Linearized gravity on branes:
from Newton’s law to cosmological perturbations

Nathalie Deruelle

Institut d’Astrophysique de Paris,
GReCO, FRE 2435 du CNRS,
98 bis boulevard Arago, 75014, Paris, France

November 2002

Contribution to the Proceedings of the Menorca ERE

Abstract

We review here how Newton's law can be approximately recovered in the simple, "paradigmatic", case of a flat $Z_2$-symmetric brane in a 5-D anti-de Sitter bulk. We then comment on the difficulties encountered so far in extending this analysis to cosmological perturbations on a Robertson-Walker brane.
I. Introduction

Since the seminal papers by Randall and Sundrum [1] there has been an increasing interest for gravity theories within spacetimes with one large extra dimension and the idea that our universe may be a four dimensional singular hypersurface, or “brane”, in a five dimensional spacetime, or “bulk”. The Randall-Sundrum scenario [1], where our universe is a four dimensional quasi-Minkowskian edge of a double-sided perturbed anti-de Sitter spacetime, was the first explicit model where the linearized Einstein equations were found to hold on the brane, apart from small $1/r^2$ corrections to Newton’s potential. This claim was thoroughly analyzed and the corrections to Newton’s law exactly calculated [2].

Cosmological models were then built, where the brane is taken to be a Robertson-Walker spacetime embedded in an anti-de Sitter bulk [3]. The perturbations of these models, in the view of calculating the microwave background anisotropies, are currently being studied and compared to the perturbations of standard, four dimensional, Friedmann universes [4].

In this contribution we review the approach advocated in [5].

II. The bulk gravitons

In the conformally Minkowskian coordinates $X^A = \{x^\mu = [T, \vec{r} = (x^1, x^2, x^3)], X^4 = w\}$ the line element of a five dimensional perturbed anti-de Sitter spacetime reads

$$ds^2|_5 = G_{AB}dX^AdX^B \quad \text{with} \quad G_{AB} = \left(\frac{L}{w}\right)^2(\eta_{AB} + \gamma_{AB})$$

where $L$ is a (positive) constant and $\gamma_{AB}$ fifteen functions of the five coordinates $X^A$. $V_5$ is defined as the part of this perturbed AdS$_5$ spacetime bounded by a 4D timelike hypersurface $\Sigma$ and $w = +\infty$ ; $\Sigma$ is such that the coordinates $X^A$ cover the whole of $V_5$.

$V_5$ is taken to be an Einstein space, that is a solution of the Einstein equations

$$R_{AB} = -\frac{4}{L^2}G_{AB}$$

where $R_{AB}$ is the Ricci tensor of the metric $G_{AB}$. These equations are, at linear order in $\gamma_{AB}$ :

$$\frac{1}{2}\left[\partial_AL\gamma^L_B + \partial_BL\gamma^L_A - \partial_AB\gamma - \Box_5\gamma_{AB} - \eta_{AB}(\partial_L\gamma^L - \Box_5\gamma)\right]$$

$$-\frac{6}{w^2}\eta_{AB}\gamma_{ww} - \frac{3}{2w}\left[\partial_A\gamma_{wB} + \partial_B\gamma_{wA} - \partial_w\gamma_{AB} + \eta_{AB}(\partial_w\gamma - 2\partial_L\gamma^L_w)\right] = 0$$

where all indices are raised with $\eta^{AB}$, where $\gamma = \gamma^L_L$ and where $\Box_5 = \partial_L\partial^L$.

In the gauge/coordinate transformation $X^A \to X^A = \tilde{X}^A + \epsilon^A(X^B)$, the metric coefficients transform at first order as $\gamma_{AB} \to \tilde{\gamma}_{AB} = \gamma_{AB} + \partial_A\epsilon_B + \partial_B\epsilon_A - 2\eta_{AB}\frac{\epsilon^4}{w}$ and it is easy to see that one can choose five functions $\epsilon^A$ such that (with tildes dropped)

$$\gamma_{Aw} = 0.$$

These conditions reduce the fifteen metric coefficients to the ten $\gamma_{\mu\nu}$ but do not fix the gauge completely, as they do not uniquely determine the five functions $\epsilon^A$. Indeed, if we perform the further coordinate transformation $X^A \to X^A = \tilde{X}^A + \epsilon^A$ with $\epsilon^4 = wd$
and $\epsilon_\mu = -\frac{1}{2}w^2 \partial_\mu d + c_\mu$, where $d(x^\nu)$ and $c_\mu(x^\nu)$ are five arbitrary functions of the four coordinates $x^\nu$, then the new metric coefficients satisfy $\tilde{\gamma}_{Aw} = 0$ as well, and

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - w^2 \partial_\mu d + \partial_\mu c_\nu + \partial_\nu c_\mu - 2\eta_{\mu\nu} d. \quad (2.4)$$

In the (hence) class of coordinate systems (2.3-4) the Einstein linearised equations (2.2) reduce to (with tildes dropped)

$$\partial_\rho \gamma^{\rho\sigma} - \Box_4 \gamma + \frac{3}{w} \partial_w \gamma = 0, \quad \partial_w (\partial_\rho \gamma_\rho^\mu - \partial_\mu \gamma) = 0, \quad \partial_{ww} \gamma - \frac{1}{w} \partial_w \gamma = 0 \quad (2.5)$$

where now all indices are raised with $\eta^{\mu\nu}$, $\gamma \equiv \gamma^{\mu^\mu}$ and $\Box_4 \equiv \partial_\mu \partial^\mu$.

The first three, constraint, equations are easily solved :

$$\gamma = -\frac{1}{6} w^2 \partial_\mu D^\mu + C, \quad \partial_\rho \gamma_\rho^\mu = -\frac{1}{6} w^2 \partial_\mu D^\rho + \partial_\mu C + D_\mu \quad (2.6)$$

where $C(x^\nu)$ and $D_\mu(x^\nu)$ are five arbitrary functions of the four coordinates $x^\nu$.

Now these five functions do not describe any perturbation of the geometry and can be chosen at will. In particular they can be set to zero. Indeed if we perform a coordinate change which transforms the metric coefficients according to (2.4), and choose $d$ and $c_\mu$ such that $\Box_4 d = -\frac{1}{6} \partial_\mu D^\mu$ and $\partial_\mu c^\rho + \Box_4 c_\mu - 2\partial_\mu d = -\partial_\mu C - D_\mu$, then in the new, barred, coordinate system : $\tilde{\gamma} = \partial_\rho \tilde{\gamma}_\rho^\rho = 0$. Hence the particular, tranverse traceless system, such that (with bars dropped)

$$\gamma_{ww} = 0, \quad \gamma_{w\mu} = 0, \quad \gamma_\rho^\rho = 0, \quad \partial_\rho \gamma_\rho^\mu = 0 \quad (2.7)$$

solves the constraint equations and reduces the ten metric coefficients $\gamma_{\mu\nu}(X^A)$ to five, which represent the five degrees of freedom of AdS$_5$ gravitational waves/gravitons.

As for the fourth, evolution, equation (2.5) it reduces in the gauge (2.7) to

$$\Box_4 \gamma_{\mu\nu} + \partial_{ww} \gamma_{\mu\nu} - \frac{3}{w} \partial_w \gamma_{\mu\nu} = 0. \quad (2.8)$$

**Remark on gaussian normal gauges.** Another gauge is frequently used : a gaussian normal gauge which belongs to the class (2.3), so that its metric coefficients are related to the transverse traceless ones by (2.4), but which is adapted to the boundary $\Sigma$ of $V_5$, whose equation, in the coordinate system (2.7) is $w = L + \zeta(x^\mu)$. This gaussian normal coordinate system $X^{(G)A}$ is defined at linear order in $\zeta$ by $X^{(G)\mu} = x^\mu + \frac{w^2}{2L} \partial_\mu \zeta - \frac{1}{2} L \partial_\mu \zeta$, and $w^G = w(1 - \frac{\zeta}{L})$. In that gauge the metric coefficients are related to the transverse traceless ones (2.7) (2.8) by

$$\gamma^G_{\mu\nu} = \gamma_{\mu\nu} - \left( \frac{w^2}{L^2} - 1 \right) \partial_\mu \zeta - 2\eta_{\mu\nu} \frac{\zeta}{L}. \quad (2.9)$$
They can be obtained alternatively by choosing $D_\mu = \frac{6}{L^2} \partial_\mu \zeta$ and $C = \Box \zeta - \frac{8}{L^2} \zeta$, so that their evolution equation (2.5) is
\[
\Box_4 \gamma^G_{\mu
u} + \partial_w \gamma^G_{\mu
u} - \frac{3}{w} \partial_w \gamma^G_{\mu
u} = - \left( \frac{w^2}{L^2} - 1 \right) \partial_{\mu\nu} \Box_4 \zeta + \frac{4}{L^2} \partial_{\mu\nu} \zeta - 2 \frac{\eta_{\mu\nu}}{L^2} \Box_4 \zeta \tag{2.10}
\]
whose general solution is (2.9). An important feature of this gaussian normal gauge is that the metric coefficients diverge for large $w$. At linear order in $\zeta$ this gauge artefact is well under control, but becomes a nuisance when one treats more elaborate models, e.g. cosmological branes or black holes on branes, as the boundary condition on the metric coefficients is no longer that they must converge when $w \to \infty$. We shall therefore stick to the gauge (2.7).

- **$AdS_5$ vs bulk gravitons.** The general solution of (2.8) depends on the boundary conditions and, as usual, we impose that it converges (more precisely is $L_2$) on its domain of definition. If the spacetime we consider spans the whole $w$ axis then
\[
\gamma_{\mu\nu}(X^A) = \mathcal{R} e \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \frac{d m}{(2\pi)^{\frac{1}{2}}} e^{i k_\rho X^\rho} e_{\mu\nu} w^2 Z_2(m w) \quad \text{with} \quad \begin{cases} k_0 = -\sqrt{k^2 + m^2} \\ k^\rho e_{\rho\mu} = 0 \\ \eta^{\rho\sigma} e_{\rho\sigma} = 0 \end{cases} \tag{2.11}
\]
where the five polarisations $e_{\mu\nu}$ are a priori arbitrary functions of $k^i$ and $m$ and where $Z_2(m w) = H_2^{(1)}(m w) + a_m H_2^{(2)}(m w)$ is an a priori arbitrary linear combination of second order Hankel functions of first and second kinds. The coefficient $a_m$ is determined by the model at hand and one usually eliminates the mode coming from $+\infty$, that is sets $a_m = 0$. The “zero modes” are the particular, bounded, solutions of (2.8) which do not depend on $w$
\[
\gamma_{\mu\nu}(0) = \mathcal{R} e \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} e^{i (k_\rho X^\rho - k T)} e_{\mu\nu}(k^i) \quad \text{with} \quad \begin{cases} k e_{0\mu} + k^i e_{i\mu} = 0 \\ \eta^{\rho\sigma} e_{\rho\sigma} = 0 \end{cases} \tag{2.12}
\]
Now, if the spacetime we consider is $V_5$ (which is delineated by the hypersurface $\Sigma$ and $w \to +\infty$) then $w$ is bounded by $\Sigma$ and does not go to $-\infty$. Equation (2.8) then possesses extra $L_2$ modes which converge exponentially as $w \to +\infty$. The general solution of (2.8) in $V_5$ is therefore the sum of (2.11) and
\[
\gamma_{\mu\nu}(X^A) = \mathcal{R} e \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \int_{-k^2}^{0} \frac{dA}{(2\pi)^{\frac{1}{2}}} e^{i k_\rho X^\rho} e_{\mu\nu} w^2 H_2^{(1)}(i \sqrt{|A|} w) \quad \text{with} \quad \begin{cases} k_0 = -\sqrt{k^2 + A} \\ k^\rho e_{\rho\mu} = 0 \\ \eta^{\rho\sigma} e_{\rho\sigma} = 0 \end{cases} \tag{2.13}
\]
The static modes are the particular solutions which do not depend on time :
\[
\gamma_{\mu\nu}^{(s)}(x^i, w) = \mathcal{R} e \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} e^{i k_\rho x^\rho} e_{\mu\nu}(k^i) w^2 H_2^{(1)}(i k w) . \tag{2.14}
\]
III. The equations for gravity on a quasi-minkowskian brane

We consider in AdS\(_5\) the hypersurface \(\Sigma\) defined, in the coordinate system (2.7), by

\[ w = \mathcal{L} + \zeta(x^\mu) \quad (3.1) \]

where the function \(\zeta(x^\mu)\) is a priori arbitrary and describes the so-called "brane-bending" effect. The induced metric on \(\Sigma\) is

\[ ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \quad \text{with} \quad h_{\mu\nu} = \gamma_{\mu\nu}|\Sigma - 2 \frac{\zeta}{\mathcal{L}} \eta_{\mu\nu} \quad (3.2) \]

where \(\gamma_{\mu\nu}(x^\mu, w)\) is a solution of (2.7) (2.8) and where the index \(\Sigma\) means that the quantity is evaluated at \(w = \mathcal{L}\). (Alternatively this induced metric can be obtained using the gaussian normal system introduced in (2.9), in which the equation of \(\Sigma\) is \(w^G = \mathcal{L}\) and \(h_{\mu\nu} = \gamma^G_{\mu\nu}|\Sigma\).)

The Randall-Sundrum brane scenario is obtained by cutting AdS\(_5\) along \(\Sigma\), by making a copy of the \(w \geq \mathcal{L} + \zeta\) side and pasting it along \(\Sigma\). Imposing that the linearised Einstein equations be valid everywhere in this new "bulk" manifold, including its edge, or brane, yields the Israel junction conditions which give the stress-energy tensor of the matter in the brane \(\Sigma\) as

\[ \kappa T_{\mu\nu} = - \frac{6}{\mathcal{L}} \delta_{\mu\nu} + \kappa S_{\mu\nu} \quad \text{with} \quad \kappa \quad \text{a coupling constant} \quad (3.3) \]

Equations (3.2) and (3.3) together with (2.7) and (2.8) completely describe gravity in the brane. They have two useful consequences

\[ \partial_{\rho} S_{\nu}^{\rho} = 0 \quad , \quad \Box \zeta = - \frac{\kappa}{6} S. \quad (3.4) \]

Let us compare them to the standard 4D Einstein equations. In order to do so, we first perform a coordinate transformation \(x^\mu \rightarrow x^\mu = x^{*\mu} + \epsilon^\mu\). Then the new metric coefficients

\[ h^*_{\mu\nu} = h_{\mu\nu}|\Sigma - 2 \frac{\zeta}{\mathcal{L}} \eta_{\mu\nu} + \partial_{\mu} \epsilon_{\nu} + \partial_{\nu} \epsilon_{\mu} \quad \text{with} \quad \Box \epsilon_{\mu} = - \frac{2}{\mathcal{L}} \partial_{\mu} \zeta \quad (3.5) \]

satisfy the harmonicity condition

\[ \partial_{\mu} \left( h^*_{\nu} - \frac{1}{2} \delta_{\nu} h^* \right) = 0. \quad (3.6) \]

Taking the d’Alembertian of (3.5) and using (3.3) (2.7) and (2.8) we get the following consequence of the brane gravity equations

\[ \Box h^*_{\mu\nu} = -16\pi G \left( S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right) - (\partial_{ww} \gamma_{\mu\nu})|\Sigma + \frac{1}{\mathcal{L}} (\partial_w \gamma_{\mu\nu})|\Sigma. \quad (3.7) \]
where we have identified $\frac{\rho}{c^2} \equiv 8\pi G$, $G$ being Newton's constant.

Now, the standard linearized Einstein equations on a 4D Minkowski background read, in harmonic coordinates

$$\Box_i h^*_{\mu\nu} = -16\pi G \left( S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right).$$

(3.8)

They hold for any type of matter (compatible with the harmonicity condition, or, equivalently with the Bianchi identity $\partial_\mu S^\mu = 0$). By contrast, the linearized equations for gravity on a brane are (3.7), provided the source $S_{\mu\nu}$ satisfies the junction condition (3.3). This proviso may restrict the type of matter which is allowed on the brane: if, for example, only zero modes are allowed in the bulk, then (3.7) reduces to the Einstein linearized equations, but, since the last term in (3.3) is absent, the derivatives $\partial_\lambda (S_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} S) = 0$ by setting $\eta_{\mu\nu}$ must be symmetric in $\lambda$ and $\mu$, a property which is not satisfied by standard matter.

In order to dwell on that point, let us decompose the stress-energy tensor of matter in Fourier space into the traditional form: $S_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \hat{S}_{\mu\nu}$ with

$$\begin{cases}
\hat{S}_{00}(t, k^i) = \hat{\rho}, & \hat{S}_{0i}(t, k^i) = -ik_i \hat{v} - \hat{\omega}_i \\
\hat{S}_{ij}(t, k^i) = \delta_{ij} \left( \hat{P} + \frac{k^2}{3} \hat{\Pi} \right) - k_i k_j \hat{\Pi} + ik_i \hat{\Pi}_j + ik_j \hat{\Pi}_i + \hat{\Pi}_{ij}
\end{cases}$$

(3.9)

where $k_i \hat{v}^i = k_i \hat{\Pi}^i = k_i \hat{\Pi}^{ij} = \hat{\Pi}^i = 0$. We Fourier transform similarly $\zeta$ and $\gamma_{\mu\nu}$. The junction conditions (3.3) then read

$$\begin{cases}
\kappa \hat{\rho} = -2k^2 \hat{\zeta} - \partial_\omega \hat{\gamma}_{l|\Sigma}^l, & \kappa \hat{P} = \frac{4k^2}{3} \hat{\zeta} + 2 \hat{\omega}^l_{l|\Sigma} \partial_\omega \hat{\gamma}_{0|\Sigma}^l \\
\kappa \hat{\Pi} = \frac{2}{k^2} \partial_\omega \hat{\gamma}_{l|\Sigma}^{l} + \frac{3}{2k^4} k^l k^m \partial_\omega \hat{\gamma}_{lm|\Sigma}^0, & \kappa \hat{\omega} = -2 \hat{\omega}_l^l - \frac{4}{k^2} \partial_\omega \hat{\gamma}_{0|\Sigma}^l \\
\kappa \hat{\omega}_i = -k_i \hat{\gamma}_{0|\Sigma}^l + \partial_\omega \hat{\gamma}_{0|\Sigma}^l, & \kappa \hat{\Pi}_i = -ik_i \frac{k^l k^m}{k^4} \partial_\omega \hat{\gamma}_{lm|\Sigma}^0 + \hat{\gamma}_{il|\Sigma}^l \\
\kappa \hat{\Pi}_{ij} = \frac{1}{2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \partial_\omega \hat{\gamma}_{l|\Sigma}^{l} - \frac{1}{2} \left( \delta_{ij} + \frac{k_i k_j}{k^2} \right) \frac{k^l k^m}{k^2} \partial_\omega \hat{\gamma}_{lm|\Sigma}
\end{cases}$$

(3.10)

(which can be further simplified using $\hat{\gamma}_{l0} = \hat{\gamma}_{0l}^l$ and $-i \hat{\gamma}_{l0}^l = k_i \hat{\gamma}_{0l}^l$). Suppose now that only the zero modes are allowed in the bulk. Then, since $\partial_\omega \hat{\gamma}_{\mu\nu|\Sigma} = 0$ in that case, the matter on the brane is forced to obey the very contrived equation of state

$$\begin{align*}
\hat{\rho} &= -k^2 \hat{\Pi}, & \hat{v} &= -\hat{\Pi}, & \hat{P} &= \hat{\Pi} + \frac{2k^2}{3} \hat{\Pi}, & \hat{\omega}_i &= \hat{\Pi}_i = \hat{\Pi}_{ij} = 0.
\end{align*}$$

(3.11)

As for the only free function $\hat{\Pi}$ it is given in terms of the brane bending function $\hat{\zeta}$ by: $\kappa \hat{\Pi} = 2 \hat{\zeta}$. 





Now, of course, the junction conditions are better seen as boundary conditions on the allowed modes in the bulk. Indeed, if matter on the brane is known, then (3.3) or (3.10) can be inverted to give \( \hat{\zeta} \) and \( \partial_w \hat{\gamma}_{\mu\nu}\rceil_\Sigma \) in terms of \( \hat{S}_{\mu\nu} \). Now, from (2.11) and (2.13) we have

\[
\partial_w \hat{\gamma}_{\mu\nu}\rceil_\Sigma = \text{Re} \int \frac{dm}{\sqrt{2\pi}} e^{-i\sqrt{k^2+m^2}T} e_{\mu\nu} mL^2 Z_1(mL) \\
+ \text{Re} \int_{-k^2}^{0} \frac{dA}{\sqrt{2\pi}} e^{-i\sqrt{k^2+AT} e_{\mu\nu} i\sqrt{|A|L^2}} H_1^{(1)}(i\sqrt{|A|L})
\]

(3.12)

which, in principle, gives by inversion the polarisations \( e_{\mu\nu} \) in terms of the brane matter source, and hence the allowed bulk gravitons. Then the induced metric on the brane (3.2) is known in terms of the matter variables and can be compared with the usual 4D Einstein result. This programme however has only been completed in the particular case of a point static source.

**IV. The \( 1/r^2 \) correction to Newton’s law**

Let us concentrate on a static, point-like source

\[
S_{00} = M \delta(\vec{r}) \quad , \quad S_{0i} = S_{ij} = 0.
\]

(4.1)

The junction conditions (3.3) or (3.10) then give, using \( \hat{\delta} = \frac{1}{(2\pi)^{7/2}} \), \( \hat{r} = \sqrt{\frac{2}{\pi} \frac{1}{k^2}} \) and

\[
\partial^i \frac{1}{r} = -\sqrt{\frac{2}{\pi} \frac{k_i k_j}{k^2}} :
\]

\[
\zeta = -\frac{\kappa M}{24\pi r} \quad , \quad \partial_w \hat{\gamma}_{00}\rceil_\Sigma = -\frac{2\kappa M}{3} \delta(\vec{r})
\]

(4.2)

\[
\partial_w \hat{\gamma}_{0i}\rceil_\Sigma = 0 \quad , \quad \partial_w \hat{\gamma}_{ij}\rceil_\Sigma = -\frac{\kappa M}{3} \delta(\vec{r}) \delta_{ij} - \frac{\kappa M}{12\pi} \partial^i \frac{1}{r}.
\]

Now, the static bulk modes are given by (2.14) : \( \hat{\gamma}_{\mu\nu}^{(s)}(k^i, w) = e_{\mu\nu}(k^i) w^2 H_2^{(1)}(ikw) \) and their \( w \)-derivatives on \( \Sigma \) by (3.12) : \( \partial_w \hat{\gamma}_{\mu\nu}^{(s)}(k^i) = e_{\mu\nu}(k^i) ikL^2 H_1^{(1)}(ikL) \). Equation (4.2) therefore gives the polarisations in terms of the brane stress-energy tensor as

\[
\begin{align*}
\left\{ e_{00}(\vec{k}) H_1^{(1)}(ikL) = -\frac{2\kappa M}{3} \frac{1}{(2\pi)^{7/2}} \frac{1}{ikL^2} \quad , \quad e_{0i}(\vec{k}) H_1^{(1)}(ikL) = 0 \\
e_{ij}(\vec{k}) H_1^{(1)}(ikL) = -\frac{\kappa M}{3} \frac{1}{(2\pi)^{7/2}} \frac{1}{ikL^2} \left( \delta_{ij} - \frac{k_i k_j}{4\pi k^2} \right)
\end{align*}
\]

(4.3)

The bulk metric is then known as

\[
\gamma_{\mu\nu}(\vec{r}, w) = \text{Re} \int \frac{d^3k}{(2\pi)^{7/2}} e^{i\vec{k}.\vec{r}} \hat{\gamma}_{\mu\nu}(\vec{k}, w) \quad , \quad \hat{\gamma}_{\mu\nu}(\vec{k}, w) = \frac{\kappa M}{3L(2\pi)^{7/2}} w^2 \frac{K_2(kw)}{kL K_1(kL)} e_{\mu\nu}
\]

(4.4)

with \( c_{00} = 2 \), \( c_{0i} = 0 \) and \( c_{ij} = \delta_{ij} - k_i k_j/k^2 \) and where \( K_\nu(z) \) is the modified Bessel function defined as \( K_\nu(z) = i^{\nu} e^{iz} H_\nu^{(1)}(iz) \).
Let us now concentrate on the $h_{00}$ component of the metric on the brane (which is the same in the $x^\mu$ coordinates and the harmonic coordinates $x^\star^\mu$). With $\zeta$ given by (4.2) it reads

$$\hat{h}_{00}(\vec{k}) = \hat{h}_{00}^\star(\vec{k}) = \tilde{h}_{00}|_\Sigma + 2\frac{\zeta}{\mathcal{L}} = \frac{\kappa M}{k^2 \mathcal{L}(2\pi)^2} \left[ 1 - \frac{2kL K_0(kL)}{3K_1(kL)} \right].$$

(4.5)

Taking the Fourier transform and integrating over angles we obtain, setting $\alpha = r/L$ and recalling that $\frac{\kappa L}{r} = 8\pi G$

$$h_{00}(\vec{r}) = \frac{2GM}{r} \left( 1 + \frac{4\pi}{3} \mathcal{K}_\alpha \right) \quad \text{with} \quad \mathcal{K}_\alpha = \lim_{\epsilon \to 0} \int_0^{+\infty} du \sin(\alpha u) \frac{K_0(u)}{K_1(u)} e^{-\epsilon u}.$$  

(4.6)

It is a (fairly) straightforward exercise to see that $\lim_{\alpha \to 0} \mathcal{K}_\alpha = \alpha^{-1} = \mathcal{L}/r$ and that $\lim_{\alpha \to \infty} \mathcal{K}_\alpha = \pi/2\alpha^2 = \pi(\mathcal{L}/r)^2/2$. At short distances the correction to Newton’s law is in $\mathcal{L}/r$, whereas as distances large compared with the characteristic scale $\mathcal{L}$ of the anti-de Sitter bulk the correction is reduced by another $\mathcal{L}/r$ factor

$$\lim_{r/\mathcal{L} \to \infty} h_{00}(\vec{r}) = \frac{2GM}{r} \left[ 1 + \frac{2}{3} \left( \frac{\mathcal{L}}{r} \right)^2 \right].$$

(4.7)

V. Cosmological branes and their perturbations

Let us now consider in (unperturbed) AdS$_5$ spacetime with line element $ds^2|_5 = \left( \frac{\mathcal{L}}{w} \right)^2 \eta_{AB} dX^A dX^B$ the hypersurface $\Sigma$ defined by

$$w = \frac{\mathcal{L}}{a(\eta)} \quad , \quad T = \int d\eta \sqrt{1 + \mathcal{L}^2 H^2}$$

(5.1)

with $H \equiv \dot{a}/a$, a dot denoting differentiation with respect to $t$ such that $dt = a(\eta)d\eta$. $\Sigma$ is then a spatially flat Robertson-Walker 4D spacetime with scale factor $a(\eta)$ and line element ($x^0 = \eta$)

$$ds^2 = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu.$$  

(5.2)

One now cuts AdS$_5$ along $\Sigma$, keeps the part between $\Sigma$ and $w \to +\infty$, copies it and pastes it along $\Sigma$. Imposing the Einstein equations to be satisfied everywhere in this bulk, including the brane $\Sigma$, yields the Israel junction conditions

$$\kappa \left( S^\mu_\nu - \frac{1}{3} \delta^\mu_\nu S \right) = 2K^\mu_\nu$$

(5.3)

where $S_{\mu\nu}$ is the stress-energy tensor of matter on the brane and $K_{\mu\nu}$ the extrinsic curvature of $\Sigma$ in AdS$_5$. For $\Sigma$ defined by (5.1) they read, setting $S^0_0 = - \left( \frac{6}{\kappa \mathcal{L}} + \rho \right)$ and $S^i_j = \delta^i_j \left( - \frac{6}{\kappa \mathcal{L}} + \rho \right)$:

$$\kappa \rho = \frac{6}{\mathcal{L}} \left( \sqrt{1 + \mathcal{L}^2 H^2} - 1 \right) \quad , \quad \dot{\rho} + 3H(\rho + P) = 0.$$  

(5.4)
At late times the first one becomes $8\pi G \rho \to 3H^2$ (with the identification $\hat{\kappa} = 8\pi G$) and the equations for gravity on the brane become the standard Friedmann equations. If matter on the brane is imposed to be a scalar field $\Phi(t)$ with potential $V(\Phi)$ and tension $\frac{6}{L}$, the junction equations (5.3) read

$$\kappa \left( \frac{\dot{\Phi}^2}{2} + V \right) = \frac{6}{L} \left( \sqrt{1 + L^2 H^2} - 1 \right) \ , \quad \ddot{\Phi} + 3H \dot{\Phi} + \frac{dV}{d\Phi} = 0 \ .$$

(5.5)

We now allow for gravitons in the bulk and perturb the position of $\Sigma$. The equations for $\Sigma$ are then taken to be

$$w = \frac{L}{a} + \frac{\zeta}{a} \sqrt{1 + L^2 H^2} \ , \quad T = \int \eta \sqrt{1 + L^2 H^2} - \frac{\zeta}{a} L H \ .$$

(5.6)

The line element on $\Sigma$ becomes

$$ds^2 = a^2(\eta) (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu dx^\nu \ .$$

(5.7)

with

$$\begin{cases}
    h_{\eta\eta} = (1 + L^2 H^2)^00|_\Sigma + \frac{2\zeta L}{\sqrt{1 + L^2 H^2}} \left( \frac{1}{L^2} + \frac{\ddot{a}}{a} \right) \\
    h_{\eta}^i = \sqrt{1 + L^2 H^2} h_{\eta 0}^i|_\Sigma \ , \quad h_{\eta}^i = \gamma_{0i}^i|_\Sigma - \frac{2\zeta}{L} \sqrt{1 + L^2 H^2} \delta_i^j .
\end{cases}$$

(5.8)

where $\gamma_{\mu\nu}|_\Sigma$ is given by (2.11) and (2.13) with $w$ and $T$ given by (5.1). Equations (5.7-8) generalize (3.2) to a Robertson-Walker brane and reduce to it when $a = 1$.

As for the spatial part of the junction conditions (5.3) it becomes, in the particular case when matter on the brane is imposed to be the perturbed scalar field $\Phi(t) + \chi(\eta, x^i)$ ($\Phi(t)$ solving (5.5))

$$\begin{cases}
    \frac{\kappa}{2} \delta \left( \tau^i_j - \frac{1}{3} \delta^i_j \tau \right) = \frac{1}{a^2} \partial^j \xi + H \delta^j_i \left( H \xi - \dot{\xi} \right) + \frac{1}{1 + L^2 H^2} \left( \frac{1}{L^2} + \frac{\ddot{a}}{a} \right) \\
    \frac{L}{2a} \left[ H(\partial_0 \gamma_j^i)|_\Sigma - \frac{1}{L^2} \sqrt{1 + L^2 H^2} (\partial_w \gamma_j^i)|_\Sigma - H(\partial_j \gamma_0^i + \partial^i \gamma_{0j})|_\Sigma \right]
\end{cases}$$

(5.9)

with

$$\frac{\kappa}{2} \delta \left( \tau^i_j - \frac{1}{3} \delta^i_j \tau \right) = \frac{\kappa}{6} \delta^i_j \left[ \ddot{\Phi} \chi + \frac{dV}{d\Phi} + \frac{\dot{\Phi}^2}{2} \right]$$

(5.9)

where on the right-hand side spatial indices are raised with $\delta^{ij}$. Equation (5.9) is the generalisation of the spatial part of (3.3) to a cosmological brane and reduces to it when $a = 1$. As for the $(00)$ and $(0i)$ parts of the junction conditions they can be replaced by the Klein-Gordon equation for the scalar field, which reads

$$\ddot{\chi} - \frac{1}{a^2} \Delta \chi + 3H \dot{\chi} + \frac{d^2 V}{d\Phi^2} \chi + (\dot{\Phi} + 3H \Phi) h_{\eta\eta} - \frac{1}{a} \dot{\Phi} \partial_i h^i_{\eta} + \frac{\dot{\Phi}}{2} (\dot{h}_{\eta\eta} + h^i_{\eta}) = 0 .$$

(5.10)
Equations (5.7-10) completely describe the perturbations of a cosmological brane when matter is imposed to reduce to a scalar field.

At that stage, one could put them in a form akin to (3.7) and compare them to the usual equations for cosmological perturbations in 4D Einstein gravity. However, as we have already seen such equations would be only a consequence of the junction conditions, and not equivalent to them. It is therefore better to stick to (5.7-10). The junction conditions (5.9-10) must be seen as boundary conditions giving the polarisations of the bulk gravitons as well as $\zeta$ in terms of $\chi$. The induced metric (5.7-8) is then in principle also known in terms of $\chi$. Such a programme has however not yet been carried out explicitly (recall that in the much simpler case of a quasi-minkowskian brane it has been carried out only when matter on the brane is a static point-like source, see Section IV).

One can nevertheless get further insight into them by introducing the spatial tensor

$$F_i^j \equiv \frac{1}{a^2} \partial_j \zeta + \frac{L}{2a} \left[ H(\partial_0 \gamma_i^j)|_\Sigma - \frac{1}{L^2} \sqrt{1 + L^2 H^2} (\partial_\omega \gamma_i^j)|_\Sigma - H(\partial_j \gamma_i^0 + \partial^i \gamma_0 j)|_\Sigma \right]$$

(5.11)

and casting (5.9) into a traceless and trace part:

$$\begin{cases}
F_i^j = \frac{1}{3} \delta_i^j F \\
F = \frac{\kappa}{2} \left[ \dot{\Phi} X + \chi \frac{dV}{d\Phi} + \frac{\dot{\Phi}^2}{2} h_{\eta\eta} \right] - 3H(H\zeta - \dot{\zeta}) - \frac{3}{2} H^2 L \sqrt{1 + L^2 H^2} \gamma_{00}|_\Sigma.
\end{cases}$$

(5.12)

In doing so, the following, partial but explicit, result can be obtained: suppose the only modes allowed in the bulk are the zero modes (2.12) and assume (without loss of generality) that $k_1 = k_2 = 0$, $k_3 \equiv k$. The transverse and traceless properties of $e_{\mu\nu}$ then imply that the five possible polarisations are characterised by $e_{11}$, $e_{12}$, $e_{13}$, $e_{23}$ and $e_{33}$, the other components being $e_{0i} = -e_{i3}$, $e_{00} = e_{33}$, and $e_{22} = -e_{11}$.

The junction conditions (5.11-12) then tell us first that $e_{13}$ and $e_{23}$ remain free and correspond to 4D gravitational waves freely propagating in the brane; they also tell us that $e_{12} = e_{11} = e_{22} = 0$, so that only $e_{33} = -e_{03} = e_{00} \equiv e(k)$ can couple to the brane scalar field; finally they give

$$\zeta(\eta, z) = \Re e \frac{iHaL}{2} \int \frac{dk}{\sqrt{2\pi}} e^{ik(z-T(\eta))} \frac{e(k)}{k}$$

(5.13)

with $T(\eta) = \int d\eta \sqrt{1 + L^2 H^2}$. As for the Klein-Gordon equation (5.10) it gives

$$\chi(\eta, z) = -\Re e \frac{i\Phi a}{2} \sqrt{1 + L^2 H^2} \int \frac{dk}{\sqrt{2\pi}} e^{ik(z-T(\eta))} \frac{e(k)}{k}.$$ 

(5.14)

The metric on the brane is then given by (5.7-8), with $\zeta$ given above and

$$\gamma_{33}|_\Sigma(\eta, z) = -\gamma_{03}|_\Sigma(\eta, z) = \gamma_{00}|_\Sigma(\eta, z) = \Re e \int \frac{dk}{\sqrt{2\pi}} e^{ik(z-T(\eta))} e(k).$$

(5.15)
Hence the perturbations of this particular cosmological brane are completely known in terms of $e(k)$. Before eventually comparing them with the standard 4D perturbations of chaotic inflationary models, one must first decide on the $k$ dependence of $e(k)$. One could try and argue that what matters is matter on the brane and, hence, that one should impose the field $\chi$ to be in its vacuum state which would amount, in practice, to choosing $\frac{e(k)}{k} \propto \frac{1}{\sqrt{2k}}$. One could on another hand argue that gravitons in the bulk should be in their vacuum state and impose $e(k) \propto \frac{1}{\sqrt{2k}}$. The right answer to this question implies a proper, non trivial, and yet to be done quantisation of the only action we dispose of, that is $\left( \int_{\text{bulk}} \sqrt{-g_5} (R + \Lambda) d^5X + \kappa \int_{\text{brane}} \sqrt{-g_4} \mathcal{L}_m d^4x \right)$.

Acknowledgements

I gratefully thank Tomas Dolezel and Joseph Katz with whom most of the results summarized here have been first obtained as well as Cedric Deffayet and Gilles Esposito-Farese for illuminating discussions.

References

[1] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690
[2] J. Garriga, T. Tanaka, Phys. Rev. Letters 84 (2000) 2778; C. Csaki, J. Erlich, T.J. Hollowood, Y. Shirman, Nucl. Phys. B581 (2000) 309; S.B. Giddings, E. Katz, L. Randall, JHEP 0003 (2000) 023; C. Csaki, J. Erlich, T.J. Hollowood, Phys. Rev. Lett. 84 (2000) 5932; C. Csaki, J. Erlich, T.J. Hollowood, Phys. Lett. B481 (2000) 107; I.Ya. Aref’eva, M.G. Ivanov, W. Muck, K.S. Viswanathan, I.V. Volovich, Nucl. Phys. B590 (2000) 273; Z. Kakushadze, Phys. Lett. B497 (2000) 125
[3] P. Binetruy, C. Deffayet, U. Ellwanger, D. Langlois, Phys. Lett. B477, 285 (2000); P. Kraus, JHEP 9912 (1999) 011, S. Mukoyama, Phys. Lett. B473 (2000) 241; D.N. Vollick, C.Q.G. 18 (2001) 1; D. Ida, JHEP 0009 (2000) 014; S. Mukohyama, T. Shiromizu, K. Maeda, Phys. Rev. D62 (2000) 024028; R. Maartens, Phys. Rev. D62 (2000) 084023; P. Kanti, K.A. Olive, M. Pospelov, Phys. Lett. B468 (1999) 31
[4] see N. Deruelle, “Cosmological perturbations of an expanding brane in an anti-de Sitter bulk”, Contribution to the Proceedings of the Porto JENAM conference, and references therein.
[5] N. Deruelle, T. Dolezel, “Grance versus shell cosmologies in Einstein and Einstein Gauss-Bonnet theories”, gr-qc/0004021; N. Deruelle, T. Dolezel, J. Katz, “Perturbations of brane world”, hep-th/0010213; N. Deruelle, J. Katz, “Gravity on branes”, gr-qc/0104007; N. Deruelle, T. Dolezel, “On linearised gravity in the Randall-Sundrum scenario”, gr-qc/0105118; N. Deruelle, “Stars on branes”, gr-qc/0111063