ON CRITICALLY COUPLED \((s_1, s_2)\)-FRACTIONAL SYSTEM OF SCHRÖDINGER EQUATIONS WITH HARDY POTENTIAL

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**Abstract.** In this article, our main concern is to study the existence of bound and ground state solutions for the following fractional system of Schrödinger equations with Hardy potentials:

\[
\begin{cases}
(-\Delta)^{s_1} u - \lambda_1 \frac{u}{|x|^{2s_1}} - u^{2^*_1-1} = \nu \alpha h(x) u^{\alpha-1} v^\beta \quad \text{in} \quad \mathbb{R}^N, \\
(-\Delta)^{s_2} v - \lambda_2 \frac{v}{|x|^{2s_2}} - v^{2^*_2-1} = \nu \beta h(x) u^{\alpha} v^{\beta-1} \quad \text{in} \quad \mathbb{R}^N,
\end{cases}
\]

where \(u, v > 0\) in \(\mathbb{R}^N \setminus \{0\}\),

where \(s_1, s_2 \in (0, 1)\) and \(\lambda_i \in (0, \Lambda_{N,s_i})\) with \(\Lambda_{N,s_i} = 2\pi^{N/2} \frac{\Gamma(N/2+s_i)}{\Gamma^2(1+s_i)} \frac{\Gamma(\min\{N, s_i\})}{\Gamma(\max\{N, s_i\})}\) \((i = 1, 2)\). By imposing certain assumptions on the parameters \(\nu, \alpha, \beta\) and on the function \(h\), we obtain ground-state solutions using the concentration-compactness principle and the mountain-pass theorem.

1. **Introduction**

The study of elliptic equations and systems involving fractional Laplacian is attracting many researchers over the last decade. The keen aspect of studying such equations is due to physical models in many different applications, e.g., geostrophic flows, crystal dislocation, water waves, etc. we refer to \([7, 13, 21]\) and the references therein for more details. In this article, we are concerned with the system of fractional Schrödinger equations with singular Hardy potential and coupled with terms up to critical power on the entire \(\mathbb{R}^N\) given below

\[
\begin{cases}
(-\Delta)^{s_1} u - \lambda_1 \frac{u}{|x|^{2s_1}} - u^{2^*_1-1} = \nu \alpha h(x) u^{\alpha-1} v^\beta \quad \text{in} \quad \mathbb{R}^N, \\
(-\Delta)^{s_2} v - \lambda_2 \frac{v}{|x|^{2s_2}} - v^{2^*_2-1} = \nu \beta h(x) u^{\alpha} v^{\beta-1} \quad \text{in} \quad \mathbb{R}^N,
\end{cases}
\]

where \(u, v > 0\) in \(\mathbb{R}^N \setminus \{0\}\),

where \(s_1, s_2 \in (0, 1)\) and \(\lambda_i \in (0, \Lambda_{N,s_i})\) with \(\Lambda_{N,s_i} = 2\pi^{N/2} \frac{\Gamma(N/2+s_i)}{\Gamma^2(1+s_i)} \frac{\Gamma(\min\{N, s_i\})}{\Gamma(\max\{N, s_i\})}\) \((i = 1, 2)\). The constant \(\Lambda_{N,s_i}\) \((i = 1, 2)\) is an optimal constant for the fractional Hardy inequality \([15, \text{Theorem} 1.1]\). Further \(2^*_i = \frac{N}{N-2s_i}\) \((2s_i < N\) and \(i = 1, 2)\) is the critical Sobolev exponent; the parameters \(\nu, \alpha, \beta\) are positive reals such that

\[
\alpha, \beta > 1 \quad \text{and} \quad \alpha + \beta \leq \min\{2^*_1, 2^*_2\},
\]

and \(h\) is a function defined on \(\mathbb{R}^N\) satisfying

\[
0 < h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
\]
For $s_1 = s_2 = 1$, the system (1.1) reduces to the coupled system of local nonlinear Schrödinger equations with singular Hardy potential

$$
\begin{align*}
-\Delta u - \lambda_1 \frac{u}{|x|^2} - u^{2^*-1} &= \nu \alpha h(x)|u|^{\alpha-2}|v|^\beta u \text{ in } \mathbb{R}^N, \\
-\Delta v - \lambda_2 \frac{v}{|x|^2} - v^{2^*-1} &= \nu \beta h(x)|u|^\alpha|v|^{\beta-2}v \text{ in } \mathbb{R}^N.
\end{align*}
$$

(1.4)

For $\nu = 0$, the system (1.4) becomes a single nonlinear elliptic equation and the author in [25, Terracini] discussed the existence of positive solutions as well as their qualitative properties. In 2009, Abdellaoui et al. (see [1]) dealt with the local system (1.4), and they obtained the existence of positive ground state solutions depending on the parameter $\nu > 0$ (large or small) and the non-negative function $h(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for $2 < \alpha + \beta < 2^*$ and $h(x) \in L^\infty(\mathbb{R}^N)$ for $\alpha + \beta = 2^*$. Later in 2014, Kang [17] proved the existence of a positive solution to (1.4) considering $h(x), \lambda_1(x), \lambda_2(x) \in C(\mathbb{R}^N)$ with additional assumptions. In 2015, Chen and Zou [9] dealt with the critical case i.e., $\alpha + \beta = 2^*$ with $h(x) = 1$, and proved the existence of positive solutions which are radially symmetric. Further, Zhong and Zou [27] proved the existence of ground state solutions by allowing $h(x)$ to change its sign with coupling parameter $\nu = 1$. Recently Colorado et al. [11] studied the problem (1.4) with $\alpha + \beta \leq 2^*$ and $0 \leq h(x) \in L^\infty(\mathbb{R}^N)$, and obtained positive ground and bound state solutions depending on the behaviour of parameter $\nu > 0$. The similar types of results were also derived by Colorado et al. [10] when $\alpha = 2$ and $\beta = 1$, $0 \leq h(x) \in L^\infty(\mathbb{R}^N)$.

While dealing with the fractional system (1.1), if we suppose $s_1 = s_2 = s$ and $\nu = 0$, then this system reduces to a fractional doubly critical equation

$$
(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = u^{2^*-1} \text{ in } \mathbb{R}^N.
$$

(1.5)

In 2016 Dipierro et al. in their paper [14, Theorem 1.5] proved the existence of a positive solution using variational approach for any $0 \leq \lambda < \Lambda_{N,s}$. Moreover, they used the moving plane method to obtain the qualitative behavior (such as radial symmetry, asymptotic behaviors etc.) of solutions of (1.5). In 2020, He and Peng [16] considered the following fractional system in $\mathbb{R}^N$

$$
\begin{align*}
(-\Delta)^s u + P(x)u - \mu_1 |u|^{2p-2}u &= \beta |v|^p|u|^{p-2}u \text{ in } \mathbb{R}^N, \\
(-\Delta)^s v + Q(x)v - \mu_2 |v|^{2p-2}v &= \beta |u|^p|v|^{p-2}v \text{ in } \mathbb{R}^N,
\end{align*}
$$

$$
u, \mu \in H^s(\mathbb{R}^N),
$$

(1.6)

where $N \geq 2$, $0 < s < 1$, $1 < p < \frac{N}{N-2s}$, $\mu_1 > 0$, $\mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant, and $P(x), Q(x)$ are continuous bounded radial functions. The authors used variational methods to obtain the existence of infinitely many non-radial positive solutions. Observe that the above system contains only the subcritical nonlinear terms and coupled terms up to subcritical power. Recently, Shen [24] considered the following fractional elliptic systems with Hardy-type singular potentials and coupled by critical homogeneous nonlinearities on the bounded domain $\Omega \subset \mathbb{R}^N$

$$
\begin{align*}
(-\Delta)^s u - \lambda_1 \frac{u}{|x|^{2s}} - |u|^{2^*-2}u &= \frac{\alpha \beta}{2s} |u|^{\alpha-2}u|v|^{\beta} + \frac{1}{2} Q_u (u,v) \text{ in } \Omega, \\
(-\Delta)^s v - \lambda_2 \frac{v}{|x|^{2s}} - |v|^{2^*-2}v &= \frac{\alpha \beta}{2s} |u|^\alpha|v|^{\beta-2}v + \frac{1}{2} Q_v (u,v) \text{ in } \Omega, \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

(1.7)

where $\lambda_1, \lambda_2 \in (0, \Lambda_{N,s})$ and $2^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. The existence of positive solutions to the systems through variational methods was ascertained for the critical case, i.e., $\alpha + \beta = 2^*$ on the bounded domain $\Omega$.

Motivated by the aforementioned articles and their results, we are interested in finding out the existence of positive solutions to the system (1.1). As far as we know, there is no literature in this direction concerning the critical case on the whole domain $\mathbb{R}^N$. The lack of compactness due to the nonlinearities in the source terms and the singular Hardy potential terms make it delicate to employ the variational methods to the problem (1.1). To deal with such kind of non-compactness, we will use the concentration compactness principle for fractional problems in unbounded domains considered in [4, Bonder et al.], [8, Chen et al.] and [23, Pucci and Temperini], and the Mountain pass theorem. These concentration compactness principles are fractional analogous to the celebrated concentration compactness principles discussed by P. L. Lions [19, 20].
Remark 1.1. In case $s_1 = s_2$, the order between the parameters $\lambda_1$ and $\lambda_2$ determines the order between the semi-trivial energy levels. Indeed, if $\lambda_2 > \lambda_1$ and $\nu$ is small enough, $J_\nu(0, z^{\lambda_2}_{\mu,s}) = \frac{s}{2} S^{N,2}_\nu(\lambda_2) < \frac{s}{2} S^{N,2}_\nu(\lambda_1) = J_\nu(z^{\lambda_1}_{\mu,s}, 0)$ i.e., the pair $(0, z^{\lambda_2}_{\mu,s})$ is a ground state solution of (1.1), see Theorem (4.3). In this case, the order of $\lambda_1$ and $\lambda_2$ plays a vital role for determining the existence of positive and ground-state semi-trivial solutions. But in the case of $s_1 \neq s_2$, the order between $\lambda_1$ and $\lambda_2$ fails to determine the order between the semi-trivial energy levels. Therefore, we find positive ground state solutions independent of the order between $\lambda_1$ and $\lambda_2$ while dealing with the system involving two different fractional Laplacians.

Our paper is organized in the following manner. First, we give some preliminary results and functional analysis settings in Section 2. Further, Section 3 deals with the results in which the functional $J_\nu$ satisfies the Palais-Smale condition for both the cases, i.e., subcritical and critical cases. Also, the local behavior of the semi-trivial solutions is given in this section depending on the parameters $\alpha, \beta$, and $\nu$. In Section 4, we prove the main results of this article concerned with positive bound and ground state solutions.

2. Preliminaries and functional setting

In this section, we give an appropriate variational setting for the system (1.1). First, we define the energy functional $J_\nu$ associated with the system (1.1) given as

\[
J_\nu(u, v) = \frac{1}{2} \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s_1}} \, dx \, dy + \frac{1}{2} \int\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N + 2s_2}} \, dx \, dy - \lambda_1 \frac{1}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_1}} \, dx - \frac{\lambda_2}{2} \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s_2}} \, dx - \frac{1}{2s_1} \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx - \frac{1}{2s_2} \int_{\mathbb{R}^N} |v|^{2^*_s} \, dx - \nu \int_{\mathbb{R}^N} (h(x)|u|^\alpha |v|^\beta \, dx,
\]

defined on the product space $\mathcal{D} = \mathcal{D}^{s_1,2}(\mathbb{R}^N) \times \mathcal{D}^{s_2,2}(\mathbb{R}^N)$. The space $\mathcal{D}^{s_i,2}(\mathbb{R}^N), (i = 1, 2)$ is the closure of $C^0_0(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

\[
\|u\|_{s_i} := \left( \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s_i}} \, dx \, dy \right)^{\frac{1}{2}}, \text{ for } i = 1, 2.
\]

We refer to the articles [5, 6] by Brasco et al. for more details about the space $\mathcal{D}^{s_i,2}(\mathbb{R}^N), (i = 1, 2)$. Further, we endow the following norm with the product space $\mathcal{D}$ given by

\[
\|(u, v)\|_{\mathcal{D}}^2 = \|u\|_{s_1}^2 + \|v\|_{s_2}^2,
\]

where

\[
\|u\|_{s_i}^2 = \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s_i}} \, dx \, dy - \lambda_i \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_i}} \, dx, \text{ for } i = 1, 2.
\]

The above norm is well defined due to the fractional Hardy inequality [15, Theorem 1.1] given by

\[
\Lambda_{N,s_i} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_i}} \, dx \leq \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s_i}} \, dx \, dy, \text{ for } i = 1, 2.
\]

where $\Lambda_{N,s_i} = 2^{N+2s_i} \Gamma^2(\frac{N+2s_i}{2}) \Gamma(\frac{N+2s_i}{2})^{-1}$ (i = 1, 2) is the sharp constant for the inequality (2.2). We can note that the norms $\|\cdot\|_{s_i}$ and $\|\cdot\|_{s_i}$ for any $\lambda_i \in (0, \Lambda_{N,s_i})$ with $i = 1, 2$ are equivalent due to the Hardy’s inequality (2.2).

Let us recall that the solutions of (1.5) arise as minimizers $z^{\lambda_i}_{\mu,s_i}, (i = 1, 2)$ of the Rayleigh quotient given by (see [14])

\[
S(\lambda_i) := \inf_{u \in \mathcal{D}^{s_i,2}(\mathbb{R}^N), u \neq 0} \frac{\|u\|_{s_i}^2}{\|u\|_{s_i}^2} = \frac{\|z^{\lambda_i}_{\mu,s_i}\|^2_{s_i}}{\|z^{\lambda_i}_{\mu,s_i}\|^2_{s_i}}, (i = 1, 2). \tag{2.3}
\]

Moreover, we have

\[
\|z^{\lambda_i}_{\mu,s_i}\|_{s_i}^2 = \|z^{\lambda_i}_{\mu,s_i}\|_{s_i}^{2^*_s} = S^{\frac{N}{2s_i}}(\lambda_i), \text{ for } i = 1, 2. \tag{2.4}
\]
If \( \lambda_i = 0 \) for \( i = 1, 2 \), then \( S(\lambda_i) = S_i \) which is known to be achieved by the extremals of the type \( C(N,s_i)(1 + |x|^2)^{-\frac{N-2}{2}} \) (see [18]), where \( C(N,s_i) \) is a positive constant depending on \( N \) and \( s_i \) only. We write \( S_1 = S_2 = S \) when \( s_1 = s_2 \). From (2.3) we have the following Sobolev embeddings

\[
\begin{aligned}
S(\lambda_1)\|u\|^2_{2,1} &\leq \|u\|^3_{3,1,s_1}, \\
S(\lambda_2)\|v\|^2_{2,2} &\leq \|v\|^3_{3,2,s_2}.
\end{aligned}
\]

(2.5)

By applying Hölder’s inequality, it follows that

\[
|J_\nu(u,v)| \leq \begin{cases} 
\frac{1}{2}\|u(v, u)\|^2 + \frac{1}{2}\|u\|_{2,1}^2 + \frac{1}{2}\|v\|_{2,2}^2 + \nu\|h\|_1 \nu \frac{1}{2}\|u\|^2_{2,1} + \nu \frac{1}{2}\|v\|^2_{2,2}, & \text{if } \frac{\alpha}{2s_1} + \frac{\beta}{2s_2} < 1, \\
\frac{1}{2}\|u\|^2_{2,1} + \frac{1}{2}\|v\|_{2,2}^2 + \nu\|h\|_1 \|u\|^2_{2,1} + \nu \|v\|^2_{2,2}, & \text{if } \frac{\alpha}{2s_1} + \frac{\beta}{2s_2} = 1.
\end{cases}
\]

(2.6)

In both cases the right-hand side is finite for every \( (u, v) \in \mathbb{D} \) due to the Sobolev embeddings given by (2.5) and thanks to (1.3). Hence, the functional \( J_\nu \) is well-defined on the product space \( \mathbb{D} \). Let us re-write the functional \( J_\nu(u,v) \) as

\[
J_\nu(u,v) = J_{\lambda_1}(u) + J_{\lambda_2}(v) - \nu \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta \, dx,
\]

(2.7)

where

\[
J_{\lambda_i}(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N-2s_i}} \, dx \ dy - \lambda_i \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_i}} \, dx - \frac{1}{2s_i} \int_{\mathbb{R}^N} |u|^{2s_i} \, dx, \text{ for } i = 1, 2.
\]

(2.8)

It is easy to verify that the functional \( J_\nu \) is Fréchet differentiable on \( \mathbb{D} \). The functional is given by

\[
J_\nu(u,v) = \frac{1}{2} A(u,v) - B(u,v) - \nu I(u,v),
\]

where \( A(u,v) = \|u(v, u)\|^2, \ B(u,v) = \frac{1}{2}\|u\|_{2,1}^2 + \frac{1}{2}\|v\|_{2,2}^2, \ I(u,v) = \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta \, dx \). If \( A, B, I \in C^1 \) on the product space \( \mathbb{D} \), then the functional \( J_\nu \) is also in \( C^1 \) on \( \mathbb{D} \). It is clear that \( A \in C^1 \) as it is the square of a norm on \( \mathbb{D} \). By following a similar approach as given in [2, Lemma 1], we can prove that the functionals \( B \) and \( I \) are in \( C^1 \) on the product space \( \mathbb{D} \).

For \( (u_0, v_0) \in \mathbb{D} \), the Fréchet derivative of \( J_\nu \) at \( (u,v) \in \mathbb{D} \) is given as follow

\[
\langle J'_\nu(u,v) | (u_0, v_0) \rangle = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|(u_0(x) - u_0(y))}{|x-y|^{N+2s_i}} \, dx \ dy
\]

\[
+ \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(v_0(x) - v_0(y))}{|x-y|^{N+2s_2}} \, dx \ dy - \lambda_1 \int_{\mathbb{R}^N} \frac{u \cdot u_0}{|x|^{2s_i}} \, dx - \lambda_2 \int_{\mathbb{R}^N} \frac{v \cdot v_0}{|x|^{2s_2}} \, dx
\]

\[
- \int_{\mathbb{R}^N} |u|^{2s_i - 2} u \cdot u_0 \, dx - \int_{\mathbb{R}^N} |v|^{2s_2 - 2} v \cdot v_0 \, dx
\]

\[
- \nu \alpha \int_{\mathbb{R}^N} h(x)|u|^{\alpha - 2} u \cdot u_0 |v|^\beta \, dx - \nu \beta \int_{\mathbb{R}^N} h(x)|u|^{\alpha}|v|^{\beta - 2} v \cdot v_0 \, dx,
\]

where \( J'_\nu(u,v) \) is the Fréchet derivative of \( J_\nu \) at \( (u,v) \), and the duality bracket between the product space \( \mathbb{D} \) and its dual \( \mathbb{D}^* \) is represented as \( \langle \cdot, \cdot \rangle \). From (2.1) and for any \( \tau > 0 \), we get

\[
J_\nu(\tau u, \tau v) = \frac{\tau^2}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s_i}} \, dx \ dy + \frac{\tau^2}{2} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s_2}} \, dx \ dy - \frac{\lambda_1 \tau^2}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_i}} \, dx
\]

\[
- \frac{\lambda_2 \tau^2}{2} \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s_2}} \, dx - \frac{\tau x_1}{2s_i} \int_{\mathbb{R}^N} |u|^{2s_i} \, dx - \frac{\tau x_2}{2s_2} \int_{\mathbb{R}^N} |v|^{2s_2} \, dx - \nu \tau^{\alpha + \beta} \int_{\mathbb{R}^N} h(x)|u|^{\alpha}|v|^{\beta} \, dx.
\]

(2.9)

Clearly, \( J_\nu(\tau u, \tau v) \to -\infty \) as \( \tau \to +\infty \) which implies that the functional \( J_\nu \) is unbounded from below on \( \mathbb{D} \). Here the concept of the Nehari manifold plays its role in minimizing the functional for finding the critical point in \( \mathbb{D} \) by using a variational approach. We introduce the Nehari manifold \( \mathcal{N}_\nu \) associated with the functional \( J_\nu \) as

\[
\mathcal{N}_\nu = \{ (u, v) \in \mathbb{D} \setminus \{(0,0)\} : \Phi_\nu(u,v) = 0 \},
\]
where
\[ \Phi_\nu(u, v) = \langle J_\nu'(u, v) \rangle(u, v). \] (2.10)
We can see that all the critical points \((u, v) \in \mathbb{D} \setminus \{(0, 0)\}\) of the energy functional \(J_\nu\) lie in the set \(\mathcal{N}_\nu\). On Nehari manifolds, we recall some well-known facts for the reader’s convenience.

Let \((u, v)\) be an element of the Nehari manifold \(\mathcal{N}_\nu\). Then the following holds:
\[ \|(u, v)\|^2_0 = \|u\|^2_{2s_1} + \|v\|^2_{2s_2} \] (2.11)
If we restrict the functional \(J_\nu\) on the Nehari manifold \(\mathcal{N}_\nu\), the functional takes the following form
\[ J_\nu|_{\mathcal{N}_\nu}(u, v) = \frac{s_1}{N}\|u\|^2_{2s_1} + \frac{s_2}{N}\|v\|^2_{2s_2} + \nu(\alpha + \beta)\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx. \] (2.12)
Now suppose that \((\tau u, \tau v) \in \mathcal{N}_\nu\) for all \((u, v) \in \mathbb{D} \setminus \{(0, 0)\}\). Then using (2.11) we get the following
\[ \|(u, v)\|^2_0 = \tau^{2\nu - 2}\|u\|^2_{2s_1} + \tau^{2\nu - 2}\|v\|^2_{2s_2} + \nu(\alpha + \beta)\tau^{\alpha + \beta - 2}\int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx. \] (2.13)
The above equation is an algebraic equation in \(\tau\) and a cautious analysis of equation (2.13) shows that this algebraic equation has a unique positive solution. Thus, we can infer that there exists a unique positive \(\tau = \tau_{u,v}\) such that \((\tau u, \tau v) \in \mathcal{N}_\nu\) for all \((u, v) \in \mathbb{D} \setminus \{(0, 0)\}\). By combining (2.11) and (1.2) we obtain that, for any \((u, v) \in \mathcal{N}_\nu\)
\[ J_\nu''(u, v)[u, v]^2 = \langle \Phi_\nu''(u, v) \rangle(u, v) \]
\[ = 2 \int_{\mathbb{R}^2N} |u(x) - u(y)|^2 dx dy + 2 \int_{\mathbb{R}^2N} |v(x) - v(y)|^2 dx dy - 2\lambda_1 \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s_1}} dx \\
- 2\lambda_2 \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2s_2}} dx - 2s_1 \int_{\mathbb{R}^N} |u|^2 s_1 dx - 2s_2 \int_{\mathbb{R}^N} |v|^2 s_2 dx - \\
- \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx \\
= 2\|(u, v)\|_0^2 - 2s_1 \|u\|^2_{2s_1} - 2s_2 \|v\|^2_{2s_2} - \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta dx. \]
Further calculations give us
\[ J_\nu''(u, v)[u, v]^2 = (2 - \alpha - \beta)\|(u, v)\|_0^2 + (\alpha + \beta)(\|u\|^2_{2s_1} + \|v\|^2_{2s_2}) - 2s_1 \|u\|^2_{2s_1} - 2s_2 \|v\|^2_{2s_2} \] (2.14)
Further, by using (2.11) we can prove the existence of a constant \(r_\nu > 0\) such that
\[ \|(u, v)\|_0 > r_\nu \text{ for all } (u, v) \in \mathcal{N}_\nu. \] (2.15)
Now by the Lagrange multiplier method, if \((u, v) \in \mathbb{D}\) is a critical point of \(J_\nu\) on the Nehari manifold \(\mathcal{N}_\nu\), then there exists a \(\rho \in \mathbb{R}\) called Lagrange multiplier such that
\[ (J_\nu|_{\mathcal{N}_\nu})'(u, v) = J_\nu'(u, v) - \rho \Phi_\nu'(u, v) = 0. \]
Thus from the above, we calculate that \(\rho(\Phi_\nu'(u, v))(u, v) = \langle J_\nu'(u, v) \rangle(u, v) = 0\). It is clear that \(\rho = 0\), otherwise the inequality (2.14) fails to hold and as a result \(J_\nu''(u, v) = 0\). Hence, there is a one-to-one correspondence between the critical points of \(J_\nu\) and the critical points of \(J_\nu|_{\mathcal{N}_\nu}\). The functional \(J_\nu\) restricted on the Nehari manifold \(\mathcal{N}_\nu\) is also written as
\[ (J_\nu|_{\mathcal{N}_\nu})(u, v) = \left(1 - \frac{1}{\alpha + \beta}\right)\|(u, v)\|_0^2 + \frac{1}{\alpha + \beta}(\|u\|^2_{2s_1} + \|v\|^2_{2s_2}) - \frac{1}{2s_1}\|u\|^2_{2s_1} - \frac{1}{2s_2}\|v\|^2_{2s_2}. \] (2.16)
Thus, combining the hypotheses (1.2) and (2.15) with (2.16), we deduce
\[ J_\nu(u, v) > \left(1 - \frac{1}{\alpha + \beta}\right)r_\nu^2 \text{ for all } (u, v) \in \mathcal{N}_\nu. \]
We come to the conclusion that the functional \(J_\nu\) restricted on \(\mathcal{N}_\nu\) is bounded from below. Hence, we continue our study to get the solution of (1.1) by minimizing the energy functional \(J_\nu\) on the Nehari manifold \(\mathcal{N}_\nu\).
Definition 2.1. If \((u_1, v_1) \in \mathbb{D} \setminus \{(0, 0)\}\) is a critical point of \(J_\nu\) over \(\mathbb{D}\), then we say that the pair \((u_1, v_1)\) is a bound state solution of (1.1). This bound state solution \((u_1, v_1)\) is said to be a ground state solution if its energy is minimal among all the bound state solutions i.e.

\[
c_\nu = J_\nu(u_1, v_1) = \min \{J_\nu(u, v) : (u, v) \in \mathbb{D} \setminus \{(0, 0)\}\} \quad \text{and} \quad J_\nu'(u, v) = 0. \tag{2.17}
\]

3. The Palais-Smale Condition

Lemma 3.1. Let us assume that (1.2) and (1.3) are satisfied and also that \(\{(u_n, v_n)\} \subset \mathcal{N}_\nu\) is a Palais-Smale sequence for \(J_\nu\) restricted on the Nehari manifold \(\mathcal{N}_\nu\) at level \(c \in \mathbb{R}\), then \(\{(u_n, v_n)\}\) is a bounded (PS) sequence for \(J_\nu\) in \(\mathbb{D}\), i.e.,

\[
J_\nu(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{in the dual space } \mathbb{D}^*. \tag{3.1}
\]

Proof. Assume that \(\{(u_n, v_n)\} \subset \mathcal{N}_\nu\) be a Palais-Smale sequence for \(J_\nu\) at level \(c\), then

\[
J(u_n, v_n) \to c \quad \text{as} \quad n \to \infty, \quad \text{i.e.,} \quad c + o(1) = J(u_n, v_n) \tag{3.2}
\]

and we recall that

\[
J(u_n, v_n) = \frac{1}{2} \|(u_n, v_n)\|^2_\mathbb{D} - \frac{1}{2s_1} \|u_n\|^{2s_1}_{2s_1} - \frac{1}{2s_2} \|v_n\|^{2s_2}_{2s_2} - \nu \int_{\mathbb{R}^N} h(x)|u_n|^\alpha |v_n|^\beta dx. \tag{3.3}
\]

For \((u_n, v_n) \in \mathcal{N}_\nu\), we have

\[
\|(u_n, v_n)\|^2_\mathbb{D} = \|u_n\|^{2s_1}_{2s_1} + \|v_n\|^{2s_2}_{2s_2} + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|u_n|^\alpha |v_n|^\beta dx \tag{3.4}
\]

By combining the above two equations, we get

\[
J(u_n, v_n) = \frac{1}{2} \|(u_n, v_n)\|^2_\mathbb{D} - \frac{1}{2s_1} \|u_n\|^{2s_1}_{2s_1} - \frac{1}{2s_2} \|v_n\|^{2s_2}_{2s_2} - \frac{1}{\alpha + \beta} \left(\|u_n\|^{2s_1}_{2s_1} - \|v_n\|^{2s_2}_{2s_2}\right)
= \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right) \|(u_n, v_n)\|^2_\mathbb{D} + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right) \|u_n\|^{2s_1}_{2s_1} + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_2}\right) \|v_n\|^{2s_2}_{2s_2}.
\]

Thus, we have

\[
c + o(1) \geq \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right) \|(u_n, v_n)\|^2_\mathbb{D}.
\]

Thus the sequence \(\{(u_n, v_n)\}\) is bounded in \(\mathbb{D}\). Furthermore, we deduce the following by considering the functional \(\Phi_\nu\) given by (2.10) with the inequalities (2.14) and (2.15)

\[
\langle \Phi_\nu'(u_n, v_n)(u_n, v_n) \rangle \leq (2 - \alpha - \beta) r_\nu^2. \tag{3.5}
\]

By the Lagrange multiplier method, we can assume the sequence of multipliers \(\{\omega_n\} \subset \mathbb{R}\) such that

\[
(J_\nu|_{\mathcal{N}_\nu})'(u_n, v_n) = J_\nu'(u_n, v_n) - \omega_n \Phi_\nu'(u_n, v_n) \quad \text{in the dual space } \mathbb{D}^*. \tag{3.6}
\]

Since \((J_\nu|_{\mathcal{N}_\nu})'(u_n, v_n)\) converges to 0 as \(n \to \infty\) in the dual space \(\mathbb{D}^*\), this implies that

\[
\langle (J_\nu|_{\mathcal{N}_\nu})'(u_n, v_n)(u_n, v_n) \rangle \to 0 \quad \text{as} \quad n \to \infty.
\]

This further implies that \(-\omega_n \langle \Phi_\nu'(u_n, v_n)(u_n, v_n) \rangle \to 0\). Finally, we know that \(\langle \Phi_\nu'(u_n, v_n)(u_n, v_n) \rangle < 0\) and hence, we have \(\omega_n \to 0\) in \(\mathbb{R}\) as \(n \to \infty\). Thus, (3.6) directly implies (3.1). \(\Box\)

Further, we prove the boundedness of the Palais-Smale sequence in \(\mathbb{D}\).

Lemma 3.2. Let us assume that (1.2) and (1.3) are satisfied and that \(\{(u_n, v_n)\} \subset \mathbb{D}\) be a (PS) sequence for the functional \(J_\nu\) at level \(c \in \mathbb{R}\). Then the sequence \(\{(u_n, v_n)\}\) is bounded in \(\mathbb{D}\).
Proof. Given \( \{ (u_n, v_n) \} \subset \mathbb{D} \) is a (PS) sequence for \( J_{\nu} \) at level \( c \), then as \( n \to \infty \)
\[
J_{\nu}(u_n, v_n) \to c \quad \text{in} \quad \mathbb{R},
\]
\[
J'_{\nu}(u_n, v_n) \to 0 \quad \text{in} \quad \mathbb{D}^*.
\]
Using (3.8), we can write
\[
\left( J'_{\nu}(u_n, v_n) \right) \frac{(u_n, v_n)}{\| (u_n, v_n) \|_{\mathbb{D}}} \to 0 \quad \text{in} \quad \mathbb{R}.
\]
Thus from the above, we have
\[
\frac{1}{2} \frac{\| (u_n, v_n) \|_{\mathbb{D}}^2}{\| (u_n, v_n) \|_{\mathbb{D}}^2} - \frac{1}{2s_1} |u_n|_{2^*_1}^{2^*_1} - \frac{1}{2} |v_n|_{2^*_2}^{2^*_2} - \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|u_n|^{\alpha}|v_n|^\beta dx = o(\| (u_n, v_n) \|_{\mathbb{D}}) \quad \text{as} \quad n \to \infty.
\]
Also, from (3.7) we obtain the following
\[
\frac{1}{2} \| (u_n, v_n) \|_{\mathbb{D}}^2 - \frac{1}{2s_1} |u_n|_{2^*_1}^{2^*_1} - \frac{1}{2} |v_n|_{2^*_2}^{2^*_2} - \nu \int_{\mathbb{R}^N} h(x)|u_n|^{\alpha}|v_n|^\beta dx = c + o(1) \quad \text{as} \quad n \to \infty.
\]
Thus, we can write
\[
J_{\nu}(u_n, v_n) - \frac{1}{\alpha + \beta} J'_{\nu}(u_n, v_n) \frac{(u_n, v_n)}{\| (u_n, v_n) \|_{\mathbb{D}}} = c + o(1) + o(\| (u_n, v_n) \|_{\mathbb{D}}) \quad \text{as} \quad n \to \infty,
\]
and hence,
\[
\left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) \frac{\| (u_n, v_n) \|_{\mathbb{D}}^2}{\| (u_n, v_n) \|_{\mathbb{D}}^2} \leq \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) \frac{\| (u_n, v_n) \|_{\mathbb{D}}^2}{\| (u_n, v_n) \|_{\mathbb{D}}^2} + \frac{1}{\alpha + \beta} \| u_n \|_{2^*_1}^{2^*_1} + \frac{1}{\alpha + \beta} \| v_n \|_{2^*_2}^{2^*_2} = c + o(1) + o(\| (u_n, v_n) \|_{\mathbb{D}}) \quad \text{as} \quad n \to \infty.
\]
Thus we can conclude that the sequence \( \{ (u_n, v_n) \} \) is bounded in \( \mathbb{D} \).

Now we derive the non-local version of Lemma 3.3 in [1].

**Lemma 3.3.** Let \( C, D > 0 \) and \( \delta \geq 2 \) be fixed. Also, assume that for any \( \nu > 0 \)
\[
T_{\nu} = \{ \varrho \in \mathbb{R}^+ \mid C \varrho^{\frac{2^*_s}{2s}} \varrho^{\frac{\delta}{2s}} \leq \varrho + D \nu \varrho^{\frac{2^*_s}{2s}} \varrho^{\frac{\delta}{2s}} \}.
\]
Then for every \( \epsilon > 0 \), there is a \( \nu_1 > 0 \) depending only on \( \epsilon, C, D, \nu, N \) and \( s \) such that
\[
\inf T_{\nu} \geq (1 - \epsilon) C^{\frac{2^*_s}{2s}} \quad \text{for all} \quad 0 < \nu < \nu_1.
\]

**Proof.** For \( \varrho \in T_{\nu} \),
\[
C \varrho^{\frac{\delta}{2s}} \varrho^{\frac{2^*_s}{2s}} \leq \varrho + D \nu \varrho^{\frac{2^*_s}{2s}} \varrho^{\frac{\delta}{2s}} = \varrho + \varrho^{\frac{2^*_s}{2s}} \varrho^{\frac{\delta}{2s}} \leq \nu\varrho^{\frac{2^*_s}{2s}} \varrho^{\frac{\delta}{2s}}.
\]
Hence, we notice that \( T_{\nu} = \{ \varrho \in \mathbb{R}^+ \mid F(\varrho) \leq \nu \} \). Also observe that \( F(C^{\frac{2^*_s}{2s}}) = 0 \) and
\[
F'(\varrho) = \varrho^{-\frac{\delta}{2s}} \left( \frac{2^*_s - \delta}{2s} \right) C \varrho^{-\frac{\delta}{2s}} - \left( \frac{2^*_s - \delta}{2s} \right) \varrho^{-\frac{\delta}{2s}}.
\]
If \( \delta \leq 2^*_s \), then \( F'(\varrho) < 0 \) i.e. the function \( F \) is strictly decreasing function. If \( \delta > 2^*_s \) and \( F'(\varrho) = 0 \) then
\[
\varrho = \left( \frac{C(\delta - 2^*_s)}{\delta - 2^*_s} \right)^{\frac{2^*_s}{\delta}} = \varrho^{\frac{2^*_s}{\delta}} \quad \text{for} \quad \varrho > \varrho^{\frac{2^*_s}{\delta}}.
\]
which implies that \( F \) has a global negative minimum at \( \varrho = \left( \frac{C(\delta - 2^*_s)}{\delta - 2^*_s} \right)^{\frac{2^*_s}{\delta}} > C^{\frac{2^*_s}{\delta}} \) and \( F \) tends to 0 as \( \varrho \to +\infty \).
In any case, \( F \) is strictly decreasing in \( (0, C^{\frac{2^*_s}{\delta}}) \) and it has only one zero at \( C^{\frac{2^*_s}{\delta}} \) with \( \lim_{\varrho \to 0^+} F(\varrho) = +\infty \) and \( F(\varrho) < 0 \) in \( (C^{\frac{2^*_s}{\delta}}, +\infty) \). Hence, \( \inf T_{\nu} = F^{-1}(\nu) \to C^{\frac{2^*_s}{\delta}} \) as \( \nu \to 0^+ \) and the conclusion follows. \( \square \)
3.1. The case $\alpha + \beta < \min\{2s_1, 2s_2\}$. In the following, we prove the Palais-Smale compactness condition of the functional $J_\nu$ at level $c$.

**Lemma 3.4.** Suppose $\alpha + \beta < \min\{2s_1, 2s_2\}$ and (1.3). Then, the functional $J_\nu$ satisfies the (PS) condition for any level $c$ satisfying

$$c < \min\left\{\frac{S_1}{N} \frac{\varphi_j}{2} (\lambda_1), \frac{S_2}{N} \frac{\varphi_k}{2} (\lambda_2)\right\}.$$  

(3.9)

**Proof.** We know by Lemma 3.2 that any (PS) sequence $\{(u_n, v_n)\}$ is bounded in $D$. So, there exists a subsequence denoted by $\{(u_n, v_n)\}$ itself and a $(\tilde{u}, \tilde{v}) \in D$ satisfying the following

$$(u_n, v_n) \rightarrow (\tilde{u}, \tilde{v}) \text{ weakly in } D,$$

$$(u_n, v_n) \rightarrow (\tilde{u}, \tilde{v}) \text{ strongly in } L^{q_1/(R^N)} \times L^{q_2/(R^N)} \text{ for } 1 \leq q_1 < 2s_1, 1 \leq q_2 < 2s_2,$$

$$(u_n, v_n) \rightarrow (\tilde{u}, \tilde{v}) \text{ a.e. in } R^N.$$  

Now by using the concentration–compactness principle of Bonder [4, Theorem 1.1], Chen [8, Lemma 4.5] and an analogous version of Pucci [23, Theorem 1.2], there exist a subsequence, still denoted as $\{(u_n, v_n)\}$, two at most countable sets of points $\{x_j\}_{j \in J} \subset R^N$ and $\{y_k\}_{k \in K} \subset R^N$, and non-negative numbers

$$\{(\mu_j, \rho_j)\}_{j \in J}, \{\tilde{\mu}_k, \tilde{\rho}_k\}_{k \in K}, \mu_0, \rho_0, \gamma_0, \bar{\mu}_0, \bar{\rho}_0 \text{ and } \bar{\gamma}_0$$

such that the following convergences hold \textit{weakly} in the sense of measures,

$$|D^{s_1} u_n| \rightarrow d\mu \geq |D^{s_1} \tilde{u}|^2 + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0,$$

$$|D^{s_2} v_n| \rightarrow d\tilde{\mu} \geq |D^{s_2} \tilde{v}|^2 + \sum_{k \in K} \tilde{\mu}_k \delta_{y_k} + \tilde{\mu}_0 \delta_0,$$

$$|u_n|^{2s_1} \rightarrow d\rho = |\tilde{u}|^{2s_1} + \sum_{j \in J} \rho_j \delta_{x_j} + \rho_0 \delta_0,$$

$$|v_n|^{2s_2} \rightarrow d\tilde{\rho} = |\tilde{v}|^{2s_2} + \sum_{k \in K} \tilde{\rho}_k \delta_{y_k} + \tilde{\rho}_0 \delta_0,$$

$$\frac{u_n}{|x|^{2s_1}} \rightarrow d\gamma = \frac{\tilde{u}}{|x|^{2s_1}} + \gamma_0 \delta_0,$$

$$\frac{v_n}{|x|^{2s_2}} \rightarrow d\tilde{\gamma} = \frac{\tilde{v}}{|x|^{2s_2}} + \bar{\gamma}_0 \delta_0,$$  

(3.10)

where $\delta_0, \delta_{x_j}, \delta_{y_k}$ are the Dirac functions at the points $0, x_j$ and $y_k$ of $R^N$ respectively. Now from inequality (1.6) of [4, Theorem 1.1] and inequalities (1.7) of [23, Theorem 1.2], we deduce the inequalities given below

$$S_1 \varphi_j^{\frac{2s_1}{s_1}} \leq \mu_j \text{ for all } j \in J \cup \{0\},$$

$$S_2 \varphi_k^{\frac{2s_2}{s_2}} \leq \tilde{\mu}_k \text{ for all } k \in K \cup \{0\},$$

(3.11)

Also, by taking $\alpha = 2s_1$ and $\alpha = 2s_2$ in the inequality (4.21) of [8, Lemma 4.5] we deduce that

$$\Lambda_{N,s_1} \gamma_0 \leq \mu_0,$$

$$\Lambda_{N,s_2} \bar{\gamma}_0 \leq \bar{\mu}_0.$$  

(3.12)

We denote the concentration of the sequence $\{u_n\}$ at infinity by the following numbers

$$\rho_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{2s_1} dx,$$

$$\mu_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |D^{s_1} u_n|^2 dx,$$

$$\gamma_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \frac{u_n^2}{|x|^{2s_1}} dx.$$  

(3.13)

In a similar way, we can define the concentrations of the sequence $\{v_n\}$ at infinity by the numbers $\tilde{\mu}_{\infty}, \tilde{\rho}_{\infty}$ and $\bar{\gamma}_{\infty}.$
Further, we assume that the function $\Psi_{j,\epsilon}(x)$ is a smooth cut-off function centered at points $\{x_j\}$, $j \in J$, satisfying
\[
\Psi_{j,\epsilon} = 1 \text{ in } B_{2}(x_j), \quad \Psi_{j,\epsilon} = 0 \text{ in } B_{r}(x_j), \quad 0 \leq \Psi_{j,\epsilon} \leq 1 \text{ and } |\nabla \Psi_{j,\epsilon}| \leq \frac{4}{\epsilon}, \quad (3.14)
\]
where $B_{r}(x_j) = \{ y \in \mathbb{R}^N : |y - x_j| < r \}$. Now, testing $J_{\nu}'(u_n, v_n)$ with $(u_n \Psi_{j,\epsilon}, 0)$ we get
\[
0 = \lim_{n \to +\infty} \langle J'_{\nu}(u_n, v_n)\rangle[(u_n \Psi_{j,\epsilon}, 0)]
\]
\[
= \lim_{n \to +\infty} \left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{2}}{|x - y|^{N+2s}} \Psi_{j,\epsilon}(x) \ dx \ dy + \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))}{|x - y|^{N+2s}} u_n(y) \ dx \ dy - \lambda_1 \int_{\mathbb{R}^{N}} \frac{u_n^{2}}{|x|^{2s}} \Psi_{j,\epsilon}(x) \ dx - \int_{\mathbb{R}^{N}} |u_n|^{2 \nu} \Psi_{j,\epsilon}(x) \ dx - \nu \alpha \int_{\mathbb{R}^{N}} h(x)|u_n|^{\alpha}|v_n|^{\beta} \Psi_{j,\epsilon}(x) \ dx \right)
\]
\[
= \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\mu - \lambda_1 \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\gamma - \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\rho + \lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))}{|x - y|^{N+2s}} u_n(y) \ dx \ dy
\]
\[- \nu \alpha \lim_{n \to +\infty} \int_{\mathbb{R}^{N}} h(x)|u_n|^{\alpha}|v_n|^{\beta} \Psi_{j,\epsilon}(x) \ dx.
\]
Notice that $0 \notin \text{supp}(\Psi_{j,\epsilon})$ for $\epsilon$ being sufficiently small and also it is given that $h \in L^{1}(\mathbb{R}^{N}) \cap L^{\infty}(\mathbb{R}^{N})$. Now we evaluate each of the integrals mentioned above when $\epsilon$ tends to 0.

\[
\int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\mu \geq \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} |D^{s_1} \tilde{u}|^{2} \ dx + \sum_{j \in J} \mu_{j} \delta_{x_{j}}(\Psi_{j,\epsilon}) + \mu_{0} \delta_{0}(\Psi_{j,\epsilon})
\]
\[= \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} |D^{s_1} \tilde{u}|^{2} \ dx + \sum_{j \in J} \mu_{j} \Psi_{j,\epsilon}(x_{j}) + \mu_{0} \Psi_{j,\epsilon}(0). \quad (3.15)
\]
Taking the limit $\epsilon \to 0$ and since $0 \notin \text{supp}(\Psi_{j,\epsilon})$ for $\epsilon$ being sufficiently small, we get
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\mu \geq \mu_{j}. \quad (3.16)
\]
and
\[
\int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\rho = \int_{\mathbb{R}^{N}} |\tilde{u}|^{2 \nu} \Psi_{j,\epsilon} \ dx + \sum_{j \in J} \rho_{j} \delta_{x_{j}}(\Psi_{j,\epsilon}) + \rho_{0} \delta_{0}(\Psi_{j,\epsilon})
\]
\[= \int_{\mathbb{R}^{N}} |\tilde{u}|^{2 \nu} \Psi_{j,\epsilon} \ dx + \sum_{j \in J} \rho_{j} \Psi_{j,\epsilon}(x_{j}) + \rho_{0} \Psi_{j,\epsilon}(0). \quad (3.17)
\]
Taking the limit $\epsilon \to 0$ and since $0 \notin \text{supp}(\Psi_{j,\epsilon})$ for $\epsilon$ being sufficiently small, we get
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\rho = \rho_{j}. \quad (3.17)
\]
Further,
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \ d\gamma = \lim_{\epsilon \to 0} \left( \int_{\mathbb{R}^{N}} \Psi_{j,\epsilon} \frac{\tilde{u}^{2}}{|x|^{2s}} \ dx + \gamma_{0} \Psi_{j,\epsilon}(0) \right) = 0. \quad (3.18)
\]
Next, we claim that:
\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))}{|x - y|^{N+2s}} u_n(y) \ dx \ dy = 0. \quad (3.19)
\]
Let
\[
\mathcal{I} = \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))}{|x - y|^{N+2s}} u_n(y) \ dx \ dy
\]
\[= \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))u_n(y)}{|x - y|^{N+2s}} \ dx \ dy.
\]
Since \( \{u_n \Psi_{j, \epsilon} \}_{n \in \mathbb{N}} \) is a bounded sequence in \( \mathcal{D}^{s_1, 2}(\mathbb{R}^N) \), by using the Hölder’s inequality we obtain

\[
\mathcal{I} \leq \left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s_1}} \, dx \, dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2N}} \frac{|\Psi_{j, \epsilon}(x) - \Psi_{j, \epsilon}(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s_1}} \, dx \, dy \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^{2N}} \frac{|\Psi_{j, \epsilon}(x) - \Psi_{j, \epsilon}(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s_1}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

By Lemma 2.2, Lemma 2.4 of Bonder et al. [4], and Lemma 2.3 of Xiang et al. [26], we obtain that

\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{|\Psi_{j, \epsilon}(x) - \Psi_{j, \epsilon}(y)|^2 |u_n(y)|^2}{|x-y|^{N+2s_1}} \, dx \, dy = 0,
\]

which implies that \( \lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathcal{I} = 0 \). Hence, the claim (3.19) is done. Now we show that

\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^N} h(x) |u_n|^{\alpha} |v_n|^{\beta} \Psi_{j, \epsilon}(x) \, dx = 0. \tag{3.20}
\]

We notice that \( \alpha + \beta < \min\{2s_1, 2s_2\} \) implies that \( \frac{\alpha}{s_1} + \frac{\beta}{s_2} < 1 \). Therefore, by applying the Hölder’s inequality, we have

\[
\int_{\mathbb{R}^N} h(x) |u_n|^{\alpha} |v_n|^{\beta} \Psi_{j, \epsilon}(x) \, dx = \int_{\mathbb{R}^N} (h(x) \Psi_{j, \epsilon}(x)) \left( |u_n|^{\alpha} |v_n|^{\beta} \right) \, dx \leq \left( \int_{\mathbb{R}^N} h(x) \Psi_{j, \epsilon}(x) \, dx \right)^{1 - \frac{\alpha}{s_1} - \frac{\beta}{s_2}} \left( \int_{\mathbb{R}^N} |u_n|^{\alpha} |v_n|^{\beta} \, dx \right)^{\frac{\alpha}{s_1}} \leq \left( \int_{\mathbb{R}^N} h(x) \Psi_{j, \epsilon}(x) \, dx \right)^{1 - \frac{\alpha}{s_1} - \frac{\beta}{s_2}} \left( \int_{\mathbb{R}^N} |u_n|^{2s_1} \Psi_{j, \epsilon}(x) \, dx \right)^{\frac{\alpha}{s_1}} \left( \int_{\mathbb{R}^N} |v_n|^{2s_2} \Psi_{j, \epsilon}(x) \, dx \right)^{\frac{\beta}{s_2}}.
\]

The first integral on the RHS of the last inequality tends to 0 as \( \epsilon \to 0 \), and the rest two integrals are bounded as the limit \( n \to \infty \). Therefore, first taking the limit as \( n \to \infty \) and then taking \( \epsilon \to 0 \), we get (3.20). Now from (3.15–3.20), one leads to the conclusion that \( \rho_j - \rho_j \leq 0 \) as \( \epsilon \to 0 \). Then using (3.11), we have

either \( \rho_j = 0 \), or \( \rho_j \geq S_1^{\frac{\alpha}{s_1}} \), for all \( j \in \mathcal{J} \) and \( \mathcal{J} \) is finite. \tag{3.21}

Analogously, we can also conclude that

either \( \bar{\rho}_k = 0 \), or \( \bar{\rho}_k \geq S_2^{\frac{\alpha}{s_2}} \), for all \( k \in \mathcal{K} \) and \( \mathcal{K} \) is finite. \tag{3.22}

Now for studying the concentration at origin, we consider a cut-off function \( \Psi_{0, \epsilon} \) which satisfies the assumption (3.14). Again, testing \( J'_\epsilon(u_n, v_n) \) with \( (u_n \Psi_{0, \epsilon}, 0) \) and following the analogous approach, we can easily deduce that \( \mu_0 - \lambda_1 \gamma_0 - \rho_0 \leq 0 \) and \( \bar{\mu}_0 - \lambda_1 \gamma_0 - \bar{\rho}_0 \leq 0 \). Moreover, using the inequality (1.7) of [23, Theorem 1.2], we have

\[
\mu_0 - \lambda_1 \gamma_0 \geq S(\lambda_1) \rho_0^{\frac{2}{s_1}} \quad \text{and} \quad \bar{\mu}_0 - \lambda_2 \gamma_0 \geq S(\lambda_2) \bar{\rho}_0^{\frac{2}{s_2}},
\]

which further implies that

either \( \rho_0 = 0 \) or \( \rho_0 \geq S^{\frac{2}{s_1}}(\lambda_1) \),

either \( \bar{\rho}_0 = 0 \) or \( \bar{\rho}_0 \geq S^{\frac{2}{s_2}}(\lambda_2) \). \tag{3.24}

Next for concentration at the point \( \infty \), we choose \( R > 0 \) large enough so that \( \{x_j\}_{j \in \mathcal{J} \cup \{0\}} \) is contained in \( B_R(0) \) and we consider a cut-off function \( \Psi_{\infty, \epsilon} \) supported in a neighbourhood of \( \infty \) satisfying the following

\[
\Psi_{\infty, \epsilon} = 0 \text{ in } B_R(0), \quad \Psi_{\infty, \epsilon} = 1 \text{ in } B^c_{R+1}(0), \quad 0 \leq \Psi_{\infty, \epsilon} \leq 1 \text{ and } |\nabla \Psi_{\infty, \epsilon}| \leq \frac{4}{R} \epsilon.
\]

Analogously, it is easy to find that \( \mu_{\infty} - \lambda_1 \gamma_\infty - \rho_\infty \leq 0 \) and \( \bar{\mu}_{\infty} - \lambda_1 \gamma_\infty - \bar{\rho}_\infty \leq 0 \) by testing \( J'_\epsilon(u_n, v_n) \) with \( (u_n \Psi_{\infty, \epsilon}, 0) \). Next, by the inequality (1.14) of [23, Theorem 1.3] we have

\[
\mu_{\infty} - \lambda_1 \gamma_\infty \geq S(\lambda_1) \rho_{\infty}^{\frac{2}{s_1}} \quad \text{and} \quad \bar{\mu}_{\infty} - \lambda_2 \gamma_\infty \geq S(\lambda_2) \bar{\rho}_{\infty}^{\frac{2}{s_2}},
\]

\[
\text{with } \epsilon = \frac{4}{R} \text{ and } 0 < R < \infty.
\]
which also further concludes that

\[
\begin{align*}
\text{either } & \rho_{\infty} = 0 \quad \text{or} \quad \rho_{\infty} \geq S_{\frac{N}{2}}^N(\lambda_1), \\
\text{either } & \tilde{\rho}_{\infty} = 0 \quad \text{or} \quad \tilde{\rho}_{\infty} \geq S_{\frac{N}{2}}^N(\lambda_2).
\end{align*}
\]

(3.27)

As we already know that

\[
c = \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right)\| (u_n, v_n) \|_D^2 + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right)\| u_n \|_{2s_1}^{2s_1} + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_2}\right)\| v_n \|_{2s_2}^{2s_2} + o(1).
\]

Now using (3.10-3.12), (3.24) and (3.27), we deduce that

\[
c \geq \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right)\left(\| (\tilde{u}, \tilde{v}) \|_D^2 + \sum_{j \in J} \mu_j + (\mu_\infty - \lambda_1 \gamma_0) + (\mu_\infty - \lambda_1 \gamma_\infty)\right)
+ \sum_{k \in K} \tilde{\mu}_k + (\bar{\mu}_0 - \lambda_2 \gamma_0) + (\bar{\mu}_\infty - \lambda_2 \gamma_\infty)
+ \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right)\left(\int_{\mathbb{R}^N} |\tilde{u}|^{2s_1} \, dx + \sum_{j \in J} \rho_j + \rho_0 + \rho_\infty\right)
+ \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_2}\right)\left(\int_{\mathbb{R}^N} |\tilde{v}|^{2s_2} \, dx + \sum_{k \in K} \tilde{\rho}_k + \tilde{\rho}_0 + \tilde{\rho}_\infty\right)
\geq \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right)\left(\| S_1 \sum_{j \in J} \rho_j \tilde{\tau}_j + S_2 \sum_{k \in K} \tilde{\rho}_k \tilde{\tau}_k \|_D^2 + S(\lambda_1) \left[\| \rho_0 \tilde{\tau}_1 + \rho_\infty \tilde{\tau}_1 \|_D^2 \right] + S(\lambda_2) \left[\| \rho_0 \tilde{\tau}_2 + \rho_\infty \tilde{\tau}_2 \|_D^2 \right]\right)
+ \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right)\left(\sum_{j \in J} \rho_j + \rho_0 + \rho_\infty\right) + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_2}\right)\left(\sum_{k \in K} \tilde{\rho}_k + \tilde{\rho}_0 + \tilde{\rho}_\infty\right).
\]

If the concentration is considered at the point \( x_j \), then \( \rho_j > 0 \) and further form above and using (3.21) we find that

\[
c \geq \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right)S_1^{1+\frac{\tilde{\tau}_1}{2s_1}} + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right)S_1^{\frac{\tilde{\tau}_1}{2s_1}} = \frac{s_1}{N} S_1^{\frac{\tilde{\tau}_1}{2s_1}}.
\]

Which contradicts the assumption on energy level given by (3.9). This leads to \( \rho_j = \mu_j = 0 \) for all \( j \in J \). Following the similar way, we further deduce that \( \tilde{\rho}_k = \bar{\mu}_k = 0 \) for all \( k \in K \). If \( \rho_0 \neq 0 \), from (3.29) and (3.24), we have

\[
c \geq \frac{s_1}{N} S_{\frac{N}{2}}^N(\lambda_1),
\]

which again contradicts the hypothesis on the energy level \( c \). Therefore, \( \rho_0 = 0 \). By the same token, we also get \( \tilde{\rho}_0 = 0 \). Arguing as above and using (3.27) we also find \( \rho_{\infty} = 0 \) and \( \tilde{\rho}_{\infty} = 0 \). Hence, there exists a subsequence that strongly converges in \( L^{2s_1}(\mathbb{R}^N) \times L^{2s_2}(\mathbb{R}^N) \). As a consequence, we have

\[
\|(u_n - \tilde{u}, v_n - \tilde{v})\|_D^2 = (J'_\nu(u_n, v_n)(u_n - \tilde{u}, v_n - \tilde{v}) + o_n(1),
\]

which infers that the sequence \( \{(u_n, v_n)\} \) strongly converges in \( D \) and the (PS) condition holds. \( \square \)

Now we consider the modified version of the problem (1.1) to deal with the positive solutions.

\[
\begin{align*}
J'_\nu(u, v) & = \frac{1}{2}\|(u, v)\|_D^2 - \frac{1}{2s_1}\| u^+ \|_{2s_1}^{2s_1} - \frac{1}{2s_2}\| v^+ \|_{2s_2}^{2s_2} - \nu \int_{\mathbb{R}^N} h(x)(u^+)\alpha(v^+)\beta \, dx.
\end{align*}
\]

(3.31)
From the above we have
\[
|J^+_v(u,v)| \leq \frac{1}{2} \|(u,v)\|_2^2 + \frac{1}{2s_1} \|u_+\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v_+\|_{s_2}^{2s_2} + \nu \int_{\mathbb{R}^N} h(x)(u^+)^\alpha(v^+)^\beta \, dx
\]
\[
= \frac{1}{2} \|(u,v)\|_2^2 + \frac{1}{2s_1} \|u\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v\|_{s_2}^{2s_2} + \nu \int_{\mathbb{R}^N} h(x)|u|^\alpha|v|^\beta \, dx.
\]
By applying Hölder’s inequality, it follows that
\[
|J^+_v(u,v)| \leq \left\{ \begin{array}{ll}
\frac{1}{2} \|(u,v)\|_2^2 + \frac{1}{2s_1} \|u\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v\|_{s_2}^{2s_2} + \nu \|h\|_1 \frac{1}{2s_1} \|u\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v\|_{s_2}^{2s_2} & \text{if } \frac{\alpha}{s_1} + \frac{\beta}{s_2} < 1, \\
\frac{1}{2} \|(u,v)\|_2^2 + \frac{1}{2s_1} \|u\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v\|_{s_2}^{2s_2} + \nu \|h\|_\infty \|u\|_{s_1}^{2s_1} + \|v\|_{s_2}^{2s_2} & \text{if } \frac{\alpha}{s_1} + \frac{\beta}{s_2} = 1.
\end{array} \right.
\]
(3.32)

In both the cases the right hand side is finite for every \((u,v) \in \mathbb{D}\) due to the the Sobolev embeddings given by (2.5) and thanks to (1.3). Hence, the functional \(J^+_v\) is well-defined on the product space \(\mathbb{D}\). It is easy to verify that the functional \(J^+_v\) is Fréchet differentiable on \(\mathbb{D}\). The functional is given by
\[
J^+_v(u,v) = \frac{1}{2} A(u,v) - B(u^+,v^+) - \nu I(u^+,v^+),
\]
where \(A(u,v) = \|(u,v)\|_2^2\), \(B(u^+,v^+) = \frac{1}{2s_1} \|u_+\|_{s_1}^{2s_1} + \frac{1}{2s_2} \|v_+\|_{s_2}^{2s_2}\), \(I(u^+,v^+) = \int_{\mathbb{R}^N} h(x)(u^+)^\alpha(v^+)^\beta \, dx\). If \(A, B, I \in C^1\) on the product space \(\mathbb{D}\), then the functional \(J^+_v\) is also in \(C^1\) on \(\mathbb{D}\). It is clear that \(A \in C^1\) as it is the square of a norm on \(\mathbb{D}\). By following a similar approach as given in [2, Lemma 1], we can prove that the functionals \(B\) and \(I\) are in \(C^1\) on the product space \(\mathbb{D}\).

For \((u_0,v_0) \in \mathbb{D}\), the Fréchet derivative of \(J^+_v\) at \((u,v) \in \mathbb{D}\) is given as follow
\[
\langle (J^+_v)'(u,v)(u_0,v_0) \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(u_0(x) - u_0(y))}{|x-y|^{N+2s_1}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v_0(x) - v_0(y))}{|x-y|^{N+2s_2}} \, dx \, dy - \lambda_1 \int_{\mathbb{R}^N} \frac{u \cdot u_0}{|x|^{2s_1}} \, dx - \lambda_2 \int_{\mathbb{R}^N} \frac{v \cdot v_0}{|x|^{2s_2}} \, dx
\]
\[- \nu \alpha \int_{\mathbb{R}^N} h(x)(u^+)^{\alpha-1} u_0 \, dx - \nu \beta \int_{\mathbb{R}^N} h(x)(v^+)^{\beta-1} v_0 \, dx.
\]

Besides, we shall denote \(\mathcal{N}^+_v\) the Nehari manifold associated to \(J^+_v\) as
\[
\mathcal{N}^+_v = \{(u,v) \in \mathbb{D}\setminus \{(0,0)\} : \langle (J^+_v)'(u,v)(u,v) \rangle = 0\}.
\]

Also for every \((u,v) \in \mathcal{N}^+_v\), we have the following
\[
\|(u,v)\|_2^2 = \|u^+\|_{s_1}^{2s_1} + \|v^+\|_{s_2}^{2s_2} + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)(u^+)^\alpha(v^+)^\beta \, dx,
\]
(3.33)
and using this we can write the functional \(J^+_v\) restricted on the Nehari manifold \(\mathcal{N}^+_v\) as
\[
J^+_v|_{\mathcal{N}^+_v}(u,v) = \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right)\|(u,v)\|_2^2 + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_1}\right)\|u^+\|_{s_1}^{2s_1} + \left(\frac{1}{\alpha + \beta} - \frac{1}{2s_2}\right)\|v^+\|_{s_2}^{2s_2}.
\]
(3.34)

In the following lemma, we prove strong convergence when certain Palais-Smale level conditions are imposed.

**Lemma 3.5.** Assume \(\alpha + \beta < \min\{s_1^*, s_2^*\}\) and (1.3). Also let \(\alpha \geq 2\), \(S_{s_1}^N(\lambda_1) \geq S_{s_2}^N(\lambda_2)\) and
\[
S_{s_1}^N(\lambda_1) + S_{s_2}^N(\lambda_2) < \min\{S_{s_1}^N, S_{s_2}^N\}.
\]
(3.35)
Then, there exists \(\nu_0 > 0\) such that, if \(0 < \nu < \nu_0\) and \(\{(u_n,v_n)\} \subset \mathbb{D}\) is a (PS) sequence for \(J^+_v\) at level \(c \in \mathbb{R}\) such that
\[
\frac{s_1}{N}S_{s_1}^N(\lambda_1) < c < \frac{\min\{s_1^*, s_2^*\}}{N} \left(\frac{S_{s_1}^N(\lambda_1) + S_{s_2}^N(\lambda_2)}{S_{s_1}^N(\lambda_1) + S_{s_2}^N(\lambda_2)}\right),
\]
(3.36)
and
\[ c \neq \frac{s_2}{N} S^{s_2} (\lambda_2) \text{ for every } l \in \mathbb{N} \setminus \{0\}, \] (3.37)
then there exists \((\tilde{u}, \tilde{v}) \in \mathbb{D}\) such that up to a subsequence \((u_n, v_n) \to (\tilde{u}, \tilde{v})\) in \(\mathbb{D}\) as \(n \to \infty\).

**Proof.** Without loss of generality suppose that \(s_1 \geq s_2\). Following Lemma 3.2 it is easy to prove that the (PS) sequence for \(J_\nu^+\) is bounded in \(\mathbb{D}\). Thus, there exist a \((\tilde{u}, \tilde{v}) \in \mathbb{D}\) and a subsequence \(\{(u_n, v_n)\}\) such that \(\{(u_n, v_n)\}\) converges weakly to \((\tilde{u}, \tilde{v})\) in \(\mathbb{D}\). Further
\[
\langle (J_\nu^+)'(u_n, v_n)|(u_n, 0) \rangle = \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(u_n(x) - u_n(y))}{|x - y|^{N+2s_1}} \, dx dy - \lambda_1 \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|^{2s_1}} \, dx.
\]
Since \(u = u^+ + u^-\) and we know that \(u^+u^- = 0\), as both \(u^+\) and \(u^-\) can not be positive simultaneously. Hence,
\[
\langle (J_\nu^+)'(u_n, v_n)|(u_n, 0) \rangle = \int_{\mathbb{R}^N} \frac{(u_n^-(x) - u_n^-(y))^2}{|x - y|^{N+2s_1}} \, dx dy + \int_{\mathbb{R}^N} \frac{(-u_n^+)(u_n^-(y) - u_n^-(y)u_n^-(x))}{|x - y|^{N+2s_1}} \, dx dy
\]
\[
- \lambda_1 \int_{\mathbb{R}^N} \frac{(u_n^-(x))^2}{|x|^{2s_1}} \, dx \geq \int_{\mathbb{R}^N} \frac{(u_n^-(x) - u_n^-(y))^2}{|x - y|^{N+2s_1}} \, dx dy - \lambda_1 \int_{\mathbb{R}^N} \frac{(u_n^-)^2}{|x|^{2s_1}} \, dx.
\]
From Hardy’s inequality (2.2), we have
\[
(1 - \lambda_1 C) \int_{\mathbb{R}^N} \frac{(u_n^-(x))^2}{|x|^{2s_1}} \, dx \leq \int_{\mathbb{R}^N} \frac{(u_n^-(x) - u_n^-(y))^2}{|x - y|^{N+2s_1}} \, dx dy - \lambda_1 \int_{\mathbb{R}^N} \frac{(u_n^-)^2}{|x|^{2s_1}} \, dx,
\]
where \(C = 1/\lambda N^{s_1} \). Now using the fact that \((J_\nu^+)'(u_n, v_n) \to 0\) in \(\mathbb{D}^*\), we have \(\langle (J_\nu^+)'(u_n, v_n)|(u_n, 0) \rangle \to 0\) as \(n \to \infty\), and hence we conclude from above inequalities that the sequence \((u_n^-) \to 0\) strongly in \(\mathbb{D}^{s_1,2}(\mathbb{R}^N)\). Proceeding in a similar way, we also have that \((v_n^-) \to 0\) strongly in \(\mathbb{D}^{s_2,2}(\mathbb{R}^N)\). Therefore, there is no loss for considering \(\{(u_n, v_n)\}\) as a non-negative (PS) sequence at level \(c\) for the functional \(J_\nu^*\).

By using the analogous approach of Lemma 3.4, we can deduce the existence of a subsequence, still denoted by \(\{(u_n, v_n)\}\), two (at most countable) sets of points \(\{x_j\}_{j \in \mathcal{J}} \subset \mathbb{R}^N\) and \(\{y_k\}_{k \in \mathcal{K}} \subset \mathbb{R}^N\) and non-negative numbers \(\{\rho_j, \rho_k\}_{j \in \mathcal{J}}, \{\rho_j, \rho_k\}_{k \in \mathcal{K}}, \rho_0, \rho_0, \gamma_0, \rho_0, \rho_0\) and \(\tilde{\rho}_0\) such that the weak* convergence given by (3.10) is satisfied as well as the inequalities (3.21-3.24) also hold.

The concentration at infinity given by the numbers \(\mu_\infty, \rho_\infty, \tilde{\rho}_\infty\) and \(\tilde{\rho}_\infty\) as in (3.13), for which (3.26) and (3.27) hold, are also defined in a similar way.

Next, we prove the strong convergence:

either \(u_n \to \tilde{u}\) in \(L^{2s_1}(\mathbb{R}^N)\) or \(v_n \to \tilde{v}\) in \(L^{2s_2}(\mathbb{R}^N)\). (3.38)

The proof follows by using the method of contradiction. So we suppose that both the sequences \(\{u_n\}\) and \(\{v_n\}\) do not converge strongly in \(L^{2s_1}(\mathbb{R}^N)\) and \(L^{2s_2}(\mathbb{R}^N)\) respectively. Thus, there exists \(j \in \mathcal{J} \cup \{0, \infty\}\) and \(k \in \mathcal{K} \cup \{0, \infty\}\) such that \(\rho_j > 0\) and \(\tilde{\rho}_k > 0\). Using the expressions (3.21-3.24) and (3.28) into the equation (3.29), we obtain
\[
c = \left(1 - \frac{1}{2} \frac{1}{\alpha + \beta} \right) \|(u_n, v_n)\|^3 + \frac{1}{\alpha + \beta} \|u_n\|^{2s_1} + \frac{1}{2s_1} \|v_n\|^{2s_2} + o(1)
\]
\[
\geq \left(1 - \frac{1}{2} \frac{1}{\alpha + \beta} \right) S(\lambda_1) \rho_j^{2s_1} + S(\lambda_2) \rho_k^{2s_2} + \left(1 - \frac{1}{2s_1} \right) \rho_j + \left(1 - \frac{1}{2s_2} \right) \tilde{\rho}_k
\]
\[
\geq \frac{s_2}{N} S^{2s_1} (\lambda_1) + \frac{s_2}{N} S^{2s_2} (\lambda_2).
\]

The aforementioned inequality contradicts assumption (3.36), so claim (3.38) is proved. Thereafter, we prove that:

either \(u_n \to \tilde{u}\) strongly in \(\mathbb{D}^{s_1,2}(\mathbb{R}^N)\) or \(v_n \to \tilde{v}\) strongly in \(\mathbb{D}^{s_2,2}(\mathbb{R}^N)\). (3.39)
We assume by the claim (3.38) that the sequence \( \{v_n\} \) strongly converges in \( L^{2_1^*}(\mathbb{R}^N) \). This implies that we have the following convergence
\[
\|u_n - \tilde{u}\|_{\lambda_1, s_1}^2 = \langle J'_p(u_n, v_n)(u_n - \tilde{u}, 0) \rangle + o_n(1),
\]
which clearly shows \( u_n \to \tilde{u} \) in \( D^{s_1, 2}(\mathbb{R}^N) \). Again, if we suppose that \( \{v_n\} \) strongly converges in \( L^{2_2^*}(\mathbb{R}^N) \), then \( v_n \to \tilde{v} \) in \( D^{s_2, 2}(\mathbb{R}^N) \). Hence, the proof of claim (3.39) is done. Further, we consider two cases to prove the strong convergence of both the components \( \{u_n\}, \{v_n\} \) in \( D^{s_1, 2}(\mathbb{R}^N) \) and \( D^{s_2, 2}(\mathbb{R}^N) \), respectively.

**Case 1:** The sequence \( \{v_n\} \) converges strongly to \( \tilde{v} \) in \( D^{s_2, 2}(\mathbb{R}^N) \).

On the contrary, assume that none of its subsequences converge. Let us assume first that the set \( \mathcal{J} \cup \{0, \infty\} \) has more than one point, by combining (3.29) with (3.21), (3.23), (3.24), (3.26) and (3.27), we find
\[
c \geq \frac{2s_1}{N} S_{N^2}^N(\lambda_1) \geq \frac{s_2}{N} S_{N^2}^N(\lambda_1) + \frac{s_2}{N} S_{N^2}^N(\lambda_2),
\]
which contradicts assumption (3.36). Thus, there can not be more than one point i.e. \( x_j, j \in \mathcal{J} \cup \{0, \infty\} \) contains only one concentration point for the sequence \( \{u_n\} \). Further, we shall show that \( \tilde{v} \neq 0 \). Again by the contradiction we assume that \( \tilde{v} \equiv 0 \), then \( \tilde{u} \geq 0 \) (as \( u_n \) is a non-negative sequence) and \( \tilde{u} \) verifies
\[
(-\Delta)^{\alpha_1} \tilde{u} - \lambda_1 \tilde{u}_{|x|^{2\alpha_1}} = \tilde{u}_2^{2_1^*} - 1 \text{ in } \mathbb{R}^N.
\]
Therefore, for some \( \mu > 0 \), we have \( \tilde{u} = z_{\mu, s_1}^1 \), and \( \int_{\mathbb{R}^N} \tilde{u}^{2_1} \, dx = S_{N^2}^N(\lambda_1) \) by (2.4). Using the fact that there is only one concentration point for the sequence \( \{u_n\} \), and combining (3.29) with (3.21), (3.23), (3.24), we deduce the following inequality
\[
c \geq \frac{s_1}{N} \left( \int_{\mathbb{R}^N} \tilde{u}^{2_1} \, dx + S_{N^2}^N(\lambda_1) \right) = \frac{2s_1}{N} S_{N^2}^N(\lambda_1) \geq \frac{s_2}{N} S_{N^2}^N(\lambda_1) + \frac{s_2}{N} S_{N^2}^N(\lambda_2).
\]
This contradicts the assumption (3.36). In case of \( \tilde{v} \equiv 0 \) and \( \tilde{u} \equiv 0 \), we have that \( \{u_n\} \) should verify the following
\[
(-\Delta)^{\alpha_1} u_n - \lambda_1 \frac{u_n}{|x|^{2\alpha_1}} = o(1) \text{ in the dual space } (D^{s_1, 2}(\mathbb{R}^N))^\prime.
\]
Which implies that
\[
\int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N + 2\alpha_1}} \, dx = \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2\alpha_1}} \, dx - \int_{\mathbb{R}^N} |u_n|^{2_1} \, dx = o_n(1),
\]
and
\[
c = J_3(u_n, v_n) + o_n(1) = J_1(u_n) + J_2(v_n) - \nu \int_{\mathbb{R}^N} h(x)(u_n)^a(v_n)^b \, dx + o_n(1)
\]
\[= J_1(u_n) + 0 - 0 + o_n(1), \text{ as } \{v_n\} \text{ strongly converges to 0 and } h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)
\]
\[= \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N + 2\alpha_1}} \, dx - \frac{\lambda_1}{2} \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2\alpha_1}} \, dx - \frac{1}{2s_1} \int_{\mathbb{R}^N} (u_n)^{2_1} \, dx + o_n(1)\right)
\]
\[= \frac{1}{2} \left( \int_{\mathbb{R}^N} (u_n)^{2_1} \, dx - \frac{1}{2s_1} \int_{\mathbb{R}^N} (u_n)^{2_1} \, dx + o_n(1)\right) \text{ by using (3.41)}
\]
\[= \frac{s_1}{N} \int_{\mathbb{R}^N} (u_n)^{2_1} \, dx + o_n(1) = \frac{s_1}{N} \rho_j, \text{ as } \tilde{u} = 0 \text{ and } \{u_n\} \text{ concentrates at one point.}
\]
Also for every \( j \in \mathcal{J} \), the sequence \( \{u_n\} \) is a positive (PS) sequence for the functional given as
\[
J_j(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N + 2\alpha_1}} \, dx - \frac{1}{2s_1} \int_{\mathbb{R}^N} (u_n)^{2_1} \, dx
\]
Using the characterization of (PS) sequence for the functional $J_\nu$ provided by [22] (See (2.6) in the proof of Theorem 1.1), we find that $\rho_0 = \ell S_{\nu}^{\frac{N}{2}}$ for some $\ell \in \mathbb{N}$, which is a contradiction to (3.35) and (3.36). Thus, $\mathcal{J} = \emptyset$. If $\{u_n\}$ is concentrating at points zero or infinity, we argue analogously for the functional $J_1$ and use the result provided by [3] (See Theorem 2.1 part (v)) to obtain the following

$$c = J_\nu(u_n, v_n) + o(1) = J_1(u_n) + o(1) = \frac{s_1 \ell}{N} S_{\nu}^{\frac{N}{2}}(\lambda_1),$$

for some $\ell \in \mathbb{N} \cup \{0\}$. Therefore, we get a contradiction to (3.36) and hence $\tilde{v} > 0$ in $\mathbb{R}^N$. Further, we may assume that there exists $\tilde{u} \neq 0$ such that $u_n \rightharpoonup \tilde{u}$ in $\mathcal{D}^{s_1, 2}(\mathbb{R}^N)$. On contrary let $\tilde{u} = 0$, then $\tilde{v}$ is a solution to the problem given by

$$(-\Delta)^{s_2} \tilde{u} - \lambda_2 \frac{\tilde{v}}{|x|^{2s_2}} = \tilde{v}^{2^*_2 - 1} \text{ in } \mathbb{R}^N. \quad (3.42)$$

Therefore, for some $\mu > 0$, we have $\tilde{v} = \tilde{s}_\mu^{\nu, s_2}$ and $\int_{\mathbb{R}^N} \tilde{v}^{2^*_2} dx = S_{\nu}^{\frac{N}{2}}(\lambda_2)$ by (2.4). As a consequence, combining (3.29) with (3.21), (3.23), (3.24), we conclude that

$$c \geq \frac{s_2}{N} \int_{\mathbb{R}^N} \tilde{v}^{2^*_2} dx + \frac{s_1}{N} S_{\nu}^{\frac{N}{2}}(\lambda_1) = \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2) + \frac{s_1}{N} S_{\nu}^{\frac{N}{2}}(\lambda_1) \geq \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2) + \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2),$$

which contradicts the assumption (3.36). We can conclude that $\tilde{u}, \tilde{v} \neq 0$. Further, we have

$$c = J_\nu(u_n, v_n) + \frac{1}{2} (J'_\nu(u_n, v_n))(u_n, v_n) + o_n(1)$$

$$= \frac{s_1}{N} \|u_n\|_2^{2s_1} + \frac{s_2}{N} \|v_n\|_2^{2s_2} + \frac{\alpha + \beta - 2}{2} \int_{\mathbb{R}^N} h(x) u_n^\alpha v_n^\beta dx + o_n(1)$$

$$= \frac{s_1}{N} \|u\|_2^{2s_1} + \frac{s_2}{N} \|v\|_2^{2s_2} + \frac{\alpha + \beta - 2}{2} \int_{\mathbb{R}^N} h(x) u^\alpha v^\beta dx,$$

by the concentration at $j \in \mathcal{J} \cup \{0, \infty\}$. Note that $\{v_n\}$ converges strongly to $\tilde{v}$ in $\mathcal{D}^{s_2, 2}(\mathbb{R}^N)$. On the other hand, using $\langle J'_\nu(u_n, v_n) \rangle(\tilde{u}, \tilde{v}) = o_n(1)$, we obtain the following expression

$$\| \tilde{u} \|_2^{2s_1} + \| \tilde{v} \|_2^{2s_2} = \int_{\mathbb{R}^N} h(x) u^\alpha v^\beta dx,$$

which clearly shows that $\tilde{u}, \tilde{v} \in \mathcal{N}_\nu$. Indeed, by combining (3.43),(3.44),(3.21),(3.23),(3.24) and (3.29), we get the following

$$J_\nu(\tilde{u}, \tilde{v}) = \frac{s_1}{N} \|\tilde{u}\|_2^{2s_1} + \frac{s_2}{N} \|\tilde{v}\|_2^{2s_2} + \frac{\alpha + \beta - 2}{2} \int_{\mathbb{R}^N} h(x) \tilde{u}^\alpha \tilde{v}^\beta dx$$

$$= c - \frac{s_1}{N} \rho_2 < \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_1) + \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2) - \frac{s_1}{N} S_{\nu}^{\frac{N}{2}}(\lambda_1) \leq \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2).$$

Thus, using the above expression we have

$$\tilde{c}_\nu = \inf_{u \in \mathcal{N}_\nu} J_\nu(u, v) < \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2).$$

But, according to Theorem 4.3, we have $\tilde{c}_\nu = \frac{s_2}{N} S_{\nu}^{\frac{N}{2}}(\lambda_2)$ provided that $\nu$ is very small. Thus, we get a contradiction to the former inequality. Hence, the proof of $u_n \rightharpoonup \tilde{u}$ strongly in $\mathcal{D}^{s_1, 2}(\mathbb{R}^N)$ is completed.

**Case 2:** The sequence $\{u_n\}$ strongly converges to $\tilde{u}$ in $\mathcal{D}^{s_1, 2}(\mathbb{R}^N)$. Our claim is that the sequence $\{v_n\}$ converges strongly to $\tilde{v}$ in $\mathcal{D}^{s_2, 2}(\mathbb{R}^N)$. Using the contradiction method, we assume that all of its subsequences do not converge. First we prove that $\tilde{u} \neq 0$. On contrary suppose that $\tilde{u} \equiv 0$, then $\{v_n\}$ is a (PS) sequence for the functional $J_2$ given by (2.8) at level $c$. Clearly for some $\mu > 0$, $\tilde{v} = \tilde{s}_\mu^{\nu, s_2}$ as $\{v_n\} \rightharpoonup \tilde{v}$ in $\mathcal{D}^{s_2, 2}(\mathbb{R}^N)$ and $\tilde{v}$ is a solution to the problem (3.42). Also, by applying the compactness theorem given by [3] and using (2.8) and (2.4), we find that

$$c = \lim_{n \to +\infty} J_2(v_n) = J_2(\tilde{s}_\mu^{\nu, s_2}) + \frac{s_2}{N} m S_{\nu}^{\frac{N}{2}}(\lambda_2) + \frac{s_2}{N} (S_{\nu}^{\frac{N}{2}}(\lambda_2) - \frac{s_1}{N} (l + 1) S_{\nu}^{\frac{N}{2}}(\lambda_2), \lambda_2),$$

for some $m \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$. This contradicts the assumptions (3.36) and (3.37). Therefore, the conclusion $\tilde{u} \neq 0$ follows immediately. On the other hand, the assumption $\tilde{v} \equiv 0$ implies that $\tilde{u}$ is a solution to the problem
(3.40) and that for some \( \mu > 0 \), we have \( \tilde{u} = s_{\mu, s_1}^1 \). Thus, we obtain
\[
c \geq \frac{s_1}{N} \int_{\mathbb{R}^N} \tilde{u}^{2^*_1} \, dx + \frac{s_2}{N} S^\infty_{2^*_2}(\lambda_2) \geq \frac{s_2}{N} S^\infty_{2^*_1}(\lambda_1) + \frac{s_2}{N} S^\infty_{2^*_2}(\lambda_2),
\]
contradicting (3.36). Hence, we conclude that \( \tilde{u}, \tilde{v} \neq 0 \). Now as \( (\tilde{u}, \tilde{v}) \) is a solution of (1.1), we have
\[
J_\nu(\tilde{u}, \tilde{v}) = \frac{s_1}{N} \int_{\mathbb{R}^N} \tilde{u}^{2^*_1} \, dx + \frac{s_2}{N} \int_{\mathbb{R}^N} \tilde{v}^{2^*_2} \, dx + \frac{\alpha + \beta - 2}{2} \int_{\mathbb{R}^N} h(x) \tilde{u}^\alpha \tilde{v}^\beta \, dx.
\] (3.44)
Since \( \{v_n\} \) does not strongly converge to \( \tilde{v} \) in \( D^2(\mathbb{R}^N) \), there exists at least \( k \in \mathbb{K} \cup \{0, \infty\} \) such that \( \tilde{p}_k > 0 \) and using again (3.43), we get
\[
c = \left( \frac{s_1}{N} \int_{\mathbb{R}^N} \tilde{u}^{2^*_1} \, dx + \frac{s_2}{N} \int_{\mathbb{R}^N} \tilde{v}^{2^*_2} \, dx + \frac{s_2}{N} \sum_{k \in \mathbb{K}} \tilde{p}_k + \tilde{p}_0 + \tilde{p}_\infty \right) + \frac{\alpha + \beta - 2}{2} \int_{\mathbb{R}^N} h(x) \tilde{u}^\alpha \tilde{v}^\beta \, dx.
\]
By using (3.22)-(3.24), (3.44) and (3.36), one has
\[
J_\nu(\tilde{u}, \tilde{v}) = c - \frac{s_2}{N} \sum_{k \in \mathbb{K}} \tilde{p}_k + \tilde{p}_0 + \tilde{p}_\infty \leq \frac{s_2}{N} S^\infty_{2^*_1}(\lambda_1) + \frac{s_2}{N} S^\infty_{2^*_2}(\lambda_2) - \frac{s_2}{N} S^\infty_2(\lambda_2) = \frac{s_2}{N} S^\infty_{2^*_1}(\lambda_1).
\] (3.45)
Further, we use the definition of \( S^\infty_{2^*_1}(\lambda_1) \) in the first equation of (1.1) which implies that
\[
\sigma_1 + \nu \alpha \int_{\mathbb{R}^N} h(x) \tilde{u}^\alpha \tilde{v}^\beta \, dx = \int_{\mathbb{R}^{2N}} \frac{(\tilde{u}(x) - \tilde{u}(y))^2}{|x - y|^{1+2s_1}} \, dx dy - \lambda_1 \int_{\mathbb{R}^N} \tilde{u}^2 \, dx \geq S(\lambda_1) \sigma_1^{2^*_1},
\] (3.46)
such that \( \sigma_1 = \int_{\mathbb{R}^N} \tilde{u}^{2^*_1} \, dx \). Then applying Hölder’s inequality leads to the following
\[
\int_{\mathbb{R}^N} h(x) \tilde{u}^\alpha \tilde{v}^\beta \, dx \leq C(h) \left( \int_{\mathbb{R}^N} \tilde{u}^{2^*_1} \, dx \right)^{\frac{\alpha}{2^*_1}} \left( \int_{\mathbb{R}^N} \tilde{v}^{2^*_2} \, dx \right)^{\frac{\beta}{2^*_2}}.
\] (3.47)
By combining (3.44) and (3.47), we can transform (3.46) into
\[
\sigma_1 + C_1 \nu \sigma_1^{\frac{n-1}{n}} \geq S(\lambda_1) \sigma_1^{\frac{n-1}{n}},
\] (3.48)
where the constant \( C_1 > 0 \) depends only on \( N, s, \alpha, \beta, h \) and independent of \( \tilde{u}, \tilde{v} \) and \( \nu \). We know that \( \tilde{v} \neq 0 \), we can choose \( \tilde{e} > 0 \) such that \( \int_{\mathbb{R}^N} \tilde{v}^{2^*_2} \, dx \geq \tilde{e} \). Now take \( e > 0 \) such a way that \( \tilde{e} \geq \epsilon S_{2^*_1}(\lambda_1) \). Since \( \alpha \geq 2 \) and (3.48) holds, therefore we can apply Lemma 3.3 to get a fixed \( \nu_0 > 0 \) such that
\[
\sigma_1 \geq (1 - \epsilon) S^\infty_{2^*_1}(\lambda_1) \quad \text{for any} \ 0 < \nu \leq \nu_0.
\]
Combining (3.44) and the last estimate, we get the following inequality
\[
J_\nu(\tilde{u}, \tilde{v}) \geq \frac{s_1}{N} (1 - \epsilon) S^\infty_{2^*_1}(\lambda_1) + \frac{s_2}{N} \frac{\epsilon}{2^*_2} \geq \frac{s_2}{N} (1 - \epsilon) S^\infty_{2^*_1}(\lambda_1) + \frac{s_2}{N} \epsilon S^\infty_{2^*_1}(\lambda_1) = \frac{s_2}{N} S^\infty_{2^*_1}(\lambda_1),
\]
contradicting the inequality (3.45). Hence, \( \{v_n\} \) converges strongly to \( \tilde{v} \) in \( D^2(\mathbb{R}^N) \). Proceeding in the same way, the result can be proved for the case \( s_1 \leq s_2 \). Finally, combining both the cases leads to the conclusion that the (PS) sequences strongly converge in \( \mathbb{D} \) to a non-trivial limit. Hence the proof is complete.

In the following lemma, we will derive the Palais-Smale compactness condition for the functional \( J^+_\nu \) associated with the modified problem (3.30) when the exponent \( \beta \geq 2 \) and \( S^\infty_{2^*_1}(\lambda_2) \geq S^\infty_{2^*_2}(\lambda_1) \).

**Lemma 3.6.** Assume \( \alpha + \beta < \min\{2s_1, 2s_2^*\} \) and (1.3), \( \beta \geq 2 \), \( S^\infty_{2^*_1}(\lambda_2) \geq S^\infty_{2^*_1}(\lambda_1) \) and \( S^\infty_{2^*_1}(\lambda_1) + S^\infty_{2^*_2}(\lambda_2) < \min\{S^\infty_{2^*_1}, S^\infty_{2^*_2}\} \).
\[
S^\infty_{2^*_1}(\lambda_1) + S^\infty_{2^*_2}(\lambda_2) = \min\{S^\infty_{2^*_1}, S^\infty_{2^*_2}\}.
\] (3.49)
Then, there exists \( \nu_0 > 0 \) such that, if \( 0 < \nu \leq \nu_0 \) and \( \{u_n, v_n\} \subset \mathbb{D} \) is a (PS) sequence for \( J^+_\nu \) at level \( c \in \mathbb{R} \) such that
\[
\frac{s_2}{N} S^\infty_{2^*_2}(\lambda_2) < c < \frac{\min\{s_1, s_2\}}{N} \left( S^\infty_{2^*_1}(\lambda_1) + S^\infty_{2^*_2}(\lambda_2) \right),
\] (3.50)
\[ c \neq \frac{s_1 I}{N} S_{N/2}^N(\lambda_1) \quad \text{for every } l \in \mathbb{N} \setminus \{0\}, \tag{3.51} \]

then \( (\tilde{u}, \tilde{v}) \in \mathbb{D} \) such that \( (u_n, v_n) \to (\tilde{u}, \tilde{v}) \in \mathbb{D} \) up to a subsequence.

Next, we give the behavior of the semi-trivial solutions based on the coupling parameter \( \nu \) (small or large) and with different values of \( \alpha, \beta \).

**Proposition 3.7.** Under hypotheses (1.2) and (1.3), the following holds:

(i) The pair \((z_{\mu,s_1}^{\lambda_1}, 0)\) is a local minimum of \( J_{\nu} \) on \( \mathcal{N}_{\nu} \) for \( \beta > 2 \) or \( \beta = 2 \) and \( \nu \) sufficiently small.

(ii) The pair \((0, z_{\mu,s_2}^{\lambda_1})\) is a local minimum of \( J_{\nu} \) on \( \mathcal{N}_{\nu} \) for \( \alpha > 2 \) or \( \alpha = 2 \) and \( \nu \) sufficiently small.

(iii) The pair \((z_{\mu,s_1}^{\lambda_1}, 0)\) is a saddle point for \( J_{\nu} \) on \( \mathcal{N}_{\nu} \) for \( \beta < 2 \) or \( \beta = 2 \) and \( \nu \) sufficiently large.

(iv) The pair \((0, z_{\mu,s_2}^{\lambda_1})\) is a saddle point for \( J_{\nu} \) on \( \mathcal{N}_{\nu} \) for \( \alpha < 2 \) or \( \alpha = 2 \) and \( \nu \) sufficiently large.

**Proof.** (i) Let \( \mu > 0, \beta > 2 \) and \((z_{\mu,s_1}^{\lambda_1} + \phi, \psi) \in \mathcal{N}_{\nu} \), i.e.,

\[
\left\| (z_{\mu,s_1}^{\lambda_1} + \phi, \psi) \right\| \mathcal{D}^s_{2,q} = \left\| z_{\mu,s_1}^{\lambda_1} + \phi \right\|_{L^2}^2 + \left\| \psi \right\|_{L^2}^2 + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|\beta dx.
\]

Let \( t = t(\phi, \psi) > 0 \) be such that \( t(z_{\mu,s_1}^{\lambda_1} + \phi) \in \mathcal{N}_{\lambda_1} \), where \( \mathcal{N}_{\lambda_1} \) denotes the Nehari manifold associated to \( J_1 \), namely \( \mathcal{N}_{\lambda_1} \) is defined by

\[
\mathcal{N}_{\lambda_1} = \left\{ u \in D^{s_1,2}(\mathbb{R}^N) \setminus \{0\} : \|u\|_{L^2}^2 = \|u\|_{L^2}^2 \right\}.
\]

Since \( t(z_{\mu,s_1}^{\lambda_1} + \phi) \in \mathcal{N}_{\lambda_1} \), we have

\[
t^2 \left\| z_{\mu,s_1}^{\lambda_1} + \phi \right\|_{L^2}^2 = t^2 \left\| z_{\mu,s_1}^{\lambda_1} + \phi \right\|_{L^2}^2
\]

Therefore, we get

\[
t = \left( \frac{\|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2}{\|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2} \right)^{\frac{1}{2}}.
\]

Combining (3.52) and (3.53), we get

\[
\begin{align*}
\frac{1}{2} & = \frac{1}{2} - \frac{\left\| \psi \right\|_{L^2}^2 - \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|\beta dx}{\|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2} \\
& = \left[ 1 - \frac{\left\| \psi \right\|_{L^2}^2 - \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|\beta dx}{\|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2} \right]^{\frac{1}{2}}.
\end{align*}
\]

Since \( h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), we have the following Hölder’s Inequality

\[
\int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|\beta dx \leq C(h) \left( \int_{\mathbb{R}^N} |z_{\mu,s_1}^{\lambda_1} + \phi|^{2\gamma_1} dx \right)^{\frac{\beta}{2\gamma_1}} \left( \int_{\mathbb{R}^N} |\psi|^{2\gamma_2} dx \right)^{\frac{\beta}{2\gamma_2}}
\]

\[
= C(h) \left\| z_{\mu,s_1}^{\lambda_1} + \phi \right\|_{L^2}^\alpha \|\psi\|_{L^2}^\beta
\]

\[
\leq C(h) \frac{1}{(S(\lambda_2))^{\gamma/2}} \left\| z_{\mu,s_1}^{\lambda_1} + \phi \right\|_{L^2}^\alpha \|\psi\|_{L^2}^\beta,
\]

\]
where the last inequality follows from the Sobolev embedding given by

$$S(\lambda_2)\|\psi\|_{L^2}^2 \leq \|\psi\|_{L^{2,2}}^2.$$  

Now

$$t^2 = \left[1 - \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 - \|\psi\|_{H^s}^2} - \nu (\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|^\beta \, dx}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} \right] \frac{2}{\alpha^2 - 2}.$$  

$$= \left[1 - \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} + A(\phi, \psi) \right] \frac{2}{\alpha^2 - 2}.$$  

Where

$$A(\phi, \psi) = \frac{\|\psi\|_{L^{2,2}}^2 + \nu (\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^{\lambda_1} + \phi|^\alpha |\psi|^\beta \, dx}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} \leq \frac{\|\psi\|_{L^{2,2}}^2 + \nu (\alpha + \beta) C(h) \|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2 |\psi|^\beta \|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} \leq \frac{C_2 \|\psi\|_{L^{2,2}}^2 + \nu (\alpha + \beta) C_1(h) \|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2 |\psi|^\beta \|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2},$$  

where $$C_2 = \frac{1}{S(\lambda_2)^{2,2}/2}.$$  

Taking $$\|\psi\|_{L^{2,2}}^2$$ as common from the above expression, we have

$$A(\phi, \psi) \leq \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} \left( C_2 \|\psi\|_{L^{2,2}}^2 + \nu (\alpha + \beta) C_1(h) \|z_{\mu,s_1}^{\lambda_1} + \phi\|_{L^2}^2 |\psi|^\beta \|\psi\|_{L^{2,2}}^2 \right).$$  

Since $$\beta > 2,$$ we can conclude that $$\|\psi\|_{L^{2,2}}^{\beta/2} \rightarrow 0$$ as $$\|(\phi, \psi)\|_D \rightarrow 0.$$ That means

$$A(\phi, \psi) \rightarrow \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} o(1)$$  

as $$\|(\phi, \psi)\|_D \rightarrow 0.$$  

Therefore,

$$t^2 = 1 - \frac{2}{\alpha^2 - 2} \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} (1 + o(1))$$  

as $$\|(\phi, \psi)\|_D \rightarrow 0.$$  

(3.54)

Similarly,

$$t^{2s_1} = 1 - \frac{2}{\alpha^2 - 2} \frac{\|\psi\|_{L^{2,2}}^2}{\|\psi\|_{L^2}^2 + \phi\|_{L^2}^2} (1 + o(1))$$  

as $$\|(\phi, \psi)\|_D \rightarrow 0.$$  

(3.55)

As $$z_{\mu,s_1}^{\lambda_1}$$ achieves the minimum of $$J_1 = J_\nu(\cdot, 0)$$ on $$\mathcal{N}_{\lambda_1},$$ the following holds i.e.,

$$J_\nu(t(z_{\mu,s_1}^{\lambda_1} + \phi), 0) - J_\nu(z_{\mu,s_1}^{\lambda_1}, 0) \geq 0.$$  

(3.56)
On the other hand, from (3.52)-(3.55) we deduce that
\[
J_\nu(z_{\mu,s}^{{\lambda}_1} + \phi, \psi) - J_\nu(t(z_{\mu,s}^{{\lambda}_1} + \phi), 0) \\
= \frac{1}{2}(1 - t^2)\|z_{\mu,s}^{{\lambda}_1} + \phi\|_{s+1}^2 - \frac{1}{2s_2}(1 - t^{2s_1}) \left[\|z_{\mu,s}^{{\lambda}_1} + \phi\|_{s+1}^2 + \|\psi\|_{s+2}^2 - \|\psi\|_{2s_2}^2\right] \\
- \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s}^{{\lambda}_1} + \phi|^{2s_1} dx + \frac{1}{2}\|\psi\|_{s+2}^2 - \nu \int_{\mathbb{R}^N} h(x)|z_{\mu,s}^{{\lambda}_1} + \phi|^{2s_1} dx \\
= \frac{1}{2}(1 - t^2)\|z_{\mu,s}^{{\lambda}_1} + \phi\|_{s+1}^2 - \frac{1}{2s_2}(1 - t^{2s_1}) \left[\|z_{\mu,s}^{{\lambda}_1} + \phi\|_{s+1}^2 + \|\psi\|_{s+2}^2 - \|\psi\|_{2s_2}^2\right] \\
- \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s}^{{\lambda}_1} + \phi|^{2s_1} dx + \frac{1}{2}\|\psi\|_{s+2}^2 - \nu \int_{\mathbb{R}^N} h(x)|z_{\mu,s}^{{\lambda}_1} + \phi|^{2s_1} dx \\
= \frac{1}{2}\|\psi\|_{s+2}^2 (1 + o(1))
\]
as \| (\phi, \psi) \|_D \to 0. Hence
\[
J_\nu(z_{\mu,s}^{{\lambda}_1} + \phi, \psi) - J_\nu(t(z_{\mu,s}^{{\lambda}_1} + \phi), 0) \geq 0
\]
(3.57)
provided \((z_{\mu,s}^{{\lambda}_1} + \phi, \psi)\) is very near to \((z_{\mu,s}^{{\lambda}_1}, 0)\) in \(D\). If \(\psi \neq 0\), the inequality given by (3.57) holds strictly.
We conclude from (3.56) and (3.57) that
\[
J_\nu(z_{\mu,s}^{{\lambda}_1} + \phi, \psi) - J_\nu(z_{\mu,s}^{{\lambda}_1}, 0) \geq 0
\]
for any \((z_{\mu,s}^{{\lambda}_1} + \phi, \psi) \in \mathcal{N}_\nu\) sufficiently closed to \((z_{\mu,s}^{{\lambda}_1}, 0)\), i.e., \((z_{\mu,s}^{{\lambda}_1}, 0)\) is a local minimum point of \(J_\nu\) in \(\mathcal{N}_\nu\).

In the case \(\beta = 2\), we obtain that
\[
J_\nu(z_{\mu,s}^{{\lambda}_1} + \phi, \psi) - J_\nu(t(z_{\mu,s}^{{\lambda}_1} + \phi), 0) \\
= \left(\frac{1}{2}\|\psi\|_{s+2}^2 - \nu \int_{\mathbb{R}^N} h(x)|z_{\mu,s}^{{\lambda}_1} + \phi|^{2s_1} dx\right)(1 + o(1)),
\]
which is still non-negative provided \(\| (\phi, \psi) \|_D\) and \(\nu\) are sufficiently small.

\[(ii)\) The proof works in the same way as \((i)\), hence we omit it.

\[(iii)\) To prove this part we suppose \(\beta < 2\) and we fix \(\nu > 0\), \(\mu > 0\) and \(v \in \mathcal{D}^{2s_2}(\mathbb{R}^N)\setminus\{0\}\). For every \(t \in \mathbb{R}\), we denote by \(s(t)\) the unique number such that \((s(t)z_{\mu,s}^{{\lambda}_1}, s(t)v) \in \mathcal{N}_\nu\) i.e.,
\[
\|s(t)z_{\mu,s}^{{\lambda}_1}\|_{s+1}^2 + \|s(t)v\|_{s+2}^2 = \|s(t)z_{\mu,s}^{{\lambda}_1}\|_{s+1}^2 + \|s(t)v\|_{s+2}^2 + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|s(t)z_{\mu,s}^{{\lambda}_1}|^{2s_1}|s(t)v|^{2s_1} dx
\]
and then we get
\[ [s(t)]^2 \left( ||z_{\mu,s_1}^2 \|_{L^2,1}^2 + ||tv||_{L^2,2}^2 + ||z_{\mu,s_1}^2 ||_{L^2,2}^2 \right) = [s(t)]^2 ||z_{\mu,s_1}^2 ||_{L^2,2}^2 +
\[ + [s(t)]^2 ||tv||_{L^2,2}^2 + [s(t)]^{\alpha + \beta} \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx,
\]
which also implies
\[ ||z_{\mu,s_1}^2 ||_{L^2,1}^2 + t^2 ||v||_{L^2,2}^2 = [s(t)]^2 ||z_{\mu,s_1}^2 ||_{L^2,2}^2 +
\[ + [s(t)]^2 ||tv||_{L^2,2}^2 + [s(t)]^{\alpha + \beta} \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx.
\]
Now putting \( t = 0 \) in the above expression, we get
\[ ||z_{\mu,s_1}^2 ||_{L^2,1}^2 = [s(0)]^2 ||z_{\mu,s_1}^2 ||_{L^2,2}^2,
\]
which implies \( s(0) = 1 \).

Now from the Implicit Function Theorem, it follows that \( s \in C^1(\mathbb{R}) \) and
\[ s'(t) = -\frac{2t||v||_{L^2,1}^2 - 2s(t) ||tv||_{L^2,2}^2 - 2s(t) \nu(\alpha + \beta)|z_{\mu,s_1}^2 ||_{L^2,2}^2}{(2s_1 - 2)||z_{\mu,s_1}^2 ||_{L^2,1}^2} \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx \]
for all \( t \in \mathbb{R} \). Hence, since \( \beta < 2 \),
\[ s'(t) = -\frac{\beta \nu(\alpha + \beta) ||z_{\mu,s_1}^2 ||_{L^2,2}^2}{(2s_1 - 2)||z_{\mu,s_1}^2 ||_{L^2,1}^2} \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx \]
as \( t \to 0 \).

Further, we have
\[ s(t) = 1 - \frac{\nu(\alpha + \beta) ||z_{\mu,s_1}^2 ||_{L^2,2}^2}{(2s_1 - 2)||z_{\mu,s_1}^2 ||_{L^2,1}^2} \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx \]
and
\[ [s(t)]^2 = 1 - \frac{\nu(\alpha + \beta) ||z_{\mu,s_1}^2 ||_{L^2,2}^2}{(2s_1 - 2)||z_{\mu,s_1}^2 ||_{L^2,1}^2} \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^2 |^\alpha |tv|^\beta \, dx \]
as \( t \to 0 \).

Recall that for any \( (tu, tv) \in \mathcal{N}_u \), the energy functional \( J_\nu \) can be written as
\[ J_\nu(tu, tv) = \frac{1}{N} ||u||_{L^2,1}^2 + \frac{2s_1}{N} ||tv||_{L^2,2}^2 + \nu(\alpha + \beta - 2) \int_{\mathbb{R}^N} h(x)|u|^\alpha |tv|^\beta \, dx \]
Combining the above with (3.58), we get
\[ J_\nu(s(t)z_{\mu,s_1}^2, s(t)tv) - J_\nu(z_{\mu,s_1}^2, 0) \]
and
\[ J_\nu(s(t)z_{\mu,s_1}^2, s(t)tv) - J_\nu(z_{\mu,s_1}^2, 0) \]
and
\[ J_\nu(s(t)z_{\mu,s_1}^2, s(t)tv) - J_\nu(z_{\mu,s_1}^2, 0) \]
and
\[ J_\nu(s(t)z_{\mu,s_1}^2, s(t)tv) - J_\nu(z_{\mu,s_1}^2, 0) \]
and
\[ J_\nu(s(t)z_{\mu,s_1}^2, s(t)tv) - J_\nu(z_{\mu,s_1}^2, 0) \]
Therefore,
\[ J_\nu(s(t)z_{\mu,s_1}^\lambda, s(t)tu) - J_\nu(z_{\mu,s_1}^\lambda, 0) < 0 \quad \text{for} \ t \neq 0 \ \text{and} \ t \ \text{is small}, \]  
(3.59)
which implies that the pair \((z_{\mu,s_1}^\lambda, 0)\) is a local strict maximum point for \(J_\nu\) along a path lying in the Nehari manifold \(\mathcal{N}_\nu\). Now by the definition of \(\mathcal{N}_{\lambda_1}, \ w \in \mathcal{N}_{\lambda_1}\) if and only if \((w, 0) \in \mathcal{N}_\nu\). Also, the minimizers of \(J_\nu(\cdot, 0)\) on \(\mathcal{N}_{\lambda_1}\) are given by the sets \(\{(z_{\alpha,s_1}^\lambda, 0) : \alpha > 0\} \). Then
\[ J_\nu(w, 0) - J_\nu(z_{\mu,s_1}^\lambda, 0) > 0, \quad \text{for all} \ w \in \mathcal{N}_{\lambda_1} \ \text{and} \ \{(z_{\alpha,s_1}^\lambda, 0) : \alpha > 0\} \]  
(3.60)
which shows that the pair \((z_{\mu,s_1}^\lambda, 0)\) is a local minimum for \(J_\nu\) restricted to \(\mathcal{N}_{\lambda_1} \times \{0\} \subset \mathcal{N}_\nu\). Thus, we deduce from (3.59) and (3.60) that \((z_{\mu,s_1}^\lambda, 0)\) is a saddle point for \(J_\nu\) in \(\mathcal{N}_\nu\).

Now we just assume that \(\nu\) is sufficiently large and follow the above argument. Then
\[ s'(t) = -2 \frac{\nu(\alpha + 2) \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^\lambda|^\alpha|u|^2 \, dx - \|u\|^2_{\lambda_2, s_2}}{(2s_2 - 2)\|z_{\mu,s_1}^\lambda\|_{\lambda_2, s_2}} t (1 + o(1)) \quad \text{as} \ t \to 0, \]
and hence
\[ J_\nu(s(t)z_{\mu,s_1}^\lambda, s(t)tu) - J_\nu(z_{\mu,s_1}^\lambda, 0) = \left( \frac{1}{2} \|u\|^2_{\lambda_2, s_2} - \nu \int_{\mathbb{R}^N} h(x)|z_{\mu,s_1}^\lambda|^\alpha|u|^2 \, dx \right) |t|^2 + o(|t|^2) \quad \text{as} \ t \to 0. \]
This implies that the inequality (3.59) holds for \(\nu\) sufficiently large.

(iv) Proceeding in the same way as in the proof of (iii), we get the required result.

3.2. The case \(\alpha + \beta = \min\{2_{s_1}, 2_r\}\). We assume some extra continuity assumptions on \(h\) to deal with this critical case. Precisely, more hypotheses on the function \(h\) are supposed to address this case. In particular,
\[ 0 \leq h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \ h \ \text{continuous near} \ 0 \ \text{and} \ \infty, \ \text{and} \ h(0) = \lim_{|x| \to +\infty} h(x) = 0. \]  
(H1)
Adding to that, we will distinguish two different cases: one with \(h\) radial case and the other when \(h\) is non-radial. To deal with the \(h\) non-radial case, we need to impose one extra assumption on \(\nu\) i.e., it should be small enough. Let us define the space of radial functions in \(\mathbb{D}\)
\[ \mathbb{D}_r := \mathbb{D}^{s_1, 2}(\mathbb{R}^N) \times \mathbb{D}^{2_r, 2}(\mathbb{R}^N) = \{(u, v) \in \mathbb{D} : u \ \text{and} \ v \ \text{are radially symmetric}\}. \]

Lemma 3.8. Assume that \(\alpha + \beta = \min\{2_{s_1}, 2_r\}\) and (H1), and \(h\) is a radial function.

(i) If \(\{(u_n, v_n)\} \subset \mathbb{D}_r\) is a PS sequence for \(J\) at level \(c \in \mathbb{R}\) such that \(c\) satisfies (3.9), then the sequence \(\{(u_n, v_n)\}\) admits a subsequence strongly converging in \(\mathbb{D}\).

(ii) If \(\alpha \geq 2\) and \(S_N^\infty(\lambda_1) \geq S_N^\infty(\lambda_2)\), and \(\{(u_n, v_n)\} \subset \mathbb{D}_r\) is a PS sequence for \(J_+^+\) at level \(c \in \mathbb{R}\) such that \(c\) satisfies (3.36) and (3.37), then there exists \(\nu_1 > 0\) and \((\tilde{u}, \tilde{v}) \in \mathbb{D}_r\) such that \((u_n, v_n) \to (\tilde{u}, \tilde{v})\) in \(\mathbb{D}_r\) up to subsequence for every \(0 < \nu \leq \nu_1\).

(iii) If \(\beta \geq 2\) and \(S_N^\infty(\lambda_2) \geq S_N^\infty(\lambda_1)\), and \(\{(u_n, v_n)\} \subset \mathbb{D}_r\) is a PS sequence for \(J_+^+\) at level \(c \in \mathbb{R}\) such that \(c\) satisfies (3.50) and (3.51), then there exists \(\nu_2 > 0\) and \((\tilde{u}, \tilde{v}) \in \mathbb{D}_r\) such that \((u_n, v_n) \to (\tilde{u}, \tilde{v})\) in \(\mathbb{D}_r\) up to subsequence for every \(0 < \nu \leq \nu_2\).

Proof. We observe that the functions \(\{(u_n, v_n)\} \subset \mathbb{D}_r\) are radial and therefore we can not have concentrations at points other than 0 or \(\infty\), otherwise, we will get a contradiction to the concentration–compactness principle by Bonder [4] as the set of concentration points is not a countable set.

Now if we want to avoid concentration at the points 0 and \(\infty\), by following the proof of Lemma 3.4 and Lemma 3.5, it is sufficient to show that (see (3.15))
\[ \lim_{k \to 0} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} h(x)|u_n|^\alpha|v_n|^\beta \Psi_{0,\epsilon}(x) \, dx = 0, \]  
(3.61)
\[ \lim_{R \to +\infty} \limsup_{n \to +\infty} \int_{|x| > R} h(x)|u_n|^\alpha|v_n|^\beta \Psi_{\infty,\epsilon}(x) \, dx = 0, \]  
(3.62)
where the cut-off function $\Psi_{0,\epsilon}$ is centred at 0 satisfying (3.14) and the cut-off function $\Psi_{\infty,\epsilon}$ supported near $\infty$ satisfying (3.25). For any $s_1, s_2 \in (0, 1)$, the assumption $\alpha + \beta = \min\{2s_1, 2s_2\}$ implies that $\frac{\alpha}{s_1} + \frac{\beta}{s_2} \leq 1$ and the equality holds if and only if $s_1 = s_2$.

If $\frac{\alpha}{s_1} + \frac{\beta}{s_2} < 1$, by using the assumption on $h$ in (H1) and the Hölder’s inequality, we get
\[
\int_{\mathbb{R}^N} h(x)|u_n|^\alpha|v_n|^\beta \Psi_{0,\epsilon}(x) \, dx = \int_{\mathbb{R}^N} h(x)\Psi_{0,\epsilon}(x) \left(1 - \frac{\alpha}{s_1} - \frac{\beta}{s_2}\right) (h(x)\Psi_{0,\epsilon}(x))^{\frac{\alpha}{s_1} + \frac{\beta}{s_2}} |u_n|^\alpha |v_n|^\beta \, dx \\
\leq \left( \int_{\mathbb{R}^N} h(x)\Psi_{0,\epsilon}(x) \, dx \right)^{1 - \frac{\alpha}{s_1} - \frac{\beta}{s_2}} \left( \int_{\mathbb{R}^N} h(x)|u_n|^{2s_1} \Psi_{0,\epsilon}(x) \, dx \right)^{\frac{\alpha}{s_1}} \\
\cdot \left( \int_{\mathbb{R}^N} h(x)|v_n|^{2s_2} \Psi_{0,\epsilon}(x) \, dx \right)^{\frac{\beta}{s_2}}.
\]

Now from (3.10) and (H1), we have
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} h(x)|u_n|^{2s_1} \Psi_{0,\epsilon}(x) \, dx = \int_{\mathbb{R}^N} h(x)|\tilde{u}|^{2s_1} \Psi_{0,\epsilon}(x) \, dx + \rho_0 h(0) \\
\leq \int_{|x| \leq \epsilon} h(x)|\tilde{u}|^{2s_1} \, dx, \text{ since } h(0) = 0.
\]

and
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} h(x)|v_n|^{2s_2} \Psi_{0,\epsilon}(x) \, dx = \int_{\mathbb{R}^N} h(x)|\tilde{v}|^{2s_2} \Psi_{0,\epsilon}(x) \, dx + \tilde{\rho}_0 h(0) \\
\leq \int_{|x| \leq \epsilon} h(x)|\tilde{v}|^{2s_2} \, dx, \text{ since } h(0) = 0.
\]

By combining the above three inequalities, we have
\[
\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} h(x)|u_n|^{\alpha}|v_n|^{\beta} \Psi_{0,\epsilon}(x) \, dx \leq \lim_{\epsilon \to 0} \left[ \left( \int_{|x| \leq \epsilon} h(x) \, dx \right)^{1 - \frac{\alpha}{s_1} - \frac{\beta}{s_2}} \left( \int_{|x| \leq \epsilon} h(x)|\tilde{u}|^{2s_1} \, dx \right)^{\frac{\alpha}{s_1}} \right] \\
\cdot \left( \int_{|x| \leq \epsilon} h(x)|\tilde{v}|^{2s_2} \, dx \right)^{\frac{\beta}{s_2}} = 0.
\]

If $\frac{\alpha}{s_1} + \frac{\beta}{s_2} = 1$, by using the fact that $h \in L^\infty(\mathbb{R}^N)$ and the Hölder’s inequality, we get
\[
\int_{\mathbb{R}^N} h(x)|u_n|^{\alpha}|v_n|^{\beta} \Psi_{0,\epsilon}(x) \, dx \leq \left( \int_{\mathbb{R}^N} h(x)|u_n|^{2s_1} \Psi_{0,\epsilon}(x) \, dx \right)^{\frac{\alpha}{s_1}} \left( \int_{\mathbb{R}^N} h(x)|v_n|^{2s_2} \Psi_{0,\epsilon}(x) \, dx \right)^{\frac{\beta}{s_2}}. \tag{3.63}
\]

Again using the above approach, we get
\[
\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} h(x)|u_n|^{\alpha}|v_n|^{\beta} \Psi_{0,\epsilon}(x) \, dx \leq \lim_{\epsilon \to 0} \left[ \left( \int_{|x| \leq \epsilon} h(x)|\tilde{u}|^{2s_1} \, dx \right)^{\frac{\alpha}{s_1}} \left( \int_{|x| \leq \epsilon} h(x)|\tilde{v}|^{2s_2} \, dx \right)^{\frac{\beta}{s_2}} \right] = 0.
\]

Similarly, we can prove (3.62) by using the assumption that $\lim_{|x| \to +\infty} h(x) = 0$. \hfill $\Box$

Next, we want to prove the Palais-Smale compactness condition for the case when $h$ is non-radial. For this purpose, we further assume that the parameter $\nu$ is sufficiently small and $s_1 = s_2 = s$.

\textbf{Lemma 3.9. Let us assume that } $\alpha + \beta = 2s$ and (H1), and $\{ (u_n, v_n) \}$ \textbf{be a (PS) sequence in } $D$ \textbf{for } $J_\nu$ \textbf{at level } $c \in \mathbb{R}$ \textbf{such that } $c$ \textbf{satisfies (3.9). Then, there exist } $(\tilde{u}, \tilde{v}) \in D$ \textbf{and } $\nu_0 > 0$ \textbf{such that } $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ \textbf{in } $D$ \textbf{up to subsequence for every } $0 < \nu \leq \nu_0$. \hfill $\Box$

\textbf{Proof.} First, we observe that the concentrations at points $0, \infty$ can be excluded due to (3.61) and (3.62) given in proof of Lemma 3.8. Therefore, we have only to take care of concentration points $x_j \neq 0, \infty$. Furthermore,
we can also assume that the index \( j \in J \cap K \). On contrary suppose that the concentration occurs at \( x_j \in \mathbb{R}^N \) with \( j \in J \cap K^c \) or \( x_k \in \mathbb{R}^N \) with \( k \in K \cap J^c \), then it is easy to prove as done before that

\[
\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} h(x) |u_n|^\alpha |v_n|^\beta \Psi_{j,\epsilon} \, dx = 0,
\]

for a smooth cut-off function \( \Psi_{j,\epsilon} \) centred at \( x_j \) satisfying (3.14). Thus, this concludes that concentrations can not occur at \( x_j \in \mathbb{R}^N \) with \( j \in J \cap K^c \) or \( x_k \in \mathbb{R}^N \) with \( k \in K \cap J^c \).

Now, we test the functional \( J'_\nu(u_n, v_n) \) with \((u_n \Psi_{j,\epsilon}, 0)\) with the assumption that \( j \in J \cap K \) and we obtain

\[
0 = \lim_{n \to +\infty} \langle J'_\nu(u_n, v_n) \mid (u_n \Psi_{j,\epsilon}, 0) \rangle \\
= \lim_{n \to +\infty} \left( \int \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \Psi_{j,\epsilon}(x) \, dx \, dy + \int \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\Psi_{j,\epsilon}(x) - \Psi_{j,\epsilon}(y))}{|x - y|^{N+2s}} u_n(y) \, dx \, dy - \lambda_1 \int \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2s}} \Psi_{j,\epsilon}(x) \, dx - \int \int_{\mathbb{R}^N} |u_n|^2 \Psi_{j,\epsilon}(x) \, dx - \nu \alpha \int \int_{\mathbb{R}^N} h(x) |u_n|^\alpha |v_n|^\beta \Psi_{j,\epsilon}(x) \, dx \right)
\]

Further, we test the functional \( J'_\nu(u_n, v_n) \) with \((0, v_n \Psi_{j,\epsilon})\) and we have the following

\[
0 = \lim_{n \to +\infty} \langle J'_\nu(u_n, v_n) \mid (0, v_n \Psi_{j,\epsilon}) \rangle \\
= \lim_{n \to +\infty} \left( \int \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x - y|^{N+2s}} \Psi_{j,\epsilon}(x) \, dx \, dy + \lambda_1 \int \int_{\mathbb{R}^N} \frac{v_n^2}{|x|^{2s}} \Psi_{j,\epsilon}(x) \, dx - \int \int_{\mathbb{R}^N} |v_n|^2 \Psi_{j,\epsilon}(x) \, dx - \nu \alpha \int \int_{\mathbb{R}^N} h(x) |u_n|^\alpha |v_n|^\beta \Psi_{j,\epsilon}(x) \, dx \right).
\]

By assumption \( h \in L^\infty(\mathbb{R}^N) \) and using the Hölder inequality (3.63), there exists some constant \( \tilde{C} > 0 \) such that the following inequality holds.

\[
\lim \limsup_{\epsilon \to 0} \int_{\mathbb{R}^N} h(x) |u_n|^\alpha |v_n|^\beta \Psi_{j,\epsilon} \, dx \leq \tilde{C} \rho_j^{\frac{\alpha}{\alpha + 2s}} \rho_j^{\frac{\beta}{\alpha + 2s}}.
\]

Hence, by letting \( \epsilon \to 0 \), from (3.64), (3.65) and (3.67) we get

\[
\mu_j - \rho_j - \nu \alpha \tilde{C} \rho_j^{\frac{\alpha}{\alpha + 2s}} \rho_j^{\frac{\beta}{\alpha + 2s}} \leq 0,
\]

\[
\bar{\rho}_j - \bar{\rho}_j - \nu \beta \tilde{C} \rho_j^{\frac{\alpha}{\alpha + 2s}} \rho_j^{\frac{\beta}{\alpha + 2s}} \leq 0.
\]

Then, by (3.11) together with (3.68) and (3.69), we find

\[
S \left( \rho_j^{\frac{\alpha}{\alpha + 2s}} + \bar{\rho}_j^{\frac{\beta}{\alpha + 2s}} \right) \leq (\rho_j + \bar{\rho}_j) (1 + 2^* \nu \tilde{C}).
\]

Therefore,

\[
S \left( \rho_j + \bar{\rho}_j \right)^{\frac{\alpha}{\alpha + 2s}} \leq (\rho_j + \bar{\rho}_j) (1 + 2^* \nu \tilde{C}).
\]
consequently, we obtain that either \( \rho_j + \bar{\rho}_j = 0 \) or \( \rho_j + \bar{\rho}_j \geq \left( \frac{s}{1 + 2^s \nu C} \right)^\frac{2}{\alpha + \beta} \). If we have a concentration at some point \( x_j \), then following the arguments of Lemma 3.4 we get

\[
c \geq \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) (\mu_j + \bar{\mu}_j) + \left( \frac{1}{\alpha + \beta} - \frac{1}{2^s} \right) (\rho_j + \bar{\rho}_j)
\]

\[
\geq S \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) (\rho_j + \bar{\rho}_j) + \left( \frac{1}{\alpha + \beta} - \frac{1}{2^s} \right) (\rho_j + \bar{\rho}_j)
\]

\[
\geq \frac{s}{N} \left( \frac{S}{1 + 2^s \nu C} \right)^\frac{2}{\alpha + \beta}
\]

If we assume that \( \nu > 0 \) is sufficiently small, then

\[
c \geq \frac{s}{N} \left( \frac{S}{1 + 2^s \nu C} \right)^\frac{2}{\alpha + \beta} \geq \frac{s}{N} \min \{S(\lambda_1), S(\lambda_2)\} \frac{2}{\alpha + \beta},
\]

which gives a contradiction to the hypothesis on level \( c \). This implies that \( \rho_j = 0 = \bar{\rho}_j \). Hence, the result follows from (3.67).

\[ \Box \]

4. EXISTENCE OF GROUND STATE AND BOUND STATE SOLUTIONS

In this section, we will state and prove the main results of the article concerning the positive ground and bound state solutions of the system (1.1). We will use the following hypotheses to prove the main results,

Either \( 2 < \alpha + \beta < \min \{2^{s_1}, 2^{s_2} \} \) and \( h \) satisfies (1.3)\n
or

\[ \alpha + \beta = \min \{2^{s_1}, 2^{s_2} \} \) and \( h \) is radial and satisfies (H1)\n
In the following theorem, we will prove the existence of a positive ground state solution of (1.1) for \( \nu \) large enough and \( h \) satisfying the assumption (4.1).

**Theorem 4.1.** If the hypothesis (4.1) is satisfied, then a positive ground state solution exists to the system (1.1) for \( \nu \) sufficiently large.

**Proof.** We know that for any \( (u, v) \in \mathbb{D} \setminus \{(0, 0)\} \), there exists a unique \( t = t(u,v) > 0 \) such that \( (tu, tv) \in \mathcal{N}_\nu \) and satisfying the algebraic equation

\[
\| (u, v) \|^2_B = t^{2^{s_1} - 2} \| u \|_{2^{s_1}}^{2^{s_1}} + t^{2^{s_2} - 2} \| v \|_{2^{s_2}}^{2^{s_2}} + \nu (\alpha + \beta) t^{\alpha + \beta - 2} \int_{\mathbb{R}^N} h(x) |u|^\alpha |v|^\beta \, dx.
\]

\[ t^{\alpha + \beta - 2} = \frac{\| (u, v) \|^2_B - t^{2^{s_1} - 2} \| u \|_{2^{s_1}}^{2^{s_1}} - t^{2^{s_2} - 2} \| v \|_{2^{s_2}}^{2^{s_2}}}{\nu (\alpha + \beta) \int_{\mathbb{R}^N} h(x) |u|^\alpha |v|^\beta \, dx},
\]

Clearly \( t = t_\nu \) tends to 0 as \( \nu \) increases to \( +\infty \) by using the fact \( \alpha + \beta > 2 \). Furthermore, we have

\[
\lim_{\nu \to +\infty} t_\nu^{\alpha + \beta - 2} \nu = \lim_{\nu \to +\infty} \frac{\| (u, v) \|^2_B - t_\nu^{2^{s_1} - 2} \| u \|_{2^{s_1}}^{2^{s_1}} - t_\nu^{2^{s_2} - 2} \| v \|_{2^{s_2}}^{2^{s_2}}}{(\alpha + \beta) \int_{\mathbb{R}^N} h(x) |u|^\alpha |v|^\beta \, dx},
\]

\[
\lim_{\nu \to +\infty} t_\nu^{\alpha + \beta - 2} \nu = \frac{\| (u, v) \|^2_B}{(\alpha + \beta) \int_{\mathbb{R}^N} h(x) |u|^\alpha |v|^\beta \, dx}, \text{ since } t_\nu \to 0 \text{ as } \nu \to +\infty.
\]
We also have the following,

\[
J_\nu(t_\nu u, t_\nu v) = \frac{1}{2} t_\nu^2 \| (u, v) \|_B^2 - \frac{t_{\nu_1}^2}{2 s_1} \int_{\mathbb{R}^N} |u|^{2^*_1} \, dx - \frac{t_{\nu_2}^2}{2 s_2} \int_{\mathbb{R}^N} |v|^{2^*_2} \, dx + \int_{\mathbb{R}^N} |u|^{2^*_1} \, dx + \int_{\mathbb{R}^N} |v|^{2^*_2} \, dx - \frac{1}{\alpha + \beta} t_\nu^2 \| (u, v) \|_B^2 +
\]

\[
+ \frac{t_{\nu_1}^2}{\alpha + \beta} \int_{\mathbb{R}^N} |u|^{2^*_1} \, dx + \frac{t_{\nu_2}^2}{\alpha + \beta} \int_{\mathbb{R}^N} |v|^{2^*_2} \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) t_\nu^2 \| (u, v) \|_B^2+
\]

\[
+ \left( \frac{1}{\alpha + \beta} - \frac{1}{2 s_1} \right) t_{\nu_1}^2 \int_{\mathbb{R}^N} |u|^{2^*_1} \, dx + \left( \frac{1}{\alpha + \beta} - \frac{1}{2 s_2} \right) t_{\nu_2}^2 \int_{\mathbb{R}^N} |v|^{2^*_2} \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{\alpha + \beta} + o_\nu(1) \right) t_\nu^2 \| (u, v) \|_B^2.
\]

The last expression is due to the fact that

\[
A = \frac{t_{\nu_1}^{-2} - \| (u, v) \|_B^2}{\| (u, v) \|_B^2} \left( \frac{1}{\alpha + \beta} - \frac{1}{2 s_1} \right) \int_{\mathbb{R}^N} |u|^{2^*_1} \, dx + \frac{t_{\nu_2}^{-2}}{\| (u, v) \|_B^2} \left( \frac{1}{\alpha + \beta} - \frac{1}{2 s_2} \right) \int_{\mathbb{R}^N} |v|^{2^*_2} \, dx = o_\nu(1),
\]

as \( \nu \to +\infty \).

Hence, \( J_\nu(t_\nu u, t_\nu v) = o(1) \) as \( \nu \to +\infty \) and there exists a \( \nu_0 > 0 \) such that, if \( \nu > \nu_0 \), where \( \nu_0 \) is sufficiently large, then

\[
\tilde{c}_\nu = \inf_{(u, v) \in \mathcal{N}_\nu} J_\nu(u, v) \leq J_\nu(t_\nu u, t_\nu v) < \min \left\{ J_\nu(z_{\nu_1}^{\lambda_1, \mu}, 0), J_\nu(0, z_{\nu_2}^{\lambda_2, \mu}) \right\}
\]

and hence from above we get the following inequality

\[
\tilde{c}_\nu < \min \left\{ \frac{s_1}{N} S^{1/2} (\lambda_1), \frac{s_2}{N} S^{1/2} (\lambda_2) \right\}.
\] (4.2)

For the case \( \alpha + \beta < \min\{2 s_1, 2 s_2\} \), we use Lemma 3.4 to ensure the existence of \((\tilde{u}, \tilde{v}) \in \mathcal{D} \) such that \( \tilde{c}_\nu = J_\nu(\tilde{u}, \tilde{v}) \). Next, we will show that the functions \( \tilde{u} \) and \( \tilde{v} \) are indeed positive. For that purpose, we define the function \( \psi(t) : (0, \infty) \to \mathbb{R} \) given by \( \psi(t) = J_\nu(t u, t v) \), for all \( t > 0 \). Then

\[
\psi(t) = \frac{A_1}{2} t^2 - \frac{A_2}{2^*_1} t^{2^*_1} - \frac{A_3}{2^*_2} t^{2^*_2} - A_4 t^\alpha t^\beta,
\]

\[
\psi'(t) = A_1 t - A_2 t^{2^*_1 - 1} - A_3 t^{2^*_2 - 1} - A_4 t^{\alpha + \beta - 1},
\]

\[
\psi''(t) = A_1 - A_2 (2 s_1 - 1) t^{2^*_1 - 2} - A_3 (2 s_2 - 1) t^{2^*_2 - 2} - A_4 t^{\alpha + \beta} (\alpha + \beta - 1) t^{\alpha + \beta - 2},
\]

\[
\psi'''(t) = -A_2 (s_1^2 - 1) t^{2^*_1 - 3} - A_3 (2 s_2 - 1) (2 s_2 - 2) t^{2^*_2 - 3}
\]

\[
- A_4 t^{\alpha + \beta} (\alpha + \beta - 1) (\alpha + \beta - 2) t^{\alpha + \beta - 3},
\]

where \( A_1 = \| (u, v) \|_B^2 \), \( A_2 = \| u \|_{2^*_1}^2 \), \( A_3 = \| v \|_{2^*_2}^2 \), \( A_4 = \int_{\mathbb{R}^N} h(x) |u|^\alpha |v|^\beta \, dx \). Clearly, \( A_1, A_2, A_3, A_4 \) are non-negative numbers. The condition \( \alpha + \beta > 2 \) implies that \( \psi'''(t) < 0 \). Therefore the function \( \psi'(t) \) is strictly concave for \( t > 0 \). Also, we have \( \lim_{t \to 0} \psi(t) = 0 \) and \( \lim_{t \to +\infty} \psi(t) = -\infty \). Moreover, the function \( \psi'(t) = 0 \) for \( t > 0 \) small enough. Hence, \( \psi'(t) \) has a unique global maximum point at \( t = t_0 \) and \( \psi'(t) \) has a unique root at \( t_0 \) for \( t > t_0 \), in particular \( \psi''(t_1) < 0 \). Also, from the equation (2.14) and \( \psi(t) \), we observe that \((tu, tv) \in \mathcal{N}_\nu \) if and only if \( \psi''(t) < 0 \).

Now we consider the function \((|\tilde{u}|, |\tilde{v}|) \in \mathcal{D} \), then from the above arguments there exists a unique \( t_2 > 0 \) such that \( t_2 (|\tilde{u}|, |\tilde{v}|) = (t_2 |\tilde{u}|, t_2 |\tilde{v}|) \in \mathcal{N}_\nu \) and \( t_2 \) satisfies the following algebraic equation

\[
|(|\tilde{u}|, |\tilde{v}|)|_B^2 = t_{\nu_1}^{-2} |\tilde{u}|_{2^*_1}^{2^*_1} + t_{\nu_2}^{-2} |\tilde{v}|_{2^*_2}^{2^*_2} + \nu(\alpha + \beta) t_2^{\alpha + \beta - 2} \int_{\mathbb{R}^N} h(x) |\tilde{u}|^\alpha |\tilde{v}|^\beta \, dx.
\]
Also we know that \((\tilde{u}, \tilde{v}) \in \mathcal{N}_\nu\), therefore
\[
\|\langle \tilde{u}, \tilde{v} \rangle \|^2 = \|\tilde{u}\|_{L^2(N)}^2 + \|\tilde{v}\|_{L^2(N)}^2 + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)|\tilde{u}|^\alpha |\tilde{v}|^\beta \, dx.
\]
Now from the inequality \(\|\langle \tilde{u}, \tilde{v} \rangle \|^2 \leq \|\langle \tilde{u}, \tilde{v} \rangle \|^2\), one finds that \(t_2 \leq 1\). Since \((\tilde{u}, \tilde{v}) \in \mathcal{N}_\nu\) and \((\tilde{u}, \tilde{v})\) is the unique maximum point of \(\psi(t) = J_\nu(t\tilde{u}, t\tilde{v}), \forall \ t > t_0\). We can deduce that
\[
\tilde{c}_\nu = J_\nu(\tilde{u}, \tilde{v}) = \max_{t > t_0} J_\nu(t\tilde{u}, t\tilde{v}) \geq J_\nu(t_2\tilde{u}, t_2\tilde{v}) = J_\nu(t_2|\tilde{u}|, t_2|\tilde{v}|) \geq \tilde{c}_\nu.
\]
Thus, we can assume that \(\tilde{u} \geq 0\) and \(\tilde{v} \geq 0\) in \(\mathbb{R}^N\). Moreover, \(\tilde{u}, \tilde{v}\) are not identically equal to 0. On contrary, if we suppose that \(\tilde{u} \equiv 0\), then \(\tilde{v}\) is a solution to the problem (3.42) and further \(\tilde{v} = \frac{z}{\nu_1, \nu_2}\), which is not possible due to the inequality (4.2). Following the same argument for \(\tilde{v} \equiv 0\), we get a contradiction to (4.2).

Furthermore, we conclude that \(\tilde{u} > 0\) and \(\tilde{v} > 0\) using the maximum principle in \(\mathbb{R}^N \setminus \{0\}\) [12, Theorem 1.2]. Hence, the existence of a positive ground state solution \((\tilde{u}, \tilde{v}) \in \mathcal{N}_\nu\) is proved.

In the case of \(\alpha + \beta = \min\{2s_1^*, 2s_2^*\}\), we follow the same argument as in the subcritical case and part (i) of Lemma 3.8 to conclude the existence of a positive ground state \((\tilde{u}, \tilde{v})\).

Next, we prove the existence of positive ground state solutions based on the order relation between \(S_{\frac{\nu}{2}}(\lambda_2)\) and \(S_{\frac{\nu}{2}}(\lambda_1)\), and considering the case \(\alpha \leq 2\) or \(\beta \leq 2\).

**Theorem 4.2.** Assume (4.1). The system (1.1) admits a positive ground state \((\tilde{u}, \tilde{v}) \in \mathcal{D}\) under one of the following hypotheses:

(i) \(S_{\frac{\nu}{2}}(\lambda_2) \geq S_{\frac{\nu}{2}}(\lambda_1), s_2 \geq s_1\) and either \(\beta < 2\) or \(\beta = 2\) and \(\nu\) large enough,

(ii) \(S_{\frac{\nu}{2}}(\lambda_1) \geq S_{\frac{\nu}{2}}(\lambda_2), s_1 \geq s_2\) and either \(\alpha < 2\) or \(\alpha = 2\) and \(\nu\) large enough.

**Proof.** (i) Under one of the hypotheses i.e. either \(\beta < 2\) or \(\beta = 2\) and \(\nu\) large enough, Proposition 3.7 states that \((z^{\lambda_1}_{\mu_1, s_1}, 0)\) is a saddle point for the functional \(J_\nu\) restricted on \(\mathcal{N}_\nu\). Since \(S_{\frac{\nu}{2}}(\lambda_2) \geq S_{\frac{\nu}{2}}(\lambda_1)\) and \(s_2 \geq s_1\), we have
\[
\tilde{c}_\nu < J_\nu(z^{\lambda_1}_{\mu_1, s_1}, 0) = \frac{s_1}{N} S_{\frac{\nu}{2}}(\lambda_1) = \min \left\{ \frac{s_1}{N} S_{\frac{\nu}{2}}(\lambda_1), \frac{s_2}{N} S_{\frac{\nu}{2}}(\lambda_2) \right\},
\]
where \(\tilde{c}_\nu\) is defined by (2.17). In the case of \(\alpha + \beta < \min\{2s_1^*, 2s_2^*\}\), the Lemma 3.4 ensures the existence \((\tilde{u}, \tilde{v}) \in \mathcal{N}_\nu\) such that \(\tilde{c}_\nu = J_\nu(\tilde{u}, \tilde{v})\). Now we can assume that \(\tilde{u}, \tilde{v} \geq 0\) and \((\tilde{u}, \tilde{v})\) are not identically 0 using the same argument as in Theorem 4.1. Then, we use the maximum principle by Pezzo and Quaas [12, Theorem 1.2] to conclude that \((\tilde{u}, \tilde{v})\) is a positive ground state solution of (1.1).

In the case of \(\alpha + \beta = \min\{2s_1^*, 2s_2^*\}\) and \(h\) radial, we use part (i) of Lemma 3.8 and deduce the existence of a positive ground state \((\tilde{u}, \tilde{v})\) of (1.1).

(ii) We can also deduce the same conclusion for this part by repeating an analogous argument.

The following theorem proves that the semi-trivial solutions are the ground state if the order relation between \(S_{\frac{\nu}{2}}(\lambda_2)\) and \(S_{\frac{\nu}{2}}(\lambda_1)\) is strict with \(\alpha \geq 2\) or \(\beta \geq 2\) and \(\nu\) sufficiently small.

**Theorem 4.3.** Assume (4.1). Then the following statements are true:

(i) If \(S_{\frac{\nu}{2}}(\lambda_1) > S_{\frac{\nu}{2}}(\lambda_2), s_1 \geq s_2\) and \(\alpha \geq 2\), then there exists \(\nu_0 > 0\) such that for any \(0 < \nu < \nu_0\) the pair \((0, z^{\lambda_1}_{\mu_1, s_2})\) is a ground state of (1.1).

(ii) If \(S_{\frac{\nu}{2}}(\lambda_2) > S_{\frac{\nu}{2}}(\lambda_1), s_2 \geq s_1\) and \(\beta \geq 2\), then there exists \(\nu_0 > 0\) such that for any \(0 < \nu < \nu_0\) the pair \((z^{\lambda_1}_{\mu_1, s_1}, 0)\) is a ground state of (1.1).

**Proof.** (i) We recall that the pair \((0, z^{\lambda_1}_{\mu_1, s_2})\) is a local minimum of \(J_\nu\) over \(\mathcal{N}_\nu\) by Proposition 3.7 provided \(\nu\) sufficiently small. Then \(c_\nu \leq J_\nu(0, z^{\lambda_1}_{\mu_1, s_2})\). Indeed, we want to prove that equality holds. On contrary,
we suppose that there exists a sequence \( \{\nu_n\} \) decreasing to 0 satisfying \( \tilde{c}_{\nu_n} < J_{\nu_n}(0, z_{\mu,s_2}) \). By assumption \( S_{NS}^{\frac{a}{N}}(\lambda_1) > S_{NS}^{\frac{b}{N}}(\lambda_2) \), \( s_1 \geq s_2 \), we have the following inequality

\[
\tilde{c}_{\nu_n} < \frac{s_2}{N} S_{NS}^{\frac{b}{N}}(\lambda_2) = \min \left\{ \frac{s_1}{N} S_{NS}^{\frac{a}{N}}(\lambda_1), \frac{s_2}{N} S_{NS}^{\frac{b}{N}}(\lambda_2) \right\},
\]

(4.3)

where \( \tilde{c}_{\nu_n} \) is given in (2.17) with \( \nu = \nu_n \). If \( \alpha + \beta < \min\{2^*_s, 2^*_{s_2}\} \), the (PS) condition holds by Lemma 3.4 at level \( \tilde{c}_{\nu_n} \). For the case \( \alpha + \beta = \min\{2^*_s, 2^*_{s_2}\} \), we apply part (i) of Lemma 3.8 for \( h \) radial to reach the same conclusion. Hence, for each \( n \in \mathbb{N} \), there exists \((\tilde{u}_n, \tilde{v}_n) \in \mathbb{D} \) which solves (1.1) such that \( \tilde{c}_{\nu_n} = J_{\nu_n}(\tilde{u}_n, \tilde{v}_n) \). By following the same argument as in Theorem 4.1, we can assume that \( \tilde{u}_n \geq 0 \) and \( \tilde{v}_n \geq 0 \). Moreover, \( \tilde{u}_n \neq 0 \) and \( \tilde{v}_n \neq 0 \) in \( \mathbb{R}^N \) as the assumption either \( \tilde{u}_n \equiv 0 \) or \( \tilde{v}_n \equiv 0 \) contradicts (4.3). Indeed, we can conclude by the maximum principle of Pezzo and Quaas [12, Theorem 1.2] that \( \tilde{u}_n > 0 \) and \( \tilde{v}_n > 0 \) in \( \mathbb{R}^N \\setminus \{0\} \). Further, Let us define the following integrals

\[
\sigma_{s_1,n} = \int_{\mathbb{R}^N} \tilde{u}_n^{2^*_s} \, dx \quad \text{and} \quad \sigma_{s_2,n} = \int_{\mathbb{R}^N} \tilde{v}_n^{2^*_s} \, dx.
\]

By (2.12), we obtain

\[
\tilde{c}_{\nu_n} = J_{\nu_n}(\tilde{u}_n, \tilde{v}_n) = \frac{s_1}{N} \sigma_{s_1,n} + \frac{s_2}{N} \sigma_{s_2,n} + \nu_n \left( \alpha + \beta - \frac{2}{2} \right) \int_{\mathbb{R}^N} h(x) \tilde{u}_n^\alpha \tilde{v}_n^\beta \, dx.
\]

From (4.3), (4.4) and (1.3), we obtain that

\[
\sigma_{s_1,n} + \sigma_{s_2,n} < S_{NS}^{\frac{a}{N}}(\lambda_2).
\]

(4.5)

We combine the definition of \( S(\lambda_1) \) with the first equation in the system (1.1) as we have given that \((\tilde{u}_n, \tilde{v}_n)\) is a solution to (1.1). We deduce

\[
S(\lambda_1)(\sigma_{s_1,n})^{\frac{N-2s_1}{N}} \leq \sigma_{s_1,n} + \nu_n \alpha \int_{\mathbb{R}^N} h(x) \tilde{u}_n^\alpha \tilde{v}_n^\beta \, dx.
\]

(4.6)

Now by the Hölder’s inequality and the inequality (4.5), one finds that

\[
\int_{\mathbb{R}^N} h(x) \tilde{u}_n^\alpha \tilde{v}_n^\beta \, dx \leq C(h) \left( \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*_s} \, dx \right)^{\frac{\alpha}{2^*_s}} \left( \int_{\mathbb{R}^N} |\tilde{v}_n|^{2^*_s} \, dx \right)^{\frac{\beta}{2^*_s}} \leq C(h)(S(\lambda_2))^{\frac{\alpha}{2^*_s}}(\sigma_{s_1,n})^{\frac{\alpha}{2^*_s}}.
\]

Using the above expression in (4.6), we get the following

\[
S(\lambda_1)(\sigma_{s_1,n})^{\frac{N-2s_1}{N}} < \sigma_{s_1,n} + \nu_n \alpha C(h)(S(\lambda_2))^{\frac{\alpha}{2^*_s}}(\sigma_{s_1,n})^{\frac{\alpha}{2^*_s}}.
\]

Since \( S_{NS}^{\frac{a}{N}}(\lambda_1) > S_{NS}^{\frac{b}{N}}(\lambda_2) \), then there exists \( \epsilon > 0 \) such that

\[
(1 - \epsilon) S_{NS}^{\frac{a}{N}}(\lambda_1) \geq S_{NS}^{\frac{b}{N}}(\lambda_2).
\]

(4.7)

Since \( \alpha \geq 2 \), by using Lemma 3.3 with \( \sigma = \sigma_{s_1,n} \), there exists a \( \nu_0 = \nu_0(\epsilon) > 0 \) such that

\[
\sigma_{s_1,n} \geq (1 - \epsilon) S_{NS}^{\frac{a}{N}}(\lambda_1) \quad \text{for any} \ 0 < \nu_n < \nu_0.
\]

By using (4.7), one finds that \( \sigma_{s_1,n} \geq S_{NS}^{\frac{a}{N}}(\lambda_2) \), which gives a contradiction to the inequality (4.5). Hence, we have

\[
\tilde{c}_\nu = \frac{s_2}{N} S_{NS}^{\frac{b}{N}}(\lambda_2) = J_\nu(0, z_{\mu,s_2}^\lambda),
\]

(4.8)

provided \( \nu \) sufficiently small. Thus, the pair \((0, z_{\mu,s_2}^\lambda)\) is a ground state of (1.1) for \( \nu \) small enough. Similarly, we can prove part (ii).

Now for dealing with the case \( \alpha + \beta = \min\{2^*_s, 2^*_{s_2}\} \) with non-radial \( h \), we set \( s_1 = s_2 = s \) and \( \nu \) sufficiently small. We have the following results.

**Theorem 4.4.** Assume that \( \alpha + \beta = 2^*_s \) with \( \nu \) sufficiently small and \( h \) is a non-radial function. Then the system (1.1) has a positive ground state solution \((\tilde{u}, \tilde{v}) \in \mathbb{D} \) under one of the following hypotheses:

(i) \( S_{NS}^{\frac{a}{N}}(\lambda_2) \geq S_{NS}^{\frac{b}{N}}(\lambda_1) \) and \( \beta < 2 \),

(ii) \( S_{NS}^{\frac{a}{N}}(\lambda_1) \geq S_{NS}^{\frac{b}{N}}(\lambda_2) \) and \( \alpha < 2 \).
Proof. The proof is direct by using the approach of Theorem 4.2 and Lemma 3.9. □

**Theorem 4.5.** Assume that \( \alpha + \beta = 2^*_s \) with \( \nu \) sufficiently small and \( h \) is a non-radial function. Then the following holds:

(i) If \( S^{\frac{N}{N-2}}(\lambda_1) > S^{\frac{N}{N-2}}(\lambda_2) \) and \( \alpha \geq 2 \), then the pair \((0, z_{\lambda_2}^{\alpha})\) is a ground state of (1.1).

(ii) If \( S^{\frac{N}{N-2}}(\lambda_2) > S^{\frac{N}{N-2}}(\lambda_1) \) and \( \beta \geq 2 \), then the pair \((z_{\lambda_1}^{\beta}, 0)\) is a ground state of (1.1).

Proof. The proof follows the approach of Theorem 4.3 and Lemma 3.9. □

In the next theorem, we show the existence of a positive bound state solution of the Mountain pass type.

**Theorem 4.6.** Assume (4.1) with \( s_1 = s_2 = s \). If

(i) Either

\[
\alpha \geq 2 \quad \text{and} \quad \frac{1}{2} < \left( \frac{N}{N-2} \right) S^{\frac{N}{N-2}}(\lambda_2) < 1,
\]

(ii) or

\[
\beta \geq 2 \quad \text{and} \quad \frac{1}{2} < \left( \frac{N}{N-2} \right) S^{\frac{N}{N-2}}(\lambda_1) < 1,
\]

then for \( \nu \) sufficiently small, there exists a Mountain pass type positive bound state solution to the problem (1.1).

Proof. (i) We start with constructing a Mountain pass level so that the functional \( J^+_\nu \) restricted on \( N^+_\nu \) satisfies the Mountain pass geometry, and the Palais-Smale condition is also satisfied at this level. So we consider the set of paths connecting continuously \((z_{\lambda_1}^{\alpha}, 0)\) to \((0, z_{\lambda_2}^{\beta})\), namely

\[
\Sigma_\nu = \{ \varphi(t) = (\varphi_1(t), \varphi_2(t)) \in C^0([0, 1], N^+_\nu) : \varphi(0) = (z_{\lambda_1}^{\alpha}, 0) \text{ and } \varphi(1) = (0, z_{\lambda_2}^{\beta}) \},
\]

and define the associated Mountain pass level as

\[
C_{\text{MP}} = \inf_{\varphi \in \Sigma_\nu} \max_{t \in [0, 1]} J^+_\nu(\varphi(t)).
\]

Assumption (4.9) implies that

\[
2 \frac{s}{N} S^{\frac{N}{N-2}}(\lambda_2) > \frac{s}{N} S^{\frac{N}{N-2}}(\lambda_1).
\]

Further, by the monotonicity of \( S(\lambda) \), we can choose \( \epsilon > 0 \) sufficiently small such that

\[
2 \frac{s}{N} (1 - \epsilon) \left( \frac{S(\lambda_1) + S(\lambda_2)}{2} \right) \frac{\alpha}{N} > 2 \frac{s}{N} S^{\frac{N}{N-2}}(\lambda_2) > \frac{s(1 + \epsilon)}{N} S^{\frac{N}{N-2}}(\lambda_1).
\]

Now we claim the existence of a \( \nu_0 = \nu_0(\epsilon) > 0 \) such that the following inequality

\[
\max_{t \in [0, 1]} J^+_\nu(\varphi(t)) \geq 2 \frac{s}{N} (1 - \epsilon) \left( \frac{S(\lambda_1) + S(\lambda_2)}{2} \right) \alpha
\]

with \( \varphi \in \Sigma_\nu, \) holds for every \( 0 < \nu < \nu_0 \). If \( \varphi = (\varphi_1, \varphi_2) \in \Sigma_\nu \), then by using identity (3.33), we have

\[
\int_{\mathbb{R}^N} \left| \varphi_1(x) - \varphi_1(y) \right|^2 + \left| \varphi_2(x) - \varphi_2(y) \right|^2 \frac{dx dy}{|x - y|^{N+2s}} - \lambda_1 \int_{\mathbb{R}^N} \frac{\varphi_1^2}{|x|^{2s}} dx - \lambda_2 \int_{\mathbb{R}^N} \frac{\varphi_2^2}{|x|^{2s}} dx
\]

\[
= \int_{\mathbb{R}^N} \left( \left( \varphi_1^+ (t) \right)^{2^*} + \left( \varphi_2^+ (t) \right)^{2^*} \right) dx + \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x) (\varphi_1^+(t))^\alpha (\varphi_2^+(t))^\beta dx,
\]

and we use (3.34) to get

\[
J^+_\nu(\varphi(t)) = \frac{s}{N} \int_{\mathbb{R}^N} \left( \left( \varphi_1^+ (t) \right)^{2^*} + \left( \varphi_2^+ (t) \right)^{2^*} \right) dx
\]

\[
+ \nu \left( \frac{\alpha + \beta - 2}{2} \right) \int_{\mathbb{R}^N} h(x) (\varphi_1^+(t))^\alpha (\varphi_2^+(t))^\beta dx.
\]
Now we define \( \sigma_s(t) = (\sigma_{1,s}(t), \sigma_{2,s}(t)) \) with \( \sigma_{j,s}(t) = \int_{\mathbb{R}^N} (\varphi_j(t))^2 \, dx \) for \( j = 1, 2 \). Observe that if \( \sigma_{j,s}(t) > 2S^{\frac{N}{2}}(\lambda_j) \), then the inequality (4.12) holds. Therefore, we assume that \( \sigma_{j,s}(t) \leq 2S^{\frac{N}{2}}(\lambda_j), j = 1, 2 \) for all \( t \in [0, 1] \). We combine the definition of \( S(\lambda) \) with (4.13) and obtain

\[
S(\lambda_1(\sigma_{1,s}(t)))^{\frac{N-2s}{2}} + S(\lambda_2(\sigma_{2,s}(t)))^{\frac{N-2s}{2}} \leq \int_{\mathbb{R}^N} \frac{|\varphi_1(x) - \varphi_1(y)|^2 + |\varphi_2(x) - \varphi_2(y)|^2}{|x-y|^{N+2s}} \, dxdy \\
- \lambda_1 \int_{\mathbb{R}^N} \frac{\varphi_1^2}{|x|^{2s}} \, dx - \lambda_2 \int_{\mathbb{R}^N} \frac{\varphi_2^2}{|x|^{2s}} \, dx \\
= \sigma_{1,s}(t) + \sigma_{2,s}(t) \\
+ \nu(\alpha + \beta) \int_{\mathbb{R}^N} h(x)(\varphi_{1}^2(t))^{\alpha}(\varphi_2^2(t))^{\beta} \, dx.
\]

Using the Hölder’s inequality, we get

\[
\int_{\mathbb{R}^N} h(x)(\varphi_{1}^2(t))^{\alpha}(\varphi_2^2(t))^{\beta} \, dx \leq C(h)(\sigma_{1,s}(t))^\frac{N-2s}{N} (\sigma_{2,s}(t))^\frac{N-2s}{N}.
\]

Also, by the definition of \( \Sigma_\nu \) and since \( \varphi = (\varphi_1, \varphi_2) \in \Sigma_\nu \), we have

\[
\sigma_s(0) = \left( \int_{\mathbb{R}^N} (z_{\mu,s}^{\lambda_1})^{2s} \, dx, 0 \right) \quad \text{and} \quad \sigma_s(1) = \left( 0, \int_{\mathbb{R}^N} (z_{\mu,s}^{\lambda_2})^{2s} \, dx \right).
\]

Thus, by the continuity of \( \sigma_s \), there is a \( t_0 \in (0, 1) \) such that \( \sigma_{1,s}(t_0) = \tilde{\sigma}_s = \sigma_{2,s}(t_0) \). Combining (4.15) and (4.16), and taking \( t = t_0 \), we deduce the following

\[
(S(\lambda_1) + S(\lambda_2))^{\frac{N-2s}{N}} \leq 2\tilde{\sigma}_s + C\nu(\alpha + \beta)\sigma_{2,s}^{\frac{N-2s}{N}}.
\]

Now by using Lemma 3.3, there exists a \( \nu_0 = \nu_0(\epsilon) \) such that

\[
\tilde{\sigma}_s \geq (1 - \epsilon) \left( \frac{S(\lambda_1) + S(\lambda_2)}{2} \right)^\frac{N}{N-2s} \quad \text{for every } 0 < \nu \leq \nu_0.
\]

Consequently, we combine (4.14) and (4.17) to get

\[
\max_{t \in [0, 1]} J^+_{\nu}(\varphi(t)) \geq \frac{s}{N} (\sigma_{1,s}(t_0) + \sigma_{2,s}(t_0)) \geq \frac{2s}{N} (1 - \epsilon) \left( \frac{S(\lambda_1) + S(\lambda_2)}{2} \right)^\frac{N}{N-2s},
\]

which proves claim (4.12). Moreover, by (4.11) and (4.12), one can state that

\[
C_{\text{MP}} > \frac{s(1+\epsilon)}{N} S^{\frac{N}{4}}(\lambda_1) = (1+\epsilon)J^+_{\nu}(z_{\mu,s}^{\lambda_1}, 0).
\]

Thus, the functional \( J^+_{\nu} \) admits a Mountain-Pass-geometry on \( \mathcal{N}_{\nu} \).

Now we show that the Palais-Smale compactness condition is satisfied at the Mountain pass level \( C_{\text{MP}} \). We consider \( \varphi(t) = (\varphi_1(t), \varphi_2(t)) = ((1-t)^{1/2}z_{\mu,s}^{\lambda_1}, t^{1/2}z_{\mu,s}^{\lambda_2}) \) for \( t \in [0, 1] \). By the definition of the Nehari manifold, there exists a continuous positive function \( \eta : [0, 1] \to (0, +\infty) \) such that the \( \eta \varphi \in \mathcal{N}_{\nu} \cap \mathcal{N}_{\nu}^c \) for \( t \in [0, 1] \). We notice that \( \eta(0) = \eta(1) = 1 \).

Now we define

\[
\sigma_s(t) = (\sigma_{1,s}(t), \sigma_{2,s}(t)) = \left( \int_{\mathbb{R}^N} (\eta \varphi_1(t))^2 \, dx, \int_{\mathbb{R}^N} (\eta \varphi_2(t))^2 \, dx \right).
\]

Then, we have

\[
\sigma_{1,s}(0) = \int_{\mathbb{R}^N} (z_{\mu,s}^{\lambda_1})^{2s} \, dx = S^{\frac{N}{2}}(\lambda_1) \quad \text{and} \quad \sigma_{2,s}(1) = \int_{\mathbb{R}^N} (z_{\mu,s}^{\lambda_2})^{2s} \, dx = S^{\frac{N}{2}}(\lambda_2).
\]

Since \( \eta \varphi(t) \in \mathcal{N}_{\nu}^c \), by using the algebraic equation (2.13), we obtain

\[
\| ((1-t)^{1/2}z_{\mu,s}^{\lambda_1}, t^{1/2}z_{\mu,s}^{\lambda_2}) \|_{\mathbb{B}}^2 = \eta^{\frac{N}{2}-2}(1-t)^{\frac{N}{2}-2}\sigma_{1,s}(0) + t^{\frac{N}{2}}\sigma_{2,s}(1) \\
+ \nu(\alpha + \beta)(\eta(t))^{\alpha+\beta-2}(1-t)^{\alpha/2+\beta/2} \int_{\mathbb{R}^N} h(x)(z_{\mu,s}^{\lambda_1})^\alpha(z_{\mu,s}^{\lambda_2})^\beta \, dx,
\]

where \( \mathcal{N}_{\nu}^c \) is the Nehari manifold of the functional \( I_{\nu} \) for \( \nu \).
and therefore,
\[ \eta^2(t)^{-2} < \frac{\|(\varphi_1(t), \varphi_2(t))\|^2}{\int_{\mathbb{R}^N} ((\varphi_1(t))^{2^*_s} + (\varphi_2(t))^{2^*_s}) \, dx} = \frac{(1-t)\sigma_{1,s}(0) + t\sigma_{2,s}(1)}{(1-t)^{2^*_s/2}\sigma_{1,s}(0) + t^{2^*_s/2}\sigma_{2,s}(1)} \]  
for every \( t \in (0,1) \). It is followed by the definition of \( \eta \), (2.16) and (4.20) that

\[ J^+_p(\eta\varphi(t)) = \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) \|\eta\varphi(t)\|^2 
+ \left( \frac{1}{\alpha + \beta} - \frac{1}{2^*_s} \right) \eta^{2^*_s}(t) \left( \int_{\mathbb{R}^N} ((\varphi_1(t))^{2^*_s} + (\varphi_2(t))^{2^*_s}) \, dx \right) 
\]

\[ = \eta^2(t) \left( \frac{1}{2} - \frac{1}{\alpha + \beta} \right) [(1-t)\sigma_{1,s}(0) + t\sigma_{2,s}(1)] 
+ \left( \frac{1}{\alpha + \beta} - \frac{1}{2^*_s} \right) \eta^{2^*_s}(t) [(1-t)^{2^*_s/2}\sigma_{1,s}(0) + t^{2^*_s/2}\sigma_{2,s}(1)] 
\]

\[ < \frac{s\eta^2(t)}{N} [(1-t)\sigma_{1,s}(0) + t\sigma_{2,s}(1)] \]

Then, by (4.20) and (4.21), and for every \( t \in (0,1) \), we obtain that

\[ J^+_p(\eta\varphi(t)) < G(t) := \frac{s}{N} [(1-t)\sigma_{1,s}(0) + t\sigma_{2,s}(1)] \left[ \frac{(1-t)\sigma_{1,s}(0) + t\sigma_{2,s}(1)}{(1-t)^{2^*_s/2}\sigma_{1,s}(0) + t^{2^*_s/2}\sigma_{2,s}(1)} \right]^{\frac{N-2s}{2s}}. \]

Clearly, the function \( G(t) \) is maximum at point \( t = \frac{1}{2} \). Also, from (4.19), we have

\[ G\left( \frac{1}{2} \right) = \frac{s}{N} (\sigma_{1,s}(0) + \sigma_{2,s}(1)) = \frac{s}{N} (S_{\hat{u}}(\lambda_1) + S_{\hat{v}}(\lambda_2)). \]

we conclude

\[ \mathcal{C}_{M_P} \leq \max_{t \in [0,1]} J^+_p(\eta\varphi(t)) < \frac{s}{N} (S_{\hat{u}}(\lambda_1) + S_{\hat{v}}(\lambda_2)). \]

If \( S_{\hat{u}}(\lambda_1) > S_{\hat{v}}(\lambda_2) \), using the separability condition (4.10) and the inequality (4.18), it follows that

\[ \frac{s}{N} S_{\hat{u}}(\lambda_2) < \frac{s}{N} S_{\hat{u}}(\lambda_1) < \mathcal{C}_{M_P} < \frac{s}{N} (S_{\hat{u}}(\lambda_1) + S_{\hat{v}}(\lambda_2)) < \frac{3s}{N} S_{\hat{u}}(\lambda_2). \]

From the above expression, it is clear that the Mountain pass level \( \mathcal{C}_{M_P} \) satisfies the assumptions of Lemma 3.5 and Lemma 3.8. Therefore, by the Mountain-Pass theorem, the functional \( J^+_p|_{\mathcal{N}_{\epsilon}^+} \) admits a Palais-Smale sequence \( \{(u_n, v_n)\} \subset \mathcal{N}_{\epsilon}^+ \) at level \( \mathcal{C}_{M_P} \).

For the subcritical case \( \alpha + \beta < 2^*_s \), from analogous versions of Lemmas 3.1 and Lemma 3.5 for the functional \( J^+_p \), we imply that the sequence \( \{(u_n, v_n)\} \) has a subsequence which strongly converges to a critical point \( (\hat{u}, \hat{v}) \) of \( J^+_p \) on \( \mathcal{N}_{\epsilon}^+ \). Therefore, it is also a critical point of \( J^+_p \) defined in \( \mathbb{D} \). Further, we have \( \hat{u}, \hat{v} \geq 0 \) in \( \mathbb{R}^N \) and \( \hat{u}, \hat{v} \neq (0,0) \). Indeed, we can conclude by the maximum principle of Pezzo and Quaas [12, Theorem 1.2] that \( \hat{u} > 0 \) and \( \hat{v} > 0 \) in \( \mathbb{R}^N \setminus \{0\} \). Hence, \( (\hat{u}, \hat{v}) \) is a bound state solution to the system (1.1). To deal with the critical case, i.e., \( \alpha + \beta = 2^*_s \), we follow the same approach for the compactness of the Palais-Smale sequence using Lemma 3.8.

(ii) Similarly, this part can be proved using Lemma 3.6 and Lemma 3.8. \( \square \)

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