 are now used to analyze two data sets. In the first case, the objective is to fit the BS_d distribution to data corresponding to biaxial fatigue-life of \( n = 46 \) metal specimens (in cycles) until failure; this data set can be found in Rieck (1989). In the second example, we fit the proposed regression model to data on the fatigue-life (in cycles \( \times 10^{-3} \)) of concrete specimens (response variable \( Y \)), where the covariate is the ratio of applied stress causing failure (covariate \( x \)); see Mills (1997). In this second data set, the number of observations is \( n = 45 \).

**Case study 1: Metal specimens**

A descriptive summary of this data provides the following sample values: 566 (median); 943.065 (mean); 1110.934 (standard deviation); 117.8 (coefficient of variation); 2.204 (coefficient of skewness); 4.682 (coefficient of kurtosis), whereas their minimum and maximum times are 125 and 5046, respectively. The histogram shown in Figure 3 and the value of the coefficient of skewness support the assumption that these data follow an asymmetrical distribution. We have assumed different discrete asymmetrical distributions to describe this data set, including the Weibull, gamma, log-normal, log-Student-\( t \), and log-power-exponential (log-PE) distributions; see Nakagawa and Osaki (1975), Abouammoh and Alhazzani (2015), and Saulo et al. (2021). Table 3 presents the Akaike (AIC) and Bayesian (BIC) information criteria. The results of Table

| \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) | \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| n = 10 | n = 10 | n = 10 | n = 10 | n = 10 | n = 10 |
| 0.5 | -0.00131(0.0011) | 0.0017(0.0069) | -0.0002(0.0004) | -0.0013(0.0026) |
| 1.5 | -0.0021(0.0157) | 0.0018(0.0566) | -0.0007(0.0062) | -0.0028(0.0215) |
| 2.5 | -0.0372(0.0531) | 0.1024(0.1312) | -0.0135(0.0255) | 0.0319(0.0561) |
| 3.0 | -0.0998(0.0918) | 0.1807(0.1908) | -0.0180(0.0433) | 0.0410(0.0765) |

| \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) | \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| n = 150 | n = 150 | n = 150 | n = 150 | n = 150 | n = 150 |
| 0.5 | -0.0003(0.0082) | 0.0023(0.0212) | -0.0027(0.0099) | -0.0007(0.0030) | 0.0011(0.0078) | -0.0009(0.0004) |
| 1.5 | -0.0087(0.0534) | 0.0160(0.1354) | -0.0072(0.0092) | -0.0043(0.0190) | 0.0039(0.0499) | -0.0015(0.0036) |
| 2.5 | -0.0195(0.1057) | 0.0208(0.2493) | -0.0034(0.0359) | -0.0128(0.0356) | 0.0170(0.0879) | 0.0024(0.0133) |
| 3.0 | -0.0270(0.1289) | 0.0209(0.2983) | 0.0063(0.0590) | -0.0166(0.0433) | 0.0143(0.1003) | 0.0050(0.0230) |

| \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) | \( \hat{\eta}_0 \) | \( \hat{\eta}_1 \) | \( \hat{\gamma} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| n = 400 | n = 400 | n = 400 | n = 400 | n = 400 | n = 400 |
| 0.5 | -0.0037(0.0531) | 0.1024(0.1312) | -0.0135(0.0255) | 0.0319(0.0561) |
| 1.5 | -0.0018(0.0566) | 0.1807(0.1908) | -0.0180(0.0433) | 0.0410(0.0765) |
| 2.5 | -0.0166(0.0433) | 0.0143(0.1003) | 0.0050(0.0230) |

Table 1. Simulated values of biases (MSEs within parentheses) of the estimators of the BS_d model (\( \beta = 2 \)).

Table 2. Simulated values of biases (MSEs within parentheses) of the estimators of the BS_d regression model (\( \eta_0 = 0.2 \) and \( \eta_0 = 1.5 \)).
3 reveal that the BS\textsubscript{d} model provides better adjustment than the other models based on the values of AIC and BIC. The estimates and standard errors (in parenthesis) for the BS\textsubscript{d} model are \( \hat{\alpha} = 1.0840(0.1130) \) and \( \hat{\beta} = 595.1987(81.6782) \), and the fitted PMF is also shown in Figure 3.

**Case study 2: Concrete specimens**

The number of cycles until failure is expected to increase inversely with the ratio of applied stress causing failure. The postulated model is given by
\[ \beta_i = \exp(\eta_0 + \eta_1 x), \quad Y_i \sim \text{BS}_d(x, \beta_i), \]

for \( i = 1, \ldots, 45 \). The maximum likelihood estimates and standard errors (in parenthesis) for \( x, \eta_0 \) and \( \eta_1 \) are \( \hat{x} = 0.4966(0.0641), \hat{\eta}_0 = 27.4913(3.2530) \) and \( \hat{\eta}_1 = -23.9647(3.5146) \), respectively. Figure 4 presents the QQ plots with envelope of the generalized Cox-Snell and randomized quantile residuals for the \( \text{BS}_d \) regression model; see Saulo et al. (2019). Note that all points are inside the bands and around the \( y = x \) line, demonstrating a very good fit of the proposed model.

### 5. Concluding remarks

The continuous Birnbaum-Saunders distribution has been widely used in several areas, besides being an alternative to the Weibull and gamma distributions. However, in many practical problems, the use of discrete distributions is more appropriate. In this sense, we have studied a discrete version of the Birnbaum-Saunders distribution. Some important properties have been presented, such as moments, quantile function and reliability. We have presented a formal proof concerning the unimodality property of discrete Birnbaum-Saunders distribution. In addition, we have proposed a new discrete Birnbaum-Saunders regression model. Monte Carlo simulations have been carried out to evaluate the behavior of the maximum likelihood estimators. Two examples with real data have illustrated the proposed methodology. The results are seen to be quite favorable to the discrete Birnbaum-Saunders distribution as well as its regression model in terms of model fitting.

### Note

1. Sen et al. (2010) derived the discrete BS but the authors did not present specific properties of the distribution. They presented only the probability mass function, survival function, hazard rate, reverse hazard rate, and maximum likelihood estimators. A brief application to real data was also presented.

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